Diagnostic of $f(R)$ under the $Om(z)$ function

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We perform the two--point diagnostic for the $Om(z)$ function proposed by Sahni et al in 2014 for the Starobinsky and Hu & Sawicki models in $f(R)$ gravity. We show that the observed values of the $Omh^2$ function can be explained in $f(R)$ models while in LCDM the $Omh^2$ function is expected to be a redshift independent number. We perform the analysis for some particular values of $\Omega_m$ finding a cumulative probability $(P(\chi^2 \leq \chi^2_{\text{model}})) P \sim 0.16$ or $\sim 0.09$ for the better cases versus a cumulative probability of $P \sim 0.98$ in the $\Lambda$CDM scenario. We also show that these models present a characteristic signature around the interval between $z \sim 2$ and $z \sim 4$, that could be confronted with future observations using the same test.

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I. INTRODUCTION

At the present the late time accelerated expansion of the Universe is a widely accepted fact supported by different independent observations [1–3]. Recently, several modifications of gravity have been proposed in order to explain this acceleration, however it is not clear if behind these generalizations of the Einstein-Hilbert action there is a modification or extension of some physical principle. Therefore, there is not a formal framework to guide the exploration of these models.

Every modification of gravity should provide similar predictions to those of the $\Lambda$CDM model, nevertheless this occurs only at some scales or in certain scenarios, e.g., the expansion of the Universe at low $z$ or in the Solar system tests. These modifications are different from each other, so every one of them should provide some signature to be distinguished from the others. Because of the lack of a formal guidance, the principal line of exploration for any model of modified gravity is to impose observational constraints on their parameters relying on the high accuracy of the current data.

The $f(R)$ gravity constitute a natural extension of General Relativity in the sense that almost all of its desirable properties remain intact. One of the changes introduced by the dependence of the gravitational action on a general function of the Ricci scalar, $f(R)$, is that we increase the number of degrees of freedom, resulting in a different set of field equations and consequently in modified dynamics for the gravitational systems. [4, 5]

Among the components of these modified gravities which can be tested are the equation of state (EOS) of the dark energy component and the growth factor. The values used in performing these tests can be obtained, in an indirect way, from astronomical measurements. These analyses depend on the value of the matter density today $\Omega_m$, for both dark and barionic components and even a small change can produce significant differences in the behavior of these models. It is worth mentioning that for $f(R)$ gravity an additional problem emerges, that is the necessity of a unique definition for the EOS of the geometric dark energy [6].

In these context, Sahni et al. [11] proposed a new test in order to distinguish between acceleration produced by a cosmological constant term and that coming from a modified gravity model. Now that results at different redshift for the Hubble parameter $H(z)/H(z = 0)$ are available from the observations of Barionic Acoustic Oscillations (BAO), it is possible to implement this test for $\Lambda$CDM as well as for modified theories of gravity.

Among the more successful models in $f(R)$ we find that of Starobinsky [14] and the proposed by Hu & Sawicki [15]. These models have proved to be viable at low redshift $z \approx 1$. In the present work we analyze them to verify their applicability at a higher redshift $z \approx 2.34$ which is is within the scale of validity for tests using the BAO data.

In the next section we will present the approach that we are using [16] in order to integrate the modified Friedman equations and the $f(R)$ models we will explore. In section III the $Omh(z)$ in introduced as well as the two-point relation $Omh^2(z_i; z_j)$ that is used as a model independent test in this paper. Section IV shows the results for the $Omh$ diagnostic for the $f(R)$ models and they are compared with the $\Lambda$CDM model. Finally, in section V we present the conclusion of this work.

II. $f(R)$ COSMOLOGY

As we have mentioned, $f(R)$ theories of gravity are the most straightforward way to extend the Hilbert-Einstein action. The dependence of the Ricci scalar is a general function which will be defined in order to reproduce ob-
sitions, the action is given by

\[ S[g_{ab}, \psi] = \int \frac{f(R)}{2\kappa} \sqrt{-g} \, d^4x + S_{\text{matter}}[g_{ab}, \psi], \quad (1) \]

where \( G = 1 \), \( c = 1 \) and \( \kappa \equiv 8\pi \). The term \( f(R) \) is an arbitrary smooth function of the Ricci scalar and \( S_{\text{matter}}[g_{ab}, \psi] \) is the usual action for matter.

