An invariant for pairs of almost commuting unbounded operators

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Abstract

For a wide class of pairs of unbounded selfadjoint operators \((A, B)\) with bounded commutator we construct a \(K\)-theoretical integer invariant \(\omega(A, B)\), which is continuous, is equal to zero for commuting operators and is equal to one for the pair \((x, -id/dx)\).

For a wide class of pairs of unbounded selfadjoint operators \((A, B)\) with bounded commutator and with operator \((1 + A^2 + B^2)^{-1}\) being compact we construct a \(K\)-theoretical integer invariant \(\omega(A, B)\), which is continuous, is equal to zero for commuting operators and is equal to one for the pair \((x, -id/dx)\). Such invariants in the case of bounded operators were constructed by T. Loring [1, 3] for matrices realizing ‘almost commutative’ torus and sphere.

Let \(A\) and \(B\) be two unbounded selfadjoint operators acting on a separable Hilbert space \(H\) with dense common domain \(D(A) \cap D(B)\). We suppose also that their compositions \(AB\) and \(BA\) have dense common domain and such that their commutator \(AB - BA\) is bounded on \(H\). Remark that as \(D(A) \cap D(B)\) is dense in \(H\), so the operators \(A \pm iB\) are densely defined and are formally adjoint to each other, hence these operators are closable. Denote by \(C\) the closure of the operator \(A + iB\), \(C = (A - iB)^*\). Put \(\Delta = 1 + (A + iB)(A - iB)^*\). \(\Delta\) is a selfadjoint invertible operator and we suppose that its inverse operator \(\Delta^{-1}\) is compact.

Remember that the group \(K^0_{\text{comp}}(\mathbb{R}^2) = K_0(C_0(\mathbb{R}^2))\) is generated by the Bott element which can be defined by a projection-valued function of the variable \(z = x + iy\) on \(\mathbb{C}\):

\[
P = P(x, y) = \begin{pmatrix}
1 & \frac{z}{1 + |z|^2} & \frac{z}{1 + |z|^2} \\
\frac{1}{1 + |z|^2} & 1 & \frac{1}{1 + |z|^2} \\
\frac{1}{1 + |z|^2} & \frac{1}{1 + |z|^2} & 1
\end{pmatrix};
\]

\(P^2 = P\). One can look at this function as at the matrix of an operator acting on the Hilbert space \(H \oplus H\) (made out of the operators of multiplication by \(x\) and by \(y\)). Change now the operators \(x\) and \(y\) in (1) by the operators \(A\) and \(B\):

\[
Q = Q(A, B) = \begin{pmatrix}
\Delta^{-1} & C\Delta^{-1} \\
\Delta^{-1}C^* & 1 - \Delta^{-1}
\end{pmatrix}.
\]
As $A$ and $B$ do not commute, there are different ways to substitute $A$ and $B$ instead of $x$ and $y$; we have fixed one of them. We will show that if $\|[A, B]\|$ is small enough then the operator $Q$ is close to a projection, namely for any $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\|[A, B]\| < \delta$ then $\|Q^2 - Q\| < \varepsilon$.

**Lemma 1** The operators $C(\lambda + C^*C)^{-1}$ and $(\lambda + CC^*)^{-1}C$ are bounded for any $\lambda > 0$ and $\|C(\lambda + C^*C)^{-1}\| \leq \frac{1}{\sqrt{\lambda}}$, $\|(\lambda + CC^*)^{-1}C\| \leq \frac{1}{\sqrt{\lambda}}$.

