Quasiregular Families Bounded in $L^p$ and Elliptic Estimates

Aimo Hinkkanen$^1$ · Gaven Martin$^2$

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Abstract
We prove that a family $\mathcal{F}$ of quasiregular mappings of a domain $\Omega$ which are uniformly bounded in $L^p$ for some $p > 0$ form a normal family. From this we show how an elliptic estimate on a functional difference implies all directional derivatives, and thus the complex gradient to be quasiregular. Consequently the function enjoys much higher regularity than apriori assumptions suggest. We present applications in the theory of Beltrami equations and their nonlinear counterparts.

Keywords
Elliptic estimate · Quasiregular mappings · Normal family · Nonlinear · Beltrami equations

Mathematics Subject Classification Primary 30C62 · Secondary 35J60

1 Introduction
The governing equations of geometric function theory and the theory of quasiconformal mappings, Teichmüller spaces and so forth are the Beltrami equations and their nonlinear counterparts, see for instance [3,7–10] and elsewhere. Beltrami equations come in several different flavours. As examples, let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be a mapping of Sobolev class $W^{1,1}_{\text{loc}}(\Omega)$ consisting of functions whose first derivatives are locally integrable, then we have

- $\mathbb{C}$-linear: $f_z = \mu(z) f_z$, with ellipticity estimate $\|\mu\|_{L^\infty(\Omega)} < 1$;
- $\mathbb{R}$-linear: $f_z = \mu(z) f_z + v(z) \overline{f_z}$, with ellipticity estimate $\| |\mu| + |v| \|_{L^\infty(\Omega)} < 1$;

1 University of Illinois at Urbana-Champaign, Urbana, IL, USA
2 Massey University, Auckland, New Zealand
• Autonomous: \( f_z = A(f_z) \), with ellipticity estimate: there is \( k < 1 \) so that for all \( \zeta, \eta \in \mathbb{C} \)

\[
|A(\zeta) - A(\eta)| \leq k|\zeta - \eta|.
\]

• Fully nonlinear: \( f_z = \mathcal{H}(z, f, f_z) \), with ellipticity estimate: there is \( k < 1 \) so that for all \( z \in \Omega \), all \( \zeta, \eta \in \mathbb{C} \)

\[
|\mathcal{H}(\zeta) - \mathcal{H}(\eta)| \leq k|\zeta - \eta|
\]

with additional conditions on \( \mathcal{H} \), see [3, Chaps. 7 and 8].

Each of these equations has a seminal application and they are all inter-related. The apriori assumption that \( f \in W^{1,1}_{\text{loc}}(\Omega) \) is so that we can even speak of \( f \) as a “solution”. Without stronger assumptions on \( \mu \) or \( \mathcal{H} \) not much can be said, but note for instance that \( \mu = 0 \) on an open set implies \( f \) is holomorphic on that set—Weyl’s Lemma. The higher regularity theory of these equations typically assumes more on \( f \), for instance \( f \in W^{1,q}_{\text{loc}}(\Omega) \) for some \( 1 < q \leq 2 \) usually depending on the ellipticity constant \( k \), and in return delivers a far nicer outcome, \( f \in W^{1,p}_{\text{loc}}(\Omega) \) for some \( p > 2 \), again depending on \( k \). Astala’s theorem [1] gives the optimal result in the \( \mathbb{C} \)-linear case and can be used to analyse other cases. Questions of existence and uniqueness are fairly well understood through the topological properties of these mappings and the well known Stoïlow factorisation theorem, [12] and the references therein, see also [3, §5.5 and §6.1]. However there are intriguing subtleties in the nonlinear case, see [4,5].

In this paper we seek general methods to go beyond the \( W^{1,p}_{\text{loc}} \) regularity to seek \( W^{2,p}_{\text{loc}} \) estimates, see our Theorem 2. Such estimates have been found before in special cases, for instance the study of the autonomous equations (for instance [5]), and these estimates have important applications (for instance [11]) and serve as a bootstrap for \( C^\infty \)-regularity. It is quite curious that an elliptic estimate such as at (1) below implies \( f_z \) itself is quasiregular (a solution to an elliptic Beltrami equation).

