THE CONTINUITY EQUATION OF THE GAUDUCHON METRICS

TAO ZHENG

Abstract. We study the continuity equation of the Gauduchon metrics and establish its interval of maximal existence, which extends the continuity equation of the Kähler metrics introduced by La Nave & Tian for and of the Hermitian metrics introduced by Sherman & Weinkove. Our method is based on the solution to the Gauduchon conjecture by Székelyhidi, Tosatti & Weinkove.

1. Introduction

Let \((M, J, g_0)\) be a closed (i.e., compact without boundary) Hermitian manifold with \(\dim_C M = n\) and the complex structure \(J\), where \(g_0\) is the Hermitian metric, i.e., a Riemannian metric with \(g_0(JX, JY) = g_0(X, Y)\) for all vector fields \(X, Y \in \mathfrak{X}(M)\) (set of all the global and smooth vector fields). Then we can define a real \((1,1)\) form \(\omega_0\) by

\[
\omega_0(X, Y) := g_0(JX, Y), \quad \forall X, Y \in \mathfrak{X}(M).
\]

This form \(\omega\) is determined uniquely by \(g\) and vice versa. In what follows, we will not distinguish these two terms and we will not emphasize the complex structure \(J\).

The Hermitian metric \(\omega_0\) is called Kähler if \(d\omega_0 = 0\), Astheno-Kähler (see [17, 18]) if \(\bar{\partial}\bar{\partial}\omega_0 = 0\), balanced (see [27]) if \(d\omega_0 = 0\), Gauduchon (see [11]) if \(\bar{\partial}\bar{\partial}\omega_0 = 0\), and strongly Gauduchon (see [29]) if \(\bar{\partial}\omega_0 = 0\) is \(\bar{\partial}\)-exact.

La Nave & Tian [20] (cf. [30]) investigate a family of Kähler metrics \(\omega := \omega(s)\) satisfying the continuity equation

\[
\omega = \omega_0 - sRic(\omega), \quad \text{for} \quad s \geq 0,
\]

where \(Ric(\omega) = -\sqrt{-1}\bar{\partial}\bar{\partial}\log \det(g_{ij})\) is the Ricci form of the Kähler metric \(\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j\) which is a real \((1,1)\) form. This continuity equation can be viewed as an alternative to the Kähler-Ricci flow in carrying out the Song-Tian analytic minimal model program [32, 33]. The Ricci curvature along the path is automatically bounded from below. This fact leads to several developments [10, 21, 25, 38, 46, 47].

Sherman & Weinkove [31] extend the continuity equation in [20] to Hermitian metrics and establish its interval of maximal existence and they also illustrate the behavior of this equation in the case of elliptic bundles over a curve of genus at least two. This equation is closely related to the Chern-Ricci flow first introduced by [14] and studied deeply by Tosatti and Weinkove (and Yang) [39, 38, 43] (see also [8, 15, 16, 22, 23, 28, 30, 45, 49]).

Li & Zheng [24] study the continuity equation of a family of almost Hermitian metrics and establish its interval of maximal existence. This continuity equation is closely related to the almost Chern-Ricci flow introduced by Chu, Tosatti & Weinkove [4] and furthermore studied by [31, 50].

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In this paper we investigate a natural analogue of (1.1) for the Gauduchon metrics, i.e., the continuity equation given by

\[
\omega_n^{n-1} := \omega_0^{n-1} - s(n-1) \left( \text{Ric}(\omega) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1} \partial \log \frac{\omega}{\bar{\omega}^{n-2}} \right) \right) > 0, \quad \text{for } s \geq 0,
\]

where \( \alpha \) is the Gauduchon metric and \( \text{Ric}(\omega) \) is the Chern-Ricci form of \( \omega \) given by (2.10) which coincides exactly with the Ricci form of \( \omega \) when \( d\omega = 0 \). If \( \omega_0 \) is the Gauduchon metric, then so is \( \omega \) given by (1.2).

When \( n = 2 \), the continuity equation (1.2) is the same as the one in Sherman & Weinkove [31] in the Hermitian case and the one in La Nave & Tian [20] in the Kähler case.

**Theorem 1.1.** Let \((M, \omega_0)\) be a closed Hermitian manifold with \( \dim_{\mathbb{C}} M = n \) and \( \omega_0 \) a Hermitian metric. Then there exists a unique family of Hermitian metrics \( \omega = \omega(s) \) satisfying (1.2) for each \( s \in [0, T) \), where \( T \) is defined by

\[
T := \sup \left\{ s \geq 0 : \exists \varphi \in C^\infty(M, \mathbb{R}) \text{ such that } \Phi_s + \sqrt{-1} \partial \bar{\partial} \varphi \wedge \alpha^{n-2} + \Re \left[ \sqrt{-1} \partial \bar{\partial} \varphi \wedge \bar{\partial}(\alpha^{n-2}) \right] > 0 \right\},
\]

with

\[
\Phi_s := \omega_0^{n-1} - s(n-1) \left( \text{Ric}(\omega_0) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1} \partial \log \frac{\omega_0}{\bar{\omega}^{n-2}} \right) \right).
\]

When \( n = 2 \), Theorem 1.1 is proved by Sherman & Weinkove [31] in the Hermitian case and by La Nave & Tian [20] in the Kähler case.

The outline of the paper is as follows. In Section 2 we establish preliminaries about Hermitian geometry for later use. In Section 3 we prove Theorems 1.1.

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### 2. Preliminaries

In this section, we collect some basic materials about Hermitian geometry (see for example [39, 51]). Let \((M, J, g)\) be a Hermitian manifold with \( \dim_{\mathbb{C}} M = n \), where \( J \) is a complex structure and \( g \) is the Hermitian metric respect to \( J \), i.e., a Riemannian metric with \( g(JX, JY) = g(X, Y) \) for any vector fields \( X, Y \in \mathfrak{X}(M) \). Then can define a real \((1,1)\) form \( \omega \) by

\[
\omega(X, Y) := g(JX, Y), \quad \forall \ X, Y \in \mathfrak{X}(M).
\]

This form is determined uniquely by \( g \) and vice versa. The Chern connection of \( g \), denoted by \( \nabla \), is the unique connection determined by \( \nabla g = \nabla J = 0 \). The torsion of \( \nabla \) is defined by

\[
T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].
\]

The curvature of \( \nabla \) is defined by

\[
R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

with

\[
R(X, Y, Z, W) := g(R(X, Y)Z, W), \quad \forall \ W, X, Y, Z \in \mathfrak{X}(M).
\]

