The KPP equation as a scaling limit
of locally interacting Brownian particles

Franco Flandoli and Ruojun Huang
January 5, 2021

Abstract

Fisher-KPP equation is proved to be the scaling limit of a system of Brownian particles with local interaction. Particles proliferate and die depending on the local concentration of other particles. Opposite to discrete models, controlling concentration of particles is a major difficulty in Brownian particle interaction; local interactions instead of mean field or moderate ones makes it more difficult to implement the law of large numbers properties. The approach taken here to overcome these difficulties is largely inspired by A. Hammond and F. Rezakhanlou [10] implemented there in the mean free path case instead of the local interaction regime.

1 Introduction

We consider the scaling limit (in a “local” interaction regime) of the empirical measure of an interacting Brownian particle system in $\mathbb{R}^d$ describing the proliferation mechanism of cells, where only approximately a constant number of particles interact with a given particle at any given time. We connect the evolution of its empirical measure process to the Fisher-KPP equation.

The Fisher-KPP equation is related to particle systems in several ways. One of them is the probabilistic representation by branching processes, see [12] which originated a large literature. Others have to do with scaling limits of interacting particles. In the case of discrete particles occupying the sites of a lattice, with local interaction some of the main classical works are [3, 5, 6]; see also the recent contributions [4, 9].

The discrete setting, as known for many other systems (see for instance [11]) offers special opportunities due to the simplicity of certain invariant or underlying measures, often of Bernoulli type; the technology in that case has become very rich and deeply developed. Different is the case of interacting diffusions, less developed. The mean field theory, for diffusions, is a flexible and elegant theory [19] but localizing the interactions is very difficult, see for instance [21, 20] as few of the attempts. When the interaction is moderate, namely intermediate between mean field and local, there are more technical opportunities, widely discovered by K. Oelschläger in a series of works including [16, 17]. Along these lines, also the Fisher-KPP equation has been obtained as a scaling limit in [8]. Let us mention also [13, 11, 14, 15, 18] for related works.

In the present work we fill in the gap and prove that the Fisher-KPP equation is also the scaling limit of diffusions, locally interacting. In a sense, this is the analog of the discrete particle results of [8] and the other references above. The proof is not based on special reference measures of Bernoulli type as in the discrete case (not invariant in the present proliferation case, but still fundamental), but it is strongly inspired by the work of [10], which deals with locally interacting diffusions in the mean-free-path regime, that we adapt to the local regime (the former requires that a particle meets a finite number of others, in the average, in the unit of time; the latter requires that a particle has a finite number of others, in the average, in its own neighbor, where interaction takes place). Compared to the discrete setting [8], where the dynamics is a superposition of simple-exclusion process (which leads to the diffusion operator) and spin-flip dynamics (leading to reaction term) and the number of particles per site is either zero or one, we have to worry about concentration of particles, one of the main difficulties for the investigation of interacting diffusions.
After this short introduction to the subject, let us give some more technical details. We start with the formal definition of a closely-related model already studied in [8] in the so-called intermediate interaction regime. Then we introduce our slightly altered model.

**Definition 1** A configuration of the system is a sequence

$$\eta = (x_i, a_i)_{i \in \mathbb{N}} \in (\mathbb{R}^d \times \{L, N\})^\mathbb{N}$$

with the following constraint: there exists $$i_{\text{max}}(\eta) \in \mathbb{N}$$ such that $$a_i = L$$ for $$i \leq i_{\text{max}}(\eta)$$, $$a_i = N$$ and $$x_i = 0$$ for $$i > i_{\text{max}}(\eta)$$.

The heuristic meaning is that particles with index $$i \leq i_{\text{max}}(\eta)$$ exist, are alive ($$= L$$), and occupy position $$x_i$$; particles with $$i > i_{\text{max}}(\eta)$$ do not exist yet ($$= N$$), but we formally include them in the description; they are placed at $$x_i = 0$$.

Test functions $$F$$ are functions on $$(\mathbb{R}^d \times \{L, N\})^\mathbb{N}$$ which depend only on a finite number of coordinates, $$F = F(x_1, ..., x_n, a_1, ..., a_n)$$ with $$(x_i, a_i) \in \mathbb{R}^d \times \{L, N\}$$ and are smooth in $$(x_1, ..., x_n) \in \mathbb{R}^{dn}$$.

**Definition 2** The infinitesimal generator $$\mathcal{L}_N$$, parametrized by $$N \in \mathbb{N}$$, is given by

$$\mathcal{L}_N F (\eta) = \sum_{i \leq i_{\text{max}}(\eta)} \frac{1}{2} \Delta x_i F (\eta) + \sum_{j \leq i_{\text{max}}(\eta)} \lambda_j^i(\eta) [F(\eta^j) - F(\eta)]$$

(1)

where, if $$\eta = (x_i, a_i)_{i \in \mathbb{N}}$$, then $$\eta^j = (x_i^j, a_i^j)_{i \in \mathbb{N}}$$ is given by

$$(x_i^j, a_i^j) = (x_i, a_i) \text{ for } i \neq i_{\text{max}}(\eta) + 1$$

$$(x_i^j, a_i^j) = (x_j, L).$$

The rate $$\lambda_j^i(\eta)$$ is given by

$$\lambda_j^i(\eta) = \left(1 - \frac{1}{N} \sum_{k \leq i_{\text{max}}(\eta)} \theta_N(x_j - x_k) \right)^+$$

(2)

where $$\theta_N$$ are smooth compact support mollifiers with a rate of convergence to the delta Dirac at zero specified in the sequel.

The heuristic behind this definition is that: i) existing particles move at random like independent Brownian motions; ii) a new particle could be created at the position of an existing particle $$j$$, with rate proportional to the empty space in a neighborhood of $$x_j$$, neighborhood described by the support of $$\theta_N$$. Our aim is to choose the scaling of $$\theta_N$$, namely the neighborhood of interaction, such that only a small finite number of particles different from $$j$$ are in that neighborhood.

In the classical studies of continuum interacting particle systems, where interactions are modulated by a potential, one usually takes

$$\theta_N(x) = N^\beta \theta(N^{\beta/d} x)$$

for some smooth compactly supported function $$\theta(\cdot)$$, where $$N$$ is the order of the number of particles in the system. The case $$\beta = 0$$ is called mean-field, since all particles interact with each other at any given time. The case $$\beta \in (0, 1)$$ is called moderate, as not all particles interact at any given time, nevertheless such number is diverging with $$N$$. The case $$\beta = 1$$ is called local, as one would expect that in a neighborhood of radius $$N^{-1/d}$$, only a constant number of particles interact. Of course, here we are assuming that particles are relatively homogeneously distributed in space at all times down to the microscopic scale (which is not always proven). For the system with generator (1), the intermediate scaling regime with $$\beta \in (0, 1/2)$$ has
been studied and its scaling limit to F-KPP equation established in [8], with earlier results [17] for a shorter range of $\beta$, and so is the mean-field case whose limit is a different kind of equation [2, 7]; our aim here is to study the local regime, subject to a modification of the rate (2).

In the local regime we associate for every positive integer $N$, a parameter $\epsilon$ such that

$$\epsilon^d N = 1.$$  \hfill (3)

We can write the rate function (2) in terms of $\epsilon$ and a given non-negative, smooth, and compactly supported function $\theta : \mathbb{R}^d \to \mathbb{R}_+$ with $\int_{\mathbb{R}^d} \theta = 1$, as follows

$$\lambda^i_N(\eta) = \left(1 - \epsilon^d \sum_k \epsilon^{-d} \theta(\epsilon^{-1}(x_j - x_k))\right)^+ = \left(1 - \sum_k \theta(\epsilon^{-1}(x_j - x_k))\right)^+,$$  \hfill (4)

whereby the proliferation part of the generator is

$$(\mathcal{L}_C F)(\eta) := \sum_j \left[1 - \sum_k \theta(\epsilon^{-1}(x_j - x_k))\right]^+ [F(\eta^j) - F(\eta)].$$

To simply notations, throughout the paper any sum is understood to be only over particles alive in the system (whose cardinality is always finite), and any double sum is understood to be over distinct particles. Thereby, we do not discuss the label $a_j$ of particle $j$.

Heuristically, the positive part on the rate (4) should be insignificant, as we would guess that if starting with a density profile not larger than 1, then subsequently the density of particles everywhere is no larger than 1. This is the case for the F-KPP equation (see (8) below). However, at the microscopic level we do not have effective control on the scale of $\epsilon$, even a posteriori. Hence in this paper we consider a slightly altered model, namely one without the positive part in the rate.

Note that now the proliferation rate can be negative, which we will interpret to mean, in terms of the proliferation part of the generator,

$$(\tilde{\mathcal{L}}_C F)(\eta) = \sum_j [F(\eta^j) - F(\eta)] + \sum_j \sum_k \theta(\epsilon^{-1}(x_j - x_k))[F(\eta^{-j}) - F(\eta)],$$  \hfill (5)

where $\eta^{-j}$ signifies deleting particle $j$ from the collection $\eta$. Thus, the infinitesimal generator of our particle system under study is

$$(\tilde{\mathcal{L}}_N F)(\eta) = \sum_i \frac{1}{2} \Delta_x F(\eta) + (\tilde{\mathcal{L}}_C F)(\eta).$$

**Condition 3** The initial density $u_0$ satisfies:

(a). It is compactly supportly in $B_0(\kappa)$, a ball of radius $\kappa$ around the origin.

(b). $0 \leq u_0(x) \leq \gamma$ for some finite constant $\gamma$ and all $x \in \mathbb{R}^d$.

In particular, $u_0 \in L^1(\mathbb{R}^d)$. At time $t = 0$, we distribute

$$i_{max}(\eta(0)) = N_0 := N \int_{\mathbb{R}^d} u_0$$

number of points independently with the same density $u_0(x)$ on $\mathbb{R}^d$, for some function $u_0$ satisfying Condition 4. In particular, $u_0(x) dx$ is the weak limit, in probability, of the initial empirical measure:

$$N^{-1} \sum_j \delta_{x_j(0)}(x) \Rightarrow u_0(x) dx.$$  \hfill (6)

We introduce the space-time normalized empirical measure

$$g^N(dt, dx) := N^{-1} dt \sum_j \delta_{x_j(t)}(dx)$$  \hfill (7)
Applying Itô formula to the process $Q$ density with respect to Lebesgue measure on $[0,T]$. This justifies (10) which implies the sequence of random measures with $u$ taking values in the space $M$. Theorem 4 which satisfies Condition 3. We take as weak formulation of (8) that for any $K \in \mathcal{C}^{1,2}_c([0,T] \times \mathbb{R}^d)$,

$$0 = \int_{\mathbb{R}^d} u_0(x)K(0,x)dx + \int_0^T \int_{\mathbb{R}^d} u(t,x)\partial_t K(t,x) dxdt + \int_0^T \int_{\mathbb{R}^d} \left[ \frac{1}{2} u(t,x)\Delta K(t,x) + (u-u^2)(t,x)K(t,x) \right] dxdt. \quad (9)$$

The choice of this weak formulation, together with the uniqueness of the F-KPP equation is discussed later, see Section 3.

**Theorem 4** For every finite $T$ and $d \geq 3$, the sequence of positive measures $\{g_N(dt, dx)\}_N$ converges weakly, in probability, in the space $\mathcal{M}$ to a limit measure $g(dt, dx)$ that is absolutely continuous with respect to Lebesgue measure on $[0,T] \times \mathbb{R}^d$, i.e. $g(dt, dx) = g(t,x)dtdx$. The density $g(t, x)$ is the unique weak solution to the Fisher-KPP equation (8), in the sense of (9).

**2 Proof of the main result**

Our proof is based on adapting the strategy of [10]. First observe that uniformly in $N$, 

$$\mathbb{E}[g^N([0,T] \times \mathbb{R}^d)] \leq e^T \int_{\mathbb{R}^d} u_0, \quad (10)$$

for every finite $T$. Indeed, our particle system is stochastically dominated from above by a system that is pure proliferation with unit rate (without killing). For the latter, it is not hard to show that the total expected number of particles grows exponentially in $t$, hence remains of order $O(N)$ on any compact time interval. This justifies [10] which implies the sequence of random measures $\{g^N(dt, dx)\}_N$ is tight in the space $\mathcal{M}$. Further, by Lemma [11] any subsequential limit of $\{g^N(dt, dx)\}_N$ has a uniformly bounded (in $N$) density with respect to Lebesgue measure on $[0,T] \times \mathbb{R}^d$.

