A $0 - 2$ LAW FOR COSINE FAMILIES WITH $\limsup_\infty$ TO $\infty$

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Abstract. For $(C(t))_{t \in \mathbb{R}}$ being a cosine family on a unital normed algebra, we show that the estimate $\limsup_{t \to \infty} \|C(t) - I\| < 2$ implies that $C(t) = I$ for all $t \in \mathbb{R}$. This generalizes the result that $\sup_{t \geq 0} \|C(t) - I\| < 2$ yields that $C(t) = I$ for all $t \geq 0$. We also state the corresponding result for discrete cosine families and for semigroups.

1. Introduction

In the recent past, laws of the form

\begin{equation}
(\text{limsup-law}) \quad \limsup_{t \to 0} \|C(t) - I\| < r \implies \lim_{t \to 0} \|C(t) - I\| = 0,
\end{equation}

\begin{equation}
(\text{sup-law}) \quad \sup_{t \in \mathbb{R}} \|C(t) - I\| < r \implies C(t) = I \forall t \in \mathbb{R},
\end{equation}

where $r > 0$ and $(C(t))_{t \in \mathbb{R}}$ is a cosine family of elements in a unital Banach algebra $A$ (with identity element $I$) were studied, see [1, 2, 3, 6] and [7, 9] for the special case where $(C(t))_{t \in \mathbb{R}}$ is strongly continuous and $A = B(X)$ is the Banach algebra of bounded operators on a Banach space $X$. For both, the limsup-law and the sup-law the largest possible constant $r$ was shown to be 2.

In this note we consider the condition

\begin{equation}
(\text{limsup-\infty-law}) \quad \limsup_{t \to \infty} \|C(t) - I\| < 2,
\end{equation}

which is weaker than the premise in the sup-law and show that

\begin{equation}
(\text{limsup-\infty-law}) \quad \limsup_{t \to \infty} \|C(t) - I\| < r \implies C(t) = I \forall t \in \mathbb{R},
\end{equation}

for $r = 2$ holds, see Theorem 2.5. A related question was raised in [8, Remark 2.6] for, more general, scaled versions of these laws. More precisely, it was asked whether for $a \geq 0$ the following holds for some $r$,

\begin{equation}
(\text{limsup-\infty-law}) \quad \limsup_{t \to \infty} \|C(t) - \cos(at)\| < r \implies C(t) = \cos(at) \forall t \in \mathbb{R}.
\end{equation}

Let us mention that ‘scaled version’ (where the unity element $I$ gets replaced by $\cos(at)I$ of limsup-law and sup-law have a different optimal constant $r = \frac{8}{3\sqrt{3}}$, see [2, 4, 5].

In the following, we show that (limsup-\infty-law) holds, using techniques by J. Esterle [6]. Finally we state the corresponding result for semigroups, for which zero-one-laws have been studied much earlier than for cosine families.

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2. A \( \limsup_{t \to \infty} \) - law

In the following, for a unital normed algebra \( A \), let \( I \) denote the identity element.

**Lemma 2.1.** Let \( (C(t))_{t \in \mathbb{R}} \) be a cosine family in a unital Banach algebra. If

\[
\limsup_{t \to \infty} \|C(t) - I\| = 0,
\]

then \( C(t) = I \) for all \( t \in \mathbb{R} \).

**Proof.** From the assumption follows that \( \lim_{t \to \infty} \|C(t) - I\| = 0 \). By d’Alembert’s defining identity for cosine families,

\[
C(t + s) + C(t - s) = 2C(t)C(s),
\]

for all \( s, t \in \mathbb{R} \). Thus, letting \( t \to \infty \), we derive \( 2I = 2C(s) \) for all \( s \in \mathbb{R} \). \( \square \)

The following lemma is a slight extension of Esterle’s Lemma 2.1 in [6], as we also allow for \( t_0 = \infty \). The proof is completely analogous the case case \( t_0 = 0 \).

