Modeling of Extreme Rainfall Recurrence Using Conditional Probability Approach

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Abstract. Forecasts for the occurrence of extreme rainfall are related to time lapse estimates until the next extreme rain event. The accuracy of these estimates depends on the choice of interevent time distributions of extreme rain events. The aim of this research is to estimate the probability of time recurrence of extreme rainfall. In this research, we observed the time between extreme rain events distributed by Weibull and Pareto. Let $T$ be the time interval for the next extreme rain and it is assumed that there has been no extreme rain at time interval $\Delta t$. To analyze the occurrence of the next extreme rain event, the concept of conditional probability $P(\Delta t | t)$ is used. $P(\Delta t | t)$ is the probability of extreme rain occurrence during the time interval $\Delta t$. The results show that by using three models, we found that the next extreme rain event is not depending on the time interval $\Delta t$ since the last extreme rain event. Other results show that for case studies of extreme rain events in South Sulawesi it was found that the time of occurrence of extreme rainfall was influenced by parameter value for each model.

1. Introduction

Extreme weather is an extreme meteorological phenomenon in history, especially the weather phenomenon that has the potential to cause disaster, destroying the social order for life, or causing human fatalities. In general, extreme weather is based on climatological distribution, where extreme events are smaller than 5% of the distribution. Indonesia as an archipelago country located in the equatorial region including a region that is very vulnerable to climate change. Changes in rainfall, sea level rise and air temperatures, as well as increased extreme climatic events in the form of floods are some of the serious impacts of climate change facing Indonesia. Climate change is caused by the increasing greenhouse gas effect that is dominantly caused by industrial pollution. These increased greenhouse gases have the effect of speeding up the process of global warming and increasing the frequency of extreme climate events[1].

Rainfall is one of the climate parameters that are affected by climate change. More intense rainfall can cause negative impacts such as disease outbreaks, health problems, natural disasters such as floods
and landslides [2]. Observation of extreme precipitation behaviour is important to prevent or minimize the negative impact it generates [3]. Extreme precipitation behaviour can be analysed by probabilistic approach ([4],[5],[6]). Studies of the occurrence time model of future extremes have been found in many literatures ([7],[8]). From past research ([9],[10]), it has been found that it is usually very difficult for a model to produce seasonal forecasts of Precipitation with Positive Skill Over Climatology, Especially for Extra tropical Regions. In this study we examined the time of extreme rain recurrence using conditional probability approach.

2. Materials and Methods

In this section we describe the procedure in estimation of next extreme rainfall recurrence by reviewing inter event time of two extreme rain events. Data retrieval is done by using annual maximum method that is taking maximum value every year. All the maximum values in each period are defined as extreme values. This method focuses on the greatest occurrence of each

\[ P(\Delta t_0|t_0) = \frac{P(t_0 < T \leq t_0 + \Delta t_0|T > t_0)}{1 - F(t_0)} = \frac{F(t_0 + \Delta t_0) - F(t_0)}{1 - F(t_0)} = \frac{\int_{t_0}^{t_0+\Delta t_0} f(s) ds}{\int_{t_0}^{\infty} f(s) ds} \]  

(1)

By \( t_0 \) denote the lapse of time since the occurrence of the last occurrence. Because of there are often stochastic fluctuations in nature, then forecasts are obtained by maximizing conditional probability \( P(\Delta t_0|t_0) \). The fundamental problem in earthquake forecasts is to predict the time interval for the next extreme rain event while time lapse since the last major earthquake is given. The criterion for maximum conditional probability is to reach maximum if

\[ \frac{\partial}{\partial \Delta t_0} P(\Delta t_0|t_0) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial (\Delta t_0)^2} P(\Delta t_0|t_0) < 0 \]  

(2)

The following outlined 3 models of the time lapse of extreme rainfall recurrence to estimate the time of the next extreme rain event with the assumption that no extreme rain has occurred at any time interval.

| No | Model | Probability Density Function |
|----|-------|-----------------------------|
| 1  | Weibull \((\alpha, \beta)\) | \( f(\tau) = (\alpha \beta)^{\beta-1} \tau^{\beta-1} e^{-\alpha \tau} \) |
| 2  | Pareto \((\alpha, x_m)\) | \( f(\tau) = \frac{\alpha x_m^{\alpha}}{(\tau + x_m)^{\alpha+1}}, \quad \alpha > 0, \quad x_m > 0 \) |
| 3  | Rayleigh \((\alpha, \tau)\) | \( f(\tau) = \frac{\tau}{\alpha^2} e^{-\frac{\tau^2}{2\alpha^2}}, \quad \tau > 0, \quad \alpha > 0 \) |
3. Result

Let \( T \) is a random variable that denotes the inter event time of successive extreme rain events. Given \( f(\tau) \) as a probability density function, \( F(\tau) \) as a cumulative distribution function. In this section we describe Weibull, Pareto, and Rayleigh models.

