A generalized Rayleigh–Taylor condition for the Muskat problem

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Abstract
In this paper we consider the evolution of two fluid phases in a porous medium. The fluids are separated from each other and also the wetting phase from air by interfaces which evolve in time. We reduce the problem to an abstract evolution equation. A generalized Rayleigh–Taylor condition characterizes the parabolicity regime of the problem and allows us to establish a general well-posedness result and to study stability properties of flat steady states. When considering surface tension effects at the interface between the fluids and if the more dense fluid lies above, we find bifurcating finger-shaped equilibria which are all unstable.

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1. Introduction

The Muskat problem [21] is a widely used model for the intrusion of water into oil sand. A linear analysis was performed in [22, 23, 25] where a relation, the so-called Rayleigh–Taylor condition, was found to determine two regimes for the problem: a stable regime, when a flat interface is stable under small deviations, and an unstable one, when fingering occurs.

Nonetheless, the existence and uniqueness of classical solutions was first proven in [26] using Newton’s iteration method. In the last decade the problem has received more interest and was studied by means of complex analysis [24], energy estimates [2, 4, 5], power series expansions [16] or abstract parabolic theory [13]. These different approaches cover a wide spectrum of questions related to the Muskat problem: local well-posedness, global existence of solution, singular solutions, stability properties of equilibria.
It is worth noting that all these papers mentioned above consider the situation when there is only one moving boundary, namely the one separating the fluids. Either one prescribes boundary conditions at two boundaries which are kept fixed during the flow or so-called far-field boundary conditions are imposed. This setting corresponds to an abstract equation with only one unknown—the interface between the fluids. In this paper we consider a more involved situation when there are two moving boundaries, one separating the two fluids and one separating the wetting phase from air (assumed to be at uniform pressure equal to zero). This aspect is more challenging and well-posedness has been established, in the absence of surface tension for fluids having the same viscosities and different densities, only recently, see [6]. In our setting the fluids are located in a porous medium (or a vertical Hele–Shaw cell) and are assumed to fill, together with the dry phase (air), the entire void medium. Moreover, we incorporate gravity and viscosity effects into the modelling as well as surface tension forces at both interfaces.

Our approach is different from that in [6], where energy methods are the main tool in solving the problem. We prove herein that the setting is in fact equivalent to an abstract nonautonomous evolution equation

\[ \partial_t Z = \Phi(t, Z), \quad Z(0) = Z_0, \]

where the variable \( Z \) parametrizes both unknown interfaces. The temporal variable \( t \) is induced into the problem by the boundary condition \( b \) for the pressure on the bottom of the cell. For this problem we find a generalized Rayleigh–Taylor condition in terms of only the boundary data \( b \), the viscosities \( \mu \) and densities \( \rho \) of the fluids of the following form

\[
\begin{align*}
\mu_+ g \rho_+ &> 0 \quad \text{and} \quad \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} (b - g \rho_+) + g (\rho_+ - \rho_-) < 0,
\end{align*}
\]

which determines the parabolic character of the problem in the absence of surface tension effects. When including surface tension forces at both interfaces we may drop condition (1.1). We steadily use in this paper the subscript \(-\) for the fluid on the bottom of the cell and \(+\) for that above. After showing that the Fréchet derivative \( \partial_Z \Phi(0) \) generates a strongly continuous and analytic semigroup, parabolic theory provides local well-posedness of the problem and the principle of linearized stability may be applied to study the stability properties of the unique flat equilibrium which is determined for a fixed amount of fluid \(+\) (this quantity is preserved by the flow) and a certain constant boundary data.

When considering surface tension effects at the interface between the fluids and if the more dense fluid lies above we rediscover the global bifurcation branches obtained in [14] which consist only of finger-shaped equilibria of the Muskat problem. The exchange of stability theorem according to Crandall and Rabinowitz [8] applies to this particular problem and we show that all small equilibria are unstable.

The outline of this paper is as follows: we describe in section 2 the mathematical model and present the main results. Section 3 is dedicated to the proof of the well-posedness result theorem 2.1, and in the subsequent section we analyse the stability properties of the unique flat equilibrium as stated in theorem 2.5. In section 5 we prove our third main result, theorem 2.7. The calculations leading to the representation of \( \partial_Z \Phi(0) \) as a Fourier multiplication operator are performed in the appendix.

2. The mathematical model and the main results

Let us start this section by presenting the mathematical model of the setting described in the introduction. Given \( m \in \mathbb{N} \) and \( \beta \in (0, 1) \) the small Hölder space \( h^{m+\beta}(S) \) stands for
the closure of the smooth functions $C^\infty(S)$ in $C^{\alpha \beta}(S)$. We let $S$ denote the unit circle and functions on $S$ are identified with $2\pi$-periodic functions on $\mathbb{R}$.

For later purposes, let $h_0^{\alpha \beta}(S)$ denote the subspace of $h^{\alpha \beta}(S)$ consisting only of even functions, $h_0^{\alpha \beta}(S)$ be the subspace of $h^{\alpha \beta}(S)$ consisting only of functions with integral mean zero, and $h_0^{\alpha \beta}(S) := h_0^{\alpha \beta}(S) \cap h^{\alpha \beta}(S)$. Furthermore, we define the set of admissible functions to be

$$U := \{ f \in C^2(S) : |f| < 1/2 \}.$$ 

Each pair $(f, h) \in U^2$ determines two open and simply connected subsets of the porous medium, seen as $S \times (-1, 2) \subset S \times \mathbb{R}$, as follows:

$$\Omega(f) := \{(x, y) : -1 < y < f(x)\},$$

$$\Omega(f, h) := \{(x, y) : f(x) < y < 1 + h(x)\}.$$ 

Let $T > 0$ and $(f, h) : [0, T] \to U^2$ be given such that, at each time $t \in [0, T]$, fluid $-$ is located at $\Omega(f(t))$ and fluid $+$ at $\Omega(f(t), h(t))$ (see figure 1). The two fluids are assumed to be of Newtonian type, incompressible, and in Darcy’s approximation. Both interfaces are supposed to move along with the fluids. The problem is governed by the following system of partial differential equations:

$$\begin{align*}
\Delta u_+ &= 0 & \text{in } \Omega(f, h), \\
\Delta u_- &= 0 & \text{in } \Omega(f), \\
\frac{k\sqrt{1 + h^2}}{\mu_+} \partial_n u_+ &= 0 & \text{on } \Gamma(h), \\
u_+ &= g\rho_+(1 + h) - \gamma_h \kappa/h & \text{on } \Gamma(h), \\
u_- &= b & \text{on } \Gamma_-, \\
u_+ - u_- &= g(\rho_+ - \rho_-)f + \gamma_f \kappa_1 & \text{on } \Gamma(f), \\
\frac{k\sqrt{1 + f^2}}{\mu_\pm} \partial_n u_\pm &= 0 & \text{on } \Gamma(f), \\
f(0) &= f_0, \\
h(0) &= h_0
\end{align*}$$

(2.1)

for $t \in [0, T]$, where $(f_0, h_0) \in U^2$ determines the initial domains occupied by the fluids. We used the variable $f$ for parametrizing the interface $\Gamma(f) := \{ y = f(x) \}$ between the two fluids and $\Gamma(h) := \{ y = 1 + h(x) \}$ separates fluid $+$ from air. The unit normal $\nu$ at $\Gamma(f)$ (respectively $\Gamma(h)$) is chosen such that, if $\tau$ is the tangent, the orthonormal basis $(\tau, \nu)$ has positive orientation. We also write $\kappa_{\Gamma(f)}$ and $\kappa_{\Gamma(h)}$ for the curvature of $\Gamma(f)$ and $\Gamma(h)$, respectively. Moreover, $\gamma_h$ (respectively $\gamma_f$) is the surface tension coefficient of the interface separating the fluids from air (respectively the fluids).

The potentials $u_\pm$ incorporate both pressure and gravity force $u_\pm := p_\pm + g\rho_\pm y$, with $g$ the gravity constant. The velocity fields $\vec{v}_\pm$, which satisfy Darcy’s law

$$\vec{v}_\pm = -\frac{k}{\mu_\pm} \nabla u_\pm,$$
are presupposed to be equal on the boundary separating the fluid phases. Hereby, \(k\) stands for the permeability of the porous medium. On the fixed boundary \(\Gamma_{-1} := \mathbb{S} \times \{-1\}\) we prescribed the value of the velocity potential \(u_-\). For a precise deduction of (2.1) we refer to [11, 13, 26].

Let \(\alpha \in (0, 1)\) be fixed for the following. A pair \((f, h, u_+, u_-)\) is called a *classical Hölder solution* of (2.1) if

\[
(f, h) \in C([0, T], \mathcal{V}) \cap C^1([0, T], (h^{1+\alpha}(\mathbb{S}))^2),
\]

\[
u_+(t) \in \text{buc}^{2+\alpha}(\Omega(f(t), h(t))) \quad \text{and} \quad u_-(t) \in \text{buc}^{2+\alpha}(\Omega(f(t)))
\]

for \(t \in [0, T]\), and if \((f, h, u_+, u_-)\) satisfies the equations of (2.1) pointwise. We defined \(\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2\) to be the subset of \(\mathcal{U}_2\) given by

\[
\mathcal{V}_1 := \{f \in h^{2+2\text{sign}(\gamma f) + \alpha(S)} : f \in \mathcal{U}\}, \quad \mathcal{V}_2 := \{h \in h^{2+2\text{sign}(\gamma h) + \alpha(S)} : h \in \mathcal{U}\},
\]

where \(\text{sign}(0) = 0\) and \(\text{sign}(\gamma) = 1\) for \(\gamma > 0\). Later on, we shall see that there are steady-state solutions to (2.1) with the property that \(h = 0\). In this particular case we extend the possible domain of the interface \(f\) to

\[
\mathcal{V}_1 := \{f \in h^{2+2\text{sign}(\gamma f) + \alpha(S)} : |f| < 1\}.
\]

The space \(\text{buc}^{2+\alpha}(\Omega(f))\) is defined as closure of the smooth functions with bounded and uniformly continuous derivatives \(\text{BUC}^\infty(\Omega(f))\) in \(\text{BUC}^{2+\alpha}(\Omega(f))\). The space \(\text{buc}^{2+\alpha}(\Omega(f, h))\) is defined similarly. Moreover, since the potentials \(u_\pm\) are determined, when knowing \((f, h)\), as solutions of elliptic problems (see section 3) we also refer to \((f, h)\) to be the solution of (2.1). The first main result of this paper is the following theorem.