**Proof.** The statement follows from the estimates

$$\left\|C(\lambda + C^*C)^{-1/2}\right\|^2 = \left\|(\lambda + C^*C)^{-1/2}C^*C(\lambda + C^*C)^{-1/2}\right\| = \sup_{t \geq 0} \frac{t}{\lambda + t} = 1,$$

$$\left\|C(\lambda + C^*C)^{-1}\right\| \leq \left\|(\lambda + C^*C)^{-1/2}\right\| \left\|(\lambda + C^*C)^{-1/2}\right\| \leq \frac{1}{\sqrt{\lambda}}.$$

Taking $C^*$ instead of $C$ we obtain the estimate

$$\left\|C^*(\lambda + CC^*)^{-1/2}\right\| \leq \frac{1}{\sqrt{\lambda}} \quad (2)$$

and taking an adjoint in (2) we get

$$\left\|(\lambda + CC^*)^{-1/2}C\right\| \leq \frac{1}{\sqrt{\lambda}}. \quad \blacksquare$$

**Lemma 2** One has $C(1 + C^*C)^{-1} = (1 + CC^*)^{-1}C$.

**Proof.** As the operator $(1 + C^*C)^{-1}$ is compact, so the Hilbert space $H$ contains the dense linear subspace $\cup_{k=1}^{\infty} H_k$ where $H_k = \text{Span} \{\xi_1, \ldots, \xi_k\}$, $\xi_k$ being the (ordered) eigenvectors of $(1 + C^*C)^{-1}$.

Let $\{\lambda_k\}$ be the ordered (increasing) set of eigenvalues of the operator $C^*C$. Let $\xi \in H_k$ for some $k$ and suppose that $\lambda \in \mathbb{C}$, $\text{Re}\lambda > \lambda_k$. Then we have

$$C(\lambda + C^*C)^{-1}\xi = \lambda^{-1}C \left(1 + \frac{1}{\lambda} C^*C\right)^{-1}\xi$$

$$= \lambda^{-1}C \left(\xi - \frac{1}{\lambda} C^*C\xi + \frac{1}{\lambda^2}(C^*C)^2\xi - \ldots\right) \quad (3)$$

$$= \lambda^{-1} \left(\xi - \frac{1}{\lambda} CC^*C\xi + \frac{1}{\lambda^2}(CC^*)^2C\xi - \ldots\right).$$

By definition

$$D(C^*C) = \{\psi \in D(C) : C\psi \in D(C^*)\}. \quad (4)$$

As the operator $C$ is closed, so the operator $C^*C$ is selfadjoint and $\text{Ran} C^*C = H$ (see [3]), therefore the eigenvectors $\xi_k$ lie in the domain of the operator $C^*C$. Hence $\xi \in D(C^*C)$. But it follows from (4) that $\xi \in D(C)$, $C\xi \in D(C^*)$. So the vector $C\xi$ lies in the domain
of operators of the form \((CC^*)^n\). As the space \(H_k\) is finite-dimensional, so the operator \(C\) is bounded on \(H_k\), \(\|C\|_{H_k} < c_k\) for some \(c_k\). The norm of the \(n\)-th term in \((\mathbf{3})\) is not greater than \(c \|\xi\| \lambda_k^n/|\lambda|^n\) and the series \((\mathbf{3})\) is convergent when \(\Re \lambda > \lambda_k\). So we have

\[
C(\lambda + C^*)^{-1}\xi = \lambda^{-1} \left( 1 - \frac{1}{\lambda} CC^* + \frac{1}{\lambda^2}(CC^*)^2 - \ldots \right) C\xi
= (\lambda + CC^*)^{-1}C\xi.
\]

Consider two analytic operator-valued functions

\[
f_1(\lambda) = C(\lambda + C^*)^{-1}, \quad f_2(\lambda) = (\lambda + CC^*)^{-1}C, \quad \Re \lambda > 0.
\]

They are bounded and they coincide on \(H_k\) for \(\Re \lambda\) big enough, hence they coincide on \(H_k\) on the whole domain of the functions \(f_i(\lambda)\), i.e. for \(\Re \lambda > 0\). But as \(\cup_{k=1}^\infty H_k \subset H\) is dense, so \(f_1(\lambda) = f_2(\lambda)\) on \(H\).

