Implementation of CR Energy SPectrum (CRESP) algorithm in PIERNIK MHD code.  
I. Spectrally resolved propagation of CR electrons on Eulerian grids.  

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Abstract  

We present an efficient algorithm to follow spectral evolution of Cosmic Rays (CR) coupled to an MHD system on Eulerian grids. The algorithm is dedicated for studies of CR energy spectrum evolution in MHD simulations of galactic interstellar medium. The base algorithm for CR transport relies on the two-moment piece-wise power-law method, known also as Coarse Grained Momentum Final Volume (CGMV), for solving the Fokker-Planck CR transport equation, with a low number of momentum-bins extending over several decades of the momentum coordinate.  

We propose extension of the method by including advection and diffusion of CR electrons on Eulerian grids. Our method involves a special treatment of momentum bins containing spectral cut-offs. Contrary to the regular bins of fixed width, those bins have variable-width and their outer edges coincide with spectral cut-offs. The cut-off positions are estimated from the particle number density and energy density in the outer bins for an assumed small value of an additional parameter representing the smallest physically significant level of CR spectral energy density.  

We perform a series of elementary tests to validate the algorithm and demonstrate, whenever possible, that results of the test simulations correspond, with a reasonable accuracy, to the results of analogous analytical solutions. In a more complex test of galactic CR-driven wind problem we obtain results consistent with expectations regarding the effects of advection, diffusion, adiabatic and synchrotron cooling of CR population.  

Keywords: Galactic cosmic rays, Magnetohydrodynamical simulations, Computational methods  

1. INTRODUCTION  

Cosmic rays (CRs) are relativistic, charged elementary particles and nuclei, pervading cosmic space in a wide range of scales: from Solar System, through interstellar medium (ISM) to intergalactic medium. The main ingredients of cosmic rays measured in the vicinity of Earth are mostly protons and alpha particles – respectively ~87% and ~12% of all CR nuclear species – and electrons, comprising for ~2% of all CR species (Tanabashi et al. 2018; Berezinskii et al. 1990). Galactic CRs most likely originate from supernova remnants (Drury et al. 1994), where they can be accelerated via diffusive shock acceleration (Blandford & Ostriker 1978), if the efficiency of SN energy conversion to CR kinetic energy is around ~10% (Blasi 2013). CRs take part in many astrophysical phenomena: cosmic protons and nuclei take part in dynamical as well as chemical evolution of galaxy (Grenier et al. 2015), e.g. quenching the process of stellar formation (Tabatabaei et al. 2018), and enriching chemical composition of the ISM via spallations, while CR electrons and other leptons
interact with magnetic and radiation fields, thus being directly linked to the observations of synchrotron radiation (Jaffe et al. 2011; Strong, A. W. et al. 2011), which they produce at the cost of its kinetic energy. Observations show that radio emission in disk galaxies varies in extent at different wavelengths, spectral indices also indicate spatial variation (Fletcher et al. 2011; Mulcahy et al. 2018) in galaxies seen: face-on – harder spectral indices in spiral arms and in edge-on – harder spectral index in the galactic plane. CR spectrum varies subtly in space and time at least in our local environment, what is observed locally in e.g. solar modulation of low energy CRs (Potgieter 2013) and measurements of Voyager spacecrafts (Cummings et al. 2016), as well as inferred from observations of distant galaxies. Some of the first numerical studies combining CR propagation with magnetohydrodynamical evolution of the interstellar medium (ISM) treated CRs as a relativistic fluid described through the momentum-integrated CR diffusion advection equation. These investigations focused on modeling Parker instability in the ISM due to buoyancy of hot CRs (e.g. Hanasz & Lesch 2003; Kuwabara et al. 2004; Heintz & Zweibel 2018) which leads to formation of magnetic loops and evacuation of CRs from the galactic disk. Hanasz et al. (2004, 2009a,b); Kulpa-Dybel et al. (2011, 2015); Siejkowski et al. (2010, 2014, 2018) have shown that CRs supplied to the ISM by SN remnants may contribute to amplification of weak small-scale magnetic fields to produce large scale fields of a few µG within 1–2 Gyr. The process named ‘cosmic ray-driven dynamo’, originally proposed by Parker (1992), is closely connected with formation of powerful galactic winds that can remove significant portions of gas away from the galaxy (see e.g. Ullig et al. 2012; Hanasz et al. 2013; Booth et al. 2013; Salem & Bryan 2014; Girichidis et al. 2016, 2018; Farber et al. 2018; Hopkins et al. 2020). Radio-observations of galactic synchrotron emission provide useful information on the strength and structure of galactic magnetic fields (see Beck 2015, for a comprehensive review), however, a unique reconstruction of the three-dimensional magnetic field structure from the observational data is not an easy task. On the other hand, theoretical models and numerical simulations provide an insight into physical processes connecting magnetic field evolution to the complex dynamics of interstellar gas and cosmic rays, but offer a limited access to useful observables, such as the spectrum of synchrotron radio-emission. These circumstances highlight the need for time-dependent spectral modeling of the CR electron population if one needs to confront MHD models of galactic ISM against observations of real galaxies. We note the efforts to interpret observational data with the help of spectrally resolved CR propagation codes, such as GALPROP (Moskalenko & Strong 1998), USINE (Putze et al. 2011), PICCARD (Werner et al. 2013), DRAGON (Gaggero et al. 2014) and SPINNAKER (Heesen et al. 2018) which provide numerical solutions for CR diffusion and advection in fixed magnetic fields and stationary background flows. We would like to emphasize, however, that our strategy, presented in this paper, is different: instead of combining CR transport with observationally based static models of galactic magnetic fields, we solve the CR transport equations together with the system of MHD equations to obtain time-dependent solutions for gas, magnetic fields, and the energy spectrum of CRs. We present Cosmic Ray Energy SPectrum (CRESP) – a new numerical algorithm designed to simulate momentum-dependent diffusion-advection propagation of CR electrons, coupled to MHD simulations of the thermal component on 3D Eulerian grids. The presented algorithm has been incorporated into the framework of PIERNIK MHD code.

The plan of the paper is as follows: in section 2 we formulate our novel method to solve the problem of CR-electron spectral evolution on Eulerian grids and in section 3 we describe initial conditions. In section 4 we describe validation tests of CRESP algorithm operating in a one-zone mode and in section 5 we present validation tests of CRESP operating together with the CR spatial transport scheme of PIERNIK. In section 6 we present test simulations of CR-electron spectrum evolution in a quasi-realistic 3D model of galactic wind and in section 7 we conclude our paper.

2. NUMERICAL ALGORITHMS FOR CR TRANSPORT EQUATION

2.1. Cosmic Ray transport equation

The time-evolution of distribution function, describing ultra relativistic CR component, can be characterized by simplified diffusion-advection equation (Skilling 1975; Blandford & Eichler 1987; Miniati 2001), which can be express as

\[
\frac{\partial f}{\partial t} = -v \cdot \nabla f + \nabla (\kappa \nabla f) + \frac{1}{3} (\nabla \cdot v) p \frac{\partial f}{\partial p} \\
+ \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 b(p)f + D_{pp} \frac{\partial f}{\partial p} \right] + j(p)
\]

where \( f = f(x,p,t) \) is the isotropic part of the distribution function of a population of CR particles, \( v \) is velocity of thermal plasma, \( \kappa \) is the diffusion tensor representing diffusive propagation of CRs in space and \( D_{pp} \) is the momentum diffusion coefficient. Similarly
as (Girichidis et al. 2020) we neglect for simplicity CR streaming process (see e.g. Sharma et al. 2010; Jiang & Oh 2018; Thomas & Pfommer 2019). We also neglect the second-order Fermi acceleration by setting $D_{pp} = 0$. The first two terms on the right-hand side represent advection and diffusion in physical space. The third term represents the adiabatic cooling or heating of CR population, depending on the sign of $\nabla \cdot \mathbf{v}$. Radiative losses $b_i(x, p)$ with 2nd order Fermi mechanism are accounted for in the fourth term and CR sources are represented by $j = j(x, p, t)$. Equation (1) describes temporal evolution of a population of energetic particles subject to strong scattering, which ensures particle distribution remains isotropic on scales much greater than particle scattering-length (Jones et al. 1999).

We are aiming to solve numerically this equation together with the full set of MHD equations describing temporal evolution of the thermal plasma in MHD approximation. The CR transport Equation (1) couples to the MHD equations via velocity field $\mathbf{v}$ (in Lagrangian derivative of the distribution function $f$ and adiabatic cooling/heating term) and magnetic field $\mathbf{B}$, if the synchrotron cooling process is considered. In a general case CRs provide an additional pressure contribution to the system. In this case the equation of gas motion should include gradient of the total CR pressure $\nabla p_{CR}$, corresponding to the pressure CRs exert on thermal gas, however the pressure from CR electron population might be neglected due to their relatively low energy density, estimated at about 1% of CR proton energy density (Strong et al. 2007).

2.2. The two-moment piece-wise power-law approximation

Starting with the papers by Falle & Giddings (1987) and Bell (1987) various numerical methods were proposed to solve the CR propagation equation (1) in momentum space. Kang & Jones (1991) solved numerically the CR transport equation using a standard discretization approximation in momentum space assuming piecewise constant values of distribution function in momentum bins. Winner et al. (2019) demonstrated that this kind of discretization requires approximately 50–100 bins per momentum decade to achieve accurate results of numerical integration of CR transport equations coupled to the system of MHD equations. Such a high spectral resolution, needed due to the steep CR spectra and large dynamical range of $f$ is, however, not feasible in multidimensional fluid-dynamical simulations involving cosmic rays.

An alternative method, based on a piece-wise power-law representation of the distribution function

$$f(p) = f_{l-1/2} \left( \frac{p}{p_{l-1/2}} \right)^{-q_l} \quad \text{for} \quad p \in [p_{l-1/2}, p_{l+1/2}]$$

has been proposed by Jones et al. (1999), where $f_{l-1/2}$ are the distribution function amplitudes, defined on left edges of momentum bins $p_{l-1/2}$ and spectral indices $q_l$ are attributed to bin interiors. The piece-wise power-law spectrum is constructed on logarythmicaly spaced momentum bins with fixed bin width equal to

$$\Delta w_l = \log_{10}(p_{l-1/2}/p_{l+1/2}) = \log(p_{\text{max}}/p_{\text{min}})/N_{\text{bin}},$$

where $N_{\text{bin}}$ is the number of bins. This approach involves the number density moment (see Appendix) of the transport equation (1) combined with an additional assumption of continuity of the distribution function. This approach leads, however, to a numerical instability pointed out by (Girichidis et al. 2020), who show that if energy is injected at one part of the spectrum, the continuity assumption enforces changes of the local slope across the entire spectrum. The resulting continuous representation then alternates between a concave and a convex spectrum. Artificial oscillations superposed to the initially smooth spectrum have a tendency to grow towards high energy bins revealing a numerical instability of the scheme.

The piece-wise power-law representation of distribution function has two degrees of freedom, which can be bound by taking the number density and energy density moments of the diffusion-advection equation (1). Miniati (2001) proposed and implemented in COSMOCR code a method which relies on the direct conversion between the two pairs of distribution function parameters in each bin: $(f_{l-1/2}, q_l)$ to the number density $n_l$ and energy density $e_l$ moments. Moreover, in absence of explicit sources these two moments are conservative quantities and therefore they are suitable for an accurate evolution with conservative transport schemes in both the spatial and momentum domains. The method is known also as Coarse–Grained Momentum finite Volume (CGMV) Jones & Kang (2005). Details of the method are presented in Appendix.

The two-moment piece-wise power-law method is particularly useful in the case of strong cooling processes such as synchrotron and inverse-Compton losses of CR electrons (Miniati 2001) or Coulomb losses of CR protons Girichidis et al. (2020), which produce cutt-offs in the distribution function. Since momentum bins span usually a wide range of values, which is the case when several decades of momentum coordinate is divided into
a few tens of bins, the cut-offs fall typically in the middle of a bin, so that only a part of the bin below the momentum cut-off is populated, while the remaining part is completely depleted. The cut-offs have to be followed accurately in order to separate the populated parts of the bins from the depleted ones. Attempts to resolve the distribution function beyond the physical cutoff lead to numerical problems, which are unavoidable if the reconstruction procedure includes the depleted parts of the bins.

