The matter effect to T-violation at a neutrino factory

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Abstract

We analyzed T-violation in neutrino oscillation by using perturbation methods with respect to $\Delta m^2_{21}L/2E$ and $\delta a(x)L/2E$, where $\delta a(x)$ represents the matter density fluctuation from its average value. We found that the matter contribution to T-violation arises from interferences between $\Delta m^2_{21}L/2E$ and $\delta a(x)L/2E$. In the 2nd order, the symmetric and asymmetric matter density fluctuations give effects to the $\sin \delta$ (intrinsic) and the $\cos \delta$ (fake) parts of T-violation. We give their analytic forms and analyze the matter contribution to the $\sin \delta$ and $\cos \delta$ terms. We found that, for $L = 3000$ km, both the

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symmetric and asymmetric matter density fluctuations give negligible contributions to T-violation, and that thus the constant (average) matter density gives a good approximation. On the other hand, we argue that, for \( L = 7000\text{km} \) or longer length, T-violation turns out to become very small due to cancellation between the 1st and the 2nd order terms. This shows that the constant (average) matter approximation is not valid.

1 Introduction

A high-intensity neutrino source based on a muon storage ring, which is now generally called a neutrino factory[1], has attracted growing interest from theorists and experimentalists[2]. One of the important physics potentials at neutrino factories is to measure a possible non-zero CP violation phase (\( \delta \)) in the 3-generation neutrino mixing matrix, the Maki-Nakagawa-Sakata (MNS) matrix[3]. To measure the phase \( \delta \) in the neutrino oscillation, one way is to compare the CP-conjugate oscillation processes, \( P(\nu_\alpha \rightarrow \nu_\beta) \) and \( P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \), and the other way is to compare the T-reversed oscillation processes, \( P(\nu_\alpha \rightarrow \nu_\beta) \) and \( P(\nu_\beta \rightarrow \nu_\alpha) \). The study of CP violation has been extensively studied[4].

T-violation has not been discussed seriously so far. It is mostly due to the difficulty of identification of \( \nu_e \) appearance at a neutrino factory, since the detection of wrong-signed electrons is hard. However, there have been some attempts to disentangle wrong-signed electrons which enables us to search for T-violation at a neutrino factory[1]. Its experimental feasibility studies are now undertaken. One of the advantages of the search for T-violation is to expect relatively small contribution from matter, whereas in the search for CP violation the fake CP-odd effects from matter dominates over the intrinsic CP violation for a long baseline length (such as more than a few thousand km) and thereby the measurement of the intrinsic CP violation becomes challenging.

T-violation in matter arises from the intrinsic contribution with matter modification, which is proportional to the CP phase, and the fake matter contribution. The constant matter density gives an effect to the intrinsic T-violation which is proportional to the
\[ \sin \delta \] term, which has been discussed extensively [5, 6, 7]. However, up to now, it has not been addressed about possible contributions from the symmetric and also asymmetric matter fluctuations deviated from the average constant density.

In this paper, we consider the symmetric and asymmetric matter fluctuations from the average density and treat them in the perturbation method developed by Koike and Sato [8], and Ota and Sato [9], where the quantities, \( \Delta m_{21}^2 L/2E \) and \( \delta a(x)L/2E \), are considered as perturbative Hamiltonians, which may be small for most of the cases. The average matter density is included in the unperturbed Hamiltonian, so that the constant matter contribution (the average matter) was taken into account. The matter fluctuation is separated into the symmetric and asymmetric terms, \( \delta a(x)sL/2E \) and \( \delta a(x)aL/2E \), respectively. Ota and Sato considered the symmetric part by using the preliminary reference earth model (PREM) [10] and analyzed the 1st order corrections for \( \Delta m_{21}^2 L/2E \) and \( \delta a(x)sL/2E \). In the 1st order, T-violation arises only through \( \Delta m_{21}^2 L/2E \). The matter fluctuation does not give any contribution to T-violation in this order.

We examined T-violation in the 2nd order perturbation of \( \Delta m_{21}^2 L/2E \) and \( \delta a(x)L/2E \). Our motivation is to obtain the analytic expression of T-violation in the 2nd order and to examine how large the 2nd order contribution from the symmetric and asymmetric fluctuations is. We found that the interference term between \( \Delta m_{21}^2 L/2E \) and \( \delta a(x)L/2E \) gives some contributions to T-violation. In particular, the symmetric matter fluctuation contributes to the \( \sin \delta \) part, while the asymmetric matter fluctuation does to the \( \cos \delta \) part. We estimated these contributions for \( L = 3000 \text{km} \) case and found that these contributions are negligibly small for most energies. As a result, the constant matter approximation works well. On the other hand, for \( L = 7000 \text{km} \) or longer, the 2nd order term becomes comparable or larger than the 1st order term, so that the matter fluctuation can not be neglected and the constant matter density approximation fails.

In Sec.2, the perturbation formula is given, and the 0th and the 1st order contributions with respect to \( \Delta m_{21}^2 L/2E \) are given in Sec.3. The general discussion on the contribution from the asymmetric matter profile to T-violation is given in Sec.4. In Sec.5, the 2nd order contribution from \( \Delta m_{21}^2 L/2E \) is given. The interference term between \( \Delta m_{21}^2 L/2E \)
and $\delta a(x)L/2E$ is presented in Sec.6 by assuming the linear dependence for $\delta a(x)$ and the numerical analysis of these interference terms is given in Sec.7. The summary is given in Sec.8.

2 The perturbation formula

The formula to evaluate the neutrino transition probabilities perturbatively with respect to the small quantities, $\Delta m_{21}^2 L/2E$ and $\delta a(x)L/2E$, has been developed by Koike and Sato[8], and Ota and Sato[9]. Ota and Sato used this formula to estimated the 1st order terms of $\Delta m_{21}^2 L/2E$ and the symmetric matter fluctuation, $\delta a(x)sL/2E$. Here, we calculate the higher order terms with respect to the symmetric, $\delta a(x)sL/2E$, and the asymmetric terms, $\delta a(x)aL/2E$. Firstly, we outline their method.

We begin with defining the neutrino mixing matrix as

$$U = e^{i\theta_y\lambda_7} \text{diag}(1, 1, e^{i\delta}) e^{i\theta_z\lambda_5} e^{i\theta_x\lambda_2}$$

$$= \begin{pmatrix}
    c_xc_z & s_xc_z & s_z \\
    -s_xc_y - c_xc_y s_z e^{i\delta} & c_xc_y - s_xc_y s_z e^{i\delta} & s_y c_x e^{i\delta} \\
    s_xc_y - c_xc_y s_z e^{i\delta} & -c_xc_y - s_xc_y s_z e^{i\delta} & c_y c_x e^{i\delta}
\end{pmatrix}, \quad (1)
$$

where $\lambda_j \ (j = 2, 5, 7)$ are Gell-Mann matrices and $c_a = \cos \theta_a$ and $s_a = \sin \theta_a$. The angles $\theta_x$, $\theta_y$ and $\theta_z$ correspond to $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, respectively, where $\theta_{ij}$ are defined in the particle data group[11]. Since the Majorana CP-violation phases are irrelevant to the neutrino oscillations (flavor oscillations)[12], we neglected them. If we multiply the irrelevant phase matrix $\text{diag}(1, 1, e^{-i\delta})$ from the right-hand side of $U$, we obtain the standard form[11]. The relation between the flavor eigenstates, $|\nu_\alpha\rangle \ (\alpha = e, \mu, \tau)$, and the mass eigenstates, $|\nu_i\rangle \ (i = 1, 2, 3)$, is given by

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle \ . \quad (2)$$

The evolution of the flavor eigenstates in matter with energy $E$ is given by

$$i \frac{d}{dx} |\nu_\beta(x)\rangle = H(x)_{\beta\alpha} |\nu_\alpha(x)\rangle \ , \quad (3)$$
where Hamiltonian $H(x)_{\beta\alpha}$ is given by

$$H(x)_{\beta\alpha} = \frac{1}{2E} \begin{pmatrix} U_{\beta i} \left( \begin{array}{c} 0 \\ \Delta m^2_{21} \\ \Delta m^2_{31} \end{array} \right) U^\dagger_{i\alpha} + \left( \begin{array}{c} a(x) \\ 0 \\ 0 \end{array} \right) \end{pmatrix}.$$  \hspace{1cm} (4)

Here $\Delta m^2_{ij} \equiv m^2_i - m^2_j$ with $m_i$ being the mass of $|\nu_i\rangle$, $G_F$ is the Fermi coupling constant and

$$a(x) \equiv 2\sqrt{2}G_F n_e(x)E = 7.56 \times 10^{-5} \left( \frac{\rho(x)}{g/cm^3} \right) \left( \frac{Y_e}{0.5} \right) \left( \frac{E}{GeV} \right) eV^2,$$ \hspace{1cm} (5)

where $n_e(x)$, $Y_e$ and $\rho(x)$ are the electron number density, the electron fraction and the matter density, respectively. For the electron fraction, we use $Y_e = 0.5$.

