CLASS GROUP STATISTICS FOR TORSION FIELDS GENERATED BY ELLIPTIC CURVES

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Abstract. For a prime $p$ and a rational elliptic curve $E$/Q, set $K = \mathbb{Q}(E[p])$ to denote the torsion field generated by $E[p] := \ker\{E \xrightarrow{p} E\}$. The class group $\text{Cl}_K$ is a module over $\text{Gal}(K/Q)$. Given a fixed odd prime number $p$, we study the average non-vanishing of certain Galois stable quotients of the mod-$p$ class group $\text{Cl}_K/p\text{Cl}_K$. Here, $E$ varies over all rational elliptic curves, ordered according to height. Our results are conditional, since we assume that the $p$-primary part of the Tate-Shafarevich group is finite. Furthermore, we assume predictions made by Delaunay for the statistical variation of the $p$-primary parts of Tate-Shafarevich groups. We also prove results in the case when the elliptic curve $E$/Q is fixed and the prime $p$ is allowed to vary.

1. Introduction

Given a family of number fields, it is natural to study the statistical variation of class groups in the family. Of significant interest is the family of imaginary quadratic extensions $K/Q$. Such investigations go back to the work of Gauss, who was interested in the determination of all imaginary quadratic fields of a given class number $h$. The problem was solved for $h = 1$ by Baker, Heegner and Stark, for $h = 2$ by Baker and Stark and for $h = 3$ by Oesterlé. Watkins in [Wat04] computed the imaginary quadratic fields for which $h \leq 100$. Soundararajan showed that the number of imaginary quadratic fields of class number $< x$ is asymptotically $\frac{3\zeta(2)}{\zeta(3)}x^2$ as $x \to \infty$. The story for real quadratic fields is significantly different. Predictions can be made for the distribution of class groups via Cohen-Lenstra heuristics for all quadratic number fields [CL84] and Cohen-Lenstra-Martinet heuristics for general number fields [MC90]. We refer to [Woo16] for a survey of these methods. The study of class numbers on average in families of number fields has notably gained significant momentum in the following works [DH71, Won99, Bha05, FK07, EV07, EPW17]. On the other hand, there has been much interest in the study of the $p$-ranks of class groups of the cyclotomic extension $\mathbb{Q}(\mu_p)$. The Herbrand-Ribet theorem (cf. [Was97]) establishes a precise relationship between generalized Bernoulli numbers and the $p$-rank of the class group of $\mathbb{Q}(\mu_p)$. Ribet’s converse relates congruences between Eisenstein series and cusp forms to the existence of $p$-cyclic unramified extensions of $\mathbb{Q}(\mu_p)$. The rank of the Eisenstein ideal is closely related to the Galois module structure of mod-$p$ class groups that arise from certain families of number fields of the form $\mathbb{Q}(N^\perp)$, where $N \equiv 1 \mod p$ is a prime, cf. [Maz77, Lec18, SS19, WWE20]. It is however
difficult to resolve distribution questions in this context, though there have been some computational experiments done (cf. [WWE20, Introduction]).

Motivated by such developments, we consider the family of number field extensions that are generated by the torsion in elliptic curves defined over \( \mathbb{Q} \). To be specific, given an elliptic curve \( E/\mathbb{Q} \), and a prime \( p \), we let \( E[p] \) denote the \( p \)-torsion subgroup of \( E(\mathbb{Q}) \). Let \( K = \mathbb{Q}(E[p]) \) be the field generated by \( E[p] \), defined to be the fixed field of the residual representation

\[
\bar{\rho} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_p)
\]
on \( E[p] \simeq \mathbb{F}_p \oplus \mathbb{F}_p \). The extension \( K/\mathbb{Q} \) in general is a nonabelian extension that contains \( \mathbb{Q}(\mu_p) \). Let \( \text{Cl}_K \) be the class group of \( K \). Note that \( G = \text{Gal}(K/\mathbb{Q}) \) acts on \( \text{Cl}_K \) and thus on the mod-\( p \) class group \( \text{Cl}_K/p\text{Cl}_K \). We study Galois stable quotients of the \( \text{Cl}_K/p\text{Cl}_K \) that are isomorphic to \( E[p] \) as a \( G \)-module. In some sense, such investigations generalize the study of the \( p \)-rank of the class group of \( \mathbb{Q}(\mu_p) \). To be precise, we study the following related questions.

1. Let \( p \) be a fixed odd prime. As \( E \) varies over all elliptic curves defined over \( \mathbb{Q} \), how often is \( \text{Hom}_G(\text{Cl}_K/p\text{Cl}_K, E[p]) \) non-vanishing?
2. Let \( E/\mathbb{Q} \) be a fixed elliptic curve. As \( p \) varies over all primes, how does the dimension of \( \text{Hom}_G(\text{Cl}_K/p\text{Cl}_K, E[p]) \) vary?

In the context of the first question above, we order elliptic curves \( E/\mathbb{Q} \) according to height. We shall assume throughout that the \( p \)-primary part of the Tate-Shafarevich group is finite. Our results also rely on heuristics for the statistical behavior of Tate-Shafarevich groups of elliptic curves. These heuristics were studied by Delaunay [Del07]. Let \( \mathcal{E}_p \) be the set of elliptic curves \( E/\mathbb{Q} \) such that \( \text{Hom}_G(\text{Cl}_K, E[p]) \neq 0 \). Given \( \Delta \in \mathbb{Z}_{>0} \) with \( \Delta \equiv 0, 1 \mod 4 \), let \( H(\Delta) \) be Hurwitz class number, see Definition 3.10. Given a set of elliptic curves \( \mathcal{S} \), we denote by \( \mathfrak{d}(\mathcal{S}) \) the lower density of Weierstrass equations for elliptic curves in \( \mathcal{S} \) (see (3.1) for the definition of \( \mathfrak{d}(\mathcal{S}) \)).

**Theorem** ([Theorem 4.4]). Let \( p \) be an odd prime, and assume that Conjecture 4.2 is satisfied. Then, we have that

\[
\mathfrak{d}(\mathcal{E}_p) > (p^{-1} + p^{-3} - p^{-4})(1 - p^{-1} - \mathfrak{d}_p - \mathfrak{d}_p'),
\]

where

\[
\mathfrak{d}_p = \begin{cases} 
\zeta(p) - 1 & \text{if } p \geq 5, \\
(\zeta(3) - 1) + (\zeta(4) - 1) + (\zeta(7) - 1) & \text{if } p = 3,
\end{cases}
\]

\[
\mathfrak{d}_p' = \begin{cases} 
\frac{p - 1}{p^2} H(1 - 4p) & \text{if } p \geq 7, \\
\frac{p - 1}{p^2} (H(1 - 4p) + H(p^2 + 1 - 6p)) & \text{if } p \leq 5.
\end{cases}
\]

We obtain the following Corollary to the above result.

**Corollary** ([Corollary 4.5]). Let \( p \) be an odd prime, and assume that Conjecture 4.2 is satisfied. Then, we have that

\[
\mathfrak{d}(\mathcal{E}_p) \geq p^{-1} + O(p^{-3/2+\epsilon}).
\]

In other words, for any choice of \( \epsilon > 0 \) there is a constant \( C > 0 \), depending on \( \epsilon \) and independent of \( p \), such that

\[
\mathfrak{d}(\mathcal{E}_p) \geq p^{-1} - C p^{-3/2+\epsilon}.
\]
Next, we study the second question, where $E$ is fixed and $p$ varies. In this context, we prove two results.

**Theorem** (Theorem 5.1). Let $E/Q$ be an elliptic curve and assume that the following conditions are satisfied,

1. $\text{III}(E/Q)$ is finite,
2. $\text{rank } E(Q) \leq 1$,
3. $E$ does not have complex multiplication.