**Theorem 2.1.** Let \((\gamma_f, \gamma_h) \in (0, \infty)^2\) be given.

There exist an open neighbourhood \(\mathcal{O}\) of the zero function in \((h^{4+\alpha}(\mathbb{S}))^2\) such that for all \((f_0, h_0) \in \mathcal{O}\) and \(b \in C([0, \infty), h^{2+\alpha}(\mathbb{S}))\) problem (2.1) possesses a unique classical Hölder solution \(\mathcal{F}(\cdot; (f_0, h_0))\) defined on a maximal time interval \([0, T(f_0, h_0))\) and which satisfies \(\mathcal{F}(0, T(f_0, h_0)) \subset \mathcal{O}\). Given \((f_0, h_0) \in \mathcal{O}\), the mapping

\[
\mathcal{F}(\cdot; (f_0, h_0)) : (0, T(f_0, h_0)) \to (h^{4+\alpha}(\mathbb{S}))^2
\]

is of class \(C^k\) if \(b\) has this property.

**Remark 2.2.** The conclusion of theorem 2.1 remains valid if \(\gamma_h = 0\) or \(\gamma_f = 0\) with the following modifications: if \(\gamma_h = \gamma_f = 0\) we have to replace \((h^{4+\alpha}(\mathbb{S}))^2\) by \((h^{2+\alpha}(\mathbb{S}))^2\) and
require that \( b \in C([0, \infty), c + \mathcal{O}_0) \) where \( \mathcal{O}_0 \) is a small neighbourhood of the zero function in \( h^{4\nu_a}(S) \) and \( c \in \mathbb{R} \) satisfies

\[
\begin{align*}
&c\mu_+ + g\rho_+\mu_- > 0, \\
&\frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} (c - g\rho_+) + g(\rho_+ - \rho_-) < 0.
\end{align*}
\]

When \((\gamma_f, \gamma_h) \in (0, \infty) \times \{0\}\) (respectively \((\gamma_f, \gamma_h) \in \{0\} \times (0, \infty)\)) we replace \( h^{4\nu_a}(S) \times h^{2\nu_a}(S) \) (respectively \( h^{2\nu_a}(S) \times h^{4\nu_a}(S) \)) and request that the constant \( c \) satisfies only equation (2.2) (respectively equation (2.3)).

Solutions have the property that the volume of fluid + is preserved by the flow.

**Lemma 2.3.** If \((f, h)\) is a solution of (2.1), the quantity \( \int_S h(t, x) - f(t, x) \, dx \) is constant in time.

**Proof.** The proof is similar to that of [10, lemma 3.1].

In order to establish similarity between our problem when \( \gamma_h > 0 \) and that in [13, 27], we determine a special solution of (2.1) in the case when the volume of fluid + is equal to \( 2\pi \), i.e.

\[
\int_S f_0 - h_0 \, dx = 0.
\]

If initially \( f(0) = h(0) = f_0 \in \mathbb{R} \) and \( b \) depends only on time, then the unique classical solution is independent of \( x \), that is \( (f(t, x), h(t, x)) = (f(t), h(t)) \), and

\[
f'(t) = \frac{-kg\rho_-}{\mu_-} f(t) + \frac{g\rho_+ - b}{\mu_+ + \mu_-}, \quad f(0) = f_0.
\]

Moreover, by lemma 3.1, \( f(t) = h(t) \) as long as the solution exists. If \( \rho_+ = \rho_- \) and \( \mu_- > \mu_+ \) we obtain from (2.3) that if \( \gamma_f = 0 \), then \( b > g\rho_+ \), thus \( f' \) is positive if \( f_0 \) is close to zero, meaning that the more viscous fluid drives the less viscous one upwards in the medium. This condition has also been found in [27] to guarantee well-posedness of the Muskat problem studied therein. Moreover, if the Atwood number

\[
A_\mu := (\mu_+ - \mu_-)/\mu_+ + \mu_-
\]

is zero, then (2.3) tells us that the more dense fluid must lay beneath in order to guarantee well-posedness of (2.1) when \( \gamma_f = 0 \).

As a corollary of theorem 2.1 we have the following result.

**Corollary 2.4.** If the fluid below is water and that above oil, and we neglect the surface force at the interface between them, we find from (2.3) an optimal value

\[
p_{\text{max}} := g(\rho_+ + \rho_-) - g(\rho_+ - \rho_-)A_\mu^{-1}
\]

for the pressure on the bottom of the porous medium below which water may drive oil upwards in a stable regime (no fingering occurs).

**Proof.** Relation (2.6) is obtained from (2.3) in view of \( u_- = 1 - g\rho_- \) on \( \Gamma_- \). The optimal value for the potential \( b \) is \( b_{\text{max}} = p_{\text{max}} - g\rho_- \), and if the boundary value \( b \) is close to this value we find that the solutions of (2.5) fulfill \( f' > 0 \), thus water drives oil upwards. This last assertion follows from

\[
g\rho_+ - b_{\text{max}} = g(\rho_+ - \rho_-)A_\mu^{-1} < 0
\]
since it is well known [3] that \( \rho_+ < \rho_- \) and \( \mu_+ > \mu_- \) (oil is less dense and more viscous than water).

We infer from (2.5) that if \( b = g\rho_+ \) and \( f_0 = 0 \), then \( f(t) = h(t) = 0 \) for all \( t \geq 0 \). Concerning the stability properties of the stationary solution \((f, h) = (0, 0)\), which is the unique flat stationary solution of (2.1) for \( b = g\rho_+ \) and which satisfies (2.4).

**Theorem 2.5.** Let \( \gamma_f, \gamma_h \in [0, \infty) \), and \( b = g\rho_+ \). Then:

(i) If \( g(\rho_- - \rho_+) + \gamma_f > 0 \), then the flat equilibrium \((f, h) = (0, 0)\) of (2.1) is exponentially stable. More precisely, there exist positive constants \( M, \delta \) and \( \sigma \) such that if \( \| (f_0, h_0) \|_{h^{2+2(\gamma_f + \gamma_h)\omega}(S) + h^{2+2(\gamma_f + \gamma_h)\omega}(S)} \leq \delta \) and \((f_0, h_0)\) satisfies (2.4), then the solution \((f, h)\) of (2.1) exists globally and

\[
\| (f(t), h(t)) \|_{h^{2+2(\gamma_f + \gamma_h)\omega}(S) + h^{2+2(\gamma_f + \gamma_h)\omega}(S)} + \| (\partial_t f(t), \partial_t h(t)) \|_{h^{2+4\omega}(S)}^2 \\
\leq Me^{-\sigma t} \| (f_0, h_0) \|_{h^{2+2(\gamma_f + \gamma_h)\omega}(S) + h^{2+2(\gamma_f + \gamma_h)\omega}(S)} \forall t \geq 0.
\]

(ii) If \( g(\rho_- - \rho_+) + \gamma_f < 0 \), then \((f, h) = (0, 0)\) is unstable.

**Remark 2.6.** When we study the stability of equilibria in theorem 2.5 and theorem 2.7 below we fixed the volume of fluid + equal to \( 2\pi r \), meaning that the initial data of (2.1) are presupposed to satisfy (2.4). This setting is imposed by lemma 2.3, since the volume of fluid + is preserved by the solutions of (2.1).

Note that if \( \gamma_f = 0 \), then the flat solution is always stable, since \( \rho_- > \rho_+ \) is exactly the condition (2.3) which guarantees well-posedness of (2.1). Concerning the unstable case, numerical experiments [17] show that the interface between the fluids becomes very ramified, and dendrite-like structures occur as time evolves if \( g(\rho_- - \rho_+) + \gamma_f < 0 \).

