Weyl-Conformally Invariant $p$-Brane Theories

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Abstract

We discuss in some detail the properties of a novel class of Weyl-conformally invariant $p$-brane theories which describe intrinsically light-like branes for any odd world-volume dimension and whose dynamics significantly differs from that of the ordinary (conformally non-invariant) Nambu-Goto $p$-branes. We present explicit solutions of the WILL-brane (Weyl-Invariant Light-Like brane) equations of motion in various gravitational backgrounds of physical relevance exhibiting the following new phenomena: (i) In spherically symmetric static backgrounds the WILL-brane automatically positions itself on (materializes) the event horizon of the corresponding black hole solutions, thus providing an explicit dynamical realization of the membrane paradigm in black hole physics; (ii) In product spaces (of interest in Kaluza-Klein context) the WILL-brane wraps non-trivially around the compact (internal) dimensions and moves as a whole with the speed of light in the non-compact (space-time) dimensions.

1 Introduction

Higher-dimensional extended objects ($p$-branes, $Dp$-branes) play an increasingly crucial role in modern non-perturbative string theory of fundamental interactions at ultra-high energies (for a background on string and brane theories, see refs.[1]). Their importance stems primarily from such basic properties as: providing explicit realization of non-perturbative string dualities, microscopic description of black-hole physics, gauge theory/gravity correspondence, large-radius compactifications of extra dimensions, cosmological brane-world scenarios in high-energy particle phenomenology, etc. .

In an independent development new classes of field theory models involving gravity, based on the idea of replacing the standard Riemannian integration measure (Riemannian volume-form) with an alternative non-Riemannian volume-form or, more generally, employing on equal footing both Riemannian and non-Riemannian volume-forms, have been proposed few years ago [2]. Since then, these new models called two-measure theories have been a subject of active research and applications [3] 1. Two-measure theories address various basic problems in cosmology and particle physics, and provide plausible solutions for a broad array of issues, such as: scale invariance and its dynamical breakdown; spontaneous generation of dimensionfull fundamental scales; the cosmological constant problem; the problem of fermionic families; applications to dark energy problem

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1For related ideas, see [4].
and modern cosmological brane-world scenarios. For a detailed discussion we refer to the series of papers [2, 3].

Subsequently, the idea of employing an alternative non-Riemannian integration measure was applied systematically to string, p-brane and Dp-brane models [5]. The main feature of these new classes of modified string/brane theories is the appearance of the pertinent string/brane tension as an additional dynamical degree of freedom beyond the usual string/brane physical degrees of freedom, instead of being introduced ad hoc as a dimensionfull scale. The dynamical string/brane tension acquires the physical meaning of a world-sheet electric field strength (in the string case) or world-volume \((p + 1)\)-form field strength (in the \(p\)-brane case) and obeys Maxwell (Yang-Mills) equations of motion or their higher-rank antisymmetric tensor gauge field analogues, respectively. As a result of the latter property the modified-measure string model with dynamical tension yields a simple classical mechanism of “color” charge confinement.

One of the drawbacks of modified-measure \(p\)-brane and \(Dp\)-brane models, similarly to the ordinary Nambu-Goto \(p\)-branes, is that Weyl-conformal invariance is lost beyond the simplest string case \((p = 1)\). On the other hand, it turns out that the form of the action of the modified-measure string model with dynamical tension suggests a natural way to construct explicitly a radically new class of Weyl-conformally invariant \(p\)-brane models for any \(p\) [6]. The most profound property of the latter models is that for any even \(p\) they describe the dynamics of inherently light-like \(p\)-branes which makes them significantly different both from the standard Nambu-Goto (or Dirac-Born-Infeld) branes as well as from their modified versions with dynamical string/brane tensions [5] mentioned above.

Before proceeding to the main exposition, which is the detailed discussion of the properties of the new Weyl-conformally invariant light-like branes, let us briefly recall the standard Polyakov-type formulation of the ordinary (bosonic) Nambu-Goto \(p\)-brane action:

\[
S = -\frac{T}{2} \int d^{p+1}\sigma \sqrt{-\gamma} \left[ \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \Lambda(p - 1) \right].
\]

(1)

Here \(\gamma_{ab}\) is the ordinary Riemannian metric on the \(p + 1\)-dimensional brane world-volume with \(\gamma \equiv \text{det} |\gamma_{ab}|\). The world-volume indices \(a, b = 0, 1, \ldots, p\); \(G_{\mu\nu}\) denotes the Riemannian metric in the embedding space-time with space-time indices \(\mu, \nu = 0, 1, \ldots, D - 1\). \(T\) is the given ad hoc brane tension; the constant \(\Lambda\) can be absorbed by rescaling \(T\) (see below Eq.(7)). The equations of motion w.r.t. \(\gamma^{ab}\) and \(X^\mu\) read:

\[
T_{ab} \equiv \left( \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu} + \frac{\Lambda}{2} (p - 1) = 0 ,
\]

(2)

\[
\partial_a \left( \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma_{\nu\lambda}^\mu = 0 \ ,
\]

(3)

where:

\[
\Gamma_{\nu\lambda}^\mu = \frac{1}{2} G^{\mu\kappa} \left( \partial\nu G_{\kappa\lambda} + \partial\lambda G_{\kappa\nu} - \partial\kappa G_{\nu\lambda} \right)
\]

(4)

is the Cristoffel connection for the external metric.

