Global Aspects of Quantizing
Yang-Mills Theory

Helmuth Hüffel\textsuperscript{1,2} and Gerald Kelnhofer\textsuperscript{3}
Institut für Theoretische Physik
Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria

Abstract

We review recent results on the derivation of a global path integral density for Yang-Mills theory. Based on a generalization of the stochastic quantization scheme and its geometrical interpretation we first recall how locally a modified Faddeev-Popov path integral density for the quantization of Yang-Mills theory can be derived, the modification consisting in the presence of specific finite contributions of the pure gauge degrees of freedom. Due to the Gribov problem the gauge fixing can be defined only locally and the whole space of gauge potentials has to be partitioned into patches. We discuss a global extension of the path integral by summing over all patches, which can be proven to be manifestly independent of the specific local choices of patches and gauge fixing conditions, respectively. In addition to the formulation on the whole space of gauge potentials we discuss the corresponding path integral on the gauge orbit space.

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2) Email: helmuth.hueffel@univie.ac.at

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The Faddeev-Popov \cite{1} path integral procedure constitutes one of the most popular quantization methods for Yang-Mills theory and is widely used in elementary particle physics. It is, however, well known that at a non-perturbative level due to the Gribov ambiguity \cite{2} a unique gauge fixing in the full space of gauge fields is not possible so that the Faddeev-Popov path integral procedure is defined only locally in field space.

Several attempts were presented to generalize the above approach in order to establish global integral representations \cite{3, 4} in the space of gauge fields. It is our aim to present a quite different argumentation based on a recently introduced generalized stochastic quantization scheme \cite{5, 6, 7, 8}.

The stochastic quantization method of Parisi and Wu \cite{9} was introduced 1981 as a new method for quantizing field theories. It is based on concepts of nonequilibrium statistical mechanics and provides novel and alternative insights into quantum field theory, see refs. \cite{10, 11}. One of the most interesting aspects of this new quantization scheme lies in its rather unconventional treatment of gauge field theories, in specific of Yang-Mills theories. We just recall that originally it was formulated by Parisi and Wu without the introduction of gauge fixing terms and the usual Faddeev-Popov ghost fields; later on a modified approach named stochastic gauge fixing was given by Zwanziger \cite{12} where again no Faddeev-Popov ghost fields where introduced. Our focus is based on extending Zwanziger’s stochastic gauge fixing scheme. By this generalized stochastic gauge fixing scheme it is possible to derive a non-perturbative proof of the equivalence between the conventional path integral formulation of this model and the equilibrium limit of the corresponding stochastic correlation functions.

The main difficulty in the previous investigations of the stochastic quantization of Yang–Mills theory for deriving a conventional field theory path integral density was to solve the Fokker–Planck equation in the equilibrium limit. In the original Parisi–Wu approach this equilibrium limit could not even be attained due to unbounded diffusions of the gauge modes. Zwanziger \cite{12, 13} suggested to introduce a specific additional nonholonomic stochastic force term to suppress these gauge modes yet keeping the expectation values of gauge invariant observables unchanged. The approach to equilibrium and the
discussion of the conditions of applicability to the nonperturbative regime, however, do not seem to have been fully completed.

Our analysis is distinguished by the above approaches by exploiting a more general freedom to modify both the drift term and the diffusion term of the stochastic process again leaving all expectation values of gauge invariant variables unchanged. Due to this additional structure of modification the equilibrium limit can be obtained immediately using the fluctuation dissipation theorem proving equivalence with the well known Faddeev-Popov path integral density. In deriving this result the gauge degrees of freedom are fully under control, no infinite gauge group volumes arise. However, this equivalence proof can be performed only locally in field space for gauge field configurations satisfying a unique gauge fixing condition and deserves an extension for global applicability. We are able to discuss such a formulation in the second half of this paper.

