A twistor formulation of the heterotic D=10 superstring with manifest (8,0) worldsheet supersymmetry

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Abstract

We propose a new formulation of the heterotic $D = 10$ Green-Schwarz superstring whose worldsheet is a superspace with two even and eight odd coordinates. The action is manifestly invariant under both target-space supersymmetry and a worldsheet reparametrisation supergroup. It contains only a finite set of auxiliary fields. The key ingredient are the commuting spinor (twistor) variables, which naturally arise as worldsheet superpartners of the target space Grassmann coordinates. These spinors parametrise the sphere $S^8$ regarded as a coset space of the $D = 10$ Lorentz group. The sphere is associated with the lightlike vector of one of the string Virasoro constraints. The origin of the on-shell $D = 10$ fermionic kappa symmetry of the standard Green-Schwarz formulation is explained. An essential and unusual feature is the appearance of the string tension only on shell as an integration constant.

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1 Introduction

The superstring is a very geometrical theory which has two apparently unrelated presentations, the worldsheet supersymmetric Neveu-Schwarz-Ramond version and the spacetime supersymmetric Green-Schwarz version, which nevertheless turn out to be equivalent (see [1] for a review). An interesting problem, and the subject of this paper, is to construct a version of the theory which has both worldsheet and space-time supersymmetry built in. To some extent the Green-Schwarz superstring does have this feature, since the GS action is invariant under a local fermionic symmetry, kappa-symmetry [2], which plays a crucial rôle. This feature of string theory occurs also in superparticles of Brink-Schwarz type [3], but until recently, there was no geometrical understanding of kappa-symmetry. In reference [4] Sorokin, Tkach, Volkov and Zheltukhin were able to re-formulate the superparticle (in $D = 3$ and 4 dimensions) in a way in which this symmetry was interpreted as worldline supersymmetry ($N = 1$ and 2, respectively). Their reformulation involved the introduction of twistor-like variables which arise essentially from solving the masslessness constraint $p^2 = 0$. Subsequently this formalism was extended to cover the $D = 6$ case ($N = 4$ worldline supersymmetry) and also general target superspaces [3]. In [3] the topological nature of the STVZ action was emphasised: it is invariant under a restricted, but non-trivial group of worldline diffeomorphisms even though it involves no worldline “supergravity” fields. A group-theoretical analysis of the twistor-like variables for the cases $D = 3, 4, 6$ and 10 was given in [7], [8]. The idea of ref. [4] has been generalised to cover superstrings in $D = 3, 4, 6$ dimensions in refs. [9], [10], [11]. These string actions can be characterised as being “semi-topological” in that they resemble the superparticle actions with respect to left diffeomorphisms, but worldsheet supergravity fields are necessary in the right sector. Note that in the cases $D = 4$ and $D = 6$ the notions of complex and quaternionic structures and the associated Grassmann analyticity (chirality in $D = 4$ and harmonic analyticity in $D = 6$) were heavily used both for the superparticle [9], [10] and for the superstring [11], [12]. Here one should also mention formulations of the $D = 10$ superstring with only $N = 2$ manifest worldsheet supersymmetry [12], [13].

In this paper we shall present an action for the heterotic superstring in 10 dimensions and show its classical equivalence (after eliminating a certain number of auxiliary fields and fixing some gauges) to the usual Green-Schwarz action. The new action is an integral over an $N = (8, 0)$ worldsheet superspace and at the same time has a $D = 10$ target superspace. Its manifest local worldsheet supersymmetry replaces kappa-symmetry of the usual superstring. We emphasise that the case $D = 10$ differs from $D = 3, 4, 6$ even for the superparticle, largely because in $D = 10$ there is no obvious replacement for the complex and quaternionic structures present in $D = 4$ and 6, respectively. A way round this difficulty was recently found in ref. [14]. There an $N = 8$ worldline supersymmetric action for the $D = 10$ superparticle was given which leads to the $N = 8$ worldline supersymmetric equations proposed in [6]. In ref. [14] the way to generalise the approach to the superstring was indicated. In this paper we complete the development of the ideas proposed in [6] and [14]. We shall focus on the $D = 10$ case, but, as in the case of the superparticle, the formalism is also applicable to $D = 3, 4, 6$.

1 Recently M. Tonin sent us his preprint [15], where a formulation of the $D = 10$ superstring is proposed. In some aspects it resembles ours, and later on in the text we shall comment on that.
The key ingredients in the construction are the twistor-like variables which parametrise the celestial sphere $S^8$ and the appearance of the string tension as a cohomological term (an integration constant) in a Lagrange multiplier superfield. Both of these are also features of the superparticle action given in [14]. The twistor-like variables arise as natural superpartners of the Grassmann target space coordinates $\theta$ with respect to worldsheet supersymmetry. Geometrically, the construction can be viewed in the following way: the string can be thought of as a $(2|8)$ $(2$ even, $8$ odd-dimensional) subsupermanifold, $M$, of the $(10|16)$ super target space, $MS$, embedded in such a way that the odd part of the tangent space to $M$ lies entirely within the odd part of the tangent space to $MS$ at any point of $M$. Thus, at each point in $M$ an eight-dimensional subspace of the odd tangent space to $MS$ is selected, and the set of such subspaces is parametrised by the coset space $Spin(1,9)/H$, where $H$ is the Borel subgroup which takes a reference subspace into itself. This coset space is the eight-sphere $[7]$. The string tension arises from the term in the action involving the supergravity two-form $B$. This part of the action is “almost” topological: variation with respect to the Lagrange multiplier field $P$ sets the pull-back of $B$ onto the worldsheet to be almost pure gauge (in fact, it is pure gauge in the left directions, which is a reflection of lightlike integrability). The Lagrange multiplier itself possesses a large abelian gauge invariance which, together with the equations of motion implies that only a single constant component survives in it. This constant is identified with the string tension.

We emphasise that the treatment of the superstring given in this paper is purely classical. Quantisation of twistor-like theories is a non-trivial subject, which deserves a separate study and will not be addressed here.

The paper is organised as follows. In Section 2 we explain how the bosonic massless particle in $D = 10$ can be formulated in terms of the new twistor variables. Unlike the original approach of [4], where only one commuting spinor (“twistor variable”) was employed, we use eight spinors satisfying an algebraic condition. They are shown to parametrise the sphere $S^8$ associated with the lightlike velocity of the particle [7], [14]. In Section 3 we extensively review the twistor formulation of the $D = 10$ superparticle of ref. [14], because it serves as the basis of the superstring theory. The worldline becomes a superspace with one even and eight odd dimensions. Then the Grassmann coordinates of the target superspace acquire superpartners with respect to worldline supersymmetry, which are just the eight spinor variables (twistors). The dynamics of the superparticle is described by equations which specify the embedding of the worldline superspace into the target one. The action simply consists of Lagrange multiplier terms, which impose these embedding conditions on shell. We also explain the identification of kappa-symmetry of the ordinary Brink-Schwarz superparticle with local $N = 8$ supersymmetry of the worldline. The coupling of the superparticle to an external Maxwell superbackground illustrates how the coupling constant (the electric charge) emerges as an integration constant of the equations of motion. Geometrically, this coupling means that the pull-back of the Maxwell one-form onto the worldline becomes pure gauge on shell. In Section 4 we generalise all

\[2\] The idea to obtain the string tension as an integration constant was first proposed in [16] in the context of Green-Schwarz type actions without worldsheet supersymmetry. Later on it was generalised to $p$-branes [17]. In this paper we shall show that worldsheet supersymmetry does in fact require such an unusual way to introduce the string tension.
these ideas to the heterotic superstring. The worldsheet is now a heterotic \((p, q) = (8, 0)\) superspace. The Wess-Zumino term of the superstring involves the two-form superfield from the supergravity background, which is treated very much like the Maxwell superfield in the particle case. The difference is that its pull-back onto the worldsheet is gauge trivial only in the left directions of the heterotic worldsheet. This time it is the string tension that appears as an integration constant.

Before proceeding further, a few words about the notation. Throughout the paper we will have to deal with four different superspaces: the \((1|8)\) or \((2|8)\) worldsheet and its tangent space, and the \((10|16)\) target space and its tangent space. In order to avoid the proliferation of different types of indices, we adopt the following notation. We denote the worldsheet by \(\mathcal{M}\) and the target superspace by \(\mathcal{M}\). We use \(M = (m, \mu)\) for coordinate indices \((m\text{ even and }\mu\text{ odd})\) and \(A = (a, \alpha)\) for tangent space indices in \(\mathcal{M}\); in \(\mathcal{M}\) we use the same conventions but the indices are underlined. The coordinates of \(\mathcal{M}\) are denoted \(z^M = (x^m, \theta^\mu)\) and those of \(\mathcal{M}\) are denoted \(z^\mathcal{M} = (x^a, \theta^{\underline{\alpha}})\). We shall use the same letters to denote corresponding quantities in \(\mathcal{M}\) and \(\mathcal{M}\). In the case that indices are displayed, it will be clear to which space we are referring from the presence or absence of underlinings; if no indices are displayed we shall distinguish quantities in \(\mathcal{M}\) by underlining them. In the case of the superstring in \(\mathcal{M}\) we shall use a light-cone split for both coordinate and tangent frames, so that \(m = (+, -)\) and \(a = (+, -)\); it should be clear from the context which type a “+” or “−” index is.

2 Massless particles, spheres and twistors

2.1 The massless \(D = 10\) particle and the sphere \(S^8\)

The action for a massless particle propagating in a \(D = 10\) target space is

\[
S = \frac{1}{2} \int_0^1 dx^- \ g \ \partial_- x^a \partial_- x^{\underline{a}}. \tag{1}
\]

Here \(x^-\) denotes the worldline time variable, \(x^a\) are the coordinates of the \(D = 10\) target space and \(g\) is the worldline metric. An important kinematical requirement is that the particle velocity of the particle \(\partial_- x^a\) does not vanish as a vector,

\[
\partial_- x^a \neq 0. \tag{2}
\]

The rôle of the metric is to make the action (1) invariant under diffeomorphisms of the worldline,

\[
x^- \rightarrow x'^-(x^-), \quad \frac{\partial x'^-}{\partial x^-} > 0; \tag{3}
\]

\[
g'(x'^-) = g(x^-) \frac{\partial x'^-}{\partial x^-}.
\]

Note that the diffeomorphisms in (3) preserve the orientation of time, hence one can regard them as local \(SO^+(1,1)\) transformations.
This action can be rewritten in an equivalent form by passing to the first-order formalism and introducing an auxiliary vector $u^a$:

$$S = \int dx^- \left[ p_\mu (\partial_- x^\mu - u^\mu) + \frac{1}{2} g u^\mu u_\mu \right].$$

(4)

Note that the variation with respect to $g$ makes the vector $u^\mu$ lightlike:

$$u^\mu u_\mu = 0.$$  

(5)

This null-vector (which coincides with the velocity of the particle) transforms as a density under the worldline diffeomorphisms (3),

$$u'(x^\tau) = u(x^-) \frac{\partial x^-}{\partial x'^\tau}.$$  

(6)

Further, we can take the time component of this vector to be positive, since the natural boundary condition $x^0(1) > x^0(0)$ together with (2) imply

$$\partial_- x^0 > 0 \rightarrow u^0 > 0.$$  

(7)

The vector $u^a$, which satisfies eqs. (3), (4) and is defined modulo the local $SO^\uparrow(1,1)$ transformations (3), parametrises an eight-dimensional sphere $S^8$. This is easily seen by choosing the $SO^\uparrow(1,1)$ gauge $u^0 = 1$ or by dividing the left-hand side of eq. (3) by $(u^0)^2$. Thus one naturally associates a sphere $S^8$ with the massless particle.