The Chern-Ricci curvature \( \text{Ric} \) is defined by

\[
\text{Ric}(X, Y) := \text{trace of the map } Z \mapsto R(X, Y)Z.
\]
Then in the real local coordinate \( x = (x^1, \cdots, x^{2n}) \) with

\[
J \left( \partial / \partial x^i \right) = \partial / \partial x^{n+i}, \quad J \left( \partial / \partial x^{n+i} \right) = -\partial / \partial x^i, \quad i = 1, \cdots, n,
\]
we have

\[
g_{ij} = g_{n+i,n+j}, \quad g_{i,n+j} = -g_{n+i,j}, \quad g_{\alpha \beta} = g_{\beta \alpha}, \quad i, j = 1, \cdots, n, \quad \alpha, \beta = 1, \cdots, 2n,
\]
where \( g_{\alpha \beta} = g \left( \partial / \partial x^\alpha, \partial / \partial x^\beta \right) \). Hence the complex local coordinate is given by

\[
z = (z^1, \cdots, z^n) = (x^1 + \sqrt{-1}x^{n+1}, \cdots, x^n + \sqrt{-1}x^{2n}).
\]

We use the notation \( \partial_i = \partial / \partial z^i, \quad \partial_j = \partial / \partial \overline{z}^j, \quad i, j = 1, \cdots, n \). One can infer that

\[
g = \sum_{i,j=1}^{n} g_{ij} \left( dz^i \otimes d\overline{z}^j + d\overline{z}^j \otimes dz^i \right),
\]

(2.2)

\[
\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\overline{z}^j,
\]

where \( g_{ij} = \frac{1}{2} \left( g_{i,j} + \sqrt{-1}g_{i,n+j} \right) \).

For each \((p, q)\) form

\[
\phi = \frac{1}{p!q!} \phi_{i_1 \cdots i_p j_1 \cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q},
\]

the Hodge star operator * with respect to the volume form \( \omega^n \) is given by (see for example [26])

(2.3)

\[
*\phi = \frac{(\sqrt{-1})^n(-1)^{np+\frac{(n-1)}{2}} \det g_{i_1 \cdots j_1} \cdots g_{i_p j_q}}{(n-p)! (n-q)! p!q!} \delta^{\overline{1} \cdots \overline{n}}_{j_1 \cdots j_q} \cdots g_{i_1 \overline{1} \cdots \overline{1}} \cdots g_{i_p \overline{q} \cdots \overline{q}} \delta_{\overline{1} \cdots \overline{n}}^{\overline{1} \cdots \overline{n}} d\overline{z}^{a_1} \wedge \cdots \wedge d\overline{z}^{a_n-q} \wedge d\overline{z}^{b_1} \wedge \cdots \wedge d\overline{z}^{b_{n-p}}.
\]

A direct calculation yields that

\[
*1 = \frac{\omega^n}{n!}, \quad *\phi = *\overline{\phi},
\]

where the second equality shows that * is a real operator.

It follows from (2.3) that

(2.4)

\[* \phi = (-1)^{p+q} \phi.
\]

Let us recall the concepts of positivity in for example [5] Chapter III].

A \((p, p)\) form \( \varphi \) is said to be positive if for any \((1, 0)\) forms \( \gamma_j, 1 \leq j \leq n-p \), then

\[
\varphi \wedge \sqrt{-1} \gamma_1 \wedge \gamma_1 \wedge \cdots \wedge \sqrt{-1} \gamma_{n-p} \wedge \gamma_{n-p}
\]

is a positive \((n, n)\) form. Any positive \((p, p)\) form \( \varphi \) is real, i.e., \( \overline{\varphi} = \varphi \). In particular, a real \((1, 1)\) form given by

(2.5)

\[
\varphi = \sqrt{-1} \phi \overline{d\overline{z}^j} \wedge d\overline{z}^j
\]

is positive if and only if \((\phi_{ij}) \) is a semi-positive Hermitian matrix and we denote \( \det \phi := \det(\phi_{ij}) \).

A real \((n-1, n-1)\) form given by

(2.6)

\[
\psi = (\sqrt{-1})^{n-1} \sum_{i,j=1}^{n} (-1)^{n(n+1)/2+i+j+1} \psi_{ij} \overline{d\overline{z}^i} \wedge \cdots \wedge \overline{d\overline{z}^i} \wedge d\overline{z}^j \wedge \cdots \wedge d\overline{z}^n
\]

\[
= (\sqrt{-1})^{n-1} \sum_{i,j=1}^{n} (-1)^n \psi_{ij} \overline{d\overline{z}^i} \wedge \cdots \wedge \overline{d\overline{z}^i} \wedge d\overline{z}^j \wedge \cdots \wedge d\overline{z}^n
\]
is positive if and only if $(\psi^T)$ is a semi-positive Hermitian matrix and we denote $\det \psi := \det(\psi^T)$. We remark that one can call a real $(1,1)$ form $\phi$ (resp. a real $(n-1,n-1)$ form $\psi$) strictly positive if the Hermitian matrix $(\phi_{\bar{\gamma}\bar{\alpha}})$ (resp. $(\psi_{\bar{\gamma}\bar{\alpha}})$) is positive definite.

For a strictly positive $(1,1)$ form $\phi$ defined as in $(2.5)$, we can deduce a strictly positive $(n-1,n-1)$ form

$$\frac{\phi^{n-1}}{(n-1)!} = (\sqrt{-1})^{n-1} \sum_{k,\ell=1}^{n} (-1)^{\frac{n(n+1)}{2}+k+\ell+1}\det(\phi_{\bar{\gamma}\bar{\alpha}})\phi^{\bar{\gamma}\bar{\alpha}} \wedge \cdots \wedge \wedge d\bar{z}^k \wedge \cdots \wedge d\bar{z}^n \wedge \cdots \wedge \wedge d\bar{z}^n$$

where $(\phi^{\bar{\gamma}\bar{\alpha}})$ is the inverse matrix of $(\phi_{\bar{\gamma}\bar{\alpha}})$, i.e., $\sum_{\ell=1}^{n} \phi^{\bar{\gamma}\bar{\alpha}} \phi_{\bar{\gamma}\bar{\alpha}} = \delta^\gamma_k$. Hence we have

$$\det \left( \frac{\phi^{n-1}}{(n-1)!} \right) = (\det \phi)^{n-1}.$$

Hence, if $\xi$ is another real $(1,1)$ form with $\det \xi \neq 0$, then we can deduce

$$\frac{\det (\ast \phi)}{\det (\ast \xi)} = \frac{\det \phi}{\det \xi}.$$

The Christoffel symbols of $\nabla$ are denoted by $\Gamma^\alpha_{\beta\gamma} := \partial_\beta \partial_\gamma (\nabla_\alpha)$, $\alpha, \beta, \gamma \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, where we use the notation that $d\bar{z}^i = d\bar{z}^i$ with $1 \leq i \leq n$. The non-zero components of the Christoffel symbols of $\nabla$ are

$$\Gamma^k_{ij} = g^{k\ell} \partial_i g_{\ell j}, \quad \Gamma^k_{ij} = \Gamma^k_{ji}, \quad 1 \leq i, j, k \leq n.$$

