Minimal areas from $q$-deformed oscillator algebras

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Abstract

On the basis of various examples we provide evidence that noncommutative spacetime involving position-dependent structure constants will give rise to deformed oscillator algebras. In turn, starting from some $q$-deformations of these algebras in a two-dimensional space for which the entire deformed Fock space can be constructed explicitly, we derive the commutation relations for the dynamical variables in noncommutative spacetime. We compute minimal areas resulting from these relations, i.e. finitely extended regions for which it is impossible to resolve any substructure in form of measurable knowledge. The size of the regions we find is determined by the noncommutative constant and the deformation parameter $q$. Any object in this type of spacetime structure has to be of membrane type or in certain limits of string type.

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1. Introduction

The idea to extend the quantization procedure from canonical variables to spacetime itself [1] traces back over 60 years. In recent years, this general possibility has become more and more appealing, especially in the context of quantum field theories as such types of spacetime structures will introduce natural cut-offs and theories on them are therefore renormalized by construction [2, 3]. In addition, almost all possible theories of quantum gravity require non-Minkowskian spacetime in one form or another [4–8].

One of the interesting consequences of these types of spacetime structures is that in many cases they lead to modifications of Heisenberg’s uncertainty relations, which in turn result in the emergence of minimal lengths. This means in such spaces one has almost inevitably definite fundamental distances below which no substructure can be resolved [9–11, 14–19]. Recently, some of us proposed [20] a consistent dynamical noncommutative spacetime...
We demand now to have a definite transformation property for the standard commutation relations for a Fock space representation:

\[ [x_0, p_{x_0}] = i\theta, \quad [x_0, p_{x_0}] = i\theta, \quad [y_0, p_{y_0}] = i\theta, \quad [y_0, p_{y_0}] = i\theta, \quad [p_{x_0}, p_{y_0}] = 0, \quad [p_{x_0}, p_{y_0}] = 0, \quad [y_0, p_{y_0}] = 0. \tag{2.1} \]

Restricting the noncommutative constant to be real, i.e. \( \theta \in \mathbb{R} \), ensures that \( x_0 \) and \( y_0 \) are the Hermitian operators. We now wish to find a representation for creation and annihilation operators in terms of the dynamical variables \( x_0, y_0, p_{x_0}, p_{y_0} \) satisfying the standard commutation relations for a Fock space representation:

\[ [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0 \quad \text{for} \quad i, j = 1, 2. \tag{2.2} \]

In order to reduce the number of unknown coefficients in a possible Ansatz for the \( a_i, a_i^\dagger \) we may take the properties of the dynamical variables under a \( PT \)-transformation as a guiding principle. These types of considerations have proved to be very fruitful, allowing even a consistent formulation of non-Hermitian systems with real eigenvalues, see e.g. [21–23] for a review or [24, 25] for recent special issues. For this purpose, we note that relations (2.1) are \( \mathcal{P}_x, \mathcal{P}_y \)-symmetric and \( \mathcal{P}_x, \mathcal{P}_y \)-symmetric in the sense that they remain invariant under a simultaneous reflection in the \( x_0 \)-direction together with a time reversal and under a simultaneous reflection in the \( y_0 \)-direction together with a time reversal, respectively,

\[
\begin{align*}
\mathcal{P}_x: \quad x_0 &\mapsto -x_0, \quad y_0 \mapsto y_0, \quad p_{x_0} \mapsto -p_{x_0}, \quad p_{y_0} \mapsto p_{y_0}, \\
\mathcal{P}_y: \quad x_0 &\mapsto x_0, \quad y_0 \mapsto -y_0, \quad p_{x_0} \mapsto p_{x_0}, \quad p_{y_0} \mapsto -p_{y_0}, \\
\mathcal{T}: \quad x_0 &\mapsto x_0, \quad y_0 \mapsto y_0, \quad p_{x_0} \mapsto p_{x_0}, \quad p_{y_0} \mapsto -p_{y_0}, \quad i \mapsto -i, \\
\mathcal{P}_x \mathcal{T}: \quad x_0 &\mapsto -x_0, \quad y_0 \mapsto y_0, \quad p_{x_0} \mapsto p_{x_0}, \quad p_{y_0} \mapsto -p_{y_0}, \quad i \mapsto -i, \\
\mathcal{P}_y \mathcal{T}: \quad x_0 &\mapsto x_0, \quad y_0 \mapsto -y_0, \quad p_{x_0} \mapsto -p_{x_0}, \quad p_{y_0} \mapsto p_{y_0}, \quad i \mapsto -i.
\end{align*}
\tag{2.3}
\]

We demand now to have a definite transformation property for the \( a_i, a_i^\dagger \), that is we would like them to be either even or odd under a \( \mathcal{P}_x, \mathcal{T} \)-transformation, i.e. \( a_i \mapsto a_i, a_i^\dagger \mapsto a_i^\dagger \) or \( a_i \mapsto -a_i, a_i^\dagger \mapsto -a_i^\dagger \), such that we can use this property to reduce the total number of
constants. Assuming that the dependence on \(x_0, y_0, p_{x_0}, p_{y_0}\) is still linear, the general operators of the form
\[
\begin{align*}
a_1 &:= \alpha_1 x_0 + i \alpha_2 y_0 + \alpha_3 p_{x_0} + \alpha_4 p_{y_0}, \\
a_2 &:= \alpha_5 x_0 + i \alpha_6 y_0 + \alpha_7 p_{x_0} + \alpha_8 p_{y_0}, \\
a_1' &:= \alpha_1 x_0 - i \alpha_2 y_0 - \alpha_3 p_{x_0} + \alpha_4 p_{y_0}, \\
a_2' &:= \alpha_5 x_0 - i \alpha_6 y_0 - \alpha_7 p_{x_0} + \alpha_8 p_{y_0},
\end{align*}
\]
(2.4)
with unknown constants \(\alpha_1, \ldots, \alpha_8 \in \mathbb{R}\) for the time being, are \(\mathcal{P}, \mathcal{T}\)-odd: \(a_i \mapsto -a_i, \ a_i' \mapsto -a_i'\) and \(\mathcal{P}, \mathcal{T}\)-even: \(a_i \mapsto a_i, \ a_i' \mapsto a_i'\) when using realization (2.3). The reverse scenario is simply achieved by \(a_j \mapsto i a_j\) for \(j = 1, \ldots, 8\).