We separate the matter density fluctuation from its average $\bar{a}$,

$$\delta a(x) \equiv a(x) - \bar{a},$$ \hspace{1cm} (6)

and consider the deviation $\delta a(x)$ as a perturbative term. That is, we solve the evolution equation by treating $\delta a(x)L/2E$ and $\Delta m^2_{21}L/2E$ as perturbative terms, because they are small for most of the cases of planned neutrino factories. The validity of this perturbation was discussed by Ota and Sato[9]. They in fact showed that the transition probability of the neutrino oscillation is well reproduced if the 1st order perturbation with respect to the symmetric matter profile, $\delta a(x)L/2E$, is taken into account, where it is assumed that the symmetric matter profile is well approximated by the preliminary reference earth model (PREM)[10], for $L = 3000$km, $L = 7332$km and $L = 12000$km.

(a) The definition of Hamiltonian

Following the work by Ota and Sato[9], we divide $H(x)$ into the unperturbed part $H_{00}$ and perturbed parts, $H_{01}$ and $H_1$

$$H = H_{00} + H_{01} + H_1(x),$$ \hspace{1cm} (7)

where

$$H_{00} = \frac{1}{2E} e^{i\theta_y \lambda_7} \text{diag}(1, 1, e^{i\delta}).$$
and

$$\tilde{U}_0 = e^{i\theta_y\lambda_\tau} \text{diag}(1, 1, e^{i\delta}) e^{i\theta_x\lambda_5} = \begin{pmatrix} c_z & 0 & s_z \\ -s_y s_z e^{i\delta} & c_y & s_y c_z e^{i\delta} \\ -s_y s_z e^{i\delta} & -s_y & c_y c_z e^{i\delta} \end{pmatrix},$$

$$\tan 2\theta_z = \frac{s_{2z}(\Delta m_{31}^2 - \Delta m_{21}^2 s_{2z}^2)}{c_{2z}(\Delta m_{31}^2 - \Delta m_{21}^2 s_{2z}^2) - \tilde{a}},$$

$$\lambda_\pm = \frac{1}{2} \left( \Delta m_{31}^2 + \Delta m_{21}^2 s_{2z}^2 + \tilde{a} \right) \pm \sqrt{\left( c_{2z}(\Delta m_{31}^2 - \Delta m_{21}^2 s_{2z}^2) - \tilde{a} \right)^2 + s_{2z}(\Delta m_{31}^2 - \Delta m_{21}^2 s_{2z}^2)^2}.$$

(9)

We also define

$$a_\pm = \left. \frac{\lambda_\pm}{2E} \right., \quad a_0 = \left. \frac{\Delta m_{21}^2 c_x^2}{2E} \right..$$

(10)

With $a_i (i = \pm, 0)$, we define

$$k_1 = a_0 - a_- , \quad k_2 = a_+ - a_0 ,$$

$$k = k_2 + k_1 = a_+ - a_- ,$$

$$\phi_{1\pm} = e^{-ia_0 L} \pm e^{-ia_- L} , \quad \phi_{2\pm} = e^{-ia_+ L} \pm e^{-ia_0 L} ,$$

$$\phi_\pm = e^{-ia_+ L} \pm e^{-ia_- L} ,$$

(11)

and

$$P(x) = \begin{pmatrix} e^{-ia_0 x} & 0 & 0 \\ 0 & e^{-ia_0 x} & 0 \\ 0 & 0 & e^{-ia_+ x} \end{pmatrix}.$$
(b) Interaction representation

In the interaction representation, the interaction Hamiltonians are given by

\[ H_{01}(x)_I \equiv e^{iH_{00}x} H_{01}(x) e^{-iH_{00}x}, \quad H_1(x)_I \equiv e^{iH_{00}x} H_1(x) e^{-iH_{00}x}, \]  

and the wave functions are presented by \( |\nu_\alpha(x)\rangle_I \equiv (e^{iH_{00}x})_{\alpha\beta} |\nu_\beta(x)\rangle \) with \( |\nu_\alpha(0)\rangle_I = |\nu_\alpha(0)\rangle \). The evolution equation of \( |\nu_\alpha(x)\rangle_I \) becomes

\[ i\frac{d}{dx} |\nu_\alpha(x)\rangle_I = (H_{01}(x)_I + H_1(x)_I)_{\alpha\beta} |\nu_\beta(x)\rangle_I. \]  

The solution is given by

\[ |\nu_\alpha(L)\rangle_I = T \left( \exp \left(-i \int_0^L dx H_{01}(x)_I + H_1(x)_I \right) \right) |\nu_\beta(0)\rangle_I, \]  

where \( T \) means the time ordered product. Then

\[ |\nu_\alpha(L)\rangle = \left[ e^{-iH_{00}L} \right]_{\alpha\beta} |\nu_\beta(L)\rangle_I \]
\[ = \left[ e^{-iH_{00}L} \right]_{\alpha\gamma} \left[ 1 + (-i) \int_0^L dx H_{01}(x)_I + (-i) \int_0^L dx H_1(x)_I \right. \]
\[ + (-i)^2 \int_0^L dx \int_0^x dy H_{01}(x)_I H_1(y)_I + (-i)^2 \int_0^L dx \int_0^x dy H_1(x)_I H_1(y)_I \]
\[ + \left. (-i)^2 \int_0^L dx \int_0^x dy (H_{01}(x)_I H_1(y)_I + H_1(x)_I H_{01}(y)_I + \cdots ) \right] \gamma_\beta |\nu_\beta(0)\rangle_I \]
\[ = \left( S_{00} + S_{01} + S_{11}^{(1)} + S_{01,01} + S_{11}^{(2)} + S_{01,1} + \cdots \right)_{\alpha\beta} |\nu_\beta(0)\rangle_I. \]  

The transition probability from one flavor eigenstate \( \alpha \) to another \( \beta \) is given by

\[ P(\nu_\alpha \to \nu_\beta) = |S(L)_{\alpha\beta}|^2 \]
\[ = |(S_{00})_{\alpha\beta}|^2 \]
\[ + 2 \text{Re} \left[ (S_{00})_{\alpha\beta} (S_{01})_{\alpha\beta}^* + (S_{00})_{\alpha\beta} (S_{11}^{(1)})_{\alpha\beta}^* \right] \]
\[ + |(S_{01})_{\alpha\beta}|^2 + |(S_{11}^{(1)})_{\alpha\beta}|^2 + 2 \text{Re} \left[ (S_{01})_{\alpha\beta} (S_{11}^{(1)})_{\alpha\beta}^* \right] \]
\[ + 2 \text{Re} \left[ (S_{00})_{\alpha\beta} (S_{01,01})_{\alpha\beta}^* + (S_{00})_{\alpha\beta} (S_{11}^{(2)})_{\alpha\beta}^* + (S_{00})_{\alpha\beta} (S_{01,1})_{\alpha\beta}^* \right] \]
\[ + \cdots . \]  

T-violation is defined by

\[ \Delta P_{\nu_\alpha \nu_\beta}^T = P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha), \]  

and is evaluated with use of the probabilities defined in Eq.(17).
3 The 0th and 1st order contribution from $H_{01}(x)$

Ota and Sato[9] calculated the 1st order contribution from $H_{01}(x)$. Here, we give a brief derivation of their results which are needed to discuss the higher order calculation.

(a) The S-matrix (0th and 1st order)

The unperturbed Hamiltonian $H_{00}$ is diagonalized explicitly and is given by

$$H_{00} \equiv \tilde{U}_0 \text{diag}(a_-, a_0, a_+)^\dagger \tilde{U}_0^\dagger,$$

(19)

where $a_\pm$ and $a_0$ are defined in Eq.(10).