Varying the above action with respect to \( g^{ab} \) we obtain the modified field equations

\[ f_R R_{ab} - \frac{1}{2} f g_{ab} - (\nabla_a \nabla_b - g_{ab} \Box) f_R = \kappa T_{ab}, \quad (2) \]

where \( f_R = \partial_R f \), \( \Box = g^{ab} \nabla_a \nabla_b \) and \( T_{ab} \) is the energy-momentum tensor for matter. The set of equations can be re-written in the following way

\[ f_R G_{ab} - f_{RR} \nabla_a \nabla_b R - f_{RRR}(\nabla_a R)(\nabla_b R) \]

\[ + g_{ab} \left[ \frac{1}{2} (R f_R - f) + f_{RR} \Box + f_{RRR}(\nabla R)^2 \right] = \kappa T_{ab}, \quad (3) \]

where \( G_{ab} = R_{ab} - g_{ab} R/2 \) is the Einstein tensor and \( (\nabla R)^2 := g^{ab}(\nabla_a R)(\nabla_b R) \).

The trace of eq. (3) yields a second order equation for the Ricci scalar

\[ \Box R = \frac{1}{3 f_{RR}} \left[ \kappa T - 3 f_{RRR}(\nabla R)^2 + 2 f - R f_R \right], \quad (4) \]

where \( T := T^a_a \). Finally, using (4) in (3) we find

\[ G_{ab} = \frac{1}{f_{RR}} \left[ f_{RRR} \nabla_a \nabla_b R + f_{RRRR}(\nabla_a R)(\nabla_b R) \right] - \frac{g_{ab}}{6} \left( R f_R + f + 2 \kappa T \right) + \kappa T_{ab}, \quad (5) \]

In this work we will consider an homogeneous, isotropic universe described by the Friedman-Robertson-Walker metric

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (6) \]

as usual, we will assume \( k = 0 \). The energy momentum tensor (EMT) is that for a fluid composed by baryons, dark matter and radiation. Under these assumptions Eqs. (4) and (5) read

\[ \ddot{R} = -3H \dot{R} - \frac{1}{3 f_{RR}} \left[ 3 f_{RRR} \dot{R}^2 + 2 f - R f_R + \kappa T \right] \quad (7) \]

\[ H^2 = \frac{1}{f_{RR}} \left[ f_{RRR} R \dot{R} - \frac{1}{6} (R f_R - f) \right] - \frac{\kappa T^2}{3 f_{RR}}, \quad (8) \]

\[ \dot{H} = -H^2 - \frac{1}{f_{RR}} \left[ f_{RRR} R \dot{R} + \frac{f}{6} + \frac{\kappa T^2}{3} \right], \quad (9) \]

\[ H = \dot{a}/a, \quad (10) \]

where \( \dot{\cdot} \equiv d/dt \). We will consider the following \( f(R) \) models:

- Starobinsky [14]:

\[ f(R) = R + \lambda R_S \left[ \left( 1 + \frac{R^2}{R_S^2} \right)^{-q} - 1 \right], \quad (11) \]

with \( q = 2 \), \( \lambda = 1 \) and \( R_S = 4.17 H_0^2 \).

- Hu–Sawicki [15]:

\[ f(R) = R - R_{HS} \frac{c_1 \left( \frac{R}{R_{HS}} \right)^n}{c_2 \left( \frac{R}{R_{HS}} \right)^n + 1}, \quad (12) \]

where the parameters are \( n = 4 \), \( c_1 \approx 1.25 \times 10^{-3} \), \( c_2 \approx 6.56 \times 10^{-5} \) and \( R_{HS} \approx 0.24 H_0^2 \).

These two models are currently the most successful among the proposed \( f(R) \) modified gravity descriptions and both pass the Solar System tests.

The numerical integration will be performed by using equations (7) and (9) together with the standard conservation equation \( \dot{\rho}_i + 3H (\rho_i + p_i) = 0 \) with \( i = 1, \ldots, 3 \) for each fluid component, namely baryons, dark matter and radiation. The Hamiltonian constriction eq. (10) is used in order to check the error in the numerical code which is acceptable (~10^{-11}).

III. THE \textit{Om} Diagnostic.

Sahni et al. proposed [11] a test to distinguish \text{$\Lambda$}CDM from modified gravity models or some other mechanism to describe the late acceleration of the Universe. The test is based on the function \text{Om} defined as

\[ O_{m}(x) = \frac{\tilde{H}(x)^2 - 1}{x^3 - 1}, \quad (13) \]

with \( x = 1 + z \) and \( \tilde{H}(x) = H(x)/H_0 \). It is worth mentioning that the \text{Om} function is constant for an accelerated expansion described by means of a Cosmological Constant term instead of a modified gravity model. In fact, in such case we have that \( O_{m}(x) = \Omega_{\text{M}}^0 \), thus \( O_{m}(x) = 0 \) and any deviation from zero would discard \text{$\Lambda$}CDM as a model for the expansion of the Universe. In a modified gravity description \text{Om}(x) is not constant but evolves with \( z \), showing a particular behavior for each theory.

