ON THE KAPPA RING OF $\overline{M}_{g,n}$

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Abstract. Let $\kappa_e(\overline{M}_{g,n})$ denote the kappa ring of $\overline{M}_{g,n}$ in codimension $e$ (equivalently, in degree $d = 3g - 3 + n - e$). For $g, e \geq 0$ fixed, as the number $n$ of the markings grows large we show that the rank of $\kappa_e(\overline{M}_{g,n})$ is asymptotic to

$$\binom{n+e}{e} \binom{g+e}{e} \sim \frac{(g+e)n^e}{e!(e+1)!}.$$  

When $g \leq 2$ we show that a kappa class $\kappa \in \kappa^*(\overline{M}_{g,n})$ is trivial if and only if the integral of $\kappa$ against all boundary strata is trivial. For $g = 1$ we further show that the rank of $\kappa_{n-d}(\overline{M}_{1,n})$ is equal to $|P_1(d, n - d)|$, where $P_1(d, k)$ denotes the set of partitions $p = (p_1, \ldots, p_\ell)$ of $d$ such that at most $k$ of the numbers $p_1, \ldots, p_\ell$ are greater than $i$.

1. Introduction

Let $\epsilon : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ denote the universal curve over the moduli space $\overline{M}_{g,n}$ of stable genus $g$, $n$-pointed curves. Let $\mathbb{L}_i \rightarrow \overline{M}_{g,n+1}$ denote the cotangent line bundle over $\overline{M}_{g,n+1}$ with fiber over a point equal to the cotangent line at the $i$th marking. Define

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{M}_{g,n+1}) \quad \text{and} \quad \kappa_i = \epsilon_*(\psi_{n+1}^i) \in A^i(\overline{M}_{g,n}).$$

The push forwards of the $\kappa$ and $\psi$ classes from the boundary strata generate the tautological ring $R^*(\overline{M}_{g,n})$ [3, 6]. The kappa ring $\kappa^*(\overline{M}_{g,n})$ is the subring of $R^*(\overline{M}_{g,n})$ generated by $\kappa_1, \kappa_2, \ldots$ over $\mathbb{Q}$. Let $\kappa^d(\overline{M}_{g,n})$ denote the $\mathbb{Q}$-module generated by the kappa classes of degree $d$ and set $\kappa_e(\overline{M}_{g,n}) = \kappa^{3g-3+n-e}(\overline{M}_{g,n})$.

Applying the localization formula of [5] to the action of $\mathbb{C}^*$ on the moduli space of stable maps from curves of genus $g$ to $\mathbb{P}^1$ we prove the following theorem.

Theorem 1. Fix the genus $g$ and the codimension $e$. As the number $n$ of the marked points grows large, the rank of $\kappa_e(\overline{M}_{g,n})$ is asymptotic to

$$\frac{(g+e)(n+e)}{(e+1)!} \sim \frac{(g+e)n^e}{e!(e+1)!}.$$  

Let $G$ be a connected graph which is decorated by assigning a genus to each one of its vertices and let the markings $1, 2, \ldots, n$ get distributed among the vertices of $G$. For a vertex $v \in V(G)$ let $g_v$ denote the genus associated
with \( v, d_v \) denote the degree of \( v \), and \( n_v \) denote the number of markings assigned to \( v \). If \( 2g_v + n_v + d_v > 2 \) for every \( v \in V(G) \), \( G \) is called a \textit{stable weighted graph}. Every stable weighted graph \( G \) describes a \textit{combinatorial cycle} \([G]\) in \( \mathcal{M}_{g,n} \) where

\[
g = |E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} g_v.
\]

The tautological ring of \([G]\) is denoted by \( R^*([G]) \). An element \( \kappa \in \kappa^* (\mathcal{M}_{g,n}) \) is called \textit{combinatorially} trivial if for all relevant stable weighted graphs \( G \) as above \( \int_{[G]} \kappa = 0 \). Let \( \kappa_0^* (\mathcal{M}_{g,n}) \subset \kappa^* (\mathcal{M}_{g,n}) \) denote the set of combinatorially trivial classes and \( \kappa_c^* (\mathcal{M}_{g,n}) \) denote the quotient \( \kappa^* (\mathcal{M}_{g,n}) / \kappa_0^* (\mathcal{M}_{g,n}) \), which sits in the short exact sequence

\[
0 \to \kappa_0^* (\mathcal{M}_{g,n}) \to \kappa^* (\mathcal{M}_{g,n}) \to \pi \to 0.
\]

A more careful examination of the localization terms in our argument proves the following theorem.

**Theorem 2.** The map \( \pi \) is an isomorphism of graded algebras for \( g \leq 2 \).

Theorem 2 is a consequence of Keel’s Theorem [7] when \( g = 0 \) and follows from Petersen’s work [9] on the structure of the tautological ring for \( g = 1 \). Our argument in genus one is, however, different from Petersen’s argument.