The S-matrix for $H_{00}$ is easily obtained as

$$S_{00} = e^{-iH_{00}L} = \tilde{U}_0 P(L) \tilde{U}_0^\dagger = \frac{1}{2} \begin{pmatrix}
\phi_+ - c_{2z} \phi_- & s_y s_{2z} e^{-i\delta} \phi_- & c_y s_{2z} e^{-i\delta} \phi_-
\phi_+ + s_y^2 c_{2z} \phi_- - c_y^2 (\phi_2_- - \phi_1-) & s_{2y} (c_y^2 \phi_2_- - s_y^2 \phi_1-) & s_2 y (c_y^2 \phi_2_- - s_y^2 \phi_1-)
\end{pmatrix},$$

(20)

where $\phi_\pm$ and $\phi_{i\pm}$ ($i = 1, 2$) are defined in Eq.(11). The 1st order term of $H_{01}$ defined in Eq.(8) is given by

$$S_{01} = \tilde{U}_0 \begin{pmatrix} 0 & A_{01} & 0 \\ A_{01} & 0 & B_{01} \\ 0 & B_{01} & 0 \end{pmatrix} \tilde{U}_0^\dagger$$

$$= \begin{pmatrix} 0 & c_y \mathcal{P}_{01} & -s_y \mathcal{Q}_{01} \\ c_y \mathcal{P}_{01} & -s_{2y} c_\delta \mathcal{Q}_{01} & (e^{i\delta} s_y^2 - e^{-i\delta} c_y^2) \mathcal{Q}_{01} \\ -s_y \mathcal{P}_{01} & (e^{-i\delta} s_y^2 - e^{i\delta} c_y^2) \mathcal{Q}_{01} & s_{2y} c_\delta \mathcal{Q}_{01} \end{pmatrix},$$

(21)

where

$$\mathcal{P}_{01} = c_{2z} A_{01} + s_{2z} B_{01}, \quad \mathcal{Q}_{01} = s_{2z} A_{01} - c_{2z} B_{01},$$

$$A_{01} = \frac{\Delta m_{21}^2 s_{2z} c_{2z-z}^2}{4E k_1} \phi_{1-}, \quad B_{01} = -\frac{\Delta m_{21}^2 s_{2z} s_{2z-z}^2}{4E k_2} \phi_{2-}.$$
The oscillation probability in the 0th order $P^{(00)}(\nu_\alpha \rightarrow \nu_\beta)$ is given by $|\langle S_{00}\rangle_{\alpha\beta}|^2$,

\[
P^{(00)}(\nu_e \rightarrow \nu_\mu) = P^{(00)}(\nu_\mu \rightarrow \nu_e) = s_y^2 s_2^2 \sin^2 \frac{kL}{2},
\]
\[
P^{(00)}(\nu_e \rightarrow \nu_\tau) = P^{(00)}(\nu_\tau \rightarrow \nu_e) = c_y^2 s_2^2 \sin^2 \frac{kL}{2},
\]
\[
P^{(00)}(\nu_\mu \rightarrow \nu_\tau) = P^{(00)}(\nu_\tau \rightarrow \nu_\mu) = s_2^2 \left\{ s_y^2 \sin^2 \frac{k_1 L}{2} + c_y^2 \sin^2 \frac{k_2 L}{2} - s_y^2 c_y^2 \sin^2 \frac{kL}{2} \right\},
\]
(23)

with $k$, $k_1$ and $k_2$ defined in Eq.(11).

The probability for antineutrinos $P^{(00)}(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$ is obtained by taking $\delta \rightarrow -\delta$ and $\bar{a} \rightarrow -\bar{a}$.

For the higher order terms, we consider only the contribution to T-violation. The 1st order term from $H_{01}$, i.e., the $\Delta m_{21}^2 L/2E$ term is solely by the CP violation phase $\delta$ and is given by

\[
\left(\Delta P^T_{\nu_e\nu_\mu}\right)_{s_\delta} = -\frac{\Delta m_{21}^2}{E} s_2 s_2 s_y s_y \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{kL}{2}. \quad (24)
\]

where we used $\sin k_1 L + \sin k_2 L - \sin kL = 4 \sin(k_1 L/2) \sin(k_2 L/2) \sin(kL/2)$, which is proved by using $k = k_1 + k_2$. This formula includes the constant matter effect.

Similarly, we find

\[
\left(\Delta P^T_{\nu_\mu\nu_\tau}\right)_{s_\delta} = \left(\Delta P^T_{\nu_\tau\nu_e}\right)_{s_\delta} = \left(\Delta P^T_{\nu_e\nu_\mu}\right)_{s_\delta},
\]
(25)

which is valid for the constant media as stated in the paper by Krastev and Petcov[5].

4 The contribution from the matter term, $H_1(x)$

In this section, we give the general formula to evaluate the n-th order effects of matter, i.e., the $(\delta a(x)L/2E)^n$ order. Then, we evaluate the contributions of up to the 3rd order and discuss the general properties of the effects.

(a) S-matrix elements
Therefore, the S-matrix is written in general by

\[ H_1(x)_I = \frac{\delta a(x)}{2E} \tilde{U}_0 \begin{pmatrix} c_z^2 & 0 & s_z c_z e^{-ikx} \\ 0 & 0 & 0 \\ s_z c_z e^{ikx} & 0 & s_z^2 \end{pmatrix} \tilde{U}_0^\dagger. \]  

Then, the n-th order matter perturbation is given by

\[
S_1^{(n)} = e^{-iH_{\text{tot}}(-i)^n} \int_0^L dx_1 \cdots \int_0^{x_{n-1}} dx_n H_1(x_1)H_1(x_2) \cdots H_1(x_n)I \\
= \tilde{U}_0 (-i)^n P(L) \left\{ \int_0^L dx_1 \cdots \int_0^{x_{n-1}} dx_n \frac{\delta a(x_1)}{2E} \cdots \frac{\delta a(x_n)}{2E} \right\} \times \left( \begin{array}{ccc} c_z^2 & 0 & s_z c_z e^{-ikx_1} \\ 0 & 0 & 0 \\ s_z c_z e^{ikx_1} & 0 & s_z^2 \end{array} \right) \cdots \left( \begin{array}{ccc} c_z^2 & 0 & s_z c_z e^{-ikx_n} \\ 0 & 0 & 0 \\ s_z c_z e^{ikx_n} & 0 & s_z^2 \end{array} \right) \right) \tilde{U}_0^\dagger. \]  

Therefore, the S-matrix is written in general by

\[ S_1^{(n)} = \tilde{U}_0 \begin{pmatrix} \mathcal{E}_1^{(n)} & 0 & \mathcal{C}_1^{(n)} \\ 0 & 0 & 0 \\ \mathcal{D}_1^{(n)} & 0 & \mathcal{F}_1^{(n)} \end{pmatrix} \tilde{U}_0^\dagger = \frac{1}{2} \begin{pmatrix} \alpha_{1,n}^{(+)} & e^{-i\delta} s_y \beta_{1,n}^{(+)} & e^{-i\delta} c_y \beta_{1,n}^{(+)} \\ e^{i\delta} s_y \beta_{1,n}^{(-)} & s_y^2 \alpha_{1,n}^{(-)} & s_y c_y \alpha_{1,n}^{(-)} \\ e^{i\delta} c_y \beta_{1,n}^{(-)} & s_y c_y \alpha_{1,n}^{(-)} & c_y^2 \alpha_{1,n}^{(-)} \end{pmatrix}, \]  

where

\[
\alpha_{1,n}^{(\pm)} = \mathcal{E}_1^{(n)} + \mathcal{F}_1^{(n)} \pm \left( c_{2z}(\mathcal{E}_1^{(n)} - \mathcal{F}_1^{(n)}) + s_{2z}(\mathcal{C}_1^{(n)} + \mathcal{D}_1^{(n)}) \right),
\]

\[
\beta_{1,n}^{(\pm)} = -s_{2z}(\mathcal{E}_1^{(n)} - \mathcal{F}_1^{(n)}) + c_{2z}(\mathcal{C}_1^{(n)} + \mathcal{D}_1^{(n)}) \pm \left( \mathcal{C}_1^{(n)} - \mathcal{D}_1^{(n)} \right). \]  

Below, we show their explicit forms, which are needed to evaluate the matter effect to T-violation up to the 3rd order. The 1st order terms are

\[
\mathcal{C}_1^{(1)} = (-i) e^{-ia_{-L}} \int_0^L dx \frac{\delta a(x)}{2E} s_z c_z e^{-ikx},
\]

\[
\mathcal{D}_1^{(1)} = (-i) e^{-ia_{+L}} \int_0^L dx \frac{\delta a(x)}{2E} s_z c_z e^{ikx},
\]

\[
\mathcal{E}_1^{(1)} = (-i) e^{-ia_{-L}} \int_0^L dx \frac{\delta a(x)}{2E} c_z^2,
\]

\[
\mathcal{F}_1^{(1)} = (-i) e^{-ia_{+L}} \int_0^L dx \frac{\delta a(x)}{2E} s_z^2.
\]
The 2nd order terms are
\[ C_1^{(2)} = (-i)^2 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{s_z c_z}{2E} \left( s_z^2 e^{-ikx} + c_z^2 e^{-ikz} \right), \]
\[ D_1^{(2)} = (-i)^2 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{s_z c_z}{2E} \left( c_z^2 e^{ikx} + s_z^2 e^{iky} \right), \]
\[ E_1^{(2)} = (-i)^2 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{c_z^2}{2E} \left( s_z^2 e^{-ikx} + c_z^2 e^{iky} \right), \]
\[ F_1^{(2)} = (-i)^2 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{s_z^2}{2E} \left( s_z^2 + c_z^2 e^{ik(x-y)} \right). \] (31)

Finally the 3rd order terms are
\[ C_1^{(3)} = (-i)^3 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{s_z c_z}{2E} \left( s_z^4 e^{-ikx} + s_z^2 c_z^2 e^{-iky} + e^{-ik(x+y+z)} + c_z^4 e^{-ikz} \right), \]
\[ D_1^{(3)} = (-i)^3 e^{-ia-L} \int_0^L dx \int_0^x dy \int_0^y dz \delta a(x) \delta a(y) \delta a(z) \frac{s_z c_z}{2E} \left( c_z^4 e^{ikx} + s_z^2 c_z^2 e^{iky} + e^{ik(x+y+z)} + s_z^4 e^{ikz} \right). \] (32)

It should be noted that \( D_1^{(n)} \) and \( F_1^{(n)} \) are derived from \( C_1^{(n)} \) and \( E_1^{(n)} \), by exchanging between \( s_z \) and \( c_z \), and also \( a_- \) and \( a_+ \).

