Emergence of Cooper pairs, d-wave duality and the phase diagram of cuprate superconductors

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BCS theory describes the formation of Cooper pairs and their instant “Bose condensation” into a superconducting state. Helium atoms are preformed bosons and, in addition to their condensed superfluid state, can also form a quantum solid, lacking phase-coherence. Here we show that the fate of Cooper pairs can be more varied than the BCS or helium paradigms. In copper-oxide d-wave superconductors (dSC) Cooper pairs are non-local objects, with both center-of-mass and relative motions. As doping decreases, the center-of-mass fluctuations force a correlated dSC into a state with enhanced diamagnetism and robust but short-ranged superconducting order. At extreme underdoping, the relative fluctuations take over and two pseudogaps — “small” (charge) and “large” (spin) — emerge naturally from the theory, as Cooper pairs “disintegrate” and charge “detaches” from spin-singlet bonds. The ensuing ground state(s) are governed by antiferromagnetic rather than by superconducting correlations. The theory is used to account for recent experiments and to draw general conclusions about the phase diagram.

Recent experiments have narrowed the field of contenders for theoretical description of high-\(T_c\) cuprates while simultaneously promoting the nature of the pseudogap state to the key conceptual issue. Two basic ideas are in play: either the pseudogap is a quantum disordered d-wave superconductor, or an entirely different form of a “competing” order, originating from the particle-hole (diagonal) channel. In this paper, I show that the pseudogap in a correlated lattice dSC is both. Several recent experiments offer important clues in this respect: the observation of enhanced quantum diamagnetism in the pseudogap state of LSCO, alongside the giant Nernst effect, points to an intimate relation between the pseudogap and a quantum disordered d-wave superconductor (dSC) — there is hardly another known microscopic mechanism which can deliver diamagnetism of this magnitude. Furthermore, the angle-resolved photoemission (ARPES) in underdoped LBCO reveals a nodal d-wave type excitation spectrum inside the pseudogap, testifying to the shared origins with the superconducting state. Importantly however, the observed quantum diamagnetism terminates at very low but finite underdoping and thereupon the magnetic signal is dominated by spin response. This is in line with the ARPES data on BSCCO, infrared ellipsometry on RBCO and the most recent STM data, which are suggestive of two pseudogaps: the nodal one, reminiscent of a superconductor and decreasing at extreme underdoping, and the antinodal one, which appears to be dominated by spin correlations and remains large.

Two kinds of new results are reported: First, guided by widely accepted microscopic features of cuprates, a theory for the long-distance charge 2\(e\) sector of the pseudogap state is constructed and shown to provide quantitative understanding of the observations in, including the measured upper critical field at \(T = 0\) and the quantum vortex liquid and solid at lower fields. Furthermore, a dSC dome in the doping-magnetic field-temperature (\(x-H-T\)) phase diagram is found to be enveloped by a larger, charge 2\(e\) Cooper pairing dome, dominated by quantum fluctuations of the superconducting order parameter \(\Psi\). This larger dome collapses to \(T = 0\) at finite \(x = x_0 \sim 0.01\), while the amplitude of the microscopic spin-singlet pairing term \(\Delta_{jk}\) remains large as \(x \to 0\). Between \(x_0\) and \(x_c \sim 0.055\), the ground state is the quantum disordered dSC, the order in \(\Psi \sim \langle \psi_{\theta jk}\rangle\) preempted by free vortex-antivortex excitations of \(\Delta_{jk}\). Two distinct pairing energy scales — \(\Psi\Delta\) for the charge and \(\Delta\) for the spin sector — provide a natural explanation for “small” and “large” pseudogaps observed by various experimental probes.

Second, a sharp distinction is drawn between the dSC (off-diagonal or particle-particle) and particle-hole (diagonal) regimes of the microscopic theory in terms of the symmetry of the action for the bond phase \(\theta_{jk}\) of \(\Delta_{jk}\): in the former case this symmetry is of the XY type, while in the latter it is related to a compact gauge symmetry, and thus much larger. A quantum phase transition between the two regimes is likely at extreme underdoping \(x \lesssim x_0\). The XY regime is dominated by the center-of-mass motion of the Cooper pairs, while the relative motion fuses into low energy physics in the “compact gauge” regime, with fundamental consequences. By analogy with its more familiar s-wave (bosonic) cousin, we dub the above sequence of events — unleashed by intensifying quantum fluctuations of \(\theta_{jk}\) on approach to the Mott insulating state — a “d-wave duality”.

