On Cornacchia’s Algorithm

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Abstract: We give an endorsement for Cornacchia’s famous algorithm. Thus we do not claim anything new but an approach which is supposed to be simpler than those of previous works written with the same aim. All variables and constants are integers.

Keywords: quadratic forms, number theoretical algorithm

1. Introduction

We consider the Diophantine equation

\[ x^2 + dy^2 = m, \]  \hspace{1cm} (1.1)

where

\[ d \geq 2, \quad m \geq 2, \quad \gcd(d, m) = 1. \]  \hspace{1cm} (1.2)

We observe that provided (1.1) admits a proper solution \( \{u, v\} \), namely

\[ u^2 + dv^2 = m, \quad \gcd\{u, v\} = 1, \]  \hspace{1cm} (1.3)

it holds that

\[ \gcd\{uv, m\} = 1, \quad uv \neq 0, \quad m \geq d + 1; \]  \hspace{1cm} (1.4)

and that \(-d\) is a quadratic residue mod \( m \), and there exists a \( w \), \( m/2 \leq w < m \), such that

\[ w^2 \equiv -d \mod m. \]  \hspace{1cm} (1.5)

In fact one may put \( w \equiv uv^{-1} \mod m, \quad vv^{-1} \equiv 1 \mod m. \)

We apply the antenaresis to the pair \( \{w, m\} \) and denote by \( \{t_j\} \) the residues thus arising. Also we expand \( w/m \) into a regular continued fraction and get the convergents \( \{C_j/D_j\} \). An explanation of these notions is to be given in the next section.

With this, Cornacchia (1908, pp. 60–66) essentially stated the following:

Theorem.

Let \( \nu \) be such that

\[ t_{\nu+1}^2 \leq m < t_\nu^2. \]  \hspace{1cm} (1.6)

Then (1.3) implies that

\[ |u| = t_{\nu+1}, \quad |v| = D_\nu. \]  \hspace{1cm} (1.7)

Algorithm.

Solely on the assumption (1.5), that is, without any prior knowledge of (1.3), find \( \nu \) that satisfies (1.6). If it holds that \( t_{\nu+1}^2 + dD_\nu^2 = m \), then this is a proper solution of (1.1) corresponding to \( w \). Otherwise (1.1) does not admit any proper solution corresponding to \( w \).

Obviously, if (1.5) is empty, then there is no need to make any further quest as far as proper solutions are concerned; but see the second example in the last section. We shall give, in the third section, a proof of the theorem and algorithm using a basic observation on the nature of finite continued fractions, which is in fact contained in the best approximation theorem of Lagrange.
(1798, pp. 55–57) but can be proved quickly with an idea of Legendre (1798, p. 27). Also, we partly follow Basilla (2004). The case \( d = 1 \) will be treated in the fourth section.

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2. Finite continued fractions

Let \( b \geq 2 \) and expand any fraction \( a/b \) into the continued fraction

\[
q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \cdots + \frac{1}{q_k}}},
\]

which means that the antenaresis applied to \( \{a, b\} \) yields the sequence of identities

\[
r_{j-1} = q_j r_j + r_{j+1}, \quad 0 \leq r_{j+1} < r_j, \quad 0 \leq j \leq k,
\]

with the convention \( r_{-1} = a, r_0 = b, r_{k+1} = 0 \); in particular \( r_k = \gcd\{a, b\} \). One may put this, in the matrix multiplication format, as

\[
(r_{j-1}, r_j) = (r_j, r_{j+1}) \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix}, \quad 0 \leq j \leq k.
\]

Writing, for \(-1 \leq j \leq k\),

\[
\begin{pmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{pmatrix} = \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
A_{-2} = 0, \quad B_{-2} = 1; \quad A_{-1} = 1, \quad B_{-1} = 0; \quad A_0 = q_0, \quad B_0 = 1,
\]

where an empty product is the unit matrix, we have, by induction,

\[
\frac{A_j}{B_j} = q_0 + \frac{1}{q_1 + q_2 + \cdots + q_j}, \quad 0 \leq j \leq k,
\]

\[
A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}, \quad -1 \leq j \leq k.
\]

Thus, we have

\[
(r_{-1}, r_0) = (r_j, r_{j+1}) \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{j-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
= (r_j, r_{j+1}) \begin{pmatrix} A_j & B_j \\ A_{j-1} & B_{j-1} \end{pmatrix},
\]

in which a use is made of the fact that \( \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \) are symmetric. Hence, we get

\[
(r_j, r_{j+1}) = (-1)^{j-1} (a, b) \begin{pmatrix} B_{j-1} & -B_j \\ -A_{j-1} & A_j \end{pmatrix}.
\]

Lemma.

