A SEQUENCE OF ALGEBRAIC INTEGER RELATION NUMBERS WHICH CONVERGES TO 4

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ABSTRACT. For $\alpha \in \mathbb{R}$, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B_\alpha = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}.$$ 

Then the subgroup $G_\alpha = \langle A, B_\alpha \rangle \subset SL_2(\mathbb{R})$ acts on the Poincare disc $\mathbb{H}$ with the boundary. By using the action, we define the generalized Farey graph $\Gamma_\alpha$. We prove that $G_\alpha$ is a free group of rank 2 if and only if the graph $\Gamma_\alpha$ is tree. By using this result and a symmetry of the generalized Farey graph $\Gamma_\alpha$, we establish the orbit test for non-freeness of the group $G_\alpha$. By applying the orbit test, we find an increasing sequence of algebraic integer relation numbers which converges to 4.

1. INTRODUCTION

For $\alpha \in \mathbb{C}$, consider the following two parabolic matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B_\alpha = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix},$$

and the subgroup generated by these matrices,

$$G_\alpha = \langle A, B_\alpha \rangle \subset SL_2(\mathbb{C}).$$

We say that $\alpha$ is free when $G_\alpha$ is isomorphic to the rank 2 free group. Otherwise, we say $\alpha$ to be relation. This terminology is firstly suggested by Kim and Koberda [11]. Our main question is characterizing complex numbers that are relation numbers. Almost all complex numbers are free as transcendental numbers are free [6]. Moreover, due to Brenner [1] and Sanov [14], it is also known that if $\alpha$ is real and $|\alpha| \geq 4$, then $\alpha$ is free. More generally, it is also true that if $\alpha$ is in the Riley slice of $\mathbb{C}$, then the group $G_\alpha$ is free and discrete [10]. For simple descriptions of free numbers, see [12], [2] and [9]. However, in the complement of the Riley slice, the characterization of free numbers has not been completed. Thus it is meaningful to understand the complement of the Riley slice. From now on, we concentrate on the complement of the Riley slice.

From [3], for algebraic conjugates $a, b \in \mathbb{C}$, the corresponding Galois conjugation gives a group isomorphism between $G_a$ and $G_b$. This implies that algebraic free numbers are dense in the complex plane. On the other hand, Ree [13] proved that relation numbers are dense in the unit open disc $\{z \in \mathbb{C} : |z| < 1\}$. Hence we focus on the case $|z| > 1$. Kim and Koberda [11] showed that there is a sequence of rational relation numbers which converges to 2. In particular, one of the most outstanding conjectures is the following.

Conjecture 1.1. If $\alpha \in \mathbb{Q}$ and $|\alpha| < 4$, then $\alpha$ is relation.

In $\mathbb{Q}$, some special cases are known as relation numbers. We say that an integer $m$ is a good numerator if $\frac{m}{n}$ is relation for every $n$ with $|\frac{m}{n}| < 4$. It is true that all integers $m$ with $1 \leq m \leq 27$ except 24 are good numerators (see [5] and [11]). For more results about rational relation numbers, see [8].
However, it is still unknown that there exists a sequence of rational relation numbers which converges to 3 or 4. More generally, it is also an open question that there exists a sequence of relation numbers which converges to 3 or 4. Our main theorem is the following.

**Theorem 6.1** There exists a sequence of polynomials $p_n(\alpha)$ satisfying the following conditions.

- The polynomial $p_n(\alpha)$ is a monic polynomial with integer coefficients.
- All roots of $p_n(\alpha)$ are relation numbers.
- Let $\alpha_n$ be the maximal root of $p_n(\alpha)$. Then the sequence $\{\alpha_n\}$ is strictly increasing and converges to 4.

This paper is organized as follows. In Section 2, we fix notations to eliminate confusion as the notation has been differed by literatures. Since there exist group isomorphisms among them, it does not matter. Next, we introduce basic notions. In Section 3, we discuss a circle action of $G_\alpha$ from the perspective of a stabilizer subgroup when $\alpha$ is free. In Section 4, by using the action of $G_\alpha$, we define the generalized Farey graph $\Gamma_\alpha$ and prove the following theorem.

**Theorem 4.15** Let $\alpha \in \mathbb{R}$. Then $\alpha$ is relation if and only if the graph $\Gamma_\alpha$ is not tree.

This gives a necessary and sufficient condition for the freeness of $G_\alpha$. In section 5, by using Theorem 4.15, we deduce the orbit test for a relation number. From this test, we obtain some rational relation numbers and present them. Finally, we will prove the main theorem, Theorem 6.1 in the last section, Section 6.

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**Theorem 4.15** Let $\alpha \in \mathbb{R}$. Then $\alpha$ is relation if and only if the graph $\Gamma_\alpha$ is not tree.

This gives a necessary and sufficient condition for the freeness of $G_\alpha$. In section 5, by using Theorem 4.15, we deduce the orbit test for a relation number. From this test, we obtain some rational relation numbers and present them. Finally, we will prove the main theorem, Theorem 6.1 in the last section, Section 6.

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### 2. Preliminaries

#### 2.1. On the relation number.

In this section, we review the convention of previous literatures and fix ours.

For $\alpha_1, \alpha_2 \in \mathbb{C}$, consider two matrices

$$A_{\alpha_1} = \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_{\alpha_2} = \begin{bmatrix} 1 & 0 \\ \alpha_2 & 1 \end{bmatrix},$$

and let $G_{\alpha_1, \alpha_2} = \langle A_{\alpha_1}, B_{\alpha_2} \rangle \subset \text{SL}_2(\mathbb{C})$.

Many previous results have been obtained in the setting $G_{2, \alpha}$ or $G_{\mu, \mu}$. Moreover, it is known that $G_{a,b}$ is isomorphic to $G_{1,ab} = G_{ab}$ for any nonzero complex numbers $a, b$ [2].

Due to the difference among definitions of the groups, there are many definitions for relation numbers consequentially. Note that terminologies including free points and non-free points were also employed in previous literatures. In this paper, we use a notation $G_\alpha = G_{1,\alpha}$. We say that $\alpha \in \mathbb{C}$ is free if $G_\alpha$ is a free group of rank 2. Otherwise, we call $\alpha$ a relation number. As mentioned before, this is firstly defined by Kim and Koberda [11]. In that paper, they use $G_q = G_{q,1}$ but we use $G_\alpha = G_{1,\alpha}$ for the convenience.

#### 2.2. A circle action of $G_\alpha$.

Now we introduce basic definitions. Recall that the group $\text{SL}_2(\mathbb{C})$ acts on the hyperbolic space $\mathbb{H}^3$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = az + b \overline{cz + d}$$

and similarly the group $\text{SL}_2(\mathbb{R})$ acts on the hyperbolic space $\mathbb{H}^2$. Thus, when $\alpha \in \mathbb{R}$, $G_\alpha$ is a subgroup of $\text{SL}_2(\mathbb{R})$, hence $G_\alpha$ acts on $\mathbb{H}^2$. For the convenience, we assume $\alpha > 0$ since $B_{\alpha}^{-1} = B_{-\alpha}$ implies $G_\alpha = G_{-\alpha}$.
Let $\mathbb{D}$ be the open unit disc on $\mathbb{C}$, that is, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The linear fractional transformation
\[ \phi(z) = i \frac{z - i}{z + i} \]
is an isometry from the upper half plane $\mathbb{H}^2$ to $\mathbb{D}$. Thus, the group SL$_2(\mathbb{R})$ acts on $\mathbb{D}$. Moreover, this action can be extended to the boundary. Hence SL$_2(\mathbb{R})$ acts on $S^1 = \partial \mathbb{D}$. Thus $G_\alpha$ also acts on $S^1$. Note that the kernel of the action of $G_\alpha$ is $\{1, -I\} \cap G_\alpha$ since PSL$_2(\mathbb{R})$ acts faithfully on the hyperbolic plane $\mathbb{H}^2$.

