ABOUT NEUTRAL KAONS AND SIMILAR SYSTEMS;
FROM QUANTUM FIELD THEORY TO EFFECTIVE MASS MATRICES

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**Abstract:** Systems of neutral interacting mesons are investigated, concerning in particular the validity of their description by an effective hamiltonian. First, I study its connection to quantum field theory and show that the spectrum of such systems cannot be reduced in general to the one of a single constant effective mass matrix. Choosing nevertheless to work in this customary formalism, one then faces several ways to diagonalize a complex matrix, which lead to different eigenvalues and eigenvectors. Last, and it is the main subject of this work, because $K^0$ and its charge conjugate $\bar{K}^0$ are also connected, in quantum field theory, by hermitian conjugation, any constant effective mass matrix is defined, in this basis, up to arbitrary diagonal antisymmetric terms; I use this freedom to deform the mass matrix in various ways and study the consequences on its spectrum. Emphasis is put on the role of discrete symmetries throughout the paper. That the degeneracy of the eigenvalues of the full renormalized mass matrix can be a sufficient condition for the outcome of $CP$ violation is outlined in an appendix. In the whole work, the dual formalism of $\mid in \rangle$ and $\langle out \mid$ states and bi-orthogonal basis, suitable for non-normal matrices, is used.

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1 INTRODUCTION

We face two pictures of neutral kaons: flavour (strangeness) eigenstates \((K_0, \overline{K}^0)\) are concerned with electroweak – for example semi-leptonic – decays, and \((K_L, K_S)\) with decays where strong interactions are also involved (for example in the disintegrations into two or three pions).

On one side, the particle-antiparticle flavour eigenstates are constrained to have the same mass by CPT symmetry and, on the other side, the \(K_L - K_S\) mass difference stands among the quantities measured with the highest precision today. So, whether there could exist a twofold (at least) set of mass eigenstates for such systems is worth investigating; which one is detected depends on which type of decay is analyzed and which quantum numbers are observed.

Along this line, the question naturally arises of the uniqueness of the effective mass matrix \(M\) attached to such systems. This paper points at ambiguities in the definition of \(M\) and eventually uses them to “deform” it.

Since the system under scrutiny is unstable, its mass matrix must be taken non-hermitian, and it is necessary to use the formalism of \(|in>| and <out| states\): the latter only coincide when the mass matrix is normal (\(CP\) conserved). However, in all cases, the same \((K^0, \overline{K}^0)\) basis can be chosen to span both Hilbert spaces \(V_{in}\) and \(V_{out}\) which accordingly can be identified.

The second section is devoted to a survey of the uncertainties and ambiguities which occur when going from quantum field theory (QFT) to an effective hamiltonian formalism for the determination of the masses and mass eigenstates of a system of interacting mesons. It is shown that, in general, the the problem of finding its mass spectrum and eigenstates cannot be reduced to the diagonalization of a single constant mass matrix. Emphasis is put on the role of discrete symmetries.

Despite this restriction, the rest of the paper is set in the formalism of a single constant effective mass matrix.

The third section is devoted to the evaluation of \(|K^0><K^0| - |\overline{K}^0><\overline{K}^0|\) by using the connection that field theory establishes between the charge conjugate of a neutral meson and its hermitian conjugate; it is shown to vanish, with a restricted meaning to be precised in the text. This introduces an ambiguity in the definition of the effective mass matrix, to which arbitrary antisymmetric diagonal terms can be added (in the \((K^0, \overline{K}^0)\) basis).

The fourth section starts by general considerations about the diagonalization of complex matrices. In particular, we show the the diagonalization by a bi-unitary transformation (used for example for the mass matrices of fermions), in which the \(|in>| eigenstates form an orthogonal basis and the <out| eigenstates another such basis, and the other procedure where |in> and <out| eigenstates form a bi-orthogonal basis are not equivalent (i.e. they lead not only to different eigenstates but to different mass eigenvalues).

Despite the lack of a definitive argument in favor of it, we make the choice to work with a bi-orthogonal basis of eigenstates, like was done in [3].

Then, we go to the neutral kaon system and use the most general parametrization of the effective mass matrix \(M\) for \(K_L\) and \(K_S\) in the \((K^0, \overline{K}^0)\) basis given in [3], which entails that the eigenstates form a bi-orthogonal basis. We first deform \(M\) into its CPT invariant form (which in this framework is always possible). We show that the sole condition of \(T\) invariance (for the starting mass matrix) is enough for the eigenstates \(K_1\) and \(K_2\) of the deformed matrix to be the \(CP\) eigenstates \((K^0 + \overline{K}^0)\) and \((K^0 - \overline{K}^0)\) (the relation between the two mass spectra is explicitly calculated): if \(T\) is broken, then the mass eigenstates can never be “deformed” into \(CP\) eigenstates; however, that the eigenstates can decay into two and three pions does not exclude that they are of a type which can be “deformed” back to \((K^0 + \overline{K}^0)\) and \((K^0 - \overline{K}^0)\). Accordingly, and since this definition appears to be free of
ambiguity, we argue in favor of defining the breaking of CP invariance (in mixing) through the one of T.

Next, we study the most general type of allowed deformation of M, irrespectively of CPT conservation. The condition that the new eigenstates match CP eigenstates is shown to bring back to the previous case of deformation to a CPT invariant mass matrix.

The question whether they can be flavour eigenstates is then raised. It is shown that this indeed can happen, but then the mass of the $K^0$ is no longer equal to that of its antiparticle, as could be expected from the allowed violation of CPT.

In the fifth section, we perform a similar analysis starting from the (diagonal) mass matrix the $(K_L, K_S)$ basis. We start by re-expressing the “vanishing” combination $|K^0\rangle < K^0| - |\bar{K}^0\rangle < \bar{K}^0|$ in terms of $|K_L^{in}\rangle, K_S^{in}\rangle, <K_L^{out}\rangle$ and $<K_S^{out}|$; various deformations are then studied.

We first investigate whether the mass spectra of the starting and deformed mass matrices can match. It is proved that, if one assumes CPT invariance at the start, masses and eigenstates match at the same time. However, if CPT invariance is not assumed, the masses can match but the eigenstates are different.

Then, we ask whether the eigenstates of the deformed mass matrix can be degenerate in mass. We show that, to any set of split $K_L$ and $K_S$ mesons can be associated another bi-orthonormal basis of degenerate eigenstates, but which belong to two different well defined “vanishing” deformations of the starting Hamiltonian. Finally we particularize the study to the case when CPT invariance is assumed at the start.

After a brief conclusion emphasizing recent attempts to treat neutral mesons in the framework of QFT, two appendices come back to the following points: 1. the relevance of mass matrices with degenerate eigenvalues in the treatment of neutral kaon-like mesons, in particular to generate mass split CP violating eigenstates; this appendix is an addendum to section 2; 2. neutral mesons are usually treated with an over-complete basis of states; using the results of section three, some simple relations linking $|K_L^{in}\rangle, |K_S^{in}\rangle, |K_L^{out}\rangle, |K_S^{out}\rangle$ and their hermitian conjugates are established.

2 FROM QUANTUM FIELD THEORY TO EFFECTIVE HAMILTONIAN

It cannot be questioned that, like other particles, all problems pertaining to neutral kaons should be tackled within the framework of quantum field theory (QFT). Nevertheless, in all textbooks and most papers, including [3], the treatment is performed in the formalism of quantum mechanics, working with an effective Hamiltonian or even simply an effective mass matrix with dimension $[m]$.

In QFT, the only coherent way to define masses of particles is by the poles of their full propagator [4]; the kinetic terms play naturally a major role in this definition. The Lagrangian having no special reason to be diagonal in a given basis, the propagator of a system of particles in interaction is, in general, a non diagonal matrix. For the sake of convenience, one works in a basis in which the (bare) kinetic terms are diagonal. Let us call $L$ the inverse full propagator for $n$ scalar fields; it is a $n \times n$ matrix that we write, in momentum space

$$L(p^2) = p^2 \mathbb{I} - M^{(2)}(p^2),$$

where $\mathbb{I}$, which factorizes the kinetic terms, is the $n \times n$ unit matrix, and the mass matrix $M^{(2)}(p^2)$ is the sum of a bare mass matrix (which does not depend on $p^2$) and terms coming from higher order
contributions (renormalized self-energies), which introduce dependences on the momenta; in particular, higher order corrections are likely to produce non-diagonal kinetic terms, which we include in $M^{(2)}(p^2)$. The superscript in $M^{(2)}$ has been put into parenthesis to avoid confusion with an ordinary power: it indicates that all elements of $M^{(2)}$ have dimension $[\text{mass}]^2$.

$L(p^2)$ is diagonalized by an $n \times n$ matrix $V(p^2)$ according to

$$V^{-1}(p^2)L(p^2)V(p^2) = p^2\mathbb{I} - M_d^{(2)}(p^2),$$

(2)

where $M_d^{(2)}(p^2)$ is diagonal, $M_d^{(2)}(p^2) = \text{diag}(m_1^{(2)}(p^2), m_2^{(2)}(p^2), \ldots, m_n^{(2)}(p^2))$. The $m_j^{(2)}(p^2)$ have dimension $[\text{mass}]^2$ and the “(“) are optional in their superscripts.

The $n$ functions $m_1^{(2)}, \ldots, m_n^{(2)}(p^2)$ are obtained by solving the characteristic equation $\det(M^{(2)}(p^2) - m^{(2)}) = 0$; the $n$ corresponding eigenvectors $\psi_j(p^2)$ are then determined by

$$M^{(2)}(p^2)\psi_j(p^2) = m_j^{(2)}(p^2)\psi_j(p^2).$$

(3)

The $(\text{mass})^2$ of the $n$ scalar fields are defined as the values of $p^2$ satisfying the $n$ self-consistent equations $(p^2 - m_1^{(2)}(p^2) = 0), \ldots, (p^2 - m_n^{(2)}(p^2) = 0)$; we call then $\mu_1^2, \mu_2^2, \ldots, \mu_n^2$:

$$\mu_i^2 = m_i^{(2)}(p^2 = \mu_i^2);$$

(4)

they satisfy accordingly the equation

$$\det \left( p^2\mathbb{I} - V^{-1}(p^2)M^{(2)}(p^2)V(p^2) \right) = \det(p^2\mathbb{I} - M^{(2)}(p^2)) = \prod_j \left( p^2 - m_j^{(2)}(p^2) \right) = 0;$$

(5)

since the $p^2$ term in $L$ is proportional to the unit matrix, the same matrix $V(p^2)$ diagonalizes both $L(p^2)$ and $M^{(2)}(p^2)$.

The (independent of $p^2$) physical mass eigenstates $\varphi_i$ of $L(p^2)$ satisfy the equation

$$M^{(2)}(\mu_i^2)\varphi_i = \mu_i^2\varphi_i.$$

(6)

Matching (6) and (3) entails

$$\varphi_i = \psi_i(p^2 = \mu_i^2).$$

(7)

However, once $p^2$ is fixed to a certain value $\mu_o^2$, each $M^{(2)}(\mu_o^2)$ is a constant $n \times n$ matrix which has itself in general $n$ distinct eigenvalues and eigenvectors $\varphi_j$ with $j \neq i$, $\varphi_j \equiv \psi_j(\mu_j^2)$ corresponds to the eigenvalue $\mu_j^2 = m_j^{(2)}(\mu_j^2)$, $\varphi_i \equiv \psi_i(\mu_i^2)$ corresponds to the eigenvalue $\mu_i^2 = m_i^{(2)}(\mu_i^2)$.\footnote{unless all eigenvalues of $M^{(2)}(\mu_o^2)$ are degenerate to $\mu_o^2$.}

$\varphi_i$ do not determine uniquely the physical eigenvectors: $\varphi_i$ is only one among the $n$ eigenvectors of $M^{(2)}(\mu_i^2)$. There is in particular no reason why the $(n - 1)$ other eigenvalues and $(n - 1)$ other eigenvectors have anything to do with the system of particles under concern.

**Can an effective constant mass matrix be naturally and uniquely introduced?**

Let us consider the matrix $M_d(p^2)$, the square root of $M_d^{(2)}(p^2)$, and

$$M_f(p^2) = V(p^2)M_d(p^2)V^{-1}(p^2);$$

(8)
$M_f(p^2)$ is also the square root of $M^{(2)}(p^2)$ since

$$M^{(2)}(p^2) = V(p^2)M_d^{(2)}(p^2)V^{-1}(p^2) = V(p^2)M_d(p^2)V^{-1}(p^2)V(p^2)M_d(p^2)V^{-1}(p^2) = (M_f(p^2))^2. \quad \text{(9)}$$

The diagonalization equation for $M_f$

$$M_f(p^2)\chi_j(p^2) = y_j\chi_j(p^2), \quad \text{(10)}$$

entails, left-multiplying by $M_f(p^2)$

$$(M_f(p^2))^2\chi_j(p^2) = (y_j)^2\chi_j(p^2), \quad \text{(11)}$$

which shows that the eigenvectors of $M_f(p^2)$ are the same as the ones of $M^{(2)}(p^2)$:

$$\chi_j(p^2) = \psi_j(p^2), \quad \text{(12)}$$

and that its eigenvalues (which of course turn out to be functions of $p^2$) are the square roots of the ones of $M^{(2)}(p^2)$:

$$(y_j(p^2))^2 = m_j^{(2)}(p^2). \quad \text{(13)}$$

Like for $M^{(2)}$, once a scale $p^2$ has been fixed to $\mu_j^2$, the resulting constant matrix $M_f(\mu_j^2)$ has $n$ eigenvalues $\mu_{j1} \equiv \sqrt{m_1^{(2)}(\mu_j^2)}, \ldots, \mu_{jn} = \sqrt{m_n^{(2)}(\mu_j^2)}$ and $n$ corresponding eigenvectors $\varphi_{j1} = \psi_1(\mu_j^2), \ldots, \varphi_{jn} = \psi_n(\mu_j^2)$ (the same as $M^{(2)}(\mu_j^2)$).

