Two-type Estimates for the Boundedness of Generalized Riesz Potential Operator in the Generalized Weighted Local Morrey Spaces

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Abstract

In this paper, we prove the Spanne-type boundedness of the generalized Riesz potential operator $I_\rho$ from the one generalized weighted local Morrey spaces $M_{1,\varphi_1}(w,\mathbb{R}^n)$ to the another one $M_{q,\varphi_2}(w,\mathbb{R}^n)$ with $w^q \in A_{1+\frac{q}{p}}$ for $1 < p < q < \infty$ and from the generalized weighted local Morrey spaces $M_{1,\varphi_1}(w,\mathbb{R}^n)$ to the weak generalized weighted local Morrey spaces $WM_{q,\varphi_2}(w,\mathbb{R}^n)$ with $w \in A_{1,q}$ for $1 < q < \infty$. We also prove the Adams-type boundedness of the operator $I_\rho$ from the weighted spaces $M_{\mu,\varphi}(w,\mathbb{R}^n)$ to the another one $M_{\mu,\varphi}(w,\mathbb{R}^n)$ with $w \in A_{\mu,q}$ for $1 < \mu < \infty$ and from the weighted spaces $M_{1,\varphi}(w,\mathbb{R}^n)$ to the weak weighted spaces $WM_{\mu,\varphi}(w,\mathbb{R}^n)$ with $w \in A_{1,q}$ for $1 < q < \infty$.

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1 Introduction

Morrey spaces \(M_{p,\lambda}(\mathbb{R}^n)\) were introduced by Morrey in \([18]\) and defined as follows: For \(0 \leq \lambda < n\), \(1 \leq p \leq \frac{n}{\lambda}\), \(f \in M_{p,\lambda}(\mathbb{R}^n)\) if \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\) and

\[
\|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < 1
\]

holds. Morrey spaces found important applications to potential theory \([1]\), elliptic equations with discontinuous coefficients \([3]\) and Shrödinger equations \([26]\).

On the other hand, on the weighted Lebesgue spaces \(L^p(w, \mathbb{R}^n)\), the boundedness of some classical operators were obtained by Muckenhoupt \([19]\), Muckenhoupt and Wheeden \([20]\), and Coifman and Fefferman \([4]\).

Recently, weighted Morrey spaces \(M_{p,\kappa}(w, \mathbb{R}^n)\) were introduced by Komori and Shirai \([13]\) as follows: For \(1 \leq p \leq \frac{n}{\kappa}\), \(0 < \kappa < 1\) and \(w\) be a weight, \(f \in M_{p,\kappa}(w, \mathbb{R}^n)\) if \(f \in L^p_{\text{loc}}(w, \mathbb{R}^n)\) and

\[
\|f\|_{M_{p,\kappa}(w,\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} w(B(x,r))^{-\frac{1}{\kappa}} \|f\|_{L^p(w,B(x,r))} < 1.
\]

They studied the boundedness of the aforementioned classical operators in these spaces. These results were extended to several other spaces in \([11]\). Weighted inequalities for fractional operators have applications to potential theory and quantum mechanics.

For a fixed \(x_0 \in \mathbb{R}^n\) the generalized weighted local Morrey spaces \(M_{p,\varphi}(w, \mathbb{R}^n)\) are obtained by replacing a function \(\varphi(x_0, r)\) instead of \(r^\lambda\) in the definition of weighted local Morrey space, which is the space of all functions \(f \in L^p_{\text{loc}}(w, \mathbb{R}^n)\) with finite norm

\[
\|f\|_{M_{p,\varphi}(w,\mathbb{R}^n)} = \sup_{r > 0} \varphi(x_0,r)^{-1} w(B(x_0,r))^{-\frac{1}{p}} \|f\chi_{B(x_0,r)}\|_{L^p(w,\mathbb{R}^n)}.
\]

During the last decades, the theory of boundedness of classical operators of the harmonic analysis in the generalized Morrey spaces \(M_{p,\varphi}(\mathbb{R}^n)\) have been well studied by now, we refer the readers to \([9, 14, 15, 21]\) and \([23]\).

For a measurable function \(\rho : (0,1) \to (0,1)\) the generalized Riesz potential operator (or generalized fractional integral operator) \(I_\rho\) is defined by

\[
I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy
\]
for any suitable function $f$ on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, then we get the Riesz potential operator $I_\alpha$. The generalized Riesz potential operator $I_\rho$ was initially investigated in [22]. Nowadays many authors have been culminating important observations about $I_\rho$ especially in connection with Morrey spaces. Nakai [22] proved the boundedness of $I_\rho$ from the generalized Morrey spaces $M_{1,\varphi}(\mathbb{R}^n)$ to the spaces $M_{1,\psi}(\mathbb{R}^n)$ for suitable functions $\varphi$ and $\psi$. The boundedness of $I_\rho$ from the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ to the spaces $M_{q,\psi}(\mathbb{R}^n)$ is studied by Eridani [5], Guliyev et al [9], Kucukaslan et al [14, 15], Kucukaslan [16, 17], Nakai [23] and Nakamura [24].

Spanne-type and Adams-type boundednesses of generalized fractional maximal operator $M_\rho$ in the generalized weighted local Morrey spaces $M_{p,\varphi}(x_0,w,\mathbb{R}^n)$ and generalized weighted Morrey spaces $M_{p,\varphi}(w,\mathbb{R}^n)$ were studied in [17]. But, Spanne-type and Adams-type boundedness of the generalized Riesz potential operator $I_\rho$ in the spaces $M_{p,\varphi}(w,\mathbb{R}^n)$ and $M_{p,\varphi}(w,\mathbb{R}^n)$ have not been studied, yet.