\[
\text{Lemma 3} \quad \text{If} \quad \|C^*C - CC^*\| < \varepsilon \quad \text{then one has}
\]

\[
\frac{\varepsilon}{2(1 + \varepsilon)} \leq \frac{\varepsilon}{2(1 + \varepsilon)}
\]

\[
\text{Proof} \quad \text{of the first assertion follows from the estimate}
\]

\[
\|((1+C^*C)^{-1} - (1 + CC^*)^{-1}) C\| = \|((1+C^*C)^{-1} - (1 + C^*C + X)^{-1}) C\|
= \|((1+C^*C)^{-1} - (1 + (1+C^*C)^{-1}X)^{-1}(1+C^*C)^{-1}) C\|
= \|((1-(1+(1+C^*C)^{-1}X)^{-1}) (1+C^*C)^{-1} C\|
\leq \left( 1 - \frac{1}{1 - \varepsilon} \right) \|1 + C^*C\|^{-1} C\| \leq \frac{\varepsilon}{(1 - \varepsilon)}.
\]

where \(X = CC^* - C^*C\). The second assertion can be proved in the same way.

\[
\text{Lemma 4} \quad \text{Let} \quad E, F \quad \text{be selfadjoint positive operators such that the operator} \quad E-F \quad \text{is densely defined on} \quad H \quad \text{and} \quad \|E - F\| < \varepsilon. \quad \text{Put} \quad f(t) = \frac{t}{(1 + t)^2}. \quad \text{Then one has}
\]

\[
\|f(E) - f(F)\| < \frac{3\varepsilon - \varepsilon^2}{(1 - \varepsilon)^2}.
\]

\[
\text{Proof} \quad \text{Put} \quad E - F = X, \quad (1 + F)^{-1}X = Y, \quad (1 + Y)^{-1} - 1 = Z. \quad \text{One has} \quad \|Y\| < \varepsilon, \quad \|Z\| < \frac{\varepsilon}{1 - \varepsilon}. \quad \text{Then}
\]

\[
E(1+E)^{-2} - F(1+F)^{-2} = (F+X)(1+F+X)^{-2} - F(1+F)^{-2}
\]
\[
\begin{align*}
&= (F + X) \left( (1 + Y)^{-1}(1 + F)^{-1}(1 + Y)^{-1}(1 + F)^{-1} \right) - F(1 + F)^{-2} \\
&= (F + X) \left( (1 + Z)(1 + F)^{-1}(1 + Z)(1 + F)^{-1} \right) - F(1 + F)^{-2} \\
&= F \left( Z(1 + F)^{-1} + (1 + F)^{-1}Z + Z(1 + F)^{-1}Z(1 + F)^{-1} \right) \\
&\quad + X \left( (1 + F)^{-2} + Z(1 + F)^{-1} + (1 + F)^{-1}Z + Z(1 + F)^{-1}Z(1 + F)^{-1} \right) \\
&\leq 2 \frac{\varepsilon}{1 - \varepsilon} + \left( \frac{\varepsilon}{1 - \varepsilon} \right)^2 + \frac{\varepsilon}{(1 - \varepsilon)^2} = \frac{3\varepsilon - \varepsilon^2}{(1 - \varepsilon)^2}. \quad \blacksquare
\end{align*}
\]

Now we are ready to show that for small enough \( \varepsilon \) the operator \( Q \) is close to a projection.

**Theorem 5** If \( \|C^*C - CC^*\| < \varepsilon \) then \( \|Q^2 - Q\| < \frac{4\varepsilon - 2\varepsilon^2}{(1 - \varepsilon)^2} \).

**Proof.** Easy calculations (using lemma 2) show that \( Q^2 - Q = \\
\begin{pmatrix}
(1 + C^*C)^{-2}C^*C - (1 + CC^*)^{-2}CC^* & C((1 + CC^*)^{-1} - (1 + C^*C)^{-1})(1 + C^*C)^{-1} \\
(1 + C^*C)^{-1}((1 + CC^*)^{-1} - (1 + C^*C)^{-1})C^* & 0
\end{pmatrix}.
\]

But the norm of the off-diagonal elements is estimated by the lemma 3:
\[
\|C((1 + CC^*)^{-1} - (1 + C^*C)^{-1})(1 + C^*C)^{-1}\| \leq \frac{\varepsilon}{1 - \varepsilon}
\]
and the norm of the first element is estimated by the lemma 3:
\[
\|(1 + C^*C)^{-2}C^*C - (1 + CC^*)^{-2}CC^*\| \leq \frac{3\varepsilon - \varepsilon^2}{(1 - \varepsilon)^2}.
\]

