Spontaneous generation of entanglement in quantum dark-soliton qubits

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We show that entanglement between two solitary qubits in quasi one-dimensional Bose-Einstein condensates can be spontaneously generated due to quantum fluctuations. Recently, we have shown that dark solitons are an appealing platform for qubits due to their appreciably long lifetime. We explore the creation of entanglement between dark soliton qubits by using the superposition of two maximally entangled states in the dissipative process of spontaneous emission. By driving the qubits with the help of Raman lasers, we observe the formation of long distance steady-state concurrence. Our results suggest that dark-soliton qubits are a good candidates for quantum information protocols based purely on matter-wave phononics.

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Introduction: Entangled states of collective quantum systems are of paramount interest in quantum mechanics and an essential ingredient in most applications of quantum information. After the exploitation of entanglement in optical and atomic setups, entanglement generation finds renewed interest in condensed matter systems. Short-distance entanglement has been envisaged for spin or charge degrees of freedom in molecules, nanotubes or quantum dots [1-4]; owing to the long-range nature of the dipolar (∼1/r3) interaction, Rydberg atoms are attractive platforms for large-distance entanglement generation [5-9]. In fact, considerably large separation between atoms is required to transport information at long distances in such systems. To achieve this purpose, a virtual boson mediating the correlation between two qubits is required. Photons are the usual candidate for this task, either for superconducting qubits coupling in the microwave range [10] or for quantum dots in the visible range [11-13]. Recently, plasmons have also been proposed to mediate qubit-qubit entanglement in plasmonic waveguides [14].

Due to their intrinsically long coherence times, ultracold gases are natural platforms for quantum information processing, quantum metrology [15], quantum simulation [16], and quantum computing. In that regard, Bose-Einstein condensates (BECs) have attracted a great deal of interest during the last decades [17-19]. The macroscopic character of the wavefunction allows BEC to display pure-state entanglement, like in the single-particle case, since all particles occupy the same quantum state. The entanglement between two cavity modes mediated by a BEC has been investigated by Ng et. al. [20]; two component BECs were realized on atom chips with full control of the Bloch sphere and spin squeezing [21, 22].

Another important feature of the macroscopic nature of BECs is the dark-soliton (DS), a structure resulting from the detailed balance between the dispersive and nonlinear effects, appearing also in other physical systems [23-25]. The dynamics and stability of DSs in BECs have been a subject of intense research over the last decades [26, 27]. The dynamical evolution of DS entanglement and how its stability is affected by quantum fluctuations has been studied in Ref. [28]. The study of collective aspects of soliton gases [29] bring DSs towards applications in many-body physics [30]. In a recent publication, we have shown that DSs can behave as qubits in quasi-1D BECs [31], being excellent candidates to store information given their appreciably long lifetimes (∼0.01 – 1s). Dark-soliton qubits thus offer an appealing alternative to quantum optics in solid-state platforms, where information processing involves only phononic degrees of freedom: the quantum excitations on top of the BEC state.

In this Letter, we report on the spontaneous generation of large-distance entanglement between two DS-qubits placed inside a quasi one-dimensional (1D) BEC. The entanglement is generated by a combination of the external driving (with the help of Raman lasers [32, 33]) and the quantum fluctuations (phonons) leading to spontaneous and collective emission. We compute the steady-state concurrence for sufficiently large distances, \(d \simeq 5\xi/2\), with \(\xi\) denoting the healing length, i.e. the size of the

![FIG. 1: (color online) a) Schematic representation of two dark-soliton qubits placed at distance \(d\) in a cigar shaped quasi one-dimensional BEC, surrounded by a dilute gas of impurities. b) Collective states of two dar- soliton qubits. Due to the coherent coupling, the two intermediate states \(|s\rangle\) and \(|a\rangle\) are maximally entangled.](image-url)
Theoretical Model. We consider two DS placed at distance $d$ in a quasi 1D BEC. The qubits are formed with the help of an extremely dilute gas surrounding the condensate, whose particles are trapped inside the potential created by the DSs, as illustrated in Fig. 1. At the mean-field level, the system is governed by the Gross-Pitaevskii and the Schrödinger equations, respectively describing the BEC and the impurities.

