A nonlinear $q$-voter model with deadlocks on the Watts–Strogatz graph

Katarzyna Sznajd-Weron\textsuperscript{1} and Karol Michal Suszczynski\textsuperscript{2}

\textsuperscript{1} Institute of Physics, Wroclaw University of Technology, Poland
\textsuperscript{2} Institute of Theoretical Physics, University of Wroclaw, Poland
E-mail: katarzyna.weron@pwr.wroc.pl

Received 5 January 2014
Accepted for publication 25 May 2014
Published 18 July 2014

Online at stacks.iop.org/JSTAT/2014/P07018
doi:10.1088/1742-5468/2014/07/P07018

Abstract. We study the nonlinear $q$-voter model with deadlocks on a Watts–Strogatz graph characterized by two parameters $k$ and $\beta$. Using Monte Carlo simulations, we obtain a so-called exit probability and an exit time. We determine how network properties, such as randomness or density of links, influence the exit properties of a model. In particular we show that the exit probability, which is the probability that the system ends up with all spins up, starting with the $p$ fraction of up-spins, has the general form $E(p) = p^n / (p^n + (1-p)^n)$. Moreover, using the finite-size scaling technique we show that the exit probability exponent $\alpha$ depends both on the parameter $q$ as well as the network structure, i.e. $k$ and $\beta$.

Keywords: interacting agent models
1. Introduction

Describing opinion dynamics has inspired many physicists to build models that could not be justified by physical phenomena (for a review of opinion dynamic models see [1, 2]). Such models are usually more caricatures than precise portraits of real social systems. However, far-reaching simplifications should not be regarded as a defect of these models. Simplicity allows not only for in-depth analysis but also for analytical treatment. First of all it allows us to describe some universal features or even to determine the most important factors that influence a given social phenomenon.

Certainly, the main challenge that persists with opinion dynamics models is the describing of complex social systems in terms of a relatively simple approach. On the other hand, such models are themselves interesting from a theoretical point of view [3]. Therefore they might be also treated as small building blocks which make a contribution to the construction of still emerging non-equilibrium statistical physics. A good example of such an interesting model is the nonlinear $q$-voter model introduced in [4] along with its modifications proposed in [5, 6]. The precise definition of the model will be given in the next section. For now, what is important is the fact that in the $q$-voter model only a unanimous group of $q$ neighbors can influence a voter. Hence this model is a simple generalization of the linear voter model [3]. In this paper we will investigate a special case of a model, which we call the $q$-voter model with deadlocks, considered already in [5] for a one dimensional lattice. We examine the role of a topology in such a model and show that increasing randomness and density of a network helps us to reach a consensus.

The second motivation for this paper came from a recent controversy on the exit probability $E(p)$ (i.e. the probability that the system ends up with all spins up starting with the $p$ fraction of up-spins) for the $q$-voter model with deadlocks. It has been shown independently in five papers [5, 7–10] that for $q=2$, which corresponds to the Sznajd model, the exit probability can be described by the following formula:

$$E(p) = \frac{p^2}{p^2 + (1-p)^2}.$$  \hspace{1cm} (1)
However, in [11] it has been suggested that the above result is valid only for finite-size systems and should approach a step-like function for the infinite system. It should be stressed that the suggestion that appeared in [11] was taken seriously and considered by others [5, 10] because the formula (1) was obtained by some approximation (different variants of the mean-field approach). The difficulties of finding the exact solution arise from the fact that the average magnetization in the \( q \)-voter model is not conserved. In [7, 8] an approximate solution was constructed by truncating the hierarchy of the rate equations of higher-order correlation functions by a decoupling scheme, known as Kirkwood approximation [12]. On the other hand, results obtained in [5, 7–9] suggest that there is no finite size dependence for \( E(p) \) in the case of the one-dimensional lattice. Recent results obtained by Timpanaro and Prado [10] for large lattices confirm the result given by (1) and indicate that the step-like function corresponds to the complete graph. It would seem, therefore, that the ambiguity associated with the exit probability for the \( q \)-voter model is explained. However, this is true only for \( q=2 \) on the one-dimensional lattice.

For a higher value of \( q \) and different network structures the problem is still open. It has been suggested in [5] that the exit probability on a one-dimensional lattice for some arbitrary value of \( q \) is given by the formula:

\[
E(p) = \frac{p^q}{p^q + (1-p)^q}.
\]  

(2)

However, recent results [10] suggest that this might be valid only for small lattices in case of \( q>2 \). We will examine this problem in this paper for different values of \( q \) and different network structures.