Then, for 100% of primes $p$, we have that $\text{Hom}_G(\text{Cl}_K/p\text{Cl}_K, E[p]) = 0$.

**Theorem** (Theorem 5.2). Let $E/Q$ be an elliptic curve and assume that the following conditions are satisfied,

1. $\text{III}(E/Q)$ is finite,
2. $\text{rank } E(Q) \geq 2$,
3. $E$ does not have complex multiplication.

Then, for 100% (i.e., for a Dirichlet density 1 set) of primes $p$, we have that

$$\text{dim Hom}_G(\text{Cl}_K/p\text{Cl}_K, E[p]) \geq \text{rank } E(Q) - 1.$$
\( \rho_1 : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p) \) with \( \bar{\rho} \). Let \( \chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times \) be the \( p \)-adic cyclotomic character, and \( \bar{\chi} : G_{\mathbb{Q}} \to \mathbb{F}_p^\times \) its mod-\( p \) reduction.

Given an integer \( r \), set \( \mathbb{Z}_p(r) := \mathbb{Z}_p(\chi^r) \) and given a \( \mathbb{Z}_p[G_{\mathbb{Q}}] \)-module \( M \), set \( M(r) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi^r) \) to denote the \( r \)-th Tate-twist of \( M \). Note that if \( M \) is an \( \mathbb{F}_p[G_{\mathbb{Q}}] \)-module, then, \( M(r) = M \otimes_{\mathbb{F}_p} \mathbb{F}_p(r) \). Let \( V_\bar{\rho} = E[p] \) be the underlying vector space for \( \bar{\rho} \). Given \( i, j \in \mathbb{Z} \) lying in the range \( 1 \leq i \leq (p-1) \) and \( 1 \leq j \leq (p-2) \), we set \( V^{i,j} = V_{p,E}^{i,j} \) to denote \( \text{Sym}^i(V_\bar{\rho})(j) \). Throughout, we shall assume that \( \bar{\rho} \) is irreducible. However, for the sake of discussion, let us consider the special case when \( \bar{\rho} \) is surjective. Then, upon choosing a basis of \( V_\bar{\rho} = E[p] \), we may identify the image of \( \bar{\rho} \) with \( GL_2(\mathbb{F}_p) \) and thus the module \( V^{i,j} \) is viewed as an irreducible representation of \( GL_2(\mathbb{F}_p) \). Note that any irreducible representation of \( GL_2(\mathbb{F}_p) \) over \( \mathbb{F}_p \) is of the form \( V^{i,j} \). A semisimple module \( M \) over \( \mathbb{F}_p[G_{\mathbb{Q}}] \) decomposes into a direct sum

\[
M \simeq \bigoplus_{i,j} \left( V^{i,j} \right)^{r_{i,j}(M)},
\]

where \( r_{i,j}(M) \geq 0 \).

Let \( K = \mathbb{Q}(E[p]) \) be the splitting field of \( \bar{\rho} \), taken to be the Galois extension of \( \mathbb{Q} \) given by \( K = \mathbb{Q}^{\ker \bar{\rho}} \). Set \( G := \text{Gal}(K/\mathbb{Q}) \), and identify \( G \) with the image of \( \bar{\rho} \). Note that when \( \bar{\rho} \) is surjective, \( G \) is identified with \( GL_2(\mathbb{F}_p) \). There is a natural action of \( G \) on the class group \( \text{Cl}_K \), and thus on the mod-\( p \) class group \( \text{Cl}_K/p\text{Cl}_K \). The study of the Galois module structure of \( \text{Cl}_K/p\text{Cl}_K \) shall be the primary focus of this paper. When \( \bar{\rho} \) is surjective, its semi-simplification decomposes into a direct sum of irreducible representations of \( GL_2(\mathbb{F}_p) \)

\[
\left( \text{Cl}_K/p\text{Cl}_K \right)^{ss} \simeq \bigoplus_{i,j} \left( V^{i,j} \right)^{n_{i,j}}.
\]

We are specifically interested in the space of homomorphisms \( \text{Hom}_G \left( \text{Cl}_K/p\text{Cl}_K, E[p] \right) \). Note that \( \text{Hom}_G \left( \text{Cl}_K/p\text{Cl}_K, E[p] \right) \neq 0 \) precisely when there is a \( G \)-stable quotient of \( \text{Cl}_K/p\text{Cl}_K \) which is isomorphic to \( E[p] \). By class field theory, this corresponds to the existence of an unramified \((\mathbb{Z}/p\mathbb{Z})^2\)-extension \( L \) of \( K \) which is Galois over \( \mathbb{Q} \), such that \( \text{Gal}(L/K) \simeq E[p] \) as a module over \( G = \text{Gal}(K/\mathbb{Q}) \). We have that

\[
n_{1,0} \geq \dim_{\mathbb{F}_p} \text{Hom}_G \left( \text{Cl}_K/p\text{Cl}_K, E[p] \right),
\]

with equality in the special case when the representation of \( G \) on \( \text{Cl}_K/p\text{Cl}_K \) is semisimple.

Given a prime number \( \ell \), let \( E \) be the Néron model of \( E \) over \( \mathbb{Z}_\ell \). Let \( E^0(\mathbb{Q}_\ell) \) be the subset of points of \( E(\mathbb{Q}_\ell) = \mathcal{E}(\mathbb{Z}_\ell) \) which reduce modulo \( \ell \) to the identity component of \( E_{/\mathbb{F}_\ell} \). Fix an absolute value \( \cdot : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times \) normalized by \( |p|_p^{-1} = p \). The Tamagawa number at \( \ell \) is set to be \( c_\ell(E) := [E(\mathbb{Q}_\ell) : E^0(\mathbb{Q}_\ell)] \), and set \( c_\ell(p)(E) := |c_\ell(E)|^{-1} \).

Therefore, \( c_\ell(E) \) is a unit in \( \mathbb{Z}_p \) if and only if \( c_\ell(p)(E) = 1 \). We denote by \( \text{Sel}_p(E/\mathbb{Q}) \) the \( p \)-Selmer group of \( E \) over \( \mathbb{Q} \) defined by

\[
\text{Sel}_p(E/\mathbb{Q}) := \ker \left\{ H^1(\mathbb{Q}/\mathbb{Q}, E[p]) \to \prod_{\ell} H^1(\mathbb{Q}_\ell/\mathbb{Q}_{\ell}, E(\mathbb{Q}_{\ell})) [p] \right\},
\]
where the restriction map for the prime $\ell$ is the composite
$$H^1(\bar{\mathbb{Q}}/\mathbb{Q}, E[p]) \to H^1(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, E[p]) \to H^1(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, E(\bar{\mathbb{Q}}_\ell))[p].$$

The Tate-Shafarevich group
$$\text{III}(E/\mathbb{Q}) := \ker \left\{ H^1(\bar{\mathbb{Q}}/\mathbb{Q}, E(\bar{\mathbb{Q}})) \to \prod_\ell H^1(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, E(\bar{\mathbb{Q}}_\ell)) \right\},$$
fits into an exact sequence
$$0 \to E(\mathbb{Q})/pE(\mathbb{Q}) \to \text{Sel}_p(E/\mathbb{Q}) \to \text{III}(E/\mathbb{Q})[p] \to 0,$$
see [Coa00, p.8, (1)].

