Moduli Spaces of Sheaves on General Blow-ups of $\mathbb{P}^2$

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Abstract

Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ general points, and $A = H - \sum \varepsilon_i E_i$ be a generic polarization with $0 < \varepsilon_i \ll 1$. We classify the Chern characters which satisfy the weak Brill-Noether property, i.e. a general sheaf in $M_A(v)$, the moduli space of slope stable sheaves with Chern character $v$, has at most one non-zero cohomology. We further give a necessary and sufficient condition for the existence of stable sheaves. Our strategy is to specialize to the case when the $m$ points are collinear.

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1 Introduction

Let $X$ be a complex smooth projective surface with an ample $\mathbb{R}$-divisor $A$, and $M_{X,A}(v)$ be the moduli space of Gieseker $A$-semistable sheaves of character $v$. Among all the fundamental problems about moduli spaces of sheaves, there are two extremely interesting ones:

(1) compute the cohomology of a general element in an irreducible component of $M_{X,A}(v)$, and

(2) classify the Chern characters $v$ for which $M_{X,A}(v)$ is non-empty.

In this paper, we classify the non-special characters and stable characters on the blow-up of $\mathbb{P}^2$ along $m$ general points by specializing to the case of $m$ collinear points and applying results from deformation theory.

1.1 Prioritary sheaves

In contrast to (semi)stable sheaves, the families of prioritary sheaves are easier to construct. Let $D$ be an effective divisor on $X$. A torsion-free coherent sheaf $E$ on $X$ is called $D$-prioritary if $\text{Ext}^2(E,E(-D)) = 0$.

For a character $v \in K(X)_\mathbb{Q}$, let $\mathcal{P}_D(v) \subseteq \text{Coh}(v)$ be the open substack of $D$-prioritary sheaves. If $X$ is some blow-up of $\mathbb{P}^2$ with an exceptional divisor $E$ and a fibre $F := H - E$, where $H$ is the pullback of a hyperplane section on $\mathbb{P}^2$, then stack $\mathcal{P}_F(v)$ of $F$-prioritary sheaves is irreducible by a theorem of Walter [Wal98]. Let $E_i$ be the exceptional divisors. It is natural to take the polarization $A$ to be $H - \sum \epsilon_i E_i$ with $0 < \epsilon_i \ll 1$. Then every $\mu_A$-semistable sheaf is automatically $H$-prioritary. Therefore, if $M_{X,A}(v)$ is nonempty, then it is an open dense substack of $\mathcal{P}_H(v)$. We prove the following result.

Theorem 1.1. (Proposition 4.3) Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ collinear points. Let $v = (r,c_1,\Delta)$ be a Chern character such that $r \geq 2$ and $\Delta \geq 0$. Then the stack $\mathcal{P}_{X,F}(v)$ is non-empty, and a general sheaf $E$ parameterized by $\mathcal{P}_{X,F}(v)$ admits a resolution of the form

$$0 \to O_X(-2H + D)^\alpha \oplus O_X(-H + D)^\beta \xrightarrow{\psi} \bigoplus_{i=1}^m O_X(-E_i + D)^{\gamma_i} \oplus O_X(D)^\delta \to E \to 0,$$

or

$$0 \to O_X(-2H + D)^\alpha \xrightarrow{\psi} O_X(-H + D)^\beta \oplus \bigoplus_{i=1}^m O_X(-E_i + D)^{\gamma_i} \oplus O_X(D)^\delta \to E \to 0$$

for some divisor $D$. In particular, the stack $M_{X,A}(v)$ is unirational when it is non-empty.

1.2 Higher rank Brill-Noether theory

The higher rank Brill-Noether theory aims to classify non-special Chern characters (Definition 1.2) on a polarized surface $(X,A)$. The applications have been found in classifying globally generated Chern characters ([CH18b]), describing effective cones of moduli spaces ([Hui16][CHW17]), and classifying Chern characters with non-empty moduli spaces $M_{X,A}(v)$ ([CH21]).

The classification of non-special Chern characters was worked out for $\mathbb{P}^2$ in [GH94], and for Hirzebruch surfaces, including $\mathbb{P}^1 \times \mathbb{P}^1$ and $F_1$, in [CH18b]. For del Pezzo surfaces and arbitrary blow-ups, partial results were obtained in [CH20] under the condition that the Chern character
v satisfies \( \chi(v) = 0 \). For del Pezzo surfaces of degree \( \geq 4 \), a classification of all non-special Chern characters is given in [LZ19]. In this paper, we classify the Chern characters for the blow-ups of \( \mathbb{P}^2 \) along \( m \) general points which satisfies the weak Brill-Noether property.

**Definition 1.2.** We say that the moduli space \( M_{X,A}(v) \) (resp. the moduli stack \( \mathcal{P}_H(v) \)) satisfies the weak Brill-Noether property (or is non-special) if there exists a sheaf \( E \in M_{X,A}(v) \) (resp. \( E \in \mathcal{P}_H(v) \)) such that \( H^i(X,E) \neq 0 \) for at most one \( i \). In this case, we also say that the character \( v \) satisfies the weak Brill-Noether property (or is non-special).

Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) distinct points \( p_1, ..., p_m \), and \( E_1, ..., E_m \) be the corresponding exceptional divisors. When \( p_1, ..., p_m \) are collinear, we have the following.

**Theorem 1.3.** (Theorem 4.7) Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points. Let \( v = (r, \nu, \Delta) \) be a Chern character such that \( r(v) \geq 2 \), and \( \Delta \geq 0 \). Write \( \nu = aH - \sum b_i E_i \), and define \( \nu' := aH - \sum b_{i,0} b_i E_i \). If \( v \) satisfies that \( (\nu, E_i) \geq -1 \) and \( (\nu', L) \geq -1 \), then a general sheaf parameterized by \( \mathcal{P}_{F_1, ..., F_m, H}(v) \) is non-special.

We will see in Section 9 that stable sheaves can deform to nearby surfaces. Applying the semicontinuity of the cohomology groups, one obtains the following result on general blow-ups.

**Theorem 1.4.** (Theorem 7.4) Let \( v = (r, \nu, \Delta) \) be a character such that \( r(v) = \alpha H - \sum b_i E_i \) with \( -1 \leq \beta_i \leq 0 \), \( \alpha - \sum \beta_i \geq -1 \), and \( \Delta(v) \geq 0 \). Let \( A = H - \sum \varepsilon_i E_i \) be a polarization on the blow-up of \( \mathbb{P}^2 \) along \( m \) general points with \( 0 < \varepsilon_i \ll 1 \). Let \( X_0 \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points, and \( X_\varepsilon \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) general points. If \( M_{X_0,A}(v) \) is non-empty, then \( v \) is a character on \( X_\varepsilon \) satisfying the weak Brill-Noether property.

### 1.3 Exceptional sheaves

Recall that an exceptional bundle is a simple vector bundle \( E \) with \( \text{Ext}^i(E, E) = 0 \) for \( i > 0 \). On \( \mathbb{P}^2 \), there is a beautiful description of the Chern characters of exceptional bundles [LP97]. When \( X \) is a del Pezzo surface, it is known that every torsion-free exceptional sheaf is locally free, constructible (see Definition 6.9) and stable with respect to the anticanonical polarization. See [KO95] for a thorough study of exceptional objects on del Pezzo surfaces.

On the blow-up of \( \mathbb{P}^2 \) along \( m \) distinct points, we don’t know whether there are non-constructible exceptional bundles. However, we still have the following result on the stability of constructible ones.

**Theorem 1.5.** (Theorem 5.2) Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) distinct collinear points, and \( A = H - \sum \varepsilon_i E_i \) with \( 0 < \varepsilon_i \ll 1 \) a generic polarization. If \( E \) is a constructible exceptional bundle, then it is \( \mu_A \)-stable.

### 1.4 Existence of stable sheaves

On \( \mathbb{P}^2 \), the existence of stable sheaves is controlled by the exceptional bundles. Drézet and Le Potier construct a function \( \delta : \mathbb{R} \to \mathbb{R} \) whose graph in the \((\mu, \Delta)\)-plane, which completely determines when \( M_H(v) \) is nonempty. If \( (\mu(v), \delta(v)) \) lies above the graph of \( \delta \), then \( M_H(v) \) is nonempty. Otherwise, \( M_H(v) \) is empty or \( v \) is semi-exceptional. See [DLP85] and [LP97] for the argument and [CH21] for a figurative illustration. The classification of semistable characters on \( \mathbb{P}^1 \times \mathbb{P}^1 \) is worked out in [Rud94]. For Hirzebruch surfaces and generic polarizations, the classification of non-empty moduli spaces is worked out by [CH21]. The existence theorems for del Pezzo surfaces of degree \( \geq 3 \) with the anti-canonical polarization is given by [LZ19].
As a consequence of the classification on \( \mathbb{P}^2 \), we are able to construct a family of stable bundles on general blow-ups of \( \mathbb{P}^2 \) by analyzing the special blow-up along collinear points. We define the weak DL-condition for a Chern character \( \nu \) in Section 8. Roughly speaking, it means that for a constructible exceptional bundle \( E \) whose slope is close to the slope of \( \nu \), the Euler characteristic \( \chi(\nu, E) \) or \( \chi(E, \nu) \) has the expected sign.

**Theorem 1.6.** (Theorem 7.3) Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) general points, \( A = H - \sum \varepsilon_i E_i \) be a polarization with \( 0 < \varepsilon_i \ll 1 \), and \( \nu = (r, \nu, \Delta) \) be a character such that \( \nu(\nu) = \alpha H - \sum \beta_i E_i \) with \( -1 \leq \beta_i \leq 0 \) and \( \Delta(\nu) \geq 0 \). Suppose that \( \alpha \notin \mathcal{C} \), where \( \mathcal{C} \) is the set of \( H \)-slopes on \( \mathbb{P}^2 \) of exceptional bundles. If \( \nu \) satisfies the weak DL-condition, then \( M_{X, A}(\nu) \neq \emptyset \).

### 1.5 The organization of the paper

In Section 2, we recall the preliminary facts needed in the rest of the paper. In Section 3, we study the basic properties of the blow-up of \( \mathbb{P}^2 \), especially the cohomology of line bundles.

Section 4 compiles some useful properties and constructions of special prioritary sheaves. We construct a family of prioritary sheaves and characterize Chern characters on the blow-up of \( \mathbb{P}^2 \) along distinct collinear points that satisfy the weak Brill-Noether property. In Section 5, we prove the stability of constructible exceptional bundles, which is crucial in the classification of stable Chern characters.

In sections 6, we define a sharp Bogomolov function and determine its basic properties. We primarily concentrate on the characters on the blow-up of \( \mathbb{P}^2 \) along collinear points. In this case, we study the stability of sheaves with respect to generic polarization in detail. In Section 7, using deformation theory, we generalize our results to general blow-ups of \( \mathbb{P}^2 \).

Finally, in Section 8, we compute its ample cone of the Hilbert scheme of points on the blow-up of \( \mathbb{P}^2 \) along distinct collinear points.

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### 2 Preliminaries

**Convention.** By a surface, we mean a connected smooth projective algebraic surface over \( \mathbb{C} \). All sheaves are coherent unless specified. For a surface \( X \) and coherent sheaves \( E \) and \( F \), we write \( h^i(X, E) = \dim H^i(X, E) \), \( \text{hom}(E, F) = \dim \text{Hom}(E, F) \), and \( \text{ext}^i(E, F) = \dim \text{Ext}^i(E, F) \).

#### 2.1 Chern characters and Riemann-Roch on surfaces

Let \( E \) be a torsion-free sheaf on a polarized surface \( (X, A) \). Let \( K(X)_{\mathbb{Q}} \) be the Grothendieck group of \( X \) with \( \mathbb{Q} \)-coefficients. The Chern character \( \text{ch}(E) = (\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E)) \) is given by

\[
\text{ch}_0(E) = r(E), \quad \text{ch}_1(E) = c_1(E), \quad \text{ch}_2(E) = \frac{c_1^2 - 2c_2}{2}.
\]

We define the total slope \( \nu \), the \( A \)-slope \( \mu_A \) and the discriminant \( \Delta \) by

\[
\nu(E) = \frac{c_1(E)}{r(E)}, \quad \mu_A(E) = \frac{c_1(E)A}{r(E)}, \quad \Delta(E) = \frac{\mu(E)^2}{2} - \frac{\text{ch}_2(E)}{r(E)}.
\]
These quantities depend only on the Chern character of $E$ (and the polarization) but not on the particular sheaf. Given a Chern character $\nu$, we define the total slope $\nu$, the $A$-slope $\mu_A$ and the discriminant $\Delta$ of $\nu$ by the same formulae.

Then we have the following Riemann-Roch formula for torsion-free sheaves $E$ and $F$:

$$\chi(E, F) = r(E)r(F)(P(\nu(F)) - \nu(E)) - \Delta(E) - \Delta(F),$$

where

$$P(\nu) := \chi(O_X) + \frac{\nu(\nu - K_X)}{2}$$

is the Hilbert polynomial of $O_X$. In particular, taking $E = O_X$, this reduces to the usual Riemann-Roch formula

$$\chi(F) = r(F)(P(\nu(F)) - \Delta(F)).$$

### 2.2 Stability

We now introduce the basic notions and properties about stability conditions. For more details, see [HL10] or [LP97].

The sheaf $E$ is called $\mu_A$-semistable if every proper subsheaf $0 \neq F \subseteq E$ of smaller rank satisfies

$$\mu_A(F) \leq \mu_A(E).$$

If the inequality is strict for every such $F$, then $E$ is called $\mu_A$-stable.

