Inhomogeneous phases at finite density in an external magnetic field

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The two-flavor quark-meson model is used as a low-energy effective model for QCD to study inhomogeneous chiral condensates at finite quark chemical potential $\mu$ in a constant magnetic background $B$. We determine the parameters of the model by matching the meson and quark masses, and the pion decay constant to their physical values using the on-shell and modified minimal subtraction schemes. We calculate the free energy in the mean-field approximation for a chiral density wave using dimensional regularization. The system has a surprisingly rich phase structure.

I. INTRODUCTION

QCD at finite temperature and baryon density has been studied in detail for several decades, largely spurred by its relevance for the early universe, heavy-ion collisions, and compact stars \cite{1,2}. At zero baryon density, one can perform lattice simulations to calculate the thermodynamic functions and the transition temperature associated with chiral symmetry restoration and deconfinement. For physical quark masses and two flavors, the transition is a crossover at a temperature of around 155 MeV \cite{3,4}. At finite $\mu_B$, the infamous sign problem prevents the use of standard Monte Carlo techniques based on importance sampling. This implies that for large parts of the phase diagram, we have to rely on model calculations. A drawback of model calculations is that some of the predictions are not robust; for example, the existence of certain phases may depend on the values of its parameters. Only at asymptotically high densities are we confident about the phase and the properties of QCD. In this limit, the ground state of QCD is the color-flavor-locked phase which is a color superconducting phase.

The Nambu-Jona-Lasinio (NJL) model and the quark-meson (QM) model are examples of low-energy models of QCD; they share some of its properties such as chiral symmetry and the breaking of it in the vacuum. While these models incorporate the chiral aspects of QCD very well, they are not confining. This has led to the introduction of the Polyakov loop, which is an approximate order parameter for confinement \cite{7}. By coupling an $SU(N_c)$ background gauge field $A_\mu$ to the chiral model one can mimic confinement in QCD in a statistical sense \cite{3}.

There are other control parameters in addition to the temperature $T$ and the baryon chemical potential $\mu_B$, for example an external magnetic field $B$. See e.g. \cite{9,21} for model calculations. QCD in a magnetic background at $\mu_B = 0$ is free of the sign problem and consequently one can perform lattice simulations. Lattice simulations of QCD in a constant magnetic background have been carried out in recent years to study the chiral condensate as a function of $T$ and to calculate the transition temperatures for chiral symmetry restoration and deconfinement \cite{22,25}. However, for finite $\mu_B$, the sign problem again prevents one from using Monte Carlo techniques.

QCD at finite $\mu_B$ and $B$ provides an example of where naive model calculations may go wrong. Using chiral perturbation theory and including the Wess-Zumino-Witten term, it has been shown that the ground state of QCD for certain values of $\mu_B$ and $B$ is a spatially modulated condensate of neutral pions \cite{26}. Recently, it has been shown that the exact solution is a chiral soliton lattice \cite{27}. The importance of this results is that it is a model-independent statement and therefore robust. The critical magnetic field $B_{\text{crit}}(\mu_B)$ has also been estimated and for $\mu_B \approx 900$ MeV, which corresponds to the onset of nuclear matter, it is approximately $10^{19}$ Gauss, which may be found in the core of magnetars. This corresponds to $|eB| = 1.88 m_\pi^2$.

Calculating the thermodynamic potential in these models, one encounters ultraviolet divergences due to vacuum fluctuations. In NJL-model calculations, one often uses a sharp three-dimensional cutoff to regulate them \cite{28}. However, in the case of inhomogeneous condensates, this typically leads to artifacts of the regularized free energy. For example, the chiral density wave is characterized by the modulus $\Delta$ and the wave vector $q$. In the limit $\Delta \to 0$, the free energy cannot depend on the wave vector. If this is the case, one must find suitable terms to subtract from the free energy to make it well defined \cite{29}. The problem with a three-dimensional cutoff is that there is an asymmetry in the values of the energies that contributes to the free energy from the different
branches of the quasiparticle branches. One can avoid this artifact in most cases by introducing a symmetric energy cutoff [15, 30], i.e. one cuts off the energy rather than the three-momentum. A convenient alternative to imposing sharp cutoffs is dimensional regularization. In Ref. [31], it was shown how one can use dimensional regularization to calculate the vacuum energy in various models for a chiral-density wave. In some cases, one still has to subtract suitable terms to make the vacuum energy well defined in the limit $\Delta \to 0$. In the present case, no such terms are necessary to subtract, as we will show.

The article is organized as follows. In the next section, we briefly discuss the quark-meson model. The free energy is calculated in the mean-field approximation. In section III, we present our results. First we discuss the case of a homogeneous chiral condensate and map out the phase diagram. We next consider the chiral density wave and inhomogeneous phases. Details of the calculations can be found in two appendices.

II. QUARK-MESON MODEL AND FREE ENERGY

A. Quark-meson model

The Euclidean Lagrangian of the two-flavor quark-meson model is

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_\mu \pi)^2 + (\partial_\mu \bar{\sigma})^2 \right] + \frac{1}{2} m^2 (\sigma^2 + \pi^2) + \frac{\lambda}{24} (\sigma^2 + \pi^2)^2 - h \sigma + \bar{\psi} \left[ D - i \gamma_5 qz \right] \psi + \frac{\bar{\psi}}{2} \partial_\mu A_\mu \right] \gamma^0 + \frac{1}{2} \left[ \gamma^5 \tau \cdot \pi \right] \psi \right) (1)$$

where $\psi$ is a flavor doublet as well as an $SU(N_c)$-plet,

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

(2)

Moreover, $\mu_B = 3 \mu = \frac{3}{2} (\mu_u + \mu_d)$ and $\mu_L = \frac{1}{2} (\mu_u - \mu_d)$ are the baryon and isospin chemical potentials in terms of the quark chemical potentials $\mu_u$ and $\mu_d$, and $\tau_i$ ($i = 1, 2, 3$) are the Pauli matrices in flavor space, $D_\mu = \partial_\mu - q_f A_\mu$ is the covariant derivative and $q_f$ is the electric charge of flavor $f = u, d$. In the case of a constant magnetic field $B$ directed along the z-axis, we choose the gauge $A_\mu = (0, 0, -B x, 0)$. In addition to the global $SU(N_c)$ symmetry, the Lagrangian (1) has, for $A_\mu = 0$, a $U(1)_B \times SU(2)_L \times SU(2)_R$ symmetry for $h = 0$ and $U(1)_B \times SU(2)_V$ for $h \neq 0$. When $\mu_u \neq \mu_d$, this symmetry is reduced to $U(1)_B \times U(1)_\tau \times U(1)_h$ for $h = 0$ and $U(1)_B \times U(1)_h$ for $h \neq 0$. The constant magnetic field breaks Lorentz invariance; it also breaks the to $SU(2)_V$ down to $U(1)_V$. In the remainder of the paper, we choose $\mu_u = \mu_d$ and $h \neq 0$. In the vacuum, the $\sigma$ field acquires a nonzero vacuum expectation value, which is denoted by $\phi_0$. In order to study inhomogeneous phases, we make an ansatz for the space-time dependence of the mesonic mean fields. In the literature, mainly one-dimensional modulations have been considered, for example chiral-density waves (CDW) and soliton lattices [32]. We opt for the simplest, namely a one-dimensional chiral-density wave, although this might not be the modulation with the lowest energy. The ansatz is

$$\phi_0(z) = \frac{\Delta}{g} \cos(qz), \quad \pi(z) = \frac{\Delta}{g} \sin(qz), \quad (3)$$

where $\Delta$ and $q$ are real parameters. The bilinear term in the Lagrangian in Minkowski space is

$$\bar{\psi} \left[ i D + \gamma_5 \mu - \Delta e^{i \gamma_5 qz} \right] \psi \right).$$