\[
\begin{align*}
\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m_1} \frac{\partial^2 \Psi}{\partial x^2} + g |\Psi|^2 \Psi + \chi |\Phi|^2 \Psi, \\
\hbar \frac{\partial \Phi}{\partial t} &= -\frac{\hbar^2}{2m_2} \frac{\partial^2 \Phi}{\partial x^2} + \chi |\psi_{\text{sol}}|^2 \Phi.
\end{align*}
\]

(1)

Here, $\chi$ is the BEC-impurity coupling constant [34], $g$ is the BEC particle-particle interaction strength, and $m_1$ and $m_2$ denote the BEC particle and impurity masses, respectively. The $i$th ($i = 1, 2$) soliton profile is $\psi_{\text{sol}}^{(i)}(x) = \sqrt{n_0} \tanh (|x - x_i|/\xi)$, where $x_i = \pm d/2$ is the position of the respective soliton, $n_0$ is the BEC linear density, and $\xi = h/\sqrt{m_1 n_0 g}$ is the healing length (of the order $0.2 - 0.7 \mu m$ in elongated $^{87}\text{Rb}$ BECs). The experimentally accessible trap frequencies amount to be $\omega_z/2\pi = (15 - 730) \text{Hz} < \omega_r/2\pi = (1 - 5) \text{kHz}$ and the corresponding length considered to be $l_z = (0.6 - 3.9) \mu m$ [37]. More recent experiments make eventual trap inhomogeneities to be much less critical by creating much larger traps, $l_z \sim 100 \mu m$ [38]. Homogeneous condensates along a trap of size $l_z \sim 70 \mu m$, largely outdoing the soliton core $\xi$ and therefore manifesting finite-size effects less important - are experimentally possible in box-shaped potentials [39]. This offers additional advantages regarding the scalability (i.e. in a multiple-soliton quantum computer), as uncontrolled phonon mediated soliton-soliton interaction appears when inhomogeneities exist [40].

Quantum fluctuations. The total condensate quantum field includes the DS wave functions and quantum fluctuations, $\Psi(x) = \sum_i \Psi_i(x)$, with $\Psi_i(x) = \psi_{\text{sol}}^{(i)}(x) + \delta \psi_i(x)$, $\delta \psi_i(x) = \sum_k (u_k^{(i)}(x) b_k + v_k^{(i)}(x) b_k^\dagger)$ and $b_k$ are the bosonic operators verifying the commutation relation $[b_k, b_l^\dagger] = \delta_{kl}$. The amplitudes $u_k(x)$ and $v_k(x)$ satisfy the normalization condition $|u_k(x)|^2 - |v_k(x)|^2 = 1$ and are explicitly given in [41]. The total Hamiltonian then reads $H = H_q + H_p + H_{\text{int}}$, where $H_q = \sum_{i=1}^2 \hbar \omega_0 \sigma_i^{(i)}$ is the qubit Hamiltonian, $\omega_0 = h(2 \nu - 1)/(2m_\xi^2)$ is the qubit gap energy, and $\nu = -1 + \sqrt{1 + 4 \chi/g}$ is a parameter controlling the number of bound states created by each DS, which operate as qubits (two-level systems) in the range $0.33 < \nu < 0.80$ [31]. The term $H_p = \sum_k \epsilon_k b_k^\dagger b_k$ represents the phonon (reservoir) Hamiltonian, where $\epsilon_k = \mu \sqrt{k^2(\xi^2 k^2 + 2)}$ is the Bogoliubov spectrum with chemical potential $\mu = gn_0$. The interaction Hamiltonian is given by

\[
H_{\text{int}} = \sum_{i,j} \chi \int dx \Phi_i \psi_{\text{sol}}^{(i)} \psi_{\text{sol}}^{(j)} \Phi_j,
\]

