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Pseudo-$\mathcal{PT}$ symmetric Dirac equation: effect of a new mean spin angular momentum operator on Gilbert damping

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Abstract

The pseudo-$\mathcal{PT}$ symmetric Dirac equation is proposed and analyzed by using a non-unitary Foldy–Wouthuysen transformations. A new spin operator $\mathcal{PT}$ symmetric expectation value (called the mean spin operator) for an electron interacting with a time-dependent electromagnetic field is obtained. We show that spin magnetization—which is the quantity usually measured experimentally—is not described by the standard spin operator but by this new mean spin operator to properly describe magnetization dynamics in ferromagnetic materials and the corresponding equation of motion is compatible with the phenomenological model of the Landau–Lifshitz–Gilbert equation (LLG).

Keywords: pseudo PT symmetry, non-Hermitian Dirac equation, Foldy–Wouthuysen transformation, Landau–Lifshitz–Gilbert equation

Supplementary material for this article is available online

In the field of micromagnetism, which provides the physical framework for understanding and simulating ferromagnetic materials, there is a fundamental unsolved problem which is the microscopic origin of the intrinsic Gilbert damping. However, this damping mechanism has been introduced phenomenologically by T L Gilbert in 1955 for describing the spatial and temporal evolution of the magnetization (known as the LLG equation), a vector field which determines the properties of ferromagnetic materials on the sub-micron length scale [1]. Let

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us stress that this equation leads to the conservation of the magnetization modulus. This phenomenological model has since been validated by numerous experimental data and constitutes the foundation of micromagnetism [2]. Moreover, magnetic damping plays a crucial role in the operation of magnetic devices. The scattering theory can be used to compute the Gilbert damping tensor [3].

Spin is a quantum concept [4] that arises naturally from the Dirac theory and is associated with the operator 
\[
\hat{\Sigma}^D \equiv \frac{i}{2} \alpha \wedge \alpha = \begin{pmatrix} \sigma & 0 \\
0 & \sigma \end{pmatrix}
\]
where \(\alpha = \begin{pmatrix} 0 & \sigma \\
\sigma & 0 \end{pmatrix}\) and \(\sigma\) are the usual 2 \(\times\) 2 Pauli matrices [5]. For a classical version of the spin (see supplementary materials). Usually, spin magnetization \(M(r, t)\) (the quantity which is experimentally measured) is defined as the expectation value of the spin angular momentum given by 
\[
\langle \hat{\Sigma}^D \rangle \equiv \frac{\mu_B}{2} \langle \Psi^D | \hat{\Sigma}^D | \Psi^D \rangle
\]
with \(\mu_B \equiv \frac{e}{2m}\) the Bohr’s magneton (\(e < 0\)) and where \(\Psi^D\) is a solution of the Dirac equation. Indeed, in most magnetic materials the orbital moment is quenched and therefore magnetism is only due to the spins [6].

While for a free electron the spin angular momentum in the Heisenberg picture is not a constant of motion \(\frac{d\hat{\Sigma}^D}{dt} \neq 0\) [5], there exists another spin operator \(\hat{\Sigma}^D\), considered to be a constant of motion, (called the mean spin operator [7]) \(\frac{d\hat{\Sigma}^D}{dt} = 0\). In the presence of an electromagnetic field, which is relevant for exploring the microscopic origin of the Landau–Lifshitz–Gilbert (LLG) equation, a satisfactory result has not yet given.

Knowing that the spinors in the Dirac theory consist of four components, it is important to check whether the Dirac equation yields physically reasonable results in the non-relativistic expansion case and to show that the Dirac equation reproduces the two-component Pauli equation. We transform the Hamiltonian in such a way that all operators of the type \(\alpha\) that couple the large to the small components will be removed. This can be achieved by a Foldy Wouthuysen transformation [7–9] which is a non-relativistic expansion of the Hamiltonian into series of the particle’s Compton wave lengths \(\lambda_C \equiv \frac{h}{m}\).

Hickey and Moodera [10] have proposed that the spin–orbit interaction, which arises from the non-relativistic expansion of the Dirac equation, may be responsible for the intrinsic ferromagnetic line width. In their work, the term containing the curl of the electric field when coupled to Maxwell’s equations lead to a time-varying magnetic induction, and the theoretical methods employed involve previously developed formalisms in which an effective non-Hermitian and time-dependent Hamiltonian is used. However, the non-Hermiticity of the Hamiltonian imposes new rules which are modified with respect to those of standard quantum mechanics. This fact was not explicitly taken into account by the authors of [10] and therefore, their derivation of the intrinsic damping process is unfortunately incorrect. Moreover, there is another fundamental issue which emerges from this work [10] concerning how to properly perform the coupling between the classical Maxwell equations and the quantum evolution resulting from the non-relativistic limit of the Dirac equation. In what follows, we show how to overcome this difficulty by using the well-known correspondence principle (CP).

