Tame and wild primes in direct products of commutative rings

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Abstract. In this article, new advances in understanding the structure (and cardinality) of prime ideals of an infinite direct product of rings are obtained. Then some applications are given. Especially among them, it is shown that if a direct product ring has a wild prime, then the set of its wild primes is infinite (uncountable). Next, the avoidance property of arbitrary direct products of avoidance rings is characterized. Finally, general new results toward computing the Zariski and flat closures of an “infinite” subset of the prime spectrum are established.

1. Introduction

If \( p \) is a prime number and \( n \geq 1 \), then \( \mathbb{Z}/p^n\mathbb{Z} \) is a local zero-dimensional ring (i.e., its prime spectrum is a singleton), but quite surprisingly the direct product ring \( \prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z} \) does not have finite Krull dimension, and also it has a huge number of prime ideals (the cardinality of its maximal ideals is uncountable and equals \( 2^\mathfrak{c} \) where \( \mathfrak{c} \) is the cardinality of the continuum). However, the structure of all prime ideals of this ring as well as \( \prod_{n \geq 1} \mathbb{Z} \) are not precisely understood yet. These examples show that fully understanding the structure of prime ideals of an infinite direct product of rings (even more specific rings such as \( \mathbb{Z} \) or \( \mathbb{Z}/p^n\mathbb{Z} \)) is a complicated problem (it is still unclear how prime ideals in an infinite direct product ring look like). In the present article, by using some algebro-topological methods, we make new advances in understanding such primes.

The following is a brief outline of the article. Extending an idea in our recent work \[12\] to the general case was the starting point of the present article. In fact, in the proof of Theorem 2.3 we applied a similar idea which was already used in the proof of \[12\] Theorem 3.10. Section 2 is devoted to investigate the structure, cardinality and various aspects and properties of the prime ideals of infinite direct products of rings. In a given direct product ring, we have precisely two quite different types of prime ideals: tame primes and wild primes. Unlike the tame primes in which their structure is completely understood, but understanding the structure of wild primes (e.g. describing all of them in terms of the other objects such as ultrafilters of the index set) is quite unknown and complicated, and every wild prime is highly non-constructive (for these reasons, we called them “wild primes”). For an infinite index set \( S \), we will observe that many (but not all) of the wild primes of an infinite direct product ring \( R = \prod_{i \in S} R_i \) can be described in...
terms of the non-principal ultrafilters of \( S \) (i.e., wild primes of the Boolean ring \( \mathcal{P}(S) \cong \prod_{k \in S} \mathbb{Z}_2 \)). Then as a consequence, we will observe that the number of wild primes of an infinite direct product of nonzero rings is very huge, at least it is of the second type uncountability (that is, strictly greater than of the cardinality of the continuum). In spite of the existence of so many wild primes, but since it will remain unclear forever how non-principal ultrafilters of \( S \) look like (even we will not be able to find an explicit example of a non-principal ultrafilter of the natural numbers), because the existence of them uses the axiom of choice and hence making it non-constructive, therefore constructing an explicit example of a wild prime of \( R \) will remain unknown as well. In fact, most of the mathematical objects which the axiom of choice has been involved in their existence, are non-constructive. It seems that in any (possibly future) description of the wild primes of an infinite direct product ring, the non-principal ultrafilters will be involved in some way. We also observe that in an infinite direct product ring \( R = \prod_{i \in S} R_i \), the direct sum ideal \( I = \bigoplus_{i \in S} R_i \) plays a vital role. Indeed, we reduce the problem of fully understanding the structure of prime ideals and many other properties of \( R \) to understand the same property for the quotient ring \( R/I \), or equivalently, for the closed subset \( V(I) \) of the prime spectrum. Then we investigate some properties of this quotient ring. Also, several interesting applications are presented.

In Section 3, we investigate the Zariski and flat closures of an “infinite” subset of the prime spectrum. Remember that for any ring \( R \), if \( E = \{p_1, \ldots, p_n\} \) is a finite set of prime ideals of \( R \), then the Zariski closure of \( E \) in \( \text{Spec}(R) \) equals \( \text{Im}(\pi^*) = \bigcap_{i=1}^n V(p_i) \) where \( \pi : R \to R/p_1 \times \cdots \times R/p_n \) is the canonical ring map.

Dually, the flat closure of \( E \) in \( \text{Spec}(R) \) equals \( \text{Im}(\pi^*) = \bigcup_{i=1}^n \Lambda(p_i) \) where \( \pi : R \to R_{p_1} \times \cdots \times R_{p_n} \) is the canonical ring map and \( \Lambda(p) = \{q \in \text{Spec}(R) : q \subseteq p\} \). But in the literature, there is no known general method (formula) to compute the Zariski and flat closures of an infinite subset of the prime spectrum. The situation in the infinite case is complicated, because of the existence of the wild primes. In this section, we establish general new results toward computing the Zariski and flat closures. Tame and wild primes play an important role in several places of this section.

2. TAME AND WILD PRIMES

In this article, all rings are commutative. If \( r \) is an element of a ring \( R \), then \( D(r) = \{p \in \text{Spec}(R) : r \notin p\} \) and \( V(r) = \text{Spec}(R) \setminus D(r) \). Every ring map \( \varphi : R \to R' \) induces a map \( \varphi^* : \text{Spec}(R') \to \text{Spec}(R) \) between the corresponding prime spectra which is given by \( p \mapsto \varphi^{-1}(p) \).

The following definition is the key concept of this article.

**Definition 2.1.** Let \((R_i)\) be a family of nonzero rings indexed by a set \( S \) and let \( R = \prod_{i \in S} R_i \) be their direct product ring. If \( p \) is a prime ideal of \( R_k \) for some \( k \in S \),
then we call $\pi_k^{-1}(p)$ a tame prime of $R$ where $\pi_k : R \to R_k$ is the projection map. By a wild prime of $R$ we mean a prime ideal of $R$ which is not tame, i.e., it is not of the form $\pi_k^{-1}(p)$.

It is obvious that $\pi_k^{-1}(p) = \prod_{i \in S} P_i$ where $P_k := p$ and $P_i = R_i$ for all $i \neq k$. If the index set $S$ is finite, then clearly every prime ideal is tame (i.e., there is no wild prime in this case).

**Remark 2.2.** Throughout this article, in order to avoid any possible confusion, for a direct product ring $R = \prod_{k \in S} R_k$ all of the factors $R_k$ are always assumed to be fixed. The notions of the tame prime and wild prime in a direct product ring $R$ are precisely dependent on the factors $R_k$. This means that, any change in the factors of $R$, then being the “tame prime” or “wild prime” of $R$ will also be changed accordingly.

**Theorem 2.3.** If the index set $S$ is infinite, then the set of tame primes of a ring $R = \prod_{i \in S} R_i$ is a non-affine open of $\text{Spec}(R)$.

**Proof.** Let $T$ be the set of tame primes of $R$. First we show that $T = \bigcup_{k \in S} D(e_k)$ where $e_k = (\delta_k)_{i \in S} \in R$ is the $k$-th unit vector and $\delta_k$ is the Kronecker delta. The inclusion $T \subseteq \bigcup_{k \in S} D(e_k)$ is obvious. To see the reverse inclusion, let $P$ be a prime ideal of $R$ with $P \in \bigcup_{k \in S} D(e_k)$. Then $e_k \not\in P$ for some $k$. It follows that $e_i \in P$ for all $i \neq k$, because $e_i e_k = 0$. We claim that $\pi_k(P)$ is a prime ideal of $R_k$. Clearly it is an ideal of $R_k$, since $\pi_k$ is surjective. It is also a proper ideal of $R_k$, because if $1_k \in \pi_k(P)$ then there exists some $a = (a_i) \in P$ such that $a_k = 1_k$, then $e_k = e_k a \in P$, a contradiction. Now suppose there are $a, b \in R_k$ such that $ab \in \pi_k(P)$. Then there exists some $x = (x_i) \in P$ such $x_k = ab$. Then consider the sequences $y = (y_i), z = (z_i)$ in $R$ where $y_i = 1$ and $z_i = x_i$ for all $i \neq k$ and $y_k = a$ and $z_k = b$. Then clearly $yz = x \in P$. Thus either $y \in P$ or $z \in P$. It follows that either $a \in \pi_k(P)$ or $b \in \pi_k(P)$. This establishes the claim. Now we show that $P = \pi_k^{-1}(q)$ where $q := \pi_k(P)$. The inclusion $P \subseteq \pi_k^{-1}(q)$ is obvious. If $r = (r_i) \in \pi_k^{-1}(q)$ then $r_k \in q$. Then clearly $r_k \in q$. So there exists some $b \in \pi_k(P)$ such that $b_k = r_k$. Then $e_k r = e_k b \in P$. Hence, $r \in P$. Thus $P$ is a tame prime of $R$, i.e., $P \in T$. Therefore $T = \bigcup_{k \in S} D(e_k)$ is an open subset of $\text{Spec}(R)$. To see that $T$ is non-affine it suffices to show that it is not quasi-compact. If $T$ is quasi-compact, then there exists a natural number $n \geq 1$ such that $T = \bigcup_{k=1}^n D(e_{i_k}) = D(e)$ where $e := \sum_{k=1}^n e_{i_k}$. Since the index set $S$ is infinite, we may choose some $d \in S \setminus \{ i_1, \ldots, i_n \}$. Then $e_d e = 0$ and so $D(e_d) = D(e_d e) = D(0)$. It follows that $e_d = 0$ which is a contradiction. 