Note that $\dim_{\mathbb{F}_p} E(\mathbb{Q})/pE(\mathbb{Q}) = \text{rank } E(\mathbb{Q}) + \dim_{\mathbb{F}_p} E(\mathbb{Q})[p]$, and since $\bar{\rho}$ is irreducible, $E(\mathbb{Q})[p] = 0$, and thus, $\dim_{\mathbb{F}_p} E(\mathbb{Q})/pE(\mathbb{Q}) = \text{rank } E(\mathbb{Q})$. Hence, we find that
$$\dim_{\mathbb{F}_p} \text{Sel}_p(E/\mathbb{Q}) = \text{rank } E(\mathbb{Q}) + \dim_{\mathbb{F}_p} \text{III}(E/\mathbb{Q})[p].$$

Due to the Cassels-Tate pairing, the $\mathbb{F}_p$-dimension of $\text{III}(E/\mathbb{Q})[p]$ is even. The following result shows that there is an explicit relationship between $\text{Cl}_{K/p} \text{Cl}_{K}$ and the $p$-torsion in the Tate-Shafarevich group $\text{III}(E/\mathbb{Q})$ and is key to constructing quotients of $\text{Cl}_{K/p} \text{Cl}_{K}$ that are isomorphic to $E[p]$.

**Theorem 2.1** (Prasad-Shekhar). Let $E$ be an elliptic curve over $\mathbb{Q}$ and $p$ be an odd prime number at which $E$ has good reduction. With respect to notation above, suppose that the following assumptions hold:

1. $c_{\ell}^{(p)}(E) = 1$ for all primes $\ell \neq p$,
2. $E(\mathbb{Q}_p)[p] = 0$.

Then, we have that
$$\dim_{\mathbb{F}_p} \text{Hom}_G \left( \text{Cl}_{K/p} \text{Cl}_{K}, E[p] \right) \geq \text{rank } E(\mathbb{Q}) + \dim_{\mathbb{F}_p} \text{III}(E/\mathbb{Q})[p] - 1.\tag{2.1}$$

In particular, under the above assumptions, if $\text{III}(E/\mathbb{Q})[p] \neq 0$, or rank $E(\mathbb{Q}) \geq 2$, then $\text{Hom} \left( \text{Cl}_{K/p} \text{Cl}_{K}, E[p] \right)$ is non-zero.

**Proof.** The inequality (2.1) follows from [PS21, Theorem 3.1]. The second assertion follows from Corollary 3.2 of loc. cit. \qed

**Definition 2.2.** Let $\mathcal{I} = \mathcal{I}_p(E)$ be the set of primes $\ell \neq p$ such that
- $E$ has split multiplicative reduction at $\ell$ and $E(\mathbb{Q}_\ell)[p]$ has rank 1,
- $E$ has non-split multiplicative reduction at $\ell$ and $E(\mathbb{Q}_\ell)[p]$ and $E(\mathbb{Q}_p^{nr})[p]$ has rank 1.

**Theorem 2.3** (Prasad-Shekhar). Let $E/\mathbb{Q}$ be an elliptic curve and $p$ an odd prime at which the conditions of Theorem 2.1 are satisfied. Let $\mathcal{I}$ be the finite set from Definition 2.2. Then, the following bound is satisfied
$$\dim_{\mathbb{F}_p} \text{Hom} \left( \text{Cl}_{K/p} \text{Cl}_{K}, E[p] \right) \leq \text{rank } E(\mathbb{Q}) + \dim_{\mathbb{F}_p} \text{III}(E/\mathbb{Q})[p] - 1 + \#\mathcal{I}.$$

**Proof.** The above result is [PS21, Theorem 4.2]. \qed
Theorem 2.1 is crucially used in the proof of Theorem 4.4, which is our main result. On the other hand, both Theorems 2.1 and 2.3 are applied to prove results in the case when \( E \) is a fixed elliptic curve and \( p \) varies over all primes at which \( E \) has good reduction, see section 5.

3. Density results for Weierstrass equations with local conditions

In this section, we recall results due to Cremona and Sadek [CS23] for the density of elliptic curves \( E/\mathbb{Q} \) satisfying local conditions at a prescribed set of primes. This set of primes may in fact be infinite. The results in this section are used to study the proportion of elliptic curves \( E/\mathbb{Q} \) ordered according to height which satisfy the conditions of Theorem 2.1. As always, we fix an odd prime \( p \). Given a tuple of integers \( a = (a_1, a_2, a_3, a_4, a_6) \) we have an associated elliptic curve \( E_a \) defined by the long Weierstrass equation

\[
Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6.
\]

The height of \( E_a \) is defined as follows

\[
ht(a) = ht(E_a) := \max_i \{|a_i| \).
\]

We let

\[
\begin{align*}
b_2 &= a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \\
b_8 &= a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2, \\
\Delta(a) &= -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6.
\end{align*}
\]

Given a ring \( R \), let

\[
\mathcal{W}(R) = R^5 = \{a = (a_1, a_2, a_3, a_4, a_6)|a_i \in R\},
\]

and let \( \Delta(a) = \Delta(E_a) \) be the associated discriminant. Given a prime \( \ell \), let \( \mathcal{W}_M(\mathbb{Z}_\ell) \) be the subset of \( \mathcal{W}(\mathbb{Z}_\ell) \) consisting of equations which are minimal in the sense of [Sil09, section VII.1]. Recall that if \( E \) is an elliptic curve over \( \overline{\mathbb{Q}}_\ell \), the Kodaira type is one of the following choices

- \( I_0 \) if \( E \) has good reduction;
- \( I_{\geq 1} \) if \( E \) has bad multiplicative reduction, of type \( I_m \) for some \( m \geq 1 \);
- Finally, if \( E \) has bad additive reduction, we have the following choices: II, III, IV, II*, III*, IV*, I_0, I_{\geq 1}, \) the latter meaning type \( I_m^{\ast} \) for some \( m \geq 1 \).

We refer to [Sil94, IV, section 9] for a comprehensive study of Kodaira types and how they may be detected via Tate’s algorithm. For \( a \in \mathcal{W}(\mathbb{Z}_\ell) \), we say that \( a \) has Kodaira type \( T \) if the associated elliptic curve \( E_a \) has Kodaira type \( T \). Note that this definition applies to non-minimal Weierstrass equations as well. Given a Kodaira type \( T \), let \( \mathcal{W}^T(\mathbb{Z}_\ell) \) be the subset of \( \mathcal{W}(\mathbb{Z}_\ell) \) of tuples \( a \) of Kodaira type \( T \), and set \( \mathcal{W}_M^T(\mathbb{Z}_\ell) = \mathcal{W}^T(\mathbb{Z}_\ell) \cap \mathcal{W}_M(\mathbb{Z}_\ell) \).

Let \( \mu \) be the Haar measure on \( \mathcal{W}(\mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^5 \), and set \( \rho^M(\ell) := \mu(\mathcal{W}_M(\mathbb{Z}_\ell)) \), and \( \rho^T(\ell) := \mu(\mathcal{W}^T(\mathbb{Z}_\ell)) \). Given a Kodaira type \( T \) at the prime \( \ell \), set

\[
\begin{align*}
\rho^M_T(\ell) &:= \mu\left(\mathcal{W}_M^T(\mathbb{Z}_\ell)\right), \\
\rho_T(\ell) &:= \mu\left(\mathcal{W}^T(\mathbb{Z}_\ell)\right).
\end{align*}
\]
According to [CS23, Proposition 2.6], we have that $\rho_T(\ell) = (1 - \ell^{-10})^{-1}\rho^M_T(\ell)$. These local densities are calculated in loc. cit., and these calculations are summarized here.