Elements in SL$_2(\mathbb{R})$ are classified into parabolic, elliptic, and hyperbolic elements. This classification is particularly useful for studying the group action on $S^1$ as it concerns fixed points on $S^1$. For the sake of completeness, we briefly recall this classification. Let $M \in$ SL$_2(\mathbb{R})$ be a matrix with $M \neq I, -I$. If $|\text{tr}(M)| = 2$, we say that $M$ is parabolic. If $|\text{tr}(M)| < 2$, $M$ is elliptic and if $|\text{tr}(M)| > 2$, $M$ is hyperbolic. In each case, we have the following properties. When $M$ is parabolic, then $M$ has a unique fixed point in $S^1$. If $M$ is elliptic, then $M$ does not have a fixed point in $S^1$. If $M$ is hyperbolic, then $M$ has distinct two fixed points in $S^1$. Thus, if a matrix $M$ fixes at least distinct three points, then $M$ must be $I$ or $-I$. When $p$ is a point in a set $X$ and a group $G$ acts on $X$, we define the stabilizer subgroup by
\[ \text{Stab}_G(p) = \{ g \in G : g \cdot p = p \}. \]

2.3. Some combinatorial notions. We summarize basic combinatorial concepts and notations that will appear in Section 4. First of all, in this paper all graphs are undirected. Let $P_n$ be the graph defined by a vertex set $V = \{0, 1, \ldots, n\}$ and an edge set $E = \{(i, i + 1) : 0 \leq i \leq n - 1\}$. A path $P$ in a graph $\Gamma$ is an image of a graph morphism $f : P_n \to \Gamma$. Thus, for a given path $P$, we can express $P$ as a finite sequence of edges $y_1, \ldots, y_n$. Here, $y_i$ is an image of an edge $(i, i + 1)$ so we call it an $i$-th edge of the path $P$. Moreover, the image of $\{0\}$ and $\{n\}$ are called the starting point of $P$ and the terminal point of $P$, respectively. We say that a path $P$ has no backtracking if $y_i \neq y_{i+1}$ for all $i$ in the sequence expression $y_1, \ldots, y_n$. If a path $P = y_1, \ldots, y_n$ has no backtracking, we define the length of $P$ as $n$.

If there exists an edge between vertices $x$ and $y$, we say that vertices $x$ and $y$ are adjacent. We say that two edges $y_1$ and $y_2$ are adjacent if their union is isomorphic to a path graph $P_2$.

An edge is said to be a loop if the starting point is equal to the terminal point where the edge is regarded as a path graph. We say that a graph $\Gamma$ has a cycle if there exists a subgraph $\Gamma'$ in $\Gamma$ such that $\Gamma'$ is isomorphic to the cycle graph. When a connected graph does not have a cycle, we say that it is tree. The following is elementary.

**Lemma 2.1.** Let $T$ be a graph. Then $T$ is tree if and only if for any two vertices $v, w$, there exists a unique path with no backtracking from $v$ to $w$.

**Proof.** See Theorem 1.5.1 in [4].

2.4. clockwise and anticlockwise maps. In Section 6, we will deal with some continuous maps from an open subset $I$ of $\mathbb{R}$ to the unit circle $S^1 = \partial \mathbb{D}$ on $\mathbb{C}$. More precisely, they are meromorphic functions restricted to $\mathbb{R}$.

restrictions of meromorphic functions to $\mathbb{R}$ and the images of them are contained in $\mathbb{R} \cup \{\infty\}$. Thus we need to identify $\mathbb{R} \cup \{\infty\}$ with $S^1$. In order to do so, let us consider $\phi$ which is the map defined in Section 2.2. Then $\phi$ is a bi-holomorphism on $\mathbb{C} \cup \{\infty\}$ which induces a homeomorphism between $\mathbb{R} \cup \{\infty\}$ and $S^1$.

The map $\phi$ is bi-holomorphic on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and a homeomorphism between $\mathbb{R} \cup \{\infty\}$ and $S^1$. Thus, we can identify $\mathbb{R} \cup \{\infty\}$ and $S^1$ by using the map $\phi$ and consider a map from an open subset $I$ of $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$ as a map from $I$ to $S^1$. For convenience, from now on, whenever we mention a map from $I$ to $S^1$, the map is also from $I$ to $\mathbb{R} \cup \{\infty\}$ by $\phi$, and vice versa.

In the proof of Theorem 6.1 the key objects are (anti)clockwise maps which wind intervals of $\mathbb{R}$ around $S^1$. Let us define it. Let $I$ be an open subset of $\mathbb{R}$. First, a map $f$ from $I$ to $S^1$ is strictly increasing and strictly decreasing at $a \in J := I - f^{-1}(\{\infty\})$ if there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq J$ and whenever $a - \varepsilon < x < y < a + \varepsilon$, $f(x) < f(y)$ and $f(x) > f(y)$, respectively. Note that if $f$ is a continuous map from $I$ to $S^1$, then $-1/f$ is also a continuous map from $I$ to $S^1$ since $-1/z$ is bi-holomorphic on the Riemann sphere.
A continuous map \( f \) from \( I \) to \( S^1 \) is anticlockwise at \( x_0 \in I \) if one of \( f \) and \(-1/f\) is strictly increasing at \( x_0 \). We simply call \( f \) an anticlockwise map when \( f \) is anticlockwise at all points in \( I \). Similarly, we can define clockwise at a point and a clockwise map. Note that if \( f \) is a (anti)clockwise map, then \( f^{-1}(\{\infty\}) \) is discrete in \( I \) by the definition.

We end this section by stating simple properties of anticlockwise and clockwise maps. We will use them in the proof of Lemma 6.3. The first two lemmas follow easily from the definition, so we only give a rigorous proof of the third lemma.

**Lemma 2.2.** Let \( I \) be an open interval in \( \mathbb{R} \). For a given anticlockwise or clockwise map \( f : I \to S^1 \) and a constant \( c_0 \in \mathbb{R} \), \( g(x) := f(x) + c_0 \) is again an anticlockwise or a clockwise map, respectively.

Here, we allow an open interval to be \((\infty, a), (a, \infty)\) or \((\infty, \infty) = \mathbb{R} \).

**Lemma 2.3.** Let \( I \) be an open interval in \( \mathbb{R} \). If \( f(x) : I \to S^1 \) is anticlockwise, then the map \( g(x) := 1/f(x) \) is clockwise, and vice versa.

**Lemma 2.4.** Let \( I \) be an open interval in \( \mathbb{R} \). Suppose \( f : I \to S^1 \) is an anticlockwise map. Then \( g : I \to S^1 \) defined by \( g(x) := f(x) + x \) is also an anticlockwise map.

**Proof.** Firstly, consider \( a \) satisfying \( f(a) \neq \infty \). Then \( g(a) \neq \infty \). By assumption, \( f \) is increasing at \( a \). Since \( g \) is also increasing at \( a \), \( g \) is anticlockwise at \( a \). Now choose \( a \) with \( f(a) = \infty \). Since \( f \) is anticlockwise, for some \( \varepsilon > 0 \), when \( a - \varepsilon < x_1 < x_2 < a + \varepsilon \), \(-1/f(x_i) < -1/f(x_2)\). Moreover, we can choose such \( \varepsilon \) satisfying \(|f(x)| > M\) on the interval \((a - \varepsilon, a + \varepsilon)\) for some positive \( M \). Note that we may assume \( M > |a + 1|, M > |a - 1| \) and \( \varepsilon < 1 \) by shrinking the open interval \((a - \varepsilon, a + \varepsilon)\). Let

\[
h(x) := -\frac{1}{f(x) + x},
\]

To show \( g(x) \) is anticlockwise at \( a \), we need to show that \( h(x) \) is strictly increasing at \( a \) because of \( g(a) = \infty \). Since \( h(a) = 0 \), we may assume \( h(x) \neq \infty \) on the interval \((a - \varepsilon, a + \varepsilon)\) by shrinking the interval if necessary. We prove this dividing two cases.

(i) Suppose \( a < x_1 < x_2 < a + \varepsilon \).
   Since \( f(a) = \infty \), we have \( 0 = -\frac{1}{f(a)} < -\frac{1}{f(x_i)} < \frac{1}{M} \) for \( i = 1, 2 \) since \(-1/f(x_i) < -1/f(x_2)\) is strictly increasing on an interval \((a - \varepsilon, a + \varepsilon)\). Thus, we have \( f(x_1), f(x_2) < 0 \). From \(-\frac{1}{f(x_1)} < -\frac{1}{f(x_2)}\), we obtain
   \[
f(x_1) < f(x_2) < -M < 0.
\]

Then we have \( f(x_1) + x_1 < f(x_2) + x_2 \). Since \( h(x) \neq \infty \) when \( a - \varepsilon < x < a + \varepsilon \), we deduce that \( f(x) + x \neq 0 \) if \( a - \varepsilon < x < a + \varepsilon \). Thus, either

\[
0 < f(x_1) + x_1 < f(x_2) + x_2, \quad \text{or} \quad f(x_1) + x_1 < f(x_2) + x_2 < 0.
\]

However, since

\[
f(x_1) + x_1 < -M + (a + \varepsilon) < -M + a + 1 < -M + |a + 1| < 0,
\]

the first case is impossible so we get \( f(x_1) + x_1 < f(x_2) + x_2 < 0 \). This implies \( 0 < h(x_1) < h(x_2) \).