**Corollary 2.4.** The set of wild primes of a ring $R = \prod_{i \in S} R_i$ is the closed subset $V(I)$ of $\text{Spec}(R)$ where $I = \bigoplus_{i \in S} R_i$.

**Proof.** Let $X$ be the set of wild primes of $R$. Then by Theorem 2.3 and its proof, $X = \bigcap_{i \in S} V(e_i) = V(I)$ where $I = \{ e_i : i \in S \} = \bigoplus_{i \in S} R_i$. 

\qed
Remark 2.5. Regarding with the above result, note that if the index set $S$ is infinite then the ideal $I$ has a particular property: $R/I$ is a flat $R$-module which is not $R$-projective. If $p$ is a wild prime and $q$ is a tame prime of $R$, then $p + q = R$. In particular, $V(I)$ is stable under generalization. That is, if $p \in V(I)$ and $q$ is a prime ideal of $R$ with $q \subseteq p$, then $q \in V(I)$. If $p \in \text{Spec}(R) \setminus V(I)$, then $p$ is a tame prime and its structure is precisely determined. In fact, $p = \prod_{i \in I} P_i$ where $P_k := \pi_k(p)$ is a prime ideal of $R_k$ for some $k$ and $P_i = R_i$ for all $i \neq k$. In this case, we have the canonical isomorphisms of rings $R/p \simeq R_k/P_k$ and $R_p \simeq (R_k)_{P_k}$.

If for each $k \in S$, $I_k$ is an ideal of a ring $R_k$, then $\bigoplus_{k \in S} I_k \subseteq \prod_{k \in S} I_k$ are ideals of $R = \prod_{k \in S} R_k$, and the quotient ring $R/\prod_{k \in S} I_k$ is canonically isomorphic to $\prod_{k \in S} R_k/I_k$.

If $\varphi : A = \prod_{i \in S} A_i \rightarrow B = \prod_{i \in S} B_i$ is the ring map induced by a family of ring maps $(\varphi_i : A_i \rightarrow B_i)$, then $\text{Ker}(\varphi) = \prod_{i \in S} \text{Ker}(\varphi_i)$ and $\text{Im}(\varphi) = \prod_{i \in S} \text{Im}(\varphi_i)$. In particular, $\varphi$ is injective if and only if each $\varphi_i$ is injective. A similar assertion holds for surjectivity, i.e., $\varphi$ is surjective if and only if each $\varphi_i$ is surjective.

Corollary 2.6. Let $\varphi : A = \prod_{i \in S} A_i \rightarrow B = \prod_{i \in S} B_i$ be the ring map induced by a family of ring maps $(\varphi_i : A_i \rightarrow B_i)$. Then a prime ideal $p$ of $B$ is a wild prime if and only if $\varphi^{-1}(p)$ is a wild prime of $A$.

Proof. If $p$ is a wild prime of $B$, then by Corollary 2.4 $\bigoplus_{i \in S} B_i \subseteq p$. It is obvious that the image of each $k$-th unit vector $e_k \in A$ under $\varphi$ is the corresponding $k$-th unit vector of $B$, i.e., $\varphi(e_k) = e_k'$. This yields that $\bigoplus_{i \in S} A_i \subseteq \varphi^{-1}(\bigoplus_{i \in S} B_i) \subseteq \varphi^{-1}(p)$.

Then, again by Corollary 2.4 $\varphi^{-1}(p)$ is a wild prime of $A$. Conversely, suppose $p$ is a tame prime of $B$. So $p = \pi_k^{-1}(p_k)$ where $p_k$ is a prime ideal of $B_k$ for some $k$. Then using the following commutative diagram:

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow \pi_k & & \downarrow \pi_k \\
A_k & \rightarrow & B_k \\
\varphi & & \varphi_k
\end{array}
$$

we get that $\varphi^{-1}(p) = (\pi_k')^{-1}(\varphi_k^{-1}(p_k))$ is a tame prime of $A$ which is a contradiction. □

Definition 2.7. For given direct product rings $A = \prod_{i \in S} A_i$ and $B = \prod_{k \in S'} B_k$ we say that a map $\text{Spec}(A) \rightarrow \text{Spec}(B)$ precisely preserves wilds if for a prime ideal $p$ of $A$, it is a wild prime of $A$ if and only if its image is a wild prime of $B$.

As an example, the map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ induced by $\varphi$ (of Corollary 2.4) precisely preserves wilds. The map $\text{Spec}(R_k) \rightarrow \text{Spec}(R)$ induced by the projection $R = \prod_{i \in S} R_i \rightarrow R_k$ is also an example of this type (provided that we do not consider $R_k$ as any infinite direct product ring, cf. Remark 2.2). Note that, under the assumptions of the above definition, then a map $\text{Spec}(A) \rightarrow \text{Spec}(B)$ precisely preserves wilds if and only if it precisely preserves tame primes (in a similar sense).
Example 2.8. Here we give two examples of maps which do not precisely preserve wilds. Assume the index set $S$ is infinite, so we may choose a wild prime $q$ in a ring $R = \prod_{i \in S} R_i$. Consider the canonical ring map $\varphi : R \to R' = \prod_{p \in \text{Spec}(R)} R/p$ given by $r \mapsto (r + p)_{p \in \text{Spec}(R)}$. Now the map $\text{Spec}(R') \to \text{Spec}(R)$ induced by $\varphi$ does not precisely preserve wilds, because $P := \pi^{-1}(0)$ is a tame prime of $R'$ where $\pi : R' \to R/q$ is the projection map, but $\varphi^{-1}(P) = q$ since $\pi \varphi : R \to R/q$ is the canonical map. As a second example, consider the (injective) canonical ring map $\psi : R \to R' = \prod_{p \in \text{Spec}(R)} R_p$ given by $r \mapsto (r/p)_{p \in \text{Spec}(R)}$. Then the map $\text{Spec}(R') \to \text{Spec}(R)$ induced by $\psi$ does not precisely preserve wilds, because $P := \pi^{-1}(qR_q)$ is a tame prime of $R'$ where $\pi : R' \to R_q$ is the projection map, but $\psi^{-1}(P) = q$ since $\pi \psi : R \to R_q$ is the canonical map. In this regard, see also Theorem 3.7.

For a set $S$ by $\mathcal{P}(S) \simeq \prod_{k \in S} \mathbb{Z}_2$ we mean the power set ring of $S$ which is a Boolean ring where $\mathbb{Z}_2 = \{0, 1\}$.

Theorem 2.9. For a ring $R = \prod_{i \in S} R_i$ we have a continuous imbedding $\text{Spec}(\mathcal{P}(S)) \to \text{Spec}(R)$ which precisely preserves wilds.