Let $P(M, G)$ be a principal fiber bundle with compact structure group $G$ over the compact Euclidean space time $M$. Let $\mathcal{A}$ denote the space of all irreducible connections on $P$ and let $\mathcal{G}$ denote the gauge group, which is given by all vertical automorphisms on $P$ reduced by the centre of $G$. Then $\mathcal{G}$ acts freely on $\mathcal{A}$ and defines a principal $\mathcal{G}$-fibration $\mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G} =: \mathcal{M}$ over the paracompact \cite{14} space $\mathcal{M}$ of all inequivalent gauge potentials with projection $\pi$. Due to the Gribov ambiguity the principal $\mathcal{G}$-bundle $\mathcal{A} \to \mathcal{M}$ is not globally trivializable.

Using this mathematical setting we start with the Parisi–Wu approach for the stochastic quantization of the Yang–Mills theory in terms of the Langevin equation

$$dA = -\frac{\delta S}{\delta A} ds + dW.$$ \hfill (1)

Here $S$ denotes the Yang–Mills action without gauge symmetry breaking terms and without accompanying ghost field terms, $s$ denotes the extra time coordinate (“stochastic time” coordinate) with respect to which the stochastic process is evolving, $dW$ is the increment of a Wiener process.

We now discuss Zwanziger’s modified formulation of the Parisi-Wu scheme: The stochastic gauge fixing procedure consists in adding an additional drift force to the
The Langevin equation (1) which acts tangentially to the gauge orbits. This additional term generally can be expressed by the gauge generator $D_A$, i.e. the covariant derivative with respect to $A$, and an arbitrary function $a$ so that the modified Langevin equation reads as follows

$$dA = \left[ -\frac{\delta S}{\delta A} + D_A a \right] ds + dW.$$  

The expectation values of gauge invariant observables remain unchanged for any choice of the function $a$. For specific choices of the – in principle – arbitrary function $a$ the gauge modes’ diffusion is damped along the gauge orbits. As a consequence the Fokker-Planck density can be normalized; we remind that this situation is different to the Parisi- Wu approach, where for expectation values of gauge variant observables no equilibrium values could be attained.

In contrast to the approach of [12] where no equilibrium distribution of the Fokker–Planck equation could be derived as well as in contrast to [13] where the full Fokker–Planck operator

$$L = \frac{\delta}{\delta A} \left[ \frac{\delta S}{\delta A} - D_A a + \frac{\delta}{\delta A} \right]$$

was needed to obtain an equilibrium distribution we present a quite different strategy: As the Fokker-Planck operator factorizes into first order differential operators the question arises whether it is possible to derive the equilibrium distribution directly by solving a simpler first order problem. However, for this to be possible a necessary integrability condition imposed on the drift term $\frac{\delta S}{\delta A} - D_A a$ has to be fulfilled. It is well known that for the Yang–Mills case this is violated.

In the following we want to clarify the relationship of this integrability condition and the underlying geometrical structure of the space of gauge potentials. We remind that any bundle metric on a principal fiber bundle which is invariant under the corresponding group action gives rise to a natural connection whose horizontal subbundle is orthogonal to the corresponding group. In the Yang-Mills theory case with respect to the natural metric on $\mathcal{A}$ this connection is given by the following Lie$\mathcal{G}$ valued one form

$$\gamma = \Delta_A^{-1} D_A^*.$$  

\(4\)
Here $D_A^*$ is the adjoint operator of the covariant derivative $D_A$, $\Delta^{-1}_A$ is the inverse of the covariant Laplacian $\Delta_A = D^*_A D_A$.

The curvature $\Omega$ of $\gamma$, $\Omega = \delta A \gamma + \frac{1}{2} [\gamma, \gamma]$, where $\delta A$ denotes the exterior derivative on $A$, however, does not vanish so that there does not exist (even locally) a manifold whose tangent bundle is isomorphic to this horizontal subbundle. Moreover this also implies that any vector field along the gauge group cannot be written as a gradient of a function.

It is our intention to modify the stochastic process (1) for the Yang–Mills theory in such a way that the factorization of the modified Fokker-Planck operator indeed allows the determination of the equilibrium distribution as a solution of a first order differential equation in a consistent manner.

