Lorentzian left invariant metrics on three dimensional unimodular Lie groups and their curvatures

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Abstract
There are five unimodular simply connected three dimensional unimodular non abelian Lie groups: the nilpotent Lie group Nil, the special unitary group SU(2), the universal covering group \( \tilde{\text{PSL}}(2, \mathbb{R}) \) of the special linear group, the solvable Lie group Sol and the universal covering group \( \tilde{\text{E}}_0(2) \) of the connected component of the Euclidean group. For each \( G \) among these Lie groups, we give explicitly the list of all Lorentzian left invariant metrics on \( G \), up to un automorphism of \( G \). Moreover, for any Lorentzian left invariant metric in this list we give its Ricci curvature, scalar curvature, the signature of the Ricci curvature and we exhibit some special features of these curvatures. Namely, we give all the metrics with constant curvature, semi-symmetric non locally symmetric metrics and the Ricci solitons.

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1. Introduction

This paper has two goals:
1. to complete the study done by Rahmani in [6] by classifying for each three dimensional simply connected unimodular Lie group \( G \) all the Lorentzian left invariant metrics on \( G \), up to automorphism of \( G \) and hence achieve a similar study to the one done in the Riemannian case in [4],
2. to study, for each class of these metrics, their curvatures.

There are five unimodular non abelian simply connected three dimensional unimodular Lie groups characterized by the signature of their Killing form: the nilpotent Lie group Nil with signature \((0, 0, 0)\), the special unitary group SU(2) with signature \((-\cdot, -\cdot, -\cdot)\), the universal covering group \( \tilde{\text{PSL}}(2, \mathbb{R}) \) of the special linear group with signature \((+\cdot, +\cdot, -\cdot)\), the solvable Lie group Sol with signature \((+\cdot, 0, 0)\) and the universal covering group \( \tilde{\text{E}}_0(2) \) of the connected component of the Euclidean group with signature \((-\cdot, 0, 0)\). Let \((G, h)\) one of these Lie groups endowed with a Lorentzian left invariant metric and \( g \) its Lie algebra with a fixed orientation. Denote by \( \langle \cdot, \cdot \rangle \) the values of \( h \) at the neutral element.

The study of Rahmani in [6] is based on a remark first made by Milnor in [7]. There exists a product \( \times : g \times g \to g \) depending on \( \langle \cdot, \cdot \rangle \) and the orientation and \( L : g \to g \) a symmetric endomorphism such that, for any \( u, v \in g \), the Lie bracket on \( g \) is given by

\[
[u, v] = L(u \times v). \tag{1}
\]

Note that \( L \) changes to \(-L\) when we change the orientation. It is well-known (see [5]) that there are four types of symmetric endomorphisms on a Lorentzian vector space. Depending on the type of \( L \), there exists \( B_1 = (e_1, e_2, e_3) \) an orthonormal basis of \( g \) with \( \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1 \) and \( \langle e_3, e_3 \rangle = -1 \) such that \( (1) \) gives one of the following forms:
1. Type $\text{diag}(a, b, c)$. There exists $a, b, c \in \mathbb{R}$ such that

$$[e_1, e_2] = -ae_3, [e_2, e_3] = be_1 \quad \text{and} \quad [e_3, e_1] = ce_2. \quad (2)$$

In this case the eigenvalues of the matrix of the Killing form in $\mathbb{B}_1$ are $[-2ab, 2ac, 2bc]$.

2. Type $\{az\}$. There exists $a, \alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ such that

$$[e_1, e_2] = -\beta e_3 - ae_3, [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = ae_2 - \beta e_3. \quad (3)$$

In this case the eigenvalues of the matrix of the Killing form in $\mathbb{B}_1$ are $[2(a^2 + \beta^2), 2a \sqrt{a^2 + \beta^2}, -2a \sqrt{a^2 + \beta^2}]$.

3. Type $\{ab\}$. There exists $a, b \in \mathbb{R}$ such that

$$[e_1, e_2] = \frac{1}{2} e_2 + \left(\frac{1}{2} - b\right) e_3, \quad [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = \left(b + \frac{1}{2}\right) e_2 + \frac{1}{2} e_3. \quad (4)$$

In this case the eigenvalues of the matrix of the Killing form in $\mathbb{B}_1$ are $[2b^2, (\sqrt{4b^2 + 1} - 1)a, -(\sqrt{4b^2 + 1} + 1)a]$.

4. Type $\{a2\}$. There exists $a \in \mathbb{R}$ such that

$$[e_1, e_2] = \frac{1}{\sqrt{2}} e_2 - ae_3, \quad [e_2, e_3] = ae_1 + \frac{1}{\sqrt{2}} e_2 \quad \text{and} \quad [e_3, e_1] = \frac{1}{\sqrt{2}} e_1 + ae_2 + \frac{1}{\sqrt{2}} e_3. \quad (5)$$

In this case the eigenvalues of $M(K, \mathbb{B}_1)$ are complicated. But $\det M(K, \mathbb{B}_1) = -8a^6$ and $\text{tr}(M(K, \mathbb{B}_1)) = 2(a^2 + 1)$ which can be used to determine the signature of $K$.

Moreover, the type of $L$ is an invariant of Lorentzian left invariant metrics on unimodular three dimensional Lie groups. Indeed, let $G$ be an unimodular three dimensional Lie group and $h_1$ and $h_2$ are two left invariant Lorentzian metrics on $G$ and $L_1$ and $L_2$ the associated endomorphisms for a fixed orientation on $g$. If there exists an automorphism $\phi : G \rightarrow G$ such that $h_1 = \phi^*(h_2)$ then $L_2 = \pm \phi \circ L_1 \circ \phi^{-1}$ where $\psi = T_0 \phi$. Thus if $L_1$ and $\pm L_2$ have two different types or have the same type and two different sets of eigenvalues then $h_1$ and $h_2$ are not isometric.

Based on what above, our method in order to fulfill our goals mentioned above goes as follows:

1. We consider a three unimodular Lie group $G$ and we fix a natural basis $\mathbb{B}_0$ of its Lie algebra $g$ (see Section 2 where these bases are given).

2. We endow $G$ with a left invariant Lorentzian metric $h_0$ and we denote by $L$ its associated endomorphism.

3. Depending on the signature of the Killing form we can determine which are the possible types of $\pm L$.

4. For each possible type there exists an orthonormal basis $\mathbb{B}_1 = (e_1, e_2, e_3)$ such that the Lie bracket has one of the forms $(2)-(6)$.

5. We find a basis $\mathbb{B}_2 = (xe_1 + ye_2 + ze_3, ve_1 + ve_2 + we_3, pe_1 + qe_2 + re_3)$ of $g$ which has the same constant structures as $\mathbb{B}_0$ and we consider the automorphism of Lie algebra $P : g \rightarrow g$ which sends the basis $\mathbb{B}_0$ to $\mathbb{B}_2$ and $\phi$ the automorphism of $G$ associated to $P$. We put $h_1 = \phi^*(h_0)$.

6. In some cases, we use another automorphism $\phi_1$ of $G$ such that $\phi_1^*(h_1)$ has a more reduced form than $h_1$.

7. Finally, we give the matrix of $\phi_1^*(h_1)$ in $\mathbb{B}_0$. The $\phi_1^*(h_1)$ obtained constitute a list of Lorentzian left invariant metrics on $G$ depending on a reduced number of parameters and each Lorentzian left invariant metric on $G$ is isometric to one in this list. We find twenty non isometric classes of such metrics which shows that the situation is far more rich than the Riemannian case $[3]$.

8. We compute for each metric in the list the Ricci curvature and the scalar curvature which determine all the curvature since we are in dimension 3. Finally, we exhibit some metrics with distinguished curvature properties.

The steps 5. and 6. are the most difficult and, to find the basis $\mathbb{B}_2$ or the automorphism $\phi$, we have used the software Maple and the expression of the groups of automorphisms of unimodular three dimensional Lie algebras given in $[3]$. All the computations have been checked by Maple.

