Degenerate bound states in the continuum in square and triangular open acoustic resonators

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We consider square and equilateral triangular open acoustic resonators with the $C_{4v}$ and $C_{3v}$ symmetries, respectively. There is a unique property of square and triangular resonators of accidental number four-fold degeneracy of eigenstates that gives rise to two-fold degenerate Friedrich-Wintgen (FW) BICs. Compared to usual FW BICs the degenerate FW BICs maintain high $Q$-factor in wide range of the size of resonators. That removes the fabrication difficulties of proper choice of resonator. The presence of degenerate BICs in triangular resonators is extremely sensitive to switch output flows by small perturbations with 100% efficiency.

I. INTRODUCTION

A new paradigm for trapping and confining of the resonant modes have emerged in recent years based on the bound states in the continuum (BICs) in wave systems. BICs, also known as trapping mode with infinite large $Q$-factor, have triggered extensive interest in photonic and acoustic communities [1–5]. The most straightforward mechanism of BICs is the symmetrical incompatibility of closed system states with propagating states of the continuum [6–10]. More interesting, Friedrich-Wintgen (FW) BICs are result of full destructive interference of two or more resonant modes competing for leakage into open channels of waveguides [4, 11, 12]. The FW BICs can be realized in open resonators by gradual change of aspect ratio of the resonator when a degeneracy of eigenfrequencies occur [12]. In acoustic systems, the FW BICs were considered by many scholars [13–17]. The experimental evidence for the FW BICs was reported by Lepetit and Kanté [18] and by Huang et al [19] in the most straightforward configuration of rectangular resonator opened to attached waveguide.

Although the last time BICs have been successfully demonstrated in different acoustic resonators fabricated by 3D printing, the $Q$-factor of quasi-BICs is sensitive to the structure imperfections. Moreover continual variation of aspect ratio of the resonator for achievement FW BICs is very challenging for any type of resonator, acoustic, metallic or dielectric. Therefore, a search of FW BICs not sensitive to the aspect ratio of resonators is highly desired. In the present paper, we advocate square or equilateral triangular resonators with the group symmetries $C_{4v}$ and $C_{3v}$, respectively. Its eigenmodes $\psi_{nm}$ and eigenfrequencies $\nu_{mn}'$ exhibit trivial degeneracy by permutation of indices $m \leftrightarrow n$. Here the eigenfrequencies are given in terms of the frequency $\omega_0 = \pi s/a$, where $s$ is the velocity of sound or light in air, and $a$ is the side size of the resonator. Therefore, one can expect that opening of the resonator by attachment of waveguides transfers these degenerate eigenmodes into superradiant mode and FW BIC for granting as it follows from the FW mechanism of BICs which occur at degeneracy of eigenfrequencies occur [4, 11, 12, 20]. Variants of attachment of waveguides to a square resonator are sketched in Fig. 1. However, for the case (a) in Fig. 1 the FW BICs occur only after deformation of square resonator into the rectangular one [19]. The reason for cancellation of FW BICs is lowering of the symmetry $C_{4v}$ of closed square resonator towards $C_{2v}$ after the (a) case opening. Quantitatively, the coupling of eigenmodes of a square resonator with evanescent modes of waveguides perturbs the eigenfrequencies of the resonator and removes the permutation degeneracy of open square resonator [21]. Next, consider the cases in which for opening of the resonators the symmetry $C_{4v}$ is preserved as sketched in Fig. 1 (b) and (c). In this case the degenerate eigenmodes $(m, n)$ and $(n, m)$ of closed resonator and their resonant counterparts of open resonator both are classified according to the different irreducible representations of the group $C_{4v}$ and therefore can not be coupled via the continuum of waveguides that cancels the FW mechanism of the BICs too. The same consideration is applied to equilateral triangular resonator with the symmetry group $C_{3v}$.

