A stronger monotonicity inequality of quantum relative entropy:
A unifying approach via Rényi relative entropy

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Abstract
In this paper, a stronger monotonicity inequality of quantum relative entropy is derived by employing a property of \( \alpha \)-Rényi relative entropy. In view of this inequality, a unifying treatment towards improvement of some quantum entropy inequalities is developed. In particular, an emphasis is put on a lower bound of quantum conditional mutual information because this lower bound gives a Pinsker-like lower bound for strong subadditivity inequality of von Neumann entropy.

Keywords: Relative entropy; Quantum channel; Strong subadditivity; Rényi relative entropy

1 Introduction
To begin with, we fix some notations that will be used in this context. Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space. A quantum state \( \rho \) on \( \mathcal{H} \) is a positive semi-definite operator of trace one, in particular, for each unit vector \( |\psi\rangle \in \mathcal{H} \), the operator \( \rho = |\psi\rangle\langle\psi| \) is said to be a pure state. The set of all quantum states on \( \mathcal{H} \) is denoted by \( D(\mathcal{H}) \). For each quantum state \( \rho \in D(\mathcal{H}) \), its von Neumann entropy is defined by \( S(\rho) := -\text{Tr} (\rho \log \rho) \). The relative entropy of two mixed states \( \rho \) and \( \sigma \) is defined by

\[
S(\rho||\sigma) := \begin{cases} 
\text{Tr} (\rho (\log \rho - \log \sigma)) , & \text{if supp}(\rho) \subseteq \text{supp}(\sigma), \\
+\infty , & \text{otherwise}.
\end{cases} \tag{1.1}
\]

A quantum channel \( \Phi \) on \( \mathcal{H} \) is a trace-preserving completely positive linear map defined over the set \( D(\mathcal{H}) \). It follows that there exists linear operators \( \{K_{\mu}\}_{\mu} \) on \( \mathcal{H} \) such that \( \sum_{\mu} K_{\mu}^+ K_{\mu} = \mathbb{1} \) and \( \Phi = \sum_{\mu} \text{Ad} K_{\mu} \), that is, for each quantum state \( \rho \), we have the Kraus representation \( \Phi(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^+ \). A well-known property of quantum relative entropy is its monotonicity under generic quantum channels. That is

\[
S(\rho||\sigma) \geq S(\Phi(\rho)||\Phi(\sigma)) . \tag{1.2}
\]

The condition of equality is an interesting and important subject. An extremely important result, i.e. the saturation of monotonicity inequality of relative entropy under a generic quantum channel, a famous result in quantum information theory, is due to Petz [12].

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Proposition 1.1. Let \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \), \( \Phi \) be a quantum channel defined over \( \mathcal{H} \). If \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \), then

\[
S(\rho||\sigma) = S(\Phi(\rho)||\Phi(\sigma)) \quad \text{if and only if} \quad \Phi^* \circ \Phi(\rho) = \rho,
\]

where \( \Phi^* = \text{Ad}_{\rho^{1/2}} \circ \Phi^* \circ \text{Ad}_{\Phi(\rho)^{-1/2}} \), and \( \Phi^* \) is the dual of \( \Phi \) with respect to Hilbert-Schmidt inner product over the operator space on \( \mathcal{H} \).

The celebrated strong subadditivity (SSA) inequality of quantum entropy, proved by Lieb and Ruskai in [3],

\[
S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}),
\]

is a ubiquitous result in quantum information theory. In fact, SSA is equivalent to monotonicity inequality of quantum relative entropy, that is they can imply each other. Based on SSA, a new concept—conditional mutual information, is proposed by mimicking classical probability theory. It measures the correlations of two quantum systems relative to a third: Given a tripartite state \( \rho_{ABC} \in \mathcal{D}(\mathcal{H}_{ABC}) \), where \( \mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \), it is defined as

\[
I(A : C|B)_\rho := S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B).
\]

Clearly conditional mutual information is nonnegative by SSA. Thus getting a lower bound of this conditional mutual information is pursued all the time. Hence characterization of vanishing conditional mutual information is a first step to this problem.