The paper is organized as follows. In Section 2, we precise the models of unimodular three dimensional Lie groups we will use in this paper. In Section 3, we perform for each unimodular three dimensional Lie groups the steps mentioned above and we give its list of left invariant Lorentzian metrics. For $\text{Nil}$ the list contains three non isometric
classes of metrics, for SU(2) one class, for \( \text{PSL}(2, \mathbb{R}) \) seven non isometric classes, for Sol seven non isometric classes and for \( \text{E}_6(2) \) three non isometric classes. These metrics are given by the formulas (11)-(31). In Section 4 we give for each class of metric found in Section 3 its Ricci curvature, scalar curvature and the signature of the Ricci curvature. We give a table describing the possible signature of the Ricci curvature and the metrics realizing these signatures and we give the different types of Ricci operators (see Proposition 4.20). Finally, we recover some known results. Namely, Lorentzian left invariant metrics on unimodular three dimensional Lie groups which are of constant curvature, Einstein, locally symmetric, semi-symmetric not locally symmetric or Ricci soliton have been determined in [1, 2, 3] by giving their Lie algebras as in [2, 3, 5]. We give their corresponding metrics in our list (see Theorems 4.1, 4.3).

2. Preliminaries

Let \((G, h)\) be a Lie group endowed with a Lorentzian left invariant metric \(h\), \(g\) its Lie algebra and \((\cdot, \cdot) = h(e)\). The Levi-Civita connection of \((G, h)\) defines a product \(L : g \times g \to g\) called the Levi-Civita product and given by Koszul’s formula

\[
2\langle L_u v, w \rangle = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [v, w], u \rangle.
\]

(6)

For any \(u, v \in g\), \(L_u : g \to g\) is skew-symmetric and \([u, v] = L_u v - L_v u\). The curvature on \(g\) is given by \(K(u, v) = L_{[u,v]} - [L_u, L_v]\). The Ricci tensor is the symmetric tensor \(\text{ric}\) given by \(\text{ric}(u, v) = tr(w \to K(u, w)v)\) and the Ricci operator \(\text{Ric} : g \to g\) is given by the relation \(\langle \text{Ric}(u), v \rangle = \text{ric}(u, v)\). The scalar curvature is given by \(s = tr(\text{Ric})\).

Recall that:
1. \((G, h)\) is called flat if \(K = 0\);
2. \((G, h)\) has constant sectional curvature if there exists a constant \(\lambda\) such that, for any \(u, v, w \in g\),

\[
K(u, v)w = \lambda \left( \langle v, w \rangle u - \langle u, w \rangle v \right).
\]
3. \((G, h)\) is called Einstein if there there exists a constant \(\lambda\) such that \(\text{Ric} = \lambda \text{Id}_g\).
4. \((G, h)\) is called locally symmetric if, for any \(u, v, w \in g\),

\[
L_u(K)(v, w) := [L_u, K(v, w)] - K(L_u v, w) - K(v, L_u w) = 0.
\]
5. \((G, h)\) is called semi-symmetric if, for any \(u, v, a, b \in g\),

\[
[K(u, v), K(a, b)] = K(K(u, v)a, b) + K(a, K(u, v)b).
\]

They are five unimodular simply connected three dimensional unimodular non abelian Lie groups:

1. The nilpotent Lie group Nil known as Heisenberg group whose Lie algebra will be denoted by \(n\). We have

\[
\text{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right. \quad \text{and} \quad n = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right. , \quad x, y, z \in \mathbb{R}
\]

and the non-vanishing Lie brackets in the canonical basis \(B_0 = (X, Y, Z)\) are given by \([X, Y] = Z\). The Killing form is trivial.

2. SU(2) = \( \left\{ \begin{pmatrix} a + bi & -c + di \\ c + di & a - bi \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1 \right\} \) and \( \text{su}(2) = \left\{ \begin{pmatrix} i z & y + ix \\ -y + xi & -i z \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\} \). In the basis

\[
B_0 = \left\{ \sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\},
\]

we have

\[
[\sigma_x, \sigma_y] = 2\sigma_z, \quad [\sigma_y, \sigma_z] = 2\sigma_x \quad \text{and} \quad [\sigma_z, \sigma_x] = 2\sigma_y.
\]

(7)

The Killing form has signature \((-,-,-)\).
3. The universal covering group $\widetilde{\text{PSL}}(2, \mathbb{R})$ of $\text{SL}(2, \mathbb{R})$ whose Lie algebra is $\text{sl}(2, \mathbb{R})$. In the basis
\[
B_0 = \left\{ X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},
\]
we have
\[
[X_1, X_2] = 2X_3, \ [X_3, X_1] = 2X_2 \quad \text{and} \quad [X_3, X_2] = 2X_1.
\] The Killing form has signature $(+, +, -)$.

4. The solvable Lie group $\text{Sol} = \left\{ \begin{pmatrix} x \ e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$ whose Lie algebra is $\text{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$.

In the basis
\[
B_0 = \left\{ X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},
\]
we have
\[
[X_1, X_2] = X_3; \ [X_1, X_3] = -X_3 \quad \text{and} \quad [X_2, X_3] = 0.
\] The Killing form has signature $(+, 0, 0)$.

5. The universal covering group $\widetilde{E}_0(2)$ of the Lie group
\[
E_0(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}, \theta, x, y \in \mathbb{R} \right\},
\]
its Lie algebra is
\[
e_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \theta, x, y, z \in \mathbb{R} \right\}.
\]

In the basis
\[
B_0 = \left\{ X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},
\]
we have
\[
[X_1, X_2] = X_3; \ [X_1, X_3] = -X_2 \quad \text{and} \quad [X_2, X_3] = 0.
\] The Killing form has signature $(-, 0, 0)$.

**Remark 1.** We can see from what above that two unimodular three dimensional non abelian Lie algebras are isomorphic if and only their Killing forms have the same signature.

### 3. Lorentzian left invariant metrics on three dimensional unimodular Lie groups

Through this section, for any $G$ among the five unimodular Lie groups described in Section 2, $B_0$ is the basis of its Lie algebra given also in Section 2, $h_0$ is a left invariant Lorentzian metric on $G$ and $L : g \rightarrow \mathfrak{g}$ the associated endomorphism given in 1.
3.1. Lorentzian left invariant metrics on Nil

We have three possibilities:

(i) \( L \) is of type \( \text{diag}(0, 0, \sqrt{\lambda}) \) with \( \lambda \neq 0 \). In the basis \( B_1 \) given in (2), we have \([e_1, e_2] = -\sqrt{\lambda}e_3\). We consider the automorphism of Lie algebras \( P : \mathfrak{n} \to \mathfrak{n} \) given by

\[
P(X) = e_1; \ P(Y) = e_2 \quad \text{and} \quad P(Z) = -\sqrt{\lambda}e_3,
\]

\( \phi : G \to G \) the associated automorphism of Lie groups. The matrix of \( \phi^*(h_0) \) is given by

\[
M(\phi^*(h_0), \mathfrak{B}_0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\lambda
\end{pmatrix}, \quad \lambda > 0. \tag{11}
\]

(ii) \( L \) is of type \( \text{diag}(\sqrt{\lambda}, 0, 0) \) with \( \lambda \neq 0 \). In the basis \( B_1 \) given in (2), we have \([e_2, e_3] = \sqrt{\lambda}e_1\). We consider the automorphism of Lie algebras \( P : \mathfrak{n} \to \mathfrak{n} \) given by

\[
P(X) = e_2; \ P(Y) = e_3 \quad \text{and} \quad P(Z) = \sqrt{\lambda}e_1,
\]

\( \phi \) its associated automorphism of Lie groups. The matrix of \( \phi^*(h_0) \) in \( \mathfrak{B}_0 \) is given by

\[
M(\phi^*(h_0), \mathfrak{B}_0) = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad \lambda > 0. \tag{12}
\]

(iii) \( L \) is of type \((002)\). In the basis \( B_1 \) given in (4), we have

\[
[e_1, e_2] = [e_1, e_1] = \frac{1}{2}(e_2 + e_3).
\]

We consider the automorphism of Lie algebras \( P : \mathfrak{n} \to \mathfrak{n} \) given by

\[
P(X) = e_1; \ P(Y) = e_2 \quad \text{and} \quad P(Z) = \frac{1}{2}(e_2 + e_3),
\]

\( \phi \) its associated automorphism of Lie groups. The matrix of \( \phi^*(h_0) \) is given by

\[
M(\phi^*(h_0), \mathfrak{B}_0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{\lambda} \\
0 & \frac{1}{\lambda} & 0
\end{pmatrix}. \tag{13}
\]

**Theorem 3.1.** Any Lorentzian left invariant metric on Nil is isometric to one of the three metrics whose matrices in \( \mathfrak{B}_0 \) are given by (11)-(13).