(2)

where $\Phi_j(x) = \sum_{l=0}^l \varphi_l^{(j)}(x) a_l^{(j)}$ describes the impurity wave function in the presence of DS potential spannable in terms of bosonic operators $a_l$, and $\varphi_0(x) = \sech (|x - x_i|/\xi)/\sqrt{2 \pi}$ and $\varphi_1(x) = i \sqrt{3} \tanh (|x - x_i|/\xi) \varphi_0(x)$ are the Wannier functions. Using the rotating wave approximation (RWA), the first order perturbed Hamiltonian can be written as

\[
H_{\text{int.}}^{(1)} = \sum_k \sum_{i=1}^2 (g^{(i)}(k) \sigma_i^{(i)} b_k + g^{(i)*}(k) \sigma_i^{(i)} b_k^\dagger) + \text{h.c.}
\]

Here, $\sigma_\uparrow = \sigma_\downarrow = a_i^\dagger a_0$ and we use the shorthand notation $g^{(i)}(k) = g_{ij}^{(i)}(k)$, where $g_{ij}^{(i)}(k) = \sqrt{n_0} \chi \int dx \varphi_i^{(j)*}(x) \varphi_i^{(j)}(x) \tanh (|x - x_i|/\xi)^\dagger u_k^{(i)}$.

The counter-rotating terms proportional to $b_k \sigma_\downarrow^{(i)}$ and $b_k^\dagger \sigma_\uparrow^{(i)}$ that do not conserve the total number of excitations are dropped by invoking the RWA. Such an approximation is well justified provided that the emission rate $\gamma$ is much smaller than the qubits transition frequency $\omega_0$, as shown in Ref. [31].

Measurement of entanglement. We derive the master equation describing the dynamics of the DS-qubits density matrix $\rho_q$. After tracing over the phonon’s degrees of freedom [42, 43], we obtain

\[
\frac{\partial \rho_q(t)}{\partial t} = -\frac{i}{\hbar} [H_q, \rho_q(t)] - i \sum_{i \neq j} \sum_{ij} \Gamma_{ij} \left[ \sigma_\downarrow^{(i)} \rho_q(t) \sigma_\uparrow^{(j)} - \frac{1}{2} \{ \sigma_\downarrow^{(i)} \sigma_\uparrow^{(j)}, \rho_q(t) \} \right]
\]

(3)

where

\[
\Gamma_{ij} = \frac{2L}{\hbar^2} \int_0^\infty \frac{dk}{k^2} g_k^{(i)*} g_k^{(j)} \delta(\omega_k - \omega_0),
\]

\[
\eta_{ij} = \frac{L}{2 \pi \hbar^2} \int_0^\infty \frac{dk}{k^2} g_k^{(i)*} g_k^{(j)} \frac{1}{(\omega_k - \omega_0)},
\]

(4)

where $L$ is the size of the condensate. For $i = j$, $\Gamma_{ii} = \gamma$ represents the spontaneous emission rate of the $i$th DS-qubit; for $i \neq j$, $\Gamma_{ij} = \Gamma$ is the collective damping resulting from the mutual exchange of phonons. The term $\eta_{ii} = \eta_{21} = \eta$ represents the phonon-induced coupling between the qubits. Both $\Gamma$ and $\eta$ display a nontrivial dependence on the distance $d$ between the DSs, as depicted in Fig. 2. Contrary to what happens for the case of qubits displaced in 1D electromagnetic reservoirs, both
parameters vanish for large separations, \( d \gg \xi \), rather than displaying a periodic dependence on \( d \) [44]. This is a consequence of the local-density approximation (LDA) performed in the computation of the functions \( u_k \) and \( v_k \), reflecting the finiteness of the soliton perturbation.

