Signatures of Chaos in Non-integrable Models of Quantum Field Theory

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We study signatures of quantum chaos in (1+1)D Quantum Field Theory (QFT) models. Our analysis is based on the method of Hamiltonian truncation, a numerical approach for the construction of low-energy spectra and eigenstates of QFTs that can be considered as perturbations of exactly solvable models. We focus on the double sine-Gordon model, also studying the massive sine-Gordon and \( \phi^4 \) model, all of which are non-integrable and can be studied by this method with sufficiently high precision from small to intermediate perturbation strength. We analyse the statistics of level spacings and of eigenvector components, both of which are expected to follow Random Matrix Theory predictions. While level spacing statistics are close to the Gaussian Orthogonal Ensemble as expected, on the contrary, the eigenvector components follow a distribution markedly different from the expected Gaussian. Unlike in the typical quantum chaos scenario, the transition of level spacing statistics to chaotic behaviour takes place already in the perturbative regime. On the other hand, the distribution of eigenvector components does not appear to change or approach Gaussian behaviour, even for relatively large perturbations. Moreover, our results suggest that these features are independent of the choice of model and basis.

Introduction. The statistical physics of non-integrable quantum systems has been successfully described by quantum chaos theory, which states that the statistics of their energy spectra and eigenvectors are described by Random Matrix Theory (RMT) i.e. they exhibit the same behaviour as matrices whose elements are chosen randomly from a Gaussian distribution. These conjectures [1–3] have been verified for a broad class of single-particle models, where they have been explained in terms of semi-classical periodic orbit theory [4, 5], while more recently research focus has been shifted to many-body systems [6–16], where RMT predictions have been verified numerically and in certain cases even analytically [17, 18]. Chaoticity tests in quantum many-body models are, however, almost exclusively limited to discrete (lattice) models, leaving continuous models unexplored. Among them, relativistic Quantum Field Theories (QFTs) and their dynamics lie at the cornerstone of important open questions of theoretical physics, like the black hole information paradox [19], making the study of ergodicity and chaos in QFT a topic of fundamental interest.

Significant recent progress has been made in this subject based on new theoretical concepts and indicators [20–22]. Nevertheless the emergence of quantum chaos in QFT remains much less understood by means of the more traditional measures of level spacing and eigenvector component statistics [23]. The study of level spacing statistics is the best way of detecting level repulsion, the characteristic property of random matrix spectra. On the other hand, the Gaussianity of the distribution of eigenvector components is important as an indication of validity of the Eigenstate Thermalisation Hypothesis (ETH) [24, 25], which explains how thermalisation emerges from the unitary dynamics of non-integrable quantum systems [26].

The main obstacle in performing chaoticity tests in QFT is that, unlike for lattice models of condensed matter physics, QFT models are continuous and thus live in an infinite dimensional Hilbert space even if finite size systems are considered. Therefore exact computation of energy spectra is not an option, at least for non-integrable models, and we inevitably resort to approximate numerical methods. The challenge is then to achieve sufficiently high accuracy in a sufficiently large part of the spectra, so that a statistical analysis is possible and reliable. An ideal method for this task is the “Truncated Conformal Space Approach” (TCSA) [29–31], more generally the Hamiltonian truncation methods [32, 33]. The TCSA is a spectral method based

![Distribution of the absolute values of eigenvector components](image-url)

FIG. 1. Statistics of level spacings and eigenvector components in the double sine-Gordon (non-integrable) and sine-Gordon (integrable) model. (a) Change in the \( r \) distribution of DSG from \((\beta_1, \beta_2) = (2.5, 2.5)\) (integrable SG point) to \((\beta_1, \beta_2) = (1.0, 2.5)\) (non-integrable point) compared to the predictions for integrable models (dashed blue curve) and to the RMT predictions (red curve), respectively. Inset shows change in the average of \( r \) when varying \( \beta_1 \). (b) Distribution of the absolute values of eigenvector components \(|c_{ij}|\) for the same two points, in log scale and log-log scale (inset). The GOE distribution of \(|c_{ij}|\) is the Gaussian [23, 27], while it is expected to be algebraic for integrable models [28]. Instead we observe that although the spectral statistics of DSG is very close to the GOE, the statistics of \(|c_{ij}|\) is not.
on the algebraic toolkit of Conformal Field Theory (CFT) and insights from Renormalisation Group theory, which can capture efficiently non-perturbative effects in the low-energy spectrum and is especially suitable for (1+1)D QFTs but also promising for extensions to higher dimensions [32, 34]. A pioneering study of quantum chaos indicators using this method has been presented in [35] for the tricritical and tetracritical Ising field theories. That work focused on level spacing statistics, demonstrating that it follows the expected Poissonian and Gaussian Orthogonal Ensemble (GOE) behaviour in, respectively, integrable and non-integrable perturbations of minimal CFTs, and observing the crossover between them for varying parameters.

In this Letter we study two independent and equally important signatures of quantum chaos, the distribution of level spacings as quantified by the consecutive level spacing ratios \( \tilde{r} \) [36, 37] and the distribution of eigenvector components. We study a class of (1+1)D models: the sine-Gordon model (SG), which is integrable, and the double sine-Gordon (DSG), massive sine-Gordon (MSG) a.k.a. massive Schwinger-Thirring model and \( \phi^4 \) model, which are all non-integrable. We first verify that level spacings follow the Poisson distribution for SG, and the GOE distribution for DSG and MSG to a very good approximation. Surprisingly, GOE behaviour is observed already in the perturbative regime, in contrast to what typically happens in single-particle models. Turning our attention to the statistics of eigenvector components, we find that, even when the level spacing distribution is close to GOE, the eigenvector distribution is markedly different from Gaussian (i.e., Porter–Thomas distribution of intensities [27]) found in RMT (Fig. 1). On the contrary, it exhibits at best exponential scaling followed by an algebraically decaying tail at large values, that contradicts the RMT prediction. Moreover, this last feature turns out to be robust, to a large extent independent of the model under study and parameter values. Given that the TCSA is an approximate method due to the truncation of the infinite dimensional Hilbert space, it is essential to reliably estimate the truncation error. This is especially important for distinguishing physical behaviour from numerical artefacts. To validate our observations we push the limits of TCSA’s potential and employ a reliable measure of truncation errors that allows us to quantify the accuracy of our data.

