A DUAL-RADIX APPROACH TO STEINER’S 1-CYCLE THEOREM

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ABSTRACT. This article presents a variety of algebraic proofs of Steiner’s 1-Cycle Theorem [12]. It also demonstrates that, under an exponential upper-bound on the iterates, the only 1-cycles in the (accelerated) 3x – 1 dynamical system are (1) and (5, 7).

1. Introduction

Within the context of the 3x + 1 Problem, Steiner’s 1-cycle Theorem [12] is a result pertaining to the non-existence of 1-cycles (or circuits): for all a, b ∈ N, Steiner shows that a rational expression of the form

(1) \( \frac{2^a - 1}{2^a + b - 3^b} \)

does not assume a positive integer value except in the case where a = b = 1. In the proof, the author appeals to the continued fraction expansion of \( \log_2 3 \), transcendental number theory, and extensive numerical computation (see [11]). This argument serves as the basis for demonstrating the non-existence of 2-cycles in [10], and the non-existence of m-cycles in [11] where m ≤ 68.

However, the author in [7] declares that the “most remarkable thing about [the theorem] is the weakness of its conclusion compared to the strength of the methods used in its proof.” This article offers alternative proofs of this theorem using a variety of algebraic approaches; assuming the upper bound on periodic iterates established in [1], these proofs exploit that fact that the denominator in the above expression is coprime to both 2 and 3: this work simultaneously analyzes the residues of the circuit elements in a 2-adic and 3-adic setting. Based on the results in [9], the first proof employs elementary modular arithmetic, the second exploits identities on weighted binomial coefficients, and the third proof analyzes the 2-adic and 3-adic digits of such rational expressions.

2. Overview

2.1. Notation. This manuscript inherits all of the notation and definitions established in [9], which we summarize here. Let \( \tau \in \mathbb{N} \), and let m and l be coprime integers exceeding 1. Let \( e, f \in \mathbb{N}^\tau \) where \( e = (e_0, \ldots, e_{\tau-1}) \) and \( f = (f_0, \ldots, f_{\tau-1}) \). For each \( u \in \mathbb{Z} \), define \( E_u = \sum_{0 \leq w < u} e_w \mod \tau \) and \( F_u = \sum_{0 \leq w < u} f_{(\tau-1-w) \mod \tau} \); we will define \( F_u \) and \( F_u \) in an analogous manner with the elements of \( f \).

This work was supported by the Naval Surface Warfare Center Dahlgren Division’s In-House Laboratory Independent Research Program.
For any integer \( a \) and positive base \( b \) \((b \geq 1)\), let \([a]_b \) denote the element\(^1\) of \([b]_0\) that satisfies the equivalence \([a]_b \equiv a \mod b\). We may also express this element as \( a \mod b \). We will also write \([a]_b^{-1} \) to denote the element in \([b]_0\) that satisfies the equivalence \([a]_b^{-1} \equiv 1 \mod b\).

We will write \((-)^n = (-1)^n\) for each \( n \in \mathbb{N}\).

2.2. **Argument Overview.** This dual-radix approach to the non-existence of circuits is based upon the following premises:

i. Let \( \tau \in \mathbb{N} \), let \( m \) and \( l \) be coprime integers exceeding 1, and let \( n \) be a periodic orbit element from a given \((m,l)\)-system of order \( \tau \) satisfying the equivalences \( \mu_\tau \equiv n \mod m^{F_\tau} \) and \( \lambda_\tau \equiv n \mod l^{E_\tau}\), where \( \mu_\tau \) and \( \lambda_\tau \) are the canonical representatives of their corresponding equivalence classes.

In [9], the equalities

\[
n = \sum_{0 \leq w < \tau} m^{F_w} \frac{E_{\tau-1-w} a_w}{E_{\tau} - m^{F_{\tau}}} = \mu_\tau + m^{F_{\tau}} \left( \frac{\mu_\tau - \lambda_\tau}{l^{E_{\tau}} - m^{F_\tau}} \right) = \lambda_\tau + l^{E_{\tau}} \left( \frac{\mu_\tau - \lambda_\tau}{l^{E_{\tau}} - m^{F_\tau}} \right)
\]

have been demonstrated for an admissible sequence of translation values \( a = (a_0, \ldots, a_{\tau-1}) \); consequently, the denominator \( l^{E_{\tau}} - m^{F_{\tau}} \) divides the sum \( \sum_{0 \leq w < \tau} m^{F_w} \frac{E_{\tau-1-w} a_w}{l^{E_{\tau}} - m^{F_{\tau}}} \) if and only if it divides the arithmetic difference of canonical representatives \( \mu_\tau - \lambda_\tau \). Furthermore, as \( \mu_\tau \in [m^{F_{\tau}}]_0 \) and \( \lambda_\tau \in [l^{E_{\tau}}]_0 \), the iterate \( n \in \mathbb{N} \) if and only if \( \mu_\tau - \lambda_\tau \in DN_0 \).

ii. In the cases where \( m = 3 \), \( l = 2 \), \( f = \mathbf{1}^\tau = a \), we apply the argument outlined in [9]: we will establish an upper bound of \( 3^\tau \) for a potential, periodic iterate value over \( \mathbb{N} \) for the \( 3x + 1 \) Problem. In this context, the authors in [1] have demonstrated that the maximal iterate \( n_{\text{max}} \) within a periodic orbit admits the upper bound

\[
n_{\text{max}} < \left( \frac{3}{2} \right)^{\tau-1} \leq \tau C \left( \frac{3}{2} \right)^{\tau-1} = o(3^{\tau-1})
\]

for some effectively computable constant \( C \) (by applying the result in [13]). A recent upper bound on \( C \) is available in [8], in which the author establishes the inequality

\[
| -E_{\tau} \log 2 + \tau \log 3 | \geq E_{\tau}^{-13.3}
\]

(in their notation, we set \( u_0 = 0 \), \( u_1 = -E_{\tau} \), and \( u_2 = \tau \); consequently, assuming \( 2E_{\tau} > 3^\tau \), we can bound\(^2\) the denominator in (2) from below

\[
1 - \frac{3^\tau}{2E_{\tau}} \geq \frac{E_{\tau}^{-13.3}}{2}.
\]

\(^1\)This element is sometimes referred to as the standard (or canonical) representative of the equivalence class \( \pi \mod b \).

