Nashian game theory is incompatible with quantum contextuality

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Abstract

In this work, we design a novel game-theoretical framework capable of capturing the defining aspects of quantum theory. We introduce an original model and an algorithmic procedure that enables to express measurement scenarios encountered in quantum mechanics as multiplayer games and to translate physical notions of causality, correlation, and contextuality to particular aspects of game theory. Furthermore, inspired by the established correspondence, we investigate the causal consistency of games in extensive form with imperfect information from the quantum perspective and we conclude that counterfactual dependencies should be distinguished from causation and correlation as a separate phenomenon of its own. Most significantly, we deduce that Nashian free choice game theory is non-contextual and hence is in contradiction with the Kochen-Specker theorem. Hence, we propose that quantum physics should be analysed with toolkits from non-Nashian game theory applied to our suggested model.

1 Introduction

The interrelation between quantum theory and game theory was established in a seminal paper by Eisert, Wilkens and Lewenstein \cite{eisert2000quantum}. This work introduced for the first time the quantization of nonzero sum games and showed that it is possible to design quantum strategies utilizing the entanglement property to generally outperform classical strategies. The embedding of quantum physics into games, via additional primitives that the agents have at their disposal, became a new research framework \cite{avin2019quantum, ukai2019quantum, durr2020quantum}. Exploring its capabilities led to novel ideas such as “coherent equilibria” \cite{duan2020coherent} and created links to other disciplines like quantum information theory and quantum algorithms \cite{ziman2019quantum}, market analysis \cite{scott2014market} and many others \cite{durr2020quantum, ukai2019quantum}.

In this paper, we investigate a yet uncharted research direction that is complementary to the mentioned studies — we go the opposite way and transfer the game-theoretic concepts to the quantum domain. In consequence, we provide an innovative approach for understanding the defining aspects of quantum theory based on game theory.

To start, we highlight the fundamental concepts and structures of game theory by studying the two examples of the promise game and of the prisoners' dilemma, which represent the two main game paradigms: dynamic games, and strategic games.

The objects that constitute the core of each game in general, that is, for both paradigms, are:

- a set \( P = \{1, \ldots, N\} \) of uniquely indexed players for some total number of players \( N \).
Figure 1: The diagram presents the promise game. Each box with the name of the player constitutes a decision node. The edges lead to possible future scenarios and eventually to outcomes of the game which are the leaves in the decision tree. Each leaf shows the payoffs of the players – the number on the left being Peter’s payoff and on the right Mary’s payoff. Each resolution of the game is a path going from root to an outcome.

- a set of possible outcomes of the game \( Z \) (in some games, also called strategy profiles).
- For each player \( i \), a payoff function \( u_i \in \mathbb{R}^Z \). This family of payoff functions determines the reward that each player gets given the outcome of the game.

The two kinds of games then differ in their structure.

The first kind of game is called a dynamic game, or game in extensive form \([11][13][18]\). Figure 1 shows the promise game, which is an example thereof. In a dynamic game, agents make their moves in turns, i.e., the consecutive choices are time-like separated events. In this scenario, we have \( P = \{1,2\}, Z = \{o_1, o_2, o_3\} \) and, for example, \( u_2(o_3) = 1 \). Additional objects are needed to organize the outcomes of the game as the leaves of a tree structures, where players make a decision at each intermediate node, going down the tree and eventually reaching one of the outcomes:

- a set \( H \) of decision nodes, each node \( n \in H \) is associated with a player \( \rho(n) \in P \),
- a set of actions \( A \), where each decision node \( n \in H \) is associated with a subset of actions \( \chi(n) \subseteq A \) that players \( \rho(n) \) can pick from.
- a successor function \( \sigma \) mapping each decision node \( h \) and action \( a \in \chi(h) \) available to \( \rho(h) \) at that node to the next node \( \sigma(h,a) \in H \cup Z \), either decision node or outcome. \( \sigma(h,a) \) is undefined for an action \( a \notin \chi(h) \).

Here \( H = \{n_1, n_2\} \), \( \rho(n_1) = 1 \), \( \rho(n_2) = 2 \), \( A = \chi(n_1) = \chi(n_2) = \{\text{cooperates}, \text{defects}\} \), \( \sigma(n_1, \text{cooperates}) = n_2 \), \( \sigma(n_1, \text{defects}) = o_1 \), \( \sigma(n_2, \text{cooperates}) = o_3 \), \( \sigma(n_2, \text{defects}) = o_2 \). We illustrate this game by the following example. Peter, the baker, starts and decides \((n_1)\) whether or not to give a loaf of bread to Mary. Afterwards, Mary, if given the bread, chooses \((n_2)\) whether or not to pay for it to Peter. If we solve this game with the Nash paradigm, it is rational for Mary not to pay for the bread \((o_2)\) as it results in a higher reward for her. Likewise, Peter, anticipating this, does not hand over the bread \((o_1)\) because he gets a higher reward.

