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COMPARING DESCENT OBSTRUCTION AND BRAUER-MANIN OBSTRUCTION FOR OPEN VARIETIES

YANG CAO, CYRIL DEMARCHE, AND FEI XU

Abstract. We provide a relation between Brauer-Manin obstruction and descent obstruction for torsors over not necessarily proper varieties under a connected linear algebraic group or a group of multiplicative type. Such a relation is also refined for torsors under a torus. The equivalence between descent obstruction and étale Brauer-Manin obstruction for smooth projective varieties is extended to smooth quasi-projective varieties, which provides the perspective to study integral points.

1. Introduction

The descent theory for tori was first established by Colliot-Thélène and Sansuc in [8] and was extended by Skorobogatov to groups of multiplicative type in [34]. In a series of papers [21], [23], [25], Harari and Skorobogatov introduced descent obstruction for a general algebraic group and compared the descent obstruction with the Brauer-Manin obstruction. By various works of Poonen [31], the second named author [14], Stoll [38] and Skorobogatov [35], it was proved that the descent obstruction is equivalent to the étale Brauer-Manin obstruction for smooth projective geometrically integral varieties. In this paper, we study the relation between the descent obstruction and the Brauer-Manin obstruction for open varieties by using new arithmetic tools developed in [2], [6], [9], [17], [22] and [27], and we extend the equivalence between the descent obstruction and the étale Brauer-Manin obstruction to smooth quasi-projective varieties.

Let $k$ be a number field, $\Omega_k$ the set of all primes of $k$ and $A_k$ the adelic ring of $k$. A variety over $k$ is defined to be a separated scheme $X$ of finite type over $k$. Fix an algebraic closure $\bar{k}$ of $k$. We denote by $X_k$ the fibre product $X \times_k \bar{k}$. Let

$$Br(X) = H^2_{\text{ét}}(X, \mathbb{G}_m) , \quad Br_1(X) = \ker(Br(X) \to Br(X_k)) \quad \text{and} \quad Br_0(X) = \text{Im}(Br(k) \xrightarrow{\pi^*} Br(X))$$

where $X \xrightarrow{\pi} \text{Spec}(k)$ is the structure morphism, and $Br_a(X) = Br_1(X)/Br_0(X)$. For any subgroup $B$ of $Br(X)$, one can define the Brauer-Manin set

$$X(A_k)^B = \{(x_v)_{v \in \Omega_k} \in X(A_k) : \sum_{v \in \Omega_k} \text{inv}_v(\xi(x_v)) = 0 \quad \text{for all} \quad \xi \in B\}$$

with respect to $B$. When $B = Br(X)$, we simply write this Brauer-Manin set as $X(A_k)^{Br}$. 

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Key words : linear algebraic group, torsor, descent obstruction, Brauer–Manin obstruction.
Suppose \( Y \xrightarrow{f} X \) is a left torsor under a linear algebraic group \( G \) over \( k \). The descent obstruction (see \([21, 23]\) and \([25]\)) given by \( f \) is defined by the following set

\[
X(A_k)^f = \{ (x_v) \in X(A_k) : ([Y](x_v)) \in \text{Im}(H^1(k, G) \to \prod_{v \in \Omega} H^1(k_v, G)) \} = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k))
\]

where \( Y^\sigma \xrightarrow{f_{\sigma}} X \) is the twist of \( Y \xrightarrow{f} X \) by a 1-cocycle representing \( \sigma \in H^1(k, G) \).

Moreover, one can define

\[
X(A_k)^\text{desc} = \bigcap_{Y \xrightarrow{f} X} X(A_k)^f
\]

following \([31]\), where \( Y \xrightarrow{f} X \) runs through all torsors under all linear algebraic groups over \( k \).

The main results in this paper are the following theorems.

**Theorem 1.1.** (Theorem 3.5) Let \( k \) be a number field, \( G \) a connected linear algebraic group or a group of multiplicative type over \( k \), and \( X \) a smooth and geometrically integral variety over \( k \). Suppose \( Y \xrightarrow{f} X \) is a left torsor under \( G \). For any subgroup \( A \subseteq \text{Br}(X) \) which contains the kernel of the natural map \( f^* : \text{Br}(X) \to \text{Br}(Y) \) we have

\[
X(A_k)^A = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k)^{f_{\sigma}(A)})
\]

where \( Y^\sigma \xrightarrow{f_{\sigma}} X \) is the twist of \( Y \xrightarrow{f} X \) by \( \sigma \) and \( \text{Br}(X) \xrightarrow{f_{\sigma}} \text{Br}(Y^\sigma) \) is the associated pull-back map, for each \( \sigma \in H^1(k, G) \).

When \( G \) is a torus, this theorem can be refined in order to get Theorem 1.1 in §4. In particular, we prove:

**Theorem 1.2.** (Corollary 4.3) Under the same assumptions as in Theorem 1.1, if \( G \) is assumed to be a torus, then

\[
X(A_k)^{\text{Br}_{\text{R}}(X)} = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k)^{\text{Br}_{\text{R}}(Y^\sigma)})
\]

and

\[
X(A_k)^{\text{Br}} = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k)^{\text{Br}_{\text{R}}(Y^\sigma) + f_{\sigma}^*(\text{Br}(X))}).
\]

This result is inspired by some lectures by Yonatan Harpaz. It should be pointed out that the first part in Theorem 1.2 was first obtained by Dasheng Wei in \([39]\): his proof uses an argument of Harari and Skorobogatov in \([20]\) together with an exact sequence due to Sansuc (see \([2, \text{Theorem 2.8}]\)). Theorem 1.2 can be applied to study strong approximation, as in \([39]\). It should be noted that in general, the image of \( \text{Br}(X) \) in \( \text{Br}(Y^\sigma) \) in Theorem 1.1 and Theorem 1.2 is not easy to describe, even under the assumption \( \overline{k}[X]^\times = \overline{k}^\times \) (see \([24, \text{Theorem 1.7(b)}]\)).

**Definition 1.3.** Let \( X \) be a variety over a number field \( k \) and let \( B \) be a subgroup of \( \text{Br}(X) \). For a finite subset \( S \) of \( \Omega_k \), we denote by \( p^S : X(A_k) \to X(A_k^S) \) the projection map, where \( A_k^S \) is the set of adeles of \( k \) without \( S \)-components.
We say that $X$ satisfies strong approximation off $S$ if $X(A_k) \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(A_k))$. 

We say that $X$ satisfies strong approximation with respect to $B$ off $S$ if $X(A_k)^B \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(A_k)^B)$. 

Corollary 3.20 in [17] provides a sufficient condition for strong approximation with Brauer-Manin obstruction to hold for a connected linear algebraic group. As an application of Theorem 1.2, we prove that this sufficient condition is also a necessary condition:

**Theorem 1.4.** (Corollary 5.3) Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $Br_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

For any variety $X$ over a number field $k$, one can define, following [31]:

$$X(A_k)^{\acute{e}t, Br} = \bigcap_{Y \to X} \bigcup_{\sigma \in H^1(k, F)} f_\sigma(Y^\sigma(A_k)^{Br}),$$

where $Y \to X$ runs through all torsors under all finite group schemes $F$ over $k$. The last two sections of the paper are devoted to the proof of the following generalization of [14] and [35]:

**Theorem 1.5.** (Corollary 6.7 and Theorem 7.6) If $X$ is a smooth quasi-projective and geometrically integral variety over a number field $k$, then $X(A_k)^{\text{desc}} = X(A_k)^{\acute{e}t, Br}$. 

Terminology and notations are standard if not explained. For any connected linear algebraic group $G$ over an field $k$ of characteristic zero, the reductive part $G^{\text{red}}$ of $G$ is defined by the exact sequence

$$1 \to R_u(G) \to G \to G^{\text{red}} \to 1$$

where $R_u(G)$ is the unipotent radical of $G$. The semi-simple part $G^{ss}$ of $G$ is defined to be the derived subgroup $[G^{\text{red}}, G^{\text{red}}]$, which is isogenous to the product of its simple factors, and the maximal toric quotient $G^{\text{tor}}$ of $G$ is defined to be $G^{\text{red}}/[G^{\text{red}}, G^{\text{red}}]$. We use $\hat{G}$ for the character group of $G$. For a topological abelian group $A$, the topological dual of $A$ is defined as $A^D = \text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ with the compact-open topology. For any ring $R$, $R^\times$ stands for the group of invertible elements of $R$. For a number field $k$, we denote by $\infty_k$ the set of all archimedean primes of $k$ and by $O_S$ the ring of $S$-integers, for any finite subset $S \subset \Omega_k$ containing $\infty_k$. For any $v \in \Omega_k$, $k_v$ is the completion of $k$ with respect to $v$, and if $v \in \Omega_k \setminus \infty_k$, $O_v$ is the integral ring of $k_v$.

The paper is organized as follows. In [2] we establish some algebraic results over an arbitrary field of characteristic zero which we need in the next sections. Then we prove Theorem 1.1 in [3] Theorem 1.2 in [4]. As an application of those results, we prove Theorem 1.4 in [5] Theorem 1.5 is proved in [6] and [7].
2. BRAUER GROUPS OF TORSORS

In this section, we assume that \( k \) is an arbitrary field of characteristic 0.

**Lemma 2.1.** Let \( H \) be a semi-simple simply connected group or a unipotent group over \( k \). Suppose \( X \) is a smooth and geometrically integral variety over \( k \). If \( Z \overset{\pi}{\to} X \) is a torsor under \( H \), then the induced map \( \text{Br}(X) \overset{\pi^*}{\to} \text{Br}(Z) \) is an isomorphism.

**Proof.** We first show that \( \text{Br}(X) \overset{\pi^*}{\to} \text{Br}(X \times_k H) \), where the map is induced by the natural projection \( X \times_k H \to X \). Using the spectral sequence

\[
H^n(k, H^q(X, \mathbb{G}_m)) \Rightarrow H^{n+q}(X, \mathbb{G}_m),
\]

one only needs to show that

\[
\bar{k}[X_k]^{\times}/\bar{k}^{\times} \xrightarrow{\cong} \bar{k}[X_k \times_k H_k]^{\times}/\bar{k}^{\times}, \quad \text{Pic}(X_k) \xrightarrow{\cong} \text{Pic}(X_k \times_k H_k), \quad \text{and} \quad \text{Br}(X_k) \overset{\cong}{\to} \text{Br}(X_k \times_k H_k).
\]

Since \( \bar{k}[H]^{\times} = \bar{k}^{\times} \) and \( \text{Pic}(H_k) = \text{Br}(H_k) = 0 \) by \([9]\) Proposition 2.6], the first two parts are true by \([32]\) Proposition 6.10]. To prove the last part, Kummer exact sequence ensures that one only needs to prove that

\[
H^2_{\text{ét}}(X_k, \mathbb{Z}/n) \overset{\cong}{\to} H^2_{\text{ét}}(X_k \times_k H_k, \mathbb{Z}/n) \tag{2.2}
\]

for all \( n \geq 1 \). This last isomorphism follows from \([37]\) Proposition 2.2] and \([13]\) Exposé XI, Théorème 4.4] with \( H^i_{\text{ét}}(H_k, \mathbb{Z}/n) = 0 \) for \( i = 1, 2 \). So we proved the required isomorphism \( \text{Br}(X) \overset{\cong}{\to} \text{Br}(X \times_k H) \).