In order to examine the general properties of these quantities, we divide the matter fluctuation \( \delta a(x) \) into the symmetric part, \( \delta a(x)_s \), and the asymmetric part, \( \delta a(x)_a \), and then expand them in terms of Fourier cosine series,
\[ \delta a(x)_s = \sum_{n \neq 0} a_{2n} e^{-iq_{2n} x}, \]
\[ \delta a(x)_a = \sum_n a_{2n+1} e^{-iq_{2n+1} x}, \quad q_n = \frac{\pi n}{L}, \] (33)
where \( a_n \) is a real number satisfying \( a_{-n} = a_n \). Here, we excluded \( n = 0 \) in the expression of \( \delta a(x)_s \), because \( \delta a(x) \) is the deviation from the average of \( a(x) \). Below, we consider the both profiles together.

The 1st order terms are
\[ C_1^{(1)} + D_1^{(1)} = \frac{s_z^2}{2E} \left( \sum_{n \neq 0} \frac{a_{2n}}{k + q_{2n}} \right) \phi_-, \]
\[ C_1^{(1)} - D_1^{(1)} = -\frac{s_z^2}{2E} \left( \sum_n \frac{a_{2n+1}}{k + q_{2n+1}} \right) \phi_+, \]
\[ E_1^{(1)} = F_1^{(1)} = 0. \] (34)
The 2nd order terms are
\[
C^{(2)}_1 + D^{(2)}_1 = \frac{s_2 \bar{z} c_2 \bar{z}}{2E^2} \left( \sum_{m+n=\text{even}} \frac{a_n a_m}{(k+q_n)(k+q_n+q_m)} \right) \phi_-, \\
C^{(2)}_1 - D^{(2)}_1 = -\frac{s_2 \bar{z} c_2 \bar{z}}{2E^2} \left( \sum_{m+n=\text{odd}} \frac{a_n a_m}{(k+q_n)(k+q_n+q_m)} \right) \phi_+, \\
E^{(2)}_1 - F^{(2)}_1 = \frac{s_2 \bar{z}}{8(2E)^2} \left\{ \left( \sum_{n,m} \frac{((-1)^n + 1)(-1)^m + 1}{(k+q_n)(k+q_m)} \right) \phi_- \\
+ 2L \left( \sum_n \frac{a_n a_{-n}}{k+q_n} \right) i\phi_+ \right\} .
\]

Finally, the 3rd order terms are
\[
C^{(3)}_1 - D^{(3)}_1 = \frac{s_2 \bar{z}}{2(2E)^3} \left\{ \frac{s_2 \bar{z}}{2} \sum_{n,m,l} \frac{((-1)^{n+m+l} - 1) a_n a_m a_l}{(k+q_n)(k+q_n+q_m)(k+q_n+q_m+q_l)} \\
- \frac{s_2 \bar{z}}{8} \left( \sum_{n,m,l} \frac{((-1)^{n+m+l} - 1) a_n a_m a_l}{(k+q_n)(k+q_m)(k+q_l)} + \sum_l \frac{((-1)^l - 1) a_l}{(k+q_l)} \sum_{m \neq n} \frac{a_n a_m}{(k+q_n)(k+q_m)} \\
+ 2 \sum_{n,m,l} \frac{((-1)^l - 1) a_l}{(k+q_l)^2} \sum_n \frac{a_n a_{-n}}{(k+q_n)} \right) \phi_+ - \frac{s_2 \bar{z}}{4} L \sum_l \left[ \frac{((-1)^l - 1) a_l}{(k+q_l)} \sum_n \frac{La_n a_{-n}}{(k+q_n)} \right] i\phi_+ \right\} .
\]

(b) The general properties

We first discuss the n-th order contributions of the matter to the transition probabilities, which are obtained by computing
\[
P^{(1,n)}(\nu_\alpha \rightarrow \nu_\beta) = \text{Re} \left( 2(S_{00})_{\alpha\beta}(S_1^{(n)*})_{\alpha\beta} + \sum_{l+m=n} (S_1^{(l)})_{\alpha\beta}(S_1^{(m)*})_{\alpha\beta} \right) .
\]

Since the \((S_1^{(n)})_{\mu\tau} = (S_1^{(n)})_{\tau\mu}\), we conclude that
\[
P^{(1,n)}(\nu_\mu \rightarrow \nu_\tau) = P^{(1,n)}(\nu_\tau \rightarrow \nu_\mu) .
\]

Therefore, the matter itself does not give any effect to T-violation for \(\nu_\mu \rightarrow \nu_\tau\) channel.

For other channels,
\[
P^{(1,n)}(\nu_\epsilon \rightarrow \nu_\mu) = \frac{s_2^2}{4} \text{Re} \left( 2s_2 \bar{z}\phi_- \beta_{1,n}^{(+)\ast} + \sum_{l+m=n} \beta_{1,l}^{(+)\ast} \right) ,
\]

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and $P^{(1,n)}(\nu_\mu \to \nu_e)$ is obtained by changing the superscript (+) to (−). The transition probabilities for $\nu_e \to \nu_\tau$ and $\nu_\tau \to \nu_e$ are obtained from those for $\nu_e \to \nu_\mu$ and $\nu_\mu \to \nu_e$, by changing the coefficient $s_y^2$ with $c_y^2$.

We consider the 1st order of the asymmetric matter fluctuation, which contributes only $C_1^{(1)} - D_1^{(1)}$, in contrast to the $C_1^{(1)} + D_1^{(1)}$ which is due to the symmetric matter profile. In addition, $E_1^{(1)} = 0$ and $F_1^{(1)} = 0$. Therefore, the asymmetric matter fluctuation gives $\alpha_{1,1} = 0$, which shows that it does not contribute to $\nu_\mu \to \nu_\tau$ channel. Also we find $\beta_{1,1}^{(\pm)} = \pm (C_1^{(1)} - D_1^{(1)})$, which is proportional to $\phi_+$. Since

$$\phi_- \phi_+^* = -2i \sin kL,$$

which implies the vanishing contribution. Thus we conclude that the asymmetric matter profile does not contribute to the transition probability in the 1st order. It contributes to the transition probability in the 2nd order.

(c) T-violation only from the matter

(c-1) The 1st order effect

The 1st order contribution to $\Delta P_{\nu_e,\nu_\mu}^T$ is from the interference between $S_{00}$ and $S_1^{(1)}$ as we see in Eq.(17). In this case, the $C_1^{(1)} - D_1^{(1)}$ in $S_1^{(1)}$ contributes to T-violation. Since $C_1^{(1)} - D_1^{(1)}$ is proportional to $\phi_+$, while $S_{00}$ to $\phi_-$, T-violation is proportional to $\text{Re}[\phi_- \phi_+^*]$, which is zero. Therefore, there is no 1st order effect to T-violation.

(c-2) The 2nd order effect

The 2nd order effect comes from $|S_1^{(1)}|^2$ and the interference between $S_{00}$ and $S_1^{(2)}$. Similarly to the 1st order contribution, $\text{Re}[\phi_- (C_1^{(2)*} - D_1^{(2)*})]$ in $S_1^{(2)}$ contributes to the asymmetry. Since $C_1^{(2)} - D_1^{(2)}$ is proportional to $\phi_+$, this term does not contribute. As for $|S_1^{(1)}|^2$ term, $\text{Re}[\{E_1^{(1)} - F_1^{(1)}\}\{C_1^{(1)*} - D_1^{(1)*}\}]$ and also $\text{Re}[\{C_1^{(1)} + D_1^{(1)}\}\{C_1^{(1)*} - D_1^{(1)*}\}]$ contribute. Since $C_1^{(1)} + D_1^{(1)}$ is proportional to $\phi_-$ and $E_1^{(1)} - F_1^{(1)}$ consists of $\phi_-$ and $i\phi_+$, they do not contribute. As a result, there is no 2nd order effect.