From the inequality
\[
\left\| \begin{pmatrix} \alpha & \beta \\ \beta^* & 0 \end{pmatrix} \right\| \leq \|\alpha\| + \|\beta\|
\]
one finally has
\[
\|Q^2 - Q\| < \frac{3\varepsilon - \varepsilon^2}{(1 - \varepsilon)^2} + \frac{\varepsilon}{1 - \varepsilon} = \frac{4\varepsilon - 2\varepsilon^2}{(1 - \varepsilon)^2} = \varepsilon'. \quad \blacksquare
\]

So we see that the point \( 1/2 \) divides the spectrum of \( Q(A, B) \) into two subsets when \( \|[A, B]\| < 0.02 \).

Denote by \( H_N \subset H \) the subspace generated by the vectors \( e_{N+1}, \ldots \) of some basis \( \{e_i\} \) of \( H \), \( H = L_N \oplus H_N \). Our ‘almost projection’ \( Q \) acts on the Hilbert space \( H \oplus H \). Consider the decomposition of this space:
\[
H \oplus H = (L_N \oplus L_N) \oplus (H_N \oplus H_N).
\]
With respect to this decomposition the operator $Q$ has the form

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12}^* & q_{22} \end{pmatrix}.$$

It follows now from compactness of $\Delta^{-1}$ that for big enough $N$ $\|q_{12}\|$ is close to zero and $q_{22}$ is close to the operator

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ (5)

Therefore the operator $q_{11}$ should be close to a projection, $\|q_{12}^2 - q_{11}\| < \varepsilon''$ for some $\varepsilon''$ and the eigenvalues of $q_{11}$ also satisfy $|\lambda^2 - \lambda| < \varepsilon''$. When $\varepsilon'' < 1/4$ then all eigenvalues of the operator $q_{11}$ can be divided into two sets:

$$S_0 = \{ \lambda \in \text{Spec } q_{11} : \lambda < 1/2 \} \quad \text{and} \quad S_1 = \{ \lambda \in \text{Spec } q_{11} : \lambda > 1/2 \}.$$

**Definition 6** Let $M_N = \#(S_1)$ be the number of eigenvalues close to one of the operator $q_{11}$ when $\varepsilon', \varepsilon'' < 1/4$. Denote by $\omega(A, B)$ the number $M_N - N$.

**Proposition 7** The definition of the number $\omega(A, B)$ depends neither on the choice of a basis in $H$ nor on the number $N$.

**Proof.** Notice that when $\varepsilon', \varepsilon'' < 1/4$ then the spectrum of restriction of the operator $Q$ to the subspace $L_N \oplus L_N$ does not contain the point $1/2$. For big enough $N, N', N < N'$ the restriction of $Q$ to the space $(L_{N'} \oplus L_N) \oplus (L_{N'} \oplus L_N)$ is close to the projection of the form (5), so on $(L_{N'} \oplus L_N) \oplus (L_{N'} \oplus L_N)$ the number of eigenvalues less than $1/2$ is equal to the number of eigenvalues bigger than $1/2$, therefore $\omega(A, B)$ does not depend on $N$. If we take two bases $\{e_i\}$ and $\{e'_i\}$ then the space $L'_{N}$ generated by $e'_1, \ldots, e'_N$ can be approximated by the space $L_{N'}$ for big enough $N'$. ■

