On the Construction and Malliavin Differentiability of Lévy Noise Driven SDE’s with Singular Coefficients

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Abstract

In this paper we introduce a new technique to construct unique strong solutions of SDE’s with singular coefficients driven by certain Lévy processes. Our method which is based on Malliavin calculus does not rely on a pathwise uniqueness argument. Furthermore, the approach, which provides a direct construction principle, grants the additional insight that the obtained solutions are Malliavin differentiable.

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1 Introduction

Consider the stochastic differential equation (SDE)

\[
X_t = x + \int_0^t b(s, X_s) ds + L_t, 0 \leq t \leq T, x \in \mathbb{R}^d, \tag{1}
\]

where \( b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a Borel-measurable function and \( L_t, 0 \leq t \leq T \) is a \( d \)-dimensional (square integrable) Lévy process, that is a process on some complete probability space \((\Omega, \mathcal{F}, \mu)\) with stationary and independent increments starting in zero (see e.g. [7]).

Using Picard iteration it is well known that there exists a unique square integrable strong solution \( X_t, 0 \leq t \leq T \) to (1) if the drift coefficient \( b \) is Lipschitz continuous and of linear growth. Here, a strong solution to (1) means that \( X_t, 0 \leq t \leq T \) is an adapted process with respect to a \( \mu \)-completed filtration \( \mathcal{F}_t, 0 \leq t \leq T \) generated by \( L_t, 0 \leq t \leq T \) having càdlàg paths and satisfying the equation (1) \( \mu \)-a.e. See e.g. [32].

In this are article, however, we are interested to study strong solutions to (1) for certain Lévy processes, when \( b \) is singular in the sense that \( b \) is bounded and \( \alpha \)-Hölder continuous, i.e.

\[
\|b\|_{C_\alpha^b} := \sup_{0 \leq t \leq T, x \in \mathbb{R}^d} |b(t, x)| + \sup_{0 \leq t \leq T} \sup_{x \neq y} \frac{|b(t, x) - b(t, y)|}{|x - y|^\alpha} < \infty.
\]

for some \( 0 < \alpha < 1 \).

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We mention that the analysis of strong solutions of SDE's with singular or non-Lipschitz coefficients is important and has been of much current interest for decades in stochastic analysis and its applications. Such solutions naturally arise e.g. from a variety of applications in the theory of controlled diffusion processes or in statistical mechanics to model interacting infinite particle systems. See e.g. [16], [18], [20] and the references therein.

The case, when \( b \) is singular and \( L_t \) is a Wiener process, has been intensively studied in the literature. A milestone in theory of SDE's is a result due to A.K. Zvonkin, [37], who constructed unique strong solutions for Wiener process driven SDE’s (1) on the real line, when \( b \) is merely bounded and measurable by employing estimates of solutions of parabolic partial differential equations and a pathwise uniqueness argument. Using similar techniques the latter result was subsequently extended to the multidimensional case ([33]). Further important generalizations of those results based on a pathwise uniqueness argument can be e.g. found in [20], [12] and [13]. We also refer to [8], where the authors use solutions to infinite-dimensional Kolmogorov equations to prove strong uniqueness of solutions to (1) for Wiener cylindrical processes \( L_t \) on Hilbert spaces, when \( b \) is bounded and measurable. Another and more direct approach to obtain strong solutions to (1) in the Wiener case, which doesn’t rely on a pathwise uniqueness argument and which is based on techniques of Malliavin calculus, was studied in [25], [24]. See also [11] in the case of Hilbert spaces.

If the driving process \( L_t \) in (1), however, is a pure jump Lévy process we observe major differences to the Gaussian case. For example, if \( L_t, 0 \leq t \leq T \) is a one-dimensional symmetric \( \alpha \)-stable process for \( 0 < \alpha < 1 \) then one can find a bounded \( \gamma \)-Hölder-continuous drift coefficient \( b \) with \( \alpha + \gamma < 1 \) such that pathwise uniqueness of solutions to (1) fails. See [34]. Similar results on non-pathwise uniqueness of solutions of SDE’s with multiplicative symmetric \( \alpha \)-stable noise were obtained by [5]. See also [4], [31] and the references therein. As for the study of weak solutions of SDE’s driven by Lévy processes we shall refer here e.g. to [3], [36] and [29]. Further, martingale problems of SDE’s driven by symmetric \( \alpha \)-stable processes were treated in [6].

In this paper we aim at introducing a new technique to construct (unique) strong solutions to (1). We illustrate this principle, which can be also applied to a variety of other Lévy processes, by considering the special case of a truncated \( \alpha \)-stable process of index \( \alpha \in (1, 2) \). Our method differs from the above mentioned ones in the sense that we do not resort to the Yamada-Watanabe principle to guarantee strong uniqueness of solutions, that is we do not require pathwise uniqueness in connection with the existence of a weak solution to find a unique strong solution to (1). In fact our approach, which provides a direct construction of strong solutions, can be regarded as a synthesis of techniques developed in [25], [24] and [10] (or [31] in the case of symmetric \( \alpha \)-stable processes) applied to Lévy processes. More precisely, we approximate the singular coefficient \( b \) in (1) by smooth functions \( b_n \) admitting a unique strong solution \( X^n_t \)

\[
X^n_t = x + \int_0^t b_n(s, X^n_s)ds + L_t, 0 \leq t \leq T, x \in \mathbb{R}^d
\]  

(2)

for each \( n \geq 1 \). Then we recast the integral \( \int_0^t b_n(s, X^n_s)ds \) in (1) by using solutions to a backward Kolmogorov equation associated with \( L_t \) in terms of a more regular expression (see [10], [31]). Finally, we apply a new compactness criterion of square integrable functionals of
Lévy processes based on Malliavin calculus to the sequence of solutions $X^n_t, n \geq 1$ to obtain a unique strong solution $X_t$ (compare [25], [24] in the Wiener process case). Moreover, our method gives the crucial additional insight that $X_t$ is Malliavin differentiable for all $t$. See [27] or [28] for more information on Malliavin calculus.

Our paper is organized as follows: In Section 3 we introduce some notation and recall some basic results from the theory of Lévy processes and Malliavin calculus which we will use throughout the article. In Section 3.1 we prove a new compactness criterion for square integrable functionals of Lévy processes and establish certain estimates of solutions of Kolmogorov type equations associated with Lévy processes. Finally, in Section 4 we apply the results of the previous section to prove our main result on the existence of a unique and Malliavin differentiable strong solution to (1) for certain Lévy processes (Theorem 18).

2 Framework

In this section we briefly introduce the mathematical framework we want to apply in the subsequent sections.

2.1 Hölder Spaces

For $\beta \in (0, 1)$ and $k, d \geq 1$, denote by $C^\beta_b(\mathbb{R}^d, \mathbb{R}^k)$ the space of bounded $\beta$–Hölder continuous functions, that is the space of continuous functions $u : \mathbb{R}^d \to \mathbb{R}^k$ such that

$$
\|u\|_{C^\beta_b(\mathbb{R}^d, \mathbb{R}^k)} := \|u\|_{\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty,
$$

where $\|u\|_{\infty} := \sup_{x \in \mathbb{R}^d} u(x)$ and $|\cdot|$ is the Euclidean norm. We also simply write $C^\beta_b(\mathbb{R}^d) = C^\beta_b(\mathbb{R}^d, \mathbb{R})$. Further, we denote by $C^{i,\beta}_b(\mathbb{R}^d)$ for $i \geq 1$ and $0 < \alpha < 1$ the Banach space of all $i$-times Fréchet differentiable functions $u : \mathbb{R}^d \to \mathbb{R}$ with $D^i u \in C^\beta_b(\mathbb{R}^d, (\mathbb{R}^d)^{(l+1)})$, $l = 1, \ldots, i$ and norm $\|\cdot\|_{C^{i,\beta}_b(\mathbb{R}^d)}$ given by

$$
\|u\|_{C^{i,\beta}_b(\mathbb{R}^d)} := \|u\|_{\infty} + \sum_{l=1}^i \|D^l u\|_{\infty} + \sup_{x \neq y} \frac{|D^i u(x) - D^i u(y)|}{|x - y|^{\beta}}.
$$

We let $C^{0,\beta}_b(\mathbb{R}^d) := C^\beta_b(\mathbb{R}^d)$. For notational convenience we also denote the norm of the Banach space $C([0, T], C^{i,\beta}_b(\mathbb{R}^d))$ by $\|\cdot\|_{C^{i,\beta}_b(\mathbb{R}^d)}$ defined as

$$
\|u\|_{C^{i,\beta}_b} := \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{C^{i,\beta}_b(\mathbb{R}^d)}.
$$
2.2 Lévy Processes

We give a concise summary of basic facts of the theory of Lévy processes. The reader may consult [7] or [28] for further information.

Given a complete probability space, $(\Omega, \mathcal{F}, P)$, we a Lévy process is defined as follows.

**Definition 1** A stochastic process $L(t) \in \mathbb{R}^d$, $t \geq 0$ is called a Lévy process if the following properties hold:

1. $L(0) = 0$ $P$-a.s.,
2. the process has independent increments, that is, for all $t > 0$ and $h > 0$, the increment $L(t + h) - L(h)$ is independent of $L(s)$ for all $s \leq t$,
3. the process has stationary increments, that is, for all $h > 0$, the increment $L(t + h) - L(h)$ has the same law as $L(h)$,
4. the process is stochastically continuous, that is, for every $t > 0$ and $\epsilon > 0$ we have that $\lim_{s \to t} P\{|L(t) - L(s)| > \epsilon\} = 0$,
5. the paths of the process are càdlàg, that is, the trajectories are right-continuous with existing left limits.

Now, define the jump of $L$ at time $t$ as

$$\Delta L(t) := L(t) - L(t^-).$$

Let $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ and let $\mathcal{B}(\mathbb{R}_0^d)$ be the Borel-$\sigma$-algebra on $\mathbb{R}_0^d$. Further, we now introduce a Poisson random measure on $\mathcal{B}([0, \infty)) \times \mathcal{B}(\mathbb{R}_0^d)$ by

$$N(t, U) := \sum_{0 \leq s \leq t} 1_U(\Delta L(s))$$

for $U \in \mathcal{B}(\mathbb{R}_0^d)$. This is the jump measure of $\eta$. The Lévy measure $\nu$ of $\eta$ is defined by

$$\nu(U) := E[N(1, U)],$$

for $U \in \mathcal{B}(\mathbb{R}_0^d)$.

