ON THE CENTER OF QUIVER-HECKE ALGEBRAS

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ABSTRACT. We compute the equivariant cohomology ring of the moduli space of framed instantons over the affine plane. It is a Rees algebra associated with the center of cyclotomic degenerate affine Hecke algebras of type $A$. We also give some related results on the center of quiver Hecke algebras and cohomology of quiver varieties.

CONTENTS

1. Introduction 2
2. Generalities 5
  2.1. The center and the trace of a category 6
    2.1.1. Categories 5
    2.1.2. Trace and center 5
    2.1.3. Operators on the trace 7
    2.1.4. Adjunction 7
    2.1.5. Operators on the center 8
  2.2. Symmetric algebras 8
    2.2.1. Kernels 8
    2.2.2. Induction and restriction 8
    2.2.3. Frobenius and symmetrizing forms 9
3. The center of quiver-Hecke algebras 9
  3.1. Quiver Hecke algebras 9
    3.1.1. Cartan datum 9
    3.1.2. Quiver Hecke algebras 10
    3.1.3. Cyclotomic quiver Hecke algebras 11
    3.1.4. Induction and restriction 12
    3.1.5. The symmetrizing form 13
  3.2. Categorical representations 14
    3.2.1. Definition 14
    3.2.2. The minimal categorical representation 15
    3.2.3. Factorization 16
    3.2.4. The isotypic filtration 21
    3.2.5. The loop operators 22
    3.2.6. The loop operators on the trace of the minimal categorical representation 22
    3.2.7. The loop operators on the center. 23

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1. Introduction

A few years ago, the cohomology ring of the Hilbert scheme of points on $\mathbb{C}^2$ was computed in [27], [39], motivated by some conjectures of Chen and Ruan on orbifold cohomology rings, which were later proved in [14]. One of the main motivations of the present work is to compute a larger class of cohomology ring of quiver varieties. More precisely, for each pair of positive integers $r, n$, one can consider the moduli space $\mathcal{M}(r, n)$ of framed instantons with second Chern class $n$ on $\mathbb{P}^2$. This is a smooth quasi-projective variety (over $\mathbb{C}$) which can be viewed as a quiver variety attached to the Jordan quiver. An $(r + 2)$-dimensional torus acts naturally on $\mathcal{M}(r, n)$ and it contains an $(r + 1)$-dimensional subtorus whose action preserves the symplectic form. One of our goal is to compute the equivariant cohomology ring of $\mathcal{M}(r, n)$ with respect to this subtorus. Since $\mathcal{M}(r, n)$ is equivariantly formal, one can easily recover the usual cohomology ring from the equivariant one. For $r = 1$, i.e., the case of the Hilbert scheme, it can be easily deduced from [39] that the equivariant cohomology ring we are interested in is the Rees algebra associated with the center of the group algebra of the symmetric group with respect to the age filtration. Here we obtain a similar description for arbitrary $r$ with the group algebra of the symmetric group replaced by a level $r$ cyclotomic quotient of the degenerate affine Hecke algebra of type $A$. 
This result has been conjectured soon after [39] was written. However, its proof requires two new ingredients which were introduced only very recently. An important tool used in [39] is Nakajima’s action of an Heisenberg algebra on the cohomology spaces of the Hilbert scheme. A similar action on the cohomology of $\mathcal{M}(r) = \bigsqcup_{n \geq 0} \mathcal{M}(r,n)$ was introduced by Baranovsky a few years ago, but it is insufficient to compute the cohomology ring. What we need in fact is the action of (a degenerate version of) a new algebra $\mathcal{W}$ which is much bigger than the Heisenberg algebra. This action was introduced recently in [38] to give a proof of the AGT conjecture for pure $N = 2$ gauge theories for the group $U(n)$. Here, we define a similar action of $\mathcal{W}$ on the center (or the cocenter) of cyclotomic quotients of degenerate affine Hecke algebras. Then we compare it with the representation of $\mathcal{W}$ on the cohomology of $\mathcal{M}(r)$ to obtain the desired isomorphism. To do this, we use categorical representation theory.

Categorical representations of Kac-Moody algebras have captured a lot of interest recently since the work of Khovanov-Lauda [22], [23], [24] and Rouquier [36]. It has been observed recently by Beliakova-Habiro-Guliyev-Lauda-Zivkovic [5], [6], that a categorical representation gives rise to some interesting structures on the center (or cocenter) of the underlying categories and not only on their Grothendieck group. More precisely, Khovanov-Lauda define a 2-Kac-Moody algebra in [24] which is a 2-category satisfying certain axioms. Their idea is that the trace of this 2-category has a natural structure of an associative algebra $Lg$ which should be some kind of loop algebra over the underlying Kac-Moody algebra $g$. The $sl_2$-case has been worked out in [5], and the $sl_n$-one in [6]. Naturally, the center (or cocenter) of 2-representations of the 2-Kac-Moody algebra gives rise to representations of $Lg$.

In this paper we first use a similar idea to investigate the center and cocenter of cyclotomic quiver-Hecke algebras associated with a Kac-Moody algebra $g$ of arbitrary type. The category of projective modules over these algebras provide minimal categorical representations of $g$, in the sense of Rouquier [36]. We compute the representation of $Lg$ on the center (or cocenter) of these minimal categorical representations. When $g$ is symmetric of finite type, we identify these $Lg$-modules with the local and global Weyl modules, which can be realized in the (equivariant) Borel-Moore homology spaces of quiver varieties by [24].

Then, in order to compute the cohomology ring of $\mathcal{M}(r,n)$, we consider another situation where $g$ is replaced by an Heisenberg algebra. On the categorical level this corresponds to the Heisenberg categorifications which have also been studied recently. But once again, instead of considering a 2-Heisenberg algebra, we focus on the particular categorical representation given by the module category of degenerate affine Hecke algebras of type $A$. The analog of $Lg$ in this case is the algebra $\mathcal{W}$ mentioned above. Probably one can generalize both situations (the Kac-Moody one and the Heisenberg one) using [17], where categorifications of some generalized (Borcherds-)Kac-Moody algebras are considered. Here, we do not go further in this direction.

Another motivation for this work is to compare the 2-Kac-Moody algebras in [24] and [36]. The definition of Rouquier contains less axioms than the definition of Khovanov-Lauda. By [18] the module category of cyclotomic quiver-Hecke algebras admits a representation of $g$ in the sense of Rouquier. By [12] it should also admits a representation of $g$ in Khovanov-Lauda’s sense, which is stronger. According to Cautis-Lauda [10], to prove this it is enough to check that the center of the category is positively graded with a one-dimensional degree zero component. These two conditions are difficult to check for minimal categorical representations. Using the representation of $Lg$ we prove that the first condition holds. The second one is more subtle. It
is equivalent to the indecomposability of the weight subcategories of the minimal categorical representations. In other words, each of these categories should have a single block. This is well-known in type A and in affine type A by the work of Brundan and Lyle-Mathas in [7], [29]. We can prove it in some new cases, using the fact that quiver varieties are connected (proved by Crawley-Boevey). But the general case is still unknown. Note that a third way to prove that minimal categorical representations are indeed representations of Khovanov-Lauda’s 2-Kac-Moody algebra is by a direct computation, see Remark A.5 in the appendix.

Now, let us describe more precisely the structure and the main results of the paper. Fix a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and a dominant integral weight $\Lambda$ of $\mathfrak{g}$.

In Section 2 we give some generalities on the centers and cocenters of linear categories. In Section 3 we introduce the cyclotomic quiver Hecke algebra of type $\mathfrak{g}$ and level $\Lambda$ over a field $k$ (of any characteristic). It is a symmetric algebra which decomposes as a direct sum $R^\Lambda = \oplus_{\alpha \in Q_+} R^\Lambda(\alpha)$, where $\alpha$ runs over the positive part of the root lattice of $\mathfrak{g}$. To $\mathfrak{g}$ we can attach another Lie algebra, $L\mathfrak{g}$, given by generators and relations. It coincides with the loop algebra of $\mathfrak{g}$ in finite types ADE. The first result is the following.

**Theorem 1.** Assume that $\mathfrak{g}$ is symmetric and that the condition (11) is satisfied. Then,

(a) there is a $\mathbb{Z}$-graded representation of $L\mathfrak{g}$ on $\text{tr}(R^\Lambda)$,

(b) if $\mathfrak{g}$ is of finite type then $\text{tr}(R^\Lambda)$ is isomorphic, as a $\mathbb{Z}$-graded $L\mathfrak{g}$-module, to the Weyl module with highest weight $\Lambda$.

Note that there are two different notions of Weyl modules for loop Lie algebras used in the literature (the local and the global ones). Both versions can indeed be recovered, see Theorem below for more details. Note also that the proof of part (b) involves the geometrical incarnation, given by Nakajima, of Weyl modules of $L\mathfrak{g}$ via the equivariant cohomology of a quiver variety $\mathcal{M}(\Lambda)$ attached to $\Lambda$.

**Theorem 2.** For any $\alpha \in Q_+$ the following hold

(a) the trace and the center of $R^\Lambda(\alpha)$ are positively graded,

(b) if $\mathfrak{g}$ is symmetric of finite type, the dimension of the degree zero subspace of $Z(R^\Lambda(\alpha))$ is one dimensional.

The proof uses a reduction to $\mathfrak{sl}_2$. Part (b) relies on the geometrical interpretation of Weyl modules. It also uses the non-degenerate pairing between the trace $\text{tr}(R^\Lambda(\alpha))$ and the center $Z(R^\Lambda(\alpha))$ of $R^\Lambda(\alpha)$ given by the symmetrizing form.

Finally, in Section 4 we focus on the Jordan quiver. In this case, instead of the cyclotomic quiver Hecke algebra, we consider a level $r$ cyclotomic quotient $R^r(n)$ of the degenerate affine Hecke algebra of $\mathfrak{S}_n$ defined over $k[h]$. Let $R^r(n)_1$ be its specialization at $h = 1$. The center of $R^r(n)_1$ has a natural filtration defined in terms of Jucy-Murphy elements. Let $\text{Rees}(Z(R^r(n)_1))$ be the corresponding Rees algebra. Set $R^r = \bigoplus_{n \in \mathbb{N}} R^r(n)$ and let $\text{tr}(R^r)'$ be a localization of $\text{tr}(R^r)$. Consider the equivariant cohomology $H^*_G(\mathcal{M}(r,n), k)$ of the quiver variety $\mathcal{M}(r,n)$ relatively to an $(r + 1)$-dimensional torus $G$ with coefficient in $k$.

**Theorem 3.** The following hold

(a) there is a level $r$ representation of $\mathcal{W}$ in $\text{tr}(R^r)'$,

(b) there is a $\mathbb{Z}$-graded algebra isomorphism $\text{Rees}(Z(R^r(n)_1)) \simeq H^*_G(\mathcal{M}(r,n), k)$. 

The proof of this theorem uses the representation of $\mathcal{W}$ on a localization $H^*_G(\mathcal{M}(r), k')$ introduced in [38].

After our paper appeared on the arXiv, A. Lauda informed us that there is some overlap between our results and his ongoing projects with collaborators.

2. Generalities

Let $k$ be a commutative noetherian ring.

2.1. The center and the trace of a category.

2.1.1. Categories. All categories are assumed to be small. A $k$-linear category is a category enriched over the tensor category of $k$-modules, a $k$-category is an additive $k$-linear category. For any $k$-linear category $\mathcal{C}$ and any $k$-algebra $k'$, let $\mathcal{C}' := k' \otimes_k \mathcal{C}$ be the $k'$-linear category whose objects are the same as those of $\mathcal{C}$, but whose morphism spaces are given by

$$\text{Hom}_{\mathcal{C}'}(a, b) = k' \otimes_k \text{Hom}_{\mathcal{C}}(a, b) \quad \forall a, b \in \mathcal{C}.$$  

We denote the identity of an object $a$ by $1_a$ or by $1$ if no confusion is possible. All the functors $F$ on $\mathcal{C}$ are assumed to be additive and/or $k$-linear. An additive and $k$-linear functor is called a $k$-functor. Let $\text{End}(F)$ be the endomorphism ring of $F$. We may denote the identity element in $\text{End}(F)$ by $F$, $1_F$ or $1$, and the identity functor of $\mathcal{C}$ by $1_\mathcal{C}$ or $1$.

The center of $\mathcal{C}$ is defined as $Z(\mathcal{C}) = \text{End}(1_\mathcal{C})$. A composition of functors $E$ and $F$ is written as $EF$ while a composition of morphisms of functors $y$ and $x$ is written as $y \circ x$.

An additive category $\mathcal{C}$ will be always equipped with its trivial exact structure, i.e., the admissible exact sequences are the split short exact sequences. Therefore, a Serre subcategory $\mathcal{I} \subset \mathcal{C}$ is a full additive subcategory which is stable under taking direct summands, and the quotient additive category $\mathcal{B} = \mathcal{C}/\mathcal{I}$ is such that

$$\text{Hom}_{\mathcal{B}}(a, b) = \text{Hom}_{\mathcal{C}}(a, b) / \sum_{c \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(c, b) \circ \text{Hom}_{\mathcal{C}}(a, c), \quad \forall a, b \in \mathcal{C}.$$  

A short exact sequence of additive categories is a sequence of functors which is equivalent to a sequence $0 \to \mathcal{I} \to \mathcal{C} \to \mathcal{B} \to 0$ as above.

Fix an integer $\ell$. By an $(\ell \mathbb{Z})$-graded $k$-category we’ll mean a $k$-category $\mathcal{C}$ equipped with a strict $k$-automorphism $[\ell]$ which we call shift of the grading. Unless specified otherwise, a functor $F$ of $(\ell \mathbb{Z})$-graded $k$-categories is always assumed to be graded, i.e., it is a $k$-functor $F$ with an isomorphism $F \circ [\ell] \simeq [\ell] \circ F$. For each integer $k \in \mathbb{N} \cap (\ell \mathbb{Z})$ we’ll abbreviate $[k] = [\ell] \circ [\ell] \circ \cdots \circ [\ell]$ ($|k/\ell|$ times) and $[-k] = [k]^{-1}$.

Let $\mathcal{C}/\ell \mathbb{Z}$ be the category enriched over the tensor category of $(\ell \mathbb{Z})$-graded $k$-modules whose objects are the same as those of $\mathcal{C}$, but whose morphism spaces are given by

$$\text{Hom}_{\mathcal{C}/\ell \mathbb{Z}}(a, b) = \bigoplus_{k \in \ell \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b[k]).$$  

Note that the center $Z(\mathcal{C}/\ell \mathbb{Z})$ is a graded ring whose degree $k$-component is equal to $\text{Hom}(1, [k])$.

Given a $\mathbb{Z}$-graded $k$-module $M$ let $M^d = \{ x \in M : \deg(x) = d \}$ for each $d \in \mathbb{Z}$. For any integer $\ell$, the $\ell$-twist of $M$ is the $(\ell \mathbb{Z})$-graded $k$-module $M^{[\ell]}$ such that $(M^{[\ell]})^d = M^{d/\ell}$ if $\ell | d$ and 0 else. Then, for each $\mathbb{Z}$-graded $k$-category $\mathcal{C}$ there is a canonical $(\ell \mathbb{Z})$-graded $k$-category
\( \mathcal{C}^{[\ell]} \) called the \( \ell \)-twist of \( \mathcal{C} \) such that \( \mathcal{C}^{[\ell]} = \mathcal{C} \) as a k-category and the shift of the grading \([\ell]\) in \( \mathcal{C}^{[\ell]} \) is the same as the shift of the grading \([1]\) in \( \mathcal{C} \). We have

\[
\text{Hom}_{\mathcal{C}^{[\ell]}}(a, b) = \text{Hom}_{\mathcal{C}}(a, b)^{[\ell]}, \quad \forall a, b.
\]

Finally, for any category \( \mathcal{C} \) we denote by \( \mathcal{C}^e \) the idempotent completion.

2.1.2. Trace and center. Let \( \mathcal{C} \) be a k-linear category and \( HH_\ast(\mathcal{C}) \) be the Hochschild homology of \( \mathcal{C} \), see [20] sec. 3.1. It is a \( \mathbb{Z} \)-graded k-module. We set \( \text{tr}(\mathcal{C}) = HH_0(\mathcal{C}) \) and \( \text{CF}(\mathcal{C}) = \text{Hom}_k(\text{tr}(\mathcal{C}), k) \). We call \( \text{tr}(\mathcal{C}) \) the cocenter or the trace of \( \mathcal{C} \) and \( \text{CF}(\mathcal{C}) \) the set of central forms on \( \mathcal{C} \). Recall that

\[
\text{tr}(\mathcal{C}) = \left( \bigoplus_{a \in \text{Ob}(\mathcal{C})} \text{End}_\mathcal{C}(a) \right) / \sum_{f, g} k \left[ f, g \right] \quad \text{for any } f : a \to b, \ g : b \to a.
\]

For any morphism \( f \) in \( \mathcal{C} \), let \( \text{tr}(f) \) denote its image in \( \text{tr}(\mathcal{C}) \).

Now, let \( A \) be any k-algebra. Unless specified otherwise, all algebras are assumed to be unital. Let \( Z(A) \) be the center of \( A \) and \( HH_\ast(A) \) be its Hochschild homology. Define \( \text{tr}(A) \) and \( \text{CF}(A) \) as above, i.e., \( \text{tr}(A) = A/[A, A] \) where \([A, A] \subset A\) is the k-submodule spanned by the commutators of elements of \( A \). For any element \( a \in A \) let \( \text{tr}(a) \) denote its image \( a + [A, A] \) in \( \text{tr}(A) \). Let \( A\text{-mod} \) and \( A\text{-proj} \) be the categories of finitely generated modules and finitely generated projective modules. For any commutative k-algebra \( R \) and any k-module \( M \) we abbreviate \( RM = R \otimes_k M \). The following is well-known.

**Proposition 2.1.** Let \( A, B \) be k-algebras and \( \mathcal{B}, \mathcal{C} \) be k-linear categories.

(\( a \)) If \( B \subset \mathcal{C} \) is full and any object of \( \mathcal{C} \) is isomorphic to a direct summand of a direct sum of objects of \( \mathcal{B} \), then the embedding \( \mathcal{B} \subset \mathcal{C} \) yields an isomorphism \( \text{tr}(\mathcal{B}) \to \text{tr}(\mathcal{C}) \).

(\( b \)) If \( \mathcal{C} = A\text{-mod} \) or \( \mathcal{A} \text{-proj} \) then \( Z(A) = Z(\mathcal{C}) \). If \( \mathcal{C} = A\text{-proj} \) then \( \text{tr}(A) = \text{tr}(\mathcal{C}) \).

(\( c \)) For any commutative k-algebra \( R \) we have \( \text{tr}(RA) = R \text{tr}(A) \).

(\( d \)) We have \( \text{tr}(A \otimes_k B) = \text{tr}(A) \otimes_k \text{tr}(B) \) and \( Z(A \otimes_k B) = Z(A) \otimes_k Z(B) \).

(\( e \)) \( Z(\mathcal{C}) \) acts on \( \text{tr}(\mathcal{C}) \) via the map \( Z(\mathcal{C}) \to \text{End}_k(\text{tr}(\mathcal{C})) \), \( a \mapsto (\text{tr}(a') \mapsto \text{tr}(aa')) \).

(\( f \)) A short exact sequence of k-categories \( 0 \to \mathcal{I} \to \mathcal{C} \to \mathcal{B} \to 0 \) yields an exact sequence of k-linear maps \( \text{tr}(\mathcal{I}) \to \text{tr}(\mathcal{C}) \to \text{tr}(\mathcal{B}) \to 0 \).

\( \Box \)

For a future use, let us give some details on part \((f)\). Assume that \( \mathcal{C} = \mathcal{C}^e \). For any object \( X \) let \( \text{add}(X) \subset \mathcal{C} \) be the smallest k-subcategory containing \( X \) which is closed under taking direct summands. Then, the functor \( \text{Hom}_\mathcal{C}(X, \bullet) \) yields an equivalence \( \text{add}(X) \to \text{End}_\mathcal{C}(X)^{\text{op}}\text{proj} \). In particular, if \( \mathcal{C} \) has a finite number of indecomposable objects \( X_1, X_2, \ldots, X_n \) (up to isomorphisms) and \( X = \bigoplus_{i=0}^d X_i \), then we have an equivalence \( \mathcal{C} \simeq \text{End}_\mathcal{C}(X)^{\text{op}}\text{proj} \).

Now, assume that \( \mathcal{C} = A\text{-proj} \), where \( A \) is a finitely generated k-algebra. Given a Serre k-subcategory \( \mathcal{I} \subset \mathcal{C} \), there is an idempotent \( e \in A \) such that \( \mathcal{I} = eAe\text{-proj} \) and the functor \( \mathcal{I} \to \mathcal{C} \) is given by \( M \mapsto Ae \otimes_{eAe} M \). Set \( \mathcal{B} = \mathcal{C}/\mathcal{I} \). Then, we have \( \mathcal{B}^e = B\text{-proj} \) where \( B = A/AeA \) and the composed functor \( \mathcal{C} \to \mathcal{B} \to \mathcal{B}^e \) is given by \( M \mapsto B \otimes_A M \). We must prove that taking the trace we get an exact sequence of k-modules \( \text{tr}(\mathcal{I}) \to \text{tr}(\mathcal{C}) \to \text{tr}(\mathcal{B}) \to 0 \).
Note that ker $j = (Ae A + [A, A])/[A, A]$ and im $i = (e Ae + [A, A])/[A, A]$. Since $aeb = e b a e + [a e, e b]$ for all $a, b \in A$, we deduce that ker $j = \text{im } i$, proving the claim.

### 2.1.3. Operators on the trace

**Definition 2.2.** Given a functor $F : \mathcal{C} \to \mathcal{C}'$ between two k-categories and a morphism of functors $x \in \text{End}(F)$, the trace of $F$ on $x$ is the linear map

$$tr_F(x) : \text{tr}(\mathcal{C}) \to \text{tr}(\mathcal{C}'), \quad \text{tr}(f) \mapsto \text{tr}(x(a) \circ F(f))$$

where $f \in \text{End}(a)$ and $x(a) \circ F(f) \in \text{End}(F(a))$.

Note that $x(a) \circ F(f) = F(f) \circ x(a)$ by functoriality. Below are some basic properties of the trace map, whose proofs are standard and are left to the reader.

**Lemma 2.3.** (a) For each $F_1, F_2 : \mathcal{C} \to \mathcal{C}'$, $x \in \text{End}(F_1 \oplus F_2)$, we have $tr_{F_1 \oplus F_2}(x) = tr_{F_1}(x_{11}) + tr_{F_2}(x_{22})$ where $x_{11} \in \text{End}(F_1), x_{22} \in \text{End}(F_2)$ are the diagonal coordinates of $x$.

(b) For two morphisms $\rho : F_1 \to F_2, \psi : F_2 \to F_1$, we have $tr_{F_1}(\psi \circ \rho) = tr_{F_2}(\rho \circ \psi)$. In particular, if $\rho : F_1 \to F_2$ is an isomorphism of functors, then for any $x \in \text{End}(F_1)$ we have $tr_{F_2}(\rho \circ x \circ \rho^{-1}) = tr_{F_1}(x)$.

(c) For each $F : \mathcal{C} \to \mathcal{C}', G : \mathcal{C}' \to \mathcal{C}''$, $x \in \text{End}(F)$ and $y \in \text{End}(G)$, we have $tr_{GF}(yx) = tr_{G}(y) \circ tr_{F}(x)$. \hfill \Box

### 2.1.4. Adjunction

Given two k-categories $\mathcal{C}_1, \mathcal{C}_2$, a pair of adjoint functors $(E, F)$ from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the datum $(E, F, \eta_E, \varepsilon_F)$ of functors $E : \mathcal{C}_1 \to \mathcal{C}_2, F : \mathcal{C}_2 \to \mathcal{C}_1$ and morphisms of functors $\eta_E : 1_{\mathcal{C}_1} \to FE$ and $\varepsilon_F : EF \to 1_{\mathcal{C}_2}$, called unit and counit, such that $(\varepsilon_F E) \circ (E \eta_E) = E$ and $(F \varepsilon_E) \circ (\eta_E F) = F$.

A pair of biadjoint functors $\mathcal{C}_1 \to \mathcal{C}_2$ is the datum $(E, F, \eta_{E'}, \varepsilon_{E'}, \eta_{F}, \varepsilon_{F})$ of functors $E : \mathcal{C}_1 \to \mathcal{C}_2, F : \mathcal{C}_2 \to \mathcal{C}_1, \mathcal{C}_1$, morphisms of functors $\eta_{E'} : 1_{\mathcal{C}_1} \to FE, \varepsilon_{E'} : EF \to 1_{\mathcal{C}_2}$ such that $(E, F, \eta_{E'}, \varepsilon_{E'})$ and $(F, E, \eta_{F}, \varepsilon_{F})$ are adjoint pairs.

**Example 2.4.** Given two pairs of adjoint functors $(E, F), (E', F')$ from $\mathcal{C}_1$ to $\mathcal{C}_2$, the direct sum $(E \oplus E', F \oplus F')$ is an adjoint pair such that

$$\eta_{E \oplus E'} = (\eta_{E}, 0, 0, \eta_{E'}) : 1_{\mathcal{C}_1} \to FE \oplus FE' \oplus F'E \oplus F''E',$$

$$\varepsilon_{E \oplus E'} = \varepsilon_{E} + \varepsilon_{E'} : EF \oplus EF' \oplus E'F \oplus E'F' \to 1_{\mathcal{C}_2}.$$  

If $E : \mathcal{C}_1 \to \mathcal{C}_2$ and $E' : \mathcal{C}_2 \to \mathcal{C}_3$, then $(E'E, FF')$ is an adjoint pair such that $\eta_{E'E} = (F \eta_{E'E}) \circ \eta_E$ and $\varepsilon_{E'E} = \varepsilon_{E'} \circ (E'\varepsilon_{E'})$.

Suppose $(E, F), (E', F')$ are two pairs of adjoint functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. For any morphism $x : E \to E'$, the left transpose $\psi x : F' \to F$ is the composition of the chain of morphisms

$$F' \xrightarrow{\eta_{E'E'}} FFE' \xrightarrow{F\psi x} FE'F' \xrightarrow{F\varepsilon_{E'}} F.$$
For any morphism \( y : F' \to F \), the right transpose \( y^\vee : E \to E' \) is the composition
\[
E \xrightarrow{E_{\eta E'}} EF' \xrightarrow{E_{\beta E'}} EFE' \xrightarrow{\varepsilon_{EE'}} E'.
\]

2.1.5. Operators on the center. Let \( C_1, C_2 \) be two \( k \)-categories, and \((E, F, \eta_E, \varepsilon_E, \eta_F, \varepsilon_F)\) a pair of biadjoint functors \( C_1 \to C_2 \). The isomorphisms \( 1_{C_1}E = E = 1_{C_1} \) yield a canonical \((Z(C_1), Z(C_2))\)-bimodule structure on \( \text{End}(E) \). Let \( Z(C_2) \to \text{End}(E) \), \( z \mapsto zE \) and \( Z(C_1) \to \text{End}(E) \), \( z \mapsto Ez \) denote the corresponding \( k \)-algebra homomorphisms.

Definition 2.5. For each \( x \in \text{End}(E) \) we define a map
\[
Z_E(x) : Z(C_2) \to Z(C_1)
\]
by sending an element \( z \in Z(C_2) \) to the composed morphism
\[
l_{C_1} \xrightarrow{\eta_E} F1_{C_2}E \xrightarrow{Fzz} F1_{C_2}E \xrightarrow{\varepsilon_F} l_{C_1}.
\]
We define \( Z_F(x) : Z(C_1) \to Z(C_2) \) for each \( x \in \text{End}(F) \) in the same manner with the role of \( E \) and \( F \) exchanged.

The proof of the following proposition is standard and is left to the reader.

Proposition 2.6. Let \((E, F, \eta_E, \varepsilon_E, \eta_F, \varepsilon_F), (E', F', \eta_{E'}, \varepsilon_{E'}, \eta_{F'}, \varepsilon_{F'})\) be two pairs of biadjoint functors. Let \( x \in \text{End}(E) \), \( x' \in \text{End}(E') \). Then, we have
(a) \( Z_E(x) : Z(C_2) \to Z(C_1) \) is \( k \)-linear,
(b) \( Z_{E \oplus E'}(x \oplus x') = Z_E(x) \circ Z_E(x') \) and \( Z_{E \oplus E'}(x \oplus x') = Z_E(x) + Z_E(x') \),
(c) the map \( Z_E : \text{End}(E) \to \text{Hom}_k(Z(C_2), Z(C_1)) \) is \((Z(C_1), Z(C_2))\)-bilinear,
(d) let \( \rho : E \to E' \) be an isomorphism with \( \rho^\vee = \rho \), then \( Z_{E'}(\rho \circ x \circ \rho^{-1}) = Z_E(x) \).

2.2. Symmetric algebras. Let \( A, B, C \) be \( k \)-algebras.

2.2.1. Kernels. There is an equivalence of categories between the category of \((A, B)\)-bimodules and the categories of functors from \( B\text{-Mod} \) to \( A\text{-Mod} \). It associates an \((A, B)\)-bimodule \( K \) with the functor \( \Phi_K : B\text{-Mod} \to A\text{-Mod} \) given by \( N \mapsto K \otimes_B N \). We say that \( K \) is the kernel of \( \Phi_K \). Since \( \Phi_K(B) = K \), the kernel is uniquely determined by the functor \( \Phi_K \). For two \((A, B)\)-bimodule \( K, K' \) we have \( \text{Hom}_{A,B}(K, K') \cong \text{Hom}(\Phi_K, \Phi_{K'}) \) given by \( f \mapsto f \otimes_B \text{id} \).

2.2.2. Induction and restriction. We’ll call \( B\)-algebra a \( k \)-algebra \( A \) with a \( k \)-algebra homomorphism \( i : B \to A \). We consider the restriction and induction functors
\[
\text{Res}_B^A : A\text{-Mod} \to B\text{-Mod}, \quad \text{Ind}_B^A = A \otimes_B - : B\text{-Mod} \to A\text{-Mod}.
\]
The pair \((\text{Ind}_B^A, \text{Res}_B^A)\) is adjoint with the co-unit \( \varepsilon : \text{Ind}_B^A \text{Res}_B^A \to 1 \) represented by the \((A, A)\)-bimodule homomorphism \( \mu : A \otimes_B A \to A \) given by the multiplication, and the unit \( \eta : 1 \to \text{Res}_B^A \text{Ind}_B^A \) represented by the morphism \( i \), which is \((B, B)\)-bilinear. Let \( A^B \) be the centralizer of \( B \) in \( A \). For any \( f \in A^B \) we set
\[
(1) \quad \mu_f : A \otimes_B A \to A, \quad a \otimes a' = af a'.
\]
2.2.3. Frobenius and symmetrizing forms. We refer to [36] for more details on this section.

Let $A$ be a $B$-algebra that is projective and finite as $B$-module. A morphism of $(B,B)$-bimodules $t : A \to B$ is called a Frobenius form if the morphism of $(A,B)$-modules $\hat{t} : A \to \text{Hom}_B(A,B)$, $a \mapsto (a' \mapsto t(a'a))$ is an isomorphism. If such a form exist, we say that $A$ is a Frobenius $B$-algebra. If we have $t(aa') = t(a'a)$ for each $a \in A$, $a' \in A^B$ then $t$ is called a symmetrizing form and $A$ a symmetric $B$-algebra.

Given $t : A \to B$ a Frobenius form, the composition of the isomorphism $A \otimes_B A \xrightarrow{\sim} \text{Hom}_B(A,B) \otimes A$ given by $a \otimes a' \mapsto t(a) \otimes a'$ and the canonical isomorphism $\text{Hom}_B(A,B) \otimes_B A \xrightarrow{\sim} \text{End}_B(A)$ yields an isomorphism $A \otimes_B A \xrightarrow{\sim} \text{End}_B(A)$. The preimage of the identity under this map is the Casimir element $\pi \in (A \otimes_B A)^A$. We have $(t \otimes 1)(\pi) = (1 \otimes t)(\pi) = 1$.

There is a bijection between the set of Frobenius forms and the set of adjunctions $(\text{Res}^A_B, \text{Ind}^A_B)$ given as follows. Given a Frobenius form $t : A \to B$, the counit $\hat{\varepsilon} : \text{Res}^A_B \text{Ind}^A_B \to 1_B$ is represented by the $(B,B)$-linear map $t : A \to B$ and the unit $\hat{\eta} : 1_A \to \text{Ind}^A_B \text{Res}^A_B$ is represented by the unique $(A,A)$-linear map $\hat{\eta} : A \to A \otimes_B A$ such that $\hat{\eta}(1_A) = \pi$. This yields an adjunction for $(\text{Res}^A_B, \text{Ind}^A_B)$. Conversely, if $\hat{\varepsilon}$ and $\hat{\eta}$ are counit and unit for $(\text{Res}^A_B, \text{Ind}^A_B)$, then the $(B,B)$-linear map $t : A \to B$ which represents $\hat{\varepsilon}$ is a Frobenius form.

Recall that $\text{tr}(A)$ is a $Z(A)$-module. We equip $\text{CF}(A)$ with the dual $Z(A)$-action. Let us recall a few basic facts.

**Proposition 2.7.** Let $A, B, C$ be $k$-algebras which are projective and finite as $k$-modules.

(a) If $t : A \to B$ and $t' : B \to C$ are symmetrizing forms then $t' \circ t : A \to C$ is again a symmetrizing form.

(b) A symmetrizing form $t : A \to k$ induces a nondegenerate $Z(A)$-bilinear form $t : Z(A) \times \text{tr}(A) \to k$ such that $t(a,a') = t(aa')$.

(c) If $t : A \to k$ is a symmetrizing form then $t(A)$ is a faithful $Z(A)$-module and $\text{CF}(A)$ is a free $Z(A)$-module of rank 1 generated by the obvious map $t : t(A) \to k$.

**Proof.** Part (a) is proved in [36] lem. 2.10. Part (b) is obvious. For (c), assume that $t : A \to k$ is a symmetrizing form. If $a,b \in Z(A)$ have the same image in $\text{End}_k(\text{tr}(A))$, then for each $a' \in \text{tr}(A)$ we have $t(aa' - ba') = 0$, from which we deduce that $a = b$ by part (b). For the second claim see, e.g., [9 lem.2.5].

3. The center of quiver-Hecke algebras

3.1. Quiver Hecke algebras. Assume that $k = \bigoplus_{n \in \mathbb{N}} k^n$ is noetherian and \mathbb{N}-graded and that $k^0$ is a field. We may abbreviate $k = k^0$ and we’ll identify $k$ with the quotient $k/k^{>0}$ without mentioning it explicitly.

3.1.1. Cartan datum. A Cartan datum consists of a finite-rank free abelian group $P$ called the weight lattice whose dual lattice, called the co-weight lattice, is denoted $P^\vee$, of a finite set of vectors $\Phi = \{\alpha_1, \ldots, \alpha_n\} \subset P$ called simple roots and of a finite set of vectors $\Phi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset P^\vee$ called simple coroots. Let $Q_+ = \mathbb{N}\Phi \subset P$ be the semigroup generated by the simple roots and $P_+ \subset P$ be the subset of dominant weights, i.e., the set of weights $\Lambda$ such that $\Lambda_i = (\alpha_i^\vee, \Lambda) \geq 0$ for all $i \in I$. We’ll call Bruhat order the partial order on $P$ such that $\lambda \leq \mu$ whenever $\mu - \lambda \in Q_+$. 
Set $I = \{1, \ldots, n\}$ and let $\langle \bullet, \bullet \rangle$ be the canonical pairing on $P^\vee \times P$. The $I \times I$ matrix $A$ with entries $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is assumed to be a generalized Cartan matrix. We'll assume that the Cartan datum is non-degenerate, i.e., the simple roots are linearly independent, and symmetrizable, i.e., there exist non-zero integers $d_i$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j$. The integers $d_i$ are unique up to an overall common factor. They can be assumed positive and mutually prime.

Let $(\bullet, \bullet)$ be the symmetric bilinear form on $h^* = \mathbb{Q} \otimes \mathbb{Z} P$ given by $(\alpha_i | \alpha_j) = d_i a_{ij}$. Let $g$ be the symmetrizable Kac-Moody algebra over $k$ associated with the generalized Cartan matrix $A$ and the lattice of integral weights $P$. Let $h, n^+ \subset g$ be the Cartan subalgebra and the maximal nilpotent subalgebra spanned by the positive root vectors of $g$. For any dominant weight $\lambda \in P_+$, let $V(\lambda)$ be the corresponding integrable simple $g$-module. For each $\lambda \in P$ let $V(\Lambda) \subset V(\Lambda)$ be the weight subspace of weight $\lambda$.

