Various stabilities of the Alexander polynomials of knots and links

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ABSTRACT
In this paper, we study distribution of the zeros of the Alexander polynomials of knots and links in $S^3$. We call a knot or link real stable (resp. circular stable) if all the zeros of its Alexander polynomial are real (resp. unit complex). We give a general construction of real stable and circular stable knots and links. We also study pairs of real stable knots and links such that the zeros of the Alexander polynomials are interlaced.

Keywords: Knot, Link, 2-bridge link, Alexander polynomial, half-plane property, real stable polynomial, circular stable polynomial, interlacing zeros

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0. Introduction

Let $\mathcal{H} \subset \mathbb{C}$ be an open right half-plane, i.e., $\{\alpha \in \mathbb{C} | \text{Re}(\alpha) > 0\}$, or an open upper half-plane, i.e., $\{\alpha \in \mathbb{C} | \text{Im}(\alpha) > 0\}$. Let $f(z_1, \cdots, z_n) \in \mathbb{C}[z_1, \cdots, z_n]$ be a polynomial in $n$ variables, $z_1, \cdots, z_n$. We say that $f(z_1, \cdots, z_n)$ is $\mathcal{H}$-stable if for any values $\alpha_j \in \mathcal{H}$, $1 \leq j \leq n$, $f(\alpha_1, \cdots, \alpha_n) \neq 0$. If $\mathcal{H}$ is an open right half-plane, then $f$ is called Hurwitz stable. If $\mathcal{H}$ is an open upper half-plane, then $f$ is called a stable polynomial, and further, if $f$ is a real polynomial, $f$ is sometimes called real stable. The theory of stable polynomials has a long history, but the recent development of this theory is very impressive and is summarized in a remarkable survey article [32].

The purpose of this paper is to provide a recent study on various stabilities of the Alexander polynomials of knots or links in $S^3$. The study was motivated by our desire to answer a question (later called conjecture) posed by Jim Hoste in 2002. He asks if the real part of each zero of the Alexander polynomial $\Delta_K(t)$ of an alternating knot $K$ is larger than $-1$. It is exactly a question whether $\Delta_K(-(t+1))$ is (strongly) Hurwitz stable for an alternating knot $K$. The question leads us to other problems on stabilities of the Alexander polynomial of a (not necessarily alternating) knot. For example, since the sequence of the coefficients of a stable univariate real polynomial under a certain condition is unimodal, we see immediately that the stable Alexander polynomial of an alternating knot satisfies Trapezoidal Conjecture, one of the outstanding conjectures that still remains open. In [22], it is shown that many 2-bridge knots or links satisfy Hostefs Conjecture. Further, a few more subtle theorems on Hurwitz stability and real stability of the Alexander polynomials of 2-bridge knots or links are proven.

In this paper, knots or links are not necessarily alternating, and we discuss stabilities of the Alexander polynomials of knots or links, and further, we discuss the third stability, called circular stability, of the Alexander polynomials of knots or links.

This paper is organized as follows. It consists of two parts. The first part, consisting of Sections 1-3 is a quick review of various types of stable polynomials. Almost all materials in this part are taken from various known sources and hence proofs are entirely omitted. The rest of the paper forms the second part of the paper. In Section 4, first we introduce new notations and various terminologies and then we prove a couple of propositions on matrices. We use these propositions as basic tools to prove many theorems in this paper. For convenience, we say that a knot or link is $\mathcal{H}$-stable if its Alexander polynomial is $\mathcal{H}$-stable. In Section 5, we review some connections between stable Alexander polynomials and various conjectures in knot theory. In Sections 6 and 7, we study real stable Alexander polynomial of a knot or link. The proto-type is a 2-bridge knot or link. As is shown in [22], a 2-bridge knot with an alternating continued fraction expansion, i.e., $[2a_1, 2a_2, \cdots, 2a_m], a_ja_{j+1} < 0,$ is always stable. By generalizing these knots, we construct more general stable knots in these sections. In Section 8, we discuss some exceptional stable 2-bridge knots.
or links which have non-alternating continued fraction expansions. We discuss the
stability of the (2 variable) Alexander polynomials of such links in Section 16. One
of the important properties of stable polynomials is the “interlacing property” of
the zeros. In Sections 9 through 11, we discuss this property, first, for 2-bridge knots
or links (Section 9) and then for a generalization of 2-bridge knots, which we name
quasi-rational knots (Sections 10 and 11). In Section 12, we study circular stable
polynomials. A real polynomial \( f(z) \) is called circular stable (or c-stable) if all the
zeros of \( f(z) \) lie on the unit circle, i.e., \( |\alpha| = 1 \) for any zero \( \alpha \) of \( f(z) \). The Alexander
polynomial of a special alternating knot is always \( c \)-stable. But, many non-special
alternating knots also have the \( c \)-stable Alexander polynomial. In this section, we
give a systematic way to construct \( c \)-stable knots or links. These knots or links
are in general not alternating. Now for convenience, we call a real polynomial \( f(z) \)
bi-stable if the zero of \( f(z) \) is either real or unit complex number. In Section 13, we
prove that 2-bridge knots of a certain special type are bi-stable. A proof of the bi-
stability of a polynomial is rather complicated. To show the real stability, a Seifert
matrix plays a key role, but to show the bi-stability, the interlacing property of the
zeros is crucial. Bi-stable knots or links appear implicitly in [15] and others. Salem
fibred knots are a special type of bi-stable knots. We briefly discuss them in this
section. In Section 14, we study a Mobius transformation \( \varphi : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\} \).
There is one special Mobius transformation \( \varphi \) that relates the Alexander polynomial \( \Delta_K(t) \) of a knot \( K \)
to Hosokawa polynomial \( \nabla_{L(K)}(t) \) of a link \( L(K) \) with an
arbitrary number of components in such a way that \( \varphi \) maps all the zeros of \( \Delta_K(t) \)
to all the zeros of \( \nabla_{L(K)}(t) \). In particular,
(1) if \( \Delta_K(t) \) is \( c \)-stable, then \( \nabla_{L(K)}(t) \) is real stable, and
(2) if \( \Delta_K(t) \) is real stable, then \( \nabla_{L(K)}(t) \) is \( c \)-stable and further,
(3) if \( \Delta_K(t) \) is bi-stable, \( \nabla_{L(K)}(t) \) is bi-stable.
Then we show that given \( \Delta_K(t) \), we can express \( \nabla_{L(K)}(t) \) in terms of the coef-
ficients of \( \Delta_K(t) \).
The (reduced) Alexander polynomial of a link depends on orientation of each com-
ponent in a very delicate manner. In fact, there exists a 2-component link \( L \) such
that one orientation gives a stable link, but reversing the orientation of one com-
ponent results in a \( c \)-stable link. We call such a link inversive. We have many inversive
Montesinos links. Therefore, in Section 15, we study the Alexander polynomials of
alternating Montesinos knots or links. We specify some class of alternating Mon-
tesinos knots or links and prove that a knot \( K \) or link in this class has the following
property:
(a) If \( K \) is a knot, then \( K \) is \( c \)-stable, stable or bi-stable.
(b) If \( K \) is a link, then \( K \) is inversive.
In Section 16, we consider a 2-component link \( L \). Let \( \Delta_L(x, y) \) be the Alexander
polynomial of \( L \). The stability problem of \( \Delta_L(x, y) \) is not an easy problem, unless
\( \Delta_L(x, y) \) is multi-affine, i.e., each variable has degree at most one in each term. If
\( \Delta_L(x, y) \) is stable, then so is \( \Delta_L(t, t) \), but in general \( t^n \Delta_L(t, t^{-1}) \) is not stable, where
$n = \deg_y \Delta_L(x, y)$. Note that $t^n \Delta_L(t, t^{-1})$ is the Hosokawa polynomial of $L$ with orientation of the second component reversed. On the other hand, $t^n \Delta_L(t, -t^{-1})$ is always stable, if $\Delta_L(x, y)$ is stable. In Section 16, we discuss mainly the stability problem of the Alexander polynomials $\Delta_L(x, y)$ of 2-bridge links $L$. In the last section, Section 17, we study inversive 2-bridge links using 2-variable Alexander polynomials $\Delta_K(x, y)$.

Appendix has three sections. In Appendix A, we study the stability problem of integer polynomials considered in knot theory, particularly, the stability of Riley polynomials associated to parabolic or dihedral representation of the knot group. Riley studied these representations of the knot groups $G(K(r))$ of 2-bridge knots $K(r)$. He defined an integer polynomial $\theta_K(z)$ associated to the parabolic representation of $G(K(r))$ to $SL(2, \mathbb{C})$. It is known [26] or [30] that $\theta_K(r)(z)$ is real stable, if $r = 1/(2n + 1)$. However, if $r \neq 1/(2n + 1)$, it is usually not stable. The second polynomial Riley studied is an integer polynomial $\varphi_{2n+1}(z)$ associated to a trace-free representation of $G(K(1/2n + 1))$ onto a dihedral group $D_{2n+1} \subset GL(2, \mathbb{C})$ (see [27]). In this section, we prove that $\varphi_{2n+1}(z)$ is real stable. Since $\varphi_{2n+1}(z)$ is not reciprocal, we cannot apply the methods we used in the previous sections and our approach here is quite different. In Appendix B, we discuss the maximal values of the real parts of the zeros of the Alexander polynomials of alternating knots. Let $\delta(K)$ be the maximal value of the real parts of the zeros of $\Delta_K(t)$. It is proved in Section 4 that even for alternating knots $K$, $\delta(K)$ is not bounded, i.e., given any positive real number $\delta_0$, there exists an alternating knot $K_0$ such that $\delta(K_0) > \delta_0$. It should be noted that for a 2-bridge knot $K$, $\delta(K) < 6$ [22, 29]. However, for alternating knots, we can modify this invariant as follows. Let $\Gamma_{2n}$ be the set of all alternating knots $K$ with $\deg \Delta_K(t) = 2n$. We conjecture that $\delta(K)$ for $K$ in $\Gamma_{2n}$ is bounded, i.e., there exists a positive real number $\delta_{2n}$ such that $\delta(K) \leq \delta_{2n}$ for $K \in \Gamma_{2n}$. Further, we conjecture that $\delta_{2n}$ can be achieved by fibred alternating knots. This seems true for 2-bridge knots. If the conjecture holds, then since the number of alternating fibred knots in each $\Gamma_{2n}$ is finite, we can determine $\delta_{2n}$ for each $n$. In this section, we prove that the conjecture holds for $n = 1$ and 2. In the last section, Appendix C, we discuss the distribution of the zeros of a series of some special type of 2-bridge knots. It seems that these examples suggest many deep properties of the distribution of the zeros of the Alexander polynomials of alternating knots and links.

Finally, we note that some of the theorems in this paper have been announced without proofs in the survey article [14].
1. Stability Property

1.1. Half-plane property

Let $\mathcal{H} \subset \mathbb{C}$ be an open half-plane such that $\partial \mathcal{H}$ contains the origin. Let $f(z_1, \cdots, z_n) \in \mathbb{C}[z_1, \cdots, z_n]$ be a polynomial in $n$ variables.

**Definition 1.1.** [	extsuperscript{4}, p.303] $f \in \mathbb{C}[z_1, \cdots, z_n]$ is said to be $\mathcal{H}$-stable if $f \equiv 0$ identically, or for any values $\alpha_j \in \mathcal{H}, 1 \leq j \leq n, f(\alpha_1, \cdots, \alpha_n) \neq 0$. If $f(z_1, \cdots, z_n) \in \mathbb{C}[z_1, \cdots, z_n]$ is $\mathcal{H}$-stable for some open half-plane, we say $f$ has a half-plane property.

There are two special cases.

**Definition 1.2.** [	extsuperscript{4}, p.303] (1) Let $\mathcal{H}$ be the right-half plane, i.e., $\mathcal{H} = \{ \alpha \in \mathbb{C} | \text{Re}(\alpha) > 0 \}$. Then an $\mathcal{H}$-stable polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ is called Hurwitzstable. In other words, $f$ is Hurwitz-stable if for any $\alpha_j \in \mathbb{C}, 1 \leq j \leq n, \text{Re}(\alpha_j) > 0, f(\alpha_1, \cdots, \alpha_n) \neq 0$. (2) Let $\mathcal{H}$ be the upper-half plane, i.e., $\mathcal{H} = \{ \alpha \in \mathbb{C} | \text{Im}(\alpha) > 0 \}$. Then an $\mathcal{H}$-stable polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ is called a stable polynomial.

**Remark 1.3.** If a real polynomial $f \in \mathbb{R}[z_1, \cdots, z_n]$ is stable, $f$ is sometimes called real stable.

From definitions we see immediately

**Proposition 1.4.** Let $f(z) \in \mathbb{R}[z]$ be a real univariate polynomial. Then (1) $f(z)$ is real stable if and only if $f(z)$ has only real zeros. (2) $f(z)$ is Hurwitz-stable if and only if for any zero $\alpha$ of $f(z)$, $\text{Re}(\alpha) \leq 0$.

**Example 1.5.** (1) $f(t) = t^4 + 7t^3 + 13t^2 + 7t + 1$ is real stable and also Hurwitz-stable. (2) $f(t) = t^4 + 2t^3 - 5t^2 + 2t + 1$ is neither real stable, nor Hurwitz-stable.

The theorem below is elementary, but useful.

**Theorem 1.6.** [	extsuperscript{32}, Lemma 2.4] The following operations preserve $\mathcal{H}$-stability in $\mathbb{C}[z_1, \cdots, z_n]$.

(a) Permutation: For any permutation $\sigma \in S_n$,

$$ f \rightarrow f(z_{\sigma(1)}, \cdots, z_{\sigma(n)}) $$

(b) Scaling: For any $c \in \mathbb{C}$, and $(a_1, \cdots, a_n) \in R_+^n$ (i.e., $a_j > 0, 1 \leq j \leq n$),

$$ f \rightarrow cf(a_1z_1, \cdots, a_nz_n) $$

(c) Diagonalization: For $\{i, j\}, 1 \leq i, j \leq n$,

$$ f \rightarrow f(z_1, \cdots, z_n) |_{z_i = z_j} $$

1.2. D-stable polynomial

There is another type of stability.

**Definition 1.7.** [	extsuperscript{2}] Let $D$ be the unit open disk in $\mathbb{C}$. A polynomial $f(z) \in \mathbb{C}[z]$ is
called \(D\)-stable if for any \(\alpha \in D\), \(f(\alpha) \neq 0\).

**Proposition 1.8.** Suppose \(f(z) \in \mathbb{R}[z]\) is reciprocal, i.e., \(f(z) = z^n f(z^{-1})\) for some \(n\). If \(f(z)\) is \(D\)-stable, then all zeros \(\alpha\) of \(f(z)\) are on the unit circle, i.e., \(|\alpha| = 1\).

**Definition 1.9.** We say that \(f(z) \in \mathbb{C}[z]\) is \(c\)-stable if for each zero \(\alpha\) of \(f\), \(|\alpha| = 1\).

**Example 1.10.** (1) \(f(t) = 2t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 2\) is \(c\)-stable, but not Hurwitz-stable. (2) \(f(t) = 2t^6 - 4t^5 + 6t^4 - 9t^3 + 6t^2 - 4t + 2\) is neither \(c\)-stable, nor Hurwitz-stable. (3) \(f(t) = 3t^4 - 12t^3 + 17t^2 - 12t + 3\) is neither \(c\)-stable, nor Hurwitz-stable.

2. Hurwitz-stability

There are two basic tools to show Hurwitz-stability of a real univariate polynomial.

**2.1. Hurwitz-Routh Criterion**

Let \(f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{R}[z]\) be a real polynomial, where \(a_0 > 0, a_j \in \mathbb{R}, 0 \leq j \leq n\). Define an \(n \times n\) matrix \(H_n\) as follows:

\[
H_n = \begin{bmatrix}
    a_1 & a_0 & 0 & 0 & \cdots & 0 \\
    a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    a_{2n-1} & a_{2n-2} & \cdots & a_{n+1} & a_n \\
\end{bmatrix}
\]

where we define \(a_j = 0\) if \(j > n\).

For \(1 \leq k \leq n\), let \(H_k\) be the first \(k \times k\) principal submatrix of \(H_n\). Namely, \(H_k\) is the \(k \times k\) submatrix consisting of the first \(k\) rows and columns of \(H_n\).

For example, \(H_1 = [a_1]\) and \(H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}\).

We say that \(f(z)\) is strongly Hurwitz-stable (or simply s-Hurwitz-stable) if any zero of \(f(z)\) has a negative real part.

**Theorem 2.1 (Hurwitz-Routh Criterion).** [20, Theorem 8.8.1] A real polynomial \(f(z) = \sum_{j=0}^n a_j z^{n-j}, a_0 > 0, a_j \in \mathbb{R}, 1 \leq j \leq n\), is strongly Hurwitz-stable if and only if \(\det H_k > 0\) for \(1 \leq k \leq n\).
Using Theorem 2.1, we can characterize strongly Hurwitz-stable polynomials with small degrees.

Example 2.2. (1) \( f(z) = a_0z + a_1, a_0 > 0 \), is s-Hurwitz-stable if and only if \( a_1 > 0 \).
(2) \( f(z) = a_0z^2 + a_1z + a_2, a_0 > 0 \), is s-Hurwitz-stable if and only if \( a_1, a_2 > 0 \).
(3) \( f(z) = a_0z^3 + a_1z^2 + a_2z + a_3, a_0 > 0 \), is s-Hurwitz-stable if and only if \( a_1, a_2, a_3 > 0 \) and \( a_1a_2 > a_0a_3 \).
(4) \( f(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4, a_0 > 0 \) is s-Hurwitz-stable if and only if (i) \( a_1, a_2, a_3, a_4 > 0 \), (ii) \( a_1a_2 > a_0a_3 \), and (iii) \( a_3(a_1a_2 - a_0a_3) > a_1^2a_4 \).

2.2. Lyapunov matrix

There is another important tool to study Hurwitz-stability of a real univariate polynomial given by Lyapunov. Let \( f(z) \) be a real polynomial of degree \( n \). Let \( M \) be a companion matrix of \( f(z) \).

Theorem 2.3 (Lyapunov, [20, Theorem 8.7.2]). \( f(z) \) is strongly Hurwitz-stable if and only if there exist two real positive definite (symmetric) matrices \( V \) and \( W \) such that

\[
V M + M^T V = -W.
\]

For convenience, we call \( V \) a Lyapunov matrix associated to \( M \). It is often quite difficult to find a Lyapunov matrix even if \( f(z) \) is known to be Hurwitz-stable.

Example 2.4. (1) \( f(z) = z + a_1 \). Then \( M = [-a_1] \). If \( a_1 < 0 \), Lyapunov matrix does not exist, since \( M \) is positive definite. If \( a_1 > 0 \), then \( V = E \) is a Lyapunov matrix associated to \( M \) and \( f(z) \) is s-Hurwitz-stable.
(2) Let \( f(z) = z^2 + a_1z + a_2 \). If \( a_1, a_2 > 0 \), then we know \( f(z) \) is s-Hurwitz-stable, see Example 2.2 (2). For example, if \( a_1 = 3 \) and \( a_2 = 4 \), i.e., \( M = \begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix} \), then

\[
V = \begin{bmatrix}
7/12 & -1/2 \\
-1/2 & 5/6
\end{bmatrix}
\]

is a Lyapunov matrix and \( W = E \).

In graph theory, this concept appears in literatures. We mention one example.

Example 2.5 ([5, Theorem 1.1] and [11, p.208]). The spanning-tree polynomial of a connected finite graph is Hurwitz-stable and also stable.

3. Stable polynomial

3.1. Multivariate stable polynomials

First, we state two basic properties of stable polynomials.

Theorem 3.1 ([22, Lemma 2.4]). The following operations preserve stability in \( \mathbb{C}[z_1, \ldots, z_n] \).
(a) Specialization: For any \( a \in \mathbb{C} \) with \( \text{Im}(a) \geq 0 \), \( f \to f(a, z_2, \ldots, z_n) \)

(b) Inversion: If \( \deg_{z_1}(f) = d \), \( f \to z_1^d f(-z_1^{-1}, z_2, \ldots, z_n) \).

(c) Differentiation (or contraction) \( f \to \frac{\partial}{\partial z_i} f(z_1, \cdots, z_n) \)

Next, the following theorems give us systematic ways to construct stable polynomials.

**Theorem 3.2 ([1] Proposition 2.4).** Let \( A_i, 1 \leq i \leq n, \) be complex, semi-positive definite \( m \times m \) matrices and \( B \) be an \( m \times m \) Hermitian matrix. Then, \( f(z_1, \ldots, z_n) = \det[z_1 A_1 + \cdots + z_n A_n + B] \) is stable.

As a consequence of Theorem 3.2, we have:

**Theorem 3.3 ([4] p.308).** Let \( Z = \text{diag}(z_1, \ldots, z_n) \) be a diagonal matrix. If \( A \) is an \( n \times n \) Hermitian matrix, then both \( \det(Z + A) \) and \( \det(E + A Z) \) are stable.

If \( n = 2 \), then the converse of Theorem 3.2 holds for a real stable polynomial.

**Theorem 3.4 ([3], Theorem 1.13).** (Characterization of real stable polynomials with two variables) Let \( f(x, y) \in \mathbb{R}[x, y] \). Then \( f \) is real stable if and only if \( f \) is written as

\[
f(x, y) = \pm \det[x A + y B + C], \tag{3.1}
\]

where \( A \) and \( B \) are positive semi-definite matrices and \( C \) is a symmetric matrix of the same order.

The following theorem claims that the stability of multivariate polynomials can be reduced to the stability of univariate polynomials.

**Theorem 3.5 ([32], Lemma 2.3).** A polynomial \( f \in \mathbb{C}[z_1, \ldots, z_n] \) is stable if and only if for any \( (a_1, \ldots, a_n) \in \mathbb{R}^n \) and \( (b_1, \ldots, b_n) \in \mathbb{R}_+^n \) (i.e., \( b_j > 0, 1 \leq j \leq n \)),

\[
f(a_1 + b_1 t, \ldots, a_n + b_n t) \in \mathbb{C}[t] \]

is stable.

If a polynomial is of special type, the stability problem could be slightly simpler.

**Theorem 3.6 ([4], Theorem 5.6).** Let \( f \in \mathbb{R}[z_1, \ldots, z_n] \) be a multi-affine polynomial, (i.e., each variable \( z_j \) has degree at most 1 in each term). Then \( f \) is stable if and only if for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) and for \( 1 \leq i, j \leq n \), \( \Delta_{ij}(f)(x_1, \ldots, x_n) \geq 0 \), where \( \Delta_{ij}(f) = \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_j} - \frac{\partial^2 f}{\partial z_i \partial z_j} \).

**Remark 3.7.** If \( f \) is not multi-affine, then in Theorem 3.4 the “only if” part holds, but the “if” part does not.

**Example 3.8 ([4], Example 5.7).** Let \( f = a_{00} + a_{01} y + a_{10} x + a_{11} x y, a_{ij} \in \mathbb{R} \). Then \( \Delta_{12}(f) = - \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \). Therefore, \( f \) is stable if and only if \( \det[a_{ij}] \leq 0 \).

**Theorem 3.9 ([33], p.1).** Suppose \( f \in \mathbb{C}[z_1, \ldots, z_n] \) is homogeneous. Then, \( f \) is Hurwitz-stable if and only if \( f \) is stable.
3.2. Real stable univariate polynomials

In this subsection, we discuss real stable univariate polynomials. We are particularly interested in them, since they have many deep properties.

**Theorem 3.10 ([41, p.307]).** Let \( f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{R}[z], a_0 \neq 0, a_j \geq 0, 0 \leq j \leq n \). Suppose \( f(z) \) is real stable. If \( a_i a_k \neq 0 \) for \( i < k \), then for any \( j, i < j < k, a_j \neq 0 \). Therefore, if \( a_n \neq 0 \), then all \( a_j \neq 0 \), for \( 1 \leq j \leq n \).

Theorem 3.10 shows that it is worth studying a sequence of the coefficients of a real stable polynomial.

**Definition 3.11 ([34, p.126]).** A sequence \( \{c_0, c_1, \ldots, c_n\} \) of positive numbers is called unimodal if there exist indices \( r, s \) such that
\[
 c_0 \leq c_1 \leq \cdots \leq c_r = c_{r+1} = \cdots = c_{r+s} \geq c_{r+s+1} \geq \cdots \geq c_n. \tag{3.2}
\]
Further, \( \{c_0, c_1, \ldots, c_n\} \) is called log-concave if
\[
 c_{j-1} c_{j+1} \leq c_j^2 \quad \text{for} \quad j = 1, 2, \ldots, n - 1. \tag{3.3}
\]
If “\( \leq \)” is replaced by “\( < \)” in (3.3), then it is called strictly log-concave.

For example, binomial coefficients \( \{\binom{n}{k}\}_{k=0}^n \) is unimodal.

The following theorem is well-known.

**Theorem 3.12 ([34, Proposition p.127]).** If a positive sequence \( \{c_0, c_1, \ldots, c_n\} \) is log-concave, then it is unimodal.

Now we have an important result.

**Theorem 3.13 ([34, p.127]).** Let \( f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{R}[z], a_0 \neq 0, a_n \neq 0 \). Suppose \( a_j \geq 0, 0 \leq j \leq n \). If \( f \) is real stable (and hence \( a_j > 0 \) for all \( j \geq 0 \)), then \( \{a_0, a_1, \ldots, a_n\} \) is strictly log-concave, and hence it is unimodal.

In this case, we have either \( a_0 < a_1 < \cdots < a_r > a_{r+1} > \cdots > a_n \) or \( a_0 < a_1 < \cdots < a_r = a_{r+1} > a_{r+2} > \cdots > a_n \).

In the rest of this paper, we study various stabilities of (mostly) the Alexander polynomials of knots and links in \( S^3 \).

4. Preliminaries

From this section on, we study polynomials of knots and links from the viewpoint of stabilities. In this paper, we make a strict distinction between a knot and a link. Namely, by a link it means a disjoint union of two or more simple closed curves in \( S^3 \). If the material can be applied on knots and links as well, we always write such as “knots (or links)” Unless specified otherwise, we assume that a link is oriented, but the orientation is not usually mentioned.
A 2-bridge knot (or link) $K$ is always represented by a rational number $r = \beta/\alpha$, where we assume $0 \leq |\beta| \leq \alpha$ and $\gcd(\alpha, \beta) = 1$. When $K$ is a link, $\alpha$ is even. When $K$ is a knot, then $\alpha$ is odd and $\beta$ is assumed to be even. Then, $r$ has a unique even continued fraction expansion of the following form, where $a_i \neq 0, 1 \leq i \leq m$.

$$r = \cfrac{1}{2a_1 - \cfrac{1}{2a_2 - \cfrac{1}{2a_3 - \cfrac{1}{\ddots - \cfrac{1}{2a_{m-1} - \cfrac{1}{2a_m}}}}}}$$

This expansion is written as $r = [2a_1, 2a_2, \ldots, 2a_m]$. We call $K$ a 2-bridge knot (or a link) of type $r = \beta/\alpha$ and denoted it by $K(r)$. For example, $2/3 = [2, 2]$ represents a trefoil knot and $2/5 = [2, -2]$ represents a figure eight knot. For 2-bridge links, we assume they are oriented as in Figure 4.1. For example, $1/4 = [4]$ represents a non-fibred link and $3/4 = [2, 2, 2]$ represents a fibred torus link.

![Fig. 4.1](image)

Now, we discuss briefly a Seifert surface and Seifert matrix of a 2-bridge knot or link $K(r)$. Since we assume that $r = [2a_1, 2a_2, \ldots, 2a_m]$, $K(r)$ has a natural Seifert surface $F$ depicted in Fig. 4.2 below.

![Fig. 4.2](image)
Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be (oriented) simple closed curves on $F$ as are shown in Fig. 4.2. Then $\alpha_j, 1 \leq j \leq m$ forms a basis for $H_1(F, \mathbb{Z})$. Let $u_{i,j} = lk(\alpha_i^+, \alpha_j)$, where $\alpha_i^+$ denotes the simple closed curve in $S^3$ that is a slight lift of $\alpha_i$ toward the positive normal direction. Then $M = [u_{i,j}]_{1 \leq i,j \leq m}$ is a Seifert matrix of $K(r)$. In this paper, we call $M$ a standard Seifert matrix of $K(r)$. It is easy to see from Fig. 4.2 that $M$ is as below left (resp. right) when $m$ is even (resp. odd).

Then the Alexander polynomial $\Delta_{K(r)}(t)$ of $K(r)$ is defined by $\Delta_{K(r)}(t) = \det(tM - M^T)$.

For 2-bridge knots and links, we have other particular forms of Seifert surfaces depicted in Fig. 4.3, where the boxes contain some full-twists. A Seifert surface for a 2-bridge knot or link is obtained by successively plumbing unknotted twisted annuli, where the shaded squares indicate the glueing squares for plumbing. The usual way is depicted in Fig. 4.3 top. This surface is isotopic to those in Fig 4.2. The surface depicted in Fig. 4.3 bottom is new. Note that two surfaces are not in general isotopic, but bound the same 2bridge knot or link if the corresponding boxes contain the same number of full-twists.

To be more precise, given $r = [2a_1, 2a_2, \ldots, 2a_n]$, let $A_1, A_2, \ldots, A_n$ be unknotted annuli such that $A_i$ has $a_i$ full-twists. In Fig 4.3 top, for, $2 \leq i \leq n$, $A_i$ is plumbed on the negative (resp. positive) side of $A_{i-1}$ if $i$ is even (resp. odd). This type of plumbed surface is said to be of chain type. Meanwhile in Fig 4.3 bottom, every annulus is plumbed on the negative side of the proceeding annulus. This type of plumbed surface is said to be of twisted chain type. A plumbed surface of twisted chain type has a Seifert matrix of the following form, which is also said to be of twisted type.
If $K$ is a link (of $\mu$ components, $\mu \geq 2$), $\Delta_K(t_1, t_2, \cdots, t_\mu)$ denotes the (multivariate) Alexander polynomial of $K$. \cite{[21]} Then $\Delta_K(t, t, \cdots, t)/(t - 1)^{\mu - 2} := \nabla_K(t)$ is called the Hosokawa polynomial of $K$. \cite{[10]} The degree of $\nabla_K(t)$ is even and $(t - 1)^{\mu - 1}\Delta_K(t)$ is called the reduced Alexander polynomial of a link $K$. We denote by $\Delta_K(t)$ the reduced Alexander polynomial of a link $K$, but we call it simply the Alexander polynomial of a link $K$. Generally, if $M$ is a Seifert matrix of a knot (resp. a link) $K$, then the Alexander polynomial (resp. reduced Alexander polynomial) is defined as $\det(tM - M^T)$. To study the zeros of a real stable polynomial $f(z) \in \mathbb{R}[z]$, it is usually assumed that the leading coefficient is positive. Since $\Delta_K(t)$ is defined up to $\pm t$, we denote by $D_K(t)$ the Alexander polynomial of a knot (or link) $K$ with a positive leading coefficient and a non-zero constant term. We may call $D_K(t)$ the normalized Alexander polynomial of $\Delta_K(t)$, or the normalization of $\Delta_K(t)$. We should note that for a 2-bridge knot or link $K(r)$ with $r = [2a_1, 2a_2, \ldots, 2a_n]$, the normalization $D_{K(r)}(t) = \Delta_{K(r)}(t)$ is given by $D_{K(r)}(t) = \varepsilon \det(tM - M^T)$, where $\varepsilon = \prod_{j=1}^{n} \frac{a_j}{|a_j|}$. For a knot $K$, $\Delta_K(t)$ is reciprocal, namely, $\Delta_K(t) = t^n\Delta_K(t^{-1})$ for some integer $n$. However, for a link, $\Delta_K(t)$ is not necessarily reciprocal, but the Hosokawa polynomial $\nabla_K(t)$ is always reciprocal.

Suppose that $f(t)$ is a reciprocal real polynomial of degree $2n$. Let $x = t + \frac{1}{t}$. Then $t^{-n}f(t)$ can be written uniquely as a real polynomial $F(x)$ in $x$ of degree $n$. For convenience, we call $F(x)$ the modified polynomial or modification of $f(t)$. We should note that if $f(t) = \Delta_K(t)$ for a knot $K$, then $F(x)$ is equivalent to the Conway polynomial of $K$. To be more precise, let $C_K(z) = a_0z^{2n} + a_1z^{2n-2} + \cdots + a_{n-1}z^2 + a_n$ be the Conway polynomial of a knot $K$. Then $F(x) = a_0(x - 2)^n + a_1(x - 2)^{n-1} + \cdots + a_{n-1}(x - 2) + a_n$. $F(x)$ is not necessarily reciprocal.

