THE HESSIAN DISCRETISATION METHOD FOR FOURTH ORDER LINEAR ELLIPTIC EQUATIONS

JÉRÔME DRONIOU, BISHNU P. LAMICHHANE, AND DEVIKA SHYLAJA

Abstract. In this paper, we propose a unified framework, the Hessian discretisation method (HDM), which is based on four discrete elements (called altogether a Hessian discretisation) and a few intrinsic indicators of accuracy, independent of the considered model. An error estimate is obtained, using only these intrinsic indicators, when the HDM framework is applied to linear fourth order problems. It is shown that HDM encompasses a large number of numerical methods for fourth order elliptic problems: finite element methods (conforming and non-conforming) as well as finite volume methods. We also use the HDM to design a novel method, based on conforming $P_1$ finite element space and gradient recovery operators. Results of numerical experiments are presented for this novel scheme and for a finite volume scheme.

Keywords: fourth order elliptic equations, numerical schemes, error estimates, Hessian discretisation method, Hessian schemes, finite element method, finite volume method, gradient recovery method.

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1. Introduction

Fourth order elliptic partial differential equations arise in various applications, such as structural engineering, thin plate theories of elasticity, thin beams, biharmonic problems, the Stokes problem, image processing, etc. A large number of schemes, such as finite element (conforming, non-conforming) and finite volume methods, have been developed for the numerical approximation of these models. The purpose of this paper is to introduce a unified analysis framework, the Hessian discretisation method (HDM), that covers most of these schemes; by highlighting key abstract properties that ensure the scheme’s convergence, the HDM also enables the design of novel schemes. We focus here on linear fourth order problem; non-linear models will be covered in a forthcoming paper.

The principle of the HDM, inspired by the Gradient Discretisation Method for 2nd order problems [10], is to first select four discrete elements (a space and three reconstruction operators), altogether called a Hessian discretisation (HD). These elements are then substituted, in the weak formulation of the model, to the corresponding continuous space and operators, giving rise to a numerical scheme; this scheme is called a Hessian scheme (HS). A few indicators only, independent of the model and related to the coercivity, consistency and limit-conformity of the HD, are required to write error estimates in $L^2$, $H^1$ and $H^2$ norms for the corresponding HS. We show that schemes of the finite element and finite volume families fit into the HDM, with proper choices of HD, and we design a novel method based on the conforming $P_1$ space and a gradient recovery operator.

The finite element (FE) method is one of the most well-known tools for solving fourth-order elliptic boundary value problems. When conforming finite elements
are used, the corresponding space must be a subspace of $H^2_0(\Omega)$. The corresponding strong continuity requirement of function and its derivatives makes it difficult to construct such a finite element, and leads to schemes with a large number of unknowns $[3, 8, 6, 26, 27]$. It is known that to consider a conforming finite element space with $C^1$ continuity for a fourth-order problem, like the plate bending problem, a polynomial of degree at least 5 with 18 parameters (Bell’s triangle) is required for a triangular element, and a bi-cubic polynomial with 16 parameters for a rectangular element (Bogner-Fox-Schmit rectangle) $[6]$. The nonconforming finite element method relaxes the continuity requirement, which has a great impact on the resulting scheme. For the fourth order problem, two interesting nonconforming elements are the Adini rectangle and the Morley triangle $[6]$. The finite element methods have been well-developed for the fourth order partial differential equation with variable constant coefficients, biharmonic problem and the bending problem, see $[1, 2, 29, 28, 23, 24, 14, 13, 20, 22, 25]$. We refer to $[12]$ and the reference therein for a discussion of finite volume methods for the biharmonic problem on general meshes. The interest of the method in $[12]$ is that it is easy to implement, computationally cheap and requires only one unknown per cell. The analysis in $[12]$ is first based on meshes that respect an adequate orthogonality property, and then generalized to general polygonal meshes. In $[21]$, a finite element method for the biharmonic equation is presented; this method is based on gradient recovery operator, where the basis functions of the two involved spaces satisfy a condition of biorthogonality. The main idea is to use the gradient recovery operator to lift the non-differentiable, piecewise-constant gradient of $P_1$ finite element functions into the $P_1$ finite element space itself; the lifted functions are thus differentiable, and can be used to compute some kind of Hessian matrix of $P_1$ finite element functions. Ensuring the coercivity of the method in $[21]$ on generic triangular/tetrahedral meshes however requires the addition of a stabilisation term. We also refer to $[4]$ for the application of the gradient recovery operator to fourth order eigenvalue problems.

We note that the interest of the HDM is that it extends the analysis beyond the setting of FE methods. It covers in particular situations where the second Strang lemma cannot be applied either because the continuous bilinear form cannot be extended to the space of discrete functions, and match there the discrete bilinear form, or even because the discrete space used in the scheme is not a space of functions (and the sum of the continuous and discrete spaces does not make sense).

The paper is organised as follows. In Section 2, we introduce the model problem and list some important examples of fourth order problems. We present the Hessian discretisation method in Section 3, together with the error estimate established in this framework. In Section 4, we present a novel scheme based on the $P_1$ FE space and a gradient recovery designed using biorthogonal systems; this scheme does not require additional stabilisation terms, as the corresponding Hessian discretisation is built to already satisfy all required coercivity properties. In Section 5, we show that the finite volume method in $[12]$ is an HDM, and that the generic error estimate established in the HDM slightly improves the estimates found in $[12]$, see Remark 5.4 below. Numerical results are presented to illustrate the theoretical convergence rate established in the HDM for the gradient recovery method and finite volume method in Section 6. In Section 7, we show that some known schemes (conforming
and non-conforming FE schemes) fit into the HDM. Finally, some technical results are gathered in an appendix.

**Notations.** A fourth order symmetric tensor $P$ is a linear map $S_d(\mathbb{R}) \to S_d(\mathbb{R})$, where $S_d(\mathbb{R})$ is the set of symmetric matrices, $d$ is the dimension; $p_{ijkl}$ denote the indices of the fourth order tensor $P$ in the canonical basis of $S_d(\mathbb{R})$. For simplicity, we follow the Einstein summation convention unless otherwise stated, i.e., if an index is repeated in a product, summation is implied over the repeated index. For $\xi \in S_d(\mathbb{R})$, using the definition of symmetric tensor, one has $P\xi \in S_d(\mathbb{R})$ and $p_{ijkl} = p_{jikl} = p_{ijlk}$. The scalar product on $S_d(\mathbb{R})$ is defined by $\xi : \phi = \xi_{ij}\phi_{ij}$. For a function $\xi : \Omega \to S_d(\mathbb{R})$, denoting the Hessian matrix by $\mathcal{H}$ we set $\mathcal{H} : \xi = \partial_{ij}\xi_{ij}$. Finally, the transpose $P^T$ of $P$ is given by $P^T = (p_{klij})$, if $P = (p_{ijkl})$. Note that $P^T\xi : \phi = \xi : P\phi$. The tensor product $a \otimes b$ of two vectors $a, b \in \mathbb{R}^d$ is the 2-tensor with coefficients $a_ib_j$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$, as is the induced norm on $S_d(\mathbb{R})$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$ is denoted by $|E|$ (note that the nature of the argument of $|\cdot|$, a vector or a set, makes it clear if we talk about the Euclidean norm or the Lebesgue measure). The norm in $L^2(\Omega)$, $L^2(\Omega)^d$ for vector-valued functions, and $L^2(\Omega; \mathbb{R}^{d \times d})$ for matrix-valued functions, is denoted by $\|\cdot\|$.

### 2. Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$ and consider the following fourth order model problem with clamped boundary conditions.

\[
\sum_{i,j,k,l=1}^d \partial_{ijkl}(a_{ijkl}\partial_{ij}\varpi) = f \quad \text{in } \Omega, \tag{2.1a}
\]

\[
\varpi = \frac{\partial\varpi}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2.1b}
\]

where $\varpi = (x_1, x_2, \ldots, x_d) \in \Omega$, $f \in L^2(\Omega)$, $n$ is the unit outer normal to $\Omega$ and the coefficients $a_{ijkl}$ are measurable bounded functions which satisfy the conditions $a_{ijkl} = a_{iklj} = a_{ijlk} = a_{klji}$ for $i,j,k,l = 1, \ldots, d$. For all $\xi, \phi \in S_d(\mathbb{R})$, we assume the existence of a fourth order tensor $B$ such that $A\xi : \phi = B\xi : B\phi$, where $A$ is the four-tensor with indices $a_{ijkl}$. We notice that $B\xi : B\phi = B^T B\xi : \phi$, so that $A = B^T B$.

Setting

\[
V = H^2_0(\Omega) = \left\{ v \in H^2(\Omega) ; v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\},
\]

the weak formulation of (2.1) is

\[
\text{Find } \varpi \in V \text{ such that } \forall v \in V, \quad \int_\Omega \mathcal{H}^B \varpi : \mathcal{H}^B v \, dx = \int_\Omega f v \, dx, \quad \tag{2.2}
\]

where $\mathcal{H}^B v = B\mathcal{H} v$. Note that $\int_\Omega \mathcal{H}^B \varpi : \mathcal{H}^B v \, dx = \int_\Omega A \mathcal{H} \varpi : \mathcal{H} v \, dx$, since $A = B^T B$. We assume in the following that $B$ is constant over $\Omega$, and that the following coercivity property holds:

\[
\exists \theta > 0 \text{ such that } \|\mathcal{H}^B v\| \geq \theta \|v\|_{H^2(\Omega)}, \forall v \in H^2_0(\Omega). \tag{2.3}
\]
Hence, the weak formulation (2.2) has a unique solution by the Lax–Milgram lemma.

**Remark 2.1.** Adapting the analysis of Section 3 to B dependent on \( x \in \Omega \) is easy, provided that the entries of \( B \) belong to \( W^{2,\infty}(\Omega) \).

2.1. **Examples.** Let us examine two specific examples of the abstract problem (2.1).

2.1.1. **Biharmonic problem.** The biharmonic problem is

\[
\Delta^2 u = f \text{ in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ in } \partial \Omega. \tag{2.4}
\]

The weak formulation of this model is given by (2.2) provided that \( B \) is chosen to satisfy

\[
\int_{\Omega} \mathcal{H}^B u : \mathcal{H}^B v \, dx = \int_{\Omega} \Delta u \Delta v \, dx.
\]

One possible choice of \( B \) is therefore to set \( B\xi = \frac{\text{tr}(\xi)}{\sqrt{d}} \text{Id} \) for \( \xi \in \mathcal{S}_d(\mathbb{R}) \) (where \( \text{Id} \) is the identity matrix), in which case \( H^B = \Delta \). Since \( \int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} \mathcal{H} u : Hv \, dx \), another possibility is to set \( B \) the identity tensor \( (B\xi = \xi) \), in which case \( H^B = \mathcal{H} \). By the Poincaré inequality, both choices satisfy (2.3).

2.1.2. **Plate problem.** The clamped plate problem [6, Chapter 6] corresponds to (2.2) with \( d = 2 \) and left-hand side

\[
\int_{\Omega} \Delta u \Delta v + (1 - \gamma)(2\partial_{12} w \partial_{12} v - \partial_{11} u \partial_{22} v - \partial_{22} u \partial_{11} v) \, dx. \tag{2.5}
\]

Here, the constant \( \gamma \) lies in the interval \((0, \frac{1}{2})\). We notice that (2.5) is equal to \( \int_{\Omega} A\mathcal{H} u : Hv \, dx \), where the tensor \( A \) has non-zero indices \( a_{1111} = 1, a_{2222} = 1, a_{1212} = (1 - \gamma), a_{2121} = (1 - \gamma), a_{1122} = \gamma \) and \( a_{2211} = \gamma \). Its ‘square root’ can be defined as the tensor \( B \) with non-zero indices \( b_{1111} = b_{2222} = \sqrt{\frac{1 + \sqrt{1 - \gamma^2}}{2}}, b_{1122} = b_{2211} = \sqrt{\frac{1 - \sqrt{1 - \gamma^2}}{2}} \) and \( b_{1212} = b_{2121} = \sqrt{1 - \gamma} \). It can be checked that (2.3) holds since, for some \( \varrho > 0 \), \( A\xi : \xi \geq \varrho^2|\xi|^2 \) for all \( \xi \in \mathcal{S}_d(\mathbb{R}) \).

3. **The Hessian discretisation method**

We present here the Hessian discretisation method, and list the properties that are required for the convergence analysis of the Hessian scheme. The error estimate is stated at the end of the section.

**Definition 3.1** (*B–Hessian discretisation*). A *B–Hessian discretisation* for clamped boundary conditions is a quadruplet \( \mathcal{D} = (X_{D,0}, \Pi_D, \nabla_D, \mathcal{H}_D^B) \) such that

- \( X_{D,0} \) is a finite-dimensional space encoding the unknowns of the method,
- \( \Pi_D : X_{D,0} \to L^2(\Omega) \) is a linear mapping that reconstructs a function from the unknowns,
- \( \nabla_D : X_{D,0} \to L^2(\Omega)^d \) is a linear mapping that reconstructs a gradient from the unknowns,
- \( \mathcal{H}_D^B : X_{D,0} \to L^2(\Omega; \mathbb{R}^{d \times d}) \) is a linear mapping that reconstructs a discrete version of \( \mathcal{H}^B(= \mathcal{H}) \) from the unknowns. It must be chosen such that \( \| \cdot \|_D := \| \mathcal{H}_D^B \cdot \| \) is a norm on \( X_{D,0} \).
Remark 3.2 (Dependence of the Hessian discretisation on B). In the (2nd order) gradient discretisation method, the definition of a gradient discretisation is independent of the differential operator. Here, our definition of Hessian discretisation depends on B, that appears in the differential operator. This is justified by the fact that some methods (such as the one presented in Section 5) are not built on an approximation of the entire Hessian of the functions, but only on some of their derivatives (such as the Laplacian of the functions). Although it might be possible to enrich these methods by adding approximations of the ‘missing’ second order derivatives (as done in [9] in the context of the GDM), it does not seem to be the most natural way to proceed, and it leads to additional technicality in the analysis. Making the definition of HD dependent on the considered model through B enables us to more naturally embed some known methods into the HDM.

Note however that a number of FE methods provide approximations of the entire Hessian of the functions (see Sections 4 and 7). For those methods, a B–Hessian discretisation is built from an Id–Hessian discretisation (that is independent of the model) by setting $H_B^D = B H_B^I$. 

If $D = (X_{D,0}, \Pi_D, \nabla_D, H_B^D)$ is a B–Hessian discretisation, the corresponding scheme for (2.1), called Hessian scheme (HS), is given by

\[
\text{Find } u_D \in X_{D,0} \text{ such that for any } v_D \in X_{D,0},
\]

\[
\int_{\Omega} H_B^D u_D : H_B^D v_D \, d\mathbf{x} = \int_{\Omega} f v_D \, d\mathbf{x}.
\]

This HS is obtained by replacing, in the weak formulation (2.2), the continuous space $V$ by $X_{D,0}$, and by using the reconstructions $\Pi_D$ and $H_B^D$ in lieu of the function and its Hessian.

We will show that the accuracy of the HS can be evaluated using only three measures, all intrinsic to the Hessian discretisation. The first one is a constant, $C_B^D$, which controls the norm of the linear mappings $\Pi_D$ and $\nabla_D$.

\[
C_B^D = \max_{w \in X_{D,0} \setminus \{0\}} \left( \frac{\|\Pi_D w\| \cdot \|\nabla_D w\|}{\|H_B^D w\|} \right). \tag{3.2}
\]

The second measure of accuracy is the interpolation error $S_B^D$ defined by

\[
S_B^D(\varphi) = \min_{w \in X_{D,0}} \left( \|\Pi_D w - \varphi\| + \|\nabla_D w - \varphi\| + \|H_B^D w - H_B^D \varphi\| \right). \tag{3.3}
\]

Finally, the third quantity is a measure of limit-conformity of the HD, that is, how well a discrete integration-by-parts formula is verified by the discrete operators:

\[
W_B^D(\xi) = \max_{w \in X_{D,0} \setminus \{0\}} \frac{1}{\|H_B^D w\|} \left| \int_{\Omega} (\mathcal{H} : B^\tau B \xi) \Pi_D w - B \xi : H_B^D w \right| \, d\mathbf{x}. \tag{3.4}
\]

Note that if $\xi \in H_B^D(\Omega)$ and $\varphi \in H_B^D(\Omega)$, integration-by-parts show that $\int_{\Omega} (\mathcal{H} : B^\tau B \xi) \varphi = \int_{\Omega} B \xi : H_H^B \varphi$. Hence, the quantity in the right-hand side of (3.4) measures a defect of discrete integration-by-parts between $\Pi_D$ and $H_B^D$.