If \( b = g\rho_+ \) and the volume of fluid + is equal to \( 2\pi r \), there also exist other stationary solutions of (2.1). They appear only in the unstable regime or sufficiently close to it, that is when \( \gamma_f > 0 \) and the more dense fluid lies above in the cell. We show that for certain small \( \gamma_f > 0 \) there exist finger-shaped stationary solutions of (2.1), and therefore we shall also refer to \((\gamma_f, f, h)\) to be a solution of (2.1). Given \( 1 \leq l \in \mathbb{N} \), we define

\[
\mathcal{P}_l := g(\rho_+ - \rho_-) / l^2.
\]

**Theorem 2.7.** Let \( b = g\rho_+ \) and \( \gamma_f (\rho_+ - \rho_-) > 0 \). If \((\gamma_f, f, h)\) is a stationary solution of (2.1) satisfying (2.4), then \( h = 0 \) and \((\gamma_f, f)\) is a solution of the Laplace–Young equation

\[
\gamma_f \kappa(f) + g(\rho_+ - \rho_-) f = 0.
\]

The solution of (2.7) is, up to a translation, even and all even solutions of (2.7) can be represented as a disjoint union

\[
\bigcup_{l=1}^{\infty} \{ (\gamma_l(\varepsilon), f_l(\varepsilon)) : \varepsilon \in \mathbb{R} \} \bigcup_{l=1}^{\infty} (\mathcal{P}_l, \mathcal{P}_l) \times \{0\} \bigcup (\mathcal{P}_l, \infty) \times \{0\},
\]

with continuous functions \((\gamma_l, f_l) : \mathbb{R} \rightarrow (0, \infty) \times \{ f \in h^{2+4\omega}(S) : \| f \|_{C(S)} < 1 \}, 1 \leq l \in \mathbb{N}\), which, near \( \varepsilon = 0 \), are real analytic and satisfy

\[
\gamma_l(\varepsilon) = \mathcal{P}_l + \frac{3g(\rho_+ - \rho_-)}{8} \varepsilon^2 + O(\varepsilon^4), \quad f_l(\varepsilon) = \varepsilon \cos(|x|) + O(\varepsilon^2).
\]

While \( \gamma_l \) is even and

\[
\lim_{|\varepsilon| \to \infty} \gamma_l(\varepsilon) = \frac{2\pi^2 g(\rho_+ - \rho_-)}{B^2(3/4, 1/2)^2},
\]

either \( \| f_l(\varepsilon) \|_{C(S)} \not\to \infty \) or \( \| f_l(\varepsilon) \|_{C(S)} \not\to \infty \).
Additionally, the equilibrium \((\gamma(\varepsilon), f(\varepsilon), 0)\) of problem (2.1) is unstable if \(|\varepsilon|\) is small. When \(l = 1\) we have to assume \(\varepsilon \neq 0\) too.

Here \(B\) stands for Euler’s beta function. Note that the stationary solutions of (2.1), which satisfy (2.4) (see in figure 2), are the same as the stationary solutions of the Muskat problem studied in \([5, 13]\), where just one moving boundary is considered (\(h\) is chosen \(a\ priori\) to be zero). For a precise description of the global bifurcation branches \((\gamma_l, f_l)\) we refer to \([14]\). It is shown there that the situation \(\|f(\varepsilon)\|_C(S) \to \infty \) may occur only for small integers \(l\).

3. The evolution equation

In order to solve problem (2.1) we re-write it as an abstract evolution equation on the unit circle. To do that we first transform system (2.1) into a system of equations on fixed domains using the unknown functions \((f, h)\). Let \(\Omega_- := \Omega(0)\) and \(\Omega_+ := \Omega(0, 0)\). Given \((f, h) \in V\) we define the mappings \(\phi_f = (\phi^1_f, \phi^2_f) : \Omega_- \to \Omega(f)\) by

\[
\phi_f(x, y) := (x, y + (1 + y)f(x)), \quad (x, y) \in \Omega_-,
\]

respectively, \(\phi_{f,h} = (\phi^1_{f,h}, \phi^2_{f,h}) : \Omega_+ \to \Omega(f, h)\)

\[
\phi_{f,h}(x, y) := (x, y(1 + h(x)) + (1 - y)f(x)), \quad (x, y) \in \Omega_+.
\]

One can easily check that \(\phi_f\) and \(\phi_{f,h}\) are diffeomorphisms for all \((f, h) \in V\). These diffeomorphisms induce pull-back and push-forward operators (see e.g. \([11]\)) which we use to transform the differential operators involved in system (2.1) into operators on the domains \(\Omega_{\pm}\) and their boundaries, respectively. Each pair \((f, h) \in V\) induces linear elliptic operators

\[
\mathcal{A}(f) : \text{buc}^{2\alpha}(\Omega_-) \to \text{buc}^{-\alpha}(\Omega_-), \quad \mathcal{A}(f)v := \Delta (v \circ \phi_f^{-1}) \circ \phi_f,
\]

\[
\mathcal{A}(f, h) : \text{buc}^{2\alpha}(\Omega_+) \to \text{buc}^{-\alpha}(\Omega_+), \quad \mathcal{A}(f, h)v := \Delta (v \circ \phi_{f,h}^{-1}) \circ \phi_{f,h},
\]

which depend, as bounded operators, analytically on \(f\) and \(h\). Denote by \(\text{tr}_0\) the trace operator with respect to \(\Gamma_0 := S \times \{0\}\). We associate problem (2.1) with the following trace operators on \(\Gamma_0\):

\[
B(f)v_- := k_m^{-1}\text{tr}_0((\nabla(v_- \circ \phi_f^{-1})(-f', 1)) \circ \phi_f), \quad v_- \in \text{buc}^{2\alpha}(\Omega_-),
\]

\[
B(f, h)v_+ := k_m^{-1}\text{tr}_0((\nabla(v_+ \circ \phi_{f,h}^{-1})(-f', 1)) \circ \phi_{f,h}), \quad v_+ \in \text{buc}^{2\alpha}(\Omega_+),
\]
which, seen as bounded operators into $h^{1+a}(\mathbb{S})$, depend analytically on $f$ and $h$ as well. Lastly, we define a boundary operator on $\Gamma_1 := \mathbb{S} \times \{1\}$. Given $(f, h) \in \mathcal{V}$, we let

$$B_1(f, h) v_+ := -k \mu_1^{-1} \text{tr}_1((\nabla (v_+ \circ \phi_{f, h}^{-1}))(\nabla h, 1) \circ \phi_{f, h}), \quad v_+ \in \text{buc}^{2+a}(\Omega_+),$$

whereby $\text{tr}_1$ is the trace operator with respect to $\Gamma_1$.

With this notation one can easily verify that if $(f, h, u_+, u_-)$ is a solution of (2.1), then $(f, h, v_+ := u_+ \circ \phi_{f, h}, v_- := u_- \circ \phi_{f})$ solves the following system of equations:

$$\begin{aligned}
A(f, h) v_+ &= 0 \quad \text{in } \Omega_+, \\
A(f) v_- &= 0 \quad \text{in } \Omega_-, \\
\partial_t h &= B_1(f, h) v_+ \quad \text{in } \Gamma_1, \\
v_+ &= \rho(f + \phi_1) h + \gamma h \kappa(h) \quad \text{on } \Gamma_0, \\
v_- &= b \quad \text{on } \Gamma_{-1}, \\
v_+ - v_- &= \rho(f + \phi_1) h + \gamma h \kappa(f) \quad \text{on } \Gamma_0, \\
\partial_t f + B(f) v_+ &= 0 \quad \text{on } \Gamma_0, \\
\partial_t f + B(f, h) v_+ &= 0 \quad \text{on } \Gamma_0, \\
f(0) &= f_0, \\
h(0) &= h_0
\end{aligned} \tag{3.1}$$

for all $t \in [0, T]$, where the transformed curvature operator $\kappa : h^{4+a}(\mathbb{S}) \to h^{2+a}(\mathbb{S})$ is defined by $\kappa(f) := f''/(1 + f^2)^{3/2}$. The notion of solution of (3.1) is defined analogously to that of (2.1). Note that the parametrization $(f, h)$ is left invariant by the transformation above. In fact, one can see [11, lemma 1.2] that each solution of (3.1) corresponds to a unique solution of (2.1).

We now introduce solution operators corresponding to the system (3.1). Given $f \in \mathcal{V}_1$ and $(q, p) \in h^{1+a}(\mathbb{S}) \times h^{2+a}(\mathbb{S})$, we let $T(f, q, p) \in \text{buc}^{2+a}(\Omega_+)$ denote the solution of the linear, elliptic mixed boundary value problem

$$\begin{aligned}
A(f) v_+ &= 0 \quad \text{in } \Omega_+, \\
B(f) v_- &= q \quad \text{on } \Gamma_0, \\
v_+ &= p \quad \text{on } \Gamma_{-1}.
\end{aligned} \tag{3.2}$$

Further on, we define $S : \mathcal{V} \times (h^{2+a}(\mathbb{S}))^2 \to \text{buc}^{2+a}(\Omega_+)$ by writing $S(f, h, p, r)$ for the unique solution of the problem

$$\begin{aligned}
A(f, h) v_+ &= 0 \quad \text{in } \Omega_+, \\
v_+ &= p \quad \text{on } \Gamma_1, \\
v_+ &= r \quad \text{on } \Gamma_0.
\end{aligned} \tag{3.3}$$

It is convenient to write $T(f, q, p) = T_1(f) q + T_2(f) p$, where

$$T_1(f) q := (A(f), B(f), \text{tr})^{-1}(0, q, 0), \quad T_2(f) p := (A(f), B(f), \text{tr})^{-1}(0, 0, p),$$

respectively $S(f, h, p, r) = S_1(f, h, p) + S_2(f, h, r)$, with

$$S_1(f, h, p) := (A(f, h), \text{tr}, \text{tr})^{-1}(0, p, 0), \quad S_2(f, h) r := (A(f, h), \text{tr}, \text{tr})^{-1}(0, 0, r).$$

The operators $T_i(f, h)$ and $S_i(f, h)$, $i = 1, 2$, are bounded linear operators and they depend, in the norm topology, analytically on $f$ and $h$ as well.
The key point of our analysis is the following observation. If \((f, h, v_+, v_-)\) is a classical solution of (3.1) for the initial data \((f_0, h_0)\), then the following must hold:

(i) \(f(0) = f_0\) and \(h(0) = h_0\),
(ii) \(v_- = T(f, -\partial_t f, b)\),
(iii) \(v_+ = S(f, h, g\rho_+(1 + h) - \gamma_h\kappa(h), \text{tr}_0 v_- + g(\rho_+ - \rho_-)f + \gamma_f\kappa(f))\),
(iv) \(\partial_t f + B(f, h)v_+ = 0\),
(v) \(\partial_t h = B_1(f, h)v_+\).

