DEGENERATE PRINCIPAL SERIES OF
METAPLECTIC GROUPS AND HOWE
CORRESPONDENCE

SOO TECK LEE AND CHEN-BO ZHU

Abstract. The main purpose of this article is to supplement the
authors’ results on degenerate principal series representations of
real symplectic groups with the analogous results for metaplectic
groups. The basic theme, as in the previous case, is that their
structures are anticipated by certain natural subrepresentations
constructed from Howe correspondence. This supplement is nec-
essary as these representations play a key role in understanding
the basic structure of Howe correspondence (and its complications
in the archimedean case), and their global counterparts play an
equally essential part in the proof of Siegel-Weil formula and its
generalizations (work of Kudla-Rallis). The full results in the meta-
plectic case also shed light on the seeming peculiarities, when the
results in the symplectic case are viewed in their isolation.

1. Introduction: classical invariant theory and its
transcendental analog

Let the complex orthogonal group $O_m$ act on the space of complex
matrices $M_{m,n}$ by matrix multiplication on the left:

$O_m \act M_{m,n}.$

The First Fundamental Theorem (FFT) of classical invariant theory
asserts that the ring of $O_m$-invariant polynomials are generated by the
fundamental invariants of degree 2:

$P(M_{m,n})^{O_m} = \langle r_{ij} \mid 1 \leq i, j \leq n \rangle,$

where $r_{ij}(X) = \sum_{k=1}^m x_{ki}x_{kj}$, for $X = (x_{ij}) \in M_{m,n}$.

Let $S^2(\mathbb{C}^n)$ denote the space of complex $n \times n$ symmetric matrices
and define

$Q : M_{m,n} \to S^2(\mathbb{C}^n),$

$X \mapsto X^t X.$

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Then FFT asserts that the pull-back map
\[ Q^* : P(S^2(C^n)) \to P(M_{m,n})^{O_m} \]
is surjective. Alternatively, we have an affine embedding:
\[ \psi : M_{m,n}/O_m \hookrightarrow S^2(C^n), \]
where \( M_{m,n}/O_m \) is the affine quotient of \( M_{m,n} \) by \( O_m \). The Second Fundamental Theorem (SFT) of classical invariant theory is then a statement on the image of \( \psi \) as an affine subvariety of \( S^2(C^n) \). Note that both \( M_{m,n} \) and \( S^2(C^n) \) carry natural actions of \( GL(n, \mathbb{C}) \), and all maps \((Q, Q^*, \psi)\) are \( GL(n, \mathbb{C}) \)-equivariant.

We now state results of Kudla-Rallis [KR1] and Lee-Zhu [LZ2], which may be viewed as transcendental analogs of FFT and SFT alluded to above.

Let
\[ O(p, q) \rtimes M_{p+q,n}^{\mathbb{R}}, \]
again by matrix multiplication on the left. Denote by \( G = \tilde{Sp}(2n, \mathbb{R}) \) the real metaplectic group of rank \( n \) and \( P \) its Siegel parabolic subgroup. The rest of notation will be explained in Section 3.

**Theorem 1.1.** (Kudla-Rallis) There is a natural topological embedding with closed image:
\[ \psi_{p,q} : S(M_{p+q,n}^{\mathbb{R}}) \to I^\alpha(\sigma) := \text{Ind}^G_P(\chi^{\alpha}), \]
as \( G \)-representations.

**Theorem 1.2.** (Lee-Zhu) describes the image of \( \psi_{p,q} \).

In [LZ2], the authors describe the image \( \Omega^{p,q} \) of \( \psi_{p,q} \) when \( p + q \) is even, in which case the representations concerned factor through the linear group \( Sp(2n, \mathbb{R}) \). In fact only the subcase of \( p - q \equiv 0 \) (mod 4) was treated in detail and the other subcase of \( p - q \equiv 2 \) (mod 4) was left to the reader.

The aim of the current article is to complete the description of \( \Omega^{p,q} \) (without any restriction on \( p \) and \( q \)). This is necessary and useful as the representations involved (the coinvariants) play a key role in understanding the basic structure of Howe correspondence (through the doubling method [HI, Ra, Ku]) and their global counterparts feature prominently in the proof of Siegel-Weil formula and its generalizations [KR3]. For a recent application of these results to first occurrence conjecture of Kudla-Rallis, see [SZ]. As an additional benefit, this full description enables us to organize the statements in a more coherent way so that their structures emerge clearer to the reader. The basic
idea is very simple: since both $\Omega^{p,q}$ and $I^\alpha(\sigma)$ are $K$-multiplicity free ($K$ being a maximal compact subgroup of $G$), and since the structure of $I^\alpha(\sigma)$ is known (Section 5, [L2] for the linear group), one can identify the image $\Omega^{p,q}$ as a $G$-representation by knowing its $K$-types. The fascinating point is that from this description, one concludes that the reducibilities of $I^\alpha(\sigma)$ are completely accounted for by the possible embeddings of $\Omega^{p,q}$'s (in a precise way). We summarize this assertion in Theorem 6.1, which should be viewed as analogous to SFT (proverbially stated) as “all relations among the fundamental invariants are generated by the obvious ones”.

Here are some words on the organization of this article. In Section 2, we review the basics of Howe duality correspondence. In Section 3, we introduce the space of coinvariants, as a special case of Howe’s construction of maximal quotients, and its embedding into the degenerate principal series $I^\alpha(\sigma)$ of the metaplectic group. In Section 4, we describe the transition coefficients of $I^\alpha(\sigma)$ and give its immediate consequences for irreducibility and complementary series. In Section 5, we give the detailed structure of $I^\alpha(\sigma)$ at points of reducibility, which are again arrived by analyzing the transition coefficients. In Section 6, we complete our description of the image of $\psi_{p,q}$ in $I^\alpha(\sigma)$. As proofs of (any new) results follow a similar line as those of [L2] or [LZ2], we shall omit them.

Finally we mention the following works which are closely related to the theme of this article: [KR1, LZ1, LZ2, LZ3, Ya] (for real groups) and [KR2, KS, Ya] (for $p$-adic groups).

2. Howe duality correspondence

In this section, we review briefly Howe’s theory of dual pair correspondence, for the case at hand.

Let $H = \text{O}(p, q)$, which acts on $M_{p+q,n}(\mathbb{R})$ by matrix multiplication on the left. We have the dualised action of $H$ on $\mathcal{Y} = L^2(M_{p+q,n}(\mathbb{R}))$, the space of square integrable functions on $M_{p+q,n}(\mathbb{R})$. As is well-known, there is a (unitary) representation $\omega$ of a metaplectic group $\tilde{\text{Sp}}(2N, \mathbb{R})$ ($N = (p + q)n$) on $\mathcal{Y}$, called the Schrödinger model of an oscillator representation. Here and after, for any real symplectic group $\text{Sp}(2N, \mathbb{R})$, $\tilde{\text{Sp}}(2N, \mathbb{R})$ denotes its metaplectic two fold cover. The oscillator representation depends on a choice of a non-trivial unitary character of $\mathbb{R}$, which we fix once for all.

The main feature of this set-up is the following: there is a reductive dual pair

$$ (H, G) = (\text{O}(p, q), \text{Sp}(2n, \mathbb{R})) \subseteq \text{Sp}(2N, \mathbb{R}), $$
namely a pair of reductive subgroups in $\text{Sp}(2N, \mathbb{R})$ which are mutual centralizers.

Let $\omega^\infty$ be the smooth representation of $\omega$ realized in $\mathcal{Y}^\infty$. For the case at hand, $\mathcal{Y}^\infty = \mathcal{S}(M_{p+q,n}(\mathbb{R}))$, the Schwartz space of rapidly decreasing functions on $M_{p+q,n}(\mathbb{R})$.

For any subgroup $E$ of $\text{Sp}(2N, \mathbb{R})$, denote by $\tilde{E}$ the preimage of $E$ under the covering map $\tilde{\text{Sp}}(2N, \mathbb{R}) \to \text{Sp}(2N, \mathbb{R})$. If $E$ is also reductive, denote by $\mathcal{R}(\tilde{E}, \omega)$ the set of infinitesimal equivalent classes of irreducible admissible representations of $\tilde{E}$ which are realizable as quotients by $\omega^\infty(\tilde{E})$-invariant closed subspaces of $\mathcal{Y}^\infty$.

**Howe Duality Theorem** ([H2]): $\mathcal{R}(\tilde{H} \cdot \tilde{G}, \omega)$ is the graph of a bijection between $\mathcal{R}(\tilde{H}, \omega)$ and $\mathcal{R}(\tilde{G}, \omega)$.

