Entropy in the interior of a Kerr black hole

Xin-Yang Wang, Jie Jiang and Wen-Biao Liu

Department of Physics, Beijing Normal University, Beijing 100875, People’s Republic of China

E-mail: wbliu@bnu.edu.cn

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Abstract

Christodoulou et al have shown that the interior volume of a Schwarzschild black hole grows linearly with time. Subsequently, their conclusion has been extended to the Reissner–Nordström (RN) and Kerr black holes. Meanwhile, the entropy of the scalar field inside a Schwarzschild black hole has also been calculated. In this paper, a general method calculating the number of quantum states of the scalar field inside the black hole is given, which can be used in an arbitrary black hole. After introducing the two important assumptions as the black-body radiation assumption and the quasi-static process assumption, the entropy of the scalar field inside a Kerr black hole is calculated using the differential form, and we find that the variation of the entropy is proportional to the variation of the Bekenstein–Hawking entropy except the ending of the black hole evaporation. Similarly, we recalculate the entropy of the scalar field inside a Schwarzschild black hole and demonstrate that the entropy inside a Kerr black hole can exactly degenerate to the Schwarzschild black hole. As well as, we find that the proportionality coefficient between the entropy of the scalar field and the Bekenstein–Hawking entropy in Schwarzschild case, which is obtained using the differential form, is half of that given in the previous literature. Furthermore, we investigate the total entropy of Kerr and Schwarzschild black holes and find that they all increase with time. It means that the black holes, evolution with Hawking radiation satisfies the second law of thermodynamics. Finally, the black hole information paradox is brought up again and discussed.

Keywords: black hole, thermodynamics, entropy, interior volume, information paradox

(Some figures may appear in colour only in the online journal)

1 Author to whom any correspondence should be addressed.

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1. Introduction

In general relativity, it is a subtle issue to define the interior volume of a black hole which has a special physical significance, and it is related to the issue of choosing a particular spatial hypersurface for the volume. Parikh [1] suggests that a reasonable definition of interior volume inside a black hole should be a slicing-invariant volume, which has been discussed by other authors [2–8]. Recently, Christodoulou et al [9] have proposed a definition of the interior volume, and introduced a special method to calculate the interior volume, which is similar to the method of calculating the geodesic equation of particles in the spacetime. Using this method, they found that the volume of the spacelike hypersurface slice in a Schwarzschild black hole reaches its maximum value as \( r = \frac{3}{2}m \), which is corresponding to the physical interior volume of a Schwarzschild black hole. Simultaneously, they also showed that the interior volume of a Schwarzschild black hole which is formed by the collapse of a spherically symmetric object is not static, and it grows like

\[
V \sim 3\sqrt{3}\pi m^3 \nu
\]

when \( \nu \gg m \), where \( \nu \) is the advanced time and \( m \) is the mass of the black hole. As we all know, a classical Schwarzschild black hole remains static when one has a view from outside, and has the constant area \( 16\pi m^2 \). However, from inside, equation (1) shows that the interior volume of a classical Schwarzschild black hole grows with time. This result has been extended to the RN [10] and the Kerr black holes [11]. For the RN black hole, since its geometry is spherical which is similar to the Schwarzschild black hole, so that the formula of interior volume is also remarkably analogue to the Schwarzschild black hole, which means that the interior volume of the RN black hole also increases linearly with the advanced time \( \nu \). However, for a Kerr black hole, the geometry is much more complicated, so that it is difficult to calculate its interior volume. To simplify this calculation, Bengtsson et al [11] directly chose an arbitrary hypersurface at constant \( r \) inside a Kerr black hole to calculate the volume of the hypersurface. Although they used a different method, their results still suggest that when \( r \) takes the special value \( r_s \), the volume of the hypersurface takes the maximal value, and corresponds to the interior volume.

Surprisingly, the interior volume of a black hole increases linearly with the advanced time \( \nu \), which can be a candidate to resolve the information paradox problem. Since the volume increases with time, a black hole can have a large amount of volume to hide this remarkably huge information at the end of the evaporation process. In this case, such a large volume may contain many models of quantum fields and the entropy of these models may relate to the Bekenstein–Hawking entropy. Naturally, one may ask that does the information hide in these fields and relate to their entropy? Therefore, it is necessary to investigate the entropy of the hidden modes of a field inside the black hole [12–14]. For a Schwarzschild black hole, the entropy has been studied in [15]. The points which were taken in [15] are objectively described as follows. (i) The Klein–Gordon equation is expanded under a time-depending background. Using the WKB approximation, the scalar field can be written as \( \Phi = \exp[-iET]\exp[iI(\lambda, \theta, \phi)] \), where \( E \) denotes the energy of the scalar modes. (ii) Although the interior volume is dynamic, the statistical properties of scalar field modes in it can still be studied. The number of quantum states with energy less than \( E \) is obtained by integrating the quantum states in the phase space. The free energy and the entropy of the interior volume can be derived from the number of quantum states by using the standard statistical method. (iii) Hawking radiation is a thermodynamic process and the Stefan–Boltzmann law fits well in it. Finally, the result verified that the entropy associated with the interior volume is proportional to the area of the horizon of the black hole.
In this paper, we propose a general method to calculate the quantum states inside a black hole, which can be smoothly applied to the more general black holes. After introducing the two important assumptions as the blackbody radiation assumption and the quasi-static process assumption, the entropy of the scalar field inside a Kerr black hole has been calculated by using the differential form. Then, we compare the variation of the entropy of the scalar field with the variation of the Bekenstein–Hawking entropy and find that they are also proportional to each other in a Kerr black hole except the late stage of black hole evaporation. Subsequently, the entropy of scalar field inside a Schwarzschild black hole is recalculated by using the differential form. The result shows that the proportionality coefficient between the entropy and the Bekenstein–Hawking entropy is half of that given in the previous literature. And we verify that if the angular momentum degenerates to zero, the proportionality coefficient in the Kerr’s case can exactly go back to the result obtained by using the differential form in a Schwarzschild black hole. Furthermore, we calculate the total entropy variation of the Kerr and Schwarzschild black holes with Hawking radiation. In the end, the black hole information paradox will be discussed in the view of the number of quantum states inside the black hole.