Closely associated to the three measures above are the notions of coercivity, consistency and limit-conformity of a sequence of Hessian discretisations.
Definition 3.3 (Coercivity, consistency and limit-conformity). Let \((D_m)_{m \in \mathbb{N}}\) be a sequence of \(B\)-Hessian discretisations in the sense of Definition 3.1. We say that

1. \((D_m)_{m \in \mathbb{N}}\) is coercive if there exists \(C_P \in \mathbb{R}^+\) such that \(C_{P_m}^B \leq C_P\) for all \(m \in \mathbb{N}\).
2. \((D_m)_{m \in \mathbb{N}}\) is consistent, if

\[
\forall \varphi \in H^2_0(\Omega), \lim_{m \to \infty} S^B_{D_m}(\varphi) = 0. \tag{3.5}
\]

3. \((D_m)_{m \in \mathbb{N}}\) is limit-conforming, if

\[
\forall \xi \in H^2(\Omega), \lim_{m \to \infty} W^B_{D_m}(\xi) = 0. \tag{3.6}
\]

Remark 3.4. As for the (2nd order) gradient discretisation method, see [10, Lemmas 2.16 and 2.17], it is easily proved that, for coercive sequences of HDs, the consistency and limit-conformity properties (3.5) and (3.6) only need to be tested for functions in dense subsets of \(H^2_0(\Omega)\) and \(H^2(\Omega)\), respectively.

Remark 3.5. If \(B = \text{Id.}\) we write \(\mathcal{H}_D\) (resp. \(C_D, S_D\) and \(W_D\)) instead of \(\mathcal{H}_D^{\text{ld}}\) (resp. \(C_D^{\text{ld}}, S_D^{\text{ld}}\) and \(W_D^{\text{ld}}\)).

We can now state our main theorem giving the error estimates.

Theorem 3.6 (Error estimate for Hessian schemes). Under Assumption (2.3), let \(\bar{\pi}\) be the solution to (2.2). Let \(D\) be a \(B\)-Hessian discretisation and \(u_D\) be the solution to the corresponding Hessian scheme (3.1). Then we have the following error estimates:

\[
||\Pi_D u_D - \bar{\pi}|| \leq C_D W^B_D(\mathcal{H}\bar{\pi}) + (C_D + 1)S^B_D(\bar{\pi}), \tag{3.7}
\]

\[
||\nabla_D u_D - \nabla \bar{\pi}|| \leq C_D W^B_D(\mathcal{H}\bar{\pi}) + (C_D + 1)S^B_D(\bar{\pi}), \tag{3.8}
\]

\[
||\mathcal{H}^B_D u_D - \mathcal{H}^B \bar{\pi}|| \leq W^B_D(\mathcal{H}\bar{\pi}) + 2S^B_D(\bar{\pi}). \tag{3.9}
\]

(Note that \(\mathcal{H}\bar{\pi} \in H^2(\Omega)\) because \(\mathcal{H}\bar{\pi} \in L^2(\Omega)^{d \times d}\) and \(\mathcal{H} : B^T B \mathcal{H}\bar{\pi} = \mathcal{H} : A \mathcal{H}\bar{\pi} = f \in L^2(\Omega)\).)

The following convergence result is a trivial consequence of the error estimates above.

Corollary 3.7 (Convergence). Let \((D_m)_{m \in \mathbb{N}}\) be a sequence of \(B\)-Hessian discretisations that is coercive, consistent and limit-conforming. Then, as \(m \to \infty\), \(\Pi_{D_m} u_{D_m} \to \bar{\pi}\) in \(L^2(\Omega)\), \(\nabla_{D_m} u_{D_m} \to \nabla \bar{\pi}\) in \(L^2(\Omega)^d\) and \(\mathcal{H}_{D_m} u_{D_m} \to \mathcal{H}^B \bar{\pi}\) in \(L^2(\Omega)^{d \times d}\).

Let us now prove Theorem 3.6.

Proof of Theorem 3.6. For all \(v_D \in \mathcal{X}_{D,0}\), the equation (2.1a) taken in the sense of distributions shows that \(f = \mathcal{H} : A \mathcal{H} \bar{\pi}\), and thus, by the Hessian scheme (3.1),

\[
\int_{\Omega} \mathcal{H}^B D u_D : \mathcal{H}^B D v_D \, dx = \int_{\Omega} f D v_D \, dx = \int_{\Omega} (\mathcal{H} : B^T B \mathcal{H} \bar{\pi}) \Pi_D v_D \, dx.
\]

Using the definition of \(W^B_D\), we infer

\[
\int_{\Omega} (\mathcal{H}^B \bar{\pi} - \mathcal{H}^B_D u_D) : \mathcal{H}^B_D v_D \, dx \leq W^B_D(\mathcal{H}\bar{\pi}) ||\mathcal{H}^B_D v_D||. \tag{3.10}
\]
Define the interpolant \( P_D : \mathcal{H}^2_u(\Omega) \to \mathcal{X}_{D,0} \) by
\[
P_D \pi = \arg\min_{w \in \mathcal{X}_{D,0}} \left( \| \Pi_D w - \pi \| + \| \nabla_D w - \nabla \pi \| + \| H^B_D w - H^B \pi \| \right)
\]
and notice that
\[
\| \Pi_D P_D \pi - \pi \| + \| \nabla_D P_D \pi - \nabla \pi \| + \| H^B_D P_D \pi - H^B \pi \| \leq S^B_D(\pi).
\] (3.11)

Introducing the term \( H^B \pi \) and using (3.10), we obtain
\[
\int_{\Omega} \left( \mathcal{H}^B_D P_D \pi - H^B_D u_D \right) : \mathcal{H}^B D v_D \, dx
= \int_{\Omega} \left( \mathcal{H}^B \pi - H^B_D u_D \right) : \mathcal{H}^B D v_D \, dx + \int_{\Omega} \left( \mathcal{H}^B_D P_D \pi - H^B \pi \right) : \mathcal{H}^B D v_D \, dx
\leq W^B_D(\mathcal{H} \pi) \| \mathcal{H}^B_D P_D \pi - H^B \pi \| \| \mathcal{H}^B D v_D \|.
\]
Choosing \( v_D = P_D \pi - u_D \), we get
\[
\| \mathcal{H}^B_D (P_D \pi - u_D) \|^2 \leq W^B_D(\mathcal{H} \pi) \| \mathcal{H}^B_D (P_D \pi - u_D) \|
+ \| \mathcal{H}^B_D P_D \pi - H^B \pi \| \| \mathcal{H}^B_D (P_D \pi - u_D) \|.
\]

Thus, by (3.11),
\[
\| \mathcal{H}^B_D P_D \pi - H^B \pi \| \leq W^B_D(\mathcal{H} \pi) + S^B_D(\pi).
\] (3.12)

A use of triangle inequality, (3.11) and (3.12) yields
\[
\| H^B_D u_D - H^B \pi \| \leq \| H^B_D u_D - H^B_D P_D \pi \| + \| H^B_D P_D \pi - H^B \pi \|
\leq W^B_D(\mathcal{H} \pi) + 2S^B_D(\pi),
\]
which is (3.9). Using the definition of \( C_D \), and (3.11) and (3.12), we obtain
\[
\| \Pi_D u_D - \pi \| \leq \| \Pi_D u_D - \Pi_D P_D \pi \| + \| \Pi_D P_D \pi - \pi \|
\leq C_D \| \mathcal{H}^B_D P_D \pi - H^B \pi \| + S^B_D(\pi)
\leq C_D W^B_D(\mathcal{H} \pi) + (C_D + 1) S^B_D(\pi).
\]

Hence, (3.7) is established, and (3.8) follows in a similar way. \(\square\)

We now aim to present particular HDMs. The first (in Section 4) is a novel scheme based on gradient recovery operators, and a particular cheap construction of these operators using biorthogonal basis. Then, we show that a finite volume method (in Section 5) and known finite element methods (in Section 7) fit into the HDM. Let us first set some notations related to meshes.

**Definition 3.8 (Polytopal mesh [10, Definition 7.2]).** Let \( \Omega \) be a bounded polytopal open subset of \( \mathbb{R}^d \) \((d \geq 1)\). A polytopal mesh of \( \Omega \) is \( \mathcal{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P}) \), where:

1. \( \mathcal{M} \) is a finite family of non-empty connected polytopal open disjoint subsets of \( \Omega \) (the cells) such that \( \overline{\Omega} = \bigcup_{K \in \mathcal{M}} K \). For any \( K \in \mathcal{M} \), \(|K| > 0\) is the measure of \( K \), \( h_K \) denotes the diameter of \( K \), \( \mathfrak{m}_K \) is the center of mass of \( K \), and \( n_K \) is the outer unit normal to \( K \).
2. \( \mathcal{F} \) is a finite family of disjoint subsets of \( \overline{\Omega} \) (the edges of the mesh in 2D, the faces in 3D), such that any \( \sigma \in \mathcal{F} \) is a non-empty open subset of a hyperplane of \( \mathbb{R}^d \) and \( \sigma \subset \overline{\Omega} \). Assume that for all \( K \in \mathcal{M} \) there exists a subset \( \mathcal{F}_K \) of \( \mathcal{F} \) such that the boundary of \( K \) is \( \bigcup_{\sigma \in \mathcal{F}_K} \sigma \). We then set \( \mathcal{M}_\sigma = \{ K \in \mathcal{M} : \sigma \in \mathcal{F}_K \} \) and assume that, for all \( \sigma \in \mathcal{F} \), \( \mathcal{M}_\sigma \) has exactly
one element and \( \sigma \subset \partial \Omega \), or \( M_\sigma \) has two elements and \( \sigma \subset \Omega \). Let \( F_{\text{int}} \) be the set of all interior faces, i.e. \( \sigma \in F \) such that \( \sigma \subset \Omega \), and \( F_{\text{ext}} \) the set of boundary faces, i.e. \( \sigma \in F \) such that \( \sigma \subset \partial \Omega \). The \((d-1)\)-dimensional measure of \( \sigma \in F \) is \( |\sigma| \), and its centre of mass is \( \sigma \).

(3) \( P = (x_K)_{K \in M} \) is a family of points of \( \Omega \) indexed by \( M \) and such that, for all \( K \in M \), \( x_K \in K \). Assume that any cell \( K \in M \) is strictly \( x_K \)-star-shaped, meaning that if \( x \in K \) then the line segment \([x_K, x]\) is included in \( K \).

The diameter of such a polytopal mesh is \( h = \max_{K \in M} h_K \).

4. Method based on Gradient Recovery Operators

4.1. General setting. Let \( V_h \) be an \( H^1_0 \)-conforming finite element space with underlying mesh \( M = M_h \). We assume that \( V_h \) contains the piecewise linear functions, and that \( M_h \) satisfies usual regularity assumptions, namely, denoting by \( \rho_K = \max\{r > 0 : B(x_K, r) \subset K\} \) the maximal radius of balls centred at \( x_K \) and included in \( K \), we assume that there exists \( \eta > 0 \) (independent of \( h \)) such that

\[
\forall K \in M, \; \eta \geq \frac{h_K}{\rho_K}. \tag{4.1}
\]

The gradient \( \nabla u \) of \( u \in V_h \) is well defined, but its second derivative \( \nabla \nabla u \) is not. In order to compute some sort of second derivatives, consider a projector \( Q_h : L^2(\Omega) \to V_h \), which is extended to \( L^2(\Omega)^d \) component-wise. Then \( \nabla u \) can be projected onto \( V_h^2 \), and the resulting function \( Q_h \nabla u \in V_h^2 \) is differentiable. We can then consider \( \nabla(Q_h \nabla u) \) as a sort of Hessian of \( u \). However, it not necessarily clear, for some interesting choices of practically computable \( Q_h \) (see Section 4.2), that this reconstructed Hessian has proper coercivity properties. We therefore also consider a function \( \mathcal{E}_h \) whose role is to stabilise this reconstructed Hessian.

Let \((V_h, Q_h, I_h, \mathcal{E}_h)\) be a quadruplet of a finite element space \( V_h \subset H^1_0(\Omega) \), a reconstruction operator \( Q_h : L^2(\Omega) \to V_h \) that is a projector onto \( V_h \) (that is, \( Q_h = \text{Id} \) on \( V_h \)), an interpolant \( I_h : H^1(\Omega) \to V_h \) and a stabilisation function \( \mathcal{E}_h \in L^\infty(\Omega)^d \) such that, with constants \( C \) not depending on \( h \),

\[
\begin{align*}
(\text{P0}) & \quad \text{[Structure of } V_h \text{ and } I_h] \quad \text{The inverse estimate } \|\nabla z\| \leq C h^{-1} \|z\| \text{ holds for all } z \in V_h \text{ and, for } \varphi \in H^2_0(\Omega), \text{ we have } \|\nabla I_h \varphi - \nabla \varphi\| \leq C h \|\varphi\|_{H^2(\Omega)}. \\
(\text{P1}) & \quad \text{[Stability of } Q_h] \quad \text{For } \phi \in L^2(\Omega), \text{ we have } \|Q_h \phi\| \leq C \|\phi\|. \\
(\text{P2}) & \quad \text{[ } Q_h \nabla I_h \text{ approximates } \nabla] \quad \text{For some space } W \text{ densely embedded in } H^3(\Omega) \cap H^1_0(\Omega) \text{ and for all } \psi \in W, \text{ we have } \|Q_h \nabla I_h \psi - \nabla \psi\| \leq C h^2 \|\psi\|_W. \\
(\text{P3}) & \quad \text{[ } H^1 \text{ approximation property of } Q_h] \quad \text{For } w \in H^2(\Omega) \cap H^1_0(\Omega), \text{ we have } \|\nabla Q_h w - \nabla w\| \leq C h \|w\|_{H^2(\Omega)}. \\
(\text{P4}) & \quad \text{[Asymptotic density of } [(Q_h \nabla - \nabla)(V_h)] \text{]} \quad \text{Setting } N_h = [(Q_h \nabla - \nabla)(V_h)]^{-1}, \text{ where the orthogonality is considered for the } L^2(\Omega)^d \text{-inner product, the following approximation property holds: } \\
& \quad \quad \inf_{\mu_h \in N_h} \|\mu_h - \varphi\| \leq C h \|\varphi\|_{H^1(\Omega)^d}, \quad \forall \varphi \in H^1(\Omega)^d, \\
(\text{P5}) & \quad \text{[Stabilisation function] } 1 \leq |\mathcal{E}_h| \leq C \text{ and, for all } K \in M, \text{ denoting by } V_h(K) = \{v|_K ; v \in V_h, K \in M\} \text{ the local FE space, } \\
& \quad \quad [\mathcal{E}_h|_K \otimes (Q_h \nabla - \nabla)(V_h(K))]' \perp \nabla V_h(K)^d, \text{ and } \\
& \quad V_h(K) \subset V_h(\Omega)^d. 
\end{align*}
\]
Remark 4.1. A classical operator $Q_h$ that satisfies these assumptions, for standard FE spaces $V_h$, is the $L^2$-orthogonal projector on $V_h$. This operator is however non-local and complicated to compute. We present in Section 4.2 a much more efficient construction of $Q_h$, local and based on biorthogonal bases.