Models, Method & Observables.—We consider the following models: the SG with Hamiltonian \( H_{SG} = H_0 + \lambda V_\beta \), the DSG \( H_{DSG} = H_0 + \lambda_1 V_\beta_1 + \lambda_2 V_\beta_2 \), the MSG \( H_{MSG} = H_0 + \lambda V_\beta + m^2 U_2 \), and the \( \phi^4 \) model \( H_{\phi^4} = H_0 + m^2 U_2 + \lambda U_4 \), where

\[
\begin{align*}
H_0 &= \frac{1}{2} \int [\dot{\phi}^2 - (\partial_x \phi)^2] \, dx, \\
U_n &= \frac{m^2}{n^2} \int \phi^2 \, dx, \\
V_\beta &= -\int \cos \beta \phi \, dx.
\end{align*}
\]

The SG is a prototypical example of integrable QFT possessing topological excitations (solitons) [31, 38–40] and is equivalent to the massive Thirring model, a model of interacting fermions [41, 42]. It has applications in condensed matter and atomic physics [43] and is of strong experimental interest [44, 45]. The DSG is the double frequency version of SG, which is non-integrable and also topologically nontrivial [46–49]. Lastly, the MSG, also non-integrable, is equivalent to the massive Schwinger-Thirring model, a theory of fermions interacting with each other and with an abelian gauge field, reducing to (1+1)D Quantum ElectroDynamics at \( \beta = \sqrt{4\pi} \) [50, 51].

All the above models can be seen as perturbations of the massless free boson CFT \( H_0 \) by some relevant operators \( V \) and as such they can be studied using the TCSA. This method yields numerical approximations of the low-energy spectrum of \( H = H_0 + \lambda V \), defined in a finite box of length \( L \), based on a simple idea of computing the matrix elements of \( V \) in an energy-truncated basis \( \{ |\Phi_n^0 \rangle : E_n^0 \leq E_{cut} \} \) of \( H_0 \) and numerically diagonalising the resulting finite matrix approximation of \( H \). In general, such an approach would not be guaranteed to converge to the exact spectrum upon increasing the truncation cutoff \( E_{cut} \). However, it is expected to do so for any \( V \) that does not couple significantly the low and high energy levels of \( H_0 \), which is precisely the case for relevant perturbations. The TCSA has already been successfully applied to the SG [31, 39, 40, 52, 53] and DSG [47, 49], while a similar Hamiltonian truncation method using the massive Klein-Gordon model \( H_{KG} = H_0 + m^2 U_2 \) to construct the truncation basis has been used for the study of the \( \phi^4 \) model [32, 54–57].

Using the TCSA we compute a low-energy part of the spectra \( E_n \) and eigenvectors \( |\Phi_n \rangle \) of these models for various parameter values and analyse their statistics. More specifically, we compute the distribution of level spacings \( s_n = E_{n+1} - E_n \), the distribution of consecutive level ratios \( r_n \) and \( \tilde{r}_n \) defined as [36]

\[
r_n = s_n/s_{n-1}, \quad \tilde{r}_n = \min(r_n, 1/r_n)
\]

and the distribution of eigenvector components \( c_{ij} = \langle \Phi_i^0 | \Phi_j \rangle \) in the TCSA basis. Since the models we study are time-reversal symmetric, the corresponding RMT ensemble is the GOE where the \( r \)-distribution is

\[
P_{GOE}(r) \propto \frac{r + r^2}{(1 + r + r^2)^{5/2}}
\]

and that of \( \tilde{r} \) is the restriction of the above to the interval \([0, 1]\), with mean value \( \langle \tilde{r} \rangle_{GOE} \approx 0.536 \) [37]. For integrable models, on the other hand, level spacings follow the Poisson distribution [58] with \( \langle \tilde{r} \rangle_P \approx 0.386 \). Compared to other tests of level spacing statistics, \( \tilde{r} \) has the advantage that it is independent of the local energy level density, therefore no ‘unfolding’ [23] of the spectra is needed for its computation. For the eigenvector
components, the RMT prediction is that they exhibit a Gaussian distribution, which results in the known Porter–Thomas distribution for the absolute values of eigenvector components [27].

To minimise truncation errors we use bases much larger than in previous studies (∼85000 states at the highest cutoff). Moreover, to ensure that our results are sufficiently accurate we verify their convergence using rather strict truncation error estimates, based on the convergence of not only the computed observables but also the energy spectra and eigenvectors themselves, as estimated by measures of their correlation at successive cutoffs (see figure captions and Supp. Mat. [59] for details).

![Density plot of $\langle \tilde{r} \rangle$ in the DSM model](image)

**FIG. 3.** Dependence of $\langle \tilde{r} \rangle$ on the perturbation strength parameters of the DSM model. (a) Plot of $\langle \tilde{r} \rangle$ for $\lambda_1, \lambda_2 \rightarrow 0$ as a function of the mixing ratio $x = \lambda_1/(\lambda_1 + \lambda_2)$. (b) Density plot of $\langle \tilde{r} \rangle$ as a function of $\lambda_1$ and $\lambda_2$. The integrable SG lines correspond to $\lambda_1 = 0$ and $\lambda_2 = 1$. Note that $\langle \tilde{r} \rangle$ is close to $\langle \tilde{r} \rangle_{\text{GOE}}$ even in the immediate vicinity of the unperturbed model (free boson CFT). The DSG parameters are $\langle \beta_1, \beta_2 \rangle = (1, 2.5)$ and the energy window is the same as in Fig. 2.