Let us now show that from (ii)–(iv) we can determine the derivative \(\partial_t f\) as a function of \(f, h\), and \(t\) only. Indeed, we plug (ii) into (iii) and (iii) into (iv) to obtain the equation

\[
\partial_t f + B(f, h)S_1(f, h)[g\rho_+(1 + h) - \gamma_h\kappa(h)] + B(f, h)S_2(f, h)[\text{tr}_0 T(f, -\partial_t f, b) + g(\rho_+ - \rho_-)f + \gamma_f\kappa(f)] = 0,
\]

which can be written equivalently as

\[
(id_{\text{htw}(\mathbb{S})} - B(f, h)S_2(f, h)\text{tr}_0 T_1(f) - B(f, h)S_2(f, h)\text{tr}_0 T_2(f)b)
\]

In the following lemma we show that the linear operator which is evaluated at \(\partial_t f\) is invertible, so that, by applying its inverse to (3.4), we obtain an equation expressing the derivative \(\partial_t f\) in dependence of \(f, h, t\).

**Lemma 3.1.** The set \(\mathcal{V}\) contains an open neighbourhood \(\mathcal{W}\) of \(0\) with the property that

\[
\mathcal{G}(f, h) := id_{\text{htw}(\mathbb{S})} - B(f, h)S_2(f, h)\text{tr}_0 T_1(f) \in \mathcal{L}(h^{1\text{tw}}(\mathbb{S}))
\]

is an isomorphism for all \((f, h) \in \mathcal{W}\).

**Proof.** The proof is based on a continuity argument. Namely, all the operators defined in this section depend analytically on their variables and then so does \(\mathcal{G}\). Thus, it suffices to show that \(\mathcal{G}(0, 0)\) is an isomorphism. To do that, we represent \(\mathcal{G}(0, 0)\) as a Fourier multiplication operator. Given \(q \in h^{1\text{tw}}(\mathbb{S})\) we let \(q = \sum_{m \in \mathbb{Z}}\hat{h}(m)e^{imx}\) denote its Fourier series expansion. A Fourier series ansatz yields for \(T_1(0)q\) the following expression:

\[
T_1(0)q(x, y) = \frac{\mu_-}{k} (1 + y)\hat{q}(0) + \frac{\mu_-}{k} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{e^{m} e^{my} - e^{-m} e^{-my}}{m(e^{m} + e^{-m})}\hat{q}(m)e^{imx}
\]

for \((x, y) \in \Omega_-\). Respectively, if \(r = \sum_{m \in \mathbb{Z}}\hat{r}(m)e^{imx}\), then \(S_2(0, 0)r\) may be expanded as follows:

\[
S_2(0, 0)r(x, y) = (1 - y)\hat{r}(0) + \frac{\sum_{m \in \mathbb{Z}\setminus\{0\}} e^{2m} e^{-my} - e^{my}}{e^{2m} - 1}\hat{r}(m)e^{imx}
\]

for \((x, y) \in \Omega_+\). Combining these two relations and taking the normal derivative yields that

\[
\mathcal{G}(0, 0)q = \frac{\mu_- + \mu_+}{\mu_+} q, \quad \forall q \in h^{1\text{tw}}(\mathbb{S}),
\]

thus \(\mathcal{G}(0, 0)\) is an isomorphism. \(\square\)

In virtue of lemma 3.1, if the pair \((f, h)\) maps into \(\mathcal{W}\), we may apply the inverse of \(\mathcal{G}(f, h)\) to (3.4), and get

\[
\partial_t f = \Phi_1(t, f, h),
\]

(3.5)
with a nonlinear and nonlocal operator $\Phi_1$ defined by the relation
\[
\Phi_1(t, f, h) := -G^{-1}(f, h) \mathcal{B}(f, h) S_2(f, h) \mathcal{T}_0 T_2(f) b
\]

Furthermore, from (ii)–(v) and (3.5) we obtain that $h$ is a solution of the equation
\[
\partial_t h = \Phi_2(t, f, h),
\]

where the operator $\Phi_2$ is given by
\[
\Phi_2(t, f, h) := B_1(f, h) S(f, h, g\rho_+(1 + h) - \gamma_h \kappa(h), g(\rho_+ - \rho_-) f + \gamma f \kappa(f))
\]

By lemma 3.1 and relations (3.5)–(3.8) we found that all the solutions $(f, h)$ of (3.1) which are contained in $W$ solve the following abstract evolution equation
\[
\partial_t Z = \Phi(t, Z), \quad Z(0) = Z_0,
\]

where $\Phi := (\Phi_1, \Phi_2)$ and we introduced the new variable $Z := (f, h)$. The following theorem is the key point in the proof of theorem 2.1.

**Theorem 3.2.** The operator $\Phi$ has the same regularity as $b$, it is analytic in the variable $Z$, and if $b(0) := c \in \mathbb{R}$, then $\partial_t \Phi_i(0)$ and $\partial_h \Phi_i(0)$, $i = 1, 2$, are Fourier multipliers with symbols
\[
(\lambda^1_i(m))_{m \in \mathbb{Z}} \text{ and } (\lambda^2_i(m))_{m \in \mathbb{Z}}, \quad i = 1, 2,
\]

respectively, given by

\[
\lambda_1^i(m) := \left[ A_\mu(c - g\rho_+) + g(\rho_+ - \rho_-) - \gamma f m^2 \right] \frac{k|m|}{(\mu_+ + \mu_-) \tanh(|m|)},
\]

\[
\lambda_2^i(m) := \left[ \frac{c(\mu_+ - \mu_-) + 2g\rho_+\mu_-}{\mu_+ + \mu_-} - g(\rho_+ - \rho_-) - \gamma f m^2 \right] \frac{k|m|}{(\mu_+ + \mu_-) \sinh(|m|)},
\]

\[
\lambda_1^3(m) := \left[ \frac{g\rho_+\mu_- + c\mu_+ + \gamma_h m^2}{\mu_+ + \mu_-} \right] \frac{k|m|}{(\mu_+ + \mu_-) \sinh(|m|)},
\]

\[
\lambda_2^3(m) := \left[ \frac{c\mu_+ + g\rho_+\mu_- + \gamma_h m^2}{\mu_+ + \mu_-} \right] \frac{k|m|}{\mu_+ \tanh(|m|)} - \frac{\mu_- \lambda^3_i(m)}{\mu_+ \cosh(m)}.
\]

**Proof.** The regularity assertion is obvious. That the first-order partial derivatives of $\Phi_i$, $i = 1, 2$, with respect to $f$ and $h$ are Fourier multipliers follows from (6.6)–(6.9), relations proven in the appendix.

We now give a proof of our first main result.

**Proof of theorem 2.1.** We verify that the assumptions of [20, theorem 8.4.1] are fulfilled by $\Phi$. Theorem 2.1 is then a consequence of this result. For continuity reasons it suffices in fact to show only that the derivative $\partial_Z \Phi(0)$ generates a strongly continuous and analytic semigroup in $L((h^{1/2})(\mathbb{S}))^2$, i.e.

\[
-\partial_Z \Phi(0) \in \mathcal{H}(h^{2+2\alpha})(\mathbb{S}) \times h^{2+2\alpha}(\mathbb{S}), (h^{1/2})(\mathbb{S}))^2
\]

for some $\beta \in (0, \alpha)$. By using the interpolation properties of the small Hölder spaces
\[
(h^{\alpha_0}(\mathbb{S}), h^{\alpha_1}(\mathbb{S}))_\theta = h^{(1-\theta)\alpha_0 + \theta \alpha_1}(\mathbb{S}),
\]

if $\theta \in (0, 1)$ and $(1-\theta)\alpha_0 + \theta \alpha_1 \not\in \mathbb{N}$, we find then all assumptions of [20, theorem 8.4.1] to be fulfilled. Here $(\cdot, \cdot)$ denotes the interpolation functor introduced by Da Prato and Grisvard [9].
Let us first note that derivative \( \partial_y \Phi_1(0) \) maps \( h^{2+2\text{sign}(\gamma_f)+\tilde{\beta}}(S) \) continuously into \( h^{1+\beta}(S) \) for some \( \tilde{\beta} \in (0, \beta) \). This property can be verified easily using [11, theorem 3.4], which is a multiplier theorem based on some generalized Marcinkiewicz conditions. Since by (3.14)

\[
h^{2+2\text{sign}(\gamma_f)+\tilde{\beta}}(S) = (h^{1+\beta}(S), h^{2+2\text{sign}(\gamma_f)+\tilde{\beta}}(S)) \times (1+2\text{sign}(\gamma_f)+\tilde{\beta}-\beta)/(2\text{sign}(\gamma_f)+1)
\]

we deduce, in virtue of theorem 1.6.1 and relation (2.2.2) in [1], that \( \partial_y \Phi(0) \) generates a strongly continuous and analytic semigroup exactly when \( \partial_y \Phi_1(0) \) and \( \partial_y \Phi_2(0) \) are generators, i.e.

\[-(\partial_y \Phi_1(0), \partial_y \Phi_2(0)) \in \mathcal{H}(h^{2+2\text{sign}(\gamma_f)+\tilde{\beta}}(S), h^{1+\beta}(S)) \times \mathcal{H}(h^{2+2\text{sign}(\gamma_f)+\tilde{\beta}}(S), h^{1+\beta}(S)).\]

When considering the surface tension effects this property holds independently of the boundary conditions, and when \( \gamma_f = 0 \) and \( \gamma_h = 0 \) this is true if

\[
\frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} (c - g \rho_+) + g (\rho_+ - \rho_-) < 0 \quad \text{and} \quad \frac{c \mu_+ + g \rho_+ \mu_-}{\mu_- + \mu_+} > 0,
\]

respectively, with \( b(0) \) sufficiently close to \( c \in \mathbb{R} \) in \( h^{2+\beta}(S) \). We refer to [11] where the generator property of a Fourier multiplier between the space of periodic and continuous functions is explicitly verified when knowing its symbol. In virtue of (3.14) the proof is completed.

