EMBEDDING CONVEX GEOMETRIES AND A BOUND ON CONVEX DIMENSION

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Abstract. The notion of an abstract convex geometry, due to [1], offers an abstraction of the standard notion of convexity in a linear space. [2] introduce the notion of a generalized convex shelling into \( \mathbb{R} \) and prove that a convex geometry may always be represented with such a shelling. We provide a new, shorter proof of their result using a recent representation theorem of [3] and deduce a different upper bound on the dimension of the shelling.

From [1], one may find several equivalent definitions of what it means for a collection of sets to constitute an abstract convex geometry. One is given below in Definition 1 stating that the collection of sets is closed under intersection, includes both the empty set and the entire set, and that given any set in the collection, there is an element outside of it which may be added to that set and yields another member of the collection. Another equivalent formulation is given in terms of the anti-exchange property, which here states that if one is given two elements and a set in the collection containing neither of those elements, then there is a larger set in the collection containing exactly one of those elements. That is, it is possible to include one of the external elements (perhaps along with additional other elements) without including the other. This is the same anti-exchange property that appears in antimatroid theory, and there is a clear equivalence between convex geometries and antimatroids. Recently, [2] have shown that any convex geometry can be represented as a “generalized convex shelling”, meaning that there is an embedding in \( \mathbb{R}^N \) so that each set in the geometry is convex if and only if its embedding is convex with respect to a fixed external set of points in \( \mathbb{R}^N \). In this paper we provide an alternate proof of this representation theorem using a recent result of [3] which represents a convex geometry through a collection of orderings. An important feature of this new proof is that it yields an upper bound on the dimension of the smallest Euclidean space into which a convex geometry may be embedded via a generalized convex shelling which may be different to the upper bound found by [2].

Definition 1. Let \( E \) be a finite set containing \( N \) points. A convex geometry on \( E \) is a collection \( \mathcal{L} \) of subsets (called convex sets) of \( E \) with the following properties.

1. \( \emptyset \) and \( E \) are in \( \mathcal{L} \)
2. If \( X, Y \in \mathcal{L} \) then \( X \cap Y \in \mathcal{L} \)
3. If \( X \in \mathcal{L} \setminus \{E\} \) then there is \( e \in E \setminus X \) so \( X \cup \{e\} \in \mathcal{L} \)

Convex geometries \( \mathcal{L}_1 \) on \( E_1 \) and \( \mathcal{L}_2 \) on \( E_2 \) are isomorphic if there is a bijection \( \psi : E_1 \to E_2 \) so \( \psi(X) \in \mathcal{L}_2 \) if and only if \( X \in \mathcal{L}_1 \).

The following definition of a generalized convex shelling for a set \( E \subset \mathbb{R}^n \) is due to [2].

Definition 2. Using \( \text{Conv}(Q) \) to denote the convex hull, let \( G, Q \subset \mathbb{R}^n \) be finite sets such that \( G \cap \text{Conv}(Q) = \emptyset \). The generalized convex shelling on \( G \) with respect to \( Q \) is

\[
\mathcal{L} = \{ X \subset G : \text{Conv}(X \cup Q) \cap G = X \}.
\]

It is easily verified that this defines a convex geometry. In [2] the converse is proved.
Theorem 1 ([2] Theorem 2.8). Any convex geometry is isomorphic to a generalized convex shelling.

Another straightforward way to define a convex geometry on $E$ uses a collection of orderings, which we denote $\succeq_i$. The orderings are reflexive, complete, antisymmetric and transitive binary relations; they are also strict, so if $a \neq b$ then for each $i$ either $a >_i b$ or $b >_i a$.

Definition 3. We say that $\mathcal{L}$ is generated by a family $\{\succeq_i\}_{i=1}^M$ of orderings on $E$ if
\[
\mathcal{L} = \{\emptyset\} \cup \{X \subseteq E : \forall y \notin X, \exists i \text{ so that } \forall x \in X, x >_i y\}
\]

The following lemma shows that the above definition produces a convex geometry from any set of orderings. This was proven in [3], but for the reader’s convenience we also demonstrate it here.

