SMALL MOMENTUM EVOLUTION OF THE EXTENDED
DRELL–HEARN–GERASIMOV SUM RULE *

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ABSTRACT

We investigate the momentum dependence of the extended Drell-Hearn-Gerasimov
sum rule. An economical formalism is developed which allows to express the extended
DHG sum rule in terms of a single virtual Compton amplitude in forward direction.
Rigorous results for the small momentum evolution are derived from chiral perturbation
theory within the one-loop approximation. Furthermore, we evaluate some higher order
contributions arising from $\Delta(1232)$ intermediate states and relativistic corrections.

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I. INTRODUCTION

Many years ago, Drell and Hearn [1] and Gerasimov [2] (DHG) suggested a sum rule for spin-dependent Compton scattering. It expresses the squared anomalous magnetic moment of the nucleon in terms of a dispersive integral over the difference of the total photonucleon absorption cross sections \( \sigma_{1/2}(\omega) \) and \( \sigma_{3/2}(\omega) \) for the scattering of circular polarized photons on polarized nucleons. The subscripts \( \lambda = 1/2 \) and \( \lambda = 3/2 \) denote the total \( \gamma N \) helicity, corresponding to states with photon and nucleon spin antiparallel or parallel. Experimentally, this sum rule has never been tested directly since up to now no measurements of the helicity cross sections have been performed. However, models for the photoabsorption cross sections [3,4,5] do indicate its approximate validity (on a qualitative level). One can now extend this sum rule to virtual photons with \( k^2 < 0 \) the four-momentum transfer of the virtual photon\(^*\) since the corresponding helicity cross sections can be parametrized in terms of the spin-dependent nucleon structure functions.

The recent data of the European Muon Collaboration [6] taken in the scaling region of large \( |k^2| \approx 10 \text{ GeV}^2 \) suggest not only that the pertinent sum rule behaves as \( 1/k^2 \) for large \( |k^2| \), but also that the sign is opposite to the DHG sum rule for real photons (which in standard notations is negative). Therefore the integral

\[
I(k^2) = \int_{\omega_{thr}}^{\infty} \frac{d\omega}{\omega} [\sigma_{1/2}(\omega, k^2) - \sigma_{3/2}(\omega, k^2)]
\] (1.1)

with \( \omega \) the virtual photon energy in the nucleon rest frame must change its sign between the photon point \( (k^2 = 0) \) and the EMC region, \( k^2 \approx -10 \text{ GeV}^2 \). A recent model predicts this turnover to happen at \( k^2 \approx -0.8 \text{ GeV}^2 \) [7] and it explains this value mainly in terms of the low-energy contribution of the \( \Delta(1232) \) resonance to the pertinent photoabsorption cross sections. Notice that the model of ref.[7] as well as the phenomenological analysis of ref.[5] seem to indicate a negative slope of \( I_p(k^2) \) in the vicinity of the photon point, \( k^2 \approx 0 \).

Here, we wish to add some new insight into the momentum dependence of the integral \( I(k^2) \) in the region of small \( k^2 \) where small means that \( \sqrt{-k^2} \) does not exceed a few pion masses. Our model–independent analysis is based on the fact that at low energies, the interactions of hadrons are governed by chiral symmetry and gauge invariance (when external photons are involved). One can systematically solve the chiral Ward-Takahashi identities of QCD via an expansion in external momenta and quark masses, which are considered small against the scale of chiral symmetry breaking, \( \Lambda_\chi \approx 1 \text{ GeV} \). This method is called chiral perturbation theory. It uses the framework of an effective lagrangian of the asymptotically observed fields. The low-energy expansion corresponds to an expansion in pion loops. In the presence of baryons, a complication arises. The nucleon (baryon) mass in the chiral limit is comparable to the chiral scale \( \Lambda_\chi \) and thus only baryon three-momenta can be considered small [8]. One can, however, restore the exact one-to-one correspondence between the loop and low-energy expansion using a