Spanne [25] and Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results, can be summarized as follows.

**Theorem A.** (Spanne, but published by Peetre [22]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{1}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator $I_\alpha$ is bounded from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, $I_\alpha$ is bounded from $M_{1,\lambda}(\mathbb{R}^n)$ to $WM_{q,\mu}(\mathbb{R}^n)$.

**Theorem B.** (Adams [1]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha - \lambda}{n - \lambda}$. Then for $p > 1$, the operator $I_\alpha$ is bounded from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\lambda}(\mathbb{R}^n)$ and for $p = 1$, $I_\alpha$ is bounded from $M_{1,\lambda}(\mathbb{R}^n)$ to $WM_{q,\lambda}(\mathbb{R}^n)$.

In the following theorems which were proved in [9], we give Spanne and Adams type results for the boundedness of operator $I_\rho$ on the generalized local Morrey spaces $M_{p,\varphi}^{(x_0)}(\mathbb{R}^n)$ and on the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$, respectively.

**Theorem C.** (Spanne type result [9]) Let $x_0 \in \mathbb{R}^n$, $1 \leq p < q < \infty$, the function $\rho$ satisfy the conditions $[3.1]$, $[3.2]$ and $[3.4]$ for $w = 1$. Let also $(\varphi_1, \varphi_2)$ satisfy the conditions

$$\text{ess inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}} \leq C \varphi_2(x_0,\frac{t}{2})t^{\frac{n}{q}},$$

$$\int_r^\infty \left( \text{ess inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}} \right) \frac{\rho(t)}{t^{\frac{n}{q}+1}} dt \leq C \varphi_2(x_0,r),$$

where $C$ does not depend on $x_0$ and $r$. Then the operator $I_\rho$ is bounded from
one generalized local Morrey spaces \( M_{\{x_0\}}^{p,\varphi_1}(\mathbb{R}^n) \) to another one \( M_{\{x_0\}}^{q,\varphi_2}(\mathbb{R}^n) \) for \( p > 1 \) and from the spaces \( M_{1,\varphi_1}(\mathbb{R}^n) \) to the weak space \( WM_{\{x_0\}}^{q,\varphi_2}(\mathbb{R}^n) \) for \( p = 1 \).

**Theorem D. (Adams type result [9])** Let \( 1 \leq p < \infty, q > p, \rho(t) \) satisfy the conditions (3.2) and (3.4) for \( w = 1 \). Let also \( \varphi(x,t) \) satisfy the conditions

\[
\sup_{r<t<\infty} \varphi(x,t) \leq C \varphi(x,r),
\]

\[
\int_r^\infty \varphi(x,t)^{\frac{1}{p}} \frac{\rho(t)}{t} dt \leq C \rho(r)^{-\frac{1}{p-1}},
\]

where \( C \) does not depend on \( x \in \mathbb{R}^n \) and \( r > 0 \). Then the operator \( I_\rho \) is bounded from the one generalized Morrey space \( M_{\{x_0\}}^{p,\varphi_1}(\mathbb{R}^n) \) to another one \( M_{\{x_0\}}^{q,\varphi_2}(\mathbb{R}^n) \) for \( p > 1 \) and from the space \( M_{1,\varphi}(\mathbb{R}^n) \) to the weak space \( WM_{\{x_0\}}^{q,\varphi_2}(\mathbb{R}^n) \) for \( p = 1 \).

In this study, by using the method given in [10], we prove the Spanne and Adams type estimates for the boundedness of generalized Riesz potential operator \( I_\rho \) on the generalized weighted local Morrey spaces \( M_{\rho,\varphi_1}^{p,\varphi}(w^p,\mathbb{R}^n) \) with \( 1 \leq p < q < \infty, w^a \in A_{\frac{1}{p}, \frac{1}{q}} \) belonging to Muckenhoupt-Weeden class \( A_{p,q} \). We find conditions on the triple \( (\varphi_1, \varphi_2, \rho) \) which ensure the Spanne-type boundedness of the operator \( I_\rho \) from one generalized weighted local Morrey spaces \( M_{\rho,\varphi_1}^{p,\varphi}(w^p,\mathbb{R}^n) \) to another \( M_{\rho,\varphi_2}^{q,\varphi}(w^q,\mathbb{R}^n) \) with \( w^a \in A_{1+\frac{1}{q}, \frac{1}{q}} \) for \( 1 < q < \infty \) and \( M_{1,\varphi_1}(w,\mathbb{R}^n) \) to the weighted weak space \( WM_{\rho,\varphi_2}^{q,\varphi}(w^q,\mathbb{R}^n) \) with \( w \in A_{1,q} \) for \( 1 < q < \infty \) (see Theorem 3.3). We also find conditions on the pair \( (\varphi, \rho) \) which ensure the Adams-type boundedness of \( I_\rho \) from \( M_{\rho,\varphi_1}^{p,\varphi}(w,\mathbb{R}^n) \) to \( M_{\rho,\varphi_2}^{q,\varphi}(w,\mathbb{R}^n) \) for \( 1 < p < q < \infty, w \in A_{1+\frac{1}{p}, \frac{1}{p}} \) and from \( M_{1,\varphi}(w,\mathbb{R}^n) \) to \( WM_{\rho,\varphi_2}^{q,\varphi}(w,\mathbb{R}^n) \) for \( 1 < q < \infty, w \in A_{1,q} \) (see Theorem 4.1).