To construct an HD based on such a quadruplet, we assume the following stronger form of (2.3):

$$\exists C_B > 0 : |B\xi| \geq C_B|\xi|, \quad \forall \xi \in S_d(\mathbb{R}). \quad (4.2)$$

Definition 4.2 (B–Hessian discretisation using gradient recovery). Under Assumption (4.2), the B-Hessian discretisation based on a quadruplet $(V_h, Q_h, I_h, \mathcal{G}_h)$ satisfying (P0)–(P5) is defined by: $X_{\mathcal{D},0} = V_h$ and, for $u \in X_{\mathcal{D},0}$,

$$\Pi_{\mathcal{D}}u = u, \quad \nabla_{\mathcal{D}}u = Q_h\nabla u \text{ and } \mathcal{H}^B_{\mathcal{D}}u = B[\nabla(Q_h\nabla u) + \mathcal{G}_h \otimes (Q_h\nabla u - \nabla u)].$$

The next theorem gives an estimate on the accuracy measures $C_B^B$, $S_B^B$ and $W_B^B$ associated with an HD $\mathcal{D}$ using gradient recovery. Incidentally, the estimate on $C_B^B$ also establishes that $\|\mathcal{H}^B_{\mathcal{D}} \|$ is a norm on $X_{\mathcal{D},0}$.

Theorem 4.3 (Estimates for Hessian discretisations based on gradient recovery). Let $\mathcal{D}$ be a $B$–Hessian discretisation in the sense of Definition 4.2, with $B$ satisfying Estimate (4.2) and $(V_h, I_h, Q_h, \mathcal{G}_h)$ satisfying (P0)–(P5). Then, there exists a constant $C$, not depending on $h$, such that

- $C_B^B \leq C$,
- $\forall \varphi \in W, S_B^B(\varphi) \leq C h \|\varphi\|_W$,
- $\forall \xi \in H^2(\Omega)^{d \times d}, W_B^B(\xi) \leq C h \|\xi\|_{H^2(\Omega)^{d \times d}}$.

Before proving this theorem, let us note the following straightforward consequence of Remark 3.4.

Corollary 4.4 (Properties of Hessian discretisation based on gradient recovery). Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of B–Hessian discretisations, with $B$ satisfying Estimate (4.2) and each $\mathcal{D}_m$ associated with $(V_{h,m}, Q_{h,m}, I_{h,m}, \mathcal{G}_{h,m})$ satisfying (P0)–(P5) uniformly with respect to $m$. Assume that $h_m \to 0$ as $m \to \infty$. Then the sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is coercive, consistent and limit-conforming.

Proof of Theorem 4.3.

- **Coercivity**: Let $v \in X_{\mathcal{D},0}$. Noticing that $|a \otimes b| = |a||b|$ for any two vectors $a$ and $b$, the definition of $\mathcal{H}^B_{\mathcal{D}}$, Property (4.2) of $B$ and $|\mathcal{G}| \geq 1$ yield

$$\|\mathcal{H}^B_{\mathcal{D}}v\|^2 \geq C_B^2 \int_{\Omega} |\nabla(Q_h\nabla v) + \mathcal{G}_h \otimes (Q_h\nabla v - \nabla v)|^2 \, dx = C_B^2 \int_{\Omega} |\nabla(Q_h\nabla v)|^2 \, dx + C_B^2 \int_{\Omega} |\mathcal{G}_h \otimes (Q_h\nabla v - \nabla v)|^2 \, dx$$

$$+ 2C_B^2 \int_{\Omega} (Q_h\nabla v : \mathcal{G}_h \otimes (Q_h\nabla v - \nabla v)) \, dx \geq C_B^2 \left(\|\nabla(Q_h\nabla v)\|^2 + \|Q_h\nabla v - \nabla v\|^2\right)$$

$$+ 2C_B^2 \sum_{K \in \mathcal{M}} \int_K \nabla(Q_h\nabla v) : \mathcal{G}_h \otimes (Q_h\nabla v - \nabla v) \, dx.$$
Since $\nabla(Q_h \nabla v)_K \in \nabla V_h(K)^d$, a use of property (P5) shows that the last term vanishes, and we have thus
\begin{equation}
\|H_D^B v\|^2 \geq C_B^2 \left( \|\nabla(Q_h \nabla v)\|^2 + \|Q_h \nabla v - \nabla v\|^2 \right),
\end{equation}
which implies
\begin{equation}
C_B^{-1} \sqrt{2} \|H_D^B v\| \geq \|\nabla(Q_h \nabla v)\| + \|Q_h \nabla v - \nabla v\|.
\end{equation}
Apply now the Poincaré inequality twice, the triangle inequality and (4.4) to obtain
\begin{align}
\|\Pi_D v\| &= |v| \leq \text{diam}(\Omega)\|\nabla v\|
&\leq \text{diam}(\Omega)\|\nabla v - Q_h \nabla v\| + \text{diam}(\Omega)\|Q_h \nabla v\|
&\leq \text{diam}(\Omega)\|\nabla v - Q_h \nabla v\| + \text{diam}(\Omega)^2 \|\nabla(Q_h \nabla v)\|
&\leq C_B^{-1} \sqrt{2} \max(\text{diam}(\Omega), \text{diam}(\Omega)^2) \|H_D^B v\|.
\end{align}
From (4.3) and the Poincaré inequality, we also have
\begin{equation}
\|\nabla_D v\| = \|Q_h \nabla v\| \leq \text{diam}(\Omega)\|\nabla(Q_h \nabla v)\| \leq \text{diam}(\Omega)C_B^{-1} \|H_D^B v\|.
\end{equation}
Estimates (4.5) and (4.6) show that $C_B^2 \leq C_B^{-1} \sqrt{2} \max(\text{diam}(\Omega), \text{diam}(\Omega)^2)$.

- **Consistency:** let $\varphi \in W \subset H^3(\Omega) \cap H_0^2(\Omega)$ and choose $v = I_h \varphi \in X_D,0$. Using the properties (P0) (which implies $|I_h \varphi - \varphi| \leq C h \|\varphi\|_{H^2(\Omega)}$ by the Poincaré inequality) and (P2), we obtain
\begin{equation}
\|\Pi_D v - \varphi\| = \|I_h \varphi - \varphi\| \leq C h \|\varphi\|_{H^2(\Omega)}
\end{equation}
and
\begin{equation}
\|\nabla_D v - \nabla \varphi\| = \|Q_h \nabla I_h \varphi - \nabla \varphi\| \leq C h^2 \|\varphi\|_W.
\end{equation}
Let us now turn to $\|H_D^B v - H^B \varphi\|$. Observe that $\nabla \nabla$ is another notation for $H$.
Using a triangle inequality, the boundedness of $B$ and $\mathcal{E}_h$ implies
\begin{align}
\|H_D^B v - H^B \varphi\| &= \|B [\nabla(Q_h \nabla v) + \mathcal{E}_h \otimes (Q_h \nabla v - \nabla v)] - B \varphi\|
&\leq \|B [\nabla(Q_h \nabla v) - \nabla \varphi\| + \|B \mathcal{E}_h \otimes (Q_h \nabla v - \nabla v)\|
&\leq C \left( \|\nabla(Q_h \nabla v) - \nabla \varphi\| + C \|Q_h \nabla v - \nabla v\| \right).
\end{align}
Introducing the term $\nabla(Q_h \nabla \varphi)$, using in sequence the triangle inequality, the inverse inequality in (P0), (P3), the projection property of $Q_h$, (P1) and (P2), we get
\begin{align}
A_1 &= \|\nabla(Q_h \nabla v - Q_h \nabla \varphi)\| + \|Q_h \nabla \varphi\| - \nabla \varphi\|
&\leq C h^{-1} \|Q_h \nabla v - Q_h \nabla \varphi\| + C h \|\varphi\|_{H^2(\Omega)}
&\leq C h^{-1} \|Q_h \nabla v - Q_h \nabla \varphi\| + C h \|\varphi\|_{H^2(\Omega)}
&\leq C h^{-1} \|Q_h \nabla I_h \varphi - \nabla \varphi\| + C h \|\varphi\|_{H^2(\Omega)} \leq C h \|\varphi\|_W.
\end{align}
To estimate $A_2$, we use the properties (P2) and (P0):
\begin{align}
A_2 &= \|Q_h \nabla v - \nabla \varphi\| + \|\varphi - \nabla v\| \leq C h^2 \|\varphi\|_W + C h \|\varphi\|_{H^2(\Omega)}.
\end{align}
The estimate on $S_D^B(\varphi)$ follows from (4.7)–(4.11).
• Limit-conformity: for $\xi \in H^2(\Omega)^{d \times d}$ and $v \in X_{D,0}$,
\[
\int_{\Omega} \left( (\mathcal{H} : B^T B) \Pi_D v - B \xi : \mathcal{H}_B^B \right) \, dx
= \int_{\Omega} \left( (\mathcal{H} : B^T B) \Pi_D v - B \nabla (Q_h \nabla v) \right) \, dx
- \int_{\Omega} B \xi : B \mathcal{S}_h \otimes (Q_h \nabla v - \nabla v) \, dx. \tag{4.12}
\]

Recall that $v = \Pi_D v$ and $A = B^T B$. Since $Q_h \nabla v \in H_0^1(\Omega)$, Lemma A.2 applied to $(\mathcal{H} : A\xi)v$ and an integration-by-parts on $B\xi : B\nabla (Q_h \nabla v) = A\xi : \nabla (Q_h \nabla v)$ show that, for any $
u_h \in N_h$, $\|(Q_h \nabla - \nabla)(V_h)\|_{(\Omega)}$.
\[
|B_1| = \left| \int_{\Omega} (\mathcal{H} : A\xi) v \, dx + \int_{\Omega} Q_h \nabla v \cdot \text{div}(A\xi) \, dx \right|
= \left| \int_{\Omega} (Q_h \nabla v - \nabla v) \cdot \text{div}(A\xi) \, dx \right|
= \left| \int_{\Omega} (Q_h \nabla v - \nabla v) \cdot (\text{div}(A\xi) - \nu_h) \, dx \right|
\leq \|Q_h \nabla v - \nabla v\| \|\text{div}(A\xi) - \nu_h\|. \tag{4.13}
\]

Take the infimum over all $\nu_h \in N_h$. Estimate (4.4) and Property (P4) yield
\[
|B_1| \leq C h \|\mathcal{H}_B^B v\| \|\text{div}(A\xi)\|_{H^1(\Omega)^d}. \tag{4.14}
\]

Let $\xi_K$ denote the average of $\xi$ over $K \in M$. By the mesh regularity assumption, $\|\xi - \xi_K\|_{L^2(K)^{d \times d}} \leq C h \|\xi\|_{H^2(K)^{d \times d}}$ (see, e.g., [10, Lemma B.6]). Moreover, since $V_h$ contains the piecewise constant functions, $\nabla V_h(K)$ contains the constant vector-valued functions on $K$ and thus, by the orthogonality condition in (P5), the Cauchy-Schwarz inequality, the boundedness of $B$ and $\mathcal{S}_h$, and (4.4),
\[
|B_2| = \left| \sum_{K \in M} \int_K B^T B \xi : \mathcal{S}_h \otimes (Q_h \nabla v - \nabla v) \, dx \right|
\leq C \sum_{K \in M} \|\xi - \xi_K\|_{L^2(K)} \|Q_h \nabla v - \nabla v\|_{L^2(K)}
\leq C h \|\xi\|_{H^2(\Omega)^{d \times d}} \|\mathcal{H}_B^B v\|. \tag{4.15}
\]

Plugging (4.14) and (4.15) into (4.12) yields
\[
\int_{\Omega} \left( (\mathcal{H} : B^T B) \Pi_D v - B \xi : \mathcal{H}_B^B \right) \, dx
\leq C h \left( \|\text{div}(A\xi)\|_{H^1(\Omega)^d} + \|\xi\|_{H^2(\Omega)^{d \times d}} \right) \|\mathcal{H}_B^B v\|.
\]

By the definition (3.4) of $W_B^B(\xi)$, this concludes the proof of the estimate on this quantity. \qed
4.2. A gradient recovery operator based on biorthogonal systems. We present here a particular case of a method based on a gradient recovery operator, using biorthogonal systems as in [21]. \( V_h \) is the conforming \( \mathbb{P}_1 \) FE space on a mesh of simplices, and \( I_h \) is the Lagrange interpolation with respect to vertices of \( \mathcal{M} \). We will build a locally computable projector \( Q_h \), that is, such that determining \( Q_h f \) on a cell \( K \) only requires the knowledge of \( f \) on \( K \) and its neighbouring cells.

Let \( B_1 := \{ \phi_1, \cdots, \phi_n \} \) be the set of basis functions of \( V_h \) associated with the inner vertices in \( \mathcal{M} \). Let the set \( B_2 := \{ \psi_1, \cdots, \psi_n \} \) be the set of discontinuous piecewise linear functions biorthogonal to \( B_1 \) also associated with the inner vertices of \( \mathcal{M} \), so that elements of \( B_1 \) and \( B_2 \) satisfy the biorthogonality relation

\[
\int_{\Omega} \psi_i \phi_j \, dx = c_{ij} \delta_{ij}, \quad c_{ij} \neq 0, \quad 1 \leq i, j \leq n, \tag{4.16}
\]

where \( \delta_{ij} \) is the Kronecker symbol and \( c_{ij} = \int_{\Omega} \psi_i \phi_j \, dx \). Let \( M_h := \text{span}\{B_2\} \). Such biorthogonal systems have been constructed in the context of mortar finite elements, and later extended to gradient recovery operators [16, 18, 21]. The basis functions of \( M_h \) can be defined on a reference element. For example, for the reference triangle, we have

\[
\hat{\psi}_1(x) := 3 - 4x_1 - 4x_2, \quad \hat{\psi}_2(x) := 4x_1 - 1, \quad \text{and} \quad \hat{\psi}_3(x) := 4x_2 - 1,
\]

associated with its three vertices \((0,0), (1,0)\) and \((0,1)\), respectively. For the reference tetrahedron, we have

\[
\hat{\psi}_1(x) := 4 - 5x_1 - 5x_2 - 5x_3, \quad \hat{\psi}_2(x) := 5x_1 - 1,
\]

\[
\hat{\psi}_3(x) := 5x_2 - 1, \quad \text{and} \quad \hat{\psi}_4(x) := 5x_3 - 1,
\]

associated with its four vertices \((0,0,0), (1,0,0), (0,1,0)\) and \((0,0,1)\), respectively. These basis functions satisfy

\[
\sum_{i=1}^{d+1} \hat{\psi}_i = 1. \tag{4.17}
\]

The projection operator \( Q_h : L^2(\Omega) \rightarrow V_h \) is the oblique projector onto \( V_h \) defined as: for \( f \in L^2(\Omega) \), \( Q_h f \in V_h \) satisfies

\[
\int_{\Omega} (Q_h f) \psi_h \, dx = \int_{\Omega} f \psi_h \, dx, \quad \forall \psi_h \in M_h. \tag{4.18}
\]

Due to the biorthogonality relation (4.16), \( Q_h \) is well-defined and has the explicit representation

\[
Q_h f = \sum_{i=1}^{n} \int_{\Omega} \psi_i f \, dx \frac{1}{c_i} \phi_i. \tag{4.19}
\]

The relation (4.18) shows \( M_h \subset [(Q_h - I)(L^2(\Omega))]^\perp \). Hence, if \( M_h \) satisfies the approximation property

\[
\inf_{\alpha_h \in M_h} \| \alpha_h - \psi \| \leq C h \| \psi \|_{H^1(\Omega)}, \quad \forall \psi \in H^1(\Omega),
\]

we know that (P4) holds. In order to get this approximation property it is sufficient that the basis functions of \( M_h \) reproduce constant functions. Let \( K \in \mathcal{M} \) be an interior element not touching any boundary vertex. Due to the property (4.17)

\[
\sum_{i=1}^{d+1} \psi_{v_i} = 1 \quad \text{on} \quad K,
\]
where \( \{ \tilde{\psi}_v \}_{i=1}^{d+1} \) are basis functions of \( M_h \) associated with the vertices \( \{ v_1, \ldots, v_{d+1} \} \) of \( K \).

