Sum Uncertainty Relations Based on \((\alpha,\beta,\gamma)\) Weighted Wigner-Yanase-Dyson Skew Information

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Abstract
We introduce \((\alpha,\beta,\gamma)\) weighted Wigner-Yanase-Dyson ((\(\alpha,\beta,\gamma\)) WWYD) skew information and \((\alpha,\beta,\gamma)\) modified weighted Wigner-Yanase-Dyson ((\(\alpha,\beta,\gamma\)) MWWYD) skew information. We explore the sum uncertainty relations for arbitrary \(N\) mutually noncommutative observables based on \((\alpha,\beta,\gamma)\) WWYD skew information. A series of uncertainty inequalities are derived. We show by detailed example that our results cover and improve the previous ones based on the original Wigner-Yanase (WY) skew information. Finally, we establish new sum uncertainty relations in terms of the \((\alpha,\beta,\gamma)\) MWWYD skew information for arbitrary \(N\) quantum channels.

Keywords Uncertainty relation · \((\alpha, \beta, \gamma)\) WWYD skew information · \((\alpha, \beta, \gamma)\) MWWYD skew information · Quantum channel

1 Introduction
As one of the most essential features of the quantum world, the uncertainty principle has been widespread concerned since Heisenberg [1] proposed the notions of uncertainties in measuring non-commuting observables. For arbitrary two observables \(A\) and \(B\), the well-known Heisenberg-Robertson [2] uncertainty relation with respect to a quantum state \(|\psi\rangle\) says that,

\[\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|, \tag{1}\]

where \([A, B] = AB - BA\) and \(\Delta \Omega = \sqrt{|\langle \psi | \Omega^2 | \psi \rangle - \langle \psi | \Omega | \psi \rangle^2|}\) is the standard deviation of an observable \(\Omega\). Many different characterizations and quantifications of quantum
uncertainty have been proposed in terms of entropy [3–11], variance [12–15], under successive measurements [16–19], and with majorization techniques [9, 20–22].

The quantum uncertainty can also be characterized by skew information. The Wigner-Yanase (WY) information and Wigner-Yanase-Dyson (WYD) skew information associated to a quantum state $\rho$ and an observable $A$ have been defined in [23]. The WYD skew information has been further extended to the generalized Wigner-Yanase-Dyson (GWYD) skew information [24]. The relationship between WY skew information and the uncertainty relation has been originally established by Luo and Zhang [25], and various types of uncertainty relations based on the WY skew information, WYD skew information and GWYD skew information have been presented [26–40].

By considering state-channel interaction, in [41] Luo and Sun defined a quantity $I_{\rho}(\Phi)$ and its dual one $J_{\rho}(\Phi)$, and explored the complementarity relation between them. Wu, Zhang and Fei introduced the non-Hermitian extension of the GWYD skew information and generalized the complementarity relation to a more general case in [42, 43].

On the other hand, another generalization of the WYD skew information, the weighted Wigner-Yanase-Dyson (WWYD) skew information, has been introduced in [44]. As its non-Hermitian extension, the modified weighted Wigner-Yanase-Dyson (MWWYD) skew information has been defined and investigated in [45]. Recently, by using the convex combination, instead of the arithmetic mean of $\rho^\alpha$ and $\rho^{1-\alpha}$, the two-parameter extension of the Wigner-Yanase skew information has been formulated [46].

Recently, the sum uncertainty relations based on the variance and WY skew information have attracted considerable attention [47–50]. In [47, 48] Chen and Fei proposed some uncertainty inequalities in terms of the sum of variances, standard deviations and the WY skew information for arbitrary $N$ mutually noncommutative observables, respectively. After that, Zhang, Gao and Yan [49] established a tighter uncertainty relation via WY skew information for arbitrary $N$ mutually noncommutative observables, which extend the results in [48]. Zhang and Fei [50] further improved the results in [49] and proposed new tighter bounds than the existing ones. Cai [51] generalized the sum uncertainty relations for WY skew information introduced in [48] to an arbitrary metric-adjusted skew information version. Ren, Li, Ye and Li [52] proposed tighter sum uncertainty relations than the ones in [51].

In [53] Fu, Sun and Luo established the uncertainty relations for two quantum channels based on the WY skew information for arbitrary operators. Afterwards, Zhang, Gao and Yan [49] generalized the uncertainty relations for two quantum channels to arbitrary $N$ quantum channels. Zhang, Wu and Fei [54] further generalized the results in [49] and proposed new bounds which are tighter than the existing ones. Cai [51] confirmed that the results in [53] also hold for all metric-adjusted skew information.

The remainder of this paper is structured as follows. In Section 2, we recall some basic concepts and propose the definitions of $(\alpha, \beta, \gamma)$ weighted Wigner-Yanase-Dyson ((\alpha, \beta, \gamma) WWYD) skew information and $(\alpha, \beta, \gamma)$ modified weighted Wigner-Yanase-Dyson ((\alpha, \beta, \gamma) MWWYD) skew information. In Section 3, we present uncertainty inequalities for arbitrary $N$ mutually noncommutative observables in terms of the $(\alpha, \beta, \gamma)$ WWYD skew information. Especially, we show that when $\alpha = \beta = \frac{1}{2}$, i.e., the $(\alpha, \beta, \gamma)$ WWYD skew information reduce to the WY skew information, the lower bounds of our inequalities improve the existing ones by a detailed example. In Section 4, we explore the $(\alpha, \beta, \gamma)$ MWWYD skew information-based sum uncertainty relations for quantum channels. Some concluding remarks are given in Section 5.
2 (\(\alpha, \beta, \gamma\)) WWYD Skew Information and (\(\alpha, \beta, \gamma\)) MWWYD Skew Information

Let \(\mathcal{H}\) be a \(d\)-dimensional Hilbert space. Denote by \(B(\mathcal{H})\), \(S(\mathcal{H})\) and \(D(\mathcal{H})\) the set of all bounded linear operators, Hermitian operators and density operators (positive operators with trace 1) on \(\mathcal{H}\), respectively. Mathematically, a quantum state and a quantum channel are represented by a density operator and a completely positive trace-preserving map, respectively.