If it holds, with a particular \( \lambda, \ -1 \leq \lambda \leq k \), that

\[
|aQ - bP| < r_\lambda, \quad Q \neq 0,
\]

(2.10)
then we have

\[ B_\lambda \leq |Q|. ~ (2.11) \]

**Proof.** Obviously we may skip the cases \( \lambda = -1, 0 \); we assume \( 1 \leq \lambda \leq k \). Following Legendre \textit{loc.cit.}, we introduce the transformation of variables

\[ P = MA_\lambda - NA_{\lambda-1}, \quad Q = MB_\lambda - NB_{\lambda-1}; \]

namely

\[ M = (-1)^\lambda(QA_{\lambda-1} - PB_{\lambda-1}), \quad N = (-1)^\lambda(QA_{\lambda-1} - PB_\lambda). ~ (2.13) \]

We have, via (2.9),

\[ aQ - bP = M(aB_\lambda - bA_\lambda) - N(aB_{\lambda-1} - bA_{\lambda-1}) = (-1)^\lambda(Mr_{\lambda+1} + Nr_\lambda). \]

(2.14)

If \( MN > 0 \), then \( |aQ - bP| = |Mr_{\lambda+1} + Nr_\lambda| \geq r_\lambda \), which is rejected by the assumption (2.10). If \( M = 0 \), then (2.13) implies \( Q = \tau B_{\lambda-1}, \quad P = \tau A_{\lambda-1} \) with a \( \tau \neq 0 \), since \( \gcd\{B_j, A_j\} = 1 \) by (2.7); hence, via (2.9), we get \( |aQ - bP| = |\tau|r_\lambda \geq r_\lambda \), which contradicts (2.10). Therefore, we may suppose that \( MN \leq 0 \), \( M \neq 0 \), and find that (2.12) implies \( |Q| = |M|B_\lambda + |N|B_{\lambda-1} \geq B_\lambda \). We end the proof.

3. **Proof of Cornacchia’s theorem and algorithm**

We specialize the discussion of the previous section by setting \( a = w, \quad b = m, \quad r_j = t_j, \quad A_j = C_j, \quad B_j = D_j \). By definition, \( t_0 = m, \quad t_1 = w, \quad t_k = 1 \); thus there exists a unique \( \nu \) that satisfies (1.6).

On the other hand, (2.9) gives that \( t_{j+1} = (-1)^j(wD_j - mC_j) \) and

\[ t_{j+1}^2 + D_j^2 \equiv (w^2 + d)D_j^2 \equiv 0 \pmod{m}, \quad -1 \leq j \leq k. \]

(3.1)

Also, we have \( u = vw - \ell m \) with an \( \ell \); and by the lemma with \( P = \ell, \quad Q = v \) we see that

for any \( \lambda \) such that \( |u| < t_\lambda, \quad -1 \leq \lambda \leq k \),

we have \( D_\lambda \leq |v| \), that is, \( dD_\lambda^2 < m \).

(3.2)

In particular, since \( |u| < \sqrt{m} < t_\nu \), we have \( t_{\nu+1}^2 + D_\nu^2 < 2m \). Hence by the congruence relation (3.1)\( _{j=\nu} \), we find that

\[ t_{\nu+1}^2 + D_\nu^2 = m. \]

(3.3)

If \( |u| > t_{\nu+1} \), then \( m = u^2 + dv^2 > t_{\nu+1}^2 + D_\nu^2 = m \), which is impossible. If \( |u| < t_{\nu+1} \), then in much the same way as above we get \( t_{\nu+2}^2 + D_{\nu+1}^2 = m \); and

\[ m^2 = |t_{\nu+1} + iD_\nu \sqrt{d}|^2 |t_{\nu+2} + iD_{\nu+1} \sqrt{d}|^2 = (t_{\nu+1}t_{\nu+2} - D_\nu D_{\nu+1})^2 + d(t_{\nu+1}D_{\nu+1} + t_{\nu+2}D_\nu)^2, \]

(3.4)

which is impossible, since \( d \geq 2 \) and \( m = t_{\nu+1}D_{\nu+1} + t_{\nu+2}D_\nu \) by (2.8)\( _{j=\nu+1} \). Hence, (1.7) is verified. We end the proof of the theorem.