The most adequate basis to solve Eq. (3) is spanned by the collective Dicke states [45], as shown in Fig. (1 b). Depicted are the ground state \( |g\rangle = |g_1,g_2\rangle \), the excited state \( |e\rangle = |e_1,e_2\rangle \), and two intermediate, maximally entangled (symmetric \( |s\rangle = (|e_1,g_2\rangle + |g_1,e_2\rangle)/\sqrt{2} \) and antisymmetric \( |a\rangle = (|e_1,g_2\rangle - |g_1,e_2\rangle)/\sqrt{2} \) states. Depending on the value of \( \Gamma \), one of the intermediate states can be decoupled from the rest of the states. In this basis, the density matrix elements are given by

\[
\rho_{ee}(t) = e^{-2\gamma t}\rho_{ee}(0)
\]
\[
\rho_{ss}(t) = e^{-(\gamma+\Gamma)t}\rho_{ss}(0)
\]
\[
\rho_{aa}(t) = e^{-(\gamma-\Gamma)t}\rho_{aa}(0)
\]
\[
\rho_{sa}(t) = e^{-(\gamma+2\mu)t}\rho_{sa}(0),
\]

with the condition \( \rho_{gg} = 1 - \rho_{ee} - \rho_{ss} - \rho_{aa} \). It can be seen from Eq. (5) that the symmetric state \( |s\rangle \) is populated, by spontaneous emission, from the state \( |e\rangle \) at the superradiant rate \( \gamma + \Gamma \), while the anti-symmetric state \( |a\rangle \) at the subradiant rate \( \gamma - \Gamma \). The quantification of the entanglement is performed by using Wootters’ concurrence formula [46],

\[
C(t) = \max(0, \sqrt{\frac{1}{\sum_{n=1}^{4} \theta_n^2}} - \sum_{a=2}^{4} \sqrt{\theta_n^2}),
\]

where \( \theta_i \)'s denotes the eigenvalues in the decreasing order of the Hermitian matrix \( \zeta = \tilde{\rho} \hat{\rho} \). Here, \( \tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y) \) describes the spin flip density matrix with \( \rho^* \) and \( \sigma_y \) being the complex conjugate of \( \rho \) and the Pauli matrix, respectively. In the following, we investigate the effect of both \( \Gamma \) and \( \eta \) in the evolution of \( C(t) \) for two different situations: (i) the system is prepared in the state \( |s\rangle + |a\rangle \)/\sqrt{2} \), from which it decays spontaneously, and (ii) the DS-qubits are continuously pumped by an external Raman laser. In the first case, analytical solutions to Eq. (5) provide

\[
C(t) = e^{-\gamma t} \sqrt{\sin^2(\Gamma t) + \sin^2(2\eta t)}.
\]

Fig. (3) shows \( C(t) \) for the initialization of the system in the superposition of maximally entangled states. The concurrence firstly displays a fast increase, being then followed by a very slow decay.

The time evolution of the initial state that is given by equal populations in the states \( |s\rangle \) and \( |a\rangle \), i.e. \( \rho_{sa}(0) = \rho_{aa}(0) = 1/2 \), can be seen in panel b) of Fig. (3). It is shown that the decay rate of the state \( |s\rangle \) becomes sub-radiant (slow decay) while the state \( |a\rangle \) decays at the superradiant rate (fast decay) at a sufficiently large distance, \( d \gtrsim 5\xi/2 \sim 2–5 \mu m \) for a typical BEC [39]. The concurrence exhibits an appreciably long lifetime (\( \sim 80 \) ms) due to the asymmetry between the two cascades, eventually reaching the value of the population of the symmetric state \( |s\rangle \), \( C(t) \approx \rho_{ss}(t) \). The major limitation to the concurrence performance discussed above could be the quantum diffusion or quantum evaporation of the dark solitons [47], a feature that has been theoretically described but not yet experimentally verified. Taking into account the latter, a maximum reduction of 20% of the total concurrence lifetime is estimated [31]. In any case, quantum evaporation is expected if important trap anisotropies are present, a limitation that we can overcome with the help of box-like or ring potentials.