Now suppose that $A$ is an integral divisor. Define the Hilbert polynomial $P_E(m)$ and the reduced Hilbert polynomial $p_E(m)$ of $E$ by

$$P_E(m) = \chi(E(m)), \quad p_E(m) = \frac{\chi(E(m))}{r(E)}.$$

Then $E$ is $A$-semistable (or Gieseker semistable) if every proper subsheaf $0 \neq F \subseteq E$ of smaller rank satisfies that

$$p_F(m) \leq p_E(m)$$

for all $m \gg 0$. If the inequality is strict for every such $F$, then $E$ is called $A$-stable (or Gieseker $A$-stable). It follows immediately from the Riemann-Roch formula that

$$\mu_A$-stable $\Rightarrow$ $A$-stable $\Rightarrow$ $A$-semistable $\Rightarrow$ $\mu_A$-semistable.

If $E$ is $\mu_A$-semistable for some polarization $A$, then $\Delta(E) \geq 0$ by Bogomolov inequality.

Every torsion free sheaf $E$ admits a Harder-Narasimhan filtration with respect to both $\mu_A$- and $A$-stability, that is there is a finite filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

such that the quotients

$$G_i = E_i/E_{i-1},$$

called Harder-Narasimhan factors, are $\mu_A$- (respectively $A$-) semistable and

$$\mu_A(G_i) > \mu_A(G_{i+1})$$

(respectively, $p_{G_i}(m) > p_{G_{i+1}}(m)$ for $m \gg 0$)

for $1 \leq i \leq n - 1$. Moreover, the Harder-Narasimhan filtration is unique.
2.3 Prioritary sheaves

In this section, we recall some results by Walter [Wal98].

**Definition 2.1.** Let \( X \) be a surface, and \( D \) be an effective divisor. A torsion free sheaf \( \mathcal{E} \) is called \( D \)-prioritary if

\[
\text{Ext}^2(\mathcal{E}, \mathcal{E}(-D)) = 0.
\]

**Lemma 2.2.** (Lemma 3.1 [CH21]) Let \( D_1 \) and \( D_2 \) be two effective divisors on a surface \( X \) such that \( D_1 - D_2 \) is effective. If a sheaf \( \mathcal{E} \) is \( D_1 \)-prioritary, then it is also \( D_2 \)-prioritary.

**Lemma 2.3.** (Lemma 4 [Wal98]) Let \( D \) be an effective Cartier divisor on a surface \( X \). If \( \mathcal{E} \) is a \( D \)-prioritary torsion-free sheaf on \( X \) of character \( \mathbf{v} \), then the restriction map \( \mathcal{P}_D(\mathbf{v}) \rightarrow \text{Coh}_D(i^*\mathbf{v}) \), given by \( \mathcal{E} \mapsto \mathcal{E}|_D \), is smooth (and therefore open) in a neighborhood of \( \mathcal{E} \), where \( i : D \rightarrow X \) is the natural inclusion.

**Definition 2.4.** A vector bundle \( \mathcal{E} \) of rank \( r \) on \( \mathbb{P}^1 \) is called balanced if \( \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^k \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)^{r-k} \) for some \( a \in \mathbb{Z} \) and \( 0 \leq k < r \).

**Lemma 2.5.** (Lemma 3 [Wal98]) Let \( r \geq 2 \) and \( 0 \leq d < r \) be integers. Let \( \text{Coh}_{\mathbb{P}^1}(r-d) \) be the stack of coherent sheaves of rank \( r \) and degree \( -d \) on \( \mathbb{P}^1 \).

1. If \( d > 0 \), then the sheaves not balanced form a closed substack of \( \text{Coh}_{\mathbb{P}^1}(r-d) \) of codimension at least 2.

2. If \( d = 0 \), then the sheaves not balanced form a closed substack of \( \text{Coh}_{\mathbb{P}^1}(r,0) \) of codimension 1.

**Corollary 2.6.** If \( \mathcal{E} \) is a general sheaf in \( \mathcal{P}_F(\mathbf{v}) \) with \( F \) a smooth rational curve, then \( \mathcal{E}|_F \) is balanced along \( F \).

**Lemma 2.7.** (Lemma 6 [Wal98]) Let \( f : X' \rightarrow X \) be the blow-up of a surface \( X \) at a point \( x \in X \), and \( E \) be the exceptional divisor in \( X' \). Suppose that \( \mathcal{E} \) is a coherent sheaf of rank \( r \) on \( X' \) such that \( \mathcal{E}|_E \simeq \mathcal{O}_E^{r-d} \oplus \mathcal{O}_E(-1)^d \) for some \( d \). Then \( f_*\mathcal{E} \) is locally free in a neighborhood of \( x \), and there are exact sequences

\[
0 \rightarrow f^*f_*\mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_E(-1)^d \rightarrow 0,
\]

\[
0 \rightarrow \mathcal{E}(-E) \rightarrow f^*f_*\mathcal{E} \rightarrow \mathcal{O}_E^{-r-d} \rightarrow 0.
\]

Moreover, for any divisor \( D \) on \( X \), we have \( \text{Ext}^2(\mathcal{E}, \mathcal{E}(f^*D)) \simeq \text{Ext}^2(f_*\mathcal{E}, f_*\mathcal{E}(D)) \). In particular, \( \mathcal{E} \) is \( f^*D \)-prioritary if and only if \( f_*\mathcal{E} \) is \( D \)-prioritary.

**Lemma 2.8.** (Proposition 2 [Wal98]) Let \( \pi : X \rightarrow C \) be a birationally ruled surface and \( F \in \text{NS}(X) \) the numerical class of a fiber of \( \pi \). Suppose \( r \geq 2 \), \( c_1 \in \text{NS}(X) \), and \( c_2 \in \mathbb{Z} \) are given. Then the stack \( \mathcal{P}_{X,F}(r,c_1,c_2) \) of \( F \)-prioritary sheaves on \( X \) of rank \( r \) and Chern classes \( c_1 \) and \( c_2 \) is smooth and irreducible.

2.4 Exceptional bundles

In this section, we recall some known results of exceptional bundles.

A coherent sheaf \( \mathcal{E} \) on \( X \) is called **exceptional** if \( \text{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C} \) and \( \text{Ext}^i(\mathcal{E}, \mathcal{E}) = 0 \) for any \( i > 0 \). The Mukai’s lemma ([Muk84] [KO95]) implies that every torsion free exceptional sheaves are locally free. Thus we call torsion free exceptional sheaves **exceptional bundles**. There do exist torsion exceptional sheaves. For example, the structure sheaves \( \mathcal{O}_E \) of exceptional divisors on the blow-up of \( \mathbb{P}^2 \) along general points are exceptional.
Let \( A = H - \sum \varepsilon_i E_i \) be a polarization on \( X \), i.e. \( \varepsilon_i > 0 \) and \( \sum \varepsilon_i < 1 \). We say that \( A \) is \textit{generic} (or \( \varepsilon \) is generic) if \((\varepsilon_1, \ldots, \varepsilon_m)\) is a generic point in the region defined by \( \varepsilon_i > 0 \) and \( \sum \varepsilon_i < 1 \). Sometimes, we prefer to take \( A = H - \varepsilon \sum E_i \), that is, \( \varepsilon_1 = \cdots = \varepsilon_m \), whence we mean \( \varepsilon \) is a generic number in \((0, 1/m)\) by saying \( A \) is generic or \( \varepsilon \) is generic.

The following lemma is proved in [CH21] on Hirzebruch surfaces. For the reader’s convenience, we give the proof on our surface here.

**Lemma 2.9.** (Lemma 6.7 [CH21]) Let \( v \in K(X)_\mathbb{Q} \) be a potentially exceptional character of rank \( r \) with \( c_1(v) = aH - \sum b_i E_i \).

1. The discriminant of \( v \) is \( \Delta = \frac{1}{2} - \frac{1}{2r^2} \).
2. The character \( v \) is primitive.
3. If \( E \) is an \( \mu_A \)-stable sheaf of discriminant \( \Delta(E) < 1/2 \), then \( E \) is exceptional.
4. If \( \varepsilon \) is generic and \( E \) is a \( \mu_A \)-semistable sheaf of character \( v \), then it is \( \mu_A \)-stable and exceptional.
5. If \( \varepsilon \) is generic and \( E \) is a \( A \)-semistable sheaf of discriminant \( \Delta(E) < \frac{1}{2} \), then it is semiexceptional.

**Proof.** (1) Solving the Riemann-Roch formula

\[
1 = \chi(v, v) = r^2(1 - 2\Delta)
\]

for \( \Delta \) proves the statement.

(2) By the Riemann-Roch formula,

\[
\chi(v) = r \left( P \left( \frac{a}{r} H - \sum \frac{b_i}{r} E_i \right) - \frac{1}{2} + \frac{1}{2r^2} \right) = \frac{1}{2r^2} \left( (a + 2r)(a + r) - \sum b_i(b_i + r) + r^2 + 1 \right).
\]

As \( \chi(v) \) is an integer, then \( \gcd(r, a, b_1, \ldots, b_m) = 1 \) and thus \( v \) is primitive.

(3) By the Riemann-Roch formula and stability, one has \( \hom(E, E) = 1 \), \( \ext^2(E, E) = 0 \), and

\[
\chi(E, E) = \frac{1}{r^2} (1 - 2\Delta) = 1 - \ext^1(E, E) > 0.
\]

Thus \( \ext^1(E, E) = 0 \) and \( E \) is exceptional.

(4) Since \( \varepsilon \) is generic and \( v \) is primitive, then \( E \) has no subsheaf of smaller rank with the same \( A \)-slope. Hence \( E \) is \( \mu_A \)-stable, and exceptional by (3).

(5) Since \( \varepsilon \) is generic, then the Jordan-Hölder factors \( gr_1, \ldots, gr_t \) of \( E \) have the same total slope and discriminant. They are also exceptional bundles, by (1), so their Chern characters are primitive, hence have the same rank, and they are the same. Thus the factors are all isomorphic, and an easy induction using \( \ext^1(gr_1, gr_1) = 0 \) shows that \( E \simeq gr_1^{\oplus t} \).

The simplest examples of exceptional bundles on blow-ups of \( \mathbb{P}^2 \) are line bundles. Now given an ordered pair of sheaves \((E, F)\), we form the evaluation and coevaluation maps

\[
ev : E \otimes \Hom(E, F) \longrightarrow F, \quad \coev : E \longrightarrow F \otimes \Hom(E, F)^*.
\]
of which is associated to the identity element of the space Hom(\(E, F\)) \(\otimes\) Hom(\(E, F\))^*. If the evaluation map is surjective, then we consider the kernel

\[
0 \to L_E F \to E \otimes \text{Hom}(E, F) \to F \to 0;
\]

if the coevaluation map is injective, then we consider the cokernel

\[
0 \to E \to F \otimes \text{Hom}(E, F)^* \to R_F E \to 0.
\]

**Definition 2.10.** The sheaf \(L_E F\) is the left mutation of \(F\) across \(E\), and the sheaf \(R_F E\) is the right mutation of \(E\) across \(F\).

If \((E, F)\) is an ordered pair of exceptional bundles, then the left and right mutations are exceptional whenever they are defined. This gives us a way of producing exceptional bundles.

**Example 2.11.** The Euler sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \otimes \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))^* \to T_{\mathbb{P}^2} \to 0
\]

implies that \(T_{\mathbb{P}^2}\) is exceptional, since it is the right mutation \(R_{\mathcal{O}(1)}\).

Start with a strong exceptional collection \(\sigma_0 = (E_0, \ldots, E_n)\) on a surface \(X\). A transformation of the exceptional collection \(\sigma_0\) is defined as a transformation of a pair of neighboring objects in this collection. One extends \(\sigma\) to an infinite periodic collection \((E_i)_{i \in \mathbb{Z}}\) by setting \(E_{i+(n+1)k} = E_i \otimes (\omega_X^*)^{\otimes k}\) for \(i = 0, 1, \ldots, n\). We can also do mutations in collections: if \((E_i, E_{i+1})\) has a surjective evaluation map (resp. injective coevaluation map), then one replaces \((E_i, E_{i+1})\) by \((L_E E_{i+1}, E_i)\) (resp. \((E_{i+1}, R_E E_{i+1})\)). When the operations are defined, we can iterate mutations. Write \(L_j(E_i)_{i \in \mathbb{Z}}\) for the left mutation \(L_{E_j}\).

On the surface \(X\) obtained by blow up \(\mathbb{P}^2\) along \(m\) distinct points, one has a standard exceptional collection

\[
\sigma_0 = (\mathcal{O}(-2H), \mathcal{O}(-H), \mathcal{O}(-E_1), \ldots, \mathcal{O}(-E_m), \mathcal{O})
\]

**Definition 2.12.** A bundle \(E\) on \(X\) is called constructible if it can be obtained by a sequence of mutations from the standard helix \(\sigma_0\).

**Theorem 2.13.** ([KO95]) All exceptional bundles and helixes on del Pezzo surfaces are constructible.

### 3 Blow-ups of the projective plane

In this section, we review some properties of blow-ups of \(\mathbb{P}^2\). We refer the reader to [Har77] and [Laz04] for definitions and details of the proof.