As to the validity of the algorithm, it suffices to show that the identity (3.3), if it holds on its own, implies that \( \gcd\{t_{\nu+1}, D_\nu\} = 1 \). To see this, we put \( w^2 + d = hm \), and get, via (2.9)\( _{j=\nu} \),

\[ t_{\nu+1}^2 + D_\nu^2 = m(hD_\nu^2 - 2wC_\nu D_\nu + mC_\nu). \]
Namely, \((h D_\nu - 2w C_\nu) D_\nu + m C_\nu^2 = 1\). Hence \(\gcd\{D_\nu, m\} = 1\), and \(\gcd\{t_{\nu+1}, D_\nu\} = 1\). This ends the endorsement of Cornacchia’s algorithm.

4. Remarks and examples

**Remark 1:** When \(d = 1\) we follow the argument of Hermite (1848). Thus we assume only that there exists a \(w, m/2 \leq w < m\), such that

\[
w^2 \equiv -1 \pmod{m}; \tag{4.1}
\]

and we adopt the specialization at the beginning of the last section. Then we choose \(\mu\) to satisfy

\[
D_\mu^2 \leq m < D_{\mu+1}^2. \tag{4.2}
\]

This is possible because \(\{D_j\}\) increase monotonically from 1 to \(k\). We note that, for \(1 \leq j \leq k\),

\[
\left| \frac{w}{m} - \frac{C_j - 1}{D_j - 1} \right| \leq \frac{1}{D_{j-1} D_j} \quad \text{or} \quad t_j \leq m/D_j, \tag{4.3}
\]

since by the construction \(w/m\) is between \(C_{j-1}/D_{j-1}\) and \(C_j/D_j\), and we have (2.7) and (2.9).

Hence we see that \(t_{\mu+1} < \sqrt{m}\), and \(t_{\mu+1}^2 + D_{\mu}^2 < 2m\). The congruence relation (3.1) holds when \(d = 1\), as well; and it implies

\[
t_{\mu+1}^2 + D_{\mu}^2 = m. \tag{4.4}
\]

As in the case \(d \geq 2\), this is a proper solution of (1.1)\(_{d=1}\).

Then we assume that we have a proper solution (1.3)\(_{d=1}\). We observe that

\[
\begin{align*}
vD_\mu - ut_{\mu+1} &\equiv vD_\mu - (-1)^\mu vu^2 D_\mu \equiv (1 + (-1)^\mu)vD_\mu \pmod{m}, \nonumber \\
uD_\mu - vt_{\mu+1} &\equiv uD_\mu - (-1)^\mu vuD_\mu \equiv (1 - (-1)^\mu)uD_\mu \pmod{m}, \quad \text{(4.5)}
\end{align*}
\]

one of which is congruent to 0 mod \(m\). Let us assume \(vD_\mu - ut_{\mu+1} \equiv 0 \pmod{m}\). Then we note that

\[
(vD_\mu - ut_{\mu+1})^2 + (uD_\mu + vt_{\mu+1})^2 = |u + iv|^2 |t_{\mu+1} + iD_\mu|^2 = m^2. \tag{4.6}
\]

This implies that either \((vD_\mu - ut_{\mu+1})^2 = 0\) or \(m^2\); and if the latter holds, then \((uD_\mu + vt_{\mu+1})^2 = 0\). Since \(\gcd\{u, v\} = 1\) and \(\gcd\{t_{\mu+1}, D_\mu\} = 1\), we conclude that

\[
\text{either} \quad |u| = D_\mu, |v| = t_{\mu+1} \quad \text{or} \quad |u| = t_{\mu+1}, |v| = D_\mu. \tag{4.7}
\]

With the remaining case, i.e., \(uD_\mu - vt_{\mu+1} \equiv 0 \pmod{m}\), we use instead \(|u - iv|^2 |t_{\mu+1} + iD_\mu|^2 = m^2\), getting (4.7) again.

It should be stressed that in the case \(d = 1\) we do not need to have (1.3); it suffices to have (1.5)\(_{d=1}\) or (4.1). However, we should first choose \(D_\mu\) instead of \(t_{\mu+1}\).

**Remark 2:** It is worth remarking that Smith (1855) showed that when dealing with \(m\) a prime \(\equiv 1 \pmod{4}\) the condition (1.5)\(_{d=1}\) is not needed to be assumed as far as one is concerned with only the existence of the representation (1.3)\(_{d=1}\). He exploited the fact that there exists a fraction \(m/h, 2 \leq h < m/2\), whose continued fraction expansion is palindromic.