It is worth comparing the entanglement generation protocol presented here with other schemes proposed in the literature, such as plasmons mediated entanglement...
in plasmonic waveguides (PW) [14, 48] and phonon mediated quantum correlation in nanomechanical resonator [49]. In the case of 1D PW, a concurrence with a lifetime of \( \sim 8 \text{ ns} \) is obtained at a distance of the order of plasmonic wavelength, \( \sim 600 \text{ nm} \) [14]. But for transient entanglement mediated by 3D PW, the concurrence lives for a short time (\( \sim 4 \text{ ns} \)) [48]. In the present investigation, the concurrence exhibits a substantially large lifetime (\( \sim 80 \text{ ms} \)) at much larger distances (\( \sim 2 - 5 \mu \text{m} \)). Moreover, the investigation of exciton-phonon coupling in hybrid systems (e.g. consisting of a semiconductor quantum dots embedded in a nanomechanical resonator) indicates that the stationary concurrence strongly depends on the resonator temperature [49]. Fortunately, in our case, thermal effects are negligible (considering BECs operating well below the critical temperature) and therefore the excitations providing the interaction between the DS-qubits (phonons) are purely quantum mechanical in nature. Here, the concurrence is generated as a result of a considerable large value of the collective damping rate \( \Gamma \), as it becomes evident in Fig. (2).

**Driven concurrence.** We now consider the steady-state entanglement generation by driving the DS qubits continuously. The system is prepared, initially, in the ground state \( |g\rangle \) to apply a resonant Raman laser (\( \omega_0 = \omega_L \)) of Rabi frequency \( \Omega_i \), to drive the \( i \)th qubit independently. The inclusion of the driving term modifies the qubit Hamiltonian \( H_q \) as

\[
H_q = \hbar \sum_{i=1}^{2} \left[ \omega_0 s_z^{(i)} - \Omega_i (s_+^{(i)} e^{-i\omega_L t} + s_-^{(i)} e^{i\omega_L t}) \right]
\]

We solve the master Eq. (3) including the driven Hamiltonian, to obtain the concurrence \( C(t) \) (see Fig. 4). We observe a finite steady state concurrence at the long distance \( d \sim 5\xi/2 \). By solving \( \rho_q(t) = 0 \), we obtain the steady-state concurrence

\[
C(\infty) = \max \left\{ 0, \frac{\Omega^2}{2\Omega^2 + \gamma^2 \left[ \Omega^2 + \frac{1}{4} \left( (\gamma + \Gamma)^2 + 4\eta^2 \right) \right]} \right\}
\]

where \( U = \Gamma + 2i\eta \). A large spot laser prepared the system in symmetric state to excite the DS qubits, equally (i.e., \( \Omega_1 = \Omega_2 \)), therefore \( C(\infty) \) obtained its maximum value at distance \( d \sim 5\xi/2 \), shown in Fig. (5). The robustness of DS qubit proposal is to achieve long distance steady state concurrence in comparison to Refs. [14, 48, 49].

Pachos and Knight [50] proposed two qubit operations, including a Toffoli gate, by combining both tunneling and adiabatic passage of condensate atoms in a 1D optical lattice. The phonon-mediated quantum logic gates in trapped ions has been investigated by Bermudez [51] where they propose that a strong driving of the qubit decouples it from external noise and enhancing the fidelity of two-qubit quantum gate. This mechanism can be extended to a variety of other systems where a strong driving protects the quantum coherence of the qubits without compromising the two-qubit couplings. Thus, the present scheme will provide a great tool to study quantum gates [52].

