A note on the extremal process of the supercritical Gaussian Free Field

Alberto Chiarini†  Alessandra Cipriani‡  Rajat Subhra Hazra§

Abstract

We consider both the infinite-volume discrete Gaussian Free Field (DGFF) and the DGFF with zero boundary conditions outside a finite box in dimension larger or equal to 3. We show that the associated extremal process converges to a Poisson point process. The result follows from an application of the Stein-Chen method from [5].

Keywords: Extremal process; Gaussian free field; point processes; Poisson approximation; Stein–Chen method.

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1 Introduction

In this article we study the behavior of the extremal process of the DGFF in dimension larger or equal to 3. This extends the result presented in [9] in which the convergence of the rescaled maximum of the infinite-volume DGFF and the 0-boundary condition field was shown. It was proved there that the field belongs to the maximal domain of attraction of the Gumbel distribution; hence, a natural question that arises is that of describing more precisely its extremal points. In dimension 2, this was carried out by [6, 7] complementing a result of [8] on the convergence of the maximum; namely, the characterization of the limiting point process with a random mean measure yields as by-product an integral representation of the maximum. The extremes of the DGFF in dimension 2 have deep connections with those of Branching Brownian Motion ([1, 2, 3, 4]). These works showed that the limiting point process is a randomly shifted decorated Poisson point process, and we refer to [15] for structural details. In $d \geq 3$, one does not get a non-trivial decoration but instead a Poisson point process analogous to the extremal process of independent Gaussian random variables. To be more precise, we let $\mathcal{E} := [0, 1]^d \times (-\infty, +\infty]$ and $V_N := [0, n-1]^d \cap \mathbb{Z}^d$ the hypercube of volume $N = n^d$. Let $(\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$ be the infinite-volume DGFF, that is a centered Gaussian field on the square lattice with covariance $g(\cdot, \cdot)$, where $g$ is the Green’s function of the simple random walk. We define the following sequence of point processes on $\mathcal{E}$:

$$\eta_n(\cdot) := \sum_{\alpha \in V_N} \xi \left( \varphi_{\alpha - \text{a}_N} \right)(\cdot)$$

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†TU Berlin, Germany. E-mail: chiarini@math.tu-berlin.de
‡Weierstrass Institute, Germany. E-mail: Alessandra.Cipriani@wias-berlin.de
§Indian Statistical Institute, Kolkata, India. E-mail: rajatmaths@gmail.com
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where \( \varepsilon_x(\cdot), x \in E \), is the point measure that gives mass one to a set containing \( x \) and zero otherwise, and

\[
b_N := \sqrt{g(0)} \left[ \frac{\log \log N + \log(4\pi)}{2\sqrt{2\log N}} \right], \quad a_N := g(0)(b_N)^{-1}. \quad (1.2)
\]

Here \( g(0) \) denotes the variance of the DGFF. Our main result is

**Theorem 1.1.** For the sequence of point processes \( \eta_n \) defined in (1.1) we have that

\[ \eta_n \xrightarrow{d} \eta, \]

as \( n \to +\infty \), where \( \eta \) is a Poisson random measure on \( E \) with intensity measure given by \( d t \otimes (e^{-z} \, dz) \) where \( d t \otimes dz \) is the Lebesgue measure on \( E \), and \( \xrightarrow{d} \) is the convergence in distribution on \( M_p(E) \).

The proof is based on the application of the two-moment method of [5] that allows us to compare the extremal process of the DGFF and a Poisson point process with the same mean measure. To prove that the two processes converge, we will exploit a classical theorem by Kallenberg.

It is natural then to consider also convergence for the DGFF \( (\psi_\alpha)_{\alpha \in \mathbb{Z}^d} \) with zero boundary conditions outside \( V_N \). For the sequences of point measures

\[ \rho_n(\cdot) := \sum_{\alpha \in V_N} \varepsilon \left( \psi_\alpha - \frac{b_N a_N}{2} \right)(\cdot) \quad (1.3) \]

we establish the following Theorem:

**Theorem 1.2.** For the sequence of point processes \( \rho_n \) defined in (1.3) we have that

\[ \rho_n \xrightarrow{d} \eta, \]

as \( n \to +\infty \) in \( M_p(E) \), where \( \eta \) is as in Theorem 1.1.

The convergence is shown by reducing ourselves to check the conditions of Kallenberg’s Theorem on the bulk of \( V_N \), where we have a good control on the drift of the conditioned field, and then by showing that the process on the whole of \( V_N \) and on the bulk are close as \( n \) becomes large.

The outline of the paper is as follows. In Section 2 we will recall the definition of DGFF and the Stein-Chen method, while Section 3 and Section 4 are devoted to the proofs of Theorems 1.1 and 1.2 respectively.