## 2 Main Results

The two main results we present here are the following.

**Theorem 1** Let \( \Omega \) be a domain in the complex plane \( \mathbb{C} \). Let \( p \) be a real number with \( p > 0 \). Let \( \mathcal{F} \) be a family of \( K \)-quasiregular mappings \( f : \Omega \to \mathbb{C} \) which is uniformly bounded in \( L^p(\Omega) \). Then the family \( \mathcal{F} \) is precompact, every sequence contains a locally uniformly convergent subsequence and each limit is \( K \)-quasiregular (possibly constant) with values in \( \mathbb{C} \).

We use this theorem to prove the following theorem which establishes very strong regularity from a standard elliptic type estimate.
Theorem 2 Let $0 \leq k < 1$ and $f : \Omega \to \mathbb{C}$ be a function of Sobolev class $W_{loc}^{1,p}(\Omega)$ for some $p > 1 + k$ and which satisfies the elliptic estimate

$$|f_\bar{z}(z + t\zeta) - f_\bar{z}(z)| \leq k|f_z(z + t\zeta) - f_z(z)|, \quad |\zeta| = 1$$

for all $0 < t < a(z)$ for some continuous function $a : \Omega \to \mathbb{R}_+, a(z) \leq \text{dist}(z, \partial\Omega)$.

Then

1. $f \in W_{loc}^{2,q}(\Omega)$ for all $q < 1 + 1/k$.
2. Each member of the $\mathbb{R}$-linear family

$$\{af_x(z) + bf_y(z) : a, b \in \mathbb{R}\}$$

is either $1 + k$-quasiregular mapping or a constant.
3. There are measurable $\mu, \nu : \Omega \to \mathbb{C}$ with $|\mu| + |\nu| \leq k$ so that both directional derivatives $f_x$ and $f_y$ satisfy the $\mathbb{R}$-linear Beltrami equation,

$$h_\bar{z} = \mu(z)h_z + \nu(z)\overline{h_z}, \quad h \in \{f_x, f_y\}.$$  

4. The complex gradient $f_z$ is itself quasiregular and satisfies the $\mathbb{R}$ linear equation

$$h_\bar{z} = \frac{\mu(z)}{1 - |\nu(z)|^2}h_z + \frac{\mu(z)\nu(z)}{1 - |\nu(z)|^2}\overline{h_z}, \quad h = f_z,$$

and thus $f_z \in W_{loc}^{1,q}(\Omega)$ for all

$$q < s = 1 + 1/k', \quad k' = \left\|\frac{|\mu|}{1 - |\nu|}\right\|_\infty \leq k.$$

The next result concerns the tangent cone of a quasiregular mapping and Hölder regularity.

Theorem 3 Let $f : \mathbb{D} \to \mathbb{C}$ be quasiregular with $f(0) = 0$. Suppose that for some $p, q > 0$ and for all $\epsilon$ sufficiently small we have

$$\int_{\mathbb{D}(0, \epsilon)} |f(z)|^p \, dz \leq C \epsilon^{2+q}$$

where $C$ is an absolute constant. Then the family of quasiregular maps

$$\mathcal{F} = \left\{ \frac{1}{\lambda^{q/p}} f(\lambda z) : \lambda \in \mathbb{D}\backslash\{0\} \right\}$$

is precompact. Every sequence contains a subsequence converging locally uniformly in $\mathbb{D}$ to a quasiregular mapping, or a constant (possibly $\infty$).
The Stoïlow factorisation theorem [3, Theorem 5.5.1, p. 179] together with the Hölder continuity properties of $K$-quasiconformal mappings tells us, as $f(0) = 0$, if the degree of $f$ at 0 is $n \geq 1$, then we have the a priori bound

$$\int_{D(0, \epsilon)} |f(z)|^p \, dz \leq C \int_{D(0, \epsilon)} |z|^{np/K} \, dz = \frac{2\pi C}{2 + np/K} \epsilon^{2+np/K}.$$  

So no matter what $p > 0$ is chosen, there is always a $q > 0$.