3.2. Lorentzian left invariant metric on SU(2)

We have one possibility, \( L \) is of type \( \text{diag}(a, b, c) \) with \( a > 0, b > 0 \) and \( c < 0 \). In the basis \( B_1 \) given in (2), we have

\[
[e_1, e_2] = -ce_3, \quad [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = be_2.
\]

We consider the automorphism \( P : \mathfrak{su}(2) \to \mathfrak{su}(2) \) given by

\[
P(\sigma_x) = \frac{2}{\sqrt{-cb}}e_1 = \sqrt{\mu}e_1; \quad P(\sigma_y) = \frac{2}{\sqrt{-ca}}e_2 = \sqrt{\mu_2}e_2 \quad \text{and} \quad P(\sigma_z) = -\frac{2}{\sqrt{ab}}e_3 = \sqrt{\mu_3}e_3,
\]

\( \phi \) its associated automorphism of Lie groups. The matrix of \( \phi^*(h_0) \) in \( \mathfrak{B}_0 \) is given

\[
M(\phi^*(h_0), \mathfrak{B}_0) = \begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_2 & 0 \\
0 & 0 & -\mu_3
\end{pmatrix}, \quad \mu_1 \geq \mu_2 > 0, \mu_3 > 0. \tag{14}
\]
We can suppose that $\mu_1 \geq \mu_2$ since $L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is an automorphism of $\mathfrak{su}(2)$ and

$$L'M(\phi^*(h_0), \mathbb{B}_0)L = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & -\mu_3 \end{pmatrix}.$$  

**Theorem 3.2.** Any Lorentzian left invariant metric on $SU(2)$ is isometric to the metric whose matrix in $\mathbb{B}_0$ is given by $(14)$.

### 3.3. Lorentzian left invariant metrics on $\text{PSL}(2, \mathbb{R})$

We have five possibilities:

1. $L$ is of type diag$\{a, b, c\}$ with $a > 0$, $b > 0$ and $c > 0$. In the basis $\mathbb{B}_1$ given in $(2)$, we have

   $$[e_1, e_2] = -ce_3, \quad [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = be_2.$$  

   We consider the automorphism $P : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ given by

   $$P(X_1) = \frac{2}{\sqrt{ab}}e_3 = \sqrt{\mu_1}e_3; \quad P(X_2) = \frac{2}{\sqrt{ab}}e_2 = \sqrt{\mu_2}e_2 \quad \text{and} \quad P(X_3) = -\frac{2}{\sqrt{ab}}e_1 = -\sqrt{\mu_3}e_1,$$

   $\phi$ its associated automorphism of Lie group. The matrix of $\phi^*(h_0)$ in $\mathbb{B}_0$ is given by

   $$M(\phi^*(h_0), \mathbb{B}_0) = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \mu_1 > 0, \mu_2 \geq \mu_3 > 0.$$  

   We can suppose that $\mu_2 \geq \mu_3$ since $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ is an automorphism of $\mathfrak{sl}(2, \mathbb{R})$ and

   $$L'M(\phi^*(h_0), \mathbb{B}_0)L = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$  

2. $L$ is of type diag$\{a, b, c\}$ with $a < 0$, $b > 0$ and $c < 0$.

   $$[e_1, e_2] = -ce_3, \quad [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = be_2.$$  

   We consider the automorphism $P : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ given by

   $$P(X_1) = \frac{2}{\sqrt{-cb}}e_1 = \sqrt{\mu_1}e_1; \quad P(X_2) = \frac{2}{\sqrt{-ab}}e_3 = \sqrt{\mu_2}e_3 \quad \text{and} \quad P(X_3) = -\frac{2}{\sqrt{-ac}}e_2 = -\sqrt{\mu_3}e_2$$

   $\phi$ its associated automorphism of Lie group. The matrix of $\phi^*(h_0)$ is given by

   $$M(\phi^*(h_0), \mathbb{B}_0) = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & -\mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \mu_1 > 0, \mu_2 > 0, \mu_3 > 0.$$  

3. $L$ is of type $\{a^2 \xi\}$ $a \neq 0$. In the basis $\mathbb{B}_1$ given in $(3)$, we have

   $$[e_1, e_2] = -\beta e_2 + a e_3, \quad [e_2, e_3] = a^2 e_1 \quad \text{and} \quad [e_3, e_1] = a e_2 - \beta e_3, \quad \beta \neq 0.$$  

   We distinguish three cases:
(a) \( \alpha > 0 \). We consider the automorphism \( P : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}) \) given by

\[
P(X_1) = \frac{2}{a \sqrt{\alpha^2 + \beta^2}} (\beta e_2 + \alpha e_1); \quad P(X_2) = \frac{2}{a \sqrt{\alpha}} e_2 = \quad \text{and} \quad P(X_3) = -\frac{2}{\sqrt{\alpha^2 + \beta^2}} e_1.
\]

\( \phi \) its associated automorphism of Lie group. The matrix of \( \phi'(h_0) \) is given by

\[
M(\phi'(h_0), B_0) = \frac{4}{a^2 \alpha \sqrt{\alpha^2 + \beta^2}} \begin{pmatrix}
\frac{\beta - \alpha^2}{\sqrt{\alpha^2 + \beta^2}} & \beta & 0 \\
\beta & \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} & 0 \\
0 & 0 & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}
\end{pmatrix}, \quad \alpha > 0, \beta > 0.
\]

We can suppose that \( \beta > 0 \) since \( L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \) is an automorphism of \( \mathfrak{sl}(2, \mathbb{R}) \) and

\[
L' M(\phi'(h_0), B_0) L = \frac{4}{a^2 \alpha \sqrt{\alpha^2 + \beta^2}} \begin{pmatrix}
\frac{\beta - \alpha^2}{\sqrt{\alpha^2 + \beta^2}} & -\beta & 0 \\
-\beta & \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} & 0 \\
0 & 0 & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}
\end{pmatrix}.
\]

(b) \( \alpha < 0 \). We consider the automorphism \( P : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}) \) given by

\[
P(X_1) = \frac{2}{a \sqrt{\alpha^2 + \beta^2}} e_2, \quad P(X_2) = -\frac{2}{\sqrt{\alpha^2 + \beta^2}} e_1 \quad \text{and} \quad P(X_3) = \frac{2}{a \sqrt{\alpha^2 + \beta^2}} (-\beta e_2 - \alpha e_3)
\]

\( \phi \) its associated automorphism of Lie group. The matrix of \( \phi'(h_0) \) is given by

\[
M(\phi'(h_0), B_0) = \frac{4}{a^2 \alpha \sqrt{\alpha^2 + \beta^2}} \begin{pmatrix}
-\frac{\beta - \alpha^2}{\sqrt{\alpha^2 + \beta^2}} & 0 & \beta \\
0 & 0 & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \\
\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} & 0 & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}
\end{pmatrix}, \quad \alpha < 0, \beta > 0.
\]

We can suppose \( \beta > 0 \) by using a same argument as in the precedent case.