**Proof of Theorem 4** Fixing any $K(t,x) \in \mathcal{C}^{1,2}_c([0,T] \times \mathbb{R}^d)$, we consider the time-dependent functional on $\eta$

$$Q(t,\eta) := \langle K(t,x), g^N(dt, dx) \rangle = N^{-1} \sum_j K(t,x_j).$$

Applying Itô formula to the process $Q(t,\eta(t))$, where $\eta(t)$ denotes the particle system configuration at time $t$, we find

$$Q(T,\eta(T)) - Q(0,\eta(0)) = N^{-1} \int_0^T \left( \sum_j \partial_t + \frac{1}{2} \sum_j \Delta x_j \right) K(t,x_j(t)) dt + N^{-1} \int_0^T \sum_j \left[ 1 - \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right] K(t,x_j(t)) dt + \tilde{M}_T$$

$$= \int_0^T \langle g^N(dt, dx), (\partial_t + \frac{1}{2} \Delta + 1)K(t,x) \rangle dt - N^{-1} \int_0^T \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) K(t,x_j(t)) K(t,x_k(t)) dt + \tilde{M}_T, \quad (11)$$

4
where \( \{ \tilde{M}_t \} \) is a martingale. We can readily control the second moment of the martingale by its quadratic variation as
\[
E[\tilde{M}_T^2] \leq 4 \int_0^T E[A_t^{(1)} + A_t^{(2)}] \, dt,
\]
where
\[
A_t^{(1)} := N^{-2} \sum_j |\nabla x_j K(t, x_j(t))|^2,
\]
\[
A_t^{(2)} := N^{-2} \sum_j \left[ 1 + \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right] K(t, x_j(t))^2.
\]
Since \( K \) is a test function and the summation signs over \( j \) have \( O(N) \) terms in expectation, combined with Lemma 5 we arrive at
\[
E \int_0^T [A_t^{(1)} + A_t^{(2)}] \, dt \leq CN^{-1}.
\]
Since \( K \in C^{1,2}([0,T] \times \mathbb{R}^d) \), we have \( Q(T, \eta(T)) = 0 \). Thus, the key is to prove the convergence of the proliferation term
\[
N^{-1}E \int_0^T \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t)))K(t, x_j(t)) \, dt
\]
to a product of densities, which will follow from Proposition 6. Indeed, applying it to (12) readily gives
\[
N^{-1} \int_0^T \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t)))K(t, x_j(t)) \, dt = \int_0^T dt \int_{\mathbb{R}^d} dw \, K(t, w)(\eta \star g^N)(t, w)^2 + err(\epsilon, \delta),
\]
where \( \eta(\cdot) := \delta^{-d} \eta(\cdot/\delta) \), and
\[
\limsup_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} E[err(\epsilon, \delta)] = 0.
\]
It now remains to send first \( \epsilon \rightarrow 0 \), then \( \delta \rightarrow 0 \) in (11), to obtain that any subsequential limit of \( \{ g^N(dt, dx) \}_N \) admits a bounded density that satisfies the F-KPP equation in the weak form (9). The proof is the same as in [10, Section 5], hence we omit it. We mention in particular that when passing the limit \( \delta \rightarrow 0 \), we use Lemma 11 that any subsequential limit (as \( \epsilon \rightarrow 0 \), equivalently, \( N \rightarrow \infty \)) of \( \{ g^N \star_x \eta^\delta \}_N \) is uniformly bounded in \( \delta \), in order to apply the dominated convergence theorem.

Finally, by Proposition 14 the F-KPP equation has a Lebesgue-a.e. unique weak solution per given initial condition \( u_0 \), thus all subsequential limits of \( \{ g^N(dt, dx) \}_N \) agree.

We start with a preliminary lemma.

**Lemma 5** For any \( d \geq 1 \) and finite \( T \), there exists finite constant \( C = C(T, u_0) \) such that
\[
E \int_0^T dt \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \leq CN.
\]

**Proof.** Applying Itô formula to the process of the total number of particles \( i_{\max}(\eta(t)) \), and then taking expectation, we get
\[
E i_{\max}(\eta(T)) = E i_{\max}(\eta(0)) + E \int_0^T dt \sum_j \left[ 1 - \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right],
\]
implying
\[
\mathbb{E} \int_0^T dt \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \leq \mathbb{E} i_{\text{max}}(\eta(0)) + \mathbb{E} \int_0^T i_{\text{max}}(\eta(t)) \, dt
\]

The RHS is dominated from above by the same quantity calculated for a particle system with pure proliferation of unit rate (and no killing), and thereby is bounded by \( e^T N \int_{\mathbb{R}^d} u_0 \). \qed

We note (12) is of the general form (taking \( J, \tilde{J} \in C^{1,2}_{c}([0, T] \times \mathbb{R}^d) \))
\[
N^{-1} \int_0^T \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) J(t, x_j(t)) \tilde{J}(t, x_k(t)) \, dt,
\]
which we claim can be approximated as follows.

**Proposition 6** For any \( d \geq 3 \) and finite \( T \), we have that
\[
N^{-1} \int_0^T dt \sum_{j,k} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) J(t, x_j(t)) \tilde{J}(t, x_k(t)) = \int_0^T dt \int_{\mathbb{R}^d} dw J(t, w) \tilde{J}(t, w) N^{-1} \sum_j \delta^d \eta \left( \frac{x_j(t) - w}{\delta} \right) N^{-1} \sum_k \delta^d \eta \left( \frac{x_k(t) - w}{\delta} \right) + \text{err}(\epsilon, \delta),
\]
for some error term that vanishes in the following limit
\[
\limsup_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{E} |\text{err}(\epsilon, \delta)| = 0,
\]
and \( \eta : \mathbb{R}^d \to \mathbb{R}_+ \) is some bump function with \( \int_{\mathbb{R}^d} \eta = 1 \).

**Proof.** As in [10], we introduce the time-dependent functional on \( \eta \), indexed by \( z \in \mathbb{R}^d \) (having in mind that \( |z| \gg \epsilon \))
\[
X_z(t, \eta) := N^{-2} \sum_{j,k} r^\epsilon(x_j - x_k + z) J(t, x_j) \tilde{J}(t, x_k),
\]
where \( J, \tilde{J} \in C^{1,2}_{c}([0, T] \times \mathbb{R}^d) \) and
\[
r^\epsilon(\cdot) = \epsilon^{2-d} r(\cdot/\epsilon).
\]
(14)

Here \( r(x) \) is a positive solution of the PDE
\[
-\Delta r(x) = \theta(x)
\]
that admits the representation
\[
r(x) = c_0(d) \int_{\mathbb{R}^d} \frac{\theta(y)}{|y - x|^{d-2}} dy,
\]
which is bounded (recall that \( \theta \) is bounded and compactly supported), where \( c_0 \) is dimensional constant. We note that by scaling
\[
-\Delta r^\epsilon(x) = \theta^\epsilon(x) := \epsilon^{-d} \theta(x/\epsilon).
\]
(15)

Applying Itô formula to the process \( (X_z - X_0)(t, \eta(t)) \), we have
\[
(X_z - X_0)(T, \eta(T)) - (X_z - X_0)(0, \eta(0)) = \int_0^T \left[ (\partial_t + \mathcal{L}_0)(X_z - X_0)(t, \eta(t)) + \tilde{\mathcal{L}}_C(X_z - X_0)(t, \eta(t)) \right] \, dt + M_T
\]
where \( \{M_t\} \) is a martingale. Specifically, the time-derivative and diffusion operator \((\partial_t + L_0)\) produce the following terms

\[
H_1 := N^{-2} \int_0^T dt \sum_{j,k} \left[ -\theta'(x_j(t) - x_k(t) + z) + \theta'(x_j(t) - x_k(t)) \right] J(t, x_j(t)) \bar{J}(t, x_k(t)),
\]

\[
H_{21} := N^{-2} \int_0^T dt \sum_{j,k} \nabla_{x_j} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] \cdot \nabla_{x_j} J(t, x_j(t)) \bar{J}(t, x_k(t))
\]
\[
\quad \quad \quad \quad + N^{-2} \int_0^T dt \sum_{j,k} \nabla_{x_k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] \cdot \nabla_{x_k} \bar{J}(t, x_k(t)) J(t, x_j(t)),
\]

\[
H_{22} := N^{-2} \int_0^T dt \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] \Delta_{x_j} J(t, x_j(t)) \bar{J}(t, x_k(t))
\]
\[
\quad \quad \quad \quad + N^{-2} \int_0^T dt \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] \Delta_{x_k} \bar{J}(t, x_k(t)),
\]

\[
H_{23} = N^{-2} \int_0^T dt \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] \partial_t J(t, x_j(t)) \bar{J}(t, x_k(t))
\]
\[
\quad \quad \quad \quad + N^{-2} \int_0^T dt \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] J(t, x_j(t)) \partial_t \bar{J}(t, x_k(t)).
\]

The martingale term is bounded via

\[
\mathbb{E}[M_t^2] \leq 4 \int_0^T \mathbb{E}[B_t^{(1)} + B_t^{(2)}] dt
\]

for

\[
B_t^{(1)} := N^{-4} \left| \nabla_{x_j} \left( \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] J(t, x_j(t)) \bar{J}(t, x_k(t)) \right) \right|^2
\]
\[
\quad \quad \quad \quad + N^{-4} \left| \nabla_{x_k} \left( \sum_{j,k} \left[ r^r(x_j(t) - x_k(t) + z) - r^r(x_j(t) - x_k(t)) \right] J(t, x_j(t)) \bar{J}(t, x_k(t)) \right) \right|^2.
\]

\[
B_t^{(2)} := N^{-4} \sum_j \left[ 1 + \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right]
\]
\[
\quad \quad \cdot \left| \sum_i \left[ r^r(x_i(t) - x_j(t) + z) - r^r(x_i(t) - x_j(t)) \right] J(t, x_i(t)) \bar{J}(t, x_j(t)) \right|^2
\]
\[
\quad \quad \quad \quad + N^{-4} \sum_j \left[ 1 + \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right]
\]
\[
\quad \quad \cdot \left| \sum_i \left[ r^r(x_i(t) - x_j(t) + z) - r^r(x_i(t) - x_j(t)) \right] J(t, x_i(t)) \bar{J}(t, x_j(t)) \right|^2.
\]

The proliferation operator \( \bar{L}_C \) produces only one term

\[
H_3 := N^{-2} \int_0^T dt \sum_j \left[ 1 - \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right]
\]
\[
\quad \quad \cdot \left| \sum_i \left[ r^r(x_i(t) - x_j(t) + z) - r^r(x_i(t) - x_j(t)) \right] J(t, x_i(t)) \bar{J}(t, x_j(t)) \right|
\]
\[
\quad \quad \quad \quad + N^{-2} \int_0^T dt \sum_j \left[ 1 - \sum_k \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \right]
\]
\[
\quad \quad \cdot \left| \sum_i \left[ r^r(x_i(t) - x_j(t) + z) - r^r(x_i(t) - x_j(t)) \right] J(t, x_i(t)) \bar{J}(t, x_j(t)) \right|.
\]
Let us separate $H_3$ into two parts:

$$H_{31} := N^{-2} \int_0^T dt \sum_{j,k} \theta^\epsilon (x_j(t) - x_k(t)) J(t, x_j(t)) \bar{J}(t, x_j(t)),$$

$$H_{32} := N^{-2} \int_0^T dt \sum_{j,k,i} \theta^{-1}(x_j(t) - x_k(t))$$

$$\cdot [r^\epsilon(x_j(t) - x_i(t) + z) - r^\epsilon(x_j(t) - x_i(t)) ] J(t, x_i(t)) \bar{J}(t, x_j(t))$$

$$+ N^{-2} \int_0^T dt \sum_{j,k,i} \theta^{-1}(x_j(t) - x_k(t))$$

$$\cdot [r^\epsilon(x_i(t) - x_j(t) + z) - r^\epsilon(x_i(t) - x_j(t)) ] J(t, x_j(t)) \bar{J}(t, x_i(t)).$$

Since $J, \bar{J} \in C^{1,2}_c([0, T] \times \mathbb{R}^d)$, we only have

$$H_4 := (X_z - X_0)(0, \eta(0))$$

$$= N^{-2} \sum_{j,k} [r^\epsilon(x_j(0) - x_k(0) + z) - r^\epsilon(x_j(0) - x_k(0))] J(0, x_j(0)) \bar{J}(0, x_k(0))$$

while $(X_z - X_0)(T, \eta(T)) = 0$.

By Lemma 8 and Lemma 9 in the limit $\epsilon \to 0$ followed by $\delta \to 0$ the only significant term is $H_1$. Thus, we have that for any $\epsilon, |z|$ small enough,

$$N^{-2} \int_0^T dt \sum_{j,k} \theta^\epsilon (x_j(t) - x_k(t)) J(t, x_j(t)) \bar{J}(t, x_k(t))$$

$$= N^{-2} \int_0^T dt \sum_{j,k} \theta^\epsilon (x_j(t) - x_k(t) + z) J(t, x_j(t)) \bar{J}(t, x_k(t)) + err(|z|, \epsilon),$$

with

$$\lim_{|z| \to 0} \limsup_{\epsilon \to 0} \mathbb{E}[err(|z|, \epsilon) = 0.$$

The LHS of (10) is exactly the quantity (12) we are interested in. The rest of the argument is essentially the same as in [10] Pages 42-43, hence we only give a sketch.