We also use the notations

$$T^\alpha_{\beta\gamma} := \partial_\gamma (T(\partial_\beta, \partial_\gamma)), \quad R_{\alpha\beta\gamma}^{\delta} := \partial_\delta (R(\partial_\alpha, \partial_\beta, \partial_\gamma)), \quad \alpha, \beta, \gamma, \delta \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}.$$

Then one infers

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}, \quad R_{ijk}^p = -\partial_j R_{ik}^p, \quad R_{ijk\ell} := R_{ijk}^p g_{p\ell}.$$

The Chern-Ricci form is defined by

$$\text{Ric}(\omega) := \sqrt{-1}R_{ij}^p dz^i \wedge d\bar{z}^j = -\sqrt{-1}d\bar{\partial} \log \det (g_{ij}).$$

3. **Proof of the Main Theorem**

In order to prove Theorem 1.1 we need reduce the equation $(1.2)$ to a complex Monge-Ampère type equation on $M$. For each $\hat{T} \in (0, T)$, the definition of $T$ yields that there is a smooth function $\varphi$ such that

$$\Phi_{\hat{T}} + \sqrt{-1}\partial \bar{\partial} \varphi \wedge \alpha^{n-2} + \Re \left[ \sqrt{-1}\partial \varphi \wedge \bar{\partial} \alpha^{n-2} \right] > 0,$$

with

$$\Phi_{\hat{T}} := \omega^{n-1}_0 - \hat{T}(n-1) \left( \text{Ric}(\omega_0) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1}\partial \log \frac{\omega_0^n}{\alpha^n} \wedge \bar{\partial} \alpha^{n-2} \right) \right).$$

Let $\Omega$ be the volume form given by

$$\Omega = \omega^{n}_0 e^{\frac{\varphi}{(n-1)T}}.$$ 

Then it follows from $(2.10)$ that

$$\omega^{n-1}_0 - \hat{T}(n-1) \left( \text{Ric}(\Omega) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1}\partial \log \frac{\Omega}{\alpha^n} \wedge \bar{\partial} \alpha^{n-2} \right) \right) > 0.$$
The convexity of the space of Hermitian matrices (cf. (2.6)) yields that
\[ \hat{\omega}_s^{n-1} := \omega_0^{n-1} - s(n - 1) \left( \text{Ric}(\Omega) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1} \partial \log \frac{\omega}{\alpha^n} \wedge \bar{\partial}(\alpha^{n-2}) \right) \right) > 0, \quad \forall \, s \in [0, T]. \]

Proposition 3.1. Let \((M, \omega_0)\) be a closed Hermitian manifold with \(\dim \mathbb{C} M = n\) and \(\omega_0\) a Hermitian metric. Then for \(s \in [0, T]\) fixed, there exists a Gauduchon metric \(\omega\) satisfying (1.2) if and only if there exists a smooth function \(u \in C^\infty(M, \mathbb{R})\) satisfying
\[
(3.2) \quad \log \frac{\det \omega^{n-1}}{e^\frac{\omega}{T} \det \omega_0^{n-1}} - u = 0, \]
with
\[
(3.3) \quad \omega^{n-1} := \hat{\omega}_s^{n-1} + s \left( \sqrt{-1} \partial \partial u \wedge \alpha^{n-2} + \Re \left( \sqrt{-1} \partial \partial u \wedge \bar{\partial}(\alpha^{n-2}) \right) \right) > 0. \]

Proof. For the ‘if’ direction, we suppose that \(\omega := \omega(s)\) satisfying (1.2). We define \(u\) by
\[ u = \log \frac{\det \omega^{n-1}}{e^\frac{\omega}{T} \det \omega_0^{n-1}}. \]
Then it follows from (2.10) and (3.1) that
\[
(n - 1)\text{Ric}(\omega) - (n - 1)\text{Ric}(\Omega) = - (n - 1)\sqrt{-1} \partial \partial \log \frac{\omega}{\omega_0} + \frac{1}{T} \sqrt{-1} \partial \bar{\partial} \phi = - \sqrt{-1} \partial \partial \log \frac{\det \omega^{n-1}}{e^\frac{\omega}{T} \det \omega_0^{n-1}} - \sqrt{-1} \partial \partial u, \]
which yields that
\[
\omega^{n-1} := \omega_0^{n-1} - s(n - 1) \left( \text{Ric}(\omega) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1} \partial \log \frac{\omega}{\alpha^n} \wedge \bar{\partial}(\alpha^{n-2}) \right) \right) = \omega_0^{n-1} - s(n - 1) \left( \text{Ric}(\Omega) \wedge \alpha^{n-2} - \Re \left( \sqrt{-1} \partial \log \frac{\omega}{\alpha^n} \wedge \bar{\partial}(\alpha^{n-2}) \right) \right) + s \sqrt{-1} \partial \partial u \wedge \alpha^{n-2} + s \Re \left( \sqrt{-1} \partial \partial u \wedge \bar{\partial}(\alpha^{n-2}) \right), \]
as desired.

For the ‘only if’ direction, if \(u\) satisfies (3.2)-(3.3), then a direct calculation, together with (2.10), yields that \(\omega\) satisfies (1.2). \hfill \Box

An immediate consequence of Proposition 3.1 is the uniqueness of solutions to the continuity equation (1.2).

Corollary 3.2. Let \((M, \omega_0)\) be a closed Hermitian manifold with \(\dim \mathbb{C} M = n\) and \(\omega_0\) a Hermitian metric. Then if \(\omega'\) and \(\omega\) are two almost Hermitian metrics solving the continuity equation (1.2) for the same \(s\) in \([0, T]\), then \(\omega' = \omega\).

Proof. For \(s = 0\), there is nothing to prove. For \(s \in (0, T)\), it suffices to prove the uniqueness of solutions to the equation (3.2) - (3.3) by Proposition 3.1. We assume that both \(u\) and \(u'\) are the solutions to (3.2) - (3.3). We set \(\theta := u' - u\). Then it follows from (3.2) that
\[
\log \frac{\det \left( \omega_0^{n-1} + s \sqrt{-1} \partial \partial \theta \wedge \alpha^{n-2} + s \Re \left( \sqrt{-1} \partial \theta \wedge \bar{\partial}(\alpha^{n-2}) \right) \right)}{\det \omega_0^{n-1}} = \theta, \]
where
\[ \omega_u^{n-1} := \omega_s^{n-1} + s\sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + s\Re (\sqrt{-1}\partial\bar{\partial}(\alpha^{n-2})) \].