However, there is a unique case of accidental or number degeneracy of eigenmodes over two-fold in the square and equilateral triangle $\nu_{mn}^2 = \nu_{m'n'}^2$. For example, in square acoustic resonator two choices of integers $m = 1, n = 6$ and $m' = 5, n' = 4$ have the same frequency $25/a^2$ [22, 23]. That brings four-fold degeneracy of a closed resonator that plays the key role in the existence of two-fold degeneracy of the FW BICs provided that open resonators preserve the group symmetries $C_{4v}$ or $C_{3v}$ as shown in Fig. 1 (b)-(d). We show that the degenerate FW BICs are classified

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according to the two-dimensional irreducible representation \( E \) of these group symmetries. Without loss of generality we focus on the acoustic resonators with Neumann boundary conditions in which BICs can be directly probed by microphone measurements of acoustic pressure inside resonators [17, 19]. Actually, resonators are three-dimensional, but the acoustic resonators allow completely disregard the third dimension if the thickness of the resonator is small enough compared to longitudinal sizes as sketched in Fig. 1 (d). In that case, the solutions are constant over the third dimension to be excluded from the solutions presented in Appendix. That opens many ways for experimental verifications of theoretical predictions outlined below in the square and equilateral triangle resonators.

II. THE SP BICS DUE TO PERMUTATION SYMMETRY IN OPEN SQUARE RESONATOR

The symmetry of the open resonators is important for classification and establishment of BICs. For dielectric resonators embedded into radiation space the total symmetry is given by the structure that determines multipole classification of radiation and the symmetry of ultra-high \( Q \) resonances (quasi-BICs) of the structure consisted of one or a few symmetrical dielectric cavities [24–26]. The symmetry of BICs in photonic crystals is given by the symmetry of the crystals [27–30]. As different from the above, the symmetry of resonators with attached directional waveguides is determined by compatibility of symmetry of closed resonator with the symmetry of waveguides. The symmetry of the total system can be lowered or coincide with the symmetry of closed resonator as illustrated in Fig. 1.

The eigenmodes and eigenfrequencies of the acoustic square resonator are collected in Appendix. There are two distinct pairs of degenerate states \( \psi_{m,n}, \psi_{n,m}, m \neq n \). The first pair of the eigenmodes with \( m - n \) odd is classified according to the two-dimensional irreducible representation \( E \). The particular case of \( m = 2, n = 3 \) is demonstrated in Fig. 6 of Appendix. The second pair with \( m - n \) even is classified according to the reducible representations. Only the linear combinations \( \psi_{m,n} \pm \psi_{n,m} \) are classified according to the irreducible one-dimensional representations \( B_1 \) and \( A_1 \) or \( B_2 \) and \( A_2 \) respectively. A particular case of eigenmodes \( \psi_{24} \) and \( \psi_{42} \) classified according to \( B_2 \) and \( A_2 \) is shown in Fig. 7 in Appendix.

Opening of the resonator transforms the real eigenfrequencies of the closed resonator into the complex eigenfrequencies. The procedure of transition from closed system onto open system can be performed by use of the Feshbach projection technique that results in the non Hermitian effective Hamiltonian [31–36]. The complex eigenvalues of this Hamiltonian respond for the position of resonances and resonant line widths and therefore provide excellent way to establish BICs as eigenmodes of effective Hamiltonian with real eigenvalues [4]. The specific form of the effective non
Hermitian Hamiltonian is given in Appendix applied to open acoustic resonators [36].