The organization of the paper is as follows. In the section 2, we will review the definition of interior volume inside a black hole and then elaborate how to select an appropriate hypersurface to represent the interior volume of a Kerr black hole. Moreover, we demonstrate that the selected hypersurface is the largest hypersurface inside the black hole. In section 3, we calculate the entropy of scalar field inside the black hole using the standard statistical method and discuss the relation between this entropy and the Bekenstein–Hawking entropy. Moreover, we investigate the total entropy variation of the Kerr black hole with Hawking radiation. In section 4, using differential form, we derive the proportionality relationship between two kinds of entropy in a Schwarzschild black hole, and compare it with the Kerr’s case when the angular momentum degenerates to zero. In the end, we also calculate the total entropy variation of the Schwarzschild black hole with Hawking radiation. In section 5, we discuss the results and try to solve some problems about the black hole information paradox. By the way, some conclusions are given.

2. The interior volume of a Kerr black hole

Christodoulou and Rovelli [9] have extended the definition of the volume inside a sphere from flat spacetime to curved spacetime. Based on it, the definition of the interior volume of a spherically symmetric black hole is given, which can be expressed as that the volume of the largest spherically symmetric hypersurface inside the black hole bounded by two-sphere in the event horizon is just the interior volume of the black hole. According to the definition, the interior volume of a Schwarzschild black hole which is formed by the collapse of a spherically symmetric object has been investigated. Through numerical analysis, a particular hypersurface that is the largest spherically symmetric one has been found. This largest hypersurface has been plotted in figure 1(a). It is shown that the largest hypersurface (red curve) in the Schwarzschild black hole is formed by three parts. The first part (1) in the Penrose diagram is a null hypersurface at constant $v$, which connects the event horizon to the spacelike hypersurface at $r = \frac{3}{2}M$. The second part (2) is a long stretch at nearly constant radius $r = \frac{3}{2}M$. The third part (3) is a hypersurface which connects the hypersurface at $r = \frac{3}{2}M$ to the center of the collapsing object $r = 0$. The first part (1) which connects the second part (2) to the event horizon is a null surface, and it does not contribute to the interior volume because its volume should be zero. The third part (3) of the largest hypersurface is entirely inside the collapsed
object. Since the spacetime inside the collapsing object acquires a timelike Killing vector field, this part is a spacelike hypersurface in this region, so it can contribute finite volume to the interior volume of the black hole. However, the contribution of this part can be ignored at large \( v \) because the second part can increase linearly with advanced time \( v \). Then, the bulk of the volume turns out to be due to a region in the vicinity of a constant value of the radial coordinate. It means that we can regard the spacelike hypersurface at \( r = \frac{3}{2} M \) as the largest hypersurface inside a Schwarzschild black hole in large \( v \). Therefore, according to the definition, the volume of the hypersurface at \( r = \frac{3}{2} M \) is the interior volume of a Schwarzschild black hole.

Recently, the definition of interior volume inside a black hole has been extended from a Schwarzschild black hole to a Kerr case [11]. Due to that the Kerr spacetime is spinning, it makes the interior of the black hole different from spherical symmetry, which can largely influence the interior of the black hole. The main reason is that the singularity inside the black hole deforms to a ring. In addition, the Kerr metric is much more complicated than the Schwarzschild metric, so it is difficult to obtain the analytical expression of interior volume using the method in [9]. Similar to [9], we think about a spacelike two-sphere in the event horizon at large advanced time \( v \). However, there is a lot of spacelike hypersurfaces bounded by this two-sphere. Among them, we select a particular hypersurface which is similar to the largest hypersurface in a Schwarzschild black hole and it is also formed by three parts. The hypersurface selected (red curve) in a Kerr black hole is plotted in the Penrose diagram of figure 1(b). It is shown that the first part of the hypersurface (1) is ‘close to null’ just inside the sphere, and joins the second part (2) \( r = \) constant hypersurface all the way down to the third part (3) which is in the matter filled region. The hypersurface is closed up at the center of the collapsing object \( r = 0 \) [11]. The volume of the first part is zero because it is null hypersurface, and the volume of the third part can be ignored at large \( v \) according to the above statement. Therefore the contribution to the interior volume of the black hole mainly comes from the spacelike \( r = \) constant hypersurface. Furthermore, it is also proved that the selected hypersurface cannot be affected by the spacetime rotation, because \( r = \) constant hypersurface is far

Figure 1. (a) Penrose diagram of a portion of the Schwarzschild geometry surrounding a spherically symmetric collapsing object. The volume of the largest hypersurface (red curve) is the interior volume of the black hole. (b) Penrose diagram of a portion of the Kerr geometry surrounding a spinning collapsing object. The hypersurface (red curve) is similar to the largest hypersurface inside a Schwarzschild black hole. The volume of this hypersurface is the interior volume of a Kerr black hole approximately.
from the singularity. Therefore, we can calculate the volume of hypersurface at $r = \text{constant}$ inside the black hole and regard it as the interior volume of the Kerr black hole approximately.