(c-3) The 3rd order effect

The 3rd order effect comes from $S_{00}S_1^{(3)*}$ and $S_1^{(1)}S_1^{(2)*}$. Since $C_1^{(3)} - D_1^{(3)}$ contains
The quantity of the bracket becomes $(\Delta m^2 | \phi_+ \rangle \langle \phi_- |)$ term. Similarly, for $S^{(2)}_0 S^{(2)*}_1$ terms, the terms contributing to T-violation are $\text{Re}[ (\mathcal{E}^{(1)}_1 - \mathcal{F}^{(1)}_1)(C^{(2)*}_1 - D^{(2)*}_1) ]$, $\text{Re}[ (C^{(1)}_1 + D^{(1)}_1)(C^{(2)*}_1 - D^{(2)*}_1) ]$ and terms that the superscript 1 and 2 are exchanged. Again it is clear that these terms do not contribute.

We conclude that there is no contribution to T-violation through the matter terms up to the 3rd order, i.e., $(\delta a(x)L/2E)^3$ terms. The vanishing of the 3rd order term is sufficient enough to deal with the actual situation. Thus we do not pursue the further investigation, although we expect that the higher terms will not contribute either.

5 The second order contribution from $H_{01}$

We consider the $(\Delta m^2_{21} L/2E)^2$ effect. For this, there are two contributions. One is from $|(S_{01})_{\beta\alpha}|^2$ and the other is from $2\text{Re}[(S_{00})_{\beta\alpha}(S_{0101})_{\beta\alpha}]$.

We first compute $S_{0101}$ defined in Eq.(17)

$$S_{0101} = \tilde{U}_0 P(L)(-i)^2 \int_0^L dx \int_0^x dy \left[ P(-x)\tilde{U}_0^\dagger H_{01} \tilde{U}_0 P(x-y)\tilde{U}_0^\dagger H_{01} \tilde{U}_0 P(y) \right] \tilde{U}_0^\dagger.$$  \hspace{1cm} (41)

The quantity of the bracket becomes $(\Delta m^2_{21}s_{2x}/4E)^2$ multiplied by

$$\begin{pmatrix} c_{z-z}^2 e^{-ik_1(x-y)} & 0 & -\frac{1}{2} s_{2(z-z)} e^{-i(k_1x+k_2y)} \\ 0 & c_{z-z}^2 e^{i(k_1x+y)} + s_{z-z}^2 e^{-ik_2(x-y)} & 0 \\ -\frac{1}{2} s_{2(z-z)} e^{i(k_2x+k_1y)} & 0 & s_{z-z}^2 e^{i(k_2x-y)} \end{pmatrix}.$$ \hspace{1cm} (42)

Then, we find

$$S_{0101} = \tilde{U}_0 \begin{pmatrix} \mathcal{E}_{0101} & 0 & \mathcal{C}_{0101} \\ 0 & \mathcal{G}_{0101} & 0 \\ \mathcal{D}_{0101} & 0 & \mathcal{F}_{0101} \end{pmatrix} \tilde{U}_0^\dagger$$

$$= \frac{1}{2} \begin{pmatrix} e^{i\delta} s_{y\alpha'} \beta_{0101}^{(-)} & s_{y\alpha'}^2 \alpha_{0101}^{(-)} + 2 s_{y\alpha} \mathcal{G}_{0101} & e^{-i\delta} s_{y\alpha'} \beta_{0101}^{(-)} \\ e^{i\delta} s_{y\alpha} \beta_{0101}^{(-)} & s_{y\alpha}^2 \alpha_{0101}^{(-)} - 2 s_{y\alpha} \mathcal{G}_{0101} & e^{-i\delta} s_{y\alpha} \beta_{0101}^{(-)} \\ s_{y\alpha'}, \alpha_{0101}^{(-)} & s_{y\alpha'} \beta_{0101}^{(-)} + 2 s_{y\alpha} \mathcal{G}_{0101} & s_{y\alpha}^2, \alpha_{0101}^{(-)} + 2 s_{y\alpha} \mathcal{G}_{0101} \end{pmatrix},$$ \hspace{1cm} (43)

where

$$\alpha_{0101}^{(\pm)} = (\mathcal{E}_{0101} + \mathcal{F}_{0101}) \pm (c_{z-z}(\mathcal{E}_{0101} - \mathcal{F}_{0101}) + s_{z-z}(\mathcal{C}_{0101} + \mathcal{D}_{0101})),$$

$$\beta_{0101}^{(\pm)} = -s_{z-z}(\mathcal{E}_{0101} - \mathcal{F}_{0101}) + c_{z-z}(\mathcal{C}_{0101} + \mathcal{D}_{0101}) \pm (\mathcal{C}_{0101} - \mathcal{D}_{0101}).$$ \hspace{1cm} (44)
We compute the 2nd order \((\Delta m^2_{21}L/2E)^2\) contribution to T-violation. Since \((S_{01,01})_{\mu\tau} = (S_{01,01})_{\tau\mu}\), there is no contribution to T-violation for the \(\nu_\mu\) and \(\nu_\tau\) oscillation channels.

For the \(\nu_e\) and \(\nu_\mu\) channels, the \(C_{01,01} - D_{01,01}\) contributes because

\[
2\text{Re}[(S_{00})_{\mu e}(S_{01,01})^*_{\mu e} - (S_{00})_{e\mu}(S_{01,01})^*_{e\mu}] = s_\theta^2 s_{2\theta} \text{Re}[\phi_- (C^*_{01,01} - D^*_{01,01})].
\]  

The same holds for the \(\nu_e\) and \(\nu_\tau\) channel.

\(C_{01,01}\) and \(D_{01,01}\) are given by

\[
C_{01,01} = -\left(\frac{\Delta m^2_{21} s_{2\theta}}{4E}\right)^2 \frac{1}{2} s_{2\theta(z-\bar{z})} \left[ \frac{1}{k k_2} \phi^- - \frac{1}{k_1 k_2} \phi^+ \right],
\]

\[
D_{01,01} = -\left(\frac{\Delta m^2_{21} s_{2\theta}}{4E}\right)^2 \frac{1}{2} s_{2\theta(z-\bar{z})} \left[ -\frac{1}{k k_1} \phi^- + \frac{1}{k_1 k_2} \phi^+ \right].
\]

Since

\[
C^*_{01,01} - D^*_{01,01} = -\left(\frac{\Delta m^2_{21} s_{2\theta}}{4E}\right)^2 \frac{1}{2} s_{2\theta(z-\bar{z})} \left[ \frac{1}{k k_2} \phi^*_- - \frac{1}{k k_1} \phi^*_1 - \frac{1}{k_1 k_2} \phi^*_- \right]
\]

\[
= -\left(\frac{\Delta m^2_{21} s_{2\theta}}{4E}\right)^2 \frac{1}{2} s_{2\theta(z-\bar{z})} \left[ \frac{1}{k k_2} - \frac{1}{k k_1} + \frac{1}{k_1 k_2} \right] \phi^*_- = 0,
\]

because \(\phi^-_2 + \phi^+_1 = \phi^-\) and \(k_2 = k - k_1\), there is no contribution to these channels.

The \(|(S_{01})_{\beta\alpha}|^2\) also give a null contribution because

\[
|(S_{01})_{e\mu}|^2 - |(S_{01})_{\mu e}|^2 = |c_y (c_{\bar{y}}A_{01} + s_{\bar{y}}B_{01})|^2 - |c_y (c_{\bar{y}}A_{01} + s_{\bar{y}}B_{01})|^2 = 0,
\]

\[
|(S_{01})_{\tau e}|^2 - |(S_{01})_{e\tau}|^2 = |s_y (c_{\bar{y}}A_{01} + s_{\bar{y}}B_{01})|^2 - |s_y (c_{\bar{y}}A_{01} + s_{\bar{y}}B_{01})|^2 = 0,
\]

\[
|(S_{01})_{\mu\tau}|^2 - |(S_{01})_{\tau\mu}|^2 = |(e^{i\delta} s_y^2 - e^{-i\delta} c_y^2)(s_{\bar{y}}A_{01} - c_{\bar{y}}B_{01})|^2
\]

\[-|(e^{-i\delta} s_y^2 - e^{i\delta} c_y^2)(s_{\bar{y}}A_{01} - c_{\bar{y}}B_{01})|^2 = 0.
\]

Therefore, we find no effect from the 2nd order term of \(H_{01}\),

\[
\Delta P^{T(01,01)}_{\nu_\mu\nu_\mu} = \Delta P^{T(01,01)}_{\nu_\mu\nu_\tau} = \Delta P^{T(01,01)}_{\nu_\tau\nu_\mu} = 0.
\]