This is the Howe quotient correspondence. Let us make it more concrete. As usual, we may twist the oscillator representation (by a character of $\tilde{H}$) so that it will factor through the standard linear action of $H$ on $\mathcal{Y}$. We will assume that this has been done (but retain the same notation) and will be concerned with $H$, rather than $\tilde{H}$. Let $\rho$ be an irreducible admissible representation of $H$, which is realizable as a quotient by a $H$-invariant closed subspace of $\mathcal{Y}^\infty$. Define $\Omega(\rho)$ to be the maximal quotient of $\mathcal{Y}^\infty$ on which $H$ acts by a representation of class $\rho$. We have (isomorphism class to mean infinitesimal equivalent class):

$$
\Omega(\rho) \cong \rho \otimes \Theta(\rho),
$$

(2.1)

where $\Theta(\rho)$ is a $\tilde{G}$-module.

Howe duality theorem asserts that $\Theta(\rho)$ is a finitely generated admissible quasisimple representation of $\tilde{G}$, and has a unique irreducible $\tilde{G}$-quotient, denoted by $\theta(\rho)$:

$$
\Theta(\rho) \twoheadrightarrow \theta(\rho).
$$

The correspondence

$$
\rho \mapsto \theta(\rho)
$$

is the Howe quotient correspondence.

**Remark**: If we take $\rho$ to be in the Casselman-Wallach class [Wal, Chapter 11], then we will have the isomorphism of Casselman-Wallach representations $\Omega(\rho) \cong \hat{\rho} \hat{\otimes} \Theta(\rho)$, where “$\hat{\otimes}$” stands for the completed projective tensor product.
3. The coinvariants and the embedding

For the dual pair \((H, G) = (O(p, q), \text{Sp}(2n, \mathbb{R}))\), let

\[ \Omega^{p,q} = \text{Howe’s maximal quotient corresponding to} \]

the trivial representation \(1_{O(p, q)}\)

\[ = \mathcal{S}(M_{p+q,n}(\mathbb{R}))_{O(p, q)} \] (the space of coinvariants).

This is a representation of \(\widetilde{G}\).

Remark: the continuous dual of \(\Omega^{p,q}\) is \(\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))_{O(p, q)}\), the space of \(O(p, q)\)-invariant tempered distributions on \(M_{p+q,n}(\mathbb{R})\). This is investigated in [KR1] and [Zh].

From now on, we shift notation and will denote \(G = \text{Sp}(2n, \mathbb{R})\).

The case of \(n = 1\) differs from the general case slightly, but since it is straightforward, we omit it and will assume that \(n \geq 2\) throughout this paper. We shall identify \(\text{Sp}(2n, \mathbb{R})\) as a set with

\[ \text{Sp}(2n, \mathbb{R}) \times \mathbb{Z}_2 = \{(g, \varepsilon) : g \in \text{Sp}(2n, \mathbb{R}), \varepsilon = \pm 1\}. \]

For \(a \in \text{GL}(n, \mathbb{R})\) and \(b \in M_n(\mathbb{R})\) such that \(b = b^t\), we let

\[ m_a = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix}, \]

\[ n_b = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}. \]

Let

\[ M = \{(m_a, \varepsilon) : a \in \text{GL}(n, \mathbb{R}), \varepsilon = \pm 1\} \]

and

\[ N = \{(n_b, 1) : b \in M_n(\mathbb{R}), b = b^t\}. \]

Then \(P = MN\) is a maximal parabolic subgroup of \(G\), called the Siegel parabolic.

Let \(\chi : M \longrightarrow \mathbb{C}^\times\) be given by

\[ \chi(m_a, \varepsilon) = \varepsilon \cdot \begin{cases} i & \text{if } \det a < 0, \\ 1 & \text{if } \det a > 0. \end{cases} \]

This is a character of \(M\) and it is of order 4. For \(\alpha = 0, 1, 2, 3\) and \(\sigma \in \mathbb{C}\), let \(\chi_\sigma^{\alpha}\) be the character of \(P\) given by

\[ \chi_\sigma^{\alpha}[(m_a, \varepsilon)(n_b, 1)] = |\det a|^\sigma \chi(m_a, \varepsilon)^\alpha. \quad (3.1) \]
Let $I^\alpha(\sigma)$ be the normalized induced representation:

$$I^\alpha(\sigma) = \text{Ind}_{\mathcal{P}}^G \chi^\alpha_\sigma.$$ 

The representation space of $I^\alpha(\sigma)$ is

$$\{ f \in C^\infty(G) : f(pg) = \delta^{\frac{1}{2}}(p)\chi^\alpha_\sigma(p)f(g), \ \forall g \in G, p \in P \},$$

and $G$ acts by right translation:

$$g \cdot f(h) = f(hg), \quad (g, h \in G).$$

Here $\delta$ denotes the modular function of $\mathcal{P}$, and is given by

$$\delta[(m_a, \varepsilon)(n_b, 1)] = |\det a|^{2\rho_n}$$

and

$$\rho_n = \frac{n + 1}{2}.$$

When $\alpha = 0$ or 2, the representation $I^\alpha(\sigma)$ descends to a representation of the linear group $\text{Sp}(2n, \mathbb{R})$.

Define the map $\psi_{p,q} : \mathcal{S}(M_{p+q,n}(\mathbb{R})) \hookrightarrow C^\infty(G)$ by

$$\psi_{p,q}(f)(g) = (\omega(g)f)(0), \quad f \in \mathcal{S}(M_{p+q,n}(\mathbb{R})), \quad g \in G.$$ 

From the well-known formula of the oscillator representation $\omega$, we see that

$$\psi_{p,q} : \mathcal{S}(M_{p+q,n}(\mathbb{R})) \hookrightarrow I^\alpha(\sigma),$$

where

$$\sigma = \frac{p + q}{2} - \rho_n, \quad \text{and} \quad \alpha \equiv p - q \pmod{4}.$$ 

(3.2)

The following is the fundamental result of Kudla-Rallis.

**Theorem 3.3.** ([KR1]) The map $\psi_{p,q}$ induces a topological embedding with closed image:

$$\Omega^{p,q} \hookrightarrow I^\alpha(\sigma)$$

**Remark:** Analogous results hold for other classical groups. See [Zh].
4. Degenerate principal series: the transition coefficients

Fix a maximal compact subgroup \( K_1 \simeq U(n) \) of \( \text{Sp}(2n, \mathbb{R}) \) and let \( K \) be the inverse image of \( K_1 \) in \( \widetilde{\text{Sp}}(2n, \mathbb{R}) \), thus a maximal compact subgroup of \( \widetilde{\text{Sp}}(2n, \mathbb{R}) \). The Lie algebra of \( \text{Sp}(2n, \mathbb{R}) \) has a Cartan decomposition

\[
\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}
\]

where \( \mathfrak{k} \) is the Lie algebra of \( K_1 \) and is isomorphic to \( u(n) \).

Let \( \Lambda^+_n = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \} \), and \( 1 = (1, \ldots, 1) \in \Lambda^+_n \).

Also let \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \), for \( 1 \leq j \leq n \).

We have the \( K \)-type decomposition:

\[
I^\alpha(\sigma)|_K = \bigoplus_{\lambda \in \Lambda^+_n} V_{2\lambda + \frac{\alpha}{2}} 1,
\]

(4.1)

where \( V_{2\lambda + \frac{\alpha}{2}} 1 \) is an irreducible \( K \)-module with highest weight \( 2\lambda + \frac{\alpha}{2} 1 \), for \( \lambda \in \Lambda^+_n \).

For \( \mu = 2\lambda + \frac{\alpha}{2} 1 \), let \( V_\mu \) be a \( K \)-type in \( I^\alpha(\sigma) \) and fix \( \gamma_\mu \) to the unique (up to a multiple) \( K \)-highest weight vector in \( V_\mu \). We consider the tensor product \( \mathfrak{p}_C \otimes V_\mu \) of \( \mathfrak{k}_C \) modules, and the \( \mathfrak{k}_C \) map

\[
m : \mathfrak{p}_C \otimes V_\mu \rightarrow I^\alpha(\sigma)|_K
\]

\[
m(p \otimes v) = p \cdot v.
\]

For each \( 1 \leq j \leq n \), there exists an element \( X_j \) in \( \mathcal{U}(\mathfrak{sp}(2n, \mathbb{C})) \) (the universal enveloping algebra of the complexified Lie algebra) with the property that \( X_j \cdot \gamma_\mu \) is the image of the unique \( \mathfrak{k}_C \) highest weight vector of weight \( \mu + 2e_j \) in \( \mathfrak{p}_C \otimes V_\mu \), valid for all \( \mu \). C.f. \cite{L2}, Section 3]. Thus \( X_j \cdot \gamma_\mu \) is a multiple of the unique \( \mathfrak{k}_C \) highest weight vector \( \gamma_{\mu+2e_j} \) in \( V_{\mu+2e_j} \). We call this multiple the transition coefficient from \( V_\mu \) to \( V_{\mu+2e_j} \). Likewise there is a transition coefficient from \( V_\mu \) to \( V_{\mu-2e_j} \). Note that the transition coefficients depend on the choice of \( \gamma_\mu \)’s.