Proof. Each $R_i$ is a nonzero ring, so it has at least a prime ideal $p_i$. For any sequence $a = (a_i) \in R = \prod_{i \in S} R_i$ setting $a^* := \{i \in S : a_i \in p_i\}$. If $M$ is a prime (maximal) ideal of the power set ring $\mathcal{P}(S)$, or equivalently, $\mathcal{P}(S) \setminus M$ is an ultrafilter of $S$, then we first show that $M^* := \{a \in R : a^* \notin M\}$ is a prime ideal of $R$. For the zero sequence we have $0^* = S \notin M$ thus $0 \in M^*$, and so $M^*$ is nonempty. If $a, b \in M^*$ then $a + b \in M^*$, because $a^* \cap b^* \subseteq (a + b)^*$ and so $(a + b)^* \notin M$. If $r \in R$ and $a \in M^*$ then $ra \in M^*$, since $a^* \subseteq r^* \cup a^* = (ra)^*$ and so $(ra)^* \notin M$. Thus $M^*$ is an ideal of $R$. Now, suppose $ab \in M^*$ for some $a, b \in R$. If none of $a$ and $b$ is a member of $M$, then $a^* \in M$ and $b^* \in M$, and so $(ab)^* = a^* \cup b^* = a^* + b^* + a^* \cap b^* \in M$, a contradiction. Hence, $M^*$ is a prime ideal of $R$. Next we show that the map $\text{Spec}(\mathcal{P}(S)) \to \text{Spec}(R)$ given by $M \mapsto M^*$ is injective. Suppose $M^* = N^*$. If $A \in M$ then consider the sequence $a = (a_i)$ where $a_i = 1$ for all $i \in A$ and otherwise $a_i = 0$. Then $a^* = S \setminus A \notin M$ thus $a^* \notin N^*$ and so $a^* \notin N$. This yields that $A \in N$. It follows that $M \subseteq N$ and so $M = N$. This map is also continuous, because the inverse image of $D(a)$ under this map equals $V(a^*) = D(1 - a^*)$. Finally, we show that this map precisely preserves wilds. If $M$ is a wild prime (i.e., non-principal maximal ideal) of $\mathcal{P}(S)$, then $\{k\} \in M$ for all $k \in S$. For the $k$-th unit vector we have $(e_k)^* = S \setminus \{k\} \notin M$ thus $e_k \in M^*$ and so $\bigoplus_{k \in S} R_k \subseteq M^*$. Thus by Corollary 2.4, $M^*$ is a wild prime of $R$. Conversely, let $M^*$ be a wild prime of $R$ where $M$ is a maximal ideal of $\mathcal{P}(S)$. If $M$ is a tame prime of $\mathcal{P}(S)$ then $M = \mathcal{P}(S \setminus \{k\})$ for some $k \in S$. It follows that $M^* = \pi_k^{-1}(p_k)$ which is a contradiction. \[\square\]

Remark 2.10. For any infinite cardinal $\kappa$, it is well known that $\kappa \cdot \kappa = \kappa$ (in a sense, every infinite cardinal is an idempotent). This shows that for any cardinals $\alpha$ and $\beta$, if one of them is infinite then $\alpha + \beta \leq \alpha \cdot \beta = \max\{\alpha, \beta\}$. 
The following result shows that the set of wild primes of a direct product ring is either empty or uncountable, according to whether the index set is finite or infinite.

**Corollary 2.11.** For a ring $R = \prod_{i \in S} R_i$ the following statements are equivalent.

(i) $R$ has a wild prime.

(ii) The index set $S$ is infinite.

(iii) The set of wild primes of $R$ has the cardinality $\geq 2^{2^{|S|}}$.

(iv) The set of tame primes of $R$ is a non-affine open of $\text{Spec}(R)$.

(v) The direct sum ideal $\bigoplus_{i \in S} R_i$ is a proper ideal of $R$.

**Proof.** The implications (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(i), (iv)$\Rightarrow$(ii) and (ii)$\Leftrightarrow$(v) are clear.

(ii)$\Rightarrow$(iii): It is well known that $\text{Spec}(\mathcal{P}(S))$ is the Stone-Čech compactification of the discrete space $S$. Therefore by \cite[Theorem 3.58]{K} or by \cite[Theorem on p.71]{K}, $|\text{Spec}(\mathcal{P}(S))| = 2^{2^{|S|}}$. It is clear that the set of tame primes of $\mathcal{P}(S)$ has the cardinality $|S|$. By Cantor’s theorem: $|S| < 2^{|S|} < 2^{2^{|S|}}$, and using Remark 2.10 we obtain that the set of wild primes of $\mathcal{P}(S)$ has the cardinality $2^{2^{|S|}}$. Then by using Theorem 2.14 the desired conclusion is deduced.

(ii)$\Rightarrow$(iv): See Theorem 2.3.

**Example 2.12.** For a ring $R = \prod_{k \in S} R_k$ the map $\text{Spec}(R) \to \text{Spec}(\mathcal{P}(S))$ given by $p \mapsto M_p := \{A \in \mathcal{P}(S) : e_A \in p\}$ is continuous and precisely preserves wilds where $e_A = (r_k) \in R$ with $r_k = 1$ for $k \in A$ and otherwise $r_k = 0$. But this map is not necessarily injective, because for finite $S$ it is clear, if $S$ is infinite and the map is injective then by Corollary 2.11 the cardinality of $\text{Spec}(R)$ will be $2^{2^{|S|}}$ which is impossible, since by \cite[Lemma 4.1]{K} we may choose the cardinality of $\text{Spec}(R_k) > 2^{2^{|S|}}$ for some $k$. This map is also a left inverse of the map of Theorem 2.14.

**Remark 2.13.** Note that if $\varphi : A \to B$ is a surjective ring map and $\varphi^{-1}(E)$ is a (prime) ideal of $A$ for some subset $E$ of $B$, then $E$ is a (prime) ideal of $B$. Every minimal (resp. maximal) wild prime of a ring $R = \prod_{i \in S} R_i$ is actually a minimal prime (resp. maximal) ideal of $R$. The same holds for minimal (resp. maximal) tame primes. In fact, if $p = \pi_k^{-1}(p_k)$ is a tame prime of $R$ for some (prime) ideal $p_k$ of $R_k$, then $p$ is a minimal prime of $R$ if and only if $p_k$ is a minimal prime of $R_k$. Similarly, $p$ is a maximal ideal of $R$ if and only if $p_k$ is a maximal ideal of $R_k$.

Also, $p$ is a finitely generated ideal if and only if $p_k$ is a finitely generated ideal.

**Theorem 2.14.** If $S$ is infinite and each $R_k$ is an integral domain, then the set of minimal wild primes of $R = \prod_{k \in S} R_k$ has the cardinality $2^{2^{|S|}}$.

**Proof.** It can be shown that $\text{Min}(R)$, the space of minimal prime ideals of $R$ is the Stone-Čech compactification of the discrete space $S$. For its proof see \cite[Theorem 3.5]{K}. Therefore by \cite[Theorem 3.58]{K} or by \cite[Theorem on p.71]{K}, $\text{Min}(R)$ has the cardinality $2^{2^{|S|}}$. It is clear that the set of minimal tame primes of $R$ has the cardinality $|S|$. Then by using Cantor’s Theorem and Remark 2.10 we get that the set of minimal wild primes of $R = \prod_{k \in S} R_k$ has the cardinality $2^{2^{|S|}}$. \hfill $\Box$
Theorem 2.15. If $S$ is infinite and each $R_k$ is a local ring, then the set of maximal wild primes of $R = \prod_{k \in S} R_k$ has the cardinality $2^{2^{|S|}}$.

Proof. It can be shown that $\text{Max}(R)$, the space of maximal ideals of $R$ is the Stone-Čech compactification of the discrete space $S$. For its proof see [7, Theorem 5.4]. Then the remainder of the argument is exactly as the above proof, with taking into account that the set of maximal tame primes of $R$ has the cardinality $|S|$.

In Theorem 2.16 the avoidance property of an arbitrary direct product of avoidance rings is characterized. First recall from [5] that an ideal $I$ of a ring $R$ has the ideal avoidance property or simply has avoidance (or, u-ideal) if whenever $I_1, \ldots, I_n$ are finitely many ideals of $R$ with $I \subseteq \bigcup_{k=1}^{n} I_k$, then $I \subseteq I_k$ for some $k$. For example, every principal ideal has avoidance. It also can be shown that every idempotent ideal has avoidance. If every ideal of a ring $R$ has avoidance, then $R$ is called an avoidance ring (or, u-ring). If $R$ is an avoidance ring, then every quotient or localization of $R$ is also an avoidance ring. The class of avoidance rings is broad. For instance, every ring containing an infinite field as a subring is an avoidance ring. Every principal ideal ring is an avoidance ring. Every Dedekind domain and more generally every Prufer domain is an avoidance ring. Every valuation ring has the avoidance property. Every Bezout ring (i.e., each finitely generated ideal is principal) and hence every Boolean ring and more generally every absolutely flat (von Neumann regular) ring is an avoidance ring. If $K$ is a finite field, then $K[x, y]$ is not an avoidance ring.

Theorem 2.16. Let $(R_i)_{i \in S}$ be a family of avoidance rings. If $R = \prod_{i \in S} R_i$ modulo the ideal $I = \bigoplus_{i \in S} R_i$ is an avoidance ring, then $R$ is an avoidance ring.

