New Characterizations of $S$-coherent rings

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Abstract

In this paper, we introduce and study the class $S$-$\mathcal{F}$-ML of $S$-Mittag-Leffler modules with respect to all flat modules. We show that a ring $R$ is $S$-coherent if and only if $S$-$\mathcal{F}$-ML is closed under submodules. As an application, we obtain the $S$-version of Chase Theorem: a ring $R$ is $S$-coherent if and only if any direct product of $R$ is $S$-flat if and only if any direct product of flat $R$-modules is $S$-flat. Consequently, we provide an answer to the open question proposed by D. Bennis and M. El Hajoui $^3$.

Key Words: $S$-coherent rings, $S$-flat modules, $S$-Mittag-Leffler modules.

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1. Introduction

Throughout this paper, $R$ is a commutative ring with identity and all modules are unitary. $S$ will always denote a multiplicative closed set of $R$. In the past few years, $S$-versions of some classical notions have been studied by many authors. In 2002, D. D. Anderson and T. Dumitrescu $^1$ introduced $S$-finite modules and $S$-Noetherian rings and extended the classical Cohen’s Theorem and Hilbert basis Theorem to some $S$-versions. In 2014, H. Kim, M. O. Kim and J. W. Lim $^5$ introduced $S$-strong Mori domains and proved that if $S$ is an anti-archimedean subset of a domain $D$, then $D$ is an $S$-strong Mori domain if and only if the polynomial ring $D[X]$ is an $S$-strong Mori domain, if and only if the $t$-Nagata ring $D[X]N_v$ is an $S$-strong Mori domain, if and only if $D[X]N_v$ is an $S$-Noetherian domain. In 2015, J. W. Lim $^12$ studied Nagata ring of $S$-Noetherian domains and locally $S$-Noetherian domains and proved that if $S$ is an anti-archimedean subset of a domain $D$, then $D$ is an $S$-Noetherian domain (respectively, locally $S$-Noetherian domain) if and only if the Nagata ring $D[X]_N$ is an $S$-Noetherian domain (respectively, locally $S$-Noetherian domain). In 2018, H. Kim and J. W. Lim $^14$ introduced $S$-$*_{w}$-principal
ideal domains and studied the local property, the Nagata type theorem, and the Cohen type theorem for $S^{* _w}$-principal ideal domains.

In 2018, D. Bennis and M. El Hajoui introduced $S$-finitely presented modules and $S$-coherent rings which are $S$-version of finitely presented modules and coherent rings and obtained an $S$-version of Chase’s result [4, Theorem 2.2] as bellow.

**Theorem 1.1.** [3, Theorem 3.8] The following assertions are equivalent:

1. $R$ is an $S$-coherent ring;
2. $(I : a)$ is an $S$-finite ideal of $R$, for every finitely generated ideal $I$ of $R$ and $a \in R$;
3. $(0 : a)$ is an $S$-finite ideal of $R$ for every $a \in R$ and the intersection of two finitely generated ideals of $R$ is an $S$-finite ideal of $R$.

Subsequently, they proposed an interesting open question as an $S$-version of Chase Theorem [4, Theorem 2.1].

**Open Question.** How to give an $S$-version of flatness that characterizes $S$-coherent rings similarly to the classical case?

One of the main purposes of this article is to characterize $S$-coherent rings inspired by this question. In Section 2 and Section 3, we introduce and study the notions of $S$-flat modules and $S$-Mittag-Leffler modules with respect to a class $Q$ (denoted by $S^{* _Q}$-ML).

In Section 4, we study $S$-Mittag-Leffler modules with respect to all flat modules (denoted by $S^{* _F}$-ML) and show the following result.

**Proposition 4.2.** An $R$-module $M$ is $S$-finitely presented if and only if it is in $S^{* _F}$-ML and finitely generated.

Then we give a new characterization of $S$-coherent rings using the class $S^{* _F}$-ML.

**Theorem 4.3.** The following assertions are equivalent for a ring $R$:

1. $R$ is an $S$-coherent ring;
2. every ideal is in $S^{* _F}$-ML;
3. every finitely generated ideal is in $S^{* _F}$-ML;
4. every submodule of projective modules is in $S^{* _F}$-ML;
5. the class $S^{* _F}$-ML is closed under finitely generated submodules;
6. the class $S^{* _F}$-ML is closed under submodules.

Utilizing this characterization, we finally obtain the following $S$-version of Chase Theorem.

**Theorem 4.4.** (the $S$-version of Chase Theorem) The following assertions are equivalent:
(1) $R$ is an $S$-coherent ring;
(2) any product of flat $R$-modules is $S$-flat;
(3) any product of projective $R$-modules is $S$-flat;
(4) any product of $R$ is $S$-flat;

2. Preliminaries

In this section, we will investigate some $S$-version of classical definitions on finitely generated modules, finitely presented modules, coherent rings and flat modules. Let $S$ be a multiplicative closed set of $R$.

