The relative Riemann–Hurwitz formula

Zhiguo Ding\textsuperscript{a} and Michael E. Zieve\textsuperscript{b}

\textsuperscript{a}Hunan Institute of Traffic Engineering, Hengyang, Hunan, China; \textsuperscript{b}Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA

\section*{ABSTRACT}
For any nonconstant $f, g \in \mathbb{C}(x)$ such that the numerator $H(x, y)$ of $f(x) - g(y)$ is irreducible, we compute the genus of the normalization of the curve $H(x, y) = 0$. We also prove an analogous formula in arbitrary characteristic when $f$ and $g$ have no common wildly ramified branch points, and generalize to (possibly reducible) fiber products of nonconstant morphisms of curves $f : A \rightarrow D$ and $g : B \rightarrow D$.

\section*{1. Introduction}
Mathematicians studying several different topics have been led to investigate the genus of algebraic curves of the form $f(x) = g(y)$. For instance, such considerations arise in algebraic topology \cite{12}, complex dynamics \cite{25}, value distribution of meromorphic functions \cite{1, 4, 16, 17, 24}, Diophantine equations \cite{2, 3, 5, 6, 9}, arithmetic dynamics \cite{7, 8}, Pell equations in polynomials \cite{2, 18}, functional equations in rational functions \cite{11, 19, 20, 26}, and coding theory \cite{21}. This has led many authors to compute the genus of some specific instance of such curves.

Besides specific instances, mathematicians have produced two general types of formulas for this genus. Ritt \cite{20} essentially determined the genus of irreducible curves of the form $f(x) = g(y)$ when $f(x)$ and $g(x)$ are nonconstant complex rational functions. His method uses the Riemann–Hurwitz formula for the projection of this curve onto the $x$-axis, and relies on computing the ramification in this projection. By examining singularities and using the Plücker formula, Kang \cite{15} proved a genus formula for irreducible curves of the form $f(x) = g(y)$ where $f(x)$ and $g(x)$ are nonconstant polynomials over an algebraically closed field of characteristic zero. The genus formulas obtained by Ritt and Kang have different forms, and are not obviously equivalent.

In this article, we prove new results generalizing these known formulas in several ways:

- We do not assume that $f(x) = g(y)$ is irreducible, but instead obtain a sum over its irreducible components.
- Our formulas are valid in positive characteristic, under some hypotheses about wild ramification.
- We do not require $f$ and $g$ to be polynomials (as did Kang) or rational functions (as did Ritt), but instead allow them to be arbitrary curve morphisms with a common target.
Our approach uses ramification, and in particular the computation of points in fiber products from [10], but it differs from Ritt’s approach in several ways. In the end, we obtain two formulas: one generalizing Ritt’s which is valid when at least one of $f$ and $g$ is tamely ramified, and one generalizing Kang’s in the more general situation where $f$ and $g$ do not have a common wildly ramified branch point.

Our main result uses the following standard notation and terminology, where $f : A \to D$ is a nonconstant morphism between smooth projective irreducible curves over an algebraically closed field $k$ of characteristic $p \geq 0$:

- $A(k)$ is the set of points on $A$ with coordinates in $k$;
- for any $P \in A(k)$, we write $e_f(P)$ for the ramification index of $P$ under $f$;
- we say that $P \in A(k)$ is tamely ramified (resp., wildly ramified) under $f$ if $p \nmid e_f(P)$ (resp., $p | e_f(P)$), and that $f$ is tamely ramified if every point in $A(k)$ is tamely ramified under $f$; in particular, if $p = 0$ then $f$ is tamely ramified;
- a branch point of $f$ is a point in $D(k)$ of the form $f(P)$ where $P \in A(k)$ satisfies $e_f(P) > 1$;
- a branch point of $f$ is wildly ramified if it has a wildly ramified $f$-preimage.

**Theorem 1.1.** Let $f : A \to D$ and $g : B \to D$ be nonconstant morphisms of smooth projective irreducible curves over an algebraically closed field $k$. Write $m$ and $n$ for the degrees of $f$ and $g$, and write $\varrho_A, \varrho_B, \varrho_D$ for the genera of $A$, $B$, $D$. Let $C_1, \ldots, C_r$ be the normalizations of the irreducible components of the fiber product of $f$ and $g$, and write $\varrho_i$ for the genus of $C_i$.