Since at the point where \( \theta \) attains its maximum (resp. minimum) one has
\[ d\theta = 0, \quad \sqrt{-1}\partial\bar{\partial}\theta \leq 0 \quad (\text{resp.} \quad \geq 0), \]
we can deduce that \( \theta \equiv 0 \), as desired. \( \square \)

Note that (3.2)-(3.3) is trivially solved when \( s = 0 \) by taking
\[ u = \log \frac{\det \omega_0^{n-1}}{e^\tau \det \omega_0^{n-1}}. \]

Fix \( s \in (0, \hat{T}] \). We define a new function \( \psi = su \) and a function
\[ G = \log e^{\phi} \det \omega_n^{n-1} \]
Then the equation (3.2)-(3.3) becomes
\[ \log \frac{\det \omega_1^{n-1}}{\det \alpha^{n-1}} = \frac{\psi}{s} + G, \]
with
\[ \omega_1^{n-1} := \omega_s^{n-1} + (\sqrt{-1}\partial\bar{\partial}\psi \wedge \alpha^{n-2} + \Re (\sqrt{-1}\partial\bar{\partial}(\alpha^{n-2}))) > 0. \]

It follows from (2.4) and (2.9) that
\[ \log \frac{\det \omega_1^{n-1}}{\det \alpha^{n-1}} = \log \frac{\det (\omega_1^{n-1}/(n-1)!) - \det (\alpha^{n-1}/(n-1)!)}{\det (\alpha^{n-1}/(n-1)!)} = \log \frac{\det (\omega_1^{n-1}/(n-1)!) - \det (\alpha^{n-1}/(n-1)!)}{\det (\alpha^{n-1}/(n-1)!)} \]

Here and henceforth, \( * \) is the Hodge star operator with respect to \( \omega_n \).

Recall that \( s \) here is fixed. Then Theorem 1.1 follows from Propositions 3.1, (3.2), (3.3), (3.4), (3.5), (3.6) and the following result.

**Theorem 3.3.** Let \( (M, \omega_h) \) be a closed almost Hermitian manifold with \( \dim \mathbb{C} M = n \) and \( \omega_h \) a Hermitian metric. Then for \( G \in C^\infty(M, \mathbb{R}) \) and \( \lambda > 0 \) a constant, there exists a unique solution \( \varphi \) to the equation
\[ \log \left( \frac{\omega + \frac{1}{(n-1)!} [(\Delta \varphi)\alpha - \sqrt{-1}\partial\bar{\partial}\varphi] + Z(d\varphi)}{\alpha^n} \right)^n = \lambda \varphi + G, \]
where \( \omega = \frac{1}{(n-1)!} * \omega_h^{n-1} \), \( \Delta \varphi = \alpha^{ji} \partial_i \partial_j \varphi \),
\[ \tilde{\omega} := \omega + \frac{1}{n-1} [(\Delta \varphi)\alpha - \sqrt{-1}\partial\bar{\partial}\varphi] + Z(d\varphi) =: \sqrt{-1} \tilde{g}_{ij} dz^i \wedge d\bar{z}^j > 0, \]
and
\[ Z(d\varphi) = \frac{1}{(n-1)!} * \Re (\sqrt{-1}\partial\bar{\partial} \varphi \wedge \bar{\partial}(\alpha^{n-2})). \]

**Proof.** For \( \lambda = 0 \), Equation (3.7) is solved by \[ (3.5) \] (cf. [51]). Here we use the method modified from [35] (cf. [9]).
We use the method of continuity to solve (3.7). We study a family of equations for \( \varphi := \varphi(t) \in C^{2,\gamma}(M, \mathbb{R}) \) for some \( \gamma \in (0, 1) \) fixed

\[
\log \left( \varpi + \frac{1}{n-1} \left[ (\Delta \varphi)\alpha - \sqrt{-1} \partial \bar{\partial} \varphi \right] + Z(d\varphi) \right)^n = \lambda \varphi + (1 - t)G_0 + tG,
\]

where \( G_0 := \log \frac{\varpi^n}{\pi^n} \). We set

\[
\mathcal{T} := \{ t \in [0, 1] : \text{there exists } \varphi \in C^{2,\gamma}(M, \mathbb{R}) \text{ solves (3.11)} \}.
\]

The definition of \( G_0 \) yields that \( 0 \in \mathcal{T} \) since we can take \( \varphi(0) = 0 \). It suffices to show that \( \mathcal{T} \) is both open and closed.

For the openness of \( \mathcal{T} \), we consider the map

\[
\Psi : [0, 1] \times C^{2,\gamma}(M, \mathbb{R}) \to C^\gamma(M, \mathbb{R}),
\]

\[
(t, \varphi) \mapsto \left( \varpi + \frac{1}{n-1} \left[ (\Delta \varphi)\alpha - \sqrt{-1} \partial \bar{\partial} \varphi \right] + Z(d\varphi) \right)^n - \lambda \varphi - (1 - t)G_0 - tG.
\]

Assume \( t_0 \in \mathcal{T} \), and that (3.10) has a corresponding solution \( \varphi_0 \). Write

\[
\tilde{\varpi}_0 := \varpi + \frac{1}{n-1} \left[ (\Delta \varphi_0)\alpha - \sqrt{-1} \partial \bar{\partial} \varphi_0 \right] + Z(d\varphi_0) =: \sqrt{-1} h_{\bar{z}j} dz^i \wedge d\bar{z}^j > 0.
\]

The derivative of \( \Psi \) in the second variable at \( (t_0, \varphi_0) \) is the linear operator \( B : C^{2,\gamma}(M, \mathbb{R}) \to C^\gamma(M, \mathbb{R}) \) given by

\[
B(u) = P(u) - \lambda u,
\]

where

\[
P(u) := \Theta^{ji} \partial_j \partial_i u + h^{ji} \left( Z_{ij}^0 \partial_\nu u + Z_{ji}^0 \partial_\nu u \right),
\]

with

\[
\Theta^{ji} = \frac{1}{n-1} \left( (\text{tr}_\omega) \alpha \alpha^{ji} - h^{ji} \right).
\]

Since \( (\Theta^{ji}) > 0 \), both \( P \) and \( B \) are strictly elliptic operators.

We define an operator \( B_0 \) by

\[
B_0 : C^{2,\gamma}(M, \mathbb{R}) \to C^\gamma(M, \mathbb{R}), \quad B_0(u) = \Delta_\Theta u - \lambda u,
\]

where \( \Delta_\Theta \) is the Laplace-Beltrami operator of the Riemannian metric \( \Theta \). Since \( \lambda > 0 \), it follows from [1] Theorem 4.18] that \( B_0 \) is an isomorphism map.

Let us consider a family of strictly elliptic operators \( B_t := (1 - t)B_0 + tB \) for \( t \in [0, 1] \). For each \( u \in C^{2,\gamma}(M, \mathbb{R}) \), the Schauder theory (see for example [13]) yields that

\[
\|u\|_{C^{2,\gamma}(M, \mathbb{R})} \leq C \left( \sup_M |B_t(u)| + \|B_t(u)\|_{C^\gamma(M, \mathbb{R})} \right)
\]

for some uniform constant \( C > 0 \) independent of \( t \). Since \( B_0 \) is an isomorphism map, it follows from (3.12) and [13] Theorem 5.2 that \( B_t \) is isomorphism for each \( t \in [0, 1] \) and so is \( B \), which, together with the Inverse Function Theorem, yields that \( \mathcal{T} \) is open.

