A SHORT EXACT SEQUENCE

I. Panin

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Let \( R \) be a semilocal integral Dedekind domain and \( K \) be its fraction field. Let \( \mu : G \to T \) be an \( R \)-group scheme morphism between reductive \( R \)-group schemes that is smooth as a scheme morphism. Assume that \( T \) is an \( R \)-torus. Then the map \( T(R)/\mu(G(R)) \to T(K)/\mu(G(K)) \) is injective, and a certain purity theorem is true. These and other results are derived from an extended form of the Grothendieck–Serre conjecture proven in the present paper for rings \( R \) as above. Bibliography: 21 titles.

1. Main results

Let \( R \) be a commutative unital ring. Recall that an \( R \)-group scheme \( G \) is said to be reductive (respectively, semisimple or simple) if it is affine and smooth as an \( R \)-scheme and, moreover, for every algebraically closed field \( \Omega \) and for every ring homomorphism \( R \to \Omega \), the scalar extension \( G_\Omega \) is a connected reductive (respectively, semisimple or simple) algebraic group over \( \Omega \). The class of reductive group schemes contains the class of semisimple group schemes, which, in turn, contains the class of simple group schemes. This notion of a reductive \( R \)-group scheme coincides with [4, Exp. XIX, Definition 2.7], and this notion of a simple \( R \)-group scheme coincides with the notion of a simple semisimple \( R \)-group scheme from [4, Exp. XIX, Definition 2.7 and Exp. XXIV, 5.3]. Here is our first main result.

**Theorem 1.1.** Let \( R \) be a semilocal integral Dedekind domain. Let \( K \) be the fraction field of \( R \). Let \( \mu : G \to T \) be an \( R \)-group scheme morphism between reductive \( R \)-group schemes that is smooth as a scheme morphism. Assume that \( T \) is an \( R \)-torus. Then the map \( T(R)/\mu(G(R)) \to T(K)/\mu(G(K)) \) is injective, and the sequence

\[
\{1\} \to T(R)/\mu(G(R)) \to T(K)/\mu(G(K)) \to \bigoplus_p \frac{T(K)}{[T(R_p) \cdot \mu(G(K))]} \to \{1\}
\]

is exact, where \( p \) runs over all nonzero prime ideals of \( R \) and \( r_p \) is the natural map (the projection to the quotient group).

Let us comment on the first assertion of the theorem. Let \( H \) be the kernel of \( \mu \). It turns out that \( H \) is a quasi-reductive \( R \)-group scheme (see Definition 1.3). There is a sequence of group sheaves \( 1 \to H \to G \to T \to 1 \), which is exact in the étale topology on \( \text{Spec} R \). Theorem 1.4 yields now the injectivity of the map \( T(R)/\mu(G(R)) \to T(K)/\mu(G(K)) \).

**Theorem 1.2.** Let \( R \) be a semilocal integral Dedekind domain. Let \( K \) be the fraction field of \( R \). Let \( G_1 \) and \( G_2 \) be two semisimple \( R \)-group schemes. Assume that the generic fibres \( G_{1,K} \) and \( G_{2,K} \) are isomorphic as algebraic \( K \)-groups. Then the \( R \)-group schemes \( G_1 \) and \( G_2 \) are isomorphic.

This theorem cannot be directly derived from [12] and [13]. Indeed, only geometrically connected group schemes are regarded there. However, to prove Theorem 1.2, we need to work with the automorphism group scheme of a semisimple \( R \)-group scheme. The latter group scheme is not geometrically connected in general.

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*St. Petersburg Department of Steklov Institute of Mathematics, St. Petersburg, Russia, e-mail: panin@pdmi.ras.ru.*

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Below, we state a theorem which asserts that an extended version of the Grothendieck–Serre conjecture holds for rings $R$ as above. It is proved in this paper. Theorem 1.2 and the first assertion of Theorem 1.1 are derived from it. To state the above-mentioned theorem, it is convenient to give the following definition.