Ruskai is the first one to discuss the equality condition of SSA, i.e. vanishing conditional mutual information. By analyzing the equality condition of Golden-Thompson inequality, she obtained the following characterization [4]:

\[
I(A : C|B)_\rho = 0 \iff \log \rho_{ABC} + \log \rho_B = \log \rho_{AB} + \log \rho_{BC}.
\]

Later on, using the relative modular approach established by Araki, Petz gave another characterization of the equality condition of SSA [5]:

\[
I(A : C|B)_\rho = 0 \iff \rho_{ABC}^{it} \rho_{BC}^{-it} = \rho_{AB}^{it} \rho_{BC}^{-it} \quad (\forall t \in \mathbb{R}),
\]

where \( i = \sqrt{-1} \) is the imaginary unit.

Hayden et al. in [6] showed that \( I(A : C|B)_\rho = 0 \) if and only if the following conditions hold:

(i) \( \mathcal{H}_B = \bigoplus_k \mathcal{H}_{b_kL} \otimes \mathcal{H}_{b_kR} \),

(ii) \( \rho_{ABC} = \bigoplus_k p_k \rho_{Ab_kL} \otimes \rho_{b_kC} \), where \( \rho_{Ab_kL} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{b_kL}) \), \( \rho_{b_kC} \in \mathcal{D}(\mathcal{H}_{b_kR} \otimes \mathcal{H}_C) \) for each index \( k \); and \( \{p_k\} \) is a probability distribution.

In order to get rid of the above-known difficult computations such as logarithm and complex exponential power of states, Zhang [7] gave another new characterization of vanishing conditional mutual information. Specifically, denote

\[
M \overset{\text{def}}{=} (\rho_{AB}^{1/2} \otimes \mathbb{1}_C)(\mathbb{1}_A \otimes \rho_B^{1/2} \otimes \mathbb{1}_C)(\mathbb{1}_A \otimes \rho_{BC}^{1/2}) \overset{\text{def}}{=} \rho_{AB}^{1/2} \rho_B^{1/2} \rho_{BC}^{1/2}.
\]

Then the following conditions are equivalent:
The conditional mutual information is vanished, i.e. $I(A : C|B)_\rho = 0$;

(ii) $\rho_{ABC} = MM^\dagger = \rho_{A|B}^{1/2} \rho_{BC}^{1/2} \rho_{B|A}^{1/2}$;

(iii) $\rho_{ABC} = M^\dagger M = \rho_{B|C}^{1/2} \rho_{AB}^{1/2} \rho_{BC}^{1/2}$.

With these characterizations of vanishing conditional mutual information. One starts to make an attempt to get a lower bound of conditional mutual information. In [8], Brandão et al. first obtained the following lower bound for $I(A : C|B)_\rho$:

$$I(A : C|B)_\rho \geq \frac{1}{8} \min_{\sigma_{AC} \in \textrm{SEP}} \|\rho_{AC} - \sigma_{AC}\|_{1-\text{LOCC}}^2,$$  \hspace{1cm} (1.8)

where

$$\|\rho_{AC} - \sigma_{AC}\|_{1-\text{LOCC}}^2 \overset{\text{def}}{=} \sup_{\mathcal{M} \in 1-\text{LOCC}} \|\mathcal{M}(\rho_{AC}) - \mathcal{M}(\sigma_{AC})\|_1.$$

They used many techniques, which is accessible by some advanced researchers, to get this result in their paper. Based on this result, they cracked a long-standing open problem in quantum information theory. That is, the squashed entanglement is faithful. Later, Li and Winter in [9] gave another approach to study the same problem and improved the lower bound for $I(A : C|B)_\rho$:

$$I(A : C|B)_\rho \geq \frac{1}{2} \min_{\sigma_{AC} \in \textrm{SEP}} \|\rho_{AC} - \sigma_{AC}\|_{1-\text{LOCC}}^2.$$  \hspace{1cm} (1.9)

A different approach is taken by Ibinson et al. in [10]. They studied the robustness of quantum Markov chains, i.e. the perturbation of states of vanishing conditional mutual information. They found that the quantum Markov chain states are not robust since when the conditional mutual information is small, the original tripartite state can deviate much from Markov chain states.