Let \( P(d) \) denote the set of partitions of \( d \) and \( P_i(d,k) \) denote the set of \( \mathbf{p} = (p_1, ..., p_\ell) \in P(d) \) such that at most \( k \) of the numbers \( p_1, ..., p_\ell \) are greater than \( i \). Combining Theorem 2 with combinatorial arguments, the following theorem is also proved in this paper.

**Theorem 3.** The rank of \( \kappa^d(\mathcal{M}_{1,n}) \) is equal to \( |P_1(d,n-d)| \).

Let us now describe our strategy for bounding the rank of the kappa ring. Let \( \pi_{g,n} : \mathcal{M}_{g,n} \to \mathcal{M}_{g,n+\ell} \) denote the forgetful map which forgets the last \( \ell \) markings. For every \( \mathbf{p} = (p_1, ..., p_\ell) \in P(d) \) let

\[
\langle \mathbf{p} \rangle_{g,n} := \left( \pi_{g,n}^\ell \right)_* \left( \prod_{i=1}^\ell \frac{1}{1 - p_i \psi_{i+n}} \right) \in \kappa^* (\mathcal{M}_{g,n}).
\]

Let \( \langle \mathbf{p} \rangle_{g,n}^j \) denote the degree \( j \) part of \( \langle \mathbf{p} \rangle_{g,n} \). The bracket classes \( \{ \langle \mathbf{p} \rangle_{g,n}^d \}_{\mathbf{p} \in P(d)} \) generate the kappa ring \( \kappa^d(\mathcal{M}_{g,n}) \) (Lemma 2.2, c.f. Proposition 3 from [4]). For every positive integer \( l \), every partition \( \mathbf{n} = (n_1, ..., n_m) \in P(2d - 2g + 2 - n - l) \) and every partition \( \mathbf{p} = (p_1, ..., p_\ell) \in P(d) \) let

\[
C^\mathbf{p}_\mathbf{n} := \frac{(-1)^{\ell+\ell}}{\operatorname{Aut}(\mathbf{p})} \prod_{i=1}^\ell \frac{p_{i-1}}{p_i!} \sum_{\phi} \prod_{i=1}^m p_{\phi(i)}^{1-n_i},
\]
where the last sum is over all injections \( \phi : \{1, ..., m\} \to \{1, ..., \ell\} \). Let
\[
J_{\text{lead}}(n) := \sum_{p \in \mathcal{P}(d)} C_{\mathcal{P}}(p)^{d}_{g,n} \in \kappa^{d}(\overline{M}_{g,n}) .
\]

The kappa classes \( J_{\text{lead}}(n) \) are trivial over \( \mathcal{M}_{g,n}^c \subset \overline{M}_{g,n} \) (the moduli space of curves of compact type) \([10]\). If \( \kappa \in \kappa^*(\overline{M}_{g,n}) \) has trivial integrals over all combinatorial cycles which only contain genus zero components, then \( \kappa \) is a linear combination of the classes \( J_{\text{lead}}(n) \). In particular, so is every element of \( \kappa_{0}^*(\overline{M}_{g,n}) \).

A stable weighted graph \( G \) is called a comb graph if \( G \) is a tree, contains a distinguished vertex \( v_\infty \), and every vertex \( v \in V(G) \setminus \{v_\infty\} \) is connected with an edge \( e_v \) to \( v_\infty \). Furthermore, the markings \( 1, ..., n \) are all assigned to \( v_\infty \). If the sum of the genera associated with the vertices of \( G \) is \( g \), we get an embedding
\[
i^G : \frac{[G]}{\text{Aut}(G)} \to \overline{M}_{g,n},
\]
of the quotient of \([G]\) by the group of automorphisms in \( \overline{M}_{g,n} \). The genus associated with \( v_\infty \) is denoted by \( g_\infty(G) \). Since every \( \kappa \in \kappa_{0}^*(\overline{M}_{g,n}) \) is a linear combination of the classes \( J_{\text{lead}}(n) \) the localization argument of Section 8 from \([10]\) gives a presentation
\[
\kappa = \sum_{G: \text{comb}} \psi_G^*(\kappa), \quad \psi_G : \left[\pi_1^{*}(\lambda_1)\pi_2^{*}(\psi)\right], \quad \psi_G : \kappa \in R^*([G]).
\]
In particular, for \( g = 1 \), there is only one comb graph \( G \) with \( g_\infty(G) < 1 \). For this comb graph \([G]\) \( \simeq \overline{M}_{1,1} \times \overline{M}_{0,n+1} \). The above argument implies that every element \( \kappa \in \kappa_{0}^*(\overline{M}_{1,n}) \) is of the form
\[
\kappa = \psi_G \left[\pi_1^{*}(\lambda_1)\pi_2^{*}(\psi)\right], \quad \text{for some} \ \psi \in R^*\left(\overline{M}_{0,n+1}\right).
\]
If a combinatorially trivial class in the tautological ring of \( \overline{M}_{1,n} \) takes the form of the right-hand-side of the above equation one can quickly conclude that \( \psi = 0 \), and thus \( \kappa = 0 \).

