FANO MANIFOLDS WITH WEAK ALMOST KÄHLER-RICCI SOLITONS

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Abstract. In this paper, we prove that a sequence of weak almost Kähler-Ricci solitons under further suitable conditions converge to a Kähler-Ricci soliton with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology. As a corollary, we show that on a Fano manifold with the modified K-energy bounded below, there exists a sequence of weak almost Kähler-Ricci solitons which converge to a Kähler-Ricci soliton with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology.

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0. Introduction

In [WZ], we studied the structure of the limit space for a sequence of Riemannian manifolds with the Bakry-Émery Ricci curvature bounded below in the Gromov-Hausdorff topology. In particular, for a sequence of weak almost Kähler-Ricci solitons \( \{(M_i, g^i, J_i)\} \), we showed that there exists a subsequence of \( \{(M_i, g^i, J_i)\} \) which converge to a metric space \( (Y, g_\infty) \) with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology. As in [CC] for Riemannian manifolds with the Ricci curvature bounded below, each tangent space on \( (Y, g_\infty) \) is a metric cone. The present paper is a continuance of [WZ]. We further prove the smoothness of the

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metric $g_\infty$ on the regular part $\mathcal{R}$ of $Y$ under further suitable conditions. Actually, $g_\infty$ is a Kähler-Ricci soliton on $\mathcal{R}$.

Inspired by a recent work of Tian and Wang on weak almost Einstein metrics [TW], we use the Kähler-Ricci flow to smooth the sequence of Kähler metrics $\{(M_i, g^i, J_i)\}$ to get the $C^\infty$-convergence. To realize this, we shall first establish a version of Perelman’s pseudolocality theorem for the Hamilton’s Ricci flow with the Bakry-Émery Ricci curvature condition, then we control the deformation of distance functions along the Kähler-Ricc flow as in [TW].

It is useful to mention that there are two new ingredients in our case compared to [TW]: One is that we modify the Kähler-Ricci flow to derive an estimate for the modified Ricci curvature (cf. Section 2); another is that we estimate the growth of the $C^0$-norm of holomorphic vector fields associated to the Kähler-Ricci solitons along the flow (cf. Section 4). The latter is usually dependent of the initial metric $g^i$ of the Kähler-Ricci flow. But for a family of Kähler metrics $g^s$ ($0 < s < 1$) constructed from solutions of a family of complex Monge-Ampère equations on a Fano manifold with the modified K-energy bounded below [WZ], we get a uniform $C^0$-norm for the holomorphic vector field (cf. Lemma 4.8) under the deformed metrics.

The following can be regarded as the main result in this paper.

**Theorem 0.1.** Let $(M, J)$ be a Fano manifold with the modified K-energy bounded below. Then there exists a sequence of weak almost Kähler-Ricci solitons on $(M, J)$ which converge to a Kähler-Ricci soliton with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology. In the other words, a Fano manifold with the modified K-energy bounded below can be deformed to a Kähler-Ricci soliton with complex codimension of singularities at least 2.

The organization of paper is as follows. In Section 1, we prove a pseudolocality theorem of Perelman for the Hamilton’s Ricci flow under the Bakry-Émery Ricci curvature condition. In Section 2, we focus on the (modified) Kähler-Ricc flow to give a local estimate for the Ricci curvature along the flow. Section 3 is devoted to estimate the distance functions along the Kähler-Ricc flow. In Section 4, we prove the main theorems in this paper, Theorem 4.2 and Theorem 4.5 (Theorem 0.1).

1. **A Version of Pseudolocality Theorem**

In this section, we prove a version of Perelman’s pseudolocality theorem with Bakry-Émery Ricci curvature condition (cf. Theorem 11.2 in [Pe]). A similar version was recently appeared in [TW]. Since our case is lack of
the lower bound of scalar curvature, in particular the lower bound of Ricci curvature, we will modify the arguments both in [Pe] and [TW].

First we recall a result about an estimate of isoperimetric constant on a geodesic ball. This result comes essentially from a lemma in [Li] and the volume comparison theorem with the Bakerly-Émergy Ricci curvature bounded below in [WW].

Lemma 1.1. Let \((M, g)\) be a Riemannian manifold with

\[
\text{Ric}(g) + \text{hess}_g f \geq -(n-1)c g, \quad |\nabla_g f| \leq A.
\]

Then for any geodesic balls in \(M\), \((B_p(s)), (B_p(r))\) with \(r \geq s\), there exists a uniform \(C = C(n)\) such that

\[
\text{ID}_{\frac{n}{n-1}}(B_p(s)) \geq C(\frac{\text{vol}(B_p(r)) - \text{vol}(B_p(s))}{v(r+s)})^{\frac{n+1}{n}},
\]

where \(v(r) = e^{2Ar} \text{vol}_e(r)\) and \(\text{vol}_e(r)\) denotes the volume of \(r\)-geodesic ball in the space form with constant curvature \(-c\).

Lemma 1.1 will be used to get a uniform Sobolev constant in the proof of following pseudolocality theorem in the Bakerly-Émergy geometry.

Theorem 1.2. For any \(\alpha, r \in [0, 1]\), there exist \(\tau = \tau(n, \alpha), \eta = \eta(n, \alpha), \epsilon = \epsilon(n, \alpha), \delta = \delta(n, \alpha)\), such that if \((M^n, g(\cdot, t)) (0 \leq t \leq (er)^2)\) is a solution of Ricci flow,

\[
\frac{\partial g}{\partial t} = -2\text{Ric}(g),
\]

whose initial metric \(g(\cdot, 0) = g_0\) satisfies

\[
\text{Ric}(g_0) + \text{hess}_{g_0} f \geq -(n-1)r^{-2}\text{vol}(g_0), \quad |\nabla f|_{g_0} \leq r^{-1}\eta,
\]

and

\[
\text{vol}(B_q(r, g_0)) \geq (1 - \delta)c_d r^n,
\]

where \(c_d\) is the volume of unit ball in the Euclidean space \(\mathbb{R}^n\), then for any \(x \in B_q(\epsilon r, g_0)\) and \(t \in (0, (er)^2]\), we have

\[
|Rm(x, t)| < \alpha t^{-1} + (er)^{-2},
\]

Moreover,

\[
\text{vol} B_x(\sqrt{t}) \geq \kappa(n) t^\frac{n}{2},
\]

where \(\kappa(n)\) is a uniform constant.
Proof. By scaling the metric, we may assume \( r = 1 \) in the theorem. As in [Pe], we use the argument by contradiction to prove (1.6). On contrary, we suppose that for some \( \alpha > 0 \), there are \( \tau_i, \eta_i, \delta_i, \epsilon_i \) which approaching zero as \( i \to \infty \), and there are a sequence of manifolds \( \{(M_i, g^i)\} \) which satisfying (1.4) and (1.5) with some points \( q_i \in M_i \) such that (1.6) doesn’t hold at \((x_i, t_i)\) for some points \( x_i \in B_{\eta_i}(\epsilon_i, g^i_0) \) some time \( \tilde{t}_i \leq \epsilon_i^2 \) along the Ricci flows \( (M_i, g^i_t) = g^i(\cdot, t) \) with \( g^i_0 \) as the initial metrics. Without the loss of generality, we may also assume that

\[
|Rm(x, t)| \leq \alpha t^{-1} + (\epsilon_i)^{-2}, \quad \forall \ t \in (0, \tilde{t}_i], \ x \in B_{\eta_i}(\epsilon_i, g^i_0).
\]

Then as showed in [Pe], for any \( A < \frac{1}{100n\epsilon_i} \), there exist points \((\tilde{x}_i, \tilde{t}_i)\) such that for any \((x, t)\) with

\[
\tilde{t}_i - \frac{1}{2}\alpha Q^{-1} \leq t \leq \tilde{t}_i, \ d_{g^i}(x, \tilde{x}_i) \leq \frac{1}{10}AQ^{-\frac{1}{2}},
\]

(1.9) \[
|Rm(x, t)| \leq 4Q,
\]

where \( Q = |Rm(\tilde{x}_i, \tilde{t}_i)| \to \infty \).

Now we consider a solution \( u_i(x, t) = (4\pi(\tilde{t}_i - t))^{-\frac{n}{2}}e^{-p_i(t, x)} \) of the conjugate heat equation associated to the flow \((M_i, g^i_t)\) which starts from a delta function \( \delta(\tilde{x}_i, \tilde{t}_i) \). Namely, \( u_i(x, t) \) satisfies

\[
\Box^* u_i(x, t) = (-\frac{\partial}{\partial t} - \Delta + R)u_i(x, t) = 0,
\]

where \( R = R(\cdot, t) \) is the scalar curvature of \( g^i_t \). Then the function

\[
v_i(x, t) = [(\tilde{t}_i - t)(2\Delta p_i - |\nabla p_i|^2 + R) + p_i - n]u_i
\]

is nonpositive. Moreover, there exists a positive constant \( \beta \) such that

(1.10) \[
\int_{B_{\tilde{x}_i}(\sqrt{\tilde{t}_i - \epsilon_i, g^i_{\tilde{t}_i}})} v_i \leq -\beta,
\]

for some \( \tilde{t}_i \in [\tilde{t} - \frac{1}{2}\alpha Q^{-1}, \tilde{t}_i] \), when \( i \) is large enough [Pe].