3.1. Case 1: Weibull Model

The Weibull model is one of the more flexible distribution opportunities than the exponential distribution model. Let \( T \) the probability density function for the probability distribution model is

\[
f(\tau) = (\alpha \beta) \tau^{\beta-1} e^{-\alpha \tau^\beta}, \quad \alpha > 0, \beta > 0.
\]  

(3)

And cumulative distribution function as

\[
F(\tau) = 1 - e^{-\alpha \tau^\beta}
\]  

(4)

Mean and variance as follows

\[
E[T] = \alpha^{-1/\beta} \Gamma\left(\frac{1}{\beta}\right) + 1.
\]  

(5)

and

\[
V[T] = \alpha^{-2/\beta} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) - \left[ \Gamma\left(\frac{1}{\beta}\right) + 1 \right]^2 \right\}.
\]  

(6)

Based on equation (1) we obtain conditional probability \( P(\Delta t_0|t_0) \) for Weibull model as follows:

\[
P(\Delta t_0|t_0) = 1 - \left( \frac{e^{-\alpha(t_0+\Delta t_0)^\beta}}{e^{-\alpha t_0^\beta}} \right).
\]  

(7)

The second term for the above equation is the exponent for the numerator. It can be written as follows

\[
\alpha(t_0 + \Delta t_0)^\beta = \alpha t_0^\beta \left( 1 + \frac{\Delta t_0}{t_0} \right)^\beta
\]  

(8)

where

\[
\left(1 + \frac{\Delta t_0}{t_0}\right)^\beta = 1 + \beta \left( \frac{\Delta t_0}{t_0} \right) + o(\Delta t_0).
\]  

(9)
If the time lapse since the last extreme rain is long, then the time interval $\Delta t_i$ will likely be small, therefore, equation (7) can be written as follows

$$\alpha(t_0 + \Delta t_0)^\beta = \alpha t_0^\beta \left[ 1 + \beta \left( \frac{\Delta t_0}{t_0} \right) \right]. \tag{10}$$

Thus, the approximation of conditional probability of Weibull for extreme rain events is written as follows

$$P(\Delta t_0|t_0) = 1 - \exp(-\alpha t_0^{\beta-1} \Delta t) \tag{11}$$

From the (9) equation we get

$$\frac{\partial}{\partial t_0} P(\Delta t_0|t_0) = \frac{\partial}{\partial t_0} \{ 1 - \exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1}) \}$$

$$= \beta \alpha (\Delta t_0)(\alpha - 1) t_0^{\alpha-2} \exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1})$$

$$= 0 \tag{12}$$

and

$$\frac{\partial}{\partial \Delta t_0} P(\Delta t_0|t_0) = \frac{\partial}{\partial \Delta t_0} \{ 1 - \exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1}) \}$$

$$= \beta \alpha t_0^{\alpha-1} \{ 1 - \exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1}) \}$$

$$= 0. \tag{13}$$

Thus, we have

$$\beta \alpha (\Delta t_0)(\alpha - 1) t_0^{\alpha-2} = 0 \quad \text{and} \quad \beta \alpha t_0^{\alpha-1} = 0. \tag{14}$$

Based on (12) we have:

$$\beta \alpha t_0^{\alpha-1} ((\alpha - 1) \Delta t_0 t_0^{-1} - 1) = 0 \tag{15}$$

Thus, we have the solution of equation (13) is

$$t_0 = (\alpha - 1) \Delta t_0 \tag{16}$$

Because of

$$\beta \alpha t_0^{\alpha-1} \{ 1 - \exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1}) \} = 0 \tag{17}$$

It can be written as

$$\exp(-\beta \alpha (\Delta t_0) t_0^{\alpha-1}) = 0 \tag{18}$$
Furthermore, the left side of the equation is expanded into the Mac Laurin series and obtained
\begin{equation}
1 - \beta \alpha (\Delta t_0) t_0^{\alpha - 1} + o(\Delta t_0)^2 = 0
\end{equation}
\tag{19}

For $\Delta t_0$ is small, $o(\Delta t_0)^2$ to be very small. Thus, we have
\begin{equation}
1 - \beta \alpha (\Delta t_0) t_0^{\alpha - 1} = 0
\end{equation}
\tag{20}

We know that $t_0 = (\alpha - 1) \Delta t$, it’s easy to find that
\begin{equation}
(\Delta t_0)_{Wei} = \frac{1}{\beta \alpha (\alpha - 1)^{\alpha - 1}}.
\end{equation}
\tag{21}

Thus, in order to correlate $t$, $\Delta t$ and $\tau$, we may assume that the recurrence time $\tau$ is approximately equal to
\[\tau = t_0 + \Delta t_0\]