The physical mass $\mu_j$ is only one among these eigenvalues, and the corresponding physical particle is one among these eigenvectors: at the physical poles $p^2 = \mu_j^2$, and naming like before

$$\varphi_j = \chi_j(p^2 = \mu_j^2) \equiv \psi_j(p^2 = \mu_j^2), \quad \text{(14)}$$

the equation

$$M_f(\mu_j^2)\varphi_j = y_j(\mu_j^2)\varphi_j, \quad \text{(15)}$$

entails

$$(M_f(\mu_j^2))^2\varphi_j = M^{(2)}(\mu_j^2)\varphi_j = (y_j(\mu_j^2))^2\varphi_j, \quad \text{(16)}$$

and, so,

$$\mu_j^2 = (y_j(\mu_j^2))^2. \quad \text{(17)}$$

It is tempting to introduce $M_f$ as an effective mass matrix and it is the only one proposed in the literature [5]. This proposition is comforted by [17] and the result stressed in [4], that the masses of particles can be consistently taken as the square roots of the poles of the full propagator $^2$.

But, as we have seen, for a system of $n$ interacting particles, there is not a single $M_f$ at work but $n$ of them, the $M_f(p^2)$’s evaluated at the $n$ poles of the full propagator $p^2 = \mu_j^2$ solving the $n$ self-consistent equations [4]. So, there are $n$ possible choices for $M_{eff}$ (see footnote [11]) and there cannot be a one to one correspondence between QFT and a constant single effective mass matrix.

It is often argued that, when dealing with nearly degenerate systems like the neutral kaon system, everything should be regular and smooth. This is forgetting that, precisely, in the case of nearly degenerate eigenvalues, a small variation of the mass matrix can have large consequences on its

\footnote{and this includes the case when these poles are complex, i.e. the case of unstable particles; the equation $y_j = \sqrt{\mu_j^{(2)}}$ for complex $\mu_j^{(2)}$ determine not only the masses (real parts) but also the widths (imaginary parts) of the unstable particles from the values of the poles of their full propagators.}
spectrum. Very close matrices, with very close eigenvalues, can have very different eigenvectors. We give below a very elementary example of this.

Consider the simplified case of a system of two nearly degenerate interacting mesons. Two possible effective $2 \times 2$ matrices are at work, $M_{f1}$ and $M_{f2}$.

Suppose that, for example in the $(K^0, \bar{K}^0)$ basis

$$
M_{f1} = \begin{pmatrix}
\rho - \epsilon & 0 \\
0 & \rho + \epsilon \\
\end{pmatrix}, \quad M_{f2} = \begin{pmatrix}
\rho & \epsilon_1 \\
\epsilon_2 & \rho \\
\end{pmatrix}
$$

(18)

with $\rho, \epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}$ and $\epsilon, \epsilon_1, \epsilon_2 \ll \rho$. The eigenvalues of $M_{f1}$ are $\rho \pm \epsilon$ and its eigenvectors are $K^0$ and $\bar{K}^0$. The eigenvalues of $M_{f2}$ are $\rho \pm \sqrt{\epsilon_1 \epsilon_2}$ and the ratio of the components of its eigenvectors are $\pm \sqrt{\epsilon_1 / \epsilon_2}$, which can be very different from the 0 and 1 which occur for the eigenvectors of $M_{f1}$.

We conclude accordingly, that for $n$ nearly degenerate particles, even if the $n$ possible effective matrices $M_{f1, \ldots, fm}$ are very close, the $(n - 1)$ spurious eigenstates of each of them have in general nothing to do with the true physical eigenstates. Only one eigenvector per effective $M_f$ matrix is a faithful description of one of the physical mass eigenstates.

Is it legitimate to consider an effective single $n \times n$ mass matrix the spectrum and eigenstates of which involve enough measurable parameters to match all possible masses and eigenstates of a system of interacting particles together with their transformation properties by discrete symmetries?

The following discussion concerns more specifically the approach of [3] which starts from a single constant effective mass matrix for the system of neutral kaons, which includes enough (measurable) parameters to fit all possibly detected mass eigenvalues and eigenvectors.

For the sake of simplicity, we proceed with a system of only two neutral interacting mesons.

From above, we have learned that two constant $2 \times 2$ matrices are involved: $M_{f1} \equiv M_f(\mu_1^2)$ and $M_{f2} \equiv M_f(\mu_2^2)$. Each of them has in general two eigenvalues and two associated eigenstates; one pair is physically relevant, and the other pair is “spurious”. So, $\mu_1$ and $\varphi_1$ are one among the two eigenvalues ($\mu_{11}, \mu_{12}$) and one among the two eigenvectors ($\varphi_{11}, \varphi_{12}$) of $M_{f1}$, $\mu_2$ and $\varphi_2$ are one among the eigenvalues ($\mu_{21}, \mu_{22}$) and one among the eigenvectors ($\varphi_{21}, \varphi_{22}$) of $M_{f2}$.

Discrete symmetries set constraints on the QFT Lagrangian, and, in particular, on the full renormalized mass matrix $M^{(2)}(p^2)$; they have to be satisfied at all values of $p^2$, so in particular by the two matrices $M^{(2)}(p^2 = \mu_1^2)$ and $M^{(2)}(p^2 = \mu_2^2)$, or their “square roots” $M_{f1}$ and $M_{f2}$. Accordingly, relations are expected between the eigenvalues and/or the eigenstates of $M_{f1}$, or between the ones of $M_{f2}$; they are likely to share many similarities since they originate from general constraints on the $q^2$-dependent mass matrix; but each of these relations connect one physical to one spurious quantity, corresponding to a given value of $p^2$; no special constraint is a priori expected between two physical parameters, since they are linked to two different values of $p^2$ and to two different constant $M_f$ mass matrices.

First, we make some remarks about $CP$ violation. Since both mass eigenstates are detected to violate $CP$, $M_{f1}$ and $M_{f2}$ are expected to be non-normal, and it should also be a property of $M_{eff}$ if we introduce such a single constant effective mass matrix.

Suppose however that both $M_{f1}$ and $M_{f2}$ have a degenerate spectrum of eigenvalues, still with $\mu_1^2 \neq \mu_2^2$, and that they have the simplest possible form, proportional to the unit matrix

$$
M_{f1} = \begin{pmatrix}
\mu_1 & 0 \\
0 & \mu_1 \\
\end{pmatrix}, \quad M_{f2} = \begin{pmatrix}
\mu_2 & 0 \\
0 & \mu_2 \\
\end{pmatrix}
$$

(19)
any linear combination of \( K^0 \) and \( \overline{K}^0 \), in particular \( CP \) violating, is a mass eigenstate for each of them; still, both are normal matrices. Furthermore, nothing prevents \( M^{(2)}(p^2) \) itself to be normal since it is only required to have degenerate eigenvalues at two values of \( p^2 \).

At the same time, a description of two split physical \( CP \) violating eigenstates by a single constant effective mass matrix demands that it be non normal. The mismatch between the two points of view is patent.

We proceed with some remarks concerning \( CPT \) and assume that it is a true symmetry of nature.

Suppose that we observe two mass split neutral mesons \( K_1 \) and \( K_2 \). Let \( K_1 \) and \( S_1 \) be the non-degenerate eigenstates of \( M_f(\mu_1^2) \) which respects the constraints of \( CPT \) invariance, and that the same occurs for \( K_2 \), \( S_2 \) and \( M_f(\mu_2^2) \); \( S_1 \) and \( S_2 \) are the “spurious” eigenstates, respectively of \( M_f^1 \) and \( M_f^2 \). Suppose furthermore that one discovers that \( K_2 = K_1 \); the natural conclusion is that \( CPT \) is violated in the \( K_1 - K_2 \) system, and any description by a single constant effective mass matrix should account for that (see for example \( \text{[3]} \)); it is however unclear whether \( M_f^1 \) and \( M_f^2 \) should have to respect the same criteria since \( S_1 \) and \( S_2 \) are devoid of physical significance. So: the problem of finding the mass spectrum and eigenstates of a system of interacting scalars cannot in general be reduced to the diagonalization of a single constant mass matrix.

It is unfortunately illusory to believe that difficulties attached to the effective Hamiltonian formalism can be avoided by working directly in QFT; indeed, computations in quantum field theory are performed with fields (quarks) which are not the particles (kaons); quark diagrams can be evaluated, but cannot be used to determine the renormalized self-energies of neutral kaons without interpolation and limiting procedures like PCAC, which introduce themselves again other types of uncertainties \(^4\).

We shall nevertheless work in the formalism of a single constant effective mass matrix, to show that there yet exist other ambiguities attached to it; they are the main subject of this work.

### 3 HILBERT SPACE AND OPERATORS

#### 3.1 Generalities

In the case of unstable particles, non-hermitian effective mass matrices must be used (see for example \( \text{[2]} \) and \( \text{[1]} \)). The experimental fact that \( CP \) is broken entails furthermore that an effective mass matrix for neutral cannot be normal; its right and left eigenvectors accordingly differ. They are commonly called respectively “in” and “out” eigenstates. In connexion with this, the formalism of “bra” \(| \cdot \rangle \) and “ket” \(< \cdot | \) is also commonly used.

The goal of this section is to establish a connection between the bra’s and ket’s \(| K^0 \rangle, | \overline{K}^0 \rangle < < K^0 |, < \overline{K}^0 | \) used for neutral kaons in the effective Hamiltonian formalism and the field operators \( \phi_{K^0}, \phi_{\overline{K}^0}, \phi_{K^0}^\dagger, \phi_{\overline{K}^0}^\dagger \) for these same kaons which appear in a Lagrangian of quantum field theory.

In a QFT Hamiltonian \( \mathcal{H} \), an (hermitian) mass term for degenerate (stable) neutral kaons would write (up to an eventual constant coefficient)

\[
\mathcal{H}_m(x) = m^2 \left( \phi_{K^0}^\dagger(x)\phi_{K^0}(x) + \phi_{\overline{K}^0}^\dagger(x)\phi_{\overline{K}^0}(x) \right).
\]

The operator \( \phi_{K^0}(x) \) destroys a \( K^0 \) at space-time point \( x \), \( \phi_{K^0}^\dagger(x) \) creates one, and \( \phi_{\overline{K}^0}(x) \) and \( \phi_{\overline{K}^0}^\dagger(x) \) do the same for \( \overline{K}^0 \) (see \( \text{[20]} \) below). These four operators have dimension \([\text{mass}]\).

\(^3\)The fact that the eigenvalues at given \( p^2 \)'s are degenerate is important since, otherwise, no normal matrix can have \( CP \) violating eigenstates. See also appendix \( \text{[A]} \).

\(^4\)See also the conclusion and references therein.
In an effective hamiltonian approach \( H(t) = \int d^3\vec{x} \mathcal{H}(x) \) and in the “bra” and “ket” formalism, an equivalent mass term writes

\[
H_m = m(\langle K^0 | + \langle \overline{K^0} |).
\]  

(21)

As the effective \( H_m \) has dimension \([\text{mass}]\), the dimension of the bra’s must be the inverse of the one of the ket’s. One can for example take both to be dimensionless.

The close similarity between the two expressions (20) and (21) suggests correspondences between bra’s, ket’s, and field operators. QFT sets hermiticity and commutation relations between the latter; we shall transpose them accordingly into equivalent relations between the bra’s and ket’s. They are at the origin of ambiguities in the definition of the effective mass matrix, which consequently also concern its eigenvectors and its spectrum.

### 3.2 Hermiticity and commutation relations

The neutral kaon \( |K^0> \) and its charge conjugate partner \( |\overline{K^0}> \equiv C |K^0> \) are defined, within the framework of the flavour SU(3) symmetry, as pseudoscalar states with quantum numbers \((6 + i7)\) and \((6 - i7)\), with strangeness respectively \((+1)\) and \((-1)\).

The states \(|6>\) and \(|7>\) are considered to form an orthonormal basis of a 2-dimensional, a priori complex, Hilbert space \(\mathcal{V}\); the latter is endowed with a natural binary product \(<|>\) satisfying

\[
< 6 | 6 > = 1 = < 7 | 7 >, \\
< 6 | 7 > = 0 = < 7 | 6 >,
\]

(22)

which entail

\[
< K^0 | \overline{K^0} > = 0 = < \overline{K^0} | K^0 >, \\
< K^0 | K^0 > = 1 = < \overline{K^0} | \overline{K^0} >.
\]

(23)

The metric matrix \([6] \Delta\) has been chosen equal to 1; hence, the basis \(\mathcal{B} \equiv \{ |6>, |7>\}\) and the reciprocal basis \(\mathcal{B} = \mathcal{B} \Delta^{-1}\) are identical: the Hilbert space \(\mathcal{V}\) and its dual are isomorphic.