For the closeness of \( \mathcal{T} \), we need a priori estimates on \( \varphi \) solving (3.10) independent of \( t \).

A uniform bound

\[
\sup_M |\varphi| \leq C
\]
follows immediately from the maximum principle. Here and henceforth, \( C \) will denote a uniform constant independent of \( t \) that may change from line to line.

We denote \( h := \lambda \varphi + (1 - t)G_0 + tG \).

**For the second order estimate**, we claim

\[
(3.14) \quad \sup_M |\partial \partial \varphi|_\alpha \leq CK,
\]

with \( K := 1 + \sup_M |\partial \varphi|^2_\alpha \).

We will deduce the estimate (3.14) by making use of \([35]\) (cf.\([34, 9]\)). The main difference is that when we apply covariant derivative to both sides of (3.7), \( \nabla_i h \) is bounded by \( O(1 + |\partial \varphi|_\alpha) \) rather than \( O(1) \), and \( \nabla_i \nabla_i h \) is bounded by \( O(\lambda_1) \) rather than \( O(1) \), which are harmless in the following arguments. For the sake of completeness, we include here a brief sketch of the proof from \([9, 35]\) motivated by \([41]\).

We need some preliminaries. For each real \((1, 1)\) form \( \xi \), we define

\[
P_\alpha(\xi) := \frac{1}{n - 1} \left( (\text{tr}_\alpha \xi) \alpha - \xi \right) = \frac{1}{(n - 1)!} \ast (\xi \wedge \alpha^{n-2}).
\]

A direct calculation yields that \( \text{tr}_\alpha \xi = \text{tr}_\alpha (P_\alpha(\xi)) \) and that

\[
\xi = (\text{tr}_\alpha (P_\alpha(\xi))) \alpha - (n - 1) P_\alpha(\xi).
\]

We set

\[
\chi_{\overline{\ell}} = (\text{tr}_\alpha \omega)(\xi_{\overline{\ell}}) - (n - 1) \omega_{\overline{\ell}},
\]

with \( P_\alpha(\chi) = \omega \), and

\[
W_{\overline{\ell}}(d\varphi) = (\text{tr}_\alpha Z(d\varphi)) \alpha_{\overline{\ell}} - (n - 1) Z_{\overline{\ell}}(d\varphi) =: W_{\overline{\ell}}^{p} \varphi_{p} + \overline{W_{\overline{\ell}}^{p} \varphi_{p}}.
\]

Note that

\[
Z(d\varphi) := P_\omega(W(d\varphi)) = \frac{1}{n - 1} \left( (\text{tr}_{\omega} W(d\varphi))\omega - W(d\varphi) \right).
\]

Then we set

\[
(3.16) \quad g_{\overline{\ell}} = \chi_{\overline{\ell}} + \varphi_{\overline{\ell}} + W_{\overline{\ell}}(d\varphi)
\]

with \( P_\alpha(g_{\overline{\ell}}) = \tilde{g}_{\overline{\ell}} \).

In orthonormal coordinates for \( \alpha \) at each given point, it follows that the component \( Z_{\overline{\ell}} \) is independent of \( \varphi_{\overline{\ell}} \) and \( u_{\overline{\ell}} \), and that \( \nabla_{\overline{\ell}} Z_{\overline{\ell}} \) is independent of \( \varphi_{\overline{\ell}}, \varphi_{\overline{\ell}}, \varphi_{\overline{\ell}}, \nabla_{\overline{\ell}} \varphi_{\overline{\ell}} \) (see \([35]\)).

Moreover, \( \nabla_{\overline{\ell}} \nabla_{\overline{\ell}} Z_{\overline{\ell}} \) is independent of \( \nabla_{\overline{\ell}} \varphi_{\overline{\ell}}, \nabla_{\overline{\ell}} \varphi_{\overline{\ell}}, \nabla_{\overline{\ell}} \varphi_{\overline{\ell}}, \nabla_{\overline{\ell}} \varphi_{\overline{\ell}} \). For each index \( p \), \( \nabla_{\overline{\ell}} Z_{\overline{\ell}} \) is independent of \( \nabla_{\overline{\ell}} \varphi_{\overline{\ell}} \). In what follows, we will use these properties directly and do not prove them again.

Denote \( \left( \tilde{B}_{i}^{j} \right) = (\tilde{g}_{\overline{\ell}} \alpha^{\overline{\ell}j}) \) which can be viewed as the endomorphism of \( T^{1,0}M \). This endomorphism is Hermitian with respect to the Hermitian metric \( \alpha \). We define

\[
\tilde{F}(\tilde{B}) = \log \det \tilde{B} = \log(\mu_1 \cdots \mu_n) =: \tilde{f}(\mu_1, \cdots, \mu_n),
\]

where \( \mu_1, \cdots, \mu_n \) denote the eigenvalues of \( \left( \tilde{B}_{i}^{j} \right) \). Then \( (3.7) \) can be rewritten as

\[
(3.17) \quad \tilde{F}(\tilde{B}) = h.
\]

For \( \tilde{f} \) and \( h \), there holds
(i) $\tilde{f}$ is defined on
\[ \Gamma_n = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, \ 1 \leq i \leq n \right\}. \]

(ii) $\tilde{f}$ is symmetric, smooth, concave and increasing, i.e., $\tilde{f}_i > 0$ for $1 \leq i \leq n$.

(iii) $\sup_{\partial \Gamma_n} \tilde{f} \leq \inf_M h$.

(iv) For each $\mu \in \Gamma_n$, we get $\lim_{t \to \infty} \tilde{f}(t\mu) = \sup_{\Gamma_n} \tilde{f} = \infty$.

(v) $h$ is bounded on $M$ because of the uniform estimate $\sup_M |\varphi| \leq C$.

