ATOMICITY RELATED TO NON-ADDITIVE INTEGRABILITY

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Abstract. In this paper we present some results concerning Gould integrability of vector functions with respect to a monotone measure on finitely purely atomic measure spaces. As an application a Radon-Nikodym theorem in this setting is obtained.

1. Introduction

In the last years, the field of non-additive measures was intensively used in a wide range of areas such as economics, social sciences, biology and philosophy and it gives a mathematical framework for describing a situation of conflict or cooperation between intelligent rational players, in order to predict the outcome of a process. In this field, the theory of (pseudo)atoms and monotonicity is used in statistics, game theory, probabilities, artificial intelligence.

Purely atomic measures were studied in literature (in different variants) due to their special form and their special properties. In this case, the entire space is assumed to be a finite collection of pairwise disjoint atoms and an atom can be viewed as a black hole. For instance, Chiţescu [7, 8] and Leung [22] established some relationships with classical problems in $L^p$ spaces, Ionaşcu and Stancu [21] have obtained concrete independent events in purely atomic probability spaces with geometric distribution, Matveev [26] has proved that every $\sigma$-finite Borel measure defined on a Rothberger space is purely atomic, Elton and Hill [12] studied the ham sandwich theorem.

The subject of our paper belongs to the non-additivity domain that was intensively studied by many authors for interesting and important properties (e.g. [1–6, 20, 23–25, 27–29]). In this paper we present some results of Gould integrability [19] on finitely purely atomic measure spaces. The idea is similar to that of [16], where mainly the measures are set-valued and the functions are scalar, while in the present research the other product by scalar is considered, namely vector valued functions and real valued and positive measures are discussed. Convergence and Radon-Nikodým theorems are also obtained in this framework.

The structure of the paper is the following: in Section 2 we give some preliminaries. Section 3 contains some results on atoms that we shall use in the sequel; examples are given to show that some of them do not hold in general in Section 4, together with different properties regarding Gould type

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integrability on finitely purely atomic measure spaces, such as: a Lebesgue type theorem of convergence and comparative results among Gould integrability, Choquet integrability, total measurability and boundedness. Finally, in Section 5 we establish a Radon-Nikodým theorem for purely atomic measures.

2. Preliminaries

$T$ is an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of $T$, $\mathcal{A}$ an algebra of subsets of $T$ and $m : \mathcal{A} \rightarrow [0, +\infty)$ an arbitrary set function, with $m(\emptyset) = 0$. If $A \subseteq T$, then $T \setminus A$ will be denoted by $A^c$.

We recall some notions that will be used throughout the paper.

Definition 2.1. ([10,11,30]) $m$ is said to be:

- a monotone measure if $m(A) \leq m(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$;
- null-additive if $m(A \cup B) = m(A)$, for every $A, B \in \mathcal{A}$, with $m(B) = 0$;
- $\sigma$-null-additive if $m(\bigcup_n A_n) = 0$ as soon as $A_n \in \mathcal{A}$ and $m(A_n) = 0$ for all $n$.
- subadditive if $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{A}$;
- finitely additive if $m(A \cup B) = m(A) + m(B)$, for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$;
- $\sigma$-subadditive if $m(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$, for every $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, so that $\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}$;

Remark 2.2. If $m$ is monotone and subadditive, then $m$ is null-additive. A subadditive monotone measure is sometimes called a submeasure ([10]).

Definition 2.3. Let $m : \mathcal{A} \rightarrow [0, +\infty)$ with $m(\emptyset) = 0$.

2.3.i) The set function $\overline{m} : \mathcal{P}(T) \rightarrow [0, +\infty]$, called the variation of $m$, is defined by $\overline{m}(E) = \sup\{\sum_{i=1}^{n} m(A_i)\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$ with $A_i \subseteq E$, for every $i \in \{1, \ldots, n\}$.

2.3.ii) $m$ is said to be of finite variation on $\mathcal{A}$ if $\overline{m}(A) < \infty$, for every $A \in \mathcal{A}$ (or, equivalently, if $\overline{m}(T) < \infty$).

2.3.iii) We consider the set function $\overline{m}$ defined by $\overline{m}(E) = \inf\{\overline{m}(A) ; E \subseteq A, A \in \mathcal{A}\}$, for every $E \in \mathcal{P}(T)$.

The statements in the following Remark easily follow by the definitions: some of them are contained in [16, Remark 2.4], though in a more abstract situation.

Remark 2.4.

2.4.i) If $E \in \mathcal{A}$, then in the definition of $\overline{m}$ the supremum could be considered over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$, so that $\bigcup_{i=1}^{n} A_i = E$.

2.4.ii) $m(A) \leq \overline{m}(A)$, $\forall A \in \mathcal{A}$. So, if $\overline{m}(A) = 0$, then $m(A) = 0$, $\forall A \in \mathcal{A}$.
2.4.iii) Suppose $m$ is monotone and let $A \in \mathcal{A}$ with $m(A) = 0$. Then $\overline{m}(A) = 0$. In consequence, if $m$ is monotone, then $\overline{m}(A) = 0$ $\iff m(A) = 0$, for every $A \in \mathcal{A}$.

2.4.iv) $\overline{m}$ and $\overline{m}$ are monotone.

2.4.v) If $m$ is subadditive ($\sigma$-subadditive respectively), then $\overline{m}$ is additive ($\sigma$-additive respectively) on $\mathcal{A}$.

2.4.vi) If $m$ is finitely additive, then $\overline{m}(E) = \sup\{m(B)|B \in \mathcal{A}, B \subseteq E\}$, for every $E \in \mathcal{P}(T)$ and $\overline{m}(A) = m(A)$ for every $A \in \mathcal{A}$.

2.4.vii) $\overline{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$.

In what follows, we give some results regarding null-additivity of $\overline{m}$ and $\overline{m}$.

**Proposition 2.5.** If $m$ is null-additive, then $\overline{m}$ is null-additive on $\mathcal{A}$.

**Proof.** Let $A, B \in \mathcal{A}$ with $\overline{m}(B) = 0$. Since $\overline{m}$ is monotone, we only have to prove that $\overline{m}(A \cup B) \leq \overline{m}(A)$. Let $\{C_i\}_{i=1}^p \subseteq \mathcal{A}$, $C_i \cap C_j = \emptyset$, $i \neq j$ so that $\bigcup_{i=1}^p C_i = A \cup B$. We have two situations.