**Definition 2.1.** Let $M$ and $N$ be $R$-modules.

1. $\tau_S(M) = \{x \in M | sx = 0 \text{ for some } s \in S\}$ is called the total $S$-torsion submodule of $M$. If $\tau_S(M) = 0$, then $M$ is called an $S$-torsion-free module; If $\tau_S(M) = M$, then $M$ is called an $S$-torsion module.

2. An $R$-homomorphism $f : M \to N$ is an $S$-monomorphism (resp. $S$-epimorphism, $S$-isomorphism) if the induced $R_S$-homomorphism $f_S : M_S \to N_S$ is a monomorphism (resp. an epimorphism, an isomorphism).

3. A sequence $0 \to M \to N \to L \to 0$ is $S$-exact if the induced sequence $0 \to M_S \to N_S \to L_S \to 0$ is exact.

**Remark 2.2.** It is easy to verify the following assertions.

1. An $R$-homomorphism $f : M \to N$ is an $S$-monomorphism if and only if $\ker(f)$ is $S$-torsion.
2. An $R$-homomorphism $f : M \to N$ is an $S$-epimorphism if and only if $\text{coker}(f)$ is $S$-torsion.
3. The class of $S$-torsion modules is closed under submodules, quotient modules, extensions and directed limits.

The following definition follows from [3].

**Definition 2.3.** (1) An $R$-module $M$ is said to be $S$-finite, if there exists a finitely generated submodule $N$ of $M$ such that $sM \subseteq N$ for some $s \in S$.

2. An $R$-module $M$ is said to be $S$-finitely presented, if there exists an exact sequence of $R$-modules $0 \to K \to F \to M \to 0$, where $K$ is $S$-finite and $F$ is a finitely generated free $R$-module.

3. A ring $R$ is $S$-Noetherian provided that every ideal of $R$ is $S$-finite.

4. A ring $R$ is $S$-coherent provided that every finitely generated ideal of $R$ is $S$-finitely presented.

**Remark 2.4.** It is easy to verify the following assertions.
(1) For any multiplicative closed set $S$, every $S$-finitely presented module is finitely generated, every finitely generated module is $S$-finite.

(2) In Definition 2.3(1), since $sM \subseteq N \subseteq M$, we obtain $N_S = M_S$.

(3) Every $S$-Noetherian ring is an $S$-coherent ring, see [3, Remark 3.4(1)].

(4) If $R$ is an $S$-coherent ring, then $R_S$ is a coherent ring. Indeed, let $I_S$ be a finitely generated ideal of $R_S$ such that $I$ is a finitely generated ideal of $R$, then there is an exact sequence $0 \to K \to R^n \to I \to 0$ such that $K$ is $S$-finite. Then $K_S$ is finitely generated as an $R_S$-ideal, and thus $R_S$ is a coherent ring.

**Definition 2.5.** Let $M$ be an $R$-module, then $M$ is said to be $S$-flat if for any finitely generated ideal $I$ of $R$, the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is an $S$-monomorphism.

Obviously, every flat module is $S$-flat. However, the converse does not hold. Indeed, let $R$ be a domain not a field, $S$ be the set of nonzero elements in $R$, then every $R$-module is $S$-flat. Thus there exists some $S$-flat module which is not flat.

Now, we give a characterization of $S$-flat modules.

**Proposition 2.6.** Let $M$ be an $R$-module, the following assertions are equivalent:

(1) $M$ is $S$-flat;
(2) for any finitely generated ideal $I$ of $R$, $\phi : I \otimes_R M \to IM$ is an $S$-isomorphism;
(3) $M_S$ is a flat $R_S$-module.

**Proof.** (1) $\Leftrightarrow$ (2) : Let $I$ be a finitely generated ideal of $R$, consider the following commutative diagram,

$$
\begin{array}{ccc}
I \otimes_R M & \xrightarrow{f} & R \otimes_R M \\
\phi \downarrow & & \cong \downarrow \\
0 & \longrightarrow & IM \longrightarrow RM.
\end{array}
$$

We have $M$ is $S$-flat if and only if $f$ is an $S$-monomorphism if and only if $\phi$ is an $S$-monomorphism.