\(^2\)We can shed the logarithms: when \(|w| < 1\), the power series expansion of \( \log(1+w) = \sum_{u \geq 1} (-1)^{u-1} \frac{w^u}{u} \) yields \(|\log(1+w)| \leq 2|w|\) when \(|w| \leq \frac{1}{2} \). See [4] (Corollary 1.6).
According to [3], in a periodic orbit over \( N \) of length \( E \tau \), the ratio \( \frac{E \tau}{\tau} \) satisfies the inequality

$$ \frac{E \tau}{\tau} \leq \log \left( 3 + \frac{1}{n_{\min}} \right) \leq 2; $$

numerical computation yields

$$ n_{\max} < \left( \frac{3}{2} \right)^{\tau-1} 2 \cdot (2\tau)^{13.3} < 3^\tau $$

when \( \tau \geq 103 \).

Thus, if \( n_{\max} > 3^\tau \) and \( n_{\max} \in N \), then \( \tau < 103 \). However, the author in [5] demonstrates that the length of a non-trivial periodic orbit (excluding 1) over \( N \) must satisfy the inequality \( 2\tau \geq E \tau \geq 35,400 \).

Thus, if \( n \in N \), then \( n < 3^\tau \), and the equalities

$$ n = \mu_\tau = \lambda_\tau $$

must hold.

iii. Within a circuit of order \( \tau \) in the (accelerated) \( 3x + 1 \) dynamical system, the maximal element equals

$$ \frac{(2^e + 1)3^{\tau-1} - 2^{e+\tau-1}}{2^{e+\tau-1} - 3^\tau} = 2 \cdot 3^{\tau-1} \left( \frac{2^e - 1}{2^{e+\tau-1} - 3^\tau} \right) - 1 $$

for some \( e \in N \) (see [2]).

When \( \tau = 1 \), we note that \( 2^e - 3 = 2^{e-1} - 1 + 2^{e-1} - 2 \geq 2^{e-1} - 1 \) for \( e \geq 2 \); thus the ratio in (1), evaluated at \( a = e - 1 \) and \( b = 1 \), is at most one. When \( e = 1 \), the left-hand side of the equality above is negative, and the ratio in (1) vanishes.

When \( \tau > 1 \), we will analyze the difference of canonical residues

$$ \mu_\tau = \left[ (2^e + 1)3^{\tau-1} - 2^{e+\tau-1} \right] [2^{e+\tau-1}]^{-1} \mod 3^\tau $$

and

$$ \lambda_\tau = \left[ (2^e + 1)3^{\tau-1} - 2^{e+\tau-1} \right] [-3^\tau]^{-1} \mod 2^E \tau; $$

we will show that the difference \( \mu_\tau - \lambda_\tau \) is non-zero (contradicting the assumption that \( n = \mu_\tau = \lambda_\tau < 3^\tau \) as per above).

We will also perform similar analyses on the maximal element of a circuit within the (accelerated) \( 3x - 1 \) dynamical system; we will show that, assuming the inequality \( n < 2^E \tau \), a circuit over \( N \) exists if and only if either \( e = 1 \), or \( \tau = e = 2 \).

\(^3\)Appealing to a similar argument outlined above, this condition holds for finitely many \( \tau \) for each fixed \( e \in N \).
3. Circuits in (3, 2)-Systems

Throughout the remainder of the manuscript, unless otherwise stated, we assume that
i. \( \tau \in \mathbb{N} \) with \( \tau \geq 2 \);
ii. \((m, l) = (3, 2)\);
iii. \( f = (1, \ldots, 1) \in \mathbb{N}^\tau \);
iv. \( e = (1, \ldots, 1, e) \) for some \( e \in \mathbb{N} \); and 
v. \( a = (a_0, \ldots, a_{\tau - 1}) \in \{-1, +1\}^\tau \).

We begin with the following assumptions.

Assumptions 3.1. Assume 3.1 and 3.3 from [9], and let \( a = 1^\tau \). Let \( N = \sum_{0 \leq w < \tau} 3^w 2^e \tau - 2^w = (2^e + 1)3^\tau - 2^e \tau - 1 \), and let \( D = 2^e + \tau - 3^\tau \) where \( D > 0 \).

Assume that

\[ n = \frac{N}{D} < \min \left( 3^\tau, 2^e \right) , \]

let \( \mu = n \mod 3^\tau \) denote the 3-residue of \( n \), and let \( \lambda = n \mod 2^e + \tau - 1 \) denote the 2-residue of \( n \).

Under these assumptions, if \( n \in \mathbb{N} \), then the chain of equalities

\[ n = \mu = \lambda \]

holds.

Our goal for the remainder of this subsection is to prove the following theorem:

Theorem 3.1. Assume 3.1.

We have the equalities

\[ \mu = \begin{cases} 3^\tau - 1 & \text{if } e \equiv 0 \pmod{2} \\ 3^\tau - 1 & \text{if } e \equiv 1 \pmod{2} \end{cases} \]

when \( \tau \equiv 0 \), and

\[ \mu = \begin{cases} 2 \cdot 3^\tau - 1 & \text{if } e \equiv 0 \pmod{2} \\ 3^\tau - 1 & \text{if } e \equiv 1 \pmod{2} \end{cases} \]

when \( \tau \equiv 1 \).

Furthermore, when \( \tau \equiv 1 \equiv e - 1 \), then

\[ \lambda = 2^e \left( \frac{3^\tau - 1}{3} \right) + \frac{2^e + \tau - 1}{3} = \frac{(2^\tau - 1)2^e - 1}{3} \]

For completeness, we have

\[ \lambda = \begin{cases} \frac{(2^e - 1)2^e - 1}{3} & \text{if } e \equiv 0 \pmod{2} \\ \frac{2^e + \tau - 1}{3} - \frac{2^e + 1}{3} & \text{if } e \equiv 1 \pmod{2} \end{cases} \]
when \( \tau \equiv \frac{1}{2} \), and
\[
\lambda_\tau = \begin{cases} 
\frac{(2^\tau - 1)2^e - 1}{3} & \text{if } e \equiv 0 \mod 2 \\
2^{e+\tau - 1} - \frac{2^e + 1}{3} & \text{if } e \equiv 1 \mod 2
\end{cases}
\]
when \( \tau \equiv 1 \). However, in order to expedite the proofs, we exclude three out of four cases when the corresponding canonical 3-residue \( \mu_\tau \) is even (assuring the inequality \( \mu_\tau \neq \lambda_\tau \)).

We exclude the remaining case with the following lemma.

**Lemma 3.2.** Assume that \( \tau \equiv \frac{1}{2} \equiv e - 1 \); furthermore, assume that
\[
\mu_\tau = 2 \cdot 3^{\tau - 1} - 1,
\]
and
\[
\lambda_\tau = \frac{(2^\tau - 1)2^e - 1}{3}.
\]

The inequality \( \mu_\tau \neq \lambda_\tau \) holds.

