\textbf{$\mathcal{N}$-fold Supersymmetry in Quantum Mechanics - Analyses of Particular Models -}

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\begin{abstract}
We investigate particular models which can be $\mathcal{N}$-fold supersymmetric at specific values of a parameter in the Hamiltonians. The models to be investigated are a periodic potential and a parity-symmetric sextic triple-well potential. Through the quantitative analyses on the non-perturbative contributions to the spectra by the use of the valley method, we show how the characteristic features of $\mathcal{N}$-fold supersymmetry which have been previously reported by the authors can be observed. We also clarify the difference between quasi-exactly solvable and quasi-perturbatively solvable case in view of the dynamical property, that is, dynamical $\mathcal{N}$-fold supersymmetry breaking.

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\end{abstract}
I. INTRODUCTION

In our previous paper [1], we have formulated in formal and abstract ways $\mathcal{N}$-fold supersymmetry in quantum mechanics [2, 3, 4, 5, 6] and investigated general properties of the models which possess this symmetry. $\mathcal{N}$-fold supersymmetry is characterized by the supercharges which are $\mathcal{N}$-th order polynomials of momentum and similar generalizations of supercharges were also investigated in different contexts [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. We have shown that $\mathcal{N}$-fold supersymmetric models have a lot of significant properties similar to the ordinary supersymmetric ones [24, 25, 26, 27] such as degenerate spectral structure between bosonic states and fermionic ones, non-renormalization theorems for the generalized Witten index and for a part of the spectra, and so on. Furthermore, we have introduced the notion of quasi-solvability to identify an important aspect of $\mathcal{N}$-fold supersymmetry and have proved the equivalence between $\mathcal{N}$-fold supersymmetry and quasi-solvability. Recently, we have further shown [1] that Type A subclass of $\mathcal{N}$-fold supersymmetry which was first introduced in Ref. [4] is equivalent to the quasi-solvable models constructed by $sl(2)$ generators [28]. Then, it has turned out that the equivalence between them for special cases which had been reported previously in Refs. [2, 17, 23] is generically holds.

Quasi-solvability means the existence of a finite dimensional invariant subspace under the action of the Hamiltonian. As a consequence, a part of the spectra can be solved by a finite dimensional algebraic equation. In the case where the subspace is physical, that is, $L^2$, these spectra give the true eigenvalues of the Hamiltonian. In this case, the system is often called quasi-exactly solvable [28, 29, 30]. On the other hand, if the subspace is not physical, solvable spectra only give perturbative eigenvalues at most and thus we have dubbed this case quasi-perturbatively solvable [1]. This distinction is quite important, especially in view of dynamical $\mathcal{N}$-fold supersymmetry breaking; $\mathcal{N}$-fold supersymmetry is broken dynamically if a system is quasi-perturbatively solvable while it is not broken if a system is quasi-exactly solvable.

In this article, we analyze particular models which can be $\mathcal{N}$-fold supersymmetric more quantitatively. Previously in Ref. [2], an asymmetric quartic double-well potential was investigated in detail by the valley method [31, 32, 33, 34, 35, 36, 37, 38, 39]. It was shown that the system possesses $\mathcal{N}$-fold supersymmetry at specific values of a parameter in the Hamiltonian where the leading Borel singularity of the perturbative corrections for the first $\mathcal{N}$-th energies disappear. This result is a consequence of a general property of $\mathcal{N}$-fold supersymmetry, namely, the non-renormalization theorem. It was also shown that the non-perturbative corrections for the first $\mathcal{N}$-th energies do not vanish even when the system becomes $\mathcal{N}$-fold supersymmetric. This result consistently reflects the fact that, in the case of an asymmetric quartic double-well potential, the solvable subspace is not physical, that is, the system is quasi-perturbatively solvable; non-vanishing non-perturbative effects break $\mathcal{N}$-fold supersymmetry dynamically.

These observations show that combining with the formal discussions in Ref. [1] quantitative analyses may give deeper and complement understanding on dynamical properties of the $\mathcal{N}$-fold supersymmetric models. As in the case of the ordinary supersymmetric models, non-perturbative analyses are quite important in the $\mathcal{N}$-fold supersymmetric case; dynamical $\mathcal{N}$-fold supersymmetry breaking can take place via purely non-perturbative effects, e.g., quantum tunneling. However, non-perturbative analyses are in general quite non-trivial even in the simple one-dimensional quantum mechanics. The valley method is one of the
most successful tools for this kind of purpose. So we fully employ it in this work.

The article organizes as follows. In the next section, we summarize the general results and properties of the Type A subclass of $\mathcal{N}$-fold supersymmetry \cite{1, 4, 5, 6}. In section II, we develop particular cases of the Type A models which are especially relevant for the analyses in this article. Sections III and IV are devoted to valley method analyses on a periodic and a sextic triple-well potential, respectively. We choose potentials to be investigated so that the systems can be Type A $\mathcal{N}$-fold supersymmetric at specific values of a parameter involved in the potentials. The way of the choice enables us to clarify the characteristic features of Type A $\mathcal{N}$-fold supersymmetry. The periodic potential is always quasi-exactly solvable when Type A $\mathcal{N}$-fold supersymmetric while the triple-well potential can be either quasi-exactly or quasi-perturbatively solvable. In both the cases we show that the disappearance of the leading Borel singularity occurs. However, the non-perturbative corrections vanish when and only when the systems are quasi-exactly solvable. Finally, we give summary in the last section.

II. GENERAL PROPERTIES OF TYPE A $\mathcal{N}$-FOLD SUPERSYMMETRY

First of all, we summarize the general results and properties of Type A $\mathcal{N}$-fold supersymmetry. For details, e.g., derivation of the results, see Refs. \cite{1, 5}. To define $\mathcal{N}$-fold supersymmetry, we introduce the following Hamiltonian $\mathbf{H}_N$ and the $\mathcal{N}$-fold supercharges,

$$\mathbf{H}_\mathcal{N} = H^\mathcal{N}(p,q)\psi\psi^\dagger + H^{\dagger \mathcal{N}}(p,q)\psi^\dagger\psi,$$

$$Q_\mathcal{N} = P_{\mathcal{N}}^\dagger(p,q)\psi, \quad Q^\dagger _\mathcal{N} = P_{\mathcal{N}}(p,q)\psi^\dagger. \quad (2.1)$$

Here $\psi$ and $\psi^\dagger$ are fermionic coordinates which satisfy,

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1, \quad (2.2)$$

and are usually represented as the following $2 \times 2$ matrix form:

$$\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (2.3)$$

The component of the $\mathcal{N}$-fold supercharges $P_{\mathcal{N}}$ is given by an $\mathcal{N}$-th order polynomial of $p = -i d/dq$ and thus expressed as,

$$P_{\mathcal{N}} = p^\mathcal{N} + w_{\mathcal{N}-1}(q)p^{\mathcal{N}-1} + \cdots + w_{1}(q)p + w_{0}(q), \quad (2.4)$$

without any loss of generality. Then, the system is said to be $\mathcal{N}$-fold supersymmetric if the Hamiltonian $\mathbf{H}_\mathcal{N}$ commutes with the $\mathcal{N}$-fold supercharges $Q_\mathcal{N}$ and $Q^\dagger _\mathcal{N}$:

$$[Q_\mathcal{N}, \mathbf{H}_\mathcal{N}] = [Q^\dagger _\mathcal{N}, \mathbf{H}_\mathcal{N}] = 0. \quad (2.5)$$

The Type A $\mathcal{N}$-fold supersymmetry is characterized by a particular class of the $\mathcal{N}$-fold
supercharges which can be expressed as the following form:\(^1\)

\[
P_N = \left(D + i \frac{N-1}{2} E(q)\right) \left(D + i \frac{N-3}{2} E(q)\right) \ldots \left(D - i \frac{N-1}{2} E(q)\right) \\
\equiv \prod_{k=-(N-1)/2}^{(N-1)/2} \left(D + i k E(q)\right), \quad D = p - i W(q). 
\] (2.7)

If we restrict \(H_N^\pm\) to be the following Schrödinger type,

\[
H_N^\pm = \frac{1}{2} p^2 + V_N^\pm(q), 
\] (2.8)

we can show \(^2\) that necessary and sufficient conditions of the Hamiltonian (2.1) with (2.8) to be Type A \(N\)-fold supersymmetric, that is, to satisfy the relation (2.6), are as the following:

\[
V_N^\pm(q) = \frac{1}{2} W(q)^2 + \frac{N^2 - 1}{24} \left(E(q)^2 - 2E'(q)\right) \pm \frac{N}{2} W'(q), 
\] (2.9a)

\[
\left(\frac{d}{dq} - E(q)\right) \frac{d}{dq} \left(\frac{d}{dq} + E(q)\right) W(q) = 0 \quad (N \geq 2), 
\] (2.9b)

\[
\left(\frac{d}{dq} - 2E(q)\right) \left(\frac{d}{dq} - E(q)\right) \frac{d}{dq} \left(\frac{d}{dq} + E(q)\right) E(q) = 0 \quad (N \geq 3). 
\] (2.9c)

**A. The Solvable Subspaces**

Owing to the relation Eq. (2.4), the \(N\)-dimensional vector spaces defined by

\[
\mathcal{V}_N^- = \ker P_N, \quad \mathcal{V}_N^+ = \ker P_N^\dagger 
\] (2.10)

are invariant under the action of \(H_N^-\) and \(H_N^+\), respectively. We can therefore define the matrices \(S^\pm\) as follows:

\[
H_N^\pm \phi^\pm_n = \sum_{m=1}^N S^\pm_{n,m} \phi^\pm_m, 
\] (2.11)

where \(\phi^\pm\) are bases of the \(\mathcal{V}_N^\pm\), respectively. It can be proved \(^3\) for general \(N\)-fold supersymmetry that the *mother Hamiltonian* \(\mathcal{H}_N\) defined by the anti-commutator of the supercharges can be expressed as,

\[
\mathcal{H}_N \equiv \frac{1}{2} \{Q_N^\dagger, Q_N\} = \frac{1}{2} \begin{pmatrix} \det \mathcal{M}_N^+(H_N^+) + p^+ P_N^\dagger & 0 \\ 0 & \det \mathcal{M}_N^-(H_N^-) + p^- P_N \end{pmatrix}, 
\] (2.12)

where,

\[
\mathcal{M}_N^\pm(\lambda) = 2(\lambda I - S^\pm), 
\] (2.13)

\(^1\) Note that \(W(q)\) in this paper is the same as \(\tilde{W}(q)\) and *not* as \(W(q)\) in the old notation of our previous paper \([1, 2, 3, 4]\). Since it is a bit troublesome to keep the unnecessary tilde, we omit it.
and \( p^\pm \) are at most \((\mathcal{N} - 1)\)th order differential operators. From the definition of the mother Hamiltonian (2.12), the elements of the subspaces \( \mathcal{V}_N^\pm \) are also characterized as the zero-modes of the mother Hamiltonian.