Suppose that $A \cap B = \emptyset$. Then $B = \bigcup_{i=1}^p (C_i \cap A^c)$. From $\overline{m}(B) = 0$ we obtain $m(B) = 0$. So $m(C_i \cap A^c) = 0$, $\forall i \in \{1, \ldots, p\}$. From the null-additivity of $m$, it results $m(C_i) = m(C_i \cap A)$, for every $i \in \{1, \ldots, p\}$. So, $\sum_{i=1}^p m(C_i) = \sum_{i=1}^p m(C_i \cap A) \leq \overline{m}(A)$ which implies that $\overline{m}(A \cup B) \leq \overline{m}(A)$.

If $A \cap B \neq \emptyset$ then we have $A \cup B = (A \setminus B) \cup B$ and $\overline{m}(A \cup B) = \overline{m}(A \setminus B) \cup B) = \overline{m}(A \setminus B) \leq \overline{m}(A)$.

**Proposition 2.6.** Suppose $\mathcal{A}$ is a $\sigma$-algebra. If $\overline{m}$ is null-additive, then $\overline{m}$ is null-additive on $\mathcal{P}(T)$.

**Proof.** Let $A, B \in \mathcal{P}(T)$ with $\overline{m}(B) = 0$. Since $\overline{m}$ is monotone, we only have to prove that $\overline{m}(A \cup B) \leq \overline{m}(A)$. Because $\overline{m}(B) = 0$, for every $n \in \mathbb{N}$ there exists $B_n \in \mathcal{A}$, $B \subseteq B_n$ so that $\overline{m}(B_n) < n^{-1}$. It results $\lim_{n \to \infty} \overline{m}(B_n) = 0$.

Without any loss of generality, we may suppose that $B_n \searrow B'$. So, $B \subseteq B'$ and $B' \in \mathcal{A}$. Since $\overline{m}$ is monotone, it follows that $\overline{m}(B') \leq \overline{m}(B_n)$ for all $n$, hence $\overline{m}(B') = 0$. Now, let $C \in \mathcal{A}$ be arbitrary with $A \subseteq C$. Then $\overline{m}(A \cup B) \leq \overline{m}(B' \cup C) = \overline{m}(C)$, whence $\overline{m}(A \cup B) \leq \overline{m}(A)$, which finishes the proof.

### 3. Atoms

In this section some properties about atomicity and total measurability are investigated. We recall that

**Definition 3.1.**

- A set $A \in \mathcal{A}$ is said to be an atom of $m$ if $m(A) > 0$ and for every $B \in \mathcal{A}$, with $B \subseteq A$, we have $m(B) = 0$ or $m(A \setminus B) = 0$.

- For further reference, atoms can be introduced also for vector-valued measures: if $m$ takes values in a Banach space $X$, a null set is an element $N \in \mathcal{A}$ such that $m(B) = 0$ for every measurable $B \subseteq N$; and an atom for $m$ is any non-null set $A \in \mathcal{A}$ such that, for every measurable $B \subseteq A$ one has that $B$ is null or $A \setminus B$ is null.
Remark 3.4. (\cite[Remark 3.8]{16}) If \(T\) is a finitely purely atomic space, then every set \(A\) \(\in\mathcal{A}\) of atoms of \(m\) is monotone, then the following statements are equivalent:

\(m\) is said to be finitely purely atomic (and \(T\) a finitely purely atomic space) if there is a finite disjoint family \(\{A_i\}_{i=1}^n \subset \mathcal{A}\) of atoms of \(m\) so that \(T = \bigcup_{i=1}^n A_i\).

Lemma 3.2. (\cite[Remark 3.7]{16}) Let \(m : \mathcal{A} \to [0, +\infty)\) be a non-negative set function, with \(m(\emptyset) = 0\) and let \(A \in \mathcal{A}\) be an atom of \(m\).

3.2.1) If \(m\) is monotone and the set \(B \in \mathcal{A}\) is so that \(B \subseteq A\) and \(m(B) > 0\), then \(B\) is also an atom of \(m\) and \(m(A \setminus B) = 0\). Moreover, if \(m\) is null-additive, then \(m(B) = m(A)\).

3.2.2) If \(m\) is monotone and null-additive, then for every finite partition \(\{B_i\}_{i=1}^n\) of \(A\), there exists only one \(i_0 \in \{1, 2, \ldots, n\}\) so that \(m(B_{i_0}) = m(A)\) and \(m(B_i) = 0\) for every \(i \in \{1, \ldots, n\}, i \neq i_0\).

Proof. It is enough to assume that the values of the multimeasure in \cite[Remark 3.7]{16} are (real) singletons. \(\square\)

Example 3.3.

3.3.i) Let \(T = \{a, b, c\}, \mathcal{A} = \mathcal{P}(T)\) and

\[
m(A) = \begin{cases} 2, & \text{if } A = T \\ 1, & \text{if } A = \{a, b\} \text{ or } A = \{c\} \\ 0, & \text{otherwise.} \end{cases}
\]

We remark that \(A_1 = \{a, b\}\) and \(A_2 = \{c\}\) are disjoint atoms of \(m\) and \(T = A_1 \cup A_2\). So \(m\) is finitely purely atomic.

3.3.ii) Let \(T\) be a countable set, \(\mathcal{A} = \{A \subseteq T \mid A\text{ is finite or } A^c \text{ is finite}\}\) and \(m : \mathcal{A} \to [0, \infty)\) defined for every \(A \in \mathcal{A}\) by

\[
m(A) = \begin{cases} 0, & \text{if } A \text{ is finite} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}
\]

Then every set \(A \in \mathcal{A}\), such that \(A^c\) is finite, is an atom of \(m\).

Remark 3.4. (\cite[Remark 3.8]{16}) If \(m\) is monotone, then the following statements are equivalent:

\(m\) is finitely purely atomic \(\iff\) \(\bar{m}\) is finitely purely atomic on \(\mathcal{A}\) \(\iff\) \(\bar{m}\) is finitely purely atomic on \(\mathcal{A}\).

If \(m\) is a null-additive monotone measure and \(A \in \mathcal{A}\) is an atom of \(m\), then \(\bar{m}(A) = m(A)\).

Indeed, let \(\{B_i\}_{i=1}^n\) be a partition of \(A\). According to Lemma 3.2.2), there is a unique \(i_0 \in \{1, \ldots, n\}\) such that \(m(B_{i_0}) = m(A)\) and \(m(B_i) = 0\) for every \(i \in \{1, \ldots, n\}, i \neq i_0\). So, we have \(\sum_{i=1}^n m(B_i) = m(A)\), which implies that \(\bar{m}(A) = m(A)\).