![Density plot of $\langle \tilde{r} \rangle$ in the DSM model](image)

**FIG. 2.** Density plot of $\langle \tilde{r} \rangle$ in the DSM model, as a function of the frequency parameters $\beta_1$ and $\beta_2$. The model is non-integrable with the exception of the three lines $\beta_1 = 0, \beta_2 = 0$ and $\beta_1 = \beta_2$, where it reduces to the SG model. The values of $\langle \tilde{r} \rangle$ are indeed close to $\langle \tilde{r} \rangle_P$ (dark blue) along the SG lines and in their vicinity, while they approach the chaotic value $\langle \tilde{r} \rangle_{\text{GOE}}$ (dark red) away from them. The point $\langle \beta_1, \beta_2 \rangle = (1.0, 2.5)$ analysed in detail in other plots belongs to the region where $\langle \tilde{r} \rangle$ is very close to the GOE value. The perturbation strength parameters are $l_1 = l_2 = 1$, and the energy window over which $\langle \tilde{r} \rangle$ is computed corresponds to the levels 1000–3000. Top: Plots for different truncation cutoffs are included for comparison. The cutoff $E_{\text{cut}}$ is reported at the lower right corner of each plot (in units of $\varepsilon = \pi/L$, which is the energy difference between the ground and first excited state of $H_0$).

**Level spacing statistics.—**We start by analysing the statistics of $r$ values. Fig. 1.a shows the distribution $P(r)$ for the DSG at two different choices of parameter values, one integrable $\langle \beta_1, \beta_2 \rangle = (2.5, 2.5)$ (SG) and one non-integrable $\langle \beta_1, \beta_2 \rangle = (1.0, 2.5)$. The parameters $\lambda_1, \lambda_2$ have been chosen so that the energy gap between the ground and first excited state is of order one in units of the inverse system size $L^{-1}$ (more precisely, $l_1 = l_2 = 1$ where $l_i = m_2/L$ and $m_2$ is the SG breather mass [59]). These values are within the perturbative regime where convergence is optimal. We observe that the two distributions agree quite well with the Poisson and GOE statistics, respectively. The change of statistics from Poisson to GOE can be demonstrated by the mean value $\langle \tilde{r} \rangle$ for varying $\beta_1$ and fixed $\beta_2$ (Fig. 1.a, inset). Starting from the Poisson value for $\beta_1 = 0$, $\langle \tilde{r} \rangle$ increases towards the GOE value, fluctuating close and below that. The complete dependence of $\langle \tilde{r} \rangle$ on both parameters $\beta_1, \beta_2$ is shown in Fig. 2 in the form of a “phase diagram”. We observe that $\langle \tilde{r} \rangle$ is close to the Poisson value along the lines where either of the two parameters is zero or when they are equal, in which cases the DSG reduces indeed to the integrable SG. In the regions between these lines, $\langle \tilde{r} \rangle$ approaches the GOE value with the difference decreasing as we move away from the SG lines.

By independently varying the perturbation strength parameters $\lambda_1, \lambda_2$ at fixed $\langle \beta_1, \beta_2 \rangle = (1, 2.5)$ (Fig. 3) we observe that $\langle \tilde{r} \rangle$ is close to $\langle \tilde{r} \rangle_{\text{GOE}}$ even in the immediate vicinity of the unperturbed model (free boson CFT), that is for $\lambda_1, \lambda_2 \rightarrow 0$, as long as the ratio $\lambda_1/\lambda_2$ is kept fixed within a window of values around one. This is somewhat surprising given that chaotic behaviour typically emerges far from integrable points and outside the perturbative regime. The chaotic $\langle \tilde{r} \rangle$ values can indeed be verified from first-order perturbation theory results (Fig. 3.a) [59]. The fluctuations of $\langle \tilde{r} \rangle$ can be partially attributed to the relatively small size of the energy window (2000 levels), since even matrices that are random by construction and of the same size display similar fluctuations. Nevertheless, it can be noticed that $\langle \tilde{r} \rangle$ values are predominantly below the theoretical value.
The values of $\langle \tilde{r} \rangle$ obtained at different truncation cutoffs are presented in Fig. 2, showing that the differences are negligible, with the best convergence achieved for small $\beta_1$ and $\beta_2$. However, even if $\langle \tilde{r} \rangle$ converges at some cutoff to the GOE value, this does not necessarily mean that this is the correct physical value, since a non-convergent spectrum is also likely to look RMT-like. For this reason, we check the convergence using an error estimate based on the average of the absolute differences of $\tilde{r}$ values between successive cutoffs and verifying that the error decreases with increasing cutoff [59]. We empirically find that increasing $\beta_1$ or the perturbation strength $l_i$ results in larger truncation errors, making the numerical data less reliable. Moreover, in TCSA convergence is achieved in the lowest part of the numerical spectra, with the truncation effects progressively increasing as we move to higher levels. Since $r$ values are computed from differences of consecutive levels, they are much more sensitive to truncation errors than the levels themselves. For the parameters of Fig. 1, a sufficiently good level of convergence of $r$ values is achieved for the lowest ~3000 levels at $E_{\text{cut}} = 42\epsilon$.

Eigenvector statistics.—Let us focus on the statistics of eigenvector components of the DSG model. Fig. 1.b shows the distribution of their absolute values in log scale for the same choice of parameter values as in Fig. 1.a, one exhibiting Poisson and the other GOE level spacing statistics. Despite the clear difference in level spacing statistics, the eigenvector distributions in the two cases are practically the same and strongly different from the Gaussian prediction of RMT. In the bulk of the distribution the scaling is at best exponential instead of Gaussian, while the tails decay slower than exponentially, matching instead with an algebraic function. This is in strong contrast with theoretical expectations for chaotic models [27]. To eliminate truncation effects, we have again restricted the analysis to the convergent lowest energy part of the eigenvector matrix $c_{ij}$.