(ii) Suppose \( a - \varepsilon < x_1 < x_2 < a \).
   Since \( f(a) = \infty \), we have \(-\frac{1}{M} < -\frac{1}{f(x_i)} < -\frac{1}{f(a)} = 0 \) for \( i = 1, 2 \) since \(-1/f(x_i) < -1/f(x_2)\) is strictly increasing on an interval \((a - \varepsilon, a + \varepsilon)\). Thus, we have \( f(x_1), f(x_2) > 0 \). From \(-\frac{1}{f(x_1)} < -\frac{1}{f(x_2)}\), we obtain
   \[
0 < M < f(x_1) < f(x_2).
\]

Thus, \( f(x_1) + x_1 < f(x_2) + x_2 \). Since \( f(x) + x \neq 0 \) when \( a - \varepsilon < x < a + \varepsilon \), we obtain either

\[
0 < f(x_1) + x_1 < f(x_2) + x_2, \quad \text{or} \quad f(x_1) + x_1 < f(x_2) + x_2 < 0.
\]

Since

\[
f(x_2) + x_2 > M + (a - \varepsilon) > M + (a - 1) > M - |a - 1| > 0,
\]

the only possible case is \( 0 < f(x_1) + x_1 < f(x_2) + x_2 \). This gives \( h(x_1) < h(x_2) < 0 \).
Therefore, the map \( h(x) \) is strictly increasing on \((a - \varepsilon, a + \varepsilon)\). Combining all cases, \( g(x) \) is anticlockwise. 

\[ \square \]

### 3. A Stabilizer Subgroup of the Circle Action

In section 2.2, we showed when \( \alpha \in \mathbb{R} \), \( G_\alpha \) acts on the circle \( S^1 \). Moreover, we classified elements into parabolic, elliptic and hyperbolic elements. In this section, we investigate stabilizer subgroups. Via this subgroups and the classification, we will give a criterion for relation numbers. Beginning from this section, \( \alpha \) is always real and positive.

**Lemma 3.1.** Let \( p \) be a point of \( S^1 \). If there exist \( g, h \in \text{Stab}_{G_\alpha}(p) \) such that \( g \) is parabolic and \( h \) is hyperbolic, then \( \alpha \) is a relation number.

**Proof.** Choose matrices \( g, h \in \text{Stab}_{G_\alpha}(p) \) such that \( g \) is parabolic and \( h \) is hyperbolic. Then \( hgh^{-1} \) is also parabolic. Moreover, because

\[
\text{hgh}^{-1}(p) = \text{hgh}^{-1}(h(p)) = \text{hg}(p) = h(p) = p,
\]

the matrix \( \text{hgh}^{-1} \) also fixes \( p \). Since a parabolic element has only one fixed point, and two elements \( g \) and \( \text{hgh}^{-1} \) has common fixed point \( p \), there exists a matrix \( B \in \text{SL}_2(\mathbb{R}) \) such that

\[
BgB^{-1} = T_a \quad \text{and} \quad Bhgh^{-1}B^{-1} = T_b
\]

where \( T_x \) means translation \( z \mapsto z + x \) so \( T_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \). Of course, \( a, b \neq 0 \) so \( \frac{b}{a} \in \mathbb{R} \).

1. The number \( \frac{b}{a} \) is irrational.
   Then the subgroup \( \langle T_a, T_b \rangle \) is isomorphic to \( \mathbb{Z}^2 \). The mapping \( X \mapsto B^{-1}XB \) gives a group isomorphism \( \langle T_a, T_b \rangle \cong \langle g, hgh^{-1} \rangle \). Recall that the group \( \langle g, hgh^{-1} \rangle \) is a subgroup of \( G_\alpha \). Thus, the group \( G_\alpha \) cannot be a free group of rank 2.

2. The number \( \frac{b}{a} \) is rational.
   Let \( \frac{b}{a} = \frac{m}{n} \) where \( m, n \in \mathbb{Z} - \{0\} \) and \( \gcd(m, n) = 1 \). This implies \( T_a^m = T_b^n \), so, \( g^m = (hgh^{-1})^n \). Therefore we obtain

\[
h^m h^{-1} g^{-m} = 1.
\]

Consider the subgroup \( H = \langle g, h \rangle \subset G_\alpha \). Then \( h^m h^{-1} g^{-m} = 1 \) gives a relation of the group \( H \). Assume \( G_\alpha \) is a free group of rank 2. Then \( H \) is also a free group. Because of a relation, \( H \) is isomorphic to \( \mathbb{Z} \). Thus, \( g, h \) are commute. However, it is a contradiction because \( g, h \) are of different types.

By above, \( \alpha \) is a relation number. 

\[ \square \]

**Lemma 3.2.** Let \( G_\alpha \) be a free group of rank 2 and \( p \) be a point of \( S^1 \). If there is a parabolic element in \( \text{Stab}_{G_\alpha}(p) \), then \( \text{Stab}_{G_\alpha}(p) \) is isomorphic to \( \mathbb{Z} \).

**Proof.** Suppose \( \text{Stab}_{G_\alpha}(p) \) contains a parabolic element. Then by Lemma 3.1, \( \text{Stab}_{G_\alpha}(p) \) does not contain hyperbolic elements. Moreover, \( \text{Stab}_{G_\alpha}(p) \) does not contain elliptic elements because an elliptic element has no fixed point in \( S^1 \). Thus the group \( \text{Stab}_{G_\alpha}(p) \) has only parabolic elements. This implies that, for \( A, B \in \text{Stab}_{G_\alpha}(p), \text{Fix}(A) = \text{Fix}(B) = \{p\} \), hence \( A \) and \( B \) commute. Since \( \text{Stab}_{G_\alpha}(p) \) is abelian, free and non-trivial, the group \( \text{Stab}_{G_\alpha}(p) \) is isomorphic to \( \mathbb{Z} \). 

\[ \square \]

**Lemma 3.3.** Let \( p \) be a point of \( S^1 \) and \( g, h \in \text{Stab}_{G_\alpha}(p) \) be hyperbolic elements. If \( \text{Fix}(g) \cap \text{Fix}(h) = \{p\} \), then \( \alpha \) is a relation number.
Proof. Since $\text{Fix}(g) \cap \text{Fix}(h) = \{p\}$, let $\text{Fix}(g) = \{p,q\}$ and $\text{Fix}(h) = \{p,r\}$ for distinct $p,q,r \in S^1$. In $\mathbb{H}^2$, we may assume $p = \infty$. Then $g = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $h = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$. Note that these are in $\text{SL}_2(\mathbb{R})$, so $ac = xz = 1$. A simple calculation gives
\[
\text{tr}((g,h)) = \text{tr}(ghg^{-1}h^{-1}) = 2acxz = 2.
\]
Thus, the commutator of $g$ and $h$ is parabolic concluding the group $\text{Stab}_{G\alpha}(p)$ contains parabolic and hyperbolic. This implies $\alpha$ is a relation number. \hfill \square

**Lemma 3.4.** Let $G\alpha$ be a free group of rank 2, and $p$ be a point of $S^1$. If there is a hyperbolic element in $\text{Stab}_{G\alpha}(p)$, then $\text{Stab}_{G\alpha}(p)$ is isomorphic to $\mathbb{Z}$.

**Proof.** Choose a hyperbolic element $h \in \text{Stab}_{G\alpha}(p)$. Say $\text{Fix}(h) = \{p,q\}$. By Lemma 3.1, $\text{Stab}_{G\alpha}(p)$ does not contain parabolic elements. Moreover, recall that $\text{Stab}_{G\alpha}(p)$ does not contain elliptic elements. Thus, only possible elements contained in $\text{Stab}_{G\alpha}(p)$ are hyperbolic. By Lemma 3.3, when $x \in \text{Stab}_{G\alpha}(p)$, $\text{Fix}(x) = \{p,q\}$. Thus, the group $\text{Stab}_{G\alpha}(p)$ is abelian. Since $\text{Stab}_{G\alpha}(p)$ is a non-trivial, abelian and free group, it must be isomorphic to $\mathbb{Z}$. \hfill \square

The following proposition follows from Lemma 3.2 and Lemma 3.4. The contraposition gives a criterion for relation numbers.

**Proposition 3.5.** Let $G\alpha$ be a free group of rank 2, and $p$ be a point of $S^1$. Then for any $p \in S^1$, the group $\text{Stab}_{G\alpha}(p)$ is either trivial or isomorphic to $\mathbb{Z}$. Thus, if the group $\text{Stab}_{G\alpha}(p)$ is neither $\mathbb{Z}$ nor trivial for some $p \in S^1$, then $\alpha$ is relation.