4. Equilibria and stability properties

As mentioned earlier, if we consider a fixed volume of fluid + equal to \( 2\pi \) and if \( b = g \rho_+ \), then \((f, h) = (0, 0)\) is the unique flat stationary solution of problem (2.1). Moreover, the reduced equation (3.9) is autonomous since \( \Phi \) does not depend on time for constant \( b \). In order to study the stability properties of this equilibrium, as stated in theorem 2.5, we shall use the principle of linearized stability, and need therefore to determine the spectrum of the derivative \( \partial \Phi(0) \). Being a generator and taking into consideration that the small Hölder space \( h^{2+\nu}(S) \) is compactly embedded into \( h^{1+\nu}(S) \), we obtain from [18, theorem III.8.29] that its spectrum consists entirely of isolated eigenvalues with finite multiplicity.

In virtue of theorems 9.1.2 and 9.1.3 in [20] we know that the trivial solution \((f, h) = (0, 0)\) is exponentially stable if the spectrum of \( \partial \Phi(0) \) is bounded away from the imaginary axis in the left half complex plane, and unstable if the infimum of the real part of all eigenvalues in the right half plane is positive. One can easily see that if \( \lambda \) is an eigenvalue of \( \partial \Phi(0) \), then it must be, for some \( m \in \mathbb{N} \), an eigenvalue of the matrix

\[
\begin{bmatrix}
\lambda^f_1(m) & \lambda^h_1(m) \\
\lambda^f_2(m) & \lambda^h_2(m)
\end{bmatrix},
\]

too, where, for \( b = g \rho_+ \), we obtained simpler expressions for the multiplier symbols:

\[
\begin{align*}
\lambda^f_1(m) &= -\left[ g(\rho_+ - \rho_-) + \gamma_f m^2 \right] \frac{km}{(\mu_+ + \mu_-) \tanh(m)}, \\
\lambda^f_2(m) &= -\left[ g(\rho_+ - \rho_-) + \gamma_f m^2 \right] \frac{km}{(\mu_+ + \mu_-) \sinh(m)}, \\
\lambda^h_1(m) &= -\left[ g \rho_+ + \gamma_h m^2 \right] \frac{km}{(\mu_- + \mu_+) \sinh(m)}, \\
\lambda^h_2(m) &= -\left[ g \rho_+ + \gamma_h m^2 \right] \frac{km}{\mu_+ + \mu_-} \frac{\mu_- (\cosh^2(m) - 1) + \mu_+ \cosh^2(m)}{\mu_+ \sinh(m) \cosh(m)}.
\end{align*}
\]
Thus, the spectrum of $\partial \Phi(0)$ consists only of the eigenvalues
\[
\Lambda_\pm(m) := \frac{\lambda_1^f(m) + \lambda_2^f(m) \pm \sqrt{(\lambda_1^f(m) - \lambda_2^f(m))^2 + 4\lambda_3^f(m)\lambda_4^f(m)}}{2},
\]
whereby $m \in \mathbb{N}$. Easily, we see that $\Lambda_+(0) = 0$, thus we find ourselves in the critical case of stability when $0$ is an eigenvalue, which makes it difficult for us to establish the stability properties of the flat solutions. This is due to the fact that the volume of fluid $+$ is preserved by the flow, and this property has not yet been included into our equations (3.9). We do this by introducing a new variable $\tilde{f} := f - h \in h_0^{2+2\text{sign}(\gamma)}S(\Sigma)$. Then
\[
\begin{align*}
\partial_t \tilde{f} &= \partial_t f - \partial_t h = \Phi_1(f, f - \tilde{f}) - \Phi_2(f, f - \tilde{f}) =: \Psi_2(f, \tilde{f}), \\
\partial_t f &= \Phi_1(f, f - \tilde{f}) =: \Psi_1(f, \tilde{f}),
\end{align*}
\]
(4.2)
and with this new variable, problem (3.9) is equivalent to
\[
\partial_t X = \Psi(X), \quad X(0) = X_0,
\]
(4.3)
where $\Psi := (\Psi_1, \Psi_2)$ and $X := (f, \tilde{f})$. The trivial solution of (3.9) corresponds to the solution $(f, \tilde{f}) = (0, 0)$ of (4.3), so that we shall study the stability properties of the trivial solution of (4.3) which, as we shall see, is more convenient. This can be seen from the following lemma.

**Lemma 4.1.** It holds that $\Psi_2(f, \tilde{f}) \in h_0^{2+2\text{sign}(\gamma)}S(\Sigma)$ for all $(f, \tilde{f})$ in a zero neighbourhood $\tilde{\mathcal{V}} \subset h_0^{2+2\text{sign}(\gamma)}S(\Sigma) \times h_0^{2+2\text{sign}(\gamma)}S(\Sigma)$.

**Proof.** Given $(f, \tilde{f})$ as above, let again $h = f - \tilde{f}$. Setting $p := g\rho_\ast(1 + h) - \gamma_0 k(h)$ and $r := \text{tr}_0 T_2(f)b + g(\rho_+ - \rho_-)f + \gamma_1 k(f)$, we infer from (3.5) that
\[
\Phi_1(f, h) = -G^{-1}(f, h)B(f, h)S(f, h, p, r),
\]
which can be reformulated as follows:
\[
\Phi_1(f, h) = B(f, h)S_0(f, h, p, r) + \text{tr}_0 T_1(f)\Phi_1(f, h) - B(f, h)S(f, h, p, r).
\]
Using this relation and (3.8), we obtain
\[
\begin{align*}
\int_S \Psi_2(f, \tilde{f}) \, dx &= \int_S B(f, h)S(f, h, 0, \text{tr}_0 T_1(f)\Phi_1(f, h)) \, dx \\
&\quad + \int_S B_1(f, h)S(f, h, 0, \text{tr}_0 T_1(f)\Phi_1(f, h)) \, dx \\
&\quad - \int_S B(f, h)S(f, h, p, r) \, dx - \int_S B_1(f, h)S(f, h, p, r) \, dx.
\end{align*}
\]
Therefore, in order to prove our claim, it will do if we show that
\[
\int_S B(f, h)S(f, h, p, r) \, dx + \int_S B_1(f, h)S(f, h, p, r) \, dx = 0
\]
for all $(f, h) \in \mathcal{V}$ and arbitrary $(p, r) \in (h_0^{2+2\ast}(\Sigma))^2$. Defining the harmonic function $u_+ := S(f, h, p, r) \circ \Phi_1^{-1}$, we then have
\[
\begin{align*}
\int_S B(f, h)S(f, h, p, r) \, dx + \int_S B_1(f, h)S(f, h, p, r) \, dx &= -\frac{k}{\mu_+} \int_{\Gamma(f)} \partial_n u_+ \, ds \\
&\quad - \frac{k}{\mu_+} \int_{\Gamma(f)} \partial_n u_+ \, ds = \frac{k}{\mu_+} \int_{\Delta(f, h)} \Delta u_+ \, d(x, y) = 0.
\end{align*}
\]
where \( n \) stands for the outward unit normal at \( \partial \Omega(f, h) \), i.e. \( n = v \) on \( \Gamma(f, h) \) and \( n = -v \) on \( \Gamma(f) \).

We now come to the proof of our second main result.

**Proof of theorem 2.5.** Studying the stability properties of the trivial solution \((f, h) = (0, 0)\) of (2.1), under the constraint (2.4), is equivalent to studying the stability of the trivial solution \((f, \tilde{f}) = (0, 0)\) of (4.3), where, by lemma 4.1, we have that

\[
\Psi : \tilde{\mathcal{V}} \rightarrow h^{1+a}(\mathbb{S}) \times h_0^{1+a}(\mathbb{S}).
\]

From (4.1) we know that every component of the matrix operator

\[
\partial \Psi(0) = \begin{bmatrix} \partial_f \Psi_1(0) & \partial_f \Psi_2(0) \\ \partial_f \Psi_1(0) & \partial_f \Psi_2(0) \end{bmatrix}
\]

is a multiplier with symbol \((\tilde{\lambda}_1^f(m))_{m \in \mathbb{Z}}, (\tilde{\lambda}_1^h(m))_{m \in \mathbb{Z} \setminus \{0\}}, (\tilde{\lambda}_2^f(m))_{m \in \mathbb{Z}}, \text{ and } (\tilde{\lambda}_2^h(m))_{m \in \mathbb{Z} \setminus \{0\}}\) respectively, given by

\[
\tilde{\lambda}_1^f(m) = \lambda_1^f(m) + \lambda_1^h(m), \quad \tilde{\lambda}_1^h(m) = -\lambda_1^h(m),
\]

\[
\tilde{\lambda}_2^f(m) = \lambda_2^f(m) + \lambda_2^h(m) - \lambda_2^h(m) - \lambda_1^h(m), \quad \tilde{\lambda}_2^h(m) = -\lambda_1^h(m) + \lambda_2^h(m)
\]

for \( m \neq 0, \tilde{\lambda}_1^f(0) = -kg\rho_- (\mu_+ + \mu_-)^{-1}, \) and \( \tilde{\lambda}_2^f(0) = 0. \) A simple computation shows that the spectrum of the Fréchet derivative \(\partial \Psi(0)\), which coincides with its point spectrum for the same reason [18, theorem III.8.29], is given by

\[
\sigma(\partial \Psi(0)) = \{ \Lambda_{\pm}(m) : 1 \leq m \in \mathbb{N} \} \cup \{-kg\rho_- (\mu_+ + \mu_-)^{-1}\}.
\]

Let us start and estimate the eigenvalues \( \Lambda_{\pm}(m) \) with \( m \in \mathbb{N} \). Since \([x \mapsto x / \tanh(x)]\) is increasing on \([0, \infty)\) we conclude that

\[
\Lambda_-(m) \leq \tilde{\lambda}_1^f(1)/2 < 0, \quad \forall m \geq 1. \tag{4.4}
\]

Let now \( g(\rho_- - \rho_+) + \gamma_f > 0. \) We show that in this case also the other eigenvalues \( \Lambda_+(m) \) are of negative sign. Indeed, it holds that

\[
\Lambda_+(m) = -4 \frac{\lambda_1^f(m)\lambda_2^h(m) - \lambda_1^h(m)\lambda_2^f(m)}{-\lambda_1^f(m) + \lambda_2^h(m) + \sqrt{(\lambda_1^f(m) - \lambda_2^h(m))^2 + 4\lambda_1^h(m)\lambda_2^f(m)}}, \tag{4.5}
\]

and, in view of

\[
\lambda_1^f(m)\lambda_2^h(m) - \lambda_1^h(m)\lambda_2^f(m) = \frac{k^2m^2(\rho_- + \gamma_f)m^2[3(\rho_- - \rho_+) + \gamma_f m^2]}{\mu_+(\mu_+ + \mu_-)}, \tag{4.6}
\]

we conclude that the spectrum of \(\partial \Psi(0)\) is bounded away from the negative half axis in \(\mathbb{R}^2\). The assertion stated in theorem 2.5(i) follows directly, with the remark that the constant \(\omega\) found there may be chosen arbitrarily in the set \(0, -\max\{\lambda : \lambda \in \sigma(\partial \Psi(0))\}\).