Numerical solutions are available in an practically unlimited range of momentum coordinate only if the strong cooling processes, such as synchrotron or Coulomb losses are neglected. Jones & Kang (2005) modeled the propagation of CRs across an Eulerian grid, using a fully hydrodynamical description, including advection and diffusion processes in physical and momentum spaces. They imposed the upper boundary condition by setting the particle number density to zero at the high end of the spectrum. At the low range of the spectrum the distribution function can be matched to the thermal particle distribution.

Miniati (2001) applied the method for the evolution of CR electron spectrum attributed to Lagrangian particles. Advection of the CR spectra resulted directly from the Lagrangian treatment of the particles, while diffusion of CRs was neglected in this approach. In absence of diffusion mixing of CR populations among fluid elements does not occur in particle codes, therefore evolution of spectral cut-offs, attributed to individual particles, can be followed directly with the aid of Equation (A4) and numerical integration of the evolution Equations (A16) and (A17) is straightforward. Following this method Mimica et al. (2009), Yang & Ruszkowski (2017) and Vaidya et al. (2018) implemented spectrally resolved CR propagation by means of (mesh-less) Lagrangian macro-particles embedded in a classical or relativistic Eulerian MHD system. However, exchange of CR spectra between different fluid elements becomes essential in the fully Eulerian treatment of CRs, i.e. when advection and diffusion of CRs are to be taken into account on Eulerian grids.

For very low momenta the timescale related to the Coulomb cooling process becomes shorter than the typical time step of MHD simulations. Similarly, for very high momenta the synchrotron and inverse Compton energy losses proceed on arbitrarily short timescales. Winner et al. (2019) presented an efficient post-processing code for Cosmic Ray Electron Spectra that are evolved in Time (CREST) on Lagrangian macro-particles. They divided the overall spectrum into three parts treated with different methods. In order to efficiently calculate the CR electron spectrum with time steps similar to the MHD time step, they used analytical solutions for low and high momenta together with the fully numerical treatment for intermediate momenta. A similar technique based on a combination of numerical integration and an analytical prescription of the Coulomb cooling has been applied by Girichidis et al. (2020) who incorporated the combination of analytical and numerical methods into the method by Miniati (2001). The use of analytical solution helps to overcome the excessive shortening of the integration time-step whenever cooling timescale is significantly shorter than the hydrodynamical timescale.

Focusing the consideration on the synchrotron cooling we note that the process operates most efficiently in the high momentum range of the spectrum. The overall momentum-dependent CR transport algorithm should be able to determine the position of the upper cut-off and should be equipped with appropriate boundary conditions in momentum space. We note that CR spectra may have different cut-offs in each spatial grid-cell, because local cooling conditions (magnetic and velocity fields) are generally different. Due to advective and diffusive propagation of the CRs across the spatial grid different populations of particles mix in every cell. The essential part of the numerical problem is to estimate an effective cut-off for the mixture of different populations inflowing from neighboring cells with different cut-offs.

Our aim is to work out a fully Eulerian algorithm, based on the piecewise power-law method, designed for momentum-dependent modeling of CR propagation in space. An obvious benefit would be a straightforward treatment of CR diffusion based in the fluid approach and accurate computation of pressure forces coupling the dynamics of CR population with thermal plasma.

2.3. Extension of the formalism to Eulerian grids

2.3.1. The environment of PIERNIK MHD code

PIERNIK is a grid-based MHD code using conservative numerical schemes. The functionality of PIERNIK includes the modeling of multiple fluids: gas, dust, magnetic field, cosmic rays, and their mutual interactions. PIERNIK is parallelized on the base of MPI library and its dataIO communication utilizes parallel HDF5 output. The MHD algorithm is based on the standard set of resistive MHD equations (Hanasz et al. 2010a,b) including uniform and current dependent resistivity (Hanasz et al. 2012a), that are solved using RTVD scheme by (Jin & Xin 1995) and (Pen et al. 2003). PIERNIK code has
been recently equipped with the HLLD Riemann solver Miyoshi & Kusano (2005) combined with the Dedner et al. (2002) algorithm which may serve as an alternative for the simple, robust but more diffusive RTVD numerical scheme. Other algorithms added recently to the code include the Adaptive Mesh Refinement (AMR) technique, multigrid (MG) Poisson solver, multigrid diffusion solver, and an N-body particle-mesh solver for a large-number point masses representing stellar and dark matter components of galaxies.

2.3.2. Spatial transport of CRs

Numerical algorithm of CR spatial transport embedded in PIERNIK code is based on formulae (A2) for CR number density and (A8) for CR energy density, respectively. No transfers of CRs between momentum bins are involved in this step.

Inter-cell fluxes along x-axis of CR number density and energy density computed at cell faces are respectively

\[
F_{x,i+\frac{1}{2},j,k,l}^{adv,n} = (v_x n_l)_{i+\frac{1}{2},j,k},
\]

\[
F_{x,i+\frac{1}{2},j,k,l}^{adv,e} = (v_x e_l)_{i+\frac{1}{2},j,k},
\]

where \(v_x\) is the x-component of thermal gas velocity, \(n_l\) and \(e_l\) are CR number density and energy density of the n-th bin and face centering of cell centered quantities relies on an interpolation consistent with the fluid advection algorithm in use. Analogous formulae are adopted for advection in the y and z directions. We note that CR advection scheme appropriate for shock discontinuities should involve CR number density (and energy density) normalized to thermal gas density (see e.g. Miniati 2007), however here we focus on the formulation of CR spectrum evolution algorithm and treat all aspects of the spatial transport in the presented simple way.

Numerical algorithm for the anisotropic, magnetic field-align diffusion relies entirely on the method presented by Hanasz & Lesch (2003). Components of the anisotropic diffusion tensor are

\[
\kappa_{ij}(p) = \kappa_{\parallel}(p) \delta_{ij} + (\kappa_{ij}(p) - \kappa_{\parallel}(p)) h_i h_j,
\]

where \(\kappa_{ij}(p)\) and \(\kappa_{\parallel}(p)\) are the parallel and perpendicular, momentum-dependent diffusion coefficients, and \(h_i = B_i/|B|\) are components of the unit vectors tangent to magnetic field lines.

To achieve numerical stability the anisotropic diffusion algorithm requires, in addition to the standard CFL timestep limitation, a careful interpolation of magnetic field components, CR number density and energy density gradients. With an appropriate use of slope limiters the algorithm turns out to be robust even in the presence of dynamical coupling between thermal gas and CR fluid. Diffusive fluxes along x-axis computed at cell faces are

\[
F_{x,i+\frac{1}{2},j,k,l}^{\text{diff},n} = (\hat{\kappa}_n^x(p_l) \nabla n_l)_{i+\frac{1}{2},j,k},
\]

\[
F_{x,i+\frac{1}{2},j,k,l}^{\text{diff},e} = (\hat{\kappa}_n^x(p_l) \nabla e_l)_{i+\frac{1}{2},j,k},
\]

where \(\hat{\kappa}_n^x(p_l)\) and \(\hat{\kappa}_n^x(p_l)\) are momentum-averaged diffusion tensors of CR number density and energy density in momentum bin No. \(l\). Similar relations, involving variations of indices \(j\) and \(k\), hold for y and z directions respectively.

In the discrete representation, the 3-dimensional conservation law reads

\[
n_{i,j,k,l}^{t+dt} = n_{i,j,k,l}^t - \frac{\Delta t}{\Delta x} \left( F_{i+1/2,j,k,l}^n - F_{i-1/2,j,k,l}^n \right)
\]

\[
- \frac{\Delta t}{\Delta y} \left( F_{i,j+1/2,k,l}^n - F_{i,j-1/2,k,l}^n \right)
\]

\[
- \frac{\Delta t}{\Delta z} \left( F_{i,j,k+1/2,l}^n - F_{i,j,k-1/2,l}^n \right) + \frac{\Delta t}{\Delta x} \left( \frac{F_e^n}{e_{i+1/2,j,k,l}} - \frac{F_e^n}{e_{i-1/2,j,k,l}} \right)
\]

\[
+ \frac{\Delta t}{\Delta y} \left( \frac{F_e^n}{e_{i,j+1/2,k,l}} - \frac{F_e^n}{e_{i,j-1/2,k,l}} \right)
\]

\[
+ \frac{\Delta t}{\Delta z} \left( \frac{F_e^n}{e_{i,j,k+1/2,l}} - \frac{F_e^n}{e_{i,j,k-1/2,l}} \right),
\]

where \(n_{i,j,k,l}^t\), \(n_{i,j,k,l}^{t+dt}\) and \(e_{i,j,k,l}^t\), \(e_{i,j,k,l}^{t+dt}\) are volume averaged CR number and energy densities of t-th bin in the cell \(i,j,k\) at times \(t\) and \(t + dt\), respectively. \(F_e^n_{i-\frac{1}{2},j,k,l}\), \(F_e^n_{i+\frac{1}{2},j,k,l}\), \(F_e^n_{i,j-\frac{1}{2},k,l}\), \(F_e^n_{i,j+\frac{1}{2},k,l}\), \(F_e^n_{i,j,k-\frac{1}{2},l}\), \(F_e^n_{i,j,k+\frac{1}{2},l}\) represent the sum of advection and diffusion fluxes through the left and right cell boundaries, in x-direction, while the remaining flux components, indexed with \(j\) and \(k\), respectively, represent transport in y- and z-directions.

The advection part of the spatial transport is realized in a way analogous to propagation of a passive tracer. The advection and diffusion steps are executed in an operator split manner. Advection of all CR bins is a part of the dimensionally split fluid update step consisting of the sequence of sweeps in x-y-z and z-y-x directions. There are two options for the choice of diffusion algorithms. The default one is the explicit, dimensionally split solver introduced by Hanasz & Lesch (2003), which executes the sequence of x-y-z or z-y-x transports before each sequence of three advection sweeps. The explicit diffusion timestep is subject to the CFL condition

\[
\Delta t_{\text{max}} = 0.5 \ C_{\text{cr-diff}} \ \min(\Delta x, \Delta y, \Delta z)^2 \ \max(\kappa_{ij}(p_l) + \kappa_{\parallel}(p_l))
\]

where \(C_{\text{cr-diff}} \leq 1\) is the Courant number for the CR diffusion process and the maximum of diffusion coefficients over the set of momentum bins is used. As alternative to the explicit method an implicit multigrid diffusion solver, which mitigates the CFL condition at some higher computational cost.
The formalism presented in the appendix provides formulae (A3) and (A10) for bin-averaged diffusion of number density and energy density respectively. The involved spatial derivatives of the distribution function, which in practical implementation lead to stability problems of the numerical algorithm and to a significant growth of computational costs (Girichidis et al. 2020). To avoid these difficulties we approximate the diffusion of number and energy densities with with their values computed for centres of momentum bins. Equal diffusion coefficients for the particle number density and energy density imply that the ratio of $e_l/n_l$ is preserved in the diffusion step and therefore the slope $q_l$ of the distribution function does not change within the bin. This leads to formation of 'teeth' in the distribution function which are apparent in the pure diffusion tests, but are usually negligible in the case of combined diffusion, advection with adiabatic and synchrotron processes (to be shown in Section 5). This approximation leads to a stable evolution of CR population, therefore we decide to use it in the forthcoming series of tests. Diffusion coefficients are dependent on momenta attributed to the bin centers and normalized as follows:

$$k_{\parallel,\perp}(p_l) = k_{10k\parallel,\perp} \left( \frac{p_l}{10^4 m_e c} \right)^{\alpha_s}.$$

where $k_{10k}$ is the parallel diffusion coefficient at the particle momentum $p = 10^4 m_e c$ which is representative for particles contributing to synchrotron radioemission at centimeter wavelengths in magnetic fields of the order of a few $\mu G$, typical for the interstellar medium. The power index $\alpha_s$ scales the momentum dependence of the diffusion coefficient. We assume $\alpha_s = 0.5$. Usually CR diffusion is dominant in the direction of magnetic field (see e.g. Berezinskii et al. (1990); Giacalone & Jokipii (1999)), hence we usually assume that perpendicular diffusion coefficient $k_{\perp} = 1\% k_{\parallel}$.

