Moser Iteration and the Large Coupling Limit

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Abstract

We consider heat semigroups of the form \( \exp(t(\Delta - \lambda I_{\Omega_0})) \) on bounded domains. Using variants of Moser iteration, we show sub-exponential decay in the “large coupling limit”, i.e. as \( \lambda \to \infty \), in compact subdomains of the “obstacle”, \( \Omega_0 \).

1 Introduction

Let \( \Omega \subset \mathbb{R}^m \), \( m \geq 3 \), be a bounded open connected subset with smooth boundary \( \Gamma \). We are given a compact inclusion, \( \Omega_0 \subset \Omega \), with boundary \( \Gamma_0 \) whose “exterior” we denote by \( \Omega_1 := \Omega \setminus \Omega_0 \). We also have the Schrödinger like operator \( A_\lambda := \Delta - \lambda I_{\Omega_0} \) with Neumann boundary conditions on \( \Gamma \). For reasonable functions \( f \), the large coupling problem is to determine the limit and convergence rate of \( f(A_\lambda) \) as the “coupling parameter” \( \lambda \to \infty \).

The case \( f(x) = e^{tx}, \ t > 0 \), corresponds to a diffusion process which is reflected on \( \Gamma \) and “killed” on entering \( \Omega_0 \). The corresponding “large coupling limit” is \( e^{tB} \) where \( B \) is the realization of \( \Delta \) in \( L^2(\Omega_1) \) with Neumann boundary conditions at \( \Gamma \) and Dirichlet boundary conditions at \( \Gamma_0 \). Using energy estimates, in [1] we showed an algebraic convergence rate in the exterior domain \( \Omega_1 \). In contrast, in this paper we are concerned with the behavior of solutions in the “obstacle”, or “killing” region, \( \Omega_0 \).

Our first result is a sub-exponential convergence rate on compact, strictly interior subregions, \( V \), of \( \Omega_0 \). Such decay is plausible in light of the Feynman–Kac formula, and the novelty here is a proof using purely “PDE techniques”. We assume that the boundary, \( \partial V \), is at least \( C^1 \), orientable and that the Sobolev–Poincare inequalities hold on \( V \). In addition, we assume the normal exponential map has non-zero “injectivity radius”. Heuristically, this means that the normal vectors exist at each point of \( \partial V \) and can be followed a uniform nonzero distance away from the surface.

In the statement that follows, \( E_0[g] \) is the extension of \( g \in L^2(\Omega_1) \) by zero into \( \Omega_0 \) while \( u_\lambda(t, x) = e^{tA_\lambda}E_0[g] \). We then have:

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Theorem 1.1. Assume \( E_0[g] \in H^1(\Omega) \), and \( r > 0 \) is the injectivity radius of \( \partial V \). Set \( a = \min\{\text{dist}(\partial V, \Gamma_0), r\} \). For any \( 0 < \nu < 1/2 \), there is a \( \lambda_0 = \lambda_0(a, \nu) \) such that for \( \lambda \geq \lambda_0 \),

\[
\sup_{t \geq 0} \|u_{\lambda t}\|_{L^2(V)}^2 \leq e^{-\lambda^\nu} \lambda^{-1} \|\nabla g\|_{L^2(\Omega_1)}^2.
\]

The proof given in §2 also shows that a similar estimate holds with a space–time \( L^2 \) norm: i.e. \( L^2(I \times V) \) in place of \( L^\infty(R_+; L^2(V)) \) for suitable intervals \( I \).

The argument is a Caccioppoli type energy estimate (Lemma 2.2 below) applied iteratively to a sequence of domains converging to \( V \) a la moser, [3]. The crux is that we iterate \( N \) times with \( N \) finite but dependent on \( \lambda \). The injectivity radius enters the argument by of the choice made for the approximating domains. This part of the argument comes for free when \( V \) is a cube or a ball since these have a natural dilation. Conversely, the general case can likely be derived from the case of cubes or balls via a covering argument. MOSER actually gives such an argument for convex domains (see [3, Theorem 2, pg 110–111]). However, the argument given here seamlessly handles the non-convex case.