The promise game is a game with perfect information, since Mary acquires the information about Peter’s choice. If Mary were presented with a opaque bag either containing or not containing the bread then it would be a game with imperfect information.

The second kind of game is called a strategic game, or game in normal form \([14][17][18]\). In strategic games, agents choose their strategies in isolation from each other. Although it is not necessarily required, this can be thought of as decisions being space-like separated. What is sufficient is to ensure with other means that there is no communication between the agents.
Figure 2: Payoffs table for the prisoners’ dilemma. The game diagram possesses a matrix rather than a tree structure, since for this game the agents are isolated from each other and do not make their moves in turns. The resolution of the games is a single entry in the table indicating the payoffs for the players in that game realisation – the number on the left being Peter’s payoff and on the right Mary’s payoff.

The eminent prisoners’ dilemma game is a member of this family.

The additional mathematical constructs needed for a strategic game are as follows:

- for each player $i \in P$, a set of strategies $S_i$ that player $i$ can choose from. Hence, we have a family of sets of strategies indexed on the players, $(S_i)_{i \in P}$.

- the set of outcomes is then simply the Cartesian product of all strategies: $Z = \prod_{i \in P} S_i$.

Each outcome is also called a strategy profile.

Analysing Figure 2, we see that $P = \{1, 2\}$, $S_1 = S_2 = \{\text{defect, cooperate}\}$, there are four outcomes in $Z = \{(\text{cooperate, cooperate}), (\text{cooperate, defect}), (\text{defect, cooperate}), (\text{defect, defect})\}$ and for example $u_1(\text{cooperate, cooperate}) = 2$.

This models the situation in which two burglars caught by police are put in separate jail cells and are interrogated. If the burglar snitches on their teammate, they defect and if they stay silent, they cooperate. In the Nash paradigm, the strategy for each agent is to defect because treating the other agent’s decision as fixed no matter whether it is to defect or cooperate, the choice to defect always gives higher payoff. However, we can spot that it is not the most rewarding outcome for both Peter and Mary.

In the above examples, we solved the games with the Nash equilibrium paradigm. In a strategic game, a Nash equilibrium $E$ is an outcome jointly reached by all the players such that, if any player had chosen another strategy assuming that others strategies are fixed (this is called a unilateral deviation), then it would have resulted in a lower payoff for them i.e., $E = (s^*_1, s^*_2, \ldots, s^*_N) \in Z$ such that

$$\forall i \forall s \in S_i : \quad u_i(E) = u_i(s^*_1, s^*_2, \ldots, s^*_i, \ldots, s^*_N) \geq u_i(s^*_1, s^*_2, \ldots, s_i, \ldots, s^*_N)$$

For dynamic games, Nash equilibria are also defined, because there is a procedure to convert any dynamic game to a strategic game with the same outcomes. In a strategic form of a dynamic game the choices, for each player, are “flattened” to a unique set of strategies that corresponds to a combination of choices the player makes in each one of their nodes; this can be seen as some sort of a masterplan that players can come up with ahead of the dynamic game, anticipating any possible play. Then, an outcome is a Nash equilibrium in the dynamic game if it is a Nash equilibrium in its strategic form. A specific instantiation of a Nash equilibrium in a dynamic game is the Subgame Perfect Equilibrium, which is obtained by a backward induction starting at the leaves and optimizing choices all the way up to the initial node.

The two categories of games above have been shown to be special cases of a wider and more general category of games, which we consider in this paper, called spacetime games with perfect information. In spacetime games, players can make decisions at any location in

|       | 2- Mary |
|-------|---------|
| defect| 1, 1    |
| cooperate| 3, 0   |
| defect| 0, 3    |
| cooperate| 2, 2   |
Minkowski spacetime. Dynamic games are spacetime games in which decisions are timelike-separated events. Strategic games are spacetime games in which decisions are space-like separated events. The additional elements needed to define spacetime games are introduced in the Methods section. An algorithm to convert any spacetime game with perfect information to a dynamic game with imperfect information, as well as to a strategic form was established in [16]. In consequence, notions of Nash equilibria, rationalizability, individual rationality, etc, are inherited by spacetime games. In a game with imperfect information, players may not be fully informed about decisions made at ancestor nodes; this is modelled with equivalence classes of decision nodes, called information sets, that make them indistinguishable. In this work, in diagrams presenting games with imperfect information, information sets are denoted with dotted lines in the dynamic game’s structure.