We also define
\[ F(A) := \tilde{F}(\tilde{B}) =: f(\lambda_1, \cdots, \lambda_n), \]
where $(A^{\alpha \beta}) = \left( g^{\alpha \beta} \right)$, which is also an endomorphism of $T^{1,0}M$ with respect to the Hermitian metric $\alpha$, and let $\lambda_1, \cdots, \lambda_n$ denote its eigenvalues. There exists a map
\[ P : \mathbb{R}^n \to \mathbb{R}^n, \quad \mu_k = \frac{1}{n-1} \sum_{i \neq k} \lambda_i, \]
induced by $P_\alpha$ above. Then we have
\[ f(\lambda_1, \cdots, \lambda_n) = \tilde{f} \circ P(\lambda_1, \cdots, \lambda_n) \]
defined on $\Gamma := P^{-1}(\Gamma_n)$. Note that $f$ satisfies the same conditions as $\tilde{f}$. Then (3.17) can also be rewritten as
\[ F(A) = h. \]

Since
\[ \tilde{f}_i = \frac{1}{\mu_i}, \quad 1 \leq i \leq n, \]
we can get
\[ f_i = \frac{1}{n-1} \sum_{k \neq i} \frac{1}{\mu_k}, \quad 1 \leq i \leq n. \]

A direct calculation shows that
\[ \mu_j = \frac{1}{n-1} \sum_{k \neq j} \lambda_k, \quad \lambda_j = \sum_{k=1}^n \mu_k - (n-1)\mu_j. \]

Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \Gamma$. From the definition of $P$, (3.20) and (3.21), we have
\[ 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n, \]
\[ \tilde{f}_1 \geq \tilde{f}_2 \geq \cdots \geq \tilde{f}_n, \]
\[ 0 < f_1 \leq f_2 \leq \cdots \leq f_n, \]
\[ \sum_{k=1}^n \lambda_k f_k = \sum_{k=1}^n \mu_k \tilde{f}_k = n. \]

where we also use the fact that $(\tilde{B}^{\alpha \beta})$ is positive definite. For $k \geq 2$, we have
\[ 0 < \frac{\tilde{f}_1}{n-1} \leq f_k \leq \tilde{f}_1. \]
and
\[ \tilde{f}_k \leq (n-1)f_1, \quad k > 1. \]

We have (see \[34, 35\])

**Proposition 3.4.** For each \( x \in M \), choose orthonormal coordinates for \( \alpha \) at \( x \), with \( g \) defined as in \((3.16)\) is diagonal with eigenvalues \( (\lambda_1, \cdots, \lambda_n) \). Then there exist uniform constants \( R > 0 \) and \( \kappa \in (0,1) \) such that if
\[ |\lambda| \geq R, \]
then for \( f(\lambda) = h \), there holds two possibilities as follows.

(a) We have
\[ \sum_k f_k(\lambda)(\chi_k - \lambda_k) > \kappa \sum_k f_k(\lambda). \]

(b) Or we have
\[ f_k(\lambda) > \kappa n \sum_{i=1}^{n} f_i(\lambda) \]
for all \( k = 1, 2, \cdots, n \).

In addition, we have
\[ F(\lambda) := \sum_{i=1}^{n} f_i(\lambda) \geq \kappa > 0. \]

We also need

**Lemma 3.5** (Gerhardt \[12\]). If \( F(A) = f(\lambda_1, \cdots, \lambda_n) \) in terms of a symmetric function of the eigenvalues, then at a diagonal matrix \( (A^j_i) \) with distinct eigenvalues we have
\[ F^{ij} = \delta_{ij} f_i, \]
\[ F^{ij,rs} = f_{ir} \delta_{ij} \delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{is} \delta_{jr}, \]
where
\[ F^{ij} = \frac{\partial F}{\partial A^j_i}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial A^j_i \partial A^r_s}. \]

We write \( F^{ij} := F^{ij}_\alpha \delta_{ij} \).

It suffices to prove
\[ \lambda_1 \leq CK. \]

Indeed, since \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i > 0 \), if \( \lambda_1 \leq CK \) then so is \( |\lambda_i|, k = 2, \cdots, n \), which yields \((3.14)\).

We consider the quantity
\[ H(x) := \log \lambda_1(x) + \varsigma(||\partial \varphi||_2^2(x)) + \psi(\varphi(x)), \quad \forall x \in M, \]
where we set
\[ \varsigma(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad \psi(s) = D_1 e^{-D_2 s}, \]
with sufficiently large uniform constants \( D_1, D_2 > 0 \) to be determined later. A direct calculation shows that
\[ \varsigma(||\partial u||_2^2) \in [0, 2 \log 2] \]
We define a strictly elliptic operator $\phi$ (3.32) and (3.31) since (3.38) and (3.39) there holds $\lambda \gg K$ at the point $x_0$ without loss of generality and hence (3.28) holds. In the followup, we will calculate at the point $x_0$ under the local coordinate $(z_1, \ldots, z_n)$ for which $\alpha$ is the identity and $(A_{ij})$ is diagonal with entries $A_{ii} = g_{ii} = \lambda_i$ for $1 \leq i \leq n$, unless otherwise indicated. Note that $(F^{ij})$ is also diagonal at the point $x_0$ (see [39]).

Since $\lambda_1$ may not be smooth at $x_0$, we introduce a smooth function $\phi$ on $M$ by (cf. [2] Lemma 5 and [11] Proof of Theorem 3.1)

\[
H(x_0) \equiv \log \phi(x) + \zeta(|\partial \phi|^2(x)) + \psi(\phi(x)), \quad \forall x \in M.
\]

Note that $\phi$ satisfies

\[
\phi(x) \geq \lambda_1(x) \quad \forall x \in M, \quad \phi(x_0) = \lambda_1(x_0).
\]

We define a strictly elliptic operator $L$ which is the same as (3.11) by

\[
L(u) = F^{ij}(A) \alpha^p (\partial_i \partial_j u + W_{pq}(du)),
\]

\[
= F^{ij}(A) \alpha^p \left( \nabla_i \nabla_j u + W_{pq}(\nabla_p u) + \overline{W_{pq}}(\nabla_p u) \right), \quad \forall u \in C^2(M, \mathbb{R}).
\]

Applying the operator $L$ defined in (3.33) to (3.31), one infers

\[
0 = \frac{1}{\lambda_1} L(\phi) = \frac{1}{\lambda_1} F^{ij} |\nabla_i \phi|^2 + \zeta |\partial \phi|^2 + \psi' L(\phi) + \psi'' F^{ij} |\nabla_i \phi|^2
\]

\[
+ \zeta'' F^{ij} \sum_p \left( (\nabla_i \nabla_j \phi)(\nabla_j \varphi) + (\nabla_j \phi)(\nabla_i \nabla_j \varphi) \right)^2.
\]

Differentiating (3.31) one can deduce

\[
0 = \nabla_i \phi + \zeta' (\nabla_j \phi)(\nabla_i \phi) + (\nabla_j \phi)(\nabla_i \nabla_j \phi) + \psi'(\nabla_i \phi).
\]

We have (see [2] Lemma 5.3)

**Lemma 3.6.** Let $\mu$ denote the multiplicity of the largest eigenvalue of $(A_{ij})$ at $x_0$, so that $\lambda_1 = \cdots = \lambda_\mu > \lambda_{\mu+1} \geq \cdots \geq \lambda_n$. Then at $x_0$, for each $i$ with $1 \leq i \leq n$, there hold

\[
\nabla_i g_{k\ell} = (\nabla_i \phi) \alpha_{k\ell}, \quad \text{for} \quad 1 \leq k, \ell \leq \mu,
\]

\[
\nabla_i \nabla_i \phi \geq \nabla_i \nabla_i g_{11} + \sum_{q > \mu} \frac{|\nabla_i g_{q1}|^2}{\lambda_1 - \lambda_q}.
\]