## 2 Preliminaries

### 2.1 The DGFF

Let \( d \geq 3 \) and denote with \( \| \cdot \| \) the \( \ell_\infty \)-norm on \( \mathbb{Z}^d \). Let \( \psi = (\psi_\alpha)_{\alpha \in \mathbb{Z}^d} \) be a discrete Gaussian Free Field with zero boundary conditions outside \( \Lambda \subset \mathbb{Z}^d \). On the space \( \Omega := \mathbb{R}^{\mathbb{Z}^d} \) endowed with its product topology, its law \( \tilde{P}_\Lambda \) can be explicitly written as

\[
\tilde{P}_\Lambda(d \psi) = \frac{1}{Z_\Lambda} \exp \left( -\frac{1}{4d} \sum_{\alpha, \beta \in \mathbb{Z}^d : \|\alpha - \beta\| = 1} (\psi_\alpha - \psi_\beta)^2 \right) \prod_{\alpha \in \Lambda} d \psi_\alpha \prod_{\alpha \in \mathbb{Z}^d \setminus \Lambda} \varepsilon_0(\psi_\alpha).
\]

In other words \( \psi_\alpha = 0 \) \( \tilde{P}_\Lambda \)-a. s. if \( \alpha \in \mathbb{Z}^d \setminus \Lambda \), and \( (\psi_\alpha)_{\alpha \in \Lambda} \) is a multivariate Gaussian random variable with mean zero and covariance \( (g_\Lambda(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}^d} \), where \( g_\Lambda \) is the Green’s

\[^1\] \( M_p(E) \) denotes the set of (Radon) point measures on \( E \) endowed with the topology of vague convergence.
function of the discrete Laplacian problem with Dirichlet boundary conditions outside \( \Lambda \). For a thorough review on the model the reader can refer for example to [16]. It is known [10, Chapter 13] that the finite-volume measure \( \psi \) admits an infinite-volume limit as \( \Lambda \uparrow \mathbb{Z}^d \) in the weak topology of probability measures. This field will be denoted as \( \varphi = (\varphi_\alpha)_{\alpha \in \mathbb{Z}^d} \). It is a centered Gaussian field with covariance matrix \( g(\alpha, \beta) \) for \( \alpha, \beta \in \mathbb{Z}^d \). With a slight abuse of notation, we write \( g(\alpha - \beta) \) for \( g(0, \alpha - \beta) \) and also \( g_\Lambda(\alpha) = g_\Lambda(\alpha, \alpha) \). \( g \) admits a so-called random walk representation: if \( \mathbb{P}_\alpha \) denotes the law of a simple random walk \( S \) started at \( \alpha \in \mathbb{Z}^d \), then

\[
g(\alpha, \beta) = \mathbb{E}_\alpha \left[ \sum_{n \geq 0} \mathbb{I}_{\{S_n = \beta\}} \right].
\]

In particular this gives \( g(0) < +\infty \) for \( d \geq 3 \). A comparison of the covariances in the infinite and finite-volume is possible in the bulk of \( V_N \): for \( \delta > 0 \) this is defined as

\[
V_N^\delta := \left\{ \alpha \in V_N : \|\alpha - \beta\| > \delta n, \forall \beta \in \mathbb{Z}^d \setminus V_N \right\}.
\] (2.1)

In order to compare covariances in the finite and infinite-volume field, we recall the following Lemma, whose proof is presented in [9, Lemma 7]).

**Lemma 2.1.** For any \( \delta > 0 \) and \( \alpha, \beta \in V_N^\delta \) one has

\[
g(\alpha, \beta) - C_d \left( \delta N^{1/d} \right)^{2-d} \leq g_{\delta N}(\alpha, \beta) \leq g(\alpha, \beta), \tag{2.2}
\]

In particular we have, \( g_{\delta N}(\alpha) = g(0) \left( 1 + O \left( N^{(2-d)/d} \right) \right) \) uniformly for \( \alpha \in V_N^\delta \).

### 2.2 The Stein-Chen method

As main tool of this article we will use (and restate here) a theorem from [5]. Consider a sequence of Bernoulli random variables \( (X_\alpha)_{\alpha \in \mathcal{I}} \) where \( X_\alpha \sim \text{Bern}(p_\alpha) \) and \( \mathcal{I} \) is some index set. For each \( \alpha \) we define a subset \( B_\alpha \subseteq \mathcal{I} \) which we consider a “neighborhood” of dependence for the variable \( X_\alpha \), such that \( X_\alpha \) is nearly independent from \( X_\beta \) if \( \beta \in \mathcal{I} \setminus B_\alpha \). Set

\[
\begin{align*}
b_1 &:= \sum_{\alpha, \beta \in B_\alpha} p_\alpha p_\beta, \\
b_2 &:= \sum_{\alpha, \beta \neq \gamma \in B_\alpha} \mathbb{E}[X_\alpha X_\beta], \\
b_3 &:= \sum_{\alpha \in \mathcal{I}} \mathbb{E}\left[ \mathbb{E}[|X_\alpha - p_\alpha| | \mathcal{H}_1] \right]
\end{align*}
\]

where

\[
\mathcal{H}_1 := \sigma \left( X_\beta : \beta \in \mathcal{I} \setminus B_\alpha \right).
\]