In the case of Type A, we can obtain analytic expressions for these bases:

\[
\phi_n^\pm(q) = h(q)^{n-1} h'(q)^{-(N-1)/2} U(q)^{\pm 1}, \quad (n = 1, \ldots, \mathcal{N}),
\]

where,

\[
U(q) = e^{\int dq W(q)},
\]

and \( h(q) \) is a solution of the following linear differential equation:

\[
h''(q) - E(q) h'(q) = 0,
\]

and thus generically given by,

\[
h(q) = c_1 \int dq e^{\int dq E(q)} + c_2.
\]

The appearance of the two arbitrary constants \( c_{1,2} \) in Eq. (2.17) reflects the fact that the spaces \( \mathcal{V}_N^\pm \) spanned by the bases Eq. (2.14) are invariant under any linear transformations on \( h(q) \). With the aid of these bases Eq. (2.14), the components of the matrices \( S_{n,m}^\pm \) defined by Eq. (2.11) can be determined (for each fixed \( n = 1, \ldots, \mathcal{N} \)) by the following recurrence relations:

\[
S_{n,N-m}^- = \frac{P_{N-m-1}(H_N^- \phi_n^- - \sum_{k=N-m+1}^N S_{n,k}^- \phi_k^-)}{P_{N-m-1}^- \phi_{N-m}^-},
\]

\[
S_{n,N-m}^+ = \frac{P_{N-m-1}^+(H_N^+ \phi_n^+ - \sum_{k=N-m+1}^N S_{n,k}^+ \phi_k^+)}{P_{N-m-1}^+ \phi_{N-n}^+},
\]

for \( m = 1, \ldots, \mathcal{N} - 1 \) with the initial conditions,

\[
S_{n,N}^- = \frac{P_{N-1}H_N^- \phi_n^-}{P_{N-1}^- \phi_N^-}, \quad S_{n,N}^+ = \frac{P_{N-1}^+H_N^+ \phi_n^+}{P_{N-1}^+ \phi_N^+}.
\]

From Eq. (2.11), the spectra \( E_n^\pm \) of the Hamiltonians \( H_N^\pm \) in the subspaces \( \mathcal{V}_N^\pm \) are given by,

\[
\det M_{N}^\pm(E_n^\pm) = 0.
\]

If \( \phi_n(q) \)'s are normalizable, linear combinations of them which diagonalize the matrix \( S \) are the true eigenstates of \( H_N \). In this case, the system is often called quasi-exactly solvable [28, 29, 30]. On the other hand, if \( \phi_n(q) \)'s are not normalizable, they have, at most, restricted meanings in the perturbation theory. In this case, the spectra determined by Eq. (2.20) only give perturbatively correct ones. For this reason, we dub the case quasi-perturbatively solvable [1]. Then, \( \mathcal{N} \)-fold supersymmetry of the total system \( H_N \) is dynamically broken when both of the systems \( H_N^\pm \) are quasi-perturbatively solvable. Otherwise, that is, at least one of the systems \( H_N^\pm \) is quasi-exactly solvable, the elements of the corresponding solvable subspace give the \( \mathcal{N} \)-fold supersymmetric physical states and therefore \( \mathcal{N} \)-fold supersymmetry is preserved.

\[2\] The definition of \( U(q) \) is also different from the one in our previous papers.
B. A Non-renormalization Theorem

A kind of the non-renormalization theorem holds for the Type A models. We first assume that we can set \( W(0) = 0 \) by the redefinition of the origin of the coordinate \( q \) and the energy. To define a perturbation theory, we then introduce a coupling constant \( g \) as,

\[
W(q) = \frac{1}{g} w(gq), \quad E(q) = g e(gq),
\]

so that, in the leading order of \( g \), the potential \( V_\pm^N \) become harmonic with frequency \( |w'(0)| \),

\[
V_\pm^N(q) = \frac{1}{2} w'(0) q^2 + O(g).
\]

From Eq. (2.14) with Eqs. (2.15), (2.17) and (2.21), we can easily see that the \( \phi_\pm^n \) behave as,

\[
\phi_\pm^n(q) = U(0) \pm 1 (q^{n-1} + O(g)) e^{\pm w'(0) q^2/2}.
\]

Here, we choose the two arbitrary constants \( c_{1,2} \) in Eq. (2.17) as \( h(0) = 0, h'(0) = 1 \). Thus, as far as \( w'(0) > 0 (< 0) \), all the \( \phi_\pm^n(\phi_\pm^+_n) \) remain normalizable in any finite order of \( g \) even if \( \phi_\pm^n(\phi_\pm^+_n) \) themselves are not normalizable. So, they stay the \( N \)-fold supersymmetric vacua in any order of the perturbation theory and therefore any perturbative corrections do not break \( N \)-fold supersymmetry.

III. SPECIAL CASES OF TYPE A \( N \)-FOLD SUPERSYMMETRY

In this section, we illustrate some special cases of the Type A \( N \)-fold supersymmetry by using the general results obtained in the previous section.

A. Exponential Type Potentials

At first, we will consider the case where \( E(q) = \lambda \) (non-zero constant). This is a trivial solution of Eq. (2.9c). From Eq. (2.9b) we yield,

\[
W(q) = C_1 e^{\lambda q} + C_2 e^{-\lambda q} + C_3.
\]

In this case, the Hamiltonians and the supercharge are given by

\[
H_{\pm N} = \frac{1}{2} p^2 + \frac{1}{2} W(q)^2 + \frac{N^2 - 1}{24} \lambda^2 \pm \frac{N}{2} W'(q), \quad P_N = \prod_{k=-N/\sqrt{2}}^{(N-1)/2} (D + ik\lambda).
\]

The function \( h(q) \) can be chosen as,

\[
h(q) = \frac{e^{\lambda q}}{\lambda}.
\]
Bases of the solvable subspaces $\mathcal{V}_N^\pm$ are calculated as,

$$
\phi_n^\pm(q) = \frac{1}{\lambda^n} \exp \left[ -\frac{1}{2}(N - 2n + 1)\lambda q \pm C_3 q \pm \frac{C_1}{\lambda} e^{\lambda q} \mp \frac{C_2}{\lambda} e^{-\lambda q} \right].
$$

(3.4)

Thus, normalizability of $\phi_n^\pm$ depends on the values of the constants $C_i$ and $\lambda$. For example, provided that all the constants $C_1, C_2$ and $\lambda$ are non-zero real numbers, either $\phi^+$ or $\phi^-$ is normalizable when $C_1 C_2 < 0$ while both of $\phi^\pm$ are not when $C_1 C_2 > 0$. The correspondence between quasi-exact solvability and $N$-fold supersymmetry in the case of the exponential type potentials Eq. (3.1) was recently discussed in Ref. [17].

The non-zero matrix elements of $S^\pm$ can be calculated as follows. The direct action of the Hamiltonians Eq. (2.8), with the Type A potentials Eq. (2.9a), on the bases Eq. (2.14) reads,

$$
H_N^\pm \phi_n^\pm = -\frac{1}{2}(n - 1)(n - 2)\hbar^2 \phi_{n-2}^\pm + \frac{1}{2}(n - 1) \left[ (N - 2)h'' + 2Wh' \right] \phi_{n-1}^\pm
$$

$$
\quad - \frac{N - 1}{12} \left[ (N - 2) \left( E' + E^2 \right) \mp 6 \left( W' + EW \right) \right] \phi_n^\pm.
$$

(3.5)

From Eqs. (3.1) and (3.3), the following relations hold:

$$
h'' = \lambda^2 h^2, 
$$

(3.6a)

$$
h'' = \lambda^2 h, 
$$

(3.6b)

$$
E' + E^2 = \lambda^2, 
$$

(3.6c)

$$
Wh' = C_1 \lambda^2 h^2 + C_3 \lambda h + C_2, 
$$

(3.6d)

$$
W' + EW = 2C_1 \lambda^2 h + C_3 \lambda.
$$

(3.6e)

Substituting the above relations (3.6) for Eq. (3.5), we obtain,

$$
S_{n,n-1}^\pm = \mp (n - 1)C_2, 
$$

(3.7a)

$$
S_{n,n}^\pm = -\frac{1}{12} \left[ (N - 1)(N - 2) + 6(n - 1)(n - N) \right] \lambda^2 \pm \frac{1}{2}(N - 2n + 1)C_3 \lambda, 
$$

(3.7b)

$$
S_{n,n+1}^\pm = \mp (n - N)C_1 \lambda^2. 
$$

(3.7c)

All the other matrix elements are zero.

The special choices $\lambda = ig, C_1 = 1/2ig, C_2 = -1/2ig$ and $C_3 = 0$ lead to,

$$
W(q) = \frac{1}{g} \sin(gq), \quad E(q) = ig,
$$

(3.8)

and correspond to the periodic potential in Ref. [3]. We note that Eq. (3.8) is incorporated with the perturbation theory defined by Eq. (2.21). We will later carry out non-perturbative analysis of this special case in section IV.

### B. Sextic Oscillator Potentials

Next, we will consider the case where,

$$
E(q) = \frac{1}{q - q_0}.
$$

(3.9)
This is also a solution of Eq. (2.9c). This special case corresponds to (one of) the cubic type in Ref. [4]. The Hamiltonians and the supercharge are given by,

\[
H_N = \frac{1}{2} p^2 + \frac{1}{2} W(q)^2 + \frac{N^2 - 1}{8(q-q_0)^2} \pm \frac{N}{2} W'(q),
\]

\[
P_N = \prod_{k=-(N-1)/2}^{(N-1)/2} \left( D + i \frac{k}{q-q_0} \right),
\]

with,

\[
W(q) = C_1(q-q_0)^3 + C_2(q-q_0) + \frac{C_3}{q-q_0}.
\]

We note that the Hamiltonians (3.10) are parity symmetric. The function \( h(q) \) can be chosen as,

\[
h(q) = \frac{(q-q_0)^2}{2}.\]

Bases of the solvable subspaces \( \mathcal{V}_N^\pm \) are calculated as,

\[
\phi_n^\pm(q) = \frac{1}{2^{n-1}} (q-q_0)^{2n-N-\frac{3}{2} \pm C_3} \exp \left[ \pm \frac{C_1}{4} (q-q_0)^4 \pm \frac{C_2}{2} (q-q_0)^2 \right].
\]

Thus, either \( \phi^+ \) or \( \phi^- \) is normalizable unless \( C_1 = C_2 = 0 \) and the corresponding system \( H_N^+ \) or \( H_N^- \) is quasi-exactly solvable. The correspondence between quasi-exact solvability and \( N \)-fold supersymmetry in the case of the sextic potential Eq. (3.10) was recently pointed out in Ref. [23].

The non-zero matrix elements of \( S^\pm \) can be obtained by Eq. (3.13). From Eqs. (3.9), (3.12) and (3.13), the following relations hold:

\[
h'' = 1,
\]

\[
E' + E^2 = 0,
\]

\[
Wh' = 4C_1h^2 + 2C_2h + C_3,
\]

\[
W' + EW = 8C_1h + 2C_2.
\]

Substituting the above relations (3.13) for Eq. (3.3) we obtain,

\[
S^\pm_{n,n-1} = \frac{1}{2} (n-1) \left[ (N-2n+2) \mp 2C_3 \right],
\]

\[
S^\pm_{n,n} = \pm (N-2n+1)C_2,
\]

\[
S^\pm_{n,n+1} = \mp 4(n-N)C_1.
\]

All the other matrix elements are zero.