For a quantum state \(\rho \in D(\mathcal{H})\) and an observable \(A \in S(\mathcal{H})\), the Wigner-Yanase (WY) skew information [23] is defined by

\[
I_\rho(A) := -\frac{1}{2} \text{Tr}((\sqrt{\rho}, A)^2) = \frac{1}{2} \|\sqrt{\rho} A\|^2, \tag{2}
\]

where \(\|\cdot\|\) denotes the Hilbert Schmidt norm, \(\|T\| = \sqrt{\text{Tr} T^* T}\). \(I_\rho(A)\) is generalized by Dyson to

\[
I^\alpha_\rho(A) := -\frac{1}{2} \text{Tr}((\rho^\alpha, A)[\rho^{1-\alpha}, A]), \quad 0 \leq \alpha \leq 1,
\]

which is now called the Wigner-Yanase-Dyson (WYD) skew information [23]. \(I^\alpha_\rho(A)\) is further generalized to [24]

\[
I^{\alpha,\beta}_\rho(A) := -\frac{1}{2} \text{Tr}((\rho^\alpha, A)[\rho^{\beta}, A][\rho^{1-\alpha-\beta}], \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, \tag{4}
\]

which is termed as generalized Wigner-Yanase-Dyson (GWYD) skew information.

Another generalization of the WYD skew information is given in [44],

\[
K^{\alpha}_\rho(A) := -\frac{1}{2} \text{Tr}\left(\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A\right]^2\right) = \frac{1}{2} \left\|\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A\right]\right\|^2, \quad 0 \leq \alpha \leq 1, \tag{5}
\]

which is called the weighted Wigner-Yanase-Dyson (WWYD) skew information. The authors in [45] proposed the modified weighted Wigner-Yanase-Dyson (MWWYD) skew information, which is the non-Hermitian extension of the WWYD skew information.

By replacing the arithmetic mean of \(\rho^\alpha\) and \(\rho^{1-\alpha}\) with their convex combination, the two-parameter extension of Wigner-Yanase skew information has been introduced,

\[
K^{\alpha,\gamma}_{\rho,\gamma}(A) := -\frac{1}{2} \text{Tr}\left(((1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, A)^2\right)
= \frac{1}{2} \left\|((1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, A]\right\|^2, \quad 0 \leq \alpha \leq 1, 0 \leq \gamma \leq 1. \tag{6}
\]

For convenience, we call it the \((\alpha, \gamma)\) weighted Wigner-Yanase-Dyson \((\alpha, \gamma)\) WWYD skew information in this paper. Note that Eq. (6) reduces to Eq. (5) and Eq. (2) when \(\gamma = \frac{1}{2}\) and \(\alpha = \frac{1}{2}\), respectively.

For a quantum state \(\rho \in D(\mathcal{H})\) and an observable \(A \in S(\mathcal{H})\), we define the \((\alpha, \beta, \gamma)\) weighted Wigner-Yanase-Dyson \((\alpha, \beta, \gamma)\) WWYD skew information as

\[
K^{\alpha,\beta,\gamma}_{\rho,\gamma}(A) := -\frac{1}{2} \text{Tr}(((1-\gamma)\rho^\alpha + \gamma \rho^{\beta}, A)(\rho^{1-\alpha-\beta})^2
= \frac{1}{2} \left\|((1-\gamma)\rho^\alpha + \gamma \rho^{\beta}, A]\right\|^2, \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, 0 \leq \gamma \leq 1. \tag{7}
\]

Note that Eq. (7) reduces to Eq. (6) when \(\beta = 1 - \alpha\).
We also define the \((\alpha, \beta, \gamma)\) modified weighted Wigner-Yanase-Dyson \((\alpha, \beta, \gamma)\) MWYD skew information with respect to a quantum state \(\rho \in \mathcal{D}(\mathcal{H})\) and an arbitrary operator \(E \in \mathcal{B}(\mathcal{H})\) (not necessarily Hermitian),

\[
K_{\rho,\gamma}^{\alpha,\beta}(\rho) = -\frac{1}{2} \text{Tr} \left[ \left( 1 - \gamma \right) \rho^{\alpha} + \gamma \rho^{\beta} + \frac{1}{1-\gamma} \left( 1 - \gamma \right) \rho^{1-\alpha-\beta} \right] \left( 1 - \gamma \right) \rho \rho^{1-\alpha-\beta} - \frac{1}{\gamma} \left[ \left( 1 - \gamma \right) \rho^{\alpha} + \gamma \rho^{\beta} \right]^{1-\alpha-\beta}, \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, \quad 0 \leq \gamma \leq 1,
\]

which is the non-Hermitian extension of the \((\alpha, \beta, \gamma)\) MWYD skew information. Note that Eq. (8) reduces to Eq. (10) in [43] when \(\gamma = \frac{1}{2}\).

Following the idea in [41], we further define the \((\alpha, \beta, \gamma)\) MWYD skew information of \(\rho\) with respect to a channel \(\Phi\) as

\[
K_{\rho,\gamma}^{\alpha,\beta}(\Phi) = \sum_{i=1}^{n} K_{\rho,\gamma}^{\alpha,\beta}(E_i),
\]

where \(\alpha, \beta \geq 0, \alpha + \beta \leq 1, \leq \gamma \leq 1\), and \(E_i(i = 1, 2, \cdots, n)\) are Kraus operators of the channel \(\Phi\), i.e., \(\Phi(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger\).