**Theorem 8** Let $A$ and $B$ commute. Then $\omega(A, B) = 0$.

**Proof.** Remember that by definition (see [3]) $A$ and $B$ commute when their spectral projections commute. It follows from the spectral theorem for two commuting operators and from the compactness of $\Delta^{-1}$ that it would be enough to consider the case of operators acting on $H = L^2(Z \times Z)$ by multiplication:

$$A(f)_{nm} = a_n f_{nm}, \quad B(f)_{nm} = b_m f_{nm},$$

where $f \in H$. Then the matrix of the projection $Q$ is $2 \times 2$-block-diagonal of the form:

$$Q = \text{diag}\{q_{nm}\} \quad \text{where} \quad q_{nm} = \begin{pmatrix} 1 & \frac{a_n + ib_m}{a_n^2 + b_m^2 + 1} \\ \frac{a_n - ib_m}{a_n^2 + b_m^2 + 1} & \frac{1}{a_n^2 + b_m^2 + 1} \end{pmatrix}.$$
But each 2×2-matrix \( q_{nm} \) is a projection unitarily equivalent to \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), hence the numbers of eigenvalues equal to zero and equal to one coincide on \( L_N \oplus L_N \) for every \( N \).

Show now that the topological invariant \( \omega(A, B) = 0 \) is continuous. Let \( \{ A_n \} \) be a sequence of closed symmetric operators such that for any \( \phi \)

We say that this sequence tends to the operator \( \omega \) if the sequences \( \{ a_n \} \) and \( \{ b_n \} \) tend to zero. By the Kato – Rellich theorem \([6]\) the operators \( A_n \) are selfadjoint on the domain of \( A \) when \( |a_n| < 1 \).

**Theorem 9** Let \( A_n \to A, B_m \to B \) and let the sequence of commutators \( [A_n, B_m] \) converge to \( [A, B] \). If \( \| [A, B] \| < 0.02 \) then \( \omega(A_n, B_m) = \omega(A, B) \) for big enough \( n \) and \( m \).

**Proof.** We start with proving continuity with respect to the first argument. The second one can be treated in the same way. Denote \( A_n - A = X_n \) and fix \( n \) so that \( a_n < \delta, b_n < \delta, [X_n, B] < \delta \). Then

\[
\| X_n \phi \| < \delta (\| A \phi \| + \| \phi \|).
\]

It follows from (6) that

\[
\| X_n (1 + A^2)^{-1/2} \phi \| < \delta \left( \| (1 + A^2)^{-1/2} \phi \| + \| (1 + A^2)^{-1/2} \phi \| \right) \leq 2\delta,
\]

hence \( \| X_n (1 + A^2)^{-1/2} \| < 2\delta \). As \( 1 + A^2 \leq 1 + (A + iB)(A - iB) + 0.02 = \Delta + 0.02 \), so \( \Delta^{-1} \leq \frac{1}{1 + 0.02} (1 + A^2)^{-1} < 2(1 + A^2)^{-1} \) and

\[
\| \Delta^{-1/2} X_n \|^2 = \| X_n \Delta^{-1} X_n \| \leq 2 \| X_n (1 + A^2)^{-1} X_n \| = 2 \| (1 + A^2)^{-1/2} X_n \|^2,
\]

therefore

\[
\| \Delta^{-1/2} X_n \| < 2\delta; \quad \| X_n \Delta^{-1/2} \| < 2\delta.
\]

Then

\[
\| \Delta^{-1/2} AX_n \Delta^{-1/2} \| < \delta, \quad \| \Delta^{-1/2} X_n A \Delta^{-1/2} \| < \delta, \quad \| \Delta^{-1/2} X_n^2 \Delta^{-1/2} \| < 4\delta^2.
\]

Estimate the matrix elements of the operator \( Q(A_n, B) - Q(A, B) \). For diagonal elements we have

\[
\| (1 + (A + X_n + iB)(A + X_n - iB))^{-1} - (1 + (A + iB)(A - iB))^{-1} \| \leq \| \Delta^{-1/2} \| \cdot \left\| (1 + \Delta^{-1/2} AX_n + X_n A + X^2 + i[X_n, B]) \Delta^{-1/2} \right\|^{-1} - 1 \| \cdot \| \Delta^{-1/2} \|
\]

\[
< \frac{1}{1 - (2\delta + 4\delta^2 + \delta)} - 1;\]
which implies also compactness of the operator \((1 + (A + X_n + iB)(A + X_n - iB))^{-1}\). A similar estimate holds for the off-diagonal elements:

\[
\left\| (A + X_n + iB)(1 + (A + X_n + iB)(A + X_n - iB))^{-1} - (A + iB)(1 + (A + iB)(A - iB))^{-1} \right\|
\]

\[
\leq \left\| X_n \Delta^{-1/2} \right\| \cdot \left\| (1 + \Delta^{-1/2}(AX_n + X_n A + X_n^2 + i[X_n, B])\Delta^{-1/2})^{-1} \right\| \cdot \left\| \Delta^{-1/2} \right\|
\]

\[
+ \left\| (A + iB)\Delta^{-1/2} \right\| \cdot \left\| (1 + \Delta^{-1/2}(AX_n + X_n A + X_n^2 + i[X_n, B])\Delta^{-1/2})^{-1} - 1 \right\| \cdot \left\| \Delta^{-1/2} \right\|
\]

\[
< 2\delta \left( \frac{1}{1 - (3\delta + 4\delta^2)} + \frac{1}{1 - (3\delta + 4\delta^2) - 1} \right).
\]

So the norms of the matrix elements of \(Q(A_n, B) - Q(A, B)\) tend to zero, hence the sequence of operators \(Q(A_n, B) - Q(A, B)\) tends to zero too.

Remark that our definition of convergence implies that for \(\alpha_0 \neq 0\) one has \(\alpha A \to \alpha_0 A\). Notice also that compactness of the operator \((1 + (A + iB)(A - iB))^{-1}\) implies compactness of the operators \((1 + (\lambda A + i\mu B)(\lambda A - i\mu B))^{-1}\) for all \(\lambda > 0, \mu > 0\). It makes the following definition correct.

**Definition 10** Let \(A, B\) be selfadjoint operators such that

i) \(AB\) and \(BA\) have dense common domain in \(H\) and \([A, B]\) is bounded on \(H\),

ii) \(A\) and \(B\) have dense common domain in \(H\) and \((1 + (A + iB)(A - iB))^{-1}\) is compact.

Then put \(\omega(A, B) = \omega(\lambda A, \mu B)\) for any \(\lambda > 0\) and \(\mu > 0\) such that \(\| [\lambda A, \mu B] \|\) is small enough.

Notice that instead of projection \([4]\) we could take any projection realizing some class of \(K_0^{\text{comp}}(\mathbb{R}^2)\). Then a pair of operators \((A, B)\) gives a map \(K_0^{\text{comp}}(\mathbb{R}) \to \mathbb{Z}\), hence pairs of operators represent \(K\)-homology classes of \(\mathbb{R}^2\).

As a corollary of the theorem \([2]\) we see that if \(T\) is a bounded selfadjoint operator on \(H\) commuting with \(B\), then \(\omega(A + T, B) = \omega(A, B)\) for any pair \((A, B)\). In particular, if \(A\) and \(B\) commute then \(\omega(A + T, B) = 0\). Remark that if the spectrum of \(A\) or \(B\) has a lacuna of big enough length then by method of \([3, 4]\) one can also obtain \(\omega(A, B) = 0\). It reflects the topological analogy: the reduced \(K\)-homology group of \(\mathbb{R}^2\) with a strip cutted out is trivial. Formulas

\[
h_1 = (1 + \Delta)^{-1}, \quad h_2 = A(1 + \Delta)^{-1}, \quad h_3 = B(1 + \Delta)^{-1}
\]

can be viewed as the formulas of the stereographic projection. They define a ‘noncommutative sphere’ given by three compact selfadjoint operators satisfying relations \(\|h_1^2 + h_2^2 + h_3^2 - h_4\| < \varepsilon\) for some \(\varepsilon > 0\) \([3]\).