The main use of the transition coefficients is as follows: if the transition coefficient from \( V_\mu \) to \( V_{\mu+2e_j} \) is nonzero, then \( V_{\mu+2e_j} \) is in the submodule generated by \( V_\mu \), and vice versa. We shall indicate by the symbol: \( V_\mu \rightarrow V_{\mu+2e_j} \); If this transition coefficient is zero, then \( \sum_{\mu_\mu' \leq \mu_j} V_{\mu'} \) is a submodule of \( I^\alpha(\sigma) \) and we say that we have a barrier to block \( j \)th rightward movement \( V_\mu \rightarrow V_{\mu+2e_j} \). Likewise for the transition coefficient from \( V_\mu \) to \( V_{\mu-2e_j} \). To summarize, the collection of
transition coefficients allows us to determine the lattice of submodules in $I^\alpha(\sigma)$, explicitly.

By constructing an appropriate choice of $\mathfrak{c}_C$ highest weight vector $\gamma_\mu$ for all $\mu$ (c.f. [L2, Section3]), the transition coefficients $A_j^+(\mu)$ from $V_\mu$ to $V_{\mu+2e_j}$, and the transition coefficients $A_j^-(\mu)$ from $V_\mu$ to $V_{\mu-2e_j}$, are computed to be

$$A_j^+(\mu) = B_j^+ - 2\lambda_j, \quad A_j^-(\mu) = 2\lambda_j - B_j^-,$$

where

$$B_j^+ = -\sigma - \rho_n - \frac{\alpha}{2} + j - 1, \quad B_j^- = \sigma - \rho_n - \frac{\alpha}{2} + j + 1. \quad (4.3)$$

**Corollary 4.4.** ([KR1]) $I^\alpha(\sigma)$ is irreducible if and only if

$$\sigma + \rho_n + \frac{\alpha}{2} \notin \mathbb{Z}.$$

The explicit form of transition coefficients also allows us to determine the complementary series. We will follow the method used in [L2]. Each $K$-type $V_{2\lambda+\frac{\alpha}{2}}1$ of $I^\alpha(\sigma)$ has a $K$-invariant inner product given by

$$\langle f_1, f_2 \rangle_\lambda = \int_K f_1(k) f_2(k) dk.$$

Since $V_{2\lambda+\frac{\alpha}{2}}1$ is an irreducible $K$-module, any $K$-invariant inner product on $V_{2\lambda+\frac{\alpha}{2}}1$ is a multiple of $\langle \cdot, \cdot \rangle_\lambda$. Thus if $\langle \cdot, \cdot \rangle$ is a $\widetilde{Sp}(2n, \mathbb{R})$-invariant inner product on $I^\alpha(\sigma)$ then there exists positive constants $\{c_\lambda\}_{\lambda \in \Lambda^+_n}$ such that

$$\langle f_1, f_2 \rangle = c_\lambda \langle f_1, f_2 \rangle_\lambda, \quad \forall f_1, f_2 \in V_{2\lambda+\frac{\alpha}{2}}1.$$

Since the $K$-types of $I^\alpha(\sigma)$ are mutually orthogonal with respect $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle$ is completely determined by the constants $\{c_\lambda\}$. Using similar arguments as the $U(n, n)$ case (see [L1, section 9]), we obtain the following:

**Lemma 4.5.** The inner product on $I^\alpha(\sigma)$ defined by the constants $\{c_\lambda\}_{\lambda \in \Lambda^+_n}$ is $\widetilde{Sp}(2n, \mathbb{R})$-invariant if and only if

$$(-\sigma - \rho_n - \frac{\alpha}{2} - 2\lambda_j + j - 1)c_{\lambda+e_j} + (-\sigma + \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1)c_\lambda = 0$$

for all $\lambda \in \Lambda^+_n$ and all $1 \leq j \leq n$. 
We let
\[ N_{\lambda,j} = -\sigma - \rho_n + \frac{\alpha}{2} - 2\lambda_j + j - 1 \]
\[ \quad - \sigma + \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1 = -\frac{c_{\lambda}}{c_{\lambda + e_j}}. \]
Then \( I^\alpha(\sigma) \) is unitarizable if and only if \( N_{\lambda,j} < 0 \) for all \( \lambda \in \Lambda_n^+ \) and for all \( j \).

Note that
\[ N_{\lambda,j} = -\sigma - (\rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1) \]
\[ \quad - \sigma + (\rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1). \]
We write \( \xi = \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1 \). Then \( N_{\lambda,j} = (\sigma - \xi)/(\bar{\sigma} + \xi) \).
Thus \( N_{\lambda,j} \) is real for all \( \lambda \) and for \( j \) if and only if either \( \text{Re}(\sigma) = 0 \) or \( \sigma \) is real. The case \( \text{Re}(\sigma) = 0 \) corresponds to the unitary axis. If \( \sigma \) is real, then
\[ N_{\lambda,j} = -\sigma - \xi < 0 \iff |\sigma| < |\xi|. \]
The minimum value of \( |\xi| \) is \( \frac{1}{2} \) if \( n + \alpha \) is even, and 0 if \( n + \alpha \) is odd. This leads to the following

**Theorem 4.6.** If \( n + \alpha \) is even, then \( I^\alpha(\sigma) \) is unitarizable for \( |\sigma| < \frac{1}{2} \).

5. **Subquotients of \( I^\alpha(\sigma) \)**

In this section, we shall give a detailed description of the module structure of \( I^\alpha(\sigma) \) when it is reducible. We shall describe all the irreducible subquotients of \( I^\alpha(\sigma) \) and determine which of them are unitarizable, i.e., possess a \( G \)-invariant positive definite inner product. We also describe the socle series and module diagram of \( I^\alpha(\sigma) \).

Let
\[ \tilde{\sigma} = \sigma + \rho_n + \frac{\alpha}{2} = \sigma + \frac{n + 1 + \alpha}{2}. \] (5.1)
By Corollary 4.4, \( I^\alpha(\sigma) \) is irreducible if and only if \( \tilde{\sigma} \notin \mathbb{Z} \). Thus we shall assume that \( \tilde{\sigma} \in \mathbb{Z} \) throughout this section.

Let \( \mu = 2\lambda + (\alpha/2)1_n \), as before. Then by the results on transition coefficients in Section 4, we have
\[ V_\mu \rightarrow V_{\mu + 2e_j} \iff 2\lambda_j \neq B_j^+ \quad \text{and} \quad V_\mu \rightarrow V_{\mu - 2e_j} \iff 2\lambda_j \neq B_j^- \]
where
\[ B_j^+ = B_j^+(\alpha, \sigma) = -\sigma - \frac{n + \alpha + 1}{2} + j - 1 = -\tilde{\sigma} + j - 1 \]
and
\[ B_j^- = B_j^-(\alpha, \sigma) = \sigma - \frac{n + \alpha + 1}{2} + j + 1 = \tilde{\sigma} - (n + \alpha) + j. \]
As discussed in Section 4, we may determine the lattice of submodules of \( I^\alpha(\sigma) \) from the transition coefficients \( \{B_j^\pm\}_{1 \leq j \leq n} \). Schematically it goes as follows:

1. Represent \( V_\mu \) by the point \( 2\lambda \in \mathbb{R}^n \) with standard coordinates \( (x_1, \ldots, x_n) \).
2. Barrier to block \( j \)-th rightward movement \( \ell_j^+ : x_j = B_j^+ \).
3. Barrier to block \( j \)-th leftward movement \( \ell_j^- : x_j = B_j^- \).
4. Analyze barriers systematically (Subsections 5.1 and 5.2).