Let \(\Gamma = \{p_1, \ldots, p_m\}\) be a set of \(m\) distinct collinear points on \(\mathbb{P}^2\). Let \(\pi : X = \text{Bl}_{\Gamma}\mathbb{P}^2 \to \mathbb{P}^2\) be the blow-up of \(\mathbb{P}^2\) along \(\Gamma\) with exceptional divisor \(E_1, \ldots, E_m\). Let \(H = \pi^*\mathcal{O}(1)\) be pull-back of the line class on \(\mathbb{P}^2\), and \(L = H - \sum_{i=1}^m E_i\) be the proper transform of the line passing through the \(m\) points. Then we have

\[
\text{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_m,
\]

and intersection numbers

\[
H^2 = 1, \quad E_i^2 = -1, \quad L^2 = 1 - m, \quad L.E_i = 1, \quad L.H = 1, \quad E_i.E_j = 0
\]
for any $i \neq j$. The canonical divisor of $X$ is $K_X = -3H + \sum_{i=1}^{m} E_i$. If $D = aH - \sum b_i E_i$, then the Riemann-Roch formula reads

$$\chi(O_X(D)) = \frac{(a+1)(a+2)}{2} - \sum_{i=1}^{m} \frac{b_i(b_i+1)}{2}.$$ 

Notice that $E_1, \ldots, E_m, L$ are effective and that $H, H - E_1, \ldots, H - E_m$ are nef. Since the cones generated by them are dual to each other, then we know that the nef cone of $X$ is generated by

$$H, H - E_1, \ldots, H - E_m,$$

and the effective cone is generated by

$$E_1, \ldots, E_m, H - E_1 - \cdots - E_m.$$

Equivalently, a divisor $D = aH - \sum b_i E_i$ is nef if and only if $a \geq \sum b_i$ and $b_i \geq 0$ for all $i$, and a divisor $D' = a'H - \sum b'_i E_i$ is effective if and only if $a' \geq 0$ and $a' \geq b'_i$ for all $i$.

### 3.1 Cohomology of line bundles

If we know the dimension of the global sections of a line bundle $O_X(D)$, then by Serre duality, one has

$$h^2(X, O_X(D)) = h^0(X, O_X(K_X - D)).$$

One can also compute $h^1(X, O_X(D))$ via Riemann-Roch formula:

$$h^1(X, O_X(D)) = h^0(X, O_X(D)) + h^0(X, O_X(K_X - D)) - \frac{D \cdot (D - K_X)}{2} - 1.$$ 

In this way, we know all the cohomology of a line bundle.

Now we compute the global sections of line bundles on $X$. Let $D = aH - \sum b_i E_i$ be a divisor with $a \geq \max\{b_1, 0\}$. If any $b_i < 0$, then chasing the exact sequence

$$0 \to O_X(D - E_i) \to O_X(D) \to O_{E_i}(b_i) \to 0,$$

one has $h^0(X, O_X(D)) = h^0(X, O_X(D - E_i))$, so one can replace $D$ by $D - E_i$. Repeating this, we may assume that $b_i \geq 0$ for any $i$.

Assume first that $D$ is nef. Chasing the exact sequence

$$0 \to O_X((n-1)H) \to O_X(nH) \to O_{H}(n) \to 0,$$

and using the vanishing of $h^i(X, O_X(nH))$ for $n \geq 0$, one obtains that

$$h^0(X, O_X(aH)) = \frac{(a+1)(a+2)}{2}.$$ 

Now consider the exact sequence

$$0 \to O_X(aH - bE_1) \to O_X(aH - (b-1)E_1) \to O_{E_1}(b-1) \to 0$$

for $1 \leq b \leq b_1$. Since $a \geq b_1$, one always has the surjection

$$H^0(X, aH - (b-1)E_1) \to H^0(E_1, O_{E_1}(b-1)).$$
As a consequence, we get

\[ h^0(X, \mathcal{O}_X(aH - b_1E_1)) = \frac{(a+1)(a+2)}{2} - \frac{b_1(b_1+1)}{2}. \]

Repeating this for \( E_2, \ldots, E_m \) and using the assumption that \( a \geq \sum b_i \), we deduce that

\[ h^0(X, \mathcal{O}_X(D)) = \frac{(a+1)(a+2)}{2} - \sum_i \frac{b_i(b_i+1)}{2}. \] (1)

If \( D \) is not nef, then the short exact sequence

\[ 0 \rightarrow \mathcal{O}_X(D - L) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_L \left( a - \sum b_i \right) \rightarrow 0 \]

together with \( a < \sum b_i \) implies that \( h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D - L)) \). Thus we can replace \( D \) by \( D - L = (a - 1) - \sum (b_i - 1)E_i \). If some coefficient \( b_i \) becomes negative, one uses the previous reduction to place \( b_i \) by 0. Repeating this, we reduce the computation to the case when \( D \) is nef.

Now we can prove the following useful properties.

**Lemma 3.1.** Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points, and \( D = aH - \sum_{i=1}^m b_iE_i \) be a divisor on \( X \).

(a) If \( a > -3 \), then \( h^2(X, \mathcal{O}_X(D)) = 0 \). In particular, any effective divisor \( D \) has \( h^2(X, \mathcal{O}_D) = 0 \).

(b) If \( D \) is a nef divisor, then \( \mathcal{O}_X(D) \) has no higher cohomology.

(c) Let \( I \) be a subset of \( \{1, 2, \ldots, m\} \) and \( j \) be any index in \( \{1, 2, \ldots, m\} \). If \( D \) is of either of the form

\[-2H + \sum_{i \in I} E_i, \quad -H + \sum_{i \in I} E_i, \quad aH - (a+1)E_j, \quad -E_j + \sum_{i \in I, i \neq j} E_i, \]

then \( \mathcal{O}_X(D) \) has no cohomology.

(d) If \( D \) is of the form

\[ D = aH - \sum_{i=1}^m b_iE_i \]

with \( a = \sum_{i=1}^m b_i - 1 \) and \( b_i \geq 0 \) for any \( i \), then \( \mathcal{O}_X(D) \) has no higher cohomology.

(e) Assume that \( H^i(X, \mathcal{O}_X(D)) = 0 \) for \( i > 0 \). If \( D.E_j \geq 0 \) (resp. \( D.H \geq -2 \)), then \( H^i(X, \mathcal{O}_X(D + E_j)) = 0 \) (resp. \( H^i(X, \mathcal{O}_X(D + H)) = 0 \)) for \( i > 0 \).

**Proof.** (a) This is because the coefficient of \( H \) in the Serre dual of \( D \) is negative.

(b) Using (1), one has that

\[ h^0(X, \mathcal{O}_X(D)) = \frac{(a+1)(a+2)}{2} - \sum_i \frac{b_i(b_i+1)}{2}. \]

Since \( h^2(X, \mathcal{O}_X(D)) = 0 \) by (a), then \( h^1(X, \mathcal{O}_X(D)) = 0 \) follows from the Riemann-Roch formula.

(c) This is also a combination of the Riemann-Roch formula, Serre duality, and the computation of global sections.
(d) Assume that \( b_1 \geq b_2 \geq \cdots \geq b_m \). If \( b_m = 0 \), then we can reduce to the \( m - 1 \) case. Thus we may assume that \( b_m > 0 \). One has

\[
\chi(O_X(D)) - \chi(O_X(D - L)) = (a + 1) - \sum_{i=1}^{m} b_i = 0,
\]

and

\[
h^0(O_X(D)) = h^0(O_X(D - L)).
\]

Since \( D - L \) is a nef divisor, thus we conclude by (b).

(e) If \( D \) is a class such that \( H^i(X, O_X(D)) = 0 \) for all \( i > 0 \) and \( E_j . D \geq 0 \), then \( H^i(X, O_X X(D + E_j)) = 0 \) for all \( i > 0 \). To see this, consider the exact sequence

\[
0 \rightarrow O_X(D) \rightarrow O_X(D + E_j) \rightarrow O_{\mathbb{P}^1}(-1) \rightarrow 0.
\]

Since \( O_X(D) \) and \( O_{\mathbb{P}^1}(-1) \) have no higher cohomology, then \( O_X(D + E_j) \) has no higher cohomology. Similar sequences imply the other statements. \( \square \)

4 Cohomology of General Sheaves in \( \mathcal{P}_{X,F}(\mathbf{v}) \)

4.1 Elementary transformations

In this section, we first recall the minimal discriminant property of type 2 elementary transformations, and then we give an explicit construction of an elementary transformation following the method in [LZ19].

Consider the projection from \( E_i \) to a general line \( H \), and let \( F_i = L - E_i \) be the class of the fiber of this projection. We write \( F \) to denote the class of one of the fibers \( F_i \) if it does not matter which fiber we choose. For a given Chern character \( \mathbf{v} = (r, c_1, \Delta) \), we can give a necessary and sufficient condition for the existence of \( H \)-prioritary sheaves with character \( \mathbf{v} \).

Write \( c_1(\mathbf{v}) = aH - \sum b_iE_i \) with

\[
a = a'r + a'', \quad b_i = b'_i + b''_i, \quad 0 \leq a'', b''_i < r, \quad \forall i = 1, ..., m.
\]

Let \( D = (a' + 2)H - \sum (b_i + 1)E_i \) be a divisor. Notice that there exists an \( F \)-prioritary sheaf with Chern character \( \mathbf{v} \) if and only if there exists one with \( \mathbf{v}' = (r, c_1 - rD, \Delta) \) since any twist of a priority sheaf is also priority.

Recall that a type 1 elementary transformation of \( \mathcal{E} \) along \( \mathcal{F} \) is the kernel of a surjective map \( \mathcal{E} \rightarrow \mathcal{F} \), and a type 2 elementary transformation of \( \mathcal{E} \) along \( \mathcal{F} \) is an extension of \( \mathcal{F} \) by \( \mathcal{E} \).

For a clearer description of the existence of priority sheaves, see Figure 1 in [CH21].

Proposition 4.1. (Proposition 4.9 [LZ19]) Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points, and \( \mathcal{E} \) be a sheaf on \( X \) of rank \( r \geq 2 \). Suppose that \( \mathcal{E} \) is a type 2 elementary transformation of \( O_X(-2H)^{\oplus(r-a')} \oplus O_X(-H)^{\oplus a''} \) along \( \bigoplus_{i=1}^m O_{E_i}(-1)^{\oplus(\Delta-r-b''_i)} \). Then \( \mathcal{E} \) is an \( H \)-prioritary sheaf, and for any \( D \in \text{Pic}(X) \), there are no \( H \)-prioritary sheaves of the same rank and total slope as \( \mathcal{E} \oplus O_X(D) \) with strictly smaller discriminant.

Now we construct type 2 elementary transformations of \( O_X(-2H)^{\oplus(r-a')} \oplus O_X(-H)^{\oplus a''} \) along \( \bigoplus_{i=1}^m O_{E_i}(-1)^{\oplus(\Delta-r-b''_i)} \). Roughly speaking, we distribute \( H \) and \( E_i \) evenly among the direct summands. Then we will see that such sheaves are \( H \)-prioritary.
Consider the $r$-tuple
\[ S = (\mathcal{O}(-2H), ..., \mathcal{O}(-2H), \mathcal{O}(-H), ..., \mathcal{O}(-H)) \]
where the number of $\mathcal{O}(-H)$ is $a''$.

1) Start with $i = 1$. Twist each coordinate by $\mathcal{O}(E_1)$ starting from left to right in $S$ until reaching the $(r - b''_i)$-th coordinate.

2) Let $S'$ be the new $r$-tuple obtained from the previous step. Reorder the coordinates of $S$ by decreasing $L$-slope. If two distinct line bundles $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ have the same $L$-slope, then $\mathcal{O}(D_1)$ sits to the left of $\mathcal{O}(D_2)$ if either
   (a) $D_1.H < D_2.H$
   (b) or $D_1.H = D_2.H$ and there exists a $j$ such that $D_1.E_i = D_2.E_i$ for all $i < j$ and $D_1.E_j > D_2.E_j$.

3) Repeat steps 1) and 2) using $E_{i+1}$.

We call such a bundle $\mathcal{E}$ a good bundle. By construction, there is a unique (up to isomorphism) good bundle $\mathcal{E}$ such that $r(\mathcal{E}) = r$ and $c_1(\mathcal{V}) = c_1$. Also, notice that good bundles are type $2$ elementary transformations.

**Lemma 4.2.** Good bundles are $H$-prioritary.

*Proof.* Notice that for any two summands $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ in the good bundles, the coefficient of $H$ in $D_2 - D_1$ is at least $-1$. Therefore we have that
\[ \text{Ext}^2(\mathcal{O}(D_1), \mathcal{O}(D_2 - H)) \approx H^2(X, \mathcal{O}(D_2 - D_1 - H)) = 0, \]
so $\text{Ext}^2(\mathcal{E}, \mathcal{E}(-H)) = 0$, $\mathcal{E}$ is $H$-prioritary. \hfill $\square$

### 4.2 Construction of a complete family of $H$-prioritary sheaves

In this section, we will construct a complete family of $H$-prioritary, hence $F_i$-prioritary, sheaves on the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points, parameterized by a rational variety. This will imply that $\mathcal{P}_H(v)$ and $\mathcal{P}_{F_i}(v)$ are unirational. In particular, if $M_{X,A}(v)$ is non-empty, then it is unirational.