A remarkable characteristic of this test is that it depends only on the values of \( H(z) \) which are determined by observations.

Shafieloo et al. [17] proposed a diagnostic by using the \text{Om} function at two different points. This way we can take observations about the determination of \( H(z) \) at several redshifts and then we can compute the two point relation given by
Later, Sahni et al. in [18] proposed a small modification to this test which allows to compare different theoretical predictions using the two-point diagnostic of Shafieloo et al. [17]. They multiply Eq. (13) by $h^2$ with $h = H_0/100\,\text{km/sec/Mpc}$ obtaining

$$\Omega_{mh}^2(z_i; z_j) = \frac{h^2(z_i) - h^2(z_j)}{(1 + z_i)^3 - (1 + z_j)^3}. \quad (15)$$

Written in this forms, this little change have the advantage that we can use that under the $\Lambda$CDM model $\Omega_M h^2 = 0.1426 \pm 0.0025$ from Planck XVI 2013 [19]. Sahni et al. [18] show that, by using this test, observations suggest that the value of $\Omega_{mh}^2$ is not constant. Taking $z_1 = 0$, $z_2 = 0.57$ and $z_3 = 2.34$ with $H(z_1) = 70.6 \pm 3.2\,\text{km/sec/Mpc}$ [19, 20], $H(z_2) = 92.4 \pm 4.5\,\text{km/sec/Mpc}$ [21] and $H(z_3) = 222 \pm 7\,\text{km/sec/Mpc}$ [19] the values for the $\Omega_{mh}^2$ two-point relation (eq. (15)) reported by Sahni are

$$\Omega_{mh}^2(z_1; z_2) = 0.124 \pm 0.045,$$
$$\Omega_{mh}^2(z_2; z_3) = 0.122 \pm 0.01,$$ \quad (16)

while for $\Lambda$CDM the value is $\Omega_{mh}^2 = 0.1426$ with a cumulative probability $P(\Lambda CDM) = 0.98$, i.e. the probability to find a sample with a $\chi^2 < \chi^2_{\text{model}} (P(\chi^2 < \chi^2_{\text{model}}))$, so the $p-value$ is given by $1 - P(\chi^2 < \chi^2_{\text{model}})$. We have considered just this value for the $\Lambda$CDM model as this is the value obtained in the best fit given by the Planck team [19].

In the next section we apply this diagnostic to two $f(R)$ models. We will present the $\Omega_{m}(z)$ function, the two-point relation $\Omega_{mh}^2(z_i; z_j)$ will be also computed in both cases. $f(R)$ results will be compared with $\Lambda$CDM in terms of the cumulative probability.

IV. $f(R)$ DIAGNOSTIC

We have introduced by now the way we will integrate the field equations in cosmology and also the test we will use as a diagnostic for two $f(R)$ models. To perform the analysis we proceed as follows. We integrate the differential equations (7) and (9) under the Ricci scalar approach fixing the initial conditions at some point in the past where it is safe to assume domination of matter and normalizing in order to fix the values of $\Omega_M^0$. (for details about cosmological integration in the context of the Ricci scalar approach see [22]). It is important to mention that we are taking into account a radiation component in the EMT. Nevertheless, this has no a significant effect in the results when integrating in the periods we are considering, i.e. $z < 5$.

The value of $\Omega_M^0$, assumed in the theoretical prediction, is crucial in order to have a better agreement with the values of the two-point relation given in [10]. In any case it is clear that, even if such changes are considered, the $\Lambda$CDM model can only give constant values for the $\Omega_{m}(z)$ function, and consequently it gives constant values for the two-point relation (14).

Here we explore different values of $\Omega_M^0$ for each $f(R)$ model: $\Omega_M^0 = 0.286$ (which corresponds to the value expected under the $\Lambda$CDM model) and also $\Omega_M^0 = 0.23, 0.24, 0.25$ and 0.26. In Fig. 1 we show the evolution of $\Omega_{m}(z)$ function by Eq. (14) as a function of the redshift for the Starobinsky model assuming different values for $\Omega_M^0$. In Fig. 2 we present the same function for the Hu & Sawicki model. The two models have a peculiar behavior that is worth mentioning. For a redshift around the value $z \sim 2$ the function $\Omega_{m}(z)$ reaches a minimum and from there it takes an stable or nearly stable value (see Fig. 3). This behavior shows a particular prediction from these models of $f(R)$ gravity that could be confronted with observations via the two-point relation.

