Stabilizing Role of Mesoscopic Fluctuations in Spin Systems

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Abstract

The occurrence of mesoscopic fluctuations in statistical systems implies, from the point of view of dynamical theory, the existence of local instabilities. However, the presence of such fluctuations can make a system, as a whole, more stable from the thermodynamic point of view. Thus, in many cases, a local dynamic instability is a requisite for the global thermodynamic stability. This idea is illustrated by several spin models.

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1 Introduction

There are two general types of fluctuations in statistical systems, microscopic and mesoscopic. The former are small oscillations about a ground state, and they define collective excitations, such as phonons, magnons, etc. This kind of fluctuations is pertinent to equilibrium state. Collective excitations characterize a set of quantum states of a statistical system, and the word microscopic reflects the microscopic nature of such fluctuations.

Contrary to these, *mesoscopic fluctuations* are such that make a macroscopic system heterogeneous or locally drive a system out of equilibrium. Locally can mean in space, or in time, or both. So, mesoscopic fluctuations are, in general, nonequilibrium and are related to an averaged description of a statistical system. The word mesoscopic, in relation to space, means that the characteristic size of such a fluctuation is much larger than the average interparticle distance but is much smaller than the size of a system itself. In relation to time, mesoscopic implies that the characteristic lifetime of a mesoscopic fluctuation is much longer than an effective oscillation period of microscopic fluctuations but much shorter than the observation time. When mesoscopic fluctuations correspond to the formation of nuclei of one thermodynamic phase inside another, they are called *heterophase fluctuations* [1]. The latter are ubiquitous in nature, and plenty of examples are described in review [1]. Recently, much attention has been paid to the study of mesoscopic fluctuations in high–temperature superconductors [2-7], where the corresponding phenomenon is often termed phase separation [3-6], although it would be more correct to call this effect *mesoscopic phase separation* in order to distinguish it from the principally different Gibbs phase separation occurring at macroscopic scales (see discussion in [1,7]).

When mesoscopic fluctuations are frozen in time, the corresponding system looks like an ensemble of clusters with different properties. Such a system can be even equilibrium [8,9]. And if these fluctuations are not frozen in time, they make the system nonequilibrium [1]. The latter case is more interesting than that of a
frozen macroscopic structure, since it involves three difficult questions: (i) How to develop a statistical description of such a nonequilibrium and nonuniform system? (ii) Can this mesoscopic state be an attractor and, if so, what kind of attractor is it? (iii) Though the existence of nonequilibrium mesoscopic fluctuations implies local instability, but is it possible that globally the system is nevertheless stable?

The answer to the first question has been done by developing a consistent statistical theory of systems with such mesoscopic fluctuations [1]. In this approach, after averaging over these fluctuations, a renormalized Hamiltonian is defined representing a set of phase replicas, each of which describes an effective equilibrium system. To deal further with the renormalized Hamiltonian, one may employ the techniques of equilibrium statistical mechanics.

As an answer to the second question, it has been conjectured that the state of a system with nonequilibrium mesoscopic fluctuations is a chaotic attractor [1]. It was also shown [10] that a uniform statistical system is structurally unstable with respect to arbitrary small random external perturbations. One can recollect as well that in dynamical theory there are plenty of examples of dynamical systems with chaotic attractors (see e.g. [11-13] and references therein).

The third question has been considered for some simple spin models [1] and analyzed in more detail for a chaotic lattice–gas model [14]. In the present paper, the study of several other less trivial spin models is given, with the aim to show that for each model there can be found a region of parameters, where mesoscopic fluctuations do make the system globally stable, that is, thermodynamically more stable than the analogous system without these fluctuations.

2 Effective Hamiltonian

This section is a very brief recollection of the main definitions we shall need in what follows for analyzing systems with mesoscopic or heterophase fluctuations. We employ the theory of such systems developed in Ref. [1]. Throughout the text, the terms mesoscopic and heterophase will be used in parallel.
Imagine that we are dealing with a system in which randomly in space and time there arise mesoscopic fluctuations. For crystals and liquids, these could be fluctuations of local space structure [15-18]. In the case of spin systems, these are fluctuations of local magnetization [1]. Following the general theory [1], we can average over such stochastic mesoscopic fluctuations and obtain an averaged effective Hamiltonian

\[ H_{\text{eff}} = \oplus_{\nu} H_{\nu}, \]  

in which the index \( \nu \) enumerates qualitatively different phases, and \( H_{\nu} \) is a phase-replica Hamiltonian representing a pure \( \nu \)-phase. Each Hamiltonian \( H_{\nu} \) is defined on a Hilbert space \( \mathcal{H}_{\nu} \) of microscopic states typical of the corresponding \( \nu \)-phase [1,19,20]. The averaged Hamiltonian (1) acts on a fiber space

\[ \mathcal{Y} = \otimes_{\nu} \mathcal{H}_{\nu}. \]  

Hamiltonian (1) depends on a set \( \{w_{\nu}\} \) of geometric phase probabilities \( w_{\nu} \), with the properties

\[ \sum_{\nu} w_{\nu} = 1, \quad 0 \leq w_{\nu} \leq 1. \]  

These phase probabilities are defined as quantities providing an absolute minimum for the free energy

\[ f = -\frac{T}{N} \ln Tr \exp (-\beta H_{\text{eff}}), \]  

where \( T \) is temperature, \( N \) is the averaged number of particles, \( \beta T = 1 \), \( k_B \equiv 1 \), and the trace is taken over the fiber space (2). The phase probabilities as functions of thermodynamic and Hamiltonian parameters are given by the solutions of the equations

\[ \frac{\partial f}{\partial w_{\nu}} = 0, \quad \frac{\partial^2 f}{\partial w_{\nu}^2} > 0, \]  

under the normalization condition (3).