**Proof.** By way of contradiction, assume \( e \) satisfies the equality
\[
2 \cdot 3^{\tau - 1} - 1 = \frac{(2^\tau - 1)2^e - 1}{3};
\]
equivalently, we require that the equality
\[
2 (3^\tau - 1) = (2^\tau - 1)2^e\]
holds. However, we have
\[
\frac{3^\tau - 1}{2} \equiv \sum_{0 \leq w < \tau} 3^w \equiv 1 \mod 2
\]
for all odd, positive \( \tau \). When \( e = 2 \), the value of \( \tau \) must satisfy the equality
\[
2^{\tau + 1} = 3^\tau + 1;
\]
equivalently, we require that
\[
2 - \frac{1}{3^\tau} = \left(\frac{3}{2}\right)^\tau;
\]
however, this equality fails to hold for \( \tau > 1 \).

\( \square \)

Lemma 3.2, Assumptions 3.1, and Theorem 3.1, along with the bounds provided in [11], [3], and [5], demonstrate the non-existence of circuits in the 3x + 1 dynamical system.

Before proceeding, we remind the reader of some elementary identities.

**Identity 3.1.** Let \( a \) and \( b \) be coprime, positive integers.

i. If \( g, h \in \mathbb{N} \) with \( h > g \), then \( b^g a \equiv b^g [a]_{b^h - g} \);

ii. \([a]_b^{-1} = \frac{b^{[a]_b} - 1}{a}\);

iii. if \( a > b \), then \([a - b]_b^{-1} = [a]_b^{-1} = \frac{b^\gamma + 1}{a - b} \) for some \( \gamma \in [a - b]_b \).
iv. if \( a > b \), then \( [a - b]^{-1} = \frac{a^\gamma + 1}{a-b} = \gamma + [a - b]^{-1} \).

**Proof.** The elementary proofs of these identities are left to the reader. Note that

\[
 b^\gamma a = b^\gamma [a]_b + b^{\gamma + h} u = b^\gamma \left( [a]_b + g + h u' \right) + b^{\gamma + h} u = b^\gamma [a]_b + b^h u'
\]

for some \( a' \in \mathbb{N} \);

iv, v: as \( a \equiv b \), we can write \( \gamma \equiv \frac{[a]^{-1}}{a-b} \equiv [b]^{-1} \).

\[ \square \]

### 3.1. Elementary Modular Arithmetic.

Our first proof of Theorem 3.1 appeals to elementary modular arithmetic.

**Proof.** We can write

\[
\mu_\tau \equiv \frac{ND^{-1}}{3^\tau - 1} 3^\tau - 1 (1 + (-1)^\tau) - 1.
\]

It follows that

\[
\mu_\tau \equiv 3^\tau - 1 (1 + (-1)^\tau) - 1.
\]

Thus, when \( e \equiv 1 \), we have \( \mu_\tau = 3^\tau - 1 \equiv 0 \). Similarly, when \( e \equiv 0 \) and \( \tau \equiv 0 \), we have \( \mu_\tau = 3^\tau - 1 \equiv 0 \). When \( \tau \equiv 1 \equiv e - 1 \), we arrive at the equality \( \mu_\tau = 2 \cdot 3^\tau - 1 \).

For the 2-residue, we begin by writing

\[
\lambda_\tau \equiv \frac{ND^{-1}}{2^e + r - 1} 2^e + r - 1 (1 + (-1)^r) - 1
\]

\[
\equiv 2^e \frac{3^\tau - 1}{2^e + r - 1} \left( -3 \right)^{r-1}
\]

When \( \tau \equiv 1 \equiv e - 1 \), we have \[ -3 \equiv \frac{3^\tau - 1}{3} \] and \[ -3 \equiv \frac{3^\tau - 1}{3} \].

As

\[
2^e \left( \frac{2^e - 1}{3} \right) + \frac{2^{e+r-1} - 1}{3} = \frac{(2^\tau - 1)2^e - 1}{3} < 2^{e+r-1},
\]

we arrive at the equality

\[
\lambda_\tau = \frac{(2^\tau - 1)2^e - 1}{3}.
\]
3.2. Weighted Binomial Coefficients. The previous approach is apparently limited; it is unclear to the author how to extrapolate this approach to admissible sequences of order \( \tau \) with an arbitrary 2-grading \( (e_0, \ldots, e_{\tau-1}) \). In this subsection, we introduce a more robust approach to identifying the 3-residues and 2-residues of the iterates of an admissible cycle in a (3, 2)-system. Moreover, we do so by connecting the residues of (3, 2)-systems to the well-known Fibonacci sequence by way of elementary equivalence identities, which we establish first.

Lemma 3.3. For \( a, b, z \in \mathbb{N} \), the equivalence

\[
\left( \sum_{0 \leq w < b} z^w \right)^a \equiv \sum_{0 \leq w < b} \binom{a-1+w}{w} z^w
\]

holds.

Proof. Define \( S_b(z) = \sum_{0 \leq w < b} z^w \), and define \( T_{a,b}(z) = \sum_{0 \leq w < b} \left( \frac{a-1+w}{w} \right) z^w \). The proof is by induction on \( b \).

When \( b = 1 \), we arrive at the equivalence \( 1^a \equiv \binom{a-1}{0} \) for all \( a, z \in \mathbb{N} \).

Assume the claim holds for \( b \in \mathbb{N} \). The identity \( S_{b+1}(z) = zS_b(z) + 1 \) allows the chain of equivalences

\[
[S_{b+1}(z)]^a \equiv \sum_{0 \leq y < b+1} \binom{a}{y} z^y [S_b(z)]^y
\]

\[
\equiv \binom{a}{0} z^0 + \sum_{1 \leq y < b+1} \binom{a}{y} z^y T_{y,b}(z).
\]

We will recast the coefficient of \( z^0 \) as \( \binom{a-1}{0} \), and we will write

\[
\sum_{1 \leq y < b+1} \binom{a}{y} z^y T_{y,b}(z) = \sum_{1 \leq y < b+1} \sum_{0 \leq u < b} z^{u+y} \binom{a}{y} \binom{y-1+u}{u}.
\]

For each \( w \in [b+1) \), the coefficient of \( z^w \) is \( \sum_{1 \leq y \leq w} \binom{a}{y} \binom{w-1}{y} = \sum_{0 \leq y < w} \binom{a}{y} \binom{w-1}{y} \), which equals \( \binom{a-1+w}{w} \) as per the Vandermonde-Chu identity.