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\* It is customary to set \( k^2 = -Q^2 \) and only use \( Q^2 \). We will not do this in the following.
non-relativistic formulation of baryon chiral perturbation theory [9]. The nucleon is considered as a very heavy (static) source and in that case, all momenta involved are small therefore restoring the consistent power counting. In what follows, we will use the non-relativistic version of baryon CHPT which was systematically investigated in ref.[10] as well as the relativistic formulation as spelled out in detail in ref.[8]. This will allow us to extract the leading term in the chiral expansion of \( I(k^2) \) and to calculate the derivative of \( I(k^2) \) around \( k^2 \simeq 0 \). This is the region where CHPT applies. Furthermore, following the suggestion of Jenkins and Manohar [11], we will also add the \( \Delta(1232) \) resonance to non-relativistic baryon CHPT. The \( \Delta(1232) \) is the lowest nucleon excitation and its closeness to the nucleon mass, \( m_{\Delta} - m \approx 2.1 M_{\pi} \), might indicate substantial contributions from it (this is also supported by phenomenological models). In fact, using these various approximation schemes, we will get a band of values for the slope of \( I(k^2) \). Our most important result, however, is that independent of the scheme we are using, we find that \( I(k^2) \) increases as \(|k^2|\) increases (around \( k^2 \simeq 0 \)). This new result should serve as a constraint for all model builders and should eventually be seen in refined phenomenological analyses or directly from the data (when they will become available).

The paper is organized as follows. In section II, we spell out an economical formalism to calculate \( I(k^2) \) in terms of a single function which possesses a right-handed cut starting at the single pion production threshold. This method is considerably simpler than the one recently proposed by Meyer [12] whose formalism involves half-off-shell nucleon form factors. In section III, we use CHPT to calculate \( I(k^2) \) for the proton and the neutron at small \( k^2 \), in the extreme non-relativistic and the fully relativistic formulation. The contribution of loops involving the \( \Delta(1232) \) isobar in the non-relativistic approach is also discussed. The numerical results and conclusions are presented in section IV.

II. SPIN–DEPENDENT COMPTON SCATTERING: FORMALISM

In this section, we outline the formalism necessary to describe the scattering of polarized (virtual) photons on polarized nucleons (protons and neutrons). Denote by \( p \) and \( k \) the four-momenta of the nucleon and photon, respectively. It is convenient to work with the two Lorentz invariants \( k^2 \) and \( \omega = p \cdot k/m \), with \( m \) the nucleon mass. The spin of the photon and nucleon can couple to the values \( 1/2 \) and \( 3/2 \) with the corresponding photoabsorption cross sections denoted by \( \sigma_{1/2}(\omega, k^2) \) and \( \sigma_{3/2}(\omega, k^2) \), in order.* In what follows, we are interested in the extended Drell-Hearn-Gerasimov sum rule, \textit{i.e.} the integral

\[
I(k^2) = \int_{\omega_{thr}}^{\infty} \frac{d\omega}{\omega} \left[ \sigma_{1/2}(\omega, k^2) - \sigma_{3/2}(\omega, k^2) \right] \quad (2.1)
\]

* For the definition of these cross sections see ref.[13] (chap.2). We omit the tilde over the symbol \( \sigma \) used in that book.
with \( k^2 \leq 0 \) and the threshold photon energy \( \omega_{thr} \) due to single pion electroproduction is given by

\[
\omega_{thr} = M_\pi + \frac{M_\pi^2 - k^2}{2m}
\]

(2.2)

where \( M_\pi \) denotes the pion mass. For real photons, the expression (2.1) becomes the celebrated DHG sum rule

\[
I(0) = \int_{\omega_{thr}}^{\infty} \frac{d\omega}{\omega} \left[ \sigma_{1/2}(\omega, 0) - \sigma_{3/2}(\omega, 0) \right] = -\frac{\pi e^2 \kappa^2}{2m^2}.
\]

(2.3)

Here, \( \kappa \) is the anomalous magnetic moment of the proton or the neutron and we use standard units, \( e^2/4\pi = 1/137.036 \). The DHG sum rule is derived under the assumption that the spin-dependent forward Compton amplitude for real photons \( f_2(\omega^2) \) satisfies an unsubtracted dispersion relation which guarantees that the right-hand side of eq.(2.3) converges. In what follows, we will make use of the same assumption for virtual photons. To set the scale for \( I(k^2) \), let us give the numerical values for the proton and the neutron,

\[
I_p(0) = -0.526 \text{ GeV}^{-2}, \quad I_n(0) = -0.597 \text{ GeV}^{-2}.
\]

(2.4)

Our main concern will be the \( k^2 \) evolution of the extended DHG sum rule, in particular around the origin \( k^2 \approx 0 \). The interest in that comes from the relation of the helicity cross sections to the spin-dependent nucleon structure functions \( G_1(\omega, k^2) \) and \( G_2(\omega, k^2) \).