The twistor approach to the particle differs from the standard one in that the sphere is described by commuting spinor (twistor) variables. The action

$$S = \int dx^- p_\mu (\partial_- x^\mu - u^\mu)$$

(8)

with the constraints (3), (4) on the vector $u^\mu$ implied is the starting point for the transition to a twistor-like particle action. One possibility, which was employed in [4], is to solve the constraint (3) in terms of a single commuting Majorana-Weyl spinor $\lambda^\alpha$:

$$u^\mu = \lambda^\alpha \gamma^\mu \lambda.$$  

(9)

To check that it satisfies (3) one should use the gamma matrix identity

$$(\gamma^\mu)_{\alpha\beta} (\gamma^\alpha \gamma^\beta) = 0.$$  

(10)

This identity holds in $D = 3, 4, 6, 10$, so the same procedure can be applied in all these special dimensions. Each time one uses the smallest possible spinor representations of the corresponding Lorentz groups $SO(1, D - 1)$, which are of real dimensions 2, 4, 8, 16, respectively. Note, however, that the case $D = 10$ essentially differs from the other three cases. To explain this difference we remark that the correspondence between $u^\mu$ and $\lambda^\alpha$ is not one-to-one: the non-vanishing null vector $u^\mu$ parametrises the spheres $S^{D-2} = S^1, S^2, S^4$ and $S^8$ in the above dimensions, while the non-vanishing spinor $\lambda^\alpha$ parametrises the spheres $S^1, S^3, S^7$ and $S^{15}$, correspondingly (the vector and the spinor are considered modulo $SO^\uparrow(1,1)$ scale transformations). The difference between the vector and spinor...
degrees of freedom is accounted for by certain gauge transformations with 0, 1, 3 and 7 parameters, respectively. Now the point is that while in the lower-dimensional cases $D = 3, 4$ and 6 these gauge transformations form the groups $Z_2, U(1)$ and $SU(2)$, the 7-parameter gauge transformations in $D = 10$ do not correspond to any group. Instead, they correspond to the sphere $S^7$ which is the fiber in the Hopf fibration $S^{15} \to S^8$. (The lower cases are the other Hopf fibrations $S^1 \to S^1, S^3 \to S^2, S^7 \to S^4$ with fibers $Z_2, S^1$ and $S^3$, respectively. These fibers are groups, since $S^1 = U(1), S^3 = SU(2)$.) In other words, the gauge transformations which act on the 16-dimensional Majorana-Weyl $D = 10$ spinor $\lambda^\alpha$ and leave the vector $u^\alpha$ (9) invariant modulo $SO^{\uparrow}(1, 1)$, do not constitute any group, their algebra is non-linear and has field-dependent structure constants. This is an undesired feature of the $D = 10$ solution (9), since we want to construct actions with linear realisations of all symmetries.

Another way to explain why the case $D = 10$ is so different from the lower ones is to note that eqs. (5) and (9) are directly related to the existence of four division algebras (see, for instance, [18]). They are equivalent to the defining identity for hypercomplex numbers: $|ab|^2 = |a|^2|b|^2$. Then the non-linearity and the field-dependence of the algebra of gauge transformations for the spinor $\lambda$ in (9) in the case $D = 10$ can be shown to arise from the non-associativity of the octonionic multiplication (see, for instance, [19]).

Fortunately, there exists another way to describe the sphere $S^8$ in terms of eight commuting Majorana-Weyl spinors [7]. That construction is purely real, it makes no use of the notion of hypercomplex numbers and all the complications mentioned above are avoided.

### 2.2 Twistor description of $S^8$

The central point in the whole construction of this paper will be to replace lightlike vectors like $u^\alpha$ by commuting spinor (“twistor”) variables in a way that maintains all symmetries linearly realised. Our twistor variables will be defined as follows:

$$\lambda_\alpha^{\underline{\alpha}}, \quad \alpha = 1, \ldots, 8; \quad \underline{\alpha} = 1, \ldots, 16,$$

where $\alpha$ is an $O(8)$ index $\underline{\alpha}$ and $\underline{\alpha}$ is a Majorana-Weyl $SO(1, 9)$ spinor index. The real $8 \times 16$ matrix (11) is defined modulo arbitrary $SO^{\uparrow}(1, 1) \times O(8)$ transformations

$$(\lambda_\alpha^{\underline{\alpha}})' = \Omega \omega^\beta_\alpha \lambda_\beta^{\underline{\alpha}}; \quad \Omega \in SO^{\uparrow}(1, 1), \quad \omega^\beta_\alpha \in O(8),$$

and is supposed to satisfy the algebraic constraint (invariant under (12))

$$\lambda_\alpha^{\underline{\alpha}}(\gamma^{\underline{\alpha}})_{\underline{\alpha}\beta} \lambda_\beta^{\underline{\beta}} = \frac{1}{8} \delta_{\alpha\beta} (\lambda_\gamma^{\underline{\alpha}} \lambda_\gamma^{\underline{\beta}}),$$

or, in matrix notation,

$$\lambda^{\underline{\alpha}} \gamma^{\underline{\alpha}} = \frac{1}{8} Tr (\lambda^{\underline{\alpha}} \lambda^{\underline{\alpha}}) 1.$$  

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3The choice of an eight-dimensional representation ($8_s, 8_c$ or $8_v$) for $\alpha$ is arbitrary at this point. Later on we shall see that in a special gauge this $O(8)$ can be identified with the $O(8)$ subgroup of the $D = 10$ Lorentz group, and the representation with, for instance, $8_s$.
In addition, the matrix (11) is required to satisfy the non-degeneracy condition

\[ \text{rank } \parallel \lambda \parallel = 8. \]  

(15)

In order to show how the twistor variables defined above parametrise the sphere \( S^8 \) we are going to use light-cone coordinates. Our conventions are as follows: the ten-dimensional Minkowski metric is \( \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) \), the light-cone components of a vector \( p^a \) are \( p^\pm = p^0 \pm p^9 \), \( \vec{p} \), hence \( p^a p_a = -p^+ p^- + (\vec{p})^2 \). We use the 16 \( \times \) 16 \( \gamma \)-matrix representation where

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^9 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{av} = \begin{pmatrix} 0 & \gamma^a v \\ (\gamma^a v)^T & 0 \end{pmatrix} \]  

(16)

and \( (\gamma^{av})_{\alpha_s \alpha_c} \) are 8 \( \times \) 8 matrices of \( O(8) \), the indices \( a_v, \alpha_s \) and \( \alpha_c \) correspond to the representations \( 8_v, 8_s \) and \( 8_c \) of \( O(8) \), respectively. Finally, a Majorana-Weyl spinor \( \Theta^\alpha \) is decomposed into \( (\Theta^{\alpha_s}, \Theta^{\alpha_c}) \). In this notation the twistor matrix \( \lambda^\alpha_\alpha \) decomposes into two 8 \( \times \) 8 matrices,

\[ \lambda^\alpha_\alpha = (u^\alpha_\alpha_s, v^\alpha_\alpha_c), \]  

(17)

in terms of which (13) reads

\[ uu^T = \frac{1}{8} Tr(uu^T) \mathbf{1}, \]  

(18)

\[ u\vec{\gamma}v^T + (u\vec{\gamma}v^T)^T = \frac{1}{4} Tr(u\vec{\gamma}v^T) \mathbf{1}, \]  

(19)

\[ vv^T = \frac{1}{8} Tr(vv^T) \mathbf{1}. \]  

(20)

Note that at least one of the matrices \( u, v \) is non-degenerate, otherwise one can derive from (18) and (20) that both of them vanish in contradiction with (15). Choosing, for instance,

\[ \text{Chart 1: } \det u \neq 0, \]  

(21)

we see that (15) means that \( u \in SO^+(1, 1) \times O(8) \). The gauge freedom (12) allows us to choose a gauge in which

\[ u^\alpha_\alpha_s = \delta^\alpha_\alpha_s, \]  

(22)

thus identifying the \( O(8) \) group of the index \( \alpha \) with the subgroup of the Lorentz group \( SO(1, 9) \). Note that this gauge breaks Lorentz invariance.

In the gauge (22) one can easily solve eq. (19) for the second matrix \( v \) in terms of an arbitrary \( O(8) \) vector \( \vec{y} \):

\[ v = \vec{y} \cdot \vec{\gamma}. \]  

(23)

Then the third equation (20) is automatically satisfied. The alternative to (21) is

\[ \text{Chart 2: } \det v \neq 0. \]  

(24)

It implies that this time \( v \in SO^+(1, 1) \times O(8) \), so it can be gauged into \( \delta^\alpha_\alpha_c \). Instead of (23) we now find

\[ u = \vec{\gamma}^T \cdot \vec{z}. \]  

(25)
In the overlapping area \((\det u \neq 0, \det v \neq 0)\) both coordinates \(\vec{y}\) and \(\vec{z}\) are non-vanishing. To find the relation between the two sets of coordinates one can consider the lightlike vector \(\lambda_\alpha \gamma^\beta \lambda_\alpha\). Under the gauge group \(\text{SO}^\dagger(1,1)\) it transforms like an \(\text{SO}^\dagger(1,1)\) density, so the ratio of any two of its components is gauge invariant. Computing these ratios on the first and the second charts one finds

\[
\vec{z} = \vec{y}/y^2,
\]

so \(\vec{y}\) and \(\vec{z}\) can be considered as stereographic coordinates on the sphere \(S^8\).

The conclusion from the discussion above is that the twistor variable \(s\) defined in (13), (15) describe the sphere \(S^8\) modulo \(\text{SO}^\dagger(1,1)\times\text{O}(8)\) gauge transformations. The \(\text{O}(8)\) vectors \(\vec{y}\) or \(\vec{z}\), corresponding to the choices (21) or (24), are the stereographic coordinates on the two charts needed to cover the sphere.