The operators defined in (2.4) satisfy the commutation relations (2.2) provided that the following four constraints on the constants hold:
\[
\begin{align*}
\alpha_1 &= \frac{\alpha_6}{2h\Delta}, \\
\alpha_4 &= \frac{\theta \alpha_6 + h\alpha_7}{2h^2\Delta}, \\
\alpha_5 &= -\frac{\alpha_2}{2h\Delta}, \\
\alpha_8 &= -\frac{\theta \alpha_2 + h\alpha_3}{2h^2\Delta},
\end{align*}
\]
(2.5)
where we abbreviated \(\Delta := \alpha_3\alpha_6 - \alpha_2\alpha_7 \neq 0\). This means we have still four almost entirely free parameters left. Inverting relations (2.4) while keeping the constraints (2.5), we can express the coordinates and the momenta in terms of the creation and annihilation operators:
\[
\begin{align*}
\alpha x &= \frac{i \theta (\alpha_2 + h\alpha_7)}{2\Delta} (a_1 + a_1') + (\theta \alpha + h\alpha_7) (a_2 + a_2'), \\
\alpha y &= \frac{i \theta (\alpha_3 - \alpha_7)}{2\Delta} (a_1 - a_1') - \frac{i \theta \alpha_3}{2\Delta} (a_2 - a_2').
\end{align*}
\]
(2.6)
It is easily verified that these operators obey (2.1) when using (2.2).

### 2.2. Oscillator algebras from string-type noncommutative spacetime

Let us now carry out a similar analysis for the situation when the underlying spacetime is dynamical, i.e. the constant \(\theta\) becomes position and possibly also momentum dependent. A set of consistent commutation relations for such a scenario was introduced in [20]
\[
\begin{align*}
[x, y] &= i\theta (1 + \tau \gamma^5), \\
[x, p_x] &= i\hbar (1 + \tau \gamma^5), \\
[y, p_y] &= i\hbar (1 + \tau \gamma^5), \\
[p_x, p_y] &= 0, \\
[x, p_y] &= 2i\tau \gamma (\theta p_x + \hbar x), \\
[y, p_x] &= 0.
\end{align*}
\]
(2.7)
Defining the analogues to the creation and annihilation operators and keeping the dependence on the dynamical variables similar as in (2.4)
\[
\begin{align*}
\hat{a}_1 &:= \alpha_1 x + i \alpha_2 y + \alpha_3 p_x + \alpha_4 p_y, \\
\hat{a}_2 &:= \alpha_5 x + i \alpha_6 y + \alpha_7 p_x + \alpha_8 p_y, \\
\hat{a}_1' &:= \alpha_1 x - i \alpha_2 y - \alpha_3 p_x + \alpha_4 p_y, \\
\hat{a}_2' &:= \alpha_5 x - i \alpha_6 y - \alpha_7 p_x + \alpha_8 p_y,
\end{align*}
\]
(2.8)
we can compute the resulting commutation relations. Keeping the constraints (2.5) and setting in addition \(\alpha_3 = 0\) we find that the standard commutation relations are deformed:
\[
\begin{align*}
[\hat{a}_1, \hat{a}_1'] &= 1 + \frac{\tau}{4\alpha_2^2} (\hat{a}_1 \hat{a}_1' + \hat{a}_1' \hat{a}_1 - \hat{a}_1 \hat{a}_1'), \\
[\hat{a}_1, \hat{a}_2] &= [\hat{a}_1', \hat{a}_2] = [\hat{a}_1', \hat{a}_2'] = \frac{\tau}{4\alpha_2^2} (\hat{a}_1 \hat{a}_2 + \hat{a}_2 \hat{a}_1 - \hat{a}_1' \hat{a}_2' - \hat{a}_1 \hat{a}_2').
\end{align*}
\]
(2.9)
The asymmetry between \(i = 1\) and \(i = 2\) in (2.9) appears odd at first sight in the light of (2.8), but it is a consequence of the non-symmetric nature of (2.7) and our choice \(\alpha_3 = 0\). Clearly when the deformation parameter \(\tau\) vanishes, we obtain the usual Fock space commutation relations (2.2).

