G-consistent price systems and bid-ask pricing for European contingent claims under Knightian uncertainty

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Abstract The target of this paper is to consider model the risky asset price on the financial market under the Knightian uncertainty, and pricing the ask (upper) and bid (lower) prices of the uncertain risk. We use the nonlinear analysis tool, i.e., G-frame work [27], to construct the model of the risky asset price and bid-ask pricing for the European contingent claims under Knightian uncertain financial market.

First, we consider the basic risky asset price model on the uncertain financial market, which we construct here is the model with drift uncertain and volatility uncertain. We describe such model by using generalized G-Brownian motion and call it as G-asset price system. We present the uncertain risk premium which is uncertain and distributed with maximum distribution $N([\mu, \mu], \{0\})$. Under G-frame work we construct G-martingale time consistent dynamic pricing mechanism, we sketch the frame work which comes from our paper [8]. We derive the closed form of bid-ask price of the European contingent claim against the underlying risky asset with G-asset price system as the discounted conditional G-expectation of the claim, and the bid and ask prices are the viscosity solutions to the nonlinear HJB equations.

Second, we consider the main part of this paper, i.e., consider the risky asset on the Knightian uncertain financial market with the price fluctuation shows as continuous trajectories. We propose the G-conditional full support condition by using uncertain capacity, and the risky asset price path satisfying the G-conditional full support condition could be approximated by its G-consistent asset price systems. We derive that the bid and ask prices of the European contingent claim against such risky asset under uncertain can be expressed by discounted of some conditional G-expectation of the claim. We give examples, such as G-Markovian processes and the geometric fractional G-Brownian motion [9], satisfying the G-conditional full support condition.

Keywords Knightian uncertain, G-asset price system, G-consistent asset price systems, G-conditional full support, uncertain risk premium, bid and ask prices, European contingent claim

JEL-classification: G10,G12,G13,D80
1 Introduction

The global economic crisis started from 2008 has revived an old philosophic idea about risk and uncertainty – Knightian uncertainty. Frank Knight formalized a distinction between risk and uncertainty in his 1921 book, Risk, Uncertainty, and Profit ([18]). As Knight saw it, an ever-changing world brings new opportunities for businesses to make profits, but also means we have imperfect knowledge of future events. Therefore, according to Knight, risk exists when an outcome can be described as draw from a probability distribution. Uncertainty, on the other hand, applies to situations where we cannot know all the information we need in order to set accurate odds, i.e., we cannot know the probability distribution for the future world. "True uncertainty," as Knight called it, is "not susceptible to measurement."

The study on uncertainty is still in its infancy. There is literatures (see [5], [4], [7], [10], [12], [13], [16], [32] and references therein) about the macroeconomic uncertainty, which caused by uncertainty shocks, such as, war, political and economical crisis, and terrorist attacks, etc. For the future uncertain macroeconomic, investors have uncertain subjective belief, which makes their consumption and portfolio choice decisions uncertain. During a disaster an asset’s fundamental value falls by a time-varying amount. This in turn generates time-varying risk premia and, thus, volatile asset prices and return predictability.

The exist linear frame work can not describe the asset price in uncertain future, uncertain risk premia, uncertain return and uncertain volatility. How to model the asset price in uncertain future, and pricing the uncertain risk becomes an open problem. In this paper, we consider using G-frame work presented by Peng in [27], which is a powerful and beautiful nonlinear analysis tool, to construct the frame work to model the future risky asset price on the Knightian uncertain financial market and pricing the bid and ask prices of the uncertain risk.

In the first part of this paper, we define G-asset price system (see Section 3.1) which describes the uncertain drift and uncertain volatility of the risky asset price under uncertain. We consider the financial market consists of the risky asset (stock) with price fluctuation \((S_t)_{t \geq 0}\) modelled by G-asset price system and the bond \((P_t)_{t \geq 0}\) satisfying

\[
dP_t = rP_t dt \quad t \in [0, T], \quad P_0 = 1,
\]

where \(T > 0\) and \(r\) is short interest rate, we assume it is constant rate without loss the technique generality. On such financial market, the risk premium of the risky asset is uncertain and we call it uncertain risk premium, the price of the uncertain risk is also uncertain (see Section 3.1). We define a deflator which implies time value and uncertain risk value. By the technique of the G-frame work, we derive the closed forms of the bid and ask prices of the European contingent claim against the underlying asset with G-asset price system as conditional G-expectations of the deflated claim.

For construct the frame work to price option against the underlying asset with the G-asset price system, we present G-Girsanov transform and define G-consistent dynamic pricing mechanism, we give the expressions of the bid and ask prices of the European contingent claim as the discounted conditional G-expectation of claim, and the bid-ask prices are the viscosity solutions to the nonlinear HJB equations, from which the upper price and lower price of the European contingent claim can be numerically computed.

The second part is our main part of this paper, we consider the uncertain financial market with the risky asset price \((S_t)_{t \geq 0}\) in the future expressed by a kind of continuous trajectories which could be approximated by G-asset price systems, and we define such G-asset price systems as G-consistent price systems. European contingent claim against the underlying asset with the continuous asset price path perhaps not be priced upper and lower prices by using G-consistent dynamic pricing mechanism. Denote the risky price path by \(S(t)\) and the portfolio process as \(\pi(t)\), and the path Riemann sum as \(\sum \pi \Delta S\). Young-Kondurar ([21], [33]) Theorem tells that the Stieltjes integral exists on certain classes of the Hölder continuous path functions:
Theorem 1 (Young-Kondurar Theorem) Suppose $\alpha$ and $\gamma$ as the Hölder exponents of the price path $(S_t)_{t \geq 0}$ and the portfolio process path $(\pi_t)_{t \geq 0}$, respectively, and $\gamma > 1 - \alpha$. Then the integral

$$I_t = \int_0^t \pi_t dS_t$$

exists almost surely as a limit of Riemann-Stieltjes sums.

Under uncertainty, if the portfolio process and the risky asset price path satisfying the above Theorem and the risky asset price path has G-consistent price systems, by using G-framework we prove that there exists G-expectation such that the bid and ask prices of the European contingent claim against such risky asset have closed forms which are expressed as the discounted conditional G-expectation of the claim. We define uncertain capacity, by which we construct G-conditional full support condition. The risky asset price path with the G-condition full support condition satisfied is proved to have the G-consistent price systems, we give examples, such as, G-Markovian process and the geometric fractional G-Brownian motion (see [9]) have the properties satisfying G-condition full support condition.