6 The interference between \(H_{01}\) and \(H_1\)

In this section, we consider the \((\Delta m^2_{21}L/2E)(\delta a(x)L/2E)\) contribution to T-violation.
The S-matrix is given by

\[
S_{01,1} = \tilde{U}_0 \begin{pmatrix}
0 & A_{01,1} & 0 \\
A'_{01,1} & 0 & B'_{01,1} \\
0 & B_{01,1} & 0
\end{pmatrix} \tilde{U}_0^\dagger,
\] (50)

where

\[
A_{01,1} = -i \frac{\Delta m_{21}^2 s_{2x}}{4E} e^{-ia_0 L} \int_0^L dx \int_0^x dy \frac{\delta a(x)}{2E} c_z \left( c_z c_{z-z} e^{-ik_1 y} - s_z s_{z-z} e^{-i(kx-k_2 y)} \right),
\]

\[
A'_{01,1} = -i \frac{\Delta m_{21}^2 s_{2x}}{4E} e^{-ia_0 L} \int_0^L dx \int_0^x dy \frac{\delta a(y)}{2E} c_z \left( c_z c_{z-z} e^{ik_1 x} - s_z s_{z-z} e^{i(ky-k_2 x)} \right),
\]

\[
B_{01,1} = -i \frac{\Delta m_{21}^2 s_{2x}}{4E} e^{-ia_0 L} \int_0^L dx \int_0^x dy \frac{\delta a(x)}{2E} s_z \left( -s_z s_{z-z} e^{-ik_2 y} + c_z c_{z-z} e^{i(ky-k_1 x)} \right),
\]

\[
B'_{01,1} = -i \frac{\Delta m_{21}^2 s_{2x}}{4E} e^{-ia_0 L} \int_0^L dx \int_0^x dy \frac{\delta a(y)}{2E} s_z \left( -s_z s_{z-z} e^{-ik_2 x} + c_z c_{z-z} e^{-i(kx-k_1 y)} \right).
\] (51)

Then, the S-matrix is given with use of \(A_{01,1}\) and \(B_{01,1}\) by

\[
S_{01,1} = \begin{pmatrix}
0 & c_y P_{01,1} & -c_y P_{01,1} \\
c_y P'_{01,1} & -s_y c_y (e^{i\delta} Q_{01,1} + e^{-i\delta} Q'_{01,1}) & e^{i\delta} s_y^2 Q_{01,1} + e^{-i\delta} s_y^2 Q'_{01,1} \\
-s_y P'_{01,1} & e^{-i\delta} s_y^2 Q_{01,1} + e^{-i\delta} s_y^2 Q'_{01,1} & s_y c_y (e^{i\delta} Q_{01,1} + e^{-i\delta} Q'_{01,1})
\end{pmatrix},
\] (52)

where

\[
P_{01,1} = (c_z A_{01,1} + s_z B_{01,1}), \quad Q_{01,1} = (s_z A_{01,1} - c_z B_{01,1}),
\]

\[
P'_{01,1} = (c_z A'_{01,1} + s_z B'_{01,1}), \quad Q'_{01,1} = (s_z A'_{01,1} - c_z B'_{01,1}).
\] (53)

(a) The contribution from the symmetric matter fluctuation

The interference terms between \(\Delta m_{21}^2 L/2E\) and the symmetric matter fluctuation, \(\delta a(x)_s\), contributes only for the term proportional to the CP phase, \(\sin \delta\). There are two contributions. One is from the \(S_{01}S_{1s}^{(1)}\) and the other is \(S_{00}S_{1s}^{*}S_{1s}^{(1)}\) terms. We take the PREM[10] as a symmetric matter. Ota and Sato expanded the PREM in the cosine series[9], which is expressed as \(\delta a(x)_s\) in Eq.(33) with appropriate values of coefficients. The \(S_{1s}^{(1)}\) is given by Eqs.(28) and (34), where we take only the even \(n\) case, i.e., from
Thus, we obtain the sin δ part of T-violation as

$$C^{(1)}_1 + D^{(1)}_1.$$ We find

$$2\text{Re}[(S_{01})_{e\mu} (S_{1s})_{e\mu}^* - (e \leftrightarrow \mu)]$$

$$= - \frac{\Delta m^2_{21} s_{2x} s_{2y} s_{2\bar{s}} s_{\delta}}{E} \left( \frac{c_{\bar{z}} c_{\bar{z}} - s_{\bar{z}} s_{\bar{z}}}{k_1} + \frac{s_{\bar{z}} s_{\bar{z}}}{k_2} \right) c_{\bar{z}}$$

$$\times \left( \sum_{n=1,2,\ldots} \frac{a_{2n} k}{E(k^2 - q_{2n}^2)} \right) \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{kL}{2}. \quad (54)$$

The term $S_{01,1s}$ is obtained once we compute $A_{01,1s}$ etc., which are given by

$$A_{01,1s} = A'_{01,1s} = \frac{\Delta m^2_{21} s_{2x}}{4E} c_{\bar{z}} \left\{ \left( \frac{c_{\bar{z}} c_{\bar{z}} - s_{\bar{z}} s_{\bar{z}}}{k_1} + \frac{s_{\bar{z}} s_{\bar{z}}}{k_2} \right) \right\}$$

$$\times \left( \sum_{n=1,2,\ldots} \frac{a_{2n} k}{E(k^2 - q_{2n}^2)} \right) \phi_1 - \frac{s_{\bar{z}} s_{\bar{z}}}{k_2} \sum_{n=1,2,\ldots} \left( \frac{a_{2n} k}{E(k^2 - q_{2n}^2)} \right) \phi_-. \quad (55)$$

Now, we find

$$2\text{Re}[(S_{00})_{e\mu} (S_{01,1s})_{e\mu}^* - (e \leftrightarrow \mu)]$$

$$= - \frac{\Delta m^2_{21} s_{2x} s_{2y} s_{2\bar{s}} s_{\delta}}{E} \left( \frac{c_{\bar{z}} c_{\bar{z}}}{k_1} + \frac{s_{\bar{z}} s_{\bar{z}}}{k_2} \right)$$

$$\times \sum_{n=1,2,\ldots} \frac{a_{2n}}{E} \left( \frac{c_{\bar{z}}^2 k_1}{k_1^2 - q_{2n}^2} - \frac{s_{\bar{z}}^2 k_2}{k_2^2 - q_{2n}^2} \right) \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{kL}{2}. \quad (56)$$

Thus, we obtain the sin δ part of T-violation as

$$\left( \Delta P^T_{\nu e \nu e} \right)_{s\bar{s}} = - \frac{\Delta m^2_{21}}{E} s_{2x} s_{2y} s_{2\bar{s}} s_{\delta} \left[ \frac{c_{\bar{z}} c_{\bar{z}}}{k_1} + \frac{s_{\bar{z}} s_{\bar{z}}}{k_2} \right] \sin \left( \frac{k_1 L}{2} \right) \sin \left( \frac{k_2 L}{2} \right) \sin \left( \frac{kL}{2} \right)$$

$$\times \left\{ 1 + \sum_{n=1,2,\ldots} \frac{a_{2n}}{E} \left( \frac{c_{\bar{z}}^2 k_1}{k_1^2 - q_{2n}^2} - \frac{s_{\bar{z}}^2 k_2}{k_2^2 - q_{2n}^2} \right) + \frac{c_{\bar{z}} c_{\bar{z}}}{k_1} \right\}. \quad (57)$$

Similarly, we confirmed that the relation

$$\left( \Delta P^T_{\nu \mu \nu \mu} \right)_{s\bar{s}} = \left( \Delta P^T_{\nu e \nu e} \right)_{s\bar{s}} = \left( \Delta P^T_{\nu e \nu e} \right)_{s\bar{s}}, \quad (58)$$

holds in the order of $(\Delta m^2_{21} L/2E)(\delta a(x)_s L/2E)$. 

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(b) The contribution from the asymmetric matter fluctuation

The interference term between $\Delta m_{21}^2 L/2E$ and the asymmetric matter fluctuation, $\delta a(x)_a$ gives the $\cos \delta$ part of T-violation. In order to estimate this contribution, we consider an extreme case, the linear fluctuation, as a typical asymmetric matter profile,

$$\delta a(x)_a = \alpha \bar{a} \left( \frac{x - \frac{L}{2}}{\frac{L}{2}} \right),$$

(59) where $\alpha$ represents the fraction of the asymmetric matter profile.

Firstly, we compute $S_{01} S_{1a}^{(1)*}$ term, where $S_{1a}^{(1)*}$ is the 1st order term from the asymmetric matter profile. For this, we estimate $C_{1}^{(1)} - D_{1}^{(1)}$ defined in Eq.(30) from $\delta a(x)_a$.