Eqs.(2) when \(p \neq 1\) imply:

\[
\Lambda \gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,
\]

(5)

which in turn allows to rewrite Eq.(2) as:

\[
T_{ab} \equiv \left( \partial_a X^\mu \partial_b X^\nu - \frac{1}{p + 1} \gamma_{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu} = 0 .
\]

(6)
Furthermore, using (5) the Polyakov-type brane action (1) becomes on-shell equivalent to the Nambu-Goto-type brane action:

\[ S = -T \Lambda^{-\frac{p-1}{2}} \int d^{p+1}\sigma \sqrt{-\det \left| \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \right|} . \]  

Let us note the following properties of standard Nambu-Goto \( p \)-branes manifesting their crucial differences w.r.t. the Weyl-conformally invariant branes discussed below. Eq.(5) tells us that: (i) the induced metric on the Nambu-Goto \( p \)-brane world-volume is non-singular; (ii) standard Nambu-Goto \( p \)-branes describe intrinsically massive modes.

2 String and Brane Models with a Modified World-Sheet/World-Volume Integration Measure

Here we briefly recall the construction of modified string and \((p-\text{and} Dp)\)-brane models with dynamical tension based on the use of alternative non-Riemannian world-sheet/world-volume volume form (integration measure density) [5].

The modified-measure bosonic string model is given by the following action:

\[ S = - \int d^2\sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \frac{\varepsilon^{ab}}{2\sqrt{-\gamma}} F_{ab}(A) \right] + \int d^2\sigma \sqrt{-\gamma} A_a J^a \]  

with the notations:

\[ \Phi(\varphi) \equiv \frac{1}{2} \varepsilon_{ij} \varepsilon^{ab} \partial_a \varphi^i \partial_b \varphi^j , \quad F_{ab}(A) = \partial_a A_b - \partial_b A_a , \]  

\( \gamma^{ab} \) denotes the intrinsic Riemannian world-sheet metric with \( \gamma = \det \| \gamma^{ab} \| \) and \( G_{\mu\nu}(X) \) is the Riemannian metric of the embedding space-time \((a, b = 0, 1; i, j = 1, 2; \mu, \nu = 0, 1, \ldots, D - 1)\). Below is the list of differences w.r.t. the standard Nambu-Goto string (in the Polyakov-like formulation):

- New non-Riemannian integration measure density \( \Phi(\varphi) \) built in terms of auxiliary world-sheet scalar fields \( \varphi^i \) \((i = 1, 2)\), independent of the world-sheet metric \( \gamma_{ab} \), instead of the standard Riemannian one \( \sqrt{-\gamma} \);
- Dynamical string tension \( T \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \) instead of \textit{ad hoc} dimensionfull constant;
- Auxiliary world-sheet gauge field \( A_a \) in a would-be “topological” term \( \int d^2\sigma \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \varepsilon^{ab} F_{ab}(A) \);
- Optional natural coupling of auxiliary \( A_a \) to external conserved world-sheet electric current \( J^a \) (see last term in (8) and Eq.(11) below).

The modified string model (8) is Weyl-conformally invariant similarly to the ordinary case. Here Weyl-conformal symmetry is given by Weyl rescaling of \( \gamma_{ab} \) supplemented with a special diffeomorphism in \( \varphi \)-target space:

\[ \gamma_{ab} \longrightarrow \gamma'_{ab} = \rho \gamma_{ab} , \quad \varphi^i \longrightarrow \varphi'^i = \varphi'^i(\varphi) \text{ with } \det \| \frac{\partial \varphi'^i}{\partial \varphi^j} \| = \rho . \]  

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The dynamical string tension appears as a canonically conjugated momentum w.r.t. $A_1$: $\pi_{A_1} \equiv \frac{\partial L}{\partial T}$, i.e., $T$ has the meaning of a \textit{world-sheet electric field strength}, and the equations of motion w.r.t. auxiliary gauge field $A_l$ is achieved as follows. Notice the following identity in 2 the modified string model with dynamical tension to non-Abelian world-sheet “color” charges. The In particular, for $J^a = 0$:

\begin{equation}
\varepsilon^{ab} \partial_b \left( \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \right) = 0 , \quad \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \equiv T = \text{const} ,
\end{equation}

one gets a \textit{spontaneously induced} constant string tension. Furthermore, when the modified string couples to point-like charges on the world-sheet (i.e., $J^0 \sqrt{-\gamma} = \sum_i e_i \delta(\sigma - \sigma_i)$ in (11)) one obtains classical charge confinement: $\sum_i e_i = 0$.

The above charge confinement mechanism has also been generalized in [5] to the case of coupling the modified string model with dynamical tension to non-Abelian world-sheet “color” charges. The latter is achieved as follows. Notice the following identity in 2D involving Abelian gauge field $A_a$:

\begin{equation}
\frac{\varepsilon^{ab}}{2\sqrt{-\gamma}} F_{ab}(A) = \sqrt{-\frac{1}{2} F_{ab}(A) F_{cd}(A) \gamma^{ac} \gamma^{bd}} .
\end{equation}

Then the extension of the action (8) to the non-Abelian case is straightforward:

\begin{equation}
S = - \int d^2x \Phi(\varphi) \left[ \frac{1}{2} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \sqrt{\frac{1}{2} \text{Tr}(F_{ab}(A) F_{cd}(A)) \gamma^{ac} \gamma^{bd}} \right] + \int d^2x \text{Tr} \left( A_a J^a \right)
\end{equation}

with $F_{ab}(A) = \partial_a A_b - \partial_b A_a + i [A_a, A_b]$, sharing the same principal property – dynamical generation of string tension as an additional degree of freedom.