Note that the barrier \( \ell_j^+ : x_j = B_j^+ \) is effective if and only if it cuts at an “even” point, i.e. \( B_j^+ \in 2\mathbb{Z} \). The parity of \( B_j^+ \) depends on the parity of \( \tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2} \) and the parity of \( j \). Similarly for the barrier \( \ell_j^- \), but the parity of \( B_j^- \) also depends on the parity of \( n + \alpha \). So our analysis will be divided into 4 cases.

| \( \tilde{\sigma} \) \( n + \alpha \) | odd | even |
|---|---|---|
| odd | Case 1a | Case 2a |
| even | Case 1b | Case 2b |

We also define the “gap” between \( \ell_j^+ \) and \( \ell_j^- \) by

\[
\text{gap} = B_j^+ - B_j^- = -2\sigma - 2.
\]  

(5.2)

From the lattice of submodules of \( I^\alpha(\sigma) \), we may give its module structure a diagrammatic representation, called the module diagram. It is a directed simple graph \( \mathcal{G} \), defined as follows: The vertex set of \( \mathcal{G} \) is the set of all irreducible subquotients in \( I^\alpha(\sigma) \), to be identified with its collection of \( K \)-types. There is a directed edge from the node \( R_1 \) to the node \( R_2 \) if and only if there are submodules \( U \) and \( V \) of \( I^\alpha(\sigma) \) such that \( V \subseteq U \) and there is a nonsplit exact sequence of infinitesimal \( \widetilde{\text{Sp}}(2n, \mathbb{R}) \)-modules \( 0 \to R_2 \to U/V \to R_1 \to 0 \). We shall also arrange the nodes in \( \mathcal{G} \) in such a way that all the edges are directed downward. Then one can recover the lattice of submodules of \( I^\alpha(\sigma) \) from the graph \( \mathcal{G} \). For a more detailed explanation on module diagram, see Section 7 of [L1].

5.1. Subquotients of \( I^\alpha(\sigma) \): \( n + \alpha \) odd.

Case 1: \( n + \alpha \) odd.
Since $\tilde{\sigma} \in \mathbb{Z}$ and $\frac{n + \alpha + 1}{2} \in \mathbb{Z}$, we have $\sigma \in \mathbb{Z}$. It follows from this and equation (5.2) that the gap $B_j^+ - B_j^-$ is even, i.e. $B_j^+ \equiv B_j^- \pmod{2}$.
This means that for each $j$, either both barriers $\ell_j^+$ and $\ell_j^-$ are effective, or both are not effective.

We consider two subcases:

Case 1a: $\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2}$ is odd

Case 1b: $\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2}$ is even.

For Case 1a, We have

$$B_j^+ = -\left(\sigma + \frac{n + \alpha + 1}{2}\right) + j - 1 \in 2\mathbb{Z} \iff j \text{ is even.}$$

Hence the effective barriers are $\ell_{2i}^+$ and $\ell_{2i}^-$ for all $1 \leq i \leq n_0/2$, where

$$n_0 = 2 \left\lfloor \frac{n}{2} \right\rfloor = \text{largest even integer less than or equal to } n. \quad (5.1.1)$$

But for Case 1b, We have

$$B_j^+ = -\left(\sigma + \frac{n + \alpha + 1}{2}\right) + j - 1 \in 2\mathbb{Z} \iff j \text{ is odd.}$$

So the effective barriers are $\ell_{2i-1}^+$ and $\ell_{2i-1}^-$ for all $1 \leq i \leq (n_1 + 1)/2$, where

$$n_1 = 2 \left\lfloor \frac{n + 1}{2} \right\rfloor - 1 = \text{largest odd integer less than or equal to } n. \quad (5.1.2)$$

Case 1a: $\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2}$ is odd

Let $i$ and $j$ be such that $0 \leq i + j \leq n_0/2$. We will define $R_{ij}(n, \sigma, \alpha)$ as follows:

(i) For $\sigma \leq -1$, let $R_{ij}(n, \sigma, \alpha)$ be the set of $2\lambda$ such that

$$2\lambda_{2i} \geq B_{2i+2}^+ \geq 2\lambda_{2i+2} \quad (5.1.3)$$

$$2\lambda_{n_0-2j} \geq B_{n_0-2j}^- \geq 2\lambda_{n_0-2j+2} \quad (5.1.4)$$

where

$$B_{2i+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + 2i + 1 = -\sigma - \frac{n + \alpha - 1}{2} + 2i,$$

$$B_{n_0-2j}^- = \sigma - \frac{n + \alpha + 1}{2} + n_0 - 2j + 1 = \sigma - \frac{n + \alpha - 1}{2} + n_0 - 2j.$$
(ii) For \( \sigma \geq 0 \), let \( R_{ij}(n, \sigma, \alpha) \) be the set of \( 2\lambda \) such that
\[
2\lambda_{2i} \geq B_{2i}^- \geq 2\lambda_{2i+2} \tag{5.1.5}
\]
\[
2\lambda_{n_0-2j} \geq B_{n_0-2j+2}^+ \geq 2\lambda_{n_0-2j+2} \tag{5.1.6}
\]
where
\[
B_{2i}^- = \sigma - \frac{n + \alpha + 1}{2} + 2i + 1 = \sigma - \frac{n + \alpha - 1}{2} + 2i,
\]
\[
B_{n_0-2j+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + n_0 - 2j + 1 = -\sigma - \frac{n + \alpha - 1}{2} + n_0 - 2j.
\]
Note that the set \( R_{ij}(n, \sigma, \alpha) \) defined above may be empty. When it is nonempty, it will be identified with the direct sum of all the \( K \)-representations \( V_{2\lambda + \alpha/2} \) in \( I^\alpha(\sigma) \) such that \( 2\lambda \in R_{ij}(n, \sigma, \alpha) \).

Remark: When \( \sigma = 0 \) and \( i + j = n_0/2 \), the two conditions (5.1.5) and (5.1.6) coincide. More precisely,
\[
R_{ij}(n, 0, \alpha) = \{2\lambda : 2\lambda_{2i} \geq B_{2i}^- \geq 2\lambda_{2i+2}\}.
\]

In what follows, \( \oplus \) denotes sum in the Grothendieck group of \( G \)-modules. If we have a direct sum of submodules, we will state it explicitly.

**Theorem 5.1.7.** (Case 1a) Assume that \( n + \alpha \) is odd, \( \sigma \) is an integer and \( \tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2} \) is odd.

(a) If \( \sigma \leq -1 \), then
\[
I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_1 \leq i + j \leq \frac{n_0}{2} \right\},
\]
where
\[
r_1 = \max \left( \frac{n_0}{2} + \sigma, 0 \right).
\]

In this case, the module diagram of \( I^\alpha(\sigma) \) can be obtained from Figure 1 by removing those \( R_{ab}(n, \sigma, \alpha) \) which are empty. In particular, the socle series of \( I^\alpha(\sigma) \) is given by
\[
\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} 
\bigoplus_{r_1 \leq i+j \leq r_1+l-1} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_0}{2} - r_1, \\
I^\alpha(\sigma) & l \geq \frac{n_0}{2} - r_1 + 1.
\end{cases}
\]

An irreducible constituent \( R_{ij}(n, \sigma, \alpha) \) of \( I^\alpha(\sigma) \) is unitarizable if and only if \( -\frac{n_0}{2} \leq \sigma \leq -1 \) and \( i + j = r_1 \).
(b) If $\sigma = 0$, then
\[
I^\alpha(0) = \bigoplus_{i+j=n_0/2} R_{ij}(n,0,\alpha)
\]
is a direct sum of irreducible unitary submodules.

(c) If $\sigma \geq 1$, then
\[
I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n,\sigma,\alpha) : r_2 \leq i+j \leq \frac{n_0}{2} \right\},
\]
where
\[
r_2 = \max \left( \frac{n_0}{2} - \sigma, 0 \right).
\]
In this case, the module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 2 by removing those $R_{ab}(n,\sigma,\alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by
\[
\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} 
\bigoplus_{\frac{n_0}{2} - l+1 \leq i+j \leq \frac{n}{2}} R_{ij}(n,\sigma,\alpha) & 1 \leq l \leq \frac{n_0}{2} - r_2, \\
I^\alpha(\sigma) & l \geq \frac{n_0}{2} - r_2 + 1.
\end{cases}
\]
An irreducible constituent $R_{ij}(n,\sigma,\alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if $1 \leq \sigma \leq \frac{n_0}{2}$ and $i+j=r_2$.