Let us now deduce Lemma 2.1 since \( \text{Pic}(H) = 0 \), \([2]\) Proposition 2.4] gives the following short exact sequence

\[
0 \to \text{Br}(X) \to \text{Br}(Z) \overset{m^*-p*_Z}{\to} \text{Br}(H \times_k Z),
\]

where \( m^* \) and \( p^*_Z \) are induced by the multiplication map \( H \times_k Z \overset{m}{\to} Z \) and the projection map \( H \times_k Z \overset{\pi}{\to} Z \) respectively. Since \( m \circ (1_H \times \text{id}) = p_Z \circ (1_H \times \text{id}) = \text{id} \), one concludes that \( m^* = p^*_Z \) by the above argument. Therefore \( \text{Br}(X) \overset{\cong}{\to} \text{Br}(Z) \). \( \square \)

Let \( H \) be a closed subgroup of an algebraic group \( G \) over \( k \), and \( Y \overset{f}{\to} X \) be a left torsor under \( H \). Let \( Z \overset{\theta}{\to} X \) be the left torsor under \( G \) defined by the contracted product \( Z = G \times^H Y \) (see \([36]\) Example 3 in p.21]): the torsor \( Z \) is the push-forward of \( Y \) by the homomorphism \( H \to G \). The projection map \( G \times_k Y \overset{prG}{\to} G \) induces the following commutative diagram

\[
\begin{array}{c}
G \times_k Y \quad \quad \xrightarrow{prG} \quad \quad G \\
\downarrow \theta \quad \quad \quad \quad \quad \quad \downarrow \\
G \quad \quad \quad \quad \quad \quad \quad G/H,
\end{array}
\]

where \( \theta \) is induced by \( prG \) via the quotient by \( H \).

**Lemma 2.4.** With the above notations, for any \( \gamma \in (G/H)(k) \), the composite map \( \theta^{-1}(\gamma) \to Z \overset{\rho}{\to} X \) is naturally a left torsor under \( H^\rho \), which is canonically isomorphic to the twist of \( Y \overset{\delta}{\to} X \) by the \( k \)-torsor \( \pi^{-1}(\gamma) \) under \( H \).
Let $G$ be a connected linear algebraic group over $k$, and $Y$ be a smooth variety over $k$. Since $G_k$ is rational over $\bar{k}$ by Bruhat decomposition, the projections $G \times_k Y \rightarrow G$ and $G \times_k Y \rightarrow Y$ induce an isomorphism
\[
\text{Br}_a(G) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(G \times_k Y)
\]
by [32] Lemma 6.6. If $P$ is a (left) torsor under $G$ over $k$ and $H^3(k, \bar{k}^\times) = 0$, the previous result generalizes to an isomorphism
\[
\text{Br}_a(P) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(P \times Y)
\] (2.5)
by [31] Lemma 5.1.

Let $G$ be a connected linear algebraic group over $k$ and let $X$ be a smooth variety over $k$ with $H^3(k, \bar{k}^\times) = 0$. Suppose that $Y \xrightarrow{f} X$ is a left torsor under $G$ and $P$ is a left $k$-torsor under $G$, associated to a cocycle $\sigma \in Z^1(k, G)$. One can consider $P$ as a right torsor under $G$ by defining a right action $x \cdot g := g^{-1}x$ (see [36] Example 2 in p.20). This right torsor is called the inverse right torsor of $P$ under $G$, and is denoted by $P'$. One can now consider the map given by the quotient of $P \times_k Y$ by the diagonal action of $G$ given by $g \cdot (p, y) := (p \circ g^{-1}, g \cdot y) = (g \cdot p, g \cdot y)$:
\[
\chi_P : P \times_k Y \rightarrow Y^\sigma := P' \times^G Y.
\]

**Definition 2.6.** With the above notation, assuming that $H^3(k, \bar{k}^\times) = 0$, consider the map
\[
\psi_\sigma = \psi_P : \text{Br}_a(Y^\sigma) \xrightarrow{\chi_P^*} \text{Br}_a(P \times_k Y) \xrightarrow{\sim} \text{Br}_a(P) \oplus \text{Br}_a(Y) \rightarrow \text{Br}_a(Y).
\]

The following lemma, which compares the algebraic Brauer groups of twists of a given torsor, can be regarded as an extension of [39] Lemma 1.3] to torsors under connected linear algebraic groups.

**Lemma 2.7.** The morphism $\psi_\sigma$ in Definition 2.6 is an isomorphism.

**Proof.** The natural morphism $(pr_P, \chi_P) : P \times_k Y \rightarrow P \times_k Y^\sigma$ is an isomorphism, and we have a commutative diagram:
\[
\begin{array}{ccc}
P \times_k Y & \xrightarrow{(pr_P, \chi_P)} & P \times_k Y^\sigma \\
pr_P \downarrow & & \downarrow pr_P \\
P & \xrightarrow{id} & P.
\end{array}
\]

Therefore $(pr_P, \chi_P)^* : \text{Br}_a(Y^\sigma \times_k P) \rightarrow \text{Br}_a(Y \times P)$ induces the identity map on the subgroups $\text{Br}_a(P) \subset \text{Br}_1(Y^\sigma \times_k P)$ and $\text{Br}_a(P) \subset \text{Br}_1(Y \times_k P)$, hence
\[
\psi_\sigma : \text{Br}_a(Y^\sigma) \rightarrow \text{Br}_a(Y^\sigma \times_k P) \xrightarrow{(pr_P, \chi_P)^*} \text{Br}_a(Y \times P) \rightarrow \text{Br}_a(Y)
\]
is an isomorphism (using the isomorphism (2.5)). \qed

Let $f : Y \rightarrow X$ be a torsor under a connected linear algebraic group $G$ over $k$ and let
\[
a_Y : G \times_k Y \rightarrow Y
\]
be the action of $G$. There is a canonical map $\lambda : \text{Br}_1(Y) \to \text{Br}_a(G)$ by [32 Lemma 6.4]. Let $e : \text{Br}_a(G) \to \text{Br}_1(G)$ be the section of $\text{Br}_1(G) \to \text{Br}_a(G)$ such that $1_G^e \circ e = 0$. If $X$ is smooth and geometrically integral, then the following diagram

$$
\begin{array}{ccc}
\text{Br}_1(Y) & \xrightarrow{\lambda} & \text{Br}_a(G) \\
\downarrow & & \downarrow p_G^e e \\
\text{Br}(Y) & \xrightarrow{a^*_Y - p_Y^*} & \text{Br}(G \times_k Y)
\end{array}
$$

(2.8)

commutes by [2 Theorem 2.8], where $G \times_k Y \xrightarrow{p_G} G$ and $G \times_k Y \xrightarrow{p_Y} Y$ are the projections. One can reformulate the commutative diagram (2.8) in the following proposition:

**Proposition 2.9.** With the above notation, one has

$$b(t \cdot x) = \lambda(b)(t) + b(x)$$

for any $x \in Y(k)$, $t \in G(k)$ and $b \in \text{Br}_1(Y)$.

**Proof.** The commutativity of diagram (2.8) implies that

$$a^*_Y - p_Y^* = p_G^* \circ e \circ \lambda : \text{Br}_1(Y) \to \text{Br}_1(G \times Y),$$

therefore one has

$$b(t \cdot x) = a^*_Y(b)(t, x) = p_Y^*(b)(t, x) + p_G^* \circ e \circ \lambda(b)(t, x) = b(x) + \lambda(b)(t)$$

as required. \qed

3. **Connected linear algebraic groups or groups of multiplicative type**

In this section, we study the relation between the descent obstruction and the Brauer-Manin obstruction for a general connected linear group or a group of multiplicative type.

First we need the following fact concerning topological groups:

**Lemma 3.1.** Let $f : M \to N$ be an open homomorphism of topological groups. If $K$ is a closed subgroup of $M$ containing $\ker(f)$, then $f(K)$ is a closed subgroup of $N$.

**Proof.** Since $K$ is a closed subgroup containing $\ker(f)$, one has

$$f(K) = f(M) \setminus f(M \setminus K).$$

Since $f$ is an open homomorphism, $f(M)$ is an open subgroup of $N$. This implies that $f(M)$ is closed in $N$. Since $f(M \setminus K)$ is open in $N$, one concludes that $f(K)$ is closed in $N$. \qed

**Remark 3.2.** The assumption $K \supseteq \ker(f)$ in Lemma 3.1 cannot be removed. For example, the projection map $pr^S : A_k \to A_k^S$ is open where $A_k^S$ is the set of adeles of $k$ without $S$-component. It is clear that $k$ is a discrete subgroup of $A_k$ by the product formula. However $k$ is dense in $A_k^S$ by strong approximation for $\mathbb{G}_a$, when $S$ is not empty.

For a short exact sequence of connected linear algebraic groups, one has the following result.
Proposition 3.3. Let
\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]
be a short exact sequence of connected linear algebraic groups over a number field \( k \). Then

1. \( \phi(G_2(A_k)^{Br_1(G_2)}) \) is a closed subgroup of \( G_3(A_k) \).