In all cases the conditions for the boundedness of \( I_\rho \) are given in terms of Zygmund-type integral inequalities on the all \( \varphi \) functions and \( r \) which do not assume any assumption on monotonicity of \( \varphi_1(x,r), \varphi_2(x,r) \) and \( \varphi(x,r) \) in \( r \).

By \( A \lesssim B \) we mean that \( A \leq CB \) with some positive constant \( C \) independent of appropriate quantities. If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \) and say that \( A \) and \( B \) are equivalent.
2 Preliminaries

Let $x \in \mathbb{R}^n$ and $r > 0$, then we denote by $B(x, r)$ the open ball centered at $x$ of radius $r$, and by $\mathcal{C}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, 1)$ almost everywhere. For a weight $w$ and a measurable set $E$, we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of $E$ by $|E|$ and the characteristic function of $E$ by $\chi_E$. If $w$ is a weight function, for all $f \in L^1_{\text{loc}}$ we denote by $L^{p}_{\text{loc}}(w, \mathbb{R}^n)$ the weighted Lebesgue space defined by the norm

$$
\|f\chi_{B(x,r)}\|_{L^p(w, \mathbb{R}^n)} = \left( \int_{B(x,r)} |f(x)|^pw(x)dx \right)^{\frac{1}{p}} < 1,
$$

when $1 \leq p < 1$ and by

$$
\|f\chi_{B(x,r)}\|_{L^1(w, \mathbb{R}^n)} = \text{ess sup}_{x \in B(y, r)} |f(x)w(x)| < 1,
$$

when $p = 1$.

We recall that a weight function $w$ belongs to the Muckenhoupt-Wheeden class $A_{p, q}$ (see [19]) for $1 < p < q < 1$, if

$$
\sup_B \left( \frac{1}{|B|} \int_B w(x)^qdx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B w(x)^{-\frac{q}{p'}}dx \right)^{\frac{1}{p'}} \leq C,
$$

where the supremum is taken with respect to all balls $B$ and $C > 0$. Note that, for all balls $B$ by Hölder’s inequality we get

$$
|B|^{\frac{1}{p}-\frac{1}{q}-1}\|w\|_{L^q(B)}\|w^{-1}\|_{L^{p'}(B)} \geq 1. \tag{2.1}
$$

If $p = 1$, $w$ is in the $A_{1, q}$ with $1 < q < 1$ if

$$
\sup_B \left( \frac{1}{|B|} \int_B w(x)^qdx \right)^{\frac{1}{q}} \left( \text{ess sup}_{x \in B} \frac{1}{w(x)} \right) \leq C,
$$

where the supremum is taken with respect to all balls $B$ and $C > 0$.

The weight function $w$ satisfies the reverse doubling condition if there exist constants $\alpha_1 > 1$ and $\alpha_2 < 1$ such that

$$
w(B(x, r)) \leq \alpha_2 w(B(x, \alpha_1 r)) \tag{2.2}
$$

for arbitrary $x \in \mathbb{R}^n$ and $r > 0$. 

Lemma 2.1. If \( w \in A_{p,q} \) with \( 1 < p < q < 1 \), then the following statements are true.

(i) \( w^q \in A_r \) with \( r = 1 + \frac{q}{p'} \).

(ii) \( w^{-p'} \in A_{r'} \) with \( r' = 1 + \frac{p'}{q} \).

(iii) \( w^p \in A_s \) with \( s = 1 + \frac{q'}{p} \).

(iv) \( w^{-q'} \in A_{s'} \) with \( s' = 1 + \frac{q'}{p} \).

We find it convenient to define the generalized weighted local Morrey spaces in the form as follows.

Definition 2.2. Let \( 1 \leq p < \infty \) and \( \varphi(x,r) \) be a positive measurable function on \( \mathbb{R}^n \times (0,\infty) \). For any fixed \( x_0 \in \mathbb{R}^n \) we denote by \( M_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n) \) the generalized weighted local Morrey space, the space of all functions \( f \in L^\text{loc}_p(w,\mathbb{R}^n) \) with finite quasinorm

\[
\|f\|_{M_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n)} = \|f(x_0 + \cdot)\|_{M_{p,\varphi}(w,\mathbb{R}^n)}.
\]

Also by \( WM_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n) \) we denote the weak generalized weighted local Morrey space of all functions \( f \in W L^\text{loc}_p(w,\mathbb{R}^n) \) for which

\[
\|f\|_{WM_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n)} = \|f(x_0 + \cdot)\|_{WM_{p,\varphi}(w,\mathbb{R}^n)} < \infty.
\]

According to this definition, we recover the weighted local Morrey space \( M_{p,\lambda}^{\{x_0\}}(w,\mathbb{R}^n) \) and weighted weak local Morrey space \( WM_{p,\lambda}^{\{x_0\}}(w,\mathbb{R}^n) \) under the choice \( \varphi(x, r) = r^{\frac{\lambda-n}{p}} \):

\[
M_{p,\lambda}^{\{x_0\}}(w,\mathbb{R}^n) = M_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n) \big|_{\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}},
\]

\[
WM_{p,\lambda}^{\{x_0\}}(w,\mathbb{R}^n) = WM_{p,\varphi}^{\{x_0\}}(w,\mathbb{R}^n) \big|_{\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}}.\]