We point out that these twistor variables admit a geometric interpretation as coordinates of a coset space \([7], [14]\). The rectangular matrix \(\lambda\) (11) can be viewed as one half of a matrix belonging to the spin group \(\text{Spin}(1,9)\) of the \(D=10\) Lorentz group:

\[
L_{\alpha \beta} = (\lambda_{\alpha \beta}, \phi_{\alpha} \bar{\omega}), \quad L \in \text{Spin}(1,9).
\]

Then conditions (13), (15) appear as that part of the defining conditions on the \(\text{Spin}(1,9)\) matrix \(L\) (27), which concern only the matrix \(\lambda\). The sphere \(S^8\) can be thought of as the coset space

\[
S^8 = \frac{\text{Spin}(1,9)}{[\text{SO}^\dagger(1,1) \times \text{O}(8)] \vec{K}}.
\]

Here the denominator is the Borel subgroup of the Lorentz group (the maximal subgroup of the Lorentz group), including eight "conformal boost" transformations \(\vec{K}\). Then the twistor variable matrix \(\lambda\) is that half of \(L\) which stays inert under the action of \(\vec{K}\). The other half, \(\phi\), is shown to be either a pure \(\vec{K}\)-gauge or is expressed in terms of \(\lambda\) via the remaining \(\text{Spin}(1,9)\) defining conditions.

### 2.3 Twistor particle

A key ingredient in the massless particle theory of subsection 2.1 was the lightlike vector \(u^\alpha\). Now we can replace that vector by a bilinear combination of the eight spinors (twistor variables) of subsection 2.2:

\[
u^\alpha = \frac{1}{8} \lambda_\alpha \gamma^\beta \lambda_\alpha.
\]

To check that it satisfies (5) one should use (13) and the gamma matrix identity (10). The time component of this vector is strictly positive,

\[
u^0 = \frac{1}{8} (\text{Tr}(uu^T) + \text{Tr}(vv^T)) > 0,
\]

in accord with (6). The specific twistor combination in (29) is \(\text{O}(8)\) invariant and \(\text{SO}^\dagger(1,1)\) covariant, so it describes \(S^8\) modulo \(\text{SO}^\dagger(1,1)\) transformations. On the other hand, in subsection 2.1 we saw that the content of the lightlike vector \(u^\alpha\) was also \(S^8 \times \text{SO}^\dagger(1,1)\). Then we conclude that the twistor combination (29) is equivalent to that vector.
After the discussion above we can propose the following twistor form of the massless particle action (4):

$$S = \int dx^- \left[ p_\alpha (\partial_- x^\alpha - \frac{1}{8} \lambda_\alpha \gamma^2 \lambda_\alpha) + p_{\alpha\beta} \lambda_{\{\alpha} \gamma^2 \lambda_{\beta\}} \right].$$  \(31\)

Here \(p_{\alpha\beta}\) is a new Lagrange multiplier, which imposes the twistor defining condition (13) on shell and the symbol \(\{\alpha\beta\}\) denotes the symmetric traceless part

$$A_{\{\alpha\beta\}} \equiv \frac{1}{2} (A_{\alpha\beta} + A_{\beta\alpha} - \frac{1}{4} \delta_{\alpha\beta} A_{\gamma\gamma}).$$  \(32\)

We emphasise that the action (31) is diffeomorphism invariant without the help of the worldline metric \(g\) of (4). Indeed, since the bilinear combinations of twistors present in (31) are \(SO^\uparrow(1,1)\) covariant, it is sufficient to identify the \(SO^\uparrow(1,1)\) parameter with \(\partial x^\gamma / \partial x^-\).

This means that the twistor variables take over the role of the one-dimensional metric. The latter reappears in the process of elimination of the twistors, which leads from (31) back to (4).

Concluding this subsection, we would like to point out there is an alternative way to show that the action (31) describes a massless particle. It consists in analysing the equations of motion

$$\delta p_\alpha:\quad \partial_- x^\alpha = \frac{1}{8} \lambda_\alpha \gamma^2 \lambda_\alpha,$$

$$\delta x^\alpha:\quad \partial_- p_\alpha = 0,$$

$$\delta \lambda_{\alpha\beta}:\quad (\gamma^2 \lambda_\alpha)_{\alpha\beta} p_\alpha = 8 p_{\alpha\beta} (\gamma^2 \lambda_\beta)_{\alpha\beta},$$

following from (31), and proving that their gauge invariant content coincides with that of the equations derived from the standard particle action (1). Using light-cone coordinates and imposing, e.g., the gauge (22), one can show that eq. (35) allows one to solve for \(p_\alpha\) in terms of the gauge invariant part of the twistors, and thus identify it with the velocity of the particle (see (33)). At the same time, the Lagrange multiplier \(p_{\alpha\beta}\) is shown to vanish on shell, up to gauge degrees of freedom contained in the transformation

$$\delta p_{\alpha\beta} = \Sigma_{\alpha\beta\gamma} (\gamma^2)_{\alpha\beta} \lambda_{\gamma\}. $$

Here the parameter \(\Sigma_{\alpha\beta\gamma}\) is symmetric and traceless with respect to its \(O(8)\) indices.

### 3 Superparticles

#### 3.1 The Brink-Schwarz superparticle

The Brink-Schwarz superparticle (in a supergravity background) is a generalisation of the bosonic one (4), where the bosonic \(D = 10\) target space is replaced by a \((10|16)\) curved target space with tangent group \(Spin(1,9)\):

$$\mathcal{M} = \{x^\mu\} \rightarrow \mathcal{M} = \{z^\underline{M} = (x^\underline{M}, \theta^\underline{M})\}. $$

\(^4\)Note that a diffeomorphism transformation is globally not exactly the same as an \(SO^\uparrow(1,1)\) one. This results in an extra boundary term (the constant invariant length of the particle trajectory) contained in the twistor variables. In other words, the twistors describe the sphere \(S^8\) and this constant.
\( M \) is equipped with preferred frames \( E^A = dz^M E^A_M(z) \) and a Lorentzian connection \( \Omega \). We take the constraints on the torsion to be those describing ten-dimensional supergravity \[20\], \[21\]; in particular, we have

\[
T_{\alpha\beta}{}^\gamma = -2i(\gamma_\alpha)_{\alpha\beta}^\gamma \tag{38}
\]

and

\[
T_{\alpha\beta}{}^\gamma = 0, \quad T_{\alpha\beta}{}^\gamma = 0. \tag{39}
\]

The superparticle can be viewed as an embedding of the one-dimensional worldline \( M \) into the target superspace \( M \). In other words, the target-superspace coordinates become worldline fields, \( z^M = z^M(x^-) \). The object

\[
E^-{}^A = \partial^- z^M E^A_M(z), \tag{40}
\]

gives the components of the tangent vector to \( M \) in the preferred basis of \( M \). Then the action of the Brink-Schwarz superparticle can be written down as the direct analogue of (1):

\[
S = \frac{1}{2} \int dx^- g E^-{}^A E^-{}^A. \tag{41}
\]

The geometric meaning of this action is to specify the embedding of \( M \) into \( M \) in such a way that the vector \( E^-{}^A \) becomes lightlike on shell.

As before, the action (41) is worldline diffeomorphism invariant due to the presence of the metric \( g \). In addition, it possesses a somewhat mysterious gauge symmetry with an anticommuting spinor parameter \( \kappa_\alpha(x^-) \). Denoting the variation of the target space coordinates referred to a preferred basis by

\[
\delta z^A = \delta z^M E_M^A \tag{42}
\]

the kappa-symmetry transformations can be written as follows [2]:

\[
\delta z^\alpha = 0, \quad \delta z^\dot{\alpha} = E^-{}^\alpha (\gamma_\alpha)^{\alpha\dot{\alpha}} \kappa_{\dot{\alpha}}, \quad \delta g = 4i g E^-{}^\alpha \kappa_{\dot{\alpha}}. \tag{43}
\]

To check the invariance of (41) one has to use the background supergravity constraints (38), (39).

This symmetry is of crucial importance for the consistency of the superparticle action. Indeed, take the simplest situation when the target superspace is flat. In this case the target space supervielbein is given by

\[
E^-{}^\alpha = \delta E^-{}^\alpha, \quad E^-{}^\dot{\alpha} = 0, \quad E^-{}^\alpha = i(\gamma_\alpha)_{\alpha\beta} \theta^\beta, \quad E^-{}^{\dot{\alpha}} = \delta^{\dot{\alpha}}. \tag{44}
\]

Then, at first sight the action has the non-linear appearance of an interacting theory, but this is not the case. The reason is as follows. The analysis of transformations [3] in light-cone coordinates shows that only 8 of the 16 components of the parameter \( \kappa \) are active, because the matrix \( E^-{}^\alpha \gamma_\alpha \) is a projector operator on shell, where the vector \( E^-{}^\alpha \) is lightlike. These 8 gauge parameters allow one to gauge away half of the components of the spinor \( \theta^\alpha \). After this has been done the action is linearised and it is easy to show that the theory is free. This very important invariance finds its natural explanation as local \( N = 8 \) supersymmetry of the worldline in the twistor version of the theory.
3.2 World-line superspace

The first step towards a twistor formulation of the superparticle is to replace the one-dimensional worldline by a (1|8) super-worldline:

$$\mathcal{M} = \{ x^- \} \rightarrow \mathcal{M} = \{ z^M = (x^-, \theta^\mu) \}. \quad (45)$$

The geometry of $\mathcal{M}$ is taken to be superconformally flat. Let $(E_A)$ be a local basis of frames for $\mathcal{M}$, $E_A = E_A^M \partial_M$, where $E_A^M$ is the inverse supervielbein, then the tangent space group acts by

$$E_A \rightarrow L_A^B E_B, \text{ where } L_A^B = \begin{pmatrix} L_- & L_+^\beta \\ 0 & L_-^\beta \end{pmatrix}. \quad (46)$$

In addition it is supposed that the distribution spanned by $(E_A)$ is non-integrable, so that the quantity $-i < [E_\alpha, E_\beta] >$ is non-singular (and positive). Here, $(E^A = dz^M E_M^A)$ are the basis one-forms dual to $(E_A)$, and $<, >$ denotes the pairing between forms and vectors, $< dz^M, \partial_N > = \delta^M_N$. In other words, the anholonomy coefficient $C_{\alpha\beta}^-$ is nonvanishing ($[E_A, E_B] = -C_{ABC} E_C$).