As concrete examples, we analyse below spin systems. Heterophase states in these systems are characterized by mesoscopic fluctuations of local magnetization. Such spatial fluctuations can be observed in experiment by means of
neutron scattering methods, like diffuse scattering, small–angle scattering, Bragg reflections, and polarized–beam scattering [21].

3 Model with Competing Interactions

Consider a one–dimensional Ising–type model modified by including two kinds of spin interactions, long–range and short–range interactions,

$$J_{ij} = \alpha I \delta_{i-j,1} + (1 - \alpha) J^0_{ij},$$

(6)

where $J^0_{ij}$ satisfies the properties

$$\lim_{N \to \infty} J^0_{ij} = 0, \quad \lim_{N \to \infty} \frac{1}{N} \sum_{i \neq j} N J^0_{ij} = J < \infty$$

and $\alpha$ is a crossover parameter [22,23]. For $\alpha = 0$, we have only a long–range interaction, while for $\alpha = 1$, we get the Ising nearest–neighbour interactions. Assume that mesoscopic fluctuations are heterophase fluctuations between two phases, ferromagnetic and paramagnetic. The averaged Hamiltonian (1) consists of the phase–replica terms

$$H_\nu = w^2_\nu \left( \frac{1}{2} NU - \frac{1}{4} \sum_{i \neq j} J_{ij} s_is_j \right),$$

(7)

where $N$ is the number of lattice cites, $U$ is a crystalline–field parameter, and $s_i = \pm 1$. Note that the crystalline field cannot be omitted since it influences the values of the phase probabilities, although it does not contain spin variables [24]. This makes the situation rather different from the case of pure monophase systems.

Each phase is characterized by an order parameter

$$\sigma_\nu \equiv \frac{1}{N} \sum_{i=1}^{N} \langle s_i \rangle_\nu,$$

(8)

where $\langle \ldots \rangle_\nu$ implies the statistical averaging over the space of typical states $\mathcal{H}_\nu$; that is, an average $\langle \hat{A} \rangle_\nu$ of an operator $\hat{A}$ means

$$\langle \hat{A} \rangle_\nu \equiv Tr_\nu \rho_\nu \hat{A}, \quad \rho_\nu \equiv \frac{\exp(-\beta H_\nu)}{Tr_\nu \exp(-\beta H_\nu)},$$
where $Tr_\nu$ means a trace over $\mathcal{H}_\nu$. Let $\nu = 1$ correspond to ferromagnetic phase and $\nu = 2$, to paramagnetic phase. Then, by definition,

$$\sigma_1 \not\equiv 0, \quad \sigma_2 \equiv 0. \quad (9)$$

This condition relates the order parameters (8) with the space $\mathcal{H}_\nu$ of typical states, making it possible to construct the latter as quantum weighted spaces [1].

Calculating the specific free energy (4), we can use for the short-range part of the Hamiltonian the transfer-matrix method (see e.g. [25]), and the long-range part, as is known [26], is asymptotically equivalent to the mean-field form. As a result, we obtain

$$f = \left( w^2 - w + \frac{1}{2} \right) U - \frac{w^2}{4} \left[ \alpha T - (1 - \alpha) J_\sigma^2 \right] -$$

$$- T \ln \left[ \cosh\varphi + \sqrt{\sinh^2\varphi + \exp(-4\varphi_1)} \right] - T \ln \left( 2\cosh\varphi_2 \right), \quad (10)$$

where $w \equiv w_1$, $\sigma \equiv \sigma_1$, and

$$\varphi \equiv w^2 \frac{(1 - \alpha)J_\sigma}{2T}, \quad \varphi_1 \equiv w^2 \frac{\alpha I}{4T}, \quad \varphi_2 \equiv (1 - w)^2 \frac{\alpha I}{4T}. \quad \text{(11)}$$

For what follows, it is convenient to introduce dimensionless quantities

$$u \equiv \frac{U}{J}, \quad g \equiv \frac{I}{J}, \quad t \equiv \frac{T}{J}. \quad \text{(11)}$$

For the probability of the ferromagnetic phase, from the first of equations (5), we find the equation

$$4g w \sigma \exp(-4\varphi_1)$$

$$\cosh\varphi \sqrt{\sinh^2\varphi + \exp(-4\varphi_1)} +$$

$$+ u(2w - 1) - 2\alpha gw - 2(1 - \alpha)w\sigma^2 + 2\alpha(1 - w)gtanh\varphi_2 = 0. \quad (12)$$

For the order parameter $\sigma \equiv \sigma_1$, defined in (8), we get

$$\sigma = \frac{\sinh\varphi}{\sqrt{\sinh^2\varphi + \exp(-4\varphi_1)}}. \quad (13)$$

The heterophase state is thermodynamically more stable than the pure state, if the inequality

$$\Delta f \equiv f(1) - f(w) > 0 \quad (14)$$
holds true. When (5) is valid but (14) is not, the heterophase state is metastable.