To gain a deeper insight into this observation, we look more closely into the structure of the eigenvector matrix $c_{ij}$. Fig. 4.a–e shows $c_{ij}$ for the DSG model at increasing values of $l = l_1 = l_2$, which controls the perturbation strength. We observe that for small $l$, the eigenvector matrix $c_{ij}$ is characterised by an approximately block-diagonal form. For increasing $l$ this block structure fades away, and $c_{ij}$ becomes more uniform, even though a pattern of fine structure is always visible.

The block structure property is easily explained by perturbation theory [59]. The CFT spectrum is organised in degenerate energy shells with the $N^{th}$ shell having energy $E_N \propto N/L$ above the ground state and degeneracy equal to the number of integer partitions of $N$. The perturbation $V$ raises the degeneracy of the unperturbed
Hamiltonian $H_0$ at first order in $\lambda$, so that for $\lambda \to 0$ the perturbed eigenvectors $|\Phi_n\rangle$ of $H_0 + \lambda V$ in each shell are equal, not to $|\Phi_n^0\rangle$, but to the eigenvectors of the matrix $\langle \Phi_0^0 | V | \Phi_n^0 \rangle$, which are linear combinations of generally all degenerate eigenvectors of the same shell. This results in the observed characteristic structure of $c_{ij}$ at small $\lambda \sim l$. For larger $\lambda$ the perturbed levels move far from their original shells and start mixing with those of other shells. At the same time, the block structure of the eigenvectors (i.e. the fingerprint of CFT) gradually disappears.

![Graph](image)

**FIG. 5.** (a) The distribution of $|c_{ij}|$ in log scale and log-log scale (inset) for integrable SG at different values of the frequency parameter $\beta$ from 0 (red) to 2.5 (blue). The exponential distribution in the bulk is therefore not a special property of the non-integrable DSG model, but is also present in the SG model which exhibits integrable spectral statistics. (b) Comparison of the distributions of $|c_{ij}|$ in the DSG (red), MSG (green) and $\phi^4$ (grey) models, always in energy windows with GOE spectral statistics. Even though the $\phi^4$ model is expanded in a different and much less special basis (KG), the distribution is still not GOE-like but clearly algebraic instead. The model parameters and corresponding $\langle r \rangle$ values are: DSG: $\beta_1 = 1.0$, $\beta_2 = 2.5$, $l = 1$, MSG: $\beta = 2.8$, $m = 0.76$, $l = 1$ ($\langle r \rangle = 0.519$), $\phi^4$: $\lambda = 1.0$, $m = 1$, $L = 7$ ($\langle r \rangle = 0.5$).

Based on these observations, we analyse how the eigenvector distribution of the DSG model depends on $l$, whether it approaches the RMT prediction as we move from weak to strong perturbation, and how it changes from shell to shell. First, we find that the distributions inside a single shell exhibit clearly exponential scaling in the bulk (Fig. 4.g), still with slower decaying tails. The slope of this exponential changes from one shell to the other. This means that the distribution shown in Fig. 1, which corresponds to a large window including many shells, is actually a superposition of many exponential distributions. From the analysis of the $l$ dependence (Fig. 4.h) we find that there is no significant change when we increase $l$ from zero to the maximum value for which our data is reliable ($l \approx 5$). In fact the distribution remains the same even for larger values of $l$, where truncation effects are non-negligible and would be expected to result in increased albeit artificial randomness. At the same time, the values of $\langle r \rangle$ in the same shell are close to the GOE value for any system size $l \gtrsim 1$ and are well convergent at least for $l \lesssim 3$ (Fig. 4.f).

Comparing the eigenvector component statistics of different models focusing on energy windows corresponding to $\langle r \rangle$ close to the GOE value, we find that they generally vary from model to model but are always different from Gaussian and at best exponential (Fig. 5.b). The $\phi^4$ model, in particular, deserves special attention. In this case, using the CFT as the unperturbed model to construct the truncation basis is not a convenient choice, and one should use instead the massive Klein-Gordon (KG) model $H_{KG} = H_0 + m^2 U_2$. In contrast to the CFT basis, in the KG basis there are no extensive degeneracies or shell structure. Nevertheless, the eigenvector distribution is once again very different from Gaussian and characterised by slowly decaying tails, as those of the DSG in Fig. 1. Lastly, comparing the single-shell eigenvector distributions for the SG model at different $\beta$ (Fig. 5.a), we find that they are similar to those of the DSG. The above results clearly show that the discrepancy between the eigenvector distributions and the RMT prediction, in particular the presence of slowly decaying tails, is robust under variations of the parameters, energy window, model and truncation basis.

**Discussion.**—We have shown that, while the level spacing statistics of the non-integrable (1+1)D QFTs we studied are well described by RMT even in the perturbative regime, their eigenvector component statistics are markedly different from the RMT predictions. Even though both of the above features emerge already in the perturbative regime, they persist unchanged beyond that, which suggests that they may be valid for any value of the perturbation parameter. Indeed, there is no indication that the scaling of the distributions changes with the perturbation strength even when the CFT shell structure disappears (Fig. 4). Moreover, the qualitative characteristics of the eigenvector component distributions for different models are similar, irrespectively of integrability and even for quite different choices of the truncation basis (Fig. 5). In particular, the latter observation rules out an explanation based on the exceptional features of the CFT basis. An interesting open question is how the observed...
discrepancy affects the validity of ETH in (1+1)D QFTs, which is currently being studied in the context of CFT [60–63]. Testing ETH using Hamiltonian truncation methods is however a more challenging problem, as it is supposed to hold in the thermodynamic limit where the perturbation strength is large and convergence of the spectrum worsens. We hope to investigate this question in the future.

The data presented in this work may be accessed at [64].