(c) \( \alpha = 0 \). We consider the automorphism \( P : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}) \) given by

\[
P(X_1) = \frac{1}{\beta} e_2 + \frac{2}{a \beta} e_1, \quad P(X_2) = \frac{2}{\beta} e_1 \quad \text{and} \quad P(X_3) = \frac{1}{\beta} e_2 - \frac{2}{a \beta} e_3
\]

\( \phi \) its associated automorphism of Lie group. The matrix of \( \phi'(h_0) \) is given by

\[
M(\phi'(h_0), B_0) = \frac{4}{a^2 \beta} \begin{pmatrix}
a^4 - 4\beta^2 & 0 & a^4 + 4\beta^2 \\
0 & 4a^4 & 0 \\
a^4 + 4\beta^2 & 0 & a^4 - 4\beta^2
\end{pmatrix}.
\]

By putting \( u = a^4 - 4\beta^2 \) and \( v = a^4 + 4\beta^2 \), we get

\[
M(\phi'(h_0), B_0) = \frac{16}{v^2 - u^2} \begin{pmatrix} u & 0 & v \\ 0 & 2(u + v) & 0 \\ v & 0 & u \end{pmatrix}, \quad v > 0, v > u.
\]

4. Let \( \text{est of type } \{ab2\} \) with \( a \neq 0 \) and \( b \neq 0 \). In the basis \( B_1 \) given in (4), we have

\[
[e_1, e_2] = \frac{1}{2} e_2 + \left( \frac{1}{2} - b \right) e_3, \quad [e_2, e_3] = ae_1 \quad \text{and} \quad [e_3, e_1] = \left( b + \frac{1}{2} \right) e_2 + \frac{1}{2} e_3.
\]
We consider the automorphism $P : \text{sl}(2, \mathbb{R}) \to \text{sl}(2, \mathbb{R})$ given by

$$P(X_1) = \left( \frac{1}{2} + \frac{1}{4b} - \frac{2}{ab} \right) e_2 + \left( -\frac{1}{2} + \frac{1}{4b} - \frac{2}{ab} \right) e_3,$$
$$P(X_2) = \left( -\frac{1}{2} + \frac{1}{4b} - \frac{2}{ab} \right) e_2 + \left( \frac{1}{2} - \frac{1}{4b} - \frac{2}{ab} \right) e_3,$$
$$P(X_3) = -\frac{2}{b} e_1.$$

$\phi$ its associated automorphism of Lie group. The matrix of $\phi'(h_0)$ is given by

$$M(\phi'(h_0), \mathbb{B}_0) = \frac{1}{2ab} \begin{pmatrix} a - 8 & -a & 0 \\ -a & a + 8 & 0 \\ 0 & 0 & \frac{8a}{b} \end{pmatrix}, \quad a \neq 0, b \neq 0. \quad (20)$$

5. Let $\mathfrak{L}$ of type $\{a3\}$ with $a \neq 0$. In the basis $\mathbb{B}_1$ given in (5), we have

$$[e_1, e_2] = \frac{1}{\sqrt{2}} e_2 - a e_3, \quad [e_2, e_3] = a e_1 + \frac{1}{\sqrt{2}} e_2 \quad \text{and} \quad [e_3, e_1] = \frac{1}{\sqrt{2}} e_1 + a e_2 + \frac{1}{\sqrt{2}} e_3.$$

We consider the automorphism $P : \text{sl}(2, \mathbb{R}) \to \text{sl}(2, \mathbb{R})$ given by

$$P(X_1) = \frac{1}{a^2} \left( \sqrt{2} e_2 - 2 a e_3 \right), \quad P(X_2) = \frac{1}{a^2} \frac{1}{\sqrt{2a^2 + 1}} \left( 2a e_1 + \sqrt{2}(2a^2 + 1)e_2 \right),$$
$$P(X_3) = \frac{2 \sqrt{2}}{\sqrt{2a^2 + 1}} e_1.$$

$\phi$ its associated automorphism of Lie group. The matrix of $\phi'(h_0)$ in $\mathbb{B}_0$ is given by

$$M(\phi'(h_0), \mathbb{B}_0) = \frac{2}{a^4(1 + 2a^2)} \begin{pmatrix} 1 - 4a^4 & (1 + 2a^2)^2 & 0 \\ (1 + 2a^2)^2 & 4a^4 + 6a^2 + 1 & 2a^3 \sqrt{2} \\ 0 & 2a^3 \sqrt{2} & 4a^4 \end{pmatrix}, \quad a \neq 0. \quad (21)$$

**Theorem 3.3.** Any Lorentzian left invariant metric on $\text{PSL}(2, \mathbb{R})$ is isometric to one of the seven metrics whose matrices in $\mathbb{B}_0$ are given by (15)-(24).

### 3.4. Lorentzian left invariant metrics on Sol

We have five possibilities:

1. $L$ is of type $\text{diag}(\alpha, \beta, 0)$ with $\alpha > 0, \beta < 0$. In the basis $\mathbb{B}_1$ given in (2), we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = a e_1 \quad \text{and} \quad [e_3, e_1] = \beta e_2, \quad \alpha > 0, \beta < 0.$$

We consider the automorphism $P : \text{sol} \to \text{sol}$ given by

$$P(X_1) = -\frac{1}{\sqrt{-a \beta}} e_3, \quad P(X_2) = e_1 - \frac{\beta}{\sqrt{-a \beta}} e_2, \quad P(X_3) = e_1 + \frac{\beta}{\sqrt{-a \beta}} e_2.$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0'(h_0)$. We have

$$M(h_1, \mathbb{B}_0) = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \alpha - \beta & \alpha + \beta \\ 0 & \alpha + \beta & \alpha - \beta \end{pmatrix}.$$
We can reduce this metric by considering the automorphism \( Q : \text{sol} \rightarrow \text{sol} \) given by

\[
M(Q, B_0) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & \sqrt{\frac{2}{\alpha^2 \beta}} \\
0 & \sqrt{\frac{2}{\alpha^2 \beta}} & 0
\end{pmatrix}.
\]

Consider \( \phi \) the automorphism of \( \text{Sol} \) associated to \( Q \). The matrix of \( \phi^*(h_1) \) is given by

\[
M(\phi^*(h_1), B_0) = M(Q, B_0)^{t} M(h_1, B_0) M(Q, B_0) = \begin{pmatrix}
\frac{1}{\alpha^2 \beta} & 0 & 0 \\
0 & 1 & \frac{\alpha + \beta}{\alpha^2 \beta} \\
0 & \frac{\alpha + \beta}{\alpha^2 \beta} & 1
\end{pmatrix}.
\]

We put \( u = \alpha + \beta \) and \( v = \alpha - \beta \) and we get

\[
M(\phi^*(h_1), B_0) = \begin{pmatrix}
\frac{1}{\alpha^2 \beta} & 0 & 0 \\
0 & 1 & \frac{\alpha + \beta}{\alpha^2 \beta} \\
0 & \frac{\alpha + \beta}{\alpha^2 \beta} & 1
\end{pmatrix}, \quad v > 0, u < v.
\] (22)

2. \( L \) is of type diag\((a, b, c)\) with \( a > 0, b = 0 \) and \( c > 0 \). In the basis \( B_1 \) given in (2), we have

\[
[e_1, e_2] = \alpha e_3, \quad [e_2, e_3] = \beta e_1 \quad \text{and} \quad [e_3, e_1] = 0, \quad \alpha < 0, \beta > 0.
\]

We consider the automorphism \( P : \text{sol} \rightarrow \text{sol} \) given by

\[
P(X_1) = -\frac{1}{\sqrt{-\alpha \beta}} e_2, \quad P(X_2) = e_3 - \frac{\beta}{\sqrt{-\alpha \beta}} e_1, \quad P(X_3) = e_3 + \frac{\beta}{\sqrt{-\alpha \beta}} e_1
\]

\( \phi_0 \) its associated automorphism of Lie group and we put \( h_1 = \phi_0^*(h_0) \). We have

\[
M(h_1, B_0) = \frac{1}{\alpha} \begin{pmatrix}
-\frac{1}{\beta} & 0 & 0 \\
0 & -\alpha - \beta & -\alpha + \beta \\
0 & -\alpha + \beta & -\alpha - \beta
\end{pmatrix}.
\]

We can reduce this metric by considering the automorphism \( Q : \text{sol} \rightarrow \text{sol} \) given by

\[
M(Q, B_0) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & \sqrt{\frac{2}{\alpha^2 \beta}} \\
0 & \sqrt{\frac{2}{\alpha^2 \beta}} & 0
\end{pmatrix}.
\]

Consider \( \phi \) the automorphism of \( \text{Sol} \) associated to \( Q \). The matrix of \( \phi^*(h_1) \) in \( B_0 \) is given by

\[
M(\phi^*(h_1), B_0) = M(Q, B_0)^{t} M(h_1, B_0) M(Q, B_0) = \begin{pmatrix}
\frac{1}{\alpha^2 \beta} & 0 & 0 \\
0 & \frac{\alpha + \beta}{\alpha^2 \beta} & -1 \\
0 & -1 & \frac{\alpha + \beta}{\alpha^2 \beta}
\end{pmatrix}.
\]