In general, an inductive use of the above procedure, which represents the elements of an appropriate subspace of \( R^*(\overline{M}_{g,n}) \) in terms of the push-forwards of other tautological classes from boundary strata, is used in this paper. Let \( G_{g,n} \) denote the stable weighted graph whose underlying graph is illustrated in Figure 1. Thus \( V(G_{g,n}) = \{v_0, ..., v_g\} \), \( G_{g,n} \) has \( 2g \) edges and all the markings \( 1, ..., n \) are assigned to \( v_0 \). The stable weighted graph \( G_{g,n} \) determines an embedding
\[
i_{g,n} : \frac{\overline{M}_{0,n+g}}{S_g} \simeq \frac{\overline{M}_{0,n+g} \times \overline{M}_{0,3} \times \cdots \times \overline{M}_{0,3}}{S_g} = \frac{[G_{g,n}]}{\text{Aut}(G_{g,n})} \to \overline{M}_{g,n}.
\]
An inductive use of the above reduction scheme shows that the rank of
\[ Q^d(g, n) := \frac{\kappa^\partial_d(M_{g,n})}{\kappa^\partial_0(M_{g,n}) \cap \psi_0^{g,n}(R^{d-2g}(\mathcal{M}_{0,n+g}))} \]
is small, compared to the rank of \( \kappa^\partial_c(M_{g,n}) \). When \( g \) is small \( Q^d(g, n) \) is trivial. Moreover, for every stable weighted graph \( G \) which determines a combinatorial cycle in \( \mathcal{M}_{0,n+g} \) there is a constant \( C_G \) and a corresponding stable weighted graph \( \pi(G) \) which determines a combinatorial cycle in \( \mathcal{M}_{g,n} \) such that
\[ \int_{|G|} \psi = C_G \int_{|\pi(G)|} \psi^{g,n}_* (\psi). \]
In particular, by Keel's Theorem [2]
\[ \kappa^\partial_0(M_{g,n}) \cap \psi^{g,n}_*(R^{d-2g}(\mathcal{M}_{0,n+g})) = 0 \]

The above scheme describes the heart of our argument in this paper. Motivated by Theorem [2] we ask the following question.

**Question 1.** Is it true that \( \kappa^*_0(M_{g,n}) \) is trivial?

The authors started an investigation of the structure of \( \kappa^*_c(M_{g,n}) \) in [2] and proved that its rank in codimension \( e \), as the number \( n \) of the marked points grows large, is asymptotic to \( \frac{(g+c)(n+c)}{(e+1)!} \). Thus, Theorem [1] provides some evidence in support of an affirmative answer to Question [1] while Theorem [2] gives an affirmative answer to the aforementioned question for \( g \leq 2 \).

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2. Combinatorial cycles and the $\psi$ classes

It is sometimes more convenient to use alternative bases for the kappa ring of $\overline{M}_{g,n}$, instead of the kappa classes. Let

$$\pi_{g,n,k}^m : \overline{M}_{g,n+m} \rightarrow \overline{M}_{g,n+k}$$

denote the forgetful map which forgets the last $m - k$ markings.

**Definition 2.1.** For every multi-set $p = (p_1 \geq p_2 \geq \ldots \geq p_m)$ of positive integers with $p_k > 1$ and $p_i = 1$ for $k < i \leq m$ define

- $|p| := m$ and $d(p) := \sum_{i=1}^{m} p_i$
- $p^- := (p_1 - 1 \geq \ldots \geq p_k - 1) \in P(d(p) - m)$
- $\psi(p) := \psi(p_1, \ldots, p_m) := (\pi_{g,n}^m)^{\ast} \left( \prod_{i=1}^{m} \psi_{p_i+1}^{n} \right) \in \kappa^d(p)(\overline{M}_{g,n})$
- $\kappa(p) := \kappa(p_1, \ldots, p_m) := \prod_{i=1}^{m} \kappa_{p_i} \in \kappa^d(p)(\overline{M}_{g,n})$
- $\langle p \rangle_{g,n,k} := \left( \pi_{g,n}^{m, k} \right)^{\ast} \left( \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} \frac{1}{1 - p_{\sigma(i)} \psi_{p_i+1}} \right) \in R^d(\overline{M}_{g,n+k})$.