**Theorem 3.1** (Cremona-Sadek). Let $\ell$ be a prime, then $\rho^M(\ell) = 1 - \ell^{-10}$. For each Kodaira type $T$, the local density $\rho^M_T(\ell)$ is given by the following values:

| $T$ | $\rho^M_T(\ell)$ |
|-----|------------------|
| $I_0$ | $(\ell - 1)/\ell$ |
| $II$ | $(\ell - 1)/\ell^3$ |
| $III$ | $(\ell - 1)/\ell^4$ |
| $IV$ | $(\ell - 1)/\ell^5$ |
| $I_0^*$ | $(\ell - 1)/\ell^6$ |
| $I_{>1}$ | $(\ell - 1)/\ell^7$ |
| $IV^*$ | $(\ell - 1)/\ell^8$ |
| $III^*$ | $(\ell - 1)/\ell^9$ |
| $II^*$ | $(\ell - 1)/\ell^{10}$ |
| $I_m$ | $(\ell - 1)^2/\ell^{m+2}$ |
| $I_{>m}$ | $(\ell - 1)/\ell^{m+1}$ |

**Proof.** The above result follows from [CS23, Propositions 2.1 (3), 2.2, 2.5]. □

Let $S$ be a subset of $W(\mathbb{Z}) \simeq \mathbb{Z}^5$, the *density* of $S$ is taken to be the following limit

$$d(S) := \lim_{x \to \infty} \frac{\#\{a \in S \mid \text{ht}(a) < x\}}{\#\{a \in \mathbb{Z}^5 \mid \text{ht}(a) < x\}},$$

(3.1)

$$= \lim_{x \to \infty} 2^{-5}x^{-16}\#\{a \in S \mid |a_i| < x^i \text{ for } i = 1, 2, 3, 4, 6\}.$$ Note that the above limit may not exist. We let $\overline{d}(S)$ and $\underline{d}(S)$ be the upper and lower limit respectively. Note that $\overline{d}(S)$ and $\underline{d}(S)$ are defined by replacing the limit by lim sup and lim inf respectively, and that these limits do exist unconditionally. If $S$ is a set of elliptic curves $E/\mathbb{Q}$, we shall by abuse of notation denote by $S$ the subset of $W(\mathbb{Z})$ consisting of all tuples $a = (a_1, a_2, a_3, a_4, a_6)$ such that $E_a \in S$. Note that since we work with long Weierstrass equations, the choice of $a$ for a given isomorphism class of elliptic curves is not unique. According to [CS23, Theorem 1.1] the proportion of integral Weierstrass equations that are globally minimal is $1/\zeta(10) = 93555/\pi^{10} \approx 99.99\%$. Let $\Phi$ be a possibly infinite set of prime numbers and for each prime $\ell \in \Phi$, let $U_\ell$ be a subset of $W(\mathbb{Z}_\ell)$ defined by a set of congruence classes. In other words, $U_\ell$ is the inverse image of a subset of $W(\mathbb{Z}/\ell^M\mathbb{Z})$ for some integer $M > 0$. Let $U$ be the family of conditions $\{U_\ell \mid \ell \in \Phi\}$. Given $N > 0$, let

$$Z_N(U) := \{a \in W(\mathbb{Z}) \mid a \in U_\ell \text{ for some prime } \ell > N\}.$$ Note that $Z_N(U)$ may not exist. We let $\overline{d}(Z_N(U))$ and $\underline{d}(Z_N(U))$ be the upper and lower limit respectively. Note that $\overline{d}(Z_N(U))$ and $\underline{d}(Z_N(U))$ are defined by replacing the limit by lim sup and lim inf respectively, and that these limits do exist unconditionally. If $Z_N(U)$ is a set of elliptic curves $E/\mathbb{Q}$, we shall by abuse of notation denote by $Z_N(U)$ the subset of $W(\mathbb{Z})$ consisting of all tuples $a = (a_1, a_2, a_3, a_4, a_6)$ such that $E_a \in Z_N(U)$. Note that since we work with long Weierstrass equations, the choice of $a$ for a given isomorphism class of elliptic curves is not unique. According to [CS23, Theorem 1.1] the proportion of integral Weierstrass equations that are globally minimal is $1/\zeta(10) = 93555/\pi^{10} \approx 99.99\%$.

**Definition 3.2.** The family $U = \{U_\ell \mid \ell \in \Phi\}$ is said to be admissible if

$$\lim_{N \to \infty} \overline{d}(Z_N(U)) = 0.$$

Let $W_U$ be the set of integral Weierstrass equations $a \in W(\mathbb{Z})$ not satisfying $U_\ell$ at any prime $\ell \in \Phi$. 

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Proposition 3.3. Let $U$ be an admissible family and $s_\ell = \mu(U_\ell)$. Then, $\sum_{\ell \in \Phi} s_\ell$ converges and

$$\mathfrak{d}(W_U) = \prod_{\ell \in \Phi} (1 - s_\ell).$$

Proof. The above result follows from [CS23, Proposition 3.4].

Definition 3.4. Given a Kodaira type $T$, let $U(T)$ be the family of conditions on the set of primes $\ell \neq p$, where $U_\ell$ consists of local Weierstrass equations $a \in W(\mathbb{Z}_\ell)$ satisfying $T$. Thus, $W_{U(T)}$ consists of integral Weierstrass equations $a \in W(\mathbb{Z})$ such that $E_a$ does not satisfy $T$ at any prime $\ell \neq p$.

Lemma 3.5. Let $T$ be a Kodaira type such that $U(T)$ is admissible, then, the density of $W_{U(T)}$ exists and equals

$$\mathfrak{d}(W_{U(T)}) = \prod_{\ell \neq p} \left(1 - \rho_T(\ell)\right) \geq 1 - \sum_{\ell \neq p} \rho_T(\ell).$$

Proof. The result follows directly from Proposition 3.3.

Theorem 3.6. Let $p$ and $\ell$ be distinct prime numbers, and $E$ an elliptic curve over $\mathbb{Q}_\ell$. The following assertions hold.

1. Suppose that $p \geq 5$, then, $p | c_\ell(E)$ if and only if $E$ has Kodaira type $T = I_{mp}$ for $m \in \mathbb{Z}_{\geq 1}$.

2. Suppose that $p = 3$, then, $p | c_\ell(E)$ if and only if $E$ has Kodaira type $T = I_{3m}$ for $m \in \mathbb{Z}_{\geq 1}$, or $T = IV$, or $T = IV^*$.

Proof. The result is well known, see [Sil94, Table 4.1, p.365].

We come to the main result of this section, which is subsequently used in the proof of our main result in the next section.

Definition 3.7. Recall that $p$ is a fixed prime number. Let $\mathcal{S}_p$ be the set of elliptic curves $E/\mathbb{Q}$ such that $p | c_\ell(E)$ for some prime $\ell \neq p$.

Theorem 3.8. Let $E$ be an elliptic curve and $p$ an odd prime number. We have the following assertions

1. if $p \geq 5$, then,

$$\overline{d}(\mathcal{S}_p) < \zeta(p) - 1$$

2. if $p = 3$, then,

$$\overline{d}(\mathcal{S}_p) < (\zeta(3) - 1) + (\zeta(4) - 1) + (\zeta(7) - 1).$$

Proof. We prove the result on a case by case basis.

(1) First, we consider the case when $p \geq 5$. Then, by Theorem 3.6 (1), we have that $p | c_\ell(E)$ if and only if the Kodaira type $T$ is $I_{mp}$ for some integer $m \geq 1$. Let $U = U(I_{mp})$ be the datum such that $U_\ell = I_{mp}$ at every prime $\ell \neq p$. It follows directly from the proof of [CS23, Theorem 4.6] that the datum $U(I_{mp})$ is
admissible, and hence, so is $U$. From Lemma 3.5, we have that
\[ d(W_U) = \prod_{\ell \neq p} \left( 1 - (1 - \ell^{-10})^{-1}(1 - \ell^{-1})\ell^{-p} \right) \]
\[ \geq \prod_{\ell \neq p} (1 - \ell^{-p}) \]
\[ \geq 1 - \sum_{\ell \neq p} \ell^{-p}. \]

Since $S_p$ is contained in the complement of $W_U$, we find that
\[ \overline{d}(S_p) \leq \sum_{\ell \neq p} \ell^{-p} < \zeta(p) - 1. \]

(2) We now consider the case when $p = 3$. By Theorem 3.6 (2), we have that $3|c_\ell(E)$ if and only if the Kodaira type $T$ is $I_{3m}$ for some integer $m \geq 1$, or, $T = IV$ or $T = IV^*$. Let $U$ be such that $U_\ell$ is the subset of $W(Z_\ell)$ with Kodaira type $I_{\geq 3}$, IV or IV* respectively. Note that $\ell^2$ divides $\Delta$ for such reduction types. Admissibility is a direct consequence of the proof of [CS23, Theorem 4.6].

