The optimal control problem for magnetoelectric actuator

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Abstract. We considered optimal control problem with a quadratic criterion for a singularly perturbed differential system describing the dynamics of a magnetoelectric actuator. We proposed a coordinate transformation that performs the decomposition of the boundary value problem of the maximum principle to the boundary value problem for slow variables and two initial problems for fast variables.

1. Introduction

We consider a mathematical model of the linear magnetoelectric power drive. Drive dynamics is described by a singularly perturbed system of differential equations. The presence of slow and fast variables makes the problem of decomposition of the system topical.

One of the approaches to the problem of separating fast and slow variables is based on the geometric and asymptotic methods of analysis, the theory of integral manifolds in particular [1-3]. The method of asymptotic decomposition was used to solve the problems of optimal control for multi-scale dynamic systems of various nature [4-10].

The optimal performance problem for a linear magnetoelectric drive was considered in [2]. An algorithm for constructing the switching points as an asymptotic expansion was developed.

We consider the problem of synthesis of an optimal control law for a linear magnetoelectric drive with a quadratic criterion. To construct the optimal control law we use the approach based on the solution of the boundary-value problem of the maximum principle.

2. Mathematical model

We consider a system which consists of a linear magnetoelectric power drive (VCM-type) and a moving mass. The drive represents a copper coil moving in the air gap of permanent magnet. A coil current \( I \) induces the actuating force – Lorenz force. The moving coil generates a voltage \( U \) – back-EMF (Figure 1).

The dynamic characteristics of the system satisfy Kirchhoff’s voltage law and mechanical equation [11–13]

\[
U(t) = I(t)R + LI(t) + K_e v(t) \\
m\ddot{v}(t) + B\dot{v}(t) = K_F I(t),
\]

(1)
where $U = U(t)$ — the voltage of the motor, $I = I(t)$ — the electric current, $L$ — the coil inductivity, $R$ — the coil resistance, $E = K_e v(t)$ — the back-EMF, $m$ — the mass of moving part (including coil), $B$ — the damping constant, $F = K_F I$ — the actuating force (Lorenz force), $x = x(t)$ — the load mass position, $v = v(t) = \dot{x}(t)$ — the load mass linear speed, the dot denotes the time derivative.

Taking into account that in this system the coil inductivity is assumed to be small we introduce a small parameter $\varepsilon = L$. The equations (1) take the form of linear singularly perturbed differential system

\begin{equation}
\dot{x} = v,
\dot{v} = -\mu v + \alpha_1 I,
\varepsilon \dot{I} = -\alpha_2 x^2 - RI + u,
\end{equation}

where $\mu = B/m$, $\alpha_1 = K_F/m$, $\alpha_2 = K_e$, $u = U(t)$ — the input variable (control).

We consider the minimization problem for a quadratic cost functional

\begin{equation}
J = \int_0^1 \left( \beta_1 x^2(t) + \beta_2 v^2(t) + \beta_3 I^2(t) + \gamma u^2(t) \right) dt,
\end{equation}

where $\beta_j \geq 0$, $j = 1, 3$, $\gamma > 0$ on the trajectories of the system (2) with initial conditions

\begin{equation}
x(0) = x_0, \quad v(0) = v_0, \quad I(0) = I_0.
\end{equation}

In accordance with the maximum principle, the control law has the form

\begin{equation}
u = -\frac{1}{\gamma q},
\end{equation}

where the adjoint variable $q$ is the solution of the system

\begin{equation}
\dot{x} = v,
\dot{v} = -\mu v + \alpha_1 I,
\dot{p}_1 = -\beta_1 x,
\dot{p}_2 = -\beta_2 v - p_1 + \mu p_2 + \alpha_2 q,
\varepsilon \dot{I} = -\alpha_2 v - RI - \frac{1}{\gamma} q,
\varepsilon \dot{q} = -\alpha_1 p_2 - \beta_3 I + R q,
\end{equation}
with boundary conditions
\[ x(0) = x_0, \quad v(0) = v_0, \quad I(0) = I_0, \quad p_1(1) = 0, \quad p_2(1) = 0, \quad q(1) = 0. \] (7)