In this paper, we want to give a unifying treatment for some entropy inequalities: Improvement of monotonicity inequality of relative entropy under generic quantum channels. The method used is quantum $\alpha$-Rényi relative entropy [11]. Once we get one of main results (Theorem 3.1), we can simply derive all improved version of some quantum entropy inequalities [12]. Note that the method taken in the present paper is much simpler than one in [13].

The paper is organized as follows. The definition and properties of quantum Rényi relative entropy are given in Section 2. Section 3 deals with the main results and its consequences with proofs. The concluding remarks are presented in Section 4.

## 2 Quantum $\alpha$-Rényi relative entropy

The quantum $\alpha$-Rényi relative entropy is defined as follows [11]:

$$S_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \rho^\alpha \sigma^{1-\alpha} \right),$$  \hspace{1cm} (2.1)

where $\rho, \sigma \in \mathcal{D} (\mathcal{H})$, and a parameter $\alpha \in (0, 1)$. The two important properties of $\alpha$-Rényi relative entropy used in this paper are listed below: it holds that

(i) $S(\rho||\sigma) = \lim_{\alpha \to 1} S_\alpha(\rho||\sigma)$, thus we denote $S(\rho||\sigma) = S_1(\rho||\sigma)$;
(ii) \( \alpha \mapsto S_\alpha(\rho||\sigma) \) is monotonically increasing on \((0,1)\).

Hence, if \( \alpha \geq \frac{1}{2} \), then \( S_{1/2}(\rho||\sigma) \leq S_\alpha(\rho||\sigma) \). Taking the limit for \( \alpha \to 1 \) in the right hand side of the inequality, we have the following important result:

\[
S(\rho||\sigma) \geq -2 \log \text{Tr} \left( \sqrt{\rho} \sqrt{\sigma} \right).
\]  

(2.2)

The same inequality is obtained by Carlen [12] using Peierls-Bogoliubov inequality and Golden-Thompson inequality. Here the method to get this inequality is pointed out to us by M. Wilde in private communications. Later, in our further investigation, we find that this inequality can imply all the improved version of some entropy inequalities obtained recently. The lower bound for relative entropy, compared with Pinsker’s bound for relative entropy:

\[
S(\rho||\sigma) \geq \frac{1}{2} \|\rho - \sigma\|_1^2,
\]

(2.3)

is very useful in our present paper.

Note that there is an identity which will be used in our treatment:

\[
S(\rho||\mu\sigma) = S(\rho||\sigma) - \log \mu, \quad \forall \mu > 0.
\]

In what follows, we start to present our main results with its proof.

### 3 Main results

In this section, the first one of our main results is proved.

**Theorem 3.1.** For two density matrices \( \rho, \sigma \in D(\mathcal{H}) \) and a quantum channel \( \Phi \) over \( \mathcal{H} \), we have

\[
S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)) \geq -2 \log \text{Tr} \left( \sqrt{\rho} \sqrt{\exp \left[ \log \sigma + \Phi^*(\log \Phi(\rho)) - \Phi^*(\log \Phi(\sigma)) \right]} \right).
\]

**Proof.** Define a state as follows:

\[
\omega = \lambda^{-1} \exp(\log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma)));
\]

with \( \lambda := \text{Tr} \left( \exp(\log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma))) \right) > 0 \). Now we get

\[
\log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma)) = \log(\lambda \omega).
\]

Thus

\[
S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)) = \text{Tr} \left( \rho (\log \rho - \log \sigma) - \Phi(\rho)(\log \Phi(\rho) - \log \Phi(\sigma)) \right)
\]

\[= \text{Tr} \left( \rho (\log \rho - \log \sigma) - \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma)) \right)
\]

\[= \text{Tr} \left( \rho (\log \rho - \log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma))) \right).
\]
There is a well-known inequality in Matrix Analysis, i.e.

\[
S(\rho || \sigma) - S(\Phi(\rho) || \Phi(\sigma)) = S(\rho || \lambda \omega) = S(\rho || \omega) - \log \lambda.
\]