Let \( \phi \) be a cut-off function which is equal to 1 on \([0, 1]\) and decreases to 0 on \([1, 2]\). Moreover, it satisfies \( \phi'' \geq -10\phi, (\phi')^2 \leq 10\phi \). Putting \( h_i = \phi(\frac{\tilde{d}(x, t)}{10A\sqrt{\tilde{t}_i}}) \), where \( \tilde{d}(x, t) = d_{g^i}(x, \tilde{x}_i) + 200n\sqrt{\tilde{t}} \). Then by Lemma 8.3 in [Pe] with the help of (1.8), we get

\[
(\frac{\partial}{\partial t} - \Delta)h_i = \frac{1}{10A\sqrt{\tilde{t}_i}}(d_t - \Delta d + \frac{100n}{\sqrt{t}})\phi' - \left(\frac{1}{10A\sqrt{\tilde{t}_i}}\right)^2 \phi'' \leq \left(\frac{1}{10A\sqrt{\tilde{t}_i}}\right)^2 10\phi, \ \forall \ t \in (0, \tilde{t}_i], \ x \in B_{\eta_i}(\epsilon, g^i_0).
\]
It follows that
\[
\frac{d}{dt} \int_{M_i} (-h_i v_i) = \int_{M_i} \square h_i (-v_i) + \int_{M_i} h_i \square^* v_i \\
\leq - \frac{1}{100A^2 \hat{t}_i} \int_{M_i} h_i v_i,
\]
where \( \square = \frac{\partial}{\partial t} - \Delta \) and we used the fact that \( \square^* v_i \leq 0 \) \[\text{Pe}\]. Thus by (1.10), we obtain

(1.11) \[
\beta(1 - A^{-2}) \leq - \int_{M_i} (h_i v_i)(0, \cdot).
\]

Similarly, we can show

(1.12) \[
\int_{M_i} (\hat{h}_i u_i)(\cdot, 0) \geq 1 - 4A^{-2},
\]
where \( \hat{h}_i = \phi(\frac{\hat{t}_i(x, t)}{5A\sqrt{\hat{t}_i}}) \). The above implies that

(1.13) \[
\int_{B_{\hat{t}_i}(20A\sqrt{\hat{t}_i}) \setminus B_{\hat{t}_i}(10A\sqrt{\hat{t}_i})} u_i(\cdot, 0) \leq 1 - \int_{M_i} \hat{h}_i u_i \leq 4A^{-2}.
\]

On the other hand, by (1.13), we see that

\[
- \int_{M_i} (h_i v_i) = \int_{M_i} \big[ -\hat{t}_i (-2\Delta p_i + |\nabla p_i|^2 - R) - p_i + n \big] h_i u_i \\
= \int_{M_i} \big[ -\hat{t}_i |\nabla \tilde{p}_i|^2 - \tilde{p}_i + n \big] \tilde{u}_i + \int_{M_i} \big[ \hat{t}_i (|\nabla h_i|^2 - R h_i) - h_i \ln h_i \big] u_i \\
\leq \int_{M_i} \big[ -\hat{t}_i |\nabla \tilde{p}_i|^2 - \tilde{p}_i + n \big] \tilde{u}_i - \hat{t}_i \int_{M_i} R \tilde{u}_i + A^{-2} + 100 \epsilon^2,
\]

where \( \tilde{u}_i = h_i u_i \) and \( \tilde{p}_i = p_i - \ln h_i \). Note that \( R + \Delta f_i \geq -(n - 1) \). Then

\[
- \int_{M_i} R \tilde{u}_i \leq n - 1 - \int_{M_i} (\nabla f_i, \tilde{u}_i) \leq n - 1 + \eta_i \int_{M_i} |\nabla \tilde{u}_i| \\
\leq n - 1 + \eta_i \sqrt{\int_{M_i} |\nabla \tilde{p}_i|^2 \tilde{u}_i \leq n + \eta_i \int_{M_i} |\nabla \tilde{p}_i|^2 \tilde{u}_i}.
\]

Hence, by (1.11), we get

\[
\int_{M_i} \big[ -\hat{t}_i (1 - \eta_i) |\nabla \tilde{p}_i|^2 - \tilde{p}_i + n \big] \tilde{u}_i \geq \beta(1 - A^{-2}) - (100 + n) \epsilon^2 - A^{-2}.
\]

Therefore, by rescaling these metrics \( g_0^i \) to \( \tilde{g}_0^i = \frac{1}{2} [\hat{t}_i (1 - \eta_i)]^{-1} g_0^i \), we derive

(1.14) \[
\int_{B_{\hat{t}_i}(20A)} \big[ -\frac{1}{2} |\nabla \tilde{p}_i|^2 - \tilde{p}_i + n \big] \tilde{u}_i \geq (1 - \eta_i) \frac{\hat{t}_i}{2} \mu > \mu_0 > 0,
\]
where \( \hat{u}_i = (2\pi)^{-\frac{n}{2}} e^{-\frac{\mu}{2}} \) and \( \mu = \beta(1-A^{-2}) - (100+n)e^2 - A^{-2} \). Normalize \( \hat{u}_i \) by multiplying a constant \( c \) so that

\[
\int_{B_{\hat{x}_i}(20A)} c\hat{u}_i = 1.
\]

By (1.12), it is easy to see that (1.14) still holds for the normalized \( \hat{u}_i \).

Next as in [TW], we introduce a functional

\[
F_i(u) = \int_{B_{\hat{x}_i}(20A)} (2|\nabla u|^2 - 2u^2 \log u - n(1 + \log \sqrt{2\pi})u^2),
\]

defined for any nonnegative functions \( u \in W^{1,2}_0(B_{\hat{x}_i}(20A), g_0^\hat{\beta}) \) with

\[
\int_{B_{\hat{x}_i}(20A)} u^2 = 1.
\]

Clearly, by (1.14), we have

(1.15) \( \lambda_i \leq F_i(\sqrt{c\hat{u}_i}) \leq -\mu_0 < 0, \)

where \( \lambda_i = \inf_{u \in W^{1,2}_0(B_{\hat{x}_i}(20A), g_0^\hat{\beta})} F_i(u) \). According to [Ro], the infinity of \( F_i(u) \) can be achieved by a minimizer \( \phi_i \) which satisfies the Euler-Lagrange equation on \( (B_{\hat{x}_i}(20A), g_0^\hat{\beta}) \),

(1.16) \(-2\Delta \phi_i(x) - 2\phi_i(x) \log \phi_i(x) - n(1 + \log \sqrt{2\pi})\phi_i(x) = \lambda_i \phi_i(x).\)

We need to estimate the \( L^\infty \)-norms and gradient norms of those \( \phi_i \). Note that \( \log x \leq \frac{n}{2} x^{\frac{2}{n}} \). Then

\[
\lambda_i + n(1 + \log \sqrt{2\pi})
\]

\[
= 2 \int_{B_{\hat{x}_i}(20A)} |\nabla \phi_i(x)|^2 - 2 \int_{B_{\hat{x}_i}(20A)} \phi_i(x)^2 \log \phi_i(x)
\]

\[
\geq 2 \int_{B_{\hat{x}_i}(20A)} |\nabla \phi_i(x)|^2 - n \int_{B_{\hat{x}_i}(20A)} \phi_i(x) \phi_i(x)^\frac{n+2}{n}
\]

\[
\geq 2 \int_{B_{\hat{x}_i}(20A)} |\nabla \phi_i(x)|^2 - n \int_{B_{\hat{x}_i}(20A)} \phi_i(x)^\frac{2n}{n-2} \phi_i(x)^\frac{n-2}{n-2} + \phi_i(x)^2
\]

\[
\geq 2 \int_{B_{\hat{x}_i}(20A)} |\nabla \phi_i(x)|^2 - a^2 (\int_{B_{\hat{x}_i}(20A)} \phi_i(x)^\frac{2n}{n-2} \phi_i(x)^\frac{n-2}{n-2} + \phi_i(x)^2).
\]

Since the Sobolev constants \( C_S \) are uniformly bounded below on \( (B_{\frac{1}{2}}(x), g_0^\beta) \) according to Lemma 1.1, by choosing the number \( a \) small enough, we see that \( \lambda_i \) is uniformly bounded below and \( \int_{B_{\hat{x}_i}(20A)} |\nabla \phi_i(x)|^2 \) is uniformly bounded. Applying the standard Moser iteration method to (1.16), we will get

(1.17) \( |\phi_i(x)| < C_1(\mu_0, n, C_S). \)
As a consequence, \( \phi_i(x) \) is an almost sub-solution of the Laplace equation. Hence, we can also get a uniform oscillation estimate for \( \phi_i(x) \) near the boundary of \( B_{\bar{x}_i}(20A) \). In fact, as in [WT], we can show that for any \( w \in \partial B_{\bar{x}_i}(20A) \)

\[
\text{Osc}_{B_{\bar{x}_i}(20A)}(\phi_i) < C\gamma^{N-1} + \frac{\gamma^{N-1} - 4^{-N+1}}{4(4\gamma - 1)},
\]

for some uniform \( C \), where \( N \geq 2 \) is any integer and the number \( \gamma \) can be chosen in the interval \( (\frac{1}{2}, 1) \).