**Definition 1.3** (quasi-reductivity). Assume that $S$ is a Noetherian commutative ring. An $S$-group scheme $H$ is said to be quasi-reductive if there is a finite étale $S$-group scheme $C$ and a smooth $S$-group scheme morphism $\lambda : H \to C$ such that its kernel is a reductive $S$-group scheme and $\lambda$ is surjective locally in the étale topology on $S$.

Clearly, reductive $S$-group schemes are quasi-reductive. Quasi-reductive $S$-group schemes are affine and smooth as $S$-schemes. There are two types of quasi-reductive $S$-group schemes that we are focusing on in the present paper. The first one is the automorphism group scheme of a semisimple $S$-group scheme. The second one is obtained as follows: take a reductive $S$-group scheme $G$, an $S$-torus $T$, and a smooth $S$-group morphism $\mu : G \to T$. Then one can check that the kernel $H$ of $\mu$ is quasi-reductive. It is an extension of a finite étale $S$-group scheme $C$ of multiplicative type via a reductive $S$-group scheme $G_0$.

Assume that $U$ is a regular scheme, $H$ is a quasi-reductive $U$-group scheme. Recall that a $U$-scheme with an action of $H$ is called a principal $H$-bundle over $U$ if $H$ is faithfully flat and quasi-compact over $U$ and the action is simple transitive, that is, the natural morphism $H \times_U H \to H \times_U H$ is an isomorphism, see [9, Sec. 6]. Since $H$ is $S$-smooth, such a bundle is trivial locally in the étale topology but, in general, not in the Zariski topology. Grothendieck and Serre conjectured that for a reductive $U$-group scheme $H$, a principal $H$-bundle $H$ over $U$ is trivial locally in the Zariski topology if it is trivial generically. A survey paper on the topic is [15].

The conjecture is true if $\Gamma(U, O_U)$ contains a field (see [7] and [18]). It is proved in [12] that the conjecture is true in general for discrete valuation rings. This result is extended in [19] to the case of semilocal Dedekind integral domains assuming that $G$ is simple, simply connected, and isotropic in a certain precise sense. In [13], the results of [12] and [19] are further extended. It is proved there that the conjecture is true in general for the case of semilocal Dedekind integral domains. The following result is a further extension of the main theorem of [13].

**Theorem 1.4.** Let $R$ be a semilocal integral Dedekind domain. Let $K$ be the fraction field of $R$. Let $H$ be a quasi-reductive group scheme over $R$. Then the map

$$H^1_{\text{ét}}(R, H) \to H^1_{\text{ét}}(K, H),$$

induced by the inclusion of $R$ into $K$, has a trivial kernel. In other words, under the above assumptions on $R$ and $G$, every principal $H$-bundle over $R$ having a $K$-rational point is trivial.

**Corollary 1.5.** Under the assumptions of Theorem 1.4, the map

$$H^1_{\text{ét}}(R, H) \to H^1_{\text{ét}}(K, H),$$

induced by the inclusion of $R$ into $K$, is injective. Equivalently, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two principal $H$-bundles isomorphic over $\text{Spec}K$, then they are isomorphic.

**Proof.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two principal $H$-bundles isomorphic over $\text{Spec}K$. Let $\text{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ be the scheme of isomorphisms of principal $H$-bundles. This scheme is a principal $\text{Aut}\mathcal{H}_1$-bundle. By Theorem 1.4, it is trivial, and we see that $\mathcal{H}_1 \cong \mathcal{H}_2$.

Theorems 1.4 and 1.2 are proved in Sec. 2. Theorem 1.1 is proved in Sec. 4.
2. Proof of Theorems 1.4 and 1.2

We begin with the following general lemma.

**Lemma 2.1.** Let $X$ be a semilocal irreducible Dedekind scheme. Let $\pi : X' \to X$ be a finite étale morphism. Let $\eta \in X$ be a generic point of $X$. Then the sections of $\pi$ over $X$ are in a bijection with the sections of $\pi$ over $\eta$.