(1.1.1) If $f$ is tamely ramified then

$$
\sum_{i=1}^r (2\varrho_i - 2) = m(2\varrho_B - 2) + \sum_{R \in D(k)} \sum_{P \in f^{-1}(R)} (e_f(P) - \gcd(e_f(P), e_g(Q))).
$$

(1.1.2) If $f$ and $g$ have no common wildly ramified branch points then

$$
\sum_{i=1}^r (2\varrho_i - 2) = m(2\varrho_B - 2) + n(2\varrho_A - 2) - mn(2\varrho_D - 2) -
\sum_{R \in D(k)} \sum_{P \in f^{-1}(R)} (e_f(P) - 1) \cdot (e_g(Q) - 1) + \gcd(e_f(P), e_g(Q)) - 1).
$$

**Remark 1.2.** In case $g$ is the identity map on $D$, the formula in (1.1.1) is the classical Riemann–Hurwitz formula for tamely ramified morphisms. For more general $g$, the formulas in (1.1.1) and (1.1.2) may be viewed as “relative” Riemann–Hurwitz formulas.

**Remark 1.3.** The summations in the above result only have finitely many nonzero summands, since the summation in the right side of (1.1.1) is zero whenever $e_f(P) = 1$, and the summation in the right side of (1.1.2) is zero when either $e_f(P) = 1$ or $e_g(Q) = 1$.

In order to make it easy to apply Theorem 1.1, we now state some special cases of this result. We begin with the case of rational functions.

**Corollary 1.4.** Let $k$ be an algebraically closed field of characteristic $p \geq 0$, and let $f, g \in k(x) \setminus k$ have degrees $m$ and $n$, respectively. Let $C_1, \ldots, C_r$ be the normalizations of the irreducible components of $f(x) = g(y)$, and write $\varrho_i$ for the genus of $C_i$.

(1.4.1) Suppose that $p \nmid e_f(P)$ for each $P \in \mathbb{P}^1(k)$. Let $\Gamma$ be the set of pairs $(P, Q) \in \mathbb{P}^1(k) \times \mathbb{P}^1(k)$ such that $f(P) = g(Q)$ and $e_f(P) > 1$. Then $\Gamma$ is finite, say $\Gamma = \{(P_j, Q_j) : 1 \leq j \leq s\}$ with
s = |\Gamma|; writing \( m_j := e_f(P_j) \) and \( n_j := e_g(Q_j) \), we have
\[
\sum_{i=1}^{r} (2q_i - 2) = -2m + \sum_{j=1}^{s} \left( m_j - \gcd(m_j, n_j) \right).
\]

(1.4.2) Let \( \Lambda \) be the set of pairs \( (P, Q) \in \PP^1(k) \times \PP^1(k) \) such that \( f(P) = g(Q) \) and \( e_f(P) > 1 \) and \( e_g(Q) > 1 \). Suppose \( \Lambda \) does not contain a pair of integers divisible by \( p \). Then \( \Lambda \) is finite, say \( \Lambda = \{(P_j, Q_j) : 1 \leq j \leq t \} \) with \( t = |\Lambda| \); writing \( m_j := e_f(P_j) \) and \( n_j := e_g(Q_j) \), we have
\[
\sum_{i=1}^{r} (2q_i - 2) = -2 + 2(m - 1)(n - 1) - \sum_{j=1}^{t} \left( (m_j - 1)(n_j - 1) + \gcd(m_j, n_j) - 1 \right).
\]

Remark 1.5. For the benefit of those whose background is analysis rather than algebraic geometry, we note the following alternate description of the \( C_i \)'s in case \( k = \CC \): if we write the numerator of the bivariate rational function \( f(x) - g(y) \) as the product \( \prod_{i=1}^{r} H_i(x, y) \) of irreducible polynomials \( H_i(x, y) \in \CC[x, y] \), then \( C_i \) is the unique compact Riemann surface (up to conformal equivalence) for which there exist a finite subset \( S_i \) of \( C_i \), and a finite subset \( T_i \) of the set \( U_i \) of zeroes of \( H_i(x, y) \) in \( \CC^2 \), such that \( C_i \setminus S_i \) is biholomorphic to \( U_i \setminus T_i \).

We now state our results for polynomials, where we incorporate the ramification over \( \infty \) into the formulas.