We next redefine the quark fields, $\psi \rightarrow e^{-\frac{i}{2} \gamma_5 \tau_3 qz} \psi$ and $\bar{\psi} \rightarrow \bar{\psi} e^{\frac{i}{2} \gamma_5 \tau_3 qz}$. This corresponds to a unitary transformation of the Dirac Hamiltonian $\mathcal{H} \rightarrow \mathcal{H}' = e^{\frac{i}{2} \gamma_5 \tau_3 qz} \mathcal{H} e^{-\frac{i}{2} \gamma_5 \tau_3 qz}$. As pointed out in Ref. [15], a field redefinition as the one above requires extra care in the presence of background gauge fields as a change of the path-integral measure must be taken into account. Using the method of Fujikawa [33], there is an extra factor of $e^{\sigma_B \mu} F_{\alpha \beta} F^{\alpha \beta}$. In a constant magnetic field, the term $e^{\sigma_B \mu} F_{\alpha \beta} F^{\alpha \beta}$ vanishes and the measure is invariant. The Dirac operator can be written as

$$D = \left[ i D + \gamma_0 \mu - \Delta + \frac{1}{2} \gamma_5 \tau_3 qz \right] \mathcal{H} + \mathcal{H}' =$$

(5)

$$(\sqrt{p_z^2 + \Delta^2} \pm \frac{q_f B}{2} (2k + 1 - \sigma_z) \right)^2 + |q_f B|(2k + 1 - \sigma_z), \quad (6)$$

where $q_f$ is the charge of the quark flavor, $\sigma_z = \pm 1$, and $k = 0, 1, 2,...$. We note that the lowest Landau level, which corresponds to $\sigma_z = 1$ and $k = 0$ is independent of the magnetic field.

At tree level, the relations between the parameters $m^2$, $\lambda$, $g^2$, and $h$ of the Lagrangian Eq. (1) and the physical observables $m_\sigma$, $m_\pi$, $m_q$, and $f_\pi$ are

$$m^2 = -\frac{1}{2} \left( m_\sigma^2 - 3m_\pi^2 \right), \quad \lambda = 3 \left( \frac{m_\sigma^2 - m_\pi^2}{f_\pi^2} \right), \quad (7)$$

$$g^2 = \frac{m_\pi^2}{f_\pi^2}, \quad h = m_\sigma^2 f_\pi. \quad (8)$$

1 A counterexample is the NJL model in 1+1 dimensions with an isospin chemical potential [30], where one must add a term $-\frac{N_f}{2} \mu_\pi^2$. 

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We also have that $f_\pi$ minimizes the tree-level potential in the vacuum, $V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4$.

Expressed in terms of physical quantities, the tree-level potential is

$$V_{\text{tree}} = \frac{1}{2}B^2 + \frac{1}{2}\left(\frac{q\Delta}{g}\right)^2 + \frac{1}{2}m^2\Delta^2 + \frac{\lambda}{24}\frac{\Delta^4}{g^4} - \frac{h}{g}$$

$$= \frac{1}{2}B^2 + \frac{1}{2}E^2\Delta^2 - \frac{1}{4}f_\pi^2(m_\sigma^2 - 3m_\pi^2)\Delta^2$$

$$+ \frac{1}{8}f_\pi^2(m_\sigma^2 - m_\pi^2)\frac{\Delta^4}{m_q^2} - \frac{\Delta}{m_q}.$$ (9)

**B. Free energy**

The free energy is calculated in the mean-field approximation, where we treat the bosonic degrees of freedom

The one-loop contribution to the effective potential from the fermions is

$$V_1 = -N_c \sum_{f,\pm,\kappa,\sigma_z} |q_f B| \int_{p_\perp} \left\{ E_\pm + T \log \left[ 1 + e^{-\beta(E_\pm - \mu)} \right] \right\}.$$ (10)

The vacuum part of the free energy is

$$V_{\text{vac}}^B = -N_c \sum_{f,\pm,\kappa,\sigma_z} \frac{|q_f B|}{4\pi} \int_{p_\perp} E_\pm.$$ (14)

Eq. (14) is ultraviolet divergent. In dimensional regularization the power divergences are set to zero and the logarithmic divergences show up as poles in $\epsilon$. All the poles, except one which is proportional to $(q_f B)^2$, are identical to the poles found when evaluating the vacuum energy $V_{\text{vac}}$ for $B = 0$. We can therefore isolate these divergences by adding and subtracting $V_{\text{vac}}$. The difference $V_{\text{vac}}^B - V_{\text{vac}}$ then contains a pole in epsilon which is proportional to $(q_f B)^2$. This divergence can be then be isolated by adding and subtracting a second divergent term. All the divergences are subsequently eliminated by minimal subtraction.

The integral representation of the energy difference $E_\pm - E_{0\pm}$

$$E_\pm - E_{0\pm} = -\frac{1}{(4\pi)^2} \int_0^\infty ds \frac{e^{-sE_\pm^2} - e^{-sE_{0\pm}^2}}{s}.$$ (15)

where $E_{0\pm}$ is the dispersion relation in the case $B = 0$, given by

$$E_{0\pm}^2 = \left( \sqrt{p_\perp^2 + \Delta^2} \pm \frac{g}{2} \right)^2 + p_\perp^2.$$ (16)

where $p_\perp^2 = p_\perp^2 + p_\sigma^2$. For notational convenience we write the dispersion relations as

$$E_\pm^2 = a_\pm^2 + M_B^2, \quad E_{0\pm}^2 = a_\pm^2 + p_\perp^2,$$ (17)

where $a_\pm^2 = (\sqrt{p_\perp^2 + \Delta^2} \pm \frac{g}{2})^2$ and $M_B^2 = |q_f B|(2k + 1 - \sigma_z)$. Using Eq. (15), the vacuum energy density difference can be written as

$$V_{\text{vac}}^B - V_{\text{vac}} = -N_c \sum_{f,\pm,\kappa,\sigma_z} \frac{|q_f B|}{4\pi} \int_{p_\perp} \int_0^\infty ds \frac{e^{-s(a_\pm^2 + p_\perp^2)}}{s}$$

$$+ \frac{N_c}{(4\pi)^2} \sum_{f,\pm,\kappa,\sigma_z} |q_f B| \int_{p_\perp} \int_0^\infty ds \frac{e^{-s(a_\pm^2 + M_B^2)}}{s^2}. \quad (18)$$

Integrating over $p_\perp$ directly in two dimensions in the first term and summing over $k$ and $\sigma_z$ in the second term, we find

$$V_{\text{vac}}^B - V_{\text{vac}} = -\frac{N_c}{(4\pi)^2} \sum_{f,\pm,\kappa,\sigma_z} \int_{p_\perp} \int_0^\infty ds \frac{|q_f B|s \coth(|q_f B|s) - 1}{s^2} e^{-su_\pm^2}. \quad (19)$$

The integral in Eq. (19) is divergent for small $s$, i.e. in the ultraviolet. Expanding the integrand in (19), it is
straightforward to see that the UV-divergence is canceled by the term

$$V^B_{\text{vac}} = \frac{2N_c}{(4\pi)^2} \sum_f \int_{p_f}^{\infty} \frac{ds}{s^2} e^{-2s|q_f B|} (q_f B)^2 e^{-sp^2} = N_c \sum_f \left( \frac{e^{\gamma^2 A^2}}{2|q_f B|} \right)^\epsilon \frac{2(q_f B)^2}{3(4\pi)^2} \Gamma(\epsilon)$$

where the extra exponential factor $e^{-2s|q_f B|}$ ensures that the integral is convergent in the infrared, i.e. for large values of $s$. Subtracting Eq. (20) from Eq. (19), we obtain the convergent result for the difference between the two vacuum energy densities

$$V^B_{\text{vac}} - V_{\text{vac}} = \frac{N_c}{(4\pi)^2} \sum_f \int_{p_f}^{\infty} \frac{ds}{s^2} \left[ \coth(|q_f B|s) - 1 \right] e^{-sa^2} - \frac{1}{3} e^{-2s|q_f B|} (q_f B)^2 e^{-sp^2}.$$  