Let us investigate the stability of the system at $T = 0$. Then, from (12), we obtain the ferromagnetic–phase probability $w(T = 0) \equiv w_0$ in the form

$$w_0 = \frac{2u - |\alpha|g}{4u - 1 + \alpha - (\alpha + |\alpha|)g}.$$  \hspace{1cm} (15)

Also, we have

$$\frac{\partial^2 f}{\partial w^2} = 2(4u - \alpha g - 1 + \alpha - |\alpha|g),$$

$$\frac{\Delta f}{J} = \frac{(2u - \alpha g - 1 + \alpha)^2(4u + 2\alpha g - 1 + \alpha)}{4(4u - 1 + \alpha)^2}, \quad \alpha < 0,$$

$$\frac{\Delta f}{J} = \frac{(2u - \alpha g - 1 + \alpha)^2}{4(4u - 2\alpha g - 1 + \alpha)}, \quad \alpha \geq 0.$$

From conditions (3), (5), and (14), it follows that the heterophase state is absolutely stable if either

$$u > \max \left\{ \frac{1}{4}(1 - \alpha - 2\alpha g), \frac{1}{2}(1 - \alpha + \alpha g) \right\}, \quad \alpha \leq 0,$$

or

$$u > \frac{1}{2}(1 - \alpha + \alpha g), \quad \alpha \geq 0.$$  \hspace{1cm} (16)

The heterophase state is metastable if either

$$\frac{1}{2}(1 - \alpha + \alpha g) < u < \frac{1}{4}(1 - \alpha - 2\alpha g), \quad \alpha < 0,$$

or

$$u < \frac{1}{2}\alpha g, \quad \alpha \geq 0.$$  \hspace{1cm} (17)

In the case when

$$\frac{1}{2}|\alpha|g < u < \frac{1}{2}(1 - \alpha + \alpha g),$$

the system at zero temperature is purely ferromagnetic, but beginning from a finite temperature $T_n$, called the nucleation temperature [1] and defined by the condition $w(T_n) = 1$, the heterophase state becomes profitable, being a mixture of ferromagnetic and paramagnetic phases.
At low temperatures, when $t \to 0$, we find the following asymptotic behaviour for the ferrophase probability

$$\frac{w}{w_0} \simeq 1 - \frac{4(1 - \alpha + \alpha g)}{4u - \alpha g - 1 + \alpha - |\alpha|g} \exp \left\{ - \frac{w_0^2}{t} (1 - \alpha + 4\alpha g) \right\} +$$

$$+ \frac{2|\alpha|g(2u - \alpha g - 1 + \alpha)}{(2u - |\alpha|g)(4u - \alpha g - 1 + \alpha - |\alpha|g)} \exp \left\{ - \frac{(1 - w_0)^2}{t} 2|\alpha|g \right\},$$

the order parameter

$$\sigma \simeq 1 - 2 \exp \left\{ - \frac{w_0^2}{t} (1 - \alpha + 4\alpha g) \right\},$$

the entropy

$$S \simeq \frac{w_0^2}{t} (1 - \alpha + \alpha g) \exp \left\{ - \frac{w_0^2}{t} (1 - \alpha + 4\alpha g) \right\} +$$

$$+ \frac{(1 - w_0)^2}{2t} |\alpha|g \exp \left\{ - \frac{(1 - w_0)^2}{t} 2|\alpha|g \right\},$$

and for the heat capacity

$$C_V \simeq \frac{w_0^4}{t^2} (1 - \alpha + \alpha g)^2 \exp \left\{ - \frac{w_0^2}{t} (1 - \alpha + 4\alpha g) \right\} +$$

$$+ \frac{(1 - w_0)^4}{4t^2} \alpha^2 g^2 \exp \left\{ - \frac{(1 - w_0)^2}{t} 2|\alpha|g \right\}.$$  

The positivity of the specific heat indicates that the heterophase state is stable with respect to thermal fluctuations.

Now, let us analyse the critical behaviour of the model. The critical temperature $t_c$ is defined by the condition $\sigma(t_c) = 0$, which gives

$$t_c = \frac{1 - \alpha}{8} \exp \left( \frac{\alpha g}{8t_c} \right). \quad (21)$$

As follows from (21), there exists a negative value of the crossover parameter $\alpha = \alpha_0$,

$$\alpha_o = - \frac{1}{eg - 1} \quad (eg > 1), \quad (22)$$

such that for $\alpha < \alpha_0$ the ferromagnetic state is impossible at all temperatures. But if $g \leq e^{-1} = 0.3679$, then a positive solution for $t_c$ is available for any
\( \alpha < 1 \). The appearance of the limiting value (22) is quite explicable. Really, negative values of \( \alpha \) correspond to the antiferromagnetic character of the short-range interaction. The presence of an interaction having the opposite sign, as compared to the ferromagnetic long-range interaction, serves as a disordered factor. The onset of ferromagnetic order is possible only if the disordereding short-range interaction is not too large. One might recollect several other examples when an ordering in a system occurs only if some limiting relations between competing interactions take place. Recall, for instance, the criteria of magnetism in the Hubbard model [27,28].