The rest of our paper is organized as follows: In Section 2 we give notations and preliminaries for the G-framework. In section 3 the risky asset price model on the uncertainty future financial market is present, which we called G-asset price system, and we propose the G-martingale time consistent dynamic pricing mechanism for the European contingent claim against the risky asset with G-asset price system. In section 4 we consider uncertain risky asset price continuous path model on uncertain financial market, which satisfying G-conditional full support condition. And we prove that such uncertain price model have G-consistent price systems, and the bid-ask prices of the European contingent claim against such uncertain risky asset can be expressed as discounted of some conditional G-expectation of the claim. We give examples of processes which satisfying G-conditional full support condition.

2 Preliminaries

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ containing constants. The space $\mathcal{H}$ is also called the space of random variables.

Definition 1 A sublinear expectation $\hat{E}$ is a functional $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying

(i) Monotonicity: $\hat{E}[X] \geq \hat{E}[Y]$ if $X \geq Y$.

(ii) Constant preserving: $\hat{E}[c] = c$ for $c \in \mathbb{R}$.

(iii) Sub-additivity: For each $X, Y \in \mathcal{H}$,

$$\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y].$$

(iv) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

In this section, we mainly consider the following type of sublinear expectation spaces $(\Omega, \mathcal{H}, \hat{E})$: if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for $\varphi \in C_{b, Lip}(\mathbb{R}^n)$, where $C_{b, Lip}(\mathbb{R}^n)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|$$

for $x, y \in \mathbb{R}$, some $C > 0, m \in \mathbb{N}$ is depending on $\varphi$. 

3
For each fixed $p \geq 1$, we take $H_0^p = \{X \in H, \hat{E}[|X|^p] = 0\}$ as our null space, and denote $H/H_0^p$ as the quotient space. We set $\|X\|_p := (\hat{E}[|X|^p])^{1/p}$, and extend $H/H_0^p$ to its completion $\hat{H}_p$ under $\| \cdot \|_p$. Under $\| \cdot \|_p$ the sublinear expectation $\hat{E}$ can be continuously extended to the Banach space $(\hat{H}_p, \| \cdot \|_p)$. Without loss generality, we denote the Banach space $(\hat{H}_p, \| \cdot \|_p)$ as $\hat{L}^p_{\Omega}(\Omega, H, \hat{E})$. For the G-frame work, we refer to [24], [25], [26], [27], [28] and [29].

In this paper we assume that $\mu, \overline{\sigma}, \overline{\sigma}$ and $\sigma$ are nonnegative constants such that $\mu \leq \overline{\mu}$ and $\sigma \leq \overline{\sigma}$.

**Definition 2** Let $X_1$ and $X_2$ be two random variables in a sublinear expectation space $(\Omega, H, \hat{E})$, $X_1$ and $X_2$ are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if
\[
\hat{E}[\phi(X_1)] = \hat{E}[\phi(X_2)] \text{ for } \forall \phi \in \mathcal{C}_{b,\text{Lip}}(\mathbb{R}^n).
\]

**Definition 3** In a sublinear expectation space $(\Omega, H, \hat{E})$, a random variable $Y$ is said to be independent of another random variable $X$, if
\[
\hat{E}[\phi(X,Y)] = \hat{E}[\hat{E}[\phi(X,Y)]|_{X=x}].
\]

**Definition 4** (G-normal distribution) A random variable $X$ on a sublinear expectation space $(\Omega, H, \hat{E})$ is called G-normal distributed if
\[
aX + b\bar{X} = \sqrt{a^2 + b^2}X \quad \text{for } a, b \geq 0,
\]
where $\bar{X}$ is an independent copy of $X$.

**Remark 1** For a random variable $X$ on the sublinear space $(\Omega, H, \hat{E})$, there are four typical parameters to character $X$
\[
\overline{\mu}_X = \hat{E}X, \quad \mu_X = -\hat{E}[-X],
\]
\[
\overline{\sigma}_X^2 = \hat{E}X^2, \quad \sigma_X^2 = -\hat{E}[-X^2],
\]
where $[\mu_X, \overline{\mu}_X]$ and $[\sigma_X^2, \overline{\sigma}_X^2]$ describe the uncertainty of the mean and the variance of $X$, respectively.

It is easy to check that if $X$ is G-normal distributed, then
\[
\overline{\mu}_X = \hat{E}X = \mu_X = -\hat{E}[-X] = 0,
\]
and we denote the G-normal distribution as $N((0), [\sigma_X^2, \overline{\sigma}_X^2])$. If $X$ is maximal distributed, then
\[
\overline{\sigma}_X^2 = \hat{E}X^2 = \sigma_X^2 = -\hat{E}[-X^2] = 0,
\]
and we denote the maximal distribution (see [27]) as $N([\mu, \overline{\mu}], \{0\})$.

**Definition 5** We call $(X_t)_{t \in \mathbb{R}}$ a $d$-dimensional stochastic process on a sublinear expectation space $(\Omega, H, \hat{E})$, if for each $t \in \mathbb{R}$, $X_t$ is a $d$-dimensional random vector in $H$.

**Definition 6** Let $(X_t)_{t \in \mathbb{R}}$ and $(Y_t)_{t \in \mathbb{R}}$ be $d$-dimensional stochastic processes defined on a sublinear expectation space $(\Omega, H, \hat{E})$, for each $t = (t_1, t_2, \ldots, t_n) \in \mathcal{T}$,
\[
F^X_{\mathcal{L}}[\varphi] := \hat{E}[\varphi(X_{\mathcal{L}})], \quad \forall \varphi \in \mathcal{C}_{1,\text{Lip}}(\mathbb{R}^{n \times d})
\]
is called the finite dimensional distribution of $X$, $X$ and $Y$ are said to be identically distributed, i.e., $X \overset{d}{=} Y$, if
\[
F^X_{\mathcal{L}}[\varphi] = F^Y_{\mathcal{L}}[\varphi], \quad \forall \mathcal{L} \in \mathcal{T} \text{ and } \forall \varphi \in \mathcal{C}_{1,\text{Lip}}(\mathbb{R}^{n \times d})
\]
where $\mathcal{T} := \{\mathcal{L} = (t_1, t_2, \ldots, t_n) : \forall m \in \mathbb{N}, t_i \in \mathbb{R}, t_i \neq t_j, 0 \leq i, j \leq n, i \neq j\}$.  