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SUPPLEMENTARY MATERIAL

A. Truncation Errors

The main challenge in TCSA is to achieve convergence of the computed quantities for increasing values of the energy cutoff $E_{\text{cut}}$. Generally, convergence of the numerically computed energy levels is relatively slow (algebraic) but it is possible that other quantities of interest converge faster than the levels themselves. Indeed, TCSA often gives correct results even when the levels have not reached their limiting values, therefore demanding convergence of the spectrum is a rather strict criterion, following which a large amount of useful data would be rejected. In the present study, in particular, the statistical properties of spectra and eigenvectors are not necessarily sensitive to the precise values of the individual energy levels. Therefore, it seems reasonable to require convergence of the statistical measures instead of the pointwise spectra themselves.

However, such a criterion may be incorrect and misleading when we compare spectral statistics, especially $\langle r \rangle$, with RMT predictions. The problem lies in the observation that the statistics of a non-convergent spectrum may resemble those of random matrices. Statistical measures may therefore appear to converge with the cutoff to the RMT predictions due to truncation errors instead of physical reasons. In fact, RMT-like statistics may be observed even for the integrable SG model, as $\langle r \rangle$ may show RMT-like behaviour for relatively low cutoff values, but switch to Poisson-like behaviour at higher cutoffs. This is indeed possible if convergence of the spectrum in the energy window under study has not been reached at a given cutoff (Fig. 6a: SG at $mL = 40$). The same can happen also for the non-integrable DSG model at some parameter values (Fig. 6b–c). Conversely, it seems reasonable that spectra whose statistics is Poisson-like have reached convergence and are reliable, since otherwise they would be RMT-like.

![Convergence of ⟨r⟩ for increasing values of the truncation cutoff $E_{\text{cut}}$.](image)

FIG. 6. Convergence of $\langle r \rangle$ for increasing values of the truncation cutoff $E_{\text{cut}}$. (a) Plots of $\langle r \rangle$ as a function of the cutoff in the DSG and SG model for different parameter values (SG: $\beta = 2.5$, DSG: $\beta_1 = 1, \beta_2 = 2.5$). The mean $\langle r \rangle$ is calculated in the energy window 1300–2000. Integrable sine-Gordon model (SG $l = 40$) can converge to the RMT value, due to non-convergence. The point used throughout the paper (DSG $l = 1$) is clearly convergent with the cutoff. When convergence is reached, the SG spectra are expected to reach Poisson statistics and indeed do so for $l = 1, 0.01$. However for the much larger value $l = 40$ the numerically computed SG spectra show RMT-like behaviour even for the maximum cutoff value 42. This is obviously due to non-convergence of the spectra at this range of cutoff values. Even though the absence of convergence can be seen in the fluctuations of $\langle r \rangle$ with the cutoff, such fluctuations cannot be used as a reliable measure of error as they are suppressed when using a larger energy window. (b–c) Density plot of $\langle r \rangle$ for the DSG as a function of the two perturbation parameters $l_1, l_2$, at two different cutoffs ($\beta_1 = 1, \beta_2 = 2$, energy window: 2000–3000). When the cutoff increases (here from 30 to 32) the red region in the upper part of the plot moves away, showing that the apparent RMT-like behaviour is due to non-convergence of the spectra. This is an indication that the values in the rest of the plot which remains unchanged are reliable, and we want a measure of truncation error that provides information on whether the computed values are physical or artificial without having to look at the whole parameter space at different cutoffs. (d) The error estimate $\langle \Delta r \rangle$ given by (4) calculated from the comparison of data for the cutoffs 30 and 32.

In order to test these statements in an unbiased way, we introduce a measure of the truncation error for spectral statistics and check how it varies for increasing cutoff values. The simplest way to estimate the error of the average of a quantity under study at a given cutoff is by averaging the absolute change of each individual contribution from one cutoff to the next. More explicitly,
considering the quantity \( \langle \tilde{r} \rangle \), we define its error estimate as
\[
\langle \Delta \tilde{r} \rangle = \frac{1}{N} \sum_{i=0}^{N} |\tilde{r}_i^{(2)} - \tilde{r}_i^{(1)}|,
\]
where the subscript denotes the index of the \( \tilde{r} \) value and the superscripts denote the two different energy cutoffs \( E_{\text{cut}}^{(1)} \) and \( E_{\text{cut}}^{(2)} \) with \( E_{\text{cut}}^{(2)} > E_{\text{cut}}^{(1)} \) at which the corresponding values are obtained. The sum is over the energy window under study, and \( N \) is the number of \( \tilde{r} \) values in that window. Based on the assumptions of TCSA, the differences \( \Delta \tilde{r}_i = |\tilde{r}_i^{(2)} - \tilde{r}_i^{(1)}| \) should decay in the limit of the cutoff going to infinity, and hence the mean value \( \langle \tilde{r} \rangle \) is expected to be quite accurate as long as we have reached a sufficiently high cutoff \( E_{\text{cut}}^{(2)} \).

But there is more to this error estimate, which makes it very reliable. First, \( \langle \Delta \tilde{r} \rangle \) is bounded, taking its maximum value when the two spectra under comparison correspond to spectra of independent random matrices. Second, it behaves analogously to the measure of correlation between the two compared spectra (Fig. 7.g–h). More specifically, when \( \langle \Delta \tilde{r} \rangle \) increases then the quantity \( 1 - \rho \), where \( \rho \) is the Pearson coefficient of correlation between the two spectra, also increases. If the spectra change with the cutoff so much that they are not correlated at all, we can say that they are maximally non-convergent. We can then estimate the maximum error by calculating the average error between two uncorrelated random matrices. This can be performed analytically for \( 3 \times 3 \) matrices by deriving the probability density of the error
\[
P(\Delta \tilde{r}) = \int_0^1 \int_0^1 P(\tilde{r}_1)P(\tilde{r}_2) \delta(\tilde{r}_1 - \tilde{r}_2 - \Delta \tilde{r})d\tilde{r}_1 d\tilde{r}_2
\]
and finding its mean. The average error can then be calculated to be \( \langle \Delta \tilde{r}_{\text{RMT}} \rangle \approx 0.29 \). We found numerically that this distribution is very general and holds for matrices of sizes at least up to \( 1000 \times 1000 \) and even for spectra of matrices that differ only by a few additional rows (Fig. 7.i).