A direct calculation, together with the Ricci identity and the first Bianchi identity, yields that (see for example [12])

\[
\nabla_i \nabla_k \nabla_j \nabla_i u = \nabla_j \nabla_i \nabla_k u - R_{k\ell i\ell} \nabla_j \nabla_p u - R_{ij k\ell} \nabla_p \nabla_j \nabla_i u - T_{kij} \nabla_k \nabla_q \nabla_i u - T_{kij} \nabla_j \nabla_q \nabla_i u.
\]

It follows from (3.16) and (3.38) that

\[
F^{ij} \nabla_i \nabla_j g_{11} = F^{ij} \nabla_i \nabla_i \chi_{11} + F^{ij} \nabla_i \nabla_i \left( W^p_{11} \nabla_p \phi + \overline{W^q_{11}} \nabla_q \varphi \right)
\]

\[
+ F^{ij} \nabla_i \nabla_j \nabla_i \phi - F^{ij} \left( T^p_{11} \nabla_i \nabla_j \nabla_i \phi + \overline{T^q_{11}} \nabla_j \nabla_i \nabla_q \nabla_j \phi \right).
\]
Differentiating both sides of (3.19) by $\nabla_{\ell}$ gives

$$\nabla_{\ell} \nabla_{k} h = F^{i} \cdot (\nabla_{i} g_{ij}) \cdot (\nabla_{i} g_{pq}) + F^{i} \cdot (\nabla_{\ell} \nabla_{k} \lambda) + \nabla_{\ell} \nabla_{k} \nabla_{i} \varphi + \nabla_{\ell} \nabla_{k} (W_{ij}(d\varphi)).$$

Substituting (3.40) with $k = \ell = 1$ into (3.39) yields

$$F^{i} \nabla_{i} \nabla_{i} g_{11}$$

$$= - F^{i} \cdot \left( \nabla_{i} g_{ij} \right) \cdot (\nabla_{i} g_{pq})$$

$$+ F^{i} \cdot (\nabla_{i} \nabla_{i} \lambda) + F^{i} \cdot (\nabla_{i} \nabla_{i} (W_{11}(d\varphi)) - \nabla_{i} \nabla_{1} (W_{ij}(d\varphi)))$$

$$+ \nabla_{1} \nabla_{1} h - F^{i} \left( T^{p}_{i1} \nabla_{i} \nabla_{p} \varphi + T^{p}_{i1} \nabla_{i} \nabla_{q} \nabla_{1} \varphi \right)$$

$$+ F^{i} \left( R_{ii}^{p} \nabla_{i} \nabla_{p} \varphi - R_{i1}^{p} \nabla_{i} \nabla_{p} \varphi - T^{p}_{i1} T^{q}_{i1} \nabla_{i} \nabla_{p} \varphi \right).$$

It follows from (3.16), (3.36) and Young’s inequality that

$$\geq - CF^{i} \nabla_{i} \nabla_{p} \varphi$$

$$\geq - CF^{i} \nabla_{i} \nabla_{p} \varphi - \sum_{q > \mu} F^{i} \left( \nabla_{i} g_{1q} \right)|\nabla_{i} \nabla_{p} u|,$$

where we use the fact that $\lambda_{1} \gg K > 1$ and that both $|\varphi_{ij}|$ and $\lambda_{q} (q > \mu)$ can be controlled by $\lambda_{1}$. It follows from (3.15), (3.22), (3.24), (3.24), (3.25) that (see the argument in [33])

$$F^{i} \nabla_{i} g_{11}$$

$$= \sum_{p > 1} F^{i} \nabla_{i} g_{pq}$$

$$\leq C \sum_{p > 1} F^{i} \nabla_{i} g_{11}$$

$$+ CF^{11} \sum_{q > 1} |\nabla_{q} g_{11}|$$

$$= C \sum_{p > 1} F^{i} \nabla_{i} g_{11} + CF^{11} \sum_{q > 1} |\nabla_{q} g_{11}|$$

$$+ C \sum_{q > 1} F^{i} \nabla_{i} g_{11} + CF^{i} |\nabla_{1} g_{11}| + C \sum_{q > 1} F^{i} \nabla_{i} g_{11}$$

$$\leq C \sum_{p > 1} F^{i} \nabla_{i} g_{11} + CF^{11} |\nabla_{1} g_{11}| + C \sum_{q > 1} F^{i} \nabla_{i} g_{11}$$

$$\leq C \sum_{p > 1} F^{i} \nabla_{i} g_{11},$$

(3.44) $F^{i} \nabla_{i} \nabla_{i} g_{11} \nabla_{p} \varphi \geq - C \left( \sum_{p} F^{i} \nabla_{i} \nabla_{p} \varphi + \lambda_{1} F \right)$
and

\[ F^{ii} \nabla_i \nabla_1 (W_{ii} (d \varphi)) \leq C \left( F^{ii} \left( |\nabla_i g_{11}| + \sum_p |\nabla_i \nabla_p \varphi| \right) + \lambda_1 F \right), \]

where we also use the fact that \( \lambda_1 \gg K > 1 \) and that \( |\varphi_{ij}| \) can be controlled by \( \lambda_1 \). Note that (3.33) and (3.45) can be found in [35] directly (cf. [51]).

Applying the operator \( L \) defined in (3.33) to \( \phi \), we can deduce from (3.28), (3.36), (3.37), (3.41), (3.32), (3.43), (3.44) and (3.45) that

\[ L(\phi) = F^{i \bar{i}} (\nabla_i \nabla \phi + W_{i \bar{i}} (\nabla \phi)) \]

\[ \geq F^{i \bar{i}} \nabla_i \nabla g_{11} + F^{i \bar{i}} W_{i \bar{i}} (\nabla g_{11}) \]

\[ + \sum_{q > \mu} F^{i \bar{i}} \frac{|\nabla_i g_{1q}|^2 + |\nabla_i g_{\bar{i} q}|^2}{\lambda_1 - \lambda_q} \]

\[ \geq - F^{i \bar{i}, pq} (\nabla_i g_{1j}) (\nabla_1 g_{pq}) \]

\[ + 2 \Re \left( F^{i \bar{i}} \bar{T}_{i \bar{i}} (\nabla_i g_{1\bar{i}}) + \sum_{q > \mu} F^{i \bar{i}} \frac{|\nabla_i g_{1q}|^2 + |\nabla_i g_{\bar{i} q}|^2}{\lambda_1 - \lambda_q} \right) \]

\[ - C F^{i \bar{i}} \left( |\nabla_i g_{11}| + \sum_p |\nabla_i \nabla_p \varphi| \right) - C \lambda_1 F \]

\[ \geq - F^{i \bar{i}, pq} (\nabla_i g_{1j}) (\nabla_1 g_{pq}) \]

\[ - C F^{i \bar{i}} \left( |\nabla_i g_{11}| + \sum_p |\nabla_i \nabla_p \varphi| \right) - C \lambda_1 F. \]

Here we remind that \( \nabla_1 \nabla_1 h = O(\lambda_1) \) is absorbed into \( C \lambda_1 F \) by (3.28).