3 Sum Uncertainty Relations for Arbitrary \(N\) Mutually Noncommutative Observables in Terms of the \((\alpha, \beta, \gamma)\) MWYD Skew Information

We now provide several sum uncertainty relations in terms of the \((\alpha, \beta, \gamma)\) MWYD skew information for arbitrary \(N\) mutually noncommutative observables.

**Theorem 1** For arbitrary \(N\) mutually noncommutative observables \(A_1, A_2, \cdots, A_N (N \geq 2)\), we have

\[
\sum_{i=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_i) \geq \max_{1 \leq i < j \leq N} \left\{ \sum_{1 \leq i < j \leq N} K_{\rho,\gamma}^{\alpha,\beta}(A_i + A_j), \sum_{1 \leq i < j \leq N} K_{\rho,\gamma}^{\alpha,\beta}(A_i - A_j) \right\},
\]

where \(\alpha, \beta \geq 0, \alpha + \beta \leq 1, \leq \gamma \leq 1\).

**Proof** By using the following equality,

\[
2(N - 1) \sum_{i=1}^{N} \|u_i\|^2 = \sum_{1 \leq i < j \leq N} \|u_i + u_j\|^2 + \sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2,
\]

we have

\[
\sum_{i=1}^{N} \|u_i\|^2 \geq \frac{1}{2(N-1)} \sum_{1 \leq i < j \leq N} \|u_i + u_j\|^2,
\]

and

\[
\sum_{i=1}^{N} \|u_i\|^2 \geq \frac{1}{2(N-1)} \sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2.
\]

Therefore,
\[
\sum_{i=1}^{N} \|u_i\|^2 \geq \max \left\{ \frac{1}{2(N-1)} \left\{ \sum_{1 \leq i < j \leq N} \|u_i + u_j\|^2, \sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2 \right\} \right\}.
\]

Setting \( u_i = \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, A_i \right] \) and \( u_j = \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, A_j \right] \), we get (10).
\[\square\]

In particular, for \( N = 2 \) from Theorem 1 we have the following Corollary 1.

**Corollary 1** For arbitrary two noncommutative observables \( A \) and \( B \), we have
\[
K_{\rho,\gamma}^{\alpha,\beta}(A) + K_{\rho,\gamma}^{\alpha,\beta}(B) \geq \max \frac{1}{2} \{ K_{\rho,\gamma}^{\alpha,\beta}(A + B), K_{\rho,\gamma}^{\alpha,\beta}(A - B) \},
\]
where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \).

Note that (11) of Corollary 1 reduces to the formula (3) of Theorem 1 in [48] when \( \alpha = \beta = \frac{1}{2} \).

**Theorem 2** For arbitrary \( N \) mutually noncommutative observables \( A_1, A_2, \cdots, A_N (N \geq 2) \), we have
\[
\sum_{i=1}^{N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i)} \geq \sqrt{K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N} A_i \right)}, \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1.
\]

and
\[
\sum_{i=1}^{N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i)} \geq \sqrt{K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N-1} A_i - A_N \right)}, \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1.
\]

**Proof** By using the norm inequality, we obtain
\[
\sqrt{K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N} A_i \right)} = \frac{1}{\sqrt{2}} \left\| \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, \sum_{i=1}^{N} A_i \right] \right\|
\]
\[
\leq \frac{1}{\sqrt{2}} \sum_{i=1}^{N} \left\| \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, A_i \right] \right\|
\]
\[
= \sum_{i=1}^{N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i)}, \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1,
\]
and
\[
\sqrt{K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N-1} A_i - A_N \right)} = \frac{1}{\sqrt{2}} \left\| \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, \sum_{i=1}^{N-1} A_i - A_N \right] \right\|
\]
\[
\leq \frac{1}{\sqrt{2}} \sum_{i=1}^{N} \left\| \rho^{\frac{1-a-\beta}{2}} \left[ (1-\gamma)\rho^a + \gamma \rho^b, A_i \right] \right\|
\]
\[
= \sum_{i=1}^{N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i)}, \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1.
\]
Setting $N = 2$ in Theorem 2, we have the following Corollary 2.

Corollary 2 For arbitrary two noncommutative observables $A$ and $B$, we have

$$\sqrt{K_{p,\gamma}^{\alpha,\beta}(A) + \sqrt{K_{p,\gamma}^{\alpha,\beta}(B)} \geq \max \left\{ \sqrt{K_{p,\gamma}^{\alpha,\beta}(A + B)}, \sqrt{K_{p,\gamma}^{\alpha,\beta}(A - B)} \right\},$$

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$.

Note that (14) of Corollary 2 reduces to (5) of Theorem 2 in [48] when $\alpha = \beta = \frac{1}{2}$.

Theorem 3 For arbitrary $N$ mutually noncommutative observables $A_1, A_2, \cdots, A_N (N > 2)$, we have

$$\sum_{i=1}^{N} K_{p,\gamma}^{\alpha,\beta}(A_i) \geq \frac{1}{N-2} \left[ \sum_{1 \leq i < j \leq N} K_{p,\gamma}^{\alpha,\beta}(A_i + A_j) - \frac{1}{(N-1)^2} \left( \sum_{1 \leq i < j \leq N} \sqrt{K_{p,\gamma}^{\alpha,\beta}(A_i + A_j)} \right)^2 \right],$$

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$.