From [14] it follows that there exists a locally finite open cover $U = \{ U_\alpha \}$ of $\mathcal{M}$ together with a set of background gauge fields $\{ A^{(a)}_0 \in A \}$ such that

$$\Gamma_\alpha = \{ B \in \pi^{-1}(U_\alpha) | D^*_A A^{(a)}_0 (B - A^{(a)}_0) = 0 \}$$

defines a family of local sections of $A \to \mathcal{M}$. Instead of analyzing Yang–Mills theory in the original field space $A$ we consider the family of trivial principal $G$-bundles $\Gamma_\alpha \times G \to \Gamma_\alpha$, which are locally isomorphic to the bundle $A \to \mathcal{M}$, where the isomorphisms are provided by the maps

$$\chi_\alpha : \Gamma_\alpha \times G \to \pi^{-1}(U_\alpha), \quad \chi_\alpha(B, g) := B^g$$

with $B \in \Gamma_\alpha$, $g \in G$ and $B^g$ denoting the gauge transformation of $B$ by $g$. Thus we transform the Parisi–Wu Langevin equation (1) into the adapted coordinates $\Psi = \begin{pmatrix} B \\ g \end{pmatrix}$. As this transformation is not globally possible the region of definition of (1) has to be restricted to $\pi^{-1}(U(A^{(a)}_0))$. Making use of the Ito stochastic calculus the above Langevin equation now reads

$$d\Psi = \left( -G^{-1}_a \delta S_\alpha \frac{\delta \Psi}{\delta \Psi} + \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (G^{-1}_a \sqrt{\det G_\alpha})}{\delta \Psi} \right) ds + E_\alpha dW$$

where $S_\alpha = \chi^*_\alpha S$ denotes the gauge invariant Yang-Mills action expressed in terms of the adapted coordinate $B$, and where the explicit forms of the vielbein $E_\alpha$ corresponding
to the change of coordinates \( A \to (B, g) \), the induced metric \( G_\alpha \), its inverse and its determinant can be found in \([7]\); for completeness we just recall that

\[
\det G_\alpha = \det(R_g^* R_g) (\det F_\alpha)^2 \left( \det \Delta_{A_0^{(\alpha)}} \right)^{-1}.
\]

Here \( \sqrt{\det(R_g^* R_g)} \) implies an invariant volume density on \( \mathcal{G} \), where \( R_g \) is the differential of right multiplication transporting any tangent vector in \( T_g \mathcal{G} \) back to the identity \( \text{id}_G \) on \( \mathcal{G} \); \( F_\alpha = D_A^{* (\alpha)} D_B \) is the Faddeev–Popov operator.

The generalized stochastic quantization procedure amounts to consider the modified Langevin equation

\[
d\Psi = \left( -G_\alpha^{-1} \frac{\delta S_\alpha}{\delta \Psi} + \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (G_\alpha^{-1} \sqrt{\det G_\alpha})}{\delta \Psi} + E_\alpha D_A a \right) ds + E_\alpha (1 + D_A b) dW,
\]

where \( A = B^\alpha \). Here \( a \) and the \( \text{Lie} \mathcal{G} \) valued one form \( b \) are à priori arbitrary and will be fixed later on.

The above Langevin equation is the most general Langevin equation for Yang–Mills theory which leads to the same expectation values of gauge invariant variables as the original Parisi–Wu equation (1): An easy way to prove this assertion is to observe that the \( a \) and \( b \) dependent terms in the modified Langevin equation (9) drop out after projecting on the gauge invariant subspace \( \Gamma_\alpha \) described by the coordinate \( B \).

Transforming back the Langevin equation (9) into the original coordinates \( A \) not only Zwanziger’s original term \( D_A a \) is appearing, but also an additional \( b \)-dependent drift term as well as a specific modification of the Wiener increment, described by the operator \( \hat{e} = 1 + D_A b \). The idea is to view \( \hat{e} \) as a vielbein giving rise to the inverse of a metric \( \hat{g} \) on the space \( \pi^{-1}(U(A_0^{(\alpha)})) \). Since any of the \( \hat{g} \) (parametrized by the not yet specified \( b \)) implies a specific connection one is likely to arrive at an analogous obstruction as mentioned above. It is therefore necessary to require that the corresponding curvature should vanish. Indeed, we know that locally there exists a flat connection \( \tilde{\gamma}_\alpha \) in our bundle namely the pull-back of the Maurer–Cartan form \( \theta = \text{ad}(g^{-1}) R_g \) on the gauge group via the local trivialization of the bundle \( \pi^{-1}(U(A_0^{(\alpha)})) \to U(A_0^{(\alpha)}) \). Explicitely \( \tilde{\gamma}_\alpha \) is given by
the following expression