Following the notations of Ioffe et al. [13]*, one can show that

\[
\sigma_{1/2}(\omega, k^2) - \sigma_{3/2}(\omega, k^2) = \frac{4\pi e^2}{2m\omega + k^2} \frac{\omega}{m} \left[ G_1(\omega, k^2) + \frac{k^2}{m\omega} G_2(\omega, k^2) \right].
\]

(2.5)

The relation of these structure functions to the spin-dependent virtual Compton amplitudes in forward direction \( S_{1,2}(\omega, k^2) \) is standard

\[
2\pi G_i(\omega, k^2) = \text{Im} S_i(\omega, k^2), \quad (i = 1, 2)
\]

(2.6)

which follows from the optical theorem. Furthermore, crossing symmetry implies that \( S_1(\omega, k^2) \) and \( G_2(\omega, k^2) \) are even functions under \( (\omega \to -\omega) \) whereas \( S_2(\omega, k^2) \) and \( G_1(\omega, k^2) \) are odd. In fact, for our purpose one does not need the information on both amplitudes \( S_1(\omega, k^2) \) and \( S_2(\omega, k^2) \) but only the particular combination entering eq.(2.5). In order to isolate this relevant combination one contracts the antisymmetric (in \( \mu \leftrightarrow \nu \)) part of the virtual Compton tensor in forward direction with polarization vectors \( \epsilon^\nu_\mu \) and \( \epsilon_\nu \) for the outgoing and incoming virtual photon, respectively. If we choose the gauge

* We use a different normalization for the nucleon spinor, \( \bar{u}u = 1 \) instead of \( \bar{u}u = 2m \).
conditions \( \epsilon \cdot p = \epsilon' \cdot p = \epsilon \cdot k = \epsilon' \cdot k = 0 \) for the polarization vectors and work in the nucleon rest-frame \( p_\mu = (m, 0, 0, 0) \) we obtain

\[
e' \, T^{\mu \nu}_{(a)} \, \epsilon_\nu = \frac{i}{2m^2} \chi \left\{ \bar{\sigma} \cdot (\vec{e}' \times \vec{e}) \left[ \omega S_1(\omega, k^2) + \frac{\omega^2}{m} S_2(\omega, k^2) \right] - \bar{\sigma} \cdot \vec{k} \vec{k}' \cdot (\vec{e}' \times \vec{e}) \frac{S_2(\omega, k^2)}{m} \right\} \chi
\]

\[
= \frac{i \omega}{2m^2} \chi \bar{\sigma} \cdot (\vec{e}' \times \vec{e}) \chi \left[ S_1(\omega, k^2) + \frac{k^2}{m \omega} S_2(\omega, k^2) \right]
\]

where \( \chi \) is a conventional two-component (Pauli) spinor. In eq.(2.7) we have exploited the fact that under the chosen gauge \( \vec{e}' \times \vec{e} \) is parallel to \( \vec{k} = \omega^2 - k^2 \). Obviously, we are projecting out the particular combination of \( S_1(\omega, k^2) \) and \( S_2(\omega, k^2) \) whose imaginary part enters the extended DHG sum rule \( I(k^2) \). In analogy to the real photon case we call this combination

\[
f_2(\omega^2, k^2) = \frac{e^2}{8\pi m^2} \left[ S_1(\omega, k^2) + \frac{k^2}{m \omega} S_2(\omega, k^2) \right]. \tag{2.8}
\]

Here, we indicated already that \( f_2(\omega^2, k^2) \) is an even function of \( \omega \) which follows from the \((\omega \to -\omega)\) crossing properties of \( S_{1,2}(\omega, k^2) \) \([13]\). The odd amplitude \( \omega f_2(\omega^2, k^2) \) can now be expressed in terms of a single function \( A(s, k^2) \) as follows

\[
2\pi(s - m^2 - k^2) f_2(\omega^2, k^2) = e^2 \left[ A(s, k^2) - A(2m^2 + 2k^2 - s, k^2) \right]. \tag{2.9}
\]