However, this property does not hold on \( K \in \mathcal{M} \) if \( K \) has one or more vertices on the boundary. We need to modify the piecewise linear basis functions of \( M_h \) to guarantee the approximation property [19, 17]. Let \( W_h \subset H^1(\Omega) \) be the lowest order FE space including the basis functions on the boundary vertices of \( \mathcal{M} \), and let \( \tilde{M}_h \) the space spanned by the discontinuous basis functions biorthogonal to the basis functions of \( W_h \). \( M_h \) is then obtained as a modification of \( \tilde{M}_h \), by moving all vertex basis functions of this latter space to nearby internal vertices using the following three steps.

1. For a basis function \( \tilde{\psi}_k \) of \( \tilde{M}_h \) associated with a vertex \( v_k \) on the boundary we find a closest internal triangle or tetrahedron \( K \in \mathcal{M} \) (that is, \( K \) does not have a boundary vertex).
2. Compute the barycentric coordinates \( \{ \alpha_{K,i} \}_{i=1}^{d+1} \) of \( v_k \) with respect to the vertices of \( K \), and modify all the basis functions \( \{ \tilde{\psi}_{K,i} \}_{i=1}^{d+1} \) of \( \tilde{M}_h \) associated with \( K \) into \( \psi_{K,i} = \tilde{\psi}_{K,i} + \alpha_{K,i} \tilde{\psi}_k \) for \( i = 1, \ldots, d+1 \).
3. Remove \( \tilde{\psi}_k \) from the basis of \( \tilde{M}_h \).

An alternative way is to modify the basis functions of all triangles or tetrahedra having one or more boundary vertices as proposed in [16].

1. If all vertices \( \{ v_i \}_{i=1}^{d+1} \) of an element \( K \in \mathcal{M} \) are inner vertices, then the linear basis functions \( \{ \psi_{v_i} \}_{i=1}^{d+1} \) of \( M_h \) on \( K \) are defined using the biorthogonal relationship (4.16) with the basis functions \( \{ \phi_{v_i} \}_{i=1}^{d+1} \) of \( V_h \).
2. If an element \( K \in \mathcal{M} \) has all boundary vertices, then we find a neighbouring element \( \tilde{K} \) which has at least one inner vertex \( v \), and we extend the support of the basis function \( \psi_v \in M_h \) associated with \( v \) to the element \( K \) by defining \( \psi_v = 1 \) on \( K \).
3. If an element \( K \in \mathcal{M} \) has only one inner vertex \( v \) and other boundary vertices, then the basis function \( \psi_v \in M_h \) associated with the inner vertex \( v \) is defined as \( \psi_v = 1 \) on \( K \).
4. If an element \( K \) has two inner vertices \( v_1 \) and \( v_2 \) and other boundary vertices, then the basis functions \( \psi_{v_1}, \psi_{v_2} \in M_h \) associated with these points are chosen to satisfy the biorthogonal relationship (4.16) with \( \phi_{v_1}, \phi_{v_2} \in V_h \), as well as the property \( \psi_{v_1} + \psi_{v_2} = 1 \) on \( K \).
5. In the three-dimensional case, we can have an element \( K \) with three inner vertices \( \{ v_i \}_{i=1}^{3} \) and one boundary vertex. In this case we define three basis functions \( \{ \psi_{v_i} \}_{i=1}^{3} \) to satisfy the biorthogonal relationship (4.16) with \( \{ \phi_{v_i} \}_{i=1}^{3} \) as well as the condition \( \sum_{i=1}^{3} \psi_{v_i} = 1 \) on \( K \).

The projection \( Q_h \) is stable in \( L^2 \) and \( H^1 \)-norms [18], and hence assumption \( (P1) \) follows. To establish \( (P2) \), we need the following mesh assumption.

\( (M) \) For any vertex \( v \), denoting by \( \mathcal{M}_v \) the set of cells having \( v \) as a vertex,

\[
\sum_{K \in \mathcal{M}_v} \frac{|K|}{S_v} (x_K - v) = O(h^2),
\]

where \( S_v \) is the support of the basis function \( \phi_v \) of \( V_h \) associated with \( v \).

This assumption is satisfied if the triangles of the mesh can be paired in sets of two that share a common edge and form an \( O(h^2) \)-parallelogram, that is, the lengths of
any two opposite edges differ only by $O(h^2)$. In three dimensions, (M) is satisfied if the lengths of each pair of opposite edges of a given element are allowed to differ only by $O(h^2)$ [5]. The following theorem establishes (P2) with $W = W^{3,\infty}(\Omega) \cap H^2_0(\Omega)$ and can be proved as in [30, 18].

**Theorem 4.5.** Let $u \in W^{3,\infty}(\Omega) \cap H^2_0(\Omega)$. Assume that the triangulation satisfies (M). Then

$$\|Q_h \nabla I_h u - \nabla u\| \leq C h^2 \| u \|_{W^{3,\infty}(\Omega)}.$$

Since $Q_h$ is a projection onto $V_h$, $Q_h I_h = I_h$. Hence, for $w \in H^2(\Omega) \cap H^1_0(\Omega)$, introducing $Q_h I_h w = I_h w$ and invoking the $H^1$-stability property of $Q_h$ [17, Lemma 1.8] leads to

$$\|\nabla Q_h w - \nabla w\| \leq \|\nabla Q_h (w - I_h w)\| + \|\nabla I_h w - \nabla w\| \leq C \|\nabla I_h w - \nabla w\|.$$

The standard approximation properties of $V_h$ then guarantee (P3). The Assumption (P4) is satisfied since $M_h \subset N_h$ ($M_h$ is obtained by combining functions in $\hat{M}_h$, that satisfies this property) and the basis functions of $M_h$ locally reproduce constant functions. To build $\mathcal{S}_h$ that satisfies (P5), divide each triangle $K \in \mathcal{M}$ into four equal triangles using the mid-points of each side, and define $\mathcal{S}_h$ as a piecewise constant function as described in Figure 1. It can be checked that this function satisfies (P5). A similar construction also works on tetrahedra (in which case $\mathcal{S}_h|_K$ is equal to 1 on the four sub-tetrahedra constructed around the vertices of $K$, and $-4$ in the rest of $K$).

![Figure 1. Values of the stabilisation function $\mathcal{S}_h$ inside a cell $K$.](image)

### 5. Finite volume method based on $\Delta$-adapted discretizations

We consider here the finite volume (FV) scheme from [12] for the biharmonic problem (2.4) on $\Delta$-adapted meshes, that is, meshes that satisfy an orthogonality property.

**Definition 5.1 ($\Delta$-adapted FV mesh).** A general mesh $\mathcal{T}$ is $\Delta$-adapted if

1. for all $\sigma \in \mathcal{F}_{\text{int}}$, denoting by $K, L \in \mathcal{M}$ the cells such that $\mathcal{M}_\sigma = \{K, L\}$, the straight line $(x_K, x_L)$ intersects and is orthogonal to $\sigma$,
2. for all $\sigma \in \mathcal{F}_{\text{ext}}$ with $\mathcal{M}_\sigma = \{K\}$, the line orthogonal to $\sigma$ going through $x_K$ intersects $\sigma$.

For such a mesh, we let $D_{K,\sigma}$ be the cone with vertex $x_K$ and basis $\sigma$, and $D_{\sigma} = \bigcup_{K \in \mathcal{M}_\sigma} D_{K,\sigma}$. For each $\sigma \in \mathcal{F}_{\text{int}}$, an orientation is chosen by defining one of the two unit normal vectors $n_\sigma$, and we denote by $K^-_{\sigma}$ and $K^+_{\sigma}$ the two adjacent control
We now define a notion of $B$–Hessian discretisation for $B = \frac{\text{tr}(\nabla^2 u)}{\sqrt{d}} \text{Id}$, in which case (2.2) corresponds to the biharmonic problem (2.4), for which the coercivity property (2.3) holds (see Section 2.1.1).

**Definition 5.2** (B–Hessian discretisation based on $\Delta$-adapted discretisation). Let $B = \frac{\text{tr}(\nabla^2 u)}{\sqrt{d}} \text{Id}$ and $T$ be a $\Delta$-adapted mesh. A $B$–Hessian discretisation is given by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{H}^B_{\mathcal{D}})$ where

- $X_{\mathcal{D},0}$ is the space of all real families $u_{\mathcal{D}} = (u_K)_{K \in \mathcal{M}}$, such that $u_K = 0$ for all $K \in \mathcal{M}$ with $\mathcal{F}_{K,\text{ext}} \neq \emptyset$.
- For $u_{\mathcal{D}} \in X_{\mathcal{D},0}$, $\Pi_{\mathcal{D}} u_{\mathcal{D}}$ is the piecewise constant function equal to $u_K$ on the control volume $K$.
- The discrete gradient $\nabla_{\mathcal{D}} u_{\mathcal{D}}$ is defined by its constant values on the cells:
  \[
  \nabla_K u_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma|(\delta_{K,\sigma} u_{\mathcal{D}})(x_{\sigma} - x_K),
  \]
  where
  \[
  \delta_{K,\sigma} u_{\mathcal{D}} = \begin{cases} u_L - u_K & \forall \sigma \in \mathcal{F}_{K,\text{int}}, \mathcal{M}_\sigma = \{K, L\} \\ 0 & \forall \sigma \in \mathcal{F}_{K,\text{ext}}. \end{cases}
  \]
- The discrete Laplace operator $\Delta_{\mathcal{D}}$ is defined by its constant values on the cells:
  \[
  \Delta_K u_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \delta_{K,\sigma} u_{\mathcal{D}} \frac{d_\sigma}{d_\sigma}.
  \]

We then set $\mathcal{H}^B_{\mathcal{D}} u_{\mathcal{D}} = \frac{\Delta_{\mathcal{D}} u_{\mathcal{D}}}{\sqrt{d}} \text{Id}$.

For $u_{\mathcal{D}}, v_{\mathcal{D}} \in X_{\mathcal{D},0}$,
\[
[u_{\mathcal{D}}, v_{\mathcal{D}}] = \sum_{\sigma \in \mathcal{F}} |\sigma| \delta_{\sigma} u_{\mathcal{D}} \delta_{\sigma} v_{\mathcal{D}} \frac{d_\sigma}{d_\sigma}
\]
defines an inner product on $X_{\mathcal{D},0}$, whose associated norm is denoted by $\|u_{\mathcal{D}}\|_{\mathcal{D}}$.

Here $\delta_\sigma$ is given by
\[
\delta_{\sigma} u_{\mathcal{D}} = \begin{cases} u_K^+ - u_K^- & \forall \sigma \in \mathcal{F}_{\text{int}} \\ 0 & \forall \sigma \in \mathcal{F}_{\text{ext}}. \end{cases}
\]

It can easily be checked that, with this Hessian discretisation, the Hessian scheme (2.2) is the scheme of [12] for the biharmonic equation. Let us examine the properties of this Hessian discretisation.

**Theorem 5.3.** Let $\mathcal{D}$ be a $B$–Hessian discretisation in the sense of Definition 5.2. Then there exists a constant $C$, depending only on $\theta \geq \theta_T$, such that
• $C_B^\mathbb{D} \leq C$.

• If $\varphi \in C^2(\Omega)$, $\Delta \varphi \in H^1(\Omega)$ and $a > 0$ is such that $\text{supp}(\varphi) \subset \{x \in \Omega; \text{dist}(x, \partial \Omega) > a\}$, then

$$S_B^\mathbb{D}(\varphi) \leq C h \|\Delta \varphi\|_{H^1(\Omega)} + Ch\|\varphi\|_{C^2(\mathbb{D})} \times \begin{cases} \frac{|\ln(a)| a^{-3/2}}{a^{-5/3}} & \text{if } d = 2, \\ \frac{h^{-5/3}}{\ln(h)} & \text{if } d = 3. \end{cases} (5.7)$$

• If $\varphi \in H^2_0(\Omega) \cap C^2(\mathbb{D})$ with $\Delta \varphi \in H^1(\Omega)$, then

$$S_B^\mathbb{D}(\varphi) \leq C h \|\Delta \varphi\|_{H^1(\Omega)} + C\|\varphi\|_{C^2(\mathbb{D})} \times \begin{cases} \frac{h^{1/4}\ln(h)}{h^{3/13}} & \text{if } d = 2, \\ \frac{h^{3/13}}{h^{3/13}} & \text{if } d = 3. \end{cases} (5.8)$$

• $\forall \xi \in H^2(\Omega)^{d \times d}$, $W_B^\mathbb{D}(\xi) \leq C h \|\text{tr}(\xi)\|_{H^2(\Omega)}$.

**Remark 5.4.** If the solution $\pi$ to (2.4) belongs to $H^4(\Omega) \cap H^2_0(\Omega)$, then $\pi \in C^2(\mathbb{D})$ and $\Delta \pi \in H^2(\Omega)$. In that case, Theorems 3.6 and 5.3 provide an $O(h^{1/4}\ln(h))$ (in dimension $d = 2$) or $O(h^{3/13})$ (in dimension $d = 3$) error estimate for the Hessian scheme based on the HD from Definition 5.2. This slightly improves the result of [12, Theorem 4.3], in which an $O(h^{1/5})$ estimate is obtained if $\pi \in C^4(\mathbb{D}) \cap H^2_0(\Omega)$.

As for the method based on gradient recovery operators, the properties of the Hessian discretisation follow from the estimates in Theorem 5.3 and from Remark 3.4.

**Corollary 5.5.** Let $(\mathbb{D}_m)_{m \in \mathbb{N}}$ be a sequence of $B$-Hessian discretisations in the sense of Definition 5.2, associated to meshes such that $h_m \to 0$ and $(\theta_{m, \tau})_{m \in \mathbb{N}}$ is bounded. Then the sequence $(\mathbb{D}_m)_{m \in \mathbb{N}}$ is coercive, consistent and limit-conforming.

Proof of Theorem 5.3.

• **Coercivity:** the discrete Poincaré inequality of [11] states that

$$\|\Pi_Dv_D\| \leq \text{diam}(\Omega)\|v_D\|_D, \quad \forall v_D \in X_D, \quad (5.9)$$

Let us first prove that

$$-\int_\Omega \Pi_D u_D \Delta_D v_D dx = [u_D, v_D]_D, \quad u_D, v_D \in X_D. \quad (5.10)$$

The definitions of $\Pi_D$ and $\Delta_D$ yield

$$-\int_\Omega \Pi_D u_D \Delta_D v_D dx = \sum_{K \in M} -|K|u_K \Delta_K v_D = -\sum_{K \in M} u_K \sum_{\sigma \in \mathcal{F}_K} \frac{|\sigma| \delta_{K, \sigma} v_D}{d_{\sigma}}. \quad (5.11)$$

For $\sigma \in \mathcal{F}_{\text{ext}}$, $\delta_{K, \sigma} v_D = 0$. Gathering the sums by edges and using (5.3) and (5.6), we obtain

$$-\int_\Omega \Pi_D u_D \Delta_D v_D dx = \sum_{K \in M} u_K \sum_{\sigma \in \mathcal{F}_{K, \text{int}}} \frac{|\sigma|(v_K - v_L)}{d_{\sigma}} = \sum_{\sigma \in \mathcal{F}_{\text{int}}} \frac{|\sigma| \delta_{\sigma} u_D \delta_{\sigma} v_D}{d_{\sigma}},$$

which establishes (5.10). Choosing $v_D = u_D$, applying the Cauchy–Schwarz inequality and using (5.9), we get

$$\|u_D\|^2 \leq \|\Pi_D u_D\|\|\Delta_D u_D\| \leq \text{diam}(\Omega)\|u_D\|_D\|\Delta_D u_D\|. \quad (5.11)$$

Thus,

$$\|u_D\|_D \leq \text{diam}(\Omega)\|\Delta_D u_D\|. \quad (5.11)$$

Combining (5.9) and (5.11), we get

$$\|\Pi_D v_D\| \leq \text{diam}(\Omega)^2\|\Delta_D u_D\|. \quad (5.12)$$
The stability of the discrete gradient [12, Lemma 4.1] yields
\[ \| \nabla_D u_D \| \leq \theta \sqrt{d} \| u_D \|_D \quad \forall u_D \in X_{D,0}. \]

Estimate (5.11) then shows that \( \| \nabla_D u_D \| \leq \text{diam}(\Omega) \theta \sqrt{d} \| \Delta_D u_D \| \), which, together with (5.12), concludes the proof of the estimate on \( C_D^B \).