4
Definition 7 A process \((B_t)_{t \geq 0}\) on the sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called a G-Brownian motion if the following properties are satisfied:

(i) \(B_0(\omega) = 0\);

(ii) For each \(t, s > 0\), the increment \(B_{t+s} - B_t\) is G-normal distributed by \(N(\{0\}, [\sigma^2, \sigma^2])\) and is independent of \((B_{t_1}, B_{t_2}, \ldots, B_{t_n})\), for each \(n \in \mathbb{N}\) and \(t_1, t_2, \ldots, t_n \in (0, t)\).

Definition 8 A process \((X_t)_{t \in \mathbb{R}}\) on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called a centered G-Gaussian process if for each fixed \(t \in \mathbb{R}\), \(X_t\) is G-normal distributed \(N(\{0\}, [\sigma^2, \sigma^2])\), where \(0 \leq \sigma \leq \sigma_t\).

Remark 2 Peng in [27] constructs G-frame work, which is a powerful and beautiful analysis tool for pricing uncertain risk under uncertainty. In [29], Peng defines G-Gaussian processes in a non-linear expectation space, \(q\)-Brownian motion under a complex-valued non-linear expectation space, and presents a new type of Feynman-Kac formula as the solution of a Schrödinger equation.

In [9], two-sided G-Brownian motion and fractional G-Brownian motion are defined. The properties of the fractional G-Brownian motion are present, such as, the similarity property and the long range dependent property in the sense of sublinearity, the properties are showed in the risky asset price fluctuations in the realistic financial market.

Definition 9 A process \((B_{\frac{1}{2}}(t))_{t \in \mathbb{R}} \in \Omega\) on the sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called a two-sided G-Brownian motion if for two independent G-Brownian motions \((B_t^{(1)})_{t \geq 0}\) and \((B_t^{(2)})_{t \geq 0}\)

\[
B_{\frac{1}{2}}(t) = \begin{cases} 
B^{(1)}(t) & t \geq 0 \\
B^{(2)}(-t) & t \leq 0 
\end{cases}
\]  

(iii) A family of continuous process under uncertainty which is corresponding with the fractional Brownian motion (fBm) provided by Kolmogorov (see [19] and [20]) and Mandelbrot (see [22]) is defined as fractional G-Brownian motion (fGBm) (see [9]):

Definition 10 Let \(H \in (0, 1)\), a centered G-Gaussian process \((B_H(t))_{t \in \mathbb{R}}\) on the sublinear space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called fractional G-Brownian motion with Hurst index \(H\) if

(i) \(B_H(0) = 0\);

(ii) \[
\begin{align*}
\hat{E}[B_H(s)B_H(t)] &= \frac{1}{\pi} \sigma^2 (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), & s, t \in \mathbb{R}^+, \\
-\hat{E}[B_H(s)B_H(t)] &= \frac{1}{\pi} \sigma^2 (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), & s, t \in \mathbb{R}^+,
\end{align*}
\]

we denote the fractional G-Brownian motion as fGBm.

We can easily check that \((B_{\frac{1}{2}}(t))_{t \in \mathbb{R}}\) is G-Brownian motion, and we denote \(B(t) = B_{\frac{1}{2}}(t)\). See [9] for the stochastic integral with respect to fGBm.

3 G-asset price system and G-martingale time consistent dynamic pricing mechanism

3.1 G-asset price model under uncertain

The first continuous-time stochastic model for a financial asset price appeared in the thesis of Bachelier [2] (1900). He proposed modelling the price of a stock with Brownian motion plus a linear
drift. The drawbacks of this model are that the asset price could become negative and the relative returns are lower for higher stock prices. Samuelson [30] (1965) introduced the more realistic model

\[ S_t = S_0 \exp \left( (\mu - \frac{\sigma^2}{2}) t + \sigma B_t \right), \]

which have been the foundation of financial engineering. Black and Scholes [6] (1973) derived an explicit formula for the price of a European call option by using the Samuelson model with

\[ S_0 = \exp (rt) \]

through the continuous replicate trade. Such models exploded in popularity because of the successful option pricing theory, as well as the simplicity of the solution of associated optimal investment problems given by Merton [23] (1973).

From then on, empirical research (see [1]) has produced the statistical evidence that is difficult to reconcile with the assumption of independent and normally distributed asset returns. Researchers have therefore attempted to build models for asset price fluctuations that are flexible enough to cope with the empirical deficiencies of the Black-Scholes model. In particular, a lot of work has been devoted to relaxing the assumption of constant volatility in the Black-Scholes model and there is a growing literature on stochastic volatility models, see e.g., Ball and Roma [3] or Frey [11] for surveys.

Knightian uncertainty can be an important factor influencing investors’ consumption and portfolio choice. Incorporating it into asset pricing models can therefore shed light on sources of asset return premiums and time variation in prices. In [8], we consider the asset price model on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F})\), which modelled the stock price with uncertain drift and uncertain volatility, i.e.,

\[ dS_t = S_t (db_t + dB_t) \]  

(5)

where \( b_t + B_t \) is generalized G-Brownian motion, \( b_t \) describes uncertain drift and is distributed with \( N([\mu, \mu], \{0\}) \), \( B_t \) is G-Brownian motion describes the uncertain volatility and is distributed with \( N(\{0\}, \sigma^2 t) \), and \( \mathcal{F}_t \) is the filtration with respect to the G-Brownian motion \( B_t \). The drift \( b_t \) can be rewritten as

\[ b_t = \int_0^t \mu_t dt \]

where \( \mu_t \) is the asset return rate (8).

**Definition 11 (G-asset price system)** If asset price process \( S_t \) on a sublinear space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F})\) satisfying (5), we call \((S_t, \hat{E})\) is G-asset price system.

If the process \( S_t \) is the asset price, and \((S_t, \hat{E})\) is G-asset price system, we define uncertain risk premium as

**Definition 12 (Uncertain risk premium)** Assume that the asset price is G-asset price system \((S_t, \hat{E})\), we define the difference between return rates of the asset and bond

\[ \hat{\vartheta}_t = \mu_t - r, \]

(6)

as uncertain risk premium of the asset.

It is easy to prove the following Proposition (8):

**Proposition 1** The uncertain risk premium of the asset, which is G-asset price system, is uncertain and distributed by \( N([\mu - r, \mu - r], \{0\}) \), where \( r \) is the interest rate of the bond.
3.2 European contingent claim pricing under G-asset price system

Consider an investor with wealth $Y_t$ in the market, who can decide his invest portfolio and consumption at any time $t \in [0, T]$. We denote $\pi_t$ as the amount of the wealth $Y_t$ to invest in the stock at time $t$, and $C(t + h) - C(t) \geq 0$ as the amount of money to withdraw for consumption during the interval $(t, t + h], h > 0$. We introduce the cumulative amount of consumption $C_t$ as RCLL with $C(0) = 0$. We assume that all his decisions can only be based on the current path information $\Omega_t$.

