Stabilities for Euler-Poisson Equations with Repulsive Forces in $\mathbb{R}^N$

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Revised 3-Jan-2010

Abstract

This article extends the previous paper in "M.W. Yuen, Stabilities for Euler-Poisson Equations in Some Special Dimensions, J. Math. Anal. Appl. 344 (2008), no. 1, 145–156.", from the Euler-Poisson equations for attractive forces to the repulsive ones in $\mathbb{R}^N$ ($N \geq 2$). The similar stabilities of the system are studied. Additionally, we explain that it is impossible to have the density collapsing solutions with compact support to the system with repulsive forces for $\gamma > 1$.

Key Words: Euler-Poisson Equations, Repulsive Forces, Stabilities, Frictional Damping, Second Inertia Function, Non-collapsing Solutions

1 Introduction

The semi-conductor models can be formulated by the isentropic Euler-Poisson equation with repulsive forces in the following form:

$$\begin{cases} 
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P + \beta \rho u = \rho \nabla \Phi, \\
\Delta \Phi(t, x) = \alpha(N) \rho,
\end{cases} \tag{1}$$

where $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 2$; $\alpha(2) = 2\pi$; For $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)}, \tag{2}$$

where $V(N)$ is the volume of the unit ball in $\mathbb{R}^N$ and $\Gamma$ is the Gamma function. As usual, $\rho = \rho(t, x)$ and $u = u(t, x) \in \mathbb{R}^N$ are the density and the velocity respectively. $P = P(\rho)$ is the pressure and $\beta \geq 0$ is the frictional damping constant. In the above system, the self-repulsive potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ through the Poisson equation. The equations (1) and (1) are the compressible Euler equations with forcing term. The equation (1) is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons themselves. Thus, we called the system (1) the Euler-Poisson equations with repulsive forces. In this case, the equations can be viewed as a semiconductor

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model. See [2], [4] for a detail about the system. For some fixed $K \geq 0$, we have a $\gamma$-law on the pressure $P(\rho)$, i.e.

$$P(\rho) = K\rho^\gamma = \frac{\rho^\gamma}{\gamma},$$

for $K > 0$, which is a common hypothesis. When $K = 0$, the pressureless system can be used as models in plasma physics [1]. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$ and $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. Additionally, the fluid is called isothermal if $\gamma = 1$.

The Poisson equation (1) can be solved as

$$\Phi(t, x) = \int_{\mathbb{R}^N} G(x - y)\rho(t, y)dy,$$

where $G$ is the Green’s function for the Poisson equation in the $N$-dimensional spaces defined by

$$G(x) = \begin{cases} |x|, & N = 1; \\ \log |x|, & N = 2; \\ \frac{-1}{|x|^{N-2}}, & N \geq 3. \end{cases}$$

If we seek solutions in radial symmetry with the radius $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$, the Poisson equation (1) is transformed to

$$r^{N-1}\Phi_{rr}(t, x) + (N - 1)r^{N-2}\Phi_r = \alpha(N)\rho r^{N-1},$$

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}}\int_0^r \rho(t, s)s^{N-1}ds.$$

We can seek the radial symmetry solutions

$$\rho(t, \bar{x}) = \rho(t, r) \text{ and } \bar{u} = \frac{\bar{x}}{r}V(t, r) = : \frac{\bar{x}}{r}V.$$

By standard computation, the Euler-Poisson equations in radial symmetry can be written in the following form:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \frac{N - 1}{r}\rho V = 0, \\
\rho(V_t + VV_r) + P_r(\rho) = \rho\Phi_r(\rho). \end{cases}$$

Perthame discovered the blowup results for 3-dimensional pressureless system with repulsive forces [9]. In short, all the results above rely on the solutions with radial symmetry:

$$V_t + VV_r = \frac{\alpha(N)}{r^{N-1}}\int_0^r \rho(t, s)s^{N-1}ds.$$

And the Emden ordinary differential equations were deduced on the boundary point of the solutions with compact support:

$$\frac{d^2R}{dt^2} = \frac{M}{R^{N-1}}, \quad R(0, R_0) = R_0 \geq 0, \quad \dot{R}(0, R_0) = 0,$$

where $\frac{dR}{dt} := V$ and $M$ is the mass of the solutions, along the characteristic curve. They showed

the blowup results for the $C^1$ solutions of the system [9].

Very recently, Yuen showed in [14] that the classical non-trivial solutions $(\rho, V)$ for the Euler or Euler-Poisson equations with repulsive forces, in radial symmetry, with compact support in $[0, R]$, where $R$ is a positive constant (which is $V(t, 0) = 0; \rho(t, r) = 0, V(t, r) = 0$ for $r \geq R$) and the initial velocity such that:

$$H_0 = \int_0^R V_0dr > 0,$$
blow up on or before the finite time \( T = 2R/H_0 \), for pressureless fluids \((K = 0)\) or \( \gamma > 1 \).

The system with attractive forces were studied in [3], [6], [7], [8], [12], and [13]. This article extends the previous paper in [13], from the Euler-Poisson equations for attractive forces with or without frictional damping to the repulsive ones in \( R^N \) \((N \geq 2)\). The similar stabilities of the system are studied.