Let us first consider the degenerated pair of eigenmodes of closed square resonator $\psi_{m,n}$, $\psi_{n,m}$ with $m - n$ odd and $m \neq n$ classified according to the two-dimensional representation $E$, for example, the pair presented in Appendix $\psi_{23}$ and $\psi_{32}$. The coupling matrix elements of these modes with the first open channel $p = 1$ and the second closed channel $p = 2$ for the case in Fig. 1 (b) are collected in Appendix. As a result, we can write the effective non-Hermitian Hamiltonian projected into these modes whose general form is given in Eq. (10) of Appendix

$$\hat{H}_{\text{eff}} = \begin{pmatrix} 5/a^2 - 2ik_1k_2|\alpha|^2 + 2|k_2|^2 & 0 \\ 0 & 5/a^2 - 2ik_1k_2|\alpha|^2 + 2|k_2|^2 \end{pmatrix}. \tag{1}$$

Here according to Table 2 of Appendix we introduced the notations $\alpha = W_{2;3,p=1,C=1} = -\frac{2}{\pi} \sin \frac{\pi}{a}$, $\beta = W_{2,3,p=2;C=1} = f(a)$. The case of Fig. 1 (c) gives the similar diagonal matrix

$$\hat{H}_{\text{eff}} = \begin{pmatrix} 5/a^2 - 4ik_1k_2|\gamma|^2 + 4|k_2|^2 & 0 \\ 0 & 5/a^2 - 4ik_1k_2|\gamma|^2 + 4|k_2|^2 \end{pmatrix}. \tag{2}$$

Here the coupling strengths $\gamma$ and $\delta$ can be evaluated only numerically by integration over thin dotted lines shown in Fig. 1 (c). Irrespectively we obtain that the degenerate pair of eigenmodes $\psi_{2m,2n+1}$ and $\psi_{2n+1,2m}$ transforms into two degenerate resonances but not BICs. That has clear physical origin. Since the symmetry of the open resonator $C_{4v}$ is preserved the eigenmodes of the closed resonator $\psi_{23}$ and $\psi_{32}$ are modified but can not be coupled through the open continuum $p = 1$ of waveguides owing to the symmetry $C_{4v}$ of the open square resonator. As a result the FW mechanism of BICs is cancelling.

Next, we consider the pair $\psi_{s,a} = \psi_{m,n} \pm \psi_{n,m}, m \neq n$ and $m - n$ even which are classified according to the one-dimensional irreducible representations $B_2$ and $A_2$. Example of these eigenfunctions $m = 2, n = 4$ are illustrated in Fig. 7 of SI. For the case in Fig. 1 (b) all coupling matrix elements with the first open channel of each waveguide equal zero as clearly seen in 7 of SI. As a result only closed channels of waveguides contribute in the effective Hamiltonian which in the space of the eigenmodes $\psi_{s,4}$ and $\psi_{4,s}$ takes the following form

$$\hat{H}_{\text{eff}} = \begin{pmatrix} 10/a^2 + 4|k_2|^2(b(a)^2 + c(a)^2) & -4|k_2|b(a)c(a) \\ -4|k_2|b(a)c(a) & 10/a^2 + 4|k_2|^2(b(a)^2 + c(a)^2) \end{pmatrix}. \tag{3}$$

Therefore we have two SP BICs $\psi_{s,a}$ shown in Fig. 2 of SI with the eigenfrequencies

$$\nu_{s,a}^2 = 10/a^2 + 4|k_2|^2 \begin{pmatrix} b(a)^2 \\ c(a)^2 \end{pmatrix}. \tag{4}$$

where the coupling constants $b(a), c(a)$ are collected in Eq. (16) of SI.

III. THE TWO-FOLD DEGENERATE BICS IN OPEN SQUARE RESONATOR

As it was said in the Introduction there is the accidental number degeneracy of two doublets each degenerated by permutation of indices. For example, eigenmodes with indices 1, 6 and 4, 5, both classified according to the two-dimensional irreducible representation $E$ have the same eigenvalue $\nu_{16}^2 = \nu_{45}^2 = 25/a^2$. The next quartet of degenerated eigenmodes is, for example, 3, 10, 10, 3 and 7, 8, 8, 7 with the eigenvalue $85/a^2$. It is reasonable to project the effective non-Hermitian Hamiltonian onto this space of the eigenmodes. Let us enumerate the eigenmodes as follows

$$\phi_1 = \psi_{1,6}, \phi_2 = \psi_{5,4}, \phi_3 = \psi_{6,1}, \phi_4 = \psi_{4,5}.$$

In order to close the second channel $p = 2$ of waveguides, we consider the eigenfrequencies of the resonator below the second cutoff of the waveguide, i.e., $\nu_{m,n}/a < 1$ if to express side sizes of resonators $a$ via the width of waveguides $d = 1$. In particular, for the eigenstates under consideration the size of resonator is to exceed $a > 5$.

