Anomalous phase fluctuations of a superfluid flowing in a random potential

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We investigate the stability of the off-diagonal long-range order of a superfluid flowing in a weak random potential. Within the classical field theory, we demonstrate that for an arbitrarily small flow velocity, the off-diagonal long-range order is destroyed in one and two dimensions. The underlying physics is that scattering of condensed atoms by disorder has the same effect as thermal excitations characterized by an effective temperature. In addition to the Landau criterion, which pertains to the stability of a superflow against a spatially localized defect, the present study reveals yet another mechanism responsible for the breakdown of a superflow by a spatially extended disorder.

\textbf{Introduction.} — Understanding the phase structures of many-body systems acted upon by random perturbations has been a central problem in statistical physics. To minimize the interaction energy, systems with large degrees of freedom tend to be ordered in the ground state. However, if such interacting systems are disturbed by random perturbations, the interplay between the interaction and randomness leads to a rich variety of nontrivial phenomena including phase transitions \cite{6,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36}.

A fundamental issue concerning this subject is to classify the universal mechanisms responsible for the breakdown of the ordered phases by random perturbations. The Hohenberg–Mermin–Wagner theorem states that thermal fluctuations destroy the long-range order in two-dimensional systems with a continuous symmetry \cite{3,4,5}. Even at zero temperature, it has been shown that the long-range order of a one-dimensional Bose gas and that of the Heisenberg antiferromagnet are destroyed by quantum fluctuations \cite{6,8}. The instability of the ordered phases against quenched disorder has also been extensively investigated. A phenomenological argument by Imry and Ma \cite{9}, which is supported by a rigorous proof by Aizenman and Wehr \cite{8}, indicates that in three dimensions, the spontaneous breaking of continuous symmetry does not occur in the presence of a random field. Examples include amorphous magnets \cite{10}, liquid crystals in porous media \cite{11}, and vortex lattices in dirty superconductors \cite{12}.

With remarkable experimental progress in ultracold atomic gases and trapped ions, the nonequilibrium phase structures of driven quantum many-body systems have attracted considerable attention \cite{13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36}. Nonequilibrium driving can destroy an ordered phase that is stable in equilibrium, or stabilize an exotic phase that has no counterpart in equilibrium. Thus, it is vital to clarify the universal mechanism responsible for the breakdown of the long-range order due to the interplay between a uniform driving and randomness.

The stability of superfluidity in a disordered environment is a long-standing problem in condensed matter physics. In the ground state, weak disorder does not affect the off-diagonal long-range order except for a small depletion of the condensate fraction. However, in the presence of strong disorder, a transition to an insulating phase takes place \cite{19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36}. In this Letter, we consider a superfluid flowing in a weak random potential. Such a system allows dissipationless and stationary flow of matter if the flow velocity is less than a certain critical value. The stability of a superflow in a random potential has been investigated theoretically \cite{28,29,30,31,32,33,34,35,36} and experimentally \cite{37,38,39,40}. In this work, we focus on the phase fluctuations of the superfluid order parameter in a stationary flow. Within the classical field theory, we demonstrate that the off-diagonal long-range order is destroyed in one and two dimensions for an arbitrarily small flow velocity. Furthermore, the correlation function of the superfluid order parameter is found to exhibit an exponential decay and a power-law decay in one and two dimensions, respectively. We argue that scattering of condensed atoms by disorder has the same effect as thermal excitations at an effective temperature. In other words, the superfluid flowing in a random potential can be identified with the corresponding uniform system at thermal equilibrium, where the long-range order is prohibited in one and two dimensions by the Hohenberg–Mermin–Wagner theorem. The stability condition of a superflow against a spatially localized defect is given by the Landau criterion \cite{37}. The present study reveals yet another universal mechanism responsible for the breakdown of a superflow by a spatially extended disorder.

\textbf{Model.} — In the classical field theory, the bosonic field operator $\hat{\psi}$ in the quantum Hamiltonian is replaced by the classical field $\Phi$. We consider a one-component superfluid described by the following classical Hamiltonian in a spatial dimension $D$, subject to the periodic boundary conditions with period $L$ in all directions:

$$H[\Phi] = \int d^D r \left[ \frac{1}{2} Z(n; \lambda) |\nabla \Phi|^2 + U(n; \kappa) \right],$$

where $Z(n; \lambda) > 0$ and $U(n; \kappa)$ are analytic functions of the density $n = |\Phi|^2$ and randomness parameters $\lambda$ and $\kappa$. Explicit forms of these functions are not needed for the general argument. The time-independent and spatially
fluctuating parameters $\lambda(r)$ and $\kappa(r)$ characterize a spatially irregular structure, such as porous media for superfluid helium \[38\] and optical speckle patterns for trapped cold atoms \[32, 33\]. We assume that $\lambda(r) = \kappa(r) = 0$, where the overline denotes the disorder average, and the spatial correlations of $\lambda(r)$ and $\kappa(r)$ decay exponentially with the distance. The dynamics of the condensate wavefunction $\Phi$ is described by

$$i\hbar \frac{\partial \Phi}{\partial t} = \delta H[\Phi] \frac{\delta \Phi^*}{\delta \Phi}.$$  \(2\)