2. If \( G'(k_\infty) \) is not compact for each simple factor \( G' \) of the semi-simple part of \( G_3 \), then one has
\[ G_3(A_k)^{Br_1(G_3)} = G_3(k) \cdot \phi(G_2(A_k)^{Br_1(G_2)}) . \]

Proof. Let \( S \) be a sufficiently large finite set of primes of \( \Omega_k \) containing \( \infty_k \) and let \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) be a smooth group scheme model of \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) over \( O_S \) with connected fibres, such that the short exact sequence of smooth group schemes
\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]
extends the given short exact sequence of their generic fibres. The set \( H^1_{et}(O_v, G_1) \) is trivial by Hensel’s lemma together with Lang’s theorem, and the following diagram
\[ \begin{array}{c}
G_3(O_v) \\
\downarrow \\
G_3(k_v)
\end{array} \xrightarrow{\partial_v} \begin{array}{c}
H^1_{et}(O_v, G_3) \\
\downarrow \\
H^1(k_v, G_3)
\end{array} \]
commutes, hence we deduce the following commutative diagram of exact sequences in Galois cohomology:
\[ \begin{array}{cccc}
G_1(k) & \xrightarrow{\psi} & G_2(k) & \xrightarrow{\phi} & G_3(k) & \xrightarrow{\partial} & H^1(k, G_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1(A_k) & \xrightarrow{(\psi_v)} & G_2(A_k) & \xrightarrow{(\phi_v)} & G_3(A_k) & \xrightarrow{(\partial_v)} & \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) .
\end{array} \]

In addition, [17, Theorem 5.1] and [32, Corollary 6.11] gives the following commutative diagram of exact sequences of topological groups and pointed topological spaces:
\[ \begin{array}{c}
G_1(A_k) \xrightarrow{\theta_1} Br_a(G_1)^D \xrightarrow{(\psi^*)^D} III^1(k, G_1) \\
\downarrow (\psi_v) \downarrow \\
1 \to \ker(\theta_2) \to G_2(A_k) \xrightarrow{\theta_2} Br_a(G_2)^D \xrightarrow{(\phi^*)^D} \ker(\theta_3) \\
\downarrow (\phi_v) \downarrow \\
1 \to \ker(\theta_3) \to G_3(A_k) \xrightarrow{\theta_3} Br_a(G_3)^D \xrightarrow{(\partial_3^*)^D} \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) ,
\end{array} \]

(3.4)
where $\text{Br}_a(G_i)^D$ is the topological dual of the discrete group $\text{Br}_a(G_i)$, for $1 \leq i \leq 3$. Since $\theta_1(G_1(A_k))$ is the kernel of the continuous map $\text{Br}_a(G_1)^D \to \text{II}^1(k, G_1)$, it is a closed subgroup of $\text{Br}_1(G)^D$. Since $(\psi^*)^D$ is a closed map, one obtains that $(\psi^*)^D(\theta_1(G_1(A_k)))$ is a closed subgroup of $\text{Br}_1(G_2)^D$. It implies that

$$\ker(\theta_2) \cdot \psi(G_1(A_k)) = \theta_2^{-1} \left[ (\psi^*)^D(\theta_1(G_1(A_k))) \right]$$

is a closed subgroup of $G_2(A_k)$ by diagram (3.4). Proposition 6.5 in Chapter 6 of [33] ensures that $\phi : G_2(A_k) \to G_3(A_k)$ is an open homomorphism of topological groups. Then $\phi(\ker(\theta_2)) = \phi(G_2(A_k)_{\text{Br}_1(G_2)})$ is closed by Lemma 3.11 and property (1) follows.

Let us now prove statement (2): Corollary 3.20 in [17] (see also the proof of Proposition 4.5 in [5]) implies that

$$\ker(\theta_3) = G_3(A_k)^{\text{Br}_1(G_3)} = G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})$$

where $G_3(k_\infty)^0$ is the connected component of identity with respect to the topology of $k_\infty$. One only needs to show that

$$G_3(A_k)^{\text{Br}_1(G_3)} \subseteq G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})$$

For any $(x_v) \in G_3(k) \cdot G_3(k_\infty)^0$, there is $h \in G_3(k)$ and $h_\infty \in G_3(k_\infty)$ such that

$$(\partial_v)(h \cdot h_\infty) = (\partial_v)(x_v),$$

because $(\partial_v)$ is a continuous map with respect to the discrete topology of $\bigoplus_{v \in \Omega_k} H^1(k_v, G_1)$. Since $\phi(\infty(G_2(k_\infty)^0)$ is open and connected, the finiteness of $H^1(k_\infty, G_1)$ gives

$$G_3(k_\infty)^0 = \phi(\infty(G_2(k_\infty)^0)).$$

Therefore

$$(h \cdot h_\infty) \in G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})$$

and one can replace $(x_v)$ by $(h \cdot h_\infty)^{-1} \cdot (x_v)$. Without loss of generality, one can therefore assume $(\partial_v)(x_v)$ is the trivial element in $\bigoplus_{v \in \Omega_k} H^1(k_v, G_1)$.

Since $\text{II}^1(k, G_1)$ is finite, one can fix $\xi_1, \ldots, \xi_n$ in $G_3(k)$ such that each element of $\text{II}^1(k, G_1) \cap \partial(G_3(k))$ is represented by one of the $\xi_i$’s. As $\partial(\infty(h_\infty)$ is trivial for any $h_\infty \in G_3(k_\infty)^0$, one concludes that

$$(x_v) \in \bigcup_{i=1}^n \xi_i \phi(\ker(\theta_2)) = \bigcup_{i=1}^n \xi_i \cdot \phi(\ker(\theta_2)) \subseteq G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})$$

by Corollary 1 in Page 50 of [33] and assertion (1). \hfill $\square$

The main result of this section is the following theorem:

**Theorem 3.5.** Let $X$ be a smooth and geometrically integral variety and let $G$ be a connected linear algebraic group or a group of multiplicative type over a number field $k$. Suppose that $f : Y \to X$ is a left torsor under $G$. If $A$ is a subgroup of $\text{Br}(X)$ which contains the kernel of the natural map $f^* : \text{Br}(X) \to \text{Br}(Y)$, then

$$X(A_k)^A = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)^A).$$
where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $f$ by $\sigma$ and $\text{Br}(X) \xrightarrow{f_\sigma^*} \text{Br}(Y^\sigma)$ is the associated pull-back morphism, for each $\sigma \in H^1(k,G)$.

**Proof.** By the functoriality of Brauer-Manin pairing, one only needs to show that

$$X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)^{f_\sigma(A_k)}).$$

It is clear that

$$(x_v) \in \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)) \iff ([Y](x_v)) \in \text{Im} \left[H^1(k,G) \to \prod_{v \in \Omega_k} H^1(k_v,G)\right].$$

(3.6)

(1) Assume that $G$ is connected.

Recall first that Hensel’s lemma together with Lang’s theorem ensures that $H^1(k,G)$ maps to $\bigoplus_{v \in \Omega_k} H^1(k_v,G)$. Since any element $P \in \text{Pic}(G)$ can be given the structure of a central extension of algebraic groups

$$1 \to \mathbb{G}_m \to P \to G \to 1$$

by [6, Corollary 5.7], one obtains a coboundary map

$$\partial_P : H^1(X,G) \to H^2(X,\mathbb{G}_m) = \text{Br}(X)$$

associated to $P$ (see [19 IV.4.4.2]). Then the map defined by

$$\Delta_{Y/X} : \text{Pic}(G) \to \text{Br}(X), \ P \mapsto \partial_P([Y])$$

appears in the following short exact sequence (see [2, Theorem 2.8])

$$\text{Pic}(G) \xrightarrow{\Delta_{Y/X}} \text{Br}(X) \xrightarrow{f_\sigma^*} \text{Br}(Y) \xrightarrow{\text{id}} \text{Br}(k_v) \subseteq \mathbb{Q}/\mathbb{Z},$$

(3.8)

For any $v \in \Omega_k$, the exact sequence (3.7) defines a coboundary map

$$\partial_{P_v} : H^1(k_v,G) \to H^2(k_v,\mathbb{G}_m) = \text{Br}(k_v).$$

One can therefore define a pairing

$$\delta_v : H^1(k_v,G) \times \text{Pic}(G) \to \text{Br}(k_v) \subseteq \mathbb{Q}/\mathbb{Z}, \ (\sigma_v, P) \mapsto \partial_{P_v}^k(\sigma_v)$$

such that the following diagram

$$(3.9)$$

commutes (see Proposition 2.9 in [9]). These pairings induce a pairing

$$(\delta_v)_{v \in \Omega_k} : \bigoplus_{v \in \Omega_k} H^1(k_v,G) \times \text{Pic}(G) \to \mathbb{Q}/\mathbb{Z}, \ ((\sigma_v)_{v \in \Omega_k}, P) \mapsto \sum_{v \in \Omega_k} \delta_v(\sigma_v, P) \in \mathbb{Q}/\mathbb{Z}$$
and a natural exact sequence of pointed sets
\[ H^1(k, G) \to \bigoplus_{v \in \Omega_k} H^1(k_v, G) \to \text{Hom}(\text{Pic}(G), \mathbb{Q}/\mathbb{Z}) \]
by [9, Theorem 3.1]. Therefore (3.6) is equivalent to the fact that \([Y](x_v) \in \bigoplus_{v \in \Omega_k} H^1(k_v, G)\) is orthogonal to Pic(G) for the pairing \(\delta_v\). The commutative diagram (3.9), together with (3.8), gives
\[ X(A_k)^{\ker(f^*)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)). \]
Since \(\ker(f^*) \subseteq A\), one has
\[ X(A_k)^A \subseteq X(A_k)^{\ker(f^*)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)). \]
Then the functoriality of the Brauer-Manin pairing implies that
\[ X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)^G(A)). \]

(2) When \(G\) is a group of multiplicative type, one obtains that (3.6) is equivalent to
\[ \sum_{v \in \Omega_k} \text{inv}_v(\chi \cup [Y])(x_v) = 0 \]
for all \(\chi \in H^1(k, \hat{G})\) by [16, Theorem 6.3]. Let
\[ \mathcal{K}_f = \{\chi \cup [Y] : \chi \in H^1(k, \hat{G})\} \]
be the subgroup of Br(X) generated by elements \(\chi \cup [Y]\), where \(\cup\) is the cup product
\[ \cup : H^1(k, \hat{G}) \times H^1(X, G) \to H^2(X, \mathbb{G}_m) = \text{Br}(X). \]
Then
\[ X(A_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)) \]
by [26, Proposition 3.1]. Functoriality of the cup product proves that the following diagram
\[ \begin{array}{ccc}
H^1(k, \hat{G}) \times H^1(X, G) & \xrightarrow{\cup} & H^2(X, \mathbb{G}_m) = \text{Br}(X) \\
\text{id} \times f^* & \downarrow & \downarrow f^* \\
H^1(k, \hat{G}) \times H^1(Y, G) & \xrightarrow{\cup} & H^2(Y, \mathbb{G}_m) = \text{Br}(Y)
\end{array} \]
is commutative. Since \(Y \rightarrow X\) becomes a trivial torsor over \(Y\), the above diagram gives \(\mathcal{K}_f \subseteq \ker(f^*)\). Since \(\mathcal{K}_f \subseteq \ker(f^*) \subseteq A\), one has
\[ X(A_k)^A \subseteq X(A_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)). \]
Then the functoriality of the Brauer-Manin pairing implies that
\[ X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y^\sigma(A_k)^{f_\sigma(A)} \right). \]

\[ \square \]

4. REFINEMENT IN THE TORIC CASE

In this section, we will refine Theorem 3.5 for torsors under tori.