(1) $\Rightarrow$ (3) : Let $I_S$ be a finitely generated ideal of $R_S$, where $I$ is a finitely generated ideal of $R$. Since $M$ is $S$-flat, the natural homomorphism $I \otimes_R M \to R \otimes_R M$ is an $S$-monomorphism. By localizing at $S$, the natural homomorphism

$$I_S \otimes_{R_S} M_S \cong (I \otimes_R M)_S \to (R \otimes_R M)_S \cong R_S \otimes_{R_S} M_S$$

is an $R_S$-monomorphism. Thus $M_S$ is a flat $R_S$-module.
(3) ⇒ (1): Let $I$ be a finitely generated ideal of $R$, then $I_S$ is a finitely generated ideal of $R_S$. Since $M_S$ is a flat $R_S$-module, the natural homomorphism $(I \otimes_R M)_S \to (R \otimes_R M)_S$ is an $R_S$-monomorphism. So, $M$ is an $S$-flat module. □

3. $S$-MITTAG-LEFFLER MODULES WITH RESPECT TO A CLASS OF $R$-MODULES

Recall the classical case in [10]. Let $Q$ be a class of $R$-modules and $M$ an $R$-module. We say that $M$ is Mittag-Leffler with respect to $Q$ if the canonical map

$$\phi_{M,Q} : M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I} (M \otimes_R Q_i)$$

is a monomorphism for any family $\{Q_i\}_{i \in I}$ of modules in $Q$. In case that $Q$ is the class of all $R$-modules, we say $M$ is Mittag-Leffler. In [11, Corollary 4.3], the authors characterized coherent rings using homological properties of Mittag-Leffler modules.

In order to characterize $S$-coherent rings, we introduce and study $S$-Mittag-Leffler modules with respect to a given class $Q$.

**Definition 3.1.** Let $Q$ be a class of $R$-modules, $M$ is said to be an $S$-Mittag-Leffler module with respect to $Q$, if the natural homomorphism $\phi_{M,Q}$ is an $S$-monomorphism.

We denote $S$-$Q$-ML to be the class of all $S$-Mittag-Leffler modules with respect to $Q$. When $Q$ is the class $\mathcal{F}$ of all flat modules (resp. the class $\mathcal{P}$ of all projective $R$-modules), we denote it by $S$-$\mathcal{F}$-ML (resp. $S$-$\mathcal{P}$-ML). Obviously, Mittag-Leffler modules with respect to $Q$ and $S$-torsion modules are all in $S$-$Q$-ML. The class $S$-$Q$-ML is closed under pure submodules, pure extensions. Any direct sum of modules is in $S$-$Q$-ML if and only if each direct summand is in $S$-$Q$-ML. If $N$ is a finitely generated submodule of $M$ in $S$-$Q$-ML, then $M/N$ is also in $S$-$Q$-ML.

Let $C$ be a class of $R$-modules, if $C$ is closed under pure submodules, direct products and direct limits, then $C$ is said to be a definable class (see [16, Theorem 3.4.7]). Let $Q$ be a class of $R$-modules. The class $Q$ which denotes the smallest definable class containing $Q$ is said to be the definable closure of $Q$. Note that $Q$ can be constructed by closing $Q$ under direct products, then under pure submodules and finally under directed limits. For the sake of simplification, we use the following symbols,

- $\lim Q$: all directed limits of modules in $Q$,
- $\text{Prod}Q$: all direct products of modules in $Q$,
- $\text{Psub}Q$: all pure submodules of modules in $Q$,
- $\overline{Q}$: the definable closure of $Q$.

**Proposition 3.2.** Let $M$ be an $R$-module, then the following assertions are equivalent:
(1) $M \in S\cdot \mathcal{Q}$-ML;
(2) $M \in S\cdot \text{P sub} \mathcal{Q}$-ML;
(3) $M \in S\cdot \text{Prod} \mathcal{Q}$-ML;
(4) $M \in S\cdot \text{lim} \mathcal{Q}$-ML;
(5) $M \in S\cdot \mathcal{Q}$-ML.

Proof. (2) ⇒ (1), (3) ⇒ (1), (4) ⇒ (1) and (5) ⇒ (1) are obvious.

(1) ⇒ (2): For any subset $\{Q'_i\}_{i \in I}$ from the class $\text{P sub} \mathcal{Q}$, there exists a family $\{Q_i\}_{i \in I}$ from $\mathcal{Q}$, such that $Q'_i$ is a pure submodule of $Q_i$, for each $i \in I$. There exist a commutative diagram,

$$
\begin{array}{ccc}
0 & \longrightarrow & M \otimes_R \prod_{i \in I} Q'_i \\
\phi_M & & \phi_{M, \mathcal{Q}} \\
& & \\
0 & \longrightarrow & \prod_{i \in I} (M \otimes_R Q'_i)
\end{array}
$$

Since $\phi_{M, \mathcal{Q}}$ is an $S$-monomorphism, $\phi_M$ is also an $S$-monomorphism. That is, $M \in S\cdot \text{P sub} \mathcal{Q}$-ML.