**Theorem 2.2** ([5, Theorem 2]). Let \( \mathcal{I} \) be an index set. Partition the index set \( \mathcal{I} \) into disjoint non-empty sets \( \mathcal{I}_1, \ldots, \mathcal{I}_k \). For any \( \alpha \in \mathcal{I} \), let \( (X_\alpha)_{\alpha \in \mathcal{I}} \) be a dependent Bernoulli process with parameter \( p_\alpha \). Let \( (Y_\alpha)_{\alpha \in \mathcal{I}} \) be independent Poisson random variables with intensity \( p_\alpha \). Also let

\[
W_j := \sum_{\alpha \in \mathcal{I}_j} X_\alpha \quad \text{and} \quad Z_j := \sum_{\alpha \in \mathcal{I}_j} Y_\alpha \quad \text{and} \quad \lambda_j := \mathbb{E}[W_j] = \mathbb{E}[Z_j].
\]

Then

\[
\| \mathcal{L}(W_1, \ldots, W_k) - \mathcal{L}(Z_1, \ldots, Z_k) \|_{TV} \leq 2 \min \left\{ 1, 1.4 \left( \min_{1 \leq j \leq k} \lambda_j \right)^{-1/2} \right\} (2b_1 + 2b_2 + b_3) \tag{2.3}
\]

where \( \| \cdot \|_{TV} \) denotes the total variation distance and \( \mathcal{L}(W_1, \ldots, W_k) \) denotes the joint law of these random variables.


3 Proof of Theorem 1.1: the infinite-volume case

Proof. We recall that \( E = [0,1]^d \times (-\infty, +\infty) \) and \( V_N = [0, n-1]^d \cap \mathbb{Z}^d \). To show the convergence of \( \eta_n \) to \( \eta \), we will exploit Kallenberg’s theorem [11, Theorem 4.7]. According to it, we need to verify the following conditions:

i) for any \( A \), a bounded rectangle\(^2\) in \([0,1]^d\), and \((x, y) \subset (-\infty, +\infty)\)

\[
E[\eta_n(A \times (x, y))] \to E[\eta((A \times (x, y))] = |A|(e^{-x} - e^{-y}).
\]

We adopt the convention \( e^{-\infty} = 0 \) and the notation \(|A|\) for the Lebesgue measure of \( A \).

ii) For all \( k \geq 1 \), and \( A_1, A_2, \ldots, A_k \) disjoint rectangles in \([0,1]^d\) and \( R_1, R_2, \ldots, R_k \), each of which is a finite union of disjoint intervals of the type \((x, y) \subset (-\infty, +\infty)\),

\[
\begin{align*}
P(\eta_n(A_1 \times R_1) = 0, \ldots, \eta_n(A_k \times R_k) = 0) \\
\to P(\eta(A_1 \times R_1) = 0, \ldots, \eta(A_k \times R_k) = 0) = \exp\left(-\sum_{j=1}^k |A_j| \omega(R_j)\right) \tag{3.1}
\end{align*}
\]

where \( \omega(dz) := e^{-z} dz \).

Let us denote by \( u_N(z) := a_N z + b_N \). The first condition follows by Mills ratio

\[
\left(1 - \frac{1}{t^2}\right) \frac{e^{-t^2/2}}{\sqrt{2\pi t}} \leq P(\mathcal{N}(0, 1) > t) \leq \frac{e^{-t^2/2}}{\sqrt{2\pi t}}, \quad t > 0. \tag{3.2}
\]

More precisely

\[
E[\eta_n(A \times (x, y))] = \sum_{\alpha \in nA \cap V_N} P(\varphi_\alpha \in (u_N(x), u_N(y)))
\]

\[
\leq \sum_{\alpha \in nA \cap V_N} \left(\frac{e^{-a_N(x)^2}}{2\pi u_N(x)^2} - \frac{e^{-a_N(y)^2}}{2\pi u_N(y)^2}\right) \left(1 - \frac{1}{u_N(y)^2}\right)
\]

\[
\leq |nA \cap V_N| \left(\frac{e^{-x+o(1)}}{N} - \frac{e^{-y+o(1)}}{N}\right) \left(1 - \frac{1}{2g(0)\log N(1 + o(1))}\right)
\]

\[
\to |A|(e^{-x} - e^{-y}). \tag{3.4}
\]

Similarly, one can plug in (3.3) the reverse bounds of (3.2) to prove the lower bound, and thus condition i).

To show ii), we need a few more details. Let \( k \geq 1 \), \( A_1, \ldots, A_k \) and \( R_1, \ldots, R_k \) be as in the assumptions. Let us denote by \( I_j = nA_j \cap V_N \) and \( I = I_1 \cup \ldots \cup I_k \). For \( \alpha \in I_j \) define

\[
X_\alpha := \left\{ \frac{u_N}{x} \in R_i \right\}
\]

and \( p_\alpha := P((\varphi_\alpha - b_N)/a_N \in R_i) \). Choose now a small \( \varepsilon > 0 \) and fix the neighborhood of dependence \( B_\varepsilon := B(\alpha, (\log N)^{2+2\varepsilon}) \) \( \cap I^3 \) for \( \alpha \in I \). Let \( W_j := \sum_{\alpha \in I_j} X_\alpha \) and \( Z_j \) be as in Theorem 2.2.

