Bell Theorem for Nonclassical Part of Quantum Teleportation Process

Marek Żukowski
Instytut Fizyki Teoretycznej i Astrofizyki, Uniwersytet Gdański, PL-80-952 Poland

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The quantum teleportation process is composed of a joint measurement performed upon two subsystems A and B (uncorrelated), followed by a unitary transformation (parameters of which depend on the outcome of the measurement) performed upon a third subsystem C (EPR correlated with system B). The information about the outcome of the measurement is transferred by classical means. The measurement performed upon the systems A and B collapses their joint wavefunction into one of the four entangled Bell states. It is shown here that this measurement process plus a possible measurement on the third subsystem (with classical channel switched off - no additional unitary transformation performed) cannot be described by a local realistic theory.

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Quantum teleportation is the operational protocol which enables one to transfer the quantum state of one system, say A, to another quantum system, C. The transfer can be obtained by performing a joint (‘Bell-state’) measurement on A and a third system B, originally EPR entangled with C, and then unitarily transforming C according to the outcome of this measurement. Teleportation separates the complete information in A into two parts: a classical part carried by the outcome c of the joint measurement on A and B, and a nonclassical part carried by the prior entanglement between B and C. No cloning of quantum information takes place; the input state of A is destroyed before it is re-created at C.

Teleportation is strongly related conceptually and experimentally to other effects, like interferometric tests of local realism involving independent sources of particles, especially entanglement swapping. In entanglement swapping, the particle A of the quantum teleportation protocol is originally entangled with some particle D. If, like in the case of quantum teleportation, a full Bell state measurement is performed on A and B, and depending on the outcome, after a classical transfer of information, a suitable unitary transformation is performed upon C, the particle D and C are in an entangled state. Thus, under such protocol, entanglement swapping can be interpreted as teleportation of entanglement (from A to C). The final state of D and C can be used in an experiment in which Bell inequalities are violated. The measurement acts on D and C can easily satisfy the necessary requirement for a Bell inequality test, namely that of spatial separation. Further, since the classical information on the outcome of the Bell measurement is needed on one side only (in the present example, in the vicinity of particle C), one may arrange the experiment in such a way that no classical information on the result of the Bell-measurement upon A and B can reach D before the local measurement on D, in the Bell inequality test, is done. In such a case, the teleportation process can be treated as just a more involved scheme of the preparation of the entangled state of D and C. This strongly suggests that there must be at least an element in the teleportation procedure which defies local and realistic interpretation.

The problem of the link of lack of link between the violations of local realism and the teleportation process has been addressed by many authors (for papers opening this discussion see ). In this work the following aspect of the problem will be discussed. As it was mentioned before, the teleportation process has its quantum and classical part. The classical part involves communication via standard classical methods, and thus cannot be suspected of adding anything interesting to the relation of the teleportation process with the Bell theorem (except for the case of entanglement swapping, as discussed above). Even worse, the classical transfer of information from the Bell-state-measuring station (operated by Alice) to the particle C makes it possible that measurements upon C (after the full teleportation protocol) can be causally linked with the events at Alice’s apparatus. Thus a Bell type analysis is absolutely excluded. Nevertheless, as it will be argued below, the quantum part of the process cannot be described by a local realistic formalism.

First one should define what is meant here by the quantum part of the process. One can employ the simplest form of amputation: the classical information link between Alice and Bob is cut. However, both parties are still allowed to perform the usual laboratory tasks for an experiment towards verification of the actuality of the teleportation process. Namely, Alice herself (or, for purists, this can be done by her friend Cecil) can prepare the particle A in any pure state, and subsequently she can make a Bell measurement on A and B. Bob, not knowing the result at Alice’s side, nor the original state of A, instead of being totally idle, performs on particle C a measurement of a (generally randomly chosen) yes-no observable.

The formal description of the above runs as follows (as in the classic case we assume all particles involved to be two-state systems). The initial three particle state is:

\[
\sin \beta |A1\rangle + \cos \beta e^{i\phi} |A2\rangle
\]

\[
\times \sqrt{\frac{1}{2}}(|B1\rangle|C1\rangle + |B2\rangle|C2\rangle),
\]

\[1\]
where $A_i$, $B_i$, $C_i$ denote the states of the three subsystems (the letter stands for the subsystem (particle) and $i = 1, 2$ is the index of two orthogonal states). The parameters $\phi$ and $\beta$ are determined by the state preparation procedure of Cecil.

