Sorted data is usually easier to compress than unsorted permutations of the same data. This motivates a simple compression scheme: specify the sorted permutation of the data along with a representation of the sorted data compressed recursively. The sorted permutation can be specified by recording the decisions made by quicksort. If the size of the data is known, then the quicksort decisions describe the data at a rate that is nearly as efficient as the minimal prefix-free code for the distribution, which is bounded by the entropy of the distribution. This is possible even though the distribution is unknown ahead of time. Used in this way, quicksort acts as a universal code in that it is asymptotically optimal for any stationary source. The Shannon entropy is a lower bound when describing stochastic, independent symbols. However, it is possible to encode non-uniform, finite strings below the entropy of the sample distribution by also encoding symbol counts because the values in the sequence are no longer independent once the counts are known. The key insight is that sparse quicksort comparison vectors can also be compressed to achieve an even lower rate when data is highly non-uniform while incurring only a modest penalty when data is random.

Sorted data is usually easier to compress than unsorted permutations of the same data. Because sequential values are in order when data is sorted, their non-negative differences can be encoded in place of the original values. And because repeated values are all contiguous, they can be encoded by including a count along with the first instance. This motivates a simple compression scheme: first describe how to permute the data into sorted order, and then describe the sorted data. The permutation is invertible, so it can
be used along with the description of the sorted data to generate the original data.

Because there are $N!$ permutations of a sequence of $N$ items, $\log_2(N!)$ bits are required to specify a particular permutation. But if there is any frequency regularity in the data, it is possible to specify the sorted permutation using fewer bits because there are fewer distinct permutations. One way to specify the sorted permutation of data that uses fewer bits when data is non-uniform is to record the decisions made by the quicksort algorithm as it recursively partitions the data around pivots. In quicksort, each item in the sequence is compared to a pivot value. Based on that comparison, it is assigned to either the left or the right partition, and the algorithm is applied recursively to each partition. The recursion terminates when the sequence cannot be partitioned, either because it is a single item or because all of the items are equal. When quicksort terminates, all of the occurrences of a given symbol are associated with a single leaf node, and the leaves are arranged in sort order [CSRL01]. Even without the original data and pivot values, the sorted sequence can be regenerated by running the quicksort algorithm again, using the recorded decisions instead of comparing data to pivots. Using the recorded decisions, the index array $X[i] = i$ is transformed by this algorithm into a permutation vector $X[i] = j$, indicating that the value in position $i$ is moved to position $j$. The inverse permutation vector, defined by $Y[X[i]] = i$ can be used to permute the sorted data back into the original order. An example quicksort partition tree is shown in figure 1. The corresponding decision bitvectors are depicted in figure 2.

Assume the initial sequence is of length $N$, and there are $M$ unique symbols. Let $c_i$ represent the count of the $i$'th symbol. When the pivots are selected so that the partitions are maximally uniform, the leaf for the $i$'th symbol is reached after approximately $\log_2(N/c_i)$ partitions. Even if the data cannot be uniformly partitioned, at most one extra partition is necessary. For each of the $c_i$ instances of a given symbol, all of the comparisons and corresponding partition decisions need to be recorded. The total number of comparisons is approximately

$$\sum c_i \log_2(N/c_i).$$

Using $p_i = c_i/N$, this can be written as the familiar

$$N \sum p_i \log_2(1/p_i),$$

showing that the number of comparisons on average is approximately equal to the entropy of the distribution. The maximally uniform quicksort tree
Figure 1: quicksort partition tree

Figure 2: decision bitvectors
is in fact equivalent to a Fano prefix code tree for the distribution, which is no more than one bit worse than the minimal Huffman code tree [CT06, P.123]. If the comparisons are recorded using bitvectors at each node, with each comparison stored in a single bit, then the description of the sorted permutation is also a description of the data that is optimal or near optimal for the distribution. Interestingly, this description efficiency is possible even though the distribution is unknown ahead of time and does not need to be encoded before the data. The only information required to decode it is the number of elements $N$, which can be encoded using a universal code in $2 \cdot \log_2(N) + 1$ bits [Eli75]. This means that quicksort acts as a universal code in that it is asymptotically optimal for any stationary source.

Perhaps surprisingly, when the distribution is nonuniform, it is possible to encode the data using even fewer bits by first encoding the symbol counts. To see why this is possible, observe that if the counts for each symbol are known, then the items in the sequences are not independent of each other. If the size of the sequence at a given node is $n$, and the number of items assigned to the right partition is $n_r$, there are $\binom{n}{n_r}$ valid selections which is less than the $2^n$ possible bitstrings of length $n$ describing the comparisons. Therefore, a description of the data that is more efficient than the sample distribution entropy may be achieved by encoding each node’s decision bitvector using $\log_2(\binom{n}{n_r})$ bits. In order to decode this, the sizes of the nodes must also be recorded. But the entire tree structure, along with the sizes for each node, can be reconstructed from just the counts for the leaves using the Fano coding algorithm [CT06].

The symbol counts are not useful when data is uniform. In that case, the information is redundant, so including it makes random data more expensive to encode. Most data is random and incompressible (by any criteria) [LV08, p.41], so the average performance suffers from including the symbol counts. However, the average extra cost is no more than $M \cdot \log(N)$ bits, while the benefit can be dramatic for data that is sparse or exhibits a lot of frequency regularity.

Encoding the partition decisions using the information theoretic limit of $\log_2(\binom{n}{n_r})$ bits per node requires some way to map between the sparse $n$-bit decision vectors and the dense $\log_2(\binom{n}{n_r})$ bit descriptions. For small values of $n$, a lookup table can be constructed by some enumeration strategy. As $n$ increases, this becomes infeasible, but a practical approximation is to divide the sequence into windows of some size $W$, which is the upper limit for the lookup table, and encode each of the $\lceil n/W \rceil$ bitvectors using $\log_2(\binom{W}{n_{ri}})$ bits, with $n_{ri}$ representing the number of ones in the $i$’th window. This requires also encoding the sequence $n_{ri}$, which can be done efficiently because it sums
In addition to the record of decisions made at each node, the full compression scheme needs to include a description of the symbols and their counts. The counts define a new, unordered, data sequence. Because the sequence of counts is less complex than the starting sequence, it can be encoded recursively using the same algorithm. The symbol values are ordered, so it is only necessary to encode their differences. Although these can be recursively compressed as well, if the differences are large and non-repeating, they may not benefit from additional compression iterations. To ensure that the recursion converges rapidly, a non-recursive, direct encoding should be used as soon as the entropy of the data exceeds some threshold.

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