Since $A \in \text{Fix}(\infty)$ and $B_{\alpha} \in \text{Fix}(0)$, we can get Corollary 3.6. Before proving this, we need to define a length of $g \in G\alpha$. This length is depending on its word representation. Let $F_2$ be the rank 2 free group with a free basis $\{x_1,x_2\}$. For a reduced word $w = x_{i_1}^{\sigma_1} \cdots x_{i_k}^{\sigma_k}$ of $F_2$, we define the length of $w$ as $k$. Here, $\sigma_i = 1$ or $2$ for $1 \leq i \leq k$. Now, consider the map $q_{\alpha} : F_2 \to G\alpha$ defined by $x_1 \mapsto A, x_2 \mapsto B_{\alpha}$. For $g \in G\alpha$, we say that $w$ is a lifting word of $g$ if $q_{\alpha}(w) = g$. If $G\alpha$ is free, then for each $g \in G\alpha$, the lifting word of $g$ is unique, but if $G\alpha$ is non-free, then there are many lifting words of $g$. Even so, after choosing a lifting word, we can define it. Thus, whenever we mention the length of $g \in G\alpha$, it refer to the length of a particular lifting word for $g$.

**Corollary 3.6.** If $\alpha$ is free, then
\[
\text{Stab}_{G\alpha}(\infty) = \{A^k : k \in \mathbb{Z}\} \text{ and } \text{Stab}_{G\alpha}(0) = \{B_{\alpha}^k : k \in \mathbb{Z}\}.
\]

**Proof.** It is immediate that $\text{Stab}_{G\alpha}(\infty) \supset \{A^k : k \in \mathbb{Z}\}$. Choose $x \in \text{Stab}_{G\alpha}(\infty) - \{A^k : k \in \mathbb{Z}\}$. Then since $B_{\alpha}^k \notin \text{Stab}_{G\alpha}(\infty)$, $x$ has a length at least 2. By Proposition 3.5, $\text{Stab}_{G\alpha}(\infty) \cong \mathbb{Z}$. Thus we can choose a generator $y$ of the group $\text{Stab}_{G\alpha}(\infty)$. Then for some $m,n \in \mathbb{Z}$, $y^m = A, y^n = x$. This gives $A^n = x^m$. Recall that the length of $x$ is at least 2, so this gives a relation in $G\alpha$. It is a contradiction because $G\alpha$ is free.

A similar argument applies to the second one. Clearly, $\text{Stab}_{G\alpha}(0) \supset \{B_{\alpha}^k : k \in \mathbb{Z}\}$. Choose $x \in \text{Stab}_{G\alpha}(0) - \{B_{\alpha}^k : k \in \mathbb{Z}\}$. Then since $A^k \notin \text{Stab}_{G\alpha}(0)$, $x$ has a length at least 2. By Proposition 3.5, $\text{Stab}_{G\alpha}(0) \cong \mathbb{Z}$. Thus we can choose a generator $y$ of the group $\text{Stab}_{G\alpha}(0)$. Then for some $m,n \in \mathbb{Z}$, $y^m = A, y^n = x$. This gives $A^n = x^m$. Recall that the length of $x$ is at least 2, so this gives a relation in $G\alpha$. It is a contradiction because $G\alpha$ is free. \hfill \square

## 4. The Generalized Farey Graph

In this section, we define a graph $\Gamma_{\alpha}$ which is a generalization of the Farey graph. The main goal of this section is to prove Theorem 4.15. First, let us define the graph $\Gamma_{\alpha}$.

**Definition 4.1.** Let $\alpha \in \mathbb{R}$. The generalized Farey graph $\Gamma_{\alpha}$ is the graph with the vertex set
\[
V = \{g \cdot 0, g \cdot \infty : g \in G\alpha\}
\]
and the edge set
\[ E = \{ g \cdot l : g \in G_\alpha \}. \]

Here, \( l \) is the geodesic in \( \overline{\mathbb{Q}} \) joining 0 and \( \infty \).

Then the graph \( \Gamma_\alpha \) has following combinatorial properties.

**Lemma 4.2.** The graph \( \Gamma_\alpha \) is connected and not locally finite.

**Proof.** Note that edges \( l, A \cdot l, B_\alpha \cdot l, A^{-1} \cdot l, B_\alpha^{-1} \cdot l \) are connected. Since \( G_\alpha \) action on \( \Gamma_\alpha \) is continuous, for any word \( X \), an edge \( X \cdot l \) is adjacent to \( XA \cdot l,XA^{-1} \cdot l, XB_\alpha \cdot l, XB_\alpha^{-1} \cdot l \). By using the mathematical induction, \( \Gamma_\alpha \) is connected. Moreover, \( \Gamma_\alpha \) is not locally finite because the vertex \( \infty \) is connected to all of integers by edges \( A^p \cdot l \).

**Lemma 4.3.** The graph \( \Gamma_\alpha \) is a simple graph.

**Proof.** It follows directly that the graph \( \Gamma_\alpha \) has no multiple edges because \( \overline{\mathbb{Q}} \) is uniquely geodesic. Moreover, it follows easily that there is no loop in \( \Gamma_\alpha \).

Thus, the graph \( \Gamma_\alpha \) shares many properties with the Farey graph. Indeed, the graph \( \Gamma_1 \) is the Farey graph. Before proving, we recall the definition of the Farey graph briefly.

**Definition 4.4.** Let \( \overline{\mathbb{Q}} = \mathbb{Q} \cup \{ \infty = \frac{1}{1} \} \). Let \( V = \overline{\mathbb{Q}} \) and
\[ E = \left\{ \left( \frac{a}{b}, \frac{c}{d} \right) \in \overline{\mathbb{Q}}^2 : |ad - bc| = 1 \right\}. \]

The graph consisting of the vertex set \( V \) and the edge set \( E \) is called the Farey graph.

**Proposition 4.5.** The graph \( \Gamma_1 \) is the Farey graph.

Recall that Theorem [4.15] states the relationship between \( G_\alpha \) and the graph \( \Gamma_\alpha \). The following lemma says that for a given word in a group \( G_\alpha \), we can construct a path in the graph \( \Gamma_\alpha \). From now on, we represent words \( g \in G_\alpha \) as follows:
\[ g = C_{i_1}^{p_i_1} \cdots C_{i_k}^{p_k} \]
where \( C_i = A \) or \( B_\alpha \), and \( \{C_i, C_{i+1}\} = \{A, B_\alpha\} \).

**Lemma 4.6.** Let \( W = C_{i_1}^{p_i_1} \cdots C_{i_k}^{p_k} \) be a word of length \( k \) in \( G_\alpha \). Then the sequence of edges
\[ l, C_{i_1}^{p_1}l, C_{i_2}^{p_2}l, \cdots, C_{i_k}^{p_k}l = Wl \]
is a path without backtracking. Thus, the length of this path is \( k + 1 \).

**Proof.** Firstly, we show that this is a path. Recall that for any edge \( y = g \cdot l \), two vertices of \( y \) are given by \( g \cdot 0 \) and \( g \cdot \infty \). Since \( A \) fixes only \( \infty \) and \( B_\alpha \) fixes only 0, for any \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\alpha \), a simple calculation gives that
\[ MA^p \cdot \infty = M \cdot \infty = \frac{a}{c} \quad \text{and} \quad MB_\alpha^p \cdot 0 = M \cdot 0 = \frac{b}{d}. \]
Moreover, note that \( MA^p \cdot 0 = M \cdot p = \frac{ap + b}{cp + d} \neq \frac{b}{d} \) and \( MB_\alpha \cdot \infty = M \cdot \frac{1}{\alpha p} = \frac{a + b \alpha a}{c + d \alpha a} \neq \frac{a}{c} \). This implies the mapping \( M \mapsto MA^p \) fixes vertex \( M \cdot \infty \) but not \( M \cdot 0 \), and the mapping \( M \mapsto MB_\alpha^p \) fixes \( M \cdot 0 \) but not \( M \cdot \infty \).

Thus, the set of edges
\[ l, C_{i_1}^{p_1}l, C_{i_2}^{p_2}l, \cdots, C_{i_k}^{p_k}l = Wl \]
is a path in \( \Gamma_\alpha \).

Now we want to prove that this path does not contain backtracking. Now let \( y = g \cdot l \) and \( y' = g C^p \cdot l \). Here \( C \) is either \( A \) or \( B_\alpha \) and is determined by the last word of \( g \). Suppose that there exists a backtracking in our path, say \( y = y' \). Then \( C^p \) must fix both 0 and \( \infty \), since \( A \) and \( B_\alpha \) already fix \( \infty \) and 0, respectively. It is impossible so our path does not have backtracking. □
With the following lemma, we prepare to prove Proposition 4.9 one direction of Theorem 4.15.

**Lemma 4.7.** Let $s$ be an element in the stabilizer of the edge $l$ in the graph $\Gamma_\alpha$. If $s$ is non-trivial, then the length of $s$ is at least 3.