Since the LHS of (13) is independent of $z$, if we introduce some smooth bump function $\eta : \mathbb{R}^d \to \mathbb{R}_+$ with $\int_{\mathbb{R}^d} \eta = 1$ (and denoting $\eta^\delta(\cdot) := \delta^{-d} \eta(\cdot / \delta)$), then we have

$$N^{-2} \int_0^T dt \sum_{j,k} \theta^\epsilon (x_j(t) - x_k(t)) J(t, x_j(t)) \bar{J}(t, x_k(t))$$

$$= N^{-2} \int_0^T dt \int_{\mathbb{R}^{2d}} dz_1 dz_2 \sum_{j,k} \theta^\epsilon (x_j(t) - x_k(t) - z_1 + z_2) \eta^\delta(z_1) \eta^\delta(z_2)$$

$$\cdot J(t, x_j(t)) \bar{J}(t, x_k(t)) + err(\delta, \epsilon)$$

with

$$\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \mathbb{E}[err(\delta, \epsilon) = 0.$$

Shifting the arguments of $J(t, \cdot), \bar{J}(t, \cdot)$ in (17) by $z_1$ and $z_2$, respectively (with the latter two in the support
of $\eta^\delta$), will cause an error of $O(\delta)$, whereby we rewrite the above expression as

$$N^{-2} \int_0^T dt \int_{\mathbb{R}^d} dz_1 dz_2 \sum_{j,k} \theta^r(x_j(t) - x_k(t) - z_1 + z_2) \eta^\delta(z_1) \eta^\delta(z_2) \cdot J(t, x_j(t) - z_1) J(t, x_k(t) - z_2) + err_1(\delta, \epsilon)$$

$$= N^{-2} \int_0^T dt \int_{\mathbb{R}^d} dw_1 dw_2 \theta^r(w_1 - w_2) \cdot J(t, w_1) J(t, w_2) \sum_j \eta^\delta(x_j(t) - w_1) \sum_k \eta^\delta(x_k(t) - w_2) + err_1(\delta, \epsilon)$$

$$= \int_0^T dt \int_{\mathbb{R}^d} dw_1 dw_2 \theta^r(w_1 - w_2) J(t, w_1) J(t, w_2) \cdot (\eta^\delta \star x g^N)(t, w_1) (\eta^\delta \star x g^N)(t, w_2) + err_2(\delta, \epsilon),$$

where the second line is a change of the order of integration, and $E|err_1(\delta, \epsilon)| = E|err(\delta, \epsilon)| + O(\delta)$. Since $\eta^\delta, J$ are all smooth, and $|w_1 - w_2| < 2C_0 \epsilon$ within the support of $\theta^r$, changing the $w_2$ to $w_1$ in the argument of $\eta^\delta, J(t, \cdot), \bar{J}(t, \cdot)$ will cause an error on the order $O(\epsilon \delta^{-2d-1})$, and we can rewrite the above

$$\int_0^T dt \int_{\mathbb{R}^d} dw_1 \left( \int_{\mathbb{R}^d} dw_2 \theta^r(w_1 - w_2) \right) J(t, w_1) J(t, w_2) \cdot (\eta^\delta \star x g^N)(t, w_1)^2 + err_2(\delta, \epsilon)$$

$$= \int_0^T dt \int_{\mathbb{R}^d} dw_1 J(t, w_1) \bar{J}(t, w_1) (\eta^\delta \star x g^N)(t, w_1)^2 + err_2(\delta, \epsilon),$$

where $E|err_2(\delta, \epsilon)| = E|err_1(\delta, \epsilon)| + O(\epsilon \delta^{-2d-1})$. The last step is due to $\int \theta^r = 1$. Note $E|err_2(\delta, \epsilon)|$ vanishes with $\epsilon$ and $\delta$ in the right order. \hfill ■

Recall (15) that we have defined $r^\epsilon(\cdot) \eta$ a positive solution of

$$-\Delta r^\epsilon = \theta^r,$$

which admits the representation

$$r^\epsilon(x) = c_0 \int_{\mathbb{R}^d} \frac{\theta^r(y)}{|x - y|^{d-1}} dy. \quad (18)$$

Let $C_0$ denote the maximum radius of the compact support of $\theta(\cdot)$. We have the following difference estimates for $r^\epsilon$ and $\nabla r^\epsilon$:

**Lemma 7** For any $d \geq 3$, there exists a finite constant $C = C(d, C_0)$ such that for any $\epsilon \in (0, 1), z \in \mathbb{R}^d$ and

$$|x| \geq \max\{2|z| + C_0 \epsilon, 2C_0 \epsilon\},$$

we have that

$$|r^\epsilon(x) - r^\epsilon(x + z)| \leq C \frac{|z|}{|x|^{d-1}},$$

$$|\nabla r^\epsilon(x) - \nabla r^\epsilon(x + z)| \leq C \frac{|z|}{|x|^{d}}.$$  

Moreover, there exists $C = C(d, C_0) < \infty$ such that for any $x \in \mathbb{R}^d$ and $\epsilon > 0$, we have that

$$r^\epsilon(x) \leq C|x|^{2-d} \wedge C\epsilon^{2-d}, \quad |\nabla r^\epsilon(x)| \leq C|x|^{1-d} \wedge C\epsilon^{1-d}.$$
Proof. This lemma is essentially already proved in [10, Lemma 3.5]; though the PDE is slightly different there, the asymptotic behavior is the same; we include the proof for completeness. Throughout the constant $C = C(d, C_0)$ changes from line to line. When

$$|x| \geq \max\{2|z| + C_0\epsilon, 2C_0\epsilon\},$$

we have, for any $y$ in the support of $\theta^\epsilon$,

$$|x - y| \geq \epsilon C_0 \geq |y|$$

$$|x - y| \geq |x| - |y| \geq |x|/2,$$

$$|x + z - y| \geq |x| - |z| - |y| \geq |x|/2.$$  

Then, using the Lipschitz property of the function $g(x) = x^{2-d}$ away from zero, the previous estimates, and $\int \theta^\epsilon = 1$, we get

$$|r^\epsilon(x + z) - r^\epsilon(x)| \leq c_0 \int_{\mathbb{R}^d} \theta^\epsilon(y) \left| \frac{1}{|x + z| - y} - \frac{1}{|x - y|} \right| dy$$

$$\leq C \int_{\mathbb{R}^d} \theta^\epsilon(y) \frac{|z|}{|x - y|} dy = C \frac{|z|}{|x|^{d-1}}.$$

Then, using the Lipschitz property of the function $g(x) = x^{2-d}$ away from zero, the previous estimates, and $\int \theta^\epsilon = 1$, we get

$$\nabla r^\epsilon(x) = c_0 (2 - d) \int_{\mathbb{R}^d} \frac{\theta^\epsilon(y)x}{|x - y|^d} dy,$$

whereby

$$|\nabla r^\epsilon(x + z) - \nabla r^\epsilon(x)| \leq C \int_{\mathbb{R}^d} \theta^\epsilon(y) \left| \frac{x}{|(x + z) - y|} - \frac{x}{|x - y|} \right| dy$$

$$\leq C \int_{\mathbb{R}^d} \theta^\epsilon(y) \frac{|xz|}{|x|^d} dy = C \frac{|z|}{|x|^d}.$$

The claimed bounds on $r^\epsilon$ and $|\nabla r^\epsilon|$ follow from the following bounds on $r$ and $|\nabla r|$, and the scaling relation:

$$r(x) \leq C|x|^{2-d} \wedge C, \quad |\nabla r(x)| \leq C|x|^{1-d} \wedge C.$$

Their derivation is similar to the difference estimates, hence we omit it. ■

Utilizing Lemma 7, we prove in the next two lemmas that various terms that come out of the application of Itô formula in the proof of Proposition 6 are minor.

Lemma 8 For any $d \geq 3$ and finite $T$, there exists some finite $C = C(T, d, C_0, \kappa, \gamma)$ such that for any $\epsilon, |z|$ small,

$$\mathbb{E}|H_{31}| \leq C(|z|^\frac{2}{\lambda} + \epsilon^2), \quad \mathbb{E}|H_{21}| \leq C(|z|^\frac{1}{\lambda} + \epsilon).$$

Proof. Let $\rho := |z|^\frac{1}{\lambda} \geq \max\{2|z| + C_0\epsilon, 2C_0\epsilon\}$ valid for $\epsilon, |z|$ small enough. Upon bounding $|J|, |\bar{J}|$ by constants, we divide the upper bound of the double
sum \(|H_{31}|\) into two parts:

\[
H_{31}^{(1)} := N^{-2}E \int_0^T \sum_{i,j: |x_i(t)-x_j(t)| \geq \rho} |r^\epsilon(x_j(t) - x_i(t) + z) - r^\epsilon(x_j(t) - x_i(t))| \\
\leq C\rho^{1-d}|z|,
\]

\[
H_{31}^{(2)} := N^{-2}E \int_0^T \sum_{i,j: |x_i(t)-x_j(t)| < \rho} |r^\epsilon(x_j(t) - x_i(t) + z) - r^\epsilon(x_j(t) - x_i(t))| \\
\leq N^{-2}E \int_0^T \sum_{j,i} 1_{|x_j(t)-x_i(t)| < \rho} (r^\epsilon(x_j(t) - x_i(t) + z) + r^\epsilon(x_j(t) - x_i(t))),
\]

where the first bound follows from Lemma 7. We proceed to analyse (for any fixed \(z\))

\[
N^{-2}E \int_0^T \sum_{j,i} 1_{|x_j(t)-x_i(t)| < \rho} r^\epsilon(x_j(t) - x_i(t) + z).
\]

Note that our particle system can be coupled with a system of pure proliferation of unit rate (with no killing), so that the former is a strict subset of the latter. Hence, it is an upper bound to compute (19) for the pure proliferation process. We proceed to do so in the rest of the proof, while abusing notations, still using the letter \(x_j(t)\) to denote particle positions (now for a different system).

Define a function \(v^\epsilon(x)\) a positive solution of the PDE:

\[-\Delta v^\epsilon(x) = r^\epsilon(x + z)1_{|x| < \rho}\]

that admits the representation

\[v^\epsilon(x) = c_0 \int \frac{|x - y|^{2-d}r^\epsilon(y + z)1_{|y| < \rho}}{dy},\]

Since \(r^\epsilon(x) \leq C|x|^{2-d}\) for any \(x \in \mathbb{R}^d, \epsilon > 0\) by Lemma 7, we have that for any \(x, \epsilon,\)

\[v^\epsilon(x) \leq C \int \frac{|x - y|^{2-d}|y|^{2-d}1_{|y| < \rho}}{dy}.
\]

Further, since by Lemma 7 we have the crude bound \(r^\epsilon(x) \leq C\epsilon^{2-d}\), we also have the crude bound \(v^\epsilon(x) \leq C\epsilon^{2-d}\) for any \(x, \epsilon\).

Now applying Itô formula to the process

\[S(t) := N^{-2} \sum_{j,i} v^\epsilon(x_i(t), x_j(t))\]

and taking expectation, we get

\[-N^{-2}E \int_0^T \sum_{j,i} \Delta v^\epsilon(x_j(t) - x_i(t))dt \]

\[= N^{-2}E \int_0^T \sum_{j,i} 1_{|x_j(t)-x_i(t)| < \rho} r^\epsilon(x_j(t) - x_i(t) + z)dt \]

\[= ES(0) + N^{-2}E \int_0^T \sum_{j,i} v^\epsilon(x_j(t) - x_i(t))dt.\]
The first term \( \mathbb{E}S(0) \leq C(|\rho| + \rho)^2 \) is identical to the one already analyzed in [10], using assumptions on \( u_0 \), which we do not repeat. We turn to the analysis of the second term.

Consider any pair of particles that ever coexisted in the system during \([0, T]\), call it \((i, j)\). They have a common interval of existence, denoted \( I_{i,j} \). Our strategy is to work with the individual integral

\[
\mathbb{E} \int_{I_{i,j}} v^r(x_j(t) - x_i(t))dt.
\]

We distinguish between two scenarios: either \( i \) and \( j \) are independent (i.e. their lineages trace back to two independent points at \( t = 0 \)); or they share a common ancestor.

