Economic law of increase of Kolmogorov complexity. Transition from financial crisis 2008 to the zero-order phase transition (social explosion)

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Abstract

In [1], a two level model of the occurrence of financial pyramid (bubbles) has been considered. We also considered the mathematical analogy of this model to Bose condensation. In the present paper, we explain why Ponzi schemes and bubbles result in a crisis in real economics. In [2], the law of increase of entropy in financial systems, and consequently increase of Kolmogorov complexity, is formulated. If this law is broken, the financial system makes a phase transition to a different state. In [3] the author considered a two level model of the zeroth-order phase transition which was interpreted in [4] as an analog of social catastrophe. In the present paper we also examine this model.

By the end of 2008, the United States has accumulated considerable debts. The debts can be counted by the same scheme as GDP, that is, as the difference of non-repayed and repayed debts in a year provided that this difference is not zero, i.e., the debt has increased owing to the velocity of debt.

Just as negative money, i.e., debts, do not occur in the definition of velocity of money and in the calculation of the GDP, so the definition of the velocity of debt does not involve positive money (earned money, etc.).

If $N_i$ is the debt resulting from one turnover (one act of borrowing from somebody and paying the dept to somebody else), then the total debt is equal to $M_i = N_i \sigma$, where $\sigma$, the number of turnovers per annum, will be called the velocity of debt (by analogy with “velocity of money”). The highest velocity of debt is observed for bubbles, or Ponzi schemes. A mortgage debt has only one turnover. For a car loan, the velocity of debt is also equal to one if the customer made a large borrowing and has not repaid the loan yet.

Thus, $\sum N_i \sigma_i \leq M$, where $M$ is the total debt (like a “negative GDP”). Hence the debts obey the same theorems of number theory as GDP (see [4] and also [5]).

If $M$ is the total debt and $\sigma$ is the average velocity of debt (analog of the temperature $\theta$ in physics), then the “negative money supply” is given by $N = M/\sigma$, where $N = \sum_{i=1}^k N_i$, $k \gg 1$, and $N_i$ corresponds to the $i$the set of subjects, which has the cardinality $\alpha_i$. If we adopt, similarly to the Baumol–Tobin model, that the velocity of debt of a single subject is proportional to the square root of its debt (cf. [10, Appendix to Chapter 13]) and take into account that the debt is equal to $i N_i$, then $N_i$ is proportional to $i$ and it is more accurate to assume that $N_i = i n_i$, where $n_i$ are distributed uniformly. Hence $\sum_{i=1}^k \alpha_i i^2 n_i = M$ and $\sum i n_i = N$. For $\alpha_i$ we assume that they decrease according to the Pareto distribution.

The critical number $N_0$ after which the ”Bose condensation phenomenon” occurs is computed in [4, 11, 34]. In these papers, the Pareto distribution was proved considered...
as negative dimension, namely, it was proved that the larger the Pareto index for the
distribution of $\alpha_i$ and the lesser the Baumol–Tobin dependence, then the larger the critical
number $N_0$ and the more difficult it can be overcome.

However, there is a substantial difference between inflation occurring if $N > N_0$ for
the case of income and the case in which $N > N_0$ for debts.

In the first case (inflation) small denominations fall out of circulation (part of the
money is abolished), and the nominal value of money is reduced. In the second case,
debts cannot be annihilated by denomination, and excessive debts for $N > N_0$ accumulate
at the ”lower level”, i.e., at the subjects who have one turnover per annum. The zero
velocity of debt will be taken into account in incomes rather than debts. On the other
hand, it follows from the theorem proved in [10, Appendix to Chapter 13] that small
velocities of both the money and the debts practically do not affect the value of $N_0$.
Hence the definition of nonzero lower level of velocity of money is not essential in the
rather schematic GDP model.

However, since the lower velocity level for debts plays an exceptional role and the
model involving the number of turnovers is too coarse for determining the least velocity
of debt, one has to single out the “longest-term debts” whose repayment spans over many
years, i.e., whose repayment velocity is small.

How the long-term project contribution to GDP is calculated? It is necessary to
calculate the excess of the cost of the project already performed part in a year over the
cost of the entire project in $L$ years minus the total expenditures in $L$ years divided by $L$.

Similarly, the rate of the debt “turnover” in the case of a long-term credit obtained
for $L$ years is equal to $1/L$. Therefore, if, without loss of generality, we assume that the
numbers of turnovers larger than 2 are integer numbers, then, for a single “turnover” (a
credit taken once), we must take the credit period (the number of years $L$) into account.
Correspondingly, the rate of the debt turnover equal to 1 splits into rates of the debt
turnover that are less than 1. Hence the debt Bose condensation accumulates precisely at
these slow rates of the debt turnovers, and the bankruptcy expects precisely them. This
follows from the mathematical theorem of the number theory. But the theorem does not
speak how this is realized in the society. Possibly, to escape the bankruptcy, the debtors
with high level of the debt velocity must again take long-term deposits. Perhaps, this
chain can be explained by economists, but the mathematical law inexorably predicts this
scenario of the debt crisis.

It is on these debts that the excessive debts exceeding $N_0$ mainly fall; as a result, the
long-term borrowers default on their debts, which fall out of the chain and descend on the
subsequent levels.

Hence mortgage is the first to collapse, and the car loans follow.

Thus, the mathematical theory, which pertains to number theory, is the same, but the
result in economics is different.

It follows from the preceding that to fight the crisis and to avert it in future one
should favor a debt structure in which the share of high-velocity debts is smaller that
in the existing structure. In [4, 11], it was proved that if the Pareto index is not too
large, then it is possible to increase the critical number by introducing a larger number
of currencies, for example, in each of the United States, by introducing its own currency
in addition to the dollar.
1 Economic Law of Increase of Kolmogorov Complexity

In his celebrated paper on complexity, A.N. Kolmogorov defined Kolmogorov complexity as a minimal code of a sequence of numbers. He also mentioned some other complexity, which was related to decoding. For the stock exchange, the definition of a code length that involves decoding is more appropriate to the situation under consideration, which leads to an arbitration-free situation in the end.

Suppose that a trader contrived a combination that brought a big gain to this trader. The market intuitively deciphers this combination (algorithm) and, thereby, annihilates it. Then, the trader invents some more complicated algorithm, and so on. In the end, the complexity of the algorithm becomes so high that it only slightly differs from a pseudo-random complexity. In the limit, this leads to an arbitration-free situation.

For this reason, it is more natural to understand complexity as the length of a code with polynomial decoding complexity. This code should be used to encode a sequence. The longer the code, the closer the situation under consideration to general position, which corresponds to the law of large numbers. If the minimum specified above tends to zero as $N \to \infty$, then the set of deals $\{N_i\}$ is close to an arbitration-free set.

In the language of probability theory, the initial probabilities corresponding to $\lambda_i$, which equal $G_i/G$ in the sense of von Mises, tend to zero as $N \to \infty$ (and, hence, as $G \to \infty$).

Therefore, the initial mathematical expectation

$$\frac{\sum g_i \lambda_i}{\sum g_i} = \frac{\sum g_i \lambda_i}{G}$$

is such that each term of the probability $p_i$ vanishes in the limit. However, the mathematical expectation tends to a constant larger than zero and the partial sums $\sum_{j=a}^i p_i$ do not tend to zero.

Similarly, the mathematical expectation $\sum N_i \lambda_i$ does not tend to zero, whereas $\frac{N_i}{N} \to 0$, but $\frac{1}{N} \sum_{j=a}^i N_i$ does not tend to zero.

This is a very important economy factor, especially in the psychological aspect (see [4, 6]).

Since the tendency of the market to an arbitration-free situation is equivalent to the eagerness of the market participants to gain profits, it follows that the entropy, if it depends on some parameters, must increase with respect to these parameters most rapidly, in the direction of steepest ascent.

