Study of the transcendence of a family of generalized continued fractions

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Abstract

We study a family of generalized continued fractions, which are defined by a pair of substitution sequences in a finite alphabet. We prove that they are stammering sequences, in the sense of Adamczewski and Bugeaud. We also prove that this family consists of transcendental numbers which are not Liouvillian. We explore the partial quotients of their regular continued fraction expansions, arriving at no conclusion concerning their boundedness.

Keywords: Continued fractions; Transcendence; Stammering Sequences.

1 Introduction

The problem of characterizing continued fractions of numbers beyond rational and quadratic has received consistent attention over the years. One direction points to an attempt to understand properties of algebraic numbers of degree at least three, but at times even this line ends up in the realm of transcendental numbers.

Some investigations \([6,7,14]\) on algebraic numbers depart from generalizations of continued fractions. This line of investigation has been tried since Euler, see \([14,6]\) and references therein, with a view to generalize Lagrange’s theorem on quadratic numbers, in search of proving a relationship between algebraic numbers and periodicity of multidimensional maps yielding a sequence of approximations to a irrational number. This theory has been further developed for instance to the study of ergodicity of the triangle map \([10]\). In fact, a considerable variety of algorithms may be called generalizations of continued fractions: for instance \([8]\), Jacobi-Perron’s and Poincaré’s algorithms in \([14]\).

We report on a study of generalized continued fractions of the form:

\[
\theta(a,b) \overset{\text{def}}{=} a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \frac{b_4}{\ddots}}}}} \quad (1.1)
\]
(where \( a_0 \geq 0, a_n \in \mathbb{N}, \) for \( n \in \mathbb{N} \) and \( b_n \in \mathbb{N}, \) for \( n \geq 0 \)), investigating a class with a regularity close, in a sense, to periodicity. This family of generalized continued fractions converges when \((a_n)\) and \((b_n)\) are finite valued sequences. They were considered formally in [9], in the context exemplified in Section 5. The stammering sequences [1], and sequences generated by morphisms [3, 4, 5], consist in a natural step away from periodic ones. Similarly to the results in [3, 4], on regular continued fractions, we prove that the family of numbers considered are transcendental.

**Theorem 1.** Suppose \((a_n)\) and \((b_n), n \geq 0,\) are fixed points of primitive substitutions, then the number \(\theta(a, b)\) is transcendental.

We also consider Mahler’s classification within this family of transcendental numbers, proving that they cannot be Liouvillian. Let us call the numbers of the family of generalized continued fractions \(\theta(a, b),\) with \(a\) and \(b\) stammering sequences coming from a primitive substitution number of type \(S_3.\)

**Theorem 2.** Type \(S_3\) numbers are either \(S\)-numbers or \(T\)-numbers in Mahler’s classification.

The paper is organized as follows. In Section 2 we prove the convergence of the generalized continued fraction expansions for type \(S_3\) numbers. In Section 3 we prove the transcendence of type \(S_3\) numbers. In Section 4 we use Baker’s Theorem to prove that they are either \(S\)-numbers or \(T\)-numbers. In Section 5 we show some inconclusive calculations on the partial quotients of the regular continued fraction of an specific type \(S_3\) number.

## 2 Convergence

We start from the analytic theory of continued fractions [15] to prove the convergence of (1.1) when \((a_n)\) and \((b_n), n \geq 0,\) are sequences in a finite alphabet.

Let \(\mathbb{R}^+ = [0, \infty]\) denote the extended positive real axis, with the understanding that \(a + \infty = \infty,\) for any \(a \in \mathbb{R}^+; a \cdot \infty = \infty,\) if \(a > 0, 0 \cdot \infty = 0\) and \(a/\infty = 0,\) if \(a \in \mathbb{R}.\) We do not need to define \(\infty/\infty.\)

Given the sequences \((a_k)\) and \((b_k)\) of non-negative (positive) integers, consider the Möbius transforms \(t_k : \mathbb{R}^+ \to \mathbb{R}^+
\)
\[t_k(w) = a_k + \frac{b_k}{w}, \quad k \in \mathbb{N}\, ,
\]
and their compositions
\[t_1 t_2 (w) = t_1 (a_j + b_j/w) = a_i + b_i/(a_j + b_j/w).
\]
This set of Möbius transforms is closed under compositions and form a semigroup.