Case 1b: $\tilde{\sigma} = \sigma + \frac{n+n_1+1}{2}$ is even

Let $i$ and $j$ be such that $0 \leq i+j \leq (n_1 + 1)/2$. We will define $R_{ij}(n,\sigma,\alpha)$ as follows:

(i) For $\sigma \leq -1$, let $R_{ij}(n,\sigma,\alpha)$ be the set of $2\lambda$ such that
\[
2\lambda_{2i-1} \geq B_{2i+1}^+ \geq 2\lambda_{2i+1} \quad (5.1.8)
\]
\[
2\lambda_{n_1-2j} \geq B_{n_1-2j}^- \geq 2\lambda_{n_1-2j+2} \quad (5.1.9)
\]
where
\[
B_{2i+1}^+ = -\sigma - \frac{n+\alpha+1}{2} + 2i,
\]
\[
B_{n_1-2j}^- = \sigma - \frac{n+\alpha+1}{2} + n_1 - 2j + 1 = \sigma - \frac{n+\alpha-1}{2} + n_1 - 2j.
\]
(ii) For $\sigma \geq 0$, let $R_{ij}(n, \sigma, \alpha)$ be the set of $2\lambda$ such that

$$2\lambda_{2i-1} \geq B_{2i-1}^{\pm} \geq 2\lambda_{2i+1} \quad (5.1.10)$$

$$2\lambda_{n_1-2j} \geq B_{n_1-2j+2}^\pm \geq 2\lambda_{n_1-2j+2} \quad (5.1.11)$$

where

$$B_{2i-1}^- = \sigma - \frac{n + \alpha + 1}{2} + 2i,$$

$$B_{n_1-2j+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + n_1 - 2j + 1 = -\sigma - \frac{n + \alpha - 1}{2} + n_1 - 2j.$$

Remark: When $\sigma = 0$ and $i + j = (n_1 + 1)/2$, the two conditions $(5.1.8)$ and $(5.1.9)$ coincide, that is, $2\lambda_{n_1-2j} = 2\lambda_{2i+2j-1-2j} = 2\lambda_{2i-1}$.

$$B_{n_1-2j+2}^- = -\frac{n + \alpha - 1}{2} + (2i + 2j - 1) - 2j = -\frac{n + \alpha + 1}{2} + 2i = B_{2i-1}^-,$$

and $2\lambda_{n_1-2j+2} = 2\lambda_{2i+2j-1-2j} = 2\lambda_{2j+1}$. Precisely,

$$R_{ij}(n, 0, \alpha) = \{2\lambda: 2\lambda_{2i-1} \geq B_{2i-1}^- \geq 2\lambda_{2i+1}\}.$$

Theorem 5.1.12. (Case 1b) Assume that $n + \alpha$ is odd, $\sigma$ is an integer and $\bar{\sigma} = \sigma + \frac{n + \alpha + 1}{2}$ is even.

(a) If $\sigma \leq -1$, then

$$I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha): r_1 \leq i + j \leq \frac{n_1 + 1}{2} \right\},$$

where

$$r_1 = \max \left( \frac{n_1 + 1}{2} + \sigma, 0 \right).$$

In this case, the module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 1 by removing those $R_{ab}(n, \sigma, \alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by

$$\text{Soc}^\ell(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{r_1 \leq i + j \leq r_1 + l - 1} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_1 + 1}{2} - r_1, \\ I^\alpha(\sigma) & l \geq \frac{n_1 + 1}{2} - r_1 + 1. \end{cases}$$

An irreducible constituent $R_{ij}(n, \sigma, \alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if

(i) $-\frac{n_1 + 1}{2} \leq \sigma \leq -1$ and $i + j = r_1$; or

(ii) $n$ is odd, $\alpha \in \{0, 2\}$ and $(i, j) \in \left\{\left(\frac{n_1 + 1}{2}, 0\right), \left(0, \frac{n_1 + 1}{2}\right)\right\}$.
If $\sigma = 0$, then
\[
I^\alpha(0) = \bigoplus_{i+j=\frac{n_1+1}{2}} R_{ij}(n, 0, \alpha)
\]
is a direct sum of irreducible unitary submodules.

If $\sigma \geq 1$, then
\[
I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_2 \leq i + j \leq \frac{n_1+1}{2} \right\},
\]
where
\[
r_2 = \max \left( \frac{n_1+1}{2} - \sigma, 0 \right).
\]

In this case, the module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 2 by removing those $R_{ab}(n, \sigma, \alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by
\[
\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} 
\bigoplus_{1 \leq l \leq \frac{n_1+1}{2} - r_2} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_1+1}{2} - r_2, \\
I^\alpha(\sigma) & l \geq \frac{n_1+1}{2} - r_2 + 1.
\end{cases}
\]

An irreducible constituent $R_{ij}(n, \sigma, \alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if
\begin{itemize}
  \item[(i)] $1 \leq \sigma \leq \frac{n_1+1}{2}$ and $i + j = r_2$; or
  \item[(ii)] $n$ is odd, $\alpha \in \{0, 2\}$ and $(i, j) \in \left\{ \left( \frac{n+1}{2}, 0 \right), \left(0, \frac{n+1}{2} \right) \right\}$.
\end{itemize}

The module diagram of $I^\alpha(\sigma)$ for $\sigma \leq -1$ and $\sigma \geq 1$ are given in Figure 1 and Figure 2 below. Here
\[
k = \begin{cases} 
\frac{n_0}{2} = \left\lceil \frac{n}{2} \right\rceil & \text{if } \tilde{\sigma} \text{ is odd}, \\
\frac{n_1+1}{2} = \left\lfloor \frac{n+1}{2} \right\rfloor & \text{if } \tilde{\sigma} \text{ is even.}
\end{cases}
\]

and we write a constituent $R_{ij}(n, \sigma, \alpha)$ simply as $R_{ij}$. 

\[(5.1.13)\]
5.2. Subquotients of $I^\alpha(\sigma)$: $n + \alpha$ even.

Case 2: $n + \alpha$ even.

Recall that $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2} \in \mathbb{Z}$. So in this case, $\sigma + \frac{1}{2} \in \mathbb{Z}$, and consequently the gap between the barriers $\ell^+_j$ and $\ell^-_j$ given by

$$\text{gap} = -2\sigma - 2$$

is odd. It follows that along each coordinate axis, exactly one barrier is effective. More precisely, for each $1 \leq j \leq n$, either $\ell^+_j$ is effective or $\ell^-_j$ is effective, but not both. Again, we consider two subcases:

Case 2a: $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$ is odd

Case 2b: $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$ is even.
For Case 2a, $\ell_j^+$ is effective when $j$ is even, and $\ell_j^-$ is effective when $j$ is odd. Case 2b is the other way round.

**Case 2a:** $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$ is odd

Since exactly one barrier is effective along each coordinate axis, the barrier partitions the $K$-types into 2 subsets. For $1 \leq r \leq \frac{n+1}{2} + 1$, let

$X_1^r = \{2\lambda : 2\lambda_{2r-1} < B_{2r-1}^-\}$,

$X_2^r = \{2\lambda : 2\lambda_{2r-1} \geq B_{2r-1}^\circ\}$,

and for $1 \leq s \leq \frac{n+1}{2} + 1$,

$Y_1^s = \{2\lambda : 2\lambda_{2s} \leq B_{2s}^+\}$,

$Y_2^s = \{2\lambda : 2\lambda_{2s} > B_{2s}^+\}$.

Now we let

$$S(n) = \{(i,j) : 0 \leq i \leq \frac{n+1}{2}, \ 0 \leq j \leq \frac{n_0}{2}\}.$$  \hspace{1cm} (5.2.1)

Then for $(i,j) \in S(n)$, we form the intersection

$$L_{i,j}(n,\sigma,\alpha) = (X_1^1 \cap \cdots \cap X_1^{i+1} \cap \cdots \cap X_1^{\frac{n+1}{2}+1}) \cap \ (Y_1^1 \cap \cdots \cap Y_1^{j+1} \cap \cdots \cap Y_1^{\frac{n_0}{2}+1}).$$

More precisely, $2\lambda \in L_{i,j}(n,\sigma,\alpha)$ if and only if

$$2\lambda_{2i-1} \geq B_{2i-1}^- \geq 2\lambda_{2i+1} \text{ and } 2\lambda_{2j} \geq B_{2j+2}^+ \geq 2\lambda_{2j+2},$$  \hspace{1cm} (5.2.2)

where

$$B_{2i-1}^- = \sigma - \frac{n + \alpha + 1}{2} + 2i - 1 + 1 = \sigma - \frac{n + \alpha}{2} + 2i - \frac{1}{2}$$

and

$$B_{2j+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + 2j + 2 - 1 = -\sigma - \frac{n + \alpha}{2} + 2j + \frac{1}{2}.$$  

Note that $L_{a,b}(n,\sigma,\alpha)$ may be empty. If it is nonempty, then we shall identify it with the direct sum of all the $K$-representations $V_{2\lambda+\frac{\alpha}{2}}^1$ with $2\lambda \in L_{a,b}(n,\sigma,\alpha)$.