If we rewrite Eq. (3.12) as,

\[
W(q) = w(q) + \frac{C_3}{q-q_0}, \quad w(q) = C_1(q-q_0)^3 + C_2(q-q_0),
\]

8
the potential parts $V^\pm_N(q)$ of the Hamiltonians (3.10) are, in terms of $w(q)$,

$$V^\pm_N(q) = \frac{1}{2} w(q)^2 + \frac{(2C_3 \pm N - 1)(2C_3 \pm N + 1)}{8(q-q_0)^2} \pm \left( \frac{N}{2} \pm \frac{C_3}{3} \right) w'(q) + \frac{2}{3} C_2 C_3. \quad (3.18)$$

We note that in the cases when $C_3 = (N \pm 1)/2$ and $-(N \pm 1)/2$, one of the potential-pair $V^\pm_N(q)$ becomes a genuine sixth order polynomial:

$$V^+_N(q) = \frac{1}{2} w(q)^2 + \frac{4N \pm 1}{6} w'(q) \left( C_3 = \frac{N \pm 1}{2} \right), \quad (3.19a)$$

$$V^-_N(q) = \frac{1}{2} w(q)^2 - \frac{4N \pm 1}{6} w'(q) \left( C_3 = -\frac{N \pm 1}{2} \right), \quad (3.19b)$$

where irrelevant constant terms are omitted. Conversely, a sextic anharmonic oscillator or a triple-well potential (with parity symmetry) can be one of the Type A $N$-fold supersymmetric pair whenever the potential can be put in one of the form of Eq. (3.19). When $C_3 = (N \pm 1)/2$, the bases Eq. (3.14) for $V^+_N(q)$ read,

$$\phi^+_n(q) = \frac{1}{2^{n-1}} (q-q_0)^{2n-\frac{3}{2}} \exp \left[ \frac{C_1}{4} (q-q_0)^4 + \frac{C_2}{2} (q-q_0)^2 \right] \left( C_3 = \frac{N \pm 1}{2} \right). \quad (3.20)$$

It is worth noting that the solvable subspace $V^+_N$ consists of the states with definite parity (odd for $C_3 = (N + 1)/2$ and even for $C_3 = (N - 1)/2$). We will later see close relation between this fact and pattern of the non-perturbative spectral shifts. When $C_3 = -(N \pm 1)/2$, the bases Eq. (3.14) for $V^-_N(q)$ are similarly,

$$\phi^-_n(q) = \frac{1}{2^{n-1}} (q-q_0)^{2n-\frac{3}{2}} \exp \left[ -\frac{C_1}{4} (q-q_0)^4 - \frac{C_2}{2} (q-q_0)^2 \right] \left( C_3 = -\frac{N \pm 1}{2} \right). \quad (3.21)$$

Again, the subspace $V^-_N$ contains only odd-parity states for $C_3 = -(N + 1)/2$ and only even-parity states for $C_3 = -(N - 1)/2$.

### C. Quartic Oscillator Potentials

In the next, we will consider the case where $E(q) = 0$. This is also a trivial solution of Eq. (2.9c). From Eq. (2.9b) we yield,

$$W(q) = C_1 q^2 + C_2 q + C_3. \quad (3.22)$$

In this case, the Hamiltonians and the supercharge are given by

$$H_{\pm N} = \frac{1}{2} p^2 + \frac{1}{2} W(q)^2 \pm \frac{N}{2} W'(q), \quad P_N = D^N. \quad (3.23)$$

The function $h(q)$ reads,

$$h(q) = q. \quad (3.24)$$

Bases of the solvable subspaces $V^\pm_N$ are calculated as,

$$\phi^\pm_n(q) = q^{n-1} \exp \left[ \pm \frac{C_1}{3} q^3 \pm \frac{C_2}{2} q^2 \pm C_3 q \right]. \quad (3.25)$$
Thus, both of $\phi^\pm$ are not normalizable and therefore the system is quasi-perturbatively solvable as far as $C_1$ is a non-zero real number. The relation between quasi-perturbative solvability and $N$-fold supersymmetry in a special case of the models was pointed out in Ref. [2].

The non-zero matrix elements of $S^\pm$ can be obtained by Eq. (3.5). From Eqs. (3.22) and (3.24), the following relations hold:

\begin{align}
  h'^2 &= 1, \\
  Wh' &= C_1 h^2 + C_2 h + C_3, \\
  W' + EW &= 2C_1 h + C_2. \\
\end{align}

Substituting the above relations (3.26) for Eq. (3.5) we obtain,

\begin{align}
  S^\pm_{n,n-2} &= -\frac{1}{2}(n-1)(n-2), \\
  S^\pm_{n,n-1} &= \mp(n-1)C_3, \\
  S^\pm_{n,n} &= \pm\frac{1}{2}(N-2n+1)C_2, \\
  S^\pm_{n,n+1} &= \mp(n-N)C_1.
\end{align}

All the other matrix elements are zero.

IV. ANALYSIS OF A PERIODIC POTENTIAL

A. The Valley Method

Before proceeding to show the results of the analyses, we briefly review the valley method [31, 32, 33, 34, 35, 36, 37, 38, 39] which is employed in this research. For more details about the method, see Ref. [2].

The main problem in quantum theories concerns with the evaluation of the Euclidean partition function:

\[ Z = \mathcal{J} \int Dq e^{-S[q]}. \]

Since the evaluation cannot be done exactly in general, one must find out a proper method which enables one to get a good estimation of the quantity. The semi-classical approximation is known to be one of the most established methods. Especially, the uses of instantons have been succeeded in analyzing non-perturbative aspects of various quantum systems which have degenerate vacua [40]. However, validity of the approximation comes into question when the fluctuations around the classical configuration contain a negative mode. Let us consider an asymmetric double-well potential as a typical example. For this potential, there is a so-called bounce solution as the classical solution which has a negative mode in the fluctuations. The negative mode contributes non-zero imaginary part of the spectra in the approximation, showing instability of the system. Since the spectra of the model must be real, the instability in the approximation must be fake.

The appearance of a negative mode indicates that the classical action does not give the minimum but rather a saddle point in the functional space. In this case, one may expect
that the quantity (4.1) is dominated by the configurations along the negative mode, which may intuitively constitute a valley in the functional space. The valley method is a natural realization of this consideration.

At first, we give a geometrical definition of the valley in the functional space \( q(\tau) \) [34]:

\[
\delta \frac{\delta q(\tau)}{\delta q(\tau)} \left[ \frac{1}{2} \int d\tau' \left( \frac{\delta S[q]}{\delta q(\tau')} \right)^2 - \lambda S[q] \right] = 0.
\]

(4.2)

The above definition (4.2) can be interpreted as follows; for each fixed “height” \( S[q] \), the valley is defined at the point where the norm of the gradient vector becomes extremal. Introducing an auxiliary field \( F(q) \), we can make the valley equation (4.2) a more perspicuous form:

\[
\frac{\delta S[q]}{\delta q(\tau)} = F(\tau),
\]

(4.3a)

\[
\int d\tau' D(\tau, \tau')F(\tau') = \lambda F(\tau).
\]

(4.3b)

where the operator \( D \) is defined as,

\[
D(\tau, \tau') = \frac{\delta^2 S[q]}{\delta q(\tau)\delta q(\tau')}. \tag{4.4}
\]

It is now evident that any solution of the equation of motion is also a solution of the valley equation (4.3) with \( F(\tau) \equiv 0 \).

Next, we separate the integration along the valley line from the whole functional integration. We parametrize the valley line by a parameter \( R \) and denote the valley configuration by \( q_R(\tau) \). We then define Faddeev-Popov determinant \( \Delta[\phi_R] \) by the following:

\[
\int dR \delta \left( \int d\tau \phi_R(\tau)G_R(\tau) \right) \Delta[\phi_R] = 1, \tag{4.5}
\]

where \( \phi_R(\tau) = q(\tau) - q_R(\tau) \) is the fluctuation over which we will be doing Gaussian integrations, and \( G_R(\tau) \) is the normalized gradient vector,

\[
G_R(\tau) = \frac{\delta S[q_R]}{\delta q_R(\tau)}/\sqrt{\int d\tau' \left( \frac{\delta S[q_R]}{\delta q_R(\tau')} \right)^2}. \tag{4.6}
\]

Inserting Eq. (4.5) into the functional integral (4.1), expanding the action \( S[q] \) around \( \phi_R(\tau) = 0 \) and integrating up to the second order term in \( \phi_R(\tau) \), we finally obtain the one-loop order result:

\[
Z = \mathcal{J} \int dR \int Dq \delta \left( \int d\tau \phi_R(\tau)G_R(\tau) \right) \Delta[\phi_R]e^{-S[q]}
\]

\[
\simeq \mathcal{J} \int \frac{dR}{\sqrt{2\pi \det' D_R}} \Delta[\phi_R]e^{-S[q_R]}, \tag{4.7}
\]

where the Jacobian \( \Delta[\phi_R] \) is given by, in this approximation,

\[
\Delta[\phi_R] = \frac{dS[q_R]}{dR}/\sqrt{\int d\tau' \left( \frac{\delta S[q_R]}{\delta q_R(\tau')} \right)^2}. \tag{4.8}
\]
In the above, \( \text{det}' \) denotes the determinant in the functional subspace which is perpendicular to the gradient vector \( G_R(\tau) \). The valley equation (4.3) ensure that the subspace does not contain the eigenvector of the eigenvalue \( \lambda \). Therefore, we can safely perform the Gaussian integrations even when we encounter a non-positive mode. The extension to the multi-dimensional valley, which will be needed when there are multiple non-positive eigenvalues, is straightforward.

In this article, we only deal with one-dimensional quantum mechanics where the Euclidean action is given by,

\[
S[q] = \int d\tau \left[ \frac{1}{2} \left( \frac{dq}{d\tau} \right)^2 + V(q) \right].
\] (4.9)

In this case, the valley equations (4.3) are explicitly written as,

\[
-\frac{d^2q(\tau)}{d\tau^2} + V'(q) = F(\tau),
\] (4.10a)

\[
\left[ -\frac{d^2}{d\tau^2} + V''(q) \right] F(\tau) = \lambda F(\tau).
\] (4.10b)

B. Valley-Instantons

At first, we will analyze a periodic potential. The form of the potential to be analyzed is the following:

\[
V(q; \epsilon) = \frac{1}{2g^2} \sin^2(gq) + \frac{\epsilon}{2} \cos(gq).
\] (4.11)

This is a periodic potential with periodicity \( 2\pi/g \) (unless \( \epsilon = 0 \)) and has two local minima at \( q = 2k\pi/g \) and \( q = (2k+1)/g \) \( (k = 0, \pm 1, \pm 2, \ldots) \) in one period, see Fig. 1.
Comparing this potential with Eqs. (3.2) and (3.8), we find that the system has Type A $\mathcal{N}$-fold supersymmetry when

$$\epsilon = \pm N.$$  \hspace{1cm} (4.12)

Since the system is defined on a bounded region, all the bases of the solvable space $\mathcal{V}_N^\pm$ are normalizable. Thus, certain linear combinations of them serve as physical eigenstates of the Hamiltonian and $\mathcal{N}$-fold supersymmetry is not broken dynamically. Therefore, we may expect that the non-perturbative corrections for certain $\mathcal{N}$ physical states will vanish and the perturbation series for the corresponding spectra will be convergent when $\epsilon = \pm N$.