- **Consistency – compact support:** The proof utilises the ideas of [12], with a few improvements of the estimates. For \( s > 0 \) we let \( \Omega_s = \{ x \in \Omega; \text{dist}(x, \partial \Omega) > s \} \).

In this proof, \( A \lesssim B \) means that \( A \leq CB \) for some constant \( C \) depending only on \( \theta \).

We first consider the case where \( \varphi \in C^2_\infty(\Omega) \) and \( \Delta \varphi \in H^1(\Omega) \), with support at distance from \( \partial \Omega \) equal to or greater than \( a \). As in [12, Proof of Lemma 4.4], let \( \psi^a \in C^\infty_c(\Omega) \), equal to 1 on \( \Omega_{3a/4} \), that vanishes on \( \Omega \setminus \Omega_{a/4} \), and such that, for all \( \alpha \in \mathbb{N}^d \), with \( |\alpha| = \sum_{i=1}^d \alpha_i \),
\[ \| \partial^a \psi^a \|_{L^\infty(\Omega)} \lesssim a^{-|\alpha|}. \]

Letting \( \psi^a_K = (\psi^a(x_K))_{K \in \mathcal{M}} \), we have \( |\Delta_D \psi^a_K| \lesssim a^{-2} \). Hence, for all \( r \in [1, \infty) \), since \( \Omega \setminus \Omega_{a/4} \) has measure \( \lesssim a \),
\[ \| \Delta_D \psi^a_K \|_{L^r(\Omega)} \lesssim a^{-2+\frac{2}{r}}. \]

Letting \( \tilde{v} = (\tilde{v}_K)_{K \in \mathcal{M}} \) be the solution of the two-point flux approximation finite volume scheme with homogeneous Dirichlet boundary conditions and source term \( -\Delta \varphi \), by [11] we have, with \( \varphi_D = (\varphi(x_K))_{K \in \mathcal{M}} \),
\[ \left( \sum_{\sigma \in F} \frac{|\sigma|}{d} (\delta_\sigma (\tilde{v} - \varphi_D))^2 \right)^{1/2} \lesssim h \| \varphi \|_{C^2(\Omega)}. \]

and, for \( q \in [1, +\infty) \) if \( d = 2 \), \( q \in [1, 6] \) if \( d = 3 \),
\[ \left( \sum_{K \in \mathcal{M}} |K| |\tilde{v}_K - \varphi(x_K)|^q \right)^{1/q} \lesssim qh \| \varphi \|_{C^2(\Omega)}. \]

We then set \( w = (\psi^a(x_K) \tilde{v}_K)_{K \in \mathcal{M}} \), that belongs to \( X_{D,0} \) if \( h \leq a/4 \). It is proved in [12, Proof of Lemma 4.4, p. 2032] that, with \( |\Delta \varphi|_K = \frac{1}{|K|} \int_K \Delta \varphi \, dx \),
\[ \Delta_K w - |\Delta \varphi|_K = (\tilde{v}_K - \varphi(x_K)) \Delta_K \psi^a_K + \frac{1}{|K|} \sum_{\sigma \in F_K} \frac{|\sigma|}{d} (\delta_{\sigma} \psi^a_K) \delta_{\sigma} (\tilde{v} - \varphi_D), \]
\[ = T_{1,K} + T_{2,K}. \]

Using Hölder’s inequality with exponents \( (q, \frac{2d}{q-2}) \), for some \( q > 2 \) admissible in (5.16), and recalling (5.14), we have
\[ \left( \sum_{K \in \mathcal{M}} |K| |T_{1,K}|^2 \right)^{1/2} \lesssim qha^{-2+\frac{2}{q}} \| \varphi \|_{C^2(\Omega)}. \]

On the other hand, we have \( |\delta_{K,\sigma} \psi^a_D| \lesssim d \sigma a^{-1} \) (see [12, Proof of Lemma 4.4]). Hence, by Cauchy–Schwarz inequality on the sum over the faces, and using the estimate \( \sum_{\sigma \in F_K} |\sigma| d \sigma \lesssim |K| \),
\[ |T_{2,K}|^2 \lesssim \frac{a^{-2}}{|K|^2} \left( \sum_{\sigma \in F_K} |\sigma| \delta_{K,\sigma} (\tilde{v} - \varphi_D) \right)^2 \lesssim \frac{a^{-2}}{|K|} \sum_{\sigma \in F_K} |\sigma| (\delta_{K,\sigma} (\tilde{v} - \varphi_D))^2. \]
Estimate (5.15) thus leads to
\[
\left( \sum_{K \in \mathcal{M}} |K| |T_{2,K}|^2 \right)^{1/2} \lesssim a^{-1} h \| \varphi \|_{C^2(\Omega)}.
\] (5.19)

Denote by $|\Delta \varphi|_D$ the piecewise constant function equal to $|\Delta \varphi|_K$ on $K \in \mathcal{M}$. Taking the $L^2$ norm of (5.17) and using (5.18) and (5.19), we arrive at, since $a^{-1} \lesssim a^{-2 - \frac{1}{2}},$
\[
\| \Delta_D w - [\Delta \varphi]|_D \|_{L^2(\Omega)} \lesssim q h a^{-2 - \frac{1}{2}} \| \varphi \|_{C^2(\Omega)}.
\]

Taking $q = |\ln(a)|$ if $d = 2$ or $q = 6$ if $d = 3$ shows that
\[
\| \Delta_D w - [\Delta \varphi]|_D \|_{L^2(\Omega)} \lesssim h \| \varphi \|_{C^2(\Omega)} \times \begin{cases} \frac{|\ln(a)|}{a^{-3/2}} & \text{if } d = 2, \\ a^{-5/3} & \text{if } d = 3. \end{cases} \] (5.20)

A classical estimate [10, Lemma B.6] gives
\[
\| [\Delta \varphi]|_D - [\Delta \varphi] \|_{L^2(\Omega)} \lesssim h \| \Delta \varphi \|_{H^1(\Omega)},
\] (5.21)

which shows that $\| \Delta_D w - \Delta \varphi \|_{L^2(\Omega)}$ is bounded above by the right-hand side of (5.7). The estimates on $\nabla \Delta_D w - \nabla \varphi$ and on $\Pi_D w - \varphi$ follow as in [12, Lemma 4.4].

**Consistency – General Case:** Consider now $\varphi \in H^2_0(\Omega) \cap C^2(\Omega)$, and take $\psi$ as above. The boundary conditions on $\varphi$ show that $|\varphi(x)| \lesssim \| \varphi \|_{C^2(\Omega)} \text{dist}(x, \partial \Omega)^2$ and $|\nabla \varphi(x)| \lesssim \| \varphi \|_{C^2(\Omega)} \text{dist}(x, \partial \Omega)$. Hence, using (5.13), $|\Omega \setminus \Omega_a| \lesssim a$ and the fact that $1 - \psi^a = 0$ in $\Omega_a$, we see that, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2,$
\[
\| \partial^\alpha \varphi - \partial^\alpha (\psi \varphi) \|_{L^2(\Omega)} \lesssim a^{1/2} \| \varphi \|_{C^2(\Omega)}.
\] (5.22)

Since $\Delta = \sum_{i=1}^d \partial_i^2$, the above estimate applies to $\Delta$ instead of $\partial^\alpha$ and, as a consequence,
\[
[\| \Delta \varphi|_D - [\Delta (\psi \varphi)]|_D \|_{L^2(\Omega)} \lesssim \| \Delta \varphi - \Delta (\psi \varphi) \|_{L^2(\Omega)} \lesssim a^{1/2} \| \varphi \|_{C^2(\Omega)}.
\] (5.23)

Consider now the interpolant $w \in X_{D,0}$ for $\psi \varphi \in C^2_0(\Omega)$ constructed above. Applying (5.20) to $\psi \varphi$ instead of $\varphi$, noting that $\| \psi \varphi \|_{L^1(\Omega) \cap C^2(\Omega)} \lesssim \| \varphi \|_{C^2(\Omega)}$ (consequence of (5.22)), and using (5.23), we obtain
\[
\| \Delta_D w - [\Delta \varphi]|_D \|_{L^2(\Omega)} \lesssim a^{1/2} \| \varphi \|_{C^2(\Omega)} + h \| \varphi \|_{C^2(\Omega)} \times \begin{cases} \frac{|\ln(a)|}{a^{-3/2}} & \text{if } d = 2, \\ a^{-5/3} & \text{if } d = 3. \end{cases} \]

Taking $a = h^{1/2}$ if $d = 2$ or $a = h^{6/13}$ if $d = 3$ leads to
\[
\| \Delta_D w - [\Delta \varphi]|_D \|_{L^2(\Omega)} \lesssim \| \varphi \|_{C^2(\Omega)} \times \begin{cases} h^{1/4} |\ln(h)| & \text{if } d = 2, \\ h^{3/13} & \text{if } d = 3. \end{cases} \]

Combined with (5.21) this shows that $\| \Delta_D w - \Delta \varphi \|_{L^2(\Omega)}$ is bounded above by the right-hand side of (5.8). The estimates on $\Pi_D w - \varphi$ and $\nabla \Delta_D w - \nabla \varphi$ follow in a similar way.

**Limit-Formality:** For $\xi \in H^B(\Omega)$ and $v_D \in X_{D,0}$, $B = \frac{\text{tr}(\xi)}{\sqrt{d}}$ Id implies
\[
\int_\Omega (H : B^\tau B \xi) \Pi_D v_D \, dx = \int_\Omega (B^\tau H : B \xi) v_D \, dx = \int_\Omega \Delta \phi \Pi_D v_D \, dx,
\]
where $\phi = \text{tr}(\xi)$. Also, by definition of $H_D^B$,
\[
\int_\Omega B \xi : H_D^B v_D \, dx = \int_\Omega \phi \Delta_D v_D \, dx.
\]
Thus, (3.4) can be rewritten as

\[
W^B_D(\xi) = \max_{v_D \in X_D \setminus \{0\}} \frac{1}{\|H^D_D\|} \left|\int_{\Omega} (\Delta \phi D v_D - \phi D v_D) \, dx\right|, \tag{5.24}
\]

where \(\phi = \text{tr}(\xi)\). Define

\[
\tilde{\delta}_\sigma \phi = \begin{cases} 
\phi(x_{K,\sigma}^+) - \phi(x_{K,\sigma}^-) & \forall \sigma \in \mathcal{F}_{\text{int}} \\
\phi(z_{\sigma}) - \phi(x_{K,\sigma}) & \forall \sigma \in \mathcal{F}_{\text{ext}},
\end{cases} \tag{5.25}
\]

where \(z_{\sigma}\) is the orthogonal projection of \(x_K\) on the hyperplane which contains \(\sigma\).

For \(\xi \in H^2(\Omega)^{d \times d}\), using the divergence theorem,

\[
\int_{\Omega} \Delta \phi D v_D \, dx = \sum_{K \in \mathcal{M}} \int_K \Delta \phi D v_D \, dx = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} v_K \int_{\sigma} \nabla \phi \cdot n_{K,\sigma} \, ds(x).
\]

Gathering over the edges and using the definition of \(\delta_\sigma\), this leads to

\[
\int_{\Omega} \Delta \phi D v_D \, dx = - \sum_{\sigma \in \mathcal{F}_{\text{ext}}} \delta_\sigma v_D \int_{\sigma} \nabla \phi \cdot n_{\sigma} \, ds(x)
= - \sum_{\sigma \in \mathcal{F}_{\text{ext}}} \delta_\sigma v_D \int_{\sigma} \left( \frac{\hat{\delta}_\sigma \phi}{d_\sigma} + \nabla \phi \cdot n_{\sigma} - \frac{\hat{\delta}_\sigma \phi}{d_\sigma} \right) \, ds(x)
= - \sum_{\sigma \in \mathcal{F}_{\text{ext}}} \delta_\sigma v_D \frac{\hat{\delta}_\sigma \phi}{d_\sigma} + \sum_{\sigma \in \mathcal{F}_{\text{ext}}} \delta_\sigma v_D \int_{\sigma} \left( \frac{\hat{\delta}_\sigma \phi}{d_\sigma} - \nabla \phi \cdot n_{\sigma} \right) \, ds(x). \tag{5.26}
\]

Since \(\delta_\sigma v_D = 0\) for any \(\sigma \in \mathcal{F}_{\text{ext}}\), (5.25), (5.3) and (5.4) imply

\[
- \sum_{\sigma \in \mathcal{F}} \delta_\sigma v_D \frac{\hat{\delta}_\sigma \phi}{d_\sigma} = - \sum_{\sigma \in \mathcal{F}_{\text{ext}}} |\sigma| \delta_\sigma v_D \left( \phi(x_{K,\sigma}^+) - \phi(x_{K,\sigma}^-) \right)
= \sum_{K \in \mathcal{M}} \phi(x_K) \sum_{\sigma \in \mathcal{F}_K} |\sigma| \delta_{K,\sigma} v_D = \sum_{K \in \mathcal{M}} |K| \phi(x_K) \Delta_K v_D.
\]

Substituting this in (5.26), we obtain

\[
\int_{\Omega} \Delta \phi D v_D \, dx = \sum_{K \in \mathcal{M}} |K| \phi(x_K) \Delta_K v_D + \sum_{\sigma \in \mathcal{F}} \delta_\sigma v_D \int_{\sigma} \left( \frac{\hat{\delta}_\sigma \phi}{d_\sigma} - \nabla \phi \cdot n_{\sigma} \right) \, ds(x). \tag{5.27}
\]

To deal with the first term, we first combine the two estimates in [10, Lemma 7.61] to see that

\[
|\phi(x_K) - \phi(y)| \leq Ch|K|^{-1/2} \|\phi\|_{H^2(K)}, \quad \forall y \in K.
\]
Hence, using the Cauchy–Schwarz inequality,
\[
\left| \sum_{K \in M} |K| \phi(x_K) \Delta_K v_D - \int_{\Omega} \phi \Delta_D v_D \, dx \right| \\
= \left| \sum_{K \in M} |K| \left( \phi(x_K) - \frac{1}{|K|} \int_K \phi(y) \, dy \right) \Delta_K v_D \right| \\
\leq Ch \| \phi \|_{H^2(\Omega)} \left( \sum_{K \in M} |K| \| \Delta_K v_D \|^2 \right)^{1/2} = Ch \| \phi \|_{H^2(\Omega)} \| \Delta_D v_D \|. \quad (5.28)
\]

Turning to the second term in the right-hand side of (5.27), we notice that the estimate on the terms $R_{K,\sigma}$ in [11, Proof of Theorem 3.4] show that
\[
\left| \delta_{\sigma} \phi d_{\sigma} - \nabla \phi \cdot n_{\sigma} \right| \leq Ch \sqrt{\frac{\sigma}{\Delta_{\sigma}}} \| \mathcal{H} \phi \|_{L^2(|L_{L,M,M}, L)^{d \times d}}.
\]

Hence, by the Cauchy–Schwarz inequality, we have
\[
\left| \sum_{\sigma \in \mathcal{F}} \delta_{\sigma} v_D \int_{\sigma} \left( \frac{\delta_{\sigma} \phi}{d_{\sigma}} - \nabla \phi \cdot n_{\sigma} \right) \, ds(x) \right| \leq Ch \| \mathcal{H} \phi \| \left( \sum_{\sigma \in \mathcal{F}} \frac{\sigma}{d_{\sigma}} \left( \delta_{\sigma} v_D \right)^2 \right)^{1/2} = Ch \| \phi \|_{H^2(\Omega)} \| v_D \|_D \leq Ch \text{diam}(\Omega) \| \phi \|_{H^2(\Omega)} \| \Delta_D v_D \| \), \quad (5.29)
\]

where we have used (5.11) in the last line. Plugging (5.28) and (5.29) into (5.27), we obtain
\[
\left| \int_{\Omega} \Delta \phi \Pi_D v_D \, dx - \int_{\Omega} \phi \Delta_D v_D \, dx \right| \leq Ch \| \phi \|_{H^2(\Omega)} \| \Delta_D v_D \|,
\]
and the estimate on $W_D(\xi)$ then follows from (5.24), recalling that $\phi = \text{tr}(\xi)$.

\[\square\]

**Remark 5.6.** The same analysis also probably applies to the second method presented in [12, Section 5], which is applicable on general polygonal meshes.