It is useful to consider the natural correspondence between Möbius transformations and \(2 \times 2\) matrices:
\[M_k = \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix}.
\]
Taking the positive real cone $C_2 = \{(x, y) \mid x \geq 0, \ y \geq 0, x + y > 0\}$ with the equivalence $(x, y) \sim \lambda (x, y)$ for every $\lambda > 0$, we have an homomorphism between the semigroup of Möbius transforms, under composition, acting on $\mathbb{R}^+$ and the algebra of matrices above (which are all invertible) acting on $C_2/\sim$.

Assume the limit

$$
\lim_{n \to \infty} t_0 t_1 t_2 \cdots t_n(0)
$$

exists as a real positive number, then it is given once we know the sequences $(a_n)$ and $(b_n)$, $n \geq 0$. In this case, it is equal to $\lim_{n \to \infty} t_0 t_1 t_2 \cdots t_{n-1}(\infty)$ as well, so that the initial point may be taken as 0 or $\infty$ in the extended positive real axis.

In terms of matrices multiplication, we have $M_0 M_1 M_2 \cdots M_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sim M_0 M_1 M_2 \cdots M_{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $C_2$. Define $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$ and

$$
\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} b_n p_{n-1} \\ b_n q_{n-1} \end{pmatrix} \quad \text{def} \quad M_0 M_1 M_2 \cdots M_n, \ n \geq 0.
$$

We have the following second order recursive formulas for $(p_n, q_n)$:

$$
p_{n+1} = a_{n+1} p_n + b_n p_{n-1}
$$

$$
q_{n+1} = a_{n+1} q_n + b_n q_{n-1}
$$

and the determinant formula

$$
p_n q_{n-1} - p_{n-1} q_n = (-1)^n b_0 \cdots b_{n-1}
$$

We recall the series associated with a continued fraction [15]:

**Lemma 3.** Let $(q_n)$ denote the sequence of denominators given in (2.2) for the continued fraction $\theta(a, b)$. Let

$$
\rho_k = -\frac{b_k q_{k-1}}{q_{k+1}}, \ k \in \mathbb{N}.
$$

Then

$$
a_0 + \frac{b_0}{a_1} \left(1 + \sum_{k=1}^{n-1} \rho_1 \rho_2 \cdots \rho_k\right) = \frac{p_n}{q_n}, \ n \geq 1.
$$

**Proof.** For $n = 1$, the sum is empty, $p_1 = a_1 a_0 + b_0$ and $q_1 = a_1$; the equality holds.

Consider the telescopic sum, for $n \geq 1$,

$$
\frac{p_1}{q_1} + \sum_{k=1}^{n-1} \left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) = \frac{p_n}{q_n}.
$$

From (2.3),

$$
\left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) = (-1)^k \frac{b_0 b_1 \cdots b_k}{q_{k+1} q_k}.
$$
Now \( \frac{b_n}{a_1} \rho_1 = \frac{b_0 b_n q_1}{q_1 q_2} = \frac{b_0 b_1}{q_1 q_2} = \frac{p_2}{q_2} - \frac{p_1}{q_1} \). Moreover

\[
\frac{p_3}{q_3} - \frac{p_2}{q_2} = \frac{b_0 b_1 b_2}{q_3 q_2} = \frac{b_0}{q_1} \left(-\frac{b_1 q_0}{q_2} \right) \left(-\frac{b_2 q_1}{q_3} \right) = \frac{b_0}{a_1} \rho_1 \rho_2.
\]