For convenience, we call a knot $K$ (or link) (real) stable, c-stable or bi-stable if the Alexander polynomial of $K$ is, respectively, (real) stable, c-stable or bi-stable. Further, we call the complex zero $\alpha$ of $\Delta_K(t)$ with $|\alpha| = 1$ a unit complex zero of $\Delta_K(t)$. If the bi-stable Alexander polynomial has both real zeros and unit complex zeros, we call it strictly bi-stable and such a knot or link is called strictly bi-stable. If any zero of $\Delta_K(t)$ is neither real nor unit complex, we say $\Delta_K(t)$ is totally unstable and a knot (or link) $K$ is called totally unstable. Therefore, a knot (or link) is classified into five classes: stable, c-stable, strictly bi-stable, totally unstable and none of them. Note that in literature, a stable knot may have appeared in a different sense. However, in this paper, we use our terminologies.
A link is never totally unstable, since $\Delta_K(1) = 0$. If a knot $K$ is totally unstable, then $\deg \Delta_K(t)$ is divisible by 4. This is because if $\alpha$ is a complex zero of $\Delta_K(t)$, then so are $\bar{\alpha}, \frac{1}{\alpha}, -\alpha$. The Hosokawa polynomial $\nabla_K(t)$ of a link can be totally unstable if $\deg \nabla_K(t)$ is a multiple of 4.

The stability problem of $\Delta_K(t)$ can be checked by using modified polynomial of $\Delta_K(t)$ as follows.

**Proposition 4.1.** Let $F(x)$ be the modified polynomial of $\Delta_K(t)$ of a knot $K$. Then $K$ is bi-stable if and only if $F(x)$ is real stable. (Therefore, if $F(x)$ does not have a real zero, $K$ is totally unstable.) Further, if $K$ is bi-stable, the number of the real zeros of $\Delta_K(t)$ is exactly twice the number of the real zeros $\alpha$ of $F(x)$ (counting multiplicity) such that $|\alpha| \geq 2$.

**Proof.** We prove a slightly more general statement. Let $f(t)$ be a real reciprocal polynomial of degree $2n$. We write

$$ f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_{2n-1} t + c_0, c_j = c_{2n-j}, 0 \leq j \leq 2n. $$

Express $f(t) = c_0 \prod_{j=1}^{2n} (t - \alpha_j)$ and $F(x) = c_0 \prod_{j=1}^{n} (x - \beta_j)$. Suppose $\beta_j$ is real and $|\beta_j| \geq 2$. Then $\beta_j$ gives two real zeros of $f(t)$, since $t - \beta_j + 1/t = 0$ has two real zeros. However, if $|\beta_j| < 2$, then $t - \beta_j + 1/t = 0$ gives two unit complex zeros. This proves the “if” part.

Conversely, suppose that $f(t)$ is bi-stable. If $\alpha_j$ is real, then $\alpha_j + \frac{1}{\alpha_j}$ is real and hence the corresponding zero of $F(x)$ is real and further, $|\alpha_j + \frac{1}{\alpha_j}| \geq 2$. If $\alpha$ is unit complex, then $\alpha_j + \frac{1}{\alpha_j} = \alpha_j + \overline{\alpha_j}$ is real, and hence the corresponding zero of $F(x)$ is real and $|\alpha_j + \frac{1}{\alpha_j}| < 2$. 

\[\square\]

**Remark 4.2.** In [35], Wu proved a similar proposition using Conway polynomial.

**Example 4.3.** Let $f(t) = t^6 - 3t^5 + 2t^4 - t^3 + 2t^2 - 3t + 1$. Then $F(x) = (x^3 - 3x) - 3(x^2 - 2) + 2x - 1 = x^3 - 3x^2 - x + 5$. $F(x)$ has three real zeros, two of which are in the interval ($-2, 2$). Therefore, $f(t)$ has two real zeros and four unit complex zeros, and hence $f(t)$ is strictly bi-stable.

In the rest of this section, we show four elementary but useful propositions. The first proposition is well-known.

**Proposition 4.4.** (Min-Max Theorem) Let $M$ be a real symmetric matrix of order $n$. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ be eigenvalues of $M$. Then,

$$ \alpha_1 \leq \min \{\text{diagonal entries of } M\} \quad \text{and} \quad \alpha_n \geq \max \{\text{diagonal entries of } M\}. $$

Since the second proposition is less known, we give a proof.

**Proposition 4.5.** (Strong Positivity Lemma) Let $M = [a_{i,j}]_{1 \leq i,j \leq n}$ be an $n \times n$
real matrix such that for $i = 1, 2, \cdots, n$,

\begin{align*}
(1) & \ a_{i,i} > 0, \\
(2) & \ a_{i,i} > |a_{i,1}| + |a_{i,2}| + \cdots + |a_{i,i-1}| + |a_{i,i+1}| + \cdots + |a_{i,n}| . \quad (4.2)
\end{align*}

Then $\det M > 0$.

For convenience, a row that satisfies (2) is called excessive.

**Proof.** If $n = 1$ or 2, the proposition is trivially true. Suppose the proposition holds for $(n-1) \times (n-1)$ matrices. Let

\begin{equation}
\begin{bmatrix}
\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\
0 & a_{12} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_{1n}
\end{bmatrix}
\end{equation}

be an $n \times n$ matrix of the form:

\begin{align*}
(1) & \ \lambda_{i,i} = 1, 1 \leq i \leq n, \\
(2) & \ \lambda_{1,k} = -\frac{a_{k,1}}{a_{1,1}}, 1 \leq k \leq n, \\
(3) & \ \lambda_{i,j} = 0, \text{ for } i \neq j \text{ or } i \neq 1. \quad (4.3)
\end{align*}

Then $PM = \begin{bmatrix}
\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\
0 & a_{12} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_{1n}
\end{bmatrix}$, where $M' = \left[ a'_{ij} \right]_{2 \leq i,j \leq n}$ is an $(n-1) \times (n-1)$ matrix of the form: For $i, j = 2, \cdots, n$, $a'_{i,j} = a_{i,j} - \frac{a_{i,1}a_{1,j}}{a_{1,1}}$.

(4.4)

We claim that $a'_{i,i} > 0$ and every row of $M'$ is excessive. If $a_{i,1} = 0$, then $a'_{i,2} = a_{i,2}, \cdots, a'_{i,n} = a_{i,n}$ and hence $a'_{i,i} = a_{i,i} > 0$, and since the $i$th row of $M$ is excessive, the $i$th row of $M'$ is also excessive. Suppose $a_{i,1} \neq 0$. We show

\begin{equation}
a'_{i,i} > |a'_{i,2}| + |a'_{i,3}| + \cdots + |a'_{i,i-1}| + |a'_{i,i+1}| + \cdots + |a'_{i,n}| . \quad (4.5)
\end{equation}

First, for $j \neq i$ and $j \geq 2$, $|a'_{i,j}| = |a_{i,j} - a_{i,1}\frac{a_{1,j}}{a_{1,1}}| \leq |a_{i,j}| + \frac{|a_{i,1}||a_{1,j}|}{a_{1,1}}$ and hence

\begin{equation}
\sum_{j=2, j \neq i}^{n} |a'_{i,j}| \leq \sum_{j=2, j \neq i}^{n} |a_{i,j}| + \frac{|a_{i,1}|}{a_{1,1}} \sum_{j=2, j \neq i}^{n} |a_{1,j}| . \quad (4.6)
\end{equation}

Next, for $i \geq 2$, since $a_{i,i} > |a_{i,1}|$ and $a_{1,1} > |a_{1,i}|$, we see

\begin{equation}
a'_{i,i} = a_{i,i} - \frac{a_{i,1}a_{1,i}}{a_{1,1}} \geq a_{i,i} - \frac{|a_{1,i}||a_{1,1}|}{a_{1,1}} > 0. \quad (4.7)
\end{equation}
From (4.6) and (4.7), we have;

\[
a_{i,i}' - \sum_{j=2, j \neq i}^{n} |a_{i,j}'| \geq a_{i,i} - \sum_{j=2, j \neq i}^{n} |a_{i,j}| - \frac{|a_{i,1}|}{a_{1,1}} \left( |a_{1,i}| + \sum_{j=2, j \neq i}^{n} |a_{1,j}| \right) \]

\[
= a_{i,i} - \sum_{j=2, j \neq i}^{n} |a_{i,j}| - \frac{|a_{i,1}|}{a_{1,1}} \left( |a_{1,i}| + \sum_{j=2, j \neq i}^{n} |a_{1,j}| \right) = a_{i,i} - \sum_{j=2, j \neq i}^{n} |a_{i,j}| - \frac{|a_{i,1}|}{a_{1,1}} \sum_{j=2}^{n} |a_{1,j}|. \tag{4.8}
\]

By assumption,

\[
a_{i,i} - \sum_{j=2, j \neq i}^{n} |a_{i,j}| > |a_{i,1}| \quad \text{and} \quad \delta = \sum_{j=2}^{n} \frac{|a_{1,j}|}{a_{1,1}} < 1. \tag{4.9}
\]

Therefore, from (4.9), we have;

\[
a_{i,i}' - \sum_{j=2, j \neq i}^{n} |a_{i,j}'| > |a_{i,1}| - |a_{i,1}| \delta > 0. \tag{4.10}
\]

This proves (4.5).

Since all rows of \(M'\) are excessive and \(a_{i,i}' > 0, 2 \leq i \leq n\), by (4.7), it follows by induction that \(\det M' > 0\) and hence \(\det M = \det (PM) = a_{1,1} \det M' > 0\). \(\square\)

**Remark 4.6.** In the above proof, even if the \(i^{th}\) row of \(M\) is not excessive, i.e., \(a_{i,i} = |a_{i,1}| + |a_{i,2}| + \cdots + |a_{i,i-1}| + |a_{i,i+1}| + \cdots + |a_{i,n}|\), a new \(i^{th}\) row of \(M'\) becomes excessive, provided \(a_{i,1} \neq 0\) and the first row is excessive. In fact, the first inequality in (4.9) becomes an equality, but the second inequality holds, and hence (4.10) holds.

Let \(P\) be a matrix representing an elementary matrix operation. Then we say, for convenience, that a matrix \(PM'P^T\) is obtained from \(M\) by applying an elementary matrix operation on the rows (and columns) of \(M\) simultaneously. For example, we say something like that a new matrix \(M'\) is obtained by interchanging the \(i^{th}\) row (and column) and the \(j^{th}\) row (and column) of \(M\) simultaneously.

Using Remark 4.6, we can obtain the same conclusion under a slightly weaker assumption than that of Proposition 4.5.

**Proposition 4.7.** (Positivity Lemma) Let \(M = [a_{i,j}]_{1 \leq i,j \leq n}\) be an \(n \times n\) real
matrix. Assume that

1. $M$ cannot be transformed into a form:
\[
\begin{bmatrix}
A & B \\
O & C
\end{bmatrix}
\text{ or } \begin{bmatrix}
A & O \\
B & C
\end{bmatrix}
\]

by a sequence of exchanges of the $i^{th}$ row (and column) and the $j^{th}$ row (and column) simultaneously, where $A$ and $C$ are square matrices.

2. For $i = 1, 2, \cdots, n$, $a_{i,i} > 0$.

3. For $i = 1, 2, \cdots, n$, $a_{i,i} \geq |a_{i,1}| + \cdots + |a_{i,i-1}| + |a_{i,i+1}| + \cdots + |a_{i,n}|$.

4. $M$ has at least one excessive row.

Then $\det M > 0$. Further, if $M$ is symmetric, then $M$ is positive definite.

**Remark 4.8.** If (4) is dropped, then $\det M \geq 0$. Suppose (1) is dropped. Then if each of the block matrices $A$ and $C$ satisfies (4.11) (1) - (4), then $\det M > 0$.

**Proof of Proposition 4.7.** We may assume without loss of generality that $M$ has been arranged by a sequence of exchanges of rows and columns, simultaneously, in such a way that

1. the first $k$ rows are excessive, but each of the remaining rows are not
2. for each $i = k+1, \cdots, n$, at least one of $a_{i,1}, a_{i,2}, \cdots, a_{i,i-1}$ is not 0.

Write $M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$, where $A$ is a $k \times k$ matrix. Starting from the first row, we can transform $A$ into an upper triangular matrix $A'$ by a sequence of row operations (without changing the value of the determinant) to get $M' = \begin{bmatrix}
A' & B' \\
C & D
\end{bmatrix}$, where $A' = \begin{bmatrix}
a_{11} & a'_{12} & \cdots & * \\
a_{22} & \cdots & \cdots & \cdots \\
O & \cdots & \cdots & \cdots \\
& & & a'_{kk}
\end{bmatrix}$. By Remark 4.6, each row of the first $k$ rows is excessive. Now by (4.12), we may assume without loss of generality that one of $a_{k+1,1}, \cdots, a_{k+1,k}$ is not 0. Apply row operations repeatedly on the $(k+1)^{st}$ row so that the first $k$ entries of the $(k+1)^{st}$ row of $M'$ become 0, and also, the new $(k+1)^{st}$ row is excessive by Remark 4.6. We can repeat this until all rows are excessive. Then apply Proposition 4.5 to show that $\det M > 0$. Furthermore, the same argument can be applied to show that all principal minors of $M$ are positive, and hence if $M$ is symmetric, then $M$ is positive definite.

Finally, we prove the following proposition.

**Proposition 4.9.** Let $M$ be a real matrix of the form:

$M = \begin{bmatrix}
A & O \\
H & B
\end{bmatrix}$ or $\begin{bmatrix}
A & H \\
O & B
\end{bmatrix}$, where $A$ is a $p \times p$ positive definite symmetric matrix and $B$ a $q \times q$ negative definite symmetric matrix and $H$ an arbitrary matrix.
Then $M^{-1}M^T$ is conjugate to a symmetric matrix in $GL(p+q, \mathbb{R})$. Therefore, the characteristic polynomial $f(t)$ of the matrix $M^{-1}M^T$ is real stable.

**Proof.** First, if $A = E_p$ and $B = -E_q$, then $M^2 = E_{p+q}$, and hence, $M^{-1}M^T = MM^T$ is symmetric. Now, consider the general case. Let $P_A$ and $P_B$ be, respectively, matrices which diagonalize $A$ and $B$. Write $P_A = \text{diag}\{a_1, a_2, \cdots, a_p\}$, $a_j > 0$, for $1 \leq j \leq p$, and $P_B = \text{diag}\{-b_1, -b_2, \cdots, -b_q\}$, $b_j > 0$, for $1 \leq j \leq q$. Let $D_a = \text{diag}\{1/\sqrt{a_1}, 1/\sqrt{a_2}, \cdots, 1/\sqrt{a_p}\}$ and $D_b = \text{diag}\{1/\sqrt{b_1}, 1/\sqrt{b_2}, \cdots, 1/\sqrt{b_q}\}$. Further, let $P = P_A \oplus P_B$ and $D = D_a \oplus D_b$. Then a simple computation shows that $DPM^TDB = M_0$, where $M_0 = \begin{bmatrix} E_p & 0 \\ H_0 & -E_q \end{bmatrix}$. Now since $M_0^2 = E_{p+q}$, it follows that $M = P^{-1}D^{-1}M_0(DT)^{-1}(PT)^{-1}$ and $M^{-1}M^T = P^T D^T M_0 DPP^{-1} D^{-1}(D^{-1})^T (P^{-1})^T = P^T D^T M_0 M_0^T (D^T)^{-1} (P^T)^{-1}$, and hence, $M^{-1}M^T$ is conjugate of a symmetric matrix $M_0M_0^T$. □

5. The Alexander polynomials of alternating knots

Before we concentrate on the study of various stabilities of knots or links, we discuss, in this section, some connection between the stability of alternating knots or links and various conjectures in Knot theory.

5.1. Hoste’s Conjecture

In 2002, based on his extensive calculations of the zeros of the Alexander polynomials, Hoste made the following conjecture.

**Conjecture 5.1 (J. Hoste, 2002).** Let $K$ be an alternating knot and $\Delta_K(t)$ the Alexander polynomial of $K$. Then for any zero $\alpha$ of $\Delta_K(t)$, $\text{Re}(\alpha) > -1$.

One of the key observations is that Conjecture 5.1 is equivalent to the following

**Conjecture 5.2.** Under the same assumption, $\Delta_K(-(t + 1)) \in \mathbb{R}[t]$ is strongly Hurwitz-stable.

Using Lyapunov matrices, the following theorem is proved.

**Theorem 5.3 ([22 Theorem 1]).** Let $K$ be a 2-bridge knot (or link). Then $\Delta_K(-(t + 3))$ and $\Delta_K(t + 6)$ are strongly Hurwitz-stable. Equivalently, any zero $\alpha$ of $\Delta_K(t)$ satisfies

$$-3 < \text{Re}(\alpha) < 6.$$  (5.1)
For other special results, see [22, Theorems 3,4 and 5].

**Remark 5.4.** A.Stoimenow proves in [29] that for a 2-bridge knot (or link) $K$, any zero $\alpha$ of $\Delta_K(t)$ satisfies

$$\left| \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right| < 2 \quad (5.2)$$

This implies

$$-1 < \text{Re}(\alpha) < 3 + \sqrt{8} = 5.8284... \quad (5.3)$$

It should be noted that for a non-alternating knot, neither a lower bound nor an upper bound of $\text{Re}(\alpha)$ exist [22, Examples 1 and 2]. Further, we think that an upper bound of $\text{Re}(\alpha)$ does exist only for a family of 2-bridge knots or links. In fact, there exists an infinite sequence of alternating stable (Montesinos) knots $K_1, K_2, \ldots, K_m, \ldots$ such that the maximal value of the zeros of $\Delta_{K_m}(t)$ is at least $m + 1$. (See Theorem [15.2] Case 3.) Therefore, in general, an upper bound of $\text{Re}(\alpha)$ does not exist, even for alternating knots. However, an upper bound may exist for some family of the Alexander polynomials. For example, let $\Gamma_n$ be the set of all Alexander polynomials (of degree $n$) of alternating knots.

**Conjecture 5.5.** There exist a real number $\delta_n > 0$ such that for any zero $\alpha$ of $\Delta_K(t)$ in $\Gamma_n$

$$\text{Re}(\alpha) \leq \delta_n \quad (5.4)$$

It is known that Conjecture 5.5 is false for non-alternating knots [22, Example 2]. Since the Alexander polynomial of an alternating knot $K$ is of the form $\Delta_K(t) = \sum_{j=n}^{2n} (-1)^j c_j t^{2n-j}$, $c_j > 0$, $0 \leq j \leq 2n$, it follows that if $\Delta_K(t)$ is real stable, then all the zeros are positive and hence Conjecture 5.1 holds. Therefore, we have:

**Theorem 5.6.** Let $K$ be an alternating knot. If $K$ is bi-stable, then Conjecture 5.1 holds for $K$.

### 5.2. Trapezoidal Conjecture

Let $\Delta_K(t) = \sum_{j=0}^{2n} (-1)^j c_j t^{2n-j}$ be the Alexander polynomial of an alternating knot $K$. Then the Trapezoidal conjecture claims:

**Conjecture 5.7.** [6] There is an integer $k$, $1 \leq k \leq n$, such that

$$c_0 < c_1 < \cdots < c_k = c_{k+1} = \cdots = c_{2n-k} > c_{2n-k+1} \cdots > c_{2n} \quad (5.5)$$

This conjecture has been proven for several families of alternating knots, but not proven in general. [9], [25] etc. Further, this conjecture does not hold for Hosokawa polynomials of alternating links. For example, a 2-bridge link $K(r), r = 11/14$ is bi-stable, and satisfies Trapezoidal conjecture, since $\Delta_K(t) = t^5 - 3t^4 + 3t^3 - 3t^2 + 3t - 1,$
but its Hosokawa polynomial $\nabla_K(t) = \Delta_K(t)/(1 - t) = t^4 - 2t^3 + t^2 - 2t + 1$ does not.

The trapezoidal property of the coefficients is quite similar to the unimodality of a sequence considered in Section 3.

For an alternating knot $K$, $\Delta_K(-t) = \sum_{j=0}^{2n} c_j t^{2n-j}$ satisfies all assumptions of Theorem 3.13 and hence the coefficient sequence is strictly log-concave. Therefore, we have

$$c_0 < c_1 < \cdots < c_n > c_{n+1} > c_{n+2} > \cdots > c_{2n}$$  \hspace{1cm} (5.6)

and hence we obtain the following:

**Theorem 5.8.** For an alternating stable knot (or link) $K$, Trapezoidal conjecture holds.

We should note that if the number of components of an alternating stable link $K$ is even, then $\deg \Delta_K(t)$ is odd, say $2n + 1$, and (5.6) should be replace by (5.7) below.

$$c_0 < c_1 < \cdots < c_n = c_{n+1} > c_{n+2} > \cdots > c_{2n+1}$$  \hspace{1cm} (5.7)

For a knot $K$, there is another necessary condition for $\Delta_K(t)$ to be stable.

**Proposition 5.9.** If $K$ is a stable knot, then the signature $\sigma(K)$ of $K$ is zero.

In fact, if the signature is not zero, $\Delta_K(t)$ has at least two zeros on the unit circle $\{24\}$. However, the converse of Proposition 5.9 is false. For example let $K(r)$ be the 2-bridge knot where $r = [2, 2, -4, -2]$. Then, $\sigma(K) = 0$ but the zeros of $\Delta_K(t) = 2 - 6t + 9t^2 - 6t^3 + 2t^4$ are $1 \pm i$ and $\frac{1 \pm i}{2}$. Further, the following example shows that for links, Proposition 5.9 does not hold.

**Example 5.10.** Let $L$ be an alternating pretzel link $P(2, 4, 4)$, oriented so that $L$ is a special alternating 3-component link. Then the reduced Alexander polynomial $\Delta_L(t) = 8(t - 1)^2$ that is stable, while $\sigma(L) = 2$.

We suspect that Hoste’s conjecture and the Trapezoidal conjecture are independent. However, for alternating knots the condition $\sigma(K) = 0$ may imply (5.6). Therefore, we propose the following conjecture.

**Conjecture 5.11.** Let $\Delta_K(t) = \sum_{j=0}^{2n} (-1)^i c_j t^{2n-j}, c_j > 0$ be the Alexander polynomial of an alternating knot $K$. If $\sigma(K) = 0$, then the coefficient sequence satisfies (5.6), i.e., $c_0 < c_1 < \cdots < c_n > c_{n+1} > c_{n+2} > \cdots > c_{2n}$. More generally, if $\sigma(K) = 2k$, then the coefficient sequence satisfies (5.8) below:

$$c_0 < c_1 < \cdots < c_{n-m-1} < c_{n-m} = \cdots = c_{n+m} > c_{n+m+1} > \cdots > c_{2n},$$  \hspace{1cm} (5.8)

where $m \leq k$. 
This conjecture is quite likely true for 2-bridge knots. However, it is false for non-alternating knots. In fact, the signature of a non-alternating knot $10_{132}$ is 0, but the Alexander polynomial is $t^4 - t^3 + t^2 - t + 1$.

**Remark 5.12.** In graph theory, the concept of unimodality has been used in [8]. Very recently, the following long-standing conjecture was proved in [17].

**Conjecture 5.13.** [8, p. 534] The sequence of the coefficients of the chromatic polynomial is unimodal.

### 6. Construction of real stable knots (I)

The first family of alternating stable knots or links was given in Theorem 6.1 below. In the proof, Seifert surfaces in Fig. 4.2 were used to show that there exists a symmetric companion matrix $M$ of the Alexander polynomial of $K(r)$, and hence, all the eigenvalues of $M$ are real.

**Theorem 6.1 ([22, Theorem 2]).** Let $r = [2a_1, 2a_2, \ldots, 2a_m]$ be an even continued fraction expansion of a rational number $r = \beta/\alpha$. If the sequence $\{a_1, a_2, \ldots, a_m\}$ alternates in sign, i.e., $a_j a_{j+1} < 0$, $1 \leq j \leq m - 1$ then the 2-bridge knot (or link) $K(r)$ is real stable.

In this section, first, we construct a new surface that is a slight generalization of a Seifert surface for a 2-bridge knot (or link), and then, using these surfaces, we define a knot (or link), called a quasi-rational knot (or link), and generalize Theorem 6.1.

#### 6.1. Quasi-rational knots

In this subsection, we define the class of “quasi-rational links” which is a generalization of 2-bridge links.

**Definition 6.2.** Let $D$ be a disk with two families $\Gamma_1 = \{\alpha_1, \ldots, \alpha_p\}$, $\Gamma_2 = \{\beta_1, \ldots, \beta_q\}$ of properly embedded arcs in $D$, where no arcs share their end points on $\partial D$. In each family, the arcs are disjoint, but arcs from different families may intersect one another. Each arc, say $\gamma$, is assigned with a non-zero integer $w(\gamma)$ called a weight. Push the interior of $\alpha_i$’s (resp. $\beta_j$’s) in the positive (resp. negative) normal direction of $D$, and along each pushed arc $\gamma'$, attach a band to $D$ with $w(\gamma)$ half-twists. The boundary of the resulting surface $F(\Gamma_1, \Gamma_2)$ is called a quasi-rational knot or link. Conventionally, the arcs in $\Gamma_2$ are depicted in dotted lines. (See Fig. 6.1.)

The surface $F$ is orientable if and only if all weights are even. In that case, $F(\Gamma_1, \Gamma_2)$ is a Murasugi sum of two Seifert surfaces $F(\Gamma_1, \emptyset)$ and $F(\emptyset, \Gamma_2)$ along
where the summands are respectively connected sums of elementary torus links. Hence by [7], \( F(\Gamma_1, \Gamma_2) \) is of minimal genus.

**Example 6.3.** As a particular example, we see in Fig. 4.2 or Fig. 6.1 that 2-bridge knots and links \( K(r) \) are quasi-rational. For weights \( \{2w(\alpha_1), 2w(\alpha_2), \ldots, 2w(\alpha_p)\} \), \( \{2w(\beta_1), 2w(\beta_2), \ldots, 2w(\beta_q)\} \), a nice Seifert matrix \( M = \begin{bmatrix} A_1 & O \\ C & A_2 \end{bmatrix} \) is obtained from \( F(\Gamma_1, \Gamma_2) \), where \( A_1 = \text{diag}\{w(\alpha_1), w(\alpha_2), \ldots, w(\alpha_p)\} \), \( A_2 = \text{diag}\{w(\beta_1), w(\beta_2), \ldots, w(\beta_q)\} \) and in \( C \), all diagonal entries are \(-1\), \( (k, k+1)\)-entries are 1 and the other entries are 0. For convenience, we say that \( M \) is of split type.

![Fig. 6.1: \( r = [2a_1, 2a_2, \ldots] \)](image)

**6.2. Stable quasi-rational knots**

In this subsection we generalize Theorem 6.1 to some classes in quasi-rational knots or links.

**Theorem 6.4.** Let \( F(\Gamma_1, \Gamma_2) \) be a Seifert surface for a quasi-rational knot or link. Suppose the weights for \( \Gamma_1 \) (resp. \( \Gamma_2 \)) are all positive (resp. negative) and even. Then \( L = \partial F(\Gamma_1, \Gamma_2) \) is alternating and stable.

**Proof.** By turning the arcs in \( \Gamma_1 \) outside of the disk \( D \), we have an alternating diagram for \( L \).

Let \( \{2w(\alpha_1), 2w(\alpha_2), \ldots, 2w(\alpha_p)\} \) and \( \{2w(\beta_1), 2w(\beta_2), \ldots, 2w(\beta_q)\} \) be weights for \( \Gamma_1 \) and \( \Gamma_2 \), where \( w(\alpha_i) \)'s are positive and \( w(\beta_j) \)'s are negative. Take a natural bases of \( H_1(F(\Gamma_1, \Gamma_2)) \), where each loop is a union of the core curve of an attached band and a curve in \( D \). The orientations of the loops are arbitrary. Then we have a Seifert matrix \( M = \begin{bmatrix} A_1 & O \\ C & A_2 \end{bmatrix} \), where \( A_1 = \text{diag}\{w(\alpha_1), w(\alpha_2), \ldots, w(\alpha_p)\} \), \( A_2 = \text{diag}\{w(\beta_1), w(\beta_2), \ldots, w(\beta_q)\} \). Since \( \Delta_L(t) = \det(tM - M^T) = (\det M) \det(tE - M^{-1}M^T) \), it follows from Proposition 4.9 that \( \Delta_L(t)/\det M \) is the characteristic polynomial of a symmetric matrix, and hence \( \Delta_L(t) \) is stable.

Finally, we see that the signature of \( L \) is \( p - q \). Since a stable knot has signature 0, it follows that if \( p \neq q \), \( L \) is not a stable knot, but a stable link.

**6.3. Examples**

In this subsection, we construct two series of stable knots, both of which are quasi-rational knots. These knots contain first two alternating non-2-bridge stable knots.
whose Alexander polynomials have the real zeros larger than 6.

**Example 6.5.** The family of knots denoted by

\[ X_n(2a_1, 2a_2, \cdots, 2a_n \mid 2b_1, 2b_2, \cdots, 2b_n) \]

is depicted in Fig. 6.2. For example, \( X_1(2 \mid -2) \) is \( 4_1 \) and \( X_2(2, 2 \mid -2, -2) \) is \( 8_{12} \) and \( X_3(2, 2, 2 \mid -2, -2, -2) \) is \( 12_{a0125} \). If all \( a_j, 1 \leq j \leq n, \) are positive and all \( b_j, 1 \leq j \leq n, \) are negative, then \( X_n \) is alternating and always stable. A Seifert surface is obtained by applying Seifert’s algorithm to Fig. 6.2 (right).

![Fig. 6.2](image.png)

**Example 6.6.** The family of knots denoted by

\[ Y_{2n+1}(2a_1, 2a_2, \cdots, 2a_{2n+1} \mid 2b_1, 2b_2, \cdots, 2b_{2n+1}) \]

is depicted in Figure 6.3, together with a Seifert matrix in the case of \( n = 3 \). For example, \( Y_1(2 \mid -2) \) is \( 4_1 \) and \( Y_3(2, 2, 2 \mid -2, -2, -2) \) is \( 12_{a1124} \). As before, if all \( a_j, 1 \leq j \leq 2n+1, \) are positive and all \( b_j, 1 \leq j \leq 2n+1, \) are negative, then \( Y_{2n+1} \) is alternating and always stable.

![Fig. 6.3](image.png)

We note that \( 12_{a0125} \) and \( 12_{a1124} \) are only alternating knots with at most 12 crossings such that the real part of the zeros is larger than 6. In fact, they are 6.904 ⋅⋅⋅ for \( 12_{a0125} \) and 7.699 ⋅⋅⋅ for \( 12_{a1124} \). Furthermore, these values increase as \( n \) increases unboundedly if all \( a_j = 1 \) and all \( b_j = -1, \) as is proved in Theorem 6.7 below.

**Theorem 6.7.** (1) Let \( K_n^{(1)} = X_n(2, 2, \cdots, 2 \mid -2, -2, \cdots, -2) \). Then \( K_n^{(1)} \) is stable and the maximal value of the zeros is at least \( n + 1 \).

(2) Let \( K_{2n+1}^{(2)} = Y_{2n+1}(2, 2, \cdots, 2 \mid -2, -2, \cdots, -2) \). Then \( K_{2n+1}^{(2)} \) is stable and the maximal value of the zeros is at least \( 2n + 1 \).
**Proof.** (1) In the proof of Theorem 6.4, the matrix \( C \) is a lower triangular matrix such that every entry under the diagonal entries and diagonal entry as well are 1, but 0 elsewhere. Since all \( a_j \) are 1 and all \( b_j \) are \(-1\), it follows from Proposition 4.9 that a companion matrix of the Alexander polynomial of \( K_n^{(1)} \) is
\[
\begin{bmatrix}
E_n & C^T \\
C & E_n + CC^T
\end{bmatrix}
= S.
\]
It is obvious that the maximal value of the diagonal entries of \( S \) is \( n + 1 \). This proves (1) by Min-Max Theorem.

(2) In this case \( C \) is like the lower left matrix in Fig.6.3. Then by the same argument, we see that the maximal value of the diagonal entries of
\[
\begin{bmatrix}
E_{2n+1} & C^T \\
C & E_{2n+1} + CC^T
\end{bmatrix}
\]

is \( 2n + 1 \). This proves (2). \( \square \)

These two quasi-rational knots are quite unlikely Montesinos knots. However, the infinite sequences \( \{K_n^{(1)}\} \) and \( \{K_{2n+1}^{(2)}\} \) give the evidence that the maximal value of the real parts of the zeros of Alexander polynomials of alternating knots is unbounded. Further, the maximal zero \( 7.699 \cdots \) of the Alexander polynomial of \( 12_{a_{1124}} \) is quite likely the number \( \delta_6 \) defined in Section 5. (See Appendix B.)

**Remark 6.8.** Since \( K_n^{(1)} \) and \( K_{2n+1}^{(2)} \) both are strongly negative amphicheiral, their Alexander polynomials \( \Delta(t) \) have the property: \( \Delta(t^2) = f(t)f(-t) \) for some \( f(t) \in \mathbb{Z}[t] \). See [10]. Therefore, their Conway polynomials are of the form: \( g(z)g(-z) \) for some \( g(z) \in \mathbb{Z}[z] \). Furthermore, it is easy to show that \( X_n(2a_1, 2a_2, \cdots , 2a_n|2b_1, 2b_2, \cdots , 2b_n) \) is strongly negative amphicheiral if \( b_j = -a_{n+1-j}, 1 \leq j \leq n \). Similarly, \( Y_{2n+1}(2a_1, 2a_2, \cdots , 2a_{2n+1}|2b_1, 2b_2, \cdots , 2b_{2n+1}) \) is strongly negative amphicheiral if \( b_j = -a_j, 1 \leq j \leq 2n + 1 \). Therefore, the Conway polynomials of these knots are of the form: \( g(z)g(-z) \) for some \( g(z) \in \mathbb{Z}[z] \) and generally these knots are neither stable nor \( c \)-stable unless all \( a_j \)'s have the same sign.