Therefore, for each value $\Omega_M^0$, we compute the two point relation $\Omega_{mh}^2(z_i; z_2)$ given by Eq. (15). The values obtained for the Starobinsky model are presented in Table I. In Table II we show the values of the two point-relation $\Omega_{mh}^2(z_i; z_2)$ obtained for the Hu & Sawicki model. As we mentioned, in the $\Lambda$CDM model, this value is constant $\Omega_{mh}^2 = 0.1426$.

The cumulative probability for each model ($P(\Lambda CDM)$) has been computed taking the $\chi^2$ values in tables I and II. The values are incorporated in column 3 of Tables I and II. We can observe the best case for the Starobinsky model is for $\Omega_M^0 = 0.24$ with a cumulative probability $P(f(R)_{SL}) = 0.16$. For the Hu & Sawicki model the

![FIG. 1. (colour online). $\Omega_{m}(z)$ function for the Starobinsky model assuming (top to bottom) $\Omega_M^0 = 0.286$ (red), $0.26$ (blue), $0.25$ (orange), $0.24$ (gray) and $0.23$ (black).](image)
| $\Omega^0_m$ | $(z_i, z_j)$ | $Omh^2(z_i, z_j)$ | $\chi^2_{St}$ | $P(f(R)_{St})$ |
|------------|-------------|----------------|-------------|--------------|
| 0.286      | $(z_1, z_2)$ | 0.145          |             |              |
| 0.26       | $(z_1, z_3)$ | 0.139          | 4.978       | 0.97         |
|            | $(z_2, z_3)$ | 0.138          |             |              |
| 0.25       | $(z_1, z_2)$ | 0.134          |             |              |
| 0.24       | $(z_1, z_3)$ | 0.127          | 0.399       | 0.47         |
|            | $(z_2, z_3)$ | 0.126          |             |              |
| 0.23       | $(z_1, z_2)$ | 0.128          |             |              |
| 0.23       | $(z_1, z_3)$ | 0.120          | 0.136       | 0.28         |
| 0.23       | $(z_2, z_3)$ | 0.119          |             |              |

TABLE I. Values for the two points two-point relation $Omh^2(z_i, z_j)$ for the Starobinsky $f(R)$ model assuming different values of $\Omega_M^0$. Column 1 shows the value of the prior for the $\Omega_M^0$ value. In column 2 we have the two points in $z$ where $Omh^2(z_i, z_j)$ is calculated ($z_1 = 0$, $z_2 = 0.57$ and $z_3 = 2.34$). In column 3 is the value of $Omh$. In column 4 is the value of $\chi^2_{St}$. In column 5 is the cumulative probability.

FIG. 2. (colour online). $Omh(z)$ function for the Hu & Sawicki model assuming (top to bottom) $\Omega_M^0 = 0.286$ (red), 0.26 (blue), 0.25 (orange), 0.24 (gray) and 0.23 (black).

FIG. 3. Evolution of the $Omh(z)$ function. Starobinsky and Hu & Sawicki models, top and bottom respectively, assuming $\Omega_M^0 = 0.25$ in both cases.

best case is when we have $\Omega_M^0 = 0.25$ with a cumulative probability $P(f(R)_{H-S}) = 0.09$.

In the $\Lambda$CDM case $\chi^2 = 7.361$ and the cumulative probability $P(\Lambda CDM) = 0.98$, these values are not listed in Tables I and II.

V. CONCLUSION

In this work we performed the $Omh^2$ test for two of the most successful $f(R)$ models taking several values
| $\Omega_M^0$ | $(z_i, z_j)$ | $\text{Om}h^2(z_i, z_j)$ | $\chi^2_{H-S}$ | $P(f(R)_{H-S})$ |
|-------------|-------------|-----------------|--------------|-----------------|
| 0.286       | $(z_1, z_2)$ | 0.138           | 3.144        | 0.92            |
|             | $(z_1, z_3)$ | 0.136           |              | 0.92            |
|             | $(z_2, z_3)$ | 0.135           |              | 0.92            |
| 0.26        | $(z_1, z_2)$ | 0.129           |              |                 |
|             | $(z_2, z_3)$ | 0.126           |              |                 |
| 0.25        | $(z_1, z_2)$ | 0.126           | 0.328        | 0.43            |
|             | $(z_2, z_3)$ | 0.126           |              | 0.43            |
| 0.24        | $(z_1, z_2)$ | 0.122           | 0.137        | 0.29            |
|             | $(z_2, z_3)$ | 0.119           |              | 0.29            |
| 0.23        | $(z_1, z_2)$ | 0.118           | 0.750        | 0.61            |
|             | $(z_2, z_3)$ | 0.115           |              | 0.61            |