Multiplicative cancelling provides the argument to deduce the formula

\[
\frac{p_{k+1}}{q_{k+1}} \rho_k = \frac{b_0}{a_1} \rho_1 \rho_2 \cdots \rho_k
\]

by induction, finishing the proof. \( \square \)

Even though \( b_n > 1 \) may occur in \((1.1)\), note that we still have \( q_{n+1} \geq 2^{(n-1)/2} \). Indeed \( q_0 = 1, q_1 = a_1 \) and \( q_2 = a_2 q_1 + b_1 q_0 > 1 \). Finally, since \( a_n \geq 1 \) and \( b_n \geq 1 \) for all \( n \in \mathbb{N} \),

\[
q_{n+1} = a_{n+1} q_n + b_n q_{n-1} \geq q_n + q_{n-1} \geq 2^{n/2-1} + \frac{2^{(n-1)/2}}{2} > 2^{(n-1)/2}.
\]

**Lemma 4.** If \((a_n)\) and \((b_n)\) are sequences on a finite alphabet \( \mathcal{A} \subset [\alpha, \beta] \subset [1, \infty) \), then the generalized continued fraction \((1.1)\) converges.

**Proof.** It follows from \((2.2)\) that \((q_n), n \geq 1\), is increasing, and

\[
|\rho_k| = \left|\frac{b_k q_{k-1}}{q_{k+1}}\right| = \left(1 + \frac{a_{k+1} q_k}{b_k q_{k-1}}\right)^{-1} < (1 + \alpha/\beta)^{-1}.
\]

Thus the series with general term \( \rho_1 \cdots \rho_k \) is bounded by a convergent geometric series. \( \square \)

**Lemma 5.** Let \((q_n)\) denote the sequence of denominators given in \((2.2)\) for a generalized continued fraction, with \((a_n)\) and \((b_n)\) sequences in a finite alphabet \( \mathcal{A} \subset [\alpha, \beta] \subset [1, \infty) \). Then \( q_n^{1/n} \) is bounded.

**Proof.** From \((2.2)\), \( q_1 = a_1 \), and \( q_{n+1} < (a_{n+1} + b_n) q_n \leq (2\beta) q_n \). Hence \( q_n^{1/n} \leq 2\beta \sqrt[n]{\alpha} \leq 2\beta^{3/2} \), where the last inequality is necessary only when \( a_1 = \beta \). \( \square \)

## 3 Transcendence

We now specialize the study of generalized continued fractions for sequences \((a_n)\) and \((b_n)\) which are generated by primitive substitutions. These sequences provide a wealth of examples of stammering sequences, defined below, following \([1]\).

Let us introduce some notation. The set \( \mathcal{A} \) is called alphabet. A word \( w \) on \( \mathcal{A} \) is a finite or infinite sequence of letters in \( \mathcal{A} \). For finite \( w \), \( |w| \) denotes the number of letters composing \( w \). Given a natural number \( k \), \( w^k \) is the word obtained by \( k \) concatenated repetitions of \( w \). Given a rational number \( r > 0 \), which is not an integer, \( w^r \) is the word \( w^{[r]}w^r \), where \( [r] \) denotes the integer part of \( r \) and \( w^r \) is a prefix of \( w \) of length \([r - [r]]|w|\), where \([q] = [q] + 1 \) is the upper integer part of \( q \).

Note that if \((a_n)\) and \((b_n)\) are sequences on \( \mathcal{A} \), then \((a_n, b_n)\) is a sequence in \( \mathcal{A} \times \mathcal{A} \), which is also an alphabet. A sequence \( a = (a_n) \) has the stammering property if it is not a periodic sequence and, given \( r > 1 \), there exists a sequence of finite words \((w_n), n \in \mathbb{N}\), such that
a) for every $n \in \mathbb{N}$, $w_n^r$ is a prefix of $a$;

b) $(|w_n|)$ is increasing.