### 3 Proof of Theorem 2

Let $U \subset V$ be relatively compact subdomains of $\Omega$ with $\overline{U} \subset V$. Then $a(z) \geq \epsilon = \epsilon(V) > 0$ on $V$. We first observe that the estimate (1) implies that the function

$$\varphi_t(z) = \frac{1}{t} \left[ f(z + t \xi) - f(z) \right]$$

is $\frac{1+k}{1-k}$-quasiregular independently of $t$—as long as $t < \epsilon$. By Astala’s Theorem [3, Theorem 13.1.5] $\varphi_t$ lies in the Sobolev space $W^{1,2}_{\text{loc}}(\Omega)$. We now appeal to the Caccioppoli type estimate [3, Theorem 5.4.2, p. 175] which informs us that there is an exponent $p = p(k) > 2$ such that

$$\int_{U} |D\varphi_t|^p \leq C_{U,V}(p) \left[ \int_{V} |\varphi_t(z)|^2 \right]^{p/2}.$$  

Since the right hand side here involves no derivatives of $\varphi_t$, and since $p \geq 2$ convexity gives us the following uniform bound independent of $t$.

$$\int_{U} \left| f(z + t \xi) - f(z) \right|^p \leq C_{U,V}(p) \left[ \int_{V} \left| f(z + t \xi) - f(z) \right|^2 \right]^{p/2} \leq C_{U,V}(p) \left[ \int_{V} |Df(z)|^2 \right]^{p/2}. \tag{2}$$

This last estimate shows that both $f_x$ and $f_y$, and hence $f_z$ and $f_{\overline{z}}$, belong to the Sobolev space $W^{1,p}_{\text{loc}}(V)$ with the uniform estimate

$$\int_{U} |Df_z|^p + \int_{U} |Df_{\overline{z}}|^p \leq C \left[ \int_{V} |Df|^2 \right]^{p/2}$$

and this is what we first wanted to establish. That is $f \in W^{2,p}_{\text{loc}}(\Omega)$.
Next, Eq. (2) implies that the family of $K$-quasiregular mappings $\{\varphi_t\}_{0 < t < \epsilon}$ is uniformly bounded in $L^2(V)$. Theorem 1 establishes the precompactness of these families. In particular as $t \to 0$ in any direction $\zeta$ the limit of any subsequence will be $K$-quasiregular (or possibly constant). In particular all directional derivatives will be $K$-quasiregular since as $f$ is differentiable almost everywhere these limits exist as $t \to 0$ for fixed $\zeta$. The directional derivatives of $f$ form an $\mathbb{R}$-linear family of quasiregular mappings,

$$f_\zeta = af_x + bf_y, \quad \zeta = a + ib, \quad a^2 + b^2 = 1.$$ 

Thus the family of directional derivatives forms an $\mathbb{R}$-linear family of quasiregular maps.

In fact the same argument with using the function $\phi_h = \frac{1}{h}[f(z + h) - f(z)]$ shows that in any sequence $h_j \to 0$ there is a convergent subsequence $\phi_{h_{jk}}$ whose limit is quasiregular or constant. The set of all such limits forms the tangent cone to $f$ and consists entirely of quasiregular mappings.

Now, it is not difficult to show that associated to an $\mathbb{R}$-linear family $\mathcal{F}$ of $\frac{1+k}{1-k}$-quasiregular mappings of a domain $\Omega$, there are measurable functions $\mu, \nu : \Omega \to \mathbb{C}$ such that each $h \in \mathcal{F}$ satisfies the $\mathbb{R}$-linear equation

$$h_\zeta = \mu(z)h_z + \nu(z)\overline{h_z}, \quad \text{a.e.} \quad \Omega \quad (3)$$

with the elliptic bounds $|\mu| + |\nu| \leq k$. See for instance [6]. In [2] there is a further discussion of this fact and uniqueness is also proved. We therefore have $f_x$ and $f_y$ satisfying Eq. (3). Then