The metric of a Kerr black hole in the Eddington–Finkelstein coordinates is [16]

$$\text{d}s^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} \text{d}v^2 + 2 \text{d}v \text{d}r + \rho^2 \text{d}\theta^2 + \frac{A \sin^2 \theta}{\rho^2} \text{d}\phi^2$$

$$- 2a \sin^2 \theta \text{d}r \text{d}\phi - \frac{4amr}{\rho^2} \sin^2 \theta \text{d}v \text{d}\phi,$$

(2)

where

$$\Delta \equiv r^2 - 2mr + a^2,$$

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta,$$

$$A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta,$$

(3)

where $a = \frac{J}{m}$ is the angular momentum of unit mass of the black hole. From the Kerr metric, the volume of an arbitrary hypersurface at constant $r$ can be obtained as

$$V_\Sigma = 2\pi v \sqrt{-\Delta} \left( \sqrt{r^2 + a^2} + \frac{\rho^2}{2a} \ln \frac{\sqrt{r^2 + a^2} + a}{\sqrt{r^2 + a^2} - a} \right).$$

(4)

This expression of volume has a maximal value when the coordinate $r$ takes a particular value $r_s$. Therefore, we can assume that the maximal value of the volume expression is the interior volume of a Kerr black hole. However, according to the definition, the volume of the largest hypersurface bounded by two-sphere in the event horizon can be considered as the interior volume of a black hole. Till now, we only need to demonstrate that the hypersurface at $r = r_s$ is the largest hypersurface inside the Kerr black hole.

According to the [15] and [17], if the trace of extrinsic curvature of a hypersurface vanishes, the variation of the volume function is automatically zero. In other words, the extrinsic curvature of hypersurface is zero, the hypersurface should be largest. Therefore, we can demonstrate whether the selected hypersurface is the largest using the value of extrinsic curvature. The extrinsic curvature of an arbitrary hypersurface at constant $r$ inside the Kerr black hole can be expressed as

$$K = -\sqrt{2} \left[ \frac{(m - r)a^2 \cos 2\theta + a^2(m - 3r) + 2r^2(3m - 2r)}{[a^2 \cos 2\theta + a^2 + 2r^2]^2} \right].$$

(5)

Based on it, we investigate the relation between the extrinsic curvature and the coordinate $r$ when $m = 1$ and $a = 0.5$. The results show that there is a zero point for the extrinsic curvature, which is corresponding to $r = r_v$. The value of $r_v$ is from 1.38465 to 1.41144 as $\theta$ varies from 0 to $\pi$. Therefore, strictly speaking, the hypersurface at constant $r$ is not the largest hypersurface in the definition of interior volume because its value of extrinsic curvature varies with the value of $\theta$. However, since the variation range of $r_v$ is very small, it can be considered that the hypersurface at constant $r$ is very close to the largest hypersurface in the black hole. Meanwhile, the results also show that the spinning of the spacetime has little influence on the largest hypersurface inside the black hole. Subsequently, according to equation (4), we calculate the value of the coordinate $r$ when the expression of volume is maximal, the result is $r_s = 1.40248$. This value falls within the range of $r_v$ which makes the value of extrinsic curvature to be zero. Therefore, we can suppose that the hypersurface at $r = r_s$ is very close to that of the largest hypersurface inside the Kerr black hole. It means that the volume of the
hypersurface at \( r = r_s \) can be approximately regarded as the interior volume of the Kerr black hole.

3. Entropy in the volume of a Kerr black hole

Although a classical black hole remains stationary and it always has the same area of event horizon from the exterior point of view, its interior volume grows with advanced time \( \nu \). The property of the interior volume is different from that of the volume of the black hole on which an outside observation is taken. The special character of the interior volume may influence the statistical quantities of the quantum field inside the black hole and it may propose a solution to the information paradox of black hole. Hence, it is significant to investigate how the special character of interior volume influences the statistical quantities of the quantum fields. In this paper, we only involve the massless scalar field inside the black hole. In order to get the statistical quantities in the phase space using the method of equilibrium statistics.

In general, we can choose the coordinate \( \{ x^1, x^2, x^3 \} \) on the maximal hypersurface at constant \( r \), on which the induced metric is

\[
d\hat{s}^2 = h_{ij} \, dx^i \, dx^j.
\]

(6)

For any point \( p \) on the hypersurface, there must exist a Gaussian normal coordinate system \( \{ T, x_i^p \}, i = 1, 2, 3 \), defined by a family of geodesics, where \( T \) is the affine parameter of the geodesic, and \( x_i^p \) denotes the spatial coordinate of point \( p \). Thus, the line element can be expressed as

\[
ds^2 = -dT^2 + h_{ij} \, dx^i \, dx^j.
\]

(7)

Now, we consider the massless scalar field \( \Phi \) in this spacetime. Actually, the line element is equivalent to the form \( ds^2 = -dT^2 + h_{ij} \, dx^i \, dx^j \), which means that the hypersurface at constant \( r \) in the the black hole is dynamical for the defined time \( T \). However, the method of equilibrium statistics to calculate the thermodynamic quantities of the system can be used only if the background is static, so that we cannot use the method to investigate the statistical property of the scalar field in such a dynamical situation. Fortunately, according to [17], when \( K = 0 \), the corresponding hypersurface is called the maximal hypersurface, where \( K \) is the trace of the extrinsic curvature of the hypersurface. From \( K = 0 \), we can derive the relation as

\[
\mathcal{L}_h \hat{\epsilon} = 0.
\]

(8)

According to this relation, we can infer \( \mathcal{L}_h \hat{\epsilon} = 0 \), where \( \hat{\epsilon} \) is the induced volume element of the hypersurface. That is to say, the maximal hypersurface at constant \( r \) does not change with the proper time, then the properties are calculated on the hypersurface at \( T = t \) which corresponds to the interior volume of the black hole. So that our statistical calculation is not affected by the nonstatic character of the metric. Next, we will use the common method in the curved spacetime to discuss the motion of the scalar field in the interior of the black hole.