We say, more briefly, that $(a_n)$ is a stammering sequence with exponent $r$. It is clear that if $(a_n)$ and $(b_n)$ are both stammering with exponents $r$ and $s$ respectively, then $(a_n, b_n)$ is also stammering with exponent $\min\{r, s\}$.

**Lemma 6.** If $u$ is a substitution sequence on a finite alphabet $\mathcal{A}$, then $u$ is stammering.

**Proof.** Denote the substitution map by $\xi : \mathcal{A} \to \mathcal{A}^+$. Since $\mathcal{A}$ is a finite set, there is a $k \geq 1$ and $\alpha \in \mathcal{A}$ such that $\alpha$ is a prefix of $\xi^k(\alpha)$. $u = \lim_{n \to \infty} \xi^{kn}(\alpha)$. Moreover, there is a least finite $j$ such that $\alpha$ occurs a second time in $\xi^j(\alpha)$. Therefore $u$ is stammering with $w \geq 1 + \frac{1}{|\xi^k(\alpha)| - 1}$. □

**Proof of Theorem 1**. From Lemma 6 $(a_n)$ and $(b_n)$ stammering sequences with exponent $r > 1$. Hence $\theta(a, b)$ has infinitely many good quadratic approximations. Let $(w_n) \in (\mathcal{A} \times \mathcal{A})^*$ be a sequence of words of increasing length characterizing $(a_n, b_n)$ as a stammering sequence with exponent $r > 1$. Consider $\psi_k(a, b)$ given by

$$\psi_k(a, b) = c_0 + \frac{d_0}{c_1 + \frac{d_2}{c_2 + \frac{d_3}{\ddots}}},$$

where $c_j = a_j$, $d_j = b_j$, for $0 \leq j < k$, and $c_j = c_j (\text{mod } k)$ and $d_j = d_j (\text{mod } k)$, for $j \geq k$. $\psi_k$ is a root of the quadratic equation

$$q_k - 1 x^2 + (q_k - p_{k-1}) x - p_k = 0,$$

which might not be in lowest terms.

Arguing as in Theorem 1 from [1], we choose $k$ from the subsequence of natural numbers given by $|w_n^r|$. Lemma 5 allows us to conclude that the generalized continued fraction $\theta(a, b)$ is transcendental if both $(a_n)$ and $(b_n)$ are stammering sequences with exponent $r > 1$. □

## 4 Quest on Liouville numbers

We address the question of Mahler’s classification of the numbers for type $S_3$ numbers. The statement of Baker’s Theorem we quote use a measure of transcendence introduced by Koksma, which is equivalent to Mahler’s, and we explain briefly, following [2], Section 2.

Let $d \geq 1$ and $\xi$ a real number. Denote by $P(X)$ an arbitrary polynomial with integer coefficients, and $H(P) = \max_{0 \leq i < l} |a_i| : P(X) = a_0 + a_1 X + \cdots + a_l X^l$ is the height of the polynomial $P$. Let $w_d(\xi)$ be the supremum of the real numbers such that the inequality

$$0 < |P(\xi)| \leq H(P)^{-w}$$

is true for infinitely many polynomials $P(X)$ with integer coefficients and degree at most $d$. Koksma introduced $\bar{w}_d(\xi)$ as the supremum of the real numbers $\bar{w}$ such that

$$0 < |\xi - \alpha| \leq H(\alpha)^{-\bar{w} - 1}$$
are true for infinitely many algebraic numbers $\alpha$ of degree at most $d$, where $H(\alpha)$ is the height of the minimal polynomial with integer coefficients which vanishes at $\alpha$.

Let $w(\xi) = \lim_{d \to \infty} \frac{w_d(\xi)}{d}$, then $\xi$ is called

- an $A$-number if $w(\xi) = 0$;
- an $S$-number if $0 < w(\xi) < \infty$;
- a $T$-number if $w(\xi) = \infty$, but $w_d(\xi) < \infty$ for every integer $d \geq 1$;
- an $U$-number if $w(\xi) = \infty$ and $w_d(\xi) = \infty$ for some $d \geq 1$.