It is worth mentioning that the estimate in Theorem 1.1 is related to the “survival” or “trapping” probability for the associated diffusion process. It is partly for this reason we opted to treat “general subdomains” \( V \) directly. The condition \( E_0[g] = 0 \) in \( \Omega_0 \) means that the probability of initially finding the particle in \( \Omega_0 \) is zero. Having said all this, we relegate further probabilistic interpretations and applications of our to another paper.

The other result is a pointwise sub-exponential bound on solutions in the interior of \( \Omega_0 \).

Theorem 1.2. Assume in addition that \( g \geq 0 \), then for \( \lambda \geq \lambda_0 \) as above and any \( s \geq 0 \),

\[
\sup_{Q_{s\gamma}} u_{\lambda t}^2 \leq C e^{-\lambda^\nu} \lambda^{-1} \|\nabla g\|_{L^2(\Omega_1)}^2,
\]

where \( Q_{s\gamma} = \{(t, x) : t \in (s, s + (a \gamma)^2), x \in V_{-\gamma a}\} \), \( 0 < \gamma < 1/2 \), and \( C = C(m, V, \Omega_0) \).

The sets \( V_\rho \) are defined in §2 and used in the proof of Theorem 1.1. As hinted earlier, they are gotten by dilating \( \partial V \). The proof of Theorem 1.2 is sketched in §3. It combines MOSER’s infinite iteration argument and the finite argument of Theorem 1.1. The assumption that \( g \geq 0 \) is not essential, but it simplifies the argument. It is also natural in view of the probabilistic interpretation of the equations.

Having outlined the explicit contents of this article, this is a good place to comment on what is implicitly contained here. All we really need is a compatible family of sets (to play the role of “balls”), a Caccioppoli type inequality and a
Sobolev inequality. Moser’s scheme is known to go through under fairly general conditions guaranteeing suitable versions of these “ingredients”. For example, in place of the Laplacian, we could consider the large coupling problem with divergence form elliptic operators; the large coupling problem for Laplace–Beltrami operator on a Riemannian manifold; or even suitable sub-elliptic operators. We refer the reader to STURM [5] for a readable discussion on this.

It is also clear from the proofs that the boundedness of Ω and Ω₀ is not used in any essential way as all of the arguments are local. This implies that appropriate versions of these results hold for unbounded domains as well. For instance, our proofs work—almost verbatim—for the case Ω := ℝᵐ and Ω₀ := \{x | xₘ > φ(x₁, . . . , xₘ₋₁)\} the region above the “graph” of a C¹ function φ.

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2 Variations on a Theme of Moser and Sub-Exponential Decay

We begin with the observation that \( u_\lambda(t, x) = e^{tA_\lambda} E₀[g] \) solves:

\[
\partial_t u_\lambda = \Delta u_\lambda(t, x) - \lambda 1_{\Omega_0}(x)u_\lambda(t, x); \quad (t, x) \in I \times \Omega,
\]

where \( I = (0, T) \) with \( 0 < T < \infty \). None of the constants in the estimates depend on \( T \), so we may adjust it as appropriate. The initial and boundary conditions are:

\[
\begin{align*}
  u_\lambda|_{t=0} &= E₀[g] := \begin{cases} g(x), & x \in \Omega₁; \\ 0, & x \in \Omega₀. \end{cases} \\
  \nabla u_\lambda(t, x) \cdot \hat{n} &= 0, \quad (t, x) \in I \times \Gamma.
\end{align*}
\]

We need two estimates which we state as Lemmas. The first is contained in [1].