We denote by \( L_\infty(w, (0,\infty)) \) the space of all functions \( g(t), t > 0 \) with finite norm

\[
\|g\|_{L_\infty(w, (0,\infty))} = \sup_{t>0} w(t)g(t)
\]

and \( L_\infty(0,\infty) \equiv L_\infty(1, (0,\infty)) \). Let \( \mathcal{S}(0,1) \) be the set of all Lebesgue-measurable functions on \( (0,1) \) and \( \mathcal{S}^+(0,1) \) its subset consisting of all non-negative functions on \( (0,1) \). We denote by \( \mathcal{S}^+(0,1;\uparrow) \) the cone of all functions in \( \mathcal{S}^+(0,1) \) which are non-decreasing on \( (0,1) \) and

\[
\mathbb{A} = \left\{ \varphi \in \mathcal{S}^+(0,1;\uparrow) : \lim_{t \to 0^+} \varphi(t) = 0 \right\}.
\]
The following theorem was proved in [9] which we will use while proving our main results.

**Theorem 2.3.** Let $w_1, w_2$ be non-negative measurable functions satisfying $0 < \|w_1\|_{L^1(t, \infty)} < 1$ for any $t > 0$. Then the identity operator $I$ is bounded from $L^1(w_1, (0, 1))$ to $L^1(w_2, (0, 1))$ on the cone $A$ if and only if

$$\left\| w_2 \left( \frac{1}{\|w_1\|_{L^1(t, \infty)}} \right) \right\|_{L^1(0, 1)} < 1.$$  

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_wg(t) := \int_{t}^{1} g(s)w(s)d\mu(s), \quad 0 < t < \infty,$$

where $w$ is weight and $d\mu(s)$ is a non-negative Borel measure on $(0, 1)$.

The following theorem was proved in [2].

**Theorem 2.4.** Let $w_1, w_2$ and $w$ be weights on $(0, \infty)$ and $w_1(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\text{ess sup}_{t>0} w_2(t)H_wg(t) \leq C \text{ ess sup}_{t>0} w_1(t)g(t)$$  \quad (2.3)$$

holds for some $C > 0$ for all non-negative and non-decreasing $g$ on $(0, 1)$ if and only if

$$B := \sup_{t>0} w_2(t) \int_{t}^{1} \frac{w(s)ds}{\text{ess sup}_{s<\tau<\infty} w_1(\tau)} < \infty.$$  \quad (2.4)$$

Moreover, the value $C = B$ is the best constant for (2.3).

**Remark 2.5.** In (2.3) and (2.4) it is assumed that $\frac{1}{t} = 0$ and $0 \cdot 1 = 0$.

### 3 Spanne-type result for the operator $I_\rho$ in the spaces $M_{p,\varphi}(w^p, \mathbb{R}^n)$

We assume that

$$\int_{1}^{1} \frac{\rho(t)}{t^n} \frac{dt}{t} < \infty,$$  \quad (3.1)$$
so that the generalized Riesz potential $I_\rho f$ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$  

In addition, we shall also assume that $\rho$ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < 1$ such that

$$\sup_{r < s \leq 2r} \frac{\rho(s)}{s^n} \leq C \int_{k_1 r}^{k_2 r} \frac{\rho(t) \, dt}{t^n}, \quad r > 0.$$ (3.2)

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C} \frac{\rho(t)}{t^n} \leq \frac{\rho(r)}{r^n} \leq C \frac{\rho(t)}{t^n},$$ (3.3)

whenever $r$ and $t$ satisfy $r, t > 0$ and $\frac{1}{2} \leq \frac{r}{t} \leq 2$.

In the sequel for the generalized Riesz potential operator $I_\rho$ we always assume that $\rho$ satisfies the conditions (3.1) - (3.3), and then denote the set of all such functions by $\tilde{G}_0$. We will write, when $\rho \in \tilde{G}_0$,

$$\tilde{\rho}(r) := Cr^n \int_r^\infty \frac{\rho(t) \, dt}{t^n}.$$  

The following lemma is valid for the operator $I_\rho$.

**Lemma 3.1.** [7] Let $w^q \in A_{1+\frac{q}{p}}$ satisfies (2.2), the function $\rho$ satisfies the conditions (3.1) - (3.3), and $f \in L_{1}^{\text{loc}}(w, \mathbb{R}^n)$. Then there exist $C > 0$ for all $B(x,r) \subset \mathbb{R}^n$ such that the inequality

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{\rho(r)}{r^n} \left( \int_{B(x,r)} w^q(x) \, dx \right)^{\frac{1}{q}} \left( \int_{B(x,r)} w(x)^{-p'} \, dx \right)^{\frac{1}{p'}} \leq C$$ (3.4)

is necessary and sufficient condition for the boundedness of generalized Riesz potential operator $I_\rho$ from $L_p(w^p, \mathbb{R}^n)$ to $WL_q(w^q, \mathbb{R}^n)$ for $1 \leq p < q < \infty$, and from $L_p(w^p, \mathbb{R}^n)$ to $L_q(w^q, \mathbb{R}^n)$ for $1 < p < q < \infty$, $w^q \in A_{1+\frac{q}{p}}$, where the constant $C$ does not depend on $f$.

The following is weighted local $L_p(\mathbb{R}^n)$-estimate for the operator $I_\rho$.  

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Lemma 3.2. Let fixed \( x_0 \in \mathbb{R}^n \), and \( 1 \leq p < q < 1 \), \( w \in A_{1+\frac{1}{p}} \) and \( \rho(t) \) satisfy the conditions (3.1) and (3.2).