For the specific choice
\[
\begin{align*}
\alpha_1 &= \alpha_2 = -\frac{\lambda_1}{h \sqrt{K_1}}, \\
\alpha_3 &= \alpha_4 = -\frac{1}{\sqrt{K_1}}, \\
\alpha_5 &= -\alpha_6 = \frac{\lambda_2}{h \sqrt{K_2}}, \\
\alpha_7 &= \alpha_8 = \frac{1}{\sqrt{K_2}}.
\end{align*}
\]
we recover the representation found in [26] when comparing with equations (57) and (58) therein and identifying the quantities \(\lambda_1, \lambda_2\) and \(K_1, K_2\) which are defined in equations (56) and (59), respectively.
2.3. Oscillator algebras from membrane-type noncommutative spacetime

We propose now a new type of deformation for the flat noncommutative spacetime (2.1):

\[ [\tilde{x}, \tilde{y}] = i\theta + i\tau (\tilde{x}\tilde{y} + \tilde{y}\tilde{x}), \quad [\tilde{x}, \tilde{p}_x] = i\hbar + i\tau \hbar (\tilde{x}\tilde{p}_x - \tilde{p}_x\tilde{x}), \quad [\tilde{y}, \tilde{p}_y] = i\hbar + i\tau \hbar (\tilde{y}\tilde{p}_y - \tilde{p}_y\tilde{y}). \]

(2.11)

In the same manner as for (2.7) we may verify that these commutation relations are consistent in the sense that the Jacobi identities are satisfied. Using the standard arguments to find a minimal length, we observe that the \( \tilde{x}, \tilde{y} \)-commutator implies a minimal length in the \( \tilde{x} \)-direction as well as in the \( \tilde{y} \)-direction, which means that the underlying object, whose substructure we cannot determine, is of a membrane structure. Once again we define creation- and annihilation-type operators analogously to (2.4) keeping the dependence on the dynamical variables the same. When specifying the coefficients such that

\[ \tilde{a}_i := \sqrt{\frac{1 - \tau}{2\theta}} (\tilde{x} + i\tilde{y}), \quad \tilde{a}_j := \sqrt{\frac{1 - \tau}{2\theta}} (\tilde{x} - i\tilde{y}). \]

(2.12)

we find the commutation relations

\[ \tilde{a}_i \tilde{a}_j - \frac{1 + \tau}{1 - \tau} \delta_{ij} \tilde{a}_i \tilde{a}_j = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = 0, \quad [\tilde{a}_i, \tilde{a}_j] = 0, \quad \text{for } i, j = 1, 2. \]

(2.13)

As expected (2.2) is recovered for \( \tau \to 0 \). These relations are very reminiscent of the \( q \)-deformed oscillator algebra studied in this context for instance in [9–18].

This example and the one in the previous subsection indicate that dynamical spacetime relations will naturally lead to deformed Fock spaces. As we have seen some of them have a very convenient and well-studied structure, such as (2.13), whereas others are rather awkward such as (2.9) and (2.10). Let us therefore now reverse the scenario and deform first the Fock space relations in a ‘nice’ way and subsequently compute the corresponding commutation relations for the dynamical variables.

3. Noncommutative spacetime from \( q \)-deformed creation and annihilation operators

Resembling relations (2.13) we \( q \)-deform the relations in (2.2) by defining a new set of creation and annihilation operators \( A_1, A_1^\dagger, A_2, A_2^\dagger \) satisfying

\[ A_1 A_2^\dagger - q^{2\delta_{ij}} A_2 A_1^\dagger = \delta_{ij}, \quad [A_1^\dagger, A_2^\dagger] = 0, \quad [A_i, A_j] = 0, \quad \text{for } i, j = 1, 2. \]

(3.1)

There exist various other possibilities to deform relations (2.2) which still lead to constructable Fock spaces, such as for instance using different functions in the first relation of (3.1), i.e., \( q^{2\delta_{ij}} \to q^{2\delta_{ij}} \), or replacing \( \delta_{ij} \) on the right-hand side of the first relation by \( q^{g(A_i A_j)} \) with \( g(x) \) being an arbitrary function as in [11, 18]. Guided by the limit \( q \to 1 \) in which we should recover relations (2.6) and the properties of these operators under a \( \mathcal{PT} \)-transformation, we expand the new set of deformed canonical variables \( X, Y, P_x, P_y \) linearly in terms of \( A_1, A_1^\dagger, A_2, A_2^\dagger \) as

\[ X = \kappa_1 (A_1^\dagger + A_1) + \kappa_2 (A_2^\dagger + A_2), \quad P_x = i\kappa_3 (A_1^\dagger - A_1) + i\kappa_4 (A_2^\dagger - A_2), \]
\[ Y = i\kappa_5 (A_1 - A_1^\dagger) + i\kappa_6 (A_2 - A_2^\dagger), \quad P_y = \kappa_7 (A_1^\dagger + A_1) + \kappa_8 (A_2^\dagger + A_2). \]

(3.2)
The constants \( k_1, \ldots, k_8 \in \mathbb{R} \) are unknown for the time being. Inverting relations (3.2) we may express the deformed creation and annihilation operators in terms of the deformed canonical variables

\[
A_1 = \frac{k_8}{\lambda} X + i \frac{k_4}{\mu} Y - \frac{k_6}{\lambda} P_x - \frac{k_2}{\lambda} P_y, \quad A_1^\dagger = \frac{k_8}{\lambda} X - i \frac{k_4}{\mu} Y + \frac{k_6}{\lambda} P_x - \frac{k_2}{\lambda} P_y,
\]

\[
A_2 = -\frac{k_7}{\lambda} X - i \frac{k_3}{\mu} Y + i \frac{k_5}{\mu} P_x + \frac{k_1}{\mu} P_y, \quad A_2^\dagger = -\frac{k_7}{\lambda} X + i \frac{k_3}{\mu} Y - i \frac{k_5}{\mu} P_x + \frac{k_1}{\mu} P_y,
\]

where we abbreviated \( \lambda := 2(k_3k_5 - k_2k_7) \neq 0 \) and \( \mu := 2(k_4k_5 - k_3k_6) \neq 0 \). Using representation (3.2) together with (3.1) we compute

\[
[X, Y] = 2i(k_1k_5 + k_2k_6) + 2i(q^2 - 1)(k_1k_5A_1^\dagger A_1 + k_2k_6A_2^\dagger A_2),
\]

\[
[X, P_x] = 2i(k_1k_3 + k_2k_4) + 2i(q^2 - 1)(k_1k_3A_1^\dagger A_1 + k_2k_4A_2^\dagger A_2),
\]

\[
[Y, P_x] = -2i(k_3k_7 + k_6k_8) + 2i(1 - q^2)(k_3k_7A_1^\dagger A_1 + k_6k_8A_2^\dagger A_2),
\]

\[
[X, P_y] = 0,
\]

\[
[Y, P_y] = 0.
\]