**Definition 13** A self-financing superstrategy (resp. substrategy) is a vector process $(Y, \pi, C)$ (resp. $(-Y, \pi, C)$), where $Y$ is the wealth process, $\pi$ is the portfolio process, and $C$ is the cumulative consumption process, such that

$$
\begin{align*}
    dY_t &= \pi_t dB_t + \pi_t \theta_t dt - dC_t, \\
    (\text{resp. } -dY_t &= -\pi_t dB_t + \pi_t \theta_t dt - dC_t)
\end{align*}
$$

where $C$ is an increasing, right-continuous process with $C_0 = 0$. The superstrategy (resp. substrategy) is called feasible if the constraint of nonnegative wealth holds

$$
y_t \geq 0, \quad t \in [0, T].
$$

We consider a European contingent claim $\xi$ written on the stock with maturity $T$, here $\xi \in L^2(Y, \Omega_T)$ is nonnegative. We give definitions of superhedging (resp. subhedging) strategy and ask (resp. bid) price of the claim $\xi$.

**Definition 14** (1) A superhedging (resp. subhedging) strategy against the European contingent claim $\xi$ is a feasible self-financing superstrategy $(Y, \pi, C)$ (resp. substrategy $(-Y, \pi, C)$) such that $Y_T = \xi$ (resp. $-Y_T = -\xi$). We denote by $\mathcal{H}(\xi)$ (resp. $\mathcal{H}'(-\xi)$) the class of superhedging (resp. subhedging) strategies against $\xi$, and if $\mathcal{H}(\xi)$ (resp. $\mathcal{H}'(-\xi)$) is nonempty, $\xi$ is called superhedgeable (resp. subhedgeable).

(2) The ask-price $X(t)$ at time $t$ of the superhedgeable claim $\xi$ is defined as

$$
X(t) = \inf \{ x \geq 0 : \exists (Y_t, \pi_t, C_t) \in \mathcal{H}(\xi) \text{ such that } Y_t = x \},
$$

and bid-price $X'(t)$ at time $t$ of the subhedgable claim $\xi$ is defined as

$$
X'(t) = \sup \{ x \geq 0 : \exists (-Y_t, \pi_t, C_t) \in \mathcal{H}'(-\xi) \text{ such that } -Y_t = -x \}.
$$

Under uncertainty, the market is incomplete and the superhedging (resp. subhedging) strategy of the claim is not unique. The definition of the ask-price $X(t)$ implies that the ask-price $X(t)$ is the minimum amount of risk for the buyer to superhedging the claim, then it is coherent measure of risk of all superstrategies against the claim for the buyer. The coherent risk measure of all superstrategies against the claim can be regard as the sublinear expectation of the claim, we have the following representation of bid-ask price of the claim.

**Theorem 2** Let $\xi \in L^2(Y, \Omega_T)$ be a nonnegative European contingent claim. There exists a superhedging (resp. subhedging) strategy $(X, \pi, C) \in \mathcal{H}(\xi)$ (resp. $(-X', \pi, C) \in \mathcal{H}'(-\xi)$) against $\xi$ such that $X_t$ (resp. $X'_t$) is the ask (resp. bid) price of the claim at time $t$.

Let $(H^t_s : s \geq t)$ be the deflator started at time $t$ and satisfy

$$
\begin{align*}
    dH^t_s &= -H^t_s [rds + \frac{\sigma_s}{\sigma_t} dB_s], \quad H^t_T = 1,
\end{align*}
$$

where $\sigma_t$ is adapted process with respect to $\mathcal{F}_t$ and $\sigma_t \in [\sigma, \bar{\sigma}]$ (see [8]).
Then the ask-price against \( \xi \) at time \( t \) is

\[
X_t = \mathcal{E}[H_t^G \xi | \Omega_t],
\]

and the bid-price against \( \xi \) at time \( t \) is

\[
X'_t = -\mathcal{E}[-H_t^G \xi | \Omega_t].
\]

Proof. See [8].

Remark 3 \((H_t^G : s \geq t)\) be the deflator started at time \( t \) satisfying [8], and

\[
H_t = \exp\{-\int_0^t r(s)\, ds + \int_0^t \frac{\partial}{\partial s} f_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial s^2} \eta_s^2 \, ds \}
\]

which is the deflator from 0 to \( t \), and implies the time value and the uncertain risk value.

### 3.3 G-Girsanov Theorem and G-martingale pricing mechanism

In this subsection we construct the G-martingale pricing frame work under G-asset price system. Define

\[
\tilde{B}_t := b_t + B_t - rt,
\]

we have the following G-Girsanov Theorem (presented in [8], [9] and [15])

**Theorem 3 (G-Girsanov Theorem)** Assume that \((B_t)_{t \geq 0}\) is G-Brownian motion and \( b_t \) is distributed with \( N(\mu, \sigma^2; \{0\}) \) on \((\Omega, \mathcal{F}, \mathcal{G}, \mathcal{F}_t)\), and \( \tilde{B}_t \) is defined by (10). There exists sublinear space \((\Omega, \mathcal{H}, \mathcal{G}, \mathcal{F}_t)\) such that \( \tilde{B}_t \) is G-Brownian motion under \( \mathcal{G} \), and

\[
\mathcal{E}[\tilde{B}_t^2] = \mathcal{E}^G[\tilde{B}_t^2], \quad -\mathcal{E}[-\tilde{B}_t^2] = -\mathcal{E}^G[-\tilde{B}_t^2].
\]

For \( t \in [0, T] \), we define G-martingale pricing mechanism as the following conditional G-expectation \( \mathcal{G}_t^G : L^2_G(\Omega_T) \rightarrow L^2_G(\Omega_T) \)

\[
\mathcal{G}_t^G[\cdot] = \mathcal{G}^G[\cdot | \mathcal{F}_t].
\]

The \( \mathcal{G}_t^G[\cdot] \) is a sublinear expectation, and has the properties, such as, sub-additivity, positive constant preserving, positive homogeneity and Chapman rule (G-Markovian Chain) (see [10]), which means that \( \mathcal{G}_t^G[\cdot] \) is a time consistent sublinear pricing mechanism (see [8]). By G-martingale decomposition Theorem [31], we can derive the following theorem (see [8])

**Theorem 4** Assume that \( \xi = \phi(S_T) \in L^2_G(\Omega_T) \) be a nonnegative European contingent claim, and \( \mathcal{G}_t^G[\cdot] \) be the G-martingale pricing mechanism. The ask price and bid price against the contingent claim \( \xi \) at time \( t \) are

\[
u^a(t, S_t) = e^{-\gamma(T-t)} \mathcal{G}_t^G \phi(S_T) \text{ and } \nu^b(t, S_t) = -e^{-\gamma(T-t)} \mathcal{G}_t^G [-\phi(S_T)].
\]

respectively.