**Proof.** Clearly, $P^n(X) = P^n(\eta)$. Since $\pi$ is finite, it is projective. Hence, $X'(X) = X'(\eta)$. □

**Corollary 2.2.** Let $X, \eta \in X$ be as in the previous lemma and $E$ be a finite étale group $X$-scheme. Then the $\eta$-points of $E$ coincide with the $X$-points of $E$.

**Corollary 2.3.** Under the assumptions of Corollary 2.2, the kernel of the pointed set map $H^1_{\text{et}}(X, E) \to H^1_{\text{et}}(\eta, E)$ is trivial.

**Proof.** Let $E$ be a principal $E$-bundle over $X$. The standard descent argument shows that the $X$-scheme $E$ is finite and étale. Thus, $E(X) = E(\eta)$. This proves the corollary. □

**Proof of Theorem 1.4.** Since $H$ is a quasi-reductive $R$-group scheme, there is a finite étale $R$-group scheme $C$ and a smooth $R$-group scheme morphism $\lambda : H \to C$ such that its kernel $G$ is a reductive $R$-group scheme and $\lambda$ is surjective locally in the étale topology on $S$. The sequence of étale sheaves $1 \to G \to H \to C \to 1$ is exact. Thus, it induces a commutative diagram of pointed set maps with exact rows

\[
\begin{array}{cccc}
C(R) & \xrightarrow{\partial} & H^1_{\text{et}}(R, G) & \xrightarrow{\alpha} & H^1_{\text{et}}(R, C) \\
\downarrow & & \downarrow & & \downarrow \\
C(K) & \xrightarrow{\partial} & H^1_{\text{et}}(K, G) & \xrightarrow{\gamma} & H^1_{\text{et}}(K, C)
\end{array}
\]

The map $\alpha$ is bijective by Corollary 2.2, the map $\delta$ has trivial kernel by Corollary 2.3, the map $\beta$ is injective by Corollary 1.5. Now a simple diagram chase shows that $\ker(\gamma) = \ast$. This proves the theorem. □

**Remark 2.4.** Both the statement of [1, Lemma 3.7] and its proof are inaccurate. The authors are forced to assume the injectivity of the map $H^1_{\text{et}}(R, G^0) \to H^1_{\text{et}}(K, G^0_R)$.

**Proof of Theorem 1.2.** The $R$-group scheme $\text{Aut} := \text{Aut}_{R-gr-sch}(G_1)$ is quasi-reductive by [5]. The $R$-scheme $\text{Iso} := \text{Iso}_{R-gr-sch}(G_1, G_2)$ is a principal $\text{Aut}$-bundle. An isomorphism $\phi : G_{1,K} \to G_{2,K}$ of algebraic $K$-groups gives a section of $\text{Iso}$ over $K$. So, $\text{Iso}_K$ is a trivial principal $\text{Aut}_K$-bundle. Hence, $\text{Iso}$ is a trivial principal $\text{Aut}$-bundle by Theorem 1.4. Thus, it has a section over $R$. So, there is an $R$-group scheme isomorphism $G_1 \cong G_2$. □

3. One Lemma

**Lemma 3.1.** Let $X$ be a regular irreducible affine scheme. Let $G$ be a reductive $X$-group scheme and $T$ be an $X$-torus. Let $\mu : G \to T$ be an $X$-group scheme morphism that is smooth as a scheme morphism. Then the kernel of $\mu$ is a quasi-reductive $X$-group scheme.