### 2.3.3. The problem to solve

Let us remind that due to the synchrotron cooling rate proportional to $p^2$ all electrons cool down below $p_{\text{cut}} \propto B^{-2} t^{-1}$ in a finite time (Kardashev 1962), therefore the whole range of momentum space above $p_{\text{cut}}$ becomes empty. Because synchrotron cooling rate depends on magnetic field strength, which in turn depends on space coordinates and time, cut-off momenta are different in different cells of the spatial grid. While CR fluid propagates according to momentum-dependent diffusion-advection equation the flux of particles through cell boundaries changes the content of momentum bins. In the presence of spatial transport of CRs the cut-off momenta of an individual cell depend not only on the local physical conditions, but also on an inflow of particles from the adjacent cells. An inflow of fresh population of high energy particles from e.g. a nearby SN remnant may change suddenly the whole spectrum in a given cell, including the change of lower and upper cut-offs. The essential problem to solve is therefore to estimate the cut-off momenta in all cells of the Eulerian grid after the spatial transport step which mixes CR populations of neighboring cells.

### 2.3.4. The proposed solution

Each cell of the spatial grid is attributed with a piece-wise power-law CR spectrum defined on a discrete, logarithmically spaced (according to Equation (3)) momentum grid, named hereafter as fixed momentum grid. To solve the problem we need an operational definition of spectral cut-offs, which is appropriate for the piece-wise representation of CR spectrum. We shall treat similarly the low- and high-energy parts of CR spectrum. We denote left and right edges of the spectrum as $p_{lo}$ and $p_{up}$, depending on spatial coordinate indexes, which we omit to simplify the notation. The spectrum limits evolve according to integrated momentum evolution equations (A20). Variations of the cutoff momenta imply that the bins of fixed grid, containing $p_{lo}$ and $p_{up}$, are filled only partially, i.e. only the high-energy part of the lowest bin and low-energy part of the highest bin are filled with particles. The cut-off momenta $p_{lo}$ and $p_{up}$ become edges of the outer variable-width bins. We define active bins as those containing CR particles. The remaining bins are inactive.

We shall estimate the cut-off momenta in each cell using the information stored in the set of known values of $(e_l, n_l)$ by solving Equation (A28) for the unknown cut-off momentum $p_{up}$ or $p_{lo}$, rather than for the slope $q_l$. We note that in the piece-wise power-law representation the distribution function values should be finite at the cut-offs. Otherwise, the slopes $q_l$ in the outer bins would be infinite.

We propose the following procedure for the bin reconstruction based on the known values of $e_l$ and $n_l$:

1. We define a new parameter $\epsilon_{\text{small}}$ denoting the smallest physically significant value of the spectral energy density

$$\epsilon \equiv \frac{de}{dp} = 4\pi cp^3 f(p).$$

We assume that $\epsilon = \epsilon_{\text{small}}$ at the outer edge of the bin, where $\epsilon_{\text{small}}$ is a free parameter which will be adopted, with some help of validation tests presented in Section 4. The values of $p$ at which $\epsilon = \epsilon_{\text{small}}$ are our estimates for $p_{lo}$ and $p_{up}$.
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2. Let us denote the number of bin containing the lower cut-off as \( l_{lo} \) and the number of bin containing the upper cut-off as \( l_{up} \). Knowing the values of \( (n_{up}, \epsilon_{up}) \) together with the left edge \( p_{l_{up}, L} \) of the uppermost bin and \( \epsilon_{small} \) we want to estimate the upper cut-off momentum \( p_{up} \). Similarly, knowing the values of \( (n_{lo}, \epsilon_{lo}) \) together with the right edge of the lowermost bin \( p_{l_{lo}, R} \) and \( \epsilon_{small} \) we want to estimate the lower cut-off momentum \( p_{lo} \). Graphical presentation of the problem for the lowermost and uppermost momentum bins is shown in Figure 1.

3. We define new variables \( \mu \equiv p_{l,R}/p_{l,L} \) and \( \phi \equiv f_{l,R}/f_{l,L} \) and use Equation (A13) to obtain

\[
q = \frac{\log_{10} \phi}{\log_{10} \mu}.
\]

(13)

4. For the lowest active bin \( p_{l_{lo}, L} = p_{lo} \) and \( f_{l_{lo}, L} = \epsilon_{small}/(4\pi cp_{lo}^3) \) and \( p_{l_{lo}, R} = p_{lo+1/2} \) is the known right (fixed) edge of the lowest bin and

\[
\mu_{lo} = \frac{p_{lo+1/2}}{p_{lo}}, \quad \phi_{lo} = \frac{4\pi cp_{lo}^3 f_{l_{lo}, R}}{\epsilon_{small}},
\]

and \( q_{lo} \) computed using Equation (13).

For the highest active bin \( p_{l_{up}, R} = p_{up} \) and \( f_{l_{up}, L} = \epsilon_{small}/(4\pi cp_{up}^3) \) and

\[
p_{l_{up}, L} = p_{l_{up}-1/2} \] is the known left (fixed) edge of the highest bin and

\[
\mu_{up} = \frac{p_{up}}{p_{l_{up}-1/2}}, \quad \phi_{up} = \frac{\epsilon_{small}}{4\pi cp_{up}^3 f_{l_{up}, L}},
\]

and \( q_{up} \) is given by Equation (13).

5. We transform Equation (A28) with the aid of Equations (14) to get

\[
\frac{\epsilon_{l_{lo}}}{n_{l_{lo}} c p_{l_{lo}+1/2}} = \begin{cases} 
\frac{1}{\mu_{lo}} \log_{10} \mu_{lo} & \text{if } q_{lo} = 3, \\
\frac{1}{\mu_{lo} - 1} \log_{10} \mu_{lo} & \text{if } q_{lo} = 4, \\
\frac{1}{\mu_{lo} - 1} & \text{otherwise},
\end{cases}
\]

(16)

Formula (A14) for particle number density \( n_{l_{lo}} \) translates to

\[
n_{l_{lo}} = \epsilon_{small} \frac{\log_{10} \mu_{lo}^{\phi_{lo}/p_{lo}}}{c} \begin{cases} 
\log_{10} \mu_{lo} & \text{if } q_{lo} = 3, \\
\frac{3 - q_{lo}}{3 q_{lo}} - 1 & \text{if } q_{lo} = 4, \\
\frac{3 - q_{lo}}{3 q_{lo}} & \text{otherwise}.
\end{cases}
\]

(17)

Similarly, for the upper boundary equation (A28) becomes

\[
n_{l_{up}} = \frac{\epsilon_{small}}{c p_{l_{up}-1/2}} \begin{cases} 
\frac{4 - q_{up}}{4 q_{up} - 1} & \text{if } q_{up} = 3, \\
\frac{3 - q_{up}}{3 q_{up}} & \text{if } q_{up} = 4, \\
\frac{3 - q_{up}}{3 q_{up}} - 1 & \text{otherwise}.
\end{cases}
\]

(18)

Equation (A14) for \( n_{l_{up}} \) translates to

\[
n_{l_{up}} = \epsilon_{small} \frac{\log_{10} \mu_{up}^{\phi_{up}/p_{up}}}{c} \begin{cases} 
\log_{10} \mu_{up}^{\phi_{up}/p_{up}} & \text{if } q_{up} = 3, \\
\frac{3 - q_{up}}{3 q_{up}} - 1 & \text{if } q_{up} = 4, \\
\frac{3 - q_{up}}{3 q_{up}} & \text{otherwise}.
\end{cases}
\]

(19)

6. Solutions to these equations are obtained with the aid of a two-dimensional Newton-Raphson-type method. For a given value of l.h.s of the equation systems (16)–(17) or (18)–(19) the numerical procedure finds solutions \( (\phi_{lo}, \mu_{lo}) \) and \( (\phi_{up}, \mu_{up}) \) which are translated into spectrum cutoffs using prescriptions:

\[
p_{lo} = p_{l_{lo}+1/2}, \quad f_{l_{lo}, R} = \frac{\epsilon_{small}}{4\pi cp_{lo}^3},
\]

(20)

for the lowest bin and

\[
p_{up} = p_{l_{up}-1/2} \mu_{up}, \quad f_{l_{up}, L} = \frac{\epsilon_{small} 1}{4\pi cp_{up}^3 \phi_{up}}
\]

(21)

for the highest bin.

2.4. Structure of the overall algorithm
Having solved the problem of determining spectrum cut-offs on Eulerian grid in Section (2.3.4), we may proceed to combine the spatial and momentum-space transport schemes to obtain the full framework for CR spectral evolution on the spatial grid. Momentum-space transport is done by CRESP module of PIERNIK code, in between two sequences of spatial transport sweeps, $x - y - z$ and $z - y - x$ (see Section 2.3.2). Transport of the CR population over momentum-grid, executed in all cells of the spatial grid, proceeds in the following order:

1. **Finding active bins.**
   
   (a) We determine the set of bins, where $n_l$ and $e_l$ are non-vanishing and tag the remaining bins empty.
   
   (b) We apply root-finding method to solve Equation (A28) for slopes $q_l$ in all non-empty bins. In the actual implementation we tabulate solutions of Equation (A28) at the simulation initialization step and use interpolation methods during the simulation.
   
   (c) The slopes $q_l$ found in the previous step are used to solve Equation (A14) for distribution function amplitudes $f_{l-1/2}$.
   
   (d) Using Equation (A13) and (12) we compute spectral energy density $\epsilon_{l-1/2}$ and $\epsilon_{l+1/2}$ at left and right edges $p_{l-1/2}$ and $p_{l+1/2}$ of the bin.
   
   (e) We compare $\epsilon_{l-1/2}$ and $\epsilon_{l+1/2}$ against $\epsilon_{\text{small}}$ and tag all bins fulfilling the condition
   \[\max(\epsilon_{l-1/2}, \epsilon_{l+1/2}) > \epsilon_{\text{small}}\] (22)
   as active bins. The remaining bins are tagged as inactive for the spectrum evolution algorithm, regardless of their content, but not for the spatial transport. For the lowermost active bin, associated with the lower cut-off we set $l_{\text{lo}}$, while $l_{\text{up}}$ is associated with the uppermost active bin.

2. **Determining cut-offs.**

   We recover the lower cutoff information for the bins containing spectral cut-offs from $n_{l_{\text{lo}}}$ and $e_{l_{\text{lo}}}$ by solving the system of Equation (14),(16), and (17) for the unknowns $p_{l_{\text{lo}}}$, and $f_{l_{\text{lo}}, R}$, as explained in Section 2.3.4. Similarly, we find the upper cut-offs from $n_{l_{\text{up}}}$ and $e_{l_{\text{up}}}$ by solving the system of Equation (15), (18), (19) for the unknown $p_{l_{\text{up}}}$ and $f_{l_{\text{up}}, L}$. Using a Newton-Raphson solver we tabulate the solutions at the initialization of the code run and use interpolated values in the time-step integration loop. The values of cut-off momenta and distribution function amplitudes are now reconstructed in all active bins.

3. **Computing changes of cut-off momenta.**

   Particle momentum changes owing to the considered processes are computed for each point of momentum grid via integration of Equation (A4) in an interval $(t, t + \Delta t)$ resulting in Solutions (A20) which are used to compute changes of cut-off momenta within the timestep. For practical reasons we use a combination of Taylor expansions of these solutions by applying the formulae (A21) and (A22).

4. **Computing particle momentum changes at bin interfaces.**

   Upstream momenta $p_{\text{ups}}$, defined in Equation (A23), are computed in a similar manner and used to determine the direction of the flow in momentum space through bin interfaces. If $p_{\text{ups}, l-1/2} > p_{l-1/2}$ the interface is marked as cooling interface, and if $p_{\text{ups}, l-1/2} < p_{l-1/2}$ is marked as heating interface.

5. **Computing particle and energy fluxes through bin interfaces.**

   Fluxes of particle and energy densities crossing the bin interfaces, $d\sigma_{\text{ups}, l-1/2}(\Delta t)$ and $d\epsilon_{\text{ups}, l-1/2}(\Delta t)$ respectively, are computed. For cooling interfaces, if the flow is directed from $l$-th to $(l-1)$-th bin, equations (A24) and (A25) are used. Similarly, for heating interfaces, if the flow is directed from $l$-th to $(l+1)$-th bin, equations (A26) and (A27) are used.

6. **Computing source terms for adiabatic changes and radiative losses.**

   Gains and losses of particle energy due to interactions of CRs with the thermal plasma and due to radiative losses are included as source terms in Equation (A18) and are quantified by the factor $R_l$ defined through Equation (A19) for the case of adiabatic process combined with synchrotron losses.