Since, in particular, the binary product has to be hermitian, the dual of a vector \(|A>\) of \(\mathcal{V}\) can be noted \(<A|\) and is the hermitian conjugate of \(|A>\)

\[
<A| = |A>^\dagger;
\]

(24)

in particular for the basis vectors 5

\[
< K^0 | = | K^0 >^\dagger, \quad < \overline{K^0} | = | \overline{K^0} >^\dagger.
\]

(25)

At the level of operators, the ‘bra’s’ \(<K^0|\) and \(<\overline{K^0}|\) belong to the dual of the Hilbert space spanned by the ‘ket’s’ \(|K^0>\) and \(|\overline{K^0}>\) (i.e. the space of linear functionals acting on the ket’s); the ket’s can also be considered to belong to the dual of the dual, i.e. the set of functionals acting on the bras; in this respect, they are themselves operators. According to \(23\), \(<\overline{K^0}|\) annihilates the ket \(|K^0>\) to yield 0; accordingly, we like to put it in a one-to-one correspondence with the operator \(\Phi_{K^0}\) which,

5Remark: particles are defined in quantum mechanics as eigenstates of mass and spin; they are the ones which should be used to build the Fock space, and it was shown in \([7]\) that, in the case of mixing when mass eigenstates become different from flavour eigenstates, the latter are not adequate to define the Fock space. However, in the case of near degeneracy, which is the case for neutral kaons, the problem becomes only conceptual and has no sizable effect.
too, destroys a $|K^0>$ state; a similar correspondence is natural to establish between $<K^0|$ and $\Phi_{K^0}$, which both destroy a $|\bar{K}^0>$: allowing for arbitrary phases $\delta$ and $\zeta$.  

$$<K^0| \sim \frac{e^{i\delta}}{\varsigma} \Phi_{K^0}, \quad <K^0| \sim \frac{e^{i\zeta}}{\varsigma} \Phi_{K^0}.$$  

(26)

where $\varsigma$ is an arbitrary (real) mass scale.

Because of (25), through hermitian conjugation, one also gets a one-to-one correspondence between $|K^0>$ and $(\Phi_{K^0})^\dagger$, and between $|\bar{K}^0>$ and $(\Phi_{K^0})^\dagger$:

$$|K^0> \sim \frac{e^{-i\delta}}{\varsigma} (\Phi_{K^0})^\dagger, \quad |K^0> \sim \frac{e^{-i\zeta}}{\varsigma} (\Phi_{K^0})^\dagger.$$  

(27)

The relation $<K^0|K^0> = 1$ of (23) is now in one to one correspondence with $(\phi_{K^0}/\varsigma)(\phi_{K^0}/\varsigma)^\dagger = 1$; acting with this product of operators for example on the vacuum shows that the correspondence that we have established is consistent and legitimate (the same of course can be done with $\bar{K}^0$).

Now, the expansions of $\Phi_{K^0}$ and $\Phi_{\bar{K}^0}$ in terms of creation $(a^\dagger, b^\dagger)$ and annihilation $(a, b)$ operators satisfying the usual commutation relations

$$[a(\vec{k}), a^\dagger(\vec{l})] = [b(\vec{k}), b^\dagger(\vec{l})] = (2\pi)^3 2k_0 \delta^3(\vec{k} - \vec{l})$$  

(28)

write ($\gamma$ is an arbitrary phase)

$$\Phi_{K^0}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} \left(a(\vec{k})e^{-i\vec{k}.x} + b^\dagger(\vec{k})e^{i\vec{k}.x}\right),$$

$$\Phi_{\bar{K}^0}(x) = e^{-i\gamma} \int \frac{d^3\vec{l}}{(2\pi)^3 2l_0} \left(b(\vec{l})e^{-i\vec{l}.x} + a^\dagger(\vec{l})e^{i\vec{l}.x}\right).$$  

(29)

According to (29) one can write

$$\Phi_{K^0} = e^{-i\gamma}(\Phi_{K^0})^\dagger,$$  

(30)

which entails, using (26), (27), $e^{-i\zeta} <K^0| = e^{-i\gamma}(e^{-i\delta} <K^0|^\dagger$, or

$$<K^0| = e^{i(\delta + \zeta - \gamma)}|\bar{K}^0>.$$  

(31)

By doing so, we have identified, as announced before, $\mathcal{V}$ and its dual.  

(30) reflects the property that, up to a phase, “creating” a $K^0$ is equivalent to “destroying” a $\bar{K}^0$, and vice-versa.

The next important step is that, using the commutation relations (28) of the $a, b, a^\dagger, b^\dagger$ operators, one finds from (29) that the following commutator vanishes

$$[\Phi_{K^0}, \Phi_{\bar{K}^0}] = 0,$$  

(32)

which, using (30), can be rewritten

$$e^{-i\gamma}(\Phi_{\bar{K}^0})^\dagger \Phi_{K^0} - e^{-i\gamma}(\Phi_{K^0})^\dagger \Phi_{K^0} = 0,$$  

(33)

or, using the correspondences (26), (27) and dropping the overall factor $e^{-i\gamma}$ (the phases $\delta$ and $\zeta$ drop out)

$$|K^0><K^0| - |\bar{K}^0><\bar{K}^0| \sim 0.$$  

(34)

6The remark at the end of this section concerning the mass term for degenerate $K^0$ and $\bar{K}^0$ shows that the normalization chosen here is adequate.
The sign "∼" in (34) must not be misinterpreted; it cannot mean a strict identity since this would
be in contradiction with the closure relation $| \bar{K}^0 > < K^0 | + | K^0 > < \bar{K}^0 | = 1$. We shall give
to it the restricted following meaning:Within the Hilbert space spanned by $K^0$ and $\bar{K}^0$, effective
Hamiltonians for the neutral kaon system can be freely deformed by adding terms proportional to
$| \bar{K}^0 > < K^0 | - | K^0 > < \bar{K}^0 |$.
Changing the effective mass matrix changes its eigenstates and, in particular, modifies their transfor-
mation properties by the discrete symmetries $C$ and $T$. Such deformations will be investigated in the
next sections.
In the common (and ambiguous) notation where $K^0$ and $\bar{K}^0$ stand for the corresponding field opera-
tors $\phi_{K^0}$ and $\phi_{\bar{K}^0}$, this is akin to saying that mass terms for neutral kaons are defined up to an arbitrary
factor $\rho (K^0 \bar{K}^0 - \bar{K}^0 K^0)^7 (\rho \in \mathbb{C}$ if the hermiticity of the Hamiltonian is no longer required, like in
the present work).
The ambiguity that has just been stressed adds up the other ones attached to the system of neutral
mesons:1- ambiguity linked to the effective hamiltonian formalism (section 2);1- ambiguities associ-
ated to the different possible procedures for diagonalizing a complex mass matrix, which we shall
overview in subsection 4.1.

4 THE MASS MATRIX IN THE $(K^0, \bar{K}^0)$ BASIS; VARIOUS DE-
FORMATIONS
In most decays, it turns out that the observed mass eigenstates are not $K^0$ and $\bar{K}^0$, but the so-called
$K_L, K_S$; they are unstable, with the consequence already mentioned that their effective mass matrix
$M$ has to be chosen non-hermitian [1], and furthermore non-normal if $CP$ is violated. The right
eigenstates (or "in") states and the left eigenstates (or "out") states – their eigenvalues being
the same – are defined by
$$M|\text{in}>=\lambda|\text{in}>, \quad <\text{out}|M=<\text{out}|\lambda.$$ (35)
Though different, the $|\text{in}>$ and $|\text{out}>$ states can be expanded on the same orthonormal
basis formed by $|K^0>$ and $|\bar{K}^0>$ (we already stressed that the two Hilbert spaces $V_{in}$ and its dual
$V_{out}$ can be identified).

4.1 Diagonalization of a complex matrix
In the present context, it is relevant to briefly recall the different ways of diagonalizing a complex
mass matrix, and to show that they correspond to different spectra and different eigenstates.

4.1.1 Bi-unitary transformation
Any complex mass matrix $M$ can be diagonalized by a bi-unitary transformation: $\forall M \in \mathbb{C}, \exists U$ and $\exists V, U^\dagger MV = D$, $UU^\dagger = 1 = VV^\dagger$.

Using the correspondences (26,27) and then (30), an effective mass term for (degenerate) $K^0$ and $\bar{K}^0$
mesons $m(|K^0 > < K^0 | + |\bar{K}^0 > < \bar{K}^0 |)$, is in one-to-one correspondence, in QFT, with
$m(\rho (K^0 \bar{K}^0 - \bar{K}^0 K^0)^7)$, in the usual notation this rewrites as the hermitian combination (the phase $\gamma$ is needed according to (30) to make the mass term hermitian)
m$ |(K^0 \bar{K}^0 + K^0 \bar{K}^0)$. In the usual notation this entails in particular that one must take $\gamma = m$.
$U$ and $V$ can be chosen to diagonalize respectively $MM^\dagger$ and $M^\dagger M$. 


$U$ and $V$ are not unique, and can in particular be adapted such that the elements of the diagonal matrix $D$ are all real. Indeed: the polar decomposition theorem \cite{6} states that any complex matrix $M$ can be written as the product $M = H \Lambda$, with $H$ hermitian $H = H^\dagger$ and $\Lambda$ unitary $\Lambda \Lambda^\dagger = 1 = \Lambda^\dagger \Lambda$. Take $V = \Lambda^\dagger U$; one has: $U^\dagger M V = U^\dagger H \Lambda \Lambda^\dagger U = U^\dagger H U = D$; since $H$ is hermitian, its eigenvalues are real and $D$ is real; $U$ is accordingly obtained by diagonalizing $H$.

Since $U$ and $V$ are unitary, the right and left eigenstates of $M$ can be tuned to form two separate orthogonal sets; the counterpart of this is that the $| \text{in} \rangle$ and $< \text{out} |$ eigenstates are not orthogonal $< \text{out} | \text{in} \rangle \neq 0$, and that their bilinear products evolve with time.

Bi-unitary transformations are specially useful to diagonalize mass matrices of fermions; $U$ acts in this case on left-handed fermions and $V$ on right-handed ones (the kinetic terms for the two chiralities are distinct). In general, the fermion masses so defined (i.e. the elements of the diagonal matrix obtained by bi-unitary transformations) do not coincide with the eigenvalues of the mass matrix (see for example \cite{8}). They however do match in the standard electroweak model because arbitrary rotations can always be performed on right-handed quarks, which can in particular absorb the unitary matrix $\Lambda$ occurring in the polar decomposition theorem (see above) and bring back the mass matrix to a hermitian one.

4.1.2 Bi-orthogonal basis

The process of diagonalization that is used in \cite{3} and in the present work (see subsection 4.2 below) is not equivalent to a bi-unitary transformation \cite{9}.

As can be seen on (49) below, neither the $U$ matrix linking $| K^\text{in}_L \rangle$ and $| K^\text{in}_S \rangle$ to $| K^0 \rangle$ and $| \overline{K}^0 \rangle$ nor $V$ linking $< K^\text{out}_L |$ and $< K^\text{out}_S |$ to $< K^0 |$ and $< \overline{K}^0 |$ is unitary; neither the $| \text{in} \rangle$ nor the $< \text{out} |$ eigenstates form an orthogonal set. The orthogonality relations involve both “in” and “out” eigenspaces, and they are now independent of time.

4.1.3 Comparison between two procedures of diagonalization

An extensive study of this question goes beyond the scope of the present work and only a few points will be sketched out here.

The first unclear one is which diagonalization procedure has the cleanest physical interpretation. Eventually which one should be used? We can propose no specific answer here.

We shall rather insist of the fact that not only the eigenstates associated to the two above procedures differ, but also their spectrum (eigenvalues) – see footnote \cite{9} –. We give below \cite{9} some remarks concerning the links between the two sets of eigenvalues in the $2 \times 2$ case.