We put \( u = \alpha + \beta \) and \( v = \beta - \alpha \) and we get

\[
M(\phi^*(h_1), B_0) = \begin{pmatrix}
\frac{1}{\alpha^2 \beta} & 0 & 0 \\
0 & \frac{\alpha + \beta}{\alpha^2 \beta} & -1 \\
0 & -1 & \frac{\alpha + \beta}{\alpha^2 \beta}
\end{pmatrix}, \quad v > 0, u < v.
\] (23)

3. \( L \) is of type \( [0; \zeta] \). In the basis \( B_1 \) given in (3), we have

\[
[e_1, e_2] = -\beta e_2 - \alpha e_3, \quad [e_2, e_3] = 0 \quad \text{and} \quad [e_3, e_1] = \alpha e_2 - \beta e_3, \quad \beta \neq 0.
\]

We distinguish two cases:
(a) $\alpha \neq 0$. We consider the automorphism $P : \text{sol} \rightarrow \text{sol}$ given by

$$P(X_1) = \frac{1}{\sqrt{\alpha^2 + \beta^2}}e_1, \quad P(X_2) = e_3 + \frac{\beta - \sqrt{\alpha^2 + \beta^2}}{\alpha}e_2, \quad P(X_3) = e_3 + \frac{\beta + \sqrt{\alpha^2 + \beta^2}}{\alpha}e_2$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0^*(h_0)$. We have

$$M(h_1, B_0) = \begin{pmatrix}
\frac{1}{\sqrt{\alpha^2 + \beta^2}} & 0 & 0 \\
0 & -\frac{2\beta(\beta - \sqrt{\alpha^2 + \beta^2})}{\alpha^2} & -2 \\
0 & \frac{2\beta(\beta + \sqrt{\alpha^2 + \beta^2})}{\alpha^2} & 0
\end{pmatrix}.$$  

We can suppose that $\beta > 0$ by using the automorphism of sol given by $L = \begin{pmatrix} -1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix}$.

We can reduce this metric by considering the automorphism $Q : \text{sol} \rightarrow \text{sol}$ given by

$$M(Q, B_0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & -\frac{\sqrt{\beta} \beta}{\alpha} & 0 \\
0 & 0 & -\frac{\sqrt{\beta} \beta}{2 \alpha}
\end{pmatrix}.$$  

Consider $\phi$ the automorphism of Sol associated to $Q$. The matrix of $\phi^*(h_1)$ in $B_0$ is given by

$$M(\phi^*(h_1), B_0) = \begin{pmatrix}
\frac{1}{\sqrt{\alpha^2 + \beta^2}} & 0 & 0 \\
0 & \frac{1}{\alpha} & 1 \\
0 & 1 & 1
\end{pmatrix}, \quad u = \alpha^2 > 0, v = \beta^2 > 0. \quad (24)$$

(b) $\alpha = 0$. We consider the automorphism $P : \text{sol} \rightarrow \text{sol}$ given by

$$P(X_1) = \frac{1}{\beta}e_1, \quad P(X_2) = e_3, \quad P(X_3) = e_2$$

$\phi$ its associated automorphism of Lie group. We have

$$M(\phi^*(h_0), B_0) = \begin{pmatrix}
\frac{1}{\alpha} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad u = \beta^2 > 0 \quad (25)$$

4. $L$ is of type $[\alpha 02]$ with $\alpha < 0$. In the basis $B_1$ given in $[4]$, we have

$$[e_1, e_2] = \frac{1}{2} e_2 + \frac{1}{2} e_3, \quad [e_2, e_3] = -a^2 e_1 \quad \text{and} \quad [e_3, e_1] = \frac{1}{2} e_2 + \frac{1}{2} e_3.$$  

We consider the automorphism $P : \text{sol} \rightarrow \text{sol}$ given by

$$P(X_1) = \frac{a}{\sqrt{2}} e_3, \quad P(X_2) = a \sqrt{2} e_1 + e_2 + e_3, \quad P(X_3) = -a \sqrt{2} e_1 + e_2 + e_3$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0^*(h_0)$. We have

$$M(h_1, B_0) = \begin{pmatrix}
2 & a \sqrt{2} & a \sqrt{2} \\
a \sqrt{2} & -2a^4 & 2a^4 \\
a \sqrt{2} & 2a^4 & -2a^4
\end{pmatrix}.$$  

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We can reduce this metric by considering the automorphism $Q : \text{sol} \rightarrow \text{sol}$ given by

$$M(Q, B_0) = \begin{pmatrix}
\frac{1}{\sqrt{2(1 + 2a^2)}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2(1 + 2a^2)}} & 0 \\
\frac{1}{\sqrt{2a^2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}. $$

Consider $\phi$ the automorphism of Sol associated to $Q$. The matrix of $\phi^*(h_1)$ is given by

$$M(\phi^*(h_1), B_0) = \begin{pmatrix} 0 & 0 & -\frac{b}{2} \\
0 & 1 & 1 \\
-\frac{2}{b} & 1 & 1
\end{pmatrix}, \quad b = a^2 > 0. \quad (26)$$

5. $L$ is of type $[0b2]$ with $b \neq 0$. In the basis $B_1$ given in [4], we have

$$[e_1, e_2] = \frac{1}{2}e_2 + \left(\frac{1}{2} - b\right)e_3, \quad [e_2, e_3] = 0 \quad \text{and} \quad [e_3, e_1] = \left(\frac{b + 1}{2}\right)e_2 + \frac{1}{2}e_3.$$

We consider the automorphism $P : \text{sol} \rightarrow \text{sol}$ given by

$$P(X_1) = \frac{2}{b}e_1, \quad P(X_2) = (2b + 1)e_2 + (1 - 2b)e_3, \quad P(X_3) = e_2 + e_3.$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0^*(h_0)$. We have

$$M(h_1, B_0) = \begin{pmatrix} \frac{1}{\sqrt{b}} & 0 & 0 \\
0 & 8b & 4b \\
0 & 4b & 0
\end{pmatrix}. $$

We can reduce this metric by considering the automorphism $Q : \text{sol} \rightarrow \text{sol}$ given by

$$M(Q, B_0) = \begin{pmatrix} 1 & 0 & 0 \\
0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2a^2}
\end{pmatrix}. $$

Consider $\phi$ the automorphism of Sol associated to $Q$. The matrix of $\phi^*(h_1)$ is given by

$$M(\phi^*(h_1), B_0) = \begin{pmatrix} \lambda^2 & 0 & 0 \\
0 & \lambda & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \lambda = \frac{1}{b} \neq 0. \quad (27)$$

6. $L$ est of type $[03]$. In the basis $B_1$ given in [5], we have

$$[e_1, e_2] = \frac{1}{\sqrt{2}}e_2, \quad [e_2, e_3] = \frac{1}{\sqrt{2}}e_2 \quad \text{and} \quad [e_3, e_1] = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_3.$$

We consider the automorphism $P : \text{sol} \rightarrow \text{sol}$ given by

$$P(X_1) = \sqrt{2}e_1, \quad P(X_2) = e_2 \quad \text{and} \quad P(X_3) = e_3.$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0^*(h_0)$. We have

$$M(h_1, B_0) = \begin{pmatrix} 2 & 0 & \sqrt{2} \\
0 & 1 & 0 \\
\sqrt{2} & 0 & 0
\end{pmatrix}. $$
We can reduce this metric by considering the automorphism $Q : \text{sol} \rightarrow \text{sol}$ given by

$$M(Q, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sqrt{a}}{\sqrt{b}} & 0 & \frac{\sqrt{c}}{\sqrt{b}} \end{pmatrix}.$$ 

Consider $\phi$ the automorphism of $\text{Sol}$ associated to $Q$. The matrix of $\phi^*(h_1)$ is given by

$$M(\phi^*(h_1), \mathbb{B}_0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (28)$$

**Theorem 3.4.** Any Lorentzian left invariant metric on $\text{Sol}$ is isometric to one of the six metrics whose matrices in $\mathbb{B}_0$ are given by (22)–(28).