Finally, to compute the Nash equilibrium, we explicitly considered unilateral deviations only. This assumption, called the Nashian free choice assumption, is however not the only possible approach to game theory. A whole spectrum of alternate non-Nashian approaches exists, such as the Perfect Prediction Equilibrium (PPE) [20], the Translucent Equilibrium [21], the Perfectly Transparent Equilibrium [25], Superrationality [22], Minimax-Rationalizability [23]. In such frameworks, for some games, there exist resolutions with potentially higher and more efficient payoffs for all the agents as compared to the Nashian framework. For the analysed games such Pareto-optimal outcomes that the Nash equilibrium fails to reach are: (1,1) in the promise game and (2,2) in the prisoners’ dilemma game. \( o_3 = (1,1) \) becomes an equilibrium in the promise game in the PPE framework because of the following reasoning. If Mary were to pick \( o_2 \) rather than \( o_3 \), Peter would have forecasted it as the rational thing for her to decide for and he would have chosen \( o_1 \). But this would be contradictory with her playing at \( n_2 \) at all. This is a reductio-ad-absurdum proof that, if she was given, by rational Peter, the possibility to play at \( n_2 \), it logically follows that she picks \( o_3 \). We provide a detailed description in the Methods section.

In this work, we use the framework of spacetime games with perfect information [16] (a concise summary thereof is given in the Methods section) which can model any intricate scenario combining the cases of decisions being both space-like and time-like separated and for which there is a well-defined equilibrium called Perfectly Transparent Equilibrium [25] (extension of PPE) that for the case of no ties in the payoffs is at most unique and Pareto-optimal. In the Results section, firstly, we propose a framework in which we establish the notions of counterfactual dependency/independence and Nashian as well as non-Nashian free choice. Secondly, we analyse the Bell experiment setup as a game, hence demonstrating the inner workings of an algorithm allowing to translate any physical scenario involving quantum measurements into a game. Furthermore, we also investigate how the optimal strategy resolutions to these games are found both using Nashian and non-Nashian assumption. Finally, analysing the Kochen-Specker theorem from the game theory perspective we conclude that the Nashian resolution corresponds to local hidden variable theories, while a non-Nashian resolution is contextual.

2 Results

We propose a mathematical framework, agnostic to the free choice assumption and consistent with game theory; our framework is capable of describing any quantum experiment scenario in which experimental setup options are selected, the results of a measurement process are obtained, and such experiments are carried out in arbitrary numbers and at arbitrary positions in spacetime. For that, we consider an input-output parameters model akin to [24] where input parameters reflect the chosen characteristics of the apparatus and output parameters correspond to measurements outcomes, but extending it to a more general framework in which input parameters need not all be jointly defined in every history, and might thus be contextual. Additionally, this more general framework allows to introduce a novel notion of symmetry between input and output parameters. Consequently, we define the following notions:
Multihistory The multihistory $\Omega$ is the set of all possible histories of the quantum experiments scenarios. Each $\omega \in \Omega$ represents one way the history could be. $\Omega$ is endowed with a $\sigma$-algebra $\mathcal{F}$, making $(\Omega, \mathcal{F})$ a measurable space. $\Omega$ is also endowed with a probability measure $P$. Each possible history $\omega \in \Omega$ is endowed with a Minkowski spacetime $\mathbb{R}^4$.

In each history, there are parameters, located in spacetime, that can take values or be undefined. A history can, thus, alternatively be seen as a history of parameter values.

Parameter An (input/output) parameter $A$ is a random variable on $\Omega$ with values in some target set $\chi(A) \cup \{\bot\}$. The special value $\bot$ is an additional element of this target set that expresses the possibility of $A$ being undefined in a history $\omega$, i.e. $A(\omega) = \bot$. Furthermore, we require that each parameter has a well-defined position in Minkowski spacetime – the same in all histories.

Let $I$ denote the set of all (input and output) parameters in a given model and let $\rho$ be a function that maps each moment of decision $i \in I$ to the agent $\rho(i) \in P$ making this decision at $i$, by picking a value in $\chi(i)$. Note that game theory also allows for nature to be a player \[32\], which typically is implemented as a random number generator. Our framework also allows moves by nature (output parameters), but does not restrict them to be necessarily made with a random number generator.