In the reference [11], the main goal was to demonstrate that there is a way to derive the LLG equation coming from a non Hermitian quantum mechanics and to spark a discussion about the connection between quantum and classical spin dynamics. Unfortunately, the quantum Heisenberg equation for a non-Hermitian Hamiltonian operator describing the damping process is not compatible with the time-evolution operators for non-Hermitian Hamiltonian operator.

From the relativistic Dirac equation, performing a Foldy–Wouthuysen (FW) transformation and using the Heisenberg equation of spin motion, Mondal et al [12–14] derive general relativistic expressions for the Gilbert damping, but the term involving the cross-product between
the magnetisation and the time-derivative of the magnetic field is purely imaginary, and therefore appears not to correspond to damping.

In the seminal work made by Dirac on relativistic quantum mechanics, the corresponding Hamiltonian would be Hermitian. We stress that the same features can be achieved when starting from non-Hermitian Hamiltonians. These features can also be obtained from theories based on non-Hermitian Hamiltonians that have been considered in different contexts. And we distinguish three separate regimes: (i) the \( \mathcal{PT} \)-symmetric regime where the eigenvalues are real, (ii) the spontaneously broken \( \mathcal{PT} \) regime where the eigenvalues are complex conjugate pairs and (iii) the regime with complex, unrelated, eigenvalues in which the \( \mathcal{PT} \)-broken regime.

Our objective here is to derive the LLG equation based on a non-Hermitian Dirac Hamiltonian when compared to the most common standard approaches [12–14].

Therefore, the Dirac equation in its fundamental representation is not unique to either Hermitian quantum mechanics or quantum field theory. By relaxing the assumption of Hermiticity and adopting instead the principles of \( \mathcal{P}\mathcal{T}\)-symmetric quantum mechanics outlined in the following paragraph, we will not make any modifications to the Dirac equation. By \( \mathcal{P}\mathcal{T}\)-symmetry we mean reflection in space, with a simultaneous reversal of time. The fundamental representation of the Dirac equation emerges completely intact, identical in every aspect to the Dirac equation derived from Hermitian theory. Before constructing the analogous 4D representation using the principles of \( \mathcal{P}\mathcal{T}\)-symmetry, let us briefly recall the notion of \( \mathcal{P}\mathcal{T}\)-symmetry.

The Hermiticity of quantum Hamiltonians depends on the choice of the inner product of the states in the physical Hilbert space. This point was first pointed out by Bender et al [16, 17]. They showed that a wide class of Hamiltonians that respect \( \mathcal{P}\mathcal{T}\)-symmetry can exhibit entirely real spectra. Since then \( \mathcal{P}\mathcal{T}\)-symmetry has been a subject of intense interest in the field of quantum mechanics.

While any evidence of \( \mathcal{P}\mathcal{T}\) symmetry has remained out of reach due to the hermitian nature of the quantum mechanics theory, optics have provided a fertile ground for observation of this property-\( \mathcal{P}\mathcal{T}\)-symmetry—since this field mainly relies on the presence of gain and loss. Note that even though \( H \) and \( \mathcal{P}\mathcal{T}\) do not continuously have identical eigenvectors, as a result of the anti-linearity of the \( \mathcal{P}\mathcal{T} \) operator. If \( H \) and \( \mathcal{P}\mathcal{T}\) do not have the same eigenvectors, we say that the \( \mathcal{P}\mathcal{T}\)-symmetry is broken. The parity operator \( \mathcal{P} \) effects the momentum operator \( p \) and the position operator \( r \) as (\( \mathcal{P}: r \rightarrow -r, \ p \rightarrow -p \)). This parity transformation has the following effect on the various vector potentials \( \mathcal{P}\mathcal{T}\)-symmetry can

The anti-linear time reversal operator \( \mathcal{T} \) has the effect of changing the sign of the momentum operator \( p \), the pure imaginary complex quantity \( i \) and the time \( (\mathcal{T}: r \rightarrow r, \ p \rightarrow -p, \ i \rightarrow -i, \ t \rightarrow -t ) \). Since \( A(\mathbf{r}, t) \) is generated by currents, which reverses signs when the sense of time is reversed, it holds that \( \mathcal{T} A(\mathbf{r}, t)(\mathcal{T})^{-1} = -A(\mathbf{r}, t) \) and \( \mathcal{T} \Phi(\mathbf{r}, t)(\mathcal{T})^{-1} = \Phi(\mathbf{r}, t) \). The two reflection operators commute with each other: \( \mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P} \).