We rewrite the system (6) in the form
\[ \dot{x}_1 = A_{11} x_1 + A_{12} x_2, \]
\[ \varepsilon \dot{x}_2 = A_{21} x_1 + A_{22} x_2, \] (8)

where
\[ x_1 = \begin{pmatrix} x \\ v \\ p_1 \\ p_2 \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ -\beta_1 & 0 & 0 & 0 \\ 0 & -\beta_2 & -1 & \mu \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ \alpha_1 & 0 \\ 0 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \]
\[ x_2 = \begin{pmatrix} I \\ q \end{pmatrix} = \begin{pmatrix} x_{2,1} \\ x_{2,2} \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -R & -\frac{1}{7} \\ -\beta_3 & R \end{pmatrix}. \]

The boundary conditions (7) take the form
\[ x_{1,1}(0) = x_0, \quad x_{1,2}(0) = v_0, \quad x_{1,3}(1) = 0, \quad x_{1,4}(1) = 0, \quad x_{2,1}(0) = I_0, \quad x_{2,2}(1) = 0. \] (9)

3. Decomposition of the system
The system (8) is a linear singular perturbed system. Using the ideas of the method of asymptotic decomposition [2], we split this system to independent "slow" and "fast" subsystems.

We make change of variables
\[ x_3 = x_1 - \varepsilon P x_4, \]
\[ x_4 = x_2 - H x_1, \] (10)

where the matrices \( H = H(\varepsilon) \) and \( P = P(\varepsilon) \) are solutions of the equations
\[ \varepsilon H (A_{11} + A_{12} H) = A_{21} + A_{22} H, \]
\[ PA_2 = \varepsilon A_1 P + A_{12}, \]
\[ A_1 = A_1(\varepsilon) = A_{11} + A_{12} H, \quad A_2 = A_2(\varepsilon) = A_{22} - \varepsilon HA_{12}. \] (11)

As a result we obtain the block-diagonal system
\[ \dot{x}_3 = A_1(\varepsilon) x_3, \] (12)
\[ \varepsilon \dot{x}_4 = A_2(\varepsilon) x_4. \] (13)

The matrices \( H = H(\varepsilon) \) and \( P = P(\varepsilon) \) can be found with any degree of accuracy as an asymptotic series in small parameter \( \varepsilon \)
\[ H = H(\varepsilon) = H_0 + \varepsilon H_1 + \ldots, \]
\[ P = P(\varepsilon) = P_0 + \varepsilon P_1 + \ldots. \] (14)
Substituting (14) into the equations (11) and equating coefficients for the same powers of \( \varepsilon \), we obtain

\[
H_0 = \frac{1}{\delta_1} \begin{pmatrix} 0 & \gamma R \alpha_2 & 0 & -\alpha_1 \\ 0 & -\beta_3 \gamma \alpha_2 & 0 & R \gamma \alpha_1 \end{pmatrix}, \quad \delta_1 = \gamma R^2 + \beta_3,
\]

\[
H_1 = \frac{1}{\delta_1^2} \begin{pmatrix} 0 & -\alpha_2 \gamma (\mu \delta_1 + \gamma R \alpha_1 \alpha_2) & 0 & -\alpha_1^2 \alpha_2 \gamma \\ 0 & -\gamma \alpha_1 (\beta_2 \delta_1 + \alpha_2^2 \beta_3 \gamma) & -\alpha_1 \gamma & \alpha_1 \gamma (\mu \delta_1 + \gamma R \alpha_1 \alpha_2) \end{pmatrix},
\]

\[
P_0 = \frac{1}{\delta_1} \begin{pmatrix} 0 & 0 & -\gamma R \alpha_1 & -\alpha_1 \\ -\gamma R \alpha_1 & -\alpha_1 & 0 & 0 \\ 0 & 0 & -\beta_3 \gamma \alpha_2 & R \gamma \alpha_2 \end{pmatrix},
\]

\[
P_1 = \frac{1}{\delta_1^2} \begin{pmatrix} \delta_1 \gamma \alpha_1 & 0 & -\gamma \alpha_1 (\mu \delta_1 + 2 \gamma R \alpha_1 \alpha_2) & -2 \gamma \alpha_2 \alpha_1^2 \\ 0 & -\gamma \alpha_1 (\beta_2 \delta_1 + 2 \alpha_2^2 \beta_3 \gamma) & \gamma \alpha_2 (\mu \delta_1 + 2 \gamma R \alpha_1 \alpha_2) & 0 \end{pmatrix}.
\]