Now Alice performs a measurement which collapses the $A - B$ system in to the four Bell states:

$$\sqrt{\frac{1}{2}} (|B1\rangle |A1\rangle + |B2\rangle |A2\rangle) = |00\rangle,$$

$$\sqrt{\frac{1}{2}} (|B1\rangle |A2\rangle + |B2\rangle |A1\rangle) = |01\rangle,$$

$$\sqrt{\frac{1}{2}} (|B1\rangle |A2\rangle - |B2\rangle |A1\rangle) = |10\rangle,$$

$$\sqrt{\frac{1}{2}} (|B1\rangle |A1\rangle - |B2\rangle |A2\rangle) = |11\rangle.$$  

Please note, that the names of the states introduced above are binary expansions of 0, 1, 2, 3. They could constitute the possible content of the classical messages of Alice to Bob, informing him about the results obtained at the Bell-state analyzer (however, this link is cut). The Alice’s measurement projects the particle C into certain orthogonal states:

$$\cos \beta’ |C1\rangle + \sin \beta’ e^{i \phi'} |C2\rangle = |0\rangle$$

and

$$- \sin \beta’ |C1\rangle + \cos \beta’ e^{i \phi'} |C2\rangle = |1\rangle$$

The probabilities of all possible eight global results (2 results of Bob times 4 results of Alice), are

$$P(00, 0) = 1/4 - P(00, 1)$$

$$= \frac{1}{8} \left(1 - \cos 2\beta \cos 2\beta’ + \sin 2\beta \sin 2\beta’ \cos (\phi - \phi’)\right),$$

$$P(01, 0) = 1/4 - P(01, 1)$$

$$= \frac{1}{8} \left(1 + \cos 2\beta \cos 2\beta’ + \sin 2\beta \sin 2\beta’ \cos (\phi + \phi’)\right).$$

$$P(10, 0) = 1/4 - P(10, 1)$$

$$= \frac{1}{8} \left(1 + \cos 2\beta \cos 2\beta’ - \sin 2\beta \sin 2\beta’ \cos (\phi + \phi’)\right),$$

$$P(11, 0) = 1/4 - P(11, 1)$$

$$= \frac{1}{8} \left(1 - \cos 2\beta \cos 2\beta’ - \sin 2\beta \sin 2\beta’ \cos (\phi - \phi’)\right).$$

Let us assign to the four possible results of Alice’s measurement, $c = 00, 01, 10$ or 11, four two-dimensional vectors (for some other non-conventional value assignments for experimental results see [5]).

\begin{equation}
\tilde{A}(00) = (-1, -1), \tilde{A}(01) = (-1, 1), \\
\tilde{A}(10) = (1, -1), \tilde{A}(11) = (1, 1).
\end{equation}

The link between the vectors and the binary numbers is obvious. The digit 0 has been replaced by $-1$ because this trick makes the subsequent derivation of a Bell inequality much easier. Please note, that this procedure differs from the usual one (i.e. assignment of certain real numbers, "eigenvalues", to certain projectors) by the fact that we ascribe to the projectors more complicated objects. The results of Bob's measurements, $i = 0$ or 1, will be described in a similar fashion, namely by ascribing numbers $I_B(0) = -1$ and $I_B(1) = +1$.

To simplify the description of the global measurement results one can introduce a suitably defined correlation function. Let us consider such a function as the average of products of the results on each side (here, vectors times numbers). E.g. the result $(00,0)$, i.e. a detection of the first Bell state, 00, by Alice and simultaneous detection of state 0 by Bob, can be ascribed $-1(-1,-1) = (1,1)$, etc. With such definitions of the values assigned to the possible pairs of the outcomes the correlation function

$$E(\beta, \phi; 0; 0) = \sum_{c=00}^{01} \sum_{i=0,1} P(c, i) I_B(i) \tilde{A}(c)$$

acquires a form of a two dimensional vector and for the explicit form of the quantum prediction reads:

$$E(\beta, \phi; 0; 0)_{QM} = \sin 2\beta \sin 2\beta’ \left(\cos \phi \cos \phi’, \sin \phi \sin \phi’\right).$$

It will be shown that this correlation function cannot be modeled by local hidden variable theories.