In conclusion, large-distance entanglement is made possible via the Raman driving two dark-soliton qubits, a recently proposed platform for quantum information processing based purely on matter waves. In this study, the qubits consists of two-level systems formed by impurities trapped at the interior of dark solitons, the stable nonlinear depressions produced in quasi one-dimensional Bose-Einstein condensates. The entanglement is mediated by the quantum fluctuations (Bogoliubov excitations, or phonons). Thanks to the large lifetimes of these solitary qubits (being of the order of 0.01 s up to 1.0s), an appreciable amount entanglement can be produced at large distances in the range of \( 2 - 5 \mu \text{m} \). Our conclusion is that dark-soliton qubits are excellent candidates for quantum applications in quantum technologies.

**Fig. 4:** (color online) Time evolution of the driven concurrence \( C(t) \) for symmetric pumping (\( \Omega_1 = \Omega_2 \)) at distance \( d \sim 5\xi/2 \) with \( \Omega = 0.25\gamma \) (dashed curve) and \( \Omega = 0.35\gamma \) (solid curve).

**Fig. 5:** (color online) Panel a) shows steady-state concurrence \( C(\infty) \) as a function of distance \( d \) between DS qubits with \( \Omega = 0.25\gamma \) (dashed curve) and \( \Omega = 0.35\gamma \) (solid curve). Panel b) depicts the variation of \( C(\infty) \) with Rabi frequency \( \Omega \).
for which information storage during large times is necessary [53, 54]. Further, we believe that our findings make dark-soliton qubits useful objects for the investigation of a new generation of dissipative processes involving matter waves [55].