On the other hand, if \( g(\rho_- - \rho_+) + \gamma_f < 0, \) one can easily observe that \( \Lambda_+(1) > 0. \) Moreover, since \( \Lambda_{\pm}(m) \rightarrow -\infty \) as \( m \rightarrow \infty, \) we conclude theorem 2.5(ii). This finishes the proof. \(\square\)
5. Finger-shaped equilibria

This last section is dedicated entirely to the proof of theorem 2.7. If the tuple \((f, h, u_+, u_-)\) is a stationary solution of (2.1), it must hold that \(u_- = b, u_+\) is constant, and

\[
\gamma_f k(f) + g(\rho_+ - \rho_-) f = u_+ - b \\
\gamma_h k(h) - g\rho_+ h = b - u_+ 
\]

on \(\mathbb{S}\). Equations (5.1) and (5.2), called Laplace–Young or capillarity equations, have been studied intensively (see [15] and references therein) subjected to certain constraints at a fixed rigid boundary. Although, when dealing with periodic solutions, we easily get, see [12], that \(h\) must be constant also in the spatial variable, and if \(\rho_+ \leq \rho_-\) then also \(f\) is constant. Whence, equations (5.1) and (5.2) may have nontrivial solutions \((f, h) \notin \mathbb{R}^2\) only when \(\gamma_f > 0\) and \(\rho_+ > \rho_-\).

We are interested in determining only the steady states \((f, h)\) of (2.1) when \(b = g\rho_+\) and which satisfy (2.4), i.e. \(\Omega(f, h)\) encloses the same volume of fluid as \(\Omega_\ast\). Assume by contradiction that \(h = c\) for some \(c \neq 0\). Since \(b - u_+ = -g\rho_- c\), we get that

\[
\gamma_f k(f) + g(\rho_+ - \rho_-) f = g\rho_+ c. 
\]

On the one hand, if \(f\) is constant, it must hold that \(f = \rho_+ c/(\rho_+ - \rho_-)\), which contradicts (2.4) for \(c \neq 0\). On the other hand, the function \(p := f - \rho_+ c/(\rho_+ - \rho_-)\) solves the equation

\[
\gamma_f f - p''/(1 + p'^2)^{3/2} + g(\rho_+ - \rho_-) p = 0. 
\]

The solutions of this equation are, up to a translation, odd. Indeed, since \(p\) is periodic and nonconstant, \(p(x_0) = 0\) must hold for some \(x_0 \in \mathbb{R}\). By translation, we may take \(x_0 = 0\). The pair \((p, q := p')\) is a global solution of the initial value problem

\[
\begin{pmatrix} p' \\ q \\ \end{pmatrix} = \begin{pmatrix} q \\ -\frac{g(\rho_+ - \rho_-)}{\gamma_f} p (1 + q^2)^{3/2} \\ \end{pmatrix}, \\
\begin{pmatrix} p \\ q \\ \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0'(0) \end{pmatrix}. 
\]

This is also true for \((\tilde{p}, \tilde{q})(x) := (-p(-x), q(-x)), x \in \mathbb{R}\). Whence, as we claimed, \(p\) is odd, so that \(\int_{-\infty}^{\infty} f dx = \rho_+ c/(\rho_+ - \rho_-)\), which contradicts again \(c \neq 0\) and (2.4). Consequently, if \((\gamma_f, f, h)\) is a solution of (2.1) and (2.4), then \(h = 0\) and \((\gamma_f, f)\) solves the problem (2.7), which implies in turn that \(f\) has an integral mean equal to 0 and an even translation by [14, theorem 3.2]. Combining that particular result with theorem 6.1 in [13] we obtain all the claims of theorem 2.7 excepting the stability assertion.

In the remaining part of this section we prove that the steady-state solution \((\gamma_f, f, h)\) of problem (2.1) is unstable provided that \(\varepsilon\) is sufficiently small. We first rediscover the global branches \((\gamma_l, f, l)\), \(1 \leq l \in \mathbb{N}\), at least locally near \((\gamma_f, 0, 0)\), by applying the theorem on bifurcations from simple eigenvalues, according to Crandall and Rabinowitz [7, theorem 1.7], to the problem

\[
\Psi(\gamma_f, f, \tilde{f}) = 0, 
\]

where \(\Psi\) is the mapping defined by (4.2). We shall refer to \((\gamma_f, 0, 0), \gamma_f > 0\), as being a trivial solution of (4.3). In order to use \(\gamma_f\) as a bifurcation argument we first establish analytic dependence of \(\Psi\) on \(\gamma_f\) and take the restriction

\[
\Psi : (0, \infty) \times \widetilde{W}_\varepsilon \subset h^{4a}(\mathbb{S}) \times h_0^{2+2\text{sign}(\gamma_\varepsilon)\text{var}}(\mathbb{S}) \rightarrow h^{1\text{var}}(\mathbb{S}) \times h_0^{1\text{var}}(\mathbb{S}), 
\]

where \(\widetilde{W}_\varepsilon := \widetilde{W} \cap (h^{4\text{var}}(\mathbb{S}) \times h_0^{2+2\text{sign}(\gamma_\varepsilon)\text{var}}(\mathbb{S}))\). That \(\Psi\) is well defined between these spaces follows by using elliptic maximum principles and lemma 4.1. When \(\gamma_f \notin \{\gamma_l : 1 \leq l \in \mathbb{N}\}, \)
we infer from (4.4)–(4.6) that all the eigenvalues of \( \partial_{(f,\tilde{f})} \Psi(\gamma_f, 0) \) are different from zero, thus \( \partial_{(f,\tilde{f})} \Psi(\gamma_f, 0) \) is an isomorphism. The implicit function theorem ensures that \((\gamma_f, 0, 0)\) is not a bifurcation point of the trivial solution. Otherwise, if \( \gamma_f = \overline{\nu}_{1} \) for some \( l \in \mathbb{N} \), then \[
abla_{\nu_{1}} \Psi(\overline{\nu}_{1}, 0) = \text{span}(\cos(lx), \cos(lx)) \]
whereby \((\cos(lx), \cos(lx))\) is the eigenvector corresponding to the eigenvalue \( \Lambda_{\nu_{1}}(m, \gamma_{f}) \), i.e. \( \Lambda_{\pm} \) depends not only on \( m \), but also on \( \gamma_{f} \). Also, the codimension of the image \( \text{Im} \partial_{(f,\tilde{f})} \Psi(\gamma_f, 0) \) is one since
\[
\left( a_{0}, \sum_{m=1}^{\infty} a_{m} \cos(mx), \sum_{m=1}^{\infty} b_{m} \cos(mx) \right) \in h^{1+\alpha}_{\nu}(S) \times h^{1+\alpha}_{0,\nu}(S)
\]
belongs to \( \text{Im} \partial_{(f,\tilde{f})} \Psi(\gamma_f, 0) \) if and only if
\[
(a_{l}, b_{l}) = \lambda \left( 1, -\frac{\mu_{\nu}(\cosh^{2}(l) - \cosh(l)) + \mu_{-}(\cosh^{2}(l) - 1)}{\mu_{\nu} \cosh(l)} \right)
\]
for some \( \lambda \in \mathbb{R} \). Moreover, one can easily check that the mixed derivative
\[
\partial_{(f,\tilde{f})} \Psi(\gamma_f, 0)(\cos(lx), \cos(lx)) = \lambda \left( \cos(lx), \frac{1 - \cosh(l)}{\cosh(l)} \right)
\]
whereby \( \lambda = -2kl^{2}((\mu_{\nu} + \mu_{-}) \sinh(l))^{-1} \), does not belong to the image of \( \partial_{(f,\tilde{f})} \Psi(\gamma_f, 0) \). We conclude by [7, theorem 1.7] the existence of a bifurcation curve
\[
(\gamma_{f}, f_{l}, \tilde{f}_{l}) : (-\delta_{l}, \delta_{l}) \rightarrow (0, \infty) \times h^{4+\alpha}_{\nu}(S) \times h^{2+2\nu_{l}}_{0,\nu}(S)
\]
consisting only of stationary solutions of (4.3). Since they all correspond to a volume of fluid + equal to 2\( \pi \), it follows that \( \gamma_{f} \) and \( \tilde{f}_{l} = f_{l} \) are, up to a parametrization, restrictions of the functions obtained in theorem 2.7.

