Abstract. The (homogeneous) Essentially Isolated Determinantal Variety is the natural generalization of generic determinantal variety, and is fundamental example to study non-isolated singularities. In this paper we study the characteristic classes on these varieties. We give explicit formulas of their Chern-Schwartz-MacPherson classes via standard Schubert calculus. As corollaries we obtain formulas for their (generic) sectional Euler characteristics, characteristic cycles and polar classes. In particular, when such variety is a hypersurfaces we compute its Milnor class and the Euler characteristics of the local Milnor fibers. We prove that for such recursive group orbit hypersurfaces the local Euler obstructions completely determine the Milnor classes.

In general for reflective group orbits, on the other hand we propose an algorithm to compute their local Euler obstructions via the Chern-Schwartz-MacPherson classes of the orbits, which can be obtained directly from representation theory. This builds a bridge from representation theory of the group action to the singularity theory of the induced orbits.

1. Introduction

The computation of geometric invariants on singular spaces has been a major task in singularity theory and algebraic geometry, and have been intensely studied for the last decades. For Whitney stratified spaces the sectional Euler characteristics (the Euler characteristics of the intersections with generic linear subspaces) of the strata, and the local Euler obstructions along the strata are key ingredients one may need to understand the such singularities. In particular when the stratified space is a hypersurface, the study of local Milnor fibers (for instance, their Euler characteristics) is the fundamental work. In this paper we consider homogeneous essentially isolated determinantal varieties. They are natural generalizations of the generic determinantal varieties, and are fundamental examples of non-isolated singularities. The goal of this paper is to compute the mentioned invariants for such spaces.

In the smooth setting, the Euler characteristic of a space is the degree of its total Chern class via Poincaré-Hopf theorem. However, in most cases $X^*_{n,k}$ is singular, and we need to employ the theory of Chern class for singular varieties defined by ManPherson in [21] over $\mathbb{C}$, and generalized to arbitrary algebraically closed field of characteristic 0 by G. Kennedy in [20]. This is the Chern-Schwartz-MacPherson class, denoted by $c_{SM}^X$ for a projective $X$, which shares the property that the degree of $c_{SM}^X$ equals the Euler characteristic of $X$ without any smoothness assumption. Two important ingredients MacPherson used to define this Chern class are the local Euler obstruction and the Chern-Mather class, denoted by $E_{UV}$ and $c_X$ respectively. They were originally defined on $\mathbb{C}$, but in [16] González-Sprinberg-Verdier proved an algebraic formula extends the definitions to arbitrary algebraically closed base field. Moreover, when $X$ is a projective hypersurface in $\mathbb{P}^n$ Parusiński and Pragacz proved that the Chern-Schwartz-MacPherson class computes the Milnor class of $X$, and the Euler characteristics of the local Milnor fibers. These are the tool box we will use in this paper.

Let $K$ be an algebraically closed field of characteristic 0. Let $M_n$, $M_n^S$ and $M_n^\Lambda$ be the spaces of $n \times n$ ordinary, symmetric and skew-symmetric matrices respectively. We will denote them by $M_n^*$, for * denotes $\emptyset$, $S$ and $\Lambda$ respectively. They have natural stratifications $M^*_n = \cup \Sigma^*_{n,i}$, where the strata are matrices of fixed corank $i$. We consider transverse maps $F: V = K^N \to M^*_n$, where transverse means the image of $F$ intersects each $\Sigma^*_{n,i}$ transversely. By homogeneous we mean that $F$ is a $K^*$-equivariant map, where $K^*$ acts on $V$ and $M^*_n$ by scalar multiplications. Then the pull back orbits of $\Sigma^*_{n,i}$ are necessarily cones, and we call their projectivizations homogeneous Essentially Isolated Determinantal Varieties. We will denote them by EIDV in short. For details we refer to [18].

In §2 we review the theory of characteristic classes for (quasi) projective varieties. We briefly recall the definitions and basic properties of the Chern-Schwartz-MacPherson class, the Chern-Mather class, the Milnor class and the Lê classes. For projective varieties these are polynomials with variable $H = c_1(O(1))$. Moreover, we recall the two involutions proposed by Aluffi in [3] and [1] separately. The first involution $J$ translates the information of Chern-Schwartz-MacPherson class of a projective variety $X$ to the information of the Euler characteristics of $X \cap L^k$ for generic codimension $k$ linear subspaces. This reduces the computation of sectional Euler characteristics to the
computation of Chern-Schwartz-MacPherson classes. The second involution $T_N$ interchanges the signed Chern-Mather classes between projective varieties and their dual varieties. The two involutions are the major tools we will use to compute the Milnor classes and to prove our algorithm for local Euler obstructions.

The main formulas to the Chern classes of EIDV are presented in [33]. First we show that, via transversal pull-back of Segre-MacPherson classes proved in [24] it’s enough to compute the Chern classes of generic determinantal varieties. For such varieties we use their canonical resolutions: the Tjurina transforms. We define the $q$ polynomials to be the pushforward of the Chern-Schwartz-MacPherson classes of the Tjurina transforms, and show that the Chern-Schwartz-MacPherson classes and the Chern-Mather classes of determinantal varieties are linear combinations of the $q$ polynomials. Our first formula (Theorem 3.2) interprets the coefficients of the $q$ polynomials by integrations of tautological classes over Grassmannians. Here by tautological we mean the Chern classes of the universal sub and quotient bundles. Thus our formula is purely combinatorial and can be easily computed by Macaulay2. We present some computed examples in Appendix.

Based on the fact that the function values at integers uniquely determine a polynomial, we also propose our equivalent formula (Theorem 3.3). For each type (ordinary, symmetric or skew-symmetric) of matrix we define the determinantal Chow (cohomology) classes $Q_{n,r}$, $Q_{n,r}^\wedge$ and $Q_{n,r}^S$. These are Chern classes of the universal bundles in the Chow (cohomology) groups of Grassmannians. Then we show that the $q$ polynomials equal the integrations of these determinantal classes with the total Chern classes over the Grassmannians.

**Theorem.** Let $S$ and $Q$ be the universal sub and quotient bundles over the Grassmannian $G(r, n)$. We define the ordinary, symmetric and skew-symmetric determinantal classes as follows.

$$Q_{n,r}(d) := \left( \sum_{k=0}^{n+r} \binom{n+r}{k} d^{n+r-k} c_k(Q^{\wedge \vee}) \right) \left( \sum_{k=0}^{n-r} d^{n-r-k} c_k(S^{\wedge \vee}) \right);$$

$$Q_{n,r}^\wedge(d) := \left( \sum_{k=0}^{n+r} \binom{n+r}{k} d^{n+r-k} c_k(\Lambda^2 Q^{\wedge}) \right) \left( \sum_{k=0}^{n-r} d^{n-r-k} c_k(\Lambda^2 S^{\vee}) \right);$$

$$Q_{n,r}^S(d) := \left( \sum_{k=0}^{n+r} \binom{n+r}{k} d^{n+r-k} c_k(Sym^2 Q^{\wedge}) \right) \left( \sum_{k=0}^{n-r} d^{n-r-k} c_k(Sym^2 S^{\vee}) \right).$$

We have the following integration formulas:

$$q_{n,r}(d) = \int_{G(r,n)} c(S^{\wedge \vee} \otimes Q) \cdot Q_{n,r}(d) \cap [G(r,n)] - d^{n+r} \binom{n}{r};$$

$$q_{n,r}^\wedge(d) = \int_{G(r,n)} c(S^{\wedge \vee} \otimes Q) \cdot Q_{n,r}^\wedge(d) \cap [G(r,n)] - d^{n+r} \binom{n}{r};$$

$$q_{n,r}^S(d) = \int_{G(r,n)} c(S^{\wedge \vee} \otimes Q) \cdot Q_{n,r}^S(d) \cap [G(r,n)] - d^{n+r} \binom{n}{r}.$$

From the definition of the determinantal classes we can observe the two-fold symmetry: the duality induced by $G(r, n) \cong G(n-r, n)$ and the symmetry induced by $d \mapsto -1-d$, which is induced by Aluffi’s projective involution. Based on the form we can directly prove that, for ordinary determinantal varieties the $q$ polynomial $q_{n,r}(H)$ is exactly the Chern-Mather class $c_M^{n,r}$, which is a classical result in [37] proved differently. This observation motivates our local Euler obstruction algorithm in [8].

For complex projective varieties the theory of MacPherson’s Chern classes are the pushdown of the theory of (Lagrangian) characteristic and conormal cycles. In [41] we apply our results to obtain formulas for the characteristic cycles of EIDV, as Chow classes of $\mathbb{P}^N \times \mathbb{P}^N$ (Proposition 4.1). For generic determinantal varieties, bases on the result in [30] and [39] we also compute their conormal cycle classes. Since the coefficients of the conormal cycle class are also the degrees of the polar classes, we also obtain explicit formulas to the polar degrees of generic determinantal varieties (Equation 7). For EIDV the conormal cycles depend on the local Euler obstruction information, thus combining with [38] we obtain an algorithm to compute the polar degrees of EIDV. We finish this section by proving an interesting observation: the characteristic cycles of the closed orbits of all singular matrices are symmetric (Proposition 4.2). Such symmetry deserves a geometric explanation.

Notice that the ordinary EIDV $X_{n,1}$, the skew-symmetric EIDV $X_{2n,2}^\wedge$ and the symmetric EIDV $X_{n,1}^S$ are hypersurfaces. In [33] and [37] we compute the Milnor classes of them, and in particular, the Euler characteristics of the local Milnor fibers when $k = \mathbb{C}$. The main result in Theorem 5.1.
Theorem. When $n \geq 3$, the Milnor classes are computed by

$$\mathcal{M}(X_{n,1}) = \frac{(-1)^n c_{n-1}}{1 + c_1 (\mathcal{O}(X_{n,1}))}, \quad \mathcal{M}(X_{2n,2}) = \frac{(-1)^{2n} c_{2n-4}}{1 + c_1 (\mathcal{O}(X_{2n,2}))}, \quad \mathcal{M}(X_{n,1}^S) = (-1)^{n+1} \frac{c_{n-2}}{1 + c_1 (\mathcal{O}(X_{n,1}))}.$$  

When $k = \mathbb{C}$ we let $F_{n,i}$, $F_{2n,2i}$, and $F_{n,i}^S$ be the local Milnor fibers. We have the following equivalent description:

$$\chi(F_{n,i}) = \chi(F_{2n,2i}) = 0; \quad i = 2, 3, \ldots, n - 1$$

$$\chi(F_{n,i}^S) = \begin{cases} 2 & i = 2 \\ 0 & i = 3, \ldots, n - 1 \end{cases}$$

The Milnor fibers of generic ordinary, symmetric and skew-symmetric determinantal hypersurfaces are concretely studied in [10]. In the paper Damon found complex models for the Milnor fibers and proved a fascinating Schubert type decomposition for the cohomology rings. Our Milnor class results are compatible with Damon’s description of the complex models.

However, we would like to point out that our method is different. First we don’t rely on the topology of $\mathbb{C}$; our proof is intersection theoretical and thus algebraic. Secondly, one can observe that the proof we present in §7 only relies on two facts: the orbital strata are dual to each other and we know their local Euler obstructions. We actually proved that (Corollary 8) for a hypersurface realizable as a closed orbit of a recursive group action (see definition in [39, Assumption 1]), the local Euler obstructions along the strata completely determine its Milnor class and the Euler characteristics of the local Milnor fibers. The same result applies to the pullback hypersurface of any homogeneous transverse map to such recursive representations.

On the other hand, for such recursive orbital hypersurfaces the Milnor class cannot determine the local Euler obstructions in general, let alone for more general group orbits. Thus the natural question is what else information do we need. In §8 we proved that, for reflective group orbits (group orbits satisfying Assumption 1) the Chern classes of the orbits completely determine the local Euler obstructions. In particular we propose an algorithm (Algorithm 1) to explicitly compute them. The algorithm is based on Aluffi’s projective duality involution defined in [1]. As explained in Remark 8, for such group orbits their Chern-Schwartz-MacPherson classes can be obtained from representation theory. In [11] the authors proved that the Chern classes actually correspond to the $K$-theoretic stable envelopes studied in [39]. Thus this algorithm provides a way to study the singularities of orbits closures directly from representation theory.

We point out here that recently Mihalcea and Singh proposed a similar algorithm for Schubert cells in cominuscule spaces in [23]. In Section 7 they proved a cohomology formula using Chern-Schwartz-MacPherson classes of Schubert cells. The formula also depends on the duality of Schubert cells, but in a different manner with ours. In fact, we believe that there should be a general notation of duality, and a general duality involution connecting the characteristic classes of dual group orbits. The further discussions will be presented elsewhere.

As shown in [7, 8], the Milnor class of a hypersurface is equivalent to its (global) Lé class. In §6 we compute the Lé class for EID Hypersurfaces. In Proposition 6.1 we show that the coefficients appearing in the Lé classes are exactly the coefficients appeared in the characteristic cycles of the smaller strata. This proposition corresponds to the local Lé classes-to-polar classes connection, and the characteristic cycle-to-conormal cycle connection.

The Appendix §11 is devoted to explicit examples. The examples are computed based on our formulas via the software Macaulay2 [17]. We highlight the patterns proved in the previous sections by the examples. We observe that all the nonzero coefficients appearing in the Chern classes are positive. Moreover, all the polynomials and sequences presented in the examples are log concave. These facts call for a conceptual, geometric explanation. Thus we close this paper with the non-negative conjecture and the log concave conjecture (Cf. [39]). The situation appears to have similarities with the case of Schubert varieties in flag manifolds, which was recently proved in [3].

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2. Preliminary

2.1. Chern-Schwartz-MacPherson Class. Let $X \subset \mathbb{P}^N$ be a projective variety. The group of constructible function is defined as the abelian group generated by indicator functions $1_V$ for all irreducible subvarieties $V \subset X$. We define the pushforward for a proper morphism $f: X \to Y$ as follows. For any closed subvariety $V \subset X$, the
Theorem 2.2. The map $c_*$ that sends the local Euler obstruction function $EU_V$ to Mather’s Chern class $c_{EU(V)}$ is the unique natural transform from $F$ to the homology functor $H_*$ satisfying the following normalization property: $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$ when $X$ is smooth.

Then in 1990 Kennedy modified Sabbah’s Lagrangian intersections and proved the following generalization.

Theorem 2.3. Replace the homology functor by the Chow functor, MacPherson’s natural transform extends to arbitrary algebraically closed field of characteristic 0.

Definition. Let $X \subset \mathbb{P}^N$ be a projective subvariety. The Chern-Schwartz-MacPherson class and the Chern-Mather class of $X$, denoted by $c^X_{SM}(H)$ and $c^X_M(H)$, are defined as the pushforward of $c_*(\mathbb{1}_X)$ and $c_*(EU_X)$ in $A_*(\mathbb{P}^N)$. If $X$ is a regular embedding with normal bundle $\mathcal{N}$, we define the Fulton-Johnson class to be $c_{FJ}(X) := \frac{c^X_{SM}(\mathcal{N})}{c^X_M(\mathcal{N})} \cap [\mathbb{P}^N]$.

