Jacobi Forms of Critical Weight and Weil Representations

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Abstract

Jacobi forms can be considered as vector valued modular forms, and Jacobi forms of critical weight correspond to vector valued modular forms of weight \( \frac{1}{2} \). Since the only modular forms of weight \( \frac{1}{2} \) on congruence subgroups of \( \text{SL}(2, \mathbb{Z}) \) are theta series the theory of Jacobi forms of critical weight is intimately related to the theory of Weil representations of finite quadratic modules. This article explains this relation in detail, gives an account of various facts about Weil representations which are useful in this context, and it gives some applications of the theory developed herein by proving various vanishing theorems and by proving a conjecture on Jacobi forms of weight one on \( \text{SL}(2, \mathbb{Z}) \) with character. (2000 Mathematics Subject Classification: 11F03 11F50 11F27)

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1 Introduction

All Siegel modular forms of degree $n$ and weight $k < \frac{n}{2}$ are singular. Similarly, every orthogonal modular form associated to a quadratic module of signature $(2, n + 1)$ and of weight $k < \frac{n}{2}$ is singular. The weights in the given ranges are called singular weights, a terminology, which was introduced by Resnikoff. Spaces of modular forms of singular weight are well-understood (cf. [Fr 91] and the literature cited therein for singular Siegel modular forms, and [Re 75] for singular orthogonal modular forms). For critical weights much less is known. Here, by critical weight, we understand the weight $\frac{n}{2}$ in the theory of Siegel modular forms of degree $n$ and in the theory of orthogonal modular forms of signature $(2, n + 1)$ respectively. [We 92] gives some results for Siegel modular forms of degree 2 and critical weight 1. More recently, it was shown in [I-S 06] that there are no cusp forms of degree 2 and weight 1 (with trivial character) on the subgroups $\Gamma_0(l)$ of $\text{Sp}(2, \mathbb{Z})$.

In a joint project Ibukiyama and the author [I-S 07] take up a more systematic study of Siegel and orthogonal modular forms of singular weight, a first result being [I-S 06]. One of the main tools used in this project is the Fourier-Jacobi expansion of the modular forms in question, more precisely, the Fourier-Jacobi expansion with respect to Jacobi forms of degree 1. For modular forms of critical weight the Fourier-Jacobi coefficients will then be Jacobi forms whose index is a symmetric positive definite matrix of size $n$ and whose weight equals $\frac{n+1}{2}$. We shall call this weight critical for Jacobi forms of degree 1 whose matrix index is of size $n$.

Jacobi forms of critical weight play also a role in recent work of Gritsenko, Sankaran and Hulek on the the geometry of moduli spaces of K3-surfaces. The explicit construction of Jacobi forms of small weight is an essential tool in their studies. In particular, they use Jacobi forms with scalar index of weight 1 (hence of critical weight) and with a nice product expansion to construct those Jacobi forms which they need. These Jacobi forms of weight 1 with product expansion are special instances of so-called thetablocks (cf. the discussion before Theorem 11 in section 5 for a more precise statement). It is an interesting question whether all Jacobi forms of small weight (and of scalar index) can be obtained by thetablocks as it is suggested by numerical experiments. For Jacobi forms of weight $1/2$ this question can be answered affirmatively using the description of these forms given in Corollary to Theorem 5 below (cf. [GSZ 07] for details). For Jacobi forms of weight 1 (and of scalar index) we are similarly lead to the problem of finding a sufficiently explicit description of these forms. Such a description will drop out as part of the considerations of this article. In fact, we shall show that all Jacobi forms of weight 1 with character $\varepsilon^8$ (see section 2) are linear combinations of
thetablocks. A more systematic theory of thetablocks will be developed in a joint article of V. Gritsenko, D. Zagier and the author [GSZ 07].

In view of the aforementioned applications it seems to be worthwhile to develop a systematic theory of Jacobi forms of critical weight. These Jacobi forms can be studied as vector valued elliptic modular forms of (critical) weight $\frac{1}{2}$. The first main tool for setting up such a theory is an explicit description of this connection. The second main tool is then the fact that elliptic modular forms of weight $\frac{1}{2}$ are theta series [Se-S 77], and the third main tool is the decomposition of the space of modular forms of weight $\frac{1}{2}$ with respect to the action of the metaplectic double cover of $SL(2, \mathbb{Z})$ [Sko 85]. The latter makes it possible to describe spaces of vector valued elliptic modular forms of weight $\frac{1}{2}$ in sufficiently explicit form. A deeper study of these explicit descriptions requires a deeper understanding of Weil representations of $SL(2, \mathbb{Z})$ (or rather its metaplectic double cover) associated to finite quadratic modules. In fact, it turns out that spaces of Jacobi forms of critical weight are always naturally isomorphic to spaces of invariants of suitable Weil representations (cf. Theorem 8 and the subsequent remark).

The present article aims to pave the way for a theory of Jacobi forms of critical weight and, in particular, for the aforementioned projects [I-S 07], [GSZ 07] and possible other applications. We propose a formal framework for such a theory, we give an account of what one can prove for Jacobi forms of critical weight within this framework, and we describe the necessary tools from the theory of Weil representations and the theory of modular forms of weight $\frac{1}{2}$ which are needed. Some of the following results are new, others are older but are not easily available elsewhere or are even unpublished.

The plan of this article is as follows: In section 3 we discuss Weil representations of finite quadratic modules. All the considerations of this section are mainly motivated by the question for the invariants of a given Weil representation. In section 4 the relation between (vector valued) Jacobi forms and vector valued modular forms is made explicit. As an application we obtain (cf. Theorem 6) a dimension formula for spaces of Jacobi forms (of arbitrary matrix index). In section 5 we finally turn to Jacobi forms of critical weight. We shall see that Jacobi forms of critical weight are basically invariants of certain Weil representations associated to finite quadratic modules (cf. Theorem 8 and the subsequent remarks). We apply our theory to prove various vanishing results in Theorems 10, 11, Corollary to Theorem 13, and to prove in Theorem 12 that Jacobi forms of weight 1 and character $\varepsilon^8$ are obtained by theta blocks. In section 6 we append those proofs which have been omitted in the foregoing sections because of their more technical or computational nature. For the convenience of the reader we insert a section 2 which contains a glossary of the main notations.
2 Notation

We use \([a,b; c,d]\) for the \(2 \times 2\) matrix with first and second row equal to \((a,b)\) and \((c,d)\), respectively. If \(F\) is an \(n \times n\) matrix and \(x\) a column vector of size \(n\), we write \(F[x]\) for \(x^tFx\). We use \(e(X)\) for \(\exp(2\pi iX)\), and \(e_m(X)\) for \(\exp(2\pi iX/m)\). For integers \(a\) and \(b\), the notation \(a|b\) indicates that every prime divisor of \(a\) is also a divisor of \(b\). For relatively prime \(a\) and \(b\) we use \((a/b)\) for the usual generalized Jacobi-Legendre Symbol with the (additional) convention \((a/2) = -1\) if \(a \equiv \pm 3 \mod 8\) and \(\beta\) is odd and \((a/2) = +1\) if \(a \equiv \pm 1 \mod 8\) or \(\beta\) is even. We summarize (in roughly alphabetical order) the most important notations of this article:

- \(\mathbb{C}(\chi)\) For a character \(\chi\) of the metaplectic cover \(\mathbb{M} = \text{Mp}(2, \mathbb{Z})\), the \(\mathbb{M}\)-module with underlying vector space \(\mathbb{C}\) and with \(\mathbb{M}\)-action \((g,z) \mapsto \chi(z)g\).
- \(\varepsilon\) The one dimensional character of \(\mathbb{M} = \text{Mp}(2, \mathbb{Z})\) given by \(\varepsilon(A, w) = \eta(A\tau)/(w(\tau)\eta(\tau))\), where \(\eta\) is the Dedekind eta function.
- \(\vartheta(\tau, z)\) The Jacobi form \(q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1} (1 - q^n)(1 - q^n\zeta)(1 - q^n\zeta^{-1})\).
- \(\tilde{\Gamma}\) For a subgroup \(\Gamma\) of \(\text{SL}(2, \mathbb{Z})\) the inverse image of \(\Gamma\) in \(\text{Mp}(2, \mathbb{Z})\) under the natural projection onto the first factor.
- \(\Gamma'\) For a subgroup \(\Gamma\) of \(\text{Mp}(2, \mathbb{Z})\) its image under the natural projection onto the first factor.
- \(\Gamma(4m)^*\) The (normal) subgroup of \(\text{Mp}(2, \mathbb{Z})\) consisting of all pairs \((A, j(A, \tau))\), where \(A\) is in the principal congruence subgroup \(\Gamma(4m)\) of matrices which are 1 modulo \(4m\), and where \(j(A, \tau)\) stands for the standard multiplier system from the theory of modular forms of half-integral weight, i.e. \(j(A, \tau) = \theta(A\tau)/\theta(\tau)\) where \(\theta(\tau) = \sum_{r \in \mathbb{Z}} \exp(2\pi i r^2)\).
- \(\mathbb{H}\) The upper half plane of complex numbers.
- \(\text{Inv}(V)\) The space of \(\text{Mp}(2, \mathbb{Z})\)-invariant elements in an \(\text{Mp}(2, \mathbb{Z})\)-module \(V\).
- \(\text{M}_k(\Gamma)\) For a subgroup \(\Gamma\) of \(\text{SL}(2, \mathbb{Z})\), the space of modular forms of weight \(k\) on \(\Gamma\); if \(k \in \frac{1}{2} + \mathbb{Z}\) then it is assumed that \(\Gamma\) is contained in \(\Gamma_0(4)\), and every \(f\) in \(\text{M}_k(\Gamma)\) satisfies \(f(A\tau)j(A, \tau)^{-2k} = f(\tau)\) for \(A \in \Gamma\) (cf. above for \(j(A, \tau)\)).
- \(\mathbb{M}\) Abbreviation for \(\text{Mp}(2, \mathbb{Z})\), see below.
The metaplectic double cover of \( SL(2, \mathbb{Z}) \), i.e. the group of all pairs \((A, w)\), where \( A \in SL(2, \mathbb{Z}) \) and \( w \) is a holomorphic function on \( \mathbb{H} \) such that \( w(\tau)^2 = c\tau + d \), equipped with the composition law \((A, w) \cdot (B, v) = (AB, w(B\tau)v(\tau))\).

\[ S \quad \text{The matrix } [0, -1; 1, 0]. \]

\[ T \quad \text{The matrix } [1, 1; 0, 1]. \]

\[ \vartheta_{F, x} \quad \text{For a symmetric, half-integral positive definite matrix } F \text{ of size } n \text{ and a column vector } x \in \mathbb{Z}^n, \]

\[ \vartheta_{F, x}(\tau, z) = \sum_{r \in \mathbb{Z}^n, r \equiv x \mod 2F\mathbb{Z}^n} e\left(\frac{1}{4}F^{−1}[r] + r^t z\right) \quad (\tau \in \mathbb{H}, \ z \in \mathbb{C}^n) \]

\[ V^* \quad \text{For a left } G\text{-module } V, \text{ the right } G\text{-module with the dual of the complex vector space } V \text{ as underlying space equipped with the } G\text{-action } (\lambda, g) \mapsto (v \mapsto \lambda(gv)). \]

\[ V^c \quad \text{For a left } G\text{-module } V, \text{ the left } G\text{-module with the dual of the complex vector space } V \text{ as underlying space equipped with the } G\text{-action } (g, \lambda) \mapsto (v \mapsto \lambda(g^{-1}v)). \]

\[ w_A \quad \text{For a matrix } A \in SL(2, \mathbb{Z}), \text{ the function } w_A(\tau) = \sqrt{a\tau + b}, \text{ where the square root is chosen in the right half plane or on the nonnegative imaginary axes.} \]

## 3 Weil Representations of \( \text{Mp}(2, \mathbb{Z}) \)

In this section we recall those basic facts about Weil representations of \( \text{Mp}(2, \mathbb{Z}) \) associated to finite quadratic modules which we shall need in the sequel. These representations have been studied by [Kl 46], [N-W 76], [Ta 67] et al.. A more complete account of this theory as well as some deeper facts which can not be found in the literature will be given in [Sko 07].

