Fundamental S-matrix
for
Vector Perturbed $WD_n$ Minimal Models

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Abstract

Kink-kink S-matrix for integrable vector perturbed $WD_n^{(k)}$ minimal models is constructed from the Boltzmann weights of $A_{2n-1}^{(2)}$ RSOS model and checked in two limit cases of $k$. 
In the last years, starting from the famous work [1] where the main principles of exact S-matrix construction for integrable perturbations of Conformal Field Theories were formulated and demonstrated on few examples, many other such integrable perturbations were found and their S-matrices were studied and classified (see the review [2]). It seems that probably all such massive integrable perturbations of Virasoro minimal models (without additional infinite symmetries) were found and S-matrices for them were constructed, as far as the spectrum of such perturbations seems to be more rich and S-matrices much more complicated for the perturbations of Conformal Field Theories with additional affine symmetries. Most of such theories (if not all of them) may be expressed as coset constructions of some Kac-Moody algebras at certain levels. Among this theories there are some integrable perturbations of WZW models [3], $Z_n$ parafermions [4], the adjoint perturbations of W-invariant theories [5] built on $A_n$ series of Lie algebras (see, for example,[6],[7]) and for other series [8].

Recently another class of integrable perturbations of W-invariant theories were found [9],[10]. In [10] it was checked by ”counting argument” [1], that vector perturbation (in the classification of primary fields according to [6]) of $W D_n^{(k)}$ minimal models are integrable and can be considered as generalization of (1,2) integrable perturbation of Virasoro minimal models to the W-invariant theories. It was shown there by explicit construction of nonlocal currents, that the model has $A_{2n-1,q}^{(2)}$ quantum group of symmetry with $q = e^{-\frac{i\pi}{2n-2+k}}$ and therefore the same group of S-matrix symmetry. The connection of the R-matrix commuting with the constructed coproduct with the known R-matrix solution for this affine group of symmetry [11] was established and possible ways of S-matrix solution for this model (kink-kink S-matrix) constructed on the base of Boltzmann weights for vector representation of $A_{2n-1}^{(2)}$-invariant RSOS model and we check them for two particular cases of $k$ - number of minimal model.

Before we present the conjectured S-matrix recall the expressions for the central charge of $W D_n^{(k)}$ minimal models ($\sim SO(2n)_k \times SO(2n)_1/SO(2n)_{k+1}$) and conformal dimension of our perturbing primary field, corresponding to the fundamental weight of vector representation of $D_n$. 

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\[ c = n \left( 1 - \frac{(2n-2)(2n-1)}{(2n-2+k)(2n-1+k)} \right) \quad (1) \]

\[ \Delta = 1 + \frac{(n-1+k)^2 + \sum_{k=2}^{n-2} k^2}{2(2n-2+k)(2n-1+k)} \quad (n \geq 4) \]

\[ \Delta = 1 + \frac{(2+k)^2}{2(4+k)(5+k)} \quad (n = 3) \]

Let us point here two remarkable facts which will be used and discussed later. In the limit \( k \to \infty \) conformal dimension of the perturbation is going to \( 1/2 \). Another feature is that for each \( n \) the central charge of the lowest minimal model \( k = 1 \) is equal to 1.

We will write down now the Boltzmann weights of \( A_{2n-1}^{(2)} \) RSOS model based on the realization for this algebra made by using the \( D_n \) loop algebra [12] (in contrast to \( C_n \) realized \( A_{2n-1}^{(2)} \) Boltzmann weights which was constructed in [13]). We will write them for the restricted model in the trigonometric limit. Let us fix some notations. \( \Lambda_i \ (0 \leq i \leq n) \) denote the fundamental weights of \( D_n^{(1)} \), and \( \rho = \Lambda_0 + ... + \Lambda_n \). Let \( \mathcal{A} \) be the set of weights in the vector representation of \( D_n \) and for \( a \in \mathcal{H} = \sum_{i=0}^{n} \mathbb{C} \Lambda_i \) we write \( \bar{a} \) to mean its classical part. In terms of the orthogonal vectors \( e_i \ (1 \leq i \leq n) \), \( (e_i, e_j) = \delta_{ij} \), \( e_{-i} = -e_i \) classical parts \( \bar{\Lambda}_i, \bar{\rho} \) and \( \mathcal{A} \) can be written as follows

\[ \bar{\Lambda}_i = e_1 + \cdots + e_i \quad (1 \leq i \leq n-2) \]