7. **Numerical integration of particle number density and energy density equations.**

   Time-integration of the number density and energy density evolution equations is performed in the first-order operator-split manner. A conservative transport step of $n_l$ and $e_l$ in momentum space, due to inter-bin fluxes is executed in the first step, according to the momentum-advection
part of Equation (A16) and (A18)

\[ n_{l}^{t+\Delta t} = n_{l}^{t} - (dn_{ups,l+1/2}(\Delta t) - dn_{ups,l-1/2}(\Delta t)), \]
\[ e_{l}^{t+\Delta t} = e_{l}^{t} - (de_{ups,l+1/2}(\Delta t) - de_{ups,l-1/2}(\Delta t)). \]

(23)

(24)

Energy density is subsequently updated with the source terms
\[ e_{l}^{t+\Delta t} = e_{l}^{(1)} (1 - \Delta t R_{l}). \]

(25)

The whole procedure is executed in all cells of the spatial grid, where the values of adiabatic process and synchrotron cooling parameters \( u_{d} \) and \( u_{b} \) depend on local conditions in individual cells of the spatial grid.

2.5. Code optimisation challenges

In order to describe some of the mentioned processes in detail, one might need to either compute \( f \) and \( q \) on the run (resulting in increase of the number of operations), or keep information on these. Our aim was not to store \( f \) and \( q \) values, but instead keep information on \( n \) and \( e \), and thus for each computational cell we need \( 2N_{\text{bin}} \) fields, still resulting in \( N_{x} \times N_{y} \times N_{z} \times 2N_{\text{bin}} \) fields on any grid. Storing information on \( f \) and \( q \) would double the memory usage by each CR component, posing a dilemma in which we either choose doubling the number of saved quantities or to recompute these quantities, impacting algorithm performance, as obtaining these quantities is computationally expensive. Press et al. (1996) points out that Newton-Raphson (N-R) convergence is fast, but has poor global convergence properties. For this reason as well as for optimization needs it is critical to provide the scheme with a good initial guess. This can be achieved by preparing solution maps for each unknown quantity \( \mu, \phi \) for both cutoffs in 2-dimensional \((\epsilon/ncp, n)\) space. Unknowns first are obtained via linear interpolation of mapped solutions, and then either refined using N-R scheme or used as is, if resolution of maps is satisfactory.

Equation (A28) used to obtain \( q_{l} \) can be parametrized with just one term, that is \( e_{l}/n_{l}c_{l}^{2} \); if quotient \( p_{r}^{1/2}/p_{l}^{1/2} = 10^{w} \) is constant; this is true for all bins except the cutoff ones. This allows us to use preliminary solution mapping method for \( q \), similar to that in Section 2.3.4. At initialization a solution array is found. Value of \( q_{l} \) obtained from the array can be used as is, or further refined using 1D numerical scheme.

3. INITIAL AND BOUNDARY CONDITIONS

3.1. Initial spectrum

The simplest initial condition for the power-law-type initial spectrum would be the pure power-law function. For convenience of implementation of the numerical algorithm we shall use the momentum coordinate and related quantities to their dimensionless form:
\[ \tilde{p} = p/(m_{e}c). \]

(26)

We may consider a simple, purely power-low form of initial distribution function
\[ f(\tilde{p}) = \begin{cases} f_{\text{init}} \left( \frac{\tilde{p}}{\tilde{p}_{\text{lo}}} \right)^{-q_{\text{init}}} & \text{for } \tilde{p} \in [\tilde{p}_{\text{lo}}, \tilde{p}_{\text{up}}] \\ 0 & \text{elsewhere} \end{cases} \]

(27)

however, the initial spectra with a sharp cutoffs (as shown in Figure 1) are impractical for numerical treatment. Realistic spectra, such as those measured in Solar System, also do not show abrupt changes in slope (e.g. Aguilar et al. 2014; Cummings et al. 2016). We assume smooth source distribution function which:

1. represents a power-law in the range \((\tilde{p}_{\text{br}}^{L}, \tilde{p}_{\text{br}}^{R})\) with slope \( q_{\text{init}} \),

2. is smooth near the two matching points \( \tilde{p}_{\text{br}}^{L} \) and \( \tilde{p}_{\text{br}}^{R} \),

3. falls mildly, outside the range \((\tilde{p}_{\text{br}}^{L}, \tilde{p}_{\text{br}}^{R})\), towards the lower and upper cut-offs \( \tilde{p}_{\text{lo}} \) and \( \tilde{p}_{\text{up}} \), where it reaches its lowest values corresponding to \( \epsilon_{\text{small}} \).

These requirements can be fulfilled by assuming that the two matching functions are second order polynomials in \( \log_{10} \tilde{p} \), while the power-law part of the spectrum is a first-order polynomial. Smooth matching of the three parts of the spectrum can be ensured by requiring continuity of the distribution function and its first derivatives at \( \tilde{p}_{\text{br}}^{L} \) and \( \tilde{p}_{\text{br}}^{R} \). The appropriate distribution function can be defined with \( \tilde{p} \) as follows:

\[ f_{\text{init}}(\tilde{p}) = \begin{cases} f_{\text{init}} \left( \frac{\tilde{p}}{\tilde{p}_{\text{lo}}} \right)^{-q_{\text{init}}} & \text{if } \tilde{p} \in (\tilde{p}_{\text{br}}^{L}, \tilde{p}_{\text{br}}^{R}) \\ f_{\text{Unit}} 10^{F(\log_{10} \tilde{p})} & \text{if } \tilde{p} \in [\tilde{p}_{\text{lo}}, \tilde{p}_{\text{br}}^{L}] \\ 0 & \text{otherwise} \end{cases} \]

(28)

where \( f_{\text{Unit}} \) is constant with value of one and has the physical dimension of distribution function \( f \) in the currently used unit system. For galactic models we use the system based on Solar mass \( M_{\odot} \), parsec pc and Megayear Myr, which implies \( f_{\text{Unit}} = 1 \text{Myr}^{-3}M_{\odot}^{-1}\text{pc}^{-6} \).

The function
\[ F(\log_{10} \tilde{p}) = \alpha_{L,R}(\log_{10} \tilde{p})^{2} + \zeta_{L,R}\log_{10} \tilde{p} + \lambda_{L,R} \]

(29)

depends on coefficients \( \alpha_{L,R}, \zeta_{L,R}, \lambda_{L,R} \) which should be adopted to ensure smooth matching of the distribution function at each of \( \tilde{p}_{\text{br}}^{L} \) and \( \tilde{p}_{\text{br}}^{R} \) by solving the system
of equations

\[
\begin{align*}
\log_{10} \left[ \frac{f_{\text{inj}}}{f_{\text{Unit}}} \left( \frac{p_{\text{lo}, \text{up}}}{m_e c} \right)^{-q_{\text{inj}}} \right] & = F \left( \log_{10} \frac{p_{\text{br}}}{m_e c} \right), \\
\log_{10} \left[ \frac{\epsilon_{\text{small}}}{4\pi c (p_{\text{lo}, \text{up}} m_e c)^2 f_{\text{Unit}}} \right] & = F \left( \log_{10} \frac{\rho_{\text{ho}}}{m_e c^2} \right), \\
\frac{d}{d\rho} \log_{10} \left[ \frac{f_{\text{inj}}}{f_{\text{Unit}}} \left( \frac{p}{\rho} \right)^{-q_{\text{inj}}} \right] & = \frac{d}{d\rho} \left[ F (\log_{10} \frac{\rho}{m_e c^2}) \right] \frac{p_{\text{br}}}{m_e c^2},
\end{align*}
\]

where \( f_{\text{inj}}, q_{\text{inj}}, \epsilon_{\text{small}}, \rho_{\text{ho}}, \rho_{\text{up}}, \rho_{\text{br}}, \frac{p_{\text{br}}}{m_e c} \) are parameters. For convenience we define other dimensionless quantities:

\[ \tilde{f} = f/f_{\text{Unit}} \quad \text{and} \quad \tilde{\epsilon}_{\text{small}} = \epsilon_{\text{small}}/\epsilon_{\text{Unit}}, \]

where \( \epsilon_{\text{Unit}} \) has the value of unity and dimension of spectral energy density of \( \epsilon \) (e.g. 1 Myr\(^{-1}\)pc\(^{-2}\)) in the currently used galactic unit system. Setting an example with \( f_{\text{inj}} = 1, q_{\text{inj}} = 4.1, \epsilon_{\text{small}} = 10^{-6}, \rho_{\text{ho}} = 10, \rho_{\text{up}} = 10^6, \frac{p_{\text{br}}}{m_e c} \approx 10^3 \), and \( \frac{p_{\text{br}}}{m_e c} \approx 10^5 \) we obtain \( \alpha_L \approx -2.46, \zeta_L \approx 10.34, \lambda_L \approx -17.05 \) for the left part of the spectrum \( \alpha_R = -6.07, \zeta_R \approx 57.18, \) and \( \lambda_R \approx -150.54 \) for the right part. The example is illustrated in Figure 2.

3.2. Injection of CR population in SN remnants

The series of tests, described in section 5, is initiated with a single SN explosion approximated by a local injection of CR protons centered at the point \((x_0, y_0, z_0) = (0, 0, 0)\) with an initial Gaussian profile of the CR proton energy density

\[
e_{\text{CRp}}(x, y, z) = A_{\text{CRp}} \times \\
\times \exp \left( -\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{r_0^2} \right),
\]

with

\[
A_{\text{CRp}} = \frac{\epsilon_{\text{CRp}} E_{\text{SN}}}{\pi^{3/2} r_0^3}
\]

\[
= 9.4 \times 10^6 \epsilon_{\text{CRp}} \left( \frac{E_{\text{SN}}}{10^{51}\text{erg}} \right) \left( \frac{r_0}{1\text{pc}} \right)^{-3} M_\odot \text{pc}^{-1} \text{Myr}^{-2}
\]

where \( r_0 \) is the SN radius determining the sphere where most of CRs are produced. We assume that 10% \((\epsilon_{\text{CRp}} = 0.1)\) of the canonical SN kinetic energy output \( E_{\text{SN}} = 10^{51}\text{erg} \) is converted to CRs. The normalization factor ensures that the spatial distribution integrated over volume brings the total energy input of CR protons equal to \( \epsilon_{\text{CRp}} E_{\text{SN}} \).

3.3. Boundary conditions for CRs on outer boundaries of the spatial grid

Boundary conditions for the thermal fluid and for magnetic field are realized in the standard way through copying the contents of respective cells to boundary cells. Periodic and internal boundary conditions for all the cosmic ray variables are realized in the same way as for the thermal fluid, i.e. through copying the the contents of a number of grid zones adjacent to the boundary (currently four layers of boundary zones on each side of the physical domain). At outer boundaries, such as the outer \( z \)-boundaries in the wind test problem described in Section 6, we apply the diode-outflow boundary conditions for the thermal fluid, which permit thermal plasma to flow out of the computational domain, but not to flow in. In the case of CRs we apply boundary conditions typical for the diffusion problem, by setting the zero value of each CR variable (number densities and energy densities in each momentum bin) on each outer boundaries of the computational domain.

4. ISOLATED CRESP TESTS
In this section we present elementary tests of isolated CRESP algorithm. We perform a series of runs, in which CR electron population in a single cell is exclusively subjected to adiabatic expansion/compression or synchrotron losses. We aim to compare the results with analytical solutions for these elementary processes and additionally compare results obtained using method described in 2.3.4, to which we refer as "approximated" approach (cutoffs are recovered at the beginning of each step) against unapproximated cutoffs case (also referred to as 'exact' cutoffs), where the cutoffs are stored in the memory. No spatial transport is accounted for at this stage, therefore no cutoff migration takes place, making the unapproximated cutoff case tests possible to carry out. We refer to the paper by Kardashev (1962), where an analysis of the effect of various modes of energy losses were studied using analytical methods. We compare numerically obtained spectra and spectrum cutoffs for synchrotron process with these analytical solutions.