Next we employ relations (3.3) and evaluate

\[
A_1^\dagger A_1 = \frac{k_2^2}{\lambda^2} X^2 + \frac{k_4^2}{\mu^2} Y^2 + \frac{k_6^2}{\lambda^2} P_x^2 + \frac{k_8^2}{\lambda^2} P_y^2 - \frac{2k_3k_2}{\lambda^2} XP_x + \frac{2k_3k_6}{\mu^2} YP_x + \frac{k_4k_8}{\lambda\mu} \left[ X, Y \right] + \frac{k_4k_8}{\lambda\mu} \left[ Y, P_x \right] - i \frac{k_6k_2}{\lambda\mu} \left[ X, P_x \right] - i \frac{k_6k_2}{\lambda\mu} \left[ P_x, P_y \right].
\]

\[
A_2^\dagger A_2 = \frac{k_2^2}{\lambda^2} X^2 + \frac{k_4^2}{\mu^2} Y^2 + \frac{k_6^2}{\lambda^2} P_x^2 + \frac{k_8^2}{\lambda^2} P_y^2 - \frac{2k_3k_1}{\lambda^2} XP_y + \frac{2k_3k_5}{\mu^2} YP_y + \frac{k_4k_7}{\lambda\mu} \left[ X, Y \right] + \frac{k_4k_7}{\lambda\mu} \left[ Y, P_x \right] - i \frac{k_6k_1}{\lambda\mu} \left[ X, P_x \right] - i \frac{k_6k_1}{\lambda\mu} \left[ P_x, P_y \right].
\]

Substituting (3.10) and (3.11) into the right-hand sides of (3.4)–(3.7) we obtain four equations for the four unknown commutators \([X, Y]\), \([X, P_x]\), \([Y, P_x]\) and \([P_x, P_y]\). Solving these equations, the resulting dynamical noncommutative relations are

\[
[X, Y] = i\theta + \frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{k_2k_5k_7 - k_3k_6k_8}{(k_2k_7 - k_3k_8)^2} X^2 + \frac{k_2k_5k_7 + k_3k_6k_8}{(k_2k_7 - k_3k_8)^2} Y^2 + \frac{k_2k_5k_7}{(k_2k_7 - k_3k_8)^2} P_x^2 - \frac{k_6k_2}{(k_2k_7 - k_3k_8)^2} XP_x \right],
\]

\[
[X, P_x] = i\hbar + \frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{k_2k_5k_7 + k_3k_6k_8}{(k_2k_7 - k_3k_8)^2} X^2 + \frac{k_3k_4}{(k_4k_5 - k_3k_6)^2} Y^2 + \frac{k_3k_4}{(k_4k_5 - k_3k_6)^2} P_x^2 + \frac{k_6k_2}{(k_2k_7 - k_3k_8)^2} XP_x \right].
\]
\[
[Y, P_y] = i\hbar - \frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_7 \kappa_8 (\kappa_6 \kappa_7 + \kappa_5 \kappa_8)}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} \right] X^2 + \frac{\kappa_6 \kappa_8 \kappa_3^2 + \kappa_5 \kappa_7 \kappa_2^2}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} Y^2 \\
+ \frac{\kappa_5 \kappa_6 (\kappa_6 \kappa_7 + \kappa_5 \kappa_8)}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} P_x^2 + \frac{\kappa_6 \kappa_8 \kappa_3^2 + \kappa_5 \kappa_7 \kappa_2^2}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} P_y^2 \\
- \frac{2\kappa_7 \kappa_8 (\kappa_2 \kappa_5 + \kappa_1 \kappa_6)}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} X P_y - \frac{2\kappa_5 \kappa_6 (\kappa_4 \kappa_5 + \kappa_3 \kappa_8)}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} Y P_y \right],
\tag{3.14}
\]
\[
[P_x, P_y] = -\frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_7 \kappa_8 (\kappa_4 \kappa_7 + \kappa_3 \kappa_8)}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} \right] X^2 + \frac{\kappa_5 \kappa_4 (\kappa_4 \kappa_7 + \kappa_3 \kappa_8)}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} Y^2 \\
+ \frac{\kappa_4 \kappa_8 \kappa_2^2 + \kappa_3 \kappa_7 \kappa_6^2}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} P_x^2 + \frac{\kappa_4 \kappa_8 \kappa_2^2 + \kappa_3 \kappa_7 \kappa_6^2}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} P_y^2 \\
- \frac{2\kappa_7 \kappa_8 (\kappa_2 \kappa_3 + \kappa_1 \kappa_4)}{(\kappa_2 \kappa_7 - \kappa_1 \kappa_8)^2} X P_x - \frac{2\kappa_5 \kappa_4 (\kappa_6 \kappa_7 + \kappa_5 \kappa_8)}{(\kappa_4 \kappa_5 - \kappa_3 \kappa_6)^2} Y P_x \right].
\tag{3.15}
\]

For the constant terms of these commutators we have implemented here the constraints
\[
\kappa_1 \kappa_5 + \kappa_2 \kappa_6 = \frac{\theta}{4} (1 + q^2),
\tag{3.16}
\]
\[
\kappa_1 \kappa_3 + \kappa_2 \kappa_4 = \frac{\hbar}{4} (1 + q^2),
\tag{3.17}
\]
\[
\kappa_5 \kappa_7 + \kappa_6 \kappa_8 = -\frac{\hbar}{4} (1 + q^2),
\tag{3.18}
\]
\[
\kappa_3 \kappa_7 + \kappa_4 \kappa_8 = 0,
\tag{3.19}
\]
in order to ensure that the limit \( q \to 1 \) for relations (3.12)–(3.15) will yield the standard commutation relations for noncommutative flat spacetime (2.1). Relations (3.8) and (3.9) remain of course unchanged.