(1) ⇒ (3): For any subset $\{Q'_i\}_{i \in I}$ from $\text{Prod} \mathcal{Q}$, there exists a family $\{Q_{i,j}\}_{i \in I, j \in J}$ from $\mathcal{Q}$, such that $Q'_i = \prod_{j \in J} Q_{i,j}$. There is a commutative diagram

$$
\begin{array}{ccc}
M \otimes_R \prod_{i \in I} Q'_i & \xrightarrow{\phi_{M, \mathcal{Q}}} & \prod_{i \in I, j \in J} (M \otimes_R Q_{i,j}) \\
\phi_M & & \\
& & \prod_{i \in I} (M \otimes_R Q'_i)
\end{array}
$$

Since $\phi_{M, \mathcal{Q}}$ is an $S$-monomorphism, $\phi_M$ is an $S$-monomorphism. Consequently, $M \in S\cdot \text{Prod} \mathcal{Q}$-ML.

(1) ⇒ (4): For any subset $\{Q'_i\}_{i \in I}$ from $\text{lim} \mathcal{Q}$, we have $Q'_i = \lim (Q_{\alpha, i}^i, f_{\beta, \alpha})_{\beta, \alpha \in J_i}$ with $Q_{\alpha}^i \in \mathcal{Q}$ for any $\alpha \in J_i$. Suppose $f_{\alpha}^i : Q_{\alpha}^i \rightarrow Q'_i$ is the canonical homomorphism. Next we will show $\phi_M : M \otimes_R \prod_{i \in I} Q'_i \rightarrow \prod_{i \in I} (M \otimes_R Q'_i)$ is an $S$-homomorphism. Let $y = \sum_{j=1}^n x'_j \otimes (q_{j,i})_{i \in I} \in \ker \phi_M$, we have $\sum_{j=1}^n x'_j \otimes q_{j,i} = 0$ for any $i \in I$. Thus, for any $i \in I$ there exist $\alpha_i \in I_i$ and $k_{1,i}, ..., k_{n,i} \in Q_{\alpha, i}$ such that $\sum_{j=1}^n x'_j \otimes k_{j,i} = 0$ in $M \otimes_R Q_{\alpha, i}$ and $f_{\alpha, i}(k_{j,i}) = q_{j,i}$ for any $j = 1, ..., n$. We have the following commutative diagram
By construction, we obtain
\[ y = \sum_{j=1}^{n} x'_j \otimes (q_j,i)_{i\in I} = (M \otimes \prod_{i\in I} f_{\alpha_i}^i)(\sum_{j=1}^{n} x'_j \otimes (k_j,i)_{i\in I}) \]
and
\[ \phi_{M,Q}(\sum_{j=1}^{n} x'_j \otimes (k_j,i)_{i\in I}) = (\sum_{j=1}^{n} x'_j \otimes k_{j,i})_{i\in I} = 0. \]
Since \( \phi_{M,Q} \) is an \( S \)-monomorphism, there exists \( s \in S \) such that \( s(\sum_{j=1}^{n} x'_j \otimes (k_j,i)_{i\in I}) = 0 \). Then \( sy = s(\sum_{j=1}^{n} x'_j \otimes (q_j,i)_{i\in I}) = s(M \otimes \prod_{i\in I} f_{\alpha_i}^i)(\sum_{j=1}^{n} x'_j \otimes (k_j,i)_{i\in I}) = (M \otimes \prod_{i\in I} f_{\alpha_i}^i)(s(\sum_{j=1}^{n} x'_j \otimes (k_j,i)_{i\in I})) = 0 \). Then \( \text{Ker}\phi_M \) is \( S \)-torsion and thus \( \phi_M \) is an \( S \)-monomorphism.

\((2) + (3) + (4) \Rightarrow (5) \) is obvious. \( \square \)

**Corollary 3.3.** Let \( M \) be an \( R \)-module, then \( M \in S{\mathcal{F}}\text{-ML} \) if and only if \( M \in S{\mathcal{P}}\text{-ML} \) if and only if \( M \in S\text{-}\{R\}\text{-ML} \).

**Proof.** Since any projective module is pure submodule of direct product of \( R \) and any flat module is direct limit of projective modules, the definable closures of \( \{R\} \) and of all projectives and all are flats are the same thing. Thus the consequence holds from Proposition 3.2. \( \square \)

For a class \( \mathcal{T} \) of \( R \)-modules, we denote \( \mathcal{T}^\top \) the class of all \( R \)-modules \( M \) such that \( \text{Tor}_1^R(T,M) = 0 \) for any \( T \in \mathcal{T} \). Let \( M \) be an \( R \)-module, \( \mathcal{C} \) a class of \( R \)-modules, \( \tau \) an ordinal. An increasing chain \( (M_\alpha|\alpha \leq \tau) \) of submodules of \( M \) is a \( \mathcal{C} \)-filtration of \( M \) provided that \( M_0 = 0 \), \( M_\tau = M \), \( M_\alpha = \bigcup_{\beta < \alpha} M_\beta \) for a limit ordinal \( \alpha \) and \( M_{\beta+1}/M_\beta \in \mathcal{C} \) for any \( \beta < \tau \). The following result which is similar to [10, Proposition 1.9] is crucial to the study of \( S \)-Mittag-Leffler modules with respect to all flat modules.