Thus

\[
S(\rho || \sigma) - S(\Phi(\rho) || \Phi(\sigma)) \geq -2 \log \text{Tr} \left( \sqrt{\rho} \sqrt{\omega} \right) - \log \lambda.
\]

Since

\[
\sqrt{\omega} = \lambda^{-1/2} \sqrt{\exp(\log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma)))},
\]

it follows that

\[
-2 \log \text{Tr} \left( \sqrt{\rho} \sqrt{\omega} \right) - \log \lambda = -2 \log \text{Tr} \left( \sqrt{\rho} \sqrt{\exp(\log \sigma + \Phi^*(\log \Phi(\rho) - \log \Phi(\sigma)))} \right).
\]

The desired inequality is obtained. \(\square\)

**Proposition 3.2.** For any given two positive semi-definite matrices \(M\) and \(N\), it holds that

\[
\| \sqrt{M} - \sqrt{N} \|_2^2 \leq \| M - N \|_1 \leq \| \sqrt{M} - \sqrt{N} \|_2 \| \sqrt{M} + \sqrt{N} \|_2,
\]

where \(\|X\|_p := (\text{Tr} (|X|^p))^{1/p}\) is Schatten \(p\)-norm for positive integers \(p\), and \(|X| = \sqrt{X^\dagger X}\).

**Proof.** There is a well-known inequality in Matrix Analysis, i.e. Audenaert’s inequality \([14]\):

\[
\text{Tr} \left( M^t N^{1-t} \right) \geq \frac{1}{2} \text{Tr} (M + N - |M - N|)
\]

for all \(t \in [0, 1]\) and positive matrices \(M, N\), implying that for \(t = \frac{1}{2}\),

\[
\text{Tr} \left( \sqrt{M} \sqrt{N} \right) \geq \frac{1}{2} \text{Tr} (M + N - |M - N|).
\]

Now that

\[
\text{Tr} \left( \sqrt{M} \sqrt{N} \right) = \frac{1}{2} \text{Tr} \left( M + N - \left( \sqrt{M} - \sqrt{N} \right)^2 \right).
\]

Thus

\[
\| \sqrt{M} - \sqrt{N} \|_2^2 \leq \| M - N \|_1.
\]

This is the famous Powers-Störmer’s inequality \([15]\). Furthermore,

\[
\| M - N \|_1 \leq \| \sqrt{M} - \sqrt{N} \|_2 \| \sqrt{M} + \sqrt{N} \|_2.
\]

Indeed, by triangular inequality and Schwartz inequality, it follows that

\[
\| M - N \|_1 = \| \frac{1}{2} \left( \sqrt{M} - \sqrt{N} \right) \left( \sqrt{M} + \sqrt{N} \right) + \frac{1}{2} \left( \sqrt{M} + \sqrt{N} \right) \left( \sqrt{M} - \sqrt{N} \right) \|_1
\]

\[
\leq \frac{1}{2} \left\| \left( \sqrt{M} - \sqrt{N} \right) \left( \sqrt{M} + \sqrt{N} \right) \right\|_1 + \frac{1}{2} \left\| \left( \sqrt{M} + \sqrt{N} \right) \left( \sqrt{M} - \sqrt{N} \right) \right\|_1
\]

\[
\leq \| \sqrt{M} - \sqrt{N} \|_2 \| \sqrt{M} + \sqrt{N} \|_2.
\]

Therefore

\[
\| \sqrt{M} - \sqrt{N} \|_2^2 \leq \| M - N \|_1 \leq \| \sqrt{M} - \sqrt{N} \|_2 \| \sqrt{M} + \sqrt{N} \|_2.
\]

This completes the proof. \(\square\)
Remark 3.3. Due to (3.3), we have
\[ \text{Tr} \left( \sqrt{M} \sqrt{N} \right) = \frac{1}{2} \text{Tr} (M + N) - \frac{1}{2} \left\| \sqrt{M} - \sqrt{N} \right\|_2^2. \]
If both the traces of $M$ and $N$ are more than one, i.e. $\text{Tr} (M), \text{Tr} (N) \leq 1$, then
\[ \text{Tr} \left( \sqrt{M} \sqrt{N} \right) \leq 1 - \frac{1}{2} \left\| \sqrt{M} - \sqrt{N} \right\|_2^2. \]
Furthermore,
\[ -2 \log \text{Tr} \left( \sqrt{M} \sqrt{N} \right) \geq -2 \log \left( 1 - \frac{1}{2} \left\| \sqrt{M} - \sqrt{N} \right\|_2^2 \right) \geq \left\| \sqrt{M} - \sqrt{N} \right\|_2^2, \] where we used the fact that $-\log(1-t) \geq t$ for $t \leq 1$.