Similar construction has also been proposed for higher-dimensional modified-measure $p$- and $Dp$-brane models whose brane tension appears as an additional dynamical degree of freedom. On the other hand, like the standard Nambu-Goto branes, they are Weyl-conformally \textit{non}-invariant and describe dynamics of \textit{massive} modes.

### 3 Weyl-Invariant Branes: Action and Equations of Motion

The identity (13) suggests how to construct \textbf{Weyl-invariant} $p$-brane models for any $p$. Namely, we consider the following novel class of $p$-brane actions:

\begin{equation}
S = - \int d^{p+1}\sigma \Phi(\varphi) \left[ \frac{1}{2} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \sqrt{F_{ab}(A) F_{cd}(A) \gamma^{ac} \gamma^{bd}} \right]
\end{equation}

\begin{equation}
\Phi(\varphi) \equiv \frac{1}{(p+1)!} \varepsilon_{i_1...i_{p+1}} \varepsilon^{a_1...a_{p+1}} \partial_{a_1} \varphi^{i_1} ... \partial_{a_{p+1}} \varphi^{i_{p+1}} ,
\end{equation}

where notations similar to those in (8) are used (here $a, b = 0, 1, \ldots, p; i, j = 1, \ldots, p + 1$).

The above action (15) is invariant under Weyl-conformal symmetry (the same as in the dynamical-tension string model (8)):

\begin{equation}
\gamma_{ab} \rightarrow \gamma'_{ab} = \rho \gamma_{ab} \quad , \quad \varphi^i \rightarrow \varphi'^i = \varphi^i(\varphi) \text{ with } \det \left| \frac{\partial \varphi'^i}{\partial \varphi^j} \right| = \rho .
\end{equation}

Let us note the following significant differences of (15) w.r.t. the standard Nambu-Goto $p$-branes (in the Polyakov-like formulation):
• New non-Riemannian integration measure density \( \Phi(\varphi) \) instead of \( \sqrt{-\gamma} \), and no “cosmological-constant” term \( (p-1)\sqrt{-\gamma} \);

• Variable brane tension \( \chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \) which is Weyl-conformal gauge dependent: \( \chi \to \rho^\frac{4}{p}(1-p)\chi \);

• Auxiliary world-volume gauge field \( A_a \) in a “square-root” Maxwell (Yang-Mills) term\(^2\); the latter is straightforwardly generalized to the non-Abelian case – \( \sqrt{-\text{Tr}(F_{ab}(A)F_{cd}(A))}\gamma^{ac}\gamma^{bd} \) similarly to (14);

• Natural optional couplings of the auxiliary gauge field \( A_a \) to external world-volume “color” charge currents \( j^a \);

• The action (15) is manifestly Weyl-conformal invariant for any \( p \); it describes intrinsically light-like \( p \)-branes for any even \( p \), as it will be shown below.

In what follows we shall frequently use the short-hand notations:

\[
(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad , \quad \sqrt{FF} \gamma \equiv \sqrt{F_{ab}F_{cd}\gamma^{ac}\gamma^{bd}} .
\]  

Employing (18) the equations of motion w.r.t. measure-building auxiliary scalars \( \varphi^i \) and w.r.t. \( \gamma^{ab} \) read, respectively:

\[
\frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) - \sqrt{FF} \gamma = M \left( = \text{const} \right) , \tag{19}
\]

\[
\frac{1}{2} (\partial_a X \partial_b X) + \frac{F_{ac}\gamma^{cd}F_{db}}{\sqrt{FF} \gamma} = 0 , \tag{20}
\]

Taking the trace in (20) implies \( M = 0 \) in Eq.(19).

Next we have the following equations of motion w.r.t. auxiliary gauge field \( A_a \) and w.r.t. \( X^\mu \), respectively:

\[
\partial_b \left( \frac{F_{cd}\gamma^{ac}\gamma^{bd}}{\sqrt{FF} \gamma} \Phi(\varphi) \right) = 0 , \tag{21}
\]

\[
\partial_a \left( \Phi(\varphi)\gamma^{ab} \partial_b X^\mu \right) + \Phi(\varphi)\gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0 , \tag{22}
\]

where \( \Gamma^\mu_{\nu\lambda} \) is the Cristoffel connection corresponding to the external space-time metric \( G_{\mu\nu} \) as in (4).

The \( A_a \)-equations of motion (21) can be solved in terms of \((p-2)\)-form gauge potentials \( A_{a_1...a_{p-2}} \) dual w.r.t. \( A_a \). The respective field-strengths are related as follows:

\[
F_{ab}(A) = -\frac{1}{\chi} \sqrt{-\gamma} \varepsilon_{abc_1...c_{p-1}} \gamma^{c_1d_1} ... \gamma^{c_{p-1}d_{p-1}} F_{d_1...d_{p-1}}(A) \gamma^{cd} (\partial_c X \partial_d X) , \tag{23}
\]

\[
\chi^2 = -\frac{2}{(p-1)^2} \gamma^{a_1b_1} ... \gamma^{a_{p-1}b_{p-1}} F_{a_1...a_{p-1}}(A) F_{b_1...b_{p-1}}(A) , \tag{24}
\]

where \( \chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \) is the variable brane tension, and:

\[
F_{a_1...a_{p-1}}(A) = (p-1)\partial_{[a_1}A_{a_2...a_{p-1}]} \tag{25}
\]

\(^2\)“Square-root” Maxwell (Yang-Mills) action in \( D = 4 \) was originally introduced in the first ref.[7] and later generalized to “square-root” actions of higher-rank antisymmetric tensor gauge fields in \( D \geq 4 \) in the second and third refs.[7].
is the \((p - 1)\)-form dual field-strength. All equations of motion can be equivalently derived from the following dual WILL-brane action:

\[
S_{\text{dual}}[X, \gamma, \Lambda] = -\frac{1}{2} \int d^{p+1} \sigma \chi(\gamma, \Lambda) \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu \nu}(X)
\]  
(26)

with \(\chi(\gamma, \Lambda)\) given in (24) above.