We find that
\[ \overline{d}(W_U) \geq \prod_{\ell \neq p} \left( 1 - \rho_{I_{\geq 3}}(\ell) - \rho_{IV}(\ell) - \rho_{IV^*}(\ell) \right) \]
\[ \geq \prod_{\ell \neq p} \left( 1 - (1 - \ell^{-10})^{-1}(1 - \ell^{-1})(\ell^{-3} + \ell^{-4} + \ell^{-7}) \right) \]
\[ \geq \prod_{\ell \neq p} \left( 1 - (\ell^{-3} + \ell^{-4} + \ell^{-7}) \right) \]
\[ \geq 1 - \sum_{\ell \neq p} \ell^{-3} - \sum_{\ell \neq p} \ell^{-4} - \sum_{\ell \neq p} \ell^{-7}. \]
\[ > 1 - (\zeta(3) - 1) - (\zeta(4) - 1) - (\zeta(7) - 1). \]

Thus, we find that
\[ \overline{d}(S_p) < (\zeta(3) - 1) + (\zeta(4) - 1) + (\zeta(7) - 1). \]

\[ \Box \]

**Definition 3.9.** Denote by $S'_p$ the set of elliptic curves $E$ for which $p \nmid \Delta(E)$ and $E(\mathbb{Q}_p)[p] \neq 0$. Note that for such elliptic curves and $p > 2$, $\tilde{E}(\mathbb{F}_p)[p] \neq 0$, where $\tilde{E}$ is the reduction of $E$.

Let $\Delta \in \mathbb{Z}_{<0}$ with $\Delta \equiv 0, 1 \mod 4$, and set
\[ B(\Delta) := \left\{ ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y] \mid a > 0, b^2 - 4ac = \Delta \right\}. \]

The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}[x, y]$, where a matrix $\sigma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ sends $x \mapsto (px + qy)$ and $y \mapsto (rx + sy)$. Thus, if $f = ax^2 + bxy + cy^2$, the matrix $\sigma$ acts by
\[ f \circ \sigma = a(px + qy)^2 + b(px + qy)(rx + sy) + c(rx + sy)^2. \]
We note that \( B(\Delta) \) is stable under the action of \( \text{SL}_2(\mathbb{Z}) \) and \( B(\Delta)/\text{SL}_2(\mathbb{Z}) \) is finite, see [Sch87] for additional details.

**Definition 3.10.** The Hurwitz class number \( H(\Delta) \) is the order of \( B(\Delta)/\text{SL}_2(\mathbb{Z}) \).

Let \( d(p) \) be the number of tuples \( a \in \mathcal{W}(\mathbb{F}_p) \) such that \( E_a(\mathbb{F}_p)[p] \neq 0 \). Note that

\[
\frac{d(p)}{\#\mathcal{W}(\mathbb{F}_p)} = \frac{d(p)}{p^3}.
\]

**Lemma 3.11.** With respect to notation above, \( \overline{d}(S'_p) \leq \frac{d(p)}{p^3} \).

**Proof.** Given \( a \in \mathcal{W}(\mathbb{F}_p) \), let \( S_a \) be the set of \( \tilde{a} \in \mathcal{W}(\mathbb{Z}) \) which reduce to \( a \). It is easy to see that \( d(S_a) = \frac{1}{p^3} \). Note that \( S'_p \) is contained in the disjoint union \( \bigsqcup_a S_a \), where \( a \) ranges over the tuples in \( \mathcal{W}(\mathbb{F}_p) \) such that \( E_a(\mathbb{F}_p)[p] \neq 0 \). The result follows from this. \( \square \)

**Proposition 3.12.** With respect to notation above, we have that

\[
\overline{d}(S'_p) \leq \begin{cases} \left( \frac{p-1}{2p^2} \right) H(1-4p) & \text{if } p \geq 7, \\ \left( \frac{p-1}{2p^2} \right) (H(1-4p) + H(p^2+1-6p)) & \text{if } p \leq 5. \end{cases}
\]

**Proof.** The number of isomorphism classes of elliptic curves \( E \) over \( \mathbb{F}_p \) such that \( E(\mathbb{F}_p)[p] \neq 0 \) is

\[
\begin{cases} H(1-4p) & \text{if } p \geq 7, \\ H(1-4p) + H(p^2+1-6p) & \text{if } p \leq 5, \end{cases}
\]

see [RS23, Corollary 3.11] for additional details. Given \( a \in \mathcal{W}(\mathbb{F}_p) \) and associated integral Weierstrass model \( E_a \), then, after a transformation \( (X,Y) \mapsto (X+r,Y+sX+t) \), we obtain an elliptic curve with a short Weierstrass equation. Note that two elliptic curves

\[
E_{c,d} : Y^2 = X^3 + cX + d \\
E_{c',d'} : Y^2 = X^3 + c'X + d'
\]

are isomorphic over \( \mathbb{F}_p \) if there exists \( s \in \mathbb{F}_p^\times \) such that

\[ c' = s^4c \text{ and } d' = s^6d. \]

It is thus easy to see that the number of Weierstrass equations in a given isomorphism class is at most \( p^3 \left( \frac{p-1}{2} \right) \). Thus, we find that

\[
d(p) \leq \begin{cases} p^3 \left( \frac{p-1}{2} \right) H(1-4p) & \text{if } p \geq 7, \\ p^3 \left( \frac{p-1}{2} \right) (H(1-4p) + H(p^2+1-6p)) & \text{if } p \leq 5. \end{cases}
\]

\( \square \)
4. Results for a fixed prime and varying elliptic curve

In this section we prove our main results. Throughout, we fix an odd prime number $p$. Given a set of rational elliptic curves $S$, we obtain a subset of $\mathcal{W}(\mathbb{Z})$ consisting of all $a$ such that $E_a \in S$. By abuse of notation, we shall refer to this set as $S$ as well. The density $\mathfrak{d}(S)$ is defined as in (3.1). Recall that the upper and lower densities, denoted $\mathfrak{D}(S)$ and $\mathfrak{d}(S)$, are always defined even if $\mathfrak{d}(S)$ need not be. Let $\mathcal{E}_p$ be the set of elliptic curves $E/\mathbb{Q}$ such that $\text{Hom}_G(\text{Cl}_K, E[p]) \neq 0$. Note that for $E \in \mathcal{E}_p$, we have in particular that $p|\# \text{Cl}_K$.

Recall from Definition 3.7 that $\mathcal{S}_p$ is the set of elliptic curves $E/\mathbb{Q}$ such that $p|c_\ell(E)$ for some prime $\ell \neq p$. Recall from Definition 3.9 that $\mathcal{S}_p'$ is the set of elliptic curves $E$ for which $p \nmid \Delta(E)$ and $E(\mathbb{Q}_p)[p] \neq 0$. Denote by $\mathcal{S}_p''$ the set of elliptic curves $E/\mathbb{Q}$ with bad reduction at $p$. Let $\mathcal{T}_p$ be the elliptic curves $E$ such that $\text{III}(E/\mathbb{Q})[p^\infty]$ is finite and $\text{III}(E/\mathbb{Q})[p] \neq 0$.