Let $\langle p \rangle = \langle p \rangle_{g,n,0}$ and let $\langle p \rangle_j^k$ denote the degree $j$ part of $\langle p \rangle$. Similarly, let $\langle p \rangle_{g,n,k}$ denote the degree $j$ part of $\langle p \rangle_{g,n,k}$ and set

$$\langle p \rangle_{g,n,k}^j := \langle p \rangle_{g,n,k}^{3g-3+n+k-j}.$$

**Lemma 2.2.** The subsets of $\mathcal{A}^d(\overline{M}_{g,n})$ defined by

$$\left\{ \psi(p) \mid p \in P(d) \right\}, \left\{ \kappa(p) \mid p \in P(d) \right\} \text{ and } \left\{ \langle p \rangle^d \mid p \in P(d) \right\}$$

are related by invertible linear transformations.

**Proof.** The transformation relating the first two sets (which is independent of $g$ and $n$) is due to Faber and is discussed in [1]. The transformation relating the first set to the third set is discussed in Proposition 3 from [4]. This later transformation only depends on $2g - 2 + n$. \hfill $\square$

Let $\Psi(d)$ denote the formal vector space over $\mathbb{Q}$ which is freely generated by the partitions $p \in P(d)$. There are surjections

$$\psi_{g,n}, \kappa_{g,n}, \langle \rangle_{g,n} : \Psi(d) \rightarrow \kappa^d(\overline{M}_{g,n})$$

which are defined by

$$\psi_{g,n} \left( \sum_{p \in P(d)} a_p p \right) := \sum_{p \in P(d)} a_p \psi(p), \quad \kappa_{g,n} \left( \sum_{p \in P(d)} a_p p \right) := \sum_{p \in P(d)} a_p \kappa(p)$$

and

$$\langle \sum_{p \in P(d)} a_p p \rangle_{g,n} := \sum_{p \in P(d)} a_p \langle p \rangle.$$

Lemma 2.2 implies that there are invertible matrices $P_d : \Psi(d) \rightarrow \Psi(d)$ and $Q_{d,m} : \Psi(d) \rightarrow \Psi(d)$ for $m \in \mathbb{Z}^+$ such that

$$\psi_{g,n} = \kappa_{g,n} \circ P_d \quad \text{and} \quad \langle \rangle_{g,n} = \kappa_{g,n} \circ Q_{d,n+2g}.$$
In [10] Pandharipande shows that associated with every pair of partitions 
\( \mathbf{m} \in \mathcal{P}(d) \setminus \mathcal{P}(d, 2g - 2 + n - d) \) and \( \mathbf{p} \in \mathcal{P}(d) \) 
there is a rational number \( C_{\mathbf{m}, \mathbf{p}} \) with the property that 

- The matrix \( (C_{\mathbf{m}, \mathbf{p}})_{\mathbf{m}, \mathbf{p}} \) is of full-rank, with rank equal to 
  \( |\mathcal{P}(d)| - |\mathcal{P}(d, 2g - 2 + n - d)| \).

- If we set 
  \( j(\mathbf{m}^-) := \sum_{\mathbf{p} \in \mathcal{P}(d)} C_{\mathbf{m}, \mathbf{p}} \mathbf{p} \in \Psi(d) \) 
the restriction of \( \langle j(\mathbf{m}^-) \rangle_{g,n} \in \kappa^* (\overline{\mathcal{M}}_{g,n}) \) to \( \mathcal{M}_{g,n}^c \) is trivial.

For \( g = 0 \), these relations among the kappa classes are all the possible relations. In other words, the kernel of the map 
\( \langle \rangle_{0,n} : \Psi(d) \to \kappa^d(\overline{\mathcal{M}}_{0,n}) \) 
is generated by \( \{ j(\mathbf{m}^-) \}_m \). This observation gives a surjection 
\( T^c_{g,n} : \kappa(\overline{\mathcal{M}}_{0,n+2g}) \to \kappa(\mathcal{M}_{g,n}^c). \) 

This map translates every kappa class over \( \overline{\mathcal{M}}_{0,n+2g} \) to a kappa class over \( \mathcal{M}_{g,n}^c \), which comes from the same formal expression. Nevertheless, the aforementioned map does not have a clear geometric meaning (at least to the authors). We will encounter such homomorphisms again when we try to obtain relations among the kappa classes in this paper.