It can be shown that the characteristic function of a Lévy process is given by the following Lévy-Khintchine formula (see e.g. [7]):

$$E[\exp(i \langle L(t), u \rangle)] = \exp(-t\Psi(u)), u \in \mathbb{R}^d, t \geq 0,$$

(3)

where $\Psi$ is the characteristic exponent

$$\Psi(u) = -\int_{\mathbb{R}^d} (e^{i(u,y)} - 1 - i \langle u, y \rangle 1_{\{|y| \leq 1\}}) \nu(dy).$$
Let us define the compensated jump measure $\tilde{N}$ by
\[
\tilde{N}(ds, dz) := N(ds, dz) - \nu(dz)dt.
\]

It turns out that Lévy processes have the following representation:

**Theorem 2 (The Lévy-Itô decomposition)** Let $L$ be a Lévy process. Then $L$ admits the following integral representation
\[
\eta(t) = at + \sigma W(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|>1} zN(ds, dz)
\]
for some $a \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d \times d}$ and a standard Wiener process $W(t), t \geq 0$.

Let us recall the infinitesimal generator $L$ of the Lévy processes $L_t, t \geq 0$:

The infinitesimal generator of $L_t, t \geq 0$ is the operator $L$, which is defined to act on suitable functions $f$ of some Banach space such that
\[
L f(x) = \lim_{t \to 0^+} \frac{E_x[f(L_t)] - f(x)}{t}
\]
exists.

### 2.3 Chaos Expansions and the Malliavin Derivative

In this subsection we briefly recall the concept of the Malliavin derivative with respect to Lévy processes as a central notion of Malliavin calculus. We refer the reader to the books [27] and [28] for more information on Malliavin calculus.

For notational convenience, we assume in this subsection $d = 1$. Consider $\Omega = \mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$. Then we know from the Bochner-Minlos-Sazonov theorem (see e.g. [35]) that there exists a probability measure $\mu$, such that
\[
\int_{\Omega} e^{i<\omega,f>} \mu(d\omega) = \exp(\int_{\mathbb{R}} \Psi(f(x))dx),
\]
for $f \in \mathcal{S}(\mathbb{R})$, where $\Psi$ is the characteristic exponent given by
\[
\Psi(u) = \int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu(dz),
\]
where $<\omega,f>$ denotes the action of $\omega \in \mathcal{S}'(\mathbb{R})$ (Schwartz distribution space) on $f \in \mathcal{S}(\mathbb{R})$ and where $\nu$ is a Lévy measure. The triple $(\Omega, F, \mu)$ is called the (pure jump) Lévy white noise probability space.

From now on we assume a square integrable Lévy process $L_t, t \geq 0$ with Lévy measure $\nu$ constructed on $(\Omega, F, \mu)$.

In what follows we want to use the chaos representation property of a square integrable Lévy process to define the Malliavin derivative with respect to such processes. To this end we need some notation:
Let us denote by \( I \) the set of all finite multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m), m \in \mathbb{N}_0 \) of non-negative integers \( a_i, i = 1, \ldots, m \), and define \( |\alpha| := \alpha_1 + \cdots + \alpha_m \). Further, let \( e_i, i \geq 1 \) be an orthonormal basis of \( L^2(\lambda \times \nu) \) (\( \lambda \) Lebesgue measure) and let for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in I \)

\[
H_{\alpha} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e_1^{\otimes \alpha_1} \cdots \hat{\otimes} e_m^{\otimes \alpha_m} ((s_1, z_1), \ldots, (s_m, z_m)) \tilde{N}(ds_1, dz_1) \cdots \tilde{N}(ds_m, dz_m),
\]

where \( \otimes \) and \( \hat{\otimes} \) denotes the tensor product and the symmetrized tensor product, respectively.

Then \( \{H_{\alpha} : \alpha \in I\} \) forms an orthogonal basis of \( L^2(\mu) \):

**Theorem 3 (Chaos expansion)** Any \( X \in L^2(\mu) \) has the unique chaos decomposition of the form

\[
X = \sum_{\alpha \in I} c_{\alpha} H_{\alpha}
\]  

(4)

with \( c_{\alpha} \in \mathbb{R} \). Moreover

\[
\|X\|_{L^2(\mu)}^2 = \sum_{\alpha \in I} \alpha! c_{\alpha}^2,
\]

where

\[
\alpha! := \alpha_1! \alpha_2! \cdots \alpha_m!
\]

for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \).

We are now ready to define the Malliavin derivative.

We define the Malliavin derivative of a square integrable functional \( X \) of a pure jump Lévy process \( L \) with chaos expansion

\[
X = \sum_{\alpha \in I} c_{\alpha} H_{\alpha}
\]  

(5)

by

\[
D_{t,z} X = \sum_{\beta \in I} \sum_{i \in \mathbb{N}} (c_{\beta+i}(\beta_i + 1)) e_i(t, z) H_{\beta},
\]

provided \( X \) belongs to the domain \( \mathbb{D}^{1,2} \subset L^2(\mu) \) given by

\[
\mathbb{D}^{1,2} : = \{ X \in L^2(\mu) \text{ with chaos expansion (5)} : \sum_{\beta \in I} \sum_{i \in \mathbb{N}} (c_{\beta+i}(\beta_i + 1))^2 \beta! < \infty \},
\]

where \( \epsilon_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the \( i \)-th position.
2.4 Fractional Sobolev Spaces

In this paper we aim at constructing strong solutions to Lévy noise driven SDE’s by using Banach spaces of functions related to fractional Sobolev spaces (or Sobolev-Slobodeckij spaces). See [1] for more information about these spaces.

Definition 4 Let $0 < \alpha < 2, 1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then, the fractional Sobolev space $W^{\alpha,p}(\Omega)$ can be defined as

$$W^{\alpha,p}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} : \|f\|_{W^{\alpha,p}(\Omega)} := \left( \|f\|_{L^p(\Omega)}^p + \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+2\alpha}} \, dx \, dy \right)^{1/p} < \infty \right\}.$$

Here,

$$[f] := \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+2\alpha}} \, dx \, dy \right)^{1/p}$$

denotes the Slobodeckij semi-norm.

The Sobolev-Slobodeckij spaces form a scale of Banach spaces, i.e. one has the continuous injections or embeddings

$$W^{k+1,p}(\Omega) \hookrightarrow W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega) \hookrightarrow W^{k,p}(\Omega), \quad k \leq s \leq s' \leq k + 1.$$

Sobolev-Slobodeckij spaces are special cases of Besov spaces. See e.g. [1].

Another approach to define fractional order Sobolev spaces $W^{\alpha,p}(\Omega)$ is

Definition 5

$$W^{\alpha,p}(\Omega) := \{ f \in L^p(\Omega) : \mathcal{F}^{-1}(1 + |\xi|^2)^\frac{\alpha}{2} \mathcal{F} f \in L^p(\Omega) \}$$

with the norm

$$\|f\|_{W^{k,p}} := \|\mathcal{F}^{-1}(1 + |\xi|^2)^\frac{k}{2} \mathcal{F} f\|_{L^p},$$

where $\mathcal{F}$ denotes the Fourier-transform. This space is also called a Bessel potential space. $\Omega$ is a domain with uniform $C^k$-boundary, $k$ a natural number and $1 < p < \infty$.

By the embeddings

$$W^{k+1,p}(\mathbb{R}^n) \hookrightarrow W^{s',p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{k,p}(\mathbb{R}^n), \quad k \leq s \leq s' \leq k + 1$$

the Bessel potential spaces form a continuous scale between these Sobolev spaces.
3 Preliminary Results

In this section we give a new compactness criterion for square integrable functionals (of pure jump) Lévy processes based on Malliavin calculus. Further, we prove some regularity results of solutions of Kolmogorov type equations associated with certain Lévy processes. We aim at employing these results in Section 4 to establish our main results on the existence and uniqueness of Malliavin differentiable strong solutions to SDEs of the form (1).

3.1 Compactness Criterion

Our construction method of solutions to (1) requires a compactness criterion for subsets of \(L^2(\mu)\). So we prove the following theorem which can be regarded as an extension of [30] from Wiener processes to (pure jump) Lévy processes.

**Theorem 6 (Compactness in \(L^2(\mu)\))** Let \(C\) be a selfadjoint compact operator on \(H \otimes L^2(\nu)\) with dense image, where \(H := L^2([0,1])\). Then for any \(c > 0\) the set

\[
\mathcal{G} = \{ G \in \mathbb{D}^{1,2} : \| G \|_{L^2(\Omega)} + \| C^{-1}DG \|_{L^2(\Omega;H \otimes L^2(\nu))} \leq c \}
\]

is relatively compact in \(L^2(\mu)\).

**Proof.** The proof is similar to that of Theorem 1 in [30]. Consider a complete orthonormal system \(\{e_i\}_{i \geq 1}\) of \(H \otimes L^2(\nu)\). Assume that \(C e_i = \beta_i e_i\) with \(\beta_i > 0\) for all \(i \geq 1\). Note that the compactness of \(C\) implies that \(\lim_{i \to \infty} \beta_i = 0\). Let \(G \in \mathbb{D}^{1,2}\) be a random variable such that

\[
\| G \|_{L^2(\Omega)} + \| C^{-1}DG \|_{L^2(\Omega;H \otimes L^2(\nu))} \leq c.
\]

Let

\[
G = \sum_{\gamma \in I} c_\gamma H_\gamma
\]

be the chaos decomposition of \(G\). Then

\[
D_{\cdot \gamma}G = \sum_{\gamma \in I} \left( \sum_k c_{\gamma + \epsilon(k)(\gamma_k + 1)} e_k(\cdot,\cdot) \right) H_\gamma,
\]

where \(\epsilon(k) \in I\) is defined by

\[
\epsilon^j = \begin{cases} 1, & \text{if } \epsilon_j = 1 \\ 0, & \text{otherwise.} \end{cases}
\]

See Section 2.3. From this we get that

\[
\| G \|^2_{L^2(\Omega)} = \sum_{\alpha} \alpha c_\alpha^2
\]
and

\[
\|C^{-1}D_\cdot G\|_{L^2(\Omega; H_0L^2(\nu))}^2 = \sum_{\gamma} \gamma! \sum_k (\gamma_k + 1)^2 \frac{1}{\beta_k^2} \gamma_k^2 \\
= \sum_{\gamma} \sum_k (\gamma - \epsilon^k)! \frac{1}{\beta_k^2} \gamma_k^2 \\
= \sum_{\gamma} \gamma^2 \gamma! \sum_k \frac{1}{\beta_k^2} \gamma_k^2 (\gamma - \epsilon^k)! \\
= \sum_{\gamma} \gamma^2 \gamma! \sum_k \frac{1}{\beta_k^2} \gamma_k^2.
\]