To get the interior gradient estimate for \( \phi_i(x) \), we will also use the Moser iteration method. For simplicity, we let \( \phi = \phi_i \) for each \( i \). First we note that by (1.16) and the estimate (1.17), it holds

\[
\langle \nabla \phi, \nabla \Delta \phi \rangle \geq -C_2(\nu, n, C_S) |\nabla \phi(x)|^2.
\]

(1.19)

Then by the Bochner identity,

\[
\frac{1}{2} \Delta |\nabla \phi|^2 = |\text{hess } \phi|^2 + R_{ij} \phi_i \phi_j + \langle \nabla \phi, \nabla \Delta \phi \rangle,
\]

we obtain

\[
\frac{1}{2} \Delta |\nabla \phi|^2 \geq |\text{hess } \phi|^2 - f_{ij} \phi_i \phi_j - (C_2 + (n - 1)\tau^2)|\nabla \phi|^2.
\]

(1.20)

Let \( \rho \) be a cut-off function on the interval \([0, 20A]\) which is supported in a subset of \([0, 20As]\), where \( s < 1 \). Then multiplying both sides of (1.20) by \( \rho(d(\bar{x}_i, .))w^p \), where \( w = |\nabla \phi|^2 \) and \( p \geq 0 \), we get

\[
\frac{2p}{(p+1)^2} \int_{B_{\bar{x}_i}(20A)} \rho(d(\bar{x}_i, .))|\nabla w|^p \frac{w^{p+1}}{2} d\mu
\]

\[
= \frac{1}{2} \int_{B_{\bar{x}_i}(20A)} \rho(d(\bar{x}_i, .))(-\Delta w)w^p - \frac{1}{2} \int_{B_{\bar{x}_i}(20A)} \langle \nabla \rho(d(\bar{x}_i, .)), \nabla w \rangle w^p
\]

\[
\leq \int_{B_{\bar{x}_i}(20A)} \rho(d(\bar{x}_i, .))w^p (-|\text{hess } \phi|^2 + f_{ij} \phi_i \phi_j + C_2 |\nabla \phi|^2)
\]

(1.21)

\[- \frac{1}{2} \int_{B_{\bar{x}_i}(20A)} w^p \langle \nabla \rho(d(\bar{x}_i, .)), \nabla w \rangle.
\]

On the other hand, using the integration by parts, we have

\[
\int_{B_{\bar{x}_i}(20A)} \rho(d(\bar{x}_i, .))w^p f_{ij} \phi_i \phi_j
\]

\[
= - \int_{B_{\bar{x}_i}(20A)} \rho f_i \phi_i \phi_j w^p + \int_{B_{\bar{x}_i}(20A)} \rho f_j \phi_i \phi_j w^p
\]

\[- p \int_{B_{\bar{x}_i}(20A)} \rho w^{p-1} f_i w_j \phi_i \phi_j - \int_{B_{\bar{x}_i}(20A)} \rho w^p f_i \phi_i \Delta \phi.
\]
Observe that
\[
\left| \int_{B_{\hat{g}_0}(20A)} \rho f_i \phi_j \phi^i \phi^j w^p \right| \leq \eta \int_{B_{\hat{g}_0}(20A)} \rho |w| w^{p+1},
\]
\[
\left| \int_{B_{\hat{g}_0}(20A)} \rho f_i \phi_j \phi^j \phi^j w^p \right| \leq 2\eta \int_{B_{\hat{g}_0}(20A)} \rho (|\text{hess} \phi|^2 + w^2) w^p,
\]
\[
\left| \int_{B_{\hat{g}_0}(20A)} \rho w^{p-1} f_i \phi_j \phi^i \phi^j w^p \right| \leq 2\eta \left( \int_{B_{\hat{g}_0}(20A)} \rho w^{p-1} |\nabla w|^2 + \int_{B_{\hat{g}_0}(20A)} \rho w^{p+1} \right),
\]
\[
\left| \int_{B_{\hat{g}_0}(20A)} \rho w f_i \phi_i \Delta \phi \right| \leq C_3 (v_0, n, C, s) \eta \int_{B_{\hat{g}_0}(20A)} \rho w^{p+1}.\]

Hence, by (1.21), we get
\[
\frac{p}{(p+1)^2} \int_{B_{\hat{g}_0}(20A)} \rho (d(\tilde{x}, .))|\nabla w|^2 \leq C_4 \int_{B_{\hat{g}_0}(20A)} (\rho + p\eta \rho + \rho') w^{p+1} + C_5 \int_{B_{\hat{g}_0}(20A)} \eta w^{p+\frac{1}{2}}.
\]

Since we may assume that \(w \geq 1\), we deduce
\[
\frac{p}{(p+1)^2} \int_{B_{\hat{g}_0}(20A)} \rho (d(\tilde{x}, .))|\nabla w|^2 \leq C_5 \int_{B_{\hat{g}_0}(20A)} (\rho + p\eta \rho + \rho') w^{p+1}, \forall \ p \geq 0.
\]

(1.22)

Note that the Sobolev constants are uniformly bounded below on \((B_{\hat{g}_0}(20A), \hat{g}_0)\). Therefore, by choosing the suitable cut-off functions \(\eta\) in (1.22), we use the iteration method to derive
\[
\|\nabla \phi_i\|_{C^0(B_{\hat{g}_0}(20A))}^2 \leq C_6 (1 + \int_{B_{\hat{g}_0}(20A)} |\nabla \phi_i|^2) < C.
\]

(1.23)

It remains to analyze the limit of \(\phi_i\). According to Corollary 4.8 in [WZ], we see that \((M_i, \hat{g}_0)\) converge to the euclidean space \(\mathbb{R}^n\) in the Gromov-Hausdorff topology. Thus by the estimates (1.17) and (1.23), there exists a subsequence of \(\phi_i\) which converge to a continuous limit \(\phi_\infty \geq 0\) on the standard \(B_0(20A) \subset \mathbb{R}^n\).

Claim 1.3. \(\phi_\infty\) is a solution of the following equation on \(B_0(20A)\),
\[
-2\Delta \phi_\infty - 2\phi_\infty \log \phi_\infty - n(1 + \log \sqrt{2\pi}) \phi_\infty = \lambda_\infty \phi_\infty,
\]
(1.24)

where \(\lambda_\infty < 0\).

As in [TW], to prove (1.24), it suffices to show that
\[
-\phi_\infty = \int_{B_0(20A)} G(z, y) \left( \frac{\lambda_\infty + n(1 + \log \sqrt{2\pi})}{2} + \log \phi_\infty \right) \log \phi_\infty.
\]
(1.25)
Here $G(z, y)$ is the Green function on the ball $B_0(20A)$, which is given by
\[
G(z, y) = \frac{1}{(n-2)nc_n} (d^{2-n}(z, y) - d^{2-n}(0, z)d^{2-n}(z^*, y)),
\]
where $z^*$ is the conjugate point of $z$.