**Proposition 4.1.** Let $p$ be an odd prime. Then, we have that

$$\mathfrak{D}(\mathcal{E}_p) \geq \mathfrak{D} \left( \mathcal{T}_p \setminus (\mathcal{S}_p \cup \mathcal{S}_p' \cup \mathcal{S}_p'') \right).$$

**Proof.** Suppose $E$ is an elliptic curve in $\mathcal{T}_p \setminus (\mathcal{S}_p \cup \mathcal{S}_p' \cup \mathcal{S}_p'')$, then, since $E \in \mathcal{T}_p$, we have that $\text{III}(E/\mathbb{Q})[p] \neq 0$, and since $E \notin \mathcal{S}_p \cup \mathcal{S}_p' \cup \mathcal{S}_p''$, we have that $p|c_\ell(E)$ for all primes $\ell \neq p$, and $E(\mathbb{Q}_p)[p] = 0$. So long as the residual representation on $E[p]$ is irreducible it follows from Theorem 2.1 that $\text{Hom}_G(\text{Cl}_K, E[p]) \neq 0$. As a result, $E$ is contained in $\mathcal{E}_p$. By a well known result of Duke [Duk97], the residual representation on $E[p]$ is irreducible for 100% of elliptic curves, and the result follows.

Throughout, we shall assume that for any elliptic curve $E/\mathbb{Q}$, the $p$-primary part of the Tate-Shafarevich group $\text{III}(E/\mathbb{Q})[p^\infty]$ is finite. The density of elliptic curves $E/\mathbb{Q}$ such that $\text{III}(E/\mathbb{Q})[p^\infty] \neq 0$ was studied by Delaunay [Del07].

**Conjecture 4.2** (Delaunay). With respect to notation above,

$$\mathfrak{D}(\mathcal{T}_p) \geq 1 - \prod_{i \geq 1} \left( 1 - p^{-2i-1} \right) > p^{-1} + p^{-3} - p^{-4}.$$  

Moreover, given a set $\mathcal{E}$ of elliptic curves defined by local congruence conditions,

$$\mathfrak{D}(\mathcal{T}_p \cap \mathcal{E}) = \mathfrak{D}(\mathcal{T}_p) \mathfrak{d}(\mathcal{E}).$$

**Remark 4.3.** For the purposes of this article, we shall use only the second lower bound predicted by the above conjecture: that is, we shall assume that

$$\mathfrak{D}(\mathcal{T}_p \cap \mathcal{E}) > \left( p^{-1} + p^{-3} - p^{-4} \right) \mathfrak{d}(\mathcal{E}),$$

where $\mathcal{E}$ is taken to be the complement of $\mathcal{S}_p \cup \mathcal{S}_p' \cup \mathcal{S}_p''$.

**Theorem 4.4.** Let $p$ be an odd prime, and assume the above conjecture. Then, we have that

$$\mathfrak{D}(\mathcal{E}_p) > (p^{-1} + p^{-3} - p^{-4})(1 - p^{-1} - \mathfrak{d}_p - \mathfrak{d}_p'),$$

where

$$\mathfrak{d}_p = \begin{cases} \zeta(p) - 1 & \text{if } p \geq 5, \\ (\zeta(3) - 1) + (\zeta(4) - 1) + (\zeta(7) - 1) & \text{if } p = 3, \end{cases}$$

and $\mathfrak{d}_p'$ is defined similarly.
\[
\vartheta'_p = \begin{cases} 
\frac{p-1}{2p^2} H(1-4p) & \text{if } p \geq 7, \\
\frac{p-1}{2p^2} \left( H(1-4p) + H(p^2 + 1 - 6p) \right) & \text{if } p \leq 5.
\end{cases}
\]

**Proof.** Let \( E \) be the set of elliptic curves in the complement of \( S_p \cup S'_p \cup S''_p \). We have from Theorem 3.8 (resp. Proposition 3.12) that \( d(S_p) = d_p \) (resp. \( d(S'_p) = d'_p \)). Furthermore, we have from Theorem 3.1 and Proposition 3.3 that \( d(S''_p) = \frac{1}{p} \). From Proposition 4.1, we have that \( \vartheta(E_p) \geq \vartheta \left( \mathcal{T}_p \setminus (S_p \cup S'_p \cup S''_p) \right) \). According to our assumption,

\[
\vartheta \left( \mathcal{T}_p \setminus (S_p \cup S'_p \cup S''_p) \right) = \vartheta(\mathcal{T}_p \cap E)
\geq \left( 1 - \prod_{i \geq 1} \left( 1 - p^{-(2i-1)} \right) \right) \vartheta(E)
> (p^{-1} + p^{-3} - p^{-4}) \vartheta(E)
= (p^{-1} + p^{-3} - p^{-4}) \left( 1 - p^{-1} - \vartheta_p - \vartheta'_p \right),
\]

and the result follows.

\[ \square \]

**Corollary 4.5.** Assume Conjecture 4.2. Then

\[ \vartheta(E_p) \geq p^{-1} + O(p^{-3/2+\epsilon}). \]

In other words, for any choice of \( \epsilon > 0 \) there is a constant \( C > 0 \), depending on \( \epsilon \) and independent of \( p \), such that

\[ \vartheta(E_p) \geq p^{-1} - C p^{-3/2+\epsilon}. \]

**Proof.** According to Theorem 4.4, we have that

\[ \vartheta(E_p) > (p^{-1} + p^{-3} - p^{-4})(1 - p^{-1} - \vartheta_p - \vartheta'_p)
\geq p^{-1} - p^{-1} \left( p^{-1} + \vartheta_p + \vartheta'_p \right). \]

We have that

\[ \zeta(p) - 1 = 2^{-p} + \sum_{n \geq 3} n^{-p} < 2^{-p} + \int_2^\infty x^{-p} dx = 2^{-p} \left( \frac{p+1}{p-1} \right), \]

and hence,

\[ \vartheta_p = O(2^{-p}) = O(p^{-1/2+\epsilon}). \]

On the other hand, it follows from known results that

\[ \vartheta'_p = O \left( p^{-1/2} \log p (\log \log p)^2 \right) = O(p^{-1/2+\epsilon}), \]

(see [LJ87, Proposition 1.9]) and the result follows. \[ \square \]
5. Results for a fixed elliptic curve and varying prime

In this section we prove results for a fixed elliptic curve $E/\mathbb{Q}$ and varying prime $p$. We shall assume throughout that $E$ does not have complex multiplication, and that the Tate-Shafarevich group $\Sha(E/\mathbb{Q})$ is finite. Given a prime $p$, we let $K_p = \mathbb{Q}(E[p])$. We make a number of observations with regard to the variation of $\text{Cl}_{K_p}/p\text{Cl}_{K_p}$, where $p$ varies over all primes $p$.

**Theorem 5.1.** Let $E/\mathbb{Q}$ be an elliptic curve, assume that the following conditions are satisfied,

1. $\Sha(E/\mathbb{Q})$ is finite,
2. $\text{rank } E(\mathbb{Q}) \leq 1$,
3. $E$ does not have complex multiplication.

Then, for 100% of primes $p$, we have that $\text{Hom}_G(\text{Cl}_{K_p}/p\text{Cl}_{K_p}, E[p]) = 0$.

**Proof.** Note that for all but finitely many primes $p$,

1. $E$ has good reduction at $p$,
2. $E(\mathbb{Q}_\ell)[p] = 0$ for all primes $\ell$ at which $E$ has bad reduction,
3. $\Sha(E/\mathbb{Q})[p] = 0$,
4. the representation on $E[p]$ is irreducible,
5. $c_\ell^{(p)}(E) = 1$ for all primes $\ell$ at which $E$ has bad reduction.