**Remark.** Let $k = \lfloor n/2 \rfloor$. Then

$$S(n) = \begin{cases} 
(i,j) : 0 \leq i \leq k+1, \ 0 \leq j \leq k & \text{n odd}, \\
(i,j) : 0 \leq i, j \leq k & \text{n even}.
\end{cases} \hspace{1cm} (5.2.3)$$

The following combinatorial result is elementary.
Lemma 5.2.4. (i) Assume that $(i, j) \in \mathbb{S}(n)$ and $i - j \leq -1$. Then $L_{i,j} \neq \emptyset$ if and only if $i - j \geq -\sigma + \frac{1}{2}$. In particular, if $\sigma \leq -1/2$, then all such $L_{i,j}$ is empty.

(ii) For $0 \leq i \leq k$, $L_{i,i}$ is always nonempty.

(iii) If $(j + 1, j) \in \mathbb{S}(n)$, then $L_{j+1,j}$ is always nonempty.

(iv) Assume that $(i, j) \in \mathbb{S}(n)$ and $i - j \geq 2$. Then $L_{i,j} \neq \emptyset$ if and only if $i - j \leq -\sigma + \frac{1}{2}$. In particular, if $\sigma \geq 1/2$, then all such $L_{i,j}$ is empty.

Theorem 5.2.5. (Case 2a) Assume that $n + \alpha$ is even, $\sigma + \frac{1}{2} \in \mathbb{Z}$, and $\bar{\sigma} = \sigma + \frac{\alpha + 1}{2}$ is odd.

(i) If $\sigma \geq 1/2$, then

$$I^\alpha(\sigma) = \bigoplus \{ L_{i,j}(n, \sigma, \alpha) : (i, j) \in \mathbb{S}(n), -1 \leq j - i \leq r_1 \},$$

where

$$r_1 = \min \left( \sigma - \frac{1}{2}, \left[ \frac{n}{2} \right] \right),$$

and each $L_{i,j}(n, \sigma, \alpha)$ which appears in the sum forms an irreducible constituent of $I^\alpha(\sigma)$. The module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 3 by removing those $L_{a,b}(n, \sigma, \alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by

$$\text{Soc}^l (I^\alpha(\sigma)) = \left\{ \bigoplus_{(i,j) \in \mathbb{S}(n), -1 \leq j - i \leq l - 2} L_{i,j}(n, \sigma, \alpha) : 1 \leq l \leq r_1 + 1, \right. \left. l \geq r_1 + 2. \right\}$$

An irreducible constituent $L_{i,j}(n, \sigma, \alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if $i = j + 1$ or $\frac{1}{2} \leq \sigma \leq \left[ \frac{n}{2} \right] + \frac{1}{2}$ and $j - i = r_1$.

(ii) If $\sigma \leq -1/2$, then

$$I^\alpha(\sigma) = \bigoplus \{ L_{i,j}(n, \sigma, \alpha) : (i, j) \in \mathbb{S}(n), 0 \leq i - j \leq r_2 \},$$

where

$$r_2 = \min \left( -\sigma + \frac{1}{2}, \left[ \frac{n+1}{2} \right] \right),$$

and each $L_{i,j}(n, \sigma, \alpha)$ which appears in the sum forms an irreducible constituent of $I^\alpha(\sigma)$. The module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 3 by removing those $L_{a,b}(n, \sigma, \alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by

$$\text{Soc}^l (I^\alpha(\sigma)) = \left\{ \bigoplus_{(i,j) \in \mathbb{S}(n), r_2 - l + 1 \leq i - j \leq r_2} L_{i,j}(n, \sigma, \alpha) : 1 \leq l \leq r_2, \right. \left. l \geq r_2 + 1. \right\}$$

An irreducible constituent $L_{i,j}(n, \sigma, \alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if $i = j$ or $-\left[ \frac{n+1}{2} \right] + \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$ and $i - j = r_2$. 
The module diagram of $I^\alpha(\sigma)$ in the case when $n + \alpha$ is even and $\sigma + \frac{n+\alpha+1}{2}$ is odd can be obtained from the following rectangle (Figure 3) by removing those $L_{a,b}(n, \sigma, \alpha)$ which are empty. Here $k = \lfloor n/2 \rfloor$ and we write a constituent $L_{a,b}(n, \sigma, \alpha)$ simply as $L_{a,b}$. Note also that when $n$ is even, the spaces $L_{k+1,j}$ ($0 \leq j \leq k$) are not defined. In this case, the rectangle below reduces to a square.

![Diagram for $I^\alpha(\sigma)$](image)

**Figure 3: Diagram for $I^\alpha(\sigma)$ when $n + \alpha$ is even and $\sigma + \frac{n+\alpha+1}{2}$ is odd**

**Case 2b: $\sigma + \frac{n+\alpha+1}{2}$ is even**

As in Case 2a, exactly one barrier is effective along each coordinate axis. But now $\ell^+_j$ is effective when $j$ is odd, and $\ell^-_j$ is effective when $j$ is even. So we have to define $L_{a,b}(n, \sigma, \alpha)$ differently. For $1 \leq r \leq \frac{n+1}{2}+1$, let

$$X^+_1 = \{2\lambda : 2\lambda_{2r-1} \leq B^+_{2r-1}\},$$

$$X^-_2 = \{2\lambda : 2\lambda_{2r-1} > B^-_{2r-1}\},$$
and for $1 \leq s \leq \frac{n}{2} + 1$,
\[ Y_1^s = \{2\lambda : 2\lambda_2s < B_{2s}\}, \]
\[ Y_2^s = \{2\lambda : 2\lambda_2s \geq B_{2s}\}. \]

We define the set $S(n)$, as in Case 2a in equation (5.2.1). Then for $(i,j) \in S(n)$, we form the intersection
\[ L_{i,j}(n,\sigma,\alpha) = (X_1^1 \cap \cdots \cap X_i^1 \cap X_i^{i+1} \cap \cdots \cap X_1^{n+1}) \cap (Y_1^1 \cap \cdots \cap Y_j^1 \cap Y_j^{j+1} \cap \cdots \cap Y_1^{\alpha+1}). \]

More precisely, $2\lambda \in L_{i,j}(\sigma,\alpha)$ if and only if
\[ 2\lambda_{2i-1} \geq B_{2i+1}^+ \geq 2\lambda_{2i+1} \quad \text{and} \quad 2\lambda_{2j} \geq B_{2j}^- \geq 2\lambda_{2j+2}. \quad (5.2.6) \]

where
\[ B_{2i+1}^+ = -\sigma - \frac{n+\alpha+1}{2} + 2i + 1 - 1 = -\sigma - \frac{n+\alpha}{2} + 2i - \frac{1}{2} \]

and
\[ B_{2j}^- = \sigma - \frac{n+\alpha+1}{2} + 2j + 1 = \sigma - \frac{n+\alpha}{2} + 2j + \frac{1}{2} \]

**Theorem 5.2.7.** (Case 2b) Assume that $n+\alpha$ is even, $\sigma + \frac{1}{2} \in \mathbb{Z}$, and $\bar{\sigma} = \sigma + \frac{n+\alpha+1}{2}$ is even.

(i) If $\sigma \geq 1/2$, then
\[ I^\alpha(\sigma) = \bigoplus \{ L_{i,j}(n,\sigma,\alpha) : (i,j) \in S(n), \ 0 \leq i - j \leq r_2 \}, \]

where
\[ r_2 = \min(\sigma + \frac{1}{2}, \left[\frac{n+1}{2}\right]), \]

and each $L_{i,j}(n,\sigma,\alpha)$ which appears in the sum forms an irreducible constituent of $I^\alpha(\sigma)$. The module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 4 by removing those $L_{a,b}(n,\sigma,\alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by
\[ \text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), 0 \leq i-j \leq l-1} L_{i,j}(n,\sigma,\alpha) & 1 \leq l \leq r_2, \\ I^\alpha(\sigma) & l \geq r_2 + 1. \end{cases} \]

An irreducible constituent $L_{i,j}(n,\sigma,\alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if $i = j$ or $\frac{1}{2} \leq \sigma \leq \left[\frac{n+1}{2}\right] - \frac{1}{2}$ and $i-j = r_2$. 
(ii) If $\sigma \leq -1/2$, then

\[ I^\alpha(\sigma) = \bigoplus \{ L_{i,j}(n, \sigma, \alpha) : (i, j) \in S(n), \ -1 \leq j - i \leq r_1 \}, \]

where

\[ r_1 = \min \left( -\sigma - \frac{1}{2}, \left[ \frac{n}{2} \right] \right), \]

and each $L_{i,j}(n, \sigma, \alpha)$ which appears in the sum forms an irreducible constituent of $I^\alpha(\sigma)$. The module diagram of $I^\alpha(\sigma)$ can be obtained from Figure 4 by removing those $L_{a,b}(n, \sigma, \alpha)$ which are empty. In particular, the socle series of $I^\alpha(\sigma)$ is given by

\[ \text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), \ r_1-l+1 \leq j-i \leq r_1} L_{i,j}(n, \sigma, \alpha) & 1 \leq l \leq r_1 + 1, \\ I^\alpha(\sigma) & l \geq r_1 + 2. \end{cases} \]

An irreducible constituent $L_{i,j}(n, \sigma, \alpha)$ of $I^\alpha(\sigma)$ is unitarizable if and only if $i = j + 1$ or $-\left[ \frac{n}{2} \right] - \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$ and $j - i = r_1$.