If the condition (3.4) is fulfill, then the inequality
\[
\|I_\rho f\chi_{B(x_0,r)}\|_{L_q(w^q,\mathbb{R}^n)} \leq \|f\chi_{B(x_0,2r)}\|_{L_p(w^p,\mathbb{R}^n)}
\]
\[
+ (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^r \|f\chi_{B(x_0,t)}\|_{L_p(w^p,\mathbb{R}^n)} (w^q(B(x_0,t)))^{-\frac{1}{n} \frac{\rho(t)}{t^n}} \frac{dt}{t} \tag{3.5}
\]
holds for the ball \( B(x_0, r) \) and for all \( f \in L_{loc}^1(\mathbb{R}^n, w) \).

If the condition (3.4) is fulfill, then for \( p = 1 \) the inequality
\[
\|I_\rho f\chi_{B(x_0,r)}\|_{W_{L_q(w^q)}} \lesssim \|f\chi_{B(x_0,2r)}\|_{L_1(w)}
\]
\[
+ (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^r \|f\chi_{B(x_0,t)}\|_{L_1(w)} (w^q(B(x_0,t)))^{-\frac{1}{n} \frac{\rho(t)}{t^n}} \frac{dt}{t} \tag{3.6}
\]
holds for the ball \( B(x_0, r) \) and for all \( f \in L_{loc}^1(\mathbb{R}^n) \).

Proof. Let \( 1 \leq p < q < 1 \) and \( w \in A_{1+\frac{1}{p}} \). For fixed \( x_0 \in \mathbb{R}^n \), set \( B \equiv B(x_0, r) \) for the ball centered at \( x_0 \) and of radius \( r \). Write \( f = f_1 + f_2 \) with \( f_1 = f\chi_{2B} \) and \( f_2 = f\chi_{\mathbb{R}^n \setminus (2B)} \). Hence, by the Minkowski inequality we have
\[
\|I_\rho f\chi_{B}\|_{L_q(w^q,\mathbb{R}^n)} \leq \|I_\rho f_1\chi_{B}\|_{L_q(w^q,\mathbb{R}^n)} + \|I_\rho f_2\chi_{B}\|_{L_q(w^q,\mathbb{R}^n)}.
\]

Since \( f_1 \in L_p(w^p,\mathbb{R}^n) \), \( I_\rho f_1 \in L_q(w^q,\mathbb{R}^n) \) and from condition (3.4) we get the boundedness of \( I_\rho \) from \( L_p(w^p,\mathbb{R}^n) \) to \( L_q(w^q,\mathbb{R}^n) \) (see Lemma 3.1) and it follows that:
\[
\|I_\rho f_1\chi_{B}\|_{L_q(w^q,\mathbb{R}^n)} \leq \|I_\rho f_1\|_{L_q(w^q,\mathbb{R}^n)} \leq C \|f_1\|_{L_p(w^p,\mathbb{R}^n)} = C \|f\chi_{2B}\|_{L_p(w^p,\mathbb{R}^n)},
\]
where constant \( C > 0 \) is independent of \( f \).

It’s clear that \( z \in B \), \( y \in B(2B) \) implies \( \frac{1}{2}|x_0-y| \leq |z-y| \leq \frac{3}{2}|x_0-y| \). Then from conditions (3.1), (3.2) and by Fubini’s theorem we have
\[
|I_\rho f_2(z)| \lesssim \int_{B(2B)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \lesssim \int_{2r}^r \int_{B(x_0,t)} |f(y)| dy \frac{\rho(t)}{t^n} \frac{dt}{t}.
\]
Applying Hölder’s inequality and from (2.1), we get
\[
\int_{B(2B)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy
\]
\[
\lesssim \int_{2r}^r \|f\chi_{B(x_0,t)}\|_{L_p(w^p,\mathbb{R}^n)} \|w^{-1}\chi_{B(x_0,t)}\|_{L_{\rho(t)(\mathbb{R}^n)}} \frac{\rho(t)}{t^n} \frac{dt}{t}
\]
\[
\lesssim \int_{2r}^r \|f\chi_{B(x_0,t)}\|_{L_p(w^p,\mathbb{R}^n)} (w^q(B(x_0,t)))^{-\frac{1}{q} \frac{\rho(t)}{t^n}} \frac{dt}{t}.
\]
Moreover, for all \( p \in [1, 1) \) the inequality
\[
\|I_p f \chi_B\|_{L_q(w^p, \mathbb{R}^n)} \lesssim (w^q(B))^{\frac{1}{q}} \int_{2r}^1 \|f \chi_{B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} (w^q(B(x_0,t)))^{-\frac{1}{q}} \frac{\rho(t)\,dt}{t^n} \tag{3.7}
\]
is valid. Thus
\[
\|I_p f \chi_B\|_{L_q(w^p, \mathbb{R}^n)} \lesssim \|f \chi_{2B}\|_{L_p(w^p, \mathbb{R}^n)}
+ (w^q(B))^{\frac{1}{q}} \int_{2r}^1 \|f \chi_{B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} (w^q(B(x_0,t))) \rho(t)\,dt\frac{t^n}{t}.
\]
On the other hand,
\[
\|f \chi_{2B}\|_{L_p(w^p, \mathbb{R}^n)} \approx \frac{r^\frac{n}{p}}{\rho(r)} \|f \chi_{2B}\|_{L_p(w^p, \mathbb{R}^n)} \int_r^3 \rho(t)\,dt\frac{t^n}{t}
\leq \frac{r^\frac{n}{p}}{\rho(r)} \int_{2r}^1 \|f \chi_{2B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} \rho(t)\,dt\frac{t^n}{t}
\lesssim (w^q(B))^{\frac{1}{q}} \|w^{-1}\chi_B\|_{L_p(w^p, \mathbb{R}^n)} \int_{2r}^1 \|f \chi_{2B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} \rho(t)\,dt\frac{t^n}{t}
\lesssim (w^q(B))^{\frac{1}{q}} \int_{2r}^1 \|f \chi_{2B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} (w^q(B(x_0,t)))^{-\frac{1}{q}} \rho(t)\,dt\frac{t^n}{t}. \tag{3.8}
\]
Hence by
\[
\|I_p f \chi_B\|_{L_q(w^p, \mathbb{R}^n)} \lesssim (w^q(B))^{\frac{1}{q}} \int_{2r}^1 \|f \chi_{B(x_0,t)}\|_{L_p(w^p, \mathbb{R}^n)} (w^q(B(x_0,t)))^{-\frac{1}{q}} \rho(t)\,dt\frac{t^n}{t}
\]
we get the inequality (3.7).
Now let $p = 1$ and $w \in A_{1,q}$. In this case by (3.4) we obtain
\[
\| I_\rho f_1 \chi_B \|_{WL_q(w^q, \mathbb{R}^n)} \leq \| I_\rho f_1 \|_{WL_q(w^q, \mathbb{R}^n)} \lesssim \| f_1 \|_{L_1(w, \mathbb{R}^n)} = \| f \chi_B \|_{L_1(w, \mathbb{R}^n)}
\]