By the simple observation that

\[
P(\eta_n(A_1 \times R_1) = 0, \ldots, \eta_n(A_k \times R_k) = 0) = P(W_1 = 0, \ldots, W_k = 0),
\]

\(^2\)A bounded rectangle has the form \( J_1 \times \cdots \times J_d \) with \( J_i = [0, 1] \cap (a_i, b_i] \), \( a_i, b_i \in \mathbb{R} \) for all \( 1 \leq i \leq d \).

\(^3\)\( B(x, r) \) denotes a ball of radius \( r \) centered at \( x \).
to prove the convergence (3.1), we can use Theorem 2.2 and show that the error bound on the RHS of (2.3) goes to 0.

First we bound $b_1$ as follows. By definition of $R_1$, $R_2$, ..., $R_k$, there exists $z \in \mathbb{R}$ such that $R_j \subset (z, +\infty)$ for $1 \leq j \leq k$. Hence for any $1 \leq j \leq k$, for any $\alpha \in I_j$ we have that

$$p_\alpha = \mathbb{P} \left( \frac{\varphi - b_N}{a_N} \in R_j \right) \leq \mathbb{P}(\varphi > u_N(z)) \leq \left( \frac{e^{\frac{u_N(z)^2}{2\alpha N}}}{\sqrt{2\pi u_N(z)}} \right) g(0).$$

The bound is independent of $\alpha$ and $j$, therefore for some $C > 0$

$$b_1 \leq C N (\log N)^{d(2+2\epsilon)} e^{-2z} N^{-2} \to 0. \tag{3.5}$$

For $b_2$ note that it was shown in [9] that for $z \in \mathbb{R}$ and $\alpha \neq \beta \in V_N$

$$\mathbb{P}(\varphi > u_N(z), \varphi > u_N(z)) \leq \left( \frac{2 - \kappa}{\kappa^{1/2}} \right) N^{-2/(2-\kappa)} \max \left\{ e^{-2z} 1_{\{z \leq 0\}}, e^{-2z/(2-\kappa)} 1_{\{z > 0\}} \right\}. \tag{3.6}$$

Here we have introduced $\kappa := \mathbb{P}_0 \left( \tilde{H}_0 = +\infty \right) \in (0,1)$ and $\tilde{H}_0 = \inf \{ n \geq 1 : S_n = 0 \}$.

Observe that for any $1 \leq j \leq k$, $\alpha \in I$ and $\beta \in B_\alpha$ one has

$$\mathbb{E}[X_\alpha X_\beta] \leq \mathbb{P}(\varphi > u_N(z), \varphi > u_N(z))$$

so that by (3.6) we can find some constant $C' > 0$ such that

$$b_2 \leq C' N^{-\kappa/(2-\kappa)} (\log N)^{d(2+2\epsilon)} \max \left\{ e^{-2z} 1_{\{z \leq 0\}}, e^{-2z/(2-\kappa)} 1_{\{z > 0\}} \right\} \to 0.$$

The error is similar to the estimate obtained in [9, Equation (8)]. Finally we need to handle $b_3$. From Section 2.2 we set for $\alpha \in I$, $H_1 := \sigma (X_\beta : \beta \in I \setminus B_\alpha)$ and we define $H_2 := \sigma (\varphi_\beta : \beta \in I \setminus B_\alpha)$. We observe that

$$b_3 = \sum_{\alpha \in I} \mathbb{E} \left[ \mathbb{E} [X_\alpha - p_\alpha | H_1] \right] \leq \sum_{\alpha \in I} \mathbb{E} \left[ \mathbb{E} [X_\alpha | H_2] - p_\alpha \right]$$

since $H_1 \subseteq H_2$ and using the tower property of the conditional expectation. Now denote by $U_\alpha := \mathbb{Z}^d \setminus (I \setminus B_\alpha)$. Let us abbreviate $u_N(R_j) := \{ u_N(y) : y \in R_j \}$. Then for $\alpha \in I_j$ and $1 \leq j \leq k$, by the Markov property of the DGFF [14, Lemma 1.2] we have that

$$\mathbb{E} [X_\alpha | H_2] = \tilde{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) \quad \mathbb{P} - a.s.$$ 

where $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ is a Gaussian Field with zero boundary conditions outside $U_\alpha$ and

$$\mu_\alpha = \sum_{\beta \in I \setminus B_\alpha} \mathbb{P}_\alpha (H_{I \setminus B_\alpha} < +\infty, S_{H_{I \setminus B_\alpha}} = \beta) \varphi_\beta.$$ 