We note that the potential (4.11) has the following symmetry:

$$V(q - \frac{\pi}{g}; \epsilon) = V(q; -\epsilon).$$ \hspace{1cm} (4.13)

Therefore, we can restrict $\epsilon$ to be positive without any loss of generality.

In the case of $\epsilon = 0$, there are (anti-)instanton solutions of the equation of motion which describe the quantum tunneling between the neighboring vacua. The instanton and anti-instanton which connect the two vacua at $q = k\pi/g$ and $q = (k + 1)\pi/g$ are given by,

$$q_0^{(I)}(\tau - \tau_0) = \frac{k\pi}{g} + \frac{1}{g} \arccos \left( -\tanh(\tau - \tau_0) \right),$$ \hspace{1cm} (4.14a)

$$q_0^{(I)}(\tau - \tau_0) = \frac{k\pi}{g} + \frac{1}{g} \arccos \left( \tanh(\tau - \tau_0) \right).$$ \hspace{1cm} (4.14b)

When $\epsilon \neq 0$, the classical solutions drastically change into the so-called bounce solutions which cause fake instability. On the other hand, the solutions of the valley equation (4.10) contain a continuously deformed (anti-)instanton which connects the two non-degenerate local minima and is called (anti-)valley-instanton [2].

The solutions of the valley equation (4.10) also contain a family of the configurations, which tends to the trivial vacuum configuration in the one limit and tends to well-separated valley-instanton and anti-valley-instanton configuration in the other limit. The latter configuration is called $I\bar{I}$-valley. The bounce solution is also realized as an intermediate configuration of this family, which is consistent with the fact that the solution of the equation of motion is also a solution of the valley equations. For details, see the numerical result in Ref. [2]. For the $I\bar{I}$-valley configuration, it turns out that $|\lambda| \ll 1$ and thus the asymptotic form of the configuration can be obtained by solving the valley equation (4.10) with perturbative expansion in $\lambda$:

$$q(\tau) = q_0(\tau) + \lambda q_1(\tau) + \cdots, \quad F(\tau) = \lambda F_1(\tau) + \lambda^2 F_2(\tau) + \cdots.$$ \hspace{1cm} (4.15)

Indeed, if we denote the distance between the valley-instanton and the anti-valley-instanton as $R$, the lambda is order $\lambda \sim O(e^{-R})$ quantity. The action of the $I\bar{I}$-valley with the boundary condition $q(\pm T/2) = 2k\pi/g \ (T \gg 1)$ is finally obtained as,

$$S^{(I\bar{I})}(R) = S^{(I\bar{I})}(\bar{R}) = 2S_0^{(I)} - \frac{\epsilon}{2} \bar{R} + \frac{\epsilon}{2} (T - \bar{R}) - \frac{8}{g^2} e^{-\bar{R}} + O(e^{-2\bar{R}}),$$ \hspace{1cm} (4.16)

while the one with $q(\pm T/2) = (2k + 1)\pi/g \ (T \gg 1)$ is,

$$S^{(I\bar{I})}(R) = S^{(I\bar{I})}(R) = 2S_0^{(I)} + \frac{\epsilon}{2} R - \frac{\epsilon}{2} (T - R) - \frac{8}{g^2} e^{-R} + O(e^{-2R}),$$ \hspace{1cm} (4.17)
where \( S_0^{(I)} \) denotes the Euclidean action of one (anti-)instanton Eq. (4.14) and amounts to,

\[
S_0^{(I)} = \frac{2}{g^2}.
\]  

(4.18)

In Eqs. (4.16) and (4.17), the fourth term can be interpreted as the interaction term between the valley-instanton and the anti-valley-instanton. Therefore, the minus sign indicates that the interaction is attractive.

The other type of the solutions emerges in this case, which is asymptotically composed of two successive valley-instantons or two successive anti-valley-instantons. We call them \( II \)-valley and \( \bar{I}I \)-valley, respectively. These configurations do not appear in the case of double-well potentials since they connect every other vacuum. The Euclidean action of them with large separation \( R \) can be also calculated in the same way as,

\[
S^{(II)}(\bar{R}) = S^{(II)}(R) = 2S_0^{(I)} - \frac{\epsilon}{2}R + \frac{\epsilon}{2}(T - \bar{R}) + \frac{8}{g^2}e^{-R} + O(e^{-2R}),
\]  

(4.19)

for the configuration with \( q(-T/2) = 2k\pi/g \) and \( q(T/2) = (2k \pm 2)\pi/g \) \( (T \gg 1) \), and,

\[
S^{(II)}(R) = S^{(II)}(R) = 2S_0^{(I)} + \frac{\epsilon}{2}R - \frac{\epsilon}{2}(T - R) + \frac{8}{g^2}e^{-R} + O(e^{-2R}),
\]  

(4.20)

for the one with \( q(-T/2) = (2k+1)\pi/g \) and \( q(T/2) = (2k+1\pm 2)\pi/g \) \( (T \gg 1) \). Note that the sign of the fourth term is plus and thus the interaction between the (anti-)valley-instantons in this case is repulsive.

C. Analysis of Two-valley Sector

From the results on the \( I\bar{I} \)-valley, the contribution of the \( I\bar{I} \)-valley to the partition function can be written as the following form:

\[
Z^{(II)} = \frac{4e^{-T/2}}{\pi g^2} \int_0^T dR (T - R)e^{-S^{(II)}(R)} = \frac{4e^{-T/2}}{\pi g^2} \int_{CV} dt \mathcal{F}(t)e^{-t/g^2},
\]  

(4.21)

where we have changed the integration variable \( R \) to \( t = g^2S^{(II)}(R) \) in the second line. The integration contour \( CV \) is \([0, 2g^2S_0^{(I)}]\) and the integrand \( \mathcal{F}(t) \) has a singularity at \( t = 2g^2S_0^{(I)} \).

The integral Eq. (4.21) contains both the perturbative contribution at \( t \sim 0 \) and the non-perturbative one at \( t \sim 2g^2S_0^{(I)} \). To separate the perturbative and the non-perturbative contribution, we deform the contour \( CV \) to the sum of \( CP \) and \( CNP \):

\[
Z^{(II)} = \frac{4e^{-T/2}}{\pi g^2} \int_{CP} dt \mathcal{F}(t)e^{-t/g^2} + \frac{4e^{-T/2}}{\pi g^2} \int_{CNP} dt \mathcal{F}(t)e^{-t/g^2}
\]  

\[
= Z_P^{(II)}(g^2) + Z_{NP}^{(II)}(g^2),
\]  

(4.22)

as is shown in Fig. 3.
We identify the first term as the formal Borel summation of the perturbation series and the second term as the non-perturbative contribution. For the non-perturbative contribution, the following analytic property holds. If we perform the analytic continuation of $Z_{NP}(|g^2|e^{i\theta})$ from $\theta = 0$ to $\theta = \pi$, the contour for $Z_{NP}$ changes from $C_{NP}(0)$ to $C_{NP}(\pi)$, as shown in Fig. 3. In the weak coupling limit, the integral of $C_{NP}(\pi)$ can be well-approximated by that of $C_V$ because the dominant contribution of the integral comes from $t \sim 2g^2 S_0^{(I)}$. Therefore, in the case of $g^2 = |g^2|e^{i\pi}$ when the interaction between valley-instantons is repulsive, the following relation holds approximately:

$$Z_{NP}(|g^2|e^{i\pi}) \simeq Z(|g^2|e^{i\pi}).$$

(4.23)

This relation coincides with what Bogomolny suggested heuristically as a method of evaluation of the instanton–anti-instanton contribution [41].

An immediate consequence of our decomposition of the perturbative and non-perturbative
contribution is,

\[ \text{Im}Z_P + \text{Im}Z_{NP} = 0, \quad (4.24) \]

since \( Z = Z_P + Z_{NP} \) is real. From the relation above, the dispersion relation becomes \[ 2 \]

\[ Z_P(g^2) = \frac{1}{2\pi i} \oint_{C_{g^2}} dz \frac{Z_P(z)}{z - g^2} \]

\[ \simeq -\frac{1}{\pi} \sum_{r=0}^{\infty} g^{2r} \int_0^\infty dz \frac{\text{Im}Z_{NP}(z)}{z^{r+1}}, \quad (4.25) \]

where \( C_{g^2} \) is the counter around \( z = g^2 \) and we have neglected the contribution from the singularities far from the origin. The last line of Eq. \( (4.25) \) gives the relation between the coefficients of the perturbation series and the imaginary part of the non-perturbative contribution. Rewriting the above relation in terms of the energy spectra, we find for the perturbative part of the spectra \( E_P(g^2) = \sum_{r=0}^{\infty} a^{(r)} g^{2r} \), that the coefficients \( a^{(r)} \) can be estimated as,

\[ a^{(r)} = -\frac{1}{\pi} \int_0^\infty d\bar{g}^2 \frac{\text{Im}E_{NP}(g^2)}{\bar{g}^{2r+2}}. \quad (4.26) \]

The situation in the case of \( \bar{I}I \)-valley is completely the same as that in the \( I\bar{I} \)-valley.

On the other hand, the situation in the case of \( II \)-valley is different, reflecting the fact that the \( II \)-valley configuration cannot be deformed continuously to the trivial vacuum configuration. The contribution of the \( II \)-valley to the partition function has quite similar form to that of the \( I\bar{I} \)-valley:

\[ Z_{II}(g^2) = 4e^{-T/2} \int_{C_V} dt \mathcal{F}(t)e^{-t/\bar{g}^2}. \quad (4.27) \]

However, the integration contour is now \( C_V = (2g^2S_0^{(I)}, g^2S^{(II)}(0]) \) and thus is disconnected to the perturbative region \( t \sim 0 \), see Fig. 4.

This means that Eq. \( (4.27) \) contains only the non-perturbative contribution,

\[ Z_{NP}(g^2) = Z(g^2). \quad (4.28) \]

Therefore, we need not separate the integration as in the case of the \( I\bar{I} \)-valley. As a consequence, the \( II \)-valley configuration does not contribute to the imaginary part and also to the large order behavior of the perturbation series, which will be confirmed in the examples in sections [IV] and [V]. Furthermore, since the interaction for the \( II \)-valley configuration is repulsive, as has been observed in Eqs. \( (4.19) \) and \( (4.20) \), the integral is dominated around \( t \sim 2g^2S_0^{(I)} \) and can be approximated by the integral on the contour \( [2g^2S_0^{(I)}, \infty) \) in the weak coupling limit. Therefore, analytic continuation is not needed. The situation in the case of the \( I\bar{I} \)-valley is completely the same as that in the \( II \)-valley.