Identity 3.2 (Fibonacci Identity). Let \( F_0 = 0 \), \( F_1 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \). The equality

\[
F_n = \sum_{0 \leq k < n} \binom{n-1-k}{k}
\]

holds.

We will use this identity to establish the residue approximation functions for (3, 2)-systems.
Lemma 3.4. Define the map $M_\tau : \mathbb{N}^\tau \times \mathbb{N}^\tau \to \mathbb{Z}$ to be

$$M_\tau = M_\tau (e,a) = \sum_{0 \leq w < u} (-)^{E_{w+1}} 3^w a_w \sum_{0 \leq y < \tau - w} \left( \frac{E_{w+1} - 1 + y}{y} \right) 3^y,$$

and define the map $\Lambda_\tau : \mathbb{N}^\tau \times \mathbb{N}^\tau \to \mathbb{Z}$ to be

$$\Lambda_\tau = \Lambda_\tau (e,a) = \sum_{0 \leq w < \tau} (-)^w 2^{E_w} a_{\tau-1-w} \sum_{0 \leq y < \eta_w} \left( \frac{w + y}{y} \right) 4^y,$$

where $\eta_w = \left\lceil \frac{E_{\tau-w}}{2} \right\rceil$.

Then, the equivalences $M_\tau \equiv \frac{3}{3^\tau} \mu_\tau$ and $\Lambda_\tau \equiv \frac{2}{2^\tau} \lambda_\tau$ hold.

Proof. We will make use of the following elementary identities involving Euler’s totient function $\phi$: we have $3^{\phi(2)} - 1 = 2$ and $2^{\phi(3)} - 1 = 3$. In light of these identities, we will appeal to Lemma 3.3: for $a,b \in \mathbb{N}$, we will write

$$\left[ 2^a \right]^{-1} \equiv \frac{1 - 3^{\phi(2)} \left\lfloor \frac{b}{\phi(2)} \right\rfloor}{2^a} \equiv (-)^a \left( \sum_{0 \leq y < b} \left( \frac{3^y}{y} \right) \right) \equiv (-)^a \sum_{0 \leq y < b} \left( \frac{a - 1 + y}{y} \right) 3^y,$$

and

$$\left[ 3^b \right]^{-1} \equiv \frac{1 - 2^{\phi(3)} \left\lfloor \frac{a}{\phi(3)} \right\rfloor}{3^b} \equiv (-)^b \left( \sum_{0 \leq y < \left\lceil \frac{a}{3} \right\rceil} 4^y \right) \equiv (-)^b \sum_{0 \leq y < \left\lceil \frac{a}{3} \right\rceil} \left( \frac{b - 1 + y}{y} \right) 4^y.$$

We derive the 3-residue approximation function as follows:

$$\mu_\tau \equiv \left\lceil ND^{-1} \right\rceil_{3^\tau} \equiv \sum_{0 \leq w < \tau} 3^w a_w \left\lceil 2^{E_w} \right\rceil_{3^\tau}^{-1} \equiv \sum_{0 \leq w < \tau} 3^w a_w \left[ 2^{E_{w+1}} \right]_{3^\tau}^{-1} \equiv \sum_{0 \leq w < \tau} (-)^{E_{w+1}} 3^w a_w \sum_{0 \leq y < \tau-w} \left( \frac{E_{w+1} - 1 + y}{y} \right) 3^y.$$
We derive the 2-residue approximation function as follows:
\[
\lambda_{\tau} \equiv \left[N D^{-1}\right]_{2\tau}
\equiv \sum_{0 \leq w < \tau} 3^{w/2} E_{\tau - 1 - w} a_w \left[-3\right]^{-1}
\equiv \sum_{0 \leq w < \tau} -2 E_{\tau - 1 - w} a_w \left[3^{\tau - w}\right]^{-1} 2^{E_{w+1}}
\equiv \sum_{0 \leq w < \tau} (\tau - 1 - w) E_{\tau - 1 - w} a_w \sum_{0 \leq y < E_{w+1}/2} \left(\frac{\tau - 1 - w + y}{y}\right)^{4y}
\equiv \sum_{0 \leq w < \tau} (-w) 2 E_w a_{\tau - 1 - w} \sum_{0 \leq y < \eta_w} \left(\frac{w + y}{y}\right)^{4y}.
\]

It will prove useful to re-index these double-sums: for example, in the 3-residue approximation, for each fixed \(w\) \(\in [\tau]_0\) the coefficient of \(3^w\) is
\[
S_w = \sum_{0 \leq y \leq w} (-)E_{y+1} \left(\frac{E_{y+1} - 1 + w - y}{w - y}\right) a_y;
\]
thus, we can write \(M_{\tau} = \sum_{0 \leq w < \tau} 3^w S_w\).

The following example will illustrate the connection between an orbit over \(\mathbb{N}\) within the 3\(x + 1\) dynamical system and the Fibonacci Sequence.

3.2.1. Example: The \((1, 4, 2)\)-Orbit in the 3\(x + 1\) Dynamical System. For this example, define \(e_w = 2\) and \(a_w = 1\) for each \(w \in [\tau]_0\). The sum \(E_{w+1} = 2(w+1) \equiv 0\) for all \(w \in [\tau]_0\), therefore, we can express the 3-residue approximation as \(M_{\tau} = \sum_{0 \leq w < \tau} 3^w S_w\), where
\[
S_w := \sum_{0 \leq y \leq w} 2(y + 1 - 1 + w - y) y^{-w - y} = \sum_{0 \leq y \leq w} 2w + 1 - y.
\]
The sequence \((S_w)_{w \geq 1}\) is the even-indexed bisection of the Fibonacci sequence \((F_w)_{w \geq 0}\) as per Identity 3.2; we have \(S_w = F_{2(w+1)}\) for \(w \geq 0\). It is known that this bisection satisfies the recurrence\(^6\) \(F_{2w} = 3F_{2(w-1)} - F_{2(w-2)}\) for \(w \geq 0\); thus, induction yields the identity \(M_{\tau} = 3^\tau F_{2(\tau-1) + 1}\) for \(\tau \in \mathbb{N}\).

For the 2-residue approximation, we have the equalities
\[
\Lambda_{\tau} = \sum_{0 \leq w < \tau} 4^w \sum_{0 \leq y \leq w} \left(\frac{w}{y}\right) (-1)^y = \sum_{0 \leq w < \tau} 4^w (1 - 1)^w = 1
\]
\(^4\)OEIS:A001906
\(^5\)The interested reader will find the elements of the odd-indexed bisection of the Fibonacci sequence in the 3-residue approximation of the same \((3, 2)\) system (i.e., “3\(x + 1\)”) where \(e_0 = 1\) and \(e_w = 2\) for \(w \in [\tau]\).
\(^6\)We assume the definition of the sequence to be \(F_{-n} = (-)^{n-1} F_n\).
for $\tau \in \mathbb{N}$.