Let $M$ be a complex matrix, which can be written, according to the polar decomposition theorem

\begin{equation}
M = H \Lambda, \quad \text{with} \quad H = H^\dagger, \quad \Lambda \Lambda^\dagger = 1 = \Lambda^\dagger \Lambda,
\end{equation}

and let $\rho_1, \rho_2$ be the real eigenvalues of $M$ obtained by a bi-unitary transformation and, in particular, by diagonalizing the hermitian $H$ (see subsection 4.1.1)

\begin{equation}
U^\dagger H U = D \equiv \begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix}.
\end{equation}

\footnote{In this respect, we disagree here with the footnote 1 on page 1 of \cite{3}; see subsection 4.1.3}
On the other side, let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of \( M \) defined by the standard equations \(^{10} \) for \(| \text{in} \rangle \) eigenstates (which will be combined with \(< \text{out} | \rangle \) eigenstates, in the rest of the work, to form a bi-orthogonal basis)

\[
M|\varphi^\text{in}_1\rangle = \lambda_1|\varphi^\text{in}_1\rangle, \quad M|\varphi^\text{in}_2\rangle = \lambda_2|\varphi^\text{in}_2\rangle.
\]

\( M \) rewrites

\[
M = UD\tilde{\Lambda}U^\dagger \quad \text{with} \quad \tilde{\Lambda} = U\Lambda U^\dagger,
\]

and we parametrize \( \tilde{\Lambda} \) in the most general way

\[
\tilde{\Lambda} = e^{i\theta} \begin{pmatrix}
\cos \chi e^{i\phi} & \sin \chi e^{i\omega} \\
-\sin \chi e^{-i\omega} & \cos \chi e^{-i\phi}
\end{pmatrix}.
\]

Evaluating the trace and the determinant of \( M \) yields the two equations:

\[
\begin{align*}
Tr(M) &\equiv \lambda_1 + \lambda_2 = Tr(UD\tilde{\Lambda}U^\dagger) = Tr(U^\dagger UD\tilde{\Lambda}) = Tr(D\tilde{\Lambda}) = e^{i\theta} \cos \chi (\rho_1 e^{i\phi} + \rho_2 e^{-i\phi}), \\
\det(M) &\equiv \lambda_1\lambda_2 = det(UD\tilde{\Lambda}U^\dagger) = det(D\tilde{\Lambda}) = \rho_1\rho_2 e^{2i\theta},
\end{align*}
\]

which yield the second order equation linking the \( \lambda \)'s and the \( \rho \)'s

\[
\lambda^2 - e^{i\theta} \cos \chi (\rho_1 e^{i\phi} + \rho_2 e^{-i\phi}) + \rho_1\rho_2 e^{2i\theta} = 0.
\]

In the degenerate case \( \rho_1 = \rho_2 = \rho \): one gets from \(^{12} \)

\[
\lambda_{\text{deg}} = e^{i\rho} \rho \left( \cos \chi \cos \phi \pm i \sqrt{1 - \cos^2 \chi \cos^2 \phi} \right) = e^{i(\theta \pm \alpha)} \rho \quad \text{with} \quad \cos \alpha = \cos \chi \cos \phi,
\]

\( \lambda_1 \) and \( \lambda_2 \) are only deduced from \( \rho_1 \) and \( \rho_2 \) by multiplication by phases; this case is trivial and uninteresting.

To get somewhat further, we simplify to the case when \( H \) is already diagonal and its two elements are very close \( \rho_1 = \rho = \rho_2 - \epsilon \); one has then \( U = 1, \tilde{\Lambda} \equiv U^\dagger \Lambda U = \Lambda \) and \( V \equiv \Lambda^\dagger U = \Lambda^\dagger \).

Developing the solutions of \(^{12} \) at first order in \( \epsilon \), one gets the following eigenvalues of \( M \)

\[
\lambda_\epsilon = e^{i\theta} \left[ \rho e^{\pm i\alpha} + \frac{\epsilon}{2 \sin \alpha} \left( \cos \phi \cos \chi \sin \alpha \mp \sin \phi \cos \alpha \pm \sin \phi \right) + i \left( -\sin \phi \cos \chi \sin \alpha \mp \cos \phi \cos \alpha \pm \cos \phi \right) \right]
\]

The \( \lambda_\epsilon \)'s are clearly not identical to the elements of \( D \) multiplied by phases; this proves that diagonalization a complex matrix by a bi-unitary transformation does not yield the same eigenvalues as looking for \(| \text{in} \rangle \) (and \(< \text{out} | \rangle \) eigenstates).

### 4.2 Bi-orthogonal mass eigenstates for neutral kaons

The mass matrix \( M \) (in the \((K^0, \overline{K^0})\) basis) \(^{10} \)

\[
M = \frac{1}{2} \begin{pmatrix}
\Lambda - c & a + b \\
-a - b & \Lambda + c
\end{pmatrix},
\]

\(^{10}\)Both anti-diagonal terms were given the wrong sign in the formula (22) of \(^{3} \). It has been corrected here: keeping the same expression for the mass matrix, the eigenstates have been adjusted accordingly, taking also in account our convention for charge conjugation of neutral kaons \( C| K^0 > = +| \overline{K^0} > \); for example the state with \( CP = -1 \) is \((K^0 + \overline{K^0})/\sqrt{2} \).
in which the parameters \(a, b, c\) and \(\Lambda\) are given by
\[
a = (\lambda_L - \lambda_S) \frac{1 + \alpha \beta}{1 - \alpha \beta}, \quad b = (\lambda_L - \lambda_S) \frac{\alpha + \beta}{1 - \alpha \beta}, \quad c = (\lambda_L - \lambda_S) \frac{\alpha - \beta}{1 - \alpha \beta},
\]
and
\[
\lambda_L = \frac{\Lambda + \sqrt{a^2 - b^2 + c^2}}{2} = m_L - \frac{i\Gamma_L}{2}, \quad \lambda_S = \frac{\Lambda - \sqrt{a^2 - b^2 + c^2}}{2} = m_S - \frac{i\Gamma_L}{2},
\]
has the following right and left eigenvectors ("in >" and "out >" states – see (35) –)
\[
|K_L^\text{in}| = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \beta)|K^0 > + (1 - \beta)|\overline{K^0} > \right),
|K_S^\text{in}| = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \alpha)|K^0 > - (1 - \alpha)|\overline{K^0} > \right),
<K_L^\text{out}| = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \alpha) <K^0 | + (1 + \alpha) <\overline{K^0} | \right),
<K_S^\text{out}| = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \beta) <K^0 | - (1 + \beta) <\overline{K^0} | \right),
\]
which are built to satisfy the orthogonality, normalization, and completeness relations of a bi-orthonormal basis
\[
<K_L^\text{out}|K_L^\text{in}| = 1 = <K_S^\text{out}|K_S^\text{in}|,
<K_L^\text{out}|K_S^\text{in}| = 0 = <K_S^\text{out}|K_L^\text{in}|, 1 = |K_L^\text{in}|<K_L^\text{out}| + |K_S^\text{in}|<K_S^\text{out}|.
\]
The corresponding Hamiltonian is
\[
\mathcal{H} = \lambda_L |K_L^\text{in}|<K_L^\text{out}| + \lambda_S |K_S^\text{in}|<K_S^\text{out}|.
\]
\(M\) becomes normal ([\(M, M^\dagger\] = 0) when \(\beta = -\alpha^* \)
\(^{11}\) in this case the "in >" and "out >" eigenstates become identical. (50) inverts into
\[
|K^0 > = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \alpha)|K_L^\text{in}| + (1 + \beta)|K_S^\text{in}| \right),
|\overline{K^0} > = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \alpha)|K_L^\text{in}| - (1 - \beta)|K_S^\text{in}| \right),
<K^0 | = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \beta) <K_L^\text{out}| + (1 + \alpha) <K_S^\text{out}| \right),
<\overline{K^0} | = \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \beta) <K_L^\text{out}| - (1 - \beta) <K_S^\text{out}| \right).
\]
The contribution of (45) to the Hamiltonian (51) proportional to \(c\) writes \(-(c/2)|K^0 > <K^0 | - |\overline{K^0} > <\overline{K^0} |\); according to (34), it can be dropped. More generally, one may add to \(M\) any such diagonal term, which has the net effect of changing \(c\) into \(c - \epsilon\) and to transform \(M\) given by (45) into
\[
N = \frac{1}{2} \begin{pmatrix}
\Lambda - (c - \epsilon) & a + b \\
\alpha - b & \Lambda + (c - \epsilon)
\end{pmatrix}.
\]

\(^{11}\)For any complex number \(h\), \(h^*\) denotes its complex conjugate.
In the process of deforming $M$ into $N$, $\alpha$, $\beta$, $\lambda_L$ and $\lambda_S$ are considered fixed, and, thus, $a$, $b$ and $c$ are fixed, too, and are given in terms of the latter by (46).

The sum of the two eigenvalues is conserved in the deformation; $\Lambda$ (see (47)) is thus left invariant and $N$ can be parametrized in terms of new parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{a}, \tilde{b}, \tilde{c}$ by

$$N = \frac{1}{2} \left( \begin{array}{cc} \Lambda - \tilde{c} & \tilde{a} + \tilde{b} \\ \tilde{a} - \tilde{b} & \Lambda + \tilde{c} \end{array} \right).$$  \hspace{1cm} (54)

The “tilde” parameters are expressed in a way analogous to (46)

$$\tilde{a} = (\lambda_1 - \lambda_2) \frac{1 + \tilde{\alpha}}{1 - \tilde{\alpha}}, \quad \tilde{b} = (\lambda_1 - \lambda_2) \frac{\tilde{\alpha} + \tilde{\beta}}{1 - \tilde{\alpha} \tilde{\beta}}, \quad \tilde{c} = (\lambda_1 - \lambda_2) \frac{\tilde{\alpha} - \tilde{\beta}}{1 - \tilde{\alpha} \tilde{\beta}},$$  \hspace{1cm} (55)

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $N$; they are expressed by

$$\lambda_1 = m_1 - \frac{i \Gamma_1}{2}, \quad \lambda_2 = m_2 - \frac{i \Gamma_2}{2},$$  \hspace{1cm} (56)

with $m_{1,2}$ (masses) and $\Gamma_{1,2}$ (widths) real, and, since the sums of the diagonal terms of $N$ and $M$ are identical, they satisfy

$$\lambda_1 + \lambda_2 = \Lambda = \lambda_L + \lambda_S.$$  \hspace{1cm} (57)

The eigenvectors $K_1, K_2$ of $N$ are accordingly parameterized in the $(K^0, \overline{K^0})$ basis by

$$| K_{1}^{\text{in}} > = \frac{1}{\sqrt{2(1 - \tilde{\alpha} \tilde{\beta})}} \left( (1 + \tilde{\beta}) | K^0 > + (1 - \tilde{\beta}) | \overline{K^0} > \right),$$

$$| K_{2}^{\text{in}} > = \frac{1}{\sqrt{2(1 - \tilde{\alpha} \tilde{\beta})}} \left( (1 + \tilde{\alpha}) | K^0 > - (1 - \tilde{\alpha}) | \overline{K^0} > \right),$$

$$< K_{1}^{\text{out}} | = \frac{1}{\sqrt{2(1 - \tilde{\alpha} \tilde{\beta})}} \left( (1 - \tilde{\alpha}) < K^0 | + (1 + \tilde{\alpha}) < \overline{K^0} | \right),$$

$$< K_{2}^{\text{out}} | = \frac{1}{\sqrt{2(1 - \tilde{\alpha} \tilde{\beta})}} \left( (1 - \tilde{\beta}) < K^0 | - (1 + \tilde{\beta}) < \overline{K^0} | \right),$$ \hspace{1cm} (58)

and the orthogonality and completeness relations satisfied by the eigenstates (58) of the deformed matrix $N$ are, by construction, the same as the ones of $M$

$$< K_{1}^{\text{out}} | K_{1}^{\text{in}} > = 1 = < K_{2}^{\text{out}} | K_{2}^{\text{in}} >,$$

$$< K_{1}^{\text{out}} | K_{2}^{\text{in}} > = 0 = < K_{2}^{\text{out}} | K_{1}^{\text{in}} >,$$

$$1 = | K_{1}^{\text{in}} > < K_{1}^{\text{out}} | + | K_{2}^{\text{in}} > < K_{2}^{\text{out}} |.$$  \hspace{1cm} (59)

The corresponding Hamiltonian is

$$\hat{H} = \lambda_1 | K_{1}^{\text{in}} > < K_{1}^{\text{out}} | + \lambda_2 | K_{2}^{\text{in}} > < K_{2}^{\text{out}} |.$$  \hspace{1cm} (60)

From the two equivalent expressions (53) and (54) of $N$, one gets two expressions for its eigenvalues $\lambda_1, \lambda_2$, respectively in terms of the “untilde” and “tilde” parameters

$$\lambda_1, \lambda_2 = \Lambda \pm \sqrt{\tilde{a}^2 - \tilde{b}^2 + \tilde{c}^2} = \Lambda \pm \sqrt{a^2 - b^2 + (c - \epsilon)^2},$$  \hspace{1cm} (61)
and, so
\[
\lambda_1 - \lambda_2 = \sqrt{\tilde{a}^2 - \tilde{b}^2 + \tilde{c}^2} = \sqrt{a^2 - b^2 + (c - \epsilon)^2}. \tag{62}
\]

This “invariance” of the mass matrix by the above deformation has the following consequences:
- the mass matrix of the neutral kaon system can always be transformed such that the criteria of \textit{CPT} invariance are fulfilled (its diagonal terms can be made identical in the \((K^0, \bar{K}^0)\) basis);
- there does not exist a one-to-one correspondence between mass matrix and observed mass eigenstates: indeed, deforming the mass matrix changes the eigenstates, and the ones \((58)\) of \(N\) are not the ones of \(M\) given by \((49)\).