Therefore, it is natural to introduce a modified Hilbert space, which is now endowed with \( \mathcal{P}\mathcal{T}\)-inner product, for the \( \mathcal{P}\mathcal{T}\)-symmetric nonself-adjoint theories. In such a Hilbert space, the time evolution becomes unitary as the Hamiltonian is self-\( \mathcal{P}\mathcal{T}\)-adjoint and the eigenfunctions form a complete set of orthonormal functions. But the norms of the eigenfunctions have alternate signs even in the new Hilbert space endowed with the \( \mathcal{P}\mathcal{T}\)-inner products. In fact, any theory having an unbroken \( \mathcal{P}\mathcal{T}\)-symmetry it exists a symmetry of the Hamiltonian associated with the fact that there are equal numbers of positive-norm and negative-norm
states [17]:

\[ \langle \psi_m, \psi_n \rangle_{P^0T^0} = \int dx \left[ P^0T^0 \psi_n(x) \right] \psi_m(x) = \int dx \psi_n^*(-x) \psi_m(x) = (-1)^n \delta_{mn} \quad (1) \]

The situation here is analogous to the problem that Dirac encountered in formulating the spinor wave equation in relativistic quantum theory [18].

This again raises an obstacle in probabilistic interpretation in spite of the system being in an unbroken \( P^0T^0 \) phase. Afterwards, a new symmetry \( C \), inherent to all \( P^0T^0 \)-symmetric non-Hermitian Hamiltonians, has been introduced [17]. \( C \) commutes with both \( H \) and \( P^0T^0 \) and fixes the problem of negative norms of the eigenfunctions when the inner products are taken with respect to \( CP^0T^0 \)-adjoint.

Does a \( P^0T^0 \)-symmetric Hamiltonian \( H \) specify a physical quantum theory in which the norms of states are positive and time evolution is unitary? The answer is that if \( H \) has an unbroken \( P^0T^0 \) symmetry, then it has another symmetry represented by a linear operator \( C \). Therefore we can construct a time-independent inner product with a positive-definite norm in terms of \( C \).

Another possibility to explain the reality of the spectrum is making use of the pseudo/quasi-Hermiticity transformations which do not alter the eigenvalue spectra. It was shown by Mostafazadeh [19] that \( P^0T^0 \)-symmetric Hamiltonians are only a specific class of the general families of the pseudo-Hermitian operators. A Hamiltonian is said to be \( \eta \)-pseudo-Hermitian if:

\[ H^\dagger = \eta H \eta^{-1}, \quad (2) \]

where \( \eta \) is a metric operator. The eigenvalues of pseudo-Hermitian Hamiltonians are either real or appear in complex conjugate pairs while the eigenfunctions satisfy bi-orthonormality relations in the conventional Hilbert space. Due to this reason, such Hamiltonians do not possess a complete set of orthogonal eigenfunctions in the conventional Hilbert space and hence the probabilistic interpretation and unitarity of time evolution have not been satisfied by these pseudo-Hermitian Hamiltonians.

However, like the case of \( P^0T^0 \)-symmetric non-Hermitian systems, the presence of the additional operator \( \eta \) in the pseudo-Hermitian theories allows us to define a new inner product in the fashion

\[ \langle \phi | \psi \rangle_\eta = \langle \eta \phi | \psi \rangle = \int (\eta \phi(x)) \psi(x) dx = \langle \phi | \eta \psi \rangle. \quad (3) \]

Later a novel concept of the pseudoparity-time (pseudo-\( P^0T^0 \)) symmetry was introduced in [15] to connect the non-Hermitian Hamiltonian \( H \) to its Hermitian conjugate \( H^\dagger \)

\[ H^\dagger = \left( P^0T^0 \right) H \left( P^0T^0 \right)^{-1} \quad (4) \]

where in the expression of the inner product (3), the metric \( \eta \) is replaced by \( P^0T^0 \). We now turn our attention to the main topic of interest, the spatial reflection and the time-reversal invariance of the Dirac equation. The complete spatial reflection (parity) transformation for spinors and the complete time-inversion operator are denoted as \( P = \gamma^0 P^0 \) and \( T = -i\alpha_1\alpha_3 T^0 = i\gamma^1\gamma^3 T^0 \) such that

\[ P\gamma^0P^{-1} = \gamma^0, \quad P\gamma^iP^{-1} = -\gamma^i \]
\[ T\gamma^0T^{-1} = \gamma^0, \quad T\alpha T^{-1} = -\alpha \quad (5) \]
where the Hermitian matrices $\beta = \gamma^0$ and $\alpha$ satisfy the ‘Dirac algebra’ $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$; $\{\alpha_i, \beta\} = 0$.