4.1. Initial spectrum

Table 1 presents all variable parameters for initial spectrum for all conducted tests. Parameters not listed in tab. 1 remain unchanged for all tests in this section: $N_{\text{bin}} = 45$, $\tilde{p}_{\text{lo}} = 10$, $\tilde{p}_{\text{up}} = 10^6$, and $\tilde{p}_{\text{br}}^{\text{R}} = 2.5 \times 10^5$.

| A1p | 1 $\times$ 10$^{-9}$ | 3.5 | -0.2 | 0 | 2 $\times$ 10$^2$ |
| A2p | 1 $\times$ 10$^{-9}$ | 4.1 | -0.2 | 0 | 2 $\times$ 10$^2$ |
| aA1p | 1 $\times$ 10$^{-9}$ | 3.5 | -0.2 | 0 | 2 $\times$ 10$^2$ |
| aA2p | 1 $\times$ 10$^{-9}$ | 4.1 | -0.2 | 0 | 2 $\times$ 10$^2$ |

Table 1. Numerical values of initial parameters chosen for various tests. All runs were conducted assuming $N_{\text{bin}} = 45$, $\tilde{p}_{\text{lo}} = 10$, $\tilde{p}_{\text{up}} = 10^6$, $\tilde{p}_{\text{br}}^{\text{R}} = 2.5 \times 10^5$ and $\tilde{\epsilon}_{\text{small}} = 10^{-6}$, $q$ denotes spectrum slope for the power-law part, $\tilde{p}_{\text{lo}}^{\text{L}}$ and $\tilde{p}_{\text{br}}^{\text{L}}$ denote spectrum matching points corresponding to particle momenta. Adiabatic tests names start with A ("p" for periodically variable $u_d$), synchrotron cooling tests with S. Names of runs with approximated cutoffs (bold) start with an additional "a".

4.2. Adiabatic evolution test

Periodic adiabatic compression/expansion experiments aim to test algorithm’s ability to preserve spectrum shape after multiple passages through bin faces and the impact of the approximation of cutoffs on the spectrum. We define time dependent velocity divergence: $u_d(t) = 1/3 \nabla \cdot \mathbf{v} = -u_{d,0} \cos(\omega t)$, which implies the following solution of Equation (A4):

$$p(t) = p_{t=0} \exp(-u_{d,0} \sin(\omega t)/\omega).$$  (34)

Adiabatic evolution should not change the overall spectrum shape, which should return to its initial shape at the end of each adiabatic cycle. Periodic adiabatic compression/expansion tests, named A for exact and aA for approximated cutoffs, were conducted for two variuos slopes $q$ in the power-law part of the spectrum (see table 1).

In Figure 3 we present spectral distribution of particles energy density and in 4 spectral indices $q$ from tests A1p, aA1p, aA2p and A2p at $t = 0, 80, 160, \text{and } 240$ in code units, with separate plots corresponding to the beginning of a new compression/expansion cycle. The spectrum should remain identical to the initial state (dark-blue). In the range described by the pure power-law ($\tilde{p}_{\text{br}}^L - \tilde{p}_{\text{br}}^R$) all spectra remain almost unaffected and preserve the initial shape throughout all 6 cycles. Beyond the pure power-law range spectra slightly diffuse with time in all tests, which is more distinct at momenta lower than $\tilde{p}_{\text{br}}^L$. Beyond $\tilde{p}_{\text{br}}^R$ in test aA1p this effect is also noticeable. We note that a slow diffusion of the spectrum is present also in the cases without cutoff approximation, but this effect does not affect the cutoffs evolution. The evolution of cutoff momenta is shown in detail in Figures 5 and 6, where analytical solutions for spectrum cutoffs from Equation (34) are confronted with results of corresponding numerical exact and approximated cutoff runs. Both numerical exact cutoffs match analytical ones almost perfectly. Approximated cutoffs evolve accordingly, however reveal slowly growing deviation from the analytical curve in time. The low momentum cutoff $\tilde{p}_{\text{lo}}$ show falling trend in both tests; in aA1p test its value decreases by 22.1% (on average 3.7% per cycle). In aA2p the decrease yields $\sim$ 21.8% test ($\sim$ 3.6% per cycle). The high momentum cutoff $\tilde{p}_{\text{up}}$ in aA1p run evolves similarly to numerical exact and analytical solutions, however also increasing in time by $\sim$ 6.1% per cycle on average ($\sim$ 36.4% total). The values of high energy cutoff obtained in run aA2p remain in good agreement with analytical and exact numerical yields, showing only slight variations ($> 1\%$ per cycle). The change in cutoff momenta is clearly a consequence of the used approximation method, described in Section 2.3.4, which implies a dependence of cutoffs on the height of the spectrum above the base level $\epsilon_{\text{small}}$ as well as on the spectrum shape. We note that this effect is much less pronounced in case of $\tilde{p}_{\text{up}}$ of aA2p test, where the high energy part of the spectrum mildly falls to $\epsilon_{\text{small}}$. 

"..."
Figure 3. Depiction of spectral energy density in periodic adiabatic tests: aA1p, aA2p, and A2p (left, middle, right) at different stages: initial state (dark-blue) and at the beginning of 2nd, 4th, and 6th (light blue, red, green) adiabatic compression/expansion cycle, corresponding to $t = 0.0$, $80.0$, $160.0$, and $240.0$ (dark-blue, blue, red, green) in code units. Vertical gray lines mark the initial cutoffs (Equation A20).

Figure 4. Comparison of changes in $q$ between tests with exact and approximated cutoffs: A1 vs. aA1 (left panel) and A2 vs. aA2 (right panel). Values of $q$ at different times are color coded. Results of tests with exact cutoffs are marked with labels 'A1' and 'A2', results obtained with approximation of momenta ('aA1' and 'aA2') are elevated by 4 units. Initial cutoffs are marked with vertical grey lines.

level. We find that the algorithm preserves very well the total CR energy density while our approximation of cutoff evolution only moderately influences the cutoff positions in the case of periodic adiabatic compression-expansion process. We compare finally total energy density: $\sum_{i=1}^{N_{bin}} e_i(t)$, figures 7 (tests A1p, aA1p) and 8 (tests A2p and Aa2p) with analytical total energy density: $\Sigma e_i(t) = \Sigma e_{i,t=0} \exp(-\nu_{aA1} \sin(\omega_d t)/\omega_d)$. In both cases numerical total energy density follows the analytical prediction well, however a small deviation from analytical solutions is present in both the exact and approximated cutoff approach.

4.3. Synchrotron cooling test

Considering particles with the same pitch-angle $\theta$ we take the formula from Kardashev (1962, Equation 5) that describes evolution of power-law energy spectrum of CR electrons distribution $N(E,t)$ subject to synchrotron losses. This solution expressed in terms of distribution function reads:

$$f(p, q, \theta, t) = \begin{cases} f(p)(1 - \beta t p)^{q-4}, & p < \frac{1}{\beta t} \\ 0, & p > \frac{1}{\beta t} \end{cases},$$

where $\beta = \frac{2\sigma_T}{m_e c^2} \frac{2(\sin \theta B)^2}{m_e c^2}$. The formula implies that for $q < 4$ the spectrum rises with time and falls abruptly to zero around $p \sim 1/\beta t$, while for $q > 4$ downward steepening of the spectrum in high-momentum range should occur. Within the current setup of isolated-cell (with no spatial transport of CR particles) we perform a series of synchrotron cooling tests with initial parameters presented in tab. 1, aiming to test algorithms’ ability to reproduce expected features and cutoff evolution. In our later tests distribution of particle momenta is as-
Figure 5. Low energy cutoff evolution in periodic adiabatic test: analytical solution (red solid line, Equation A20) vs. exact A1p (green) and approximated spectrum cutoffs: runs aA1p, aA2p, (dark-blue, blue, and azure). \( \tilde{p}_{lo}(t) \) manifest falling trend in both approximated cases; in aA1p test it falls by 22.1% (on average 3.7% per cycle) and by \( \sim 21.8\% \) in all 6 cycles in aA2 test (\( \sim 3.6\% \) per cycle).

Figure 6. High energy cutoff evolution in periodic adiabatic test: analytical solution (red solid line, Equation A20) with exact A1p (green) and with approximated spectrum cutoffs: runs aA1p and aA2p (light-blue, dark-blue). In aA1p there is continuous increase by \( \sim 6.1\% \) per cycle, 36.4% in total, and good agreement with analytical curve in aA2p case.

Figure 7. Comparison of total energy density evolution in periodic compression/expansion A1p and aA1p. Approximated and exact cutoff cases show almost perfect agreement. Apparent small dissipation, with similar rate in both runs.

Figure 8. Comparison of total energy density evolution in periodic compression/expansion tests A2p and aA2p. Almost perfect agreement in both cases and small deviation, with similar rate in both cases is apparent.

sumed to be isotropic, hence pitch-angle integrated synchrotron cooling coefficient \( u_b \) from Equation (A4) is used, however here for simplicity we assume \( \theta = \pi/2 \). These experiments, named S- (exact) and aS- (approximated cutoffs), were conducted for \( q < 4 \) (S1, aS1) and
We compare spectra at times 0.54 and 5.97 Myr, which correspond to periods needed for \( \tilde{p}_{\text{up}} = 10^6 \) to cool by factor 10 and 100 in the presence of \( B_L = 5 \mu \text{G} \) (Equation A20). After \( t = 0.54 \) Myr we expect \( \tilde{p}_{\text{up}} = 10^5 \) and find \( \tilde{p}_{\text{up}, S1} = 1.01 \times 10^5 \) for the run with exact cutoff momenta and \( \tilde{p}_{\text{up}, aS1} = 1.53 \times 10^5 \), \( \tilde{p}_{\text{up}, aS2} = 1.22 \times 10^5 \) for the runs with approximated cutoff momenta. For \( t = 5.97 \) Myr we expect \( \tilde{p}_{\text{up}} = 10^4 \) and find the numerical values to be \( 1.05 \times 10^4 \), \( 1.85 \times 10^4 \), and \( 1.53 \times 10^4 \) respectively. The departures of the approximated solutions with respect to the exact and approximated one are comparable to the bin width, therefore we consider the approximation error as satisfactory small. Spectral distribution of particle energy densities resulting from tests S1–aS1 and S2–aS2 are compared in Figure 9. The analytical curve of energy density is obtained using A15 and 35, plotted with green dash-dotted line on the figures. In both cases the evolution of high-energy end of the spectra reproduces the analytical results of Kardashev (1962). For \( q < 4 \) spectrum rises with time and falls to zero abruptly near \( 1/\beta t \) (Figure 9, top), while for \( q > 4 \) a gradual steepening of the spectrum with time can be observed near the high momentum cutoff (Figure 9, bottom). Spectral features expected for the synchrotron cooling from Equation (35) are reproduced by the numerical algorithm with a satisfactory accuracy (analytical curve plotted with green). Spectra with exact cutoff momenta follow the analytical curve closely, with precision limited by resolution. Spectrum from aS1 runs also reproduce the analytical result well, however with less precision, as near the cutoff spectrum falls to value imposed by \( \epsilon_{\text{small}} \). In the case aS2 the mild fall of the spectrum due to the synchrotron cooling is followed by the fall of the cutoff to \( \epsilon_{\text{small}} \) level, thus the departure from analytical cutoff is smaller in this case.

Evolution of \( \tilde{p}_{\text{up}} \) in tests S1, aS1, and aS2 is presented in Figure 10. In both aS- runs \( \tilde{p}_{\text{up}} \) shows fluctuations resulting from the cutoff transitioning to the bin of lower number, which causes the new cutoff momentum value to slightly exceed the value prior to transition. We note that in both cases spectrum cutoffs first evolve almost perfectly following the analytical curve, however they deviate later from analytical values. The departures from analytical cutoff in aS- tests grow with time rather slowly, what indicates that synchrotron cooling naturally limits the numerical diffusivity of the cut-off approximation algorithm. The approximated momenta closely follow the analytical curve and relative differences between numerical and approximated cutoff values are small. The code produces results consistent with predictions of Kardashev (1962) with accuracy close to the bin width, as well as with numerical results of Miniati (2001), who conducted similar test for CR electrons with \( q_{\text{init}} = 4.3 \).

5. TESTS OF CRESP WITH PIERNIK

In this section we present tests of the CRESP algorithm embedded in the PIERNIK MHD code. The fluid transport algorithms of PIERNIK are used in advection and diffusion steps of CRs on the spatial Eulerian grid, while CRESP algorithm is used in each spatial cell to transport CRs on the momentum grid. We prepared a series of tests aiming to check the interoperation of CRESP algorithms with transport of the entire spectrum on the spatial grid by the fluid transport algorithms of PIERNIK. We shall switch on and off individual processes: advection, diffusion, adiabatic process, and synchrotron losses. The present setup assumes instantaneous injections of CR protons and electrons in isolated supernova remnants. CR protons play only a passive role and are subject to anisotropic diffusion, advection, and adiabatic process, modeled with the aid of momentum integrated diffusion-advection equation (A12) using the algorithms described in (Hanasz & Lesch 2003; Hanasz et al. 2012b). CR electrons evolve according to momentum-dependent diffusion equation, integrated by means of the piece-wise power-law method by Miniati (2001), with the code extension, presented in this paper, designed for time-dependent transport on the \((x, y, z, p)\)-grid of CR particles subject to the adiabatic process and synchrotron cooling mechanism.