\[ \tilde{\gamma}_\alpha = \text{ad}(g^{-1})F^{-1}_\alpha D^{*}_{A_0^{(\alpha)}} \text{ad}(g), \quad A = B^g. \]  

(10)

The associated horizontal subbundle \( \tilde{\mathcal{H}} \) is built by all those vectors in the tangent space in \( A \in \pi^{-1}(U(A_0^{(\alpha)})) \) which can be written in the form \( \tau = \text{ad}(g^{-1})\zeta_B \), where \( A = B^g \) and \( \zeta_B \) is a tangent vector of \( \Gamma_\alpha \) in point \( B \). The fact that the curvature corresponding to the connection \( \tilde{\gamma}_\alpha \) is vanishing implies that the horizontal subbundle \( \tilde{\mathcal{H}} \) is integrable. The connection \( \tilde{\gamma}_\alpha \) cannot be extended to a globally defined flat connection on the whole bundle \( A \to \mathcal{M} \), however, due to its nontriviality.

We determine the value of \( b \) by fixing the metric \( \tilde{g} \) in such a way that \( \tilde{\gamma}_\alpha \) is exactly the induced connection imposed by itself. This implies that the horizontal subbundle \( \tilde{\mathcal{H}} \) is orthogonal to the gauge orbits with respect to the metric \( \tilde{g} \) and -in this sense- the gauge fixing surface is orthogonal to the gauge orbits.

The determination of \( a \) can in fact be given a suggestive meaning, too: we have the freedom to totally exchange the drift term of the \( g \)-field component of the Langevin equation (9) by the damping term \( -(\tilde{G}^{-1})^{-1}_\alpha \frac{\delta S_{\tilde{g}}[g]}{\delta g} \) and add a judiciously chosen Ito-term \( \frac{1}{\sqrt{\text{det} G}} \delta((\tilde{G}^{-1})^{-1}_\alpha \sqrt{\text{det} G}) \) as well. Here we introduced a new vielbein \( \tilde{E}_\alpha \) and a new (inverse) metric \( \tilde{G}^{-1}_\alpha \)

\[ \tilde{E}_\alpha = E_\alpha(1 + D_A b), \quad \tilde{G}^{-1}_\alpha = \tilde{E}_\alpha^* \tilde{E}_\alpha, \]  

(11)

with \( A = B^g \); furthermore \( S_{\tilde{g}}[g] \) is an arbitrary damping function with the property that

\[ \int g \sqrt{\det(R_g^* R_g)} e^{-S_{\tilde{g}}[g]} < \infty. \]  

(12)

Due to these choices for \( a \) and \( b \) we firstly obtain a well damped Langevin equation for \( g \) and secondly can recast the Langevin equation (9) into the geometrically distinguished form

\[ d\Psi = \left[ -\tilde{G}^{-1}_\alpha \frac{\delta S_{\tilde{g}}^{\text{tot}}}{\delta \Psi} + \frac{1}{\sqrt{\text{det} G_\alpha}} \frac{\delta((\tilde{G}^{-1}_\alpha \sqrt{\text{det} G_\alpha}))}{\delta \Psi} \right] ds + \tilde{E}_\alpha dW. \]  

(13)

Here

\[ S_{\tilde{g}}^{\text{tot}} = \chi^*_\alpha S + pr^*_g S_g \]  

(14)
denotes a total Yang-Mills action defined by the original Yang-Mills action $S$ without
gauge symmetry breaking terms and by the above $S_G; pr_G$ is the projector $\Gamma_\alpha \times G \to G$. The
associated Fokker–Planck equation can be derived in a straightforward manner where
now the Fokker-Planck operator $L[\Psi]$ is appearing in the factorized form

$$L[\Psi] = \frac{\delta}{\delta \Psi} \tilde{G}_\alpha^{-1} \left[ \frac{\delta S_{\text{tot}}}{\delta \Psi} - \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (\sqrt{\det G_\alpha})}{\delta \Psi} + \frac{\delta}{\delta \Psi} \right].$$