In what follows, we will denote by

$$\hat{H}^D(t) = i(c\alpha.(\hat{p} + ieA(r, t)) - e\Psi(r, t) + mc^2\beta)$$

(6)

the non-Hermitian Dirac Hamiltonian for a single electron in the presence of a classical time-dependent external electromagnetic field defined by $(A(r, t), \Psi(r, t))$. The associated non-Hermitian Dirac equation for a single electron in the presence of a classical time-dependent external electromagnetic field reads

$$ih\frac{\partial \Psi^D(r, t)}{\partial t} = \hat{H}^D(t)\Psi^D(r, t) = i(\hat{T} + \hat{V} + mc^2\beta)\Psi^D(r, t),$$

(7)

where $\Psi^D(r, t) = \text{col}(u(r, t), v(r, t))$, is bispinors verifying the pseudo-orthogonality relation $\langle \Psi^D|\Psi^D \rangle_{PT} = I$, $\hat{V} \equiv -e\Phi$ and $\hat{T} \equiv c\alpha.\hat{r} = c\alpha.(\hat{p} + ieA(r, t))$ is the kinetic energy which produces a coupling between small and large components of the Dirac wavefunction $\Psi^D$, $\alpha$, $\beta$ and $\gamma^5$ (see equation (15)) are the Dirac matrices [20]. The $PT$ symmetry condition

$$(PT)\hat{H}^D(t)(PT)^{-1} = \hat{H}^D(t),$$

(8)

connects the non-Hermitian Dirac Hamiltonian $\hat{H}^D(t)$ to its Hermitian conjugate $\hat{H}^D(t)$. This observation leads us to introduce a novel concept of the pseudo-parity-time (pseudo-$PT$) symmetry, where $(PT)$ is interpreted as a metric. Thus as the case of pseudo-hermiticity, the bispinor $\Pi^D(r, t) = \text{col}(u(r, t), v(r, t))$, verifies the pseudo-orthogonality relation $\langle \Psi^D|\Psi^D \rangle_{PT} = I$. Note that, there is another situation which differs from that described above, also called pseudo-PT symmetry which means that the system can have a real eigenvalues whether or not the original system is $PT$-symmetric [21, 22].

As we are dealing with the non-relativistic expansion of the Dirac equation, the following decomposition $\Psi^D(r, t) = \Pi(r, t) \times \chi^D(u, t)$ [23] can be used, where $\chi^D(u, t)$ is the time-dependent bi-spinor representing the spin state oriented in the direction defined by $u$, and $\Pi(r, t)$ the scalar part of the wave function.

We argue that, in the non-relativistic limit, the operator $\chi^D$ is the one which must be interpreted as the spin operator in the Pauli theory and used to define the magnetization as $M(r, t) \equiv \mu_B \langle \chi^D|\Sigma^D|x^D \rangle_{PT} = \mu_B \langle \chi^D|\Sigma^D|\chi^D \rangle_{PT} = \mu_B \langle \chi^D|U\Sigma^U\chi^U \rangle_{PT} = \mu_B \langle \chi^FW|\Sigma^FW|\chi^FW \rangle_{PT}$ where $\chi^FW$ is the spin part of the Dirac bi-spinor wave function $\Psi^D$ in a non-relativistic expansion, obtained by using the FW transformation [7] and $U$ is the associated operator (see the definition in the following). It worth mentioning that, according to the above definition, expanding the spin operator (to a consistent order in $\hbar/mc$) and using the Dirac representation of the wave function is equivalent to expanding the wave function (also to a consistent order in $\hbar/mc$ using the FW transformation) and keeping the original spin operator in the Dirac representation. In this work we have chosen to expand the mean value operator. The classical magnetization, $M(r, t)$, is obtained by using the CP. Indeed, we will show in what follows that the equation of motion of the mean spin operator for an electron interacting with a time-dependent electromagnetic field leads to the LLG equation of motion revealing thus its microscopic origin.