Corollary 1.6. Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \), and let \( f, g \in k[x] \setminus k \) have degrees \( m \) and \( n \), respectively. Let \( C_1, \ldots, C_r \) be the normalizations of the irreducible components of \( f(x) = g(y) \), and write \( q_i \) for the genus of \( C_i \).

(1.6.1) Suppose \( p \nmid m \) and \( p \nmid e_f(x) \) for each \( x \in k \). Let \( \Gamma^* \) be the set of pairs \( (P, Q) \in k \times k \) such that \( f(P) = g(Q) \) and \( e_f(P) > 1 \). Then \( \Gamma^* \) is finite, say \( \Gamma^* = \{(P_j, Q_j) : 1 \leq j \leq s \} \) with \( s = |\Gamma^*| \); writing \( m_j := e_f(P_j) \) and \( n_j := e_g(Q_j) \), we have
\[
\sum_{i=1}^{r} (2q_i - 2) = -m + \gcd(m, n) + \sum_{j=1}^{s} \left( m_j - \gcd(m_j, n_j) \right).
\]

(1.6.2) Let \( \Lambda^* \) be the set of pairs \( (P, Q) \in k \times k \) such that \( f(P) = g(Q) \) and \( e_f(P) > 1 \) and \( e_g(Q) > 1 \). Suppose \( \Lambda^* \cup \{(m, n)\} \) does not contain a pair of integers divisible by \( p \). Then \( \Lambda^* \) is finite, say \( \Lambda^* = \{(P_j, Q_j) : 1 \leq j \leq t \} \) with \( t = |\Lambda^*| \); writing \( m_j := e_f(P_j) \) and \( n_j := e_g(Q_j) \), we have
\[
\sum_{i=1}^{r} (2q_i - 2) = -1 + (m - 1)(n - 1) - \gcd(m, n) - \sum_{j=1}^{t} \left( (m_j - 1)(n_j - 1) + \gcd(m_j, n_j) - 1 \right).
\]

Remark 1.7. The formula in (1.6.1) is not always valid if one only assumes the hypotheses of (1.6.2): for instance, if \( f(x) = x^8 - x \) and \( g(y) = y \) where \( p > 0 \) then the left side of the formula in (1.6.1) is \(-2\) but the right side is \(-p - 1\). However, the formulas in (1.4.1) and (1.6.1) are useful when they do apply, since often there is a point \( R \in k \) for which the pairs in \( f^{-1}(R) \times g^{-1}(R) \)
contribute enough to the right sides of these formulas in order to force the left sides to be large. For instance, special cases of these formulas are used in this way in [2, 3, 6, 8, 11, 19, 20].

Several authors have proven or stated special cases of these results. For instance:

- The case \( k = \mathbb{C} \), \( r = 1 \), and \( g_1 = 0 \) of (1.4.1) was shown by Ritt [20, §1], who had previously treated a special case [19, pp. 60–61]. Ritt’s proof immediately extends to the case that \( r = 1 \) and \( f(x) \) and \( g(x) \) are tamely ramified [13, Prop. 2].
- The case \( \text{char}(k) = 0 \) and \( r = 1 \) of (1.6.2) is the main result of [15]; a further specialization of this case is [18, Thm. 1.5].
- The case \( k = \mathbb{F}_2 \), \( f(x) = x^m \), and \( g(y) = y^n + y \) of (1.6.1) is the main result of [21].

Our formulas also yield much simpler proofs of various known results, such as [14, Thm. 1.1 and 1.3] and [24, Thm. 1 and 2]. We note that the former reference uses the second main theorem of Nevanlinna theory to prove certain results in case \( k = \mathbb{C} \), and our formulas immediately yield stronger results in this case which also extend to arbitrary \( k \).

This article is organized as follows. In the next section, we recall some known results which will be used in this article. We conclude in Section 3 with proofs of Theorem 1.1, Corollary 1.4, and Corollary 1.6.

2. Previous results

In this section, we list the previous results used in this paper. We use the following notation and conventions:

- \( k \) is an algebraically closed field of characteristic \( p \geq 0 \);
- all curves are assumed to be smooth, projective, and irreducible;
- if \( C \) is a curve over \( k \), then \( g_C \) denotes the genus of \( C \), and \( C(k) \) denotes the set of \( k \)-rational points on \( C \);
- if \( f : C \to D \) is a nonconstant morphism between curves over \( k \), and \( P \in C(k) \), then \( e_f(P) \) denotes the ramification index of \( P \); if in addition \( f \) is separable then \( d_f(P) \) denotes the different exponent of \( P \) under \( f \).