We find

$$C_{1}^{(1)} - D_{1}^{(1)} = \frac{\alpha \bar{a} s_{2\bar{z}}}{2EL} \left( \frac{L}{2k} \phi_+ - i \frac{1}{k^2} \phi_- \right).$$

(60)

Now the contribution from the interference term between $S_{01}$ and $S_{1a}^{(1)}$ is obtained by using Eqs.(21) and (28) as

$$2\text{Re}[(S_{01})_{e\mu} (S_{1})_{e\mu}^* - (e \leftrightarrow \mu)] = - \frac{\alpha \bar{a}}{2EL} \frac{\Delta m_{21}^2}{E} s_{2\bar{z}} s_{2\bar{y}} s_{2\bar{z}} c_\delta \left( \frac{c_{z-z-\bar{z}}}{k_1} + \frac{s_{z-z-\bar{z}}}{k_2} \right) \times \left( \frac{1}{k^2} - \frac{L}{2} \cot \frac{kL}{2} \right) \sin \frac{k_1L}{2} \sin \frac{k_2L}{2} \sin \frac{kL}{2}. \quad (61)$$

Next we consider the $S_{00} S_{01,1a}$ contribution. By performing the integrations, we find

$$A_{01,1a} = -A'_{01,1a} = \frac{\alpha \bar{a}}{4EL} \frac{\Delta m_{21}^2}{4E} s_{2\bar{z}} \left( \frac{c_{z-z-\bar{z}}}{k_1} + \frac{s_{z-z-\bar{z}}}{k_2} \right) \left( \frac{L}{k_1} \phi_1^+ - i \frac{2}{k_1^2} \phi_1^- \right) \right) \right),$$

(61)

$$B_{01,1a} = -B'_{01,1a} = \frac{\alpha \bar{a}}{4EL} \frac{\Delta m_{21}^2}{4E} s_{2\bar{z}} \left( - \frac{c_{z-z-\bar{z}}}{k_1} + \frac{s_{z-z-\bar{z}}}{k_2} \right) \left( \frac{L}{k_2} \phi_2^+ - i \frac{2}{k_2^2} \phi_2^- \right) + \frac{c_{z-z-\bar{z}}}{k_1} \left( \frac{L}{k} \phi_+ - i \frac{2}{k^2} \phi_- \right) \right) \right),$$

(62)
Now $T$-violation for the $\nu_e$ and $\nu_\mu$ oscillation channels is given by

$$2\text{Re}[(S_{00})_{e\mu}^*(S_{01,1})_{e\mu}^* - (e \leftrightarrow \mu)]$$

$$= -\frac{\alpha a}{2EL} \frac{\Delta m_{21}^2 s_{2x} s_{2y} s_{2z} c_\delta}{E} \left( \frac{c_{z-\bar{z}} c_{\bar{z}}}{k_1} + \frac{s_{z-\bar{z}} s_{\bar{z}}}{k_2} \right) \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{k L}{2}$$

$$\times \left\{ - \left( \frac{c_{\bar{z}}}{k_1^2} + \frac{s_{\bar{z}}}{k_2^2} \right) + \frac{L}{2} \left( \frac{c_{\bar{z}}}{k_1} \cot \frac{k_1 L}{2} + \frac{s_{\bar{z}}}{k_2} \cot \frac{k_2 L}{2} \right) \right\}.$$  (63)

By adding these contributions, we obtain the asymmetric matter effect to $T$-violation,

$$(\Delta P_{T\nu_e\nu_\mu})_{c\delta} = -\frac{\alpha a}{2EL} \frac{\Delta m_{21}^2 s_{2x} s_{2y} s_{2z} c_\delta}{E} \left( \frac{c_{z-\bar{z}} c_{\bar{z}}}{k_1} + \frac{s_{z-\bar{z}} s_{\bar{z}}}{k_2} \right) \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{k L}{2}$$

$$\times \left\{ \left( \frac{1}{k^2} - \frac{c_{\bar{z}}^2}{k_1^2} - \frac{s_{\bar{z}}^2}{k_2^2} \right) - \frac{L}{2} \left( \frac{1}{k} \cot \frac{k L}{2} - \frac{c_{\bar{z}}}{k_1} \cot \frac{k_1 L}{2} - \frac{s_{\bar{z}}}{k_2} \cot \frac{k_2 L}{2} \right) \right\}.  \quad (64)$$

Similarly, we confirmed

$$(\Delta P_{T\nu_e\nu_\mu})_{c\delta} = (\Delta P_{T\nu_\mu\nu_e})_{c\delta} = (\Delta P_{T\nu_\tau\nu_e})_{c\delta} = (\Delta P_{T\nu_\tau\nu_\mu})_{c\delta}.  \quad (65)$$

From Eqs.(58) and (65), we find that $T$-violation is independent of flavor in the 2nd order perturbation.

### 7 Numerical analysis

In order to analyze the matter effect, we use the PREM profile for the symmetric matter density distribution.

(a) $L = 3000$ km

(a-1) The $T$-violation (intrinsic) asymmetry

Firstly, we estimate how large the $T$-violation asymmetry is. In Fig.1, we plotted $(\Delta P_{T\nu_e\nu_\mu})_{c\delta}/2P(\nu_e \rightarrow \nu_\mu)$ as a function of the neutrino energy $E$, with $\delta = \pi/4$, $s_{2x} = s_{2y} = 1$, $s_z = 0.1$. Here, we used our formula in Eq.(57) for $(\Delta P_{T\nu_\mu\nu_e})_{c\delta}$ and the 0th order term for $P(\nu_e \rightarrow \nu_\mu)$. For the symmetric matter profile, we use $\bar{\rho} = 3.31602$ g/cm$^3$ and the Fourier coefficients, $a_2$, $a_4$, $a_6$ and $a_8$ are $-0.045$, $-0.048$, $-0.047$ and $-0.044$ g/cm$^3$.
respectively, which are derived by Ota and Sato[9] from PREM. We neglect the \( n \geq 10 \) terms. As we can see from this figure, we can expect about \( 4 \sim 10\% \) effect for \( E > 5\text{GeV} \).

(a-2) The matter-modified T-violation in the symmetric matter profile

In the 2nd order perturbation, the symmetric matter gives the contribution to the \( \sin \delta \) of T-violation and the combined formula is given in Eq.(57).

In Fig.2, the comparison between our result and the vacuum case of T-violation,

\[
(\Delta P^T_{\nu_e,\nu_e})_{\text{vacuum}} = -s_{2x}s_{2y}s_{2z}c_{\delta} s_{\delta} \sin \frac{\Delta m^2_{31} L}{4E} \sin \frac{\Delta m^2_{31} L}{4E} \sin \frac{\Delta m^2_{32} L}{4E} .
\]  

(66)

is made, for (a) \( 3\text{GeV} < E < 30\text{GeV} \) and (b) \( 1\text{GeV} < E < 3\text{GeV} \). The values of parameters are the same as those in Fig.1. The solid line shows the vacuum case and the dash-dotted line shows the matter-modified T-violation with the addition of the 1st and the 2nd order terms. The 2nd order contribution is shown by the dotted line, but it is hard to see because it is almost zero in this scale. That is, the 2nd order term is negligibly small and we can safely use the formula given by taking the average density. There is the matter enhancement around \( E = 6\text{GeV} \) which is consistent with the discussion by Parke and Weiler[7].

(a-3) The matter-modified T-violation in the asymmetric matter profile

In Fig.3, we plotted \( (\Delta P)_{\alpha} \), the fake contribution to T-violation from the asymmetric matter fluctuation, for (a) \( 3\text{GeV} < E < 30\text{GeV} \) and (b) \( 1\text{GeV} < E < 3\text{GeV} \). The values of parameters are the same as those in Fig.1. In addition, we considered the 10% asymmetric matter fluctuation, \( \alpha = 0.1 \). We observe that for \( E > 5\text{GeV} \), the contribution is much less than 1% of the intrinsic T-violation and for \( 1\text{GeV} < E < 5\text{GeV} \), the contribution is at most 3%. Thus we conclude that the asymmetric matter contribution is negligibly small for most energies. The small contribution for T-violation asymmetry from the asymmetric matter may be understood by observing that the content of the curly parenthesis in Eq.(64) vanishes when \( |k_1 L| << 1 \), \( |k_2 L| << 1 \) and \( |k L| << 1 \).

We expect that the linear shape for the asymmetric matter is the biggest deviation from the symmetric matter. In order to see the shape dependence of the contribution, we consider the cosine shapes and discuss which cosine shape in the Fourier series will
give the largest contribution to T-violation. That is, we consider the asymmetric matter fluctuation by
\[
\delta a(x) = -\frac{4}{\pi^2} \alpha_n \bar{a} \cos(q_{2n+1}x) ,
\]
where \(q_{2n+1} = (2n+1)\pi/L\). The coefficient, \(4/\pi^2\) is attached by normalizing to the linear shape, \((x - L/2)/L = -4/\pi^2 \sum_{n=0,1} (2n+1)^{-2} \cos(2n+1)\pi x/L\). Now we compare the contribution for various \(n\) by taking \(\alpha_n = 0\). We find
\[
\left( \Delta P_{\nu_\mu,\nu_\mu}^T \right)_n = \frac{4\alpha_n \bar{a} \Delta m_{21}^2 s_{2x} s_{2y} s_{2z} c_\delta}{2\pi^2 E^2} \left( \frac{c_{2x} c_{2z} - \bar{c}_{2z}}{k_1} + \frac{s_{2z} s_{2z} - \bar{s}_{2z}}{k_2} \right) \sin \frac{k_1 L}{2} \sin \frac{k_2 L}{2} \sin \frac{kL}{2} \times \left\{ \frac{k}{k^2 - q_{2n+1}^2} \cot \frac{kL}{2} - \frac{c_{2x}^2 k_1}{k_1^2 - q_{2n+1}^2} \cot \frac{k_1 L}{2} - \frac{s_{2z}^2 k_2}{k_2^2 - q_{2n+1}^2} \cot \frac{k_2 L}{2} \right\}.
\]

Firstly, we checked that the \(n = 0\) case with \(\alpha_0 = 0.1\) agrees with the linear shape case with \(\alpha = 0.1\). We confirmed that the difference around edges does not make any difference and the agreement is quite good. Here, we compare \(n = 1, 2, 3\) cases with \(n = 0\) case in Fig.4, with the same values of parameters as in Fig.1. That is, we plotted \(\left( \Delta P \right)_n/(\Delta P)_{n=0}\) for \(n = 1, 2, 3\) with \(\alpha_n = 0.1\) for \(E > 5\text{GeV}\). We found that as \(n\) becomes larger, the contribution becomes smaller. The \(n = 1, 2\) and 3 cases give about 1/2, 1/5 and 1/10 times smaller than the \(n = 0\) case, respectively. Thus, we conclude that the linear shape case gives the largest contribution to T-violation.