7. Construction of real stable knots (II)

In this section, we show a more general construction of stable knots or links.

7.1. Positive or negative disks

Let \( D \) be a disk and divide \( D \) into small domains by a (not necessarily connected) plane graph \( G \). Let \( \{v_1, v_2, \cdots , v_n, v_1^\circ, v_2^\circ, \cdots , v_k^\circ\} \) and \( \{e_1, e_2, \cdots , e_\ell\} \) be, respectively, the set of vertices and edges of \( G \), where \( v_j^\circ, 1 \leq j \leq k \), is a vertex on \( \partial D \), and \( e_j, 1 \leq j \leq \ell \), is not a part of \( \partial D \) and \( e_j \) does not intersect \( \partial D \) except its ends. We call \( v_j \) an interior vertex and \( v_j^\circ \) a boundary vertex. Any part of the boundary of \( D \) is not considered as an edge of \( G \).
We assume

1. To every interior vertex \( v_j \) of \( G \), there is a path in \( G \) that connects \( v_j \) to some boundary vertex \( v_j^\circ \).

2. The valency of \( v_j \) is at least 2.

Now each edge \( e_j, 1 \leq j \leq \ell \), has a weight, \( w(e_j) = m_j \), a non-zero integer, as before. If all weights are even, we call such a graph an even graph. If some weights are odd, then we assume that \( G \) satisfies condition (7.2) below. Let \( d_1, d_2, \ldots, d_m \) be the domains in which \( G \) divides \( D \) and \( \{ e_{j_1}, e_{j_2}, \ldots, e_{j_s} \} \) be the set of all edges on the boundary of \( d_j, 1 \leq j \leq m \). Then for \( 1 \leq j \leq m \),

\[
\sum_{i=1}^s w(e_{j_i}) \equiv 0 \pmod{2}
\]  

(7.2)

If \( G \) is an even graph, then (7.2) is satisfied automatically. A weighted graph \( G \) that satisfies (7.2) is called \textit{admissible}. If every edge \( G \) has positive (or negative) weight, \( G \) is called a positive (or negative) graph. Now, first, we replace each interior vertex with a small disk and then, as we did in the previous section, replace each edge \( e_j, 1 \leq j \leq \ell \), with a narrow band of \( w(e_j) = m_j \) half-twists. The resulting surface is denoted by \( F(G) \) and is called the surface representing a weighted graph \( G \). The projection of the 1-skeleton of \( F(G) - \text{int}(D) \) on \( D \) is \( G \). \( F(G) \) is orientable if and only if \( G \) is admissible. See Fig. 7.1.

![Fig. 7.1](image)

7.2. Stable alternating knots and links

We prove the following theorem that is a generalization of Theorem 6.4.

\textbf{Theorem 7.1.} Let \( \{ F(G_1^+), \ldots, F(G_p^+) \} \) and \( \{ F(G_1^-), \ldots, F(G_q^-) \} \) be respectively the set of surfaces representing even positive and negative graphs \( \{ G_1^+ \} \) and \( \{ G_1^- \} \).
Suppose $K$ is obtained as the boundary of a Murasugi sum of these surfaces, where gluing is only allowed between surfaces of positive graphs and those of negative graphs. Then $K$ is alternating and real stable.

Proof. From a construction, a Seifert matrix $M$ of $K$ is of the form of (7.3) below, where $A$ is the direct sum of positive definite symmetric matrices and $C$ is the direct sum of negative definite symmetric matrices, and $B$ is obtained from the information of the gluing process of a Murasugi sum.

\[
M = \begin{bmatrix}
A & O \\
B & C
\end{bmatrix}
\] (7.3)

Then by Proposition 4.9, $\Delta_K(t)/\det(M)$ is the characteristic polynomial of a symmetric matrix, and hence $\Delta_K(t)$ is stable. This proves Theorem 7.1. ☐

7.3. Pseudo-positive or pseudo-negative disk

Let $G$ be a weighted even plane graph on a disk $D$ that divides $D$ as before. Suppose $G$ is neither positive nor negative. Let $F(G)$ be the surface obtained from $G$. We call $G$ a pseudo-positive (or pseudo-negative) if the Seifert matrix obtained from $F(G)$ is positive definite (or negative definite). In general, $F(G)$ is not alternating. However, we have the following theorem.

Theorem 7.2. Let $\{G_i^+\}, 1 \leq i \leq p$ and $\{G_j^-\}, 1 \leq j \leq n$ be the sets of even pseudo-positive and pseudo-negative graphs, respectively. Let $\{F(G_1^+), \cdots, F(G_p^+)\}$ and $\{F(G_1^-), \cdots, F(G_n^-)\}$ be respectively the sets of surfaces representing $\{G_i^+\}$ and $\{G_j^-\}$. Suppose $K$ is obtained as the boundary of a Murasugi sum of these surfaces, where gluing is only allowed between surfaces of pseudo-positive graphs and those of pseudo-negative graphs. Then $K$ is real stable.

Since a proof is essentially the same as that of Theorem 7.1 we omit the details. We note that $K$ is not necessarily alternating.

Remark 7.3. If an even graph $G$ is neither positive nor negative, then a Seifert matrix $M$ of $F(G)$ may not be positive definite or negative definite. However, we may change at most $s$ weights of $G$ so that $F(G)$ becomes pseudo-positive or pseudo-negative, where $s$ is the first Betti number of $H_1(F(G); \mathbb{Z})$. For details, see Section 12.

7.4. Example

Let $G_1$ and $G_2$ be even weighted graphs on disks as in Fig.7.2. Let $F(G) = F(G_1) \ast F(G_2)$, a Murasugi sum of $F(G_1)$ and $F(G_2)$. Then $K = \partial F(G)$ is not stable, but bi-stable, since $\Delta_K(t) = t^6 - 4t^4 + 7t^3 - 4t^2 + 1$ has two real and four unit complex
zeros. Now change $G_1$ to $G'_1$ by giving three new even weights to $G_1$ as shown in Fig 7.2. Then $G'_1$ is pseudo-positive and $K' = \partial F(G'_1) * F(G_2)$ is stable. In fact, 
$$\Delta_{K'}(t) = t^6 - 15t^5 + 60t^4 - 93t^3 + 60t^2 - 15t + 1$$ is stable. 

8. Exceptional stable knots and links

If the sequence of a continued fraction expansion of $r$ alternates in sign, then $K(r)$ is stable. (Theorem 6.1 or [22, Theorem 2]). However, the converse is not necessarily true. There are many stable 2-bridge knots with non-alternating sequences. For example, if $r = [2, -2, -2, -8, 2]$, $K(r)$ is stable, since 
$$\Delta_{K(r)}(t) = (2t^2 - 5t + 2)^2.$$ 
Further, for 2-bridge links, it is possible to construct systematically such exceptional stable links. In this section, we show some of these knots and links.

8.1. Exceptional stable knots

We begin with the following proposition.

**Proposition 8.1.** Let $r = [2a, -2, -2b, 2c]$, where $a, b, c > 0$. Then $K(r)$ is stable if $bc \geq 2a(c + 1)$.

*Proof.* First we see that 
$$\Delta_{K(r)}(t) = At^4 - Bt^3 + Ct^2 - Bt + A,$$ where $A = abc$, $B = 4abc + bc - ac + a$ and $C = 6abc + 2bc - 2ac + 2a + 1$. Consider the modified polynomial $f(x)$ of $\Delta_{K(r)}(t)$, where 
$$f(x) = Ax^2 - Bx + (C - 2A).$$ Since the discriminant $d$ of $f(x)$ is $bc(bc - 2a(c + 1)) + a^2(c - 1)^2$, it follows that $d \geq 0$ if $bc \geq 2a(c + 1)$. Let $\alpha$ and $\beta$ be two real zeros of $f(x)$. We claim that $|\alpha|, |\beta| > 2$. In fact, $\alpha$ and $\beta$ are given as 
$$\frac{-B + \sqrt{d}}{2abc} = 2 + \frac{bc - a(c - 1) \pm \sqrt{d}}{2abc}.$$ But since $bc \geq 2a(c + 1)$, we see that $bc - a(c - 1) \geq 0$ and also $(bc - a(c - 1))^2 > d$, and hence $\alpha, \beta > 2$. \qed

**Example 8.2.** If $a = 1, b = 4$ and $c = 1$, then $K(r)$ is stable.

A similar argument can be applied to show the following proposition.

**Proposition 8.3.** Let $r = [2a, 2b, -2b, -2a], a, b > 0$. Then $K(r)$ is stable if and only if $a \geq 4b$. Here, $f(x) = 1 + 2a^2 - 4ab + 4a^2b^2 + (-a^2 + 2ab - 4a^2b^2)x + a^2b^2x^2$. 

Fig. 7.2
Example 8.4. (1) For \( r = [8, 2, -2, -8], \) \( \Delta_K(r)(t) = (4t^2 - 9t + 4)^2 \) is stable.
(2) Let \( r = [10, 2, -2, -10], \) \( \Delta_K(r)(t) = 25t^4 - 115t^3 + 181t^2 - 115t + 25 \) is stable.

8.2. Exceptional stable links

In this sub-section, we study exceptional stable 2-bridge links.

Proposition 8.5. Let \( r = [2a, 2b, -2c], a, b, c > 0. \) Then \( K(r) \) is stable if and only if \( a \geq c. \)

Proof. We see that \( \Delta_K(r) = (t - 1)(At^2 - Bt + A), \) where \( A = abc \) and \( B = 2abc + a - c. \) Therefore \( g(t) = At^2 - Bt + A \) has two real zeros if the discriminant \( d = (a - c)(4abc + a - c) \geq 0. \) Since \( 4abc + a - c > 0, \) the proposition follows easily.

Now we construct exceptional stable links systematically.

Definition 8.6. Let \( r = [2a_1, 2a_2, \ldots, 2a_n] \) be a sequence of non-zero integers.
(1) \( N(r) \) denotes \( tM(r) - M(r)^T, \) where \( M(r) \) is the \( n \times n \) matrix \( \{m_{ij}\} \) such that for all \( k, \) \( m_{k,k} = a_k, m_{k,k+1} = 1, \) and other entries are 0. See (1).
(2) we write \( -r = [-2a_1, -2a_2, \ldots, -2a_n], \) and \( r^{-1} = [2a_n, 2a_{n-1}, \ldots, 2a_1]. \)

Lemma 8.7. For a given sequence \( r = [2a_1, 2a_2, \ldots, 2a_n], \) we have;
\[
\det N(r) = \det N(r^{-1}) = (-1)^n \det N(-r) = (-1)^n \det N(-r^{-1}).
\]

Proof. This lemma can be simply proven by induction on \( n, \) but understood better in terms of Alexander polynomials of 2-bridge knots and links. Actually, \( \det N(r) \) coincides with the Alexander polynomial \( \Delta_K(r)(t). \) And hence \( \det N(r), \) \( \det N(r^{-1}), \) \( \det N(-r) \) and \( \det N(-r^{-1}) \) are equivalent up to multiplication of \( \pm 1 \) and powers of \( t. \) In this case, the differences are detected by their constant terms, which are respectively \( (-1)^n \prod_{i=1}^n a_i, (-1)^n \prod_{i=1}^n a_i, (-1)^n \prod_{i=1}^n (-a_i), \) and \( (-1)^n \prod_{i=1}^n (-a_i). \)

For given two sequences \( r = [2a_1, 2a_2, \ldots, 2a_n] \) and \( s = [2b_1, 2b_2, \ldots, 2b_n], \) and an integer \( k, \) let \( [r, 2k, s] \) denote \([2a_1, 2a_2, \ldots, 2a_n, 2k, 2b_1, 2b_2, \ldots, 2b_n].\)

Theorem 8.8. Given \( r = [2a_1, 2a_2, \ldots, 2a_n], \) let \( T_1 = [r, 2k, r], T_2 = [r, 2k, -r], T_3 = [r, 2k, r^{-1}] \) and \( T_4 = [r, 2k, -r^{-1}]. \) Then for any integer \( k \neq 0, \) we have;
(1) \( \Delta_K(T_i)(t) = \Delta_K(r)(t)f(t), \) where \( f(t) \) is an integer polynomial. \((1 \leq i \leq 4)\)
(2) \( \Delta_K(T_{4i})(t) = k(t - 1)[\Delta_K(r)(t)]^2. \)

Corollary 8.9. \( \Delta_K(T_i)(t) \) is stable if and only if \( \Delta_K(r)(t) \) is stable.

For other cases, \( T_1, T_2 \) and \( T_3, \) generally \( \Delta_K(T_i)(t) \) is not stable unless the sequence of \( T_j \) alternates in sign. For example, if \( m \) is odd and \( r \) is an alternating
sequence with $2a_1 > 0$ and $k < 0$, then $K(T_1)$ and $K(T_3)$ are stable.

**Example 8.10.** (1) Let $s = [4, 2, -2]$. By Proposition 8.5, $K(s)$ is stable. Thus, for $r = [4, 2, -2, -4, 2, -2, -4]$, $K(r)$ is stable by Theorem 8.8 and further, for $r' = [r, 2k, -r^{-1}]$, $K(r')$ is also stable, if $k \neq 0$.

(2) Let $s = [2, -2, -8, 2]$. Then $K(s)$ is stable by Proposition 8.1. Therefore, for $r = [2, -2, -8, 2, -2, -8, 2, -2]$, $K(r)$ is stable.

Theorem 8.8 is a corollary of the following lemma. For a given matrix $M$, $M_{ij}$ denotes the matrix obtained from $M$ by deleting the $i$th row and $j$th column.

**Lemma 8.11.** For sequences $r = [2a_1, 2a_2, \cdots, 2a_n]$ and $s = [2b_1, 2b_2, \cdots, 2b_n]$, let $A$ (resp. $B$) denote $tM(r) - M(r)^T$ (resp. $tM(s) - M(s)^T$), where $M(r)$ and $M(s)$ are as in Definition 8.6. Then we have: $\Delta_{K([r, 2k, s])}(t) = \det N([r, 2k, s]) = k(t - 1) \det A \det B + t (\det A_{n,n} \det B + \det A \det B_{1,1})$.

**Proof of Theorem 8.8.** (1) If $s$ is equal to $r, r^{-1}, -r$ or $-r^{-1}$, then $\det A = \varepsilon \det B$, where $\varepsilon$ equals 1 or $-1$ according to Lemma 8.7. Since $\Delta_{K(r)}(t) = \det A$, we have the conclusion by Lemma 8.11. (2) If $r = r^{-1}$, then (i) $\det A \det B = (-1)^n \det A \det A$, (ii) $\det A_{n,n} \det B = (-1)^n \det A_{n,n} \det A$, and (iii) $\det A \det B_{1,1} = \det A (-1)^{n-1} \det A_{n,n}$. Therefore, by Lemma 8.11 we have the conclusion.

The following formula is often used in this paper. A proof is an exercise.

**Proposition 8.12.** Let $A$ and $B$ be square matrices of sizes $n$ and $m$. Let $M$ be the matrix obtained from $A \oplus B$ by changing the $(\alpha, n+\beta)$-entry to $x$ and $(n+\gamma, \delta)$-entry to $y$, where $1 \leq \alpha, \delta \leq n$ and $1 \leq \beta, \gamma \leq m$. Then we have:

$$\det M = \det A \det B - (-1)^{\alpha+\beta+\gamma+\delta} xy \det A_{n,\delta} \det B_{\gamma,\beta} (8.1)$$
Proof of lemma 8.11. By (8.1), we have the following:

\[
\det N([r, 2k, s]) = \det \begin{bmatrix} A & t & \vdots & t \\ -1 & k(t-1) & \vdots & t \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \vdots & \vdots & B \end{bmatrix} 
\]

\[
= \det A \det \begin{bmatrix} k(t-1) & t \\ -1 & B \end{bmatrix} + t \det A_{n,n} \det B 
\]

\[
= \det A \{ (k(t-1) \det B + t \det B_{1,1}) + t \det A_{n,n} \det B \}
\]

\[
= k(t-1) \det A \det B + t(\det A_{n,n} \det B + \det A \det B_{1,1})
\]

\(\blacksquare\)

Question 8.13. To what extent does the stability property of the Alexander polynomials of an alternating knot \(K\) reflect the topological properties of \(K\)?

Problem 8.14. Characterize stable alternating knots and links.

9. Interlacing property (I) 2-bridge knots

For a series of stable real polynomials, the interlacing property of two sets of zeros is an interesting and important property. First, in this section, we prove a simple, but useful basic theorem in this paper (Theorem 9.4). We begin with a definition.

Definition 9.1 ([4, p.310]). Let \(f, g \in \mathbb{R}[z]\) be univariate polynomials. Suppose \(f, g\) are real stable. Let \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\) and \(\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m\) be the zeros of \(f\) and \(g\), respectively. Then we say that the zeros \(\{\alpha_j\}\) and \(\{\beta_k\}\) are interlaced, (or we simply say that \(f\) and \(g\) are interlaced) if the following conditions are satisfied.

(i) \(|m - n| \leq 1,
(ii) they can be ordered so that

(a) if \(n = m\), then

\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_n \leq \beta_n, \text{ or }
\]

\[
\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n,
\]

(b) if \(n = m + 1\), then

\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_m \leq \beta_m \leq \alpha_{m+1} (= \alpha_n),
\]

\[
= k(t-1) \det A \det B + t(\det A_{n,n} \det B + \det A \det B_{1,1})
\]
(c) if \( m = n + 1 \), then
\[
\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_{n+1}(= \beta_m).
\]

**Definition 9.2 ([32, p.56]).** For \( f, g \in \mathbb{C}[z] \), we define the Wronskian \( W[f, g] \) as
\[
W[f, g] = f'g - fg'.
\]
For \( f(\neq 0), g(\neq 0) \in \mathbb{R}[z] \), we say that two real stable \( f, g \) are in proper position (denoted by \( f \ll g \)) if \( W[f, g] \leq 0 \) on all real values.

If the zeros of \( f \) and \( g \) are interlaced, then either \( W[f, g] \leq 0 \) or \( W[f, g] \geq 0 \) on all real values and hence \( f \ll g \) or \( g \ll f \) ([32], p.56).

**Theorem 9.3 ([32, p.57]). (Hermite-Kakeya-Obreschkoff Theorem)** Let \( f, g \in \mathbb{R}[z] \). Then all non-zero polynomials in \( \{af + bg| a, b \in \mathbb{R}\} \) are real-rooted if and only if (1) \( f, g \) are real stable and (2) \( f \ll g, g \ll f \) or \( f = g = 0 \).

Now first we prove the following theorem. Although Theorem 9.3 is a strong tool, our proof is not a simple application of Theorem 9.3.

**Theorem 9.4.** Let \( s = [2a_1, -2a_2, \cdots, (-1)^{k-1}2a_k, \cdots, (-1)^{n-2}2a_{n-1}] \), \( s' = [2a_1, -2a_2, \cdots, (-1)^{n-2}2a_{n-1}] \), where \( a_j > 0 \), for \( 1 \leq j \leq n \). Then \( \Delta_{K(s)}(t)\Delta_{K(s')}(t) \) is simple, and \( \Delta_{K(s)}(t) \) and \( \Delta_{K(s')}(t) \) are interlaced.

Note that by reversing the sequences, the same conclusion of Theorem 9.4 holds for the case \( s' = [-2a_2, 2a_3, \cdots, (-1)^{n-1}2a_n] \).

**Proof of Theorem 9.4.** Our proof is by induction. We use Seifert matrices of twisted chain type (4.1).

Case 1. \( n = 2 \). \( \Delta_{K(s')}(t) = a_1(t-1) \) and \( \Delta_{K(s)}(t) = -a_1a_2 + t + 2a_1a_2t - a_1a_2t^2 \).
Since \( \Delta_{K(s)}(1) = 1 \), \( \Delta_{K(s)}(t)\Delta_{K(s')}(t) \) is simple and \( \Delta_{K(s)}(t) \) has two real zeros, \( \alpha_1 \) and \( \alpha_2 \) with \( \alpha_1 < 1 < \alpha_2 \).

Case 2. \( n = 3 \). Write \( s = [2a, -2b, 2c], s' = [2a, -2b], s'' = [2a], a, b, c > 0 \). By a Seifert matrix \( M = \begin{bmatrix} a & 1 & 0 \\ 0 & -b & 1 \\ 0 & 0 & c \end{bmatrix} \) for \( K(s) \), we have
\[
\Delta_{K(s)}(t) = c(t-1)\Delta_{K(s')}(t) + t\Delta_{K(s'')}(t)
= (t-1)\{c(-ab + (1 + 2ab)t - abt^2) + at\}
\]
Consider two curves \( y_1 = c(-ab + (1 + 2ab)t - abt^2) \) and \( y_2 = -at \). From the observation in case \( n = 2 \), we see that these two curves intersect in two points at say, \( t = \beta_1 \) and \( t = \beta_2 \) such that \( \beta_1 < \alpha_1 < 1 < \alpha_2 < \beta_2 \). See Fig.9.1. Since the zeros of \( \Delta_{K(s)}(t) \) are \( \beta_1, 1 \) and \( \beta_2 \), we have the conclusion for \( n = 3 \).
Fig. 9.1

Note that for alternating stable knots and links, all the zeros of the Alexander polynomials are positive, because the coefficients are non-zero and have alternating signs. Moreover, since the Alexander polynomials are reciprocal, each zero less than 1 has its counterpart greater than one, and vice versa.

Case 3. General case.
Assume inductively that $\Delta_{K(s')}(t)\Delta_{K(s'')(t)}$ is simple and $\Delta_{K(s')}(t)$ and $\Delta_{K(s'')(t)}$ are interlaced, where for simplicity, we write the sequences as follows:

\[ s = [2a_1, -2a_2, \ldots, (-1)^{n-2}a_n], \]
\[ s' = [2a_1, -2a_2, \ldots, (-1)^{n-2}2a_{n-1}], \]
\[ s'' = [2a_1, -2a_2, \ldots, (-1)^{n-3}2a_{n-2}]. \]

Use Seifert matrices $M_s, M_{s'}, M_{s''}$ of twisted chain type and call

\[ f_n(t) := \det(tM_s - M_s^T), \]
\[ f_{n-1}(t) := \det(tM_{s'} - M_{s'}^T), \]
\[ f_{n-2}(t) := \det(tM_{s''} - M_{s''}^T). \]

Expand $\det(tM_s - M_s^T)$ at the last row and column, and we have:

\[ f_n(t) = (-1)^{n-1}a_n(t-1)f_{n-1} + tf_{n-2} \quad (9.1) \]

Consider two curves $y_1 = (-1)^{n-1}a_n(t-1)f_{n-1}$ and $y_2 = -tf_{n-2}$. We show that $y_1$ and $y_2$ intersect in $n$ points and (i) $f_n f_{n-1}$ is simple and (ii) $f_n(t)$ and $f_{n-1}(t)$ are interlaced. First, note that the leading coefficients of $y_1$ and $y_2$ have the same sign.

Case 3.1. $n$ is even, say $2m$. Fig. 9.2 depicts the case $m$ is even (in particular $m = 4$). The case $m$ is odd is similar, and the arguments are the same. By induction hypothesis $f_{n-1}f_{n-2}$ is simple, and since $K(s')$ is a link and $f_{n-1}$ is simple, $(t-1)$ divides $f_{n-1}$ exactly once. Hence $t = 1$ is the double zero of $y_1$. Meanwhile, since $K(s'')$ is a knot, $(t-1)$ does not divide $f_{n-2}$. Also, by induction hypothesis, $f_{n-1}$ and $f_{n-2}$ have interlaced zeros. If $m$ is even (resp. odd), then the leading coefficients of $y_1$ and $y_2$ are both positive (resp. negative). The sets of the zeros of $f_{n-1}$ and $y_1$
are the same and the set of the zeros of \( y_2 \) coincides with that of \( f_{n-2} \) with 0 added. Then two curves \( y_1 \) and \( y_2 \) intersect exactly \( n \) times and (i) \( f_n f_{n-1} \) is simple and (ii) \( f_n \) and \( f_{n-1} \) are interlaced, as shown in Fig. 9.2. Since \( \deg f_n = n \). We have the conclusion.

Fig. 9.2

Fig. 9.3
Case 3.2. $n$ is odd, say $2m + 1$. Fig. 9.3 depicts the case $m$ is even (in particular $m = 4$). The case $m$ is odd is similar, and the arguments are the same. $K(s')$ is a knot, and hence $f_{n-1}(1) \neq 0$. By induction hypothesis, (i) $f_{n-1}f_{n-2}$ is simple and (ii) $f_{n-1}$ and $f_{n-2}$ have interlaced zeros. If $m$ is even (resp. odd), then the leading coefficients of $y_1$ and $y_2$ are both positive (resp. negative). $y_1$ and $y_2$ have interlacing zeros, except sharing 1 in common. Then two curves $y_1$ and $y_2$ intersect exactly $n$ times and (i) $f_n f_{n-1}$ is simple and (ii) $f_n$ and $f_{n-1}$ are interlaced.

In proving Theorem 9.4 we have the following theorem:

**Theorem 9.5.** Let $s = [2a_1, -2a_2, \ldots, (-1)^{k-1}2a_k, \ldots, (-1)^{n-1}2a_n]$ and $r = [2a_1, -2a_2, \ldots, (-1)^{k-1}2a_k, \ldots, (-1)^{n-1}2(a_n - 1)]$, (or $r = [2(a_1 - 1), -2a_2, \ldots, (-1)^{k-1}2a_k, \ldots, (-1)^{n-1}2a_n], a_1 > 1$, where $a_j > 0$, for $1 \leq j \leq n$ and $a_n > 1$. Let $\{\alpha_j, 1 \leq j \leq \lceil \frac{n}{2} \rceil \}$ and $\{\beta_j, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \}$ be the zeros of $\Delta_K(r)(t)$ and $\Delta_K(s)(t)$ respectively in $(0, 1)$. Then $\{\alpha_j \}$ and $\{\beta_j \}$ are disjoint and are interlaced, i.e., $0 < \alpha_1 < \beta_1 < \cdots < \alpha_{\lceil \frac{n}{2} \rceil} < \beta_{\lfloor \frac{n}{2} \rfloor} < 1$.

**Proof.** As in the proof of Theorem 9.4 $f_n = (-1)^{n-1}a_n(t - 1)f_{n-1} + tf_{n-2}$. If we replace $a_n$ by $a_{n-1}$, the curve $y_1$ is squeezed toward the $t$-axes, while the intersection points with the $t$-axes are fixed. Since $y_2$ is irrelevant to $a_n$, each of the zeros of $f_n$ less than (resp. more than) 1 is moved toward (but never beyond) its left (resp. right) neighbour. Therefore, we have the conclusion.

Theorem 9.5 is generalized as follows:

**Theorem 9.6.** Let $s = [2a_1, -2a_2, \ldots, (-1)^{k-1}2a_k, \ldots, (-1)^{n-1}2a_n]$ and $r = [2a_1, -2a_2, \ldots, (-1)^{k-1}2(a_k - 1), \ldots, (-1)^{n-1}2a_n]$, where $a_j > 0$, $1 \leq j \leq n$ and $a_k > 1, 1 \leq k \leq n$. Let $\delta(K)$ be the maximal value of the zeros of $\Delta_K(t)$. Then $\delta(K(s)) < \delta(K(r))$.

**Proof.** We prove the theorem for the case where $n$ is even, say, $n = 2m$ and $k$ is odd. The same argument works for the other cases.

Let $s_1 = [2a_1, -2a_2, \ldots, (-1)^{k-2}2a_{k-1}], s_2 = [(-1)^{k-1}2a_k, \ldots, (-1)^{n-1}2a_n], r_2 = [(-1)^{k-1}2(a_k - 1), \ldots, (-1)^{n-1}2a_n]$. We use Seifert matrices $M(s), M(r), M(s_1), M(s_2), M(r_2), M(s'_1) and M(s'_2)$ of twisted chain type. Let $f_n = det(tM(s) - M(s)'T), f_n = det(tM(r) - M(r)'T), f_{k-1} = det(tM(s_1) - M(s_1)'T), g_{n-k+1} = det(tM(s_2) - M(s_2)'T), h_{n-k+1} = det(tM(r_2) - M(r_2)'T), f_{k-2} = det(tM(s'_1) - M(s'_1)'T), g_{n-k} = det(tM(s'_2) - M(s'_2)'T)$. Now by Proposition 8.12 we have $f_n = f_{k-1}g_{n-k+1} + tf_{k-2}g_{n-k}$ and $f_n = f_{k-1}h_{n-k+1} + tf_{k-2}g_{n-k}$. Let $\alpha, \beta$ and $\gamma$ be respectively the smallest zeros of $f_{k-1}, g_{n-k+1}$ and $h_{n-k+1}$. Then $\gamma < \beta$ be Theorem 9.5. Let $y_1 = f_{k-1}g_{n-k+1}, z_1 = f_{k-1}h_{n-k+1}$ and $y_2 = -tf_{k-2}g_{n-k}$. We note that the signs of the leading coefficients of $y_1$ and $z_1$ are both $(-1)^m$ and that of $f_{k-2}g_{n-k}$ is $(-1)^{m-1}$. Also, the smallest positive zero of $y_2$ is larger than $\alpha$ or $\beta$ by Theorem 9.4. Consider
the intersection of three curves $y_1, z_1$ and $y_2$. The smallest value of the intersection will be seen from the diagrams below. See Fig 9.4 for the case $\alpha \leq \gamma$, and Fig 9.5 for the case $\gamma < \alpha$.

![Diagrams](image)

(1) $m$ is even. (2) $m$ is odd.

Fig. 9.4

In each case, $d(K(r)) < d(K(s))$, where $d(K)$ denotes the smallest (positive) zero of $\Delta_K(t)$, and hence $\delta(K(r)) > \delta(K(s))$. \hfill $\square$

As an immediate consequence of Theorem 9.6, we have the following theorem:

**Theorem 9.7.** Let $F_n$ be the set of stable 2-bridge knots or links $K(r)$ with $r = [2a_1, -2a_2, \ldots, (-1)^{n-1}2a_n], a_j > 0, 1 \leq j \leq n$. Then $\delta(K(r))$ is maximal if and only if $r = [2, -2, \ldots, (-1)^{n-1}2]$, i.e., $K(r)$ is fibred.

**Example 9.8.** (1) Let $s = [4, -2, 2, -6, 4, -2], s' = [4, -2, 2, -6, 4]$. Then the zeros of $\Delta_K(s)$ are approximately $\{0.2866, 0.4550, 0.7654, 1.3065, 2.1976, 3.4888\}$ and those of $\Delta_K(s')$ are approximately $\{0.2877, 0.6179, 1.0000, 1.6183, 3.4761\}$.

(2) Let $s = [4, -2, 2, -6, 4, -2, -4], s' = [4, -2, 2, -6, 4, -2, 4]$. Then the zeros of $\Delta_K(s)$ are approximately $\{0.2857, 0.3535, 0.6148, 0.8171, 1.2237, 1.6265, 2.8287, 3.4999\}$ and those of $\Delta_K(s')$ are approximately $\{0.2859, 0.3716, 0.6772, 1.1, 1.4767, 2.6913, 3.4973\}$

**Remark 9.9.** For exceptional stable 2-bridge knots or links, the interlacing property may not hold. For example, let $s = [10, 2, -2, -10]$ and $s' = [10, 2, -2]$. Then $K(r)$ and $K(s')$ are both stable, but they are not interlaced. In fact, the zeros of $\Delta_K(s)(t)$ are approximately $\{0.4923, 0.7592, 1.3172, 2.0313\}$, but those of $\Delta_K(s')(t)$ are approximately $\{0.4202, 1, 2.3797\}$. Therefore, they are not interlaced.
10. Interlacing property (II) Quasi-rational knots $X_n$ 

In the following two sections, we prove the interlacing property for two series of alternating stable knots $X_n$ and $Y_{2n+1}$ considered in Section 6.3. The idea of our proof is similar to the proof of Theorem 9.4, but we need a lot of computations. The first series of knots $X_n$ shows us that the zeros of the Alexander polynomials of alternating knots is unbounded. On the other hand, the Alexander polynomial of each knot in the second series of knots $Y_{2n+1}$ is not irreducible. Nevertheless, the maximal value of the zeros of a factor (of degree 4) of the Alexander polynomial of $Y_3$ is quite likely equal to $\delta_4$ defined in Section 5.1.

Now consider a series of stable knots $X_n(a, b) = X_n(2a_1, \cdots, 2a_n \mid -2b_1, \cdots, -2b_n), a_j, b_j > 0, 1 \leq j \leq n$. We prove

**Theorem 10.1.**

(1) $X_n(a, b)$ is a stable alternating knot of genus $n$.