4 Intrinsically Light-Like Branes. WILL-Membrane

Let us consider the \(\gamma^{ab}\)-equations of motion (20). \(F_{ab}\) is an anti-symmetric \((p + 1) \times (p + 1)\) matrix, therefore, \(F_{ab}\) is not invertible in any odd \((p + 1)\) – it has at least one zero-eigenvalue vector \(V^a\) \((F_{ab} V^b = 0)\). Therefore, for any odd \((p + 1)\) the induced metric

\[
g_{ab} \equiv (\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu \nu}(X)
\]  
(27)

on the world-volume of the Weyl-invariant brane (15) is singular as opposed to the ordinary Nambu-Goto brane (where the induced metric is proportional to the intrinsic Riemannian world-volume metric, cf. Eq.(5)):

\[
(\partial_a X \partial_b X) V^b = 0 , \quad \text{i.e.} \quad (\partial_V X \partial_V X) = 0 , \quad (\partial_\perp X \partial_V X) = 0 ,
\]  
(28)

where \(\partial_V \equiv V^a \partial_a\) and \(\partial_\perp\) are derivates along the tangent vectors in the complement of the tangent vector field \(V^a\).

Thus, we arrive at the following important conclusion: every point on the world-surface of the Weyl-invariant p-brane (15) (for odd \((p + 1)\)) moves with the speed of light in a time-evolution along the zero-eigenvalue vector-field \(V^a\) of \(F_{ab}\). Therefore, we will name (15) (for odd \((p + 1)\)) by the acronym WILL-brane (Weyl-Invariant Light-Like-brane) model.

Henceforth we will explicitly consider the special case \(p = 2\) of (15), i.e., the Weyl-invariant light-like membrane model. The associated WILL-membrane dual action (particular case of (26) for \(p = 2\)) reads:

\[
S_{\text{dual}} = -\frac{1}{2} \int d^3 \sigma \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} (\partial_a X \partial_b X) \quad , \quad \chi(\gamma, u) \equiv \sqrt{-2\gamma^{cd}\partial_c u \partial_d u},
\]  
(29)

where \(u\) is the dual “gauge” potential w.r.t. \(A_a\):

\[
F_{ab}(A) = -\frac{1}{2\chi(\gamma, u)} \gamma^{cd} \partial_d u \gamma^{ef} (\partial_e X \partial_f X). \quad (30)
\]

\(S_{\text{dual}}\) is manifestly Weyl-invariant (under \(\gamma_{ab} \to \rho \gamma_{ab}\)).

The equations of motion w.r.t. \(\gamma^{ab}, u\) (or \(A_a\)), and \(X^\mu\) read accordingly:

\[
(\partial_a X \partial_b X) + \frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) \left( \frac{\partial_a u \partial_b u}{\gamma^{ef}} - \gamma_{ab} \right) = 0 ,
\]  
(31)

\[
\partial_a \left( \sqrt{-\gamma} \gamma^{ab} \partial_b u \chi(\gamma, u) \right) = 0 ,
\]  
(32)

\[
\partial_a \left( \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma_{\nu \lambda} = 0 .
\]  
(33)
The last factor in brackets on the l.h.s. of Eq. (31) is a projector implying that the induced metric 
\( g_{ab} \equiv (\partial_a X \partial_b X) \) has zero-mode eigenvector \( V^a = \gamma^{ab} \partial_b u \).

The invariance under world-volume reparametrizations allows to introduce the following standard (synchronous) gauge-fixing conditions:

\[
\gamma^{0i} = 0 \quad (i = 1, 2) \quad , \quad \gamma^{00} = -1 .
\]

Using (34) we can easily find solutions of Eq. (32) for the dual “gauge potential” \( u \) in spite of its high non-linearity by taking the following ansatz:

\[
u(\tau, \sigma^1, \sigma^2) = T_0 \sqrt{2 \tau} ,
\]

Here \( T_0 \) is an arbitrary integration constant with the dimension of membrane tension. In particular:

\[
\chi \equiv \sqrt{-2 \gamma^{ab} \partial_a u \partial_b u} = T_0
\]

The ansatz (35) means that we take \( \tau \equiv \sigma^0 \) to be evolution parameter along the zero-eigenvalue vector-field of the induced metric on the brane (\( V^a = \gamma^{ab} \partial_b u = \text{const} \ (1, 0, 0) \)).

The ansatz for \( u \) (35) together with the gauge choice for \( \gamma_{ab} \) (34) brings the equations of motion w.r.t. \( \gamma_{ab}, u \) (or \( A_a \) and \( X^\mu \)) in the following form (recall (\( \partial_a X \partial_b X \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu \nu} \)):

\[
(\partial_0 X \partial_0 X) = 0 \quad , \quad (\partial_0 X \partial_i X) = 0 ,
\]

\[
(\partial_i X \partial_j X) - \frac{1}{2} \gamma_{ij}^{kl} (\partial_k X \partial_l X) = 0 ,
\]