3.2. Case 2 : Pareto Model

Let $T$ as random variable denote interevent time of two extreme rain event, then probability density function for interevent time of successive extreme rain events is Pareto distributed (KPW02):
\begin{equation}
f(\tau) = \frac{\alpha x_m^\alpha}{\tau^{\alpha + 1}}, \tau > x_m, \quad \alpha > 0; \quad x_m > 0.
\end{equation}
\tag{22}

and cumulative distribution function as
\begin{equation}
F(\tau) = \int_0^\tau \frac{\alpha x_m^\alpha}{u^{\alpha + 1}} du.
\end{equation}
\tag{23}

with mean and variance are
\begin{equation}
E[T] = \begin{cases} \frac{\alpha x_m}{\alpha} & , \quad \alpha < 1 \\ \infty & , \quad \alpha = 1 
\end{cases}
\end{equation}
\tag{24}

and
\begin{equation}
Var[T] = \frac{\alpha x_m^2}{(\alpha - 1)^2(\alpha - 2)}.
\end{equation}
\tag{25}

For conditional probability of extreme rainfall occurrence $P(\Delta t_0|t_0)$, we have
\begin{equation}
P(\Delta t_0|t_0) = 1 - \frac{t_0^\alpha}{(t_0 + \Delta t_0)^\alpha} = 1 - t_0^\alpha (t_0 + \Delta t_0)^{-\alpha}
\end{equation}
\tag{26}
or
\[ P(\Delta t_0|t_0) = 1 - t_0^\alpha \left\{ t_0 \left( 1 + \frac{\Delta t_0}{t_0} \right) \right\}^{-\alpha} = 1 - \left\{ 1 + \left( \frac{\Delta t_0}{t_0} \right) \right\}^{-\alpha}. \] (27)

If time interval \( t \) since last extreme rain event, it can be assumed that interval \( \Delta t_0 \) is small with respect to \( t_t \). Then using the three first-order terms of the binomial series expansion

\[ \left\{ 1 + \left( \frac{\Delta t_0}{t_0} \right) \right\}^{-\alpha} \approx 1 + \frac{\alpha(\Delta t_0)}{t_0} - \frac{\alpha(\alpha + 1)(\Delta t_0)^2}{2t_0^2}, \quad \left| \frac{\Delta t_0}{t_0} \right| < 1 \] (28)

Then we have

\[ P(\Delta t_0|t_0) = \frac{\alpha(\Delta t_0)}{t_0} - \frac{\alpha(\alpha + 1)(\Delta t_0)^2}{2t_0^2}. \] (29)

Evaluate the Pareto model, we have:

\[
\frac{\partial}{\partial \Delta t_0} P(\Delta t_0|t_0) = \frac{\partial}{\partial \Delta t_0} \left\{ \frac{\alpha(\Delta t_0)}{t_0} - \frac{\alpha(\alpha + 1)(\Delta t_0)^2}{2t_0^2} \right\} \\
= \frac{\alpha}{t_0} - \frac{\alpha(\alpha + 1)(\Delta t_0)}{t_0^2} \\
= 0
\] (30)

So we have,

\[ \frac{\alpha}{t_0} - \frac{\alpha(\alpha + 1)(\Delta t_0)}{t_0^2} = 0 \] (31)

Furthermore, evaluate to second derivative, we have

\[
\frac{\partial^2}{(\partial \Delta t_0)^2} P(\Delta t_0|t_0) = \frac{\alpha(\alpha + 1)}{t_0^2}
\] (32)

from which we obtain the predictive formula to estimate the time interval in the Pareto model as follows:

\[ (\Delta t_0)_{\text{pareto}} = \frac{\alpha}{t_0} \frac{t_0^2}{\alpha(\alpha + 1)} = \frac{t_0}{\alpha + 1}. \] (33)

### 3.3. Case 3: Rayleight Model

Assume a Pareto power-law probability distribution to represent the recurrence times on one region. The Pareto distribution has been found to describe a wide variety of economic, social, and physical phenomena. The probability density function for Pareto distribution is
\[ f(\tau) = \frac{\tau}{\alpha^2} \exp \left( -\frac{\tau^2}{2\alpha^2} \right), \quad \tau > 0, \quad \alpha > 0. \quad (34) \]

and the cumulative distribution function of \( T \) is:

\[
F(\tau) = \int_0^\infty \frac{\tau}{\alpha^2} \exp \left( -\frac{\tau^2}{2\alpha^2} \right) d\tau = 1 - e^{-\left( \frac{\tau^2}{2\alpha^2} \right)}. \quad (35)
\]