Consider first the independent case, where \( x_i(t) \) and \( x_j(t) \) have the law of two independent Brownian motions, with initial density \( u_0 \) (to be clear, particles \( i \) and \( j \) may not exist since \( t = 0 \), but their laws can be described this way). The difference \( x_j(t) - x_i(t) \) thus has the law of a Brownian motion with variance 2, with initial density \( u_0 * \bar{u}_0 \), where \( \bar{u}_0(x) := u_0(-x) \). By Condition [3], \( u_0 \) is compactly supported in a ball of radius \( \kappa \) around the origin, and bounded by \( \gamma_i \), implying the same for \( u_0 * \bar{u}_0 \). We may write, by [20],

\[
\mathbb{E} \int_{I_{i,j}} v^r(x_j(t) - x_i(t))dt \\
\leq C \int_0^T \int_{\mathbb{R}^d} |x - y|^{2-d} |y + z|^{2-d} 1_{|y| < \rho} (e^{\kappa \Delta} (u_0 * \bar{u}_0))(x) dxdydt.
\]

Observe that for some \( C = C(d, T, \gamma) > 0 \), we have that for any \( x \in \mathbb{R}^d \) and \( t \in [0, T] \),

\[
(e^{\kappa \Delta} (u_0 * \bar{u}_0))(x) = \mathbb{E}[(u_0 * \bar{u}_0)(x + W_t)]
\]

\[
\leq \gamma \mathbb{E}[1_{B(0, \kappa)}(x + W_t)] \leq \gamma \mathbb{P}(|W_t| \geq |x| - \kappa)
\]

\[
\leq \gamma \mathbb{P}(|W_t| \geq |x|) \leq C e^{(\kappa - |x|)/C},
\]

where \( \{W_t\} \) stands for a Brownian motion with variance 2 in \( \mathbb{R}^d \), whereby the previous integral reduces to

\[
CT \int_{\mathbb{R}^d} |x - y|^{2-d} |y + z|^{2-d} 1_{|y| < \rho} e^{(\kappa - |x|)/C} dxdy
\]

\[
= C \int_{|y| < \rho} |y + z|^{2-d} dy \int_{\mathbb{R}^d} |x - y|^{2-d} e^{-|x|/C} dx.
\]

Since we have, uniformly in \(|y| < \rho \leq 1\), both

\[
\int_{|x| < 2} |x - y|^{2-d} e^{-|x|/C} dx \leq \int_{|x| < 3} |x - y|^{2-d} dx \leq C,
\]

and

\[
\int_{|x| \geq 2} |x - y|^{2-d} e^{-|x|/C} dx \leq C \int_{|x| \geq 2} |x|^{2-d} e^{-|x|/C} dx \leq C,
\]

we arrive at

\[
\mathbb{E} \int_{I_{i,j}} v^r(x_j(t) - x_i(t))dt \leq C \int_{|y| < \rho} |y + z|^{2-d} dy \leq C(\rho + |z|)^2.
\]

In the second scenario, where \( i \) and \( j \) share a common ancestor, we just bound crudely, using \( v^r(x) \leq C e^{4-d} \) for any \( x, \epsilon \),

\[
\mathbb{E} \int_{I_{i,j}} v^r(x_j(t) - x_i(t))dt \leq C T e^{2-d}.
\]
Now since we start with \( N_0 := N \int u_0 \) independent points, each of which creates an independent lineage. Each lineage starts with one particle and its members have unit proliferation rate. In a compact interval \([0, T]\), the number of offsprings in each lineage is a Poisson random variables with constant mean \( \epsilon^T \). Let us denote, among all the pairs \((i, j)\) that ever coexisted in the pure proliferation system, by \( N_a \) the number of pairs that are independent, \( N_b \) the number of those correlated. Then

\[
E N_a \leq C(T)N^2, \quad E N_b \leq C(T)N,
\]

and further \( N_a, N_b \) are independent of the Brownian trajectories. Indeed, the exponential clocks can even be programmed in advance.

Utilizing such independence, we can sum the individual integrals over all pairs, and get

\[
N^{-2}E \int_0^T \sum_{j,i} \nu^\epsilon(x_j(t) - x_i(t)) dt \leq C \frac{(\rho + |z|)^2 E N_a}{N^2} + C \frac{\epsilon^2d E N_b}{N^2} = C(\rho + |z|)^2 + C \epsilon^2.
\]

Combining estimates on \( H^{(1)}_{31} \) and \( H^{(2)}_{31} \) and with \( \rho = |z|^1 \), we get the claimed bound on \( H_{31} \).

An alternative and more rigorous argument using multi-indices to represent a pure proliferation system is given in the Appendix, Proposition 15 for reader’s convenience.

We next deal with the double sum \( H_{21} \), whose analysis is similar. Upon bounding \(|J|, |\bar{J}|, |\nabla J|, |\nabla \bar{J}|\) by constants we again divide the upper bound of the first sum in \(|H_{21}|\) into two parts (the second sum is analogous which we omit):

\[
H^{(1)}_{21} := N^{-2}E \int_0^T dt \sum_{i,j: |x_i(t) - x_j(t)| \geq \rho} |\nabla r^\epsilon(x_j(t) - x_i(t) + z) - \nabla r^\epsilon(x_j(t) - x_i(t))|
\leq C \rho^{-d}|z|,
\]

\[
H^{(2)}_{21} := N^{-2}E \int_0^T dt \sum_{i,j: |x_i(t) - x_j(t)| < \rho} |\nabla r^\epsilon(x_j(t) - x_i(t) + z) - \nabla r^\epsilon(x_j(t) - x_i(t))|
\leq N^{-2}E \int_0^T dt \sum_{j,i} 1_{\{|x_j(t) - x_i(t)| < \rho\}} (|\nabla r^\epsilon(x_j(t) - x_i(t) + z)| + |\nabla r^\epsilon(x_j(t) - x_i(t))|) ,
\]

where the first bound follows from Lemma \( \square \). We proceed to analyse (for fixed \( z \))

\[
N^{-2}E \int_0^T dt \sum_{j,i} 1_{\{|x_j(t) - x_i(t)| < \rho\}} |\nabla r^\epsilon(x_j(t) - x_i(t) + z)|.
\]

Again as an upper bound, we calculate the same quantity for the system of pure proliferation with unit rate (with no killing), which dominates our system from above under an obvious coupling.

Consider a function \( w^\epsilon(x) \) a positive solution of the PDE:

\[
-\Delta w^\epsilon(x) = |\nabla r^\epsilon(x + z)|1_{\{|x| < \rho\}}
\]

that admits the representation

\[
w^\epsilon(x) = c_0 \int_{\mathbb{R}^d} |\nabla r^\epsilon(y + z)||y - x|^{2-d}1_{\{|y| < \rho\}}dy.
\]

Since \( |\nabla r^\epsilon(x)| \leq C|x|^{-d} \) for any \( x \in \mathbb{R}^d, \epsilon > 0 \) by Lemma \( \square \) we have that

\[
w^\epsilon(x) \leq C \int_{\mathbb{R}^d} |y - x|^{2-d}|y + z|^{1-d}1_{\{|y| < \rho\}}dy.
\]
Further, from the crude bound $|\nabla r^\epsilon(x)| \leq C\epsilon^{1-d}$ for any $x, \epsilon$ by Lemma 4, we also have the crude bound $w^\epsilon(x) \leq C\epsilon^{1-d}$ for any $x, \epsilon$.

Now applying Itô formula to the process
\[ \tilde{S}(t) := N^{-2} \sum_{j,i} w^\epsilon(x_j(t) - x_i(t)) \]
and taking expectation, we get
\[
- N^{-2} \mathbb{E} \int_0^T \sum_{j,i} \Delta w^\epsilon(x_j(t) - x_i(t)) dt \\
= N^{-2} \mathbb{E} \int_0^T dt \sum_{j,i} 1_{|x_j(t) - x_i(t)| \leq \rho} |\nabla r(x_j(t) - x_i(t) + z)| \\
= \mathbb{E} \tilde{S}(0) + N^{-2} \mathbb{E} \int_0^T \sum_{j,i} w^\epsilon(x_j(t) - x_i(t)) dt.
\]
The first term $\mathbb{E} \tilde{S}(0) \leq C(|z| + \rho)$ is again already analysed in [10] which we do not repeat, and we focus on the second term.

Consider any pair of particles $(i, j)$ that ever coexisted, we distinguish between two cases: either they are independent, or they share a common ancestor. Inspecting this part of the argument before, we readily get the following estimates. In case $(i, j)$ are independent, we have that
\[ \mathbb{E} \int_{I_{ij}} w^\epsilon(x_j(t) - x_i(t)) dt \leq C(\rho + |z|); \]
and in case $(i, j)$ share an ancestor, we bound crudely
\[ \mathbb{E} \int_{I_{ij}} w^\epsilon(x_j(t) - x_i(t)) dt \leq C T \epsilon^{1-d}. \]
Combining all pairs, with the same argument we get
\[ N^{-2} \mathbb{E} \int_0^T \sum_{j,i} w^\epsilon(x_j(t) - x_i(t)) dt \leq C(\rho + |z| + \epsilon). \]
Combining $H_{21}^{(1)}$ and $H_{21}^{(2)}$ and with $\rho = |z|^{2/3}$, we get the claimed bound on $H_{21}$. The more rigorous argument in the Appendix, Proposition 15 also applies here.

**Lemma 9** For any $d \geq 3$ and finite $T$, there exists some finite $C = C(T, d, C_0, \kappa, \gamma)$ such that for any $\epsilon, |z|$ small,
\[ \mathbb{E} |H_{32}| \leq C(|z|^{\frac{2}{3d-1}} + \epsilon^2 \log 1/\epsilon). \]

**Proof.** Taking the same $\rho = |z|^{2/3}$ as in the proof of Lemma 8 upon bounding $|\tilde{J}|$ by a constant, we divide the upper bound of the first triple sum in $|H_{32}|$ into two parts (the second sum is analogous which we omit):
\[
H_{32}^{(1)} := N^{-2} \mathbb{E} \int_0^T dt \sum_{k,j,i: |x_j(t) - x_k(t)| \geq \rho} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \\
\cdot |r^\epsilon(x_j(t) - x_i(t) + z) - r^\epsilon(x_j(t) - x_i(t))||J|(t, x_i(t)) \\
\leq N^{-2} C \rho^{1-d} |z| \mathbb{E} \int_0^T dt \sum_{j,k,i} \theta(\epsilon^{-1}(x_j(t) - x_k(t)))
\]
by Lemma 7 and

\[ H_{32}^{(2)} := N^{-2} \mathbb{E} \int_0^T dt \sum_{k,j,i} 1_{(x_i(t) - x_j(t)) < \rho} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \]

\[ \cdot |r^\epsilon(x_j(t) - x_i(t) + z) - r^\epsilon(x_j(t) - x_i(t))| J(t,x_i(t)) \]

\[ = N^{-2} \mathbb{E} \int_0^T dt \sum_{k,j,i} 1_{(x_i(t) - x_j(t)) < \rho} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) \]

\[ \cdot (r^\epsilon(x_j(t) - x_i(t) + z) + r^\epsilon(x_j(t) - x_i(t))) J(t,x_i(t)). \]

As in the proof of Lemma 8 we once again upper bound the above quantities by calculating them for the system of pure proliferation of unit rate (with no killing), in the rest of the proof. We first analyse \( H_{32}^{(2)} \) and consider (for fixed \( z \))

\[ N^{-2} \mathbb{E} \int_0^T dt \sum_{k,j,i} 1_{(x_i(t) - x_j(t)) < \rho} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) r^\epsilon(x_j(t) - x_i(t) + z) J(t,x_i(t)). \]

We focus on each individual integral involving a triple of particles \((i,j,k)\) that ever coexisted during some random \( I_{ijk} \subset [0,T] \)

\[ \mathbb{E} \int_{I_{ijk}} dt \sum_{k,j,i} 1_{(x_i(t) - x_j(t)) < \rho} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) r^\epsilon(x_j(t) - x_i(t) + z) J(t,x_i(t)), \quad (21) \]

and we distinguish among three scenarios: either (a). \((i,j,k)\) are mutually independent (i.e. their lineages trace back to three independent points at \( t = 0 \)); or (b). \( i \) is independent of \((j,k)\) while the latter pair is correlated (i.e. \((j,k)\) share a common ancestor); or (c). \( i \) is correlated with at least one of \( j \) and \( k \).