Proof Employing the following inequality [47],

$$\sum_{i=1}^{N} \| u_i \|^2 \geq \frac{1}{N-2} \left[ \sum_{1 \leq i < j \leq N} \| u_i + u_j \|^2 - \frac{1}{(N-1)^2} \left( \sum_{1 \leq i < j \leq N} \| u_i + u_j \| \right)^2 \right] \geq \frac{1}{2(N-1)} \sum_{1 \leq i < j \leq N} \| u_i + u_j \|^2,$$

and setting $u_i = \rho^{\frac{\gamma - \beta}{2}} [(1 - \gamma) \rho^\alpha + \gamma \rho^\beta, A_i]$ and $u_j = \rho^{\frac{\gamma - \beta}{2}} [(1 - \gamma) \rho^\alpha + \gamma \rho^\beta, A_j]$, we obtain (15). □

Note that (15) of Theorem 3 reduces to (12) of Theorem 4 in [48] when $\alpha = \beta = \frac{1}{2}$.

Theorem 4 For arbitrary $N$ mutually noncommutative observables $A_1, A_2, \cdots, A_N (N > 2)$, we have

$$\sum_{i=1}^{N} \sqrt{K_{p,\gamma}^{\alpha,\beta}(A_i)} \geq \frac{1}{N-2} \left[ \sum_{1 \leq i < j \leq N} \sqrt{K_{p,\gamma}^{\alpha,\beta}(A_i + A_j)} - \sqrt{K_{p,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N} A_i \right) \right],$$

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$.

Proof By using the following inequality [47, 55, 56],
\[
\sum_{i=1}^{N} \|u_i\| \geq \frac{1}{N-2} \left( \sum_{1 \leq i < j \leq N} \|u_i + u_j\| - \left\| \sum_{i=1}^{N} u_i \right\| \right) \\
\geq \left\| \sum_{i=1}^{N} u_i \right\|,
\]

with \( u_i = \rho^{1-\alpha-\beta} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_i \right] \) and \( u_j = \rho^{1-\alpha-\beta} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_j \right] \), we obtain (16). \( \square \)

Note that (16) of Theorem 4 reduces to (14) of Theorem 5 in [48] when \( \alpha = \beta = \frac{1}{2} \). Moreover, from the proof of Theorem 4, it can be seen that the right hand side of (16) is tighter than the right hand side of (12).

**Theorem 5** For arbitrary \( N \) mutually noncommutative observables \( A_1, A_2, \ldots, A_N \) \((N \geq 2)\), we have

\[
\sum_{i=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_i) \geq \frac{1}{N} K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{i=1}^{N} A_i \right) + \frac{2}{N^2(N-1)} \left( \sum_{1 \leq i < j \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i - A_j)} \right)^2,
\]

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \).

**Proof** According to the lemma in [49], we have

\[
\frac{2}{N^2(N-1)} \left( \sum_{1 \leq i < j \leq N} \|u_i - u_j\| \right)^2 \leq \frac{1}{N} \sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2 \\
= \sum_{i=1}^{N} \|u_i\|^2 - \frac{1}{N} \left\| \sum_{i=1}^{N} u_i \right\|^2,
\]

that is,

\[
\sum_{i=1}^{N} \|u_i\|^2 \geq \frac{1}{N} \left\| \sum_{i=1}^{N} u_i \right\|^2 + \frac{2}{N^2(N-1)} \left( \sum_{1 \leq i < j \leq N} \|u_i - u_j\| \right)^2.
\]

By substituting \( u_i \) and \( u_j \) with \( \rho^{1-\alpha-\beta} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_i \right] \) and \( \rho^{1-\alpha-\beta} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_j \right] \), respectively, we obtain (17). \( \square \)

Note that (17) of Theorem 5 reduces to (12) of Theorem 1 in [49] when \( \alpha = \beta = \frac{1}{2} \).

**Theorem 6** For arbitrary \( N \) mutually noncommutative observables \( A_1, A_2, \ldots, A_N \) \((N \geq 2)\), we have
\[
\sum_{i=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_i) \geq \frac{1}{2(N-1)} \left[ \frac{2}{N(N-1)} \left( \sum_{1 \leq i < j \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i + A_j)} \right)^2 + \sum_{1 \leq i < j \leq N} K_{\rho,\gamma}^{\alpha,\beta}(A_i - A_j) \right], \ \alpha, \beta \geq 0, \ \alpha + \beta \leq 1, \ 0 \leq \gamma \leq 1.
\]

(18)

and

\[
\sum_{i=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_i) \geq \frac{1}{2(N-1)} \left[ \frac{2}{N(N-1)} \left( \sum_{1 \leq i < j \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i + A_j)} \right)^2 + \sum_{1 \leq i < j \leq N} K_{\rho,\gamma}^{\alpha,\beta}(A_i + A_j) \right], \ \alpha, \beta \geq 0, \ \alpha + \beta \leq 1, \ 0 \leq \gamma \leq 1.
\]

(19)

**Proof** By using the following equality,

\[
2(N - 1) \sum_{i=1}^{N} \|u_i\|^2 = \sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2 + \sum_{1 \leq i < j \leq N} \|u_i + u_j\|^2
\]

and the Cauchy-Schwarz inequality, we obtain

\[
\sum_{1 \leq i < j \leq N} \|u_i + u_j\|^2 \geq \frac{2}{N(N-1)} \left( \sum_{1 \leq i < j \leq N} \|u_i + u_j\| \right)^2,
\]

and

\[
\sum_{1 \leq i < j \leq N} \|u_i - u_j\|^2 \geq \frac{2}{N(N-1)} \left( \sum_{1 \leq i < j \leq N} \|u_i - u_j\| \right)^2,
\]

respectively. Therefore, we have

\[
\sum_{i=1}^{N} \|u_i\|^2 \geq \frac{1}{2(N-1)} \left[ \frac{2}{N(N-1)} \left( \sum_{1 \leq i < j \leq N} \|u_i \pm u_j\| \right)^2 + \sum_{1 \leq i < j \leq N} \|u_i \mp u_j\|^2 \right].
\]

The inequalities (18) and (19) follow by replacing \(u_i\) and \(u_j\) with \(\rho^{1-\alpha} = [(1 - \gamma)\rho^a + \gamma\rho^b, A_i] \) and \(\rho^{1-\beta} = [(1 - \gamma)\rho^a + \gamma\rho^b, A_j] \), respectively. \(\Box\)

As a special case, when \(\alpha = \beta = \frac{1}{2}\), (18) and (19) of Theorem 6 reduce to (12) and (13) of Theorem 2 in [50], respectively. Note also that Theorem 5 and Theorem 6 are identical when \(N = 2\).