Notice that when $X$ is smooth, the Chern-Mather class, Chern-Schwartz-MacPherson class and the Fulton-Johnson class all equal to the total Chern class $i_* (c(T_X) \cap [X])$.

The property of characteristic classes can also be generalized to motivic settings. For definitions, properties and examples we refer to [6]. In [11] the authors propose an axiomatic approach for such classes; recently in [4] the authors applied such theory on pointed Brill-Noether problems. In this paper we only consider ordinary characteristic classes.

2.2. Involutions of Chern Classes. In this subsection we introduce the key ingredient that will be used in the paper: the two involutions defined by Aluffi in [1] [3]. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial. For any $d \geq 0$ we define the following transformations $I_d: \mathbb{Z}[x] \to \mathbb{Z}[x]$ and $J: \mathbb{Z}[x] \to \mathbb{Z}[x]$ by

$$I_d: f(x) \mapsto f(-1 - x) - f(-1)((1 + x)^{d+1} - x^{d+1})$$

$$J: f(x) \mapsto \frac{2f(-1 - x) - f(0)}{1 + x}$$

Proposition 2.3. One can observe the following properties for $I_d$ and $J$ by direct computations:

1. For any polynomial $f$ with no constant term, $I_d(I_d(f)) = J(J(f)) = f$. Thus $I_d$ and $J$ are involutions on the set of polynomials.

2. The involutions $I_d$ and $J$ are linear, i.e., $I_d(a f + bg) = a I_d(f) + b I_d(g)$; $J(a f + bg) = a J(f) + b J(g)$.

3. The involution $I_d$ takes $H f(H)$ to $(-1 - H) I_d(f) - f(-1) H((1 + H)^{d+1} - H^{d+1})$.

Let $X \subset \mathbb{P}(V)$ be a projective variety of dimension $n$. For any $r \geq 0$ we define

$$X_r = X \cap H_1 \cap \cdots \cap H_r$$

to be the intersection of $X$ with $r$ generic hyperplanes. Let $\chi(X_r) = \int_{X_r} c_{sm}(X_r)$ be its Euler characteristic, we define $\chi(X_r) = \sum_i \chi(X_r) \cdot (-1)^i$ to be the corresponding sectional Euler characteristic polynomial. On the other hand, write $c^X_{SM} = \sum_{i \geq 0} \gamma_{N,i} H^i$ we define the $\gamma$ polynomial $\gamma_X(t) := \sum_i \gamma_i t^i$ by switching the variable from $H^i$ to $[\mathbb{P}]$. The polynomials $\chi_X(t)$ and $\gamma_X(t)$ are polynomials of degree $\leq n$. 
Theorem 2.4 ([13] [34]). The involution $J$ connects the Chern-Schwartz-MacPherson class with sectional Euler characteristics:

$$J(\gamma_X(t)) = \chi_X(t); \quad J(\chi_X(t)) = \gamma_X(t).$$

The involution $I_{\dim V - 1}$, on the other side, takes the signed Chern-Mather class of $X$ to the signed Chern-Mather class of $X^\vee$. More precisely, we have

$$(1) \dim X^\vee c_M^X(H) = (-1)^{\dim X} I_{\dim V - 1}(c_M^X(H)).$$

For the definition of projective dual varieties we refer to [13] [34] for more details.

2.3. Milnor Classes of Hypersurfaces. For complex hypersurface singularities, the most classical invariant is the (local) Milnor fibration introduced by John Milnor in his celebrating paper [28]. Let $M$ be a smooth $n+1$-dimensional complex manifold, and let $X \subseteq M$ be a hypersurface defined by a section $s \in H^0(M, L)$ for some line bundle $L$. For any point $x \in X$, there exists a neighborhood $x \in U$ such that $f_x : (U_x, x) \subseteq (\mathbb{C}^{n+1}, 0) \rightarrow \mathbb{C}$ is an analytic function germ with $V(f_x) = X \cap U$. Milnor showed that for $\epsilon$ small enough, and for such $\epsilon$ there exists $\delta(\epsilon)$ such that when $\delta < \delta(\epsilon)$ the restriction

$$f_x : B_\epsilon^\circ(x) \cap f_x^{-1}(\partial D_\delta) \rightarrow \partial D_\delta$$

is a smooth locally trivial fibration. Here $B_\epsilon(x)$ is the (closed) ball of radius $\epsilon$ centered at $x$ with boundary $S_\epsilon(x) \cong S^{2n+1}$, and $D_\delta$ is the (closed) disk in $\mathbb{C}$ centered at $0$. The diffeomorphism type is independent of $\epsilon$ and $\delta$, and we define its fiber (up to diffeomorphism) to be the (local) Milnor Fiber of $X$ at $x$:

$$F_x \cong F_{x, \delta} := B_\epsilon \cap f_x^{-1}(\delta).$$

We define the $\mu$-invariant at $x$ to be the signed Euler characteristic of the reduced homology of $F_x$, i.e., $\mu(x) := (-1)^{\dim X}(\chi(F_x) - 1)$.

When $x$ is an isolated singularity, Milnor showed that the homotopy type of $F_x$ is the wedge of $n-1$ spheres and thus the Milnor number $\chi(F_x)$ equals the number of spheres. When $x$ is non-isolated, the local fibration theorem still holds and the Milnor fiber is well-defined. However, the above proposition doesn’t apply in general. For example, in [32] [33] [35] they showed that for non-isolated singularity with small dimensional singular set, the Milnor fiber is still homotopic to a banquet of spheres, but probably of different dimensions. In [9] Damon shows that for matrix hypersurfaces, the Milnor fibers are never homotopic to banquet of spheres. Nevertheless, we still have the following property:

Proposition 2.5. For any Whitney stratification $S = \{S_\alpha | \alpha \in I\}$ of $X$, the $\mu$-invariant is constant along the strata $S_\alpha$. When $x$ is a smooth point, we have $\chi(F_x) = 1$, and thus $\mu(x) = 0$.

Thus the $\mu$-invariant is a constructible function supported on the singularity locus of $X$. We denote it by $\hat{\mu}_X$ standing for the reduced homology theory. We then consider the characteristic class $c_\ast(\hat{\mu}_X)$. In [28] the authors defined the Milnor class of $X$ as follows:

$$\mathcal{M}(X) := (-1)^{\dim X}(c_{F,J}(X) - c_{sm}(X)) = (-1)^{\dim X}(\frac{c(TM)}{c(L)} \cap [X] - c_{sm}(X)).$$

Theorem 2.6 ([28] [24]). The following equality justifies the name:

$$c_\ast(\hat{\mu}_X) = c(L) \cap \mathcal{M}(X).$$

Proposition 2.7. The following propositions are due to [28].

1. The Milnor class $\mathcal{M}(X)$ is supported on the singular locus of $X$.
2. When $X$ has only isolated singularity $\{p_1, p_2, \cdots, p_m\}$, the Milnor class of $X$ is the class of singular points weighted by the Milnor numbers, i.e., $\mathcal{M}(X) = \sum_{i=1}^m \mu_i[x_i]$.

The Milnor fiber is the most important object in the study of hypersurface singularities. To study the Milnor fibers of non-isolated singularities in [22] Massey introduced the (local) Lê cycles and use them to describe the topological information of the Milnor fibers. Let $(X, 0)$ be a hypersurface germ with singularity locus $X_s$. For any point $p$ in $X$ Massey defined local analytic cycles $\Lambda_k(x)$ for $k$ ranging from 0 to $\dim X_s$, as the intersection of the local Polar varieties with complementary linear spaces. He named them Lê cycles. The multiplicities of these cycles at $p$ count the number of handles one has to attach to the corresponding dimension ball to obtain the diffeomorphism type of the local Milnor fiber at $p$. The local multiplicities then derive Morse type inequalities for the Betti numbers of the Milnor fibers. In [13] Gaffney and Gassler proved that the class of the Lê cycles admit an algebraic intersection formula, thus they extends to global hypersurface singularities, and are viewed as classes in
the Chow groups or the Borel-Moore homology groups. Later in [8] [7], for any hypersurface \( X \) in some projective smooth variety \( M \) the authors defined the global Lé class \( \Lambda(X) = \sum_k \Lambda_k(X) \). This is a class in \( A_*(M) \), and for any point \( p \) in \( X \) and any dimension \( k \) there are (global) cycles representing the dimensional \( k \) piece \( \Lambda_k(X) \). These cycles restrict to Massey’s local Lé cycles in some analytic neighborhood of \( p \). In [7] they proved that the Lé class determines the Milnor class, and vice versa, it is determined by the Milnor classes.

**Theorem 2.8.** Write \( M(X) = \sum_k \mathcal{M}_k(X) \) and \( \Lambda(X) = \sum_k \Lambda_k(X) \) be the Milnor class and the Lé class, where the grading is indexed by dimensions. Then we have

\[
\mathcal{M}_k(X) = \sum_{j \geq 0} \sum_{i \geq k + j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(O(X))^j H^{i-k-j} \cap \Lambda_i(X); \\
\Lambda_k(X) = \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} H^j \cap (\mathcal{M}_{k+j}(X) + c_1(O(X)) \mathcal{M}_{k+j+1}(X))
\]

**Remark 2.** In [12] the author defined another class \( \tilde{\Lambda}(X) \) to fit in the ‘wrong’ formula originally proposed in [8]. The definition involves the Segre class \( s(X_*, M) \), and is different from the class \( \Lambda(X) \).

### 2.4. Essentially Isolated Determinantal Varieties

The Essentially Isolated Determinantal Singularities (EIDS) was introduced in [13], as a generalization of determinantal type singularities. Let \( K \) be a characteristic 0 algebraically closed field. Let \( M_n, M_n^S \) and \( M_n^\wedge \) be the space of \( n \times n \) ordinary, symmetric and skew-symmetric matrices over \( K \) respectively. When the matrix type is not specified, we use \( * \) to denote the upper-script. We consider maps \( F = (f_{i,j})_{n \times n} : K^{N+1} \to M_n^* \) that intersect transversely along all the rank strata \( \Sigma_{n,k}^* \) of \( M_n^* \). Here \( \Sigma_{n,k}^* \) denotes the stratum consisting matrices of rank \( n - i \). In this paper we always assume that \( F \) is homogeneous, i.e., \( f_{i,j,s} \) are homogeneous polynomials of degree \( d \). We consider the projectivization map \( F : \mathbb{P}(K^{N+1}) \to \mathbb{P}(M_n^*) \). Let \( \tau_{n,i}^* \) be the projectivization of \( \Sigma_{n,k}^* \), and let \( \tau_{n,i}^\wedge \) be its closure. We define \( X_{n,i}^*: = F^{-1}(\tau_{n,i}^*) \subset \mathbb{P}^N \) as the preimage of \( \tau_{n,i}^* \). We call these varieties the Essentially Isolated Determinantal varieties, and throughout this paper we will use EIDV in short. We call the varieties \( \tau_{n,i}^* \) generic determinantal varieties. In particular \( X_{n,1}^* \), \( X_{n,1}^S \) and \( X_{2n,2}^\wedge \) are irreducible hypersurfaces in \( \mathbb{P}^{N-1} \), and we will call them EID hypersurfaces.

**Proposition 2.9.** The following properties follow naturally from affine to projective setting.

1. The map \( F \) intersect transversely to the strata \( \tau_{n,i}^* \).
2. Let \( X_{n,i}^* \) be the preimage of \( \tau_{n,i}^* \) for \( i \geq k \), then they form a stratification of \( X_{n,k}^* \).
3. \( X_{n,k}^* \) is smooth on \( X_{n,k}^\wedge \), its singular locus is inside \( X_{n,k+1}^* \).
4. The tautological line bundle of \( \mathbb{P}(M_n^*) \) pulls back to the \( d \)-tensor tautological line bundle of \( \mathbb{P}^N \), i.e., \( F^*(O_{\mathbb{P}(M_n^*)}(1)) = O_{\mathbb{P}^N}(d) \).

For detailed definitions and more properties we refer to [13] [15].

**Example 1.** The following two maps

\[
F : \mathbb{C}^4 \to M_{2,3} : \begin{bmatrix} x_1 \\ \cdots \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_3 + x_4 & x_1 \\ x_4 & x_1 & x_2 \end{bmatrix} ; G : \mathbb{C}^4 \to M_{2,3} : \begin{bmatrix} x_1 \\ \cdots \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}
\]

are both EIDS of degree 1. The following map

\[
P : \mathbb{C}^5 \to M_{2,3} : \begin{bmatrix} x_1 \\ \cdots \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 + x_2^2 \\ x_3 x_4 \\ x_3 x_5 + x_2^2 \\ x_3^2 + x_4^2 \\ x_5^2 \end{bmatrix}
\]

is an EIDS of degree 2.

### 3. Characteristic Class of EIDS

In this section we compute the Chern-Schwartz-MacPherson classes of the EIDV. First we show that it’s enough to compute the Chern classes for generic determinantal varieties.

**Theorem 3.1 (Reduction to Generic Rank Loci).** For \( * \) substituted by \( \emptyset, S \) and \( \wedge \), which correspond to ordinary, skew-symmetric and symmetric cases, we have the following formulas:

\[
X_{n,k}^*_{csm} (H) = \frac{(1 + H)^{N+1}}{(1 + dH)^{\dim M_n^*}} \cdot \tau_{n,k}^* (dH).
\]
Proof. We consider the pullback of characteristic classes from determinantal varieties to EIDS. As shown in [36], the Chern-Schwartz-MacPherson classes don’t behave very well under pull back, i.e., Verdier-Riemann-Roch for Chern-Schwartz-MacPherson classes fails in general. However, under our transversality assumption on $F$ the Verdier-Riemann-Roch holds for our case. This is due to the pullback property of the Segre-MacPherson class defined by T. Ohmoto in [27].

Definition. Let $X \to M$ be a closed embedding into a smooth projective variety. We define the Segre-MacPherson class of $X$ to be

$$s^{SM}(X, M) := \text{Dual}(c(TM)^{-1} \cap c_*(X)) \in A^*(M).$$

Here Dual denotes the Poincaré dual of the ambient space $A^*(M) \sim A_*(M)$.

Remark 3. The Segre-MacPherson class $s^{SM}(X, M)$ are operational Chow classes that act on the Chow group of the ambient space $M$. The class and it's Poincaré dual are not necessarily supported on $X$, they only describe the information after the pushforward to the ambient space.

The Segre-MacPherson classes behave very well under transverse pullbacks. Let $f : M \to N$ be a morphism of Whitney stratified smooth compact complex varieties, and assume that $f$ intersects transversely with the strata of any closed subvariety $Y$ of $N$. Ohmoto in [27] proved that

$$f^*(s^{SM}(Y, N)) = s^{SM}(f^{-1}(Y), M).$$

Since we require transversality in the definition of EIDS, we then have

$$c_{sm}^* \in A_*(\mathbb{P}^N) = \frac{c(F^*c_0(M^*_n)(1))^{\dim M^*_n}}{c(1)} \cap F^*c^*_{sm} = \frac{1 + dH}{(1 + H)^N} \cdot c^*_{sm}(dH)$$

This shows that the computation of the Chern classes of EIDS is equivalent to the computation of Chern classes of determinantal varieties, for which we have the following.