In the last section, we exclude the possibility of collapsing solutions for this system. The non-existence of collapsing solutions can be shown by the simple argument for the energy function:

\[
E(t) = \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} P - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for } \gamma > 1.
\]  
(13)

**Theorem 1** For the classical solutions with compact support of the Euler-Poisson equations with repulsive force, (1), in \( R^N \) \((N \geq 2)\) with \( \gamma > 1 \) or without pressure \((K = 0)\), there is no collapsing phenomenon where part of the density \( \rho(t, x) \) collapses to a point.

2 Stabilities

In this section, we study the stabilities of the Euler-Poisson equations with repulsive forces, (1), in \( R^N \) \((N \geq 2)\). The total energy can be defined by:

\[
\begin{cases}
E(t) = \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} P - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for } \gamma > 1, \\
E(t) = \int_\Omega \left( \frac{1}{2} \rho |u|^2 - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for without pressure}.
\end{cases}
\]  
(14)

For the system, we have the energy estimate:

**Lemma 2** For the Euler-Poisson equations, (1), suppose the solutions \((\rho, u)\) have compact support in \( \Omega \). We have,

\[
\dot{E}(t) = -\beta \int_\Omega \rho |u|^2 dx \leq 0.
\]  
(15)

Initially, Sideris used the second inertia function

\[
H(t) = \int_\Omega \rho(t, x) |x|^2 dx,
\]  
(16)

to study instability results for the Euler equations [10]. After that in [8], the instability result of the Euler-Poisson equations with attractive forces, in radial symmetry, was obtained for \( \gamma \geq 4/3 \) and \( N = 3 \). For the corresponding cases in \( R^N \), with non-radial symmetry, were studied [3], [13]. By the standard computation for energy method, it is clear to have the following lemma:

**Lemma 3** Consider \((\rho, u)\) is a solution with compact support in \( \Omega \) for the Euler-Poisson equations, (1) with \( \beta = 0 \). We have

\[
\begin{cases}
\dot{H}(t) = 2 \int_\Omega \left[ \rho |u|^2 + NP \right] dx - \frac{N-2}{2} \rho \Phi \right] dx, \quad \text{for } N \geq 3; \\
\dot{H}(t) = 2 \int_\Omega (\rho |u|^2 + 2P) dx + M^2, \quad \text{for } N = 2.
\end{cases}
\]  
(17)

By applying the above lemma, we could have:

**Theorem 4** Suppose \((\rho, u)\) is a global classical solution in the Euler-Poisson equations, (1) with \( \gamma > 1 \), without frictional damping \((\beta = 0)\). We have

(i) for \( N \geq 3 \),

\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{\inf(2, N(\gamma - 1), N - 2)}{M} E \right]^{1/2};
\]  
(18)
(2) For \( N = 2 \),
\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \sqrt{\frac{1}{2M}};
\]
(19)

(3) For \( N \geq 2 \),
\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{NK \gamma^{-1}}{|\Omega|^{(\gamma - 1)}} \right]^{1/2},
\]
(20)

with \( R(t) = \max_{x \in \Omega(t)} \{|x|\} \). Here
\[
M = \int_{\Omega} \rho(t, x) dx,
\]
(21)
is the total mass which is constant for any classical solution and \( |\Omega| \) is the fixed volume of \( \Omega \).

**Proof.** (1) For \( N \geq 3 \), we have the positive energy function \( E \geq 0 \). We can get from Lemma 3,
\[
\dddot{H}(t) = 2 \left\{ \int_{\Omega} \left[ \rho |u|^2 + NP \right] dx - \frac{N - 2}{2} \int_{\Omega} \rho^2 dx \right\} \geq 2 \inf(2, N(\gamma - 1), N - 2)E.
\]
(22)

That is
\[
H(t) \geq H(0) + \dddot{H}(0)t + \inf(2, N(\gamma - 1), N - 2)Et^2.
\]
(23)

On the other hand, we obtain,
\[
H(0) + \dddot{H}(0)t + \inf(2, N(\gamma - 1), N - 2)Et^2 \leq H(t) \leq R(t)^2 M.
\]
(24)

That is,
\[
O\left(\frac{1}{t} \right) + \inf(2, N(\gamma - 1), N - 2)E \leq \frac{R(t)^2 M}{t^2},
\]
(25)

\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{\inf(2, N(\gamma - 1), N - 2)E}{M} \right]^{1/2}.
\]
(26)

For \( N = 2 \), we have
\[
\dddot{H}(t) = 2 \int_{\Omega} (\rho |u|^2 + 2P) dx + M^2 \geq M^2,
\]
(27)

\[
\frac{M^2}{2} t^2 + C_0 t + C_1 \leq H(t) \leq R(t)^2 M,
\]
(28)

\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \sqrt{\frac{1}{2M}}.
\]
(29)