### 3.1.2. Quiver Hecke algebras.

Fix an element $c_{i,j,p,q} \in k$ for each $i, j \in I$, $p, q \in \mathbb{N}$ such that $\deg(c_{i,j,p,q}) = -2d_i(a_{ij} + p) - 2d_jq$ and $c_{i,j,-a_{ij},0}$ is invertible. Fix a matrix $Q = (Q_{ij})_{i,j \in I}$ with entries in $k[u, v]$ such that
\[
Q_{ij}(u, v) = Q_{ji}(v, u), \quad Q_{ii}(u, v) = 0, \quad Q_{ij}(u, v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^p v^q \text{ if } i \neq j.
\]

**Definition 3.1.** The quiver Hecke algebra (or QHA) of rank $n \geq 0$ associated with $A$ and $Q$ is the $k$-algebra $R(n; Q, k)$ generated by $e(\nu)$, $x_k$, $\tau_l$ with $\nu \in I^n$, $k, l \in [1, n]$, $l \neq n$, satisfying the following defining relations:

(a) $e(\nu) e(\nu') = \delta_{\nu, \nu'} e(\nu)$, $\sum_{\nu} e(\nu) = 1$,

(b) $x_k x_l = x_l x_k$, $x_k e(\nu) = e(\nu) x_k$,

(c) $\tau_l e(\nu) = e(s_l(\nu)) \tau_l$, $\tau_{k-l} = \tau_k \tau_l$ if $|k-l| > 1$,

(d) $\tau_l^2 e(\nu) = Q_{\nu,\nu+1}(x_l, x_{l+1}) e(\nu)$,

(e) $(\tau_{k-l} x_l - x_{l(k-l)} \tau_l) e(\nu) = \delta_{\nu_k, \nu_{k+1}} (\delta_{l,k+1} - \delta_{l,k}) e(\nu)$,

(f) $(\tau_{k-1} \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \delta_{\nu_k, \nu_{k+2}} \partial_{k,k+2} Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu)$,

where $\partial_{k,l}$ is the Demazure operator on $k[x_1, x_2, \ldots, x_n]$ which is defined by
\[
\partial_{k,l}(f) = (f - (k, l)(f))/(x_k - x_l).
\]

The algebra $R(n; Q, k)$ is free as a $k$-module. It admits a $\mathbb{Z}$-grading given by
\[
\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = 2d_{\nu_k}, \quad \deg(\tau_k e(\nu)) = -d_{\nu_k} a_{\nu_k, \nu_{k+1}}.
\]

For $\alpha \in Q_+$ such that $\text{ht}(\alpha) = n$, we set
\[
I^n = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n; \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \alpha\}.
\]

The idempotent $e(\alpha) = \sum_{\nu \in I^n} e(\nu)$ is central in $R(n; Q, k)$. Given $\nu, \nu' \in I^n$ we write $\nu \nu' \in I^{n+m}$ for their concatenation. Set $e(\alpha, \alpha') = \sum_{\nu \in I^n} e(\nu \nu')$ and $e(\alpha, \nu') = \sum_{\nu \in I^n} e(\nu \nu')$.

The quiver Hecke algebra of rank $\alpha$ is the algebra
\[
R(\alpha; Q, k) = e(\alpha) R(n; Q, k) e(\alpha).
\]
3.1.3. **Cyclotomic quiver Hecke algebras.** Given a dominant weight $\Lambda \in P_+$ we set

$$I_{\Lambda} = \{(i, p); i \in I, p = 1, \ldots, \Lambda_i\}.$$ 

For a future use, let

$$I_{\Lambda} \to I, \quad t \mapsto i_t$$

denote the canonical map such that $(i, p) \mapsto i$. Fix a family of commuting formal variable

$$\{c_t; t \in I_{\Lambda}\}.$$  

Let $k^\Lambda$ be the $\mathbb{N}$-graded ring given by

$$k^\Lambda = k[c_t; t \in I_{\Lambda}], \quad \deg(c_{ip}) = 2pd_i.$$  

We'll abbreviate $k = k^1$ and we'll write $c_{i0} = 1$.

Now, fix a $\mathbb{N}$-graded $k$-algebra $k$. Let $c_t$ denote both the element in $k$ above and its image
in $k$ by the canonical map $k \to k$ (which is homogeneous of degree 0). Then, set

$$a^\Lambda_i(u) = \sum_{p=0}^{\Lambda_i} c_{ip} u^{\Lambda_i-p} \in k[u].$$

The monic polynomial $a^\Lambda_i(u)$ is called the $i$-th cyclotomic polynomial associated with $k$.

For each $\alpha \in Q_+$ and $1 \leq k \leq \text{ht}(\alpha)$, we set

$$a^\Lambda_{\alpha}(x_k) = \sum_{\nu \in I^\alpha} a^\Lambda_{\nu}(x_k) e(\nu).$$

Note that $a^\Lambda_{\alpha}(x_k)e(\nu)$ is a homogeneous element of $R(\alpha; Q, k)$ with degree $2d_{\nu_k} \Lambda_{\nu_k}$.

**Definition 3.2.** The cyclotomic quiver Hecke algebra (or CQHA) of rank $\alpha$ and level $\Lambda$ is the quotient $R^\Lambda(\alpha; Q, k)$ of $R(\alpha; Q, k)$ by the two-sided ideal generated by $a^\Lambda_{\alpha}(x_1)$.

To simplify notation, we write $R(\alpha) = R(\alpha; k) = R(\alpha; Q, k)$ and $R^\Lambda(\alpha) = R^\Lambda(\alpha; k) = R^\Lambda(\alpha; Q, k)$. We may also write $R = \bigoplus_{\alpha} R(\alpha)$, $R(k) = \bigoplus_{\alpha} R(\alpha; k)$, $R^\Lambda = \bigoplus_{\alpha} R^\Lambda(\alpha)$, etc. The following is proved in [16, cor. 4.4, thm. 4.5].

**Proposition 3.3.** The $k$-algebra $R^\Lambda(\alpha; k)$ is free of finite type as a $k$-module. \hfill $\square$

**Remark 3.4.** A morphism of $\mathbb{N}$-graded $k$-algebras $k \to h$ yields canonical graded $h$-algebra isomorphisms $h \otimes_k R(\alpha; k) \to R(\alpha; h)$ and $h \otimes_k R^\Lambda(\alpha; k) \to R^\Lambda(\alpha; h)$.

**Example 3.5.** (a) Set $R^\Lambda(\alpha) = R^\Lambda(\alpha; k)$. We call $R^\Lambda(\alpha)$ the global (or universal) CQHA.

(b) If $k = k$ then $a^\Lambda_i(u) = u^{\Lambda_i}$ for each $i$. We call $R^\Lambda(\alpha; k)$ the local (or restricted) CQHA.

(c) For each $i \in I$ we fix an element $c_i \in k$ of degree $2d_i$. Let $k'$ denote the new $k$-algebra structure on $k$ such that the corresponding cyclotomic polynomial is $a^\Lambda_i(u - c_i)$. Set $Q'_{ij}(u, v) = Q_{ij}(u - c_i, v - c_j)$. Then, the assignment $e(\nu), x_k e(\nu), \tau c(\nu) \mapsto e(\nu), (x_k + c_k) e(\nu), \tau e(\nu)$ extends uniquely to a $k$-algebra isomorphism $R^\Lambda(\alpha; Q, k) \sim R^\Lambda(\alpha; Q', k')$. In particular, fix $i \in I$ and assume that $\Lambda = \omega_i$ is the $i$-th fundamental weight. Assume also that condition (11) below is satisfied. Set $a^\Lambda_i(u) = u + c_i$. Then, we have a $k$-algebra isomorphism

$$R^\omega(\alpha; Q, k) \simeq k \otimes_k R^\omega(\alpha; Q, k).$$
3.1.4. Induction and restriction. Let \( i \in I \) and \( \alpha \in Q_+ \) of height \( n \). Set \( \lambda = \Lambda - \alpha \) and \( \lambda_i = \langle \alpha_i^\vee, \lambda \rangle \).

We have a \( \mathbb{Z} \)-graded \( k \)-algebra embedding \( \iota_i : R(\alpha) \hookrightarrow R(\alpha + \alpha_i) \) given by \( e(\nu), x_k, \tau_i \mapsto e(\nu, i), x_k, \tau_i \) for each \( \nu \in I^\alpha \) with \( 1 \leq k \leq n \) and \( 1 \leq l \leq n - 1 \). It induces a \( \mathbb{Z} \)-graded \( k \)-algebra homomorphism \( \iota_i : R(\alpha) \to R(\alpha + \alpha_i) \).

The restriction and induction functors form an adjoint pair \((F^i, E^i)\) with

\[
\begin{align*}
F^i : R^i(\alpha + \alpha_i) \text{-grmod} & \to R^i(\alpha) \text{-grmod}, \quad N \mapsto e(\alpha, i)N, \\
E^i : R^i(\alpha) \text{-grmod} & \to R^i(\alpha + \alpha_i) \text{-grmod}, \quad M \mapsto R^i(\alpha + \alpha_i)e(\alpha, i) \otimes R^i(\alpha)M.
\end{align*}
\]

The counit \( \varepsilon'_{i,\lambda} : F^i_i E^i_i 1_{\lambda} \to 1_{\lambda} \) and the unit \( \eta'_{i,\lambda} : 1_{\lambda} \to E^i_i F^i_i 1_{\lambda} \) are represented respectively by the multiplication map \( \mu \) and the map \( \iota_i \)

\[
\varepsilon'_{i,\lambda} : R^i(\alpha)e(\alpha - \alpha_i, i) \otimes_{R^i(\alpha - \alpha_i)} e(\alpha - \alpha_i, i)R^i(\alpha) \to R^i(\alpha),
\]

\[
\eta'_{i,\lambda} : R^i(\alpha) \to e(\alpha, i)R^i(\alpha + \alpha_i)e(\alpha, i).
\]

Finally, let \( \sigma'_{i,j,\lambda} : F^i_j E^j_i 1_{\lambda} \to E^i_j F^j_i 1_{\lambda} \) be the morphism represented by the linear map

\[
R^i(\alpha - \alpha_j + \alpha_i)e(\alpha - \alpha_j, i) \otimes R^i(\alpha - \alpha_j) e(\alpha - \alpha_j, j)R^i(\alpha) \to e(\alpha - \alpha_j + \alpha_i, j)R^i(\alpha + \alpha_i)e(\alpha, i),
\]

\[
x \otimes y \mapsto x \tau_{\nu} y.
\]

For \( j = i \), the element \( \tau_{\nu} \in R^i(\alpha + \alpha_i) \) centralizes the subalgebra \( e(\alpha - \alpha_i, i^2)R^i(\alpha - \alpha_i)e(\alpha - \alpha_i, i^2) \), so we have \( \sigma'_{ii,\lambda} = \mu_{\tau_{\nu}}, \) see [11].

**Theorem 3.7** ([11]). For each \( \alpha \in Q_+ \) of height \( n \), we have

(a) if \( \lambda_i \geq 0 \), then the following morphism of endofunctors on \( R^i(\alpha) \text{-Mod} \) is an isomorphism

\[
\rho'_{i,\lambda} = \sigma'_{ii,\lambda} \sum_{k=0}^{\lambda_i-1} (E^i_k x^k) \circ \eta'_{i,\lambda} : F^i_i E^i_i 1_{\lambda} \oplus \bigoplus_{k=0}^{\lambda_i-1} kx^k \otimes 1_{\lambda} \to E^i_i F^i_i 1_{\lambda},
\]

(b) if \( \lambda_i \leq 0 \), then the following morphism of endofunctors on \( R^i(\alpha) \text{-Mod} \) is an isomorphism

\[
\rho'_{i,\lambda} = (\sigma'_{ii,\lambda}, \varepsilon'_{i,\lambda} \circ (F^i_k x^0), \ldots, \varepsilon'_{i,\lambda} \circ (F^i_k x^{-\lambda_i-1})) : F^i_i E^i_i 1_{\lambda} \to E^i_i F^i_i 1_{\lambda} \oplus \bigoplus_{k=0}^{-\lambda_i-1} k(x^{-1})^k \otimes 1_{\lambda}.
\]

The theorem can be rephrased as follows.
• assume $\lambda_i \geq 0$: for any $z \in e(\alpha, i)R^A(\alpha + \alpha_i)e(\alpha, i)$ there are unique elements $\pi(z) \in R^A(\alpha)e(\alpha + \alpha_i, i) \otimes R^A(\alpha - \alpha_i, i) e(\alpha - \alpha_i, i)R^A(\alpha)$ and $p_k(z) \in R^A(\alpha)$ such that

$$z = \mu_{\tau_n}(\pi(z)) + \sum_{k=0}^{\lambda_i-1} p_k(z) x_k^{n+1}. \tag{6}$$

• assume $\lambda_i \leq 0$: for any $z \in e(\alpha, i)R^A(\alpha + \alpha_i)e(\alpha, i)$ and any $z_0, \ldots, z_{-\lambda_i-1} \in R^A(\alpha)$, there is a unique element $y \in R^A(\alpha)e(\alpha - \alpha_i, i) \otimes R^A(\alpha - \alpha_i, i) e(\alpha - \alpha_i, i)R^A(\alpha)$ such that

$$\mu_{\tau_n}(y) = z, \quad \mu_{\pi_k}(y) = z_k, \quad \forall k \in [0, -\lambda_i - 1]. \tag{7}$$

For a future use, let us introduce the following notation. Assume that $\lambda_i \leq 0$ and that $z \in e(\alpha, i)R^A(\alpha + \alpha_i)e(\alpha, i)$. For each $\ell \in [0, -\lambda_i - 1]$ let

$$\tilde{z}, \tilde{\pi}_\ell \in R^A(\alpha)e(\alpha - \alpha_i, i) \otimes R^A(\alpha - \alpha_i, i) e(\alpha - \alpha_i, i)R^A(\alpha)$$

be the unique elements such that

$$\mu_{\tau_n}(\tilde{z}) = z, \quad \mu_{\pi_k}(\tilde{z}) = 0, \quad \mu_{\tau_{\pi_k}}(\tilde{\pi}_\ell) = 0, \quad \mu_{\pi_k}(\tilde{\pi}_\ell) = \delta_{k, \ell}. \tag{8}$$

**Theorem 3.8 ([13]).** The pair $(E'_i, F'_i)$ is adjoint with the counit $\varepsilon'_{i, \lambda} : E'_iF'_i1_{\lambda} \to 1_{\lambda}$ and the unit $\eta'_{i, \lambda} : 1_{\lambda} \to F'_iE'_i1_{\lambda}$ represented by the morphisms

$$\varepsilon'_{i, \lambda} : e(\alpha, i)R^A(\alpha + \alpha_i)e(\alpha, i) \to R^A(\alpha)$$

$$\eta'_{i, \lambda} : R^A(\alpha) \to R^A(\alpha)e(\alpha - \alpha_i, i) \otimes R^A(\alpha - \alpha_i, i) e(\alpha - \alpha_i, i)R^A(\alpha)$$

such that

- $\varepsilon'_{i, \lambda}(z) = p_{\lambda_i-1}(z)$ if $\lambda_i > 0$ and $\mu_{\pi_{\lambda_i-1}}(\tilde{z})$ if $\lambda_i \leq 0$,
- $\eta'_{i, \lambda}(1) = -\pi(x_{n+1}^{\lambda_i})$ if $\lambda_i \geq 0$ and $\tilde{\pi}_{-\lambda_i-1}$ if $\lambda_i < 0$.

We abbreviate $\varepsilon'_i = \varepsilon'_{i, \lambda}$, $\eta'_i = \eta'_{i, \lambda}$, $\varepsilon'_i = \varepsilon'_{i, \lambda}$, $\eta'_i = \eta'_{i, \lambda}$, etc, when $\lambda$ is clear from the context.

**Corollary 3.9.** The linear maps $\varepsilon'_i$, $\eta'_i$ are homogeneous of degree zero. The linear maps $\varepsilon'_i$, $\eta'_i$ are homogeneous of degree $2d_i(1 - \lambda_i)$, $2d_i(1 + \lambda_i)$ respectively. The linear map $\sigma'_{i, j}$ is homogeneous of degree $-d_ia_{ij}$. \(\square\)

### 3.1.5. The symmetrizing form

For each $\alpha \in Q_+$ we set

$$d_{\Lambda, \alpha} = (\Lambda|\Lambda) - (\Lambda - \alpha|\Lambda - \alpha).$$

We’ll need the following result from [12] rem. 3.19).

**Proposition 3.10.** The $k$-algebra $R^A(\alpha)$ is symmetric and admits a symmetrizing form $t_{\Lambda, \alpha}$ which is homogeneous of degree $-d_{\Lambda, \alpha}$. \(\square\)

The definition of $t_{\Lambda, \alpha}$ is given in Definition [A.6]. We’ll abbreviate $t_\alpha = t_{\Lambda, \alpha}$ and $t_\Lambda = \sum_\alpha t_\alpha$. Since we have not found any proof of the proposition in the literature, we have given one in Appendix [A].
3.2. Categorical representations. Let $k$ be an $\mathbb{N}$-graded commutative ring as in Section 3.1. Write $\mathfrak{g}_k = k \otimes_k \mathfrak{g}$. Fix an integer $\ell$.

3.2.1. Definition. For each $\lambda \in P$, let $\mathcal{C}_\lambda$ be an $(\ell\mathbb{Z})$-graded $k$-category. Set $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$ and denote by $1_\lambda$ the obvious functor $1_\lambda : \mathcal{C} \to \mathcal{C}_\lambda$. For each $i, j \in I$, $\lambda \in P$ we fix

- a $\mathbb{Z}$-graded $k$-algebra homomorphism $k^{[\ell]}_\lambda \to \mathbb{Z}(\mathcal{C}/\ell\mathbb{Z})$,
- a functor $1_{\lambda - \alpha_i} F_i = F_i 1_\lambda$ with a right adjoint $1_{\lambda} E_i[\ell d_i (1 - \lambda_i)] = E_i 1_{\lambda - \alpha_i} [\ell d_i (1 - \lambda_i)]$,
- morphisms of functors $x_i 1_\lambda, F_i 1_\lambda \to F_i 1_{\lambda}[2 \ell d_i]$ and $\tau_{ij} 1_{\lambda}, F_i F_j 1_\lambda \to F_j F_i 1_{\lambda}[-\ell d_i a_{ij}]$.

Thus $\mathcal{C}/\ell\mathbb{Z}$ is a $k$-category and the functors $F_i 1_\lambda, E_i 1_\lambda$ are $k$-linear. Let

$$e_i 1_\lambda : F_i E_i 1_\lambda \to 1_{\lambda}[\ell d_i (1 + \lambda_i)], \quad \eta_i 1_\lambda : 1_{\lambda} \to E_i F_i 1_{\lambda}[rd_i (1 - \lambda_i)]$$

be the counit and the unit of the adjoint pair $(1_{\lambda} F_i, E_i 1_{\lambda}[-\ell d_i (1 + \lambda_i)])$. We'll abbreviate

$$E_i = \bigoplus_\lambda E_i 1_\lambda, \quad F_i = \bigoplus_\lambda F_i 1_\lambda, \quad F_\alpha = \bigoplus_{\nu \in I^\alpha} F_\nu,$$ 

where $F_\nu = F_{\nu_1} F_{\nu_2} \cdots F_{\nu_n}$ for $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$. Next, we define the following morphisms

- $\sigma_{ij} = (E_j F_i e_j) \circ (E_i \tau_{ji} E_i) \circ (\eta_j F_i E_i) : F_i E_i \to E_j F_i$,
- $\rho_{i1} = \sigma_{ii} 1_{\lambda} + \sum_{l=0}^{\lambda_i-1} (\varepsilon_i 1_{\lambda}) \circ (x^l_i E_i 1_{\lambda}) : F_i E_i 1_{\lambda} \to E_i F_i 1_{\lambda} \oplus \bigoplus_{l=0}^{\lambda_i-1} 1_{\lambda}[\ell d_i (1 + 2l + \lambda_i)]$ if $\lambda_i \leq 0$,
- $\rho_{i1} = \sigma_{ii} 1_{\lambda} + \sum_{l=0}^{\lambda_i-1} (\varepsilon_i 1_{\lambda}) \circ (x^l_i E_i 1_{\lambda}) \circ (\eta_i 1_{\lambda}) : F_i E_i 1_{\lambda} \oplus \bigoplus_{l=0}^{\lambda_i-1} 1_{\lambda}[\ell d_i (1 + 2l - \lambda_i)] \to E_i F_i 1_{\lambda}$ if $\lambda_i \geq 0$.

Definition 3.11. A categorical representation of $\mathfrak{g}_k$ of degree $\ell$ in $\mathcal{C}$ is a tuple $C_\lambda, E_i, F_i, \varepsilon_i, \eta_i, \tau_{ij}$ as above such that the following hold

- the assignment $e(\nu) \mapsto 1_{F_\nu}, x_k e(\nu) \mapsto x_{\nu_k} 1_{F_\nu}, \tau_{ij} e(\nu) \mapsto \tau_{ij} x_{\nu_{ij}} 1_{F_\nu}$ for each $\nu \in I^\alpha$ defines a $\mathbb{Z}$-graded $k^{[\ell]}$-algebra homomorphism $R(\alpha; k)^{[\ell]} : \End_{\mathcal{C}/\ell\mathbb{Z}}(F_\alpha)$,
- the morphisms $\rho_{i1}, \sigma_{ij}, i \neq j$, are isomorphisms.

Morphisms of categorical representations are defined in the obvious way.

We'll call the map $R(\alpha; k)^{[\ell]} : \End_{\mathcal{C}/\ell\mathbb{Z}}(F_\alpha)$ the canonical homomorphism associated with the categorical representation of $\mathfrak{g}_k$ in $\mathcal{C}$.

Unless specified otherwise, a categorical representation will be of degree 1. Note that, given a categorical representation of $\mathfrak{g}_k$ in $\mathcal{C}$, there is a canonical categorical representation of $\mathfrak{g}_k$ of degree $\ell$ in $\mathcal{C}^{[\ell]}$ called its $\ell$-twist such that the $\mathbb{Z}$-graded $k^{[\ell]}$-algebra homomorphism

$$R(\alpha; k)^{[\ell]} : \End_{\mathcal{C}/\ell\mathbb{Z}}(F_\alpha) = \End_{\mathcal{C}/\mathbb{Z}}(F_\alpha)^{[\ell]}$$

is equal to the homomorphism $R(\alpha; k) : \End_{\mathcal{C}/\mathbb{Z}}(F_\alpha)$ associated with the $\mathfrak{g}_k$-action on $\mathcal{C}$.

We'll also use the following definitions

- $\mathcal{C}$ is integrable if $E_i, F_i$ are locally nilpotent for all $i$,
- $\mathcal{C}$ is bounded above if the set of weights of $\mathcal{C}$ is contained in a finite union of cones of type $\mu - Q_+$ with $\mu \in P$,
- the highest weight subcategory $\mathcal{C}^{\text{hw}} \subset \mathcal{C}$ is the full subcategory given by

$$\mathcal{C}^{\text{hw}} = \{ M \in \mathcal{C} : E_i (M) = 0, \forall i \in I \}.$$
Remark 3.12. (a) Taking the left transpose of the morphisms of functors

\[ x_i 1_\lambda : F_i 1_\lambda \rightarrow F_i 1_\lambda[2\ell d_i], \quad \tau_{ij} 1_\lambda : F_i F_j 1_\lambda \rightarrow F_j F_i 1_\lambda[-\ell d_i a_{ij}] \]

we get the morphisms of functors

\[ 1_\lambda^\vee x_i : 1_\lambda E_i \rightarrow 1_\lambda E_i[2\ell d_i], \quad 1_\lambda^\vee \tau_{ij} 1_\lambda : 1_\lambda E_i E_j \rightarrow 1_\lambda E_j E_i[-\ell d_i a_{ij}] \]

We'll abbreviate \( x_i = \langle x_i \rangle \) and \( \tau_{ij} = \langle \tau_{ij} \rangle \).

(b) Forgetting the grading at each place we define as above a categorical representation of \( \mathfrak{g}_k \) in a (not graded) \( k \)-category \( C \).

(c) For each short exact sequence of \( \mathbb{Z} \)-graded \( k \)-categories \( 0 \rightarrow I \rightarrow C \rightarrow B \rightarrow 0 \) such that \( I \rightarrow C \) is a morphism of categorical representations of \( \mathfrak{g}_k \), there is a unique categorical representation of \( \mathfrak{g}_k \) on \( B \) such that \( C \rightarrow B \) is morphism of categorical representations.

(d) Given a categorical representation of \( \mathfrak{g}_k \) on \( C \), there is a unique categorical representation of \( \mathfrak{g}_k \) on \( C^c \) such that the canonical fully faithful functor \( C \rightarrow C^c \) is a morphism of categorical representations. Recall that the objects of the idempotent completion \( C^c \) are the pairs \( (M, e) \) where \( M \) is an object of \( C \) and \( e \) is an idempotent of \( \text{End}_C(M) \), and that \( \text{Hom}_{C^c}((M, e), (N, f)) = f \text{Hom}_C(M, N) e \). Then, we have \( F_i(M, e) = (F_i(M), F_i(e)), E_i(M, e) = (E_i(M), E_i(e)) \) and \( x_i 1_\lambda, \tau_{ij} 1_\lambda \) are defined in a similar way.

3.2.2. The minimal categorical representation. Fix a dominant weight \( \Lambda \in P_+ \) and an \( \mathbb{N} \)-graded \( k \)-algebra \( k \). Given \( \alpha \in \mathbb{Q}_+ \) we write \( \lambda = \Lambda - \alpha \). Recall that we abbreviate \( R^\Lambda(\alpha) = R^\Lambda(\alpha; k) \).

Let \( k \mathcal{A}^\Lambda_\alpha = R^\Lambda(\alpha) \)-grmod be the \( \mathbb{Z} \)-graded abelian \( k \)-category consisting of the finitely generated \( \mathbb{Z} \)-graded \( R^\Lambda(\alpha) \)-modules and let \( k \mathcal{V}^\Lambda_\alpha = R^\Lambda(\alpha) \)-grproj be the full subcategory formed by the projective \( \mathbb{Z} \)-graded modules. When there is no confusion we’ll abbreviate \( \mathcal{A}^\Lambda_\alpha = k \mathcal{A}^\Lambda_\alpha \) and \( \mathcal{V}^\Lambda_\alpha = k \mathcal{V}^\Lambda_\alpha \). Let \( \mathcal{A}^\Lambda, \mathcal{V}^\Lambda \) be the categories

\[ \mathcal{A}^\Lambda = \bigoplus_\alpha \mathcal{A}^\Lambda_\alpha, \quad \mathcal{V}^\Lambda = \bigoplus_\alpha \mathcal{V}^\Lambda_\alpha. \]

Fix an integer \( \ell \). Let \( \mathcal{V}^{\Lambda,[\ell]} = (\mathcal{V}^\Lambda)^{[\ell]} \) be the \( \ell \)-twist of \( \mathcal{V}^\Lambda \) and \( R^\Lambda(\alpha)^{[\ell]} \) be the \( \ell \)-twist of \( R^\Lambda(\alpha) \). Thus, \( R^\Lambda(\alpha)^{[\ell]} \) is a \((\ell \mathbb{Z})\)-graded \( k^{[\ell]} \)-algebra and \( \mathcal{V}^{\Lambda,[\ell]}_\alpha \) is the category of finitely generated projective \((\ell \mathbb{Z})\)-graded modules, i.e.,

\[ \mathcal{V}^{\Lambda,[\ell]}_\alpha = R^\Lambda(\alpha)^{[\ell]} \)-grproj.

Definition 3.13. The minimal categorical representation of \( \mathfrak{g}_k \) of highest weight \( \Lambda \) and degree \( \ell \) is the representation on \( \mathcal{V}^{\Lambda,[\ell]}_\alpha \) given by

- \( E_i 1_\lambda = E_i'[-\ell d_i(1 + \lambda_i)] \),
- \( F_i 1_\lambda = F_i' \),
- \( \varepsilon_i 1_\lambda = \varepsilon_i' 1_\lambda \), \( \eta_i 1_\lambda = \eta_i' 1_\lambda \),
- \( x_i 1_\lambda \in \text{Hom}(F_i 1_\lambda, F_i 1_\lambda[2\ell d_i]) \) is represented by the right multiplication by \( x_{n+1} \) on \( R^\Lambda(\alpha + \alpha_i e(\alpha, i)) \),
- \( \tau_{ij} 1_\lambda \in \text{Hom}(F_i F_j 1_\lambda, F_j F_i 1_\lambda[-\ell d_i a_{ij}]) \) is represented by the right multiplication by \( \tau_{n+1} \) on \( R^\Lambda(\alpha + \alpha_i + \alpha_j e(\alpha, j)) \).
The category $\mathcal{A}_\lambda^{A_1^{|\ell|}}$ is Krull-Schmidt with a finite number of indecomposable projective objects. The category $\mathcal{V}_{\lambda}^{A_1^{|\ell|}}/(\ell\mathbb{Z})$ is the category of $(\ell\mathbb{Z})$-graded finitely generated projective $R_{\lambda}^{A_1^{|\ell|}}$-modules with morphisms which are not necessarily homogeneous. We’ll call it the category of all $(\ell\mathbb{Z})$-gradable projective modules.

**Example 3.14.** We’ll abbreviate $e = sl_2$. Assume that $g = e$, $\Lambda = k\omega_1$ and $\alpha = n\alpha_1$ with $k, n \geq 0$. In this case we write $\mathcal{V}^{k} = \mathcal{V}_{1}^{1}$ and $\mathcal{V}_{-2n}^{k} = \mathcal{V}_{1}^{1}$. Consider the polynomial ring $Z^{k} = k[c_1, \ldots, c_k]$ with $\deg(c_p) = 2p$ for all $p$. Let $H_k^{k}$ be the global cyclotomic affine nil Hecke algebra of rank $n$ and level $k$, i.e., the $\mathbb{Z}$-graded $Z^{k}$-algebra denoted by $H_{n,k}$ in [21, sec. 4.3.2]. Note that $H_k^{k}$ is $Z^{k}$ and $k = Z^{k}$. Given an $\mathbb{N}$-graded $Z^{k}$-algebra $k$, the cyclotomic quiver Hecke algebra $R_{\lambda}^{k}(\alpha; k)$ is isomorphic to $k \otimes_{Z^{k}} H_k^{k}$ as a $\mathbb{Z}$-graded $k$-algebra by [21, lem. 4.27]. In particular, we have $R_{\lambda}^{k}(\alpha) = H_k^{k}$. For each integer $\ell$, we abbreviate $Z^{k,|\ell|} = (Z^{k})^{[\ell]}$ and $H_k^{k,|\ell|} = (H_k^{k})^{[\ell]}$. We have

\begin{equation}
\mathcal{V}^{k,|\ell|} = \bigoplus_{n > 0} (k^{[\ell]} \otimes_{Z^{k,|\ell|}} H_k^{k,|\ell|})-\text{grproj}.
\end{equation}

We’ll also identify $R_{\lambda}^{k}(\alpha; k)$ with the local cyclotomic affine nil Hecke algebra of rank $n$ and level $k$, which is the quotient $H_k^{k} = k \otimes_{Z^{k}} H_k^{k}$ of $H_k^{k}$ by the ideal $(c_1, \ldots, c_k)$.

3.2.3. Factorization. Fix a dominant weight $\Lambda \in P_+$. Recall the map $I_\Lambda \to I$ introduced in (2). For any $t \in I_\Lambda$ we abbreviate $\omega_t = \omega_{t_i}$ and $d_t = d_{t_i}$. Set $h = k[y_t; t \in I_\Lambda]$ where $y_t$ is a formal variable of degree $\deg(y_t) = 2d_t$. The graded ring $h$ has a natural structure of $\mathbb{N}$-graded $k$-algebra such that the element $c_{ip} \in h$ is given by $c_{ip} = c_p(y_{t_1}, \ldots, y_{t_{I_\Lambda}})$. The corresponding cyclotomic polynomials are

\begin{equation}
a^\Lambda_t(u) = \prod_{p=1}^{\Lambda_t}(u + y_{ip}), \quad \forall i \in I.
\end{equation}

Let $h'$ be the fraction field of $h$. We’ll consider the condition

\begin{equation}
Q_{ij}(u, v) = r_{ij} (u - v)^{-a_{ij}} \text{ for some } r_{ij} \text{ such that } r_{ij} = (-1)^{a_{ij}} r_{ji} \text{ for all } i \neq j.
\end{equation}

**Theorem 3.15.** If the condition (11) is satisfied, then the following hold

(a) for each integer $n > 0$, there is an $h'$-algebra isomorphism

\[R_{\lambda}^{k}(n; h') \to \bigoplus_{(n_\lambda)} \text{Mat}_{\mathbb{C}^{n_\lambda}} \otimes_{\mathbb{C}} \mathcal{V}^{\omega_t}(n_\lambda, h'),\]

where $(n_\lambda)$ runs over the set of $I_\Lambda$-tuples of non-negative integers with sum $n$,

(b) there is an isomorphism $h' \otimes_k \mathcal{V}^{\lambda} \to h' \otimes_k \bigoplus_{t \in I_\Lambda} \mathcal{V}^{\omegaimg{1}{T}}(\text{not graded})$ categorical representations taking the functor $F_\alpha$ to $\bigoplus_{(\alpha_t)} \otimes_{I_\Lambda} F_{\alpha_t}$, the sum being over all $I_\Lambda$-tuples $(\alpha_t)$ with sum $\alpha$. The canonical homomorphism $\otimes_{t \in I_\Lambda} R(\alpha_t; h') \to \text{End}(\bigotimes_{t \in I_\Lambda} F_{\alpha_t})$ is the composition of the inclusion $\bigotimes_{t \in I_\Lambda} R(\alpha_t; h') \subset R(\alpha; h')$ underlying (a) and of the canonical homomorphism $R(\alpha; h') \to \text{End}(F_\alpha)$.
Proof. Let \( n = \text{ht}(\alpha) \). Fix \( M \in R^\Lambda(\alpha; \mathbf{h}')\mod \) and \( g(u) \in \mathbf{h}'[u] \). From \cite[p. 715-716]{16} we get
\[
g(x_a) e(\nu) M = 0 \Rightarrow Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) g(x_{a+1}) e(s_a(\nu)) M = 0, \quad \forall a \in [1, n), \quad \forall \nu \in I^n.
\]
Set \( Q(u, v) = \prod_{i \neq j} Q_{ij}(u, v) \). We deduce that
\[
(12) \quad g(x_a) e(\nu) M = 0 \Rightarrow Q(x_a, x_{a+1}) g(x_{a+1}) e(s_a(\nu)) M = 0, \quad \forall a \in [1, n).
\]
Now, assume that the polynomial \( Q(u, v) \in \mathbf{h}'[u, v] \) has the following form
\[
(13) \quad Q(u, v) = r \prod_{\lambda \in S} (u - v - \lambda)
\]
for some finite family \( S \) of elements of \( \mathbf{h}' \) and some element \( r \in \mathbf{h}' \). Let \( \text{sp}_{e(\nu)M}(x_a) \subset \mathbf{h}' \) be the set of \( \lambda \in \mathbf{h}' \) such that the operator \( x_a - \lambda \text{id} \in \text{End}_{\mathbf{h}'}(e(\nu) M) \) is not invertible. Since \( x_a, x_{a+1} \) commute with each other, from (12), (13) we deduce that
\[
\text{sp}_{e(\nu)M}(x_a) = S \cup \{0\}.
\]
Next, recall that \( g(x_1)(e(\nu) M) = 0 \) if \( g(u) = a_{\nu_a}^\Lambda(u) \). We deduce that
\[
\text{sp}_{e(\nu)M}(x_a) \subseteq \{ -y_{\nu_a, p} ; p = 1, \ldots, \Lambda_a \} - NS, \quad \forall a \in [1, n].
\]
Assume further that the condition (11) holds. Then we have \( S = \{0\} \), hence
\[
(14) \quad \text{sp}_{e(\nu)M}(x_a) \subseteq \{ -y_{\nu_a, p} ; p = 1, \ldots, \Lambda_a \}, \quad \forall a \in [1, n].
\]
In the rest of the proof we write \( \tilde{I} = I_\Lambda \) to simplify the notation. For each \( n \)-tuple \( \tilde{\nu} \in \tilde{I}^n \) set
\[
M_{\tilde{\nu}} = \{ m \in M ; (x_k + y_{\tilde{\nu}_k})^D m = 0, \forall k \in [1, n], \forall D \gg 0 \}.
\]
Considering the decomposition of the regular module, we deduce that there is a complete collection of orthogonal idempotents \( \{ e(\tilde{\nu}) ; \tilde{\nu} \in \tilde{I}^n \} \) in \( R^\Lambda(\alpha, \mathbf{h}') \) such that \( e(\tilde{\nu}) M = M_{\tilde{\nu}} \). The map \( \tilde{I} \to I \) in \( \cite{2} \) yields, in the obvious way, a map
\[
\tilde{I}^n \to I^n, \quad \tilde{\nu} \mapsto \nu.
\]
The following properties are immediate
\[
- e(\tilde{\nu}) e(\nu') = e(\nu') e(\tilde{\nu}) = \delta_{\nu, \nu'} e(\tilde{\nu}) \text{ for each } \nu' \in I^n,
- x_l e(\tilde{\nu}) = e(\tilde{\nu}) x_l,
- \varphi_k e(\tilde{\nu}) = e(s_k(\tilde{\nu})) \varphi_k,
- \tau_k e(\tilde{\nu}) = e(\tilde{\nu}) \tau_k \text{ if } \tilde{\nu}_k = \nu_{k+1},
\]
where \( k, l, \mu, \nu \) run over the sets \([1, n], [1, n], \tilde{I}^n \) and \( I^n \) respectively. In particular, the idempotents \( e(\tilde{\nu}) \) with \( \tilde{\nu} \in \tilde{I}^n \) refine the idempotents \( e(\nu) \) with \( \nu \in I^n \).