D. Multi-valley Calculus

Utilizing the knowledge of the (anti-)valley-instantons and the interactions between them obtained previously and applying the manipulation discussed in subsection [IVC], we will
evaluate the partition function $Z = \text{tr} \ e^{-HT}$ by summing over those configurations made of several (anti-)valley-instantons which satisfy a boundary condition in $T$. The periodic boundary condition for a configuration $q(\tau)$ is in general given by

$$q(\tau + T) = q(\tau).$$

(4.29)

For a system which has a periodic potential like Eq. (4.11), however, the condition (4.29) can be relaxed and be replaced with,

$$q(\tau + T) = q(\tau) + \frac{2k\pi}{g} \ (k = 0, \pm 1, \pm 2, \ldots).$$

(4.30)

This condition restricts the number of the valley-instantons to be even, $2n$. The non-perturbative contributions from the multi-valley configurations satisfying Eq. (4.30) can be calculated by the extension of the technique developed in Ref. [42]. We divide the time interval $0 \leq \tau \leq T$ into $n$ regions and put a valley-instanton pair on each of the region. In order to distinguish what kind of pairs, we introduce the indices $\epsilon_i$ and $\bar{\epsilon}_i$ for the $i$-th region (mod $n$) as follows:

(i) $\epsilon_i = 1, \ \bar{\epsilon}_i = 1 \ \text{for} \ II$-valley,

(ii) $\epsilon_i = 1, \ \bar{\epsilon}_i = -1 \ \text{for} \ I\bar{I}$-valley,

(iii) $\epsilon_i = -1, \ \bar{\epsilon}_i = 1 \ \text{for} \ II$-valley,

(iv) $\epsilon_i = -1, \ \bar{\epsilon}_i = -1 \ \text{for} \ I\bar{I}$-valley.

In this way, the allowed configurations for a given valley-instanton number $2n$ are exhausted by the allowed combinations of the set $\{\epsilon_i, \bar{\epsilon}_i\}$ ($i = 1, \ldots, n$). Combining the results on well-separated valley-instanton pairs with the above conventions, the well-separated multiple...
valley-instanton action for given $n$ and $\{\epsilon_i, \tilde{\epsilon}_i\}$ is expressed as,

$$S_n = 2nS_0^{(l)} + \frac{8}{g^2} \sum_{i=1}^{n} \epsilon_i \tilde{\epsilon}_i e^{-R_i} + \frac{8}{g^2} \sum_{i=1}^{n} \tilde{\epsilon}_i \epsilon_{i+1} e^{-\tilde{R}_i} + \frac{\epsilon}{2} \sum_{i=1}^{n} (R_i - \tilde{R}_i) + i \frac{\eta}{2} \sum_{i=1}^{n} (\epsilon_i + \tilde{\epsilon}_i),$$

(4.31)

where $R_i$ is the distance between the $(2i-1)$-th and $2i$-th (anti-)valley-instanton and $\tilde{R}_i$ the one between the $2i$-th and the $(2i+1)$-th (anti-)valley-instanton (mod $n$) and $\epsilon_{n+1} = \epsilon_1$, see Fig. 5.

![Diagram](image)

**FIG. 5:** The collective coordinates $R_i$ and $\tilde{R}_i$ for a $2n$ valley-instantons configuration.

The sum of the contributions from the $2n$ valley-instantons configuration can be written as,

$$Z_{NP} = \sum_{n=1}^{\infty} \alpha^{2n} J_n,$$

(4.32)

where $\alpha^2$ denotes the contribution of the Jacobian and the $R$-independent part of the determinant for one valley-instanton-pair and is calculated as, in this case,

$$\alpha^2 = \frac{4}{\pi g^2} e^{-4/g^2}.$$

(4.33)
The term $J_n$ is given by,

$$J_n = \frac{T}{n} \int_0^\infty \left( \prod_{i=1}^n dR_i \right) \left( \prod_{i=1}^n d\tilde{R}_i \right) \delta \left( \sum_{i=1}^n (R_i + \tilde{R}_i) - T \right) \sum_{\epsilon_i, \tilde{\epsilon}_i = \pm} \exp \left( -\frac{1-\epsilon}{2} \sum_{i=1}^n R_i - \frac{1-\epsilon}{2} \sum_{i=1}^n \tilde{R}_i - \frac{8}{g^2} \sum_{i=1}^n \epsilon_i \tilde{\epsilon}_i e^{-R_i} - \frac{8}{g^2} \sum_{i=1}^n \tilde{\epsilon}_i \epsilon_{i+1} e^{-\tilde{R}_i} - i\tilde{\vartheta} \sum_{i=1}^n (\epsilon_i + \tilde{\epsilon}_i) \right). \tag{4.34}$$

To calculate the sum over the set $\{\epsilon_i, \tilde{\epsilon}_i\}$, we introduce the following transfer matrices:

$$T(R_i) = \begin{pmatrix} \exp\left(-\frac{8}{g^2} e^{-R_i} - i\vartheta\right) & \exp\left(\frac{8}{g^2} e^{-R_i}\right) \\ \exp\left(-\frac{8}{g^2} e^{-R_i} + i\vartheta\right) & \exp\left(\frac{8}{g^2} e^{-R_i}\right) \end{pmatrix}, \tag{4.35a}$$

$$\tilde{T}(\tilde{R}_i) = \begin{pmatrix} \exp\left(-\frac{8}{g^2} e^{-\tilde{R}_i}\right) & \exp\left(\frac{8}{g^2} e^{-\tilde{R}_i}\right) \\ \exp\left(-\frac{8}{g^2} e^{-\tilde{R}_i}\right) & \exp\left(\frac{8}{g^2} e^{-\tilde{R}_i}\right) \end{pmatrix}. \tag{4.35b}$$

Then, using these matrices we have,

$$J_n = \frac{T}{2\pi i n} \int_{-i\infty-\eta}^{i\infty-\eta} ds e^{-Ts} \text{tr} \left( \prod_{i=1}^n \int_0^\infty dR_i e^{(s-\frac{1}{2}+\frac{i}{2})R_i} T(R_i) \int_0^\infty d\tilde{R}_i e^{(s-\frac{1}{2}+\frac{i}{2})\tilde{R}_i} \tilde{T}(\tilde{R}_i) \right),$$

$$= \frac{T}{2\pi i n} \int_{-i\infty-\eta}^{i\infty-\eta} ds e^{-Ts} \text{tr} \left[ T(s, \epsilon, \vartheta)^n \right], \tag{4.36}$$

where,

$$T(s, \epsilon, \vartheta) = \begin{pmatrix} K(s-\epsilon/2)e^{-i\vartheta} & I(s-\epsilon/2) \\ I(s-\epsilon/2) & K(s-\epsilon/2)e^{i\vartheta} \end{pmatrix} \begin{pmatrix} K(s+\epsilon/2) & I(s+\epsilon/2) \\ I(s+\epsilon/2) & K(s+\epsilon/2) \end{pmatrix}. \tag{4.37}$$

In the above, $K$ and $I$ are defined by,

$$K(s) = \int_0^\infty dRe^{(s-\frac{1}{2})R - \frac{8}{g^2} e^{-R}} \simeq \left( \frac{8}{g^2} \right)^{s-\frac{1}{2}} \Gamma \left( -s + \frac{1}{2} \right), \tag{4.38a}$$

$$I(s) = \int_0^\infty dRe^{(s-\frac{1}{2})R + \frac{8}{g^2} e^{-R}} \simeq \left( -\frac{8}{g^2} \right)^{s-\frac{1}{2}} \Gamma \left( -s + \frac{1}{2} \right), \tag{4.38b}$$

where the manipulation explained in subsection IV C is utilized for estimating each of the integration. The calculation of the trace in Eq. (4.36) can be done by diagonalizing $T$. If we denote the two eigenvalues of $T$ as $t_{\pm}$, we immediately yield,

$$J_n = \frac{T}{2\pi i n} \int_{-i\infty-\eta}^{i\infty-\eta} ds e^{-Ts} \left( t_+(s)^n + t_-(s)^n \right). \tag{4.39}$$

From Eq. (4.37), $t_\pm$ is evaluated as,

$$t_\pm(s) = K(s+\epsilon/2)K(s-\epsilon/2) \cos \vartheta + I(s+\epsilon/2)I(s-\epsilon/2) \pm \sqrt{\left[ I(s+\epsilon/2)^2 - K(s+\epsilon/2)^2 \right] K(s-\epsilon/2)^2 \sin^2 \vartheta} + \left[ K(s+\epsilon/2)I(s-\epsilon/2) + I(s+\epsilon/2)K(s-\epsilon/2) \cos \vartheta \right] \cos \vartheta} \right)^{1/2}. \tag{4.40}$$
Finally, combining Eqs. (4.32) and (4.39) we obtain the non-perturbative contribution to the partition function:

$$Z_{\text{NP}} = -\frac{T}{2\pi i} \int_{i\infty - \eta}^{i\infty - \eta} ds e^{-Ts} \ln \left(1 - \alpha^2 t_+(s)\right) \left(1 - \alpha^2 t_-(s)\right),$$  
(4.41)

E. Non-perturbative Contributions

From the results in Eqs. (4.38), (4.40) and (4.41), the non-perturbative contributions to the spectra are determined by the following equation:

$$\alpha^2 \beta_\pm(E, \epsilon, \vartheta) \left(\frac{8}{g^2}\right)^{(E - \frac{1}{2})^2} \Gamma \left(-E + \frac{1}{2} - \frac{\epsilon}{2}\right) \Gamma \left(-E + \frac{1}{2} + \frac{\epsilon}{2}\right) = 1,$$  
(4.42)

where,

$$\beta_\pm(E, \epsilon, \vartheta) = \cos \vartheta + (-)^{(E - \frac{1}{2})^2} \pm \sqrt{(-)^{(E - \frac{1}{2})^2} + (-)^{(E - \frac{1}{2} + \frac{\epsilon}{2})^2} + (-)^{(E - \frac{1}{2})^2} \left[\cos \vartheta - \sin^2 \vartheta\right].}$$  
(4.43)

We will solve the above equation by the series expansion in $\alpha$:

$$E_n = E_n^{(0)} + \alpha E_n^{(1)} + \alpha^2 E_n^{(2)} + \cdots, \quad E_n^{(0)} = n + \frac{1}{2} \pm \frac{\epsilon}{2},$$  
(4.44)

where $E_{n_\pm}$ stands for the spectra corresponding to, in the limit $g \to 0$, the eigenfunctions of the shallower potential wells and $E_{n_\pm}$ for the ones corresponding to the eigenfunctions of the deeper potential wells. For $\epsilon \neq N$ ($N = 1, 2, 3, \ldots$), the 1st order contributions vanish and the leading 2nd order contributions are calculated as follows:

$$E_n^{(2)} = 2 \left[\cos \vartheta + (-)^{\pm \epsilon}\right] \left(-\right)^{\frac{n_\pm + 1}{n_\pm}} \left(\frac{8}{g^2}\right)^{2n_\pm \pm \epsilon} \Gamma(-n_\pm \mp \epsilon).$$  
(4.45)