For fixed \( R > 0 \) define the set

\[
A_R = \left\{ \alpha \in \mathcal{I} : \sum_k \frac{1}{\beta_k^2} \alpha_k < R \right\}.
\]

Since \( \lim_{i \to \infty} \beta_i = 0 \) and \( \alpha_i \in \mathbb{N}_0, i \geq 1 \) for \( \alpha \in \mathcal{I} \) we see that the set \( A_R \) only has finitely many elements. On the other hand we obtain

\[
\|G\|_{L^2(\Omega)}^2 = \sum_{\alpha} \alpha! c_\alpha^2 = \sum_{\alpha \in A_R} \alpha! c_\alpha^2 + \sum_{\alpha \notin A_R} \alpha! c_\alpha^2 = \frac{R}{R} \sum_{\alpha \notin A_R} \alpha! c_\alpha^2 + \sum_{\alpha \in A_R} \alpha! c_\alpha^2 \\
\leq \frac{1}{R} \sum_{\alpha \in A_R} \alpha! c_\alpha^2 \sum_k \frac{1}{\beta_k^2} \alpha_k + \sum_{\alpha \in A_R} \alpha! c_\alpha^2 \\
\leq \frac{1}{R} \sum_{\alpha} \alpha! c_\alpha^2 \sum_k \frac{1}{\beta_k^2} \alpha_k + \sum_{\alpha \in A_R} \alpha! c_\alpha^2 \\
= \frac{1}{R} \|C^{-1}D_\cdot G\|_{L^2(\Omega; H_0L^2(\nu))}^2 + \sum_{\alpha \in A_R} \alpha! c_\alpha^2.
\]

Let \( \epsilon > 0 \). Since \( A_R \) is finite we can find \( C_\alpha^j, \alpha \in A_R, 1 \leq j \leq n(R, \epsilon) \), such that for all \( G \in \mathcal{G} \)

\[
\inf_j \left\{ \sum_{\alpha \in A_R} \alpha! c_\alpha^j c_{\alpha}^j \right\} < \frac{\epsilon}{2}.
\]

Define

\[
G^j := \sum_{\alpha \in A_R} c_{\alpha}^j H_\alpha,
\]

and replace \( G \) in \( 6 \) by \( G - G^j \). Then we see that

\[
\inf_j \|G - G^j\|_{L^2(\Omega)}^2 \leq \frac{1}{R} \|C^{-1}D_\cdot G\|_{L^2(\Omega; H_0L^2(\nu))}^2 + \frac{\epsilon}{2} \leq \epsilon,
\]

\( j = 1, \ldots, n(R, \epsilon) \) for \( R \geq 2\frac{\epsilon}{\epsilon} \). So the \( L^2(\Omega) \)-balls with center \( G^j, j = 1, \ldots, n(R, \epsilon) \) and radius \( \epsilon \) cover \( \mathcal{G} \). \( \blacksquare \)
3.2 Some Regularity Results

3.2.1 Kolmogorov Type Equations Associated with Lévy Processes

In this subsection we want to prove some regularity results for Kolmogorov type equations associated with certain Lévy processes. The latter results will be used to recast the drift term $\int_0^t b(s, X_s) ds$ in the SDE (1) in terms of a more regular expression which enables us to compute certain estimates with respect to the Malliavin derivative of approximating solutions to $X$ (see Section 4).

We need the following lemma:

**Lemma 7** Let $L_t, 0 \leq t \leq T$ be a Lévy process and let $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function such that $\phi(\cdot, x)$ is continuous for all $x$ and such that $\phi(t, \cdot) \in \text{Dom}(L)$ for all $t$, where $L$ is the generator of $L_t, 0 \leq t \leq T$. Consider

$$u(t, x) = \int_0^t E[\phi(s, x + L_{t-s})] ds.$$  \hspace{1cm} (7)

Then $u$ solves

$$\frac{\partial u}{\partial t} = L_t u + \phi$$

with $u(0, x) = 0$.

**Proof.** Denote by $\{P_t\}_{t \geq 0}$ the strongly continuous semigroup on $C_\infty(\mathbb{R}^d)$ (space of continuous functions vanishing at infinity) associated with our Lévy process $L_t, 0 \leq t \leq T$, that is

$$P_t(f)(x) := E[f(x + L_t)]$$

for $f \in C_\infty(\mathbb{R}^d)$. See e.g. [2]. If $f \in \text{Dom}(L)$ we know that $P_t f$ solves the heat equation

$$\frac{d}{dt} P_t f = L P_t f.$$  

Then it follows from the linearity of the operator $L$ that

$$u(t, x) = \int_0^t E[\phi(s, x + L_{t-s})] ds,$$

solves the Kolmogorov equation

$$\frac{\partial u}{\partial t}(t, x) = L u(t, x) + \phi(t, x),$$

with $u(0, x) = 0$ for all $x$. \hfill \blacksquare

**Remark 8** We mention that the Schwartz test function space $S(\mathbb{R}^d)$ is contained in $\text{Dom}(L)$. See [2].
In what follows we want to consider Lévy processes $L_t$, $0 \leq t \leq T$ given by truncated $\alpha$-stable processes of index $\alpha \in (0,2)$, that is Lévy processes, whose characteristic exponent is given by

$$\Psi(u) = \int_{\mathbb{R}^d} (1 - \cos(u \cdot y))\nu(dy),$$

with Lévy measure

$$\nu(dy) = 1_{\{|y| \leq 1\}} \frac{1}{|y|^{d+\alpha}} dy.$$

See e.g. [15] for further properties of this process.

Note that the infinitesimal generator $L$ of the process $L$ is given by

$$L f(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x)) - 1_{\{|y| \leq 1\}} y \cdot Df(x) \nu(dy)$$

for $f \in C^\infty_c(\mathbb{R}^d)$ (space of infinitely differentiable functions with compact support). See e.g. [2].

We need the following auxiliary result:

**Lemma 9** Let $f \in W^{r_1,\ldots,r_d}_1(\mathbb{R}^d)$, $d \geq 2$ and let $r_1, \ldots, r_d \in \mathbb{N}$ such that

$$\sum_{i=1}^d \frac{1}{r_i} = 1.$$

Then

$$\|\hat{f}\|_{L^1(\mathbb{R}^d)} \leq C \sum_{j=1}^d \|\frac{\partial^{r_j}f}{\partial x_j^{r_j}}\|_{L^1(\mathbb{R}^d)},$$

where $\hat{f}$ denotes the Fourier-transform of $f$.

**Proof.** See Remark 1 in [17].

**Theorem 10** Let $L_t$, $0 \leq t \leq T$ be a $d$-dimensional truncated $\alpha$-stable process for $\alpha \in (1,2)$ and $d \geq 2$. Suppose that $\phi \in C([0,T], C^\beta_b(\mathbb{R}^d))$ for $\beta \in (0,1)$ satisfies that $\alpha + \beta > 2$. Then there exists a $u \in C([0,T], C^\beta_b(\mathbb{R}^d)) \cap C^1([0,T], C_b(\mathbb{R}^d))$ such that

$$\frac{\partial u}{\partial t} = Lu + \phi,$$

with $L$ defined as in (9) and such that

$$\|Du\|_{C^\beta_b} \leq C(T) \|\phi\|_{C^\beta_b},$$

where

$$C(T) \rightarrow 0 \text{ for } T \searrow 0,$$

as well as

$$\|D^2u\|_{\infty} \leq M \|\phi\|_{C^\beta_b}$$

for a constant $M$. 

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Proof. We subdivide the proof into two parts:

(A) We first want to show that $u$ defined by (7) in Lemma 7 admits the estimates (11) and (12):

We recall that $Lt, 0 \leq t \leq T$ has the characteristic exponent

$$\Psi(u) = \int_{|y| \leq 1} \frac{1 - \cos(u \cdot y)}{|y|^{d+\alpha}} \, dy.$$ 

So we get

$$t\Psi(t^{\frac{1}{d}}u) = \int_{|y| \leq 1} \frac{1 - \cos(t^{\frac{1}{d}}u \cdot y)}{|y|^{d+\alpha}} \, dy$$

$$= \int_{|r| \leq t^{\frac{1}{d}}} \frac{t \cdot t^{\frac{1}{d}} \cdot t^{\frac{d+\alpha}{d}} \cdot 1 - \cos(u \cdot r)}{|r|^{d+\alpha}} \, dr$$

$$= \int_{|r| \leq t^{\frac{1}{d}}} \frac{1 - \cos(u \cdot r)}{|r|^{d+\alpha}} \, dr$$

$$\geq \int_{|r| \leq T^{\frac{1}{d}}} \frac{1 - \cos(u \cdot r)}{|r|^{d+\alpha}} \, dr : = \tilde{\Psi}(u). \quad (13)$$

Observe that

$$\tilde{\Psi}(u) \sim |u|^\alpha \quad (14)$$

nearby infinity. Because of (13) and (14) we can apply the Fourier inversion formula and obtain for the probability density function $p_t(x)$, the representation

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixu} e^{-t\Psi(u)} \, du.$$

Hence

$$p_t(t^{\frac{1}{d}}x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\frac{t}{d}x} e^{-t\Psi(z)} \, dz$$

$$= \frac{t^{-\frac{d}{\alpha}}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixu} e^{-t\Psi(t^{\frac{1}{d}}u)} \, du.$$ 

Because of (13) and (14) we are allowed to differentiate $p_t(\cdot)$ and get

$$\frac{\partial}{\partial x_i}(p_t(t^{\frac{1}{d}}x)) = t^{\frac{1}{d}} \left( \frac{\partial}{\partial x_i} p_t \right)(t^{\frac{1}{d}}x)$$

$$= \frac{t^{-\frac{d}{\alpha}}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixu}(-i)(u_i) e^{-t\Psi(t^{\frac{1}{d}}u)} \, du. \quad (15)$$