Lemma 3.16. For each \( M \in R^\Lambda(\alpha; \mathbf{h}')\mod \), the map \( \varphi_k e(\tilde{\nu}) : e(\tilde{\nu}) M \to e(s_k(\tilde{\nu})) M \) is invertible whenever \( \tilde{\nu}_k \neq \nu_{k+1} \).

Proof. The lemma is an immediate consequence of the following relations, for each \( \nu \in I^n \),
\[
\varphi_k^2 e(\nu) = 1 \text{ if } \nu_k = \nu_{k+1}, \quad \varphi_k^2 e(\nu) = Q_{\nu_k, \nu_k+1}(x_k, x_{k+1}) e(\nu) \text{ if } \nu_k \neq \nu_{k+1}.
\]
Set $\bar{Q}_+ = \bar{N}I$. Fix an element $\bar{\alpha} = \sum_{t \in I_\Lambda} a_t \cdot t$ in $\bar{Q}_+$. Set $ht(\bar{\alpha}) = \sum_i a_i$, and assume that $ht(\bar{\alpha}) = n$. The map $\bar{I} \to I$ in (2) yields a map $\bar{Q}_+ \to Q_+$. We’ll also assume that $\bar{\alpha}$ maps to $\alpha$, i.e., that $\sum_{t \in I_\Lambda} a_t \cdot i_t = \alpha$.

Now, consider the set $\bar{I}^\alpha = \{ \bar{\nu} \in \bar{I}^n ; \sum_k \bar{\nu}_k = \bar{\alpha} \}$. We’ll say that two $n$-tuples $\bar{\nu}, \bar{\nu}' \in \bar{I}^n$ are equivalent, and we write $\bar{\nu} \sim \bar{\nu}'$, if we have $\bar{\nu}, \bar{\nu}' \in \bar{I}^\alpha$ for some $\bar{\alpha} \in \bar{Q}_+$. Then, we define the idempotent $e(\bar{\alpha}) \in R^A(\alpha, \mathfrak{h}')$ by $e(\bar{\alpha}) = \sum_{\bar{\nu} \in \bar{I}^\alpha} e(\bar{\nu})$.

Next, fix a total order on $\bar{I}$ and set $\bar{I}_+^\alpha = \{ \mu \in \bar{I}^n ; i < j \Rightarrow \mu_i \leq \mu_j \}$. For any tuple $\bar{\nu} \in \bar{I}^n$ there is a unique element $\bar{\nu}^+ \in \bar{I}_+^\alpha$ such that $\bar{\nu} \sim \bar{\nu}^+$. Let $\bar{\alpha}^+$ be the unique element in $\bar{I}_+^\alpha \cap \bar{I}^\alpha$. The part (a) of the theorem is a consequence of the following lemma.

**Lemma 3.17.** The following hold

(a) $e(\bar{\nu}) R^A(\alpha, \mathfrak{h}') e(\bar{\nu}') = 0$ unless $\bar{\nu} \sim \bar{\nu}'$,

(b) $e(\bar{\alpha}) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}) \simeq \text{Mat}_{\bar{I}_+} (e(\bar{\alpha}^+) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}^+))$ as $\mathfrak{h}'$-algebras,

(c) $e(\bar{\alpha}^+) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}^+) \simeq \bigotimes_{t \in \bar{I}} R^{\mathfrak{h}^+}(\alpha, \mathfrak{h}')$ as $\mathfrak{h}'$-algebras.

**Proof.** Let $\tilde{I}^\alpha \subset \bar{I}^n$ be the inverse image of $I^n$ by the map (15). It is not difficult to prove that

- the $\mathfrak{h}'$-algebra $R^A(\alpha, \mathfrak{h}')$ is generated by the set of elements

\[
\{ \tau_h e(\bar{\nu}), \varphi_k e(\bar{\nu}), x_l e(\bar{\nu}) ; h, k \in [1, n] \},
\]

- the $\mathfrak{h}'$-algebra $e(\bar{\alpha}^+) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}^+)$ is generated by the subset

\[
\{ \tau_h e(\bar{\alpha}^+), x_l e(\bar{\alpha}^+) ; h, l \in [1, n] \},
\]

and the element

\[
\varphi_{l_1} \varphi_{l_2} \cdots \varphi_{l_j} = \varphi_w = \tau_{l_1} \tau_{l_2} \cdots \tau_{l_j}
\]

depends only on $\bar{\nu}$ and not on the choice of $l_1, l_2, \ldots, l_j$. Further, it is invertible by Lemma 3.16. We deduce that there is an $\mathfrak{h}'$-algebra isomorphism

\[
\text{Mat}_{\bar{I}^\alpha} (e(\bar{\alpha}^+) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}^+)) \to e(\bar{\alpha}) R^A(\alpha, \mathfrak{h}') e(\bar{\alpha}),
\]

Here $E_{\bar{\nu}, \bar{\nu}'}(m)$ is the elementary matrix with a $m$ at the spot $(\bar{\nu}, \bar{\nu}')$ and 0’s elsewhere. Part (b) is proved.

To prove (c) we must check that the map

\[
\bigotimes_{t \in \bar{I}} R(a_t, \mathfrak{h}') \to e(\bar{\alpha}^+) R(A, \mathfrak{h}') e(\bar{\alpha}^+),
\]

is an isomorphism.
factors to an $\mathfrak{h}$-algebra isomorphism

\[ (21) \quad \bigotimes_{i \in I} R^{\omega_i}(a_i, \mathfrak{h}) \cong e(\tilde{\alpha}^+) R^{\Lambda}(\alpha, \mathfrak{h}) e(\tilde{\alpha}^+). \]

To do that, in order to simplify the notation, we’ll assume that

\[ \# I = 3, \quad \Lambda = 2\omega_i + \omega_j, \quad i \neq j \in I. \]

The proof of the general case is very similar. Write $I = \{a, b, c\}$ with $a < b < c$ such that the map (2) takes $a, b, c$ to $i, j, k$ respectively. We have $\tilde{\alpha}^+ = (a_n b_n c_0)$ with $n_a + n_b + n_c = n$. To simplify we’ll also assume that $n_a, n_b, n_c > 0$. We have

\[ a_i^k(u) = (u + y_a)(u + y_b), \quad a_j^k(u) = u + y_c, \quad a_k^k(u) = 1, \quad \forall k \neq i, j. \]

First, we must prove that the following relations hold in $e(\tilde{\alpha}^+) R^{\Lambda}(n, \mathfrak{h}') e(\tilde{\alpha}^+)$

- $\cdot (x_1 + y_a)(x_1 + y_b)e(\alpha^+) = 0$ if $\alpha_1^+ = i, \text{ and } e(\alpha^+) = 0$ else,
- $\cdot (x_1 + y_a + y_b)e(\alpha^+) = 0$ if $\alpha_1^+ = i + n_a, \text{ and } e(\alpha^+) = 0$ else,
- $\cdot (x_1 + n_a + y_b)e(\alpha^+) = 0$ if $\alpha_1^+ = j, \text{ and } e(\alpha^+) = 0$ else.

The first one is obvious, because

\[ (x_1 + y_a)(x_1 + y_b)e(\alpha^+) = 0 \text{ if } \alpha_1^+ = i, \quad (x_1 + y_a)e(\alpha^+) = 0 \text{ if } \alpha_1^+ = j, \quad e(\alpha^+) = 0 \text{ else} \]

and $(x_1 + y_b)e(\alpha^+), (x_1 + y_c)e(\alpha^+)$ are invertible in $e(\tilde{\alpha}^+) R^{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}^+)$. To prove the second relation, note that

\[ \varphi_1 \varphi_2 \cdots \varphi_n (x_1 + n_a + y_b)e(\tilde{\alpha}^+) = (x_1 + y_b)e(\tilde{\mu}) \varphi_1 \varphi_2 \cdots \varphi_n . \]

where $\tilde{\mu} = s_1 s_2 \cdots s_{n_a}(\tilde{\alpha}^+)$. Since $\mu_1 = \alpha_1^+ + n_a$, we deduce that

\[ \varphi_1 \varphi_2 \cdots \varphi_n (x_1 + n_a + y_b)e(\tilde{\alpha}^+) = 0 \text{ if } \alpha_1^+ = i, \quad \text{and } \varphi_1 \varphi_2 \cdots \varphi_n e(\tilde{\alpha}^+) = 0 \text{ else}. \]

Further, by Lemma 3.16 the operator

\[ \varphi_1 \varphi_2 \cdots \varphi_n e(\tilde{\alpha}^+) : e(\tilde{\alpha}^+) R^{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}^+) \to e(\tilde{\mu}) R^{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}^+) \]

is invertible. Thus, we have

\[ (x_1 + n_a + y_b)e(\tilde{\alpha}^+) = 0 \text{ if } \alpha_1^+ = i, \quad \text{and } e(\tilde{\alpha}^+) = 0 \text{ else}, \]

proving the second relation. The third one is proved in a similar way, using the product of intertwiners $\varphi_1 \varphi_2 \cdots \varphi_n + n_b$ instead of $\varphi_1 \varphi_2 \cdots \varphi_n$.

The relations above imply that the homomorphism (21) is well-defined. We must check that it is invertible. The surjectivity is immediate using the set of generators of $e(\tilde{\alpha}^+) R^{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}^+)$ in (17). Let us concentrate on the injectivity. It is enough to construct a left inverse to (21). To do that, recall that $(a), (b)$ yields an isomorphism

\[ (22) \quad \bigoplus_{\tilde{a}} \text{Mat}_{\tilde{a}}(e(\tilde{\alpha}^+) R^{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}^+)) \cong R^{\Lambda}(\alpha, \mathfrak{h}') \]

where the sum is over the set of all $\tilde{a} \in \tilde{Q}_+$ which map to $\alpha$ by (2).
Claim 3.18. (a) Fix \( \tilde{v} \in \tilde{I}^0 \) and \( w \in S_n \) as in (19). For each \( h, k \in [1, n] \), \( l \in [1, n] \) such that \( \tilde{v}_h = \tilde{v}_{h+1} \) and \( \tilde{v}_k \neq \tilde{v}_{k+1} \), the map (22) satisfies the following relations

\[
E_{\tilde{\nu}, \tilde{\nu}}(x_{w^{-1}(t)}) \in \nu^+ \mapsto x_t e(\tilde{\nu}), \quad E_{s_k(\tilde{\nu}), \tilde{\nu}}(\omega) \in \nu^+ \mapsto \varphi_k e(\tilde{\nu}), \quad E_{\tilde{\nu}, \tilde{\nu}}(\tau_{w^{-1}(h)} e(\tilde{\nu})) \mapsto \tau_h e(\tilde{\nu}).
\]

(b) The assignment

\[
x_t e(\tilde{\nu}) \mapsto E_{\tilde{\nu}, \tilde{\nu}}(x_{w^{-1}(t)}), \quad \varphi_k e(\tilde{\nu}) \mapsto E_{s_k(\tilde{\nu}), \tilde{\nu}}(1), \quad \tau_h e(\tilde{\nu}) \mapsto E_{\tilde{\nu}, \tilde{\nu}}(\tau_{w^{-1}(h)}),
\]

where \( h, k, l, \tilde{\nu} \) are as above, yields an \( \mathfrak{h}' \)-algebra homomorphism

\[
R_{\Lambda}(\alpha, \mathfrak{h}') \to \bigoplus_{\tilde{\alpha}} \text{Mat}_{\tilde{\alpha}} \left( \bigotimes_{t \in \tilde{I}} R^{\omega t}(a_t, \mathfrak{h}') \right).
\]

Proof. Part (a) follows from (20) and from the following computations

\[
x_t e(\tilde{\nu}) = x_t \pi_{\tilde{\nu}} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1} = \pi_{\tilde{\nu}} x_{w^{-1}(t)} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1}
\]

\[
\varphi_k e(\tilde{\nu}) = \varphi_k \pi_{\tilde{\nu}} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1} = \varphi_k \pi_{\tilde{\nu}} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1}
\]

\[
\tau_h e(\tilde{\nu}) = \tau_h \pi_{\tilde{\nu}} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1} = \tau_h \pi_{\tilde{\nu}} e(\tilde{\nu}) \pi_{\tilde{\nu}}^{-1}
\]

Here, we have used the equality \( w^{-1}(h + 1) = w^{-1}(h) + 1 \), which follows from the definition of \( w \) in (18), and the equalities \( s_h(\tilde{\nu}) = \tilde{\nu} \) and \( s_{w^{-1}(h)} e(\tilde{\nu}) = \tilde{\nu} \), which follow from the identities \( s_h = s_h+1 \) and \( \tilde{\nu} = w(\tilde{\nu}) \).

Now, let us concentrate on (b). Since the elements \( x_t e(\tilde{\nu}), \varphi_k e(\tilde{\nu}), \tau_h e(\tilde{\nu}) \) above generate \( R_{\Lambda}(\alpha, \mathfrak{h}') \) by (19), it is enough to check that the defining relations of \( R_{\Lambda}(\alpha, \mathfrak{h}') \) given in Section 3.1.3 are satisfied. This is obvious. Note that the element \( \tau_{w^{-1}(h)} \) belongs indeed to \( \bigotimes_{t \in \tilde{I}} R^{\omega t}(a_t, \mathfrak{h}') \) because \( s_{w^{-1}(h)}(\tilde{\nu}) = \tilde{\nu} \).

Composing (22) and (23), we get an \( \mathfrak{h}' \)-algebra homomorphism

\[
\bigoplus_{\tilde{\alpha}} \text{Mat}_{\tilde{\alpha}} \left( e(\tilde{\alpha}) R_{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}) \right) \to \bigoplus_{\tilde{\alpha}} \text{Mat}_{\tilde{\alpha}} \left( \bigotimes_{t \in \tilde{I}} R^{\omega t}(a_t, \mathfrak{h}') \right).
\]

For each \( \tilde{\alpha} \) it restricts to an \( \mathfrak{h}' \)-algebra homomorphism

\[
e(\tilde{\alpha}) R_{\Lambda}(\alpha, \mathfrak{h}') e(\tilde{\alpha}) \to \bigotimes_{t \in \tilde{I}} R^{\omega t}(a_t, \mathfrak{h}')
\]
which is a left inverse to \([21]\).

\[
\square
\]

Finally, the part \((b)\) of the theorem follows easily from the isomorphism in \((a)\).

\[
\square
\]

3.2.4. The isotypic filtration. Recall that \(\mathfrak{c} = \mathfrak{sl}_2\). A \((P \times \mathbb{Z})\)-graded \(k\)-category \(\mathcal{C}\) is a direct sum of categories of the form \(\mathcal{C} = \bigoplus_{\lambda \in P} \mathcal{C}_\lambda\) where each \(\mathcal{C}_\lambda\) is a \(\mathbb{Z}\)-graded \(k\)-category. We’ll call \(\lambda\) the \(P\)-degree of \(\mathcal{C}_\lambda\).

Given \(i \in I\) there is an \(\mathfrak{sl}_2\)-triple \(\mathfrak{c}_i \subset \mathfrak{g}\). In this section, we study the restriction of a categorical \(\mathfrak{g}_k\)-representation to such an \(\mathfrak{sl}_2\)-triple. To simplify the notation, in this section we fix an element \(i \in I\) and identify \(\mathfrak{c} = \mathfrak{c}_i\).

By a \((P \times \mathbb{Z})\)-graded categorical \(\mathfrak{c}_k\)-representation on \(\mathcal{C}\) we’ll mean a representation such that for each \(\lambda \in P\) the \(\mathfrak{k}\)-subcategory \(\mathcal{C}_\lambda\) has the weight \(\lambda_1\) relatively to the \(\mathfrak{c}\)-action. In particular, if \(\mathcal{C}\) is an integrable categorical \(\mathfrak{g}_k\)-representation \(\mathfrak{g}_k\), restricting the \(\mathfrak{g}\)-action on \(\mathcal{C}\) to \(\mathfrak{c}_i\) yields an integrable \(P \times \mathbb{Z}\)-graded categorical \(\mathfrak{c}\)-representation on \(\mathcal{C}\) of degree \(d_i\).

Now, fix an integer \(k \in \mathbb{N}\). Given a \((P \times \mathbb{Z})\)-graded \(k\)-category \(\mathcal{M}\) such that \(\mathcal{M}_\mu = 0\) whenever \(\mu_i \neq k\) and a \(\mathbb{Z}\)-graded \(k\)-algebra homomorphism \(Z^{k,[d_i]} \rightarrow Z(\mathcal{M}/\mathbb{Z})\), we equip the tensor product \(\mathcal{V}^{k,[d_i]} \otimes Z^{k,[d_i]} \mathcal{M}\) with the \((P \times \mathbb{Z})\)-graded categorical \(\mathfrak{c}\)-representation of degree \(d_i\) such that

- \(\mathfrak{c}\) acts on the left factor,
- the summand \(\mathcal{V}^{k,[d_i]} \otimes Z^{k,[d_i]} \mathcal{M}_\mu\) has the \(P\)-degree \(\mu - n\alpha_i\) for each \(n \in \mathbb{N}\).

**Proposition 3.19.** Fix an integrable categorical representation of \(\mathfrak{g}_k\) on \(\mathcal{C}\) of degree 1 which is bounded above. For each \(i \in I\) there is a decreasing filtration \(\cdots \subseteq \mathcal{C}_{\geq 1} \subseteq \mathcal{C}_{\geq 0} = \mathcal{C}\) of \(\mathcal{C}\) by full \((P \times \mathbb{Z})\)-graded integrable categorical \(\mathfrak{c}\)-representations of degree \(d_i\) which are closed under taking direct summands and such that

- for each \(k \geq 0\) there is a \(P \times \mathbb{Z}\)-graded \(k\)-category \(\mathcal{M}_k\) with a \(\mathbb{Z}\)-graded \(k\)-algebra homomorphism \(k \otimes_k Z^{k,[d_i]} \rightarrow Z(\mathcal{M}_k/\mathbb{Z})\) and a \(k\)-linear equivalence of \((P \times \mathbb{Z})\)-graded categorical \(\mathfrak{c}_k\)-representations \((\mathcal{C}_{\geq k}/\mathcal{C}_{>k})^c \simeq (\mathcal{V}^{k,[d_i]} \otimes Z^{k,[d_i]} \mathcal{M}_k)^c\) of degree \(d_i\),
- for each \(\lambda \in P\) we have \(\mathcal{C}_{\geq k} \cap \mathcal{C}_\lambda = 0\) for \(k\) large enough.

**Proof.** Since \(\mathcal{C}\) is bounded above, by a standard argument we may assume that the set of weights of \(\mathcal{C}\) is contained in a cone \(\mu - Q_+\) for some \(\mu \in P\), see e.g., [30 lem. 2.1.10]. Next, for each coset \(\pi \in P/\mathbb{Z}\alpha_i\), the \(\mathfrak{g}\)-action on \(\mathcal{C}\) yields a \(P \times \mathbb{Z}\)-graded categorical \(\mathfrak{c}\)-representation on \(\mathcal{C}_\pi = \bigoplus_{\lambda \in \pi} \mathcal{C}_\lambda\) of degree \(d_i\). Since \(\mathcal{C}\) decomposes as the direct sum of \(k\)-categories \(\mathcal{C} = \bigoplus \mathcal{C}_\pi\), it is enough to define a filtration of \(\mathcal{C}_\pi\) satisfying the properties above for each \(\pi\). Note that \(\pi \cap (\mu - Q_+)\) is a cone of the form \(\nu - N\alpha_i\) for some weight \(\nu \in \mu - Q_+\). Thus the claim follows from [37 thm. 4.22], applied to the \(\mathfrak{c}\)-representation on \(\mathcal{C}_\pi\).

\[
\square
\]
The grading is given by deg($x$)

\[ C_{\geq k, \lambda} = C_{\geq k} \cap C_{\lambda} \]

We also define

\[ C_k = (\mathcal{V}^k_{\geq 0} \otimes \mathcal{Z}^k_{\geq 0}, \mathcal{M}_k)^c, \quad C_{k, \lambda} = \bigoplus_{n, \mu} (\mathcal{V}^k_{\geq 0} \otimes \mathcal{Z}^k_{\geq 0}, \mathcal{M}_{k, \mu})^c, \]

where the sum runs over all $\mu \in P$ and $n \in \mathbb{N}$ such that $\lambda = \mu - n \alpha_i$. Recall that we have assumed that $\mathcal{M}_{k,\mu} = 0$ whenever $\mu_i \neq k$. Note also that $C^h_{k} = \mathcal{M}_k^c$, and that $C^h_{k}$ is the full subcategory of $(C/C_{\geq k})^c$ consisting of the objects $M$ such that $E_i(M) = 0$.

### 3.2.5. The loop operators

Consider a categorical representation of $\mathfrak{g}_k$ of degree $\ell$ on a $\mathbb{Z}$-graded $\mathbb{k}$-category $C$. For each $i \in I$ and $r \in \mathbb{N}$, we define $\mathbb{k}$-linear operators $x^\pm_{ir}$ on the $\mathbb{Z}$-graded $\mathbb{k}$-module $\text{tr}(C/\mathbb{Z})$ such that the maps

\[ x^+_i : \text{tr}(C_\lambda/\mathbb{Z}) \to \text{tr}(C_{\lambda+\alpha_i}/\mathbb{Z}), \quad x^-_i : \text{tr}(C_\lambda/\mathbb{Z}) \to \text{tr}(C_{\lambda-\alpha_i}/\mathbb{Z}), \]

are given by

\[ x^+_i = \text{tr}_{E_i}(x^+_i), \quad x^-_i = \text{tr}_{F_i}(x^-_i), \]

see Definition 3.2 for the notation. The map $x^\pm_{ir}$ is homogeneous of degree $2r d_i$. Let us quote the following fact for future use.

**Proposition 3.20.** The maps $x^\pm_{ir}$ are functorial, i.e., a morphism $C \to C'$ of categorical representations of $\mathfrak{g}_k$ yields a $\mathbb{k}$-linear map $\text{tr}(C/\mathbb{Z}) \to \text{tr}(C'/\mathbb{Z})$ which intertwines the operators $x^\pm_{ir}$ on $\text{tr}(C/\mathbb{Z})$ and on $\text{tr}(C'/\mathbb{Z})$.

### 3.2.6. The loop operators on the trace of the minimal categorical representation

Fix a dominant weight $\Lambda \in P_+$. For $\alpha \in Q_+$ we write $\lambda = \Lambda - \alpha$. Recall the base ring $\mathbb{k}$ from Section 3.1.3. Let $\mathbb{k}$ be an $\mathbb{N}$-graded $\mathbb{k}$-algebra. In this section we consider the particular case of the minimal categorical representation $\mathcal{V}^\Lambda$.

**Definition 3.21.** Let $L_{\mathfrak{g}}$ be the $\mathbb{N}$-graded Lie $\mathbb{k}$-algebra generated by elements $x^\pm_{ir}$, $h_{ir}$ with $i \in I$, $r \in \mathbb{N}$ satisfying the following relations

(a) $[h_{ir}, h_{js}] = 0$,
(b) $[x^+_{ir}, x^-_{js}] = \delta_{ij} h_{i,r+s}$,
(c) $[x^\pm_{ir}, x^\pm_{js}] = \pm a_{ij} x^\pm_{j,r+s}$,
(d) $\sum_{p=0}^{m} (-1)^p \binom{m}{p} [x^\pm_{i,r+p}, x^\pm_{j,s+m-p}] = 0$ with $i \neq j$ and $m = -a_{ij}$,
(e) $[x^\pm_{ir}, x^\pm_{i,s}] = 0$,
(f) $[x^\pm_{i,r}, x^\pm_{i,r_1}, \ldots, x^\pm_{i,r_m}, x^\pm_{j,r_0}] = 0$ with $i \neq j$, $r_p \in \mathbb{N}$ and $m = 1 - a_{ij}$.

The grading is given by $\text{deg}(x^\pm_{ir}) = \text{deg}(h_{ir}) = 2r$.

There is a unique Lie $\mathbb{k}$-algebra anti-involution $\varpi$ of $L_{\mathfrak{g}}$ such that

\[ \varpi(h_{ir}) = h_{ir}, \quad \varpi(x^\pm_{ir}) = x^\mp_{ir}. \]

We define the $\ell$-twist of $L_{\mathfrak{g}}$ to be the Lie $\mathbb{k}$-subalgebra $L_{\mathfrak{g}}^{[\ell]} \subset L_{\mathfrak{g}}$ generated by the set of elements $\{x^\pm_{i,r\ell}, h_{i,r\ell} ; i \in I, r \in \mathbb{N} \}$. We write $L_{\mathfrak{g}}^{[\ell]} = \mathbb{k}^{[\ell]} \otimes \mathbb{k} L_{\mathfrak{g}}^{[\ell]}$. 

Theorem 3.22. If \( g \) is symmetric and (11) is satisfied, then the operators \( x^\pm_{i,r} \) with \( i \in I, r \in \mathbb{N} \) define a \( \mathbb{Z} \)-graded representation of \( Lg_k^{[\ell]} \) on \( tr(V^\alpha) \).

Proof. It is enough to set \( \ell = 1 \). The proof is similar to a proof in [31, 32]. However, our setting differs from loc. cit. and cannot be reduced to it, because, in our case, \( g \) may have any symmetric type and because we do not require that all axioms of a strong categorical representation to be satisfied by \( V^\alpha \). We have written an independent proof in Appendix [A].

\[ \square \]

Remark 3.23. (a) If \( g \) is not symmetric or (11) is not satisfied, then a more general version of the theorem is given in Proposition [3.5] below.

(b) Assume that \( g \) is symmetric and (11) holds. If \( g \) is simply laced, then the relations (d) and (e) are equivalent to the following: the bracket \([x^r_{i,r}, x^s_{j,s}]\) depends only on the sum \( r + s \) and not on the integers \( r \) or \( s \). If \( g \) is of finite type then \( Lg \) is the current algebra \( g[t] \) with \( \deg(t) = 2 \). In general \( Lg \) is bigger than \( g[t] \). For instance if \( g \) is not of type \( A_1^{(1)} \) it contains the center of the universal central extension of \( g[t] \), which is infinite dimensional. See [31, sec. 3], [40, sec. 13] and [13, sec. 1.3] for more details.

3.2.7. The loop operators on the center. Since the functors \( E_i, F_i \) on \( V^\alpha \) are biadjoint, we also have operators

\[
Z^+_i : Z(V^\alpha Z) \to Z(V^\alpha_{\lambda + s_i} Z), \quad Z^-_i : Z(V^\alpha Z) \to Z(V^\alpha_{\lambda - s_i} Z)
\]

defined by

\[
Z^+_i = Z_{F_i}(x^+_i), \quad Z^-_i = Z_{E_i}(x^-_i),
\]

see Definition [2.3] for the notation.

The proposition below gives some more explicit description of the operators \( x^\pm_{i,r} \) and \( Z^\pm_{i,r} \) in terms of the algebras \( R^\alpha(\alpha) \). Fix \( v_k, v_k^\vee \) such that

\[
\hat{\eta}_{i,\lambda}(1) = \sum_k v_k^\vee \otimes v_k, \quad v_k^\vee \in R^\alpha(\alpha) e(\alpha - \alpha_i, i), \quad v_k \in e(\alpha - \alpha_i, i) R^\alpha(\alpha).
\]

Proposition 3.24. For each \( \alpha \in Q_+ \) of height \( n \) and each \( f \in R^\alpha(\alpha), g \in Z(R^\alpha(\alpha)) \), we have

(a) \( \text{tr}(V^\alpha/Z) = \text{tr}(R^\alpha(\alpha) Z(V^\alpha/Z) = Z(R^\alpha) \) as a \( \mathbb{Z} \)-graded \( k \)-module and \( k \)-algebra respectively,

(b) \( x^-_i(\text{tr}(f)) = \text{tr}(e(\alpha, i) x^+_{n+1} f) \in \text{tr}(R^\alpha(\alpha + \alpha_i)), \)

(c) \( x^+_i(\text{tr}(f)) = \sum_k \hat{\epsilon}_{i,\lambda + \alpha_i}(x^+_i v_k f v_k^\vee) \in \text{tr}(R^\alpha(\alpha - \alpha_i)), \)

(d) \( Z^+_i(g) = \hat{\epsilon}_{i,\lambda + \alpha_i}(x^+_i g e(\alpha - \alpha_i, i)) \in Z(R^\alpha(\alpha - \alpha_i)), \)

(e) \( Z^-_i(g) = \mu((1 \otimes x^+_{n+1} g) \hat{\eta}_{i,\lambda - \alpha_i}(1)) \in Z(R^\alpha(\alpha + \alpha_i)). \)

The proof is standard and left to the reader. We only make a few comments.

First, the element \( f \) in (b) is viewed as an element of \( R^\alpha(\alpha + \alpha_i) \) via the map \( \iota_i : R^\alpha(\alpha) \to R^\alpha(\alpha + \alpha_i) \). Hence \( e(\alpha, i) x^+_{n+1} \) belongs to \( R^\alpha(\alpha + \alpha_i) \) and \( x^-_i(f) \) to \( \text{tr}(R^\alpha(\alpha + \alpha_i)). \)

Next, the equality (c) should be interpreted in the following way: \( f \) is identified with the \( R^\alpha(\alpha) \)-module endomorphism of \( R^\alpha(\alpha) \) given by \( m \mapsto mf \) and \( E_\beta \) is represented by the bimodule \( e(\alpha - \alpha_i, i) R^\alpha(\alpha) \) which is a projective \( R^\alpha(\alpha - \alpha_i) \)-module by [16, thm. 4.5]. Then
Explicitly, since \((\hat{\epsilon}_i^\prime E_i) \circ (E_i \hat{\eta}_i^\prime) = E_i\), we have \(m = \sum_k \hat{\epsilon}_i^\prime (mv_k^\prime) v_k\). Hence there is a surjective \(R^A(\alpha - \alpha_i)\)-module morphism

\[
p : \bigoplus_k R^A(\alpha - \alpha_i) \to e(\alpha - \alpha_i, i) R^A(\alpha), \quad (m_k)_k \mapsto \sum_k m_k v_k,
\]

with a splitting \(i\) given by

\[
i : e(\alpha - \alpha_i, i) R^A(\alpha) \to \bigoplus_k R^A(\alpha - \alpha_i), \quad m \mapsto (\hat{\epsilon}_i^\prime (mv_k^\prime))_k,
\]
i.e., we have \(p \circ i = 1\). Consider the endomorphism \(\bar{\varphi}\) given by

\[
\bar{\varphi} = i \circ \varphi \circ p : \bigoplus_k R^A(\alpha - \alpha_i) \to \bigoplus_k R^A(\alpha - \alpha_i), \quad (m_k)_k \mapsto (\hat{\epsilon}_i^\prime (x_n^r \sum_i m_l v_l f v_k^\prime))_k.
\]

Since \(m_l \in R^A(\alpha - \alpha_i)\), it commutes with \(x_n^r\). Since \(\hat{\epsilon}_i^\prime\) is \(R^A(\alpha - \alpha_i)\)-linear, we deduce that \(\hat{\epsilon}_i^\prime (x_n^r \sum_i m_l v_l f v_k^\prime) = \sum_i m_l \hat{\epsilon}_i^\prime (x_n^r v_l f v_k^\prime)\). So \(\hat{\varphi}\) is the right multiplication by the matrix \((\hat{\epsilon}_i^\prime (x_n^r v_l f v_k^\prime))_{l,k}\). Therefore we have

\[
\text{tr}(\varphi) = \text{tr}(\varphi \circ p \circ i) = \text{tr}(\hat{\varphi}) = \sum_k \hat{\epsilon}_i^\prime (x_n^r v_k f v_k^\prime).
\]

We obtain \(x_n^r(\text{tr}(f)) = \sum_k \hat{\epsilon}_i^\prime (x_n^r v_k f v_k^\prime)\).

Finally, in parts (d), (e) we have \(Z_{ir}^+(g) = Z_{F_i}(x_n^r g)\) and \(Z_{ir}^-(g) = Z_{F_i}(x_{n+1}^r g)\). Note that \(e(\alpha - \alpha_i, i) = \eta_i^\prime,_{\lambda + \alpha_i}(1)\) and \(\mu = \xi_i^\prime,_{\lambda - \alpha_i}\).

Let \(r(\alpha, i)\) be as in \([62]\). The following proposition relates the operator \(x_n^r(\text{tr}(f))\) to \(\text{tr}(V^A/Z)\) with the transpose of the operator \(Z_{ir}^\pm\) on \(Z(V^A/Z)\) under the pairing given by the symmetrizing form \(t_A\) in Proposition 3.10

\[
Z(V^A/Z) \times \text{tr}(V^A/Z) \to k, \quad (a, b) \mapsto t_A(ab).
\]

\[
(25) \quad Z(V^A/Z) \times \text{tr}(V^A/Z) \to k, \quad (a, b) \mapsto t_A(ab).
\]

**Proposition 3.25.** Let \(f \in \text{tr}(R^A(\alpha))\), \(g \in Z(R^A(\alpha + \alpha_i))\) and \(h \in Z(R^A(\alpha - \alpha_i))\). We have

\[
(a) \quad t_{\alpha + \alpha_i}(g x_n^r(f)) = r(\alpha, i) t_{\alpha}(Z_{ir}^+(g) f),
\]

\[
(b) \quad t_{\alpha - \alpha_i}(h x_n^r(f)) = r(\alpha - \alpha_i, i)^{-1} t_{\alpha}(Z_{ir}^-(h) f).
\]
Proof. Write \( f = \text{tr}(f') \) with \( f' \in R^h(\alpha) \). Write \( \hat{\eta}^i_1(1) = \sum_k v_k^i \otimes v_k \) as above. By \[63\] we have
\[
\begin{align*}
t_{\alpha+\alpha_i}(g x_{ir}^{-}(f)) &= t_{\alpha+\alpha_i}(g e(\alpha, i) x_{r+1} f') \\
&= r(\alpha, i) t_{\alpha} \hat{\varepsilon}^{i}_{\alpha, \lambda}(e(\alpha, i) x_{r+1} g f') \\
&= r(\alpha, i) t_{\alpha}(Z_{ir}^{+}(g) f) \\
t_{\alpha-\alpha_i}(h x_{ir}^{+}(f)) &= t_{\alpha-\alpha_i}(\sum_k h \hat{\varepsilon}^{i}(x_{r}^{+} v_k f' v_k^{i})) \\
&= t_{\alpha-\alpha_i}(\sum_k \hat{\varepsilon}^{i}(hx_{r}^{+} v_k f' v_k^{i})) \\
&= r(\alpha - \alpha, i, i)^{-1} t_{\alpha}(\sum_k h x_{r}^{+} v_k f' v_k^{i}) \\
&= r(\alpha - \alpha, i, i)^{-1} t_{\alpha}(\sum_k v_k^{i} h x_{r}^{+} v_k f') \\
&= r(\alpha - \alpha, i, i)^{-1} t_{\alpha}(\mu((1 \otimes x_{r}^{+} h) \hat{\eta}^i_{\alpha, \lambda}(1)) f') \\
&= r(\alpha - \alpha, i, i)^{-1} t_{\alpha}(Z_{ir}^{-}(h) f).
\end{align*}
\]

\( \square \)

3.2.8. The loop operators and the isotypic filtration. In this section we consider an integrable categorical representation of \( \mathfrak{g}_k \) of degree 1 on a \( \mathbb{Z} \)-graded \( k \)-category \( \mathcal{C} \) which is bounded above. As a preparation for Section \[5\] we study the behavior of the loop operators on \( \text{tr}(\mathcal{C}/\mathbb{Z}) \) with respect to the isotypic filtration.