It is interesting that the crossover behaviour of the critical temperature, between the mean-field value \( t_c = \frac{1}{8} (\alpha = 0) \) and the short-range case \( t_c = 0 (\alpha = 1) \), is nonmonotonic. The maximum of (21) occurs at

\[
t_{\text{max}} = \frac{g}{8(1 + \ln g)}, \quad \alpha_{\text{max}} = \frac{\ln g}{1 + \ln g}. \tag{23}
\]

The ratio of this maximum to the mean-field critical temperature \( t_c = 1/8 \), that is,

\[
8t_{\text{max}} = \frac{g}{1 + \ln g},
\]

can become arbitrary large for \( g \gg 1 \).

The presence of mesoscopic fluctuations, as well as the antiferromagnetic short-range interaction, can change the second-order phase transition to the first-order one. The region of first-order phase transition is defined either by the inequalities

\[
u_0 < u < u_t \quad (\alpha > \alpha_1), \tag{24}
\]
or by the inequalities

\[
u_0 < u < u_0 - |u_t - u_0| \quad (\alpha_0 < \alpha < \alpha_1), \tag{25}
\]

where

\[
u_t = u_0 + \frac{12(1 - \alpha)(\alpha g + 2t_c)^2}{12t_c^2 - (1 - \alpha)^2},
\]

\[
u_0 = \frac{4\alpha(1 - \alpha)g + 4t_c^2 - (1 - \alpha)^2}{4(1 + t_c - \alpha)^2}, \tag{26}
\]
The value $u = u_t$, defined in (26), is a function of the parameters $\alpha$ and $g$. The equation $u = u_t(\alpha, g)$ describes a surface on which the order of phase transition changes. This is called a tricritical surface. On the latter, the critical indices also change by a jump. Thus, considering the asymptotic behaviour of the specific heat $C_V \propto |\tau|^{-\alpha}$, order parameter $\sigma \propto |\tau|^{\beta}$, and susceptibility $\chi \propto |\tau|^{-\gamma}$, when approaching the critical point $t_c$, so that $\tau \equiv (t - t_c)/t_c \to -0$, we obtain

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1 \quad (u \neq u_t)$$

outside the tricritical surface, and

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{4}, \quad \gamma = 1 \quad (u = u_t)$$
on the tricritical surface.

One more critical index can be introduced [1] for the phase probability $w$ as

$$w - \frac{1}{2} \propto |\tau|^{\varepsilon}.$$  

This index $\varepsilon$ is specific for heterophase systems. In our case we find that this index also jumps on the tricritical surface $u = u_t(\alpha, g)$,

$$\varepsilon = \begin{cases} 1, & u \neq u_t \\ \frac{1}{2}, & u = u_t. \end{cases}$$

Despite the seeming simplicity of the model considered, it displays quite non-trivial behaviour. One of the most interesting features is a strongly nonmonotonic dependence of the critical temperature on the crossover parameter $\alpha$. Also, in the space of three parameters, $u$, $\alpha$, and $g$, there is a region, where mesoscopic fluctuations make the system thermodynamically more stable than that system without such fluctuations.

### 4 Heterophase Spin Glass

Consider a generalization of the Sherrington–Kirkpatrick spin–glass model [29] to the case of a system with mesoscopic fluctuations. We shall keep in mind
paramagnetic fluctuations inside the spin–glass phase [30]. Let interactions $J_{ij}$ between spins at sites $i, j = 1, 2, \ldots, N$ be distributed by the Gaussian law

$$
P(J_{ij}) = \frac{1}{\sqrt{2\pi N}} \exp \left\{-\frac{N}{2J^2} \left(J_{ij} - \frac{J_0}{N}\right)^2 \right\}.
$$

For simplicity, we put $J_0 \equiv 0$. The average over interactions for a function $A\{J_{ij}\}$ of a set $\{J_{ij}\}$ of interactions $J_{ij}$ is defined as

$$
[A\{J_{ij}\}]_{av} \equiv \int A\{J_{ij}\} \prod_{i \neq j} P(J_{ij}) dJ_{ij}.
$$

For an Ising–like system with mesoscopic fluctuations, following the general renormalization procedure [1], we have an effective Hamiltonian (1) with the phase–replica terms

$$
H_{\nu}\{J_{ij}\} = w_{\nu}^2 \left(\frac{1}{2}NU - \sum_{i \neq j} J_{ij}s_is_j\right),
$$

in which $U$ is a crystalline–field constant and $s_i = \pm 1$. To distinguish phases, we need an order parameter. For spin glasses, this is the Edwards–Anderson [31] order parameter

$$
q_{\nu} \equiv \langle s_i^2 \rangle_{av}^{\nu},
$$

where $\langle \ldots \rangle_{\nu}$ means a statistical averaging with the Hamiltonian (33) over the space $\mathcal{H}_{\nu}$ of states typical of a phase with the parameter (34), and $[\ldots]_{av}$ denotes the averaging (32) over interactions. Let the spin–glass phase be indexed by $\nu = 1$ and the paramagnetic phase, by $\nu = 2$. Then, by definition,

$$
q_1 \neq 0, \quad q_2 \equiv 0.
$$

The ferromagnetic phase, because of $J_0 \equiv 0$, is absent.