First in the scenario (c), using \( r^\epsilon(x) \leq C\epsilon^{2-d} \) for all \( x,\epsilon \) and the boundedness of \( J \), we have (21) is bounded by

\[ C\epsilon^{2-d} \mathbb{E} \int_{I_{ijk}} \theta(\epsilon^{-1}(x_j(t) - x_k(t))) dt. \]

For the scenarios (a) and (b), we define a function \( A^\epsilon(x,y,w) \) a positive solution of the PDE:

\[ -\frac{1}{2}(\Delta_x + \Delta_y + \Delta_w) A^\epsilon(x,y,w) = 1_{(x-y) < \rho} \theta(\epsilon^{-1}(y-w)) r^\epsilon(y-x+z) J(x) \]

that admits the representation

\[ A^\epsilon(x,y,w) = c_2(d) \int_{\mathbb{R}^{3d}} \left( |x-x'|^2 + |y-y'|^2 + |w-w'|^2 \right)^{\frac{2-3d}{2}} \]

\[ \cdot 1_{(x'-y') < \rho} \theta(\epsilon^{-1}(y'-w')) |r^\epsilon(y'-x'+z) J(x') dx' dy' dw', \]

where \( c_2 \) is a dimensional constant, and we recall \((|x-x'|^2 + |y-y'|^2 + |w-w'|^2)^{\frac{2-3d}{2}}\) is the Green kernel in the three-variable case, and we denoted, by an abuse of notation

\[ J(x) := \sup_t J(t,x). \]

Since \( r^\epsilon(x) \leq C|x|^{2-d} \) for any \( x,\epsilon \) by Lemma 7 we have that

\[ A^\epsilon(x,y,w) \leq C \int_{\mathbb{R}^{3d}} \left( |x-x'|^2 + |y-y'|^2 + |w-w'|^2 \right)^{\frac{2-3d}{2}} \]

\[ \cdot 1_{(x'-y') < \rho} \theta(\epsilon^{-1}(y'-w')) |y' - x' + z|^{2-d} J(x') dx' dy' dw' \]

\[ \leq C \int_{\mathbb{R}^{3d}} |x-x'|^{2-3d} |y-y'|^{2-3d} |w-w'|^{2-3d} \]

\[ \cdot 1_{(x'-y') < \rho} \theta(\epsilon^{-1}(y'-w')) |y' - x' + z|^{2-d} J(x') dx' dy' dw', \quad (22) \]
using the arithmetic-geometric means inequality $3^{3/2}|abc| \leq (a^2 + b^2 + c^2)^{3/2}$ for any $a, b, c \in \mathbb{R}$.

Now applying Itô formula to the process

$$Q(t) := A^r(x_i(t), x_j(t), x_k(t))$$

(with the understanding that if $t \notin I_{ijk}$, then $Q(t) = 0$) and taking expectation, we get

$$- \mathbb{E} \int_{I_{ijk}} \frac{1}{2}(\Delta x_j + \Delta x_i + \Delta x_k)A^r(x_i(t), x_j(t), x_k(t))dt$$

$$= \mathbb{E} \int_{I_{ijk}} dt 1_{\{|x_i(t) - x_j(t)| < \rho\}} \theta(\epsilon^{-1}(x_j(t) - x_k(t)))\sigma^r(x_j(t) - x_i(t) + z)\|J(t)\|$$

$$= \mathbb{E} Q(0) + 3\mathbb{E} \int_{I_{ijk}} A^r(x_i(t), x_j(t), x_k(t))dt.$$

The first term $\mathbb{E} Q(0) \leq C(|\rho| + \rho)^2$ is already analyzed in [10], using assumptions on $u_0$, which we do not repeat. We focus on the second term.

If we are in scenario (a), where $(i, j, k)$ is an independent triple each starting at $t = 0$ with density $u_0$, we have, by [22],

$$\mathbb{E} \int_{I_{ijk}} A^r(x_i(t), x_j(t), x_k(t))dt$$

$$\leq C \int_0^T \cdots \int_{\mathbb{R}^d} |x - x'|^{2/3-d}|y - y'|^{2/3-d}|w - w'|^{2/3-d} \prod_{\ell=x, y, w} (e^{t\Delta/2}u_0)(\ell) dx dy dw$$

$$\cdot |y' - x'|^2 - z^{2-d}J(\epsilon^{-1}(y' - w'))dx' dy' dw' dt.$$  

Since $J, \theta$ as well as the indicator function all have compact supports, we have that all the $x', y', w'$ that contribute to the integral are compactly supported in a ball of some radius $K$ around the origin (without loss of generality assume $K \geq \kappa$). Hence, for such $(x', y', w')$ and fixed $t$,

$$\int_{\mathbb{R}^d} |x - x'|^{2/3-d}|y - y'|^{2/3-d}|w - w'|^{2/3-d} \prod_{\ell=x, y, w} (e^{t\Delta/2}u_0)(\ell) dx dy dw$$

$$= \prod_{\ell=x, y, w} \int_{\mathbb{R}^d} |\ell' - \ell|^{2/3-d} (e^{t\Delta/2}u_0)(\ell) d\ell$$

$$\leq C \left( K^{2/3} + \int_{|x| > 2K} |x - x'|^{2/3-d}(e^{t\Delta/2}u_0)(x)dx \right)$$

$$\cdot \left( K^{2/3} + \int_{|y| > 2K} |y - y'|^{2/3-d}(e^{t\Delta/2}u_0)(y)dy \right)$$

$$\cdot \left( K^{2/3} + \int_{|w| > 2K} |w - w'|^{2/3-d}(e^{t\Delta/2}u_0)(w)dw \right),$$  

(23)

where we used that $||e^{t\Delta/2}u_0||_{L^\infty} \leq ||u_0||_{L^\infty} \leq \gamma$, and

$$\int_{|x| \leq 2K} |x - x'|^{2/3-d}(e^{t\Delta/2}u_0)(x)dx \leq C\gamma K^{2/3},$$

for fixed $|x'| < \kappa \leq K$ (and the analogous ones for $y, w$) in the analysis of the individual integrals in $x, y, w$. Now we integrate [22] in $t$ over $[0, T]$. Opening the product, it consists of the sum of several integrals. We
have a term
\[
\int_0^T \int \int \int_{|x|,|y|,|w| > 2K} |x - x'|^{2/3-d} |y - y'|^{2/3-d} |w - w'|^{2/3-d} \prod_{\ell=x,y,w} (e^{t\Delta/2} u_0)(\ell) \, dx \, dy \, dw \\
\leq C \int \int \int_{|x|,|y|,|w| > 2K} |xyw|^{2/3-d} |xyw|^{2/3-d} \, dx \, dy \, dw \leq C(d),
\]
where we used that for any $|x_1|, |y_1|, |w_1| \leq \kappa \leq K$ and $|x|, |y|, |w| > 2K$,
\[
\int_0^\infty \int \prod_{\ell=x,y,w} (e^{t\Delta/2} \delta_\ell)(\ell) \leq C(|x - x_1|^2 + |y - y_1|^2 + |w - w_1|^2)^{1-3d/2} \\
\leq C(|x|^2 + |y|^2 + |w|^2)^{1-3d/2} \leq C|xyw|^{2/3-d}.
\]
We also have three terms one of which is (the other two are analogous)
\[
K^{2/3} \int_0^T \int \int |x - x'|^{2/3-d} |y - y'|^{2/3-d} \, dt \prod_{\ell=x,y} (e^{t\Delta/2} u_0)(\ell) \, dx \, dy \\
\leq CK^{2/3} \int \int |xy|^{2/3-d} |xy|^{1-d} \, dx \, dy \leq C(d),
\]
where we used that for any $|x_1|, |y_1| \leq \kappa \leq K$ and $|x|, |y| > 2K$,
\[
\int_0^\infty \int \prod_{\ell=x,y} (e^{t\Delta/2} \delta_\ell)(\ell) \leq C(|x - x_1|^2 + |y - y_1|^2)^{1-d} \leq C|xy|^{1-d}.
\]
We also have three terms one of which is (the other two are analogous)
\[
K^{4/3} \int \int |x - x'|^{2/3-d} (e^{t\Delta/2} u_0)(x) \, dx \leq CK^{4/3} \int \int |x - x'|^{2/3-d} |x|^{2-d} \, dx \leq C(d).
\]
Taken together it justifies the finiteness of (22), and we are left with
\[
\mathbb{E} \int_{I_{ijk}} A'(x_i(t), x_j(t), x_k(t)) dt \\
\leq C \int \int \int_{\mathbb{R}^d} |y' - x' + z|^{2-d} |J|(x') P_{\{x' - y' < \rho\}} \theta(e^{-1}(y' - w')) \, dx' \, dy' \, dw' \\
\leq Ce^d \int \int_{\mathbb{R}^d} |y' - x' + z|^{2-d} |J|(x') P_{\{x' - y' < \rho\}} \, dx' \, dy' \, dw' \\
\leq Ce^d (|z| + \rho)^2,
\]
where we again used $\int \theta' = 1$ and that those $x'$ that contribute to the integral are compactly supported in $\mathbb{B}_0(K)$. This finishes the analysis of scenario (a).

If we are in scenario (b) where $i$ is independent of $(j, k)$ and the latter pair is correlated, we write, by (22),
\[
\mathbb{E} \int_{I_{ijk}} A'(x_i(t), x_j(t), x_k(t)) dt \\
\leq C \int_0^T \int \int \int_{\mathbb{R}^d} |x - x'|^{2/3-d} |y' - y'|^{2/3-d} |w - w'|^{2/3-d} (e^{t\Delta/2} u_0)(x) \, q_t(y, w) \, dx \, dy \, dw \, dt \\
\cdot |y' - x' + z|^{2-d} |J|(t, x') P_{\{x' - y' < \rho\}} \theta(e^{-1}(y' - w')) \, dx' \, dy' \, dw'
\]

17
where \( q_t(y, w) \) denotes the joint density of \((x_j(t), x_k(t))\) at time \( t \). We compute the integral in \( w' \) first: we have already noted that all the \( w' \) that contribute to the integral are compactly supported in \( B_0(K) \), hence for fixed \( y' \in B_0(K) \), if \(|w| \leq K + 1\), we have that

\[
\int_{\mathbb{R}^d} |w - w'|^{2/3 - d} \theta(\epsilon^{-1}(y' - w')) dw' \leq ||\theta||_{L^\infty} \int_{|w - w'| \leq 2K + 1} |w - w'|^{2/3 - d} dw' \leq C(\theta, d) K^{2/3},
\]

whereas if \(|w| > K + 1\), then \(|w - w'| > 1\) and

\[
\int_{\mathbb{R}^d} |w - w'|^{2/3 - d} \theta(\epsilon^{-1}(y' - w')) dw' \leq \int_{\mathbb{R}^d} \theta(\epsilon^{-1}(y' - w')) dw' = e^d.
\]

Thus, in both cases a constant bound holds. Now we are left to deal with

\[
C \int_0^T \int \ldots \int_{\mathbb{R}^{4d}} |x - x'|^{2/3 - d} |y - y'|^{2/3 - d} (e^{t \Delta/2} u_0)(x) q_t(y, w) \, dx\, dy\, dw\, dt
\]

\[
\cdot |y' - x' + z|^{2-d} |J|(x') \cdot \frac{1}{1} |z'| \cdot dy' \cdot dx'.
\]

Notice that

\[
\int_{\mathbb{R}^d} q_t(y, w) dw
\]

is the marginal density of \( x_j(t) \) at time \( t \), which is equal to \( e^{t \Delta/2} u_0 \). Thus, we have that

\[
C \int_0^T \ldots \int_{\mathbb{R}^{4d}} |x - x'|^{2/3 - d} |y - y'|^{2/3 - d} (e^{t \Delta/2} u_0)(x) (e^{t \Delta/2} u_0)(y) \, dx\, dy\, dt
\]

\[
\cdot |y' - x' + z|^{2-d} |J|(x') \frac{1}{1} |z'| \cdot dx' \cdot dy'.
\]

The computation of

\[
\int_0^T \int \ldots \int_{\mathbb{R}^{4d}} |x - x'|^{2/3 - d} |y - y'|^{2/3 - d} (e^{t \Delta/2} u_0)(x) (e^{t \Delta/2} u_0)(y) \, dx\, dy\, dt \leq C(d, T, K)
\]

we have already seen. Lastly, we are left with the integral

\[
C \int_{\mathbb{R}^{2d}} |y' - x' + z|^{2-d} |J|(x') \frac{1}{1} |z'| \cdot dx' \cdot dy' \leq C(|z| + \rho)^2.
\]

This finishes the analysis of scenario (b).