Let us, first, consider the case of sidewall connection of waveguides as shown in Fig. 1 (b). The coupling matrix elements of these eigenmodes with open channel $p = 1$ and closed channel $p = 2$ are collected in Tables II and III of Appendix. As a result, we obtain for the effective Hamiltonian in truncated Hilbert space of the eigenfunctions
where according to Tables 2 and 3 in Appendix

\[ \epsilon = |k_2|^2 (g(a)^2 - h(a)^2), \quad u = 2|k_2|g(a)h(a), \quad \gamma_1 = \frac{4k_1}{a^2}, \quad \gamma_2 = \frac{2k_1}{\pi^2} \sin^2 \left( \frac{2\pi}{a} \right) , \]

and according to Eq. (4) in Appendix \( k_1 = \nu = \nu_{16} = 5/a, |k_2| = 7/a \).

It is easy to fulfill Eq. (7) if the parameters \( u, \gamma_1, \gamma_2, \epsilon \) were independent. However, for the present case all constants depend on only the square size \( a \) as given in Eq. (6). As Comsol MultiPhysics shows in Fig. 2 (a) the condition (7) is not fulfilled for variation \( a \) that defines the solution as quasi FW BIC although with extremely large \( Q \)-factor around 600000. In view material losses of 3d printed acoustic resonators which restrict the \( Q \) factor by order of \( 10^3 \) [19] one can consider the solution in Fig. 2 (b) as the FW BIC. Remarkably, compared to usual FW BICs with sharp peak in the \( Q \)-factor for variation of the sizes of resonator the degenerate FW BICs maintain extremely high \( Q \)-factor in wide range of the size. That relieves experimentalists from the fabrication difficulties of proper choice of the resonators.

Fig. 2 (c) clearly shows that this BIC is composed of eigenmodes (1,6) and (5,4). The FW BIC composed of the eigenmodes (6,1) and (4,5) differs from this solution by 90° rotation.

In order to restore true degenerate FW BICs, we attach waveguides at the vertices of the square as shown in Fig. 1 (c) which also give equal couplings of eigenmodes of the square resonator with the continua of waveguides. However similar to side coupling of square with waveguides the evanescent modes remove the number degeneracy of modes 1, 6 and 5, 4. Numerics reveals two degenerate FW BICs with the frequency \( \nu = 0.832051 \) occur for variation of square size \( a = 6.8818 \), the first of which is shown in Fig. 3. One can see that FW BIC is superposed of two eigenmodes (1,6) and (5,4). The second degenerate FW BIC is obtained by 90° rotation and both BICs are classified according to the two-dimensional representation \( E \).
Figure 3. The same as in Fig. 2 but for vertex connection of resonator to waveguides as shown in Fig. 1 (c).

IV. DEGENERATE BICS IN EQUILATERAL TRIANGLE

Less obvious cases of the number degeneracy exist in the equilateral triangle with the eigenmodes and eigenfrequencies presented in Appendix. Similar to a square resonator, all eigenmodes are two-fold degenerate relative to \( m \leftrightarrow n \), giving rise to the FW BICs. However, there are also exceptional cases of the four-fold accidental number degeneracy, for example, for \( m = -11, n = -19 \) and \( m = -16, n = -17 \). Pressure profiles of these four eigenmodes with the lowest frequency \( \nu \approx 40 \) are shown in Appendix. Respectively, with opening of the triangular resonator with three attached waveguides these four eigenmodes are transformed into two superradiant modes and two FW BICs classified according to the two-dimensional irreducible representation \( E \) of group symmetry \( C_{3v} \). Similar to open square resonator, only vertex attachment of waveguides allows existence of degenerate BICs, as shown in Fig. 4. The resonant eigenmodes of the open triangular resonator can be found only numerically as distinctive from the case of the square open resonator.