3.1. Some special limits

Keeping all the constants generic in the algebra (3.12)–(3.15) will make the handling very cumbersome. However, using the fact that we still have four \( \kappa s \) free at our disposal allows us to extract some special limiting cases in order to obtain some more tractable algebras.

3.1.1. Dependent X- and Y-directions. Considering (3.2) the first natural limit is to reduce the number of free parameters to 4, e.g. \( \kappa_1, \ldots, \kappa_4 \), and introduce some dependence for the coefficients in the \( Y \)-direction and those in the \( X \)-direction. Considering the representation (3.3) we impose
\[
\kappa_5 = \kappa_1, \quad \kappa_6 = -\kappa_2, \quad \kappa_7 = -\kappa_3 \quad \text{and} \quad \kappa_8 = \kappa_4,
\tag{3.20}
\]
such that without activating the constraints (3.16)–(3.19) the eight unknown constants are already limited to 4. The four constraints (3.16)–(3.19) are not independent for these choices as (3.17) and (3.18) become identical. The remaining three constraints read
\[
\kappa_1^2 - \kappa_2^2 = \frac{\theta}{4} (1 + q^2), \quad \kappa_1 \kappa_3 + \kappa_2 \kappa_4 = \frac{\hbar}{4} (1 + q^2) \quad \text{and} \quad \kappa_3^2 = \kappa_2^2,
\tag{3.21}
\]
which means that we have still one constant at our disposal. The algebra (3.12)–(3.15), (3.8) and (3.9) simplifies to
\[ [X, Y] = i\theta + i\frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_1\kappa_4 - \kappa_2\kappa_3}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (X^2 + Y^2) - \frac{2\kappa_1\kappa_2}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (XP_y - YP_x) \right], \quad (3.22) \]

\[ [X, P_x] = \hbar + i\frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (X^2 + Y^2) + \frac{\kappa_2}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (P_x^2 + P_y^2) \right], \quad (3.23) \]

\[ [Y, P_y] = \hbar + i\frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (X^2 + Y^2) + \frac{\kappa_1}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (P_x^2 + P_y^2) \right], \quad (3.24) \]

\begin{align*}
[X, P_y] &= -\frac{q - q^{-1}}{q + q^{-1}} \left[ \frac{\kappa_1\kappa_4 - \kappa_2\kappa_3}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (P_x^2 + P_y^2) - \frac{2\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3} (XP_y - YP_x) \right], & (3.25) \\
[X, P_x] &= 0, & (3.26) \\
[Y, P_y] &= 0. & (3.27) 
\end{align*}

The conditions \( \lambda \neq 0, \mu \neq 0 \) now coincide and have translated into \( \kappa_1\kappa_4 + \kappa_2\kappa_3 \neq 0 \). Our choice of constants has achieved that the terms \( XP_y \) and \( YP_x \) have combined into the angular momentum operator \( L_z \).

### 3.2. Membrane- and string-type relations

As one of the \( \kappa \) is still not fixed we can simplify the commutation relations (3.22)–(3.27) further by setting \( \kappa_2 = 0 \), such that all three unknown left are fixed by the remaining three relations

\[ \kappa_4^2 = \frac{\theta}{4} (1 + q^2), \quad \kappa_1\kappa_3 = \frac{\hbar}{4} (1 + q^2) \quad \text{and} \quad \kappa_3^2 = \kappa_4^2. \quad (3.28) \]

We may now implement the constraints (3.28) in the algebra (3.22)–(3.27) and eliminate all constants \( \kappa_i \) being left with a purely \( q \)-deformed algebra:

\[ [X, Y] = i\theta + i\frac{q - q^{-1}}{q + q^{-1}} (X^2 + Y^2), \quad (3.29) \]

\[ [X, P_x] = i\hbar + i\frac{q - q^{-1}}{q + q^{-1}} \theta (X^2 + Y^2), \quad (3.30) \]

\[ [Y, P_y] = i\hbar + i\frac{q - q^{-1}}{q + q^{-1}} \theta (X^2 + Y^2), \quad (3.31) \]

\[ [P_x, P_y] = i\frac{q - q^{-1}}{q + q^{-1}} \left[ P_x^2 + P_y^2 + 2\frac{\hbar}{\theta} (XP_y - YP_x) \right], \quad (3.32) \]

\[ [X, P_y] = 0, \quad (3.33) \]

\[ [Y, P_x] = 0. \quad (3.34) \]

These relations reduce to (2.11) for \( q = \pm \sqrt{(1 + \tau)/(1 - \tau)} \). Note further that the \( q \)-deformation and the \( \theta \)-deformation originally introduced in the space–space commutation relations have become intrinsically linked through the constraints. We can no longer take the limit \( \theta \to 0 \) separately without taking also the limit \( q \to 0 \). However, the limit \( q \to 0 \) may still be taken separately and we recover (2.1).