**Definition 2.3.** A weighted graph \( G \) is a finite connected graph with the set \( V(G) \) of vertices, the set \( E(G) \) of edges and a weight function 
\( \epsilon = \epsilon_G : V(G) \to \mathbb{Z}^{\geq 0} \times 2^{\{1,\ldots,n\}}, \)
where \( 2^{\{1,\ldots,n\}} \) denotes the set of subsets of \( \{1,\ldots,n\} \). For \( i \in V(G) \) denote the degree of \( i \) by \( d_i = d(i) \) and let \( \epsilon(i) = (g_i, I_i) \). \( G \) is called a stable weighted graph if \( \{I_i\}_{i \in V(G)} \) is a partition of \( \{1,\ldots,n\} \) and for every vertex \( i \in V(G) \), \( 2g_i + |I_i| + d_i > 2 \). Define \( n(G) = n \) and 
\( g(G) := \left( \sum_{i \in V(G)} g_i \right) + |E(G)| - |V(G)| + 1. \)

Associated with a stable weighted graph \( G \) there is a natural map 
\( \iota_G : \mathcal{C}(G) = \prod_{i \in V(G)} \overline{\mathcal{M}}_{g_i, |I_i| + d_i} \to \overline{\mathcal{M}}_{g(G), n(G)} \)
which is an embedding after we mod out the source by its automorphisms. Thus, a stable weighted graph \( G \) determines a combinatorial cycle 
\( [G] := (\iota_G)_* \mathcal{C}(G) \in A_d (\overline{\mathcal{M}}_{g(G), n(G)}) \).
where $d = 3g(G) - 3 + n(G) - \lvert E(G) \rvert$.

Let $H = H_{g,n}$ be a stable weighted graph with a single vertex $v$, $g$ self edges from $v$ to itself, and with $\epsilon(v) = (0, \{1, \ldots, n\})$. $H$ determines a homomorphism $\imath^H : \overline{M}_{0,n+2g} \to \overline{M}_{g,n}$. If $G$ is a stable weighted graph with $g(G) = 0$ and $n(G) = n + 2g$ then

$$\int_{\hat{G}[G]} \langle q \rangle_{g,n} = \frac{1}{249 \times g!} \int_{[G]} \langle q \rangle_{0,n+2g} \, \forall \, q \in P(d).$$

**Remark 2.4.** Let $\kappa = \langle \psi \rangle_{g,n}$ for some $\psi \in \Psi(d)$ is such that $\int_{\hat{G}[G]} \kappa = 0$ for all stable weighted graph $G$ with $g(G) = 0$ and $n(G) = n + 2g$. By the above observation, $\langle \psi \rangle_{0,n+2g}$ has trivial integral over all combinatorial cycles, and is thus trivial by Keel’s Theorem \cite{Keel}. By Pandharipande’s result this implies that $\psi = \sum_m a_m j(m^-)$, and thus

$$\kappa = \sum_m a_m \langle j(m^-) \rangle_{g,n}.$$

Fix the stable weighted graph $G$ and $\psi(p) = \psi(p_1, \ldots, p_k)$ with $n = n(G)$, $g = g(G)$, $d(p) + \lvert E(G) \rvert = 3g - 3 + n$ and $V(G) = \{1, \ldots, m\}$. Let

$$Q = \{ (h, r) \in \mathbb{Z}^\geq \times \mathbb{Z}^\geq \mid 2h + r > 2 \}.$$

The **modified weight** multi-set associated with $G$ is the multi-set

$$q_G := \{ (\theta_G(i) \in Q \mid i \in \{1, \ldots, m\} \}, \quad \text{where}\quad \theta_G(i) := (g, m_i = \lvert I_i \rvert + d_i), \quad \forall \, 1 \leq i \leq m.$$  

The integral

$$\langle \psi(p) , [G] \rangle = \int_{[G]} \psi(p) = \int_{(x^m)_{g,n,k}} \prod_{j=1}^k \psi^{\theta_j + 1}_{n+j} \in \mathbb{Q}$$

only depends on the multi-set $q_G$ \cite{Pandharipande}. We denote the value of the above integral by $\langle \psi(p), q_G \rangle_{g,n}$, or just $\langle \psi(p), q \rangle$ if there is no confusion. We denote by $Q(d;g,n)$ the set of all multi-sets $q = (\theta_i)_{i=1}^m$ such that $q = q_G$ for some stable weighted graph $G$ with $g = g(G)$, $n = n(G)$ and $d = 3g - 3 + n - \lvert E(G) \rvert$. 

A kappa class $\kappa \in \kappa^{d}(\overline{M}_{g,n})$ is combinatorially trivial if $\langle \kappa, q \rangle = 0$ for all $q \in Q(d;g,n)$.