On the other hand we know that

$$E[\phi(s, x + L_t)] = \int_{\mathbb{R}^d} \phi(s, x + u)p_t(u) \, du$$

$$= \int_{\mathbb{R}^d} \phi(s, u)p_t(u - x) \, du.$$ 

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Lemma 9 we find where

In order to give an estimate of the $L^1$–norm of $\frac{\partial}{\partial x_i}(p_t(t\frac{1}{\alpha}u))$, $i = 1, \ldots, d$ in (15) we want to apply Lemma 9. Without loss of generality let us consider the case $d = 2$. Then, using Lemma 9 we find

$$
\left\| \frac{\partial}{\partial x_i}(p_t(t\frac{1}{\alpha}u)) \right\|_{L^1(\mathbb{R}^d)} \leq C t^{-\frac{d}{\alpha}} \sum_{j=1}^{d} \frac{\partial^2}{\partial u_j^2} \eta_i(u).
$$

where

$$
\eta_i(u) = u_i e^{-t \Psi(t\frac{1}{\alpha}u)}.
$$

Let $i \neq j$. Then

$$
\frac{\partial^2}{\partial u_j^2} \eta_i(u) = u_i t^{1-\frac{2}{\alpha}} \frac{\partial^2}{\partial u_j^2} \Psi(t\frac{1}{\alpha}u) e^{-t \Psi(t\frac{1}{\alpha}u)}
$$

$$
+ u_i (t^{1-\frac{1}{\alpha}} \frac{\partial}{\partial u_j} \Psi(t\frac{1}{\alpha}u))^2 e^{-t \Psi(t\frac{1}{\alpha}u)}.
$$

We know that

$$
\frac{\partial}{\partial u_j} \Psi(u) = \int_{|y| \leq 1} \frac{\sin(u \cdot y)}{|y|^{d+\alpha}} dy
$$

$$
= \int_{|r| \leq |u|} \frac{1}{|u|^{d+1}} |u|^{d+\alpha} r_j \sin\left(\frac{u}{|u|} \cdot r\right) dr
$$

$$
= |u|^{-1} \int_{|r| \leq |u|} r_j \frac{\sin\left(\frac{u}{|u|} \cdot r\right)}{|r|^{d+\alpha}} dr.
$$

On the other hand we see that

$$
\left| \int_{|r| \leq |u|} r_j \frac{\sin\left(\frac{u}{|u|} \cdot r\right)}{|r|^{d+\alpha}} dr \right| \leq \int_{|r| \leq |u|} |r_j| \frac{\sin\left(\frac{u}{|u|} \cdot r\right)}{|r|^{d+\alpha}} dr
$$

$$
\leq \int_{|r| \leq |u|} |r| \frac{\sin\left(\frac{u}{|u|} \cdot r\right)}{|r|^{d+\alpha}} dr
$$

$$
\leq \int_{|r| \leq |u|} \frac{\sin\left(\frac{u}{|u|} \cdot r\right)}{|r|^{d+\alpha-1}} dr
$$

\leq M < \infty
$$
for all \( u \). So
\[
\left| t^{1 - \frac{1}{\alpha}} \frac{\partial}{\partial u_j} \Psi(t^{-\frac{1}{\alpha}} u) \right| \leq M t^{1 - \frac{1}{\alpha}} (t^{-\frac{1}{\alpha}})^{\alpha - 1} |u|^{\alpha - 1} = M |u|^{\alpha - 1}
\]
(17)
for all \( u \). Further we have that
\[
\left| \frac{\partial^2}{\partial u_j^2} \Psi(u) \right| \leq \int_{|y| \leq 1} y_j^2 \frac{\cos(u \cdot y)}{|y|^{d+\alpha}} dy 
\]
\[
\leq \int_{|y| \leq 1} \frac{1}{|y|^{d+\alpha-2}} dy = C < \infty
\]
(18)
for all \( u \). Using (13), (14) and (18), it follows that
\[
\left| \frac{\partial^2}{\partial x_i \partial u_j} \eta_i \right|_{L^1(\mathbb{R}^d)} \leq t^{1 - \frac{d}{\alpha}} C \left\| u e^{-\varphi(u)} \right\|_{L^1(\mathbb{R}^d)}
\]
\[
+ M \left\| u |u|^{2(\alpha - 1)} e^{-\varphi(u)} \right\|_{L^1(\mathbb{R}^d)}
\]
Similarly, we can treat the case \( i = j \) and find
\[
\left\| \frac{\partial}{\partial x_i} \left( p(t^{-\frac{1}{\alpha}} \cdot) \right) \right\|_{L^1(\mathbb{R}^d)} \leq C_1 t^{1 + \frac{1 - d}{\alpha}} + C_2 t^{-\frac{d}{\alpha}}
\]
\[
\leq C t^{-\frac{d}{\alpha}}
\]
for constants \( C_1, C_2 \) and \( C < \infty \). This estimate and (10) then give
\[
\left| \frac{\partial}{\partial x_i} E[\phi(s, x + L_t)] \right| \leq C \|\phi\|_\infty t^{\frac{d-1}{\alpha}} t^{-\frac{d}{\alpha}}
\]
\[
= C \|\phi\|_\infty t^{-\frac{d}{\alpha}}
\]
for all \( x \) and \( s \).

So
\[
\left| \frac{\partial}{\partial x_i} u(t, x) \right| \leq C \|\phi\|_\infty \int_0^t (t - s)^{-\frac{1}{\alpha}} ds
\]
\[
\leq C(T) \|\phi\|_\infty
\]
for all \( x, s \) with
\[
C(T) \to 0 \text{ for } T \searrow 0.
\]
Using the same arguments just as above we also get
\[
\left| \frac{\partial}{\partial x_i} u(t, x) - \frac{\partial}{\partial x_i} u(t, y) \right| \leq C(T) \|\phi\|_{C^2_b}
\]
\[
\frac{1}{|x - y|^\beta}
\]
for all \( x \neq y \) and \( s \).

Let us now derive an estimate with respect to \( D^2 u \). We observe that

\[
\frac{\partial^2}{\partial x_i^2} E[\phi(s, x + L_t)] = \int_{\mathbb{R}^d} \phi(s, u) \frac{\partial^2}{\partial x_i^2} p_t(u - x) du \\
= \int_{\mathbb{R}^d} \phi(s, u + x) \frac{\partial^2}{\partial x_i^2} p_t(u) du \\
= \int_{\mathbb{R}^d} \phi(s, t^{\frac{1}{\alpha}} u + x) t^{\frac{d}{\alpha}} \frac{\partial^2}{\partial x_i^2} p_t(t^{\frac{1}{\alpha}} u) du \\
= \int_{\mathbb{R}^d} \phi(s, t^{\frac{1}{\alpha}} u + x) t^{\frac{d-2}{\alpha}} \frac{\partial^2}{\partial x_i^2} (p_t(t^{\frac{1}{\alpha}} u)) du. \tag{19}
\]

On the other hand it follows from (15)

\[
\frac{\partial^2}{\partial x_i^2} (p_t(t^{\frac{1}{\alpha}} x)) = -\frac{t^{-d}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixu} u_i^2 e^{-t\Psi(t^{-\frac{1}{\alpha}} u)} du. \tag{20}
\]

Then it follows from Lemma 6, (19) and (20) by using the same arguments as above that

\[
\left| \frac{\partial^2}{\partial x_i^2} E[\phi(s, x + L_t)] \right| \leq C \| \phi \|_\infty t^{\frac{d-2}{\alpha}} t^{-\frac{d}{\alpha}} \\
= C \| \phi \|_\infty t^{-\frac{\alpha}{\alpha-1}} \tag{21}
\]

for all \( x, s \). The case of mixed partial derivatives can be treated similarly and we obtain

\[
\| D^2 P_t \phi \|_\infty \leq C \| \phi \|_\infty t^{-\frac{\alpha}{\alpha-1}} \tag{22}
\]

for all \( \phi \in C_b(\mathbb{R}^d) \) as well as

\[
\| D^2 P_t \phi \|_\infty \leq C \| D \phi \|_\infty t^{-\frac{1}{\alpha}} \tag{23}
\]

for all \( \phi \in C^1_b(\mathbb{R}^d) \), where we used the semi-group notation

\[
(P_t \phi)(x) = E[\phi(x + L_t)].
\]

Further, from interpolation theory (see e.g. [22]), it is known that

\[
\left( C_b(\mathbb{R}^d), C^1_b(\mathbb{R}^d) \right)_{\beta, \infty} = C^d_b(\mathbb{R}^d).
\]

So using (22) and (23) in connection with Theorem 1.1.6 in [22] one finds that

\[
\| D^2 P_t \phi \|_\infty \leq C \left( \frac{1}{t^{\frac{d-2}{\alpha}}} \right) \| \phi \|_{C^d_b(\mathbb{R}^d)} \\
= C \left( \frac{1}{t^{\frac{d-2}{\alpha}}} \right) \| \phi \|_{C^d_b(\mathbb{R}^d)} \tag{24}
\]
for all $\phi \in C^2_\beta(\mathbb{R}^d)$, where $C$ is a constant depending on $\beta$. Since by assumption $\alpha + \beta > 2$ we get that
\[ \|D^2u\|_\infty \leq C \|\phi\|_{C^0_\beta} \]
for all $\phi \in C([0,T], C^0_\beta(\mathbb{R}^d))$.

(B) We aim at showing that $u$ defined by (7) actually solves the equation (10) for such $\phi$ as stated in the theorem.

We observe that
\[ |f(y + x) - f(x) - y \cdot Df(x)| = \left| \int_0^1 (Df(x + \theta y) - Df(x)) \cdot y d\theta \right| \leq \|D^2f\|_\infty |y|^2 \]
for all $x$ and $y$ with $|y| \leq 1$. Using this inequality we see that $\mathcal{L}f \in C_\beta(\mathbb{R}^d)$, if $f \in C^2(\mathbb{R}^d)$.

So it follows from part (A) and the inequality (26) that $Lu$ is well-defined. Further, one has that $C^2(\mathbb{R}^d) \subset \text{Dom}(\mathcal{L})$, where $C^2(\mathbb{R}^d) := C^2_\beta(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. See e.g. [2]. Hence, if $\phi \in C([0,T], C^2(\mathbb{R}^d))$ then we know by Lemma 7 that $u$ given by (7) satisfies (10).

Let us now assume that $\phi \in C([0,T], C^2(\mathbb{R}^d))$. Choose a $\varphi \in C^\infty_c(\mathbb{R}^d)$ with $\varphi(0) = 1$ and set $\phi_n(t,x) = \varphi(x/n)\phi(t,x)$ for $0 \leq t \leq T, x \in \mathbb{R}^d$ and $n \geq 1$. We know from the proofs in (A) that $u \in C([0,T], C^0_\beta(\mathbb{R}^d))$. So using this one verifies that $\phi_n, u_n \in C([0,T], C([0,T], C^2(\mathbb{R}^d)))$ for all $n$, where $u_n(t,x) := \varphi(x/n)u(t,x)$. Hence $u_n$ satisfies (10) for all $n \geq 1$. Further, one sees that $\phi_n \rightarrow \phi, \ D\phi_n \rightarrow D\phi$ pointwise and that for a constant $C$: $\|\phi_n\|_2 \leq C$ for all $n \geq 1$. On the other hand by (10) we obtain pointwise convergence of $Du_n$ to $Du$. So using dominated convergence in connection with the estimates with (26) and (25) we find that $Lu_n$ converges pointwise to $Lu$ for $n \rightarrow \infty$. From this we can see that $u$ for $\phi \in C([0,T], C^2(\mathbb{R}^d))$ solves (10).