By a finite quadratic module \( M \) we understand a finite abelian group \( M \) endowed with a quadratic form \( Q_M : M \rightarrow \mathbb{Q}/\mathbb{Z} \). Thus, by definition, we have \( Q_M(ax) = a^2Q_M(x) \) for all \( x \) in \( M \) and all integers \( a \), and the application \( B_M(x, y) := Q_M(x + y) - Q_M(x) - Q_M(y) \) defines a \( \mathbb{Z} \)-bilinear map \( B_M : M \times M \rightarrow \mathbb{Q}/\mathbb{Z} \). All quadratic modules occurring in the sequel will be assumed to be non-degenerate if not otherwise stated. Recall that \( M \) is called non-degenerate if \( B_M(x, y) = 0 \) for all \( y \) is only possible for \( x = 0 \).
Denote by $\mathbb{C}[M]$ the complex vector space of all formal linear combinations $\sum_x \lambda(x) e_x$, where $e_x$, for $x \in M$, is a symbol, where $\lambda(x)$ is a complex number and where the sum is over all $x$ in $M$. We define an action of $(T, w_T)$ and $(S, w_S)$ on $\mathbb{C}[M]$ by

$$(T, w_T) e_x = e(Q_M(x)) e_x$$

$$(S, w_S) e_x = \sigma |M|^{-\frac{1}{2}} \sum_{y \in M} e_y e(-B_M(y, x)),$$

where

$$\sigma = \sigma(M) = |M|^{-\frac{1}{2}} \sum_{x \in M} e(-Q_M(x)).$$

This can be extended to an action of the metaplectic group $\mathbb{M}$ [Sko 07], and we shall use $W(M)$ to denote the $\mathbb{M}$-module with underlying space $\mathbb{C}[M]$. This action factors through $SL(2, \mathbb{Z})$ if and only if $\sigma^4 = 1$ [Sko 07]; in general, $\sigma$ is an eighth root of unity. That these formulas define an action of $SL(2, \mathbb{Z})$ if $\sigma^4 = 1$ is well-known (cf. e.g. [N 76]).

It follows immediately from the defining formulas for the Weil representations that the pairing $\{-, -\} : W(M) \otimes W(-M) \to \mathbb{C}$ given by $\{e_x, e_y\} = 1$ if $x = y$ and $\{e_x, e_y\} = 0$ otherwise is invariant under $\mathbb{M}$. Here $-M$ denotes the quadratic module with the same underlying group as $M$ but with the quadratic form $x \mapsto -Q_M(x)$. The $\mathbb{M}$-invariance of this pairing is just another way to state that the matrix representation of $\mathbb{M}$ afforded by $W(M)$ with respect to the basis $e_x \ (x \in M)$ is unitary. The perfect pairing induces a natural isomorphism of $W(M)^c$ and $W(-M)$.

A standard example for a quadratic module is the determinant group $D_F$ of a symmetric non-degenerate half-integral matrix $F$. By half-integral we mean that $2F$ has integer entries and even integers on the diagonal. The quadratic module has $D_F = \mathbb{Z}^n / 2F\mathbb{Z}^n$ as underlying abelian group. The quadratic form on $D_F$ is the one induced by the quadratic form $x \mapsto \frac{1}{4}F^{-1}[x]$ on $\mathbb{Z}$. We shall henceforth write $W(F)$ for $W(D_F)$. Special instances are the quadratic modules $D_m = (\mathbb{Z}/2m, x \mapsto \frac{x^2}{4m})$ for integers $m \neq 0$ and their associated Weil representations $W(m)$. The decomposition of $W(m)$ into irreducible $\mathbb{M}$-modules was given in [Sko 85, Theorem 1.8, p.22]. We shall recall this in section 6.

The level $l$ of a quadratic module is the smallest positive integer such that $lM = 0$ and $lQ_M = 0$. The level of $D_F$ coincides with the level of $2F$ as defined in the theory of quadratic forms, i.e. it coincides with the smallest integer $l > 0$ such that $lF^{-1}/2$ is half-integral. If $l$ denotes the level of $M$ and if $\sigma(M)^4 = 1$, i.e. if $W(F)$ can be viewed as $SL(2, \mathbb{Z})$-module, then the
group $\Gamma (l)$ acts trivially on $W(M)$; if $\sigma (M)^4 \neq 1$ then $l$ is divisible by 4 and $\Gamma (l)^*$ acts trivially on $W(M)$ [Sko 07].

By $O(M)$ we denote the orthogonal group of a quadratic module $M$, i.e. the group of all automorphisms of the underlying abelian group of $M$ such that $Q_M \circ \alpha = Q_M$. The group $O(m)$, for an integer $m > 0$, is the group of left multiplications of $\mathbb{Z}/2m$ by elements $a$ in $(\mathbb{Z}/2m)^*$ satisfying $a^2 = 1$. Its order equals the number of prime factors of $m$. The group $O(M)$ acts on $\mathbb{C}[M]$ in the obvious way. It is easily verified from the defining equations for the action of $(S, w_S)$ and $(T, w_T)$ on $\mathbb{C}[M]$ that the action of $O(M)$ intertwines with the action of $M$ on $W(M)$. In particular, if $H$ is a subgroup of $O(M)$ then the subspace $W(M)^H$ of elements in $W(F)$ which are invariant by $H$ is a $M$-submodule of $W(M)$.

It will turn out that spaces of Jacobi forms of critical weight are intimately related to the spaces of invariants of Weil representations. For a Weil representation $W(M)$ we use $\text{Inv}(M)$ for the subspace of elements in $W(M)$ which are invariant under the action of $M$. For a matrix $F$, we also write $\text{Inv}(F)$ for $\text{Inv}(D_F)$. If $\sigma (M)^4 \neq 1$ then the action of $M$ on $W(M)$ does not factor through $\text{SL}(2, \mathbb{Z})$, hence $(1, -1)$ does not act trivially on $W(M)$. It can be checked (or cf. [Sko 07]) that $(1, -1)$ acts as nontrivial homothety, whence $\text{Inv}(M) = 0$.

If $\sigma (M)$ is a fourth root of unity then the question for $\text{Inv}(M)$ is much more subtle. Roughly speaking there will be invariants if $M$ is big enough, and the spaces $\text{Inv}(M)$ fall into several natural categories according to certain local invariants of quadratic modules [Sko 07].

There is one obvious way to construct invariants. Namely, suppose that $M$ contains an isotropic self-dual subgroup $U$, i.e. a subgroup $U$ such that $Q_M(x) = 0$ for all $x$ in $U$ and such that the dual $U^*$ of $U$ equals $U$ (where, for a submodule $U$, the dual $U^*$ is, by definition, the submodule of all $y$ in $M$ satisfying $B_M(x, y) = 0$ for all $x$ in $U$). Then the element $I_U := \sum_{x \in U} e_x$ is invariant under $\text{SL}(2, \mathbb{Z})$ (as follows immediately from the defining equations for the action of $S$ and $T$ and the fact that these matrices generate $\text{SL}(2, \mathbb{Z})$).

Note that here $|M|$ must be a perfect square (since, for a subgroups $U$ of $M$ one always has an isomorphism of abelian groups $M/U \cong U^*$). Also, it is not hard to check that here $\sigma (M) = 1$.

There is one important case where this construction exhausts all invariants. We cite some of the results of [Sko 07] which will clarify this a bit and which will supplement the considerations in section 5. For a prime $p$, let $M(p)$ be the quadratic module with the $p$-part of the abelian group $M$ as underlying space, equipped with the quadratic form inherited from $M$. 
We set \( \sigma_p(M) := \sigma\left(M(p)\right) \). If \( F \) is half-integral and non-degenerate then

\[
\sigma_p(D_F) = \begin{cases} 
  e_8\left(p\text{-excess}(2F)\right) & \text{for } p \geq 3, \\
  e_8\left(-\text{oddity}(2F)\right) & \text{for } p = 2.
\end{cases}
\]

(For \( p\text{-excess} \) and oddity cf. [Co-S 88, p. 370].) The proof will be given in [Sko 07]; we use these formulas in this article only for the case where \( F \) is a scalar matrix, say \( F = (n) \). Here these formulas read

\[
\sigma_p(D_n) = \begin{cases} 
  \sqrt{\left(-\frac{4}{q}\right)\left(-\frac{n/q}{q}\right)} & \text{if } p \neq 2, \\
  e_8(-n/q)\left(-\frac{n/q}{2q}\right) & \text{if } p = 2,
\end{cases}
\]

where \( q \) denotes the exact power of \( p \) dividing \( n \). These identities follow directly from the well-known theory of Gauss sums.

**Theorem 1.** Let \( M \) be a quadratic module whose order is a perfect square and such that \( \sigma_p(M) = 1 \) for all primes. Then \( \text{Inv}(M) \) is different from zero. Moreover, \( \text{Inv}(M) \) is generated by all \( I_U = \sum_{x \in U} e_x \), where \( U \) runs through the isotropic self-dual subgroups of \( M \).

The proof of this theorem is quite tedious. The theorem can be reduced to a special case of a more general theory concerning invariants of the Clifford-Weil groups of certain form rings [N-S-R 07, Theorem 5.5.7]. A more direct proof tailored to the Weil representations considered here will be given in [Sko 07]. It is not hard to show that the assumptions of Theorem 1 are necessary for \( \text{Inv}(M) \) being generated by invariants of the form \( I_U \) (in this article we do not make use of this). In general, if \( M \) does not satisfy the hypothesis of Theorem 1, there might still be invariants. However, there is one important case, where this is not the case.

