Lattice Boltzmann-Langevin Equations

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Abstract

Intrinsic fluctuations around the solution of the lattice Boltzmann equation are described or modeled by addition of a white Gaussian noise source. For stationary states a fluctuation-dissipation theorem relates the variance of the fluctuations to the linearized Boltzmann collision operator and the pair correlation function.

1 Introduction

The discussions and contributions at this meeting have shown a growing interest in fluctuations both in the microscopic lattice gas approach, as well as in the phenomenological lattice Boltzmann approach. The emphasis was on unstable systems (phase separation, spinodal decomposition, pattern formation) where fluctuations drive the system away from a spatially uniform equilibrium state. So far the noise considered in connection with the lattice Boltzmann approach was introduced in a rather ad hoc fashion. It may be additive or multiplicative, external or intrinsic. There do not seem to be any systematic studies that make the connection with the multitude of results for fluctuations and transport in the continuous case (see Bixon and Zwanzig [1969], Fox and Uhlenbeck [1970], Ernst and Cohen [1981] and Marchetti and Dufty [1983]). The goal of this paper is to provide this link and to show that the well-eblished results on intrinsic fluctuations, derived from the Boltzmann-Langevin equation in the continuous case, can be extended to the lattice Boltzmann equation.

The Boltzmann equation is a deterministic equation for the average occupation \( f(\vec{r}, \vec{c}, t) \) of a one particle state specified by position \( \vec{r} \) and velocity \( \vec{c} \). When derived from an underlying microdynamics, it describes the ‘slow’ time...
evolution of $f(\vec{r}, \vec{c}, t)$ on coarse grained spatial and temporal scales of the order of the mean free path $\ell_o$ and time $t_o$ between collisions. The deviations of the mesoscopic occupation, $n(\vec{r}, \vec{c}, t)$, from this average Boltzmann value reflect fluctuations due to other ‘fast’ degrees of freedom that are averaged out in obtaining the Boltzmann equation. A measure of these fluctuations is given by their correlation functions, which also can be calculated from the microdynamics. A unified picture of both the transport and fluctuations is given by a Boltzmann-Langevin (B-L) equation. The existence of other degrees of freedom is recognized in the B-L equation by adding a stochastic source that generates fluctuations in the solutions to this equation relative to the deterministic solution. This is a particularly convenient representation for simulation of noise effects on the Boltzmann equation. The characteristics of this stochastic source should be determined from the microdynamics for consistency with the direct evaluation of correlation functions using methods of statistical mechanics and kinetic theory. Studies along these lines are in progress [Dufty and Ernst 1994].

In other contexts, a Boltzmann equation is simply postulated as a mathematical model to simulate and study macroscopic fluid properties. Its extension to a corresponding B-L equation to describe the effects of noise now has no underlying microdynamics to characterize that noise. To assist in this characterization, we establish in the next section the general relationship of the noise variance to the pair correlation function at equal time and different times. The form of this relationship is independent of any underlying microdynamics. This connection of the noise to correlation functions provides a somewhat more physical basis for suggesting the noise characteristics. Furthermore, known results about the correlation functions for systems with a microdynamics can be ‘borrowed’ for application to the mathematical modeling as well.

Our primary interest here is to make the connections just mentioned and to discuss their content at a phenomenological level. The B-L equation is defined in the next section and it is assumed that the noise is white (uncorrelated in time) and essentially characterized by its variance (this is strictly the case only for Gaussian noise). Equations for the correlation of fluctuations at equal and different times are derived from the B-L equation, under the condition that the fluctuations are small. It is observed that these equations are the precise analog of those derived for a real low density gas (see Ernst and Cohen [1981], Marchetti and Dufty [1983]). For the special case of fluctuations about a stationary state a fluctuation-dissipation relation is obtained, defining the noise variance in terms of the stationary state correlations and the linear Boltzmann operator. For non-stationary states additional information is required. The analysis applies as well to unstable states up to times for which the fluctuations remain small. In section 2 we give a concrete example of a BGK-Langevin description of fluctuations about a steady state. Finally, in section 3 we show that hydrodynamic equations with noise can be obtained consistently from the B-L description. The expected Landau-Lifshitz form is confirmed.
2 B-L Equation and Correlation Functions