$$2(f_\zeta)_\bar{z} = (f_x - if_y)_\bar{z} = \mu(z)(f_x)_\bar{z} + v(z)\overline{(f_x)_\bar{z}} - i\left[\mu(z)(f_y)_z + v(z)(f_y)_\bar{z}\right]$$

$$= \mu(z)(f_x)_\bar{z} + v(z)(\overline{f_x}_z) - i\left[\mu(z)(f_y)_z + v(z)(f_y)_\bar{z}\right]$$

$$= 2\left(\mu(z)(f_x)_z + v(z)(\overline{f_x}_z)\right) = 2\left(\mu(z)(f_x)_z + v(z)(\overline{(f_x)_\bar{z}})\right).$$

Hence we obtain the following equation for $f_z$.

$$(1 - |v(z)|^2)(f_z)_\bar{z} = \mu(z)(f_x)_z + v(z)\overline{\mu(z)(f_x)_z}.$$ 

This now establishes all the claims of the theorem. □

Of course the most natural way that the elliptic estimate (1) is produced is from the nonlinear autonomous Beltrami equation $f_\bar{z} = A(f_\bar{z})$ with the elliptic Lipschitz estimate

$$|A(\zeta) - A(\eta)| \leq k|\zeta - \eta|, \quad k < 1, \quad \zeta, \eta \in \mathbb{C}.$$ 

Actually, the method of “frozen coefficients” enables the nonlinear Beltrami equation $f_\bar{z} = H(f, f_\bar{z})$ to be studied in this was as well, see [4].

We remark that in the preceding proof we have made use of the possibility of interchanging the order of differentiation which we need in the sense of distributions.
We could have started by multiplying by a test function and integrating by parts a few times, but these ideas are relatively routine and only add complexity. We have therefore omitted these details.

4 Proof of Theorem 3

It is clear each member of $\mathcal{F}$ is quasiregular with the same distortion bounds. We wish to prove a uniform bound in some $L^p(D)$, $p > 0$. We compute that

$$\int_D \left| \frac{f(\lambda z)}{\lambda^q/p} \right|^p \, dz = \frac{1}{|\lambda|^q} \int_D |f(\lambda z)|^p \, dz = \frac{1}{|\lambda|^{2+q}} \int_{D(0,|\lambda|)} |f(z)|^p \, dz \leq C.$$ 

So we obtain a uniform bound in $L^p$ on the quasiregular family $\{ f(\lambda z) : \lambda \in \mathbb{D}\setminus\{0\} \}$, $s = q/p$.

5 Quasiregular Families Bounded in $L^p$

We now present the proof of our main result, Theorem 1.

The conclusion of the theorem is already known to be valid if the elements of $\mathcal{F}$ are uniformly bounded in $L^\infty(\Omega)$. We also note that if $0 < p < 1$, then $L^p(\Omega)$ is not a Banach space, but of course the $L^p$-norm is well defined.

Let $\{f_k\}_{k=1}^\infty(\Omega)$ be a sequence in $\mathcal{F}$. The Stoïlow factorisation theorem [3, Theorem 5.5.1, p. 179], allows us to write

$$f_k = \varphi_k \circ g_k,$$

where each $g_k : \mathbb{C} \to \mathbb{C}$ is $K$-quasiconformal with $g_k(\infty) = \infty$ and $\varphi_k : g_k(\Omega) \to \mathbb{C}$ is holomorphic. Let $a_i$, $i = 1, 2$ be distinct points of $\Omega$ and $\psi_k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a Möbius transformation such that $\psi_k(g_k(a_i)) = a_i$, $i = 1, 2$ and $\psi_k(\infty) = \infty$. Writing

$$f_k = \varphi_k \circ \psi_k^{-1} \circ \psi_k \circ g_k$$

we see that we can replace $g_k$ by $\psi_k \circ g_k$ so that we could (and will) in fact assume each $g_k$ fixes two points of $\Omega$ and the point at infinity. Then the family $\{g_k : \Omega \to \mathbb{C}\}$ is precompact and passing to a subsequence we have $g_k \to g$ locally uniformly in $\Omega$, where $g : \Omega \to \mathbb{C}$ is $K$-quasiconformal (in fact, we may choose $g : \mathbb{C} \to \mathbb{C}$). Similarly we write $\varphi_k$ for $\varphi_k \circ \psi_k^{-1}$.