For a partition $n = (n_1, \ldots, n_k) \in \mathbb{P}(n)$ of length $k = \lvert n \rvert$, let

$$q_0(n) = \{(0, n_1 + 2), \ldots, (0, n_k + 2)\} \in Q(n - k; 1, n), \quad \text{and}$$

$$q_1(n) = \{(1, l), (0, n_1 + 2), \ldots, (0, n_k + 2)\} \in Q(n + l - k; 1, n + l), \quad l \geq 1.$$  

Every element of $Q(d; 1, n)$ is of the form $q_1(n)$ for some non-negative integer $l$ and some $n \in \mathbb{P}(n - l; n - d)$. Here $\mathbb{P}(d; k)$ denotes the set of the partitions of $d$ into precisely $k$ parts.
If \( n, m \in P(d) \) define \( n < m \) if \( n \) refines \( m \). This partial ordering may be extended to a total ordering on \( P(d) \). We fix one such total ordering and will refer to it as the refinement ordering. Define

\[ p : Q(d; g, n) \to P(d, 3g - 2 + n - d) \]

by sending \( q = \{(g_i, n_i)\}_{i=1}^k \to \{3g_i - 3 + n_i\}_{i=1}^k \). If \( \langle \psi(n), q \rangle \) is non-trivial for some \( q \in Q(d; g, n) \) then \( n \) refines \( p(q) \) [2].

3. Localization and moduli space of stable maps to \( \mathbb{P}^1 \)

3.1. The vanishing cycles. Fix the integers \( m \geq k \geq 0 \) and let

\[ \overline{M}_{g,n+m}(\mathbb{P}^1, d) \]

denote the moduli space of stable maps of degree \( d \) from curves of genus \( g \) with \( n + m \) marked points to \( \mathbb{P}^1 \) and denote by

\[ \epsilon : \overline{M}_{g,n+m}(\mathbb{P}^1, d) \to \overline{M}_{g,n+k} \]

the homomorphism which forgets the map and the last \( m - k \) markings.

Let \( \mathbb{C}^* \) act on \( V = \mathbb{C} \oplus \mathbb{C} \) by

\[ \zeta(z_1, z_2) = (z_1, \zeta z_2) \quad \forall \zeta \in \mathbb{C}^*, \ (z_1, z_2) \in \mathbb{C} \oplus \mathbb{C}. \]

Let \( p_0 = [0 : 1] \) and \( p_\infty = [1 : 0] \) denote the fixed points of the corresponding action on \( \mathbb{P}^1 = \mathbb{P}(V) \). For every line bundle \( L \to \mathbb{P}(V) \), an equivariant lifting of the \( \mathbb{C}^* \) action to \( L \) is determined by the weights \( l_0 \) and \( l_\infty \) of the fiber representations \( L_0 = L|_{p_0} \) and \( L_\infty = L|_{p_\infty} \). The canonical lift of the action for \( T_{\mathbb{P}^1} \) has weights \([l_0, l_\infty] = [1, -1]\).

Let \( \pi : U_{g,n+m}(\mathbb{P}(V), d) \to \overline{M}_{g,n+m}(\mathbb{P}(V), d) \) denote the universal curve and \( \mu : U_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V) \) denote the universal map. The action of \( \mathbb{C}^* \) on \( \mathbb{P}(V) \) induces \( \mathbb{C}^* \) actions on \( U_{g,n+m}(\mathbb{P}(V), d) \) and \( \overline{M}_{g,n+m}(\mathbb{P}(V), d) \) compatible with \( \pi \) and \( \mu \). Let

\[ [\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^\text{vir} \in A_{2g+2d-2+n+m}^*(\overline{M}_{g,n+m}(\mathbb{P}(V), d)) \]

denote the \( \mathbb{C}^* \)-equivariant virtual fundamental class of \( \overline{M}_{g,n+m}(\mathbb{P}(V), d) \) [5].

Following [10] we consider three types of equivariant chow classes over the moduli space \( \overline{M}_{g,n+m}(\mathbb{P}(V), d) \):

- The linearization \([0, 1]\) on \( O_{\mathbb{P}(V)}(-1) \) defines the \( \mathbb{C}^* \) action on the rank \( d + g - 1 \) bundle

\[ \mathbb{R} = \mathbb{R}^1 \pi_* (\mu^* O_{\mathbb{P}(V)}(-1)) \to \overline{M}_{g,n+m}(\mathbb{P}(V), d). \]

We denote the top Chern class of this bundle by

\[ c_{\text{top}}(\mathbb{R}) \in A_{d+g-1}^* (\overline{M}_{g,n+m}(\mathbb{P}(V), d)). \]
• For each marking $i$, let $\psi_i \in A^{1,*}_C(\mathcal{M}_{g,n+m}(\mathbb{P}(V), d))$ denote the first Chern class of the canonically linearized cotangent line corresponding to the $i^{th}$ marking.