6. Numerical results

In this section, we present the results of some numerical experiments for the gradient recovery (GR) method and finite volume (FV) method presented in Sections 4 and 5. All these tests are conducted on the biharmonic problem $\Delta^2 \overline{u} = f$ on $\Omega = (0, 1)^2$, with clamped boundary conditions and for various exact solutions $\overline{u}$.

6.1. Numerical results for Gradient Recovery method. Three examples are presented to illustrate the theoretical estimates of Theorem 3.6 on the Hessian discretisation described in Section 4.2. The considered FE space $V_h$ is therefore the conforming $P_1$ space, and the implementation was done following the ideas in [20]. The following relative errors, and related orders of convergence, in $L^2(\Omega)$, $H^1(\Omega)$
and $H^2(\Omega)$ norms are presented:

$$
\text{err}_D(u) := \frac{\|\Pi Du - u\|}{\|u\|}, \quad \text{err}(\nabla u) := \frac{\|\nabla u_D - \nabla u\|}{\|\nabla u\|},
$$

$$
\text{err}_D(\nabla u) := \frac{\|\nabla Du - \nabla u\|}{\|\nabla u\|} = \frac{\|Q_h \nabla u_D - \nabla u\|}{\|\nabla u\|},
$$

$$
\text{err}_D(H^2 u) := \frac{\|H^2 Du - H^2 u\|}{\|H^2 u\|} = \frac{\|\nabla (Q_h \nabla u_D) - H^2 u\|}{\|H^2 u\|},
$$

where $u_D$ is the solution to the Hessian scheme (3.1).

We provide in Table 1 the mesh data: mesh sizes $h$, numbers of unknowns (that is, the number of internal vertices) $n_u$, and numbers of non-zero terms $n_{nz}$ in the square matrix of the system.

**Table 1.** (GR) Mesh size, number of unknowns and number of non-zero terms in the square matrix

| $h$     | $n_u$ | $n_{nz}$ |
|---------|-------|----------|
| 0.176777 | 9     | 79       |
| 0.088388 | 49    | 1203     |
| 0.044194 | 225   | 7011     |
| 0.022097 | 961   | 32835    |
| 0.011049 | 3969  | 141315   |
| 0.005524 | 16129 | 585603   |

6.1.1. Example 1. The exact solution is chosen to be $u(x, y) = x^2(x - 1)^2y^2(y - 1)^2$.

To assess the effect of the stabilisation function $\mathcal{S}_h$ on the results, we multiply it by a factor $r$ that takes the values 0.1, 1, 10, and 100.

The errors and orders of convergence for the numerical approximation to $u$ are shown in Tables 2–5. It can be seen that the rate of convergence is quadratic in $L^2$-norm and linear in $H^1$-norm (see $\text{err}(\nabla u)$). However, using gradient recovery operator, a quadratic order of convergence in $H^1$ norm is recovered (see $\text{err}_D(\nabla u)$).

The rate of convergence in energy norm is linear (see $\text{err}_D(H^2 u)$), as expected by plugging the estimates of Theorem 4.3 into Theorem 3.6. We also notice a very small effect of $r$ on the relative errors and rates.

**Table 2.** (GR) Convergence results for the relative errors, Example 1, $r = 0.1$

| $n_u$ | $\text{err}_D(u)$ | Order | $\text{err}(\nabla u)$ | Order | $\text{err}_D(\nabla u)$ | Order | $\text{err}_D(H^2 u)$ | Order |
|-------|-------------------|-------|------------------------|-------|-------------------------|-------|------------------------|-------|
| 9     | 9.274702          | -     | 31.591906              | -     | 0.568338                | -     | 0.595635               | -     |
| 49    | 0.220065          | 5.3971| 0.682922               | 5.5317| 0.164105                | 1.7921| 0.266927               | 1.1580|
| 225   | 0.066997          | 1.7160| 0.201282               | 1.7625| 0.049395                | 1.7322| 0.128410               | 1.0557|
| 961   | 0.019135          | 1.8079| 0.088805               | 1.1805| 0.013697                | 1.8505| 0.062164               | 1.0460|
| 3969  | 0.005133          | 1.8983| 0.040845               | 1.1205| 0.003623                | 1.9185| 0.030457               | 1.0293|
| 16129 | 0.001331          | 1.9474| 0.019422               | 1.0724| 0.000933                | 1.9568| 0.015059               | 1.0161|

6.1.2. Example 2. We consider here the transcendental exact solution $u = x^2(x - 1)^2y^2(y - 1)^2(\cos(2\pi x) + \sin(2\pi y))$, and $r = 0.1, 1$ and 10. Tables 6–8 presents the numerical results. The same comments as in Example 1 can be made about the rates of convergence. Past the coarsest meshes, we also notice as in Example 1 that $r$ only has a small impact on the relative errors.
| \( \nu \) | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\psi) \) | Order | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\nabla^2 \psi) \) | Order |
|---|---|---|---|---|---|---|---|---|
| 9  | 1.056930 | - | 3.254044 | - | 0.567670 | - | 0.582647 | - |
| 49 | 0.214195 | 2.2947 | 0.482666 | 2.7531 | 0.167145 | 1.7640 | 0.267188 | 1.1248 |
| 225 | 0.067498 | 1.6660 | 0.200108 | 1.2703 | 0.049952 | 1.7425 | 0.128511 | 1.1060 |
| 961 | 0.019240 | 1.8107 | 0.088607 | 1.1743 | 0.013806 | 1.8553 | 0.062184 | 1.0473 |
| 3969 | 0.005156 | 1.8999 | 0.048835 | 1.1186 | 0.003646 | 1.9209 | 0.030460 | 1.0296 |
| 16129 | 0.001336 | 1.9482 | 0.019421 | 1.0722 | 0.000938 | 1.9581 | 0.015060 | 1.0162 |

Table 4. (GR) Convergence results for the relative errors, Example 1, \( r = 10 \)

| \( \nu \) | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\psi) \) | Order | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\nabla^2 \psi) \) | Order |
|---|---|---|---|---|---|---|---|---|
| 9  | 0.661894 | - | 0.778521 | - | 0.583641 | - | 0.586174 | - |
| 49 | 0.236529 | 1.4846 | 0.449484 | 0.7925 | 0.195127 | 1.5807 | 0.274630 | 1.0970 |
| 225 | 0.072610 | 1.7038 | 0.197892 | 1.1836 | 0.055493 | 1.8140 | 0.129911 | 1.0768 |
| 961 | 0.020393 | 1.9335 | 0.088413 | 1.1621 | 0.014907 | 1.8463 | 0.062418 | 1.0575 |
| 3969 | 0.005382 | 1.9154 | 0.048804 | 1.1156 | 0.003877 | 1.9429 | 0.030949 | 1.0335 |
| 16129 | 0.001387 | 1.9564 | 0.019417 | 1.0714 | 0.000990 | 1.9695 | 0.015064 | 1.0174 |

Table 5. (GR) Convergence results for the relative errors, Example 1, \( r = 100 \)

| \( \nu \) | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\psi) \) | Order | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\nabla^2 \psi) \) | Order |
|---|---|---|---|---|---|---|---|---|
| 9  | 0.784444 | - | 0.805690 | - | 0.701021 | - | 0.695247 | - |
| 49 | 0.499420 | 0.9381 | 0.456340 | 0.8201 | 0.386868 | 0.8576 | 0.408218 | 0.7680 |
| 225 | 0.123166 | 1.7330 | 0.199370 | 1.1947 | 0.108498 | 1.8342 | 0.157333 | 1.3757 |
| 961 | 0.031509 | 1.9667 | 0.088447 | 1.1726 | 0.026358 | 2.0414 | 0.066443 | 1.2436 |
| 3969 | 0.007812 | 2.0121 | 0.040790 | 1.1166 | 0.006356 | 2.0521 | 0.031019 | 1.0990 |
| 16129 | 0.001934 | 2.0139 | 0.019414 | 1.0711 | 0.001552 | 2.0340 | 0.015130 | 1.0357 |

Table 6. (GR) Convergence results for the relative errors, Example 2, \( r = 0.1 \)

| \( \nu \) | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\psi) \) | Order | \( \text{err}_2(\nabla \psi) \) | Order | \( \text{err}_2(\nabla^2 \psi) \) | Order |
|---|---|---|---|---|---|---|---|---|
| 9  | 89.04689 | 183.46172 | - | 1.211097 | - | 1.614525 | - |
| 49 | 0.821060 | 6.7538 | 3.403174 | 5.7532 | 0.235295 | 2.3638 | 0.501568 | 1.6866 |
| 225 | 0.076841 | 3.4246 | 0.337917 | 3.3314 | 0.050832 | 2.2107 | 0.172316 | 1.5414 |
| 961 | 0.017830 | 2.1076 | 0.114315 | 1.5637 | 0.013579 | 1.9044 | 0.079638 | 1.1135 |
| 3969 | 0.004565 | 1.9655 | 0.052228 | 1.1301 | 0.003638 | 1.9002 | 0.039166 | 1.0239 |
| 16129 | 0.001168 | 1.9662 | 0.025518 | 1.0333 | 0.000949 | 1.9391 | 0.019457 | 1.0093 |

6.1.3. Example 3. Here, \( \psi(x,y) = x^3 y^3 (1-x)^3 (1-y)^3 (e^{x+y} \sin(2\pi x) + \cos(2\pi x)) \) and \( r = 0.1, 1 \), and 10. The results presented in Tables 9–11 are similar to those obtained for Examples 1 and 2.

6.2. Numerical results for FVM. In this section, we present numerical results based on the finite volume method presented in Section 5. As noticed, this scheme requires only one unknown per cell, and is therefore easy to implement and computationally cheap. The schemes were first tested on a series of regular triangular meshes (mesh1 family) and then on square meshes (mesh2 family), both taken from
TABLE 7. (GR) Convergence results for the relative errors, Example 2, $r = 1$

| $n_U$ | $\text{err}_{D}(\widehat{\tau})$ | Order | $\text{err}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(H\widehat{\tau})$ | Order |
|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|
| 9     | 0.222667                       |       | 19.376883                      |       | 1.058048                       |       | 1.333720                       |       |
| 49    | 0.475973                       |       | 4.4247                         |       | 1.467316                       |       | 0.229176                       |       |
| 225   | 0.034399                       |       | 2.6775                         |       | 0.313397                       |       | 0.057555                       |       |
| 961   | 0.017711                       |       | 2.0706                         |       | 0.112806                       |       | 0.013591                       |       |
| 3969  | 0.005457                       |       | 1.9615                         |       | 0.052162                       |       | 0.003640                       |       |
| 16129 | 0.001164                       |       | 1.9657                         |       | 0.025515                       |       | 0.000949                       |       |

TABLE 8. (GR) Convergence results for the relative errors, Example 2, $r = 10$

| $n_U$ | $\text{err}_{D}(\widehat{\tau})$ | Order | $\text{err}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(H\widehat{\tau})$ | Order |
|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|
| 9     | 1.413122                       |       | 2.511434                       |       | 0.845365                       |       | 0.894504                       |       |
| 49    | 0.313425                       |       | 2.1727                         |       | 0.878752                       |       | 1.5319                         |       |
| 225   | 0.066842                       |       | 2.2293                         |       | 0.262354                       |       | 1.7439                         |       |
| 961   | 0.016897                       |       | 1.9840                         |       | 0.109794                       |       | 1.2567                         |       |
| 3969  | 0.004376                       |       | 1.9492                         |       | 0.052012                       |       | 1.0779                         |       |
| 16129 | 0.001123                       |       | 1.9621                         |       | 0.025506                       |       | 1.0280                         |       |

TABLE 9. (GR) Convergence results for the relative errors, Example 3, $r = 0.1$ 

| $n_U$ | $\text{err}_{D}(\widehat{\tau})$ | Order | $\text{err}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(H\widehat{\tau})$ | Order |
|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|
| 9     | 164.35430                      |       | 6.1230                         |       | 0.323374                       |       | 2.2013                         |       |
| 49    | 6.9153                         |       | 2.358209                       |       | 0.232374                       |       | 1.517095                       |       |
| 225   | 2.8602                         |       | 0.447143                       |       | 2.3989                         |       | 0.048701                       |       |
| 961   | 2.4459                         |       | 0.125296                       |       | 1.8354                         |       | 0.010361                       |       |
| 3969  | 2.1074                         |       | 0.053941                       |       | 1.2159                         |       | 0.002643                       |       |
| 16129 | 2.0167                         |       | 0.026457                       |       | 1.0278                         |       | 0.000692                       |       |

TABLE 10. (GR) Convergence results for the relative errors, Example 3, $r = 1$ 

| $n_U$ | $\text{err}_{D}(\widehat{\tau})$ | Order | $\text{err}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(\nabla \widehat{\tau})$ | Order | $\text{err}_{D}(H\widehat{\tau})$ | Order |
|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|--------------------------------|-------|
| 9     | 16.990965                      |       | 3.5113                         |       | 0.950590                       |       | 0.990455                       |       |
| 49    | 4.0744                         |       | 1.490046                       |       | 3.5113                         |       | 0.228477                       |       |
| 225   | 2.5326                         |       | 0.414243                       |       | 1.8468                         |       | 0.048056                       |       |
| 961   | 2.4005                         |       | 0.122315                       |       | 1.7599                         |       | 0.010349                       |       |
| 3969  | 2.0975                         |       | 0.053813                       |       | 1.1846                         |       | 0.002646                       |       |
| 16129 | 2.0153                         |       | 0.026452                       |       | 1.0246                         |       | 0.000693                       |       |

[15]. To ensure the correct orthogonality property (see Definition 5.1), the point $x_K \in K$ is chosen as the circumcenter of $K$ if $K$ is a triangle, or the center of mass of $K$ if $K$ is a rectangle. As a result, for triangular meshes, the $L^2$ error, $\text{err}_{D}(\widehat{\tau})$, is calculated using a skewed midpoint rule, where we consider the circumcenter of each cell instead of its center of mass. We denote the relative $H^2$ error by

$$\text{err}_{D}(\Delta \tau) := \frac{\|\Delta_{D} \tau_{D} - \Delta \tau\|}{\|\Delta \tau\|}.$$
Table 11. (GR) Convergence results for the relative errors, Example 3, $r = 10$

| $h$   | $\nu$ | $\text{err}_{D}(u)$ | Order | $\text{err}_{D}(\nabla u)$ | Order | $\text{err}_{D}(\nabla \nabla u)$ | Order | $\text{err}_{D}(\nabla \nabla \nabla u)$ | Order |
|-------|-------|----------------------|-------|---------------------------|-------|--------------------------------|-------|--------------------------------|-------|
| 9     | 1.097695 | 1.097695  | -     | 0.109189  | -     | 0.792818  | -     |                                   |       |
| 49    | 0.351280 | 0.351280  | 1.6438 | 0.969172  | 1.0935 | 0.9762    | 1.9762 | 0.409436  | 0.9533 |
| 225   | 0.073936 | 0.073936  | 2.2483 | 0.306858  | 1.6592 | 0.205661  | 1.9762 | 0.809189  | 0.7928 |
| 961   | 0.007356 | 0.007356  | 2.0624 | 0.113622  | 1.4333 | 0.046151  | 2.1558 | 0.186959  | 1.1309 |
| 3969  | 0.000935 | 0.000935  | 2.0068 | 0.053444  | 1.0882 | 0.010414  | 2.1478 | 0.083455  | 1.1637 |
| 16129 | 0.000288 | 0.000288  | 2.0068 | 0.026437  | 1.0155 | 0.002689  | 2.1478 | 0.020526  | 1.0204 |

The $H^1$ and $H^2$ errors ($\text{err}_{D}(\nabla u)$ and $\text{err}_{D}(\Delta u)$) are computed using the usual midpoint rule. For comparison with the gradient recovery method (see Table 1), the details of mesh size $h$, number of unknowns $\nu$ and the number of non-zero terms in the system square matrix $\text{nnz}$ for the finite volume method are also provided in the following tables.