**Proposition 3.4.** Let \( \mathcal{T} \) be a class of \( R \)-modules that are \( S \)-Mittag-Leffler with respect to \( \mathcal{Q} \subseteq \mathcal{T}^\top \), then any module isomorphic to a direct summand of a \( \mathcal{T} \cup \mathcal{P} \)-filtered module is an \( S \)-Mittag-Leffler module with respect to \( \mathcal{Q} \).
Proof. We imitate the proof given by [10, Proposition 1.9] with some changes. We can also assume that $\mathcal{T}$ contains $\mathcal{P}$. Indeed, all projective modules are in $S:\mathcal{Q}$-ML and $(\mathcal{T} \cup \mathcal{P})^{\top} = \mathcal{T}^{\top}$.

Let $M$ be a $\mathcal{T}$-filtered $R$-module, $\tau$ be an ordinal such that there exists an $\mathcal{T}$-filtration $\{M_\alpha | \alpha \leq \tau\}$. We prove by induction that $M_\alpha$ are all in $S:\mathcal{Q}$-ML.

Firstly, let $\alpha$ be a successor ordinal, $\{Q_i | i \in I\}$ be a set of modules in $\mathcal{Q}$. Consider the exact sequence $0 \to M_{\alpha-1} \to M_\alpha \to M_\alpha/M_{\alpha-1} \to 0$, we obtain the following commutative diagram of exact sequences.

\[
\begin{array}{ccc}
M_{\alpha-1} \otimes_R \prod_{i \in I} Q_i & \longrightarrow & M_\alpha \otimes_R \prod_{i \in I} Q_i \\
\phi_{M_{\alpha-1}} & & \phi_{M_\alpha} \\
0 & \longrightarrow & \prod_{i \in I} (M_{\alpha-1} \otimes_R Q_i)
\end{array}
\]

Note that $f$ is an monomorphism as $M_\alpha/M_{\alpha-1} \in \mathcal{T} \subseteq \mathcal{Q}^{\top}$. Since $\phi_{M_{\alpha-1}}$ and $\phi_{M_\alpha/M_{\alpha-1}}$ are $S$-monomorphisms, we have $\phi_{M_\alpha}$ is an $S$-monomorphism.

Secondly, let $\alpha$ be a limit ordinal such that $M_\beta$ be in $S:\mathcal{Q}$-ML for any $\beta < \alpha$, we show that $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ is $S$-Mittag-Leffler with respect to $\mathcal{Q}$. Let $\{Q_i | i \in I\}$ be a set of modules in $\mathcal{Q}$ and $x \in \text{Ker}(\phi_{M_\alpha})$, then there exists $\beta < \alpha$ and $y \in M_\beta \otimes_R \prod_{i \in I} Q_i$ such that $x = (\varepsilon_\beta \otimes_R \prod_{i \in I} Q_i)(y)$, where $\varepsilon_\beta : M_\beta \to M_\alpha$ is the natural monomorphism. Consider the following commutative diagram,

\[
\begin{array}{ccc}
M_\beta \otimes_R \prod_{i \in I} Q_i & \longrightarrow & M_\alpha \otimes_R \prod_{i \in I} Q_i \\
\phi_{M_\beta} & & \phi_{M_\alpha} \\
\prod_{i \in I} (M_\beta \otimes_R Q_i) & \longrightarrow & \prod_{i \in I} (M_\alpha \otimes_R Q_i)
\end{array}
\]

Note that $\phi_{M_\beta}$ is an $S$-monomorphism and $\prod_{i \in I} (\varepsilon_\beta \otimes_R Q_i)$ is a monomorphism since $\text{Tor}_1^R(M_\alpha/M_\beta, Q_i) = 0$. Thus $y \in \text{Ker}(\phi_{M_\beta} \circ \prod_{i \in I} (\varepsilon_\beta \otimes_R Q_i))$ is $S$-torsion, and there is some $s \in S$ such that $sy = 0$. Therefore, $sx = s(\varepsilon_\beta \otimes_R \prod_{i \in I} Q_i)(y) = (s\varepsilon_\beta(y)) \otimes_R \prod_{i \in I} Q_i = (\varepsilon_\beta(sy)) \otimes_R \prod_{i \in I} Q_i = 0$. Consequently, $\phi_{M_\alpha}$ is an $S$-monomorphism.