(notice that Eqs. (38) look exactly like the classical (Virasoro) constraints for an Euclidean string theory with world-sheet parameters (\( \sigma^1, \sigma^2 \));

\[
\partial_0 \left( \sqrt{\gamma^{(2)}} \gamma^{kl} (\partial_k X \partial_l X) \right) = 0 , \quad \text{(39)}
\]

where \( \gamma^{(2)} = \text{det} |\gamma_{ij}| \) (the above equation is the only remnant from the \( A_a \)-equations of motion (21));

\[
\Box^{(3)} X^\mu + \left( -\partial_0 X^\mu \partial_0 X^\lambda + \gamma^{kl} \partial_k X^\nu \partial_l X^\lambda \right) \Gamma_{\nu \lambda}^\mu = 0 , \quad \text{(40)}
\]

where:

\[
\Box^{(3)} \equiv - \frac{1}{\sqrt{\gamma^{(2)}}} \partial_0 \left( \sqrt{\gamma^{(2)}} \partial_0 \right) + \frac{1}{\sqrt{\gamma^{(2)}}} \partial_i \left( \sqrt{\gamma^{(2)}} \gamma^{ij} \partial_j \right) . \quad \text{(41)}
\]

We can also extend the WILL-brane model (15) via a coupling to external space-time electromagnetic field \( A_\mu \). The natural Weyl-conformal invariant candidate action reads (for \( p = 2 \)):

\[
S = - \int d^3 \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu \nu} - \sqrt{F_{ab} F_{cd} \gamma^{ac} \gamma^{bd}} \right] - q \int d^3 \sigma \varepsilon^{abc} A_\mu \partial_a X^\mu F_{bc} . \quad \text{(42)}
\]

The last Chern-Simmons-like term is a special case of a class of Chern-Simmons-like couplings of extended objects to external electromagnetic fields proposed in ref.[8].

Instead of the action (42) we can use its dual one (similar to the simpler case Eq. (15) versus Eq. (29)):

\[
S_{\text{WILL-brane}}^{\text{dual}} = - \frac{1}{2} \int d^3 \sigma \chi(\gamma, u, A) \sqrt{-\gamma} \gamma^{ab} (\partial_a X \partial_b X) , \quad \text{(43)}
\]
where the variable brane tension \( \chi \equiv \frac{\Phi(x)}{\sqrt{\gamma}} \) is given by:

\[
\chi(\gamma, u, A) \equiv \sqrt{-2\gamma^{cd} (\partial_c u - q\mathcal{A}_c) (\partial_d u - q\mathcal{A}_d)} , \quad \mathcal{A}_a \equiv A_a \partial_a X^\mu .
\] (44)

Here \( u \) is the dual “gauge” potential w.r.t. \( A_a \) and the corresponding field-strength and dual field-strength are related as (cf. Eq.(30)):

\[
F_{ab}(A) = -\frac{1}{2\chi(\gamma, u, A)} \sqrt{-\gamma^{abc}\gamma^{cd}} (\partial_d u - q\mathcal{A}_d) \gamma^{ef}(\partial_e X \partial_f X) .
\] (45)

The corresponding equations of motion w.r.t. \( \gamma^a, u \) (or \( A_a \)), and \( X^\mu \) read accordingly:

\[
(\partial_a X \partial_b X) + \frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) \left( \frac{(\partial_a u - q\mathcal{A}_a)(\partial_b u - q\mathcal{A}_b)}{\gamma^{ef}(\partial_e u - q\mathcal{A}_e)(\partial_f u - q\mathcal{A}_f)} - \gamma^{ab} \right) = 0 ;
\] (46)

\[
\partial_a \left( \frac{\sqrt{-\gamma^{ab}(\partial_b u - q\mathcal{A}_b)}}{\chi(\gamma, u, A)} \gamma^{cd}(\partial_c X \partial_d X) \right) = 0 ;
\] (47)

\[
\partial_a \left( \chi(\gamma, u, A) \sqrt{-\gamma^{ab} \partial_b X^\mu} \right) + \chi(\gamma, u, A) \sqrt{-\gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda}} - q\varepsilon^{abc} F_{bc} \partial_a X^\nu \left( \partial_{\lambda} \mathcal{A}_a - \partial_{\nu} \mathcal{A}_a \right) \mathcal{G}^a_{\lambda} = 0 .
\] (48)

5 **\( \text{WILL} \)-Membrane Solutions in Various Gravitational Backgrounds**

5.1 **\( \text{WILL} \)-Membrane in a PP-Wave Background**

As a first non-trivial example let us consider \( \text{WILL} \)-membrane dynamics in an external background generalizing the plane-polarized gravitational wave (pp-wave):

\[
(ds)^2 = -dx^+ dx^- - F(x^+, x^I)(dx^+)^2 + h_{IJ}(x^K)dx^I dx^J ,
\] (49)

(for the ordinary pp-wave \( h_{IJ}(x^K) = \delta_{IJ} \)), and let us employ in (37)–(41) the following natural ansatz for \( X^\mu \) (here \( \sigma^0 \equiv \tau, I = 1, \ldots, D - 2 \)):

\[
X^- = \tau , \quad X^+ = X^+(\tau, \sigma^1, \sigma^2) , \quad X^I = X^I(\sigma^1, \sigma^2) .
\] (50)

The non-zero affine connection symbols for the generalized pp-wave metric (49) are: \( \Gamma^-_+ = \partial_+ F, \quad \Gamma^-_I = \partial_I F, \quad \Gamma^I_+ = \frac{1}{2} h^{IJ} \partial_J F \), and \( \Gamma^I_J \) – the ordinary Christoffel symbols for the metric \( h_{IJ} \) in the transverse dimensions.