Proof. It will be enough to show that $R$ satisfies the condition (c) of the technical result [6, Theorem 2.6] which asserts that a ring $R$ is an avoidance ring if and only if for each maximal ideal $M$ of $R$, either the field $R/M$ is infinite or $R_M$ is a Bezout ring. Let $M$ be a maximal ideal of $R$. If $I \subseteq M$ then by hypothesis and [5, Theorem 2.6], either the field $(R/I)/(M/I) \simeq R/M$ is infinite or the localization $(R/I)_{M/1} \simeq (R/I)_M \simeq R_M/IR_M$ is a Bezout ring. Clearly $I$ is generated by the idempotents $e_k$ with $k \in S$. So its extension $IR_M$ is generated by the elements $e_k/1$. Since each $1 - e_k \in R \setminus M$, so $e_k/1 = 0$. Thus $IR_M = 0$. Hence, $R$ satisfies the condition (c) in this case. Now assume $M$ does not contain $I$. Thus by Theorem 2.3 (or, by Corollary 2.4), $M$ is a tame prime of $R$. So there exists some $k \in S$ such that $M = \prod M_i$ where $M_k = \pi_k(M)$ is a maximal ideal of $R_k$ and $M_i = R_i$ for all $i \neq k$. Now if $R/M \simeq R_k/M_k$ is a finite field, then by using [6, Theorem 2.6] for the avoidance ring $R_k$, we have $R_M \simeq (R_k)_{M_k}$ is a Bezout ring.

One can easily generalize the avoidance property to the setting of modules: we say that an $R$-module $M$ has the avoidance property or simply has avoidance if $M$ is equal to a finite union of $R$-submodules must be equal to one of them. For example, every invertible fractional ideal has avoidance. By a um-ring (see also [5]) we mean a ring $R$ such that every $R$-module has avoidance. For example, every ring containing an infinite field as a subring is a um-ring. Every um-ring is an
avoidance ring, but its converse does not hold. For example, the ring of integers $\mathbb{Z}$ is an avoidance ring which is not a um-ring.

**Theorem 2.17.** Let $(R_i)$ be a family of um-rings. If $R = \prod_i R_i$ modulo the ideal $I = \bigoplus_i R_i$ is a um-ring, then $R$ is a um-ring.

**Proof.** It is proved exactly like Theorem 2.16 by applying [5] Theorem 2.3 instead of [5] Theorem 2.6. \qed

By Theorem 2.16 every finite product of avoidance rings is an avoidance ring. Similarly, by Theorem 2.17 every finite product of um-rings is a um-ring. Of course, one can prove these observations directly by more elementary methods.

Our next goal is to investigate nilpotents, Krull dimensions and wild primes of infinite products. First note that the Jacobson radical is well behaved with the direct products. More precisely, for a ring $R = \prod_k R_k$, we have $\mathfrak{J}(R) = \prod_k \mathfrak{J}(R_k)$.

But the nil-radical is not well behaved with the direct products (which leads to some interesting observations). In fact, $\mathfrak{N}(R) \subseteq \prod_k \mathfrak{N}(R_k)$. Also $\bigoplus_k \mathfrak{N}(R_k) \subseteq \mathfrak{N}(R)$.

These inclusions may be strict. For example, in the ring $\prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$ with $p$ a prime number, the sequence $(a_1, a_2, a_3, \ldots)$ is a member of $\prod_{n \geq 1} \mathfrak{N}(\mathbb{Z}/p^n\mathbb{Z})$ but it is not nilpotent where each $a_n = p + p^n\mathbb{Z}$. To see the strictness of the second inclusion, consider the sequence $b = (b_n)$ with $b_1 = 0$ and $b_n = p^{n-1} + p^n\mathbb{Z}$ for all $n \geq 2$, then $b^2 = 0$ but $b \notin \bigoplus_{n \geq 1} \mathfrak{N}(\mathbb{Z}/p^n\mathbb{Z})$.

Note that for any ring $R$, the minimal spectrum $\operatorname{Min}(R)$ is Zariski dense in $\operatorname{Spec}(R)$. Also note that if $Y$ is a dense subspace of a topological space $X$, then a subset of $Y$ is dense in $Y$ if and only if it is dense in $X$. We have then the following result.

**Theorem 2.18.** For a ring $R = \prod_k R_k$ the following assertions are equivalent.

(i) $\mathfrak{N}(R) = \prod_k \mathfrak{N}(R_k)$.

(ii) The set of minimal tame primes of $R$ is Zariski dense in $\operatorname{Spec}(R)$.

**Proof.** It is straightforward, but we provide a proof for the sake of completeness.

(i)$\Rightarrow$(ii): It suffices to show that every nonempty Zariski open $U$ of $\operatorname{Spec}(R)$ meets the set of minimal tame primes of $R$. Since $U$ is nonempty, so there exists some $a = (a_k) \in R$ such that $\emptyset \neq D(a) \subseteq U$. This shows that $a$ is not nilpotent. Thus by hypothesis, there exists some $k$ such that $a_k \notin \mathfrak{N}(R_k)$. So $a_k \notin p$ for some minimal prime ideal $p$ of $R_k$. Then $\pi_k^{-1}(p)$ is a minimal tame prime of $R$ with $\pi_k^{-1}(p) \in D(a)$.

(ii)$\Rightarrow$(i): Take $a = (a_k) \in \prod_k \mathfrak{N}(R_k)$. If $a$ is not nilpotent, then $D(a)$ is nonempty. Then by hypothesis, there is a minimal tame prime $\pi_k^{-1}(p)$ of $R$ such that $\pi_k^{-1}(p) \in D(a)$ where $p$ is a (minimal) prime ideal of $R_k$ for some $k$. It follows that $a_k \notin p$ which is a contradiction, because $a_k$ is nilpotent. \qed

The following result is well known. The equivalence (i)$\iff$(ii) can be found in [4] Proposition 2.6. We give a new proof for this equivalence.
Theorem 2.19. If each $R_i$ is a zero-dimensional ring, then for $R = \prod_{i \in S} R_i$ the following assertions are equivalent.

(i) $\dim(R) = 0$.
(ii) $\mathfrak{M}(R) = \prod_{i \in S} \mathfrak{M}(R_i)$.
(iii) $\dim(R)$ is finite.

Proof. (i)$\Rightarrow$(ii): By hypothesis, $\mathfrak{M}(R) = \mathfrak{J}(R) = \prod_{i \in S} \mathfrak{J}(R_i) = \prod_{i \in S} \mathfrak{M}(R_i)$.

(ii)$\Rightarrow$(i): For any ring $R$, we have $\dim(R) = \dim(R/\mathfrak{M}(R))$. By hypothesis, the ring $R/\mathfrak{M}(R)$ is canonically isomorphic to $R' := \prod_{i \in S} R_i/\mathfrak{M}(R_i)$. But each $R_i/\mathfrak{M}(R_i)$ is a reduced zero-dimensional ring, and so it is a von Neumann regular ring. It is easy to see that the every direct product of von Neumann regular rings is von Neumann regular, and every von Neumann regular ring is zero-dimensional. Thus, $\dim(R) = \dim(R') = 0$.

(i)$\Rightarrow$(iii): There is nothing to prove.

(iii)$\Rightarrow$(ii): See [2, Theorem 3.4].

Corollary 2.20. If a ring $R$ has a principal maximal ideal $Rx$ such that $x$ is a non zerodivisor of $R$, then the Krull dimensions of $\prod_{n \geq 1} R/(x^n)$ and $\prod_{n \geq 1} R$ are not finite.

Proof. Each $a_n := x + (x^n) \in R/(x^n)$ is nilpotent, whereas the sequence $a = (a_1, a_2, a_3, \ldots)$ is not nilpotent, because if $a^d = 0$ for some $d \geq 1$, then $x^d \in (x^{d+1})$ and so $x^d = rx^{d+1}$ for some $r \in R$, but $x^d$ is non zerodivisor so we get that $x$ is an invertible in $R$ which is a contradiction. Also each $R/(x^n)$ is zero-dimensional. Thus by Theorem 2.19, the Krull dimension of $\prod_{n \geq 1} R/(x^n)$ is not finite. The canonical surjective ring map $\prod_{n \geq 1} R \rightarrow \prod_{n \geq 1} R/(x^n)$ induces an injective map between the corresponding prime spectra. It follows that the Krull dimension of $\prod_{n \geq 1} R$ is not finite.