The integral in Eq. (21) is is convergent in the ultraviolet and in the infrared. The integral over $p_z$ can therefore be evaluated in $d = 1$ dimensions. For $\Delta = 0$, this can be done explicitly. Integrating over $p_z$ and summing over $\pm$ yields

$$V^B_{\text{vac}} - V_{\text{vac}} = \frac{2N_c}{(4\pi)^2} \int_{p_f}^{\infty} \frac{ds}{s^2} \left[ \left| q_f B \right| \frac{\coth(|q_f B|s) - 1}{s} \right] e^{-sa^2} - \frac{1}{3} e^{-2s|q_f B|} (q_f B)^2 e^{-sp^2}.$$  

This integral is independent of $q$ showing that the vacuum energy is well defined.

We next consider the divergent part of the vacuum energy, which is given by

$$V_{\text{vac}} = -2N_c \int_p (E_{0+} + E_{0-})$$

where the integral is analogous to Eq. (12), but now in $d = 3 - 2\epsilon$ dimensions. Introducing the variable $u = \sqrt{p_u^2 + \Delta^2}$ and integrating over angles in the $(p_x, p_y)$ plane, we can write Eq. (23) as

$$V_{\text{vac}} = -\frac{16N_c(e^{\gamma^2 A^2})^\epsilon}{(4\pi)^2 \Gamma(1 - \epsilon)} \sum_{\pm} \int_{\Delta}^{\infty} \frac{du}{\sqrt{u^2 - \Delta^2}} \int_0^{\infty} dp_\perp \sqrt{(u + \frac{q}{2})^2 + p_\perp^2} \gamma^{1-2\epsilon}. $$

The strategy is to isolate the ultraviolet divergences in Eq. (24) by expanding the integrand in powers of $q$ and identifying appropriate subtraction terms $\text{sub}(u, p_\perp)$. Integrating the subtraction terms can be done in dimensional regularization, while the integral of $E_{\pm} - \text{sub}(u, p_\perp)$ is finite and can be calculated directly in three dimensions. The subtraction terms $\text{sub}(u, p_\perp)$ is found by expanding Eq. (24) through order $q^4$. This yields

$$\text{sub}(u, p_\perp) = 2\sqrt{u^2 + p_\perp^2} + \frac{q^2 p_\perp^2}{4(u^2 + p_\perp^2)^2} + \frac{q^4 p_\perp^2 (4u^2 - p_\perp^2)}{64(u^2 + p_\perp^2)^2}. $$

We can then write $V_{\text{vac}} = V_{\text{div}} + V_{\text{fin}}$, where

$$V_{\text{div}} = -\frac{16N_c(e^{\gamma^2 A^2})^\epsilon}{(4\pi)^2 \Gamma(1 - \epsilon)} \sum_{\pm} \int_{\Delta}^{\infty} \frac{du}{\sqrt{u^2 - \Delta^2}} \int_0^{\infty} dp_\perp \text{sub}(u, p_\perp) \gamma^{1-2\epsilon}, $$

$$V_{\text{fin}} = -\frac{16N_c(e^{\gamma^2 A^2})^\epsilon}{(4\pi)^2 \Gamma(1 - \epsilon)} \sum_{\pm} \int_{\Delta}^{\infty} \frac{du}{\sqrt{u^2 - \Delta^2}} \int_0^{\infty} dp_\perp \left[ \sqrt{(u + \frac{q}{2})^2 + p_\perp^2} - \frac{1}{2} \text{sub}(u, p_\perp) \right] \gamma^{1-2\epsilon}. $$

The integral $V_{\text{fin}}$ can now be calculated directly in three dimensions. After integrating over $p_\perp$, we find

$$V_{\text{fin}} = -\frac{16N_c}{3(4\pi)^2} \int_{\Delta}^{\infty} \frac{du}{\sqrt{u^2 - \Delta^2}} (u - \frac{q}{2})^2 \left[ (u - \frac{q}{2}) - \left| u - \frac{q}{2} \right| \right]. $$
This finally yields

\[ V_{\text{fin}} = -\frac{32N_c}{3(4\pi)^2} \int_\Delta^\infty \frac{u \, du}{\sqrt{u^2 - \Delta^2}} (u - \frac{q}{2})^3 \theta(\frac{q}{2} - \Delta) \]

\[ = \frac{N_c}{3(4\pi)^2} \left[ q\sqrt{\frac{q^2}{4} - \Delta^2(26\Delta^2 + q^2)} - 12\Delta^2(\Delta^2 + q^2) \log \frac{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \Delta^2}}{\Delta} \right] \theta(\frac{q}{2} - \Delta) \]

\[ = f(\Delta, q) . \quad (29) \]

We next integrate \( V_{\text{div}} \) using dimensional regularization. Again this is done by first integrating over \( p_\perp \) and then over \( u \). This yields

\[ V_{\text{div}} = \frac{2N_c}{(4\pi)^2} \left( \frac{\epsilon \gamma \Lambda^2}{\Delta^2} \right)^\epsilon \left[ 2\Delta^4 \Gamma(-2 + \epsilon) + q^2 \Delta^2 \Gamma(\epsilon) + \frac{q^4}{12}(-1 + \epsilon)\Gamma(1 + \epsilon) \right] . \quad (30) \]

The divergent parts of the vacuum energy are given by Eqs. (20) and (30), and require renormalization. In the \( \overline{\text{MS}} \) scheme, the poles in \( \epsilon \) are removed by multiplying the \( B^2 \) term, the mass parameter and the couplings in the tree-level potential (9) by \( Z_A, Z_{m^2}, Z_\lambda, Z_{g^2}, \) and \( Z_h \), respectively, where

\[ Z_A = 1 - N_c \sum_f \frac{4g_f^2}{3(4\pi)^2} \epsilon, \quad Z_{m^2} = 1 + \frac{4g^2 N_c}{(4\pi)^2} \epsilon, \quad Z_\lambda = 1 + \frac{8N_c}{(4\pi)^2} \left[ \lambda g^2 - 6g^4 \right] , \quad (31) \]

\[ Z_{g^2} = 1 + \frac{4g^2 N_c}{(4\pi)^2} \epsilon, \quad Z_h = 1 + \frac{2g^2 N_c}{(4\pi)^2} \epsilon . \quad (32) \]