The existence of degenerate resonances and, in particular, BICs opens a way of highly effective manipulating out power flows by small perturbations. In particular, that can be done by slight violation of symmetry of the system, say, by slight local pressure onto the resonator walls. In the present letter, we apply a local perturbation in the form of pencil of triangular or rectangular cross-sections at the center of the resonator which can be rotated by the angle \( \theta \) relative to the resonator. Fig. 5 demonstrates the striking result of the symmetry incompatibility of the resonator and perturbation. Fig. 5 (a) shows that perturbation which preserves the symmetry \( C_3 \) of the total system can not manipulate by acoustic flows irrespective of the angle \( \theta \). While the perturbation of the symmetry \( C_2 \) whose symmetry is not compatible with the symmetry of the total system drastically changes the output acoustic flows.

Figure 4. (a) Degenerate BICs with the frequency \( \nu_{\text{BIC}} = 0.8938 \) classified according to the two-dimensional irreducible representation \( E \) of group symmetry \( C_{3v} \). \( a = a_c = 14.1939 \). (b) The \( Q \)-factor dependence on side size of triangle \( a \).
Figure 5. Transmittance from input waveguide 1 into the output waveguides 2 and 3 in the triangular resonator with rotated (a) triangular pencil and (b) rectangular pencil. The frequency of input wave is tuned onto the frequency of degenerate FW BICs shown in Fig. 4.

V. CONCLUSIONS

We considered square and triangular resonators with the symmetries whose eigenmodes are classified to the irreducible representations of group symmetries $C_{4v}$ and $C_{3v}$ among of which there is the two-dimensional representation $E$. One can preserve the symmetries of open resonators owing to proper connection of waveguides to the resonator as sketched in Fig. 1. The Hamiltonian of closed resonator $H_B$ transforms into non-Hermitian effective Hamiltonian $H_{eff}$ after the Feshbach projection of the total system into space of eigenfunction of $H_B$ [31]. Respectively, the eigenmodes of $H_B$ with real eigenvalues are substituted by resonant modes, which are the eigenmodes of $H_{eff}$ with complex eigenvalues. Both Hamiltonians commute with the symmetry group transformations, and therefore one can expect that there are two-fold degenerate resonant states classified according to the two-dimensional irreducible representation $E$. However, analytical consideration in Section 3 explicitly shows these resonant states can not be true FW BICs because of the absence of interaction through the continuum of waveguides. In order to realize the degenerate FW BICs, we explore the unique property of square and triangular resonators of numeric accidental degeneracy of eigenmodes. As a result, we obtain the four-fold degeneracy of eigenmodes of $H_B$ which transform into the two-fold degenerate FW BICs and two superradiant resonances. Shaw has presented even more unique cases of number eight-fold degeneracy for $m = 5, n = 34; m = 10, n = 33; m = 13, n = 32; m = 24, n = 25$ with higher eigenfrequencies [22]. Respectively, we can expect the four-fold degenerate FW BICs or go above the first cutoff $\nu = 1$ towards the FW BICs embedded into a few continua of the next propagating bands of waveguides [38]. In general, degenerate BICs can also occur in open systems symmetrical, for example, relative to axial rotations. Then the Hilbert space of total system splits into a direct sum of spaces specified by the azimuthal index $m$. Respectively, the BICs, if they exist, are degenerate relative to $\pm m$ because of time-reversal symmetry. Examples of such degenerate BICs were reported in the periodical array of dielectric spheres [39] and disks [40]. However, these BICs are not degenerate in each Hilbert subspace specified by the azimuthal index $m$.