If $H(E, N)$ depends on $k$ additional $x_1, \cdots, x_k$, i.e., $H = H(x_1, \cdots, x_k)$, (the dependence on $E$ and $N$ is not indicated), then

$$\dot{x}_i = c(H) \frac{\partial H}{\partial x_i}$$

$$\frac{\partial H}{\partial t} = \sum_i \frac{\partial H}{\partial x_i} \frac{\partial x_i}{\partial t} = c(H) \sum_i \left( \frac{\partial H}{\partial x_i} \right)^2, \quad c(H) > 0 \quad H|_{t=0} = H(x_1, \cdots, x_k).$$

(1)

For simplicity, suppose that $C(H) = 1$. Otherwise, setting $F(H) = \int C(H) dH$, we obtain $\frac{\partial F}{\partial t} = (\nabla F)^2$. This implies

$$H(x, t) = \min_{\xi} \left( \frac{(x - \xi)^2}{2t} + H(\xi) \right).$$
Remark 1. Performing geometric quantization \[7\], we obtain

\[ \hat{H}(x,t) = \ln \frac{1}{\sqrt{t^k}} \int e^{-\frac{(x-\xi)^2}{2t}} e^{H(\xi)} d\xi, \quad \xi \in \mathbb{R}^k. \]

This asymptotically coincides with the tunnel canonical operator on the Lagrangian manifold \( p_i = \frac{\partial M}{\partial x_i} \) shifted along the trajectories \( \dot{x} = p \) and \( \dot{p} = 0 \). The quantization of Eq. (1) leads to solution of the heat equation \[8\]

\[ N \frac{\partial u}{\partial t} = \mathcal{H}(\ln u^2) \Delta u \quad u|_{t=0} = H(x_1, \ldots, x_n), \tag{2} \]

Quantizing this equation (as a Bose system) and setting \( u = \Psi^+ \) and \( u^2 = \Psi \cdot \Psi^- \), we obtain a linear equation whose asymptotics is again the canonical operator.

The prices \( \lambda_i(x_1, \ldots, x_k) \) vary according to the equation

\[ \frac{d\lambda_i}{dt} = \sum_j \frac{\partial \lambda_i}{\partial x_j} dx_j = \sum_j \frac{\partial \lambda_i(x_1, \ldots, x_k)}{\partial x_j} C(H) \frac{\partial H}{\partial x_j}. \tag{3} \]

Thus, if we know the change of some (specific) price, we can find the function \( C(H) \) and, thereby, solve the problem completely.

A similar situation may arise when, e.g., barriers between countries (such as customs tariffs) decrease and prices tend to even according to the law of steepest ascent of entropy, because profiteers begin to use the arising arbitration (difference in prices) at a furious pace, which leads to an arbitration-free situation most rapidly. Such was the situation in the Soviet Union in the late 1980s when the iron curtain collapsed \[4\].

On September 30, 2008, the National Debt Clock in Manhattan ran out of digits as the United States public debt exceeded $10 trillion, a significant symbol of the current financial and economical crisis. There are many diverse factors behind the crisis, and they are naturally a subject of broad interest. However, apart from economics laws, which are widely discussed nowadays, there is a purely mathematical law contributing to the disaster, an inexorable law of numbers, which economists, let alone the general public, fail to recognize.

Suppose we want to deposit two kopecks in two different banks (see \[4\]); then we can say that there are three possibilities: (1) put both kopecks into one of the banks; (2) put both kopecks into the other bank; (3) one kopeck in one bank, the other, in the other one. Here it is of no consequence which of the two coins we deposit in the first bank and what is its year of issue. Now imagine a situation in which we are depositing one kopeck and one pence instead of two kopecks. In that case, we have four options rather than three, because it is significant which coin we placed in what bank, and so the variant in which the coins are placed in different banks yields two different options: (a) one kopeck to bank 1 and one pence to bank 2; (b) one kopeck to bank 2 and one pence to bank 1.

These two cases give rise to two different laws in number theory (see \[9\]). In the second case, the corresponding law implies that the total number of cents (and hence of dollars and of millions of dollars), i.e., the total amount of money in circulation, cannot be arbitrarily large. There is a threshold, and if the amount of money in a given currency exceeds that threshold, then an economic disaster occurs. The same thing pertains to negative amounts of money, i.e., debts.

Mathematically, this kind of disaster can be described as an analog of “Bose condensation”: should there be too many particles in a system, all the excessive particles would collapse into the ground state. The condensation effect is indeed observed in finance: for
example, if the inflation rate is too high (too much money has been issued), then the lower denominations (like cents, pence, or kopecks) die away, i.e., are withdrawn from circulation.

It is important to note that the condensation phenomenon is solely due to the fact that particles are indistinguishable—there is no condensation at all if the particles are distinguishable (and so obey the Gibbs statistics). One should simultaneously use as many currencies as possible. For example, if, in addition to dollar, its own currency were introduced in each of the United States, then the sum of the thresholds for all states would be much greater than the threshold in the case of a single currency; it would be much harder to exceed this new threshold, and hence the disaster would at least be postponed. By the same pattern, all national currencies should have been retained along with the euro. Then the violation of a country’s threshold could cause a crisis in that country but would not affect the other economies, just as the 1998 default in Russia did not bring down the world economy. (Unfortunately, the dollar largely circulates outside the United States, and hence the world on the whole is much more affected this time).

Thus, the introduction of $K$ currencies has raised $N_{cr}$ by the factor of approximately $K^{1/2}$. Or, the introduction of concurrently existing $K$ currencies increases the crisis threshold by the factor $K^{1/2}$. The outcome will be more or less the same regardless of whether these new currencies are nationwide or function only within their respective states.

The above described transition from the MMM pyramid (“bubbles”) leads to entropy increase, therefore, to the Kolmogorov complexity. Hence “bubbling” leads to decrease of complexity. Likewise, transition to only one currency leads to decrease of complexity, and this is contributing to a crisis. Development of advanced technologies results in increase of complexity, therefore it hinders a crisis.

Thus, a general law of economics is the law of increase of complexity\(^1\). Otherwise, a crisis is imminent.

## 2 Nonstationary financial averaging

Consider a random variable $\lambda$ which takes $l$ different values $\lambda_1, \ldots, \lambda_l$. In what follows, we assume that the values of this random variable satisfy the following condition: if $k_1, \ldots, k_l$ are integers such that

$$\sum_{i=1}^{l} k_i = 0, \quad \sum_{i=1}^{l} \lambda_i k_i = 0. \quad (4)$$

then $k_i = 0$ for all $i = 1, 2, \ldots, l$.

In [13], the notion of financial averaging and financial averaging axioms were introduced. Suppose given a convex function $f(x)$ having inverse $f^{-1}(x)$, where $x \in \mathbb{R}$ and a set of nonnegative numbers $p_i$, where $i = 1, \ldots, l$; we call them weights in what follows.

The first axiom [13] is that the financial averaging of the random variable $\lambda$ corresponding to the function $f(x)$ and the set of weights $p_i$, $i = 1, \ldots, l$ is the expression

$$E_{f,p}(\lambda) = f^{-1} \left( \sum_{i=1}^{l} p_i f(\lambda_i) \right). \quad (5)$$

\(^1\)The analogy to the above-mentioned effect in statistical physics is given e.g., in [12].
We modify the fourth axiom as follows: For any set of weights $p_i$, any set of numbers $\lambda_i$, and any number $a$, the following equality holds:

$$E_{f,p}(\tilde{\lambda}) = Ca + E_{f,p}(\lambda),$$

(6)

where $\tilde{\lambda}_i = \lambda_i + a$, and $C$ is a number not depending on $a$. Then, the following assertion is valid.

**Proposition 1.** Condition (6) implies that the function $f(x)$ has the form

$$f(x) = A \exp(-\beta x), \quad \text{or} \quad f(x) = Ax + D,$$

(7)

where the numbers $\beta$ and $A$ are nonzero.

This assertion is proved by the method described in [13].

Hereafter, we consider the case of a nonlinear function $f(x)$. According to Proposition 1, the financial averaging of a random variable $\lambda$ corresponding to a set of weights $p_i$ has the form

$$E_p(\lambda) = -\frac{1}{\beta} \ln \left( \sum_{i=1}^{l} p_i e^{-\beta \lambda_i} \right).$$

(8)

Properly, Axioms 2 and 3 [13] do not hold; thus, we regard this averaging as non-stationary and consider the process stabilizing it. The stabilization procedure does not essentially use the fourth axiom.