Here $H_\Lambda := \inf \{ n \geq 0 : S_n \in \Lambda \}$, $\Lambda \subset \mathbb{Z}^d$. Now as in [9, Equation (10)] one can show, using the Markov property, that

$$\text{Var} [\mu_\alpha] \leq \sup_{\beta \in I \setminus B_\alpha} g(\alpha, \beta) \leq \frac{c}{(\log N)^{2(1+\sigma)(d-2)}}$$

for some $c > 0$. Hence we get that there exists a constant $c' > 0$ (independent of $\alpha$ and $j$) such that

$$\mathbb{P} \left( |\mu_\alpha| > (u_N(z))^{1-\epsilon} \right) \leq c' \exp \left( - (\log N)^{(2d-5)(1+\epsilon)} \right). \tag{3.7}$$

Recalling that $R_j \subset (z, +\infty)$ for all $1 \leq j \leq k$, this immediately shows that for $d \geq 3$

$$\sum_{j=1}^k \sum_{\alpha \in I_j} \mathbb{E} \left[ \tilde{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) - p_\alpha \right] 1_{\{|\mu_\alpha| > (u_N(z))^{1-\epsilon}\}} \to 0.$$
So to show that \( b_3 \to 0 \) we are left with proving

\[
\sum_{j=1}^{k} \sum_{\alpha \in I_j} \mathbb{E} \left[ \left| \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha \in u_N (R_j)) - p_\alpha \right| \mathbb{I}_{\{ |\mu_\alpha| \leq (u_N(z))^{1-\epsilon} \}} \right] \to 0. \tag{3.8}
\]

We now focus on the term inside the summation. For this, first we write \( R_j = [r_m, \infty) \) with \(-\infty < w_1 < r_1 < w_2 < \cdots < r_m \leq \infty \) for some \( m \geq 1 \). Hence, we can expand the difference in the absolute value of (3.8) as follows:

\[
\begin{align*}
&\left( p_\alpha - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha \in u_N (R_j)) \right) \\
&= \sum_{l=1}^{m} \left( \mathbb{P} (\varphi_\alpha \in (u_N(w_l), u_N (r_l)]) - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha \in (u_N(w_l), u_N (r_l)]) \right) \\
&= \sum_{l=1}^{m} \left( \mathbb{P} (\varphi_\alpha > u_N(w_l)) - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w_l)) \right) \\
&\quad - \sum_{l=1}^{m} \left( \mathbb{P} (\varphi_\alpha > u_N(r_l)) - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(r_l)) \right) \tag{3.9}
\end{align*}
\]

(this if \( r_l = +\infty \) for some \( l \), we conventionally set \( \mathbb{P} (\varphi_\alpha > u_N (r_l)) = 0 \) and similarly for the other summand). Using the triangular inequality in (3.8), it turns out that to finish it is enough to show that for an arbitrary \( w \in \mathbb{R} \),

\[
\sum_{\alpha \in I} \mathbb{E} \left[ \left| \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w)) - \mathbb{P}(\varphi_\alpha > u_N(w)) \right| \mathbb{I}_{\{|\mu_\alpha| \leq (u_N(z))^{1-\epsilon}\}} \right] \to 0. \tag{3.10}
\]

For this, first we show that on \( Q := \left\{ \mathbb{P}(\varphi_\alpha > u_N(w)) > \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w)) \right\} \)

\[
T_{1,2} = \sum_{\alpha \in I} \mathbb{E} \left[ \left( \mathbb{P}(\varphi_\alpha > u_N(w)) - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w)) \right) \mathbb{I}_{\{|\mu_\alpha| \leq (u_N(z))^{1-\epsilon}\}} \mathbb{I}_Q \right] \to 0. \tag{3.11}
\]

This follows from the same estimates of \( T_{1,2} \) and Claim 6 of [9]. Indeed on \( Q \cap \left\{ |\mu_\alpha| \leq (u_N(z))^{1-\epsilon} \right\} \)

\[
\begin{align*}
&\sum_{\alpha \in I} \left( \mathbb{P}(\varphi_\alpha > u_N(w)) - \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w)) \right) \\
&\leq \sum_{\alpha \in I} \frac{\sqrt{g(0)} e^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi u_N(w)}} \left( 1 - (1+o(1)) \left( \sqrt{g_{U_\alpha}(\alpha) u_N(w)} e^{1-\frac{g(0)}{g_{U_\alpha}(\alpha)}} \frac{u_N(w)^2}{g(0) u_N(w)(1+o(1))} \right) \right) \\
&\leq C N \frac{\sqrt{g(0)} e^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi u_N(w)}} o(1) = o(1).
\end{align*}
\]

Similarly one can show that on the complementary event \( Q^c \) (recall (3.11) for the definition of \( Q \))

\[
T_{1,1} = \sum_{\alpha \in I} \mathbb{E} \left[ \left( \tilde{P}_{U_\alpha} (\psi_\alpha + \mu_\alpha > u_N(w)) - \mathbb{P}(\varphi_\alpha > u_N(w)) \right) \mathbb{I}_{\{|\mu_\alpha| \leq (u_N(z))^{1-\epsilon}\}} \mathbb{I}_{Q^c} \right] = o(1).
\]