After renormalization, the vacuum energy in the mean-field approximation reads

\[ V = \frac{1}{2} B_{\text{str}}^2 + \frac{1}{2} g_{\text{str}}^2 \frac{\Delta^2}{24} - h_{\text{str}} \frac{\Delta}{24} + N_c \sum_f \frac{2(q_f B)^2}{3(4\pi)^2} \log \frac{\Lambda^2}{2|q_f B|} \]

\[ + \frac{2N_c \Delta^2 q^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Delta^2} + \frac{2N_c \Delta^4}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{\Delta^2} + 3 \right] - \frac{N_c q_f^4}{6(4\pi)^2} + f(\Delta, \frac{q}{2}) \]

\[ + \frac{N_c}{(4\pi)^2} \sum_{f, \pm} \int_0^\infty ds \int_0^{\infty} \frac{d \bar{s}}{s^2} \left[ [q_f B s \coth ([q_f B s] - 1)] e^{-s a_{\perp}^2} - \frac{1}{3} e^{-2s|q_f B| (q_f B s^2 - s^2)} \right] , \quad (33) \]

where the subscript \( \text{str} \) indicates that the coupling are running. The running field, mass and couplings constants satisfy the following renormalization group equations

\[ \Lambda \frac{dB_{\text{str}}^2(\Lambda)}{d\Lambda} = -N_c \sum_f \frac{4q_f^2 B_{\text{str}}^2(\Lambda)}{3(4\pi)^2} , \quad (34) \]

\[ \Lambda \frac{dm_{\text{str}}^2(\Lambda)}{d\Lambda} = \frac{8N_c m_{\text{str}}^2(\Lambda) g_{\text{str}}^2(\Lambda)}{(4\pi)^2} , \quad (35) \]

\[ \Lambda \frac{dg_{\text{str}}^2(\Lambda)}{d\Lambda} = \frac{8N_c g_{\text{str}}^2(\Lambda)}{(4\pi)^2} , \quad (36) \]

\[ \Lambda \frac{d\lambda_{\text{str}}(\Lambda)}{d\Lambda} = \frac{16N_c \lambda_{\text{str}}(\Lambda) g_{\text{str}}^2(\Lambda)}{(4\pi)^2} - 6g_{\text{str}}^4(\Lambda) \quad (37) \]

\[ \Lambda \frac{dh_{\text{str}}(\Lambda)}{d\Lambda} = \frac{4N_c g_{\text{str}}^2(\Lambda) h_{\text{str}}(\Lambda)}{(4\pi)^2} . \quad (38) \]
The solutions to Eqs. (34)–(38) are

\[
B^2_{\sigma\pi}(\Lambda) = \frac{B_0^2}{1 + \sum_f \frac{4g_f^2 N_c}{3(4\pi)^2} \log \Lambda^2_{m_f^q}},
\]

(39)

\[
m^2_{\sigma\pi}(\Lambda) = \frac{m_0^2}{1 - \frac{4g_0^2 N_c}{(4\pi)^2} \log \Lambda^2_{m_0^q}},
\]

(40)

\[
g^2_{\sigma\pi}(\Lambda) = \frac{g_0^2}{1 - \frac{4g_0^2 N_c}{(4\pi)^2} \log \Lambda^2_{m_0^q}},
\]

(41)

\[
\lambda_{\sigma\pi}(\Lambda) = \frac{\lambda_0}{1 - \frac{4g_0^2 N_c}{(4\pi)^2} \log \Lambda^2_{m_0^q}},
\]

(42)

\[
h_{\sigma\pi}(\Lambda) = \frac{h_0}{1 - \frac{2g_0^2 N_c}{(4\pi)^2} \log \Lambda^2_{m_0^q}},
\]

(43)

where \(B_0, m_0^2, \lambda_0, g_0^2, \) and \(h_0\) are constants. The parameters \(m_0^2, \lambda_0, g_0^2, \) and \(h_0\) are the values of the running parameters at the scale \(\Lambda_0\), where \(\Lambda_0\) satisfies

\[
\log \frac{\Lambda^2_s}{m_0^2} + F(m_0^2) + m_0^2 F'(m_0^2) = 0.
\]

(44)

In Appendix B, we derive the relations between the parameters in the on-shell and \(\overline{\text{MS}}\) schemes. The parameters in the \(\overline{\text{MS}}\) can then be expressed in terms of the physical quantities \(m_0^2\) etc. Evaluating these parameters at \(\Lambda = \Lambda_0\) gives \(m_0\) etc as functions of \(m_0^2\), \(m_0^2\), \(m_0\), and \(f_\pi\). For example, evaluating \(g^2_{\sigma\pi}(\Lambda)\) at \(\Lambda = \Lambda_0\), yields \(g_0 = \frac{m_0^2}{f_\pi}\). The other parameters can be found in the same manner. Inserting the solutions (39)–(43) into (33) and expressing the parameters in terms of physical quantities, we finally obtain the renormalized vacuum energy in the large-\(N_c\) limit

\[
V_1 = \frac{1}{2} B_0^2 \left\{ 1 + \frac{2g_0^2}{3(4\pi)^2} \log \frac{m_0^2}{2[q_f B]} + \frac{1}{2} f_\pi^2 q^2 \right\} \left\{ 1 - \frac{4m_0^2 N_c}{f_\pi^2} \log \frac{\Delta^2_{m_0^q}}{m_0^2} \right\}
\]

(45)

\[
+ \frac{3}{4} m_0^2 f_\pi^2 \left\{ 1 - \frac{4m_0^2 N_c}{f_\pi^2} m_0^2 F'(m_0^2) \right\} \frac{\Delta^2_{m_0^q}}{m_0^2}
\]

\[
- \frac{1}{4} m_0^2 f_\pi^2 \left\{ 1 + \frac{4m_0^2 N_c}{f_\pi^2} \left[ \left( 1 - \frac{4m_0^2}{m_0^2} \right) F(m_0^2) + \frac{4m_0^2}{m_0^2} F(m_0^2) - m_0^2 F'(m_0^2) \right] \right\} \frac{\Delta^2_{m_0^q}}{m_0^2}
\]

\[
+ \frac{1}{8} m_0^2 f_\pi^2 \left\{ 1 - \frac{4m_0^2 N_c}{f_\pi^2} \left[ \frac{4m_0^2}{m_0^2} \log \frac{\Delta^2_{m_0^q}}{m_0^2} - \frac{3}{2} \right] - \left( 1 - \frac{4m_0^2}{m_0^2} \right) F(m_0^2) + F(m_0^2) + m_0^2 F'(m_0^2) \right\} \frac{\Delta^4_{m_0^q}}{m_0^4}
\]

\[
- \frac{1}{8} m_0^2 f_\pi^2 \left\{ 1 - \frac{4m_0^2 N_c}{f_\pi^2} m_0^2 F'(m_0^2) \right\} \frac{\Delta^4_{m_0^q}}{m_0^4} - m_0^2 f_\pi^2 \left\{ 1 - \frac{4m_0^2 N_c}{f_\pi^2} m_0^2 F'(m_0^2) \right\} \frac{\Delta^4_{m_0^q}}{m_0^4} - N_c q^4 \frac{\Delta^4_{m_0^q}}{6(4\pi)^2}
\]

\[
+ \frac{N_c}{3(4\pi)^2} \int_{\rho_{\pm}} \int_{s_\pm} \int_{0}^{\infty} \frac{ds}{s^2} \left\{ \left[ q_f B |s \coth(|q_f B|s) - 1 \right] e^{-s_\pm} - \frac{1}{3} e^{-2s_\pm |q_f B|^2 B} e^{-s_\pm} \right\}.
\]

III. RESULTS AND DISCUSSION

In the numerical results presented below, we use a sigma mass of \(m_\sigma = 600\) MeV, a pion mass of \(m_\pi = 140\) MeV, a quark mass of \(m_q = 300\) MeV, and a pion decay constant \(f_\pi = 93\) MeV.