\[
\approx \frac{r^n}{\rho(r)} \| f \chi_{2B} \|_{L_1(w, \mathbb{R}^n)} \int_r^1 \frac{\rho(t) dt}{t^n}
\]

\[
\leq \frac{r^n}{\rho(r)} \int_{2r}^1 \| f \chi_{2B(x_0,t)} \|_{L_1(w, \mathbb{R}^n)} \frac{\rho(t) dt}{t^n}
\]

\[
\lesssim (u^q(B)) \frac{1}{q} \| \chi_{B} \|_{L_1(w, \mathbb{R}^n)} \int_{2r}^1 \| f \chi_{2B(x_0,t)} \|_{L_1(w, \mathbb{R}^n)} \frac{\rho(t) dt}{t^n}
\]

\[
\lesssim (u^q(B)) \frac{1}{q} \int_{2r}^1 \| f \chi_{2B(x_0,t)} \|_{L_1(w, \mathbb{R}^n)} \| \chi_{B(x_0,t)} \|_{L_1(w, \mathbb{R}^n)} \frac{\rho(t) dt}{t^n}
\]

\[
\lesssim (u^q(B)) \frac{1}{q} \int_{2r}^1 \| f \chi_{2B(x_0,t)} \|_{L_1(w, \mathbb{R}^n)} (u^q(B(x_0,t))) \frac{\rho(t) dt}{t^n}. \quad (3.9)
\]

Then from (3.8) and (3.9) we get the inequality (3.6). \qed

The following theorem one of the main result of our paper, in which we prove the Spanne-type estimate for the boundedness of generalized fractional integral operator $I_\rho$ from generalized weighted local Morrey spaces $M_{p,\varphi_1}^{(x_0)}(w^p, \mathbb{R}^n)$ to $M_{q,\varphi_2}^{(x_0)}(w^q, \mathbb{R}^n)$.

**Theorem 3.3.** Let fixed $x_0 \in \mathbb{R}^n$, $1 \leq p < q < \infty$, $w \in A_{1+\frac{1}{p}, \infty}$, the function $\rho$ satisfy the conditions (3.1), (3.2) and (3.3). Let also $(\varphi_1, \varphi_2)$ satisfy the conditions

\[
\text{ess inf}_{t < s < \infty} \varphi_1(x_0, s) s^{\frac{n}{q}} \leq C \varphi_2(x_0, \frac{t}{2}) \frac{t^{\frac{n}{q}}}{2}, \quad (3.10)
\]

\[
\int_r^\infty \text{ess inf}_{t < s < \infty} \varphi_1(x_0, s) (w^p(B(x_0, s)))^{\frac{1}{q}} \rho(t) \frac{dt}{t^{\frac{n}{q}}} \leq C \varphi_2(x_0, r), \quad (3.11)
\]

where $C$ does not depend on $x$ and $r$. Then the operator $I_\rho$ is bounded from $M_{p,\varphi_1}^{(x_0)}(w^p, \mathbb{R}^n)$ to $M_{q,\varphi_2}^{(x_0)}(w^q, \mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}^{(x_0)}(w, \mathbb{R}^n)$ to $WM_{q,\varphi_2}^{(x_0)}(w^q, \mathbb{R}^n)$ for $p = 1$. Moreover, for $p > 1$

\[
\| I_\rho f \|_{M_{q,\varphi_2}^{(x_0)}(w^q, \mathbb{R}^n)} \lesssim \| f \|_{M_{p,\varphi_1}^{(x_0)}(w^p, \mathbb{R}^n)},
\]

and for $p = 1$

\[
\| I_\rho f \|_{WM_{q,\varphi_2}^{(x_0)}(w^q, \mathbb{R}^n)} \lesssim \| f \|_{M_{1,\varphi_1}^{(x_0)}(w, \mathbb{R}^n)}.
\]
Proof. Let \( p > 1 \). Then by Theorems 2.3, 2.4, Lemma 3.2 and conditions (3.10)-(3.11) we have that