(2) Let $K_n = X_n(2, 2, \cdots, 2| -2, -2, \cdots, -2)$. Then for $n \geq 2$, $\Delta_{K_n}(t)\Delta_{K_{n-1}}(t)$ is simple and $\Delta_{K_n}(t)$ and $(t-1)\Delta_{K_{n-1}}(t)$ are interlaced.

(3) Let $\alpha_n$ be the maximal value of the zeros of $\Delta_{K_n}(t)$. Then $\alpha_n \geq n + 1$.

We suspect that (2) holds for $X_n(a, b)$ with any $a_j > 0$ and $b_j > 0, 1 \leq j \leq n$. Therefore we conjecture:

**Conjecture 10.2.** Let $a_j > 0$ and $b_j > 0, 1 \leq j \leq n$. Then for $n \geq 2$, $\Delta_{X_n}(t)\Delta_{X_{n-1}}(t)$ is simple, and $\Delta_{X_n}(t)$ and $(t-1)\Delta_{X_{n-1}}(t)$ are interlaced.

Now since (1) and (3) are already proved in Theorem 6.7 (1), we prove only (2) in this section. For simplicity, we denote by $G(n)$ the normalization of $\Delta_{K_n}(t)$.

As the first step, we prove

**Proposition 10.3.** Let $\lambda(t) = 2t^2 - 5t + 2$.

(1) For $n \geq 2$, $G(n) = \lambda(t)G(n-1) - (t-1)^4G(n-2)$.

(2) For $n \geq 0$, $\lambda(t)G(n)$.

(3) For $n \geq 0$, $t-1 \cdot G(n)$, where we define $G(0) = 1$.

**Proof.** Since $K_n$ is a knot, (3) holds trivially. Now Proposition 10.3 holds for $n = 1$ and 2. In fact, $K_1 = 4_1$ and $K_2 = 8_{12}$ and $\Delta_{K_2}(t) = t^4 - 7t^3 + 13t^2 - 7t + 1 = (2t^2 - 5t + 2)(t^2 - 3t + 1) - (t - 1)^4 = \lambda(t)G(1) - (t - 1)^4G(0)$. Suppose $n \geq 3$. A Seifert matrix $M$ of $K_n$ is given in a proof of Theorem 6.7 (1). It is of the form:

$$M = \begin{bmatrix} E_n & O \\ C & -E_n \end{bmatrix},$$

where $C$ is a lower triangular matrix defined in the proof of Theorem 6.7 (1). Since $\Delta_{K_n}(t) = \det[Mt - MT^T]$, we see that $G(n) = \det N$, where $N = \begin{bmatrix} (t - 1)E_n & Ct \\ Ct & (t - 1)E_n \end{bmatrix}$. To compute $G(n)$, first expand $\det N$ along the $n^{th}$ row and we obtain by Proposition 8.12 that $G(n) = \det N = (t - 1)\det N_1 -$
If \( n = (\bullet \bullet \) This proves (1). Using (1), (2) follows by induction.

While the zeros of \( \beta \) \( N \) the next to last column of \( \text{det} \)
the last row, and we obtain \( \text{det} \) \( N \) \( G \) of the form:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\]

Next, subtract the next to last row of \( \text{det} \) \( N_1 \) from the last row and then subtract the next to last column of \( \text{det} \) \( N_1 \) from the last column and then expand it along the last row, and we obtain \( \text{det} \) \( N_1 = 2(t-1)G(n-1) - (t-1)^2G(n-2) \). Therefore, \( G(n) = (t-1) \det N_1 - tG(n-1) = (t-1) \{ 2(t-1)G(n-1) - (t-1)^2G(n-2) \} - tG(n-1) = 2(t-1)^2 - \lambda(t)G(n-1) - (t-1)^4G(n-2) = \lambda(t)G(n-1) - (t-1)^4G(n-2). \)

This proves (1). Using (1), (2) follows by induction.

To prove Theorem \( \text{10.1} \) (2), let \( \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < 1 \) be the zeros of \( (t-1)G(n-1) \) in \( [0,1] \) and let \( \beta_1 < \beta_2 < \cdots < \beta_n \) be the zeros of \( G(n) \) in \( [0,1] \). Then it suffices to prove the following proposition.

**Proposition 10.4.** (1) For \( n \geq 1 \), \( G(n)G(n-1) \) is simple. (2) \( \{ \alpha_j, 1 \leq j \leq n-1 \} \cup \{ 1 \} \) and \( \{ \beta_j, 1 \leq j \leq n \} \) are interlaced, namely,

\[
\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_{n-1} < \beta_n < 1. \tag{10.1}
\]

(3) \( a \) If \( n \) is even, say \( 2m \), then \( \alpha_m < \frac{1}{2} < \beta_{m+1} \), and

\( b \) if \( n \) is odd, say \( 2m+1 \), then \( \beta_{m+1} < \frac{1}{2} < \alpha_{m+1} \).

We note that \( \frac{1}{2} \) (and 2) are the zeros of \( \lambda(t) \) and also we need (3) to show (2) by induction.

**Proof.** We use induction.

Case \( n = 1 \). The zeros of \( G(1) = t^2 - 3t + 1 \) in \( [0,1] \) is \( \beta_1 = 0.38 \cdots \). Since \( G(0) = 1 \), we have \( \beta_1 = 1 \) and further \( \beta_1 < \frac{1}{2} \). This proves Proposition \( \text{10.4} \) when \( n = 1 \).

Case \( n = 2 \). The zeros of \( G(2) \) in \( [0,1] \) are \( \beta_1 = 0.228 \cdots \) and \( \beta_2 = 0.5449 \cdots \), while the zeros of \( (t-1)G(1) \) in \( [0,1] \) are \( \alpha_1 = 0.38 \cdots \) and 1. Therefore, \( \beta_1 < \alpha_1 < \beta_2 < 1 \) and \( \alpha_1 < \frac{1}{2} < \beta_2 \).

Case \( n \geq 3 \). Suppose \( G(n)G(n-1) \) is simple. Let \( \{ \alpha_j, 1 \leq j \leq n-1 \} \cup \{ 1 \} \) be the zeros of \( (t-1)G(n-1) \) in \( [0,1] \) and \( \{ \beta_j, 1 \leq j \leq n \} \) be the zeros of \( G(n) \) in \( [0,1] \).

Inductively, we assume that they are interlaced, namely,

\[
\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_{n-1} < \beta_n < 1. \tag{10.2}
\]
Consider the zeros of \( G(n+1) \) in \([0, 1]\). Since \( G(n+1) = \lambda(t)G(n) - (t-1)^4 G(n-1) \), the zeros of \( G(n+1) \) in \([0, 1]\) are determined by the intersection of two curves \( y_1 = \lambda(t)G(n) \) and \( y_2 = (t-1)^4 G(n-1) \). Note that \( \lambda(t)G(n) \) is simple and since \((t-1)/G(n-1)\), 1 is the zero of \( y_2 \) of exactly order 4.

Case (a) \( n+1 \) is even, say \( 2m \). Then by induction, since \( n \) is odd, we have

\[
\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_m < \frac{1}{2} < \alpha_m < \beta_{m+1} < \cdots < \alpha_{n-1} < \beta_n < 1.
\] (10.3)

Note that \( y_1(0) = 2 \) and \( y_2(0) = 1 \), since \( G(k) \) is monic for all \( k \geq 1 \).

When \( m \) is even (resp. odd), two curves are depicted in Fig. 10.1 top (resp. bottom).

There are exactly \((n+1)\) points of intersection \( \{\gamma_j, 1 \leq j \leq n+1\} \) in \([0, 1]\) which are the zeros of \( G(n+1) \) in \([0, 1]\):

\[
\gamma_1 < \beta_1 < \gamma_2 < \beta_2 < \gamma_3 < \cdots < \gamma_m < \beta_m < \frac{1}{2} < \gamma_{m+1} < \beta_{m+1} < \cdots < \beta_n < \gamma_{n+1} < 1.
\]

Thus \( G(n+1)G(n) \) is simple and \( \{\gamma_j\} \) and \( \{\beta_j\} \cup \{1\} \) are interlaced and \( \beta_m < \frac{1}{2} < \gamma_{m+1} \). Proposition 10.4 is now proved for this case.

Case (b) \( n+1 \) is odd, say \( 2m+1 \). Then by induction, since \( n \) is even, we see

\[
\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_m < \frac{1}{2} < \beta_{m+1} < \cdots < \alpha_{n-1} < \beta_n < 1.
\]

When \( m \) is even (resp. odd), two curves are depicted in Fig. 10.2 top (resp. bottom).
There are \((n + 1)\) points of intersection \(\{\gamma_j, 1 \leq j \leq n + 1\}\) in \([0, 1]\) and
\[
\gamma_1 < \beta_1 < \gamma_2 < \beta_2 < \cdots < \beta_m < \gamma_m < \frac{1}{2} < \beta_{m+1} < \gamma_{m+1} < \cdots < \beta_n < \gamma_{n+1} < 1.
\]

Now we covered all cases and a proof is completed.

**Example 10.5.** Given below is the list of the zeros of the Alexander polynomials for \(G(k), k = 1, 2, \ldots, 8\). The table satisfies Theorem 10.1.

\[
\begin{array}{ccccccccc}
0 & 0.032 & 0.040 & 0.194 & 0.052 & 0.228 & 0.343 \\
& 0.070 & 0.269 & 0.382 & 0.446 & 0.098 & 0.320 & 0.426 & 0.481 & 0.515 \\
& 0.145 & 0.382 & 0.474 & 0.519 & 0.544 & 0.560 & 0.228 & 0.458 & 0.526 & 0.557 & 0.573 & 0.582 & 0.589 \\
& 0.382 & 0.544 & 0.578 & 0.591 & 0.597 & 0.601 & 0.603 & 0.605 & 2.618 & 1.838 & 1.730 & 1.692 & 1.674 & 1.664 & 1.658 & 1.654 \\
2.618 & 1.838 & 1.730 & 1.692 & 1.674 & 1.664 & 1.658 & 1.654 & 4.390 & 2.186 & 1.900 & 1.797 & 1.746 & 1.717 & 1.699 \\
6.904 & 2.618 & 2.109 & 1.927 & 1.838 & 1.786 & 10.193 & 3.129 & 2.349 & 2.077 & 1.943 & 14.273 & 3.719 & 2.618 & 2.242 \\
19.155 & 4.390 & 2.915 & 24.841 & 5.144 & 31.333
\end{array}
\]
11. Interlacing property (III) Quasi-rational knots \( \{Y_{2n+1}\} \)

In this section, we discuss a slightly different sense of interlacing property of the second series of stable alternating knots. The Alexander polynomials of knots in this series may have multiple zeros and thus, they are not irreducible. Nevertheless, some zero has the largest value up to 12 crossing knots.

Let \( Y_{2n+1}(2a_1, 2a_2, \cdots, 2a_{2n+1} \mid 2b_1, 2b_2, \cdots, 2b_{2n+1}) \) be a quasi-rational knot obtained as the boundary of the surface constructed in Example 6.6. Define a series of alternating quasi-rational knots \( Y_n \) by \( Y_{2n+1}(-2, -2, \cdots, -2 | 2, 2, \cdots, 2) \). Then \( Y_1 \) is the knot 4_1 and \( Y_2 \) is the 12 crossing alternating knot 12_{a1124}. We note that the largest zero of \( Y_2 \) is \( 7_{69853} \), which attains the largest real part of all the zeros of Alexander polynomials of alternating knots up to 12 crossings.

11.1. Conway polynomials of \( Y_n \)

In this subsection, we give inductive formulae for the Conway polynomials of \( Y_n \). (For the Conway polynomial, see [21].) Denote by \( W_n \) the link obtained from \( Y_n \) by removing band corresponding the middle vertical edge of the graph in Fig.6.3. Denote by \( c_n(z) \) (resp. \( d_n(z) \)) the Conway polynomial of \( Y_n \) (resp. \( W_n \)). For simplicity, we write a function \( f_n(z) \) of \( z \) as \( f_n \).

**Proposition 11.1.** Let \( a(z) = 1 - z^2 \), \( b(z) = 1 + z^2 \). Then we have:

\[

c_1 = a, \\
c_2 = a(a^2 - 4z^2), \text{ and for } n \geq 1, \\
c_{n+2} = a^2(2c_{n+1} - b^2 c_n).
\] (11.1)

**Proof.** In (11.2) below, we can write \( d_n \) (resp. \( d_{n+1} \)) in terms of \( c_n \) and \( c_{n+1} \) (resp. \( c_{n+1} \) and \( c_{n+2} \)). Substituting these into (11.3), we have the conclusion. Using skein trees, we prove Lemma 11.2 at the end of this subsection. \( \square \)

**Lemma 11.2.**

\[

d_1 = z, \text{ and for } n \geq 1, \\
c_{n+1} = (3z^4 - 4z^2 + 1)c_n + 2z(z^4 - 1)d_n \\
d_{n+1} = 2z(1 - z^2)c_n + (1 - z^4)d_n.
\] (11.2) (11.3)

We know that the Conway polynomials \( c_n \) have a factor \( a = 1 - z^2 \). In the proposition below, we determine the exponent of \( a \) in \( c_n \).

**Proposition 11.3.** Let \( f_{2m-1} = \frac{c_{2m-1}}{a^{2m-1}} \), \( f_{2m} = \frac{c_{2m}}{a^{2m}} \). Then for \( m \geq 1 \), we have
the following, where \( a(z) = 1 - z^2 \), \( b(z) = 1 + z^2 \).

\[
\begin{align*}
f_{2m-1}, f_{2m} &\in \mathbb{Z}[z] \\
a \not| f_{2m-1} \text{ and } a \not| f_{2m},
\end{align*}
\]

(11.4)

\[
\begin{align*}
f_{2m+1} &= 2f_{2m} - b^2f_{2m-1} \\
f_{2m+2} &= 2a^2f_{2m+1} - b^2f_{2m}
\end{align*}
\]

(11.6)

(11.7)

Proof. First we prove (11.4) by induction. For \( c_1 \) and \( c_2 \), the claim is trivial. Assume that the claim holds up to \( 2m \). By (11.4) and induction hypothesis, we have

\[
c_{2m+1} = a^2(2c_{2m} - b^2c_{2m-1})
\]

\[
= a^2(2a^{2m-1}f_{2m} - b^2a^{2m-1}f_{2m-1})
\]

\[
= a^{2m+1}(2f_{2m} - b^2f_{2m-1}),
\]

(11.8)

and hence \( c_{2m+1} \) has factor \( a^{2m+1} \). We also have

\[
c_{2m+2} = a^2(2c_{2m+1} - b^2c_{2m})
\]

\[
= a^2(2a^{2m+1}f_{2m+1} - b^2a^{2m-1}f_{2m})
\]

\[
= a^{2m+1}(2a^2f_{2m+1} - b^2f_{2m}),
\]

(11.9)

and hence \( c_{2m+2} \) has factor \( a^{2m+1} \). Therefore, we have (11.4). By (11.8) and (11.9), we have (11.10) and (11.11). Finally, we prove (11.5), using the fact that if a polynomial \( f(t) \) is divided by \( a = 1 - z^2 \) then \( f(1) = 0 \). Let \( e_n = f_n(1) \), and we prove that \( e_k \neq 0 \) for \( k \geq 1 \). By putting \( z = 1 \) in (11.10) and (11.11), we have

\[
e_{2m+1} = 2e_{2m} - 4e_{2m-1}
\]

(11.10)

(11.11)

Since \( e_1 = 1, e_2 = -4 \), by (11.11) we have \( e_{2m} = (-4)^m \) and hence \( e_{2m} \neq 0 \). Then by (11.10), \( e_{2m+1} = 2(-4)^m - 4e_{2m-1} \) and hence \( e_{2m+1} \) is positive (resp. negative) if \( m \) is even (resp. odd), and in any case, \( e_{2m+1} \neq 0 \). In fact, we see that \( e_{2m-1} = (-4)^{m-1}(2m-1) \).

The following lemma is used to prove Lemma 11.2. The equation (11.12) means the relation of the Conway polynomials of three knots or links that differ only locally as depicted in the diagrams. In Lemma 11.3 and its proof, we adopt such a convention. Lemma 11.4 below reduces the Conway polynomial of a diagram with two or more parallel bands (of positive writhe) to those with one and zero bands.

Lemma 11.4.

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{fig_x1} \\
\includegraphics[width=1cm]{fig_x2}
\end{array} &= 2z \begin{array}{c}
\includegraphics[width=1cm]{fig_x3} \\
\includegraphics[width=1cm]{fig_x4}
\end{array} - z^2 \begin{array}{c}
\includegraphics[width=1cm]{fig_x5} \\
\includegraphics[width=1cm]{fig_x6}
\end{array} \\
\begin{array}{c}
\includegraphics[width=1cm]{fig_x7} \\
\includegraphics[width=1cm]{fig_x8}
\end{array} &= 3z^2 \begin{array}{c}
\includegraphics[width=1cm]{fig_x9} \\
\includegraphics[width=1cm]{fig_x10}
\end{array} - 2z^3 \begin{array}{c}
\includegraphics[width=1cm]{fig_x11} \\
\includegraphics[width=1cm]{fig_x12}
\end{array}
\end{align*}
\]

(11.12)

(11.13)
Proof. First, we have \[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + z \] and hence \[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} - z \].
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + z = \frac{\partial}{\partial z} + z = z \left( \frac{\partial}{\partial z} - z \right) + z \frac{\partial}{\partial z} \].
In fact, we can inductively prove \[ \frac{\partial}{\partial z} \cdots \frac{\partial}{\partial z} = nz^{n-1} \frac{\partial}{\partial z} - (n-1)z^n \].

To prove Lemma 11.2 using skein trees, notice the following (see Figure 11.1).
Suppose that a band \( b_1 \) crosses over exactly one other band \( b_2 \), then untwisting \( b_1 \) results in cutting both \( b_1 \) and \( b_2 \). Suppose a band \( b_1 \) crosses over exactly two bands \( b_2 \) and \( b_3 \). Then untwisting \( b_1 \) results in removing \( b_1 \) and merging the band \( b_2 \) and \( b_3 \).

(1) \[ \text{Fig. 11.1} \]

Proof of Lemma 11.2. We depict skein trees for \( Y_n \)'s and \( W_n \)'s. The knots \( Y_n \)'s and links \( W_n \)'s are represented by diagrams in Figures 11.2 through 11.5. Each dotted arc indicates the site where an arc is removed, and hence dotted arcs are not counted as arcs. Horizontal arcs are assigned with weight \(-2\) and non-horizontal ones with weight 2, except for the ones with label 4 in Fig. 11.4. Recall that a diagram represents a knot or link on the boundary of a Seifert surface obtained from a disk by attaching twisted bands along the arcs, where horizontal arcs contribute bands on the top side, and non-horizontal arcs on the back side.

First, we prove (11.3). We have a skein tree as in Fig. 11.2, where the sites of crossing changes and splicing are marked with *. In the first step, we use Fig. 11.1 (1).
In the left of the bottom row, we have two parallel bands. We apply Lemma 11.4 but since the writhe is negative, we replace $z$ by $-z$. See Fig. 11.3. Then we have (11.3).

Next, we prove (11.2). We have a skein tree as in Fig. 11.4. In the first step, we use Fig. 11.1 (2) and have a band with two full-twists. On the way, we have the connected sum of $Y_n$ and the 2-bridge link $K(\frac{1}{4})$, whose Conway polynomial is equal to that of $Y_n$ multiplied by $-2z$.

Now we have completed a proof of Lemma 11.2. □
11.2. Alexander polynomials of $Y_n$

In the previous subsection we investigated the Conway polynomial $c_n(z)$ of $Y_n$. Namely, let $g = \ldots < \beta_1$ of $Y_n$. By Proposition 11.3 we inductively have the following:

**Corollary 11.5.** $\deg f_1(z) = 1$, $\deg f_2(z) = 4$. For $m > 1$, $\deg f_{2m-1}(z) = 4(m-1)$, $\deg f_{2m}(z) = 4m$. The leading coefficient of $f_k$ is equal to 1, for $k \geq 1$.

The Alexander polynomial of $Y_n$ is obtained by putting $z = \sqrt{t - \frac{1}{\sqrt{t}}}$ in $c_n(z)$.

**Proposition 11.6.** Let $h_n(t)$ be the normalized Alexander polynomial of $Y_n$. Let $\mu(t) = 1 - 3t + t^2$ and $\rho(t) = 1 - t + t^2$. Let $g_{2m-1}(t) = f_{2m-1}(\sqrt{t - \frac{1}{\sqrt{t}}})$ and $g_{2m}(t) = f_{2m}(\sqrt{t - \frac{1}{\sqrt{t}}})$.

Then we have:

\[
\begin{align*}
    h_{2m-1}(t) &= \mu(t)^{2m-1} g_{2m-1}(t) \\
    h_{2m}(t) &= \mu(t)^{2m-1} g_{2m}(t)
\end{align*}
\]  

Moreover, $\mu(t) \nmid g_{2m-1}(t)$, $\mu(t) \nmid g_{2m}(t)$.

**Proposition 11.7.** $\deg g_{2m-1}(t) = 4(m-1)$, $\deg g_{2m} = 4m$, $g_k(t)$’s are monic and reciprocal.

\[
\begin{align*}
    g_1 &= 1, g_2 = 1 - 10t + 19t^2 - 10t^5 + t^9 \\
    g_{2m+1} &= 2g_{2m} - \rho(t)^2 g_{2m-1} \\
    g_{2m+2} &= 2\mu(t)^2 g_{2m+1} - \rho(t)^2 g_{2m}
\end{align*}
\]

Now we study the interlacing property of $g_k$ and $g_{k+1}$. The following is the main theorem of this section.

**Theorem 11.8.** For $m \geq 1$, we have the following, where $\mu_0 \approx 0.382$ is the zero of $\mu(t) = 1 - 3t + t^2$ in $[0, 1]$.

1. $g_{2m-1} g_{2m}$ is simple,
2. $(t - 1)\mu(t) g_{2m-1}$ and $g_{2m}$ are interlaced.

Namely, let $\alpha_1 < \alpha_2 < \cdots < \alpha_{2(m-1)}$ be the zeros of $g_{2m-1}$ in $[0, 1]$, and $\beta_1 < \beta_2 < \cdots < \beta_{2m}$ be the zeros of $g_{2m}$ in $[0, 1]$. Then, $\beta_1 < \alpha_1 < \beta_2 < \cdots < \alpha_{m-1} < \beta_m < \mu_0 < \beta_{m+1} < \alpha_m < \cdots < \alpha_{2(m-1)} < \beta_{2m} < 1.
(3) \( g_{2m}g_{2m+1} \) is simple.
(4) \((t-1)g_{2m} \) and \( \mu(t)g_{2m+1} \) are interlaced. Namely, let \( \gamma_1 < \gamma_2 < \cdots < \gamma_{2m} \) be the zeros of \( g_{2m+1} \) in \([0, 1] \). Then, \( \gamma_1 < \beta_1 < \gamma_2 < \cdots < \gamma_m < \beta_m < \mu_0 < \beta_{m+1} < \gamma_{m+1} < \cdots < \beta_{2m} < \gamma_{2m} < 1 \).

**Example 11.9.** We list \( g_k \)’s with \( k \) up to 6. Note that in general, \( g_k(0) = g_k(1) = 1. \)

\[
\begin{align*}
g_1 &= 1, \\
g_2 &= 1 - 10t + 19t^2 - 10t^3 + t^4, \\
g_3 &= 1 - 18t + 35t^2 - 18t^3 + t^4, \\
g_4 &= 1 - 36t + 266t^2 - 784t^3 + 1107t^4 - 784t^5 + 266t^6 - 36t^7 + t^8, \\
g_5 &= 1 - 52t + 458t^2 - 1424t^3 + 2035t^4 - 1424t^5 + 458t^6 - 52t^7 + t^8, \\
g_6 &= (1 - 10t + 19t^2 - 10t^3 + t^4)(1 - 68t + 522t^2 - 1552t^3 + 2195t^4 - 1552t^5 + 522t^6 - 68t^7 + t^8).
\end{align*}
\]

Fig. 11.6 depicts adjacent pairs of \( g_k \)’s.

---

**Proof of Theorem 11.8.** We prove the theorem by induction. The zeros of \( g_2 \) in \([0, 1] \) are approximately 0.129 and 0.662, and those of \( g_3 \) are approximately 0.063 and 0.765. See Fig. 11.7. Since \( \mu_0 \approx 0.38 \), the claim is true for the pairs \((g_1, g_2)\) and \((g_2, g_3)\), i.e., \( g_1g_2 \) is simple and \( \mu(t)(t-1)g_1 \) and \( g_2 \) are interlaced, also, \( g_2g_3 \) is simple and \((t-1)g_2 \) and \( \mu(t)g_3 \) are interlaced.

---
For general cases, we first fix $m$ and assume (1) and (2) of the statement of Theorem 11.8, prove (3) and (4) for that $m$, and then prove (1) and (2) with $m$ replaced by $m + 1$.

By Proposition 11.7, $g_{2m+1} = 2g_{2m} - (1 - t + t^2)^2g_{2m-1}$. So we examine intersection points of $y_1(t) = (1 - t + t^2)^2g_{2m-1}$ and $y_2(t) = 2g_{2m}$. See Fig. 11.8 for $m$ odd (here $m = 5$). For the case $m$ being even, the figure is similar.

Since $0 < 1 - t + t^2 \leq 1$ in $[0, 1]$, $y_1$ and $g_{2m-1}$ have the same real zeros, and the graph of $y_1$ is obtained from that of $g_{2m-1}$ by moving it toward the $x$-axis, fixing the points of intersection with the $x$- and $y$-axes. By assumption, $\{\text{zeros of } y_1\} \cup \{\mu_0\} \cup \{1\}$ and $\{\text{zeros of } y_2\}$ are interlaced. Since $g_{2m-1}(0) = g_{2m-1}(1) = g_{2m}(0) = g_{2m}(1) = 1$, we have $y_1(0) = y_1(1) = 1, y_2(0) = y_2(1) = 2$. Let $\gamma_1 < \cdots < \gamma_{2m}$ be the zeros of $g_{2m+1}$ in $[0, 1]$. Then we see that $\gamma_1 < \beta_1 < \alpha_1 < \cdots < \alpha_{m-1} < \gamma_m < \beta_m < \mu_0 < \beta_{m+1} < \gamma_{m+1} < \alpha_m < \beta_{m+2} < \gamma_{m+2} < \cdots < \alpha_{2m-2} < \beta_{2m} < \gamma_{2m} < 1$. Therefore, we have (3) and (4) of Theorem 11.8 for the fixed $m$.

Next assume (3) and (4) holds for a fixed $m$. Then we examine $g_{2m+1}$ and $g_{2m+2}$ and prove (1) and (2) with $m$ replaced by $m + 1$. By proposition 11.7, $g_{2m+2} = 2(1 - 3t + t^2)^2g_{2m+1} - (1 - t + t^2)^2g_{2m}$. So we examine intersection points of $y_1(t) = (1 - t + t^2)^2g_{2m}$ and $y_2(t) = 2(1 - 3t + t^2)^2g_{2m+1}$. Note that $\mu_0$ is a zero of $y_2$ of order 2. See Fig. 11.9 below for $m$ odd (here $m = 5$).
For the case \( m \) being even, the figure is similar. Let \( \delta_1 < \cdots < \delta_{2m+2} \) be the zeros of \( g_{2m+2} \) in \([0, 1]\). Then we see that \( \delta_1 < \beta_1 < \delta_2 < \beta_2 < \cdots < \delta_m < \beta_m < \delta_{m+1} < \mu_0 < \delta_{m+2} < \beta_{m+1} < \cdots < \delta_{2m+1} < \beta_{2m} < \delta_{2m+2} < 1 \). Therefore, we have (1) and (2) with \( m \) replaced by \( m + 1 \).

Proof of Theorem 11.8 is now complete. \( \square \)

**Question 11.10.** If the zeros of \( \Delta_{K_1}(t) \) and \( \Delta_{K_2}(t) \) are interlaced, how are \( K_1 \) and \( K_2 \) related geometrically?

### 12. \( c \)-stable knots and links

It is well-known [24] that if the absolute value of the signature of a knot \( K \) is equal to the degree of the Alexander polynomial, then all the zeros of \( \Delta_K(t) \) are on the unit circle and hence, \( K \) is \( c \)-stable. However, the converse is not necessarily true, even for 2-bridge knots. In this section, first we discuss \( c \)-stable 2-bridge knots and links, and then we show some general construction of \( c \)-stable knots or links.

#### 12.1. Regular and exceptional \( c \)-stable 2-bridge knots and links

We begin with the following proposition:

**Proposition 12.1.** Let \( r = [2a_1, 2a_2, \ldots, 2a_n] \). If all \( a_i \)'s have the same sign, then \( K(r) \) is \( c \)-stable.
Proof. Suppose that all $a_j$’s are positive. Let $M$ be a Seifert matrix of twisted chain type. (See Fig 4.3.) Then $M + M^T$ is positive definite by Positivity lemma (Proposition 4.7), and hence $|\sigma(K(r))| = \deg \Delta_{K(r)}(t)$. Therefore, $K(r)$ is $c$-stable.

The converse of Proposition 12.1 does not hold. The simplest counter-example is $K(r)$, where $r = [2, 8, -2, -2]$. In fact $\Delta_{K(r)}(t) = (2 - 3t + 2t^2)^2$ and hence $K(r)$ is $c$-stable, but $\sigma(K(r)) = 0$. In fact, we have the following proposition.

**Proposition 12.2.** Let $r = [2, 2k, -2, -2]$. Then we have:

1. If $k < 0$, then $K(r)$ is strictly bi-stable.
2. If $k = 1, 2, 3$, then $K(r)$ is totally unstable.
3. If $k \geq 4$, then $(r)$ is $c$-stable.

Proof. First, we see that $\Delta_{K(r)}(t) = k(t - 1)^2(t^2 - t + 1) + t^2$, hence the modification of $\Delta_{K(r)}(t)$ is $f(x) = k(x - 1)(x - 2) + 1$. The conclusion follows by checking the intersection of two curves $y_1 = (x - 1)(x - 2)$ and $y_2 = -\frac{t}{k}$.

**Conjecture 12.3.** Let $r_m = [\overbrace{2, 2, \ldots, 2, 2k}, \overbrace{-2, -2, \ldots, -2}]$. Then if $m$ is even, $K(r_m)$ is $c$-stable for sufficiently large $k$. To be more precise, there exists a positive integer $N_m$ such that (1) if $m$ is even, then $K(r_m)$ is $c$-stable for $k \geq N_m$ and (2) if $m$ is odd, then $K(r_m)$ is $c$-stable for $k \leq -N_m$.

We can show that $N_2 = 4, N_3 = 3$ and $N_4 = 7$ (See Appendix C). These knots $K(r_m)$ are exceptional $c$-stable knots.

Example 12.4 below gives an exceptional $c$-stable link.

**Example 12.4.** Let $r = [2, 2, 2, -6, -2]$. Then $K(r)$ is a $c$-stable link. In fact, $\Delta_{K(r)}(t) = (t - 1)(3t^4 - 6t^3 + 7t^2 - 6t + 3)$ is $c$-stable.

### 12.2. Construction of $c$-stable quasi-rational knots and links

Let $K$ be a quasi-rational knot or link such that a Seifert matrix of $K$ is of the form $M = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$, where $A = \text{diag}\{a_1, a_2, \ldots, a_p\}, a_j > 0, 1 \leq j \leq p$ and $C = \text{diag}\{c_1, c_2, \ldots, c_q\}, c_j > 0, 1 \leq j \leq q$. Let $B = [b_{i,j}]_{1 \leq i \leq p, 1 \leq j \leq q}$.

**Proposition 12.5.** Suppose $M$ satisfies the following conditions:

1. $a_k > \frac{1}{2} \{ |b_{1,k}| + |b_{2,k}| + \cdots + |b_{q,k}| \}$ for $k = 1, 2, \ldots, p$,
2. $c_\ell > \frac{1}{2} \{ |b_{1,\ell}| + |b_{2,\ell}| + \cdots + |b_{q,\ell}| \}$ for $\ell = 1, 2, \ldots, q$.

Then $K$ is $c$-stable.
Proof. Since a symmetric matrix \( \hat{M} = M + M^T \) is positive definite, the signature of \( \hat{M} \) is equal to \( p + q \) and also \( p + q \) is the degree of the Alexander polynomial of \( K \). Hence, \( K \) is c-stable. \( \square \)

Note that \( K \) is generally non-alternating.

Remark 12.6. Conditions (1) and (2) in Proposition 12.5 are sufficient conditions for knots \( X_n = X_n(2a_1, 2a_2, \ldots, 2a_n|2b_1, 2b_2, \ldots, 2b_n) \) defined in Example 6.5 to be c-stable. Suppose that all \( a_j > 0 \) and \( b_j > 0 \). Then if \( X_n \) is c-stable, at least (1) or (2) is necessary. In fact, \( K_3 = X_3(2, 2, 2|2, 2, 2) \) is not c-stable, but strictly bi-stable. Here, \( \Delta_{K_3}(t) = t^6 - 4t^4 + 7t^3 - 4t^2 + 1 \) and \( K_3 \) is not alternating. On the other hand, \( X_3(4, 2, 2|2, 2, 2) \) is c-stable.