Imagine that a hidden variable $\lambda$ specifies the future results of the experiments of Alice and Bob. The product of such predictions reads

$$I_B(0; 0; 0)_{QM}$$

where, $I_B(0; 0; 0) = \pm 1$ is the local hidden variable (LHV) prediction for the result of the measurement by Bob (for the given value of the hidden parameter $\lambda$, and the local observable defined by $\phi’$) and the vector $\tilde{A}(\phi, \lambda)$, which is the LHV prediction for the Alice’s result, depends on $\lambda$ and $\phi$, and takes one of the four values [6]. The local hidden variable prediction for the correlation function is an average of [6] over a certain (properly normalized) distribution $\rho(\lambda)$, namely
Now let us assume that Alice can set the values of the phase $\phi$ which prepares the state of particle $A$ at 0 or 90 degrees, whereas Bob can play with $\phi'$ at -45 and +45 degrees.

To show that $E(\phi;\phi')_{QM}$ cannot be modeled by $E(\phi;\phi')_{LHV}$ the geometric approach of [8] will be used. It is based on the following simple observation. Assume that one knows the components of a certain vector $q$ (the known vector) belonging to some vector space, whereas about a second vector $h$ (the test vector) one is only able to establish that its scalar product with $q$ satisfies the inequality $\langle h|q \rangle < ||q||^2$. The immediate implication is that these two vectors cannot be equal: $q \neq h$.

To form a vector for such an argument, one can take the values of the quantum correlation function at the $2 \times 2 = 4$ pairs of the possible settings of the macroscopic parameters controlled by Alice and Bob ($\phi, \phi'$). In this way a "super-vector" $\vec{V}_{QM}$ is built. The first component of the super-vector, for the settings $(0^\circ, -45^\circ)$, reads:

$$\vec{V}_{1QM} = E(0; -45)_{QM} = \left(\sqrt{1/2}, 0\right), \quad (18)$$

the second for $(0, 45)$

$$\vec{V}_{2QM} = E(0; 45)_{QM} = \left(\sqrt{1/2}, 0\right), \quad (19)$$

the third one at $(90, -45)$

$$\vec{V}_{3QM} = E(90; -45)_{QM} = \left(0, -\sqrt{1/2}\right), \quad (20)$$

the fourth $(90, 45)$

$$\vec{V}_{4QM} = E(90; 45)_{QM} = \left(0, \sqrt{1/2}\right). \quad (21)$$

The square of the norm of such a super vector, denoted by $\parallel \vec{V}_{QM} \parallel^2$, can be defined as the sum of the squares of the norms of all the components, where the square of the norm of a component is in turn the sum of the squares of its two components. Therefore one has

$$\parallel \vec{V}_{QM} \parallel^2 = \sum_{i=1}^{4} |\vec{V}_{iQM}|^2 = 2. \quad (22)$$

We shall now estimate the scalar product of the quantum super-vector with analogous super-vector $\vec{V}_{LHV}$ which has the structure characteristic for (deterministic) local hidden variables. Of course, the aforementioned scalar product is defined in a way compatible with the norm (i.e. it is a sum of the products of the respective components, and the product of two components is again the sum of the products of the respective elements of the components):

$$\langle \vec{V}_{QM}, \vec{V}_{LHV} \rangle = \sum_{i=1}^{4} \vec{V}_{iQM} \cdot \vec{V}_{iLHV} \quad (23)$$

with $\vec{V}_{iLHV}$ being equal to the value of $E(\phi, \phi')_{LHV}$ for appropriate pairs of settings. As it is usual in the proofs of the Bell theorem, it is better first to consider the hidden variable prediction for a single specified $\lambda$ and only later average this over the distribution of the hidden variables.

Thus, what we should do [8] is to estimate the scalar product of the super-vector constructed out of hidden-variable predictions for the specified $\lambda$ with the quantum super-vector (defined above). The hidden variable super-vector for a specific $\lambda$, which will be denoted by $\vec{H}(\lambda)$, has the following components:

$$\vec{H}(\lambda)_{1} = I_{B_{LHV}}(-45, \lambda)(A(0, \lambda)_{1}, A(0, \lambda)_{2}), \quad (24)$$

$$\vec{H}(\lambda)_{2} = I_{B_{LHV}}(45, \lambda)(A(0, \lambda)_{1}, A(0, \lambda)_{2}), \quad (25)$$

$$\vec{H}(\lambda)_{3} = I_{B_{LHV}}(-45, \lambda)(A(90, \lambda)_{1}, A(90, \lambda)_{2}), \quad (26)$$

$$\vec{H}(\lambda)_{4} = I_{B_{LHV}}(45, \lambda)(A(90, \lambda)_{1}, A(90, \lambda)_{2}). \quad (27)$$