**Proposition 4.3.** Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ collinear points. Let $v = (r, c_1, \Delta)$ be a Chern character such that $r \geq 2$ and $\Delta \geq 0$. Then the stack $\mathcal{P}_{X,F}(v)$ is non-empty, and a general sheaf $\mathcal{E}$ parameterized by $\mathcal{P}_{X,F}(v)$ admits a resolution of the form
\[ 0 \rightarrow \mathcal{O}_X(-2H + D)^\alpha \oplus \mathcal{O}_X(-H + D)^\beta \xrightarrow{\alpha} \bigoplus_{i=1}^m \mathcal{O}_X(-E_i + D)^{\gamma_i} \oplus \mathcal{O}_X(D)^\delta \rightarrow \mathcal{E} \rightarrow 0, \]
or
\[ 0 \rightarrow \mathcal{O}_X(-2H + D)^\alpha \xrightarrow{\alpha} \mathcal{O}_X(-H + D)^\beta \oplus \bigoplus_{i=1}^m \mathcal{O}_X(-E_i + D)^{\gamma_i} \oplus \mathcal{O}_X(D)^\delta \rightarrow \mathcal{E} \rightarrow 0 \]
for some divisor $D$. If the coefficient of $E_i$ in $c_1 - rD$ is $b_i$, then the exponents are given by
\[ \alpha = -\chi(v(-D - H)), \quad \delta = \chi(v(-D)), \quad \gamma_i = b_i, \quad \beta = \left| r + \alpha - \delta - \sum \gamma_i \right|. \]
In particular, the stack $\mathcal{P}_{X,F}(v)$ is unirational.

Proof. We first show that we can choose $D$ such that $\alpha, \beta, \gamma, \delta$ given above are all non-negative. Write $D = cH - \sum d_i E_i$ and $c_1 - rD = (a - c)H - \sum b_i E_i$. We first fix the coefficients $d_i$ by making $0 \leq b_i < r$.

Write $a = a' + a''$ such that $0 \leq a'' < r$. By the Riemann-Roch formula, we have that

$$\chi(v(-(a' + 2)H + \sum d_i E_i)) = r \left( \frac{a'' - 2r}{r} \left( \frac{a'' - 2r}{r} + 3 \right) + 1 - \sum_i \frac{b_i}{r} \left( \frac{b_i}{r} + 1 \right) - \Delta \right) < 0.$$ 

Thus we can choose $c$ to be the largest integer such that $\chi(v(-D)) \geq 0$ but $\chi(v(-D-H)) < 0$. Setting

$$U := \mathcal{O}_X(-2H + D)^{\alpha} \oplus \mathcal{O}_X(-H + D)^{\beta}, \quad \text{and} \quad V := \bigoplus_{i=1}^{m} \mathcal{O}_X(-E_i + D)^{\gamma_i} \oplus \mathcal{O}_X(D)^{\delta},$$

one has that the sheaf $\mathcal{H}om(U, V)$ is globally generated. Since rank $V - \text{rank} U = r \geq 2$, then a general cokernel $E = E_a$ is a vector bundle, whose Chern character is given by $\chi(E) = v$. The same argument applies to the exact sequence

$$0 \to O_X(-2H)^{\alpha} \to O_X(-H)^{\beta} \oplus \bigoplus_{i=1}^{m} O_X(-E_i)^{\gamma_i} \oplus O_X^\delta \to E \to 0.$$ 

The rest of the statement is the contents of the next two lemmas, which are proved in [LP97] for $\mathbb{P}^2$ and in [CH18b] for blow-ups of $\mathbb{P}^2$ with $\chi(v) = 0$.

\[\square\]

**Lemma 4.4.** A general cokernel $E$ constructed as above is $H$-prioritary, and hence $F_1$-prioritary for all $i$.

Proof. We may assume that $E$ is a vector bundle. To check that $E$ is $F$-prioritary, we need to show that $\text{Ext}^2(E, \mathcal{O}_X(-H)) = 0$. Applying $\text{Ext}(\cdot, \cdot)$ to the exact sequence

$$0 \to O_X(-2H-H)^{\alpha} \oplus O_X(-H-H)^{\beta} \to \bigoplus_{i=1}^{m} O_X(-E_i-H)^{\gamma_i} \oplus O_X(-H)^{\delta} \to \mathcal{E}(-H) \to 0,$$

we notice that it suffices to prove that $\text{Ext}^2(E, \mathcal{O}_X(-E_i-H)) = 0$ and that $\text{Ext}^2(E, \mathcal{O}_X(-H)) = 0$ for $i = 1, \ldots, m$. Now applying $\text{Ext}(\cdot, \mathcal{O}_X(-E_i-H))$ to the sequence

$$0 \to O_X(-2H)^{\alpha} \oplus O_X(-H)^{\beta} \to \bigoplus_{i=1}^{m} O_X(-E_i)^{\gamma_i} \oplus O_X^\delta \to E \to 0,$$

one obtains

$$\text{Ext}^1(O_X(-2H)^{\alpha} \oplus O_X(-H)^{\beta}, O_X(-E_i-H)) \to \text{Ext}^2(E, O_X(-E_i-H))$$

$$\to \text{Ext}^2(\bigoplus_{i} O_X(-E_i)^{\gamma_i} \oplus O_X^\delta, O_X(-E_i-H)).$$

This gives $\text{Ext}^2(E, O_X(-E_i-H)) = 0$. Similarly, one can apply $\text{Ext}(\cdot, \mathcal{O}_X(-H))$ to obtain

$$\text{Ext}^1(O_X(-2H)^{\alpha} \oplus O_X(-H)^{\beta}, O_X(-H)) \to \text{Ext}^2(E, O_X(-H))$$

$$\to \text{Ext}^2(\bigoplus_{i} O_X(-E_i)^{\gamma_i} \oplus O_X^\delta, O_X(-H)).$$

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yielding $\operatorname{Ext}^2(\mathcal{E}, \mathcal{O}_X(-H)) = 0$. The same argument applies to the exact sequence
\[
0 \rightarrow \mathcal{O}_X(-2H) \alpha \rightarrow \mathcal{O}_X(-H) \beta \oplus \bigoplus_{i=1}^m \mathcal{O}_X(-E_i) \gamma_i \oplus \mathcal{O}_X^4 \rightarrow \mathcal{E} \rightarrow 0.
\]

\[\square\]

**Lemma 4.5.** Let
\[
\mathcal{U} = \mathcal{O}_X(-2H) \alpha \oplus \mathcal{O}_X(-H) \beta \quad \text{and} \quad \mathcal{V} = \bigoplus \mathcal{O}_X(-E_i) \gamma_i \oplus \mathcal{O}_X^4
\]
or
\[
\mathcal{U} = \mathcal{O}_X(-2H) \alpha \quad \text{and} \quad \mathcal{V} = \mathcal{O}_X(-H) \beta \oplus \bigoplus \mathcal{O}_X(-E_i) \gamma_i \oplus \mathcal{O}_X^4
\]
be as above. Then the open dense subset $S \subset \operatorname{Hom}(\mathcal{U}, \mathcal{V})$ parameterizing locally free $F$-prioritary sheaves is a complete family of $F$-prioritary sheaves.

**Proof.** We only prove for the first case, and the second is the same. We need to check that the Kodaira-Spencer map
\[
\kappa : T_s S = \operatorname{Hom}(U, V) \rightarrow \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})
\]
is surjective. As the map $\kappa$ factors as the composition of two maps
\[
\operatorname{Hom}(U, V) \xrightarrow{\phi} \operatorname{Hom}(U, \mathcal{E}) \xrightarrow{\psi} \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}),
\]
where $\phi$ and $\psi$ are given by applying $\operatorname{Ext}(U, \cdot)$ and $\operatorname{Ext}(\cdot, \mathcal{E})$, respectively:
\[
\operatorname{Hom}(U, V) \xrightarrow{\phi} \operatorname{Hom}(U, \mathcal{E}) \rightarrow \operatorname{Ext}^1(U, U), \quad \operatorname{Hom}(U, \mathcal{E}) \xrightarrow{\psi} \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow \operatorname{Ext}^1(\mathcal{V}, \mathcal{E}).
\]
Notice that we have
\[
\operatorname{Ext}^1(U, U) = 0, \quad \operatorname{Ext}^1(V, V) = 0, \quad \text{and} \quad \operatorname{Ext}^2(V, U) = 0.
\]
Applying $\operatorname{Ext}(V, \cdot)$ to the sequence
\[
0 \rightarrow \mathcal{O}_X(-2H) \alpha \oplus \mathcal{O}_X(-H) \beta \xrightarrow{\delta} \bigoplus \mathcal{O}_X(-E_i) \gamma_i \oplus \mathcal{O}_X^4 \rightarrow \mathcal{E} \rightarrow 0,
\]
one obtains that
\[
\operatorname{Ext}^1(V, V) \rightarrow \operatorname{Ext}^1(\mathcal{V}, \mathcal{E}) \rightarrow \operatorname{Ext}^2(\mathcal{V}, U).
\]
Thus we conclude that $\operatorname{Ext}^1(V, \mathcal{E}) = 0$ and consequently $\kappa$ is surjective.

\[\square\]

### 4.3 Brill-Noether property

In this section, we will give a sufficient condition for the character $v$ to satisfy the weak Brill-Noether property.

By semicontinuity, if $\mathcal{E}$ is any sheaf with at most one cohomology, then the cohomology also vanishes for the general sheaf in any component of $\mathcal{P}_H(v)$ that contains $\mathcal{E}$. If moreover, the moduli space $M_A(v)$ is non-empty, then the general sheaf in $M_A(v)$ has at most one non-zero cohomology.

**Lemma 4.6.** ([CH18b]) Let $\mathcal{L}$ be a line bundle on a smooth surface $X$. Let $\mathcal{V}$ be a torsion-free sheaf on $X$, and let $\mathcal{V}'$ be a general elementary modification of $\mathcal{V}$ at a general point $p \in X$,
defined as the kernel of a general surjection \( \phi : \mathcal{V} \to \mathcal{O}_p \):

\[
0 \to \mathcal{V}' \to \mathcal{V} \xrightarrow{\phi} \mathcal{O}_p \to 0.
\]

1. If \( \mathcal{V} \) is \( \mathcal{L} \)-prioritary, then \( \mathcal{V}' \) is \( \mathcal{L} \)-prioritary.

2. The sheaves \( \mathcal{V} \) and \( \mathcal{V}' \) have the same rank and \( c_1 \), and

\[
\chi(\mathcal{V}') = \chi(\mathcal{V}) - 1,
\]

\[
\Delta(\mathcal{V}') = \Delta(\mathcal{V}) + \frac{1}{r}.
\]

3. We have \( H^2(X, \mathcal{V}) \cong H^2(X, \mathcal{V}') \).

4. If at least one of \( H^0(X, \mathcal{V}) \) or \( H^1(X, \mathcal{V}) \) is zero, then at least one of \( H^0(X, \mathcal{V}') \) or \( H^1(X, \mathcal{V}') \) is zero. In particular, if \( H^2(X, \mathcal{V}) = 0 \) and \( \mathcal{V} \) is non-special, then \( H^2(X, \mathcal{V}') = 0 \) and \( \mathcal{V}' \) is also non-special.

**Theorem 4.7.** Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points. Let \( \nu = (r, \nu, \Delta) \) be a Chern character such that \( r(\mathcal{V}) \geq 2 \) and \( \Delta \geq 0 \). Write \( \nu = ah - \sum b_i E_i \), and define \( \nu' := ah - \sum_{i > 0} b_i E_i \). If \( \nu \) satisfies that \( (\nu, E_i) \geq -1 \) and \( (\nu', L) \geq -1 \), then \( \mathcal{E}_{H, F_1, \ldots, F_m}(\mathcal{V}) \) is non-special.

**Proof.** If \( \nu = \nu' \), then we can find good bundles \( \mathcal{E} = \oplus \mathcal{O}(D_i) \) such that \( r(\mathcal{E}) = r \) and \( \nu(\mathcal{E}) = \nu \). Then \( \mathcal{E} \) has no higher cohomology by our computation of cohomology of line bundles. Thus we can find a non-special sheaf in \( \mathcal{E}_H(\mathcal{V}) \). It then follows that a general sheaf parameterized by \( \mathcal{E}_{H, F_1, \ldots, F_m}(\mathcal{V}) \) is non-special.

If \( \nu \neq \nu' \), we may assume that \( -1 \leq (\nu, E_m) = -d_m/r < 0 \). Consider the map \( \pi : X_m \to X_{m-1} \) contracting \( E \). Let \( \mathcal{E} \) be a general sheaf in \( \mathcal{E}_{H, F_1, \ldots, F_m}(\mathcal{V}) \). Then in particular \( \mathcal{E} \) is locally free and balanced along \( E \). Thus \( \mathcal{E}|_E \cong \mathcal{O}_E^a \oplus \mathcal{O}_E(-1)^{-a} \) for some \( a \). Taking the push-forward of the resolution

\[
0 \to \mathcal{O}_X(-2H + D)^\alpha \xrightarrow{\pi} \mathcal{O}_X(-H + D)^\beta \oplus \bigoplus_{i=1}^m \mathcal{O}_X(-E_i + D)^\gamma_i \oplus \mathcal{O}_X(D)^\delta \to \mathcal{E} \to 0
\]

of \( \mathcal{E} \), one gets

\[
0 \to \mathcal{O}_{X_{m-1}}(-2H + D')^\alpha \xrightarrow{\pi} \mathcal{O}_{X_{m-1}}(-H + D')^\beta \oplus \bigoplus_{i=1}^{m-1} \mathcal{O}_{X_{m-1}}(-E_i + D')^\gamma_i \oplus \mathcal{O}_{X_{m-1}}(D')^\delta
\]

\[
\longrightarrow \pi_* \mathcal{E} \longrightarrow R^1 \pi_* \mathcal{O}_X(-2H + D)^\alpha = 0
\]

because \( (-2H + D)|_E \cong \mathcal{O}_E(-1) \), where \( D' = \pi_* D \). In particular, the higher direct image \( R^i f_* \mathcal{E} \) vanish for all \( i > 0 \) so that the cohomology of \( \mathcal{E} \) is the cohomology of \( \pi_* \mathcal{E} \). Moreover, \( \pi_* \mathcal{E} \) is locally free and prioritary with respect to \( H, F_1, \ldots, F_{m-1} \) and admits a desired resolution. Notice that the rational map \( \pi_* : \mathcal{E}_H(\mathcal{V}) \to \mathcal{E}_{H, F_1, \ldots, F_m}(\mathcal{V}) \) defined by \( \mathcal{E} \mapsto \pi_* \mathcal{E} \) is dominant: the tangent map is

\[
\text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^1(\pi_* \mathcal{E}, \pi_* \mathcal{E}) \cong \text{Ext}^1(\pi^* \pi_* \mathcal{E}, \mathcal{E}),
\]

whose cokernel is

\[
\text{Ext}^2(\mathcal{E}_H(\mathcal{V}), \mathcal{E}) \simeq H^0(\mathcal{E}^* \otimes K_{X|E_m})(-1)) \equiv H^0(X, \mathcal{O}_{E_m}(-1)^{r-d} \oplus \mathcal{O}_{E_m}(-2)^d) = 0.
\]

Thus we reduce to the case when \( \nu = \nu' \).
5 Stability of exceptional bundles

Corollary 4.8. Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points, and $v = (r, c_1, \Delta)$ be a Chern character on $X$ such that $r(v) \geq 2$, $\Delta \geq 0$, and $c_1$ is nef. Then $\mathcal{P}_F, \ldots, F_m, H(v)$ is non-special.