Theorem 3.2 (Main Formula I). Denote $S$ and $Q$ to be the universal sub and quotient bundle over the Grassmanian $G(k, n)$. For $k \geq 1$, $i, p = 0, 1 \cdots e^*$, we define the following Schubert integrations:

$$A_{i,p}(n, k) := \int_{G(k,n)} c(S^i \otimes Q)c_i(Q^v)n)c_{p-i}(S^v) \cap [G(k,n)]$$

$$A^S_{i,p}(n, k) := \int_{G(k,n)} c(S^i \otimes Q)c_i(Sym^2Q^v)s_{k(2n-k-1)}^{-1}l_{p-p-i}(Sym^2Q^v) \cap [G(k,n)]$$

$$A^\wedge_{i,p}(n, k) := \int_{G(k,n)} c(S^i \otimes Q)c_i(\wedge^2Q^v)s_{k(2n-k-1)}^{-1}l_{p-p-i}(\wedge^2Q^v) \cap [G(k,n)];$$

and the following binomials:

$$B_{i,p}(n, k) := \binom{n(n-k)-p}{i-p}; \quad B^S_{i,p}(n, k) := \binom{(n-k+1)^2}{i-p}; \quad B^\wedge_{i,p}(n, k) := \binom{(n-k+1)^2}{i-p}.$$ 

Here $e = n(n - r)$, $e^S = \binom{n-r+1}{2}$ and $e^\wedge = \binom{n-r}{2}$ correspond to the ranks of the vector bundles. Let $H$ be the hyperplane class in $\mathbb{P}(M^*_n)$, we define the following $q$ polynomials for $k \geq 1$:

$$q_{n,k} := \sum_{l=0}^{n^2-1} \sum_{p=0}^{n(n-r)} \sum_{i=0}^{p} A_{i,i,p}(n, k) \cdot B^S_{p,i}(n, k) \cdot H^l;$$

$$q^S_{n,k} := \sum_{l=0}^{n^2-1} \sum_{p=0}^{n(n-r)} \sum_{i=0}^{p} A_{i,i,p}(n, k) \cdot B^S_{p,i}(n, k) \cdot H^l;$$

$$q^\wedge_{n,k} := \sum_{l=0}^{n^2-1} \sum_{p=0}^{n(n-r)} \sum_{i=0}^{p} A_{i,i,p}(n, k) \cdot B^\wedge_{p,i}(n, k) \cdot H^l.$$
For ordinary rank loci, when $k \geq 1$ we have:

\[
\tau^*_{n,k} = q_{n,k}; \quad \tau^S_{n,k} = \sum_{r=k}^{n-1} (-1)^{r-k} \binom{r}{k} \cdot q_{n,r}.
\]

For symmetric rank loci, when $k \geq 1$ we have

\[
\tau^S_{n,k} = \sum_{r=k}^{n-1} (-1)^{r-k} \binom{r}{k} \cdot q^S_{n,r}.
\]

The Chern-Mather classes are given as follows. When $A = 2k$ is even we have

\[
\tau^S_{A,B} = \sum_{r=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r}{k} \cdot \left( \sum_{i=2r}^{B-1} (-1)^{i-2r} \binom{i}{2r} \cdot q^S_{B,i} \right) + \sum_{r=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r}{k} \cdot \left( \sum_{i=2r+1}^{B-1} (-1)^{i-2r-1} \binom{i}{2r+1} \cdot q^S_{B,i} \right).
\]

When $A = 2k + 1$ is odd, we have

\[
\tau^S_{A,B} = \sum_{r=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r}{k} \cdot \left( \sum_{i=2r}^{B-1} (-1)^{i-2r} \binom{i}{2r} \cdot q^S_{B,i} \right).
\]

For skew-symmetric rank loci, we define $E_i$ to be the Euler numbers appearing as the coefficients of the Taylor expansion

\[
\frac{1}{\cosh(x)} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n.
\]

For $k \geq 1$ we then have:

\[
\tau^S_{n,k} \equiv \mathbb{P}(Q^n); \quad \tau^\wedge_{n,k} \equiv \mathbb{P}(\wedge^2 Q^n); \quad \tau^S_{n,k} \equiv \mathbb{P}(\text{Sym}^2 Q^n).
\]

Proof of the Theorems. Recall that for all three cases, set $* = \emptyset, * = \wedge$ and $* = S$, and set $\mathbb{P}^N$ by $\mathbb{P}(M_n), \mathbb{P}(M^\wedge_n)$ and $\mathbb{P}(M^S_n)$ we have commutative diagrams of Tjurina transforms:

\[
\begin{array}{ccc}
\hat{\tau}_{n,k} & \longrightarrow & G(k,n) \times \mathbb{P}^N \\
\downarrow p & & \downarrow \\
\tau_{n,k} & \longrightarrow & \mathbb{P}^N.
\end{array}
\]

The first projection $p$ is a resolution of singularity, and is isomorphic over $\tau^S_{n,k}$. The second projections $q$ identifies the Tjurina transforms with projectivized bundles:

\[
\hat{\tau}_{n,k} \cong \mathbb{P}(Q^n); \quad \hat{\tau}^\wedge_{n,k} \cong \mathbb{P}(\wedge^2 Q^n); \quad \hat{\tau}^S_{n,k} \cong \mathbb{P}(\text{Sym}^2 Q^n).
\]

First we show that $q^*_{n,k}$ polynomials are exactly the pushforward of the classes $p_*(\hat{\tau}^S_{n,k})$ in the projective spaces $\mathbb{P}(M_n)$. Write $p_*(\hat{\tau}^S_{n,k}) = \sum_{l} \gamma^S_l H^l \in A_*(\mathbb{P}(M_n^*)), \text{ and denote } N^* = \dim \mathbb{P}(M_n^*)$. The coefficients $\gamma^S_l$ thus can be computed as $\gamma^S_l = \int_{\mathbb{P}(M^*_n)} H^{N^*-l} \cap p_*(\hat{\tau}^S_{n,k})$. Notice that the pull back of the hyperplane bundle $O_{\mathbb{P}(M^*_n)}(1)$ on $\mathbb{P}(M^*_n)$ to $\tau_{n,k}$ agrees with the tautological line bundle $O_{\tau_{n,k}}(1)$, thus we denote $O(1)$ for both of them. Since $\int_X \alpha = \int_Y f_\alpha$ for any class $\alpha$ and any proper morphism $f: X \rightarrow Y$, by the projection formula we have (omitting the obvious pullbacks):

\[
\gamma^S_l = \int_{\mathbb{P}(M^*_n)} H^{N^*-l} \cap p_*(\hat{\tau}^S_{n,k}) = \int_{\hat{\tau}_{n,k}} c_1(O(1))^{N^*-l} \cap \hat{\tau}^S_{n,k}
\]

\[
\gamma^S_l = \int_{\hat{\tau}_{n,k}} c_1(O(1))^{N^*-l} (\mathbb{P}(\tau_{n,k})) \cap [\hat{\tau}_{n,k}]
\]

\[
\gamma^S_l = \int_{\hat{\tau}_{n,k}} c(S^\vee \otimes Q)(E_{n} \otimes O(1)) c_1(O(1))^{N^*-l} \cap [\hat{\tau}_{n,k}]
\]
Here $E_*$ denotes the vector bundles $Q^{\vee n}$, $\text{Sym}^2Q^{\vee}$ and $\wedge^2Q^{\vee}$ for three types of matrices respectively. The last equation comes from the standard Euler sequence of projective bundles. Expand the tensor $c(E_* \otimes \mathcal{O}(1))$ using [13, Example 3.2.2], and then combine the definition of Segre classes we have

$$\gamma_i = \sum_{p=0}^{r} \sum_{p=0}^{n} \left( \begin{array}{c} e^* - i \\ p - i \end{array} \right) c(S^{\vee} \otimes Q)c_i(E_*)c_1(\mathcal{O}(1))^N \cdot \gamma_i^{*} \cdot H^{N+i}$$

Theorem 3.3 (Equivalent formula II). Let $S$ and $Q$ be the universal sub and quotient bundles over the Grassmannian $G(r, n)$. We define $Q^S_n(d)$ to be the following Chow (cohomology) classes (we omit the obvious \cap [G(r, n)] here):

$$Q_n^r(d) := \left( \sum_{k=0}^{r} (1 + d)^{n(r-k)} k_{c_k}(Q^{\vee n}) \right) \left( \sum_{k=0}^{n} d^{n(r-k)} k_{c_k}(S^{\vee n}) \right);$$

$$Q_n^\wedge (d) := \left( \sum_{k=0}^{r} (1 + d)^{n(r-k)} k_{c_k}(\wedge^2Q^{\vee}) \right) \left( \sum_{k=0}^{r} d^{n(r-k)} k_{c_k}(\wedge^2S^{\vee}) \right);$$

$$Q_n^\wedge (d) := \left( \sum_{k=0}^{r} (1 + d)^{n(r-k)} k_{c_k}(\text{Sym}^2Q^{\vee}) \right) \left( \sum_{k=0}^{r} d^{n(r-k)} k_{c_k}(\text{Sym}^2S^{\vee}) \right).$$

We have the following integration formals:

$$q_{n,r}(d) = \int_{G(r, n)} c(S^{\vee} \otimes Q) \cdot Q_{n,r}(d) \cap [G(r, n)] - d^{n^2 \left( \begin{array}{c} n \\ r \end{array} \right)};$$

$$q_{n,r}^\wedge (d) = \int_{G(r, n)} c(S^{\vee} \otimes Q) \cdot Q_{n,r}^\wedge (d) \cap [G(r, n)]; - d^{n^2 \left( \begin{array}{c} n \\ r \end{array} \right)};$$

$$q_{n,r}^\wedge (d) = \int_{G(r, n)} c(S^{\vee} \otimes Q) \cdot Q_{n,r}^\wedge (d) \cap [G(r, n)] - d^{n^2 \left( \begin{array}{c} n \\ r \end{array} \right)}.$$

Proof. Recall that $q_{n,k}^*(H) = \sum_{l=0}^{N^+} \gamma_l^* H^l$ are defined as the pushforward $p_* c_{sm}^* H^l$. Here $N^+ = \dim \mathbb{P}(M^*)$ are the dimensions of the projective spaces. One then has $\gamma_i^* = \int_{\gamma_i} c_{sm}^* H^{N^* + i}$. This shows that

$$q_{n,k}^*(\frac{1}{d}) = \sum_{l=0}^{N^+} \sum_{l=0}^{N^+} \int_{\gamma_i} c_{sm}^* H^{N^* + i} = \sum_{l=0}^{N^+} \int_{\gamma_i} c_{sm}^* H^{N^* + i}.$$
Here $E^*$ stands for $Q^\vee$, $\wedge^2 Q^\vee$ and $\text{Sym}^2 Q^\vee$ when $* = \emptyset$, $* = \wedge$ and $* = S$ respectively. The vector bundles $S$ and $Q$ denote the universal sub and quotient bundles over the Grassmannian $G(k, n)$. To compute above integration we will need the following Lemma.

**Lemma 1.** Let $E$ be a rank $e$ vector bundle over $X$, let $p: \mathbb{P}(E) \to X$ be the projective bundle. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological bundle. We denote its Chern class $c_1(\mathcal{L})$ by $H$. Then for any integer $d$ we have:

$$
d \cdot p_* \left( \frac{c(E \otimes \mathcal{L})}{1 - d \cdot c_1(\mathcal{L})} \right) = \left( \sum_{k=0}^{e} \sum_{i=0}^{k} \binom{e - i}{k - i} c_i(E) \cdot c_1(\mathcal{L})^{k - i} \right) \left( \sum_{k=0}^{\infty} d^k s_k(E) \right) - 1.
$$

**Proof.**

$$
c(E \otimes \mathcal{L}) = \sum_{k=0}^{e} \left( \sum_{i=0}^{k} \binom{e - i}{k - i} c_i(E) \cdot c_1(\mathcal{L})^{k - i} \right)
= \sum_{k=0}^{e} \left( \sum_{j=k}^{e} \binom{e - j + k}{k} c_{j-k}(E) \right) H^k
= \sum_{k=0}^{e} \left( \sum_{j=0}^{e-k} \binom{e - j}{k} c_j(E) \right) H^k
$$

Thus for $\frac{c(E \otimes \mathcal{L})}{1 - d \cdot c_1(\mathcal{L})}$ we have

$$
\frac{c(E \otimes \mathcal{L})}{1 - d \cdot c_1(\mathcal{L})} = \sum_{l=0}^{\infty} c(E \otimes \mathcal{L}) \cdot d^l H^l = \sum_{l=0}^{\infty} \sum_{k=0}^{e} \left( \sum_{j=0}^{e-k} \binom{e - j}{k} c_j(E) \right) d^l H^{k+l}.
$$

Since we are pushing forward the Chern classes to the base $X$, by the definition of Segre class we only concern with $H^{e-1}$ part. The coefficient for $H^{e-1}$ is

$$
\sum_{k=0}^{e-1} \left( \sum_{j=0}^{e-k} \binom{e - j}{k} c_j(E) \right) d^{e-1-k} = \frac{1}{d} \left( \sum_{k=0}^{e} d^k (1 + d)^{e-k} c_k(E) - c_0(E) \right);
$$

and the coefficient for $H^{e+l}$, $l \geq 0$ is

$$
\sum_{k=0}^{e} \left( \sum_{j=0}^{e-k} \binom{e - j}{k} c_j(E) \right) d^{e+l-k} = \sum_{k=0}^{e} d^{k+l} (1 + d)^{e-k} c_k(E)
$$

Thus we have

$$
d \cdot p_* \left( \frac{c(E \otimes \mathcal{L})}{1 - d \cdot c_1(\mathcal{L})} \right) = \sum_{k=0}^{e} d^k (1 + d)^{e-k} c_k(E) s_0(E) - c_0(E) s_0(E)
+ \sum_{l \geq 0} \left( \sum_{k=0}^{e} d^{k+l+1} (1 + d)^{e-k} c_k(E) s_{l+1}(E) \right)
= \left( \sum_{k=0}^{e} d^k (1 + d)^{e-k} c_k(E) \right) \left( \sum_{k=0}^{\infty} d^k s_k(E) \right) - 1
$$