The classical field theory is valid when the quantum depletion due to the interaction is negligible and almost all the bosons occupy a single orbit. However, in one dimension, special care is required because quantum fluctuations destroy the off-diagonal long-range order even at zero temperature, and there exists a finite length scale $\xi_0$, which describes the correlation of the order parameter. If $\xi_0$ is considerably larger than the system size $L$, we expect that the classical field approximation is valid. Such a condition is well satisfied in the experiments of quasi-one-dimensional ultracold atomic gases \[34, 39\].

From Eqs. (1) and (2), we obtain the continuity equation $\delta n = -\nabla \cdot j$, where the current density is given by $j = -(i/2)Z(n; \lambda)(\Phi^* \nabla \Phi - \Phi \nabla \Phi^*)$. We concentrate on a stationary state: $i\hbar \delta \partial \Phi / \partial t = \mu \Phi$, where $\mu$ is the chemical potential. If the flow velocity is smaller than the critical velocity, a stable stationary solution of Eq. (2) exists. We denote such a solution as $\Phi(r) = \sqrt{\bar{n}(r)} e^{iK \cdot r} \bar{\psi}(r)$, where $K$ is the mean momentum and $\bar{\psi}(r)$ represents the phase fluctuation. We assume that $\bar{\psi}(r) = 0$ and the typical amplitude of $\bar{\psi}$ is proportional to those of the random parameters $\lambda$ and $\kappa$ if $|\lambda|$ and $|\kappa|$ are sufficiently small. From the continuity equation, we have

$$\nabla \cdot [Z(n; \lambda) n(\nabla \phi + K)] = 0.$$  \(3\)

For a given realization of disorder, $n$ and $Z$ can be split into their spatial averages and the deviations from them: $n(r) = \bar{n} + \delta n(r)$ and $Z(n(r); \lambda(r)) = \bar{Z} + \delta Z(r)$, where $\bar{n} \equiv L^{-D} \int d^D r n(r)$ and $\bar{Z} \equiv L^{-D} \int d^D r Z(n(r); \lambda(r))$. To the leading order, Eq. (3) reads

$$\nabla^2 \phi(r) + K \cdot \nabla \delta \bar{n}(r) = 0,$$  \(4\)

where $\delta \bar{n}(r) \equiv \delta n(r) + \bar{n} \delta Z(r)/\bar{Z}$. By introducing the Fourier transform $\varphi_q = L^{-D/2} \int d^D r \varphi(r) e^{-i q \cdot r}$, where $q = 2\pi n/L$, we have

$$|\varphi_q|^2 = \frac{\langle (K \cdot q)^2 \rangle}{\varphi_0^2} - \frac{\bar{n}}{\bar{n}} |\delta \bar{n}|^2,$$  \(5\)

Next, we consider the correlation function of density fluctuations, specifically, $g_n(r - r') = \overline{\delta n(r) \delta n(r')}$, whose Fourier transform gives $|\delta \bar{n}|^2 = \int d^D q g_n(q) e^{-i q \cdot r}$. We assume that $g_n(r)$ consists of an exponentially decaying part and a negative offset which is inversely proportional to the volume: $g_n(r) \simeq \sigma_n^2 e^{-|r|/\xi_n} - \kappa L^{-D}$, where $\sigma_n^2 \equiv \overline{\delta n(0)^2} + \kappa L^{-D}$ is the amplitude of the density fluctuations and $\xi_n$ is the correlation length. The constant $\kappa > 0$ is determined from the trivial condition $\int d^D r g_n(r) = 0$, which follows from the definition of $\delta \bar{n}(r)$. Because $\xi_n$ can depend on the relative angle between $r$ and $K$, it is convenient to define $\xi_n$ as $\int d^D r (g_n(r) + \kappa L^{-D}) = \sigma_n^2 \xi_n^D$. Then, $|\delta \bar{n}|^2$ converges to $\sigma_n^2 \xi_n^D$ as $q$ approaches zero, and we have

$$|\varphi_q|^2 \sim \frac{(K \cdot q)^2}{\varphi_0^2} \frac{\bar{n}}{\bar{n}} - 2 \sigma_n^2 \xi_n^D$$  \(6\)

for small $q$. Remarkably, $|\varphi_q|^2$ diverges in the long wavelength region as $\propto |q|^{-2}$. This aspect implies that the amplitude of the phase fluctuations, $\sigma_n^2 \equiv \int d^D r (g_n(r) + \kappa L^{-D}) = \sigma_n^2 \xi_n^D$ diverges in one and two dimensions as $\sigma_n^2 \sim \bar{n}$ and $\sim \ln \bar{n}$, respectively.