Given \( i \in I \) and \( k \geq 0 \), the \( i \)-th isotypic filtration of \( \mathcal{C} \) yields \( P \times \mathbb{Z} \)-graded integrable \( \mathcal{C}_{\geq k} \), \( \mathcal{C}_k \) of degree \( d_i \) such that \( \mathcal{C}_k = (\mathcal{C}_{\geq k}/\mathcal{C}_{> k})^c \). By Proposition 2.1(f), Theorem 3.22 and Proposition 3.20 it also yields an exact sequence of \( \mathbb{Z} \)-graded \( k \otimes_k \mathcal{L}(d_i) \)-modules
\[
\text{tr}(\mathcal{C}_{> k}/\mathbb{Z}) \to \text{tr}(\mathcal{C}_{\geq k}/\mathbb{Z}) \to \text{tr}(\mathcal{C}_k/\mathbb{Z}) \to 0.
\]
(26)

Let \( h_k : \mathcal{C}_{\geq k} \to \mathcal{C} \) be the canonical inclusion. Consider the \( \mathbb{Z} \)-graded \( k \otimes_k \mathcal{L}(d_i) \)-submodule \( \text{tr}(\mathcal{C}/\mathbb{Z})_{\geq k} \) of \( \text{tr}(\mathcal{C}/\mathbb{Z}) \) given by \( \text{tr}(\mathcal{C}/\mathbb{Z})_{\geq k} = \text{tr}(h_k)(\text{tr}(\mathcal{C}_{\geq k}/\mathbb{Z})) \). Since \( \text{tr}(\mathcal{C}/\mathbb{Z})_{< k} \subseteq \text{tr}(\mathcal{C}/\mathbb{Z})_{\geq k} \), we have an exact sequence
\[
0 \to \text{tr}(\mathcal{C}/\mathbb{Z})_{< k} \to \text{tr}(\mathcal{C}/\mathbb{Z})_{\geq k} \to \text{tr}(\mathcal{C}/\mathbb{Z})_k \to 0
\]
(27)
and the map \( \text{tr}(h_k) \) factors to a surjective \( \mathbb{Z} \)-graded \( k \otimes_k \mathcal{L}(d_i) \)-module homomorphism
\[
\text{tr}(\mathcal{C}_k/\mathbb{Z}) = \text{tr}(\mathcal{V}^{k,[d_i]}_{\mathbb{Z}^k,d_i}) \otimes_k \mathcal{M}_k/\mathbb{Z} \to \text{tr}(\mathcal{C}/\mathbb{Z})_k.
\]
(28)

Now, assume that \( \mathcal{M}_k^{c_{k,\mu}} = B_{k,\mu} \)-grproj for some \( \mathbb{Z} \)-graded \( k \otimes_k \mathcal{Z}^{k,[d_i]} \)-algebra \( B_{k,\mu} \) and set \( B_k = \bigoplus_{\mu} B_{k,\mu} \). From (9), (24) we deduce that
\[
\mathcal{C}_k = \bigoplus_{n \in \mathbb{N}} (\mathcal{H}^{k,[d_i]}_{\mathbb{Z}^k,d_i} B_k) \text{-grproj}.
\]
(29)
This yields a $\mathbb{Z}$-graded $k$-vector space isomorphism

$$\text{tr}(C_k/\mathbb{Z}) \simeq \bigoplus_{n \in \mathbb{N}} \text{tr}(H^{k,[d]}) \otimes_{\mathbb{Z}^{k,[d]}} \text{tr}(B_k)$$

and the $L^{[d]}_k$-action on $\text{tr}(C_k/\mathbb{Z})$ is given by explicit formulas as in Section 3.2.9 below.

Finally, for a future use we'll write

$$\text{tr}(C_{\lambda} \cap C_{\lambda}) \simeq \bigoplus_{n \geq 0} \text{tr}(H^{k,[d]}) \otimes_{\mathbb{Z}^{k,[d]}} \text{tr}(B_k)$$

and we define $\text{tr}(C_{\lambda})_k$ in the obvious way.

Remark 3.26. We conjecture that the left arrow in sequence (26) is injective if $C = V^A$. Given an idempotent $e$ of a finite dimensional algebra $A$ such that $AeA$ is a stratifying ideal of $A$ in the sense of Cline, Parshall and Scott, we have a long exact sequence by \cite[thm. 3.1]{21}

$$\cdots \to HH_1(B) \to HH_0(eAe) \to HH_0(A) \to HH_0(B) \to 0$$

where $B = A/AeA$. Now, set $g = sl_3$ and let $\Lambda = \theta = \alpha_1 + \alpha_2$ be the highest positive root. Then $A = R^\Lambda(\theta)$ has a primitive idempotent $e$ such that $C_{>0} = eAe$-proj, $C = A$-proj and $C_0 = B$-proj. In this case, the left arrow in (26) is indeed injective, although the ideal $AeA$ is not a stratifying ideal of $A$.

3.2.9. The $sl_2$-case. We'll use the same notation as in Example 3.14. Thus, we have $\Lambda = k\omega_1$, $\alpha = n\alpha_1$ and $d_{k,n} = 2(nk - n)$. Let $k$ be a $\mathbb{Z}^{k,[d]}$-algebra. Recall (9) yields

$$V^{k,[d]} = \bigoplus_{n \geq 0} (k \otimes_{\mathbb{Z}^{k,[d]}} H^{k,[d]})_{-\text{grproj}}.$$ 

For a future use, let us quote the following.

Proposition 3.27. (a) The $k \otimes L^{[d]}_k$-module $\text{tr}(V^{k,[d]})$ is generated by $\text{tr}(V^{k,[d]})$.

(b) We have $\text{tr}(H^{k,[d]}) = V \otimes_{k} Z^{k,[d]}$ as a $\mathbb{Z}$-graded $Z^{k,[d]}$-module, with $V^d = 0$ if $d \notin [0, \ell d_{k,n}]$ and $V^{\ell d_{k,n}} = k$.

Proof. It is enough to set $\ell = 1$ and $k = Z^k$. Then, we have

$$V^k = \bigoplus_{n=0}^k V^{k-2n}, \quad V_{k-2n}^k = H^{k}_n_{-\text{grproj}}.$$ 

Fix formal variables $y_1, \ldots, y_k$ of degree 2 such that $c_p = c_p(y_1, \ldots, y_k)$ for $p = 1, 2, \ldots, k$ and

$$Z^k = k[y_1, y_2, \ldots, y_k]^{S_n}_n.$$ 

We have $Z^k$-linear isomorphisms, see e.g., \cite{36},

$$Z(H^{k}_n) \simeq \begin{cases} k[y_1, y_2, \ldots, y_k]^{S_n} & \text{if } n \in [0, k], \\ 0 & \text{else.} \end{cases}$$

Since the $Z^k$-algebra $H^{k}_n$ is symmetric, we have $\text{tr}(H^{k}_n) \simeq Z(H^{k}_n)^*$ where $(\bullet)^*$ is the dual as a $Z^k$-module. This proves part (b).
By Proposition 3.25 under the isomorphism $\text{tr}(\mathbb{H}_n^k) \simeq Z(\mathbb{H}_n^k)^*$ the operators $x_{ir}^+, x_{ir}^-$ on $\bigoplus_n \text{tr}(\mathbb{H}_n^k)$ are identified with the transpose of the operators $Z_{ir}^+, Z_{ir}^-$ on $\bigoplus_n Z(\mathbb{H}_n^k)$. The formulas in \cite{36, 37} for the symmetrizing form of $\mathbb{H}_n^k$ imply that the Bernstein operators are given by the following explicit formulas for all $f \in Z(\mathbb{H}_n^k)$

$$Z_{ir}^-(f) = \partial_n \circ \cdots \circ \partial_{s_n} \left( y_{n+1}^r f \prod_{p=n+2}^k (y_{n+1} - y_p) \right) \in Z(\mathbb{H}_{n+1}^k),$$

$$Z_{ir}^+(f) = \partial_{s_{k-1}} \circ \cdots \circ \partial_{s_n} \left( y_n^r f \prod_{p=1}^{n-1} (y_p - y_n) \right) \in Z(\mathbb{H}_{n-1}^k).$$

Now, for each $p = 1, \ldots, k-1$, let $\partial_{s_p}$ be the Demazure operator on $k[y_1, y_2, \ldots, y_k]$ associated with the simple reflection $s_p = (p, p+1)$, which is defined by $\partial_{s_p}(f) = (f - s_p(f))/(y_{p+1} - y_p)$. Then, the $Z^k$-algebra $Z(\mathbb{H}_n^k)$ is equipped with the symmetrizing form $\partial_{w_n} : Z(\mathbb{H}_n^k) \to Z^k$, where $w_n \in S_k$ is the represent of minimal length in the longest coset in $S_k/S_n \times S_{k-n}$. It yields a non-degenerate bilinear form

$$\tau_n^k : Z(\mathbb{H}_n^k) \times Z(\mathbb{H}_n^k) \to Z^k, \quad (a, b) \mapsto \partial_{w_n}(ab).$$

The bilinear form $\tau_n^k$ identifies $Z(\mathbb{H}_n^k)^*$ with $Z(\mathbb{H}_n^k)$ as a $Z^k$-module.

Finally, taking the transpose of $Z_{ir}^-$, $Z_{ir}^+$ with respect to $\tau_n^k$, we get the following formulas for the operators $x_{ir}^+, x_{ir}^-$, which are viewed as $Z^k$-linear operators on $\bigoplus_n Z(\mathbb{H}_n^k)$:

$$x_{ir}^+(f) = \sum_{p=n}^k (p, n)(y_n^r f) \in Z(\mathbb{H}_{n-1}^k),$$

$$x_{ir}^-(f) = \sum_{p=1}^{n+1} (p, n+1)(y_{n+1}^r f) \in Z(\mathbb{H}_{n+1}^k).$$

We can now prove part (a) of the proposition. For each $n \geq 0$, let

$$\Lambda^+(n) = \{ \lambda \in \mathbb{N}^n ; \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \}$$

be the set of dominant weights. We abbreviate $x_{i\lambda}^- = x_{i\lambda_1}^- x_{i\lambda_2}^- \cdots x_{i\lambda_n}^-$. We must check that

$$(31) \quad \sum_{\lambda \in \Lambda^+(n)} x_{i\lambda}^-(Z^k) = Z(\mathbb{H}_n^k).$$

This follows from the equality $Z(\mathbb{H}_n^k) = \sum_{\lambda \in \Lambda^+(n)} Z^k \cdot h_\lambda(y_1, y_2, \ldots, y_n)$ and the formula

$$x_{i\lambda}^-(f) = f \cdot h_\lambda(y_1, y_2, \ldots, y_n), \quad \forall f \in Z^k.$$
3.3. The center and the cocenter of the minimal categorical representation. Let \( g \) be the symmetrizable Kac-Moody algebra over \( k \) associated with the Cartan datum \( (\mathcal{P}, \mathcal{P}^\vee, \Phi, \Phi^\vee) \). Fix a dominant weight \( \Lambda \in \mathcal{P}_+ \). Let \( k \) be an \( \mathbb{N} \)-graded \( k \)-algebra as in Section 3.1. We equip \( \text{tr}(R^\Lambda), Z(R^\Lambda) \) with the \( \mathbb{Z} \)-gradings such that for each \( d \in \mathbb{Z} \) we have

\[
\text{tr}(R^\Lambda)^d = \{\text{tr}(x); x \in R^\Lambda, d\}\}
\]

\[
Z(R^\Lambda)^d = R^\Lambda, d \cap Z(R^\Lambda).
\]

3.3.1. The grading of the cocenter. In this section we set \( k = k \). For \( \alpha \in Q_+ \) we write \( \lambda = \Lambda - \alpha \) and \( R^\Lambda(\alpha) = R^\Lambda(\alpha; k) \).

**Theorem 3.28.** Assume that \( k = k \). Then, for each \( \alpha \in Q_+ \) we have

(a) if \( \text{tr}(R^\Lambda(\alpha))^d \neq 0 \), then \( d \in [0, d_{\Lambda,\alpha}] \),

(b) if \( \text{tr}(R^\Lambda(\alpha))^{d_{\Lambda,\alpha}} = 0 \), then \( V(\lambda)_\Lambda = 0 \),

(c) \( \dim \text{tr}(R^\Lambda(\alpha))^0 = \dim V(\lambda)_\Lambda \).

**Proof.** For each \( i \in I \) and \( k \geq 0 \) the \( i \)-th isotypic filtration of \( \mathcal{V}^\Lambda \) yields \( \mathcal{P} \times \mathbb{Z} \)-graded categorical \( \mathfrak{g} \)-representations \( \mathcal{V}_{\geq k}^\Lambda, \mathcal{V}_k^\Lambda \) of degree \( d \) such that \( \mathcal{V}_{\geq 0}^\Lambda = V^\Lambda, \mathcal{V}_{\geq k}^{\Lambda, k} = 0 \) if \( k \) is large enough (depending on \( \lambda \)) and \( \mathcal{V}_k^\Lambda = (\mathcal{V}_{\geq k}^\Lambda / \mathcal{V}_{> k}^\Lambda)^c \). Further, taking the trace we get \( \mathbb{Z} \)-graded \( L\mathcal{V}_{\geq k}^\Lambda \)-submodules \( \text{tr}(R^\Lambda)_{\geq k} \subset \text{tr}(R^\Lambda) \) such that \( \text{tr}(R^\Lambda)_{\geq 0} = \text{tr}(R^\Lambda) \) and \( \text{tr}(R^\Lambda(\alpha))_{\geq k} = 0 \) if \( k \) is large enough.

Now, fix an indecomposable object \( P \in \mathcal{V}_\Lambda^\Lambda \) and an element \( v_P \in \text{tr}(\text{End}_{\mathcal{V}_\Lambda^\Lambda}(\mathcal{P})) \) which is homogeneous of degree \( d \). Let \( v \) be the image of \( v_P \) in \( \text{tr}(R^\Lambda(\alpha)) \). We must prove that either \( d \in [0, d_{\Lambda,\alpha}] \) or \( v = 0 \). Since \( R^\Lambda(0) \approx k \), we may assume that \( \alpha \neq 0 \). The Grothendieck group of \( \mathcal{V}_\Lambda^\Lambda / \mathbb{Z} \) is isomorphic to a \( \mathbb{Z} \)-lattice in \( V(\lambda)_\Lambda \) by [16]. Since \( \alpha \neq 0 \), this weight subspace does not contain any highest weight vector for the \( \mathfrak{g} \)-action on \( V(\lambda)_\Lambda \).

Let \( \epsilon_i \) be the Kashiwara function on the set of simple objects in \( \mathcal{A}_\Lambda / \mathbb{Z} \), which is defined as in [22, sec. 3.2]. For each indecomposable objects \( P, Q \in \mathcal{C} \) we have \( \epsilon_i(\text{top}(P)) > \epsilon_i(\text{top}(Q)) \) if and only if there is an integer \( k \) such that \( P \in \mathcal{C}_{\geq k} \) and \( Q \notin \mathcal{C}_{\geq k} \).

Now, we may choose \( i \in I \) such that the integer \( m = \epsilon_i(\text{top}(P)) \) is positive. Let \( k_0 \) be maximal such that \( P \in \mathcal{V}^\Lambda_{\geq k_0} \) and set \( m = \epsilon_i(\text{top}(P)) \). Note that \( \epsilon_i(\text{top}(Q)) \) is bounded above as \( Q \) runs over the set of all indecomposable objects in \( \mathcal{V}_\Lambda^\Lambda \) and that \( \epsilon_i(\text{top}(Q)) > m \) if and only if \( Q \in \mathcal{V}^\Lambda_{\geq k_0} \).

The proof is an induction on \( \alpha \) and a descending induction on \( m \). We’ll assume that

- \( \deg(\text{tr}(R^\Lambda(\beta))) \subset [0, d_{\Lambda,\beta}] \) for each \( \beta \in Q_+ \) such that \( \beta < \alpha \),
- \( \deg(\text{tr}(R^\Lambda(\alpha))_k) \subset [0, d_{\Lambda,\alpha}] \) for each \( k > k_0 \),

and we must check that \( \deg(\text{tr}(R^\Lambda(\alpha))_{k_0}) \subset [0, d_{\Lambda,\alpha}] \).

By [28], [31] there is a finitely generated \( \mathcal{P} \times \mathbb{Z} \)-graded \( \mathbb{Z}_{k_0, [d]} \)-algebra \( B_{k_0} \) such that \( B_{k_0, 0} = 0 \) if \( \mu_i \neq k_0 \) and a surjective \( \mathbb{Z} \)-graded \( L\mathcal{V}_{\geq k_0}^\Lambda \)-module homomorphism

\[
\text{tr}(\mathcal{V}^\Lambda_{k_0} / \mathbb{Z}) \simeq \bigoplus_{n \in \mathbb{N}} \text{tr}(\mathcal{H}_{n_{k_0, [d]}}) \otimes_{\mathbb{Z}_{k_0, [d]}} \text{tr}(B_{k_0}) \rightarrow \text{tr}(R^\Lambda)_{k_0}.
\]
Let $M_{k_0,\mu}$ be the image of $\text{tr}(\mathbb{H}^{k_0,[d_i]}) \otimes_{Z^{k_0,[d_i]}} \text{tr}(B_{k_0,\mu})$. Set $M_{k_0} = \bigoplus_{\mu} M_{k_0,\mu}$. Recall that $\text{tr}(\mathbb{H}^{k_0,[d_i]}) = Z^{k_0,[d_i]}$ and that $\sum_{\mu \in \Lambda^+(n)} x_{i\mu}^- (\text{tr}(\mathbb{H}^{k_0,[d_i]})) = \text{tr}(\mathbb{H}^{k_0,[d_i]})$ by (31). Thus, there is a (unique) surjective $\mathbb{Z}$-graded $k$-vector space homomorphism

\[(32) \quad \bigoplus_{n \in \mathbb{N}} \text{tr}(\mathbb{H}^{k_0,[d_i]}) \otimes_{Z^{k_0,[d_i]}} M_{k_0} \rightarrow \text{tr}(R^A)_{k_0}\]

such that $x_{i\mu}^-(f) \otimes u \mapsto f x_{i\mu}^-(u)$ for each $f \in k$, $\mu \in \Lambda^+(n)$, $u \in M_{k_0}$. Further, by definition of $k_0$ we have $k_0 = \lambda_i + 2m$ and (32) yields a surjective map

$$\text{tr}(\mathbb{H}^{k_0,[d_i]}) \otimes_{Z^{k_0,[d_i]}} M_{k_0,\lambda+m\alpha_i} \rightarrow \text{tr}(R^A(\alpha + (n - m)\alpha_i))_{k_0}.$$ 

Now, a short computation yields

$$d_i d_{k_0,m} + d_{\lambda,\alpha-m\alpha_i} = d_{\lambda,\alpha}.$$ 

Thus by Proposition 3.27 to prove part (a) of the theorem we must check that

$$\text{deg}(M_{k_0,\lambda+m\alpha_i}) \subset [0, d_{\lambda,\alpha-m\alpha_i}].$$

Since $B_{k_0,\lambda+m\alpha_i}$ is the endomorphism ring of an object of $\mathcal{V}^{\Lambda}_{k_0,\lambda+m\alpha_i}$ and $\text{tr}(B_{k_0,\lambda+m\alpha_i})$ maps onto $M_{k_0,\lambda+m\alpha_i}$, we have

$$\text{deg}(M_{k_0,\lambda+m\alpha_i}) \subset \text{deg}(\text{tr}(R^A(\alpha - m\alpha_i))).$$

Since $m > 0$, we have $\text{deg}(\text{tr}(R^A(\alpha - m\alpha_i))) \subset [0, d_{\lambda,\alpha-m\alpha_i}]$ by the inductive hypothesis. This finishes the proof of (a).

To prove part (b), note that by Proposition 3.10 the symmetrizing form on $R^A(\alpha)$ yields a non-degenerate bilinear form $Z(R^A(\alpha))^0 \times \text{tr}(R^A(\alpha))^{d_{\lambda,\alpha}} \rightarrow k$. We deduce that

$$V(\Lambda)_\lambda \neq 0 \Rightarrow R^A(\alpha) \neq 0 \Rightarrow Z(R^A(\alpha))^0 \neq 0 \Rightarrow \text{tr}(R^A(\alpha))^{d_{\lambda,\alpha}} \neq 0.$$ 

Finally, let us prove (c). We identify $g$ with its image in $Lg$. The operators $x_{i0}^\pm$ with $i \in I$ define a representation of $g$ on $\text{tr}(R^A)^d$ for each $d$. Since $\text{tr}(\mathbb{H})^0 = k$ for each $n \in [0, k]$, the map (32) yields a surjective $\mathfrak{e}$-module homomorphism $V(k) \otimes_k (M_{k_0})^0 \rightarrow \text{tr}(R^A)_{k_0}$. Here $V(k)$ is the $k + 1$-dimensional representation of $\mathfrak{e}$. See the proof of Proposition 3.27 for more details. Thus, the proof of (a) above implies that the representation of $g$ on $\text{tr}(R^A)^0$ is cyclic. Hence $\text{tr}(R^A)^0 \simeq V(\Lambda)$ as a $g$-module.

\[\square\]

### 3.3.2. The grading of the center

We use the same notation as in Section 3.3.1. The pairing (25) gives a non-degenerate bilinear form

$$Z(R^A(\alpha))^d \times \text{tr}(R^A(\alpha))^{d_{\lambda,\alpha}-d} \rightarrow k.$$ 

From Theorem 3.28 we deduce that

**Corollary 3.29.** If $k = k$ then we have $Z(R^A(\alpha))^d = 0$ for any $d \notin [0, d_{\lambda,\alpha}]$.

\[\square\]

We have $Z(R^A(\alpha))^0 \neq \{0\}$ whenever $V(\Lambda)_\lambda \neq 0$.

**Conjecture 3.30.** If $k = k$ then we have $Z(R^A(\alpha))^0 \simeq \text{tr}(R^A(\alpha))^{d_{\lambda,\alpha}} \simeq k$ whenever $V(\Lambda)_\lambda \neq 0$. 

If \( \mathfrak{g} \) is symmetric of finite type then the conjecture holds, see Remark \( \text{3.38} \) below.

### 3.3.3. The cocenter is a cyclic module

Consider a categorical representation of \( \mathfrak{g}_k \) on a \( \mathbb{Z} \)-graded \( \mathbf{k} \)-category \( \mathcal{C} \). Then, the \( \mathbf{k} \)-linear operator \( x^+_r \) on \( \text{tr}(\mathcal{C}/\mathbb{Z}) \) is well-defined for each \( i \in I \), \( r \in \mathbb{N} \). Let \( h : \text{tr}(\mathcal{C}^\text{bw}/\mathbb{Z}) \to \text{tr}(\mathcal{C}/\mathbb{Z}) \) be the trace of the canonical embedding \( \mathcal{C}^\text{bw}/\mathbb{Z} \subset \mathcal{C}/\mathbb{Z} \). Let \( \text{tr}(\mathcal{C}/\mathbb{Z})^{\text{cyc}} \subset \text{tr}(\mathcal{C}/\mathbb{Z}) \) be the \( \mathbf{k} \)-submodule generated by the image of \( h \) under the action of all operators \( x^+_r \).

**Proposition 3.31.** If \( \mathcal{C} = \mathcal{V}^{\Lambda} \) then we have \( \text{tr}(\mathcal{C}/\mathbb{Z}) = \text{tr}(\mathcal{C}/\mathbb{Z})^{\text{cyc}} \).

The \( i \)-th isotypic filtration of \( \mathcal{C} \) yields a categorical \( \mathfrak{e}_z^k \)-representation \( C_k \) for each \( k \geq 0 \). Let us quote the following consequence of \((\text{3.30})\) and Proposition \( \text{3.27} \).

**Lemma 3.32.** For all \( i \in I \), \( k \geq 0 \) we have \( \text{tr}(\mathcal{C}_k/\mathbb{Z}) = \text{tr}(\mathcal{C}_k/\mathbb{Z})^{\text{cyc}} \).

Now, we can prove Proposition \( \text{3.31} \).

**Proof.** Fix \( \alpha \in Q_+ \) and set \( \lambda = \Lambda - \alpha \). Fix an indecomposable object \( P \in \mathcal{C}_\Lambda^\alpha \) and an element \( v_P \in \text{End}_{\mathcal{V}^{\Lambda}/\mathbb{Z}}(P) \). Let \( v \) be the image of \( v_P \) in \( \text{tr}(\mathcal{R}^\alpha) \). We must prove that \( v \in \text{tr}(\mathcal{C}_\lambda/\mathbb{Z})^{\text{cyc}} \). Since \( \mathcal{C} = \mathcal{V}^{\Lambda} \), we have \( \text{tr}(\mathcal{C}/\mathbb{Z}) = \text{tr}(\mathcal{R}^\alpha) \). Since \( \text{tr}(\mathcal{R}^\alpha(0)) \subset \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \), we may assume that \( \alpha \neq 0 \). Let \( i, m, k_0 \) be as in the proof of Theorem \( \text{3.28} \). The proof is an induction on \( \alpha \) and a descending induction on \( m \).

We have \( v \in \text{tr}(\mathcal{R}^\alpha(i))_{\geq k_0} \). We’ll assume that

- \( \deg(\text{tr}(\mathcal{R}^\beta)) \subset \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \) for each \( \beta \in Q_+ \) such that \( \beta < \alpha \),
- \( \text{tr}(\mathcal{R}^\alpha(i))_{\geq k_0} \subset \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \).

We must check that \( \text{tr}(\mathcal{R}^\alpha(\alpha))_{\geq k_0} \subset \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \).

Fix \( \beta \) as above and fix an element \( x \in U(\mathfrak{L}_k) \) of weight \( \alpha - \beta \). Consider the following commutative diagram whose rows are exact sequences of \( \mathfrak{L}_k \)-modules

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{tr}(\mathcal{R}^\alpha(i))_{\geq k_0} & \overset{f}{\longrightarrow} & \text{tr}(\mathcal{R}^\alpha(\alpha))_{\geq k_0} & \overset{g}{\longrightarrow} & \text{tr}(\mathcal{R}^\alpha(\alpha))_{k_0} & \longrightarrow 0 \\
& & x & \downarrow & x & & x \\
& & \text{tr}(\mathcal{R}^\beta(\beta))_{\geq k_0} & \overset{g}{\longrightarrow} & \text{tr}(\mathcal{R}^\beta(\beta))_{k_0} & \longrightarrow 0.
\end{array}
\]

Now, set \( \bar{v} = g(v) \in \text{tr}(\mathcal{R}^\alpha(\alpha))_{k_0} \). By definition of \( \text{tr}(\mathcal{R}^\alpha(\alpha))_{k_0} \), there is a surjective \( U(\mathfrak{L}_k) \)-module homomorphism \( \text{tr}(\mathcal{V}^\alpha/\mathbb{Z}) \to \text{tr}(\mathcal{R}^\alpha(\alpha))_{k_0} \). Hence, by Lemma \( \text{3.32} \), we can choose \( \beta, x \) such that there is an element \( \bar{u} \in \text{tr}(\mathcal{R}^\beta(\beta))_{k_0} \) with \( \bar{v} = x \cdot \bar{u} \). Then, \( u \in \text{tr}(\mathcal{R}^\beta(\beta))_{\geq k_0} \) such that \( g(u) = \bar{u} \). Then, we have \( v = x \cdot u \). Fix some \( v' \in \text{tr}(\mathcal{R}^\alpha(\alpha))_{\geq k_0} \).

Finally, we can apply both recursive hypotheses. We deduce that \( u \in \text{tr}(\mathcal{R}^\beta(\beta)) \subset \text{tr}(\mathcal{R}^\beta)^{\text{cyc}} \) and \( v' \in \text{tr}(\mathcal{R}^\alpha(\alpha))_{\geq k_0} \subset \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \). Hence, we have \( v = v - x \cdot u + x \cdot u = v' + x \cdot u \in \text{tr}(\mathcal{R}^\alpha)^{\text{cyc}} \).

\[\Box\]

### 3.4. The symmetric case

Let \( k \) be a commutative \( \mathbb{N} \)-graded ring as in Section \( \text{3.1} \). Let \( \mathfrak{g} \) be the symmetrizable Kac-Moody algebra over \( k \) associated with the Cartan datum \( (P, P', \Phi, \Phi') \).
3.4.1. Weyl modules. Assume that \( \mathfrak{g} \) is of finite type. Fix a dominant weight \( \Lambda \). The local Weyl module \( W(\Lambda) \) (over \( k \)) is the \( \mathbb{Z} \)-graded \( L\mathfrak{g} \)-module generated by a nonzero element \( |\Lambda\rangle \) of degree 0 with the following defining relations

- \( n[t] \cdot |\Lambda\rangle = 0 \),
- \( (f_i)^{\Lambda + 1} \cdot |\Lambda\rangle = 0 \),
- \( h \cdot |\Lambda\rangle = \langle h, \Lambda \rangle |\Lambda\rangle \) for all \( h \in \mathfrak{h} \),
- \( t\mathfrak{h}[t] \cdot |\Lambda\rangle = 0 \).

The global Weyl module \( W(\Lambda) \) is the \( \mathbb{Z} \)-graded \( L\mathfrak{g} \)-module generated by a nonzero element \( |\Lambda\rangle \) satisfying the first three relations above. Consider the formal series \( \Psi_i(z) = \sum_{r \geq 0} \Psi_{ir} z^r \), \( i \in I \), given by \( \Psi_i(z) = \exp \left( - \sum_{r \geq 1} h_{ir} z^r / r \right) \). Then, there is a unique \( k \)-module structure on \( W(\Lambda) \) such that the representation of \( L\mathfrak{g} \) is \( k \)-linear and we have \( \Psi_{ip} \cdot |\Lambda\rangle = c_{ip} |\Lambda\rangle \) for each \( (i, p) \in I \times \mathbb{N} \). For any \( k \)-algebra homomorphism \( k \rightarrow k \) we set \( W(\Lambda, k) = k \otimes_k W(\Lambda) \). The Weyl modules are universal in the following sense. Let \( M \) be a \( \mathbb{Z} \)-graded integrable \( L\mathfrak{g}_k \)-module containing an element \( m \) of weight \( \Lambda \) which is annihilated by \( n \). Then there is a unique \( k \)-algebra structure on \( k \) and a unique \( \mathbb{Z} \)-graded \( L\mathfrak{g}_k \)-module homomorphism \( W(\Lambda, k) \rightarrow M \) such that \( |\Lambda\rangle \mapsto m \).

Let \( \Lambda_{\min} \) be the unique minimal element in the poset \( \{ \lambda \in P_+ ; \lambda \leq \Lambda \} \). Let \( \mathfrak{h}' \) be as in Section 3.2.3. The following is well-known, see [32], [26, thm. 1.1].

**Proposition 3.33.** (a) \( \dim_k(W(\Lambda)) < \infty \).

(b) \( W(\Lambda) \) is a free \( k \)-module of finite rank.

(c) \( \text{top}(W(\Lambda)) = W(\Lambda)^0 \cong V(\Lambda) \) as a \( \mathbb{Z} \)-graded \( \mathfrak{g} \)-module.

(d) if \( \mathfrak{g} \) is symmetric, then

\[
\text{soc}(W(\Lambda)) = W(\Lambda)^{d_{\Lambda, \Lambda_{\min}}[\Lambda]} \cong V(\Lambda_{\min})[-d_{\Lambda, \Lambda_{\min}}]
\]

as a \( \mathbb{Z} \)-graded \( \mathfrak{g} \)-module.

(e) \( W(\Lambda, k) \rightarrow W(\Lambda) \) as a \( \mathbb{Z} \)-graded \( L\mathfrak{g} \)-module.

(f) \( \Lambda = \omega_i \Rightarrow W(\Lambda) \cong k \otimes_k W(\Lambda) \) a \( \mathbb{Z} \)-graded \( L\mathfrak{g}_k \)-module.

(g) \( W(\Lambda, \mathfrak{h}') \cong \mathfrak{h}' \otimes_{\mathfrak{h}} \otimes_i W(\omega_i)^{\otimes_{\Lambda_i}} \) as \( L\mathfrak{g}_\mathfrak{h}' \)-modules such that \( |\Lambda\rangle \mapsto 1 \otimes \otimes_i (w_{\omega_i})^{\otimes_{\Lambda_i}} \).

Now, assume that \( \mathfrak{g} \) is symmetric and (11) is satisfied. We consider the \( \mathbb{Z} \)-graded representation of \( L\mathfrak{g}_k \) on \( \text{tr}(R^A) \) in Theorem 3.22.

**Theorem 3.34.** If \( \mathfrak{g} \) is symmetric of finite type and (11) is satisfied, then there is a unique \( \mathbb{Z} \)-graded \( L\mathfrak{g}_k \)-module isomorphism \( W(\Lambda, k) \rightarrow \text{tr}(R^A) \) such that \( |\Lambda\rangle \mapsto |\Lambda\rangle \).

**Proof.** From Proposition 3.31 we deduce that the element \( |\Lambda\rangle = \text{tr}(1) \) of \( \text{tr}(R^A(0)) \) is a generator of the \( L\mathfrak{g}_k \)-module \( \text{tr}(R^A) \). Thus, it is enough to prove that there is a \( \mathbb{Z} \)-graded \( L\mathfrak{g}_k \)-module isomorphism \( W(\Lambda) \rightarrow \text{tr}(R^A) \) such that \( |\Lambda\rangle \mapsto |\Lambda\rangle \). To do this, note that, since \( W(\Lambda) \) is universal, there is a unique \( \mathbb{Z} \)-graded \( L\mathfrak{g}_k \)-module homomorphism

\[
\phi^A : W(\Lambda) \rightarrow \text{tr}(R^A), \quad |\Lambda\rangle \mapsto |\Lambda\rangle.
\]

By Proposition 3.31 we deduce that \( \phi^A \) is onto.
First, we consider the map \( \phi^A : W(\Lambda) \to \text{tr}(R^A(k)) \) given by \( \phi^A = 1 \otimes \phi^A \). Since \( \phi^A \) is surjective, the map \( \phi^A \) is also surjective. To prove that it is injective, we must check that \( \phi^A(\text{soc}(W(\Lambda))) \neq 0 \). Since \( \phi^A \) is surjective and \( \text{soc}(W(\Lambda)) \simeq V(\Lambda_{\min})[-d_{\Lambda,\Lambda_{\min}}] \) as a \( \mathbb{Z} \)-graded \( L_\mathfrak{g} \)-module, it is enough to prove that \( \text{tr}(R^A(k))^{\underline{d}_{\Lambda,\Lambda_{\min}}} \neq 0 \). The weight subspace \( V(\Lambda)_{\Lambda_{\min}} \) is non-zero. Thus the injectivity of \( \phi^A \) follows from Theorem 3.28(b).

Now, we prove that \( \phi^A \) is injective. To do so, since \( W(\Lambda) \) is a free \( k \)-module and since \( \phi^A \) is invertible, it is enough to check that \( \text{tr}(R^A) \) is free as a \( k \)-module. To do so, note that by Theorem 3.15 and Example 3.3(c), the \( h' \)-algebras \( R^A(h') \) and \( h' \otimes_k \otimes_i R^w_i(k)^{\otimes i} \) are Morita equivalent. We deduce that there is an \( h' \)-linear isomorphism

\[
(35) \quad \text{tr}(R^A(h')) \rightarrow h' \otimes_k \bigotimes_i \text{tr}(R^w_i(k))^{\otimes i}, \quad |\Lambda| \mapsto 1 \otimes \bigotimes_i |\omega_i|^{\otimes i}.
\]

Further, by Theorem 3.15 the map (35) is \( L_{\mathfrak{g}h} \)-linear. Next, by Proposition 3.33(g) we have an \( L_{\mathfrak{g}h} \)-module isomorphism

\[
(36) \quad W(\Lambda, h') \rightarrow h' \otimes_k \bigotimes_i W(\omega_i)^{\otimes i}, \quad |\Lambda| \mapsto 1 \otimes \bigotimes_i |\omega_i|^{\otimes i}.
\]

Since the maps (35), (36) are \( L_{\mathfrak{g}h} \)-linear, from the unicity of the morphism \( \phi^A \) in (34) we deduce that the following square is commutative

\[
\begin{array}{ccc}
W(\Lambda, h') & \overset{36}{\longrightarrow} & h' \otimes_k \bigotimes_i W(\omega_i)^{\otimes i} \\
1 \otimes \phi^A & \downarrow & 1 \otimes \bigotimes_i (\phi^w_i)^{\otimes i} \\
\text{tr}(R^A(h')) & \overset{35}{\longrightarrow} & h' \otimes_k \bigotimes_i \text{tr}(R^w_i(k))^{\otimes i}.
\end{array}
\]

Since \( \phi^w_i \) is an isomorphism for each \( i \), this implies that the map \( 1 \otimes \phi^A \) is also an isomorphism.