And the ask price \( u^a(t, x) \) and bid price \( u^b(t, x) \) against the contingent claim \( \xi \) are the viscosity solutions to the following nonlinear HJB (proved in [31])

\[
\partial_t u^a(t, x) + rx\partial_x u^a(t, x) + G(x^2 \partial^2_x u^a(t, x)) - r u^a(t, x) = 0, \quad (t, x) \in [0, T) \times R,
\]

\[
\begin{align*}
\text{and} \\
u^a(T, x) = \phi(x).
\end{align*}
\]
\[
\partial_t u^b(t,x) + r x \partial_x u^b(t,x) - G(-x^2 \partial_x u^b(t,x)) - ru^b(t,x) = 0, \quad (t,x) \in [0,T) \times R,
\]
\[
u^b(T,x) = \phi(x),
\]
where the sublinear function \(G(\cdot)\) is defined as follows
\[
G(\alpha) = \frac{1}{2}(\sigma^2\alpha^+ - \sigma^2\alpha^-), \quad \forall \eta, \alpha \in R.
\]

### 4 G-consistent price systems and G-consistent bid-ask pricing under Uncertainty

#### 4.1 G-consistent price systems

We consider the risky asset price \(S_t\) on the uncertain financial market which shows as continuous trajectory, for example, some path perhaps satisfying the SDE driven by fGBm (see [9]) or be a G-stochastic integral with average moving integrant kernel, etc. Such process has some properties, for example, if the price is driven by fGBm with Hurst exponent \(H \in (0,1)\), the price process is G-asset price system for \(H = 1/2\) and have long range dependence if \(H > 1/2\). In this section we consider to study a type of price process which has G-consistent price systems

**Definition 15** Assume that \(S_t\) be a continuous price path on the sublinear space \((\Omega, \mathcal{H}, \hat{\mathbb{E}}, (\mathcal{F}_t)_{t \geq 0})\), where \(\mathcal{F}_t\) is the filtration with respect to the process \(S_t\). For any \(\varepsilon > 0\) if there exists a G-asset price system \((\hat{S}_t, \hat{\mathbb{E}})\), such that
\[
(1 + \varepsilon)^{-1} \leq \frac{\hat{S}_t}{\bar{S}_t} \leq 1 + \varepsilon, \quad \text{for all } t \in [0,T],
\]
we call \((\hat{S}_t, \hat{\mathbb{E}})\) \(\varepsilon\)-G-consistent price systems, and call the continuous price process \(S_t\) has G-consistent price systems.

Denote \(R_{++} = (0, \infty)\), \(C[u, v]\) be the set of \(R\)-valued continuous functions on \([u, v]\) and \(C_+\) \([u, v]\) be all the functions \(f(t) \in C[u, v]\) with \(f(u) = x\). For \(x \in R_{++}\), we denote \(C_+\) \([u, v]\) be the set of \(R_{++}\)-valued continuous functions on \([u, v]\) starting at \(x\).

**Definition 16** On the sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}}, (\mathcal{F}_t)_{t \geq 0})\), for \(A \in \mathcal{F}(\Omega)\) we define the capacity of \(A\) as
\[
c[A] = \hat{\mathbb{E}}[I_A],
\]
where \(I_A\) is the indicator function
\[
I_A := I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}
\]

Let us define \(S_t := S_{tT}\) for \(t > T\), let \(c(\cdot, \omega)\) be the capacity of the \(C^+[0, T]\)-valued random variable \((S_t)_{t \in [0,T]}\).
\[
c((S_t)_{t \in [0,T]}) := \hat{\mathbb{E}}[I_{(S_t)_{t \in [0,T]}}].
\]

**Definition 17** (G-Conditional Full Support) A continuous \(R_{++}\)-valued process \((S_t)_{t \in [0,T]}\) satisfies G-Conditional Full Support (GCF$\$S) if, for all \(t \in [0,T]\),
\[
supp c(S|\cdot|_{t,T}|\mathcal{F}_t) = C^+_{S_t}[t, T].
\]
Lemma 1

Let \( S_t := S_T \) for \( t > T \), let \( c(\cdot, \omega) \) be the capacity of the \( C^+[0,T] \)-valued random variable \( (S_t)_{t \in [0,T]} \), and let \( c^*(\cdot, \omega) \) be the \( \mathcal{F}_T \)-conditional capacity of the \( C^+[0,T] \)-valued random variable \( (S_t)_{t \in [0,T]} \).

With the similar argument in [14] (proof of Lemma 2.9 in [14] p.26 Appendix), we derive the following lemma.

Lemma 1

The \( G \)-conditional full support condition (GCFS) implies the \( G \)-strong conditional full support condition (GSCFS), hence they are equivalent.

Theorem 5

Let \( (S_t)_{t \in [0,T]} \) be an adapted positive price process on sublinear space \( (\Omega, \mathcal{H}, \hat{E}, \mathcal{F}_t) \) satisfying GCFS condition. Then for all \( \varepsilon > 0 \), there exist \( G \) expectation \( E^G[\cdot] \) and \( G \)-asset price system \( ((\hat{S}_t)_{t \in [0,T]}, E^G) \) in \( G \) expectation space \( (\Omega, \mathcal{H}, E^G, \mathcal{F}_t) \) such that

\[
|S_t - \hat{S}_t| \leq \varepsilon, \quad \text{for all } t \in [0,T].
\]

Proof. For \( \forall \varepsilon > 0 \), we define the increasing sequence of stopping times

\[
\tau_0 = 0, \quad \tau_{n+1} = \inf\{t \geq \tau_n : \frac{S_t}{S_{\tau_n}} \notin ((1 + \varepsilon)^{-1}, 1 + \varepsilon)\} \wedge T. \quad (19)
\]

For \( n > 1 \), we set

\[
R_n = \begin{cases} 
\text{sign}(S_{\tau_n} - S_{\tau_{n-1}}), & \text{if } \tau_n < T, \\
0, & \text{if } \tau_n = T. 
\end{cases} \quad (20)
\]

