Darboux transformation of coherent states

V. G. Bagrov, B. F. Samsonov and L. A. Shekoyan
Tomsk State University, Tomsk, Russia

Abstract

It is proved that the Darboux transformation of the system of coherent states of a free particle leads to the states that may be treated as coherent states of soliton-like potentials.

1 Introduction

In quantum mechanics a nice recipe is known which permit us to multiply exactly solvable quantum mechanical problems. This method is known under three different names. The first name is SUSI QM [1]. It is originated from quantum field theory. The second name is the factorization method proposed by Schrödinger [2]. The third name is the Darboux transformation method [3]. This method is extensively studied in soliton theory [4]. It was Darboux who gave first the more clear formulation of it. This method now is a part of a more general scheme known as the method of transformation operators [5].

In this paper Darboux transformation applied to the free particle Schrödinger equation. It is proved that transformed free particle coherent states are coherent states of soliton-like potentials. We use the definition of coherent states given by Klauder [6].

2 Formalism

Let us know a general solution of the Schrödinger equation

\[(i\partial_t - h_0)\psi(x, t) = 0, \quad h_0 = -\partial_x^2 + V_0(x, t), \quad x \in [a, b].\]  

(1)

We want to find the solutions of another equation

\[(i\partial_t - h_0)\varphi(x, t) = 0, \quad h_1 = -\partial_x^2 + V_1(x, t), \quad x \in [a, b].\]  

(2)

The problem of the search for the solutions of the equation (2) can be reduced to the problem of looking for such an operator \(L\) that participates in the following intertwining relation

\[L(i\partial_t - h_0) = (i\partial_t - h_1)L.\]  

(3)

Given the operator \(L\) which satisfy this relation we easily find function \(\varphi(x, t) = L\psi(x, t)\). It is clear that for an arbitrary Hamiltonian \(h_1\) the problem of looking for the operator \(L\) is not more easy then the problem of searching for the solutions of the Schrödinger equation (2). It turns out to be that to make this method suitable for giving useful information it is sufficient to restrict the operator \(L\) by any class of operators. It is natural to look for the operator \(L\) as a differential operator. In this

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case \( h_1 \) can not be an arbitrary Hamiltonian and equation (3) becomes the equation for \( L \) and \( V_1 \).

Let \( L \) be an operator of the first degree in derivatives \( \partial_x \) and \( \partial_t \). If we want to apply \( L \) only to the solutions of the equation (1) we can replace \( \partial_t = -i h_0 \) and it becomes of the second degree in \( \partial_x \). Therefore if we want that \( L \) be of the first degree in \( \partial_x \) we should take the following form for it

\[
L = L_0(x, t) + L_1(x, t) \partial_x.
\]

(4)

The equation (3) results then in the equation for the functions \( L_0, L_1, \) and \( V_1 \). It is remarkable that this system can be integrated and it has a very nice and simple solution \( 4 \)

\[
L = L_1(t)(-u_x / u + \partial_x), \quad L_1(t) = \exp[2 \int dt \text{Im}(\ln u)_{xx}],
\]

(5)

\[
A = -(\ln |u|^2)_{xx}.
\]

(6)

The function \( u \) called the transformation function is a solution to the initial Schrödinger equation subject to the condition \( (\ln u/\overline{u})_{xxx} = 0 \).

3 Coherent states

**Definition 1.** Every system of states described by vectors \( |\psi_z\rangle \) is called the system of coherent states if the following conditions are fulfilled:

(i) \( |\psi_z\rangle \in H_0 \) where \( H_0 \) is a Hilbert space;

(ii) \( z \in \mathcal{D} \subseteq \mathbb{C} \);

(iii) \( \mathcal{D} \) is a domain endowed with a measure \( \mu(z, \bar{z}) \), \( z, \bar{z} \in \mathcal{D} \) which is defined and finite on a class of Borel sets of \( \mathcal{D} \) and guaranties the following resolution of the identity operator \( I \) on \( H_0 \):

\[
\int_{\mathcal{D}} d\mu(\psi_z)|\psi_z\rangle\langle\psi_z| = I;
\]

(7)

(iv) \( \forall z \in \mathcal{D}, |\psi_z\rangle \) belong to a domain of definition of a Hamiltonian \( h_0 \) on \( H_0 \) and are solutions to the Schrödinger equation

\[
(i \partial_t - h_0)|\psi_z\rangle = 0.
\]

(8)

Consider the vectors \( \varphi_z(x, t) = L\psi_z(x, t) \). The question that arises in this respect is the following. Whether the vectors \( \varphi_z \) may be interpreted as coherent states of the transformed system. It is clear that all the properties of coherent states formulated in the Definition 1 are fulfilled except may be for the resolution of the identity. We shall consider now coherent states of soliton potential.