Due to the positivity of $\tilde{G}_\alpha$ the fluctuation dissipation theorem applies and the (non-
normalized) equilibrium Fokker–Planck distribution can be obtained by direct inspection
of the first order differential operator on the right hand side of the Fokker-Planck operator as

$$\mu_\alpha e^{-S_{\text{tot}}}, \quad \mu_\alpha = \sqrt{\det G_\alpha}.$$  

(15)

It is the basic idea of the stochastic quantization scheme to interpret an equilibrium limit
of a Fokker–Planck distribution as Euclidean path integral measure. Although our above
result for the path integral measure implies unconventional finite contributions along the
gauge group (arising from the $pr_G^* S_G$ term) it is equivalent to the usual Faddeev–Popov
prescription for Yang–Mills theory. This follows from the fact that for expectation values
of gauge invariant observables these contributions along the gauge group are exactly
cancelled out due to the normalization of the path integral, see below. We stress once
more that due to the Gribov ambiguity the usual Faddeev–Popov approach as well as
-presently- our modified version are valid only locally in field space.

In order to compare expectation values on different patches we consider the diffeo-
morphism in the overlap of two patches

$$\phi_{\alpha_1 \alpha_2} : (\Gamma_{\alpha_1} \cap \pi^{-1}(U_{\alpha_2})) \times G \to (\Gamma_{\alpha_2} \cap \pi^{-1}(U_{\alpha_1})) \times G \quad \phi_{\alpha_1 \alpha_2}(B, g) := (B^{\omega_{\alpha_2}(B)}^{-1}, g).$$

(17)

Here $\omega_{\alpha_2} : \pi^{-1}(U_{\alpha_2}) \to G$ is uniquely defined by $A^{\omega_{\alpha_2}(A)}^{-1} \in \Gamma_{\alpha_2}$. To the density $\mu_\alpha$
there is associated a corresponding twisted top form on $\Gamma_\alpha \times G$ (see e.g. [15]) which for
simplicity we denote by the same symbol. Using for convenience a matrix representation
of $G_\alpha$ [7] we straightforwardly verify that

$$\phi_{\alpha_1 \alpha_2}^* \mu_{\alpha_2} = \mu_{\alpha_1}.$$

(18)
This immediately implies that in overlap regions the expectation values of gauge invariant observables \( f \in C^\infty(\mathcal{A}) \) are equal when evaluated in different patches.

Finally we propose the definition of the global expectation value of a gauge invariant observable \( f \in C^\infty(\mathcal{A}) \) by summing over all the elements \( e_\alpha \) of a partition of unity on \( \mathcal{M} \) (the existence of a partition of unity is guaranteed by the result of [14]) so that

\[
\langle f \rangle = \frac{\sum_\alpha \int_{\Gamma_\alpha \times \mathcal{G}} \mu_\alpha e^{-S_{\text{tot}}^\alpha} \chi_\alpha^*(f \pi^* e_\alpha)}{\sum_\alpha \int_{\Gamma_\alpha \times \mathcal{G}} \mu_\alpha e^{-S_{\text{tot}}^\alpha} \chi_\alpha^* \pi^* e_\alpha}.
\]

Due to (18) it is trivial to prove that the global expectation value \( \langle f \rangle \) is independent of the specific choice of the locally finite cover \( \{U_\alpha\} \), of the choice of the background gauge fields \( \{A_0^{(\alpha)}\} \) and of the choice of the partition of unity \( e_\alpha \), respectively.