Remark 4.9. On a smooth del Pezzo surface, we expect that $v$ is non-special for $v = (r, c_1, \Delta)$ such that $\Delta \geq 0$ and that $(\nu.C) \geq -1$ for any negative curve $C$. On our surface $X = X_m$, this already fails for line bundles. Consider for example be a Chern character on $X$ such that $\Delta \geq 0$ and that $(\nu.C) \geq -1$ and $(D.L) \geq -1$. We have $h^0(D) = h^0(H) = 3$ and $\chi(D) = 6 - 4 = 2$, so that $h^1(D) = 1$.

5 Stability of exceptional bundles

In this section, we prove the stability of the constructible exceptional bundles. Although it is unknown whether all exceptional bundles on the blow-up of $\mathbb{P}^2$ along $m$ collinear points are constructible, we will see in the next two sections that the constructible ones are sufficient to give us a description of stable characters.

Proposition 5.1. Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ distinct points (not necessarily collinear). If $\mathcal{E}$ is an exceptional bundle which is balanced on $E_i$ for any $i$, then $\pi_*\mathcal{E}$ is semi-exceptional.

Proof. Let $\mathcal{E}$ be an exceptional bundle balanced on every $E_i$. By twisting with some line bundle, we may assume that $\mathcal{E}|_{E_i} = \mathcal{O}_{E_i}^{r-d_i} \oplus \mathcal{O}_{E_i}(-1)^{d_i}$. Let $\pi : X \to \mathbb{P}^2$ be the blow-down map. We first show that $R^i\pi_*\mathcal{E} = 0$ for any $i > 0$. For each exceptional divisor $E = E_i$, by the theorem on formal functions, the cohomology vanishes if and only if $\lim H^1(E_n, \mathcal{E}|_{E_n}) = 0$, where $E_n$ is the closed subscheme of $X$ defined by $\mathcal{O}(-nE)$. For each $n \geq 1$, there are exact sequences on $X$

$$0 \to \mathcal{O}_E(n) \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$$

Tensoring with $\mathcal{E}$ and induction on $n$ gives the result.

Now consider the short exact sequence

$$0 \to \pi^*\pi_*\mathcal{E} \to \mathcal{E} \to \bigoplus_{i=1}^m \mathcal{O}_{E_i}(-1)^{d_i} \to 0.$$

Applying the functor Hom($\cdot, \mathcal{E}$), one gets the long exact sequence

$$\text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^1(\pi^*\pi_*\mathcal{E}, \mathcal{E}) \to \bigoplus_{i=1}^m \text{Ext}^2(\mathcal{O}_{E_i}(-1)^{d_i}, \mathcal{E}) \to \text{Ext}^2(\mathcal{E}, \mathcal{E}) \to \text{Ext}^2(\pi^*\pi_*\mathcal{E}, \mathcal{E}) \to 0.$$

We have that

$$\text{Ext}^i(\pi^*\pi_*\mathcal{E}, \mathcal{E}) = H^i(X, \pi^*(\pi_*\mathcal{E})^\vee \otimes \mathcal{E}) = H^i(\mathbb{P}^2, \pi_*((\pi_*\mathcal{E})^\vee \otimes \mathcal{E})) = \text{Ext}^i(\pi_*\mathcal{E}, \pi_*\mathcal{E}),$$

and that

$$\text{Ext}^2(\mathcal{O}_{E_i}(-1), \mathcal{E}) \simeq \text{Hom}(\mathcal{E}, \mathcal{O}_{E_i}(-2))^* = 0$$

as $\mathcal{E}_{E_i} \simeq \mathcal{O}_{E_i}^{r-d_i} \oplus \mathcal{O}_{E_i}(-1)^{d_i}$. Since $\mathcal{E}$ is exceptional, then $\pi_*\mathcal{E}$ is semi-exceptional by the long exact sequence.

\[\square\]
**Theorem 5.2.** Let \( X \) be the blow-up of \( \mathbb{P}^2 \) along \( m \) distinct collinear points, and \( A = H - \sum \varepsilon_i E_i \) with \( 0 < \varepsilon_i \ll 1 \) a generic polarization. If \( \mathcal{E} \) is a constructible exceptional bundle, then it is \( \mu_A \)-stable.

**Proof.** Notice first that constructible exceptional bundles are balanced on every \( E_i \). It suffices to show that \( \mathcal{E} \) is \( \mu_A \)-semistable. Suppose otherwise, one has a destabilizing subsheaf \( \mathcal{F} \subseteq \mathcal{E} \). Write \( \nu(\mathcal{E}) = \alpha H - \sum \beta_i E_i \) and \( \nu(\mathcal{F}) = \alpha' H - \sum \beta'_i E_i \), then we have

\[
\alpha - \sum \varepsilon_i \beta_i < \alpha' - \sum \varepsilon_i \beta'_i.
\]

It follows that \( \alpha \leq \alpha' \) by taking \( \varepsilon_i \) sufficiently small. However, since \( \pi_* \mathcal{E} \) is an exceptional bundle on \( \mathbb{P}^2 \), in particular \( \mu_H \)-stable, then we get a contradiction. \hfill \Box

**Remark 5.3.** For a general exceptional bundle \( \mathcal{E} \), consider the exact sequence

\[
0 \to \mathcal{E}(-H) \to \mathcal{E} \to \mathcal{E}|_H \to 0.
\]

As \( \text{ext}^1(\mathcal{E}, \mathcal{E}) = \text{ext}^2(\mathcal{E}, \mathcal{E}) = 0 \), then applying \( \text{Ext}(\mathcal{E}, \cdot) \) to the sequence, one gets

\[
\text{Ext}^2(\mathcal{E}, \mathcal{E}(-H)) \simeq \text{Ext}^1(\mathcal{E}, \mathcal{E}|_H).
\]

Notice that there exists a line \( l_0 \in |H| \) such that \( \mathcal{E}|_{l_0} \) is balanced if and only if \( \mathcal{E} \) is \( H \)-prioritary:

\[
\text{Ext}^1(\mathcal{E}, \mathcal{E}|_H) \simeq \text{Ext}^1(\mathcal{E}|_H, \mathcal{E}|_H) = H^1(l_0, (\mathcal{E} \otimes \mathcal{E})|_{l_0}).
\]

**Example 5.4.** On our polarized surface \((X, A)\) with \( \pi: X \to \mathbb{P}^2 \) the blow-up of \( \mathbb{P}^2 \) along \( m \) collinear points and \( A = H - \varepsilon \sum E_i \) an ample class, consider the bundle \( \mathcal{E} := \pi^* \mathcal{T}_{\mathbb{P}^2} \). Notice that

\[
\text{Ext}^* (\mathcal{E}, \mathcal{E}) = \text{Ext}^* (\mathcal{O}, \pi^*(\mathcal{T}_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}) \simeq H^* (X, \pi^*(\mathcal{T}_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}))
\]

\[
= H^* (\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}) \simeq \text{Ext}^* (\mathcal{T}_{\mathbb{P}^2}, \mathcal{T}_{\mathbb{P}^2}),
\]

implying that \( \mathcal{E} \) is exceptional since \( \mathcal{T}_{\mathbb{P}^2} \) is. We will show that \( \mathcal{E} \) is \( \mu_A \)-stable for any sufficiently small \( \varepsilon > 0 \). Suppose we have a sub-line bundle \( \mathcal{O}_X(D) \) of \( \mathcal{E} \) with \( D = aH - \sum b_i E_i \), we claim that \( a \leq 1 \). If not, consider the short exact sequence

\[
0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X(H - D) \to \mathcal{O}_H(H - D) \simeq \mathcal{O}_{\mathbb{P}^1}(1 - a) \to 0.
\]

Taking the induced long exact sequence on cohomology, we see that the map

\[
H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X(H - D))
\]

given by multiplication of a non-zero element in \( H^0(X, \mathcal{O}_X(H)) \) is injective. Now consider the pull-back of the Euler sequence on \( \mathbb{P}^2 \) twisted by \( \mathcal{O}(-D) \):

\[
0 \to \mathcal{O}_X(-D) \xrightarrow{(\varepsilon_0, \varepsilon_1, \varepsilon_2)} \mathcal{O}_X(H - D)^{\oplus 3} \to \mathcal{E}(-D) \to 0.
\]

As a consequence, the induced exact sequence on cohomology

\[
0 = H^0(X, \mathcal{O}_X(H - D))^{\oplus 3} \to H^0(\mathcal{E}(-D)) \to H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X(H - D))^{\oplus 3}
\]

implies that \( H^0(X, \mathcal{E}(-D)) = 0 \), which is a contradiction.
We can choose \( \varepsilon_0 \) sufficiently small such that

\[
\mu_A(\mathcal{O}_X(D)) < 3/2 = \mu_A(T_{\mathbb{P}^2})
\]

for all sub-line bundles \( \mathcal{O}_X(D) \) of \( \mathcal{E} \) with \( a = 0 \) or \( a = 1 \) as there are only finitely many of them. Now we claim that \( \mathcal{E} \) is \( \mu_A \)-stable. It suffices to check that there does not exist any possible destabilizing sub-line bundle \( \mathcal{O}_X(D) \) with \( a < 0 \). Suppose that \( \mathcal{O}_X(D) \) is such a line bundle: \( D = aH - \sum b_iE_i \) with \( a < 0 \) and

\[
\mu_A(\mathcal{O}_X(D)) = a - \varepsilon_0 \sum b_i \geq \frac{3}{2}.
\]

If \( b_i > 0 \) for some \( i \), then we may replace \( D \) by \( D + E_i \); notice that \( \mu_A(\mathcal{O}_X(D)) < \mu_A(\mathcal{O}_X(D + E_i)) \) and that

\[
H^0(\mathcal{E}(D)) = \mu_A(\mathcal{O}_X(D))
\]

due to the short exact sequence

\[
0 \rightarrow \mathcal{E}(-D - E_i) \rightarrow \mathcal{E}(-D) \rightarrow \mathcal{E}(-D)|_{E_i} \rightarrow 0
\]

and \( \mathcal{E}(-D)|_{E_i} \simeq \mathcal{O}_{\mathbb{P}^1}(-b_i) \oplus \mathcal{O}_{\mathbb{P}^1}(-b_i) \). Now we reduce to the case when \( b_i \leq 0 \) for all \( i \). Consider the short exact sequence

\[
0 \rightarrow \mathcal{E}(-D - L) \rightarrow \mathcal{E}(-D) \rightarrow \mathcal{E}(-D)|_L \rightarrow 0.
\]

We see that \( \mathcal{E}|_L = \pi^*T_{\mathbb{P}^2}|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \) by performing the pull-back of the Euler sequence on \( \mathbb{P}^2 \) restricted to \( L \). It thus follows that

\[
\mathcal{E}(-D)|_L \simeq \mathcal{O}_L(1 - a + \sum b_i) \oplus \mathcal{O}_L(2 - a + \sum b_i).
\]

Observe that \( 2 - a + \sum b_i < 0 \): suppose not, then one gets

\[
2 + \sum b_i \geq a \geq \varepsilon_0 \sum b_i + \frac{3}{2}, \quad (1 - \varepsilon_0)(\sum b_i) \geq -\frac{1}{2},
\]

which is impossible. Hence we arrive at the case when \( a \geq 0 \) by performing these two kinds of reduction. This case follows from our choice of \( \varepsilon_0 \).

In fact, \( \mathcal{E} \) is \( \mu_A \)-stable for any \( \varepsilon \) such that \( A \) is ample, i.e. \( 0 < \varepsilon < 1/m \). By our argument above, it suffices to check the case where \( a = 0 \) and \( a = 1 \). When \( a = 0 \), notice that \( \mathcal{E}(-2E_i) \) has no sections because every non-zero tangent vector field on \( \mathbb{P}^2 \) vanishes at a point of multiplicity at most one. Thus the sub-line bundle with maximal \( A \)-slope is \( \mathcal{O}_X(\sum E_i) \), which satisfies that \( \mu_A(\mathcal{O}_X(\sum E_i)) = m\varepsilon < 1 < 3/2 \), and hence does not destabilize \( \mathcal{E} \) either.