**Lemma 2.2.** Let \( (c(t))_{t \in \mathbb{R}} \) be a complex-valued cosine family and \( t_0 \in \{0, \infty\} \). Then, we have one of the following situations.

(i) \( \limsup_{t \to t_0} |c(t) - 1| = \infty \),

(ii) \( \limsup_{t \to t_0} |c(t) - 1| = 2 \),

(iii) \( \limsup_{t \to t_0} |c(t) - 1| = 0 \).

Moreover, in case (iii) it follows that

\[
c(t) = \begin{cases} 
1 & \text{if } t_0 = \infty, \\
\cos(at) & \text{if } t_0 = 0,
\end{cases}
\]

for some \( a \geq 0 \).

**Proof.** As mentioned the proof is analogous to the one in [6, Lemma 2.1]. In case (iii) and \( t_0 = \infty \), it follows by Lemma 2.1 that \( c(t) = 1 \) for all \( t \in \mathbb{R} \). \( \square \)

**Proposition 2.3.** Let \( (C(t))_{t \in \mathbb{R}} \) be a cosine family on a unital Banach algebra \( A \). If 

\[
\limsup_{t \to \infty} \rho(C(t) - I) < 2,
\]

then \( \rho(C(t) - I) = 0 \) for all \( t \in \mathbb{R} \).

**Proof.** Let \( \hat{A} \) denote the space of characters on \( A \). For all \( t \in \mathbb{R} \) we have that

\[
\rho(C(t) - I) = \sup_{\chi \in \hat{A}} |\chi(C(t)) - 1| = \sup_{\chi \in \hat{A}} |\chi(C(t)) - 1|.
\]

Thus, by the assumption we get that \( \limsup_{t \to \infty} |\chi(C(t)) - 1| < 2 \) for \( \chi \in \hat{A} \). As \( (\chi(C(t)))_{t \in \mathbb{R}} \) is a complex-valued cosine family, Lemma 2.2 then implies that \( \chi(C(t)) = 1 \) for all \( t \in \mathbb{R} \) and \( \chi \in \hat{A} \). Using this in (2.3), we deduce that \( \rho(C(t) - I) = 0 \) for all \( t \in \mathbb{R} \). \( \square \)

As pointed by Esterle [6], for a commutative unital Banach algebra \( A \), for \( x \in A \) with \( \|x\| \leq 1 \) we can define

\[
\sqrt{I - x} := \sum_{n=0}^{\infty} (-1)^n \alpha_n x^n,
\]
where \((-1)^n\alpha_n\), with \(\alpha_0 = 1\), \(\alpha_n = \frac{1}{n!\pi^2}((-\frac{1}{2})\cdots(-\frac{1}{2}+n+1))\), \(n > 0\), are the Taylor coefficients of the function \(z \rightarrow \sqrt{1-z}\) at the origin (with convergence radius equal to 1). Since \((-1)^{n-1}\alpha_n > 0\) for \(n \geq 1\),

\[
(2.5) \quad \|I - \sqrt{I-x}\| \leq \sum_{n=1}^{\infty} |\alpha_n\| x^n = \sum_{n=1}^{\infty} (-1)^{n-1}\alpha_n x^n = 1 - \sqrt{1-\|x\|}.
\]

**Lemma 2.4** (Esterle 2015, [4]). Let \((C(t))_{t \in \mathbb{R}}\) be a cosine family in a unital Banach algebra. If \(\|C(2s) - I\| \leq 2\) and that \(\rho(C(s) - I) < 1\) for some \(s \in \mathbb{R}\), then,

\[
C(s) = \sqrt{I - \frac{C(2s)}{2}},
\]

where the square root is defined as described above.

With the above preparatory results, the limit sup-law is now easy to show. The proof can be done analogously to the one in [6, Theorem 3.2], which in turn can be seen as an elegant refinement of the technique used in the three-lines-proof in [11].

**Theorem 2.5.** Let \((C(t))_{t \in \mathbb{R}}\) be a cosine family in a unital Banach algebra \(A\). Then, \(\limsup_{t \to \infty} \|C(t) - I\| < 2\) implies that \(C(t) = I\) for all \(t \in \mathbb{R}\).