It was shown by Koksma that $w_\ast$ and $w$ provide the same classification of numbers. Liouville numbers are precisely those for which $w_1(\xi) = \infty$, they are $U$-numbers of type 1.

**Theorem (Baker).** Let $\xi$ be a real number and $\epsilon > 0$. Assume there is an infinite sequence of irreducible rational numbers $(p_n/q_n)_{n \in \mathbb{N}}$, $(p_n/q_n) = 1$, ordered such that $2 \leq q_1 < q_2 < \cdots$ satisfying

$$\left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{2+\epsilon}}.$$ 

Additionally, suppose that

$$\limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} < \infty,$$

then there is a real number $c$, depending only on $\xi$ and $\epsilon$ such that

$$w_\ast(\xi) \leq \exp \exp(c d^2).$$

for every $d \in \mathbb{N}$. Consequently, $\xi$ is either an $S$-number or a $T$-number.

**Proof of Theorem.** We note that the hypothesis of irreducibility is lacking for type $S_3$ numbers. Let us write $d_n = (p_n, q_n)$. By eq. (2.3), $d_n = b_0 \ldots b_{n-1}$. Recall that, for primitive substitutions in $\mathcal{A} = \{\alpha, \beta\} \subset \mathbb{N}$, there is a frequency $\nu$, which is uniform in the sequence (b) [11], for which $b_k = \beta$. Thus, $d_n \approx \beta^1 \alpha^{n-1}$ for large $n$. If, for every $n \in \mathbb{N}$, there is a number $\theta$ such that $d_n < (\frac{q_n}{d_n})^\theta$, then

$$0 < \left| \frac{p_n/d_n}{q_n/d_n} \right| < \frac{1}{q_n^{2+\epsilon}} = \frac{1}{d_n^{2+\epsilon}(q_n/d_n)^{2+\epsilon}}.$$ 

In this case, from the estimates of $q_n$ and $d_n$, the limit

$$\limsup_{n \to \infty} \frac{\log(q_{n+1}/d_{n+1})}{\log(q_n/d_n)} < \infty.$$ 

We would conclude from Theorem [4] that type $S_3$ contains either $S$-numbers or $T$-numbers and no Liouville numbers.

From the analysis of Section [2] keeping its notations,

$$(-1)^n \frac{d_n}{q_n q_{n-1}} = \frac{b_0}{a_1} \rho_1 \ldots \rho_{n-1}.$$
Now $|\rho_k| = \left(1 + \frac{a_k+b_k}{b_kq_{k-1}}\right)^{-1}$, and since $q_k \leq 2\beta q_{k-1}$, we conclude that

$$|\rho_k| > \left(1 + \frac{2\beta^2}{\alpha}\right)^{-1}.$$ 

Therefore, recalling that $q_{n-1} \geq 2^{(n-3)/2}$

$$\frac{d_n}{q_n} > q_{n-1}(1 + 2\beta^2/\alpha)^{-(n+1)} \frac{b_0}{a_1} \Rightarrow \frac{q_n}{d_n} < (1 + 2\beta^2/\alpha)^{n-1} 2^{-(n-3)/2} \frac{\beta}{\alpha}.$$ 

Therefore, we want to determine the existence of a solution for $\theta$ for the inequality

$$d_n < \left(\frac{q_n}{d_n}\right)^\theta$$

considering that $d_n \approx \beta^n \alpha^{1-\nu n}$, we obtain the inequality

$$\beta^n \alpha^{1-\nu n} < (1 + 2\beta^2/\alpha)^{\theta(n-1)/2} 2^{-\theta(n-3)/2} \frac{\beta}{\alpha}.$$ 