When \( a = 1 \), Notice that every section of \( T_{\mathbb{P}^2}(-1) \) vanishes at one point, hence the divisor corresponding to any section of \( \mathcal{E}(\mathcal{E})(-H) \) is either trivial or \( E_i \). In particular, the sub-line bundle with maximal \( A \)-slope is \( \mathcal{O}_X(H + E_i) \), whose \( A \)-slope is \( 1 + \varepsilon < 3/2 \), and hence does not destabilize \( \mathcal{E} \).

However, in the case when \( a = 1 \), if we take our polarization to be \( A = H - (1 - \delta)E_1 - \sum_{i \geq 2} \varepsilon_i E_i \) such that \( 0 < \delta \ll 0 \) and \( 0 < \varepsilon \ll 0 \) and \( 1 - \delta + (m - 1)\varepsilon < 1 \), then \( \mathcal{O}(H + E_1) \) destabilizes \( \mathcal{E} \). This suggests that we cannot expect that the exceptional bundles, even the constructible ones, to be \( \mu_A \)-stable for any polarization \( A = H - \sum \varepsilon_i E_i \).
6 Existence of Stable Sheaves

Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points. In this section, we first computationally determine whether the moduli space $M_{X,A}(v)$ is nonempty. Then for a generic polarization and an arbitrary character except for one special case, we will give an equivalent condition for the existence of stable sheaves with this character.

6.1 Generic polarization and sharp Bogomolov inequalities

In this section we introduce functions of the slope which provide sharp Bogomolov-type inequalities for various stabilities. We follow the treatment in [CH21].

For a $\mu A$-stable exceptional bundle $E$, we define a function

$$DLP_{A,E}(\nu) = \begin{cases} 
    P(\nu - \nu(E)) - \Delta(E), & \text{if } \frac{1}{2}K_X.A \leq (\nu - \nu(E)).A < 0 \\
    P(\nu(E) - \nu) - \Delta(E), & \text{if } 0 < (\nu - \nu(E)).A \leq -\frac{1}{2}K_X.A \\
    \max\{P(\pm(\nu(E) - \nu)) - \Delta(E)\}, & \text{if } (\nu - \nu(E)).A = 0 \end{cases}$$

**Definition 6.1.** Let $A = H - \sum \epsilon_iE_i$ be a polarization and $E_A$ be the set of $\mu_A$-stable exceptional bundles on $X$. Define functions

$$DLP_{A}(\nu) = \sup_{E \in E_A} DLP_{A,E}(\nu),$$

and

$$DLP_{A}^{<r}(\nu) = \sup_{E \in E_A, r(E) < r} DLP_{A,E}(\nu).$$

**Proposition 6.2.** Let $\epsilon$ be generic.

(i) If $E$ is an $A$-semistable exceptional bundle on $X$ of rank $r$, then

$$\Delta(E) \geq DLP_{A}^{<r}(\nu(E)).$$

(ii) If $E$ is an $A$-semistable non-semiexceptional bundle on $X$, then

$$\Delta(E) \geq DLP_{A}(\nu(E)).$$

**Proof.** If $E$ is a $\mu_A$-semistable sheaf with

$$\frac{1}{2}K_X.A \leq \mu_A(E) - \mu_A(V) < 0,$$

when $\text{hom}(V,E) = \text{hom}(E,V(K_X))$ by stability and duality. Therefore $\chi(V,E) \leq 0$ and

$$\Delta(E) \geq P(\nu(E) - \nu(V)) - \Delta(V).$$

Likewise, if

$$0 < \mu_A(E) - \mu_A(V) \leq -\frac{1}{2}K_X.A,$$

then the inequality $\chi(E,V) \leq 0$ provides a lower bound

$$\Delta(E) \geq P(\nu(V) - \nu(E)) - \Delta(V).$$
Notice that if \( \varepsilon \) is generic, then \((\nu - \nu(V)).A = 0\) happen only when \(\nu = \nu(V)\). Suppose that \(\mathcal{E}\) is \(A\)-semistable of total slope \(\nu(\mathcal{E}) = \nu(V)\). If \(\Delta(\mathcal{E}) = \Delta(V) < 1/2\), then \(\mathcal{E}\) is semistable. If \(\Delta(\mathcal{E}) \neq \Delta(V)\), then either \(\text{hom}(\mathcal{E}, V) = 0\) or \(\text{hom}(V, \mathcal{E}) = 0\) by \(A\)-semistability, and in either case Riemann-Roch implies

\[
\Delta(\mathcal{E}) \geq \text{DLP}_{A,V}(\nu) = \frac{1}{2} + \frac{1}{2r(V)^2}.
\]

Thus if \(A\) is generic, then \(\Delta \geq \text{DLP}_{H,E}(\nu)\) whenever there is an \(A\)-semistable sheaf of total slope \(\nu\) and discriminant \(\Delta\) satisfying \(\|\nu - \nu(V)\|.A\| \leq -\frac{1}{2}K_X.A\).

**Definition 6.3.** Let \(A = H - \sum \varepsilon_i E_i\) be a polarization and \(v\) be a Chern character. Define

\[
\delta_{A-s}^v(\nu) = \inf \left\{ \Delta \geq \frac{1}{2} : \text{there is a } A\text{-stable sheaf of total slope } \nu \text{ and discriminant } \Delta \right\}.
\]

We similarly define functions \(\delta_{A}^s, \delta_{A}^{ss}, \delta_{A}^{-ss}\).

It is immediate that

\[
\delta_{A}^{-ss}(\nu) \leq \delta_{A}^s(\nu) \leq \delta_{A}^{ss}(\nu) \leq \delta_{A}^{-ss}(\nu).
\]

Now let us compare the various \(\delta\)-functions in the case where the polarization \(A = H - \sum \varepsilon_i E_i\) is generic.

**Theorem 6.4.** (Theorem 9.2 [CH21]) Let \(\nu \in \text{Pic}(X)_\mathbb{Q}\), and \(A\) be a generic polarization. Then

\[
\delta_{A}^{ss}(\nu) = \delta_{A}^s(\nu) = \delta_{A}^{-ss}(\nu).
\]

If moreover there is no \(A\)-stable exceptional bundle of total slope \(\nu\), then these numbers also equal \(\delta_{A}^{-s}(\nu)\).

The main result about existence of sheaves with discriminant above \(\delta_{A}^{-s}(\nu)\) is the following:

**Theorem 6.5.** (Theorem 9.7 [CH21]) Let \(v = (r, \nu, \Delta) \in K(X)\) and \(A\) be any polarization.

1. If \(\Delta > \delta_{A}^{-s}(\nu)\), then there are \(A\)-stable sheaves of character \(v\).
2. If there is a non-exceptional \(A\)-stable sheaf of character \(v\), then \(\Delta \geq \delta_{A}^{-s}(\nu)\).
3. If there is a \(A\)-stable sheaf of slope \(\nu\) and discriminant \(\delta_{A}^{-s}(\nu) > \frac{1}{2}\), then non-exceptional \(A\)-stable sheaves of character \(v\) exist if and only if \(\Delta \geq \delta_{A}^{-s}(\nu)\).

### 6.2 Harder-Narasimhan filtration

Let \(X = X_m\) be the blow-up of \(\mathbb{P}^2\) along \(m\) collinear points, and \(A = H - \sum \varepsilon_i E_i\) be a polarization of \(X\), where \(\varepsilon_i > 0\) is a rational number such that \(\sum \varepsilon_i < 1\).

Let \(\mathcal{E}\) be a complete family of torsion-free coherent sheaves on \((X, A)\), which is both \(H\)-prioritary and \(F_i\)-prioritary for all \(i\), parameterized by a smooth algebraic variety \(S\). Consider the \(r(\varepsilon)A\)-Harder-Narasimhan filtration of a general sheaf \(\mathcal{E}\), where \(r(\varepsilon)\) is the smallest positive integer such that \(r(\varepsilon)A\) is an integral divisor. Suppose this Harder-Narasimhan filtration has length \(l\), and the \(r(\varepsilon)A\)-semistable quotients \(\text{gr}_{i,s}\) have corresponding \(r(\varepsilon)A\)-Hilbert polynomial \(P_i\), reduced \(r(\varepsilon)A\)-Hilbert polynomial \(p_1 > \cdots > p_l\), and Chern characters \(\text{gr}_{i} = (r_i, \nu_i, \Delta_i)\).

The next lemma is useful in bounding the polarization in Section 6.3. We include the proof here.
Lemma 6.6. (Lemma 5.1 [CH21]) A general sheaf $E_s$ in this family satisfies that

$$0 \leq \mu_{\text{max}}(E_s) - \mu_{\text{min}}(E_s) \leq 1.$$  

Proof. First suppose $C$ is a smooth rational curve, and then general $E_s|_C$ is a locally free sheaf. Recall that if $E_s/S$ is a complete family of $\mathcal{O}(C)$-prioritary sheaves which are locally free along $C$, then the general $E_s$ has restriction $E_s|_C$ which is balanced so that

$$\mu_{\text{max}}(E_s|_C) - \mu_{\text{min}}(E_s|_C) \leq 1.$$  

Observe that $\mu_{\text{max}}_{\mathcal{O}_C}(E_s) \leq \mu_{\text{max}}(E_s|_C)$. Indeed, suppose $F \subseteq E_s$ is a subsheaf. Then

$$\mu_{\mathcal{O}_C}(F) = \mu(F|_C) \leq \mu_{\text{max}}(E_s|_C).$$

Analogously we have $\mu_{\text{min}}_{\mathcal{O}_C}(E_s) \geq \mu_{\text{min}}(E_s|_C)$, and we conclude that

$$\mu_{\text{max}}_{\mathcal{O}(C)}(E_s) - \mu_{\text{min}}_{\mathcal{O}(C)}(E_s) \leq 1$$

holds for a general $s \in S$. (Even if $L$ is not ample, we write for example $\mu_{\text{max},L}(E)$ for the maximum $L$-slope of a subsheaf of $E$, if it exists. For $L = \mathcal{O}(C)$, the above restriction argument shows the maximum exists.)

Now observe that if $E_s/S$ is a complete family of $(F_1, ..., F_m, H)$-prioritary sheaves, then

$$mu_{\text{max},A}(E_s) - \mu_{\text{max},A}(E_s) = \mu_A(\text{gr}_{1,s}) - \mu_A(\text{gr}_{l,s})$$

$$= (\nu_1 - \nu_l)(1 - \sum \varepsilon_i)H + \sum \varepsilon_i F_i$$

$$\leq (1 - \sum \varepsilon_i) + \sum \varepsilon_i = 1.$$  

\[\square\]

Lemma 6.7. (Lemma 5.2 [CH21]) With the notation above, we have $\chi(\text{gr}_i, \text{gr}_j) = 0$ for all $i < j$.

The following theorem provides an algorithm to determine stable characters inductively. We will use it to determine a class of special characters that cannot be detected by the weak DL condition given in Definition 6.11. This is proved for Hirzebruch surfaces in [CH21]. We repeat the argument here for the reader’s convenience.

Theorem 6.8. (Theorem 5.3 [CH21]) Suppose $w_1, ..., w_k \in K(X)$ are characters of positive rank satisfying the following properties:

1. $w_1 + \cdots + w_k = v$.
2. $q_1 > \cdots > q_k$, where $q_i$ is the reduced $A$-Hilbert polynomial corresponding to $w_i$.
3. $\mu_A(w_1) - \mu_A(w_k) \leq 1$.
4. $\chi(w_i, w_j) = 0$ for $i < j$.
5. The moduli space $M_A(w_i)$ is nonempty for each $i$.

Then $k = l$ and $\text{gr}_i = w_i$ for each $i$.

Proof. Pick $A$-semistable sheaves $W_i \in M_A(w_i)$ for each $i$, and consider the sheaf

$$\mathcal{U} := \bigoplus_i W_i$$
so that $\mathcal{U}$ has character $v$ and the Harder-Narasimhan filtration of $\mathcal{U}$ has factors $\mathcal{W}_1, ..., \mathcal{W}_k$. Then by assumption

$$\mu_{\max, A}(\mathcal{U}) - \mu_{\min, A}(\mathcal{U}) = \mu_A(\mathcal{W}_1) - \mu_A(\mathcal{W}_k) \leq 1,$$

so that

$$\text{Ext}^2(\mathcal{W}_i, \mathcal{W}_j(-F_n)) \cong \text{Hom}(\mathcal{W}_j, \mathcal{W}_i(K_X + F_n))^* = 0,$$

$\mathcal{U}$ is both $H$-prioritary and $F_n$-prioritary for each $n$.