Under the hypothesis of the above result, the ring $\prod_{n \geq 1} R/(x^n)$ and so by Corollary 2.6, the ring $\prod_{n \geq 1} R$ have strict chains of wild primes of any finite length. Also, the Krull dimensions of $\prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$ and $\prod_{n \geq 1} \mathbb{Z}$ as well as $\prod_{n \geq 1} K[x]/(x^n)$ and $\prod_{n \geq 1} K[x]$ are not finite where $K$ is a field.

Remark 2.21. In Theorem 2.9, the structure of so many (uncountable number of) wild primes of an infinite direct product ring $R = \prod_{i \in S} R_i$ are precisely described in terms of the non-principal ultrafilters of $S$ (i.e., wild primes of the power set ring of $S$) and a given family $(p_i)$ of prime ideals of $(R_i)$. But note that every wild prime of $R$ does not fall into this description. More precisely, the proof of Theorem 2.9 shows that every tame prime of $R$ is of the form $M^*$. Also, a maximal ideal $M$ of $\mathcal{P}(S)$ is principal if and only if $M^*$ is a tame prime of $R$. But every wild prime of $R$ is not necessarily of the form $M^* = M[p_i]$ with $M$ a maximal ideal of $\mathcal{P}(S)$ and with $(p_i)$ a family of prime ideals of $(R_i)$. For example, consider $R = \prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$,
then each $a_n = p + p^n \mathbb{Z} \in \mathbb{Z}/p^n \mathbb{Z}$ is nilpotent, so for every maximal ideal $M$ of $\mathcal{P}(S)$ with $S = \{1, 2, 3, \ldots\}$ then the sequence $(a_1, a_2, a_3, \ldots)$ is a member of $M^*$, but it is not nilpotent, hence not contained in some prime ideal $p$ of $R$. So $p$ is a wild prime and it is not of the form $M^*$.

For each $k \in S$, let $p_k$ be a fixed prime ideal of a ring $R_k$ and let $T$ be the set of all sequences $r = (r_k)$ in $R = \prod_{k \in S} R_k$ such that the set $\Omega(r) = \{k \in S : r_k \notin p_k\}$ is cofinite (i.e., its complement $\Omega(r)^c = S \setminus \Omega(r)$ is finite). It is clear that $\Omega(ab) = \Omega(a) \cap \Omega(b)$ for all $a, b \in R$. In particular, $T$ is a multiplicative subset of $R$. For each $k \in S$, $1 - e_k \in T$. So if $p$ is a prime ideal of $R$ with $p \cap T = \emptyset$, then $p$ is a wild prime of $R$.

In the following results, by $U$ we mean the set of all $r = (r_k)$ in $R = \prod_{k \in S} R_k$ such that its support $S(r) = \{k \in S : r_k \neq 0\}$ is cofinite.

**Lemma 2.22.** If each $R_k$ is an integral domain, then the ring $R = \prod_{k \in S} R_k$ modulo $\bigoplus_{k \in S} R_k$ can be canonically imbedded in $U^{-1}R$.

**Proof.** We observed that $U$ is a multiplicative subset of $R$. We show that $\ker \pi = \bigoplus_{k \in S} R_k$ where $\pi : R \to U^{-1}R$ is the canonical ring map. If $a \in \ker \pi$ then $ab = 0$ for some $b \in U$. This yields that $S(a) \subseteq S(b)^c$ which is finite. Thus $a \in I := \bigoplus_{k \in S} R_k$.

To see the reverse inclusion, take $a \in I$. Then consider the sequence $b = (b_k) \in R$ such that $b_k$ is either 0 or 1, according to whether $k \in S(a)$ or $k \notin S(a)$. Then clearly $ab = 0$, and $b \in U$ because $S(b)^c = S(a)$ is finite. So we obtain an injective morphism of rings $\varphi : R/I \to U^{-1}R$ given by $a + I \mapsto a/1$. \(\square\)

**Remark 2.23.** Remember that if $\varphi : A \to B$ is an injective ring map and $p$ is a minimal prime ideal of $A$, then there exists a prime ideal $q$ of $B$ which lying over $p$, i.e., $p = \varphi^{-1}(q)$. Indeed, consider the following commutative (pushout) diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\pi} & & \downarrow{\pi} \\
A_p & \to & A_p \otimes_A B
\end{array}
\]

where $\pi$ and the unnamed arrows are the canonical maps. Clearly $A_p \otimes_A B \simeq B_p$ is a nonzero ring, since $\varphi$ is injective. So it has a prime ideal $P$. The contraction of $P$ under the canonical ring map $A_p \to A_p \otimes_A B$ equals $pA_p$, because $p$ is a minimal prime of $A$. Let $q$ be the contraction of $P$ under the canonical ring map $B \rightarrow A_p \otimes_A B$ which is a prime ideal of $B$. Now we have $\varphi^{-1}(q) = \pi^{-1}(pA_p) = p$.

**Corollary 2.24.** If each $R_k$ is an integral domain and $p$ is a minimal wild prime of $R = \prod_{k \in S} R_k$, then $p \cap U = \emptyset$.

**Proof.** By Lemma 2.22, we have an injective ring map $\varphi : R/I \to U^{-1}R$ where $I = \bigoplus_{k \in S} R_k$. Clearly $p/I$ is a minimal prime ideal of $R/I$. So by Remark 2.23 there exists a prime ideal $q$ of $R$ such that $q \cap U = \emptyset$ and $p/I = \varphi^{-1}(U^{-1}q)$. But
the composition of $\varphi$ with the canonical map $R \to R/I$ gives us the canonical map $R \to U^{-1}R$. It follows that $p = q$.

As a second proof, suppose $a \in p \cap U$. It is well known that $ab$ is nilpotent for some $b \in R \setminus p$. It follows that $ab = 0$, because $R$ is a reduced ring. Thus $S(b) \subseteq S \setminus S(a)$ which is finite. This shows that $b \in I \subseteq p$ which is a contradiction. \qed

In the following two results, recall that $T$ denotes the set of all $r = (r_k)$ in $R = \prod_{k \in S} R_k$ such that $\Omega(r) = \{k \in S : r_k \notin m_k\}$ is cofinite.

**Theorem 2.25.** If each $R_k$ is a local ring with the maximal ideal $m_k$, then the ring $R = \prod_{k \in S} R_k$ modulo $\bigoplus_{k \in S} R_k$ is canonically isomorphic to $T^{-1}R$.

**Proof.** First we show that $\text{Ker } \pi = \bigoplus_{k \in S} R_k$ where $\pi : R \to T^{-1}R$ is the canonical ring map. If $a \in \text{Ker } \pi$ then $ab = 0$ for some $b \in T$. This yields that $S(a) = \{k \in S : a_k \neq 0\} \subseteq \Omega(b)^c$ which is finite. So $a \in I := \bigoplus_{k \in S} R_k$. To see the reverse inclusion, take $a \in I$. Then consider the sequence $b = (b_k) \in R$ such that $b_k$ is either 0 or 1, according to whether $k \in S(a)$ or $k \notin S(a)$. Then clearly $ab = 0$, and $b \in T$ because $\Omega(b)^c = S(a)$ is finite. Thus we obtain an injective morphism of rings $\varphi : R/I \to T^{-1}R$ given by $a + I \mapsto a/1$. The image of each $b = (b_k) \in T$ under the canonical ring map $R \to R/I$ is invertible, because consider the sequence $b' = (b'_k) \in R$ where $b'_k := b_k^{-1}$ if $k \in \Omega(b)$ and otherwise $b'_k := 0$, then $1 - bb' = \sum_{k \in \Omega(b)} c_k \in I = \text{Ker } \pi$. Thus $c(1 - bb') = 0$ for some $c \in T$. It follows that $\varphi(ab') = a/b$. Hence, $\varphi$ is surjective. \qed

**Corollary 2.26.** If each $(R_k, m_k)$ is a local ring, then a prime ideal $p$ of $R = \prod_{k \in S} R_k$ is a wild prime if and only if $p \cap T = \emptyset$.

**Proof.** The implication “$\Rightarrow$” is an immediate consequence of the above result. The reverse implication holds more generally. \qed

The following result also immediately follows from the above theorem.

**Corollary 2.27.** If each $R_k$ is a field, then the ring $R = \prod_{k \in S} R_k$ modulo $\bigoplus_{k \in S} R_k$ is canonically isomorphic to $U^{-1}R$.