VI. APPENDIX

A. The eigenmodes of square resonator classified according to irreducible representations of square symmetry group $C_{4v}$

In the acoustic square resonator with the Neumann boundary conditions result in the following eigenmodes

$$\psi_{m,n}(x,y) = \sqrt{\frac{(2-\delta_{m,1})(2-\delta_{n,1})}{a}} \cos \left( \frac{\pi (m-1)x}{a} \right) \cos \left( \frac{\pi (n-1)y}{a} \right)$$

(8)
Table I. The characters of irreducible representations of group symmetry $C_{4v}$

| $C_{4v}$ | $C_2$ | $2C_4$ | $\sigma_x$, $\sigma_y$, $\sigma_{xy}$, $\sigma'_{xy}$ | Basic modes |
|----------|-------|-------|---------------------------------|-------------|
| $A_1$    | 1     | 1     | 1                               | $z$         |
| $A_2$    | 1     | 1     | -1                              | $J_z$       |
| $B_1$    | 1     | 1     | -1                              | $x^2 - y^2$ |
| $B_2$    | 1     | 1     | -1                              | $xy$        |
| $E$      | 2     | -2    | 0                               | $x, y$      |

with the eigenfrequencies

$$\nu^2_{m,n} = \frac{\omega^2_{m,n}}{\omega^2_0} = (m - 1)^2 + (n - 1)^2, \quad m, n = 1, 2, 3, \ldots,$$

where $\omega_0 = \pi s/a$, $s$ is the velocity of sound in air and $a$ is the size of square.

The group of symmetry $C_{4v}$ of square consists of rotations $C_4$ and $C_2$, two mirror reflections $\sigma_v$ along the square axis $x$ and $y$ and $\sigma'_v$ along the diagonals of square [41, 42]. Tabl. I shows the irreducible representations of each symmetry transformation and their characters [42]. Fig. 6 shows that the eigenmodes $\psi_{m,2n+1}$ and $\psi_{2n+1,m}$ belong to the two-dimensional irreducible representation $E$. However the eigenmodes $\psi_{2m,2m+2n}$ and $\psi_{2m+2n,2m}$ are classified by the reducible representations. As Fig. 7 shows only the linear combinations $\psi_{2m+2n,2m} + \psi_{2m,2m+2n}$ and $\psi_{2m+2n,2m} - \psi_{2m,2m+2n}$ are classified according to the irreducible one-dimensional representations $B_2$ and $A_2$ respectively. Similarly, the linear combinations $\psi_{2m+2n+2,2m+1} + \psi_{2m+1,2m+2n+2}$ and $\psi_{2m+2n+2,2m+1} - \psi_{2m+1,2m+2n+2}$ are classified according to the irreducible one-dimensional representations $B_1$ and $A_1$ respectively. In Fig. 8 we present

Figure 6. Symmetry group $C_{4v}$ transformations of the eigenmode $\psi_{23}(x, y)$.

an example of eigenmodes $\psi_{16}$ and $\psi_{54}$ degenerated accidentally [22, 23].

B. Effective non Hermitian Hamiltonian

The procedure of the Feshbach projection of the total Hilbert space of the total system closed resonator plus waveguides with Neumann boundary conditions onto Hilbert space of the eigenmodes of closed resonator is described
Figure 7. Symmetry group $C_{4v}$ transformations for the eigenmode $\psi_{24}(x,y)$.

\begin{align*}
\psi_{24} & \quad \tilde{C}_4 \psi_{24} = -\psi_{42} \\
\sigma_x \psi_{24} & \quad \sigma_y \psi_{24} = \psi_{42} \\
\sigma_y \psi_{24} & \quad \sigma_{xy} \psi_{24} = \psi_{42} \\
\sigma_{xy} \psi_{24} & \quad \psi_{42} + \psi_{24} \\
& \quad \psi_{42} - \psi_{24}
\end{align*}

in Ref. [36]. In application to the square acoustic resonator we have

$$\hat{H}_{\text{eff}} = \nu^2 \delta_{m,m'} \delta_{n,n'} - \sum_{p=1}^{\infty} \sum_{C} i k_p \tilde{W}_{C_p} \tilde{W}^\dagger_{C_p},$$