**Remark 3:** An effective way to adopt prior to any use of Cornacchia’s algorithm is to restrict oneself to the cases of \(m\) being square-free and prime powers. Then in the general case the identity \(|x + iy \sqrt{d}|^2 = x^2 + dy^2\) is to be exploited. It is important not to restrict oneself to prime powers.
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only, since the product of primes, none of which has the representation (1.3), may admit the representation. An example is given below.

Example 1: We consider the situation \( d = 5 \) and \( m = 435629 \). Since \( 2^{m-1} \not\equiv 1 \mod m \), we see that \( m \) is a composite number. The \( \rho \) method of Pollard, for instance, gives the decomposition \( m = p_1p_2 \), with primes \( p_1 = 367 \), \( p_2 = 1187 \). Since \( p_1, p_2 \equiv 7 \mod 20 \), they are not expressible by the quadratic form \( x^2 + 5y^2 \), but the product \( m = p_1p_2 \) does admit a proper representation by the form, according to a well-known criterion which can be traced back to Fermat (1658: 1894, p. 405). Let us confirm this by Cornacchia’s algorithm. Following Lagrange (1768, p. 500), we put \( w_l \equiv \pm(-5)^{(p_l+1)/4} \mod p_l \), \( l = 1, 2 \); namely \( w_1 \equiv \pm27 \mod p_1 \), \( w_2 \equiv \pm282 \mod p_2 \). Then \( w_l^2 \equiv -5 \mod p_l \). Checking that

\[
\frac{367}{1187} = 0 + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5} \Rightarrow 207 \cdot 367 - 64 \cdot 1187 = 1,
\]

we have that \( w \equiv 207 \cdot 367 \cdot w_2 - 64 \cdot 1187 \cdot w_1 \mod m \) satisfies \( w^2 \equiv -5 \mod m \), or \( w = 231183, 386057 \). Then the algorithm yields the solutions \( t_{\nu+1}^2 + 5D_\nu^2 = m \) with

\[
\begin{align*}
w &= 231183: & t_\nu &= 1385, & t_{\nu+1} &= 228, & D_\nu &= 277, \\
w &= 386057: & t_\nu &= 1450, & t_{\nu+1} &= 123, & D_\nu &= 290.
\end{align*}
\]

Further, as to the equation \( x^2 + 5y^2 = p_1^2 \), we use the Schönemann–Hensel lifting, and from \( 27 \mod p_1 \) obtain \( w_\nu = 109760, w_\nu^2 \equiv -5 \mod p_1^2 \). The antenaresis applied to the pair \( \{w_\nu,p_1^2\} \) yields \( 362^2 + 5 \cdot 27^2 \equiv p_1^2 D \) with this, we exploit \( (362 + 27\sqrt{5}i)(228 - 277\sqrt{5}i) \), and find the following proper representation of \( p_1^2 p_2 \):

\[
119931^2 + 5 \cdot 94118^2 = 58674434381.
\]

The other combination \( (362 + 27\sqrt{5}i)(228 + 277\sqrt{5}i) \) does not lead to a proper representation, a phenomenon which can be explained by a use of the theory of ideals in \( \mathbb{Q}(\sqrt{-5}) \).

Example 2: Consider \( x^2 + 7y^2 = 4p \) where \( p = 9241 \) is a prime. This equation does not have any proper solution, since if \( x, y \) are both odd, then \( x^2 + 7y^2 \equiv 0 \mod 8 \), and if there is a solution, then \( x, y \) are both even. Thus, Cornacchia’s algorithm should not work with this situation, even though we have (1.5), that is, \( w^2 \equiv -7 \mod 4p \), \( w = 24899 \). Applying the antenaresis to the pair \( \{w,4p\} \), we find that \( t_{\nu+1} = 52, D_\nu = 144 \), and \( t_{\nu+1}^2 + 7D_\nu^2 \neq 4p \) indeed. Hence, one should follow the prescription indicated in the third remark above. We have \( 6417^2 \equiv -7 \mod p \), and the algorithm gives \( t_{\nu+1} = 13, D_\nu = 36 \), and \( 13^2 + 7 \cdot 36^2 = p \), which yields the improper solution \( 26^2 + 7 \cdot 72^2 = 4p \). By the way, the quadratic residue 6417 mod \( p \) is readily obtained with the algorithm of Tonelli (1891).

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