**Proof.** Let $s \in G_\alpha$ be a non-trivial element in the stabilizer of the geodesic $l$. Then $s$ either fixes both 0 and $\infty$, or maps $0 \mapsto \infty$ and $\infty \mapsto 0$. Suppose $s$ fixes both 0 and $\infty$. This implies that $s$ has the form \[
\begin{bmatrix}
a & 0 \\
0 & \frac{1}{a}
\end{bmatrix}.
\] In this case, the length of $s$ must be at least 3 since
\[
A^{p_1}B^{p_2}_\alpha = \begin{bmatrix} 1 + p_1p_2\alpha & p_1 \\
p_2\alpha & 1 \end{bmatrix}
\] and $B^{p_1}_\alpha A^{p_2} = \begin{bmatrix} 1 & p_2 \\
p_1\alpha & 1 + p_1p_2\alpha \end{bmatrix}$.

Now suppose $s$ maps $0 \mapsto \infty$ and $\infty \mapsto 0$. Then $s$ has the form \[
\begin{bmatrix} 0 & a \\
\frac{1}{a} & 0
\end{bmatrix}.
\] By the calculation above, $s$ has the length at least 3. \qed

**Remark 4.8.** The lemma above does not hold for any edge $y$ in $\Gamma_\alpha$. For example, consider $\alpha = 1$ and edge $y = B_1 \cdot l$. Then $M = AB_1^{-2} = \begin{bmatrix} -1 & 1 \\
-2 & 1 \end{bmatrix}$ fixes an edge $y$.

**Proposition 4.9.** Let $\alpha \in \mathbb{R}$. If $\alpha$ is relation, then the graph $\Gamma_\alpha$ is not tree.

**Proof.** Suppose $\alpha$ is a relation number, say $W = C_{p_1}^1 \cdots C_{p_k}^k = I$. Since $W$ is in the stabilizer of $l$, by Lemma 4.7 we get $k \geq 3$. Consider the path
\[l, C_{p_1}^1 \cdot l, C_{p_2}^1 \cdot l, \cdots, C_{p_k}^1 \cdots C_{p_k}^k \cdot l = W \cdot l.
\] Recall that the length of this path is $k + 1$. Since $k \geq 3$ and this path does not have backtracking, this path gives a cycle in the graph $\Gamma_\alpha$. Thus, $\Gamma_\alpha$ is not tree. \qed

Hence, Proposition 4.9 gives one part of Theorem 4.15. To prove the converse, we will show that if $\alpha$ is free, then the graph $\Gamma_\alpha$ must be tree. In the case that $\alpha$ is free, we can extract a graph structure of $\Gamma_\alpha$.

**Lemma 4.10.** When $\alpha$ is free, the edge stabilizer in the graph $\Gamma_\alpha$ is trivial.

**Proof.** Choose an element $s \in \text{Stab}_{G_\alpha}(l)$. Then either $s$ fixes both 0 and $\infty$, or is of the form \[
\begin{bmatrix}
0 & a \\
\frac{1}{a} & 0
\end{bmatrix}.
\] However, since \[
\begin{bmatrix}
0 & a \\
\frac{1}{a} & 0
\end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\
0 & -1 \end{bmatrix},
\] $s$ must fix both 0 and $\infty$. From Corollary 3.6 we have that
Proof. By acting on $\Gamma_\alpha$, say $g \cdot l$. Then note that
\[ \text{Stab}_{\alpha}(g \cdot l) = \{ gxg^{-1} : x \in \text{Stab}_{\alpha}(l) \}. \]
The stabilizer subgroup $\text{Stab}_{\alpha}(l)$ is trivial implies that $\text{Stab}_{\alpha}(g \cdot l)$ is also trivial. 

To study the structure of the graph $\Gamma_\alpha$, we want to know that which two vertices or two edges are adjacent. When $\alpha$ is free, the answer is quite simple.

**Lemma 4.11.** When $\alpha$ is free, the set of edges adjacent to vertices $\infty$ and 0 are
\[ \{ A^k \cdot l : k \in \mathbb{Z} \} \text{ and } \{ B^k_{\alpha} \cdot l : k \in \mathbb{Z} \}, \]
respectively.

**Proof.** Firstly, assume that $g \cdot l$ is an edge adjacent to $\infty$. Then either $g \cdot \infty = \infty$ or $g \cdot 0 = \infty$. Suppose $g \cdot \infty = \infty$.

Then $g \in \text{Stab}_{\alpha}(\infty)$. By Corollary 3.6, $g = A^k$ for some $k \in \mathbb{Z}$.

Suppose $g \cdot 0 = \infty$. Consider $M = gB_\alpha g^{-1}$.

Then $M \in \text{Stab}_{\alpha}(\infty)$. Again by Corollary 3.6, we obtain that $gB_\alpha g^{-1} = A^k$ for some $k$. This gives a relation, so it is a contradiction. Thus, when $g \cdot l$ is an edge adjacent to $\infty$, $g$ must be in the set $\{ A^k : k \in \mathbb{Z} \}$.

Now we assume $g \cdot l$ is an edge adjacent to 0. Then either $g \cdot 0 = 0$ or $g \cdot \infty = 0$. Suppose $g \cdot 0 = 0$. Then $g \in \text{Stab}_{\alpha}(0)$. By Corollary 3.6, $g = B^k_{\alpha}$ for some $k \in \mathbb{Z}$.

Suppose $g \cdot \infty = 0$. Consider $M = gA g^{-1}$.

Then $M \in \text{Stab}_{\alpha}(0)$. Again by Corollary 3.6, we obtain that $gA g^{-1} = B^k_{\alpha}$ for some $k$. This gives a relation, so it is a contradiction. 

**Corollary 4.12.** When $\alpha$ is free, for given an edge $g \cdot l$, the set of edges adjacent to the edge $g \cdot l$ is
\[ \left\{ gA^k \cdot l : k \in \mathbb{Z} - \{0\} \right\} \cup \left\{ gB^k_{\alpha} \cdot l : k \in \mathbb{Z} - \{0\} \right\}. \]

**Proof.** By acting $g^{-1}$, it suffices to show that the set of edges adjacent to the edge $l$ is given by
\[ \left\{ A^k \cdot l : k \in \mathbb{Z} - \{0\} \right\} \cup \left\{ B^k_{\alpha} \cdot l : k \in \mathbb{Z} - \{0\} \right\}. \]

This easily follows from Lemma 4.11.

Thus, when we are given a path in $\Gamma_\alpha$, say that the first edge is $l$ and the last edge is $g \cdot l$, the length of the path is related to the length of $g$.

**Lemma 4.13.** Suppose that a group $G_\alpha$ is a free group of rank 2. Let $D_1 \cdot l, D_2 \cdot l, \ldots, D_k \cdot l$ be a path without backtracking in $\Gamma_\alpha$. Then
\[ \prod_{i=1}^{k} D_i = A^{p_0} B^{q_0}_{\alpha} : p, q \in \mathbb{Z} - \{0\} \].

Moreover, if $\prod_{i=1}^{k} D_i = A^{p_0}$ for some nonzero integer $p_0$, then $\prod_{i=1}^{k} D_i = B^{q_0}_{\alpha}$ for some nonzero integer $q_0$, and vice versa.

**Proof.** The first part follows directly from Corollary 4.12. To prove the second part, suppose not. We may assume $g \cdot l, gA^n \cdot l, gA^m \cdot l$ forms a path in $\Gamma_\alpha$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then endpoints of $g \cdot l$ are $g \cdot 0 = \frac{a}{b}$ and $g \cdot \infty = \frac{d}{c}$. A simple calculation shows
\[ gA^n \cdot 0 = \frac{an + b}{cn + d}, \quad gA^n \cdot \infty = \frac{a}{c}, \quad gA^m \cdot 0 = \frac{am + b}{cm + d}, \quad gA^m \cdot \infty = \frac{a}{c}. \]
Note that $\frac{am + b}{cn + d} = \frac{am + b}{cn + d}$ if and only if $(ad - bc)(m - n) = 0$. Thus, three edges $g \cdot l, gA^n \cdot l, A^m \cdot l$ cannot be a path since the degree of a vertex $g \cdot \infty$ is 3. The degree of vertices in a path of which length is 3, must be 1 or 2.

**Corollary 4.14.** Assume that the group $G_\alpha$ is a free group of rank 2. Let $l = g_1 \cdot l, g_2 \cdot l, \cdots, g_k \cdot l$ be a path of length $k$ beginning at $l$. Then the length of the word $g_k \in G_\alpha$ is $k - 1$.

**Proof.** It is obtained from Lemma 4.13

Now let us prove the main result in this section.