Using the equation of motion of the massless scalar field, we obtain

\[
P^\mu P_\mu = g^{\mu\nu} P_\mu P_\nu = g^{00} E^2 + h^{ij} P_i P_j = -E^2 + h^{ij} P_i P_j = 0,
\]

(9)

in which \( g^{\mu\nu} \) is the inverse metric of spacetime on the hypersurface at constant \( r \) and \( h^{ij} \) is the inverse induced metric which is non-diagonal in general case. Since \( h^{ij} P_i P_j \) is a quadratic
form, we can always find a similarity transformation which can diagonalize the induced metric. Completing the transformation, we can obtain

$$E^2 - \lambda_1 P'_1^2 - \lambda_2 P'_2^2 - \lambda_3 P'_3^2 = 0,$$

(10)

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the diagonal elements of the diagonalizable induced metric respectively, and $P'_1$, $P'_2$ and $P'_3$ are the elements of the eigenstate of the diagonalizable induced metric. Thus the number of quantum states with energy less than $E$ can be obtained as

$$g(E) = \frac{1}{(2\pi)^3} \int dx_1 dx_2 dx_3 dP'_1 dP'_2 dP'_3 \frac{1}{\lambda_1} \sqrt{E^2 - \lambda_2 P'_2^2 - \lambda_3 P'_3^2} = \frac{E^3}{(2\pi)^3} \frac{1}{3} \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}},$$

(11)

where the relation $P'_1 = \sqrt{\frac{1}{\lambda_1} \sqrt{E^2 - \lambda_2 P'_2^2 - \lambda_3 P'_3^2}}$ is used in the second line, the integral formula $\int \int \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dx \, dy = \frac{2\pi}{2} ab$ is used in the third line. We can see that the number of quantum states is proportional to the interior volume of a Kerr black hole. This result appears to be very similar to the result in the flat spacetime. However, their physical significance is very different. In the flat spacetime, because the volume bounded by a closed spacelike hypersurface is a constant, then the number of quantum states which is proportional to the volume is a constant. However, the volume inside the black hole is no longer a constant and it can increase with the advanced time $\nu$, then the number of quantum states of the scalar field inside the black hole can also increase with the advanced time. This property of the quantum states is the special characteristic of the curved spacetime, and it will influence the results in the following.

Subsequently, we continue to calculate the free energy of the scalar field at some inverse temperature $\beta$ as

$$F(\beta) = \frac{1}{\beta} \int d\beta(E) \ln(1 - e^{-\beta E}) = \frac{V_\Sigma}{12\pi^2} \int \frac{E^3 dE}{e^{\beta E} - 1} = -\frac{\pi^2 V_\Sigma}{180 \beta^3},$$

(12)

Furthermore, the entropy is obtained as

$$S_\Sigma = \beta^2 \frac{\partial F}{\partial \beta} = \frac{\pi^2 V_\Sigma}{45 \beta^3},$$

(13)

where $\beta$ is the inverse of the scalar field’s temperature. This entropy is similar to that in the flat phase space. However, it also increases with the advanced time $\nu$ because it is proportional to the interior volume of a black hole.

The above calculation leads to the conclusion that the number of quantum states and the entropy of the scalar field inside the black hole are independent on the specific form of metrics. So that the method of calculating these properties can be applicable to any other black holes. Therefore, this method has its own merit in calculating the entropy of the black hole in general cases.
Next, the two important assumptions will be taken into account [15, 18, 19], which can be summarized as follows:

(a) The black space body radiation assumption. The Hawking radiation of a black hole can be seen as black body radiation. Then, the radiation temperature at infinity can be considered as the temperature of event horizon.

(b) The quasi-static process assumption $\frac{dm}{dv} \ll 1$, which means that the evaporation process is slow enough. Although the Hawking temperature is changing slowly, because of this condition the thermal equilibrium between the scalar field inside the black hole and the event horizon is preserved in this adiabatic process.

According to the assumption (a), the temperature of event horizon can be regarded as the Hawking temperature. And then based on the assumption (b), the temperature of both the scalar field and the horizon can be regarded as equal when they can establish a thermal equilibrium in an infinitesimal process. Therefore, the temperature of the scalar field inside the black hole can be considered as the Hawking temperature, and $\beta$, the inverse of the temperature, can be expressed as

$$\beta = \frac{1}{T} = \frac{2\pi \left( m + \sqrt{m^2 - a^2} \right)^2 + a^2}{\sqrt{m^2 - a^2}}.$$  \hspace{1cm} (14)

Now we consider the interior volume of a Kerr black hole with the Hawking radiation. According to the assumption (a), the lost mass rate of a Kerr black hole can be given by the Stefan–Boltzmann law [20, 21]

$$\frac{dm}{dv} = -\frac{1}{\gamma} T^4 A, \quad \gamma > 0,$$  \hspace{1cm} (15)

where $\gamma$ is a positive constant that depends on the number of quantized matter fields coupling with gravity, and its value does not influence the following discussion. In [10], it has been discussed that the large volume remains until the final stage of black hole evaporation. At this point, the radiation can last. Thus, for a black hole with mass $m(v)$ and angular momentum $a(v)$, we have

$$dv = -\gamma \frac{4\pi^3 \left( 2m(v)^2 + 2m(v) \sqrt{m(v)^2 - a(v)^2} \right)^3}{[m(v)^2 - a(v)^2]^2} dm,$$  \hspace{1cm} (16)

in which, due to the Hawking radiation, the mass is not constant and changes with the advanced time $v$.