Finally, we describe the module diagram of $I^\alpha(\sigma)$ in the case when both $n+\alpha$ and $\sigma + \frac{n+\alpha+1}{2}$ are even. It can be obtained from the following rectangle (Figure 4) by removing those $L_{a,b}(n, \sigma, \alpha)$ which are empty. As before, $k = \left[ \frac{n}{2} \right]$, and we write a constituent $L_{a,b}(n, \sigma, \alpha)$ simply as $L_{a,b}$. When $n$ is even, $L_{k+1,j}$ ($0 \leq j \leq k$) is empty so that this rectangle reduces to a square.
6. **The Image of \( \psi_{p,q} \) in \( I^\alpha(\sigma) \)**

In this section, we will describe the image of \( \psi_{p,q} \) in \( I^\alpha(\sigma) \):

\[
\psi_{p,q} : \Omega^{p,q} \hookrightarrow I^\alpha(\sigma),
\]

where

\[
\sigma = \frac{p + q}{2} - \frac{n + 1}{2}, \quad \text{and} \quad \alpha \equiv p - q \pmod{4}.
\]

We allow \((p,q) = (0,0)\), in which case we understand \( \Omega^{0,0} \) to be the trivial representation of \( G = \widetilde{Sp}(2n, \mathbb{R}) \).

Denote \( m = p + q \). Observe that

\[
n + \alpha \equiv n + m \pmod{2},
\]

and

\[
\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2} = \frac{m + \alpha}{2} \equiv p \pmod{2}.
\]

Recall that our analysis of \( I^\alpha(\sigma) \) is divided into 4 cases:
Correspondingly, our analysis of $\Omega^{p,q}$ will also be divided into 4 cases:

| $\sigma + n + m$ | odd | even |
|------------------|-----|------|
| odd              | Case 1a | Case 2a |
| even             | Case 1b | Case 2b |

Throughout this section, we assume that we are at a point of reducibility, namely

$$\sigma + \frac{n + \alpha + 1}{2} \in \mathbb{Z}.$$ 

If $\frac{p+q-(n+1)}{2} = \sigma$ and $p - q \equiv \alpha \pmod{4}$, we say that we have a possible embedding of $\Omega^{p,q}$ (into $I^\alpha(\sigma)$).

We shall give the detailed description of $\Omega^{p,q}$ in $I^\alpha(\sigma)$ in the following two subsections. We end this section with a corollary which gives conceptual underpinning to our results.

**Theorem 6.1.** *The relationship between $\Omega^{p,q}$’s and $I^\alpha(\sigma)$ is as follows.*

(a) $-\rho_n \leq \sigma < 0$: The irreducible submodules of $I^\alpha(\sigma)$ are given by the possible embeddings of $\Omega^{p,q}$’s, and all of them are unitary.

(b) Unitary axis ($\sigma = 0$) (when $n + \alpha$ is odd):

$$I^\alpha(0) = \bigoplus_{\substack{p+q=n+1 \\text{\scriptsize (mod 4)}}} I^\alpha(\sigma) = \bigoplus_{\substack{p-q=\alpha \\text{\scriptsize (mod 4)}}} \Omega^{p,q}.$$

(c) $\sigma > 0$: The reducibilities of $I^\alpha(\sigma)$ are completely accounted for by the possible embeddings of $\Omega^{p,q}$’s.

See Part b) of Theorems 6.1.2, 6.1.3, 6.2.2 and 6.2.3 for the precise statements.

6.1. **The image of $\psi_{p,q}$ in $I^\alpha(\sigma)$: $m$ and $n$ different parity.**

**Case 1:** $n + m$ odd.

We consider two subcases:

Case 1a: $p$ is odd  

Case 1b: $p$ is even.

**Case 1a:** we write

$$p = 2i + 1, \quad q = 2j + \epsilon,$$
so that \( p + q = m \). Here

\[
\epsilon = \begin{cases} 
1 & \text{if } m \text{ even}, \\
0 & \text{if } m \text{ odd}.
\end{cases}
\]

Note that

\[
i + j = \frac{m - 1 - \epsilon}{2} = \frac{m + n_0 - (n + 1)}{2} = \frac{n_0}{2} + \sigma.
\]

**Case 1b:** we write

\[
p = 2i, \quad q = 2j + \epsilon',
\]

so that \( p + q = m \). Here

\[
\epsilon' = \begin{cases} 
0 & \text{if } m \text{ even}, \\
1 & \text{if } m \text{ odd}.
\end{cases}
\]

Note that

\[
i + j = \frac{m - \epsilon'}{2} = \frac{m + (n_1 + 1) - (n + 1)}{2} = \frac{n_1 + 1}{2} + \sigma.
\]

In accordance with 5.1.13, we let

\[
k = \begin{cases} 
\frac{n_0}{2} = \left\lfloor \frac{n}{2} \right\rfloor & \text{if } p \text{ is odd}, \\
\frac{n_1 + 1}{2} = \left\lfloor \frac{n + 1}{2} \right\rfloor & \text{if } p \text{ is even}.
\end{cases}
\]

Recall that we have the following module diagrams:

---

**Case 1:** Module diagram for \( I^\alpha(\sigma) \) \((-k \leq \sigma \leq 0)\); \( r_1 = k + \sigma \)

---

**Case 1:** Module diagram for \( I^\alpha(\sigma) \) \((1 \leq \sigma \leq k)\); \( r_2 = k - \sigma \)
We introduce one notation. For an irreducible subquotient $R_{s,t}$ of $I^\alpha(\sigma)$ where $\sigma > 0$, denote by $\langle R_{s,t} \rangle$ the submodule of $I^\alpha(\sigma)$ generated by $R_{s,t}$:

$$\langle R_{s,t} \rangle = \oplus \{ R_{i,j} : i \geq s, j \geq t \}.$$  \hfill (6.1.1)

Recall that

$$\sigma = \frac{m - (n + 1)}{2}, \quad \text{and} \quad p - q \equiv \alpha \pmod{4}.$$  

**Theorem 6.1.2.** (Case 1a) Assume that $n + m$ is odd, and $p$ is odd.

(a) If $\sigma \leq 0$, then

$$\Omega^{\sigma q} = R_{\frac{n}{2}, \frac{q - \epsilon}{2}}.$$  

(b) If $\sigma \geq 1$, then

$$\Omega^{\sigma q} = \langle R_{s,t} \rangle,$$

where $s = \max \left( 0, \frac{n - q}{2} \right)$, and $t = \max \left( 0, \frac{n + 1 - \epsilon - p}{2} \right)$.

**Remarks:**

(a) For $(i, j) = (\frac{p - 1}{2}, \frac{q - \epsilon}{2})$, we have noted that $i + j = \frac{n}{2} + \sigma = r_1$. Thus $R_{i,j}$ is at the bottom layer of the module diagram of $I^\alpha(\sigma)$, namely is an irreducible submodule. The collection of $R_{i,j}$’s exhausts the set of all irreducible submodules of $I^\alpha(\sigma)$.

(b) If $q \leq n$, and $p \leq n + 1 - \epsilon$, then $s = \max \left( 0, \frac{n - q}{2} \right) = \frac{n - q}{2}$, $t = \max \left( 0, \frac{n + 1 - \epsilon - p}{2} \right) = \frac{n + 1 - \epsilon - p}{2}$, and $s + t = \frac{n - \epsilon - \sigma}{2} = \frac{n}{2} - \sigma = r_2$. Such $R_{s,t}$’s are exactly those at the top layer of the module diagram of $I^\alpha(\sigma)$, namely irreducible quotients. The rest of $R_{s,t}$’s are those on the “left boundary” ($s = 0$) or the “right boundary” ($t = 0$).