6.2.1. Example 1. In the first example, we choose the right hand side load function $f$ such that the exact solution is given by $u(x,y) = x^2y^2(1-x)^2(1-y)^2$. Tables 12 and 13 show the relative errors and order of convergence rates for the variable $u_D$ on triangular and square grids. As seen in the table, we obtain linear (in $H^1$-like norm) and sub-linear convergence rates (in $H^2$-like norm) for triangular grids, and quadratic order of convergence for square grids. This behaviour has already been observed in [12]. With respect to $L^2$ norm, quadratic (or slightly better) order of convergence is obtained. These numerical order of convergence are better than the orders of convergences from the theoretical analysis, see Remark 5.4. This is somehow expected as, due to the difficulty of finding a proper interpolant for this very low-order method [12], the theoretical rates are much below than the actual rates.

Table 12. (FV) Convergence results, Example 1, triangular grids ($\text{mesh}1$ family)

| $h$   | $\nu$ | $\text{nnz}$ | $\text{err}_{D}(u)$ | Order | $\text{err}_{D}(\nabla u)$ | Order | $\text{err}_{D}(\Delta u)$ | Order |
|-------|-------|---------------|----------------------|-------|---------------------------|-------|--------------------------------|-------|
| 0.250000 | 56    | 392          | 0.137345  | -     | 2.1150        | 0.131915 | 0.9585    | 0.071457 | 1.1828 |
| 0.125000 | 224   | 1896         | 0.031705  | 2.1150 | 0.066136  | 0.9961 | 0.038596  | 0.8886 |
| 0.062500 | 896   | 8264         | 0.007400  | 2.0991 | 0.033067  | 1.0000 | 0.022662  | 0.7682 |
| 0.031250 | 3584  | 34440        | 0.0001691 | 2.1297 | 0.016528  | 1.0005 | 0.014158  | 0.6786 |
| 0.015625 | 14336 | 143552       | 0.0000556 | 2.1244 | 0.008262  | 1.0004 | 0.009281  | 0.6092 |

Table 13. (FV) Convergence results, Example 1, square grids ($\text{mesh}2$ family)

| $h$   | $\nu$ | $\text{nnz}$ | $\text{err}_{D}(u)$ | Order | $\text{err}_{D}(\nabla u)$ | Order | $\text{err}_{D}(\Delta u)$ | Order |
|-------|-------|---------------|----------------------|-------|---------------------------|-------|--------------------------------|-------|
| 0.353553 | 16    | 56           | 0.328639  | -     | 0.417244  | -     | 0.260189  | -     |
| 0.176777 | 64    | 472          | 0.081325  | 2.0147 | 0.107481 | 1.9568 | 0.062624  | 2.0548 |
| 0.088388 | 256   | 2552         | 0.020161  | 2.0121 | 0.026808 | 2.0034 | 0.015430  | 2.0210 |
| 0.044194 | 1024  | 11704        | 0.005028  | 2.0035 | 0.006694 | 2.0018 | 0.003842  | 2.0057 |
| 0.022097 | 4096  | 49976        | 0.001256  | 2.0009 | 0.001673 | 2.0005 | 0.000960  | 2.0015 |
| 0.011049 | 16384 | 206392       | 0.000314  | 2.0002 | 0.000418 | 2.0001 | 0.000240  | 2.0004 |
6.2.2. Example 2. In this example, we perform the numerical experiment for the exact solution given by $u(x, y) = x^2y^2(1-x)^2(1-y)^2(\cos(2\pi x) + \sin(2\pi y))$. The errors in the energy norm, $H^1$ norm and the $L^2$ norm, together with their orders of convergence, are presented in Tables 14 and 15. The results are similar to those for Example 1.

### Table 14. (FV) Convergence results, Example 2, triangular grids (mesh1 family)

| $h$    | nu  | nnz  | $\text{err}_P(\overline{\Pi})$ | Order | $\text{err}_P(\nabla \overline{\Pi})$ | Order | $\text{err}_P(\Delta \overline{\Pi})$ | Order |
|--------|-----|------|---------------------------------|-------|---------------------------------|-------|---------------------------------|-------|
| 0.250000 | 56  | 392  | 0.418276                        | -     | 0.533799                        | -     | 0.274105                        | -     |
| 0.125000 | 224 | 1896 | 0.075761                        | 2.4649| 0.204870                        | 1.3816| 0.101375                        | 1.4350|
| 0.062500 | 896 | 8264 | 0.013663                        | 2.4712| 0.093729                        | 1.1281| 0.044254                        | 1.1958|
| 0.031250 | 3584| 34440| 0.003218                        | 2.0862| 0.046056                        | 1.0251| 0.021393                        | 1.0127|
| 0.015625 | 14336| 140552| 0.000784                        | 2.0365| 0.022932                        | 1.0060| 0.011500                        | 0.9315|
| 0.007813 | 57344| 567816| 0.000191                        | 2.0414| 0.011454                        | 1.0015| 0.006323                        | 0.8630|

### Table 15. (FV) Convergence results, Example 2, square grids (mesh2 family)

| $h$    | nu  | nnz  | $\text{err}_P(\overline{\Pi})$ | Order | $\text{err}_P(\nabla \overline{\Pi})$ | Order | $\text{err}_P(\Delta \overline{\Pi})$ | Order |
|--------|-----|------|---------------------------------|-------|---------------------------------|-------|---------------------------------|-------|
| 0.353553 | 16  | 56   | 1.333981                        | -     | 0.745194                        | -     | 0.773521                        | -     |
| 0.176777 | 64  | 472  | 0.223384                        | 2.5781| 0.135128                        | 2.4633| 0.175192                        | 2.1425|
| 0.088388 | 256 | 2552 | 0.050527                        | 2.1444| 0.030239                        | 2.1599| 0.042123                        | 2.0563|
| 0.044194 | 1024| 11704| 0.012331                        | 2.0347| 0.007339                        | 2.0427| 0.010416                        | 2.0158|
| 0.022097 | 4096| 49976| 0.003065                        | 2.0086| 0.001821                        | 2.0109| 0.002597                        | 2.0041|
| 0.011049 | 16384| 206392| 0.000765                        | 2.0021| 0.000454                        | 2.0027| 0.000649                        | 2.0010|

6.2.3. Example 3. The numerical results obtained for $u(x, y) = x^3y^3(1-x)^3(1-y)^3(\exp(x)\sin(2\pi x) + \cos(2\pi y))$ are shown in Tables 16 and 17 respectively. As in Examples 1 and 2, the theoretical rates of convergence are confirmed by these numerical outputs, except that on this test a real linear order of convergence is attained in the $H^2$-like norm.

### Table 16. (FV) Convergence results, Example 3, triangular grids (mesh1 family)

| $h$    | nu  | nnz  | $\text{err}_P(\overline{\Pi})$ | Order | $\text{err}_P(\nabla \overline{\Pi})$ | Order | $\text{err}_P(\Delta \overline{\Pi})$ | Order |
|--------|-----|------|---------------------------------|-------|---------------------------------|-------|---------------------------------|-------|
| 0.250000 | 56  | 392  | 0.637895                        | -     | 0.825992                        | -     | 0.423933                        | -     |
| 0.125000 | 224 | 1896 | 0.050763                        | 3.6515| 0.220328                        | 1.9065| 0.096604                        | 2.1337|
| 0.062500 | 896 | 8264 | 0.013330                        | 1.9291| 0.097939                        | 1.1697| 0.045854                        | 1.0750|
| 0.031250 | 3584| 34440| 0.003160                        | 2.0765| 0.047945                        | 1.0305| 0.021417                        | 1.0983|
| 0.015625 | 14336| 140552| 0.000786                        | 2.0084| 0.023857                        | 1.0070| 0.010550                        | 1.0215|
| 0.007813 | 57344| 567816| 0.000196                        | 2.0016| 0.011914                        | 1.0017| 0.005257                        | 1.0049|

Comparing Table 1 and the Tables for FV, we see that the GR method based on biorthogonal reconstruction has only few unknowns (number of internal vertices) but leads to a large stencil for each of them whereas the FV has more unknowns (number of cells) but produces a much sparser matrix. Looking for example at the finest GR mesh and the finest triangular FV mesh, we notice that the meshes have similar sizes $h$ and the matrices have similar complexity $\text{nnz}$, but the FV
Table 17. (FV) Convergence results, Example 3, square grids (mesh2 family)

| $h$   | $\nu$ | $\text{nnz}$ | $\text{err}_2(\mathbf{u})$ | Order | $\text{err}_2(\nabla \mathbf{u})$ | Order | $\text{err}_2(\Delta \mathbf{u})$ | Order |
|-------|-------|-------------|-----------------|-------|-----------------|-------|-----------------|-------|
| 0.353553 | 16   | 56         | 2.478402        | -     | 1.405462        | -     | 1.140625        | -     |
| 0.176777 | 64   | 472        | 0.242959        | 3.35063 | 0.113945        | 3.6246 | 0.196693        | 2.5358 |
| 0.088388 | 256  | 2552       | 0.050784        | 2.25835 | 0.022495        | 2.3406 | 0.049149        | 2.0007 |
| 0.044194 | 1024 | 11704      | 0.012212        | 2.05612 | 0.005577        | 2.0120 | 0.012217        | 2.0083 |
| 0.022097 | 4096 | 49976      | 0.003025        | 2.01332 | 0.001396        | 1.9982 | 0.003049        | 2.0026 |
| 0.011049 | 16384| 206392     | 0.000755        | 2.00332 | 0.000349        | 1.9993 | 0.000762        | 2.0007 |

accuracy in $L^2$- and $H^2$-like norms is much better than the GR method; this is expected since the FV method has a number of unknowns $\nu$ more than 3.5 times larger than that of GR. However, the super-convergence property of the gradient reconstruction gives a clear advantage to GR for the $H^1$-like norm. For a similar number of unknowns $\nu$ (which means a matrix that is much cheaper to solve for the FV method than the GR method, due to a reduced $\text{nnz}$), the FV method still has a clear advantage in the $L^2$ norm over the GR method, but similar accuracy in the $H^2$-like norm (compare the results for the 5th mesh in the mesh1 family with the finest mesh used for the GR method); the GR method however still preserves a clear lead on the $H^1$-like norm error.

7. CLASSICAL FE SCHEMES FITTING INTO THE HDM

We show here that some known FE schemes fit into the Hessian discretisation method, that is, they are Hessian schemes for particular choices of Hessian discretisations.

7.1. Conforming methods. For conforming finite elements, we require our finite element space $V_h$ to be a subspace of the underlying Hilbert space $H^2_0(\Omega)$. We can then define a Hessian discretisation by $X_D^0 = V_h$ and, for $v \in X_D^0$, $\Pi_D v = v$, $\nabla_D v = \nabla v$ and $H_B^D v = H_B v$. The estimates on $C_B^D$, $S_B^D$ and $W_B^D$ easily follow:

- $C_B^D$ is bounded by the constant of the continuous Poincaré inequality in $H^2_0(\Omega)$.
- Standard approximation properties (see, e.g., [10]) yield, for almost-affine families of FE, estimates on the interpolation error $S_B^D$.
- Integration-by-parts in $H^2_0(\Omega)$ shows that $W_B^D(\xi) = 0$ for all $\xi \in H^2_0(\Omega)$.

We briefly describe hereafter three finite elements which meet this requirement. The reader is referred to [10] for details.

The Argyris triangle: The Argyris triangle is a $C^1$ element which uses a complete polynomial of degree five. The degrees of freedom consist of function values and first and second derivatives at the vertices in addition to normal derivatives at the midpoints of the sides. One difficulty with the Argyris triangle is that there are 21 degrees of freedom per triangle. A modification to the Argyris triangle is the Bell’s element which suppresses the values of the normal slopes at the nodes at the three midpoint sides, reducing the number of degrees of freedom to 18 per element.

Hsieh-Clough-Toucher triangles: In the Hsieh-Clough-Tocher (HCT) triangle, the triangle is first decomposed into three triangles by connecting the barycenter of the given triangle with each of its vertices. On each of the subtriangles a cubic polynomial is constructed so that the resulting function is $C^1$ on the original triangle. There are a total of 12 degrees of freedom per triangle, which consist of
the function values and first partial derivatives at the three vertices of the original
triangle in addition to the normal derivative at the midpoints of the sides of the
original triangle.

7.2. An example of non-conforming method: the Adini rectangle. Assume
that Ω can be covered by mesh M made up of rectangles (we restrict the presen-
tation to d = 2 for simplicity). The element K consists of a rectangle with vertices
\{a_i, 1 \leq i \leq 4\}; the space \(P_K\) is given by \(P_K = P_3 \oplus \{x_1 x_2^2\} \oplus \{x_1^2 x_2\}\), by which we mean polynomials of degree \(\leq 4\) whose only fourth-degree terms are those involving
\(x_1 x_2^2\) and \(x_1^2 x_2\). Thus \(P_3 \subset P_K\). The set of degrees of freedom in each cell is

\[\Sigma_K = \left\{ p(a_i), \frac{\partial p}{\partial x_1}(a_i), \frac{\partial p}{\partial x_2}(a_i); 1 \leq i \leq 4, p \in P_K \right\}.\]

The global approximation space is then given by

\[V_h = \{ v_h \in L^2(\Omega); v_h|_K \in P_K \forall K \in M, v_h\text{ and } \nabla v_h\text{ are continuous at the vertices of elements in } M, v_h\text{ and } \nabla v_h\text{ vanish at vertices on } \partial \Omega \}.\]

Note that \(V_h \subset H^1_0(\Omega) \cap C^0(\overline{\Omega})\).

Definition 7.1 (Hessian discretisation for the Adini rectangle). Each \(v_B \in X_{D,0}\)
is a vector of three values at each vertex of the mesh (with zero values at boundary
vertices), corresponding to function and gradient values, \(\Pi_D v_B\) is the function
such that \((\Pi_D v_B)|_K \in P_K\) and its gradient takes the values at the vertices dictated by
\(v_B\), \(\nabla_D v_B = \nabla(\Pi_D v_B)\) and \(H_B^B(\Pi_D v_B) = H_M^B(\Pi_D v_B)\) is the broken \(H^B\) (\(H_D\) is the
broken \(H\)).

We assume that the mesh is regular, that is, (4.1) holds with \(\eta\) not depending on
the mesh.

Theorem 7.2. Let \(D\) be a \(B\)-Hessian discretisation in the sense of Definition
7.1 with \(B\) satisfying the coercive property. Then, there exists a constant \(C\), not
depending on \(D\), such that

- \(C_B^B \leq C\),
- \(\forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega), S_B(\varphi) \leq Ch\|\varphi\|_{H^1(\Omega)}\),
- \(\forall \xi \in H^2(\Omega)^{d \times d}, W_B^B(\xi) \leq Ch\|\xi\|_{H^2(\Omega)^{d \times d}}\).

The properties of Hessian discretisations built on the Adini rectangle follow from
this theorem and Remark 3.4.

Corollary 7.3. Let \((D_m)_{m \in \mathbb{N}}\) be a sequence of \(B\)-Hessian discretisations built on
the Adini rectangle, such that \(B\) is coercive and the underlying sequence of meshes
are regular and have a size that goes to \(0\) as \(m \to \infty\). Then the sequence \((D_m)_{m \in \mathbb{N}}\)
is coercive, consistent and limit-conforming.

Proof of Theorem 7.2.
In this proof, \(C > 0\) denotes a generic constant that can change from one line to
the other but depends only on \(\Omega, d, B\) and \(\eta\).