Following [14], we associate the random variable under consideration with an n-dimensional space $L$ and an operator $\Lambda$ in this space. The operator $\Lambda$ has eigenvectors $e_i$, where $i = 1, 2, \ldots, l$, which form an orthonormal basis in the space $L$. The eigenvalue corresponding to the eigenvector $e_i$ equals $\lambda_i$. Consider a series of $M$ identical trials whose outcomes are occurrences of the values $\lambda_1, \ldots, \lambda_n$ of the random variable. According to [14], this probability problem corresponds to the space $L_M$ being the tensor product of $M$ copies of $L$. The vectors in the space $L_M$ have the form

$$\Psi = \sum_{i_1=1}^{l} \ldots \sum_{i_M=1}^{l} \psi(i_1, \ldots, i_M) e_{i_1} \otimes \cdots \otimes e_{i_M},$$

(9)

where $\psi(i_1, \ldots, i_M)$ is an arbitrary function of the set of discrete variables $i_s = 1, \ldots, l$, $s = 1, \ldots, M$.

To the series of trials under consideration, there corresponds, in addition to the space $L_M$, the operator $\hat{\Lambda}_M$ acting on vectors (9) as

$$\hat{\Lambda}_M \Psi = \sum_{i_1=1}^{l} \ldots \sum_{i_M=1}^{l} \left( \sum_{s=1}^{M} \lambda_{i_s} \right) \psi(i_1, \ldots, i_M) e_{i_1} \otimes \cdots \otimes e_{i_M}.$$

(10)

In further considerations, we use the notation

$$M_s(i_1, \ldots, i_M) = \sum_{s=1}^{M} \delta_{i_s i_s},$$

(11)

where $\delta_{i_s i_s}$ is the Kronecker symbol. Let us define projectors $\hat{P}_{(M)}$

$$\hat{P}_{(M)} \Psi = \sum_{i_1=1}^{l} \ldots \sum_{i_M=1}^{l} \left( \prod_{i=1}^{l} \delta_{M_s(i_1, \ldots, i_M)} \right) \psi(i_1, \ldots, i_M) e_{i_1} \otimes \cdots \otimes e_{i_M}.$$

(12)
Here, \( \{M\} \) denotes an arbitrary set of nonnegative integers \( M_1, \ldots, M_n \) satisfying the condition
\[
\sum_{i=1}^{l} M_i = M. \tag{13}
\]

It is easy to show that operators \( \{13\} \) satisfy the equality
\[
\hat{P}_{\{M\}} \hat{P}_{\{M'\}} = \prod_{i=1}^{l} \delta_{M_i, M'_i} \hat{P}_{\{M\}} \tag{14}
\]
thus, these operators are indeed projectors. The operator \( \hat{H}_M \) defined by \( \{10\} \) is expressed in terms of the projectors \( \hat{P}_{\{M\}} \) as follows:
\[
\hat{H}_M = \sum_{\{M\}} \left( \sum_{i=1}^{l} \lambda_i M_i \right) \hat{P}_{\{M\}}. \tag{15}
\]

In the subspace \( L_{\{M\}} \), onto which the space \( L_M \) is projected by \( \hat{P}_{\{M\}} \), we distinguish the basis formed by vectors of the form
\[
E_{i_1, \ldots, i_M} = e_{i_1} \otimes \cdots \otimes e_{i_M}, \tag{16}
\]
where \( e_i \) occurs \( M_i \) times, or, in other words, the set \( i_1, \ldots, i_M \) satisfies the conditions
\[
M_i(i_1, \ldots, i_M) = M_i. \tag{17}
\]

Consider the following system of vectors \( \Phi_{\{M\}} \): \[
\Phi_{\{M\}} = \hat{P}_{\{M\}} \mathbf{e} \otimes \cdots \otimes \mathbf{e}, \quad \mathbf{e} = \sum_{i=1}^{l} e_i. \tag{18}\]

Vectors \( \{18\} \) are equal to the sum of all vectors \( \{16\} \) from the subspace \( L_{\{M\}} \); in addition,
\[
\Phi_{\{M\}} = \frac{1}{M!} \left( \prod_{i=1}^{l} M_i! \right) \text{Symm}_{i_1, \ldots, i_M} E_{i_1, \ldots, i_M}, \tag{19}\]
where \( \text{Symm}_{i_1, \ldots, i_M} \) is the operator of symmetrization over the indices \( i_1, \ldots, i_M \).

We endow the space \( L_M \) with the norm
\[
\|\Psi\| = \sum_{i_1}^{l} \cdots \sum_{i_M=1}^{l} |\psi(i_1, \ldots, i_M)|. \tag{20}\]

3 The evolution process

In this section, we define the notions of evolution and data reduction.

**Definition 1.** The one-step evolution of an ensemble consisting of \( M \) elements, i.e., of a vector \( \Psi \in L_M \), is the following vector belonging to the space \( L_M \):
\[
\Psi_1 = \exp \left( -\beta \hat{H}_M \right) \Psi, \tag{21}\]
where \( \beta \) is a parameter.
Definition 2. The data reduction, or factorization, is the operation that maps each vector \( \Psi \in \mathcal{L}_M \) to another vector according to the rule

\[
R(\Psi) = \sum_{\{M\}} \| \tilde{P}_{\{M\}} \| \Psi \| \Phi_{\{M\}}. \tag{22}
\]

where summation is over all sets of nonnegative integers \( M_1, \ldots, M_n \), satisfying condition (13).

To each set of nonnegative numbers \( g_i \), where \( i = 1, \ldots, l \) we assign the following vector of the space \( \mathcal{L}_M \):

\[
\Psi_g = g \otimes \cdots \otimes g, \tag{23}
\]

where \( g \) is the vector from \( L \), defined by

\[
g = \sum_{i=1}^{l} g_i e_i. \tag{24}
\]

Proposition 2. If \( g_i = p_i e^{-\beta \lambda_i} \), then the expression

\[
-\frac{1}{M \beta} \ln (\| \Psi_g \|) \tag{25}
\]

coincides with financial averaging (8).

Proof. Let us write vector (23) in the form

\[
\Psi_g = \left( \sum_{i_1=1}^{l} g_{i_1} e_{i_1} \right) \otimes \cdots \otimes \left( \sum_{i_M=1}^{l} g_{i_M} e_{i_M} \right) = \sum_{i_1=1}^{l} \cdots \sum_{i_M=1}^{l} g_{i_1} \cdots g_{i_M} e_{i_1} \otimes \cdots \otimes e_{i_M}. \tag{26}
\]

Substitution of (26) into (20) yields

\[
\| \Psi_g \| = \sum_{i_1=1}^{l} \cdots \sum_{i_M=1}^{l} g_{i_1} \cdots g_{i_M} = \left( \sum_{i=1}^{l} g_i \right)^M. \tag{27}
\]

Taking into account that \( g_i = p_i e^{-\beta \lambda_i} \) and applying (27), we obtain

\[
-\frac{1}{M \beta} \ln (\| \Psi_g \|) = -\frac{1}{\beta} \ln \left( \sum_{i=1}^{l} p_i e^{-\beta \lambda_i} \right). \tag{28}
\]

The right-hand side of (28) coincides with the right-hand side of (8), which proves the statement.

In what follows, we do not use the axiomatics of nonlinear averaging; instead, we consider the evolution of an arbitrary (nonnormalized) probability distribution of \( g_i \).

For any set of nonnegative \( g_i \), where \( i = 1, \ldots, l \) we define the evolution of vectors \( \Psi_g(n) \in \mathcal{L}_{\{M\}} \) by the recursive formulas

\[
\Psi_g(n+1) = R \left( \exp \left( -\beta \hat{R}_M \right) \Psi_g(n) \right), \quad n = 0, 1, \ldots, \quad \Psi_g(0) = \Psi_g. \tag{29}
\]
We use $\psi_g(i_1, \ldots, i_M; n)$ to denote the coefficients in expansion (29) of the vector $\Psi_g(n)$ in basis (16). Thus, $\psi_g(i_1, \ldots, i_M; n)$ and $\Psi_g(n)$ are related by

$$\Psi_g(n) = \sum_{i_1=1}^l \cdots \sum_{i_M=1}^l \psi_g(i_1, \ldots, i_M; n) e_{i_1} \otimes \cdots \otimes e_{i_M}. \quad (30)$$

The function $\psi_g(i_1, \ldots, i_M; n)$ can be regarded as a nonnormalized distribution function of the factored ensemble after $n$ steps under the condition that, at the initial moment, the distribution function of the factored ensemble has the form $\psi_g(i_1, \ldots, i_M; 0) = g_{i_1} \ldots g_{i_M}$. In this case, $g_i$ has the meaning of a nonnormalized distribution function for one system in the factored ensemble.