The Fibonacci sequence appears within the 2-residue approximation for the following proof of Theorem 3.1. In order to expedite the derivation of this 2-residue, we will first prove the following lemma.

**Lemma 3.5.** For $a \in \mathbb{N}$, let $F_a$ denote the $a$-th Fibonacci number; furthermore, for $k \in \mathbb{N}_0$, define $\sigma(a,k) = 2^{a+1} - \binom{a}{k}$, and define $S(k) = \sum_{0 \leq i < k} \sigma(2k - i, i + 1)$.

We have the equality $S(0) = 0$, and, for $k > 0$, the equality

$$S(k) = F_{2k+2} + 2F_{2k+1} - 3$$

holds.

**Proof.** Assume the conditions within the statement of the lemma. Clearly, $S(0) = 0$. As per Identity 3.2, when $k > 0$, we will write

$$S(k) = \sum_{0 \leq i < k} \left[ 2^{2k - i + 1} - \binom{2k - i}{i + 1} \right]$$

$$= \sum_{1 \leq i < k+1} \left[ 2^{2k + 2 - i} - \binom{2k + 1 - i}{i} \right]$$

$$= 2 \left[ F_{2k+3} - \binom{2k + 2}{0} - \binom{k + 1}{k + 1} \right] - F_{2k+2} - \binom{2k + 1}{0}$$

$$= F_{2k+2} + 2F_{2k+1} - 3.$$  

We proceed with the proof of the theorem.

**Proof.** First, we will demonstrate the equality

$$M_\tau = -1 + 3^{\tau-1}(-1)^{\tau-1} \left[ 1 + (-1)^\tau \right];$$

afterwards, by assuming $\tau \equiv 1 \equiv e - 1$, we will show that

$$\Lambda_\tau = 2^\tau \left( \frac{2^{\tau-1} - 1}{3} + \frac{2^\tau - 1}{3} \right) + 2^\tau (F_{\tau-2} - 1).$$

In circuits, we have

$$E_w = \begin{cases} w & w < \tau \\ e + \tau - 1 & w = \tau, \end{cases}$$
and $E_w = e + w - 1$ for $w \in [\tau)$. Thus, when $w < \tau - 1$, we have

$$S_w = \sum_{0 \leq y \leq w} (-1)^{E_{y+1}} \left( E_{y+1} \frac{1 + w - y}{w - y} \right)$$

$$= \sum_{0 \leq y \leq w} (-1)^{y+1} \left( \frac{w}{w - y} \right)$$

$$= - \sum_{0 \leq y \leq w} (-1)^{w+y} \left( \frac{w}{y} \right)$$

$$= -(1 - 1)^w$$

$$= \begin{cases} 0 & w > 0 \\ -1 & w = 0 \end{cases} ;$$

when $w = \tau - 1 \geq 1$, we have

$$S_{\tau-1} = \sum_{0 \leq y \leq \tau - 1} (-1)^{E_{y+1}} \left( E_{y+1} \frac{1 + \tau - 1 - y}{\tau - 1 - y} \right)$$

$$= \sum_{0 \leq y \leq \tau - 2} (-1)^{y+1} \left( \frac{\tau - 1}{\tau - 1 - y} \right) + (-1)^{e+\tau-1} \left( \frac{e + \tau - 2}{0} \right)$$

$$= -(1 - 1)^{\tau-1} + (-1)^{\tau-1} \left( \frac{\tau - 1}{\tau - 1} \right) + (-1)^{e+\tau-1} \left( \frac{e + \tau - 2}{0} \right)$$

$$= (-1)^{\tau-1} [1 + (-1)^{e}] .$$

It follows that

$$M_\tau = -1 + 3^{\tau-1} (-1)^{\tau-1} [1 + (-1)^{e}] .$$

Thus, when $e \equiv 1$, we have $\mu_\tau = 3^{\tau} - 1$. Similarly, when $e \equiv 0$ and $\tau \equiv 0$, we have $\mu_\tau = 3^{\tau-1} - 1$.

When $\tau \equiv 1 \equiv e - 1$, we arrive at the equality $\mu_\tau = 2 \cdot 3^{\tau-1} - 1$. Continuing with these parity conditions, we let $T_w$ denote the sum $\sum_{0 \leq y \leq \left\lceil \frac{E_w}{2} \right\rceil} (w+y)4^y$. We write

$$\Lambda_\tau = \sum_{0 \leq w < \tau} (-1)^w 2T_w$$

$$= T_0 + \sum_{1 \leq w < \tau} (-1)^w 2T_w$$

$$= \sum_{0 \leq y \leq \frac{e + w - 1}{2}} y 4^y + \sum_{1 \leq w < \tau} (-1)^w 2E_w \left( \begin{array}{c} w \\ 0 \end{array} \right) + \sum_{1 \leq w < \tau} (-1)^w 2T_w \left[ T_w \left( \begin{array}{c} w \\ 0 \end{array} \right) \right] .$$
We proceed with the first two sums in this expression. When \( e + \tau - 1 \equiv 0 \), we can write

\[
T_0 = \sum_{0 \leq y < \frac{e+\tau-1}{2}} \binom{y}{y} 4^y = \frac{2^{e+\tau-1} - 1}{3};
\]

furthermore, as \( \tau - 1 \equiv 0 \), we can also write

\[
\sum_{1 \leq w < \tau} \left[ T_w - \binom{w}{0} \right] \equiv 2^e \sum_{0 \leq w < \tau-1} (-1)^w 2^w \equiv 2^e \sum_{0 \leq w < \frac{\tau-1}{2}} \left[ 2^{2w+1} - 2^{2w} \right] \equiv 2^e \sum_{0 \leq w < \frac{\tau-1}{2}} 4^w \equiv 2^e \left( \frac{2^{\tau-1} - 1}{3} \right).
\]

What remains to be shown is that

\[
\sum_{1 \leq w < \tau} (-)^w 2^E_w \left[ T_w - \binom{w}{0} \right] \equiv 2^e \Lambda_{\tau} \equiv 2^e (F_{\tau-2} - 1).
\]