Finally, we must check that \( \phi^A \) is an isomorphism. To do so, note that by construction the map \( \phi^A \) preserves the weight decomposition of \( W(\Lambda), \text{tr}(R^A) \). Therefore, the claim follows from the following lemma

**Lemma 3.35.** Let \( \psi : M \rightarrow N \) be a \( \mathbb{Z} \)-graded \( k \)-module homomorphism such that \( M, N \) are both finitely generated. Assume that the maps \( 1 \otimes \psi : k \otimes_k M \rightarrow k \otimes_k N \) and \( 1 \otimes \psi : h' \otimes_k M \rightarrow h' \otimes_k N \) are invertible. Then \( \psi \) is also invertible.

\[\square\]

### 3.4.2. Equivariant homology.
For any complex algebraic variety \( X \) and any commutative ring \( k \) let \( H_*(X, k) \) be the Borel-Moore homology with coefficients in \( k \). Given an action of a complex linear algebraic group \( G \) on \( X \), let \( H_*^G(X, k) \) be the \( G \)-equivariant Borel-Moore homology. We'll assume that \( X \) admits a locally closed \( G \)-equivariant embedding into a smooth projective \( G \)-variety. We define it as in [28 sec. 2.8], but we assign the degree as in [12], so that the fundamental class \([X]\) of \( X \) has degree 2 \( \dim X \) if \( X \) is pure dimensional.
Alternatively, let $D^G(X, k)$ be the equivariant derived category of constructible complexes on $X$ with coefficients in $k$, see [I]. Let $k_X$ and $k^D_X$ be the constant, and the dualizing sheaf on $X$. These are objects of $D^G(X, k)$. If $M$ is in $D^G(X, k)$ the $i$-th equivariant cohomology of $Y$ with coefficients in $M$ is by definition $H^i_G(X, M) = \text{Ext}^i(k_X, M)$. In particular, the $G$-equivariant cohomology and Borel-Moore homology of $X$ are defined by

$$H^i_G(X, k) = H^i_G(X, k_X), \quad H^i_G(X, k) = H^{-i}_G(X, k^D_X).$$

Note that with our conventions, one can have $H^i_G(X, k) \neq 0$ for $i < 0$. We’ll abbreviate

$$H^i(X, k) = H^i_{(1)}(X, k), \quad H_i(X, k) = H^{(1)}_i(X, k).$$

The action of the cohomology on the Borel-Moore homology yields a map

$$\cap : H^i_G(X, k) \otimes_k H^i_G(X, k) \to H_{2 \dim X-i}^G(X, k).$$

If $X$ is smooth and pure dimensional, the cap product with $[X]$ yields an isomorphism

$$H^i_G(X, k) \to H_{2 \dim X-i}^G(X, k).$$

### 3.4.3. Quiver varieties.

Assume that $g$ is symmetric. Following Nakajima, to each $\Lambda \in P_+$ and $\alpha \in Q_+$ we associate a quiver variety $\mathcal{M}(\Lambda, \alpha)$. It is a complex quasi-projective variety equipped with an action of the complex algebraic group $G_\Lambda = \prod_{i \in I} \text{GL}(\Lambda_i)$. Recall that

- $\mathcal{M}(\Lambda, \alpha)$ is nonsingular, symplectic, possibly empty and equipped with a $G_\Lambda$-equivariant projective morphism to an affine variety $p : \mathcal{M}(\Lambda, \alpha) \to \mathcal{M}_0(\Lambda, \alpha)$,
- the variety $\mathcal{M}_0(\Lambda, \alpha)$ has a distinguished point denoted by 0 such that the closed subvariety $\mathcal{L}(\Lambda, \alpha) = p^{-1}(0)$ of $\mathcal{M}(\Lambda, \alpha)$ is Lagrangian,
- if $\mathcal{M}(\Lambda, \alpha) \neq \emptyset$ then its dimension is $d_{\Lambda, \alpha}$.

We put $\mathcal{M}(\Lambda) = \bigsqcup_\alpha \mathcal{M}(\Lambda, \alpha)$ and $\mathcal{L}(\Lambda) = \bigsqcup_\alpha \mathcal{L}(\Lambda, \alpha)$. We have a canonical $k$-algebra isomorphism $H^*_{G_\Lambda}(\bullet, k) = k$. Under this isomorphism the equivariant Euler class of the $p$-th fundamental representation of $\text{GL}(\Lambda_i)$ maps to $c_p$, for any $p \geq 0$. We define

$$H^d_{G_\Lambda}(\mathcal{M}(\Lambda), k) = \bigoplus_{d \geq 0} H^d_{G_\Lambda}(\mathcal{M}(\Lambda), k),$$

$$H^d_{G_\Lambda}(\mathcal{L}(\Lambda), k) = \bigoplus_{d \geq 0} \bigoplus_{\alpha} H^d_{G_\Lambda,d_{\Lambda,\alpha}-d}(\mathcal{L}(\Lambda, \alpha), k).$$

The following is well-known.

**Proposition 3.36.** We have

(a) $H^d_{[\alpha]}(\mathcal{L}(\Lambda), k) = 0$ if $d \notin (2\mathbb{Z}) \cap [0, d_{\Lambda,\alpha}]$,

(b) $H^d_{[\alpha]}(\mathcal{L}(\Lambda), k)$ is free over $k$ and $H^d_{[\alpha]}(\mathcal{L}(\Lambda), k) = k \otimes_k H^d_{[\alpha]}(\mathcal{L}(\Lambda), k)$,

(c) there is a perfect $k$-bilinear pairing $H^d_{G_\Lambda}(\mathcal{M}(\Lambda), k) \times H^d_{G_\Lambda,d_{\Lambda,\alpha}-d}(\mathcal{L}(\Lambda), k) \to k$,

(d) there is a $\mathbb{Z}$-graded $L_{\mathfrak{g}k}$-representation on $H^d_{[\alpha]}(\mathcal{L}(\Lambda), k)$ which is isomorphic to $W(\Lambda)$.

Under this isomorphism $H^d_{[\alpha]}(\mathcal{L}(\Lambda, \alpha), k)$ maps to $W(\Lambda)_{\Lambda-\alpha}$. 


Proof. Parts (b), (c) are proved in [33, thm. 7.3.5] for equivariant $K$-theory, but the same proof applies also to equivariant Borel-Moore homology. Part (d) follows from [34].

From Theorem 3.34 we deduce that

**Theorem 3.37.** If $\mathfrak{g}$ is symmetric of finite type and (11) is satisfied, then there are $\mathbb{Z}$-graded $L_{\mathfrak{g}k}$-module isomorphisms \( \text{tr}(R^A) \simeq k \otimes_k H^G_{[\ast]}(\Lambda, k) \).

\[ \square \]

**Remark 3.38.** (a) Since $\Lambda(\Lambda, \alpha)$ is connected, we deduce from Theorem 3.37 that $\text{tr}(R^A(\alpha))^{d_{\Lambda, \alpha}} \simeq k$ whenever $V(\Lambda) \neq 0$ if $\mathfrak{g}$ is symmetric and of finite type.

(b) We equip $Z(R^A)$ with the $L_{\mathfrak{g}k}$-representation dual to $\text{tr}(R^A)$ relatively to the pairing (25) and the anti-involution $\varpi$ of $L_{\mathfrak{g}k}$. The action of $x^+_i$ on $Z(R^A)$ is described in Proposition 3.26. Next, we equip $H^*_{G_A}(\mathfrak{M}(\Lambda), k)$ with the $L_{\mathfrak{g}k}$-representation dual to $H^G_{[\ast]}(\Lambda, k)$ relatively to the pairing in (c) above and the anti-involution $\varpi$. Then, from Theorem 3.37 we deduce that there is $\mathbb{Z}$-graded $L_{\mathfrak{g}k}$-module isomorphism

\[ Z(R^A) \simeq k \otimes_k H^*_{G_A}(\mathfrak{M}(\Lambda), k). \]

4. **The Jordan quiver**

Hopefully, the results above can be generalized to quivers with loops using the generalized quiver-Hecke algebras introduced in [17]. In this section we consider the particular case of the Jordan quiver. In this particular case the quiver-Hecke algebra in [17] is the degenerate affine Hecke algebra of the symmetric group.

From now on, let $k$ be a commutative domain, $k = k[h, y_1, \ldots, y_r]$ and $k'$ be the fraction field of $k$. Write $M' = k' \otimes_k M$ for any $k$-module $M$. The ring $k$ is $\mathbb{Z}$-graded with $\deg(y_p) = \deg(h) = 2$. If $M$ is free $\mathbb{Z}$-graded of finite rank, let $\text{grdim}(M)$ be its graded rank. It is the unique element in $\mathbb{N}[t, t^{-1}]$ such that $\text{grdim}(k[d]) = t^{-d}$ and $\text{grdim}(M \oplus N) = \text{grdim}(M) + \text{grdim}(N)$.

4.1. **The quiver-Hecke algebra.** For any integer $n > 0$ the QHA of rank $n$ associated with the Jordan quiver is the degenerate affine Hecke $k$-algebra $R(n)$, which is generated by elements $\tau_1, \ldots, \tau_{n-1}, x_1, \ldots, x_n$ with the defining relations

(a) $x_k x_l = x_l x_k$,
(b) $\tau_k \tau_l = \tau_l \tau_k$ if $|k - l| > 1$,
(c) $\tau_k^2 = 1$,
(d) $\tau_k x_l - x_{s_k(l)} \tau_k = h (\delta_{l,k+1} - \delta_{l,k})$,
(e) $\tau_k \tau_{k+1} = \tau_k \tau_{k+1} \tau_k$.

We write $R(0) = k$. For any integer $r \geq 0$, the CQHA of rank $n$ and level $r$ is the quotient $R^n(n)$ of $R(n)$ by the two-sided ideal generated by the element $\prod_{p=1}^r (x_1 - y_p)$. The $k$-algebras $R(n)$, $R^n(n)$ are $\mathbb{Z}$-graded, with $\deg(t_k) = 0$ and $\deg(x_k) = 2$.

The canonical map $R(n) \to R^n(n)$ yields a $\mathbb{Z}$-graded $k$-algebra homomorphism $Z(R(n)) \to Z(R^n(n))$. Let $Z(R^n(n))^{\mathbb{Q}M}$ be its image.
We equip $Z(R^n(n))$, $Z(R^n(n))^{JM}$ with the grading such that

$$Z(R^n(n))^d = R^n(n)^d \cap Z(R^n(n)), \quad (Z(R^n(n))^{JM})^d = R^n(n)^d \cap Z(R^n(n))^{JM}.$$  

The canonical inclusion $R(n) \to R(n+1)$ factors to a $(R^n(n), R^n(n))$-bilinear map $\iota : R^n(n) \to R^n(n+1)$. We have the following.

**Proposition 4.1.** (a) $R^n(n) = \bigoplus_{r_1,\ldots, r_n} k x_1^{r_1} \cdots x_n^{r_n} w$ where $r_1 + \cdots + r_n < r$ and $w \in S_n$,

(b) $R^n(n+1) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n+1} R^n(n) (j, n+1) x_j^k$;

(c) for each $z \in R^n(n+1)$ there are unique elements $\pi(z) \in R^n(n) \otimes_{R^n(n-1)} R^n(n)$ and $p_k(z) \in R(n)$ such that $z = \mu_{\tau_n}(\pi(z)) + \sum_{k=0}^{r-1} p_k(z) x_{n+1}^k$, yielding a $(R^n(n), R^n(n))$-bimodule isomorphism

$$R^n(n+1) = R^n(n) \tau_n R^n(n) \oplus \bigoplus_{k=0}^{r-1} R^n(n) x_{n+1}^k;$$

(d) the canonical map $Z(R(n))' \to Z(R^n(n))'$ is surjective and we have

$$\sum_{n \geq 0} \dim(ZR^n(n)') q^{2n} = \prod_{j \geq 1} (1 - q^{2j})^{-r},$$

(e) $Z(R^n(n))^{JM}$ is a free $k$-module of finite rank such that

$$\sum_{n \geq 0} \text{grdim}(ZR^n(n))^{JM} q^{2n} = \prod_{p=1}^{r} \prod_{i=1}^{\infty} (1 - q^{2i} t^{2i(ri+p-1-r)})^{-1}.$$  

**Proof.** Parts (a), (b) are well-known. The proof of part (c) is similar to [25, lem. 5.6.1], where the case $\hbar = 1$ is done. Indeed, by part (b) it is enough to show that

$$\bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^n(n) (j, n+1) x_j^k = R^n(n) \tau_n R^n(n).$$

By (b) we have $R^n(n) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^n(n-1) (j, n) x_j^k$. Since $\tau_n$ commutes with $R^n(n-1)$ and $\tau_n(j,n) = (j,n)(j,n+1)$ we deduce $R^n(n) \tau_n R^n(n) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^n(n) (j, n+1) x_j^k$. Part (d) is proved in [24].

Let us concentrate on (e). First, we prove that $Z(R^n(n))^{JM}$ is free of finite rank as a $k$-module. To do that, set $k_1 = k[y_1, \ldots, y_r]$ and consider the $k_1$-algebras

$$R(n)_1 = R(n)/(\hbar - 1), \quad R^n(n)_1 = R^n(n)/(\hbar - 1).$$

Then, the assignment $\tau_k \mapsto \tau_k, x_k \mapsto x_k \otimes \hbar, y_p \mapsto y_p \otimes \hbar$ yields a $k$-algebra homomorphism

$$R^n(n) \to R^n(n)_1 \otimes_{k_1} k = R^n(n)_1[\hbar].$$

It restricts to an inclusion

$$Z(R^n(n))^{JM} \subset Z(R^n(n)) \subset Z(R^n(n)_1)[\hbar].$$

Now, for any $n$-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of non-negative integers, let

$$p_{\mu}(x_1, \ldots, x_n) = \sum_{\nu} x_1^{\nu_1} \cdots x_n^{\nu_n} \in Z(R^n(n)_1)$$

and
where $\nu$ runs over the set of all $n$-tuples which are obtained from $\mu$ by permuting its entries.

Let $\mathcal{P}_n(r)$ be the set of all partitions $\mu$ such that $\ell(\mu) + \sum_i [\mu_i/r] \leq n$. By [7, thm. 3.2], the canonical map $R(n)_1 \to R^r(n)_1$ yields a surjection $Z(R(n)_1) \to Z(R^r(n)_1)$. Further, the elements $p_\mu(x_1, \ldots, x_n)$ where $\mu$ runs over the set $\mathcal{P}_n(r)$, form a $k_1$-basis of $Z(R^r(n)_1)$. Therefore, under the inclusion (38), the elements $p_\mu(x_1, \ldots, x_n) \otimes h^{[\mu]}$ where $\mu \in \mathcal{P}_n(r)$ yield a $k$-basis of $Z(R^r(n))^{JM}$. We deduce that $Z(R^r(n))^{JM}$ is free of finite rank as a $k$-module and that

$$\sum_{n \geq 0} \text{grdim}(ZR^r(n))^{JM} q^{2n} = \sum_{n \geq 0} \mu \in \mathcal{P}_n(r) t^{2|\mu|} q^{2n}.$$  

To compute the right hand side, note that [7, p. 243] yields a bijection

$$\varphi : \Lambda^+(n) \to \mathcal{P}_n(r)$$

where $\Lambda^+(n)$ is the set of $r$-partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$. Further, if $\varphi(\lambda) = \mu$ then

$$|\mu| = r|\lambda| - (r + 1)\ell(\lambda) + \sum_{p=1}^r p \ell(\lambda^{(p)}).$$

We deduce that

$$\sum_{n \geq 0} \text{grdim}(ZR^r(n))^{JM} q^{2n} \sum_{n \geq 0} \sum_{\lambda \in \Lambda^+(n)} t^{2r|\lambda|-2(r+1)\ell(\lambda)+2} \sum_{p=1}^r p \ell(\lambda^{(p)}) q^{2|\lambda|},$$

$$= \prod_{p=1}^r \sum_{\lambda \in \Lambda^+(n)} t^{2r|\lambda|+2(r-1)\ell(\lambda)} q^{2|\lambda|},$$

$$= \prod_{p=1}^r \prod_{i=1}^\infty (1 - q^{2i} t^{2(r-1-p-1)})^{-1}. $$

4.2. The Lie algebra $\mathcal{W}$. Let $\mathcal{W}$ be the Lie $k'$-algebra generated by elements $C_k, D_{-1,k}, D_{0,k+1}, D_{1,k}$ with $k \geq 0$, modulo the following definition relations

(a) $[D_{0,t+1}, D_{0,k+1}] = 0,$
(b) $[D_{0,t+1}, D_{1,k}] = h D_{1,t+k}$ and $[D_{0,t+1}, D_{-1,k}] = -h D_{-1,t+k},$
(c) $3[D_{12}, D_{11}] - [D_{13}, D_{10}] + h^2 [D_{11}, D_{10}] = 0,$
(d) $3[D_{-1,2}, D_{-1,1}] - [D_{-1,3}, D_{-1,0}] + h^2 [D_{-1,1}, D_{-1,0}] = 0,$
(e) $[D_{10}, [D_{10}, D_{11}]] = [D_{-1,0}, [D_{-1,0}, D_{-1,1}]] = 0,$
(f) $[D_{-1,k}, D_{1,l}] = E_{k+l},$
(g) $C_k$ is central,

where the series $E(z) = \sum_{k \geq 0} E_k z^k$ is given by the following formula

$$E(z) = C_0 + \sum_{k \geq 1} (\gamma_h(D_{0,k+1}) + \gamma_{-h}(D_{0,k+1}) + C_k) z^k,$$

(39)

$$\gamma_t(D_{0,k+1}) = \sum_{p=1}^k \binom{k}{p} D_{0,k-p+1} t^{p-1}. $$
Let \( \mathcal{W}_{<0}, \mathcal{W}_{\geq 0}, \mathcal{W}_0 \subset \mathcal{W} \) be the Lie subalgebras generated by \( \{ D_{-l,k} : l < 0 \}, \{ D_{-l,k} : l \geq 0 \}, \) and \( \{ D_{0,k+1}, C_k \} \) respectively.

There is a unique Lie \( \mathbb{k} \)-algebra anti-involution \( \varpi \) of \( \mathcal{W} \) such that
\[
\varpi(C_k) = C_k, \quad \varpi(D_{l,k}) = D_{-l,k}.
\]

The Lie \( \mathbb{k} \)-algebra \( \mathcal{W} \) is \( \mathbb{Z} \)-graded with \( \text{deg}(D_{l,k}) = 2l \) and \( \text{deg}(C_k) = 0 \). A representation \( V \) is diagonalizable if the operator \( D_{0,1}/\hbar \) is diagonalizable with integral eigenvalues. Then \( V \) is \( \mathbb{Z} \)-graded and its degree \( 2n \) component \( V_n \) is the eigenspace associated with the eigenvalue \( n \). We'll say that a diagonalizable representation is quasi-finite if the degree \( 2n \) component is finite dimensional for each \( n \). Finally, we define the character of a quasi-finite representation \( V \) to be the formal series in \( \mathbb{N}((q)) \) given by
\[
\text{ch}(V)(q) = \sum_{n \in \mathbb{Z}} q^{2n} \dim(V_n).
\]

Given a linear form \( \Lambda : \mathcal{W}_0 \to \mathbb{k}' \) and a module \( V \), an element \( v \in V \) is primitive of weight \( \Lambda \) if \( \mathcal{W}_0 \) acts on \( v \) by \( \Lambda \) and \( \varpi(\mathcal{W}_{<0}) \) by zero. We call \( \Lambda(C_0) \) the level of \( \Lambda \).

Let \( M(\Lambda) \) be the Verma module with the lowest weight \( \Lambda \). It is the diagonalizable module induced from the one-dimensional \( \mathcal{W}_{\geq 0} \)-module spanned by a primitive vector \( |\Lambda\rangle \). Let \( V(\Lambda) \) be the top of \( M(\Lambda) \). It is an irreducible diagonalizable module. We call \( \Lambda(C_0) \) the lowest weight and \( \Lambda(C_0) \) the level of \( M(\Lambda), V(\Lambda) \). We call \( |\Lambda\rangle \) the highest weight vectors of \( M(\Lambda), V(\Lambda) \).

4.3. The loop operators on the center and the cocenter. For each \( r, n \in \mathbb{N} \) with \( r \neq 0 \) we set \( \mathcal{C}_n^r = R^r(n)\text{-grproj} \). Write \( \mathcal{C}^r = \bigoplus_n \mathcal{C}_n^r \). The restriction and induction functors form an adjoint pair \((F, E)\) with
\[
E : R^r(n+1)\text{-grmod} \to R^r(n)\text{-grmod}, \quad N \mapsto N,
\]
\[
F : R^r(n)\text{-grmod} \to R^r(n+1)\text{-grmod}, \quad M \mapsto R^r(n+1) \otimes_{R^r(n)} M.
\]

Let \( \varepsilon : FE \to 1 \) and \( \eta : 1 \to EF \) be the counit and unit of the adjoint pair \((F, E)\). They are represented respectively by the multiplication map \( \mu \) and the canonical map \( \iota \)
\[
\varepsilon : R^r(n) \otimes_{R^r(n-1)} R^r(n) \to R^r(n),
\]
\[
\eta : R^r(n) \to R^r(n+1).
\]

**Proposition 4.2.** (a) The pair \((E, F)\) is adjoint with the counit \( \hat{\varepsilon} : EF \to 1 \) and the unit \( \hat{\eta} : 1 \to FE \) represented by the morphisms \( \hat{\varepsilon} : R^r(n+1) \to R^r(n), \hat{\eta} : R^r(n) \to R^r(n) \otimes_{R^r(n-1)} R^r(n) \) such that \( \hat{\varepsilon}(z) = p_{r-1}(z) \) and \( \hbar \hat{\eta}(1) = \pi(x_{n+1}^r) \).

(b) The \( \mathbb{k} \)-algebra \( R^r(n) \) is a symmetric algebra. The symmetrizing form \( t_{r,n} : R^r(n) \to \mathbb{k} \) is the unique \( \mathbb{k} \)-linear map sending the element \( x_1^{r_1} \cdots x_w^{r_w} \) to \( 1 \) if \( r_1 = \cdots = r_n = r - 1 \) and \( w = 1 \) and to 0 otherwise. We have \( t_{r,n} = \hat{\varepsilon} \circ \cdots \circ \hat{\varepsilon} \) (\( n \) times).

**Proof.** See [25, lem. 5.7.2] for (a) and [8, thm. A.2] for (b).

We have \( \text{tr}(C^r/\mathbb{Z}) = \bigoplus_n \text{tr}(R^r(n)) \). We equip \( \text{tr}(C^r/\mathbb{Z}) \) with the \( \mathbb{Z}^2 \)-grading such that \( \text{tr}(R^r(n)) \) has the weight \( 2n \) and the order given by the degree of elements of \( R^r(n) \). For
each \( k \in \mathbb{N} \), we define \( k \)-linear operators \( x^\pm_k \) on \( \text{tr}(C^r/\mathbb{Z}) = \bigoplus_n \text{tr}(R^r(n)) \) of weights \( \pm 2 \) and order \( 2k \) such that the maps
\[
x^+_k : \text{tr}(R^r(n)) \to \text{tr}(R^r(n - 1)), \quad x^-_k : \text{tr}(R^r(n)) \to \text{tr}(R^r(n + 1)),
\]
are given, for each \( f \in \text{End}_{C^r}(a) \) and \( a \in C^r \), by
\[
x^+_k(f) = \text{tr}(x^k(a) \circ (E(f))), \quad x^-_k(f) = \text{tr}(x^k(a) \circ (F(f))).
\]
Here \( x \in \text{End}(F) \) is represented by the right multiplication by \( x_{n+1} \) on \( R^r(n + 1) \), and \( x \in \text{End}(E) \) is the left transposed endomorphism.

**Theorem 4.3.** The assignment \( C_k \mapsto p_k(y_1, \ldots, y_r) \), \( D_{\pm 1,k} \mapsto x^\pm_k \) defines a representation of level \( r \) of \( \mathcal{W} \) on \( \text{tr}(C^r/\mathbb{Z})' \).

**Proof.** First, we check that the operators \( x^-_0, x^-_1, x^-_2, x^-_3 \) satisfy the relations (c), (e) of \( D_{10}, D_{11}, D_{12}, D_{13} \) in the definition of \( \mathcal{W} \). For each \( k \in \mathbb{N} \) and \( f \in R^r(n) \) we have \( x^-_k(\text{tr}(f)) = \text{tr}(f x^k_{n+1}) \), where the element \( f \) in the right hand side is identified with its image by the map
\[
\iota : R^r(n) \to R^r(n + 1).
\]

Thus, using the relations \( \tau_{n+1} x_{n+2} \tau_{n+1} = x_{n+1} + \hbar \tau_{n+1} \) and \( \tau^2_{n+1} = 1 \) we get
\[
[x^-_1, x^-_0](\text{tr}(f)) = \text{tr}(f x_{n+2} - f x_{n+1})
= \text{tr}(f x_{n+2}^2 - f x_{n+1})
= \text{tr}(f \tau_{n+1} x_{n+2} \tau_{n+1} - f x_{n+1})
= \hbar \text{ tr}(f \tau_{n+1})
\]
Similarly, using the relations
\[
\tau_{n+1} x_{n+2}^2 \tau_{n+1} = x_{n+1}^2 + \hbar x_{n+1} x_{n+2} + \hbar x_{n+1} x_{n+2} \tau_{n+1},
\tau_{n+1} x_{n+2}^2 \tau_{n+1} = x_{n+1}^3 + \hbar (x_{n+2}^2 + x_{n+1} x_{n+2} + x_{n+1}^2) \tau_{n+1}
\]
we deduce that
\[
[x^-_2, x^-_1](\text{tr}(f)) = \hbar \text{ tr}(f x_{n+1} x_{n+2} \tau_{n+1}),
[x^-_3, x^-_0](\text{tr}(f)) = \hbar \text{ tr}(f x_{n+2} + x_{n+1} x_{n+2} + x_{n+1}^2) \tau_{n+1} + 1).
\]
The relation (e) follows from
\[
[x^-_0, [x^-_0, x^-_1]](\text{tr}(f)) = \hbar \text{ tr}(f (\tau_{n+2} - \tau_{n+1}))
= \hbar \text{ tr}(f (\tau_{n+2}^2 - \tau_{n+1}^2))
= \hbar \text{ tr}(f (\tau_{n+1} \tau_{n+2} - \tau_{n+2} \tau_{n+1})),
= 0.
\]
To prove (e) we introduce the element \( \varphi_l = (x_l - x_{l+1}) \tau_l + \hbar \). We have
\[
\varphi_l^2 = \hbar^2 - (x_l - x_{l+1})^2, \quad \varphi_l x_k = x_{s_l(k)} \varphi_l, \quad \varphi_l \tau_l = -\tau_l \varphi_l.
\]
We deduce that
\[
(3[x^+_2, x^-_1] - [x^-_3, x^-_0] + \hbar^2[x^-_1, x^-_0])(f) = \\
= \hbar \, \tr(f(3x_{n+1}x_{n+2} - x^2_{n+2} - x_{n+1}x_{n+2} - x^2_{n+1} - \hbar^2)\tau_{n+1}) \\
= \hbar \, \tr(f((x_{n+1} - x_{n+2})^2 - \hbar^2)\tau_{n+1}) \\
= -\hbar \, \tr(f\varphi^2_{n+1}\tau_{n+1}) \\
= h \, \tr(f\varphi^2_{n+1}\tau_{n+1}) \\
= 0.
\]
We prove that the operators \(x^+_0, x^+_1, x^+_2, x^+_3\) satisfy the relations (d), (e) of \(D_{-1,0}, D_{-1,1}, D_{-1,2}, \ D_{-1,3}\) in a similar way.

Next, we prove the relations (a), (b) and (f). To do so, for each \(l \geq 0\), consider the element \(p_l(x_1, \ldots, x_n) \in Z(R^r(n))\) given by \(p_l(x_1, \ldots, x_n) = \sum_{i=1}^n x^i_1\) if \(n > 0\) and \(0\) if \(n = 0\). Let \(x^0_{l+1}\) be the \(k\)-linear operator on \(tr(C^r/\mathbb{Z})\) given by
\[
(41) \quad x^0_{l+1}(f) = \hbar \, \tr(f p_l(x_1, \ldots, x_n)), \quad \forall f \in R^r(n).
\]
Then, the relation (a) is obvious. Let us concentrate on (b). We have
\[
[x^0_{l+1}, x^+_k](f) = \hbar \, \tr(f \left( x^k_{n+1}p_l(x_1, \ldots, x_{n+1}) - x^k_{n+1}p_l(x_1, \ldots, x_n) \right)) \\
= \hbar \, x^0_{l+k}(f).
\]
The relation \([x^0_{l+1}, x^+_k] = \hbar x^0_{l+k}\) is proved in a similar way.

Finally, let us prove the relation (f) in the definition of the Lie algebra \(\mathcal{W}\). Let
\[
\hat{\varepsilon} : R^r(n+1) \to R^r(n), \quad \hat{\eta} : R^r(n) \to R^r(n) \otimes_{R^r(n-1)} R^r(n)
\]
be as above. For each \(k \in \mathbb{N}\) we consider the following elements in \(Z(R^r(n))\)
\[
B^k_{\pm,n} = \hat{\varepsilon}(x^r_{n+1-k}), \\
B^k_{-n} = \begin{cases} 
-\hbar^2\mu_{x^r_{n}}(\hat{\eta}(1)) & \text{if } k \geq r + 1, \\
-p_{r-k}(x^r_{n+1}) & \text{if } 1 \leq k \leq r, \\
1 & \text{if } k = 0.
\end{cases}
\]
We may abbreviate \(B^k_{\pm} = B^k_{\pm,n}\). Consider the formal series \(B_{\pm}(z) = \sum_{k \in \mathbb{N}} B^k_{\pm} z^k\). For each \(k, l \in \mathbb{N}\) we define the operator \(E_{k,l} = [x^+_k, x^-_l]\) on \(tr(C^r/\mathbb{Z})\).

**Lemma 4.4.** The following hold
(a) \(\hat{\varepsilon}(\tau_n a \tau_n) = \hat{\varepsilon}(a)\) for each \(a \in R^r(n)\),
(b) \(B^+_{\pm}(z) B^-_{\pm}(z) = 1\),
(c) \(E_{k+l} = E_{k,l}\) depends only on \(k + l\) and we have \(E_l = \sum_{k=0}^l (r-k) B^{l-k}_{\pm} B^k_{\pm}\),
(d) \(E(z) = r - z \frac{d}{dz} \log B^-_{\pm}(z)\),
(e) \(B^-_{\pm}(z) = \prod_{k=1}^n (1 - \hbar^2 z^2/(1 - z z_k)^2) \prod_{p=1}^n (1 - z y_p)\).
Proof. For part (a), note that $a = \mu_{r_0}(\pi(a)) + \sum_{k=0}^{r-1} p_k(a) x_n^k$. Thus, a direct computation yields

$$
\tau_n a \tau_n = \mu_{r_0+r-1}(\pi(a)) - \sum_{k=0}^{r-1} p_k(a) \hbar \sum_{s+t=k-1} x_n^s \tau_n x_n^t + \sum_{k=0}^{r-1} p_k(a) x_{n+1}^{-}- \hbar^2 \sum_{k=0}^{r-1} p_k(a) \sum_{t=0}^{k-1} \sum_{a=0}^{t-1} x_n^{k-1-t+a} x_{n+1}^{t-1-a}.
$$

We deduce that $p_{r-1}(\tau_n a \tau_n) = p_{r-1}(a)$.

Next, a similar computation as in the proof of Lemma B.1 (a), (c) yields

$$
\pi(x_{n+1}^k) = \sum_{s=0}^{k-r} (\varepsilon(x_{n+1}^{k-s}) \otimes x_n^s \otimes 1 \otimes 1) \pi(x_{n+1}^s),
$$

and the equality in part (b).

The proof of part (c) is similar to Proposition B.5 (b), we briefly indicate the key steps. First, the isomorphism in Proposition 4.1 (42), (43)

$$
\pi(x_{n+1}^k) = \sum_{s=0}^{r-a-1} B_{+n-s,n}^k B_{-,n}^s,
$$

and it is equal to the sum of the trace of $\rho^{-1}(x^k x^l)$ restricted to each direct factor of $G$. The restriction of $\rho^{-1}(x^k x^l)$ to $FE1_n$ is represented by

$$
R^n(n) \otimes R^n(n-1) R^n(n) \rightarrow R^n(n) \otimes R^n(n-1) R^n(n), \quad z \mapsto \pi(x_{n+1}^k \mu_{r_0}(z) x_{n+1}^l).
$$

We have

$$
\pi(x_{n+1}^k \mu_{r_0}(z) x_{n+1}^l) = \mu_{x_{n+1}^k} \tau_n (z) + \hbar \sum_{a=0}^{k+l-1} \mu_{x_{n+1}^a} (z) \pi(x_{n+1}^{k+l-1-a})
$$

Hence the restriction of $\rho^{-1}(x^k x^l)$ to $FE1_n$ is the endomorphism

$$
x^l x^k + \hbar \sum_{a=0}^{k+l-1} \pi(x_{n+1}^{k+l-1-a}) \varepsilon(1 \otimes x_n^a),
$$

and by (42) its trace is equal to

$$
x_l x_k + \sum_{s=r+1}^{k+l} (r-s) B_{+n,s}^{l+k-s} B_{-,n}^s.
$$
The restriction of $\rho^{-1}(x^k x^l)\rho$ to the $a$-th copy of $1_n$ is represented by the map $R^r(n) \to R^r(n)$, $z \mapsto z p_a(x^{k+l+a})$. By (43) it is equal to $\sum_{s=0}^{r-1-a} B_{+}^{k+s} B_{-}^{r-s}$. We conclude that

$$x_k^+ x_l^- = x_k^- x_l^+ + \sum_{s=r+1}^{k+l} (r-s)B_{+}^{l+k-s} B_{-}^{s} + \sum_{a=0}^{r-1} \sum_{s=0}^{r-1-a} B_{+}^{k+s} B_{-}^{s}.$$ 

To prove (d), note that, using (b), we get

$$-z \frac{d}{dz} \log B_-(z) = -\left( \sum_{k \geq 0} k B_{+}^{k} z^k \right) \left( \sum_{k \geq 0} B_{+}^{k} z^k \right) = \sum_{l \geq 0} \sum_{k=0}^{l} \left( -kB_{+}^{k} B_{-}^{l-k} \right) z^l = -r + E(z).$$

Finally we concentrate on (e). We have

$$\tau_n^a x_n^k \tau_n = x_n^{k+1} - h \sum_{p+q=k-1} x_n^p \tau_n^q - h^2 \sum_{a=0}^{k-2} (a+1)x_n^a x_n^{k-2-a}.$$ 

Using (a), we deduce that

$$\hat{e}(x_n^k) = \hat{e}(\tau_n x_n^k \tau_n) = \hat{e}(x_n^{k+1}) - h^2 \sum_{a=0}^{k-2} (a+1)x_n^a \hat{e}(x_n^{k-2-a}).$$

Thus, we have

$$B_{+}^{k+r+1} B_{-}^{n-1} = B_{+}^{k+r+1} - h^2 \sum_{a=0}^{k-2} (a+1)x_n^a B_{+}^{k-2-a-r+1}.$$ 

This yields

$$B_{+}^{n-1}(z) = (1 - h^2 z^2 (1 - zx_n)^{-2}) B_{+}^{n}(z).$$

Hence by (b) we get

$$B_{-}^{n}(z) = (1 - h^2 z^2 (1 - zx_n)^{-2}) B_{-}^{n-1}(z).$$

for $n \geq 1$. By induction it remains to compute $B_{-}^{0}(z)$. Since

$$x_1^r = -\sum_{a=1}^{r} (-1)^a e_a(y_1, \ldots, y_r) x_1^{r-a},$$

we have

$$\pi(x_1^r) = 0, \quad B_{-}^{a} = (-1)^a e_a(y_1, \ldots, y_r) \forall a \in [1, r], \quad B_{-}^{a} = 0 \forall a > r.$$ 

Therefore, we have $B_{-}^{0}(z) = \prod_{a=1}^{r}(1 - zy_a)$. \hfill $\square$
We can now finish the proof of the relation (f) of \( \mathcal{W} \). According to Lemma [4.4], the formal series \( E(z) = \sum_{l \geq 0} E_l z^l \) is given by

\[
E(z) = r - z \frac{d}{dz} \log \left( \prod_{p=1}^{r} (1 - z y_p) \prod_{k=1}^{n} (1 - z (x_k - h))(1 - z (x_k + h))(1 - z x_k)^{-2} \right).
\]

Comparing this with formula (41) and the identity

\[
\sum_{k \geq 1} (za)^k = -z (\frac{d}{dz}) \log(1 - za),
\]

we deduce that

\[
E(z) = r - \sum_{k \geq 1} \sum_{p=1}^{n} \left( 2x_p^k - (x_p - h)^k - (x_p + h)^k \right) z^k + \sum_{k \geq 1} \sum_{p=1}^{r} y_p^k z^k.
\]

This implies that

\[
E(z) = r + \sum_{k \geq 1} \left( \gamma_{h}(D_{0,k+1}) + \gamma_{-h}(D_{0,k+1}) + p_k(y_1, \ldots, y_r) \right) z^k,
\]

\[
\gamma_{t}(D_{0,k+1}) = \sum_{p=1}^{k} \binom{k}{p} D_{0,k-p+1} t^{p-1}.
\]

This finishes the proof of the theorem.

Set \( Z(C_r/\mathbb{Z}) = \bigoplus_n Z(R^r(n)) \). The symmetrizing form \( t_r = \bigoplus_n t_{r,n} \) on \( \bigoplus_n R^r(n)' \) yields a non-degenerate \( k' \)-bilinear form

\[ Z(C_r/\mathbb{Z})' \times \text{tr}(C_r/\mathbb{Z})' \to k', \quad (a, b) \mapsto t_r(ab). \]

Taking the transpose with respect to this bilinear form and twisting the action by the anti-involution \( \varpi \) in (40), we get a representation of \( \mathcal{W} \) on \( Z(C_r/\mathbb{Z})' \) of level \( r \). Let \( |r\rangle \in Z(C_r/\mathbb{Z})' \) denote the unit of \( Z(R^r(0))' = k' \). We define a weight \( \Lambda_r \) of level \( r \) of \( \mathcal{W} \) by the formula

\[
\Lambda_r(C_k) = p_k(y_1, \ldots, y_r), \quad \Lambda_r(D_{0,k+1}) = 0, \quad \forall k \geq 0.
\]

**Proposition 4.5.** The following hold

(a) \( |r\rangle \) is a primitive vector of \( Z(C_r/\mathbb{Z})' \) of weight \( \Lambda_r \),

(b) \( Z(C_r/\mathbb{Z})' \) is quasifinite of character \( \prod_{j=1}^{r} (1 - q^2j)^{-r} \).