Consider the functions

$$F(n, g, M) = -\frac{1}{M\beta(n+1)} \ln (\|\Psi_g(n)\|), \quad (31)$$

and

$$w_i(n, g, M) = \frac{1}{\|\Psi_g(n)\|} \sum_{i_2=1}^l \cdots \sum_{i_M=1}^l \psi_g(i, i_2, \ldots, i_M; n). \quad (32)$$

Since $\psi_g(i_1, \ldots, i_M)$ is a distribution function for the factored ensemble of $M$ systems, formula (32) defines a distribution function for one system. Consider the limit of distribution functions (31) and (32) for one system of the factored ensemble as the number $M$ of its elements tends to infinity.

**Theorem.** As $M \to \infty$ the functions $F(n, g, M)$ and $w_i(n, g, M)$ approach the following limits:

$$\lim_{M \to \infty} F(n, g, M) = -\frac{1}{\beta} \ln \left( \sum_{i=1}^l e^{-n\beta\lambda_i/(n+1)} g_i^{1/(n+1)} \right) \equiv \tilde{F}(n, g), \quad (33)$$

$$\lim_{M \to \infty} w_i(n, g, M) = \frac{e^{-n\beta\lambda_i/(n+1)} g_i^{1/(n+1)}}{\sum_{j=1}^l e^{-n\beta\lambda_j/(n+1)} g_j^{1/(n+1)}} \equiv \tilde{w}_i(n, g). \quad (34)$$

**Proof.** Vector (23) can be represented in the form

$$\Psi_g = \sum_{\{M\}} \left( \prod_{i=1}^l g_i^{M_i} \right) \Phi_{\{M\}}. \quad (35)$$

Substituting (35) into (29), we obtain

$$\Psi_g(n) = (M!)^l \sum_{\{M\}} \left( \prod_{i=1}^l g_i^{M_i} e^{-n\beta\lambda_i M_i} \right) \frac{1}{(M_i!)^n} \Phi_{\{M\}}. \quad (36)$$

This implies

$$\psi_g(i_1, \ldots, i_M; n) = (M!)^l \left( \prod_{i=1}^l \frac{g_i e^{-n\beta\lambda_i M_i}^{M_i}}{(M_i!)^n} \right), \quad (37)$$
where $M_i$ are expressed in terms of $i_1, \ldots, i_M$ according to (11). Substituting (37) into (20) and summing (37) over all but one subscripts, we obtain

$$\parallel \Psi_g(n) \parallel = (M!)^{n+1} \sum_{\{M\}} \prod_{i=1}^{l} \frac{g_i^{M_i} e^{-n \beta \lambda_i M_i}}{(M_i!)^{n+1}},$$

(38)

$$\sum_{i_2=1}^{l} \sum_{i_M=1}^{l} \psi_g(i, i_2, \ldots, i_M; n) = (M!)^{n+1} \sum_{\{M\}} \left( \prod_{i=1}^{l} \frac{g_i^{M_i} e^{-n \beta \lambda_i M_i}}{(M_i!)^{n+1}} \right) \left( \sum_{j=1}^{l} \frac{M_j}{M} \right).$$

(39)

To determine the asymptotic behavior of the sums in (38), (39) as $M \to \infty$, we apply the Laplace method. This method gives the equalities

$$\lim_{M \to \infty} \frac{1}{M} \ln (\parallel \Psi_{p,\lambda}(n) \parallel) = (n+1) \ln \left( \sum_{i \in I} e^{-\beta \lambda_i/(n+1)} \right),$$

(40)

$$\lim_{M \to \infty} \frac{1}{\parallel \Psi_{p,\lambda}(n) \parallel} \sum_{i_2=1}^{l} \sum_{i_M=1}^{l} \psi_{p,\lambda}(i, i_2, \ldots, i_M; n) = \frac{e^{-n \beta \lambda_i/(n+1)} g_i^{1/(n+1)}}{\sum_{j=1}^{l} e^{-n \beta \lambda_j/(n+1)} g_j^{1/(n+1)}}. $$

(41)

The substitution of (40) and (41) into (31) and (32) yields (33) and (34). This completes the proof of the theorem.

Thus, by virtue of (34), the normalized distribution $\tilde{w}_i(0, g) = \frac{g_i}{\sum_{i} g_i}$ transforms into $\tilde{w}_i(n, g)$ in $n$ evolution steps.

Now, consider the behavior of $\tilde{F}(n, g)$ (33) and $\tilde{w}_i(n, g)$ (34) as $n \to \infty$, i.e., the limit distribution. Let $I$ be a nonempty subset of the set $1, \ldots, l$. Suppose that the $g_i$ satisfy the condition

$$g_i > 0, \quad \text{for} \quad i \in I, \quad g_i = 0, \quad \text{for} \quad i \notin I.$$  

(42)

According to (33) and (34), for any $g_i$ satisfying (42), we have

$$\lim_{n \to \infty} \tilde{F}(n, g) = -\frac{1}{\beta} \ln \left( \sum_{i \in I} e^{-\beta \lambda_i} \right),$$

(43)

$$\lim_{n \to \infty} \tilde{w}_i(n, g) = \frac{e^{-\beta \lambda_i}}{\sum_{j=1}^{l} e^{-\beta \lambda_j}} \equiv \rho_i.$$  

(44)

If the set $I$ coincides with the set of $1, \ldots, l$, expressions (43) and (44) are, respectively, the free energy and Gibbs distribution at temperature $1/\beta$ of a system with nondegenerate energy levels $\lambda_1, \ldots, \lambda_l$. If $I$ is smaller than $1, \ldots, l$, then expressions (43) and (44) describe the distribution of the system only over a part of levels, rather than over all levels. Note also that, if the $g_i$ satisfy the condition

$$g_i = \begin{cases} Ae^{-\beta \lambda_i} & \text{for} \quad i \in I, \\ 0 & \text{for} \quad i \notin I, \end{cases}$$

(45)

where $A$ is an arbitrary positive number, then $\tilde{w}_i(n, g)$ (34) coincides with $\rho_i$ (44) for any $n$. Thus, the full and partial Gibbs distributions are invariant with respect to the evolution described above.
Remark 2. Consider the special case where the random variable $\lambda$ takes the same values as the polynomial

$$E(N_1) = \sum_{q=0}^{p} A_q N_1^q,$$  \hspace{1cm} (46)

where $N_1$ ranges over $0, 1, \ldots, N$. Consider the satisfiability of condition (4) in this case.

Proposition 3. If $p < N$, then condition (4) holds for no values of the coefficients $A_q$, and if $p \geq N$, then condition (4) holds for generic coefficients $A_q$. 

4 Zeroth-order phase transition

4.1 Zeroth-order phase transition: general considerations

4.1.1 Main large parameters in thermodynamics

Thermodynamics studies steady-state processes in which, independently of its initial state, the system comes to a state that remains the same in what follows. But if such a state still varies, then this is a thermodynamic variation only if the process occurs extremely slowly [15]. In other words, we slightly vary the state of the system and wait until it returns to equilibrium and again is independent of the initial state.

Thus, a large parameter, time, is invisibly present in this case, i.e., we observe the system at huge time intervals. On the other hand, thermodynamics is the limit macroscopic theory obtained from the microscopic statistical physics as the number of particles $N$ tends to infinity. Hence there are two large parameters, and many things depend on their ratio. In turn, classical statistical physics is the limit of quantum statistics as $h \to 0$ ($h$ is the Planck parameter). Thus, already three large parameters "meet" here: time scale, number of particles, and $1/h$.