If \(m\) is a finitely purely atomic subadditive monotone measure, then \(m\) is of finite variation.

Indeed, suppose \(T = \bigcup_{i=1}^p A_i\), where \(\{A_i\}_{i=1}^p \subset \mathcal{A}\) are pointwise disjoint atoms of \(m\). Then

\[
\bar{m}(T) = \bar{m}\left(\bigcup_{i=1}^p A_i\right) = \sum_{i=1}^p \bar{m}(A_i) = \sum_{i=1}^p m(A_i) < \infty.
\]
In order to state our next theorems, a result of [24] will be presented. In the sequel let $T$ be a locally compact Hausdorff topological space, $K$ be the lattice of all compact subsets of $T$, $\mathcal{B}$ be the Borel $\sigma$-algebra (that is the smallest $\sigma$-algebra containing $K$) and $\tau$ be the class of all open sets. For a study on this subject it is possible to see also [31].

**Definition 3.5.** A set function $m : \mathcal{B} \to [0, +\infty)$ is called regular if for each set $A \in \mathcal{B}$ and each $\varepsilon > 0$, there exist $K \in K$ and $D \in \tau$ such that $K \subseteq A \subseteq D$ and $m(D \setminus K) < \varepsilon$.

**Theorem 3.6.** [24, Theorem 4.6] Let $m : \mathcal{B} \to [0, +\infty)$ be a regular null-additive monotone set function. If $A \in \mathcal{B}$ is an atom of $m$, then there exists a unique point $a \in A$ such that $m(A) = m(\{a\})$ and $m(A \setminus \{a\}) = 0$.

**Proposition 3.7.** Suppose $m : \mathcal{B} \to [0, +\infty)$ is finitely purely atomic, regular, null-additive and monotone. Then there exists a finite family $\{A_i\}_{i=1}^n \subset \mathcal{A}$ of pairwise disjoint atoms of $m$ so that $T = \bigcup_{i=1}^n A_i$.

**Proof.** By Theorem 3.6, there are unique $a_1, a_2, \ldots, a_n \in T$ such that $a_i \in A_i$ and $m(A_i \setminus \{a_i\}) = 0$, for every $i \in \{1, \ldots, n\}$. Then we have

$$0 \leq m(T \setminus \{a_1, \ldots, a_n\}) \leq m(A_1 \setminus \{a_1\}) + \ldots + m(A_n \setminus \{a_n\}) = 0,$$

which implies $m(T \setminus \{a_1, \ldots, a_n\}) = 0$. Now, since $m$ is null-additive it follows $m(T) = m(\{a_1, \ldots, a_n\})$. \qed

We finish this section by presenting a property of total measurability on atoms. Let $\mathcal{A}$ be an algebra of subsets of $T$, $m : \mathcal{A} \to [0, +\infty)$ a non-negative set function with $m(\emptyset) = 0$ and $(X, \| \cdot \|)$ a Banach space.

**Definition 3.8.** A partition of $T$ is a finite family of nonvoid sets $P = \{A_i\}_{i \leq n} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $\bigcup_{i \leq n} A_i = T$. Let $P = \{A_i\}_{i \leq n}$ and $P' = \{B_j\}_{j \leq m}$ be two partitions of $T$, $P'$ is said to be finer than $P$, denoted by $P \leq P'$ (or, $P' \geq P$), if for every $j \in \{1, \ldots, m\}$, there exists $i_j \in \{1, \ldots, n\}$ so that $B_j \subseteq A_{i_j}$. The common refinement of two partitions $P = \{A_i\}_{i \leq n}$ and $P' = \{B_j\}_{j \leq m}$ is the partition $P \vee P' = \{A_i \cap B_j\}_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}}$.

We denote by $\mathcal{P}$ the class of all partitions of $T$ and if $A \in \mathcal{P}(T)$ is fixed, by $\mathcal{P}_A$ the class of all partitions of $A$.

**Definition 3.9.** A vector function $f : T \to X$ is said to be:

- \textit{m-totally-measurable (on $T$)} if for every $\varepsilon > 0$ there exists a finite family $\{A_i\}_{i=0}^n \subset \mathcal{A}$ of pairwise disjoint sets, with $\{A_1, \ldots, A_n\} \subset \mathcal{A} \setminus \{\emptyset\}$, such that the following two conditions hold:

\begin{equation}
\begin{cases}
m(A_0) < \varepsilon \\
\sup_{t,s \in A_i} \| f(t) - f(s) \| = \text{osc}(f, A_i) < \varepsilon, \text{ for every } i \in \{1, \ldots, n\}.
\end{cases}
\end{equation}

- \textit{m-totally-measurable on $B \in \mathcal{A}$} if the restriction $f|_B$ of $f$ to $B$ is $m$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $m_B = m|_{\mathcal{A}_B}$.
Example 3.10. Every simple function $f : T \to X, f = \sum_{i=1}^{n} a_i 1_{A_i}$ (where 
$\{A_i\}_{i=1}^{n}$ is a partition of $T$ and $1_{A_i}$ is the characteristic function of $A_i$), is 
$m$-totally-measurable.

Remark 3.11.
3.11.1) In the condition ($*$) of [19, Definition 4.2] for a totally-measurable 
function $f : T \to \mathbb{R}$, instead of $m$ it is used $m^*$. For a vector measure 
$m : \mathcal{A} \to X$, $m^*$ is defined by:

$$m^*(E) = \sup\{\|m(A)\|; A \in \mathcal{A}, A \subseteq E\}, \forall E \in \mathcal{P}(T).$$

We remark that if $m : \mathcal{A} \to [0, +\infty)$ is a non-negative set function, then 
$$m^*(E) = \sup\{m(A); A \in \mathcal{A}, A \subseteq E\}, \forall E \in \mathcal{P}(T).$$

Thus, if $A \in \mathcal{A}$, then $m(A) \leq m^*(A)$, which implies that $m^*(E) \leq m^*(E)$, for every $E \in \mathcal{P}(T)$. In consequence, if $f$ is $m$-totally-
measurable according to our Definition 3.2, then $f$ is also totally-
measurable according to [19, Definition 4.2 ] (which we call $m^*$-totally-measurable).

We also observe that if $m$ is finitely additive (a subadditive monotone 
measure respectively), then, according to Remark 2.4.vii), we have 
$m^*(E) = m^*(E)$, for every $E \in \mathcal{P}(T)$ ($E \in \mathcal{A}$ respectively). So, in 
these cases, the two definitions coincide.