### 3.5. Lorentzian left invariant metrics on $\widetilde{E}_0(2)$

There are three possibilities:

1. $L$ is of type $\text{diag}(a, b, c)$ with $a > 0$, $b > 0$ and $c = 0$. In the basis $\mathbb{B}_1$ given in (2), we have

$$[e_1, e_2] = 0, \ [e_2, e_3] = a e_1 \quad \text{and} \quad [e_3, e_1] = b e_2, \quad \alpha > 0, \beta > 0.$$ 

We consider the automorphism $P : e_0(2) \rightarrow e_0(2)$ given by

$$P(X_1) = \frac{1}{\sqrt{a \beta}} e_3, \ P(X_2) = e_1 - \frac{\beta}{\sqrt{a \beta}} e_2, \ P(X_3) = e_1 + \frac{\beta}{\sqrt{a \beta}} e_2$$

$\phi_0$ its associated automorphism of Lie group and we put $h_1 = \phi_0^*(h_0)$. We have

$$M(h_1, \mathbb{B}_0) = \frac{1}{\alpha} \begin{pmatrix} -\frac{1}{\alpha} & 0 & 0 \\ 0 & \alpha + \beta & \alpha - \beta \\ 0 & \alpha - \beta & \alpha + \beta \end{pmatrix}.$$ 

We can reduce this metric by considering the automorphism $Q : e_0(2) \rightarrow e_0(2)$ given by

$$M(Q, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{\alpha \beta}} & 0 & \frac{\sqrt{c}}{\sqrt{b}} \\ \frac{1}{\sqrt{\alpha \beta}} & \frac{\sqrt{c}}{\sqrt{b}} & \frac{\sqrt{c}}{\sqrt{b}} \end{pmatrix}.$$ 

Consider $\phi$ the automorphism of $\widetilde{E}_0(2)$ associated to $Q$. The matrix of $\phi^*(h_1)$ is given by

$$M(\phi^*(h_1), \mathbb{B}_0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & v \end{pmatrix}, \ u = \alpha \beta > 0, \ v = \beta^2 > 0. \quad (29)$$

2. $L$ is of type $\text{diag}(a, b, c)$ with $a > 0$, $b = 0$ and $c < 0$. In the basis $\mathbb{B}_1$ given in (2), we have

$$[e_1, e_2] = \alpha e_3, \ [e_2, e_3] = \beta e_1 \quad \text{and} \quad [e_3, e_1] = 0, \ \alpha > 0, \beta > 0.$$ 

We consider the automorphism $P : e_0(2) \rightarrow e_0(2)$ given by

$$P(X_1) = \frac{1}{\sqrt{a \beta}} e_2, \ P(X_2) = e_3 - \frac{\beta}{\sqrt{a \beta}} e_1, \ P(X_3) = e_3 + \frac{\beta}{\sqrt{a \beta}} e_1$$

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\( \phi_0 \) its associated automorphism of Lie group and we put \( h_1 = \phi_0'(h_0) \). We have
\[
M(h_1, B_0) = \frac{1}{\alpha} \begin{pmatrix}
\beta & 0 & 0 \\
0 & -\alpha + \beta & -\alpha - \beta \\
0 & -\alpha - \beta & -\alpha + \beta
\end{pmatrix}.
\]

We can reduce this metric by considering the automorphism \( Q : e_0(2) \rightarrow e_0(2) \) given by
\[
M(Q, B_0) = \begin{pmatrix}
\frac{1}{2\sqrt{a}} & 0 & \frac{\sqrt{a}}{2} \\
0 & \frac{\sqrt{a}}{2} & 0 \\
\frac{\sqrt{a}}{2} & 0 & -\frac{\sqrt{a}}{2}
\end{pmatrix}.
\]

Consider \( \phi \) the automorphism of \( \widetilde{E}_0(2) \) associated to \( Q \). The matrix of \( \phi^*(h_1) \) is given by
\[
M(\phi^*(h_1), B_0) = \begin{pmatrix}
0 & -1 & 0 \\
-1 & -u & 0 \\
0 & 0 & v
\end{pmatrix}, \quad u = \alpha \beta > 0, v = \beta^2 > 0.
\] (30)

3. \( L \) is of type \( \{a02 \} \) with \( a > 0 \). In the basis \( B_1 \) given in (1), we have
\[
[e_1, e_2] = \frac{1}{2} e_2 + \frac{1}{2} e_3, [e_2, e_3] = a^* e_1 \quad \text{and} \quad [e_3, e_1] = \frac{1}{2} e_2 + \frac{1}{2} e_3.
\]

We consider the automorphism \( P : e_0(2) \rightarrow e_0(2) \) given by
\[
P(X_1) = \frac{\sqrt{a}}{a} e_3, P(X_2) = a \sqrt{2} e_1 + e_2 + e_3, P(X_3) = -a \sqrt{2} e_1 + e_2 + e_3
\]

\( \phi_0 \) its associated automorphism of Lie group and we put \( h_1 = \phi_0'(h_0) \). We have
\[
M(h_1, B_0) = -\frac{1}{a^2} \begin{pmatrix}
2 & a \sqrt{2} & a \sqrt{2} \\
a \sqrt{2} & -2a^4 & 2a^4 \\
a \sqrt{2} & 2a^4 & -2a^4
\end{pmatrix}.
\]

We can reduce this metric by considering the automorphism \( Q : e_0(2) \rightarrow e_0(2) \) given by
\[
M(Q, B_0) = \begin{pmatrix}
\frac{1}{2\sqrt{a}} & 0 & \frac{\sqrt{a}}{2} \\
0 & \frac{\sqrt{a}}{2} & 0 \\
\frac{\sqrt{a}}{2} & 0 & -\frac{\sqrt{a}}{2}
\end{pmatrix}.
\]

Consider \( \phi \) the automorphism of \( \widetilde{E}_0(2) \) associated to \( Q \). The matrix of \( \phi^*(h_1) \) is given by
\[
M(\phi^*(h_1), B_0) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & u
\end{pmatrix}, \quad u = a^4 > 0.
\] (31)

**Theorem 3.5.** Any Lorentzian left invariant metric on \( \widetilde{E}_0(2) \) is isometric to one of the three metrics whose matrices in \( B_0 \) are given by (29)–(31).

4. **Curvature of Lorentzian left invariant metrics on unimodular three dimensional Lie groups**

In this section, we give for each metric whose matrix is given by one of the formulas (29)–(31) its Ricci tensor, its signature and the scalar curvature.
4.1. Curvature of Lorentzian left invariant metrics on Nil

There are three classes of metrics on Nil given by the formulas (11)-(13). Here are their Ricci curvature and scalar curvature.

Proposition 4.1. 1. The Ricci curvature and the scalar curvature of the metric (11) are given by

\[
M(\text{ric}, B_0) = \frac{1}{2} \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^2
\end{pmatrix}, \quad s = \frac{1}{2} \lambda.
\]

In particular, \( \text{ric} > 0 \) and \( s > 0 \).

2. The Ricci curvature and the scalar curvature of the metric (12) are given by

\[
M(\text{ric}, B_0) = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad s = \frac{1}{2} \lambda.
\]

In particular, \( \text{ric} > 0 \) and \( s > 0 \).

3. The metric (13) is flat.

4.2. Curvature of Lorentzian left invariant metrics on SU(2)

There is one class of metrics on SU(2) given by the formula (14). Here is its Ricci curvature and scalar curvature.

Proposition 4.2. The Ricci curvature and the scalar curvature of the metric on SU(2) given by (14) are given by

\[
M(\text{ric}, B_0) = \text{diag} \left[ -\frac{2(\mu_1 - \mu_2 - \mu_3)(\mu_1 + \mu_2 + \mu_3)}{\mu_3}, \frac{2(\mu_1 + \mu_2 + \mu_3)(\mu_1 - \mu_2 + \mu_3)}{\mu_1 \mu_2}, -\frac{2(\mu_1 - \mu_2 - \mu_3)(\mu_1 - \mu_2 + \mu_3)}{\mu_1 \mu_2} \right],
\]

\[
s = \frac{2((\sqrt{\mu_1} + \sqrt{\mu_2})^2 + \mu_3)((\sqrt{\mu_1} - \sqrt{\mu_2})^2 + \mu_3)}{\mu_1 \mu_2 \mu_3} > 0.
\]

Moreover, the signature of \( \text{ric} \) is given by

\[
\text{sign}(\text{ric}) = \begin{cases} 
(+, +, +) & \text{if } \mu_1 < \mu_2 + \mu_3, \\
(+, +, -) & \text{if } \mu_1 > \mu_2 + \mu_3, \\
(+, 0, 0) & \text{if } \mu_1 = \mu_2 + \mu_3.
\end{cases}
\]

4.3. Curvature of Lorentzian left invariant metrics on \( \tilde{\text{PSL}}(2, \mathbb{R}) \)

There are seven classes of metrics on \( \tilde{\text{PSL}}(2, \mathbb{R}) \) given by the formulas (15)-(21). Here are their Ricci curvature and scalar curvature.