From (3.33) and (3.46), one can infer that

\[ 0 \geq - \frac{1}{\lambda_1} F^{j \bar{j}, pq} (\nabla_j g_{1j}) (\nabla_1 g_{pq}) - \frac{1}{\lambda_1^2} F^{i \bar{i}} |\nabla_i \phi|^2 \]

\[ + \zeta' L(|\partial \varphi|_2^2) + \zeta'' F^{i \bar{i}} \sum_p \left( (\nabla_i \nabla_p \varphi) \partial_p \varphi + (\nabla_p \varphi) \nabla_i \nabla_p \varphi \right) \]

\[ + \psi' L(\varphi) + \psi'' F^{i \bar{i}} |\nabla_i \varphi|^2 - \frac{C}{\lambda_1} F^{i \bar{i}} \left( |\nabla_i g_{11}| + \sum_p |\nabla_i \nabla_p \varphi| \right) - C F. \]

On the other hand, we can still have (see [9] Lemma 5.2])

**Lemma 3.7.** There exists a uniform constant \( C > 0 \) such that

\[ L(|\partial \varphi|_2^2) = \sum_k F^{i \bar{i}} \left( |\nabla_i \nabla_k \varphi|^2 + |\nabla_i \nabla_k \varphi|^2 \right) + 2 \Re \left( \sum_k (\nabla_k \varphi) \nabla_k h \right) \]

\[ + F^{i \bar{i}} (\nabla_k \varphi) \bar{T}_{i \bar{i}} (\nabla_i \nabla_k \varphi) + F^{i \bar{i}} (\nabla_k \varphi) T_{i \bar{i}} \nabla_i \nabla_k \varphi + O(|\partial u^2_\varphi|) F \]

\[ \geq \sum_k F^{i \bar{i}} \left( |\nabla_i \nabla_k \varphi|^2 + (1 - \varepsilon)|\nabla_i \nabla_k \varphi|^2 \right) \]

\[ + 2 \Re \left( \sum_k (\nabla_k \varphi) \nabla_k h \right) - C \varepsilon^{-1}|\partial \varphi|_\alpha^2 F, \]
where \( \varepsilon \) is an arbitrary constant with \( \varepsilon \in (0, 1/2) \).

Note that the term

\[
2\Re \left( \sum_k (\nabla_k \varphi) (\nabla_k h) \right)
\]

which is \( O(K) \) can be absorbed into \( C\varepsilon^{-1} |\partial \varphi| \| \mathcal{F} \) with a larger uniform constant \( C \) by (3.28) since we assume that \( K \gg 1 \). Since \( \varepsilon' \geq 1/(4K) \), it follows from (3.48) with \( \varepsilon = 1/3 \) and Young’s inequality that

\[
(3.49) \quad \varepsilon' L(|\partial \varphi|_\alpha^2) - \frac{C}{\lambda_1} \sum_p |\nabla_i \nabla_p \varphi| \geq \frac{1}{6K} \sum_k F^{ij} \left( |\nabla_i \nabla_k \varphi|^2 + |\nabla_i \nabla_k \varphi|^2 \right) - C \mathcal{F},
\]

where we also use the fact that \( \lambda_1 \gg K > 1 \).

Hence we can deduce from (3.47) and (3.49) that

\[
(3.50) \quad 0 \geq -\frac{1}{\lambda_1} F^{ji, \rho q} (\nabla_i g_{\rho q}) (\nabla_j g_{\rho q}) - \frac{1}{\lambda_1} F^{ji} |\nabla_i \varphi|^2 + \frac{1}{6K} \sum_k F^{ij} \left( |\nabla_i \nabla_k \varphi|^2 + |\nabla_i \nabla_k \varphi|^2 \right) + \varepsilon'' F^{ii} \left( \sum_p \left( (\nabla_i \nabla_p \varphi)(\nabla_i \nabla_p \varphi) + (\nabla_i \nabla_p \varphi)(\nabla_i \nabla_p \varphi) \right) \right)^2 + \psi' L(\varphi) + \varepsilon'' F^{ij} |\nabla_i \varphi|^2 - \frac{C}{\lambda_1} F^{ij} |\nabla_i g_{11}| - C \mathcal{F}.
\]

From (3.30), we know that \( \nabla_i \phi = \nabla_i g_{11} \). This, together with (3.47), yields that (3.50) is the same as [35] Equation (3.7)], yields that (3.50) is the same as [35] Equation (3.28) essentially after changing \( \nabla_p \nabla \varphi \) and \( \nabla_i g_{11} \) into \( \partial_i \partial_p \varphi \) and \( \partial_i g_{11} \) respectively, and changing the coefficient of \( \mathcal{F} \) into a larger uniform constant (only replacing \( \tilde{H}_k \) in [35] with \( \nabla_k \phi \) in [35] by \( \varsigma \) and \( F^{i\bar{k} \bar{k}} u_{k\bar{k}} \) in [35] by \( L(\varphi) \)). After changing these notations, we can repeat the argument in [35] word for word to get

\[
(3.51) \quad \lambda_1 \leq CK,
\]

by replacing \( \tilde{H}_k = 0 \) in [35] by (3.35) and replacing the paragraph between [35] Inequality (3.53)(not containing) and [35] Inequality (3.54)(containing) by

\[
\psi' L(\varphi) = \psi' L(\varphi) = \psi' F^{k\bar{k}}(g_{k\bar{k}} - \chi_{k\bar{k}}).
\]

For the first order estimate, we need prove

\[
(3.52) \quad \sup_M |\partial u|_g \leq C.
\]

We use the blowup argument in [34, 40] originated from [6]. Since \( h \) still uniformly bounded, the argument in [34, 42, 9] still works without any modification. We omit the details here.

Given (3.13), (3.14) and (3.52), \( C^{2, \gamma} \)-estimate for some \( 0 < \gamma < 1 \) follows from the Evans-Krylov theory [7, 19, 44] (see also [37]). Differentiating the equations and using the Schauder theory (see for example [13]), we then deduce uniform a priori \( C^k \) estimates for all \( k \geq 0 \).

Uniqueness follows from the maximum principle as in the arguments in Corollary 3.2.
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School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

E-mail address: zhengtao08@amss.ac.cn