2. Model

The original \( q \)-voter model introduced in [4] on the one dimensional lattice has been defined as follows:

- Each \( i \)-th site of a graph of a size \( N \) is occupied by a voter \( S_i = \pm 1 \)
- Initially there is a probability \( p \) of finding a voter in a state \( +1 \) and a probability \( 1-p \) of finding a voter in a state \( -1 \)
- The system evolves according to the following algorithm:

  (a) At each elementary time step \( t \), choose one spin \( S_i \), located at site \( i \), at random
  (b) Choose \( q \) neighbors (\( q \)-panel) of site \( i \)
  (c) If all \( q \) neighbors are in the same state then \( S_i \) takes the same state as its neighbors
  (d) Otherwise, if the \( q \) neighbors are not unanimous then take \( S_i \rightarrow -S_i \) with probability \( \epsilon \)
  (e) Time is updated \( t \rightarrow t + \frac{1}{N} \)
In [5] it was proposed to study a one-dimensional model with $\epsilon = 0$, which for $q=2$ corresponds to the Sznajd model. It should be noted that for $\epsilon = 0$ the evolution of the system is hampered due to the existence of deadlock configurations. Deadlocks should be understood as configurations in which there is no possibility of an evolution due to the lack of a unanimous $q$-panel. In the case of a one dimensional lattice and $q=2$ there is only one deadlock configuration—an antiferromagnetic state $+-+-+\ldots$. For $q=3$ there are already many more, e.g. $+-+-+-\ldots$, $+-+-+++\ldots$ or $+-+-+-++\ldots$, etc. However, if initially there is at least one $q$-panel, evolution will reach one of two final absorbing states. The nonlinear $q$-voter model with deadlocks has been found to be interesting for several reasons:

- For $q=2$ it reduces to the Sznajd model for opinion dynamics and in this case the analytical formula for an exit probability has been found independently by [7–9, 11]
- The exit probability does not depend on a system size as reported by [5, 7–9]. This result should be treated with caution, taking into account recent results obtained by [10] for large lattices. It seems that additional simulations are needed to explain this contradiction.
- In a case of random noise, a system undergoes a phase transition which changes its type from continuous to discontinuous at $q=5$ [6].

To investigate the role of the network topology we have decided to use the model introduced by [14], mainly because it allows us to study various structures—from regular lattices with different sizes of neighborhood, through small-world networks to random graphs. The Watts–Strogatz (WS) algorithm, that we have used, is defined as follows:

- Start with a 1D lattice of size $N$ with periodic boundary conditions in which each node is connected to its $2k$ neighbors
- Then with probability $\beta$ replace each edge by a randomly chosen edge

For $\beta=0$ we deal with regular lattices—e.g. for $k=1$ we have a simple one-dimensional lattice with interactions only to the nearest neighbors, and for $k=(N-1)/2$ we have a complete graph. With increasing $\beta$ we increase the randomness of the network going through the small-world (for $\beta=0.01–0.1$) to the random graph for $\beta=1$. Summarizing, we have one parameter $q$ that defines the model itself and two parameters $k$ and $\beta$ that describe the network properties.

3. Results

3.1. Exit probability on a complete graph

An important property of a system with absorbing states is the so-called exit probability [3]. In our case the exit probability $E(p)$ should be understood as a probability of the absorbing state with all spins ‘+1’ as a function of the initial probability $p$ of finding
a spin in a state $+1$. For a complete graph, which corresponds also to the mean-field treatment, the evolution of the probability of ‘up’-spins is given by:

$$p(t + \Delta t) = p(t) + p^q(t)(1 - p(t)) - (1 - p(t))^q p(t)$$

(3)

Fixed points can be easily found from the condition $p(t + \Delta t) = p(t) = p^*$, i.e.:

$$(p^*)^q (1 - p^*) - (1 - p^*)^q p^* = p^* (1 - p^*) [(p^*)^{q-1} - (1 - p^*)^{q-1}] = 0.$$  

(4)