**Theorem 2.** Let \( M \) be a quadratic module whose order is a power of the prime \( p \). Suppose \( \dim_{\mathbb{F}_p} M \otimes \mathbb{F}_p \leq 2 \). Then \( \text{Inv}(M) \neq 0 \) if and only if \( |M| \) is a perfect square and \( \sigma(M) = 1 \)

The proof of this theorem will be given in 6. In general, as soon as \( \dim_{\mathbb{F}_p} M \otimes \mathbb{F}_p > 2 \) the space of invariants of \( D_F \) is nontrivial [Sko 07].

If \( M \) and \( N \) are quadratic modules, we denote by \( M \perp N \) the orthogonal sum of \( M \) and \( N \), i.e. the quadratic module whose underlying abelian group is the direct sum of the abelian groups \( M \) and \( N \) and whose quadratic form is given by \( x \oplus y \mapsto Q_M(x) + Q_N(y) \). It is obvious that every \( M \) is the orthogonal sum of its \( p \)-parts. From the product formula for quadratic forms
[Co-S 88, p. 371] the sum of the numbers $p$-excess$(2F)$, taken over all odd $p$, minus the oddity of $2F$ plus the signature of $2F$ add up to 0 modulo 8. Hence we obtain $\sigma(D_F) = e_8(-\text{signature}(2F))$.

A nice functorial (and almost obvious) property is that the $M$-modules $W(M \perp N)$ and $W(M) \otimes W(N)$ are isomorphic. In particular, $W(M)$ is isomorphic to $\bigotimes W(M(p))$, taken over, say, all $p$ dividing the exponent of $M$. If $l$ denotes the level of $M$ then, for each prime $p$, the level of $M(p)$ equals the $p$-part $l_p$ of $l$, and then $W(M), W(M(2))$ and $W(M(p))$, for odd $p$, factor through $\Gamma(l)^* \Gamma(l_2)^* \Gamma(l_p)$, respectively (here we view $M(p)$ as $\text{SL}(2, \mathbb{Z})$-module which is possible since $\sigma(M(p))^4 = 1$ as is obvious from the definition of $\sigma$ in terms of Gauss sums). Since $M$ is isomorphic to the product of the groups $\mathbb{M}/\Gamma(l_2)^*$ and $\text{SL}(2, \mathbb{Z})/\Gamma(l_p)$ we deduce that in fact $W(M)$, viewed as $\mathbb{M}/\Gamma(l)^*$-module, is naturally isomorphic to the outer tensor product of the $\mathbb{M}/\Gamma(l_2)^*$-module $W(M(2))$ and the $\text{SL}(2, \mathbb{Z})/\Gamma(l_p)$-modules $W(M(p))$. In particular, we have a natural isomorphism

$$\text{Inv}(M) \cong \bigotimes_{p|l} \text{Inv}(M(p)).$$

The preceding Theorem thus implies

**Theorem 3.** Let $F$ be half-integral. Suppose that $\dim_{\mathbb{F}_p} D_F \otimes \mathbb{F}_p \leq 2$ for all primes $p$. Then $\text{Inv}(F) \neq 0$ if and only if $\det(2F)$ is a perfect square and $\sigma_p(D_F) = 1$ for all primes $p$.

**Remark.** If $F$ is positive-definite, say, of size $n$, then $\sigma(D_F) = e_8(-n)$. Thus, if $\dim_{\mathbb{F}_p} D_F \otimes \mathbb{F}_p \leq 2$ for all primes $p$ then, by the theorem, $\text{Inv}(F) = 0$ unless $\det(D_F)$ is a perfect square and $n$ is divisible by 8.

The meaning of the numbers $\sigma_p(M)$ becomes clearer if one introduces the Witt group of finite quadratic modules (see [Sch 84, Ch. 5, §1], or [Sko 07] for a discussion more adapted to the current situation). This group generalizes the well-known Witt group of quadratic spaces, say, over the field $\mathbb{F}_p$, which can be viewed as special quadratic modules. We call two quadratic modules $M$ and $N$ Witt equivalent if they contain isotropic subgroups $U$ and $V$, respectively, such that $U^*/U$ and $V^*/V$ are isomorphic as quadratic modules. Here, for an isotropic subgroup $U$ of a quadratic module $M$, we use $U^*/U$ for the quadratic module with underlying group $U^*/U$ (as quotient of abelian groups) and quadratic form $x + U \mapsto Q_M(x)$ (note that $Q_M(x)$, for $x \in U^*$ does depend on $x$ only modulo $U$). Note that a quadratic module $M$ is Witt

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1In the literature, this formula is sometimes cited as Milgram’s formula.
equivalent to the trivial module 0 if and only if it contains an isotropic self-dual subgroup. It is not hard to see that Witt equivalence defines indeed an equivalence relation, and that the orthogonal sum \( \perp \) induces the structure of an abelian group on the set of Witt equivalence classes. One can prove then [Sko 07] (but this can also be read off from [Sch 84, Ch. 5, §1,2]):

**Theorem 4.** Two quadratic modules \( M \) and \( N \) are Witt equivalent if and only if their orders are equal up to a rational square and \( \sigma_p(M) = \sigma_p(N) \) for all primes \( p \).

That Witt equivalent modules have the same order in \( \mathbb{Q}^*/\mathbb{Q}^{*2} \) and the same \( \sigma_p \)-invariants is obvious (for proving equality of the sigma invariants, say, for modules of prime power order, split, for an isotropic submodule of \( M \), the sum in the definition of \( \sigma(M) \) into a double sum over a complete set of representatives \( y \) for \( M/U \) and a sum over \( x \) in \( U \)). The converse statement is not needed in this article and for its proof we refer the reader to [Sko 07] or [Sch 84].

The connection between Witt equivalence and Weil representations is given by the following functorial property of quadratic modules. If \( U \) is an isotropic subgroup of \( M \) then \( U^*/U \) is again non-degenerate, and hence we can consider its associated Weil representation. The map \( e_{x+U} \mapsto \sum_{y \in x+U} e_y \) defines an \( M \)-equivariant embedding of \( W(U^*/U) \) into \( W(M) \). Again, this is an immediate consequence of the defining equations for the action of \((T,1)\) and \((S,w_S)\).

## 4 Jacobi Forms and Vector Valued Modular Forms

If \( \Gamma \) denotes a subgroup of \( \text{SL}(2, \mathbb{Z}) \) we use \( J_n(\Gamma) \) for the *Jacobi group* \( J_n(\Gamma) = \Gamma \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n) \). Thus, \( J_n(\Gamma) \) consists of all pairs \((A,(\lambda,\mu))\) with \( A \in \Gamma \), and \( \lambda,\mu \in \mathbb{Z}^n \), equipped with the composition law

\[
(A,(\lambda,\mu)) \cdot (A',(\lambda',\mu')) = (AA',(\lambda,\mu)A' + (\lambda',\mu')).
\]

Here and in the following elements of \( \mathbb{Z}^n \) will be considered as column vectors. Moreover, we use \((\lambda,\mu)A\) for \((\lambda a + \mu c, \lambda b + \mu d)\) if \( A = [a,b;c,d] \).

We identify \( \Gamma \) with the subgroup \( \Gamma \ltimes (0 \times 0) \), and for \( \lambda,\mu \in \mathbb{Z}^n \) we use \([\lambda,\mu]\) for the element \((1,(\lambda,\mu))\) of \( J_n(\Gamma) \). Then any element \( g \in J_n(\Gamma) \) can be written uniquely as \( g = A[\lambda,\mu] \) with suitable \( A \in \Gamma \) and \( \lambda,\mu \in \mathbb{Z}^n \).

Let \( F \) be a symmetric, half-integral \( n \times n \) matrix. For every integer \( k \), we have an action of the Jacobi group \( J_n(\Gamma) \) on functions \( \phi \) defined on \( \mathbb{H} \times \mathbb{C}^n \)
which is given by the formulas:

\[ \phi|_{k,F} A(\tau, z) = \phi \left( A\tau, \frac{z}{c\tau + d} \right) (c\tau + d)^{-k} e \left( \frac{-cF[z]}{c\tau + d} \right), \]

\[ \phi|_{k,F}[\lambda, \mu](\tau, z) = \phi(\tau, z + \lambda \tau + \mu) e \left( \tau F[\lambda] + 2z^t F\lambda \right) \]

where \( A = [a, b; c, d] \in \Gamma \) and \( \lambda, \mu \in \mathbb{Z}^n \).

For the following it is convenient to admit also half-integral \( k \). To this end we consider the group \( J_n(\Gamma) = \Gamma \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n) \) for subgroups \( \Gamma \) of \( \mathbb{M} \), which is defined as in the case of a subgroup of \( \text{SL}(2, \mathbb{Z}) \) with respect to the action \( ((A, w), x) \mapsto x A \) of \( \mathbb{M} \) on \( \mathbb{Z}^2 \). For half-integral \( k \), we then define \( \phi|_{k,F}(A, w) \) as in the formulas above but with the factor \((c\tau + d)^{-k}\) replaced by \( w(\tau)^{-2k} \). In this way the symbol \( |_{k,F} \) defines a right action of \( J_n(\Gamma) \) on functions defined on \( \mathbb{H} \times \mathbb{C}^n \). If \( k \) is integral this action factors through the action of \( J_n(\Gamma') \) defined above, where \( \Gamma' \) denotes the projection of \( \Gamma \) onto its first coordinates.

**Definition.** Let \( F \) be a symmetric, half-integral positive definite \( n \times n \) matrix, let \( k \) be a half-integral integer and, for a subgroup \( \Gamma \) of finite index in \( \mathbb{M} \), let \( V \) be a complex finite dimensional \( \Gamma \)-module. A *Jacobi form of weight \( k \) and index \( F \) with typus \((\Gamma, V)\)* is a holomorphic function \( \phi : \mathbb{H} \times \mathbb{C}^n \to V \) such that the following two conditions hold true:

(i) For all \( J_n(\Gamma) \) and all \( \tau \in \mathbb{H}, z \in \mathbb{C}^n \), one has \( (\phi|_{k,F} g)(\tau, z) = g(\phi(\tau, z)) \), where we view \( V \) as a \( J_n(\Gamma) \)-module by letting act \( \mathbb{Z}^n \times \mathbb{Z}^n \) trivially on \( V \).

(ii) For all \( \alpha \in \mathbb{M} \) the function \( \phi|_{k,F} \alpha \) possesses a Fourier expansion of the form

\[ \phi|_{k,F} \alpha = \sum_{l \in \mathbb{Q} \setminus \mathbb{Z}, r \in \mathbb{Z}^n \atop 4l - F^{-1}[r] \geq 0} c(l, r) q^l e(z^t r). \]

We shall use \( J_{k,F}(\Gamma, V) \) for the complex vector space of Jacobi forms of weight \( k \) and index \( F \) of typus \((\Gamma, V)\)\(^2\). If, for a Jacobi form \( \phi \), in condition (i) of the definition, for all \( \alpha \), the stronger inequality \( 4l - F^{-1}[r] > 0 \) holds true then we call \( \phi \) a cusp form. The subspace of cusp forms in \( J_{k,F}(\Gamma, V) \) will be denoted by \( J_{k,F}^{\text{cusp}}(\Gamma, V) \).