**Theorem 7** For arbitrary \(N\) mutually noncommutative observables \(A_1, A_2, \ldots, A_N (N \geq 2)\) and a quantum state \(\rho\), let \(G\) be an \(N \times N\) matrix with entries \(G_{ij} = \text{Tr}(X_j X_i)\), where

\[
X_j = i\rho^{\frac{1-\alpha}{2}} [((1 - \gamma)\rho^a + \gamma\rho^b, A_i)]/\|\rho^{\frac{1-\alpha}{2}} [((1 - \gamma)\rho^a + \gamma\rho^b, A_i)]\|, i = \sqrt{-1}.\]

We have
\[
\sum_{j=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_j) \geq \frac{1}{\lambda_{\text{max}}(G)} \sum_{j=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(A_j), \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, \quad 0 \leq \gamma \leq 1,
\]

where \(\lambda_{\text{max}}(G)\) denotes the maximal eigenvalue of \(G\).

**Proof** It is obvious that \(G\) is a positive semi-definite matrix. Noting that

\[
K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{j=1}^{N} A_j \right) = -\frac{1}{2} \text{Tr} \left[ \left( (1 - \gamma)\rho^{\alpha} + \gamma \rho^{\beta} \sum_{j=1}^{N} A_j \right)^{\frac{1}{2} - \alpha - \beta} \right]
\]

we prove that (20) holds. \(\Box\)

In particular, the Theorem 6 in [48] is a special case of our Theorem 7 when \(\alpha = \beta = \frac{1}{2}\).

When \(\alpha = \beta = \frac{1}{2}\), the \((\alpha, \beta, \gamma)\) WWYD skew information reduces to the WY skew information. We next compare our uncertainty relations with the existing ones. For \(\alpha = \beta = \frac{1}{2}\), we denote by \(LB_0, LB_1, LB_2\) and \(LB_3\) the right hand sides of (15), (17), (18) and (19), respectively.

**Example 1** Given a qubit state \(\rho = \frac{1}{2} (1 + r \cdot \sigma)\), where \(r = (x,y,z)\) is the Bloch vector satisfying \(|r| \leq 1\), \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) with \(j \in \{1,2,3\}\) the Pauli matrices, and \(r \cdot \sigma = \sum_{j=1}^{3} r_j \sigma_j\). The eigenvalues of \(\rho\) are \(\lambda_{1,2} = (1 \mp \sqrt{t})/2\), where \(t = |r|^2\).

The sum of the skew information of three Pauli operators is given by

\[
K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_1) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_2) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_3) = 2(1 - \sqrt{1 - t}). \quad (21)
\]

From (18) and (15) we have, respectively,

\[
K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_1) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_2) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_3) \geq (1 - \sqrt{1 - t}) \left( 1 + \frac{xy + xz + yz}{2t} \right) + \frac{1}{12} \beta^2 \quad (22)
\]

and

\[
K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_1) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_2) + K_{\rho,\gamma}^{\frac{1}{2},\frac{1}{2}}(\sigma_3) \geq (1 - \sqrt{1 - t}) \left( 4 - \frac{2(xy + xz + yz)}{t} \right) - \frac{1}{4} \beta^2, \quad (23)
\]

where
\[
\alpha = \sqrt{1 - \sqrt{1 - t} \left( \sqrt{1 + \frac{z^2 + 2xy}{t}} + \sqrt{1 + \frac{y^2 + 2xz}{t}} + \sqrt{1 + \frac{x^2 + 2yz}{t}} \right)}. \tag{24}
\]

(19) and (17) give rise to
\[
K_{p,x}^{\frac{3}{5}}(\sigma_1) + K_{p,y}^{\frac{3}{5}}(\sigma_2) + K_{p,z}^{\frac{3}{5}}(\sigma_3) \geq (1 - \sqrt{1 - t}) \left( 1 - \frac{xy + xz + yz}{2t} \right) + \frac{1}{12} \alpha^2 \tag{25}
\]
and
\[
K_{p,x}^{\frac{1}{3}}(\sigma_1) + K_{p,y}^{\frac{1}{3}}(\sigma_2) + K_{p,z}^{\frac{1}{3}}(\sigma_3) \geq \frac{2}{3} (1 - \sqrt{1 - t}) \left( 1 - \frac{xy + xz + yz}{t} \right) + \frac{1}{9} \alpha^2, \tag{26}
\]
respectively, where
\[
\beta = \sqrt{1 - \sqrt{1 - t} \left( \sqrt{1 + \frac{z^2 - 2xy}{t}} + \sqrt{1 + \frac{y^2 - 2xz}{t}} + \sqrt{1 + \frac{x^2 - 2yz}{t}} \right)}. \tag{27}
\]