We begin with the Riemann–Hurwitz formula [23, Thm. 3.4.13]:

**Lemma 2.1.** If \( f : C \to D \) is a separable nonconstant morphism of curves over \( k \), then

\[
2g_C - 2 = \deg(f) \cdot (2g_D - 2) + \sum_{P \in C(k)} d_f(P).
\]

The ramification index [23, Def. 3.1.5] and different exponent [23, Def. 3.4.3] are related by Dedekind’s different theorem, e.g. cf. [22, Prop. III.13] or [23, Thm. 3.5.1]:

**Lemma 2.2.** If \( f : C \to D \) is a separable nonconstant morphism of curves over \( k \), then for any \( P \in C(k) \) we have \( d_f(P) \geq e_f(P) - 1 \), with equality holding if and only if \( P \) is tamely ramified under \( f \).

The next result describes the ramification index and different exponent in a tower of curves [23, Prop. 3.1.6 and Cor. 3.4.12]:

**Lemma 2.3.** Let \( f : C \to D \) and \( g : D \to E \) be nonconstant morphisms of curves over \( k \). Then for any \( P \in C(k) \) we have

\[
e_{g\circ f}(P) = e_f(P) \cdot e_g(f(P)).
\]
If in addition both \( f \) and \( g \) are separable then
\[
d_{g \circ f}(P) = e_f(P) \cdot d_g(f(P)) + d_f(P).
\]

We will use the following Fundamental Equality [23, Thm. 3.1.11]:

Lemma 2.4. If \( f : C \to D \) is a nonconstant morphism of curves over \( k \), then for any \( Q \in D(k) \) we have
\[
\deg(f) = \sum_{P \in f^{-1}(Q)} e_f(P).
\]

We will also use the following version of Abhyankar’s lemma [23, Thm. 3.9.1 and Prop. 3.10.2]:

Lemma 2.5. Let \( f : A \to D \) and \( g : B \to D \) be nonconstant morphisms of curves over \( k \), and let \( C \) be the normalization of a component of the fiber product of \( f \) and \( g \), with induced maps \( \phi : C \to A \) and \( \psi : C \to B \). For any \( S \in C(k) \) such that \( \phi(S) \) is tamely ramified under \( f \), we have
\[
e_{f \circ \phi}(S) = \text{lcm}(e_f(\phi(S)), e_g(\psi(S))).
\]

The next result was proved in [10]:

Lemma 2.6. Assume \( f : A \to D \) and \( g : B \to D \) are nonconstant morphisms of curves over \( k \). Let \( C_1, \ldots, C_r \) be the normalizations of the irreducible components of the fiber product of \( f \) and \( g \), with induced morphisms \( \phi_i : C_i \to A \) and \( \psi_i : C_i \to B \). Then for any pair \( (P,Q) \in A(k) \times B(k) \) such that \( f(P) = g(Q) \) and \( p \nmid \text{gcd}(e_f(P), e_g(Q)) \) we have
\[
\sum_{i=1}^{r} |\phi_i^{-1}(P) \cap \psi_i^{-1}(Q)| = \text{gcd}(e_f(P), e_g(Q)).
\]

3. Proofs of main results

Corollaries 1.4 and 1.6 follow immediately from Theorem 1.1, so we need only prove the latter result. Throughout this section, we assume the hypotheses and notation from Theorem 1.1. For \( i \in \{1, 2, \ldots, r\} \), let \( \phi_i : C_i \to A \) and \( \psi_i : C_i \to B \) be the maps arising from the fiber product. We begin by combining the Riemann–Hurwitz formula for the various maps \( \psi_i \) into a single formula. We express this combined formula in terms of a summation whose summands will turn out to be convenient for explicit computation in the setting of Theorem 1.1.