\(^2\)It is sometimes useful to consider more general types of Jacobi forms. In particular, the definition does not include the case of Jacobi forms of half-integral scalar index, the basic example for such type being \( \vartheta(\tau, z) \) (cf. section 2). However, \( \vartheta(\tau, 2z) \) defines an element of \( J_{1/2,2}(\mathbb{M}, \mathbb{C}(z^3)) \) and thus our omission merely amounts to ignoring a certain additional invariance with respect to the bigger lattice \( \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \).
If \( V \) is a \( \Gamma \)-module for a subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \), then we may turn \( V \) into a \( \tilde{\Gamma} \)-module \( \tilde{V} \) by setting \( (A, w)v := Av \), and we simply write \( J_{k,F}(\Gamma, V) \) for the space \( J_{k,F}(\tilde{\Gamma}, \tilde{V}) \). Note that in this case \( J_{k,F}(\Gamma, V) = 0 \) unless \( k \) is integral (since then the element \((1, -1)\) of \( \tilde{\Gamma} \) acts trivially on \( \tilde{V} \) and it acts as multiplication by \((-1)^{2k}\) on \( J_{k,F}(\Gamma, V) \) by the very definition of the operator \( |k,F\)). Thus, if \( V \) is a \( \Gamma \)-module for a subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \), we may confine our considerations to integral \( k \) and then the first condition in the definition of \( J_{k,F}(\Gamma, V) \) is equivalent to the statement that \( \phi|_{k,F}A(\tau, z) = A(\phi(\tau, z)) \) for all \( A \in \Gamma \). Similarly, if \( V \) is an irreducible \( \Gamma \)-module for a subgroup \( \Gamma \) of \( \text{M} \) then, for integral \( k \), we have \( J_{k,F}(\Gamma, V) = 0 \) unless the action of \( \Gamma \) on \( V \) factors through an action of \( \Gamma' \) (since, if the action of \( \Gamma \) does not factor then \( \Gamma \) contains \((1, -1)\), which must act nontrivially on \( V \), and, since \( V \) is irreducible, the central element \((1, -1)\) acts then as multiplication by \(-1\)). Finally, we may in principle always confine to irreducible \( V \). Indeed, if \( V = \bigoplus V_j \) is the decomposition of the \( \Gamma \)-module \( V \) into irreducible parts, then \( J_{k,F}(\Gamma, V) \cong \bigoplus J_{k,F}(\Gamma, V_j) \).

If \( C(1) \) denotes the trivial \( \Gamma \)-module for a subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \) or \( \text{M} \) we simply write \( J_{k,F}(\Gamma) \) for \( J_{k,F}(\Gamma, C(1)) \), and we write \( J_{k,F} \) for \( J_{k,F}(\Gamma) \) if \( \Gamma \) is the full modular group. Note that for a positive integer \( m \) and integral \( k \), the space \( J_{k,m}(\Gamma) \) coincides with the usual space of Jacobi forms of weight \( k \) and index \( m \) on \( \Gamma \) as defined in [E-Z 85].

It is an almost trivial but useful observation that, in the theory of vector valued Jacobi or modular forms, one can always restrict to forms on the full group \( \text{M} \). In fact, one has

**Lemma 1.** Let \( \Gamma \) be a subgroup of \( \text{M} \) and let \( V \) be a \( \Gamma \)-module. Then there is a natural isomorphism

\[
J_{k,F}(\Gamma, V) \cong J_{k,F}(\text{M}, \text{Ind}^\text{M}_\Gamma V).
\]

(Here \( \text{Ind}^\text{M}_\Gamma V = \mathbb{C}[\text{M}] \otimes_{\mathbb{C}[\Gamma]} V \) denotes the \( \text{M} \)-module induced by \( V \).)

**Proof.** Consider the natural map \( \phi \mapsto \sum_{g \in \text{M}/\Gamma} g \otimes \phi|_{k,F}g^{-1} \). It is easily verified that the summands do not depend on the choice of the representatives \( g \) and that this map defines an isomorphism as claimed in the theorem.

Jacobi forms may be viewed as vector valued elliptic modular forms. For stating this more precisely we denote by \( M_k \) the space of holomorphic functions \( h \) on \( \mathbb{H} \) such that \( h \) is a modular form of weight \( k \) on some subgroup \( \Gamma \) of \( \text{M} \) (where we assume that \( \Gamma \) is contained in \( \Gamma_0(4) \) if \( k \) is not integral). Note that \( M_k \) is a \( \text{M} \)-module with respect to the action \( |k \). By \( M_k^{\text{cusp}} \) we denote the submodule of cusp forms. Using the \( \text{M} \)-module \( W(F) \) introduced in section 3 we then have
Theorem 5. For a subgroup $\Gamma$ of $\mathbb{M}$, let $V$ be a finite dimensional $\Gamma$-module. Assume that the image of $\Gamma$ under the associated representation is finite. Then, for any half-integral $k$ and any half-integral $F$ there is a natural isomorphism

$$J_{k,F}(\Gamma, V) \cong (M_{k-\frac{n}{2}} \otimes W(F)^*) \otimes_{\mathbb{C}[\mathbb{M}]} \text{Ind}_{\Gamma}^{\mathbb{M}} V.$$ 

Proof. Using the isomorphism of the preceding lemma we can assume that $V$ is a $\mathbb{M}$-module.

Denote by $J_{k,F}$ the space of all functions $\phi$, where $\phi$ is in $J_{k,F}(\Gamma)$ for some subgroup $\Gamma$. The space $J_{k,F}$ is clearly a $\mathbb{M}$-module under the action $|_{k,F}$. The bilinear map $(\phi, v) \mapsto \phi \cdot v$ induces an isomorphism

$$J_{k,F} \otimes_{\mathbb{C}[\mathbb{M}]} V \cong J_{k,F}(\mathbb{M}, V).$$

For verifying the surjectivity of this map we need that some subgroup $\Gamma$ of finite index in $\mathbb{M}$ acts trivially on $V$ (or equivalently that the image of $\mathbb{M}$ under the representation afforded by the $\mathbb{M}$-module $V$ is finite). Namely, if we write a $\phi \in J_{k,F}(\Gamma, V)$ with respect to some basis $e_j$ of $V$ as $\phi = \sum_j \phi_j \cdot e_j$, then, by our assumption, the $\phi_j$ are scalar valued Jacobi forms on $J_{\frac{k}{2},F}(\Gamma)$.

Assume now that $\psi$ is an element of $J_{k,F}(\Gamma)$. Then it is well-known (and follows easily from the invariance of $\psi$ under $\mathbb{Z}^n \times \mathbb{Z}^n$) that we can expand $\psi$ in the form

$$\psi(\tau, z) = \sum_{x \in \mathbb{Z}^n/2F\mathbb{Z}^n} h_x(\tau) \vartheta_{F,x}(\tau, z)$$

with functions $h_x$ which are holomorphic in $\mathbb{H}$, and where the $\vartheta_{F,x}$ denote the theta series defined in section 2. The space of functions $\Theta(F)$ spanned by the $\vartheta_{F,x}$ is a $\mathbb{M}$-(right)module with respect to the action $|_{\frac{k}{2},F}$ and is acted on trivially by $\Gamma(4N)^*$ for some integer $N$ (see e.g. [Kl 46]). Accordingly, the $h_x$ are then invariant under $\Gamma(4N)^* \cap \Gamma$ under the action $|_{k-\frac{n}{2}}$ (one needs here also the linear independence of the $\vartheta_{F,x}(\tau, z)$, where $x$ runs through a set of representatives for $\mathbb{Z}^n/2F\mathbb{Z}^n$). In fact, one easily deduces from the regularity condition for Jacobi forms at the cusps (i.e. from condition (ii) of the definition) that the $h_x$ are elements of $M_{k-\frac{n}{2}}$. Thus, the bilinear map $(h, \theta) \mapsto h \cdot \theta$ induces an isomorphism of $\mathbb{M}$-right modules

$$M_{k-\frac{n}{2}} \otimes \Theta(F) \cong J_{k,F}.$$ 

Finally we note that $\Theta(F)$ is isomorphic as $\mathbb{M}$-module to $W(F)^*$ via the map $W(F)^* \ni \lambda \mapsto \sum_{e \in D_F} \lambda(e) \vartheta_{F,e}$ (cf. [Kl 46]). From this the theorem is now immediate. 

$\square$
Remark. A review of the foregoing proof shows that in the statement of the theorem we may replace $M_{k-\frac{1}{2}}$ by $M_{k-\frac{1}{2}}^{\text{cong}}$, the subspace of all $h$ in $M_{k-\frac{1}{2}}$ which are modular forms on some congruence subgroup of $\mathbb{M}$, provided $\Gamma(4N)^*$, for some $N$, acts trivially on $V$. This remark will become important in the next section.

We use the isomorphism of Theorem 5 to define the subspace $J_{k,F}^{\text{Eis}}(\Gamma, V)$ of Jacobi-Eisenstein series as the preimage of the subspace which is obtained by replacing $M_{k-\frac{1}{2}}$ on the right hand side of this isomorphism by the space of Eisenstein series $M_{k-\frac{1}{2}}^{\text{Eis}}$ in $M_{k-\frac{1}{2}}$ (i.e. by the orthogonal complement of the cusp forms in $M_{k-\frac{1}{2}}$).

Theorem 5 implies that there are no Jacobi forms of index $F$ and weight $k < \frac{1}{2}$ (since then $M_{k-\frac{1}{2}} = 0$). For $k = \frac{1}{2}$ we have $M_{k} = \mathbb{C}$, and hence the theorem implies

Corollary. There exists a natural isomorphism

$$J_{\frac{n}{2},F}(\Gamma, V) \cong \text{Hom}_{\mathbb{M}}(W(F), \text{Ind}_{\Gamma}^{\mathbb{M}} V).$$

The corollary reduces the study of $J_{\frac{n}{2},F}(\Gamma, V)$ to the problem of describing the decomposition of $W(F)$ and $\text{Ind}_{\Gamma}^{\mathbb{M}} V$ into irreducible parts.