Thus, it follows from Theorem 2.3, that we have that $\text{Hom}_G(\text{Cl}_{K_p}/p\text{Cl}_{K_p}, E[p]) = 0$ provided $E(\mathbb{Q}_p)[p] = 0$. Let $\tilde{E}$ be the reduction of $E$ over $\mathbb{F}_p$. The formal group $\tilde{E}(\mathbb{Z}_p)$ of $E$ at $p$ has no nontrivial $p$-torsion. Thus, if $p$ is a prime of good reduction, the natural map $E(\mathbb{Q}_p) \to \tilde{E}(\mathbb{F}_p)$ induces an injection $E(\mathbb{Q}_p)[p] \to \tilde{E}(\mathbb{F}_p)[p]$.

A prime $p$ at which $E$ has good reduction is called an anomalous prime if $\tilde{E}(\mathbb{F}_p)[p] \neq 0$. It is well known that for a non-CM elliptic curve, 100% of primes are non-anomalous (see for instance [Mur97]). Thus, $E(\mathbb{Q}_p)[p] = 0$ for 100% of primes $p$, and thus the result follows. \qed

**Theorem 5.2.** Let $E/\mathbb{Q}$ be an elliptic curve, assume that the following conditions are satisfied,

1. $\Sha(E/\mathbb{Q})$ is finite,
2. $\text{rank } E(\mathbb{Q}) \geq 2$,
3. $E$ does not have complex multiplication.

Then, for 100% of primes $p$, we have that

$$\dim \text{Hom}_G(\text{Cl}_{K_p}/p\text{Cl}_{K_p}, E[p]) \geq \text{rank } E(\mathbb{Q}) - 1.$$ 

**Proof.** It follows from the proof of Theorem 5.1 that for 100% of primes $p$, all of the following conditions are satisfied:

1. $E$ has good reduction at $p$,
2. the representation on $E[p]$ is irreducible,
3. $c_\ell^{(p)}(E) = 1$ for all primes $\ell$ at which $E$ has bad reduction,
4. $\tilde{E}(\mathbb{Q}_p)[p] = 0$. 


It follows from Theorem 2.1 that
\[
\dim \text{Hom}_G(\text{Cl}_{K_p}/p\text{Cl}_{K_p}, E[p]) \geq \text{rank } E(\mathbb{Q}) - 1.
\]

6. Computational results

In this section we present some computations of \(\text{Cl}_{K/3}\text{Cl}_{K_p}\), where \(K = \mathbb{Q}(E[3])\), for certain families of elliptic curves \(E\). Due to the difficulty of computing statistics for larger extensions, we primarily restricted ourselves to the case that \(\text{Gal}(K/\mathbb{Q}) \subset \text{GL}_2(\mathbb{F}_3)\) is the normalizer of a split Cartan subgroup. This is the smallest irreducible subgroup of \(\text{GL}_2(\mathbb{F}_3)\). We include also a few additional calculations where the Galois group is the normalizer of a non-split Cartan subgroup.

We note that the hypotheses of [PS21] are often violated in these cases. Specifically, a majority of such \(E\) have bad reduction at 3. In addition, all ramification at 3 is tame as the Galois group has order prime to 3: thus any such \(E\) for which \(a_3(E) \equiv 1 \pmod{3}\) has 3 as a local torsion prime. We computed our examples without regard to these considerations. We comment only that we verified that every computation which violates the lower bound of [PS21] does not satisfy their hypotheses.

6.0.1. Normalizer of a split Cartan: Varying \(j\)-invariants. We use the formula of Zywina [Zyw15, Theorem 1.2]
\[
j_2(t) = 27\frac{(t+1)^3(t-3)^3}{t^3}
\]
such that any elliptic curve with such a \(j\)-invariant has \(\text{Gal}(\mathbb{Q}(E[3])/\mathbb{Q})\) contained in the normalizer of a split Cartan subgroup. For any \(t\), let \(E_t\) denote the elliptic curve of smallest conductor with \(j\)-invariant \(j_2(t)\) and let \(K_t\) denote the field \(\mathbb{Q}(E_t[3])\). We computed \(\text{Sel}_3(E_t/\mathbb{Q})\) as well as the Galois module structure of \(\text{Cl}_{K_t}/3\) for the 599 elliptic curves in the set
\[
S_0 = \{E_{a/b} \mid |a| \leq 50, 1 \leq b \leq 10, [K_t : \mathbb{Q}] = 8\}.
\]

Let \(L_t\) denote the unique biquadratic subfield of \(K_t\); note that \(L_t\) necessarily contains \(\mathbb{Q}(\sqrt{-3})\). In each case there was \(d_t \geq 0\) such that there was an isomorphism
\[
\text{Cl}_{K_t}/3 \cong \text{Cl}_{L_t}/3 \oplus E_t[3]^{d_t}
\]
as \(\text{Gal}(K_t/\mathbb{Q})\)-modules. Note that \(\text{Gal}(K_t/\mathbb{Q})\) acts on \(\text{Cl}_{L_t}/3\) via its abelianization.

Our interest here is primarily in comparing \(\text{Sel}_3(E_t/\mathbb{Q})\) and \(d_t\). We briefly report the data for \(\text{Cl}_{L_t}/3\):

| \(\dim \text{Cl}_{L_t}/3\) | \(\# t\) |
|-------------------------|-------|
| 0                      | 447   |
| 1                      | 123   |
| 2                      | 28    |
| 3                      | 1     |

The only curve in our sample with \(\dim \text{Cl}_{L_t}/3 = 3\) was \(E_{-46/3}\) with \(L_t = \mathbb{Q}(\sqrt{-3}, \sqrt{2917})\).

Turning to the other quantities, we have the following.
dim Sel₃(Eₜ/Q) | d₁ | # t
--- | --- | ---
0 | 0 | 103
0 | 1 | 26
1 | 0 | 228
1 | 1 | 52
1 | 2 | 2
2 | 0 | 95
2 | 1 | 74
2 | 2 | 2
3 | 0 | 4
3 | 1 | 12
3 | 2 | 1

We remark that the restriction to E with specified 3-torsion fields obviously biases the distribution of 3-Selmer groups. Although it is unwise to extrapolate too much from this limited data, we note that for t with dim Sel₃(Eₜ/Q) ≤ 1, the dimension of Clₗₜ/3 appears to be roughly the same (being nonzero around 19% of the time) whether the Selmer group has dimension zero or one. By contrast, for Selmer groups of dimension two, the proportion having d₁ ≥ 1 is much larger at 44%. For Selmer groups of dimension three it is much larger than that, in this limited sample.

Granting the limits of this data set, it certainly appears that Sel₃(Eₜ/Q) has a significant effect on d₁, but there are additional influences as well which are not currently clear.

6.0.2. Normalizer of a split Cartan: Fixed j-invariants. We also investigated two quadratic twist families. Let E₁ denote the elliptic curve

\[ y^2 + xy + y = x^3 - 141x + 624 \]

with j(E₁) = 857375/8 and conductor 10082 = 2 ⋅ 71². For any squarefree t, let E₁,t denote the quadratic twist of E₁ by t and let K₁,t denote the field Q(E₁,t[3]). Then Gal(K₁,t/Q) is the normalizer of a split Cartan subgroup of GL₂(𝔽₃). We note that each K₁,t contains the biquadratic field \( L₁ = \mathbb{Q}(\sqrt{-3}, \sqrt{-71}) \) which has trivial 3-class group.