3.11.2) If $f : T \to X$ is $m$-totally-measurable on $T$, then $f$ is $m$-totally-
measurable on every $A \in \mathcal{A}$.

3.11.3) If $m$ is null-additive and monotone, and $A \in \mathcal{A}$ is an atom for $m$, then 
$m_A$ is finitely additive, where $m_A$ is the restriction of $m$ to 
$A \cap \mathcal{A}$, and so $m_A = m^*_A$. (This allows to avoid the request that $m$ is 
of finite variation in the Theorem 4.13 and subsequent Corollary).

3.11.4) If $m$ is null-additive and monotone, and $A \subseteq T$ is an atom for $m$, then 
a function $f : T \to X$ is totally measurable on $A$ if and only if 
$$\inf_{U \in \mathcal{U}} \text{osc}(f, U) = 0,$$

where $\mathcal{U}$ is the family of all atoms contained in $A$.

3.11.5) Suppose $m$ is subadditive and let $\{A_i\}_{i=1}^{p} \subseteq \mathcal{A}$. If $f : T \to X$ 
is $m$-totally-measurable on every $A_i, i \in \{1, \ldots, p\}$, then $f$ is $m$-
totally-measurable on $\bigcup_{i=1}^{p} A_i$.

Theorem 3.12. Suppose $f : T \to X$ is a vector function and $m : \mathcal{B} \to [0, +\infty)$ is a regular null-additive monotone measure. If $A \in \mathcal{B}$ is an atom 
of $m$, then $f$ is $m$-totally-measurable on $A$.

Proof. By Theorem 3.6, there exists a unique point $a \in A$ so that $m(A \setminus 
\{a\}) = 0$. We observe that the partition $\mathcal{P}_A = \{A \setminus \{a\}, \{a\}\}$ assures the 
m-total measurability of $f$ on $A$. \hfill \Box

Observe that an analogous result was given in [16, Theorem 4.9] for compact metric spaces. By 3.11.3) and Theorem 3.12, we immediately get:
Corollary 3.13. If $f : T \to X$ is a vector function and $m : \mathcal{B} \to [0, +\infty)$ is a finitely purely atomic regular subadditive monotone measure, then $f$ is $m$-totally-measurable on $T$.

4. Gould integrability on atoms

In this section some properties regarding Gould integrability on finitely purely atomic monotone measure spaces are established: a Lebesgue type theorem of convergence and comparative results between Gould integrability and total measurability. The Gould integral was defined in [19] for real functions with respect to a finitely additive vector measure taking values in a Banach space. Different generalizations and topics on Gould integrability were introduced and studied in [9, 13–17, 32–35]. In what follows, a Banach space. Different generalizations and topics on Gould integrability were introduced and studied in [9, 13–17, 32–35]. In what follows, $m : \mathcal{A} \to [0, +\infty)$ is a set function with $m(T) > 0$. For an arbitrary vector function $f : T \to X$, $(\sigma(P))_{P \in (\mathcal{P}, \subseteq)}$ is convergent in $X$, where $\mathcal{P}$ is ordered by the relation “$\subseteq$” given in Definition 3.8.

If $(\sigma(P))_{P \in (\mathcal{P}, \subseteq)}$ is convergent, then its limit is called the Gould integral of $f$ on $T$ with respect to $m$, denoted by $(G) \int_T f dm$ (shortly $\int_T f dm$).

If $B \in \mathcal{A}$, $f$ is said to be $m$-integrable on $B$ if the restriction $f|_B$ of $f$ to $B$ is $m$-integrable on $(B, \mathcal{A}_B, m_B)$.

Remark 4.2. By Definition 4.1, the following statements hold:

- $f$ is $m$-integrable on $T$ if and only if there exists $\alpha \in X$ such that for every $\varepsilon > 0$, there exists a partition $P_\varepsilon$ of $T$, so that for every other partition of $T$, $P = \{A_i\}_{i=1}^n$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i \in \{1, \ldots, n\}$, we have $||\sigma(P) - \alpha|| < \varepsilon$.

- Let $B, C \in \mathcal{A}$ satisfy $B \cap C = \emptyset$. If $f : T \to X$ is $m$-integrable on $B$ and $C$, then $f$ is $m$-integrable on $B \cup C$ and $\int_{B \cup C} f dm = \int_B f dm + \int_C f dm$.

Example 4.3.

- Let $T$ be a finite set, $\mathcal{A} = \wp(T)$, $m : \mathcal{A} \to [0, +\infty)$ and $f : T \to \mathbb{R}$ be arbitrary. Then $f$ is Gould $m$-integrable and $\int_T f dm = \sum_{t \in T} f(t)m(\{t\})$.

- If $m : \mathcal{A} \to [0, +\infty)$ is finitely additive and $f : T \to \mathbb{R}$ is simple, $f = \sum_{i=1}^n a_i \cdot 1_{A_i}$, then $f$ is Gould $m$-integrable and $\int_T f dm = \sum_{i=1}^n a_i \cdot m(A_i)$.

- Let $T = \mathbb{N}$, $p \in \mathbb{N}$ be fixed, $\mathcal{A}$ and $m$ defined as in Example 3.3.i) and let $f : \mathbb{N} \to \mathbb{R}$ be defined for every $x \in T$ by

$$f(x) = \begin{cases} x, & x \in \{0, \ldots, p\} \\ 0, & x \geq p + 1 \end{cases}$$

As we remarked in Example 3.3.i), $T$ is an atom of $m$. Then $f$ is Gould $m$-integrable on $T$ and $\int_T f dm = 0$. 

Theorem 4.4. Suppose $m : \mathcal{B} \to [0, +\infty)$ is a regular null-additive monotone measure and let $f : T \to X$ be any vector function. For every atom $A \in \mathcal{B}$, $f$ is $m$-integrable on $A$, and $\int_A f dm = f(a)m(A)$, where $a \in A$ is the single point resulting by Theorem 3.6.