In the absence of noise, the lattice Boltzmann equation describes deterministic dynamics for the occupation number, \( n_i(\vec{r}, t) \), of a single particle state at node \( \vec{r} \) in velocity channel \( \vec{c}_i \) where \( i \) labels the allowed velocity states. The \( n_i(\vec{r}, t) \) are continuous real variables on the interval \([0, a]\) where \( a \) is the maximum occupancy of a state (in the case of an underlying microdynamics with Fermi exclusion rules, a value \( a > 1 \) implies that these occupation numbers are coarse grained averages over several nodes). The corresponding B-L equation with noise is,

\[
n(x, t + 1) = V(x \mid n(t)) + S_x F(x, t),
\]

where a simplified notation \( n(x, t) = n_i(\vec{r}, t) \) has been introduced. The generator for the deterministic dynamics, \( V(x \mid n(t)) \), is given by,

\[
V(x \mid n(t)) = S_x(n(x, t) + I(x \mid n(t))),
\]

where \( I(x \mid n(t)) = I_i(\vec{r} \mid n(t)) \) is the non-linear Boltzmann collision operator, and \( S_x \) is the free streaming operator defined by \( S_x f(\vec{r}) = f(\vec{r} - \vec{c}_i) \) for any function of \( \vec{r} \). Finally, the stochastic source, \( F(x, t) \), is assumed to represent Gaussian white noise with zero mean value, statistically independent of the occupation numbers, \( \langle n(x, t) F(y, \tau) \rangle = 0 \) for \( \tau \leq t \), and covariance specified by a noise intensity matrix, \( B(x, y) \),

\[
\langle F(x, t) F(y, t') \rangle = \delta(t, t') B(x, y \mid n(t)).
\]

The average is taken over the noise source. Furthermore the noise does not generate any mass, momentum or energy, so that \( F \) is orthogonal to the summational invariants. We note that the noise matrix \( B(x, y \mid n) \) in general can depend on the state at time \( t \); this is an example of multiplicative noise. Once the noise matrix has been specified, the above provides a complete description of fluctuations and transport.

To simplify the notation, let \( \mathbf{n} \) denote the vector whose components are \( n(x, t) \). Similarly, \( \mathbf{V}(\mathbf{n}) \), \( \mathbf{F} \), and \( \mathbf{B}(\mathbf{n}) \) denote the vectors and matrix with elements \( V(x \mid n) \), \( F(x) \), and \( B(x, y \mid n) \), respectively. The B-L equation is then,

\[
\mathbf{n}(t + 1) = \mathbf{V}(\mathbf{n}(t)) + \mathbf{S} \mathbf{F}(t), \quad \langle \mathbf{F}(t) \mathbf{F}(t') \rangle = \delta(t, t') \mathbf{B}(\mathbf{n}(t)).
\]

The solution to the deterministic (noise-free) Boltzmann equation, \( \mathbf{f}(t) \), is obtained from

\[
\mathbf{f}(t + 1) = \mathbf{V}(\mathbf{f}(t)).
\]

The fluctuations around this deterministic solution are measured by \( \delta \mathbf{n}(t) = \mathbf{n}(t) - \mathbf{f}(t) \). The corresponding pair correlation functions are defined by,

\[
\mathbf{g}(t) = \langle \delta \mathbf{n}(t) \delta \mathbf{n}(t) \rangle, \quad \mathbf{c}(t, t') = \langle \delta \mathbf{n}(t) \delta \mathbf{n}(t') \rangle.
\]
Closed equations for these correlation functions follow in two steps: first the B-L equation is applied to $g(t+1)$ and $c(t+1,t')$; next the resulting averages are expanded to lowest order in $\delta n(t)$, for small fluctuations,

$$g(t+1) = L(t)g(t)L^T(t) + SB(t)S^T$$  \hspace{1cm} (7)

$$c(t+1,t') = L(t)c(t,t').$$  \hspace{1cm} (8)

Here $B(t) \equiv B(f(t))$ and $L(t)$ is the matrix,

$$L(t) = \frac{d}{dn} V(n) \big|_{f(t)}.$$

Both the equal time and two time correlation functions obey linear equations governed by the matrix $L(t)$ whose form is determined from the generator, $V(f)$, for the given Boltzmann equation. They depend on the specific nonequilibrium solution to the Boltzmann equation, $f(t)$, being considered. For the special case where $f(t) = f_o$ is the equilibrium state, this result forms the starting point in the calculation of the Green-Kubo formulas for lattice gas cellular automata (LGCA) in the mean field approximation [Brito et al 1991], as well as for the dynamic structure factor in such systems [Grosfils et al 1993].