In view of these properties of the functions $g_k$, to obtain the conclusion of the theorem it suffices to prove that the functions $\varphi_k$ form a normal family in any relatively compact subset of $g(\Omega)$, which we now proceed to do.

Let the domain $W$ be a relatively compact subset of $g(\Omega)$. Then let the domain $U$ be a relatively compact subset of $\Omega$ such that $\overline{W} \subset g(U)$. Then there is a domain $V \subset g(\Omega)$ such that $\overline{g(U)} \subset V$ and $\overline{V}$ is a compact subset of $g(\Omega)$. Since $g_k \to g$
uniformly on $U$, we have $W \subset g_k(U) \subset V$ for all large $k$; we may assume that this is true for all $k$.

Now each $g_k$ is $K$-quasiconformal in $\Omega$ and so by Astala’s Theorem $J(w, g_k) \in L^q(U)$ for each $q \in [1, \frac{K}{K-1})$.

We now need the following lemma.

**Lemma 1** Let $\Omega$ be a planar domain and $U$ be a relatively compact disk in $\Omega$. Suppose $g_k, g : \Omega \to \mathbb{C}$ is a sequence of $K$-quasiconformal mappings with $g_k \to g$ locally uniformly on $\Omega$. Then there is a constant $C = C(q, K)$ such that for every $1 \leq q < \frac{K}{K-1}$

$$\int_U J(w, g_k)^q \, dw \leq C \int_U J(w, g)^q \, dw.$$  \hfill (5)

**Proof** We suppose that $U = B = \mathbb{D}$ is a relatively compact disk in $\Omega$, the general result follows in an elementary manner. Under the hypotheses we have $g_k \to g$ uniformly on $B$ and for all sufficiently large $k$, $|g_k(B)| \leq |g(B)| + 1 < \infty$. The local uniform convergence on $\Omega$ implies weak convergence of the Jacobians,

$$\int_{\Omega} \varphi(z) \, J(z, g_k) \to \int_{\Omega} \varphi(z) \, J(z, g)$$

for every $\varphi \in C^\infty_0(\Omega)$. See Corollary, p. 141, and Theorem 9.1, p. 216 in [13]. We recall that if $J$ is the Jacobian of a $K$-quasiconformal mapping and $\omega = J^s, \frac{1}{K-1} < s < \frac{K}{K-1}$, then $\omega$ is an $A_p$-weight for all

$$p > \begin{cases} 1 + s(K - 1) & 0 \leq s < \frac{K}{K-1}, \\ 1 - \frac{1}{K}(K - 1) & \frac{1}{K-1} < s \leq 0, \end{cases}$$  \hfill (6)

see [3, Theorem 13.4.2]. With $s = q$ we have [3, (13.56)]

$$\frac{1}{|B|} \int_B J(z, g_k)^q \leq C(K, q) \left( \frac{|g_k(B)|}{|B|} \right)^q \leq \left( \frac{|g(B)| + 1}{|B|} \right)^q$$  \hfill (7)

with $C(K, q)$ finite. Thus the sequence $\{J(z, g_k)\}$ is uniformly bounded in $L^q(B)$ and we may extract a weakly convergent subsequence in $L^q(B)$. Since $J(z, g)$ also lies in $L^q(B)$ this weak limit must in fact be $J(z, g)$ and any weakly convergent subsequence will have the same limit. The result follows. \hfill $\square$

We can now use Hölder’s inequality to see that if $q' = \frac{q}{q - 1} > K$ and $s > 0$, we have

$$\int_W |\varphi_k(z)|^s \, dz \leq \int_{g_k(U)} |\varphi_k(z)|^s \, dz$$

$$= \int_{g_k(U)} |(f_k \circ g_k^{-1})(z)|^s \, dz$$

$$= \int_U |f_k(w)|^s J(w, g_k) \, dw$$
Thus if \( f_k = \varphi_k \circ g_k \) is uniformly bounded in \( L^{sq'}(U) \) for some \( q' > K \), then the sequence \( \{ \varphi_k \} \) is uniformly bounded in \( L^s(W) \).