**Theorem 4.15.** Let $\alpha \in \mathbb{R}$. Then $\alpha$ is relation if and only if the graph $\Gamma_\alpha$ is not tree.

**Proof.** The only remaining part is to show that if $\Gamma_\alpha$ is not tree, then $\alpha$ is relation. We shall prove if $G_\alpha$ is a free group of rank 2, then the graph $\Gamma_\alpha$ is tree. Suppose $G_\alpha$ is free and $\Gamma_\alpha$ is not tree. Since $\Gamma_\alpha$ is not tree, there is a cycle. We may assume the cycle contains $l$, so we can say that the cycle is $l = C_0 \cdot l, \cdots, C_k \cdot l = l$. Here, $k \geq 3$. Lastly, we also may assume $C_i \cdot l \neq l$ for all $0 < i < k$.

From Lemma 4.10 $C_0 = C_k = I$. In cycle, consider the path

$$l = C_0 \cdot l, \cdots, C_{k-1} \cdot l.$$

Then the length of this path is $k$. By Corollary 4.14, the length of $C_{k-1}$ is $k - 1$. Since the edge $C_{k-1} \cdot l$ is adjacent to $C_k \cdot l = l$, the length of $C_{k-1}$ is 1 by Lemma 4.13. Thus, since $G_\alpha$ is a free group of rank 2, we have $k - 1 = 1$. However, $k \geq 3$ so it is impossible. Therefore, the theorem is proved.

### 5. Applications: The Orbit Test for Relation Numbers

In this section, we construct the method to find a cycle in the graph $\Gamma_\alpha$. By Theorem 4.15 this gives the non-freeness of the group $G_\alpha$. According to this method, it suffices to check whether the $G_\alpha - \text{orbit of 0 or } \infty$ contains $\frac{1}{2}$ in order to verify that $\alpha$ is relation. In other words, if the graph $\Gamma_\alpha$ has $\frac{1}{2}$ as a vertex, then $\alpha$ is relation. Firstly, we establish the orbit test, and next we give new examples of relation numbers.

The first lemma below says that the graph $\Gamma_\alpha$ is symmetric with respect to the geodesic $l$. Let $W_\alpha$ be the set of all words consisting of $A, B_\alpha$ and its formal inverses $A^{-1}, B_\alpha^{-1}$. Consider the map $h : W_\alpha \rightarrow W_\alpha$ defined by

$$A \leftrightarrow A^{-1}, B_\alpha \leftrightarrow B_\alpha^{-1}.$$ 

Then the map $h$ induces a map $\hat{h} : G_\alpha \rightarrow G_\alpha$. We abuse the notation so we are given the map $h : G_\alpha \rightarrow G_\alpha$.

**Lemma 5.1.** Let $X \in G_\alpha$. If $X \cdot 0 = a \neq \infty$, then $h(X) \cdot 0 = -a$. Similarly, if $X \cdot \infty = b \neq \infty$, then $h(X) \cdot \infty = -b$.

**Proof.** We shall see that for any word $X$ in $G_\alpha$, $X \cdot 0 + h(X) \cdot 0 = 0$ and $X \cdot \infty + h(X) \cdot \infty = 0$ whenever $X \cdot 0 \neq 0$ and $X \cdot \infty \neq 0$, respectively. Suppose $X \cdot 0 = a \neq \infty$. We use the mathematical induction. Clearly $A \cdot 0 + A^{-1} \cdot 0 = 0, B_\alpha \cdot 0 + B_\alpha^{-1} \cdot 0 = 0$, so it holds when $X = A, A^{-1}, B_\alpha$ or $B_\alpha^{-1}$. Assume that $X \cdot 0 + h(X) \cdot 0 = 0$. Put $X \cdot 0 = a$ so $h(X) \cdot 0 = -a$. Then

$$A^{-1}h(X) \cdot 0 = A^{-1} \cdot (-a) = -a - 1 = -AX \cdot 0,$$

thus $AX \cdot 0 + h(AX) \cdot 0 = 0$. Moreover,

$$B_\alpha^{-1}h(X) \cdot 0 = B_\alpha^{-1} \cdot (-a) = -a \frac{a}{a\alpha + 1} = -BX \cdot 0$$

implies $BX \cdot 0 + h(BX) \cdot 0 = 0$. Now let us prove that $X \cdot \infty + h(X) \cdot \infty = 0$. Obviously we have $B \cdot \infty + B^{-1} \cdot \infty = 0$. Now assume $X \cdot \infty = a \neq \infty$. Then the precisely same argument gives that $X \cdot \infty + h(X) \cdot \infty = 0$ whenever $X \cdot \infty \neq \infty$.

**Lemma 5.2.** Let $X \in G_\alpha$. Suppose $X \cdot 0 = \frac{1}{2}$ (or $X \cdot \infty = \frac{1}{2}$). Then there exists $Y \in G_\alpha$ with $Y \neq X$ such that $Y \cdot 0 = \frac{1}{2}$ or $(Y \cdot \infty = \frac{1}{2}$, respectively).
Lemma 2.1. Suppose that the graph \( G \) contains a cycle, clearly we may assume that a terminal point of the path \( P \) is \( \Gamma \)-relation.

Proof. For given \( X \), take

\[
Y = Ah(X).
\]

Assume \( X \cdot 0 = \frac{1}{2} \). Then by Lemma 5.1 we have \( h(X) \cdot 0 = -\frac{1}{2} \). Thus

\[
Ah(X) \cdot 0 = A \left( -\frac{1}{2} \right) = \frac{1}{2}
\]

and the same argument shows \( Ah(X) \cdot \infty = \frac{1}{2} \) whenever \( X \cdot \infty = \frac{1}{2} \). □

The last ingredient is quite combinatorial.

Lemma 5.3. Let \( X \) be a connected graph. Choose a vertex \( v \) in \( X \). Suppose that there exist distinct two edges \( e_1, e_2 \) starting at \( v \). Suppose two paths \( P_1, P_2 \) in \( X \) are given with following properties.

- The starting points of \( P_1, P_2 \) is \( v \) and terminal points of \( P_1, P_2 \) are same.
- The first edge of \( P_1 \) and \( P_2 \) is \( e_1 \) and \( e_2 \), respectively.
- Two paths \( P_1, P_2 \) have no backtracking.

Then \( X \) is not tree.

Proof. Consider union of two paths \( S := P_1 \cup P_2 \). We shall see that \( S \) contains a cycle. If \( P_1 \) or \( P_2 \) already contains a cycle, clearly \( S \) contains cycle, so we may assume \( P_1 \) and \( P_2 \) do not contain any cycles. Moreover, since \( P_1, P_2 \) do not have backtracking, we can consider \( P_1, P_2 \) as an injective image of path graph \( P_n \) in \( X \).

(Of course, the length of \( P_1, P_2 \) is not necessarily same.)

Let \( w \) be a terminal point of \( P_1 \) and \( P_2 \). Then two vertices \( v, w \) are linked by two distinct paths \( P_1, P_2 \). By Lemma 2.1, \( S \) is not tree, so \( X \) is not tree. □

Now we give a proof of the orbit test.

Proposition 5.4. Suppose that the \( G_\alpha \)-orbit of 0 or \( \infty \) contains the number \( \frac{1}{2} \), that is, for some \( X \in G_\alpha \), \( X \cdot 0 = \frac{1}{2} \) or \( X \cdot \infty = \frac{1}{2} \). Then \( \alpha \) is relation.

Proof. For \( X = C_1^{p_1} \cdots C_k^{p_k} \in G_\alpha \), we define the first letter of \( X \) as \( C_1 \). For \( X = C_1^{p_1} \cdots C_k^{p_k} \), consider the path corresponding \( X \), \( P(X) \) defined by

\[
P(X) = C_1^{p_1} \cdot l, C_1^{p_1} C_2^{p_2} \cdot l, \cdots, X \cdot l.
\]

Note that this path has no backtracking due to Lemma 4.6. Let \( P_1 = P(X), P_2 = P(Ah(X)) \). By assumption, we may assume that a terminal point of the path \( P_1 \) is \( \frac{1}{2} \). Then by Lemma 5.2 the path \( P_2 \) passes through the vertex \( \frac{1}{2} \).

Firstly, suppose either \( p_1 = 1, -1 \) or \( C_1 = B_\alpha \). Then the first letter of \( X \) is different from \( Ah(X) \). This implies a starting point of \( P_1 \) and \( P_2 \) are different. Note that if a starting point of \( P_1 \) is 0, then a starting point of \( P_2 \) is \( \infty \), and vice versa. Thus, by using concatenation \( l \) and \( P_1 \) or \( P_2 \), we can extend \( P_1 \) and \( P_2 \) to \( Q_1 \) and \( Q_2 \), respectively. Here, we may assume starting points of \( Q_1 \) and \( Q_2 \) are same as 0. Note that the first edge of \( Q_1, Q_2 \) are different. Since both \( Q_1 \) and \( Q_2 \) pass through the vertex \( 1/2 \), by restriction, we may assume a terminal point of \( Q_1, Q_2 \) are same as \( \frac{1}{2} \). Then by Lemma 5.3, \( \Gamma_\alpha \) is not tree. This gives \( \alpha \) is relation.