The starting point is a general strongly correlated microscopic Hamiltonian which almost certainly underlies the essential physics of cuprate superconductors:

\[
\hat{H} = - \sum_{\langle ij \rangle, \sigma} \hat{c}_{i \sigma}^\dagger \hat{t} \hat{c}_{j \sigma} + \sum_{\langle ij \rangle} \hat{\Delta}_{ij} [\hat{c}_{i \uparrow}^\dagger \hat{c}_{j \downarrow} - \hat{c}_{i \downarrow}^\dagger \hat{c}_{j \uparrow}] + (\text{h.c.})
+ \frac{2}{J_{\text{eff}}} \sum_{\langle ij \rangle} \hat{\Delta}_{ij} \hat{\Delta}_{ij} + \sum_i U_p \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} + \sum_{\langle i,j \rangle} W_{ij} \hat{n}_{i\uparrow} \hat{n}_{j\downarrow}. \tag{1}
\]

In \(\hat{c}_{i \sigma}^\dagger\) (\(c_{i \sigma}\)) creates (annihilates) electrons of spin \(\sigma\) on site \(i\) of the \(\text{CuO}_2\) lattice, \(\hat{n}_{i\sigma} \equiv \hat{c}_{i \sigma}^\dagger \hat{c}_{i \sigma}\), \(\hat{n} \equiv \hat{n}_{i\uparrow} + \hat{n}_{i\downarrow}\), \(\hat{t}\) is an effective hopping, \(\hat{\Delta}_{ij}\) is the Hubbard-Stratonovich “pairing” operator, the integration over which yields the superexchange term \(J_{\text{eff}} \sum_{\langle ij \rangle} \hat{S}_i \cdot \hat{S}_j - \frac{1}{2} \hat{n}_i \hat{n}_j\), and \(U_p\) is an arbitrarily large...
repulsion, a purely mathematical tool introduced to suppress double occupancy [10]. \{W_{ij}\} are the extended-range (nearest, next-nearest,...) Coulomb and possibly phonon-induced interactions. At half-filling (\(x = 0\)), \(\hat{n}_i \equiv 1\) and all but the “pairing” terms in (1) drop out; the result is the \(S = \frac{1}{2}\) Heisenberg Hamiltonian and the ground state is the Neel-Mott antiferromagnet (AF). At \(x > 0\), the precise nature of the ground state of strongly interacting Hamiltonians (1) is unknown but we assume a domain of its parameters where a correlated BCS-type d-wave superconductivity is stabilized, as established in a myriad of experiments [1].

We anchor our analysis to the off-diagonal saddle point of \(\{\Delta_{jk}\} \equiv |\Delta_{jk}| = \Delta_d = \pm \Delta\) at \((x,y)\) bonds. As \(x\) decreases toward \(x_c\) and beyond, the increasing quantum disorder in \(\theta_{jk}\) progressively destabilizes the dSC ground state. Nevertheless, we persist in following this progression all the way to the Heisenberg AF at \(x = 0\), which, in a sense precisely defined below, is an “infinitely” strongly quantum fluctuating dSC.

To follow this path of growing fluctuations in \(\theta_{jk}\), we integrate out the fermions in (1) about the above saddle point. This is (much!) easier said than done: the presence of gapless nodal fermions in a d-wave superconductor makes the resulting action severely non-analytic. Still, as long as we stay focused on physics intrinsically tied to Cooper pairs and steer clear of nodal quasiparticles (or gapless spinons discussed in [3]), the following (bond) phase-only action for the correlated dSC captures the essentials:

\[
S_{XY}^\text{d} = \int \mathcal{D}\varphi_i \exp(-\int dt \mathcal{L}_{\text{XY}}^\text{d}),
\]

\[
\mathcal{L}_{\text{XY}}^\text{d} = i \sum_i f_i \varphi_i + \frac{\kappa_0}{2} \sum_{i,j} \varphi_i^2 - J \sum_{nn} \cos(\varphi_i - \varphi_j) - J_1 \sum_{nnn} \cos(\varphi_i - \varphi_j) - J_2 \sum_{bnnn} \cos(\varphi_i - \varphi_j) - \sum_{\square} K \cos(\square \theta) + \mathcal{L}_{\text{nodal}}[\cos(\varphi_i - \varphi_j)] + \mathcal{L}_{\text{core}}.
\] (2)