Notice that although in the expression we have $\sum_{k=0}^{\infty} d^k s_k(E)$, this is actually a finite sum. When the degree of the Segre class exceeds the dimension of $X$, it then equals 0. \qed
Back to our case: the base space $X = G(r, n)$ is the Grassmannian. For the ordinary rank loci $* = \emptyset$, the vector bundle $E* = Q^{\vee n}$ has rank $n(n-r)$, and the ambient space $P(M_n)$ has dimension $N = n^2 - 1$. Thus we have

$$d^{n^2} \cdot q_{n,r}(\frac{1}{d}) = d \cdot d^{n^2-1} \cdot q_{n,r}(\frac{1}{d}) = d \cdot \int_{P(Q^{\vee n})} \frac{c(S^\vee \otimes Q)c(Q^{\vee n} \otimes L)}{1 - d \cdot c_1(L)} \left( \sum_{k=0}^{(n-r)} d^k (1 + d)^{n(n-r)-k} c_k(Q^{\vee n}) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$

$$= \int_{G(r,n)} c(S^\vee \otimes Q) \left( \sum_{k=0}^{(n-r)} d^k (1 + d)^{n(n-r)-k} c_k(Q^{\vee n}) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$

For the skew-symmetric rank loci $* = \wedge$, the bundle $E* = \wedge^2 Q^\vee$ is of rank $\binom{n-r}{2}$ and we have $N^\wedge = \binom{n}{2} - 1$. Thus one obtains

$$d^{\binom{n}{2}} \cdot q^\wedge_{n,r}(\frac{1}{d}) = d \cdot d^{\binom{n}{2}-1} q^\wedge_{n,r}(\frac{1}{d}) = d \cdot \int_{P(\wedge^2 Q^\vee)} \frac{c(S^\vee \otimes Q)c(\wedge^2 Q^\vee \otimes L)}{1 - d \cdot c_1(L)} \left( \sum_{k=0}^{\binom{n-r}{2}} d^k (1 + d)^{n(n-r)-k} c_k(\wedge^2 Q^\vee) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$

$$= \int_{G(r,n)} c(S^\vee \otimes Q) \left( \sum_{k=0}^{\binom{n-r}{2}} d^k (1 + d)^{n(n-r)-k} c_k(\wedge^2 Q^\vee) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$

Here we take $A_r = \binom{n}{2} - \binom{n-r}{2} = \binom{r}{2} + r(n-r)$. Notice that we have

$$c(\wedge^2 Q^\vee)c(S^\vee \otimes Q^\vee)c(\wedge^2 S^\vee) = 1; \ A + r = \binom{r}{2} + r(n-r).$$

Define $Q^\wedge_{n,r}(d)$ to be the following Chow (cohomology) class

$$\left( \sum_{k=0}^{\binom{n-r}{2}} (1 + d)^{\binom{n-r}{2}-k} c_k(\wedge^2 Q^\vee) \right) \left( \sum_{k=0}^{\binom{r}{2}} d^{\binom{r}{2}-k} c_k(\wedge^2 S^\vee) \right) \left( \sum_{k=0}^{r(n-r)} d^{r(n-r)-k} c_k(S^\vee \otimes Q^\vee) \right),$$

then the formula can be written as

$$q^\wedge_{n,r}(d) = \int_{G(r,n)} c(S^\vee \otimes Q) \cdot Q^\wedge_{n,r}(d) - d^{\binom{r}{2}} \left( \begin{array}{c} n \\ r \end{array} \right).$$

For the symmetric rank loci $* = S$, $E* = Sym^2 Q^\vee$ is of rank $\binom{n-r+1}{2}$ and $N^S = \binom{n+1}{2} - 1$. Thus we have

$$d^{\binom{n+1}{2}} \cdot q^S_{n,r}(\frac{1}{d}) = d \cdot d^{\binom{n+1}{2}-1} q^S_{n,r}(\frac{1}{d}) = d \cdot \int_{P(Sym^2 Q^\vee)} \frac{c(S^\vee \otimes Q)c(Sym^2 Q^\vee \otimes L)}{1 - d \cdot c_1(L)} \left( \sum_{k=0}^{\binom{n-r+1}{2}} d^k (1 + d)^{n(n-r+1)-k} c_k(Sym^2 Q^\vee) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$

$$= \int_{G(r,n)} c(S^\vee \otimes Q) \left( \sum_{k=0}^{\binom{n-r+1}{2}} d^k (1 + d)^{n(n-r+1)-k} c_k(Sym^2 Q^\vee) \right) - \left( \begin{array}{c} n \\ r \end{array} \right)$$
Substitute $d$ by $d^{-1}$ we then have
\[
q^S_{n,r}(d) + d^{(n+1)}\binom{n}{r} \\
= \int_{G(r,n)} c(S^r \otimes Q) \left( \sum_{k=0}^{(n-r+1)/2} (1 + d)^{(n-r+1)/2-k} c_k(Sym^2 Q^r) \right) \left( \sum_{k=0}^{\infty} d^{B_r-k} s_k(Sym^2 Q^r) \right).
\]
Here we take $B_r = \binom{n+1}{2} - \binom{n-r+1}{2}$. Notice that we have
\[
c(Sym^2 Q^r)c(S^r \otimes Q)c(Sym^2 S^r) = 1; \quad B_r = \left( \frac{r + 1}{2} \right) + r(n - r).
\]
Define $Q^S_{n,r}(d)$ to be the following Chow (cohomology) class
\[
\left( \sum_{k=0}^{(n-r+1)/2} (1 + d)^{(n-r+1)/2-k} c_k(Sym^2 Q^r) \right) \left( \sum_{k=0}^{r(n-r)} d^{r(n-r)-k} c_k(S^r \otimes Q^r) \right),
\]
then the formula can be written as
\[
q^S_{n,r}(d) = \int_{G(r,n)} c(S^r \otimes Q) \cdot Q^S_{n,r}(d) - d^{(n+1)}\binom{n}{r}.
\]
This complete the proof of the Theorem. \qed

Recall that for a projective variety, Aluffi’s $J$ involution interchanges the Chern-Schwartz-MacPherson $\gamma$ polynomial and the sectional Euler characteristics polynomial. Here the sectional Euler characteristic polynomial $\chi_X(t)$ is defined as follows: $\chi_X(t) := \sum_{k \geq 0} \chi(X \cap L^k) \cdot (-t)^k$ for $L^k$ being a generic codimension $k$ linear subspace. For generic determinantal varieties we define the $\Gamma$ polynomials as follows.

\[
d \cdot \Gamma_n, r (d) = \int_{G(r,n)} c(S^r \otimes Q) \left( \sum_{k=0}^{n(n-r)} d^k (1 + d)^{n(n-r)-k} c_k(Q^n) \right) \left( \sum_{k=0}^{\infty} d^k s_k(Q^n) \right) - \binom{n}{r},
\]
\[
d \cdot \Gamma^\wedge_n, r (d) = \int_{G(r,n)} c(S^r \otimes Q) \left( \sum_{k=0}^{n-r} d^k (1 + d)^{n-r}-k c_k(S^r) \right) \left( \sum_{k=0}^{\infty} d^k s_k(S^r) \right) - \binom{n}{r},
\]
\[
d \cdot \Gamma^S_n, r (d) = \int_{G(r,n)} c(S^r \otimes Q) \left( \sum_{k=0}^{n-r+1} d^k (1 + d)^{n-r+1}-k c_k(Sym^2 Q^r) \right) \left( \sum_{k=0}^{\infty} d^k s_k(Sym^2 Q^r) \right) - \binom{n}{r}.
\]
The $\Gamma$ polynomials are related to $q$ polynomials by $d \mapsto d^{-1}$, since the Chern-Schwartz-MacPherson $\gamma$ polynomials are related to Chern-Schwartz-MacPherson classes by $H \mapsto H^{-1}$. We have the following result.

**Corollary 1.** For any integer $d$, following the proof in Formula II we have:

\[
\chi^\circ_{r, n, k} (d) = \sum_{r=k}^{n-1} (-1)^{r-k} \binom{r}{k} \cdot \frac{d \cdot \Gamma_n, r (-1-d) + \Gamma_n, r (0)}{1+d},
\]
\[
\chi^S_{r, n, k} (d) = \sum_{r=k}^{n-1} (-1)^{r-k} \binom{r}{k} \cdot \frac{d \cdot \Gamma^S_{n, r} (-1-d) + \Gamma^S_{n, r} (0)}{1+d},
\]
\[
\chi^\wedge_{r, n, k} (d) = \begin{cases} \sum_{r=k}^{n-1} \frac{(-2r)^2}{2k} E_{2r-2k} \cdot \frac{d \Gamma_{2n-2r} (-1-d) + \Gamma^\wedge_{2n-2r} (0)}{1+d} & A = 2n, B = 2k \\
\sum_{r=k}^{n-1} \frac{(2r+1)^2}{2k+1} E_{2r-2k+1} \cdot \frac{d \Gamma_{2n+1, 2r+1} (-1-d) + \Gamma^\wedge_{2n+1, 2r+1} (0)}{1+d} & A = 2n + 1, B = 2k + 1 \end{cases}
\]

**Proof.** The proof is a direct application of Aluffi’s involution formula. The evaluations here is valid due to the fact that $\frac{t \cdot f(-1-t)+f(0)}{1+t}$ is actually a polynomial for any $f(t)$, instead of the truncation of the first $N$ terms from an infinite power series. \qed

The form of $q_{n,r}(d)$ in the Formula II is symmetric with the $d \mapsto -1 - d$ substitution, this induces the following interesting application:
Corollary 2. The polynomial $q_{n,r}(H)$ equals the Chern-Mather class of $\tau_{n,k}$, i.e.,

$$q_{n,r}(H) = c_M^{\tau_{n,k}}(H)$$

Proof. Substituting $d$ by $-1 - d$ one obtains the following:

$$q_{n,r}(-1 - d)$$

$$= \int_{G(n,r)} c(S^n \otimes Q) \left( \sum_{k=0}^{n(r-n)} (-d)^{n(r-n)-k} c_k(Q^n) \right) \left( \sum_{k=0}^{n} (-d)^{n-k} c_k(S^n) \right) - (-1 - d)^2 \left( \begin{array}{c} n \\ r \end{array} \right)$$

$$= \int_{G(n-r,n)} c(S^n \otimes Q) \left( \sum_{k=0}^{n(r-n)} (-d)^{n(r-n)-k} c_k(S^n) \right) \left( \sum_{k=0}^{n} (-1 - d)^{n-k} c_k(Q^n) \right) - (-1 - d)^2 \left( \begin{array}{c} n \\ n-r \end{array} \right)$$

$$= (-1)^n q_{n,n-r}(d) + (-1)^n d^n \left( \begin{array}{c} n \\ n-r \end{array} \right) - (-1 - d)^2 \left( \begin{array}{c} n \\ n-r \end{array} \right)$$

Recall that the dimension of $\tau_{n,r}$ is $n^2 - r^2 - 1$. Since above computation holds for any $d$, thus as polynomial we have

$$q_{n,r}(-1 - H) = q_{n,n-r}(1 + H)^n - H^n = q_{n,n-r}(H).$$

Here $q_{n,r}(H) = (-1)^{\dim \tau_{n,r}} q_{n,r}(H)$ is the signed polynomial. Recall from Theorem 2.4 that the projective duality operation $I$ takes the signed Chern-Mather class of $\tau_{n,r}$ to the signed Chern-Mather class of $\tau_{n-n-r}$. Moreover, the dual variety of $\tau_{n,k}$ is exactly $\tau_{n,n-k}$. Thus we have $I(c_k) = c_{n-k}$ for $c_k$ being the polynomial $(-1)^{\dim \tau_{n,k}} c_M^{\tau_{n,k}}$. Since $p_*(1_{\tau_{n,r}})$ is a constructible function, there are integers $\beta^r_k$ that $q_{n,r}(H) = \sum_{k=0}^{n-1} \beta^r_k c_M^{\tau_{n-k}}(H)$. Then the involution equality shows that

$$I(\hat{q}_{n,r}(H)) = I(\sum_{k=0}^{n-1} \beta^r_k c_k) = \sum_{k=0}^{n-1} \beta^r_{n-k} = \hat{q}_{n,n-r}(H) = \sum_{k=0}^{n-1} \beta^r_k c_k.$$ 

Notice that $\hat{q}_{n,r}$ is a resolution of $\tau_{n,r}$, thus we have $\beta^r_0 = 1$. Since the lowest degree terms all the polynomials $c_i$ have different degrees, expand the above equalities for $r = 1, 2, \ldots, n-1$ one can see that there is a unique solution

$$\beta^A_B = \begin{cases} 1 & A = B \\ 0 & A \neq B \end{cases}$$

This completes the proof. \hfill \Box

Remark 4. The result is classical from either the computation of local Euler obstructions [19], or the irreducibility of the intersection cohomology sheaf complex in [19]. However this proof is in a different flavor: it shows that the Chern-Schwartz-MacPherson classes completely determines the local Euler obstructions. As we will show in [5] this philosophy works in a more general setting: the reflective projective varieties. This corollary is the motivation of [5] of the paper.

Remark 5. However we fail to obtain such an easy proof for the Chern-Mather classes of symmetric and skew-symmetric determinantal varieties. This is due to the term $\sum_{k=0}^{n} d^{-k} c_k(S^n \otimes Q^n)$, when substituting $d$ by $-1 - d$ we lose the symmetry. The fact that the involution still interchanges the signed Chern-Mather classes indicate that there should be more symmetry and vanishing patterns hide behind the integrations. It would be interesting to use Schubert calculus to give a similar proof to the symmetric and skew-symmetric determinantal varieties.

4. Characteristic Cycles and Polar Degrees

In this section we take $K = \mathbb{C}$. In complex category, the theory of Chern classes can be thought of the pushdown of the theory of characteristic cycles of constructible sheaves. Consider the embedding $i: X \subset M$ of a $d$-dimensional variety into a $m$-dimensional complex manifold. The conormal space of $X$ is defined as the dimension $m$ subvariety of $T^* M$:

$$T^*_X M := \{ (x, \lambda) | x \in X_{sm} \cap (T_x X) = 0 \} \subset T^* M$$
This is a conical Lagrangian subvariety of $T^*M$. In fact, the conical Lagrangian subvarieties of $T^*M$ supporting inside $X$ are exactly the conormal spaces of closed subvarieties $V \subset X$. For a proof we refer to [21] Lemma 3. Let $L(X)$ be the free abelian group generated by the conormal spaces $T^*_V M$ for subvarieties $V \subset X$, and we call an element of $L(M)$ a (conical) Lagrangian cycle of $X$. We say a Lagrangian cycle is irreducible if it equals the conormal space of some subvariety $V$.

This group is independent of the embedding: the group $L(X)$ is isomorphic to the group of constructible functions $F(X)$ by the group morphism $Eu$ that sends $(−1)^{\dim V} T^*_V M$ to $Eu_V$. However, the fundamental classes $[T^*_X M]$ depends on the Chow ring of the ambient space. When the embedding $M$ is specified, we call $[T^*_X M] \in \mathbb{A}(T^*M)$ the Conormal cycle class of $X$ in $M$. We define the projectivized conormal cycle class of $X$ to be $Con(X) := [\mathbb{P}(T^*_X M)]$, which is a $m−1$-dimensional cycle in the total space $\mathbb{P}(T^*M)$.