In the models under study we indeed observe that the error estimate is maximal for the least convergent spectra (Fig. 7.g–h). Using this error we can now distinguish between spectra that exhibit RMT-like spectral statistics due to physical reasons or due to truncation artefacts. If the average error is close to the maximum value corresponding to the difference between two uncorrelated random matrices, then the spectra must be random due to non-convergence. If instead we observe very small average errors of the order of, say, 10% of the maximum value, then we can trust that the randomness is due to genuine physical reasons. It should be noted that this measure is still quite conservative, because, as the \( r \) values are computed from differences of consecutive levels, they are very sensitive to truncation errors and converge much slower than the absolute spectra themselves. Moreover, passing from one cutoff to the next the ordering of a non-negligible number of levels changes, which means that the corresponding differences \( \Delta \tilde{r}_i \) are larger than the actual values resulting in an overestimation of the error.

Equipped with a reliable truncation error estimate, we can now test quantitatively the quality of TCSA convergence by analysing how the error changes for increasing values of the cutoff (Fig. 8). We generally
Moreover, we find that lower left corner of the phase diagram shown in Fig. 2. The spectra and eigenvectors corresponding to well convergent windows (Fig. 7.g–h) with the lowest 3000 to 4000 levels verify that convergence is achieved first at lower energy in the DSG case and convergence for \( \langle r \rangle \). Based on this convergence tests we find that the best value, \( \Delta \hat{r} \), starts decreasing from the maximum value at lower cutoffs and keeps decreasing while at the same time fluctuating as the cutoff increases further. Based on this convergence tests we find that the best convergence for \( \langle r \rangle \) is achieved in the area around the lower left corner of the phase diagram shown in Fig. 2. Moreover, we find that \( \langle r \rangle \) converges quite well for \( \Delta \hat{r} \). We observe that as the cutoff increases the error decreases at every point in parameter space and the area where the error is smaller than the 0.05 threshold expands from smaller to larger \( \beta_1, \beta_2 \).

The proposed error measure improves considerably our ability to confidently distinguish physical from numerically artificial results. We now realise that the intuitive guess that non-convergent spectra are RMT-like is not always correct. Even though we do observe that this behaviour is the typical case, we can also clearly find exceptions to this rule: Poisson-like spectral statistics can be observed even in the region with maximal non-convergence (Fig. 6.c).

While we have so far focused on \( r \) values, we can estimate in a similar way the convergence of eigenvector statistics by comparing how much each eigenvector changes from cutoff to cutoff. We empirically found that a reliable test for the comparison of two vectors can be based on the correlation between their cumulative squared coefficients, which can be used as the fingerprint of a vector. More explicitly, for each numerically computed vector \( |\Phi_j\rangle \) we construct the list \( \sum_{k=1}^{N} |c_{k,j}|^2 \) for \( i = 1, \ldots, N \) where \( N \) is the length of the shortest between the two compared vectors, and then compare the two lists by computing the Pearson correlation coefficient. It should be noted in passing that this comparison function is also useful for tracking the eigenvectors as we incrementally change a parameter (physical parameter of the Hamiltonian or the cutoff) and identifying changes in the energy level ordering between two consecutive steps. To this end, we pairwise compare all eigenvectors of one matrix with those of the other and match them based on which pairs show the highest correlation. Typically, such ordering changes are limited to first or second neighbours. In this way we can correct the order of the eigenvectors and therefore obtain a more accurate estimate of the convergence error.

Using this error estimate we found that eigenvector statistics are convergent for a wider range of parameter values than \( r \) values. This can be easily understood from the fact that the \( r \) values are more sensitive to small changes in the energy level values. We found that, at the maximum cutoff \( E_{\text{cut}}/\varepsilon = 42 \) used here and for the values of \( \beta_i \) as in Fig. 4, eigenvector statistics are well convergent for up to \( \sim 3000 \) levels at \( l = 5 \).

**B. Quantum chaos and perturbed CFT**

As shown in the main text, the emergence of chaotic level spacing statistics in the DSG occurs already at infinitesimal values of the perturbation strength, as long as the two cosine perturbations are mixed with approximately equal coefficients (Fig. 3.a). This observation motivates us to study the problem using first order perturbation theory. Like any CFT, the spectrum of the free massless boson field theory, which is the unperturbed model, exhibits extensive degeneracies, in which case perturbation theory tells us that at first order the perturbation affects each of the degenerate energy shells independently from the others. This means that at first perturbative order we can analyse the spectral statistics working on finitely dimensional Hilbert spaces (those of the degenerate energy shells), that is, without having to deal with the problems arising from an infinite Hilbert space dimension and the need to truncate it in an efficient way. This is especially convenient as it allows us to derive exact spectra and statistics, reach higher energy shells and study the changes in spectral statistics for increasing shell number. In this appendix, we present more detailed results for the statistics of level spacings and eigenvectors of the DSG model at first order in perturbation theory.