Next, we define formally the concept of counterfactual dependency in the proposed framework. Counterfactual dependencies \[26, 27\] are needed when we consider alternate values parameters could have taken, and what alternate history this would have corresponded to. In plain English they can be expressed as subjective conditionals, such as: “if Mary had measured the spin along the $z$ axis, then the outcome would have been 1”. We already encountered such statements in the Introduction where we analysed the promise game from the non-Nashian perspective. To give the counterfactual dependency formal meaning, we begin with employing the notion of closest history utilised in game theory \[23\]:

Closest history Given a history $\omega \in \Omega$, a parameter $A$ (which may or may not be defined in $\omega$) and a value $a$ that $A$ can take, we assume (as an axiom) that there is exactly one closest history in which $A = a$ is true, and denote this history $f_{A=a}(\omega)$.

Then, the counterfactual dependency between two (input or output) parameters of the quantum system can thus be formally defined as follows:

Counterfactual dependency Given a history $\omega \in \Omega$, two parameters $A$ and $B$ (which may or may not be defined in $\omega$) and two values $a$ and $b$ that $A$ and $B$ can take, $A = a$ is said to counterfactually imply $B = b$ in $\omega$, which is denoted $A = a \Rightarrow B = b$, if $B = b$ is true in $f_{A=a}(\omega)$.

This definition implies that there is no restriction in the spacetime location of counterfactually dependent variables, in particular, in general, a spacetime variable may be counterfactually dependent on another located in its future light cone, or on another that is spacelike-separated.

We also define the notion of counterfactual independence of $A$ from $B$.

Counterfactual independence Given two parameters $A$ and $B$, $A$ is said to be counterfactually independent from $B$ if

$$\forall \omega \in \Omega, ((A(\omega) = a \neq \bot \wedge B(\omega) = b \neq \bot) \implies \forall b' \in \chi(B), B = b' \Rightarrow A = a)$$

Counterfactual independence means that, in any history $\omega$ in which both parameters are defined, in the closest history $f_{B=b'}(\omega)$ in which $B$ takes any other value $b'$, $A$ has the same
Figure 3: Figure illustrates the notions of Nashian and non-Nashian free choice. For the spacetime diagrams time is denoted as a vertical direction.

A parameter \( A \) is said to be **freely chosen in the Nash sense** if any parameter \( P \) that is not in its future light cone is counterfactually independent from \( A \).

A parameter \( A \) is said to be **freely chosen in the non-Nashian sense** if for every \( a \in \chi(A) \) there exists a history \( \omega \) in which \( A(\omega) = a \).

At that place, we would like to model a Bell-type experiment from the game theory point of view. Consider a game with four players \( P = \{1(\text{Peter}), 2(\text{Mary}), 3(\text{Ulysses}), 4(\text{Valentina})\} \). Peter and Mary are experimental physicists sharing an entangled pair of particles and deciding which measurement settings are used for the experiment, i.e. they determine the input parameters of the system. Peter can choose from \( \{a_1, a_2\} \) and Mary from \( \{b_1, b_2\} \). Ulysses’ and Valentina’s choices model the measurement outcomes (output parameters) which, for both of them, can be either \(-1\) or \(1\). Although Ulysses and Valentina do not represent human players, from the perspective of game theory they can be treated as agents that emulate what happens to the particles that are being measured. If we assume Nashian free choice, then because of the Free Will Theorem (Conway, Kochen [30,31]), Ulysses and Valentina posses Nashian free choice as Peter and Mary do.
Figure 4: Figure presents a causal structure for a Bell type experiment. It is a Directed Acyclic Graph where nodes represent (input/output) parameters and edges correspond to future-directed time-like separation for the relevant pairs of parameters. For readability, only the irreflexive and transitive reduction of the causal dependencies is shown. In that setting six input-output parameters are specified with the following map from moments of decisions (parameters) to players: $\rho(I_1) = \text{Peter}$, $\rho(I_2) = \text{Mary}$, $\rho(I_3) = \rho(I_4) = \text{Ulysses}$, $\rho(I_5) = \rho(I_6) = \text{Valentina}$. Agents’ payoffs are left open.

We further impose that both events corresponding Peter’s and Ulysses’ choices are space-like separated from both events representing Mary’s and Valentina’s decisions. Hence, the causal structure of this setup can be depicted as in Figure 4. Note that, we explicitly distinguish Ulysses’ choice in the history for which Peter chooses $a_1$ from his decision in the history for which Peter chooses $a_2$ by modelling these two decisions with two different parameters $I_3$ and $I_4$. The same for Valentina. In the language used in [16] (please see the Methods section for more details), we can say that the contingency coordinate for $I_3$ to be defined is $I_1 = a_1$ (similarly for other output parameters).