Matrices \( A_1 \) and \( A_2 \) up to terms of order \( O(\varepsilon) \) will take the form \( A_1 = A_1(\varepsilon) = A_1^{(0)} + \varepsilon A_1^{(1)}, \)

\[
A_2 = A_2(\varepsilon) = A_2^{(0)} + \varepsilon A_2^{(1)},
\]

\[
A_1^{(0)} = \frac{1}{\delta_1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\mu \delta_1 + \gamma R \alpha_1 \alpha_2 & 0 & -\alpha_1^2 \\ 0 & -\beta_1 & 0 & 0 \\ 0 & -\beta_2 \delta_1 + \alpha_2^2 \beta_3 \gamma & -1 (\mu \delta_1 + \gamma R \alpha_1 \alpha_2) & 0 \end{pmatrix},
\]

\[
A_1^{(1)} = \frac{\gamma \alpha_1 \alpha_2}{\delta_1^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mu \delta_1 + \gamma R \alpha_1 \alpha_2 & 0 & -\alpha_1^2 \\ 0 & 0 & 0 & 0 \\ 0 & -\beta_2 \delta_1 + \alpha_2^2 \beta_3 \gamma & -1 (\mu \delta_1 + \gamma R \alpha_1 \alpha_2) & 0 \end{pmatrix},
\]

\[
A_2 = A_2(\varepsilon) = \begin{pmatrix} -R + \varepsilon \gamma R \delta_2 & -\frac{1}{\gamma} + \varepsilon \delta_2 \\ -\beta_3 + \varepsilon \gamma R \delta_2 & R - \varepsilon \gamma R \delta_2 \end{pmatrix}, \quad \delta_2 = \frac{\alpha_1 \alpha_2}{\delta_1}.
\]

Taking into account the structure of the matrix \( A_2(\varepsilon) \), we transform the fast subsystem (13) to block-diagonal form by change of variables

\[
x_4 = \left( \begin{array}{cc} 1 \\ \frac{K}{M+MK} \end{array} \right) z,
\]

(15)
where \( z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \), \( M \) and \( K \) satisfy the equations

\[
M[a_{11}(\varepsilon) + a_{12}(\varepsilon)M] = a_{21}(\varepsilon) + a_{22}(\varepsilon)M, \\
KN_2(\varepsilon) = N_1(\varepsilon)K + a_{22}(\varepsilon),
\]

\[ \text{(16)} \]

\( N_1 = N_1(\varepsilon) = a_{11}(\varepsilon) + a_{12}(\varepsilon)M, \quad N_2 = N_2(\varepsilon) = a_{22}(\varepsilon) - Ma_{12}(\varepsilon), \)

\( a_{ij}(\varepsilon) \) — the elements of the matrix \( A_2(\varepsilon) \).

The functions \( M(\varepsilon), K(\varepsilon) \) can be found from these equations as an asymptotic series in small parameter

\[ M = M_0 + \varepsilon M_1 + \ldots, \quad K = K_0 + \varepsilon K_1 + \ldots. \]

We get

\[ M = -R_\gamma + \sqrt{\gamma} \delta_1 + O(\varepsilon^2), \quad K = -\frac{1}{2\sqrt{\gamma} \delta_1} + O(\varepsilon^2). \]

As a result, the system (13) is reduced to the form

\[ \begin{align*}
\varepsilon \dot{z}_1 &= N_1(z)z_1, \\
\varepsilon \dot{z}_2 &= N_2(z)z_2. \\
N_1(z) &= -\frac{1}{\sqrt{\gamma} \delta_1} (\delta_1 - \varepsilon \alpha_1 \alpha_2 \gamma) + O(\varepsilon^2) > 0, \\
N_2(z) &= \frac{1}{\sqrt{\gamma} \delta_1} (\delta_1 - \varepsilon \alpha_1 \alpha_2 \gamma) + O(\varepsilon^2) < 0.
\end{align*} \]

4. Decomposition of boundary conditions

For splitting the boundary conditions, we apply the resulting coordinates transformation