We define the correlation function of the superfluid order parameter by $C(r - r') \equiv e^{-iK \cdot (r - r')}[\Phi(r)\Phi^*(r')]$, where $\Phi(r)$ is the stationary solution of Eq. (2). Because the density correlation $g_n(r)$ rapidly decays exponentially, for $|r| \gg \xi_n$, $C(r)$ is approximated as $\overline{\delta n(r') \delta n(0)}$. In terms of the mean square relative displacement $B(r) \equiv \langle (\varphi(r) - \bar{\psi}(0))^2 \rangle$, the correlation function is rewritten as $C(r) \simeq \bar{n} \exp[-(1/2)B(r)]$, where we have retained only the second cumulant of the phase fluctuations. From Eqs. (4) and (5), the asymptotic behavior of the mean square relative displacement in the long distance $|r| \gg \xi_n$ can be calculated as $B(r) \sim K^2 \bar{n} \xi_n^2$ in one dimension and $B(r) \sim (2\pi)^{-1} K^2 \bar{n} \xi_n^2$ in two dimensions \[40\]. Thus, we have

$$C(r) \sim \begin{cases} e^{-|r|^2/\xi_n^2}, & (D = 1), \\ (|r|/\xi_n)^{-\eta}, & (D = 2), \end{cases}$$  \(7\)

where the inverse correlation length $l_c^{-1}$ and the exponent $\eta$ are given by

$$l_c^{-1} = \frac{K^2 \sigma_n^2 \xi_n}{2 \bar{n}^2}, \quad \eta = \frac{K^2 \sigma_n^2 \xi_n}{4\pi \bar{n}^2}.$$  \(8\)

In two dimensions, although $C(r)$ is anisotropic, the exponent $\eta$ is independent of the direction of $r$. Thus, in one and two dimensions the condensate density $n_0 \equiv \lim_{|r| \to \infty} C(r)$ vanishes; in other words, the off-diagonal long-range order is absent.

The asymptotic behavior of the correlation function Eq. (4) is the same as that of a uniform Bose gas at thermal equilibrium with temperature $T$. In such a case, the inverse correlation length in one dimension and the exponent in two dimensions are given by

$$l_c^{-1} = \frac{mk_b T}{\hbar^2}, \quad \eta = \frac{mk_b T}{2\pi \hbar^2},$$  \(9\)
where \( m \) is the mass of the atom. Comparing Eq. (8) with Eq. (9), we are led to introduce an effective temperature

\[
k_{\text{B}} T_{\text{eff}} = \frac{\hbar^2 K^2 \sigma_n^2 \tilde{\xi} D}{2mn}.
\]

(10)

This effective temperature can be rewritten as follows:

\[
k_{\text{B}} T_{\text{eff}} = \left( \frac{\hbar^2 K^2}{2m} \right) \cdot \tilde{\xi} \tilde{\rho}_n \cdot \left( \frac{\sigma_n^2}{\tilde{\rho}_n^2} \right),
\]

where the first, second, and third terms represent the kinetic energy of the condensed atoms, the number of atoms within the correlation length, and the amplitude of the density fluctuations, respectively. The last term \( \sigma_n^2/\tilde{\rho}_n^2 \) is interpreted as the scattering probability of the condensed atom by the random medium. For example, suppose a Bose gas flows in a weak random potential \( V(r) \). Fermi’s golden rule implies that the transition rate between plane wave states with momenta \( k \) and \( k' \) is proportional to \( |\langle k' | V(r) | k \rangle|^2 \). Because the amplitude of the density fluctuations is proportional to that of the random potential, \( \sigma_n^2/\tilde{\rho}_n^2 \) is proportional to the number of atoms scattered out of the condensate. Thus, Eq. (10) implies that the scattering process due to the random potential is equivalent to virtual thermal excitations. This process does not lead to an actual heating of the system because neither injection nor dissipation of energy is involved.