In a seminal work, Foldy and Wouthuysen (FW) solved the problem of finding a canonical transformation that allows to obtain a two-component theory in the low-energy limit (Pauli approximation), in the case of the Dirac equation coupled to an electromagnetic field [7]. Unfortunately, contrary to the free-electron case, the solution cannot be expressed in a closed
form. However, FW showed how to obtain successive approximations of this transformation as a power series expansion in powers of the Compton wave length of the particle \( \lambda c \equiv \frac{\lambda}{mc} \). This procedure, generally restricted to the second-order in \( 1/m \), is presented in many textbooks on relativistic quantum mechanics [8, 20, 24, 25] and has been extended to fifth order in powers of \( 1/m \) [26].

In what follows, the symbol \( [\hat{C}, \hat{D}] \) (\( \{\hat{C}, \hat{D}\} \)) denotes the commutator (anticommutator) of the operators \( \hat{C} \) and \( \hat{D} \). We shall also use the following notations: \( \hat{X} \equiv X(r, t) \) and \( \hat{Y} \equiv Y(r, t) \).

In the Hermitian Dirac representation

\[
\hat{H}^{AD}(t) = \left( e\alpha \cdot (\hat{p} - eA(r, t)) + e\Phi(r, t) + mc^2 \beta \right),
\]

the Heisenberg equation of motion for the spin operator reads as follows

\[
\frac{d\hat{\Sigma}^{D}}{dt} = \frac{i}{\hbar} \left[ \hat{H}^{AD}, \hat{\Sigma}^{D} \right] = -\frac{2e}{\hbar} \left[ \alpha \wedge (\hat{p} - eA(r, t)) \right].
\]

It is well established that the expectation value onto \( |\Psi^{D}\rangle \) of the above equation does not lead to the LLG equation for the magnetization. However, as it will be shown in the following, the latter can be obtained by using the non unitary FW transformation and the new definition of the magnetization as an expectation value of mean spin operator. The first- and second-order terms of the FW expansion in powers of \( 1/m \) correspond, respectively, to the precessional motion of the magnetization around an effective magnetic field and its damping.

Since the Hamiltonian \( \hat{H}^{D}(t) \) has a similar structure to the one in the Dirac case, by analogy with the latter, we use for \( \hat{U}(t) \) the form \( e^{\hat{S}^{D}t} \) where \( \hat{S} \) is a non self-adjoint operator. Therefore, the transformation \( \hat{\Psi}^{FW}(r, t) = e^{\hat{S}^{D}t} \hat{\Psi}^{D}(r, t) \equiv \hat{U}(t) \hat{\Psi}^{D}(r, t) \) leads to a new Hamiltonian

\[
\hat{H}^{FW}(t) = e^{\hat{S}^{D}t} \left( \hat{H}^{D}(t) - i\hbar \frac{\partial}{\partial t} \right) e^{-\hat{S}^{D}t},
\]

where \( \hat{S} \) is a non self-adjoint operator. More generally, any operator in the FW representation, that is not explicitly time dependent, \( \hat{O}^{FW} \) will be transformed in the Dirac representation as \( \hat{O}^{D}(t) = U^{-1}(t) \hat{O}^{FW} U(t) \).

The most natural extension of the Ehrenfest equation to non-Hermitian pseudo-\( PT \)-symmetric systems is by replacing a Hermitian \( \hat{H}^{AD} \) with a non-Hermitian one. The structure of the Ehrenfest equation does not change, having assumed that part of the action of \( T \) (i.e. \( T^{-1} \)) is to send \( t \rightarrow -t \), i.e.; we show that it anticommutes with the operator \( \partial/\partial t \), consequently, the operator \( PT \) commutes with \( i\partial/\partial t \). Which immediately leads us to deduce the Ehrenfest equation of motion for the diagonal matrix element of an operator \( \hat{O}^{D}(t) \)

\[
\frac{d}{dt} \langle \Psi^{D} | \hat{O}^{D} | \Psi^{D} \rangle_{PT} = \left\langle \Psi^{D} \left| \frac{i}{\hbar} \left[ \hat{H}^{FW}, \hat{O}^{FW} \right] \Psi^{FW} \right| \right\rangle_{PT}.
\]

It is straightforward to show that the equation of motion (12) leads to

\[
\frac{d}{dt} \langle \Psi^{FW} | \hat{O}^{FW} | \Psi^{FW} \rangle_{PT} = \left\langle \Psi^{FW} \left| \frac{i}{\hbar} \left[ \hat{H}^{FW}, \hat{O}^{FW} \right] \Psi^{FW} \right| \right\rangle_{PT}.
\]