• With $\text{ev}_i : \mathcal{M}_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V)$ denoting the $i$-th evaluation map and with the $\mathbb{C}^*$-linearization $[1, 0]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let

$$\rho_i = c_1(\text{ev}_i^* \mathcal{O}_{\mathbb{P}(V)}(1)) \in A^{1,*}_C(\mathcal{M}_{g,n+m}(\mathbb{P}(V), d)),$$

while with the $\mathbb{C}^*$ linearization $[0, -1]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$ we let

$$\tilde{\rho}_i = c_1(\text{ev}_i^* \mathcal{O}_{\mathbb{P}(V)}(1)) \in A^{1,*}_C(\mathcal{M}_{g,n+m}(\mathbb{P}(V), d)).$$

Note that in the non-equivariant limit $\rho_i^2 = 0$, and that $\epsilon$ is equivariant with respect to the trivial action on $\mathcal{M}_{g,n+k}$.

Fix the cycle dimension $e$ and the sequence $\mathbf{n} = (n_1, \ldots, n_m)$ with

$$\sum_{i=1}^{m} n_i = d + g - 1 - e - l, \quad l > 0,$$

which determines a partition in $P(d + g - 1 - e - l; m)$, denoted by $\mathbf{n}$ by slight abuse of the notation. The partition $\mathbf{n}$ is of the form $\mathbf{n} = \mathbf{m}^-$, where $\mathbf{m} \in P(d) \setminus P(d, \max\{e - g + 1, k - 1\})$.

Let $I(\mathbf{n}) = I(\mathbf{n}; d, g, n, k)$ denote the $\mathbb{C}^*$-equivariant push-forward

$$\epsilon_*(\rho_{n+1}^I \prod_{i=1}^{m} \rho_{i+n} \psi_{i+n}^{n_i} \prod_{j=1}^{n} \tilde{\rho}_j \ c_{top}(\mathbb{R}) \cap [\mathcal{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir}).$$

Since the degree of

$$\rho_{n+1}^I \left( \prod_{i=1}^{m} \rho_{i+n} \psi_{i+n}^{n_i} \right) \left( \prod_{j=1}^{n} \tilde{\rho}_j \right) c_{top}(\mathbb{R})$$

is $2d + 2g - e - 2 + n + m$ and the cycle dimension of the virtual fundamental class is $2d + 2g - 2 + n + m$, the cycle dimension of the class $I(\mathbf{n})$ is

$$e = (2d + 2g - 2 + n + m) - (2d + 2g - e - 2 + n + m).$$

In other words, $I(\mathbf{n}) \in A^{2g-3+n+k-e}_{\mathbb{C}^*}(\mathcal{M}_{g,n+k})$. Since the exponent of $\rho_{n+1}$ is at least 2, $I(\mathbf{n})$ vanishes in the non-equivariant limit.

3.2. The localization terms. The virtual localization formula of [5] may be used to calculate $I(\mathbf{n})$ in terms of the tautological classes on $\mathcal{M}_{g,n+k}$. The sum in the localization formula is over connected decorated graphs $\Gamma$ (indexing the $\mathbb{C}^*$-fixed loci of $\mathcal{M}_{g,n+m}(\mathbb{P}(V), d)$). Every vertex of $\Gamma$ either lies over $p_0$ or over $p_{\infty}$, and is labelled by a genus. The edges of the graph lie over $\mathbb{P}^1$ and are labelled with degrees (of the maps corresponding to the
edges). The total sum of these degrees is equal to $d$. The graphs carry $n + m$
markings over their vertices. For a vertex $v$ of $\Gamma$ let $d(v)$ denote the degree of $v$.

If a graph $\Gamma$ has a vertex $v$ over $p_0$ with $d(v) > 1$, $v$ yields a trivial Chern
root of the bundle $\mathbb{R}$ with trivial weight 0 in the numerator of the localization
formula, by our choice of linearization on the bundle $\mathbb{R}$. Hence the contribution of such graphs to the sum in the localization formula is trivial. Thus, only comb graphs $\Gamma$ contribute to $I(n)$. Every comb graph contains
a set $V_0 = V_0(\Gamma)$ of vertices which lie over $p_0$, and each $v \in V_0$ is connected
by an edge to a unique vertex $v_\infty$ which lies over $p_\infty$. The linearization of
the classes $\rho_{n+1}, \ldots, \rho_{n+m}$ and $\tilde{\rho}_1, \ldots, \tilde{\rho}_n$ implies that the first $n$ markings lie
on $v_\infty$ and the last $m$ markings are placed on the vertices in $V_0$. For every $v \in V_0$ let $g_v$ denote the genus associated with $v$ and let $I_v \subset \{1, \ldots, m\}$
determine the subset of the last $m$ markings which is associated with $v$. Note
that the genus associated with $v_\infty$ is $g_\infty = g - \sum_{v \in V_0} g_v$. Denote the degree
associated with the edge connecting $v$ to $v_\infty$ by $p_v$. The fixed locus associated
with the decorated comb graph $\Gamma$ is thus determined by a multi-set
$\{ (g_v, p_v, I_v) \}_{v \in V_0}$ such that $\sum g_v \leq g$, $\sum_v p_v = d$ and $\{ I_v \}_{v}$ is a partition
of $\{1, \ldots, m\}$. We abuse the notation and use $\Gamma$ to refer to this associated
multi-set. The partition $(p_v)_{v \in V_0(\Gamma)}$ of $d$ is denoted by $\mathbf{p}_\Gamma$.