With the spacetime structure of the game set, parameters and the contingency dependencies between them defined we can use the algorithm established in [16] to convert the above described scenario to an extensive form game with imperfect information (see Figure 5). First of all, let us emphasize that structure of the obtained game is observer-independent since space-like and time-like separation relations of the decision points are invariant under Lorentz transformations. Secondly, let us observe that the game can be seen as a process in which agents make decisions in turns. However, a crucial aspect must be added to this view, namely that agents making a decision do not know in which history they are located. They only know their moments of decision which are equivalence classes formed on the set of decision points. This notation is formally captured with information sets in games with imperfect information.

Let us consider how the presented Bell-type game is solved for an equilibrium in both cases of Nashian and non-Nashian (Perfectly Transparent Equilibrium [25]) free choice assumptions. We introduce some concrete payoffs, whereby the choice of payoff structure is a free parameter of each model instantiated from our framework.

In Figure 6 we see the resolution of the game in the Nashian framework. The equilibrium suchly found is stable against unilateral deviations. For example, if Peter had chosen $a_2$ instead of $a_1$, keeping the other players’ choices fixed it would have led to the outcome $(1, 13, 15, 15)$ in which he would have obtained 1 which is less than 10. The same reasoning applies to the other players. Note that, in the Nash paradigm, the choices have to be specified for all the decision nodes including unreached ones, and an assignment of choices is called a Nash strategy. We will see later on that this feature does correspond to non-contextuality.
We now analyse how the same game is solved when the non-Nashian form of free choice is assumed. Figure 7 presents the forward induction process which leads to the PTE solution. The principles utilized in the elimination procedure correspond to the ones introduced in the Methods section for the PPE framework. The detailed description of the elimination process can also be found in the Methods section. It is important to point out that contrary to the Nashian case, decisions are only assigned on a path leading to the equilibrium outcome. Interpreting this fact in a physics context we can say that only the events happening in the actual history are elements of reality. Additionally to that we can conclude that the resolution for the PTE concept is inherently contextual. Finally, a forward induction reasoning is computationally more efficient than the backward induction used in Nash game theory that requires starting from every single possible future; indeed, in a forward induction, only decisions on the actual equilibrium path history ever get assigned, and this happens in a dynamics going top-down, which allows a partial resolution of a game up to a specific period of time, pausing, and then resuming at will.

Eventually, let us analyse the Kochen-Specker theorem [10] from the game theory perspective taking into account defined notions and concepts. We focus here on one particular quantum scenario example (introduced by Mermin [33]) that does exhibit contextuality. Consider Mermin’s square:

$$\begin{align*}
1 \otimes \sigma_z & \quad \sigma_z \otimes I & \quad \sigma_z \otimes \sigma_z \\
\sigma_x \otimes I & \quad I \otimes \sigma_z & \quad \sigma_z \otimes \sigma_x \\
-\sigma_x \otimes \sigma_z & \quad -\sigma_z \otimes \sigma_x & \quad \sigma_y \otimes \sigma_y
\end{align*}$$

(1)

Each row and each column of the square consists of three mutually commuting operators, each with eigenvalues equal to +1 and −1. Furthermore, the operators in each row multiply out to the $I \otimes I$ and the operators in each column multiply out to $-I \otimes I$. The key point of Mermin’s reasoning concluding that the proposed setup cannot be described by any Local Hidden Variable theory is to observe that there does not exist an assignment of +1s and −1s
Figure 6: Figure presents the Nashian resolution of a Bell-type scenario. We assign payoffs to each of the outcomes in a form of number tuples for which the first number is the Peter’s, the second Mary’s, the third Ulysses’ and the fourth Valentina’s payoff. Bold arrows show which choices the agents make. Note that decisions are also defined for unreached nodes. The resolution of the game is a path following bold arrows from the root of the tree to one of the leaves. The found equilibrium is $(10, 15, 10, 10)$.

Figure 7: Figure presents the process of finding the non-Nashian resolution of a Bell-type scenario. Subfigures a), b), c) and d) show consecutive steps in a forward induction procedure. To discard some outcomes, possible decision choices of agents inscribed in bold frames are considered. The eliminated outcomes are marked with dark grey colour and are no longer considered. We assign payoffs to each of the outcomes in a form of number tuples for which the first number is the Peter’s, the second Mary’s, the third Ulysses’ and the fourth Valentina’s payoff. Bold arrows show which choices the agents make. Note that decisions are undefined for unreached nodes. The resolution of the game is a path following bold arrows from the root of the tree to one of the leaves. The found equilibrium is $(14, 16, 11, 11)$. 