We remark at this point that (5) and (7) through (9) are the direct analog of corresponding results from the kinetic theory of real gases at low density. The difference here is that the form of $B(t)$ is given explicitly from the theory there, whereas here it is as yet unspecified.

Before discussing the content of these equations it may be useful to see how they can be applied to determine the noise matrix in a special case. Consider fluctuations around a stationary (or quasi-stationary) solution to the Boltzmann equation, $f_o$. For consistency, the noise matrix must be chosen such that the equal time correlation functions are also stationary. Consequently, (6) gives,

$$SB_oS^T = g - L_o g L_o^T \quad \text{or} \quad B_o = S^T g S - (1 + \Omega)g(1 + \Omega^T),$$

where $B_o$ and $L_o$ are given by $B$ and $L$ evaluated at the stationary state $f_o$, and $L_o = S(1 + \Omega)$ has been expressed in terms of the linearized collision operator $\Omega$. This is the classical fluctuation-dissipation theorem extended to lattice gas automata. It expresses the noise matrix in terms of the stationary state correlations and the Boltzmann collision operator linearized around the stationary state. A more explicit form is given in the example at the end of this section.

How would (10) be used to implement a simulation of noise in a B-L equation? First the model for the Boltzmann equation is specified by using collision rules for a LGCA or by a BGK model; next the equal time correlation matrix is determined for the model; finally, solutions to the B-L equation are simulated for initial states near $f_o$, selecting the noise $F$ from a Gaussian whose half width $B_o$ is determined from the fluctuation-dissipation relation. In this simulation of
the B-L equation in the stationary state the equal time correlation function \( g \) is used as input to determine the noise statistics \( B_{ij} \), and the two time correlation function \( c(t, t') = \delta n(t)\delta n(t') \) can be measured by simulations, even for models without any underlying microdynamics. The simulated values should agree with the theoretical value calculated from (8).

To illustrate the procedure sketched below (10) we consider two examples: a standard LGCA that obeys the conditions of semi-detailed balance (SDB), and a BGK-model. In the LGCA all equilibrium fluctuations are totally uncorrelated, i.e.,

\[
g(x, y) = \delta(x, y)g_i \quad (x = \vec{r} \ i; y = \vec{r}' \ j)
\]

with a known value, \( g_i = f_{oi}(1 - f_{oi}) \) for the fluctuations in a single particle state because of the Fermi exclusion rule. Furthermore, the collision rules are strictly local such that the collision matrix in (10) has the form \( \Omega(x, y) = \delta(\vec{r}, \vec{r}')\Omega_{ij} \).

The noise strength (10) reduces to

\[
B(x, y) = \delta(\vec{r}, \vec{r}')B_{ij} = -\delta(\vec{r}, \vec{r}')[\Omega_{ij}g_j + \Omega_{ji}g_i + \Omega_{i\ell}g_\ell\Omega_{j\ell}] .
\]

Let us compare these results with those of Bixon and Zwanzig [1969], and Fox and Uhlenbeck [1970] for the strength of the fluctuating term in the continuous case. The first two terms are direct analogs of the corresponding terms in the continuous case. The term quadratic in \( \Omega \) is a typical lattice effect, similar to the so called ‘propagation part’ of the transport coefficients, as occurring in lattice gas automata (see e.g. Ernst and Dufty [1991]) and in the lattice Boltzmann approach. This will be shown in section 3.

As a second example we consider a typical athermal BGK-model equation,

\[
f_i(\vec{r} + \vec{c}_i, t + 1) - f_i(\vec{r}, t) = -\frac{1}{\tau}\left[f_i(\vec{r}, t) - f_i^e\right],
\]

where the local equilibrium distribution \( f_i^e \) depends on the local density \( \rho(\vec{r}, t) = \sum_i f_i(\vec{r}, t) \) and the local flow velocity \( \rho(\vec{r}, t)\vec{u}(\vec{r}, t) = \sum_i c_i f_i(\vec{r}, t) \). We linearize this equation around basic equilibrium \( f_{oi} \) with vanishing flow velocity, where \( f_{oi} \) equals \( f \) for a rest particle (\( i = 0 \)) and equals \( f \) for a moving particle. The collision operator is diagonal and can be identified as,

\[
\Omega_{ii} = -\frac{1}{\tau}\left[1 - \frac{df_i^e}{d\rho} - (\frac{c_i}{\rho}) \cdot \frac{df_i^e}{d\vec{u}}\right]_o,
\]

where the subscript \( (o) \) indicates that the derivatives are taken at \( \vec{u} = 0 \).