Define the following Random Walk with retirement

\[
X_n = X_0(1 + \varepsilon)^{\sum_{i=1}^n R_i} \quad (21)
\]

which adapted to the discretized filtration \( \mathcal{F}_{\tau_n} \). With the similar argument in [14] (Lemma A.1. p. 27), we have that \( S_n \) satisfying GCFS implies

\[
c(R_{n+1} = z|\mathcal{F}_{\tau_n}) > 0, \quad \text{for } z = 0, \pm 1, n > 0, \text{on } \{\tau_n < T\}. \quad (22)
\]

Denote \( \tilde{X}_n \) be terminal value of \( X \), we define the following continuous path from \( X \) as

\[
\tilde{S}_t := \hat{E}[X_n|\mathcal{F}_t], \quad t \in [0,T]. \quad (23)
\]

For \( \sigma > \alpha \geq 0 \), define a sublinear function \( G(\cdot, \cdot) \) as follows

\[
G(\eta, \alpha) = (\beta^+ \eta^+ - \mu \eta^-) + \frac{1}{2} (\sigma^2 \alpha^+ - \sigma^2 \alpha^-), \quad \forall \eta, \alpha \in R. \quad (24)
\]
For given $\varphi \in C_{b,1,p}(R)$, we denote $u(t,x)$ as the viscosity solution of the following G-equation (see (25))

$$
\partial_t u - G(\partial_x u, \partial_{xx} u) = 0, \quad (t,x) \in (0,\infty) \times R,
$$

$$
u(0,x) = \varphi(x).
$$

For $\omega \in \Omega$ consider the process $\tilde{B}_i(\omega) := (\ln \frac{S_i}{S_0})(\omega, t \in [0,\infty)$, we define $E^G[\cdot]: \mathcal{G} \to R$ as

$$
E^G[\varphi(\tilde{B}_i)] = u(t,0),
$$

and for each $s,t \geq 0$ and $\eta_1, \cdots, \eta_N \in [0,t]$.

$$
E^G[\varphi(\tilde{B}_{\eta_1}, \cdots, \tilde{B}_{\eta_N}, \tilde{B}_{i+s} - \tilde{B}_i)] := E^G[\varphi(\tilde{B}_{\eta_1}, \cdots, \tilde{B}_{\eta_N})]
$$

where $\varphi(x_1, \cdots, x_N) = E^G[\varphi(x_1, \cdots, x_N, \tilde{B}_i)]$.

For $0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots < t_N < +\infty$, we define $G$ conditional expectation with respect to $\Omega_i$ as

$$
E^G[\varphi(\tilde{B}_{i_1}, \tilde{B}_{i_2} - \tilde{B}_{i_1}, \cdots, \tilde{B}_{i_{i+1}} - \tilde{B}_{i_i}, \cdots, \tilde{B}_{i_N} - \tilde{B}_{i_{N-1}})]|\mathcal{F}_i]
$$

where $\varphi(x_1, \cdots, x_i) = E^G[\varphi(x_1, \cdots, x_i, \tilde{B}_{i+1} - \tilde{B}_{i_1}, \cdots, \tilde{B}_{i_N} - \tilde{B}_{i_{N-1}})]$.

We consistently define a sublinear expectation $E^G$ on $\mathcal{G}$. Under sublinear expectation $E^G$ we define above, the corresponding canonical process $(\tilde{B}_i)_{i \geq 0}$ is a generalized G-Brownian motion and $(\tilde{S}_i)$ is G-asset price system on the sublinear space $(\Omega, \mathcal{G}, E^G, (\mathcal{F}_i)_{i \geq 0})$. We call $E^G[\cdot]$ as G-expectation on $(\Omega, \mathcal{G}, E^G[\cdot])$.

Denote $L^p_G(\Omega, P \geq 1)$ as the completion of $\mathcal{G}$ under the norm $\|X\|_p = (E^G[|X|^p])^{1/p}$, and similarly we can define $L^p_G(\Omega, \mathcal{G})$. The sublinear expectation $E^G[\cdot]$ can be continuously extended to the space $(\Omega, L^p_G(\Omega, \mathcal{G}))$.

For fix $t \in [0,T]$, define $\underline{\tau} = \max\{\tau_n: \tau_n \leq t\}$ and $\overline{\tau} = \min\{\tau_n: \tau_n > t\}$. We have

$$(1 + \varepsilon)^{-1} \leq \frac{S_t}{S_{\overline{\tau}}} \leq \frac{S_{\underline{\tau}}}{S_t} \leq 1 + \varepsilon, \quad \forall t \in [0,T]$$

and therefore

$$(1 + \varepsilon)^{-2} \leq \frac{S_{\overline{\tau}}}{S_t} \leq (1 + \varepsilon)^2, \quad \forall t \in [0,T].$$

From construction (24), for $n \geq 0$ on $\{\tau_n < T\}$ we have $\tilde{S}_{\tau_n} = X_n, S_{\tau_n} = X_n$. On $\{\tau_n = T\}$, we have

$$(1 + \varepsilon)^{-1} \leq \frac{\tilde{S}_{\tau_n}}{S_{\tau_n}} \leq (1 + \varepsilon), \quad \forall n \geq 0.$$

Therefore, we have

$$
\frac{\tilde{S}_t}{S_t} = \frac{E^G[\tilde{S}_t|\mathcal{F}_j]}{S_t} = E^G[\frac{\tilde{S}_t}{S_t} | \mathcal{F}_j]
$$

which implies

$$
(1 + \varepsilon)^{-3} \leq \frac{\tilde{S}_t}{S_t} \leq (1 + \varepsilon)^3.
$$

From which we complete the proof of the Theorem. $\square$
4.2 Bid-Ask Pricing

On the uncertain financial market if the price process \((S_t)_{t \in [0, T]}\) of the risky asset is continuous trajectory satisfying GCFS condition in the G-expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F}_t)\), we correct the definition \((\Box)\) of the self-finance superhedging (resp. subhedging) strategy \((Y, \theta, C)\) (resp. \((-Y, \theta, C)\)) as following

\[
Y_t = \int_0^t (Y_s - \theta_s S_s) \, rds + \int_0^t \theta_s dS_s - C_t,
\]

(resp. \(-Y_t = \int_0^t (-Y_s - \theta_s S_s) \, rds + \int_0^t \theta_s dS_s - C_t\),

where \(\theta_t = \frac{\pi_t}{S_t}\), the integral \(\int_0^t \theta_s dS_s\) is in a pointwise Riemann-stieltjes sense, and \(C_t\) is a right continuous, nondecreasing cost process with \(C_0 = 0\), i.e.,

\[
dY_t = (Y_t - \theta_t S_t) \, rdt + \theta_t dS_t - dC_t.
\]

The ask-price \(X_e(t)\) at time \(t\) of the superhedgeable claim \(\xi\) is defined as

\[
X_e(t) = \inf \{ x \geq 0 : \exists (Y_t, \theta_t, C_t) \in \mathcal{H}_e(\xi) \text{ such that } Y_t = x \},
\]

and the bid-price \(X'_e(t)\) at time \(t\) of the subhedgeable claim \(\xi\) is defined as

\[
X'_e(t) = \sup \{ x \geq 0 : \exists (-Y_t, \theta_t, C_t) \in \mathcal{H}'_e(\xi) \text{ such that } -Y_t = -x \}.
\]

Theorem 6 Let \((S_t)_{t \in [0, T]}\) be an \(R_+\) -valued continuous process on the sublinear expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F}_t)\) satisfying the G-conditional full support assumption (GCFS) and the nonnegative European contingent claim \(\xi = g(S_T)\) in \(L_+^2(\Omega_T)\).