**Proof.** Consider the coradical $\text{Corad}(G)$ of $G$ together with the canonical $X$-group morphism $\alpha : G \to \text{Corad}(G)$. By the universal property of the $X$-group morphism $\alpha$, there is a unique $X$-group morphism $\tilde{\mu} : \text{Corad}(G) \to T$ such that $\mu = \tilde{\mu} \circ \alpha$. Since $\mu$ is surjective locally for the étale topology, so is $\tilde{\mu}$. Let $\ker(\tilde{\mu})$ be the kernel of $\tilde{\mu}$, and let $H := \alpha^{-1}(\ker(\tilde{\mu}))$ be the scheme-theoretic preimage of $\ker(\tilde{\mu})$. Clearly, $H$ is a closed $X$-subgroup scheme of $G$ which is the kernel of $\mu$. We must check that $H$ is quasi-reductive.
The $X$-group scheme $\ker(\mu)$ is of multiplicative type. Hence, there is a finite $X$-group scheme $M$ of multiplicative type and a faithfully flat $X$-group scheme morphism $\can : \ker(\mu) \to M$ that has the following property: for any finite $X$-group scheme $M'$ of multiplicative type and an $X$-group morphism $\phi : \ker(\mu) \to M'$ there is a unique $X$-group morphism $\psi : M \to M'$ with $\psi \circ \can = \phi$. It is known that the kernel of $\can$ is an $X$-torus; denote it by $T^0$. Since $\mu$ is smooth, so is $\mu$. Thus, the $X$-group scheme $\ker(\mu)$ is an $X$-smooth scheme. It follows that $M$ is étale over $X$.

Let $\beta = \alpha|_H : H \to \ker(\mu)$, and let $G^0 := \beta^{-1}(T^0)$ be the scheme-theoretic preimage of $T^0$. Clearly, $G^0$ is a closed $X$-subgroup scheme of $H$ which is the kernel of the morphism $\can \circ \beta : H \to M$. Let $\gamma = \beta|_{G^0} : G^0 \to T^0$.

The $X$-group scheme $M$ is finite and étale. The morphism $\can$ is smooth. The morphism $\beta$ is smooth as a base change of the smooth morphism $\alpha$. Thus, $\lambda := \can \circ \beta$ is smooth. It is also surjective locally in the étale topology on $X$, because $\can$ and $\beta$ have this property.

By construction, $G^0 = \ker(\lambda)$. So, to prove that $H$ is quasi-reductive, it remains to check the reductivity of $G^0$.

The $X$-group scheme $G^0$ is affine as a closed $X$-subgroup scheme of the reductive $X$-group scheme $G$. We prove now that $G^0$ is smooth over $X$. Indeed, the morphism $\gamma$ is smooth as a base change of the smooth morphism $\alpha$. The $X$-scheme $T^0$ is smooth, since it is an $X$-torus. Thus, the $X$-scheme $G^0$ is smooth.

Write $X$ as $\text{Spec}S$ for a regular integral domain $S$. It remains to verify that for every algebraically closed field $\Omega$ and for every ring homomorphism $S \to \Omega$, the scalar extension $G^0_\Omega$ is a connected reductive algebraic group over $\Omega$. First, recall that $\ker(\alpha)$ is a semisimple $S$-group scheme. It is the $S$-group scheme $G^{ss}$ in the notation of [5]. Clearly, $\ker(\gamma) = \ker(\alpha)$. Thus, $\ker(\gamma) = G^{ss}$ is semisimple $S$-group scheme. Since the morphism $\gamma$ is smooth for every algebraically closed field $\Omega$ and for every ring homomorphism $S \to \Omega$, we have an exact sequence of smooth algebraic groups over $\Omega$:

$$1 \to G^{ss}_\Omega \to G^0_\Omega \to T^0_\Omega \to 1.$$ The groups $T^0_\Omega$, $G^{ss}_\Omega$ are connected. Hence, the group $G^0_\Omega$ is connected too. We know already that it is affine.

Finally, we check that the unipotent radical $U$ of $G^0_\Omega$ is trivial. Since there is no nontrivial $\Omega$-group morphisms $U \to T^0_\Omega$, we conclude that $U \subset G^{ss}_\Omega$. Since $G^{ss}_\Omega$ is semisimple, one has $U = \{1\}$. This completes the proof of the reductivity of the $R$-group scheme $G^0$. Thus, the $R$-group scheme $H$ is quasi-reductive. This proves the lemma.  

\[ \square \]