**Theorem 4.1.** Let \( f : Y \to X \) be a torsor under a torus \( G \) over a number field \( k \). Assume that \( X \) is smooth and geometrically integral. Let \( \ker(f^*) \subseteq A \subseteq Br(X) \) be a subgroup, and for all \( \sigma \in H^1(k,G) \), let \( B_{\sigma} \subseteq Br_1(Y^\sigma) \) be a subgroup such that

\[ f^* - 1 \left( \sum_{\sigma \in H^1(k,G)} \psi_{\sigma}(\tilde{B}_{\sigma}) \right) \subseteq A, \]

where \( Br_a(Y^\sigma) \xrightarrow{\psi_{\sigma}} Br_a(Y) \) is the morphism of Definition 2.6 and \( \tilde{B}_{\sigma} \) is the image of \( B_{\sigma} \) in \( Br_a(Y^\sigma) \).

Then one has

\[ X(A_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y^\sigma(A_k)^{B_{\sigma} + f_\sigma(A)} \right) \]

where \( Y^\sigma f_{\sigma} \to X \) is the twist of \( Y f \to X \) by \( \sigma \).

**Proof.** Since

\[ \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y^\sigma(A_k)^{B_{\sigma} + f_\sigma(A)} \right) \subseteq \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y^\sigma(A_k)^{f_\sigma(A)} \right) \subseteq X(A_k)^A \]

by the functoriality of Brauer-Manin pairing, one only needs to prove the converse inclusion.

Step 1. We first prove the result when \( \hat{G} \) is a permutation Galois module. In this case, Shapiro Lemma and Hilbert 90 gives \( H^1(K,G) = \{1\} \) for any field extension \( K/k \). This implies that

\[ X(A_k)^A = f \left( Y(A_k)^{f(A)} \right) \]

by the functoriality of Brauer-Manin pairing.

Let \( (x_v) \in X(A_k)^A \). Then there is \( (y_v) \in Y(A_k)^{f(A)} \) such that \( (x_v) = f((y_v)) \).

By Proposition 6.10 (6.10.3) in [32], the natural sequence

\[ Br_1(X) \xrightarrow{f^*} Br_1(Y) \xrightarrow{\lambda} Br_a(G) \]

is exact, and it induces the exact sequence

\[ (f^*)^{-1}(B) \xrightarrow{\lambda^D} B \xrightarrow{\omega} Br_a(G) \]

for any subgroup \( B \subseteq Br_1(Y) \). Therefore the following sequence

\[ Br_a(G)^D \xrightarrow{\lambda^D} B^D \xrightarrow{(f^*)^D} ((f^*)^{-1}(B))^D \]
is exact. Assuming \((f^*)^{-1}(B) \subseteq A\), one has \((f^*)^D((y_v)) = 0\), where we (abusively) identify \((y_v)\) with its image in \(B^D\) via the Brauer-Manin pairing. By the aforementioned exactness, there is \(\xi \in \text{Br}_a(G)^D\) such that \(\lambda^D(\xi) = (y_v)\). Since \(\text{III}^1(k,G) = \{1\}\), Theorem 2 in [22] implies that every element in \(\text{Br}_a(G)^D\) is given by an element in \(G(A_k)\) via the Brauer-Manin pairing. Namely, there is \((g_v) \in G(A_k)\) such that

\[
b(g_v) = \lambda(b)(g_v)
\]

for all \(b \in B\). Then \((g_v)^{-1} \cdot (y_v) \in Y(A_k)^{B^*f^*(A)}\) by Proposition 2.9 and \((x_v) = f((g_v)^{-1} \cdot (y_v))\).

Step 2. We now prove the case of an arbitrary torus \(G\). By Proposition-Definition 3.1 in [6], there is a short exact sequence of tori

\[
1 \to G \to T_0 \xrightarrow{q} T_1 \to 1,
\]

such that \(T_0\) is a permutation Galois module and \(T_1\) is a coflasque Galois module. Since

\[
H^3(k, T_1) \cong \prod_{v \in \infty_k} H^3(k_v, T_1) \cong \prod_{v \in \infty_k} H^1(k_v, T_1) = \{1\}
\]

(see for instance Proposition 5.9 in [27]), the map \(\text{Br}_1(T_0) \to \text{Br}_1(G)\) is surjective.

Let \(Z \xrightarrow{\rho} X\) be the torsor under \(T_0\) defined by \(Z := T_0 \times^G Y\). We have a morphism of torsors under \(G\):

\[
Y \xrightarrow{\epsilon_0 \times \text{id}_Y} T_0 \times_k Y \xrightarrow{\chi} Z = T_0 \times^G Y
\]

where \(\epsilon_0 \in T_0(k)\) is the unit element, \(p_0\) is the projection map and \(\theta\) is given as in (2.3). For simplicity, denote by \(i := \chi \circ (\epsilon_0 \times \text{id}_Y) : Y \to Z\) the composite morphism defined in the previous diagram.

Then Proposition 6.10 (6.10.3) in [32] gives the following commutative diagram of exact sequences:

\[
\begin{array}{ccc}
\text{Br}_1(T_1) & \xrightarrow{q^*} & \text{Br}_1(T_0) & \longrightarrow & \text{Br}_a(G) \\
\theta^* & & \downarrow p_0^* & & \downarrow \text{id} \\
\text{Br}_1(Z) & \xrightarrow{i^*} & \text{Br}_1(T_0 \times_k Y) & \longrightarrow & \text{Br}_a(G).
\end{array}
\]

Since the following sequence

\[
\text{Br}_1(T_0) \xrightarrow{p_0^*} \text{Br}_1(T_0 \times_k Y) \xrightarrow{(\epsilon_0 \times \text{id}_Y)^*} \text{Br}_a(Y) \to 1
\]

is exact by Lemma 6.6 in [32], the surjectivity of the map \(\text{Br}_1(T_0) \to \text{Br}_1(G)\) implies that the morphism

\[
i^* : \text{Br}_1(Z) \to \text{Br}_1(Y)
\]

is surjective, by a simple diagram chase.
Lemma 2.4 implies that for any \( t \in T_1(k) \), the composite morphism \( \theta^{-1}(t) \to Z \xrightarrow{\rho} X \) is canonically isomorphic to the twist \( f_t : Yq^{-1}(t) \to X \) of \( f : Y \to X \) by the Spec\((k)\)-torsor \( q^{-1}(t) \) under \( G \).

Denote by \( i_t : \theta^{-1}(t) \to Z \) the closed immersion. Then \( f_t = \rho \circ i_t \) for any \( t \in T_1(k) \).

Let \( \chi_t \) be the restriction of \( \chi \) to \( q^{-1}(t) \times_k Y \) for any \( t \in T_1(k) \). Then the following diagram

\[
\begin{array}{c}
q^{-1}(t) \times_k Y \xrightarrow{\chi_t} Yq^{-1}(t) \\
\downarrow j_t \times \text{id}_Y \quad \quad \downarrow i_t \\
Y \xrightarrow{e_0 \times \text{id}_Y} T_0 \times_k Y \xrightarrow{\chi} Z \\
\downarrow p_0 \quad \quad \quad \downarrow \theta \\
G \xrightarrow{q} T_0 \xrightarrow{q} T_1
\end{array}
\]

is commutative, where \( j_t : q^{-1}(t) \to T_0 \) is the closed immersion of the fiber of \( q \) at \( t \). Therefore Definition 2.6 implies that we have a commutative triangle:

\[
\begin{array}{c}
\text{Br}_a(Z) \xrightarrow{i_t^*} \text{Br}_a(Yq^{-1}(t)) \\
\downarrow i_t^* \sim \downarrow \psi_{q^{-1}(t)} \\
\text{Br}_a(Y)
\end{array}
\]

i.e. that \( \psi_{q^{-1}(t)} \circ i_t^* = i_t^* \).

Let

\[
B = i_t^{-1} \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \overline{B_{q^{-1}(t)}} \right) \right) \subset \text{Br}_a(Y)
\]

where \( \overline{B_{q^{-1}(t)}} \) is the image of \( B_{q^{-1}(t)} \) in \( \text{Br}_a(Yq^{-1}(t)) \) and \( \psi_{q^{-1}(t)} \) is given by Definition 2.6 for all \( t \in T_1(k) \).

Since \( i^* \circ \rho^* = f^* \), we have

\[
\rho^{q^{-1}(t)}(B) = f^{q^{-1}(t)} \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \overline{B_{q^{-1}(t)}} \right) \right) \subseteq A,
\]

hence step 1 applied to the torsor \( Z \xrightarrow{\rho} X \) under \( T_0 \) implies that

\[
X(A_k)^A = \rho \left( Z(A_k)^{B+\rho^*(A)} \right). \tag{4.2}
\]

Let \( (x_v) \in X(A_k)^A \). By (4.2), there is \( (z_v) \in Z(A_k)^{B+\rho^*(A)} \) such that \( (x_v) = \rho((z_v)) \). Since

\[
\theta^* \circ i^* \circ \theta^*(\text{Br}_1(T_t)) = (e_0 \times \text{id}_Y)^* \circ p_0^* \circ q^* \circ \theta^*(\text{Br}_1(T_t)) = \text{Br}_0(Y)
\]

and \( i^* (\text{Br}_0(Z)) = \text{Br}_0(Y) \), one gets \( \theta^* (\text{Br}_1(T_t)) \subseteq \text{Br}_0(Z) + B \) (by construction, \( B \) contains \( \text{ker}(i^* : \text{Br}_1(Z) \to \text{Br}_1(Y)) \)). Functoriality of the Brauer-Manin pairing now gives

\[
\theta((z_v)) \in T_1(A_k)^{\text{Br}_1(T_t)}.
\]
By Proposition 3.3, there are \( \alpha \in T_1(k) \) and \( (\beta_v) \in T_0(A_k)^{Br_1(T_0)} \) such that \( \theta((z_v)) = \alpha \cdot q(\beta_v) \). Therefore \( (\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha) \), hence \( (\beta_v)^{-1} \cdot (z_v) \in Z(A_k)^{B+\rho^*(A)} \).

Since \( i^*: Br_1(Z) \to Br_1(Y) \) is surjective, one has

\[
\psi_{q^{-1}(\alpha)} \circ i^*_\alpha(B) = i^*(B) = \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \widetilde{B}_{q^{-1}} \right) \supseteq \psi_{q^{-1}(\alpha)} \left( \widetilde{B}_{q^{-1}(\alpha)} \right),
\]

where \( \widetilde{B} \) is the image of \( B \) in \( Br_a(Z) \). It implies that \( i^*(B) + Br_0(\theta^{-1}(\alpha)) \supseteq B_{q^{-1}(\alpha)} \) by Lemma 2.7 and

\[
(\beta_v)^{-1} \cdot (z_v) \in \left[ \theta^{-1}(\alpha)(A_k) \right]^{i^*(B)+\rho^*(A)} \subseteq \left[ \theta^{-1}(\alpha)(A_k) \right]^{B_{q^{-1}(\alpha)}+\rho^*(A)}
\]
as desired. \( \square \)

The first part of the following result is also proved in Theorem 1.7 of [39].