Comparing the lower bound \(LB_2\) (\(LB_3\)) on the right hand of the inequality (18) ((19)) with the bound \(LB_0\) (\(LB_1\)) on the right hand of inequality (15) ((17)), we obtain \(LB_2 - LB_0 = (1 - \sqrt{1 - t})\gamma\) and \(LB_3 - LB_1 = (1 - \sqrt{1 - t})\gamma_1\), respectively, where
\[
\gamma = -3 + \frac{5(xy + xz + yz)}{2t} + \frac{1}{3} \left( \sqrt{1 + \frac{z^2 - 2xy}{t}} + \sqrt{1 + \frac{y^2 - 2xz}{t}} + \sqrt{1 + \frac{x^2 - 2yz}{t}} \right)^2 \tag{28}
\]
and
\[
\gamma_1 = \frac{1}{3} + \frac{xy + xz + yz}{6t} - \frac{1}{36} \left( \sqrt{1 + \frac{z^2 + 2xy}{t}} + \sqrt{1 + \frac{y^2 + 2xz}{t}} + \sqrt{1 + \frac{x^2 + 2yz}{t}} \right)^2. \tag{29}
\]

Note that \(1 - \sqrt{1 - t} > 0\). Denote \(x = \sqrt{t} \cos \theta\), \(y = \sqrt{t} \sin \theta\), \(z = \sqrt{\pi} \sin \varphi\) and \(z = \sqrt{\pi} \cos \theta\) with \(\theta \in [0, \pi]\) and \(\varphi \in [0, 2\pi]\). We have
\[
\gamma = -3 + \frac{5}{2} (\sin^2 \theta \sin \varphi \cos \varphi + \sin \theta \cos \theta \cos \varphi + \sin \theta \cos \varphi \sin \varphi) + \frac{1}{3} \left( \sqrt{1 + \cos^2 \theta - 2 \sin^2 \theta \sin \varphi \cos \varphi} + \sqrt{1 + \sin^2 \theta \sin^2 \varphi - 2 \sin \theta \cos \theta \cos \varphi} + \sqrt{1 + \sin^2 \theta \cos^2 \varphi - 2 \sin \theta \cos \theta \sin \varphi} \right)^2 > 0, \tag{30}
\]
and
\[
\gamma_1 = \frac{1}{3} + \frac{1}{6} (\sin^2 \theta \sin \varphi \cos \varphi + \sin \theta \cos \theta \cos \varphi + \sin \theta \cos \varphi \sin \varphi) - \frac{1}{36} \left( \sqrt{1 + \cos^2 \theta + 2 \sin^2 \theta \sin \varphi \cos \varphi} + \sqrt{1 + \sin^2 \theta \sin^2 \varphi + 2 \sin \theta \cos \theta \cos \varphi} + \sqrt{1 + \sin^2 \theta \cos^2 \varphi + 2 \sin \theta \cos \theta \sin \varphi} \right)^2 > 0, \tag{31}
\]
see Fig. 1.
Figure 1 shows that our lower bound $LB_3$ ($LB_4$) is larger than the lower bound given in [48] ([49]) except $t = 0$ for the case of spin-$\frac{1}{2}$. In [50] the authors illustrated by an example that their lower bounds are better than the ones given in [49] for the case of a special qubit state with Bloch vector $r = (\frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta, 0)$. Here, in our example we have considered the general case of arbitrary qubit states.

Now, we give two theorems in which the lower bounds consist of the sum uncertainty relations of different size $k$.

**Theorem 8** For $N$ mutually noncommutative observables $A_1, \ldots, A_N$, $2 \leq k < N$, we have

$$\sum_{i=1}^{N} K_{\rho,\gamma}(A_i) \geq \left( \frac{N - 2}{k - 1} \right)^{-1} \left[ \sum_{1 \leq i_1 < \cdots < i_k \leq N} K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{j=1}^{k} A_{i_j} \right) - \left( \frac{N - 2}{k - 2} \right) \left( \frac{N - 1}{k - 1} \right)^{-2} \sum_{1 \leq i_1 < \cdots < i_k \leq N} K_{\rho,\gamma}^{\alpha,\beta} \left( \sum_{j=1}^{k} A_{i_j} \right) \right] \sum_{1 \leq i_1 < \cdots < i_k \leq N} \left( \sum_{j=1}^{k} A_{i_j} \right), \quad \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1. \tag{32}$$

**Proof** It is a direct result of the following inner product inequality proved in the Theorem 1 in [51],

$$\sum_{i=1}^{N} \|u_i\|^2 \geq \left( \frac{N - 2}{k - 1} \right)^{-1} \left[ \sum_{1 \leq i_1 < \cdots < i_k \leq N} \|u_{i_1} + \cdots + u_{i_k}\|^2 - \left( \frac{N - 2}{k - 2} \right) \left( \frac{N - 1}{k - 1} \right)^{-2} \sum_{1 \leq i_1 < \cdots < i_k \leq N} \|u_{i_1} + \cdots + u_{i_k}\|^2 \right].$$

By substituting $u_i$ and $u_j$, $(1 \leq j \leq k)$ with $\rho^{\frac{1+\gamma-\beta}{2}} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_i \right]$ and $\rho^{\frac{1+\gamma-\beta}{2}} \left[ (1 - \gamma)\rho^\alpha + \gamma \rho^\beta, A_{i_j} \right]$, respectively, we obtain (32). \(\square\)

We observe that Theorem 3 is a special case of Theorem 8 when $k = 2$. Similarly, we can give a variation of the Theorem 4 for the sum uncertainty relation of $\sqrt{K_{\rho,\gamma}^{\alpha,\beta}(A_i)}$.

**Theorem 9** For $N$ mutually noncommutative observables $A_1, \ldots, A_N$, we have...
It is a direct result of the following inner product inequality, which has been proved in the Theorem 2 in [51]. Replacing \( u_i \) and \( u_{ij} \) (\( 1 \leq j \leq N - 1 \)) by \( \frac{1}{\|u_1\|} - \frac{1}{\|u_2\| - \beta} \sqrt{\left( \frac{1}{\|u_1\| \|u_2\|} + \frac{1}{\|u_2\| \|u_1\|} \beta \right)} \), \( A_i \) and \( A_{ij} \), respectively, we get (33) immediately.