4. Proof of Theorem 1.1

Proof of the first assertion of Theorem 1.1. Let $H$ be the kernel of $\mu$. Since $\mu$ is smooth, the sequence of group schemes

$$1 \to H \to G \to T \to 1$$

gives rise to a short exact sequence of group sheaves in the étale topology. In turn, this sequence of sheaves induces a long exact sequence of pointed sets. So, the boundary map $\partial : T(R) \to H^1_\text{ét}(R, H)$ fits in the commutative diagram

$$\begin{array}{ccc}
T(R)/\mu(G(R)) & \longrightarrow & H^1_\text{ét}(R, H) \\
\downarrow & & \downarrow \\
T(K)/\mu(G(K)) & \longrightarrow & H^1_\text{ét}(K, H).
\end{array}$$

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Clearly, the horizontal arrows have trivial kernels. The right vertical arrow has trivial kernel by Lemma 3.1 and Theorem 1.4. Thus, the left vertical arrow has trivial kernel too. Since it is a group homomorphism, it is injective.

To prove that the sequence (1) is exact in the middle term, we need some preparations. First, consider the covariant functor $\mathcal{F}$ on the category of commutative $R$-algebras that takes an $R$-algebra $S$ to $\mathcal{F}(S) := \mathbb{T}(S)/\mu(\mathbb{G}(S))$. The following result holds. Its proof repeats word for word the proof of [14, Lemma 4.0.9].

**Lemma 4.1.** Under the notation and assumptions of Theorem 1.1, set $\mathbb{H} = \ker(\mu)$. Then the boundary map $\partial : \mathbb{T}(K)/\mu(\mathbb{G}(K)) \rightarrow H^1_{\text{ét}}(K, \mathbb{H}_K)$ is injective.

Consider the following group and the following pointed set:

\[
\mathcal{F}_{\text{nr}, R}(K) = \bigcap_p \text{Im}[\mathcal{F}(R_p) \rightarrow \mathcal{F}(K)],
\]

\[
H^1_{\text{ét}}(K, \mathbb{H}_K)_{\text{nr}, R} = \bigcap_p \text{Im}[H^1_{\text{ét}}(R_p, \mathbb{H}) \rightarrow H^1_{\text{ét}}(K, \mathbb{H}_K)],
\]

where $p$ runs over all nonzero prime ideals of $R$. Clearly, one has the following inclusion: $\partial_K(\mathcal{F}_{\text{nr}, R}(K)) \subseteq H^1_{\text{ét}}(K, \mathbb{H}_K)_{\text{nr}, R}$. Consider now a commutative diagram of the form

\[
\begin{array}{ccc}
1 & \longrightarrow & \mathcal{F}(R) & \stackrel{\partial}{\longrightarrow} & H^1_{\text{ét}}(R, \mathbb{H}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbb{G}) \\
& & \downarrow_{\epsilon} & & \rho & & \eta \\
1 & \longrightarrow & \mathcal{F}(K) & \stackrel{\partial_K}{\longrightarrow} & H^1_{\text{ét}}(K, \mathbb{H}_K) & \longrightarrow & H^1_{\text{ét}}(K, \mathbb{G}_K) \\
& & a & \longrightarrow & \xi & \longrightarrow & *
\end{array}
\]

in which all the maps are canonical and the horizontal lines are exact sequences of pointed sets. The map $\eta$ has trivial kernel by the main result of [13], since $\mathbb{G}$ is reductive. The map $\partial_K$ is injective by Lemma 4.1. Using Zariski’s patching of principal bundles on Spec($R$), we conclude that the image of $\rho$ coincides with $H^1_{\text{ét}}(K, \mathbb{H}_K)_{\text{nr}, R}$.