Now we start with \( N_0 = N \int u_0 \) independent points which create independent lineages, and the size of each lineage is an independent Poisson random variable with constant mean \( e^T \). Let us denote, among all triples that ever coexisted in the pure proliferation system, by \( N_a, N_b \) the number of those triples that fall into scenario (a), (b). Then,

\[
E N_a \leq C(T) N^3, \quad E N_b \leq C(T) N^2.
\]

Further, fixing a pair \((j, k)\), let us denote by \( N_c(j, k) \) the number of \( i \) that are correlated with it in the system of pure proliferation, then all the \( N_c(j, k) \) are bounded by a random variable \( \tilde{N} \), which is twice the maximum among the population sizes of the \( N_0 \) lineages. For the maximum of a sequence of independent Poisson variables, we have that

\[
E \tilde{N} \leq C(T) \log N.
\]
Combining the three scenarios and utilizing the independence between Brownian trajectories and $N_a, N_b, \hat{N}$, we get

$$N^{-2} \mathbb{E} \int_{t=k} dt \sum_{j<k,i} 1_{\{x_i(t) - x_j(t) \leq \rho \}} \theta(e^{-1}(x_j(t) - x_k(t))) (x_j(t) - x_i(t) + z) J(t, x_i(t)) \leq C \frac{e^d(|z| + \rho)^2 \mathbb{E} N_a}{N^2} + C \frac{(|z| + \rho)^2 \mathbb{E} N_b}{N^2} + C \frac{\rho^2}{N^2} \mathbb{E} \int_0^T dt \sum_{j,k} \theta(e^{-1}(x_j(t) - x_k(t))) \mathbb{E} \hat{N}$$

using also Lemma 5. This gives the bound on the term $H_{32}^{(2)}$.

Going back to $H_{32}^{(1)}$, we note the number of particles in the system of pure proliferation is the sum (call it $\mathcal{V}_N$) of $N_0$ independent Poisson variables with mean $e^T$, with $\mathbb{E} \mathcal{V}_N = e^T N_0$, independent of the Brownian trajectories, hence

$$H_{32}^{(1)} \leq N^{-2} C \rho^{-d} |z| \mathbb{E} \int_0^T dt \sum_{k,j} \theta(e^{-1}(x_j(t) - x_k(t))) \leq N^{-2} C \rho^{-d} |z| \mathbb{E} \int_0^T dt \sum_{k,j} \theta(e^{-1}(x_j(t) - x_k(t))) \mathbb{E} \mathcal{V}_N \leq CN^{-1} \rho^{-d} |z| \mathbb{E} \int_0^T dt \sum_{k,j} \theta(e^{-1}(x_j(t) - x_k(t))) \leq C \rho^{-d} |z|$$

by Lemma 5. Combining both $H_{32}^{(1)}, H_{32}^{(2)}$ and with $\rho = |z|^{\frac{1}{d+2}}$ yields the claimed estimate on $H_{32}$.

It is possible to formulate the proof alternatively and more rigorously using multi-indices similarly to the Appendix, which is done for Lemma 8 in the present case, we omit it for brevity.

**Remark 10** The rest of the terms that come out of Itô formula, namely $H_{21}, H_{23}, E_4^{(1)}, E_4^{(2)}$, and $H_4$, all can be bounded in similar fashion as either Lemma 8 or Lemma 5, thus we omit the details for brevity.

**Lemma 11** Any subsequential limit of $\{g^N(dt, dx)\}_N$ has a uniformly bounded (in $N$) density with respect to Lebesgue measure on $[0,T] \times \mathbb{R}^d$.

**Proof.** We roughly follow the strategy of [13] Lemma 4.1, 4.2. Taking a smooth approximation $\{\psi_n\}$ to the function $(x - k - 1)_+$, where $k = k(d, T, u_0)$ is a constant to be determined, we consider a smooth function $\psi : \mathbb{R} \to \mathbb{R}_+$ that is non-decreasing and whose derivative vanishes for $x \leq k$ and is at most 1 for $x > k$. For simplicity, we write $g^N \ast_x \eta^\delta =: f^\delta(t, x) dt$.

Applying Itô formula to the process $\int_{\mathbb{R}^d} \psi(f^\delta(t, x)) dx$, and then taking expectation, we have that

$$\mathbb{E} \int_{\mathbb{R}^d} \psi(f^\delta(T, x)) dx \leq \mathbb{E} \int_{\mathbb{R}^d} \psi(f^\delta(0, x)) dx + \mathbb{E} \int_0^T \int \frac{1}{2} \sum_j \Delta_j \psi(f^\delta(t, x)) dx dt + \mathbb{E} \int_0^T \sum_j [\psi(f^\delta(t, x) + \epsilon \eta^\delta(x - x_j)) - \psi(f^\delta(t, x))] dx dt$$

$$\leq \mathbb{E} \int_{\mathbb{R}^d} \psi(f^\delta(0, x)) dx + C \frac{\epsilon^d}{\delta x^2} \int |\nabla \eta|^2 dx + \mathbb{E} \int_0^T \sum_j \epsilon \eta^\delta(x - x_j) 1_{\{f^\delta(t, x) > k\}} dx dt$$

$$= \mathbb{E} \int_{\mathbb{R}^d} \psi(f^\delta(0, x)) dx + C \frac{\epsilon^d}{\delta x^2} \int \int f^\delta(t, x) 1_{\{f^\delta(t, x) > k\}} dx dt,$$  

(24)
where the analysis of the diffusion term is already done in [10, page 46]. We argue that the RHS converges to zero as $N \to \infty$ for some constant $k = k(T, d, u_0)$ and fixed $\delta$. With that we can get the desired conclusion by repeating the argument of [10, Lemma 4.2]. In particular, the constant $k+1$ is the uniform (in $N$) upper bound on the density.

To this end, fixing $t$ and $x$ and we consider $B_\delta(x)$, the $\delta$-ball around $x$. We denote by $Z(t, x, \delta)$ the number of particles in a binary Branching Brownian motion of unit rate that fall into $B_\delta(x)$ at time $t$, starting with a single particle at $t = 0$ according to distribution with density $u_0$.

Recall that our particle system starts with $N_0 = N \int u_0$ number of independent points with density $u_0$, and each point generates its own lineage, dominated from the above by a binary Branching Brownian motion. If we denote $Z^{(i)}(t, x, \delta)$ the number of particles in our system that fall into $B_\delta(x)$ at time $t$ that come from the $i$-th lineage, for $i = 1, 2, \ldots, N_0$, then $Z^{(i)}$ are dominated by i.i.d. copies $Z_i$ of $Z$.

Note that each $Z_i$ is a Poisson variable with constant mean $e^T$. By Chernoff’s bound for sum of independent Poisson variables, for fixed $\delta, t, x$, some $C = C(d, T)$, any $N$ and $s > EZ$ large enough, we have that

$$
\mathbb{P}\left(f^\delta(t, x) \geq \frac{s\delta^{-d}\|\eta\|}{\int u_0}\right) \leq \mathbb{P}\left(N_0^{-1} \sum_{i=1}^{N_0} Z^{(i)} \geq s\right)
$$

$$
\leq \mathbb{P}\left(N_0^{-1} \sum_{i=1}^{N_0} Z_i \geq s\right) \leq Ce^{-CN\delta}\log(s-EZ).
$$

Further, due to the joint continuity in time and space of Brownian motion transition probabilities, $\mathbb{E}[Z(t, x, \delta)]$ is continuous in $(t, x)$. By the boundedness and compact support of the initial density $u_0$, this quantity decays to zero as $|x| \to \infty$. Further, the limit as $\delta \to 0$ of $\delta^{-d}\mathbb{E}[Z(t, x, \delta)]$ exists and is nothing but the occupation density of the Branching Brownian motion. Thus,

$$
k_1(d, T) := \sup_{t \in [0, T], x \in \mathbb{R}^d, \delta \in (0, 1]} \delta^{-d}\mathbb{E}[Z(t, x, \delta)] < \infty.
$$

In particular, upon choosing $k := 2k_1\|\eta\|_\infty(\int u_0)^{-1}$, we have by [25] that for some $C = C(k)$ finite,

$$
\mathbb{E}[f^{\delta}(t, x)1_{\{f^{\delta}(t, x) > k\}}] = \int_0^\infty \mathbb{P}(f^{\delta}(t, x) > u)du \leq Ce^{-CN}.
$$

To conclude, we apply the dominated convergence theorem, as $f^{\delta}(t, x)1_{\{f^{\delta}(t, x) > k\}}$ is dominated by $f^{\delta}(t, x)$, and the latter is integrable as can be checked

$$
\mathbb{E}\int_0^T \int f^{\delta}(t, x)dxdt \leq e^T \int u_0, \quad \forall \delta > 0.
$$

This shows the third term of [24] converges to zero as $N \to \infty$, and the first two terms do so as well (those are identical to [10]).

3 Uniqueness of weak solutions of F-KPP equation

Denote by $L^1_+(\mathbb{R}^d)$ the set of nonnegative integrable functions.

As a preliminary, let us recall that we have denoted by $g^N(dt, dx)$ the space-time empirical measure and we have remarked that it has converging subsequences. Thus assume that $g^{N_k}(dt, dx)$ weakly converges to a space-time finite measure $\mu(dt, dx)$:

$$
\lim_{k \to \infty} \int_0^T \int_{\mathbb{R}^d} K(t, x) g^{N_k}(dt, dx) = \int_0^T \int_{\mathbb{R}^d} K(t, x) \mu(dt, dx)
$$
for every bounded continuous function $K$. Moreover, we know that there exists

$$u \in L^\infty ([0, T] ; L^\infty (\mathbb{R}^d) \cap L^1_+ (\mathbb{R}^d))$$

such that

$$\int_0^T \int_{\mathbb{R}^d} K(t, x) \mu(t, dx) = \int_0^T \int_{\mathbb{R}^d} K(t, x) u(t, x) \, dxdt.$$  

Moreover, we also know that the finite measure $\mu^N_0 (dx)$ defined on $\mathbb{R}^d$ by $\mu^N_0 (dx) = \frac{1}{N} \sum_j \delta_{x_j(t)} (dx)$ converges weakly to a finite measure $\mu_0 (dx)$ with density $u_0 \in L^1_+ (\mathbb{R}^d)$:

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(x) \mu^N_0 (dx) = \int_{\mathbb{R}^d} \phi(x) u_0 (x) \, dx$$

for every bounded continuous function $\phi$. Finally, let us also introduce the notation $\mu^N_t (dx)$ for the family of finite measures on $\mathbb{R}^d$, indexed by $t$, such that

$$g^N (dt, dx) = \mu^N_t (dx) \, dt$$

namely, more explicitly, $\mu^N_t (dx) = \frac{1}{N} \sum_j \delta_{x_j(t)} (dx)$. We have, for $K \in C^1_c ([0, T] \times \mathbb{R}^d)$ (compact support is here a restriction only in space, since we take $[0, T]$ closed),

$$\left\langle \mu^N_t, K(T) \right\rangle = \left\langle \mu^N_0, K(0) \right\rangle + \int_0^T \left\langle g^N_t, \left( \partial_t + \frac{1}{2} \Delta + 1 \right) K(t) \right\rangle \, dt$$

$$- N^{-1} \int_0^T \sum_{j,k} \theta (\epsilon^{-1} (x_j(t) - x_k(t))) K(t, x_j(t)) \, dt + M_T.$$  

Notice that we may express the first time integral by means of $g^N (dt, dx)$, but we cannot do the same for the term $\left\langle \mu^N_t, K(T) \right\rangle$, since this is not a space-time integral. Therefore we have

$$\left\langle \mu^N_t, K(T) \right\rangle = \left\langle \mu^N_0, K(0) \right\rangle + \int_0^T \int_{\mathbb{R}^d} \left( \partial_t + \frac{1}{2} \Delta + 1 \right) K(t, x) g^N (dt, dx)$$

$$- N^{-1} \int_0^T \sum_{j,k} \theta (\epsilon^{-1} (x_j(t) - x_k(t))) K(t, x_j(t)) \, dt + M_T.$$  

Now, from the convergence property of $g^N_k$ above, we have

$$\lim_{k \to \infty} \int_0^T \int_{\mathbb{R}^d} \left( \partial_t + \frac{1}{2} \Delta + 1 \right) K(t, x) g^N_k (dt, dx)$$

$$= \int_0^T \int_{\mathbb{R}^d} \left( \partial_t + \frac{1}{2} \Delta + 1 \right) K(t, x) u(t, x) \, dxdt.$$  

Similarly, we have

$$\lim_{k \to \infty} \left\langle \mu^N_k, K(0) \right\rangle = \int_{\mathbb{R}^d} K(0, x) u_0 (x) \, dx.$$  

The last term is the one carefully analyzed in the paper. But we cannot state anything about the convergence of $\left\langle \mu^N_k, K(T) \right\rangle$, unless we investigate the tightness of the empirical measures $\mu^N$ in $C ([0, T] ; M^*_+ (\mathbb{R}^d))$, a fact that we prefer to avoid.