Lemma 1 ([3] Claim 1). If $\mathcal{L}$ is generated by a family of orderings $\{\succeq_i\}_{i=1}^m$, then it is a convex geometry.

Proof. The only non-trivial part of Definition 1 to verify is (3). Take $X \in \mathcal{L} \setminus \{E\}$ and define a relation on $E \setminus X$ by $a \succeq_X b$ if $\forall i \exists y_i \in X \cup \{b\}$ such that $a \succeq_i y_i$. This is easily seen to be reflexive and transitive. To prove antisymmetry assume $a \neq b$ are in $E \setminus X$. Since $X \in \mathcal{L}$ this implies $\exists i$ so $\forall x \in X, x >_i a$. For this $i$, either $b >_i a$ and thus $a \not\succeq_X b$, or $a >_i b$ and therefore also $\forall x \in X, x >_i b$, the two of which imply that $b \not\succeq_X a$. Hence $\succeq_X$ is antisymmetric and is a partial order. Now take $z$ a maximal element in $E \setminus X$ according to $\succeq_X$. If $y \notin X \cup \{z\}$ then $z \succeq_X y$, and since $X \in \mathcal{L}$ there is $i$ so $\forall x \in X, x >_i z$. By the above reasoning $z \succeq_X y$ implies that $\forall x \in X, x >_i z >_i y$, so $X \cup \{z\}$ is convex. This verifies (3) of Definition 1 and completes the proof. \hfill \Box

The following converse was proved in [3] by a maximality argument.

Theorem 2 ([3] Claim 1). If $\mathcal{L}$ is a convex geometry on $E$ then there are orderings $\succeq_i$ on $E$ such that $\mathcal{L}$ is obtained as in Definition 3.

The purpose of this note is to give a proof of Theorem 1 from Theorem 2. We first define a map which realizes the orderings as the orders on coordinate directions in $\mathbb{R}^M$. To do so, for each $i$ arrange $E$ using the $i$th order as $x_{i1} >_i x_{i2} >_i \cdots >_i x_{in}$ and for $x \in E$ let $j_i(x)$ be the unique choice such that $x = x_{j_i(x)}$. That is, $j_i(x)$ denotes $x$’s place according to the $i$th ordering. Then define $F_i : E \to \mathbb{R}$ by $F_i(x) = -M^{i(x)}$ and let $F(x) = (F_1(x), \ldots, F_M(x)) : E \to \mathbb{R}^M$. We have replicated the orderings $\succeq_i$ from $E$ using the coordinate directions on $F(E)$, so the following is obvious.

Lemma 2. On $\mathbb{R}^M$ define $(x_1, \ldots, x_M) >_i (y_1, \ldots, y_M)$ to mean that $x_i > y_i$. Then $\{\succeq_i\}$ are orderings on $F(E)$ and the convex geometry $\mathcal{L}_1$ they generate is isomorphic to $\mathcal{L}$ on $E$.

There is a hull operation for $\mathcal{L}_1$ which we call Pos (an abbreviation of positive sector). If $P \subseteq F(E)$ then $\text{Pos}(P) = \{x : \forall i \exists p(i) \in P \text{ with } x \succeq_i p(i)\}$. Clearly $\text{Pos}(P) \cap F(E) \subseteq \mathcal{L}_1$ and $P \in \mathcal{L}_1 \iff P = \text{Pos}(P) \cap F(E)$. We will also need another hull operation, which we define by $\text{ExtConv}(P) = \{x : \exists p \in \text{Conv}(P) \text{ with } x \succeq p, \forall i\}$, from which we can define a collection $\mathcal{L}_2$ of subsets of $F(E)$ by $P \in \mathcal{L}_2 \iff P = \text{ExtConv}(P) \cap F(E)$. It is not immediately clear whether $\mathcal{L}_2$ is a convex geometry, but in fact we have the following.

Theorem 3. $\text{Pos}(P) \cap F(E) = \text{ExtConv}(P) \cap F(E)$ for any $P \subseteq F(E)$. Equivalently, $\mathcal{L}_1 = \mathcal{L}_2$.