**Bose gas flowing in a random potential.** — As a simple example of a superflow in a random environment, let us consider a weakly interacting Bose gas flowing in a weak random potential. The Hamiltonian is given by

\[
H[\Phi] = \int d^2r \left[ \frac{\hbar^2}{2m} |\nabla \Phi(r)|^2 + V(r) n(r) + g \frac{1}{2} n(r)^2 \right],
\]

(11)

where \( g (> 0) \) is the strength of the interaction. The mean-zero random potential \( V(r) \) satisfies \( V(r) V(r') = C_R(r - r') \), where \( C_R(r) \) is short-ranged with a characteristic length scale \( \xi_k \); for example, \( C_R(r) = V_0^2 \exp(-r^2/2\xi_k^2) \) \cite{41}. We assume that the amplitude of the random potential \( V_0 \) is considerably smaller than the mean interaction energy \( g \tilde{\rho}_n \).

From Eqs. (2) and (11), the stationary state satisfies the time-independent GP equation:

\[
- \frac{\hbar^2}{2m} \nabla^2 \Phi(r) + [V(r) + g n(r)] \Phi(r) = \mu \Phi(r).
\]

(12)

For a given mean density \( \bar{n} \), the chemical potential \( \mu \) is determined from the condition \( L^{-2} \int d^2r (\bar{n} + n(\Phi)) = \bar{n} \), where \( n(\Phi) \) is the solution of Eq. (12). We consider a stationary state of the form \( \Phi(r) = \sqrt{\bar{n}(r)} e^{i K_r \cdot r + i \varphi(r)} \). The density \( n(r) \), the phase \( \varphi(r) \), and the chemical potential \( \mu \) are expanded with respect to the disorder strength: \( n(r) = \bar{n} + n^{(1)}(r) + n^{(2)}(r) + \ldots, \varphi(r) = \varphi^{(1)}(r) + \varphi^{(2)}(r) + \ldots, \) and \( \mu = \mu^{(0)} + \mu^{(1)} + \mu^{(2)} + \ldots \). Here \( n^{(s)}(r), \varphi^{(s)}(r), \) and \( \mu^{(s)} \) \((s = 1, 2, \ldots)\) are \( O(V^s) \). From Eq. (12), we have \( \mu^{(0)} = g \tilde{\rho}_n + \hbar^2 K^2 / 2m \) and \( \mu^{(1)} = 0 \). Substituting the above expansions into Eq. (12), we obtain

\[
\varphi^{(1)}_q = - \frac{i v q}{\hbar c^2 \xi_k^2} - \frac{ivq}{\sqrt{2} \tilde{\rho}_n^2} + \frac{\tilde{\xi}^2 q^2}{2} V_q,
\]

(13)

from the first-order perturbation theory, where \( v = \hbar K/m, c = \sqrt{gn/m}, \) and \( \xi_k = \hbar / \sqrt{2mg} \) denote the flow velocity, the sound velocity, and the healing length, respectively \cite{41}. In the same manner as in Eq. (7), the inverse correlation length \( \xi_q^{-1} \) in one dimension and the exponent \( \eta \) in two dimensions can be calculated as

\[
\xi_q^{-1} = \frac{v^2 \tilde{C}_R(0)}{2 \hbar^2 (c^2 - v^2)^{3/2}}, \quad \eta = \frac{v^2 \tilde{C}_R(0)}{4 \pi \hbar^2 c (c^2 - v^2)^{3/2}},
\]

(14)

where \( \tilde{C}_R(q) = \int d^2r C_R(r) e^{-iq \cdot r} \).

**Numerical experiments.**— To confirm the asymptotic behavior of the correlation function \( \tilde{C}_R(q) \), we perform numerical simulations by solving the GP equation. For simplicity, we consider the Bose–Hubbard model in one- and two-dimensional square lattices. If the number of atoms per site is sufficiently large, the classical field theory is applicable. The discrete GP equation is given by

\[
-J \sum_{k \in \mathcal{N}_j} \Phi_k + U_j \Phi_j + g |\Phi_j|^2 \Phi_j = \mu \Phi_j,
\]

(15)
where $N_j$ denotes the set of the nearest neighbor sites of $j$. The potential $U_j$ is randomly chosen from a uniform distribution on the interval $[-W,W]$. We calculate the stationary solution of the form $\Phi_j = \sqrt{\rho_j} e^{iK \cdot \mathbf{R}_j + i\varphi_j}$, where $\mathbf{R}_j$ is the coordinate of $j$, and $\mathbf{K}$ is assumed to be parallel to the positive $x$-direction $[13]$. In the following, we set $J = g = 1$ and $\bar{n} \equiv L^{-D} \sum_j |\Phi_j|^2 = 1$.