Choose a sequence $z_k \to z, z_k^* \to z^*$. By the Laplacian comparison for the distance functions on $(B_{x_i}(20A), \hat{g}_0^i)$ [WW],
\[
\Delta d(z_k, .) \leq (n-1)\tau_i \coth \tau_i d(z_k, .) + 2\eta_i \leq \frac{n-1}{d(z_k, .)} + (n-1)\tau_i + 2\eta_i,
\]
we have
\[
\Delta d^{2-n}(z_k, .) + (n-2)d^{1-n}(z_k, .)((n-1)\tau_i + 2\eta_i) \geq 0.
\]

It follows
\[
\int_{B_{x_k}(20A) \setminus z_k} |\Delta d^{2-n}(z_k, .)| \leq \int_{B_{x_k}(20A) \setminus \{z_k\}} |\Delta d^{2-n}(z_k, .) + (n-2)d^{1-n}(z_k, .)((n-1)\tau_i + 2\eta_i)|
\]
\[
+ \int_{B_{x_k}(20A) \setminus \{z_k\}} (n-2)d^{1-n}(z_k, .)((n-1)\tau_i + 2\eta_i).
\]

By a direct computation, we obtain
\[
\int_{B_{x_k}(20A) \setminus \{z_k\}} |\Delta d^{2-n}(z_k, .)| \to 0, \text{ as } k \to \infty.
\]

Note that
\[
\int_{B_{x_k}(20A)} d^{2-n}(z_k, y)\Delta \phi_k(y) = (n-2)nc_n\phi(z_k) + \int_{B_{x_k}(20A) \setminus \{z_k\}} \phi_k(y)\Delta d^{2-n}(z_k, y).
\]

Hence we derive that
\[
\lim_{k \to \infty} \int_{B_{x_k}(20A)} d^{2-n}(z_k, y)\Delta \phi_k(y) = (n-2)nc_n\phi(z).
\]

Similarly, since $z_k^*$ is outside $B_{x_k}(20A)$, we have
\[
\lim_{k \to \infty} \int_{B_{x_k}(20A)} d^{2-n}(z_k^*, y)\Delta \phi_k(y) = 0.
\]
Combining (1.27) and (1.28), we get
\[-\phi_\infty(z) = -\lim_{k \to \infty} \int_{B_k(20A)} (d^{2-n}(z_k, y) - d^{2-n}(x_k, z_k) d^{2-n}(z^*_k, y)) \Delta \phi_k(y) \]
\[= \lim_{k \to \infty} \int_{B_k(20A)} (d^{2-n}(z_k, y) - d^{2-n}(x_k, z_k) d^{2-n}(z^*_k, y)) \times \left( \frac{\lambda_k + n(1 + \log \sqrt{2\pi})}{2} + \log \phi_k \right) \phi_k \]
\[= \int_{B_0(20A)} G(z, y) \left( \frac{\lambda_\infty + n(1 + \log \sqrt{2\pi})}{2} + \log \phi_\infty \phi_\infty \right).\]
The claim is proved.

By the estimates (1.18), \(\phi_\infty\) is in fact in \(C_0(B_0(20A))\). Thus by (1.24), we get
\[F(\phi_\infty) = \int_{B_0(20A)} (2|\nabla \phi_\infty|^2 - 2\phi_\infty^2 \log \phi_\infty - n(1 + \log \sqrt{2\pi}) \phi_\infty^2) = \lambda_\infty < 0,\]
which is a contradiction to the Log-Sobolev inequality in \(\mathbb{R}^n\) [Gr]. The proof of (1.6) is completed. \(\square\)

To obtain (1.7), it suffices to estimate the lower bound of the injective radius at \(x\). This can be done by using the same blowing-up argument as in the proof of (1.6) (cf. [Pe], [TW]). We leave it to the readers.

2. A Ricci curvature estimate

In this section, we prove several technical lemmas which will be used in next sections. From now on we assume that \(M\) is an \(n\)-dimensional Fano manifold with a reductive holomorphic vector field \(X\) [TZ1]. As in [TZ3], we consider the following modified Kähler-Ricci flow,
\[\frac{\partial}{\partial t} g = -\text{Ric}(g) + g + L_X g,\] (2.1)
with a \(K_X\)-invariant initial Kähler metric \(g_0\) in \(2\pi c_1(M)\), where \(K_X\) is the one-parameter compact subgroup generated by \(\text{im}(X)\). Thus \(L_X g\) is a real valued complex hessian tensor. If we scale \(g_0\) by \(\frac{1}{\lambda}\), where \(0 < \lambda \leq 1\), then (2.1) becomes
\[\frac{\partial}{\partial t} g = -\text{Ric}(g) + \lambda g + \lambda L_X g.\] (2.2)
Clearly, the flow is solvable for any \(t > 0\) and \(\omega_{g_t} \in \frac{2\pi}{\lambda} c_1(M)\), where \(g_t = g(\cdot, t)\).
By a direct computation from the flow (2.2), we see that
\[
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - R_{ik} R_{kj} + R_{ik} R_{ijkl} - \lambda \Delta \theta_{ij} + \frac{\lambda}{2} (R_{ik} \theta_{kj} + R_{kj} \theta_{ik}) - \lambda R_{ijkl} \theta_{lk}.
\] (2.3)
and
\[
\frac{\partial}{\partial t} \theta_{ij} = L_X (-\text{Ric}(g) + \lambda g + \lambda L_X g),
\]
where \( \theta = \theta_{gt} \) is a potential of \( X \) associated to \( g_t \) such that \( \theta_{ij} = L_X g_t \).

Thus if we let \( H = \text{Ric}(g) - \lambda g - \lambda L_X g, \) then we have
\[
\frac{\partial}{\partial t} H = \Delta H + \lambda L_X H + \Lambda(H, Rm),
\] (2.4)
where \( \Lambda \) is a linear operator with bounded coefficients with respect to the metric \( g_t \) and \( Rm = Rm(\cdot, t) \) is the sectional curvature of \( g_t \).

Moreover, we have

**Lemma 2.1.**
\[
\frac{\partial}{\partial t} (R - \lambda \Delta \theta - n \lambda) = \lambda (R - \lambda \Delta \theta - n \lambda) + \Delta (R - \lambda \Delta \theta - n \lambda) - \lambda \Delta \frac{\partial}{\partial t} \theta + |\text{Ric}(g) - \lambda g - \lambda \sqrt{-1} \partial \bar{\partial} \theta|^2.
\] (2.5)

The following lemma is a consequence of Theorem 1.2 in Section 1.

**Lemma 2.2.** Let \( g = g_t \) be a solution of (2.2) with \( \omega_{g_0} = \frac{2\pi}{n} c_1(M) \). Suppose that there exists a small \( \delta \leq \delta_0 << 1 \) such that \( g_0 \) satisfies:

i) \( \text{Ric}(g_0) + \lambda L_X g_0 \geq -(n - 1) \delta^2 g_0; \)

ii) \( |X|_{g_0}(x) \leq \frac{\delta}{\lambda}, \forall x \in B_q(1, g_0); \)

iii) \( \text{vol}(B_q(1, g_0)) \geq (1 - \delta) c_n. \)

Then
\[
|Rm(x, t)| \leq 4t^{-1}, \forall x \in B_q(\frac{3}{4}, g_0), t \in (0, 2\delta]
\]
and
\[
\text{vol}(B_x(\sqrt{t}, g(t))) \geq \kappa(n) t^n,
\]
where \( \kappa = \kappa(n) \) is a uniform constant.

By Lemma 2.1 and Lemma 2.2, we prove...
Lemma 2.3. Let \( g = g_t \) be a solution of (2.2) with \( \omega_{g_0} \in 2\pi c_1(M) \). Suppose that for any \( t \in [-2, 1] \) (we may replace \( t \) by \( t - 3 \)), \( g_t \) satisfies:

i) \( \text{inj}(g, g_t) \geq 1; \)

ii) \( |Rm(x, t)| \leq 1 \) and \( |X|_{g_t} \leq \frac{A}{\lambda}, \forall x \in B_q(1, g_t). \)

Then

\[
|\text{Ric}(g) - \lambda g - \lambda L_X g|(q, 0) \leq C(A, n)\left\{ \int_{-2}^{1} dt \int_M \left| R - n\lambda - \Delta \theta |\omega_{g_t}^n \right|^2 \right\}^{\frac{1}{2}}.
\] (2.6)

Proof. Putting \( h = |H| \), by (2.4), we get

\[
(\frac{\partial}{\partial t} - \Delta)h \leq \frac{\Lambda_1(H, H, Rm)}{h} + \lambda X(h) + \frac{\Lambda_2(\sqrt{-1}\partial \bar{\partial} \theta, H, H)}{h},
\] (2.7)

where \( \Lambda_1, \Lambda_2 \) are two linear operators with bounded coefficients with respect to the metric \( g_t \). Note that under the conditions i) and ii) in the lemma the Sobolev constants are uniformly bounded below on \( B_q(\frac{1}{2}, g_0) \). Then using the method of Moser iteration, we obtain

\[
|\text{Ric}(g) - \lambda g - \lambda L_X g|(q, 0) \leq C(A, n)\left\{ \int_{-2}^{1} dt \int_M \left| R - n\lambda - \Delta \theta |\omega_{g_t}^n \right|^2 \right\}^{\frac{1}{2}}.
\] (2.8)

On the other hand, we see that there exist some \( t_1 \in [-2, -1] \) and \( t_2 \in [0, 1] \) such that

\[
\begin{align*}
\int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_{t_1}}^n &\leq \int_{-2}^{-1} dt \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_t}^n, \\
\int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_{t_2}}^n &\leq \int_0^1 dt \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_t}^n.
\end{align*}
\]

Then integrating (2.1) in Lemma 2.1 it follows

\[
\begin{align*}
&\int_{t_1}^{t_2} dt \int_M |\text{Ric}(g) - \lambda g - \lambda L_X g|^2 \omega_{g_t}^n \\
&\leq \int_{t_1}^{t_2} dt \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_t}^n \\
&+ \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_{t_1}}^n + \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_{t_2}}^n \\
&\leq 3 \int_{-2}^{1} dt \int_M |R - \lambda \Delta \theta - n\lambda |\omega_{g_t}^n.
\end{align*}
\] (2.9)
Hence by \((2.8)\), we derive
\[
h(q,0) \leq C(A,m) \left\{ \int_{t_1}^{t_2} dt \int_M |R - \lambda \Delta \theta - n \lambda |\omega^n_{g_t}|^\frac{1}{2} \right\}
\leq 3C(A,m) \left\{ \int_{t_1}^{t_2} dt \int_M |R - \lambda \Delta \theta - n \lambda |\omega^n_{g_t}|^\frac{1}{2} \right\}.\]
\[
\square
\]

**Lemma 2.4.** Under the conditions of Lemma 2.2 and \(|X|_{g_t} \leq \frac{A}{\sqrt{t}}\), we have
\[
|Ric(g) - \lambda g - \lambda L_X g|(x,s) \leq C(n,A)s^{-\frac{n+2}{2}} \left\{ \int_0^{2s} dt \int_M |R - n \lambda - \lambda \Delta \theta|\omega^n_{g_t}|^\frac{1}{2} \right\},
\]
for \(0 < s \leq \delta\).