**Proof**

It is a direct result of the following inner product inequality,

\[
\sum_{i=1}^{N} \|u_i\| \geq \sum_{1 \leq i_1 < \cdots < i_{N-1} \leq N} \|u_{i_1} + \cdots + u_{i_{N-1}}\| - (N - 2) \left\| \sum_{i=1}^{N} u_i \right\|,
\]

which has been proved in the Theorem 2 in [51]. Replacing \( u_i \) and \( u_{ij} \) (\( 1 \leq j \leq N - 1 \)) by \( \rho^{\frac{1+\delta}{2}} [(1 - \gamma)\rho^a + \gamma \rho^b, A_i] \) and \( \rho^{\frac{1+\delta}{2}} [(1 - \gamma)\rho^a + \gamma \rho^b, A_j] \), respectively, we get (33) immediately.

From the above results we have, for arbitrary \( N \) mutually noncommutative observables \( A_1, A_2, \cdots, A_N \),

\[
\sum_{i=1}^{N} \sqrt{K^{a,b}_{\rho,\gamma}(A_i)} \geq \max\{\text{thm1, thm3, thm5, thm6, thm7, thm8}\};
\]

\[
\sum_{i=1}^{N} \sqrt{K^{a,b}_{\rho,\gamma}(A_i)} \geq \max\{\text{thm2, thm4, thm9}\},
\]

where thm 1, thm 3, thm 5, thm 6, thm 7, thm 8, thm 2, thm 4 and thm 9 stand for the lower bounds in Theorem 1, Theorem 3, Theorem 5, Theorem 6, Theorem 7, Theorem 8, Theorem 2, Theorem 4 and Theorem 9, respectively.

### 4 Sum Uncertainty Relations for Quantum Channels in Terms of \((a, b, \gamma)\) MWWYD Skew Information

In this section, we explore the uncertainty relations for arbitrary \( N \) quantum channels in terms of \((a, b, \gamma)\) MWWYD skew information. By using the same techniques, it can be seen that the results in Section 2 also hold if the observables \( A_i \in \mathcal{S}(\mathcal{H}) \) (\( i = 1, 2, \cdots, N \)) are replaced by \( E_i \in \mathcal{B}(\mathcal{H}) \) (\( i = 1, 2, \cdots, N \)) (which are not necessarily Hermitian). Therefore, the results in this section are easy consequences by imitating the proofs of Theorem 1, Theorem 3, Theorem 4, Theorem 5 and Theorem 6 in Section 3, Theorem 2 in [53] and the definition of \((a, b, \gamma)\) MWWYD skew information with respect to quantum channels in Eq. (9). Hence we only sketch the proof of Theorem 11 and omit the proofs of the rest theorems.

**Theorem 10** Let \( \Phi_1, \cdots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_i(\rho) = \sum_{t=1}^{n} E_i^t \rho (E_i^t)^\dagger, \ t = 1, 2, \cdots, N \ (N > 2) \), we have
\[
\sum_{i=1}^{N} K_{\rho,\tau}^{\alpha,\beta}(\Phi_i) \geq \max_{\pi,\sigma \in S_n} \frac{1}{2(N-1)} \left\{ \sum_{1 \leq i < j \leq N} \left( K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau} \pm E_{\sigma(j)}^{\tau}) \right) \right\},
\]

(34)

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \). \( S_n \) is the n-element permutation group and \( \pi, \sigma \in S_n \) are arbitrary n-element permutations.

**Proof** By using the inequality in the proof of Theorem 1, we obtain

\[
\sum_{i=1}^{N} K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau}) \geq \frac{1}{2(N-1)} \left\{ \sum_{1 \leq i < j \leq N} \left( K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau} \pm E_{\pi(j)}^{\tau}) \right) \right\},
\]

(35)

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \). By the definition of Eq. (9), the conclusion follows immediately. \( \square \)

**Theorem 11** Let \( \Phi_1, \cdots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_i(\rho) = \sum_{j=1}^{n} E_{j}^{\tau}(\rho(E_{j}^{\tau})^{\dagger}) \), \( t = 1, 2, \cdots, N \) \((N > 2)\), we have

\[
\sum_{i=1}^{N} K_{\rho,\tau}^{\alpha,\beta}(\Phi_i) \geq \max_{\pi,\sigma \in S_n} \frac{1}{N-2} \left\{ \sum_{1 \leq i < j \leq N} \left( K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau} \pm E_{\pi(j)}^{\tau}) \right) \right\}.
\]

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \). \( S_n \) is the n-element permutation group and \( \pi, \sigma \in S_n \) are arbitrary n-element permutations.

In particular, Theorem 2 in [49] is a special case of our Theorem 11 with \( \alpha = \beta = \frac{1}{2} \).

**Theorem 12** Let \( \Phi_1, \cdots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_i(\rho) = \sum_{j=1}^{n} E_{j}^{\tau}(\rho(E_{j}^{\tau})^{\dagger}) \), \( t = 1, 2, \cdots, N \) \((N > 2)\), we have

\[
\sum_{i=1}^{N} \sqrt{K_{\rho,\tau}^{\alpha,\beta}(\Phi_i)} \geq \max_{\pi,\sigma \in S_n} \frac{1}{N-2} \left\{ \sum_{1 \leq i < j \leq N} \sqrt{K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau} \pm E_{\pi(j)}^{\tau})} \right\}.
\]

(36)

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \). \( S_n \) is the n-element permutation group and \( \pi, \sigma \in S_n \) are arbitrary n-element permutations.