**Proof of the exactness of the sequence (1) in the middle term.** We must prove the following equality: $\text{Im}[\mathcal{F}(R) \rightarrow \mathcal{F}(K)] = \mathcal{F}_{\text{nr}, R}(K)$. Obviously, $\text{Im}[\mathcal{F}(R) \rightarrow \mathcal{F}(K)] \subseteq \mathcal{F}_{\text{nr}, R}(K)$. It remains to check the opposite inclusion. Take an element $a \in \mathcal{F}_{\text{nr}, R}(K)$ and set $\xi = \partial_K(a)$. As mentioned above, $\xi$ is in $H^1_{\text{ét}}(K, \mathbb{H}_K)_{\text{nr}, R}$. We already know that $\xi$ can be lifted to an element $\tilde{\xi}$ in $H^1_{\text{ét}}(R, \mathbb{H})$. Let $\tilde{\zeta}$ be the image of $\tilde{\xi}$ in $H^1_{\text{ét}}(R, \mathbb{G})$. Note that $\eta(\tilde{\zeta}) = *$. Since the kernel of $\eta$ is trivial, we see that $\tilde{\zeta} = *$. Hence, there is an element $\tilde{a}$ in $\mathcal{F}(R)$ such that $\partial(\tilde{a}) = \tilde{\xi}$. The injectivity of $\partial_K$ yields the equality $\epsilon(\tilde{a}) = a$. The exactness of the sequence (1) in the middle term is proved. $\Box$

In the rest of the proof, we establish the surjectivity of the map $\sum p$. Clearly, it is sufficient to prove the surjectivity of the map

\[
\mathbb{T}(K) \stackrel{\sum p}{\longrightarrow} \bigoplus_p \mathbb{T}(K)/\mathbb{T}(R_p),
\]

where $p$ runs over all nonzero prime ideals of $R$ and $p'$ is the quotient map. The rest of the proof will be given in scheme-theoretic notation. Namely, set $X = \text{Spec}R$, $\mathcal{O} = \Gamma(X, \mathcal{O}_X)$. 423
Thus, $\mathcal{O} = R$. For each closed point $x$ in $X$, write $\mathcal{O}_x$ for $\mathcal{O}_{X,x}$ (the local ring of the point $x$ on the scheme $X$).

Consider a finite étale Galois morphism $\pi : \tilde{X} \to X$ such that the torus $T$ splits over $\tilde{X}$ and $\tilde{X}$ is irreducible. Set $\mathcal{O} = \Gamma(X, \mathcal{O}_{\tilde{X}})$, and let $\tilde{K}$ be the fraction field of the ring $\mathcal{O}$. For each closed point $x \in X$, consider a ring $\mathcal{O}_x$ that is the semilocal ring $\mathcal{O}_{\tilde{X}, \tilde{x}}$ of the finite closed set $\tilde{x} = \pi^{-1}(x)$ in $\tilde{X}$. Let $\text{Gal} := \text{Aut}(\tilde{X}/X)$ be the Galois group of $\tilde{X}/X$.

Since the torus $T$ splits over $\tilde{X}$, we have a short exact sequence of Gal-modules

$$\{1\} \to \mathbf{T}(\tilde{\mathcal{O}}) \to \mathbf{T}(\tilde{K}) \to \bigoplus_x \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x) \to \{1\},$$

where $x$ runs over the set of all closed points of the scheme $X$. This short exact sequence of Gal-modules gives rise to a long exact sequence of Gal-cohomology groups of the form

$$\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{\text{in}} \mathbf{T}(K) \to \bigoplus_x \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)^{\text{Gal}} \to H^1(\text{Gal}, \mathbf{T}(\tilde{\mathcal{O}})) \xrightarrow{H^1(\text{in})} H^1(\text{Gal}, \mathbf{T}(\tilde{K})).$$

We claim that the map $H^1(\text{in})$ is a monomorphism. Indeed, the group $H^1(\text{Gal}, \mathbf{T}(\tilde{\mathcal{O}}))$ is a subgroup of the group $H^1_{\text{ét}}(X, \mathbf{T})$, and the group $H^1(\text{Gal}, \mathbf{T}(\tilde{K}))$ is a subgroup of the group $H^1_{\text{ét}}(\text{Spec} \mathcal{O}, \mathbf{T}_{\mathcal{O}})$. By Theorem 1.4, the group map $H^1_{\text{ét}}(X, \mathbf{T}) \to H^1_{\text{ét}}(\text{Spec} \mathcal{O}, \mathbf{T}_{\mathcal{O}})$ is injective. Thus, $H^1(\text{in})$ is injective too. So, we have a short exact sequence of the form

$$\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{\text{in}} \mathbf{T}(K) \to \bigoplus_x \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)^{\text{Gal}} \to \{1\}.$$