The Hamiltonian of the free massless boson field theory with Dirichlet boundary conditions is (up to an irrelevant additive constant)

\[
H_0 = \sum_{n=1}^{\infty} E_n a_n^\dagger a_n
\]
TABLE I. Number of states of the degenerate energy shells.

| shell no. | $N$ | number of states |
|-----------|-----|------------------|
| 10        | 22  | 505              |
| 11        | 24  | 793              |
| 12        | 26  | 1224             |
| 13        | 28  | 1867             |
| 14        | 30  | 2811             |
| 15        | 32  | 4186             |
| 16        | 34  | 6168             |
| 17        | 36  | 9005             |

where $a_n, a_n^\dagger$ are the ladder operators corresponding to the sinusoidal harmonic eigenfunctions $\sqrt{2/L}\sin(k_n x)$ with wavenumbers $k_n = n\pi/L$, $n = 1, 2, \ldots$, satisfying canonical commutation relations $[a_n, a_{n'}^\dagger] = \delta_{n,n'}$, and the single particle dispersion relation is $E_n = k_n$ (in units where the speed of light is $c = 1$ and $\hbar = 1$). The eigenstates of $H_0$ are therefore

$$|\{\nu_n\}⟩ = \prod_{n=1}^{\infty} \frac{(a_n^\dagger)^{\nu_n}}{\sqrt{\nu_n!}} |0⟩$$  \hspace{1cm} (7)

with eigenvalues

$$E(\{\nu_n\}) = \frac{\pi}{L} \sum_{n=1}^{\infty} n\nu_n$$  \hspace{1cm} (8)

Evidently, owing to the linearity of the dispersion relation, all energy eigenvalues of $H_0$ are integer multiples of $\pi/L$ and can be classified into shells of degenerate levels with energy $E = N\pi/L$ for all positive integers $N$ and degeneracy equal to the number $P(N)$ of integer partitions of $N$. For large $N$ the number $P(N)$ increases rapidly ($\log(P(N)) \propto \sqrt{N}$). For this reason the dimensions of CFT degenerate shells increase with the energy faster than in few-body quantum mechanical systems (e.g. hydrogen atom). We will denote the subspace corresponding to the shell with energy $E = N\pi/L$ as $\mathcal{H}_N$. Taking into account the additional restriction to a single symmetry sector (see Sec. C), the resulting shell sizes are listed in Tab. I.

Let us consider a perturbation of $H_0$ by an operator $V$ that raises the degeneracy of the energy levels at first order in the perturbation parameter $\lambda$. This is true for the DSG interaction. From degenerate perturbation theory we know that the perturbed energy eigenstates of $H = H_0 + \lambda V$ corresponding to the shell $\mathcal{H}_N$ are

$$|\Phi_{N,j}⟩ = \sum_{j=1}^{P(N)} c_{i,j} |\Phi_{N,i}^0⟩ + \mathcal{O}(\lambda)$$  \hspace{1cm} (9)

with energy eigenvalues

$$E_{N,j} = E_N + \lambda E_{N,j}^{(1)} + \mathcal{O}(\lambda^2)$$  \hspace{1cm} (10)

where $|\Phi_{N,i}^0⟩ = \sum_{j=1}^{P(N)} c_{i,j} |\Phi_{N,j}^0⟩$ and $E_{N,j}^{(1)}$ are the eigenvectors and corresponding eigenvalues of the restriction $V_N$ of $V$ in the subspace $\mathcal{H}_N$. In the limit $\lambda \to 0$ the perturbed eigenstates are linear combinations of states of the shell they originate from and only those.

In the following we will analyse the dependence of

FIG. 9. Histograms of $\bar{r}$ values for the DSG in-shell spectrum for four different values of the mixing parameter $x$ showing the change from Poisson at $x = 0$ to GOE statistics at $x = 0.5$. The histograms are computed from spectra of the shell $N = 36$ (9005 levels).

FIG. 10. (a) Variation of $\overline{(\bar{r})}$ of the DSG in-shell spectra as a function of the parameter $x$ of mixing of the two cosine perturbations. The curves correspond to the shells with $N = 22, 24, 26, \ldots, 36$ (10th to 17th shell). (b) A sample of DSG energy levels for varying mixing parameter $x$ from 0 to 1. We observe the level repulsion for most values of $x$ except close to the edges where the DSG reduces to the SG model. The sample corresponds to the 10th energy shell and the unfolding procedure has been applied to uniform the level density.
spectral statistics on the mixing parameter $x$ and shell number $N$, keeping $\beta_1$ and $\beta_2$ fixed at the values used in the main text, i.e. 1 and 2.5 respectively. Fig. 9 shows histograms of $\hat{r}$ based on in-shell spectra for the 17th shell ($N = 36$) at various values of the mixing parameter, demonstrating the transition from Poisson statistics at $x = 0$ to GOE at $x \approx 1/2$ where $\langle \hat{r} \rangle$ is maximal. Owing to the larger spectrum size, the statistical fluctuations of the histograms are reduced compared to Fig. 1 of the main text, allowing a more accurate comparison with the theoretical curves. In Fig. 10.a showing $\langle \hat{r} \rangle$ as a function of $x$ in different shells, it is evident that as we move from lower to higher shells, $\langle \hat{r} \rangle$ approaches the GOE value closer and in a broader window of $x$ values expanding from the middle $x \approx 1/2$. This suggests that at higher energy levels the RMT behaviour of level spacings not only persists but also becomes more prominent. In-shell eigenvector statistics exhibit identical scaling as that shown in Fig. 4 in the main text, which is computed from spectra at intermediate values of the perturbation strength.

As pointed out in the main text, the above general features of spectra and eigenvectors observed in the limit $\lambda \to 0$ persist for all values of the perturbation parameters for which convergence was achieved in our study. Fig. 11 shows that the spectra exhibit a distinct shell structure characteristic of first-order perturbation theory for all $l \lesssim 1$, while the shells start sensing each other at $l \approx 1$ and visibly mix with each other at larger $l$. In particular, at the maximum value $l = 5$ for which we have achieved convergence of the spectrum and eigenvectors (lowest $\sim 3000$ levels), the effects of the interaction are strong enough to result in a spectral density that is clearly different from that of first-order perturbation theory. Yet the level spacing and eigenvector statistics for $l = 5$ exhibit the same characteristics as for $l \to 0$.