For \( N \geq 2 \), we obtain
\[
M = \int_{\Omega} \rho dx \leq \left( \int_{\Omega} \rho^\gamma dx \right)^{1/\gamma} |\Omega|^{(\gamma - 1)/\gamma},
\]
(30)

and
\[
\dddot{H}(t) = 2 \left\{ \int_{\Omega} \left[ \rho |u|^2 + NP \right] dx - \frac{N - 2}{2} \int_{\Omega} \rho^\gamma dx \right\} \geq 2 \int_{\Omega} NP dx.
\]
(31)

From the inequality \( \Box \), it is clear to have
\[
\dddot{H}(t) \geq 2NK |\Omega|^{-1-\gamma} M^\gamma > 0,
\]
(32)

\[
H(0) + \dddot{H}(0)t + NK |\Omega|^{-1-\gamma} M^\gamma t^2 \leq H(t) \leq R(t)^2 M,
\]
(33)

\[
O\left(\frac{1}{t} \right) + NK |\Omega|^{-1-\gamma} M^\gamma \leq \frac{R(t)^2 M}{t^2}.
\]
(34)
This gives
\[
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left( \frac{NKM^{\gamma-1}}{\Omega^{(\gamma-1)}} \right)^{1/2}.
\] (35)

The proof is completed. ■

By applying the similar method, in the above section to the system, (1), with frictional damping constant \((\beta > 0)\), the below theorem is followed:

**Theorem 5** Suppose \((\rho, u)\) is a global classical solution with compact support in the system, (1), with frictional damping \((\beta > 0)\). We have for \(N \geq 2\),

\[
\lim_{t \to \infty} \inf \frac{R(t)}{t^{1/2}} \geq \left( \frac{2\beta NKM^{\gamma-1}}{\Omega^{(\gamma-1)}} \right)^{1/2},
\] (36)

with \(R(t) = \max_{x \in \Omega(t)} \{|x|\}\).

**Proof.** For \(N \geq 2\), we get,

\[
\dot{H}(t) = \int_{\Omega} 2x \rho u dx,
\] (37)

and

\[
\ddot{H}(t) = 2 \int_{\Omega} x \left[ \nabla \cdot (\rho u \otimes u) - \nabla P + \rho \nabla \Phi - \beta \rho u \right] dx.
\] (38)

Additionally, it can be arranged as the following:

\[
\dot{H}(t) = 2 \int_{\Omega} [x - \nabla \cdot (\rho u \otimes u) - \nabla P + \rho \nabla \Phi] dx - \beta H(t),
\] (39)

\[
\ddot{H}(t) + \frac{1}{\beta} \dot{H}(t) = 2 \left\{ \int_{\Omega} \rho |u|^2 + NP \right\} dx - \frac{N - 2}{2} \int_{\Omega} \rho \Phi dx \geq 2 \int_{\Omega} NP dx \geq 2NK |\Omega|^{1-\gamma} M^\gamma > 0.
\] (40)

Therefore, we are able to obtain the inequality,

\[
C_3 + C_4 e^{-\beta t} + 2\beta NK |\Omega|^{1-\gamma} M^\gamma t \leq H(t) \leq R(t)^2 M,
\] (41)

\[
O\left(\frac{1}{t}\right) + 2\beta NK |\Omega|^{1-\gamma} M^\gamma \leq \frac{R(t)^2 M}{t}.
\] (42)

This gives

\[
\lim_{t \to \infty} \inf \frac{R(t)}{t^{1/2}} \geq \left( \frac{2\beta NKM^{\gamma-1}}{\Omega^{(\gamma-1)}} \right)^{1/2}.
\] (43)

The proof is completed. ■

### 3 Non-existence of Collapsing Solution

In this section, we explain the idea that there is no possibility to have a density collapsing solution, with compact support for the Euler-Poisson equations with repulsive forces:

We restate their energy for \(\gamma > 1\) or the pressureless fluids is:

\[
E(t) = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} P - \frac{1}{2} \rho \Phi \right) dx \geq - \int_{\Omega} \frac{1}{2} \rho \Phi dx.
\] (44)

When a \(\delta\)-shock exists for the density function \(\rho(t, x)\), the potential energy function, with \(N \geq 3\), becomes to be infinite:

\[
- \int_{\Omega} \rho \Phi dx = \int_{\Omega} \rho(t, x) \int_{\Omega} \rho(t, y) \frac{\rho(t, y)}{|x-y|^{N-2}} dy dx = \lim_{\epsilon \to 0^+} \int_{\Omega} \delta(t, x) \int_{\Omega} \frac{\delta(t, y)}{\epsilon^{N-2}} dy dx = \infty.
\] (45)
With $N = 2$, the situation is similar:

$$-\int_{\Omega} \rho \Phi \, dx = -\int_{\Omega} \delta(t, x) \int_{\Omega} \delta(t, y) \ln |x - y| \, dy \, dx = \infty. \quad (46)$$

Therefore, the total energy of the $\delta$-shock density solutions must be infinite. However, for the classical solutions with compact support, the initial energy is finite. By the energy estimate in Lemma 2 we have,

$$E(t) \leq E(0). \quad (47)$$

If the total energy is finite, it is impossible to obtain the density collapsing solutions. It is clear to have Theorem 1.

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