This shows that \( b_3 \to 0 \). Hence from Theorem 2.2 it follows that

\[
\mathbb{P}(W_1 = 0, \ldots, W_k = 0) - \prod_{j=1}^{k} \mathbb{P}(Z_j = 0) = o(1),
\]

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having used the independence of the $Z_j$’s. Notice that by definition $Z_j$ is a Poisson random variable with intensity $\sum_{a \in I_j} P((\varphi_a - b_N)/a_N \in R_j)$. Decomposing $R_j$ as a union of finite intervals and using Mills ratio, similarly to the argument leading to (3.4), one has

$$P(Z_j = 0) \to \exp(-|A_j|\omega(R_j))$$

(recall $\omega(R_j) = \int_{R_j} e^{-z \cdot d \, z}$). Hence it follows that

$$\prod_{j=1}^k P(Z_j = 0) \to \exp \left( - \sum_{j=1}^k |A_j|\omega(R_j) \right), \quad (3.12)$$

which completes the proof of ii) and therefore of Theorem 1.1.

4 Proof of Theorem 1.2: the finite-volume case

We will now show the theorem for the field with zero boundary conditions. As remarked in the Introduction, since on the bulk defined in (2.1) we have a good control on the conditioned field, we will first prove convergence therein, and then we will use a converging-together theorem to achieve the final limit. We will first need some notation used throughout the Section: first, we consider $(\psi_{\alpha})_{\alpha \in \mathbb{V}_N}$ with law $\bar{P}_N := \bar{P}_{V_N}$. We also use the shortcut $g_N(\cdot, \cdot) = g_N(\cdot, 0)$. We will need the notation $C_K(E)$ for the set of positive, continuous and compactly supported functions on $E = [0, 1]^d \times (-\infty, +\infty]$. We first begin with a lemma on the point process convergence on bulk. Define a point process on $E$ by

$$\rho^\delta_n(\cdot) := \sum_{\alpha \in \mathbb{V}_N^\delta} \varepsilon \left( \frac{1}{2} \frac{\psi_{\alpha} - b_N}{a_N} \right)(\cdot). \quad (4.1)$$

**Lemma 4.1.** Let $\delta > 0$. On $\mathcal{M}_{p}(E)$, $\rho^\delta_n \Rightarrow \rho^\delta$ where $\rho^\delta$ is a Poisson random measure with intensity $d t_{(\delta, 1 - \delta)^d} \otimes (e^{-x \cdot d \, x})^4$.

**Proof.** We will show i) and ii) of Page 4 (and from which we will borrow the notation starting from now).

i) We begin with an upper bound on $\bar{E}_N[\rho^\delta_n(A \times (x, y))]:$

$$\sum_{\alpha \in \mathbb{V}_N^\delta} \bar{P}_N(\psi_a > u_N(x)) - \bar{P}_N(\psi_a > u_N(y))$$

$$\leq \sum_{\alpha \in \mathbb{V}_N^\delta} \frac{e^{-u_N(x)^2}}{2\pi u_N(x)} \sqrt{g_N(\alpha)} - \frac{e^{-u_N(y)^2}}{2\pi u_N(y)} \sqrt{g_N(\alpha)} (1 + o(1))$$

$$\leq \sum_{\alpha \in \mathbb{V}_N^\delta} \frac{e^{-u_N(x)^2}}{2\pi u_N(x)} \sqrt{g(0)(1 + c_n)} - \frac{e^{-u_N(y)^2}}{2\pi u_N(y)} \sqrt{g(0)(1 + c_n)}$$

$$\to e^{-x \cdot e^{-y}} |A \cap [\delta, 1 - \delta]^d| \quad (4.2)$$

We stress that in the second step the error term $c_n := O(n^{2-d})$ coming from Lemma 2.1 guarantees the convergence in the last line. The lower bound follows similarly.

ii) To show the second condition we again use Theorem 2.2. Let $A_1, \ldots, A_k$ and $R_1, \ldots, R_k$ be as in proof of Theorem 1.1. Let $I_j := nA_j \cap V_N$ and $I := I_1 \cup \cdots \cup I_k$. For $\epsilon > 0$ we are setting $B_\epsilon := B(\alpha, (\log N)^2(1+\epsilon)) \cap I$. Note that, albeit slightly different, we are using the same notations for the neighborhood of dependence and the index sets of Section 3.