3. Zariski and flat closures

First remember that for any ring $R$, there exists a (unique) topology over $\text{Spec}(R)$ such that the collection of $D(a) \cap V(b)$ with $a, b \in R$ forms a sub-basis for its opens. This topology is called the patch (or, constructible) topology. It follows that the collection of $D(a) \cap V(I)$ with $a \in R$ and $I$ a finitely generated ideal of $R$ forms a basis for the opens of the patch topology. There exists also a (unique) topology over $\text{Spec}(R)$ such that the collection of $V(a)$ with $a \in R$ forms a sub-basis for its opens. It is called the flat (or, inverse) topology. Thus the collection of $V(I)$ with $I$ a finitely generated ideal of $R$ forms a basis for the opens of the flat topology. For further information on the patch and flat topologies see e.g. [10], [11] and [13].
In the following result, we give a slightly different proof to our recent result [13, Theorem 3.1]. By \( \gamma(E) \) we mean the patch closure of \( E \) in \( \text{Spec}(R) \). Also, \( \kappa(p) \) denotes the residue field of \( R \) at \( p \), i.e., \( \kappa(p) := R_p / pR_p \).

**Lemma 3.1.** Let \( R \) be a ring and \( E \) a subset of \( \text{Spec}(R) \). Then \( \gamma(E) = \text{Im}(\pi^*) \) where \( \pi : R \to \prod_{p \in E} \kappa(p) \) is the canonical ring map.

**Proof.** Clearly \( E \subseteq \text{Im}(\pi^*) \). This yields that \( \gamma(E) \subseteq \text{Im}(\pi^*) \), because it is well known that the patch closed subsets of \( \text{Spec}(R) \) are precisely of the form \( \text{Im}(\varphi^*) \) where \( \varphi : R \to A \) is a ring map. To see the reverse inclusion, we act as follows. Clearly \( R' := \bigcap_{p \in E} \kappa(p) \) is a von Neumann regular ring, and so it is a reduced zero-dimensional ring. The Zariski and patch topologies over the prime spectrum of a zero-dimensional ring are the same. Thus by Theorem 2.18, \( \gamma(E') = \text{Spec}(R') \) where \( E' := \{ \pi^{-1}(0) : p \in E \} \) denotes the set of tame primes of \( R' \) and \( \pi_p : R' \to \kappa(p) \) is the projection map. It is obvious that the map \( \pi^* : \text{Spec}(R') \to \text{Spec}(R) \) is continuous with respect to the patch topology, and \( \pi^*(E') = E \). It is also easy to see that if \( f : X \to Y \) is a continuous map of topological spaces and \( E \subseteq X \), then \( f(E) \subseteq f(E) \). Using these observations, then we have \( \text{Im}(\pi^*) = \pi^*(\text{Spec}(R')) = \pi^*(\gamma(E')) \subseteq \gamma(\pi^*(E')) = \gamma(E) \). \(\square\)

In the following result, \( \overline{E} \) denotes the Zariski closure of \( E \) in \( \text{Spec}(R) \).

**Theorem 3.2.** Let \( R \) be a ring and \( E \) a subset of \( \text{Spec}(R) \). Then \( \text{Im}(\pi^*) \subseteq \overline{E} \) where \( \pi : R \to \prod_{p \in E} R/p \) is the canonical ring map.

**Proof.** If \( P \) is a prime ideal of \( R' := \prod_{p \in E} R/p \) then \( P' \subseteq P \) for some minimal prime ideal \( P' \) of \( R' \). It is easy to see that \( \kappa(p) \) is the field of fractions of the integral domain \( R/p \), so the ring map \( \varphi : R' \to \prod_{p \in E} \kappa(p) \) induced by the canonical injective ring maps \( R/p \to \kappa(p) \) is injective. Thus by Remark 2.23 there exists a prime ideal \( P'' \) of \( R'' := \prod_{p \in E} \kappa(p) \) which laying over \( P' \), i.e., \( P'' = \varphi^{-1}(P'') \). But \( \varphi : R \to R'' \) is the canonical ring map. Using this and Lemma 3.1 we get that \( \pi^{-1}(P') \in \gamma(E) \). But \( \gamma(E) \subseteq \overline{E} \), because the patch topology is finer than the Zariski topology. It follows that \( \pi^{-1}(P) \in \overline{E} \), since every Zariski closed subset of the prime spectrum is stable under specialization. \(\square\)

In Remark 3.30 we will observe that the above inclusion may be strict.

In what follows, \( cl(E) \) denotes the closure of \( E \) in \( \text{Spec}(R) \) with respect to the flat topology. The following result can be viewed as the dual of Theorem 3.2.

**Theorem 3.3.** Let \( R \) be a ring and \( E \) a subset of \( \text{Spec}(R) \). Then \( \text{Im}(\pi^*) \subseteq cl(E) \) where \( \pi : R \to \prod_{p \in E} R_p \) is the canonical ring map.

**Proof.** If \( P \) is a prime ideal of \( R' := \prod_{p \in E} R_p \) then \( P \subseteq M \) for some maximal ideal \( M \) of \( R' \). The ring map \( \varphi : R' \to R'' := \prod_{p \in E} \kappa(p) \) induced by the canonical ring maps \( \varphi_p : R_p \to \kappa(p) \) is surjective and \( \text{Ker}(\varphi) = \prod_{p \in E} \text{Ker}(\varphi_p) = \prod_{p \in E} pR_p = \prod_{p \in E} \mathfrak{m}(R_p) = \)
Regarding with Theorem 3.2, it is clear that Remark 3.6.

of prime ideals of \( R \) then maximal, we get that \( V \) proper nonzero ideal can be written (uniquely) as a finite product of prime ideals.\[ \text{Spec}(R) = \{ \text{prime ideals of } R \} \]

Proof. If \( V(a) \) is infinite for some \( a \in R \setminus \mathfrak{N}(R) \). Then \( V(a) = \overline{E} = \text{Spec}(R) \). This shows that \( a \) is nilpotent which is a contradiction. Conversely, let \( E \) be an infinite subset of \( \text{Spec}(R) \). It suffices to show that each nonempty Zariski open \( U \) of \( \text{Spec}(R) \) meets \( E \). But there exists some \( a \in R \) with \( D(a) \subseteq U \) such that \( D(a) \) is nonempty. This shows that \( a \) is not nilpotent. If \( D(a) \cap E = \emptyset \) then \( E \subseteq V(a) \) which is impossible, since \( E \) is infinite.

(ii): Assume every infinite subset of \( \text{Spec}(R) \) is flat dense. Suppose \( E := D(a) \) is infinite for some \( a \in R \setminus U(R) \). Then \( D(a) \) is flat dense. Suppose \( E := D(a) \) is a finite set for all \( a \in R \setminus U(R) \). Then \( D(a) = \text{cf}(E) = \text{Spec}(R) \). This shows that \( a \) is invertible in \( R \) which is a contradiction. Conversely, let \( E \) be an infinite subset of \( \text{Spec}(R) \). It will be enough to show that each nonempty flat open \( U \) of \( \text{Spec}(R) \) meets \( E \). Since \( U \) is nonempty, so there exists a finitely generated ideal \( I = (a_1, \ldots, a_n) \) of \( R \) such that \( \emptyset \neq V(I) \subseteq U \). It follows that \( a_i \in R \setminus U(R) \) for all \( i \). If \( V(I) \cap E = \emptyset \) then \( E \subseteq V(I)^c = \left( \bigcap_{i=1}^{n} V(a_i) \right)^c = \bigcup_{i=1}^{n} D(a_i) \) which is impossible, because \( E \) is infinite.

Corollary 3.5. If \( R \) is a PID or more generally a Dedekind domain, then every infinite subset of \( \text{Spec}(R) \) is Zariski dense.

Proof. By Proposition 3.4(i), it suffices to show that \( V(a) \) is a finite set for all nonzero \( a \in R \). We may assume \( a \) is a nonunit of \( R \) (because if \( a \) is invertible in \( R \) then \( V(a) = \emptyset \)). There exists a finite set \( \{ p_1, \ldots, p_n \} \) (with cardinality \( \leq n \)) of prime ideals of \( R \) such that \( Ra = \prod_{i=1}^{n} p_i \), because in a Dedekind domain every proper nonzero ideal can be written (uniquely) as a finite product of prime ideals. Using this and the fact that in a Dedekind domain every nonzero prime ideal is maximal, we get that \( V(a) = \{ p_1, \ldots, p_n \} \).