The statement that $\mathcal{M}$ is superconformally flat (which can be easily proven given the above structure) means that there exist local coordinates in which the frames take the standard form of flat superspace, i.e.

$$E_\alpha = \partial_\alpha + i\theta_\alpha \partial_-, \quad E_- = \partial_. \quad (47)$$

When using these basis vectors as derivatives we shall write them as $D_\alpha$ and $D_-$ respectively. By construction, we have the flat $N=8$ supersymmetry algebra

$$\{ D_\alpha, D_\beta \} = 2i\delta_{\alpha\beta} D_-, \quad [D_-, D_\alpha] = 0. \quad (48)$$

The class of diffeomorphisms which preserves the form of the frames (47) up to tangent space transformations of the form (46) is easily found. Making a change of coordinates, we have

$$D_A = D_A z^M \frac{\partial}{\partial z^M}. \quad (49)$$

Taking the $\alpha$ component of this equation and demanding that $D_\alpha$ transform into itself, as dictated by the structure of the tangent space group, we find

$$D_\alpha x^\alpha - iD_\alpha \theta^\beta \theta'^\beta = 0. \quad (50)$$

This equation shows that the diffeomorphisms in the $\theta$ direction are expressible in terms of the $x^-$ transformation. We then have

$$D_\alpha = L_\alpha^\beta D'_\beta, \quad (51)$$

where

$$L_\alpha^\beta = D_\alpha \theta'^\beta. \quad (52)$$
Differentiating (52) and taking the symmetric part we get the integrability condition

\[ L_\alpha L_{\beta\gamma} = \delta_{\alpha\beta} L_{\gamma}, \quad (53) \]

where

\[ L_{\gamma} = D_{\gamma} x' - D_{\theta^{\alpha}} \theta'_{\alpha}. \quad (54) \]

Equation (53) states that \( L_{\alpha} \) is a matrix belonging to the group \( SO^+(1,1) \times O(8) \); note also that \( L_{\gamma} \) is positive, i.e. if we regard \( x^- \) as a time variable, the allowed coordinate changes preserve the time orientation. The "–" component of (49) yields no new constraints on the transformation parameters.

For infinitesimal transformations \( \delta z^M = z'^M - z^M \) eq. (53) is linearised and can easily be solved:

\[ \delta \theta_\alpha = -\frac{i}{2} D_\alpha \Lambda^-, \quad (55) \]

where \( \Lambda^-(z) \) is an arbitrary superfield parameter. Inserting this solution into eq. (54), we can solve for \( \delta x^- \) too:

\[ \delta x^- = \Lambda^- - \frac{1}{2} \theta_\alpha D_\alpha \Lambda^-. \quad (56) \]

For the derivatives, we have, with \( \delta D_A = D'_A - D_A \),

\[ \delta D_\alpha = \frac{i}{2} (D_\alpha D_\beta) \Lambda^- D_\beta, \quad (57) \]

\[ \delta D_- = -(D_- \Lambda^-) D_- + \frac{i}{2} (D_- D_\alpha \Lambda^-) D_\alpha, \quad (58) \]

and these transformations are thus explicitly seen to be of the form (46). Thus the group of transformations involves one parameter, \( \Lambda^- \), which is an unconstrained superfield. This group clearly contains \( x^- \)-space diffeomorphisms and local supersymmetry transformations.

### 3.3 Embedding the worldline in the target superspace

The motion of the superparticle can be regarded as a specific embedding of the worldline \( \mathcal{M} \) in the target superspace \( \mathcal{M} \). This means that the coordinates of \( \mathcal{M} \) are defined as worldline superfields, \( z^M = z^\mathcal{M}(z^\mathcal{M}) \). To describe this embedding it is convenient to introduce the matrix

\[ E_A^A = E_A^M(z) \partial_M z^\mathcal{M}(z) E_M^A(z). \quad (59) \]

This matrix can be viewed in two ways, as the pull-back of the coframe \( E_A^A \) referred to a preferred basis in \( \mathcal{M} \), or as the push-forward of the frame \( E_A \) to \( \mathcal{M} \) referred to the preferred basis of \( \mathcal{M} \).

Now we shall specify the way the embedding of \( \mathcal{M} \) in \( \mathcal{M} \) is done. First of all, we remark that the odd-odd part \( E_{\alpha}^{\alpha} \) of the matrix (59) looks very much like the twistor matrix \( \lambda_{\alpha}^{\alpha} \). Therefore we impose on \( E_{\alpha}^{\alpha} \) the same restriction as the one on \( \lambda_{\alpha}^{\alpha} \) in (13),

\[ \delta_E^{(\alpha}(\gamma^{\alpha)}_{\alpha\beta} E_{\beta})^\beta = 0. \quad (60) \]
This, together with a non-degeneracy condition similar to (15),
\[
\text{rank } \parallel E \parallel = 8, \tag{61}
\]
allows to identify the lowest-order component of the worldline super field \( E_{\alpha}^{\underline{\alpha}} \) with the twistor variables:
\[
\lambda_{\alpha}^{\underline{\alpha}} = E_{\alpha}^{\underline{\alpha}}|_{\theta=0}. \tag{62}
\]
One can say that the twistor variables emerge as the superpartners of the target-space odd coordinates with respect to \( N = 8 \) worldline supersymmetry, \( \mathbb{H} \):
\[
\theta^{\underline{\alpha}}(z) = \theta^{\underline{\alpha}} + \theta_{\alpha} \lambda_{\alpha}^{\underline{\alpha}} + \ldots \tag{63}
\]
As our second embedding condition we demand that the pull-back of a target-space vector onto a worldline spinor vanish:
\[
E_{\alpha}^{\underline{a}} = 0. \tag{64}
\]
Geometrically, this means that \( \mathcal{M} \) is embedded in \( \mathcal{M} \) in such a way that the odd part of the tangent space to \( \mathcal{M} \) lies entirely within the odd part of the tangent space to \( \mathcal{M} \) at any point of \( \mathcal{M} \). Thus, at each point in \( \mathcal{M} \) an eight-dimensional subspace of the odd tangent space to \( \mathcal{M} \) is selected, and the set of such subspaces is parametrised by the coset space \( Spin(1,9)/H \), where \( H \) is the Borel subgroup which takes a reference subspace into itself. As explained earlier, this coset space is the eight-sphere.

Equation (64) is a differential equation which requires an integrability condition. We shall show that, when the constraints on background supergravity are taken into account, this condition is just (60). First we separate the non-linear terms in (64) by writing down the vielbeins \( E_M^{\underline{a}} = \delta_M^{\underline{a}} + \text{n.l.t.} \). Then eq. (64) becomes
\[
D_{\alpha} x^{\underline{m}} = J_{\alpha}^{\underline{m}}(z^\mathcal{M}), \tag{65}
\]
where we have put all the non-linear terms in the right-hand side of the equation. Using the flat algebra (48), one finds the integrability condition on the non-linear source:
\[
D_{(\alpha} J_{\beta)}^{\underline{m}} = 0. \tag{66}
\]
To find the covariant version of (66) we shall use the first of Cartan’s structure equations for \( \mathcal{M} \):
\[
T^A = dE_A^A + E_B^A \Omega^A_{BA}. \tag{67}
\]
If we pull this back to \( \mathcal{M} \), we can write out the component form as
\[
D_A E_B^{\underline{C}} - (-)^{AB} D_B E_A^{\underline{C}} - [E_A, E_B]^{\underline{C}} E_C^{\underline{C}} - \Omega_A^{\underline{C}} E_C^{\underline{C}} - (\Omega_B^{\underline{C}} e^{-\Omega_B^{\underline{C}}}) = -T_{AB}^{\underline{C}}, \tag{68}
\]
\(^5\)The possibility to interpret these superpartners as Lorentz harmonic variables for the case \( D = 10 \) has been pointed out in [10], [7].
\(^6\)This equation was proposed in ref. [7] (in the case of a flat background). An analogous equation, but written down in octonionic notation and in a non-linear and \( O(8) \) non-covariant gauge, was studied in ref. [19].
where
\[ T_{AB}^C = (\mathcal{T}^{A} + \mathcal{T}^{B} - \mathcal{T}^{C}) \]
and similarly for \( \Omega_{AB}^C \). Putting in (68) \( \mathcal{A} = \alpha, \mathcal{B} = \beta, \mathcal{C} = \gamma \), taking the symmetric traceless part in \( \alpha, \beta \) (as in (66)) and using once again (64), we obtain
\[ E_{(\alpha} \mathcal{T}_{\alpha\beta} \mathcal{E}_{\beta)}^\gamma = 0. \]  
Comparing this with (60) we see that they are the same, provided the background supergravity constraint (38) holds. Actually, the compatibility of the integrability condition (70) with the sphere-defining condition (60) on the twistor variables not only follows from that supergravity constraint, but also implies it (up to purely conventional parts). One can show this by decomposing all the matrices in (70) into \( O(8) \) representations and analysing the corresponding algebraic relations. This is a manifestation of the so-called lightlike integrability principle [21], [22].

Another important consequence of the embedding equations (60), (64) is obtained by putting in (68) the same values of the indices as before, but this time taking the trace in \( \alpha, \beta \):
\[ E_{-\alpha} \mathcal{E}_{-\alpha} = \frac{1}{8} \mathcal{E}_{\alpha} \mathcal{E}_{\alpha}. \]  
Following the arguments of section 2.3, one sees that this vector is lightlike,
\[ E_{-\alpha} \mathcal{E}_{-\alpha} = 0. \]  
The lowest-order component in the \( \theta \) expansion of eq. (72) coincides with the equation following from the variation with respect to the worldline metric in the Brink-Schwarz superparticle action (41). As before, the twistor variables describe the sphere \( S^8 \) associated with this lightlike vector.

The conclusion of this subsection is that by an appropriate embedding of the (1|8) worldline into the (10|16) target superspace we have found a natural place for the twistor variables. We also obtained an important ingredient of the superparticle on-shell dynamics, the lightlike vector (72).

### 3.4 The twistor superparticle action

The action for the twistor superparticle consists of two Lagrange multiplier terms corresponding to the embedding equations (60) and (64):
\[ S = \int dx^\mu d^8 \theta \left[ iP_{\alpha\beta} \mathcal{E}_{\alpha\beta} + P_{\alpha\beta} \mathcal{E}_{(\alpha} \mathcal{E}_{\beta)} \right]. \]  
This action is invariant under the special superdiffeomorphisms of subsection 3.2. Indeed, in (73) the spinor worldline indices transform like the spinor derivatives \( D_\alpha \), i.e. homogeneously, see (51). It is then easy to find suitable transformations of the Lagrange multipliers which can compensate for this, as well as for the transformation of the supervolume. Once again, we remark that no worldline supergravity fields are needed to achieve the diffeomorphism invariance of the twistor action (cf. subsection 2.3).
We shall now show that the component content of the action (73) reduces to that of the target-space supersymmetric version of the twistor action (31). The transition to the Brink-Schwarz action (41) will then follow the pattern of Section 2.