**Corollary 4.3.** Let \( X \) be a smooth and geometrically integral variety. If \( f: Y \to X \) is a torsor under a torus \( G \) over a number field \( k \), then

\[
X(A_k)^{Br_1(X)} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma \left( Y^\sigma(A_k)^{Br_1(Y^\sigma)} \right)
\]

and

\[
X(A_k)^{Br} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma \left( Y^\sigma(A_k)^{Br_1(Y^\sigma)+f^*_\sigma(Br(X))} \right).
\]

**Proof.** To get the first equality, apply Theorem 14 to \( A = Br_1(X) \) and \( B_\sigma = Br_1(Y^\sigma) \) for each \( \sigma \in H^1(k, G) \). Since Pic\( (G_k) = 0 \), Proposition 6.10 in [32] gives

\[
f^{-1} \left( \sum_{\sigma \in H^1(k, G)} \psi_\sigma \left( \widetilde{B}_\sigma \right) \right) \subseteq f^{-1}(Br_a(Y)) \subseteq Br_1(X) = A,
\]
as required.

The second equality follows from Theorem 4.1 by taking \( A = Br(X) \) and \( B_\sigma = Br_1(Y^\sigma) \) for each \( \sigma \in H^1(k, G) \). \( \square \)

## 5. An Application

In this section, we apply the previous results to study the necessary conditions for a connected linear algebraic group to satisfy strong approximation with Brauer-Manin obstruction.

When \( X \) is affine, the set \( X(k) \) is discrete in \( X(A_k) \) by the product formula. Therefore if such an \( X \) satisfies strong approximation off \( S \), then \( \prod_{v \in S} X(k_v) \) is not compact. However this necessary condition for strong approximation is no longer true for strong approximation with Brauer-Manin obstruction if \( Br(X)/Br(k) \) is not finite. For example, a torus \( X \) always satisfies strong approximation with Brauer-Manin obstruction off \( \infty_k \), \( X \) being anisotropic over \( k_\infty \) or not: see [22] Theorem 2. When \( X \) is a semi-simple linear algebraic group, the necessary and sufficient condition for \( X \) to satisfy strong approximation with Brauer-Manin obstruction is
given by Proposition 6.1 in [5]. In this section, we extend this result to a general connected linear algebraic group.

The following lemma explains that strong approximation with Brauer-Manin obstruction for a general connected linear algebraic group can be reduced to the reductive case.

**Lemma 5.1.** Let $G$ be a connected linear algebraic group over a number field $k$. If $\pi : G \rightarrow G_{\text{red}}$ is the quotient map, then $G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} = \pi (G(A_k)^{\text{Br}_1(G)})$.

In particular, for any finite subset $S$ of $\mathcal{O}_k$, $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $G_{\text{red}}$ satisfies strong approximation with respect to $\text{Br}_1(G_{\text{red}})$ off $S$.

**Proof.** By applying Lemma 2.1 of Chapter III in [33], Proposition 6.5 in Chapter 6 of [30] and the functoriality of Brauer-Manin pairing. Then by assumption there is $x \in G(k) \cap \pi^{-1}(M)$. It implies that $\pi(x) \in M \cap G_{\text{red}}(k)$, as required.

Conversely, suppose $G_{\text{red}}$ satisfies strong approximation with respect to $\text{Br}_1(G_{\text{red}})$ off $S$. For any open subset

$$M = \prod_{v \in S} G_{\text{red}}(k_v) \times \prod_{v \notin S} M_v$$

of $G_{\text{red}}(A_k)$ such that $M \cap \left[ G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} \right] \neq \emptyset$, one has that

$$\pi^{-1}(M) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v)$$

with $\pi^{-1}(M) \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$ by the first part. Then by assumption there is $x \in G(k) \cap \pi^{-1}(M)$. It implies that $\pi(x) \in M \cap G_{\text{red}}(k)$, as required.

Conversely, suppose $G_{\text{red}}$ satisfies strong approximation with respect to $\text{Br}_1(G_{\text{red}})$ off $S$. For any open subset

$$N = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} N_v$$

of $G(A_k)$ such that $N \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$, we have

$$\pi(N) = \prod_{v \in S} G_{\text{red}}(k_v) \times \prod_{v \notin S} \pi(N_v)$$

and this set is an open subset of $G_{\text{red}}(A_k)$, with $\pi(M) \cap \left[ G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} \right] \neq \emptyset$: here we use Proposition 6 of §2.1 of Chapter III in [33], Proposition 6.5 in Chapter 6 of [30] and the functoriality of Brauer-Manin pairing. Then by assumption there is $y \in G_{\text{red}}(k) \cap \pi(N)$. Using Proposition 6 of §2.1 of Chapter III in [33] one more time, one concludes that $\pi^{-1}(y)$ is isomorphic to $R_a(G)$ as an algebraic variety, hence it satisfies strong approximation off $S$. Since

$$\pi^{-1}(y) \cap N = \prod_{v \in S} \pi^{-1}(y)(k_v) \times \prod_{v \notin S} (\pi^{-1}(y)(k_v) \cap N) \neq \emptyset,$$

there is $z \in \pi^{-1}(y)(k) \cap N \subset G(k) \cap N$, as desired. \hfill \Box

The main result of this section is the following statement:
Theorem 5.2. Let $G$ be a connected linear algebraic group over a number field $k$ and let $G^{sp} := G/R(G)$, where $R(G)$ is the solvable radical of $G$. If $\pi : G \to G^{sp}$ is the quotient map, then

$$G^{sp}(A_k)^{Br_1(G^{sp})} = \pi \left( G(A_k)^{Br_1(G)} \right) \cdot G^{sp}(k).$$

In particular, if $G$ satisfies strong approximation with respect to $Br_1(G)$ off a finite subset $S$ of $\Omega_k$, then $G^{sp}$ satisfies strong approximation with respect to $Br_1(G^{sp})$ off $S$.

Proof. For the first part, by functoriality of the Brauer-Manin pairing, one only needs to prove

$$G^{sp}(A_k)^{Br_1(G^{sp})} \subseteq \pi \left( G(A_k)^{Br_1(G)} \right) \cdot G^{sp}(k).$$

By Lemma 5.1 we can assume that $G$ is reductive. Then $R(G)$ is a torus contained in the center of $G$ (see Theorem 2.4 in Chapter 2 of [30]) and $\pi : G \to G^{sp}$ is a torsor under $R(G)$. By Corollary 4.3 for any $(x_v) \in G^{sp}(A_k)^{Br_1(G^{sp})}$, there are $\sigma \in H^1(k,R(G))$ and $(y_v) \in G^\sigma(A_k)^{Br_1(G^\sigma)}$ such that $(x_v) = \pi_{\sigma}(y_v)$. Since $G^\sigma(k) \neq \emptyset$ by Corollary 8.7 in [32] (see also Theorem 5.2.1 in [36]), there is $\gamma \in G^{sp}(k)$ such that $\partial(\gamma) = \sigma$, where $\partial$ is the coboundary map in the following exact sequence in Galois cohomology:

$$1 \to R(G)(k) \to G(k) \to G^{sp}(k) \xrightarrow{\partial} H^1(k,R(G)) \to H^1(k,G).$$

In addition, the choice of an element $\bar{\gamma} \in G(k)$ such that $\pi(\bar{\gamma}) = \gamma$ defines a commutative diagram defined over $k$:

$$
\begin{array}{ccc}
G^\sigma & \xrightarrow{\bar{\gamma}} & G \\
\pi_{\sigma} \downarrow & & \downarrow \pi \\
G^{sp} & \xrightarrow{\gamma} & G^{sp}
\end{array}
$$

(see for instance Example 2 of p.20 in [36]). This implies that

$$\pi_{\sigma} \left( G^\sigma(A_k)^{Br_1(G^\sigma)} \right) = \pi \left( G(A_k)^{Br_1(G)} \right) \cdot \gamma,'$$

as desired.

Suppose now that $G$ satisfies strong approximation with respect to $Br_1(G)$ off $S$. For any open subset

$$M = \prod_{v \in S} G^{sp}(k_v) \times \prod_{v \notin S} M_v$$

of $G^{sp}(A_k)$ such that $M \cap G^{sp}(A_k)^{Br_1(G^{sp})} \neq \emptyset$, the first part implies that there is $g \in G^{sp}(k)$ such that

$$\pi^{-1}(M \cdot g) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v \cdot g),$$

with $\pi^{-1}(M \cdot g) \cap G(A_k)^{Br_1(G)} \neq \emptyset$. Since $G$ satisfies strong approximation with algebraic Brauer-Manin obstruction off $S$, there exists $x \in G(k) \cap \pi^{-1}(M \cdot g)$. This implies that $\pi(x) \cdot g^{-1} \in M \cap G^{sp}(k)$ as required. □
Corollary 5.3. Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $\Br_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

Proof. By Theorem 2.3 and Theorem 2.4 of Chapter 2 in [30], the quotient map

$$G^{\text{red}} \to G/R(G) = G^{qs}$$

induces an isogeny $G^{ss} \to G^{qs}$. One side follows from Corollary 3.20 in [17]. The other side follows from Theorem 5.2 and Proposition 6.1 in [5]. \qed

Remark 5.4. All the results in this section involve the group $\Br_1(G)$, and they remain true with $\Br_1(G)$ replaced by $\Br(G)$. Indeed, there is a sufficiently large subset $S$ of $\Omega_k$ containing $\infty_k$ such that $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of $G^{ss}$, therefore Corollary 3.20 in [17], Proposition 2.6 in [9] and the functoriality of Brauer-Manin pairing gives the following inclusions:

$$G(\mathbb{A}_k)^{\Br_1(G)} = G(k) \cdot \rho(\prod_{v \in S} G^{\text{scu}}(k_v)) \subseteq G(\mathbb{A}_k)^{\Br(G)} \subseteq G(\mathbb{A}_k)^{\Br_1(G)},$$

where $G^{\text{scu}} = G^{\text{sc}} \times \text{Gal} G$ with the projection map $G^{\text{scu}} \to G$ and $G^{\text{sc}}$ is the simply connected covering of $G^{ss}$. In particular, we have $G(\mathbb{A}_k)^{\Br(G)} = G(\mathbb{A}_k)^{\Br_1(G)}$.

6. Comparison I, $X(\mathbb{A}_k)^{\text{desc}} \subseteq X(\mathbb{A}_k)^{\text{et,Br}}$

Let $Y \xrightarrow{f} X$ be a left torsor under a linear algebraic group $G$ over a number field $k$. The fundamental problem to define the descent obstruction for strong approximation with respect to $Y \xrightarrow{f} X$ is to decide whether the set

$$X(\mathbb{A}_k)^f = \left\{ (x_v) \in X(\mathbb{A}_k) : ([Y](x_v)) \in \text{Im} \left( H^1(k, G) \to \prod_v H^1(k_v, G) \right) \right\} = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(\mathbb{A}_k))$$

is closed or not in $X(\mathbb{A}_k)$. We already know that this is true when $G$ is either connected or a group of multiplicative type, by Theorem 5.5. For a general linear algebraic group $G$, this result is proved by Skorobogatov in Corollary 2.7 of [35], when $X$ is assumed to be proper over $k$. The proof depends on Proposition 5.3.2 in [36] or Proposition 4.4 in [23], which are not true for open varieties, as explained in the following example.