Corollary 3.4. For two arbitrary bipartite states $\rho_{AB}, \sigma_{AB} \in D(\mathcal{H}_{AB})$ with $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, it holds that
\[ S(\rho_{AB}||\sigma_{AB}) - S(\rho_A||\sigma_A) \geq -2 \log \text{Tr} \left( \sqrt{\rho_{AB}} \sqrt{\exp(\log \rho_{AB} - \log \sigma_{AB} + \log \rho_A)} \right) \]
\[ \geq \left\| \sqrt{\rho_{AB}} - \sqrt{\exp(\log \sigma_{AB} - \log \sigma_A + \log \rho_A)} \right\|_2^2. \] In particular, $S(\rho_{AB}||\sigma_{AB}) = S(\rho_A||\sigma_A)$ if and only if $\log \rho_{AB} - \log \rho_A = \log \sigma_{AB} - \log \sigma_A$.

Proof. In Theorem 3.1 letting $\rho = \rho_{AB}$ and $\sigma = \sigma_{AB}$, and the quantum channel $\Phi = \text{Tr}_B$, a partial trace over system $B$, gives the first inequality. The second inequality follows from Remark 3.3 due to the fact that $\text{Tr} (\exp(\log \sigma_{AB} - \log \sigma_A + \log \rho_A)) \leq 1$. We are done.

Corollary 3.5. For an arbitrary tripartite state $\rho_{ABC}$, we have that
\[ I(A : C|B)_{\rho} \geq -2 \log \text{Tr} \left( \sqrt{\rho_{ABC}} \sqrt{\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})} \right) \]
\[ \geq \left\| \sqrt{\rho_{ABC}} - \sqrt{\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})} \right\|_2^2. \] In particular, the conditional mutual information is vanished if and only if $\log \rho_{ABC} + \log \rho_B = \log \rho_{AB} + \log \rho_{BC}$.

Proof. Note that the quantum conditional mutual information $I(A : C|B)_{\rho}$ can be rewritten as follows:
\[ I(A : C|B)_{\rho} = S(\rho_{ABC}||\omega_{ABC}) - \log \lambda, \]
where $\lambda \omega_{ABC} = \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})$ and $\lambda = \text{Tr} (\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}))$.

By using (2.2), we get
\[ I(A : C|B)_{\rho} \geq -2 \log \text{Tr} \left( \sqrt{\rho_{ABC}} \sqrt{\omega_{ABC}} \right) - \log \lambda \]
\[ = -2 \log \text{Tr} \left( \sqrt{\rho_{ABC}} \sqrt{\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})} \right). \] This is the first inequality. The second approach to get the first inequality is by using Theorem 3.1. Indeed, denote by $\Phi = \text{Tr}_A, \rho = \rho_{ABC}$ and $\sigma = \rho_{AB} \otimes \rho_C$. It follows that
\[ I(A : C|B)_{\rho} = S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)). \]
By employing Theorem 3.1, we have

\[ I(A : C|B)_\rho \geq 2 \log \text{Tr} \left( \sqrt{\rho_{ABC}} \sqrt{\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})} \right). \]

The second inequality follows directly from Remark 3.3 due to the fact that

\[ \text{Tr} \left( \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}) \right) \leq 1. \]

Now if the conditional mutual information is vanished, then

\[ \| \sqrt{\rho_{ABC}} - \sqrt{\exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B)} \|_2 = 0, \]

that is, \( \sqrt{\rho_{ABC}} = \sqrt{\exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B)} \), which is equivalent to the following:

\[ \rho_{ABC} = \exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B). \]

By taking logarithm over both sides, it is seen that \( \log \rho_{ABC} = \log \rho_{AB} + \log \rho_{BC} - \log \rho_B \), a well-known equality condition of strong subadditivity obtained by Ruskai in [4]. The proof is finished.