**Proof.** By Lemma 2.2, we know that for \(x \in B_q(3^4, g_0)\) and \(t \in (0,2\delta]\),
\[
|Rm(x,t)| \leq t^{-1} \text{ and vol}(B_x(\sqrt{t})) \geq \kappa(n)t^n.
\]
Then the injective radius estimate in [CGT] implies that
\[
inj(x,t) \geq \xi(n)\sqrt{t}.
\]
Let \(l = \xi(n)^{-1}s^{-\frac{1}{2}}\). By scaling the metric \(g_t\) as
\[
\tilde{g}_t = l^2 g(l^{-2}t + s), t \in [-2,1],
\]
\(\tilde{g}_t\) satisfies
\[
\frac{\partial}{\partial t} \tilde{g} = -Ric(\tilde{g}) + \frac{\lambda}{l^2} \tilde{g} + \frac{\lambda}{l^2} L_X \tilde{g}.
\]
Moreover, \(\tilde{g}_t\) satisfies the conditions i) and ii) in Lemma 2.3 for any \(t \in [-2,1]\) while \(\lambda\) is replaced by \(\frac{\lambda}{l^2}\).

Note that
\[
|X|_{\tilde{g}_t} = l|X|_g \leq \frac{2Cl}{\lambda \sqrt{s}} = \frac{2C\xi(n)l^2}{\lambda}.
\]
Applying Lemma 2.3 to \(\tilde{g}_t\), we have
\[
|Ric(\tilde{g}) - \frac{\lambda}{l^2} \tilde{g} - \frac{\lambda}{l^2} L_X \tilde{g}|(x,0) \leq C(n,A) \left\{ \int_{-2}^{1} dt \int_M |R(\tilde{g}) - n \lambda \frac{1}{l^2} tr_{\tilde{g}}(L_X \tilde{g})|\omega^n_{\tilde{g}_t}|^\frac{1}{2} \right\}.
\]
Observe that
\[
|Ric(g) - \lambda g - \lambda L_X g|_{g_t}(x,s) = l^2 |Ric(\tilde{g}) - \frac{\lambda}{l^2} \tilde{g} - \frac{\lambda}{l^2} L_X \tilde{g}|_{g_t}(x,0)\]
and
\[ \int_{s-l-2}^{s+2l-2} dt \int |R - n\lambda - \lambda \Delta \theta| \omega_{g_t}^n \]
\[ = l^{-n} \int_{-2}^{1} dt \int |R(g) - n\frac{\lambda}{l^2} - \frac{\lambda}{l^2} tr_g(L_X g)| \omega_{g_t}^n. \]
Thus we get
\[ |\text{Ric}(g) - \lambda g - \lambda L_X g|_g(x, s) \]
\[ \leq C(n, A)s^{-n+2} \{ \int_{s-l-2}^{s+2l-2} dt \int |R - n\lambda - \lambda \Delta \theta| \omega_{g_t}^n \}^\frac{1}{2}, \]
which implies (2.10). \(\square\)

3. Estimate for the distance functions
We are going to compare the distance functions between the initial metric \(g_0\) and \(g_\delta\) in the flow (2.2). The following lemma is due to Perelman for the normalized Ricci flow \([Pe]\).

**Lemma 3.1.** Let \(g_t = g(\cdot, t) (0 \leq t \leq 1)\) be a solution of rescaled Ricci flow on \(M^n\) (in our case, \(M\) is Kähler),
\[ \frac{d}{dt} g = -\text{Ric}(g) + \lambda g, \quad g(0, \cdot) = g_0, \]
where \(0 < \lambda \leq 1\). Let \(x_1, x_2\) be two points in \(M\). Suppose that at time \(t \geq 0\),
\[ \text{Ric}(g_t)(x) \leq (2n - 1)K, \quad \forall x \in B_{x_1}(r_0, g_t) \cup B_{x_2}(r_0, g_t) \]
for some \(r_0 > 0\). Then
\[ \frac{d}{dt} d_{g_t}(x_1, x_2) \geq \lambda d_{g_t}(x_1, x_2) - 2(2n - 1)(\frac{2}{3}Kr_0 + r_0^{-1}). \]
**Proof.** Without loss of generality, we may assume that \(t = 0\). Putting
\[ \tilde{g}_t = (1 - \lambda t)g(\log(1 - \lambda t)) - \lambda, \quad (0 \leq t < \frac{1}{\lambda}), \]
then \(\tilde{g} = \tilde{g}_t\) satisfies the Hamilton Ricci flow,
\[ \frac{\partial}{\partial t} \tilde{g} = -\text{Ric}(\tilde{g}). \]
Since \(\tilde{g}_0 = g_0\), by applying Lemma 8.3 in \([Pe]\), we have
\[ \frac{d}{dt} d_{\tilde{g}_t}|_{t=0} \geq -2(2n - 1)(\frac{2}{3}Kr_0 + r_0^{-1}). \]
Note that
\[ d_t = -\lambda d + d_t. \]
Hence (3.2) is true. \(\square\)
By Lemma 3.1 together with Lemma 2.4 in Section 2, we give a lower bound estimate for the distance functions along the flow as follows.

**Proposition 3.2.** Under the assumption of Lemma 2.4, we have that for two points $x_1, x_2$ in $B_q(\frac{1}{2}, g_0)$,

$$d_{g_t}(x_1, x_2) \geq d_{g_0}(x_1, x_2) - \frac{C_0}{\lambda}(\sqrt{t} + t^{-\frac{n}{2}} E_t^{\frac{1}{2}}), \quad \forall \ t \in (0, \delta],$$

where $C_0$ is a uniform constant and $E = \int_0^2 dt \int_M |R - \lambda \Delta \theta - n\lambda| \omega_\theta^n$. In particular, when $E \leq \delta^{n+1}$,

$$d_{g_t}(x_1, x_2) \geq d_{g_0}(x_1, x_2) - C_0 \lambda E_t^{\frac{1}{2}(n+1)}.$$  \hfill (3.3)

**Proof.** Let $\Phi(t)$ be a one parameter subgroup generated by $\text{real}(X)$. Then $\hat{g}_t = \Phi(-t)^* g_t$ is a solution of the normalized flow (3.1). Applying Lemma 2.2 for two points $y_1 = \Phi(-t)x_1$ and $y_2 = \Phi(-t)x_2$ by choosing $r_0 = \sqrt{t}$, together with Lemma 3.1 we have

$$\frac{d}{dt} d_{\hat{g}_t}(y_1, y_2) \geq \frac{\lambda}{2} d_{\hat{g}_t}(y_1, y_2) - C_1 t^{-\frac{1}{2}}.$$  \hfill (3.4)

It follows

$$d_{\hat{g}_t}(y_1, y_2) \geq d_{\hat{g}_0}(y_1, y_2) - 2C_1 \sqrt{t}.$$  \hfill (3.5)

As a consequence, we derive

$$d_{g_t}(x_1, x_2) = d_{\hat{g}_t}(y_1, y_2)$$

$$\geq d_{\hat{g}_0}(y_1, y_2) - 2C_1 \sqrt{t}$$

$$\geq d_{\hat{g}_0}(x_1, x_2) - 2\|X\|_{g_0} t - 2C_1 \sqrt{t}$$

$$\geq d_{g_0}(x_1, x_2) - \frac{C_2 \sqrt{t}}{\lambda}.$$  \hfill (3.5)

On the other hand, integrating (2.2), we get from Lemma 2.4

$$\log \frac{d_{g_t}(x_1, x_2)}{d_{g_t}(x_1, x_2)} \geq -C_3 \int_t^\delta s^{-\frac{n+2}{2}} E_t^{\frac{1}{2}} ds \geq -C_4 E_t^{\frac{1}{2}} t^{-\frac{n}{2}}, \quad \forall t > 0.$$  \hfill (3.6)

Hence combining (3.5) and (3.6), we obtain

$$d_{g_t}(x_1, x_2) \geq d_{g_0}(x_1, x_2) e^{-C_4 E_t^{\frac{1}{2}} t^{-\frac{n}{2}}}$$

$$\geq \left(d_{\hat{g}_0}(x_1, x_2) - \frac{C_2 \sqrt{t}}{\lambda}\right) e^{-C_4 E_t^{\frac{1}{2}} t^{-\frac{n}{2}}}$$

$$\geq d_{\hat{g}_0}(x_1, x_2) - \frac{C_4}{\lambda}(\sqrt{t} + t^{-\frac{n}{2}} E_t^{\frac{1}{2}}).$$

When $E \leq \delta^{n+1}$, we can choose $t = E^{\frac{1}{n+1}}$ to get (3.4). \hfill \square
Next we use the above proposition to give an upper bound estimate for the distance functions by using a covering argument as in [TW].