In advection and diffusion tests the domain size is \( 1 \times 1 \times 0.0625 \) kpc with spatial resolution \( 128 \times 128 \times 8 \) grid cells and with periodic boundary conditions in all directions. For adiabatic expansion test we assume domain with size \( 1 \) kpc \(^3\) with \( 128^3 \) cells and outflow boundary conditions. We set initial spectrum of CRs extending over 5 decades in the momentum space \( (\tilde{p} \text{ in range } 5 - 8.5 \times 10^5) \), with total 23 bins. The initial spectrum has amplitude \( \tilde{p}_{\text{init}} = 10^{-9} \), slope \( q = 4.1 \) and matching points at \( \tilde{p}_{\text{br}}^L = 100 \) and \( \tilde{p}_{\text{br}}^R = 5 \times 10^5 \) beyond which the spectrum is defined by means of a 2nd order polynomials in \( \log_{10}(p) \) – see section 3.1. Spectrum cutoffs are approximated using method described in section 2.3.4, with \( \epsilon_{\text{small}} = 10^{-10} \). CRs are injected by single SN explosion with a radial profile given by Equation (31).

5.1. Advection test

To test the pure advection of the whole energy spectrum in space we set up velocity field \( v_x = 10 \) pc Myr\(^{-1}\), \( v_z = v_y = 0 \) in a periodic simulation domain and with diffusive transport and synchrotron losses switched off. The
velocity field is uniform ($\nabla \cdot \mathbf{u} = 0$), thus we expect that spectrum shape does not change during the advective transport. The spatial and spectral distribution of energy density are shown in Figure 12 for $t = 40$ Myr and 100 Myr. As demonstrated in the figure the spectrum shape remains practically unchanged during the simulation time, except a small reduction of the distribution function amplitude due to a relatively weak numerical diffusion; comparing $n_{\text{tot}}$ and $\epsilon_{\text{tot}}$ for times 0 and 100 Myr we get relative errors of $\approx 0.2$ for both quantities.

5.2. Adiabatic expansion test

The adiabatic expansion test is aiming at validation of the CR spectrum evolution under the action of adiabatic expansion of the underlying plasma and in the absence of diffusion and synchrotron losses. We assume that spatial distribution of CRs in the domain in this test is
The aim of the diffusion test is to validate the momentum-dependent diffusion part of CR spatial propagation algorithm in absence of any other propagation effects. Diffusion coefficients are computed with formula (11), where $\kappa_{\parallel(10k)} = 3 \times 10^{26} \text{ cm}^2 \text{s}^{-1} \approx 10^3 \text{ pc}^2 \text{ Myr}^{-1}$. $\kappa_{\perp}(p_l) = 10^{-3} \kappa_{\parallel}(p_l)$ with momentum dependence given by $\alpha_k = 1/7$. Magnetic field is set to $b_x = b_y = 1 \mu\text{G}$ (vectors shown), synchrotron cooling is switched off for this test. The diffusion test was performed with periodic boundary conditions. With the initial Gaussian distribution of CRs in space (Equation 31) of $n$ and $e$, as in (Hanasz & Lesch 2003), we expect the CR population to diffuse diagonally, according to the relation:

$$e_l(r_D, t) = \frac{e_{\text{CR}eR_E\text{SN}}}{\pi^{3/2} r_0^3} \times \left[ \frac{\int_{P_{\text{inj}}} P_{l+1/2}^d 4\pi p^2 T(p) f_{\text{inj}}(p) dp}{\int_{P_{\text{inj}}} 4\pi p^2 T(p) f_{\text{inj}}(p) dp} \right] \times \exp \left( \frac{-r_D^2}{r_0^2 + 4\kappa_{\parallel}(p) t} \right),$$

(36)

where $r_D$ is the diagonal distance from the injection center and the remaining symbols have same meaning as in Equation 33. Results of the test are presented in Figure 14 for $t = 25 \text{ Myr}$, showing spatial distribution of the total energy density, spatial profiles of selected bins along a diagonal line of the domain, and spectra obtained at three marked positions. Momentum-dependent diffusion with coefficients defined in Equation (11), treated separately from other propagation effects,
Figure 12. Advection test results, same as Figure 11 for \( t \approx 40\,\text{Myr} \) (left) and \( t = 100 \, \text{Myr} \) (right), with velocity field vectors annotated. After 100 Myr of evolution the resulting spectrum is almost undistinguishable from the initial one, represented by the dashed black line onto which the resulting spectrum is superimposed.

Figure 13. Distribution of the total CR energy density in adiabatic expansion test (with velocity field vectors annotated) and its spectra, averaged for the marked region (centered at \([200, 200, 0]\) pc) at \( t = 0 \) (black, dashed) and \( t = 100\,\text{Myr} \): accounting only for the isotropic outflow – expansion only (light blue, solid) and with adiabatic shift (blue, solid).

causes particles occupying separate bins to diffuse at different rates. We remind that in the present implementation of the algorithm we approximate the diffusion coefficients by their bin-centered values, which is equal for CR number density and energy density, instead of computing corresponding bin-weighted values which would be naturally different. Therefore, diffusion taken alone retains the slopes \( q_i \) which are determined for a given ratio \( e_i/n_i \) from equation (A28). Preservation of spectral indices leads to emergence of the saw-tooth pattern of the distribution function apparent in the right panel of Figure 13. Spectral indices of individual bins remain roughly unchanged, except near the cutoffs and lower momentum matching point. A more detailed discussion of this point can be found in Girichidis et al. (2020).

Implication of the momentum dependent diffusion is the apparent softening of the spectrum near the injection point and hardening at larger distances. Profiles of CR energy density are consistent with exact results, given by Equation (36), with numerical values showing small underestimates of the peak amplitude, identically to the results of similar 2D test for one CR component, presented in Hanasz & Lesch (2003), as the same anisotropic diffusion algorithm is used here and a small perpendicular diffusion \( \kappa_\perp = 0.001\kappa_\parallel \) has been used to improve stability of the present algorithm. Away from the center (+ and * points) the spectrum hardens, as expected, due to the growth of diffusion coefficients with momentum. At point * spectrum is distinguishably dominated by the particles injected at the upper part of the momentum range. Though the slopes \( q_i \) being preserved in the bins, we observe hardening of the overall spectrum, relative to the initial one with increasing distance from the center, while in the source center that spectrum becomes softer than the initial spectrum.

6. CR-ELECTRON SPECTRUM EVOLUTION IN GALACTIC WIND

In this section we analyze evolution of CR electron spectrum in a galactic wind driven by CRs. We present results of 3D simulations of an outflow driven by CR protons, supplemented with spectrally resolved CR electrons. This experiment demonstrates CR spectral evolution in a stratified box. We assume that:

1. the CR-proton component, described by momentum-integrated diffusion-advection Equation (A12), is dynamically coupled with thermal gas and magnetic fields stratified by vertical gravitational field of galactic disk using the method described by Hanasz & Lesch (2003).

2. the CR-electron component evolves according to the momentum-dependent diffusion-advection Equation (1) using the numerical algorithm developed in this paper on the base of the piece-wise power law two-moment method (Miniati 2001).
3. the simulation volume represents a stratified box with domain sizes \((L_x, L_y, L_z) = (0.5\text{kpc}, 1\text{kpc}, 9\text{kpc})\) and grid resolution \((n_x, n_y, n_z) = (24 \times 48 \times 432)\), with periodic boundary conditions used in horizontal directions and outflow-diode boundary conditions (see Section 3.3) at the lower and upper computational box boundaries.

Initial parameters for the ISM reflect galactic disk properties, including vertical gravitational field model at galactocentric radius of 5kpc, as taken from Ferriere (1998); Hanasz et al. (2009a), yielding SN rate of 130 kpc\(^{-2}\)Myr\(^{-1}\), speed of sound 7 km s\(^{-1}\), density of the ISM in the disk plane \(\rho_0 = 0.125 \text{ M}_\odot \text{ pc}^{-3}\) (equivalent to hydrogen number density of \(\approx 5 \text{ cm}^{-3}\)). We solve the hydrostatic equilibrium equation for the external gravitational field to get vertical distributions of gas and horizontal magnetic field, yielding \(B_y \approx 1\mu\text{G}\) at \(z = 0\).

For electrons we set diffusion coefficients according to formula (11), assuming \(\kappa_{10k||} = 3 \times 10^{27} \text{ cm}^2 \text{s}^{-1} \approx 10^4 \text{ pc}^2 \text{ Myr}^{-1}\) and \(\kappa_{1} = 0.1\kappa_{||}\) with momentum dependence power index \(\alpha_{e} = 0.0\) and 0.5, with constant value of \(\kappa_{||} = 10^4 \text{ pc}^2 \text{ Myr}^{-1}\) and \(\kappa_{\perp} = 0.1\kappa_{||}\) for CR protons.

No cosmic rays are assumed in the initial state. SNe inject CR protons and electrons to the ISM with efficiencies \( \epsilon_{\text{CRT}} = 0.1\) and \( \epsilon_{\text{CRE}} = 0.01\) (defined in Equation (31–33)), referring to the canonical kinetic energy output at a single SN equal to \(E_{\text{SN}} = 10^{51}\text{erg}\), with radius \(r_0 = 50\text{pc}\). We use the source spectrum of CR electrons defined by formula (30), assuming the following set of parameters: \(\tilde{f}_{\text{init}} = 10^{-2}\), \(q_{\text{init}} = 4.1\), \(\tilde{p}_0 = 5\), \(\tilde{p}_{\text{up}} = 8.5 \times 10^{5}\), and matching points at \(\tilde{p}_{\text{lo}}^{\text{B}} = 100\) and \(\tilde{p}_{\text{lo}}^{\text{D}} = 5 \times 10^5\) with \(\varepsilon_{\text{small}} = 10^{-8}\). To analyze the effects of separate processes we run a series of tests. The aim is to compare separately the effects of advection, diffusion, adiabatic, and synchrotron processes on the spectra and to compare the effects of energy-dependent and energy-independent diffusion. Selected processes, alternating parameters and test names for CR-driven galactic wind simulation are presented in Table 2.

| Adv. | Diff. | Synch. | Adiab. | Name | \(\alpha = 0.0\) | \(\alpha = 0.5\) |
|------|-------|--------|--------|------|----------------|----------------|
| +    | -     | -      | A      | A    | B0.0           | B0.5           |
| +    | +     | -      | B0.0   | B0.5 |
| +    | +     | +      | C0.0   | C0.5 |
| +    | +     | +      | D0.0   | D0.5 |

Table 2. Collection of tests including various transport modes (Adv. – advective, diff. – diffusive) and spectrum evolution processes (Synch. – synchrotron, Adiab. – adiabatic).
Advective transport is required to creation of the wind and is therefore included in every test. In test A the only considered process is the advective transport; we expect the spectral slopes \( q \) to remain unchanged. In test B we include advective and diffusive transport with synchrotron process. We expect spectrum to manifest steepening at the highest energies due to synchrotron losses. We also expect softer spectra at \( z \approx 0 \) and harder above the disk (\( z > 1 \) kpc) in case B0.5, relative to case B0.0, due to the momentum dependence of diffusion coefficient. In test C adiabatic process is considered together with advective and diffusive transport. The spectrum is expected to reveal a shift the spectrum towards lower energies due to wind expansion. In test D all the mentioned processes are included and we expect to see the combination of all the mentioned effects.

6.1. Results

In Figure (15) we present domain distribution of CR-electrons for \( t \approx 200 \) Myr and in Figure 16 we show corresponding spectra of CR electrons which are averaged over horizontal plains to eliminate fluctuations resulting from varying local conditions. The spectra taken at positions marked with horizontal white lines are presented for the disk plane (\( z = 0 \)kpc) and for an elevated location at \( z = 2 \) kpc. For comparison’s sake we chose bins no. 8 and 15, with respective Lorentz factor \( \tilde{\rho}_8 \in [2.15 \times 10^2, 4.64 \times 10^2] \) for the 8-th bin, and \( \tilde{\rho}_{15} \in [4.64 \times 10^4, 10^5] \) for the 15-th bin; these ranges are highlighted in the spectra shown in Figure 16. In the momentum-dependent diffusion cases (\( \alpha_k = 0.5 \)) this implies the growth of the parallel diffusion coefficient by a factor \( \kappa_{\parallel\perp} \tilde{\rho}_{15} = 4.64 \times 10^4 / \kappa_{\parallel\perp} \tilde{\rho}_8 = 2.15 \times 10^2 = 15 \). We summarize the test simulation results as follows:

1. In the pure advection test A the source spectrum is being injected in random SN explosions and propagated throughout the domain without a change of shape and only with a drop of amplitude at higher altitudes. In this test the CR electron population extends to the domain limits, however mostly occupies the disk plane. The slope of the main part of the spectra taken at \( z = 0 \) and 2 kpc remains unchanged, what is consistent with the expectation that the advection process alone does not change the shape of the spectrum. The only difference appears on the low energy end owing to a numerical inaccuracy of the algorithm near the low energy end of the spectrum.