To this end, for each \( k \in \mathbb{N} \), we will define

\[
\hat{\Lambda}_{2k+1} = \sum_{1 \leq w < 2k-1} (-)^w 2^{w-1} \sum_{1 \leq y < \frac{2k+1-w}{2}} \binom{w+y}{y} 4^y;
\]

we will show that

\[
\sum_{1 \leq w < \tau} (-)^w 2^E_w \left[ T_w - \binom{w}{0} \right] = 2^e \hat{\Lambda}_\tau = 2^e + \tau - 1 (F_{\tau-2} - 1).
\]

Assume the notation from the statement of Lemma 3.5. We will demonstrate the chain of equalities

\[
\hat{\Lambda}_{2k+1} = \hat{\Lambda}_{2k-1} + 4^{k-1} S (k-1) = 4^k (F_{2k-1} - 1)
\]

inductively for \( k \in \mathbb{N} \). Firstly, we have

\[
\hat{\Lambda}_3 = 0 = 4^0 S (0) = 4^0 (F_1 - 1)
\]

for \( k = 1 \). Assuming the inductive claim, we proceed with the chain of equalities for \( k \geq 2 \):

\[
\hat{\Lambda}_{2k+1} = \sum_{1 \leq w < 2k-1} (-)^w 2^{w-1} \sum_{1 \leq y < \frac{2k+1-w}{2}} \binom{w+y}{y} 4^y
\]

\[
= \hat{\Lambda}_{2k-1} + A_k + B_k,
\]
where
\[ A_k = \sum_{1 \leq w < 2k-1} (-)^{w} 2^{w-1} \binom{w + \left\lceil \frac{2k-1-w}{2} \right\rceil}{w + \left\lfloor \frac{2k-1-w}{2} \right\rfloor} 4^{\left\lfloor \frac{2k-1-w}{2} \right\rfloor}, \]
and
\[ B_k = \sum_{2k-1 \leq w < 2k+1} (-)^{w} 2^{w-1} \sum_{1 \leq y < \left\lceil \frac{2k-1-w}{2} \right\rceil} \binom{w + y}{y} 4^{y}. \]

Firstly, the sum \( B_k = \sum_{2k-1 \leq w < 2k+1} (-)^{w} 2^{w-1}, \emptyset = 0 \), and the sum
\[ A_k = \sum_{1 \leq w < 2k-1} (-)^{w} 2^{w-1} \binom{k + w + \left\lfloor \frac{1-w}{2} \right\rfloor}{k + \left\lfloor \frac{1-w}{2} \right\rfloor} 4^{k + \left\lfloor \frac{1-w}{2} \right\rfloor} \]
\[ = \sum_{1 \leq w < 2k-1} \left[ 2^{2w-1} \binom{k + w}{k-w} - 2^{2w-2} \binom{k-1 + w}{k-w} \right] 4^{k-w} \]
\[ = 4^{k-1} \sum_{1 \leq w < k-1} \left[ 2^{k + w} - (k-1 + w) \right] \]
\[ = 4^{k-1} \sum_{1 \leq w < k-1} \left[ 2^{2k - w} - (2k - 1 - w) \right] \]
\[ = 4^{k-1} \sum_{0 \leq w < k-1} \left[ 2^{2k - 1 - w} - (2k - 2 - w) \right] \]
\[ = 4^{k-1} S(k-1). \]

Thus, with Lemma 3.5 and the inductive hypothesis, we can write
\[ \hat{\Lambda}_{2k+1} = \hat{\Lambda}_{2k-1} + 4^{k-1} S(k-1) \]
\[ = 4^{k-1} [F_{2k-3} - 1 + F_{2k} + 2F_{2k-1} - 3] \]
\[ = 4^{k-1} [F_{2k-3} + F_{2k-1} + 3F_{2k-1} - 4] \]
\[ = 4^k [F_{2k-1} - 1] \]
as required.

Consequently, when \( \tau \equiv 1 \equiv e - 1, \) the 2-approximation
\[ \Lambda_\tau = 2^e \left( \frac{2^\tau - 1}{3} \right) + \frac{2^{e+\tau-1} - 1}{3} + 2^{e+\tau-1} (F_{\tau-2} - 1), \]
and we conclude that
\[ \lambda_\tau = 2^e \left( \frac{2^\tau - 1}{3} \right) + \frac{2^{e+\tau-1} - 1}{3} = \frac{(2^\tau - 1)2^e - 1}{3}. \]
Note that the approach within this subsection exploits the serendipitous pair of identities $3^{\phi(2)} - 1 = 2$ and $2^{\phi(3)} - 1 = 3$. In general, Euler’s Theorem allows one to write

$$m^{\phi(l)} - 1 = \left[-l\right]_{m^{\phi(l)}}^{-1} l$$

and

$$l^{\phi(m)} - 1 = \left[-m\right]_{l^{\phi(m)}}^{-1} m;$$

however, for arbitrary, coprime $m$ and $l$ exceeding 1, the terms $\left[-l\right]_{m^{\phi(l)}}^{-1}$ and $\left[-m\right]_{l^{\phi(m)}}^{-1}$ may prevent one from executing the approach above in an analogous manner.

3.3. Dual-Radix Modular Division. The approach in this section, based on the work in [9], demonstrates a different method of proving Theorem 3.1 using dual-radix modular division.

Proof. Under the assumption that

$$e_w = \begin{cases} 1 & w \in [\tau - 1)_0 \\ e & w = \tau - 1, \end{cases}$$

we have the following initial conditions for the recurrence in Theorem 4.4 in [9]. For $w \in [\tau)_0$, the 3-adic digit $d_{w,0} \equiv \frac{2^{e_w} - 1}{3};$ thus, we have

$$d_{w,0} = \begin{cases} 2 & w \in [\tau - 1)_0 \\ 1 + e \mod 2 & w = \tau - 1; \end{cases}$$

furthermore, the 2-adic digit $b_{w,0} \equiv \frac{2^{e_w} - 1}{3};$ thus, we have

$$b_{w,0} = \begin{cases} 2^{\lceil \frac{e_w}{2} \rceil - 1} & w = 0 \\ 1 & w \in [\tau - 1]. \end{cases}$$

For $u > 0$, the equivalences

$$d_{v,u} \equiv \frac{2^{e_v} - 1}{3} \left[d_{v+1,u-1} - b_{v+u,u-1}\right]$$

and

$$b_{v,u} \equiv \frac{2^{e_v} - 1}{3} \left[d_{v-u,u-1} - b_{v-1,u-1}\right]$$

yields, by induction on $u$, the equalities $d_{v,u} = 2[2 - 1] = 2$ for $v < \tau - 1 - u$, and $b_{v,u} = 1[2 - 1] = 1$ for $v > u$.