Finally, consider the case $\phi \in C([0,T], C^0_\beta(\mathbb{R}^d))$. Here we apply an approximation argument which can be found e.g. in the book [19]: Let $\varphi_n(x) = n^d \varphi(xn)$, where $\varphi \in C^\infty_c(\mathbb{R}^d)$ such that $\varphi(x) \in [0,1]$ for all $x$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Define
\[ \phi_n(t,x) = (\phi(t, \cdot) * \varphi_n)(x) \]
($f * g$ convolution of functions $f$ and $g$). One obtains that $\phi_n \in C([0,T], C^\infty(\mathbb{R}^d))$ and
\[ \|\phi_n\|_{C^0_\beta} \leq \|\phi\|_{C^0_\beta} \]
for all $n$. Further, we have that
\[ \phi_{n_k}(t, \cdot) \rightarrow \phi(t, \cdot) \text{ in } C^0(K) \]
for all $t$, any compact set $K \subset \mathbb{R}^d$ and $0 < \delta < \beta$ for a subsequence $n_k(t), k \geq 1$ depending on $t$ and $K$. See e.g. [19]. Let $K_m, m \geq 1$ be an increasing sequence of compact sets such that $\bigcup_{m \geq 1} K_m = \mathbb{R}^d$. Then for each $K_m$ there exists a subsequence $n_k(m,t), k \geq 1$ such that (25) holds. Then by choosing a diagonal sequence $n_k(t), k \geq 1$ with respect to $n_k(1,t), k \geq 1, n_k(2,t), k \geq 1, ..., n_k(t), n \geq 1$ we conclude from (10) in connection with dominated convergence that
\[ (P_t \phi_{n_k}(t, \cdot))(x) \rightarrow (P_t \phi(t, \cdot))(x), \]
\[(DP_r \phi_n^*(t, \cdot))(x) \to (DP_r \phi(t, \cdot))(x)\]
pointwise in \(x\) for all \(r, t\). So using (22), (26) and dominated convergence we get
\[(LP_r \phi_n^*(t, \cdot))(x) \to (LP_r \phi(t, \cdot))(x)\]
pointwise in \(x\) for all \(r, t\). On the other hand we can argue as above and find that
\[\frac{\partial}{\partial r} P_r \phi_n^*(t, \cdot) = LP_r \phi_n^*(t, \cdot).\]
By employing dominated convergence we obtain that
\[\frac{\partial}{\partial r} P_r \phi(t, \cdot) = LP_r \phi(t, \cdot).\]
Then, using the proof of Lemma 7 we see that \(u\) satisfies (10) for \(\phi \in C([0, T], C^\beta_b(R^d))\).
Finally, by applying (21), (24), (26) and
\[u(t, x) = \int_0^t ((Lu)(s, x) + \phi(s, x))ds,\] (29)
in connection with dominated convergence, we see that \(u \in C^1([0, T], C^2_b(R^d)) \cap C^1([0, T], C^1_b(R^d))\).

**Theorem 11** Let \(L_t, 0 \leq t \leq T\) be a \(d\)-dimensional truncated \(\alpha\)-stable process for \(\alpha \in (1, 2)\) and \(d \geq 2\). Require that \(\varphi \in C([0, T], C^\beta_b(R^d))\) for \(\beta \in (0, 1)\) with \(\alpha + \beta > 2\). Then there exists \(u \in C([0, T], C^2_b(R^d)) \cap C^1([0, T], C^1_b(R^d))\) satisfying the backward Kolmogorov equation
\[\frac{\partial u}{\partial t} + b \cdot \nabla u + Lu = -\varphi \text{ on } [0, T],\]
\[u|_{t=T} = 0.\] (30)
Moreover
\[\|Du\|_{C^\beta_b} \leq C(T) \|\varphi\|_{C^\beta_b},\] (31)
where
\[C(T) \to 0 \text{ for } T \searrow 0,\]
as well as
\[\|D^2u\|_{\infty} \leq M \|\varphi\|_{C^\beta_b}\] (32)
for a constant \(M\).

**Proof.** We want to use Picard iteration based on (10) to construct a solution to (30) (compare Theorem 2.8 in [9] in the case of Brownian motion): Let \(u^0 = 0\) and define for \(n \geq 0\)
\[\frac{\partial u^{n+1}}{\partial t} + Lu^{n+1} = -(b \cdot \nabla u^n) - \varphi \text{ on } [0, T],\]
\[u^{n+1}|_{t=T} = 0.\] (33)
Since \( u \) in Theorem 10 belongs to \( C([0, T], C^2_b(\mathbb{R}^d)) \cap C^1([0, T], C_b(\mathbb{R}^d)) \), we see from (11) that
\[
\| b \cdot \nabla u + \varphi \|_{C^\beta_b} \leq \| \varphi \|_{C^\beta_b} + 2 \| b \|_{C^\beta_b} \| \nabla u \|_{C^\beta_b} \leq \| \varphi \|_{C^\beta_b} + 2 \| b \|_{C^\beta_b} C(T) \| \phi \|_{C^\beta_b} < \infty
\]
for \( \phi \in C([0, T], C^\beta_b(\mathbb{R}^d)) \). So it follows from Theorem 10 that we obtain in each iteration step a solution
\[
u^{n+1} \in C([0, T], C^2_b(\mathbb{R}^d)) \cap C^1([0, T], C_b(\mathbb{R}^d))
\]
Let us now choose a \( T > 0 \) in Theorem 10 such that
\[
2C(T) \| b \|_{C^\beta_b} \leq \frac{1}{2}.
\]
Then, using the estimates (21) and (24) in the proof of Theorem 10 we find for all \( n \geq 0 \) that
\[
\| \nabla u^{n+1} - \nabla u^n \|_{C^\beta_b} = \| \nabla (u^{n+1} - u^n) \|_{C^\beta_b} \leq C(T) \| b \cdot \nabla u^n + \varphi \|_{C^\beta_b} = C(T) \| b \cdot (\nabla u^n - \nabla u^{n-1}) \|_{C^\beta_b} \leq 2C(T) \| b \|_{C^\beta_b} \| \nabla u^n - \nabla u^{n-1} \|_{C^\beta_b} \leq \frac{1}{2} \| \nabla u^n - \nabla u^{n-1} \|_{C^\beta_b} \leq \ldots \leq (\frac{1}{2})^n \| \nabla u^n \|_{C^\beta_b} \leq (\frac{1}{2})^n C(T) \| \varphi \|_{C^\beta_b},
\]
\[
\| u^{n+1} - u^n \|_{C^\beta_b} \leq K \| b \cdot (\nabla u^n - \nabla u^{n-1}) \|_{C^\beta_b} \leq 2K \| b \|_{C^\beta_b} \| \nabla u^n - \nabla u^{n-1} \|_{C^\beta_b} \leq 2K \| b \|_{C^\beta_b} (\frac{1}{2})^{n-1} C(T) \| \varphi \|_{C^\beta_b} \leq K \| \varphi \|_{C^\beta_b} (\frac{1}{2})^n
\]
as well as
\[
\|D^2 u^{n+1} - D^2 u^n\|_\infty = \|D^2(u^{n+1} - u^n)\|_\infty \\
\leq C(T) \|(b \cdot \nabla u^n) + \varphi - ((b \cdot \nabla u^{n-1}) + \varphi)\|_{C_b^\beta} \\
= C(T) \|b \cdot (\nabla u^n - \nabla u^{n-1})\|_{C_b^\beta} \\
\leq 2C(T) \|b\|_{C_b^\beta} \|\nabla u^n - \nabla u^{n-1}\|_{C_b^\beta} \\
\leq \frac{1}{2} \|\nabla u^n - \nabla u^{n-1}\|_{C_b^\beta} \\
\leq (\frac{1}{2})^n C(T) \|\varphi\|_{C_b^\beta}.
\]

So we get that
\[
\|\nabla u^{n+p} - \nabla u^n\|_{C_b^\beta} \\
\leq \sum_{j=0}^{p-1} \|\nabla u^{n+p-j} - \nabla u^{n+p-j-1}\|_{C_b^\beta} \\
\leq C(T) \|\varphi\|_{C_b^\beta} \sum_{j=0}^{p-1} (\frac{1}{2})^{n+p-j-1} \\
= C(T) \|\varphi\|_{C_b^\beta} (\frac{1}{2})^n \sum_{j=0}^{p-1} (\frac{1}{2})^j
\]
and similarly
\[
\|u^{n+p} - u^n\|_{C_b^\beta} \leq K \|\varphi\|_{C_b^\beta} \left(\frac{1}{2}\right)^n \sum_{j=0}^{p-1} (\frac{1}{2})^j
\]
and
\[
\|D^2 u^{n+p} - D^2 u^n\|_\infty \leq C(T) \|\varphi\|_{C_b^\beta} \left(\frac{1}{2}\right)^n \sum_{j=0}^{p-1} (\frac{1}{2})^j
\]
for all \(n, p \geq 0\).

So we see that there exists a function \(u\) such that
\[
u^n \rightarrow u, \ \nabla u^n \rightarrow \nabla u, \ D^2 u^n \rightarrow D^2 u
\]
uniformly in \(t, x\).

Using (26) and dominated convergence, we also observe that
\[
\mathcal{L} u_n \rightarrow \mathcal{L} u
\]
uniformly in \(t, x\). Passing to the limit we obtain the equation
\[
u(t, x) = \int_t^T ((\mathcal{L} u)(s, x) + b(s, x)(\nabla u)(s, x) + \varphi(s, x)) ds, \ n \geq 0 \quad (34)
\]
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Finally, by employing (21), (24), (26) and (34) in connection with dominated convergence, we get that \( u \in C^1([0, T], C_b(\mathbb{R}^d)) \).

**Remark 12** We mention that the statements of Theorem 10 and 11 are valid for all dimensions \( d \geq 1 \). The case \( d = 1 \) can be shown by using the inequality
\[
\|\widehat{u}\|_{L^1(\mathbb{R})} \leq C \|u\|_{H^s(\mathbb{R})}
\]
for \( s > \frac{1}{2} \), where \( \widehat{u} \) is the Fourier transform of \( u \in H^s(\mathbb{R}) = W^{s, 2}(\mathbb{R}) \) and \( C \) a universal constant. See [1] and Section 2.4.

Using Theorem 11 we can rewrite \( \int_0^t b(s, X_s)ds \) in (1) in terms of a more regular expression.