Proof. Fix any partition $P$ of $A$, finer than $P_0 := \{\{a\}, A \setminus \{a\}\}$. Then $P$ is of the type $P = \{\{a\}, B_1, ..., B_n\}$, where $m(B_i) = 0$ for all $i = 1, ..., n$ (since $m$ is monotone and $m(A \setminus \{a\}) = 0$). So $\sigma_f(P) = f(a)m(\{a\}) = f(a)m(A)$ thanks to Remark 3.4. Since this quantity is constant for every $P$ finer than $P_0$, this is the announced integral. \hfill \Box

More generally, with a quite similar proof, one has

Corollary 4.5. Suppose $m : \mathcal{B} \to [0, +\infty)$ is a finitely purely atomic regular null-additive monotone measure, with $T = \bigcup_{i=1}^n A_i$, and $\{A_i\}_{i=1}^n \subset \mathcal{A}$ are pairwise disjoint atoms of $m$. Then any vector function $f : T \to X$ is $m$-integrable on $T$, and $\int_T f dm = \sum_{i=1}^n f(a_i)m(A_i)$, where $a_i \in A_i$ is the single point resulting by Theorem 3.6, for every $i \in \{1, ..., n\}$.

Observe also that, when $X = \mathbb{R}^+$ and the measure $m$ is also null continuous ($m(\cup_n A_n) = 0$ for every increasing sequence $(A_n)_n \in \mathcal{B}$, with $m(A_n) = 0$ for every $n$), the Gould integral of a non negative function $f$ is equal to its Choquet integral thanks to [24, Corollary 4.9]. This result can also be extended in the following way:

Proposition 4.6. Let $f : T \to \mathbb{R}^+$ be a measurable, bounded function and $m : \mathcal{A} \to [0, \infty[$ be a monotone, null additive, atomic measure. If $A \in \mathcal{A}$ is an atom then

$$(G) \int_A f dm = (C) \int_A f dm.$$  

Proof. First of all let $t_0 = \sup\{t \geq 0 : m(\{f > t\}) = m(A)\}$. Obviously $t_0 < \infty$ since $f$ is bounded. So, $m(\{f > t\}) = m(A)$ for every $t < t_0$ and $m(\{f \leq t\}) = m(A)$ for every $t > t_0$. So it is

$$(C) \int_A f dm = \int_0^{\infty} m(\{f > t\}) dt = \int_0^{t_0} m(\{f > t\}) dt = t_0 m(A).$$  

We prove now that $f$ is Gould integrable. Let $\varepsilon > 0$ be fixed and let $t_1 < t_0 < t_2$ be such that $(t_2 - t_1)m(A) \leq \varepsilon$.

Let $\Pi_i = \{\{x \in A : f \leq t_i\}, \{x \in A : f > t_i\}\}, i = 1, 2$, so

$$m(A) = m(\{x \in A : f \leq t_1\}) = m(\{x \in A : f \leq t_2\}) = m(\{x \in A : f > t_1\} \cap \{x \in A : f \leq t_2\}).$$

Let $\Pi$ finer that $\Pi_1 \lor \Pi_2$. By Lemma 3.2.2) it is $t_1 m(A) \leq \sigma(f, \Pi) = f(\xi)m(A) \leq t_2 m(A)$. Then

$$|\sigma(f, \Pi) - t_0 m(A)| \leq (t_2 - t_1)m(A) \leq \varepsilon$$

and then $f$ is Gould integrable. \hfill \Box
Corollary 4.7. (Lebesgue type) Suppose $m : \mathcal{B} \to [0, +\infty)$ is a regular null-additive monotone measure and let $f, f_n : T \to X$ be arbitrary functions. If $A \in \mathcal{B}$ is an atom of $m$, then
\[
\lim_{n \to \infty} \int_A f_n dm = \int_A f dm
\]
if and only if
\[
\lim_{n \to \infty} f_n(a) = f(a),
\]
where $a \in A$ is the single point resulting from Theorem 3.6.

In case $m$ is not regular, the results in Theorems 3.6 and 4.4 are not valid, in general. In order to give an example, we recall the concept of filter.

Definition 4.8. Let $Z$ be any fixed set. A family $\mathcal{U}$ of subsets of $Z$ is called a filter in $Z$ if and only if

4.8.a): $\emptyset \notin \mathcal{U}$,

4.8.b): $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{U}$ and

4.8.c): $A \in \mathcal{F}, B \supset A \Rightarrow B \in \mathcal{U}$.

A filter $\mathcal{U}$ is an ultrafilter if for every $A \subset Z$ then $A \in \mathcal{U}$ or $Z \setminus A \in \mathcal{U}$.

Given any filter $\mathcal{U}$ of subsets of $Z$, the dual ideal of $\mathcal{U}$ is the family of all complements of elements from $\mathcal{U}$. Usually the dual ideal will be denoted by $\mathcal{I}_U$. If the dual ideal $\mathcal{I}_U$ contains all finite subsets of $Z$, we say that $\mathcal{U}$ is a free filter.

Consider now the following example:

Example 4.9. Let $T = [0, 1]$ with the $\sigma$-algebra $\mathcal{A} = \mathcal{P}([0, 1])$, and choose any free ultrafilter $\mathcal{U}$ in $\mathcal{P}([0, 1])$. Then define $m(A) = 1$ if $A \in \mathcal{U}$, and 0 otherwise. Clearly, each element of $\mathcal{U}$ is an atom; but, since the ultrafilter is free, for every point $a \in [0, 1]$ one has $m(\{a\}) = 0$, and therefore, for any mapping $f$, the quantity $f(a)m(\{a\})$ is always null.

In general, when $m$ is null-additive and monotone, and $A$ is an atom for $m$, integrability of a mapping $f : T \to X$ in the set $A$ is strictly related with its total measurability in $A$.

Concerning the relationships between Gould integrability and total measurability, we recall the following:

- if $m : \mathcal{A} \to X$ is a finitely additive vector measure and $f : T \to \mathbb{R}$ is bounded, then $f$ is Gould $m$-integrable if and only if $f$ is $m^*$-totally-measurable ( [19]);

- if $m : \mathcal{A} \to [0, +\infty)$ is a submeasure of finite variation and $f : T \to \mathbb{R}$ is bounded, then $f$ is Gould $m$-integrable if and only if $f$ is $m$-totally-measurable ( [18]).

An interesting result in the present framework is the following.