A. Homogeneous case

First we restrict ourselves to a homogeneous condensate. The vacuum part of the thermodynamic potential is then found by setting \(q = 0\) in Eq. (45). In that case, one can integrate over \(p_\pm\) and \(s\) explicitly, which yields
\[ V_1 = \frac{1}{2} B^2 \left\{ 1 + N_c \sum_f \frac{2 q_f^2}{3(4\pi)^2} \log \frac{m^2_q}{2|q_f B|} \right\} + \frac{3}{4} m^2 \left( -\frac{4 m^2_N c}{(4\pi)^2 f^2}\right) F'(m^2) \left( 1 - \frac{4 m^2_N c}{(4\pi)^2 f^2} \right) \frac{\Delta^2}{m^2} \]

\[ -\frac{1}{4} m^2 \left( -\frac{4 m^2_N c}{(4\pi)^2 f^2}\right) \left( 1 + \frac{4 m^2 N c}{(4\pi)^2 f^2} \right) \left( \frac{1}{m^2} - \frac{4 m^2 N c}{(4\pi)^2 f^2} \right) \frac{\Delta^2}{m^2} \]

\[ + \frac{1}{8} m^2 \left( -\frac{4 m^2 N c}{(4\pi)^2 f^2}\right) \left( 1 - \frac{4 m^2 N c}{(4\pi)^2 f^2} \right) \left( \frac{m^2}{m^2} \log \frac{2 m^2}{m^2} - \frac{3}{2} \right) \left( \frac{1}{m^2} - \frac{4 m^2 N c}{(4\pi)^2 f^2} \right) \frac{\Delta^2}{m^2} \]

\[ - \sum_f \frac{8 N_c |q_f B|^2}{(4\pi)^2} \left( \frac{1}{\sqrt{\mu^2 - \Delta^2 - M^2_B}} \log \frac{\mu^2 - \Delta^2 - M^2_B}{\sqrt{\Delta^2 + M^2_B}} \right) \]

where \( x_f = \frac{\Delta^2}{2|q_f B|} \), \( \zeta(a, x_f) \) is the Hurwitz zeta-function, and \( \zeta(1,0)(-1, x_f) = \frac{\partial \zeta(a, x_f)}{\partial a} \big|_{a=-1} \).

The finite-density contribution \( V_{\text{den}} \) is given by the zero-temperature limit of the logarithmic terms in Eq. (13). In the homogeneous case, it reduces to

\[ V_{\text{den}} = -N_c \sum_{f,k,\sigma} \frac{|q_f B|}{2\pi} \int_{p_z} (\mu - E) \theta(\mu - E) , \]

where \( E = \sqrt{p_z^2 + \Delta^2 - M^2_B} \). Integrating over \( p_z \) from \( p_z = 0 \) to \( p_z = p_F = \sqrt{\mu^2 - \Delta^2 - M^2_B} \), we obtain

\[ V_{\text{den}} = -N_c \sum_{f,k,\sigma} \frac{|q_f B|}{4\pi^2} \left( \mu \sqrt{\mu^2 - \Delta^2 - M^2_B} - \sqrt{\Delta^2 + M^2_B} \log \frac{\mu + \sqrt{\mu^2 - \Delta^2 - M^2_B}}{\sqrt{\Delta^2 + M^2_B}} \right) \times \theta(\mu - \sqrt{\Delta^2 - M^2_B}) . \]

The quark density is given by

\[ \rho = - \frac{\partial V_{\text{den}}}{\partial \mu} = N_c \sum_{f,k,\sigma} \frac{|q_f B|}{2\pi} \sqrt{\mu^2 - \Delta^2 - M^2_B} \theta(\mu - \sqrt{\Delta^2 - M^2_B}) . \]

The sum over Landau levels is cut off due to the theta function and the highest Landau level included in the sum is for each quark flavor given by

\[ k_{\text{max}, f} = \frac{\mu^2 - \Delta^2}{2|q_f B|} . \]

In Fig. [1] we show \( \Delta \) as a function of the magnetic field in in units of \( m^2 \) in the vacuum, i.e., for \( \mu = 0 \). The quark mass is increasing as a function of \( B \), which implies that the system shows magnetic catalysis. Magnetic catalysis in the vacuum is a robust result, which has been found in lattice simulations [21] as well as model calculations, see [31] for a review.

In Fig. [2] we show \( \Delta \) as a function of \( |eB| \) for \( \mu = 330 \) MeV. We notice the oscillations in the parameters \( \Delta \) as \( |eB| \) increases. Such oscillatory behavior is known from cold dense systems in an external magnetic field and caused by the discrete nature of the Landau levels. For constant chemical potential the maximum number of Landau levels that are summed over in [18] decreases with \( \frac{1}{|eB|} \). The successive large and small jumps in \( \Delta \) are related to summing over one less Landau level of the up- and down-quark due to their different electric charge. For a large enough magnetic field only the lowest Landau level contributes and the oscillations stop.

In Fig. [3] we show the full phase diagram in the homogeneous case. One can clearly see the critical lines associated with the different Landau levels. The critical chemical potential \( \mu_{c,0} \), which indicates the transition from the vacuum phase to the one with non-zero quark density, becomes lower with increasing magnetic field strength. It has been shown in [21] that the true ground state of QCD in a magnetic field is a chiral soliton lattice, above a critical value of \( B \) that depends on \( \mu_B \). This critical value \( B_c(\mu) \) has been estimated in Ref. [27] and is shown in Fig. [3] as a black line.

At this point, it is appropriate to compare the quark-meson model to the NJL model and point out some dif-
FIG. 1. $\Delta$ as a function $|eB|$ in units of $m_{\pi}^2$ for $\mu = 0$.

FIG. 2. $\Delta$ as a function $|eB|$ in units of $m_{\pi}^2$ for $\mu = 330$ MeV.

ferences. At zero magnetic field $B$, the NJL model exhibits spontaneous symmetry breaking only if the coupling $G$ is larger than some critical coupling $G_c = 4\pi^2/\Lambda^2$, where $\Lambda$ is a sharp momentum cutoff which is used to regulate the fermionic vacuum fluctuations. Thus, in the NJL model vacuum fluctuations induce spontaneous symmetry breaking, which is in contrast to the QM model, as symmetry breaking has been implemented at tree level using a negative mass parameter $m^2$ in the potential.

Moreover, for $G < G_c$ an arbitrarily small magnetic field induces symmetry breaking. This is often referred to as dynamical chiral symmetry breaking (DCSB) and was first observed in the NJL model in 2+1 dimensions [35]. The mechanism behind was later explained [36]; the magnetic field reduces the dynamics of the two spatial transverse directions, leaving an effectively 1+1 dimensional system. This is reminiscent of the formation of a gap in superconductors. We expect DCSM to be present in the QM model as well but this has not been studied since in phenomenological applications, symmetry breaking is implemented at tree level.

B. Inhomogeneous case

In the inhomogeneous case, the finite-density contribution to the potential reads

$$V_{\text{den}} = -N_c \sum_{f,\pm,k,\sigma_z} \frac{|q_f B|}{2\pi} \int_{p_z} (\mu - E_\pm) \theta(\mu - E_\pm) \tag{51}$$

where $E_\pm$ is given by Eq. (6). In section II we showed that the vacuum energy is independent of $q$ in the limit $\Delta \rightarrow 0$, which indicates that it is meaningful. Setting $\Delta = 0$ in the dispersion relation (6) and evaluating the integral over $p_z$ in Eq. (51), one finds that $V_{\text{den}}$ is independent of $q$ as well. Thus the full thermodynamic potential given by the sum of Eqs. (45) and (51) is well defined.