**Lemma 3.3.** Let \((M, g(\cdot, t), q)\) \((0 \leq t \leq 1)\) be a solution of \((2.2)\) as in Lemma \(2.4\). Let \(\Omega = B_q(1, g_0), \Omega' = B_q\left(\frac{1}{2}, g_0\right)\). For every \(l < \frac{1}{2}\), we define

\[
A_{+, l} = \sup_{B_x(s, g_0) \subset \Omega, s \leq l} c_n^{-1}s^{-2n} \text{vol}_{g_0}(B_x(s, g_0))
\]

and

\[
A_{-, l} = \inf_{B_x(s, g_0) \subset \Omega', s \leq l} c_n^{-1}s^{-2n} \text{vol}_{g_0}(B_x(s, g_0)).
\]

Then for any \(x_1, x_2 \in \Omega'' = B_q\left(\frac{1}{4}, g_0\right)\), it holds

\[
d_{g_0}(x_1, x_2) \leq r + \frac{C_0}{\lambda} A_{+, 4r} \left\{ \frac{A_{+, r}}{A_{-, r}} - 1 \right\} \left( \frac{1}{\lambda} + r^{-\frac{n}{2}} E^{\frac{1}{2(n+1)}} \right)r,
\]

where \(r = d_{g_0}(x_1, x_2) \leq \frac{1}{8}\) and \(E << r^{(n+1)}\).

**Proof.** By Proposition \(3.2\), we see that

\[
B_{x_1}(r - \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}, g_0) \subset B_{x_1}(r, g_0),
\]

deinition of \(B_x(s, g_0)\) satisfies

\[
\frac{d}{dt} \text{vol}(g_t) = (-R + n \lambda + \lambda \Delta \theta) \text{vol}(g_t),
\]

it is easy to see that there is a ball \(B_{x_0}(s_0, g_0)\) such that

\[
\text{vol}_{g_0}(B_{x_0}(s_0, g_0)) \leq \text{vol}_{g_0}(B_{x_1}(r, g_0)) - \text{vol}_{g_0}(B_{x_1}(r - \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}, g_0)).
\]

(3.9)

Observe that

\[
B_{x_0}(s_0, g_0) \supseteq B_{x_0}(s_0 - \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}, g_0).
\]

we have

\[
A_{-, r}(s_0 - \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}})^{2n} \leq \text{vol}_{g_0}(B_{x_0}(s_0 - \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}, g_0)) \leq \text{vol}_{g_0}(B_{x_0}(s_0, g_0)).
\]
Thus plugging the above inequality into (3.9) together with (3.8) and the fact that
\[
\text{vol}_{g_0}(B_{x_1}(r, g_0)) \leq A_{+, r} r^{2n},
\]
we obtain
\[(3.10) \quad s_0 \leq \left\{ \frac{A_{+, r}}{A_{-, r}} - 1 \right\} + \frac{C_0}{\lambda} r^{-1} E^{\frac{1}{2(n+1)}} r + \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}.
\]

On the other hand, since
\[
B_{x_2}(3s_0, g_0) \cap B_{x_1}(r - C_0 \lambda E^{\frac{1}{2(n+1)}}, g_0) \neq \emptyset,
\]
we see that there exists some point
\[
x_3 \in B_{x_2}(3s_0, g_0) \cap B_{x_1}(r - C_0 \lambda E^{\frac{1}{2(n+1)}}, g_0).
\]

Claim 3.4. There is a uniform constant \(C_1 = C_1(n)\) such that
\[(3.11) \quad d_{g_3}(x_2, x_3) \leq C_1 A_{+, 4r} \max\{s_0, 3C_0 \lambda E^{\frac{1}{2(n+1)}}\}.
\]
Combining (3.11) with (3.10), we will finish the proof of (3.7) because of the triangle inequality
\[
d_{g_3}(x_1, x_2) \leq d_{g_3}(x_1, x_3) + d_{g_3}(x_2, x_3).
\]

To prove Claim 3.4, we first assume that
\[(3.12) \quad s_0 > \frac{3C_0}{\lambda} E^{\frac{1}{2(n+1)}}.
\]

Let \(\gamma\) be the minimizing geodesic curve which connecting \(x_2\) and \(x_3\) in \((M, g_0)\). Choose \(N\) geodesic balls \(B_{z_i}(s_0, g_0)\) in \((M, g_0)\) such that \(B_{z_i}(\frac{s_0}{2}, g_0)\) are disjoint. Since
\[
B_{z_1}(\frac{T_0}{2}, g_0) \subset B_{z_1}(\frac{s_0}{2} + \frac{C_0}{\lambda} E^{\frac{1}{2(n+1)}}, g_0)
\subset B_{z_1}(s_0, g_0) \subset B_{x_2}(4s_0, g_0) \subset B_{x_1}(\frac{1}{2}, g_0),
\]
we have
\[
N A_{-, r}(\frac{s_0}{2})^{2n} \leq \sum_{i=1}^{N} \text{vol}_{g_3}(B_{z_i}(\frac{s_0}{2}, g_0)) \leq \text{vol}_{g_3}(B_{x_2}(4s_0)) \leq \text{vol}_{g_3}(B_{x_2}(4s_0)) + E \leq A_{+, 4r} (4s_0)^{2n} + E.
\]
Noticing that by the Bishop volume comparison and Lemma 2.2, we see that
\[
A_{-, r} \geq C(n, \delta) = C(n).
\]
By (3.12), it follows
\[
N \leq C' A_{+, 4r}.
\]
Since
\[ d_{g_\delta}(x_2, x_3) \leq 2Ns_0, \]
we deduce (3.11) from (3.10) immediately.

Secondly, we assume that
\[ s_0 \leq 3C_0 \lambda E \frac{1}{2(n+1)}. \]
In this case, we can copy the above argument of geodesic balls covering to prove (3.11) while the radius \( s_0 \) of balls is replaced by \( \frac{3C_0}{\lambda} E \frac{1}{2(n+1)} \). The claim is proved. □

**Proposition 3.5.** Let \((M, g(\cdot, t), q) \) \((0 \leq t \leq 2\delta)\) be a solution of (2.2) as in Lemma 2.4. Then for two points \(x_1, x_2 \in \Omega'' \) with \(r = d(x_1, x_2, g_0) \leq \frac{1}{8} \), we have
\[ d(x_1, x_2, g_\delta) \leq r + C_0 \frac{1}{\lambda} E \frac{1}{6n(n+1)} r, \]
if \(E << r^{6(n+1)}.\)

**Proof.** By the Bishop volume comparison and Lemma 2.2, we see that
\[ A_{-,r} \geq 1 - Ar, \]
for some uniform constant \(A\), where \(r \leq \delta << 1\). Also by the volume comparison in [WW], we have
\[ A_{+,r} \leq 1 + Ar^2, \quad \forall \ r \leq 1. \]
Applying Lemma 3.3 to any two points \(x_1, x_2 \in \Omega'' \) with \(d_{g_0}(x_1, x_2) = r \leq \delta << 1\), we get
\[ d_{g_\delta}(x_1, x_2)r^{-1} \leq 1 + \frac{C_0}{\lambda}(r^{-\frac{1}{n}} + r^{-\frac{1}{2n}} E^{4n(n+1)}). \]
For general two points \(x_1, x_2\) with \(d(x_1, x_2, g_0) = l \leq \frac{1}{8}\), we divide the minimal geodesic curve which connecting \(x_1\) and \(x_2\) into \(N\) parts with the same length \(\frac{l}{N} \leq \delta\). Thus by (3.14), we obtain
\[ \frac{d(x_1, x_2, g_\delta)}{N^{-1}l} \leq N\{1 + \frac{C_0}{\lambda}\{(N^{-1}l)^{-\frac{1}{n}} + (N^{-1}l)^{-\frac{1}{2n}} E^{4n(n+1)}\}\}. \]
Choosing \(N \sim lE^{-\frac{1}{6(n+1)}},\) we derive (3.13). □
4. Almost Kähler Ricci solitons

In this section, we are able to prove the smoothness of the regular part of the limit space for a sequence of weak almost Kähler-Ricci solitons studied in [WZ]. Recall the definition of weak almost Kähler-Ricci solitons.