Here, we introduced the Mandelstam variable \( s = (p + k)^2 \) which is related to \( \omega \) via \( \omega = (s - m^2 - k^2)/2m \). The function \( A(s, k^2) \) appearing in eq.(2.9) can always be chosen such that it has only a right-handed cut starting at the single pion production threshold \( s = (m + M_\pi)^2 \). Under the assumption that \( f_2(\omega^2, k^2) \) fulfills an unsubtracted dispersion relation (in \( \omega \)) or equivalently that \( A(s, k^2) \) fulfills a once-subtracted dispersion relation (in \( s \), subtracted at an arbitrary point \( s_0 \)) we can make use of the previous equations and calculate the extended DHG sum rule \( I(k^2) \) as

\[
I(k^2) = 8\pi \frac{\text{Im} f_2(\omega^2, k^2)}{(m + M_\pi)^2} ds \frac{\text{Im} A(s, k^2)}{s - m^2}
\]

\[
= 4e^2 \frac{\text{Im} A(s, k^2)}{(s - m^2)(s - m^2 - k^2)} \tag{2.10}
\]

\[
= \frac{4\pi e^2}{k^2} \left[ A(m^2 + k^2, k^2) - A(m^2, k^2) \right].
\]

This equation is our basic result. It is completely general and allows one to calculate the extended DHG sum rule \( I(k^2) \) from a single function \( A(s, k^2) \) which can be easily computed from the virtual Compton tensor in forward direction. To repeat it, eq.(2.10) was derived under the assumption that \( A(s, k^2) \) obeys a once-subtracted dispersion
relation. That this is not a too strong assumption e.g. can be seen from the fact that in the relativistic formulation of baryon CHPT to one-loop \( A(s, k^2) \) indeed has this analytical property. However, a general proof for this is not yet available. In this sense the situation is analogous to \( f_2(\omega^2, 0) \) where the validity of an unsubtracted dispersion relation can not yet be proven in general. In the following section, we will use CHPT (in the one–loop approximation) to evaluate \( A(s, k^2) \) and to calculate \( I(k^2) \) for \( k^2 \) in the vicinity of zero (this is where CHPT applies).

III. CHIRAL EXPANSION

At low energies, any QCD Green function can be systematically expanded in powers of small momenta and quark (pion) masses. This is done within the framework of an effective chiral lagrangian of the asymptotically observed fields, here the nucleons, pions and photons. The low-energy expansion amounts to an expansion in (pion) loops of the effective theory. In the presence of baryons, a complication arises due to the baryon mass which is non-vanishing in the chiral limit and therefore adds a new scale to the theory. In that case there is in general no strict one-to-one correspondence between the low energy and loop expansion. Stated differently, there is no guarantee that all next-to-leading order corrections at order \( q^3 \) (with \( q \) denoting a generic small momentum) are given completely by the one loop graphs. All calculations performed so far, however, indicate that the leading non-analytic terms (in the quark masses) which arise due to infrared singularities in the chiral limit of vanishing pion mass are indeed produced. Furthermore one also gets in the one loop approximation an infinite tower of higher order terms [8] which spoil the one-to-one mapping between low-energy and loop expansion. To overcome these difficulties, it was recently proposed to use a heavy fermion effective field theory, i.e. considering the baryons as very heavy [9] and to expand the theory in inverse powers of the baryon mass. In that case, the \( n \)-loop contributions are suppressed by relative powers of \( q^{2n} \) (with \( q \) a genuine small momentum) and a consistent counting scheme emerges. Furthermore, in this framework one can easily couple in the \( \Delta(1232) \) resonance since one does not encounter the usual problems with the relativistic spin-3/2 particle [11]. Nevertheless, we have to stress that the baryon mass \( m \) comparable to the chiral symmetry breaking scale \( \Lambda_\chi \) is not very large. Therefore, an expansion in powers of \( M_\pi/m \) is a priori not to be expected to converge very fast. Such \( M_\pi/m \) suppressed contributions are partly resummed in the relativistic approach. Of course the evaluation of all \( M_\pi/m \) corrections is necessary to judge the quality of the chiral expansion. Furthermore, once the spin–3/2 decuplet is included, one has an extra non–vanishing scale in the chiral limit (the average octet–decuplet mass splitting) which again complicates the low energy structure.

The basic \( \pi N \gamma \) lagrangian in the relativistic formulation of baryon CHPT to leading order (\( \mathcal{O}(q) \)) reads