As a specific example we take the BGK-model defined in Eq.(15) of Ernst and Bussemaker (see these proceedings) and obtain,

\[
\Omega_{oo} = -\frac{dc^2}{\tau c_o^2}, \quad \Omega_{ii} = -\frac{1}{\tau}\left[1 - \frac{d}{b c_o^2}(c^2_o + c^2_o)\right],
\]

(15)
where $i$ refers to a moving particle. Furthermore, $d$ is the spatial dimension of the lattice, $|\vec{c}_i| = c_o$ the lattice distance, and $b$ the coordination number. We also used the speed of sound from the above reference, i.e.,

$$c_s^2 = \frac{dp}{d\rho} = \frac{bc_o^2 \frac{df}{d\rho}}{d} = \frac{c_o^2}{d}(1 - \frac{df}{d\rho}).$$

(16)

So far we have specified the matrix elements of the collision operator to be used in the expression (12) for the noise strength. Next we address the choice of equilibrium fluctuation $g_i = \langle (\delta n_i)^2 \rangle$. In lattice gas automata the exclusion rule for occupation numbers makes the equilibrium fluctuations $g_i$ similar to those for an ideal Fermi gas. In the mathematical modeling of the lattice Boltzmann approach the positive function $g_i$ can be chosen freely by lack of an underlying microscopic model. However, the choice $g_i = \rho(\frac{df_i}{d\rho})$, which resembles the result for a Maxwell-Boltzmann gas, is closest in spirit to the BGK-model. This is so because the local equilibrium distribution function $f^e_i$, used in typical lattice gas applications [Chen et al 1992], depends on the local average density $\rho$ and the fluid flow velocity $\vec{u}$, and resembles a local Maxwell-Boltzmann distribution expanded to terms of $O(u^2)$ for small velocities. In the typical model under consideration we have for the rest particles $g_o = \rho(1 - \frac{dc_s^2}{c_o^2})$ and for moving particles $g_i = (\rho d/b)(\frac{c_s^2}{c_o^2})$. This completes the determination of the linear BGK-Langevin equation.

3 Noise in Fluid Dynamics

In this section we calculate the correlation function in equilibrium of the fluctuating part of the single particle distribution function in the limit of large spatial and temporal scales, for systems where the noise strength is determined by $B_o(x, y)$ in (12). From that result one can compute the fluctuations in the pressure tensor or in the heat current, if energy is also conserved in the lattice Boltzmann equation. The results obtained will be compared with the Landau-Lifshitz formulas for the correlation strength in fluctuating hydrodynamics.

For the analysis of this section it is convenient to work with Fourier and Laplace transforms, defined through,

$$\tilde{\delta n}_i(\vec{k}, z) = \sum_{t=0}^{\infty} \sum_{\vec{r}} e^{-i\vec{k} \cdot \vec{r} - zt} \delta n_i(\vec{r}, t) = \sum_{t=0}^{\infty} e^{-zt} \tilde{n}_i(\vec{k}, t).$$

(17)

For fluctuations around equilibrium the B-L equation becomes,

$$e^{i\vec{k} \cdot \vec{v}_i + z} \tilde{n}_i(\vec{k}, z) = (\delta_{ij} + \Omega_{ij}) \tilde{n}_i(\vec{k}, z) + \tilde{F}_i(\vec{k}, z).$$

(18)

In vector and matrix notation the formal solution of (18) reads

$$\tilde{n}(\vec{k}, z) = \tilde{n}_0(\vec{k}, z) + \tilde{f}(\vec{k}, z),$$

(19)
where the sure (deterministic) part is \( \tilde{n}_0(\vec{k}, z) \equiv \Delta(\vec{k}, z) e^{i\vec{k} \cdot \vec{z}} n(\vec{k}, 0) \) and the fluctuating part is \( \tilde{f}(\vec{k}, z) \equiv \Delta(\vec{k}, z) \tilde{F}(\vec{k}, z) \). Here \( \Delta_{ij}(\vec{k}, z) \) and \( [\exp(i\vec{k} \cdot \vec{c})]_{ij} = \delta_{ij} \exp(i\vec{k} \cdot \vec{c}_i) \) are matrices, and \( \tilde{n}(\vec{k}, z) \), etc are vectors with components \( \tilde{n}_i(\vec{k}, z), etc \). The matrix \( \Delta \) is defined as
\[
\Delta(\vec{k}, z) = [e^{i\vec{k} \cdot \vec{z}} - 1 - \Omega]^{-1}. \tag{20}
\]