In thermodynamics, sufficiently large time periods are also often considered, and in these periods only a part of the system comes to equilibrium; this is the so-called "local equilibrium." For example, when plasma is heated by a magnetic field, the ions begin to obey the Maxwell distribution after a large observation period, and the entire system comes to the thermodynamic equilibrium only after a significantly large period of time.

4.1.2 The Gibbs formula

Let us consider the Gibbs formula, which has the following form in quantum statistics.

Suppose that a system is characterized by the Hamiltonian operator $\hat{H}$ in a Hilbert space (in particular, in the Fock space) $\Phi$. Suppose that the operator $\hat{H}$ has a nonnegative discrete spectrum $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$. Then the free energy depending on the temperature $\theta$ is determined as

$$E = \theta \ln \sum_{n=0}^{\infty} e^{-\lambda_n/\theta} \delta_n,$$

where $\delta_n$ is the multiplicity of the eigenvalue $\lambda_n$.

We note that one more parameter appears in the Gibbs postulate, namely,

$$E = \theta \ln \lim_{M \to \infty} \sum_{n=0}^{M} e^{-\lambda_n/\theta} \delta_n.$$

This is one of the most important facts, since $\lim_{M \to \infty}$ and $\lim_{N \to \infty}$ do not commute! It turns out that one must first pass to the limit as $M \to \infty$ and then as $N \to \infty$.

If we speak about local equilibrium, then, as a rule, the Gibbs formula deals with a subset $\{\lambda_n\}$ of the set $\{\lambda_n\}$. This must be done in the study of the above problem concerning the ion distribution in plasma on heating. This is also done in glass and in several other physical problems.

4.1.3 "Friction" operator

At temperature $\theta = 0$, the Gibbs formula gives the value of the lowest eigenvalue, which, in physics, is called the "ground state."
This is a law which has not been stated accurately and which was called the “energetic efficiency” by N. N. Bogolyubov and others. The lower energy is energetically more efficient. For example, if a system is perturbed by an operator \( \hat{V} (\Phi \rightarrow \Phi) \) and this operator is sufficiently small, then the transition matrix from the state \( \lambda_n \) to the state \( \lambda_m \) is determined by the matrix element \( \langle \Psi_n | \hat{V} | \Psi_m \rangle \), where \( \Psi_n \) and \( \Psi_m \) are the eigenfunctions corresponding to \( \lambda_n \) and \( \lambda_m \), respectively (see [16]).

The square of this matrix element is the probability of transition from the state with energy \( \lambda_n \) to the state with energy \( \lambda_m \). But if \( \lambda_n < \lambda_m \), then this transition is “not energetically efficient” and hence, as if from the viewpoint of statistical physics and thermodynamics, is unrealizable, i.e., from the mathematical viewpoint, it must be set to be zero. Then the transition matrix is not a self-adjoint matrix with zero entries above the diagonal. This means that friction is taken into account. A pendulum oscillates and stops in the end if friction is taken into account, i.e., it comes to the “ground state.” The “friction” operator was considered in more detail in the author’s paper [17].

4.1.4 Phase transitions

The derivative of \( E(\theta) \) with respect to \( \theta \) is called the entropy, and when the entropy has a jump at a point \( \theta_0 \), it is said that a first-order phase transition occurs; when the second-order derivative has a jump, it is said that a second-order phase transition occurs, etc.

But, of course, it is not the function itself that has a jump, but the leading term of its asymptotics as \( N \to \infty \). In experiments, this “jump” can sometimes be actually extended in time, but we have agreed to consider variations in sufficiently large periods of time. In these periods, not only thermodynamic, but also dynamical processes can occur. We neglect them and consider only the time periods in which the system comes to equilibrium with the “energetic efficiency” taken into account [15].

The author discovered the zeroth-order phase transition both in the theory of superfluidity and superconductivity and in economics (a stock price break-down, a default, etc.), and, quite unexpectedly, it has turned out that, in view of the natural axiomatics (see [19]), the zeroth-order phase transition is related to quantum statistics and thermodynamics. This phase transition has not been noticed by physicists, and, perhaps, it contradicts their ideas that the free energy can be determined up to a constant.

4.1.5 Metastable state

We consider a simple example of semiclassical approximation of the one-dimensional Schrödinger equation

\[-h^2 y''_n + u(x) y_n = \lambda_n y_n, \quad y_n(x) \in L_2, \quad h \ll 1,\]

where \( u(x) = (x^2 - 1)^2 + qx \) and \( q > 0 \) is a constant.

In classical mechanics, this equation gives the picture of “two cups with a barrier between them.” If a particle is at the bottom of the right-hand cup with higher walls, then it can get into the other cup with lower walls only if the barrier disappears.

From the viewpoint of the Gibbs postulate, as \( \theta \to 0 \), the particle must be in the deeper well. Nevertheless, it is obvious that if \( h \ll 1 \), then the particle will stay in the well with higher walls for a very long time. In this case, the summation in the Gibbs formula must be performed over the subset of eigenvalues corresponding to the well with higher walls whose eigenfunctions tend to zero as \( h \to 0 \) outside this well. Moreover, the
temperature \( \theta \) must not be too high, so that the eigenvalues above the barrier do not play any role in the Gibbs formula.

Such a state at a local minimum of the potential well \( u(x) \) is an example of a metastable state.

If we consider the matrix element of the transition from the lower level corresponding to the well with higher walls to the lowest level \( \lambda_0 \) (corresponding to the bottom of the deep well), then it turns out to be exponentially small with respect to \( h \), but any transition to higher levels is forbidden according to the “energetic efficiency” law (i.e., we consider perturbations by “friction”).

4.1.6 Superfluidity

N. N. Bogolyubov used Landau’s ideas (see the footnote on p. 219 in [20]) to show that superfluidity is not a motion of the fluid, not a dynamics, but a state of the fluid such as, for example, ice or vapor for water.

This state corresponds to a metastable state of the system such that any transition from this state to the normal state is almost forbidden: it is exponentially small as \( N \to \infty \).

N. N. Bogolyubov proved this rigorously under the assumption that the system of Schrödinger equations is periodic; in other words, the Schrödinger equations were considered on a torus. The spectrum of superfluid velocities (the energy levels corresponding to the relevant momenta) was discrete. This readily distinguishes the state of superfluidity from the hydrodynamics of fluids. In the limit as the torus radius tends to infinity, the spectrum does not become, as usual, a continuous spectrum, but becomes an everywhere dense point spectrum.

In the author’s opinion, the fact that the spectrum is everywhere dense pointwise in the limit can be easily explained from the physical viewpoint.

Indeed, if the system is in a state with a superfluid velocity \( v \), then its transition to a state with larger velocity is forbidden by the “energy state efficiency” law, and its transition to a state with any lesser velocity is forbidden by the fact that any decrease in the velocity contradicts the notion of superfluidity. From the mathematical viewpoint, this means that the larger the torus radius, the less (exponentially less) is the transition from one state to another.

4.1.7 Zeroth-order phase transition

1. The author explained the spouting effect discovered in 1938 by J. Allen and H. Jones in which a superfluid was “flowing” through a capillary of diameter \( 10^{-4} \) cm (in fact, this superfluid was at the superfluid level of a metastable state).

The author used the two-level model to show that, at a point heated (by light) till the phase transition temperature \( \theta_c \), the heat capacity becomes infinite, the entropy has a jump, and the free energy decreases to its lower value, i.e., to the point attained by the curve issuing from the ground state heated to the temperature \( \theta_c \). This means that a zeroth-order phase transition occurs. In the present paper, we show that the same picture also appears in N. N. Bogolyubov’s model of a weakly nonideal Bose gas [22].

This phenomenon can be easily explained if we assume that superfluidity is not a thermodynamic state, but the motion of a fluid. At \( \theta = \theta_c \), the fluid becomes viscous and cannot penetrate through a thin capillary.
But the point is that this is not a motion, but a state, and so we have a zeroth-order phase transition.

The following question arises. What will happen to superfluidity not in a capillary, but in a rather thick pipe? Where is the zeroth-order phase transition? At $\theta = \theta_c$, the fluid passes from the superfluid state to the state of the ordinary fluid with viscosity and begins to flow according to the usual laws of hydrodynamics. After a while, the motion stops, and the transition from the superfluid state to the state of a fluid at rest is a thermodynamic transition. All the intermediate flow is hydrodynamics, and must be neglected for our time scale.