(10)

where $k_p$ are the propagating momenta of the $p$-th channel in waveguides of width $d$. In what follows all dimensions are measured in terms of $d$, i.e., $d = 1$. Then

$$\nu^2 = k_p^2 + (p - 1)^2, p = 1, 2, 3, \ldots$$

(11)

with

$$\phi_p(x, y) = \sqrt{2 - \delta_{p,1}} \cos(\pi(p - 1)y) e^{ik_p x}.$$  

(12)

The index $w$ in Eq. (10) sorts waveguides. For attachments of waveguides shown in Fig. 1 of the main text the coupling matrix $\tilde{W}_{C_p}$ can be evaluated analytically

$$W_{mn,p,C=1,3} = \int_{-1/2}^{1/2} \psi_{mn}(x = \mp a/2, y) \phi_p(y) dy,$$

$$W_{mn,p,C=2,4} = \int_{-1/2}^{1/2} \psi_{mn}(x, y = \pm a/2) \phi_p(x) dx.$$  

(13)

(14)

After integration in geometry shown in Fig. 1 (b) of the paper we obtain for the waveguide 1:

$$W_{mn,p,C=1} = \sqrt{2-\delta_{m,1}}(2-\delta_{n,1}) (2-\delta_{p,1}) \left\{ \sin\left[\frac{\pi}{a(n-1+a(p-1))}\right] \cos\left[\frac{\pi}{a(n-1+a(p-1))}\right] \right\}$$

$$+ \frac{\sin\left[\frac{\pi}{a(n-1+a(p-1))}\right] \cos\left[\frac{\pi}{a(n-1+a(p-1))}\right]}{n-1-a(p-1)},$$

(15)

Some particular coupling matrix elements relevant for the paper are collected in Tables II and III where according
The eigenfrequencies of equilateral triangle equal for the Neumann boundary conditions
\[ \nu_{mn}^2 = \omega_{mn}^2/\omega_0^2 = \frac{16}{27}(m^2 + n^2 - mn), \quad m, n = 0, \pm 1, \pm 2, \ldots, \]

The eigenmodes of equilateral triangular billiard

to Eq. (15) we denote

\begin{align*}
    b(a) &= \frac{4\sqrt{2}}{\pi(a^2 - 9)} \sin \frac{\pi a}{2} \cos \frac{3\pi}{2a}, \\
    c(a) &= \frac{4\sqrt{2}}{\pi(a^2 - 4)} \sin \frac{\pi a}{2} \cos \frac{\pi}{2a}, \\
    f(a) &= \frac{4\sqrt{2}}{\pi(a^2 - 4)} \cos \frac{\pi a}{2} \cos \frac{\pi}{2a}, \\
    g(a) &= \frac{4a}{\pi(a^2 - 25)} \cos \frac{5\pi}{2} \sin \frac{\pi a}{2}, \\
    h(a) &= \frac{4\sqrt{2}}{\pi(a^2 - 16)} \cos \frac{2\pi a}{a} \cos \frac{\pi a}{2}. \quad (16)
\end{align*}

C. Eigenmodes of equilateral triangular billiard

Figure 8. The number degenerated eigenfunctions of square.