**Theorem 13** Let \( \Phi_1, \cdots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_i(\rho) = \sum_{j=1}^{n} E_{j}^{\tau}(\rho(E_{j}^{\tau})^{\dagger}) \), \( t = 1, 2, \cdots, N \) \((N \geq 2)\), we have

\[
\sum_{i=1}^{N} K_{\rho,\tau}^{\alpha,\beta}(\Phi_i) \geq \max_{\pi,\sigma \in S_n} \left\{ \frac{1}{N} \sum_{i=1}^{n} K_{\rho,\tau}^{\alpha,\beta}(\sum_{j=1}^{n} E_{\pi(j)}^{\tau}) \right\} + \frac{2}{N^2(N-1)} \left\{ \sum_{i=1}^{n} \left( \sum_{1 \leq i < j \leq N} \sqrt{K_{\rho,\tau}^{\alpha,\beta}(E_{\pi(i)}^{\tau} \pm E_{\pi(j)}^{\tau})} \right) \right\}.
\]

(37)
where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, $S_n$ is the n-element permutation group and $\pi, \pi_s \in S_n$ are arbitrary n-element permutations.

In particular, when $\alpha = \beta = \frac{1}{2}$, (37) of Theorem 14 reduces to (32) of Theorem 3 in [49].

**Theorem 14** Let $\Phi_1, \ldots, \Phi_N$ be $N$ quantum channels with Kraus representations $\Phi_i(\rho) = \sum_{j=1}^{N} E_i^j \rho (E_i^j)\dagger$, $i = 1, 2, \ldots, N$ $(N \geq 2)$, we have

$$\sum_{i=1}^{N} K_{a,\beta}^{a,\beta}(\Phi_i) \geq \max_{\pi, \pi_s \in S_n} \frac{1}{2(N-1)} \left[ \sum_{i=1}^{n} \left( \sum_{1 \leq \gamma \leq N} \sqrt{K^{a,\beta}_{\pi,\pi_s}(E_{\pi(\gamma)}^i - E_{\pi_s(\gamma)}^i)^2} \right) \right].$$

(38)

and

$$\sum_{i=1}^{N} K_{a,\beta}^{a,\beta}(\Phi_i) \geq \max_{\pi, \pi_s \in S_n} \frac{1}{2(N-1)} \left[ \sum_{i=1}^{n} \left( \sum_{1 \leq \gamma \leq N} \sqrt{K^{a,\beta}_{\pi,\pi_s}(E_{\pi(\gamma)}^i - E_{\pi_s(\gamma)}^i)^2} \right) \right].$$

(39)

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, $S_n$ is the n-element permutation group and $\pi, \pi_s \in S_n$ are arbitrary n-element permutations.

For arbitrary $N$ matrices (not necessarily Hermitian) $E_1, \ldots, E_N$, we call an $N \times N$ matrix $G$ a covariance matrix of $E_1, \ldots, E_N$ if the entries of $G$ are given by

$$G_{jk} = \text{Tr} \left( \frac{((1-\gamma)^{p^s + \gamma q^s}, E_{\gamma})^{1-\alpha-\beta}}{\|\rho - E_{\gamma}^s\| (1-\gamma)^{p^s + \gamma q^s}, E_{\gamma})^{1-\alpha-\beta}} \right),$$

where $\| \cdot \|$ is a norm on the $d$-dimensional complex linear space of matrices $M_d(C)$. It can be verified that $G$ is a positive semi-definite complex matrix. Analogizing the idea in [53], we obtain the following theorem.

**Theorem 15** Let $\Phi_1, \ldots, \Phi_N$ be $N$ quantum channels with Kraus representations $\Phi_i(\rho) = \sum_{j=1}^{N} E_i^j \rho (E_i^j)\dagger$, $i = 1, 2, \ldots, N$ $(N \geq 2)$, we have

$$\sum_{i=1}^{N} K_{a,\beta}^{a,\beta}(\Phi_i) \geq \frac{1}{\lambda_{\max}(G)} \left( \sum_{i=1}^{n} \sum_{i=1}^{N} E_i^j \right).$$

(40)

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, and $G$ is the $Nn \times Nn$ covariance matrix of $\{E_i^j\}_{1 \leq i \leq N, 1 \leq \gamma \leq \gamma}$. Note that (40) of Theorem 15 reduces to (2) of Theorem 2 in [53] when $\alpha = \beta = \frac{1}{2}$ and $N = 2$. 

\[ Springer \]
5 Conclusions

We have introduced the \((\alpha,\beta,\gamma)\) weighted Wigner-Yanase-Dyson ((\(\alpha,\beta,\gamma\) WWYD) skew information and the \((\alpha,\beta,\gamma)\) modified weighted Wigner-Yanase-Dyson ((\(\alpha,\beta,\gamma\) MWWYD) skew information, which are more general than the previous concepts. We have derived sum uncertainty relations for \(N\) mutually noncommutative observables based on the \((\alpha,\beta,\gamma)\) WWYD skew information, which includes the results in \[48]\ as special cases. Following the idea in Ref. \[48, 51\], we have derived other forms of lower bounds using the sum of the \((\alpha,\beta,\gamma)\) WWYD skew information for any other number of observables less than \(N\). It is found that when \(a = \beta = \frac{1}{2}\), for the spin-\(\frac{1}{2}\) case and the observables of Pauli operators \(\sigma_1,\sigma_2,\sigma_3\), our lower bounds \(LB_2\) and \(LB_3\) are tighter than the existing ones. Finally, we have also explored sum uncertainty relations for quantum channels in terms of the \((\alpha,\beta,\gamma)\) MWWYD skew information. The results in this paper cover the ones in \[48\] and \[49\] for WY skew information, and may shed some new light on the study of skew information-based sum uncertainty relations for observables and quantum channels.

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