We begin by analysing the dynamical content of eq. (64), which follows from the first term in (73). One can show that it produces algebraic equations for almost all of the components of the superfield \( x^\alpha(z) \), except for the lowest-order one \( x^\alpha|_{\theta=0} \). Indeed, by dropping all the non-linear terms in (65) one obtains the homogeneous form of eq. (64), which is \( D^\alpha x^\alpha = 0 \). It is not hard to verify that it sets equal to zero all the components of the superfield \( x^\alpha(z) \), except for the lowest-order one, which satisfies the equation \( \partial x^\alpha|_{\theta=0} = 0 \). The same pattern is repeated in the inhomogeneous equation (65), provided the integrability condition (66), or rather its covariant version (60) holds. Thus, we conclude that the only non-trivial component of (64) is the lowest-order component of eq. (71).

Since all the algebraic component equations contained in (74) are introduced in (73) by Lagrange multipliers (the components of \( P_{\alpha a} \)), one can drop them in the action. Thus the only surviving component of the first term in \( S \) is

\[
\int dx^- p^- (E^- z - \frac{1}{8} \lambda_\alpha \gamma^\alpha \lambda_\alpha).
\]  

(74)

Here we have used the notation (62) and have denoted the only relevant component of the Lagrange multiplier by \( p^- = (D^\gamma)_\alpha P_{\alpha a}|_0 \) and the lowest-order component of \( E^- z \) by \( E^- z = E^- z|_0 \).

Next we discuss the second term in \( S \), which introduces the \( S^8 \)-defining constraint (60). The analysis of this equation goes along the lines of subsection 2.2. Using light-cone notation, one can decompose the matrix \( E^\alpha z \) into two \( 8 \times 8 \) matrices

\[
E^\alpha = (E_{\alpha s}, F_{\alpha c}),
\]

which satisfy equations similar to (18)-(21):

\[
EE^T = \frac{1}{8} Tr(EE^T) \ 1,
\]  

(76)

\[
E^\gamma F^T + (E^\gamma F^T)^T = \frac{1}{4} Tr(E^\gamma F^T) 1,
\]  

(77)

\[
FF^T = \frac{1}{8} Tr(FF^T) \ 1.
\]  

(78)

To understand the implications of eq. (76) we note the following. Suppose that the matrix \( E_{\alpha s} \) is non-degenerate (this corresponds to choosing one of the two charts on the sphere \( S^8 \)). This matrix \( E_{\alpha s} \) transforms homogeneously under the finite diffeomorphisms of subsection 3.3.

\[
E_{\alpha s} = (D_\alpha \theta^\gamma_\beta) E^\beta_{\gamma s}.
\]  

(79)

In (53) we have seen that the parameter matrix in (79) satisfies the same condition as \( E \) in (76). On the other hand, the matrix \( D\theta \) satisfies the differential constraint

\[
D_{(\alpha} (D_{\beta)} \theta^\gamma_\gamma) = 0.
\]  

(80)
It turns out that the matrix $E$ satisfies the same type of constraint. This can be derived from (68), (39) and (64). Using the non-degeneracy of $E_{\alpha\alpha}$, one is able to fix the light-cone gauge

$$E_{\alpha\alpha} = \delta_{\alpha\alpha} \rightarrow \theta^{\alpha \alpha}(z) = \theta^{\alpha \alpha}.$$  \hspace{1cm} (81)

This gauge fixes most of the worldline superdiffeomorphisms and identifies the $O(8)$ automorphism group of worldline supersymmetry with the $O(8)$ subgroup of the tangent-space $D = 10$ Lorentz group, thus breaking Lorentz invariance. Note that in the action we shall not need to fix that gauge, thus preserving all the gauge symmetry of the worldline, including its local $N = 8$ supersymmetry. For us the essential conclusion from the discussion above will be that eq. (76) is purely auxiliary.

Equation (77) allows one to solve for the matrix $F$ in terms of an $O(8)$ vector-valued superfield $Y_{a}(z)$:

$$F = \vec{Y} \cdot \vec{\gamma}.$$  \hspace{1cm} (82)

This equation is auxiliary too. To see that one should look at it in the gauge (81) and in the linearised limit with respect to background supergravity:

$$D_{a} \theta_{\alpha e} = (\gamma_{ae})_{\alpha\alpha} Y_{ae}.$$  \hspace{1cm} (83)

It is clear that part of $\theta^{ae}$ is eliminated, and the rest defines the components of $Y$. The essential point is that the only component of the left-hand side of eq. (83), which could possibly give rise to an equation of motion, is absorbed by a component of $Y$:

$$8i \partial_{-} \theta_{\alpha e} \mid_{0} = (\gamma_{ae})_{\alpha\alpha} D_{a} Y_{ae} \mid_{0}.$$  \hspace{1cm} (84)

Finally, the third equation (78) is not independent, it is solved by (82). As we explained in subsection 2.2, the whole argument above can be turned around by choosing the other chart on $S^{8}$, where $\det F \neq 0$.

So, we have seen that equations (76)-(78) are purely auxiliary. They put algebraic restrictions on the components. In particular, the twistor variables (62) satisfy their defining condition (13). Therefore in the component action we can keep only the Lagrange multiplier term imposing that constraint. Putting this together with (74), we obtain the component form of the action (73):

$$S = \int dx^{-} \left[ p_{a} \left( \mathcal{E}_{-a} - \frac{1}{8} \lambda_{\alpha} \gamma^{a} \lambda_{\alpha} \right) + p_{a} q_{\alpha} \lambda_{\alpha} \right].$$  \hspace{1cm} (85)

We see that this is the target-space supersymmetrised version of the twistor action (31). To obtain the Brink-Schwarz action (41), we have to vary with respect to the Lagrange multiplier $p_{a}q_{\alpha}$ and $p_{a}$, and then eliminate the twistor variables replacing them by the lightlike vector $u^{a} = 1/8 \lambda_{\alpha} \gamma^{a} \lambda_{\alpha}$. In order to account for the lightlike condition on that vector, we need to introduced a new Lagrange multiplier, the worldline metric $g$:

$$S = \frac{1}{2} \int dx^{-} g \mathcal{E}_{-a} \mathcal{E}_{-a}.$$  \hspace{1cm} (86)

In the process of obtaining the component action (85) the Lagrange multiplier superfields in (73) were almost entirely eliminated. To a large extent this is due to a powerful
abelian gauge invariance:

$$\delta P_{\alpha a} = (D_\beta \delta_a^b + \Omega_{\beta a}^b - \frac{1}{2} E_{\beta}^c T_{ca}^b) \Sigma_{\alpha \beta b}, \quad \delta P_{\alpha \beta a} = \Sigma_{\alpha \beta b},$$

(87)

where $\Sigma_{\alpha \beta a}$ is an arbitrary parameter which is completely symmetric and traceless with respect to its worldline indices. The necessary and sufficient condition for this transformation to be a symmetry of the superspace superparticle action is that the supergravity constraint (38) should hold (see [14] for more details). Note that the transformations (87) contain the component gauge transformation (36), the rôle of which was explained in subsection 2.2.

Concluding this subsection, we would like to compare our approach to the one recently proposed by Tonin [15]. Although he only discusses the superstring, part of his action can be applied to the superparticle as well. He writes down an equation of motion of the type of eq. (64) and derives from it the analogue of (60), which defines his space of twistor variables. The principal difference is that he uses complex worldsheet Grassmann variables (his index $\alpha$ is a $U(4)$ one, and not an $O(8)$ one). As a result, his twistor variables parametrise a 20-dimensional coset space,

$$\frac{\text{Spin}(1,9)}{[SO^\uparrow(1,1) \times U(4)] \mathcal{K}},$$

(88)

which is called the space of projective pure spinors (see [23], [22]). As we saw earlier, the massless particle in ten dimensions is naturally associated with the sphere $S^8$, which is represented by the coset (28), so pure spinors are not necessary for our purposes. We shall comment on further substantial differences between the approach of Tonin and ours in the section devoted to the superstring.

### 3.5 Kappa-symmetry from worldline supersymmetry

The superspace action $S$ (73) is invariant under the worldline superdiffeomorphisms of subsection 3.2. In the process of deriving the component form of $S$ we have just eliminated a number of auxiliary fields. Therefore we can be sure that the action (86) has retained its original symmetry. In particular, it must be invariant under local $N = 8$ supersymmetry, which is part of the diffeomorphism supergroup. The corresponding transformations of the dynamical variables $z^M|_0$ can be obtained as follows:

$$\delta z^A = \delta z^M E_M^A|_0 = \epsilon_\alpha D_\alpha z^M E_M^A|_0 = \epsilon_\alpha \mathcal{E}_\alpha A.$$

(89)

The local supersymmetry parameter $\epsilon_\alpha(x^-)$ is the lowest-order term in the superdiffeomorphism transformation $\delta \theta^\alpha = \epsilon_\alpha(x^-) + \ldots$. It is precisely this parameter that can be used to

---

7Note that eq. (64) already appeared in ref. [7] (see also [19]).

8At the same time, pure spinors seem to have an advantage from the point of view a lightlike integrability interpretation of the constraints of $D = 10$ supergravity, see [22]. Future development of the subject may show if one would need pure spinors for some applications. If so, switching from $S^8$ to the space of pure spinors is easy, since the space (88) is locally isomorphic to the tensor product of $S^8$ with the compact coset $O(8)/U(4)$. Then one could add a set of new $O(8)$ harmonic variables to our twistor variables.
gauge away half of the Grassmann coordinates of target superspace, \( \theta^a(x^-) = \theta^a(z)|_{\theta=0} \) (cf. the gauge (81)). Using (64), one finds from (89):

\[
\delta z^a = 0, \quad \delta z^\alpha = \epsilon_\alpha \lambda_{\alpha a}.
\]

These transformations are Lorentz covariant due to the presence of the twistor variables \( \lambda_{\alpha a} \), which “push forward” the worldline \( O(8) \) spinor parameter \( \epsilon_\alpha \) to a target-space Lorentz spinor. However, to obtain (90) we have eliminated the twistors from (85), so we have to do the same in (91). In (90) this was easy, because the twistors appeared only in specific quadratic combinations. In (91) they appear linearly, so we need to express them in terms of the dynamical variables. This can be done using the results of subsection 2.2 and the “–” and transverse projections of the equation \( \mathcal{E}_- \alpha - 1/8 \lambda_{\alpha a} \gamma^a \lambda_\alpha = 0 \) following from the action (85). The result is

\[
\lambda_{\alpha a} = (u_{aa}, v_{ac}), \quad u_{aa} = \delta_{aa} \sqrt{\frac{\mathcal{E}_-}{2}}, \quad v_{ac} = \frac{1}{\sqrt{2 \mathcal{E}_-}} \mathcal{E}_- \cdot (\tilde{\gamma} \epsilon)^{ac}.
\]

Here we have gauged away the \( O(8) \) part of the matrix \( u \), thus actually breaking Lorentz invariance by identifying the worldline group \( O(8) \) with the subgroup of the target-space Lorentz group. Further, in deriving (91) we had to make the non-covariant assumption

\[
\mathcal{E}_- > 0.
\]