Proposition 4.3. The Ricci curvature and the scalar curvature of the metric (15) on \( \tilde{\text{PSL}}(2, \mathbb{R}) \) are given by

\[
M(\text{ric}, B_0) = \text{diag} \left[ -\frac{2(\mu_1^2 - (\mu_2 - \mu_3)^2)}{\mu_2 \mu_3}, -\frac{2(\mu_2^2 - (\mu_1 - \mu_3)^2)}{\mu_1 \mu_3}, -\frac{2(\mu_3^2 - (\mu_2 - \mu_1)^2)}{\mu_1 \mu_2} \right],
\]

\[
s = \frac{2((\sqrt{\mu_1} + \sqrt{\mu_2})^2 - \mu_3)((\sqrt{\mu_1} - \sqrt{\mu_2})^2 - \mu_3)}{\mu_1 \mu_2 \mu_3}.
\]

When \( \mu_1 = \mu_2 = \mu_3 = \mu \) then \( \text{ric} = -\frac{2}{\mu^2} h \) and in fact the metric has constant sectional curvature \( \frac{-1}{\mu^2} \). The possible signatures of the Ricci curvature are \((+, +, +), (+, +, -), (+, 0, 0), (-, 0, 0)\).
Proposition 4.4. The Ricci curvature and the scalar curvature of the metric \((16)\) on \(\text{PSL}(2, \mathbb{R})\) are given by

\[
M(\text{ric}, B_0) = \text{diag}
\begin{bmatrix}
\frac{2(\mu_1 - \mu_2 - \mu_3)(\mu_1 + \mu_2 + \mu_3)}{\mu_2 \mu_3}, & \frac{2(\mu_1 + \mu_2 + \mu_3)(\mu_1 - \mu_2 + \mu_3)}{\mu_1 \mu_3}, & \frac{2(\mu_1 - \mu_2)(\mu_1 - \mu_2 + \mu_3)}{\mu_1 \mu_2} \\
\end{bmatrix},
\]

\[
s = \frac{2((\sqrt{\mu_1} + \sqrt{\mu_2})^2 + \mu_3)((\sqrt{\mu_1} - \sqrt{\mu_2})^2 + \mu_3)}{\mu_1 \mu_2 \mu_3} > 0.
\]

Moreover, the signature of \(\text{ric}\) is given by

\[
\text{sign}(\text{ric}) = \begin{cases}
(+, +, +) & \text{if } \mu_1 < \mu_2 - \mu_3 < \mu_2 + \mu_3, \\
(+, -, -) & \text{if } \mu_1 > \mu_2 - \mu_3, \\
(-, 0, 0) & \text{if } \mu_1 = \mu_2 - \mu_3 \text{ or } \mu_1 = \mu_2 + \mu_3.
\end{cases}
\]

Proposition 4.5. The Ricci curvature and the scalar curvature of the metric \((17)\) are given by

\[
M(\text{ric}, B_0) = \begin{bmatrix}
\frac{2a^2 - 2a^2(\alpha^2 - a^2 + 4\beta^2)}{a^2 \alpha^3} & \frac{2a^2 - 4\alpha^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}} & 0 \\
\frac{2a^2 - 2a^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}} & 0 & 0 \\
0 & 0 & -\frac{2a^2 - 4\alpha^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}}
\end{bmatrix},
\]

\[
s = \frac{1}{2} a^4 - 2a^3 \alpha - 2\beta^2.
\]

The signature of \(\text{ric}\) is \((+, -, -)\) if \(a^2 \neq 2a\) and \((-0, 0)\) if \(a^2 = 2a\). The operator of Ricci is of type \([a \alpha^2]\).

Proposition 4.6. The Ricci curvature and the scalar curvature of the metric \((18)\) are given by

\[
M(\text{ric}, B_0) = \begin{bmatrix}
\frac{2a^2 - a^2 \alpha}{a^2} & 0 & \frac{2a^2 - 4\alpha^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}} \\
0 & \frac{2a^2 - 4\alpha^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}} & 0 \\
\frac{2a^2 - 2a^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}} & 0 & \frac{2a^2 - 2a^2 \beta}{a^2 \sqrt{\alpha^2 + \beta^2}}
\end{bmatrix},
\]

\[
s = \frac{1}{2} a^4 + 2a^3 \alpha - 2\beta^2.
\]

The signature of \(\text{ric}\) is \((+, -, -)\). The operator of Ricci is of type \([a \alpha^2]\).

Proposition 4.7. The Ricci curvature and the scalar curvature of the metric \((19)\) are given by

\[
M(\text{ric}, B_0) = \begin{bmatrix}
\frac{4a}{a^2} & 0 & \frac{4a}{a^2} \\
\frac{4a}{a^2} & 0 & \frac{4a}{a^2 - 2a} \\
\frac{4a}{a^2 - 2a} & \frac{4a}{a^2} & 0
\end{bmatrix},
\]

\[
s = \frac{u}{2}.
\]

The signature of \(\text{ric}\) is \((+, -, -)\).

Proposition 4.8. The Ricci curvature and the scalar curvature of the metric \((20)\) are given by

\[
M(\text{ric}, B_0) = \frac{1}{4b} \begin{bmatrix}
(a + 2b - 8)(a - 2b) & 4b^2 - a^2 & 0 \\
4b^2 - a^2 & (a + 2b + 8)(a - 2b) & 0 \\
0 & 0 & -\frac{4b}{a}
\end{bmatrix},
\]

\[
s = \frac{1}{2} a(a - 4b).
\]

The Ricci curvature has signature \((+, -, -)\) if \(a \neq 2b\) and \((-0, 0)\) if \(a = 2b\).
Proposition 4.9. The Ricci curvature and the scalar curvature of the metric (21) are given by

\[ M(\text{ric}, B_0) = \begin{pmatrix}
\frac{2u^2}{v^2} & 0 & 0 \\
0 & -\frac{1}{2}u^2 & -\frac{1}{2}uv \\
0 & -\frac{1}{2}uv & -\frac{1}{2}u^2
\end{pmatrix}, \quad s = \frac{1}{2}v^2.
\]

The operator Ric is of type \( a3 \). From the relations \( \det(M(\text{ric}, B_0)) = 8 \) and \( \text{tr}(M(\text{ric}, B_0)) = -\frac{2(\alpha^2+9)}{\alpha^2} \), we deduce that the signature of \( \text{ric} \) is \((+, -, -)\).

4.4. Curvature of Lorentzian left invariant metrics on \( \text{Sol} \)

There are six classes of metrics on \( \text{Sol} \) given by the formulas (22) – (28). Here are their Ricci curvature and scalar curvature.

Proposition 4.10. The Ricci curvature and the scalar curvature of the metric (22) are given by

\[ M(\text{ric}, B_0) = \begin{pmatrix}
\frac{2u^2}{\sqrt{\alpha}v} & 0 & 0 \\
0 & -\frac{1}{2}u^2 & -\frac{1}{2}uv \\
0 & -\frac{1}{2}uv & -\frac{1}{2}u^2
\end{pmatrix}, \quad s = \frac{1}{2}v^2.
\]

The signature of \( \text{ric} \) is given by

\[ \text{sign}(\text{ric}) = \begin{cases}
(-, 0, 0) & \text{if } u = 0, \\
(-, -, 0) & \text{if } u = -v, \\
(-, -, +) & \text{if } u > 0, \text{ or } u < 0, u > v, \\
(-, -, -) & \text{if } u < 0, u < v.
\end{cases}
\]

Proposition 4.11. The Ricci curvature and the scalar curvature of the metric (23) are given by

\[ M(\text{ric}, B_0) = \begin{pmatrix}
\frac{2u^2}{\sqrt{\alpha}v^2} & 0 & 0 \\
0 & -\frac{1}{2}u^2 & -\frac{1}{2}uv \\
0 & -\frac{1}{2}uv & -\frac{1}{2}u^2
\end{pmatrix}, \quad s = \frac{1}{2}v^2.
\]

This metric is flat if \( u = 0 \). For \( u \neq 0 \), the signature of \( \text{ric} \) is given by

\[ \text{sign}(\text{ric}) = \begin{cases}
(+, -, -) & \text{if } u > 0, \\
(+, +, +) & \text{if } u < 0, u > v, \\
(+, +, -) & \text{if } u < 0, u < v.
\end{cases}
\]

Proposition 4.12. The Ricci curvature and the scalar curvature of the metric (24) are given by

\[ M(\text{ric}, B_0) = \begin{pmatrix}
\frac{2u^2}{\sqrt{\alpha}v^2} & 0 & 0 \\
0 & 2v & 2v \\
0 & 2v & -2u
\end{pmatrix}, \quad s = -2v.
\]

The Ricci curvature has signature \((+, -, -)\) and the Ricci operator is of type \([a, z3]\).