By expanding \( \hat{S} = \hat{S}_1 m^{-1} + \hat{S}_2 m^{-2} + \hat{S}_3 m^{-3} + \cdots \) with \( \hat{S}_1 = \frac{\hbar}{mc^2} (\alpha \cdot \hat{p})(\alpha \cdot \Phi) \), \( \hat{S}_2 = -\frac{\hbar^2}{mc^4} (\alpha \cdot \Phi)(\alpha \cdot \Phi) + iecH^2 (\alpha \cdot \Phi)E \), \( \hat{S}_3 = -\frac{\hbar^3}{mc^6} (\alpha \cdot \Phi)(\alpha \cdot \Phi)(\alpha \cdot \Phi) + iecH^3 (\alpha \cdot \Phi)E \), \( H \equiv -\nabla \Phi - \frac{\hbar^2}{mc^2} (\alpha \cdot \Phi)E \), the mean spin operator is computed using the inverse FW transformation of the spin operator as \( \hat{\Sigma}^{D} = \hat{S} \equiv \hat{S}_1 m^{-1} + \hat{S}_2 m^{-2} + \hat{S}_3 m^{-3} + \cdots \).
\[ e^{-\left(\hat{\mathcal{S}}_{1/m}\hat{\mathcal{S}}_{2/m} + \hat{\mathcal{S}}_{3/m}\right)} \sum_d e^{\left(\hat{\mathcal{S}}_{1/m}\hat{\mathcal{S}}_{2/m} + \hat{\mathcal{S}}_{3/m}\right)} \text{ and may be expanded in power series of } (1/m) \]

leading to \[ \hat{\mathcal{S}}_0 = \hat{\mathcal{S}}^D, \]
\[ \hat{\mathcal{S}}_1 = -\frac{i\beta}{c}(\alpha \times \hat{\pi}), \]
\[ \hat{\mathcal{S}}_2 = \frac{1}{8e^2} \left( -4ie\mathbf{B} + 2e\left(\hat{\mathcal{S}}^D \times \mathbf{B}\right) - 4\left(\hat{\pi} \times \left(\hat{\mathcal{S}}^D \times \hat{\pi}\right)\right) \right) - \frac{e\hbar}{2c^3}(\alpha \times \mathbf{E}), \]
\[ \hat{\mathcal{S}}_3 = \frac{\beta}{48e^3} \left[ (\alpha, \hat{\pi}), \left[ (\alpha, \hat{\pi}), \left[ (\alpha, \hat{\pi}), \hat{\mathcal{S}}^D\right]\right]\right] + \frac{\beta}{6c^3} \left[ (\alpha, \hat{\pi}) (\alpha, \hat{\pi}) (\alpha, \hat{\pi}), \hat{\mathcal{S}}^D\right] - \frac{e\hbar^2\beta}{4c^5}(\alpha \times (\beta, \mathbf{E})), \]

where \( \mathbf{B} = \nabla \times \mathbf{A} \). The free case which is investigated in [7] (may be obtained in closed form in this case) is recovered from the above formula by substituting \( \mathbf{E} = \mathbf{B} = 0, i\beta \to \beta \) and \( i\hat{\pi} \to \hat{p} \) leading to

\[ \hat{\mathcal{S}}^D = \hat{\mathcal{S}}^D - \frac{i\beta (\alpha \wedge \hat{p})}{E_p} - \frac{\left(\hat{p} \wedge \left(\hat{\mathcal{S}}^D \wedge \hat{p}\right)\right)}{E_p(E_p + mc)} \]

where \( E_p = \sqrt{m^2c^2 + p^2} \).

The pseudo-\( PT \) equation of motion (12) for the mean spin operator is

\[ \frac{d}{dt} \langle \psi^D | \hat{\mathcal{S}}^D | \psi^D \rangle_{PT} = \langle \psi^D \frac{e\beta}{m} \left(\hat{\mathcal{S}}^D \wedge \mathbf{B}\right) - \frac{e\hbar}{4mc^2} \left(\hat{\mathcal{S}}^D \wedge \partial_t \mathbf{B}\right) - \frac{ie}{2mc^2} \left(\hat{\mathcal{S}}^D \wedge (\mathbf{E} \wedge \hat{\pi})\right) + \frac{1}{4\hbar mc^2} \left[ \left[ (\alpha, \hat{\pi}), \hat{\mathcal{S}}^D \right], (\alpha, \hat{\pi}) \right] \right] - \hat{\pi}(m^{-3}) \langle \psi^D \rangle_{PT}. \]