The quark density can be calculated explicitly and
The vacuum phase extends from dashed line) as functions of the chemical potential and the wave vector \( q \) for zero magnetic field.\(^2 \) The ansatz (3) assumes that the wave vector \( q \) (red dashed line) as functions of \( \mu \) for for \(|eB| = m^2_\mu\). Again, the vacuum phase exists for \( \mu = 0 \) to \( \mu = \mu_{c,0} = 300 \text{ MeV} \), whereafter several homogeneous phases with finite quark density appear. The inhomogeneous phase starts at \( \mu = \mu_{c,1} = 321 \text{ MeV} \). One can clearly see the successive jumps in the order parameters \( \Delta \) and \( q \). For a constant magnetic field the number of relevant Landau levels increases with \( \mu^2 \). When the next Landau level is included in the sum, the values of the chiral condensate and wave vector jump. It turns out that \( \Delta \) starts increasing again past \( \mu = 415 \), and beyond \( \mu = 505 \) we can no longer find a minimum of the effective potential. This is not worrisome as one cannot trust the model for chemical potentials this large anyway. Finally, the effects of a magnetic field on the inhomogeneous phase has been studied before in [15] using the NJL model. There it has been found that for non-zero magnetic field the wave vector increases linearly up to the onset of a strong inhomogeneous phase. In contrast to these results we find zero wave vector for all \( \mu \) smaller than \( \mu_{c,1} \).

- FIG. 4. \( \Delta \) (blue solid line) and the wave vector \( q \) (red dashed line) as functions of the chemical potential \( \mu \) for \( B = 0 \).

- FIG. 5. \( \Delta \) (blue solid line) and the wave vector \( q \) (red dashed line) as functions of \( \mu \) for \(|eB| = m^2_\mu\).

In Fig. 6 we show \( \Delta \) (blue solid line) and the wave vector \( q \) (red dashed line) as functions \(|eB|\) for \( \mu = 330 \text{ MeV} \). We have chosen a value for \( \mu \) that lies in the inhomogeneous phase for \( B = 0 \). \( \Delta \) is oscillating just like in the homogeneous case and the value of \( \Delta \) for a given \(|eB|\) is larger than in the homogeneous case (see Fig. 2).

The ansatz \(^3 \) assumes that the wave vector \( q \) is parallel to the magnetic field \( B \), which is only a special case. The most general case has the wave vector pointing in an arbitrary direction, allowing for a nonzero component \( q_\perp \). In this case the rotational symmetry is completely broken. Then, however, the spectrum is not known, which prevents us from carrying out a complete analysis of the problem. In Ref. [15], the authors used perturbation theory for two nearly degenerate levels to calcu-

---

2 The explicit expressions for the free energy and the quark density for \( B = 0 \) can be found in Ref. [19].
FIG. 6. $\Delta$ (blue solid line) and the wave vector $q$ (red dashed line) as functions $|eB|$ for $\mu = 330$ MeV.

late the correction to the energy levels where the perturbation is a function of $q_{\perp}$. Using these results, they numerically calculated the second derivative of the thermodynamic potential $\frac{\partial^2 U}{\partial q_{\perp}^2}$ in the NJL model. It turns out that this quantity is positive everywhere, indicating stability of the thermodynamic potential with respect to $q_{\perp}$.

The specific integrals, we need are

\[
\int_p \sqrt{u^2 + p_{\perp}^2} = -\left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2}\right)^\epsilon \frac{\Delta^4}{(4\pi)^2} \Gamma(-2 + \epsilon) = -\left(\frac{\Lambda^2}{\Delta^2}\right)^\epsilon \frac{\Delta^4}{2(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{3}{2} + \mathcal{O}(\epsilon)\right],
\]

\[
\int_p \frac{p_{\perp}^2}{u^2 + p_{\perp}^2} = \left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2}\right)^\epsilon \frac{4\Delta^2}{(4\pi)^2} \Gamma(\epsilon) \left(\frac{\Lambda^2}{\Delta^2}\right)^\epsilon = \frac{4\Delta^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right],
\]

\[
\int_p \frac{p_{\perp}^2 (4u^2 - p_{\perp}^2)}{u^2 + p_{\perp}^2} = \left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2}\right)^\epsilon \frac{16}{3(4\pi)^2} (-1 + \epsilon) \Gamma(1 + \epsilon) = -\frac{16}{3(4\pi)^2} + \mathcal{O}(\epsilon),
\]

where $u = \sqrt{\Delta^2 + p_{\perp}^2}$.

We also need some integrals in $D = 4 - 2\epsilon$ dimensions. Specifically, we need the integrals

\[
A(m^2) = \int_p \frac{1}{p^2 - m^2} = \frac{im^2}{(4\pi)^2} \left(\frac{\Delta^2}{m^2}\right)^\epsilon \left[\frac{1}{\epsilon} + 1 + \mathcal{O}(\epsilon)\right],
\]

\[
B(p^2) = \int_k \frac{1}{(k^2 - m_q^2)((k + p)^2 - m_q^2)} = \frac{i}{(4\pi)^2} \left(\frac{\Delta^2}{m_q^2}\right)^\epsilon \left[\frac{1}{\epsilon} + F(p^2) + \mathcal{O}(\epsilon)\right],
\]

\[
B'(p^2) = \frac{i}{(4\pi)^2} F'(p^2),
\]

where we have defined the functions

\[
F(p^2) = -\int_0^1 dx \log \left[\frac{p^2}{m_q^2} x(x - 1) + 1\right] = 2 - 2\arctan \left(\frac{1}{p^2}\right),
\]

\[
F'(p^2) = \frac{4m_q^2 r}{p^2 (4m_q^2 - p^2)} \arctan \left(\frac{1}{p^2}\right) - \frac{1}{p^2}.
\]

**Appendix B: Parameter fixing**

In this appendix we use the on-shell renormalization scheme to relate the model parameters to physical ob-

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**Appendix A: Integrals in dimensional regularization**

In order to calculate the vacuum energy for $B = 0$, we need a number of integrals in three dimensions. These integrals are divergent in the ultraviolet and regularized using dimensional regularization. In analogy with Eq. (12), we define the dimensionally regulated integral in $d = 3 - 2\epsilon$ dimension as

\[
\int_p \frac{e^{\gamma_E} \Lambda^2}{4\pi} \int \frac{d^d p_{\perp}}{2\pi} \int \frac{d^d-1 p_{\perp}}{(2\pi)^{d-1}},
\]

where $p_{\perp}^2 = p_x^2 + p_y^2$. 
parameters

\[ m_B^2 = Z_m^2 m^2, \quad \lambda_B = Z_\lambda \lambda, \quad g_B^2 = Z_g g^2, \]
\[ h_B = Z_h h, \quad f_{\pi, B}^2 = Z_{f_\pi} f_{\pi}^2, \quad B_B^2 = Z_A B^2 \]
\[ \sigma_B = Z_{\sigma}^{1/2} \sigma, \quad \pi_B = Z_{\pi}^{1/2} \pi, \quad \psi_B = Z_{\psi}^{1/2} \psi, \]
where \( Z_\sigma = 1 + \delta Z_\sigma \) and so on. The renormalization constants of the model parameters are given in terms of those of the physical parameters and from Eq. (2.7) and (2.8) we find

\[ \delta m^2 = - \frac{1}{2} (\delta m^2 - 3 \delta m^2), \quad (B1) \]
\[ \delta \lambda = 3 \frac{\delta m^2}{f_\sigma^2} - \lambda \frac{\delta f_\sigma^2}{f_\sigma^2}, \quad (B2) \]
\[ \delta g^2 = \frac{\delta m^2}{f_\pi^2} - g^2 \frac{\delta f_\pi^2}{f_\pi^2}. \quad (B3) \]

Considering that \( h = m_\pi f_\pi - t \), where \( t = 0 \) is the tree-level tadpole, we find

\[ \delta h = \delta m^2 f_\pi + \frac{1}{2} m^2 f_\pi \delta f_\pi^2 - \delta t. \quad (B4) \]