The free energy (4) writes

$$
f = f_1 + f_2, \quad f_{\nu} = [f_{\nu}\{J_{ij}\}]_{av},
$$

where

$$
f_{\nu}\{J_{ij}\} = -\frac{T}{N} \ln \text{Tr}_{\nu} \exp (-\beta H_{\nu}\{J_{ij}\}),
$$
and \( \mathcal{H}_\nu = \{ s_i \mid i = 1, 2, \ldots, N; q_\nu \} \) is the space of states typical of a phase with the order parameter (34). With the replica trick [31], one has
\[
[\ln Z\{J_{ij}\}]_{av} = \lim_{n \to 0} \frac{1}{n} \left( [Z^n\{J_{ij}\}]_{av} - 1 \right) = \lim_{n \to 0} \frac{\partial}{\partial n} [Z^n\{J_{ij}\}]_{av}.
\]

Using this, for the free energy (36), we obtain
\[
f = \frac{U}{2} \left[ w^2 + (1 - w)^2 \right] - \frac{1}{4} \beta J^2 w^4 (1 - q)^2 - \frac{1}{4} \beta J^2 (1 - w)^4 - T \int_{-\infty}^{+\infty} p(x) \ln \left[ 2 \cosh(\beta J w^2 q^{1/2} x) \right] dx - T \ln 2,
\]
where \( w \equiv w_1, q \equiv q_1 \), and
\[
p(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x^2 \right).
\]

For the spin–glass order parameter (34), with \( \nu = 1 \), we get
\[
q = \int_{-\infty}^{+\infty} p(x) \tanh^2 \left( \beta J w^2 q^{1/2} x \right) dx.
\]

The trivial solution \( q = 0 \) of (38) is excluded by condition (35). From the first of eqs. (5), we get an equation
\[
w^3(1 - q)^2 - (1 - w)^3 - u(2w - 1)t = 0
\]
for the probability \( w \) of the spin–glass phase, where
\[
u \equiv \frac{U}{J}, \quad t \equiv \frac{T}{J}.
\]

Among three roots of eq. (39), we have to choose that one satisfying the normalization condition (3). In comparison to the standard spin glass [31,32], the phase probability \( w \) plays the role of an additional order parameter.

Let us analyse the thermodynamic characteristics of the heterophase spin glass. It is convenient to introduce the notation
\[
u_0 \equiv 2 \sqrt{\frac{2}{\pi}} = 1.595769.
\]
At low temperatures, the spin–glass order parameter behaves as

\[ q \simeq 1 - \frac{u_0}{2} t - u_0(u_0 - u)t^{4/3} \quad (u < u_0), \]

\[ q \simeq 1 - \frac{u_0}{2} t - \frac{1}{\pi} t^2 \quad (u \geq u_0), \] (42)

where \( t \to 0 \). The asymptotic, as \( t \to 0 \), behaviour of the phase probability is

\[ w \simeq 1 - (u_0 - u)t^{1/3} \quad (u < u_0), \]

\[ w \simeq 1 \quad (u \geq u_0). \] (43)

For the specific heat and entropy, we find

\[ C_V \simeq \frac{1}{6}(u_0 - u)^{4/3}t^{-2/3} \quad (u < u_0), \]

\[ C_V \simeq \frac{(\pi^3 - 6)}{24\pi} u_0 t \quad (u \geq u_0), \] (44)

as \( t \to 0 \), and, respectively,

\[ S \simeq -\frac{1}{4}(u - u_0)^{4/3}t^{-2/3} \quad (u < u_0), \]

\[ S \simeq \ln 2 - \frac{1}{2\pi} = 0.53399 \quad (u \geq u_0). \] (45)

As follows from these expressions, the ground state is a pure spin–glass phase: \( w = 1 \) at \( t = 0 \).

When the crystal–field parameter \( u < u_0 \), the system is unstable at low temperatures, since \( C_V \to \infty \) and \( S \to -\infty \), in analogy with the Sherrington–Kirkpartick case [29]. But for \( u \geq u_0 \), mesoscopic paramagnetic fluctuations stabilize the system making the behaviour of the specific heat and entropy normal.