Proposition 12.7. Let \( X_n = X_n(2a_1, 2a_2, \cdots, 2a_n|2b_1, 2b_2, \cdots, 2b_n) \) be a quasi-rational knot defined in Example 6.5. Suppose
(1) \( a_1 \geq n/2, a_2 \geq (n-1)/2, \cdots, a_k \geq (n - k + 1)/2, \cdots, a_n \geq 1/2, \) and
(2) \( b_1 \geq 1/2, b_2 \geq 2/2, \cdots, b_k \geq k/2, \cdots, b_n \geq n/2. \)
Then \( X_n \) is c-stable.

Proof. Let \( M \) be a Seifert matrix given in Section 6.3. Since \( a_j \) and \( b_j \) are integers, it follows that \( a_n \geq 1 \) and \( b_1 \geq 1 \). Therefore, the \( n \)th row and \((n + 1)^{\text{st}}\) row are excessive. Apply the proof of Positivity Lemma on the matrix \( M + M^T \) to show that \( M + M^T \) is positive definite. \( \square \)

Proposition 12.8. Let \( Y_{2n+1} = Y_{2n+1}(2a_1, 2a_2, \cdots, 2a_{2n+1}|2b_1, 2b_2, \cdots, 2b_{2n+1}) \) be a quasi-rational knot defined in Example 6.5. Suppose
(1) \( a_1 \) and \( a_{2n+1} \geq 1, a_2 \) and \( a_{2n} \geq 2, \cdots, a_k \) and \( a_{2n+2-k} \geq k, \cdots, a_{n+1} \geq n + 1, \)
(2) \( b_1 \) and \( b_{2n+1} \geq 1, b_2 \) and \( b_{2n} \geq 2, \cdots, b_k \) and \( b_{2n+2-k} \geq k, \cdots, b_{n+1} \geq n + 1. \)
Then \( Y_{2n+1} \) is c-stable.

Proof. \( M + M^T \) satisfies all conditions of Positivity Lemma. \( \square \)

12.3. General construction of c-stable knots and links

The previous propositions show that given an arbitrary quasi-rational knot or link, we can make it c-stable by changing the number of full twists on some bands.

In this subsection, we generalize this result to that we can construct a c-stable knot or link from a given Seifert surface.

In case of Seifert surfaces specified by graphs as before, we can construct a c-stable knot or link with the same underlying graph.

Theorem 12.9. Let \( F \) be a Seifert surface for a knot or link \( K \), with \( \text{rank}H_1(F, \mathbb{Z}) = n. \) Suppose that a system of mutually disjoint \( n \) arcs \( \alpha_1, \ldots, \alpha_n \)
properly embedded in $F$ is specified so that $F \setminus \bigcup_i \alpha_i$ is a disk. Let $\tilde{F}$ be a Seifert surface obtained by full-twisting $F$ along each arc $\alpha_i$, $k_i$ times. Denote by $\tilde{K}$ the knot or link $\partial \tilde{F}$. Then, there exist $N_i \in \mathbb{N}$ ($i = 1, 2, \ldots, n$) such that if $k_i \geq N_i$ for each $i$, then $\tilde{K}$ is $c$-stable.

Proof. Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of embedded loops in $F$ such that for each $i$, $\alpha_i \cap (\bigcup \ell_i)$ is a single transverse point in $\ell_i$. Note that such a system $L$ is unique up to isotopy since $F \setminus \bigcup_i \alpha_i$ is a disk. Then $L$ with an arbitrary orientation gives Seifert matrices $S$ for $F$ and $\tilde{S}$ for $\tilde{F}$. Since twisting $F$ along $\alpha_i$ affects only the self-linking number of $\ell_i$, $\tilde{S} - S$ is a diagonal matrix whose $i$th diagonal entry is $k_i$. Let $M$ be the symmetric matrix $\tilde{S} + \tilde{S}^T = (m_{i,j})$. If each $k_i$ is large enough, we have $m_{i,i} > 0$ and $m_{i,i} > \Sigma_{j \neq i} |m_{i,j}|$ and hence by Strong Positivity Lemma (Proposition 4.5), $M$ is positive definite. Then the signature $\sigma(M)$ is equal to $n$. By [24], $\Delta_{\tilde{K}}(t)$ has at least $n$ of its zeros on the unit circle, and hence $n \leq \deg \Delta_{\tilde{K}}(t)$. Since $n = 2g(\tilde{F})$, we have $\deg \Delta_{\tilde{K}}(t) \leq n$. Therefore, $\deg \Delta_{\tilde{K}}(t) = n$ and the conclusion follows. 

Note that if $M$ is positive definite, we have $n \leq \deg \Delta_{\tilde{K}}(t) \leq 2g(\tilde{K}) \leq n$, and hence $\tilde{F}$ is a minimal genus Seifert surface for $\tilde{K}$.

Before we discuss some application of Theorem 12.9 we prove one proposition.

**Proposition 12.10.** Let $G$ be a positive (or negative) admissible connected planar graph on a disk $D$. Suppose that $G$ satisfies (7.1). Let $F(G)$ be the surface representing $G$. Then $K = \partial F(G)$ is alternating and $c$-stable.

We should note that $G$ is not necessarily an even graph.

Proof. Since a diagram is special alternating, $K$ is special alternating. Now let $M$ be a Seifert matrix obtained from $F(G)$. Then $M + M^T$ is positive (or negative) definite by Positivity lemma and hence $K$ is $c$-stable.

Now take finitely many disks $D_1, D_2, \ldots, D_n$ each of which has a positive admissible graph $G_j$ ($1 \leq j \leq n$) that satisfies (7.1). Consider a Murasugi sum $F$ of surfaces $F(G_1), F(G_2), \ldots, F(G_n)$ glued by an arbitrary fashion. Then the knot $K = \partial F$ is generally not $c$-stable, but by Theorem 12.9, we can make $K$ to be $c$-stable, by changing at most $s$ weights in $\{G_1, G_2, \ldots, G_n\}$, where $s = \text{rank}H_1(F; \mathbb{Z})$.

**Example 12.11.** The knot or link in the left is not $c$-stable, but by changing at most four weights, it becomes $c$-stable.

![Fig. 12.1](image-url)
12.4. *Interlacing property of zeros on the unit circle*

In this sub-section, we define the interlacing property for two c-stable real polynomials.

**Definition 12.12.** Let \( f(t) \) and \( g(t) \) be c-stable real polynomials, and let \( \{\alpha_j, 1 \leq j \leq n\} \) and \( \{\beta_k, 1 \leq k \leq m\} \) be, respectively, the unit complex zeros of \( f(t) \) and \( g(t) \) with a property that \( \text{Im}(\alpha_j) \geq 0 \) and \( \text{Im}(\beta_k) \geq 0 \). Then we say that \( f(t) \) and \( g(t) \) are interlaced if \( \{\text{Re}(\alpha_j), 1 \leq j \leq n\} \) and \( \{\text{Re}(\beta_k), 1 \leq k \leq m\} \) are interlaced.

As a typical example, we prove the following proposition:

**Proposition 12.13.** Let \( r_n = [\frac{2}{n}, \cdots, \frac{2}{n}] \). Then \( \Delta_{K(r_n)}(t) \) and \( \Delta_{K(r_n-1)}(t) \) are interlaced.

**Proof.** The unit complex zeros \( \{\alpha_k\} \) of \( \Delta_{K(r_n)}(t) \) with \( \text{Im}(\alpha_j) \geq 0 \) are: (1) if \( n \) is even, say \( 2m \), then \( \{\alpha_k\} = \{e^{\frac{2k\pi}{2m}}, 0 \leq k \leq m-1\} \), and (2) if \( n \) is odd, say \( 2m+1 \), then \( \{\alpha_k\} = \{e^{\frac{2k\pi}{2m+1}}, 0 \leq k \leq m\} \).

Then, the proposition follows from inequalities below.

\[
\begin{align*}
(1) \cos \frac{2k\pi}{2m+2} &> \cos \frac{(2k+1)\pi}{2m+1} > \cos \frac{(2k+2)\pi}{2m+2}, 0 \leq k \leq m-1. \\
(2) \cos \frac{(2k+1)\pi}{2m+3} &> \cos \frac{(2k+2)\pi}{2m+2} > \cos \frac{(2k+3)\pi}{2m+3}, 0 \leq k \leq m-1
\end{align*}
\]  

\(\blacksquare\)

The following theorem is the c-stable version of Theorem 9.4 and is proved by using modified Alexander polynomials instead of Alexander polynomials. Therefore, the details are omitted.

**Theorem 12.14.** Let \( r = [2a_1, 2a_2, \cdots, 2a_n], a_j > 0, 1 \leq j \leq n \) and \( s = [2a_1, 2a_2, \cdots, 2a_{n-1}] \). Then are \( \Delta_{K(r)}(t) \) and \( \Delta_{K(s)}(t) \) interlaced.

**Problem 12.15.** Characterize c-stable alternating knots and links.

13. Bi-stable knots and links

13.1. *Bi-stable 2-bridge knots and links*

A bi-stable knot has not only real zeros, but also unit complex zeros. Therefore, we could say that it combines two parts, one is a c-stable part and another is a real stable part. From this point of view, the following theorem is not surprising, although a proof is not straightforward.

**Theorem 13.1.** Let \( r = [2a_1, 2a_2, \cdots, 2a_{2m}, 2b_1, -2b_2, 2b_3, -2b_4, \cdots, -2b_{2p}], \) (or \( r = [2b_1, -2b_2, 2b_3, -2b_4, \cdots, -2b_{2p}, 2a_1, 2a_2, \cdots, 2a_{2m}] \)), where \( a_j > 0, 1 \leq j \leq 2m \) and \( b_k > 0, 1 \leq k \leq 2p \). Then \( \Delta_{K(r)}(t) \) and \( \Delta_{K(r)}(t) \) are interlaced.
2m and \(b_k > 0, 1 \leq k \leq 2p\). Then \(K(r)\) is bi-stable. The number of the real zeros is \(2p\) and that of the unit complex zeros is \(2m\).

Proof. Since the signature of \(K(r)\) is \(2m\), it follows that the number of the unit complex zeros is at least \(2m\). Therefore, it suffices to show that the number of the real zeros is (at least) \(2p\). First we prove the following lemma.

**Lemma 13.2.** (1) Let \(r' = [2a_1, 2a_2, \ldots, 2a_{2m-1}]\). Then \((t - 1)\) divides \(\Delta_{K(r')}(t)\), but \((t - 1)^2\) does not. (2) Let \(r^* = [-2b_2, 2b_3, -2b_4, \ldots, -2b_{2p}]\). Then \((t - 1)\) divides \(\Delta_{K(r')}(t)\), but \((t - 1)^2\) does not.

Proof. (1) \(\Delta_{K(r')}(t)/(t - 1) = \Delta_{K(r')}(t, t)\), where \(\Delta_{K(r')}(x, y)\) denotes the 2-variable Alexander polynomial of a 2-component link \(K(r')\). Then \(|\Delta_{K(r')}(1, 1)|\) is the absolute value of the linking number \(\ell\) between two components of \(K(r')\). \[\] Since \(|\ell| = |a_1 + a_3 + \cdots + a_{2m-1}| > 0\), \(\Delta_{K(r')}(t, t)\) is not divisible by \(t - 1\). (2) \(K(r^*)\) is stable, and all the zeros are simple.

**Lemma 13.3.** Let \(D_K(t)\) be the normalization of \(\Delta_K(t)\). Then we have

\[
D_{K(r)}(t) = D_{K(r_1)}(t)D_{K(r_2)}(t) + tD_{K(r')}D_{K(r')}(t), \tag{13.1}
\]

where \(r_1 = [2a_1, 2a_2, \ldots, 2a_{2m}]\), \(r_2 = [2b_1, -2b_2, 2b_3, -2b_4, \ldots, -2b_{2p}]\), and \(r'\) and \(r^*\) are given in Lemma [13.2].

Proof. Using a twisted chain type Seifert surface of \(K(r)\), we have a Seifert matrix \(M\) of the form: \(M = \begin{bmatrix} A & B \\ O & C \end{bmatrix}\), where \(A, C\) are Seifert matrices of \(K(r_1)\) and \(K(r_2)\), respectively, and \(B\) has only 1 at the \((2m, 2m + 1)\)-entry and 0 elsewhere (see [13.1]). Then it is easy to see that \(D_{K(r)}(t)\) has the required form. \(\square\)

We return to a proof of Theorem [13.1] We know now

(1) \(f_m(t) = D_{K(r_1)}(t)\) is \(c\)-stable and hence \(f_m > 0\) for any real \(t\).

(2) \(D_{K(r')}\) is \(c\)-stable, and has only one real zero that is 1,

and hence we can write \(D_{K(r')} = (t - 1)g_m(t)\) and \(g_m(t) > 0\) for any real \(t\),

(3) \(D_{K(r_2)}D_{K(r')}\) is simple,

(4) \(D_{K(r_2)}(t)\) is stable and has \(2p\) positive real zeros, say, \(\beta_1 < \beta_2 < \cdots < \beta_p\), in \([0, 1]\),

(5) \(D_{K(r')} = (t - 1)h_p(t)\) is stable and has \((2p - 1)\) real zeros, say,

\[
\alpha_1 < \alpha_2 < \cdots < \alpha_{p-1} < \alpha_p = 1 \text{ in } [0, 1], \text{ and } h_p(1) \neq 0.
\]

Further, \(\{\beta_j, 1 \leq j \leq p\}\) and \(\{\alpha_j, 1 \leq j \leq p\}\) are interlaced, i.e.,

\[
\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_{p-1} < \beta_p < \alpha_p = 1. \tag{13.2}
\]

Using these notations, we can write

\[
D_{K(r)}(t) = f_m(t)D_{K(r_2)}(t) + t(t - 1)^2g_m(t)h_p(t), \text{ and } g_m(1) \neq 0 \neq h_p(1). \tag{13.3}
\]
Now we calculate the number of real zeros of $D_{K(r)}(t)$. From (13.3), we see that the real zeros of $D_{K(r)}(t)$ are determined by the intersection $\{\gamma_j\}$ of two curves $y_1 = f_m(t)D_{K(r_2)}(t)$ and $y_2 = -t(t-1)^2g_m(t)h_p(t)$. Using the fact that $\{\alpha_j\}$ and $\{\beta_j\}$ are interlaced, we have a graph below. Note that $f_m(t), g_m(t) > 0$ for any real $t$, and $t = 1$ is a double zero for $y_2$. Further, $y_1(0) > 0$ and $y_2'(0) < 0$.

(1) If $p$ is even, we have Fig. 13.1.

From the graph, we see that there are (at least) $p$ points of intersection $\{\gamma_j, 1 \leq j \leq p\}$ in $[0, 1]$ and

$$\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3 < \cdots < \beta_{p-1} < \gamma_{p-1} < \beta_p < \gamma_p < 1.$$ 

(2) If $p$ is odd, then we have the following graph.

Therefore, $D_{K(r)}(t)$ has at least (and hence exactly) $2p$ real zeros. □

**Example 13.4.** Let $r = [4, 2, 6, 2, -6, 2, -4]$. $\Delta_{K(r)}(t)$ has $p$ real zeros and four unit complex zeros.

**13.2. Exceptional bi-stable knots and Salem knots**

A fibred knot (or link) $K$ is called a *Salem fibred knot (or link)* ([13]), if $\Delta_K(t)$ is bi-stable and has exactly two real ($\neq 1$) zeros. A typical example of a Salem fibred knot is a 2-bridge knots $K(r_m)$ by Theorem [13.1], where $r_m = [2, 2, \cdots, 2, -2]$, $m$ odd, (or abbreviated $[(2)^m, -2]$). Modifying $K(r_m)$, we obtain a series of exceptional bi-stable knots given below.

**Proposition 13.5.** Let $r(m, n) = [(2)^m, -2, (2)^n], m \geq n \geq 0, m + n$ being odd. Then $K(r(m, n))$ is a Salem fibred knot.
Proof. By induction on $m$ and $n$, it is shown easily that the normalized Alexander polynomial $D_{m,n}(t)$ of $K(r(m,n))$ is given by the following formula:

$$D_{m,n}(t) = \sum_{k=0}^{n} (-1)^k(4k + 1)t^k + (4n + 3)\sum_{k=n+1}^{m}(-1)^k t^k + \sum_{j=0}^{n}(-1)^{m+j+1}(4n + 1 - 4j)t^{m+j+1}$$

Since $D_{m,n}(1) = -1$, $D_{m,n}(t)$ has at least two real zeros. Further, since $\sigma(K(r(m,n))) = m + n - 1$, $D_{m,n}(t)$ has at least $m + n - 1$ unit complex zeros and hence $K(r(m,n))$ is a Salem fibred knot. \qed

Let $\mu(K)$ denote the maximal absolute value of the real zeros of $\Delta_K(t)$. We note that if $K$ is a Salem fibred knot, $\mu(K)$ is equal to Mahler measure of $\Delta_K(t)$.

Now our computation suggests that for $m \geq 1$,

1. $\mu(K(r(m + 2,0))) < \mu(K(r(m,0)))$ and
2. $\mu(K(r(m + 2,1))) > \mu(K(r(m,1)))$. Further,
3. $\mu(K(r(m + n,0))) < \mu(K(r(m,0))) < \mu(K(r(m + n,1)))$. Finally
4. $\lim_{m \to \infty} \mu(K(r(m,0))) = 2$, and
5. $\lim_{m \to \infty} \mu(K(r(m,1))) = 3.41421$ (13.4)

Beside these bi-stable knots, Hironaka showed two more Salem fibred 2-bridge knots [15].

1. $K_1 = K(s_1), s_1 = [(2)^5, (-2)^3]$, and $\mu(K_1) \doteq 1.63557$,
2. $K_2 = K(s_2), s_2 = [(2)^9, (-2)^5]$ and $\mu(K_2) \doteq 1.42501$.

We find three more sporadic Salem fibred 2-bridge knots.

3. $K_3 = K(s_3), s_3 = [(2)^6, -2, 2, -2, -2]$, and $\mu(K_3) \doteq 3.94748$,
4. $K_4 = K(s_4), s_4 = [(2)^4, (-2)^3, 2]$ and $\mu(K_4) \doteq 2.38215$,
5. $K_5 = K(s_5), s_5 = [(2)^6, (-2)^5, (2)^3]$ and $\mu(K_5) \doteq 1.80017$ (13.5)

We suspect that there exist other Salem fibred 2-bridge knots. However, contrary to knots, we find many Salem fibred 2-bridge links and we will study these links in a separate paper.

13.3. General bi-stable knots and links

A 2-bridge knot in Theorem 13.1 is given as a quasi-rational knot shown in Fig 13.3.
The first half part \([2a_1, 2a_2, \cdots, 2a_{2m}]\) is a special alternating knot and it is also represented by an admissible positive graph \(G_0\). For example, the surface \(F(G_0)\) of \(K(r), r = [2a, 2b]\) is represented by \(G_0\). See Fig. 13.4.

Therefore, for example, \(F(G)\) of \(K(r), r = [2a_1, 2a_2, 2b_1, -2b_2, 2b_3, -2b_4]\) is represented by a graph \(G\) below.

This observation suggests us a construction of general bi-stable knots as in Proposition 13.6 below. See Fig. 13.6 for example, where a bi-stable knot is depicted by a graph, and the zeros are plotted.

**Proposition 13.6.** Let \(G_0\) be an admissible positive (or negative) graph on a disk. Attach \(p\) mutually disjoint positive (or negative) arcs to \(\partial D\) and then \(p\) negative (or positive) arcs to \(\partial D\) from the back side in such a way that the first (or the last) arc crosses exactly one edge of \(G_0\), where the \(2p\) arcs attached to \(\partial D\) represent a
A crucial point of this construction is the interlacing property of a 2-bridge knot \(K(r), r = [2b_1, -2b_2, 2b_3, \ldots, -2b_p]\). Therefore, a knot \(K(r)\) may be replaced by other stable knots which have some kind of interlacing property.

In Fig 13.7 below, a 2-bridge knot \(K(r)\) is replaced by a stable knot \(K_3 = X_3(2, 2, 2 - 2, -2, -2)\). The knot \(K\) thus obtained is bi-stable. In fact, \(\Delta_K(t) = -3 + 44t - 235t^2 + 662t^3 - 1161t^4 + 1387t^5 - 1161t^6 + 662t^7 - 235t^8 + 44t^9 - 3t^{10}\) is bi-stable. The zeros are plotted in Fig 13.7 right.

However, \(K(r)\) may not be replaced by an exceptional stable 2-bridge knot \(K'\). If the c-stable part \([2a_1, 2a_2, \ldots, 2a_{2m}]\) is replaced by an exceptional c-stable 2-bridge knot, then the knot is generally not bi-stable. For example, neither \([2, 8, -2, -2, 4, -2, 6, -8]\) nor \([2, 8, -2, -2, 10, 2, -2, -10]\) is bi-stable, where \([2, 8, -2, -2]\) is exceptionally c-stable and \([10, 2, -2, -10]\) is exceptionally stable. However, it is interesting to see that if the second term 8 in both cases is replaced by a sufficiently large positive integer, the knots become bi-stable.
14. Mobius Transformations

In this section, we study the image of the zeros of the Alexander polynomial of a knot by a Mobius transformation $\varphi$. We begin with the definition of a special Mobius transformation $\varphi$ that is used in this section.

Let $\varphi : C \cup \{\infty\} \rightarrow C \cup \{\infty\}$ be a Mobius transformation given by

$$\varphi(z) = \frac{1 - zi}{z - i}. \quad (14.1)$$

$\varphi$ has the following properties.

1. $\varphi$ is one to one and $\varphi^{-1}$ is given by $\varphi^{-1}(z) = \frac{1 + zi}{z + i}$.
2. $\varphi$ keeps two points $z = \pm 1$ fixed,
3. $\varphi(0) = i, \varphi(-i) = 0$ and $\varphi(i) = \infty$, and
4. $\varphi^2(z) = 1/z. \quad (14.2)$

We can easily check the following lemma.

**Lemma 14.1.** (1) $\varphi$ maps the interior of the unit circle centred at 0 onto the upper half-plane, and the exterior of the unit circle onto the lower half-plane. (2) $\varphi$ maps the unit circle onto the real line and vice versa.

We easily check the following lemma.

**Theorem 14.3.** Let $f(t)$ be a reciprocal real polynomial of even degree, say $2n$. Assume that 0 and $\pm i$ are not zeros of $f(t)$. Then there exists a reciprocal real polynomial $f^*(t)$ of the same degree $2n$ satisfying the following conditions.

1. the zeros of $f^*(t)$ are exactly the image of the zeros of $f(t)$ under $\varphi$, namely, if $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ are the zeros of $f(t)$, then $\varphi(\alpha_1), \varphi(\alpha_2), \cdots, \varphi(\alpha_{2n})$ are exactly the zeros of $f^*(t)$.
2. If $f$ is an integer polynomial, then so is $f^*$. 

The following simple property of $\varphi$ is crucial to our purpose. A proof follows from easy computations, and hence we omit the details.

**Proposition 14.2.** For any $\alpha \in C$, $\alpha \neq 0, \pm i$,

1. $\varphi(\alpha) + \varphi(\frac{1}{\alpha}) = \frac{4}{(\alpha + \frac{1}{\alpha})}$.
2. $\varphi(\alpha)\varphi(\frac{1}{\alpha}) = 1.$

In particular, if $\alpha (\neq 0)$ is real or $|\alpha| = 1, \alpha \neq \pm i$, then $\alpha + \frac{1}{\alpha}$ and $\varphi(\alpha) + \varphi(\frac{1}{\alpha})$ are both real.

The main theorem in this section is the following:

**Theorem 14.3.** Let $f(t)$ be a reciprocal real polynomial of even degree, say $2n$. Assume that 0 and $\pm i$ are not zeros of $f(t)$. Then there exists a reciprocal real polynomial $f^*(t)$ of the same degree $2n$ satisfying the following conditions.

1. the zeros of $f^*(t)$ are exactly the image of the zeros of $f(t)$ under $\varphi$, namely, if $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ are the zeros of $f(t)$, then $\varphi(\alpha_1), \varphi(\alpha_2), \cdots, \varphi(\alpha_{2n})$ are exactly the zeros of $f^*(t)$.
2. If $f$ is an integer polynomial, then so is $f^*$. 

Before we prove the theorem, we mention a couple of corollaries.

**Corollary 14.4.** Let $\Delta_K(t)$ be the Alexander polynomial of a knot $K$ and degree $\Delta_K(t) = 2n$. Then we have the following:

1. $\Delta_K^*(t)$ is a reciprocal integer polynomial of the same degree, $2n$. Therefore, $\Delta_K^*(t)$ is the Hosokawa polynomial of some link $K^*$ (with an arbitrary number of components).
2. $|\Delta_K^*(1)| = 2^n$ and $|\Delta_K^*(-1)| = 2^n|\Delta_K(-1)|$.
3. $\Delta_K^{**}(t) = 2^{2n}\Delta_K(t)$.

**Corollary 14.5.** If $\Delta_K(t)$ is stable, then $\Delta_K^*(t)$ is $c$-stable. If $\Delta_K(t)$ is $c$-stable, then $\Delta_K^*(t)$ is stable. Further, if $\Delta_K(t)$ is bi-stable, so is $\Delta_K^*(t)$.

Now we proceed to a proof of Theorem 14.3.

Write

$$f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_{2n}, \quad (14.3)$$

where $c_0 > 0$ and $c_j = c_{2n-j}, 0 \leq j \leq 2n$.

Let $\alpha_1, 1/\alpha_1, \alpha_2, 1/\alpha_2, \cdots, \alpha_n, 1/\alpha_n$ be all the zeros of $f(t)$. Then we can write

$$f(t) = c_0 \prod_{j=1}^{n} (t - \alpha_j)(t - \frac{1}{\alpha_j}) \quad (14.4)$$

**Lemma 14.6.** Let $A_j = \alpha_j + \frac{1}{\alpha_j}, 1 \leq j \leq n$. Then $\lambda = \prod_{j=1}^{n} A_j$ is a real number.

**Proof.** Since $\alpha_j$ is a zero of $f(t)$, so is $\bar{\alpha_j}$ and hence $(\alpha_j + \frac{1}{\alpha_j})(\bar{\alpha_j} + \frac{1}{\bar{\alpha_j}})$ is a real number. \hfill \square

Later we will see that $\lambda c_0$ is an integer and show the following:

$$f^*(t) = \lambda c_0 \prod_{j=1}^{n} (t - \varphi(\alpha_j))(t - \varphi(\frac{1}{\alpha_j})) \quad (14.5)$$

Now consider $F(t) = f(t)/c_0 = t^{2n} + \frac{c_1}{c_0} t^{2n-1} + \cdots + \frac{c_{2n}}{c_0}$. Then

$$F(t) = \prod_{j=1}^{n} (t - \alpha_j)(t - \frac{1}{\alpha_j}) = \prod_{j=1}^{n} (t^2 - A_j t + 1). \quad (14.6)$$
For \(0 \leq k \leq n\), define \(X_k = \sum_{j_1, \ldots, j_k} A_{j_1} A_{j_2} \cdots A_{j_k}\), where the summation runs over all \(j_1, j_2, \ldots, j_k\) such that \(1 \leq j_1 < j_2 < \cdots < j_k \leq n\). In particular, \(X_0 = 1\) and \(X_n = \lambda\). By expanding the right hand side of (14.6), we have the following system of relations.

Case I. \(n = 2m\).

For \(k = 0, 1, 2, \cdots, m\),

\[
c_{2k} = \binom{n}{k} X_0 + \binom{n-2}{k-1} X_2 + \cdots + \binom{n-2k}{k-k} X_{2k},
\]

(14.7)

and for \(k = 1, 2, \cdots, m\),

\[
-c_{2k-1} = \binom{n-1}{k-1} X_1 + \binom{n-3}{k-2} X_3 + \cdots + \binom{n-(2k-1)}{k-k} X_{2k-1}
\]

(14.8)

For simplicity, let \(M_0\) and \(N_0\) be, respectively, the coefficient matrices of the system of relations of (14.7) and (14.8). Namely, \(M_0\) and \(N_0\) are lower triangular integer matrices of sizes respectively \(m + 1\) and \(m\), and each with determinant 1.

\[
M_0(X_0, X_2, \cdots, X_{2m})^T = \frac{1}{c_0}(c_0, c_2, \cdots, c_{2m})^T \quad \text{and}
\]

\[
N_0(X_1, X_3, \cdots, X_{2m-1})^T = -\frac{1}{c_0}(c_1, c_3, \cdots, c_{2m-1})^T,
\]

(14.9)

and hence

\[
(X_0, X_2, \cdots, X_{2m})^T = \frac{M_0^{-1}}{c_0}(c_0, c_2, \cdots, c_{2m})^T \quad \text{and}
\]

\[
(X_1, X_3, \cdots, X_{2m-1})^T = -\frac{N_0^{-1}}{c_0}(c_1, c_3, \cdots, c_{2m-1})^T.
\]

(14.10)

Case II. \(n = 2m + 1\).

The same argument shows the following

For \(k = 0, 1, 2, \cdots, m\),

\[
c_{2k} = \binom{n}{k} X_0 + \binom{n-2}{k-1} X_2 + \cdots + \binom{n-2k}{k-k} X_{2k},
\]

(14.11)

and

\[
-c_{2k+1} = \binom{n-1}{k} X_1 + \binom{n-3}{k-1} X_3 + \cdots + \binom{n-(2k+1)}{k-k} X_{2k+1}
\]

(14.12)

Using coefficient matrices \(M_1\) and \(N_1\) of these systems of relations, we can write

\[
M_1(X_0, X_2, \cdots, X_{2m})^T = \frac{1}{c_0}(c_0, c_2, \cdots, c_{2m})^T \quad \text{and}
\]

\[
N_1(X_1, X_3, \cdots, X_{2m+1})^T = -\frac{1}{c_0}(c_1, c_3, \cdots, c_{2m+1})^T.
\]

(14.13)
Here $M_1$ and $N_1$ are $(m + 1) \times (m + 1)$ lower triangular integer matrices with determinant 1 and hence

$$(X_0, X_2, \cdots, X_{2m})^T = \frac{M_1^{-1}}{c_0}(c_0, c_2, \cdots, c_{2m})^T,$$

and

$$(X_1, X_3, \cdots, X_{2m+1})^T = -\frac{N_1^{-1}}{c_0}(c_1, c_3, \cdots, c_{2m+1})^T \quad (14.14)$$

Now we study $f^*(t)$:

$$f^*(t) = \lambda c_0 \prod_{j=1}^{n} (t - \varphi(\alpha_j))(t - \varphi(\frac{1}{\alpha_j}))$$

$$= \lambda c_0 \prod_{j=1}^{n} (t^2 - (\varphi(\alpha_j) + \varphi(\frac{1}{\alpha_j}))t + \varphi(\alpha_j)\varphi(\frac{1}{\alpha_j}))$$

$$= \lambda c_0 \prod_{j=1}^{n} (t^2 - \frac{4}{A_j}t + 1). \quad (14.15)$$

We write it as

$$f^*(t) = \lambda c_0 (d_0 t^{2n} + d_1 t^{2n-1} + \cdots + d_{2n}), d_0 = 1. \quad (14.16)$$

If we compare (14.15) with (14.6), we see immediately the following relations.

Case (I) $n = 2m$.

For $k = 0, 1, 2, \cdots, m$,

$$\frac{d_{2k}}{d_0} = \binom{n}{k} X_0 + \frac{n - 2}{k - 1} \sum_{j_1, j_2} \frac{4^2}{A_{j_1}A_{j_2}} + \frac{n - 4}{k - 2} \sum_{j_1, \cdots, j_4} \frac{4^4}{A_{j_1} \cdots A_{j_4}} + \cdots$$

$$+ \frac{n - 2k - 1}{k - k} \sum_{j_1, \cdots, j_{2k}} \frac{4^{2k}}{A_{j_1} \cdots A_{j_{2k}}}, \quad (14.17)$$

and hence,

$$\lambda d_{2k} = \binom{n}{k} X_n + \left( \frac{n - 2}{k - 1} \right) 4^2 X_{n-2} + \left( \frac{n - 4}{k - 2} \right) 4^4 X_{n-4} + \cdots$$

$$+ \left( \frac{n - 2k - 1}{k - k} \right) 4^{2k} X_{n-2k} \quad (14.18)$$

Similarly, for $k = 1, 2, \cdots, m$,

$$-\lambda d_{2k-1} = \binom{n - 1}{k - 1} 4X_{n-1} + \left( \frac{n - 3}{k - 2} \right) 4^3 X_{n-3} + \cdots$$

$$+ \left( \frac{n - (2k - 1)}{k - k} \right) 4^{2k-1} X_{n-(2k-1)}. \quad (14.19)$$
Let $P_\ell$ be a diagonal matrix of order $\ell$ of the form:

$$P_\ell = \text{diag}\{1, 4^2, 4^4, \ldots, 4^{2(\ell-1)}\}.$$ 

Then (14.18) and (14.19) can be written as

$$\lambda(d_0, d_2, \cdots, d_{2m})^T = M_0 P_{m+1} (X_n, X_{n-2}, \cdots, X_0)^T, \text{ and}$$

$$-\lambda(d_1, d_3, \cdots, d_{2m-1})^T = N_0 \widehat{P_m} (X_{n-1}, X_{n-3}, \cdots, X_1)^T,$$  

(14.20)

where $\widehat{P}_\ell = 4P_\ell$.