Consider the set of 608 elliptic curves

\[ S₁ = \{ E₁,t \mid 1 ≤ t ≤ 1000, t \text{ squarefree} \} . \]

In each case we computed a Gal(K₁,t/Q)-isomorphism

\[ Cl_{K₁,t}/3 ≅ E₁,t[3]^{d₁,t} \]

for some d₁,t. We computed the following data.

| dim Sel₃(E₁ₜ/Q) | d₁ | # t |
|---|---|---|
| 0 | 0 | 341 |
| 0 | 1 | 1 |
| 1 | 0 | 74 |
| 1 | 1 | 31 |
| 1 | 2 | 2 |
| 2 | 1 | 140 |
| 2 | 2 | 18 |
| 3 | 2 | 1 |
We note here that the dependence of $d_{1,t}$ on $\text{Sel}_3(E_{1,t}/\mathbb{Q})$ is more clear than in the case of varying $j$-invariants: the lower bound of $|PS21|$ (although as noted the hypotheses are often violated in this data set) appears to control most of the behavior of $d_{1,t}$.

We computed analogous data for the elliptic curve $E_2$ given by

$$y^2 = x^3 - 83667346875x - 10711930420406250$$

with $j(E_2) = -42875/8$ and conductor $6962 = 2 \cdot 59^2$. With notation as above, we consider the set of 608 quadratic twists

$$S_2 = \{ E_{2,t} \mid 1 \leq t \leq 1000, \text{t squarefree} \}.$$

Each 3-division field field $K_{2,t}$ has Galois group the normalizer of a split Cartan subgroup and contains the biquadratic field $L_2 = \mathbb{Q}(\sqrt{-3}, \sqrt{-59})$. This field has class group $\mathbb{Z}/3$, and for each $t$ we computed a $\text{Gal}(K_{2,t}/\mathbb{Q})$-isomorphism

$$\text{Cl}_{K_{2,t}} / 3 \cong \text{Cl}_{L_2} / 3 \oplus E_{2,t}[3]^{d_{2,t}}$$

for some $d_{2,t}$.

| $\dim \text{Sel}_3(E_{2,t}/\mathbb{Q})$ | $d_{2,t}$ | $\# t$ |
|--------------------------------------|---------|------|
| 0                                   | 0       | 54   |
| 0                                   | 1       | 18   |
| 1                                   | 0       | 332  |
| 1                                   | 1       | 165  |
| 2                                   | 0       | 18   |
| 2                                   | 1       | 13   |
| 2                                   | 2       | 2    |
| 3                                   | 2       | 6    |

This behavior appears somewhat less straightforward than for $S_1$. The distribution of $\text{Sel}_3(E_{2,t}/\mathbb{Q})$ is obviously quite different than that of $\text{Sel}_3(E_{1,t}/\mathbb{Q})$ and the effect on $d_{2,t}$ is less clear. We do note that the distributions of $d_{1,t}$ and $d_{2,t}$ ignoring Selmer groups are similar, which is somewhat curious.

6.0.3. **Normalizer of a non-split Cartan: fixed $j$-invariant.** We compiled very limited data in one case with mod 3 image of Galois the normalizer of a non-split Cartan subgroup. Consider the elliptic curve $E_3$ given by

$$y^2 = x^3 + x^2 - 2x - 8$$

with $j(E_3) = -64$ and conductor $1568 = 2^5 \cdot 7^2$. We consider its set of 61 twists

$$S_3 = \{ E_{3,t} \mid 1 \leq t \leq 100, \text{t squarefree} \}.$$

For each $t$ we have an isomorphism of $\text{Gal}(K_{3,t}/\mathbb{Q})$-modules

$$\text{Cl}_{K_{3,t}} / 3 \cong E_{3,t}[3]^{d_{3,t}}$$

for some $d_{3,t}$. 
There is too little data here to provide anything more than wild speculations; unfortunately the computations rapidly became very time consuming beyond this point.

References

[Bha05] Manjul Bhargava. The density of discriminants of quartic rings and fields. *Annals of Mathematics*, pages 1031–1063, 2005.

[CL84] Henri Cohen and Hendrik W Lenstra. Heuristics on class groups of number fields. In *Number Theory Noordwijkerhout 1983*, pages 33–62. Springer, 1984.

[Coa00] John Henry Coates. Galois cohomology of elliptic curves. *Lecture Notes at the Tata Institute of Fundamental Research No. 88*, 2000.

[CS23] John E. Cremona and Mohammad Sadek. Local and global densities for Weierstrass models of elliptic curves. *Math. Res. Lett.*, 30(2):413–461, 2023.

[Del07] Christophe Delaunay. Heuristics on class groups and on Tate-Shafarevich groups: the magic of the cohen-lenstra heuristics. *Ranks of elliptic curves and random matrix theory*, 341:323–340, 2007.

[DH71] Harold Davenport and Hans Arnold Heilbronn. On the density of discriminants of cubic fields. II. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 322(1551):405–420, 1971.

[Duk97] William Duke. Elliptic curves with no exceptional primes. *Comptes rendus de l’Académie des sciences. Série I, Mathématique*, 325(8):813–818, 1997.

[EPW17] Jordan Ellenberg, Lillian Pierce, and Melanie Wood. On ℓ-torsion in class groups of number fields. *Algebra & Number Theory*, 11(8):1739–1778, 2017.

[EV07] Jordan S Ellenberg and Akshay Venkatesh. Reflection principles and bounds for class group torsion. *International Mathematics Research Notices*, 2007, 2007.

[FK07] Étienne Fouvry and Jürgen Klüners. On the 4-rank of class groups of quadratic number fields. *Inventiones mathematicae*, 167(3), 2007.

[Lec18] Emmanuel Lecouturier. On the Galois structure of the class group of certain Kummer extensions. *J. Lond. Math. Soc. (2)*, 98(1):35–58, 2018.

[LJ87] Hendrik W Lenstra Jr. Factoring integers with elliptic curves. *Annals of mathematics*, pages 649–673, 1987.

[Maz77] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977. With an appendix by Mazur and M. Rapoport.

[MC90] Jacques Martinet and Henri Cohen. Étude heuristique des groupes de classes des corps de nombres. 1990.

[Mur97] V Kumar Murty. Modular forms and the Chebotarev density theorem II. *London Mathematical Society Lecture Note Series*, pages 287–308, 1997.

[PS21] Dipendra Prasad and Sudhanshu Shekhar. Relating the Tate–Shafarevich group of an elliptic curve with the class group. *Pacific Journal of Mathematics*, 312(1):203–218, 2021.

[RS23] Anwesh Ray and R. Sujatha. Arithmetic statistics for the fine Selmer group in Iwasawa theory. *Res. Number Theory*, 9(3):Paper No. 59, 25, 2023.

[Sch87] R. Schoof. Nonsingular plane cubic curves over finite fields. *Journal of combinatorial theory, Series A*, 46(2):183–211, 1987.

[Sil94] Joseph H Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151. Springer Science & Business Media, 1994.

[Sil09] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer, 2009.
[SS19] Karl Schaefer and Eric Stubley. Class groups of Kummer extensions via cup products in Galois cohomology. *Trans. Amer. Math. Soc.*, 372(10):6927–6980, 2019.

[Was97] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Wat04] Mark Watkins. Class numbers of imaginary quadratic fields. *Mathematics of Computation*, 73(246):907–938, 2004.

[Won99] Siman Wong. On the rank of ideal class groups. In *Proc. Fourth Canad. Number Theory Conf*, pages 377–383, 1999.

[Woo16] Melanie Matchett Wood. Asymptotics for number fields and class groups. In *Directions in number theory*, pages 291–339. Springer, 2016.

[WWE20] Preston Wake and Carl Wang-Erickson. The rank of Mazur’s Eisenstein ideal. *Duke Math. J.*, 169(1):31–115, 2020.

[Zyw15] David Zywina. On the possible images of the mod ℓ representations associated to elliptic curves over Q. *arXiv preprint arXiv:1508.07660*, 2015.

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