Now we can construct a complete family $\mathcal{U}_t/\Sigma$ parameterized by a smooth, irreducible variety $\Sigma$ such that $\mathcal{U} = \mathcal{U}_{t_0}$ for some $t_0 \in \Sigma$. Let $d \gg 0$ be sufficiently large and divisible, let $\chi = \chi(\mathcal{O}_X(-dA), \mathcal{U})$, and consider the universal family of quotients $\mathcal{U}_t/\Sigma$ on $\Sigma = \text{Quot}(\mathcal{O}_X(-dA)^\chi, \text{ch} (\mathcal{U}))$ parameterizing quotients

$$0 \rightarrow K_t \rightarrow \mathcal{O}_X(-dA)^\chi \rightarrow \mathcal{U}_t \rightarrow 0.$$

Let $t_0 \in \Sigma$ be the point corresponding to the canonical evaluation

$$\mathcal{O}_X(-dA) \otimes \text{Hom}(\mathcal{O}_X(-dA), \mathcal{U}) \rightarrow \mathcal{U}.$$

Then the tangent space to $\Sigma$ at a point $t$ corresponding to the previous short exact sequence is $\text{Hom}(K_t, \mathcal{U}_t)$, and $\Sigma$ is smooth at $t$ if $\text{Ext}^1(K_t, \mathcal{U}_t) = 0$. Applying $\text{Hom}(\cdot, \mathcal{U}_t)$ to the exact sequence, one gets

$$\text{Hom}(K_t, \mathcal{U}_t) \rightarrow \text{Ext}^1(\mathcal{U}_t, \mathcal{U}_t) \rightarrow \text{Ext}^1(\mathcal{O}_X(-dA)^\chi, \mathcal{U}_t) \rightarrow \text{Ext}^1(K_t, \mathcal{U}_t) \rightarrow \text{Ext}^2(\mathcal{U}_t, \mathcal{U}_t).$$

By passing to the open subset parameterizing locally free sheaves if necessary, we have

$$\text{ext}^2(\mathcal{U}_t, \mathcal{U}_t) = \text{hom}(\mathcal{U}_t, \mathcal{U}_t(K_X)) = 0$$

by our assumptions on the slopes. Since $d \gg 0$, we have $\text{Ext}^1(\mathcal{O}_X(-dA)^\chi, \mathcal{U}_t) = 0$ by Serre vanishing and boundedness of the Quot scheme. Therefore $\text{Ext}^1(K_t, \mathcal{U}_t) = 0$ and $\Sigma$ is smooth at $t$, including at $t = t_0$. Furthermore, the Kodaira-Spencer map at $t$ is the natural map

$$T_t \Sigma = \text{Hom}(K_t, \mathcal{U}_t) \rightarrow \text{Ext}^1(\mathcal{U}_t, \mathcal{U}_t),$$

so the universal family on $\Sigma$ is complete at $t$, including at $t = t_0$. We have thus constructed the required complete family $\mathcal{U}_t/\Sigma$.

Let $Q_i$ be the $A$-Hilbert polynomial corresponding to $w_i$. Then by the same computation as in the previous lemma, the Schatz stratum $S_A(Q_1, ..., Q_k) \subset \Sigma$ is smooth at $t_0$ of codimension 0. Therefore the stratum is dense in $\Sigma$, and the general sheaf $\mathcal{U}_t$ has an $A$-Harder-Narasimhan filtration with quotients of character $w_i$. Thus $\text{gr}_i = w_i$ and $k = l$.

### 6.3 Classification of stable characters

In this section, we will give an equivalent condition for the existence of $\mu_A$-stable bundles of character $v$ for some polarization $A = H - \sum \varepsilon_i E_i$.

**Definition 6.9.** A torsion-free coherent sheaf $\mathcal{E}$ (or Chern character) satisfies the strong Drézet-Le Potier condition (abbr. as strong DL condition) if

$$\text{(1)}$$

$$\text{(2)}$$

$$\text{(3)}$$
(a) for every $\mu_A$-stable sheaf $F$ satisfying $r(F) < r(E)$ and

$$\mu_A(E) \leq \mu_A(F) \leq \mu_A(E) - A.K_X,$$

we have $\chi(F, E) \leq 0$;

(b) for every $\mu_A$-stable sheaf $F$ satisfying $r(F) < r(E)$ and

$$\mu_A(E) + A.K_X \leq \mu_A(F) \leq \mu_A(E),$$

we have $\chi(E, F) \leq 0$.

**Lemma 6.10.** Suppose $E$ is a non-exceptional $\mu_A$-stable sheaf with $r(E) \geq 2$. Then $\Delta(E) \geq 1/2$ and $E$ satisfies strong DL condition.

**Proof.** We have that $\text{ext}^2(E, E) = 0$ and $\text{hom}(E, E) = 1$ by stability. As $E$ is not exceptional, then

$$r(E)^2(1 - 2\Delta(E)) = \chi(E, E) = 1 - \text{ext}^1(F, F) \leq 0,$$

which gives $\Delta(E) \geq 1/2$. Now if $F$ is a $\mu_A$-stable bundle such that $r(F) < r(E)$ and

$$\mu_A(E) \leq \mu_A(F) \leq \mu_A(E) - A.K_X,$$

then we have $\text{hom}(F, E) = 0$ and $\text{ext}^2(F, E) = \text{hom}(E, F(K_X)) = 0$ by stability; which implies that $\chi(F, E) \leq 0$. The other condition is similar.

**Definition 6.11.** A torsion-free coherent sheaf $E$ (or Chern character) satisfies the weak Drézet-Le Potier condition (abbr. as weak DL condition) if

(a) for every exceptional bundle $F$ which is constructible and satisfies $r(F) < r(E)$ and

$$\mu_A(E) \leq \mu_A(F) \leq \mu_A(E) - A.K_X,$$

for some polarization $A$, we have $\chi(F, E) \leq 0$;

(b) for every exceptional bundle $F$ which is constructible and satisfies $r(F) < r(E)$ and

$$\mu_A(E) \leq \mu_A(F) \leq \mu_A(E) - A.K_X,$$

for some polarization $A$, we have $\chi(E, F) \leq 0$.

**Remark 6.12.** If $E$ is a $\mu_A$-stable sheaf for some polarization $A$, then $E$ satisfies both strong and weak DL conditions thanks to the stability condition and Serre duality.

Let $v = (r, \nu, \Delta)$ be a character such that $\nu(v) = \alpha H - \sum \beta_i E_i$ with $-1 \leq \beta_i \leq 0$ and $\Delta(v) \geq 0$. Suppose that $\alpha \notin \mathcal{C}$, where $\mathcal{C}$ is the set of $H$-slopes on $\mathbb{P}^2$ of exceptional bundles.

**Proposition 6.13.** Let $v = (r, \nu, \Delta)$ be as above, and $E \in \mathcal{P}_H(v)$ be a general sheaf. If $E$ satisfies the weak DL-condition, then $\pi_* E$ is $\mu_H$-stable.

**Proof.** Since $E$ is general, then one has $E|_{E_i} \simeq O_{E_i}^{-d_i} \oplus O_{E_i}(-1)^{d_i}$ for every $i = 1, ..., m$. In particular, $\pi_* E$ is a vector bundle on $\mathbb{P}^2$ which fits in the short exact sequence

$$0 \to \pi^* \pi_* E \to E \to \bigoplus_{i=1}^m O_{E_i}(-1)^{d_i} \to 0.$$
We can conclude that \( \pi \) satisfies the DL-condition. We have that \( P \) is \( \text{Ext} \)-stable. By considering the exceptional bundle \( F \) given by
\[
\phi \nu \delta \alpha \beta \sum \mu \epsilon \gamma \delta = \mu A(E) - \mu A(E') \leq 1.
\]
Similarly, for any exceptional bundle \( V \) on \( \mathbb{P}^2 \) of such that \( r(V) < r \) and \( 0 < \mu_H(V) - \mu_H(\pi \ast E) \leq 2 \), we have that
\[
\chi(\pi \ast V, V) \leq 0
\]
between the exceptional bundle \( F \) on \( X \) given by
\[
0 \rightarrow \pi \ast V \rightarrow F \rightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{E_i}(1)r(V) \approx \bigoplus_{i=1}^{m} \text{Ext}^1(O_{E_i}(1), \pi \ast V) \otimes \mathcal{O}_{E_i}(1) \rightarrow 0.
\]
We can conclude that \( \pi \ast E \) is \( H \)-stable. Since \( \pi \ast E \) is general, then it is \( \mu_H \)-stable.

**Theorem 6.14.** Let \( v = (r, \nu, \Delta) \) be as above, and \( E \in \mathcal{P}_H(v) \) be a general sheaf. If \( E \) satisfies the weak DL-condition, then \( E \) is \( \mu_A \)-stable for \( 0 < \varepsilon_i \ll 1 \), where \( A = H - \sum \varepsilon_i E_i \).

**Proof.** Recall that for any subsheaf \( E' \) of \( E \) and any polarization \( A = H - \sum \varepsilon_i E_i \), one has
\[
\mu_A(E') - \mu_A(E) \leq 1.
\]
Write \( \nu(E) = \alpha H - \sum \beta_i E_i \) and \( \nu(E) = \alpha' H - \sum \beta_i' E_i \), then the inequality comes down to saying that
\[
\alpha' - \alpha - \sum \varepsilon_i (\beta_i' - \beta_i) \leq 1.
\]
Setting \( \varphi_i \rightarrow 1 \) and \( \varphi_j \rightarrow 0 \), one obtains that
\[
\beta_i - \beta_i' \leq \alpha - \alpha' + 1.
\]
Notice that for any subsheaf \( E' \) of \( E \), since \( \pi \ast E \) is \( \mu_H \)-stable, then \( \alpha' < \alpha \). Thus \( \alpha - \alpha' \) has a strictly positive lower bound \( \delta \) since \( r \) is a finite number. Now let \( \sum \varepsilon_i \) satisfy that
\[
\delta > \frac{\sum \varepsilon_i}{1 - \sum \varepsilon_i}.
\]
Then we claim that $E$ is $\mu_A$-stable for $A = H - \sum \varepsilon_i E_i$. Indeed, for any subsheaf $E'$ of $E$, one has

$$
\mu_A(E) - \mu_A(E') = \alpha - \alpha' - \sum \varepsilon_i (\beta_i - \beta'_i) \\
\geq (\alpha - \alpha') - \sum \varepsilon_i (\alpha - \alpha' + 1) \\
= (1 - \sum \varepsilon_i) (\alpha - \alpha') - \sum \varepsilon_i \\
\geq (1 - \sum \varepsilon_i) \delta - \sum \varepsilon_i > 0.
$$

Now let us consider the case when $\mu_H(\pi^* E) = \mu_H(V)$ for some exceptional bundle $V$. Consider the Harder-Narasimhan filtration of $\pi^* E$. Either none of the Harder-Narasimhan factors is isomorphic to $V$, or every Harder-Narasimhan factor is isomorphic to $V$. In the former case, one can run the same argument as above to show that $E$ is $\mu_A$-stable for $0 < \varepsilon_i \ll 1$ if $E$ satisfies the weak DL-condition.

For the latter case, a general element in $\mathcal{P}_H(v)$ fits in an exact sequence

$$0 \to \pi^* V \oplus N \to E \to \bigoplus_{i=1}^m \mathcal{O}_{E_i}(-1)^d_i \to 0.$$ 

If $E$ is not $\mu_A$-semistable, then every destabilizing sheaf $E'$ is of the form

$$0 \to \pi^* V \oplus N' \to E' \to \bigoplus_{i=1}^m \mathcal{O}_{E_i}(-1)^{d'_i} \to 0,$$

where $N > N'$, $d'_i \geq d_i$ for every $i$, and for at least one $i$, $d'_i > d_i$.

Now let $v = v(E)$ be a character with $E$ a general sheaf fitting in an exact sequence

$$0 \to \pi^* V \oplus N \to E \to \bigoplus_{i=1}^m \mathcal{O}_{E_i}(-1)^d_i \to 0.$$ 

It follows from Theorem 8.3 that $E$ is $\mu_A$-stable if and only if there exists an integer $K \geq 2$ and

1. $N_1, \ldots, N_k \geq 1$ such that $\sum N_i = N$, 
2. $d'_i \geq 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$ such that $\sum_j d'_i = d_i$, $d'_i \geq d_{i+1}^j$, and for every $j$ at least one strict inequality $d'_i > d_{i+1}^j$ holds,

satisfying the following conditions

a) $M_A(v_i)$ is non-empty for any $i$, where $v_i$ is the character of a sheaf given by an extension

$$0 \to \pi^* V \oplus N_i \to E_i \to \bigoplus_{j=1}^m \mathcal{O}_{E_j}(-1)^{d_j} \to 0,$$

and

b) $\chi(v_i, v_j) = 0$ for any $i < j$. 

The condition b) comes down to saying that

\[ \frac{1}{r_0^2} = \sum_{i=1}^{m} \frac{d_i^2}{N_j r_0} - \left( \frac{d_i}{N_j r_0} - \frac{d_j}{N_j r_0} \right)^2 \]

for any \( k < j \), where \( r_0 \) is the rank of \( V \).

7 Stable characters on general blow-ups of \( \mathbb{P}^2 \)

In this section, we will apply some results in deformation theory to illustrate that the weak Brill-Noether property and the non-emptiness of the moduli space of stable sheaves proven before actually hold on a blow-up of \( \mathbb{P}^2 \) along \( m \) general points.