Corollary 3.6. It holds that

\[ I(A : C|B)_\rho \geq \frac{1}{4} \| \rho_{ABC} - \exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B) \|_1^2. \]  

(3.10)

Proof. It follows from Proposition 3.2 that

\[ \frac{1}{\| \sqrt{\rho} + \sqrt{\sigma} \|_2^2} \| \rho - \sigma \|_1^2 \leq \| \sqrt{\rho} - \sqrt{\sigma} \|_2^2 \leq \| \rho - \sigma \|_1. \]  

(3.11)

In view of the fact that \( \| \sqrt{\rho} + \sqrt{\sigma} \|_2 \in [\sqrt{2}, 2] \), we have

\[ \frac{1}{4} \leq \frac{1}{\| \sqrt{\rho} + \sqrt{\sigma} \|_2^2} \leq \frac{1}{2}. \]

Applying (3.11) to (3.9) in Corollary 3.5 we get the desired inequality.

4 Concluding remarks

The lower bound in (3.10) is clearly independent of any measurement, compared with (1.8) and (1.9). Since the 1-norm is decreasing under generic quantum channels, in particular partial trace, it follows that

\[ E_{sq}(\rho_{AC}) \geq \frac{1}{8} \| \rho_{AC} - \text{Tr}_B(\exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B)) \|_1^2, \]

(4.1)

where \( E_{sq} \) is an entanglement measure, i.e. squashed entanglement, defined by

\[ E_{sq}(\rho_{AC}) = \inf_B \left\{ \frac{1}{2} I(A : C|B)_\rho : \text{Tr}_B(\rho_{ABC}) = \rho_{AC} \right\}. \]

(4.2)

Finding some properties of the following operators is a very interesting subject:

\[ \exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B), \quad \text{Tr}_B(\exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B)). \]
For instance, a generalized Lie-Trotter product formula \[16\]: for any \(k\) matrices \(A_1, \ldots, A_k\), it holds that
\[
\lim_{n \to \infty} (\exp(A_1/n) \exp(A_2/n) \cdots \exp(A_k/n))^n = \exp(A_1 + A_2 + \cdots + A_k).
\]

This leads to the following identities:
\[
\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}) = \lim_{n \to \infty} \left( \rho_{AB}^{1/2n} \rho_B^{-1/2n} \rho_{BC}^{1/2n} \rho_B^{-1/2n} \rho_{AB}^{1/2n} \right)^n = \lim_{n \to \infty} \left( \rho_{BC}^{1/2n} \rho_B^{-1/2n} \rho_{AB}^{1/2n} \rho_B^{-1/2n} \rho_{BC}^{1/2n} \right)^n.
\]

We wonder whether the sequence \(\text{Tr} \left( \left( \rho_{AB}^{1/2n} \rho_B^{-1/2n} \rho_{BC}^{1/2n} \rho_B^{-1/2n} \rho_{AB}^{1/2n} \right)^n \right)\) is monotone in \(n\) and is no more than one. In any way, the computation of the above-mentioned two operators is very difficult. Any progress on this problem will help us understanding the meaning of our lower bound for squashed entanglement.

There is a big challenge in proving the following inequality:
\[
\text{Tr} \left( \exp (\log \sigma + \Phi^*(\log \Phi(\rho)) - \Phi^*(\log \Phi(\sigma))) \right) \leq 1.
\]

We leave it for the future research. Once it is proved, then it would be true that
\[
S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)) \geq \frac{1}{4} \left\| \rho - \exp (\log \sigma + \Phi^*(\log \Phi(\rho)) - \Phi^*(\log \Phi(\sigma))) \right\|_1^2.
\]

There is another big open question:
\[
S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)) \geq \frac{1}{4} \left\| \rho - \Phi^*_\rho \circ \Phi(\rho) \right\|_1^2.
\]

Further investigation on this topics is pursuing. We wish these results derived in this paper shed new light over related subjects in quantum information theory.

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