2. Results of test B include advective and diffusive transport together with synchrotron cooling. As expected, the synchrotron cooling effect steepen the spectra in the high energy range: in both cases B0.0 and B0.5 the effect is apparent beyond \( \tilde{\rho} \sim 2.15 \times 10^4 \), producing mild break. Diffusion allows particles to travel further from the acceleration sites, thus the amplitudes of spectra averaged at \( z = 2 \) kpc are higher than in the test A, at the expense of CR population in the central part, which have lower amplitudes in comparison to test A. The differences in slopes in low and moderate energy range are owed to different momentum dependence of diffusion coefficients: in the disk plane of case B0.0 the slope remains identical as in the injection spectrum, while the spectrum of the case B0.5 steepens with distance, similarly to the results of tests presented in section 5.3.

3. Test C includes the adiabatic process together with advective and diffusive transport. In the case C0.0 energy spectrum of CR particles at the disk mid-plane shows no difference in the slope, relative to A0.0 and to B0.0 up to high momentum range, however at higher \( z \) the spectra are adiabatically shifted towards lower energies – most prominent is the shift of the spectrum peak. This effect was expected as the result of adiabatic expansion associated with the divergent galactic wind outflow due to the overall CR vertical pressure provided by the CR population. In case C0.5 spectrum above the disk reveals combined effects of momentum-dependent diffusion and adiabatic cooling. Diffusion dominates in the upper part of momentum range, while it is less efficient at the higher momenta, therefore peak of the spectrum is smaller at the higher altitude. The fluctuation of the slope in near the lower cut-off is an artefact.

4. Test D includes all the considered processes. Vertical slices and the spectra corresponding to test D show combined effects of all considered processes: advection, diffusion together with adiabatic and synchrotron processes. The spectra of cases D0.0 and D0.5 show a mild break near \( \tilde{\rho} \approx 2.15 \times 10^4 \) similar to the one appearing in test B, which can be attributed to the synchrotron cooling process. The high energy cutoff for case D0.5 at \( z = 2 \) kpc locates higher than the one of D0.0 revealing the fact that the higher diffusion coefficient in the momentum-dependent diffusion case allows particles to escape faster from the strong mid-plane magnetic field region. Similar, albeit less pronounced effect is apparent in test B. The opposite statement is valid for the spectra at \( z = 0 \). Similarly to cases B0.5 and C0.5, momentum de-
pendent diffusion softens the spectra in the disk and hardens above the disk. Adiabatic expansion effect shifts the spectrum towards lower momenta similarly as in test C.

6.2. Impact on performance

To investigate the computational costs of the new CRESP algorithm we conducted a series of short cosmic ray-driven wind runs with different combinations of active algorithms. The first setup consists of only one CR species – protons in momentum integrated approach (without CRESP module compiled), the second assuming CR electron spectrum initialized and subject to momentum-dependent diffusion, but without spectrum evolution, and the third assuming both the momentum-dependent diffusion and energy losses. Tests were performed using the same setup as tests from preceding section, but with domain size set $0.5 \times 1 \times 2$ kpc and resolution $12 \times 24 \times 48$. Each test consisted of 50 steps and was performed on one processor. To investigate performance of CRESP algorithm we assume varying three cases $N_{\text{bin}} = 10, 20, \text{ and } 30$. The resulting total execution times are presented in Figure 17. The shortest period (46.9s) corresponds to pure MHD case with CR protons, the longest period (668.4s) to case with 30 bins considering all possible processes. Inclusion of CRESP module with only diffusion results in significant execution time increase, as the number of fluid components increases $2 \times N_{\text{bin}}$-fold. Comparison with wall times of most complex setup shows that adding spectral evolution results in only slight increase of wall time by $\sim 6-15$ percent.

7. SUMMARY AND CONCLUSIONS

We described the CRESP module of PIERNIK MHD code, dedicated to cosmic ray spectrum studies in galactic context. We generalized the two-moment piecewise power-law method of CR propagation in momentum-space, including fast cooling processes, to the form which is applicable for Eulerian grids in three spatial dimensions. We achieved this goal by providing an operational definition of spectral cut-offs, which can serve as a specific kind of boundary conditions suitable for the transport of the power-law type spectrum of relativistic particles subject to fast cooling processes. The synchrotron emission process naturally tend to evacuate all particles from high energy bins, producing steep spectra near the upper cut-off.

We proposed a novel approach extending the method with specific boundary conditions defined on movable boundaries in momentum space. The construction described in Section 2.3 allowed us to estimate the positions of spectral cut-offs solely on the base of the particle number density and energy density in the outer bins of the spectrum. The solution enabled binding the algorithms of CR spectral evolution with advection and diffusion in space.

We performed a series of elementary tests of the algorithm to validate its individual elements and their common operation the case of CR-driven galactic wind setup. The tests have proven robustness of the algorithm. Despite some apparent, but non-essential inaccuracies, our algorithm provides results which closely correspond to analytical solutions. We obtained a reasonable evolution of CR spectrum in the case of CR-driven wind setup, consistent with expectations concerning the effects of advection, diffusion, adiabatic and synchrotron cooling of CR electron population. Presented here performance tests and other tests, not reported in this paper, indicate that the algorithm is fully suitable for production runs including synchrotron-emitting CR electrons in global galactic-scale MHD simulations on timescales as long as several Gyr.

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Figure 15. Distribution of CR electron energy density in physical domain at \( t = 200 \) Myr, white lines mark the regions from which \( z \)-averaged spectra were taken. Test names shown in bottom right are listed in table (2) together with other settings of individual tests. The leftmost panel shows distribution of total CR energy density (summed over all bins) from test A. The next panels compare energy distributions in test D for 8-th and 15-th bins, encompassing momentum ranges \( \tilde{p}_8 \in [215, 464) \) and \( \tilde{p}_{15} \in [4.65 \times 10^4, 10^5) \), respectively. Vectors show magnetic field directions and relative strengths.

A. NUMERICAL ALGORITHM FOR SOLVING CR TRANSPORT EQUATION FOR PIECEWISE POWER-LAW DISTRIBUTION FUNCTION

In this appendix we summarize the piecewise power-law method proposed by Miniati (2001) for numerical solving of momentum-dependent CR transport equation Eqn. (1). We adopt the notation to the case of CR transport on a 3D Eulerian grid supplemented with a 1D grid representing momentum of CR particles. We assume an isotropic distribution function.
Let us consider evolution of two moments: number density and energy density of the isotropic distribution function $f(x, p)$ of CRs in a section $(p_L, p_R)$ of momentum axis. The evolution equation for CR number density

$$n^{LR} = \int_{p_L}^{p_R} 4\pi p^2 f(p) dp$$

in momentum range $(p_L, p_R)$, resulting from Eqn. (1), reads

$$\frac{\partial n^{LR}}{\partial t} = -\nabla \cdot (vn^{LR}) + \nabla \left( (\langle \kappa^{LRn} \rangle \nabla n^{LR}) + \left[ \frac{1}{3} (\nabla \cdot v)p + b_l(p) \right] 4\pi p^2 f(p) \right) + Q^{LR}(p),$$

where $\langle \kappa^{LRn} \rangle$ is the momentum-averaged diffusion coefficient of CR particles.

$$\langle \kappa^{LRn} \rangle \equiv \frac{\int_{p_L}^{p_R} p^2 \kappa(p) \nabla f(p) dp}{\int_{p_L}^{p_R} p^2 \nabla f(p) dp}.$$

The first and second terms on the RHS of Eqn. (A2) represent the advection and diffusion transport of CRs. The third term accounts for evolution of particle momentum due to the adiabatic process and radiative cooling mechanisms.
In absence of any energy loss processes the adiabatic condition
\[-\left(\frac{dp}{dt}\right) = b_{\text{tot}}(p) = \frac{1}{3}(\nabla \cdot \mathbf{u}) p + \frac{4}{3 m_e^2 c^2} u_B p^2 \equiv u_d p + u_p p^2,\] (A4)
where \(u_B\) is magnetic field energy density, \(\sigma_T\) is the Thomson’s cross section and \(m_e\) is the electron mass and we limit radiative loss processes to synchrotron losses in the present paper. The symbols \(u_p\) and \(u_d\) are introduced to simplify the notation. The last term accounts for CR sources
\[Q_{\text{CR}} = \int_{p_L}^{p_R} 4\pi p^2 j(p) dp.\] (A5)

For \(p_L = 0\) and \(p_R = \infty\) we obtain the momentum-integrated diffusion-advection equation for CR particle number density
\[\frac{\partial n_{\text{CR}}}{\partial t} = -\nabla \cdot (\mathbf{v} \mathbf{n}_{\text{CR}}) + \nabla \left( (\kappa_e) \nabla n_{\text{CR}} \right) + Q_{\text{CR}}.\] (A6)

Similarly, energy density of CR particles
\[e_{\text{CR}} = \int_{p_L}^{p_R} 4\pi p^2 T(p) f(p) dp\] (A7)
is described by the following equation resulting from Eqn. (1)
\[\frac{\partial e_{\text{CR}}}{\partial t} = -\nabla \cdot (\mathbf{v} e_{\text{CR}}) + \nabla \left( (\kappa_e) \nabla e_{\text{CR}} \right) + \left[ \frac{1}{3} (\nabla \cdot \mathbf{v}) p + b_3(p) \right] 4\pi p^2 f(p) T(p) \bigg|_{p_L}^{p_R} \]
\[-\int_{p_L}^{p_R} \left( \frac{1}{3} (\nabla \cdot \mathbf{v}) p + b_3(p) \right) 4\pi p^2 f(p) \frac{cp}{\sqrt{p^2 + m_e^2 c^2}} dp + S_{\text{LR}},\] (A8)
where \(T(p) \equiv \sqrt{p^2 c^2 + m_e^2 c^2} - m_e c^2\) is the particle kinetic energy and \(m\) is the particle rest-mass. We assume relativistic energy range, therefore we substitute \(T(p) \simeq p c\). The first two RHS terms represent spatial advection and diffusion, respectively, the third one describes advection in the momentum space, the fourth term represents adiabatic compression/expansion and radiative losses. The last term
\[S_{\text{LR}} = \int_{p_L}^{p_R} 4\pi p^2 j(p) T(p) dp\] (A9)
represents sources of CRs. The momentum averaged diffusion coefficient of CR energy density is defined as
\[\langle \kappa^{\text{LR}} \rangle = \frac{\int_{p_L}^{p_R} p^2 T(p) \kappa(p) \nabla f(p) dp}{\int_{p_L}^{p_R} p^2 T(p) \nabla f(p) dp}.\] (A10)
The CR pressure contribution from particles in the momentum range \([p_L, p_R]\) can be computed as
\[P_{\text{CR}}^{\text{LR}} = \frac{4\pi}{3} \int_{p_L}^{p_R} p^3 v(p) f(p) dp \approx \frac{4\pi}{3} \int_{p_L}^{p_R} p^3 c f(p) dp.\] (A11)
The latter approximate equality indicates that the expression is valid only in the ultrarelativistic limit \((p \gg m c)\). For \(p_L = 0\), \(p_R = \infty\) and \(b_3 = 0\) we get the momentum-integrated form of the diffusion-advection equation for CR energy density
\[\frac{\partial e_{\text{CR}}}{\partial t} = -\nabla \cdot (\mathbf{v} e_{\text{CR}}) + \nabla \left( (\kappa_e) \nabla e_{\text{CR}} \right) - P_{\text{CR}} (\nabla \cdot \mathbf{v}) + S_{\text{CR}}(p).\] (A12)
In absence of any energy loss processes the adiabatic condition \(P_{\text{CR}} = (\gamma_{\text{CR}} - 1) e_{\text{CR}}\) where \(\gamma_{\text{CR}}\) is the adiabatic index of CR gas, can be used to complete the system of equations. We keep the subscript CR, whenever it is necessary, to avoid confusion with similar symbols characterizing the thermal gas component.