\[
\mathcal{L} = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi \pi} \\
\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}(i\not{D} - m + \frac{g_A}{2}\not{\gamma}_5)\Psi \\
\mathcal{L}^{(2)}_{\pi \pi} = \frac{F^2}{4}\text{Tr}[\nabla_\mu U\nabla^\mu U^\dagger + M_\pi^2(U + U^\dagger)]
\]
where \( U = \exp[i \vec{r} \cdot \vec{\pi}/F] \) embodies the Goldstone bosons, \( u = \sqrt{U} \) and \( u_\mu = i u^\dagger \nabla_\mu U u^\dagger \) with \( \nabla_\mu \) the pertinent covariant derivative. The isospinor \( \Psi \) contains the proton and neutron fields. The superscript \((i)\) denotes the chiral power of the corresponding terms, it counts derivatives and meson masses. The construction of this effective lagrangian is unique. Let us point out that it contains four parameters. These are the pion decay constant \( F \), the axial-vector coupling \( g_A \) and the nucleon mass (all in the chiral limit) and the leading term in the quark mass expansion of the pion mass, \( M_\pi = \sqrt{2mB} \). Here, \( \hat{m} = \frac{1}{2}(m_u + m_d) \) is the average light quark mass and \( B = -\langle 0 | \bar{u} u | 0 \rangle / F^2 \) is the order parameter of the spontaneous chiral symmetry breaking. Calculating tree diagrams with this effective lagrangian, one reproduces the well-known current algebra results. To restore unitarity, one has to consider pion loops in addition. To give all corrections at next-to-leading order in the chiral expansion one has to work out all one loop diagrams constructed from the vertices in \( L \) and furthermore one has to add the tree graph contribution from the most general chirally symmetric counterterm lagrangian \( L^{(2)}_{\pi N} + L^{(3)}_{\pi N} + L^{(4)}_{\pi \pi} \). For the (spin-dependent) Compton tensor under consideration here, however, no such counterterm can contribute. As stressed in ref.[10], we are dealing with a pure loop effect (within the one-loop approximation).

As already noted, in eq.(3.1) the troublesome nucleon mass term appears. In the extreme non-relativistic limit, it can be eliminated in the following way. Decompose the baryon four momentum as \( p_\mu = mv_\mu + l_\mu \) with \( v_\mu \) the four-velocity \((v^2 = 1)\) and \( l_\mu \) a small off-shell momentum \((v \cdot l \ll m, )\) and write \( \Psi \) in terms of eigenstates of the velocity projection operator

\[
\Psi = e^{-im \cdot \hat{v} \cdot \hat{x}}(H + h)
\]

with \( \hat{v}H = H \) and \( \hat{v}h = -h \). Eliminating now the "small" component \( h \) via its equation of motion, one ends up with

\[
L^{(1)}_{\pi N} = \hat{H}(iv \cdot D + g_A S \cdot u)H + O(1/m)
\]

(3.3)

Here, \( S_\mu = \frac{i}{2} \gamma_5 \sigma_{\mu \nu} v^\nu \) is the covariant spin operator which obeys \( S \cdot v = 0 \). The nucleon mass term has disappeared allowing for a consistent chiral power counting scheme. All one loop contributions are order \( q^3 \). Furthermore, one has to expand the tree contributions from the vertices of eq.(3.1) in \( 1/m \) appropriately to collect all terms up to and including order \( q^3 \). For a more detailed discussion of these topics, see ref.[10]. One can furthermore add the \( \Delta(1232) \), which is a spin-3/2 field, very easily in the extreme non-relativistic limit. For details on the couplings of the \( \Delta(1232) \) see the appendix. Here, we just note that the mass splitting \( m_\Delta - m \) stays finite in the chiral limit. Therefore loops with intermediate \( \Delta(1232) \) states will count as order \( q^4 \) and higher (since the counterterm contributions start only at order \( q^5 \)).