**Lemma 2.1.** If \( g \in H₀¹(Ω₁) \), then

\[
\sup_{t \geq 0} \|u_\lambda(t, \cdot)\|^2_{L²(Ω₀)} \leq \frac{1}{\lambda} \|\nabla g\|^2_{L²(Ω₁)}; \quad \|u_\lambda\|^2_{L²(Ω₀)} \leq \frac{1}{2\lambda}\|g\|^2_{L²(Ω₁)}
\]

The proof is a standard energy argument: multiply the PDE by the solution or one of its derivatives—in this case \( u_\lambda \) and \( \partial_t u_\lambda \)—and integrate in space, time or both. Similar arguments are shown later so we omit this proof.

The other ingredient is a Caccioppoli type estimate. As mentioned in the introduction, these are usually given on cubes, balls and their parabolic counterparts. Our unusual statement requires some notation.
Viewing $\partial V$ as a $C^1$ compact, embedded, oriented surface in $\mathbb{R}^m$, as in Lee [2, pg. 255–257], we can define the map

$$\exp^\perp: \partial V \times \mathbb{R} \to \mathbb{R}^m; \quad (x, s) \mapsto x + s\hat{n}(x),$$

(3)

where $\hat{n}(x)$ is the unit outward normal vector field on $\partial V$. Let $\partial V_\rho = \exp^\perp(\partial V, \rho)$ and define

$$V_\rho = \begin{cases} V \cup \left( \bigcup_{0 \leq s < \rho} \exp^\perp(\partial V, s) \right), & \rho > 0; \\ V \setminus \left( \bigcup_{\rho < s < 0} \exp^\perp(\partial V, s) \right), & \rho < 0; \end{cases}$$

The injectivity radius $r$ is the supremum over all $|\rho| > 0$ for which $\partial V$ is $C^1$ diffeomorphic to $\partial V_\rho$. Note that $\text{dist}(V_\rho, V_\rho + \sigma) = |\sigma|$.

Lemma 2.2. With $a = \min\{\text{dist}(\partial V, \Gamma_0), r\}$ and for any $0 \leq \rho < \rho + \sigma < a$,

$$\|u\|_{L^\infty(\mathbb{R}_+; L^2(V_\rho))} \leq \frac{4}{\lambda a^2} \|u\|_{L^\infty(\mathbb{R}_+; L^2(V_{\rho + \sigma}))};$$

$$\|u\|_{L^2(I \times V_\rho)} \leq \frac{4}{\lambda a^2} \|u\|_{L^2(I \times V_{\rho + \sigma})}.$$

Before proving this lemma, let us show how it implies the theorem.

Proof of Theorem 1.1 Define $U_0 := \Omega_0$ and $U_j = V_{\gamma a(1 - j/N)}$ for $1 \leq j \leq N$, any $0 < \gamma < 1$ and some integer $N$ to be determined soon. Thus

$$V =: U_N \in U_{N-1} \in \ldots \in U_1 \in U_0 := \Omega_0,$$

with $\text{dist}(U_j, U_{j-1}) \geq \frac{\gamma a}{N}$. With $X(U)$ either of the spaces $L^\infty(\mathbb{R}_+; L^2(U))$ or $L^2(I \times L^2(U))$, Lemma 2.2 implies

$$\|u\|_{X(U_j)}^2 \leq \frac{4}{\lambda \text{dist}(U_j, U_{j-1})^2} \|u\|_{X(U_{j-1})}^2.$$

Iterating this $N$ times and applying Lemma 2.1 gives

$$\|u\|_{X(U_N)}^2 \leq \left( \frac{4}{\lambda (\gamma a/N)^2} \right)^N \|u\|_{X(U_0)}^2 \leq \frac{\|g\|_{H^1(\Omega_1)}}{\lambda \frac{4N^2}{\gamma^2 a^2}} \left( \frac{4N^2}{\gamma^2 a^2} \right)^N,$$