\[ \bar{\Lambda}_{n-1} = 1/2 (e_1 + \cdots + e_{n-1} - e_n) \]

\[ \bar{\Lambda}_n = 1/2 (e_1 + \cdots + e_{n-1} + e_n) \]

\[ a = (L-a_1-a_2-1)\Lambda_0 + \sum_{i=1}^{n-1} (a_i-a_{i+1}-1)\Lambda_i + (a_{n-1}+a_n-1)\Lambda_n, \quad (4) \]

\begin{align*}
L &> a_1 + a_2, \quad a_1 > a_2 > \cdots > a_n, \quad a_{n-1} + a_n > 0
\end{align*}

where \( a_i \in \mathbb{Z} \) or \( a_i \in \mathbb{Z} + \frac{1}{2} \) and \( L = 2n-2 + k, \quad (k = 1, 2, ...) \) - is the number of minimal unitary \( WD \) model which we perturbe. It can be easily seen that
\[
\bar{a} + \bar{\rho} = \sum_{i=1}^{n} a_i e_i, \quad a_\mu = \langle a + \rho, e_\mu \rangle, \quad -n \leq \mu \leq n.
\] (5)

It was shown in [12] that Boltzmann weights (here we write them in the trigonometric limit, while in [12] they are written in general elliptic form) for this RSOS model take the form

\[
[x] = \sin \omega x, \quad [x]_+ = \cos \omega x, \quad \omega = \frac{\pi}{L}
\]

\[
W_u \begin{pmatrix} a & a + e_\mu \\ a + e_\mu & a + 2e_\mu \end{pmatrix} = \frac{[1 + u][n + u]_+}{[1][n]_+} \quad (\mu \neq 0)
\] (6)

\[
W_u \begin{pmatrix} a & a + e_\mu \\ a + e_\mu & a + e_\mu + e_\nu \end{pmatrix} = \frac{[a_{\mu\nu} - u][n + u]_+}{[a_{\mu\nu}][n]_+} \quad (\mu \neq \pm \nu)
\]

\[
W_u \begin{pmatrix} a & a + e_\nu \\ a + e_\mu & a + e_\mu + e_\nu \end{pmatrix} = \left( \frac{[a_{\mu\nu} + 1][a_{\mu\nu} - 1]}{[a_{\mu\nu}]^2} \right)^{1/2} \frac{u[n + u]_+}{[1][n]_+} \quad (\mu \neq \pm \nu)
\]

\[
W_u \begin{pmatrix} a & a + e_\nu \\ a + e_\mu & a \end{pmatrix} = (G_{a,\mu} G_{a,\nu})^{1/2} \frac{u[a_{\mu-\nu} + 1 - n - u]_+}{[a_{\mu-\nu} + 1][n]_+} \quad (\mu \neq \nu)
\]

\[
W_u \begin{pmatrix} a & a + e_\mu \\ a + e_\mu & a \end{pmatrix} = \frac{[2a_\mu + 1 - u][n + u]_+}{[2a_\mu + 1][n]_+} + \frac{[u][2a_\mu + 1 - n - u]_+}{[2a_\mu + 1][n]_+} G_{a,\mu} \quad (\mu \neq 0)
\]

\[
= \frac{[2a_\mu + 1 - 2n - u][n - u]_+}{[2a_\mu + 1 - 2n][n]_+} - \frac{[u][2a_\mu + 1 - n - u]_+}{[2a_\mu + 1 - 2n][n]_+} H_{a,\mu}
\]

where \( a_{\mu\nu} = a_\mu - a_\nu, a_{\mu-\nu} = a_\mu + a_\nu \),

\[
G_{a,\mu} = G_{a + e_\mu} / G_a = \begin{cases} 
\Pi_{k \neq 0, \pm \mu} \frac{[a_{\mu k} + 1]}{[a_{\mu k}]} & \mu \neq 0 \\
1 & \mu = 0
\end{cases}
\] (7)

\[
G_a = \prod_{1 \leq i < j \leq n} [a_i - a_j][a_i + a_j]
\]

\[
H_{a,\mu} = \sum_{k \neq \mu} \frac{[a_\mu + a_k + 1 - 2n]}{[a_\mu + a_k + 1]} G_{a,k}
\]
Unitarity and crossing relations for these Boltzmann weights read as

\[
\sum_g W_u \begin{pmatrix} a & g \\ c & d \end{pmatrix} W_{-u} \begin{pmatrix} a & b \\ g & d \end{pmatrix} = \delta_{bc} \rho \left[ \eta u + \eta (1 - u) \right] + \left[ \eta 1 - \eta (1 - u) \right] = \delta_{bc} \rho(u) \tag{8}
\]

\[
W_u \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left( \frac{G_b G_c}{G_d G_a} \right)^{1/2} W_{k/2-1-u} \begin{pmatrix} c & a \\ d & b \end{pmatrix} \tag{9}
\]