This is possible, since the lightlike vector \( \mathcal{E}_- \) has a twistor origin, so \( \mathcal{E}_- \mathcal{E}_+ = (\tilde{\mathcal{E}})^2 \geq 0 \) and \( \mathcal{E}_- + \mathcal{E}_+ = \mathcal{E}_0^2 > 0 \). This means that at least one of the components \( \mathcal{E}_- \) or \( \mathcal{E}_+ \) is strictly positive. Actually, the assumption (92) corresponds to one of the charts on \( S^8 \); the other choice \( \mathcal{E}_+ > 0 \) corresponds to the second chart and results in an alternative form of (91). So, putting (91) in (90), we obtain

\[
\delta z^a = 0, \quad \delta z^\alpha = \sqrt{\frac{\mathcal{E}_-}{2}} \epsilon^\alpha, \quad \delta z^c = \frac{1}{\sqrt{2 \mathcal{E}_-}} \mathcal{E}_- \cdot (\tilde{\gamma} \epsilon)^c.
\]

Finally, using the background torsion constraints, one can verify that the action (86) is invariant under (93), provided the metric transforms as follows \( 10 \)

\[
\delta g = -\frac{2\sqrt{2i}}{\sqrt{\mathcal{E}_-}} g \epsilon^a \mathcal{E}_-^{-1} \epsilon^a.
\]

So, we see two distinct possibilities. The first one is the twistor action, where the local \( N = 8 \) supersymmetry transformations (91) are Lorentz covariant. The other one is the action (80), which only involves the standard fields of the Brink-Schwarz superparticle, but where supersymmetry is non-covariant. One can say that in the twistor version Lorentz

\[9\]
A gauge of the type (81) would be too strong here, because it fixes the \( N = 8 \) worldline local supersymmetry.

\[10\]
In fact, with the assumption (92) one can show that the metric in (86) originates from a component of the Lagrange multiplier \( p^- \) in (83), \( g = p^- / \mathcal{E}_- \). Then this implies the transformation law (94).
covariance is due to the \( O(8) \) gauge degrees of freedom present in the twistor variables. Remarkably, by adding new gauge degrees of freedom (this time fermionic ones), one can restore the Lorentz covariance of the supersymmetry transformations (93), (94). Namely, one can replace the 8 parameters \( \epsilon^\alpha \) by a twistor projection of a full 16-component \( D = 10 \) spinor parameter \( \kappa^\alpha_a \):

\[
\epsilon^\alpha = \lambda^\alpha_a \kappa^a_a.
\]  

(95)

or, using the explicit solution (91),

\[
\epsilon^\alpha_a = \sqrt{\frac{\mathcal{E}^-}{2}} \kappa^\alpha_a + \frac{1}{\sqrt{2\mathcal{E}^-}} \kappa^\alpha_c (\vec{\gamma} \cdot \vec{\mathcal{E}}^-)_{a\alpha c}. \tag{96}
\]

Putting this in the transformations laws (93) and (94), one sees that they take the Lorentz-covariant form (43) of kappa symmetry (up to trivial terms proportional to the equations of motion). The conclusion is that both the twistor and the Brink-Schwarz superparticle actions have local \( N = 8 \) worldline supersymmetry, but its Lorentz-covariant realisations are different.

### 3.6 Maxwell coupling

A very interesting feature of the twistor superparticle is revealed when coupling it to a background Maxwell superfield \( A^M(z) \). The latter undergoes abelian gauge transformations

\[
\delta A^M(z) = \partial^M a(z) \tag{97}
\]

and satisfies the constraints

\[
F_{a\bar{b}} = 0, \quad F^a_b = (\gamma_b F)^a_{\bar{b}} \tag{98}
\]

(in \( D = 10 \) these constraints are on-shell). Here

\[
F_{A\bar{B}} = (-)^{A(B+N)} F_{\bar{B} A} = F_{\bar{A} A} \tag{99}
\]

and \( F_{MN} = \partial_M A_N - (-)^{MN} \partial_N A_M \) is the field-strength tensor.

Consider now the pull-back of \( A^M(z) \) to the worldline

\[
A_M = \partial^M z^M A^M. \tag{100}
\]

It undergoes the gauge transformations

\[
\delta A_M = \partial^M z^M \partial_M a(z) = \partial_M a(z). \tag{101}
\]

One can show that the pull-back of the field-strength \( F_{AB} = (-)^{A(B+N)} E_{B} \bar{A} E_{\bar{A}} A F_{AB} \) vanishes as a consequence of the constraints (98) and the superparticle equations of motion. Indeed, using (93) and (94) one finds

\[
F_{a\bar{b}} = E^a_{\alpha} E^b_{\bar{b}} F_{a\bar{b}} = 0. \tag{102}
\]
Further, from (64), (71), (60) and (98) one obtains

$$F_{\alpha} = E_{\alpha} + \frac{1}{8}(E_{\beta} \gamma^{\beta} E_{\beta})(\gamma_{\alpha} F_{\alpha}) = 0. \quad (103)$$

So,

$$F_{AB} = 0 \leftrightarrow F_{MN} = 0. \quad (104)$$

Now we can write down the superparticle-Maxwell coupling term

$$S_{Maxw} = \int dx^{-8} \theta (A_{M} - \partial_{M} Q) P^{M} \cdot (105)$$

Here $P^{M}$ and $Q$ are Lagrange multipliers. The equation following from the variation with respect to $P^{M}$ is

$$A_{M} = \partial_{M} Q. \quad (106)$$

Comparing it with (101), one can say that on shell the pull-back $A_{M}$ becomes a pure gauge. There is an integrability condition for equation (106), and it is just (104). In fact, we can turn this argument the other way around. Demanding integrability for (106), we arrive at (104). From there, using the properties of the twistor variables, we can derive the super-Maxwell constraints (98). This is another example of lightlike integrability.

The implications of the equation of motion for $Q$ are very interesting. It reads

$$\partial_{M} P^{M} = \partial_{-} P^{-} + \partial_{\mu} P^{\mu} = 0. \quad (107)$$

It is easy to see that the general solution of eq. (107) is

$$P^{M} = \partial_{N} \Sigma^{MN} + \theta^{8} \delta_{-} e, \quad e = \text{const}. \quad (108)$$

Here $\Sigma^{MN}(z)$ is an arbitrary worldline superfield with graded antisymmetry in $M, N$ and $e$ is an integration constant. The origin of this constant term has to do with the non-trivial cohomology of superspace. Indeed, the second term in (107) contains a Grassmann derivative, so in its $\theta$ expansion the highest-order term $\theta^{8}$ is missing. This yields the constraint $\partial_{-} e = 0$ on the highest-order component of $P^{-}$. The $\Sigma$ term in (108) corresponds to a gauge symmetry of the action (105), which follows from the constraint (104). Using this gauge freedom one can fix an on-shell gauge for (107), where it reduces to just one constant component:

$$P^{-} = e \theta^{8}, \quad P^{\mu} = 0. \quad (109)$$

In the process of fixing that gauge one does not have to use parameters with time derivatives, so the gauge is globally achievable and can be inserted in the action (103). The result is the usual component superparticle-Maxwell coupling:

$$S_{Maxw} = \int dx^{-8} \theta^{8} e A_{-}(z) = \int dx^{-8} e \partial_{-} z^{M} A_{M}(z) |_{0}. \quad (110)$$

We see that if the constant $e$ is non-vanishing, it has the meaning of an electric charge. In the original superspace action (103) the electric charge was not manifestly present, it was represented by a worldline field (the last component of the Lagrange multiplier $P^{-}$). Only on shell did it become a constant and acquire its standard physical meaning.

Finally we remark that the geometric meaning of the Lagrange multiplier $Q$ is that of a Kaluza-Klein coordinate. Indeed, under the Maxwell gauge transformation (101) it is shifted by $\delta Q = a(z)$ and neither the fields nor the gauge parameters depend on it.
4 Heterotic superstrings

4.1 The Green-Schwarz superstring

The principal difference between the Brink-Schwarz superparticle and the Green-Schwarz superstring is that the one-dimensional worldline of the former becomes a two-dimensional worldsheet for the latter:

\[ \mathcal{M} = \{ x^- \} \rightarrow \mathcal{M} = \{ x^m = (x^+, x^-) \}. \]  (111)

This allows one to write down an action which incorporates a sigma-model term similar to (11), and a new, Wess-Zumino term involving the two-form of background supergravity:

\[ S = \int d^2 x \left( e^\Phi \sqrt{-g} g^{mn} E_m \mathring{a} E_n \mathring{a} \partial_- z^N \partial_+ z^M B_{MN} \right). \]  (112)

Here \( g^{mn} \) is the metric of the two-dimensional worldsheet, \( E_m \mathring{a} = \partial_m z^M E_M \mathring{a} \) is a vielbein pull-back, \( \Phi(z) \) is the \( D = 10 \) supergravity dilaton superfield and \( B_{MN}(\mathring{z}) \) is the two-form superfield. The field-strength of the latter, the three-form

\[ H_{ABC} = (-)^{(A+B)(K+\bar{C})+A(\bar{B}\bar{N})} E_C \mathring{K} E_{\bar{B}N} \mathring{E}_A \mathring{M} (\partial_M B_{NK} + \text{gradedcycle}), \]  (113)

is supposed to satisfy the supergravity constraints

\[ H_{a\bar{b}\bar{\gamma}} = 0 \]  (114)

and

\[ H_{a\bar{b}\bar{\gamma}} = -2ie^{\Phi(\gamma_\mathring{a})} \bar{\gamma}_\mathring{a}. \]  (115)

Note that the sigma-model term in (112) needs the two-dimensional metric for diffeomorphism invariance. Because of the specific, Weyl invariant combination \( \sqrt{-g} g^{mn} \) only two of the three components of that metric actually appear in (112). The variation with respect to them gives rise to the two Virasoro constraints

\[ E_{-\mathring{a}} E_{-\mathring{a}} = 0, \quad E_{+\mathring{a}} E_{+\mathring{a}} = 0 \]  (116)

(in the gauge \( g^{mn} = \eta^{mn} \)). Note that the metric does not appear in the Wess-Zumino term, which is a topological invariant. The kappa symmetry (13) of the superparticle can be generalised to the case of the superstring action (112), taking into account the new background constraints (114), (115). Once again, it plays the crucial rôle of rendering the theory free when the background is flat. By supersymmetrising the worldsheet and thus introducing twistor variables, we shall be able to replace kappa-symmetry by local supersymmetry of the worldsheet. The first of the Virasoro constraints (116) will be obtained via the twistor mechanism, while for the second one we shall still need one component of the worldsheet metric.
4.2 Two-dimensional super-worldsheet

Our first step is to replace the worldsheet (111) by a (2|8) super-worldsheet:

\[ M = \{x^m\} \rightarrow M = \{z^M = (x^m, \theta^\mu)\}. \]  

It is natural to choose the supersymmetry of the worldsheet to be of the heterotic type \( N = (8,0) \), i.e. the algebra of the flat covariant derivatives is given by

\[ \{D_\mu, D_\nu\} = 2i\delta_{\mu\nu}\partial_-, \quad D_\mu = \partial_\mu + i\theta_\mu\partial_- . \]  

So, supersymmetry only affects the left-handed even coordinate \( x^- \), as well as the odd ones \( \theta^\mu \), but does not affect the right-handed coordinate \( x^+ \).