### C. Details of the Hamiltonian Truncation Method

As mentioned in the main text, Hamiltonian truncation methods have been extensively used for the derivation of the spectra of almost all of the models studied in the present work. Details on the application of TCSA to SG and DSG can be found in [31, 39, 40, 52] and [47, 49] respectively, while the application to MSG is a relatively straightforward extension of the method. Details on Hamiltonian truncation in the $\phi^4$ model can be found in [32, 54, 55, 57]. Here we provide further information relevant for the analysis of spectral statistics in these models. More specifically, we discuss the discrete symmetries of the models and restriction to a single symmetry sector, provide information on the size of the truncated bases we used and hints on the efficient construction of the Hamiltonian matrices. Lastly, we give explicit formulas for the parameters of the models as used here.

**Massless free boson basis**

We express the SG, DSG and MSG Hamiltonians in the truncated free boson CFT eigenstate basis ordering the states by their energy. We assume Dirichlet boundary conditions in a box of length $L$. Both the unperturbed and perturbed Hamiltonians are invariant under two discrete $\mathbb{Z}_2$ transformations: field reflection $\phi \to -\phi$ and space reflection $x \to L - x$. For the study of spectral statistics it is necessary to eliminate any symmetries of the model by restricting the spectrum to a single symmetry sector (energy levels belonging to different symmetry sectors are independent and can clearly cross with each other). Out of the four symmetry sectors we choose the one containing the ground state. The truncated basis sizes of the selected symmetry sector are listed in Tab. II. The Hamiltonian matrix is dense in this basis and to derive the spectra we use exact diagonalisation. The construction time of the Hamiltonian matrix can be reduced considerably using
with
\[
V_\beta = -\int_0^L \cos \beta \phi(x) dx
\]
\[
= -\frac{\pi}{L} \int_0^L \frac{1}{2} \left( V_+(x,t) + V_-(x,t) \right) dx
\]
(13)
where
\[
V_\beta(x,t) = e^{i \beta \phi(x)}
\]
is the vertex operator. As in the standard TCSA notation, the perturbation strength $\lambda$ has been re-parametrised in favour of the first sine-Gordon breather mass in units of the inverse system size $l = m_\beta L$, which is equal to the energy gap between the ground and first excited state in the thermodynamic limit
\[
\lambda(\beta, l) = \frac{\kappa(p(\beta))}{2} \left( \frac{\pi}{l} \right)^{\frac{p(\beta)-1}{p(\beta)+1}}
\]
(15)
where
\[
\kappa(p) = \frac{2}{\pi} \frac{\Gamma \left( \frac{p}{p+1} \right)}{\Gamma \left( \frac{1}{p+1} \right)} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{p+1}{2} \right)}{2 \Gamma \left( \frac{p}{2} \right)} \right]^{2/(p+1)}
\]
(16)
and
\[
m_\beta = 2M \sin(\pi p/2)
\]
(17)
is the first breather mass, where $M$ is the soliton mass.

To construct the double sine-Gordon Hamiltonian with frequencies $\beta_1$ and $\beta_2$, we add the corresponding sine-Gordon Hamiltonians. More explicitly, when we vary $\beta_{1,2}$ we adjust the coefficients so as to keep the values of $l_1, l_2$ fixed and equal $l_1 = l_2 = l$, and compute the spectra of the dimensionless Hamiltonian matrix
\[
H_{DSG} = 2H_0(l) + \lambda(\beta_1, l)V_{\beta_1} + \lambda(\beta_2, l)V_{\beta_2}
\]
When we vary $l_1$ and $l_2$ at fixed $\beta_1$ and $\beta_2$ (Fig. 6), we use the Hamiltonian matrix
\[
H_{DSG} = l_1H_0(l_1) + l_2H_0(l_2) + l_1\lambda(\beta_1, l_1)V_{\beta_1} + l_2\lambda(\beta_2, l_2)V_{\beta_2}
\]
When studying the massive Schwinger-Thirring model, we write the Hamiltonian as
\[
H_{MSG} = l_1H_0(l_1) + l_2\lambda(\beta, l)V_{\beta} + \frac{m^2}{2} \int_0^L \phi^2 : dx.
\]

### Table II. Truncated basis sizes and corresponding computing time for the construction of the Hamiltonian matrices for the cutoff values $E_{cut}/\varepsilon$ used in the present study.

| cutoff | number of states | CPU time [days] |
|--------|-----------------|-----------------|
| 32     | 12170           | 0.7             |
| 34     | 18338           | 2               |
| 36     | 27343           | 4               |
| 38     | 40369           | 8               |
| 40     | 59061           | 17              |
| 42     | 85674           | 36              |

To achieve sufficiently good convergence the $\phi^4$ Hamiltonian should be expressed in the KG instead of the massless free boson basis [32, 56]. The perturbation operator $U_\beta$ (more generally any $U_n$ operator) is sparse in this basis, and so the construction of the corresponding matrix can be programmed efficiently with the use of dictionaries associating each basis state with its order in the basis. This way we can benefit from the logarithmic lookup time and considerably speed up the calculation. For easier comparison with earlier literature [32, 56] we focused on the case of periodic boundary conditions. This means that we now have translational invariance symmetry, apart from the $\mathbb{Z}_2$ symmetry under field reflection $\phi \rightarrow -\phi$. We again restrict ourselves to a single symmetry sector, the one containing the ground state. For the basis truncation we set the total momentum cutoff at 42 and the system size is chosen to be $mL = 7$, where $m$ is the KG mass parameter. This choice corresponds to a basis size of 1,504,767. To compute the spectra and eigenstates of the $\phi^4$ Hamiltonian, we employed sparse matrix diagonalisation techniques, which allowed us to obtain the lowest $\sim 500$ eigenvalues and eigenstates.

### Klein Gordon basis

To construct the SG, DSG and MSG Hamiltonian matrices

The free boson CFT Hamiltonian for a system of length $L$ with Dirichlet boundary conditions, expressed in dimensionless form in units of a mass scale $M$, is
\[
H_0/M = \frac{\pi}{l} \sum_{n=1}^\infty m a_n^\dagger a_n
\]
(11)
where $l = ML$. The sine-Gordon Hamiltonian is defined as
\[
H_{SG} = H_0 + \lambda V_\beta
\]
(12)