4.3 Deformation of \(M\) into a mass matrix fulfilling the criteria of \textit{CPT} invariance

The mass matrix \(N_{\text{CPT}}\) below is obtained from \(M\) by taking \(\epsilon = c\) in \((53)\):
\[
N_{\text{CPT}} = \frac{1}{2} \begin{pmatrix} \Lambda & a + b \\ a - b & \Lambda \end{pmatrix}; \tag{63}
\]

\(a\) and \(b\) are given by \((45)\) in terms of \(\alpha, \beta, \lambda_L\) and \(\lambda_S\) which are considered to be fixed. The eigenvalues of \(N_{\text{CPT}}\) \((63)\) are
\[
\lambda_1^{\text{CPT}} = \frac{\Lambda + \sqrt{a^2 - b^2}}{2}, \quad \lambda_2^{\text{CPT}} = \frac{\Lambda - \sqrt{a^2 - b^2}}{2}; \tag{64}
\]

their knowledge enables to calculate its eigenvectors, which are parametrized according to \((58)\) by \(\tilde{\alpha}\) and \(\tilde{\beta}\). One finds explicitly
\[
\tilde{\alpha}_{\text{CPT}} = \tilde{\beta}_{\text{CPT}} = \frac{\sqrt{a + b} - \sqrt{a - b}}{\sqrt{a + b} + \sqrt{a - b}} \frac{(\lambda_L - \lambda_S) \neq 0}{1 + \alpha \beta + \sqrt{1 + \alpha^2 \beta^2 - \alpha^2 - \beta^2}} \tag{65}
\]

and, for the normalization factor \(\tilde{n}\) (see \((58)\))
\[
\tilde{n}_{\text{CPT}} \equiv \sqrt{2(1 - \tilde{\alpha}_{\text{CPT}} \tilde{\beta}_{\text{CPT}})} = \sqrt{2(1 - \tilde{a}^2_{\text{CPT}})} = \frac{2 \sqrt{2(a^2 - b^2)}}{\sqrt{a + b} + \sqrt{a - b}}. \tag{66}
\]

That \(\tilde{\alpha}_{\text{CPT}} = \tilde{\beta}_{\text{CPT}}\) entails, through \((55)\), that \(\tilde{c}\) occurring in \((54)\) vanishes, \(\tilde{c} = 0\), such that \(N_{\text{CPT}}\) can finally be parametrized by (see \((54)\))
\[
N_{\text{CPT}} = \frac{1}{2} \begin{pmatrix} \Lambda & \tilde{a}_{\text{CPT}} + \tilde{b}_{\text{CPT}} \\ \tilde{a}_{\text{CPT}} - \tilde{b}_{\text{CPT}} & \Lambda \end{pmatrix}, \tag{67}
\]

which yields \(\tilde{a}_{\text{CPT}} = a\) and \(\tilde{b}_{\text{CPT}} = b\) by comparison with \((63)\).

Its eigenvalues \((64)\) are then also given, from the characteristic equation of \((67)\), by
\[
\lambda_1^{\text{CPT}} = \frac{\Lambda + \sqrt{\tilde{a}^2_{\text{CPT}} - \tilde{b}^2_{\text{CPT}}}}{2}, \quad \lambda_2^{\text{CPT}} = \frac{\Lambda - \sqrt{\tilde{a}^2_{\text{CPT}} - \tilde{b}^2_{\text{CPT}}}}{2}. \tag{68}
\]

One has in particular
\[
\tilde{a}^2_{\text{CPT}} - \tilde{b}^2_{\text{CPT}} = a^2 - b^2, \tag{69}
\]
and we evaluate both sides of (69). Since $\tilde{a}_{CPT} = \tilde{\lambda}_{CPT}$ (see (70)), $\tilde{a}_{CPT}$ and $\tilde{b}_{CPT}$ are given, according to (55), by

$$\tilde{a}_{CPT} = \frac{\lambda_1^{CPT} - \lambda_2^{CPT}}{1 - \alpha\beta^2}, \quad \tilde{b}_{CPT} = \frac{\lambda_1^{CPT} - \lambda_2^{CPT}}{1 - \alpha\beta^2}. \quad (70)$$

such that $\tilde{a}_{CPT}^2 - \tilde{b}_{CPT}^2 = (\lambda_1^{CPT} - \lambda_2^{CPT})^2$. As far as the r.h.s. of (69) is concerned, (66) entails that $(\alpha^2 - \beta^2) = (\lambda_1 - \lambda_2)^2(1 + \alpha^2\beta^2 - \alpha^2 - \beta^2)/(1 - \alpha\beta)^2$.

Finally, (69) yields the ratio of the splittings between the eigenvalues of the “$CPT$ invariant” and “$CPT$ non-invariant” mass matrices $N_{CPT}$ and $M$

$$\frac{\lambda_1^{CPT} - \lambda_2^{CPT}}{\lambda_L - \lambda_S} = \frac{(1 + \alpha^2\beta^2 - \alpha^2 - \beta^2)^{1/2}}{1 - \alpha\beta}. \quad (71)$$

The two mass splittings vanish simultaneously; they become identical only for $\alpha = \beta$, but this trivial case (except when $\alpha = \beta \rightarrow 1$, see below) corresponds (see (46)) to $c = 0$, i.e. starting from an already $CPT$ invariant mass matrix which does not undergo any deformation.

We now answer the two questions: 1- can the eigenstates of the deformed mass matrix be $CP$ eigenstates? 2- can they be flavour eigenstates?

- The condition for $K_1$ and $K_2$ to be $CP$ eigenstates (see (58)) is $\tilde{a}_{CPT} = 0$ and $\tilde{b}_{CPT} = 0$. (65) then entails $b = 0$, which can only occur either if $\lambda_L = \lambda_S$, but this first case corresponds to a trivially diagonal $M$ ($a = b = c = 0$) or if $\beta = -\alpha$ (see (46)); the masses of the eigenstates of the deformed mass matrix are then given by (61): $\lambda_1^{CP}, \lambda_2^{CP} = (\lambda_L + \lambda_S \pm \alpha)/2$.

This is summarized as follows: by deforming their mass matrix into its equivalent $CPT$ invariant form, any $K_L$ and $K_S$ mass eigenstates of the following form

$$\begin{align*}
| K_{L}^{in} > &= \frac{1}{\sqrt{2(1 + \alpha^2)}} \left( | K^0 + \overline{K^0} > - \alpha | K^0 - \overline{K^0} > \right) \\
| K_{S}^{in} > &= \frac{1}{\sqrt{2(1 + \alpha^2)}} \left( | K^0 - \overline{K^0} > + \alpha | K^0 + \overline{K^0} > \right) \\
< K_{L}^{out} | &= \frac{1}{\sqrt{2(1 + \alpha^2)}} \left( < K^0 + \overline{K^0} | - \alpha < K^0 - \overline{K^0} | \right) \\
< K_{S}^{out} | &= \frac{1}{\sqrt{2(1 + \alpha^2)}} \left( < K^0 - \overline{K^0} | + \alpha < K^0 + \overline{K^0} | \right),
\end{align*} \quad (72)$$

with masses $\lambda_L$ and $\lambda_S$, can be transformed into $CP$ invariant eigenstates with masses

$$\lambda_1 = \lambda_L + \alpha^2 \lambda_S, \quad \lambda_2 = \lambda_L - \alpha^2 \lambda_L, \quad \frac{\lambda_1^{CP} - \lambda_2^{CP}}{\lambda_L - \lambda_S} = \frac{1 - \alpha^2}{1 + \alpha^2}. \quad (73)$$

(we have used the expression for $a$ corresponding to $\beta = -\alpha$ given by (65): $a_{CP} = (\lambda_L - \lambda_S)(1 - \alpha^2)/(1 + \alpha^2)$). Note that $(\alpha + \beta) = 0$ is the condition for $T$ invariance outlined in [3]; so, when $T$ invariance is satisfied, the $CP$ eigenstates ($K^0 \pm \overline{K^0}$) are always mass eigenstates.

Defining $CP$ violation in mixing (“indirect” $CP$ violation) by the property that the mass eigenstates are linear combinations of $CP$ eigenstates appears ambiguous; in the present framework, the decays $K_L \rightarrow 2\pi$ and/or $K_S \rightarrow 3\pi$, which have always been linked with $CP$ violation in mixing, do not provide a sufficient characterization of the latter. \footnote{The states (72) are expected to decay into final states of various $CP$; nevertheless, by a transformation which adds a “vanishing” contribution to the effective Lagrangian, the corresponding mass matrix can be deformed into another one the eigenstates of which are the $CP$ eigenstates ($K^0 \pm \overline{K^0}$).}
On the other side, the detection of $T$ violation has been proved\textsuperscript{13} [3] by the CPLEAR collaboration\textsuperscript{10}; from what has been shown above, this proves that the mass eigenstates can never be cast into $CP$ eigenstates, and thus characterizes $CP$ violation in mixing\textsuperscript{14}.

Remark that, in addition to the normalization conditions (50), (72) entails that, if $\alpha$ is furthermore real

$$\alpha \in \mathbb{R} \Rightarrow \begin{cases} <K_L^{|in}|K_S^{|in}> = 1 = <K_L^{|in}|K_L^{|in}> > 0, \\ <K_L^{|out}|K_L^{|out}> = 1 = <K_S^{|out}|K_S^{|out}> > 0; \end{cases}$$

(74)

this is in agreement with the general condition for $M$ to be normal in (4.2), which transforms into the condition of reality for $\alpha$ in the case $\beta = -\alpha$ under concern. Then, $|in>$ and $|out>$ eigenstates become identical.

- $K_1$ and $K_2$ can match the flavour eigenstates $K^0$ and $\overline{K}^0$ if $\tilde{\alpha}_{CPT} = \pm 1$ and $\tilde{\beta}_{CPT} = \pm 1$. The two cases $\tilde{\beta}_{CPT} = -\tilde{\alpha}_{CPT} = \pm 1$ are excluded by (65) which requires $\tilde{\alpha}_{CPT} = \tilde{\beta}_{CPT}$.

The case $\tilde{\alpha}_{CPT} = \pm 1 = \tilde{\beta}_{CPT}$ needs some remarks. It corresponds, by (65), either to $a = b$ or to $a = -b$, that is, in both cases, to a triangular $M$. The eigenvectors of $M$ are $K^0$ or $\overline{K}^0$ (and not $K^0$ and $\overline{K}^0$), which corresponds to $\alpha$ or $\beta$ equal to $\pm 1$. $N_{CPT}$ is a triangular matrix with its two eigenvalues (its two diagonal entries) equal to $(\lambda_L + \lambda_S)/2$. The two points to be noticed are: both $|K_1^{|in}>$ and $|K_2^{|in}>$ are proportional to $|K^0>$, while both $<K_1^{|out}|$ and $<K_2^{|out}|$ are proportional to $<K^0|$: at first sight, $<K_1^{|out}|K_1^{|in}> = 0 = <K_2^{|out}|K_2^{|in}>; however, the explicit calculation shows that the normalization $\tilde{n}_{CPT}$ (66) of the new eigenstates goes to 0, and the Hamiltonian (65) is given, as expected, by

$$\tilde{H}_{CPT} \frac{\alpha_{CPT}=\tilde{\alpha}_{CPT}=\pm 1}{\tilde{\alpha}_{CPT}=\pm 1} = \frac{\lambda_L + \lambda_S}{2} \left( |K^0> <K^0| + |\overline{K}^0> <\overline{K}^0| \right).$$

(75)

4.4 General deformation

We now study the more general case of the deformation of $M$ (45) into $N$ (53), for any $\epsilon$.

(62) provides, through (46), the relation – in general non linear – between the two mass splittings $(\lambda_1 - \lambda_2)$ (corresponding to the “deformed” mass matrix) and $(\lambda_L - \lambda_S)$ (corresponding to the original mass matrix), and the parameters $\alpha$, $\beta$ and $\epsilon$

$$\lambda_1 - \lambda_2 = \sqrt{(\lambda_L - \lambda_S)^2 - 2\epsilon(\lambda_L - \lambda_S)\frac{\alpha - \beta}{1 - \alpha\beta} + \epsilon^2}$$

(76)

For $(\lambda_L - \lambda_S) = 0$, $(\lambda_1 - \lambda_2) = \pm \epsilon$, but this trivial case corresponds by (46) to $a = b = c = 0$ and, hence, to $M$ diagonal. For $(\lambda_L - \lambda_S) \neq 0$, $(\lambda_1 - \lambda_2)$ can vanish for $\epsilon = (\lambda_L - \lambda_S)(\alpha - \beta \pm \sqrt{\alpha^2 + \beta^2 - \alpha^2\beta^2 - 1})/(1 - \alpha\beta)$.

After some algebra, the coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ determining the new eigenvectors (see [53]) are determined to be

$$\tilde{\alpha} = \frac{\sqrt{a^2 - b^2 + (c - \epsilon)^2 - (a - b) + (c - \epsilon)}}{\sqrt{2(a - b)(a + \sqrt{a^2 - b^2 + (c - \epsilon)^2})}}, \quad \tilde{\beta} = \frac{\sqrt{a^2 - b^2 + (c - \epsilon)^2 - (a - b) - (c - \epsilon)}}{\sqrt{2(a - b)(a + \sqrt{a^2 - b^2 + (c - \epsilon)^2})}},$$

(77)

\textsuperscript{13}In the framework of a single constant effective mass matrix
\textsuperscript{14}However, no sign of $T$ violation has been observed in the decay $K^+ \rightarrow \pi^0 \mu\nu$ [11].
which yields the normalization factor (see (58))

\[ n \equiv \sqrt{2(1 - \tilde{\alpha}\tilde{\beta})} = 2 \sqrt{\sqrt{a^2 - b^2 + (c - \epsilon)^2} \over a + \sqrt{a^2 - b^2 + (c - \epsilon)^2}}. \quad (78) \]

(72) gives back, as expected, (65) when \( \epsilon = c \).

From the expressions (\( \tilde{\alpha} \) and \( \tilde{\beta} \)), the ones of \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) can be determined via (58); this entirely determines the matrix \( N \) (54) in terms of \( a, b, c \), and \( \epsilon \), or in terms of \( \alpha, \beta, \lambda_L, \lambda_S \) and \( \epsilon \).

Thanks to the expressions (77), the question whether one (or both) eigenstate(s) of the deformed mass matrix can be \( CP \) eigenstate(s) or flavour eigenstate(s) can be answered.