Composing the two operations we obtain a group homomorphism $ch: F(X) \to A_{m−1}(\mathbb{P}(T^*M))$ sending $Eu_V$ to $(−1)^{\dim V} Con(V)$. The cycle class $ch(1_X)$ is called the Characteristic Cycle class of $X$, and denoted by $Ch(X)$. The ‘casting the shadow’ process discussed in [2] relates the $ch(1_X)$ with $c_{\varphi}$, and $ch(Eu_X)$ with $c_{\varphi}$.

**Proposition 4.1.** Let $X_n^s \subset \mathbb{P}^N$ be an EIDS of type $s$, for $s$ being $0$, $S$ or $\wedge$. Let $X_n^s \subset \mathbb{P}^N$ be the Chern-Mather class and Chern-MacPherson-Schwartz class in $\mathbb{A}(\mathbb{P}^N)$ respectively, as computed in [4]. Let $d_{n,k}^s$ be the dimension of $X_n^s$, then the projectivized conormal cycle $Con(X_n^s)$ equals:

$$Con(X_n^s) = (−1)^{d_{n,k}^s} \sum_{l=0}^{N-1} \sum_{l=j−1}^{N-1} (−1)^l \beta_l \left( \frac{l+1}{j} \right) h_1^{N+1−j} h_2^j \cap [\mathbb{P}^N \times \mathbb{P}^N].$$

The characteristic cycle of $X_n^s$ is given by

$$Ch(X_n^s) = (−1)^{d_{n,k}^s} \sum_{l=0}^{N-1} \sum_{l=j−1}^{N-1} (−1)^l \gamma_l \left( \frac{l+1}{j} \right) h_1^{N+1−j} h_2^j \cap [\mathbb{P}^N \times \mathbb{P}^N];$$

**Proof.** Firstly, note that when $M = \mathbb{P}^N$ we have the following diagram

$$\begin{array}{ccc}
P = \mathbb{P}(T^*M) & \xrightarrow{j} & \mathbb{P}^N \times \mathbb{P}^N \\
\downarrow \quad pr_1 & & \downarrow \quad pr_2 \\
M = \mathbb{P}^N & \xrightarrow{pr_2} & \mathbb{P}^N = M^* \\
\end{array}$$

Let $L_1, L_2$ are the pull backs of the line bundle $O_{\mathbb{P}^N}(1)$ of $\mathbb{P}^N$ from projections $pr_1$ and $pr_2$. Then we have $O_P(1) = j^*(L_1 \otimes L_2)$, and $j_!\mathbb{P}(T^*M] = c_1(L_1 \otimes L_2) \cap [\mathbb{P}^N \times \mathbb{P}^N]$ is a divisor in $\mathbb{P}^N \times \mathbb{P}^N$. Thus both the characteristic cycle and the conormal cycles can be realized as polynomials in $h_1 = c_1(L_1)$ and $h_2 = c_1(L_2)$, as classes in $\mathbb{A}^N(\mathbb{P}^N \times \mathbb{P}^N)$.

For any constructible function $\varphi \in F(X)$, we define the signed class $\mathcal{E}_i(\varphi) \in \mathbb{A}(\mathbb{P}^N)$ as $\{ \mathcal{E}_i(\varphi) \} = (−1)^r \{ c_{\varphi} \}$. Here for any class $C \in \mathbb{A}(M)$, $C_r$ denotes the $r$-dimensional piece of $C$. As proved in [2] Lemma 4.3, this class is exactly the shadow of the characteristic cycle $ch(\varphi)$. For $i = 1, 2$, let $h_i = c_1(L_i) \cap [\mathbb{P}^N \times \mathbb{P}^N]$ be the pull backs of hyperplane classes. Write $c_{\varphi}(\mathcal{E}_i(\varphi)) = \sum_{l=0}^{N} \gamma_l h_1^{N-l}$ as a polynomial of $H$, then by the structure theorem for projective bundles we have inversely:

$$ch(\varphi) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} (−1)^k \gamma_k \left( \frac{k+1}{j} \right) h_1^{N+1−j} h_2^j$$

as a class in $\mathbb{P}^N \times \mathbb{P}^N$. Set $\varphi$ to be $1_X$ and $Eu_X$ one obtains the proposition. □

Proved in [29, 21 Remark 2.7], the multiplicities appeared in the expression of the projectivized conormal cycle $Con(X)$ are exactly the polar degrees of $X$. Write $c_{M,n,k}^s = \sum_{l=0}^{N} \beta_l h_1^{N-l}$, then we obtain a formula for the polar degrees of $X_n^s$:

$$P_j = (−1)^{d_{n,k}^s} \sum_{l=j−1}^{N-1} (−1)^l \beta_l \left( \frac{l+1}{j} \right).$$
The sum of the polar degrees is also a very interesting invariant. It is called the generic Euclidean distance degree of $X$, and denoted by $gED(X)$. We refer to [4] for more details. The generic Euclidean distance degree of $X_{n,k}^{\ast}$ is given by

$$gED(X_{n,k}^{\ast}) = \sum_{l=0}^{d_{n,k}^{\ast}} \sum_{i=0}^{l} (-1)^i \left( \frac{d^* + 1 - i}{d^* + 1 - l} \right) \beta_{d^* - i}.$$ 

We define the following 'flip' operation in $A_{n-1}(\mathbb{P}^N \times \mathbb{P}^N)$. For any class $\alpha = \sum_{i=0}^{n} \delta_i h_i h_2^{n-i}$, its flip $\alpha^\dagger$ is defined as $\alpha^\dagger := \sum_{i=0}^{n} \delta_i h_i'^h h_2^{n-i}$. In other word, we just switch the powers of $h_1$ to $h_2$. This 'flip' process is compatible with addition: $(\alpha + \beta)^\dagger = \alpha^\dagger + \beta^\dagger$. Aluffi’s projective duality involution shows that

**Proposition 4.2.** For any projective subvariety $X \subset \mathbb{P}^N$ with dual variety $X^\vee$ we have $\text{Con}(X^\vee) = \text{Con}(X)^\dagger$. Moreover, one can see that the $i$th polar degree of $X$ equals the $(\dim X - i)$th polar degree of $X^\vee$, and hence $gED(X) = gED(X^\vee)$.

In particular, for generic determinantal varieties we have the following symmetry proposition.

**Proposition 4.3.** The characteristic cycles of $\tau_{n,1}^S$, $\tau_{2n,2}^\wedge$ and $\tau_{2n+1,3}^\wedge$ are symmetric:

$$\text{Ch}(\tau_{n,1}^S) = \text{Ch}(\tau_{n,1}^{S \dagger}); \quad \text{Ch}(\tau_{2n,2}^\wedge) = \text{Ch}(\tau_{2n,2}^{\wedge \dagger}); \quad \text{Ch}(\tau_{2n+1,3}^{\wedge}) = \text{Ch}(\tau_{2n+1,3}^{\wedge \dagger}).$$

**Proof.** First we prove for skew-symmetric case. Recall that $\text{Ch}(X) = \text{ch}(1_X)$ and $\text{Con}(X) = (-1)^{\dim X} \text{ch}(\text{Eu}_X)$, thus from Theorem 3.2 we have

$$\text{Ch}(\tau_{2n,2}^\wedge) = \sum_{i=1}^{n-1} (-1)^{i-1} \cdot (-1)^{2i(2n-2i) + (2n-2i) - 1} \text{Con}(\tau_{2n,2i}^{\wedge}) = \sum_{i=1}^{n-1} (-1)^{n-1} \text{Con}(\tau_{2n,2i}^{\wedge}).$$

We have shown that $\text{Con}(\tau_{m,n,i}) = \text{Con}(\tau_{m,n,n-i})^\dagger$, thus

$$(\text{Con}(\tau_{2n,2i}) + \text{Con}(\tau_{2n,2n-2i})) = \text{Con}(\tau_{2n,2i}^{\wedge}) + \text{Con}(\tau_{2n,2n-2i}^{\wedge}) = \text{Con}(\tau_{m,n,n-i}) + \text{Con}(\tau_{m,n,i}).$$

is symmetric. Thus we have

$$\text{Ch}(\tau_{2n,2}^\wedge) = \sum_{i=1}^{n-1} (-1)^{n-1} \text{Con}(\tau_{m,n,i})$$

$$= (-1)^{n-1} \left( \text{Con}(\tau_{m,n,1}) + \text{Con}(\tau_{m,n,n-1}) + \text{Con}(\tau_{m,n,2}) + \text{Con}(\tau_{m,n,n-2}) + \cdots \right).$$

is a sum of symmetric terms, and hence is symmetric. The proof for $\text{Ch}(\tau_{2n+1,3}^{\wedge})$ and $\text{Ch}(\tau_{n,1}^S)$ follows from the same argument, by computing the base change between indicator functions and Euler obstruction functions using Equation (1)(2) and (5)(6) in Theorem 3.2.

## 5. Milnor Classes of EID Hypersurfaces

In this section we compute the Milnor classes of the EID Hypersurfaces $X_{n,1}$, $X_{2n,2}^\wedge$ and $X_{n,1}^S$ respectively. The main result is the following.

**Theorem 5.1** (Milnor Class of EID Hypersurface). When $n \geq 3$, the Milnor class of the ordinary EID Hypersurface $X_{n,1}$, the skew-symmetric EID Hypersurface $X_{2n,2}^\wedge$ and the symmetric EID Hypersurface $X_{n,1}^S$ are given by

$$\mathcal{M}(X_{n,1}) = \frac{(-1)^{n-1} \text{Ch}_c(X_{n,1}^{2n,2})(H)}{1 + c_1(\mathcal{O}(X_{n,1}^{2n,2}))},$$

$$\mathcal{M}(X_{2n,2}^\wedge) = (-1)^{\binom{2n}{2}} \frac{\text{Ch}_c(X_{2n,2}^{2n,4})(H)}{1 + c_1(\mathcal{O}(X_{2n,2}^{2n,4}))},$$

$$\mathcal{M}(X_{n,1}^S) = (-1)^{\binom{n+1}{2}} \frac{\text{Ch}_c(X_{n,1}^{2n,2})(H) - 2 \text{Ch}_c(X_{n,1}^{2n,2})(H)}{1 + c_1(\mathcal{O}(X_{n,1}^{2n,2}))}. $$

When $n = 2$, the hypersurfaces are all smooth and thus we have

$$\mathcal{M}(X_{2,1}) = \mathcal{M}(X_{4,2}^\wedge) = \mathcal{M}(X_{2,1}^S) = 0.$$
Recall that the Milnor class is equivalent to the Euler characteristics of the local Milnor fibers. Let $F_{n,i}$, $F_{2n,2i}$ and $F_{n,1}$ be the local Milnor fibers of the hypersurfaces $X_{n,1}$, $X_{2n,2}$ and $X_{S,n}$ at the strata $X_{n,i}$, $X_{2n,2i}$ and $X_{S,n,i}$ respectively. Then we have the following equivalent description:

$$
\chi(F_{n,i}) = 0; \quad i = 2, 3, \ldots, n - 1
$$

$$
\chi(F_{2n,2i}) = 0; \quad i = 2, 3, \ldots, n - 1
$$

$$
\chi(F_{S,n,i}) = \begin{cases} 
2 & i = 2 \\
0 & i = 3, \ldots, n - 1
\end{cases}
$$

Proof. We follow the proof in Theorem 5.1 to reduce the proof to generic determinantal hypersurfaces $\tau_{n,1}$, $\tau_{2n,2}$ and $\tau_{n,1}$. Since the map $F$ intersects transversely along the strata $\tau_{n,i}$, the Segre-MacPherson classes of $\tau_{n,i}$ pulls back to Segre-MacPherson classes of $X_{n,i}$. We denote $T$ by $T_{P(V)}$, and $T$ by $T_{P(M)}$ for the corresponding tangent bundles. Let $H = c_1(O_{P(V)}(1))$ and $h = c_1(O_{P(M)}(1))$ denotes the hyperplane classes respectively, we have $F^*(h) = dH$. Assume that the theorem holds true for generic determinantal hypersurfaces. For ordinary case (by omitting the obvious pullbacks) we have

$$
c_F(X_1) = \frac{c(T)}{c(O(\mathcal{O}(dn)))} \cap [P(V)] = \frac{c(T)}{c(T)} \frac{c(T)}{c(\mathcal{O}(dn))} \cap [P(V)]
$$

$$
= \frac{c(T)}{c(T)} F^*(\frac{c(T)}{c(\mathcal{O}(n))}) \cap [P(V)] = \frac{c(T)}{c(T)} F^*(c_F(\tau_{n,1})).
$$

For $c_{sm}^X$ we have

$$
c_{sm}^X = c(T) s^{SM}(X_1, P(V)) = c(T) F^*(s^{SM}(\tau_{n,1}, P(M))
$$

$$
= c(T) F^*(\frac{c_{sm1}}{c(T)}) = \frac{c(T)}{c(T)} F^*(c_{sm1}).
$$

The same argument shows that

$$
c_{sm}^X = \frac{c(T)}{c(T)} F^*(c_{sm2}).
$$

Thus we have:

$$
c_F(X_1) - c_{sm}^X + c(\mathcal{O}(dn))^{-1} c_{sm}^X = \frac{c(T)}{c(T)} F^*(c_F(\tau_{n,1}) - c_{sm1} + c(\mathcal{O}(n))^{-1} c_{sm2}) = 0.
$$

Thus

$$
c_F(X_1) = c_{sm}^X = \frac{c_{sm}^X}{1 + dnH} = \frac{c_{sm}^X}{c(\mathcal{O}(X_{n,1}))}.
$$

The proof for skew-symmetric and symmetric case are the same.

Thus it amounts to prove the result for generic determinantal hypersurfaces $\tau_{n,1}$, $\tau_{2n,2}$ and $\tau_{S,n}$. The proof is quite technical and we will leave it to the next section. \end{proof}

Remark 6. When $k = \mathbb{C}$ in [11] Damon found Lie-type complex models for the Milnor fibers of generic determinantal hypersurfaces. Based in these models he compute the cohomology rings of the Milnor fibers and proved the fascinating Schubert decomposition theorem. Our result is compatible with [11] since $\chi(SL_m) = \chi(SL_{2n}/Sp_n) = 0$, $\chi(SL_2/SO_2) = 2$ and $\chi(SL_m/\mathbb{SO}_m) = 0$ for $m > 2$.