It is straightforward to show that the solution does not depend on the form of the pp-wave front \( F(x^+ , x^I) \) and reads:

\[
X^+ = X^+_0 = \text{const} \quad , \quad \gamma_{ij} \text{ are } \tau - \text{independent} ;
\] (51)

\[
(\partial_i X^J \partial_j X^J - \frac{1}{2} \gamma_{ij} \gamma^{kl} \partial_k X^I \partial_l X^J) h_{IJ} = 0
\] (52)

\[
\frac{1}{\sqrt{\gamma(2)}}\partial_i \left( \sqrt{\gamma(2)} \gamma_{ij} \partial_j X^I \right) + \gamma^{kl} \partial_k X^J \partial_l X^K \Gamma^I_{JK} = 0
\] (53)

The latter two equations for the transverse brane coordinates describe a string moving in the \( (D - 2) \)-dimensional Euclidean-signature transverse space.
5.2 WILL-Membrane in a Product-Space Background

Here we consider WILL-membrane moving in a general product-space $D = (d + 2)$-dimensional gravitational background $\mathcal{M}^d \times \Sigma^2$ with coordinates $(x^\mu, y^m)$ ($\mu = 0, 1, \ldots, d - 1$, $m = 1, 2$) and Riemannian metric $(ds)^2 = f(y)g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n$. The metric on $\mathcal{M}^d$ is of Lorentzian signature and $\Sigma^2$ will be taken as a sphere for simplicity.

We assume that the WILL-brane wraps around the “internal” space $\Sigma^2$ and use the following ansatz (recall $\tau \equiv \sigma^0$):

$$X^\mu = X^\mu(\tau), \quad Y^m = \sigma^m, \quad \gamma_{mn} = a(\tau) g_{mn}(\sigma^1, \sigma^2) \quad (54)$$

Then the equations of motion and constraints (37)–(41) reduce to:

$$\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu}(X) = 0, \quad \frac{1}{a(\tau)} \partial_\tau \left( a(\tau) \partial_\tau X^\mu \right) + \partial_\tau X^\nu \partial_\tau X^\lambda \Gamma^\mu_{\nu\lambda} = 0 \quad (55)$$

where $a(\tau)$ is the conformal factor of the space-like part of the internal membrane metric (last Eq.(54)). Eqs.(55) are of the same form as the equations of motion for a massless point-particle with a world-line “einbein” $e = a^{-1}$ moving in $\mathcal{M}^d$. In other words, the simple solution above describes a membrane living in the extra “internal” dimensions $\Sigma^2$ and moving as a whole with the speed of light in “ordinary” space-time $\mathcal{M}^d$.

Let us particularly emphasize the fact that, although the WILL-brane is wrapping the extra (compact) dimensions in a topologically non-trivial way (cf. second Eq.(54)), its modes remain massless from the projected $d$-dimensional space-time point of view. This is a new phenomenon from the point of view of Kaluza-Klein theories: here we have particles (membrane modes), which acquire non-zero quantum numbers due to non-trivial winding, while at the same time these particles (modes) remain massless. In contrast, one should recall that in ordinary Kaluza-Klein theory (for a review, see [9]), non-trivial dependence on the extra dimensions is possible for point particles or even standard strings and branes only at a very high energy cost (either by momentum modes or winding modes), which implies a very high mass from the projected $d$-dimensional space-time point of view.

5.3 WILL-Membrane in Spherically-Symmetric Backgrounds

Let us consider general $SO(3)$-symmetric background in $D = 4$ embedding space-time:

$$(ds)^2 = -A(z, t)(dt)^2 + B(z, t)(dz)^2 + C(z, t) \left( (d\theta)^2 + \sin^2 \theta (d\phi)^2 \right). \quad (56)$$

The usual ansatz:

$$X^0 \equiv t = \tau, \quad X^1 \equiv z = z(\tau, \sigma^1, \sigma^2), \quad X^2 \equiv \theta = \sigma^1, \quad X^3 \equiv \phi = \sigma^2 \quad (57)$$

yields:

(i) equations for $z(\tau, \sigma^1, \sigma^2)$:

$$\frac{\partial z}{\partial \tau} = \pm \sqrt{\frac{A}{B}}, \quad \frac{\partial z}{\partial \sigma^i} = 0 \quad (58)$$
(ii) a restriction on the background itself (comes from the gauge-fixed equations of motion for the dual gauge potential \( u (39) \)):

\[
\frac{dC}{d\tau} \equiv \left( \frac{\partial C}{\partial t} \pm \sqrt{\frac{A}{B}} \frac{\partial C}{\partial z} \right) \bigg|_{t=\tau, z=z(\tau)} = 0 ;
\]

(iii) an equation for the conformal factor \( a(\tau) \) of the internal membrane metric:

\[
\partial_{\tau} a + \left( \frac{\sqrt{AB} \pm \partial_z A}{\sqrt{AB}} \right) a(\tau) - \frac{\partial C}{A} \bigg|_{t=\tau, z=z(\tau)} = 0 .
\]

Eq.(59) tells that the (squared) sphere radius \( R^2 \equiv C(z, t) \) must remain constant along the WILL-brane trajectory.

In particular, let us take static spherically-symmetric gravitational background in \( D = 4 \):

\[
(ds)^2 = -A(r)(dt)^2 + B(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2] .
\]

Specifically we have:

\[
A(r) = B^{-1}(r) = 1 - \frac{2GM}{r}
\]

for Schwarzschild black hole,

\[
A(r) = B^{-1}(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}
\]

for Reissner-Nordström black hole,

\[
A(r) = B^{-1}(r) = 1 - kr^2
\]

for (anti-) de Sitter space, etc.