The stability properties of the equilibrium \((\gamma(\epsilon), f_{l}(\epsilon), 0)\) for (2.1) under the constraint (2.4) are equivalent with that of the steady-state solution \((\gamma(\epsilon), f_{l}(\epsilon), \tilde{f}_{l}(\epsilon))\) of problem (4.3). For our purposes, theorem 2.7, it suffices in fact to show that \((\gamma(\epsilon), f_{l}(\epsilon), \tilde{f}_{l}(\epsilon))\) is an unstable stationary solution of the abstract Cauchy problem
\[
\partial_{\nu} X = \Psi(\gamma_f, X), \quad X(0) = X_{0},
\]
where \( \Psi \) is the restriction (5.3).

Indeed, if \( \epsilon \) small and \( l \geq 2 \), then \((\gamma(\epsilon), f_{l}(\epsilon), \tilde{f}_{l}(\epsilon))\) is an unstable solution of (5.4) since the eigenvalue \( \Lambda_{\nu}(1, \overline{\nu}_{1}) \) of \( \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \) is positive.

For the stability of the stationary solution \((\gamma(\epsilon), f_{l}(\epsilon), \tilde{f}_{l}(\epsilon))\), when \( |\epsilon| \) is small and \( |\epsilon| \neq 0 \), it is important how the eigenvalue \( \Lambda_{\nu}(1, \overline{\nu}_{1}) \) (which is equal to 0) perturbs for small \( \epsilon \). Our main tool is the exchange of stability theorem [8, theorem 1.16] according to Crandall and Rabinowitz. The assumptions of this theorem are satisfied by \( \Psi \) since:

1. \( \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \) is a Fredholm operator of index 0 with a one-dimensional kernel;
2. \( \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0)(\cos(x), \cos(x)) \notin \text{Im} \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \);
3. \( \cos(x), \cos(x) \notin \text{Im} \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \).

Let \( J \) denote the inclusion
\[
h^{4+\alpha}_{\nu}(S) \times h^{2+2\nu_{l}}_{0,\nu}(S) \hookrightarrow h^{1+\alpha}_{\nu}(S) \times h^{1+\alpha}_{0,\nu}(S),
\]
in the terminology of [8], (a), (b) and (c) mean that 0 is a \( \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \)-simple eigenvalue and a \( J \)-simple eigenvalue of \( \partial_{(f,\tilde{f})} \Psi(\overline{\nu}_{1}, 0) \). By choosing \( \delta_{l} \) sufficiently small, we obtain from [8, theorem 1.16] four continuously differentiable functions \( \lambda : (\overline{\nu}_{1} - \delta_{l}, \overline{\nu}_{1} + \delta_{l}) \rightarrow \mathbb{R} \),
\[ \mu : (-\delta_1, \delta_1) \to \mathbb{R}, u : (\mathcal{V}_1 - \delta_1, \mathcal{V}_1 + \delta_1) \to h^{3+\mu}_{0,e}(S) \times h^{2+2\text{sign} \gamma_f + \alpha}_{0,e}(S) \text{ and } v : (-\delta_1, \delta_1) \to h^{3+\mu}_{0,e}(S) \times h^{2+2\text{sign} \gamma_f + \alpha}_{0,e}(S) \] such that
\[
\partial_t \psi \Psi (y_f, 0) u (y_f) = \lambda (y_f) u (y_f) \quad \text{for } y_f \in (\mathcal{V}_1 - \delta_1, \mathcal{V}_1 + \delta_1),
\]
\[
\partial_t \psi \Psi (\mathcal{V}_1 (\epsilon), f_1 (\epsilon), \tilde{f}_1 (\epsilon)) w (\epsilon) = \mu (\epsilon) w (\epsilon) \quad \text{for } \epsilon \in (-\delta_1, \delta_1),
\]
\[
\lambda (\mathcal{V}_1) = \mu (0) = 0, \text{ and } u (\mathcal{V}_1) = w (0) = (\cos (x), \cos (x)).
\]
Moreover, \( \lambda (\mathcal{V}_1) \neq 0 \) and
\[
\lim_{\epsilon \to 0, \mu (\epsilon) \neq 0} \frac{-\epsilon \gamma' (\epsilon) \lambda (\mathcal{V}_1)}{\mu (\epsilon)} = 1.
\]
Since \( \lambda (y_f) \) is an eigenvalue of \( \partial_t \psi \tilde{\psi} \Psi (y_f, 0) \) and \( \lambda (\mathcal{V}_1) = 0 \) we get, by continuity, that \( \lambda (y_f) = \Lambda_\ast (1, y_f) \) for all \( |y_f| > \mathcal{V}_1 | < \delta_1 \). Moreover, \( \Lambda_\ast (1, y_f) \) is positive for \( y_f < \mathcal{V}_1 \), and negative if \( y_f > \mathcal{V}_1 \), thus \( \lambda (\mathcal{V}_1) < 0 \). In order to determine the sign of the eigenvalue \( \mu (\epsilon) \), which is the perturbation of the eigenvalue 0 of \( \partial_t \psi \tilde{\psi} \Psi (\mathcal{V}_1, 0) \), we need to specify the sign of \( \gamma' (\epsilon) \). From Theorem 2.7 we obtain in view of \( \gamma' (0) = 0 \) and \( \gamma'' (0) > 0 \), that \( \gamma' (\epsilon) \) and \( \epsilon \) have the same sign, thus \( \mu (\epsilon) \) is a positive eigenvalue, and we are done by [20, theorem 9.1.3].

**Appendix**

We end this paper with a detailed proof of theorem 3.2. Since the diffeomorphisms used in section 3 to transform the original problem (2.1) into (3.1) are given explicitly in terms of \( f \) and \( h \), we obtain by direct computation the following expressions for the elliptic and trace operators defined therein:

\[
A (f) = \frac{\partial^2}{\partial x^2} - 2 \frac{(1 + y) f'}{1 + f} \frac{\partial^2}{\partial x \partial y} + \left( \frac{(1 + y)^2 f'^2}{(1 + f)^2} + \frac{1}{(1 + f)^2} \right) \frac{\partial^2}{\partial y^2}
\]

\[
- \left( (1 + y) \frac{(1 + f) f'' - 2 f'^2}{(1 + f)^2} \right) \frac{\partial}{\partial y},
\]

\[
A(f, h) = \frac{\partial^2}{\partial x^2} - 2 \frac{y h' + (1 - y) f'}{1 + h - f} \frac{\partial^2}{\partial x \partial y} + \left( \frac{(y h' + (1 - y) f')^2 + 1}{(1 + h - f)^2} \right) \frac{\partial^2}{\partial y^2}
\]

\[
- \left( \frac{y h'' + (1 - y) f''}{1 + h - f} \right) - 2 \frac{(h' - f')(y h' + (1 - y) f')}{(1 + h - f)^2} \frac{\partial}{\partial y},
\]

\[
B(f) = \frac{k}{\mu_-} \left( \frac{1 + f'^2}{1 + f} \frac{\partial}{\partial y} - f' \frac{\partial}{\partial x} \right),
\]

\[
B(f, h) = \frac{k}{\mu_+} \left( \frac{1 + f'^2}{1 + h - f} \frac{\partial}{\partial y} - f' \frac{\partial}{\partial x} \right),
\]

\[
B_i(f, h) = - \frac{k}{\mu_i} \left( \frac{1 + h'^2}{1 + h - f} \frac{\partial}{\partial y} - h' \frac{\partial}{\partial x} \right).
\]

In the following we take \( b (0) = c \in \mathbb{R} \) and show that \( \partial_t \Phi_i (0) \) and \( \partial_t \Phi_i (0) \), \( i = 1, 2 \), are Fourier multiplication operators. We shall also determine the symbol of these operators which are the main ingredients when proving theorems 2.1 and 2.5.
A.1. The derivative $\partial_f \Phi_1(0)$

Let us start by determining the symbol of the operator $\partial_f \Phi_1(0)$. Putting $h = 0$ in (3.6) yields that

$$
\Phi_1(0, f, 0) := -G^{-1}(f, 0)B(f, 0)S_2(f, 0)\text{tr}_0 T_2(f) c
- G^{-1}(f, 0)B(f, 0)S_2(f, 0) \left[ g(\rho_+ - \rho_-) f + \gamma f_k(f) \right]
- G^{-1}(f, 0)B(f, 0)S_1(f, 0) g \rho_v.
$$

We show first that the derivative of $\Phi := G^{-1}$ with respect to $f$ in $0$ is a Fourier multipler. Indeed, by the chain rule we have, due to $\Phi\mathcal{F}(f, 0)G(f, 0) = \text{id}_{\mathcal{L}^2(S)}$, that $\partial_f \Phi(0, 0)[f]M = -G^{-1}(0, 0)\partial_f G(0, 0)[f]G^{-1}(0, 0)M$ for all $f \in h^2(\text{sign}(\gamma) + a) (\mathbb{S})$ and $M \in \mathcal{B}$. So, we need to determine

$$
\partial_f G(0, 0)[f]M = -\partial_f B(0, 0)[f]S_2(0, 0)\text{tr}_0 T_1(0)M
- B(0, 0)\partial_f S_2(0, 0)[f]\text{tr}_0 T_1(0) M - B(0, 0)S_2(0, 0)\text{tr}_0 \partial T_1(0)[f] M.
$$