The sure part of (13) approaches the Chapman-Enskog hydrodynamic solution of the Boltzmann equation for large spatial and temporal scales. The fluctuating part of the distribution function has correlations given by,
\[
V^{-1} \langle \tilde{f}_i(\vec{k}, z) \tilde{f}_j^*(\vec{k}, z') \rangle = V^{-1} \Delta_{it}(\vec{k}, z) \langle \tilde{F}_t(\vec{k}, z) \tilde{F}_{tm}(\vec{k}, z') \rangle \Delta_{jm}^*(\vec{k}, z'), \tag{21}
\]
where the asterisk represent complex conjugation and where \( V \) is the number of nodes in the lattice. The correlation function of the \( \tilde{F} \)’s is obtained by taking Fourier and Laplace transforms of (13), and using translational invariance with the result,
\[
V^{-1} \langle \tilde{f}_i(\vec{k}, z) \tilde{f}_j^*(\vec{k}, z') \rangle = \Delta_{it}(\vec{k}, z) B_{tm} \Delta_{jm}^*(\vec{k}, z')/[1 - e^{-z - z'}]. \tag{22}
\]

As we are interested in hydrodynamic space and time scales, we take the limit \( \vec{k} \to 0, z \to 0 \). Then \( \Delta(\vec{k}, z) \to -1/\Omega \), where the inverse of \( \Omega \) is only defined in the subspace orthogonal to the zero-eigenfunctions, the so-called collisional invariants. Next we invert the transforms in (24) to find the fluctuations \( \tilde{f}_i(\vec{r}, t) \) in the distribution function as,
\[
\langle \tilde{f}_i(\vec{r}, t) \tilde{f}_j^*(\vec{r}', t') \rangle \simeq \delta(\vec{r}, \vec{r}') \delta(t, t') \Omega_{it}^{-1} B_{tm} \Omega_{jm}^{-1} = -\delta(\vec{r}, \vec{r}') \delta(t, t') \left[ \frac{1}{\Omega} + \frac{1}{2} \right]_{ij} g_j + \left[ \frac{1}{\Omega} + \frac{1}{2} \right]_{ji} g_i. \tag{23}
\]

The second equality is a consequence of (13). This is the final result for the fluctuations in the distribution functions on hydrodynamic space and time scales. The equation also shows that the term \( \propto \Omega^2 \) in (13) and (12) directly transforms into the two terms in (23) containing \( \frac{1}{\Omega} \), which constitute the so called ‘propagation’ part of the transport coefficients.

The result (23) enables us to calculate the strength of the fluctuations in the dissipative currents in the fluctuating hydrodynamic equations, such as the stress tensor or the heat current in lattice gases with energy conservation. We illustrate this by calculating the fluctuations in the shear stress \( \tilde{\Pi}_{xy}(\vec{r}, t) = \sum_i c_x c_y f_i(\vec{r}, t) \) with the help of (23). This yields,
\[
\langle \tilde{\Pi}_{xy}(\vec{r}, t) \tilde{\Pi}_{xy}(\vec{r}', t') \rangle = 2 \langle c_x | c_x \rangle \nu \delta(\vec{r}, \vec{r}') \delta(t, t'), \tag{24}
\]
where \( \nu \) is the kinematic viscosity, given by
\[
\nu = -\langle c_x c_y | \left( \frac{1}{\Omega} + \frac{1}{2} \right) | c_x c_y \rangle / \langle c_x | c_x \rangle. \tag{25}
\]
The brackets are defined as,
\[ \nu = \langle a \mid M \mid b \rangle = a_i M_{ij} g_j b_j, \]  

which includes the definition of the inner product for \( M = 1 \). This expression is the standard formula for the kinematic viscosity as obtained from a lattice Boltzmann equation [Brito et al 1991]. In lattice gas automata the occupation numbers obey an exclusion principle, so that \( g_i = \langle (\delta n_i)^2 \rangle_o = f^o_i (1 - f^o_i) \). In athermal lattice gases with or without rest particles the single particle state distribution function \( f^o_i = f \) is independent of the speed \( |\vec{c}_i| \) and equal to the average density \( \rho \) per node, divided by the number of allowed velocity channels.