Proof. Let $p(i) \in P$ be such that $p(i)_i = \min_{p \in P} p_i$, where the subscript denotes the $i$th coordinate. That is, $p(i)$ is the minimally-ranked member of $P$ in coordinate $i$. There is exactly one such $p(i)$ for each $i$, because $>_i$ is strict on $F(E)$, however it is possible that $p(i) = p(j)$ for some $i \neq j$. It is clear that $\text{Pos}(P) = \{x : \forall i, x_i \succeq p(i)_i\}$. \hfill \Box
If \( x \in \text{ExtConv}(P) \) there are elements \( p^k \in P \) so \( x_i \geq \sum_k \alpha_k(p^k)_i \), where \( \alpha_k \geq 0, \sum \alpha_k = 1 \). But then \( (p^k)_i \geq p(i)_i \), so \( x_i \geq p(i)_i \) for each \( i \), and thus \( x \in \text{Pos}(P) \) and the inclusion \( \supset \) is proved.

For the converse let \( p = M^{-1} \sum_i p(i) \in \text{Conv}(P) \). Recall from the definition of \( F(E) \) that \( p(i)_j \leq -M \) for all \( i, j \), so that \( M p_j \leq -M(M - 1) + p(j)_j \). Now let \( x \in \text{Pos}(P) \cap F(E) \). If \( x \in P \) then we are done. If not, then by strictness of the orderings \( \succ \) we have \( x \succ y \) and therefore \( Mx_j \geq p(j)_j \).

Combining this with our first estimate yields \( M p_j \leq -M(M - 1) + Mx_j \), so that \( p_j \leq x_j \). This is true for all \( j \), so \( x \geq y \) for all \( j \) and thus \( x \in \text{ExtConv}(P) \). \( \square \)

Finally we relate \( \text{ExtConv}(P) \) to the geometry given by a generalized convex shelling. Specifically, let \( e_i \) be the \( i \)-th coordinate vector in \( \mathbb{R}^M \), let \( Q \) be the set of points \( \{0\} \cup \{\lambda e_i : 1 \leq i \leq M\} \) and let \( L \) be the convex geometry on \( F(E) \) given by the convex shelling of \( F(E) \) with respect to \( Q \) as in Definition 2. This is legitimate because \( \text{Conv}(Q) \) is in the positive sector, so cannot intersect \( F(E) \).

**Theorem 4.** There exists \( \lambda \) large enough such that \( \text{Conv}(P \cup Q) \cap F(E) = \text{ExtConv}(P) \cap F(E) \) for any \( P \subset F(E) \). Equivalently \( L_2 = L_3 \).

**Proof.** Since \( P \subset F(E) \) is in the negative sector all points of \( Q \) are in \( \text{ExtConv}(P) \), thus \( \text{Conv}(P \cup Q) \subset \text{ExtConv}(P) \) and we must prove the converse for the intersection with \( F(E) \). In the case \( M = 1 \) this is clear with \( Q = \{0\} \), i.e. \( \lambda = 0 \), so we may assume \( M \geq 2 \).

If \( x \in F(E) \cap \text{ExtConv}(P) \) then there is \( y \in \text{Conv}(P) \) so \( x_j \geq y_j + M^2 - M \) for all \( j \). Moreover, as \( y_j \geq -M^N \),

\[(1 - M^{-N})y_j = y_j - M^{-N}y_j \leq y_j + 1 \leq x_j - (M^2 - M - 1) \leq x_j - 1\]

Define \( u_j = x_j - (1 - M^{-N})y_j \geq 1 \) and make these the components of \( u \), so that \( x = (1 - M^{-N})y + u \) with \( y \in \text{Conv}(P) \). This is a convex combination of \( y \) and \( M^N u \). We know \( M^N u \) is in the positive sector so it is clear we may make \( \lambda \) large enough that \( M^N u \in \text{Conv}(Q) \), thus completing the proof. In particular \( u_j \leq -M - (1 - M^{-N})(-M^N) < M^N \), so \( \lambda = M^{2N+1} \) suffices. \( \square \)

Evidently Lemma 1, Theorem 3 and Theorem 4 together show that for a suitable choice of \( \lambda \) we have our original geometry \( L \) is isomorphic to \( L_1 = L_2 = L_3 \), with this last being a generalized convex shelling. Therefore, from Theorem 2 we then have Theorem 1.