Then there exists G-expectation \(E^G\), such that the ask and bid prices of the European contingent claim \(\xi = g(S_T)\) at time \(t\) are given by

\[
X_e(t) = e^{-r(T-t)} E^G[g(S_t)|\mathcal{F}_t] \quad X'_e(t) = -e^{-r(T-t)} E^G[-g(S_t)|\mathcal{F}_t]
\]

respectively.

Proof. By Theorem 5 for \(\forall \varepsilon > 0\), there exist \(\varepsilon\)-G-consistent price systems \((\tilde{S}_t, \tilde{E})\) satisfying

\[
(1 + \varepsilon)^{-1} \leq \frac{\tilde{S}_t}{S_t} \leq 1 + \varepsilon.
\]

For small enough \(\varepsilon\), we denote the family of the \(\varepsilon\)-G-consistent price systems as

\[
\mathcal{Z}_\varepsilon := \{(\tilde{S}_t, \tilde{E}) : \tilde{S}_t \text{ is a G-asset price, } 1 - \varepsilon \leq \frac{\tilde{S}_t}{S_t} \leq 1 + \varepsilon, t \in [0, T]\}.
\]

For fix \(\varepsilon\), with the corresponding \(\varepsilon\)-G-consistent price system \((\tilde{S}_t, \tilde{E})\) there exist adapt processes \(\tilde{\delta}_{t,1}\) and \(\tilde{\delta}_{t,2}\) satisfying

\[
\tilde{\delta}_{t,1} \in [-\varepsilon, \varepsilon], \quad \tilde{\delta}_{t,2} \in [-2\varepsilon, 2\varepsilon] \quad \tilde{\delta}_{t,1}\tilde{\delta}_{t,2} \neq 0, \text{ a.s.}
\]
such that
\[ S_t = (1 + \delta_t) S_t, \quad dS_t = 2\delta_t d\bar{S}_t, \quad \forall t \in [0, T]. \]

Denote \( \bar{B} \) as the G-Brownian motion on the sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\) distributed with \( N(\{0\}, \sigma^2 I)\), from the construction of the process \( \bar{S}_t \) in Theorem 5, process \( \bar{S} \) is a G-asset price process
\[ d\bar{S}_t = \bar{S}_t(d\bar{b}_t + d\bar{B}_t), \]
where \( \bar{b} \) is distributed with \( N(\mu, \mathbb{P}), \{0\}) \) in \((\Omega, \mathcal{F}, \mathbb{E})\).

By G-Girsanov transform \([8], [9], [15]\) there exists a G-expectation space \((\Omega, \mathcal{F}, \mathbb{E}_G, \mathcal{G}_t)\) such that
\[ B_t^G := \bar{B}_t + \bar{b}_t - \int_0^t \frac{1 + \delta_s}{2\delta_s} r_s ds, \quad t \geq 0 \]
is a G-Brownian motion in \((\Omega, \mathcal{F}, \mathbb{E}_G, \mathcal{G}_t)\).

Define process \( X_t \) as follows
\[ X_t := e^{-rt} \mathbb{E}[g(S_T) | \mathcal{G}_t], \]
we have that \( e^{-rt} X_t = \mathbb{E}[e^{-rT} g(S_T) | \mathcal{F}_t] \) is a G-martingale in \((\Omega, \mathcal{F}, \mathbb{E}_G, \mathcal{G}_t)\), by the G-martingale representation Theorem \([31]\)
\[ e^{-rt} X_t = \mathbb{E}[e^{-rT} g(S_T)] + \int_0^t \beta_s dB_s^G - K_t, \]
where \( \beta_t \in L_G^0 [0, T], \) \( K_t \) is a continuous, increasing process with \( K_0 = 0 \), and \( \{K_t\}_{t \in [0, T]} \) is a G-martingale. Define
\[ \theta_t := \frac{e^{rt} \beta_t}{2\delta_s^2 S_t}, \quad C_t = \int_0^t e^{rt} K_s ds \]
we derive that
\[ dX_t = rX_t dt + 2\theta_t \delta_t d\bar{S}_t d\bar{B}_t^G - dC_t \]
which implies that
\[ X_t = g(S_T) + \int_0^T (X_s - \theta_s S_s) r_s ds + \int_0^T \theta_s dS_s - (C_T - C_t). \]

Thus, we prove that \((\mathbb{E}[e^{-rT} g(S_T) | \mathcal{F}_t], \theta_t, C_t) \) is a superhedging strategy against the claim \( \xi = g(S_T) \).

For given any superhedging strategy \((\bar{X}_t, \bar{\theta}_t, \bar{C}_t) \) against the claim \( \xi = g(S_T) \)
\[ \bar{X}_t = g(S_T) + \int_0^T (\bar{X}_s - \bar{\theta}_s S_s) r_s ds + \int_0^T \bar{\theta}_s dS_s - (\bar{C}_T - \bar{C}_t). \]
From \([29], [30]\) and G-Girsanov transform \([31]\), the above equation can be rewritten as
\[ d(e^{-rt} \bar{X}_t) = 2e^{-rt} \bar{\theta}_t \delta_t \bar{S}_t d\bar{B}_t^G - e^{-rt} d\bar{C}_t \]
\[ e^{-rt} \bar{X}_t = e^{-rt} g(S_T) - \int_0^T 2e^{-rt} \bar{\theta}_s \delta_s \bar{S}_s d\bar{B}_s^G + \int_0^T e^{-rt} d\bar{C}_t \]
take G conditional expectation with respect to \( \mathcal{F}_t \), notice that the cost function \( \bar{C}_t \) is nonnegative and nondecreasing process, we have that \( \bar{X}_t \geq e^{-r(t-s)} \mathbb{E}[g(S_T) | \mathcal{F}_s] = X_s \), which prove that \( X_t(t) = e^{-r(T-t)} \mathbb{E}[g(S_T) | \mathcal{F}_t] \).