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but no confusion should arise. Observe that there exists \( z \in \mathbb{R} \) such that for all \( 1 \leq j \leq k, R_j \subset (z, \infty) \); we have

\[
p_{\alpha} = \hat{P}_N \left( \frac{\psi_{\alpha} - b_N}{a_N} \in u_N(R_j) \right) \leq \hat{P}_N (\psi_{\alpha} > u_N(z)) \leq e^{-\frac{u_N(z)^2}{2g(0)}} \sqrt{\frac{2\pi u_N(z)}{g(0)}} (3.2)
\]

where we have also used the fact that \( g_N(0) \leq g(0) \). The bound on \( b_1 \) (cf. Theorem 2.2) follows exactly as in (3.5) and yields that, for some \( C > 0 \),

\[
b_1 \leq CN(\log N)^{d(2+2\epsilon)} e^{-2z^2 N^{-2}} 0.
\]

The calculation of \( b_2 \) can be performed similarly using the covariance matrix of the vector \((\psi_{\alpha}, \psi_{\beta})\), \( \alpha \neq \beta \in V_N^g \) and Lemma 2.1. This gives that for some \( C, C' > 0 \) independent of \( \alpha, \beta \in V_N^g \)

\[
b_2 \leq \sum_{\alpha \in \lambda} \sum_{\beta \in B_{\alpha}} \frac{C}{\log N} \exp \left( -\frac{u_N(z)^2}{g(0) + g(\alpha - \beta)} \left( 1 + O \left( N(2-d)/d \right) \right) \right)
\]

\[
\leq C' N^{-\kappa/(2-\kappa)} (\log N)^{2d(1+\epsilon)} \max \left\{ e^{-2z^2} 1_{\{z \leq 0\}}, e^{-2z^2/(2-\kappa)} 1_{\{z > 0\}} \right\} \rightarrow 0
\]

(cf. [9, Equation (8)]). Note the estimate for \( b_2 \) is exactly same as in the infinite volume case.

We will now pass to \( b_3 \). We repeat our choice of \( \mathcal{H}_1 = \sigma(X_{\beta} : \beta \in \mathcal{T} \setminus B_{\alpha}) \) and \( \mathcal{H}_2 = \sigma(\psi_{\beta} : \beta \in \mathcal{T} \setminus B_{\alpha}) \) so that \( b_3 \) becomes

\[
\sum_{j=1}^k \sum_{\alpha \in \lambda} \hat{E}_N \left[ \hat{E}_N [X_{\alpha} - p_{\alpha}[\mathcal{H}_1]] \right] = \sum_{j=1}^k \sum_{\alpha \in \lambda} \hat{E}_N \left[ \hat{E}_N [X_{\alpha} | \mathcal{H}_2] - p_{\alpha} \right].
\]

We define \( U_{\alpha} := V_N \setminus (\mathcal{T} \setminus B_{\alpha}) \). By the Markov property of the DGFF

\[
\hat{E}_N [X_{\alpha} | \mathcal{H}_2] = \hat{P}_{U_{\alpha}}(\xi_{\alpha} + h_{\alpha} \in u_N(R_j)) \Rightarrow \hat{P}_N a.s.
\]

for \((\xi_{\alpha})_{\alpha \in \mathbb{Z}^d} \) a DGFF with law \( \hat{P}_{U_{\alpha}} \) and \((h_{\alpha})_{\alpha \in \mathbb{Z}^d} \) is independent of \( \xi \). From [9, Equation (28)] we can see that, for any \( \alpha \in V_N^g \) and \( N \) large enough such that \( B(\alpha, (\log N)^{2(1+\epsilon)}) \subseteq V_N \)

\[
\text{Var}[h_{\alpha}] = \sum_{\beta \in \mathcal{T} \setminus B_{\alpha}} \mathbb{P}_\beta (H_{\mathcal{T} \setminus B_{\alpha}} < +\infty, S_{H_{\mathcal{T} \setminus B_{\alpha}}} = \beta) g_N(\alpha, \beta)
\]

\[
\leq \sup_{\beta \in \mathcal{T} \setminus B_{\alpha}} g_N(\alpha, \beta) \leq C (\log N)^{2(1+\epsilon)(d-2)}.
\]

This yields

\[
\sum_{j=1}^k \sum_{\alpha \in \lambda} \hat{E}_N \left[ \hat{P}_{U_{\alpha}}(\xi_{\alpha} + h_{\alpha} > u_N(R_j)) - p_{\alpha} \right] 1_{\{|h_{\alpha}| > (u_N(z))^{-1-\epsilon}\}} \rightarrow 0.
\]

It then suffices to show

\[
\sum_{j=1}^k \sum_{\alpha \in \lambda} \hat{E}_N \left[ \hat{P}_{U_{\alpha}}(\xi_{\alpha} + h_{\alpha} > u_N(R_j)) - p_{\alpha} \right] 1_{\{|h_{\alpha}| \leq (u_N(z))^{-1-\epsilon}\}} \rightarrow 0.
\]