We named these relations ‘membrane type’ as relation (3.29) will give rise to a minimal length in the \( X \)- and \( Y \)-directions in a simultaneous measurement which we will explain in more detail below. As it stands, relation (3.29) will lead to the same minimal length in either
direction. This by no means unavoidable and can be overcome by taking another limit of the algebra (3.12)–(3.15), (3.8) and (3.9). Setting for instance \( \kappa_2 = \kappa_6 = 0 \) without any additional constraints besides (3.16)–(3.19), which in this case read

\[
\kappa_1 \kappa_5 = \frac{\theta}{4} (1 + q^2), \quad \kappa_1 \kappa_3 = \frac{\hbar}{4} (1 + q^2), \quad \kappa_5 \kappa_7 = -\frac{\hbar}{4} (1 + q^2), \quad \kappa_3 \kappa_7 = -\kappa_4 \kappa_8,
\]

the algebra simplifies considerably

\[
[X, Y] = i\theta + i \frac{q - q^{-1}}{q + q^{-1}} \left( \frac{\kappa_5}{\kappa_1} X^2 + \frac{\kappa_1}{\kappa_5} Y^2 \right),
\]

\[
[X, P_x] = i\hbar + i \frac{q - q^{-1}}{q + q^{-1}} \left( \frac{\kappa_3}{\kappa_1} X^2 + \frac{\kappa_1 \kappa_3}{\kappa_5^2} Y^2 \right),
\]

\[
[Y, P_y] = i\hbar - i \frac{q - q^{-1}}{q + q^{-1}} \left( \frac{\kappa_5 \kappa_7}{\kappa_1} X^2 + \frac{\kappa_7}{\kappa_5} Y^2 \right),
\]

\[
[P_x, P_y] = -\frac{q - q^{-1}}{q + q^{-1}} \left[ \kappa_4 \kappa_7 + \kappa_5 \kappa_8 \right] \left( \frac{\kappa_7}{\kappa_4 \kappa_5} X^2 + \frac{\kappa_3}{\kappa_4 \kappa_5} Y^2 \right) + \frac{\kappa_5}{\kappa_4} P_x^2 + \frac{\kappa_7}{\kappa_8} P_y^2 - 2 \frac{\kappa_4 \kappa_7}{\kappa_1 \kappa_8} X P_y - 2 \frac{\kappa_3 \kappa_8}{\kappa_4 \kappa_5} X P_x.
\]

\[
[X, P_x] = 0,
\]

\[
[Y, P_y] = 0.
\]

We note that in (3.36) we have now different coefficients in front of the \( X^2 \)- and \( Y^2 \)-terms and may achieve unequal minimal length in either direction, although they are not entirely independent being related by the first relation in (3.35).

Taking now a less trivial limit, we may obtain string-like relations from (3.36)–(3.41) similar to those proposed in [20]. Parameterizing \( q = e^{2\tau \kappa_5} \) with \( \tau \in \mathbb{R}^+ \) and taking the limit \( \kappa_5 \to 0 \) we obtain yet simpler relations. As we have still many free parameters left in (3.39) we have several choices. With respect to the constraints (3.35) we can take for instance \( \kappa_3 = \hbar / \theta \kappa_5, \kappa_4 = \hbar^2 / \theta \kappa_5, \kappa_8 = (1 + q^2) / (4 \kappa_5) \) and derive the simple ‘string-type’ relations

\[
[X, Y] = i\theta (1 + \tau Y^2), \quad [X, P_x] = i\hbar (1 + \tau Y^2), \quad [X, P_y] = 0,
\]

\[
[P_x, P_y] = i\tau \frac{\hbar^2}{\theta} Y^2, \quad [Y, P_y] = i\hbar (1 + \tau Y^2), \quad [Y, P_x] = 0.
\]

Arguing in the same way as in [20], we obtain now from the first relation in (3.42) a minimal length in the \( Y \)-direction in a simultaneous \( X, Y \)-measurement as the commutator \([X, Y]\) is identical. The remaining commutators are, however, different.

There are of course plenty of other possible limits compatible with the constraints (3.16)–(3.19), which we do not present here.

4. Minimal areas and minimal lengths

As mentioned, one of the interesting physical consequences of noncommutative spacetime, especially when it is dynamical, is the emergence of minimal lengths in simultaneous measurements of two observables. The standard noncommutative spacetime relations (2.1) give rise to additional uncertainties similar to the usual Heisenberg uncertainty relations, meaning for instance that the two position operators \( x_0 \) and \( y_0 \) can never be known with

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complete precision at the same time, where \( \theta \) plays the role of \( \hbar \) when compared with the conventional spacetime relations. When the underlying algebra becomes a dynamical noncommutative spacetime structure, the consequences are more severe and one finds that the position operators \( X \) or \( Y \) can never be known, that is even when giving up the entire knowledge about the canonical conjugate partner \( Y \) or \( X \), respectively. Thus \( X \) or \( Y \) are said to be bound by some absolute minimal length \( \Delta X_0 \) or \( \Delta Y_0 \), which is the highest possible precision to which these quantities can be resolved.

Minimal lengths have been known and studied for some time [9–11, 14–18] in simultaneous \( x, p \)-measurements as a consequence of a deformation of the \( x, p \)-commutator. In [20] it was demonstrated explicitly that they also result in simultaneous \( x, y \)-measurements as a consequence of the dynamical noncommutativity of spacetime. Whereas the algebra investigated in [20] only gave rise to a minimal length in one direction, i.e. ‘string like’ objects, we demonstrate here that the algebras provided in section 3 will lead to minimal lengths in two directions, i.e. minimal areas. Objects in these types of spaces are ‘membrane like’, meaning that there exists a finitely extended region about whose substructure it is impossible to obtain any measurable knowledge.