Theorem 4.10. If $m : \mathcal{A} \to [0, +\infty)$ is finitely additive, and $f : T \to X$ is a bounded $m$-totally measurable function, then $f$ is Gould-integrable.
Proof. Denote by $M$ any positive constant dominating $\|f\|$, and fix arbitrarily $\varepsilon > 0$. Then there exists a partition $P = \{A_0, A_1, \ldots, A_n\}$ of $T$, such that $m(A_0) < \varepsilon(4M)^{-1}$ and

$$\|f(t_j) - f(\tau_j)\| \leq \frac{\varepsilon}{2m(T)}$$

for all points $t_j, \tau_j \in A_j$, $j = 1, \ldots, n$. Now choose arbitrarily two partitions $P', P''$ finer than $P$, and, without loss of generality, assume that $P''$ is finer than $P'$. Then

$$\sigma_f(P') - \sigma_f(P'') = \sum_{A \in P'} f(t_A)m(A) - \sum_{B \in P''} f(\tau_B)m(B) =$$

$$= \sum_{A \in P'} \sum_{B \in P''} (f(t_A) - f(\tau_B))m(B).$$

Therefore

$$\|\sigma_f(P') - \sigma_f(P'')\| \leq \sum_{A \in P'} \sum_{B \in P''} f(t_A) - f(\tau_B)m(B).$$

Now, split the summation above into two parts: summands for which the sets $A$ are contained in $A_0$ and the remaining summands. Then we have

$$\sum_{A \subseteq A_0, B \subseteq A} \|f(t_A) - f(\tau_B)\|m(B) \leq 2Mm(A_0) \leq \varepsilon/2,$$

$$\sum_{A \not\subseteq A_0, B \subseteq A} \|f(t_A) - f(\tau_B)\|m(B) \leq \frac{\varepsilon}{2M(T)} \sum_{B \in P''} m(B) \leq \varepsilon/2.$$

Thus

$$\|\sigma_f(P') - \sigma_f(P'')\| \leq \varepsilon.$$

Since $P'$ and $P''$ were chosen arbitrarily finer than $P$, this proves Gould integrability, thanks to completeness of $X$. \hfill $\square$

In the sequel, we obtain some comparative results between Gould integrability and total measurability for the case when $m$ has weaker properties.

**Theorem 4.11.** Suppose $m : A \to [0, +\infty)$ is a null-additive monotone set function which has atoms. If $f : T \to X$ is $m$-totally-measurable on an atom $A \in A$, then $f$ is $m$-integrable on $A$.

Proof. The proof is more direct than that given in [16, Theorem 4.13] and therefore it is added here. Let $A \in A$ be an atom of $m$. According to Lemma 3.2, if $\{A_i\}_{i=1}^n$ is a partition of $A$, then there exists only one set, for instance, without any loss of generality, $A_1$, so that $m(A_1) = m(A) > 0$ and $m(A_2) = \ldots = m(A_n) = 0$. From this it clearly follows that $m$ is finitely additive when restricted to the measurable subsets of $A$. Moreover, thanks to 3.11.4), $f$ is certainly bounded on some atom $U \subseteq A$. Then, thanks to Theorem 4.10, $f$ is Gould integrable on $U$, and it follows easily also integrability on $A$, since $m(A \setminus U) = 0$. \hfill $\square$

By Theorem 4.11 and Remark 4.2, we get:
Corollary 4.12. Suppose $m$ is a finitely purely atomic null-additive monotone measure. If $f$ is $m$-totally-measurable on $T$, then $f$ is $m$-integrable on $T$.

Theorem 4.13. Let $m : A \to [0, +\infty)$ be a null-additive monotone set function and $f : T \to X$ an arbitrary function. If $f$ is $m$-integrable on an atom $A \in \mathcal{A}$ of $m$, then $f$ is $m$-totally-measurable on $A$.

Proof. Since $f$ is $m$-integrable on $A$, for every $\varepsilon > 0$, there is $P_\varepsilon \in \mathcal{P}_A$, $P_\varepsilon = \{C_1, ..., C_n\}$, so that
\[
\left\| \sum_{i=1}^{n} f(t_i) m(C_i) - \sum_{i=1}^{n} f(s_i) m(C_i) \right\| < \varepsilon,
\]
for every $t_i, s_i \in C_i$, $i \in \{1, 2, ..., n\}$. Since $A$ is an atom, suppose $m(A \setminus C_1) = 0, m(C_i) = 0, i \in \{2, 3, ..., n\}$. It follows that, by 3.11.3, $m$ is finitely additive on $A$ and $m(A \setminus C_1) = 0$ and so $f$ is $m$-totally-measurable on $A$. \hfill \Box

According to Theorems 4.11 and 4.13, the following result is obtained:

Corollary 4.14. Suppose $m : A \to [0, +\infty)$ is a null-additive monotone set function and $A \in \mathcal{A}$ is an atom of $m$. Then $f$ is $m$-integrable on $A$ if and only if $f$ is $m$-totally-measurable on $A$. Moreover, if this is the case, the integral of $f$ on $A$ is equal to $xm(A)$, where $x$ is the unique element in $X$ such that
\[
\{x\} = \bigcap_{U \in \mathcal{U}} f(U),
\]
where $U$ is the filter of all atoms $U \subset A$.

Proof. The only fact to prove is the conclusion about the integral. Without loss of generality, we shall assume that $m(A) = 1$. Since $f$ is totally measurable on $A$, by 3.11.4, it follows that, for every integer $n$ there exists an atom $U_n \subset A$ such that $\text{osc}(f, U_n) \leq n^{-1}$. Without loss of generality, the atoms $U_n$ can be chosen to be decreasing. Therefore, choosing arbitrarily an element $u_n \in U_n$ for every $n$, the sequence $(f(u_n))_n$ is Cauchy in $X$ and therefore convergent to some element $x$. We shall prove now that $x$ is the integral of $f$ in $A$. Indeed, fix arbitrarily $\varepsilon > 0$ and pick any integer $n$ larger than $\frac{1}{\varepsilon}$, and such that $\|x - u_n\| \leq \varepsilon$. Then, consider any partition $P$ of $A$, finer than $(U_n, A \setminus U_n)$: setting $P = (A_1, ..., A_k)$, and choosing points $a_i \in A_i$, $i = 1, ..., k$ one has
\[
\sigma(f,P) = f(a_j),
\]
where $A_j$ is the unique element of $P$ contained in $U_n$ and belonging to $\mathcal{U}$. Now, we have
\[
\|f(a_j) - x\| = \|f(a_j) - x\| \leq (\|x - f(u_n)\| + \|f(u_n) - f(a_j)\|) \leq 2\varepsilon
\]
since both $u_n$ and $a_j$ belong to $U_n$. This clearly shows that $x$ is the integral of $f$ on $A$. Finally, let us prove that $x$ is the unique point of $X$ belonging to the intersection of all the sets $f(U)$, as $U$ runs among the atoms contained in $A$. Indeed, choosing the sequence $(U_n)$ as above, one clearly has $x \in f(U_n)$ for every $n$. Since the diameter of the set $f(U_n)$ coincides with $\text{osc}(f, U_n)$ and $\text{diam}(f(U_n)) = \text{diam}(f(U_n))$, there is at most one point in common to all...
the sets \( f(U_n) \). Finally, if there exists an element \( U \in \mathcal{U} \) such that \( x \notin f(U) \), the same procedure as above can be repeated, replacing \( U_n \) with \( U_n \cap U \) and choosing points \( u_n \in U_n \cap U \), showing that the limit of the sequence \( f(u_n) \) is still \( x \), contradiction.