In the large-\( N_c \) limit there are no loop corrections to the quark mass or quark field renormalization, therefore \( \delta Z_\psi = 0 \) and \( \delta m_q = 0 \), which leads to

\[ \frac{\delta g^2}{g^2} = - \frac{\delta f_\sigma^2}{f_\sigma^2}. \quad (B5) \]

Similarly, in the large-\( N_c \) limit, the corrections to the pion-quark vertex vanish. This implies

\[ \frac{\delta g^2}{g^2} = - \delta Z_\pi, \quad (B6) \]

which allows us to write

\[ \frac{\delta f_\pi^2}{f_\pi^2} = \delta Z_\pi. \quad (B7) \]

In the OS scheme, the renormalized mass of each particle is equal to the physical one, which is given by the pole of the respective propagator. This gives the OS renormalization conditions for the sigma and pion masses

\[ \delta m_{\sigma, \pi}^2 = -i \delta \Sigma_{\sigma, \pi}(p^2 = m_{\sigma, \pi}^2). \quad (B8) \]

In addition, the residues of the propagators at the poles are equal to one, which gives the renormalization conditions for the fields:

\[ \delta Z_{\sigma, \pi} = i \frac{\partial}{\partial p^2} \delta \Sigma_{\sigma, \pi}(p^2)|_{p^2=m_{\sigma, \pi}^2} \quad (B9) \]

The self-energies in the large-\( N_c \) limit are

\[ \delta \Sigma_{\sigma}(p^2) = -8 g^2 N_c \left[ A(m_\sigma^2) - \frac{1}{2} (p^2 - 4 m_\sigma^2) B(p^2) \right] + \frac{4 \lambda g f_\sigma N_c m_q}{m_\sigma^2} A(m_q^2), \quad (B10) \]
\[ \delta \Sigma_{\pi}(p^2) = -8 g^2 N_c \left[ A(m_\pi^2) - \frac{1}{2} p^2 B(p^2) \right] + \frac{4 \lambda g f_\sigma N_c m_q}{3 m_\sigma^2} A(m_q^2), \quad (B11) \]

where the last term in both equations is the tadpole contribution. Their derivatives, i.e. the wave function renormalization counterterms are

\[ \delta Z_\sigma = 4 i g^2 N_c \left[ B(m_\sigma^2) + (m_\sigma^2 - 4 m_q^2) B'(m_\sigma^2) \right] \quad (B12) \]
\[ \delta Z_\pi = 4 i g^2 N_c \left[ B(m_\pi^2) + m_\pi^2 B'(m_\pi^2) \right]. \quad (B13) \]

The tadpole counterterm is determined from the vanishing one-point function and reads

\[ \delta t = -8 i N_c g^2 f_\pi A(m_q^2). \quad (B14) \]
With this we find the OS renormalization constants of the model parameters

\[
\delta m^2_{\text{OS}} = 8ig^2N_c \left[ A(m^2_\sigma) + \frac{1}{4} (m^2_\sigma - 4m^2_\pi)B(m^2_\sigma) - \frac{3}{4}m^2_\pi B(m^2_\pi) \right] \\
= \delta m^2_{\text{div}} + \frac{4g^2N_c}{(4\pi)^2} \left[ m^2_\sigma \log \frac{\Lambda^2}{m^2_\sigma} - 2m^2_\sigma - \frac{1}{2} (m^2_\sigma - 4m^2_\pi)F(m^2_\sigma) + \frac{3}{2}m^2_\pi F(m^2_\pi) \right], \\
\delta \lambda_{\text{OS}} = -\frac{12ig^2N_c}{f^2_\pi} (m^2_\sigma - 4m^2_\pi)B(m^2_\sigma) + \frac{12ig^2N_c}{f^2_\pi} m^2_\pi B(m^2_\pi) - 4i\lambda g^2N_c \left[ B(m^2_\pi) + m^2_\pi B'(m^2_\pi) \right] \\
= \delta \lambda_{\text{div}} + \frac{12g^2N_cm^2_\sigma}{(4\pi)^2f^2_\pi} \left[ \left( 1 - \frac{4m^2_\sigma}{m^2_\pi} \right) \left( \log \frac{\Lambda^2}{m^2_\pi} + F(m^2_\pi) \right) + \log \frac{\Lambda^2}{m^2_\pi} + F(m^2_\pi) + m^2_\pi F'(m^2_\pi) \right] \\
- \frac{12g^2N_cm^2_\pi}{(4\pi)^2f^2_\pi} \left[ 2 \log \frac{\Lambda^2}{m^2_\pi} + 2F(m^2_\pi) + m^2_\pi F'(m^2_\pi) \right], \\
\delta g^2_{\text{OS}} = -4ig^4N_c \left[ B(m^2_\pi) + m^2_\pi B'(m^2_\pi) \right] \\
= \delta g^2_{\text{div}} + \frac{4g^4N_c}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m^2_\pi} + F(m^2_\pi) + m^2_\pi F'(m^2_\pi) \right], \\
\delta h_{\text{OS}} = -2ig^2N_cm^2_\pi f_\pi \left[ B(m^2_\pi) - m^2_\pi B'(m^2_\pi) \right] \\
= \delta h_{\text{div}} + \frac{2g^2N_cm^2_\pi f_\pi}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m^2_\pi} + F(m^2_\pi) - m^2_\pi F'(m^2_\pi) \right], \\
\delta Z_{\sigma}\text{div} = -\frac{4g^2N_c}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m^2_\sigma} + F(m^2_\sigma) + (m^2_\sigma - 4m^2_\pi)F'(m^2_\pi) \right], \\
\delta Z_{\pi}\text{div} = -\frac{4g^2N_c}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m^2_\pi} + F(m^2_\pi) + m^2_\pi F'(m^2_\pi) \right],
\]

where we have defined the divergent quantities

\[
\begin{align*}
\delta m^2_{\text{div}} &= \frac{4g^2N_c}{(4\pi)^2} \left( \frac{m^2_\sigma}{\epsilon} \right), \\
\delta \lambda_{\text{div}} &= \frac{8N_c}{(4\pi)^2} \left( \lambda g^2 - 6g^4 \right), \\
\delta g^2_{\text{div}} &= \frac{4g^4N_c}{(4\pi)^2} \left( \frac{1}{\epsilon} \right), \\
\delta Z_{\sigma}\text{div} &= -\frac{4g^2N_c}{(4\pi)^2} \left( \frac{1}{\epsilon} \right), \\
\delta Z_{\pi}\text{div} &= -\frac{4g^2N_c}{(4\pi)^2} \left( \frac{1}{\epsilon} \right), \\
\delta h &= \frac{2g^2hN_c}{(4\pi)^2} \left( \frac{1}{\epsilon} \right).
\end{align*}
\]

The photon self-energy in the vacuum is

\[
\Pi_{\mu\nu}(p^2) = i \left( p^2 g_{\mu\nu} - p_\mu p_\nu \right) \Pi(p^2),
\]

where \( \Pi(p^2) \) in the large-\( N_c \) limit is given by the quark-loop contribution

\[
\Pi(p^2) = -\frac{i8N_c}{(4\pi)^2} \sum_f g^2_f \left[ \frac{1}{6} \left[ \frac{1}{\epsilon} + \log \left( \frac{\Lambda^2}{m^2_\sigma} \right) \right] - \frac{1}{3} \int dx (1-x) \log \left( 1 - x(1-x) \frac{p^2}{m^2_\pi} \right) \right].
\]

The \( B \)-field renormalization constant is then determined by the renormalization condition

\[
\delta Z_{A}\text{div} = -i\Pi(p^2 = 0) = -\frac{4N_c}{3(4\pi)^2} \sum_f g^2_f \left[ \frac{1}{\epsilon} + \log \left( \frac{\Lambda^2}{m^2_\sigma} \right) \right].
\]