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to the nine operators in the square such the product of these numbers for each row is equal to 1 and the product of considered values for each column is equal to \(-1\). It means that the values cannot be defined out of the context in which they are measured. The context for the operator \(I \otimes \sigma_z\) is either the first row or the first column of Mermin’s square.

Let us build a game modelling the encountered quantum feature of contextuality. Consider the following setup. Mary as an experimental physicist picks one of the six contexts – one of rows or one of the columns. For each one of the contexts, a measurement outcome is then produced for each one of the three operators in this context, which are eight possible outcomes within each context. Furthermore, the measurement outcomes corresponding to the same operator are linked with the same output parameter. Let us assume that it is possible to analyse this game using the Nashian free choice framework. We will show that such an assumption leads to a contradiction directly related to the Mermin’s argument.

First, we define such payoffs for the outcomes that the Nashian equilibrium exists. If it is a Nash equilibrium then it is possible to consider unilateral deviations from it. Because of the Kochen-Specker theorem, we know that there exists a context (say, without loss of generality, the first row), and there exists an operator in this context (say, without loss of generality, \(I \otimes \sigma_z\)) such that, had Mary picked the first column instead as a context, the measurement outcome for this operator would have been different. Hence, there exists an operator whose measurement outcome is counterfactually dependent on the choice of context. Under the Nashian game theory paradigm, which has the notion of the strategic form of a game, it is possible for Ulysses to assign a value to all his decisions ahead of the game (this is called Ulysses’s strategy), that is, in the past of Mary's choice. This, however, is in contradiction with the introduced notion of Nashian free choice which says any event in the past of a decision must be counterfactually independent from it. Therefore, it implies that Nashian free choice cannot generally hold and that contextuality is incompatible with the Nashian free choice of a context with respect to the measurement outcome assigned to each operator. Another way of formulating this is that contextuality requires dropping the Nashian free choice assumption, i.e., dropping Nashian free choice must follow from the Kochen-Specker theorem in the introduced mathematical framework.

3 Discussion

As a necessary step for defining the Nashian and non-Nashian free choice we have defined counterfactual dependency. We would like to compare this concept firstly with causation and secondly with correlation. Hence we aim to show that counterfactual dependency has further importance than being a piece of terminology.

For the purpose of our discussion we define causation in terms of special relativity: a parameter \(A\) defined is said to be causally dependent on another parameter \(B\) if \(B\) is in the past light cone (including the boundary but not the tip) of \(A\). (Note that this statement has physically meaningful interpretation only in the histories in which both \(A\) and \(B\) are defined.)

Distinguishing the notion of counterfactual dependency as a separate construct becomes instrumental when analysing a Bell-type experiment. Suppose Peter and Mary, who share an entangled pair of particles, perform measurements that are space-like separated which outcomes are either \(-1\) or \(+1\). They obtain opposite results \(A = -B\) (\(A\) for Peter, \(B\) for Mary). Parameter \(A\) is space-like separated from parameter \(B\), hence it is causally independent from \(B\) – no need to invoke cause and effect. However, \(A\) is counterfactually dependent from \(B\) because if Mary had measured a different value, Peter would have measured a different value as well.

As for correlations, in our framework, they correspond to the use of statistics on the values of input and output parameters that actually happen and get collected on some record medium.
within the one and same history.

The use of conditional probabilities on random variables that do not jointly take values within
the same history might be prone to misinterpretation. When we consider several alternate
histories at once we reason in a counterfactual way. Let us illustrate this point by an example.
Consider a sequence of two experiments on non-commuting observables, say, the spin along
the y axis and then along the x axis. Both can return either the eigenvalues \(1\) or \(-1\).

Let us assign a parameter \(A\) to the outcome of the first experiment, and a parameter \(B\) to the
outcome of the second experiment. Note that the latter parameter is obtained by taking the
union of the supports of the second measurement outcome over all possible outcomes of the
first experiment. Then one can express the law of total probability, for the case of the second
measurement returning \(+1\), as follows:

\[
P(B = 1) = P(B = 1 | A = 1)P(A = 1) + P(B = 1 | A = -1)P(A = -1)
\]

Within our framework, conditioning \(B\) on \(A\) has natural meaning with respect to the in-
troduced measurable space \((\Omega, \mathcal{F})\). Indeed, \(P(B = 1 | A = 1)\) corresponds to the parameter
\(B_1 = 1\) only defined in the history for which \(A = 1\). And \(P(B = 1 | A = -1)\) corresponds
to the parameter \(B_{-1} = 1\) only defined in the history for which \(A = -1\). Thus, each of the
conditionals corresponds to histories in which either \(B_1\) or \(B_{-1}\) is defined. Hence, within our
framework the law of total probability is expressed as follows:

\[
P(B = 1) = P((A = 1 \land B_1 = 1) \lor (A = -1 \land B_{-1} = 1))
\]