**Proof.** By Proposition 2.4, we have that \(\rho(C(t) - I) = 0\) for \(t \in \mathbb{R}\). Thus, we can apply Lemma 2.4 and Eq. (2.5) so that for all \(s \in \mathbb{R}\),

\[
\|I - C(s)\| \leq 1 - \sqrt{1 - \frac{\|I - C(2s)\|}{2}} \leq 1.
\]

With \(S := \limsup_{s \to \infty} \|C(s) - I\|\), this yields that

\[
S \leq 1 - \sqrt{1 - \frac{S}{2}} \leq 1,
\]

which implies that \(S = 0\). Hence, Lemma 2.1 concludes the assertion. \(\square\)

**Remark 2.6.** After finishing this note, Esterle pointed out that, alternatively, [5] Theorem 2.3 implies that for a bounded cosine sequence with \(\rho(C(1) - I) = 0\), it follows that \(C(t) = \cos(at)I\) for all \(t \in \mathbb{R}\) and some \(a \in \mathbb{R}\). Thus, \(\limsup_{t \to \infty} \|C(t) - I\| < 2\) implies \(C(t) = I\) for all \(t \in \mathbb{R}\) and therefore, the use of Lemma 2.4 can be omitted.

**Remark 2.7.** It is clear that Theorem 2.5 generalizes the sup-law with \(r = 2\). We remark that the known proofs of the sup-law, see [3, 9], which use a diagonalisation argument and the limsup-law, cannot be generalized to the assertion of Theorem 2.5.

2.1. **A discrete lim-sup-law.** For discrete cosine families, or cosine sequences \((C(n))_{n \in \mathbb{Z}}\), the following was proved in [9] (There, it was formulated for the special case of \(C : \mathbb{Z} \to B(X)\) for a Banach space \(X\). However, the proof is the same for general Banach-algebra-valued cosine families).

**Theorem 2.8** ([9]). Let \((C(n))_{n \in \mathbb{Z}}\) be a discrete cosine family in a unital Banach algebra. Then,

\[
\sup_{n \in \mathbb{N}} \|C(n) - I\| < \frac{3}{2} \implies C(n) = I \ \forall n \in \mathbb{Z}.
\]
The proof is based on an elegant idea of Arendt, which can directly be applied to weaken the sup to lim sup in the theorem.

**Theorem 2.9.** Let \((C(n))_{n \in \mathbb{Z}}\) be a discrete cosine family in a unital Banach algebra. Then,

\[
\limsup_{n \to \infty} \|C(n) - I\| < \frac{3}{2} \implies C(n) = I \quad \forall n \in \mathbb{Z}.
\]

This result is optimal as can be seen by \(C(n) = \cos\left(\frac{2n\pi}{3}\right)\) which yields \(\limsup_{n \to \infty} \|C(n) - I\| = \frac{3}{2}\), see [9, Theorem 3.2].

3. The corresponding semigroup result

Let us finally state the corresponding result for (discrete) semigroups in a unital normed algebra, which is a corollary of a well-known result by Wallen [10].

**Theorem 3.1.** Let \(\{T_n\}_{n \in \mathbb{N}}\) be a semigroup in a normed unital algebra. Then,

\[
\limsup_{n \to \infty} \|T_n - I\| < 1 \implies T_n = I \quad \forall n \in \mathbb{N}.
\]

**Proof.** If \(\limsup_{n \to \infty} \|T_n - I\| < 1\), then \(\liminf_{n \in \mathbb{N}} \frac{1}{n} \sum_{j=1}^{n} \|T_j - 1\| < 1\). By Wallen [10], the assertion follows. □

**Remark 3.2.** Clearly, Theorem 3.1 implies that for a semigroup on \([0, \infty)\), \((T(t))_{t \geq 0}\), we have that

\[
\limsup_{t \to \infty} \|T(t) - I\| < 1 \implies T(t) = I \quad \forall t \geq 0.
\]

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