However, our method is different. Firstly our method is algebraic from intersection theory, thus can generalize to algebraically closed field of characteristic 0. Secondly we would like to remark here that, as one may observe in [11] our proof only use the following two properties: 1 the orbits $\tau_{n,k}$ are dual to each other; 2 the local Euler obstructions $Eu_{\tau_{n,k}}(\tau_{n,r}^*)$ are known to be binomials. Thus our method have actually proved the following stronger result:

Corollary 3 (Milnor Class of Recursive Group Orbit Hypersurface). Let $V_n$ be a sequence of recursive group $G_n$ actions defined in [39], and let $\mathcal{O}_{n,k}$ be the recursive group orbits. If $\mathcal{O}_{n,1}$ are hypersurfaces, then the Milnor classes of $\mathcal{O}(\mathcal{O}_{n,1})$ are completely determined by the local Euler obstructions $\{e_{k,r} = Eu_{\mathcal{O}_{n,k}}(\mathcal{O}_{n,r})\}$. Moreover, for any transverse map $F: \mathbb{P}^N \to \mathbb{P}(V_n)$, the Milnor class of $F^{-1}(\mathcal{O}(\mathcal{O}_{n,1}))$ is also determined by $\{e_{k,r}\}$.\end{proof}
Remark 7. For such recursive group orbit hypersurfaces their local Euler obstructions are enough to determine the Milnor classes. The converse, however, is not true: the knowledge of the local Milnor numbers are not enough. They only reveal partial information. Thus the natural question is that, for such cases, what else information one will need to obtain the local Euler obstructions. In § we will prove that, the knowledge of the Chern-Schwartz-MacPherson classes of all the orbits would be sufficient.

In terms of sectional Euler characteristic we have the following:

Corollary 4. Let \( L_r \subset \mathbb{P}(V) \) be a generic linear subspace of codimension \( r \). Then we have:

\[
-n \cdot \chi(r_{n,0}^r \cap L_r) = \chi(r_{n,1}^r \cap L_r \setminus L_{r-1}) \\
-n \cdot \chi(r_{2n,0}^r \cap L_r) = \chi(r_{2n,2}^r \cap L_r \setminus L_{r-1}) \\
-n \cdot \chi(r_{n,0}^S \cap L_r) = \chi(r_{n,1}^S \cap L_r \setminus L_{r-1}) - 2 \cdot \chi(r_{n,2}^S \cap L_r \setminus L_{r-1}).
\]

The last equation holds for \( n \geq 5 \).

Proof. Recall that the involution \( \mathcal{J} \) interchanges the Chern-Schwartz-MacPherson \( \gamma \) polynomial and sectional Euler characteristics polynomial. Then the corollary follows from the definition of \( \mathcal{J} \) mentioned in Theorem [2.3].

6. Lé Classes of EID Hypersurfaces

In this section we compute the global Lé classes of EID Hypersurfaces \( X_{n,1} \), \( X_{2n,2}^a \), and \( X_{n,1}^S \).

Proposition 6.1. Write \( \Lambda(X_{n,1}) = \sum_k \Lambda_k(X_{n,1}) \), \( \Lambda(X_{2n,2}^a) = \sum_k \Lambda_k^a(X_{2n,2}^a) \) and \( \Lambda(X_{n,1}^S) = \sum_k \Lambda_k^S(X_{n,1}^S) \) as sums of dimensional pieces. For \( i = 1, 2, 3 \) we write

\[
Ch(X_{n,1}) = \sum_k \lambda_i h_i^{2-1-k} h_2^k; \quad Ch(X_{2n,2}^a) = \sum_k \lambda_i h_i^{(n-1)-1-k} h_2^k; \quad Ch(X_{n,1}^S) = \sum_k \lambda_i h_i^{(n+1)-1-k} h_2^k.
\]

as the corresponding characteristic cycles defined in §. Then we have

\[
\Lambda_k(X_{n,1}) = (-1)^{n-1} \lambda_{2,k}; \quad \Lambda_k(X_{2n,2}^a) = (-1)^{(n^2-1)} \lambda_{2,k}; \quad \Lambda_k(X_{n,1}^S) = (-1)^{(n+1)} \lambda_{2,k} - 2 \lambda_{3,k}.
\]

Proof. First we prove the ordinary matrix case. We write \( c_{sm}^{X_{2n,2}} = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_d[\mathbb{P}^d] \) in the dimension grading form, where \( d = \dim X_{n,2} \). Then we have

\[
(-1)^{n-1} M_k(X_{n,1}) = \left( \frac{X_{n,2}^{c_{sm}^{X_{2n,2}}}}{1 + nH} \right)_k = \left( (1 - nH + n^2 H^2 - \cdots) \cap c_{sm}^{X_{2n,2}} \right)_k \equiv \left( \sum_{i=0}^{d-k} a_{k+i}(-n)^i \right)[\mathbb{P}^k]
\]

Thus from the Milnor-Lé correspondence formula in Theorem 5.1 we have

\[
(-1)^{n-1} \Lambda_k(X_{n,1}) = \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} \cap \left( H^j \cap M_{k+j}(X_{n,1}) + nH^{j+1} \cap M_{k+j+1}(X_{n,1}) \right) = \left( \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} \left( \sum_{i=0}^{d-k-j} a_{k+j+i}(-n)^i + n \cdot \sum_{i=0}^{d-k-j-1} a_{k+j+i+1}(-n)^i \right) \right)[\mathbb{P}^k]
\]

\[
= \left( \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} \left( \sum_{i=0}^{d-k-j} a_{k+j+i}(-n)^i - \sum_{i=0}^{d-k-j-1} a_{k+j+i+1}(-n)^{i+1} \right) \right)[\mathbb{P}^k]
\]

\[
= \left( \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} \left( \sum_{i=0}^{d-k-j} a_{k+j+i}(-n)^i - \sum_{i=1}^{d-k-j} a_{k+j+i}(-n)^i \right) \right)[\mathbb{P}^k]
\]

\[
= \left( \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{j} a_{k+j} \right)[\mathbb{P}^k]
\]
Proposition 7.2

7.1. Stabilization Property. First we show that, similar to the local Euler obstructions, the local Milnor numbers of the determinantal hypersurfaces only depend on the relative position of the strata.

Proposition 7.1. Let $F_{n,k}^*$ be the local Milnor fiber of $\tau_{n,1}^*$ along the stratum $\tau_{n,k}^o$. (When $\ast = \wedge$, we multiply the subscripts by 2.) Then for any $r \leq n - k + 1$, the local Milnor fibers $\chi(F_{n,k}^*)$ and $\chi(F_{n-r,k}^*)$ are homotopy equivalent.

Proof. Consider the map

$$F: M_{n-r}^* \rightarrow M_{n}; \quad A_{n-r \times n-r} \mapsto \begin{bmatrix} I_r & 0 \\ 0 & A \end{bmatrix}$$

One can see that

$$F^{-1}(\Sigma_{n,o}^*) = \begin{cases} \Sigma_{n-r,i}^o & i \leq n - r \\ 0 & i > n - r \end{cases}$$

The map $F$ preserves the codimensions of $\Sigma_{n,o}^*$, and is transverse to all the strata of $M_{n}^*$. Let $A_k^* \in \Sigma_{n-r-k}^o$ be any matrix of corank $r$, the germ $(M_{n-r}^*, A_k^* \tau_{n-k}^1) \cong (M_{n-k}^*, 0)$ can then be viewed as the ‘normal’ slice of $M_{n}^*$ at the point $F(A_k^*) \in \Sigma_{n,k}^o$. Thus locally around $F(A_k^*)$ we have isomorphism of germs

$$(M_{n-r}^*, F(A_k^*)) \cong (M_{n-r}^*, A_k^*) \times \mathbb{C}^N \cong (M_{n-k}^*, 0) \times \mathbb{C}^{N'}$$

for $N$ and $N'$ being the complementary dimensions. Moreover, the local expansion of the $n \times n$ determinant function restrict to the $n - r \times n - r$ determinant function. Thus both the local Milnor fibers of $\Sigma_{n,1}^*$ at $\Sigma_{n,k}^o$ and the local Milnor fiber of $\Sigma_{n-r,1}^*$ at $\Sigma_{n-r,k}^o$ are homotopic to the Milnor fiber of $\Sigma_{n-k,1}^*$ at 0. Since we are looking at local behavior, this passes to projectivizations.

We denote $F_k$ to be the Milnor fiber of $\Sigma_{n,1}^*$ at 0, and denote $\chi_k := \chi(F_k) - 1$ to be the reduced Euler characteristic.

7.2. Proof via Involution.

7.2.1. Ordinary Case. First we consider the ordinary case. Fix $n$ to be an integer $\geq 3$, and set $d = \dim(\mathbb{P}(M_{n}^*)) = n^2 - 1$. The determinantal hypersurface $\tau_{n,1}$ has $\tau_{n,2}$ as singular locus, and admits natural rank stratification $\sum_{i=1}^{n-1} \tau_{n,i}$. Moreover, the varieties $\tau_{n,i}$ are exactly the dual varieties of $\tau_{n-i,n}$. We will denote the Chern-Mather classes $c_{n,i}^* \in A_*(\mathbb{P}^{n-1})$ by $c_i$, as polynomials in $H = c_1(O(1))$. The Chern-Mather involution $I_d$ then takes $c_i$ to $(-1)^n c_{n-i}$. 

Recall that the Milnor class of $\tau_{n,1}$ equals $(-1)^{n^2-2}(c_F(\tau_{n,1}) - c_{n,1}^*)$, thus we just need to prove the following identity:

$$P_n(H) = P(H) := (1 + nH)c_{n,1}^* - c_{n,2}^* - \bigg((1 + H)^n - H^{n^2}\bigg) nH = 0.$$ 

Notice that $c_{n,1}^*$ and $c_{n,2}^*$ have dimension $< n^2 - 2$; then $H^2$ divides both $c_{n,1}^*$ and $c_{n,2}^*$. Thus $P(0) = c_{n,1}^*(0) - c_{n,2}^*(0) - 0 = 0$.

Theorem 5.3 shows that $c_{n,1}^*$ and $c_{n,2}^*$ can be written as the linear combinations:

$$
c_{n,1}^* = c_1 - c_2 + c_3 - \cdots + (-1)^n c_{n-1},

$$

$$
c_{n,2}^* = c_2 - 2c_3 + 3c_4 - \cdots + (-1)^{n-1}(n-2)c_{n-1}.$$

Proposition 7.2. We have the following evaluations:

$$c_i(-1) = (-1)^n \binom{n}{i}.$$ 

Moreover we have $P(-1) = 0$. 

To simplify notations we will denote \((1 + H)^{n^2} - H^{n^2}\) by \(A_{n^2}\). Then by computation one has
\[
\mathcal{I}_d(nA_{n^2}H) = \mathcal{I}_{n^2-1}(nA_{n^2}H) = (-1)^n nA_{n^2}H.
\]
Thus the identity can be rewritten as
\[
P(H) := (1 + nH) \sum_{i=1}^{n-1} (-1)^i c_i - \sum_{i=2}^{n-1} (-1)^i (i - 1) c_i - nHA = 0.
\]

**Proposition 7.3.** We have the following involution property:
\[
\mathcal{I}_d(P) = (-1)^{n-1} P.
\]

**Proof.** We prove this statement for odd and even \(n\) separately. Assume that \(P\) is a non-zero polynomial. Write \(A = A_{n^2}\) for short.

- **Case I:** \(n\) is odd. We need to show that \(\mathcal{I}(P) - P = 0\). Write \(c_{sm}\) classes into \(c_i\)'s and then apply the involution we get:
  \[
  \mathcal{I}_d(P) = \mathcal{I}_d[(1 + nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - (c_2 - 2c_3 + 3c_4 - \cdots + (n-2)c_{n-1}) - nHA]
  = (1 - n - nH)(c_{n-1} - c_{n-2} + c_{n-3} - \cdots - c_1) - nA(c_1 - c_2 - \cdots - c_{n-1} - (1))
  + [-c_{n-2} + 2c_{n-3} - 3c_{n-4} \cdots +(n-3)c_2 - (n-2)c_1 + nAH]
  = (-1 + n + nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - nA(c_1 - c_2 - \cdots - c_{n-1} - (1))
  - [n - 2](c_1 - n - 3)c_2 + \cdots - 2c_{n-3} + c_{n-2}] + nAH.
  
  Thus the subtraction \(\mathcal{I}(P) - P\) equals:
  \[
  \mathcal{I}_d(P) - P = (1 + n + nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - nA(c_1 - c_2 - \cdots - c_{n-1} - (1))
  + [(n-2)c_1 - (n-3)c_2 + \cdots - 2c_{n-3} + c_{n-2}] + nAH
  - [c_{n-2} - 2c_{n-3} + 3c_{n-4} \cdots +(n-3)c_2 - (n-2)c_1] - nAH
  = (1 - n - nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - nA(c_1 - c_2 - \cdots - c_{n-1} - (1))
  + [c_{n-2} + 2c_{n-3} + 3c_{n-4} \cdots +(n-3)c_2 - (n-2)c_1] + nAH
  
  Here the last step is due to Prop. 7.2 from which we have
  \[
  c_1 - c_2 - \cdots - c_{n-1} - (1) = \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} = (1 - 1)^n + 1 - (1)^n = 0.
  \]

- **Case II:** \(n\) is even. We need to show that \(\mathcal{I}(F) + F = 0\).
  \[
  \mathcal{I}(P) = \mathcal{I}[(1 + nH)(c_1 - c_2 + c_3 - \cdots + c_{n-1}) - (c_2 - 2c_3 + 3c_4 - \cdots - (n-2)c_{n-1}) - nHA]
  = (1 - n - nH)(c_{n-1} - c_{n-2} - c_{n-3} - \cdots + c_1) - nA(c_1 - c_2 - \cdots + c_{n-1} - (1))
  - [c_{n-2} - 2c_{n-3} + 3c_{n-4} \cdots +(n-3)c_2 - (n-2)c_1] - nAH
  
  Thus we have
  \[
  \mathcal{I}(P) + P = (1 - n - nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - nA(c_1 - c_2 - \cdots - c_{n-1} - (1))
  + [(n-2)c_1 - (n-3)c_2 + \cdots - 2c_{n-3} + c_{n-2}] - nAH
  + [(1 + nH)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - (c_2 - 2c_3 + 3c_4 - \cdots + (n-2)c_{n-1}) - nHA]
  = (2 - n)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) + (n-2)(c_1 - c_2 + c_3 - \cdots - c_{n-1}) - 2nHA
  
  Here the last step is due to Prop. 7.2
  \[
  c_1 - c_2 - \cdots - c_{n-1} - (1) = (1 - 1)^n - 1 - (1)^n = -2.
  \]
Now we prove that \( P = 0 \). We prove by induction on \( n \). When \( n = 3, 4 \), due to Theorem 5.3 this can be done by standard Schubert calculus. We refer to Macaulay2 \([17]\) for detailed computations. Now assume that we have proved \( P_n(H) = 0 \). Recall that

\[
0 = P_n(H) = nH((1 + H)^{n^2} - H^{n^2}) - (1 + nH)c_{sm}^{n^1} + c_{sm}^{n^2}
\]

\[
= (-1)^{n^2} - (1 + nH) \cdot \mathcal{M}(\tau_{n,1}) + c_{sm}^{n^2}
\]

\[
= \left( \sum_{k=2}^{n-1} (\chi(F_{n+1,k}) - 1) \cdot c_{sm}^{n,k} \right) + c_{sm}^{n^2}
\]

\[
= \sum_{k=2}^{n-1} \chi(F_k) \cdot c_{sm}^{n,k}
\]

Since the lowest term of \( c_{sm}^{n,k} \) are their fundamental classes, the fact that all the orbits are of different dimensions shows that

\[
\chi(F_k) = 0; \quad k = 2, 3, \cdots, n - 1.
\]

Thus for \( n + 1 \times n + 1 \) matrices we have

\[
P_{n+1}(H) = (n + 1)H((1 + H)^{(n+1)^2} - H^{(n+1)^2}) - (1 + (n + 1)H)c_{sm}^{n+1,1} + c_{sm}^{n+1,2}
\]

\[
= (-1)^{(n+1)^2} - (1 + (n + 1)H) \cdot \mathcal{M}(\tau_{n+1,1}) + c_{sm}^{n+1,2}
\]

\[
= \left( \sum_{k=2}^{n} (\chi(F_{n+1,k}) - 1) \cdot c_{sm}^{n+1,k} \right) + c_{sm}^{n+1,2}
\]

\[
= \sum_{k=2}^{n} \chi(F_k) \cdot c_{sm}^{n+1,k} = \chi(F_n) \cdot c_{sm}^{n+1,n}
\]

Set \( d = (n + 1)^2 - 1 \). Recall that we have just proved that \( I_d(P_{n+1}(H)) = (-1)^n P_{n+1}(H) \), and that \( \tau_{n+1,n} \) is actually smooth. This shows that

\[
I_d(P_{n+1}(H)) = I_d(\chi(F_n) \cdot c_{sm}^{n+1,n}) = \chi(F_n) \cdot I_d(c_{sm}^{n+1,n})
\]

\[
= \chi(F_n) \cdot I_d(c_{sm}^{n+1,n})
\]

\[
= (-1)^m \chi(F_n) \cdot c_{sm}^{n+1,1}
\]

\[
= (-1)^n P_{n+1}(H) = (-1)^n \chi(F_n) \cdot c_{sm}^{n+1,n}
\]

for some \( m \). However, \( c_{M}^{n+1,1} \) is a polynomial starting with \( H \) while \( c_{sm}^{n+1,n} \) is a polynomial starting with \( H^{(n+1)^2} - 2n - 1 \). Thus we must have \( \chi(F_n) = 0 \). This completes the proof for ordinary rank loci.