Corollary 3.5. The following statements hold.

1. The class $S:\mathcal{F}$-ML is closed under $S:\mathcal{F}$-ML-filtration.
2. Let $(M_\alpha | \alpha \leq \tau)$ be a chain with each $M_\alpha \in S:\mathcal{F}$-ML. Then $M = \bigcup_{\beta < \alpha} M_\beta$ is also in $S:\mathcal{F}$-ML.

Proof. (1) Since any flat module is in $\mathcal{T}^{\top}$, (1) follows from Proposition 3.4 immediately by putting $\mathcal{Q} = \mathcal{F}$ and $\mathcal{T} = S:\mathcal{F}$-ML.
(2) If $\alpha$ is a successor ordinal, then $M = M_{\alpha-1}$. If $\alpha$ is a limit ordinal, then the proof of Proposition 3.4 on the limit case remains valid. □

4. Characterizing $S$-Coherent rings using $S$-$F$-ML

In this section, we utilize the class $S$-$F$-ML of $S$-Mittag-Leffler modules with respect to all flat modules to characterize $S$-Coherent rings, and then we give an $S$-version of Chase theorem using $S$-flat modules as an application. Firstly, we build a connection among finitely generated modules, $S$-Mittag-Leffler modules with respect to all flat modules and $S$-finitely presented modules.

**Lemma 4.1.** Let $\mathcal{F}$ be a set of flat modules containing $R$, $\phi_{M,\mathcal{F}}$ the natural homomorphism.

1. If $\phi_{M,\mathcal{F}}$ is an $S$-epimorphism, then $M$ is $S$-finite.
2. Moreover, if $M$ is finitely generated and $\phi_{M,\mathcal{F}}$ is an $S$-isomorphism, then $M$ is $S$-finitely presented.

**Proof.** (1) If $\phi_{M,\mathcal{F}}$ is an $S$-epimorphism, we consider the exact sequence

\[
\begin{array}{c}
M \otimes_R R^M \xrightarrow{\phi_{M,\mathcal{F}}} M^M \xrightarrow{\phi_{M,\mathcal{F}}} T \xrightarrow{0} \\
\text{Im} \phi_M
\end{array}
\]

with $T$ an $S$-torsion module. Let $x = (m)_{m \in M} \in M^M$, there is some $s \in S$ such that $sx \in \text{Im} \phi_{M,\mathcal{F}}$. Subsequently, for any $i \in M$, there exists $m_j \in M, r_{j,i} \in R$ such that $sx = \phi_{M,\mathcal{F}}(\sum_{j=1}^n m_j \otimes (r_{j,i})_{i \in M}) = (\sum_{j=1}^n m_j r_{j,i})_{i \in M}$. Set $K = \langle \{m_j | j = 1, \ldots, n\} \rangle$ be a finitely generated submodule of $M$. Now, for any $m \in M$, $sm = \sum_{j=1}^n m_j r_{j,m} \in K$, thus $sM \subseteq K \subseteq M$ and then $M$ is $S$-finite.

(2) Let $0 \to K \to F \to M \to 0$ be an exact sequence, where $F$ is a finitely generated free $R$-module. Consider the following commutative diagram of exact sequences,

\[
\begin{array}{c}
K \otimes_R \prod_{i \in I} R \xrightarrow{\phi_{K,\mathcal{F}}} F \otimes_R \prod_{i \in I} R \xrightarrow{\phi_{F,\mathcal{F}}} M \otimes_R \prod_{i \in I} R \xrightarrow{0} \\
0 \xrightarrow{\phi_{K,\mathcal{F}}} \prod_{i \in I} (K \otimes_R R) \xrightarrow{\phi_{F,\mathcal{F}}} \prod_{i \in I} (F \otimes_R R) \xrightarrow{\phi_{M,\mathcal{F}}} 0
\end{array}
\]

Since $\phi_{F,\mathcal{F}}$ is an isomorphism and $\phi_{M,\mathcal{F}}$ is an $S$-isomorphism, then $\phi_{K,\mathcal{F}}$ is an $S$-epimorphism, thus $K$ is $S$-finite by (1). □
Proposition 4.2. An $R$-module $M$ is $S$-finitely presented if and only if it is in $S$-$\mathcal{F}$-ML and finitely generated.