For $k \geq \frac{n}{2} + 2$, the last theorem makes it possible to derive an explicit formula for the dimension of the space of Jacobi forms of weight $k$. In fact, the isomorphism of the theorem can be rewritten as

$$J_{\frac{n}{2},F} \cong M_{k-\frac{1}{2}} \otimes_{\mathbb{C}[\mathbb{M}]} (W(F) \otimes \text{Ind}_{\Gamma}^{\mathbb{M}} V).$$

But here (using the obvious map $h \otimes v \mapsto h \cdot v)$ the right hand side can be identified with the space $M_{k-\frac{1}{2}}(\rho)$ of holomorphic maps $h : \mathbb{H} \rightarrow W(F) \otimes \text{Ind} V$ which satisfy the transformation law $h(A\tau)w(t)^{-2k} = \rho(\alpha)(h(\tau))$ for all $(A, w)$ in $\mathbb{M}$, where $\rho$ denotes the representation afforded by the $\mathbb{M}$-module $W(F) \otimes \text{Ind} V$, and which satisfy the usual regularity conditions at the cusps. The dimensions of the spaces $M_{k}(\rho)$ have been computed in [Sko 85, p. 100] (see also [Eh-S 95, Sec. 4.3]) using the Eichler-Selberg trace formula. In our context these formulas read as follows:

Theorem 6. Let $F$ be a half-integral positive definite $n \times n$ matrix, let $k \in \frac{1}{2}\mathbb{Z}$, and, for a subgroup $\Gamma$ of $\mathbb{M}$, let $V$ be a $\Gamma$-module such that the associated
representation of $\Gamma$ has finite image in $GL(V)$. Then one has

$$\dim J_{k,F}(\Gamma, V) - \dim M_{2+n-k}^{\text{cusp}} \otimes_{\mathbb{C}[M]} X(i^{n-2k})^c$$

$$= \frac{k - \frac{n}{2} - 1}{2} \dim X(i^{n-2k}) + \frac{1}{4} \text{Re} \left( e^{\pi i (k - \frac{n}{2})/2} \text{tr}((S, w_S), X(i^{n-2k})) \right)$$

$$+ \frac{2}{3\sqrt{3}} \text{Re} \left( e^{\pi i(2k-n+1)/6} \text{tr}((ST, w_{ST}), X(i^{n-2k})) \right) - \sum_{j=1}^r (\lambda_j - \frac{1}{2}).$$

Here $X(i^{n-2k})$ denotes the $\mathbb{M}$-submodule of all $v$ in $W(F) \otimes \text{Ind}^\mathbb{M}_V$ such that $(-1, i)v = i^{n-2k}v$; moreover, $0 \leq \lambda_j < 1$ are rational numbers such that $\prod_{j=1}^r (t - e^{2\pi i \lambda_j}) \in \mathbb{C}[t]$ equals the characteristic polynomial of the automorphism of $X(i^{n-2k})$ given by $v \mapsto (T, 1)v$. (Recall that $T = [1, 1; 0, 1]$ and $S = [0, -1; 1, 0]$).

Note that the traces on the space $X(i^{n-2k})$ occurring on the right hand side of the dimension formula can be easily computed from the explicit formulas for the action of $(S, w_S)$ and $(T, w_T)$ on $W(F) \otimes \text{Ind} V$. Note also that the second term occurring on the left hand side vanishes for $k \geq \frac{n}{2} + 2$. In general, this second term may be interpreted as the dimension of the space $J_{2+n-k,F}(\Gamma, V)$ of skew-holomorphic Jacobi forms, which can be defined by requiring that Theorem 5 holds true for skew-holomorphic Jacobi forms if one replaces the $\mathbb{M}$-module $M_k$ on the right hand side by $M_k^c$.

It remains to discuss the case of weights $k = \frac{n}{2} + 1, \frac{n}{2} + 1, \frac{n}{2} + \frac{3}{2}$. For weight $k = \frac{n}{2} + 1$ the second term on the left hand side refers to elliptic modular forms of weight 1. Accordingly, this term is in general unknown.

If $k = \frac{n}{2} + \frac{3}{2}$ then the second term on the left hand side equals the dimension of the space of cusp forms in $J_{2+n-k,F}(\Gamma, V)$, which can be investigated by the methods of the next section. However, we shall not pursue this any further in this article. Finally, the case of critical weight $k = \frac{n+1}{2}$ will be considered by different methods in the next section.

To end this section we remark that Theorem 5 remains true if we replace on both sides of the isomorphism the space of Jacobi forms and modular forms by the respective subspaces of cusp forms (as can be easily read off from the proof of the theorem). Moreover, Theorem 6, which gives an explicit formula for $\dim J_{k,F}(\Gamma, V) - \dim J_{2+n-k,m}^{\text{cusp}}(\Gamma, V)$, remains true if the left hand side is replaced by $\dim J_{k,F}^{\text{cusp}}(\Gamma, V) - \dim J_{2+n-k,F}(\Gamma, V)$ and if we subtract from the right hand side the dimension of the maximal subspace of $X(i^{n-2k})$ which is invariant under $(T, 1)$. For deducing this from Theorem 5 we refer the reader again to [Sko 85, p. 100]).
5 Jacobi Forms of Critical Weight

In this section we study the spaces \( J_{n+1,F}(\Gamma, V) \), where \( n \) denotes the size of \( F \). We are mainly interested in the cases \( J_{n+1,F} \) and \( J_{1,N}(\varepsilon^a) \). Here \( \varepsilon \) denotes the character of \( \mathbb{M} = \text{Mp}(2, \mathbb{Z}) \) given by \( \varepsilon(A, w) = \eta(A\tau)/(w(\tau)\eta(\tau)) \), where \( \eta(\tau) \) is the Dedekind eta-function. It is a well-known fact that \( \varepsilon \) generates the group of one dimensional characters of \( \mathbb{M} \). At the end of this section we add a remark concerning the case where \( \Gamma = \Gamma_0(l) \) and where \( V \) is the trivial \( \Gamma \)-module \( \mathbb{C}(1) \); a more thorough discussion of this case with complete proofs will be given in [I-S 07].

We assume that \( \Gamma \) is a congruence subgroup. By Theorem 5 and the subsequent remark we need first of all a description of the \( \Gamma \)-module \( M_{\frac{n}{2}}^{\text{cong}} \) of all modular forms of weight \( \frac{1}{2} \) on congruence subgroups of \( \Gamma \). Starting with the observation that \( M_{\frac{1}{2}} \) is generated by theta series (cf. [Se-S 77]) the decomposition of the \( \mathbb{M} \)-module \( M_{\frac{1}{2}} \) into irreducible parts was calculated in [Sko 85, p. 101]. As an immediate consequence of the result loc. cit. we obtain

\[
M_{\frac{1}{2}}(\Gamma(4m)) \cong \bigoplus_{\substack{l|m \\ m/l \text{ squarefree}}} (W(l)^*)^*,
\]

\[
M_{\frac{1}{2}}^{\text{Eis}}(\Gamma(4m)) \cong \bigoplus_{\substack{l|m \\ m/l \text{ squarefree}}} (W(l)^{O(l)})^*
\]

Here \( \epsilon \) denotes the element \( \epsilon : x \mapsto -x \) of \( O(l) \), and \( W(l)^{\epsilon} \) and \( W(l)^{O(l)} \) denote the subspaces of elements of \( W(l) \) which are invariant under \( \epsilon \) and \( O(l) \), respectively. (For deducing this from [Sko 85, p. 101, Theorem 5.2] one also needs [Sko 85, Theorem 1.8, p. 22]).

If we insert this into the isomorphism of Theorem 5, we obtain

**Theorem 7.** For a congruence subgroup \( \Gamma \) of \( \mathbb{M} \), let \( V \) be a finite dimensional \( \Gamma \)-module, and let \( F \) be half-integral of size \( n \) and level \( f \). Assume that, for some \( m \), the group \( \Gamma((4m)^*) \) acts trivial on \( V \) and that \( f \) divides \( 4m \). Then there are natural isomorphisms

\[
J_{n+1,F}(\Gamma, V) \longrightarrow \bigoplus_{\substack{l|m \\ m/l \text{ squarefree}}} \left( W((l \oplus F)^{\epsilon \times 1})^* \otimes_{\mathbb{C}[\mathbb{M}]} \text{Ind}_{\Gamma}^{\mathbb{M}} V, \right.
\]

\[
J_{n+1,F}^{\text{Eis}}(\Gamma, V) \longrightarrow \bigoplus_{\substack{l|m \\ m/l \text{ squarefree}}} \left( W((l \oplus F)^{O(l) \times 1})^* \otimes_{\mathbb{C}[\mathbb{M}]} \text{Ind}_{\Gamma}^{\mathbb{M}} V. \right.
\]
Here $O(l) \times 1$ denotes the subgroup of $O((l) \oplus F)$ of all elements of the form $(x, y) \mapsto (\alpha(x), y)$ ($x \in D(l)$, $y \in D(F)$, $\alpha \in O(l)$), and $\epsilon \times 1$ denotes the special element of $O(l) \times 1$ given by $(x, y) \mapsto (-x, y)$.

Thus the description of Jacobi forms of critical weight reduces to a problem in the theory of finite dimensional representations of $\mathbb{M}$. Actually, the description of Jacobi forms of critical weight reduces to an even more specific problem, namely to the problem of determining the invariants of Weil representations associated to finite quadratic modules. To make this more precise we rewrite the first isomorphism of Theorem 7 as

**Theorem 8.** Under the same assumptions as in Theorem 7 the applications

$$\sum_j e_{x,j} \otimes e_{y,j} \otimes w_j \mapsto \sum_j \vartheta_{l,x,j}(\tau, 0) \vartheta_{F,y,j}(\tau, z) w_j$$

define an isomorphism

$$J_{n+1, F}(\Gamma, V) \leftarrow \bigoplus_{l|m \text{ squarefree}} \text{Inv} \left( W(-l)^{\epsilon} \otimes W(-F) \otimes \text{Ind}^M_{\mathbb{M}} V \right).$$

Here we used that, for any group $G$, any $G$-right module $A$ and $G$-left module $B$, the spaces $A \otimes_{C[G]} B$ and $(A' \otimes B)^G$ are naturally isomorphic (where $A'$ denotes the $G$-left module with underlying space $A$ and action $(g, a) \mapsto a \cdot g^{-1}$), we used the isomorphism (of vector spaces) of $W(G)^c$ with $W(-G)$ (cf. section 3), and we wrote out explicitly the isomorphism constructed in the proof of Theorem 5.