Lemma 3.1. If \( f \) is separable then
\[
\sum_{i=1}^{r} \left(2g_i - 2 \right) = m(2g_B - 2) + \sum_{i=1}^{r} \sum_{S \in C_i(k)} d_{\phi_i}(S)
\]
\[
= m(2g_B - 2) + n(2g_A - 2) - mn(2g_D - 2) - \sum_{i=1}^{r} \sum_{S \in C_i(k)} \left( e_{\phi_i}(S) \cdot d_f(\phi_i(S)) - d_{\phi_i}(S) \right). \tag{3.3}
\]

If both \( f \) and \( g \) are separable then each \( S \in C_i(k) \) satisfies
\[
e_{\phi_i}(S) \cdot d_f(\phi_i(S)) - d_{\phi_i}(S) = d_{f \circ \phi_i}(S) - d_{\phi_i}(S) - d_{\psi_i}(S), \tag{3.4}
\]
which yields the symmetric equation...
\[
\sum_{i=1}^{r} (2\vartheta_i - 2) = m(2\vartheta_B - 2) + n(2\vartheta_A - 2) - mn(2\vartheta_D - 2)
\]
\[
- \sum_{i=1}^{r} \sum_{S \in C_i(k)} \left( d_{f \circ \varphi_i}(S) - d_{\psi_i}(S) - d_{\phi_i}(S) \right).
\]

(3.5)

**Proof.** Since \( f \) is separable, also each \( \psi_i \) is separable. Riemann–Hurwitz for \( \psi_i \) says that
\[
2\vartheta_i - 2 = \deg(\psi_i) \cdot (2\vartheta_B - 2) + \sum_{S \in C_i(k)} d_{\psi_i}(S).
\]
By summing over all \( i \), and using the fact that \( \sum_{i=1}^{r} \deg(\psi_i) = m \), this yields (3.2). Riemann–Hurwitz for the map \( f : A \to D \) says that
\[
2\vartheta_A - 2 = m(2\vartheta_B - 2) + \sum_{P \in A(k)} d_f(P).
\]
(3.6)

For each \( P \in A(k) \), **Lemma 2.4** yields
\[
\sum_{i=1}^{r} \sum_{S \in \phi_i^{-1}(P)} e_{\phi_i}(S) = \sum_{i=1}^{r} \deg(\phi_i) = n.
\]
(3.7)

Multiply (3.6) by \( n \), and then substitute (3.7), to get
\[
n(2\vartheta_A - 2) - mn(2\vartheta_D - 2) = \sum_{P \in A(k)} n d_f(P)
\]
\[
= \sum_{P \in A(k)} \sum_{i=1}^{r} \sum_{S \in \phi_i^{-1}(P)} e_{\phi_i}(S) \cdot d_f(P)
\]
\[
= \sum_{i=1}^{r} \sum_{S \in C_i(k)} e_{\phi_i}(S) \cdot d_f(\phi_i(S)).
\]

Upon subtracting the right side of this equation from the left, and adding the difference to the right side of (3.2), we obtain (3.3).

If both \( f \) and \( g \) are separable then also each \( \phi_i \) is separable, so by **Lemma 2.3** we have
\[
e_{\phi_i}(S) \cdot d_f(\phi_i(S)) = d_{f \circ \varphi_i}(S) - d_{\phi_i}(S)
\]
for each \( S \in C_i(k) \). Subtracting \( d_{\phi_i}(S) \) from both sides yields (3.4), and combining (3.3) with (3.4) yields (3.5).

In light of **Lemma 3.1**, in order to prove **Theorem 1.1** we must compute the summands involving different exponents in the right sides of (3.2), (3.3) and (3.5). We do this in the next result, under suitable hypotheses about tame ramification.

**Lemma 3.8.** Pick \( S \in C_i(k) \) for some \( i \), and put \( P := \phi_i(S) \) and \( Q := \psi_i(S) \). If \( P \) is tamely ramified under \( f \) then
\[
d_{\phi_i}(S) = \frac{e_f(P)}{\gcd(e_f(P), e_f(Q))} - 1
\]
(3.9)

and
By (3.9) the right side equals
\[
gcd(e_f(P), e_g(Q)) \cdot \left( e_{\phi_i}(S) \cdot d_f(P) - d_{\psi_i}(S) \right)
= (e_f(P) - 1) \cdot (e_g(Q) - 1) + gcd(e_f(P), e_g(Q)) - 1. \tag{3.10}
\]