**Definition 4.1.** We call a sequence of Kähler metrics \( \{(M_i, g^i, J_i)\} \) weak almost Kähler-Ricci solitons if there are uniform constants \( \Lambda \) and \( A \) such that

\[
\begin{align*}
  i) & \quad \text{Ric}(g^i) + L_{X_i}g^i \geq -\Lambda^2 g^i, \quad im(L_{X_i}g^i) = 0; \\
  ii) & \quad |X_i|_{g^i} \leq A; \\
  iii) & \quad \lim_{i \to \infty} \|\text{Ric}(g^i) - g^i + L_{X_i}g^i\|_{L^1_{\text{M}}(g^i)} = 0.
\end{align*}
\]

Here \( \omega_{g^i} \in 2\pi c_1(M_i, J_i) \) and \( X_i \) are reductive holomorphic vector fields on Fano manifolds \( (M_i, J_i) \).

We now assume that \( \text{vol}_{g^i}(B_{p^i}(1)) \geq v > 0 \), for some \( p^i \in M_i \).

Let \( g^i_t = g^i(\cdot, t) \) be a solution of the Kähler-Ricci flow (2.1) on \( (M_i, J_i) \) with \( g^i \) the initial metric. Suppose that \( g^i_t \) satisfies

\[
|X_i|_{g^i_t} \leq \frac{B}{\sqrt{t}}
\]

and

\[
\int_0^1 dt \int_{M_i} |R(g^i_t) - \Delta \theta_{g^i_t} - n \omega_{g^i_t}^n| \to 0, \quad \text{as} \ i \to \infty.
\]

Here \( B \) is a uniform constant. We note that (4.2) and (4.3) have been used in Lemma 2.4, Proposition 3.2 and Proposition 3.5, respectively. Under the assumption \((4.1)+(4.3)\), we prove

**Theorem 4.2.** Let \( \{(M_i, g^i, J_i)\} \) be a sequence of weak almost Kähler-Ricci solitons. Suppose that \( g^i \) satisfy the conditions \((4.1)+(4.3)\). Then there exists a subsequence of \( \{g^i\} \) which converge to a Kähler-Ricci soliton with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology.

**Proof.** It was proved in [WZ] that under the condition \((4.1)+(4.3)\) there exists a subsequence of \( \{g^i\} \) which converge to a metric space \( (Y, g_\infty) \) with complex codimension of singularities of \( Y \) at least 2. Denote \( \mathcal{R} \) as the regular part of \( Y \). We want to show that \( \mathcal{R} \) is an open manifold and \( g_\infty \) is in fact a Kähler-Ricci soliton for some complex structure on \( \mathcal{R} \).
Let $y_0 \in \mathcal{R}$. This means that the tangent cone $T_{y_0}$ at $y_0$ is isometric to $\mathbb{R}^{2n}$. Then by the Volume Convergence Theorem 4.10 in [WZ], it is easy to see that for any $\delta > 0$ there exists $r_0 < 1$ such that

$$
\text{vol}(B_{y_0}(r)) > (1 - \delta)c_n r^{2n}, \quad \forall \ r < r_0.
$$

Again by the above convergence theorem together with the monotonicity of volume [WW], there exists an $\epsilon > 0$ such that for any $y \in B_{y_0}(\epsilon, g_\infty)$ it holds

$$
\text{vol}(B_y(r)) > (1 - \delta)c_n r^{2n}, \quad \forall \ r < r_0.
$$

Claim 4.3. $y \in \mathcal{R}$ for any $y \in B_{y_0}(\epsilon, g_\infty)$.

For a fixed $r$, we choose a sequence of geodesic balls $B_{y_i}(r) \subset M_i$ which converge to $B_y(r)$ in the Gromov-Hausdorff topology. Then by (4.4), for $i$ large enough, we have

$$
\text{vol}(B_{y_i}(r)) > (1 - \delta)c_n r^{2n}.
$$

Scale $g^i$ to $\tilde{g}^i = \frac{1}{r}g^i$ and we consider the solution $\tilde{g}^i(\cdot, t) = \tilde{g}^i_t$ of flow (2.2) with the initial metric $\tilde{g}^i$, where $\lambda = r$. By applying Proposition 3.2 and Proposition 3.3 to each ball $B_{y_i}(1, \tilde{g}^i)$, we obtain

$$
|d_{\tilde{g}^i}(x_1, x_2) - d_{\tilde{g}^i_s}(x_1, x_2)| \leq CE\frac{1}{n(n+1)}, \quad \forall \ x_1, x_2 \in B_{y_i}(1, \tilde{g}^i),
$$

where

$$
E = \frac{1}{r^{n-1}} \int_0^{\frac{4}{r}} dt \int_M |R(\tilde{g}^i_t) - \lambda \Delta \theta_{\tilde{g}^i_t} - n\lambda|\omega_{\tilde{g}^i_t}^n \to 0, \quad \text{as} \ i \to \infty.
$$

On the other hand, since the curvature are uniformly bounded in $B_{y_i}(1, \tilde{g}^i)$ by Lemma 2.2, $B_{y_i}(1, \tilde{g}^i)$ converge to a smooth metric ball $B_{y_\infty}(1, \check{g}_\infty)$ by the regularity of $\check{g}_\infty$. Hence by (4.6), we derive

$$
s^{-1}d_{GH}(B_y(s, g_\infty), B_{y_\infty}(s, g_\infty)) \leq L s^2, \quad \forall \ s \leq \frac{r}{4},
$$

where $L$ is a uniform constant and $g_\infty = r\check{g}_\infty$. This means that the tangent cone at $y$ is isometric to $\mathbb{R}^{2n}$, so the claim is proved.

By the above claim, we see that there exists a small $r$ for any $y \in \mathcal{R}$ such that $B_y(r) \subset \mathcal{R}$ and (4.3) is satisfied. Then following the argument in the proof of Claim 4.3, there exists a sequence of geodesic balls $(B_{y_i}(r), g_{\tilde{g}^i}) \subset M_i$ which converge to $B_y(r)$ in $C^\infty$-topology. Consequently, the potentials $\theta_{\tilde{g}^i}$ of $X_i$ restricted on $(B_{y_i}(r), g_{\tilde{g}^i})$ converge to a smooth function $\theta_\infty$ defined on $B_y(r)$. Namely,

$$
\lim_{i \to \infty} \Psi_i^*(\theta_{\tilde{g}^i}) = \theta_\infty.
$$
where $\Psi_i$ are diffeomorphisms from $B_y(r)$ to $B_q(r)$ such that $\Psi_i^*(g_i)$ converge to $g_\infty$ and $\Psi_i^*J_i$ converge to some limit complex structure $J_\infty$ on $B_y(r)$. By the regularity of flow (2.2) and the condition (4.3), $\theta_\infty$ satisfies in $B_y(r)$,
\begin{equation}
\Delta \theta_\infty = R(g_\infty) - n \quad \text{and} \quad \partial \partial \theta_\infty = 0.
\end{equation}
Moreover, by (2.10) in Lemma 2.4, we get
\begin{equation}
\text{Ric}(g_\infty) - g_\infty - \sqrt{-1} \partial \bar{\partial} \theta_\infty = 0, \quad \text{in} \quad B_y(r).
\end{equation}
Hence, $\theta_\infty$ can be extended to a potential of holomorphic vector field $X_\infty$ on $(\mathcal{R}, J_\infty)$, and consequently $g_\infty$ is a Kähler-Ricci soliton on $\mathcal{R}$. \hfill \Box

Remark 4.4. It seems that the limit space $Y$ in Theorem 4.2 is actually a normal algebraic variety as showed in recent papers by Tian, Chen-Donaldson-Sun to solve the Yau-Tian-Donaldson conjecture for Kähler-Einstein metrics [T2], [CDS].