- Like previously, the condition for \( K_1 \) and \( K_2 \) to be \( CP \) eigenstates (see (58)) is \( \tilde{\alpha}_{CP T} = 0 \) and \( \tilde{\beta}_{CP T} = 0 \). This entails in particular \( \tilde{\alpha} - \tilde{\beta} = 0 \), which, by (77), can only be achieved for \( \epsilon = c \); this brings us back to subsection 4.3.
- \( K_1 \) and \( K_2 \) can match the flavour eigenstates \( K^0 \) and \( \bar{K}^0 \) if \( \tilde{\alpha}_{CP T} = \pm 1 \) and \( \tilde{\beta}_{CP T} = \pm 1 \). The case \( \tilde{\beta} = \tilde{\alpha} = \pm 1 \) requires \( \epsilon = c \) for the same reason as above that \( \tilde{\beta} - \tilde{\alpha} \) has to vanish, and this brings us back again to subsection 4.3. The cases \( \tilde{\beta} = -\tilde{\alpha} = \pm 1 \) yield, using (77)

\[
\begin{align*}
\tilde{\alpha} + \tilde{\beta} = 0 \Rightarrow & \quad \sqrt{a^2 - b^2 + (c - \epsilon)^2} = a - b; \\
\Rightarrow & \quad \{ \tilde{\alpha} = -\tilde{\beta} = \pm 1 \Rightarrow \epsilon = c \pm \sqrt{2(a - b)(2a - b)} \}.
\end{align*}
\]

(79)

The two equations (79) above can only be satisfied- either if \( a = b \), which entails \( \epsilon = c \) and bring us back to the last case and subsection 4.3- or if \( a = 0 \), which entails \( \epsilon = c \pm \sqrt{2} \). \( a = 0 \) can only be achieved either if \( \lambda_L = \lambda_S \), or if \( \beta = -1/\alpha \). The case \( \lambda_L = \lambda_S \) is uninteresting since it also corresponds to \( b = 0 = c \) and to a degenerate diagonal \( M \). Restricting to \( \beta = -1/\alpha \), and using (46), one then gets

\[ \epsilon = (\lambda_L - \lambda_S)\alpha^2 + 1 \pm \sqrt{2(\alpha^2 - 1)} \over 2\alpha. \quad (80) \]

In this last case, the “flavour” and \( CP \) violating \( K_L \) and \( K_S \) eigenstates can both be defined as “mass eigenstates”. It can be summarized as follows: by deforming their mass matrix from \( M \) (45) to \( N \) (53) with \( \epsilon \) given by (80), any \( K_L \) and \( K_S \) mass eigenstates of the form (see (49) with \( \beta = -1/\alpha \))

\[
\begin{align*}
| K_L^{in} > & = {1 \over 2} \left( | K^0 + \bar{K}^0 > - \frac{1}{\alpha} | K^0 > - | \bar{K}^0 > \right) \\
| K_S^{in} > & = {1 \over 2} \left( | K^0 - \bar{K}^0 > + \alpha | K^0 > + | \bar{K}^0 > \right) \\
< K_L^{out} | & = {1 \over 2} \left( < K^0 + \bar{K}^0 | - \alpha < K^0 > - < \bar{K}^0 > \right) \\
< K_S^{out} | & = {1 \over 2} \left( < K^0 - \bar{K}^0 | + \alpha < K^0 > + < \bar{K}^0 > \right),
\end{align*}
\]

(81)

with masses \( \lambda_L \) and \( \lambda_S \), can be transformed into flavour mass eigenstates \( K^0 \) and \( \bar{K}^0 \) with masses

\[
\begin{align*}
\lambda_1, \lambda_2 = & \quad 2\alpha(\lambda_L + \lambda_S) \pm (\alpha^2 - 1)(\lambda_L - \lambda_S) \over 4\alpha, \\
\lambda_1 - \lambda_2 = & \quad \pm b, \quad \lambda_L - \lambda_S = \pm {1 - \alpha^2 \over 2\alpha} = \pm {1 - \beta^2 \over 2\beta},
\end{align*}
\]

(82)

(we have used the r.h.s. of (62)).
For $|\alpha| = 1$, $M$ becomes normal since the condition $\beta = -1/\alpha$ under concern then matches the one in (4.2) and $|in>$ and $|out>$ states become identical.

$\lambda_1$ and $\lambda_2$ are in general non-degenerate (since they correspond to the masses of a particle and its antiparticle, CPT is broken, as expected since the diagonal terms of the deformed mass matrix $N$ are not equal for $\epsilon$ given by (80)); they become degenerate only for $\alpha = \pm 1$, which is the trivial case when the starting $K_L$ and $K_S$ are themselves flavour eigenstates.

5 THE MASS MATRIX IN THE ($K_L, K_S$) BASIS; VARIOUS DEFORMATIONS

In this section, we start from the mass matrix in the ($K_L, K_S$) basis, which has the simplest possible form since it is, by definition, diagonal. We deform it by adding contributions proportional to (34), after rewriting it in terms of the $K_L$ and $K_S |in>$ and $<out|$ states.

We then investigate whether the eigenvalues of the deformed mass matrix can match those of the starting one, and, last, whether the eigenvalues of the deformed mass matrix can be degenerate.

5.1 Back to an operatorial identity

(34) rewrites in terms of $K_L$ and $K_S |in>$ and $<out|$ states (use (52))

$$\mathcal{H}_0 \equiv \frac{\rho}{\rho} \left( \beta - \alpha \right) \left( |K_L^{in}><K_L^{out}| - |K_S^{in}><K_S^{out}| \right) + (1 - \alpha^2) |K_L^{in}><K_S^{out}| + (1 - \beta^2) |K_S^{in}><K_L^{out}| \sim 0,$$

(83)

where $\rho$ is an arbitrary mass parameter.

In the ($K_L, K_S$) basis, by definition, whatever the parameters $\alpha, \beta$, the kaon mass matrix $M$ becomes diagonal and writes

$$\mathcal{M}_0 = \begin{pmatrix} \lambda_L & 0 \\ 0 & \lambda_S \end{pmatrix};$$

(84)

according to (83), it may be freely transformed into

$$\mathcal{M}(\rho) = \mathcal{M}_0 + \rho \begin{pmatrix} \beta - \alpha & 1 - \alpha^2 \\ 1 - \beta^2 & \alpha - \beta \end{pmatrix},$$

(85)

the eigenvalues of which are

$$\mu_1, \mu_2 = \frac{\lambda_L + \lambda_S \pm \sqrt{(\lambda_L - \lambda_S)^2 - 4\rho ((\lambda_L - \lambda_S)(\alpha - \beta) - \rho(1 - \alpha\beta)^2)}}{2}.$$

(86)

One parametrizes $\mathcal{M}(\rho)$ by

$$\mathcal{M}(\rho) = \frac{1}{2} \begin{pmatrix} \Lambda - \xi & \xi + b \\ \xi - b & \Lambda + \xi \end{pmatrix},$$

(87)
which defines unambiguously $a$, $b$ and $c$. For $\mu_2 \neq \mu_1$, one can introduce two parameters $\alpha$ and $\beta$

\begin{align*}
    a &= (\mu_2 - \mu_1) \frac{1 + \alpha \beta}{1 - \alpha \beta}, \quad b &= (\mu_2 - \mu_1) \frac{\alpha + \beta}{1 - \alpha \beta}, \quad c &= (\mu_2 - \mu_1) \frac{\alpha - \beta}{1 - \alpha \beta} \quad (89)
\end{align*}

such that the eigenvectors of $M(\rho)$ write

\begin{align*}
    | K_1^{\text{in}} \rangle &= \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \beta) | K_L^{\text{in}} \rangle + (1 - \beta) | K_S^{\text{in}} \rangle \right), \\
    | K_1^{\text{in}} \rangle &= \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 + \alpha) | K_L^{\text{in}} \rangle - (1 - \alpha) | K_S^{\text{in}} \rangle \right), \\
    < K_2^{\text{out}} | &= \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \alpha) < K_L^{\text{out}} | + (1 + \alpha) < K_S^{\text{out}} | \right), \\
    < K_2^{\text{out}} | &= \frac{1}{\sqrt{2(1 - \alpha \beta)}} \left( (1 - \beta) < K_L^{\text{out}} | - (1 + \beta) < K_S^{\text{out}} | \right). \quad (90)
\end{align*}

One defines the dimensionless parameters $u$, $v$ and $w$ by

\begin{align*}
    u &= \frac{a}{\mu_1 - \mu_2} = \frac{1 + \alpha \beta}{1 - \alpha \beta}, \quad v &= \frac{b}{\mu_1 - \mu_2} = \frac{\alpha + \beta}{1 - \alpha \beta}, \quad w &= \frac{c}{\mu_1 - \mu_2} = \frac{\alpha - \beta}{1 - \alpha \beta}; \quad (91)
\end{align*}

they are constrained by (88) to live on the 3-dimensional sphere

\begin{align*}
    u^2 - v^2 + w^2 = 1. \quad (92)
\end{align*}

$\alpha$ and $\beta$ are determined by (89) and write

\begin{align*}
    \alpha &= \frac{v + w}{u + 1}, \quad \beta &= \frac{v - w}{u + 1}, \quad (93)
\end{align*}

and satisfy also, because of (92) the equation $\alpha \beta = (u - 1)/(u + 1)$.

For any $u \neq \pm 1$, $\alpha$ and $\beta$ can be parametrized by

\begin{align*}
    \alpha &= \sqrt{\frac{1 - u}{1 + u}} (\sinh \eta + \cosh \eta), \\
    \beta &= \sqrt{\frac{1 - u}{1 + u}} (\sinh \eta - \cosh \eta), \quad (94)
\end{align*}

where we have made the change of variables

\begin{align*}
    \cosh \eta &= \frac{w}{\sqrt{1 - u^2}}, \quad \sinh \eta = \frac{v}{\sqrt{1 - u^2}}. \quad (95)
\end{align*}
So, for any complex $u \neq \pm 1$ and $\eta$, all matrices $M(\rho)$ (87) which rewrite, using (89), (91) and (92),

$$M = \frac{1}{2} \begin{pmatrix} \lambda_L + \lambda_S - (\mu_1 - \mu_2)\sqrt{1 - u^2} \cosh \eta & (\mu_1 - \mu_2)(u + \sqrt{1 - u^2} \sinh \eta) \\ (\mu_1 - \mu_2)(u - \sqrt{1 - u^2} \sinh \eta) & \lambda_L + \lambda_S + (\mu_1 - \mu_2)\sqrt{1 - u^2} \cosh \eta \end{pmatrix}$$  \hspace{1cm} (96)

and which can be obtained by deformation of $M_0$, have the same eigenvalues $\mu_1$ and $\mu_2$ (86), and their eigenvectors are given by

$$|K^{\text{in}}_1> = \frac{1}{2} \left( \sqrt{1 + u}(|K^{\text{in}}_L> + |K^{\text{in}}_S>) + \sqrt{1 - u} (\sinh \eta - \cosh \eta)(|K^{\text{in}}_L> - |K^{\text{in}}_S>) \right),$$

$$|K^{\text{in}}_2> = \frac{1}{2} \left( \sqrt{1 + u}(|K^{\text{in}}_L>-|K^{\text{in}}_S>) + \sqrt{1 - u} (\sinh \eta + \cosh \eta)(|K^{\text{in}}_L>+|K^{\text{in}}_S>) \right),$$

$$<K^{\text{out}}_1| = \frac{1}{2} \left( (\sqrt{1 + u}(<K^{\text{out}}_L|+<K^{\text{out}}_S|) - \sqrt{1 - u} (\sinh \eta + \cosh \eta)(<K^{\text{out}}_L|-<K^{\text{out}}_S|) \right),$$

$$<K^{\text{out}}_2| = \frac{1}{2} \left( (\sqrt{1 + u}(<K^{\text{out}}_L|-<K^{\text{out}}_S|) - \sqrt{1 - u} (\sinh \eta - \cosh \eta)(<K^{\text{out}}_L|+<K^{\text{out}}_S|) \right).$$  \hspace{1cm} (97)

### 5.2 Matching the mass spectra

Among all possible cases, the ones when $\mu_1, \mu_2$ eventually match $\lambda_L$ and $\lambda_S$ deserve a special investigation. This can occur for $\rho = \tilde{\rho}$ with

$$\tilde{\rho} = (\lambda_L - \lambda_S) \frac{\alpha - \beta}{(1 - \alpha \beta)^2} = \frac{c}{1 - \alpha \beta},$$  \hspace{1cm} (98)

where we have used (46) for the last identity on the r.h.s.

Since $\tilde{\rho}$ only vanishes (i.e. the two mass spectra match and there is no deformation) when $\beta = \alpha$, i.e. when the starting mass matrix respects $CPT$ invariance, any non-vanishing deformation of a $CPT$ invariant mass matrix alters its mass spectrum.

The question arises of which eigenstates correspond in general to $\rho = \tilde{\rho}$. Parameterizing, as before, $\tilde{M} \equiv M(\tilde{\rho})$ by

$$\tilde{M} = \frac{1}{2} \begin{pmatrix} \Lambda - \hat{c} & \hat{a} + \hat{b} \\ \hat{a} - \hat{b} & \Lambda + \hat{c} \end{pmatrix},$$  \hspace{1cm} (99)

equations (89), (91) and (92) are changed into their equivalent with $\mu_1$ replaced by $\lambda_L$, $\mu_2$ replaced by $\lambda_S$, and underlined parameters replaced by “hatted” ones.