\[
\| I_p f \|_{M_{p,q,2}^{(x_0)}(w^q,\mathbb{R}^n)} = \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| I_p f \|_{L_q(w^q, B(x_0, 2r))} \\
= \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| I_p f \chi_{B(x_0, 2r)} \|_{L_q(w^q, \mathbb{R}^n)} \\
\lesssim \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| \| f \|_{L_p(w^p, B(x_0, 2r))} \\
+ \sup_{r > 0} \| \varphi_2(x_0, r)^{-1} \int_r^1 \| f \|_{L_p(w^p, B(x_0, 2t))} \frac{\rho(t)}{t^{n+1}} dt \].
\]

\[
\approx \sup_{r > 0} \| \varphi_1(x_0, r)^{-1}(w^p(B(x_0, r)))^{-\frac{1}{p}} \| \| f \|_{L_p(w^p, B(x_0, r))} \\
= \| f \|_{M_{p^q_0,2}^{(x_0)}(w,\mathbb{R}^n)}.
\]

Now let \( p = 1 \) then

\[
\| I_p f \|_{W_{M_{p,q,2}^{(x_0)}(w^q,\mathbb{R}^n)}} = \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| I_p f \|_{W_{L_q(w^q, B(x_0, 2r))}} \\
= \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| I_p f \chi_{B(x_0, 2r)} \|_{W_{L_q(w^q, \mathbb{R}^n)} } \\
\lesssim \sup_{r > 0} \| \varphi_2(x_0, r)^{-1}(w^q(B(x_0, r)))^{-\frac{1}{q}} \| \| f \|_{L_1(w, B(x_0, 2r))} \\
+ \sup_{r > 0} \| \varphi_2(x_0, r)^{-1} \int_r^1 \| f \|_{L_1(w, B(x_0, 2t))} \frac{\rho(t)}{t^{n+1}} dt \].
\]

\[
\approx \sup_{r > 0} \| \varphi_1(x_0, r)^{-1}(w(B(x_0, r)))^{-\frac{1}{p}} \| \| f \|_{L_1(w, B(x_0, r))} \\
= \| f \|_{M_{p^q_0,1}^{(x_0)}(w,\mathbb{R}^n)}.
\]

Hence the proof is completed. \( \square \)

**Corollary 3.4.** In the case \( w \equiv 1 \) from Theorem 3.3 we get Theorem C, in which we give Spanne-type result for generalized Riesz potential operator \( I_p \) on generalized local Morrey spaces \( M_{p,q,2}^{(x_0)}(\mathbb{R}^n) \) which was proved in [12] (Theorem 16, p.6).

**Corollary 3.5.** In the case \( \rho(t) = t^\alpha, w \equiv 1, x \equiv x_0 \) from Theorem 3.3 we get Spanne-type result for Riesz potential operator \( I_\alpha \) on generalized Morrey spaces \( M_{p,\varphi}(\mathbb{R}^n) \) which was proved in [12] (Theorem 5.4, p.338).
Corollary 3.6. In the case $\rho(t) = t^\alpha$, $w \equiv 1$ and $\varphi(x_0, t) = t^{\frac{\lambda}{p-\lambda}}$, $0 < \lambda < n$ from Theorem 3.3 we get Spanne result for Riesz potential operator $I_\alpha$ on local Morrey spaces $M_{p,\lambda}^{\varphi}(\mathbb{R}^n)$ which is variant of Theorem A proved in [25].

4 Adams-type result for the operator $I_\rho$ in the spaces $M_{p,\varphi}(w)$

The following theorem is Adams-type estimate for generalized Riesz potential operator $I_\rho$ on generalized weighted Morrey spaces $M_{p,\varphi}(w, \mathbb{R}^n)$.

Theorem 4.1. Let $1 \leq p < \infty$, $q > p$, $w \in A_{1+\frac{q}{p}}$, $\rho(t)$ satisfy the conditions (3.1), (3.2) and (3.4). Let also $\varphi(x, t)$ satisfy the conditions

\begin{equation}
\begin{aligned}
c^{-1} \varphi(x, r) &\leq \varphi(x, t) \leq c \varphi(x, r), \\
\int_r^\infty \frac{\text{ess inf}_{t<s<\infty} \varphi(x, s)}{w(B(x, s))^{\frac{1}{q}}} \rho(t) \frac{dt}{t} &\leq C \left(\rho(r)\right)^{-\frac{p}{q-p}},
\end{aligned}
\end{equation}

where $C$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $I_\rho$ is bounded from $M_{p,\varphi}^{\frac{1}{p}}(w, \mathbb{R}^n)$ to $M_{q,\varphi}^{\frac{1}{q}}(w, \mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi}(w, \mathbb{R}^n)$ to $WM_{q,\varphi}^{\frac{1}{q}}(w, \mathbb{R}^n)$ for $p = 1$. Moreover, for $p > 1$

$$\|I_\rho f\|_{M_{q,\varphi}^{\frac{1}{q}}(w, \mathbb{R}^n)} \lesssim \|f\|_{M_{p,\varphi}^{\frac{1}{p}}(w, \mathbb{R}^n)},$$

and for $p = 1$

$$\|I_\rho f\|_{WM_{q,\varphi}^{\frac{1}{q}}(w, \mathbb{R}^n)} \lesssim \|f\|_{M_{1,\varphi}(w, \mathbb{R}^n)}.$$