**Remark.** Next, consider a decomposition $\text{Ind}^M_{\mathbb{M}} V = \bigoplus_j W_j$ into irreducible $\mathbb{M}$-submodules $W_j$. By the results in [N-W 76] every irreducible representation of $\mathbb{M}$ is equivalent to a subrepresentation of a Weil representation $W(M)$ for a suitable finite quadratic module $M^3$. Hence we may replace the $W_j$ by

$^3$Actually it is proven in [N-W 76] that, for prime powers $m$, every finite dimensional irreducible $SL(2, \mathbb{Z}/m)$-module is contained in a Weil representation associated to some finite quadratic module. Since $SL(2, \mathbb{Z}/m)$, for arbitrary $m$, is the direct product of the groups $SL(2, \mathbb{Z}/p^n)$, where $p^n$ runs through the exact prime powers dividing $m$, and since the category of Weil representations $W(M)$ is closed under tensor products we can dispense with the assumption of $m$ being a prime power. Hence every irreducible $SL(2, \mathbb{Z})$-module which is acted on trivially by some congruence subgroup is isomorphic to a submodule of some $W(M)$. Finally, if $\rho$ is an irreducible representation of $\mathbb{M}$ which does not factor through a representation of $SL(2, \mathbb{Z})$, i.e. which satisfies $\rho(1, -1) \neq 1$, but which factors through some congruence subgroup $\Gamma(4N)^*$ then $\rho/\epsilon$ factors through a representation of $SL(2, \mathbb{Z}/4N')$ for some $N'$. Hence by the preceding argument, $\rho/\epsilon$ is afforded by a submodule of a Weil representation $W(M)$. But $\epsilon$ is afforded by a submodule in $D(6)$ (cf. section 6), and hence $\rho$ is afforded by a submodule of the Weil representation $D(6) \otimes W(M)$. 17
submodules of Weil representations \( W(M_j) \), and at the end we find a natural injection

\[
J_{\frac{n+1}{2},F}(\Gamma, V) \hookrightarrow \bigoplus_j \bigoplus_{l|m \atop m/l \text{ squarefree}} \text{Inv}(D_{-l} \perp D_{-F} \perp M_j).
\]

Here the precise image can also be characterized by the action of the groups \( O(-l) \) on \( D_{-l} \) and \( O(M_j) \) on \( W(M_j) \) (and certain additional intertwiners of the \( \mathbb{M} \)-action), so that the last isomorphism could be written in an even more explicit form. We shall not pursue this any further in this article.

We note a special case of Theorem 7. Namely, if \( V \) is the trivial \( \mathbb{M} \)-module then the right hand side of, say, the first formula of Theorem 7 becomes the space of elements in \( W((l) \oplus F)^* \) which are invariant under \( \mathbb{M} \) and \( \epsilon \times 1 \). Using again the natural isomorphism between the spaces \( W((l) \oplus F)^c \) and \( W((-l) \oplus (-F)) \) (cf. sect. 3) we thus obtain

**Theorem 9.** For any half-integral \( F \) of size \( n \) and level dividing \( 4m \) there are natural isomorphisms

\[
J_{\frac{n+1}{2},F} \cong \bigoplus_{l|m \atop m/l \text{ squarefree}} \text{Inv}((-l) \oplus (-F))^{\epsilon \times 1},
\]

\[
J_{\frac{n+1}{2},F}^{\text{Eis}} \cong \bigoplus_{l|m \atop m/l \text{ squarefree}} \text{Inv}((-l) \oplus (-F))^{O(-l) \times 1}.
\]

(Here \( O(-l) \times 1 \) and \( \epsilon \times 1 \) are as explained in Theorem 7.)

Note that \( O(-l) \times 1 \) acts on \( \text{Inv}((-l) \oplus (-F)) \) since the action of \( O(-l) \times 1 \) intertwines with the action of \( \mathbb{M} \) on the space \( W((-l) \oplus (-F)) \). Note also that this Theorem is a trivial statement if \( n \) is even since then both sides of the claimed isomorphism are 0 for trivial reasons (consider the action of \((-1,1))\). If the size of \( F \) is odd then the level of \( F \) is divisible by 4, and hence we may choose \( 4m \) equal to the level of \( F \).

If we take \( F \) equal to a number \( m \), then \( W((-l) \oplus (-m)) \) does not contain nontrivial invariants (see the remark following Theorem 3.). The theorem thus implies \( J_{1,\mathbb{N}} = 0 \), a result which was proved in [Sko 85, Satz 6.1]. More generally, Theorem 9 and the remark following Theorem 3 imply

**Theorem 10.** Let \( F \) be half-integral of size \( n \). If \( n \not\equiv 7 \mod 8 \) and \( F \) has at most one nontrivial elementary divisor then \( J_{\frac{n+1}{2},F} = 0 \).
Note that we cannot dispense with the assumption \( n \not\equiv 7 \mod 8 \). A counterexample can be easily constructed as follows. Let \( 2G \) denote a Gram matrix of the \( E_8 \)-lattice then
\[
\theta(\tau, z) := \sum_{x \in \mathbb{Z}^8} e(\tau G[x] + 2z'Gx) \quad (z \in \mathbb{C}^8)
\]
defines an element of \( J_{4,G} \). If \( M \) is an integral \( 8 \times 7 \)-matrix then \( F := M^tGM \) is half-integral positive definite of size 7 and it is easily checked that
\[
\theta|U_M(\tau, w) := \theta(\tau, Mw) \quad (w \in \mathbb{C}^7)
\]
defines a nonzero element of \( J_{4,F} \), hence a Jacobi form of critical weight.

Suitable choices of \( G \) and \( M \) yield an \( F \) with exactly one elementary divisor, i.e. an \( F \) satisfying the second assumption of the theorem.

For doing explicit calculations it is worthwhile to write out explicitly the isomorphisms of Theorem 9. If \( v = \sum_{x,y} \lambda(x,y) e(x,y) \) is an element of \( \text{Inv}((-l) \oplus (-F)) \) then
\[
\phi_v(\tau, z) := \sum_{x \in \mathbb{Z}^8/2l} \sum_{y \in \mathbb{Z}^n/2FZ^n} \lambda(x,y) \vartheta_{l,x}(\tau,0) \vartheta_{F,y}(\tau,z)
\]
defines a Jacobi form in \( J_{4,F} \). It vanishes unless \( \lambda(x,y) \) is even in \( x \), i.e. unless \( v \) is invariant under \( \epsilon \times 1 \), and it defines an Eisenstein series if \( \lambda(x,y) \) is invariant under \( O(-l) \times 1 \).

We now turn to the case \( J_{1,N}(\epsilon^a) \). Using the Jacobi forms
\[
\vartheta(\tau, az) = q^{1/8}(\zeta^{a/2} - \zeta^{-a/2}) \prod_{n \geq 1} (1 - q^n)(1 - q^n\zeta^a)(1 - q^n\zeta^{-a})
\]
for positive natural numbers \( a \) and trying to build \( \text{thetablocks} \), i.e. Jacobi forms which are products or quotients of these forms and powers of the

\[2G = \begin{pmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]
and \( 2F \) equal to the matrix which is obtained by deleting the last row and column of \( 2G \), which has 4 as sole nontrivial elementary divisor.

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\(^4\text{One can take}

Dedekind eta-function, it turns out [GSZ 07] that there are indices \( N \) and nontrivial Jacobi forms in \( J_{1,N}(\epsilon^a) \) if \( a \equiv 2, 4, 6, 8, 10, 14 \mod 24 \). Moreover, extensive computer aided search suggests that for all other \( a \) there are indeed no Jacobi forms, and that all Jacobi forms of weight 1 can be obtained by the indicated procedure built on the \( \psi(\tau, az) \). Note that in fact \( J_{1,N} = 0 \) for all \( N \) [Sko 85, [Satz 6.1]. As an application of the theory developed so far we shall prove in section 6

**Theorem 11.** For all positive integers \( N \), one has \( J_{1,N}(\epsilon^{16}) = 0 \).

Actually, we shall prove more. Namely,

**Theorem 12.** For every positive integers \( N \), the space \( J_{1,N}(\epsilon^8) \) is spanned by the series

\[
\vartheta_{\rho} := \sum_{\alpha \in \mathcal{O}} \left( \frac{x(\alpha)}{3} \right) q^{\frac{\|\alpha\|^2}{3}} \zeta^{y(\alpha \rho)}
\]

Here \( \mathcal{O} = \mathbb{Z}[\frac{1+i\sqrt{-3}}{2}] \), and we use \( x(\alpha) = \alpha + \overline{\alpha} \) and \( y(\alpha) = (\alpha - \overline{\alpha})/\sqrt{-3} \). Moreover, \( \rho \) runs through all numbers in \( \mathcal{O} \) with \( |\rho|^2 = N \).

**Remark.** Let \( \rho = \frac{p + i\sqrt{-3}}{2} \) be a number in \( \mathcal{O} \) with \( |\rho|^2 = N \). By multiplying \( \rho \) by a suitable 6th root of unity (which will not change \( \vartheta_{\rho} \)) we may assume that \( q \geq |p| > 0 \). For \( q = |p| \) one has \( \vartheta_{\rho} = 0 \). For \( q > |p| \), one can show by elementary transformations of the involved series [GSZ 07] that \( \vartheta_{\rho} \) has the factorization

\[
\vartheta_{\rho}(\tau, z) = -\vartheta(\tau, \frac{q + p}{2} z) \vartheta(\tau, \frac{q - p}{2} z) \vartheta(\tau, qz) / \eta(\tau).
\]

Similar theorems can be produced for the spaces \( J_{1,N}(\epsilon^a) \) for arbitrary even integers \( a \) modulo 24. Since the analysis of the invariants in the corresponding Weil representations becomes quite involved this will eventually be treated elsewhere.

Finally, we mention the case \( \Gamma = \Gamma_0(l) \) acting trivially on \( V = \mathbb{C}(1) \). On investigating the representation \( \text{Ind}_V \) occurring on the right hand side of Theorem 7 it is possible to deduce the following [I-S 07]

**Theorem 13.** Let \( F \) be a symmetric, half-integral and positive definite matrix of size \( n \times n \) with odd \( n \). Then, for every positive integer \( l \), we have

\[
J_{\frac{n+1}{2},F}(\Gamma_0(l)) = J_{\frac{n+1}{2},F}(\Gamma_0(l_1))
\]

Here \( l_1 \) is the first factor in the decomposition \( l = l_1 l_2 \), where \( l_1 | \det(2F)^\infty \) and where \( l_2 \) is relatively prime to \( \det(2F) \).
For $n = 1$, i.e. for the case of ordinary Jacobi forms in one variable, the theorem was already stated and proved in [I-S 06].

As immediate consequence we obtain

**Corollary.** Suppose $J_{n+1,F} = 0$. Then we have $J_{n+1,F}(\Gamma_0(l)) = 0$ for all $l$ which are relatively prime to $\det(2F)$.

This corollary might be meaningful for the study of Siegel modular forms of critical weight on subgroups $\Gamma_9(l)$. In [I-S 06] we used it to prove that there are no Siegel cusp forms of weight one on $\Gamma_0(l)$, for any $l$.

### 6 Proofs

In this section we append the proofs of Theorems 2, 11 and 12.