Figure 1(a) shows the spatial distribution of the phase $\varphi_j$ for a stationary state in two dimensions. Note that the phase fluctuations are more strongly correlated for $\varphi_2$ we set $J = g = 1$ and $\bar{n} \equiv L^{-D} \sum_j |\Phi_j|^2 = 1$.

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Next, we consider the case of moderately strong disorder, in which deviations from the perturbative results should be significant. Figure 2 (a) shows the correlation function $C(r)$ in one dimension for different system sizes. Although the finite-size effect is not small, $C(r)$ exhibits exponential decay in the small-$r$ region, whose width increases with the system size. Figure 2 (c) shows the inverse correlation length $\xi^{-1}$ obtained from $C(r)$ for $L = 500$ plotted against the flow momentum $K$ up to the critical flow momentum $K_c$, above which the stationary solution of Eq. (15) does not exist. Figure 2 (b) shows the correlation function $C(r)$ in two dimensions for the direction parallel to the flow momentum. We have confirmed that, in contrast to the one-dimensional case, the finite-size effect is rather small in two dimensions. In addition, the power-law decay of $C(r)$ can be observed for $r > 5$. Figure 2 (d) shows the exponent $\eta$ obtained from $C(r)$ for $L = 100$. The dashed curves in Fig. 2 (c) and (d) respectively show the inverse correlation length and the exponent given by Eq. (14). The deviation between the numerical and theoretical values increases as the flow momentum $K$ approaches the critical momentum $K_c$. From Fig. 2 we conclude that Eq. (2) holds even for a moderately strong disorder, while the inverse correlation length $\xi^{-1}$ in one dimension and the exponent $\eta$ in two dimensions can be modified from the corresponding perturbative values $\xi_0$ or $\xi_1$.

**Concluding remarks.**— We have demonstrated that the interplay between a uniform flow and disorder destroys the off-diagonal long-range order in one and two dimensions. Here, we consider an experimental setup to confirm the present findings. For trapped cold atomic gases, the correlation function $C(r)$ can be estimated from the interference between two independent condensates [14]. In Ref. [15], the power-law decay of $C(r)$ was observed for a two-dimensional trapped Bose gas. The superflow in a random potential can be realized by imposing a moving optical speckle pattern on an atomic cloud confined in a trapping potential at rest. We expect that the correlation function observed in this setup coincides with that of the uniform Bose gas at thermal equilibrium with the temperature defined by Eq. (10).

The absence of the off-diagonal long-range order suggests that the critical velocity vanishes in the thermodynamic limit $L \to \infty$. Note that this does not contradict the Landau criterion, which predicts that the critical velocity is given by the sound velocity, because it concerns the stability of a superflow against a spatially localized defect and not the spatially extended disorder considered in this work. Although the critical velocity of a disordered superfluid has been discussed in several theoretical works [28, 31], most of these studies focus on specific systems described by the one-dimensional GP equation with Gaussian random potentials. The universal behavior of the correlation function given by Eqs. (2) and (8) suggests that there exists a universal relationship among the critical velocity, disorder strength, system size, and spatial dimension, which is independent of the details of the Hamiltonian.

It is interesting to investigate the analogy between the breakdown of the superflow at the critical velocity and the equilibrium phase transition to the normal fluid at the critical temperature. When the flow velocity exceeds the critical velocity, the superflow becomes unstable and the system undergoes a transition to a turbulent state, which is manifested by the proliferation of vor-

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**FIG. 2:** (a) Correlation function $C(r)$ in one dimension with disorder strength $W = 0.5$ and flow momentum $K = 0.2$ for different system sizes: $L = 200$ (top), 400, 600, 800, and 1000 (bottom). The horizontal and vertical axes are shown in linear and log scales, respectively. (b) Correlation function $C(r)$ in two dimensions for the direction parallel to the flow momentum with disorder strength $W = 1$ and systems size $L = 500$ for different flow momentum values: $K = 0.1$ (top), 0.125, 0.15, 0.175, 0.2, 0.225, and 0.25 (bottom). The horizontal and vertical axes are shown in log scales. (c) Inverse correlation length $\xi^{-1}$ in one dimension for disorder strength $W = 0.2, 0.3$, and 0.5. The dashed curves indicate the theoretical values given by Eq. (14). (d) Exponent $\eta$ in two dimensions for disorder strength $W = 0.5, 1$, and 2.
of Science. Furthermore, we recall that the two-dimensional Bose gases exhibit the so-called Berezinskii–Kosterlitz–Thouless (BKT) transition, which is driven by the unbinding of the vortex-antivortex pairs. Since the effect of the disorder and flow can be cast into the effective temperature, we speculate that the vortex dissociation picture in the BKT transition is also responsible for the breakdown of the superflow in a random environment. Thus, the notion of the effective temperature introduced here may help bridge the conceptual gap between the nonequilibrium laminar–turbulent transition and the equilibrium topological phase transition.