It is well known that the soliton potential

\[
V_1(x) = -2a^2 \text{sech}^2 ax, \quad a > 0
\]

(9)
can be obtained from the free particle Schrödinger equation, \( V_0(x, t) = 0 \) with the help of the Darboux transformation. The Darboux transformation operator has the form \( L = -a \text{th} ax + \partial_x \). This operator together with its conjugate \( L^+ = -a \text{th} ax - \partial_x \) factorizes the free particle Hamiltonian \( h_0 = L^+ L - a^2 \). So, the operator \( g_0 = L^+ L = h_0 + a^2 \) is strictly positive definite and well defined in \( H_0 \) with the well-defined domain of definition \( D_0 \).
The discrete basis of the Hilbert space $H_0$ is defined with the help of the raising and lowering operators
\begin{equation}
   a = (i-t)\partial_x + ix/2, \quad a^+(i+t)\partial_x - ix/2, \quad (10)
\end{equation}
\begin{equation}
   a^+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a\psi_n = \sqrt{n}\psi_{n-1}, \quad a\psi_0 = 0, \quad \psi_n = \psi_n(x,t). \quad (11)
\end{equation}

The free particle coherent states may be defined as such solutions of the equation
\begin{equation}
   L\psi_0 = \psi_0, \quad (12)
\end{equation}

These states satisfy the Definition 1 with the measure $d\mu = dx dy/\pi$, $z = x + iy$.

The momentum operator $p_x$ and the Hamiltonian $h_0$ are expressed in terms of $a$ and $a^+$
\begin{equation}
   p_x = \frac{1}{2}(a + a^+), \quad h_0 = -p_x^2 = \frac{1}{4}(a + a^+)^2. \quad (14)
\end{equation}

Consider the lineal $L_0 = \text{span}\{\psi_n\}$ which is the space of the finite linear combinations of the functions $\psi_n$. Then $H_0 = \bar{L}_0$. (Bar over $L_0$ means the closure with respect to the norm generated by the conventional scalar product that we will label with the indice zero).

The operators $a$ and $a^+$ are completely defined on the lineal $L_0$ by the formulas (13). Therefor we can consider $L_0$ as the initial domain of definition of $g_0$. Since the operator $h_0 = -\partial_x^2$ initially defined on $L_0$ has the deficiency indices equal zero it has the unique self adjoint extension which coincides with its closure $\bar{h}_0$, $\bar{h}_0 = \bar{h}_0^+$ with the domain of definition $D_0 \subset H_0$. The operator $g_0 = h_0 + a^2$ is essentially self adjoint as well, $\bar{g}_0 = \bar{g}_0^+$ and it has the same domain of definition $D_0$. The spectrum of $\bar{g}_0$ is purely continuous. The eigenfunctions $\psi_p(x,t)$ of the momentum operator $p$ are the eigenfunctions of $\bar{g}_0$ as well $\bar{g}_0\psi_p = N_p^2\psi_p$, $N_p^2 = p^2 + a^2$.

The Hamiltonian of the soliton potential, $h_1 = -\partial_x^2 + V_1(x)$, is essentially self-adjoint in $H_0$ and it has a mixed spectrum. It has a single discrete spectrum level $E_{-1} = -a^2$ with the eigenfunction
\begin{equation}
   \varphi_{-1}(x,t) = (a/2)^{1/2}e^{-ia^2t}\cosh^{-1}(ax) \quad (15)
\end{equation}

Its continuous spectrum is the same that those of the hamiltonian $h_0$. Let $\varphi_p = \varphi_p(x,t)$ be the continuous spectrum eigenfunctions of $h_1$, $h_1\varphi_p = p^2\varphi_p$.

It is easy to see that the action of the operator $L$ on the basis functions $\psi_n$ is well defined and gives the functions
\begin{equation}
   \varphi_n(x,t) = L\psi_n(x,t) \quad (16)
\end{equation}

which are solutions to the Schrödinger equation with the soliton potential.

Let us consider the orthogonal decomposition $L^2(\mathbb{R}) = L_0^2 \oplus L_1^2$ where $L_0^2 = \text{span}\varphi_{-1}$. The functions $\varphi_n$ are the basis functions in the space $L_1^2$. The relation (14) defines the action of $L$ in the lineal $L_0$. The natural question that arises at this level is the following: What is the maximal domain of definition of $L$. Our analysis shows that the following lemma is valid.

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Lemma 1. The operator $L$ has such an extension $\bar{L}$ that it domain of definition is $D_0'$ and it domain of values is $H_1$ where $D_0' = D_0\sqrt{g_0}$ and $H_1 = \bar{L}_1$, $\bar{L}_1 = \text{span}\{\varphi_n\}$, $n = 0, 1, \ldots$, and the closure is taken with respect to the norm generated by the scalar product 

$$\langle \varphi_a | \varphi_b \rangle_1 \equiv \langle \psi_a | g_0 | \psi_b \rangle_0, \quad \psi_{a,b} \in L_0, \quad \varphi_{a,b} \in L_1$$

Corollary 1. Every $\varphi \in H_1$ may be presented in the form $\varphi = \bar{L}\psi$, $\psi \in D_0'$, $D_0' \supset D_0$