We start with the description of the decomposition of Weil representations of rank 1 modules into irreducible parts. For an odd prime power $q = p^\alpha \geq 1$ let $L_q$ be the quadratic module $(\mathbb{Z}/q, x^2/q)$. The submodule $U := p^{\lceil\alpha/2\rceil}L_q$ is isotropic, for its dual one finds $U^* = p^{\lfloor\alpha/2\rfloor}L_q$ and, for $\alpha \geq 2$, the quotient module $U^*/U$ is isomorphic (as quadratic module) to $L_q/p^2$. Hence $W(L_q/p^2)$ embeds naturally as $\mathbb{M}$-submodule into $W(L_q)$ (see the discussion at the end of section 3). Let $W_1(L_q)$ be the orthogonal complement of $W(L_q/p^2)$ (with respect to the $\mathbb{M}$-invariant scalar product $(\cdot, \cdot)$ given by $(e_x, e_y) = 1$ if $x = y$ and $(e_x, e_y) = 0$ otherwise). Then $W_1(L_q)$ is invariant under $\mathbb{M}$ and under $O(L_q)$. The latter group is generated by the involution $\alpha : x \mapsto -x$. For $\epsilon = \pm 1$, let $W_1^\epsilon(L_q)$ be the $\epsilon$-eigenspace of $\alpha$ viewed as involution on $W_1(L_q)$. Since $\alpha$ intertwines with $\mathbb{M}$ these eigenspaces are $\mathbb{M}$-submodules of $W(L_q)$.

By induction we thus obtain the decomposition

$$W(L_q) = \bigoplus_{d^2 | q} (W_1^+(L_q/d^2) \oplus W_1^-(L_q/d^2)) \oplus \begin{cases} W(L_1) & \text{if } q \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

of $W(L_q)$ into $\mathbb{M}$-submodules. Note that $W(L_1) \cong \mathbb{C}(1)$.

Finally, for an integer $a$ which is relatively prime to $q$ we let $L_q(a)$ be the quadratic module which has the same underlying abelian group as $L_q$ but has quadratic form $Q_{L_q(a)}(x) = aQ_{L_q}(x)$. With $W_1^\epsilon(L_q(a))$ being defined similarly as before we obtain a corresponding decomposition of $W(L_q(a))$ as for $W(L_q)$.

Note that $\sigma(L_q(a)) = 1$ if $q$ is a square, and otherwise $\sigma(L_q(a)) = \left(\frac{-a}{p}\right)$ if $p \equiv 1 \mod 4$ and $\sigma(L_q(a)) = \left(\frac{-a}{p}\right)i$ if $p \equiv 3 \mod 4$. We may restate this
as $\sigma((L_q(a))) = \left(\frac{-a}{p}\right)^\alpha \sqrt{\left(\frac{-4}{p}\right)^\alpha}$ for $q = p^\alpha$. In particular, $W(L_q(a))$ can be viewed as $\text{SL}(2, \mathbb{Z})$-module. The level of $L_q(a)$ equals $q$, whence $\Gamma(q)$ acts trivially on $W(L_q(a))$.

**Lemma 2.** The $\text{SL}(2, \mathbb{Z})$-modules $W_1^r(L_q(a))$ are irreducible. The exact level of $W_1^r(L_q(a))$ equals $q$ (i.e. $q$ is the smallest positive integer such that $\Gamma(q)$ acts trivially on $W_1^r(L_q(a))$). Two $\mathcal{M}$-modules $W_1^r(L_q(a))$ and $W_1^{r'}(L_q(a'))$ are isomorphic if and only if $q = q'$, $\epsilon = \epsilon'$ and $aa'$ is a square modulo $p$.

**Proof.** It is easy to see that the modules $W_1^r(L_q(a))$ are nonzero. Thus the given decomposition of $W(L_q(a))$ contains $\sigma_0(q)$ non zero parts. Hence, for proving the irreducibility of these parts is suffices to show that the number of irreducible components of $W(L_q(a))$ is exactly $\sigma_0(q)$. But the number of irreducible components equals the number of invariant elements of $W(L_q(a)) \otimes W(L_q(-a))$. i.e. the dimension of $\text{Inv} \left( L_q(a) \perp -L_q(a) \right)$. The latter space is spanned by the elements $I_U := \sum_{x \in U} e_x$, where $U$ runs through the isotropic self-dual modules of $L_q(a) \perp -L_q(a)$ (cf. Theorem 1).

Alternatively, for avoiding the use of Theorem 1, which we did not prove in this article, (but see [N-S-R 07, Theorem 5.5.7]), one can show that $W(L_q(-a)) = \text{Inv} \left( L_q(a) \perp -L_q(a) \right)$ is isomorphic to the permutation representation of $\text{SL}(2, \mathbb{Z})$ given by the right action of $\text{SL}(2, \mathbb{Z})$ on the row vectors of length 2 with entries in $\mathbb{Z}/q$ (see e.g. [N-W 76, § 3]). The number of invariants equals thus the number of orbits of this action, which in turn are naturally parameterized by the divisors of $q$ (all vectors $(x, y)$ with $\gcd(x, y, q) = d$, for fixed $d|q$ constitute an orbit).

We already saw that $\Gamma(q)$ acts trivially on $W_1^r(L_q(a))$. That $q$ is the smallest integer with this property follows from the fact that $e(a/q)$ is an eigenvalue of $T$ acting on $W_1^r(L_q(a))$. The latter can directly be deduced from the definition of $W_1^r(L_q(a))$ (see also [Sko 85, p. 33] for details of this argument).

Finally, if $W_1^r(L_q(a))$ and $W_1^{r'}(L_q(a'))$ are isomorphic then, by comparing the levels of the $\text{SL}(2, \mathbb{Z})$-modules in question, we conclude $q = q'$. The eigenvalues of $T$ on each of these spaces are of the form $e_q(ax^2)$ and $e_q(a'x'^2)$, respectively, for suitable integers $x$, with at least one $x$ relatively prime to $q$; whence $aa'$ must be a square modulo $q$. But then $W_1^r(L_q(a'))$, (and hence, by assumption, $W_1^r(L_q(a'))$, is isomorphic to $W_1^{r'}(L_q(a))$. This finally implies $\epsilon = \epsilon'$, since $-1 = S^2$ acts by $e_x \mapsto \sigma^2 e_{-x} = \sigma^2(a \cdot e_x)$ on $L_q(a)$ (where $\sigma = \sigma(L_q(a))$) as follows from the explicit formula for the action of $S$. □

Similarly, we can decompose the representations associated to the modules $D_{2^\alpha}$ or, more generally, to the modules $D_{2^\alpha}(a) := (\mathbb{Z}/2q, ax^2/4q)$,
isotropic subgroups of $W$ embeds naturally into a where however, that since $\sigma$ holds true with suitable modifications when replacing $L_q(a)$ by $D_{2^s}(a)$. Note, however, that $D_{2^s}(a)$ does not factor through a representation of $SL(2,\mathbb{Z})$ since $\sigma(D_{2^s}(a)) = e_8(-a)$.5

**Lemma 3.** Let $q$ be a power of 2. The $SL(2,\mathbb{Z})$-modules $W_r^f(D_q(a))$ are irreducible. The exact level of $W_r^f(D_q(a))$ equals $4q$ (i.e. $4q$ is the smallest positive integer such that $\Gamma(q)^\ast$ acts trivially). Two $\mathbb{M}$-modules $W_r^f(D_q(a))$ and $W_r^f(D_q'(a'))$ are isomorphic if and only if $q = q'$, $\epsilon = \epsilon'$ and $aa'$ is a square modulo $4q$.

The proof is essentially the same as for the preceding lemma and we leave it to the reader.

Note that the modules $D_m$, for arbitrary nonzero $m$, and, more generally, the modules $D_m(a) := (\mathbb{Z}/2m, ax^2/4m)$, for $a$ relatively prime to $m$, have as $p$-parts modules of the form $L_q(a')$ and $D_{2^s}(a'')$, whence $W(D_m(a))$ is isomorphic to the outer tensor product of $SL(2,\mathbb{Z})/\Gamma(q)$-modules $W(L_q(a'))$ and a suitable $\mathbb{M}/\Gamma(4 \cdot 2^s)$-module $W(D_{2^s}(a''))$. We conclude that the $\mathbb{M}$-module $W(D_m(a))$ decomposes as

$$W(D_m(a)) \cong \bigoplus_{f \text{squarefree}} W_r^f(D_{m/d^2}(a)).$$

Here $W_r^f(D_m(a))$ is the subspace of all $v = \sum_{x \in D_m} \lambda(x)e_x$ in $W(D_m(a))$ such that, for all isotropic submodules $U$ of $D_m(a)$ and all $y \in U^\ast$, one has $\sum_{x \in y + U} \lambda(x) = 0$, and such $gv = \chi_f(g)v$ for all $g$ in $O(m)$. Here $\chi_f$ denotes that character of $O(m)$ which maps $g_p$ to $-1$ if $p | f$ and to $1$ otherwise, where $g_p$ is the orthogonal map which corresponds to the residue class $x$ in $\mathbb{Z}/2m$ (under the correspondence described in section 3) which satisfies $x \equiv -1 \mod 2p^a$ and $x \equiv +1 \mod 2m/p^a$ with $p^a$ being the exact power of $p$ dividing $m$. Note that, for a fixed $U$, the vanishing condition can be restated as $v$ being orthogonal to the image of the quadratic module $U^\ast/U$ under the embedding of $W(U^\ast/U)$ into $W(D_m(a))$ and with respect to the scalar product as described above. Note also that $U_d := W^d D_m$ runs through all isotropic subgroups of $D_m$ if $d$ runs through all positive integers whose square

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5For $\alpha = 0$, this formula holds only true for $\alpha = \pm 1$, which, however, can always be assumed without loss of generality.
divides \( m \), and that, for such \( d \), the quadratic module \( U_d^*/U_d \) is isomorphic to \( D_{m/d^2} \).

The decomposition of \( W(m) \) was already given (and essentially deduced by the same methods as explained here) in [Sko 85, Theorem 1.8, p. 22].

After these preparations we can now give the proofs of Theorems 2 and 11.

**Proof of Theorem 2.** Suppose first of all that \( p \) is odd. The assumptions on \( M \) imply a decomposition \( M \cong L_{p^\alpha}(a) \perp L_{p^\beta}(b) \) with, say, \( 0 \leq \alpha \leq \beta \) [N 76, Satz 3]. If \( |M| \) is a square and \( \sigma(M) = 1 \) then \( \alpha, \beta \) are both even or they are both odd and \( \left( \frac{-ab}{p} \right) = \sigma(M) = 1 \). In the first case \( M \) contains the trivial submodule as follows immediately from Lemma 2. In the second case we may assume \( b = -a \); but then, as follows again from Lemma 2, \( A := W(L_{p^\alpha}(a)) \) is a \( SL(2, \mathbb{Z}) \)-submodule of \( B := W(L_{p^\beta}(-b)) \), whence \( W(M) \cong A \otimes B \) contains nonzero invariants.

Conversely, if \( \alpha \) and \( \beta \) do not have the same parity, or if \( -ab \) is not a square modulo \( p \), then by Lemma 2, \( A \) and \( B \) have no irreducible representation in common, whence \( W(M) \cong A \otimes B \) contains no invariants.