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SUPPLEMENTAL MATERIAL

Calculation of the mean square relative displacement

We calculate the mean square relative displacement \( B(r) \equiv (\varphi(r) - \varphi(0))^2 \). In terms of the Fourier transform of \( \varphi(r) \), \( B(r) \) is written as

\[
B(r) = 2 \int \frac{d^D q}{(2\pi)^D} |\varphi_q|^2 |1 - \cos q \cdot r|.
\]

(S1)

Since \( \varphi(r) \) is real, we have \( |\varphi_q|^2 = |\varphi_{-q}|^2 \), which leads to \( B(r) = B(-r) \). In two dimensions, \( B(r) \) is not isotropic in the presence of a flow. We recall that \( |\varphi_q|^2 \) is related to \( |\delta_n q|^2 \) by Eq. (5) in the main text.

In one dimension, \( B(r) \) is calculated as

\[
B(r) \simeq 2K^2\bar{n}^{-2} \int_{-\infty}^{\infty} dq \frac{1 - \cos q}{q^2} = 4K^2\bar{n}^{-2} \int_{0}^{\infty} dq' \frac{1 - \cos q'}{q'^2},
\]

(S2)

where we have changed the integration variable as \( q' = q|r| \).

In two dimensions, \( B(r) \) is rewritten as

\[
B(r) \simeq 2\pi^{-1} K^2\bar{n}^{-2} \int_{0}^{\xi} dq_x \int_{0}^{\xi_x} dq_y \frac{q_x^2 (1 - \cos q \cdot r)}{(q_x^2 + q_y^2)^2},
\]

(S3)

where \( K \) is assumed to be parallel to the positive \( x \)-direction. First, let us consider the case in which \( r \) is parallel to \( K \). Then, \( B(r) \) is calculated as

\[
B(r) \simeq \pi^{-1} K^2\bar{n}^{-2} \int_{0}^{\xi} dq_x \int_{0}^{\xi_x} dq_y \frac{1 - \cos q_x |r|}{(q_x^2 + q_y^2)^2} \left[ \tan^{-1}(\xi^{-1}_x q_x^{-1}) + \frac{\xi_n q_x}{1 + (\xi_n q_x)^2} \right].
\]

(S5)

For \( |r| \gg \xi_n \), the dominant contribution to the \( q_x \)-integral comes from \( q_x \simeq |r|^{-1} \ll \xi_n^{-1} \). Thus, the quantity in the square brackets in Eq. (S5) becomes \( \pi/2 \) and we have

\[
B(r) \simeq (2\pi)^{-1} K^2\bar{n}^{-2} \int_{0}^{\xi} dq_x \frac{1 - \cos q_x |r|}{q_x} \simeq (2\pi)^{-1} K^2\bar{n}^{-2} \text{ln}(|r|/\xi_n).
\]

(S6)

In a similar manner, if \( r \) is perpendicular to \( K \), \( B(r) \) is calculated as

\[
B(r) \simeq \pi^{-1} K^2\bar{n}^{-2} \int_{0}^{\xi} dq_y \int_{0}^{\xi_y} dq_y \frac{q_y^2 (1 - \cos q_y |r|)}{(q_x^2 + q_y^2)^2} \left[ \tan^{-1}(\xi^{-1}_x q_x^{-1}) - \frac{\xi_n q_y}{1 + (\xi_n q_y)^2} \right].
\]

(S7)
Perturbative calculation for a Bose gas flowing in a random potential

We calculate the correlation function $C(r)$ for a Bose gas flowing in a weak random potential. The stationary state satisfies the time-independent Gross-Pitaevskii (GP) equation:

$$\frac{-\hbar^2}{2m} \nabla^2 \Phi(r) + [V(r) + gn(r)]\Phi(r) = \mu\Phi(r). \tag{S8}$$