We will first identify the 3-adic digits of the 3-residue of $n (= n_0)$. When $e \equiv 1$, we have the initial condition $d_{\tau-1,0} = 2$. Thus, for $u \in [\tau)$, we have

$$d_{\tau-1-u,u} \equiv \frac{2^{e_{\tau-1-u}}}{3} \left[d_{\tau-u,u-1} - b_{\tau-1,u-1}\right]$$

$$\equiv \frac{2}{3} \left[2 - 1\right]$$

$$\equiv 2.$$
Consequently, we have

\[ \mu_\tau = \sum_{0 \leq w < \tau} 3^w d_{0,w} = 2 \left( \frac{3^\tau - 1}{2} \right) = 3^\tau - 1. \]

When \( e \equiv 0 \), we have the initial condition \( d_{\tau-1,0} = 1 \), and

\[ d_{\tau-2,1} \equiv \frac{1}{3} \left[ 2^1 \right]^{-1} [d_{\tau-1,0} - b_{\tau-1,0}] \equiv \frac{1}{3} \left[ 2^1 \right]^{-1} [1 - 1] \equiv 0. \]

By induction, for \( u \in [\tau) \) where \( u \equiv 0 \), we have

\[ d_{\tau-1-u,u} \equiv \frac{1}{3} \left[ 2^{e-1-u} \right]^{-1} [d_{\tau-u,u-1} - b_{\tau-1,u-1}] \equiv \frac{2}{3} [0 - 1] \equiv \frac{1}{3} 1. \]

For \( u \equiv 1 \), we have

\[ d_{\tau-1-u,u} \equiv \frac{1}{3} \left[ 2^{e-1-u} \right]^{-1} [d_{\tau-u,u-1} - b_{\tau-1,u-1}] \equiv \frac{2}{3} [1 - 1] \equiv 0. \]

Thus,

\[ d_{0,\tau-1} = \begin{cases} 0 & \tau \equiv 0 \\ 1 & \tau \equiv 1. \end{cases} \]

Thus, when \( \tau \equiv 0 \), the 3-adic residue

\[ \mu_\tau = \sum_{0 \leq w < \tau} 3^w (2) = 3^{\tau-1} - 1 \equiv 0; \]

and, when \( \tau \equiv 1 \), the 3-adic residue

\[ \mu_\tau = 2 \left( \frac{3^{\tau-1} - 1}{2} \right) + 3^{\tau-1} = 2 \cdot 3^{\tau-1} - 1. \]

We will now determine the 2-adic digits of \( n \) when \( \tau \equiv 1 \equiv e - 1 \): when \( e \equiv 0 \), the 2-adic digit

\[ b_{0,0} = \frac{2^e - 1}{3}, \]

and the digit

\[ b_{0,1} \equiv 2^{e-2} \left[ -3 \right]^{-1} [d_{\tau-1,0} - b_{\tau-1,0}] \equiv (1) \cdot [1 - 1] \equiv 0. \]

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For \( u \in [\tau) \) where \( u \equiv 0 \), we have

\[
b_{0,u} \equiv 2^{[d_{\tau,u,u-1} - b_{\tau-1,u-1}] \equiv (1) \cdot [0 - 1] \equiv 1},
\]

and, when \( u \equiv 1 \), we have

\[
b_{0,u} \equiv 2^{[d_{\tau,u,u-1} - b_{\tau-1,u-1}] \equiv (1) \cdot [1 - 1] \equiv 0}.
\]

Thus, when \( \tau \equiv 1 \equiv e - 1 \), the 2-adic residue

\[
\lambda_{\tau} = b_{0,0} + \sum_{1 \leq u < \tau} 2^{E_{\tau}} b_{0,u}
\]

\[
= \frac{2^e - 1}{3} + 2^e \sum_{2 \leq u < \tau} 2^{u-1}[u \equiv 0]
\]

\[
= \frac{2^e - 1}{3} + 2^{e+1} \sum_{0 \leq u \leq \tau - 2} 2^{u}[u \equiv 0]
\]

\[
= \frac{2^e - 1}{3} + 2^{e+1} \sum_{0 \leq u \leq \tau - 2} 4^{u}
\]

\[
= \frac{2^e - 1}{3} + 2^{e+1} \left( \frac{4^{\tau-1} - 1}{3} \right)
\]

\[
= \frac{2^{e+\tau} - 2^e - 1}{3} = \frac{2^e \left( 2^{\tau-1} - 1 \right)}{3} + \frac{2^{e+\tau-1} - 1}{3}.
\]

□

3.4. Circuits in the 3x \(-1\) Dynamical System. We conclude this article by applying the previous analyses to the 3x \(-1\) dynamical system; now, we will consider the case where \( a_w = -1 \) for all \( w \in [\tau)_0 \).

We will extend the argument in [1] to the case where \( 3^\tau > 2^{E_{\tau}} \): the magnitude of the numerator of a maximal iterate in a periodic orbit can be bound from above as follows:

\[
\left| (2^e + 1) 3^{\tau-1} - 2^{E_{\tau}} \right| = 3^\tau \left[ \frac{2^e + 1}{3} - \frac{2^{E_{\tau}}}{3^\tau} \right] < 3^{\tau-1} (2^e + 1) .
\]

We can bound the denominator \( 3^\tau - 2^{E_{\tau}} \) from below by appealing to the inequality (3) once again\(^7\) to conclude that the maximal iterate within a periodic orbit in the 3x \(-1\)

---

\(^7\)The changing of the signs of \( u_1 \) and \( u_2 \) does not alter the bound.
dynamical system satisfies the inequality
\[
n_{\text{max}} < \frac{2^e + 1}{3} < \left( \frac{2^e + 1}{3} \right) 2(e + \tau - 1)^{13.3} = o(2^{e+\tau-1})
\]
for any fixed \( e \in \mathbb{N} \). Thus, we will reuse the notation of the previous section and begin with the following assumptions.

**Assumptions 3.2.** Assume 3.1, except that now we assume that \( N = 2^e + \tau - 1 - (2^e + 1)3^{\tau - 1} < 0 \), and \( D = 2^{e+\tau-1} - 3^\tau < 0 \).

As before, under these assumptions, if \( n \in \mathbb{N} \), then the chain of equalities
\[
n = \mu_\tau = \lambda_\tau
\]
holds.

Our goal for the remainder of this subsection is to prove the following theorem:

**Theorem 3.6.** Assume 3.2.

*The 3-residue*
\[
\mu_\tau = \begin{cases} 
2 \cdot 3^{\tau - 1} + 1 & e \equiv 0 \\
1 & e \equiv 1
\end{cases}
\]
when \( \tau \equiv 0 \), and
\[
\mu_\tau = \begin{cases} 
3^{\tau - 1} + 1 & e \equiv 0 \\
1 & e \equiv 1
\end{cases}
\]
when \( \tau \equiv 1 \).