In [2], \{\varphi_i\} are identical to the bond phases \{\theta_{jk}\} of \{\Delta_{jk}\}, with subscript \(i\) referring to the center of each bond \((jk)\), the first term is the charge \(2e\) Berry phase, \(nn\) and \(r(b)nn\) are nearest and (inequivalent!) next-nearest neighbors on the \{i\} lattice, \(\square\) indicates plaquettes of the CuO_2 lattice, and \(\theta \equiv \theta_{12} - \theta_{23} + \theta_{34} - \theta_{41}\) around each plaquette. The meaning of (2) and its relation to (1) are detailed in [2] and in Fig. 1. We will use (2), and lean on (1) when necessary, to analyze the low-energy physics of cuprates.

To extract the long distance effective theory of quantum fluctuations in \{\exp[i\varphi_i] \equiv \exp[i\theta_{ij}]\}, we introduce two real Hubbard-Stratonovich fields, \{\Pi_{ij}(\tau)\} and \{Q_{ij}(\tau)\}, to decouple terms like \(\exp\left(J \sum_{(i,j)} \cos(\varphi_i - \varphi_j)\right)\) in (2) as

\[
\prod_i \int d\Pi_i dQ_i \exp\left[2 \sum_{ij} (\Pi_{ij} \cos \varphi_i + Q_i \sin \varphi_i) - \sum_{i,j} (\Pi_{ij}, Q_i) \langle i | J \sum_{\alpha=x,y} \cos k_{\alpha} a^{-1} | j \rangle (\Pi_{j}, Q_j)^T \right].
\] (3)

Letting \(\gamma_{\tau} = \kappa_0/8J^2\tau^2\), \(\gamma = \alpha g^2 J^2 \gamma_{d\tau}^2\), \(\alpha = (1/2Jd\tau_{v})^{-1}\), \(J_d = J + 1/2J_1 + 1/2J_2\), and \(J = J_1 + J_2\). Note that, consistent with our strategy, \(\mathcal{L}_{\text{nodal}}\) produces only higher (third order, non-analytic) derivatives in (5).

The physical meaning of (5) is important: \(\mathcal{L}_{\text{QGL}}^d\) represents the low energy, long-distance charge \(2e\) sector of (1) in the regime of its parameters and doping characterized by strong quantum superconducting fluctuations. While the procedure that led us to (5) from (2), via (3, 4), bears a superficial resemblance to the standard Gor’kov’s derivation of the
Ginzburg-Landau functional from the BCS theory, the actual physics is different: by setting $\Psi$ to a complex constant in the standard Gor’kov-Ginzburg-Landau functional and focusing on the quadratic term, one is probing the weak-coupling instability of the electronic system toward formation of Cooper pairs, i.e., a finite complex $\Delta$, accompanied by full phase coherence. Since the pairing susceptibility of a typical normal system diverges as $T \to 0$, such instability is always present as long as there is an attractive coupling constant in a given angular momentum channel. In contrast, $\Psi$ in (5) is testing for phase order in $\exp(i\theta_j)$ in the system in which large $|\Delta_{jk}| \neq 0$ is already established. The susceptibility to such phase order is typically finite, something the reader can easily check by using (3) and applying our procedure to the simple quantum XY model [16]. This fact of much theoretical importance constitutes the foundation of various dual descriptions of strongly correlated systems [7][12][13]: a single effective action (2) describes both ordered and quantum disordered ground states -- depending on the strength of its coupling constants -- just like the original microscopic Hamiltonian. Obviously, (5) as it follows from (2) is only a coarse-grained caricature of (1). Nevertheless, its key features faithfully reflect general properties of the underlying microscopic theory.

The first among these are Mott correlations which, as $x \to 0$, suppress double occupancy and drive kinetic energy toward zero. This implies $J_d(x) \to 0$: it becomes impossible to establish phase coherence between $\Delta_{ij}$ on different bonds (Fig. 1). This is reflected in $\alpha(x \to 0) \to \infty$ [5]. In the language of [5] $x = 0$ acts as an infinite “temperature” and thus the familiar Heisenberg AF at half-filling is “infinitely far” removed from a dSC (and vice versa!). This statement is made more precise from the standpoint of the microscopic theory (1) later in the text.