**Corollary 13 (Representation of \( \int_0^t b(s, X_s)ds \))** Retain the assumptions of Theorem 11 for \( \phi = -b \) in (1). Suppose the drift coefficient \( b \) admits the existence of a strong solution \( X \) to (1). Then we have the following representation:
\[
\int_0^t b(s, X(s))ds = u(0, x) - u(t, X(t)) + \int_0^t \int_{\mathbb{R}^d} \{u(s, X(s^-) + \gamma(z)) - u(s, X(s^-))\} \tilde{N}(ds, dz),
\]
where
\[
\gamma(z) := 1_{\{|z| \leq 1\}}z.
\]

**Proof.** Let \( u \) be the solution to the backward Kolmogorov equation in Theorem 11 for \( \phi = -b \). Then, using Itô’s Lemma, we obtain:
\[
\frac{\partial u}{\partial t} + bu + Lu = -b
\]
we get
\[
\int_0^t b(s, X(s))ds = u(0, x) - u(t, X(t)) + \int_0^t \int_{\mathbb{R}^d} \{u(s, X(s^-) + \gamma(z)) - u(s, X(s^-))\} \tilde{N}(ds, dz).
\]

\[\blacksquare\]
4 Construction of Solutions to SDE’s via the Compactness Criterion for $L^2(\mu)$

In this section we want to apply the compactness criterion in Theorem 6 in connection with the results of the previous section to construct strong solutions to the SDE (1), when $L_t, 0 \leq t \leq T$ is a truncated $\alpha$–stable process of index $\alpha \in (1, 2)$ and the drift coefficient $b \in C([0, T], C^\beta_b(\mathbb{R}^d))$ such that $\alpha + \beta > 2$.

To this end, we aim defining a self adjoint operator $A$ on $L^2((0, \tau)) \otimes L^2(\nu)$ (for fixed $\tau > 0$) which admits a compact inverse $A^{-1}: L^2((0, \tau)) \otimes L^2(\nu) \rightarrow L^2((0, \tau)) \otimes L^2(\nu)$. More precisely, the operator $A$ is constructed as follows:

Let the function $p$ (potential) be given by

$$p(t, x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ \frac{1}{|x|^{d+\alpha} + \alpha} & \text{if } \frac{1}{2} < |x| < 1 \\ \frac{1}{|x|^{1/2}} & \text{if } |x| \geq 1 \end{cases}$$

for some $\delta > 0$ such that $\alpha + \delta < 2$.

Consider now for fixed $\tau > 0$ the symmetric form $E$ on $L^2((0, \tau)) \otimes L^2(\nu)$ defined as

$$E(f, g) = \int_0^\tau \int_0^\tau \int_{|y|<1} \int_{|x|<1} \frac{(f(t_1, x) - f(t_2, y))(g(t_1, x) - g(t_2, y))}{(|t_1 - t_2| + |x - y|)^{d+1+2s}} dxdydt_1dt_2$$

$$+ \int_0^\tau \int_{|x|<1} p(t, x) f(t, x) g(t, x) dxdt$$

for functions $f, g$ in the (dense) domain $D(E) \subset L^2((0, \tau)) \otimes L^2(\nu)$ and a fixed $s$ with

$$0 < s < \frac{1}{2},$$

where

$$D(E) = \{ f : \|f\|_{L^2((0, \tau))}^2 + \int_0^\tau \int_0^\tau \int_{|y|<1} \int_{|x|<1} \frac{|f(t_1, x) - f(t_2, y)|^2}{(|t_1 - t_2| + |x - y|)^{d+1+2s}} dxdydt_1dt_2$$

$$+ \int_0^\tau \int_{|x|<1} p(t, x) |f(t, x)|^2 dxdt < \infty \}.$$
for all \( f \in D(\mathcal{E}) \), that is the form \( \mathcal{E} \) is bounded from below by a positive number, we also have that \( D(\mathcal{E}) = D(T_{\mathcal{E}}^{1/2}) \). See [13].

Let us now define the operator \( A \) in Theorem 6 as

\[
A = T_{\mathcal{E}}^{1/2}.
\]

(37)

We want to show that \( A \) has a discrete spectrum and a compact operator inverse \( A^{-1} \). To verify this we prove that \( T_{\mathcal{E}} \) has a discrete spectrum with existing compact operator inverse.

Before we proceed we briefly introduce some notation: Consider now a general symmetric closed form \( \mathcal{E} \) bounded from below by a positive number with a domain \( D(\mathcal{E}) \) that is dense in the Hilbert space \( H = L^2((0, \tau)) \otimes L^2(\nu) \). We assume here that \( \nu \) is a Lévy measure with Lebesgue-density \( w \). Let \( \Omega \) be an open subset of \((0, \tau) \times \mathbb{R}^d \) and we assume that \( \Omega \) is the union of an increasing sequence of open sets \( \Omega_k \subset \Omega, k \geq 1 \). Further, we denote by \( H_\mathcal{E}(\Omega_k) \) the inner product space with respect to \( \{ f1_{\Omega_k} : f \in D(\mathcal{E}) \} \) and the inner product \( (f, g)_\mathcal{E} = \mathcal{E}(f, g) \). Similarly, we define space \( H_\mathcal{E}(\Omega) \).

We need the following auxiliary result:

**Lemma 14** Let \( \Omega \) be as above and assume that the identity maps \( i_k : H_\mathcal{E}(\Omega_k) \rightarrow L^2(\Omega_k, dt \times \nu), k \geq 1 \) are compact. Suppose there is a positive-valued function \( p \) on \( \Omega \) and a sequence \( \varepsilon_k, k \geq 1 \) of positive numbers with \( \varepsilon_k \rightarrow 0, k \rightarrow \infty \) such that

\[
w(x)/p(x) < \varepsilon_k
\]

for a.e. \( x \in \Omega - \Omega_k \) and

\[
\int_{\Omega - \Omega_k} p(t, x) |f(t, x)|^2 dx \leq \mathcal{E}(f, f)
\]

(39)

for all \( f \in D(\mathcal{E}) \). Then \( T_\mathcal{E} \) has a discrete spectrum and a compact inverse \( T_\mathcal{E}^{-1} \).

**Proof.** See Lemma 1 in [21]. \( \blacksquare \)

We now choose the function \( p \) in Lemma 14 as in (35) and we assume that \( \nu \) is the Lévy measure of a truncated \( \alpha \)-stable Lévy process. Further, suppose that \( \Omega_k \subset \Omega := (0, \tau) \times (U_1(0) - \{0\}) \) with \( \pi_2(\Omega - \Omega_k) \) is bounded away from \( y = 0 \) and \( \{ y : |y| = 1 \}, k \geq 1 \) \( (\pi_2((t, y)) = y \) projection onto the spatial component) such that each \( \Omega_k \) is of class \( C^{0,1} \) with bounded boundary and \( \Omega_k \nearrow \Omega \) and such that (35) is fulfilled. Then we observe that \( L^2(\Omega_k, \lambda) \) (\( \lambda \) Lebesgue measure on \( \mathbb{R}^{d+1} \)) and \( L^2(\Omega_k, dt \times \nu) \) coincide and that their corresponding norms are equivalent for each \( k \). So the latter, the definition of \( \mathcal{E} \) in (36) in connection with (35) and compactness results for fractional spaces \( W^{s,p}(\Omega) \) (see e.g. [11] or [26]) imply that the identity maps \( i_k : H_\mathcal{E}(\Omega_k) \rightarrow L^2(\Omega_k, dt \times \nu), k \geq 1 \) are compact. Finally, we also see that condition (39) is an immediate consequence of the definition of \( \mathcal{E} \). Hence, it follows from Lemma 14 that \( T_\mathcal{E} \) has a discrete spectrum and a compact inverse \( T_\mathcal{E}^{-1} \). Using this we find that the operator \( A \) in (37) satisfies the assumptions of Theorem 6.

In order to apply Theorem 6 to the construction of solutions to the SDE (11) we need the following estimate with respect to the operator \( A \) in (37):
Lemma 15 Let $b \in C([0,T], C^\infty_b(\mathbb{R}^d))$. Further, let $X$ be the unique strong solution to (1) with respect to the drift coefficient $b$. Then for sufficiently small $T < \infty$ we have that

$$E[\|AD_x X(\tau)\|_{L^2((0,\tau))}^2] \leq K \exp(TM H_1(\|b\|_{C^\infty_b}^2))$$

for all $0 < \tau \leq T$, where $K, M < \infty$ are constants being independent of $b$ and where $H_1$ is a non-negative continuous function given by

$$H_1(y) := \frac{(y + 1)^2}{(1 - C^2(T)y)^2}, 0 \leq y < \frac{1}{C^2(T)}.$$ 

for a constant $C(T)$ with $C(T) \rightarrow 0$ as $T \downarrow 0$.

Proof. We know from Corollary [13] that we can rewrite the SDE (1) as

$$X(t) = x + \int_0^t b(s, X(s))ds + L_t = u(0, x) - u(t, X(t)) + \int_0^t \int_{\mathbb{R}^d} \{u(s, X(s^-) + \gamma(z)) - u(s, X(s^-))\} \tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R}^d} \gamma(z) \tilde{N}(ds, dz),$$

where $u \in C([0,T], C^2_b(\mathbb{R}^d)) \cap C^1([0,T], C^1_b(\mathbb{R}^d))$ is the solution to the backward Kolmogorov equation [30] in Theorem [11] and where

$$\gamma(z) := 1_{\{|z| \leq 1\}} z.$$ 

So it follows from the properties of the Malliavin derivative $D$ associated with our Lévy process (see e.g. [28]) that for all $0 \leq l \leq t$ and $y$

$$D_{t,y} X(t) = u(t, X(t)) - u(t, X(t) + D_{t,y} X(t)) + \int_0^t \int_{\mathbb{R}^d} \{u(s, X(s^-) + \gamma(z) + D_{t,y} X(s^-)) - u(s, X(s^-) + \gamma(z)) - (u(s, X(s^-) + D_{t,y} X(s^-)) - u(s, X(s^-))) \tilde{N}(ds, dz) + u(l, X(l^-) + \gamma(y)) - u(l, X(l^-)) + \gamma(y)$$

for all $0 < \tau \leq T$. And now we can estimate

$$E[\|AD_x X(\tau)\|_{L^2((0,\tau))}^2] \leq K \exp(TM H_1(\|b\|_{C^\infty_b}^2)).$$
On the other hand, by repeated use of the mean value theorem we get

\[ D_{l_1,y_1}X(t) - D_{l_2,y_2}X(t) = u(t, X(t) + D_{l_2,y_2}X(t)) - u(t, X(t) + D_{l_1,y_1}X(t)) \]