Subsequently, the interior volume of a Kerr black hole is calculated. Equation (4) can be written as

$$V_{\Sigma} = 2\pi f \left( \frac{r}{m}, \frac{a}{m} \right) m^3 v,$$  \hspace{1cm} (17)

where

$$f \left( \frac{r}{m}, \frac{a}{m} \right) = \sqrt{2 \frac{r}{m} - \left( \frac{r}{m} \right)^2 - \left( \frac{a}{m} \right)^2 - \left( \frac{r}{m} \right)^2 + m \left( \frac{r}{m} \right)^2 \ln \left( \frac{\sqrt{(\frac{r}{m})^2 + (\frac{a}{m})^2} + \frac{a}{m}}{\sqrt{(\frac{r}{m})^2 + (\frac{a}{m})^2} - \frac{a}{m}} \right)}.$$

(18)
According to the previous statements, the hypersurface is at \( r = r_* \), which corresponds to the interior volume of a black hole. Therefore, for equation (17), we just need to find out the special value of the \( f \left( \frac{r_*}{m}, \frac{a}{m} \right) \) as \( \frac{r_*}{m} = \left( \frac{m}{r_*} \right)_s \), which corresponds to the interior volume of a Kerr black hole. According to equation (18), we find the relationship between \( \frac{r_*}{m} \) and \( f \left( \frac{r_*}{m}, \frac{a}{m} \right) \), as shown in figure 2.

From figure 2, we can see that \( f \left( \frac{r_*}{m}, \frac{a}{m} \right) \) indeed reaches its maximal value at \( \left( \frac{r_*}{m} \right)_s = 1.402 \), which corresponds to the interior volume of the Kerr black hole when \( \frac{a}{m} = 0.5 \).

At the end of this section, when we calculate the relationship between the entropy of scalar field and the Bekenstein–Hawking entropy, two difficulties emerge. Firstly, when we investigate the variation of entropy in an interval, the integral of the equation (16) is unavoidable. This integral is very complicated, so we cannot obtain the analytic solution. More importantly, the assumption (b) is no longer valid when we consider the evaporation in an interval, so the temperature of the scalar field inside the black hole cannot be considered as the Hawking temperature and the method of equilibrium statistics cannot be used. Secondly, when we calculate the entropy of scalar field using the method of equilibrium statistics, the temperature of the scalar field varies with the mass of black hole in the evaporation process, so that we cannot calculate the entropy of scalar field using the method of equilibrium statistics.

However, if we consider an infinitesimal process to investigate the variation of the entropy of the scalar field, we can naturally avoid the two difficulties above. According to the assumptions (a) and (b), the temperature of the scalar field inside the black hole can be treated as the Hawking temperature. Meanwhile, in an infinitesimal process, the relationship between the scalar field and the horizon can be regarded as thermal equilibrium, so the temperature of the black hole can be regarded as a constant which is exactly the value at the beginning of the infinitesimal process. Since the Hawking temperature is a function of the mass, the mass can also be considered a constant in the infinitesimal process. Hence, the variation of both interior volume and the entropy only depends on the variation of advanced time \( \dot{v} \), meanwhile, the variation of the mass can be ignored. For this reason, the method of equilibrium statistics in the infinitesimal process can be used to discuss the variation of the scalar field’s entropy. From the above, we can use the differential form instead of the integral form to investigate the entropy of the scalar field and the Bekenstein–Hawking entropy when the Hawking radiation is considered. In the following, we set the constant mass and angular momentum at the beginning of an infinitesimal process as \( m_0(\dot{v}) \) and \( a_0(\dot{v}) \) respectively.

According to the above statement, we differentiate two sides of the equation (17) and substitute both the maximal value of \( f \left( \frac{r_*}{m}, \frac{a}{m} \right) \) and equation (16) into it, then the differential form of the interior volume is expressed as

\[
\dot{V}_\Sigma = -64\pi^3 f_{\text{max}} \left( \frac{a_0(\dot{v})}{m_0(\dot{v})} \right) \left[ 1 + \sqrt{1 - \left( \frac{a_0(\dot{v})}{m_0(\dot{v})} \right)^2} \right] \left[ 1 - \frac{a_0(\dot{v})}{m_0(\dot{v})} \right]^2 m_0^3(\dot{v}) \dot{m}(\dot{v}), \tag{19}
\]

where \( \dot{V}_\Sigma \) and \( \dot{m}(\dot{v}) \) represent \( \frac{d{V}_\Sigma}{d\dot{v}} \) and \( \frac{d{m}(\dot{v})}{d\dot{v}} \) respectively, \( f_{\text{max}} \left( \frac{a_0(\dot{v})}{m_0(\dot{v})} \right) \) is the maximal value when \( \left( \frac{r_*}{m} \right)_s = \left( \frac{r_*}{m} \right)_s \) and \( \dot{m}(\dot{v}) < 0 \). Substituting both equations (14) and (19) into (13), we can obtain the differential form of the entropy in the infinitesimal process.
Next, we calculate the variation of the Bekenstein–Hawking entropy and compare it with the variation of the entropy of the scalar field in the infinitesimal process. The Bekenstein–Hawking entropy is defined as \[ S_{BH} = \frac{A}{4}, \] where \( A = 4\pi (r_+^2 + a^2) \) is the area of event horizon of a Kerr black hole. Using the differential form, the variation of the Bekenstein–Hawking entropy in this process can be expressed as

\[
\dot{S}_{BH} = 4\pi \left( 1 + \sqrt{1 - \left( \frac{m_0(v)}{m_0(v)} \right)^2} \right) m_0(v) \dot{m}(v),
\]  