Proof. For the “only if” part, $M$ is finitely generated by Remark 2.4 (1). Now, we show $M$ is in $S$-$\mathcal{F}$-ML. Let $0 \to K \to F \to M \to 0$ be an exact sequence, where $K$ is $S$-finite and $F$ is a finitely generated free $R$-module. Consider the following commutative diagram of exact sequences,

$$
\begin{array}{cccc}
K \otimes_R \prod_{i \in I} R & \longrightarrow & F \otimes_R \prod_{i \in I} R & \longrightarrow & M \otimes_R \prod_{i \in I} R & \longrightarrow & 0 \\
\phi_{K,F} & \approx & \phi_{M,F} & \\
0 & \longrightarrow & \prod_{i \in I}(K \otimes_R R) & \longrightarrow & \prod_{i \in I}(F \otimes_R R) & \longrightarrow & \prod_{i \in I}(M \otimes_R R) & \longrightarrow & 0.
\end{array}
$$

To prove $\phi_{K,F}$ is an $S$-monomorphism, we only need to show $\phi_{K,F}$ is an $S$-epimorphism. Since $K$ is $S$-finite, there exists a finitely generated submodule $K'$ of $K$ such that $sK \subseteq K' \subseteq K$ for some $s \in S$. The natural commutative diagram

$$
\begin{array}{cccc}
K' \otimes_R \prod_{i \in I} F_i & \longrightarrow & K \otimes_R \prod_{i \in I} F_i \\
\phi_{K,F} & \approx & \phi_{K,F} & \\
\prod_{i \in I}(K' \otimes_R F_i) & \longrightarrow & \prod_{i \in I}(K \otimes_R F_i)
\end{array}
$$

induces the following commutative diagram by localizing at $S$,

$$
\begin{array}{cccc}
K'_S \otimes_R \prod_{i \in I} F_i & \longrightarrow & K_S \otimes_R \prod_{i \in I} F_i \\
\phi_{K,F} & \approx & \phi_{K,F} & \\
(\prod_{i \in I}(K' \otimes_R F_i))_S & \longrightarrow & (\prod_{i \in I}(K \otimes_R F_i))_S.
\end{array}
$$

For any $k_i \in K, q_i \in F_i (i \in I)$ and $s \in S$, we obtain $\frac{(k_i \otimes_R q_i)_i \in I}{st} = \frac{s(k_i \otimes_R q_i)_i \in I}{st} = \frac{(sk_i \otimes_R q_i)_i \in I}{st} \in (\prod_{i \in I}(K' \otimes_R F_i))_S$, thus $f$ is an epimorphism (moreover, since $f$ is a monomorphism, $f$ is an isomorphism). Thus $\phi_{K,F}^S$ is an epimorphism, and then $\phi_{K,F}$ is an $S$-epimorphism.

For the “if” part, let $M$ be a finitely generated $R$-module in $S$-$\mathcal{F}$-ML. Consider the exact sequence $0 \to K \to F \to M \to 0$ with $F$ a finitely generated free module, then there is a commutative diagram of exact sequences,

$$
\begin{array}{cccc}
K \otimes_R \prod_{i \in I} F_i & \longrightarrow & F \otimes_R \prod_{i \in I} F_i & \longrightarrow & M \otimes_R \prod_{i \in I} F_i & \longrightarrow & 0 \\
\phi_{K,F} & \approx & \phi_{F,F} & \approx & \phi_{M,F} & \\
0 & \longrightarrow & \prod_{i \in I}(K \otimes_R F_i) & \longrightarrow & \prod_{i \in I}(F \otimes_R F_i) & \longrightarrow & \prod_{i \in I}(M \otimes_R F_i) & \longrightarrow & 0.
\end{array}
$$
Note that $\phi_{M,\mathcal{F}}$ is an $S$-monomorphism and $\phi_{F,\mathcal{F}}$ is an isomorphism, thus $\phi_{K,\mathcal{F}}$ is an $S$-epimorphism. Then $K$ is $S$-finite by Lemma 4.1(1) and $M$ is $S$-finitely presented. \hfill \Box

In [11, Corollary 4.3], Izurdiaga proved a ring $R$ is coherent if and only if the class of Mittag-Leffler $R$-modules with respect to all flat modules is closed under submodules. Similarly, we give an $S$-version of Izurdiaga’s result on $S$-coherent rings.

**Theorem 4.3.** The following assertions are equivalent for a ring $R$:

1. $R$ is an $S$-coherent ring;
2. every ideal is in $S\mathcal{F}$-ML;
3. every finitely generated ideal is in $S\mathcal{F}$-ML;
4. every submodule of projective modules is in $S\mathcal{F}$-ML;
5. the class $S\mathcal{F}$-ML is closed under finitely generated submodules;
6. the class $S\mathcal{F}$-ML is closed under submodules.

**Proof.**

(2) $\Rightarrow$ (3), (6) $\Rightarrow$ (4), (4) $\Rightarrow$ (2), (6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (3) are obvious.  

(3) $\Leftrightarrow$ (1): By Proposition 4.2.