The reason for this choice is as follows. The target and worldsheet manifolds are closely related to each other since one of them is embedded into the other. Given a point on the worldsheet \( M \) we can always choose a frame in the target space \( \mathcal{M} \) with two axes in the tangent plane to the worldsheet. In this frame the \( N = 1 D = 10 \) target-space supersymmetry algebra \( \{D_\alpha, D_\beta\} = 2i(\gamma^a)_{\alpha\beta}\partial_a \) reads

\[ \{D_{\alpha s}, D_{\beta s}\} = 2i\delta_{\alpha s\beta s}\partial_-, \quad \{D_{\alpha c}, D_{\beta c}\} = 2i\delta_{\alpha c\beta c}\partial_+, \quad \{D_{\alpha s}, D_{\beta c}\} = 2i(\gamma^a)_{\alpha s\beta c}\partial_a, \]  

where \( \alpha_s, \alpha_c \) and \( a_v \) are indices of the 8\( s \), 8\( c \) and 8\( v \) representations of \( O(8) \), correspondingly. In general, the supersymmetry transformations in the target space induce some supersymmetry on the worldsheet. From (119) it is perfectly clear that only one half of the 16 generators \( D_{\alpha} \), either \( D_{\alpha s} \) or \( D_{\alpha c} \) (but not both!) do not generate translations in the transverse direction, which are orthogonal to the worldsheet. So, the supersymmetry induced on the worldsheet is of the type \( (8,0) \). It should be emphasized that the heterotic nature of the worldsheet is a direct consequence of the chiral structure of \( N = 1 D = 10 \) supersymmetry. There are two inequivalent 16-dimensional Majorana-Weyl spinors in \( D = 10, 16 \) and \( 16' \), and \( N = 1 \) supersymmetry involves only one of them. Since \( 16 \) and \( 16' \) are related by the time reflection \( T \) one can say that \( T \) is broken in a maximal way. The same is true for the worldsheet \( (8,0) \) supersymmetry (118).

Another argument in favour of the above choice of the worldsheet is related to the \( \kappa \)-symmetry of the ordinary Green-Schwarz superstring. This symmetry effectively involves 8 anticommuting gauge parameters, which we are going to interpret as local worldsheet supersymmetry later on.

The geometry of \( M \) is defined by two requirements. First, the Grassmann worldsheet derivative should transform homogeneously, which means that the tangent space group action is given by (cf. (14))

\[ L^B_A = \begin{pmatrix} L^+ & -L^- & L^0 \\ 0 & L^- & -L^0 \\ 0 & 0 & L^0 \end{pmatrix} \]  

Second, there should be no objects invariant under (120) and the worldsheet superdiffeomorphism group. These requirements are sufficient for the existence of local coordinates in which the frames take the standard form of flat superspace (cf. (17)), i.e.

\[ E_+ = \partial_+ \]  

\[ E_- = \partial_- \]  

\[ E_0 = \partial_0 . \]
This form is preserved up to local tangent space rotations by a group of generalised superconformal coordinate transformations. Indeed, repeating the argument of subsection 3.2, we see that $E_\alpha$ and $E_-$ are preserved by left-handed superdiffeomorphisms similar to (55)-(58). Clearly, these involve an unconstrained left-handed even diffeomorphism parameter $\delta x^- = \lambda(x^+, x^-)$. However, the right-handed transformations compatible with the frames (121) have to be restricted to conformal ones, i.e. $\delta x^+ = \lambda(x^+)$. If we write down a superstring action which only has this symmetry, we are going to miss the second of the Virasoro constraints (116). Therefore we shall need a slightly bigger group for our purposes. This can be achieved by making coordinate and tangent space transformations on the above frames to bring them to the form

\begin{align*}
E_\alpha &= \partial_\alpha + i\theta_\alpha \partial_- + E_\alpha^+ \partial_+ \\
E_- &= \partial_- - \frac{i}{8} D_\alpha E_\alpha^+ \partial_+ \\
E_+ &= \partial_+.
\end{align*}

(122)

When using these basis vectors as derivatives we shall write them as $D_A$. By construction, they satisfy the flat algebra (48), which implies the constraint

$$D_{\{\alpha} E_{\beta\}}^+ = 0.$$  

(123)

Once more, we require that the allowed worldsheet superdiffeomorphisms preserve the form of the frames (122) up to tangent space transformations of the form (120), cf. (49). The resulting constraints are (50) and

$$D_\alpha x^{\prime+} - D_\alpha \theta^\beta E^{\prime+}_\beta = 0.$$  

(124)

The new equation (124) gives the transformation rule of $E_\alpha^+$.

In infinitesimal form, the transformation laws of the derivatives $D_\alpha, D_-$ are the same as in the particle case, see (53)-(58). To these we have to add the shift of the new worldsheet coordinate,

$$\delta x^+ = \Lambda^+(z)$$  

(125)

with an arbitrary superfield parameter, the transformation law of $E_\alpha^+$,

$$\delta E_\alpha^+ = D_\alpha \Lambda^+ + \frac{i}{2} D_\alpha D_\beta \Lambda^- E_{\beta}^+$$  

(126)

and that of the derivative $D_+$,

$$\delta D_+ = -[D_+ \Lambda^+ + \frac{i}{2} (D_+ D_\alpha \Lambda^-) E_\alpha^+ + \frac{i}{8} (D_+ \Lambda^-) D_\beta E_{\beta}^+] D_+$$

$$- (D_+ \Lambda^-) D_- + \frac{i}{2} (D_+ D_\alpha \Lambda^-) D_\alpha.$$  

(127)

Clearly, $D_A$ transform in accordance with (120).

The group of superdiffeomorphisms above involves two parameters, $\Lambda^+, \Lambda^-$, both of which are unconstrained superfields. This group clearly contains $x$-space diffeomorphisms and local supersymmetry transformations. It should be emphasised that this group is
larger than the \((8,0)\) superconformal group. The latter is obtained by setting \(E_+\) equal to zero and requiring that the derivative \(D_+\) transforms into itself, i.e. by restricting the tangent space group such that \(L_+^\alpha = L_+^\alpha = 0\). From (126) and (127) we find that these restrictions lead to the holomorphic parameters \(\partial_+ \Lambda^- = 0\) \(\rightarrow\) \(\Lambda^- = \Lambda^- (x^-)\), \(D_\alpha \Lambda^+ = D_- \Lambda^+ = 0\) \(\rightarrow\) \(\Lambda^+ = \Lambda^+ (x^+)\) of the superconformal group. The superconformal symmetry is characteristic for all actions with dimensionless coupling constants. However, our action has an additional Chern-Simons structure which explains the larger group.

Note also that if only \(E_+\) is set equal to zero, but the derivative \(D_+\) is not required to transform homogeneously, the parameter \(\Lambda^-\) remains unconstrained while the parameter \(\Lambda^+\) is restricted as before, \(\Lambda^+ = \Lambda^+ (x^+)\). This defines an intermediate generalised superconformal group. The gauge \(E_\alpha^+ = 0\) is not globally possible and is thus not allowed in the action. Nevertheless, there exists a slightly weaker Wess-Zumino gauge for \(E_\alpha^+\):

\[
E_\alpha^+ |_{WZ} = i \theta_\alpha g_{--} (x).
\]

In this gauge the covariant derivative \(D_+\) (see (122)) becomes

\[
D_+ = \partial_+ + g_{--} \partial_+.
\]

To achieve that gauge one uses only those parameters in the decomposition of \(\Lambda^+\), which enter (126) without space-time derivatives. The remaining gauge field \(g_{--} (x)\) is the component of the two-dimensional metric corresponding to the purely bosonic right diffeomorphisms \(\delta x^+ = \Lambda^+ (x^-)\). In fact, the necessity to keep the gauge field \(g_{--} (x)\) (which is responsible for the second Virasoro constraint in (116)) in the formalism made us introduce the zweibein \(E_\alpha^+\) in the otherwise flat derivatives (122).

### 4.3 The twistor superstring action

The twistor superstring action consists of three terms,

\[
S = S_1 + S_2 + S_3.
\]

The first of them is a replica of the superparticle action (13)

\[
S_1 = \int d^2 x d^8 \theta \left[ i p_\alpha^a E_\alpha^a + p_\alpha \beta \gamma E_{\{\alpha \gamma \beta } \right].
\]

Its invariance under the worldsheet diffeomorphisms of subsection 4.2 is once again due to the homogeneous transformation laws for the spinor derivatives \(D_\alpha\). The meaning of the two equations of motion introduced by Lagrange multipliers is the same as in the superparticle case, and so is the component content of the action (131) (cf. (83)):

\[
S_1 = \int d^2 x \left[ p_\alpha^a \left( E_\alpha^a - \frac{1}{8} \lambda_\alpha \gamma^a \lambda_\alpha \right) + p_\alpha \beta \gamma \lambda_{\{\alpha \gamma \beta} \right].
\]

Note that the vector \(E_\alpha^a = E_\alpha^a |_0\) contains the covariant derivative (129) (in the Wess-Zumino gauge (128)). Eliminating the twistor variables from (132) by varying with respect to the Lagrange multipliers \(p_\alpha^a\) and \(p_\alpha \beta \gamma\), we obtain

\[
S_1 = \int d^2 x \ g_{++} E^a_\alpha E_{-a}^a.
\]
where \(g_{++}(x)\) is a new Lagrange multiplier, needed to impose the lightlike condition on the vector \(\mathcal{E}_{-\underline{\underline{\alpha}}}\),
\[
\mathcal{E}_{-\underline{\underline{\alpha}}} \mathcal{E}_{-\underline{\underline{\alpha}}} = 0. \tag{134}
\]
Actually, this is one of the two Virasoro constraints for the string, so the Lagrange multiplier \(g_{++}\) is the second component of the worldsheet metric (the first one is \(g_{--}\) from (129)).

The second term in the superstring action is
\[
S_2 = \int d^2 x d^8 \theta P_{\alpha \beta} D_{(\alpha} E_{\beta)}^+. \tag{135}
\]
It enforces the constraint (123) on the only non-trivial zweibein \(E_+\alpha\) on shell. This allows one to treat it as an unconstrained superfield off shell. The solution to this constraint (in the Wess-Zumino gauge (128)) was found in (129), and one easily sees that the only essential component \(g_{--}\) of \(E_+\alpha\) does not appear in \(S_2\). Therefore \(S_2\) is a purely auxiliary term in the action, and does not contribute to the component superstring action.