As can be seen there are three fixed points $p^* = 0, 1/2, 1$. This can be easily checked by calculating the following derivative:

$$\frac{d}{dp} (p + p^q(1-p) - (1-p)^q p) \bigg|_{p=p^*},$$

(5)

where $p^* = 0$ and $p^* = 1$ are stable, but $p^* = 1/2$ is an unstable fixed point. Therefore on a complete graph for $p < 1/2$ the system eventually reaches the absorbing state $p^* = 0$, and for $p > 1/2$ the system reaches $p^* = 1$. This means that the exit probability for the $q$-voter model on the complete graph with arbitrary value $q$ is a step-like function:

$$E(p) = \begin{cases} 0 & \text{for } p < 1/2 \\ 1 & \text{for } p > 1/2 \end{cases}$$

(6)

Timpanaro and Prado have recently proposed a much more rigorous approach to show that the exit probability is a step-like function for a complete graph [10]. Their and our results confirm that the results obtained by [15] within a unifying frame (GUF) coincides with the mean-field approach and may not be true for arbitrary topology.

### 3.2. Results on regular graphs

For $\beta = 0$, a broad class of WS networks reduces to regular graphs with various sizes of neighborhood given by $k$. In this section we examine the role of $k$ which from a social
point of view measures the density of the social group. In particular we would like to answer the following questions:

(a) How does parameter $k$ influence the exit probability? Results on a complete graph suggest that exit probability should become steeper with increasing $k$.

(b) How does $k$ influence exit time, i.e. is a consensus reached faster for smaller or larger values of $k$?

(c) How do the results scale with the system size for $k>1$? For $k=1$ no finite size dependence has been noted in one-dimensional voter, Sznajd and $q$-voter models [5, 7–9], as well as several Ising-like models with so-called inflow dynamics [13].

In agreement with our predictions, it is seen in figure 1 that the steepness of the exit probability slope increases with $k$. This was expected because we have found that for $k=(N-1)/2$ (complete graph) $E(p)$ is a step-like function. It is also the case that for $q=2$ and arbitrary value of $k$ simulation data can be fitted by (see the left panel in figure 1):

$$E(p) = \frac{p^\alpha}{p^\alpha + (1-p)^\alpha},$$

Figure 2. Exit probability (top panels) and exit time (bottom panels) as a function of the initial probability $p$ of spin $+1$. In the left panels results for $q=2$, $k=4$, $\beta=0$ and several system sizes $N=100, 200, 400, 800, 1600$ are presented. In the right panels a comparison between the results for $q=2$, $k=8$ with $q=4$, $k=4$ are presented.
where $\alpha = k/2 + 3/2$. Parameter $k$ also influences exit time, which should be understood as the time needed to reach an absorbing state [3]. As seen the exit time significantly decreases with $k$, which means that consensus is reached faster in more dense societies.

One of the most interesting questions is related to the finite-size effects. All previous results [5, 7–10] show that the exit probability does not depend on the system size for $q=2$ on a one-dimensional lattice. Interestingly, analogous results have been very recently obtained for a larger class of Ising-like models—no finite size dependence in the exit probability has been found [13]. In figure 2 we present the exit probability and exit time as a function of the initial probability $p$ of spin $+1$ for $q=2$, $k=4$, $\beta=0$ and several system sizes $N=100, 200, 400, 800, 1600$. It is seen that the exit probability does not depend on the system size. It is also seen that the exit time (EP) nearly scales with the system size as $L^{-2}$, though the scaling relation is not exact, in contrast to the voter and Sznajd models on a one-dimensional lattice with $k=1$ [8]. Interestingly, a comparison between the results for $q=2$, $k=8$ and $q=4$, $k=4$ that are presented in the right panels of figure 2 suggests that the exit probability depends on $kq$, instead of two parameters $k$ and $q$. This conjecture has been examined for other values of $k$ and $q$ (see figure 3). It can be seen in figure 3 that the relation $EP(q, k) = EP(qk)$ is roughly valid. On the other hand, the exit time (or in other words the consensus time) clearly decreases with $k$. One might expect that the exit time for different system sizes could be somehow rescaled by $k$, for example $\tau(k, N) = \tau(k/N)$. To check this expectation we have conducted simulations for several values of $q$ and ten pairs $(k, N)$, for $k/N=100$ and $N=100, 200, ..., 1000$. Unfortunately, no simple relation has been found.