Following the standard arguments we will now compute these quantities by starting with the well-known relation

\[
\Delta A \Delta B \geq \frac{1}{2} |<[A, B]>|,
\]

which holds for any two observables \( A \) and \( B \), which are Hermitian with respect to the standard inner product. In order to determine the range of validity for this inequality we simply have to minimize \( f(\Delta A, \Delta B) := \Delta A \Delta B - \frac{1}{2} |<[A, B]>| \) as a function of \( \Delta B \) to find the absolute minimal length \( \Delta A_0 \). This means we need to solve the two equations \( \partial_{\Delta B} f(\Delta A, \Delta B) = 0 \) and \( f(\Delta A, \Delta B) = 0 \) for \( \Delta A =: \Delta A_{\text{min}} \) and subsequently compute the smallest value for \( \Delta A_{\text{min}} \) in order to obtain the absolute minimal length \( \Delta A_0 \). In case we obtain a minimal length for both these observables, we define the minimal area and its smallest possible value of four times the product, that is \( \Delta(AB)_{\text{min}} \) and \( \Delta(AB)_0 \), respectively.

For definiteness we choose now \( \theta \in \mathbb{R}^+ \) and carry out the analysis for the algebra (3.36)—(3.41) starting with a simultaneous \( X, Y \)-measurement. When \( q^2 > 1 \), the imaginary parts of all terms of the commutator \([X, Y]\) are positive due to the first constraint in (3.35). The absolute value for \(|<[X, Y]>|\) is therefore simply \( \text{Im} \, |<[X, Y]>| \). When \( q^2 < 1 \), we use \(|A - B| \geq A - B \) for \( A, B > 0 \) to drop the absolute value. Using furthermore that the mean-squared deviation about the expectation value \( \langle X \rangle \) is given by \( \Delta X^2 = \langle X^2 \rangle - \langle X \rangle^2 \) and similarly for \( X \leftrightarrow Y \), we compute

\[
\Delta X_{\text{min}} = \frac{\sqrt{2q^2 - 1} \left(k_1^2 \langle X \rangle^2 + k_2^2 \langle Y \rangle^2\right) + \theta (q^4 - 1) k_1 k_5}{2q k_5}, \quad (4.2)
\]

\[
\Delta Y_{\text{min}} = \frac{\sqrt{2q^2 - 1} \left(k_1^2 \langle X \rangle^2 + k_2^2 \langle Y \rangle^2\right) + \theta (q^4 - 1) k_1 k_5}{2q k_1}, \quad (4.3)
\]

such that the absolute minimal lengths result to

\[
\Delta X_0 = \frac{k_1}{q} \sqrt{|q^2 - 1|} \quad \text{and} \quad \Delta Y_0 = \frac{k_5}{q} \sqrt{|q^2 - 1|}; \quad (4.4)
\]

hence \( \langle X \rangle = \langle Y \rangle = 0 \). Together with the first constraint in (3.35) the absolute minimal area in the \( X, Y \)-plane results to

\[
\Delta(XY)_0 = \theta |q^2 - q^{-2}|. \quad (4.5)
\]
This means the size of the minimal area is independent of the free parameters \( \kappa_1 \) and \( \kappa_5 \). We can also make \( \Delta Y_0 \) a function of \( \Delta X_0 \) and compute for given \( \Delta X_0 \) the corresponding minimal length \( \Delta Y_0 \) or vice versa. Note that it is impossible to achieve any of the minimal lengths to vanish without the other becoming infinitely large. We illustrate this in figure 1, where we plot \( \Delta Y_0(\Delta X_0) = \pm \theta |q^2 - q^{-2}|/(4\Delta X_0) \) for a specific value of \( \theta \) and various values of \( q \). The two minimal areas indicated in the figure have the same size.

For a simultaneous \( X, P_x \)-measurement we compute similarly the minimal momentum in the \( X \)-direction:

\[
(\Delta P_x)_{\text{min}} = \sqrt{(q^2 - 1)^2((Y)^2 + (Y^2))\kappa_3^2 \kappa_2^2 + \hbar |q^2 - 1|\kappa_3 \kappa_2^2 + (X)^2(q^2 - 1)^2\kappa_3 \kappa_2^2}
\]

such that the corresponding absolute value turns out to be

\[
(\Delta P_x)_0 = 2\kappa_3 \frac{\sqrt{|q^2 - 1|}}{q^2 + 1}.
\]

There is no minimal length for \( X \) in this case as we can tune \( \Delta X \) to be as small as we wish by enlarging \( \Delta P_x \).

Similarly we compute for a simultaneous \( Y, P_y \)-measurement the minimal momentum in the \( Y \)-direction:

\[
(\Delta P_y)_{\text{min}} = \sqrt{(q^2 - 1)^2((X)^2 + (X^2))\kappa_7^2 \kappa_5^2 + \hbar |q^4 - 1|\kappa_5 \kappa_7^2 + (Y)^2(q^2 - 1)^2\kappa_5 \kappa_7^2}
\]

with a corresponding absolute value

\[
(\Delta P_y)_0 = 2\kappa_7 \frac{\sqrt{|q^2 - 1|}}{q^2 + 1}.
\]
By the same reasoning as in the previous case there is also no minimal length for \( Y \) in this case as \( \Delta Y \) can be taken to be as small as desired by enlarging \( \Delta P_y \).