\[ + \int_0^t \int_{\mathbb{R}^d} \{ u(s, X(s^-)) + \gamma(z) + D_{l_1,y_1}X(s^-) \} - u(s, X(s^-) + D_{l_1,y_1}X(s^-)) \}

\[ - u(s, X(s^-) + D_{l_2,y_2}X(s^-)) + u(s, X(s^-) + D_{l_2,y_2}X(s^-)) \} \tilde{N}(ds, dz) \]

\[ + u(l_1, X(l_1^-) + \gamma(y_1)) - u(l_1, X(l_1^-)) \]

\[ - u(l_2, X(l_2^-) + \gamma(y_2)) + u(l_2, X(l_2^-)) \]

\[ + \gamma(y_1) - \gamma(y_2) \]

\[ = u(t, X(t) + D_{l_2,y_2}X(t)) - u(t, X(t) + D_{l_1,y_1}X(t)) \]

\[ + \int_0^t \int_{\mathbb{R}^d} \{ \int_0^1 (Du(s, X(s^-) + D_{l_1,y_1}X(s^-) + \theta \gamma(z)) \}

\[ - Du(s, X(s^-) + D_{l_2,y_2}X(s^-) + \theta \gamma(z))d\theta \gamma(z) \}

\[ + u(l_1, X(l_1^-) + \gamma(y_1)) - u(l_1, X(l_1^-)) \]

\[ - u(l_2, X(l_2^-) + \gamma(y_2)) + u(l_2, X(l_2^-)) \]

\[ + \gamma(y_1) - \gamma(y_2) \]

\[ = \int_0^1 Du(t, X(t) + \theta(D_{l_2,y_2}X(t) - D_{l_2,y_2}X(t)))(D_{l_2,y_2}X(t) - D_{l_1,y_1}X(t)) d\theta \]

\[ + \int_0^t \int_{\mathbb{R}^d} \{ \int_0^1 \int_0^1 D^2u(s, X(s^-) + \gamma(z) + \theta(D_{l_1,y_1}X(s^-) - D_{l_2,y_2}X(s^-)) + \theta \gamma(z)) \}

\[ [(D_{l_1,y_1}X(s^-) - D_{l_2,y_2}X(s^-)), \gamma(z)] d\sigma d\theta \} \tilde{N}(ds, dz) \]

\[ + u(l_1, X(l_1^-) + \gamma(y_1)) - u(l_1, X(l_1^-)) \]

\[ - u(l_2, X(l_2^-) + \gamma(y_2)) + u(l_2, X(l_2^-)) \]

\[ + \gamma(y_1) - \gamma(y_2) \].

On the other hand, by repeated use of the mean value theorem we also have that

\[ u(l_1, X(l_1^-) + \gamma(y_1)) - u(l_2, X(l_2^-) + \gamma(y_2)) \]

\[ = \int_0^1 \left( \frac{\partial}{\partial \theta} u(l_1 + \theta(l_1 - l_2), X(l_1^-) + \gamma(y_1) + \theta(X(l_1^-) - X(l_2^-) + \gamma(y_1) - \gamma(y_2))) \right) \]

\[ \frac{\partial}{\partial y} u(l_1 + \theta(l_1 - l_2), X(l_1^-) + \gamma(y_1) + \theta(X(l_1^-) - X(l_2^-) + \gamma(y_1) - \gamma(y_2))) \]

\[ \cdot (l_1 - l_2), (X(l_1^-) - X(l_2^-) + \gamma(y_1) - \gamma(y_2))^T d\theta. \]  

(40)

Further, since

\[ X(l_1^-) - X(l_2^-) = \int_{l_1}^{l_2} b(s, X(s))ds + \int_{l_1}^{l_2} \int_{\mathbb{R}^d} z\tilde{N}(ds, dz) \]

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for $l_1 \leq l_2$ Itô’s isometry yields

\[ E[|X(l_1^-) - X(l_2^-)|^2] \leq C(|l_1 - l_2|^2 \|b\|_{\infty}^2 + |l_1 - l_2|). \]

Using the latter, (10), the estimates (31), (32) in Theorem 11 and (26) in connection with (34) we get that

\[
E[|u(l_1, X(l_1^-) + \gamma(y_1)) - u(l_2, X(l_2^-) + \gamma(y_2))|^2] \\
\leq K \|b\|_{C^0_d}^2 (|l_1 - l_2|^2 + |l_1 - l_2| \|b\|_{\infty}^2 + |l_1 - l_2| + |\gamma(y_1) - \gamma(y_2)|^2) \\
\leq L \|b\|_{C^0_d}^2 (1 + \|b\|_{C^0_d}^2)(|l_1 - l_2| + |\gamma(y_1) - \gamma(y_2)|).
\]

In the same way we also obtain that

\[ E[|u(l_1, X(l_1^-)) - u(l_1, X(l_1^-))|^2] \leq CL \|b\|_{C^0_d}^2 (1 + \|b\|_{C^0_d}^2) |l_1 - l_2|. \]

By employing the Itô isometry and once again the estimates (31), (32) for $T$ with $C(T) \|b\|_{C^0_d} \leq \frac{1}{2}$ in Theorem 11 we then find

\[
E[D_{t_1, y_1}X(t) - D_{t_2, y_2}X(t)]^2(1 - C^2(T) \|b\|_{C^0_d}^2) \\
\leq K \{ \|D^2u\|_{\infty}^2 \int_{\mathbb{R}^d} |\gamma(z)|^2 \nu(dz) \int_0^t E[D_{t_1, y_1}X(s^-) - D_{t_2, y_2}X(s^-)]^2 ds \\
+ \|b\|_{C^0_d}^2 (1 + \|b\|_{C^0_d}^2)(|l_1 - l_2| + |\gamma(y_1) - \gamma(y_2)|) \}
\]

Hence

\[
E[D_{t_1, y_1}X(t) - D_{t_2, y_2}X(t)]^2 \\
\leq M \{ \|b\|_{C^0_d}^2 \int_0^t E[D_{t_1, y_1}X(s^-) - D_{t_2, y_2}X(s^-)]^2 ds \\
+ \frac{\|b\|_{C^0_d}^2 (\|b\|_{C^0_d}^2 + 1)}{(1 - C^2(T) \|b\|_{C^0_d}^2)} (|l_1 - l_2| + |\gamma(y_1) - \gamma(y_2)|) \}
\]

\[
\leq MH_1(\|b\|_{C^0_d}^2) \{ \int_0^t E[D_{t_1, y_1}X(s) - D_{t_2, y_2}X(s)]^2 ds \\
+ (|l_1 - l_2| + |\gamma(y_1) - \gamma(y_2)|) \},
\]

where

\[ H_1(y) := \frac{(y + 1)^2}{(1 - C^2(T)y)}, \quad 0 \leq y < \frac{1}{C^2(T)}. \]

Therefore we get

\[
\int_0^T \int_0^T \int_{|y| < 1} \int_{|x| < 1} E[D_{t_1, y_1}X(t) - D_{t_2, y_2}X(t)]^2 dy_1 dy_2 dl_1 dl_2 \\
\leq MH_1(\|b\|_{C^0_d}^2) \{ \int_0^t \int_0^T \int_{|y| < 1} \int_{|x| < 1} E[D_{t_1, y_1}X(s) - D_{t_2, y_2}X(s)]^2 dy_1 dy_2 dl_1 dl_2 \\
+ \int_0^T \int_0^T \int_{|y| < 1} \int_{|x| < 1} (|l_1 - l_2| + |y_1 - y_2|)^{d+1+2s} dy_1 dy_2 dl_1 dl_2 \}.
\]
Similarly, we find

\[ E[|D_{l,y}X(t)|^2] \]
\[ \leq MH_1(\|b\|^2_{C^0}) \left\{ \int_0^t E[|D_{l,y}X(s)|^2]ds + |\gamma(y)|^2 \right\}. \]

So

\[ \int_0^\tau \int_0^\tau p(l, y)E[|D_{l,y}X(t)|^2]dyl \]
\[ \leq MH_1(\|b\|^2_{C^0}) \left\{ \int_0^t \int_0^\tau \int_0^\tau p(l, y)E[|D_{l,y}X(s)|^2]dyl ds \right. \]
\[ + \int_0^\tau \int_0^\tau p(l, y) |\gamma(y)|^2) dydl \}, \quad (42) \]

where the potential \( p \) is defined as in (35). By combining the estimates (41) and (42) we obtain

\[ E[\|AD_{.,X}(t)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2] \]
\[ \leq MH_1(\|b\|^2_{C^0}) \left\{ \int_0^t E[\|AD_{.,X}(s)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2] ds \right. \]
\[ + K \}, \]

where

\[ K := \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau \frac{1}{(c_1 - c_2 + y_1 - y_2)^{2s}} dy_1 dy_2 dl_1 dl_2 \]
\[ + \int_0^\tau \int_0^\tau p(l, y) |\gamma(y)|^2) dydl \]
\[ < \infty, \]

since \( 0 < s < \frac{1}{2} \). By Picard iteration one verifies that \( E[\|AD_{.,X}(t)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2] < \infty \).

So we can apply Gronwall’s Lemma and get

\[ E[\|AD_{.,X}(t)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2] \leq K \exp(TH_1(\|b\|^2_{C^0})). \]

Thus the proof follows.

We also want to employ the following result, whose proof can be found in [23]:

**Theorem 16** Let \( \Phi^\prime \) be the topological dual of a countably Hilbertian nuclear space \( \Phi \). Further, consider the Skorohod space \( D([0, T], \Phi^\prime) \) of functions \( f : [0, T] \to \Phi \) with right-continuous paths and existing left limits. Then a set \( A \subset D([0, T], \Phi^\prime) \) is relatively compact if and only if the set \( \{ f(\cdot)[\phi] : f \in A \} \) is relatively compact in \( D([0, T], \mathbb{R}) \) for all \( \phi \in \Phi \).
Let us now consider a function $b \in C([0,T];C^\beta_b(\mathbb{R}^d))$. Then we know from the proof of Theorem 10 that there exists $b_n \in C([0,T],C^\infty(\mathbb{R}^d)), n \geq 1$ such that

$$\|b_n\|_{C^\beta_b} \leq \|b\|_{C^\beta_b}$$

for all $n$. Further, we have that

$$b_{n_k(t)}(t,\cdot) \rightarrow b(t,\cdot) \text{ in } C^\delta(K)$$

for all $t$, any compact set $K \subset \mathbb{R}^d$ and $0 < \delta < \beta$ for a subsequence $n_k(t), k \geq 1$ depending on $t$ and $K$. See also p. 37 in [19].