**Theorem 6.1.3.** (Case 1b) Assume that $n + m$ is odd, and $p$ is even.

(a) If $\sigma \leq 0$, then

$$\Omega^{\sigma q} = R_{\frac{n}{2}, \frac{q - \epsilon'}{2}}.$$  

(b) If $\sigma \geq 1$, then

$$\Omega^{\sigma q} = \langle R_{s,t} \rangle,$$

where $s = \max \left( 0, \frac{n + 1 - q}{2} \right)$, and $t = \max \left( 0, \frac{n + 1 - \epsilon' - p}{2} \right)$.  

Remarks:
(a) For \((i, j) = (\frac{p}{2}, \frac{q-\epsilon'}{2})\), we have noted that \(i + j = \frac{n+1}{2} + \sigma = r_1\). Thus \(R_{i,j}\) is at the bottom layer of the module diagram of \(I^\alpha(\sigma)\), namely is an irreducible submodule. The collection of \(R_{i,j}\)'s exhausts the set of all irreducible submodules of \(I^\alpha(\sigma)\).
(b) If \(q \leq n+1\), and \(p \leq n+1-\epsilon'\), then \(s = \max \left(0, \frac{n+1-\epsilon'}{2} - p\right) = \frac{n+1-\epsilon'}{2}, \quad t = \max \left(0, \frac{n+1-\epsilon'}{2} - p\right) = \frac{n+1-\epsilon'}{2}, \quad s + t = \frac{n+1-\epsilon'}{2} - \sigma = \frac{n+1}{2} - \sigma = r_2\). Such \(R_{s,t}\)'s are exactly those at the top layer of the module diagram of \(I^\alpha(\sigma)\), namely irreducible quotients. The rest of \(R_{s,t}\)'s are those on the “left boundary” \((s = 0)\) or the “right boundary” \((t = 0)\).

6.2. The image of \(\psi_{p,q}\) in \(I^\alpha(\sigma)\): \(m\) and \(n\) same parity.

Case 2: \(n + m\) even.

We consider two subcases:

**Case 2a:** \(p\) is odd  
**Case 2b:** \(p\) is even.

We introduce one similar notation in this case. For an irreducible subquotient \(L_{s,t}\) of \(I^\alpha(\sigma)\) where \(\sigma > 0\), denote by \(\prec L_{s,t} \succ\) the submodule of \(I^\alpha(\sigma)\) generated by \(L_{s,t}\):

\[
\prec L_{s,t} \succ = \bigoplus \{L_{i,j} : i \geq s, j \leq t\}, \quad \text{(Case 2a),}
\bigoplus \{L_{i,j} : i \leq s, j \geq t\}, \quad \text{(Case 2b).}
\]

(6.2.1)

As in Subsection 5.2 we let \(k = \lfloor n/2 \rfloor\). First recall the module diagram of \(I^\alpha(\sigma)\) for Case 2a:

\[
\begin{array}{cccccc}
L_{k,k} & L_{k-1,k-1} & \cdots & L_{1,1} & L_{0,0} \\
L_{k+1,k} & L_{k,k-1} & & & & \\
& L_{k,k-1} & L_{2,1} & L_{1,0} & \\
& & \cdots & & & \\
& L_{k+1,k+1-r_2} & \cdots & L_{r_2,0} & \\
\end{array}
\]

Case 2a: Diagram for \(I^\alpha(\sigma)\) when \(n + m\) is even and \(p\) is odd; \(\sigma \leq -\frac{1}{2}; \quad r_2 = -\sigma + \frac{1}{2} \leq \left\lfloor \frac{n+1}{2} \right\rfloor\)
Case 2a: Diagram for $I^\alpha(\sigma)$ when $n + m$ is even and $p$ is odd; $\sigma \geq \frac{1}{2}$; $r_1 = \sigma - \frac{1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor$.

**Theorem 6.2.2.** (Case 2a) Assume that $n + m$ is even, and $p$ is odd.

(a) If $\sigma \leq -\frac{1}{2}$, then

$$\Omega^{p,q} = L_{\frac{n+1-q}{2}, \frac{p-1}{2}}.$$ 

(b) If $\sigma \geq \frac{1}{2}$, then

$$\Omega^{p,q} \triangleleft L_{s,t} \triangleright,$$

where $s = \max\left(0, \frac{n+1-q}{2}\right)$, and $t = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, \frac{p-1}{2}\right)$.

Remarks:

(a) For $(i, j) = \left(\frac{n+1-q}{2}, \frac{p-1}{2}\right)$, we have $i - j = \frac{n+2-m}{2} = -\sigma + \frac{1}{2} = r_2$. Thus $L_{i,j}$ is at the bottom layer of the module diagram of $I^\alpha(\sigma)$, namely an irreducible submodule. The collection of $L_{i,j}$'s exhausts the set of all irreducible submodules of $I^\alpha(\sigma)$.

(b) If $q \leq n + 1$, and $\frac{p-1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor$, then $s = \max\left(0, \frac{n+1-q}{2}\right) = \frac{n+1-q}{2}$, $t = \min\left(\left\lceil \frac{n}{2} \right\rceil, \frac{p-1}{2}\right) = \frac{p-1}{2}$, and $t - s = m - (n + 2) = \sigma - \frac{1}{2} = r_1$. Such $L_{s,t}$'s are exactly those at the top layer of the module diagram of $I^\alpha(\sigma)$, namely irreducible quotients. The rest of $L_{s,t}$'s are those on the “left boundary” ($t = k = \left\lceil \frac{n}{2} \right\rceil$) or the “right boundary” ($s = 0$). Note that when $n$ is even, the submodule $L_{k+1,k}$ is empty.

Next recall the module diagram of $I^\alpha(\sigma)$ for Case 2b:
Case 2b: Diagram for $I^\alpha(\sigma)$ when $n + m$ and $p$ are even; $\sigma \leq -\frac{1}{2}$; $r_1 = -\sigma - \frac{1}{2} \leq \lfloor \frac{n+1}{2} \rfloor$

Case 2b: Diagram for $I^\alpha(\sigma)$ when both $n + m$ and $p$ are even; $\sigma \geq \frac{1}{2}$; $r_2 = \sigma + \frac{1}{2} \leq \lfloor \frac{n+1}{2} \rfloor$

**Theorem 6.2.3.** (Case 2b) Assume that $n + m$ is even, and $p$ is even.

(a) If $\sigma \leq -\frac{1}{2}$, then
$$\Omega_{p,q} = L_{\frac{p}{2}, \frac{n-q}{2}}.$$ 

(b) If $\sigma \geq \frac{1}{2}$, then
$$\Omega_{p,q} = \prec L_{s,t} \succ,$$ 
where $s = \min \left( \left\lfloor \frac{n+1}{2} \right\rfloor, \frac{p}{2} \right)$, and $t = \max \left( 0, \frac{n-q}{2} \right)$.

Remarks:

(a) For $(i, j) = \left( \frac{p}{2}, \frac{n-q}{2} \right)$, we have $j - i = \frac{n-m}{2} = -\sigma - \frac{1}{2} = r_1$. Thus $L_{i,j}$ is at the bottom layer of the module diagram of $I^\alpha(\sigma)$, namely is an irreducible submodule. The collection of $L_{i,j}$'s exhausts the set of all irreducible submodules of $I^\alpha(\sigma)$. 

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(b) If \( \frac{p}{2} \leq \left\lfloor \frac{n+1}{2} \right\rfloor \) and \( q \leq n \), then \( s = \min \left( \left\lfloor \frac{n+1}{2} \right\rfloor, \frac{p}{2} \right) \), \( t = \max \left( 0, \frac{n-q}{2} \right) = \frac{n-q}{2} \), and \( s - t = \frac{m-n}{2} = \sigma + \frac{1}{2} = r_2 \). Such \( L_{s,t}'s \) are exactly those at the top layer of the module diagram of \( I^\alpha(\sigma) \), namely irreducible quotients. The rest of \( L_{s,t}'s \) are those on the “left boundary” (\( t = 0 \)) or the “right boundary” (\( s = \left\lfloor \frac{n+1}{2} \right\rfloor \)). Note that when \( n \) is even, the subquotient \( L_{k+1,j} \) is empty.

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10
LOWER KENT RIDGE ROAD, SINGAPORE 119076
E-mail address: matleest@nus.edu.sg

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10
LOWER KENT RIDGE ROAD, SINGAPORE 119076
E-mail address: matzhucb@nus.edu.sg