Choosing $N$ as the integer part of $\lambda^\nu$ with $0 < \nu < 1/2$, we see that $\left( \frac{4N^2}{\gamma^2 a^2} \right)^N \leq \left( \frac{4 \lambda^{2\nu}}{\gamma^2 a^2} \right)^{\lambda^\nu}$. The theorem follows once we realize that $\frac{4 \lambda^{2\nu}}{\gamma^2 a^2} \leq e^{-1}$ for $\lambda \geq \left( \frac{4}{\gamma^2 a^2} \right)^{1/2\nu}$. \( \square \)
We can fiddle with \( N \) as long as \( N(\lambda) = o(\sqrt{\lambda}) \) at the possible cost of a larger \( \lambda_0 \). One amusing example is \( N(\lambda) = \lambda^{k^2 - \frac{k}{2} (\log \lambda)^{\frac{k}{2}}} \) for any natural number \( k \). With so much leeway, the best choice of \( N(\lambda) \) is an interesting question. A related observation is that the number of approximating domains in this argument was finite and \( \sigma_j = \text{dist}(U_j, U_{j-1}) \) was chosen to be constant in the proof. A natural idea is to try to choose \( \sigma_j \) such that \( \sum_{j=1}^{\infty} \sigma_j = \gamma a \) and to estimate \( \lim_{n \to \infty} \prod_{j=1}^{n} \frac{4}{\lambda \sigma_j} \).

However \( \sigma_j = o(1/j) \) since the infinite sum converges. This implies that for fixed \( \lambda \) the infinite product will diverge. Nevertheless in §3 we’ll see that the full version of Moser’s scheme deals with this.

Right now, we turn to the

**Proof of Lemma 2.2** Let \( 0 \leq \eta(x) \leq 1 \) be a smooth function with \( \eta = 1 \) on \( V_\rho \) and vanishing outside \( V_{\rho + \sigma} \). We multiply the PDE by \( \eta^3 u_\lambda \) and integrate by parts using the fact that \( \eta^3 u_\lambda \) is compactly supported in \( \Omega_0 \) to get

\[
\int_{\Omega_0} \partial_t u_\lambda \eta^2 dx = \int_{\Omega_0} \Delta u_\lambda \eta^2 dx - \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 dx,
\]

\[
= - \int_{\Omega_0} \nabla u_\lambda \cdot \nabla (u_\lambda \eta^2) dx - \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 dx,
\]

\[
= - \int_{\Omega_0} |\nabla u_\lambda|^2 \eta^2 dx - \int_{\Omega_0} 2 \eta u_\lambda \nabla \eta \cdot \nabla u_\lambda dx - \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 dx.
\]

For simplicity, we drop the integration measure. Cauchy’s inequality “with \( \epsilon \)” implies

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} u_\lambda^2 \eta^2 + \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 + \int_{\Omega_0} |\nabla u_\lambda|^2 \eta^2 \leq 2 \int_{\Omega_0} \eta u_\lambda \nabla \eta \cdot \nabla u_\lambda,
\]

\[
\leq \frac{1}{2} \int_{\Omega_0} |\nabla u_\lambda|^2 + 2 \int_{\Omega_0} u_\lambda^2 |\nabla \eta|^2,
\]

which after some rearrangement gives

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} u_\lambda^2 \eta^2 + \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 + \frac{1}{2} \int_{\Omega_0} |\nabla u_\lambda|^2 \eta^2 \leq 2 \int_{\Omega_0} u_\lambda^2 |\nabla \eta|^2. \quad (4)
\]

The positivity of the term involving the gradient on the left side implies that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} u_\lambda^2 \eta^2 + \lambda \int_{\Omega_0} u_\lambda^2 \eta^2 \leq 2 \int_{\Omega_0} u_\lambda^2 |\nabla \eta|^2,
\]

and Gronwall’s Lemma shows that

\[
\| (u_\lambda \eta)(t) \|^2_{L^2(\Omega_0)} \leq 4 \int_0^t \int_{\Omega_0} e^{-2\lambda(t-s)} u_\lambda^2 |\nabla \eta|^2 \, dx \, dt.
\]
Since $\eta = 1$ on $V_{\rho}$, $\nabla \eta$ is supported in $V_{\rho+\sigma}$, and $|\nabla \eta| \leq 2/\sigma$, we get
\[
\|u_\lambda(t)\|_{L^2(V_\rho)}^2 \leq \|(u_\lambda \eta)(t)\|_{L^2(\Omega_0)}^2 \leq \frac{4}{\lambda \sigma^2} \|u_\lambda\|_{L^\infty(\mathbb{R}^+;L^2(V_{\rho+\sigma}))}^2,
\]
and taking the supremum over $t$ on the left proves the “$L^\infty$ part” of the Lemma.