Next, assume \( C_1 = A \) and \( |p_1| > 1 \). In this case, note that the first letter of \( X \) and \( Ah(X) \) are \( A \). Thus, starting points of \( P_1 \) and \( P_2 \) are \( \infty \). Since both \( P_1 \) and \( P_2 \) pass through the vertex \( 1/2 \), by restriction, we may assume a terminal point of \( P_1 \) and \( P_2 \) are same as \( \frac{1}{2} \). Clearly, these two paths do not have backtracking. For using Lemma 5.3, we need to show that the first edge of \( P_1 \) and \( P_2 \) are different. Note that first edge of \( P_1, P_2 \) is \( A^{p_1} \cdot l \) and \( A^{1-p_1} \cdot l \), respectively. Since \( |p_1| > 1 \), the first edge of \( P_1, P_2 \) are different. Thus, \( \Gamma_\alpha \) is not tree. This concludes \( \alpha \) is relation. □

By \( A^n \), if the \( G_\alpha \)-orbit of 0 or \( \infty \) contains a number \( \frac{2n+1}{2} \) for some \( n \in \mathbb{Z} \), then \( \alpha \) is relation.
Remark 5.5. Here are examples. The most trivial example is that \( G_\alpha \) is relation when \( \alpha = \frac{1}{n} \) for nonzero integer \( n \), because \( B_n^\alpha \cdot -\infty = \frac{1}{2} \). Moreover, we can prove that \( \alpha = \frac{3}{n} \) and \( \alpha = \frac{5}{n} \) is relation by checking \( B_\alpha \cdot -\infty = \frac{n}{2} \) and \( B_\alpha^2 A^{-1} \cdot -\infty = \frac{1}{2} \), respectively. One of non-trivial and so far unknown relation numbers is \( 41 \). A matrix calculation says that

\[
B_\alpha^2 AB_\alpha^{-1} A \cdot 0 = \frac{45}{2}
\]

This gives the non-freeness of \( G_{\frac{1}{2}} \). The following table is new examples for relation numbers with denominator \( \leq 30 \) including previous example.

| \( \alpha \) | Orbit value | \( \alpha \) | Orbit value |
|--------|-------------|--------|-------------|
| \( \frac{41}{18} \) | \( B^{-2}AB^{-1}A \cdot 0 = \frac{45}{2} \) | \( \frac{33}{19} \) | \( B^{-3}AB^{-1}A^2 \cdot 0 = \frac{171}{2} \) |
| \( \frac{13}{20} \) | \( B^{-2}A^2B^{-1}A^2B^{-1}A^2 \cdot 0 = \frac{105}{2} \) | \( \frac{35}{27} \) | \( B^{-2}A^{-1}BA^{-2}BA^{-3} \cdot 0 = \frac{2893}{2} \) |
| \( \frac{43}{24} \) | \( B^2A^4B^{-1}AB^{-1}A^{-1} \cdot 0 = \frac{51}{2} \) | \( \frac{41}{25} \) | \( B^2A^{-1}B^{-2}A^{-2} \cdot 0 = \frac{35}{2} \) |
| \( \frac{57}{28} \) | \( B^2AB^{-1}A^2 \cdot 0 = \frac{175}{2} \) | \( \frac{33}{26} \) | \( BA^{-1}B^4A^3 \cdot 0 = \frac{497}{2} \) |
| \( \frac{35}{26} \) | \( BA^{-1}B^{-3}BA^{-3} \cdot 0 = \frac{1001}{2} \) | \( \frac{59}{26} \) | \( B^{-1}AB^{-1}A^2 \cdot 0 = \frac{65}{2} \) |
| \( \frac{35}{27} \) | \( B^{-1}A^{-1}AB^{-1}A^{-1} \cdot 0 = \frac{27}{2} \) | \( \frac{43}{27} \) | \( B^{-2}A^2B^{-1}A^2 \cdot 0 = \frac{135}{2} \) |
| \( \frac{33}{28} \) | \( BA^{-2}BA^{-3} \cdot 0 = \frac{21}{2} \) | \( \frac{37}{30} \) | \( B^{-1}A^{-1}BA^{-1} \cdot 0 = \frac{45}{2} \) |

Table 1. New rational relation numbers (\( \alpha \) is omitted.)

Recall that the orbit test can be applied when \( \alpha \) is irrational. By using the test, we can also prove that \( \alpha = 2 + \sqrt{2} \) is relation since \( B_\alpha A^{-1}B_\alpha \cdot -\infty = \frac{1}{2} \).

We do not know that the converse of the orbit test holds. We expect that the converse is false, but we cannot find any counterexamples.

6. PROOF OF THE MAIN THEOREM

In this section, we prove our main theorem, Theorem 6.1. Before we deal with lemmas and prove the main theorem, we explain notations which we will use. Putting

\[
(B_\alpha A^{-1})^n = \left[\begin{array}{cc} 1 & -1 \\ \alpha & 1 - \alpha \end{array}\right]^n = \begin{bmatrix} \ell_n(\alpha) & u_n(\alpha) \\ m_n(\alpha) & l_n(\alpha) \end{bmatrix},
\]

we can express \( (B_\alpha A^{-1})^n \cdot 0 = u_n(\alpha)/l_n(\alpha) \). Moreover, we obtain

\[
\begin{bmatrix} u_{n+1}(\alpha) \\ l_{n+1}(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \alpha & 1 - \alpha \end{bmatrix} \begin{bmatrix} u_n(\alpha) \\ l_n(\alpha) \end{bmatrix}
\]

with \( u_1(\alpha) = -1 \) and \( l_1(\alpha) = 1 - \alpha \). Then it is easy to show that \( u_n(\alpha) \) is a degree \( n - 1 \) polynomial and \( l_n(\alpha) \) is a degree \( n \) polynomial. Both have integer coefficients and leading coefficients of them are \((-1)^n\).

By the orbit test, Proposition 5.4 in particular, when \( u_n(\alpha)/l_n(\alpha) = 1/2 \), \( \alpha \) is relation. In other words, letting

\[
p_n(\alpha) := (-1)^{n+1} \left( 2u_n(\alpha) - l_n(\alpha) \right),
\]

\( p_n(\alpha) = 0 \) implies \( \alpha \) is relation. Note that \( p_n(\alpha) \) is a monic polynomial with integer coefficients. We will show that the polynomial \( p_n(\alpha) \) satisfies the conditions of the following theorem.

**Theorem 6.1.** There exists a sequence of polynomials \( p_n(\alpha) \) satisfying

- Each polynomials \( p_n(\alpha) \) is a monic polynomial with integer coefficients.
• All roots of $p_n(\alpha)$ are relation numbers.
• Let $\alpha_n$ be the maximal root of $p_n(\alpha)$. Then the sequence $\{\alpha_n\}$ is strictly increasing and converges to 4.

We have shown that the first two conditions. Thus, the only remaining part is to prove the last condition. First, we shall show that the sequence consisting of the maximal root of $p_n(\alpha)$ is increasing (Proposition 6.4). Then we prove that the sequence $\{\alpha_n\}$ converges to 4 (Theorem 6.1).

**Lemma 6.2.** For all positive integer $n$, $0 < (B_4A^{-1})^n \cdot 0 < \frac{1}{2}$.

**Proof.** From the Jordan decomposition, we obtain

$$B_4A^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} = SJS^{-1} \text{ where } J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = -A^{-1}, S = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$  

Thus, we have $\left( (B_4A^{-1})^n \right) = S J^n S^{-1} = (-1)^n \begin{bmatrix} 1 - 2n^n \alpha & n \\ -4n & 1 + 2n \end{bmatrix}$. Hence, $(B_4A^{-1})^n \cdot 0 = \frac{n}{2n+1} < \frac{1}{2}$.  

The key ingredient is the following lemma.

**Lemma 6.3.** The map $c_n : \mathbb{R} \to \mathbb{R} \cup \{\infty\} = S^1$ by $\alpha \mapsto (B_\alpha A^{-1})^n \cdot 0$ is a clockwise map.