Therefore the only choice is to assume that

$$K(T) = 0.$$  

21
Definition 12. Assume \( u_0 \in L^\infty (\mathbb{R}^d) \cap L_+^1 (\mathbb{R}^d) \). We say that
\[
   u \in L^\infty ([0, T); L^\infty (\mathbb{R}^d) \cap L_+^1 (\mathbb{R}^d))
\]
is a weak solution of the Cauchy problem
\[
   \partial_t u = \frac{1}{2} \Delta u + u (1 - u) \quad (26)
\]
if
\[
   0 = \int_{\mathbb{R}^d} K (0, x) u_0 (x) \, dx + \int_0^T \int_{\mathbb{R}^d} \left( \partial_t + \frac{1}{2} \Delta + 1 \right) K (t, x) u (t, x) \, dx dt
\]
\[
   - \lim_{k \to \infty} N^{-1} \int_0^T \sum_{j,k} \theta (\epsilon^{-1} (x_j (t) - x_k (t))) K (t, x_j (t)) \, dt
\]
(assuming also the fact that the martingale goes to zero). This motivates the next definition.

Lemma 13. Given \( \phi \in C_c^1 (\mathbb{R}^d) \), the measurable bounded function \( t \mapsto \left< u (t), \phi \right> \) has a continuous modification, with value equal to \( \left< u_0, \phi \right> \) at time zero. Moreover, denoting by \( t \mapsto \left< u (t), \phi \right> \) the continuous modification, we have
\[
   \left< u (t), \phi \right> = \left< u_0, \phi \right> + \frac{1}{2} \int_0^t \left< u (s), \Delta \phi \right> ds + \int_0^t \left< u (s) (1 - u (s)), \phi \right> ds \quad (27)
\]
for all \( t \in [0, T] \).

Proof. Given \( t_0 \in [0, T) \), \( h > 0 \) such that \( t_0 + h \leq T \) and \( \phi \in C_c^1 (\mathbb{R}^d) \), consider the function \( K (t, x) = \phi (x) \chi (t) \), where \( \chi (t) \) is equal to 1 for \( t \in [0, t_0] \), \( 1 - \frac{1}{h} (t - t_0) \) for \( t \in [t_0, t_0 + h] \), zero for \( t \in [t_0 + h, T] \). This function is only Lipschitz continuous in time but it is not difficult to approximate it by a \( C^1 \) function of time (this is not really needed, since Lipschitz continuity in time of the test functions would be sufficient in the definition above). We have \( \partial_t K (t, x) = \phi (x) \chi' (t) \) where \( \chi' (t) \) is equal to zero outside \([t_0, t_0 + h]\) and to \(-h^{-1}\) inside, with only lateral derivatives at \( t_0 \) and \( t_0 + h \). Using this test function above we get
\[
   0 = \int_{\mathbb{R}^d} \phi (x) u_0 (x) \, dx + \int_0^T \chi' (t) \int_{\mathbb{R}^d} \phi (x) u (t, x) \, dx dt
\]
\[
   + \int_0^T \chi (t) \left( \left< u (t), \frac{1}{2} \Delta \phi \right> + \left< u (t) (1 - u (t)), \phi \right> \right) dt
\]
the function equal to \( t \).

Therefore we get

where we have denoted by \( v(t) \) the bounded measurable function \( \int_{\mathbb{R}^d} \phi(x) u_0(x) \, dx \). Since \( u(t) \) is bounded, the function equal to \( (t-t_0) \left( \langle u(t), \frac{1}{2}\Delta \phi \rangle + \langle u(t)(1-u(t)), \phi \rangle \right) \) for \( t \in [t_0, T] \) and equal to zero at \( t_0 \) is continuous at \( t = t_0 \), hence

\[
\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} (t-t_0) \left( \langle u(t), \frac{1}{2}\Delta \phi \rangle + \langle u(t)(1-u(t)), \phi \rangle \right) dt = 0.
\]

By Lebesgue differentiability theorem, the following limit exists for a.e. \( t_0 \):

\[
\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} v(t) dt = v(t_0).
\]

Therefore we get

\[
v(t_0) = \int_{\mathbb{R}^d} \phi(x) u_0(x) \, dx + \int_{t_0}^{t_0+h} \left( \langle u(t), \frac{1}{2}\Delta \phi \rangle + \langle u(t)(1-u(t)), \phi \rangle \right) dt
\]

for a.e. \( t_0 \). The right-hand-side of this identity is a continuous function of \( t_0 \), hence the function \( v \) has a continuous modification. And its value at \( t_0 = 0 \) is \( \int_{\mathbb{R}^d} \phi(x) u_0(x) \, dx \).

We can now prove the main result of this section.

**Proposition 14** Two weak solutions of the Cauchy problem \((22)\) coincide a.s.

**Proof. Step 1.** Let \( u \) be a weak solution. Let \( e^{t/2} \Delta \) be the heat semigroup, defined for instance on bounded measurable functions. In this step we are going to prove that

\[
u(t) = e^{t/2} \Delta u_0 + \int_0^t e^{(t-s)/2} \Delta [u(s)(1-u(s))] \, ds. \tag{28}
\]

Let \( (\theta_\epsilon)_{\epsilon \in (0,1)} \) be classical smooth compact support mollifiers; set

\[
\phi_\epsilon(t) = \theta_\epsilon \ast u(t).
\]

Given \( \psi \in C_0^1(\mathbb{R}^d) \), take \( \phi = \phi_\epsilon \ast \psi \) in \((27)\), where \( \phi_\epsilon(x) = \theta_\epsilon(-x) \). Then, being

\[
\langle u(t), \phi_\epsilon \ast \psi \rangle = \langle \theta_\epsilon \ast u(t), \psi \rangle = \langle u_\epsilon(t), \psi \rangle
\]

\[
\langle u_0, \phi_\epsilon \ast \psi \rangle = \langle \theta_\epsilon \ast u_0, \psi \rangle
\]

\[
\langle u, \Delta \phi_\epsilon \ast \psi \rangle = \langle u \ast \phi_\epsilon, \Delta \psi \rangle = \langle \theta_\epsilon \ast u_\epsilon, \Delta \psi \rangle = \langle \Delta u_\epsilon(t), \psi \rangle
\]

we get

\[
\langle u_\epsilon(t), \psi \rangle = \langle \theta_\epsilon \ast u_0, \psi \rangle + \frac{1}{2} \int_0^t \langle \Delta u_\epsilon(s), \psi \rangle \, ds + \int_0^t \langle \theta_\epsilon \ast [u(s)(1-u(s))], \psi \rangle \, ds
\]

23
and therefore
\[ u_\varepsilon(t) = \theta_\varepsilon * u_0 + \frac{1}{2} \int_0^t \Delta u_\varepsilon(s) \, ds + \int_0^t \theta_\varepsilon * [u(s)(1 - u(s))] \, ds \]
which implies that \( t \mapsto u_\varepsilon(t, x) \) is differentiable, for every \( x \in \mathbb{R}^d \). With classical arguments we can rewrite the equation in the form
\[ u_\varepsilon(t) = e^{t \frac{\Delta}{2}} \theta_\varepsilon * u_0 + \int_0^t e^{(t-s) \frac{\Delta}{2}} \theta_\varepsilon * [u(s)(1 - u(s))] \, ds. \]
Notice that \( e^{t \frac{\Delta}{2}} \) is defined by a convolution with a smooth kernel, for \( t > 0 \), and thus by commutativity between convolutions we have \( e^{t \frac{\Delta}{2}} \theta_\varepsilon * u_0 = \theta_\varepsilon * e^{t \frac{\Delta}{2}} u_0 \) and similarly under the integral sign. Hence we can also write
\[ u_\varepsilon(t) = \theta_\varepsilon * e^{t \frac{\Delta}{2}} u_0 + \int_0^t \theta_\varepsilon * e^{(t-s) \frac{\Delta}{2}} [u(s)(1 - u(s))] \, ds. \]

Given \( \phi \in C_c(\mathbb{R}^d) \), we deduce
\[ \langle u(t), \theta_\varepsilon * \phi \rangle = \langle e^{t \frac{\Delta}{2}} u_0, \theta_\varepsilon - * \phi \rangle + \int_0^t \langle e^{(t-s) \frac{\Delta}{2}} [u(s)(1 - u(s))], \theta_\varepsilon - * \phi \rangle \, ds. \]
Since \( \theta_\varepsilon - * \phi \) converges uniformly to \( \phi \), from dominated convergence theorem we deduce
\[ \langle u(t), \phi \rangle = \langle e^{t \frac{\Delta}{2}} u_0, \phi \rangle + \int_0^t \langle e^{(t-s) \frac{\Delta}{2}} [u(s)(1 - u(s))], \phi \rangle \, ds \]
and thus we get \((28)\).

**Step 2.** Assume that \( u(i) \) are two weak solutions. Then, from \((28)\),
\[ u^{(1)}(t) - u^{(2)}(t) = \int_0^t e^{(t-s)\frac{\Delta}{2}} \left( u^{(1)}(s) - u^{(2)}(s) \right) \left( 1 - u^{(1)}(s) - u^{(2)}(s) \right) \, ds \]

hence
\[ \|u^{(1)}(t) - u^{(2)}(t)\|_\infty \leq \int_0^t \|u^{(1)}(s) - u^{(2)}(s)\|_\infty \left( 1 + \|u^{(1)}(s)\|_\infty + \|u^{(2)}(s)\|_\infty \right) \, ds. \]
Since, by assumption, \( u^{(i)} \) are bounded, we deduce \( \|u^{(1)}(t) - u^{(2)}(t)\|_\infty = 0 \) by Gronwall lemma.

**Appendix**

The following proposition formalizes part of the proof of Lemma 3 and is given here for completeness and as a reference.

**Proposition 15** Let \( d \geq 1 \) and \( (x_i(t)) \) be the pure proliferation model with unit rate (no killing), with \( N \) initial particles distributed independently with density \( u_0 \); assume \( u_0 \) is bounded with compact support in a ball \( B(0, R) \). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a bounded non-negative function and let \( T > 0 \) be given. Then there are constants \( C_T, c > 0 \), independent of \( N \) and \( f \), such that
\[ E \left[ \sum_{i,j} f(x_i(t) - x_j(t)) \right] \leq C_T N \|f\|_\infty + N^2 C_T^2 \|u_0\|^2 \|f\|_\infty^2 e^{cR} \int f(x) e^{-c|x|} \, dx \]
where the sum is extended to all particles alive at time \( t \). The constant \( C_T \) is the average number of alive particles at time \( T \), when starting from a single initial particle; the constant \( c \) depends only on the Brownian motion.
Proof. Step 1. Essential for the proof is the fact that the exponential clocks of proliferation can be modeled
a priori, therefore let us write a few details in this direction for completeness. Particles, previously indexed
by \( t \), will be indexed below by a multi-index \( a \) of the form
\[
\mathcal{N} = \{a(1),...,a_n\}
\]
with \( n \) positive integer, \( a_1 \in \{1,...,N\}, a_2,...,a_n \in \{1,2\} \) (if \( n \geq 2 \)). Denote by \( \mathcal{N} \) the set of all such
multi-indexes. Given \( a \in \mathcal{N} \), we denote by \( n(a) \) the length of the string \( a = (a_1,...,a_n) \) defining \( a \). We set
\[
a^{-1} = (a_1,...,a_{n-1})
\]
when \( n \geq 2 \). The heuristic idea behind these notations is that \( a_1 \) denotes the progenitor at time \( t = 0 \);
\( a_2,...,a_n \) describe the subsequent story, where particle \( a \) is a direct descendant of particle \( a^{-1} \).

Each particle lives for a finite random time. On a probability space \((\Omega,\mathcal{F},\mathbb{P})\), assume to have a
countable family of independent Exponential r.v.'s \( \tau^a \) of parameter \( \lambda = 1 \), indexed by \( a \in \mathcal{N} \). The time \( \tau^a \)
is the life span of particle \( a \); its interval of existence will be denoted by \([T^a_0, T^a_f]\) with \( T^a_0 = T_0^a + \tau^a \). The
random times \( T^a_0 \) are defined recursively in \( n \in \mathbb{N} \): if \( n(a) = 0 \), \( T^a_0 = 0 \); if \( n(a) > 0 \),
\[
T^a_0 = T^a_{0-1} + \tau^a_{0-1} = T^a_f - 1.
\]
We may now define the set of particles alive at time \( t \): it is the set
\[
\mathcal{N} = \{a \in \mathcal{N} : t \in [T^a_0, T^a_f]\}.
\]

Initial particles have a random initial position in the space \( \mathbb{R}^d \): we assume that on the probability space
\((\Omega,\mathcal{F},\mathbb{P})\) there are r.v.'s \( X_1,...,X_N \) distributed with density \( \rho_0 \) independent among themselves and with
respect to the random times \( \tau^a \), \( a \in \mathcal{N} \).