For $\epsilon = N$ ($N = 1, 2, 3, \ldots$), all the harmonic spectra $E_n^{(0)}$ of the shallower wells and the higher harmonic spectra $E_n^{(0)}$ of the deeper wells degenerate for $n_- = n_+ + N$, see Fig. 3 for $\epsilon = N = 1, 2$. Between these degenerate states, resonant tunneling enhances the non-perturbative corrections, and results in order $\alpha^1$ contributions to the spectra $E_n$ and $E_{n_\pm}$ with $n_- = n_+ + N$:

$$E_n^{(1)} = \pm \sqrt{\frac{2 \left[\left(1 - (-)^N \cos \vartheta\right)\left(\frac{8}{g^2}\right)^{n_+ + n_-}}{n_+! n_-!}},}$$  
(4.46)

$$E_n^{(2)} = \frac{E_n^{(1)^2}}{2} \left[\frac{2 \ln(-)}{1 + (-)^N \cos \vartheta} + 2 \ln \left(\frac{8}{g^2}\right) - \psi(n_+ + 1) - \psi(n_- + 1)\right],$$  
(4.47)

where $\psi(z) = d \ln \Gamma(z)/dz$ is the digamma function. For the other spectra, say, the lower $E_n$ with $n_- < N$, the contributions are the same as Eq. (4.45). We see from Eq. (4.45) the non-perturbative corrections for these lower $E_n$ vanish, at least up to order $\alpha^2$, at $\vartheta = 0 (\pi)$ when $\epsilon = N$ is odd(even) and thus, from Eq. (4.12), the system has $N$-fold
supersymmetry. This means that when the system is $\mathcal{N}$-fold supersymmetric with odd(even) $\mathcal{N}$, among the physical states for each of the lower $\mathcal{N}$ spectral bands, the state which satisfies the periodic(anti-periodic) boundary condition does not receive non-perturbative correction. From Eq. (3.4) in this case, these physical states are surely the elements of the solvable subspace $\mathcal{V}^-$. Therefore, the results are consistent with the fact that $\mathcal{N}$-fold supersymmetry in this case is not broken dynamically.

\begin{center}
\textbf{FIG. 6:} Degeneracies of the harmonic spectra for (a) $\epsilon = 1$ and (b) $\epsilon = 2$.
\end{center}

Finally, we make a remark on the resultant equation (4.42). The origin of the disappearance of the non-perturbative corrections discussed above comes from the factor $\beta \pm$ in Eq. (4.42). Indeed, to make the l.h.s. of Eq. (4.42) finite when $\beta \pm = 0$, the gamma functions must diverge adequately. This happens only when $E = E_{n\pm}^{(0)}$ for certain values of $n\pm$, and therefore the non-perturbative corrections must vanish.\(^3\) From the derivation of Eq. (4.42), we can see that the appearance of the factor $\beta \pm$ is achieved by taking into account both of the repulsive and attractive interactions between the valley-instantons properly. Therefore, we guess that naive application of the dilute-gas approximation can hardly lead to the correct results even for the ground-state energy.

\section*{F. Large Order Behavior of the Perturbation Series}

The large order behavior of the perturbation series in $g^2$ for the spectra can be estimated by the same way as in the case of the double-well potential. From the non-perturbative contributions Eqs. (4.45)–(4.47), we can easily see that the imaginary parts of them are continuous, at least up to the order $\alpha^2$, in $\epsilon$ and yield,

$$\text{Im}E_{n\pm} \sim -\alpha^2 \frac{2\pi}{n\pm! \Gamma(n\pm + 1 \pm \epsilon)} \left( \frac{8 g^2}{\gamma^2} \right)^{2n\pm \pm \epsilon},$$

(4.48)

which are valid for arbitrary $\epsilon$. Then, if we expand the spectra in power of $g^2$ such that,

$$E_{n\pm} = E_{n\pm}^{(0)} + \sum_{r=1}^{\infty} a_{n\pm}^{(r)} g^{2r},$$

(4.49)

\(^3\) Note that the argument does not depend on the perturbative expansion in $\alpha$. 

21
the large order behavior of the coefficients $a^{(r)}$ for sufficiently large $r$ are calculated as, using Eqs. (4.26) and (4.48),

$$a_{n\pm}^{(r)} \sim A_{n\pm}(\epsilon) 4^{-r} \Gamma(r + 2n_{\pm} + 1 \pm \epsilon),$$  

where,

$$A_{n\pm}(\epsilon) = \frac{2}{\pi n_{\pm}! \Gamma(n_{\pm} + 1 \pm \epsilon)}. \quad (4.51)$$

Equation (4.50) shows that the perturbative coefficients diverge factorially unless the prefactor $A(\epsilon)$ vanish. From Eq. (4.51), we can find the disappearance of the leading divergence takes place only when $\epsilon = \mathcal{N}$ ($\mathcal{N} = 1, 2, 3, \ldots$). Comparing the results with Eq. (4.12) and taking the symmetry (4.13) into account, we see that the above cases completely coincide with the case where the system possesses Type A $\mathcal{N}$-fold supersymmetry. Therefore, the results of the valley method analyses are consistent with a consequence of Type A $\mathcal{N}$-fold supersymmetry, that is, the non-renormalization theorem.

V. ANALYSIS OF A TRIPLE-WELL POTENTIAL

In this section, we will analyze a sextic triple-well potential. The form of the potential to be analyzed is the following:

$$V(q) = \frac{1}{2} q^2 (1 - g^2 q^2)^2 + \frac{\epsilon}{2} (1 - 3g^2 q^2). \quad (5.1)$$

This has three local minima at $q = 0$ and $q \simeq \pm 1/g$ for $\epsilon g^2 \ll 1$, see Fig. 7.

![FIG. 7: The form of the triple-well potential investigated in this section.](image)

Comparing this potential with Eq. (3.19), $C_1$, $C_2$ and $q_0$ in $w(q)$ being,

$$C_1 = -g^2, \quad C_2 = 1, \quad q_0 = 0, \quad (5.2)$$
we can easily see that the system has Type A $\mathcal{N}$-fold supersymmetry when

$$\epsilon = \pm \frac{4\mathcal{N} \pm 1}{3}. \quad (5.3)$$

More precisely, the system becomes one of the $\mathcal{N}$-fold supersymmetric pair $H^\pm_\mathcal{N}$; $H^+_\mathcal{N}$ when $\epsilon = (4\mathcal{N} \pm 1)/3$ and $H^-_\mathcal{N}$ when $\epsilon = -(4\mathcal{N} \pm 1)/3$. As has been explained in subsection IIIB, $\mathcal{N}$-fold supersymmetry does not break in the cubic type because either bases of the solvable subspace $\mathcal{V}_\mathcal{N}^+$ or those of $\mathcal{V}_\mathcal{N}^-$ are normalizable in general. Since $C_1 < 0$ in this case, bases of the solvable subspace $\mathcal{V}_\mathcal{N}^+$ are normalizable and physical while those of $\mathcal{V}_\mathcal{N}^-$ are not. Therefore, we may expect that the non-perturbative corrections for certain $\mathcal{N}$ states will vanish when $\epsilon = (4\mathcal{N} \pm 1)/3$ while those will not when $\epsilon = -(4\mathcal{N} \pm 1)/3$, though the perturbation series for the corresponding spectra will be in both the cases convergent.

As far as we know, little has been investigated for triple-well potentials. We have found only two references [43, 44] on the subject, both of which employed the dilute-gas approximation. However, from the consideration mentioned at the end of subsection IV E, we intend to analyze beyond the dilute-gas approximation using the same technique of the valley method as in section IV.\footnote{Indeed, the results of Ref. [43, 44] for the lowest three energies are different from ours and are not consistent with $\mathcal{N}$-fold supersymmetry.}

In the case of $\epsilon = 0$, the three local minima of the potential have the same potential value. Thus, there are (anti-)instanton solutions of the equation of motion which describe the quantum tunneling between the neighboring vacua:

$$q_0^{(I)}(\tau - \tau_0) = \pm \frac{1}{g} \left( \frac{1}{1 + e^{2(\tau - \tau_0)}} \right)^{1/2}, \quad q_0^{(I)}(\tau - \tau_0) = \pm \frac{1}{g} \left( \frac{1}{1 + e^{2(\tau - \tau_0)}} \right)^{1/2}. \quad (5.4)$$

When $\epsilon \neq 0$, the solutions of the valley equation now become the (anti-)valley-instantons. In this case, there are three kinds of the solutions of the valley equation which are asymptotically composed of two (anti-)valley-instantons. Contrary to the periodic potential in section IV, there are two different $I\bar{I}$-valley or $I\bar{I}$-valley configurations in this case since the curvature at the central potential bottom (at $q = 0$) is different, even at the leading order of $g^2$, from the one at the side potential bottoms (at $q \simeq \pm 1/g$); the $I\bar{I}$ ($I\bar{I}$)-valley which satisfy $q(\pm T/2) = 0$ ($T \gg 1$) are different from the ones which satisfy $q(\pm T/2) \simeq 1/g$ or $-1/g$ ($T \gg 1$). The Euclidean action of the former with large separation $R$ can be calculated by the perturbative expansion in $\lambda \sim O(e^{-2R})$ as follows:

$$S^{(I)}(R) = S^{(I)}(R) = 2S_0^{(I)} - \epsilon R + \frac{\epsilon}{2}(T - R) - \frac{1}{g^2} e^{-2R} + O(e^{-4R}), \quad (5.5)$$

while the one of the latter with large separation $\tilde{R}$ can be calculated in the same way as,

$$S^{(I)}(\tilde{R}) = S^{(I)}(\tilde{R}) = 2S_0^{(I)} + \frac{\epsilon}{2} \tilde{R} - \epsilon (T - \tilde{R}) - \frac{2}{g^2} e^{-\tilde{R}} + O(e^{-2\tilde{R}}), \quad (5.6)$$

where $S_0^{(I)}$ denotes the Euclidean action of one (anti-)instanton Eq. (5.4) and amounts to,

$$S_0^{(I)} = \frac{1}{4g^2}. \quad (5.7)$$
The other type is $II$-valley or $II$-valley. The Euclidean action of them with large separation $\tilde{R}$ can be also calculated in the same way as,

$$S^{(II)}(\tilde{R}) = S^{(\bar{II})}(\tilde{R}) = 2S_0^{(I)} + \frac{\epsilon}{2} \tilde{R} - \epsilon(T - \tilde{R}) + \frac{2}{g^2} e^{-\tilde{R}} + O(e^{-2\tilde{R}}). \quad (5.8)$$