While the exact calculation of $t^*(x)$, $\Delta_d(x)$, and $J_d(x)$ is beyond reach, the above trend is manifest in various approximations ($t^* \sim x t$, $\Delta_d \sim J_{\text{eff}}$ as $x \to 0$ [5][17][18]). In any case, the precise form of $J_d(x)$ is not needed for our considerations: we can expand $\alpha(x) \approx \alpha_0(x_0 - x)$ about $\alpha(x = x_0) = (1/2) J_d(x_0) \tau_v(x_0) - 1 = 0$, the above arguments clearly implying that $x_0$ is finite. A reasonable estimate gives $x_0 \sim \Delta_d/t$, where $t$ is the bare hopping.

With these simplifications, we use (5) to address certain basic features of the cuprates’ $x$-$H$-$T$ phase diagram. In uniform $H$, we follow Abrikosov and expand the quadratic terms in $\left(\Psi, J_{\text{eff}} \right)$ in charge 2e Landau levels (LLs) (distinct from single electron LLs!) $\Phi(r, \tau) = \sum_0 \Psi_0(r, \tau), j = 0, 1, 2, \ldots$

$$\mathcal{L}_{\text{GL}}^{d} \to -\sum_{j=0} \left[ \gamma_j |\Psi_j|^4 + \alpha_j(x, H, |\Psi_j|^2) + \frac{1}{4} |\Psi|^4 \right],$$

where $\alpha_j = \alpha_0(x_0 - x + (2j + 1)H/\Delta_d)$. $\alpha_j=0(x, H) = 0$ determines the upper critical field

$$H_{c2}(x, T \to 0) = H_{c2}^0(x - x_0), \quad H_{c2}^0 = \alpha_0/4\pi e\gamma.$$

In Fig. 2(a) we compare (7) to (6) and find good agreement with the choice $x_0 \approx 0.01$ and $H_{c2}^0 \approx 9.4$ Tesla/% of doping.

The upper critical field (7) is a mean-field concept – it describes the line in the $H$-$x$ phase diagram along which $S_{\text{GL}}^{d} \Phi(r, \tau)$ has an absolute minimum for a single configuration of $\Phi(r, \tau) = \Psi_{\text{mf}}(r)$. At $H = 0$, $\Psi_{\text{mf}} = \sqrt{2\alpha_0(x - x_0)} \neq 0$, implying $\langle \exp(i\theta_j) \rangle \neq 0$, for $x > x_0$. For $x < x_0$, however, the minimum of $S_{\text{GL}}^{d}$ is at $\langle \exp(i\theta_j) \rangle = 0$. Thus, for low $x$, the above mean-field theory of $S_{\text{GL}}^{d}$ is intrinsically different from both the standard BCS theory and the slave boson mean-field approximation and the related Gutzwiller-projected BCS wavefunction [3][17][18].

This difference stems from an important physical information captured by (5): in a BCS-style weak-coupling theory, it is the kinetic energy part of (1) that is large at the point of the quantum phase transition, while the amplitude $\Delta_d \to 0$. Thus, the moment the self-consistent solution $\Delta_d \neq 0$ appears, the phase coherence between different bonds is automatically established. $\Psi_{\text{mf}}$ is immediately finite and proportional to $\Delta_d$. Incidentally, this is the likely physics behind the closing of the dSC dome on the overdoped side of the phase diagram in Fig. 2(b), at $x = x_{uc}$. Beyond $x_{uc}$, there are no superconducting fluctuations in the ground state; the system is in its underlying normal state, most likely the Fermi liquid.

In the underdoped regime, near $x_0$, the situation is reversed. At low $x$, we are dealing with rapidly growing Mott correlations and the ensuing suppression of coherent motion of $2e$ charge. Now, it is the kinetic energy that is becoming small while the spin-singlet pairing $|\Delta|$ remains large. However, the superconductivity now does not appear the moment $t^* \neq 0$ -- in a strongly correlated system the kinetic energy must rise to a finite fraction of $\Delta_d$ before even a mean-field solution $\Psi_{\text{mf}} \neq 0$ becomes possible at $x = x_0$. This crucial feature of the microscopic theory (1) is faithfully captured by Eq. (5) and the mean-field approximation $\Psi(\rho, \tau) \to \Psi_{\text{mf}}$. For $x < x_0$, even though $\Delta_d$ remains large ($\sim J_{\text{eff}}$), $\Psi_{\text{mf}}$ collapses to zero (Fig. 2), reflecting enormous quantum fluc-

![Figure 2](image-url)
tations in \( \{ \theta_{ij} \} \), unrestrained by the coherence between different bonds. In contrast, the slave boson and the Gutzwiller-projected mean-field theories \( [5, 17, 19] \) produce a dSC the moment kinetic energy, \( \sim x^2 \), becomes finite.