Matching (88) for $\rho = \tilde{\rho}$ with (99) yields the equations linking $\alpha$ and $\beta$ (the coefficients of the eigenvectors (49) of $M$, in the $(K^0, \bar{K}^0)$ basis), to $\hat{\alpha}$ and $\hat{\beta}$ (the coefficients of the eigenvectors of the deformed matrix in the $(K_L(\alpha, \beta), K_S(\alpha, \beta))$ basis).

$$\frac{\hat{\alpha} - \hat{\beta}}{1 - \hat{\alpha} \hat{\beta}} = 2 \left( \frac{\alpha - \beta}{1 - \alpha \beta} \right)^2 - 1,$$

$$\frac{1 + \hat{\alpha} \hat{\beta} + \hat{\alpha} + \hat{\beta}}{1 - \hat{\alpha} \hat{\beta}} = 2 \left( \frac{\alpha - \beta}{1 - \alpha \beta} \right) \left( \frac{1 - \alpha^2}{(1 - \alpha \beta)^2} \right),$$

$$\frac{1 + \hat{\alpha} \hat{\beta} - (\hat{\alpha} + \hat{\beta})}{1 - \hat{\alpha} \hat{\beta}} = 2 \left( \frac{\alpha - \beta}{1 - \alpha \beta} \right) \left( \frac{1 - \beta^2}{(1 - \alpha \beta)^2} \right).$$  \hspace{1cm} (100)
Calling

\[
\begin{align*}
\theta &= 2 \left( \frac{\alpha - \beta}{1 - \alpha \beta} \right)^2 - 1, \\
\phi &= \frac{(\alpha - \beta)^2 - \alpha^2 - \beta^2}{(1 - \alpha \beta)^2}, \\
\omega &= \left( \alpha - \beta \right) \frac{\beta^2 - \alpha^2}{(1 - \alpha \beta)^2},
\end{align*}
\]

(101)

\(\theta, \phi\) and \(\omega\) are of course non independent and satisfy the constraint

\[
\theta^2 + \phi^2 - \omega^2 - 1 = 0.
\]

One finds explicitly

\[
\begin{align*}
\hat{\alpha} &= \frac{(\alpha - \beta) - 1 + \beta^2}{(\alpha - \beta) + 1 - \beta^2}, \\
\hat{\beta} &= \frac{-(\alpha - \beta) + 1 - \alpha^2}{(\alpha - \beta) + 1 - \alpha^2}.
\end{align*}
\]

(103)

As expected, for a \(CPT\) invariant starting mass matrix \((\alpha = \beta)\), the deformed matrix matches the starting one (we have seen above that matching the mass spectrum requires \(\hat{\rho} = 0\)), and their eigenstates also match (one finds \(\hat{\alpha} = -1, \hat{\beta} = 1\)).

For \(\beta \neq \alpha\), the deformed mass matrix can keep the same mass spectrum as the starting one, but exhibits different mass eigenstates.

### 5.3 Degenerate “deformed” eigenstates

The values \(\rho_d\) of \(\rho\) which lead to degenerate mass eigenstates for the deformed mass matrix, with mass \(\mu = (\lambda_L + \lambda_S)/2\) are given by

\[
\rho_d^2 - (\lambda_L - \lambda_S) \frac{\alpha - \beta}{(1 - \alpha \beta)^2} \rho_d + \frac{1}{4} \left( \frac{\lambda_L - \lambda_S}{1 - \alpha \beta} \right)^2 = 0.
\]

(104)

#### 5.3.1 The general case

The solutions of (104) are

\[
\rho_{d\pm} = (\lambda_L - \lambda_S) \frac{(\alpha - \beta) \pm \sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2}}{2(1 - \alpha \beta)^2}.
\]

(105)

Since \(\mu_2 = \mu_1\), the parametrization is no longer possible; the eigenstates of the deformed mass matrix have to be investigated directly, in particular their normalization and orthogonality which is no longer guaranteed.

The deformed mass matrix writes

\[
\begin{align*}
\mathcal{M}_{\text{degen}}^{\pm} &= \frac{\lambda_L + \lambda_S}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda_L - \lambda_S}{2} \begin{pmatrix} 1 - (\alpha - \beta)A_\pm & (1 - \alpha^2)A_\pm \\ (1 - \beta^2)A_\pm & -(1 - (\alpha - \beta)A_\pm) \end{pmatrix} \\
&= \frac{\lambda_L + \lambda_S}{2} D + \frac{\lambda_L - \lambda_S}{2} \Delta
\end{align*}
\]

(106)
with

$$A_{\pm} = \frac{(\alpha - \beta) \pm \sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2}}{(1 - \alpha \beta)^2}. \quad (107)$$

The determinant and the sum of the two diagonal terms of $\Delta$ vanishes, ensuring that the eigenvalues of $M_{\pm}^{\text{degen}}$ are $(\lambda_L + \lambda_S)/2$.

Noting respectively the right and left eigenvectors

$$| K_{\pm}^\text{in} > = \frac{1}{n_{\pm}} (x_{\pm} | K_L^\text{in} > + y_{\pm} | K_S^\text{in} >), \quad | K_{\pm}^\text{out} | = \frac{1}{n_{\pm}} (z_{\pm} < K_L^\text{out} | + t_{\pm} < K_S^\text{out} |) \quad (108)$$

the coefficients $x, y, z, t$ must satisfy the conditions

$$\frac{y_{\pm}}{x_{\pm}} = -\frac{1 - (\alpha - \beta) A_{\pm}}{(1 - \alpha^2) A_{\pm}} = \frac{(1 - \beta^2) A_{\pm}}{1 - (\alpha - \beta) A_{\pm}}; \quad \frac{t_{\pm}}{z_{\pm}} = -\frac{1 - (\alpha - \beta) A_{\pm}}{(1 - \beta^2) A_{\pm}} = \frac{(1 - \alpha^2) A_{\pm}}{1 - (\alpha - \beta) A_{\pm}}, \quad (109)$$

which immediately entail that the two following bilinear products vanish

$$< K_{\pm}^\text{out} | K_{\pm}^\text{in} > = 0; \quad (110)$$

since the “in” and “out” spaces have both shrunk to 1-dimensional spaces (see footnote 18). (109) shows that no suitable normalization and orthogonality condition can be achieved for any single choice of $A$ ($A_+$ or $A_-$).

However, between eigenstates of different deformations of the starting mass matrix, one gets

$$< K_{\pm}^\text{out} | K_{\pm}^\text{in} > = 2\frac{x_{\pm} z_{\pm}}{n_{\pm} \tilde{n}_{\pm}}, \quad < K_{\pm}^\text{out} | K_{\pm}^\text{in} > = 2\frac{x_{\pm} z_{\pm}}{n_{\pm} \tilde{n}_{\pm}}; \quad (111)$$

both bilinear products can be normalized to 1 by the simplest choice

$$n_+ = n_- = \tilde{n}_+ = \tilde{n}_- = \sqrt{2}, \quad x z = 1 = x z; \quad (112)$$

the last equation of (112) together with (109) entail

$$y_+ t_- = 1 = y_- t_+. \quad (113)$$

A bi-orthonormal basis is formed by the four vectors $| K_{\pm}^\text{in} >, | K_{\pm}^\text{in} >, < K_{\pm}^\text{out} |$ and $< K_{\pm}^\text{out} |$, corresponding to the same mass but to two different deformations of the starting mass matrix, respectively into $M_{\pm}^{\text{degen}}$ and $M_{\mp}^{\text{degen}}$.

One gets the closure relation

$$| K_{\text{in}}^- > < K_{\text{out}}^+ | + | K_{\text{in}}^+ > < K_{\text{out}}^- | = | K_{\text{in}}^\text{in} > < K_{\text{out}}^\text{out} | + | K_{\text{in}}^\text{in} > < K_{\text{out}}^\text{out} | \equiv \frac{| K^0 > < K^0 | + | K^0 > < K^0 |}{1}, \quad (114)$$

---

18To the matrix $M_{\pm}^{\text{degen}}$ normally correspond two right eigenvectors $| K_{\text{in}}^\text{in} >$ and $| K_{\text{in}}^\text{out} >$, and two left eigenvectors $< K_{\text{in}}^- |$ and $< K_{\text{in}}^- |$; however, in the present case, $| K_{\text{in}}^\text{in} >$ and $| K_{\text{in}}^\text{out} >$, are the same, that we note $| K_{\text{in}} >$, and so is the case of $< K_{\text{in}}^- |$ and $< K_{\text{in}}^- |$ that we note $< K_{\text{in}}^- |$. The same occurs for the eigenvectors of the matrix $M_{\pm}^{\text{degen}}$, that we note $| K^- >$ and $< K^- |$. 
and

\[ | K_{\pm}^{\text{in}} > < K_{\pm}^{\text{out}} | - | K_{\pm}^{\text{in}} > < K_{\pm}^{\text{out}} | \]

\[ = - \frac{\sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2}}{1 - \beta^2} | K_{L}^{\text{in}} > < K_{S}^{\text{out}} | - \frac{1 - \beta^2}{\sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2}} | K_{S}^{\text{in}} > < K_{L}^{\text{out}} | \]

\[ = - \frac{1}{2(1 - \alpha \beta)} \left[ \sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2} \left( (1 - \beta)^2 | K_0^0 > < K_0^0 | - (1 + \beta)^2 | K_0^0 > < K_0^0 | \right) \right. \]

\[ + \frac{1 - \beta^2}{\sqrt{(\alpha - \beta)^2 - (1 - \alpha \beta)^2}} \left( (1 + \alpha^2) | K_0^0 > < K_0^0 | - (1 - \alpha^2) | K_0^0 > < K_0^0 | \right) \right] ; \]  

(115)

in the last line of (115) we have dropped the terms proportional to (| K_0^0 > < K_0^0 | - | K_0^0 > < K_0^0 |) according to [34].

This subsection can be summarized as follows: To any set of split \( K_L \) and \( K_S \) mesons described by a Hamiltonian \( \mathcal{H} \) can be uniquely associated another bi-orthonormal basis of, now degenerate, states which are the eigenstates of two different well defined deformations of \( \mathcal{H} \), \( \mathcal{H} + \delta \mathcal{H} \pm \); the variations \( \delta \mathcal{H} \pm \) vanish by the field theory constraints linking \( K_0^0 \) to \( K_0^0 \). There still exists a closure relation which matches the one for \( K_0^0 \) and \( K_0^0 \), but which mixes the two deformed Hamiltonians.

5.3.2 The case of a CPT invariant starting mass matrix

We study here the case \( \alpha = \beta \), corresponding to a CPT conserving starting mass matrix \( M \); one then gets

\[ \rho_{d}^{CPT} = \pm i \frac{\lambda_L - \lambda_S}{2} \frac{1}{1 - \alpha^2} , \quad A_{\pm}^{CPT} = \pm i \frac{1}{1 - \alpha^2} \frac{y_{\pm}}{x_{\pm}} = \pm i , \quad t_{\pm} = \pm i . \]  

(116)

which corresponds, in the \( (K_L, K_S) \) basis, to the deformed mass matrices

\[ \mathcal{M}_{\pm}^{\text{degen}} = \begin{pmatrix} \lambda_L & \pm \frac{1}{2}(\lambda_L - \lambda_S) \\ \pm \frac{1}{2}(\lambda_L - \lambda_S) & \lambda_S \end{pmatrix} , \]  

(117)

with eigenvalues \( \mu_1 = \mu_2 = (\lambda_L + \lambda_S)/2 \). The corresponding eigenvectors are 19

\[ | K_{\pm}^{\text{in}} > = \frac{x_{\pm}}{\sqrt{2}} | K_{L}^{\text{in}} > \pm i | K_{S}^{\text{in}} > ) , \quad | K_{\pm}^{\text{out}} > = \frac{1}{\sqrt{2}} \frac{1}{x_{\mp}} (< K_{L}^{\text{out}} | \pm i < K_{S}^{\text{out}} | ) , \]  

(18)

or, writing \( x_{\pm} = \sigma_{\pm} e^{i\chi_{\pm}} , \sigma_{\pm} \in \mathbb{R} \)

\[ | K_{\pm}^{\text{in}} > = \frac{\sigma_{\pm} e^{i\chi_{\pm}}}{\sqrt{2}} ( | K_{L}^{\text{in}} > \pm i | K_{S}^{\text{in}} > ) \]

\[ = \frac{e^{i\chi_{\mp}}}{2\sqrt{1 - \alpha^2}} \left( (1 \pm i\alpha) | K_0^0 + \overline{K_0^0} > + (\alpha \pm i) | K_0^0 - \overline{K_0^0} > \right) , \]

\[ < K_{\pm}^{\text{out}} | = \frac{1}{\sigma_{\pm} \sqrt{2}} e^{-i\chi_{\mp}} (< K_{L}^{\text{out}} | \pm i < K_{S}^{\text{out}} | ) \]

\[ = \frac{e^{-i\chi_{\mp}}}{2\sqrt{1 - \alpha^2}} \left( (1 \mp i\alpha) < K_0^0 + \overline{K_0^0} > + (\alpha \pm i) < K_0^0 - \overline{K_0^0} > \right) . \]