**Proof.** Part (a) follows from the formula (41), which implies that \( x_{l+1}^{0}(|r\rangle) = 0 \) for all \( l \geq 0 \). Part (b) follows from Proposition [4.4d].
4.4. The cohomology ring of the moduli space of framed instantons. Let $\mathcal{M}(r,n)$ be the moduli space of framed rank $r$ torsion free sheaves on $\mathbb{P}^2$ with fixed second Chern class $n$. Set $\mathcal{M}(r) = \bigsqcup_n \mathcal{M}(r,n)$. First, let us review a few basic facts on $\mathcal{M}(r)$. See [35, sec. 3] for more details.

The group $\text{GL}(r) \times \text{GL}(2)$ acts on $\mathcal{M}(r)$ in the obvious way: $\text{GL}(r)$ acts by changing the framing and $\text{GL}(2)$ via the tautological action on $\mathbb{P}^2$ which preserves the line at infinity. Let $T \subset \text{GL}(r)$, $A \subset \text{GL}(2)$ be the maximal tori, and let $\mathbb{C}^\times \subset A$ be the hyperbolic torus $\{ \text{diag} (t, t^{-1}); t \in \mathbb{C}^\times \}$. Set $G = T \times \mathbb{C}^\times$ and $G_A = T \times A$.

We identify the $\mathbb{Z}$-graded $k$-algebra $H^*_G(\bullet,k)$ with $k$ in the obvious way. Let $h = H^*_G(\bullet,k)$ and write $h'$ for the fraction field of $h$. From now on, let $k$ be a field of characteristic zero.

The $G_A$-variety $\mathcal{M}(r,n)$ is equivariantly formal and smooth of dimension $d_{r,n} = 2rn$. Thus $H^*_G(\mathcal{M}(r),k)$ is a free $\mathbb{Z}$-graded $k$-module isomorphic to $H^*(\mathcal{M}(r),k) \otimes k$. The $G_A$-action yields an $\alpha$-partition of $\mathcal{M}(r)$ into affine spaces in the sense of DeConcini-Lusztig-Procesi. We deduce that

$$\sum_{d,n \geq 0} \dim H^d(\mathcal{M}(r,n),k)q^{2n}t^d = \sum_{n \geq 0} \text{grdim} H^*_G(\mathcal{M}(r,n),k)q^{2n} = \prod_{p=1}^\infty \prod_{i=1}^r \left( 1 - q^{2i}t^{2(ri+p-1-r)} \right)^{-1}.$$  \hspace{1cm} (47)

In particular, note that the odd cohomology of $\mathcal{M}(r)$ vanishes, hence the $k$-algebra $H^*_G(\mathcal{M}(r),k)$ is commutative. Let $|r|$ be the unit of $H^*_G(\mathcal{M}(r,0),k) \simeq k$. First, we prove the following.

**Proposition 4.6.** (a) There is a representation of the Lie $k'$-algebra $\mathcal{W}$ on $H^*_G(\mathcal{M}(r),k)'$ which is isomorphic to $V(\Lambda_r)$. This representation is quasifinite of character $\prod_{j \geq 1} (1 - q^{2j})^{-r}$.

(b) There is a unique isomorphism $\psi : Z(C'/\mathbb{Z})' \to H^*_G(\mathcal{M}(r),k)'$ of representations of $\mathcal{W}'$ which takes the element $|r|$ to $|r|$.

(c) For each $n \in \mathbb{N}$, the element $\psi(1) \in H^*_G(\mathcal{M}(r,n),k)$ is invertible and the map $\phi' : Z(R'(n))' \to H^*_G(\mathcal{M}(r,n),k)'$ given by $\phi'(\bullet) = \psi(1)^{-1} \cup \psi(\bullet)$ is a $k'$-algebra isomorphism.

**Proof.** For each $n,k \in \mathbb{N}$, we consider the locus

$$\mathfrak{B}(r,n+k,n) \subset \mathcal{M}(r,n+k) \times \mathbb{C}^2 \times \mathcal{M}(r,n)$$

of triples $(\mathcal{E},x,\mathcal{F})$ such that $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}/\mathcal{E}$ is a length $k$ sheaf supported at $x$. For each $\gamma \in H^*_G(\mathfrak{B}(r,n+k,n),k)$ the correspondence $\mathfrak{B}(r,n+k,n)$ defines two maps

$$\Theta_+^{\gamma} : H^*_G(\mathcal{M}(r,n),k) \to H^*_G(\mathcal{M}(r,n+k),k),$$

$$\Theta_-^{\gamma} : H^*_G(\mathcal{M}(r,n+k),k) \to H^*_G(\mathcal{M}(r,n),k).$$

The map $\Theta_-$ uses localized equivariant cohomology, because the projection $\mathfrak{B}(r,n+k,n) \to \mathcal{M}(r,n)$ is not proper.

Let $\tau_{n+k,n}$, $\tau_n$ be the tautological bundles on $\mathcal{M}(r,n+k) \times \mathcal{M}(r,n)$, $\mathcal{M}(r,n)$. Write $c_i$ for the $i$-th equivariant Chern class. The obvious map

$$H^*_G(\mathcal{M}(r,n+k) \times \mathcal{M}(r,n),k) \times H^*_G(\mathfrak{B}(r,n+k,n),k) \to H^*_G(\mathfrak{B}(r,n+k,n),k)$$

is denoted by $(a,b) \mapsto a \otimes b$. 
Now, we define the action of $C_k$, $D_{1,k}$, $D_{-1,k}$, $D_{0,k+1}$ on an element of $H^*_{G_A} (\mathcal{M}(r, n), k)'$ by

\begin{align}
D_{0,1} &= \hbar n, \\
C_k &= p_k(y_1, \ldots, y_r), \\
D_{1,k} &= -\hbar^2 \Theta_+(c_1(\tau_{n+1,n})^k \otimes 1), \\
D_{-1,k} &= (-1)^{r-1} \Theta_-(c_1(\tau_{n+1,n})^k \otimes 1), \\
\sum_{k \geq 0} D_{0,k+2} z^k &= -\hbar (d/dz) \log \left( 1 + \sum_{k \geq 1} c_k(\tau_n)(-z)^k \right) \cup \bullet.
\end{align}

(48)

The operators $D_{-1,k}$, $D_{1,k}$, $D_{0,k+1}$ in above are equal to the operators $\hbar^k D_{-1,k}$, $\hbar^k D_{1,k}$, $\hbar^{k+1} D_{0,k+1}$ in [38, (3.17)] respectively. The reason for this normalization by powers of $\hbar$ is to give the term $\hbar$ in the relations (b), (c), (d) of $\mathcal{W}$ in Section 4.2 which does not appear in the corresponding relations in [2].

By [38, cor. 3.3] and [2], the formulas (48) yield a representation of $h' \otimes_k \mathcal{W}$ on

$$H^*_{G_A} (\mathcal{M}(r), k)' = h' \otimes_{\hbar} H^*_{G_A} (\mathcal{M}(r), k).$$

Note that [38] uses equivariant homology rather than equivariant cohomology, but since $\mathcal{M}(r)$ is smooth its equivariant homology and cohomology are isomorphic by Poincaré duality. We must check that the formulas (48) give indeed a representation of $\mathcal{W}$ on $H^*_{G_A} (\mathcal{M}(r), k)'$.

The representation of $h' \otimes_k \mathcal{W}$ in [38] depends on parameters $y_1, \ldots, y_r, x, y$ which are generators of the field extension $h'$ of $k$. Note that $y_p$ is denoted by the symbol $e_p$ in [38]. The representation of $\mathcal{W}$ we consider here is a specialization along the hyperplane $x = -y = \hbar$ of some integral form of the representation in [38]. We must check that the representation in [38] specializes effectively.

To prove this, note that the main results of [38] are obtained by explicit computations in the $h'$-basis of $H^*_{G_A} (\mathcal{M}(r), k)'$ formed by the fundamental classes of the fixed points of $\mathcal{M}(r)$ under the action of the torus $G_A$. For these computations it is essential that the fixed points are isolated. Now, it is well-known, and easy to prove, that the fixed points sets $\mathcal{M}(r)^G$ and $\mathcal{M}(r)^{G_A}$ are the same. Indeed, one can easily check that the explicit formulas for the representation of $h' \otimes_k \mathcal{W}$ in [38, cor. 3.3] in the basis of $H^*_{G_A} (\mathcal{M}(r), k)'$ have no poles along the hyperplane $x = y$. This follows from the formulas [38, (3.17), (D.1)-(D.3)].

Note however that the series $E(z)$ in (39) differs from the corresponding one in [38] (1.70)]. We must check that these formulas are compatible. To do that, let us review quickly the proof in [38, p. 326-327]. Our setting differs from [38] because in loc. cit. we assumed that both parameters $x, y$ are generic, while here we have $x = h = -y$ with $\hbar$ generic.

Let $\{a_i; i \in I\}$ and $\{b_j; j \in J\}$ be as in [38, p. 327]. Then, the computation there implies that the element $[D_{-1,k}, D_{1,l}] = E_{k+l}$ depends only on $k + l$ and yields the following identity

$$E(z) = \sum_{k \geq 0} \left( \sum_{i \in I} a_i - \sum_{j \in J} b_j \right) z^k.$$
Fix some formal variables \( x_1, x_2, \ldots, x_n \) such that \( c_i(\tau_n) = e_i(x_1, x_2, \ldots, x_n) \) for each \( i \in [1, n] \). Then, the same argument as in [38, p. 327] using [38, lem. D.1] implies that

\[
E(z) = r - \sum_{k \geq 1} \sum_{p=1}^{n} \left( 2x_p^k - (x_p - h)^k - (x_p + h)^k \right) z^k + \sum_{k \geq 1} \sum_{p=1}^{r} y_p^k z^k.
\]

We deduce that, see (45),

\[
E(z) = r + \sum_{k \geq 1} \left( \gamma_h(D_{0,k+1}) + \gamma_{-h}(D_{0,k+1}) + p_k(y_1, \ldots, y_r) \right) z^k,
\]

(49)

\[
\gamma_t(D_{0,k+1}) = \sum_{p=1}^{k} (\frac{k}{p}) D_{0,k-p+1} t^{p-1}.
\]

We have proved that the formulas in (48) define a representation of \( \mathcal{W} \) on \( H^*_G(\mathcal{M}(r), k)' \). Now, we must check that this representation is irreducible and is isomorphic to \( V(\Lambda_r) \).

The irreducibility follows from the main result of [38]. More precisely, it is proved in [38, thm. 8.33] that the representation of \( \mathcal{W} \) on \( H^*_{G_A}(\mathcal{M}(r), k)' \) gives rise to a representation of the \( W \)-algebra of the affine Kac-Moody algebra \( \hat{\mathfrak{gl}}_r \) on \( H^*_{G_A}(\mathcal{M}(r), k)' \). See, e.g., [1] and [15] for some background on \( W(\hat{\mathfrak{gl}}_r) \). It is also proved there that the \( W(\hat{\mathfrak{gl}}_r) \)-module \( H^*_{G_A}(\mathcal{M}(r), k)' \) is isomorphic to the Verma module with highest weight and level given respectively by

\[
a/x - \rho(1 + y/x) \quad \text{and} \quad -y/x - r.
\]

Here we have set \( a = (y_1, y_2, \ldots, y_r) \) and \( \rho = (0, -1, \ldots, 1 - r) \). Then, the irreducibility of \( H^*_{G_A}(\mathcal{M}(r), k)' \) as a \( W(\hat{\mathfrak{gl}}_r) \)-module is well-known, because a Verma module with a generic highest weight is irreducible. The same argument proves that \( H^*_G(\mathcal{M}(r), k)' \) is irreducible as a \( W(\hat{\mathfrak{gl}}_r) \)-module. To prove that it is also irreducible as a \( \mathcal{W} \)-module use [38, thm. 8.22] as in [38, cor. 8.29].

Next, we must identify the representation of \( \mathcal{W} \) on \( H^*_G(\mathcal{M}(r), k)' \) with \( V(\Lambda_r) \). To do that, it is enough to prove that the element \( |r \rangle \) of \( H^*_G(\mathcal{M}(r), k)' \) is primitive of weight \( \Lambda_r \). The equality [38, (3.9)] yields

\[
c_k(\tau_n) \cup |r \rangle = 0, \quad \forall k \geq 1.
\]

Thus, from (48) we deduce that \( D_{0,k+1}(|r \rangle) = 0 \) for each \( k \geq 0 \).

Finally, we must check the character formula in (a). It is well-known, and follows easily by counting the (isolated) fixed points in \( \mathcal{M}(r,n) \). This finishes the proof of (a).

Now, let us concentrate on (b). Since the element \( |r \rangle \) of \( Z(C^*/\mathbb{Z})' \) is primitive of weight \( \Lambda_r \) and since \( M(\Lambda_r) \) has a simple top isomorphic to \( V(\Lambda_r) \), there is a unique surjective \( \mathcal{W} \)-module homomorphism from the submodule \( M \subseteq Z(C^*/\mathbb{Z})' \) generated by \( |r \rangle \) to \( H^*_G(\mathcal{M}(r), k)' \) such that \( |r \rangle \mapsto |r \rangle \). Since \( H^*_G(\mathcal{M}(r), k)' \) and \( Z(C^*/\mathbb{Z})' \) have the same character, we deduce that

\[
Z(C^*/\mathbb{Z})' = M = H^*_G(\mathcal{M}(r), k)'.
\]

Let \( \psi \) be the unique \( \mathcal{W} \)-module isomorphism

\[
\psi : Z(C^*/\mathbb{Z})' \to H^*_G(\mathcal{M}(r), k)', \quad |r \rangle \mapsto |r \rangle.
\]
Finally, let us prove part (c). By restriction, the map $\psi$ yields a $k'$-linear isomorphism

$$\psi : Z(R(r)'; n) \rightarrow H^*_G(M(r, n), k')$$

for each $n \in \mathbb{N}$. We must prove that $\psi(1)$ is invertible in $H^*_G(M(r, n), k')$ and the map

$$\phi' : Z(R(r)'; n) \rightarrow H^*_G(M(r, n), k'), \quad \phi'(\bullet) = \psi(1)^{-1} \cup \psi(\bullet)$$

is a $k'$-algebra isomorphism. To do so, we consider the diagram

$$\begin{array}{ccc}
Z(R(r); n) & & Z(R(r); n') \\
\downarrow a' & & \downarrow b' \\
H^*_G(M(r, n), k') & \rightarrow & Z(R(r); n')
\end{array}$$

The map $b'$ is the $k'$-algebra homomorphism induced by the canonical map $R(n) \rightarrow R(r)$. It is surjective by Proposition 4.1. The map $a'$ is the $k'$-algebra homomorphism given by

$$a'(e_i(x_1, \ldots, x_n)) = c_i(\tau_n), \quad \forall i \in [1, n].$$

Note that by definition of the representation of $W$ on $H^*_G(M(r, n), k')$, the formula (48) yields

$$h^{-1} D_{0,k+1} = a'(p_k(x_1, \ldots, x_n)) \cup \bullet \text{ on } H^*_G(M(r, n), k').$$

Next, by definition of the representation of $W$ on $Z(C'/\mathbb{Z})'$, the formula (11) in the proof of Theorem 4.3 yields

$$h^{-1} D_{0,k+1} = b'(p_k(x_1, \ldots, x_n)) \cdot \bullet \text{ on } Z(R(r); n').$$

From (51), (52), since $\psi$ is $W$-linear, we deduce that

$$\psi(b'(p_k(x_1, \ldots, x_n)) \cdot \bullet) = a'(p_k(x_1, \ldots, x_n)) \cup \psi(\bullet).$$

Now, an easy induction using (53) yields

$$\psi b'(z) = a'(z) \cup \psi(1), \quad \forall z \in Z(R(r); n').$$

We also deduce that

$$\psi(z z') \cup \psi(1) = \psi(z) \cup \psi(z'), \quad \forall z, z' \in Z(R(r); n').$$

Now, since $b'$ and $\psi$ are surjective, the equality (54) implies that the element $\psi(1)$ is invertible in the (commutative) $k'$-algebra $H^*_G(M(r, n), k')$. Thus, the map $\phi'$ above is well-defined and it is a $k'$-algebra homomorphism by (55). It is clearly bijective because it is injective and both sides are finite dimensional of the same dimension over $k'$. Further, we have a commutative diagram

$$\begin{array}{ccc}
Z(R(r); n) & & Z(R(r); n') \\
\downarrow a' & & \downarrow b' \\
H^*_G(M(r, n), k') & \rightarrow & Z(R(r); n')
\end{array}$$

Part (c) of the proposition is proved.

We can now prove the following, which is one of the main results of this paper.
Theorem 4.7. The canonical map \( Z(R(n)) \to H_G^*(\mathcal{M}(r,n),k) \) is a surjective \( k \)-algebra homomorphism. It factors to a \( k \)-algebra isomorphism \( Z(R^r(n))^{JM} \to H_G^*(\mathcal{M}(r,n),k) \).

Proof. We define the maps \( a : Z(R(n)) \to H_G^*(\mathcal{M}(r,n),k) \) and \( b : Z(R(n)) \to Z(R^r(n)) \) as in the triangle (57) above. Thus, we have \( a' = k' \otimes a \) and \( b' = k' \otimes b \). We claim that there is a \( k \)-linear map \( \phi \) making the following triangle to commute

\[
\begin{array}{ccc}
Z(R(n)) & \xrightarrow{a} & H_G^*(\mathcal{M}(r,n),k) \\
& & \phi \\
& & Z(R^r(n))^{JM} \\
& \xleftarrow{b} &
\end{array}
\]

(57)

To prove this, it is enough to check that \( \text{Ker}(b) \subset \text{Ker}(a) \). Since the triangle (57) commutes, we have \( \text{Ker}(b') \subset \text{Ker}(a') \). Thus, since \( Z(R(n)) \) is free as a \( k \)-module, the map \( x \mapsto 1 \otimes x \) yields an inclusion

\[
\text{Ker}(b) \subset (1 \otimes Z(R(n))) \cap \text{Ker}(a').
\]

Finally, since \( Z(R(n)) \) is free as a \( k \)-module, the map \( x \mapsto 1 \otimes x \) yields an isomorphism

\[
\text{Ker}(a) \simeq 1 \otimes \text{Ker}(a) \\
\simeq (1 \otimes Z(R(n))) \cap \text{Ker}(a').
\]

The claim is proved. Note that, since the map \( b' \) is surjective and the triangles (56), (57) commute, we have \( \phi' = k' \otimes \phi \). Thus, since \( \phi' \) is injective, we deduce that \( \phi \) is also injective.

Now, recall that Proposition (4.1) and (4.7) yield

\[
\begin{align*}
\sum_{n \geq 0} \text{grdim}(ZR^r(n))^{JM} q^{2n} &= \sum_{n \geq 0} \text{grdim} H_G^*(\mathcal{M}(r,n),k) q^{2n}.
\end{align*}
\]

We deduce that the map \( \phi \) is an isomorphism \( Z(R^r)^{JM} \to H_G^*(\mathcal{M}(r,n),k) \).

The theorem above can be reformulated in the following way. Set \( k_1 = k[y_1, \ldots, y_r] \) and consider the \( k_1 \)-algebras

\[
R(n)_1 = R(n)/(\hbar - 1), \quad R^r(n)_1 = R^r(n)/(\hbar - 1).
\]

Recall the inclusion \( Z(R^r(n))^{JM} \subset R^r(n)_1[\hbar] \) in (38). By (7) the canonical map \( R(n)_1 \to R^r(n)_1 \) yields a surjection \( Z(R(n)_1) \to Z(R^r(n)_1) \). Since \( Z(R(n)_1) \) is \( \mathbb{N} \)-graded, this yields an increasing separated and exhaustive \( \mathbb{N} \)-filtration \( F_* \) of \( Z(R^r(n)_1) \). Let \( \text{Rees}(Z(R^r(n)_1)) \) be the corresponding Rees algebra, i.e.,

\[
\text{Rees}(Z(R^r(n)_1)) = \sum_{d \geq 0} F_{2d}(Z(R^r(n)_1)) \otimes \hbar^d \subset Z(R^r(n)_1)[\hbar].
\]

By construction, the map (38) identifies the \( k \)-algebras \( Z(R^r(n))^{JM} \) and \( \text{Rees}(Z(R^r(n)_1)) \). We deduce the following

Corollary 4.8. There is a \( k \)-algebra isomorphism \( \text{Rees}(Z(R^r(n)_1)) \simeq H_G^*(\mathcal{M}(r,n),k) \).
Remark 4.9. In the particular case $r = 1$ the corollary was already known and follows from [39].
Appendix A. The symmetrizing form

Fix a dominant weight $\Lambda \in P_+$. Let $\alpha \in Q_+$ and $i, j \in I$. Set $\lambda = \Lambda - \alpha$ and $\lambda_i = \langle \alpha_i^\vee, \lambda \rangle$.

A.1. Bubbles. Assume that $\alpha$ has the height $n$.

Definition A.1. For each $k \in \mathbb{N}$ the bubble $B^k_{\pm i, \lambda}$ is the element of $R^\Lambda(\alpha)$ given by

- if $\lambda_i \geq 0$ we set

$$B^k_{\pm i, \lambda} = \begin{cases} 
\varepsilon'_{i, \lambda}(x_{n+1}^{\lambda_i-1+k} e(\alpha, i)) & \text{if } k \geq -\lambda_i + 1, \\
1 & \text{if } k = \lambda_i = 0,
\end{cases}$$

$$B^k_{-i, \lambda} = \begin{cases} 
\mu_{x_n^{\lambda_i-1+k}}(\tilde{\eta}_{i, \lambda}(1)) & \text{if } k \geq \lambda_i + 1, \\
-p_{\lambda_i-k}(x_{n+1}^{\lambda_i} e(\alpha, i)) & \text{if } 1 \leq k \leq \lambda_i, \\
1 & \text{if } k = 0,
\end{cases}$$

- if $\lambda_i \leq 0$ we set

$$B^k_{i, \lambda} = \begin{cases} 
\varepsilon'_{i, \lambda}(x_{n+1}^{\lambda_i-1+k} e(\alpha, i)) & \text{if } k \geq -\lambda_i + 1, \\
-p_{x_n^{\lambda_i}}(\tilde{\pi}_{-\lambda_i-k}) & \text{if } 1 \leq k \leq -\lambda_i, \\
1 & \text{if } k = 0,
\end{cases}$$

$$B^k_{-i, \lambda} = \begin{cases} 
\mu_{x_n^{\lambda_i-1+k}}(\tilde{\eta}_{i, \lambda}(1)) & \text{if } k \geq \lambda_i + 1, \\
1 & \text{if } k = \lambda_i = 0.
\end{cases}$$

Note that $B^0_{\pm i, \lambda} = 1$ in all cases. We set by convention $B^k_{\pm i, \lambda} = 0$ if $k < 0$.

Lemma A.2. The elements $B^k_{\pm i, \lambda}$ are homogenous central element in $R^\Lambda(\alpha)$ of degree $2k$.

Proof. The central and homogenous property follows from the fact that $\varepsilon'_i, p_k, \eta'_i$ are homogenous $R^\Lambda(\alpha)$-bilinear morphisms and that the element $x_{n+1}$ centralizes $R^\Lambda(\alpha)$ in $R^\Lambda(\alpha + \alpha_i)$. The degree is given by an explicit computation.

\hfill $\square$

Remark A.3. In Khovanov-Lauda’s diagrammatic categorification, the element $B^k_{+i, \lambda}$ corresponds to a clockwise bubble with a dot of multiplicity $\lambda_i - 1 + k$ and $B^k_{-i, \lambda}$ corresponds to a clockwise bubble with a dot of multiplicity $-\lambda_i - 1 + k$.

A.2. A useful lemma. Assume that $\alpha$ has the height $n - 1$. Let $\lambda' = \lambda - \alpha_j$ and $\lambda'_i = \langle \alpha_i^\vee, \lambda' \rangle = \lambda_i - a_{ij}$. Consider the morphisms

\[
\begin{array}{llllllllll}
\Xi_{i,j,\lambda} : E'_i F'_i 1_\lambda & \xrightarrow{E'_i \eta'_i F'_i} & E'_i E'_j F'_j F'_i 1_\lambda & \xrightarrow{\tau\tau} & E'_j E'_i F'_i F'_j 1_\lambda & \xrightarrow{E'_j E'_i \tilde{\eta}_{i, \lambda'} F'_i} & E'_j F'_j 1_\lambda, \\
\iota_{i,j,\lambda} : E'_i F'_i 1_\lambda & \xrightarrow{\iota'_{i, \lambda}} & 1_\lambda & \xrightarrow{\eta'_{i, \lambda}} & E'_j F'_j 1_\lambda.
\end{array}
\]
The morphism $X_{i,j,\lambda}$ is represented by the composition
\[
e(\alpha,i)R^A(\alpha + \alpha_i)e(\alpha,i) \xrightarrow{\iota_j} e(\alpha,ij)R^A(\alpha + \alpha_i + \alpha_j)e(\alpha,ij) \xrightarrow{\tau_n(\bullet)\tau_n} e(\alpha,j)R^A(\alpha + \alpha_j)e(\alpha,j),
\]
and $\mathbb{I}_{i,j,\lambda}$ is represented by
\[
e(\alpha,i)R^A(\alpha + \alpha_i)e(\alpha,i) \xrightarrow{\epsilon_{i,j}'\lambda'} R^A(\alpha) \xrightarrow{\iota_j} e(\alpha,i)R^A(\alpha + \alpha_i)e(\alpha,i).
\]
In other words, given $a \in e(\alpha,i)R^A(\alpha + \alpha_i)e(\alpha,i)$ we have
\[
X_{i,j,\lambda}(a) = \epsilon_{i,j}'\lambda'\tau_n(\iota_j a)\tau_n, \quad \mathbb{I}_{i,j,\lambda}(a) = \iota_j\epsilon_{i,j}'\lambda(a).
\]
Note that, since $\iota_j : R^A(\beta) \to R^A(\beta + \alpha_j)$ is the canonical embedding for any $\beta \in Q_+$, we write $\iota_j(b) = be(\beta,j)$ or simply $\iota_j(b) = b$ for any $b \in R^A(\beta)$. Note also that, since $B_{\pm,i,\lambda}^k \in Z(R^A(a))$, it can be viewed as an element in $\text{End}(1_\lambda)$. Thus $x^tB_{\pm,i,\lambda}^kx^t$ defines an endomorphism of $E^t_1F^t_i = E^t_iF^t_i'$ for each $r, s, t \in \mathbb{N}$.

**Lemma A.4.** The following hold
1. If $i \neq j$ then $X_{i,j,\lambda} = c_{i,j,-a_{ij},0} \mathbb{I}_{i,j,\lambda}$,
2. $X_{i,i,\lambda} = -\mathbb{I}_{i,i,\lambda} + \sum g_1 + g_2 + g_3 = -\lambda_i' - 1 x^g_1 B_{\pm,i,\lambda}^{g_2} x^{g_3}$.

Note that if $\lambda_i' \geq 0$ the sum over $g_1, g_2, g_3$ is empty, hence $X_{i,i,\lambda} = -\mathbb{I}_{i,i,\lambda}$.

**Proof.** Let us prove part (a). First, assume $\lambda_i > 0$. Then
\[
a = \mu_{\tau_n-1}(\pi(a)) + \sum_{k=0}^{\lambda_i-1} p_k(a)x^k_n
\]
with $\pi(a) \in R^A(\alpha)e(\alpha - \alpha_i,i) \otimes R^A(\alpha - \alpha_i) e(\alpha - \alpha_i,i)R^A(\alpha)$ and $p_k(a) \in R^A(\alpha)$. We have
\[
\tau_n(\iota_j a)\tau_n = \mu_{\tau_n-1}e(\alpha - \alpha_i,ij)\pi(a) + \sum_{k=0}^{\lambda_i-1} p_k(a)\tau_n x^k_n e(\alpha,ij)\tau_n.
\]
The relations (d), (e), (f) in the definition of QHA yields
\[
\mu_{\tau_n-1}e(\alpha - \alpha_i,ij)\pi(a) = \mu_{\tau_n-1}e(\alpha - \alpha_i,ij)\pi(a)
\]
\[
\tau_n x^k_n e(\alpha,ij)\tau_n = x^k_{n+1}\tau_n^2 e(\alpha,ji) = x^k_{n+1}Q_{ji}(x_n, x_{n+1})e(\alpha,ji).
\]
Since $\lambda_i' = \lambda_i - a_{ij} \geq \lambda_i > 0$, we have $X_{i,j,\lambda}(a) = p_{\lambda_i'}(\tau_n a)\tau_n$, the coefficient of $x^k_{n+1}$ is at most $-a_{ij} - 1$, which is less than $\lambda_i' - 1 = \lambda_i - a_{ij} - 1$, and since $p_{\lambda_i'}(\mu_{\tau_n-1}e(\alpha - \alpha_i,ij)\pi(a)) = 0$, we deduce $p_{\lambda_i'}(\mu_{\tau_n-1}e(\alpha - \alpha_i,ij)\pi(a))$. 

Next, the degree of \( x_{n+1} \) in \( x_{n+1}^{k} Q_{i,j}(x_{n}, x_{n+1}) = x_{n+1}^{k} Q_{ij}(x_{n+1}, x_{n}) \) is less or equal to \( k - a_{ij} \) with the coefficient of \( x_{n+1}^{k-a_{ij}} \) given by \( c_{i,j,-a_{ij},0} \), therefore

\[
p_{\lambda_{i}^{'}, 1} \left( \sum_{k=0}^{\lambda_{i} - 1} p_{k}(a) \tau_{n} x_{n}^{k} e(\alpha, ij) \tau_{n} \right) = t_{j}(p_{\lambda_{i} - 1}(a)) c_{i,j,-a_{ij},0} = c_{i,j,-a_{ij},0} (t_{j} \circ \hat{\varepsilon}_{i}(a)).
\]

It follows that \( X_{i,j,\lambda} = c_{i,j,-a_{ij},0} \Pi_{i,j,\lambda} \).

Now, we consider the case \( \lambda_{i} \leq 0 \). Let \( \tilde{a} \in R^{\lambda}(a)e(\alpha - \alpha_{i}, i) \otimes R^{\lambda}(a_{ii}) e(\alpha - \alpha_{i}, i) R^{\lambda}(a) \) such that \( \mu_{\tau_{i}^{-1}}(\tilde{a}) = a, \mu_{x_{n}^{k}}(\tilde{a}) = 0 \) for \( k \in [0, -\lambda_{i} - 1] \). We have

\[
\tau_{n} t_{j}(a) \tau_{n} = \mu_{\tau_{n}^{-1} \tau_{n} e(\alpha - \alpha_{i}, ij)}(\tilde{a})
\]

\[
= \mu_{\tau_{n}^{-1} \tau_{n} + \frac{Q_{i,j}(x_{n-1, x_{n}}) - Q_{i,j}(x_{n+1, x_{n}})}{x_{n-1, x_{n+1}}}} e(\alpha - \alpha_{i}, ij)(\tilde{a}).
\]

Assume that \( \lambda_{i}, \lambda_{i}' \leq 0 \). Then \( X_{i,j,\lambda}(a) = \mu_{x_{n}^{k}}(\tau_{n} t_{j}(a) \tau_{n}) \). We claim that

\[
(58) \quad \tau_{n} t_{j}(a) \tau_{n} = (1 \otimes \tau_{n}^{-1} \otimes \tau_{n}^{-1} \otimes 1)(t_{j} \otimes t_{j}) \tilde{a}.
\]

Denote the right hand side by \( b \). Concretely write \( \tilde{a} = \sum_{r} \tilde{a}_{r}^{\prime} \otimes \tilde{a}_{r}^{\prime\prime} \) with \( \tilde{a}_{r}^{\prime} \in R^{\lambda}(a)e(\alpha - \alpha_{i}, i) \), \( \tilde{a}_{r}^{\prime\prime} \in (\alpha - \alpha_{i}, i) R^{\lambda}(a) \), then \( b \in \sum_{r} \tilde{a}_{r}^{\prime} e(\alpha - \alpha_{i}, ij) \tau_{n}^{-1} \otimes \tau_{n}^{-1} e(\alpha - \alpha_{i}, ij) \tilde{a}_{r}^{\prime\prime} \).

To prove (58), note that the degree of \( x_{n-1} \) in \( \frac{Q_{i,j}(x_{n-1, x_{n}}) - Q_{i,j}(x_{n+1, x_{n}})}{x_{n-1, x_{n+1}}} e(\alpha - \alpha_{i}, ij) \) is less than or equal to \( -a_{ij} - 1 \leq -\lambda_{i} - 1 \), hence

\[
\frac{Q_{i,j}(x_{n-1, x_{n}}) - Q_{i,j}(x_{n+1, x_{n}})}{x_{n-1, x_{n+1}}} e(\alpha - \alpha_{i}, ij) \tilde{a} = 0.
\]

We deduce

\[
\mu_{\tau_{n}}(b) = \mu_{\tau_{n}^{-1} \tau_{n} e(\alpha - \alpha_{i}, ij)}(\tilde{a}) = \tau_{n} t_{j}(a) \tau_{n},
\]

\[
\mu_{x_{n}^{k}}(b) = \mu_{x_{n}^{k} \tau_{n}^{-1} \tau_{n} e(\alpha - \alpha_{i}, ij)}(\tilde{a}) = \mu_{x_{n}^{k} Q_{i,j}(x_{n-1, x_{n}} e(\alpha - \alpha_{i}, ij)}(\tilde{a}).
\]

Next, using the fact \( \mu_{x_{n}^{k}}(\tilde{a}) = 0 \) for \( k \in [0, -\lambda_{i} - 1] \), \( \varepsilon_{\lambda_{i},\lambda}(a) = \mu_{x_{n}^{-\lambda_{i}}}(\tilde{a}) \) and the degree of \( x_{n-1} \) in \( Q_{i,j}(x_{n-1, x_{n}}) \) is at most \( -a_{ij} \) with the coefficient of \( x_{n-1}^{-a_{ij}} \) equals \( c_{i,j,-a_{ij},0} \), we get

\[
\mu_{x_{n}^{-\lambda_{i}} Q_{i,j}(x_{n-1, x_{n}} e(\alpha - \alpha_{i}, ij)}(\tilde{a}) = c_{i,j,-a_{ij},0} t_{j} \frac{Q_{i,j}(x_{n-1, x_{n}}) - Q_{i,j}(x_{n+1, x_{n}})}{x_{n-1, x_{n+1}}} e(\alpha - \alpha_{i}, ij) \tilde{a} = 0 \text{ if } k \in [0, -\lambda_{i} - 1 + a_{ij}].
\]

Formula (58) follows and we have \( X_{i,j,\lambda}(a) = \mu_{x_{n}^{k}}(b) = c_{i,j,-a_{ij},0} \Pi_{i,j,\lambda}(a) \).