Remark 3.6. Regarding with Theorem 3.2 it is clear that \( E \subseteq \text{Im}(\pi^*) \). Thus \( \overline{E} = \text{Im}(\pi^*) \) if and only if \( \text{Im}(\pi^*) \) is stable under specialization (e.g. \( \pi \) has the going-up property). If \( E \) is a finite set then \( \overline{E} = \text{Im}(\pi^*) = \bigcup_{p \in E} V(p) \). But here we show with an example that the inclusion \( \text{Im}(\pi^*) \subseteq \overline{E} \) may be strict for some infinite subsets. To this end, let \( S \) be a proper infinite set of prime numbers and take
Recall that by a C.P. (or, compactly packed) ring we mean a ring which satisfies infinite prime avoidance. Dually, by a P.Z. (or, properly zipped) ring we mean a ring \( R \) which satisfies infinite prime absorbance: for any prime ideal \( p \) of \( R \) and for any infinite set \( \{ p_k \} \) of prime ideals of \( R \), if \( \bigcap_k p_k \subseteq p \) then \( p_k \subseteq p \) for some \( k \).

It is well known that a Dedekind domain is C.P. if and only if its class group has torsion. For more information on C.P. rings and P.Z. rings we refer the interested reader to [6] and [9].

Further analysis of the example given in the above remark leads us to the following result.

**Theorem 3.7.** Let \( R \) be a Dedekind domain with torsion class group and \( E \) an infinite set of maximal ideals of \( R \). Then the following assertions hold.

(i) \( \text{Im}(\pi^*) = E \cup \{ 0 \} \) and for every wild prime \( P \) of \( R' := \prod_{p \in E} R/p \) we have \( \pi^{-1}(P) = 0 \) where \( \pi : R \to R' \) is the canonical ring map.

(ii) \( \text{Im}(\pi^*) = c\ell(E) = E \cup \{ 0 \} \) and for every wild prime \( P \) of \( R' := \prod_{p \in E} R_p \) we have \( \pi^{-1}(P) = 0 \) where \( \pi : R \to R' \) is the canonical ring map.

**Proof.** (i): First we show that \( \text{Im}(\pi^*) \subseteq E \cup \{ 0 \} \). Take \( p \in \text{Im}(\pi^*) \). Suppose \( p \notin E \cup \{ 0 \} \). By [6, Theorem 2.2], \( R \) has the infinite prime avoidance property. So we may choose some \( a \in p \) such that \( a \notin \bigcup_{q \in E} q \). There exists a prime ideal \( p' \) of \( R' \) such that \( p = \pi^{-1}(p') \). It follows that \( \pi(a) = (a + q)_{q \in E} \in p' \). Note that each \( a + q \) is a nonzero element of the field \( R/q \). This shows that \( \pi(a) \) is invertible in \( R'/q \) which is a contradiction. Therefore \( \text{Im}(\pi^*) \subseteq E \cup \{ 0 \} \). It is clear that \( E \subseteq \text{Im}(\pi^*) \).

Since \( E \) is infinite, \( R' \) has wild primes. Hence, to conclude the assertion, it will be enough to show that if \( P \) is a wild prime of \( R' \) then \( \pi^{-1}(P) \neq 0 \). If \( \pi^{-1}(P) \) is not zero, then \( \pi^{-1}(P) = q \) for some \( q \in E \). We may choose some \( a \in q \) such that \( a \notin \bigcup_{p \notin q} p \). This yields that \( \pi(a) = (a + p)_{p \in E} \in P \). If \( p \neq q \) then \( a + p \) is a nonzero element of the field \( R/p \). Then consider the sequence \( x = (x_p) \in R' \) where \( x_q = 1 \) and \( x_p = (a + p)^{-1} \) for all \( p \in E \) with \( p \neq q \). Then \( 1 - e_q = x\pi(a) \in P \) which is a contradiction where \( e_q \) is the \( q \)-th unit vector of \( R' \).

(ii): The inclusion \( \text{Im}(\pi^*) \subseteq c\ell(E) \) follows from Theorem 3.3. It is easy to check that \( c\ell(E) \subseteq E \cup \{ 0 \} \), because \( R \) has the infinite prime avoidance property. It is obvious that \( E \subseteq \text{Im}(\pi^*) \). Similarly above, since \( E \) is infinite, so \( R' \) has wild primes. Thus to conclude the assertion, it suffices to show that if \( P \) is a wild prime of \( R' \) then \( \pi^{-1}(P) \neq 0 \). If \( \pi^{-1}(P) \neq 0 \) then \( \pi^{-1}(P) = q \) for some \( q \in E \). We may choose some \( a \in q \) such that \( a \notin \bigcup_{p \notin q} p \). This yields that \( \pi(a) = (a/p)_{p \in E} \in P \). Consider the sequence \( x = (x_p) \in R' \) where \( x_q = 0 \) and \( x_p = 1/a \) for all \( p \in E \) with \( p \neq q \). Then \( 1 - e_q = x\pi(a) \in P \) which is a contradiction. \( \square \)
Note that every PID satisfies the hypothesis of the above theorem. In fact, the class group of a PID is zero, so it trivially has torsion. It is also well known that every Abelian group is isomorphic to the class group of a Dedekind domain (see [1, Theorem 7]). Hence, Dedekind domains with torsion class groups are ubiquitous. For instance, if \( n \geq 1 \) then there exists a Dedekind domain whose class group is isomorphic to the torsion additive group \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \) (in fact, if \( R \) is a ring of nonzero characteristic, then the additive group of \( R \) has torsion, and the exponent of this group is the characteristic of \( R \)).

The following result is the dual of Theorem 3.7.

**Theorem 3.8.** Let \((R, m)\) be a reduced local P.Z. ring with \( \dim(R) = 1 \) and \( E \) an infinite set of minimal primes of \( R \). Then the following assertions hold.

(i) \( \text{Im}(\pi^*) = E \cup \{m\} \) and for every wild prime \( P \) of \( R^i := \prod_{p \in E} R_p \) we have \( \pi^{-1}(P) = m \) where \( \pi : R \to R' \) is the canonical ring map.

(ii) \( \text{Im}(\pi^*) = \overline{E} = E \cup \{m\} \) and for every wild prime \( P \) of \( R' := \prod_{p \in E} R/p \) we have \( \pi^{-1}(P) = m \) where \( \pi : R \to R' \) is the canonical ring map.

**Proof.** (i): First we show that \( \text{Im}(\pi^*) \subseteq E \cup \{m\} \). Let \( P \) be a prime ideal of \( R' \) and setting \( q := \pi^{-1}(P) \). Suppose \( q \notin E \cup \{m\} \). Since \( R \) is a P.Z. ring, so there exists some \( a \in \bigcap_{p \in E} p \) such that \( a \notin q \). For each \( p \in E \), \( R_p \) is a reduced ring and so \( a/1 \in pR_p = 0 \). This means that \( \pi(a) = (a/1)_{p \in E} \) is the zero sequence. But \( \pi(a) \notin P \) which is a contradiction. Thus \( \text{Im}(\pi^*) \subseteq E \cup \{m\} \). It is clear that \( E \subseteq \text{Im}(\pi^*) \). Since \( E \) is infinite, \( R' \) has wild primes. Hence, to conclude the assertion, it suffices to show that if \( P \) is a wild prime of \( R' \) then \( \pi^{-1}(P) = m \). If \( \pi^{-1}(P) \neq m \) then \( \pi^{-1}(P) = q \) for some \( q \in E \). Since \( R \) is a P.Z. ring, so we may choose some a \( \in \bigcap_{p \in E} p \) such that \( a \notin q \). Thus \( \pi(a) = (a/1)_{q \in E} \notin P \) and \( a/1 \) is invertible in \( R_q \). Also, \( a/1 = 0 \) in \( R_p \) for all \( p \in E \) with \( p \neq q \). Then consider the sequence \( x = (x_q)_{q \in E} \) in \( R' \) with \( x_q = 1/a \) and \( x_p = 1 \) for all \( p \in E \) with \( p \neq q \). Thus \( x \) is invertible in \( R' \) and so \( x \notin P \). It follows that the \( q \)-th unit vector \( e_q = x\pi(a) \notin P \) which is a contradiction.