Similarly, we can prove \( X_t'(t) = -e^{-r(T-t)} \mathbb{E}[\mathbb{E}[g(S_T) | \mathcal{F}_t]], \) \( \square \)
4.3 Examples

Example 1 G-Markovian processes

Denote $\hat{E}_t[\cdot] := \hat{E}[\cdot | F_t]$, we consider continuous nonnegative G-Markovian processes $\{S_t\}_{t \in [0,T]}$ in $(\Omega,\mathcal{F},\mathbb{P})$, defined by
\begin{equation}
\hat{E}[\phi(S_t) | F_t] = \phi(S_t), \quad s \geq t, \quad \forall \phi \in C_{b,Lip}(R).
\end{equation}

The G-Markovian property implies the GCFS
\begin{equation}
supp c(S|_{[v,T]} | F_v) = supp c(S|_{[v,T]} | F_v) = C_{S_v} [v,T], \quad 0 \leq v \leq T.
\end{equation}

Example 2 Processes Driven by Fractional G-Brownian Motion

Denote $B_t^H$ as fractional G-Brownian motion with Hurst index $H \in (0,1)$, which is defined in [9] as a centered G-Gaussian process with stationary increment in the sense of sublinear
\begin{align*}
\mathbb{K}(t,s) := \mathbb{E}[B_t^H B_s^H] &= \frac{\Gamma^2}{2} (t^{2H} + s^{2H} + |t-s|^{2H}), \\
\mathbb{K}(t,s) := -\mathbb{E}[-B_t^H B_s^H] &= \frac{\Gamma^2}{2} (t^{2H} + s^{2H} + |t-s|^{2H}).
\end{align*}

The moving representation of the fractional G-Brownian motion (see Theorem 1 in [9]) is
\begin{equation}
B_H(t,\omega) = C_H^w \int_t^T ([t-s]^{H-1/2} - (-s)^{H-1/2}) dB(s,\omega),
\end{equation}
where $C_H^w = \frac{(2H \sin \pi H(2H))^{1/2}}{\Gamma(H+1/2)}$ and $(B_t)_{t \in R}$ is a two-sided G-Brownian motion.

Denote $R_H(t,s) = \frac{1}{2} (t^{2H} + s^{2H} + |t-s|^{2H})$, then there exists square-integrable kernel $K_H(t,s)$ such that
\begin{equation}
B_t^H = \int_0^t K_H(t,s) dB_s,
\end{equation}
where $(B_t)_{t \in [0,T]}$ is G-Brownian motion. $(B_t)_{t \in [0,T]}$ generate the same filtration as $(B_t^H)_{t \in [0,T]}$, and $K_H(t,s)$ is as following
\begin{equation}
K_H(t,s) = \begin{cases} 
C_{H,1} \left( \frac{1}{2} H - \frac{1}{2} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2} - H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} du \right), & H < \frac{1}{2} \\
C_{H,2} s^{\frac{1}{2} - H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du, & t > s, \ H \geq 12
\end{cases}
\end{equation}
where
\begin{align*}
C_{H,1} &= \left( \frac{2H}{\Gamma(1-2H,2H+1/2)} \right)^{1/2} \\
C_{H,2} &= \left( \frac{H(2H-1)}{\Gamma(2-2H,2H+1/2)} \right)^{1/2}
\end{align*}
and $\beta$ denotes the Beta function.
It is easy to check that for any $v \in [0, T]$, the process $(B^H_t)_{t \in [v, T]}$ is G-Gaussian, conditionally on $F_v$ in the sense of finite-dimensional distributions (see \cite{29}), and its conditional G-expectation and conditional increment function in the sense of sublinear are given

$$E[B^H_t|F_v] = \int_0^v K_H(t, s)dB_s, \quad t \geq v,$$

$$\hat{E}[B^H_tB^H_s|F_v] = \sigma^2 \int_v^{\min(t,s)} K_H(t, u)K_H(s, u)du, \quad t, s \geq v,$$

$$-\hat{E}[-B^H_tB^H_s|F_v] = \sigma^2 \int_v^{\min(t,s)} K_H(t, u)K_H(s, u)du, \quad t, s \geq v.$$

Then the law of $(B^H_t)_{t \in [v, T]}$ conditional on $F_v$ is identical with the law of $E[B^H_t|F_v] + X_v$, where $(X_t)_{t \in [v, T]}$ is still a fractional G-Brownian motion start from $v$, i.e., is a centered G-Gaussian process with continuous path on $[v, T]$. With a similar argument as above, $X_t = \int_t^T K_H(t, s)dB_s$. We just need to prove that the centered G-Gaussian process $(X_t)_{t \in [v, T]}$ has full support as follows

$$\text{supp } c(X|v, T)|F_v) = C_0[v, T], \quad (39)$$

where $c(X|v, T)|F_v)$ is the capacity on $X|v, T)|F_v$. By using the properties of the capacity, we have the similar result as Theorem 3 in \cite{17}.

**Theorem 7** For the centered G-Gaussian process described by

$$X_t = \int_v^T K_H(t, s)dB_s,$$

the support of $(X_t)_{t \in [v, T]}$ satisfying

$$\text{supp } c(X|v, T)|F_v) = \overline{H(K_H)},$$

where $H(K_H)$ is reproducing kernel Hilbert space define by

$$H(K_H) := \{ f(t) \in C_0([v, T], R) : f(t) = \int_v^T K_H(t, s)g(s)ds, \text{ for some } g \in L^2[v, T] \}.$$

Define the kernel operator $K_H$ as

$$(K_H g)(t) := \int_0^t K_H(t, s)g(s)ds, \quad g \in L^2[0, T], t \in [0, T],$$

then for $g \in C_0[v, T], K_H : C_0[v, T] \rightarrow C_0[v, T]$ is continuous and has a dense range (see \cite{14}), and $H(K_H)$ is norm-dense in $C_0([v, T], R)$, thus we have

**Theorem 8** Processes $(S_t)_{t \in [0, T]}$ in $(\Omega, \mathcal{H}, \hat{E}, F_t)$ driven by fractional G-Brownian motion $B^H_t$

$$dS_t = S_t(b(t)dt + dB^H_t)$$

where $b(t)$ is a deterministic continuous function, $H \in (0, 1)$, then $(S_t)_{t \in [0, T]}$ satisfies the G-conditional full support condition (GCFS).
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