An interesting consequence is that any bounded real function is integrable in a purely finitely atomic space. Indeed, we have

**Corollary 4.15.** Let \( m : \mathcal{A} \to [0, +\infty) \) be a null-additive finitely purely atomic monotone set function. Then every bounded measurable mapping \( f : T \to \mathbb{R} \) is \( m \)-integrable and \( m \)-totally measurable.

**Proof.** Of course, it is enough to prove integrability of \( f \) on each atom \( T \), so we shall assume that \( T \) itself is an atom for \( m \). Let us denote by \( \mathcal{U} \) the filter of those elements \( U \) of \( \mathcal{A} \) such that \( m(U) = m(T) \). Thanks to the previous result, and to the Remark 3.11, it is enough to prove that

\[
\inf_{U \in \mathcal{U}} \sup_{t \in U} f(t) = \inf_{U \in \mathcal{U}} \sup_{t \in U} f(t).
\]

Of course, the left-hand term is not greater than the right-hand one. So, by contradiction, let us assume that a real number \( u \) exists, such that

\[
\sup_{U \in \mathcal{U}} \inf_{t \in U} f(t) < u < \inf_{U \in \mathcal{U}} \sup_{t \in U} f(t).
\]

Now, let \( B := \{ t \in T : f(t) < u \} \). Since \( f \) is measurable, \( B \) is too, and then either \( m(B) = m(T) \) or \( m(B^c) = m(T) \). In the first case \( B \in \mathcal{U} \) but \( \sup_{t \in B} f(t) \leq u \), contradicting (2). In the second case, \( B^c \in \mathcal{U} \) but \( \inf_{t \in B^c} f(t) \geq u \), thus contradicting (2) again. Then (1) is true, and integrability of \( f \) is proved. Then, thanks to Corollary 4.14, \( f \) is \( m \)-totally measurable on every atom \( A \), and therefore on the whole of \( T \). \( \square \)

However, as soon as \( X \) is infinite-dimensional, there exist \( X \)-valued bounded measurable maps on some atomic space \( T \) that are not integrable, as the following example shows.

**Example 4.16.** Let \( X \) be any infinite-dimensional Banach space, and denote by \( B_X \) its unit ball. Of course, \( B_X \) is not compact, so there exist an \( \varepsilon > 0 \) and a sequence \( (x_n)_n \) in \( B_X \), such that \( \|x_n - x_m\| \geq \varepsilon \) whenever \( n \neq m \). Denote by \( Y \) the countable set \( \{ x_n : n \in \mathbb{N} \} \) and let \( \varphi : \mathbb{N} \to Y \) be any bijection. Now, let \( T \) be the space \( \mathbb{N} \) endowed with the \( \sigma \)-algebra \( 2^\mathbb{N} \). Moreover, let \( \mathcal{U} \) be any free ultrafilter on \( T \), and \( m : 2^T \to \{ 0, 1 \} \) be the ultrafilter measure associated to \( \mathcal{U} \), i.e. such that \( m(E) = 1 \) if \( E \in \mathcal{U} \) and \( m(E) = 0 \) otherwise. Now, the function \( \varphi \) above, mapping \( T \) into \( X \), is clearly bounded and measurable (in the sense that the inverse image of any Borel subset of \( X \) is in the \( \sigma \)-algebra fixed in \( T \), but is not integrable, since the set \( \varphi(U) \) has diameter larger than \( \varepsilon \) for every \( U \subset T \) with \( m(U) = 1 \).

The following result states that, under several supplementary conditions, any sequence of totally-measurable bounded vector functions is uniformly bounded almost everywhere.

**Theorem 4.17.** Let \( \mathcal{A} \) be a \( \sigma \)-algebra and \( m : \mathcal{A} \to [0, +\infty) \) be finitely purely atomic and \( \sigma \)-null-additive. Suppose that for every \( n \in \mathbb{N} \), \( f_n : T \to X \) is \( m \)-integrable on \( T \). If there is \( K > 0 \) so that \( \| \int_A f_n dm \| \leq K \), for every
Let \( f \) be a null-additive, monotone, and \( \sigma \)-null-additive, then \( f \) is still an atom for \( A \). Thanks to the Corollary 4.14, for every \( n \) there exists an atom \( U_n \subset T \) such that \( \text{diam}(f_n(U_n)) \leq 1 \) and \( x_n \in f_n(U_n) \); hence \( f_n(U_n) \subset B(x_n, 1) \subset B(0, K+1) \) for all \( n \). Consider now \( U := \bigcap_n U_n \). Since \( m(U_n^c) = 0 \) and \( m \) is \( \sigma \)-null-additive, it follows that \( m(U_n^c) = 0 \), and therefore \( U \) is an atom. Of course, for every \( n \), \( f_n(U) \subset f_n(U_n) \subset B(0, K+1) \) and this concludes the proof. \( \square \)

**Theorem 4.18.** Let \( A \) be a \( \sigma \)-algebra and \( m : A \to [0, +\infty) \) be finitely purely atomic and \( \sigma \)-null-additive. Suppose that for every \( n \in \mathbb{N} \), \( f_n : T \to X \) is \( m \)-integrable on \( T \). Moreover, assume that, for every atom \( A \), there exists in \( X \) the limit

\[
\lim_n \int_A f_n dm = x(A).
\]

Then, for every atom \( A \) there exists an atom \( U \subset A \) such that \( f_n \) uniformly converges in \( U \) to the constant \( x(A) \).