Let us define the operator $\bar{L}^+$ in the space $H_1$. For this purpose let us consider the functions $\psi \in D_0 \subset D_0'$ and for every $\varphi = \bar{L}\psi$, $\psi \in D_0$ define $\bar{L}^+\varphi \equiv \bar{g}_0\psi$. Denote $D_1$ the domain of definition of $\bar{L}^+$. Domain $D_1$ consists of all $\varphi \in H_1$ of the form $\varphi = \bar{L}\psi$, $\psi \in D_0$. Domain $D_1$ is dense in $H_1$. We have established the validity of the following lemmas

Lemma 2. $\bar{g}_0 = \bar{L}^+\bar{L}$, $\bar{g}_1 = \bar{h}_1 + a^2 = \bar{L}\bar{L}^+$

Lemma 3. $\bar{L}^+$ is adjoint to $\bar{L}$ with respect to the scalar products $\langle \cdot | \cdot \rangle_0$ and $\langle \cdot | \cdot \rangle_1$

Lemma 4. $\bar{L} = \bar{L}^{++}$.

Corollary 2. The operator $\bar{L}$ is closed.

The operator $\bar{L}$ has a natural extension to the continuous spectrum eigenfunctions $\psi_p$ of the momentum operator and $\bar{L}\psi_p = N_p\varphi_p$, $N_p^2 = p^2 + a^2$, $p \in \mathbb{R}$.

The operator $\bar{L}^+$ is invertible in $H_1$. Introduce an operator $M$ by the relation

$$ML^+\varphi = \varphi, \quad \varphi \in D_1, \quad M = (\bar{L}^+)^{-1}$$

(17)

Lemma 5. The bases $\{\eta_n = M\psi_n\}$ and $\{\varphi_n = L\psi_n\}$ are biorthogonal Riesz bases [9].

Theorem 1. Operator $U = \bar{L}\bar{g}_0^{-1/2}$ realizes the isometric mapping of the domain $D_0'$ onto $D_1$. Operator $U^+ = U^{-1} = \bar{g}_0^{-1/2}\bar{L}^+$ realizes the inverse mapping. Operators $U$ and $U^+$ have the following resolutions in terms of the generalized eigenvectors $|\psi_p\rangle$ and $|\varphi_p\rangle$:

$$U = \int dp |\varphi_p\rangle \langle \psi_p|, \quad U^+ = \int dp |\psi_p\rangle \langle \varphi_p|.$$  (18)
Corollary 3. From (18) it follows the spectral representation for $\bar{L}$ and $\bar{L}^+$

$$\bar{L} = \int dpN_p|\varphi_p\rangle\langle\psi_p|, \quad \bar{L}^+ = \int dpN_p|\psi_p\rangle\langle\varphi_p|$$

and the similar representation for $M$ and $M^+$

$$M = \int dpN_p^{-1}|\varphi_p\rangle\langle\psi_p|, \quad M^+ = \int dpN_p^{-1}|\psi_p\rangle\langle\varphi_p|.$$ 

Operators $M$ and $M^+$ are bounded and factorize the operators $\bar{g}_0^{-1}$ and $\bar{g}_1^{-1}$: $M^+M = \bar{g}_0^{-1}$, $MM^+ = \bar{g}_1^{-1}$.

Remark 1. The representation $\bar{L} = U\bar{g}_0^{1/2}$ is a canonical representation of the closed operator $\bar{L}$ and $M = U\bar{g}_0^{-1/2}$ is the similar representation of the bounded operator $M$. These representations are called polar factorizations as well.

Theorem 2. The states associated with the vectors $\eta_z = M\psi_z = \Phi\sum_n a_n z^n\eta_n$ are coherent states in the sense of the Definition 1. The measure $d\mu_\eta = d\mu_\eta(z, \bar{z})$ which realizes the resolution of the identity in terms of the vectors $\eta_z$ gives a solution to the problem of moments in the complex plane

$$a_n a_k \int d\mu_\eta|\Phi|^2 z^n\bar{z}^k = S_{nk}$$

and has the form

$$d\mu_\eta = \omega_\eta(x)dxdy, \quad \omega_\eta(x) = \frac{1}{\pi}(x^2 + a^2 - \frac{1}{4}), \quad z = x + iy$$

Consider now the vectors $\varphi_z = L\psi_z = \Phi\sum_n a_n z^n\varphi_n$.

Theorem 3. The states associated with the vectors $\varphi_z$ satisfy all the conditions of the Definition 1. The measure $d\mu_\varphi = d\mu_\varphi(z, \bar{z})$ which realizes the identity resolution in terms of the vectors $\varphi_z$ gives a solution to the problem of moments on the complex plane

$$a_n a_k \int d\mu_\varphi|\Phi|^2 z^n\bar{z}^k = S_{nk}^{-1}$$

and has the form $d\mu_\varphi = dy\nu(x)$, $z = x + iy$. The measure $d\nu(x)$ is defined by its Fourier transform

$$d\tilde{\nu}(t) = \rho(t)dt, \quad \rho(t) = \exp(it^2/8 - a|t|)/(2\pi a)$$

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