(In order) to check this result, we have applied to the above equation the direct FW transformation. It leads to

\[ \frac{d}{dt} \langle \psi^D | \hat{\mathcal{S}}^D | \psi^D \rangle_{PT} = \frac{d}{dt} \langle \psi^D | e^{\left(\hat{\mathcal{S}}_{1/m} + \hat{\mathcal{S}}_{2/m} + \hat{\mathcal{S}}_{3/m}\right)} \sum_d e^{\left(\hat{\mathcal{S}}_{1/m} + \hat{\mathcal{S}}_{2/m} + \hat{\mathcal{S}}_{3/m}\right)} | \psi^D \rangle_{PT} \]
\[ \equiv \frac{d}{dt} \langle \psi^D | \hat{\mathcal{S}} \psi^D \rangle_{PT} = \langle \psi^D | \frac{i}{\hbar} \left(\hat{H}^D, \hat{\mathcal{S}}^D\right) | \psi^D \rangle_{PT} \]

where \( \hat{\mathcal{S}}^D \equiv \hat{\mathcal{S}} \) and the expression of \( \hat{H}^D \) is given by
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\[ H_{FW} = i \beta mc^2 - ie \dot{\Phi} + i \beta \frac{\vec{\pi}^2}{2m} + i \beta \frac{\vec{\pi}^4}{8m^3c^2} + \beta \frac{e \hbar^2}{16m^4c^2} (\bar{\pi}, \partial_t \vec{E}) \]

\[ - \beta \frac{e \hbar}{2m} \left[ B - \beta \frac{\hbar}{4mc^2} \left( (\nabla \wedge \vec{E}) + \frac{2i}{\hbar} \vec{E} \wedge \bar{\pi} \right) \right] \]

\[ + \beta \frac{e \hbar}{8m^2c^2} \left( \partial_t \vec{E} \wedge \bar{\pi} + \bar{\pi} \wedge \partial_t \vec{E} \right) \]

\[ - i \beta \left( \frac{e \hbar}{2m} \right)^2 \frac{B^2}{2mc^2} + \frac{ie \hbar^2}{8mc^2} \nabla \cdot \vec{E} + \partial(m^{-3}). \]  

(17)

Now, the equation of motion for the mean spin operator (15) can be written in a simpler way being obtained as the pseudo-P\( \mathcal{T} \) expectation value a spin Dirac states \( |\chi^D\rangle \) associated to \( H^D \), one gets

\[ \frac{d}{dr} (\chi^D | \hat{\Sigma}^D | \chi^D)_{P\mathcal{T}} = \frac{e \beta}{m} (\chi^D | \hat{\Sigma}^D | \chi^D)_{P\mathcal{T}} \wedge B - \frac{e \hbar}{4mc^2} (\chi^D | \hat{\Sigma}^D | \chi^D)_{P\mathcal{T}} \wedge \partial_t B \]

\[ - \frac{e}{2mc^2} (\chi^D | \hat{\Sigma}^D | \chi^D)_{P\mathcal{T}} \wedge \left( E \wedge (\frac{1}{m} \bar{\pi} | \chi^D)_{P\mathcal{T}} \right) + \partial(m^{-3}). \]  

(18)

The above expression has been obtained by using \( \langle \psi^D | \hat{\Sigma}^D | \psi^D \rangle_{P\mathcal{T}} = \langle \chi^D | \hat{\Sigma}^D | \chi^D \rangle_{P\mathcal{T}} \). It is of interest to mention that the non-diagonal terms which appear at the second line of (15) are due to the Zitterbewegung phenomenon \[5\]. They cancel out when they are pseudo averaged out in a Dirac states.

Let us note that, it is more convenient to use the FW representation to find the evolution of spin, (indeed) from equation (7) where \( \frac{d}{dt} (\langle \psi^D | \hat{\Sigma}^D | \psi^D \rangle_{P\mathcal{T}} = \frac{d}{dt} (\langle \chi^D | \hat{\Sigma}^D | \chi^D \rangle_{P\mathcal{T}} \wedge \partial_t B \), we get

\[ \frac{d}{dt} (\chi^FW | \hat{\Sigma} | \chi^FW)_{P\mathcal{T}} = \frac{e \beta}{m} (\chi^FW | \hat{\Sigma} | \chi^FW)_{P\mathcal{T}} \wedge \left( B + \frac{1}{2c^2} \left[ E \wedge \left( \frac{i \beta}{m} \bar{\pi} \right) \right] \right) \]

\[ - \frac{e \hbar}{4mc^2} \left( \hat{\Sigma} \wedge (\nabla \wedge \vec{E}) \right) | \chi^FW \rangle_{P\mathcal{T}} \]  