In the vicinity of the critical point

\[ t_c = \frac{1}{4} \quad (q = 0), \] (46)

we have for the spin–glass parameter

\[ q \simeq |\tau| \quad \left( \tau \equiv \frac{t - t_c}{t_c} \to -0 \right) \] (47)
and for the phase probability

\[ w \simeq \frac{1}{2} - \frac{\tau^2}{4(u-3)}. \]  

(48)

From the stability condition

\[ \frac{\partial^2 f}{\partial w^2} \simeq 2J(u-3) > 0 \quad (t \to t_c), \]  

(49)

we conclude that the second–order transition between the spin–glass and paramagnetic phases occurs if \( u > 3 \). The value \( u = u_t = 3 \) corresponds to a tricritical point, where the second–order transition changes for the first order transition being realized for \( u < 3 \). The critical index for the phase probability, defined in (29), is

\[ \varepsilon \equiv \lim_{\tau \to 0} \frac{\ln|w - \frac{1}{2}|}{\ln|\tau|} = 2 \quad (u < 3). \]  

(50)

In this way, mesoscopic paramagnetic fluctuations, when \( u > u_0 \), stabilize the Sherrington–Kirkpatrick mean–field spin glass, making its specific heat finite and entropy positive. In order to check whether the heterophase spin glass becomes absolutely stable, one has to consider as well the sign of magnetic susceptibility. According to our analysis [30], the latter is positive, at least in the critical region. The mean–field glass, as is known, can be made stable by invoking, for the Edwards–Anderson order parameter, solutions with a broken replica symmetry [33]. Generalizing this type of spin glass to the case including mesoscopic fluctuations, it is possible to show [30] that, again, for sufficiently large crystal–field parameter \( u \), the free energy of the heterophase spin glass becomes lower than of a pure spin glass. That is, mesoscopic fluctuations can make the spin–glass system thermodynamically more stable.

5 Systems with Magnetic Reorientations

The appearance of coexisting magnetic phases with different directions of magnetization is characteristic of spin–reorientational transitions in small or zero
external magnetic fields. The existence of such mixed states is well documented by a large number of experiments and have been discussed in detail in books [34,35] and reviews [1,36,37]. The standard theoretical description of magnetic reorientations, going through intermediate mixed states, is done by means of Landau expansions involving a set of fitting functions taken from experimental data [34,35]. This is a purely phenomenological treatment giving no physical insight. Another approach to systems with magnetic reorientations can be based on the theory of mesoscopic fluctuations [1], with taking account of phase fluctuations corresponding to different angle phases with mutually orthogonal magnetizations [24,38,39]. Following such a microscopic approach, it is possible to show that the balance between phase probabilities and, as a result, magnetic reorientations are governed by the tendency of a system to reach the state of an absolute thermodynamic stability by allowing the appearance of mesoscopic fluctuations.

Consider a system in which there can coexist four different phases, three of them, magnetic, having orthogonal to each other nonzero magnetizations and one, paramagnetic, with zero magnetization. These phases will be enumerated by the index \( \nu = 1, 2, 3, 4 \), and the notation \( \{ \nu \} = \{ \alpha, 4 \} \), where \( \alpha = 1, 2, 3 \), will be used. To separate the phases, we need an order parameter, whose definition, as usual for spin systems, is based on the average spin operator

\[
\vec{S} \equiv \frac{1}{N} \sum_{i=1}^{N} \vec{S}_i = \{ S^\alpha \},
\]

in which \( \alpha = 1, 2, 3 \) and \( \vec{S}_i \) is a spin operator at a lattice site \( i = 1, 2, \ldots, N \). The vector order parameter of a \( \nu \)-phase is defined as

\[
\vec{\eta}_\nu \equiv \langle \vec{S} \rangle_\nu = \{ \eta^\alpha_\nu \},
\]

so that

\[
\eta^\beta_\alpha = \delta_{\alpha,\beta} \eta_\alpha , \quad \eta^4_\alpha \equiv 0,
\]

where

\[
\eta_\alpha \equiv \langle S^\alpha \rangle_\alpha \neq 0.
\]
In another way, we could write
\[ \eta_\alpha = \eta_\alpha \vec{e}_\alpha, \quad \eta_4 \equiv 0, \tag{55} \]
where \( \eta_\alpha \) is given by (54) and \( \vec{e}_\alpha \) is a unit vector along the \( \alpha \)-axis. The order parameters (55) define the directions of magnetization for the related phases, three of which are magnetic and one is paramagnetic.

Following the general theory [1] for a spin system with anisotropic interactions \( J^\alpha_{ij} \), after averaging over random phase configurations, we come to an effective Hamiltonian (1) with the phase-replica terms
\[ H_\nu = N w_\nu K + w_\nu^2 \left( NU - \sum_{i \neq j}^{N} \sum_{\alpha=1}^{3} J^\alpha_{ij} \langle S^\alpha_i S^\alpha_j \rangle_\nu \right), \tag{56} \]
where \( K \) is a mean kinetic energy per site for electrons and ions. Recall [24] that the term \( NU \) is the total potential energy, not including spin operators, of electrons and ions in a crystalline lattice. Therefore, the parameter \( U \) can be called the crystal-field parameter, structural constant, configurational energy per site, or lattice energy per site. In general, the values \( K, U \) and \( J^\alpha_{ij} \) can also depend on the type of a phase, but, for simplicity, we assume that they are the same for all thermodynamic phases.