Now, assume that \( \lambda_{i} \leq 0 \) and \( \lambda_{i}' > 0 \). Then \( X_{i,j,\lambda}(a) = p_{\lambda_{i}' - 1}(\tau_{n} t_{j}(a) \tau_{n}) \). We have

\[
p_{\lambda_{i}' - 1}(\mu_{\tau_{n}^{-1} \tau_{n} e(\alpha - \alpha_{i}, ij)}(\tilde{a}) = 0.\]

Recall \( Q_{i,j}(u, v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^{p} v^{q} \), with \( p \leq -a_{ij} \),

\[
\frac{Q_{i,j}(x_{n-1, x_{n}}) - Q_{i,j}(x_{n+1, x_{n}})}{x_{n-1, x_{n+1}}} = \sum_{p,q \geq 0} c_{i,j,p,q} x_{n-1}^{p} - x_{n+1}^{p} x_{n}^{q} x_{n-1}^{-q} x_{n+1}^{-q} = \sum_{p,q \geq 0} c_{i,j,p,q} \sum_{r_{1}+r_{2}=p-1} x_{n-1}^{r_{1}} x_{n+1}^{r_{2}} x_{n}^{q}.
\]
The height of $\alpha$ is $n - 1$, hence $x_n, x_{n+1}$ centralize $R^\lambda(\alpha)$. We deduce
\[
\sum_{p,q \geq 0} c_{i,j,p,q} \left( \sum_{r_1 + r_2 = p-1} \mu_{x_{n-1}^r}(\tilde{a}) x_{n+1}^q e(\alpha, ji) \right).
\]
Now $\mu_{x_{n-1}^r}(\tilde{a}) \neq 0$ only if $r_1 > -\lambda_i$, and $p_{x_{n-1}^r}(\mu_{x_{n-1}^r}(\tilde{a}) x_{n+1}^q) \neq 0$ only if $r_2 > \lambda_i' - 1 = \lambda_i - a_{ij} - 1$. Hence $r_1 + r_2 = p - 1 \leq -a_{ij} - 1$ implies that
\[
p_{x_{n-1}^r}(\mu_{x_{n-1}^r}(\tilde{a}) x_{n+1}^q) e(\alpha, ji) = c_{i,j,-a_{ij},0} \mu_{x_{n-1}^r}(\tilde{a}) e(\alpha, j).
\]
We get $X_{i,j,i}(a) = c_{i,j,-a_{ij},0} \pi_{i,j,i}(a).$

Now, we concentrate on part (b) in the case $\lambda_i' \geq 0$. Since $\lambda_i' = \lambda_i - 2$, we have $\lambda_i \geq 2$. Thus
\[
a = \mu_{\tau_{n-1}}(\pi(a)) + \lambda_i - 1 \sum_{k=0} p_k(a) x_n^k
\]
with $\pi(a) \in R^\lambda(\alpha)e(\alpha - \alpha_i, i) \otimes_{R^\lambda(\alpha - \alpha_i)} e(\alpha - \alpha_i, i) R^\lambda(\alpha)$ and $p_k(a) \in R^\lambda(\alpha)$. We have
\[
\tau_n^{i_1}(\tau_n) = \mu_{\tau_{n-1}\tau_n}(e(\alpha - \alpha_i, i^3)) (\tau_n) e(\alpha, i^2),
\]
\[
X_{i,j,i}(a) = p_{x_{n-1}^r}(\tau_n^{i_1}(\tau_n) e(\alpha, i^2))
\]
Since $Q_{ij} = 0$, the relation (f) in the definition of QHA yields
\[
\mu_{\tau_{n-1}\tau_n}(e(\alpha - \alpha_i, i^3)) (\tau_n e(\alpha, i^2)) = \mu_{\tau_{n-1}\tau_n}(e(\alpha - \alpha_i, i^3)) (\pi(a)),
\]
Hence it is killed by $p_{x_{n-1}^r}$ when $\lambda_i' > 0$. The relation (d), (e) for QHA implies
\[
\tau_n x_n^k \tau_n e(\alpha, i^2) = x_n^{k+1} \tau_n e(\alpha, i^2) - \frac{x_n^{k+1} - x_n^k}{x_n - x_n^{k+1}} \tau_n e(\alpha, i^2)
\]
\[
= - \sum_{r_1 + r_2 = k-1} x_n^{r_1} x_n^{r_2} \tau_n e(\alpha, i^2)
\]
\[
(60)
\]
If $\lambda_i' > 0$, we deduce that
\[
p_{x_{n-1}^r}(\tau_n x_n^k \tau_n e(\alpha, i^2)) = -p_{\lambda_i-3} \left( \sum_{g_1+g_2 = k-2} (g_1+1)x_n^{g_1} x_n^{g_2} \right) = \begin{cases} 0 & \text{if } k < \lambda_i - 1, \\ -1 & \text{if } k = \lambda_i - 1. \end{cases}
\]
By consequence $X_{i,j,i}(a) = -p_{\lambda_i-1}(e(\alpha, i) = -\tau_i \circ \bar{\tau}_{i} \circ \bar{\tau}_{i}(a) = -\pi_{i,j,i}(a)$.

If $\lambda_i' = 0$, we deduce that
\[
\tau_n^{i_1}(\tau_n) = \mu_{\tau_{n-1}\tau_n}(e(\alpha - \alpha_i, i^3)) (\tau_n e(\alpha, i^2)) - p_1(a) \tau_n e(\alpha, i^2)
\]
\[
= \mu_{\tau_n} \left( (1 \otimes_{\tau_{n-1} \otimes_{\tau_{n-1}}} (\tau_i \otimes \tau_i)) e(\alpha, i^2) - \tau_i(p_1(a)) \otimes e(\alpha, i^2) \right).
\]
Hence
\[
\varepsilon'_{i,\lambda'}(\tau_n\ell_i(a)\tau_n) = \mu_1 \left( (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\ell_i \otimes \ell_i)\pi(a) - \ell_i(p_1(a)) \otimes e(\alpha, i^2) \right) \\
= -\ell_i(p_1(a)).
\]
Here in the second equality we have used the fact $\tau_{n-1}^2 e(\alpha - \alpha_i, i^3) = 0$. Since $\lambda_i = 2$, we have $\varepsilon'_{i,\lambda}(a) = p_1(a)$. So we get again $\mathbb{X}_{i,\lambda}(a) = -\mathbb{I}_{i,\lambda}(a)$.

Finally, we prove on part (b) for $\lambda'_i < 0$. By assumption we have $\lambda'_i = \lambda_i - 2 < 0$. First, if $\lambda_i = 1$, then $a = \mu_{\tau_{n-1}}(\pi(a)) + p_0(a)$ and $\varepsilon'_{i,\lambda}(a) = p_0(a)$ as in (59). The same computation as in the previous lemma yields
\[
\tau_n\ell_i(a)\tau_n = \mu_{\tau_{n-1}\tau_{n-1}}(\alpha - \alpha_i, i^3)(\pi(a)).
\]
Let $b = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\ell_i \otimes \ell_i)\pi(a)$. Then $\tau_n\ell_i(a)\tau_n = \mu_{\tau_n}(b)$ and $\mu_{x_n^0}(b) = 0$. Since $\lambda'_i = -1$, we deduce $b = \tau_n\ell_i(a)\tau_n$ and
\[
\varepsilon'_{i,\lambda'}(\tau_n\ell_i(a)\tau_n) = \mu_{x_n}(b) \\
= \mu_{\tau_{n-1}x_{n-1}e(\alpha - \alpha_i, i^2)}(\pi(a)) \\
= \mu_{\tau_{n-1}e(\alpha - \alpha_i, i^2)}(\pi(a)) \\
= a - \ell_i(p_0(a)).
\]
We conclude $\mathbb{X}_{i,\lambda}(a) = -\mathbb{I}_{i,\lambda}(a) + a = -\mathbb{I}_{i,\lambda}(a) + B^0_{\lambda'_i,\lambda'}a$.

It remains to consider the case $\lambda_i \leq 0$. Let $\tilde{a} \in R^\Lambda(\alpha) e(\alpha - \alpha_i, i) \otimes R^\Lambda(\alpha - \alpha_i) e(\alpha - \alpha_i, i) R^\Lambda(\alpha)$ such that $\mu_{\tau_{n-1}}(\tilde{a}) = a$, $\mu_{x_{n-1}^k}(\tilde{a}) = 0$ for $k \in [0, -\lambda_i - 1]$. We have
\[
\tau_n\ell_i(a)\tau_n = \mu_{\tau_{n-1}\tau_{n-1}e(\alpha - \alpha_i, i^3)}(\tilde{a}) \\
= \mu_{\tau_{n-1}\tau_{n-1}e(\alpha - \alpha_i, i^3)}(\tilde{a}).
\]
Let $b = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\ell_i \otimes \ell_i)(\tilde{a}) \in R^\Lambda(\alpha + \alpha_i)e(\alpha, i) \otimes R^\Lambda(\alpha)e(\alpha, i) R^\Lambda(\alpha + \alpha_i)$. Then $\mu_{\tau_n}(b) = \tau_n\ell_i(a)\tau_n$, and $\mu_{x_n^0}(b) = \mu_{\tau_{n-1}x_{n-1}^ke(\alpha - \alpha_i, i^2)}(\tilde{a})$. A computation similar to (60) yields
\[
\tau_{n-1}x_{n-1}^ke(\alpha - \alpha_i, i^2) = \left( \sum_{g_1 + g_2 = k-1} x_{n-1}^{g_1} x_n^{g_2} - \sum_{g_1 + g_2 = k-2} (g_2 + 1) x_{n-1}^{g_1} x_n^{g_2} \right) e(\alpha - \alpha_i, i^2).
\]
Therefore for $0 \leq k \leq -\lambda'_i$, we have
\[
\mu_{x_n^k}(b) = \sum_{g_1 + g_2 = k-1} x_{n-1}^{g_1} \mu_{\tau_{n-1}}(\tilde{a}) x_n^{g_2} - \sum_{g_1 + g_2 = k-2} (g_2 + 1) \mu_{x_{n-1}^{g_1}}(\tilde{a}) x_n^{g_2} \\
= \sum_{g_1 + g_2 = k-1} x_{n-1}^{g_1} ax_n^{g_2} - \delta_{k=-\lambda'_i} \mu_{x_{n-1}}(\tilde{a}).
\]
Here we have used the fact $\mu_{\tau_{n-1}}(\tilde{a}) = a$ and $\mu_{x_{n-1}^k}(\tilde{a}) = 0$ for $0 \leq k \leq -\lambda_i - 1$ in the second equality. Finally, recall from (77) that there are elements $\tilde{\pi}_\ell \in R^\Lambda(\alpha + \alpha_i)e(\alpha, i) \otimes R^\Lambda(\alpha)e(\alpha, i) R^\Lambda(\alpha + \alpha_i)$ for $0 \leq \ell \leq -\lambda'_i - 1$ such that $\mu_{\tau_{n}}(\tilde{\pi}_\ell) = 0$ and $\mu_{x_{n}^k}(\tilde{\pi}_\ell) = \delta_{k,l}$. Set
\[
c = b - \sum_{k=0}^{-\lambda'_i - 1} \left( \sum_{g_1 + g_2 = k-1} x_{n-1}^{g_1} \tilde{\pi}_k x_n^{g_2} \right).
Then $\mu_{\tau_n}(c) = \mu_{\tau_n}(b) = \tau_n\iota_i(a)\tau_n$ and $\mu_{x_n^k}(c) = 0$ for $0 \leq k \leq -\lambda'_i - 1$. Hence $c = \tau_n\iota_i(a)\tau_n$, and $X_{x_i,\lambda}(a)$ is equal to

$$
\mu_{x_n^\lambda_i}(c) = \mu_{x_n^\lambda_i}(b) - \sum_{k=0}^{-\lambda'_i-1} \left( \sum_{g_1+g_2=k-1} x_n^{g_1} a_{x_n^\lambda_i}(\tilde{\pi}_k) x_n^{g_2} \right)
$$

$$
= -\mu_{x_n^\lambda_i}(\tilde{a}) e(\alpha, i) + \sum_{g_1+g_2=-\lambda'_i-1} x_n^{g_1} a_{x_n^\lambda_i} x_n^{g_2} - \sum_{g_1=-1} \sum_{g_2=-\lambda'_i-1-g_3} x_n^{g_1} a_{x_n^\lambda_i}(\tilde{\pi}_{-\lambda'_i-g_3}) x_n^{g_2}
$$

$$
= -\mathbb{I}_{i,i,\lambda}(a) + \sum_{g_1+g_2+g_3=-\lambda'_i-1} x_n^{g_1} B_{+1,\lambda_i}^{g_1}(a).
$$

In the second equality, we have substituted $g_3 = -\lambda'_i - k$. In the third equality, we have used the definition of $B_{+1,\lambda_i}^{g_1}$ for $0 \leq g_3 \leq -\lambda'_i$ in Definition A.1. \qed

**Remark A.5.** Assume that the $Q$-cyclicity condition in [10, (2.4), (2.5)] holds for $\mathcal{V}^A$, i.e., the endomorphisms $x_i \in \text{End}(E_i^\vee)$, $\tau_{ij} \in \text{End}(E_i^\vee E_j^\vee)$ are such that

$$
x_i^\vee = \tau_{ij} x_i, \quad \tau_{ij}^\vee = \tau_{ij}.
$$

See the notation in Section 2.1.4. Then, under the adjunction isomorphism $\text{Hom}(E_i^\vee F_i', E_j^\vee F_j') = \text{Hom}(F_j E_i, F_j E_i)$, part (a) of the previous lemma gives the second equality in the mixed relation [10, (2.16)]. For $i = j$, under the same adjunction, part (b) gives the second equalities in [10, (2.22), (2.24), (2.26)]. The other relations in [10 sec. 2.6.3] can be checked similarly. Since the computations are quite lengthy and will not be needed, we omit the details here. Finally, the fake bubbles relations [10, (2.20)] is proved in Lemma A.2(a) below. Therefore, assuming the $Q$-cyclicity condition, we have proved that $\mathcal{V}^A$ carries a representation of the Khovanov-Lauda’s 2-Kac-Moody algebra. The $Q$-cyclicity condition can probably be proved by similar computations as in [118]. We have not checked this.

A.3. **Proof of Proposition 3.10** Assume that $\text{ht}(\alpha) = n$. For $\nu = (\nu_1, \ldots, \nu_n) \in I^\alpha$ and $k \in [1, n]$ we set $\nu(k) = (\nu_1, \ldots, \nu_k)$. Consider the map

$$
\hat{e}_\nu = \hat{e}_{\nu_1} \circ \cdots \circ \hat{e}_{\nu_n} : e(\nu) R^A(\alpha) e(\nu) \to k.
$$

Recall that $\hat{e}_{\nu_k}$ is a map

$$
\hat{e}_{\nu_k} : e(\nu(k)) R^A(\alpha - \sum_{j=k+1}^n \alpha_{\nu_j}) e(\nu(k)) \to e(\nu(k-1)) R^A(\alpha - \sum_{j=k}^n \alpha_{\nu_j}) e(\nu(k-1)).
$$

Next, we define an invertible element $r_\nu \in k^\circ_0$ by

$$
r_\nu = \prod_{k<j} r_{\nu_k,\nu_l} \text{ where } r_{ij} = \begin{cases} c_{ii,j,-a_{ij},0} & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases}
$$

(61)
Definition A.6. We define a $k$-linear map $t_{\Lambda, \alpha} : R^\Lambda(\alpha) \to k$ by setting for all $\nu, \nu' \in I^\alpha$

$$t_{\Lambda, \alpha}(e(\nu) \cdot e(\nu')) = \begin{cases} 0 & \text{if } \nu \neq \nu', \\ r_\nu \hat{\varepsilon}_\nu(e(\nu) \cdot e(\nu')) & \text{if } \nu = \nu'. \end{cases}$$

For $\nu \in I^\alpha$ we'll abbreviate

$$r(\alpha, \nu_n) = \prod_{k=1}^{n-1} r_{\nu_k, \nu_n}. \tag{62}$$

By Corollary 3.9 the map $t_\alpha$ is homogenous of degree $-d_{\Lambda, \alpha}$. Note that $r_\nu = r(\alpha, \nu_n) r_{\nu(n-1)}$. Therefore we have

$$t_\alpha(a) = r(\alpha - \alpha_{\nu_n}, \nu_n) t_{\alpha-\alpha_{\nu_n}}(\hat{\varepsilon}_{\nu_n}(a)), \quad \forall a \in e(\nu) R^\Lambda(\alpha) e(\nu). \tag{63}$$

We will prove that $t_\alpha$ is a symmetric form. By Theorem 3.8 the form $t_\alpha$ is nondegenerate. We must prove that for each $w, z \in R^\Lambda(\alpha)$ we have

$$t_\alpha(zw) = t_\alpha(wz). \tag{64}$$

Without loss of generality, we may assume

$$z \in e(\nu) R^\Lambda(\alpha) e(\mu), \quad w \in e(\mu) R^\Lambda(\alpha) e(\nu), \tag{65}$$

the other cases being trivial. We will prove (64) by induction on the height of $\alpha$. Assuming it holds for all $\alpha$ of height $n-1$, let us prove it for $\alpha$ of height $n$. We will write $\nu_n = i$, $\mu_n = j$, $\beta = \alpha - \alpha_i - \alpha_j$ and $\lambda = \Lambda - (\alpha - \alpha_i)$.

First, consider the case when $z$ belongs to the image of the map

$$\sigma_{ij} = \mu_{\tau_{n-1}} : R^\Lambda(\alpha - \alpha_i) e(\beta, j) \otimes_{R^\Lambda(\beta)} e(\beta, i) R^\Lambda(\alpha - \alpha_j) \to e(\alpha - \alpha_i, i) R^\Lambda(\alpha) e(\alpha - \alpha_j, j)$$

for $\beta = \alpha - \alpha_i - \alpha_j$ and $w$ belongs to the image of $\sigma_{ji}$. Note that this is always the case if $i \neq j$ or if $i = j$ and $\lambda_i \leq 0$. In this situation, up to taking a linear combination, we may write

$$z = \iota_i(z') \tau_{n-1} \iota_j(z''), \quad w = \iota_j(w') \tau_{n-1} \iota_i(w''), \tag{66}$$

where $\iota_s$ is the canonical embedding $R^\Lambda(\alpha - \alpha_s) \to e(\alpha - \alpha_s, s) R^\Lambda(\alpha) e(\alpha - \alpha_s, s)$ for $s = i, j$ and

$$z' \in e(\nu^{(n-1)}) R^\Lambda(\alpha - \alpha_i) e(\xi, j), \quad z'' \in e(\xi, i) R^\Lambda(\alpha - \alpha_j) e(\mu^{(n-1)})$$

$$w' \in e(\mu^{(n-1)}) R^\Lambda(\alpha - \alpha_j) e(\eta, i), \quad w'' \in e(\eta, j) R^\Lambda(\alpha - \alpha_i) e(\nu^{(n-1)})$$

for some $\xi, \eta \in I^\beta$. By (63) and $R^\Lambda(\alpha - \alpha_i)$-bilinearity of $\hat{\varepsilon}_i$ we have

$$t_\alpha(zw) = r_\nu \hat{\varepsilon}_\nu(\iota_i(z') \tau_{n-1} \iota_j(z'' w')) = r_\nu \hat{\varepsilon}_\nu(\iota_j(z' w') \tau_{n-1} \iota_i(z''))$$

$$= r(\alpha - \alpha_i, i) t_{\alpha-\alpha_i}(z' \hat{\varepsilon}_i(\tau_{n-1} \iota_j(z'' w') \tau_{n-1})) = r(\alpha - \alpha_i, i) t_{\alpha-\alpha_i}(z' \hat{\varepsilon}_i(z'' w') \tau_{n-1} \tau_{n-1}).$$

Next, Lemma A.3 yields

$$\hat{\varepsilon}_i(\tau_{n-1} \iota_j(z'' w') \tau_{n-1}) = r_{ij} \iota_j \hat{\varepsilon}_i(z'' w') + \delta_{ij} \sum_{g_1, g_2} x_{g_1} g_1^2 B_{g_1 \lambda} z'' w' x_{g_2} g_2.$$
Therefore \( t_\alpha(zw) = A(z, w) + B(z, w) \), where

\[
A(z, w) = r(\alpha - \alpha_i, i) r_{ij} t_{\alpha-\alpha_i}(\hat{\xi}_j(z''w')w'') \\
B(z, w) = \delta_{ij} r(\alpha - \alpha_i, i) t_{\alpha-\alpha_i}(\sum_{g_1+g_2+g_3=-\lambda_i-1} z' x_{n-1}^{g_1} B_{+i,\lambda}^{g_2} z'' x_{n-1}^{g_3} w'').
\]

Thus, the formula (64) follows from the identities (67)

\[
A(z, w) = A(w, z), \quad B(z, w) = B(w, z).
\]

Let us first prove (67) for \( A(z, w) \). We have

\[
t_{\alpha-\alpha_i}(\hat{\xi}_j(z''w')w'') = t_{\alpha-\alpha_i}(t_j \circ \hat{\xi}_i(z''w')w''z') \\
= (\prod_{p=1}^{n-2} r_{\xi_p,j}) t_\beta(\hat{\xi}_j(t_j \circ \hat{\xi}_i(z''w')w''z')) \\
= (\prod_{p=1}^{n-2} r_{\xi_p,j}) t_\beta(\hat{\xi}_i(z''w')\hat{\xi}_j(w''z')).
\]

Here, the first equality is because \( t_{\alpha-\alpha_i} \) is symmetric by induction, the second one is given by (63) since \( \hat{\xi}_i(z''w')w''z' \in e(\xi, j) R^\lambda(\alpha - \alpha_j) e(\xi, j) \), the third one is the \( R^\lambda(\beta) \)-bilinearity of \( \hat{\xi}_j \). Now, observe that \( r_{ij}(\prod_{p=1}^{n-1} r_{\xi_p,j}) = \prod_{p=1}^{n-1} r_{\xi_p,j} \). We deduce

\[
A(z, w) = r(\alpha - \alpha_i, i) r(\alpha - \alpha_j, j) t_\beta(\hat{\xi}_i(z''w')\hat{\xi}_j(w''z'))
\]

Exchanging \( z \) and \( w \) means also exchanging \( i \) and \( j \), \( \nu \) and \( \mu \). So the right hand side is symmetric with respect to \( z \) and \( w \). We deduce \( A(z, w) = A(w, z) \).

Next, since \( t_{\alpha-\alpha_i} \) is symmetric by the inductive hypothesis and since \( B_{+i,\lambda}^{g_2} \) belongs to the center of \( R^\lambda(\alpha - \alpha_i) \), we have

\[
B(z, w) = \delta_{ij} r(\alpha - \alpha_i, i) t_{\alpha-\alpha_i}(\sum_{g_1+g_2+g_3=-\lambda_i-1} B_{+i,\lambda}^{g_2} z'' x_{n-1}^{g_1} w' x_{n-1}^{g_3} w'').
\]

Exchanging \( z \) and \( w \) means also exchanging \( i \) and \( j \), \( \nu \) and \( \mu \). But here \( B(z, w) \neq 0 \) only when \( i = j \). In this situation \( r(\alpha - \alpha_i, i) = r(\alpha - \alpha_j, j) \). We conclude that \( B(z, w) = B(w, z) \). Hence, we have proven (64) when \( z, w \) are both of the form (60).

If \( z \) and \( w \) are not of the form (60), then we must have \( i = j \) and \( \lambda_i > 0 \) as discussed in the paragraph before (63). In this situation \( z \in e(\alpha - \alpha_i, i) R^\lambda(\alpha) e(\alpha - \alpha_i, i) \) can be uniquely written as \( z = \mu_{n-1}(\pi(z)) + \sum_{k=0}^{\lambda_i-1} p_k(z) x_n^k \), see Theorem 3.7. Similar for \( w \). By linearity of \( t_\alpha \) the remaining cases to be considered are

1. \( z = z_k x_n^k, \quad w = w' x_{n-1}^l \),
2. \( z = z_k x_n^k, \quad w = w_l x_n^l \),

for \( z_k, w_l \in R^\lambda(\alpha - \alpha_i), k, l \in [0, \lambda_i - 1] \), and \( w' \in R^\lambda(\alpha - \alpha_i) e(\beta, i) \), \( w'' \in e(\beta, i) R^\lambda(\alpha - \alpha_i) \).

Note that in both cases we have

\[
t_\alpha(zw - wz) = r_\nu \hat{\xi}_\nu(zw) - r_\mu \hat{\xi}_\mu(wz) \\
= r(\alpha, i) t_{\alpha-\alpha_i}(\hat{\xi}_i(zw - wz)).
\]
Here, in the last equality, we used (63) and the fact that \( i = j \). By induction, to prove \( t_\alpha \) symmetric, it is enough to prove \( \hat{\varepsilon}_i(zw - wz) \) belongs to the commutator of \( R^\Lambda(\alpha - \alpha_i) \).

We do this case by case

- We have \( zw = z_k w'(x_n^k \tau_{n-1}) e(\beta, i^2) w'' \). Now
  \[
  x_n^k \tau_{n-1} e(\beta, i^2) = (\tau_{n-1} x_{n-1} + \sum_{p+q=k-1} x_{n-1}^p x_n^q) e(\beta, i^2).
  \]

  Hence \( \hat{\varepsilon}_i(zw) = p_{\lambda_i-1}(zw) = 0 \). Similarly \( \hat{\varepsilon}_i(wz) = 0 \).

- We have \( zw - wz = (z_k w_l - w_l z_k) x_{n}^{k+l} \). Hence \( \hat{\varepsilon}_i(zw - wz) = [z_k, w_l] \hat{\varepsilon}_i(x_{n}^{k+l}) \). Note that \( \hat{\varepsilon}_i(x_{n}^{k+l}) \) belongs to the center of \( R^\Lambda(\alpha - \alpha_i) \). So \( \hat{\varepsilon}_i(zw - wz) \) belongs to the commutator of \( R^\Lambda(\alpha - \alpha_i) \).

The proof of Proposition 3.10 is now complete.

\( \square \)

**APPENDIX B. RELATIONS**

In this section we prove Theorem 3.22.

### B.1. A useful lemma

Let \( z \in e(\alpha, i) R^\Lambda(\alpha + \alpha_i) e(\alpha, i) \). Recall that

- if \( \lambda_i \geq 0 \), then \( z \) can be uniquely written as
  \[
  z = \mu_{\tau_n}(\pi(z)) + \sum_{k=0}^{\lambda_i-1} p_k(z) x_{n+1}^k,
  \]

  where \( \mu_{\tau_n}(\pi(z)) \) is a unique polynomial of degree \( \lambda_i - 1 \) in \( z \).

- if \( \lambda_i \leq 0 \), there are \( \tilde{z}, \tilde{\pi}_k \) such that \( \mu_{\tau_n}(\tilde{z}) = z \), \( \mu_{x_n^p}(\tilde{\pi}_k) = 0 \) and \( \mu_{x_n^p}(\pi_k) = \delta_{k,p} \) for \( k, p \in [0, -\lambda_i - 1] \).

To prove Theorem 3.22 we’ll need the following technical result.

**Lemma B.1.** For each \( r \in \mathbb{N} \), we have

(a) if \( \lambda_i > 0 \) then

\[
\pi(x_{n+1}^r e(\alpha, i)) = \sum_{a=0}^{r-\lambda_i} (B_{+i,\lambda}^{r-\lambda_i-a} \otimes x_n^a \otimes 1 \otimes 1)(-\tilde{n}_i(1)),
\]

\[
p_k(x_{n+1}^r e(\alpha, i)) = \sum_{a=0}^{\lambda_i-k-1} B_{+i,\lambda}^{r-k-a} B_{-i,\lambda}^a, \quad \forall k \in [0, \lambda_i - 1],
\]

(b) if \( \lambda_i \leq 0 \) then

\[
\mu_{x_n^p}(\tilde{z}) = \sum_{p=0}^{r+\lambda_i} \tilde{\varepsilon}_{i,\lambda}^p (z x_{n+1}^p) B_{+i,\lambda}^{r+\lambda_i-p}, \quad \forall z \in e(\alpha, i) R^\Lambda(\alpha + \alpha_i) e(\alpha, i),
\]

\[
\mu_{x_n^p}(\tilde{\pi}_k) = \sum_{a=0}^{\lambda_i-1} B_{+i,\lambda}^a B_{-i,\lambda}^{r-a}, \quad \forall k \in [0, -\lambda_i - 1],
\]

(c) \[\sum_{a=0}^{r} B_{+i,\lambda}^{r-a} B_{-i,\lambda}^a = \delta_{r,0}.\]
Proof. First, assume $\lambda_i > 0$. To simplify notation, we write $x_{n+1}^r = x_{n+1}^r e(\alpha, i)$. We have

\[ x_{n+1}^{r+1} = x_{n+1}^{r+1} \mu_{\tau_n}(\pi(x_{n+1}^r)) + \sum_{k=0}^{\lambda_i-1} \mu_{\tau_n}(\pi(x_{n+1}^r)) + \sum_{k=0}^{\lambda_i-1} p_k(x_{n+1}^r) x_{n+1}^{k+1} \]

Therefore we deduce

\[ \mu_{\tau_n}(\pi(x_{n+1}^r)) + \sum_{k=0}^{\lambda_i-1} p_k(x_{n+1}^r) x_{n+1}^{k+1} \]

Here in the last equality we have used $(x_{n+1}^r - \tau_n x_{n+1} - 1)e(\alpha - \alpha_i, i^2) = 0$ and $\mu_1 = \varepsilon'$. It follows that

\[ \pi(x_{n+1}^{r+1}) = (1 \otimes x_n \otimes 1 \otimes 1) \pi(x_{n+1}^r) + p \lambda_i-1(x_{n+1}^r) \pi(x_{n+1}^r), \]

\[ p_0(x_{n+1}^{r+1}) = \varepsilon'_i(\pi(x_{n+1}^r)) + p \lambda_i-1(x_{n+1}^r) p_0(x_{n+1}^r), \]

\[ p_k(x_{n+1}^{r+1}) = p_{k-1}(x_{n+1}^r) + p \lambda_i-1(x_{n+1}^r) p_k(x_{n+1}^r), \quad \forall k \in [1, \lambda_i - 1]. \]

Now, recall that $p \lambda_i-1(x_{n+1}^{r+1}) = \varepsilon'_i(x_{n+1}^{r+1}) = B_{+i,\lambda}^r, p_k(x_{n+1}^r) = B_{+i,\lambda}^{r-k}$ and $\varepsilon'_i = -\pi(x_{n+1}^r)$. If $r < \lambda_i$ the first equality in part (a) is trivial with both sides being zero. If $r \geq \lambda_i$, it follows recursively from \((68)\).

Next, by applying recursively \((70)\) we obtain

\[ p_k(x_{n+1}^{r+1}) = p_0(x_{n+1}^{r+1}) - \sum_{a=0}^{k-1} B_{+i,\lambda}^{r-a} B_{-i,\lambda}^{\lambda_i-k+a}. \]

Substitute $p_0(x_{n+1}^{r+1})$ using \((69)\) gives

\[ p_k(x_{n+1}^{r+1}) = \varepsilon'_i(\pi(x_{n+1}^{r+1})) - \sum_{a=0}^{k} B_{+i,\lambda}^{r-a} B_{-i,\lambda}^{\lambda_i-k+a}. \]

Apply this to the special case $k = \lambda_i - 1$ we get

\[ \varepsilon'_i(\pi(x_{n+1}^r)) = \sum_{a=0}^{\lambda_i} B_{+i,\lambda}^{r+a} B_{-i,\lambda}^a. \]

Therefore we deduce

\[ p_k(x_{n+1}^{r+1}) = \sum_{a=0}^{\lambda_i} B_{+i,\lambda}^{r+a} B_{-i,\lambda}^a - \sum_{a=0}^{\lambda_i-1} B_{+i,\lambda}^{r+a} B_{-i,\lambda}^a \]

\[ = \sum_{a=0}^{\lambda_i-1} B_{+i,\lambda}^{r+a} B_{-i,\lambda}^a. \]

This proves the second equality in part (a) for $r \geq \lambda_i$. 

58. P. SHAN, M. VARAGNOLO, E. VASSEROT
On the other hand, we have
\[ \varepsilon_i'(\pi(x_{n+1}^r)) = \sum_{a=0}^{r-\lambda_i} B_{+,i,\lambda}^{r-\lambda_i-a} \mu_{x_n^a}(-\tilde{\eta}_i'(1)) = - \sum_{a=\lambda_i+1}^{r+1} B_{+,i,\lambda}^{r+1-a} B_{-,i,\lambda}^a. \]
by the first equality in part (a). Combined with (71) it gives part (c) for \( r > 0 \). The case \( r = 0 \) is obvious.

Finally, for \( r \leq \lambda_i - 1 \) we have \( p_k(x_{n+1}^r e(\alpha, i)) = \delta_{k,r} \) and
\[ \sum_{a=0}^{\lambda_i-k-1} B_{+,i,\lambda}^{r-k-a} B_{-,i,\lambda}^a = \sum_{a=0}^{r-k} B_{+,i,\lambda}^{r-k-a} B_{-,i,\lambda}^a = \delta_{r,k} \]
by part (c). We deduce the second equality in part (a) for \( r \leq \lambda_i - 1 \).

The case \( \lambda_i < 0 \) is proved by a computation of similar style. We only indicate some key steps. First, one checks by a direct computation that
\[ \widetilde{zx}_{n+1} = (1 \otimes x_n \otimes 1 \otimes 1) \tilde{z} - \varepsilon_i'(x_{n+1}) \tilde{\eta}_i'(1). \]
Applying it recursively we get
\[ \widetilde{zx}_{n+1} = (1 \otimes x_n^r \otimes 1 \otimes 1) \tilde{z} - \sum_{p=0}^{r-1} \varepsilon_i'(x_{n+1}) (1 \otimes x_n^{r-1-p} \otimes 1 \otimes 1) \tilde{\eta}_i'(1). \]
The first equality in (b) is obtained by applying \( \mu_{x_n^r} \) to both sides of the above equality with \( r \) replaced by \( r + \lambda_i \). To prove the second equality, observe that for \( k, p \in [0, -\lambda_i - 2] \), write
\( A = (1 \otimes x_n \otimes 1 \otimes 1) \tilde{x}_{k+1} \) we have \( \mu_{x_n^r}(A) = 0 \), \( \mu_{x_n^r}(A) = \delta_{k,p} \), and \( \mu_{x_n^r-\lambda_i-1}(A) = B_{-,i,\lambda}^{-\lambda_i-k-1} \). We deduce that
\[ \tilde{x}_{k} = \sum_{p=0}^{-\lambda_i-1-k} B_{-,i,\lambda}^{-\lambda_i-k-1-p} (1 \otimes x_n^p \otimes 1 \otimes 1) \tilde{\eta}_i'(1). \]
Now, apply \( \mu_{x_n^r} \) to both sides we get the second equality in (b). Finally, to get (c), observe that the first equality applied to \( z = 1 \) yields
\[ \mu_{x_n^r}(1) = \sum_{p=-\lambda_i+1}^{r+1} B_{+,i,\lambda}^p B_{-,i,\lambda}^{r+1-p}. \]
On the other hand, it is easy to check that \( -\tilde{1} = (1 \otimes x_n \otimes 1 \otimes 1) \tilde{x}_0 + B_{+,i,\lambda}^{-\lambda_i} \tilde{\eta}_i'(1). \) Hence
\[ -\mu_{x_n^r}(1) = \mu_{x_n^r+1}(\tilde{x}_0) + B_{+,i,\lambda}^{-\lambda_i} B_{-,i,\lambda}^{r+1+\lambda_i} = \sum_{p=0}^{-\lambda_i} B_{+,i,\lambda}^p B_{-,i,\lambda}^{r+1-p} \]
by the second equality in (b). Combining the two equalities gives (c). \( \square \)
B.2. The Cartan loop operators. Consider the following formal power series

$$B_{\pm i, \lambda}(z) = \sum_{k \geq 0} B_{\pm i, \lambda}^k z^k \in Z(R^A(\alpha))[[z]].$$

The aim of this section is to express the coefficients of the formal series $B_{\pm i, \lambda}(z)$ as elements in the image of the canonical map $Z(R(\alpha)) \to Z(R^A(\alpha))$, see Proposition B.3 for details. To do that, fix formal variables $y_{i,1}, \ldots, y_{i,\Lambda_i}$ of degree 2 such that

$$a^i(u) = \prod_{p=1}^{\Lambda_i} (u + y_{ip}) \quad \text{with} \quad c_{ip} = c_p(y_{i,1}, \ldots, y_{i,\Lambda_i}).$$

Consider the following formal series in $k[[u, v]]$

$$q_{ij}(u, v) = \begin{cases} u^{-a_{ij}} Q_{ij}(u^{-1}, v)/c_{i,j,-a_{ij},0} & \text{if } i \neq j, \\ (1 - uv)^{-2} & \text{else.} \end{cases}$$

**Lemma B.2.** For each $\alpha \in Q_+$ of height $n$, we have

(a) $B_{+,i,\lambda}(z) B_{-,i,\lambda}(z) = 1$,

(b) $(B_{+,i,\lambda}(z) - q_{ij}(z, x)^{n+1} B_{+,i,\lambda+\alpha_j}(z)) e(\alpha - \alpha_j, j) = 0$ for each $i, j$.

**Proof.** Part (a) will be proved in Lemma B.1(c) below. Now, we concentrate on (b). By (a) it is enough to prove it for $B_{+,i,\lambda}(z)$. Write $\beta = \alpha - \alpha_j$ and $\lambda' = \Lambda - \beta = \lambda + \alpha_j$.