**Definition 4.** For a convex geometry \( L \), its dimension \( \text{dim}(L) \) is defined as the lowest dimension for which the convex geometry is isomorphic to a generalized convex shelling.

The proofs of Lemma 1 and Theorems 3 and 4 together show that any convex geometry \( L \) represented by \( k \) orderings may be embedded into \( \mathbb{R}^k \) as a generalized convex shelling. On the other hand, the proof of Theorem 2.8 in [2] presents an embedding into \( \mathbb{R}^{|E|} \). Therefore, we have the following corollary.

**Corollary 1.** For a convex geometry \( L \) over \( E \) represented by \( k \) orderings, \( \text{dim}(L) \leq \min(|E|, k) \).

The upper bounds in the corollary are trivially optimal when \( k = 1 \). The following examples show that for \( k > 1 \) it is possible for the bound \( \text{dim}(L) \leq k \) to be optimal or very far from optimal. In the first example the bound \( \text{dim}(L) \leq k \) from our construction is much better than that which comes from the argument in [2].

**Example 1** (Optimal dimension bound from number of orderings). Let \( E = \{a, b, c\} \) and \( L = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, E\} \). Observe that \( L \) can be represented by the two orderings \( c \prec_1 b \prec_1 a \) and \( b \prec_2 c \prec_2 a \), thus \( \text{dim}(L) \leq 2 \). If it were possible to embed \( E \) in \( \mathbb{R} \) so as to obtain \( L \) from the
generalized convex shelling with respect to a set \( Q \) we would have both \( \text{Conv}(Q) \cap E = \emptyset \) and \( a \in \text{Conv}((b) \cup Q) \cap \text{Conv}((c) \cup Q) \). It follows easily that if \( I_1 \) and \( I_2 \) are the components of \( \mathbb{R} \setminus \{a\} \) then \( \emptyset \neq Q \subset I_1 \) and \( \{b, c\} \subset I_2 \), so without loss of generality we may suppose that \( q < a < b < c \) for all \( q \in Q \). Then \( \text{Conv}(c) \cup Q) \cap E = E \) in contradiction to the fact that \( \{a, c\} \) is convex. We conclude that \( \mathcal{L} \) cannot be given by a generalized convex shelling in \( \mathbb{R} \) and therefore \( \dim(\mathcal{L}) = 2 \).

Comparing this to Corollary 1 we see that in this case \( 2 = \dim(\mathcal{L}) = k < |E| = 3 \).

**Example 2** (Non-optimal dimension bound from number of orderings). Let \( E \) be a finite subset of the unit circle in \( \mathbb{R}^2 \) with the standard convexity, so \( \mathcal{L} \) is the power set \( 2^E \). Clearly \( \dim(\mathcal{L}) \leq 2 \). Suppose \( \mathcal{L} \) is represented by orderings \( \prec_1, \ldots, \prec_k \). If \( x \in E \) then \( E \setminus \{x\} \) is convex, so from Definition 3 there is \( j_x \) so that \( x \) is minimal with respect to \( \prec_{j_x} \). Evidently \( x \mapsto j_x \) is injective, so \( k \geq |E| \), and both can be made arbitrarily large by comparison with \( \dim(\mathcal{L}) \leq 2 \).

The set of orderings that generates a convex geometry is typically non-unique, so some representations will provide a tighter bound on the dimension of the convex geometry than others. The tightest such bound is provided when a minimum number of orderings is used. Thus \( \min\{k : \succ_1, \ldots, \succ_k \} \) represents \( \mathcal{L} \) is a statistic which stratifies convex geometries according to a notion of complexity. From the examples it is apparent that this statistic is generally distinct from \( \dim(\mathcal{L}) \), though there are cases where the two values coincide and it would be interesting to know a characterization of such geometries. It might also be interesting to know whether there are cases where \( \dim(\mathcal{L}) = |E| \neq 1 \) so the bound from [2] is optimal; we do not know of any, and conjecture that none exist.

**References**

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