Case II. $n = 2m + 1$.

The same argument shows

$$\lambda(d_0, d_2, \cdots, d_{2m})^T = M_1 P_{m+1} (X_{2m+1}, X_{2m-1}, \cdots, X_1)^T, \text{ and}$$

$$-\lambda(d_1, d_3, \cdots, d_{2m+1})^T = N_1 \widehat{P_m} (X_{2m}, X_{2m-2}, \cdots, X_0)^T.$$  

(14.21)

Let $Q_\ell$ be an $\ell \times \ell$ matrix (that is the mirror of the identity matrix), namely, $Q_\ell = [q_{i,j}]_{1 \leq i, j \leq \ell}$, where $q_{i,j} = 1$, if $i + j = \ell + 1$, and $q_{i,j} = 0$, otherwise. Using $Q_\ell$, we have the final result.

Case (a) $n = 2m$. $(X_n, X_{n-2}, \cdots, X_0)^T = Q_{m+1}(X_0, X_2, \cdots, X_n)^T$ and $(X_{n-1}, X_{n-3}, \cdots, X_1)^T = Q_m(X_1, X_3, \cdots, X_{n-1})^T$, and hence, combining (14.20) and (14.21), we have

$$\lambda_0(d_0, d_2, \cdots, d_{2m})^T = M_0 P_{m+1} Q_{m+1} M_0^{-1}(c_0, c_2, \cdots, c_{2m})^T \text{ and}$$

$$-\lambda_0(d_1, d_3, \cdots, d_{2m-1})^T = N_0 \widehat{P_m} Q_m (-N_0^{-1})(c_1, c_3, \cdots, c_{2m-1})^T.$$  

(14.22)

(14.23)

Case (b) $n = 2m + 1$.

Similarly, we have

$$\lambda_0(d_0, d_2, \cdots, d_{2m})^T = M_1 P_{m+1} Q_{m+1} M_0^{-1}(c_0, c_2, \cdots, c_{2m})^T \text{ and}$$

$$-\lambda_0(d_1, d_3, \cdots, d_{2m+1})^T = N_1 \widehat{P_m} Q_m (-N_0^{-1})(c_1, c_3, \cdots, c_{2m+1})^T.$$  

(14.24)

(14.25)

To be more precise, let $f(t) = \sum_{j=0}^{2n} c_j t^{2n-j}, c_0 > 0$, and $f^*(t) = \sum_{j=0}^{2n} a_j t^{2n-j}$. Then $a_j, 0 \leq j \leq n$, is obtained by the following formulas.

Case (a) $n = 2m$.

$$ (a_0, a_2, \cdots, a_{2m})^T = M_0 P_{m+1} Q_{m+1} M_0^{-1}(c_0, c_2, \cdots, c_{2m})^T \text{ and}$$

$$ (a_1, a_3, \cdots, a_{2m-1})^T = N_0 \widehat{P_m} Q_m N_0^{-1}(c_1, c_3, \cdots, c_{2m-1})^T.$$  

(14.26)
Case (b) $n = 2m + 1$.

$$
(a_0, a_2, \cdots, a_{2m})^T = -M_1 P_{m+1} Q_{m+1} N_1^{-1}(c_1, c_3, \cdots, c_{2m+1})^T \quad \text{and} \\
(a_1, a_3, \cdots, a_{2m+1})^T = -N_1 P_{m+1} Q_{m+1} M_1^{-1}(c_0, c_2, \cdots, c_{2m})^T.
$$

(14.27)

This proves Theorem 14.3 (1).

Since all the matrices involved in the proof are integer matrices, it follows that if $f(t)$ is an integer polynomial, then so is $f^*(t)$. This proves (2).

To prove (3) and (4), we compute $f(\pm 1)$ and $f^*(\pm 1)$. Since $f(t) = c_0 \prod_{j=1}^{n} (t^2 - A_j t + 1)$, it follows that $f(1) = c_0 \prod_{j=1}^{n} (2 - A_j)$ and $f(-1) = c_0 \prod_{j=1}^{n} (2 + A_j)$.

Meanwhile, $f^*(t) = \lambda c_0 \prod_{j=1}^{n} (t^2 - \frac{4}{A_j} t + 1)$, and hence $f^*(1) = \lambda c_0 \prod_{j=1}^{n} (2 - \frac{4}{A_j}) = \lambda c_0 \prod_{j=1}^{n} (2 + A_j - 4) = 2^n (-1)^n f(1)$. Similarly, $f^*(-1) = \lambda c_0 \prod_{j=1}^{n} (2 - \frac{4}{A_j}) = c_0 \prod_{j=1}^{n} (2 A_j + 4) = c_0 2^n \prod_{j=1}^{n} (A_j + 2) = 2^n f(-1)$. This proves (3) and (4).

To show (5), first we note that $\varphi^2(z) = 1/z$. Thus the set of the zeros of $f^{**}(t)$ and that of $f(t)$ are identical. Therefore, $f(t)$ divides $f^{**}(t)$ or $f^*(t)$ divides $f(t)$. However, $f^{**}(1) = 2^n f^*(1) = 2^n f(1)$ and hence, $f^{**}(t) = 2^n f(t)$. Finally, the uniqueness is evident. A proof of Theorem 14.3 is now completed. 

**Example 14.7.** Let $f(t) = \sum_{j=0}^{2n} c_j t^{2n-j}, c_0 > 0$ and $f^*(t) = \sum_{j=0}^{2n} a_j t^{2n-j}$.

(1) (i) Let $n = 1$ and $m = 0$. Then $M_1 = N_1 = P_1 = Q_1 = [1]$, and hence $a_0 = -c_1$ and $a_1 = -4c_0$. For example, if $f(t) = t^2 - 3t + 1$, then $f^*(t) = 3t^2 - 4t + 3$.

(ii) Let $n = 3$ and $m = 1$. Then $M_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and $N_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and hence

$$
\begin{bmatrix}
a_0 \\
a_2
\end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\
c_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 \\
a_3 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ -40 & -8 \end{bmatrix} \begin{bmatrix} c_0 \\
c_2 \end{bmatrix}.
$$

For example, if $f(t) = t^6 - t^5 + t^3 - t + 1$, then $f^*(t) = -(3t^6 - 12t^5 - 7t^4 + 40t^3 - 7t^2 - 12t + 3)$.

(2) (i) Let $n = 2$ and $m = 1$. Then $M_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $N_0 = [1]$. Therefore,

$$
\begin{bmatrix}
a_0 \\
a_2
\end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 12 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\
c_2 \end{bmatrix} \quad \text{and} \quad (a_1) = (4c_1). \quad \text{For example, if } f(t) = t^4 - t^3 + t^2 - t + 1, \text{ then } f^*(t) = -(t^4 + 4t^3 - 14t^2 + 4t + 1).
$$

(ii) Let $n = 4$ and $m = 2$. Then $M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 2 & 1 \end{pmatrix}$, and $N_0 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, and hence,

$$
\begin{bmatrix}
a_0 \\
a_2 \\
a_4
\end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ -56 & 8 & 4 \\ 140 & 20 & 6 \end{bmatrix} \begin{bmatrix} c_0 \\
c_2 \\
c_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 \\
a_3 \\
a_5 \end{bmatrix} = \begin{bmatrix} -12 & 4 & 12 \\ 28 & 12 & c_3 \end{bmatrix}.
$$

For example, if $f(t) = \sum_{j=0}^{8} (-1)^j t^{8-j}$, then $f^*(t) = t^8 + 8t^7 - 44t^6 - 40t^5 + 166t^4 - 40t^3 - 44t^2 + 8t + 1$.

**Remark 14.8.** Even if $f(t)$ is monic, $f^*(t)$ is not necessarily monic. Furthermore,
even if \( \{c_0, c_1, c_2, \cdots, c_{2n}\} \) alternates in sign, \( \{a_0, a_1, a_2, \cdots, a_{2n}\} \) may not alternate in sign.

**Question 14.9.** Let \( K \) be a c-stable knot and \( K^* \) be a stable link obtained by \( \varphi \). What can we say on \( K^* \)? Does there exist a geometric way to construct \( K^* \) from \( K \)?

### 15. Montesinos knots

In this section, we study the various stabilities of alternating Montesinos knots or links. It is not surprising to see that many Montesinos knots or links are quasi-rational, and hence, their stability properties can be determined by our method discussed earlier.

Now we begin with a well-known result of a characterization of alternating Montesinos knots or links. Let \( K = M(e \mid \beta_1/\alpha_1, \beta_2/\alpha_2, \cdots, \beta_n/\alpha_n) \) be a Montesinos knot or link. We assume that \( n \geq 3 \). Montesinos knots or links have two classes.

**Class (I)**

1. \( \beta_i/\alpha_i > 0 \) for any \( i, 1 \leq i \leq n \), and \( e \geq 0 \), or
2. \( \beta_i/\alpha_i < 0 \) for any \( i, 1 \leq i \leq n \), and \( e \leq 0 \).

**Class (II)**

\( 0 < \# \{ \beta_i/\alpha_i > 0, 1 \leq i \leq n \} < n \) and \( e = 0 \).

The following proposition is well-known. (See [23].)

**Proposition 15.1.** A Montesinos knot (or link) \( K \) is alternating if and only if \( K \) belongs to Class (I).

Since we are interested in various stabilities of alternating knots or links, we study the special classes of Montesinos knots or links described in the following theorem.

**Theorem 15.2.** Let \( K = M(e \mid \beta_1/\alpha_1, \beta_2/\alpha_2, \cdots, \beta_n/\alpha_n) \), \( n \geq 3 \), be a Montesinos knot or link. We assume the following conditions:

1. \( \beta_i/\alpha_i > 0 \) for any \( i, 1 \leq i \leq n \),
2. \( e \geq 0 \),
3. At most one \( \alpha_i \), say \( \alpha_1 \), can be even.
4. \( \beta_i \equiv 0 \) (mod 2), \( 1 \leq i \leq n \), unless \( \alpha_i \) is even.
5. Let \( [2a_1^{(i)}, 2a_2^{(i)}, \cdots, 2a_m^{(i)}] \) be the even continued fraction expansion of \( \beta_i/\alpha_i, 1 \leq i \leq n \). For each \( i \), the sequence \( \{2a_1^{(i)}, 2a_2^{(i)}, \cdots, 2a_m^{(i)}\} \) alternates in sign. In particular, \( 2a_1^{(i)} > 0 \) for \( 1 \leq i \leq n \).

Then we have the following conclusion.

**Case 1.** If all \( \alpha_i, 1 \leq i \leq n \) are odd and \( e \) is odd, then \( K \) is a special alternating knot, and hence \( K \) is c-stable.

**Case 2.** If all \( \alpha_i, 1 \leq i \leq n \), are odd and \( e \) is even, then \( K \) is a 2-component link and \( K \) is inversive. (A 2-component link \( K \) is said to be inversive if the original (oriented) link is c-stable (or stable), but if the orientation of one component is reversed, the resulting (oriented) link becomes a stable (or c-stable) link.)
Case 3. If $\alpha_1$ is even and others are odd and $e$ is even, then $K$ is a knot and is stable. If $e$ is 2 and all $|a_j^{(i)}| = 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, then the maximal value of the zero of $\Delta_K(t)$ is at least $n + 1$.

Case 4. If $\alpha_1$ is even and others are odd, and $e$ is odd, then $K$ is a knot and $K$ is bi-stable.

Remark 15.3. We should note that our cases do not contain all alternating Montesinos knots. Since we are interested in stability of the Alexander polynomial, the assumption (5) is crucial in the theorem. Any knot or link in our list has some stability properties.

Now, proofs of the first three cases are easy. For the first case, $K$ has a special alternating as in Fig. 15.1 and hence, $K$ is $c$-stable.

For the second case, one orientation gives us a special alternating diagram, and hence it is $c$-stable. If we reverse orientation of one component, the diagram shows that $K$ is a quasi-rational links discussed in Section 6, and hence, it is stable. See Figure 15.2 and 15.3.
For the third case, \( K \) is also quasi-rational knot discussed in Section 6, and hence, it is stable. Since the second statement can be proven by applying the same argument used in the proof of Theorem \[6.7\] (1), we omit the details. See Fig. 15.4.
The last case is the most complicated case. (See Fig. 15.5.)

Let \( r_j = \frac{\theta_j}{\alpha_j}, 1 \leq j \leq n \), and write:

\[
    r_1 = [2a_1^{(1)}, -2a_2^{(1)}, \ldots, (-1)^{k-1}2a_k^{(1)}, \ldots, 2a_{2m+1}^{(1)}], \quad \text{where } a_k^{(1)} > 0, 1 \leq k \leq 2m_1 + 1.
\]

Let \( r_0 = [-e, r_1] = [-e, 2a_1^{(1)}, -2a_2^{(1)}, \ldots, (-1)^{k-1}2a_k^{(1)}, \ldots, 2a_{2m+1}^{(1)}] \). Since \( e \) is odd, we see from the diagram that \( K(r_0) \) is a special alternating knot.

Further, we see that \( K \) is a Murasugi sum of \( K(r_0) \) and the connected sum of remaining \((n-1)\) 2-bridge knots \( K(r_2) \# K(r_3) \# \cdots \# K(r_n) \). Since \( K(r_0) \) is \( e \)-stable and \( K(r_2) \# K(r_3) \# \cdots \# K(r_n) \) is stable, it is not surprising that a Murasugi sum of these knots is bi-stable, but a proof is not immediate. The rest of this section will be devoted to a proof of this case.

First, from the diagram, we see that a Seifert matrix \( M \) of \( K \) is a direct sum of \( M_j, j = 0, 2, 3, \ldots, n \), except the first column, where \( M_j \) is a Seifert matrix of \( K(r_j) \) of twisted chain type.
In fact, $M$ is of the form above and has the following properties.

(1) The diagonal entries of $M_0$ is quite different from those of $M_j, j \geq 2$. They are
$$\left\{ -\frac{e+1}{2}, -1, \ldots, -1, -\left( a_2^{(1)} + 1 \right), -1, \ldots, -\left( a_4^{(1)} + 1 \right), -1, \ldots, -1, \right.$$ $$-\left( a_6^{(1)} + 1 \right), \ldots, -\left( a_{2m_1}^{(1)} + 1 \right), -1, \ldots, -1 \}.$$ 

(2) Other non-zero entries of $M_0$ are those of the one line above the diagonal, all of which are 1.

(3) The diagonal entries of $M_j, j \geq 2$, are $\left\{ a_1^{(j)}, -a_2^{(j)}, \ldots, -a_{2m_j}^{(j)} \right\}$.

(4) The size $\rho_0$ of $M_0$ is $\rho_0 = 1 + \sum_{j=1}^{1(2)} (2a_j^{(1)} - 1) + m_1 = \sum_{j=1}^{1(2)} 2a_j^{(1)}$,
while the size $\rho_j$ of $M_j$ is $\rho_j = 2m_j, 2 \leq j \leq n$.

(5) The extra 1 on the first column of $M$ appears only on the first row of each block matrix $M_2, M_3, \ldots, M_n$, namely 1 appears on the
$$(\rho_0 + 1, 1)-, (\rho_0 + \rho_2 + 1, 1)-, \ldots, (\sum_{j=0, j\neq 1}^{n} \rho_j + 1, 1)-\text{entries of } M. \quad (15.1)$$

Now to study the Alexander polynomial of $K$, we consider the determinant of $tM - M^T$ that is of the form
$$\begin{vmatrix}
\begin{array}{cccc}
tM_0 - M_0^T & -10 \cdots 0 & \cdots & \cdots \\
0 & tM_2 - M_2^T & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
0 & \vdots & \cdots & tM_p - M_p^T
\end{array}
\end{vmatrix}$$
By expanding it along the first row and then the first column, we can show easily that
\[
\Delta_K(t) = \det(Mt - M^T) \\
= \prod_{j=0, j \neq 1}^n \det(tM_j - M_j^T) \\
+ t \det \hat{M}_0 \sum_{j=2}^n \det(tM_2 - M_2^T) \cdots \det \hat{M}_j \cdots \det(tM_n - M_n^T) \\
= \Delta_K(r_0)(t)\Delta_K(r_2)(t) \cdots \Delta_K(r_n)(t) \\
+ t \det \hat{M}_0 [\sum_{j=2}^n \Delta_K(r_2)(t) \cdots \Delta_K(r_j)(t) \cdots \Delta_K(r_n)(t)], \tag{15.2}
\]
where \(\hat{M}_j, j = 0, 2, 3, \cdots, n\), is the matrix obtained from \(tM_j - M_j^T\) by deleting the first row and column and \(\hat{r}_j = [-2a_{2}^{(j)}, 2a_{3}^{(j)}, \cdots, -2a_{2m_j}^{(j)}], j \geq 2\).

For simplicity, we denote
\[
f_0 = \det(tM_0 - M_0^T) \quad \text{and} \quad \hat{f}_0 = -\det \hat{M}_0,
\]
and for \(j = 2, 3, \cdots, n\),
\[
f_j = (-1)^{m_j} \det(tM_j - M_j^T) \quad \text{and} \quad \hat{f}_j = (-1)^{m_j} \det \hat{M}_j.
\]
Since the leading coefficients of \(f_0, \hat{f}_0, f_j\) and \(\hat{f}_j\) are all positive, these polynomials are normalizations of \(\Delta_K(r_0)(t), \det \hat{M}_0, \Delta_K(r_j)(t)\) and \(\Delta_K(r_j)(t)\), respectively. Using these polynomials, we rewrite (15.2) as follows.
\[
\Delta_K(t) = f_0(t)f_2(t) \cdots f_n(t)(-1)^{m_2 + \cdots + m_n} \\
+ t\hat{f}_0(t)\hat{f}_2(t)f_3(t) \cdots f_n(t)(-1)(-1)^{m_2 + \cdots + m_n} \\
+ t\hat{f}_0(t)f_2(t)\hat{f}_3(t) \cdots f_n(t)(-1)(-1)^{m_2 + \cdots + m_n} + \cdots \\
+ t\hat{f}_0(t)f_2(t)f_3(t) \cdots f_{n-1}(t)\hat{f}_n(t)(-1)(-1)^{m_2 + \cdots + m_n}. \tag{15.3}
\]
Therefore, the normalization \(F\) of \(\Delta_K(t)\) is
\[
F = f_0(t)f_2(t) \cdots f_n(t) - t\hat{f}_0(t)\{\hat{f}_2(t)f_3(t) \cdots f_n(t) \\
+ f_2(t)\hat{f}_3(t)f_4(t) \cdots f_n(t) + \cdots + f_2(t)f_3(t) \cdots f_{n-1}(t)\hat{f}_n(t)\}.
\]
Further, \(\hat{f}_0, \hat{f}_2, \cdots, \hat{f}_n\) are Alexander polynomials of links and hence, they are divisible by \(t - 1\). Let \(f_0^* = \frac{\hat{f}_0}{t-1}\) and \(f_j^* = \frac{\hat{f}_j}{t-1}, 2 \leq j \leq n\). Then \(f_0^*\) and \(f_j^*\) are respectively reciprocal and \(f_j^*)(t)\) is also stable. We can write
\[
F = f_0 f_2 \cdots f_n - t(t-1)^2 f_0^* \{\sum_{j=2}^n f_2^* \cdots f_j^* \cdots f_n\}.
\]
Let \( F_1 = f_0 f_2 \cdots f_n \) and \( F_2 = t(t-1)^2 f_0^n \{ \sum_{j=2}^{n} f_j \} \), and further, for \( 2 \leq k \leq n \), \( F_{2,k} = t(t-1)^2 f_0^n f_2 \cdots f_k \cdots f_n \). Since \( \deg F = \deg \sum_{j=0,j\neq1}^{n} f_{j} \) and \( |\sigma(K)| = \rho_0 = \deg f_0 \), it suffices to show that \( F \) has at least \( 2q = \sum_{j=2}^{n} \deg f_j \) real zeros. The proof will be divided into two parts. In the first part we consider the case where no zeros of \( f_2(t), \cdots, f_n(t) \) are in common, i.e., \( f_2(t) \cdots f_n(t) \) is simple. Note that each \( f_j(t) \), \( j \neq 0 \), has no multiple zeros. In the second part, we consider the case where these Alexander polynomials have some zeros in common.

(1) Case (1). \( f_2 f_3 \cdots f_n \) is simple.

We will show that two curves \( y_1 = F_1 \) and \( y_2 = F_2 \) intersect at least \( q = \sum_{j=2}^{n} \rho_j / 2 \) points in \([0,1]\). Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_q \) be all the (real) zeros of \( f_2 f_3 \cdots f_n \) in \([0,1]\). Note that \( \deg F_1 = \deg F_2 + 1 \) and \( F_1(1) \neq 0 \).

First we see that (1)\( F_1(0) > 0 \) and (2) \( F_2(0) = 0 \) and \( F_2'(0) > 0 \).

In fact, (1) \( F_1(0) = \prod_{j \neq 1} f_j(0) > 0 \), since the leading coefficient of \( f_j, j \neq 1 \), is positive and \( f_j \) is reciprocal. (2) follows, since \( f_j*(0) > 0 \) for \( j \neq 1 \).

Now, suppose \( \gamma_1 \) is the zeros of \( f_k \). Then \( f_k(\gamma_1) = 0 \), but \( f_k^*(\gamma_1) \neq 0 \), since \( f_k f_k^* \) is simple. Therefore, \( F_{2,k}(\gamma_1) \neq 0 \), but \( F_{2,j}(\gamma_1) = 0 \) for \( j \neq k \). Further, since \( F_{2,k}^*(0) > 0 \), it follows that \( F_{2,k}(\gamma_1) > 0 \), and hence \( F_1 \) and \( F_2 \) intersect in \([0,1]\).

Next, we prove inductively the following lemma.

**Lemma 15.4.** (1) \( F_1(t) \leq 0 \) in \([\gamma_{2k-1}, \gamma_{2k}], 1 \leq k \leq \lfloor \frac{q}{2} \rfloor \), and \( F_1(t) \geq 0 \) in \([\gamma_{2k+1}, \gamma_{2k+1}], 1 \leq k \leq \lfloor \frac{q-1}{2} \rfloor \).

(2) \( F_2(\gamma_{2k+1}) > 0, 0 \leq k \leq \lfloor \frac{q-1}{2} \rfloor \) and \( F_2(\gamma_{2k}) < 0, 1 \leq k \leq \lfloor \frac{q}{2} \rfloor \).

Therefore, \( F_1 \) and \( F_2 \) intersect in \([\gamma_i, \gamma_{i+1}], 1 \leq i \leq q - 1 \), and hence \( F_1 \) and \( F_2 \) intersect at least \( q \) points in \([0,1]\).

Note that \( F_1 \) and \( F_2 \) do not intersect in \([\gamma_q, \gamma_{q+1}] \). 1.

**Proof.** Since (1) is obvious, we prove only (2) by induction on \( k \). Since we already showed that \( F_2(\gamma_1) > 0 \), we consider \( F_2(\gamma_2) \). Suppose \( \gamma_2 \) is the zero of \( f_k(t) \). Then \( F_{2,j}(\gamma_2) = 0 \) for \( j \neq s \). We know that \( \gamma_1 \) is the zero of \( f_k \).

(1) If \( s = k \), then \( f_k^*(\gamma_2) \neq 0 \) and \( F_{2,k}(\gamma_1) > 0 \). Since \( \gamma_1 \) and \( \gamma_2 \) are the zeros of \( f_k(= f_s) \), and \( f_k \) and \( (t-1)f_k^* \) are interlaced, we see that \( f_k^* \) has the zero \( \beta_0 \) in \([\gamma_1, \gamma_2] \) and hence \( F_{2,k} \) crosses the \( t \)-axis at \( \beta_0 \). Therefore, \( F_{2,k}(\gamma_2) < 0 \). Since \( F_{2,j}(\gamma_2) = 0 \) for \( j \neq k \), it follows that \( F_2(\gamma_2) < 0 \).

(2) If \( s \neq k \), then \( F_{2,j}(\gamma_2) = 0 \) for \( j \neq s \) and \( F_{2,s}(\gamma_2) < 0 \), since \( F_{2,s}(\gamma_2) \neq 0 \) and \( F_{2,s}(\gamma_1) < 0 \). Therefore, \( F_2(\gamma_2) < 0 \).

Now consider \( F_2(\gamma_m), 1 \leq m \leq q \). Suppose \( \gamma_m \) is the zero of \( f_p \).

Case (1) \( \gamma_m \) is the smallest zeros of \( f_p \). Then \( f_p \) is not 0 at \( \gamma_1, \gamma_2, \cdots, \gamma_{m-1} \) and it is obvious that (i) if \( m \) is even, then \( F_{2,p}(\gamma_m) < 0 \), and \( F_{2,s}(\gamma_m) = 0 \) for \( s \neq p \) and hence \( F_2(\gamma_m) < 0 \), and (ii) if \( m \) is odd, then \( F_{2,p}(\gamma_m) > 0 \), and \( F_{2,s}(\gamma_m) = 0 \) for \( s \neq p \) and hence \( F_2(\gamma_m) > 0 \).
Case (2) There exists $\gamma_h, h < m$, that is the closest zero of $f_p$ to $\gamma_m$, i.e., $f_p(\gamma_h) = 0$, but $f_p(\gamma_{h+1}) \neq 0, \ldots, f_p(\gamma_{m-1}) \neq 0$. If $h$ is even, by induction assumption, $F_{2,p}(\gamma_h) < 0$. Further, if $m$ is even, there are an odd number of zeros $\gamma_{h+1}, \ldots, \gamma_{m-1}$ between $\gamma_h$ and $\gamma_m$, and $F_{2,p}$ crosses the t-axis at these points. However, since $f_p$ and $(t-1)f_p^*$ are interlaced, there is exactly one zero of $f_p^*$, say $\beta_1$, in $[\gamma_h, \gamma_m]$, i.e., $\gamma_h < \beta_1 < \gamma_m$, and $F_{2,p}$ must cross the t-axis at $\beta_1$ as well. Thus $F_{2,p}(\gamma_m) < 0$. Since $F_{2,\ell}(\gamma_m) = 0$ for $\ell \neq p$, we have $F_2(\gamma_m) < 0$.

The same argument works for other cases where (a) $h$ is even and $m$ is odd, and (b) $h$ is odd and $m$ is even or odd. This proves Lemma 15.4 and Theorem 15.2 for non-multiple zero case.

Case (II) $f_2 f_3 \cdots f_n$ is not simple.

Let $\gamma_1 < \gamma_2 < \cdots < \gamma_{2d}$ be all distinct real zeros of $f_2 f_3 \cdots f_n$. Let $p_k$ be the multiplicity of $\gamma_k, 1 \leq k \leq 2d$ and $a_j$ the leading coefficient of $f_j$. Note $a_j > 0, 2 \leq j \leq n$. Since $f_2 f_3 \cdots f_n = a_2 a_3 \cdots a_n \prod_{k=1}^{2d} (t - \gamma_k)^{p_k}$, we can write

$$F = f_0 f_2 f_3 \cdots f_n - t(t-1)^2 f_0^n \sum_{j=2}^{n} f_2 \cdots f_j^* \cdots f_n$$

$$= a_2 a_3 \cdots a_n f_0 \prod_{k=1}^{2d} (t - \gamma_k)^{p_k} - t(t-1)^2 f_0^n \sum_{j=2}^{n} a_2 a_3 \cdots a_n f_j^* \prod_{k=1}^{2d} (t - \gamma_k)^{p_k} / f_j$$

$$= a_2 a_3 \cdots a_n \prod_{k=1}^{2d} (t - \gamma_k)^{p_k-1} [f_0 \prod_{k=1}^{2d} (t - \gamma_k) - t(t-1)^2 f_0^n \sum_{j=2}^{n} f_j^* \prod_{k=1}^{2d} (t - \gamma_k) / f_j].$$

Since $f_j$ is simple and $\{\gamma_k\}$ is the set of all distinct zeros of $f_2 f_3 \cdots f_n$, we see that $\prod_{k=1}^{2d} (t - \gamma_k) / f_j$ is a (real) polynomial that is denoted by $g_j$. Therefore to prove Theorem 15.2, it suffices to show that $G = f_0 \prod_{k=1}^{2d} (t - \gamma_k) - t(t-1)^2 f_0^n \sum_{j=2}^{n} f_j^* g_j$ has $2d$ real zeros, or equivalently, two curves $y_1 = G_1(t) = f_0 \prod_{k=1}^{2d} (t - \gamma_k)$ and $y_2 = G_2(t) = t(t-1)^2 f_0^n \sum_{j=2}^{n} f_j^* g_j$ have $d$ points of intersection in $[0, 1]$.

Let $G_{2,j} = t(t-1)^2 f_0^n f_j^* g_j$ and hence $G_2 = \sum_{j=1}^{n} G_{2,j}$. Note that $f_0$ and $f_0^*$ do not have any real zeros. Suppose that $\gamma_j$ is the zero of $f_{j_1}, f_{j_2}, \ldots, f_{j_p}$, i.e., $\gamma_j$ is the zeros of $f_2 \cdots f_n$ of multiplicity $p_j$. Since the zeros of $g_k$ consist of all real zeros of $G_1$ except those of $f_k$, it follows that $G_{2,k}(\gamma_j) \neq 0$ if and only if $\gamma_j$ is the zero of $f_k$. Therefore, as it was proved in Lemma 15.4, we can prove inductively that to each $\lambda, 1 \leq \lambda \leq p_j, G_{2,j_1}(\gamma_j) > 0$ or $< 0$, according as $j$ is odd or even. Since $G_{2,\ell}(\gamma_j) = 0$, if $\ell \neq j_1, 1 \leq \lambda \leq p_j$, it follows that $G_{2}(\gamma_j) = \sum_{\lambda=1}^{p_j} G_{2,j_1}(\gamma_j) > 0$ or $< 0$, according as $j$ is odd or even. Therefore, $y_1 = G_1$ and $y_2 = G_2$ intersect in $[\gamma_{j-1}, \gamma_j]$ and finally, two curves $y_1 = G_1$ and $y_2 = G_2$ intersect at least $d$ points in $[0, 1]$.

A proof of Theorem 15.2 is now complete. \[ \square \]
16. Multivariate stable link polynomials

In this section, we study the stability of the 2-variable Alexander polynomial \( \Delta_{K(r)}(x, y) \) of a 2-bridge link \( K(r) \), \( r = \beta/2\alpha, 0 < \beta < 2\alpha \). Let \( \mathcal{H} \) be the upper half-plane of \( \mathbb{C} \). If the reduced Alexander polynomial of \( K(r) \), i.e., \( \Delta_{K(r)}(t) = (t - 1)\Delta_{K(r)}(t, t) \) is not real stable, then \( \Delta_{K(r)}(x, y) \) is not \( \mathcal{H} \)-stable. Therefore, we only need to consider a real stable link. If the continued fraction expansion of \( r \) gives an alternating sequence, then \( K(r) \) is real stable. However, for such links, \( \mathcal{H} \)-stability problem is completely solved (Proposition 16.5). Therefore, we should study exceptional stable links. In this section, we solve the \( \mathcal{H} \)-stability problem for the simplest exceptional stable 2-bridge links.

Now we begin with a definition.

**Definition 16.1.** For a positive integer \( n \), we define \( G_n = \frac{x^n - y^n}{x - y} \) and \( G_{-n} = \frac{-1}{(xy)^n}G_n \). In particular, we define \( G_0 = 0 \).

It is easy to see that \( G_n \) is \( \mathcal{H} \)-stable if and only if \( |n| \leq 2 \).

The following proposition is well-known.

**Proposition 16.2.** Let \( r = [2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2a_n, 2b_n, 2a_{n+1}] \). Then \( \Delta_{K(r)}(x, y) \) is given by

\[
\Delta_{K(r)}(x, y) = \sum_{0 \leq m \leq n} b_{j_1} b_{j_2} \cdots b_{j_m} (x - 1)^m (y - 1)^m G_{\mu_1} G_{\mu_2} \cdots G_{\mu_{m+1}}, \tag{16.1}
\]

where the summation is taken over all indices \( j_k \) such that \( 1 \leq j_1 < j_2 < \cdots < j_m \leq n \) and \( \mu_1 = a_1 + a_2 + \cdots + a_{j_1}, \mu_2 = a_{j_1+1} + \cdots + a_{j_2}, \cdots, \mu_k = a_{j_{k-1}+1} + a_{j_{k-1}+2} + \cdots + a_{j_k}, \cdots, \mu_{m+1} = a_{j_m+1} + a_{j_m+2} + \cdots + a_n \).