However, in a single history, one has no access as experimental physicist to the actual measur-
able space. What is done in practice is to actually consider an iteration of the experiment, say,
one million times. Then, the corresponding dependency graph describing the game must be
organized as a Directed Acyclic Graph with two million levels, with each history containing two
million output parameter values, interpreted as samples of the two same underlying random
variables. Under the assumption that the one million pairs of variables are i.i.d. (indepen-
dent and identically distributed), the two ways introduced ways of looking at the law of total
probability are the same.

At this point, as a main takeaway of the paper, we would like to address the fact that Nashian
game theory, as established in the Results section, corresponds to Local Hidden Variables the-
ories and hence is incompatible with quantum contextuality. If we were to stay in the Nashian
free choice framework, it would be possible, from the game theory perspective, to eliminate the
obtained contradiction by introducing chance moves for all the agents [32] (agents’ decisions
governed by random processes). This approach, although it resolves the raised issue, puts
tight restrictions for research at the intersection of game theory and quantum theory because
most of the game-theoretic concepts are not applicable to scenarios evolving in a random way:
game theory, and the concept of rationality, is useful only if a game contains at least some
moves that are not chance moves. A game with only chance moves is better modeled with
concepts from the theory of Bayesian networks [34].

That is why, we would like to propose to circumvent the brought up incompatibility by re-
placing the Nashian free choice with the non-Nashian free choice assumption. It has been
proven that Nashian free choice violations and locality violations are equivalent resources in
Bell experiments [28]. That result strengthens our proposal to investigate further quantum
theory from a perspective where the free choice assumption is weakened. Let us also further
observe that previously established constraints of non-extensibility of quantum theory to im-
proved predictive power theories [29] does not hold in non-Nashian framework.

Finally, we note that the conclusions of our reasoning are independent on the exact form of
agents’ payoffs. Hence, in this work, we left them open. However, if the payoffs were set, a
fixed point equilibrium might be found in a procedure resembling the considerations revoking
to the least action principle. We would consider alternate histories which can be seen as perturbations of the examined history and by studying their properties we would obtain the actual history that is Pareto-optimal for all agents.

4 Methods

**Spacetime games structure.** The additional objects needed to provide a structure for the outcomes of a spacetime game are as follows:

- a set $H$ of decision nodes, each node $n \in H$ is associated with a player $\rho(n) \in P$,
- a set of actions $A$, where each decision node $n \in H$ is associated with a subset of actions $\chi(n) \subseteq A$ that players $\rho(n)$ can pick from.
- A relation $R$ that forms a directed acyclic graph over the nodes $H$.
- A label function $\sigma$ mapping each edge $(n, m) \in R$ to an action $\sigma(n, m) \in \chi(n)$ available to $\rho(n)$ at that node.

The relation $R$ constitutes the causal dependency graph of the spacetime game based on the Minkowski locations of the decisions. $\sigma$ is referred to as the contingency coordinate system, which tells under which conditions on past choices a future decision takes or does not take place.

The way the game is played is that agents at the source nodes (with no previous nodes) make their decisions. Then, the decision making propagates down the directed acyclic graph following the arrows. However, a decision is only made at a node (it is said to be active) if all its parents are active, and the decisions at the parents match the label on the edge connecting to that parent.

Specifically, a possible outcome $z$ is a partial function from $H$ to $A$ so that two conditions apply:

First, all root decision nodes have an assignment:

$$\forall j \in H, (\exists i \in H \text{ s.t. } (i, j) \in R \implies z(j) \in \chi(j)(\iff z(j) \neq \bot))$$

Second, non-root decision nodes have an assignment or not depending on whether the assignment of their parent decision nodes matches the label on the edge:

$$\forall (i, j) \in R, z(i) = \sigma(i, j) \iff z(j) \in \chi(j)(\iff z(j) \neq \bot)$$

$Z$ is then defined, as all games, as the set of all such outcomes, and the payoff function is then defined on $Z$.

**PPE equilibrium framework.** At that point we present a brief description of the game theoretic notion of Perfect Prediction Equilibrium (PPE), first introduced in [20]. It is an equilibrium concept applicable to finite games in extensive form with perfect information that allows to incorporate non-Nashian free choice into game theory. For dynamic games with strict preferences, the PPE always exists. Additionally to that, it is unique and Pareto-optimal. We illustrate the inner-workings of the elimination algorithm used to solve for PPE by first introducing two high-level principles it utilizes and then by showing a step-by-step procedure how it operates on a concrete example of a $\Gamma$ – game.