We would like now to construct the S-matrix for kinks on the base of the solutions of Yang-Baxter equation written above. We denote a kink state by \( K_{ab}(\theta) \), where \( a \) and \( b \) are two vacua of the theory and \( \theta \) is the rapidity of the kink, and, according to the main feature of two dimensional integrable field theory, we need only to consider the S-matrix for the process \( K_{ac}(\theta_1) + K_{cd}(\theta_2) \rightarrow K_{ab}(\theta_2) + K_{bd}(\theta_1) \), since all the other S-matrix elements are determined in terms of these. The idea is well known \[5\][8]: we look for the S-matrix of the scattering process of kinks in the form

\[
S_u \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y(u) W_{\eta u} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( \frac{G_a G_d}{G_b G_c} \right)^{u/2} \tag{10}
\]

with some scalar function \( Y \) to be found, where \( u \) is connected to the rapidity difference of the incoming kinks \( \theta \) by \( u = \theta/\pi i \), and \( \eta \) – some constant.

The unitarity constraint can be satisfied by virtue of the relation \[8\] provided

\[
Y(u)Y(-u) = 1/\rho(\eta u) \tag{11}
\]

The crossing relation is satisfied provided \( \eta \) is equal to the crossing parameter \( \eta = k/2 - 1 \) and

\[
Y(u) = Y(1 - u) \tag{12}
\]
The nontrivial feature of the crossing parameter of the Boltzmann weights for $A_n^{(2)}$ algebras based on the orthogonal groups, is the presence of special point $k = 2$, when it is equal to zero. This property objects to a construction of physical scattering theory directly on the base of these Boltzmann weights for the perturbed $k = 2$ minimal model, although at the first glance there is no something special in the $k = 2$ case. Having no an answer on this question, we meanwhile are going to propose the fundamental S-matrix solution for all other $k$, supposing that $k = 2$ case is the point of “regime change” for the S-matrix.

3. The system of functional equations (11) and (12) can be solved by standard iteration procedure, which has the ambiguity of the first step giving rise to the well known CDD ambiguity of the solution. In what follows we will consider, without lost of generality, the particular case $n = 3$.

For the lowest minimal model $k = 1$ we chose the first ”test” function for the iteration as a product of two gamma functions divided by $\cos$, which gives the following solution for $Y$:

$$Y(u) = \frac{1 \Gamma(\frac{1}{5}(1 - u/2)) \Gamma(1 - \frac{1}{5}(1 + u/2))}{3 + u/2} \prod_{l=0}^{\infty} \frac{\Gamma(\frac{1}{5}(\frac{1}{2} - l + u/2)) \Gamma(\frac{1}{5}(-l - u/2)}{\Gamma(\frac{1}{5}(\frac{1}{2} - l - u/2)) \Gamma(\frac{1}{5}(-l + u/2)} (13)$$

This choice can be argued by the following check. One can see that the S-matrix of vector perturbed $WD_{n}^{(k)}$ minimal theories for $k = 1$ should take the form of the (nonrestricted) Sine-Gordon S-matrix at the special value of its coupling constant, since the central charge for this $k$ is equal to 1 for each $n$. Moreover, since the dimension of perturbation for $k = 1$ is inverse even number, we should expect to have SG S-matrix at reflectionless point. Indeed, in the case $k = 1$ and, for example, $n = 3$ the restriction condition (4) leaves only four possibilities for the choice of $(a_3, a_2, a_1)$: $(3, 1, 0)$; $(2, 1, 0)$; $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $(\frac{5}{2}, \frac{3}{2}, -\frac{1}{2})$. The first set of $a_i$ corresponds to the highest weight of vector $(v)$ representation of $D_3$, the second – of scalar $(0)$, and fourth and third – to the highest weights of two spinor representations $(s$ and $c$). Using the admissibility condition, we have the following four
nonzero Boltzmann weights in this case

\[
W_u \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad W_u \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}, \quad W_u \begin{pmatrix} s & c \\ c & s \end{pmatrix}, \quad W_u \begin{pmatrix} c & s \\ s & c \end{pmatrix}
\] (14)