We should note that (16.1) is slightly different from the original formula given in [19], since the orientation of one component of \( K(r) \) in [19] is different form ours.

**Example 16.3.** (1) If \( r = [2a_1] \), then \( \Delta_{K(r)}(x, y) = G_{a_1} \).

(2) If \( r = [2a_1, 2b_1, 2a_2] \), then \( \Delta_{K(r)}(x, y) = b_1 (x - 1) (y - 1) G_{a_1} G_{a_2} + G_{a_1+a_2} \).

(3) If \( r = [2a_1, 2b_1, 2a_2, 2b_2, 2a_3] \), then

\[
\Delta_{K(r)}(x, y) = b_1 b_2 (x - 1)^2 (y - 1)^2 G_{a_1} G_{a_2} G_{a_3} + \sum (x - 1)(y - 1) \{b_1 G_{a_1} G_{a_2+a_3} + b_2 G_{a_1+a_2} G_{a_3} \} + G_{a_1+a_2+a_3}.
\]

Now the following simple proposition gives us a strong restriction on \( \mathcal{H} \)-stability for 2-component links.

**Proposition 16.4.** Let \( K = K_1 \cup K_2 \) be a 2-component link. Suppose that \( \Delta_K(x, y) \) is not a constant. (If \( \Delta_K(x, y) \) is a constant, then \( K \) is always \( \mathcal{H} \)-stable.) If \( K \) is \( \mathcal{H} \)-stable, then both \( K_1 \) and \( K_2 \) are stable and further, \( \|k(K_1, K_2)\| \leq 2 \).

**Proof.** By Theorem 3.1(a), if \( \Delta_K(x, y) \) is \( \mathcal{H} \)-stable, then \( \Delta_K(x, 1) \) and \( \Delta_K(1, y) \) are \( \mathcal{H} \)-stable. Further, \( \Delta_K(x, 1) = \frac{x - 1}{x - 1} \Delta_K(x) \) and \( \Delta_K(1, y) = \frac{y - 1}{y - 1} \Delta_K(y) \) [31],

\[
\text{Proposition 16.5.}
\]
where $\ell = lk(K_1, K_2)$, and hence, $\Delta_{K_1}(x), \Delta_{K_2}(y)$ and \(\frac{y^{m-1}}{x+y}\) are $H$-stable. The proposition follows immediately. \hfill \Box

**Proposition 16.5.** Let $r = [2a_1, -2b_1, 2a_2, -2b_2 \ldots, 2a_n, -2b_n, 2a_{n+1}]$, $a_j, b_j > 0$. Then $K(r)$ is $H$-stable if and only if (1) $n = 0$ and $a_1 = 1$ or $2$, or (2) $n = 1$ and $a_1 = a_2 = 1$.

**Proof.** (a) Suppose $K(r)$ is $H$-stable. Then $|\ell| = |lk(K_1, K_2)| \leq 2$. Since $\ell = \sum_{j=1}^{n+1} a_j$, we have (1) or (2). (b) Suppose (1) or (2) holds. If (1) holds, then $\Delta_{K}(x, y) = 1 + x + y$, and both are $H$-stable. Suppose (2) holds. Then $r = [2, -2b_1, 2]$ and $\Delta_{K}(x, y) = -b_1(x + y) + (x + y) = -b_1 + (b_1 + 1)(x + y) - b_1 x y$. Since $\Delta_{K}(x, y)$ is multi-affine, it is $H$-stable if (and only if) \[
\det \begin{bmatrix} -b_1 & b_1 + 1 & -b_1 \\ b_1 + 1 & -b_1 & \end{bmatrix} < 0 \quad \text{(Example 3.3)}.
\]
Since $b_1 > 0$, obviously, the determinant is negative. This proves Proposition 16.5. \hfill \Box

We now consider exceptional stable links.

**Proposition 16.6.** Let $r = [2a, 2b, -2c], a, b, c > 0$ and $a > c$. (1) Suppose $a = c$. Then $K(r)$ is $H$-stable if and only if $a = 1$ or $2$. (2) Suppose $a > c$. Then $K(r)$ is not $H$-stable, unless $(a, c) = (2, 1)$.

**Proof.** (1) Suppose $a = c$. Then $\Delta_{K}(x, y) = b(x - 1)(y - 1)G_a$. If $a \geq 3$, then $G_a$ is not $H$-stable and hence $K(r)$ is not $H$-stable. However, if $a = c = 1$ or $2$, then each factor is $H$-stable and hence $K(r)$ is $H$-stable. (2) Suppose $a > c$. Since $lk(K_1, K_2) = a - c, c = a - 1$ or $c = a - 2$. From (16.1), we have $\Delta_{K(r)}(x, y) = b(x - 1)(y - 1)G_a G_c - G_a - G_c = b(x - 1)(y - 1) \frac{x^a y^b x^{-c} y^c}{x-y} + \frac{x^a y^c x^{-c} y^c}{x-y}$. Let $F(x, y) = (xy)^c \Delta_{K(r)}(x, y) = b(x - 1)(y - 1) \frac{x^a y^b x^{-c} y^c}{x-y} + \frac{x^a y^c x^{-c} y^c}{x-y}$. If $\Delta_{K(r)}(x, y)$ is $H$-stable, so is $\Delta_{K(r)}(x, -1)$ by Theorem 3.1 (a) and hence $f(x) = F(x, -1) = 2b(x - 1) \frac{x^{-c} (x-1)^a}{x+1} + \frac{x^{-c} (x-1)^b}{x+1} \frac{x^{-c} (x-1)^a}{x+1}$. We show that if $a \geq 3$, $f(x)$ is not (real) stable. Since $f(x)$ is reciprocal, consider the modified polynomial $g(z)$ of $f(x)$.

Case(a) $a$ is even, say $2m \geq 4$.

Then $f(x) = 2b(x - 1) \frac{x^{2m-1} - x^{2m-1} + 1}{x+1} - x^{2m-1}$ and hence $g(z)$ is written as $g(z) = 2b g_1(z) g_2(z) - 1$, where $g_1(z)$ is the modification of $f_1(x) = (x - 1) \frac{x^{2m-1}}{x+1}$ and $g_2(z)$ is that of $f_2(x) = \frac{x^{2m-1} + 1}{x+1}$. Since both $f_1$ and $f_2$ are $c$-stable, all zeros of $g_1(z)$ and $g_2(z)$ are in $[-2, 2]$ and further, since $m \geq 2$, at least one zero of $g_1(z)$ and $g_2(z)$ are in $(-2, 2)$. Therefore it is impossible that all points of intersection of two curves $z_1 = g_1(z) g_2(z)$ and $z_2 = \frac{1}{2b}, b > 0$ are outside of $(-2, 2)$, and hence $f(x)$ cannot be stable.

Case(b) $a$ is odd, say $2m + 1, m \geq 1$. 


Then \( f(x) = 2b(x - 1) \frac{x^{2m+1} + 1}{x+1} \frac{x^{2m-1}}{x+1} + x^{2m} \) and the modification is: \( g(z) = 2bg_1(z)g_2(z) + 1 \), where \( g_1(z) \) is the modification of \( \frac{x^{2m+1} + 1}{x+1} \) and \( g_2(z) \) is that of \((x - 1)\frac{x^{2m-1}}{x+1}\). Therefore, all the zeros of \( g_1(z) \) are in \((-2, 2)\). As is proved in Case (a), \( f(x) \) cannot be stable.

Case(II). \( c = a - 2 \) and \( a \geq 3 \). Then

\[
\Delta_{K(r)}(x, y) = b(x - 1)(y - 1)G_2G_{-a-2} + G_2
= b(x - 1)(y - 1) \frac{x^a - y^a}{x - y} - \frac{x^{a-2} - y^{a-2}}{(xy)^{a-2}} + 1, \]

and

\[
f(x) = (-1)^{a-2}x^{a-2} \Delta_{K(r)}(x, -1)
= 2b(x - 1) \frac{x^a - (-1)^a x^{a-2} - (-1)^{a-2}}{x + 1} + (x - 1)(-1)^{a-2}x^{a-2}.
\]

Case(a) \( a \) is even, say \( 2m \geq 4 \). Then \( f(x) = 2b(x - 1) \frac{x^{2m+1} + 1}{x+1} \frac{x^{2m-1}}{x+1} + (x - 1)x^{2m-2} \) and hence \( h(x) = \frac{f(x)}{x^{2m-1}} = 2b(x - 1) \frac{x^{2m-1}}{x+1} + x^{2m-2} \) is reciprocal. The modification \( \lambda(z) \) of \( h(x) \) is \( \lambda(z) = 2b(z - 2) \lambda_1(z) \lambda_2(z) + 1 \), where \( \lambda_1(z) \) and \( \lambda_2(z) \) are the modifications of \( \frac{x^{2m-1}}{x^2-1} \) and \( \frac{1}{x^{2m-2}} \), respectively. Since all the zeros of \( \lambda_1(z) \) and \( \lambda_2(z) \) are in \((-2, 2)\), \( h(x) \) cannot be real stable.

Case(b) \( a \) is odd, say \( 2m + 1 \).

The same argument works to show that \( h(x) \) is not real stable and hence \( \Delta_{K(r)}(x, y) \) is not \( \mathcal{H} \)-stable. \( \square \)

If \((a, c) = (2, 1)\), we have the following proposition.

**Proposition 16.7.** Let \( r = [4, 2, -2] \). Then \( K(r) \) is \( \mathcal{H} \)-stable.

**Proof.** \( xy\Delta_{K(r)}(x, y) = -(x - 1)(y - 1)G_2G_1 + xyG_1 = -(x - 1)(y - 1)(x + y) + xy = -(xy - x - y)(x + y - 1) \). Each factor is \( \mathcal{H} \)-stable by Example 3.8 and hence \( K(r) \) is \( \mathcal{H} \)-stable. \( \square \)

**Question 16.8.** For \( r = [4, 2b, -2] \) and \( b \geq 2 \), is \( K(r) \) \( \mathcal{H} \)-stable ?

Finally, we prove that the 2-variable Alexander polynomial of a 2-bridge link has the same property as Theorem 8,8 (2). Using this, we can systematically obtain exceptional \( \mathcal{H} \)-stable 2-component 2-bridge links.

**Theorem 16.9.** Let \( s = [2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2a_n, 2b_n, 2a_{n+1}] \), \( a_j \neq 0 \neq b_j, 1 \leq j \leq n + 1 \). Let \( r = [s, 2k, -s^{-1}], k \neq 0 \). Then,

\[
\Delta_{K(r)}(x, y) = k(x - 1)(y - 1)[\Delta_{K(s)}(x, y)]^2,
\]

and hence, \( K(r) \) is \( \mathcal{H} \)-stable if and only if \( K(s) \) is \( \mathcal{H} \)-stable.
Proof. Consider a sequence of integers

\[ A = \{a_1, a_2, b_2, \cdots, a_n, b_n, \cdots, a_{2n+1}, b_{2n+1}, a_{2n+2}\}. \]

Take an ordered subset

\[ C = \{b_1, b_2, \cdots, b_{2n+1}\}. \]

Let \( \tilde{C} \) be the set of all ordered subset of \( C \), i.e., \( \tilde{C} \ni U = \{b_{j_1}, b_{j_2}, \cdots, b_{j_k}\} \), where \( 1 \leq j_1 < j_2 < \cdots < j_k \leq 2n + 1 \). To each set \( U \) in \( \tilde{C} \), we define a mapping \( \rho_{2n+1} : \tilde{C} \rightarrow \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \) as follows.

\[
\rho_{2n+1}(U) = b_{j_1}b_{j_2} \cdots b_{j_k}(x - 1)^k(y - 1)^k G_{\mu_1}G_{\mu_2} \cdots G_{\mu_{k+1}},
\]

where \( \mu_1 = a_1 + a_2 + \cdots + a_{j_1}, \mu_2 = a_{j_1+1} + \cdots + a_{j_2}, \cdots, \mu_{k+1} = a_{j_k+1} + a_{j_k+2} + \cdots + a_{2n+2}. \)

For example, \( \rho_{2n+1}(\emptyset) = G_{a_1+a_2+\cdots+a_{2n+2}} \) and \( \rho_{2n+1}(C) = b_1b_2 \cdots b_{2n+1}(x - 1)^{2n+1}(y - 1)^{2n+1}G_{a_1}G_{a_2} \cdots G_{a_{2n+2}}. \)

Now to each \( U \), we call \( U^* = \{b_{2n+2-j_k}, b_{2n+2-j_{k-1}}, \cdots, b_{2n+2-j_1}\} \) the dual of \( U \). We use these concepts to prove the theorem.

In the following, we assume that

\[
\text{for } 1 \leq j \leq n + 1, a_{n+1+j} = -a_{n+2-j}, \quad \text{and} \quad (16.4)
\]

\[
\text{for } 1 \leq j \leq n, b_{n+1+j} = -b_{n+1-j}. \quad (16.5)
\]

Therefore, \( A \) becomes

\[ \{a_1, b_1, a_2, b_2, \cdots, a_{n+1}, b_{n+1}, -a_{n+1}, -b_n, -a_n, \cdots, -b_2, -a_2, -b_1, -a_1\} \]

and we can write

\[
\rho_{2n+1}(U^*) = b_{2n+2-j_k}b_{2n+2-j_{k-1}} \cdots (x - 1)^k(y - 1)^k G_{-\mu_{k+1}}G_{-\mu_k} \cdots G_{-\mu_1}. \quad (16.6)
\]

Then we prove

**Claim 1.** \( \rho_{2n+1}(U) + \rho_{2n+1}(U^*) = 0 \), and hence \( \sum_{U} \rho_{2n+1}(U) = 0 \), where the summation is taken over all \( U \) that does not contain \( b_{n+1} \).

**Proof.** If some \( \mu_i = 0 \), then \( G_{\mu_i} = G_{-\mu_i} = 0 \) and hence \( \rho_{2n+1}(U) = \rho_{2n+1}(U^*) = 0 \). Therefore we may assume that none of \( \mu_i \) is 0. Let \( m \) be the number of negative elements in \( U \). Then that number in \( U^* \) is \( k - m \). Let \( q \) be the number of negative integers in the set \( \{\mu_{j_1}, \cdots, \mu_{j_k+1}\} \). Then that number in \( U^* \) is \( k + 1 - q \). Therefore the number of occurrence of \( -1 \) in \( \rho_{2n+1}(U) \) is \( m + q \), while that in \( \rho_{2n+1}(U^*) \) is \( k - m + k + 1 - q = m + q + 1 \) (mod 2). Next we count the exponent of the factor \( 1/2 \)

\[
\text{in } \rho(U) \text{ and } \rho(U^*). \] Suppose \( \mu_{\ell_1}, \mu_{\ell_2}, \cdots, \mu_{\ell_q} \) are negative. Then in \( \rho_{2n+1}(U) \), the exponent of \( \frac{1}{2^g} \) is \( |\mu_{\ell_1}| + |\mu_{\ell_2}| + \cdots + |\mu_{\ell_q}| \), while in \( \rho_{2n+1}(U^*) \) that is \( \sum_{\lambda \neq \ell_j} \mu_\lambda \). Since \( \sum_{\lambda \neq \ell_j} \mu_\lambda - \sum_{j=1}^q |\mu_{\ell_j}| = \sum_{j=1}^{2n+2} a_j = 0 \), we see that \( \rho_{2n+1}(U) + \rho_{2n+1}(U^*) = 0 \). \( \square \)

Now consider a short sequence \( A_0 = \{a_1, b_1, a_2, b_2, \cdots, a_n, b_n, a_{n+1}\} \), the first half part of \( A \). Let \( B = \{b_1, b_2, \cdots, b_n\} \) be an ordered set and \( \tilde{B} \) be the set of all ordered subset of \( B \). Then we have a mapping \( \rho_n : \tilde{B} \rightarrow \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \).
Given $U = \{b_{j_1}, b_{j_2}, \ldots, b_{j_p}, b_{n+1}, b_{j_{p+2}}, \ldots, b_{j_k}\} \in \tilde{C}$, we can define two sets $W_+$ and $W_-$ in $\tilde{B}$ as follows. 

$W_+ = \{b_{j_1}, b_{j_2}, \ldots, b_{j_p}\}$ and $W_- = \{b_{2n+2-j_k}, b_{2n+2-j_k-1}, \ldots, b_{2n+2-j_{p+2}}\}$.

Then we claim

**Claim 2.** $\rho_{2n+1}(U) = b_{n+1}(x-1)(y-1)\rho_n(W_+)|\rho_n(W_-)|^{-\frac{1}{(xy)^\alpha}}$, where $\alpha = a_1 + a_2 + \ldots + a_{n+1}$.

**Proof.** Since $U \ni b_{n+1}$, we can write

$$\rho_{2n+1}(U) = b_{n+1}(x-1)(y-1) \{b_{j_1}, \ldots, b_{j_p}, (x-1)p(y-1)^p G_{\mu_1} \cdots G_{\mu_{p+1}}\}$$

$$\{b_{j_{p+2}}, \ldots, b_{j_k}(x-1)^{k-p}p^{-1}(y-1)^{k-p-1}G_{\mu_{p+2}} \cdots G_{\mu_{k+1}}\}.$$ 

Therefore, it suffices to show that

$$(x-1)^{k-p-1}(y-1)^{k-p-1} \{b_{j_{p+2}}, \ldots, b_{j_k}\} G_{\mu_{p+2}} \cdots G_{\mu_{k+1}} = \rho_n(W_-)|\frac{1}{(xy)^\alpha}|.$$  \hspace{2cm} (16.7)

Since

$$\rho_n(W_-) = (x-1)^{k-p-1}(y-1)^{k-p-1} \{b_{2n+2-j_k}, \ldots, b_{2n+2-j_{p+2}}\} G_{\mu_{k+1}} \cdots G_{\mu_{p+2}},$$

as done before, we compare the number of occurrences of $-1$ and the exponent of $\frac{1}{xy}$, in LHS and RHS of (16.7). First, let $d$ be the number of negative $b_j$ in LHS. Then that number in RHS is $k-p-1-d$. Let $q$ be the number of negative $\mu_\lambda$ in the set $\{\mu_{p+2}, \ldots, \mu_{k+1}\}$ in LHS. Then that number in RHS is $k-p-q$, and hence the sign of RHS is opposite to that of LHS.

Next, we count the exponent of $\frac{1}{xy}$. Let $-\nu_1, -\nu_2, \ldots, -\nu_q$ be all negative members in $\{\mu_{p+2}, \ldots, \mu_{k+1}\}$, and $\nu_{q+1}, \ldots, \nu_{k-p}$ be all positive members. Then the exponent of $\frac{1}{xy}$ in LHS is exactly $\nu_1 + \nu_2 + \ldots + \nu_q$, while that in RHS is $\nu_{q+1} + \ldots + \nu_{k-p}$. Since $\nu_{q+1} + \ldots + \nu_{k-p} - (\nu_1 + \ldots + \nu_q) = \sum_{j=1}^{n+1} a_j = \alpha$, Claim 2 follows.

Claim 2 implies easily the following

**Claim 3.** If $U \in \tilde{C}$ contains $b_{n+1}$, then $U^* \ni b_{n+1}$ and $\rho_{2n+1}(U) = \rho_{2n+1}(U^*)$.

From Claims 1-3, we have,

**Claim 4.** $\sum_{U \in \tilde{C}} \rho_{2n+1}(U) = b_{n+1}(x-1)(y-1)|\sum_{V \in \tilde{B}} \rho_n(V)|^2|\frac{1}{(xy)^\alpha}|.$

Hence Theorem [16.9] follows.

**Example 16.10.** Let $s = [4, 2, -2]$. Then $K(s)$ is $\mathcal{H}$-stable by Proposition [16.7] and hence for $r = [s, 2k, -s^{-1}], k \neq 0$, $K(r)$ is $\mathcal{H}$-stable.

If $K(s)$ is a 2-bridge knot, Theorem [16.9] does not hold, but $\Delta_{K(r)}(x, y)$ will be of a nice form. The following theorem is proven by applying a similar argument used in the proof of Theorem [16.9]. The detail will appear in a separate paper.

**Theorem 16.11.** Let $s = [2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2a_n, 2b_n], a_j \neq 0 \neq b_j, 1 \leq j \leq n,$ and $r = [s, 2k, -s^{-1}], k \neq 0$. Then
\[ \Delta_{K(r)}(x, y) = G_k f(x, y) f(y, x), \quad (16.8) \]

where \( f(x, y) \in \mathbb{Z}[x, y] \) and \( f(t, t) = \Delta_{K(s)}(t) \).

We should note that if \( |k| \geq 3 \), then \( G_k \) is not \( \mathcal{H} \)-stable. Therefore, for \( K(r) \) to be \( \mathcal{H} \)-stable, \( k \) must be \( \pm 1 \) or \( \pm 2 \).

17. Inversive links

A 2-component link \( K \) is called **inversive** if the original link is stable (or \( c \)-stable), but reversing the orientation of one component results in a \( c \)-stable (or stable) link. We see in Section 15 that some Montesinos links are inversive (Theorem 15.2, Case 2). In this section, we study these links using 2-variable Alexander polynomial \( \Delta_K(x, y) \).

17.1. **Standard inversive links**

From the definition, the following proposition is immediate.

**Proposition 17.1.** Let \( K \) be a 2-component link and \( \Delta_K(x, y) \) the Alexander polynomial. Then \( K \) is inversive if and only if

1. \( \Delta_K(t, t) \) is stable (or \( c \)-stable) and
2. \( t^n \Delta_K(t, t^{-1}) \) is \( c \)-stable (or stable) \((17.1)\)

**Remark 17.2.** Note that (2) in (17.1) is equivalent to (2') below, since \( \Delta_K(x, y) = x^m y^n \Delta_K(x^{-1}, y^{-1}) \) for some integers \( n \) and \( m \).

\( (2') \) \( t^n \Delta_K(t^{-1}, t) \) is \( c \)-stable (or stable).

For convenience, we call \( \Delta_K(x, y) \) **inversive** if \( \Delta_K(x, y) \) satisfies (1) and (2) in (17.1).

**Proposition 17.3.** If a 2-bridge link \( K(s) \) is inversive, then \( K(r) \) is inversive, where \( r = [s, 2k, -s^{-1}], k \neq 0 \).

The simplest inversive 2-bridge link is \( K(s), s = [2a], a \neq 0 \). Therefore we have the following corollary.

**Corollary 17.4.** Let \( r = [2a, 2k, -2a], \) where \( k \neq 0 \) and \( a > 0 \). Then \( K(r) \) is inversive.

If \( K(s) \) is a 2-bridge knot, then \( K(r), r = [s, 2k, -s^{-1}] \), may not be inversive.

**Example 17.5.** Let \( s = [2, -2] \) and \( r = [2, -2, 2, 2, -2] \). Then \( K(r) \) is not inversive. In fact, \( \Delta_{K(r)}(x, y) = (1 - (2x + y) + xy)(1 - (x + 2y) + xy) \) and \( \Delta_{K(r)}(t, t) = (1 - 3t + t^2)^2 \) is stable, but \( \Delta_{K(r)}(t, t^{-1}) = (1 - 2t + 2t^2)(2 - 2t + t^2) \) is not \( c \)-stable.
Now consider the general case. For convenience, we denote by \( K^* \) the 2-component link obtained from \( K \) by reversing the orientation of one component of \( K \).

**Proposition 17.6.** Let \( r = [2a_1, -2a_2, \ldots, (-1)^2 a_{2n+1}], a_j > 0 \) for \( 1 \leq j \leq 2n + 1 \). Then \( K(r) \) is inversive.

Proof. First, by Theorem 6.1 \( \Delta_{K(r)}(t) = (t - 1)\Delta_{K(r)}(t, t) \) is stable. Further, we see that a diagram of \( K(r) \) is alternating and the diagram of \( K^*(r) \) is special alternating, and hence \( \Delta_{K^*(r)}(t) \) is c-stable. Therefore, \( K(r) \) is inversive. \( \square \)

If \( r = [2a_1, 2a_2, \ldots, 2a_{2n+1}], a_j > 0 \) for \( 1 \leq j \leq 2n + 1 \), then \( K(r) \) is c-stable, by Proposition 12.1. But, generally, \( K^*(r) \) is not stable, and hence \( K(r) \) is not inversive. The following proposition, however, gives one sufficient condition for \( K(r) \) to be inversive.

**Proposition 17.7.** Let \( r = [2a_1, 2a_2, \ldots, 2a_{2n+1}], a_j > 0 \) for \( 1 \leq j \leq 2n + 1 \). If \( a_{2k+1} = 1 \) for all \( k, 1 \leq k \leq n \), then \( K^*(r) \) is stable and hence \( K(r) \) is inversive.

Proof. Write \( r = \frac{\beta}{2a} \). We may assume without loss of generality that \( 0 < \beta < 2a \). Denote \( r^* = \frac{2a - \beta}{2a} \). Then it is known [12, Proposition 3.17] that \( K(r^*) \) is equivalent to the mirror image of \( K^*(r) \). Therefore, \( K^*(r) \) is stable if (and only if) \( K(r^*) \) is stable.

Now, the even continued fraction expansion of \( r^* \) is called the dual of (the even continued fraction expansion of) \( r \) in [12] Theorem 3.5] and there is an algorithm to find the expression of \( r^* \) [12] p.7. Using this algorithm, we can show that if all \( a_{2k+1} = 1, 1 \leq k \leq n \), then \( r^* = [2, -(2a_2 - 2), 2, -(2a_4 - 2), \ldots, 2, -(2a_{2n} - 2), 2] \). Therefore, \( K(r^*) \) is stable and \( K(r) \) is inversive. \( \square \)

**Remark 17.8.** In Proposition 17.7 if we assume that all \( a_{2k} = 1 \), for \( 1 \leq k \leq n \), then both \( K(r) \) and \( K(r^*) \) are c-stable. In fact, it is easy to show that all entries of \( r^* \) are positive.

**Example 17.9.** (1) Let \( r = [2, 4, 2, 6, 2] = \frac{60}{117} \). Then \( r^* = \frac{40}{117} = [2, -2, 2, -4, 2] \) and hence \( K(r^*) \) is stable. Since \( K(r) \) is c-stable, \( K(r) \) is inversive.

(2) Let \( r = [4, 4, 2] = \frac{7}{16} \). Then \( r^* = \frac{10}{16} = [2, 2, 2, 2] \). \( K(r^*) \) is not c-stable. In fact, \( \Delta_{K(r^*)}(x, y) = x^2y^2 - (2x^2y + 2xy^2) + 3xy - (2x + 2y) + 1 \), and hence \( \Delta_{K(r^*)}(t, t) = t^4 - 4t^3 + 3t^2 - 4t + 2 \). Then the modified polynomial \( f(x) \) of \( \Delta_{K(r^*)}(t, t) \) has two real zeros, one of which is in \((-2, 2)\) and another is larger than 2, and hence \( \Delta_{K(r^*)}(t, t) \) is strictly bi-stable, and \( K(r) \) is not inversive.

(3) Let \( r = [4, 2, 2, 2, 4] = \frac{13}{12} \). Then \( r^* = \frac{29}{32} = [2, 2, 6, 2, 2] \) and hence \( K(r^*) \) is c-stable.
17.2. Exceptional inversive links

If the original 2-bridge link $K(r)$ is an exceptional stable link, then $K(r^*)$ (and hence $K^*(r)$) may not be c-stable. However, the following proposition shows that for some exceptional 2-bridge link, $K^*(r)$ is c-stable.

**Proposition 17.10.** let $r = [4, 2k, -2]$, $k > 0$. Then $K(r)$ is inversive.

*Proof.* Since $\Delta_{K(r)}(x, y) = k(x-1)(y-1)(x+y) - xy$, we see that $\Delta_{K(r^*)}(t, t) = t^2 \Delta_{K(r)}(t, t^{-1}) = kt^4 - 2kt^3 + (2k + 1)t^2 - 2kt + k$. The modified polynomial $f(x)$ of $\Delta_{K(r^*)}(t, t)$ is $f(x) = kx^2 - 2kx + 1$, and both zeros of $f(x)$ are real and are in $(0, 2)$ and hence $\Delta_{K(r^*)}(t, t)$ is c-stable and $K(r)$ is inversive. \[\square\]

This is a rather exceptional case. For example, for $r = [6, 2, -2]$, $K(r)$ is not inversive. However, in general, we can prove the following theorem.

**Theorem 17.11.** Let $r = [2a, 2k, -2c]$, $a > c$, $k > 0$. If $k$ is sufficiently large, then $K(r)$ is inversive. More precisely, there exists a positive integer $N(a, c)$ such that if $k \geq N(a, c)$, then $K(r)$ is inversive.

*Proof.* Since $\Delta_{K(r)}(x, y) = k(x-1)(y-1)G_4G_{-c} + G_{a-c}$, a simple calculation shows that $\Delta_{K(r^*)}(t, t) = k(t-1)^{t^a-1} \frac{t^c-1}{t^c-1} + \frac{t^{2(a-c)}-1}{t^c-1}$. and the modification $f(x)$ of $\Delta_{K(r)}(t, t)$ is of the form: $f(x) = k(x-2)f_1(x)f_2(x) + g(x)$, where $f_1$, $f_2$ and $g$ are, respectively, the modifications of $\frac{t^{2a}}{t^c-1}$, $\frac{t^c-1}{t^c-1}$ and $\frac{t^{2(a-c)}-1}{t^c-1}$. We note that all the zeros of $f_1$, $f_2$ and $g$ are in $(-2, 2)$. Consider two graphs $z_1 = (x - 2)f_1f_2$ and $z_2 = -\frac{g(x)}{k}$. If $k \rightarrow \infty$, then $z_2 \rightarrow 0$ and hence if $k$ is sufficiently large, the points of intersection of two curves are almost the zeros (not 2) of $z_1$ and hence, $\Delta_{K(r^*)}(t, t)$ is c-stable when $k$ is sufficiently large. Therefore $K(r)$ is inversive if $k$ is sufficiently large. \[\square\]

**Problem 17.12.** Determine $N(a, c)$.

We should note that if $a = c$ then $N(a, a) = 1$.

**Example 17.13.** It is easy to show that $N(3, 1) = 3$ and $N(3, 2) = 2$.

**Question 17.14.** Can a 2-component inversive link $K$ be characterized by the Alexander polynomial $\Delta_K(x, y)$?

**Appendix A: Representation polynomials**

There are various integer polynomials associated to representations of the knot group into $GL(2, \mathbb{C})$. In this section, we discuss two particular representations of $G(K)$, namely, a parabolic representation of $G(K(r))$, the group of a 2-bridge knot $K(r)$, to $SL(2, \mathbb{C})$ and a trace-free representation of $G(K(r))$ to a dihedral group $D_{2n+1} \subset GL(2, \mathbb{C})$. 
A.1. Parabolic representation

Let \( \theta_r(z) \) be the parabolic representation polynomial (Riley polynomial) of \( G(K(r)) \) to \( SL(2,\mathbb{C}) \). (See [26].) Suppose \( r = \frac{1}{2n+1} \), and hence \( K(r) \) is a torus knot of type \((2,2n+1)\). Then \( \theta_r(z) = \sum_{k=0}^{n} \left( \frac{n+k}{2k} \right) z^k \).

**Theorem A.1.** [26],[30] If \( r = \frac{1}{2n+1} \), \( \theta_r(z) \) is real stable. In fact, all the zeros of \( \theta_r(z) \) are simple and they are

\[
\alpha_k = -4 \sin^2 \left( \frac{2k-1}{2k+1} \right), \quad 1 \leq k \leq n
\]  

(A.1)

**Remark A.2.** For a rational number \( r = \frac{\beta}{\alpha}, 0 < \beta < \alpha, \alpha \) odd, \( \theta_r(z) \) is an integer polynomial of degree \( \alpha - 1 \) and generally, \( \theta_r(z) \) is not reciprocal.

**Example A.3.** (1) Let \( r = \frac{2}{5} \). Then \( \theta_r(z) = 1 - z + z^2 \) is not stable, but \( c \)-stable.  
(2) Let \( r = \frac{5}{7} \). Then \( \theta_r(z) = 1 + 2z + z^2 + z^3 \) is not stable.

**Problem A.4.** Characterize \( r \) so that \( \theta_r(z) \) is stable.

For a 2-bridge link \( K(r), \frac{q}{2n} \), Riley polynomial is defined in a slightly different manner. Let \( G(K(r)) = \langle x, y | Wy = yW \rangle \) be a presentation of the group of \( K(r) \), where \( x \) and \( y \) are (oriented) meridian generators. Then \( W \) is of the form:

\[
W = x^{\varepsilon_j} y^{\eta_j} x^{\varepsilon_{n-j}} y^{\eta_{n-j}} x^{\varepsilon_n},
\]  

(A.2)

where (1) \( |\varepsilon_j| = |\eta_j| = 1 \) for all \( j \), and (2) \( \varepsilon_j = \varepsilon_{n-j+1} \) for \( 1 \leq j \leq n \), and \( \eta_j = \eta_{n-j} \) for \( 1 \leq j \leq n-1 \).