Using the expansions found in the proof of lemma 3.1, we find that $T_1(0)M = \mu_- k^{-1} M(1 + y)$ and $S_2(0, 0)\text{tr}_0 T_1(0) M = \mu_- k^{-1} M(1 - y)$, hence

$$
\partial_f S_2(0, 0)[f]M = \frac{M\mu_-}{\mu_+} f.
$$

Concerning the first two terms of (6.1) we first determine an expansion for $\partial T_1(0)[f] M$ and $\partial_f S_2(0, 0)[f] M$. We start with $\partial_f S_2(0, 0)[f] M$. Elliptic estimates yield that the function $\partial_f S_2(0, 0)[f] M$ is the solution of the Dirichlet problem

$$
\begin{cases}
\Delta w = -\partial_f A(0, 0)[f]S_2(0, 0) M = -M(1 - y) f'' & \text{in } \Omega_1, \\
w = 0 & \text{on } \Gamma_1, \\
w = 0 & \text{on } \Gamma_0,
\end{cases}
$$

and, as in the proof of lemma 3.1, we get that

$$
\partial_f S_2(0, 0)[f] M = - \sum_{m \in \mathbb{Z}[0]} M \left( \frac{e^{my} - e^{2my}}{e^{2m} - 1} + (1 - y) \right) \hat{f}(m)e^{imx}.
$$

Hence

$$
- B(0, 0)\partial_f S_2(0, 0)[f]\text{tr}_0 T_1(0) M = \frac{M\mu_-}{\mu_+} \sum_{m \in \mathbb{Z}} \frac{m}{\tanh(m)} \hat{f}(m)e^{imx} - \frac{M\mu_-}{\mu_+} f.
$$

Consider now the function $\partial T_1(0)[f] M$, which is the solution of

$$
\begin{cases}
\Delta w = -\partial A(0)[f]T_1(0) M = \mu_- k^{-1} M(1 + y) f'' & \text{in } \Omega_-, \\
\partial_y w = -\mu_- k^{-1} \partial B(0)[f]T_1(0) M = \mu_- k^{-1} M f & \text{on } \Gamma_0, \\
w = 0 & \text{on } \Gamma_1.
\end{cases}
$$

Expanding

$$
f = \sum_{m \in \mathbb{Z}} \hat{f}(m)e^{imx} \quad \text{and} \quad w(x, y) = \sum_{m \in \mathbb{Z}} w_m(y)e^{imx},
$$

we find that $w_m$ is the solution of the following problem:

$$
\begin{cases}
w''_{m} - m^2 w_m = -M\mu_- k^{-1} \hat{f}(m)m^2(1 + y) & \text{in } -1 < y < 0, \\
w_m(0) = M\mu_- k^{-1} \hat{f}(m), \\
w_m(-1) = 0,
\end{cases}
$$

Putting
which has the solution \( w_m(y) = M\mu_-k^{-1}\hat{f}(m)(1 + y) \) for all \( m \in \mathbb{Z} \). Whence
\[
\partial T_1(0) f M = M\mu_-k^{-1}(1 + y)f,
\]
and, with the convention that \( m/\tanh(m) = 1 \) if \( m = 0 \), we determine that
\[
- B(0, 0)S_2(0, 0)\partial T_1(0) f M = M\mu_-\sum_{m \in \mathbb{Z}} \frac{m}{\tanh(m)}\hat{f}(m)e^{imx}. \quad (6.4)
\]
Relations (6.1)–(6.4) fuse, in view of lemma 3.1, to
\[
\partial f \Phi_1(0, 0)M = -\frac{2M\mu_-\mu_+}{(\mu_- + \mu_+)^2} \sum_{m \in \mathbb{Z}} \frac{m}{\tanh(m)}\hat{f}(m)e^{imx} \quad (6.5)
\]
for all \( f \in h^{2+2\text{sign}(\gamma_f)}u(\mathbb{S}) \) and \( M \in \mathbb{R} \). In order to determine the derivative \( \partial f \Phi_1(0) \) two more steps must be carried out: we must find the expansions of the derivatives \( \partial f T_2(0) \) and \( \partial f S_1(0, 0) \). Since \( T_2(f)M = M \), we get that \( \partial T_2(0)f M = 0 \), and proceeding similarly as we did before, yields
\[
\partial f S_1(0, 0)f M = \sum_{m \in \mathbb{Z}} M\left(\frac{e^{my} - e^{2m}e^{-my}}{e^{2m} - 1} + (1 - y)\right)\hat{f}(m)e^{imx}. \quad (6.6)
\]
Combining all these relations, we finally obtain for \( \partial f \Phi_1(0, 0) \) that
\[
\partial f \Phi_1(0)f M = \sum_{m \in \mathbb{Z}} \lambda_1^f(m)\hat{f}(m)e^{imx},
\]
whereby \((\lambda_1^f(m))_{m \in \mathbb{Z}}\) is given by (3.10).

A.2. The derivative \( \partial f \Phi_2(0) \)

The expansions found in the previous subsection are very useful when studying \( \partial f \Phi_2(0, 0) \), since (3.8) we have
\[
\Phi_2(0, f, 0) := B_1(f, 0)S_1(f, 0)\rho_+ - B_1(f, 0)S_2(f, 0)\partial T_1(0)\Phi_1(0, f, 0)
\]
\[
+ B_1(f, 0)S_2(f, 0)\partial T_2(0)c + B_1(f, 0)S_2(f, 0)\gamma f[k(f) + g(\rho_+ - \rho_-)f].
\]
In view of \( \Phi_1(0) = k(c - \rho_+)/\mu_+ \) we obtain the following representation
\[
\partial f \Phi_2(0)f = \sum_{m \in \mathbb{Z}} \lambda_2^f(m)\hat{f}(m)e^{imx}, \quad (6.7)
\]
with symbol \((\lambda_2^f(m))_{m \in \mathbb{Z}}\) given by (3.11).

A.3. The derivative \( \partial h \Phi_1(0) \)

In order to determine a representation of the Fréchet derivative \( \partial h \Phi_1(0) \) we have to first investigate the partial derivatives with respect to \( h \) of the solution operators defined in section 3.
By (3.6), we get for \( f = 0 \) that
\[
\Phi_1(0, 0, h) := - \mathcal{G}^{-1}(0, h)B(0, h)S_2(0, h)\partial T_2(0)c
\]
\[
- \mathcal{G}^{-1}(0, h)B(0, h)S_1(0, h)[g(\rho_+(1 + h) - \gamma k(h)].
\]
Differentiating the relation \( \mathcal{F}(0, h)\mathcal{G}(0, h) = \text{id}_{h^{1+\text{sign}(\gamma_f)}}(0, h) \) yields at \( h = 0 \) that
\[
\partial h \mathcal{F}(0, 0)[h]M = -\mathcal{G}^{-1}(0, 0)\partial h \mathcal{G}(0, 0)[h]\mathcal{G}^{-1}(0, 0)M
\]
Generalized Rayleigh–Taylor condition

for all $M \in \mathbb{R}$. We infer from the definition of $G$ that

$$\partial_h G(0, 0)[h] = -\partial_h B(0, 0)[h]S_2(0, 0)tr_0T_1(0) - B(0, 0)\partial_h S_2(0, 0)[h]tr_0T_1(0).$$

Given $h \in h^{2+2\text{sign}(\gamma)}(S)$, the partial derivative $\partial_h S_2(0, 0)[h]M$ is the solution of the linear Dirichlet problem

$$\begin{aligned}
\Delta w &= -\partial_h A(0, 0)[h]S_2(0, 0)M = -Myh'' \quad \text{in } \Omega_+, \\
w &= 0 \\
&w = 0
\end{aligned}$$

on $\Gamma_1$, on $\Gamma_0$,

thus $\partial_h S_2(0, 0)[h]M$ expands as follows:

$$\partial_h S_2(0, 0)[h]M = \sum_{m \in \mathbb{Z}\{0\}} M \left( \frac{e^{my} - e^{-my}}{e^m - e^{-m}} - y \right) \hat{h}(m)e^{imx},$$

and similarly

$$\partial_h S_1(0, 0)[h]M = -\sum_{m \in \mathbb{Z}\{0\}} M \left( \frac{e^{my} - e^{-my}}{e^m - e^{-m}} - y \right) \hat{h}(m)e^{imx}.$$

It then follows easily that

$$\partial_h F(0, 0)[h]M = \frac{\mu - \mu_+ M}{(\mu_+ + \mu_-)^2} \sum_{m \in \mathbb{Z}} \frac{m}{\sinh(m)} \hat{h}(m)e^{imx}.$$ 

By definition, $S_1(0, 0)p = (\Delta, \text{tr}, \text{tr})^{-1}(0, p, 0)$ for all $p \in h^{2+\text{sign}(\gamma)}(S)$, and one can easily check that if $p = \sum_{m \in \mathbb{Z}} \hat{p}(m)e^{imx}$, then

$$S_1(0, 0)p = y\hat{p}(0) + \sum_{m \in \mathbb{Z}\{0\}} \frac{e^{my} - e^{-my}}{e^m - e^{-m}} \hat{p}(m)e^{imx}.$$

Summarizing, we obtain that

$$\partial_h \Phi_1(0)[h] = \sum_{m \in \mathbb{Z}} \lambda^1_h(m) \hat{h}(m)e^{imx},$$

(6.8)

with symbol $(\lambda^1_h(m))_{m \in \mathbb{Z}}$ given by relation (3.12).

A.4. The derivative $\partial_h \Phi_2(0, 0)$

Putting $f = 0$ in (3.8) yields

$$\Phi_2(0, h, 0) = B_1(0, h)S_1(0, h)\left[ g\rho_4(1+h) - \gamma_h\kappa(h) \right] - B_1(0, h)S_2(0, h)tr_0T_1(0)\Phi_1(0, h, 0) + B_1(0, h)S_2(0, h)tr_0T_2(0).$$

Using the relations derived above, finally yields

$$\partial_h \Phi_2(0)[h] = \sum_{m \in \mathbb{Z}} \lambda^2_h(m) \hat{h}(m)e^{imx},$$

(6.9)

with symbol $(\lambda^2_h(m))_{m \in \mathbb{Z}}$ given by relation (3.13).
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