2. Let us consider the example studied in Sec. 1.5 in more detail. We slowly vary the constant $q$. We note that, in thermodynamics with the field taken into account, there are two more thermodynamic quantities: the field intensity and the charge. Thus, a variation in $q$ is a variation in a thermodynamic variable.

We shall show that the passage of $q$ through the zero point results in a zeroth-order phase transition. Indeed, resonance occurs at $q = 0$: the eigenfunctions are already not concentrated in one of the wells and the probability (the square of each eigenfunction) is identically distributed over both wells. The number of eigenvalues is “doubled” and the Gibbs distribution has a jump. It is precisely here that one can see the role of time. A certain amount of time is required for half the function, decreasing outside the first well, to be “pumped” into the second well. But the Gibbs formula does not take this into account. It may happen that we will have to wait for this transition for a long time, as it was in the preceding example with a thick pipe.

If $q$ becomes negative, then all the eigenfunctions remain both in the first and in the second well, and the Gibbs formula is taken over all the eigenvalues, rather than over some of them.

3. Finally, we consider the effect of transition into the turbulent flow for fluid helium, which, in fact, is very close to Landau’s idea concerning energy pumping between large and small vortices.

As will be shown in another paper, it is precisely the resonance that occurs between vortices of these two types that results in the zeroth-order phase transition, which sharply changes the thermodynamic parameters from the viewpoint of thermodynamics in which we take into account the large scales of time between transitions and, naturally, the averaging over these times.

4. Since only the values of bank notes are important, while their serial numbers do not play any role, and the interchange of two bank notes of the same denomination is not an operation, bank notes obey the Bose statistics. The averaging of gains is a nonlinear operation, as well as addition. As was shown above, the only nonlinear “arithmetics” (the semiring) that satisfies the natural axioms of averaging for Bose particles (bank notes) has the form $a \oplus b = \ln(e^a + e^b)$.

This lead to a formula of Gibbs type. A variation in $\beta = 1/T$ can be treated as a variation in the rouble purchasing power caused, for example, by printing a lot of new bank notes. After this, for a period of time, the balance (equilibrium) is again established.

Ordinary financial efficiency plays the role of energetic efficiency. The zeroth-order phase transition is either a default or a crisis \[19, 23\].
4.2 An exactly solvable model

First, we consider the one-dimensional Schrödinger equation for a single particle on the circle

\[ \hat{H}\psi_k(x) = E\psi_k(x), \quad \psi_k(x - L) = \psi_k(x), \]  

(47)

where \( \psi_k(x) \) is the wave function, \( x \) takes values on the circle, and \( \hat{H} \) is a differential operator of the form

\[ \hat{H} = \epsilon(-i\hbar\partial/\partial x), \quad \epsilon(z) \in \mathbb{C}^\infty. \]  

(48)

Here \( \hbar \) is the Planck constant, \( \psi_k(x) = \exp(ip_kx) \), where \( p_k = 2\pi\hbar k/L \), \( k \) is an arbitrary integer, and the corresponding eigenvalues are equal to

\[ E_k = \epsilon(p_k). \]  

(49)

We pass from the Hamiltonian function \( \epsilon(p) \) to a discrete function \( \tilde{\epsilon}(p) \) of the form

\[ \tilde{\epsilon}(p) = \epsilon(n\Delta p) \quad \text{for} \quad \Delta p\left(n - \frac{1}{2}\right) \leq p < \Delta p\left(n + \frac{1}{2}\right), \]  

(50)

where \( n \) is an arbitrary integer and \( \Delta p \) is a positive constant. Then the Hamiltonian (48) changes appropriately, and we denote the new Hamiltonian by \( \hat{\tilde{H}} \). The eigenfunctions of this operator coincide with \( \psi_k(x) \), and the eigenvalues (49) become

\[ \tilde{E}_k = \tilde{\epsilon}(p_k). \]  

(51)

In what follows, we assume that the constant \( \Delta p \) takes the form

\[ \Delta p = \frac{2\pi\hbar G}{L}, \]  

(52)

where \( G \) is a positive integer.

In this case, it follows from (50) that the set of energy levels (51) is the set of \( G \)-fold degenerate energy levels

\[ \lambda_n = \tilde{E}_{Gn} = \epsilon(p_{Gn}). \]  

(53)

The Schrödinger equation for \( N \) noninteracting particles has the form

\[ \hat{H}_N\Psi(x_1, \ldots, x_N) = E\Psi(x_1, \ldots, x_N), \]  

(54)

where \( \Psi(x_1, \ldots, x_N) \) is a symmetric function of the variables \( x_1, \ldots, x_N \) (bosons). The Hamiltonian \( \hat{H}_N \) is given by the formula

\[ \hat{H}_N = \sum_{j=1}^{N} \hat{H}_j, \]  

(55)

where \( \hat{H}_j \) is the Hamiltonian of the particle with number \( j \), which has the form

\[ \hat{H}_j = \tilde{\epsilon}\left(-i\hbar \frac{\partial}{\partial x_j}\right). \]  

(56)

The complete orthonormal system of symmetric eigenfunctions of the Hamiltonian (55) has the form

\[ \Psi_{\{N\}}(x_1, \ldots, x_N) = \text{Symm}_{x_1,\ldots,x_N} \psi_{\{N\}}(x_1, \ldots, x_N), \]  

(57)
where \( \text{Symm}_{x_1, \ldots, x_N} \) is the symmetrization operator over the variables \( x_1, \ldots, x_N \), \( \{N\} \) is the set of nonnegative integers \( N_k, k \in \mathbb{Z} \), satisfying the condition
\[
\sum_{k=-\infty}^{\infty} N_k = N, \tag{58}
\]
and the function \( \psi_{\{N\}}(x_1, \ldots, x_N) \) is equal to
\[
\psi_{\{N\}}(x_1, \ldots, x_N) = \prod_{s=1}^{N} \psi_{k_s}(x_s). \tag{59}
\]
Here the indices \( k_1, \ldots, k_N \) are expressed in terms of the set \( \{N\} \) using the conditions
\[
k_s \leq k_{s+1} \quad \text{for all } 1 \leq s \leq N - 1, \quad \sum_{s=1}^{N} \delta_{k_k} = N_k \quad \text{for all } k \in \mathbb{Z}, \tag{60}
\]
and \( \delta_{k_k'} \) is the Kronecker symbol. The eigenvalues of the Hamiltonian (55) are
\[
E(\{N\}) = \sum_{k=-\infty}^{\infty} \tilde{E}_k N_k. \tag{61}
\]
We consider interparticle interactions of the following type. We assume that the particles interact in pairs and the interaction operator for particles with numbers \( j \) and \( k \) has the form
\[
\hat{V}_{jk} = -\frac{V}{N} W(\hat{H}_j - \hat{H}_k), \tag{62}
\]
where \( V > 0 \) is the interaction parameter and the function \( W(\xi) \) is given by the formula
\[
W(\xi) = \begin{cases} 
1 & \text{for } |\xi| < D, \\
0 & \text{for } |\xi| \geq D. \end{cases} \tag{63}
\]
Here \( D > 0 \) is the parameter of the width of interaction with respect to energy. The operator (63) corresponds to the interaction under which particles in a pair attract each other and radiate the energy quantum \(-V/N\) if the difference between their energies is less than \( D \) and do not interact at all if the difference between their energies is larger than \( D \). The Hamiltonian of the system of \( N \) particles with interaction (63) has the form
\[
\hat{H}_N = \sum_{j=1}^{N} \hat{H}_j + \sum_{j=1}^{N} \sum_{k=j+1}^{N} \hat{V}_{jk}. \tag{64}
\]
In view of (62), the sums in (64) commute, and hence the set of eigenfunctions of the Hamiltonian (64) coincides with (57). It also follows from (62) that the corresponding eigenvalues are equal to
\[
E(\{N\}) = \sum_{k=-\infty}^{\infty} \tilde{E}_k N_k - \frac{V}{2N} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} W(\tilde{E}_k - \tilde{E}_l)(N_kN_l - \delta_{kl}N_k). \tag{65}
\]
In what follows, we assume that the interaction width is sufficiently small, satisfying the condition
\[
D < \min_{n \neq m} |\lambda_n - \lambda_m|. \tag{66}
\]
The set of energy levels \( \tilde{E}_k, k \in \mathbb{Z} \), coincides with the \( G \)-fold degenerate set of levels (53). Hence, by (66), the set of energy levels (55) of the system of \( N \) particles under study can be written as

\[
E(\{\tilde{N}\}) = \sum_{n=-\infty}^{\infty} \lambda_n \tilde{N}_n - \frac{V}{2N} \sum_{n=-\infty}^{\infty} \tilde{N}_n (\tilde{N}_n - 1),
\]

where the level \( E(\{\tilde{N}\}) \) has the multiplicity

\[
\Gamma(\{\tilde{N}\}) = \prod_{n=-\infty}^{\infty} \frac{(G + \tilde{N}_n - 1)!}{(G - 1)! \tilde{N}_n!}. \tag{68}
\]