### 7.2.2. Skew-symmetric and Symmetric Case

The skew-symmetric case is exactly the same with ordinary case: they have the same number of orbits and share the same local Euler obstructions. The only difference is the degree: the determinant of a \( 2n \times 2n \) skew-symmetric matrix is the square of the Phaffian, thus the degree of \( \tau_{2n,2}^S \) is \( n \). The rest of the proof is just a duplicate of the one in the ordinary case, and we leave the details to the interested readers.

The case of symmetric case is slightly different, and we provide it here. In this subsection we define \( d_m = \binom{m+1}{2} - 1 = \dim \mathbb{P}(M_m^S) \). The hypersurface \( \tau_{m,1}^S \) has rank stratification \( \cup_{i=1}^{m-1} \tau_{m,i}^S \). Denote \( A = A_m \) to be the polynomial \( c_{sm}^{p_{dm}} = ((1 + H)^{d_m+1} - H^{d_m+1}) \). By simple computation we have \( I_d(mA_m H) = (-1)^{d_m+1}mA_m H \). We define the following polynomial \( P_m(H) = P(H) \), and our goal is to show that \( P = 0 \).

\[
P_m(H) = P(H) := mH((1 + H)^{d_m+1} - H^{d_m+1}) - (1 + mH) \cdot \tau_{sm}^S + \tau_{sm}^S - 2c_{sm}^S.
\]

The local Euler obstructions along the strata are different with the ordinary case: they depend on the oddness of \( m \). Since the even case is essentially the same with the odd case, we only discuss here when \( m \) is odd. Set
m = 2n + 1, and denote $c_i$ to be the Chern-Mather class $c_{M}^{\tau_1}$. From Equation (2)(3) of Theorem 3.2 we obtain the following:

$$c_{sm}^{\tau_2 n+1} = \sum_{k=0}^{n-1} (-1)^k (c_{2k+1} + c_{2k+2})$$

$$= c_1 + c_2 - c_3 - c_4 + c_5 + \cdots + (-1)^{n+1} c_{2n}$$

$$c_{sm}^{\tau_2 n+1.2} = \sum_{k=1}^{n-1} (-1)^k c_{2k+2} = c_2 - c_4 + \cdots + (-1)^{n-1} c_{2n}$$

$$c_{sm}^{\tau_2 n+1.3} = \sum_{k=1}^{n-1} (-1)^{k+1} (kc_{2k+1} + kc_{2k+2})$$

$$= c_3 + c_4 - 2c_5 - 2c_6 + \cdots + (-1)^n(n-1) c_{2n-1} + (-1)^n(n-1) c_{2n}$$

Set $d = d_{2n+1} = \binom{2n+2}{2} - 1$, then we have

$$P(H) := (2n+1)A_{2n+1}H \left((1 + H)^{d+1} - H^{d+1}\right) - (1 + (2n+1)H) \cdot c_{sm}^{\tau_2 n+1} - c_{sm}^{\tau_2 n+1.2} + 2c_{sm}^{\tau_2 n+1.3}$$

$$= (2n+1)A_{2n+1}H - (1 + (2n+1)H) \left(\sum_{k=0}^{n-1} (-1)^k (c_{2k+1} + c_{2k+2})\right)$$

$$- \sum_{k=1}^{n-1} (-1)^{k+1} ((2k-1) \cdot c_{2k} - (2k) \cdot c_{2k+1}) + (-1)^n(2n-1)c_{2n}$$

A computation on the dimension shows that the involution $I_d$ takes $c_i$ to $(-1)^{n+1} c_{2n+1-i}$. Moreover, from the local Euler obstruction knowledge in [39] we have (for some $N$ and $N'$):

$$c_{sm}^{\tau_2 n+1} (-1) = (-1)^{N+1+N'} \cdot 2.$$

Thus the involution takes $P$ to the following

$$I_{d}(P) = (-1)^{d+1}(2n+1)A_{2n+1}H + (2n + (2n+1)H) \cdot \left(\sum_{k=0}^{n-1} (-1)^{k+n+1} (c_{2n-2k} + c_{2n-2k-1})\right)$$

$$- \sum_{k=1}^{n-1} (-1)^{n+k} ((2k-1) \cdot c_{2n+1-2k} - (2k) \cdot c_{2n-2k}) - (2n-1)c_1$$

$$= (-1)^{n+1}(2n+1)A_{2n+1}H + \sum_{k=1}^{n-1} (-1)^k ((2n-2k) \cdot c_{2k} - (2n-2k-1) \cdot c_{2k+1}) - (2n-1)c_1$$

$$+ (-1)^{n+1}(2n + (2n+1)H) \left(\sum_{k=1}^{n} (-1)^{k+1} (c_{2k} + c_{2k-1})\right) + (-1)^n 2 \cdot (2n+1)A_{2n+1}H$$
Thus we have

\[-(1)^{n+1} \cdot I_d(P)\]

\[= -(2n+1)A_{2n+1}H + (2n + (2n + 1)) \left( \sum_{k=0}^{n-1} (-1)^{k+1}(c_{2k} + c_{2k+1}) \right) \]

\[+ \sum_{k=1}^{n-1} (-1)^{n+1-k} ((2n - 2k) \cdot c_{2k} - (2n - 2k - 1) \cdot c_{2k+1}) + (-1)^n(2n - 1)c_1\]

\[= -(2n+1)A_{2n+1}H + (1 + (2n + 1))H \left( \sum_{k=0}^{n-1} (-1)^{k+1}(c_{2k} + c_{2k+1}) \right) \]

\[+ (2n - 1) \left( \sum_{k=1}^{n-1} (-1)^{k+1}(c_{2k} + c_{2k+1}) \right) \]

\[+ \sum_{k=1}^{n-1} (-1)^{k+1} (2k \cdot c_{2n-2k} - (2k - 1) \cdot c_{2n-2k+1})\]

\[= -(2n + 1)A_{2n+1}H + (1 + (2n + 1))H \left( \sum_{k=0}^{n-1} (-1)^{k+1}(c_{2k} + c_{2k+1}) \right) \]

\[+ \sum_{k=1}^{n-1} (-1)^{k+1} (2k - 1) \cdot c_{2k} - (2k) \cdot c_{2k+1}) + (-1)^n(2n - 1)c_{2n}\]

\[= -P\]

Similar argument for \(m = 2n\) shows that \((-1)^{n+1}I_{(m+1)/2}^{-1}(P_{2n}(H)) = P_{2n}(H)\). In conclusion we have shown that, for symmetric determinantal case we have

\[(-1)^{i+1}I_{(m+1)/2}^{-1}(I_{m}(H)) = (-1)^{m}(P_{m}(H)).\]

The rest of the proof follow the from the ordinary case line by line: the same induction argument for \(m\) implies that \(P_{m}(H) = 0\). This completes the proof for the symmetric case \(* = S\).

8. Local Euler Obstructions of Reflective Group Orbits

As mentioned at the end of \(\S3\) in this section we propose an algorithm to compute the local Euler obstructions in the following situations. Let \(G\) be a connected algebraic group and \(V\) be a linear \(G\) representation of dimension \(N\) with finite orbits.

**Assumption 1.** Let \(O_1, \ldots, O_s\) be a maximal flag of orbits in \(V\), and label the dual orbits labeled by the duality correspondence: \(\mathbb{P}(O_k)\) is dual to \(\mathbb{P}(O_{s-k})\). Assume that the following holds:

1. \(G\) action contains the scalar multiplication, then the orbits are necessarily cones.
2. For any \(i\) we have \(O_i = \cup_{j \geq i} O_j\) and \(O'_i = \cup_{j \geq i} O'_j\).
3. All the orbits \(O_i\) and \(O'_i\) are purely dimensional.

The assumptions forces \(O_s = O'_s = \emptyset\). We call such group orbits Reflective.

**Proposition 8.1.** For any \(i\) we denote \(S_i = \mathbb{P}(O_i)\) and \(S'_i = \mathbb{P}(O'_i)\) to be the projectivized orbits in \(\mathbb{P}(V)\) and \(\mathbb{P}(V^*)\). Let \(I = I_{N-1}\) be the Aluffi’s projective duality involution in \(\mathbb{P}(V)\). Let \(c_{sm}^S = c_{sm}^S(H)\) and \(c_{sm}^S = c_{sm}^S(H)\) be the Chern-Schwartz-MacPherson polynomials in \(H\). For any \(r\), assume that there are not-all-zeros integers \(\{\alpha_i | i = r, \cdots, s - 1\}\) and \(\{\beta_j | j = s - r, \cdots, s - 1\}\) such that

\[I_{N-1} \left( \sum_{i \geq r} \alpha_i \cdot S_{sm}^i \right) = \sum_{k \geq s-r} \beta_k \cdot S_{sm}^k, \quad \alpha_r = \beta_{s-r} = 1.\]

Then we have

\[\alpha_i = Eu_{S_i}(S_i); \quad \beta_i = Eu_{S'_i}(S'_i).\]
Proof. Denote \( m_i \) and \( m_j' \) to be the signed Chern-Mather class polynomials: \( m_i : = ( -1 )^{\dim S_i} c_{SM_i}^S(H) \) and \( m_j' : = ( -1 )^{\dim S_j'} c_{SM_j'}^S(H) \). For any \( r \) there are unique integers \( E_{r,i} \) and \( E_{s-r,j} \) such that \( E_{r,r} : = ( -1 )^{\dim S_r} E_{s-r,s-r} = ( -1 )^{\dim S_{s-r}} \) and

\[
c_{SM_r}^S(H) = \sum_{i=r}^{s-1} E_{r,i} \cdot ( -1 )^{\dim S_i} c_{SM_i}^S(H) = \sum_{i=r}^{s-1} E_{r,i} m_i;
\]

\[
c_{SM_{s-r}}^S(H) = \sum_{j=s-r}^{s-1} E'_{s-r,j} \cdot ( -1 )^{\dim S_j'} c_{SM_j'}^S(H) = \sum_{j=s-r}^{s-1} E'_{s-r,j} m'_j.
\]

Since \( \bar{S}_k \) is dual to \( S_{s-k} \), recall from \([1]\) that the involution \( I_{N-1} \) then takes \( m_k \) to \( m'_{s-k} \). Thus the involution equality can be written as

\[
I_{N-1} \left( \sum_{k=r}^{s-1} \alpha_k \cdot c_{SM_k}^S(H) \right) = I_{N-1} \left( \sum_{k=r}^{s-1} \alpha_k \cdot \sum_{i=k}^{s-1} E_{k,i} m_i \right)
\]

\[
= \sum_{k=r}^{s-1} \alpha_k \cdot \sum_{i=k}^{s-1} E_{k,i} m'_{s-i} = \sum_{k=1}^{s-r-1} \sum_{i=k}^{s-1} \alpha_i E_{i,s-k} m'_k + m'_{s-r}.
\]

Since \( \alpha_r = \beta_{s-r} = 1 \), we have

\[
m'_{s-r} + \sum_{k=s-r+1}^{s} \sum_{i=s-r}^{k} \beta_j E'_{j,k} m'_k = \sum_{k=1}^{s-r-1} \sum_{i=r}^{s-k} \alpha_i E_{i,s-k} m'_k + m'_{s-r}.
\]

Since each orbit \( S'_j \) are of different dimensions, the smallest degrees of \( m'_j \) are thus all different. Thus the equality forces all the coefficients of \( m'_k \) to be 0, and then we obtain linear systems

\[
\begin{cases}
\sum_{i=r}^{s-k} \alpha_i E_{i,s-k} = 0 & k \leq s - r - 1 \\
\sum_{j=s-r}^{s-k} \beta_j E'_{j,k} = 0 & k \geq s - r - 1 \\
\alpha_r = \beta_{s-r} = 1
\end{cases}
\]

The linear system is then expressed by a \( s \times ( s + 1 ) \) upper triangular matrix with 1 or \(-1\) along the diagonal, thus the solution is of dimension 1. Then our initial condition \( \alpha_r = \beta_{s-r} = 1 \) guarantees the uniqueness of the solution. Moreover, notice that \( \alpha_i = Eu_{S_i}(S_i) \); and \( \beta_j = Eu_{S_{s-r}}(S'_j) \) is indeed a solution: the involution does interchanges the signed Chern-Mather classes. Thus by the uniqueness of the solution we completes the proof.

This proposition shows that for reflective group orbits the Chern-Schwartz-MacPherson classes of the orbits and the dual orbits completely determine the local Euler obstructions. Based on it we then propose the following algorithm:

**Algorithm 1** (Algorithm to compute the Local Euler Obstructions). The following algorithm for local Euler obstructions.

**Step 0** For each pairs of orbits \( O_i \) and \( O_j \), notice that \( Eu_{O_i}(O_j) = Eu_{S_i}(S_j) \) passes to the projective setting for \( j \leq s \).