All of them are representatives of the last type of nonzero Boltzmann weights in (13) and explicit calculation gives the same expression for all of them

\[
W(u) = \frac{[3 - u][1 + u]}{[1][3]}
\]

and for this case the crossing factor \((G_aG_d/G_bG_c)^{u/2}\) turns out to be equal to one for all of four types of S-matrix. Such a trivial tensor structure is compatible with the tensor structure of soliton-antisoliton SG S-matrix [14]

\[
S(\theta, \xi) = S_0(\theta, \xi)R(\theta, \xi) = S_0(\theta, \xi)
\]

\[
= \begin{pmatrix}
sh(\frac{\pi}{\xi}(\theta - i\pi)) & -sh(\frac{ix^2}{\xi}) & -sh(\frac{xi}{\xi}) & -sh(\frac{ix^2}{\xi}) \\
-sh(\frac{xi}{\xi}) & -sh(\frac{ix^2}{\xi}) & sh(\frac{\pi}{\xi}(\theta - i\pi)) \\
S(\theta, \xi) & S_R(\theta, \xi) & S_T(\theta, \xi) & S_0(\theta, \xi) \\
S_T(\theta, \xi) & S_R(\theta, \xi) & S_T(\theta, \xi) & S(\theta, \xi)
\end{pmatrix}
\] (15)

\[
S_0(\theta, \xi) = \frac{1}{sh(\frac{\pi}{\xi}(\theta - i\pi))} exp \left[-i \int_0^\infty \frac{dx}{x} \frac{\sin(x\theta)}{ch(\frac{\pi x}{\xi})} \frac{\sin(\frac{\pi x}{\xi})}{sh(\frac{3\pi x}{\xi})} \right]
\] (16)

at \(\xi = \pi/5\) reflectionless point. Moreover, full equivalence of these two S-matrices, including the infinite product of \(Y\) and exponential of integral of [16], also can be established.

(After we rewrite the cosines in the infinite product [13] as the product of two gamma
functions, partial cancelation of gamma functions takes place and we leave with the infinite CDD type product of 8 gamma functions, which, after some algebra, together with other factors, gives the $S_0$ of (16).

In the case $k > 2$ solution for $Y$ should be taken in the following form (as before we write the answer for $n = 3$ case)

$$
Y(u) = \frac{[1][3]_+}{[1 - \eta u][3 - \eta u]_+} \prod_{l=0}^{\infty} \frac{[2(l + 1) - k(l + \frac{1}{2}) - \eta u][2l + 3 - k(l + 1) + \eta u]}{[2(l + 1) - k(l + \frac{1}{2}) + \eta u][2l + 3 - k(l + 1) - \eta u]}
$$

This choice is dictated by the following argument. Since in the limit $k \to \infty$ the model under consideration takes the form of free fermions–$SO(2n)_1$ Kac-Moody perturbed by the field of conformal dimension $1/2$ (see (2)), we expect the trivial limit $(-1)$ for the S-matrix. The proposed solution for S-matrix seems to be the only one with this property. Now we will describe this limit of S-matrix.

First of all, as it was pointed out in [5], for the parameters $a_\mu$ this limit means that $a_\mu, a_{\mu\nu} \to \infty$, $a_\mu/k, a_{\mu\nu}/k \to 0$. One can show that the infinite product in $Y$ goes to 1 in the limit $k \to \infty$. It can easily be checked that the prefactor before the infinite product in $Y$ together with the five types of Boltzmann weights (5) gives the zero limit for all of them except for the first and the third one, for which the limit is equal to -1, giving the $-1$ limit for the S-matrix.

4. We expect the mass spectrum, particle content (higher kinks and breathers) and structure of full S-matrix to be rather complicated for general case. (Even the lowest model of general case ($n = 3, k = 3$) has 24 fundamental particles.) In the same way as it was shown in [13], one can show, that the problem of spectral decomposition for our R-matrix is equivalent to this problem for two arbitrary representations obtained by tensor product of vector representation of $D_n$ algebra, which is unknown in general case. Some examples of bootstrap for $SO(n)$ symmetric R-matrix in the simplest particular cases have been shown in [13] and led to a complicated picture.
Another interesting question is the understanding of $k = 2$ case and construction of the S-matrix for it. It was checked by [16] that the naive regularization of zero crossing parameter leads to a trivial S-matrix.

So needless to say that there are many points in the S-matrix construction which should be understood.

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