Here \( \{\tilde{N}\} \) denotes the set of nonnegative integers \( \tilde{N}_n, n \in \mathbb{Z} \), satisfying the condition

\[
\sum_{n=-\infty}^{\infty} \tilde{N}_n = N. \tag{69}
\]

Let us consider the partition function for the system of \( N \) bosons with Hamiltonian (64). Since the energy levels and their multiplicities are given by formulas (67) and (68), respectively, the partition function at temperature \( \theta \) takes the form

\[
Z(\theta, N) = \sum_{\{\tilde{N}\}} \Gamma(\{\tilde{N}\}) \exp\left(-E(\{\tilde{N}\})\theta\right). \tag{70}
\]

Here the summation is performed over all sets \( \{\tilde{N}\} \) with condition (69) taken into account.

In what follows, we assume that \( G \) depends on \( N \) and the following condition is satisfied:

\[
\lim_{N \to \infty} \frac{G}{N} = g > 0. \tag{71}
\]

By \( \tilde{F}(\{\tilde{N}\}, \theta) \) we denote a function of the form

\[
\tilde{F}(\{\tilde{N}\}, \theta) = E(\{\tilde{N}\}) - \theta \ln(\Gamma(\{\tilde{N}\})), \tag{72}
\]

and by \( \{\tilde{N}^0\} \) the set of nonnegative numbers \( \tilde{N}_n^0, n \in \mathbb{Z} \), for which the function (72) is minimal under condition (69).

Now, we consider the problem of finding the minimal value of the function (72) under condition (69). In the limit as \( N \to \infty \) and under condition (71), the point of minimum has the form

\[
\tilde{N}_n(\theta) = N(m_n(\theta) + o(1)), \tag{73}
\]

where \( m_n(\theta), n \in \mathbb{Z} \), is determined by the system of equations

\[
\lambda_n - Vm_n + \theta \ln\left(\frac{m_n}{g + m_n}\right) = \mu(\theta), \quad n \in \mathbb{Z}, \tag{74}
\]

and \( \mu(\theta) \) can be found from the equation

\[
\sum_{n=-\infty}^{\infty} m_n = 1. \tag{75}
\]
The substitution of (73) into (72) and then the use of the asymptotic Stirling formula give the following relation for the specific free energy:

\[
f(\theta) \equiv \lim_{N \to \infty} f(\theta, N) = \lim_{N \to \infty} \frac{\tilde{F}(\{\tilde{N}^0\}, \theta)}{N} = \sum_{n=-\infty}^{\infty} \left( \lambda_n m_n - \frac{V}{2} m_n^2 \right) + \left( \theta m_n \ln \left( \frac{m_n}{g} \right) - \theta(g + m_n) \ln \left( 1 + \frac{m_n}{g} \right) \right),
\]

(77)

where, for brevity, we omit the argument \( \theta \) of \( m_n(\theta) \). We introduce the notation

\[
\omega_n = \lambda_n - V m_n.
\]

(78)

**Remark 3.** Remark In the notation (78), the system of Eqs. (74) and (75) takes the form

\[
\omega_n(\theta) = \lambda_n - V \frac{g}{\exp((\omega_n - \mu)/\theta) - 1}, \quad n \in \mathbb{Z},
\]

(79)

\[
\sum_{n=-\infty}^{\infty} \frac{g}{\exp((\omega_n - \mu)/\theta) - 1} = 1.
\]

(80)

The system of Eqs. (79) and (80) exactly coincides with the temperature Hartree equations [24] for the system of \( N \) bosons with Hamiltonian (64).

We shall study the solutions of Eqs. (74) and (75). For \( \theta = 0 \), the system has many solutions, which we number by the integer \( l \):

\[
m^{(l)}_n = \delta_{ln}, \quad n, l \in \mathbb{Z}.
\]

(81)

Among all the numbers \( l \), we choose those that satisfy the condition

\[
\nu_{nl} \equiv \lambda_n - \lambda_l + V > 0 \quad \text{for all} \quad n \neq l.
\]

(82)

For these numbers, there exist solutions of Eqs. (74) and (75) converging to (81) as \( \theta \to 0 \). The asymptotics of these solutions as \( \theta \to 0 \) has the form

\[
m^{(l)}_n \sim g \exp \left( \frac{-\nu_{nl}}{\theta} \right) \quad \forall n \neq l, \quad 1 - m^{(l)}_l \sim g \sum_{n \neq l} \exp \left( \frac{-\nu_{nl}}{\theta} \right).
\]

(83)

Thus, depending on the spectrum \( \lambda_n, n \in \mathbb{Z} \), for sufficiently small values of the temperature \( \theta \), the system of Eqs. (74) and (75) has many solutions. These solutions, along with the point of global minimum, also contain points of local minimum of the function (72). The values of the function (72) at the points of local minimum are equal to the free energy of metastable states. We consider the function (72) at \( \theta = 0 \). In this case, it coincides with the energy spectrum of system (67). We consider the energy of the system for the case in which almost all particles are at the energy level \( \lambda_l \), which means that following conditions holds:

\[
\tilde{N}_n \ll N \quad \forall n \neq l.
\]

(84)

Deriving \( \tilde{N}_l \) from relation (69) and substituting the result into (67), we see that, in view of (84), the energy spectrum of the system in this domain has the form

\[
E(\tilde{N}) \approx \lambda_l N - \frac{VN}{2} + \sum_{n \neq l} (\lambda_n - \lambda_l + V) \tilde{N}_n.
\]

(85)
To the Hamiltonian \( (64) \), there correspond the Hartree equation and the appropriate system of variational equations. To each \( l \in \mathbb{Z} \), there corresponds a solution of the Hartree equation, and this solution describes the state

\[ \tilde{N}^{(l)}_n = N\delta_{nl}. \]  

Moreover, the eigenvalues of the system of variational equations corresponding to this solution of the Hartree equation coincide with

\[ \nu_{nl} = \lambda_n - \lambda_l + V, \quad n \neq l. \]  

In \([25, 26]\), it was shown that if the eigenvalues of the system of variational equations for the solution of the Hartree equation are real and nonnegative, then such a solution corresponds to the ground state or to a metastable state of the system. This means that to all \( l \) for which condition \( (82) \) holds at \( \theta = 0 \), there correspond metastable states of the system of bosons. As in the case of zero temperature, to solutions of Eqs. \( (74) \) and \( (75) \) for \( \theta \neq 0 \), there corresponds a temperature metastable state if the point \( (73) \) is a point of local minimum. Now, we note that, for very large temperatures, the system of Eqs. \( (74) \) and \( (75) \) has only one solution corresponding to the global minimum of the function \( (72) \).

The asymptotics of this solution as \( \theta \to \infty \) has the form

\[ n_m(\theta) \sim g \frac{e^{-\lambda_m/\theta}}{\sum_{l=-\infty}^{\infty} e^{-\lambda_l/\theta}}. \]

The uniqueness of the solution at large temperatures means that all metastable states disappear with increasing temperature. The temperature at which a metastable state disappears is critical for this state.