Proof. Let $1 < p < \infty$, $q > p$, $w \in A_{1+\frac{q}{p}}$ and $f \in M_{p,\varphi}^{\frac{1}{p}}(w, \mathbb{R}^n)$. Write $f = f_1 + f_2$, where $B = B(x, r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)}$. Then we have

$$I_\rho f(x) = I_\rho f_1(x) + I_\rho f_2(x).$$

For $I_\rho f_1(y)$, $y \in B(x, r)$, following Hedberg’s trick (see for instance [27], p. 354), we obtain

$$|I_\rho f_1(y)| \lesssim Mf(x)\rho(r),$$

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(see [9], for more detail). Thus by taking $L_q(w, \mathbb{R}^n)$—norm we get
\[
\|I_{\rho}f\|_{L_q(B(x,r))}(w, \mathbb{R}^n) \leq w(B(x,r))^{1 \over q} \left( \int_{B(y,2r)} \left( \rho(\|y-z\|) \frac{1}{|y-z|^n}|f(z)| \right)^q dz \right)^{1/q}.
\]

For $I_{\rho}f(y)$, $y \in B(x,r)$ from (2.1) we have
\[
|I_{\rho}f(y)| \lesssim \rho(\|y-z\|) |y-z|^n |f(z)| dz
\]
\[
\lesssim \int_{2r}^1 \|f\|_{L_p(w, \mathbb{R}^n)} w(B(x,r))^{-1 \over q} \rho(t) dt \frac{dt}{t^n}.
\]

Then from condition (4.2) and inequality (4.3) for all $y \in B(x,r)$ we get
\[
|I_{\rho}f(y)| \lesssim \rho(r) Mf(x) + \int_{2r}^1 \|f\|_{L_p(w, \mathbb{R}^n)} w(B(x,r))^{-1 \over q} \rho(t) dt \frac{dt}{t^n}
\]
\[
\lesssim \rho(r) Mf(x) + \rho(r)^{-1 \over q} \|f\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)}.
\]

Hence choosing $\rho(r) = \left( \frac{\|f\|_{M_{p,\phi}^1(w, \mathbb{R}^n)}}{Mf(x)} \right)^{-\frac{2-p}{p}}$ for all $y \in B(x,r)$, we have
\[
|I_{\rho}f(y)| \lesssim (Mf(x))^{\frac{p}{q}} \|f\|^{1-p \over q}_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)}.
\]

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator $M$ in $M_{p,\phi}^{1 \over p}(\mathbb{R}^n)$ provided in [11] in virtue of condition (11), hence, for $1 < p < q < \infty$ we get
\[
\|I_{\rho}f\|_{M_{q,\phi}^{1 \over q}(w, \mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x, t)^{-1 \over q} w(B(x,t))^{-1 \over q} \|I_{\rho}f\|_{L_q(w, B(x,t))}
\]
\[
\lesssim \|f\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)} \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x, t)^{-1 \over q} w(B(x,t))^{-1 \over q} \|Mf\|_{L_p(w, B(x,t))}^{\frac{q}{p}}
\]
\[
= \|f\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)} \left( \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x, t)^{-1 \over q} w(B(x,t))^{-1 \over q} \|Mf\|_{L_p(w, B(x,t))} \right)^{\frac{q}{p}}
\]
\[
= \|f\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)} \|Mf\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)}^{\frac{q}{p}}
\]
\[
\lesssim \|f\|_{M_{p,\phi}^{1 \over p}(w, \mathbb{R}^n)},
\]

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and for $1 < q < \infty$

$$\|I_\rho f\|_{W^{q,q}_{\varphi}(w,\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x,t)^{-\frac{1}{q}}w(B(x,t))^{-\frac{1}{q}}\|I_\rho f\|_{W^{q,q}_{\varphi}(w,\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}_{1,\varphi}(w,\mathbb{R}^n)}^{1-\frac{1}{q}} \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x,t)^{-1}w(B(x,t))^{-1}\|Mf\|_{W^{1,1}_{\varphi}(w,\mathbb{R}^n)}^{\frac{1}{q}}$$

$$= \|f\|_{\mathcal{M}_{1,\varphi}(w,\mathbb{R}^n)}^{1-\frac{1}{q}} \left( \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x,t)^{-1}w(B(x,t))^{-1}\|Mf\|_{W^{1,1}_{\varphi}(w,\mathbb{R}^n)} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{\mathcal{M}_{1,\varphi}(w,\mathbb{R}^n)}.$$ 

Hence the proof is completed. \[\square\]

**Corollary 4.2.** In the case $w \equiv 1$ from Theorem 4.1 we get Theorem D, in which we give Adams type result for generalized Riesz potential operator $I_\rho$ on generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ which was proved in [9] (Theorem 22, p.7).

**Corollary 4.3.** In the case $\rho(t) = t^\alpha, w \equiv 1, x \equiv x_0$ from Theorem 4.1 we get Adams type result for Riesz potential operator $I_\alpha$ on generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ which was proved in [12] (Theorem 5.7, p.182).

**Corollary 4.4.** In the case $\rho(t) = t^\alpha, w \equiv 1$ and $\varphi(x_0,t) = t^{\frac{\lambda-n}{p}}$, $0 < \lambda < n$ from Theorem 4.1 we get Adams result for Riesz potential operator $I_\alpha$ on local Morrey spaces $M_{p,\lambda}^{(x_0)}(\mathbb{R}^n)$ which is variant of Theorem B proved in [22].

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**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this article.
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