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[1] J. R. Weber et al., Proc. Natl. Acad. Sci. U.S.A. 107, 8513 (2010).
[2] Makhlkin, G. Schon, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
[3] S. D. Franceschc, L. Kouwenhoven, C. Schonenberger, and W. Wernsdorfer, Nature Nanotech. 5, 703 (2010).
[4] R. Hanson, L. P. Kouwenhoven, L. R. Petta, S. Tarucha, and L. M. K. Vandersypen, Rev. Mod. Phys. 79, 1217 (2007).
[5] J. Gillet, G. S. Agarwal, and T. Bastin, Phys. Rev. A 81, 013837 (2010).
[6] M. D. Lukin et al., Phys. Rev. Lett. 87, 037901 (2001).
[7] L. Saelen, S. I. Simonsen, and J. P. Hansen, Phys. Rev. A 83, 015401 (2011).
[8] E. Urban et al., Nat. Phys. 5, 110 (2009).
[9] C. Hettich, C. Schmitt, J. Zitzmann, S. Kuhn, I. Gerhardt, and V. Sandoghdar, Science 298, 385 (2002).
[10] Majer et al., Nature 449, 443 (2007).
[11] E. Gallardo et al., Phys. Rev. B 81, 193301 (2010).
[12] A. Imamoglu et al., Phys. Rev. Lett. 83, 4204 (1999).
[13] A. Laucht et al., Phys. Rev. B 82, 075305 (2010).
[14] A. Gonzalez-Tudela, D. Martin-Cano, E. Moreno, L. Martin-Moreno, C. Tejedor, and F. J. Garcia-Vidal, Phys. Rev. Lett. 106, 020501 (2011).
[15] A. Srensen, L.-M. Duan, J. I. Cirac, P. Zoller, Nature 409, 63 (2000).
[16] I. Buluta, F. Nori, Science 326, 108 (2009).
[17] K.B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995).
[18] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science 269, 198 (1995).
[19] A.S. Parkins and D.F. Walls, Phys. Rep. 303, 1 (1998).
[20] H. T. Ng and S. Bose, New Jour. Phys. 11, 043009 (2009).
[21] P. Bhi et al. Nature Phys. 5, 592 (2009).
[22] M. Riedel et al. Nature 464, 1170 (2010).
[23] J. Demsclag et al., Science 287, 97 (2000).
[24] Y. S. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals (Academic Press, San Diego, USA, 2003).
[25] S. Burger et al., Phys. Rev. Lett. 83, 5198 (1999).
[26] J. Dziarmaga, Z. P. Karkuszewski, and K. Sacha, J. Phys. B: At. Mol. Opt. Phys. 36, 1217 (2003).
[27] B. Jackson, N. P. Proukakis, and C. F. Barenghi, Phys. Rev. A 75, 051601 (2007).
[28] R. V. Mishmash and L. D. Carr, Phys. Rev. Lett. 103, 140403 (2009).
[29] G. A. El and A. M. Kamchatnov, Phys. Rev. Lett. 95, 204101 (2005).
[30] H. Terças, D. D. Solnyshkov and G. Malpuech, Phys. Rev. Lett. 110, 035302 (2013); ibid 113, 036403 (2014).
[31] M. I. Shaukat, E. V. Castro and H. Terças, Phys. Rev. A 95, 053618 (2017).
[32] H. Rong, R. Jones, A. Liu, O. Cohen, D. Hak, A. Fang and M. Paniccia, Nature. 433, 7027 (2005).
[33] V. Galitski and I. B. Spielman, Nature 49, 494 (2013).
[34] Here, the discussion is restricted to repulsive interactions (g > 0) where the dark solitons are assumed to be not disturbed by the presence of impurities. To achieve this, the impurity gas is chosen to be sufficiently dilute, i.e. $|\Phi|^2 \gg |\Phi|^2$, and much less massive than the BEC particles (experimental realization can be found in [31]).
[35] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972); ibid 37, 823 (1973).
[36] G. Huang, J. Szeftel, and S. Zhu, Phys. Rev. A 65, 053605 (2002).
[37] N. Parker, Numerical Studies of Vortices and Dark Solitons in atomic Bose Einstein Condensates, Ph.D Thesis (2004).
[38] P. Krüger, S. Hofferberth, I. E. Mazets, I. Lesanovsky, and J. Schmiedmayer, Phys. Rev. Lett. 105, 265302 (2010).
[39] A. L. Gaunt, T. F. Schmidtutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, Phys. Rev. Lett. 110, 200406 (2013).
[40] A. J. Allen, D. P. Jackson, C. F. Barenghi, and N. P. Proukakis, Phys. Rev. A 83, 013613 (2011).
[41] J. C. Martinez, Euro. Phys. Lett 96, 14007 (2011).
[42] Z. Ficek and R. Tanas, Phys. Rep. 372, 369 (2002).
[43] R. H. Lehmberg, Phys. Rev. A 2, 883 (1970); 889 (1970).
[44] M. Scully and M. Zubairy, Quantum Optics, Cambridge University Press (1997).
[45] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[46] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[47] J. Dziarmaga, Phys. Rev. A 70, 063616 (2004).
[48] S. Ali Hassan Gangeraj, A. Nemilentsau, G. W. Hanson and S. Hughes, Opt. Express 23, 22330 (2015).
[49] Y. He and M. Jiang, Opt. Comm. 382 , 580 (2017).
[50] J. K. Pachos and P. L. Knight, Phys. Rev. Lett. 91, 107902 (2003).
[51] A. Bermudez, P. O. Schmidt, M. B. Plenio and A. Retzker, Phys. Rev. A 85, 040302(R) (2012).
[52] D. Dzotian, A. S. Srensen, and M. Fleischhauer, Phys. Rev. B 82, 075427 (2010).
[53] J. Borregaard, P. Km, E. M. Kessler, M. D. Lukin, and A. S. Srensen, Phys. Rev. A 92, 012307 (2015).
[54] Z. Jin, S. L. SU, A. I. Zhu, H. F. Wang and S. H. Zhang, Opt. Express 25, 88 (2017).
[55] A. F. Alharbi and Z. Ficek, Phys. Rev. A 82, 054103 (2010).