Let us consider the behavior of the entropy and the heat capacity of metastable states when we approach the critical temperature. We consider the metastable state to which there corresponds a solution of Eqs. \( (74) \) and \( (75) \) converging to \( (81) \) as \( \theta \to 0 \) for some \( l \) for which \( (82) \) holds. In what follows, we assume that the function \( \epsilon(p) \) satisfies the condition \( \epsilon(0) < \epsilon(p) \) for \( p \neq 0 \). Then, according to \( (53) \), we have \( \lambda_0 < \lambda_l \) for \( l \neq 0 \). Hence the solution of Eqs. \( (74) \) and \( (75) \) converging to \( (81) \) as \( \theta \to 0 \) for \( l = 0 \) corresponds to the temperature ground state of the system of \( N \) bosons. Moreover, condition \( (82) \) becomes equivalent to the condition \( \lambda_l - \lambda_0 < V \).

We assume that this inequality holds for \( l \neq 0 \). The condition that the corresponding solution \( m_n^{(l)}(\theta) \) of Eqs. \( (74) \) and \( (75) \) determines a point of local minimum of the function \( (72) \) under condition \( (69) \) by formula \( (73) \) can be written as the following system of inequalities:

\[ \alpha_n^{(l)}(\theta) \equiv -V + \frac{\theta g}{m_n^{(l)}(\theta)(g + m_n^{(l)}(\theta))} > 0 \quad \forall n \neq l, \]

\[ \alpha_l^{(l)}(\theta) \equiv -V + \frac{\theta g}{m_l^{(l)}(\theta)(g + m_l^{(l)}(\theta))} < 0, \quad -\sum_{n \neq l} \frac{\alpha_n^{(l)}(\theta)}{\alpha_n^{(l)}(\theta)} < 1. \]  

We note that these inequalities hold for \( (83) \) as \( \theta \to 0 \). Inequalities \( (89) \) follow from the condition that the second variation of the function \( (72) \) is positive; the variation is calculated under condition \( (69) \).

The metastable state disappears at the temperature at which the last inequality in \( (89) \) becomes an equality. We denote this critical temperature by \( \theta_c^{(l)} \). It follows from \( (89) \) and
Eqs. (74) and (75) for $\theta < \theta_c^{(l)}$ that $m_n^{(l)}(\theta)$ is an increasing function of the variable $\theta$ for $n \neq l$ and $m_{l}^{(l)}(\theta)$ is a decreasing function of $\theta$. Moreover, we see that

$$m_{l}^{(l)}(\theta) > m_n^{(l)}(\theta) > m_{n'}^{(l)}(\theta)$$

(90)

if $\lambda_n < \lambda_{n'}$ and $n, n' \neq l$.

From (77) we obtain the following expression for the specific entropy of a metastable state in the limit as $N \to \infty$:

$$s^{(l)}(\theta) = \sum_{n=-\infty}^{\infty} \left( g + m_n^{(l)} \right) \ln \left( \frac{1 + m_n^{(l)}}{g} - m_n^{(l)} \ln \left( \frac{m_n^{(l)}}{g} \right) \right),$$

(91)

where, for brevity, we omit the argument $\theta$ of $m_n^{(l)}(\theta)$. Differentiating (91), we obtain

$$\frac{\partial s}{\partial \theta} = \sum_{n \neq l} \frac{\partial m_n^{(l)}(\theta)}{\partial \theta} \ln \left( \frac{g + m_n^{(l)}}{m_n^{(l)}} \frac{m_{l}^{(l)}}{g + m_{l}^{(l)}} \right) > 0.$$  

(92)

The last inequality follows from the properties of $m_n^{(l)}(\theta)$. Since the last inequality in (89) becomes an equality at the critical temperature, we can show that, as $\theta \to \theta_c^{(l)} - 0$, the solutions of Eqs. (74) and (75) behave as follows:

$$m_n^{(l)}(\theta) - m_{n'}^{(l)}(\theta)^{c^{(l)}} < m_{n'}^{(l)}(\theta) \quad \forall n \in \mathbb{Z},$$  

(93)

where $C^{(l)}$ is a negative number. The substitution of (93) into (91) shows that the derivative of the specific entropy with respect to the temperature (this quantity is equal to the heat capacity divided by the temperature) tends to infinity as $\theta \to \theta_c^{(l)} - 0$ according to the law $1/\sqrt{\theta_c^{(l)} - \theta}$. This means that the projection of the Lagrangian manifold corresponding to the metastable state on the $\theta$-axis is not uniquely determined in a neighborhood of the critical temperature [13, 27]. The derivative of the temperature with respect to the entropy vanishes as the critical temperature is approached. Therefore, as well as in view of (92), as was already pointed out, the Lagrangian manifold is uniquely projected on the $s$-axis. We note that it follows from the properties of $m_n^{(l)}(\theta)$ that the following conditions hold for $\theta < \theta_c^{(l)}$:

$$m_0^{(l)}(\theta) < m_n^{(l)}(\theta) \quad \forall n \neq 0, l,$$

$$\alpha_0^{(l)}(\theta) < \alpha_n^{(l)}(\theta) \quad \forall n \neq 0, l.$$  

(94)
Inequalities (94) mean that the potential barrier between the energy levels $\lambda_l$ and $\lambda_0$ is less than the energy barrier between $\lambda_l$ and $\lambda_n$ for $n \neq 0, l$. This means that, as the critical temperature is attained, the system of bosons under study which is in the metastable state with number $l$ changes its state in a jump and passes to the temperature ground state. This is a zeroth-order phase transition, since not only the entropy and heat capacity, but also the free energy have jumps.

If the quantity
\[
\min_{n \neq 0} |\lambda_n - \lambda_0| = \delta
\] (95)
is small, then the difference between the free energies in the zeroth-order phase transition from the lowest metastable state to the ground state is also small, and the heat capacity in this transition has a singularity. The asymptotics of the partition function near the critical point is given by the asymptotics of the canonical operator in a neighborhood of a focal point. This asymptotics has the form of an Airy type function of an imaginary argument and has a singularity as $N \to \infty$. The parameters $\delta \ll 1$, $N \gg 1$, and $L \gg 1$ can be chosen so that the form of the phase transition point coincides exactly with the $\lambda$-point [28–33].

If adequate measures increasing the Kolmogorov complexity of the Hartley entropy are not taken, the financial system still does not necessarily result in a zero-order phase transition (social explosion).

Now, let us return to the problem about deposits in a pyramid and banks, i.e., to the assumptions that the bills of the same value with different numbers are identical. The phase transition related to the disappearance of the condensate has the following meaning in this case. As $\beta$ (price) decreases, the actions cease to be bought and sold at some moment $\beta_0$, i.e., nobody trades them at $\beta < \beta_0$, although, seemingly, it is more profitable to sell them at any price, and therefore, somebody can speculate in resale, thereby reducing the price of the actions to zero. Nevertheless, this paradoxical fact is observed in practice and mentioned in the literature.

Then the corresponding shares will be involved in a complicated barter exchange, which makes for an increase in the entropy. The same happens to other equities, in particular, to currencies provided that the inflation rate is sufficiently high. All in all, this ensures that the entropy and complexity increase and can prevent zero-order phase transition, which occurs only at some threshold temperature and for some nonlinear interaction swinging the equilibrium system.

Now we consider the following model. Assume that the number of people is $n = n_1 + n_2$. The culture (velocity, energy) level $\lambda_1$ typical of $n_1$ is twice less than the level $\lambda_2$ typical of $n_2$. The quantities $\lambda_1 = 1$, $\lambda_2 = 2$, and $1 < \gamma < 2$ determine the quadratic interaction
\[
\mathcal{E}(N_1) = \left( N_1 + 2N_2 - \gamma \frac{N_1^2}{2N} - \gamma \frac{N_2^2}{2N} \right) + \frac{M}{N_0} \ln \left( \frac{(N_1 + n_1 - 1)!(N - N_1 + n_2 - 1)!}{(n_1 - 1)!N_1!(n_2 - 1)!(N - N_1)!} \right),
\]
where $n_1$ of people have $N_1$ units of money and $n_2$ of people have $N_2$ units of money, and $N_1 \ll N_2$. For a sufficiently large $M/N_0$ (an analog of temperature), there is a phase transition to the state with $N_1 = N$ and $N_2 = 0$ with an enormous outburst of the kinetic energy (in physics, this is the Allen–Jones fountain effect [4, 17]).

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