To get the “$L^2$ part”, we integrate (4) from 0 to $T$ to get
\[
\frac{1}{2}\|u_\lambda(t)\|_{L^2(V_\rho)}^2 + \lambda \|u_\lambda\|_{L^2(I \times V_\rho)}^2 + \frac{1}{2}\|
abla u_\lambda\|_{L^2(I \times V_\rho)}^2 \leq \int_0^T \int_{\Omega_0} u_\lambda^2 |\nabla \eta|^2 \, dx \, dt,
\]
from which we get
\[
\|u_\lambda\|_{L^2(I \times V_\rho)}^2 \leq \frac{4}{\lambda \sigma^2} \|u_\lambda\|_{L^2(I \times V_{\rho+\sigma})}^2.
\]

Though we squeezed two estimates from the proof above, it has a bit more to give:

1. One refinement—which we won’t pursue here—is to capitalize on the fact that $\nabla \eta$ is actually supported in $V_{\rho+\sigma}\setminus V_{\rho}$ and not on the whole of $V_{\rho+\sigma}$.

2. Using the $L^\infty$ estimate in $t$ after the application of the Gronwall’s Lemma does not take advantage of the explicit integral representation.

3. We completely ignored the gradient term in the energy inequalities.

The third issue is examined in §3 with the help of the Poincare–Sobolev inequality which relates $L^p$ norms of the function and its gradient. Here is a sketch of an attempt to deal with the second refinement. The starting point is the following estimate derived in the course of the proof of Lemma 2.2
\[
\|(u_\lambda \eta)(t)\|_{L^2(\Omega_0)}^2 \leq 4 \int_0^t \int_{\Omega_0} e^{-2\lambda(t-s)} u_\lambda^2 |\nabla \eta|^2 \, dx \, dt.
\]
In anticipation of an iterative argument, we let $\varphi_j(t) = \|u_\lambda(t, \cdot)\|_{L^2(V_j)}^2$. With an appropriate choice of $\eta$, this becomes
\[
\varphi_j(t) \leq \frac{4}{\sigma^2} \int_0^t e^{-2\lambda(t-s)} \varphi_{j-1}(s) \, ds
\]
If we iterate this once we get
\[
\varphi_j(t) \leq \left(\frac{4}{\sigma^2}\right)^2 \int_0^t e^{-2\lambda(t-s)} \int_0^s e^{-2\lambda(s-\tau)} \varphi_{j-2}(\tau) \, d\tau \, ds
\]
\[
\leq \left(\frac{4}{\sigma^2}\right)^2 \int_0^t \int_0^s e^{-2\lambda(t-\tau)} \varphi_{j-2}(\tau) \, d\tau \, ds
\]
And by induction
\[ \varphi_N(t) \leq \left( \frac{4}{\sigma^2} \right)^N \int_{t_{N-1}}^t \int_{t_{N-2}}^{t_{N-1}} \cdots \int_{t_1}^{t_2} e^{-2\lambda(t-t_N)} \varphi_0(t_N) \, dt_N \, dt_{N-1} \cdots dt_1 \]