### A.1. Piece-wise power-law distribution function

Following Miniati (2001) we assume a piecewise power-law distribution function defined on a section \(p_{i-1/2}, p_{i+1/2}\) of momentum axis. It is convenient to choose logarithmic scaling for particle bins with fixed bin width equal to \(\Delta w_l = \log_{10}(p_{i-1/2}/p_{i+1/2}) = \log(p_{\text{max}}/p_{\text{min}})/N_{\text{bin}},\) where \(N_{\text{bin}}\) is the number of bins. We assume that a piece-wise power-law distribution function characterizes the spectrum of CRs in each cell \((i, j, k)\) of the spatial grid and assign symbol \(l\) as the index of actual bin. For the sake of clarity of notation we omit the label 'CR' and spatial indexes
\( f(p) = f_{i-1/2} \left( \frac{p}{p_{i-1/2}} \right)^{-q_i} \) for \( p \in [p_{i-1/2}, p_{i+1/2}] \)

where the distribution function amplitudes \( f_{i-1/2} \) are defined on left edges of momentum bins \( p_{i-1/2} \) and spectral indices \( q_i \) are attributed to bin interiors. The number density of CR particles in the \( l \)-th bin is

\[
n_l = \int_{p_{i-1/2}}^{p_{i+1/2}} 4\pi p^2 f(p) dp = 4\pi f_{i-1/2} p_{i-1/2}^3 \times \left\{ \begin{array}{ll} \frac{1}{3-q_i} \left( \frac{p_{i+1/2}}{p_{i-1/2}} \right)^{3-q_i} - 1 & \text{if } q_i \neq 3, \\ - \log \left( \frac{p_{i+1/2}}{p_{i-1/2}} \right) & \text{if } q_i = 3. \end{array} \right.
\]

Similarly, the energy density of particles in the \( l \)-th bin given Eqn. (A7) in the relativistic limit is

\[
e_l = \int_{p_{i-1/2}}^{p_{i+1/2}} 4\pi c p^3 f(p) dp = 4\pi c f_{i-1/2} p_{i-1/2}^4 \times \left\{ \begin{array}{ll} \frac{1}{4-q_i} \left( \frac{p_{i+1/2}}{p_{i-1/2}} \right)^{4-q_i} - 1 & \text{if } q_i \neq 4, \\ - \log \left( \frac{p_{i+1/2}}{p_{i-1/2}} \right) & \text{if } q_i = 4. \end{array} \right.
\]

The above expression differentiate between \( q_i = 3, q_i = 4 \) and other cases. Number density \( n_l \) and energy density \( e_l \) are stored in a global 4D array for each cell and bin \((i, j, k, l)\), which is updated by the spatial transport and spectrum evolution algorithms, while distribution function \( f \) and spectral index \( q \) are recomputed whenever the spectrum evolution algorithm is invoked. Evolution of CR spectrum in momentum space is governed by the third terms of eqs. (A2) and (A8) including adiabatic process (cooling and heating) synchrotron, inverse Compton, and other loss mechanisms. The corresponding evolution equations for particle number density and energy density on a discrete 1D momentum grid can be represented as

\[
n_{l+\Delta t} = n_l + \left( n_{l+1/2} - n_{l-1/2} \right) \Delta t,
\]

and

\[
e_{l+\Delta t} = e_l + \left( e_{l+1/2} - e_{l-1/2} \right) \Delta t - \left( \frac{\Delta t}{2} R_l \right) t_{l+\Delta t},
\]

where the term proportional to \( R_l \) has been integrated implicitly and \( n_{l+1/2}, n_{l-1/2}, e_{l+1/2}, e_{l-1/2} \) are inter-bin fluxes of particle concentration and particle energy density integrated over the timestep \( \Delta t \), resulting from particle acceleration, adiabatic compression or expansion and radiative cooling processes. A simple first-order approximation to the solution of energy equation (A8)

\[
e_{l+\Delta t} = e_l + \left( e_{l+1/2} - e_{l-1/2} \right) - \Delta t R_l t_{l+\Delta t}^2 e_l,
\]

and spectral index \( q \) is attributed to bin interiors. The number density of CR particles in the momentum grid while the energy equation is a conservation law with source terms. Due to heating and cooling processes particles are only located on the momentum grid and the total number of particles is conserved. A part of the particles energy is relocated while the other part is radiated out of the system or exchanged with thermal plasma via adiabatic compression or expansion. To compute the fluxes we first integrate of Eqn. (A4) in the time interval \((t, t + \Delta t)\). For the synchrotron and adiabatic processes taken separately we get

\[
p(t+\Delta t)_{\text{sync}} = p(t) \left( 1 + u_b p t \Delta t \right)^{-1} \quad \text{and} \quad p(t+\Delta t)_{\text{adiab}} = p(t) \exp(-u_d \Delta t).
\]

with its Taylor expansion up to 3rd order being

\[
p^{t+\Delta t} \approx p^t \left( 1 - u_b p^t \Delta t + \frac{1}{2} (u_b p^t \Delta t)^2 - \frac{1}{6} (u_b p^t \Delta t)^3 \right)
\]

Taylor expansion of equation A4 for adiabatic cooling part up to 3rd order yields:

\[
p^{t+\Delta t} \approx p^t \left( 1 - u_d \Delta t + \frac{1}{2} u_d^2 \Delta t^2 - \frac{1}{6} u_d^3 \Delta t^3 \right)
\]

Computation of particle and energy fluxes relies on the analysis of particle flow through interfaces of momentum bins. We identify particles crossing the interface located at \( p_{i-1/2} \) within the time interval \([t, t+\Delta t]\). We define the upstream
momentum \( p_{\text{ups}} \), depending on \( \Delta t \), through the requirement that particles located at \( p_{\text{ups}} \) at time \( t \), cross the interface \( p_{l-1/2} \) at \( t + \Delta t \). By setting \( p(t) = p_{\text{ups},l-1/2} \) and \( p(t + \Delta t) = p_{l-1/2} \) we find

\[
p_{\text{ups},l-1/2}\,\text{synch} = p_{l-1/2} \left( 1 - u_b p_{l-1/2} \Delta t \right)^{-1} \quad \text{and} \quad p_{\text{ups},l-1/2}\,\text{adiab} = p_{l-1/2} \exp(u_d \Delta t),
\]

for the synchrotron and adiabatic processes respectively. Particle and energy fluxes can be computed now by integration from (A14).

In the practical implementation we replace the strict condition of Equation (A28) can be solved for unknown values of

\[
\epsilon_{q} \mid_{\text{ups}} - \epsilon_{q} \mid_{l-1/2} = 4 \pi f_{l-1/2} p_{l-1/2} \frac{3 \epsilon_{q} - 1}{3 \epsilon_{q} - 1} - \frac{1}{3 \epsilon_{q} - 1} \left( \frac{p_{l-1/2}}{p_{\text{ups},l-1/2}} \right)^{3 \epsilon_{q} - 1} - 1 \quad \text{if} \quad q_{l} \neq 3,
\]

\[
\epsilon_{q} \mid_{\text{ups}} - \epsilon_{q} \mid_{l-1/2} = 4 \pi f_{l-1/2} p_{l-1/2} \frac{4 \epsilon_{q} - 1}{4 \epsilon_{q} - 1} - \frac{1}{4 \epsilon_{q} - 1} \left( \frac{p_{l-1/2}}{p_{\text{ups},l-1/2}} \right)^{4 \epsilon_{q} - 1} - 1 \quad \text{if} \quad q_{l} \neq 4,
\]

(24a) and in the case of heating (\( p_{\text{ups},l-1/2} < p_{l-1/2} \)) the analogous formulae are

\[
\epsilon_{q} \mid_{\text{ups}} - \epsilon_{q} \mid_{l-1/2} = 4 \pi f_{l-1/2} p_{l-1/2} \frac{3 \epsilon_{q} - 1}{3 \epsilon_{q} - 1} - \frac{1}{3 \epsilon_{q} - 1} \left( \frac{p_{l-1/2}}{p_{\text{ups},l-1/2}} \right)^{3 \epsilon_{q} - 1} - 1 \quad \text{if} \quad q_{l} \neq 3,
\]

\[
\epsilon_{q} \mid_{\text{ups}} - \epsilon_{q} \mid_{l-1/2} = 4 \pi f_{l-1/2} p_{l-1/2} \frac{4 \epsilon_{q} - 1}{4 \epsilon_{q} - 1} - \frac{1}{4 \epsilon_{q} - 1} \left( \frac{p_{l-1/2}}{p_{\text{ups},l-1/2}} \right)^{4 \epsilon_{q} - 1} - 1 \quad \text{if} \quad q_{l} \neq 4,
\]

(26a)

where \( p_{\text{ups},l-1/2} \) is the upstream momentum for the bin interface placed at \( p_{l-1/2} \).

A2. Reconstruction of the distribution function

In the piece-wise power-law approximation the whole spectrum of CR particles at each cell of spatial grid can be described equivalently by two sets of discretised quantities:

\[(n_{i}, \epsilon_{i}) \Leftrightarrow (f_{i-1/2}, q_{i}), \quad i = 1, N_{\text{bin}},\]

The slopes \( q_{i} \) of the piecewise power-law distribution function can be retrieved by taking the ratio of particle energy density \( \epsilon_{i} \) to the number density \( n_{i} \) given by eqs. (A14) and (A15):

\[
\begin{align*}
\frac{\epsilon_{i}}{n_{i} c p_{l-1/2}} &= \begin{cases} 
\frac{p_{l+1/2}}{p_{l-1/2}} \left( \frac{p_{l+1/2}}{p_{l-1/2}} \right)^{3 \epsilon_{i} - 1} - 1, & \text{if } q_{i} = 3 \\
\frac{p_{l+1/2}}{p_{l-1/2}} \left( \frac{p_{l+1/2}}{p_{l-1/2}} \right)^{3 \epsilon_{i} - 1} - 1, & \text{if } q_{i} = 4 \\
\frac{p_{l+1/2}}{p_{l-1/2}} ^{3 \epsilon_{i} - 1} - 1, & \text{otherwise.}
\end{cases}
\end{align*}
\]

Equation (A28) can be solved for unknown values of \( q_{i} \) with the aid of a a Newton-Raphson-type numerical method. In the practical implementation we replace the strict condition of \( q_{i} = 3 \) by \( q_{i} - 3 < \epsilon_{q} \) and \( q_{i} = 4 \) by \( q_{i} - 4 < \epsilon_{q} \), where \( \epsilon_{q} = 10^{-2} \) seems to be an optimal choice. If the root-finder attempts to converge to a steep solution \( |q_{i}| > q_{\max} \) then \( q_{i} = q_{\text{big}} \) or \( q_{i} = -q_{\text{big}} \) is used to avoid floating point exceptions in subsequent computations. The value of \( q_{\text{big}} \) is typically set to 30. With the computed values of \( q_{i} \) the distribution function amplitudes \( f_{l-1/2} \) can be obtained from (A14).
In order to ensure stability of the numerical scheme defined by eqs. (A16-A18) we have to restrict the timestep of numerical integration by the condition that particles cannot move more than one bin-width in momentum space

\[
\max_{1 \leq l \leq N_{\text{bin}}} \left| \log_{10} \frac{p_{\text{ups},l-1/2}}{p_{l-1/2}} \right| \leq C_{\text{CRspectr}} \Delta w
\]  

(A29)

where \( C_{\text{CRspectr}} \leq 1 \) is a Courant like number controlling the evolution of CR population in momentum space, where \( p_{\text{ups},l-1/2} \) given by formulae (A23) and \( \Delta w = \log_{10} \left( \frac{p_{l+1/2}}{p_{l-1/2}} \right) \). For practical reasons we approximate (A23) to the linear order in \( \Delta t \) and obtain the timestep limit for each individual process

\[
\Delta t_{\text{adiab}} \leq C_{\text{CRspectr}} \log_{10} \frac{\Delta w}{|u_d|},
\]  

(A30)

and

\[
\Delta t_{\text{synch}} < \frac{\Delta w}{p_{\text{up}} u_b} = C_{\text{CRspectr}} \frac{\Delta w}{p_{\text{up}} u_b}.
\]  

(A31)

To follow accurately spectrum evolution in presence of adiabatic and synchrotron processes we impose \( C_{\text{CRspectr}} \approx 0.1 - 0.2 \).

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