Minimizing the free energy (4), we get an equation
\[ w_\nu = \left( \frac{4}{N} \sum_{\nu=1}^{N} \frac{U - B_\nu}{U - B_\mu} \right)^{-1} \tag{57} \]
for the phase probabilities \( w_\nu \), where
\[ B_\nu \equiv \frac{1}{N} \sum_{i \neq j}^{N} \sum_{\alpha=1}^{3} J^\alpha_{ij} \langle S^\alpha_i S^\alpha_j \rangle_\nu. \tag{58} \]
From the second of eqs. (5), we have the stability condition
\[ U > \frac{1}{2} \left( \sup_\alpha \{ B_\alpha \} + B_4 \right), \tag{59} \]
while the inequality \( 0 < w_\nu < 1 \) is valid when
\[ U > \sup_\alpha \{ B_\alpha \}. \tag{60} \]
For ferromagnets, the interactions $J^\alpha_{ij}$ are positive, thence $B^\nu > 0$, and for antiferromagnets, $J^\alpha_{i\alpha}$ are negative, hence $B^\nu < 0$. Therefore, heterophase fluctuations appear easier in antiferromagnets than in ferromagnets, as follows from (59) and (60).

To proceed further, we need to invoke some approximation. In what follows, we use the mean-field decoupling

$$\langle S^\alpha_i S^\alpha_j \rangle^\nu = \langle S^\alpha_i \rangle^\nu \langle S^\alpha_j \rangle^\nu,$$

which yields

$$B_\alpha = J_\alpha S^2 \eta^2_\alpha, \quad B_4 = 0; \quad J_\alpha \equiv \frac{1}{N} \sum_{i \neq j} J^\alpha_{ij},$$

where $S$ is a spin value and $\eta_\alpha$ is given by (54). For $S = 1/2$, we have

$$\eta_\alpha = \frac{1}{2} \tanh(\beta w^2 \eta_\alpha), \quad \eta_4 \equiv 0. \quad (61)$$

The free energy (4) becomes

$$f = \sum_{\alpha=1}^3 \left\{ w^2_\alpha (U + J_\alpha \eta^2_\alpha) - T \ln[2 \cosh(\beta w^2 \eta_\alpha)] \right\} + w^2_4 U - T \ln 2. \quad (62)$$

In this way, the thermodynamic behaviour of the system is defined by the set of seven coupled equations: four equations for the phase probabilities (57) and three nontrivial equations for the order parameter (61). Among all admissible solutions, one has to choose those satisfying all stability conditions and providing an absolute minimum of the free energy (62). In addition to the case when the phase probabilities are found from (57), we need to consider the cases when one or several phase probabilities are put zero [38]. This implies that we have to compare fifteen types of solutions:

1. $w_1 \neq 0, \quad w_2 \neq 0, \quad w_3 \neq 0, \quad w_4 \neq 0$
2. $w_1 = 0, \quad w_2 \neq 0, \quad w_3 \neq 0, \quad w_4 \neq 0$
3. $w_1 \neq 0, \quad w_2 = 0, \quad w_3 \neq 0, \quad w_4 \neq 0$
4. $w_1 \neq 0, \quad w_2 \neq 0, \quad w_3 = 0, \quad w_4 \neq 0$
5. $w_1 \neq 0, \quad w_2 \neq 0, \quad w_3 \neq 0, \quad w_4 = 0$
6. $w_1 = 0, \quad w_2 = 0, \quad w_3 \neq 0, \quad w_4 \neq 0$
7. $w_1 \neq 0, \quad w_2 = 0, \quad w_3 = 0, \quad w_4 \neq 0$
8. $w_1 \neq 0, \quad w_2 = 0, \quad w_3 \neq 0, \quad w_4 = 0$
9. $w_1 = 0, \quad w_2 \neq 0, \quad w_3 = 0, \quad w_4 \neq 0$
10. $w_1 = 0, \quad w_2 = 0, \quad w_3 \neq 0, \quad w_4 = 0$
11. $w_1 = 0, \quad w_2 \neq 0, \quad w_3 = 0, \quad w_4 = 0$
12. $w_1 \neq 0, \quad w_2 = 0, \quad w_3 = 0, \quad w_4 = 0$
13. $w_1 = 0, \quad w_2 \neq 0, \quad w_3 \neq 0, \quad w_4 = 0$
14. $w_1 \neq 0, \quad w_2 \neq 0, \quad w_3 = 0, \quad w_4 = 0$
15. $w_1 \neq 0, \quad w_2 \neq 0, \quad w_3 \neq 0, \quad w_4 = 0$
(5) \( w_1 \equiv 0, \ w_2 \equiv 0, \ w_3 \neq 0, \ w_4 \neq 0; \)

(6) \( w_1 \equiv 0, \ w_2 \neq 0, \ w_3 \equiv 0, \ w_4 \neq 0; \)

(7) \( w_1 \neq 0, \ w_2 \equiv 0, \ w_3 \equiv 0, \ w_4 \neq 0; \)

(8) \( w_1 \equiv 0, \ w_2 \equiv 0, \ w_3 \equiv 0, \ w_4 \equiv 1; \)

(9) \( w_1 \neq 0, \ w_2 \neq 0, \ w_3 \neq 0, \ w_4 \equiv 0; \)