19(115) rewrites: \( | K_{\pm}^{\text{in}} > < K_{\pm}^{\text{out}} | - | K_{\pm}^{\text{in}} > < K_{\pm}^{\text{out}} | = i ( | K_{L}^{\text{in}} > < K_{S}^{\text{out}} | - | K_{S}^{\text{in}} > < K_{L}^{\text{out}} | ) . \)
In the three cases \( \{ \alpha = 0, \pm i \} \) below, either degenerate \( CP \) eigenstates originate from split states of the form \((K^0 \pm \bar{K}^0) \pm \iota (K^0 + \bar{K}^0)\), or split \( CP \) eigenstates are transformed into degenerate \((K^0 \pm \bar{K}^0) \pm \iota (K^0 + \bar{K}^0) \) states:

* for \( \alpha = 0 \), \( K_L \) and \( K_S \) are the \( CP \) eigenstates \( K^0 \pm \bar{K}^0 \), and the degenerate states of (119) become

\[
\begin{align*}
| K^\text{in}^\alpha \rangle &= \frac{\e^{\i \chi \pm \iota \kappa}}{\sqrt{2}} \left( | K^0 + \bar{K}^0 > \pm \iota i | K^0 - \bar{K}^0 > \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\e^{-\i \chi \mp \iota \kappa}}{2 \sigma \mp} \left( < K^0 + \bar{K}^0 | \pm \iota i < K^0 - \bar{K}^0 | \right); \quad (120)
\end{align*}
\]

* for \( \alpha = i \)

\[
\begin{align*}
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 + \bar{K}^0 > + \iota | K^0 - \bar{K}^0 > \right), \\
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 - \bar{K}^0 > + \iota | K^0 + \bar{K}^0 > \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 + \bar{K}^0 | - \iota i < K^0 - \bar{K}^0 | \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 - \bar{K}^0 | - \iota i < K^0 + \bar{K}^0 | \right); \quad (121)
\end{align*}
\]

and the degenerate eigenstates of the deformed mass matrices are

\[
\begin{align*}
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 + \bar{K}^0 > \right), \\
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 - \bar{K}^0 > \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 + \bar{K}^0 | \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 - \bar{K}^0 | \right); \quad (122)
\end{align*}
\]

* for \( \alpha = -i \)

\[
\begin{align*}
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 + \bar{K}^0 > - \iota i | K^0 - \bar{K}^0 > \right), \\
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 - \bar{K}^0 > - \iota i | K^0 + \bar{K}^0 > \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 + \bar{K}^0 | + \iota i < K^0 - \bar{K}^0 | \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 - \bar{K}^0 | + \iota i < K^0 + \bar{K}^0 | \right); \quad (123)
\end{align*}
\]

and the degenerate eigenstates of the deformed mass matrices are

\[
\begin{align*}
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 + \bar{K}^0 > \right), \\
| K^\text{in}^\alpha \rangle &= \frac{\iota}{\sqrt{2}} \left( | K^0 - \bar{K}^0 > \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 + \bar{K}^0 | \right), \\
< K^\text{out}^\alpha |_{\alpha = 0} &= \frac{\iota}{\sqrt{2}} \left( < K^0 - \bar{K}^0 | \right); \quad (124)
\end{align*}
\]

This example is the last that we propose to illustrate the many aspects that the effective mass matrix for neutral kaons can take; of course this study is not exhaustive and cannot be; a continuous range of different values is indeed possible, due to the relations between the states of the basis which are provided by the commutation relations of QFT, and, in particular in the \((K_L, K_S)\) basis, because the corresponding bi-orthogonal set of states is over-complete (see appendix B). It is only maybe useful to emphasize again the dominant role of discrete symmetries to distinguish between the different possible eigenstates.
6 CONCLUSION

This work points at several ambiguities occurring in the usual treatment of interacting neutral mesons. After casting doubts on the validity of introducing a single constant effective mass matrix, it was shown that, even if one puts aside the corresponding arguments, new ambiguities arise which, in particular, allow for deformations of the effective mass matrix; one of the main consequences of them all is that the characterization of $CP$ violation becomes itself blurred.

This adds to the longing for a fundamental explanation and theory of $CP$ violation, which could go beyond the simple parameterization which we have been standing on for many years. It would be satisfactory to exhibit, for example, a mechanism similar to the phenomenon called ”frustration” in solid state physics, leading to an impossibility for mass eigenstates to align along $CP$ conserving directions.

The ultimate goal is of course a treatment $ab\ initio$ by QFT of the systems of interacting mesons. The way to go is still long, but beautiful attempts at such a program have appeared recently. The works [12] [13], which in particular uncover a non-trivial structure of the vacuum, specially deserve to be mentioned, and it is to be hoped that, in a close future, they will be improved and completed. We refer the reader to [5] for their analysis.

Last, I mention that this work generalizes [14], which already uncovered ambiguities in the description of the neutral kaon system by a constant effective mass matrix.

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A MASS MATRICES WITH DEGENERATE EIGENVALUES

This appendix refers more specifically to section 2.

$M^{(2)}(p^2)$ is the full renormalized mass matrix of the neutral kaon system (see (1)). Suppose that it degenerate eigenvalues for all $p^2$, i.e. that they satisfy

$$m^{(2)}_1(p^2) = m^{(2)}_2(p^2), \forall p^2.$$  \hspace{1cm} (125)

The two physical masses $\mu_1^2$ and $\mu_2^2$ are defined by the two equations $m^{(2)}_1(p^2) = \mu_1^2, m^{(2)}_2(p^2) = \mu_2^2$, which reduce to a single one with solution(s) $\mu^2$ when (125) is satisfied: if its solution $p^2 = \mu^2$ is unique, there exists a single $M_f \equiv (M^{(2)}(\mu^2))^{1/2}$ and the physical states are always degenerate: if it has several solutions $p^2 = \mu^2$, there exist as many constant matrices $M_f \equiv M_f(p^2 = \mu^2)$ and each of them has degenerate eigenvalues;

If one writes, in the $(K^0, \bar{K}^0)$ basis, such a generic $2 \times 2$ constant mass matrix $M_{f\alpha} = \begin{pmatrix} a_{\alpha} & b_{\alpha} \\ c_{\alpha} & d_{\alpha} \end{pmatrix}$, with $a, b, c, d \in \mathbb{C}$, the conditions that its two eigenvalues coincide is $(a_{\alpha} - d_{\alpha})^2 + 4b_{\alpha}c_{\alpha} = 0$. Let us take for example $d_{\alpha} = a_{\alpha} - 2i\sqrt{b_{\alpha}c_{\alpha}}$; the physical mass is given by $\mu_{\alpha} = a_{\alpha} - i\sqrt{b_{\alpha}c_{\alpha}}$.

The right eigenvector $\begin{pmatrix} u_{\alpha} \\ v_{\alpha} \end{pmatrix} \equiv (u_{\alpha} | K^0 > + v_{\alpha} | \bar{K}^0 >)$ of $M_{f\alpha}$ must satisfy $u_{\alpha}\sqrt{c_{\alpha}} = iv_{\alpha}\sqrt{b_{\alpha}}$, and its left eigenvector $(z_{\alpha} t_{\alpha}) \equiv (z_{\alpha} < K^0 | + t_{\alpha} < \bar{K}^0 |)$ must satisfy $z_{\alpha}\sqrt{b_{\alpha}} = it_{\alpha}\sqrt{c_{\alpha}}$.

Can all physical eigenstates be $CP$ eigenstates?

The only possible situation is that they appear in two different constant effective matrices with degenerate eigenvalues $M_{f1}$ and $M_{f2}$, and in such a way that the right eigenstate of $M_{f1}$ do not match the one of $M_{f2}$, since we have supposed that their masses are different $\mu_1 \neq \mu_2$.

The two above conditions on $u_{\alpha}, v_{\alpha}, z_{\alpha}, t_{\alpha}$ entail that, if $u_{\alpha}/v_{\alpha} = 1$, i.e. if the right eigenstate is proportional to $| K^0 + \bar{K}^0 >$, at the same times one gets $z_{\alpha}/t_{\alpha} = -1$, such that the left eigenstate is proportional to $< K^0 - \bar{K}^0 |$; this occurs for each $M_{f\alpha}$. The corresponding picture is that $| K^0 + \bar{K}^0 >$ and $< K^0 - \bar{K}^0 |$ have mass $\mu_1$, and that $| K^0 + \bar{K}^0 >$ and $< K^0 + \bar{K}^0 |$ have mass $\mu_2$.

This situation is paradoxical since the two elements of each pair of $| in >$ and $< out |$ eigenstates, which are supposed to represent the same particle (with a given mass), have exactly opposite $CP$ properties, and are furthermore orthogonal, forbidding a suitable normalization and closure relation. Rejecting this possibility, one concludes that:

When the renormalized mass matrix for neutral kaons has degenerate (p² dependent) eigenvalues, it is impossible that all physical eigenstates are CP eigenstates; it is thus a sufficient condition for indirect CP violation.

Assuming that the full renormalized mass matrix for the neutral kaon system has degenerate (p² dependent) eigenvalues is not unreasonable: the basis fundamental degeneracy of the neutral kaon system keeps, at this level, unbroken. That the self-consistent equation defining the physical masses have several distinct solutions is also realistic since $m^{(2)}(p^2)$ can be a complicated function of $p^2$.

---

20If they appear in more than two, certainly one CP eigenstate will occur with two different masses, which is impossible. The eventual other $M_{f\alpha}$ lead to spurious eigenvalues and eigenstates.
The mass splitting and indirect \( CP \) violation arise through the self-consistent procedure which mixes kinetic and mass terms.

Remark: it can also happen that, in general \( m_1^2(p^2) \neq m_2^2(p^2) \), except at the two physical masses \( \mu_1^2 \) and \( \mu_2^2 \), at which, consequently, \( M_f(p^2 = \mu_1^2) \) and \( M_f(p^2 = \mu_2^2) \) have degenerate eigenvalues; the same argumentation as above holds, though this situation looks much less natural.

## B AN OVER-COMPLETE BASIS

A consequence of the fact that the maximal number of independent states is two (\( V_{in} \equiv V_{out} \) is a 2-dimensional complex vector space) is that the mass eigenstates \( | K_{in}^0 >, | K_{in}^0 >, | K_{out}^0 >, | K_{out}^0 > \) and their hermitian conjugates are not independent \(^{21}\). The link (24) between states and their duals, together with (52) yields the relations

\[
(1 + \beta) < K_0^0 | = | K_0^0 >^\dagger \quad \Rightarrow \quad \frac{(1 - \alpha^*) < K_0^0 | + (1 - \beta^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} = \frac{(1 - \alpha^*) < K_0^0 | + (1 - \beta^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}}
\]

(126)

\[
(1 - \beta) < K_0^0 | - (1 - \alpha) < K_0^0 | \quad \Rightarrow \quad \frac{(1 + \alpha^*) < K_0^0 | - (1 + \beta^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} \quad (127)
\]

which exemplifies the non independence of \( < K_0^0 |, < K_0^0 |, < K_0^0 |, < K_0^0 | \).

Other relations can be deduced from (31) and (52) and using the correspondence (24)

\[ (1 - \beta) < K_0^0 | - (1 - \alpha) < K_0^0 | \quad \Rightarrow \quad \frac{e^{-i\gamma} < K_0^0 | + (1 + \alpha^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} \]

\[ \Rightarrow \quad \frac{e^{-i\gamma} < K_0^0 | + (1 + \alpha^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} \quad (128) \]

\[ (1 + \beta) < K_0^0 | + (1 + \alpha) < K_0^0 | \quad \Rightarrow \quad \frac{e^{-i\gamma} < K_0^0 | - (1 - \beta^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} \]

\[ \Rightarrow \quad \frac{e^{-i\gamma} < K_0^0 | - (1 - \beta^*) < K_0^0 |}{\sqrt{2(1 - \alpha \beta)}} \quad (129) \]

Similar equations can be deduced which only involve \( | in > \) states and their hermitian conjugates.

Relations complementary to (128) and (129) can be obtained from (31) and (52) without using (24):

\[ < K_0^0 | = e^{-i\gamma} | K_0^0 > \quad \Rightarrow \quad (1 - \beta) < K_0^0 | - (1 - \alpha) < K_0^0 | = e^{-i\gamma} \left( (1 - \alpha) | K_0^0 > + (1 - \beta) | K_0^0 > \right) \]

(130)

\(^{21}\) (31) shows that the neutral kaon system is spanned by only two real degrees of freedom, associated respectively with the combinations \( (K_0^0 + e^{i\gamma} K_0^0) \) and \( -i(K_0^0 - e^{i\gamma} K_0^0) \).
\[ < K^0 | = e^{-i\gamma} | \overline{K^0} > \]
\[ (1 + \beta) < K^\text{out}_L | + (1 + \alpha) < K^\text{out}_S | = e^{-i\gamma} ( (1 + \alpha) | K^\text{in}_L > - (1 + \beta) | K^\text{in}_S > ) . \]

(131)

One reminds that both \(| \text{in} >\) and \(< \text{out} |\) states can be expanded on the same \((K^0, \overline{K^0})\) basis, and that one can go, for example, from \(| K^\text{cin}_L >\) to \(< K^\text{cin}_L |\) by hermitian conjugation. \((128)\) and \((129)\) are more transparent than \((130)\) and \((131)\) in terms of “states” since they only involve \(< \text{out} |\) states and their hermitian conjugates.
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