One sees that the breaking up (3.9) can be performed also here replacing \( \varphi_{\alpha} \) and \( \psi_{\alpha} \) (with their laws) with \( \psi_{\alpha} \) and \( \xi_{\alpha} \) (with their laws) respectively, and \( \mu_{\alpha} \) with \( h_{\alpha} \). Accordingly, it is enough to show that

\[
\sum_{\alpha \in \lambda} \hat{E}_N \left[ \hat{P}_{U_{\alpha}}(\xi_{\alpha} + h_{\alpha} > u_N(w)) - \hat{P}_N(\psi_{\alpha} > u_N(w)) \right] 1_{\{|h_{\alpha}| \leq (u_N(z))^{-1-\epsilon}\}} \rightarrow 0
\]
for all $w \in \mathbb{R}$. To this aim, we choose for any $w \in \mathbb{R}$ the event
\[ Q' := \left\{ \tilde{P}_N(\psi_\alpha > u_N(w)) > \tilde{P}_{U_\alpha}(\xi_\alpha + h_\alpha > u_N(w)) \right\} \]
and we proceed as in (3.11) with the help of Lemma 2.1 to show (4.6). Given this, the convergence $b_3 \to 0$ is finally ensured. Hence we can conclude that
\[ \| \mathcal{L}(W_1, \ldots, W_k) - \mathcal{L}(Z_1, \ldots, Z_k) \|_{TV} \to 0 \]
where $Z_j$ are i.i.d. Poisson of mean $p_\alpha$. By Mills ratio, as in (4.2) we see that
\[ P(Z_j = 0) \to \exp\left(-|A_j \cap [\delta, 1-\delta]| e^{-x} \omega(R_j)\right). \]
From this it follows that the two conditions i) and ii) of Kallenberg’s Theorem are satisfied, and thus we obtain the convergence to a Poisson point process with mean measure given in i).

\textbf{Proof of Theorem 1.2.} $\mathcal{M}_p(E)$ is a Polish space with metric $d_p$:
\[ d_p(\mu, \nu) = \sum_{i \geq 1} \min\left\{ \frac{\| \mu(f_i) - \nu(f_i) \|}{2^i}, 1 \right\} \], \quad \mu, \nu \in \mathcal{M}_p(E) 
for a sequence of functions $f_i \in C^+_K(E)$ (cf. [12, Section 3.3]). Therefore we are in the condition to use a converging-together theorem [13, Theorem 3.5], namely to prove that $\rho_n \overset{d}{\to} \eta$ it is enough to show the following:

(a) $\rho_n^\delta \overset{d}{\to} \rho^\delta$, as $n \to +\infty$.
(b) $\rho^\delta \overset{d}{\to} \eta$ as $\delta \to 0$.
(c) For every $\epsilon > 0$,
\[ \lim_{\delta \to 0} \lim_{n \to +\infty} \tilde{P}_N\left( \| \rho_n(f) - \rho_n^\delta(f) \| > \epsilon \right) = 0. \quad (4.7) \]
Note that by Lemma 4.1, (a) is satisfied. For $f \in C^+_K(E)$, the Laplace functional of $\rho^\delta$ is given by (cf. [12, Prop. 3.6])
\[ \Psi^\delta(f) := E\left[ \exp\left(-\rho^\delta(f)\right) \right] = \exp\left(-\int_E \left(1 - e^{-f(t,x)}\right) t_{|t|,1-\delta} e^{-x} \mathrm{d}x \right). \]
Hence by the dominated convergence theorem we can exchange limit and expectation as $\delta \to 0$ to obtain that
\[ \Psi^\delta(f) \to \exp\left(-\int_E \left(1 - e^{-f(t,x)}\right) \mathrm{d} e^{-x} \mathrm{d}x \right) \]
and the right hand side is the Laplace functional of $\eta$ at $f$. This shows (b).

Hence to complete the proof it is enough to show (4.7). Thanks to the definition of the metric $d_p$ it suffices to prove that for $f \in C^+_K(E)$ and for $\epsilon > 0$
\[ \limsup_{\delta \to 0} \lim_{n \to +\infty} \tilde{P}_N\left( |\rho_n(f) - \rho_n^\delta(f) | > \epsilon \right) = 0. \]
Without loss of generality assume that the support of $f$ is contained in $[0, 1]^d \times [z_0, +\infty)$ for some $z_0 \in \mathbb{R}$. Choosing $n$ large enough such that $u_N(z_0) > 0$ and $g_N(\alpha) \leq g(0)$, we
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obtain that

\[
\tilde{E}_N \left[ \left| \rho_n(f) - \rho_n^d(f) \right| \right] \leq \tilde{E}_N \left[ \sum_{\alpha \in V_N \setminus V_N^d} f \left( \frac{\psi_\alpha - b_N}{a_N} \right) \mathbb{I} \left\{ \frac{\psi_\alpha - b_N}{a_N} > z_0 \right\} \right] 
\leq \sup_{z \in E} \left| f(z) \right| \sum_{\alpha \in V_N \setminus V_N^d} \tilde{P}_N \left( \frac{\psi_\alpha - b_N}{a_N} > z_0 \right) \leq C \sum_{\alpha \in V_N \setminus V_N^d} e^{-u_N(z_0)/g(0)} \sqrt{2\pi u_N(z_0)/g(0)} \leq C' (1 - (1 - 2\delta)^d) e^{-z_0}
\]

as \( n \to +\infty \) for some positive constants \( C, C' \). Now letting \( \delta \to 0 \) the result follows and this completes the proof.

\[\square\]

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