For a given mean density $\bar{n}$, the chemical potential $\mu$ is determined from the condition

$$L^{-D} \int d\nu(r) = \bar{n}, \tag{S9}$$

where $\nu(r)$ is the solution of Eq. [SS]. Substituting $\Phi(r) = \sqrt{n(r)}e^{iK\cdot r + i\varphi(r)}$ into Eq. [S8], we have

$$-\frac{\hbar^2}{2m} \nabla^2 \sqrt{n(r)} - 2\sqrt{n(r)}(K \cdot \nabla)\varphi(r) - \sqrt{n(r)}|\nabla\varphi(r)|^2 - K^2 \sqrt{n(r)} + (gn(r) + V(r) - \mu)\sqrt{n(r)} = 0, \tag{S10}$$

$$\nabla \cdot [n(r)(\nabla\varphi(r) + K)] = 0, \tag{S11}$$

where the second equation is the continuity equation. We expand the density $n(r)$, the phase $\varphi(r)$, and the chemical potential $\mu$ with respect to the disorder strength as

$$n(r) = \bar{n} + n^{(1)}(r) + n^{(2)}(r) + \cdots,$$

$$\varphi(r) = \varphi^{(1)}(r) + \varphi^{(2)}(r) + \cdots,$$

$$\mu = \mu^{(0)} + \mu^{(1)} + \mu^{(2)} + \cdots \tag{S12}$$

where $n^{(s)}(r)$, $\varphi^{(s)}(r)$, and $\mu^{(s)}$ ($s = 1, 2, \cdots$) are $O(V^s)$. From Eqs. [SS] and [S9], we have

$$\mu^{(0)} = gn + \frac{\hbar^2 K^2}{2m}, \quad \mu^{(1)} = 0. \tag{S13}$$

Inserting Eq. [S12] into [S10] and [S11] and keeping only the leading-order terms, we have

$$-\frac{\hbar^2}{2m} [\nabla^2 n^{(1)}(r) - 4\bar{n}K\partial_x \varphi^{(1)}(r) - K^2 n^{(1)}(r)] + [3gn - \mu^{(0)}]n^{(1)}(r) + 2\bar{n}V(r) = 0, \tag{S14}$$

$$\bar{n} \nabla^2 \varphi^{(1)}(r) + K\partial_x n^{(1)}(r) = 0, \tag{S15}$$

where $K$ is assumed to be parallel to the positive $x$-direction, i.e., $K = Ke_x$. The Fourier transforms of Eqs. [S14] and [S15] read

$$\frac{\hbar^2}{2m} [|q|^2 n^{(1)}_q + 4\bar{n}Kiq_x \varphi^{(1)}_q] + 2g\bar{n}n^{(1)}_q + 2\bar{n}V_q = 0, \tag{S16}$$

$$-\bar{n}|q|^2 \varphi^{(1)}_q + Kiq_x n^{(1)}_q = 0. \tag{S17}$$

In terms of the flow velocity $v = \hbar K/m$, the sound velocity $c = \sqrt{gn/m}$, and the healing length $\xi_h = \hbar/\sqrt{2mg}$, $n^{(1)}(r)$ and $\varphi^{(1)}(r)$ are written as

$$n^{(1)}_q = -\frac{\bar{n}|q|^2}{mc^2[|q|^2 - (v/c)^2 q_x^2 + \xi_h^2|q|^4/2]} V_q, \tag{S18}$$

$$\varphi^{(1)}_q = -\frac{ivq_x}{\hbar c^2[|q|^2 - (v/c)^2 q_x^2 + \xi_h^2|q|^4/2]} V_q. \tag{S19}$$
First, let us consider the one-dimensional case. Substituting Eq. (S19) into Eq. (S1), we have

\[ B(r) = 2 \left( \frac{v}{\hbar c^2} \right)^2 \int_{-\infty}^{\infty} dq \frac{1 - \cos qr}{2\pi} \frac{1 - \cos qr}{q^2 [1 - (v/c)^2 + \xi^2 q^2/2]^2} \]

\[ \simeq 4 \left( \frac{v}{\hbar c^2} \right)^2 \frac{\tilde{C}_R(0)}{[1 - (v/c)^2/2]} \int_0^{\xi^{-1}} dq \frac{1 - \cos qr}{2\pi q^2} \]

\[ = \frac{v^2 \tilde{C}_R(0)}{\hbar^2 (c^2 - v^2)/2} |r|, \quad (S20) \]

where \( \tilde{C}_R(q) \) is the Fourier transform of the disorder correlator,

\[ \tilde{C}_R(q) = \int d^d r C_R(r)e^{-iq \cdot r}. \quad (S21) \]

In the second line of Eq. (S20), we have assumed that \(|r| \gg \xi \equiv \max(\xi_h, \xi_R)\), where \( \xi_R \) is the correlation length of the random potential. Finally, we obtain the inverse correlation length

\[ l_c^{-1} = \frac{v^2 \tilde{C}_R(0)}{2\hbar^2 (c^2 - v^2)/2} \quad (S22) \]