If \( p = 2 \) and \( M \cong D_{2\alpha}(a) \perp D_{2\beta}(b) \) with, say, \( \alpha \leq \beta \), then \( \sigma(M) = e_8(-a - b) \) (using the convention that \( a \) or \( b \) is chosen to be \( \pm 1 \) if \( \alpha \) or \( \beta \) are equal to zero, respectively). Hence \( \sigma(M) = 1 \) if and only if \( b \equiv -ax^2 \pmod{4 \cdot 2^\beta} \) for some \( x \). We can therefore follow the same line of reasoning as before to deduce the claim.

Finally, if \( p = 2 \) and \( M \) is not of the form as in the preceding argument then [N 76, Satz 4]

\[
M \cong \left( (\mathbb{Z}/2^\alpha)^2, \frac{xy}{2^\alpha} \right) \quad \text{or} \quad M \cong \left( (\mathbb{Z}/2^\alpha)^2, \frac{x^2 + xy + y^2}{2^\alpha} \right).
\]

In particular, \( |M| \) is a square. In the first case \( \sigma(M) = 1 \) and \( W(M) \) possesses an invariant, namely \( I_U = \sum_{x \in U} e_x \), where, e.g. \( U = \mathbb{Z}/2^\alpha \times 0 \). In the second case \( \sigma(M) = (-1)^\alpha \). If \( \alpha \) is even then \( I_V = \sum_{x \in V} e_x \), for \( V = 2^{\alpha/2}M \), is an invariant. If \( \alpha \) is odd then \( W(M) \) possesses no invariants (see the complete decomposition of \( W(M) \) in [N-W 76, pp. 519], which is called \( N_\alpha \) loc. cit.). This completes the proof of the theorem.

**Proof of Theorems 11 and 12.** The \( \mathbb{M} \)-module \( L_3(s) \) \( (s = \pm 1) \) is one-dimensional, and \( T \) acts on it by multiplication with \( e_3(s) \). Hence the character afforded by \( L_3(s) \) equals \( e^{8s} \). By Theorem 8 the space \( J_{1,N} \) embeds injectively into the direct sum of the spaces \( \text{Inv}(M_l) \), where

\[
M_l := D_{-l} \perp D_{-N} \perp L_3(s)
\]
and where $l$ runs through the divisors of $N' := \text{lcm}(3, N)$ such that $N'/l$ is squarefree.

Suppose that $\text{Inv}(M_l) \neq 0$. Since $\text{Inv}(M_l) = \bigotimes_k \text{Inv}(M_l(p))$, where, $M_l(p)$, for a prime $p$, denotes the $p$-part of $M_l$, we conclude $\text{Inv}(M_l(p)) \neq 0$. For $p \neq 3$ this implies, by Theorem 3, that the $p$-parts of $\vert M_l \vert = 3NI$ are perfect squares and that $\sigma_p(M_l) = 1$. From the first condition and since $N'/l$ is squarefree we conclude $l = N'$ or $l = N'/3$. But $\sigma_2(M_l) = e_8(-l/q - N/q) \left( \frac{4N/q^2}{2^q} \right)$, where $q$ is the exact power of 2 dividing $N$, and this is real if only if $ln/q^2$ is not a square mod 4. Thus $l = N/3$ or $l = 3N$ accordingly as 3 divides $N$ or not. For this $l$ and $p \neq 3$ we have $\sigma_p(M_l) = \left( \frac{-3}{q} \right)$, where $q_p$, for any $p$, denotes the exact power of $p$ dividing $N$. Since $\sigma_p(M_l) = 1$ we find that $\left( \frac{-3}{q} \right) = 1$ for all $p \neq 3$. In particular, $N/q_3 \equiv +1 \mod 3$. Finally, if, say, $3 \mid N$ and $l = 3N$, we find

$$M_l(3) = \left( \mathbb{Z}/3q_3 \times \mathbb{Z}/3q_3 \times \mathbb{Z}/q_3, \frac{-N/q_3 x^2 - 3N/q_3 y^2 + q_3 s z^2}{3q_3} \right).$$

If $N/q_3 \equiv +1 \mod 3$ then, for $s = -1$, this quadratic module contains no nontrivial isotropic element, hence $T$ does not afford the eigenvalue 1 on $W(M_l(3))$. Hence $\text{Inv}(M_l(3)) \neq 0$ implies $s = +1$. The same holds true, by a similar argument, if $N$ is not divisible by 3. Note that, for $s = +1$ and $N/q_3 \equiv +1 \mod 3$ we have $\sigma_3(M_l) = 1$.

Summing up we thus have proved that $J_{1,N}(\varepsilon^{8s}) = 0$ unless $s = +1$ and $\left( \frac{-3}{q} \right) = 1$ for all $p \neq 3$. Moreover, if the latter conditions hold true then

$$J_{1,N}(\varepsilon^{8}) \cong \begin{cases} \text{Inv}(M_{3N})^{+,-} & \text{if } 3 \nmid N \\ \text{Inv}(M_{N/3})^{+,-} & \text{if } 3 \mid N \end{cases}.$$

Here the superscript indicates the subspace of all elements in $W(M_{3N})$ resp. $W(M_{N/3})$ which are even in the first and odd in the third argument.

By Theorem 1 the space $\text{Inv}(M_{3N})$ is spanned by the special elements $e_U = \sum_{m \in U} e_m$, where $U$ runs through the isotropic self-dual subgroups of $M_{3N}$. Accordingly, $J_{1,N}(\varepsilon^{8})$ is the spanned by the Jacobi forms

$$\vartheta_U(\tau, z) := \sum_{(x,y,z) \in U} \vartheta_{3N,x}(\tau, 0) \vartheta_{N,y}(\tau, z) \left( \frac{z}{3} \right).$$

Here we used the application of Theorem 8 (and the identification $L_3 \cong \mathbb{C}(\varepsilon^{8})$ given by $e_z - e_{-z} \mapsto (\frac{z}{3})$).
The statements of the last paragraph still hold true if \( N \) is divisible by 3. In fact, a literal application of Theorem 8 and the preceding considerations imply that \( J_{1,N}(\varepsilon^8) \) is spanned by the the Jacobi forms which are given by the same formula as the \( \vartheta_U \) but with \( \vartheta_{3N,x} \) replaced by \( \vartheta_{N/3,x} \) and where \( U \) runs through the isotropic self-dual subgroups of \( M_{N/3} \). But the quadratic module \( D_{-N/3} \) is isomorphic to \( X^*/X \) (via \( x \mapsto 3x + X \)), where \( X^* = 3\mathbb{Z}/6N \). This map induces an embedding of \( \mathbb{M} \)-modules \( W(-N/3) \to W(-3N) \) (via \( (x) \mapsto \sum_{y=3x \mod 2N} 2N(y); \) see end of section 3), and it induces a map \( U \mapsto U' \) from the set of isotropic self-dual subgroups of \( M_{n/3} \) into the set of isotropic self-dual subgroups of \( M_{3N} \) (via pullback). Note that this map is one to one (since \(-x^2 - 3y^2 + 4Nz^2 \equiv 0 \mod 12N \) implies \( 3|x \)). Finally, the diagram formed by the maps \( \text{Inv}(M_{N/3}) \ni U \mapsto e_U \mapsto \vartheta_U \) and \( \text{Inv}(M_{N/3}) \ni U \mapsto U' \mapsto \vartheta_{U'} \) commute as follows on using the formulas

\[
\vartheta_{N/3,x} = \sum_{y=3x \mod 2N} \vartheta_{3N,y}.
\]

It remains to determine, for arbitrary \( N \), the isotropic self-dual subgroups \( U \) of \( M_{3N} \). The map \( U \ni (x,y,z) \mapsto (x,y) \) is injective (since \(-\frac{x^2 + 3y^2}{4N} + z^2 \in 3\mathbb{Z}\)) and thus maps \( U \) onto an isotropic subgroup \( U' \) of the (degenerate) quadratic module \( M := (\mathbb{Z}/6N \times \mathbb{Z}/2N, -\frac{x^2 + 3y^2}{4N}) \) of order \( |U'| = 6N \). For determining the set \( S \) of isotropic subgroups \( U' \) of \( M \) whose order is \( 6N \) let \( \mathcal{O} \) be the maximal order of \( \mathbb{Q}(\sqrt{-3}) \). The image of the map \( \mathcal{O} \ni \alpha = \frac{x + \sqrt{-3}y}{4N} \mapsto (x + 6NZ, y + 2NZ) \) contains the isotropic vectors of \( M \) (since \(-\frac{x^2 + 3y^2}{4N} \in \mathbb{Z} \) implies that \( x \) and \( y \) have the same parity) and its kernel equals \( 2\sqrt{-3}NZ[\sqrt{-3}] \). The \( U' \) in \( S \) thus correspond to the subgroups \( 2\sqrt{-3}NZ[\sqrt{-3}] \subset I \subset \mathcal{O} \) of index \( N \) in \( \mathcal{O} \) and such that \( N||\alpha|^2 \) for all \( \alpha \in I \).

Let \( I \) be such a subgroup and let \( I' := \mathcal{O} - \mathcal{O} \) be the \( \mathcal{O} \)-ideal generated by \( I \). Clearly \( N||\alpha|^2 \) for all \( \alpha \in I' \). Since \( I' = \mathcal{O} \rho \) for some \( \rho \) we find \( N||\rho|^2 \) and \( |\rho|^2 \leq N \). But then \( |\rho|^2 = N \) and \( I' = I \). Thus all isotropic self-dual subgroups \( U \) of \( M_{3N} \) are of the form

\[
U = \left\{ (x(\alpha) + 6NZ, y(\alpha) + 2NZ, \psi(\alpha)) : \alpha \in \mathcal{O} \rho \right\},
\]

where \( \mathcal{O} \rho \) is an ideal of norm \( N \), where \( \psi \) is a group homomorphism \( \mathcal{O} \rho \to \mathbb{Z}/3 \) such that \(-|\alpha|^2/N + \psi(\alpha)^2 \in 3\mathbb{Z} \), and where we use \( x(\alpha) = \alpha + \overline{\alpha} \), \( y(\alpha) = (\alpha - \overline{\alpha})\sqrt{-3} \). It is easily checked that the only possible \( \psi \) are \( \psi(\alpha) = \epsilon x(\alpha/\rho) \) (\( \epsilon = \pm 1 \)). In view of the formula for \( \vartheta_U \) it suffices to consider only those \( U = U_\rho \) where \( \psi(\alpha) = x(\alpha/\rho) \). If we write \( \vartheta_\rho \) for \( \vartheta_U \) then by a simple calculation

\[
\vartheta_\rho = \sum_{\alpha \in \mathcal{O}} \left( \frac{x(\alpha)}{3} \right) q^{|\alpha|^2/3} \zeta^{y(\alpha\rho)},
\]

26
We have thus have proved that $J_1,N(ε^8)$ is indeed spanned by the series $ϑ_ρ$ as stated in Theorem 12, where $O_ρ$ runs through the ideals of norm $N$ in $O$.

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