The analysis for a simultaneous \( P_x, P_y \)-measurement is less straightforward due to the appearance of the angular momentum term. We first note that

\[
|\{P_x, P_y\}| \geq \left| \frac{q^2 - 1}{q^2 + 1} \right| \left( |k_4 k_7 + k_3 k_8| \left( \frac{k_7}{k_8 k_1^2} \langle X^2 \rangle - \frac{k_3}{k_4 k_2^2} \langle Y^2 \rangle \right) + \frac{k_8}{k_4} \langle p_x^2 \rangle + \frac{k_4}{k_8} \langle p_y^2 \rangle - 2 \frac{k_4 k_7}{k_1 k_8} \langle Y P_x \rangle - 2 \frac{k_3 k_8}{k_4 k_5} \langle X P_y \rangle \right),
\]

(4.10)

where for definiteness we assumed that \( k_3^2 < k_4^2 \). Using next the estimate \(|\langle AB\rangle| \leq \Delta A \Delta B + |\langle A \rangle \langle B \rangle|\) we compute

\[
\Delta P_x \Delta P_y \geq \frac{1}{2} \left| \frac{q^2 - 1}{q^2 + 1} \right| \left( \frac{k_8}{k_4} \Delta P_x^2 + \frac{k_4}{k_8} \Delta P_y^2 - 2 \frac{k_4 k_7}{k_1 k_8} \Delta Y \Delta P_x - 2 \frac{k_3 k_8}{k_4 k_5} \Delta X \Delta P_y + \lambda \right),
\]

(4.11)

with

\[
\lambda = \frac{k_8}{k_4} \langle P_x \rangle^2 + \frac{k_4}{k_8} \langle P_y \rangle^2 + |k_4 k_7 + k_3 k_8| \left( \frac{k_7}{k_8 k_1^2} \langle X^2 \rangle - \frac{k_3}{k_4 k_2^2} \langle Y^2 \rangle \right) - 2 \frac{k_4 k_7}{k_1 k_8} |\langle P_y \rangle| - 2 \frac{k_3 k_8}{k_4 k_5} |\langle P_y \rangle|.
\]

(4.12)

When varying inequality (4.11) in the same manner as the above expressions, we find

\[
(\Delta P_x)_{\text{min}} = -\left| \frac{q^2 - 1}{q^2 + 1} \right| \frac{k_8}{k_4} \Delta X - \left( \frac{q^2 - 1}{q^2 + 1} \right) \frac{k_4 k_7}{k_1 k_8} \frac{k_4}{k_8} \Delta Y
\]

\[
\pm \frac{|q^2 - q^2|}{4} \left\{ \frac{k_4}{k_8} \Delta X^2 + \frac{k_2 k_4^2}{k_1^2 k_8^2} \Delta Y^2 + \frac{2 |x_4 x_1|}{k_5} |k_1 \Delta X \Delta Y|}{k_5 |q^2 - 1| (q^2 + 1)^{-1} + \frac{4 q^2 \lambda_4}{k_5 (q^2 - 1)^2}}.
\]

(4.13)

and

\[
(\Delta P_y)_{\text{min}} = -\left( \frac{q^2 - 1}{q^2 + 1} \right)^2 \frac{k_3}{k_1 k_2 k_5} \Delta X - \left( \frac{1 - q^4}{4 q^2} \right) \frac{k_4 k_7}{k_1 k_8} \frac{1}{k_1 k_3 k_5} \Delta Y
\]

\[
\pm \frac{|q^2 - q^2|}{4} \left\{ \frac{k_4}{k_8} \Delta X^2 + \frac{k_2 k_4^2}{k_1^2 k_8^2} \Delta Y^2 + \frac{2 |x_4 x_1|}{k_5} |k_3 \Delta X \Delta Y|}{k_5 (q^2 - 1)^2 |1 - q^4|^{-1} + \frac{4 q^2 \lambda_8}{k_4 (q^2 - 1)^2}}.
\]

(4.14)

We can minimize this expression further with a subsequent \( X, Y \)-measurement. This is, however, a matter of interpretation if one would like to view measurements as a pairwise succession or whether this should be considered as a simultaneous measurement of four quantities. A further option would be to exploit the explicit occurrence of the \( L_z \)-operator and take this complication here as a hint that the angular momentum variables are possibly a more natural set of variables. We leave this problem for future investigations. Similar expressions are obtained for the choice \( k_3^2 > k_4^2 \).

5. Conclusions

Minimal lengths and minimal areas result when the \([x, y]\)-commutator involves a position-dependent structure constant in one or two directions, respectively. Taking such types of
noncommutative spacetime relations as a starting point the closure and consistency of the Jacobi identities leads to deformations of the remaining commutation relations involving other phase space variables. On the basis of two simple examples representing each of these cases in sections 2.2 and 2.3, we demonstrated that it is almost unavoidable that the corresponding oscillator algebras will be deformed. Since some of the resulting deformed oscillator algebras are rather awkward, we reverse the logic in the following section and start with those being ‘nice’, by which we mean that they are well-studied oscillator algebras with the useful physical property that the entire Fock spaces associated with them is explicitly constructable. We derived some very general commutation relations (3.12)–(3.15) for the dynamical variables. Since these relations are rather cumbersome, we investigated some specific limits leading to simplified and more tractable variants whose properties can be discussed more transparently. All of these special limits led to minimal lengths in a two-dimensional space and mostly to minimal areas which we have calculated explicitly (4.5).

There are some obvious further problems following from our considerations. First of all it would be very interesting to explore the consequences of taking different types of deformations as starting points and derive the resulting dynamical commutation relations. Secondly it would be interesting to consider explicit models on these types of spacetime structures and thirdly but not last a generalization to a three-dimensional space would be highly interesting. The latter will almost unavoidably lead to minimal volumes.

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