(22)

in which \( \dot{m}(v) \) is also negative. Substituting equations (22) into (20), we obtain the differential relationship between the entropy of interior volume and the Bekenstein–Hawking entropy

\[
\dot{S}_\Sigma = -\frac{\pi^2}{180} \gamma F \left( \frac{a_0(v)}{m_0(v)} \right) S_{BH},
\]  

(23)
where

\[
F \left( \frac{a_0(v)}{m_0(v)} \right) = f_{\text{max}} \left( \frac{a_0(v)}{m_0(v)} \right) \left[ 1 - \sqrt{1 - \left( \frac{a_0(v)}{m_0(v)} \right)^2} \right] \left( \frac{a_0(v)}{m_0(v)} \right)^{-2}.
\]  

(24)

According to equation (24), we have the relationship between the \(a_0(v)\) and \(m_0(v)\), as shown in the figure 3.

Figure 3 shows that the function \(F \left( \frac{a_0(v)}{m_0(v)} \right)\) can be approximated as a constant in the early stage of the evaporation, which means that the variation of entropy from quantum theory in the interior volume is proportional to the variation of the Bekenstein–Hawking entropy. It means that the expansion of interior volume is proportional to the shrink of event horizon for a Kerr black hole space. This is an intriguing result. However, the function \(F \left( \frac{a_0(v)}{m_0(v)} \right)\) with the change of mass reduces greatly at the end of the evaporation of the black hole, which means the variation of the two kinds of entropy violate the proportional relationship. The value of \(F \left( \frac{a_0(v)}{m_0(v)} \right)\) is not suitable for the late stage of the black hole evaporation. Since the process can no longer be considered as a very slow process in the late stage of black hole evaporation, the assumption (b) is not satisfied. Based upon the method of calculation, if we want to calculate the entropy of the scalar field inside the black hole at the end of the Hawking radiation, the variation of the mass of the black hole cannot be ignored in the infinitesimal process. In other words, when we differentiate two sides of the equation (17), the mass of the black hole cannot be considered as a constant. Since the area of the event horizon becomes very small, the effect of quantum gravity cannot be ignored at the end of the black hole evaporation as well. For this reason, the calculation of the scalar field’s entropy inside the black hole at the end of the evaporation process becomes very difficult. Therefore, we will discuss the situation of the late stage of the black hole evaporation in subsequent research work.

Finally, we investigate the total entropy variation of the Kerr black hole. Considering Hawking radiation, the entropy variation of the scalar field inside the Kerr black hole and the variation of Bekenstein–Hawking entropy have been given as equations (20) and (22) respectively. So the total variation of the entropy can be obtained as

\[
\dot{S}_{\text{total}} = \dot{S}_\Sigma + \dot{S}_{\text{BH}} = \left[ \frac{4\pi + \frac{180\pi - \int_{m_0(v)}^{\infty} m_0(v) \dot{m}(v) \, dv}{45\sqrt{1 - \left( \frac{\dot{m}(v)}{m_0(v)} \right)^2}} }{\pi^3} \right] m_0(v) \dot{m}(v).
\]  

(25)

Since \(\frac{1}{\gamma} \sim 10^{-5}\) [21] and \(\dot{m}(v) < 0\), it is easy to see

\[
\dot{S}_{\text{total}} > 0.
\]  

(26)

If the black hole can be regarded as a thermodynamic system, its evolution with Hawking radiation satisfies the second law of thermodynamics.

4. Entropy in the volume of a Schwarzschild black hole

In this section, we return back to the Schwarzschild black hole. The relationship between the entropy of scalar field inside the black hole and the Bekenstein–Hawking entropy has been studied in [15]. However, the method used to calculate the entropy of scalar field inside the
black hole is flawed in the literature, because it chose an unreasonable temperature to calculate the entropy of scalar field. The primary reason beneath this issue is that when the Hawking radiation is considered, the black hole’s mass varies with time, which leads to a change in the temperature of the black hole as time goes on. Therefore, the method of equilibrium statistics cannot be applied in this situation. To solve this problem, according to the assumptions (a) and (b), we use the differential form which is similar to the Kerr case to calculate the relationship between the variation of the entropy of the scalar field inside the black hole and the variation of the Bekenstein–Hawking entropy in the infinitesimal process. The proportionality coefficient between the two kinds of entropy can be obtained more reasonably by using the differential form, and the result obtained in the previous literature are corrected.