(10) \( w_1 \equiv 0, \ w_2 \neq 0, \ w_3 \neq 0, \ w_4 \equiv 0; \)

(11) \( w_1 \neq 0, \ w_2 \equiv 0, \ w_3 \neq 0, \ w_4 \equiv 0; \)

(12) \( w_1 \neq 0, \ w_2 \neq 0, \ w_3 \equiv 0, \ w_4 \equiv 0; \)

(13) \( w_1 \equiv 1, \ w_2 \equiv 0, \ w_3 \equiv 0, \ w_4 \equiv 0; \)

(14) \( w_1 \equiv 0, \ w_2 \equiv 1, \ w_3 \equiv 0, \ w_4 \equiv 0; \)

(15) \( w_1 \equiv 0, \ w_2 \equiv 0, \ w_3 \equiv 1, \ w_4 \equiv 0. \)

Leaving aside thermodynamics of this model [38,39], we would like to concentrate here on the fact that choosing the most stable solutions makes it possible to describe various reorientation transitions.

Let us arrange the exchange integrals in the order of their magnitude so that

\[ 0 < J_1 < J_2 < J_3. \]

The reorientation temperatures are defined by the equations

\[ \eta_\alpha(T_\alpha) = 0 \quad (\alpha = 1, 2, 3). \] (63)

The largest reorientation temperature is the critical temperature

\[ T_c \equiv \sup_\alpha T_\alpha \] (64)

for a ferromagnet–paramagnet phase transition. When this transition is of first order, the transition temperature will be denoted by \( T_0 \). The temperature at which a pure state transforms into a mixture is called [1] the nucleation temperature. The latter is defined as

\[ T_n \equiv \inf_\nu T_\nu^n, \quad w_\nu(T_\nu^n) = 0, \] (65)

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where \( \nu = 1, 2, 3, 4 \).

The arising sequence of phase transitions can be classified according to the value of the parameter \( U \) corresponding to the potential lattice energy per site. When \( U < 0 \), no heterophase states appear, and no reorientation transitions occur. The sole transition is the ferromagnet–paramagnet phase transition at \( T_c = \frac{1}{2} J_3 \) to which the notation

\[
[0 \ 0 \ 1] \leftarrow 2, T_c \rightarrow [0 \ 0 \ 0]
\]

can be ascribed, where the numbers inside the square brackets means the existence of a nonzero or zero magnetization along an axis \( \alpha = 1, 2, 3 \), respectively, and the number between the arrows shows the phase–transition order at a temperature \( T_c \).

When \( 0 < U \leq U_0 \), where the value \( U_0 \) depends on magnitudes of \( J_\alpha \), then there appears the sequence of phase transitions

\[
[0 \ 0 \ 1] \leftarrow 1, T_n \rightarrow [1 \ 1 \ 1] \leftarrow 1, T_0 \rightarrow [0 \ 0 \ 0],
\]

in which \( T_n = T_1 = T_2 \). Both the nucleation and ferromagnet–paramagnet transitions are of first order.

Increasing further the lattice–energy parameter, in an interval \( U_0 < U \leq U_1 \), we get the sequence

\[
[0 \ 0 \ 1] \leftarrow 1, T_n \rightarrow [1 \ 0 \ 1] \leftarrow 2, T_2 \rightarrow [1 \ 1 \ 1] \leftarrow 1, T_0 \rightarrow [0 \ 0 \ 0]
\]

of phase transitions, with \( T_n = T_1 \). At the value \( U = U_1 \), the nucleation temperature \( T_n \) corresponds to a tricritical point.

For \( U_1 < U \leq U_2 \), we have

\[
[0 \ 0 \ 1] \leftarrow 2, T_n \rightarrow [0 \ 0 \ 1] \leftarrow 2, T_1 \rightarrow [1 \ 0 \ 1] \leftarrow 2, T_2 \rightarrow [1 \ 1 \ 1] \leftarrow 1, T_0 \rightarrow [0 \ 0 \ 0].
\]

The nucleation becomes a second–order transition.

When \( U_2 < U \leq U_3 \), then the sequence of phase transitions simplifies to just one transition

\[
[0 \ 0 \ 1] \leftarrow 1, T_0 \rightarrow [0 \ 0 \ 0]
\]
between ferromagnetic and paramagnetic phases. For $U = U_3$, the transition temperature $T_0$ corresponds to a tricritical point.

Finally, for $U_3 < U < \infty$, we have the sequence

$$[1 1 1] \leftarrow 2, T_1 \rightarrow [0 1 1] \leftarrow 2, T_2 \rightarrow [0 0 1] \leftarrow 2, T_c \rightarrow [0 0 0].$$

As is seen, the existence of heterophase mesoscopic fluctuations makes it possible to get a rich variety of reorientation magnetic transitions which are impossible in the pure case. Thus, mesoscopic fluctuations, at the same time, provide global stability for a system and make its physics much richer. Such fluctuations can also lead to specific features of thermodynamic characteristics, related to pre-transitional phenomena [1]. Probably, these fluctuations could be responsible for the existence of two length scales at structural and magnetic phase transitions observed with high resolution X–ray scattering techniques, particularly using synchrotron sources [40].

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