Let \( \varphi : x \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) and \( y \rightarrow \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \) be a parabolic representation of a free group \( F(x, y) \), generated by \( x \) and \( y \), in \( SL(2,\mathbb{C}) \), and \( z \) is a complex number that is determined later. Then \( \varphi \) defines a parabolic representation \( \varphi_r \) of \( G(K(r)) \) in \( SL(2,\mathbb{C}) \) if and only if

\[
\varphi(Wy) = \varphi(yW).
\]  

(A.3)

Let \( \varphi(W) = \begin{bmatrix} a_r(z) & b_r(z) \\ c_r(z) & d_r(z) \end{bmatrix} \). Then a simple computation shows that \( \textbf{[A.3]} \) is equivalent to

\[
z = 0 \text{ or } b_r(z) = 0 \text{ and } a_r(z) = d_r(z).
\]  

(A.4)

We prove first that always \( a_r(z) = d_r(z) \). To prove this, we need the following simple lemma. For convenience, we call a matrix \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C}) \) is of
\[ D \text{-type if } a = d. \]

**Lemma A.5.** If each of \( M \) and \( N \) in \( \text{GL}(2, \mathbb{C}) \) is of \( D \)-type, then \( NMN \) is also of \( D \)-type.

Now \( \varphi(x) \) and \( \varphi(y) \) are of \( D \)-type, and so are \( \varphi(x^{-1}) \) and \( \varphi(y^{-1}) \). Since the length of \( W \) is \( 2n - 1 \), \( W \) has the central element. If \( n \) is even, say \( 2m \), it is \( y^{2m} \). Since \( \varphi(y^{2m}) \) is of \( D \)-type and \( \varepsilon_m = \varepsilon_{m+1} \), we see by Lemma A.5 that \( \varphi(x^{2m}y^{2m}x^{2m-1}) \) is of \( D \)-type. Further, \( \varphi(y^{2m-1}(x^{2m}y^{2m}x^{2m-1})y^{2m+1}) \) is also of \( D \)-type. By repeating this process, we see that \( \varphi(W) = \varphi(x^{2m}y^{2m} \cdots y^{2m-1}x^{2m}) \) is of \( D \)-type. If \( n \) is odd, say \( 2m + 1 \), then the central element is \( x^{2m+1} \), and the same argument works well. If \( z = 0 \), then \( \varphi \) is an abelian representation, and hence we ignore it. But each zero of \( b_r(z) \) gives a (non-abelian) parabolic representation \( \varphi_r \). \( b_r(z) \) is called the \textit{Riley polynomial} \( \theta_r(z) \) of \( K(r) \). The degree of \( \theta_r(z) \) is \( n - 1 \).

**Example A.6.** Let \( r = 1/2n, n \geq 2 \). Then \( K(r) \) is an elementary torus link, and it follows from [A.6] and [13, Prop. 2.4] that the Riley polynomial \( \theta_r(z) \) of \( K(r) \) is given by (A.5) below.

\[
\theta_r(z) = \sum_{j=0}^{n-1} \binom{n+j}{2j+1} z^j. \tag{A.5}
\]

It is known that \( \theta_r(z) \) is real stable. In fact, the zeros of \( \theta_r(z), r = \frac{1}{2n} \) are \( -4\sin^2 \frac{r \pi}{2n}, r = 1, 2, \ldots, n - 1 \). The following proposition confirms a conjecture by Dan Silver [28].

**Proposition A.7.** Let \( r = q/2n, 0 < q < 2n, \gcd(q, 2n) = 1 \). Then \(|\theta_r(0)| = \|lk(K(r))\|\), where \( \|lk(K(r))\| \) denotes the linking number between two components of a 2-bridge link \( K(r) \).

**Proof.** \( \theta_r(0) \) is determined by \( \varphi_r(W) \) evaluated at \( z = 0 \) which is \( \prod_{j=1}^{n} \varphi_r(x^{\varepsilon_j}) = \left[ \frac{1}{\sum_{j=1}^{n} \varepsilon_j} \right] \). Since \( \sum_{j=1}^{n} \varepsilon_j \) is equal to \( lk(K(r)) \), Proposition A.7 follows. \qed

If \( \|lk(K(r))\| = 0 \), then Dan Silver also conjectures that the absolute value of the coefficient \( c_1 \) of \( z \) of \( \theta_r(z) \) is the wrapping number of \( K(r) \). However, examples below show that it is not correct. Note that if \( \|lk(K(r))\| = 0 \), then \( n \) is a multiple of 4.

**Example A.8.** (1) Let \( r = 9/16 = [2, 4, -2] \). Then \( \theta_r(z) = z(2 + z^2)(2 - 4z + 4z^2 - 2z^3 + z^4) \), but the wrapping number is 2. (2) Let \( r = 11/24 = [2, -6, -2] \). Then \( \theta_r(z) = 6z + 18z^2 + 35z^3 + 48z^4 + 56z^5 + 44z^6 + 36z^7 + 16z^8 + 10z^9 + 2z^{10} + z^{11} \), but the wrapping number is 2.
A.2. Dihedral representation

It is well-known that there is a trace-free representation $\xi$ of a dihedral group $D_p = \langle x, y | x^2 = y^2 = (xy)^p = 1 \rangle$, $p = 2n + 1$, in $GL(2, \mathbb{C})$. $\xi$ is given by $\xi(x) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\xi(y) = \begin{bmatrix} -1 & 0 \\ \omega & 1 \end{bmatrix}$, where $\omega$ is a zero of the integer polynomial $\varphi_p(z) = \sum_{k=0}^{2n+1} \frac{(n+k)}{2k+1} z^k$. See [26].

Example A.9. $\varphi_3(z) = z + 3$, $\varphi_5(z) = z^2 + 5z + 5$, $\varphi_7(z) = z^3 + 7z^2 + 14z + 7$, $\varphi_9(z) = z^4 + 9z^3 + 27z^2 + 30z + 9 = (z + 3)(z^3 + 6z^2 + 9z + 3)$.

Using $\xi$, we prove in [11] some properties of the twisted Alexander polynomial of a 2-bridge knot associated to a dihedral representation. In this subsection, we prove for any odd number $2n + 1$, $\varphi_{2n+1}(z)$ is stable. Our proof is different from those of other parts in this paper.

Theorem A.10. Let $p = 2n + 1$, $n \geq 1$, and $\varphi_p(z) = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \frac{(n+k)}{2k} z^k$. Let $\zeta = e^{\frac{2\pi i}{p+1}}$. Then $z_\varphi = \zeta^k - \zeta^{-k} - 2, 1 \leq k \leq n$, are the zeros of $\varphi_p(z)$.

Proof. Since $\zeta^k + \zeta^{-k} - 2 = (\sqrt{\zeta} - \frac{1}{\sqrt{\zeta}})^{2k}$, it suffices to show that

$$\varphi_{2n+1}(z_1) = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \frac{(n+k)}{2k} (\sqrt{\zeta} - \frac{1}{\sqrt{\zeta}})^{2k} = 0.$$ (A.6)

Now we expand the right side of (A.6) and let $A_k^{(n)}$ denote the term of $\zeta^k, k = 0, \pm 1, \pm 2, \ldots, \pm n$. Namely,

$$\varphi_{2n+1}(z_1) = A_0^{(n)}z^{-n} + \cdots + A_{-1}^{(n)}z^{-1} + A_0^{(n)} + A_1^{(n)} + A_2^{(n)}z^2 + \cdots + A_n^{(n)}z^n.$$

Then we see that

$$A_0^{(n)} = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \frac{(n+k)}{2k} (-1)^k \binom{2k}{k}.$$

$$A_1^{(n)} = A_{-1}^{(n)} = \sum_{k=1}^{n} \frac{2n+1}{2k+1} \frac{(n+k)}{2k} (-1)^k \binom{2k}{k+1}$$

$$\vdots$$

$$A_m^{(n)} = A_{-m}^{(n)} = \sum_{k=m}^{n} \frac{2n+1}{2k+1} \frac{(n+k)}{2k} (-1)^{k+m} \binom{2k}{k+m}$$

$$\vdots$$

$$A_n^{(n)} = A_{-n}^{(n)} = \frac{2n+1}{2n+1} \binom{2n}{2k} (-1)^{n} \binom{2n}{2n} = 1.$$ (A.7)

Therefore, to prove $\varphi_{2n+1}(z) = 0$, it suffices to show

$$A_0^{(n)} = A_1^{(n)} = \cdots = A_n^{(n)} = 1.$$
We prove these equalities by applying generating function theory.
First we show that \( A_0^{(n)} = 1 \). Consider the generating function \( F_0(x) \) of \( A_0^{(n)} \):

\[
F_0(x) = \sum_{n \geq 0} A_0^{(n)} x^n = \sum_{n \geq 0} \left( x^n \sum_{k \geq 0} \binom{n + k}{2k} \frac{(-1)^k}{k} \frac{2n + 1}{2k + 1} \right). \tag{A.8}
\]

We prove that \( F_0(x) = \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots \).
Now, by interchanging the order of summations in (A.8), we have

\[
F_0(x) = \sum_{k \geq 0} \left( \frac{2k}{k} \frac{(-1)^k}{2k + 1} x^{-k} \left( \sum_{n \geq 0} \frac{(n + k + 1)}{2k} x^{n+k} \right) \right). \tag{A.9}
\]

Note \( \sum_{n \geq 0} (2n + 1) \binom{n+k}{2k} x^{n+k} = \sum_{n \geq k} (2n + 1) \binom{n+k}{2k} x^{n+k} \). Let \( n + k = r \). Then \( 2n + 1 = 2(r - k) + 1 = 2r - 2k + 1 \) and

\[
\sum_{n \geq k} (2n + 1) \binom{n+k}{2k} x^{n+k}
= \sum_{r \geq 2k} \frac{(2r - 2k + 1)}{2k} \binom{r}{2k} x^r
= x^{2k} \left( \frac{(2k+1)}{2k+1} \binom{2k+1}{1} x \right.
+ \left. (2k+5) \binom{2k+2}{2} x^2 + \cdots + (2k+2m+1) \binom{2k+m}{m} x^m + \cdots \right)
= x^{2k} \sum_{m \geq 0} (2k+2m+1) \binom{2k+m}{m} x^m. \tag{A.9}
\]

We need the following lemma.

**Lemma A.11.** (1) \([W, p.50, (2.5.7)]\) \( \sum_{m \geq 0} \binom{2k+m}{m} x^m = \frac{1}{(1-x)^2e^{2kx}}. \)
(2) \([W, p.50, (2.5.11)]\) \( \sum_{k \geq 0} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}. \)
(3) \([W, p.54, (2.5.15)]\) \( \sum_{k \geq 0} \binom{2k+m}{k} x^k = \frac{1}{\sqrt{1-4x}}\binom{1-\sqrt{1-4x}}{2k} x^m. \)
(4) \([W, p.32]\) Let \( P(y) \) be a polynomial and \( f = \sum_{n \geq 0} a_n x^n \). Then
\( \sum_{n \geq 0} P(n) a_n x^n = P(x \frac{d}{dx}) f. \)
Example A.12. If $P(y) = 2y + 2k + 1$ and $f = \sum_{m \geq 0} \binom{2k+m}{m} x^m = (1 - x)^{-(2k+1)}$, Then $P(m) = 2m + 2k + 1$ and

$$\sum_{m \geq 0} (2m + 2k + 1) \binom{2k+m}{m} x^m = 2x \frac{df}{dx} + (2k+1)f = 2(-2k+1)(-1)(1 - x)^{-(2k+2)}x + (2k + 1)(1 - x)^{-(2k+1)} = \frac{(2k+1)(x+1)}{(1 - x)^{2k+2}}.$$ 

Using Example A.12 we have from (A.9)

$$\sum_{n \geq 0} (2n+1) \binom{n+k}{2k} x^{n+k} = x^{2k} \frac{(2k+1)(x+1)}{(1 - x)^{2k+2}}. \quad (A.10)$$

Therefore,

$$F_0(x) = \sum_{k \geq 0} \binom{2k}{k} \frac{(-1)^k}{2k+1} x^{-k} \left\{ \frac{x^{2k}(2k+1)(x+1)}{(1 - x)^{2k+2}} \right\} = \frac{x+1}{(1 - x)^2} \sum_{k \geq 0} \binom{2k}{k} (-1)^k \left\{ \frac{x}{(1 - x)^2} \right\}^k.$$ 

Let $y = \frac{x}{(1-x)^2}$. Then $F_0(x) = \frac{x+1}{(1-x)^2} \sum_{k \geq 0} (-1)^k \binom{2k}{k} y^k$. By Lemma A.11 (2), we see that $\sum_{k \geq 0} (-1)^k \binom{2k}{k} y^k = \frac{1}{\sqrt{1 + 4y}}$. Since $1 + 4y = 1 + \frac{4x}{(1-x)^2} = \frac{(1+x)^2}{(1-x)^2}$, we have $\sqrt{1 + 4y} = \frac{1+x}{1-x}$, and hence,

$$F_0(x) = \frac{x+1}{(1-x)^2} \frac{1-x}{1+x} = \frac{1}{1-x} = 1 + x + x^2 + \cdots.$$ 

Therefore, for any $n$, the coefficient of $F_0(x)$ is 1, i.e., $A_0^{(n)} = 1$.

Next, for $m \geq 1$, we show $A_{m}^{(n)} = 1$. Our approach is almost the same, but we need a slight change in the process. Now, $A_{m}^{(n)} = \sum_{k \geq m} (-1)^{k+m+1} \binom{n+k}{2k} \binom{2k}{k} (k+m)$. As before, we interchange the order of summations of the generating function $F_m(x)$.
We show \( A_m(n) \) as:

\[
F_m(x) = \sum_{n \geq 0} x^n \left\{ \sum_{k \geq m} (-1)^{k+m} \frac{2n+1}{2k+1} \binom{n+k}{2k} \binom{2k}{k+m} \right\}
\]

\[
= \sum_{k \geq m} \frac{(-1)^{k+m}}{2k+1} \binom{2k}{k+m} x^{-k} \left\{ \sum_{n \geq 0} \frac{2n+1}{2k+1} \binom{n+k}{2k} x^{n+k} \right\}
\]

\[
= \sum_{k \geq m} \frac{(-1)^{k+m}}{2k+1} \binom{2k}{k+m} x^{-k} \frac{x^{2n}(x+1)(2k+1)}{(1-x)^{2k+2}} \quad \text{by (4.10)}
\]

\[
= \frac{x+1}{(x-1)^2} \sum_{k \geq m} (-1)^{k+m} \binom{2k}{k+m} \frac{x^k}{(1-x)^{2k}}.
\]

We show \( F_m(x) = \frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + \cdots \). We note that

\[
\sum_{k \geq m} (-1)^{k+m} \binom{2k}{k+m} \frac{x^k}{(1-x)^{2k}}
\]

\[
= \frac{2m}{2m} \frac{x^m}{(1-x)^{2m}} - \frac{2m+2}{2m+1} \frac{x^{m+1}}{(1-x)^{2(m+1)}} + \cdots
\]

\[
= \frac{x^m}{(1-x)^{2m}} \left\{ \frac{2m}{2m} \frac{x}{(1-x)^2} - \frac{2m+2}{2m+1} \frac{x}{(1-x)^2} + \cdots \right\}
\]

\[
= \frac{x^m}{(1-x)^{2m}} \sum_{k \geq 0} (-1)^k \binom{2k+2m}{2k} \left\{ \frac{x}{(1-x)^2} \right\}^k.
\]

Again, let \( y = \frac{x}{(1-x)^2} \). Then \( \sqrt{1+4y} = \frac{1+x}{1-x} \) and \( 1 - \sqrt{1+4y} = 1 - \frac{1+x}{1-x} = \frac{2x}{1-x} \) and \( \frac{1+\sqrt{1+4y}}{2y} = 1 - x \). Then by Lemma [A.11] (3), we have

\[
\sum_{k \geq m} (-1)^{k+m} \binom{2k}{k+m} \frac{x^k}{(1-x)^{2k}}
\]

\[
= \frac{x^m}{(1-x)^{2m}} \cdot \frac{1}{\sqrt{1+4y}} \left\{ \frac{1 - \sqrt{1+4y}}{-2y} \right\}^{2m}
\]

\[
= \frac{x^m}{(1-x)^{2m}} \cdot \frac{1-x}{1+x} \left( \frac{1}{1-x} \right)^{2m}
\]

\[
x^{m(1-x)}.
\]

Therefore \( F_m(x) = \frac{x^{m+1}}{(1-x)^2} \frac{1+x}{1-x} = \frac{x^m}{1-x} = x^m(1+x+x^2+\cdots) \). Thus if \( n < m \), then \( A_m(n) = 0 \), and if \( n \geq m \), then \( A_m(n) = 1 \), i.e., for any \( n \geq m \), \( A_m(n) = 1 \). \( \square \)
Appendix B: Determination of $\delta_4$

Let $\Gamma_{2n}$ be the set of all Alexander polynomials $\Delta_K(t)$ of alternating knots $K$ of genus $n$, i.e., $\deg \Delta_K(t) = 2n$. Let $\delta_{2n}(K)$ be the maximal value of $\text{Re}(\alpha)$ of the zero $\alpha$ of $\Delta_K(t)$ and $\delta_{2n} = \max_{\Gamma_{2n}} \delta_{2n}(K)$.

**Conjecture B.1.** $\delta_{2n}$ exists for any $n \geq 1$, and further, there is a fibred stable alternating knot $K_n$ such that $\delta_{2n}(K_n) = \delta_{2n}$.

In this section, we prove Conjecture B.1 for $n = 2$. Conjecture B.1 is trivially true for $n = 1$. In fact, $\delta_2 = 2.618 \cdots$ that is a zero of $\Delta_{K_1}(t) = t^2 - 3t + 1$, where $K_1$ is $4_1$.

**Theorem B.2.** Let $K_2 = 8_{12}$. Then $\delta_4 = \delta_4(K_2) = 4.3902 \cdots$.

Note that $K_2$ is a fibred stable alternating knot with $\Delta_{K_2}(t) = t^4 - 7t^3 + 13t^2 - 7t + 1$.

**Proof of Theorem B.2.** Let $K$ be an alternating knot. Since $\delta_4(K_2)$ is the maximal value among those of fibred knots of genus 2, we may assume that $K$ is a non-fibred alternating knot. Write $\Delta_K(t) = at^4 - bt^3 + ct^2 - bt + a$, where $a, b, c > 0$ and further, $a \geq 2$. To prove Theorem B.2, we need the following theorem due to Jong.

**Theorem B.3.** [18] Let $\Delta_K(t) = at^4 - bt^3 + ct^2 - bt + a$, where $a, b, c > 0$, be the Alexander polynomial of an alternating knot $K$ of genus 2. Then the following holds.
1. if $\sigma(K) = 0$, then $3a - 1 \leq b \leq 6a + 1$,
2. if $|\sigma(K)| = 2$, then $2a + 1 \leq b \leq 6a - 1$,
3. if $|\sigma(K)| = 4$, then $2a - 1 \leq b \leq 4a - 2$.

Now there are three cases.

Case 1. $\Delta_K(t)$ has four complex zeros, none of which is a unit complex. Then, $\sigma(K) = 0$. Let $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ be all zeros of $\Delta_K(t)$, where $\alpha \beta = 1$ and $\overline{\alpha \beta} = 1$. First, the real part of each zero is positive. In fact, if $\text{Re}(\alpha) < 0$, then the real parts of all zeros are negative, since $\alpha \beta = \overline{\alpha \beta} = 1$. Therefore, $\alpha + \overline{\alpha} + \beta + \overline{\beta} = 2\text{Re}(\alpha) + 2\text{Re}(\beta) < 0$, but $\alpha + \overline{\alpha} + \beta + \overline{\beta} = b/a > 0$, a contradiction. Now suppose $\text{Re}(\alpha) \geq \delta_4$. Then $b/a = \alpha + \overline{\alpha} + \beta + \overline{\beta} = 2\text{Re}(\alpha) + 2\text{Re}(\beta) > 2\delta_4 > 8$, but by Theorem B.3 $b/a \leq 6 + \frac{1}{a} < 7$, a contradiction. Therefore, $\delta_4(K) < \delta_4$.

Case 2. $K$ is $c$-stable. Trivially, $\delta_4(K) < 1$ and hence $\delta_4(K) < \delta_4$. If $|\sigma(K)| = 4$, then $K$ is $c$-stable and hence we may assume hereafter that $|\sigma(K)| \leq 2$, and further, $\Delta_K(t)$ has at least two real zeros. Therefore, the last case is the following:

Case 3. $K$ is bi-stable, but not $c$-stable. From the above remark, we see that $\delta_4(K)$ is the maximal real zero of $\Delta_K(t)$. To show that $\delta_4(K) < \delta_4$, first we consider the modified polynomial $f(x)$ of $\Delta_K(t)$. Write $\Delta_K(t) = at^4 - bt^3 + (2b - 2a - \varepsilon)t^2 - bt + a$, where $\varepsilon = \pm 1$. Then
\[
f(x) = ax^2 - bx + (2b - 4a - \varepsilon) = (x - 2)(ax - (b - 2a)) - \varepsilon.
\]
Since $\delta_4(K)$ is less than the maximal real zero of $f(x)$, we compute the real zeros of $f(x)$. Now the real zeros of $f(x)$ are determined by the intersection of two curves $y_1 = (x - 2)(ax - (b - 2a))$ and $y_2 = \varepsilon$. Since $y_1(0) = 2(b - 2a) \geq 2$, by Theorem B.3 we have the following graphs: (1) $\frac{b - 2a}{a} \leq 2$, (2) $\frac{b - 2a}{a} \geq 2$

![Fig. B.1](image)

The maximal real zero $\gamma$ of $f(x)$ (if exists) is given by

$$\gamma = \frac{b}{2a} + \sqrt{\frac{d}{4a^2}},$$

where $d = (b - 4a)^2 + 4a\varepsilon$.

From this formula, we should note that when $a$ is fixed, $\gamma$ gets larger as $b$ gets larger. Therefore, to obtain the maximal real zero of $f(x)$, $b$ should be the maximal possible value.

Subcase (a) $\varepsilon = 1$. From Fig. B.1, we see that $f(x)$ has two zeros, one is larger than 2, but the other is less than 2. Therefore, $\Delta_K(t)$ has two unit complex zeros and two real zeros, and hence $|\sigma(K)| = 2$. By Theorem B.3 (2), we see that $2a + 1 \leq b \leq 6a - 1$. When $b = 6a - 1$, $d = 4a^2 + 1$ and hence

$$\gamma = \frac{b}{2a} + \sqrt{\frac{d}{4a^2}} = \frac{b}{2a} + \sqrt{1 + \frac{1}{4a^2}}.$$

Since $a \geq 2$, we have $\gamma = \frac{6a - 1}{2a} + \sqrt{1 + \frac{1}{4a^2}} \leq 3 + \sqrt{1.0625} = 4.03077 < \delta_4$, and hence $\delta_4(K) < \delta_4$.

Subcase (b) $\varepsilon = -1$. Since $f(x)$ has a real zero larger than 2, Fig B.1 (1) cannot occur. Therefore, from Fig B.1 (2), $f(x)$ has two real zeros greater than 2 and hence $\Delta_K(t)$ has four real zeros, and $|\sigma(K)| = 0$. Then by Theorem B.3 (1), we have that $3a - 1 \leq b \leq 6a + 1$. When $b = 6a + 1$, $d = 4a^2 + 1$ and $\gamma = \frac{6a + 1}{2a} + \sqrt{1 + \frac{1}{4a^2}}$. Since $a \geq 2$, it follows that $\gamma \leq 3 + \frac{1}{4} + \sqrt{1.0625} = 4.281 \cdot \cdot \cdot < \delta_4$.

A proof of Theorem B.2 is now complete.

Appendix C: Distribution of the zeros.

In this section, we discuss distribution of the zeros of the Alexander polynomials of two infinite sequences of 2-bridge knots. Namely, they are vertical and horizontal extensions of the 2-bridge knot $[2, 2, 2, 2, -2, -2, -2, -2]$. 
Let \( r(k) = [2, 2, 2k, -2, -2, -2, -2], k \neq 0 \). For simplicity, \( K(r(k)) \) will be denoted by \( K(k) \). The type of the zeros of the Alexander polynomial of \( K(k) \) depends on \( k \). More precisely, we prove the following theorem.

**Theorem C.1.** Case 1. \( k > 0 \).

1. If \( k = 1 \) or \( 2 \), then \( \Delta_{K(k)}(t) \) is totally unstable, i.e., every zero is a non-unit complex number.
2. If \( k = 3, 4, 5 \) or \( 6 \), then \( \Delta_{K(k)}(t) \) has four unit complex zeros and four non-unit complex zeros, and hence \( \Delta_{K(k)}(t) \) has no real zeros.
3. If \( k \geq 7 \), then \( \Delta_{K(k)}(t) \) has eight unit complex zeros, and hence \( K(k) \) is \( c \)-stable.

Therefore, in this case, \( \Delta_{K(k)}(t) \) does not have real zeros.

Case 2. \( k < 0 \).

For all \( k \), \( \Delta_{K(k)}(t) \) has two real zeros and six unit complex zeros, and hence \( K(k) \) is strictly bi-stable.

**Example C.2.** For any \( k \neq 0 \), \( \Delta_{K(k)}(t) = k^{2}t^{4} - 5kt^{3} + (8k + 1)t^{2} - 7kt + 1 \). We plot the zeros around the unit circle in Fig. C.1 for \( k = -4, -3, -2, -1, 1, 2, \ldots, 8 \).

![Fig. C.1](image)

**Remark C.3.** If \( k > 0 \), then \( \sigma(K(k)) = 0 \). But, if \( k \geq 7 \), \( K(k) \) is \( c \)-stable. Therefore, \( c \)-stable alternating knots are not necessarily special alternating. If \( k < 0 \), then \( |\sigma(K(k))| = 2 \), but \( \Delta_{K(k)}(t) \) has always more than two unit complex zeros.

**Proof of Theorem C.1.** First, using a standard Seifert matrix of \( K(k) \), we can show that \( \Delta_{K(k)}(t) = kf(t) + t^{4} \), where \( f(t) = (t-1)^{2}(t^{2}+1)(t^{4}-t^{3}+t^{2}-t+1) \). Consider the modification \( F(x) \) of \( \Delta_{K(k)}(t) \) : \( F(t) = kx(x-2)(x^{2}-x-1) + 1 \). To prove the theorem, we study the intersection of two curves, \( y_{1} = g(x) = x(x-2)(x^{2}-x-1) \) and \( y_{2} = -1/k, k \neq 0 \). By simple calculations, \( y_{1} \) is depicted in Fig. C.2. Then we see (1) two curves \( y_{1} \) and \( y_{2} = -1/k, k \leq -1 \), intersect in exactly four points, only one of which has \( x \)-coordinate greater than \( 2 \) and others in \((-2, 2)\). This proves Case 2.

(2) Suppose \( k > 0 \). If \( k = 1 \) or \( 2 \), two curves do not intersect and hence, \( K(k) \) is totally unstable. If \( k = 3, 4, 5 \) or \( 6 \), then two curves intersect in two points with \( x \)-coordinate in \((-2, 2)\). If \( k \geq 7 \), two curves intersect in four points with \( x \)-coordinate between \(-2 \) and \( 2 \), and hence all the zeros are unit complex. This proves Theorem C.1. \( \Box \)
The previous sequence can be considered as a vertical extension of the original 2-bridge knot $K(1) = [2, 2, 2, -2, -2, -2, -2, -2]$. The next sequence is a horizontal extension of $K(1)$. Consider the sequence $r[n] = [2, \ldots, 2, -2, \ldots, -2]$, $n$ consecutive 2’s followed by $n$ consecutive –2’s, with $n \geq 1$. $K(r[n])$ will be denoted by $K[n]$.

We prove the following theorem.

**Theorem C.4.** (1) If $n$ is odd, then $\Delta_{K[n]}(t)$ has two real zeros, and other are non-unit complex zeros.

(2) If $n$ is even, then $\Delta_{K[n]}(t)$ is totally unstable.

**Proof.** First, using a standard Seifert matrix, it is easy to show by induction on $n$ that

$$\Delta_{K[n]}(t) = (t+1)^2\Delta_{K[n]}(t) = (t^{2n+1} - 1)(t-1) + (-1)^n4t^{n+1}. \quad \text{(C.1)}$$

To prove the theorem, we need the following lemma.

**Lemma C.5.** $\Delta_{K[n]}(t)$ does not have a unit complex zeros.

**Proof.** Obviously, ±1 is not the zeros of $\Delta_{K[n]}(t)$. Now, we express $\Delta_{K[n]}(t)$ in a different form:

$$(t + 1)^2\Delta_{K[n]}(t) = (t^{2n+1} - 1)(t-1) + (-1)^n4t^{n+1}. \quad \text{(C.2)}$$

Suppose $\Delta_{K[n]}(t)$ has a unit complex zero $\alpha = e^{i\theta}, \theta \neq 0, \pi$. Then $(e^{(2n+1)\theta} - 1)e^{i\theta} - 1 = (-1)^n4e^{(n+1)i\theta}$ and hence, $|e^{(2n+1)i\theta} - 1|e^{i\theta} - 1| = 4. This is impossible, since $|e^{(2n+1)i\theta} - 1| \leq 2$ and $|e^{i\theta} - 1| < 2$.\[\square\]

We return to a proof of the theorem.

Case 1. $n$ is odd, say $2m + 1$. Then

$$\Delta_{K[n]}(t) = (t - 1)^2(t^{2m} + t^{2m-2} + \cdots + t^2 + 1)(t^{2m} - t^{2m-1} + \cdots + t^2 - t + 1) - t^{2m+1}. \quad \text{(C.3)}$$

Consider the modified polynomial $F(x)$ of $\Delta_{K[n]}(t)$. Then $F(x) = (x -
2) \( g(x)h(x) - 1 \), where \( g(x) \) and \( h(x) \) are, respectively, the modified polynomials of \( f_1(t) = t^{2m} + t^{2m-2} + \cdots + t^2 + 1 \) and \( f_2(t) = t^{2m} - t^{2m-1} + \cdots + t^2 - t + 1 \). Since \( f_1(t) \) and \( f_2(t) \) have only unit complex zeros, all the zeros of \( g(x) \) and \( h(x) \) are real in \((-2, 2)\). Then the zeros of \( \Delta_{K[n]}(t) \) are determined by intersection of two curves \( y_1 = (x-2)g(x)h(x) \) and \( y_2 = 1 \). See Fig. C.3 (1). We see that \( y_1 \) and \( y_2 \) intersect in one point \( P(p_1, p_2) \), with \( p_1 > 2 \). If \( y_2 \) intersects \( y_1 \) at another point, say \( Q(q_1, q_2) \), then \(-2 < q_1 < 2\) and the corresponding zeros of \( \Delta_{K[n]}(t) \) are unit complex. This is impossible by Lemma [C.5](#).

![Fig. C.3](image-url)

Case 2. \( n \) is even, say \( 2m \). Then again, we have \( \Delta_{K[n]}(t) = (t - 1)^2(t^{2m-2} + t^{2m-4} + \cdots + t^2 + 1)(t^{2m} - t^{2m-1} + \cdots + t^2 - t + 1) + t^{2m} \).

From this form, we see easily that \( \Delta_{K[n]}(t) \) has no real zeros, since \( \Delta_{K[n]}(t) > 0 \) if \( t \) is real. Now we did as before, consider the modified polynomial \( F(x) \) of \( \Delta_{K[n]}(t) \) and see that \( F(x) = (x-2)g(x)h(x) + 1 \), where \( g(x) \) and \( h(x) \) are the modifications of \( \frac{t^{2m-1}}{t^2-1} \) and \( \frac{t^{2m+1}+1}{t+1} \), respectively. Both have only real zeros in \((-2, 2)\). Consider the intersection of \( y_1 = (x-2)g(x)h(x) \) and \( y_2 = -1 \). Since \( \deg y = 2m \), the graph appears as in Fig. C.3 (2). As we proved earlier, \( y_1 \) and \( y_2 \) do not intersect, otherwise \( \Delta_{K[n]}(t) \) would have a unit complex zero. \( \square \)

Now, none of the zeros of \( \Delta_{K[n]}(t) \) is unit complex, but these zeros seem to be distributed in a narrow strip containing the unit circle. More precisely, we propose the following conjecture.

**Conjecture C.6.** Let \( \alpha = \frac{3 - \sqrt{5}}{2} \). \( \alpha \) is one of the real zeros of \( \Delta_{K[1]}(t) = t^2 - 3t + 1 \). Let \( C \) be the circle with centre at \( \left( \frac{\alpha + 1}{2}, 0 \right) \) and radius \( \frac{\alpha + 1}{2} \). Then, for \( n \geq 1 \), all the zeros of \( \Delta_{K[n]}(t) \) with length < 1 lie in a narrow lunar domain bounded by the unit circle and \( C \).
Remark C.7. If Hoste’s conjecture is true, then none of the zeros of the Alexander polynomial of an alternating knot is in the interior of the circle with centre at $(-1/2,0)$ and radius $1/2$.

Example C.8. Fig. C.5 below depicts the zeros for the cases $n = 1, \ldots, 10$ around the unit circle.

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