In [WZ], it was showed that there exists a sequence of weak almost Kähler-Ricci solitons $g^s$ ($s < 1$) on a Fano manifold $(M, g, J)$ if the modified K-energy $\mu(\cdot)$ is bounded below. Here $\mu(\cdot)$ is defined for any $K_X$-invariant Kähler potential $\phi$ by ([TZ2]),
\begin{equation}
\mu(\phi) = -\frac{n}{V} \int_0^1 \int_M \psi [\text{Ric}(\omega_\psi) - \omega_\psi - \sqrt{-1} \partial \bar{\partial} \theta_{\omega_\psi} + \sqrt{-1} \partial (h_{\omega_\psi} - \theta_{\omega_\psi}) \wedge \partial \theta_{\omega_\psi}] \times e^{\theta_{\omega_\psi}} \omega_\psi^{n-1} \wedge dt.
\end{equation}
In fact, such $g^s$ are a family of Kähler metrics induced by the Kähler potential solutions $\phi_s$ of a family of complex Monge-Ampère equations, which are equivalent to a family of Ricci curvature equations,
\begin{equation}
\text{Ric}(\omega_{\phi_s}) = s \omega_{\phi_s} + (1 - s)\omega_g + L X \omega_{\phi_s}.
\end{equation}
(4.10) are also equivalent to equations,
\begin{equation}
h_{\omega_{\phi_s}} - \theta_{\omega_{\phi_s}} = -(1 - s)\phi_s,
\end{equation}
(4.11) where $h_{\omega_{\phi_s}}$ are the Ricci potentials of $\omega_{\phi_s}$.

In the following, we need to verify the conditions (4.2) and (4.3) for $g^s$. We note that (4.1) is true for $g^s$ [WZ]. Thus as an application of Theorem 4.2 we prove that

Theorem 4.5. There exists a sequence of weak almost Kähler-Ricci solitons $\{g^s\}$ ($s_i \to 1$) which converge to a Kähler-Ricci soliton with complex codimension of singularities at least 2 in the Gromov-Hausdorff topology. In the other words, a Fano manifold with the modified K-energy bounded below can be deformed to a Kähler-Ricci soliton with complex codimension of singularities at least 2.
Lemma 4.6.

\[(4.12) \quad h_{g_s} - \theta_{g_s} \to 0, \text{ as } s \to 1.\]

Consequently,

\[(4.13) \quad |h_{g_s}| \leq C.\]

Proof. Recall the two functionals \(I\) and \(J\) defined for \(K_X\)-invariant Kähler potentials by ([Zh], [TZ1]),

\[I(\phi) = \int_M \phi(e^{\theta_{g_0}} \omega_0^n - e^{\theta_{\phi}} \omega_\phi^n)\]

and

\[J(\phi) = \int_0^1 \int_M \phi_t(e^{\theta_{g_0}} \omega_0^n - e^{\theta_{\phi}} \omega_\phi^n) dt.\]

It was showed for the potential \(\phi_s\) in [TZ1] that

\[-\frac{d}{ds} \mu(\phi_s) = (1 - s) \frac{d}{ds} (I - J)(\phi_s).\]

Then

\[(I - J)(\phi_s) = -\frac{\mu(\phi_s)}{1 - s} + \int_0^s \frac{\mu(\phi_\tau)}{(1 - s)^2} d\tau ds.\]

Since \(\mu(\phi_s)\) is monotone and bounded below, \(\lim_{s \to 1^-} \mu(\phi_s)\) exists. By 'Hôpital's rule, it is easy to see that

\[\lim_{s \to 1^-} (1 - s) \int_0^s \frac{\mu(\phi_\tau)}{(1 - \tau)^2} d\tau = \lim_{s \to 1^-} \mu(\phi_s).\]

Thus we get

\[\lim_{s \to 1^-} (1 - s)(I - J)(\phi_s) = 0.\]

On the other hand, by using the Green formula [Ma] (also see [CTZ]), there exists a uniform constant \(C\) such that

\[\text{osc}(\phi_s) \leq \|\phi_s\|_{C^0} \leq I(\phi_s) + C.\]

It follows that

\[(1 - s)\|\phi_s\|_{C^0} \leq (1 - s)(c(I - J)(\phi_s) + C) \to 0. \text{ as } s \to 1.\]

Hence by (4.11), we obtain (4.12). (4.13) is a direct consequence of (4.12) since \(\theta_{g_s}\) are uniformly bounded [Zh]. \(\square\)

Lemma 4.7. Let \(g_t^s = g^s(\cdot, t)\) be a solution of the Kähler-Ricci flow (2.1) with the above \(g^s\) as an initial metric. Then

\[(4.14) \quad |X|_{g_t^s} \leq \frac{B}{\sqrt{t}}.\]
Proof. Let \( u_t \) be the Kähler potential of \( g^s_t \). Namely, it is defined by
\[
\omega_{g^s_t} = \omega_{g^s} + \sqrt{-1} \partial \bar{\partial} u.
\]
According to Lemma 4.3 in [CTZ], we have
\[
|\nabla (\frac{\partial}{\partial t} u)|_{g^s_t} \leq e^{2 \parallel h_{g^s} - \theta_{g^s} \parallel_{C^0}} \sqrt{t}, 0 < t \leq 1.
\]
Since \( \tilde{g}^s_t = \Phi_t^* (g^s_t) \) is a solution of the Kähler-Ricci flow,
\[
\frac{\partial}{\partial t} g = - \text{Ric}(g) + g,
\]
where \( \Phi(-t) \) is an one parameter subgroup generated by \( \text{real}(X) \), we also have for the Kähler potential \( \tilde{u} \) of \( \tilde{g}^s_t \),
\[
|\nabla (\frac{\partial}{\partial t} \tilde{u})|_{\tilde{g}^s_t} \leq e^{2 \parallel h_{g^s} - \theta_{g^s} \parallel_{C^0}} \sqrt{t}, \forall \ 0 < t \leq 1.
\]
Note that
\[
\frac{\partial}{\partial t} u = \Phi_t^* (\frac{\partial}{\partial t} \tilde{u}) + \theta_{g^s_t} + m(t).
\]
We get
\[
|X|_{g^s_t} = |\nabla \theta_{g^s_t}|_{g^s_t} \leq |\nabla (\frac{\partial}{\partial t} \tilde{u})|_{g^s_t} + |\nabla (\frac{\partial}{\partial t} u)|_{g^s_t}.
\]
Now (4.14) follows from (4.13) immediately. \( \square \)

Lemma 4.8. Let \( g^s_t = g^s(\cdot, t) \) be a solution of the Kähler-Ricci flow as in Lemma 4.4. Then

\[
\int_0^1 dt \int_M |R(g^s_t) - \Delta \theta_{g^s_t} - n|\omega^n_{g^s_t} \to 0, \ as \ s \to 1.
\]

Proof. First by (4.10), we note that
\[
(\Delta + X)(h_{g^s} - \theta_{g^s}) \geq -(1 - s)n - (1 - s)|X(\phi_s)| \geq -(1 - s)(c_1 + n),
\]
where \( c_1 = \sup \{ \parallel X(\phi) \parallel_{C^0(M)} \} K_X - \text{invariant Kähler potential} \ \phi \) is a bounded number [Zh]. By the Maximum Principle, it follows that (cf. Lemma 4.2 in [CTZ]),
\[
(\Delta + X)(h_{g^s_t} - \theta_{g^s_t}) \geq -(1 - s)(c_1 + n)\epsilon_t, \ \forall \ 0 < t.
\]
The above implies that (cf. Lemma 4.4 in [CTZ]),
\[
\int_M |\nabla (h_{g^s_t} - \theta_{g^s_t})|^2 e^{\theta_{g^s_t} \omega^n_{g^s_t}} \leq 2e^2(c_1 + n)(1 - s)\parallel h_{g^s} - \theta_{g^s} \parallel_{C^0(M)}, \ \forall \ 0 < t \leq 1.
\]
Hence by (4.14), we get
\[
\int_0^1 dt \int_M |X(h_{g^s_t} - \theta_{g^s_t})| e^{\theta_{g^s_t}} \omega_{g^s_t}^n \leq \int_M |\nabla (h_{g^s_t} - \theta_{g^s_t})| e^{\theta_{g^s_t}} \omega_{g^s_t}^n \int_0^1 |X|_{g^s_t} dt \\
\leq C(1-s)^{\frac{1}{2}} \int_0^1 \frac{1}{\sqrt{t}} dt \to 0, \quad \text{as } s \to 1.
\]

Therefore,
\[
\int_0^1 dt \int_M |R(g^s_t) - \Delta \theta_{g^s_t} - n| e^{\theta_{g^s_t}} \omega_{g^s_t}^n \leq \int_0^1 dt \int_M |\Delta (h_{g^s_t} - \theta_{g^s_t}) + X(h_{g^s_t} - \theta_{g^s_t}) + (1-s)(c_1 + n)| e^{\theta_{g^s_t}} \omega_{g^s_t}^n \\
+ \int_0^1 dt \int_M |X(h_{g^s_t} - \theta_{g^s_t})| e^{\theta_{g^s_t}} \omega_{g^s_t}^n + V(1-s)(c_1 + n) \\
= \int_0^1 dt \int_M |X(h_{g^s_t} - \theta_{g^s_t})| e^{\theta_{g^s_t}} \omega_{g^s_t}^n + 2V(1-s)(c_1 + n) \\
\to 0, \quad \text{as } s \to 1.
\]

This finishes the proof of Lemma 4.8. \qed

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