Proposition 4.13. The Ricci curvature and the scalar curvature of the metric (25) are given by

\[ M(\text{ric}, B_0) = \begin{pmatrix}
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s = -2u.
\]
Proposition 4.14. The Ricci curvature and the scalar curvature of the metric \((26)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
-2 & b & 0 \\
 b & -\frac{1}{2}b^2 & -\frac{1}{2}b^2 \\
 0 & -\frac{1}{2}b^2 & -\frac{1}{2}b^2
\end{pmatrix}, \quad s = \frac{1}{2}b^2.
\]

The Ricci operator is of type \(\{ab2\}\). From the relations \(\det(M(\text{ric}, B_0)) = \frac{4}{7}\) and \(\text{tr}(M(\text{ric}, B_0)) = -2 - b^2\), we deduce that the signature of \(\text{ric}\) is \((+, -, -)\).

Proposition 4.15. The Ricci curvature and the scalar curvature of the metric \((27)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{2}{3} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s = 0.
\]

We have \(\text{Ric}^2 = 0\).

Proposition 4.16. The Ricci curvature and the scalar curvature of the metric \((28)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s = 0.
\]

We have \(\text{Ric}^2 = 0\) and this metric is semi-symmetric.

4.5. Curvature of Lorentzian left invariant metrics on \(\tilde{E}_0(2)\)

There are seven classes of metrics on \(\tilde{E}_0(2)\) given by the formulas \((29)-(31)\). Here are their Ricci curvature and scalar curvature.

Proposition 4.17. The Ricci curvature and the scalar curvature of the metric \((29)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
\frac{x-u}{2v} & \frac{y^2-u^2}{2v} & 0 \\
\frac{y^2-u^2}{2v} & \frac{x-u}{2v} & 0 \\
0 & 0 & \frac{y^2-x^2}{2v}
\end{pmatrix}, \quad s = \frac{(u - v)^2}{2v}.
\]

This metric is flat when \(u = v\),

\[
\text{sign}(\text{ric}) = (+, +, -) \text{ if } u < v \quad \text{and} \quad \text{sign}(\text{ric}) = (+, -, -) \text{ if } u > v.
\]

Proposition 4.18. The Ricci curvature and the scalar curvature of the metric \((30)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
\frac{x+y}{2v} & \frac{y^2-x^2}{2v} & 0 \\
\frac{y^2-x^2}{2v} & \frac{x+y}{2v} & 0 \\
0 & 0 & \frac{y^2-x^2}{2v}
\end{pmatrix}, \quad s = \frac{(u + v)^2}{2v}.
\]

The signature of \(\text{ric}\) is given by

\[
\text{sign}(\text{ric}) = (+, -, -) \text{ if } u < v, \text{sign}(\text{ric}) = (+, +, +) \text{ if } u > v \quad \text{and} \quad \text{sign}(\text{ric}) = (+, 0, 0) \text{ if } u = v.
\]

Proposition 4.19. The Ricci curvature and the scalar curvature of the metric \((31)\) are given by

\[
M(\text{ric}, B_0) = \begin{pmatrix}
1 & \mu & 0 \\
\mu & 0 & 0 \\
0 & 0 & -\mu
\end{pmatrix}, \quad s = \frac{u}{2}.
\]

The operator of Ricci is of type \(\{ab2\}\) and the signature of \(\text{ric}\) is \((+, -, -)\).
The following Table gives the possible signatures of Ricci curvature of Lorentzian left invariant metrics on unimodular three dimensional Lie groups and the metrics realizing these signatures.

| Signature of Ricci curvature | Metrics realizing this signature | Remarks |
|------------------------------|---------------------------------|---------|
| (0, 0, 0)                    | (13), (23) \( \mu = 0 \), (29) \( \mu = v \) | These metrics are flat |
| (+, +, +)                    | (11), (12), (14), (15), (16), (23), (30) | |
| (−, −, −)                    | (23) | |
| (+, −, −)                    | (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (29), (30), (31) | The metric (15) has negative constant sectional curvature for \( \mu_1 = \mu_2 = \mu_3 \), The metric (21) is a shrinking Ricci soliton |
| (+, +, −)                    | (23), (29) | |
| (−, −, 0)                    | (23) | |
| (+, 0, 0)                    | (13), (15), (23) | |
| (−, 0, 0)                    | (15), (19), (21), (22), (23), (25), (27), (28) | The metrics (27) and (28) are steady Ricci soliton and semi-symmetric not locally symmetric, \( (20) \ a = b \neq 0 \) is a shrinking Ricci soliton |

The following proposition gives the type of the Ricci operators.

**Proposition 4.20.**
1. The metrics (17), (18) and (24) have their Ricci operators of type \( \{a\bar{z}\} \).
2. The metric (21) has its Ricci operator of type \( \{a3\} \).
3. The metrics (27) and (28) have their Ricci operators of type \( \{002\} \) and satisfies \( \text{Ric}^2 = 0 \).
4. The metrics (26) and (31) have their Ricci operators of type \( \{ab2\} \).
5. All the others have their Ricci operators diagonalizable.

Lorentzian left invariant metrics on unimodular three dimensional Lie groups which are of constant curvature, Einstein, locally symmetric, semi-symmetric not locally symmetric or Ricci soliton have been determined in \([1, 2, 3]\) by giving their Lie algebras as in (2)-(5). Our study permits a precise description of these metrics. We give their corresponding metrics in our list.

**Theorem 4.1** \([3]\). Let \( h \) be a Lorentzian left invariant metric on a unimodular three dimensional Lie group. Then the following assertions are equivalent:
1. The metric \( h \) is locally symmetric.
2. The metric \( h \) is Einstein.
3. The metric \( h \) has constant sectional curvature.

Moreover, a metric satisfying one of these assertions is either flat and is isometric to the metric (13), (23) \( \mu = 0 \) or (29), \( \mu = v \); or it has a negative constant sectional curvature and is isometric to (15), \( \mu_1 = \mu_2 = \mu_3 \).

**Theorem 4.2** \([1, 3]\). The metrics (27) and (28) are the unique, up to an automorphism, Lorentzian semi-symmetric not locally symmetric left invariant metrics on a unimodular three dimensional Lie group.

**Theorem 4.3** \([2]\).
1. The metric (27) satisfies the relation
   \[
   L_X h + \text{ric}(h) = 0, \quad X = -\frac{1}{\lambda} X_1
   \]

   and hence it is a steady Ricci soliton.
2. The metric (28) satisfies the relation
   \[
   L_X h + \text{ric}(h) = 0, \quad X = -X_3
   \]

   and hence it is a steady Ricci soliton.
3. The metric \( a = b \neq 0 \) satisfies the relation

\[
L_X h + \text{ric}(h) = -\frac{1}{2} b^2 h, \quad X = \frac{b^2}{4} X_3
\]

and hence it is a shrinking Ricci soliton.

4. The metric \( a \neq 0 \) satisfies the relation

\[
L_X h + \text{ric}(h) = -\frac{1}{2} a^2 h, \quad X = \frac{a}{2 \sqrt{2}} X_1 + \frac{(2a^3 - a) \sqrt{2}}{4 \sqrt{2a^2 + 1}} X_2 - \frac{a^2}{\sqrt{2a^2 + 1}} X_3
\]

and hence it is a shrinking Ricci soliton.

Moreover, these metrics are the only Lorentzian left invariant Ricci solitons, up to automorphism, on three dimensional unimodular Lie groups.

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