A. Multi-valley Calculus

The evaluation of the partition function $Z = \text{tr} e^{-HT}$ by summing over multi-valley-instanton configurations can be done in the same manner as those for the double-well and the periodic potentials. One can easily see that in order to incorporate with the periodic boundary condition in $T$, the number of the valley-instantons in a period $T$ must be even. For a given number $2n$ of the valley-instantons, however, there are still several configurations. If we regard a configuration as $n$ valley-instanton pairs, we have four kinds of pair, $II$-, $\bar{I}I$-, $\bar{II}$-, and $\bar{I}\bar{I}$-valleys. We denote the number of the $II$- and $\bar{II}$-valley as $n_{II}$ and $n_{\bar{II}}$, respectively, and that of the others as $n_{I\bar{I}}$. Contrary to the periodic potential case, the particle must come back to the start point after the period $T$ passes in this case. Therefore, we must impose Eq. (4.29) rather than Eq. (4.30). This condition results in $n_{II} = n_{\bar{II}}$. As a consequence we have,

$$2n_{II} + n_{I\bar{I}} = n. \quad (5.9)$$

This restriction shows that for a given $n$ there are $[n/2] + 1$ variety of the $n_{II}$ value. For $n$ and $n_{II}$ fixed, however, the configuration is not determined uniquely yet. There remains a freedom of the permutation of the pairs. The number of cases can be calculated if one notices that the configuration is uniquely determined as far as the position of the $II$- and $\bar{II}$-valleys among $n$ area is fixed. We denote a set of the position as $\{i_{II}\}$. It is therefore clear that for given $n$ and $n_{II}$ there are $nC_{2n_{II}}$ configurations of the multiple valley-instantons. Combining the results on well-separated valley-instanton pairs with the above considerations, the well-separated multi-valley-instanton action for given $n$, $n_{II}$ and $\{i_{II}\}$ is expressed as,

$$S^{\{i_{II}\}}_{n,n_{II}} = 2nS_0^{(I)} - \epsilon \sum_{i=1}^{n} R_i + \frac{\epsilon}{2} \sum_{i=1}^{n} \tilde{R}_i$$

$$- \frac{1}{g^2} \sum_{i=1}^{n} e^{-2R_i} + \frac{2}{g^2} \sum_{i \in \{i_{II}\}} e^{-\tilde{R}_i} - \frac{2}{g^2} \sum_{i \notin \{i_{II}\}} e^{-\tilde{R}_i}, \quad (5.10)$$

where $R_i$ is the distance between the $(2i-1)$-th and $2i$-th (anti-)valley-instanton and $\tilde{R}_i$ the one between the $2i$-th and the $(2i + 1)$-th (anti-)valley-instanton (mod $n$), see Fig. 8.

The sum of the contributions from the $2n$ valley-instantons configuration can be written as,

$$Z_{NP} = \sum_{n=1}^{\infty} \alpha^{2n} J_n, \quad J_n = \sum_{n_{II}=0}^{[n/2]} \sum_{\{i_{II}\}} S^{\{i_{II}\}}_{n,n_{II}}, \quad (5.11)$$

where $\alpha^2$ denotes the contribution of the Jacobian and the $R$-independent part of the determinant for one valley-instanton-pair and is calculated as, in this case,

$$\alpha^2 = \frac{\sqrt{2}}{\pi g^2} e^{-1/2g^2}. \quad (5.12)$$
The collective coordinates $R_i$ and $\tilde{R}_i$ for a $2n$ valley-instantons configuration. The term $J_{n,n_{II}}^{\{i_{II}\}}$ is given by,

$$J_{n,n_{II}}^{\{i_{II}\}} = \frac{T}{n} \int_0^\infty \left( \prod_{i=1}^n dR_i \right) \left( \prod_{i=1}^n d\tilde{R}_i \right) \delta \left( \sum_{i=1}^n (R_i + \tilde{R}_i) - T \right) \exp \left[ -(1 - \epsilon) \sum_{i=1}^n R_i - \left( \frac{1}{2} + \frac{\epsilon}{2} \right) \sum_{i=1}^n \tilde{R}_i ight]
+ \frac{1}{g^2} \sum_{i=1}^n e^{-2R_i} - \frac{2}{g^2} \sum_{i \in \{i_{II}\}} e^{-\tilde{R}_i} + \frac{2}{g^2} \sum_{i \notin \{i_{II}\}} e^{-\tilde{R}_i} \right].$$

In the above expression, we notice that for $n$ and $n_{II}$ fixed, the contribution $J_{n,n_{II}}^{\{i_{II}\}}$ does not depend on the choice of the set $\{i_{II}\}$. This means the following equality,

$$\sum_{\{i_{II}\}} J_{n,n_{II}}^{\{i_{II}\}} = \left( \frac{n}{2n_{II}} \right) J_{n,n_{II}},$$

(5.14)
where $J_{n,n_{11}}$ is the contribution $J_{n,n_{11}}^{(i_{11})}$ for a specific $i_{11}$ and is evaluated as,

$$J_{n,n_{11}} = \frac{T}{n} \int_0^\infty \left( \prod_{i=1}^n dR_i \right) \left( \prod_{i=1}^n d\tilde{R}_i \right) \delta \left( \sum_{i=1}^n (R_i + \tilde{R}_i) - T \right) \exp \left[ -(1 - \epsilon) \sum_{i=1}^n R_i - \left( \frac{1}{2} + \frac{\epsilon}{2} \right) \sum_{i=1}^n \tilde{R}_i \right] + \frac{1}{g^2} \sum_{i=1}^n e^{-2R_i} - \frac{2}{g^2} \sum_{i=1}^{2n_{11}} e^{-\tilde{R}_i} + \frac{2}{g^2} \sum_{i=2n_{11}+1}^n e^{-\tilde{R}_i} \right]$$

$$= \frac{T}{2\pi i n} \int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} d\tilde{R} \exp \left[ (s - 1 + \epsilon)R + \frac{1}{g^2} e^{-2R} \right]$$

$$\simeq \frac{1}{2} \left( \frac{1}{g^2} \right)^{\frac{s-1}{2}+\frac{1}{2}} \Gamma \left( -\frac{s}{2} + \frac{1}{2} - \frac{\epsilon}{2} \right), \quad (5.16a)$$

$$K_{-}(s) = \int_0^\infty dR \exp \left[ \left( s - 1 + \epsilon \right)R - \frac{2}{g^2} e^{-R} \right]$$

$$\simeq \left( \frac{2}{g^2} \right)^{s-\frac{1}{2}+\frac{1}{2}} \Gamma \left( -s + \frac{1}{2} \right), \quad (5.16b)$$

$$K_{+}^{(1)}(s) = \int_0^\infty dR \exp \left[ \left( s - 1 - \epsilon \right)R + \frac{2}{g^2} e^{-R} \right]$$

$$\simeq \left( -\frac{g^2}{2} \right)^{s-\frac{1}{2}+\frac{1}{2}} \Gamma \left( s - \frac{1}{2} + \epsilon \right), \quad (5.16c)$$

In the last expression for $J_{n,n_{11}}$, several $K$’s are defined by,

$$Z_{NP} = \sum_{n=1}^{\infty} \sum_{n_{11}=0}^{[n/2]} \left( \frac{n}{2n_{11}} \right) J_{n,n_{11}}$$

$$= -\frac{T}{4\pi i} \int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} d\tilde{R} \ln \left( 1 - \alpha^2 K_{-}(s) K_{+}^{(1)}(s) \right) \left( 1 - \alpha^2 K_{-}(s) K_{+}^{(2)}(s) \right), \quad (5.17)$$

where,

$$K_{+}^{(\pm)}(s) = K_{+}^{(2)}(s) \pm K_{+}^{(1)}(s). \quad (5.18)$$

**B. Non-perturbative Contributions**

From the results in Eqs. (5.16)–(5.18), the non-perturbative contributions to the spectra are determined by the following equation:

$$\alpha^2 \beta_{\pm}(E, \epsilon) \left( \frac{2}{g^2} \right)^{E-\frac{1}{2}+\frac{1}{2}} \Gamma \left( -E + \frac{1}{2} + \frac{\epsilon}{2} \right) \left( -\frac{1}{g^2} \right)^{E-\frac{1}{2}+\frac{1}{2}} \Gamma \left( -\frac{E}{2} + \frac{1}{2} - \frac{\epsilon}{2} \right) = 1, \quad (5.19)$$
where,
\[ \beta_{\pm}(E, \epsilon) = \frac{(-E - \frac{1}{2} - \frac{\epsilon}{2} \pm 1)}{2}. \]  
(5.20)

We will solve the above equation by the series expansion in \( \alpha \):
\[ E_{n_0} = E_{n_0}^{(0)} + \alpha E_{n_0}^{(1)} + \alpha^2 E_{n_0}^{(2)} + \cdots, \quad E_{n_0}^{(0)} = n_0 + \frac{1}{2} + \frac{\epsilon}{2}, \]  
(5.21a)
\[ E_{n_{\pm}} = E_{n_{\pm}}^{(0)} + \alpha E_{n_{\pm}}^{(1)} + \alpha^2 E_{n_{\pm}}^{(2)} + \cdots, \quad E_{n_{\pm}}^{(0)} = 2n_{\pm} + 1 - \epsilon, \]  
(5.21b)

where \( E_{n_0} \) stands for the spectra corresponding to, in the limit \( g \to 0 \), the eigenfunctions of the center potential well and \( E_{n_{\pm}} \) for the ones corresponding to the parity eigenstates obtained by the linear combinations of the eigenfunctions of the each side potential well. For \( \epsilon \neq \pm(2N + 1)/3 \) (\( N = 0, 1, 2, \ldots \)), the 1st order contributions vanish and the leading 2nd order contributions are calculated as follows;
\[ E_{n_0}^{(2)} = -\frac{1}{n_0!} \left( \frac{2}{g^2} \right)^{n_0} \left( -\frac{1}{g^2} \right)^{n_0} \Gamma \left( -n_0 + 1 + \frac{3}{4} \epsilon \right), \]  
(5.22)
\[ E_{n_{\pm}}^{(2)} = -\left( -\frac{1}{2} - \frac{3}{4} \epsilon \right) \frac{1}{n_{\pm}!} \left( \frac{2}{g^2} \right)^{2n_{\pm} + \frac{1}{2} - \frac{3}{4} \epsilon} \left( \frac{1}{g^2} \right)^{n_{\pm}} \Gamma \left( -2n_{\pm} + \frac{1}{2} + \frac{3}{2} \epsilon \right). \]  
(5.23)

In this case, degeneracies of the harmonic oscillator spectra for the each potential well only occur between the both side wells. The different non-perturbative contributions for \( n_{\pm} \) in Eq. (5.23) show the splitting of the degeneracies via the quantum tunneling as in the case of symmetric double-well potentials.

When \( \epsilon = (4N + 1)/3 \) (\( N = 0, 1, 2, \ldots \)), all the even-parity central harmonic spectra \( E_{n_0}^{(0)} \) and the higher side harmonic spectra \( E_{n_{\pm}}^{(0)} \) degenerate for \( n_{\pm} = m_0 + N \), see Fig. 9(a) for \( \epsilon = 5/3 \) (\( N = 1 \)).