### 3.3. Results on a WS graph

We now investigate the model on a WS graph for $\beta=0.01$ and different values of $k$ and $q$. Similarly as for $\beta=0$, the steepness of the exit time increases both with $q$ and $k$, and the exit time strongly decreases with $k$ (see figure 4). However, contrary to $EP$ for $\beta=0$,
finite-size effects are clearly seen for any value of $q$ (see figure 5). Using the finite-size technique we were able to determine the scaling exponent $\nu$. As usual, we choose the scaling variable $x=(p-p^*)N^{1/\nu}$ [16], where $\nu$ is the critical exponent describing the correlation length and $p^*$ is the critical value, i.e. in our case $p^*=0.5$. In figure 5 we present results for $q=2$ and two values of $k$. Data for different system sizes $N=100, 200, 400, 800, 1600$ collapse into a single curve, though the scaling exponent is not universal and depends on the network structure.

Now we are ready to examine the role of the second parameter, which describes the level of randomness i.e. $\beta$.

In figure 6 we present results for $k=4, q=4$ and two system sizes $N=100, 800$ for several values of $\beta$. First of all, it is seen that the steepness of the exit probability slope increases and simultaneously the exit time decreases with $\beta$. However, it is seen that the EP for the small system $N=100$ differs from the one for the larger system ($N=800$). The dependence between the system size $N$ and $EP$ is much weaker for $q=2$ than for larger values of $q$. This is an interesting result taking into account results obtained recently by [10]. For regular lattices, i.e. $\beta=0$, the dependence between the exit probability and the system size was very difficult or even impossible to observe. Only results on the very large system sizes ($10^5$–$10^7$), that required redefinition of the simulation algorithm as proposed in [10], have shown a small dependence on the system size for $q>2$. For the system sizes investigated in this paper, i.e. $L=100, 200, 400, 800,$
1600, this dependence was almost invisible. However, for $\beta > 0$ the difference between results for $q = 2$ and $q > 2$ are very clear (see upper panel in figure 6).

The second interesting result is related to the exit (consensus) time. Results are presented on a semi-log scale (bottom panel in figure 6) to allow for the comparison between different values of $\beta$. The exit time clearly decreases with increasing randomness of the network. A similar result has been found in the case of a linear voter model, which corresponds to $q = 1$—for networks of finite size the ordering process takes a time shorter than on a regular lattice of the same size [17]. Certainly shorter paths help to reach consensus faster, although our research does not allow us to determine if this is the only property of the network which helps to achieve a consensus. Probably the effect of the network structure is more complex.

4. Discussion

We have investigated a special case of a $q$-voter model (with $\epsilon = 0$) on a WS network described by parameters $k$ and $\beta$. We have shown that the exit probability is the
A nonlinear q-voter model with deadlocks on the Watts–Strogatz graph

doi:10.1088/1742-5468/2014/07/P07018

S-shaped function for arbitrary values of parameters $q, k$ and $\beta$ and can be fitted by an analytical formula (7).

It should be recalled that even for $\beta=0, k=1$ and $q=2$ only approximate calculations are available, although surprisingly four independent analytical approaches [7–9, 11] give exactly the same result (1) which is in perfect agreement with Monte Carlo results [5, 10].

In the case of heterogeneous networks, i.e. for $\beta>0$, one could try to apply a powerful technique known as the heterogeneous mean-field (HMF) approach. Recently HMF has been applied to the q-voter model without deadlocks (i.e. for $\epsilon>0$) [18]. It has been argued that for $q=2$ and any network structure, the application of HMF leads to the trivial result as long as $\epsilon=1/2$. In such a case the flipping probability $f(x) = x^q + \epsilon[1-x^q-(1-x)^q]$ is a linear function of the fraction $x$ of neighbors in the opposite state. For values of $q>2$, the application of HMF is hampered by the nonlinearity of evolution equations, even if one considers the system only for the critical value of $\epsilon$, i.e. without the drift term [18]. The case with $\epsilon=0$ is already difficult for $q=2$, as mentioned in the introduction and discussed in [7, 8], not only because of non-linearity but also because of the presence of a drift term in the evolution equation. Therefore, deriving an analytical formula (7) is a challenge for the future.