First, assume that $i = j$. Recall that

$$\varphi_ne(\beta, i^2) = (x_n\tau_n - \tau_n x_n) e(\beta, i^2) \in R^A(\alpha + \alpha_i).$$

We have

$$x_{n+1}^k e(\beta, i^2) = \varphi_n x_{n+1}^k e(\beta, i^2) = (x_n\tau_n - \tau_n x_n) x_n^k (x_n\tau_n - \tau_n x_n) e(\beta, i^2) = (x_n^k x_n^{k+1} \tau_n - \tau_n x_n^{k+2} + x_n (x_n x_n \tau_n + (\tau_n x_n^{k+1} \tau_n - x_n) e(\beta, i^2)).$$

By Lemma A.4(b), for all $\ell \in \mathbb{N}$ we have

$$\hat{\varepsilon}_{i,\lambda}(\tau_n x_n^\ell e(\beta, i^2)) = \mathbb{X}_{i,\lambda,\sigma}(x_n^\ell e(\beta, i))$$

$$= (- I_{i,\lambda,\sigma} + \sum_{g_1+g_2+g_3=-\lambda_i-1} x^{g_1} B_{+,i,\lambda}^{g_2} x^{g_3} n e(\beta, i))$$

$$= -\hat{\varepsilon}_{i,\lambda}(x_n^\ell e(\beta, i)) + \sum_{g_1+g_2+g_3=-\lambda_i-1} x^{\ell+g_1} B_{+,i,\lambda}^{g_2} x^{g_3}.$$

If $k \geq -\lambda_i + 1$ then $k - 2 \geq -\lambda'_i + 1$, we have

$$B_{+,i,\lambda}^k = \hat{\varepsilon}_{i,\lambda}(x_n^{\lambda_i-1+k} e(\alpha, i)), \quad B_{+,i,\lambda}^l = \hat{\varepsilon}_{i,\lambda}(x_n^{\lambda_i-3+l} e(\beta, i)), \quad \forall \ l \geq k - 2.$$
In this situation

\[ B_{+i,i}^k e(\beta, i) = \varepsilon_{i,i}(x_{n+1}^{\lambda_i-1+k} e(\alpha, i)) e(\beta, i) \]

\[ = \varepsilon_{i,i}'(x_{n+1}^{\lambda_i-1} e(\beta, i^2)) \]

\[ = -2x_n \varepsilon_{i,i}(x_n^{\lambda_i+k} e(\beta, i)) + \varepsilon_{i,i}'(x_n^{\lambda_i+k+1} e(\beta, i)) \]

\[ + x_n^2 \varepsilon_{i,i}'(x_n^{\lambda_i+k-1} e(\beta, i)) \]

(74)

In particular, this yields

\[ B_{+i,i}(z) e(\beta, i) = (1 - x_n z)^2 B_{+i,i}(z) e(\beta, i) \]

if \( \lambda_i < 0 \) we must also check (74) for \( k \leq -\lambda_i \). Consider the element

\[ A_k = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(t_i \otimes t_i)(\tilde{\pi}_{-\lambda_i-k} - 2x_n \tilde{\pi}_{-\lambda_i-k} - 2x_n^2 \tilde{\pi}_{-\lambda_i-(k-2)}) \]

in \( R^\Lambda(\alpha) e(\beta, i) \otimes R^\Lambda(\beta) e(\beta, i) R^\Lambda(\alpha) \). Here \( \tilde{\pi}_{-\lambda_i-l} \in R^\Lambda(\beta) e(\beta - \alpha_i, i) \otimes R^\Lambda(\beta-\alpha_i) e(\beta-\alpha_i, i) R^\Lambda(\beta) \) is the element defined in (8) for \( l \in [1, -\lambda_i] \), and we have set \( \tilde{\pi}_{-1} = -e(\beta, i) \) and \( \tilde{\pi}_{-2} = -x_n e(\beta, i) \), where \( e(\beta, i) \) and \( x_n e(\beta, i) \) are viewed as elements in \( e(\beta, i) R(\alpha) e(\beta, i) \). One can check by direct computation that \( \mu_{\tau_n}(A_k) = 0, \mu_{x_n}(A_k) = \delta_{a_i,-\lambda_i-k} e(\beta, i) \) for \( a \in [0, -\lambda_i - 1] \), and that

\[ \mu_{x_n-\lambda_i}(A_k) = (B_{+i,i}^k - 2x_n B_{+i,i}^{k-1} + x_n^2 B_{+i,i}^{k-2}) e(\beta, i). \]

It follows that \( A_k = \tilde{\pi}_{-\lambda_i-k} e(\beta, i) \) and

\[ B_{+i,i}^k e(\beta, i) = \mu_{x_n-\lambda_i}(A_k) = (B_{+i,i}^k - 2x_n B_{+i,i}^{k-1} + x_n^2 B_{+i,i}^{k-2}) e(\beta, i). \]

The proof for part (b) in the case \( i = j \) is complete.

Finally, assume that \( i \neq j \). By relation (d) in QHA, we have

\[ \tau_n x_n^k \tau_n e(\beta, ji) = x_n^k \tau_n^2 e(\beta, ji) = \sum_{p,q} c_{i,j-p,q} x_{n+1}^{p+k} x_n^q e(\beta, ji). \]

Applying \( \varepsilon_{i,i}' \) to both sides of the equality, we get

\[ c_{i,j-a_ij,0} \varepsilon_{i,i}'(x_n^k) e(\beta, j) = \sum_{p,q} c_{i,j-p,q} \varepsilon_{i,i}'(x_{n+1}^{p+k} e(\alpha, i)) x_n^q e(\beta, j) \]

by Lemma 4(a). Hence if \( k \geq -\lambda_i + 1 \) we get

\[ c_{i,j-a_ij,0} B_{+i,i}^{k-a_ij} e(\beta, j) = \sum_{p,q} c_{i,j-p,q} B_{+i,i}^{k+p} x_n^q e(\beta, j). \]

Now, assume \( k \leq -\lambda_i \). Then \( k - a_{ij} \leq -\lambda_i' \). In this case \( B_{+i,i}^{k-a_{ij}} = -\mu_{x_{n-1}^{\lambda_i'-1}}(\tilde{\pi}_{-\lambda_i-k}) \) with \( \tilde{\pi}_{-\lambda_i-k} \in R^\Lambda(\beta) e(\beta - \alpha_i, i) \otimes R^\Lambda(\beta-\alpha_i) e(\beta - \alpha_i, i) R^\Lambda(\beta) \). Set

\[ A_k = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(t_j \otimes t_j)(\tilde{\pi}_{-\lambda_i-k}). \]
Then we have the following equalities in \( e(\alpha, i)R^\Lambda(\alpha + \alpha_i)e(\alpha, i) \),

\[
\mu_{\tau_n}(A_k) = \mu_{\tau_n-1}\tau_{n-1}(\beta - \alpha_i, ij) (\tilde{\pi}_{-\lambda_i-k})
\]

\[
= \mu_{(\tau_n-1)\tau_{n-1}-\sum_{p,q} c_{i,j,p,q}(\sum_{a=0}^{p-1} c_{\alpha_0}^{p-a} x_n^{p-a})e(\beta - \alpha_i, ij)} (\tilde{\pi}_{-\lambda_i-k})
\]

\[
= - \sum_{p,q} c_{i,j,p,q} (\sum_{a=0}^{p-1} x_n^{p-a}) e(\beta, ji),
\]

because \( \mu_{\tau_n-1}(\tilde{\pi}_{-\lambda_i-k}) = \tau_n \mu_{\tau_n-1}(\tilde{\pi}_{-\lambda_i-k}) \tau_n = 0 \). Since \( \mu_{x_n^{a_i}}(\tilde{\pi}_{-\lambda_i-k}) = \delta_{a_i, -\lambda_i-k} \) for \( a \in [0, -\lambda_i' - 1] \), for any \( p \in [0, -a_{ij}] \) we have

\[
\sum_{a=0}^{p-1} x_n^{p-1-a} e(\beta, ji), \quad \text{if } p > -\lambda_i - k + 1
\]

otherwise.

Next, for any positive integer \( l \) we have

\[
\mu_{x_n^{1}}(A_k) = \mu_{x_n^{1}}(\tilde{\pi}_{-\lambda_i-k})
\]

\[
= - \sum_{p,q} c_{i,j,p,q} x_n^{p-1} e(\beta, ji), \quad \text{if } p > -\lambda_i - k + 1
\]

otherwise.

It follows that

\[
A_k = - \sum_{p=-\lambda_i-k+1}^{\infty} \sum_{q} c_{i,j,p,q} x_n^{p+1} e(\beta, ji) + \sum_{p=0}^{\infty} \sum_{q} c_{i,j,p,q} x_n^{p+1} e(\beta, ji) \tilde{\pi}_{-\lambda_i-p-k}
\]

in \( R^\Lambda(\alpha) e(\alpha - \alpha_i, i) \otimes R^\Lambda(\alpha - \alpha_i) e(\alpha - \alpha_i, i) R^\Lambda(\alpha) \). Apply \( \mu_{x_n^{1}} \) to both sides of the equation, by (75) and Definition A.1.1 we get

\[
c_{i,j,-a_{ij}} B_{+i,\lambda}^{k-a_{ij}} e(\beta, j) = \sum_{p,q} c_{i,j,p,q} B_{+i,\lambda}^{k+p} x_n^q e(\beta, j).
\]

The proof for part (b) is now complete. \( \square \)

We can now prove the main result of this section.

**Proposition B.3.** For each \( \alpha \in Q_+ \) of height \( n \) we have

\[
B_{\pm i,\lambda}(z) = z^{\mp \Lambda_i} a_i^{\pm 1} \prod_{\nu \in I \pm 1} q_{i\nu} (z, x_\nu)^{\mp 1} e(\nu).
\]
Proof. The relation \(a_i^\Lambda(x_1) = 0\) yields \(x_1^\Lambda = -\sum_{k=0}^{\Lambda_i-1} e_k(y_{i,1}, \ldots, y_{i,\Lambda_i})x_1^{\Lambda_i-k}\). Therefore, for \(k \in [1, \Lambda_i]\) we have \(B_{-i,\Lambda}^k = -\mathcal{P}_{\Lambda_i-k}(x_1^\Lambda) = e_k(y_{i,1}, \ldots, y_{i,\Lambda_i}),\) and for \(k > \Lambda_i\), since \(\eta_{i,\Lambda} = -\pi(x_1^\Lambda) = 0\), we deduce \(B_{-i,\Lambda}^k = 0\). So we have

\[
B_{-i,\Lambda}(z) = \sum_{p=0}^{\Lambda_i} B_{-i,\Lambda}^p z^p = \sum_{p=0}^{\Lambda_i} e_p(y_{i,1}, \ldots, y_{i,\Lambda_i}) z^p = \prod_{p=1}^{\Lambda_i} (1 + y_{i,p} z).
\]

We deduce that \(B_{\pm i,\Lambda}(z) = \prod_{p=1}^{\Lambda_i} (1 + y_{i,p} z)^{\mp 1}\). Now, by Lemma B.2 we have

\[
B_{\pm i,\Lambda}(z) = B_{\pm i,\Lambda}(z) \sum_j q_{ij}(z, x_n)^{\mp 1} e(\alpha - \alpha_j, j),
\]

\[
= B_{\pm i,\Lambda}(z) \sum_{\nu \in I^\Lambda} \prod_{k=1}^n q_{\nu k}(z, x_k)^{\mp 1} e(\nu).
\]

□

B.3. **Proof of Theorem 3.22.** Consider the operator \(h_{ir} \in \text{End}_k(\text{tr}(C/Z))\) which acts on \(\text{tr}(R^\Lambda(\alpha))\) by multiplication by the central element

\[
h_{ir,\lambda} = \sum_{k=0}^r (\lambda_i - k) B_{+i,\Lambda}^{r-k} B_{-i,\Lambda}^k.
\]

in \(R^\Lambda(\alpha)\). Then, define the following formal series

\[
(76) \quad \Psi_i(z) = \sum_{r \geq 0} \psi_{ir} z^r = \exp \left( - \sum_{r \geq 1} h_{ir} z^r / r \right), \quad H_i(z) = \sum_{r \geq 0} h_{ir} z^r.
\]

The following holds.

**Lemma B.4.** The operator \(\psi_{ir}\) acts on \(\text{tr}(R^\Lambda(\alpha))\) by multiplication by \(B_{-i,\Lambda}^r\).

**Proof.** By definition, the operator \(h_{ir}\) acts by multiplication by the element

\[
h_{ir,\lambda} = \sum_{k=0}^r (\lambda_i - k) B_{+i,\Lambda}^{r-k} B_{-i,\Lambda}^k.
\]

Since \(B_{\pm i,\Lambda}(z) = \sum_{k \geq 0} B_{\pm i,\Lambda}^k z^k\) and since \(B_{+i,\Lambda}(z) B_{-i,\Lambda}(z) = 1\) by Lemma B.2 we deduce that

\[
H_{i,\Lambda}(z) = \lambda_i - z \frac{d}{dz} \log B_{-i,\Lambda}(z).
\]

Now, from (76) we get

\[
H_i(z) = h_{i0} - z \frac{d}{dz} \log \Psi_i(z).
\]

Hence, the formal series \(\Psi_i(z)\) acts on \(\text{tr}(R^\Lambda(\alpha))\) by multiplication by \(B_{-i,\Lambda}(z)\). □
Finally, let $a_{i,j,p,q} \in k$ be such that
\[ g_{ij}(u,v) = \sum_{p,q \geq 0} a_{i,j,p,q} u^p v^q. \]
We can now prove the following.

**Proposition B.5.** For each $i, j \in I, r, s \in \mathbb{N}$, we have

(a) $[h_{ir}, h_{js}] = 0$,
(b) $[h_{ir}, h_{js}] = \delta_{ij} h_{ir+s}$,
(c) $\psi_{ir} x_{js} = \sum_{p,q \geq 0} a_{i,j,p,q} x_{j,s+q} \psi_{i,r-p}$ and $x_{js} \psi_{ir} = \sum_{p,q \geq 0} a_{i,j,p,q} \psi_{i,r-p} x_{j,s+q}$,
(d) $\sum_{p,q \geq 0} c_{i,j,p,q} [x_{i,r+p}, x_{j,s+q}] = 0$ if $i \neq j$,
(e) $[x_{i,r}, x_{i,s}] = 0$,
(f) $[x_{i,r_1}, [x_{i,r_2}, \ldots, x_{i,r_m}, x_{j,s}] \ldots] = 0$ with $i \neq j$, $r_p \in \mathbb{N}$, $m = 1 - a_{ij}$.

**Proof.** The first relation is obvious. Let us concentrate on part (b). If $i \neq j$ we have an isomorphism $\sigma_{j,i,\lambda}: F_2 E_1 \lambda \simeq E_2 F_1 \lambda$ such that $\sigma_{j,i,\lambda}(x_i^x x_j^y) = x_j^{-1} x_i$. Hence by Lemma 2.3 we have $\text{tr}_{F_2 E_1 \lambda}(x_i^x x_j^y) = \text{tr}_{E_2 F_1 \lambda}(x_j^y x_i^x)$, which is $[x_{i,r}, x_{j,s}] = 0$.

Now consider the case $i = j$. First, assume $\lambda_i > 0$. Let $G = F_1 E_1 \lambda \oplus 1_{\rho_i,\lambda}$. Recall the isomorphism of functors $\rho_{i,\lambda}: G \rightarrow E_1 F_1 \lambda$. By Lemma 2.3 we have

\[ x_i^x x_j^y = \text{tr}_{E_1 \lambda}(x_i^x x_j^y) = \text{tr}_{G}(\rho_{i,\lambda}^{-1}(x_i^x x_j^y) \rho_{i,\lambda}) \]

and it is equal to the sum of the trace of $\rho_{i,\lambda}^{-1}(x_i^x x_j^y) \rho_{i,\lambda}$ restricted to each direct factor of $G$. The restriction of $\rho_{i,\lambda}^{-1}(x_i^x x_j^y) \rho_{i,\lambda}$ to $F_1 E_1 \lambda$ is represented by

\[ R^\lambda(\alpha) e(\alpha - \alpha_i, i) \otimes R^\lambda(\alpha - \alpha_i) e(\alpha - \alpha_i, i) \rightarrow R^\lambda(\alpha) e(\alpha - \alpha_i, i) \otimes R^\lambda(\alpha - \alpha_i) e(\alpha - \alpha_i, i) \rightarrow \pi(x_{n+1} \mu_{\tau_n}(z) x_{n+1}^s).

Now, we have

\[ \pi(x_{n+1} \mu_{\tau_n}(z) x_{n+1}^s) = \pi(\mu_{x_{n+1}^{r+s}} x_{n+1}^s) \]

\[ = \pi(\mu_{x_{n+1}^{r+s}} x_{n+1}^s) + \sum_{p=0}^{r+s-1} \mu_{x_{n+1}^{r+s-1-p}}(z) x_{n+1}^p \]

\[ = (1 \otimes x_{n+1}^s) \otimes (x_{n+1}^s \otimes 1)(z) + \sum_{p=\lambda_i}^{r+s-1} \mu_{x_{n+1}^{r+s-1-p}}(z) x_{n+1}^p \]

\[ = (1 \otimes x_{n+1}^s) \otimes (x_{n+1}^s \otimes 1)(z) + \sum_{p=\lambda_i}^{r+s-1} \mu_{x_{n+1}^{r+s-1-p}}(z) (B^{-\lambda_i-a}_{p+1,\lambda} \otimes x_{n+1}^a) \otimes (1 \otimes 1) \eta_i^1(1). \]

Here we used the relation (e) of QHA to get the second equality, and Lemma B.1 for the last equality. It yields that the restriction of $\rho_{i,\lambda}^{-1}(x_i^x x_j^y) \rho_{i,\lambda}$ to $F_1 E_1 \lambda$ is the endomorphism

\[ x_i^y x_i^x = \sum_{p=\lambda_i}^{r+s-1} \sum_{a=0}^{p-\lambda_i} (B^{-\lambda_i-a}_{p+1,\lambda} F_i x_i^a) \otimes \eta_i^1(1 \otimes 1) \circ (F_i x_i^{r+s-1-p}), \]
and its trace is equal to

$$\text{tr}_{F_iE_i1_\lambda}(x_i^sx_i^r) - \text{tr}_{1_\lambda} \left( \sum_{p=0}^{r+s-1} B_{p+i,\lambda}^{\rho_{i,\lambda}-a} \rho_{i,\lambda}^{p-\lambda_i} \varepsilon_i' \circ (F_i x_i^{r+s-1-p+a}) \circ \hat{\eta}_i' \right) =$$

$$= x_{is}x_{ir}^{+} - \sum_{p=\lambda_i}^{r+s-1} B_{p+i,\lambda}^{\rho_{i,\lambda}-a} \sum_{a=0}^{p-\lambda_i} B_{p+s-a+i,\lambda}^{r+s-a} a_{i,\lambda} - x_{is}x_{ir}^{+} (\lambda_i - a) B_{r+s-a+i,\lambda}^{a} a_{i,\lambda}.$$  (77)

The restriction of $\rho_{i,\lambda}^{-1}(x_i^s x_i^r) \rho_{i,\lambda}$ to the $k$-th copy of $1_\lambda$ is represented by the map $R^A(\alpha) \rightarrow R^A(\alpha)$, $z \mapsto p_k(x_{r+k}^s z x_{r+k}^s) = z p_k(x_{r+k}^s)$. By the second equality in Lemma (B.1)(a) it is equal to $\sum_{a=0}^{\lambda_i-k} B_{r+s-a+i,\lambda}^{a} a_{i,\lambda}$. Combined with (77) we obtain

$$x_{is}x_{ir}^{+} = \text{tr}_G(\rho_{i,\lambda}^{-1}(x_i^s x_i^r) \rho_{i,\lambda})$$

Now, assume $\lambda_i < 0$. Let $G = E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus (-\lambda_i)}$ and consider the isomorphism $\rho_{i,\lambda} : F_i E_i 1_\lambda \rightarrow G$. By Lemma (2.3) we have

$$x_{is}x_{ir}^{+} = \text{tr}_{F_iE_i}(x_i^s x_i^r) = \text{tr}_G(\rho_{i,\lambda}(x_i^s x_i^r) \rho_{i,\lambda}^{-1})$$

and it is equal to the sum of the trace of the restriction of $\rho_{i,\lambda}(x_i^s x_i^r) \rho_{i,\lambda}^{-1}$ to each direct factor of $G$. The restriction of $\rho_{i,\lambda}(x_i^s x_i^r) \rho_{i,\lambda}^{-1}$ to $E_i F_i 1_\lambda$ is represented by

$$e(\alpha, i) R^A(\alpha + \alpha_i) e(\alpha, i) \rightarrow e(\alpha, i) R^A(\alpha + \alpha_i) e(\alpha, i)$$

$$z \mapsto \mu_{x_{\lambda_i}^s x_{\lambda_i}^r}(\tilde{z}).$$

By the relation (e) of QHA and the definition of $\tilde{z}$ we have

$$\mu_{x_{\lambda_i}^s x_{\lambda_i}^r}(\alpha - \alpha, i)^2(\tilde{z}) = x_{s+1}^s z x_{s+1}^s - \sum_{p=-\lambda_i}^{r+s-1} \mu_{x_{\lambda_i}^s}(\tilde{z}) x_{s+1}^{r+s-1-1} \circ p.$$

The restriction of $\rho_{i,\lambda}(x_i^s x_i^r) \rho_{i,\lambda}^{-1}$ to the $k$-th copy of $1_\lambda$ is represented by

$$R^A(\alpha) \rightarrow R^A(\alpha), \quad a \mapsto a \mu_{x_{\lambda_i}^s x_{\lambda_i}^r}(\tilde{z}).$$

Now, relation (b) follows from Lemma (B.1)(b) and the fact that

$$\text{tr}_{E_i F_i 1_\lambda} \left( (E_i x_i^s) \circ \eta_{i,\lambda}' \circ \varepsilon_{i,\lambda}' \circ (E_i x_i^b) \right) = \text{tr}_{1_\lambda} \left( \varepsilon_{i,\lambda}'(E_i x_i^{a+b}) \circ \eta_{i,\lambda}' \right)$$

using similar computation as in the previous case. We leave the details to the reader.
Next, we prove (c). Consider the formal series $X_j^-(w) = \sum_{s \geq 0} x_j^s w^s$. From Proposition B.3 and Lemma B.4 we deduce that for each $f \in R^A(\alpha)$ we have

$$X_j^-(w)(\text{tr}(f)) = \text{tr} \left( \sum_{s \geq 0} x_j^s w^s f e(\alpha, j) \right),$$

$$\Psi_i(z)X_j^-(w)\Psi_i(z)^{-1}(\text{tr}(f)) = \text{tr} \left( q_{ij}(z, x_{n+1}) \sum_{s \geq 0} x_j^s w^s f e(\alpha, j) \right).$$

This yields the first equation of (c). The second one is obtained in a similar way.

Let us now prove the relation (d). Consider the endomorphism $x_i^r x_j^s \tau_{ji} \tau_{ij}$ on $E_i E_j$. We have $x_i^r x_j^s \tau_{ji} \tau_{ij} = \tau_{ji} x_j^s x_i^r \tau_{ij}$. Therefore

$$\text{tr}_{E_i E_j}(x_i^r x_j^s \tau_{ji} \tau_{ij}) = \text{tr}_{E_i E_j}(\tau_{ji} x_j^s x_i^r \tau_{ij}) = \text{tr}_{E_j E_i}(x_j^s x_i^r \tau_{ij} \tau_{ji}).$$

Next, by relation (d) in QHA we have

$$\tau_{ji} \tau_{ij} = Q_{ij}(x_i, x_j) = \sum_{p,q} c_{i,j,p,q} x_i^p x_j^q = Q_{ji}(x_j, x_i) = \tau_{ij} \tau_{ji}.$$

Put this back into the equation above we get

$$\sum_{p,q} c_{i,j,p,q} \left( \text{tr}_{E_i E_j}(x_i^{r+p} x_j^{s+q}) - \text{tr}_{E_j E_i}(x_j^{s+q} x_i^{r+s}) \right) = 0,$$

which is the relation (d).

Next, let us prove (e). The functor $E_i^2$ acting on $R^A(\alpha)$ is represented by the bimodule $e(\alpha - 2\alpha_i, i^2)R^A(\alpha)$. A morphism $x_i^r x_i^s$ on $E_i^2$ is represented by the left multiplication by $x_i^{r-n} x_i^s$ if $\alpha$ has height $n$. It is enough to consider the case $n = 2$. The intertwiner $\varphi_1 = (x_1 - x_2)\tau_1 + 1$ is such that $\varphi_1(x_i^r x_i^s) = e(i^2)$ and $\varphi_1 e(i^2) = e(i^2)$. It follows that $\text{tr}_{E_i^2}(x_i^r x_i^s) = \text{tr}_{E_i^2}(x_i^s x_i^r)$. Hence $x_i^r x_i^s = x_i^s x_i^r$. The proof for $x_i^-$ is similar.

Finally let us prove the relation (f). By (e), it is equivalent to prove the following relation

$$\sum_{w \in S_n} [x_{i,r_{w(1)}}^\pm, [x_{i,r_{w(2)}}^\pm, \ldots [x_{i,r_{w(m)}}^\pm, x_{j,r_{w(1)}}^\pm] \ldots] = 0 \text{ with } i \neq j, r_p \in \mathbb{N} \text{ and } m = 1 - a_{ij}.$$

We will prove it for $x_i^+$, the proof for $x_i^-$ is similar. First, recall that the functor $E_i^{(a)}$ is the image of the divided power operator

$$e_{i,a} = x_{1}^{a-1} \cdots x_{a-1} x_{w_0} \in R(aa_i)$$

by the canonical morphism $R(aa_i) \to \text{End}(E_i^{a})$. The element $e_{i,a}$ is an idempotent, and we have $E_i^{a} \simeq (E_i^{(a)})^{\oplus a}$. See, e.g., [23] for details.

Given $i, j \in I$ with $i \neq j$, we write $m = 1 - a_{ij}$. In [23, prop. 6] an isomorphism of functors

$$\alpha' : \bigoplus_{a=0}^{\lfloor \frac{m}{2} \rfloor} E_i^{(2a)} E_j E_i^{(m-2a)} \cong \bigoplus_{a=0}^{\lfloor \frac{m-1}{2} \rfloor} E_i^{(2a+1)} E_j E_i^{(m-2a-1)}$$

and a quasi-inverse $\alpha''$ are constructed.
If \( a_{ij} = 0 \), we have \( \alpha' = \tau_{ji} : E_j E_i \to E_i E_j \) and \( \alpha'' = c_{ij,0,0}^{-1} \tau_{ij} \). Since \( i \neq j \), we have
\[
x_{j,i}^+ x_{i,r}^+ = \text{tr}_{i,j} x_{i}^+(x_{i,r}^+) = \text{tr}_{i,j} (\alpha''(x_{i,r}^+)\alpha') = \text{tr}_{i,j} (x_{i,r}^+) = x_{i,r}^+ x_{i,s}^+,
\]
yielding relation \((f)\) in this case.

Assume now \( a_{ij} < 0 \). Then the maps \( \alpha' \), \( \alpha'' \) are given by
\[
\alpha' = \sum_{a=0}^{[m+1]} \alpha_{(2a,m-2a)}^+ + \sum_{a=0}^{[m]} \alpha_{(2a,m-2a)},
\]
\[
\alpha'' = \sum_{a=0}^{[m]} \alpha_{(2a+1,m-1-2a)}^- - \sum_{a=0}^{[m+1]} \alpha_{(2a+1,m-1-2a)},
\]
where for \( a \in [0,m] \) and \( b = m-a \), we have
\[
\alpha_{a,b}^+ = (e_{i,a+1} E_j e_{i,b-1}) \circ \tau_{a+1} \circ \tau_{a+2} \cdots \circ \tau_{a+b} \circ \tau_a : E_i^{(a)} E_j E_i^{(b)} \to E_i^{(a+1)} E_j E_i^{(b-1)},
\]
\[
\alpha_{a,b}^- = (e_{i,a-1} E_j e_{i,b+1}) \circ \tau_{a} \circ \tau_{a-1} \cdots \circ \tau_1 \circ \tau_a : E_i^{(a)} E_j E_i^{(b)} \to E_i^{(a-1)} E_j E_i^{(b+1)}.
\]

Here \( \tau_a : E_i^{(a)} E_j E_i^{(b)} \to E_i^{(a-1)} E_j E_i^{(b+1)} \) is the canonical embedding, and \( \tau_k \) acts by \( \tau \) on the \( k \)-th and \( k+1 \)-th copy of \( E \) in the sequence \( E_i^{(a)} E_j E_i^{(b)} \).

Given any integers \( r_1, \ldots, r_m, s \in \mathbb{N} \), we define for each \( a \in [0,m] \) a morphism
\[
\Xi_a = \sum_{w \in \mathcal{S}_m} \prod_{i=1}^{m} x_i^{r_{w(1)}} \cdots x_i^{r_{w(a)}} x_i^{s} \cdots x_i^{r_{w(m)}} \in \text{End}(E_i^{(a)} E_j E_i^{(b)}).
\]
Note that \( \Xi_a \) is symmetric in the first \( a \)-tuple of \( x_i \)'s and also in the last \( b \)-tuple of \( x_i \)'s, hence it commutes with the divided power operator \( e_{i,a} E_j e_{i,b} \). By consequence \( \Xi_a \) restricts to a well defined endomorphism of \( E_i^{(a)} E_j E_i^{(b)} \), which we denote again by \( \Xi_a \). Further, since \( E_i^{(a)} E_j E_i^{(b)} \simeq (E_i^{(a)} E_j E_i^{(b)}) \otimes \mathbb{K} \), we have \( \text{tr}_{E_i^{(a)} E_j E_i^{(b)}}(\Xi_a) = (a!) \text{tr}_{E_i^{(a)} E_j E_i^{(b)}}(\Xi_a) \).

**Claim B.6.** We have
\[
(b) \quad \alpha_{a,b}^+ \Xi_a = \Xi_{a+1} \alpha_{a,b}^+, \quad \alpha_{a,b}^- \Xi_a = \Xi_{a-1} \alpha_{a,b}^-.
\]

Let us show how to deduce the relation \((f)\) from this claim. Part \((b)\) implies that the following diagram commute
\[
\begin{array}{ccc}
\bigoplus_{a=0}^{[m]} E_i^{(2a)} E_j E_i^{(m-2a)} & \xrightarrow{\alpha''} & \bigoplus_{a=0}^{[m]} E_i^{(2a+1)} E_j E_i^{(m-2a-1)} \\
\bigoplus_{a=0}^{[m]} E_i^{(2a)} E_j E_i^{(m-2a)} & \xrightarrow{\alpha'} & \bigoplus_{a=0}^{[m]} E_i^{(2a+1)} E_j E_i^{(m-2a-1)}.
\end{array}
\]

Therefore
\[
\sum_{a=0}^{[m]} \text{tr}_{E_i^{(2a)} E_j E_i^{(m-2a)}}(\Xi_{2a}) = \sum_{a=0}^{[m]} \text{tr}_{E_i^{(2a+1)} E_j E_i^{(m-2a-1)}}(\Xi_{2a+1}).
\]
Hence by part (a) of the claim we get \( \sum_{w \in \mathcal{S}_m} [x_{i,r(w(1))}^+, \ldots, x_{i,r(w(m-1))}^+, x_{j,s}^+] = 0 \) as desired.

It remains to prove the claim. Let us introduce some more notation. For \( a \in [0, m] \) let \( \Gamma^a = \{ k = (k_1, \ldots, k_a) \in \mathbb{N}_a \mid 1 \leq k_1 < k_2 < \cdots < k_a \leq m \} \), and for \( k \in \Gamma^a \) we set \( \kappa^b = (l_1, \ldots, l_b) \in \Gamma^b \) such that \( \{k_1, \ldots, k_a\} \cup \{l_1, \ldots, l_b\} = \{r_1, \ldots, r_m\} \). Let \( \mathcal{S}^{(a,b)} \) be the set of minimal representatives of the left cosets \( \mathcal{S}_m / \mathcal{S}_a \times \mathcal{S}_b \). Then the map \( w \mapsto w(1, 2, \ldots, m) \) yields a bijection \( \mathcal{S}^{(a,b)} \simeq \Gamma^a \). Let \( w_b \) be the longest element in \( \mathcal{S}_b \). Given any sequence \( r_1, \ldots, r_m \) and \( s \in \mathbb{N} \) we have

\[
[x_{i,r_1}^+, [x_{i,r_2}^+, \ldots, [x_{i,r_m}^+, x_{j,s}^+] \ldots]] = m \sum_{a=0}^m (-1)^b \sum_{y \in \mathcal{S}^{(a,b)}_{w_b}} x_{i,r_{y(1)}}^+ \cdots x_{i,r_{y(a)}}^+ x_{j,s}^+ \cdots x_{i,r_{y(m)}}^+.
\]

It follows that

\[
\sum_{w \in \mathcal{S}_m} [x_{i,r(w(1))}^+, [x_{i,r(w(2))}^+, \ldots, [x_{i,r(w(m))}^+, x_{j,s}^+] \ldots]] = m \sum_{a=0}^m (-1)^b \sum_{y \in \mathcal{S}^{(a,b)}_{w_b}} x_{i,r_{wy(1)}}^+ \cdots x_{i,r_{wy(a)}}^+ x_{j,s}^+ \cdots x_{i,r_{wy(m)}}^+.
\]

To check the relation in (b), without loss of generality we may assume \( \alpha = m\alpha_i + \alpha_j \). Then \( \Xi_a \) is represented by the left multiplication on \( j^{(a)} j^{(m-a)} P \) by

\[
\sum_{w \in \mathcal{S}_m} x_{1}^{r_{w(1)}} \cdots x_{a}^{r_{w(a)}} x_{a+1}^{r_{w(a+1)}} \cdots x_{m+1}^{r_{w(m)}}.
\]
and \( \alpha_{a,b}^+ \) represented by \((1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^a j^b) \tau_{a+1} \tau_{a+2} \cdots \tau_m \). Since \( \Xi_a \) is symmetric in \( \{x_{a+2}, \ldots, x_{m+1}\} \), we have \( e(i^a j^b) \tau_k \Xi_a = e(i^a j^b) \Xi_{a \tau_k} \) for any \( k \in [a+2, m] \). Next,

\[
e(i^{a+1} j^{b-1}) \tau_{a+1} \Xi_a = \tau_{a+1} e(i^{a+1} j^{b-1}) \Xi_a
\]

\[
= \sum_{w \in \mathcal{G}_m} x_w^{r(a+1)} \cdots x_a^{r(a+1)} (x_{a+2}^{r(a+1)}) \cdots x_m^{r(m)} e(i^a j^b)
\]

\[
= \Xi_{a+1} \tau_{a+1} e(i^a j^b)
\]

Finally, since \( \Xi_{a+1} \) is symmetric in \( \{x_1, \ldots, x_{a+1}\} \) and in \( \{x_{a+3}, \ldots, x_{a+b+1}\} \), the divided power operator \( 1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1} \) commute with \( \Xi_{a+1} \). To summarize, we have

\[
\alpha_{a,b}^+ \Xi_a = (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^{a+1} j^{b-1}) \tau_{a+1} \tau_{a+2} \cdots \tau_{a+b} \Xi_a
\]

\[
= (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^{a+1} j^{b-1}) \tau_{a+1} \Xi_a \tau_{a+2} \cdots \tau_{a+b}
\]

\[
= (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) \Xi_{a+1} \tau_{a+2} \cdots \tau_{a+b}
\]

\[
= \Xi_{a+1} \alpha_{a,b}^+ ,
\]

which is the first equality in part \((b)\), the proof for the second one is similar.

Finally, we can deduce Theorem 3.22 for \( \ell = 1 \) as a special case of Proposition B.5. Indeed, assume \( g \) is symmetric and (I1) holds. Then, we have

\[
q_{ij}(u, v) = (1 - uv)^{-a_{ij}}, \quad \forall i, j .
\]

We must check that the relations \((c), (d)\) in Proposition B.5 can be re-written as

\[
(c) \quad [h_{ir}, x_j^{\pm}] = \pm a_{ij} x_{j,r+s}^{\pm} ,
\]

\[
(d) \quad \sum_{p=0}^{m} (-1)^p (\frac{m}{p}) [x_i^{\pm}, r_{i+p}, x_j^{\pm}, s_{i+s+m-p}] = 0 \text{ with } i \neq j \text{ and } m = -a_{ij}.
\]

The relation \((d)\) is obvious, because by (I1) we have \( c_{i,j,p,q} = \delta_{q,-a_{ij}-p} r_{ij} (-1)^p (-a_{ij}) \) for all \( p, q \) and \( r_{ij} \) is invertible. Let us prove \((c)\). Given \( \alpha \) of height \( n \) and \( \lambda = \Lambda - \alpha \), by Proposition B.3 we have

\[
B_{-i, \lambda}(z) = \sum_{\nu \in I^\alpha} \prod_{p=1}^{\Lambda_i} (1 + y_p z) \prod_{k=1}^{n} (1 - z x_k)^{-a_{\nu k}} e(\nu).
\]

By Lemma B.4 we have

\[
\exp \left( - \sum_{r \geq 1} h_{ir, \lambda} z^r / r \right) = B_{-i, \lambda}(z).
\]

Since the \( e(\nu) \)'s are orthogonal idempotents in \( R^\Lambda(\alpha) \), we have

\[
\exp \left( - \sum_{r \geq 1} h_{ir, \lambda} z^r / r \right) = \sum_{\nu \in I^\alpha} \exp \left( - \sum_{r \geq 1} h_{ir, \lambda} e(\nu) z^r / r \right) e(\nu) .
\]
Therefore, we have
\[
\sum_{r \geq 1} h_{ir, \lambda} e(\nu) z^r = z (d/dz) \log \left( \prod_{p=1}^{A_i} (1 + y_{ip} z) \prod_{k=1}^{n} (1 - z x_k)^{-a_{i,\nu k}} \right) e(\nu)
\]
\[
h_{ir, \lambda} = \left( \sum_{p=1}^{A_i} (-y_{ip})^r - \sum_{k=1}^{n} a_{i,\nu k} x_k^r \right) e(\nu) \quad \forall r \geq 1.
\]

In particular, we deduce that
\[
h_{ir, \lambda - \alpha_j} e(\alpha, j) = (h_{ir, \lambda} - a_{ij} x_{n+1}^r) e(\alpha, j).
\]
Hence for any \( f \in \mathcal{R}^\lambda(\alpha) \) we have
\[
h_{ir} x_{js}^- tr(f) = tr(h_{ir, \lambda - \alpha_j} x_{n+1}^s f e(\alpha, j))
\]
\[
= tr(h_{ir, \lambda} x_{n+1}^s f e(\alpha, j)) - a_{ij} tr(x_{n+1}^{r+s} e(\alpha, j))
\]
\[
= x_{js}^- h_{ir} - a_{ij} x_{j,r+s}^-.
\]
The proof is complete.

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