4 Concluding Remarks

We offer several remarks to summarize and put the above results in context.

1) The noise considered here has been assumed to be white and Gaussian, and the resulting Boltzmann-Langevin equations define a discrete time Markov process. It is entirely characterized once the deterministic dynamics (i.e., the form of the Boltzmann collision operator \( V \)) and the noise intensity \( B \) is specified. Non-Gaussian noise could be considered as well, and most of the results obtained here for pair correlations still hold. However, higher order correlations would be different.

2) For stationary and quasi-stationary states the noise intensity has been determined by the consistency between the solution to the linearized Boltzmann-Langevin equation for equal time pair correlations and a specified stationary value for this correlation. The important new result here is the fluctuation-dissipation theorem (10). It relates the fluctuations to the matrix elements of the linear collision operator and the equilibrium value of the correlations \( g(x, y) \), which are assumed to be given. The relation is somewhat similar to that for continuous gases, but there are some typical lattice effects, caused by the discreteness of time and closely related to the so-called ‘propagation’ viscosities of lattice gases.

3) The linear Boltzmann-Langevin equations imply associated hydrodynamic equations with noise. The latter occurs as an additional component to the stress tensor. The fluctuation formula for the stress tensor (24), as derived from the Langevin noise added to the lattice Boltzmann equation, is in complete agreement with the results of Landau and Lifschitz for the noise in fluid dynamics. Furthermore, the explicit form of (24) identifies the noise intensity in terms of the Boltzmann value of the transport coefficient.

4) The examples at the end of section 2 concern models satisfying the conditions of SDB, where the equilibrium distributions are completely factorized and position and velocity correlations are completely absent, as in (11). However, this is not the case in LGCA’s violating SDB, where computer simulations have
shown [Hénon 1992, Bussemaker and Ernst 1992] the existence of (strong) on-node velocity correlations and (weaker) off-node spatial correlations in thermal equilibrium. The existence of such correlations is not yet understood. However, in driven diffusive systems [Zia et al 1993, Garrido et al 1990], where detailed balance is broken by imposing an external bias field or by boundary conditions, the absence of detailed balance does indeed give rise to long spatial correlations.

5) As long as the analytic structure of the equilibrium correlation function $g(x, y)$ is unknown, the fluctuation-dissipation relation (10) is still valid but does not provide any information about the noise strength, $B(x, y)$. The same remarks apply when dealing with stationary non-equilibrium states, supporting stationary temperature gradients or shear rates. For real fluids the equal time correlation functions in such states are long ranged, and have been calculated from both Langevin and kinetic theories [Schmitz 1988] for which the noise strength is specified. As this information is not yet available for LGCA’s further exploitation of (10) is not possible. Similar limitations apply to (11) that describes fluctuations around non-stationary states. To calculate the correlation functions it is necessary to know the noise strength as a function of the non-equilibrium state, $f(t)$. Again, this information is available for real fluids from kinetic theory but the corresponding analysis for LGCA’s is incomplete [Ernst and Dufty 1994].

6) So far we have discussed stable systems. For unstable systems (phase separation, spinodal decomposition, pattern formation) the fundamental quantity is the equal time correlation function $g(x, y, t)$ or its Fourier transform, the time dependent structure factor $S(\vec{k}, t)$. In the Cahn-Hilliard-Cook theory [Langer 1992] the Langevin equation has been used to study the onset of instability, the wavelength of maximum growth, and initial patterns. Such studies have been done for real systems as well as for LGCA’s, quenched into a spatially uniform but metastable or unstable state. For some LGCA’s simulations and analytic results (see Alexander et al [1992], Bussemaker and Ernst [1993], Ernst and Bussemaker 1994]) on $S(\vec{k}, t)$ and $g(x, y, t)$ are becoming available, but studies on fluctuations around such states have yet to be done. Once $B(x, y \mid f(t))$ is understood on the basis of a microscopic model, that information may be used for mathematical modeling of $g(x, y, t)$ via the fluctuation-dissipation relation (11).

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