*The 2-residue*
\[
\lambda_\tau = \begin{cases} 
\frac{2^e (2^\tau + 1) + 1}{3} & e \equiv 0 \\
\frac{2^e + 1}{3} & e \equiv 1
\end{cases}
\]
when \( \tau \equiv 0 \), and
\[
\lambda_\tau = \begin{cases} 
\frac{2^e (2^{\tau - 1} + 1) + 1}{3} & e \equiv 0 \\
\frac{2^e + 1}{3} & e \equiv 1
\end{cases}
\]
when \( \tau \equiv 1 \).

Analogous to Lemma 3.2, the following lemma will aid in identifying circuits within the \( 3x - 1 \) Dynamical System.

**Lemma 3.7.** Assume that the 3-residue is
\[
\mu_\tau = \begin{cases} 
2 \cdot 3^{\tau - 1} + 1 & e \equiv 0 \\
1 & e \equiv 1
\end{cases}
\]
when $\tau \equiv 0$, and

$$\mu_\tau = \begin{cases} 
3^{\tau-1} + 1 & e \equiv 0 \\
1 & e \equiv 1
\end{cases}$$

when $\tau \equiv 1$. Moreover, assume that the $2$-residue is

$$\lambda_\tau = \begin{cases} 
\frac{2^e(2^{\tau+1})}{3} + 1 & e \equiv 0 \\
\frac{2^{\tau+1}}{3} & e \equiv 1
\end{cases}$$

when $\tau \equiv 0$, and

$$\lambda_\tau = \begin{cases} 
\frac{2^e(2^{\tau+1})}{3} + 1 & e \equiv 0 \\
\frac{2^{\tau+1}}{3} & e \equiv 1
\end{cases}$$

when $\tau \equiv 1$.

The equality $\mu_\tau = \lambda_\tau$ holds if and only if either i.) $e = 1$ or ii.) $e = \tau = 2$.

Proof. When $e \equiv 1$, we require that the equality $2^e + 1 = 1$ holds; consequently, we require that $e = 1$ (irrespective of the parity of $\tau$).

When $e \equiv 0$ and $\tau \equiv 0$, we require that the equality

$$2 \cdot 3^{\tau-1} + 1 = \frac{2^e (2^\tau + 1) + 1}{3}$$

holds. Equivalently, we require that $2 \cdot 3^{\tau-1} + 1 = 2^e (2^\tau + 1) + 1$; after simplifying, we require that $\frac{3^{\tau+1} - 1}{2^e - 1} = 2^\tau + 1$. When $\tau \equiv 0$, the numerator on the left-hand side $9^\tau + 1 \equiv 2$; thus, it follows that we require that $e = 2$. The equality $3^\tau = 2^\tau + 1$ holds only when $\tau = 2$ as per a result of Gersonides\(^8\) on harmonic numbers.

When $e \equiv 0$ and $\tau \equiv 1$, we have $\mu_\tau \equiv 0$ and $\lambda_\tau \equiv 1$.

We offer one proof of Theorem 3.6.

Proof. We can write

$$\mu_\tau \equiv \frac{-N \left[3^\tau - 2^{\tau+\tau-1}\right]^{-1}}{3^\tau}$$

$$\equiv \frac{\left[(2^e + 1)3^{\tau-1} - 2^{\tau+\tau-1}\right]^{-1}}{3^\tau}$$

$$\equiv \frac{\left[-2^{\tau-1}\right]^{-1} + \left[-2^{\tau+\tau-1}\right]^{-1}}{3^{\tau-1}} 3^{\tau-1} + 1.$$}

It follows that

$$\mu_\tau \equiv 3^{\tau-1}(-1)^\tau [1 + (-1)^\tau] + 1.$$

\(^8\)Levi Ben Gerson, 1342 AD. See [6].
For the 2-residue, we begin by writing

\[
\lambda_\tau \equiv \frac{\tau}{2^{e+\tau-1} - N} \left[ 2^{e+\tau-1} \left(2^e + 1\right)3^{\tau-1} - 2^{e+\tau-1} \right]^{-1} \equiv 2^{e+\tau-1} \left(2^e + 1\right)3^{\tau-1} - 2^{e+\tau-1} \cdot [3]_{2^{e+\tau-1}}^{-1}.
\]

We have the identities \( [3]_{2^{e+\tau-1}}^{-1} = \frac{2^{e+\tau-1} + 2^{e+\tau-1} + 2^{e+\tau-1}}{3} \) and \( [3]_{2^{e+\tau-1}}^{-1} = \frac{2^{e+\tau-1} + 2^{e+\tau-1} + 2^{e+\tau-1}}{3} \).

We complete the proof by cases.

i. \((e \equiv 0, \tau \equiv 0)\) \(\mu_\tau = 2 \cdot 3^{\tau-1} + 1\), and \(\lambda_\tau = \left[ 2^e \left(\frac{2^{\tau-1} + 1}{3}\right) + 2^{e+\tau-1} \right] \mod 2^{e+\tau-1} = 2^{e+\tau-1} + 2^{e+\tau-1}
\]

ii. \((e \equiv 0, \tau \equiv 1)\) \(\mu_\tau = 3^{\tau-1} + 1\), and \(\lambda_\tau = \left[ 2^e \left(\frac{2^{\tau-1} + 1}{3}\right) + 2^{e+\tau-1} \right] \mod 2^{e+\tau-1} = 2^{e+\tau-1} + 2^{e+\tau-1}
\]

iii. \((e \equiv 1, \tau \equiv 0)\) \(\mu_\tau = 1\), and \(\lambda_\tau = \left[ 2^e \left(\frac{2^{\tau-1} + 1}{3}\right) + 2^{e+\tau-1} \right] \mod 2^{e+\tau-1} = 2^{e+\tau-1} + 2^{e+\tau-1}
\]

iv. \((e \equiv 1, \tau \equiv 1)\) \(\mu_\tau = 1\), and \(\lambda_\tau = \left[ 2^e \left(\frac{2^{\tau-1} + 1}{3}\right) + 2^{e+\tau-1} \right] \mod 2^{e+\tau-1} = 2^{e+\tau-1} + 2^{e+\tau-1}.
\]

Thus, under the assumption that \(n < 2^{e+\tau-1}\), the only circuits within the \(3x+1\) dynamical system are (1) and (7).

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