![FIG. 9: Degeneracies of the harmonic spectra for (a) \( \epsilon = 5/3 \) and (b) \( \epsilon = -1 \).](image)

It is interesting, however, that the interference due to the quantum tunneling only occurs between the same (even-)parity states. As a consequence, \( E_{2m_0} \) and \( E_{n_{\pm}} \) satisfying \( n_{\pm} = m_0 + N \),
\( m_0 + \mathcal{N} \) acquire order \( \alpha^1 \) contributions as follows:

\[
E_{n_{+}/2m_0}^{(1)} = \pm \sqrt{\frac{2}{n_{+}!(2m_0)!}} \left( \frac{2}{g^2} \right)^{2m_0} \left( \frac{1}{g^2} \right)^{n_{+}}, \quad (5.24)
\]

\[
E_{n_{+}/2m_0}^{(2)} = \frac{E_{n_{+}/2m_0}^{(1)^2}}{4} \left[ \ln \left( -\frac{2}{g^2} \right) + \ln \left( \frac{2}{g^2} \right) + \ln \left( -\frac{1}{g^2} \right) - \psi(n_{+} + 1) - 2\psi(2m_0 + 1) \right]. \quad (5.25)
\]

For the other spectra, say, \( E_{n_{-}} \), \( E_{2m_0+1} \) and the lower \( E_{n_{+}} \) with \( n_{+} < \mathcal{N} \), the contributions are the same as Eqs. (5.22) and (5.23). When \( \epsilon = -(4\mathcal{N} - 1)/3 \) (\( \mathcal{N} = 1, 2, 3, \ldots \)), all the side harmonic spectra \( E_{0_{\pm}}^{(0)} \) and the higher even-parity central harmonic spectra \( E_{2m_0}^{(0)} \) degenerate for \( m_0 = n_{\pm} + \mathcal{N} \), see Fig. 8(b) for \( \epsilon = -1 \) (\( \mathcal{N} = 1 \)). In this case, the interference also occurs only between the same (even-)parity states. The contributions for \( E_{2m_0} \) and \( E_{n_{+}} \) satisfying \( m_0 = n_{+} + \mathcal{N} \) are given by the same as Eqs. (5.24) and (5.25). For the other spectra, say, \( E_{n_{-}}, E_{2m_0+1} \) and the lower \( E_{2m_0} \) with \( m_0 < \mathcal{N} \), the contributions are the same as Eqs. (5.22) and (5.23).

When \( \epsilon = (4\mathcal{N} - 1)/3 \) (\( \mathcal{N} = 1, 2, 3, \ldots \)), all the odd-parity central harmonic spectra \( E_{2m_0+1}^{(0)} \) and the higher side harmonic spectra \( E_{n_{\pm}}^{(0)} \) degenerate for \( n_{\pm} = m_0 + \mathcal{N} \), see Fig. 10(a) for \( \epsilon = 1 \) (\( \mathcal{N} = 1 \)).

![FIG. 10: Degeneracies of the harmonic spectra for (a)\( \epsilon = 1 \) and (b)\( \epsilon = -5/3 \).](image)

In this case, only the odd-parity states interfere and yield order \( \alpha^1 \) contributions for \( n_{-} = m_0 + \mathcal{N} \):

\[
E_{n_{-}/2m_0+1}^{(1)} = \pm \sqrt{\frac{2}{n_{-}!(2m_0 + 1)!}} \left( \frac{2}{g^2} \right)^{2m_0+1} \left( \frac{1}{g^2} \right)^{n_{-}}, \quad (5.26)
\]

\[
E_{n_{-}/2m_0+1}^{(2)} = \frac{E_{n_{-}/2m_0+1}^{(1)^2}}{4} \left[ \ln \left( -\frac{2}{g^2} \right) + \ln \left( \frac{2}{g^2} \right) + \ln \left( -\frac{1}{g^2} \right) - \psi(n_{-} + 1) - 2\psi(2m_0 + 2) \right]. \quad (5.27)
\]
The contributions for the other spectra, say, \( E_{n+}, E_{2m_0} \) and the lower \( E_{n-} \) with \( n_- < N \), are given by the same as Eqs. (5.22) and (5.23). When \( \epsilon = -(4N + 1)/3 \) (\( N = 0, 1, 2, \ldots \)), all the side harmonic spectra \( E_{m_0}^{(0)} \) and the higher odd-parity central harmonic spectra \( E_{2m_0+1}^{(0)} \) degenerate for \( m_0 = n_+ + N \), see Fig. 10(b) for \( \epsilon = -5/3 \) (\( N = 1 \)). Only the odd-parity states interfere in the same way and the non-perturbative contributions for \( E_{2m_0+1} \) and \( E_{n-} \) satisfying \( m_0 = n_- + N \) are the same as Eqs. (5.26) and (5.27). Again, the expressions for the other spectra, say, \( E_{n+}, E_{2m_0} \) and the lower \( E_{2m_0+1} \) with \( m_0 < N \), are given by Eqs. (5.22) and (5.23).

From the whole result obtained here, we see that the non-perturbative corrections vanish only when \( \epsilon = (4N \pm 1)/3 \) (\( N = 1, 2, 3, \ldots \)). More precisely, when \( \epsilon = (4N + 1)/3 \), Eq. (5.23) is applied for the even-parity states labeled by the quantum number \( n_- \) and results in \( E_{n_-}^{(2)} = 0 \) for all \( n_- < N \). Similarly, when \( \epsilon = (4N - 1)/3 \), Eq. (5.23) is applied for the odd-parity states labeled by the quantum number \( n_+ \) and results in \( E_{n_+}^{(2)} = 0 \) for all \( n_+ < N \). It should be noted that in the case of \( \epsilon = -(4N \pm 1)/3 \) (\( N = 1, 2, 3, \ldots \)) the non-perturbative corrections do remain although the models are \( N \)-fold supersymmetric, reflecting the fact that they are only quasi-perturbatively solvable but are not quasi-exactly solvable. These results are just what we have expected from the general properties of \( N \)-fold supersymmetry.

C. Large Order Behavior of the Perturbation Series

The large order behavior of the perturbation series in \( g^2 \) for the spectra can be estimated by the same way as in the case of the double-well and periodic potentials. From the non-perturbative contributions Eqs. (5.22)–(5.27), we can easily see that the imaginary parts of them are continuous, at least up to the order \( \alpha^2 \), in \( \epsilon \) and yield,

\[
\text{Im} E_{n_0} \sim -\alpha^2 \frac{\pi}{n_0! \Gamma (\frac{3}{2} + \frac{2}{3} + \frac{4}{3} \epsilon)} \left( \frac{2}{g^2} \right)^{n_0} \left( \frac{1}{g^2} \right)^{\frac{2n_0}{3} + \frac{1}{4} + \frac{4}{3} \epsilon},
\]

(5.28a)

\[
\text{Im} E_{n_\pm} \sim -\alpha^2 \frac{\pi}{n_! \Gamma (2n_+ + \frac{3}{2} - \frac{3}{2} \epsilon)} \left( \frac{2}{g^2} \right)^{2n_+ \frac{1}{2} - \frac{2}{3} \epsilon} \left( \frac{1}{g^2} \right)^{n_+},
\]

(5.28b)

which are valid for arbitrary \( \epsilon \). Then, if we expand the spectra in power of \( g^2 \) such that,

\[
E_{n_0/n_\pm} = E_{n_0/n_\pm}^{(0)} + \sum_{r=1}^{\infty} a^{(r)}_{n_0/n_\pm} g^{2r},
\]

(5.29)

the large order behavior of the coefficients \( a^{(r)} \) for sufficiently large \( r \) are calculated as, using Eqs. (1.20) and (5.28),

\[
a^{(r)}_{n_0} \sim A_{n_0}(\epsilon) 2^r \Gamma \left( r + \frac{3}{2} n_0 + \frac{3}{4} + \frac{3}{4} \epsilon \right),
\]

(5.30a)

\[
a^{(r)}_{n_\pm} \sim A_{n_\pm}(\epsilon) 2^r \Gamma \left( r + 3n_\pm + \frac{3}{2} - \frac{3}{2} \epsilon \right),
\]

(5.30b)
where,

$$A_{n_0}(\epsilon) = \frac{\sqrt{2}}{\pi} \frac{2^{2n_0} + \frac{4}{3} + \frac{1}{3} \epsilon}{n_0! \Gamma \left( \frac{m_0}{2} + \frac{3}{4} + \frac{3}{4} \epsilon \right)},$$  \hspace{1cm} (5.31a)$$

$$A_{n_\pm}(\epsilon) = \frac{\sqrt{2}}{\pi} \frac{2^{5n_\pm + 2 - \epsilon}}{n_\pm! \Gamma \left( 2n_\pm + \frac{3}{2} - \frac{3}{2} \epsilon \right)}. \hspace{1cm} (5.31b)$$

Equations (5.30) show that the perturbative coefficients diverge factorially unless the prefactor $A(\epsilon)$'s vanish. From Eq. (5.31), we can find the disappearance of the leading divergence takes place only when $\epsilon = \pm (2n + 1)/3 \ (n = 1, 2, 3, \ldots)$. More precisely, we obtain the following results:

1. $\epsilon = (4N \pm 1)/3 \ (N = 1, 2, 3, \ldots)$
   
   $$A_{n_\pm}(\epsilon) = 0 \ \text{for} \ n_\pm < N.$$  

2. $\epsilon = -(4N + 1)/3 \ (N = 1, 2, 3, \ldots)$
   
   $$A_{2m_0+1}(\epsilon) = 0 \ \text{for} \ m_0 < N.$$  

3. $\epsilon = -(4N - 1)/3 \ (N = 1, 2, 3, \ldots)$
   
   $$A_{2m_0}(\epsilon) = 0 \ \text{for} \ m_0 < N.$$  

Comparing these results with Eq. (5.3), we see that the above cases completely coincide with the case where the system possesses Type A $\mathcal{N}$-fold supersymmetry. Again, the results of the valley method analyses are consistent with the non-renormalization theorem.

VI. SUMMARY

In this article, we have made non-perturbative analyses on the models which can be $\mathcal{N}$-fold supersymmetric at specific values of the parameter. Combining the results obtained in this article with the ones in Ref. [2], we get the following:

1. For all the potentials investigated (double-well, triple-well, periodic), the leading divergence of the perturbation series disappears when and only when they are $\mathcal{N}$-fold supersymmetric. The non-renormalization theorem ensures that $\mathcal{N}$-fold supersymmetry is sufficient for the disappearance of the divergence. The results indicate that it may also be necessary.

2. The non-perturbative corrections to the spectra for certain states vanish when and only when the models are quasi-exactly solvable (triple-well, periodic).

3. For the quasi-perturbatively solvable potentials (double-well, triple-well), the non-perturbative corrections remain although they are $\mathcal{N}$-fold supersymmetric.
As was mentioned in Ref. [28] the quasi-solvable models constructed by $sl(2)$ generators do not always have normalizable solvable states. Although the conditions on the normalizability of the models were fully investigated in Ref. [45], it remains unclear what is the role of the partial algebraization of the models without normalizable solvable states. The results listed above provide an answer to this problem. Even though the solvable wave functions are not normalizable, they can be normalizable and thus make sense in the perturbation theory. In this case, the spectra corresponding to the solvable states also make sense in the perturbation theory. As was shown in Ref. [1], the perturbation series for them are convergent since they are the solutions of a finite order algebraic equation. However, the fact that the solvable states and the corresponding spectra make sense only in the perturbation theory inevitably means the existence of the non-perturbative effects, which is in contrast to the case of the quasi-exactly solvable models. That is why we have called the case quasi-perturbatively solvable.

Finally, we would like to mention about applicability of the dilute-gas approximation. As has been mentioned previously, the dilute-gas approximation cannot give proper results, that is, consistent results with $\mathcal{N}$-fold supersymmetry, for both the potentials Eqs. (4.11) and (5.1). Therefore, it seems that the success of the dilute-gas approximation for the symmetric double-well potential is rather exceptional and applicability of it is quite limited.

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