These structures can equally be translated into the original field space \( \mathcal{A} \). With the help of the partition of unity the locally defined densities \( \mu_\alpha \) as well as \( e^{-S_{\text{tot}}^\alpha} \) can be pieced together to give a globally well defined twisted top form \( \Omega \) on \( \mathcal{A} \)

\[
\Omega := \sum_\alpha \chi_\alpha^{-1*} (\mu_\alpha e^{-S_{\text{tot}}^\alpha}) \pi^* e_\alpha.
\]

The global expectation value (19) then reads

\[
\langle f \rangle = \frac{\int_\mathcal{A} \Omega f}{\int_\mathcal{A} \Omega}
\]

which due to the discussion from above is independent of all the particular local choices.

Let us extract the gauge invariant part of the local Fokker–Planck densities (16)

\[
\det \mathcal{F}_\alpha (\det \Delta_{A_0^{(\alpha)}})^{-1/2} e^{-\chi_\alpha S}.
\]

By using (18) we can prove that their projections on \( \mathcal{M} \) on overlapping sets of \( \mathcal{U} \) are agreeing so that they are giving rise to a globally well defined top form \( \tilde{\Omega} \) on \( \mathcal{M} \). One can furthermore show that the above expectation values of gauge invariant functions \( f \) can identically be rewritten as corresponding integrals over the gauge orbit space \( \mathcal{M} \) with respect to \( \tilde{\Omega} \)

\[
\langle f \rangle = \frac{\int_\mathcal{M} \tilde{\Omega} f}{\int_\mathcal{M} \tilde{\Omega}}.
\]
We note that this last expression shows agreement with the formulation proposed by Stora [3] upon identification of $\tilde{\Omega}$ with the Ruelle-Sullivan form [16].

Whereas in [3] the global definition (23) of expectation values on $\mathcal{M}$ appeared as the starting point for a path integral formulation of Yang-Mills theory in the whole space of gauge potentials it appears now as our final result; we aimed at its direct derivation within the stochastic quantization approach: First we derived a local path integral measure on $\Gamma_\alpha \times \mathcal{G}$ in terms of the probability density $\mu_\alpha e^{-S_{\text{tot}}}$ which assured gaussian decrease along the gauge fixing surface as well as along the gauge orbits. The inherent interrelation of the field variables on the patches $\Gamma_\alpha \times \mathcal{G}$ subsequently led to simple relations of the local densities in the overlap regions and eventually to the global path integral formulations (19), (21) and (23), respectively.

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**References**

[1] L. Faddeev and V. Popov, Phys. Lett. **25B**, 29 (1967).

[2] V. Gribov, Nucl. Phys. **B139**, 1 (1978).

[3] R. Stora, in *BRs Symmetry*, edited by M. Abe et al. (Universal Academy Press, Tokyo, 1996), p. 1, p. 335; hep-th/961114; hep-th/96115; hep-th/96116.

[4] C. Becchi and C. Imbimbo, in *BRs Symmetry*, edited by M. Abe et al. (Universal Academy Press, Tokyo, 1996), p. 265;
C. Becchi, S. Giusto and C. Imbimbo; hep-th/9811018.

[5] H. Hüffel and G. Kelnhofer, Phys. Lett. **B408** (1997) 241

[6] H. Hüffel and G. Kelnhofer, Ann. of Phys. **266**, 417 (1998).
[7] H. Hüffel and G. Kelnhofer, Ann. of Phys. **270**, 231 (1998).

[8] H. Hüffel and G. Kelnhofer, Univ. of Vienna preprint UWThPh-1998-65, hep-th/9901165.

[9] G. Parisi and Wu Yongshi, Sci.Sin. **24** (1981) 483

[10] P. Damgaard and H. Hüffel, Phys. Rep. **152** (1987) 227

[11] M. Namiki, “Stochastic Quantization”, Springer, Heidelberg, 1992

[12] D. Zwanziger, Nucl. Phys. **B192** (1981) 25

[13] L. Baulieu and D. Zwanziger, Nucl. Phys. **B193** (1981) 163

[14] P. Mitter and C. Viallet, Comm. Math. Phys. **79**, 457 (1981).

[15] R. Bott and L.Tu, *Differential Forms in Algebraic Topology* (Springer, New York, 1982).

[16] D. Ruelle and D. Sullivan, Topology **14**, 319 (1975);
A. Connes, *Non Commutative Geometry* (Academic Press, New York, 1994).