Since the bare parameters are independent of the renormalization scheme we can immediately find the \( \overline{\text{MS}} \) parameters

\[12\]
from $m_{\text{str}}^2 = m^2 + \delta m_{\text{str}}^2 - \delta m_{\text{str}}^2$ etc., which gives

$$B_{\text{str}}^2 = B^2 + i N_c \sum_f \frac{4(q_f B)^2}{3(4\pi)^2} B(0) - \delta B_{\text{str}}^2 = B^2 - N_c \sum_f \frac{4(q_f B)^2}{3(4\pi)^2} \log \frac{\Lambda^2}{m_{\pi}^2}, \quad (B24)$$

$$m_{\text{str}}^2 = m^2 + 8ig^2 N_c \left[ A(m_{\pi}^2) + \frac{1}{2} (m_{\sigma}^2 - 4m_{\pi}^2) B(m_{\sigma}^2) - \frac{3}{2} m_{\pi}^2 B(m_{\pi}^2) \right] - \delta m_{\text{str}}^2$$

$$= m^2 + \frac{4g^2 N_c}{(4\pi)^2} \left[ m_{\sigma}^2 \log \frac{\Lambda^2}{m_{\pi}^2} - 2m_{\pi}^2 + \frac{1}{2} \left( m_{\pi}^2 - 4m_{\pi}^2 \right) F(m_{\pi}^2) + \frac{3}{2} m_{\pi}^2 F'(m_{\pi}^2) \right], \quad (B25)$$

$$\lambda_{\text{str}} = \lambda - \frac{12ig^2 N_c}{f_{\pi}^2} (m_{\pi}^2 - 4m_{\pi}^2) B(m_{\sigma}^2) + \frac{12ig^2 N_c f_{\pi}^2 B(m_{\sigma}^2) - 4ig^2 N_c \left[ B(m_{\pi}^2) + m_{\pi}^2 B'(m_{\pi}^2) \right]}{f_{\pi}^2}$$

$$= \lambda + \left\{ \frac{12g^2 N_c}{(4\pi)^2 f_{\pi}^2} \left[ m_{\sigma}^2 \log \frac{\Lambda^2}{m_{\pi}^2} + m_{\sigma}^2 \left( \frac{1}{2} + F(m_{\pi}^2) \right) \right] \right\}, \quad (B26)$$

$$g_{\text{str}}^2 = g^2 - 4ig^2 N_c \left[ B(m_{\pi}^2) + m_{\pi}^2 B'(m_{\pi}^2) \right] - \delta g_{\text{str}}^2$$

$$= \frac{m_{\pi}^2}{f_{\pi}^2} \left\{ 1 + \frac{4g^2 N_c}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{\pi}^2} + F(m_{\pi}^2) \right] \right\}, \quad (B27)$$

$$h_{\text{str}} = h - 2ig^2 N_c m_{\pi}^2 f_{\pi} \left[ B(m_{\pi}^2) - m_{\pi}^2 B'(m_{\pi}^2) \right] - \delta h_{\text{str}}$$

$$= h + \frac{2g^2 h N_c m_{\pi}^2 f_{\pi}}{(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{\pi}^2} + F(m_{\pi}^2) \right], \quad (B28)$$

where the integrals $A(m_{\pi}^2)$ and $B(p^2)$ are defined in Eqs. $A5$-$A6$, and functions $F(p^2)$ and $F'(p^2)$ are defined in Eqs. $A7$-$A8$.

[1] M. G. Alford, A. Schmitt, K. Rajagopal, T. Schäfer, Rev. Mod. Phys. 80, 1455 (2008).
[2] K. Fukushima and T. Hatsuda, Rept. Prog. Phys. 74, 014001 (2011).
[3] Y. Aoki, Z. Fodor, S. Katz, and K. Szabo, Phys. Lett. B 643, 46 (2006).
[4] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S.D. Katz et al., JHEP 0906, 088 (2009).
[5] S. Borsanyi et al. (Wuppertal-Budapest Collaboration), JHEP 1007, 073 (2010).
[6] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H. Ding et al., Phys. Rev. D 85, 054503 (2012).
[7] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B 210, 423 (1982).
[8] K. Fukushima, Phys. Lett. B 591, 277 (2004).
[9] D. Ebert, K. G. Klimenko, M. A. Vdovichenko, and A. S. Vshivtsev, Phys. Rev. D 61, 025005 (2000).
[10] T. Inagaki, D. Kimura, and T. Murata, Prog. Theor. Phys. Suppl. 153, 321 (2004).
[11] E. J. Ferrer, V. de la Incera, C. Manuel, Phys. Rev. Lett. 95, 152002 (2005).
[12] J. L. Noronha and I. A. Shovkovy, Phys. Rev. D 76, 105030 (2007), Erratum: Phys. Rev. D 86, 049901 (2012).
[13] K. Fukushima and H. J. Warringa, Phys. Rev. Lett. 100, 032007 (2008).
[14] D. P. Menezes, M. B. Pinto, S. S. Avancini, A. Perez Martinez, and C. Providencia, Phys. Rev. C 93, 035807 (2009).
[15] I. E. Frolov, V. Ch. Zhukovsky, and K. G. Klimenko, Phys. Rev. D 82, 076002 (2010).
[16] P. G. Allen and N. N. Scoccola, Phys. Rev. D 88, 094005 (2013).
[17] A. G. Grunfeld, D. P. Menezes, M. B. Pinto, and N. N. Scoccola, Phys. Rev. D 90, 044024 (2014).
[18] R. Yoshiike, K. Nishiyama, and T. Tatsumi, Phys. Lett. B 751, 123 (2015).
[19] G. Cao and A. Huang, Phys. Rev. D 93, 076007 (2016).
[20] K. Nishiyama, S. Karasawa, and T. Tatsumi, Phys. Rev. D 92, 036008 (2015).
[21] T. Tatsumi, K. Nishiyama, and S. Karasawa, Phys. Lett. B 743, 66 (2015).
Katz, and A. Schafer, Phys. Rev. D 86, 071502 (2012).
[25] F. Bruckmann, G. Endrodi, T. G. Kovacs, JHEP 13 04, 112 (2013).
[26] D. T. Son and M. A. Stephanov, Phys. Rev. D 77, 014021 (2008).
[27] T. Brauner and S. Kadam, JHEP 17 03, 015 (2017).
[28] S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).
[29] J. O. Andersen and T. Brauner, Phys. Rev. D 81, 096004 (2010).
[30] D. Ebert, N. V. Gubina, K. G. Klimenko, S. G. Kurbanov, and V. Ch. Zhukovsky, Phys. Rev. D 84, 025004 (2011).
[31] P. Adhikari and J. O. Andersen, Phys. Rev. D 95, 036009 (2017).
[32] M. Buballa and S. Carignano, Prog. Part. Nucl. Phys. 81, 39 (2015).
[33] K. Fujikawa, Phys. Rev. D 21, 2848 (1980).
[34] I. A. Shovkovy, Lect. Notes Phys. 871, 13 (2013).
[35] K. G. Klimenko, Theor. Math. 89, 1161 (1992); Z. Phys. C 54, 323 (1992); K. G. Klimenko, A. S. Vshivtsev, and B. V. Magntisky, Nuovo Cimento A 107, 439 (1994).
[36] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. 73, 3499 (1994).
[37] E. V. Gorbar, V. A. Miransky, and I. A. Shovkovy. Phys. Rev. C 80 032801 (2009).
[38] T. D. Cohen, Phys. Rev. Lett. 91, 222001 (2003).
[39] P. Adhikari, J. O. Andersen, and P. Kneschke, Phys. Rev. D 95, 036017 (2017).