We also start from the interior volume of a Schwarzschild black hole and it can be expressed as \[ V_\Sigma \sim 3\sqrt{3}\pi m^2 v. \] (27)

By adopting the method of equilibrium statistics, we can obtain the entropy of scalar field in the volume. The expression of entropy is the same as equation (13). According to the assumption (a), a Schwarzschild black hole evaporation can also be seen as black body radiation. Therefore, we can use the Stefan–Boltzmann law

\[
\frac{dm}{dv} = -\frac{1}{\gamma} T^4 A
\] (28)

to calculate the change of mass with the advanced time \( v \), where \( A = 16\pi m^2(v) \) is the area of event horizon. According to the assumptions (a) and (b), the temperature of the scalar field can be regarded as the Hawking temperature, which is expressed as

![Figure 3. Plot of \( \frac{m(v)}{m_0(v)} \) versus \( F\left(\frac{m(v)}{m_0(v)}\right) \). This figure shows that \( F\left(\frac{m(v)}{m_0(v)}\right) \) can be approximated as a constant at the beginning of the evaporation of the black hole, but the value of \( F\left(\frac{m(v)}{m_0(v)}\right) \) decreases rapidly with the change of mass at the end of black hole evaporation.](image)
Substituting equations (29) into (28), we have

\[ dv = - \frac{2}{8\pi m(v)} \gamma \pi^3 m_0^2(v) \dot{m}(v). \]  

(30)

According to the assumption (b), the Hawking radiation is very slow and the evaporation process can be regarded as a quasi-static process. Hence, the variation of the interior volume of the black hole only depends on the advanced time \( v \). Differentiating two sides of the equation (27) and substituting equation (30) into it, we can obtain the interior volume in an infinitesimal process which can be expressed as

\[ \dot{V}_\Sigma = - \frac{2}{8\pi} \gamma \pi^3 m_0^2(v) \dot{m}(v). \]  

(31)

Substituting equations (31) into (13), we can obtain the differential form of entropy inside the black hole

\[ \dot{S}_\Sigma = - \frac{\sqrt{3}}{240} \gamma m_0(v) \dot{m}(v). \]  

(32)

From the definition of the Bekenstein–Hawking entropy, we can directly derive its differential form as

\[ \dot{S}_{\text{BH}} = 8\pi m_0(v) \dot{m}(v). \]  

(33)

Combining equations (33) with (32), we find the differential relationship between the entropy of interior volume and the Bekenstein–Hawking entropy, which can be expressed as

\[ \dot{S}_\Sigma = - \frac{\sqrt{3}}{240} \dot{S}_{\text{BH}}. \]  

(34)
Finally, if we set $a = 0$, the results in a Kerr black hole are expected to degenerate to the results in a Schwarzschild black hole. In this approach, we verify that it can obtain the reasonable result by using the differential form to study the Hawking radiation in these two kinds of black hole. Therefore, the limit of equation (24) can be expressed as

$$
\lim_{a \to 0} f \left( \frac{r}{m(v)} \cdot \frac{a(v)}{m(v)} \right) \left[ 1 - \sqrt{1 - \left( \frac{a(v)}{m(v)} \right)^2} \right] \left( \frac{a(v)}{m(v)} \right)^{-2}
$$

= \frac{r}{m(v)} \sqrt{2r - \left( \frac{r}{m(v)} \right)^2}

= f \left( \frac{r}{m(v)} \right),

(35)

where $\frac{r}{m(v)}$ must take special value, because it corresponds to the interior volume of the black hole. The relationship between the $\frac{r}{m(v)}$ and the value of the limit can be shown in figure 4.

Figure 4 illustrates that the function $f \left( \frac{r}{m(v)} \right)$ has maximal value at $\frac{r}{m(v)} = \frac{3}{2}$, and this value is exactly the same as the Schwarzschild case. Substituting the maximal value of $f \left( \frac{r}{m(v)} \right)$ into the equation (23), we obtain the proportional relation between two types of entropy

$$
\dot{S}_\Sigma = -\frac{\sqrt{3\pi^2/}\gamma}{240} \dot{S}_{BH}.

(36)

Comparing equations (36) with (34), we find that they are exactly the same. This means the proportional relation in the Kerr case can totally degenerate to the Schwarzschild case. This complete degeneration relationship reflects the interior volume of a Kerr black hole can be degenerated to that of a Schwarzschild black hole when $a = 0$ and it also reflects the fact that the proportional relationship between the variation of the two kinds of entropy for a Schwarzschild black hole is a special case for a Kerr black hole.

Finally, we can also investigate the total entropy variation of the Schwarzschild black hole with Hawking radiation. Summing equations (32) and (33), the total entropy variation can be given as

$$
\dot{S}_\text{total} = \dot{S}_\Sigma + \dot{S}_{BH} = \left( 8\pi - \frac{\sqrt{3\pi^2/\gamma}}{30} \right) m_0(v) \dot{m}(v).

(37)

Since $\frac{1}{7} \sim 10^{-5}$ [21] and $\dot{m}(v) < 0$, equation (37) also satisfies the condition

$$
\dot{S}_\text{total} > 0.

(38)

The evolution of the Schwarzschild black hole with Hawking radiation also satisfies the second law of thermodynamics.

5. Discussions and conclusions

According to the two important assumptions, we have adopted the differential form for discussing the variation of entropy inside a Kerr black hole in an infinitesimal process, and have obtained the relationship between the variation of entropy of the scalar filed inside the black hole and the variation of the Bekenstein–Hawking entropy. The analysis and discussion of the result show that the proportionality coefficient of the two kinds of entropy is approximated
as a constant except the late stage of the evaporation process. In other words, the variation of entropy of the scalar field is proportional to the variation of the Bekenstein–Hawking entropy. However, the two kinds of entropy do not satisfy this relation at the end of the evaporation, since the assumption (b) does not hold.