Another task that could be considered in the future is the exact relation between the finite-size scaling exponent $\nu$ and the parameters of the model. Here, it has been found that for $\beta=0, q=2$ and some arbitrary value of $k$ the exit probability does not...
depend on the system size. The finite-size effects are also almost invisible for larger values of $q$, but recent results for very large lattices [10] suggest we be careful with the formulation of final conclusions. Our caution is also dictated by the results for $\beta > 0$, which have shown that for $q = 2$ EP depends much less on the system size than for $q > 2$. Nevertheless, taking into account the results obtained in this paper and earlier papers [5, 10] we may safely say that for $\beta = 0$ and the arbitrary values of $q$ and $k$ the finite-size effects are very weak. This agrees also with recent results for the broad class of Ising-type models. Roy et al [13] have investigated the exit probability in several one dimensional Ising-like models with so-called inflow dynamics, such as the Ising–Glauber model at temperature $T = 0$. Using Monte Carlo simulations they have found that a general form for the exit probability is given, as in our case, by equation (7), where the exponent $\alpha$ depends on model details (range of interactions, asymmetry or fluctuations present in a given dynamics). Moreover, they have shown that for the inflow dynamics in one dimension, the exit probability does not depend on the lattice size. However, for $\beta > 0$ the finite-size analysis provides much less trivial results. Using finite-size scaling technique we were able to determine scaling exponents $\nu$ for several sets of parameters $(q, k, \beta)$, though the general dependence is not yet uncovered.

Summarizing, up till now the $q$-voter model with deadlock has been investigated only on the one-dimensional lattice with nearest neighbors. Here we have examined both the role of the density of the network given by $k$ and its randomness characterized by $\beta$. We have been able to show that the exit time decreases with $k$ and $\beta$. From a social point of view this means that the consensus is reached faster in societies with larger numbers of links and shorter paths. Moreover, we were able to show that, for any values of parameters, the exit probability can be described by the general relation (7), with the exponent $\alpha$ that depends on model parameters analogously, as in case of Ising-like models with inflow dynamics [13]. Interestingly, it became apparent that, for $\beta = 0$ and arbitrary values of $q$ and $k$, the EP does not depend on the system size or depends very weakly for $q > 2$. On the contrary, for $\beta > 0$ non-trivial finite-size scaling has been found. Although we leave some problems open, as stressed above, we believe that our paper will contribute to the understanding of non-equilibrium phenomena and opinion spreading on networks. On the other hand we hope that the question posed here will motivate others to study $q$-voter models with deadlocks more deeply.

Acknowledgments

This work was supported by funds from the National Science Centre (NCN) through grant no. 2011/01/B/ST3/00727.

References

[1] Castellano C, Fortunato S and Loreto V 2009 Rev. Mod. Phys. 81 591–646
[2] Galam S 2012 Sociophysics: a Physicist’s Modeling of Psycho-Political Phenomena (New York: Springer)
[3] Krapivsky P L, Redner S and Ben-Naim E 2010 A Kinetic View of Statistical Physics (Cambridge: Cambridge University Press)
[4] Castellano C, Muñoz M A and Pastor-Satorras R 2009 Phys. Rev. E 80 041129
[5] Przybyla P, Sznajd-Weron K and Tabiszewski M 2011 Phys. Rev. E 84 031117
[6] Nyczka P, Sznajd-Weron K and Cislo J 2012 Phys. Rev. E 86 011105
A nonlinear q-voter model with deadlocks on the Watts–Strogatz graph

[7] Lambiotte R and Redner S 2008 *Europhys. Lett.* **82** 18007
[8] Slanina F, Sznajd-Weron K and Przybyla P 2008 *Europhys. Lett.* **82** 18006
[9] Castellano C and Pastor-Satorras R 2011 *Phys. Rev. E* **83** 046113
[10] Timpanaro A M and Prado C P C 2014 *Phys. Rev. E* **89** 052808
[11] Galam S and Martins A C R 2011 *Europhys. Lett.* **95** 48005
[12] Mobilia M and Redner S 2003 *Phys. Rev. E* **68** 046106
[13] Roy P, Biswas S and Sen P 2014 *Phys. Rev. E* **89** 030103
[14] Watts D J and Strogatz S H 1998 *Nature* **393** 440
[15] Galam S 2005 *Europhys. Lett.* **70** 705–11
[16] Landau D P and Binder K 2005 *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge: Cambridge University Press)
[17] Castellano C, Vilone D and Vespignani A 2003 *Europhys. Lett.* **63** 153
[18] Moretti P, Liu S, Castellano C and Pastor-Satorras R 2013 *J. Stat. Phys.* **151** 113–30

doi:10.1088/1742-5468/2014/07/P07018