(b) \(L = 7332\text{km}\) case

We use \(\bar{\rho} = 4.21498\text{g/cm}^3\) and the Fourier coefficients, \(a_2, a_4, a_6\) and \(a_8\) are \(-0.31, -0.13, -0.035\) and \(0.01\text{g/cm}^3\), respectively\[9\]. In Fig.5, our formula for the \(\sin \delta\) part in Eq.(57) is plotted in comparison with the vacuum contribution. The solid line shows the vacuum case, while the 1st and the 2nd order terms are shown by the dotted and the dashed lines for \(5\text{GeV}< E < 20\text{GeV}\), respectively. We observe that the 2nd order term is comparable to the 1st order term, and moreover they cancel each other. The net contribution shown by the dash-dotted line is quite small. In Fig.6, the asymmetric matter contribution (\(\cos \delta\) part) is shown in the solid line, in comparison with the \(\sin \delta\)
part. As we see from the figure, the $\cos \delta$ part is as comparable as the $\sin \delta$ due to the severe cancellation between the 1st and the 2nd order terms, though it is much smaller than the vacuum case.

From the above analysis, we observe the followings for $L = 7332\text{km}$:

1. We may need to calculate the 3rd order contribution to obtain the accurate formula for the $\sin \delta$ part in order to check that T-violation is really as small as we obtained.

2. The matter fluctuation is no more neglected, because $\delta a(x)L/2E \sim 1/3$ and the convergence of the perturbation is not fast. This is because the PREM distribution has the $\sin \pi x/L$ like structure for $L = 7332\text{km}$ and thus the symmetric matter profile can not be approximated by the constant (average) density distribution. Therefore, if the PREM distribution is correct, the fluctuation from the average medium really gives the important contribution for a small quantity such as T-violation.

8 Summary

In this paper, we gave the analytical expressions of the contributions from the symmetric and asymmetric matter density fluctuations to T-violation in the 2nd order perturbation with respect to $\delta m_{21}^2 L/2E$ and $\delta a(x)L/2E$. We found that the contribution to T-violation arises from the interference between $\delta m_{21}^2 L/2E$ and $\delta a(x)L/2E$. The matter fluctuation only does not give any contribution to T-violation, which we confirmed by calculating up to the 3rd order terms of $\delta a(x)L/2E$. The symmetric and the asymmetric matter fluctuations give the effect to the $\sin \delta$ and the $\cos \delta$ terms, respectively. These analytic formula are quite accurate for the distance less than $L = 3000\text{km}$ and can be used to discuss T-violation analytically.

By analyzing these formula numerically, we found the following results: Both the contributions from the symmetric and asymmetric matter density fluctuations of the order of less than 10% to T-violation are small for $L = 3000\text{km}$ or a shorter length. The use of the average matter density is sufficient for the practical use. Therefore, we conclude that the observation of T-violation with $L = 3000\text{km}$ or a shorter length will
give a quite clear method to determine the CP violation angle, $\delta$.

For the $L = 7332\text{km}$, the situation changes drastically. We found that the 2nd order term from the symmetric matter fluctuation of the order of less than 10% is comparable to the 1st order term which includes the effect from the constant (average) matter density, and moreover they cancel each other. The net contribution to the $\sin \delta$ term becomes as small as the contribution from the asymmetric matter fluctuation. This shows that the symmetric matter fluctuation becomes important for $L = 7332\text{km}$. This is because the PREM distribution has the $\sin \pi x/L$ like structure for $L = 7332\text{km}$ and it is not approximated by the constant (average) distribution. Therefore, if the PREM distribution is correct, the constant (average) density approximation does not work for $L = 7332\text{km}$ and a longer distance. For $L = 7332\text{km}$, we may need to evaluate the 3rd order term to obtain the accurate formula, which is now under the study.

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Figure 1: The energy dependence of the T-violation asymmetry for the $\nu_e$ to $\nu_\mu$ channel for $L = 3000\text{km}$. $(\Delta P)_{s_\delta}$ represents T-violation with the symmetric matter profile which is proportional to $\sin \delta$ as given in Eq.(57), and $P = P(\nu_e \rightarrow \nu_\mu)$ in oscillation probability in the 0-th order. For the symmetric matter profile, the PREM distribution is used. In this plot, we use $\sin 2\theta_x = \sin 2\theta_y = 1$, $\sin \theta_z = 0.1$, $\Delta m^2_{21} = 5 \cdot 10^{-5}\text{eV}^2$, $\Delta m^2_{31} = 3 \cdot 10^{-3}\text{eV}^2$ and $\delta = \pi/4$. 
Figure 2: The energy dependence of T-violation $(\Delta P)_{s\bar{s}}$ (the dash-dotted line) in comparison with the vacuum case (solid line) with $L = 3000\text{km}$ for (a) $3\text{GeV} < E < 30\text{GeV}$ and (b) $1\text{GeV} < E < 3\text{GeV}$. We used the PREM distribution for the symmetric matter profile. The dashed line shows the 1st order term, which include the constant (average) matter contribution, while the contribution from the 2nd order symmetric matter is indicated by the dotted line. The 2nd order term gives negligible contribution and it is hard to see in this scale. The difference between $(\Delta P)_{s\bar{s}}$ and the vacuum case around $E = 6\text{GeV}$ is the matter effect due to the constant (average) density.
Figure 3: The energy dependence of the asymmetric matter contribution (a linear shape) to T-violation, which is proportional to $\cos \delta$, $(\Delta P)_{c\delta}$, with $L = 3000\text{km}$ for (a) $3\text{GeV} < E < 30\text{GeV}$ and (b) $1\text{GeV} < E < 3\text{GeV}$. We assumed the 10% asymmetry of the average density. The oscillation parameters are the same as those in Fig.1. By comparing this with Fig.2, we see the asymmetric matter contribution is negligibly small for $E > 5\text{GeV}$ and about 3% for $1\text{GeV} < E < 5\text{GeV}$. 
Figure 4: The shape dependence of T-violation for the asymmetric matter. Shapes are expressed by $\cos((2n+1)\pi x/L)$. We confirmed that T-violation from the $n = 0$ case agrees with that from the linear shape case if we choose an appropriate normalization. We plotted the ratios of the contributions from $n = 1, 2, 3$ and $n = 0$. We observe that the contribution from the higher $n$ is suppressed by about 1/2, 1/5 and 1/10 in comparison with the $n = 0$ case. This shows that the linear shape or $n = 0$ case will give the biggest contribution to T-violation.
Figure 5: The energy dependence of T-violation from the symmetric matter, $(ΔP)_{sδ}$ (the dash-dotted line) in comparison with the vacuum case (solid line) with $L = 7332 \text{km}$ for $5 \text{GeV} < E < 30 \text{GeV}$. We used the PREM distribution for the symmetric matter. The dashed line shows the 1st order term, which include the constant (average) matter contribution, while the contribution from the 2nd order symmetric matter is indicated by the dotted line. The oscillation parameters are the same as those in Fig.1. There is severe cancellation between the 1st and the 2nd order terms, and the net contribution becomes much smaller. This shows that the matter fluctuation gives a sizable effect to the sin $δ$ term and the constant (average) approximation for the symmetric matter does not give a good approximation.
Figure 6: The comparison between T-violation from the symmetric matter \(((\Delta P)_s, \delta)\), the dash-dotted line) and from the asymmetric matter \(((\Delta P)_a, \delta)\), the solid line) for \(L = 7332\) km. We used the PREM for the symmetric matter and assumed the asymmetry of 10% in comparison with the average density. The oscillation parameters are the same as those in Fig.1. The \(\sin \delta\) part is not the dominant term in contrast to \(L = 3000\) km case. This is due to the severe cancellation between the 1st term (including the effect from the constant (average) matter) and the 2nd order term from the symmetric matter fluctuation.