(ii): The inclusion \( \text{Im}(\pi^*) \subseteq \overline{E} \) follows from Theorem 3.2. Since \( R \) is a P.Z. ring, so the inclusion \( \overline{E} \subseteq E \cup \{m\} \) is also clear. To conclude the assertion, it suffices to show that if \( P \) is a wild prime of \( R' \) then \( \pi^{-1}(P) = m \). Suppose \( \pi^{-1}(P) = q \) for some \( q \in E \). We may choose some \( a \in \bigcap_{p \in E, p \neq q} p \) such that \( a \notin q \). Thus \( \pi(a) = (a/1)_{p \in E, p \neq q} \notin P \). Also, \( 1 - e_q \notin P \). This yields that \( 0 = (1 - e_q)\pi(a) \notin P \) which is a contradiction.

As a sort of the dual of Corollary 3.3, we have the following result.

**Corollary 3.9.** If \((R, m)\) is a local P.Z. ring with \( \dim(R) = 1 \), then every infinite subset of \( \text{Spec}(R) \) is flat dense.

**Proof.** Let \( E \) be an infinite subset of \( \text{Spec}(R) \). If \( m \in E \) then \( c\ell(E) = \text{Spec}(R) \), because \( c\ell(E) \) is stable under the generalization. Now assume \( m \notin E \). Without loss of generality, we may assume \( R \) is reduced, because the canonical ring map \( R \to R/\mathfrak{m} \) induces a homeomorphism between the corresponding prime spectra (with
respect to the flat topology) and $R/\mathfrak{m}$ is still a local P.Z. ring of Krull dimension one. Then by Theorem 3.3(i), $\text{Im}(\pi^*) = E \cup \{m\}$ where $\pi : R \to \prod_{p \in E} R_p$ is the canonical ring map. By Lemma 3.1 (or, by Theorem 3.3), $\text{Im}(\pi^*) \subseteq c(E)$. Thus $m \in c(E)$. But $c(E)$ is stable under generalization, and hence $c(E) = \text{Spec}(R)$. □

Note that in the following result, the index set $S$ is nonempty.

**Lemma 3.10.** Let $T = K[x_i : i \in S]$, with $K$ a ring, $I = (x_i x_k : i, k \in S, i \neq k)$ and $I_k = (x_i : i \in S, i \neq k)$ for all $k \in S$. Then the following assertions hold.

(i) $I = \bigcap_{k \in S} I_k$. In particular, $K$ is reduced if and only if $T/I$ is reduced.

(ii) $K$ is an integral domain if and only if $\text{Min}(I) = \{I_k : k \in S\}$.

(iii) If $K$ is a Noetherian domain, then $\text{dim}(T/I) = \text{dim}(K) + 1$.

(iv) If $S$ is infinite and $K$ is a field, then $R := (T/I)_m$ is a reduced local P.Z. ring with $\text{dim}(R) = 1$ and infinitely many minimal primes where $m = (x_i : i \in S)$.

**Proof.** The statements (i) and (ii) are straightforward, but we provide a proof for the sake of completeness.

(i): It is clear that $I \subseteq I_k$ for all $k \in S$. Thus $I \subseteq \bigcap_{k \in S} I_k$. To see $f \in I$ it suffices to show that in the expansion of $f$ for every monomial which appears, at least two distinct variables with positive powers are involved. If not, then there exist $g(x_d) \in K[x_d]$ (for some $d \in S$) and $h \in T$ such that $f = g(x_d) + h$ and there is no monomial of the form $x_d^n$ (with $n \geq 0$) in the expansion of $h$. It follows that $h \in I_d$ and so $g(x_d) \in I_d$ which is a contradiction, since for each nonzero $r \in R$ and $n \geq 0$, then $rx_d^n \notin I_d$. Therefore $I = \bigcap_{k \in S} I_k$. If $K$ is reduced then $T/I_k \simeq K[x_k]$ is also reduced, and so $I_k$ is a radical ideal for all $k \in S$. Thus the ideal $I$ is also radical, because the intersection of every family of radical ideals is a radical ideal. This yields that $T/I$ is reduced. Conversely, if $T/I$ is reduced then $I$ is a radical ideal. Now if $a \in K$ is nilpotent then $a^n = 0 \in I$ for some $n \geq 1$, and so $a \in \sqrt{I} = I$. It follows that $a = 0$.

(ii): If $K$ is an integral domain, then $I_k$ is a prime ideal of $T$ for all $k$. If $p$ is a prime ideal of $T$ with $I \subseteq p \subseteq I_k$ for some $k$, then for each $i$ with $i \neq k$ we have $x_i x_k \in I$. Since $x_k \notin p$, so $x_i \in p$. Thus $p = I_k$. To see the reverse inclusion, take $p \in \text{Min}(I)$. We may assume $|S| \geq 2$. Then $p \neq (x_k : k \in S)$. Thus $x_k \notin p$ for some $k$. But for each $i$ with $i \neq k$ we have $x_i x_k \in I$ and so $x_i \in p$. This shows that $I_k \subseteq p$ and so $I_k = p$. Conversely, since $S$ is nonempty so we may choose some $k \in S$. By hypothesis, $I_k$ is a prime ideal of $T$ thus $T/I_k \simeq K[x_k]$ is an integral domain, and so every its subring, especially $K$, is an integral domain.

(iii): We know that the Krull dimension of any ring $R$ is the supremum of all $\text{dim}(R/p)$ where $p$ ranges over the set of minimal primes of $R$. Also, since $K$ is a Noetherian ring then $T/I_k \simeq K[x_k]$ has the Krull dimension $\text{dim}(K) + 1$. Now by using these observations and (ii), we get that $\text{dim}(T/I) = \text{dim}(K) + 1$.

(iv): If $K$ is a field then by using (iii), we have $1 \leq \text{dim}(R) \leq \text{dim}(T/I) = 1$. Hence, the non-maximal prime ideals of the local ring $R$ are precisely the minimal primes $P_k := I_kT_m/IT_m$ with $k \in S$. Since $S$ is infinite, thus $R$ has infinitely many minimal primes. By (i), $T/I$ and so every its localization, especially $R$, are reduced. It remains to show that $R$ is a P.Z. ring. Suppose $\bigcap_{k \in S'} P_k \subseteq P_I$ where $S'$ is a subset of $S$ and $d \in S$. It suffices to show that $d \in S'$. We have $\bigcap_{k \in S'} P_k = (\bigcap_{k \in S'} I_kT_m)/IT_m$
and clearly \( pT_m \subseteq \bigcap_{k \in S'} I_k T_m \) where \( p = (x_i : i \in S \setminus S') \). It follows that \( p \subseteq I_d \).

This yields that \( d \in S' \) which completes the proof. \( \square \)

In addition to the above result, Hochster’s characterization of spectral spaces provides us further examples of (reduced) local P.Z. rings with Krull dimension one and infinitely many minimal primes. In fact, by applying [9, Proposition 5.5] to every (nonfield) Dedekind domain \( A \) with torsion class group we obtain a reduced local P.Z. ring \( R \) such that \( \text{Spec}(A) \) endowed with the Zariski topology is isomorphic to \( \text{Spec}(R) \) equipped with the flat topology, and this homeomorphism reverses prime orders. In particular, \( \text{dim}(R) = \text{dim}(A) = 1 \). If moreover, \( A \) has infinitely many maximal ideals, then the resulting ring \( R \) will have infinitely many minimal primes.

**Remark 3.11.** Regarding with Theorem 3.3, it is clear that \( E \subseteq \text{Im}(\pi^*) \). Thus \( \text{cl}(E) = \text{Im}(\pi^*) \) if and only if \( \text{Im}(\pi^*) \) is stable under generalization (e.g. \( \pi \) has the going-down property). If \( E \) is a finite set then \( \text{cl}(E) = \text{Im}(\pi^*) = \bigcup_{p \in E} \Lambda(p) \). But here we show with an example that the inclusion \( \text{Im}(\pi^*) \subseteq \text{cl}(E) \) may be strict for some infinite subsets. To this end, let \( (R, \mathfrak{m}) \) be a reduced local P.Z. ring with \( \text{dim}(R) = 1 \) and infinitely many minimal primes, and let \( E \) be a proper infinite set of minimal primes of \( R \). Then by Corollary 3.9 \( E \) is flat dense in \( \text{Spec}(R) \). But by Theorem 3.8(i), \( \text{Im}(\pi^*) = E \cup \{ \mathfrak{m} \} \).

We conclude this article by proposing the following problem. In Corollary 2.11 we observed that if the index set \( S \) is infinite then with \( I = \bigoplus_{i \in S} R_i \) we have \( 2^{2^{|S|}} \leq |\text{Spec}(R/I)| \). Finding the precise cardinality of \( \text{Spec}(R/I) \) or at least an optimal upper bound is still unknown for us.

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