Mean and variance of random variable \( T \) respectively

\[
E[T] = \alpha \sqrt{\frac{\pi}{2}} = 1.253\alpha \quad \text{dan} \quad V[T] = \left( \frac{4 - \pi}{2} \right) \alpha^2 = 0.429\alpha^2
\]

\[
E[Y] = \alpha \int_0^\infty \frac{s}{\sqrt{2\pi} \alpha^2} \exp \left( -\frac{s^2}{2\alpha^2} \right) ds = 1.253\alpha \quad \text{and} \quad V[Y] = \left( \frac{4 - \pi}{2} \right) \alpha^2 = 0.429\alpha^2 \quad (36)
\]

Furthermore, conditional probability of Rayleigh for extreme rainfall occurrence is \( P(\Delta t_0 | t_0) \) as follows

\[
\int_{t_0}^{t_0+\Delta t_0} f(s)ds = \int_{t_0}^{t_0+\Delta t_0} \frac{s}{\alpha^2} \exp \left( -\frac{s^2}{2\alpha^2} \right) ds = e^{-\frac{(t_0+\Delta t_0)^2}{2\alpha^2}} + e^{-\frac{t_0^2}{2\alpha^2}} \quad (37)
\]

and

\[
\int_{t_0}^{\infty} f(s)ds = \int_{t_0}^{\infty} \frac{s}{\alpha^2} \exp \left( -\frac{s^2}{2\alpha^2} \right) ds = e^{-\frac{t_0^2}{2\alpha^2}} \quad (38)
\]

Therefore, we have:

\[
P(\Delta t_0 | t_0) = 1 - e^{-\frac{2t_0\Delta t_0 + (\Delta t_0)^2}{2\alpha^2}}. \quad (39)
\]

Furthermore, The criterion for a maximum conditional probability is

\[
\frac{\partial}{\partial t} P(\Delta t_0 | t_0) = \frac{\partial}{\partial t_0} \left\{ 1 - e^{-\frac{2t_0\Delta t_0 + (\Delta t_0)^2}{2\alpha^2}} \right\} = \frac{\Delta t_0}{\alpha^2} e^{-\frac{2t_0\Delta t_0 + (\Delta t_0)^2}{2\alpha^2}} = 0 \quad (40)
\]
and

\[ \frac{\partial}{\partial \Delta t_0} P(\Delta t_0 | t_0) = \frac{\partial}{\partial \Delta t} \left\{ \frac{1 - e^{-\frac{2t_0 \Delta t + (\Delta t_0)^2}{2\alpha^2}}}{\alpha^2} \right\} \]

\[ = \frac{t_0 + \Delta t_0}{\alpha^2} e^{-\frac{2t_0 \Delta t + (\Delta t_0)^2}{2\alpha^2}} = 0. \]  

(41)

By equation (40) and (41) we have:

\[ \frac{\Delta^2}{\alpha^2} = 0 \quad \text{and} \quad \frac{t_0^2 + \Delta t_0^2}{\alpha^2} = 0 \]

(42)

so that

\[ t_0 = -2 \Delta t_0 \quad \text{and} \quad e^{\frac{2t_0 \Delta t_0 + (\Delta t_0)^2}{\alpha^2}} = 0, \]

(43)

Expanding \( e^{\frac{2t_0 \Delta t_0 + (\Delta t_0)^2}{2\alpha^2}} \) in MacLaurin series, we get

\[ 1 - \frac{2t_0 (\Delta t_0) + (\Delta t_0)^2}{2\alpha^2} + o(\Delta t_0)^2 = 0 \]

(44)

Where \( o(\Delta t_0)^2 \) becomes negligible for small \( \Delta t_i \). Thus, for small \( \Delta t_i \), we can write

\[ 1 - \frac{2t_0 (\Delta t_0) + (\Delta t_0)^2}{2\alpha^2} = 0 \]

(45)

or equivalently we can write

\[ 1 - t_0 \left( \frac{\Delta t_0}{\alpha^2} \right) - \frac{(\Delta t_0)^2}{2\alpha^2} = 0 \]

(46)

We know that \( \frac{(\Delta t_0)^2}{\alpha^2} = \gamma \), so

\[ 1 - \frac{(\Delta t_0)^2}{2\alpha^2} = 0 \]

(47)

Thus,

\[ (\Delta t_0)_{\text{Rayleigh}} = \pm \left( \sqrt{2} \right) \alpha \]

(48)

Of the three models tested, it was found that the length of time interval until the occurrence of the next extreme rain did not depend on the length of the time interval since the last event. It is only influenced by each parameter of the model.
4. Conclusion

A conditional probability approach to estimate the extreme precipitation in the future time has been described. The results show that by using three models, we found that the next extreme rain event is not depending on the time interval $\Delta t$ since the last extreme rain event. Other results show that for case studies of extreme rain events in South Sulawesi it was found that the time of occurrence of extreme precipitation was influenced by parameter value for each model. An interesting feature of the conditional probability forecast method which need to be examined the future research is the dependence of the extreme precipitation recurrence on the model is dependence on other models besides the exponential family distribution.

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