**Proof.** By Lemma 2.5 we have
\[
e_{f \circ \phi_i}(S) = \text{lcm}(e_f(P), e_g(Q)) = \frac{e_f(P) \cdot e_g(Q)}{gcd(e_f(P), e_g(Q))}.
\]
Lemma 2.3 yields
\[
e_{\phi_i}(S) \cdot e_f(P) = e_{f \circ \phi_i}(S) = e_{g \circ \psi_i}(S) = e_{\psi_i}(S) \cdot e_g(Q),
\]
so that
\[
e_{\phi_i}(S) = \frac{e_f(Q)}{gcd(e_f(P), e_g(Q))} \quad \text{and} \quad e_{\psi_i}(S) = \frac{e_f(P)}{gcd(e_f(P), e_g(Q))}.
\]
In particular, S is tamely ramified under $\psi_i$, so Lemma 2.2 implies that
\[
d_{\phi_i}(S) = e_{\phi_i}(S) - 1 = \frac{e_f(P)}{gcd(e_f(P), e_g(Q))} - 1,
\]
which is (3.9). Lemma 2.2 also gives $d_f(P) = e_f(P) - 1$, so that
\[
gcd(e_f(P), e_g(Q)) \cdot \left( e_{\phi_i}(S) \cdot d_f(P) - d_{\psi_i}(S) \right)
= e_g(Q) \cdot (e_f(P) - 1) - \left( e_f(P) - gcd(e_f(P), e_g(Q)) \right)
= (e_g(Q) - 1) \cdot (e_f(P) - 1) + gcd(e_f(P), e_g(Q)) - 1,
\]
which is (3.10). \qed

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** If f is tamely ramified then in particular f is separable, so that (3.2) yields
\[
\sum_{i=1}^{r} (2\delta_i - 2) = m(2\delta_B - 2) + \sum_{i=1}^{r} \sum_{S \in C_i(k)} d_{\phi_i}(S).
\]
By (3.9) the right side equals
\[
m(2\delta_B - 2) + \sum_{i=1}^{r} \sum_{S \in C_i(k)} \left( \frac{e_f(\phi_i(S))}{gcd(e_f(\phi_i(S)), e_g(\psi_i(S)))} - 1 \right)
= m(2\delta_B - 2) + \sum_{R \in D(k)} \sum_{P \in f^{-1}(R)} \sum_{Q \in e^{-1}(R)} \left( \frac{e_f(P)}{gcd(e_f(P), e_g(Q))} - 1 \right) \cdot N_{P,Q}
\]
where
\[
N_{P,Q} := \sum_{i=1}^{r} |\phi_i^{-1}(P) \cap \psi_i^{-1}(Q)|
\]
for all $P \in A(k)$ and $Q \in B(k)$ satisfying $f(P) = g(Q)$. Substituting the value $N_{P,Q} = gcd(e_f(P), e_g(Q))$ from Lemma 2.6 yields (1.1.1).
Likewise, if $g$ is tamely ramified then the combination of (3.3), (3.10) and Lemma 2.6 yields the conclusion of (1.1.2). Since the conclusion of (1.1.2) is symmetric in $f$ and $g$, it follows that this conclusion is also true if $g$ is tamely ramified.

It remains to prove (1.1.2) when neither $f$ nor $g$ is tamely ramified. Thus $f$ has a wildly ramified branch point, which by the hypothesis of (1.1.2) is not a wildly ramified branch point of $g$, so that $g$ is separable. Likewise $f$ is separable. For any $i$ and any $S \subseteq C_i(k)$, put $P := \phi_i(S)$ and $Q := \psi_i(S)$. Since $f(P) = g(Q)$ but $f$ and $g$ have no common wildly ramified branch points, at least one of $e_f(P)$ and $e_g(Q)$ is not divisible by $p$. If $p \nmid e_f(P)$ then (3.4) and (3.10) yield

$$d_{f,\phi_i}(S) - d_{\phi_i}(S) - d_{\psi_i}(S) = e_{\phi_i}(S) \cdot d_f(P) - d_{\psi_i}(S) = \frac{(e_f(P) - 1) \cdot (e_g(Q) - 1) + \gcd(e_f(P), e_g(Q)) - 1}{\gcd(e_f(P), e_g(Q))}.$$  

If $p \mid e_g(Q)$ then the same argument yields the same conclusion. Thus the above equation is true in any case. Combining it with (3.5) and Lemma 2.6 yields (1.1.2).

\[\square\]

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