**Lemma 17** Suppose that $X^n_t, 0 \leq t \leq T, n \geq 1$ are the unique strong solutions to (1) with respect to the drift coefficients $b_n$ in (43). Then there exists a subsequence $(n_k)_{k \geq 1}$ which only depends on (a sufficiently small) $T$ such that for all $0 \leq t \leq T$, $X^n_t$ converges in $L^2(\Omega)$ for $k \rightarrow \infty$.

**Proof.** We know that

$$X^n_t = x + \int_0^t b_n(s, X^n_s)ds + L_t, 0 \leq t \leq T.$$ 

Let $\delta > 0$ and consider a finite partition

$$0 = t_0 < t_1 < ... < t_n = T$$

with $t_j - t_{j-1} \geq \delta$ for all $j = 1, ..., n$.

Then we have

$$X^n_{t_j} - X^n_{t_{j-1}} = \int_{t_{j-1}}^{t_j} b_n(s, X^n_s)ds + L_{t_j} - L_{t_{j-1}}$$

for all $j = 1, ..., n$.

Now let $f$ be an element of the Lévy-Hida test function space $(\mathcal{S}) \subset L^2(\Omega)$. Denote by $(\mathcal{S})^*$ its topological dual (Lévy-Hida distribution space). See e.g. [28] and the references therein for further information on these spaces. Then

$$ \langle (X^n_{t_1} - X^n_{t_2}), f \rangle_{(\mathcal{S})^*,(\mathcal{S})} = E[(X^n_{t_1} - X^n_{t_2}) f],$$

where $\langle \cdot, \cdot \rangle_{(\mathcal{S})^*,(\mathcal{S})}$ is the dual pairing. So using (46) we get

$$E[(X^n_{t_1} - X^n_{t_2}) f] = \int_{t_{j-1}}^{t_j} E[b^{(i)}_n(s, X^n_s) f]ds + E[(L_{t_j} - L_{t_{j-1}}) f]$$

for all $j$. Thus we it follows form Hölder’s inequality and Itô’s isometry that

$$\left| E[(X^n_{t_1} - X^n_{t_2}) f] \right| \leq C |t_j - t_{j-1}| (E[f^2])^{1/2}$$

for all $j$. This completes the proof.
for all $i$ and $j$ and a constant $C$ depending on $\|b\|_{C_b^\alpha}$ and the Lévy measure $\nu$. So

$$\sup_{n \geq 1} \omega^T((X^{n,i}, f)_{(S)^*}, \delta) \to 0 \text{ for } \delta \searrow 0,$$

where $\omega^T$ is the modulus given by

$$\omega^T(g, \delta) := \inf \max \sup \{|g(t) - g(s)| : s, t \in [t_{j-1}, t_j]\},$$

where the infimum is taken over partitions $\{t_j\}$ of the form (35).

So $(X^{n,i}, f)_{(S)^*}$ is relatively compact in $D([0, T]; \mathbb{R})$ for all $f \in (S)$. Since $(S)^*$ is the dual of a countably Hilbertian nuclear space $(S)$, we can apply Theorem 16 and find that there exists for all $i$ a subsequence $(n_{k,i})_{k \geq 1}$ which only depends on (a sufficiently small) $T$ such that $X^{n_{k,i},i}$ converges in $D([0, T]; (S)^*)$.

On the other hand it follows from Lemma 15 and (43) that for sufficiently small $T < \infty$ we have

$$E[\|AD \cdot X^n(\tau)\|_{L^2([0, \tau]) \otimes L^2(\nu)}^2] \leq K \exp(TM \|b\|_{C_b^\alpha}^2)$$

for all $0 < \tau \leq T$, where $K, M < \infty$ are constants being independent of $b$ and where $H_1$ is a non-negative continuous function on some interval $[0, M]$ with $0 \leq \|b\|_{C_b^\alpha}^2 < M$.

Then, applying Theorem 8 to the sequence $X^{n_{k,i},i}_t$ we find that for all $t$ and $i$ there exists a subsequence $m_l = m^{k,i}_l, l \geq 1$ of $n_{k,i} \geq 1$ and a $\bar{X}^i_t \in L^2(\Omega)$ such that

$$X^{n_{m_l,i},i}_t \to \bar{X}^i_t \text{ for } l \to \infty \quad (47)$$

in $L^2(\Omega)$.

Let us show that

$$X^{n_{k,i},i}_t \to \bar{X}^i_t \text{ for } k \to \infty \text{ in } L^2(\Omega)$$

for all $t, i$. To this end we argue by contradiction. Assume that there exists for some $t, i$ a $\varepsilon > 0$ and a subsequence $\varphi_{l,i}, l \geq 1$ such that

$$\left\|X^{\varphi_{l,i}}_t - \bar{X}^i_t\right\|_{L^2(\Omega)} \geq \varepsilon.$$

On the other hand we know by Theorem 8 that there exists a subsequence $\phi_{r,i}, r \geq 1$ of such that

$$X^{\varphi_{l,i}}_t \to \bar{Y}^i_t \text{ for } r \to \infty \text{ in } L^2(\Omega).$$

But since

$$X^{n_{k,i},i}_t \to \bar{X}^i_t \text{ for } k \to \infty \text{ in } (S)^*$$

because of (17), we see that

$$\bar{Y}^i_t = \bar{X}^i_t.$$

But this leads to the contradiction

$$\left\|X^{\varphi_{l,i}}_t - \bar{X}^i_t\right\|_{L^2(\Omega)} \geq \varepsilon.$$
So the proof follows. ■

We are coming to the main result of our paper on SDE’s with time-homogeneous drift coefficients:

**Theorem 18** Suppose that \( L_t, 0 \leq t \leq T \) is a \( d \)-dimensional truncated \( \alpha \)-stable process for \( \alpha \in (1, 2) \) and \( d \geq 2 \). Require that \( b \in C_b^\beta (\mathbb{R}^d) \) for \( \beta \in (0, 1) \) such that \( \alpha + \beta > 2 \). Then there exists for sufficiently small \( T > 0 \) a unique strong solution \( X \) to the SDE

\[
    dX_t = b(X_t) dt + dL_t, 0 \leq t \leq T, X_0 = x. 
\]

Moreover, \( X_t \) is Malliavin differentiable for all \( 0 \leq t \leq T \).

**Proof. 1. Existence:** By (44) (see also the proof of Theorem 10) we find a subsequence \( n_k^*, k \geq 1 \) such that

\[
    b_{n_k^*}(y) \to b(y) \quad \text{as} \quad k \to \infty
\]

for all \( y \). Consider now the sequence of unique strong solutions \( X_k^k \) to

\[
    X_k^k = x + \int_0^t b_{n_k^*}(X_s^k) ds + L_t
\]

with respect to the drift coefficients \( b_{n_k^*}, k \geq 1 \) in (43). Then we know from Lemma 17 that there exists a subsequence \( (n_k)_{k \geq 1} \) which only depends on (a sufficiently small) \( T \) such that for all \( 0 \leq t \leq T \):

\[
    X_{t_k}^{n_k} \to X_t \quad \text{in} \quad L^2(\Omega)
\]

for \( k \to \infty \). On the other hand we obtain by dominated convergence that

\[
    E[(\int_0^t b_{n_k^*}(X_{s_k}^k) ds - \int_0^t b(X_s) ds)^2] \\
    = E[(\int_0^t (b_{n_k^*}(X_{s_k}^k) - b_{n_k^*}(X_s) + b_{n_k^*}(X_s) - b(X_s)) ds)^2] \\
    \leq C(E[\int_0^t (b_{n_k^*}(X_{s_k}^k) - b_{n_k^*}(X_s))^2 ds] + E[\int_0^t (b_{n_k^*}(X_s) - b(X_s))^2 ds]) \\
    \leq C \|b\|^2_{C_b^\beta} (E[\int_0^t |X_{s_k}^k - X_s|^\beta ds] + E[\int_0^t (b_{n_k^*}(X_s) - b(X_s))^2 ds]) \\
    \to 0 \quad \text{as} \quad k \to \infty.
\]

So by passing to the limit in \( L^2(\Omega) \) on both sides of (49) we get

\[
    X_t = x + \int_0^t b(X_s) ds + L_t, 0 \leq t \leq T.
\]
2. **Uniqueness**: Suppose that there are two solutions $X^1$ and $X^2$ to (49). Then it follows from Corollary 13 and the mean value theorem that

$$X^1(t) - X^2(t) = \int_0^t (b(X^1(s)) - b(X^2(s)))ds = u(t, X^2(t)) - u(t, X^1(t))$$

$$+ \int_0^t \int_{\mathbb{R}^d} \{u(s, X^1(s^-) + \gamma(z)) - u(s, X^1(s^-)) - u(s, X^2(s^-) + \gamma(z)) + u(s, X^2(s^-))\} \tilde{N}(ds, dz)$$

$$= u(t, X^2(t)) - u(t, X^1(t))$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_0^1 \int_0^1 D^2 u(s, \theta \gamma(z) + \tau(X^1(s^-) - X^2(s^-)))$$

$$[(X^1(s^-) - X^2(s^-)), \gamma(z)] d\theta d\tau \tilde{N}(ds, dz).$$

Using the Itô isometry and the estimates (31), (32) we obtain

$$E[|X^1(t) - X^2(t)|^2] \leq K \frac{\|b\|_{C^\beta_b}^2}{(1 - C^2(T) \|b\|_{C^\beta_b}^2)} \int_0^t E[|X^1(s) - X^2(s)|^2] ds.$$ 

Hence Gronwall’s Lemma gives

$$X^1 = X^2.$$

The Malliavin differentiability of $X_t$ is a consequence of the fact (see Lemma 15) that

$$E[\|D_{\cdot}, X^{nk}(\tau)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2]$$

$$\leq CE[\|AD_{\cdot}, X^{nk}(\tau)\|_{L^2((0,\tau)) \otimes L^2(\nu)}^2]$$

$$\leq K \exp(TM H_1(\|b\|_{C^\beta_b}^2)),$$

$$k \geq 1, 0 < \tau \leq T$$

and Lemma 1.2.3 in [27].

**Remark 19** The proof of Theorem 18 and the preceding results which are formulated with respect to time-inhomogeneous coefficients $b$ show that we may choose in Theorem 18 drift coefficients of the form

$$b(t, x) = \sum_{i=1}^m f_i(t)b_i(x),$$

where $f_i, i = 1, \ldots, m$ are continuous functions and $b_i \in C^\beta_b(\mathbb{R}^d), i = 1, \ldots, m.$
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