Example 6.1. The short exact sequence of linear algebraic groups

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \to 1,$$

where $f(x) = x^2$, can be viewed as torsor over $\mathbb{G}_m$ under $\mu_2$. For any $\sigma \in H^1(k, \mu_2) \cong k^\times/(k^\times)^2$, the twist $\mathbb{G}_m^\sigma$ of $\mathbb{G}_m$ by $\sigma$ is given by the equation $x = a_\sigma y^2$ in $\mathbb{G}_m \times_k \mathbb{G}_m$, where $a_\sigma$ is an element in $k^\times$ representing the class $\sigma$ by the above isomorphism. It is clear that $\mathbb{G}_m^\sigma \cong \mathbb{G}_m$ as varieties over $k$, hence it always contains adelic points.

We use the same definition of an integral model as in [28].
Definition 6.2. Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. An integral model of $X$ over $O_S$ is a faithfully flat separated $O_S$-scheme $X_S$ of finite type such that $X_S \times_{O_S} k \cong X$.

The replacement for Proposition 5.3.2 in [36] or Proposition 4.4 in [23] is the following proposition:

Proposition 6.3. Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Fix an integral model $X_S$ of $X$ over $O_S$. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set

$$\left\{ [\sigma] \in H^1(k, G) : f_\sigma(Y^{\sigma}(A_k)) \cap \prod_{v \in S} X(k_v) \times \prod_{v \notin S} X_S(O_v) \neq \emptyset \right\}$$

is finite.

Proof. It follows from the same argument as the proof of Proposition 4.4 in [23]. □

One can now extend Corollary 2.7 in [35] to open varieties by using the above replacement for Proposition 4.4 in [23].

Proposition 6.4. Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set $X(A_k)^f$ is closed in $X(A_k)$.

Proof. Take an integral model $X_{S_0}$ of $X$ over $O_{S_0}$, where $S_0$ is a finite subset of $\Omega_k$ containing $\infty_k$. Then

$$\left\{ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \backslash S} X_{S_0}(O_v) \right\}_{S}$$

is an open covering of $X(A_k)$ (see Theorem 3.6 in [11]), where $S$ runs through all finite subsets of $\Omega_k$ containing $S_0$. By Proposition 6.3 and Corollary 2.5 in [35], the set

$$X(A_k)^f \cap \left[ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \backslash S} X_{S_0}(O_v) \right]$$

is closed in $\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \backslash S} X_{S_0}(O_v)$, therefore the set $X(A_k)^f$ is closed in $X(A_k)$. □

Applying Proposition 6.3 one can also extend Lemma 2.2 and Theorem 1.1 in [35] to open varieties. For any variety over a number field $k$, and following [35], we write

$$X(A_k)^{\text{desc}} = \bigcap_{Y \xrightarrow{f} X} X(A_k)^f,$$

where $Y \xrightarrow{f} X$ runs through all torsors under all linear algebraic groups over $k$ (see also [11]).
**Lemma 6.5.** Let $X$ be a (not necessarily proper) variety and let $Y \to X$ be a torsor over a number field $k$. For any $(P_v) \in X(A_k)^{\text{desc}}$, there is a twist $Y' \to X$ of $Y \to X$ such that the following property holds:

For any surjective $X$-torsor morphism $Z \to Y'$ (see Definition 2.1 in [35]), there is a twist $Z' \to Y'$ of $Z \to Y'$ such that $(P_v)$ lies in the image of $Z'(A_k)$.

**Proof.** There are a finite subset $S_0$ of $\Omega_k$ containing $\infty_k$ and an integral model $X_{S_0}$ over $O_{S_0}$ such that

$$ (P_v) \in \prod_{v \in S_0} X(k_v) \times \prod_{v \in \Omega_k \setminus S_0} X_{S_0}(O_v) $$

(see for instance Theorem 3.6 in [11]), hence Proposition 6.3 implies that there are only finitely many twists of a given torsor over $X$ such that $(P_v)$ lifts as an adelic point of this torsor. As pointed out in the proof of Lemma 2.2 in [35], the finite combinatorics in the first part of the proof of Proposition 5.17 in [38] are still valid. It concludes the proof. □

**Proposition 6.6.** Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \xrightarrow{f} X$ is a left torsor under a finite group scheme $F$ over $k$, then

$$ X(A_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma \left( Y^\sigma(A_k)^{\text{desc}} \right). $$

**Proof.** One only needs to modify the proof of Theorem 1.1 in [35] by replacing Lemma 2.2 in [35] with Lemma 6.5 Corollary 2.7 in [35] with Proposition 6.4. Moreover, since $f$ is finite, the induced map $Y(A_k) \xrightarrow{f} X(A_k)$ is topologically proper by Proposition 4.4 in [11]. This implies that $f^{-1}((P_v))$ is compact. □

Recall that, following [31], one can define for any variety $X$ over a number field $k$, the set

$$ X(A_k)^{\text{et,Br}} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma \left( Y^\sigma(A_k)^{\text{Br}} \right), $$

where $Y \xrightarrow{f} X$ runs over all torsors under all finite groups $F$ over $k$ (see [11]). Since the induced map $Y(A_k) \xrightarrow{f} X(A_k)$ is topologically closed for any finite morphism $Y \xrightarrow{f} X$ by Proposition 4.4 in [11], one concludes that $X(A_k)^{\text{et,Br}}$ is closed in $X(A_k)$ by the same argument as in Proposition 6.4.

**Corollary 6.7.** If $X$ is a smooth quasi-projective variety over a number field $k$, then

$$ X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{et,Br}} \subseteq X(A_k)^{\text{Br}}. $$

**Proof.** One only needs to show that $X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{et,Br}}$. For any torsor $Y \xrightarrow{f} X$ under a finite group scheme $F$, Proposition 6.6 gives the equality

$$ X(A_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma \left( Y^\sigma(A_k)^{\text{desc}} \right). $$
Since $X$ is quasi-projective, $Y^\sigma$ is quasi-projective as well. By a theorem of Gabber (see [12]), one has

$$Y^\sigma(A_k)^{\text{desc}} \subseteq Y^\sigma(A_k)^{\text{Br}}$$

(see the proof of Lemma 2.8 in [35]) and the result follows. □

7. COMPARISON II, $X(A_k)^{et, Br} \subseteq X(A_k)^{\text{desc}}$

In this section, we prove the inclusion $X(A_k)^{et, Br} \subseteq X(A_k)^{\text{desc}}$ for open varieties, which implies Theorem 1.5. The strategy of proof is the same as in [14].

The second named author would like to thank Laurent Moret-Bailly warmly for finding a mistake and for suggesting the following alternative proof of Lemma 4 in [14] (which already appeared in [15]). The statement of this lemma is correct, but the proof in [14] uses a result of Stoll (see [38]) that is not. Note that in contrast with [14], all torsors (unless explicitly mentioned) are assumed to be left torsors.

**Lemma 7.1.** Let $X$ be a smooth geometrically connected $k$-variety. Let $(P_v) \in X(A_k)^{et, Br}$ and let $Z \to X$ be a torsor under a finite $k$-group $F$.

Then there is a cocycle $\sigma \in Z^1(k, F)$ and a connected component $X'$ of $Z^\sigma$ over $k$ such that the restriction of $g_\sigma$ to $X'$ is a torsor $X' \to X$ under the stabilizer $F'$ of $X'$ for the action of $F^\sigma$, and the point $(P_v)$ lifts to a point $(Q_v') \in X'(A_k)^{Br}$.

In particular, $X'$ is geometrically integral.

**Proof.** By assumption, the point $(P_v)$ lifts to some point $(Q_v) \in Z^\sigma(A_k)^{Br}$ for some cocycle $\sigma$ with values in $F$. Since $Z^\sigma$ is smooth, $Z^\sigma$ is a disjoint union of connected components over $k$. By Proposition 3.3 in [28], there is a $k$-connected component $X'$ of $Z^\sigma$ such that $(Q_v)_{v \in \Xi} \in P_\Xi(X'(A_k)^{Br})$, where $\Xi$ is the set of all complex places of $k$, $A_k^\Xi$ is the ring of adeles without $\Xi$-components and $P_\Xi$ is the projection from $X'(A_k)$ to $X'(A_k^\Xi)$. Since for $v \in \Xi, Z^\sigma \times_k k_v$ is a trivial torsor under the finite constant group scheme $F^\sigma \times_k k_v$, we have $g_\sigma(X'(k_v)) = X(k_v)$ for all $v \in \Xi$. Hence one can assume that $Q_v \in X'(k_v)$ for $v \in \Xi$, so that we have $(Q_v) \in X'(A_k)^{Br}$.

Since $X'$ is connected and $X'(A_k) \neq \emptyset$, the proof of Lemma 5.5 in [38] implies that $X'$ is geometrically connected. Eventually, $X'$ being geometrically connected guarantees that the variety $X'$ is an $X$-torsor under the stabilizer $F'$ of $X'$ in $F^\sigma$. □

Let us continue the proof of the aforementioned inclusion. Let $X$ be a smooth and geometrically integral $k$-variety, and $(P_v) \in X(A_k)^{et, Br}$. We need to prove that $(P_v) \in X(A_k)^{\text{desc}}$.

For a linear algebraic group $G$ over $k$, one has the following short exact sequence of algebraic groups over $k$:

$$1 \to H \to G \to F \to 1,$$

where $H$ is the connected component of $G$ and $F$ is finite over $k$. This induces the following diagram of short exact sequences

$$
\begin{array}{cccccc}
1 & \to & H & \to & G & \to & F & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & T & \to & G' & \to & F & \to & 1
\end{array}
$$
where $T$ denotes the maximal toric quotient of $H$ and $G'$ is the quotient of $G$ by the kernel of $H \to T$.

Let $Y \to X$ be a torsor under $G$ and let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under $F$. If $\sigma \in Z^1(k, F)$ is a $1$-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and to the point $(P_v)$, we want to show that the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$, as in Proposition 5 in [14]. The obstruction to lift $\sigma$ to a cocycle in $Z^1(k, G)$ gives a natural cohomology class $\eta_\sigma \in H^2(k, \kappa_\sigma)$ by (5.1) in [18] (see also (7.7) in [1]). Where $\kappa_\sigma$ is a natural $k$-kernel on $H_k'$ associated to $\sigma$. Lemma 6 in [14] implies that there is a canonical map $H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma)$ such that the class $\eta_\sigma$ is neutral if and only if its image $\eta'_\sigma \in H^2(k, T^\sigma)$ is zero.