Let \( X_0 \) be a smooth projective surface, and let \( \mathcal{E}_0 \) be a coherent sheaf on \( X_0 \). Let \( X \) be a deformation of \( X_0 \) over a local Artin ring \( C \). By a deformation of \( \mathcal{E}_0 \) over \( X \) we mean a coherent sheaf \( \mathcal{E} \) on \( X \), flat over \( C \), together with a map \( \mathcal{E} \rightarrow \mathcal{E}_0 \) such that the induced map

\[ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} \rightarrow \mathcal{E}_0 \]

is an isomorphism. We know that if \( C \) is the ring of dual numbers \( D \), and \( X = X_0 \times_C D \) is the trivial deformation of \( X_0 \), then such deformations \( \mathcal{E} \) always exist, and they are classified by \( \text{Ext}_{X_0}^1(\mathcal{E}_0, \mathcal{E}_0) \). Now we consider the more general situation over a sequence

\[ 0 \rightarrow J \rightarrow C' \rightarrow C \rightarrow 0, \]

where \( C \) is a local Artin ring with the residue field \( C \), \( C' \) is another local Artin ring mapping to \( C \), and \( J \) is an ideal with \( m_C \cdot J = 0 \) so that \( J \) can be considered as a \( C \)-vector space. Suppose we are given \( X_0, \mathcal{E}_0, X, \mathcal{E} \) as above, and further suppose we are given an extension \( X' \) of \( X \) over \( C' \). We ask for an extension \( \mathcal{E}' \) of \( \mathcal{E} \) over \( C' \), that is, a coherent sheaf \( \mathcal{E}' \) on \( X' \), flat over \( C' \), together with a map \( \mathcal{E}' \rightarrow \mathcal{E} \) inducing an isomorphism \( \mathcal{E}' \otimes_{C'} C \rightarrow \mathcal{E} \). We only need to treat the case of a vector bundle since a general sheaf in \( \mathcal{P}_H(v) \) is locally free, in which case \( \mathcal{E} \) and \( \mathcal{E}' \) will also be locally free.

**Theorem 7.1.** ([Har10]) In the situation as above, assume that \( \mathcal{E}_0 \) is locally free. Then:

1. If an extension \( \mathcal{E}' \) of \( \mathcal{E} \) over \( X' \) exists, then \( \text{Aut}(\mathcal{E}'/\mathcal{E}) = J \otimes_C \text{Hom}_{X_0}(\mathcal{E}_0, \mathcal{E}_0) \).
2. Given \( \mathcal{E} \), there is an obstruction in \( J \otimes_C \text{Ext}_{X_0}^1(\mathcal{E}_0, \mathcal{E}_0) \) to the existence of \( \mathcal{E}' \).
3. If an \( \mathcal{E}' \) exists, then the set of all such is a torsor under the action of \( J \otimes_C \text{Ext}_{X_0}^1(\mathcal{E}_0, \mathcal{E}_0) \).

**Corollary 7.2.** Let \( X_0 \) be the blow-up of \( m \) collinear points and \( \mathcal{E}_0 \) a general member in a prioritary stack \( \mathcal{P}_H(v) \), which is \( \mu_A \)-stable for some \( A = H - \sum \varepsilon_i E_i \). Then

1. an extension \( \mathcal{E}' \) always exists, and
2. \( \text{Aut}(\mathcal{E}'/\mathcal{E}) = J \).

Now let \( \mathcal{H} \) be the Hilbert scheme of \( m \) points on \( \mathbb{P}^2 \), and \( \mathcal{U} \subseteq \mathcal{H} \times \mathbb{P}^2 \) the universal family. Let \( \mathcal{X} \rightarrow \mathcal{H} \) the blow-up of \( \mathcal{H} \times \mathbb{P}^2 \) along \( \mathcal{U} \). This is the family parameterizing blow-ups of \( \mathbb{P}^2 \) along \( m \) points. Let \( h_0 \in \mathcal{H} \) be the point corresponding to a blow-up of \( \mathbb{P}^2 \) along \( m \) distinct collinear points. Let \( \mathcal{H}' \) be the open subscheme of \( \mathcal{H} \) parameterizing distinct points of \( \mathbb{P}^2 \). Then for \( 0 < \varepsilon_i \ll 1 \), the divisor \( A = H - \sum \varepsilon_i E_i \) is ample on the surface \( \mathcal{X}_h \) for any \( h \in \mathcal{H}' \). Given a \( \mu_A \)-stable sheaf \( \mathcal{E}_{h_0} \) on \( \mathcal{X}_{h_0} \), by Corollary 9.2, one can always deform \( \mathcal{E}_{h_0} \) to the nearby surfaces,
Theorem 7.3. Let $v = (r, \nu, \Delta)$ be a character such that $\nu(v) = \alpha H - \sum \beta_i E_i$ with $-1 \leq \beta_i \leq 0$ and $\Delta(v) \geq 0$. Suppose that $\alpha \notin \mathcal{E}$, where $\mathcal{E}$ is the set of $H$-slopes on $\mathbb{P}^2$ of exceptional bundles. If $v$ satisfies the weak DL-condition, then $M_{X_h, A}(v) \neq \emptyset$ for general $0 < \varepsilon_i \ll 1$ and general $h \in \mathcal{H}'$, where $A = H - \sum \varepsilon_i E_i$.

Theorem 7.4. Let $v = (r, \nu, \Delta)$ be a character such that $\nu(v) = \alpha H - \sum \beta_i E_i$ with $-1 \leq \beta_i \leq 0$, $\alpha - \sum \beta_i \geq -1$, and $\Delta(v) \geq 0$. Let $A = H - \sum \varepsilon_i E_i$ be a polarization on any surfaces $X_h$, $h \in \mathcal{H}'$. If $M_{X_h, A}(v)$ is non-empty, then $v$ is a character on $X_h$ satisfying weak Brill-Noether property for general $h \in \mathcal{H}'$.

8 Birational geometry of Hilbert schemes of $X_m$

It is interesting to figure out the birational properties of moduli spaces. In [CH18a], Coskun and Huizenga reveal the relation between the strong Bogomolov inequalities and the nef cone of the moduli space of sheaves on a surface. The ample cone of Hilbert schemes of points is described in [BC13] and [BHL+16]. In this section, we will study the ample cone of the Hilbert schemes of points on the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points.

Let $X = X_m$ be the blow-up of $\mathbb{P}^2$ along $m$ collinear points. Let $X[n]$ denote the Hilbert scheme parameterizing zero-dimensional schemes of length $n$. Let $X^{(n)} = X^n/\mathfrak{S}_n$ denote the n-th symmetric product of $X$. There is a natural morphism $h : X^{[n]} \to X^{(n)}$, called the Hilbert-Chow morphism, that maps a zero dimensional scheme $Z$ of length $n$ to its support weighted by multiplicity.

In [Fog73], Fogarty determined the Picard group of $X^{[n]}$ in terms of the Picard group of $X$. A line bundle $\mathcal{L}$ on $X$ naturally determines a line bundle $\mathcal{L}^{[n]}$ as follows: $\mathcal{L}$ gives rise to a line bundle $\mathcal{E} \otimes \cdots \otimes \mathcal{L}$ on $X^n$, which is invariant under the action of the symmetric group $\mathfrak{S}_n$, therefore $\mathcal{E} \otimes \cdots \otimes \mathcal{L}$ descends to a line bundle $\mathcal{L}_{X^{(n)}}$ on $X^{(n)}$: the pull-back $\mathcal{L}^{[n]} := h^*\mathcal{L}_{X^{(n)}}$ gives the desired line bundle on $X^{[n]}$.

Let $B$ be the class of the exceptional divisor of the Hilbert-Chow morphism, which parameterizes non-reduced schemes in $X^{[n]}$. Fogarty proves in [Fog73] that if the irregularity $q(X) = 0$, then

$$\text{Pic}(X^{[n]}) \simeq \text{Pic}(X) \oplus \mathbb{Z} \frac{B}{2}$$

As a consequence, the Néron-Severi space $N^1(X^{[n]})$ is spanned by $N^1(X)$ and $B$.

Let $\mathcal{L}$ be an ample line bundle on $X$. Then $\mathcal{L}^{[n]}$ is an ample line bundle on $X^n$ invariant under the $\mathfrak{S}_n$-action. As a result, $\mathcal{L}^{[n]}$ descends to an ample bundle $\mathcal{L}^{(n)}$ on $X^{(n)}$. Since the Hilbert-Chow morphism is birational, then the induced line bundle $\mathcal{L}^{[n]}$ is big and nef on $X^{[n]}$. However, since $\mathcal{L}^{[n]}$ has degree zero on the fibres of the Hilbert-Chow morphism, then it is not ample and hence it lies on the boundary of the nef cone of $X^{[n]}$.

Given a line bundle $\mathcal{L}$ on $X$, we consider the short exact sequence

$$0 \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}|_Z \to 0,$$

which induces an inclusion $H^0(X, \mathcal{L} \otimes \mathcal{I}_Z) \subseteq H^0(X, \mathcal{L})$. Suppose that $N > n$, then the inclusion induces a rational map

$$\psi_{\mathcal{L}} : X^{[n]} \dashrightarrow \text{Gr}(N - n, N).$$
Denote by $D_L(n) := \psi_L^*O_{Gr(N-n,N)}(1)$ the pull-back of $O_{Gr(N-n,N)}(1)$. By the Grothendieck-Riemann-Roch Theorem, one can show that the class of $D_L(n)$ is

$$D_L(n) = L[n] - \frac{B}{2}. $$

As $O_{Gr(N-n,N)}(1)$ is very ample, then the base locus of $D_L(n)$ is contained in the indeterminacy locus of $\psi_L$. If $\psi_L$ is a morphism, then $D_L(n)$ is base point free and in particular nef.

Definition 8.1. A line bundle $\mathcal{L}$ on $X$ is called $k$-very ample if the restriction map

$$H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L}|_Z)$$

is surjective for every zero dimensional scheme $Z$ of length at most $k + 1$.

Let $\mathcal{L}$ be an $(n-1)$-very ample line bundle on a surface $X$ and assume that $h^0(X, \mathcal{L}) = N$ and $h^i(X, \mathcal{L}) = 0$ for $i > 0$. Then $H^i(X, \mathcal{L} \otimes I_Z) = 0$ for any $i > 0$ and any $Z \in X^n$. Let

$$\Xi_n \subseteq X^n \times X$$

be the universal family and let $\pi_1$ and $\pi_2$ be the natural projections. By cohomology and base change, the $\pi_1^*(\pi_2^*L \otimes I_{\Xi_n})$ is a vector bundle of rank $N-n$ on $X^n$. By the universal property of the Grassmannian, the map $\psi_L : X^n \rightarrow Gr(N-n,N)$ is a morphism. It follows that $D_L(n) = L[n] - \frac{B}{2}$ is base point free.

Lemma 8.2. ([BS88]) Let $\mathcal{L}_i$ be a $k_i$-ample line bundle on a projective smooth surface $X$, where $i = 1, \ldots, n$. Then $\mathcal{L}_1 \otimes \cdots \otimes L_n$ is $(k_1 + \cdots + k_n)$-ample.

Proposition 8.3. Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points. Then the divisor $aH + \sum b_i(H - E_i)$ with $a, b_1, \ldots, b_m \geq 1$ is very ample. In particular, the divisor $(n-1)(aH + \sum b_i(H - E_i))$ with $a, b_1, \ldots, b_m \geq 1$ is $(n-1)$-very ample.

Proof. This is immediate from the criterion that a divisor is very ample if and only if it separates points and tangent directions ([Har77]).

Theorem 8.4. Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ distinct collinear points. Then the nef cone of $X^n$ is the cone $\alpha H[n] - \sum \beta_i E_i[n] + \gamma \frac{B}{2}$ satisfying the inequalities

$$\gamma \leq 0, \quad \beta_i + (n-1)\gamma \geq 0, \quad \text{and} \quad \alpha + (n-1)\gamma \geq \sum \beta_i. $$

Proof. Let $R$ be a general fibre of the Hilbert-Chow morphism over the singular locus of $X^{(n)}$. Then the curve $R$ has the intersection number $(R.B) = -2$ and $(R.L[n]) = 0$ for any line bundle $\mathcal{L}$ on $X$. As a consequence, the coefficient of $B$ in any nef line bundle on $X^{(n)}$ is non-positive.

Let $C$ be a curve in $X$ that admits a $g^1_n$. The morphism $f : C \rightarrow \mathbb{P}^1$ defined by the $g^1_n$ induces a curve $C(n)$ in $X^n$. Now let $L$ be the special line $H - E_1 - \cdots - E_m$ on $X$, and then the induced curve $L(n)$ satisfies the intersection numbers

$$(L(n).H[n]) = 1, (L(n).E_i[n]) = 1, (L(n), B/2) = n - 1. $$

It follows that

$$\alpha + (n-1)\gamma \geq \sum \beta_i. $$

Similarly, intersecting with $E_i(n)$, one obtains that

$$\beta_i + (n-1)\gamma \geq 0. $$
On the other hand, let $D = \alpha H^n - \sum \beta_i E_i^n + \gamma \frac{B}{2}$ be a divisor satisfying the inequalities. Then we may write

$$D = \left( \alpha - \sum \beta_i + (n-1)\gamma \right) H^n - \gamma \left( (n-1)(H + \sum (H - E_i))[n] - \frac{B}{2} \right) + \sum (\beta_i - (n-1))(H - E_i)[n]$$

as the sum of three nef divisors. Thus we conclude that the nef cone of $X^n$ is given by the above inequalities.

\[\square\]

Corollary 8.5. The Hilbert scheme $X^n$ is log Fano.

Proof. Choose a boundary divisor $\Delta = (1-\delta)L^n + \varepsilon \frac{B}{2}$ such that $0 < \delta \ll 1$ and $0 < (n-1)\varepsilon < \delta$. Then the divisor

$$-K_{X^n} - \Delta = -K_X^n - \Delta = (2 + \delta)H^n - \sum \delta E_i^n - \varepsilon \frac{B}{2}$$

is ample. As $X^n$ is smooth, then $(X^n, \Delta)$ is a klt pair, and hence log Fano. \[\square\]

Question 8.6. Let $X$ be the blow-up of $\mathbb{P}^2$ along $m$ collinear points, and $A = H - \sum \varepsilon_i E_i$ a polarization. Is the moduli space $M_A(v)$ a log Fano variety?

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