1st principle. Consider one of the possible outcomes $o$, and an agent $p$ making a decision parameterized as $D$ on the path to $o$. The path from $D$ to $o$ is called a causal bridge. If it is so that, had the player made a different decision at $D$ than the one leading to $o$, it would guarantee them a higher payoff no matter what choice other players in the future make, then the outcome $o$ is logically impossible (ad-absurdum proof) and discarded. This preemption
process is said, in the original publication \cite{20}, to break the causal bridge to \( o \) because its anticipation would not cause it. Figure 8a) provides a schematic view of how this principle is applied in practice. Let us make clear that there is no causal effect on the past involved. The reasoning that agents use is purely based on counterfactual statements (for more details please refer to the Results section). Any agent playing after the agent \( p \) would not have chosen the path ultimately leading to \( o \) because agent \( p \) would have anticipated that and hence would have deviated.

\textbf{2nd principle.} Consider all the outcomes that have not been eliminated yet. Note that all the discarded outcomes are no longer taken into account. If a player \( p \) decides between two decisions \( D_1 \) and \( D_2 \), and all the payoffs connected to the outcomes following \( D_1 \) are higher than the maximum payoff for the outcomes subsequent to \( D_2 \), then it is rational for this agent to pick \( D_1 \). We visualize this concept by means of Figure 8b).

As an example, let us consider an extensive form game without chance moves, with perfect information played between Peter and Mary who have strict preferences. We call this specific example \( \Gamma \) – game (see Figure 9). This particular game does not possess any physical or game-theoretic interpretation. It was created solely for pedagogical reasons of demonstrating how we solve a game for the equilibrium employing two introduced aforementioned principles. In the diagram (see Figure 9) we see what are the consecutive steps in the process of solving for PPE. The sequential steps are labeled by integer indices and are annotated with a description whether they use the 1st or the 2nd principle. For more detailed description please refer to \cite{20}.

The algorithm for finding an equilibrium with non-Nashian free choice assumption also exists generally for any games in extensive form with imperfect information, and in particular for spacetime games, assuming payoffs are in generic position (no ties between payoffs for the same agent). The generalisation of PPE concept is called Perfectly Transparent Equilibrium (PTE) \cite{25}.

\textit{Resolution of a Bell-type game in the non-Nashian framework.} In this paragraph, we explain in details, how it is possible to solve the game presented in Figure 7 for the non-Nashian equilibrium (of PTE type).

We start the discussion by realising that Peter and Mary take a decision in all the possible histories. We do not consider yet the choices of Ulysses and Valentina because their opportunities depend on the what Peter and Mary do before them. That is why in Figure 7a) Peter and Mary are put in bold frames to emphasize that we firstly examine the situation from their point of view. Let us observe that the minimal payoff that Peter could get if he chose \( a_1 \) is 6. Hence, from the 1st principle introduced in the paragraph explaining the PPE concept, we conclude that \( (1, 13, 15, 15) \) together with four other outcomes in the right part of the tree is discarded. Outcome \((6, 1, 16, 16)\) is eliminated using the same principle but
Figure 9: Figure presents the $\Gamma$ – game. It is a dynamic game with two players: Peter and Mary. To solve for the equilibrium, a process of forward induction utilizing the 1st and the 2nd principle of the PPE framework is implemented. The consecutive steps of the algorithms are indicated with round indexed tokens. The PPE for the $\Gamma$ – game is $O_{11} = (3, 2)$. Image source: [20].

considering Mary’s decision. If it had been the solution Mary would have deviated to $b_1$. Now, the minimum payoff for Peter for the $a_1$ choice has changed to 11. Subsequently, we eliminate three more outcomes leaving the right branch empty (see $7b$). Hence, we know that Peter chooses $a_1$. Now, we can add to considerations Ulysses’ perspective since we know what is the contingency coordinate of his decision point. That is why we put Ulysses in bold frame. Next round of elimination once again based on the 1st principle is depicted in subfigure $7c$). Accordingly, we conclude that Mary chooses $b_1$ and that now also Valentina gets to choose. In the last round, Ulysses eliminates the outcome $(10, 15, 10, 10)$ (utilizing the 1st principle) and consequently Valentina chooses $(14, 6, 11, 11)$ because it gives her higher payoff (1st principle). This outcome is the Perfectly Transparent Equilibrium for the considered game.

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