Now fix any \( q \) with \( 1 \leq q < K/(K-1) \). This determines \( q' = \frac{q}{q-1} > K \). Then choose \( s = p/q' > 0 \). Hence \( sq' = p \). By the assumption of the theorem, we see that the numbers \( \int_W |\varphi_k(z)|^s \, dz \) are uniformly bounded. Now it follows from the lemma below that the functions \( \varphi_k(z) \) are locally uniformly bounded in \( W \) and therefore that all the other claims of the theorem are valid.

**Lemma 2** Let \( \Omega \) be a domain in \( \mathbb{C} \). Suppose that \( s \) and \( M \) are positive real numbers. Let \( \mathcal{F} \) be a family of holomorphic functions \( f : \Omega \to \mathbb{C} \) such that one of (a) and (b) below holds.

(a) For all \( f \in \mathcal{F} \), we have

\[
\int_\Omega \log^+ |f(x + iy)| \, dx \, dy \leq M.
\]

(b) For all \( f \in \mathcal{F} \), we have

\[
||f||_s = \left( \int_\Omega |f(x + iy)|^s \, dx \, dy \right)^{1/s} \leq M.
\]

Then the functions in \( \mathcal{F} \) are locally uniformly bounded in \( \Omega \), \( \mathcal{F} \) is a normal family in \( \Omega \), and the limit of every locally uniformly convergent sequence of functions in \( \mathcal{F} \) is holomorphic in \( \Omega \).

**Proof** Let \( f \in \mathcal{F} \).

Suppose that \( b \in \Omega \) and set \( d = \text{dist}(b, \partial \Omega) > 0 \). For \( 0 < r < d \), write

\[
L(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(b + re^{i\theta})| \, d\theta,
\]

\[
I_s(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(b + re^{i\theta})|^s \, d\theta.
\]

Since \( \log^+ |f| \) and \( |f|^s \) are subharmonic in the disk \( B(b, d) = \{ z : |z - b| < d \} \), we have

\[
\log^+ |f(b)| \leq L(r, f),
\]

\[
|f(b)|^s \leq I_s(r, f)
\]

for every \( r \in (0, d) \). Also, since \( \log^+ |f| \) and \( |f|^s \) are subharmonic in \( B(b, d) \), the functions \( L(r, f) \) and \( I_s(r, f) \) are increasing functions of \( r \) for \( 0 < r < d \). Denote

\[ Springer \]
the annulus \( \{ z : d/2 < |z - b| < d \} \) by \( A \). Applying each of the above inequalities with \( r = d/2 \), and then multiplying both sides by \( r \) and integrating with respect to \( r \) from \( d/2 \) to \( d \) we get

\[
(3/8)d^2 \log^+ |f(b)| = (1/2)(d^2 - (d/2)^2) \log^+ |f(b)|
\leq \int_{d/2}^{d} L(d/2, f) \, dr \leq \int_{d/2}^{d} L(r, f) \, r \, dr
= \frac{1}{2\pi} \int_A \log^+ |f(x + iy)| \, dx \, dy \leq \frac{M}{2\pi}
\]

if the assumption (a) is satisfied, and similarly

\[
(3/8)d^2 |f(b)|^s \leq \int_{d/2}^{d} I_s(r, f) \, r \, dr
= \frac{1}{2\pi} \int_A |f(x + iy)|^s \, dx \, dy \leq \frac{M^s}{2\pi}
\]

if the assumption (b) is satisfied.

Thus

\[
|f(b)| \leq \exp \left( \frac{4M}{3\pi d^2} \right)
\]

in case (a), and

\[
|f(b)| \leq M \left( \frac{4}{3\pi d^2} \right)^{1/s}
\]

in case (b). This proves that the functions in \( F \) are locally uniformly bounded in \( \Omega \), and the remaining claims now follow from standard results in complex analysis. This completes the proof of the lemma.

\[\square\]

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