- Coercivity: Since \(V_h \subset H^1_0(\Omega)\), for \(v \in X_{D,0}\), the Poincaré inequality yields
  \(\|\Pi_D v\| \leq \text{diam}(\Omega)\|\nabla_D v\|\), which gives us part of the estimate on \(C_B^B\). Define the
  broken Sobolev space
  \[H^1(M) = \{ v \in L^2(\Omega); \forall K \in M, v|_K \in H^1(K) \}\]
and endow it with the dG norm

$$\|w\|_{dG}^2 := \|\nabla \mathcal{M} w\|^2 + \sum_{\sigma \in \mathcal{F}} \frac{1}{h_\sigma} \|\|w\|\|_{L^2(\sigma)}^2,$$

(7.1)

where

$$h_\sigma = \begin{cases} \min(h_K, h_L) & \text{if } \sigma \in \mathcal{F}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\} \\ h_K & \text{if } \sigma \in \mathcal{F}_{\text{ext}}, \mathcal{M}_\sigma = K, \end{cases}$$

and the jump of $w$ is

$$\jump{w} = \begin{cases} w|_{K} - w|_{L} & \text{if } \sigma \in \mathcal{F}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\} \\ w|_{K} & \text{if } \sigma \in \mathcal{F}_{\text{ext}}, \mathcal{M}_\sigma = K. \end{cases}$$

If $\jump{w} = 0$ at the vertices of $\sigma$ then, by the Poincaré inequality in $H^1_0(\sigma)$ (Lemma A.1),

$$\|\jump{w}\|_{L^2(\sigma)} \leq C h_\sigma \|\nabla \mathcal{M} w\|_{L^2(\sigma)^d}.$$

(7.2)

If $\sigma \in \mathcal{F}_{\text{int}}$ with $\mathcal{M}_\sigma = \{K, L\}$ then $\jump{w} = 0$ at the vertices of $\sigma$, and (7.2) combined with the trace inequality [7, Lemma 1.46] therefore give

$$\|\jump{w}\|_{L^2(\sigma)} \leq C_{\text{tr}} h_\sigma \left( h_K^{-1/2} \|\nabla \mathcal{M} w\|_{L^2(K)^d} + h_L^{-1/2} \|\nabla \mathcal{M} w\|_{L^2(L)^d} \right),$$

(7.3)

where $C_{\text{tr}}$ depends only on $d$ and the mesh regularity parameter $\eta$. Take $v \in X_{D,0}$. Since $\nabla_D v$ is continuous at the vertices of $\mathcal{M}$ and $\nabla_D v$ vanish at vertices along $\partial \Omega$, choosing $w = \nabla_D v$ in (7.2) and (7.3) yields

$$\|\nabla_D v\|_{dG}^2 \leq C_{\text{tr}} h_\sigma \left( h_K^{-1/2} \|\nabla \mathcal{M} (\nabla_D v)\|_{L^2(K)^{d \times d}} + h_L^{-1/2} \|\nabla \mathcal{M} (\nabla_D v)\|_{L^2(L)^{d \times d}} \right).$$

Recalling the definition (7.1) of the dG norm, the above inequality and the coercivity property of $B$ yield

$$\|\nabla_D v\|_{dG}^2 \leq \|\nabla \mathcal{M} (\nabla_D v)\|^2 + 2 C_{\text{tr}} \sum_{\sigma \in \mathcal{F}} h_\sigma \left( h_K^{-1} \|\nabla \mathcal{M} (\nabla_D v)\|_{L^2(K)^{d \times d}} + h_L^{-1} \|\nabla \mathcal{M} (\nabla_D v)\|_{L^2(L)^{d \times d}} \right) \leq \|\nabla \mathcal{M} (\nabla_D v)\|^2 + C \sum_{K \in \mathcal{M}} \|\nabla \mathcal{M} (\nabla_D v)\|_{L^2(K)^{d \times d}} \leq C \|\mathcal{H}_M(\nabla_D v)\|^2 \leq C \varrho^{-2} \|\mathcal{H}_M(\nabla_D v)\|^2 = C \varrho^{-2} \|\mathcal{H}_B(\nabla_D v)\|^2.$$

Using the fact that $\|w\| \leq C \|w\|_{dG}$ whenever $w$ is a broken polynomial on $\mathcal{M}$ (see [7, Theorem 5.3]), we infer that $\|\nabla_D v\| \leq C \varrho^{-1} \|\mathcal{H}_B(\nabla_D v)\|$, which concludes the estimate on $C_B$.

- **Consistency**: Consistency follows from the affine property of the family of Adini rectangles. Using [6, Theorem 3.1.5, Chapter 3], for $\varphi \in H^3(\Omega) \cap H^2_D(\Omega)$, we obtain

$$\inf_{w \in X_{D,0}} \|\mathcal{H}_B w - \mathcal{H}_B \varphi\| \leq C h |\varphi|_{3,\Omega}, \quad \inf_{w \in X_{D,0}} \|\nabla_D w - \nabla \varphi\| \leq C h^2 |\varphi|_{3,\Omega}$$

and

$$\inf_{w \in X_{D,0}} \|\Pi_D w - \varphi\| \leq C h^3 |\varphi|_{3,\Omega},$$

which implies $S_B^B(\varphi) \leq C h |\varphi|_{3,\Omega}$.
\textbf{Limit-Conformity:} for $\xi \in H^2(\Omega)^{d \times d}$ and $v_\Omega \in X_{D,0}$, cellwise integration-by-parts (see Lemma A.2) yields

\[
\int_\Omega (\mathcal{H} : B^T B \xi) \Pi_{D^2} \, dx = \sum_{K \in \mathcal{M}} \int_K (\mathcal{H} : A\xi) \Pi_{D^2} \, dx
= \int_\Omega A\xi : \mathcal{H}_{D^2} \, dx - \sum_{K \in \mathcal{M}} \int_{\partial K} (A\xi \nabla_{D^2}) \cdot \nabla_{D^2} \, ds(x)
+ \sum_{K \in \mathcal{M}} \int_{\partial K} (\text{div}(A\xi) \cdot n_K) \Pi_{D^2} \, ds(x).
\]

For $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, let $n_{K,\sigma}$ be the unit vector normal to $\sigma$ outward to $K$. For all $\sigma \in \mathcal{F}$, we choose an orientation (that is, a cell $K$ such that $\sigma \in \mathcal{F}_K$) and we set $n_{\sigma} = n_{K,\sigma}$. We then set $[w] = w_K - w_L$ if $\sigma \in \mathcal{F}_{\text{int}}$ with $\mathcal{M}_\sigma = \{K, L\}$, and $[w] = w_K$ if $\sigma \in \mathcal{F}_{\text{ext}}$ with $\mathcal{M}_\sigma = K$. Then

\[
\int_\Omega (\mathcal{H} : A\xi) \Pi_{D^2} \, dx - \int_\Omega A\xi : \mathcal{H}_{D^2} \, dx
= -\sum_{\sigma \in \mathcal{F}} \int (A\xi n_{\sigma}) \cdot \|\nabla_{D^2}\| \, ds(x) + \sum_{\sigma \in \mathcal{F}} \int_{\mathcal{F}} (\text{div}(A\xi) \cdot n_{\sigma}) \Pi_{D^2} \|\nabla_{D^2}\| \, ds(x).
\]

Since $\Pi_{D^2} \in H^1_0(\Omega) \cap C(\Omega)$, $\Pi_{D^2} = 0$. Let $\Lambda_K$ denote the $Q_1$ interpolation operator associated with the values at the four vertices of $K$, and $\Lambda_h$ be the patched interpolator such that $(\Lambda_h)_K = \Lambda_K$ for all $K$. $\Lambda_h(\nabla_{D^2})$ takes the values of $\nabla_{D^2}$ at the vertices, so it is continuous at internal vertices and vanishes at the boundary vertices. Hence, for any $\sigma \in \mathcal{F}$, $\|[\Lambda_h(\nabla_{D^2})]\|$ vanishes on $\sigma$ since it is linear on this edge and vanishes at its vertices. As a consequence,

\[
\int_\Omega (\mathcal{H} : A\xi) \Pi_{D^2} \, dx - \int_\Omega A\xi : \mathcal{H}_{D^2} \, dx
= -\sum_{\sigma \in \mathcal{F}} \int (A\xi n_{\sigma}) \cdot \|\nabla_{D^2} - \Lambda_h(\nabla_{D^2})\| \, ds(x)
= -\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int A\xi n_{K,\sigma} \cdot \left(\nabla_{D^2} - \Lambda_K(\nabla_{D^2})\right) \, ds(x). \tag{7.5}
\]

Setting $\varphi = A\xi n_{K,\sigma}$ and $w = \nabla_{D^2}$, a change of variables yields

\[
\int_{\sigma \in \mathcal{F}_K} \varphi \cdot (w - \Lambda_K(w)) \, ds(x) = |\sigma| \int_{\hat{K}} \hat{\varphi} \cdot (\hat{w} - \Lambda_{\hat{K}}(\hat{w})) \, ds(x), \tag{7.6}
\]

where $\hat{K}$ is the reference finite element. Let $\mathcal{F}_K = \{\sigma_1', \sigma_2', \sigma_1'', \sigma_2''\}$ such that $|\sigma_1'| = |\sigma_1''| = h_1$ and $|\sigma_2'| = |\sigma_2''| = h_2$. Let us consider

\[
\delta_{1,K}(\phi, v) = \int_{\sigma_1'} \phi(v - \Lambda_K(v)) \, ds(x) - \int_{\sigma_2'} \phi(v - \Lambda_K(v)) \, ds(x), \tag{7.7}
\]

for $\phi \in H^1(K)$ and $v \in \partial_1 \mathcal{F}_K$. The steps in [6, Theorem 6.2.3] show that $\delta_{1,K}(\phi, v) \leq C(h) |\phi|_{1,K} |v|_{1,K}$. For the sake of completeness, let us briefly recall the argument. Using changes of variables, $\delta_{1,K}(\phi, v) = h_1 \delta_{1,\hat{K}}(\hat{\phi}, \hat{v})$. Since $\mathcal{F}_0 \subset Q_1$, which is preserved by $\Lambda_K$, for all $\hat{v} \in \mathcal{F}_0$ and $\hat{\phi} \in H^1(\hat{K})$ we have $\delta_{1,\hat{K}}(\hat{\phi}, \hat{v}) = 0$ (first polynomial invariance). Let us now prove that the same relation holds if
\( \hat{\phi} \in \mathbb{P}_0 \) and \( \hat{v} \in \partial_1 P_2 \). Since \( \hat{\phi} \in \mathbb{P}_0 \), its value on \( \hat{K} \) is a constant, say, equal to \( a_0 \). Since \( \hat{v} \in \partial_1 P_2 \) we have
\[
\hat{v} = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1^2 + b_4 x_1 x_2 + b_5 x_2^2 + b_7 x_3^3.
\]
Taking the values at the four vertices, we get
\[
\Lambda_{\hat{K}} \hat{v} = b_0 + (b_1 + b_3) x_1 + (b_2 + b_5 + b_7) x_2 + (b_4 + b_6) x_1 x_2.
\]
Assuming without loss of generality that \( \sigma'_1 \) is the line \( x_1 = 1 \) and \( \sigma''_1 \) is the line \( x_1 = 0 \), we infer
\[
(\hat{v} - \Lambda_{\hat{K}} \hat{v}) |_{x_1 = 0} = -(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_3^3,
\]
\[
(\hat{v} - \Lambda_{\hat{K}} \hat{v}) |_{x_1 = 1} = -(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_3^3.
\]
The relation \( \delta_{1,\hat{K}}(\hat{\phi}, \hat{v}) = 0 \) (second polynomial invariance) then follows from
\[
\int_{\sigma'_1} \hat{\phi}(\hat{v} - \Lambda_{\hat{K}} \hat{v}) \, ds(x) = \int_0^1 a_0 (-(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_3^3) \, dx_2
\]
\[
= \int_{\sigma''_1} \hat{\phi}(\hat{v} - \Lambda_{\hat{K}} \hat{v}) \, ds(x).
\]

The bilinear form \( \delta_{1,\hat{K}}(\hat{\phi}, \hat{v}) \) is continuous over the space \( H^1(\hat{K}) \times \partial_1 P_{\hat{K}} \) by the trace theorem. Using the bilinear lemma [6, Theorem 4.2.5], we deduce from the two polynomial invariances the existence of a constant \( C \) such that \( |\delta_{1,\hat{K}}(\hat{\phi}, \hat{v})| \leq C|\hat{\phi}|_{1,\hat{K}}|\hat{v}|_{1,\hat{K}} \) for all \( \hat{\phi} \in H^1(\hat{K}), \hat{v} \in \partial_1 P_{\hat{K}} \). A direct change of variables shows that
\[
|\hat{\phi}|_{1,\hat{K}} \leq C|\phi|_{1,K} \quad \text{and} \quad |\hat{v}|_{1,\hat{K}} \leq C|v|_{1,K}.
\]
Since \( \delta_{1,K}(\phi, v) = h_1 \delta_{1,\hat{K}}(\hat{\phi}, \hat{v}) \), we infer \( \delta_{1,K}(\phi, v) \leq C h_1 |\phi|_{1,K} |v|_{1,K} \). Similarly, \( \delta_{2,K}(\phi, v) \leq C h_1 |\phi|_{1,K} |v|_{1,K} \) (considering integrals over \( \sigma'_2 \) and \( \sigma''_2 \)). Hence, from (7.5), (7.6) and (7.7),
\[
\left| \int \mathcal{H} : A\xi \Pi_D v_D \, dx - \int \Omega A\xi : \mathcal{H}_D v_D \, dx \right| \leq C \|\xi\|_{H^2(\Omega)} h_1 \|\mathcal{H}_D v_D\|.
\]
The proof of the estimate on \( W^B_D(\xi) \) is complete. \( \square \)

**Appendix A. Technical results**

**Lemma A.1** (Poincaré inequality along an edge). Let \( \sigma \) be an edge of a polygonal cell, \( w \in H^1(\sigma) \) and assume that \( w \) vanish at a point on the edge \( \sigma \in \mathcal{F} \). Then there exists \( C > 0 \) such that
\[
\|w\|_{L^2(\sigma)} \leq h_\sigma \|\partial w\|_{L^2(\sigma)},
\]
where \( \partial \) denotes the derivative along the edge and \( h_\sigma \) is the length of the edge.

**Proof.** Let \( m \) denote the point on the edge \( \sigma \) which satisfies \( w(m) = 0 \). For \( m < x \), we get
\[
w(x) = w(m) + \int_m^x \partial w(y) \, dy = \int_m^x \partial w(y) \, dy.
\]
A use of Cauchy-Schwarz inequality yields
\[
|w(x)| \leq |x - m|^{1/2} \left( \int_m^x |\nabla w|^2 \, dy \right)^{1/2} \leq \sqrt{h_\sigma} \left( \int_\sigma |\partial w|^2 \, dy \right)^{1/2}.
\]
Squaring this yields \(|w(x)|^2 \leq h_r \int \vert \partial w \vert^2 \, dy\) and integrating over the edge concludes the proof. □

**Lemma A.2** (Integration by parts). Let \(P\) be a fourth order tensor. For \(\xi \in H^2(\Omega)^{d \times d}\) and \(\phi \in H^1(\Omega)\), we have

\[
\int_{\Omega} (\mathcal{H} : P\xi)\phi = -\int_{\Omega} \nabla \phi \cdot \text{div}(P\xi) + \int_{\partial \Omega} \text{div}(P\xi \cdot n)\phi.
\]

For \(\psi \in H^2(\Omega)\),

\[
\int_{\Omega} P\xi : \mathcal{H}\psi = -\int_{\Omega} \nabla \psi \cdot \text{div}(P\xi) + \int_{\partial \Omega} (\text{div}(P\xi n)) \cdot \nabla \psi.
\]

For \(\zeta \in H^1(\Omega)^d\),

\[
\int_{\Omega} P\xi : \nabla \zeta = -\int_{\Omega} \text{div}(P\xi) \cdot \zeta + \int_{\partial \Omega} (\text{div}(P\xi n)) \cdot \zeta.
\]

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