Table II. Coupling matrix elements with the first open channel \( p = 1 \)

\begin{table}[h]
\centering
\begin{tabular}{c|cccc}
\hline
\text{modes } m, n \backslash C & 1 & 2 & 3 & 4 \\
\hline
2, 3 & \frac{1}{2} \sin \frac{\pi}{2} & 0 & \frac{1}{2} \sin \frac{\pi}{2} & 0 \\
3, 2 & 0 & \frac{1}{2} \sin \frac{\pi}{2} & 0 & -\frac{1}{2} \sin \frac{\pi}{2} \\
2, 4 & 0 & 0 & 0 & 0 \\
4, 2 & 0 & 0 & 0 & 0 \\
1, 6 & 0 & \frac{\sqrt{2}}{a} \sin \frac{2\pi}{a} & 0 & \frac{1}{\sqrt{2}} \sin \frac{\pi}{2} \\
5, 4 & 0 & \frac{1}{\sqrt{2}} \sin \frac{2\pi}{a} & 0 & \frac{1}{\sqrt{2}} \sin \frac{\pi}{2} \\
6, 1 & \frac{\sqrt{2}}{a} \sin \frac{2\pi}{a} & 0 & \frac{1}{\sqrt{2}} \sin \frac{\pi}{2} & 0 \\
4, 5 & \frac{1}{\sqrt{2}} \sin \frac{2\pi}{a} & 0 & \frac{1}{\sqrt{2}} \sin \frac{\pi}{2} & 0 \\
\hline
\end{tabular}
\end{table}
where $\omega_0$ is defined in Eq. (9) with the following conditions: $m + n$ is a multiple of 3 [43]. The eigenmodes are of the form

$$\psi_{mn} = f_{mn} + f_{m,m-n} + f_{-n,m-n} + f_{-n,-m} + f_{n,-m} + f_{n-m,n}$$

$$f_{mn}(x,y) = \exp(2\pi i/3)(nx + (2n - m)y/\sqrt{3}).$$

(18)

In Fig. 9 we show patterns of the eigenmodes which are four fold degenerate due to permutation symmetry $m \leftrightarrow n$ and accidental number degeneracy at $m = -16, n = -17$ and $m = 11, n = -19$. 

![Figure 9. Four fold degenerate eigenvalues of equilateral triangle with eigenfrequencies $\nu^2_{-11,-19} = \nu^2_{-16,-17} = 161.778.$](image)

There are also other cases of the four fold degeneracy however with higher eigenfrequencies with $m = -13, n = -23, m = -23, n = -13$ and $m = -17, n = -22, m = -22, n = -17$ with the eigenvalue $\nu^2_{-13,-23} = 226.312$. The group of symmetry $C_{3v}$ of equilateral triangle consists of rotations $C_3$, mirror reflections $\sigma_v$ along the diagonals of resonator [42]. Tabl. IV shows the irreducible representations of each symmetry transformation and their characters [42].

### Table III. Coupling matrix elements with the second closed channel $p = 2$

| modes $m, n \backslash C$ | 1 | 2 | 3 | 4 |
|--------------------------|---|---|---|---|
| 2, 3                     | 0 | -$f(a)$ | 0 | $f(a)$ |
| 3, 2                     | -$f(a)$ | 0 | $f(a)$ | 0 |
| 2, 4                     | -$b(a)$ | -$c(a)$ | $b(a)$ | $c(a)$ |
| 4, 2                     | $c(a)$ | $b(a)$ | -$c(a)$ | -$b(a)$ |
| 1, 6                     | $g(a)$ | 0 | $g(a)$ | 0 |
| 5, 4                     | -$h(a)$ | 0 | -$h(a)$ | 0 |
| 6, 1                     | 0 | $g(a)$ | 0 | $g(a)$ |
| 4, 5                     | 0 | $h(a)$ | 0 | $h(a)$ |
Table IV. The characters of irreducible representations of group symmetry $C_{3v}$.

| $C_{3v}$ | $2C_{3v}$ | $3z_{v}$ | basic modes |
|----------|-----------|----------|-------------|
| $A_{1}$  | 1         | 1        | $z$         |
| $A_{2}$  | 1         | -1       | $J_{z}$     |
| $E$      | 2         | 0        | $x, y$      |

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