The last and most interesting term in the twistor superstring action involves the two-form \(B_{MN}(\bar{z})\) of background supergravity. It is constructed in close analogy with the superparticle-Maxwell coupling term (105) of subsection 3.6. The Lagrange multiplier in that term yielded the equation of motion (106), which meant that the pull-back of the Maxwell one-form \(A_M\) to the worldsheet was a pure gauge. The integrability condition (104), which made that possible, followed from the constraints (98) on the field strength. Finally, on shell the Lagrange multiplier itself reduced to a constant, the electric charge. In the case of the supergravity two-form we shall follow the same strategy.

The two-form \(B_{MN}(\bar{z})\) is defined up to abelian gauge transformations,
\[
\delta B_{MN} = \partial_M b_N(\bar{z}) - (-)^{MN} \partial_N b_M(\bar{z}). \tag{136}
\]
Consequently, the pull-back
\[
B_{MN} = (-)^{M(N+K)} \partial_N z^K \partial_M z^M B_{MN} \tag{137}
\]
undergoes the transformations
\[
\delta B_{MN} = \partial_M b_N - (-)^{MN} \partial_N b_M, \quad b_M = \partial_M z^M b_M(\bar{z}). \tag{138}
\]
Following the analogy with eq. (106), we could try to make \(B_{MN}\) pure gauge on shell,
\[
B_{MN} = \partial_M Q_N, \tag{139}
\]
where \(\{\}\) means graded antisymmetrisation. However, this would put too strong a restriction on the background. The integrability condition for (139) is
\[
0 = \partial_M B_{NK} = H_{MINK} = (-)^{(M+N)(K+K)+M(N+N)} \partial_K z^K \partial_N z^N \partial_M z^M H_{MINK}(\bar{z}) \tag{140}
\]
\[
\rightarrow H_{ABC} = (-)^{(A+B)(C+K)+A(B+N)} E^K_C E^N_B E^M_A H_{MINK} = 0.
\]
One can show that it is not compatible with the supergravity constraints (114), (115) on the three-form. Indeed, using the equation of motion (64) coming from the term (131) of the superstring action and applying the constraint (114), one finds that

\[ H_{\alpha\beta\gamma} = 0. \] (141)

Further, the constraint (115) together with (60) and (72) imply

\[ H_{\alpha\beta} = -2iE_\alpha^\alpha E_\beta^\beta E_\gamma^\gamma e^\Phi = -2i\delta_{\alpha\beta}E_-^\alpha E_-^\beta e^\Phi = 0. \] (142)

Similarly,

\[ H_{\alpha\beta} = \frac{1}{8}\delta_{\alpha\beta}H_{\gamma\gamma} = -2i\delta_{\alpha\beta}E_-^\alpha E_+^\beta e^\Phi \neq 0. \] (143)

The last component of \( H_{ABC} \) can be found from the Bianchi identity

\[ (D_\alpha + \partial_+ E_{(\alpha}^+) H_{\beta\gamma})_+ - \partial_+ H_{\alpha\beta\gamma} = -2i\delta_{(\alpha\beta}H_{\gamma)++}. \] (144)

With the help of (141) and (143) this gives

\[ H_{\alpha+} = \frac{i}{16}(D_\alpha + \partial_+ E_{\alpha}^+)H_{\beta\beta} \neq 0. \] (145)

We see that the components (143) and (145) of the three-form pull-back do not vanish, so the integrability condition (140) is too strong. This suggests that the correct on-shell condition on the two-form pull-back should be somewhat weaker than (139). The right choice is the “almost flat” pull-back

\[ B_{MN} + E_{[M}^E_{N]} - e^\Phi E_-^\alpha E_+^\alpha \partial_{MN} \] (146)

where \( E_M^\pm \) are elements of the zweibein matrix on the worldsheet. The integrability condition for (146) is

\[ H_{KMN} + \partial_{[K}(E_{M}^+ E_{N]} - e^\Phi E_-^\alpha E_+^\alpha \partial_{MN}) = 0. \] (147)

In a tangent space basis it reads

\[ H_{ABC} = \frac{i}{16}(-1)^{(A+B)(C+K)+A(B+N)} E_{[C}^K E_B^N D_A) \left(E_{N}^+ E_{K}^- H_{\delta\delta}^+\right), \] (148)

where the term with \( e^\Phi \) has been replaced by the three-form component from (143). To check that (148) holds one should use the expressions (141), (142), (143), the identity (143) and the constraint (123).

As in the case of the superparticle-Maxwell coupling, the on-shell “almost flat” condition (140) will be obtained from a Lagrange multiplier term in the superstring action:

\[ S_3 = \int d^2 x d^8 \theta \left[B_{MN} + E_{[M}^E_{N]} - e^\Phi E_-^\alpha E_+^\alpha \partial_{MN}\right] P^{MN}. \] (149)
It is invariant under the diffeomorphisms of subsection 4.2. Indeed, the terms with $B$ and $Q$ transform as curved worldsheet supertensors. The same applies to the term with $\Phi$. The easiest way to see this is as follows. Using (64) and (72), one can complete the trace in the term $E_M ^+ E_+ a = E_M ^+ E_A ^a = E_M ^a$, so this term is covariant. The other term, $E_N ^- E_- a$, is covariant as well, since $E_- a$ transforms into itself, $\delta E_- a = -(D_- \Lambda) E_- a$ (see (58) and use (64)), and the zweibein $E_N ^-$ compensates for this.

To find the component content of the action term (149) we have to repeat the argument of subsection 3.6. We first look at the equations of motion following from the variation with respect to the Lagrange multiplier $Q_M$:

\[ \partial_N P^{NM} = 0. \] (150)

It is easy to see that this equation has the general solution

\[ P^{MN} = \partial_K \Sigma^{KMN} + \theta^8 \delta_+ [M \delta_- N] T, \] (151)

where $\Sigma^{KMN}(z)$ is a totally (graded) antisymmetric superfield and $T$ is a constant,

\[ \partial_+ T = \partial_- T = 0. \] (152)

The origin of the cohomology term with $T$ in (151) can be traced back to the “+” and “−” projections of (150):

\[ \partial_- P^{+-} + \partial_\mu P^{\mu+} = 0, \quad \partial_+ P^{+-} + \partial_\mu P^{\mu-} = 0. \] (153)

In both of them the second terms contain the odd derivative $\partial_\mu$, therefore the $\theta^8$ term is missing in their expansions. This leads to the constraints (152) on the highest-order term in $P^{+-}$.

Further, the arbitrary superfield $\Sigma$ in (151) actually corresponds to a gauge symmetry of the action. This follows from the integrability condition (147). Therefore one can gauge away almost everything in $P^{MN}$ but its highest-order constant component. After this has been done, the component form of $S_3$ can be obtained very easily:

\[ S_3 = \int d^2 x d^8 \theta \theta^8 T(B_{+-} + e^\Phi E_+ a E_- a)|0 \]
\[ = T \int d^2 x \left( \partial_- z^N \partial_+ \dot{z}^M B_{MN} + e^\Phi \dot{E}_+ ^a \dot{E}_- ^a \right), \] (154)

where $z$ denotes the lowest-order component of the superfield $z(z)$. Clearly, this term has retained its original symmetries, notably the local $N = (8, 0)$ worldsheet supersymmetry. Of course, in (152) we have assumed that $T \neq 0$, otherwise the action would be trivial.

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13 At this point it becomes clear why we did not need to covariantise the derivative $D_+$ in (122). The only occurrence of a worldsheet tangent index “+” in the action is in the term $E_M ^+ E_+ a$, and the presence of any zweibeins $E_+ ^N$ in it would be irrelevant, as we have just seen.

14 It is not hard to see that this is a gauge of the Wess-Zumino type, i.e. no parameters with space-time derivatives are used.
Finally, we put together the component expressions of the terms $S_1$ (133) and $S_3$ (154), redefine the Lagrange multiplier $g_{++} \rightarrow Te^{\Phi}g_{++}$ and insert the expression for the covariant derivative $D_-$ in the Wess-Zumino gauge (129). The result is

$$S = T \int d^2x \left\{ \partial_- z^M \partial_+ z^N B_{MN} + e^{\Phi} \left[ e_{++} e_{-+} (1 + 2g_{++}g_{--}) + e_{-+} e_{++} g_{++} + e_{++} e_{-+} g_{--}(1 + g_{++}g_{--}) \right] \right\},$$

where $e_{\pm} = \partial_{\pm} z^M E_{M}^a |_{\theta=0}$ now involve only partial derivatives $\partial_{\pm}$. Up to a conventional redefinition of the two-dimensional metric, the action (155) is that of the Green-Schwarz superstring (112).

At this point we can show the main difference between our approach to the superstring and that of Tonin [15]. He does not use the non-trivial mechanism of generating the string tension as an integration constant on shell, but introduces it by hand. This prevents him from writing down a worldsheet superspace Wess-Zumino term. Instead, he proposes an essentially component term similar to eq. (154), thus loosing manifest worldsheet supersymmetry. Further, to prove supersymmetry he is forced to put gauge non-covariant restrictions directly on the two-form. As we have shown, using the Wess-Zumino term (149) of the Chern-Simons type allowed us to maintain all the symmetries manifest, and at the same time to clearly exhibit the geometry of the superstring action.

## 5 Conclusions

In the present paper we developed a new geometric formulation of the heterotic superstring in ten dimensions. Its main feature is that all the inherent symmetries are linearly realised and manifest. In particular, kappa-symmetry of the usual Green-Schwarz formulation is replaced by worldsheet supersymmetry. We also established the equivalency between the two formulations at the classical level. An issue of crucial importance is whether the new action is can be covariantly quantised, thus hopefully overcoming the well-known difficulties in the Lorentz-covariant quantisation of the Green-Schwarz action.

A further open question is how to generalise the twistor approach to superstrings of the non-heterotic type, which have full $N = (8,8)$ worldsheet supersymmetry, as well as to other super p-branes. This would involve dealing with other dimensions of the target space, notably $D = 11$. It is not clear at present if the twistor-like approach can be extended beyond the special dimensions $D = 3, 4, 6, 10$.

To complete the heterotic superstring one needs a matter sector of chiral fermions (or bosons). At the moment we do not know how to formulate it, keeping both $N = (8,0)$ worldsheet supersymmetry and target space Lorentz symmetry manifest and having a compact matter Yang-Mills group. Perhaps trying to couple the superstring to an external super-Maxwell field may provide the key to this problem. Another, probably related task is to study in detail the meaning and implications of the principle of lightlike integrability in the new context (throughout the text we have made several remarks about that). It is not impossible that the new construction may also inspire a solution of the long-standing problem of off-shell super-Yang-Mills and supergravity in ten dimensions.
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