Let us now turn to the calculation of \( I(k^2) \) for small \( k^2 \). In Fig.1.a we show the pertinent Feynman diagrams which contribute in the heavy mass limit (with intermediate nucleons only). We work in the Coulomb gauge \( e^e \cdot v = \epsilon \cdot v = 0 \) which is very economical in the calculation of photon-nucleon processes since most diagrams (those
with an isolated photon-nucleon vertex) are then identical to zero. The integral \( I(k^2) \) takes the form
\[
I(k^2) = I(0) + \bar{I}(k^2)
\] (3.4)
with \( I(0) = -\pi e^2 \kappa^2 / 2m^2 \) the DHG sum rule value for real photons. In the heavy mass formulation of baryon CHPT the leading term of the chiral expansion of \( \bar{I}(k^2) \) is given completely by the one loop graphs in Fig.1a. All higher order corrections to \( \bar{I}(k^2) \) are suppressed by further powers of the pion mass \( M_\pi \) and \( k^2 \). Some (but not all) of these corrections will be generated from loop diagrams with \( \Delta(1232) \) intermediate states or in the relativistic version of baryon CHPT. The leading term of the chiral expansion of \( \bar{I}(k^2) \) can be given in closed form
\[
\bar{I}(k^2) = \frac{e^2 g_A^2}{4\pi F^2} \left[-1 + \sqrt{1 + \frac{4}{\rho}} \ln\left(\sqrt{1 + \frac{\rho}{4}} + \frac{\sqrt{\rho}}{2}\right)\right] = \frac{e^2 g_A^2}{48\pi F^2} \rho + \mathcal{O}(\rho^2)
\] (3.5)
with \( \rho = -k^2 / M_\pi^2 > 0 \). We see that the slope of \( I(k^2) \) at \( k^2 = 0 \) is negative and singular in the chiral limit, i.e. it diverges like \( 1 / M_\pi^2 \). This behaviour is a direct consequence of the chiral structure of QCD which governs the low-energy strong interaction phenomena. Furthermore, \( \bar{I}(k^2) \) is equal for both proton and neutron (within the \( \mathcal{O}(q^3) \) approximation to the virtual Compton tensor). We should also add here that presently the usual DHG sum rule value \( I(0) \) for real photons can not be obtained through a dispersive integral like eq.(2.10) within the one-loop approximation of CHPT. In the heavy mass formulation this term arises from real \( 1/m^2 \) suppressed tree graphs involving the anomalous magnetic moment \( \kappa \) (in the chiral limit). In the relativistic version of baryon CHPT the anomalous magnetic moment of the nucleon is generated from one loop diagrams and it is non-vanishing in the chiral limit. In order to obtain a term proportional to \( \kappa^2 \) like \( I(0) \) one necessarily has to go to the level of two-loop graphs. This problem of how \( I(0) \) can be obtained from a dispersion relation for loop amplitudes does, however, not affect our discussion of the \( k^2 \) dependence of \( I(k^2) \). Extending the effective lagrangian to the \( \Delta(1232) \) resonance as spelled out in the appendix we have to calculate the diagrams of Fig.1b. These amount to some higher order \( (q^n, n \geq 1) \) corrections to eq.(3.5) which we include because of the phenomenological importance of this resonance (a complete evaluation of all \( \mathcal{O}(q) \) corrections to \( I(k^2) \) corresponding to \( \mathcal{O}(q^4) \) for the virtual Compton tensor goes beyond the scope of this paper). A straightforward calculation gives for the sum of nucleon and \( \Delta(1232) \) one-loop diagrams
\[
\bar{I}(k^2) = \frac{e^2 g_A^2}{4\pi F^2} \left[\frac{r}{\sqrt{r^2 - 1}} \ln(r + \sqrt{r^2 - 1}) - \int_0^1 dx \frac{r}{\sqrt{r^2 - 1 - \rho x(1-x)}} \ln\left(\frac{r}{\sqrt{1 + \rho x(1-x)}} + \frac{r^2}{1 + \rho x(1-x) - 1}\right)\right]
\] (3.6)
with \( r = (m_\Delta - m)/M_\pi \simeq 2.1 \). Obviously, \( \bar{I}(0) = 0 \) in agreement with the celebrated low-energy theorem of Low, Gell-Mann and Goldberger [15]. As a check one can show that in the limit \( m_\Delta - m \to \infty \) one recovers the result of eq.(3.5). Again there is no splitting between proton and neutron sum rules, \( i.e. \bar{I}(k^2) = \bar{I}_p(k^2) = \bar{I}_n(k^2) \). The slope of the extended DHG sum rule at the photon point is given as

\[
I'(0) = \frac{e^2 g_A^2}{4\pi F^2 M_\pi^2} \frac{r^2 \sqrt{r^2 - 1} - r \ln(r + \sqrt{r^2 - 1})}{(r^2 - 1)^{3/2}}. \tag{3.7}
\]

In the relativistic formulation matters are different. First one has to calculate many more Feynman diagrams. These generate some of the \( M_\pi/m \) suppressed higher order corrections and naturally lead to a splitting between proton and neutron for the momentum dependence of the extended DHG sum rule, \( i.e. \bar{I}_p(k^2) \neq \bar{I}_n(k^2) \). What is conceptually most important is that in the relativistic version of baryon CHPT one can indeed show that the amplitude function \( A(s, k^2) \) obeys a once-subtracted dispersion relation. Using now the definitions of the various loop functions as given in ref.[14] extended to \( k^2 \leq 0 \), the following expressions can be deduced for \( \bar{I}_p(k^2) \) and \( \bar{I}_n(k^2) \)