We have also the complex $\{1\} \to \mathbf{T}(\mathcal{O}) \xrightarrow{\text{in}} \mathbf{T}(K) \to \bigoplus_x \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ and $\gamma_x = \bigoplus_x \gamma_x$ where $\gamma_x : \mathbf{T}(K)/\mathbf{T}(\tilde{\mathcal{O}}_x) \to [\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{\text{Gal}}$. This short exact sequence of the complex and the above short exact sequence. We claim that this morphism is an isomorphism. This claim completes the proof of the theorem.

To prove this claim, it suffices to prove that $\gamma$ is an isomorphism. Since the map $\mathbf{T}(K) \to \bigoplus_x [\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)]^{\text{Gal}}$ is an epimorphism, so is the map $\gamma$. It remains to prove that $\gamma$ is a monomorphism. To do this, it suffices to check that for any closed point $x \in X$ the map $\mathbf{T}(K)/\mathbf{T}(\tilde{\mathcal{O}}_x) \to \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ is a monomorphism. We will write $\epsilon_x$ for the latter map. We prove below that $\ker(\epsilon_x)$ is a torsion group and the group $\mathbf{T}(K)/\mathbf{T}(\tilde{\mathcal{O}}_x)$ has no torsion. These two claims show that the map $\epsilon_x$ is indeed injective.

To prove that $\ker(\epsilon_x)$ is a torsion group, recall that there are norm maps $N_{\tilde{\mathcal{O}}_x/\mathcal{O}_x} : \mathbf{T}(\tilde{\mathcal{O}}_x) \to \mathbf{T}(\mathcal{O}_x)$ and $N_{\tilde{K}/K} : \mathbf{T}(\tilde{K}) \to \mathbf{T}(K)$ (see [14, Sec. 2]). These maps induce a homomorphism

$$N_x : \mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x) \to \mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$$

such that $N_x \circ \epsilon_x = \text{the multiplication by } d$, where $d$ is the degree of $\tilde{K}$ over $K$. Thus, $\ker(\epsilon_x)$ is killed by the integer $d$.

We show now that the group $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ has no torsion. Take an element $a_K \in \mathbf{T}(K)$ and suppose that its class in $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ is a torsion element. Let $\tilde{a}_K$ be the image of $a_K$ in $\mathbf{T}(\tilde{K})$. Since $T$ splits over $\tilde{K}$, we see that $\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ is torsion-free. Thus, the class of $\tilde{a}_K$ in $\mathbf{T}(\tilde{K})/\mathbf{T}(\tilde{\mathcal{O}}_x)$ vanishes. So, there is a unique element $\tilde{a}$ in $\mathbf{T}(\tilde{\mathcal{O}}_x)$ whose image in $\mathbf{T}(\tilde{K})$ is $\tilde{a}_K$. Moreover, $\tilde{a}$ is a Gal-invariant element in $\mathbf{T}(\tilde{\mathcal{O}}_x)$, because $\tilde{a}_K$ comes from $\mathbf{T}(K)$. Since $\mathbf{T}(\tilde{\mathcal{O}}_x)^{\text{Gal}} \cong \mathbf{T}(\mathcal{O}_x)$, there is a unique element $a \in \mathbf{T}(\mathcal{O}_x)$ whose image in $\mathbf{T}(\mathcal{O}_x)$ is $\tilde{a}$. Clearly, the image of $a$ in $\mathbf{T}(K)$ is the element $a_K$. Thus, the class of $a_K$ in $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ vanishes. So, the group $\mathbf{T}(K)/\mathbf{T}(\mathcal{O}_x)$ is torsion-free.
The injectivity of $\epsilon_x$ is proved. This completes the proof of Theorem 1.1.

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