**Proof.** Recall that $(B_\alpha A^{-1})^n \cdot 0 = \frac{u_n(\alpha)}{l_n(\alpha)}$. We will prove by the mathematical induction on $n$. For $n = 1$, just $c_1(\alpha) = \frac{u_1(\alpha)}{l_1(\alpha)} = \frac{1}{\alpha - 1}$. Thus, when $\alpha \neq 1$, $c_1(\alpha)$ is decreasing. At $\alpha = 1$, $-\frac{1}{c_1(\alpha)} = 1 - \alpha$ is strictly decreasing. Thus, $c_1(\alpha)$ is a clockwise map.

Now assume that $c_n(\alpha) = \frac{u_n(\alpha)}{l_n(\alpha)}$ is clockwise. Then the map $c_{n+1}$ can be expressed as

$$c_{n+1}(\alpha) = \frac{u_{n+1}(\alpha)}{l_{n+1}(\alpha)} = \frac{u_n(\alpha) - l_n(\alpha)}{\alpha + (1 - \alpha)l_n(\alpha)} = \frac{1}{\alpha + \frac{l_n(\alpha)}{u_n(\alpha)} - 1}.$$  

By Lemma 2.2 the map $\frac{u_n(\alpha)}{l_n(\alpha)} - 1$ is clockwise. By Lemma 2.3, the map $\frac{1}{\alpha + \frac{1}{u_n(\alpha)} - 1}$ is anticlockwise. Then from Lemma 2.4 the map $\frac{1}{\alpha + \frac{1}{u_n(\alpha)} - 1} + \alpha$ is again anticlockwise. Lastly, using Lemma 2.5 the map $\frac{1}{\alpha + \frac{1}{u_n(\alpha)} - 1}$ is clockwise, as desired.

**Proposition 6.4.** Let $\alpha_n$ be the maximal root of $p_n(\alpha)$. Then $\alpha_n < \alpha_{n+1}$.

**Proof.** We use the mathematical induction on $n$. Since $p_1(\alpha) = \alpha - 3$ and $p_2(\alpha) = \alpha^2 - 5\alpha + 5$, the maximal roots are 3 and $(5 + \sqrt{5})/2$ respectively. Since $3 < \frac{5 + \sqrt{5}}{2}$, our claim holds for $n = 1$. Let $\alpha_n$ be the maximal root of the polynomial $p_n(\alpha)$. We want to show that $\alpha_n < \alpha_{n+1}$. Since $\alpha_1 = 3$, we may assume $\alpha_0 > 3$. Moreover, recall that $\alpha_n < 4$ because $\alpha_n$ is a relation number. By definition, we have $(B_{\alpha_n}A^{-1})^n \cdot 0 = 1/2$. Then

$$(B_{\alpha_n}A^{-1})^{n+1} \cdot 0 = \begin{bmatrix} 1 & -1 \\ \alpha_n & 1 - \alpha_n \end{bmatrix} \cdot \frac{1}{2} = \frac{1}{\alpha_n - 2}.$$  

Since $3 < \alpha_n < 4$, we have $\frac{1}{2} < (B_{\alpha_n}A^{-1})^{n+1} \cdot 0 < 1$. From Lemma 6.2

$$0 < (B_4A^{-1})^{n+1} \cdot 0 < \frac{1}{2}.$$  

From the notation in Lemma 6.3

$$0 < c_{n+1}(4) < \frac{1}{2} < c_{n+1}(\alpha_n) < 1.$$
Now consider the image of a closed interval \([\alpha_n, 4]\) under the map \(c_{n+1}\). We claim that the image must contains \(\frac{1}{4}\). Since the interval \([\alpha_n, 4]\) is compact and connected, so is the image. Thus, candidates of the image are \(S_1 := \{r \in S^1 : c_{n+1}(4) \leq r \leq c_{n+1}(\alpha_n)\}\), \(S_2 := S^1 - S_1\) and whole of the circle \(S^1\).

To prove that the number \(\frac{1}{4}\) is contained in the image, it suffices to show that the image cannot be \(S_2\). By Lemma 6.5, \(c_{n+1}\) is a clockwise map. By the definition, we may assume \(c_{n+1}\) is strictly decreasing at \(\alpha_n\) because \(c_{n+1}(\alpha_n) \neq \infty\). Thus, \(c_{n+1}(\alpha)\) is decreasing on the interval \([\alpha_n, \alpha_n + \varepsilon_0]\) for some \(\varepsilon_0 > 0\). Since \(\alpha_n < 4\), we can choose \(\varepsilon_1\) such that \(\alpha_n + \varepsilon_1 < 4\). Moreover, we can choose \(\varepsilon_2 < c_{n+1}(\alpha) - c_{n+1}(4)\). Since \(c_{n+1}(\alpha)\) is continuous, there exists a \(\delta > 0\) such that

\[
|c_{n+1}(\alpha_n) - c_{n+1}(y)| < \varepsilon_2
\]

whenever \(|\alpha_n - y| < \delta\). Let \(\varepsilon\) be the minimum of \(\varepsilon_0, \varepsilon_1, \delta\). Then by construction, the image \(c_{n+1}([\alpha_n, \alpha_n + \varepsilon])\) must contain a point in \(S_1\). Contradiction. Thus, \(\frac{1}{4}\) is contained in the image of the interval \([\alpha_n, 4]\) under \(c_{n+1}(\alpha)\).

This implies there exists \(\alpha_+\) such that \(\alpha_n < \alpha_+ < 4\) and \(p_{n+1}(\alpha_+) = 0\). Since \(\alpha_{n+1}\) is the maximal root of \(p_{n+1}(\alpha)\), we have

\[
\alpha_n < \alpha_+ \leq \alpha_{n+1}.
\]

To prove that the sequence \(\{\alpha_n\}\) converges to 4, we use the notion of rotation numbers.

**Lemma 6.5.** For \(3 < \alpha < 4\), denote the set of \(\alpha\) satisfying that the set

\[
\{(B_{\alpha}A^{-1})^n \cdot 0 : n \in \mathbb{N}\}
\]

is dense in \(S^1\), by \(D\). Then the set \(D\) is dense in the open interval \((3, 4)\).

**Proof.** Recall that the matrix \(B_{\alpha}A^{-1} = \begin{bmatrix} 1 & -1 \\ \alpha & 1 - \alpha \end{bmatrix}\) is elliptic and the rotation number \(\omega(B_{\alpha}A^{-1})\) is given by \(\frac{1}{\pi} \cos^{-1} \frac{2 \alpha}{\pi} - \frac{2 \alpha}{\pi}\). Since the map \(\alpha \mapsto \frac{1}{\pi} \cos^{-1} \frac{2 \alpha}{\pi}\) is a homeomorphism, the set

\[
\left\{ \alpha \in (3, 4) : \frac{1}{\pi} \cos^{-1} \frac{2 \alpha}{\pi} \in \mathbb{R} - \mathbb{Q} \right\}
\]

is dense in \((3, 4)\). Since \(B_{\alpha}A^{-1}\) is a smooth map, by a classification of the circle action, we have the fact that \(\omega(B_{\alpha}A^{-1}) \in \mathbb{R} - \mathbb{Q}\) if and only if the set \(\{(B_{\alpha}A^{-1})^n \cdot 0 : n \in \mathbb{Z}\}\) is dense in \(S^1\). For the classification, see \([7]\). Moreover, when \(\{(B_{\alpha}A^{-1})^n \cdot 0 : n \in \mathbb{Z}\}\) is dense in \(S^1\), the set \(\{(B_{\alpha}A^{-1})^n \cdot 0 : n \in \mathbb{N}\}\) is also dense in \(S^1\). □
Proof of the Main Theorem 6.1. Choose any \( \varepsilon \) such that \( 0 < \varepsilon < 1 \). To complete proof, we have to prove that there exists \( N \in \mathbb{N} \) such that \( 4 - \varepsilon < \alpha_N \) since the sequence \( \{ \alpha_n \} \) is increasing. By Lemma 6.5 the set \( \{ (B_aA^{-1})^n \cdot 0 : n \in \mathbb{N} \} \) is dense in \( S^1 \) for some \( 4 - \varepsilon < a < 4 \). Now choose \( N \) such that
\[
\frac{1}{2} < (B_aA^{-1})^N \cdot 0 < 1.
\]
By Lemma 6.2, we obtain \( (B_aA^{-1})^N \cdot 0 < \frac{1}{2} \). Thus, the image of closed interval \([a, 4]\) under \( c_N \) must contains \( 1/2 \) since the map \( c_N \) is clockwise. Recall the proof of Lemma 6.3. Hence, this implies that for some \( a < a_0 < 4 \), \( (B_{a_0}A^{-1})^N \cdot 0 = 1/2 \). By construction, \( a_0 \) is one of solution of \( p_N(\alpha) \). Since
\[
4 - \varepsilon < a < a_0 \leq \alpha_N < 4,
\]
we complete the proof. \( \square \)

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