How are the above mean-field arguments generalized to \( T \neq 0 \)? Various coefficients in \( [5] \) have temperature dependencies that are difficult to determine directly from a strongly interacting theory \( [1] \). Again, a simple Gaussian correction to the quadratic term in \( [6] \alpha_j \sim \alpha_j(x, H, T) \approx \alpha_0(x_0 - x + (2 + 1)H/H_x^2 + T^2/\Theta^2) \) will suffice for our needs and gives

\[
H_{22}(x, T) = H_{22}(x - x_0 - T^2/\Theta^2).
\]

A corollary of \( [8] \) is that, at \( H = 0 \), \( T_{mf} = \Theta \sqrt{x-x_0} \ll T_{\Delta} \sim |\Delta| \) defines the line in the \( x-T \) phase diagram below which \( \Psi_{mf}(x, T) = 2 \alpha_0(x - x_0 - T^2/\Theta^2) \) becomes finite (Fig. 2(b)). The estimate from the undoped side of “Nernst dome” \( [6] \) yields \( \Theta \sim 3.7 K/\sqrt{\%} \) in LSCO.

The above mean-field transition to \( \Psi_{mf}(r) \neq 0 \) from \( \Psi_{mf}(r) = 0 \) is only a crossover: Eqs. \( [7] \) define the undoped side of the “Nernst dome” (or “diamagnetic dome”) in the \( x-H-T \) phase diagram \( [6] \) along which enhanced diamagnetism and (anti)vortex fluctuations become observable (see Fig. 2). These are but a couple of manifestations of the rise of superconducting correlations and of \( |\Psi(r, \tau)| \) in \( [5] \) sampling large local values even in absence of the long-range off-diagonal order. Actually, much is known about the fluctuations in \( [5, 6] \) (see \( [19, 20] \) and references therein). At \( H = 0 \), the quantum phase transition to the true superconducting ground state is shifted to \( x_c \), \( x_c - x_0 \sim \Theta \), to which \( \Psi_{mf}(x, T) \) depends \( \Theta \sim 0.045 \) from our fit to \( [6] \). At finite \( T \), the true superconducting transition line emanates from \( x_c \) as

\[
T_c(x) \sim \rho_s \sim \alpha_{d}(x)^{1/2} \sim \alpha_{d}(x)^{1/2} \sim |x - x_c|^{2v},
\]

where \( \rho_s(x), \alpha_{d}(x), \alpha_{d}(x), \xi_{sc}(x), \nu, \) and \( z \) are the superfluid density, penetration depth, diamagnetic susceptibility, correlation length, its exponent, and dynamical exponent, respectively, all associated with the quantum superconducting transition at \( x_c \). Eq. \( [3] \) predicts \( \nu = \nu_{XY} \approx 0.667 \) and \( z = 1 \) but a more detailed treatment \( [17] \), incorporating the Berry phase \( [2] \) can lead to different values. The regime between the mean-field crossover and the true transition is dominated by quantum vortex-antivortex fluctuations (Fig. 2(b) \( [21] \)).

Similarly, the true transition for \( H \neq 0 \) in \( [5] \) takes place only far below \( H_{22}(x, T) \) \( [6] \), at \( H_m(x, T) \). The \( x-H-T \) phase diagram between \( H_{22}(x, T) \) and \( H_m(x, T) \) is occupied by the quantum vortex liquid (Fig. 2(a)), exhibiting strong vortex fluctuations and enhanced diamagnetism. Along \( H = H_m(x, T) \), this quantum vortex liquid freezes into a solid. and any weak pinning ensures superconductivity. The shape of the \( H = H_m(x, T) \) line depends on whether one is in the “low” or “high” field regime of \( [6, 19] \). For \( H \gg H_b \), the quartic term in \( [6] \) is ineffective in mixing the LLs of the quadratic part; in the opposite limit, \( H \ll H_b \), this mixing is strong and the \( H \neq 0 \) behavior is dictated by the quantum vortex-antivortex unbinding at \( H = 0 \) \( [19, 20] \). An estimate \( [19] \) gives \( H_b \sim (Gi/3)H_c \approx 4 \) Tesla in LSCO, about midway through the fluctuation region explored in \( [6] \).