TABLE II. Values for the two points two-point relation $\text{Om}h^2(z_1, z_2)$ for the Hu-Sawicki $f(R)$ model assuming different values of $\Omega_M^0$. Column 1 shows the value of the prior for the $\Omega_M^0$ value. In column 2 we have the two points in $z$ where $\text{Om}h^2(z_i, z_j)$ is calculated ($z_1 = 0, z_2 = 0.57$ and $z_3 = 2.34$). In column 3 is the value of $\text{Om}h$. In column 4 is the value of $\chi^2_{H-S}$. In column 5 is the cumulative probability.

of the current density $\Omega_M^0$. The results obtained show that the two considered $f(R)$ models have, in general, the behaviour that is expected from observations. In the case of ΛCDM this behaviour cannot be present because $\text{Om}h^2$ is expected to be a redshift independent number.

In the $f(R)$ models we use in this paper, the evolution of the $\text{Om}(z)$ function is appropriate to have the values expected under the two-point relation, as we can observe for all the elections of $\Omega_M^0$. It is found that the cumulative probability is better than in the ΛCDM case. In particular, for the Starobinsky model with $\Omega_M^0 = 0.24$ the cumulative probability $P(f(R)_{S}) = 0.16$ and for the Hu-Sawicki case we find $P(f(R)_{H-S}) = 0.09$ taking $\Omega_M^0 = 0.25$. This both results are suitable values for this type of statistical test.

The asymptotic behavior of $\text{Om}h^2$ in these $f(R)$ models is necessary to reach the values from $\text{Om}h^2(z_1, z_3)$ to $\text{Om}h^2(z_2, z_3)$ which remain (almost) constant. This behavior is not possible using the ΛCDM model. It is remarkable that the evolution of the $\text{Om}h^2$ function from $z \sim 2$ to $z \sim 4$ could be tested by using future observations. Such evolution does not decrease in a monotonic way, it presents a change of sign of the $\text{Om}h^2(z_1, z_2)$ and this signature could be used in order to rule out or give support to these models.

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[1] S. Perlmutter et al., Astrophys. J. 517, 565 (1999).
[2] A. G. Riess et al., Astron. J. 116, 1038 (1998).
[3] R. Amanullah et al. (Supernova Cosmology Project), Astrophys. J. 716, 712 (2010).
[4] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007); Phys. Rep. 505, 59 (2011).
[5] S. Capozziello, M. De Laurentis, and V. Faraoni, arXiv: 0909.4672.
[6] S. Capozziello, and M. De Laurentis, arXiv: 1108.6266; T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rep. 513, 1 (2011).
[7] S. Capozziello, and M. Francaviglia, Gen. Relativ. Gravit. 40, 357 (2008).
[8] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).
[9] A. De Felice, and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010).
[10] L. G. Jaime, L. Patiño, and M. Salgado, Phys. Rev. D 89, 084010 (2014).

[11] V. Sahni, A. Shafieloo, and A. Starobinsky, Phys. Rev. D 78, 103502 (2008).

[12] L. Anderson et al. Mon. Not. R. Astron. Soc. 439, 83 (2008).

[13] T. Delubac et al. arXiv: 1404.1801;

[14] A. A. Starobinsky, JETP Lett. 86, 157 (2007).

[15] W. Hu, and I. Sawicki, Phys. Rev. D 76, 064004 (2007).

[16] L. G. Jaime, L. Patiño, and M. Salgado, Phys. Rev. D 83, 024039 (2011).

[17] A. Shafieloo, V. Sahni, and A. Starobinsky, Phys. Rev. D 86, 103527 (2012).

[18] V. Sahni, A. Shafieloo, and A. Starobinsky, arXiv: 1406.2209;

[19] Planck collaboration arXiv: 1303.5076;

[20] A. G. Riess, L. Macri and S. Csertano, Astrophys. J. 730, 119 (2011).

[21] L. Samushia, et al. Mon. Not. R. Astron. Soc. 429, 1514 (2013).

[22] L. G. Jaime, L. Patiño, and M. Salgado, arXiv: 1206.1642.