**Theorem 11** Let \(A = x, B = -i\frac{d}{dx}\). Then \(\omega(A, B) = 1\).
Proof. Multiply the operators $A$ and $B$ by $\sqrt{\lambda}$ to make their commutator small enough. As $\|[A,B]\| = 2\lambda$, so we can take $\lambda$ so that $\frac{16\lambda - 32\lambda^2}{(1 - 4\lambda)^2} < 1/4$ (then $\|Q^2 - Q\| < 1/4$). It will be easier to make calculations in the basis of $H$ consisting of eigenvectors $\phi_n$ of the operator $C^*C = \lambda(x^2 - d^2/\lambda^2 - 1)$. Then one has

$$C\phi_n = \sqrt{n}\lambda\phi_{n-1}, \quad C^*\phi_n = \sqrt{(n+1)}\lambda\phi_{n+1}, \quad \Delta\phi_n = (2n\lambda + 1)\phi_n,$$

and the operator $Q$ can be written in the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{\lambda}{2\lambda+1} & \frac{\sqrt{\lambda}}{4\lambda+1} & 0 & 0 & 0 & \cdots \\
0 & \frac{\sqrt{\lambda}}{2\lambda+1} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{2\lambda}{4\lambda+1} & \frac{\sqrt{\lambda}}{4\lambda+1} & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{\sqrt{\lambda}}{4\lambda+1} & \frac{2\lambda}{4\lambda+1} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{\sqrt{\lambda}}{6\lambda+1} & \frac{4\lambda}{6\lambda+1} & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{4\lambda}{6\lambda+1} & \frac{\sqrt{\lambda}}{6\lambda+1} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

To calculate the number of eigenvalues close to one we notice first that the matrix $q_{11}$ (which is the restriction of $Q$ to the space $L_N \oplus L_N$) decomposes into direct sum of $N - 1$ two-dimensional blocks of the form

$$\begin{pmatrix}
\frac{2(n-1)\lambda}{2(n-1)\lambda+1} & \frac{\sqrt{n}\lambda}{2n\lambda+1} \\
\frac{\sqrt{n}\lambda}{2n\lambda+1} & \frac{2n\lambda}{2n\lambda+1}
\end{pmatrix}
$$

and of two one-dimensional blocks: 1 and $\frac{2N\lambda}{2N\lambda+1}$. When $N$ is big enough the last eigenvalue is close to one: $\lim_{N \to \infty} \frac{2N\lambda}{2N\lambda+1} = 1$. So it remains to show that out of the two eigenvectors of each matrix of the form (7) one is close to zero and another is close to one. Let $x^2 - px + q$ be the characteristic polynomial for the matrix (7). Then

$$1 - p = \frac{2\lambda}{(2n\lambda + 1)(2(n-1)\lambda + 1)} \leq \frac{2\lambda}{2\lambda + 1} < 2\lambda,$$

$$q = \frac{2\lambda}{(2n\lambda + 1)^2(2(n-1)\lambda + 1)} \leq \frac{2\lambda}{(2\lambda + 1)^2} < 2\lambda,$$

hence when $\lambda$ tends to zero the characteristic polynomial tends to $x^2 - x$ and that gives us necessary estimates for the eigenvalues, so every block of the form (7) gives equal numbers of eigenvalues in the sets $S_0$ and $S_1$. But the two scalar blocks give us $M_N = N + 1$.  

Remark that the construction of the $K$-theoretical invariant can be generalized to the case of pairs of operators on Hilbert $C^*$-modules over $C^*$-algebras. This generalization includes the case of families of pairs of operators $(A_x, B_x)$, $x \in X$ satisfying pointwise the conditions of definition [10]. We refer to [2] for basic facts about Hilbert $C^*$-modules. Let $H_A$ be a Hilbert $A$-module and let $(A, B)$ be a pair of unbounded selfadjoint operators on $H_A$ such that
i) compositions $AB$ and $BA$ have dense common domain in $H_A$ and the operator $AB - BA$ is bounded,

$ii)$ $A$ and $B$ have dense common domain in $H_A$ and the operator $(k + (A + iB)(A - iB))^{-1}$ is $A$-compact on $H_A$.

Then the invariant $\omega(A, B)$ can be defined as an element of $K_0(A)$ given by $\omega(A, B) = M_N - N$ where $M_N$ is the $K$-theory class of the spectral projection of $Q$ corresponding to the interval $(-\infty, 1/2)$.

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