At high fields, \( H \gg H_b \), \( [6] \) is dominated by the lowest Landau level (LLL) \( \{ j \} = 0 \). \( S_{QGL} \rightarrow S_{QGL-LLL} = \int d(\tau_\tau) d^2(\tau/a) L_{QGL-LLL}[\Psi_0(r, \tau), \Psi_0(r, \tau)], \)

where

\[
L_{QGL-LLL} = \gamma |\Psi_0|^2 + \alpha_j = 0(x, H, T)|\Psi_0|^2 + \frac{1}{4}|\Psi_0|^4.
\]

with \( \alpha_j = \alpha_0(x_0 - x + H/H_x^2 + T^2/\Theta^2) \). The functional integral \( \int D\Psi \exp(-S_{QGL-LLL}) \) is confined to the LLL, \( \Psi \approx \Psi_0(r, \tau) = \Phi(\tau) \prod_i (z_i - z_i(\tau))) \exp(-|z|^2/4), \) where \( z = (x + iy)/\ell, \) \( \ell = \sqrt{2e/4H} \) is the quantum zeros (vortices) of \( \Psi(r, \tau) \). A prominent feature of \( [10] \) is that the physical quantities computed from it exhibit single-parameter scaling: ground state energy, magnetization, low T thermodynamics, etc., are all universal functions of \( g(x, H, T) = \) const. \( \times (x_0 - x + H/H_x^2 + T^2/\Theta^2)^{2/3} \) only. Various scaling functions are computed in \( [19] \). In particular, the quantum vortex liquid-solid transition in \( \{ z_i(\tau) \} \) observed in \( [6] \) takes place at \( g(x, H, T) = g_m < 0 \), where \( g_m \) is a universal number, \( g_m \approx -3 \). This leads to

\[
H_m(x, T) \sim (x - x_0 - T^2/\Theta^2)^{3/2}.
\]

Note that both \( [11] \) and \( [12] \) are convex lines while the observed \( H_m(x, T \rightarrow 0) \) is concave (Fig. 2(a)). The likely cause is pinning of quantum vortices by the CuO_2 lattice (or disorder), absent from continuum theory \( [5] \). This explanation is supported by the \( H_m(x) \) curves in \( [6] \), turning convex at a finite, but still low, \( T \sim 4 K \ll |\Delta| \) suggestive of thermal depinning and the \( H_m(x) \) lines for different \( T \) all terminating at the same \( x_c \approx 0.055 \) as \( H \rightarrow 0 \). The number of data points in \( [6] \) is insufficient for a detailed fit to \( [11, 12] \) – it would be interesting to test these predictions in further experiments.

The physics captured by \( [5] \) and discussed above reflects the center-of-mass motion of Cooper pairs. To display this more clearly consider slowly changing phase \( \theta_{jk} \), far from the cores of vortex or antivortex defects. Since electrons live on the sites of CuO_2 lattice it is convenient to define two sets of site variables: phases \( \phi_j \) as \( e^{i\phi_j} = \sum_k e^{i\theta_{jk}k}/\sum_k e^{i\theta_{jk}k} \) where \( k \) runs over four bonds emanating from site \( j \), and complex numbers \( \{ s_j \} \) as \( s_j \equiv e^{i\theta_{j1}} = e^{i\theta_{j2} + e^{i\theta_{j3}} + e^{i\theta_{j4}} + e^{i\theta_{j1}}} \) where \( \omega \) denotes a “virtual” site at the center of each (1234) plaquette of the CuO_2 lattice. Up to and including second order derivatives one can invert these definitions to find \( e^{i\theta_{jk}} = e^{i\theta_{jk}^{\Delta}} + e^{i\theta_{jk}^{\Psi}} \), where \( e^{i\theta_{jk}^{\Delta}} = (e^{i\theta_{jk}^{\Delta}} + e^{i\theta_{jk}^{\Delta}})/|e^{i\theta_{jk}^{\Delta}} + e^{i\theta_{jk}^{\Delta}}| \).
and \( \theta_{jk}^\alpha \) is the “irreducible” part of \( \theta_{jk} \) which cannot be reduced to site phases. By inserting the above expression for \( \theta_{jk} \) into (1) and simply repeating the steps that led from (2) to (5), one arrives at precisely the same phase structure as in (5), except that now \( \phi(r, \tau) \) and \( \nabla \phi(r, \tau) - (2e) \mathbf{A} \) replace the corresponding derivatives of \( \Psi \). Evidently, \( \phi_j \) plays the role of the overall center-of-mass phase of a Cooper pair. Importantly, \( \theta_{jk}^\alpha \) does not appear at the second order level in derivatives and is naturally interpreted as corresponding to the relative motion of the paired spin-singlet bonds in (1).