**Proof.** Again, without loss of generality, we shall assume that \( T \) is an atom for \( m \). Let us denote \( x_n := \int_T f_n dm \), and \( x := \lim_n x_n \). Thanks to the Corollary 4.14, for every \( n \) there exists an atom \( U_n \) such that \( \text{diam}(f_n(U_n)) \leq \frac{1}{n} \) and \( x_n \in f_n(U_n) \). Without loss of generality, we shall assume that \( U_n \subset U_{n-1} \) for each \( n \). Since \( m \) is \( \sigma \)-null-additive, then \( U := \bigcap U_n \) is still an atom: we shall prove now that the functions \( f_n \) uniformly converge on \( U \) to the constant \( x \). Fix \( t \in U \) and \( n \). Since \( f_n(t) \subset f_n(U_n) \), we get that \( \|f_n(t) - x_n\| \leq \frac{1}{n} \). Then, for every positive \( \varepsilon \), an integer \( n_0 \) exists, such that \( |x_n - x| \leq \varepsilon \) as soon as \( n > n_0 \); therefore, when \( n \geq n_0 \),

\[
\|f_n(t) - x\| \leq \varepsilon
\]

for all \( t \in U \). \( \square \)

5. **Radon-Nikodým Theorem**

In this section we shall investigate the behavior of the integral measure with respect to a finitely purely atomic measure \( m \), and shall deduce the existence of a Radon-Nikodým derivative, under mild conditions.

**Proposition 5.1.** Let \( m : A \to [0, +\infty) \) be a null-additive, monotone, and finitely purely atomic measure. For every Gould integrable mapping \( f : T \to X \), denote by \( \mu \) the integral measure of \( f \), i.e.

\[
\mu(B) = \int_B f dm.
\]

Then \( \mu \) is finitely additive, finitely purely atomic and absolutely continuous with respect to \( m \), i.e. \( m(E) = 0 \) implies that \( \mu(E) = 0 \) for \( E \in A \).

**Proof.** First of all, thanks to the Remark 4.2, \( \mu \) is finitely additive. Since \( m \) is monotonic, it is obvious that \( m(E) = 0 \) implies that \( \int_E f dm = 0 \), so \( \mu \) is absolutely continuous with respect to \( m \).

Now, let \( A \) be an atom for \( m \), and assume that \( A \) is not an atom for \( \mu \). If \( A \)
is not null for \( \mu \) then there exists a measurable \( B \subset A \) such that \( \mu(B) \neq 0 \) and \( \mu(A \setminus B) \neq 0 \): but \( B \) or \( A \setminus B \) is null for \( m \), and so also for \( \mu \), so the only possibility is that \( A \) is null for \( \mu \).

This shows that each atom \( A \) for \( m \) is also an atom for \( \mu \), unless \( \mu(B \cap A) = 0 \) for all \( B \in \mathcal{A} \), and this implies that \( \mu \) is purely finitely atomic.

**Proposition 5.2.** Let \( m \) and \( f \) as above, and denote by \( A_1, A_2, \ldots, A_k \) the atoms of \( m \). For each \( i = 1, \ldots, k \), let \( a_i \) denote the integral of \( f \) in \( A_i \). Then, for every set \( B \in \mathcal{A} \),

\[
\mu(B) = \sum_{i=1}^{k} \frac{a_i}{m(A_i)} m(B \cap A_i)
\]

**Proof.** Indeed, for every \( B \in \mathcal{A} \), one has

\[
\mu(B) = \int_B f \, dm = \sum_{i=1}^{k} \mu(B \cap A_i).
\]

Now, for each index \( i \), there are two possibilities: either \( A_i \) is an atom for \( \mu \), or \( a_i = \mu(A_i) = 0 \). Let \( H \) denote the set of indexes \( i \) for which \( A_i \) is an atom for \( \mu \), and \( H_B \) denote the subset of \( H \) of those indexes for which \( \mu(A_i \cap B) = \mu(A_i) \). Then clearly

\[
\mu(B) = \sum_{i \in H_B} \mu(B \cap A_i) = \sum_{i \in H_B} \mu(A_i) = \sum_{i \in H_B} a_i.
\]

Now, if \( i \notin H \), we have \( a_i = 0 \) and so

\[
\sum_{i=1}^{k} \frac{a_i}{m(A_i)} m(B \cap A_i) = \sum_{i \in H} \frac{a_i}{m(A_i)} m(B \cap A_i).
\]

Furthermore, if \( i \in H \setminus H_B \), this means that \( \mu(A_i \cap B) = 0 \) and so \( \mu(A_i \setminus B) = \mu(A_i) \), hence \( m(A_i \setminus B) > 0 \) and therefore \( m(A_i \cap B) = 0 \). For this reason

\[
\sum_{i \in H} \frac{a_i}{m(A_i)} m(B \cap A_i) = \sum_{i \in H_B} \frac{a_i}{m(A_i)} m(B \cap A_i) = \sum_{i \in H_B} a_i.
\]

This concludes the proof. \( \square \)

Now we can state the following version of the Radon-Nikodým theorem.

**Theorem 5.3.** Let \( m : \mathcal{A} \to [0, +\infty) \) be a null-additive, monotone, and finitely purely atomic measure. Let also \( \mu : \mathcal{A} \to X \) be any finitely purely atomic finitely additive measure, absolutely continuous with respect to \( m \). Then there exists a function \( f : T \to X \), Gould integrable with respect to \( m \), and satisfying

\[
\int_B f \, dm = \mu(B),
\]

for all \( B \in \mathcal{A} \).

**Proof.** Let \( A_1, \ldots, A_k \) denote the atoms of \( m \). Then each \( A_i \) is also an atom for \( \mu \), unless \( \mu(B \cap A_i) = 0 \) for all \( B \in \mathcal{A} \). So, define \( f : T \to X \) as follows:

\[
f = \sum_{i=1}^{k} \frac{\mu(A_i)}{m(A_i)} 1_{A_i}.
\]
Of course, \( f \) is simple, and therefore integrable. Moreover, for each set \( B \in \mathcal{A} \)
\[
\int_B f \, dm = \sum_{i=1}^{k} \frac{\mu(A_i)}{m(A_i)} m(B \cap A_i).
\]
As before, let \( H \) denote the set of indexes \( i \) for which \( A_i \) is an atom for \( \mu \), and \( H_B \) denote the subset of \( H \) of those indexes for which \( \mu(A_i \cap B) = \mu(A_i) \). Then \( \mu(B) = \sum_{i \in H_B} \mu(A_i) \), and
\[
\int_B f \, dm = \sum_{i \in H} \frac{\mu(A_i)}{m(A_i)} m(B \cap A_i) = \sum_{i \in H_B} \frac{\mu(A_i)}{m(A_i)} m(B \cap A_i) = \sum_{i \in H_B} \mu(A_i) = \mu(B)
\]
as desired. \( \square \)

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