In the case of (61) Eqs.(58)–(59) reduce to:

\[
\frac{\partial r}{\partial \tau} = \pm A(r) , \quad \frac{\partial r}{\partial \sigma^i} = 0 , \quad \frac{\partial r}{\partial \tau} = 0
\]

yielding:

\[
r = r_0 \equiv \text{const} , \quad \text{where} \quad A(r_0) = 0 .
\]

Further, Eq.(60) implies for the intrinsic WILL-membrane metric:

\[
\|\gamma_{ij}\| = c_0 e^{\mp \tau/r_0} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\sigma^1) \end{pmatrix},
\]

where \( c_0 \) is an arbitrary integration constant.

From (66) we conclude that the WILL-membrane with spherical topology (and with exponentially blowing-up/deflating radius w.r.t. internal metric, see Eq.(67)) automatically “sits” on (materializes) the event horizon of the pertinent black hole in \( D = 4 \) embedding space-time. This conforms with the well-known general property of closed light-like hypersurfaces in \( D = 4 \) (i.e., their section with the hyper-plane \( t=\text{const} \) being a compact 2-dimensional manifold) which automatically serve as horizons [10]. On the other hand, let us stress that our WILL-membrane model (29) provides an explicit dynamical realization of event horizons.
6 Coupled Einstein-Maxwell-WILL-Membrane System: WILL-Membrane as a Source for Gravity and Electromagnetism

We can extend the results from the previous section to the case of the self-consistent Einstein-Maxwell-WILL-membrane system, i.e., we will consider the WILL-membrane as a dynamical material and electrically charged source for gravity and electromagnetism. The relevant action reads:

\[
S = \int d^4x \sqrt{-G} \left[ \frac{R(G)}{16\pi G_N} - \frac{1}{4} F_{\mu\nu}(A) F_{\kappa\lambda}(A) G^{\mu\kappa} G^{\nu\lambda} \right] + S_{\text{WILL-brane}},
\]

where \( F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the space-time electromagnetic field-strength, and \( S_{\text{WILL-brane}} \) indicates the Willis-membrane action coupled to the space-time gauge field \( A_\mu \) – either (42) or its dual (43).

The equations of motion for the WILL-membrane subsystem are of the same form as Eqs.(46)–(48). The Einstein-Maxwell equations of motion read:

\[
R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi G N \left( T^{(EM)}_{\mu\nu} + T^{(brane)}_{\mu\nu} \right),
\]

where:

\[
T^{(EM)}_{\mu\nu} \equiv F_{\mu\kappa} F_{\nu\lambda} G^{\kappa\lambda} - G_{\mu\rho} \frac{1}{4} F_{\rho\kappa} F_{\sigma\lambda} G^{\rho\sigma} G^{\kappa\lambda},
\]

\[
T^{(brane)}_{\mu\nu} \equiv -G_{\mu\kappa} G_{\nu\lambda} \int d^3\sigma \frac{\delta(4)}{\sqrt{-G}} \chi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\kappa \partial_b X^\lambda,
\]

\[
j^\mu \equiv q \int d^3\sigma \delta(4) \left( x - X(\sigma) \right) \varepsilon^{abc} F_{bc} \partial_a X^\mu.
\]

We find the following self-consistent spherically symmetric stationary solution for the coupled Einstein-Maxwell-WILL-membrane system (68). For the Einstein subsystem we have a solution:

\[
(ds)^2 = -A(r)(dt)^2 + A^{-1}(r) (dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta) (d\phi)^2],
\]

consisting of two different black holes with a common event horizon:

- Schwarzschild black hole inside the horizon:
  \[
  A(r) \equiv A_-(r) = 1 - \frac{2GM_1}{r}, \quad \text{for} \quad r < r_0 \equiv r_{\text{horizon}} = 2GM_1.
  \]

- Reissner-Norström black hole outside the horizon:
  \[
  A(r) \equiv A_+(r) = 1 - \frac{2GM_2 + GQ^2}{r^2}, \quad \text{for} \quad r > r_0 \equiv r_{\text{horizon}},
  \]

where \( Q^2 = \frac{8\pi q^2 r^4_{\text{horizon}}}{r^2} \equiv 128\pi q^2 G^4 M_1^4 \).

For the Maxwell subsystem we have \( A_1 = \ldots = A_{D-1} = 0 \) everywhere and:

- Coulomb field outside horizon:
  \[
  A_0 = \frac{\sqrt{2} q r^2_{\text{horizon}}}{r}, \quad \text{for} \quad r \geq r_0 \equiv r_{\text{horizon}}.
  \]
No electric field inside horizon:

\[ A_0 = \sqrt{2} q r_{\text{horizon}} = \text{const}, \quad \text{for} \quad r \leq r_0 = r_{\text{horizon}}. \quad (78) \]

Using the same (synchronous) gauge choice (34) and ansatz for the dual “gauge potential” (35), as well as taking into account (77)–(78), the WILL-membrane equations of motion (46)–(48) acquire the form (recall \( (\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu \nu} ) \):

\[ (\partial_0 X \partial_0 X) = 0, \quad (\partial_0 X \partial_i X) = 0, \quad (79) \]

\[ (\partial_i X \partial_j X) - \frac{1}{2} \gamma_{ij} \gamma^{kl} \left( \partial_k X \partial_l X \right) = 0, \quad (80) \]

(these constraints are the same as in the absence of coupling to space-time gauge field (37)–(38));

\[ \partial_0 \left( \sqrt{\gamma^{(2)}} \gamma^{kl} \left( \partial_k X \partial_l X \right) \right) = 0, \quad (81) \]