For large $n$, it is sufficient to solve

$$\beta^n \alpha^{1-\nu} < \frac{1}{2^{\theta/2}} \left(1 + \frac{2\beta^2}{\alpha}\right)^\theta,$$

which clearly has the solution $\theta = 1$, since $\alpha < \beta$ and $0 < \nu < 1$, implying $\beta^n \alpha^{1-\nu} < \beta$. We conclude that type $S_3$ consists only of $S$-numbers or $T$-numbers. □

5 Example: partial quotients of a corresponding regular continued fraction

We now examine one specific example: a generalized continued fraction associated with the period doubling sequence. The period doubling sequence, which we denote by $\omega_n$, is the fixed point of the substitution $\xi(\alpha) = \alpha \beta$ and $\xi(\beta) = \alpha \alpha$ on the two lettered alphabet $\{\alpha, \beta\}$. It is also the limit of a sequence of foldings, and called a folded sequence [5].

We make some observations and one question about the partial quotients of the corresponding regular continued fraction representing the real number that the generalized continued fraction given by (1.1) when both sequences (a) and (b) are given by the period doubling sequence: $a_n = b_n = \omega_n$.

We choose to view the period doubling sequence as the limit of folding operations. The algebra of matrices with fixed determinant will play a role. A folding is a mapping

$$F_p : \mathcal{A}^* \rightarrow \mathcal{A}^*$$

$$w \mapsto wp\bar{w}$$

where $\bar{w}$ equals the word $w$ reversed: if $w = a_1 \ldots a_n, a_i \in \mathcal{A}$, then $\bar{w} = a_n \ldots a_1$, and $p \in \mathcal{A}^*$. It is clear that

$$\omega = \lim_{n \rightarrow \infty} (F_a \circ \cdots \circ F_a)^n(a),$$

where $\bar{w}$ equals the word $w$ reversed: if $w = a_1 \ldots a_n, a_i \in \mathcal{A}$, then $\bar{w} = a_n \ldots a_1$, and $p \in \mathcal{A}^*$. It is clear that

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where $\bar{w}$ equals the word $w$ reversed: if $w = a_1 \ldots a_n, a_i \in \mathcal{A}$, then $\bar{w} = a_n \ldots a_1$, and $p \in \mathcal{A}^*$. It is clear that

$$\omega = \lim_{n \rightarrow \infty} (F_a \circ \cdots \circ F_a)^n(a),$$
see also [5], where the limit is understood in the product topology (of the discrete topology) in $\mathcal{A}^N \cup \mathcal{A}^*$.

Let $\theta$ denote the number whose generalized continued fraction is obtained from the substitution of the letters $\alpha$ and $\beta$ by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 3 \\ 1 & 0 \end{pmatrix}$$

respectively. It corresponds to the choice $\{1, 3\}$ for the alphabet where the sequences $(a)$ and $(b)$ take values.

Now we use Raney transducers [12] to describe the computation of some partial quotients of the regular continued fraction converging to $\theta$. A transducer $\mathcal{T} = (Q, \Sigma, \delta, \lambda)$, or two-tape machine, is defined by a set of states $Q$, an alphabet $\Sigma$, a transition function $\delta : Q \times \sigma \to Q$, and an output function $\lambda : Q \times \sigma \to \Sigma$, where $\sigma \subset \Sigma$ (a more general definition is possible [5], but this is sufficient for our purposes).

The states of Raney’s transducers are column and row (or doubly) balanced matrices over the non-negative integers with a fixed determinant. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is column balanced if $(a - b)(c - d) < 0$ [12].

Figure 1 shows the Raney transducer for determinant 3 doubly balanced matrices. In the text, we use the abbreviations: $\beta_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\beta_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then $Q = \{\beta_1, \beta_2, \beta_3\}$, $\Sigma = \{L, R\}$, where

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the transition function $\delta$ and the output function are indicated in the graph.