(4) $\Rightarrow$ (6): Let $M$ be an $R$-module in $S\mathcal{F}$-ML, $K$ a submodule of $M$ and consider the following pull-back diagram,

\[
\begin{array}{cccccc}
0 & & 0 & & & \\
\downarrow & & \downarrow & & & \\
K' & \equiv & K' & & & \\
\downarrow & & \downarrow & & & \\
0 & \longrightarrow K & \longrightarrow & Q & \longrightarrow & P & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & & \\
\end{array}
\]

with $P$ projective. By (4), $K'$ is in $S\mathcal{F}$-ML and thus $Q$ is in $S\mathcal{F}$-ML. As the middle row split, $K$ is also in $S\mathcal{F}$-ML.

(2) $\Rightarrow$ (4): As $S\mathcal{F}$-ML is closed under direct summand, we just consider the free module case. Assume $L$ is a submodule of $R^{(\beta)}$, where $\beta$ is an ordinal, we prove (4) by induction on $\beta$.

If $\beta$ is a successor ordinal, consider the exact sequence

\[0 \to L \cap R^{(\beta-1)} \to L \cap R^{(\beta)} \to J \to 0\]
for some ideal \( J \) of \( R \). Since the ideal \( J \) is in \( S-\mathcal{F}-\text{ML} \) by (2) and \( L \cap R^{(\beta-1)} \) is in \( S-\mathcal{F}-\text{ML} \) by induction, thus \( L = L \cap R^{(\beta)} \) is in \( S-\mathcal{F}-\text{ML} \).

If \( \beta \) is a limit ordinal, then \( L = \bigcup_{\gamma<\beta} (L \cap R^{(\gamma)}) \in S-\mathcal{F}-\text{ML} \) by \( L \) is a direct unions of modules in \( S-\mathcal{F}-\text{ML} \).

(3) \( \Rightarrow \) (2): By [11, Corollary 3.6], we just need to prove any \((\aleph_0, S-\mathcal{F}-\text{ML})\)-free module (which is a direct union of modules in \( S-\mathcal{F}-\text{ML} \)) belong to \( S-\mathcal{F}-\text{ML} \). This can exactly be deduced from Corollary 3.5(2).

In [4, Theorem 2.1], Chase proved that a ring \( R \) is coherent if and only if any product of flat \( R \)-modules is flat. We extend it to the \( S \)-version and obtain the promised result.

**Theorem 4.4. (the \( S \)-version of Chase Theorem)** The following assertions are equivalent:

1. \( R \) is an \( S \)-coherent ring;
2. any product of flat \( R \)-modules is \( S \)-flat;
3. any product of projective \( R \)-modules is \( S \)-flat;
4. any product of \( R \) is \( S \)-flat;

**Proof.** (1) \( \Rightarrow \) (2): Let \( R \) be an \( S \)-coherent ring, \( F_i \) be flat \( R \)-modules, \( J \) a finitely generated ideal, we have \( J \in S-\mathcal{F}-\text{ML} \) by Theorem 4.3. Consequently, the natural homomorphism \( J \otimes_R (\prod_{i \in I} F_i) \to \prod_{i \in I} J \otimes_R F_i \cong \prod_{i \in I} (JF_i) = J(\prod_{i \in I} F_i) \) is an \( S \)-isomorphism, thus \( \prod_{i \in I} F_i \) is an \( S \)-flat module from Proposition 2.6.

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are obvious.

(4) \( \Rightarrow \) (1): Let \( J \) be a finitely generated ideal of \( R \), \( 0 \to J \to R \to R/J \to 0 \) an exact sequence. Consider the following commutative diagram,

\[
\begin{array}{ccccccccc}
\phi_{J,R} & & & & f & & & & \phi_{R,R} & & & & \phi_{M,R} & & & & \approx \\
J \otimes_R \prod_{i \in I} R & \longrightarrow & R \otimes_R \prod_{i \in I} R & \longrightarrow & R/J \otimes_R \prod_{i \in I} R & \longrightarrow & 0 \\
0 & \longrightarrow & \prod_{i \in I} (J \otimes_R R) & \longrightarrow & \prod_{i \in I} (R \otimes_R R) & \longrightarrow & \prod_{i \in I} (R/J \otimes_R R) & \longrightarrow & 0.
\end{array}
\]

Since \( \prod_{i \in I} R \) is an \( S \)-flat module, then \( f \) is an \( S \)-monomorphism. Thus \( \ker(f) = \ker(\phi_{J,R}) \) is \( S \)-torsion and then \( \phi_{J,R} \) is an \( S \)-monomorphism and thus \( J \in S-\mathcal{F}-\text{ML} \) from Corollary 3.3. Consequently, \( R \) is \( S \)-coherent from Theorem 4.3.

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