We now apply the open descent theory and the extended type developed by Harari and Skorobogatov in [26] to establish the analogue of Lemma 7 in [14] for open varieties. As in the proof of [14], the torsor $Y \to Z$ under $H$ induces a torsor $W \to Z$ under $T$ by the natural map $H^1(Z, H) \to H^1(Z, T)$. Instead of using the type of the torsor $\varpi$ that was used in [14], we consider the so-called "extended type" of the torsor $\varpi$ that was introduced by Harari and Skorobogatov (see Definition 8.2 in [26]). For a variety $Z$ over $k$, let $KD'(Z)$ denote the complex of Galois modules $[\bar{k}(Z)^*/\bar{k}^* \to \text{Div}(Z_k)]$ in the derived category $D^b_{\text{et}}(k)$ of bounded complexes of étale sheaves over $\text{Spec}(k)$. One can associate to the torsor $W \to Z$ under $T$ a canonical morphism in this derived category

$$\lambda_W : \hat{T} \to KD'(Z),$$

called the extended type of $\varpi$. This induces a morphism in the derived category of bounded complexes of abelian groups

$$\lambda^\sigma_W : \hat{T}^\sigma \to KD'(Z^\sigma)$$

for the above $\sigma \in Z^1(k, F)$.

Lemma 7.2. The morphism $\lambda^\sigma_W : \hat{T}^\sigma \to KD'(Z^\sigma)$ is a morphism in the derived category of bounded complexes of étale sheaves over $\text{Spec}(k)$.

Proof. The natural left actions of $F$ on both $T$ and $Z$ induces right actions of $F$ on $\hat{T}$ and on $KD'(Z)$.

We first prove that the morphism $\lambda_W$ is $F$-equivariant for those actions.

Let $f \in F(\bar{k})$. We denote by $f_z : Z_k \to Z_k$ the morphism of $k$-varieties defined by $z \mapsto f \cdot z$. This morphism induces a natural morphism in the derived category $f_z^* : KD'(Z_k) \to KD'(Z_k)$. Similarly, the element $f$ defines a natural morphism of $\bar{k}$-tori $f_T : T_k \to T_k$ such that $f_T(t) := gtg^{-1}$, where $g \in G'(\bar{k})$ is any point lifting $f \in F(\bar{k})$. This morphism $f_T$ induces a morphism of abelian groups $\hat{f}_T : \hat{T} \to \hat{T}$ such that $\hat{f}_T(\chi) := \chi \circ f_T$. 
One needs to prove that the following diagram

\[
\begin{array}{ccc}
\hat{T} & \xrightarrow{\lambda_{W_k}} & KD'(Z_k) \\
\downarrow{\hat{f}_T} & & \downarrow{f_Z} \\
\hat{T} & \xrightarrow{\lambda_{W_k}} & KD'(Z_k)
\end{array}
\]

is commutative.

Let \( f_{T,*}W_k \) be the push-forward of the torsor \( W_k \to Z_k \) under \( T_k \) by the \( k \)-morphism \( \hat{f}_T: T_k \to T_k \) and let \( f_Z^*W_k \) be the pullback of the torsor \( W_k \to Z_k \) under \( T_k \) by the \( k \)-morphism \( f_Z: Z_k \to Z_k \).

Then functoriality of the extended type gives:

\[ f_Z^* \circ \lambda_{W_k} = \lambda_{f_Z^*W_k} \quad \text{and} \quad \lambda_{f_{T,*}W_k} = \lambda_{W_k} \circ \hat{f}_T. \]

To prove the required commutativity \( f_Z^* \circ \lambda_{W_k} = \lambda_{f_Z^*W_k} \), it is enough to show that the torsors \( f_Z^*W_k \to Z_k \) and \( f_{T,*}W_k \to Z_k \) under \( T_k \) are isomorphic. Indeed, we have the following commutative diagram

\[
\begin{array}{ccc}
T_k \times W_k & \xrightarrow{g} & W_k \\
\downarrow{\pi \circ p_W} & & \downarrow{=} \\
Z_k & \xrightarrow{f_Z} & Z_k,
\end{array}
\]

where \( p_W \) denotes the projection on \( W_k \) and the morphism \( g \) is defined by \( (t, w) \mapsto (tg) \cdot w \). This diagram induces a natural \( Z_k \)-morphism \( \phi: T_k \times W_k \to f_Z^*W_k \). Consider now the right action of \( T_k \) on \( T_k \times W_k \) defined by \( (s, w) \cdot t := (sf_T(t), t^{-1} \cdot w) = (sgtg^{-1}, t^{-1} \cdot w) \). Then the morphism \( \phi \) is \( T_k \)-invariant under this action, hence it induces a \( Z_k \)-morphism \( \psi: f_{T,*}W_k \to f_Z^*W_k \). One can check by a simple computation that \( \psi \) is \( T_k \)-equivariant, i.e. that \( \psi \) is a morphism of (left) torsors over \( Z_k \) under \( T_k \). It concludes the proof of the required commutativity, hence the morphism \( \lambda_W \) is \( F \)-equivariant.

By definition of the twists \( T^\sigma \) and \( Z^\sigma \), the fact that \( \lambda_W \) is \( F \)-equivariant implies that the morphism \( \lambda_W^\sigma \) is Galois equivariant, i.e. that \( \lambda_W^\sigma \) is a morphism in the derived category of bounded complexes of étale sheaves over \( \text{Spec}(k) \). \( \square \)

By Proposition 8.1 in [26], there is a natural exact sequence of abelian groups

\[
H^1(k, T^\sigma) \to H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(T^\sigma, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)
\]

where the map \( \lambda \) is the extended type. Let \( \lambda^\sigma = \psi^* \circ \lambda_W^\sigma \), where \( \psi: X' \to W \) is the inclusion of the \( k \)-connected component given by Lemma 7.1 and \( KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X') \) is the map induced by \( \psi \).

The following lemma, which is an analogue of Lemma 8 in [14], is a crucial step for proving the main result of this section. We give here a more conceptual proof than that in [14], where a similar statement was proven by cocycle computations under the assumption that \( \bar{k}[X]^\times = \bar{k}^\times \).
Lemma 7.3. With the above notation, one has
\[ \partial(\lambda'_\sigma) = 0 \] if and only if \( \eta'_\sigma = 0 \).

Proof. In the following proof, we work over the small étale site of \( \text{Spec}(k) \).

Recall that we are given a cocycle \( \sigma \in Z^1(k, F) \) as in Lemma 7.1; one can associate to \( \sigma \) a \( \text{Spec}(k) \)-torsor \( U \) under \( F \) with a point \( u_0 \in U(\overline{k}) \). This torsor \( U \) is naturally a homogeneous space of the group \( G' \) with geometric stabilizer isomorphic to \( T_k \). Section IV.5.1 in [19] implies that the element \( \eta'_\sigma \in H^2(k, T^\sigma) \) is the class of the \( \text{Spec}(k) \)-gerbe \( E_\sigma \) banded by \( T^\sigma \) such that for all étale schemes \( S \) over \( \text{Spec}(k) \), the category \( E_\sigma(S) \) is defined as follows: the objects of \( E_\sigma(S) \) are triples \( (P, p, \alpha) \) where \( P \to S \) is a torsor under \( G' \), \( p \in P(S_k) \) and \( \alpha : P \to U_S \) is a \( G' \)-equivariant \( S \)-morphism. The morphisms of \( E_\sigma(S) \) between triples \( (P, p, \alpha) \) and \( (P', p', \alpha') \) are given by morphisms of torsors \( P \to P' \) over \( S \) under \( G' \) that commute with \( \alpha \) and \( \alpha' \).

Similarly, one can associate to the morphism \( \lambda'_\sigma \) a \( \text{Spec}(k) \)-gerbe banded by \( T^\sigma \) that will be the obstruction for the morphism \( \lambda'_\sigma \) to be the extended type of a torsor over \( X' \) under \( T^\sigma \). The morphism \( \lambda'_\sigma \) induces a morphism \( \lambda'_\sigma : \widehat{T^\sigma} \to KD'(X'_k) \) in \( D_{\text{et}}(k) \). By construction, \( \lambda'_\sigma \) is the extended type of the torsor \( Y_0 := W_k \times_{Z_k} X'_k \) over \( X'_k \) under \( T^\sigma_k = T_k \).

We now define \( \mathcal{L}_\sigma \) to be the fibered category defined as follows: for all étale schemes \( S \) over \( \text{Spec}(k) \), the objects of the category \( \mathcal{L}_\sigma(S) \) are pairs \( (V, \varphi) \), where \( V \to X'_k \) is a torsor under \( T^\sigma_k \) of extended type \( \lambda_V \) compatible with \( \lambda'_\sigma \) and \( \varphi : V_k \to Y_0 \times_k S_k \) is an isomorphism of torsors over \( X' \times_k S_k \) under \( T^\sigma_k \). Given two such objects \( (V, \varphi) \) and \( (V', \varphi') \), a morphism between \( (V, \varphi) \) and \( (V', \varphi') \) in the category \( \mathcal{L}_\sigma(S) \) is a pair \( (\alpha, t) \), where \( \alpha : V \to V' \) is a morphism of torsors over \( X'_k \) under \( T^\sigma_k \) and \( t \in T^\sigma(S_k) \) such that the diagram

\[
\begin{array}{ccc}
V_k & \xrightarrow{\pi} & V'_k \\
\varphi \downarrow & & \varphi' \downarrow \\
Y_0 \times_k S_k & \xrightarrow{t} & Y_0 \times_k S_k
\end{array}
\]

commutes.

One can check that \( \mathcal{L}_\sigma \) is a stack for the étale topology over \( \text{Spec}(k) \), and the fact that this is a gerbe is a consequence of the exact sequence of Proposition 8.1 in [26]

\[ H^1(S, T^\sigma) \to H^1(X'_S, T^\sigma) \xrightarrow{\lambda} \text{Hom}_S(T^\sigma, KD'(X'_S)) \xrightarrow{\delta} H^2(S, T^\sigma) \]

(which holds provided that \( S \) is integral, regular and noetherian).

The band of this gerbe is the abelian band represented by \( T^\sigma \).

In addition, it is clear that \( \mathcal{L}_\sigma \) is neutral if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if there exists a torsor over \( X' \) under \( T^\sigma \) of type \( \lambda'_\sigma \) if and only if \( \partial(\lambda'_\sigma) = 0 \).

Let us now construct an equivalence of gerbes between \( E_\sigma \) and \( \mathcal{L}_\sigma \).