For instance, if $RL^2R$ is the input word on state $\beta_2$, the output word is $L^2R^4$ and the final state is $\beta_1$. Any infinite word in $\Sigma^N$ can be read by $\mathcal{T}$, but not every finite word can be fully read by $\mathcal{T}$, for instance, $L^3$ in state $\beta_1$ will produce $L^3$, but $L^2$ will stay in the reading queue in state $\beta_1$. Algebraically, these two examples are written as

$$\beta_2RL^2R = L\beta_3LR = LLR\beta_1R = L^2RR^3\beta_1 = L^2R^4\beta_1$$
$$\beta_1L^3 = L^3\beta_1L^2$$

As explained in [13], Theorem 1 in [9] or even Theorem 5.1 in [12], one may use the transducer $\mathcal{T}$ to commute the matrices $A$ and $B$ to get an approximation to the continued fraction of $\theta$.

Introducing the matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we note the following relations: $A = RJ$, $B = \beta_1RJ$, $\beta_2J = J\beta_1$.

The homomorphism between $\mathcal{A}^*$ and the semigroup of matrices generated by $\{A, B\}$, as Moebius transforms, is the basis for discovering some curious properties of the regular continued fraction of $\theta$.

The sequence of matrices

$$ABA, \ ABAAABA, \ ABAAABABABAAABA,$$
Figure 1: Transducer $\mathcal{T}$.

which corresponds to $\mathcal{T}_b(a)$, $\mathcal{T}_a \circ \mathcal{T}_b(a)$, $\mathcal{T}_b \circ \mathcal{T}_a \circ \mathcal{T}_b(a)$, yields the beginning of the regular continued fraction expansion of $\theta$.

A step by step calculation shows the basic features in the use of $\mathcal{T}$:

\[
ABAAABA = RJ(\beta_1 RJ)RJ(\beta_1 RJ)RJ \\
= R\beta_2 (JRJ)R(JRJ)R\beta_2 (JRJ)RJ \\
= R\beta_2 LRRL\beta_2 LRJ \\
= RL^3 \beta_2 RLRL^3 \beta_2 RJ \\
= RL^3 L_3 \beta_3 RL^3 \beta_2 RJ \\
= RL^4 RL_2 L^3 \beta_2 RJ \\
= RL^4 RLL^9 \beta_2 RJ = RL^4 RL^{10} \beta_2^2 RJ .
\]

This means that the continued fraction of $\theta$ begins as $[1; 4, 1, k, \cdots]$, with $k \geq 10$. 
Writing $T = ABAABA$, upon the next folding

$$TBT = RL^4RL^{10}β_2^2RJ(β_1RJ)RL^4RL^{10}β_2^2RJ$$

$$= RL^4RL^{10}β_2^2Rβ_2LRL^4RL^{10}β_2^2RJ$$
$$= RL^4RL^{10}β_2^2RL^3Lβ_3L^3RL^{10}β_2^2RJ$$
$$= RL^4RL^{10}β_2^2RL^4LRβ_1L^2RL^{10}β_2^2RJ$$
$$= RL^4RL^{10}β_2^2RL^5RRL^3β_2L^{10}β_2^2RJ$$
$$= RL^4RL^{10}β_2^2Lβ_3L^4R^2L^{32}β_2^2RJ$$
$$= RL^4RL^{10}β_2^2Lβ_3L^4R^2L^{32}β_2^2RJ$$

Now we have the knowledge that the beginning of the regular continued fraction expansion of $θ = [1; 4, 1, 17, 1, 1, 1, 1, 2, a, \ldots]$, with $a \geq 3$.

Instructions for the transducer are stuck on the right of this factorization to be read for the next folding. Note that only states $β_1$ and $β_2$ will remain, since a transition from $β_3$ is always possible given any finite word in $Σ^∗$.

An inductive prediction as to whether the high power in the beginning, $L^{17}$, will consistently increase upon (sufficient) repetitions of foldings is out of reach. This observation poses the question: are the partial quotients of regular continued fraction of $θ$ bounded? Similar calculations have been done with a simpler choice of the alphabet, that is, $\{1,2\}$, where the transducer has only two states (doubly balanced matrices with determinant 2).

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