\[
\bar{I}_p(k^2) = \frac{e^2 g_A^2 m^2}{4\pi F^2 k^2} \int_0^1 dx \int_0^1 dy \left\{ \frac{y^2}{2} \ln \frac{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(2 - 2m^2 + k^2 y)}{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)} - 2y \ln \frac{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)}{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(1-y)^2} \right\}
\]

\[
\bar{I}_n(k^2) = \frac{e^2 g_A^2 m^2}{4\pi F^2 k^2} \int_0^1 dx \int_0^1 dy \left\{ \frac{y^2}{2} \ln \frac{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(1-x + x)}{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(1-y)^2} + 2y \ln \frac{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(1-x + x)}{M_\pi^2(1-y) + m^2 y^2 + k^2 x(1-y)(1-y)^2} \right\}
\]

As an important analytical check we can again verify that \( \bar{I}_p(0) = \bar{I}_n(0) = 0 \) and one can show that in the limit \( m \to \infty \) both \( \bar{I}_p(k^2) \) and \( \bar{I}_n(k^2) \) tend to \( \bar{I}(k^2) \) as given in
eq. (3.5). With this we have collected all formulae necessary to study $I(k^2)$ for both the proton and the neutron.

IV. RESULTS AND DISCUSSION

First, we must fix parameters. Throughout, we use $F = 93$ MeV, $M_\pi = 139.57$ MeV, $m = 938.27$ MeV and $g_A = 1.26$. In the case of the $\Delta(1232)$ resonance, we use the $SU(4)$ relation among coupling constants $g_{\pi N\Delta} = 3g_{\pi N}/\sqrt{2}$ with $g_{\pi N} = g_A m/F$ given by the Goldberger-Treiman relation. The mass splitting between nucleon and $\Delta(1232)$ has a value of $m_\Delta - m = 293$ MeV.

Consider now the proton. We will first discuss the slope of $I_p(k^2)$ at the photon point, $k^2 = 0$. In the heavy mass limit with only intermediate nucleon states we find

$$\frac{dI_p(k^2)}{dk^2}\bigg|_{k^2=0} = -\frac{e^2 g_A^2}{48\pi F^2 M_\pi^2} = -5.7 \text{ GeV}^{-4}$$

(4.1)

This value is decreased by 16% when the $\Delta(1232)$ resonance is included in the one loop graphs as inspection of eq. (3.7) reveals. Therefore the $\Delta(1232)$ does not play a major role in determining the slope of $I_p(k^2)$ in our approach. Much more drastic is the effect of the relativistic $M_\pi/m$ suppressed terms. In the fully relativistic calculation where many (but not all) of such terms are included we find $I_p'(0) = -2.2 \text{ GeV}^{-4}$ for the proton and $I_n'(0) = -1.7 \text{ GeV}^{-4}$ for the neutron. In Fig. 2, we show $\tilde{I}_p(k^2)$ for $-k^2 \leq 0.25 \text{ GeV}^2$. In the heavy mass limit half of the value of $I_p(0)$ (in magnitude) is reached at $k^2 \simeq 0.06 \text{ GeV}^2$. The crossover where $I_p(k^2)$ goes from negative to positive values takes place at $k^2 \simeq -0.15 \text{ GeV}^2$. This is a very low value compared to previous phenomenological analysis but compared to the pion mass scale $M_\pi^2$ it is already quite large, $k^2 \simeq -7.7 M_\pi^2$. Therefore one can no longer trust the one loop approximation in that region of $k^2$ where the sign change of $I_p(k^2)$ takes place. Including some higher order chiral corrections through loops with $\Delta(1232)$ resonances, the momentum dependence of $\tilde{I}_p(k^2)$ becomes softer and the corresponding numbers decrease by roughly 30%. The zero of $I_p(k^2)$ is now shifted to a higher value of $k^2 \simeq -0.23 \text{ GeV}^2$. In the relativistic formulation of CHPT where in addition to the leading terms also many higher order corrections are included, $\tilde{I}_p(k^2)$ is much smaller than in the case of infinite nucleon mass. This phenomenon, that higher order relativistic correction are quite large was also observed in previous calculations of the nucleon electromagnetic polarizabilities [14]. However, since the $M_\pi/m$ corrections generated in the one–loop approximation of relativistic baryon CHPT are by no means complete, one can not draw any conclusions about the convergence of the chiral expansion at the moment.