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} E & \varepsilon D \\ H & V \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}, \]

\[ \text{(18)} \]

\[ D = D(\varepsilon) = \begin{pmatrix} D_{11}(\varepsilon) & D_{12}(\varepsilon) \\ D_{21}(\varepsilon) & D_{22}(\varepsilon) \end{pmatrix} = P \begin{pmatrix} 1 \\ M \end{pmatrix} \begin{pmatrix} K \\ 1 + MK \end{pmatrix}, \quad H = H(\varepsilon) = \begin{pmatrix} H_{11}(\varepsilon) & H_{12}(\varepsilon) \\ H_{21}(\varepsilon) & H_{22}(\varepsilon) \end{pmatrix}, \]

\[ V = V(\varepsilon) = \begin{pmatrix} V_{11}(\varepsilon) & V_{12}(\varepsilon) \\ V_{21}(\varepsilon) & V_{22}(\varepsilon) \end{pmatrix} = (E + \varepsilon HP) \begin{pmatrix} 1 \\ M \end{pmatrix} \begin{pmatrix} K \\ 1 + MK \end{pmatrix}, \quad w = w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}, \]

\[ w_1 = w_1(t) = \begin{pmatrix} w_{1,1}(t) \\ w_{1,2}(t) \end{pmatrix}, \quad w_2 = w_2(t) = \begin{pmatrix} w_{1,3}(t) \\ w_{1,4}(t) \end{pmatrix}, \quad z = z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}. \]

Taking into account that \( N_1 < 0, \quad N_2 > 0, \) and neglecting terms of order \( O(\varepsilon^{-1/\varepsilon}) \), we can rewrite the boundary conditions (9) in the form

\[ w_1(0) + \varepsilon D_{11} z_1(0) = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}, \]

\[ H_{11} w_1(0) + H_{12} w_2(0) + V_{11} z_1(0) = I_0, \]

\[ \text{(19)} \]
Expressing \( z_1(0) \), \( z_2(1) \) from (19), (20), we obtain the boundary value problem for slow variables

\[
\dot{w} = A_1 w, \\
w_1(0) + \varepsilon D_{11} (V_{11})^{-1} [I_0 - H_{11} w_1(0) - H_{12} w_2(0)] = \left( \begin{array}{c} x_0 \\ v_0 \end{array} \right), \\
-\varepsilon D_{22} (V_{22})^{-1} H_{22} w_1(1) + (E - \varepsilon D_{22} (V_{22})^{-1} H_{22}) w_2(1) = 0
\]

and two initial problems for fast variables

\[
\varepsilon \dot{z}_1 = N_1 z_1, \\
z_1(0) = (V_{11})^{-1} [I_0 - H_{11} w_1(0) - H_{12} w_2(0)], \\
\varepsilon \dot{z}_2 = N_2 z_2, \\
z_2(1) = -(V_{22})^{-1} [H_{21} w_1(1) + H_{22} w_2(1)].
\]

Thus, using the linear transformation (18), we split the singularly perturbed boundary value problem (6), (9) into a regularly perturbed boundary-value problem (21) and two singularly perturbed initial problems (22) (23).

With an accuracy of order \( O(\varepsilon) \) the boundary conditions of the problem (21) take the form

\[
w_{1,1}(0) = x_0, \\
w_{1,2}(0) - \varepsilon \sqrt{\frac{\gamma_1}{\delta_1}} (\delta_1 I_0 + R \gamma_2 w_{1,2}(0) + \alpha_1 w_{1,4}(0)) - v_0 = 0, \\
w_{1,3}(1) = 0, \\
w_{1,4}(1) + \varepsilon \alpha_2 (\alpha_2 \beta_3 w_{1,2}(1) - \alpha_1 R w_{1,4}(1)) = 0
\]

As an illustration, let us consider the problem for the following characteristic values of the parameters (Figures 2-8):

\[
R = 2, \; \; L = 0.001, \; \; \alpha_1 = 2, \; \; \alpha_2 = 2, \; \; \beta_1 = 10, \; \; \beta_2 = 0, \; \; \beta_3 = 0, \; \; \gamma = 0.1.
\]
Figures 2, 3 show the dynamics of slow variables \( w_{1,1}(t) - w_{1,4}(t) \), figures 4, 5 — the fast variables \( z_1(t) \), \( z_2(t) \), which are the boundary layer functions.

Figures 6–8 show the optimal trajectory and optimal control law.
5. Conclusions
The method of asymptotic decomposition allows us to split a singularly perturbed boundary value problem to boundary value problem for slow variables and two regular Cauchy problems for fast variables with conditions on the left and right ends of the interval. The solutions of fast subsystems are the boundary layer functions.

6. References
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