Particles move as Brownian motions: we assume that on the probability space \((\Omega,\mathcal{F},\mathbb{P})\) there is a
countable family of independent Brownian motions \( W^a \), \( a \in \mathcal{N} \), independent among themselves and with
respect to the random times \( \tau^a \), \( a \in \mathcal{N} \) and the initial positions \( X_1^a,...,X_N^a \). The position \( x^a_t \) of particle \( a \)
during its existence interval \([T^a_0, T^a_f]\) is defined recursively in \( n \in \mathbb{N} \) as follows: if \( n(a) = 0 \), \( x^a_t = X^a_t + W^a_t \)
for \( t \in [T^a_0, T^a_f] \); if \( n(a) > 0 \)
\[
x^a_t = x^{a-1}_t + W^a_{t-T^a_0} \quad \text{for } t \in [T^a_0, T^a_f].
\]

Step 2. Given \( k \in \{1,...,N\} \) and \( a = (a_1,...,a_n) \in \mathcal{N} \), the process \( x^a_t \) is formally defined only for
\( t \in [T^a_0, T^a_f] \). Call \( \mathbb{E}^a_t \) the related process, defined for all \( t \geq 0 \) as follows: for each \( b = (a_1,...,a_m) \)
with \( m \leq n \), on the interval \([T^b_0, T^b_f]\) it is given by \( x^b_t \); and on \([T^a_0, T^a_f) \) it is given by \( x^{a-1}_t + W^a_{t-T^a_0} \). The process \( \mathbb{E}^a_t \)
is a Brownian motion with initial position \( X^a_0 \). More precisely, if \( \mathcal{G} \) denotes the \( \sigma \)-algebra generated by
the family \( \{\tau^a ; a \in \mathcal{N} \} \), then the law of \( \mathbb{E}^a_t \) conditioned to \( \mathcal{G} \) is the law of a Brownian motion with initial
position \( X^a_0 \).

Step 3. With the notations of Step 1 above, we have to handle
\[
\mathbb{E} \left[ \sum_{a,b \in \mathcal{N}} f(x^a_t - x^b_t) \right] = \sum_{a,b \in \mathcal{N}} \mathbb{E} \left[ l_{a,b} \mathbb{E} f(x^a_t - x^b_t) \right].
\]
As explained in the previous step, let us denote the components of \( a, b \) as \( a = (a_1,...a_n) \), \( b = (b_1,...b_m) \), with
integers \( n, m > 0 \), \( a_1, b_1 \in \{1,...,N\} \) and all the other entries in \( \{1,2\} \). Then
\[
\sum_{a,b \in \mathcal{N}} \mathbb{E} \left[ l_{a,b} \mathbb{E} f(x^a_t - x^b_t) \right] = \sum_{a,b \in \mathcal{N}, a_1 = b_1} \mathbb{E} \left[ l_{a,b} \mathbb{E} f(x^a_t - x^b_t) \right] + \sum_{a,b \in \mathcal{N}, a_1 \neq b_1} \mathbb{E} \left[ l_{a,b} \mathbb{E} f(x^a_t - x^b_t) \right].
\]
In the following computation, when we decompose a multi-index $a = (a_1, \ldots, a_n)$ in the form $(a_1, a')$ we understand that $a'$ does not exist in the case $n = 1$, while $a' = (a_2, \ldots, a_n)$ if $n \geq 2$. We simply bound

\[
\sum_{a, b \in \Lambda^N : a_1 = b_1} \mathbb{E}[1_{a \in \Lambda^N} 1_{b \in \Lambda^N} f(x^a_t - x^b_t)]
= \sum_{a_1 = 1}^N \sum_{a', b' \in I} \mathbb{E}[1_{(a_1, a') \in \Lambda^N} 1_{(a_1, b') \in \Lambda^N} f(x^a_t - x^b_t)]
\leq \|f\|_\infty \sum_{a_1 = 1}^N \sum_{a', b' \in I} \mathbb{E}[1_{(a_1, a') \in \Lambda^N} 1_{(a_1, b') \in \Lambda^N}]
= N \|f\|_\infty \sum_{a', b' \in I} \mathbb{E}[1_{(1, a') \in \Lambda^1} 1_{(1, b') \in \Lambda^1}]
\]

where the last identity is due to the fact that the quantity $\mathbb{E}[1_{(a_1, a') \in \Lambda^N} 1_{(a_1, b') \in \Lambda^N}]$ is independent of $a_1$: then it is equal to

\[
N \|f\|_\infty \mathbb{E}\left[\sum_{a, b \in \Lambda^N} 1\right] = N \|f\|_\infty \mathbb{E}[|\Lambda^1|]
\]

where $\Lambda^1$ is the set of indexes relative to the case of a single initial particle, and the identity holds because the presence of more initial particles does not affect the expected values of the previous expression; finally the previous quantity is equal to

\[
N \|f\|_\infty \mathbb{E}\left[\sum_{a, b \in \Lambda^N} 1\right] = N \|f\|_\infty \mathbb{E}[|\Lambda^1|]
\]

where we have denoted by $|\Lambda^1|$ the cardinality of the set $\Lambda^1$. We set $C_T = \mathbb{E}[|\Lambda^1|]$, which is finite and we get one addend of the inequality stated in the proposition.

Concerning the other sum,

\[
\sum_{a, b \in \Lambda^N : a_1 \neq b_1} \mathbb{E}[1_{a \in \Lambda^N} 1_{b \in \Lambda^N} f(x^a_t - x^b_t)]
\]

\[
= \sum_{a_1 \neq b_1} \sum_{a', b' \in I} \mathbb{E}[1_{(a_1, a') \in \Lambda^N} 1_{(b_1, b') \in \Lambda^N} f(x^a_t - x^b_t)]
\leq N^2 \sum_{a', b' \in I} \mathbb{E}[1_{(1, a') \in \Lambda^2} 1_{(2, b') \in \Lambda^2} f(\bar{x}^a_t - \bar{x}^b_t)]
\]

where the last inequality, involving a system with only two initial particles, can be explained similarly to what done above. Recalling the notation of Step 2 above, the previous expression is equal to

\[
= N^2 \sum_{a', b' \in I} \mathbb{E}[1_{(1, a') \in \Lambda^2} 1_{(2, b') \in \Lambda^2} f(\bar{x}^{(1, a')}_t - \bar{x}^{(2, b')}_t)].
\]

Now we use the fact that the laws of processes indexed by 1 and 2 are independent and the law of $\bar{x}^{(1, a')}_t$ conditioned to $\mathcal{G}^1$ is a Brownian motion with initial position $X^{(1, a')}_0$, where $\mathcal{G}^1$ is the $\sigma$-algebra generated by the family $\{\tau^{(1, a')}; (1, a') \in \Lambda^1\}$; and similarly for $\bar{x}^{(2, b')}_t$ with respect to $\mathcal{G}^2$, similarly defined. Thus, after
taking conditional expectation with respect to $G^1 \lor G^2$ inside the previous expected value, we get that the previous expression is equal to

$$
N^2 \sum_{a',b' \in I} P ((1,a') \in \Lambda^1_t) P ((2,b') \in \Lambda^2_t) \mathbb{E} \left[ f \left( W^1_t - W^2_t + X^1_0 - X^2_0 \right) \right]
$$

where $W^i_t, i = 1, 2$ are two independent Brownian motions in $\mathbb{R}^d$, independent also of $X^1_0, X^2_0$. We may simplify the previous expression to

$$
N^2 \left( \sum_{a' \in I} P ((1,a') \in \Lambda^1_t) \right)^2 \mathbb{E} \left[ f \left( \sqrt{2}W_t + X^1_0 - X^2_0 \right) \right]
$$

where $W_t$ is a Brownian motion in $\mathbb{R}^d$ independent of $X^1_0, X^2_0$. One has

$$
\sum_{a' \in I} P ((1,a') \in \Lambda^1_t) = \mathbb{E} \left[ \sum_{a \in \Lambda^1_t} 1 \right] = \mathbb{E} [ |\Lambda^1_T| ] \leq \mathbb{E} [ |\Lambda^1_T| ] = C_T.
$$

Moreover, denoting $\overline{u_0}(x) = u_0(-x)$,

$$
\mathbb{E} \left[ f \left( \sqrt{2}W_t + X^1_0 - X^2_0 \right) \right] = \int \mathbb{E} \left[ f \left( \sqrt{2}W_t + x \right) \right] (\overline{u_0} * u_0)(x) dx
= \langle e^{\Delta t} f, \overline{u_0} * u_0 \rangle = \langle f, e^{\Delta t} (\overline{u_0} * u_0) \rangle
= \int f(x) \mathbb{E} \left[ (\overline{u_0} * u_0) \left( \sqrt{2}W_t + x \right) \right] dx.
$$

Now we may estimate

$$
\mathbb{E} \left[ (\overline{u_0} * u_0) \left( \sqrt{2}W_t + x \right) \right] \leq \|u_0\|_\infty^2 \mathbb{E} \left[ 1_{B(0,R)} \left( \sqrt{2}W_t + x \right) \right]
= \|u_0\|_\infty^2 \mathbb{P} \left( \sqrt{2}W_t \in B(x,R) \right)
\leq \|u_0\|_\infty^2 \mathbb{P} \left( \left| \sqrt{2}W_t \right| \geq |x| - R \right)
\leq \|u_0\|_\infty^2 e^{cR} e^{-c|x|}
$$

for some constant $c = c(d,T) > 0$. Therefore, summarizing,

$$
N^2 \left( \sum_{a' \in I} P ((1,a') \in \Lambda^1_t) \right)^2 \mathbb{E} \left[ f \left( \sqrt{2}W_t + X^1_0 - X^2_0 \right) \right]
\leq N^2 C_T^2 \|u_0\|_\infty^2 e^{cR} \int f(x) e^{-c|x|} dx.
$$

\[\blacksquare\]

References

[1] V. Bansaye and S. Méléard. *Stochastic models for structured populations*, volume 1 of *Mathematical Biosciences Institute Lecture Series. Stochastics in Biological Systems*. Springer, Cham; MBI Mathematical Biosciences Institute, Ohio State University, Columbus, OH, 2015. Scaling limits and long time behavior.
[2] N. Champagnat and S. Méléard. Invasion and adaptive evolution for individual-based spatially structured populations. *J. Math. Biol.*, 55(2):147–188, 2007.

[3] A. De Masi, P. A. Ferrari, and J. L. Lebowitz. Reaction-diffusion equations for interacting particle systems. *J. Statist. Phys.*, 44(3-4):589–644, 1986.

[4] A. De Masi, T. Funaki, E. Presutti, and M. E. Vares. Fast-reaction limit for Glauber-Kawasaki dynamics with two components. *ALEA Lat. Am. J. Probab. Math. Stat.*, 16(2):957–976, 2019.

[5] A. De Masi and E. Presutti. *Mathematical methods for hydrodynamic limits*, volume 1501 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[6] J. Farfán, C. Landim, and K. Tsunoda. Static large deviations for a reaction-diffusion model. *Probab. Theory Related Fields*, 174(1-2):49–101, 2019.

[7] F. Flandoli and M. Leimbach. Mean field limit with proliferation. *Discrete Contin. Dyn. Syst. Ser. B*, 21(9):3029–3052, 2016.

[8] F. Flandoli, M. Leimbach, and C. Olivera. Uniform convergence of proliferating particles to the FKPP equation. *J. Math. Anal. Appl.*, 473(1):27–52, 2019.

[9] T. Funaki and K. Tsunoda. Motion by mean curvature from Glauber-Kawasaki dynamics. *J. Stat. Phys.*, 177(2):183–208, 2019.

[10] A. Hammond and F. Rezakhanlou. The kinetic limit of a system of coagulating Brownian particles. *Arch. Ration. Mech. Anal.*, 185(1):1–67, 2007.

[11] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[12] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.

[13] S. Méléard and S. Roelly-Coppoletta. A propagation of chaos result for a system of particles with moderate interaction. *Stochastic Process. Appl.*, 26(2):317–332, 1987.

[14] M. Métivier. Quelques problèmes liés aux systèmes infinis de particules et leurs limites. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 426–446. Springer, Berlin, 1986.

[15] G. Nappo and E. Orlandi. Limit laws for a coagulation model of interacting random particles. *Ann. Inst. H. Poincaré Probab. Statist.*, 24(3):319–344, 1988.

[16] K. Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 69(2):279–322, 1985.

[17] K. Oelschläger. On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately interacting stochastic processes. *Probab. Theory Related Fields*, 82(4):565–586, 1989.

[18] A. Stevens. The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems. *SIAM J. Appl. Math.*, 61(1):183–212, 2000.

[19] A.-S. Sznitman. Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[20] K. Uchiyama. Pressure in classical statistical mechanics and interacting Brownian particles in multi-dimensions. *Ann. Henri Poincaré*, 1(6):1159–1202, 2000.

[21] S. R. S. Varadhan. Scaling limits for interacting diffusions. *Comm. Math. Phys.*, 135(2):313–353, 1991.