(once again the same equation as in the absence of coupling to space-time gauge field (39));

\[ \Box^{(3)} X^\mu + \left( -\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl} \partial_k X^\nu \partial_l X^\lambda \right) \Gamma^\mu_{\nu \lambda} - q \frac{\gamma^{kl} \left( \partial_k X \partial_l X \right)}{\sqrt{2} \chi} \partial_0 X^\nu \left( \partial_\lambda A_\nu - \partial_\nu A_\lambda \right) G^{\lambda \mu} = 0. \quad (82) \]

Here \( \chi \equiv T_0 - \sqrt{2} q A_0 \) with \( A_0 \) as in Eqs.(77),(78) is the variable brane tension coming from Eqs.(35),(44); \( X^0 \equiv t, X^1 \equiv r, X^2 \equiv \theta, X^3 \equiv \phi \); and:

\[ \Box^{(3)} \equiv -\frac{1}{\chi \sqrt{\gamma^{(2)}}} \partial_0 \left( \chi \sqrt{\gamma^{(2)}} \partial_0 \right) + \frac{1}{\chi \sqrt{\gamma^{(2)}}} \partial_i \left( \chi \sqrt{\gamma^{(2)}} \gamma^{ij} \partial_j \right). \quad (83) \]

A self-consistent solution to Eqs.(79)–(82) reads:

\[ X^0 \equiv t = \tau, \quad \theta = \sigma^1, \quad \phi = \sigma^2, \quad (84) \]

\[ r(\tau, \sigma^1, \sigma^2) = r_{\text{horizon}} = \text{const}, \quad A_\pm (r_{\text{horizon}}) = 0, \quad (85) \]

i.e., the WILL-membrane automatically positions itself on the common event horizon of the pertinent black holes. Furthermore, inserting (84)–(85) in the expression (72) for the WILL-membrane energy-momentum tensor \( T_{\mu \nu}^{(\text{brane})} \), the Einstein equations (69) entail the following important matching conditions for the space-time metric components along the WILL-membrane surface:

\[ \left. \frac{\partial A_+}{\partial r} \right|_{r = r_{\text{horizon}}} - \left. \frac{\partial A_-}{\partial r} \right|_{r = r_{\text{horizon}}} = -16\pi G \chi. \quad (86) \]

Condition (86) in turn yields relations between the parameters of the black holes and the WILL-membrane (\( q \) being its surface charge density):

\[ M_2 = M_1 + 32\pi q^2 G^3 M_1^3 \quad (87) \]

and for the brane tension \( \chi \):

\[ \chi \equiv T_0 - 2q^2 r_{\text{horizon}} = q^2 G M_1, \quad \text{i.e.} \quad T_0 = 5q^2 G M_1 \quad (88) \]

The matching condition (86) corresponds to the so called statically soldering conditions in the theory of light-like thin shell dynamics in general relativity [11]. Unlike the latter model, where the membranes are introduced \textit{ad hoc}, the present WILL-brane models provide a systematic dynamical description of light-like branes (as sources for both gravity and electromagnetism) from first principles starting with concise Weyl-conformally invariant actions (42), (68).
7 Conclusions and Outlook

In the present work we have discussed a novel class of Weyl-invariant $p$-brane theories whose dynamics significantly differs from ordinary Nambu-Goto $p$-brane dynamics. The principal features of our construction are as follows:

- Employing alternative non-Riemannian integration measure (volume-form) (16) on the $p$-brane world-volume independent of the intrinsic Riemannian metric.
- Acceptable dynamics in the novel class of brane models (Eqs.(15),(42)) naturally requires the introduction of additional world-volume gauge fields.
- By employing square-root Yang-Mills actions for the pertinent world-volume gauge fields one achieves manifest Weyl-conformal symmetry in the new class of $p$-brane theories for any $p$.
- The brane tension is not a constant dimensionful scale given ad hoc, but rather it appears as a composite world-volume scalar field (Eqs.(24),(29),(44)) transforming non-trivially under Weyl-conformal transformations.
- The novel class of Weyl-invariant $p$-brane theories describes intrinsically light-like $p$-branes for any even $p$ (WILL-branes).
- When put in a gravitational black hole background, the WILL-membrane ($p = 2$) automatically sits on (“materializes”) the event horizon.
- When moving in background product-spaces (“Kaluza-Klein” context) the WILL-membrane describes massless modes, even though the membrane is wrapping the extra dimensions and therefore acquiring non-trivial Kaluza-Klein charges.
- The coupled Einstein-Maxwell-WILL-membrane system (68) possesses self-consistent solution where the WILL-membrane serves as a material and electrically charged source for gravity and electromagnetism, and it automatically “sits” on (materializes) the common event horizon for a Schwarzschild (in the interior) and Reissner-Nordström (in the exterior) black holes. Thus our model (68) provides an explicit dynamical realization of the so called “membrane paradigm” in the physics of black holes [12].
- The WILL-branes could be good representations for the string-like objects introduced by ’t Hooft in ref.[13] to describe gravitational interactions associated with black hole formation and evaporation, since as shown above the WILL-branes locate themselves automatically in the horizons and, therefore, they could represent degrees of freedom associated particularly with horizons.

The novel class of Weyl-conformal invariant $p$-branes discussed above suggests various physically interesting directions for further study such as: quantization (Weyl-conformal anomaly and critical dimensions); supersymmetric generalization; possible relevance for the open string dynamics (similar to the role played by Dirichlet- $(Dp)$-branes); WILL-brane dynamics in more complicated gravitational black hole backgrounds (e.g., Kerr-Newman).

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