This brings us to the second feature that transpires from (5) and concerns extreme underdoping \( x < x_0 \); in this regime \( \alpha \to +\infty \) and off-diagonal correlations are maximally suppressed. Such limit of extreme quantum fluctuations is different for the d-wave bond phase \( \theta_{jk} \) than for the site phase in an s-wave (bosonic) XY duality [12, 13]. In the s-wave case, the dual opposite of a superconductor is a Wigner crystal of Cooper pairs. This remains an entirely center-of-mass affair, with the quantum disorder in the corresponding site phase promoted by suppressed density fluctuations of positionally ordered Cooper pairs. But off-diagonal correlations are still strong, as exemplified by quantum diamagnetism and vortex-antivortex fluctuations, and the ground state is still well described by the XY-like phase action; these properties are broadly similar to the intermediate “Quantum vorticity” state in Fig. 2(b) although there are important differences as well, like the presence of spinful nodal fermions in the latter.

By contrast, in the d-wave case, the extreme quantum disorder limit \( (x \to 0) \) of the action (2) does not have the XY symmetry: \( t^* \to 0 \) and the XY parts of (2) disappear, leaving behind only the plaquette-type action, since terms like \( K \) include only \( \Delta \)'s (Fig. 1). This demise of the XY behavior in (2) and (1) – occasioned by Mott suppression of kinetic energy – translates into collapse of off-diagonal correlations and is identified here as the microscopic physics behind the observed “two pseudogaps” in underdoped cuprates [9, 10, 11], as alluded to earlier in the text (Fig. 2(b)). To see this explicitly, we consider below the off-diagonal, “anomalous” electron propagator following from (1):

\[
F(r - r', \tau' - \tau) = \langle c_j^\uparrow \tau (r, \tau) c_j^\downarrow (r', \tau') \rangle. 
\]

Finite \( F \) implies superconducting off-diagonal long-range order.

We can actually compute \( F \) with the accuracy needed to make contact with the recent experiments [9, 10, 11]. First, within the “Quantum vorticity” state in Fig. 2(b) it suffices to keep only the leading relevant derivatives and we can replace \( e^{i\phi_j + e^{i\phi_k}} \) with \( e^{i\phi_j + e^{i\phi_k}} \), as explained above. Next, we define new fermions \( f_{\sigma} \) as \( c_{\sigma} = f_{\sigma} e^{i\phi_j} \), \( f_{\sigma} = e^{-i\phi_j} c_{\sigma} \). This leads to:

\[
F(r - r', \tau' - \tau) \approx \langle \Psi \rangle \langle f_j^\dagger (r, \tau) f_j^\dagger (r', \tau') \rangle, 
\]

accurate up to leading derivatives, where \( \langle \Psi \rangle \) is nothing but the expectation value of the superconducting order parameter computed from (5). The precise structure of the off-diagonal \( f \)-fermion propagator (13) is a complex problem, as \( f \) fermions behave as neutral d-wave quasiparticles interacting via gauge fields generated by spacetime derivatives of the phase \( \frac{1}{2} \phi \); nevertheless, for the purposes of overall energetics and at the level of resolution available in [9, 10, 11], such propagator behaves simply as – a somewhat smeared version of – the ordinary BCS d-wave anomalous propagator. Furthermore, Refs. [9, 10, 11] are sensitive only to short range off-diagonal order and thus \( \langle \Psi \rangle \) should be replaced by \( \Psi_{\text{mf}} \). After the Fourier transform we finally obtain:

\[
F(k, \omega) \approx \Psi_{\text{mf}} \frac{\Delta_k}{-\omega^2 + (v_F \cdot (k - k_F))^2 + |\Delta_k|^2} 
\]

where \( \Delta_k = \Delta \cos(2\phi_k) \), \( \phi_k \) is the angle going around the Fermi surface and \( v_F, k_F \) are the Fermi velocity and momentum, respectively. From (14) it follows naturally that \( \Psi_{\text{mf}} \Delta \) should be interpreted as the “small” pseudogap, measuring charge \( 2e \) off-diagonal correlations and collapsing near \( x_0 \), while the spin-singlet pairing gap \( \Delta \) itself remains large (Fig. 2(b)).