Using the \( L^\infty \) estimate on \( \varphi_0(t) \) and carrying out the integration we get
\[ \varphi_N(t) \leq \frac{1}{\lambda} \| \nabla g \|_{L^2(\Omega_0)}^2 \left( \frac{2}{\sigma^2 \lambda} \right)^N \left[ 1 - e^{-2M_{N-1}(2\lambda t)} \right] \]
or equivalently
\[ \| (u_\lambda(t, \cdot)) \|_{L^2(V)}^2 \leq \frac{1}{\lambda} \| \nabla g \|_{L^2(\Omega_0)}^2 \left( \frac{2}{\sigma^2 \lambda} \right)^N e^{-2M_R N-1(2\lambda t)} \]

where \( M_k(s) = \sum_{j=0}^k \frac{s^j}{j!} \) is the \( k \)-th degree Maclaurin expansion of \( e^s \) and \( R_k(s) \) is the corresponding remainder. From here it is possible to find asymptotic formulas but we will not pursue this here.

### 3 Moser Iteration all the Way

This section is devoted to the proof of Theorem 1.2. It turns out that it is implied by the following “mean value inequality” also sometimes called a “reverse Hölder inequality”:

\[ \sup_{Q} u_\lambda^2 \leq C \int \int_{(s, s+a^2 \gamma^2) \times \mathcal{V}} u_\lambda^2 \, dx \, dt. \]

The argument is simple: the quantity on the right is controlled by \( \| u_\lambda \|_{L^2(I \times \mathcal{V})}^2 \) which decays exponentially by Theorem 1.1. The inequality is called a mean value inequality because in the case of a sphere, the constant, \( C \) is proportional to the reciprocal of the volume of the region of integration.

The proof of the mean value inequality follows MOSER’s proof of a similar result in [3]. It addresses two issues raised earlier: (i) the fact that we have ignored the gradient thus far, and (ii) the fact we have iterated our estimates only a finite number of times. Partly because our formulation is slightly different, partly to keep this note self-contained, and partly because it is so neat, we include a sketch of the proof. The discussion leans heavily on the exposition in SALOFF-COSTE [4, pg 445-447].

We begin with the following identity for \( p \geq 1 \) and non-negative \( u_\lambda \)
\[ \frac{1}{2p} \int_{\Omega_0} \frac{d}{dt} \left( (u_\lambda^p \eta)^2 \right) + \chi^2 \int_{\Omega_0} |\nabla (u_\lambda^p \eta)|^2 + \chi^2 (1-p)^2 \int_{\Omega_0} u_\lambda^{2p-2} \eta^2 |\nabla u_\lambda|^2 + \lambda \chi^2 \int_{\Omega_0} u_\lambda^{2p} \eta^2 \]
\[ = \chi \int_{\Omega_0} u_\lambda^{2p} \eta^2 + \chi^2 \int_{\Omega_0} |\nabla \eta|^2 u_\lambda^{2p}, \]
where \( \chi = \chi(t) \) is a smooth function of \( t \) and \( \eta \) we have encountered before. This can be derived by multiplying the differential equation by \( u_\lambda^{2p-1} \chi^2 \eta^2 \) and integrating by parts. We omit the details, but when the algebraic dust settles the above identity emerges. Note that when \( p = 1 \), \( \chi \equiv 1 \), Cauchy’s inequality allows us to recover the main inequality, (4), in the proof of Lemma 2.2.

It’s time to specify \( \chi \) and \( \eta \). With \( 0 < \sigma < \overline{\sigma} \), we choose \( 0 \leq \eta(x) \leq 1 \) with \( \eta = 1 \) on \( V_{p+\sigma} \) and vanishing outside \( V_{p+\overline{\sigma}} \) and \( 0 \leq \chi(t) \leq 1 \) with \( \chi = 1 \) on \( (-\infty, s + \sigma^2) \) and vanishing outside \( (s + \sigma^2, \infty) \). Let \( I_\sigma = (s, s + \sigma^2) \) and integrate the identity from \( s \) to any \( t \in I_\sigma \) to get

\[
\sup_{t \in I_\sigma} \int_{V_{p+\sigma}} u_\lambda^{2p} + 2p \int_{I_\sigma \times V_{p+\sigma}} |\nabla (u_\lambda^p)|^2 \leq \frac{8p}{(\sigma - \sigma)^2} \int_{I_\sigma \times V_{p+\sigma}} u_\lambda^{2p} \tag{5}
\]