In summary, we have presented a novel formalism to calculate the momentum dependence of the extended DHG sum rule at finite $k^2 \leq 0$. A single amplitude function $A(s, k^2)$ which enters the spin-dependent virtual Compton tensor in forward direction is sufficient to evaluate $I(k^2)$, as long as $A(s, k^2)$ fulfills a once–subtracted dispersion relation. We have used baryon chiral perturbation theory to investigate the behaviour of the extended DHG sum rule $I(k^2)$ in the vicinity of $k^2 = 0$. We could give a (rather wide) range of values for the slope $I_p'(0)$. Eventually, this prediction will be tested experimentally, at present we consider it as a constraint following from the chiral structure of QCD which will be useful for phenomenological analysis and model-building.
APPENDIX: THE $\Delta(1232)$ IN THE HEAVY MASS FORMULATION

Here, we discuss briefly the description of the $\Delta(1232)$ resonance in the heavy mass formulation following ref.[11]. To leading order (up to $O(q)$ the relevant effective lagrangian reads (we write down only those terms which are actually needed for our purpose)

$$\mathcal{L}^{(1)}_{\pi N\Delta} = -i\bar{T}^{\mu a} v \cdot D \bar{T}^{\mu a} T^{\mu a} + \frac{3g_A}{2\sqrt{2}}(\bar{T}^{\mu a} u^{a} H + \bar{H} u^{a} T^{\mu a}). \quad (A.1)$$

The Rarita-Schwinger spinor $T^{a}_{\mu}$ with $a$ an isospin index and $\mu$ a Lorentz index incorporates the four charge states of the $\Delta(1232)$ as follows

$$T^{1}_{\mu} = \frac{1}{\sqrt{2}} \left( \frac{\Delta^{++} - \Delta^{0}/\sqrt{3}}{\Delta^{0}/\sqrt{3} - \Delta^{-}} \right)_{\mu}, \quad T^{2}_{\mu} = \frac{i}{\sqrt{2}} \left( \frac{\Delta^{++} + \Delta^{0}/\sqrt{3}}{\Delta^{0}/\sqrt{3} + \Delta^{-}} \right)_{\mu}, \quad T^{3}_{\mu} = -\frac{\sqrt{2}}{3} \left( \frac{\Delta^{+}}{\Delta^{0}} \right)_{\mu}. \quad (A.2)$$

Furthermore in the heavy mass limit this field is subject to the constraint $v_{\mu} T^{\mu a} = 0$. In (A.1) $\delta m = m_{\Delta} - m$ stands for the mass splitting of nucleon and $\Delta(1232)$ and $v^{a}_{\mu} = \frac{i}{2} \text{Tr}(\pi^{a} u^{\dagger} \nabla_{\mu} U u^{\dagger}) = -\partial_{\mu} \pi^{a}/F - e e^{a3b} A_{\mu} \pi^{b}/F + \ldots$ gives rise to the chiral couplings of pions and photons to the $N\Delta$ system. We already exploited the $SU(4)$ relation $g_{\pi N\Delta} = 3g_{\pi N}/\sqrt{2}$ with $g_{\pi N} = g_A m/F$ between the $\pi N\Delta$ and $\pi NN$ coupling constant. The empirical information on the $\Delta \rightarrow \pi N$ decay width confirms that this relation holds very well within a few percent. In the heavy mass limit the propagator of the $\Delta(1232)$ reads

$$P^{\mu\nu} = \frac{i}{v \cdot l - \delta m} \left[ v^{\mu} v^{\nu} - g^{\mu\nu} - \frac{4}{3} S^{\mu} S^{\nu} \right] \quad (A.3)$$

where $S^{\mu}$ is the covariant spin operator of heavy mass approach satisfying $v \cdot S = 0$. Let us finally remark that this formulation of $\Delta(1232)$ couplings is completely equivalent to the usual isobar model as discussed in ref.[16] for the special choice $v_{\mu} = (1, 0, 0, 0)$. This corresponds to the standard non-relativistic description.

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FIGURE CAPTIONS

Fig.1. a) One loop diagrams contributing to the spin-dependent Compton tensor in the heavy mass formulation of CHPT. Dashed lines denote pions.
   b) One loop Compton graphs including the \( \Delta(1232) \) resonance in the heavy mass approach (denoted by a thick line).

Fig.2. The momentum dependence of the extended DHG sum rule \( \tilde{I}_p(k^2) \). The solid line gives the one–loop result in the heavy mass limit of baryon CHPT. The dashed line is obtained from one–loop graphs involving nucleons as well as \( \Delta(1232) \) resonances. The dashed–dotted line gives the result of the relativistic version of baryon CHPT to one loop.