The above collapse of “small” pseudogap \( \Psi_{\text{mf}} \Delta \) at \( x < x_0 \) implies that vortex-antivortex pairs, the staple of XY-type physics, are displaced as relevant excitations. To see what replaces them, we transform \( c_{\sigma} \) to \( \sigma \) on one of the sublattices of the CuO\(_2\) lattice, turning the spin-singlet pairing \( \langle \Delta e^{i\varphi_j} c_{\downarrow}^\dagger \sigma_k \rangle \) into hopping \( \langle \Delta e^{i\varphi_j} c_{\downarrow}^\dagger \sigma_k \rangle \). After this transformation, \( \langle \pm \theta_{jk} \rangle \) on alternating bonds are cast in the role of a gauge field \( \{a_{jk}\} \) coupled to the staggered “charge”, switching from +1 to -1 between the two sublattices. In this language, the plaquette terms in (2) are nothing but the action for \( \{a_{jk}\} \), invariant under the compact gauge transformations \( a_{jk} \to a_{jk} + c_j - c_k \). This is a much larger symmetry than the XY one of (2) at finite \( x \) and \( t^* \). The relevant excitations of a compact gauge theory are monopoles and antimonopoles in \( \{a_{jk}\} \) – in turn, monopoles in \( \{a_{jk}\} \) are closely related to the relative motion of Cooper pairs and the bond phases \( \{\theta_{jk}^\alpha\} \) defined earlier. This is easily appreciated by observing that the plaquette term in (2) always equals \( \pm 1 \) when \( \theta_{jk} \) is restricted to \( \theta_{jk}^\alpha \) alone. Only by allowing unrestricted fluctuations of \( \theta_{jk} \) and including the relative phases \( \theta_{jk}^\alpha \), can one successfully proliferate monopoles and antimonopoles in \( a_{jk} \). With (11) serving as the microscopic model, the monopoles will be in their plasma phase and thus confining. Actually, despite apparent mathematical opaqueness of our argument, this is just one among many disguises of the traditional Heisenberg-Mott AF [3]. This opaqueness is an unavoidable consequence of our persistence in following the fate of a dSC all the way to the half-filling, the route along which the quantum fluctuations in the phase \( \theta_{jk}^\alpha \) of its gap function progressively become enormous and ultimately unrestrained. In this sense, it is the standard Heisenberg-Mott AF that stands at the far end \( (x = 0) \) of the “d-wave dual road” for correlated dSC [23].

How is the XY symmetry recovered at finite \( x \) and \( t^* \)? The \( J_1 \) term in (2) provides an illustration (Fig. 1): it moves charge \( 2e \) “Cooper pair” from one spin-singlet bond to the next, a process vital for establishing local superconducting phase coherence. In the gauge theory language this is a Higgs term:
cos(θ_{ij} - θ_{kl}) \rightarrow \cos(\alpha_{ij} + \alpha_{kl})! Thus, the charge 2e motion and the associated kinetic energy correspond to the symmetry breaking Higgs terms in the gauge theory dialect. The presence of Higgs terms suppresses free monopoles in \{a_{ij}\} and promotes its gauge sector to physical reality – the result is the conservation of vorticity in \{θ_{ij}\} and the XY symmetry of its action [2]. Thus, Copper pairs “emerge” from the underlying dynamics of (1) within the “Quantum vorticity” region in Fig. 2(b), as dual partner of well-defined vortex-antivortex excitations. One is tempted to speculate that a quantum transition between the “compact” and “XY” regimes takes place at finite x ∼ x_0, with the AF+x ground state(s) in Fig. 2(b) regulated by monopoles and antimonopoles in a_{jk} and reflecting the relative motion, instead of vortices and antivortices associated with the Cooper pair center-of-mass fluctuations in θ_{jk}. Such ground states are the true “competing orders” of a dSC, arising from the particle-hole (diagonal) channel, and likely include non-uniform AF and stripes [2,4].

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