Next, we consider the two-dimensional case. In a manner similar to the previous section, we have

\[ B(r) = 2 \left( \frac{v}{\hbar c^2} \right)^2 \int d^2 q \frac{q^2 (1 - \cos q \cdot r) \tilde{C}_R(q)}{(2\pi)^2 [(1 - (v/c)^2)q_x^2 + \xi^2 q_y^2/2]^2} \]

\[ \simeq 2 \left( \frac{v}{\hbar c^2} \right)^2 \tilde{C}_R(0) \int_{|r| < \xi^{-1}} d^2 q \frac{q_x^2 (1 - \cos q \cdot r)}{(2\pi)^2 [(1 - (v/c)^2)q_x^2 + \xi^2 q_y^2/2]^2} \]

\[ \simeq \frac{v^2 \tilde{C}_R(0)}{2\pi \hbar^2 c (c^2 - v^2)^{3/2}} \ln(|r|/\xi), \quad (S23) \]

and we obtain the exponent

\[ \eta = \frac{v^2 \tilde{C}_R(0)}{4\pi \hbar^2 c (c^2 - v^2)^{3/2}}. \quad (S24) \]

**Numerical method to calculate the stationary state**

We perform numerical experiments by solving the time-independent discrete GP equation:

\[ -J \sum_{k \in \mathcal{N}_j} \Phi_k + U_j \Phi_j + g |\Phi_j|^2 \Phi_j = \mu \Phi_j, \quad (S25) \]

where the potential \( U_j \) is randomly chosen from a uniform distribution on the interval \([-W, W]\). We seek for the stationary solution of the form \( \Phi_j = \sqrt{\rho_j} e^{iK \cdot R_j + i\varphi_j} \), where \( R_j \) is the coordinate of site \( j \) and \( K \) is assumed to be parallel to the positive \( x \)-direction. To obtain such a solution numerically, we consider the following time-dependent GP equation:

\[ i \frac{d\Phi_j}{dt} = -J \sum_{k \in \mathcal{N}_j} e^{i\theta(i)(R_k - R_j)} e^{i\varphi} \Phi_k + U_j \Phi_j + g |\Phi_j|^2 \Phi_j. \quad (S26) \]

At the initial time \( t = 0 \), we start from the ground state of the Hamiltonian, which is obtained by solving the imaginary time GP equation. Then, the phase parameter \( \theta(t) \) is gradually increased up to a given value \( K \) during a time interval \( T \): \( \theta(t) = K t / T \). If \( K \) is smaller than the critical momentum \( K_c \), one reaches some stationary solution at \( t = T \). The time interval \( T \) is chosen to be sufficiently long so that the solution is well converged. Above \( K_c \), there exists no stationary solution that is continuously connected to the ground state of the Hamiltonian. Equation (S26)
is numerically solved by the Runge-Kutta method with a time discretization $dt = 0.1$. To compare numerical results with perturbative ones, the following replacement is required:

$$\frac{\hbar^2}{2m} |\mathbf{q}|^2 \rightarrow 2J \sum_{\mu} (1 - \cos q_{\mu}), \quad \frac{m}{\hbar} v \rightarrow K, \quad \hbar c \rightarrow \sqrt{2J g n}. \quad (S27)$$

In Figs. 1 (b) and (c) of the main text, we show the amplitude of the phase fluctuations $\sigma_{\phi}^2 \equiv L^{-D} \sum_j \tilde{\phi}_j^2$. For the one-dimensional case (b), the time interval $T$ is set to $10^4$. The disorder average is taken over 100, 50, 20, and 10 realizations of the random potential for different system sizes $L = 20 - 1000$, $L = 2000$, $L = 5000$, and $L = 10000$, respectively. For the two-dimensional case (c), the time interval $T$ is set to $2 \times 10^3$. The disorder average is taken over 100, 50, 20, and 10 realizations of the random potential for different system sizes $L = 10 - 50$, $L = 100$, $L = 200$, and $L = 500$, respectively.

In Fig. 2 of the main text, we show the correlation function $C(r)$. For the one-dimensional case (a), the disorder average is taken over 100 realizations of the random potential. The inverse correlation length $l_c^{-1}$ in Fig. 2 (c) is obtained by fitting $C(r)$ for $L = 500$ by $e^{-r/l_c}$ in the small-$r$ region. For the two-dimensional case (b), the disorder average is taken over 10 realizations of the random potential. The exponent $\eta$ in Fig. 2 (d) is obtained by fitting $C(r)$ for $L = 100$ by $r^{-\eta}$ for $r > 5$. 