We have used the support properties of \( \chi \) and \( \eta \) and the fact that \( |\chi'| \leq 2/(\bar{\sigma} - \sigma)^2 \) and \( |\nabla \eta|^2 \leq 2/(\bar{\sigma} - \sigma)^2 \). The \( p \) in the fraction on the right side of the inequality can probably be eliminated, but we proceed as is. Later on, we will use \( Q_\sigma := I_\sigma \times V_{p+\sigma} \) to further simplify the notation.

Hölder’s inequality implies that

\[
\int_U w^{2(1 + \frac{\alpha}{m})} \leq \left( \int_U w^2 \right)^{\frac{\alpha}{m}} \left( \int_U w^{\frac{m-\alpha}{m}} \right)^{\frac{m-\alpha}{m}},
\]

while the Poincare–Sobolev inequality for compactly supported functions implies

\[
\left( \int_U w^{\frac{m-\alpha}{m}} \right)^{\frac{m-\alpha}{m}} \leq \kappa \int_U |\nabla w|^2,
\]

and \( \kappa = \kappa(U, m) \) is the best constant for the inequality. Hence, for any interval \( J \),

\[
\int_J \int_U w^{2(1 + \frac{\alpha}{m})} \leq \kappa \sup_{t \in J} \left( \int_U w^2 \right)^{\frac{\alpha}{m}} \int_U |\nabla w|^2.
\]

Applying the above inequality with \( J = I_\sigma \) and \( U = V_{p+\sigma} \) and using (5) gives

\[
\int_{Q_\sigma} u_\lambda^{2p\theta} \leq 2^{\frac{\alpha}{m}} \kappa \left( \frac{4p}{(\sigma - \sigma)^2} \right)^{\frac{\alpha}{m}} \int_{Q_\sigma} u_\lambda^{2p} \theta, \tag{6}
\]

with \( \theta = 1 + \frac{2}{m} \) and \( \kappa \) is the Poincare–Sobolev constant for \( V \). To iterate this, we put \( \rho = -2a_\gamma \), with \( 0 < \gamma < 1/2 \), and define \( \sigma_i = a_\gamma(1 + 2^{-i}) \) for \( i = 0, 1, 2 \ldots \). Note that \( \sigma_0 = 2a_\gamma \) and \( \lim_i \sigma_i = a_\gamma \). Applying (6) with \( p = \theta^i \) and \( \sigma = \sigma_{i+1} \) and \( \overline{\sigma} = \sigma_i \) we get:

\[
\int_{Q_{\sigma_{i+1}}} u_\lambda^{2\theta^{i+1}} \leq 2^{\frac{\alpha}{m}} \kappa \left( \frac{4\theta^i 4^{i+1}}{(a_\gamma)^2} \right)^{\frac{\alpha}{m}} \int_{Q_{\sigma_i}} u_\lambda^{2\theta^i} \theta.
\]
It then follows that
\[
\left( \iint_{Q_{\lambda}^{i+1}} u_{\lambda}^{2^{i+1}} \right)^{\theta^{-i-1}} \leq (2^{\frac{m}{2}} \kappa) \sum_{j=0}^{i+1} \theta^{-j} \left( \frac{16}{(a\gamma)^2} \right) \sum_{j=0}^{i} \theta^{-j} \int_{Q_{\lambda}^{0}} u_{\lambda}^{2}.
\]
Sending \( i \to \infty \) and playing with several geometric series gives
\[
\sup_{Q_{\lambda}^{i}} u_{\lambda}^{2} \leq \frac{C(m) \kappa^m}{(a\gamma)^{m+2}} \iint_{Q_{\lambda}^{0}} u_{\lambda}^{2},
\]
for some explicit constant \( C(m) \) depending only on \( m \).

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