We also have adopted the differential form to calculate the variation of scalar field entropy inside the Schwarzschild black hole when the Hawking radiation is considered, and have calculated the proportional relation between it and the variation of the Bekenstein–Hawking entropy. Meanwhile, setting $a = 0$, we expect that the result of the Kerr case can completely degenerate to the Schwarzschild case to verify the results obtained. By comparing the two results, it reveals that, if the angular momentum degenerates to zero, the proportional relation between the variation of two types of entropy in the Kerr black hole is exactly the same as the result calculated by the differential form in the Schwarzschild black hole. This conclusion reflects that with the Hawking radiation, if we use the differential form to calculate the proportional relation between the variation of the entropy of the scalar field inside the black hole and the variation of the Bekenstein–Hawking entropy, we will obtain the reasonable results.

At the end of sections 3 and 4, we investigate the total entropy variation of Kerr and Schwarzschild black holes. The results show that the total entropy of two types of black holes all increases with the advanced time. If the black hole can be considered as a thermodynamic system, its evolution satisfies the second law of thermodynamics. Since the evaporation process is very slow in the early stage of Hawking radiation, the Bekenstein–Hawking entropy changes slowly. However, the entropy of the scalar field inside the black hole increases linearly with the advanced time. It implies that the variation of the entropy inside the black is more significant than the variation of the Bekenstein–Hawking entropy. Therefore, the total entropy of the black hole can increase with Hawking radiation.

In the previous literature [15], it is unreasonable to use integral method to study the proportional relation between the two types of entropy. There are two reasons for the unreasonableness. Firstly, the quasi-static assumption cannot be used in the interval. Secondly, the changing temperature with the advanced time $\tau$ is replaced by the initial temperature. Based on these two reasons, the unreasonable proportional relation can be obtained. However, if the differential form is used, the unreasonable situation can be naturally avoided. The proportional coefficient obtained by the differential method is half of that obtained by the integral method. We corrected the result in the previous literature by improving the calculation method.

According to previous calculations, the interior volume of a Kerr black hole grows with the advanced time $\tau$ for $\tau \gg m$. If the massless scalar field is added inside the black hole, the number of the quantum states of the scalar field is proportional to the interior volume. While the interior volume increases with the advanced time $\tau$, so does the number of quantum states. It is really a clue to think about the black hole information paradox [25–40].

In the process of black hole evaporation, the mass loss of black hole is due to Hawking radiation, then the surface area of the event horizon decreases, which accounts for the decrease of the Bekenstein–Hawking entropy. However, since the entropy of the scalar field in the interior of black hole is proportional to the interior volume of the black hole, the entropy of the scalar filed also increases while the interior volume increases with the advanced time $\tau$. The relationship between these two types of entropy is expressed as equations (23) and (34).

It is often assumed that the maximal number of quantum states contained in a black hole surface is [41–43]

$$N_{BH} = e^{S_{BH}},$$

(39)
where \( S_{\text{BH}} = \frac{A}{4} \) is the Bekenstein–Hawking entropy and \( A \) is the area of the event horizon. When the evaporation happens, the number of quantum states on the black hole surface gradually reduces due to the decrease of the Bekenstein–Hawking entropy. According to equation (11), the number of quantum states inside the black hole increases with the expansion of the interior volume, and the entropy of the scalar field in the interior volume also increases. When the black hole evaporation eventually stops, the number of quantum states inside the black hole is much more than the number of quantum states on the surface. Hence, the results in this paper are consistent with [44]. According to [10], the large volume inside the black hole should be able to contain all the information, even though the area of event horizon shrinks to a very small size. In the end, the black hole becomes ‘a remnant’ [45, 46], and it has enough room to store the information. Based on this fact, we propose that the information loss claimed in the process of black hole evaporation is actually stored in the interior of the black hole as the form of quantum states of the scalar field.

In this paper, the interior volume of a Kerr black hole has been investigated. Moreover, some explanation about why the statistical quantities can be calculated in the dynamical background is given. Involving the massless scalar field inside the black hole, we have proposed a more general method to calculate the number of quantum states in the interior volume of a Kerr black hole, and found that the entropy of a Kerr black hole is deeply related to the Bekenstein–Hawking entropy. Based on the two assumptions, the proportional relation between the two types of entropy has been calculated by using the differential form. It is found that they are proportional to each other except the late stage of the black hole evaporation. Moreover, the proportional relation in Schwarzschild black hole has been recalculated. It is found that the proportionality coefficient of the two types of entropy is half of the result obtained in the previous literature and the proportionality coefficient in a Kerr black hole can completely degenerate to the Schwarzschild case when the angular momentum degenerated to zero. It means that the reasonable result can be obtained by using the differential form and the result obtained in the previous literature is corrected. Furthermore, the total entropy variation of the Kerr and Schwarzschild black holes has been calculated. The results show that the total entropy of two types of black hole increases with the advanced time. The evolution of black holes with Hawking radiation satisfies the second law of thermodynamics. Finally, we have explained the black hole information paradox from the number of quantum states inside the black hole. Early literature reveals that the number of quantum states on the black hole surface is bounded. However, our calculation have shown that the number of quantum states of the scalar field inside the black hole is much more than that on the surface. When Hawking radiation exists, the number of quantum states on the surface gradually decreases. On the contrary, the number of quantum states inside the black hole increases. Consequently, we propose that the information so-called lost in the process of black hole evaporation is stored in the black hole as the form of quantum states. Hence, the information has never been lost.

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ORCID iDs

Wen-Biao Liu @ https://orcid.org/0000-0001-5271-2307
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