For all étale \( \text{Spec}(k) \)-schemes \( S \), consider the functor

\[ m_S : E_\sigma(S) \to \mathcal{L}_\sigma(S) \]

that maps an object \( (P, p, \alpha) \) to the object \( (V, \varphi) \), where \( V \) is defined to be the contracted product \( V := (P \times^S W_S) \times_{Z_S} X'_S \) and \( \varphi : V_k \to Y_0 \times_k S_k = (W_k \times_{Z_k} X'_k) \times_{Z_k} S_k \) is induced by the
point \( p \in P(S_k) \). Indeed, by construction, we have a natural map \( P \times_{S} W_S \to U_S \times_{S} Z_S = Z_{\overline{S}} \), and a simple computation proves that this map is a torsor under \( T^\sigma \) of extended type compatible with \( \lambda_W^\sigma \).

By definition, the functor \( m_S \) sends a morphism \( \varphi : (P, p, \alpha) \to (P', p', \alpha') \) to the morphism \((\overline{\varphi}, t_0)\) such that \( \overline{\varphi} : (P \times_{S} W_S) \times_{Z_S} X_S \to (P' \times_{S} W_S) \times_{Z_S} X'_S \) is the morphism induced by the morphism of torsors \( \varphi : P \to P' \), and \( t_0 \in T^\sigma(S_{\overline{S}}) \) is the element such that \( p' = t_0 \cdot \varphi(p) \) as \( S_{\overline{S}} \)-points in \((P' \times_{S} W_S) \times_{Z_S} X'_S \).

Finally, one checks that the collection of functors \( m_S \) defines a morphism of gerbes \( m : \mathcal{E}_\sigma \to \mathcal{L}_\sigma \) banded by the identity of \( T^\sigma \), which implies that \( \eta'_\sigma := [\mathcal{E}_\sigma] = [\mathcal{L}_\sigma] \in H^2(k, T^\sigma) \).

Therefore, \( \eta'_\sigma = 0 \) if and only if \( \mathcal{E}_\sigma(k) \neq \emptyset \) if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if \( \partial(\lambda'_\sigma) = 0 \). \( \square \)

The immediate consequence of Lemma \( \ref{7.3} \) is the following result which extends Proposition \( \ref{5} \) in \( \cite{14} \) to open varieties.

**Proposition 7.4.** Let \( X \) be a smooth geometrically integral \( k \)-variety. Let \((P_v) \in X(A_k)^{\text{ét},Br}\) and let \( Y \to X \) be a torsor under a linear \( k \)-group \( G \). Let

\[
1 \to H \to G \to F \to 1
\]

be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \), which is a torsor under \( F \). Let \( \sigma \in Z^1(k, F) \) be a 1-cocycle applied to the torsor \( Z \to X \) and the point \((P_v)\).

Then the cocycle \( \sigma \in Z^1(k, F) \) lifts to a cocycle \( \tau \in Z^1(k, G) \).

**Proof.** As mentioned above, Construction \((5.1) \) in \( \cite{18} \) (see also \((7.7) \) in \( \cite{1} \)) gives a class \( \eta_\sigma \) of \( H^2(k, \kappa_\sigma) \) such that \( \sigma \) can be lifted to \( Z^1(k, G) \) if and only if \( \eta_\sigma \) is neutral, where \( \kappa_\sigma \) is a \( k \)-kernel on \( H_k \). By \((6.1.2) \) in \( \cite{1} \) and Lemma \( \text{6} \) in \( \cite{14} \), there is a canonical map \( H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma) \) such that the class \( \eta_\sigma \) is neutral if and only if its image \( \eta_\sigma \in H^2(k, T^\sigma) \) is zero. By Lemma \( \ref{7.3} \) one only needs to show that \( \partial(\lambda'_\sigma) = 0 \) where \( \lambda'_\sigma = \psi^* \circ \lambda_W^\sigma \), with \( KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X') \) given by Lemma \( \ref{7.1} \) and \( \lambda_W^\sigma \) defined by Lemma \( \ref{7.2} \).

By Lemma \( \ref{7.1} \), we know that \( X'(A_k)^{Br} \neq \emptyset \). Therefore the map \( \lambda \) in the exact sequence (see Proposition \( \ref{8.1} \) in \( \cite{26} \))

\[
H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(\widehat{T^\sigma}, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)
\]

is surjective by Corollary \( \ref{8.17} \) in \( \cite{26} \). Hence the map \( \partial \) is the zero map and \( \partial(\lambda'_\sigma) = 0 \), which concludes the proof. \( \square \)

**Remark 7.5.** The proof of Proposition \( \ref{7.4} \) also gives the following result:

Let \( X \) be a smooth geometrically integral \( k \)-variety and let \( Y \to X \) be a torsor under a linear algebraic \( k \)-group \( G \). Let

\[
1 \to H \to G \to F \to 1
\]

be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \).

If \( \sigma \in H^1(k, F) \) satisfies \( Z^\sigma(A_k)^{Br}(Z^\sigma) \neq \emptyset \), then \( \sigma \) can be lifted to \( H^1(k, G) \).

One can now prove the main result of this section:
Theorem 7.6. If $X$ is a smooth and geometrically integral variety over a number field $k$, then
\[ X(A_k)^{\text{â‚„ Br}} \subseteq X(A_k)^{\text{desc}}. \]

Proof. Since the statement 2 of Theorem 2 in [21] (which we apply to $X'$) holds for any geometrically integral variety (without any assumption on $\bar{k}[X']^\times$), the proof of this theorem using Proposition 7.4 is exactly the same as the proof of Theorem 1 using Proposition 5 in [14] (see in particular [14], p. 244-245).

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References

[1] M. Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. 72 (1993), 217-239.
[2] M. Borovoi and C. Demarche, Manin obstruction to strong approximation for homogeneous spaces, Comment. Math. Helv. 88 (2013), 1-54.
[3] M. Borovoi and J. van Hamel, Extended Picard complexes and linear algebraic groups, J. Reine Angew. Math. 627 (2009), 53-82.
[4] Y. Cao and F. Xu, Strong approximation with Brauer-Manin obstruction for toric varieties, arXiv:1311.7655 (2013).
[5] ______, Strong approximation with Brauer-Manin obstruction for groupic varieties, arXiv:1507.04340v4 (2015).
[6] J.-L. Colliot-Thélène, Résolutions flasques des groupes linéaires connexes, J. reine angew. Math. 618 (2008), 77-133.
[7] J.-L. Colliot-Thélène and D. Harari, Approximation forte en famille, to appear in J. reine angew. Math.
[8] J.-L. Colliot-Thélène and J.-J. Sansuc, La descente sur les variétés rationnelles, II, Duke Math. J. 54 (1987), 375-492.
[9] J.-L. Colliot-Thélène and F. Xu, Brauer-Manin obstruction for integral points of homogeneous spaces and representations by integral quadratic forms, Compositio Math. 145 (2009), 309-363.
[10] ______, Strong approximation for the total space of certain quadric fibrations, Acta Arithmetica 157 (2013), 169-199.
[11] B. Conrad, Weil and Grothendieck approaches to adelic points, Enseign. Math. 58 (2012), 61-97.
[12] A. J. de Jong, A result of Gabber, Available at \texttt{http://www.math.columbia.edu/~dejong/papers}.
[13] M. Artin, Comparaison avec la cohomologie classique: cas d’un préschéma lisse, SGA 4, Lecture Notes in Mathematics 305, Springer-Verlag, 1973.
[14] C. Demarche, Obstruction de descente et obstruction de Brauer-Manin étale, Algebra Number Theory 3 (2009), 237-254.
[15] ______, Méthodes cohomologiques pour l’étude des points rationnels sur les espaces homogènes, PhD thesis, University Paris-Sud XI (2009), Available at \texttt{https://webusers.imj-prg.fr/~cyril.demarche/these/these.pdf}.
[16] ______, Suites de Poitou-Tate pour les complexes de tores à deux termes, Int. Math. Res. Not. (2011), 135-174.
[17] Le défaut d’approximation forte dans les groupes linéaires connexes, Proc. London Math. Soc. **102** (2011), 563-597.

[18] Y. Z. Flicker, C. Scheiderer, and R. Sujatha, Grothendieck’s theorem on non-abelian $H^2$ and local-global principles, J. Amer. Math. Soc. **11** (1998), 731-750.

[19] J. Giraud, Cohomologie non-abélienne, Die Grundlehren der mathematischen Wissenschaften, vol. 179, Springer-Verlag, 1971.

[20] A. Grothendieck, Le groupe de Brauer, I, II, III, Dix exposés sur la cohomologie des schémas, North-Holland, 1968.

[21] D. Harari, Groupes algébriques et points rationnels, Math. Ann. **322** (2002), 811-826.

[22] Le défaut d’approximation forte pour les groupes algébriques commutatifs, Algebra & Number Theory **2** (2008), 595-611.

[23] D. Harari and A. N. Skorobogatov, Non-abelian cohomology and rational points, Compos. Math. **130** (2002), 241-273.

[24] The Brauer group of torsors and its arithmetic applications, Ann. Inst. Fourier, Grenoble (2003), 1987-2019.

[25] Non-abelian descent and the arithmetic of Enriques surfaces, Intern. Math. Res. Notices **52** (2005), 3203-3228.

[26] Descent theory for open varieties, London Mathematical Society Lecture Note Series **405** (2013), 250-279.

[27] D. Harari and T. Szamuely, Arithmetic duality theorem for 1-motives, J. reine angew. Math. **578** (2005), 93-128.

[28] Q. Liu and F. Xu, Very strong approximation for certain algebraic varieties, Math. Ann. **363** (2015), 701-731.

[29] J.S. Milne, Étale cohomology, Princeton University Press, 1980.

[30] V.P. Platonov and A.S. Rapinchuk, Algebraic groups and number theory, Academic Press, 1994.

[31] B. Poonen, Insufficiency of the Brauer-Manin obstruction applied to étale covers, Ann. of Math. **171** (2010), 2157-2169.

[32] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. reine angew. Math. **327** (1981), 12-80.

[33] J. P. Serre, Cohomologie Galoisienne, Lecture Notes in Mathematics, vol. 5, Springer, Berlin, 1965.

[34] A. N. Skorobogatov, Beyond the Manin obstruction, Invent. Math. **135** (1999), 399-424.

[35] Descent obstruction is equivalent to étale Brauer-Manin obstruction, Math. Ann. **344** (2009), 501-510.

[36] Torsors and rational points, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, 2001.

[37] A. N. Skorobogatov and Y. G. Zarhin, The Brauer group and the Brauer-Manin set of products of varieties, J. Eur. Math. Soc. **16** (2014), 749-768.

[38] M. Stoll, Finite descent obstructions and rational points on curves, Algebra Number Theory **1** (2007), 349-391.

[39] Dasheng Wei, Open descent and strong approximation, arXiv.1604.00610v2 (2016).
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