For the scalar product $\langle \vec{V}_{QM}, \vec{H}(\lambda) \rangle$, since $I_{B}(\phi', \lambda) = \pm 1$ and $A(\phi, \lambda)_{i} = \pm 1$, one gets:

$$-2\sqrt{1/2} \leq \langle \vec{V}_{QM}, \vec{H}(\lambda) \rangle = \sqrt{1/2}(A(0, \lambda)_{1}[I_{B_{LHV}}(-45, \lambda) + I_{B_{LHV}}(45, \lambda)] + A(90, \lambda)_{2}[I_{B_{LHV}}(45, \lambda) - I_{B_{LHV}}(-45, \lambda)]) \leq 2\sqrt{1/2}. \quad (28)$$

Thus if one now averages this inequality over the distribution of the hidden variables $\rho(\lambda)$, the following relation emerges

$$-\sqrt{2} \leq \langle \vec{V}_{QM}, \vec{V}_{LHV} \rangle \leq \sqrt{2} < \parallel \vec{V}_{QM} \parallel^2 = 2. \quad (29)$$

This implies simply that $\vec{V}_{LHV} \neq \vec{V}_{QM}$, which means in turn nothing else than that no local hidden variable correlation function can reproduce the quantum prediction (i.e. we have a Bell theorem for the process). Please note that the appropriate Bell inequality is given here by the first two inequalities in [24].

This method can still be expanded to cover much more settings of the variables, here only the simplest case was presented. It is an interesting fact that needs further investigation, that the Bell inequality presented here is violated by the same factor $\sqrt{2}$ as the CHSH inequality for the usual Bell theorem involving a pair of particles in a maximally entangled state. This may imply that the quantum component of the teleportation process cannot be described in a local and realistic way as long as the initial state of $B$ and $C$ neither admits such models.
The present result also explains why the current local hidden variable model explaining the low detection efficiency teleportation [9] cannot be extended into high efficiency case. Simply, had this been possible, such model would constitute a LHV model of the process considered here, what by (29) is impossible. Also, for the same reason, considerations with toy-models like those in [10] cannot be extended in such a way that they can fully reproduce the quantum teleportation process. Nevertheless, the interesting conclusions reached in [11], namely that one can model the teleportation process with specific local hidden variables and classical communication channel, requiring the transfer of on average 2.19 bits, are not in a disagreement with the present result.

The inequality (29) can serve as the Bell-type inequality for the experiment of Boschi et al [12]. In this experiment the system A was replaced by the polarization degree of freedom of one of the photons of the EPR entangled pair (the pair was entangled in linear momentum directions). In this way measurement discriminating between the four correlated states of polarization and momentum direction of a single photon, which are formally equivalent to (2-5), can be performed with standard quantum interferometric techniques. Thus all observables involved in the present scheme found their representation in the experiment. However, due to the angles chosen in the experiment, one cannot directly apply the inequality (29). Nevertheless the obtained, very high, visibility of the two-particle fringes, is well above the threshold (71%) indicated by (29). This indirectly rules out a LHV model for the experiment (of course, provided one accepts the fair sampling assumption).

In the teleportation experiment involving all three particles (with A emitted independently of the emission of the EPR pair B and C) [13], due to fundamental technical limitations, one currently cannot distinguish between all four states (2-5), and therefore the inequality cannot be applied. However, the extension of the experiment to the teleportation of entanglement, i.e. entanglement swapping process [14], [15], results in entangling previously independent photons, on which in turn a Bell-type experiment can be performed. Such experiment is possible on a subensemble of events for which only one of the states (2-5) was measured, i.e. entanglement swapping does not need a full Bell-state measurement to be indescribable by local realistic theories. Unfortunately, the visibility in [14] was around 65%, i.e. within a zone for which one can build explicit LHV models [16].

Thus, higher visibility realization of entanglement swapping would constitute an important fact in the empirical knowledge on the nature of quantum teleportation.

The presented results cannot be applied directly to the teleportation experiment involving continuous variables of [17]. However one can speculate that the recent result of ref. [18], concerning the Bell theorem for the original EPR state, may, after suitable extensions, lead to the same conclusions.

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