**Step 1** For each projectivized orbit \( S_i \) we compute its Chern-Schwartz-MacPherson class \( c_{SM_i}^S(H) \).

**Step 2** For each projectivized orbit \( S'_j \) we compute its Chern-Schwartz-MacPherson class \( c_{SM_j'}^S(H) \).

**Step 3** For each \( r \) we set up the following linear system

\[
\sum_{k=r}^{s-1} x_k c_{SM_k}^S(H) - \sum_{k=s-r}^{s-1} y_k c_{SM_k}^S(H) = 0; x_r = y_{s-r} = 1.
\]

This is a linear system since \( c_{SM_k}^S(H) \) and \( c_{SM_k}^S(H) \) are polynomials in \( H \), and the equality gives at least \( s + 1 \) linear equations concerning the coefficients of powers of \( H \). As proved in the previous Proposition, the solution \( \{ x_k, y_k \} \) are the local Euler obstructions of \( O_i \) and \( O_{s-r} \) at each stratum.

**Step 4** The local Euler obstruction of \( O_r \) at \( O_s = \{ 0 \} \) come from the algebraic Brasselet-Lê-Seade type formula proved in [19].

\[
Eu_{O_r}(0) = \sum_{k \geq r} ( -1 )^{\dim S_k} c_{SM_k}^S(-1) \cdot Eu_{O_r}(O_k).
\]
Same argument applies to $Eu_{\Sigma^2_{n-r}}(0)$.

In fact, this algorithm works in a more general setting.

**Definition.** We say a projective variety $X \subset \mathbb{P}^N$ is reflective if $X$ admits a finite stratification $X = \bigcup_{i=1}^n S_i$, such that

1. The dual variety of $S_n$, denoted by $S'_1$, admits the stratification $S'_1 = \bigcup_{i} S'_i$ such that $S'_i$ is dual to $S_{n+1-i}$.
2. $S_j \subset S_i$ and $S'_j \subset S'_i$ for any $j > i$.

**Corollary 5.** The Proposition 8.1 and algorithm works for reflective projective varieties.

**Proof.** In fact one can see that the proof of Proposition 8.1 only involves the duality assumption and the dimension counts. Thus the same proof works here. \hfill $\square$

**Example 2.** We use the following example to illustrate the algorithm. We consider the symmetric rank stratification $M^S_3 = \bigcup_{i=0}^3 \Sigma^S_{3,1}$. From the computation in Theorem 3.2 we have

$$
\begin{align*}
\tau^S_{3,0} &= 3H^4 + 6H^3 + 6H^2 + 3H + 1, \\
\tau^S_{3,1} &= 3H^5 + 6H^4 + 10H^3 + 9H^2 + 3H, \\
\tau^S_{3,2} &= 3H^6 + 6H^5 + 4H^4; \\
\tau^S_{3,3} &= (-1) = -1
\end{align*}
$$

The orbits $\tau^S_{3,1}$ is dual to $\tau^S_{3,2}$, with dimension 7 and 4 respectively. Then we have

$$
c^S_{3m} \cdot (1 - H) + ((1 + H)^6 - H^6) + x \cdot c^S_{3m} \cdot (1 - H) + x \cdot ((1 + H)^6 - H^6) = c^S_{3m} \cdot (H).
$$

Expand the polynomials we get

$$3H^5 + 6H^4 + 4H^3 + x \cdot (3H^5 + 6H^4 + 10H^3 + 9H^2 + 3H) = 3H^5 + 6H^4 + 4H^3.$$

There is a unique solution $x = 0$. This shows that $Eu_{\Sigma^S_{3,1}}(\Sigma^S_{3,2}) = 0$. The Brasselet-Lê-Seade formula then gives

$$Eu_{\Sigma^S_{3,1}}(0) = Eu_{\Sigma^S_{3,1}}(\Sigma^S_{3,2}) \cdot \tau^S_{3,1} \cdot c^S_{3m} \cdot (-1) + Eu_{\Sigma^S_{3,1}}(\Sigma^S_{3,2}) \cdot \tau^S_{3,1} \cdot c^S_{3m} \cdot (-1) = (-1) \cdot (-1) \cdot 0 = 1.$$

**Remark 8.** As pointed out in [31] [37], for such group orbits their Chern-Schwartz-MacPherson classes can be obtained from representation theory. More precisely, they proved that the equivariant Chern-Schwartz-MacPherson classes are characterized by certain Axioms. The equivariant classes are expressed as polynomials in the weights of the $G$ action, the axioms then provide constraint equations these Chern polynomials have to satisfy from studying localizations and restrictions. The solutions are weight functions from representation theory. In fact, in [31] they proved that for such orbits the solutions correspond to the $K$-theoretic stable envelops studied in [23]. Thus this algorithm provides a way to study the singularities of orbits closures directly from representation theory.

9. Conjecture

We close this paper with the following conjectures:

**Conjecture 1** (Positivity). All the coefficients appeared in $c^S_{3m}$, $c^S_{n,k}$ and $c^S_{n,k}$ are non-negative.

This was proved for Schubert cells in flag manifold in [35]. We don’t know a proof for the determinantal varieties.

**Conjecture 2** (Log Concave). For $\ast$ being $\emptyset$, $\land$ and $S$, the coefficients appeared in $c^S_{sm}$, $\text{Con}(\tau^S_{n,k})$ and $\text{Ch}(\tau^S_{n,k})$ are log concave.

10. Appendix: Examples of Chern Classes

10.1. Skew-Symmetric Matrix.
10.1.1. \( n = 6 \). The total space is \( \mathbb{P}(\mathbb{M}_6^\wedge) = \mathbb{P}^{14} \).

\[
q_{6,2}^\wedge = 90H^{14} + 405H^{13} + 1290H^{12} + 2925H^{11} + 4878H^{10} + 6225H^9 + 6318H^8 + 5217H^7
+ 3504H^6 + 1863H^5 + 744H^4 + 207H^3 + 36H^2 + 3H
\]
\[
q_{6,4}^\wedge = 15H^{14} + 60H^{13} + 170H^{12} + 330H^{11} + 438H^{10} + 394H^9 + 234H^8 + 84H^7 + 14H^6
\]

Thus
\[
c_{sm}^\wedge = q_{6,2}^\wedge - 6q_{6,4}^\wedge

= 15H^{12} + 90H^{11} + 315H^{10} + 750H^9 + 1287H^8 + 1638H^7 + 1571H^6 + 1140H^5 + 621H^4
+ 248H^3 + 69H^2 + 12H + 1
\]
\[
c_{sm}^\wedge = 45H^{13} + 270H^{12} + 945H^{11} + 2250H^{10} + 3861H^9 + 4914H^8 + 4713H^7
+ 3420H^6 + 1863H^5 + 744H^4 + 207H^3 + 36H^2 + 3H
\]
\[
c_{sm}^\wedge = 15H^{14} + 60H^{13} + 170H^{12} + 330H^{11} + 438H^{10} + 394H^9 + 234H^8 + 84H^7 + 14H^6
\]

One can observe that
\[
3Hc_{sm}^\wedge = c_{sm}^\wedge .
\]

The characteristic cycles and conormal cycles are computed as:

| Table | \( h_1^4h_2 \) | \( h_1^3h_2^2 \) | \( h_1^2h_2^3 \) | \( h_1^1h_2^4 \) | \( h_1^0h_2^5 \) | \( h_1^9h_6^2 \) | \( h_1^8h_7^2 \) | \( h_1^7h_8^2 \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( Ch(\tau_{6,2}^\wedge) \) | -3 | -6 | -12 | -24 | -48 | -82 | -108 | -108 |
| \( Ch(\tau_{6,4}^\wedge) = Con(\tau_{6,4}^\wedge) \) | 3 | 6 | 12 | 24 | 48 | 68 | 66 | 42 |
| \( Con(\tau_{6,2}^\wedge) \) | 0 | 0 | 0 | 0 | 0 | -14 | -42 | -66 |

One can observe the duality in \( Con(\tau_{6,2}^\wedge) \) and \( Con(\tau_{6,4}^\wedge) \), since they are projective dual to each other. One can also observe the symmetry of \( Ch(\tau_{6,2}^\wedge) \), as proved in Proposition 4.3.

10.1.2. \( n = 7 \). The total space is \( \mathbb{P}(\mathbb{M}_7^\wedge) = \mathbb{P}^{20} \).

\[
q_{7,3}^\wedge = 210H^{20} + 1155H^{19} + 4690H^{18} + 14175H^{17} + 32970H^{16} + 61299H^{15} + 94698H^{14}
+ 125139H^{13} + 142898H^{12} + 139839H^{11} + 115038H^{10} + 77777H^9 + 42238H^8
+ 17965H^7 + 5782H^6 + 1330H^5 + 196H^4 + 14H^3
\]
\[
q_{7,5}^\wedge = 21H^{20} + 105H^{19} + 385H^{18} + 1015H^{17} + 1939H^{16} + 2695H^{15} + 2719H^{14} + 1960H^{13}
+ 966H^{12} + 294H^{11} + 42H^{10}
\]

Thus we have
\[
c_{sm}^\wedge = 105H^{18} + 945H^{17} + 4830H^{16} + 17220H^{15} + 46053H^{14} + 95991H^{13} + 159726H^{12}
+ 215523H^{11} + 238056H^{10} + 216153H^9 + 161252H^8 + 98315H^7 + 48482H^6 + 19019H^5
+ 5789H^4 + 1327H^3 + 210H^2 + 21H + 1
\]
\[
c_{sm}^\wedge = q_{7,3}^\wedge - 10q_{7,5}^\wedge

= 105H^{19} + 840H^{18} + 4025H^{17} + 13580H^{16} + 34349H^{15} + 67508H^{14} + 105539H^{13}
+ 133238H^{12} + 136899H^{11} + 114618H^{10} + 77777H^9 + 42238H^8 + 17965H^7 + 5782H^6
+ 1330H^5 + 196H^4 + 14H^3
\]
\[
c_{sm}^\wedge = 21H^{20} + 105H^{19} + 385H^{18} + 1015H^{17} + 1939H^{16} + 2695H^{15} + 2719H^{14} + 1960H^{13}
+ 966H^{12} + 294H^{11} + 42H^{10}
\]

The characteristic cycles and conormal cycles are computed as:
One can observe that observe the symmetry of Ch\(^{\tau_{7,5}}\) and Con\((\tau_{7,5})\), since they are projective dual to each other. One can also observe the symmetry of Ch\(^{\tau_{7,3}}\) proved in Proposition 4.3.

### 10.2. Symmetric Matrices.

#### 10.2.1. \(n = 3\).

The total space is \(P^5\).

- \(q_{3,1}^S = 9H^5 + 18H^4 + 18H^3 + 9H^2 + 3H\)
- \(q_{3,2}^S = 3H^5 + 6H^4 + 4H^3\)

Thus we have

- \(c_{sm}^{\tau_{3,0}} = 3H^4 + 6H^3 + 6H^2 + 3H + 1\)
- \(c_{sm}^{\tau_{3,1}} = q_{3,1}^S - 2q_{3,2}^S\)
- \(c_{sm}^{\tau_{3,2}} = 3H^5 + 6H^4 + 10H^3 + 9H^2 + 3H\)
- \(c_{sm}^{\tau_{3,3}} = 3H^5 + 6H^4 + 4H^3\)

One can observe that

\[3H \cdot c_{sm}^{\tau_{3,0}} = c_{sm}^{q_{3,1}} + 2 \cdot c_{sm}^{q_{3,2}}.\]

The characteristic cycles and conormal cycles are computed as:

| \(Ch(\tau_{3,1})\) | \(h_1^2h_2\) | \(h_1^3h_2\) | \(h_1^4h_2\) | \(h_1^5h_2\) | \(h_1^6h_2\) |
|------------------|-------------|-------------|-------------|-------------|-------------|
| \(Ch(\pi_{3,2})\) | 3 | 6 | 8 | 6 | 3 |
| \(Ch(\tau_{3,1}) = Con(\tau_{3,2})\) | 3 | 6 | 4 | 0 | 0 |
| \(Con(\tau_{3,1})\) | 0 | 0 | 4 | 6 | 3 |

The symmetry of Ch\(^{\tau_{3,1}}\) is proved in Proposition 4.3 and the duality of Con\((\tau_{3,1})\) and Con\(^{\tau_{3,2}}\) come from projective duality.

#### 10.2.2. \(n = 4\).

The total space is \(P(M^4) = P^9\).

- \(q_{4,1}^S = 24H^9 + 84H^8 + 184H^7 + 264H^6 + 264H^5 + 184H^4 + 84H^3 + 24H^2 + 4H\)
- \(q_{4,2}^S = 18H^9 + 54H^8 + 92H^7 + 96H^6 + 72H^5 + 40H^4 + 10H^3\)
- \(q_{4,3}^S = 4H^9 + 12H^8 + 16H^7 + 8H^6\)

Thus we have

- \(c_{sm}^{\tau_{4,0}} = 3H^8 + 12H^7 + 34H^6 + 60H^5 + 66H^4 + 46H^3 + 21H^2 + 6H + 1\)
- \(c_{sm}^{\tau_{4,1}} = q_{4,1}^S - 2q_{4,2}^S + 3q_{4,3}^S\)
- \(c_{sm}^{\tau_{4,2}} = 12H^8 + 48H^7 + 96H^6 + 120H^5 + 104H^4 + 64H^3 + 24H^2 + 4H\)
- \(c_{sm}^{\tau_{4,3}} = q_{4,2}^S - 3q_{4,3}^S\)
- \(c_{sm}^{\tau_{4,3}} = 6H^9 + 18H^8 + 44H^7 + 72H^6 + 72H^5 + 40H^4 + 10H^3\)

One can observe that

\[4H \cdot c_{sm}^{\tau_{4,0}} = c_{sm}^{q_{4,1}} + 2 \cdot c_{sm}^{q_{4,2}}.\]

The characteristic cycles and conormal cycles are computed as:
Table

| $\text{Ch}(\tau_{4,1}^h)$ | $h_1^0h_2^1$ | $h_1^0h_2^2$ | $h_1^2h_2^1$ | $h_1^2h_2^2$ | $h_1^4h_2^2$ | $h_1^2h_2^3$ | $h_1^4h_2^3$ | $h_1^6h_2^3$ | $h_1^4h_2^4$ |
|--------------------------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $\text{Ch}(\tau_{4,2}^S)$ = $\text{Con}(\tau_{4,2}^S)$ | 4 | 12 | 26 | 38 | 42 | 38 | 26 | 12 | 4 |
| $\text{Ch}(\tau_{4,3}^S)$ = $\text{Con}(\tau_{4,3}^S)$ | -4 | -12 | -16 | -8 | 0 | 0 | 0 | 0 | 0 |
| $\text{Con}(\tau_{4,1}^S)$ | 0 | 0 | 0 | 0 | 0 | 8 | 16 | 12 | 4 |

In fact this gives another example that $E_{u_{4,2}}(\tau_{4,3}^S) = 1$, but $\tau_{4,2}^S$ is singular at $\tau_{4,3}^S$.

**Observation 1.** All the sequences appeared above are log concave.

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