(19)

Concerning the damping process, as previously explained, the only term of importance in the expression (15) is \( - \frac{e \hbar}{4mc^2} (\hat{\Sigma} \wedge \partial_t B \) which has been obtained from the commutator \( [(\alpha \times \vec{E}), (\alpha, \bar{\pi})] \) coming from \( H^D, \hat{\Sigma}^D \) and the partial derivative with respect to time of the mean spin angular momentum operator at second order in \( 1/m \) \[27\] given in equation (14). In addition, classical Maxwell equations have been also employed. Similarly to the Breit Hamiltonian, which is obtained from the classical Darwin Lagrangian (which also originates from Maxwell equations) by using the CP \[25\], we here resort to the same procedure (in its inverse form, from quantum to classical) for the Maxwell equations. According to this principle, the quantum counterparts \( \hat{f}, \hat{g} \) of classical observables \( f, g \) satisfy \( \langle \hat{f}, \hat{g} \rangle = i h \{ f, g \}_{pq} \) where \( \langle \hat{f}, \hat{g} \rangle \) is the expectation value of the commutator and the symbol \( \{ \} \) denotes the Poisson bracket \[28–31\]. Let us take for instance the Maxwell–Faraday equations, we have

\[ \nabla \wedge \vec{E}(r, t) = - \frac{\partial \vec{B}(r, t)}{\partial t} \]  

(20)
which can be rewritten as

$$\epsilon_{ijk} \{ p_i, E_j \} p_k = -\frac{\partial B(r, t)}{\partial t}, \quad (21)$$

and using the CP one gets

$$\epsilon_{ijk} \left[ \hat{p}_i, E_j \right] p_k = -\frac{\partial B(r, t)}{\partial t}, \equiv -\partial B.$$ 

Consequently, by using our definition of the magnetization $M(r, t) \equiv \mu_B \langle \chi_D | \Sigma_D | \chi_D \rangle_{PT} \equiv \mu_B \langle \chi_{FW} | \Sigma_{FW} | \chi_{FW} \rangle_{PT}$ and the CP, the equation of motion (18) may be rewritten for the electron part as

$$\frac{dM(r, t)}{dt} = \frac{e}{m} M(r, t) \wedge B(r, t) - \frac{e \hbar}{4m^2c^2} M(r, t) \wedge \partial_t B(r, t)$$

$$+ \frac{e}{2mc^2} M(r, t) \wedge (E(r, t) \wedge v) + \partial(m^{-3}). \quad (23)$$

The above equation constitutes the main result of this work.

Moreover, if the electron is embedded in a magnetically polarizable medium, defined by its magnetic polarizability $\chi_m$, then $\partial_t B(r, t) = \frac{\partial M}{\partial t}$ generates a time-dependent magnetic induction according to the relation $\partial_t B(r, t) = \frac{1}{\chi_m} \partial M$ and the equation (23) can be rewritten as

$$\frac{dM(r, t)}{dt} = -\gamma M(r, t) \wedge B_{\text{eff}}(r, t) - \frac{\alpha_G}{M} \left( M(r, t) \wedge \frac{\partial M(r, t)}{\partial t} \right) \quad (24)$$

with $\gamma = -\frac{e}{m} > 0$ the gyromagnetic ratio for an isolated electron, $B_{\text{eff}} \equiv B - \frac{1}{2c^2} v \wedge E$ and $\alpha_G \equiv \frac{e \hbar}{4m^2c^2} \chi_m$. The first term describes the precessional motion of the magnetization vector around the direction of the effective magnetic field and the second term represents its damping, characterized by the Gilbert’s constant $\alpha_G$.

Let us stress that the first term in the right-hand side of equation (24) can be retrieved from the non-relativistic expansion of the Bargmann–Michel–Telegdi’s equation [24, 32, 33] which represents the relativistic equation of motion of a classical magnetic dipole moment [34]. However, the damping term cannot be obtained from this classical description due to its quantum origin.

In summary, the mean spin angular momentum operator introduced for the first time by Foldy and Wouthuysen for the case of a free electron has been extended to the non-Hermitian or precisely to a pseudo $\mathcal{PT}$-symmetric case of an electron interacting with a time-dependent electromagnetic field. The expectation equation of the motion of the latter leads to the LLG equation revealing thus its microscopic origin. We therefore argue that the expectation value of the pseudo-mean spin operator with the new definition of $\mathcal{PT}$-inner product must be used instead of the usual one to properly describe the dynamics of the spin magnetization.
Data availability statement

No new data were created or analysed in this study.

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