The group $S_\Gamma$ of permutations $\sigma : V_0 \to V_0$ of the vertices in $V_0 = V_0(\Gamma)$
acts on the multi-set associated with $\Gamma$ by sending $\{(g_v, p_v, I_v)\}_{v \in V_0}$ to
$\{(g_{\sigma(v)}, p_{\sigma(v)}, I_{\sigma(v)})\}_{v \in V_0}$. We denote the image of $\Gamma$ under the action of $\sigma \in S_\Gamma$ by $\sigma(\Gamma)$. The automorphism group of $\Gamma$ consists of the permutations $\sigma$ of the vertices such that for every vertex $v \in V_0$ either $I_v = I_{\sigma(v)} = \emptyset$ and $g_{\sigma(v)} = g_v$, or $\sigma(v) = v$. We denote the group of automorphisms of $\Gamma$ by $\text{Aut}(\Gamma)$.

If $I_v = \{i_1, \ldots, i_{k_v}\}$ the fixed locus corresponding to $\Gamma$ contains a product
factor $\overline{\mathcal{M}}^{[1]} \cong \overline{\mathcal{M}}_{g_v, k_v+1}$, provided that $2g_v + k_v > 1$. The subset $I_v$ labels
$k_v$ of the markings on $\overline{\mathcal{M}}^{[1]}$ and we use the vertex $v$ itself to label the last marking
on this moduli space. The classes $\psi_{i_{j+n}}$ carry trivial $\mathbb{C}^*$ weight. Moreover, the integrand term $\epsilon_{\text{top}}(\mathbb{R})$ yields a factor $\lambda_{g_v}$ on $\overline{\mathcal{M}}^{[1]}$. Thus, we obtain the class $\lambda_{g_v, \psi_{i_{j+n}}}(n) \in A^*(\overline{\mathcal{M}}^{[1]})$ where

$$
\psi_{i_{j+n}} := \prod_{i=1}^{k_v} \psi_{i_{j+n}} \in A^{n_{i_1} + \cdots + n_{i_{k_v}}} \left( \overline{\mathcal{M}}^{[1]} \right)
$$

over this product factor, which is trivial unless

$$
\sum_{j=1}^{k_v} (n_{i_j} - 1) \leq 2g_v - 2.
$$
ON THE KAPPA RING OF $\mathcal{M}_{g,n}$

In particular, $(g_v, k_v) \neq (0, i)$ with $i > 1$. In other words, if $g_v = 0$ the vertex $v$ can accommodate at most one of the markings from $\{n + 1, \ldots, n + m\}$.

Let $\mathcal{M}^\infty_{g,n} = \mathcal{M}_{g, n+|V_0(\Gamma)|}$, where the last $|V_0(\Gamma)|$ markings are again labelled by the vertices in $V_0(\Gamma)$. Denote the subset of genus zero vertices in $V_0$ with no markings on them by $V^0 = V^0(\Gamma)$, the subset of genus zero vertices $v$ with one marking by $V^1 = V^1(\Gamma)$ and set $V^2 = V^2(\Gamma) = V_0 \setminus (V^0 \cup V^1)$. For $v \in V^1$, if $I_v$ consists of the single element $i \in \{1, \ldots, m\}$ we set $n_v = i$.

The contribution of the fixed locus corresponding to $\Gamma$ to $I(n)$ may be computed following [10]. The only difference is that in this case

- The contribution from the deformation of the source (i.e. smoothing the nodes) adds an extra product factor

$$\prod_{v \in V^2(\Gamma)} \frac{1}{l + \psi_v}.$$  

A power $\psi_v^{m_v}$ of $\psi_v$ thus appears in the contribution of $\Gamma$ to $I(n)$ for smoothing the node corresponding to the vertex $v$.

- The deformation of the map contributes a factor of $e(E^* \otimes 1)$ over each one of the components in the fixed locus which are mapped to $p_0$ (i.e. over each $\mathcal{M}^{v,\Gamma}$ with $v \in V^2(\Gamma)$). The Euler class of $E^* \otimes 1$ over the product factor corresponding to a vertex $v \in V^2(\Gamma)$ can contribute via a lambda-class $(-1)^{g_v-h_v} \lambda_{h_v}$ for some integer $0 \leq h_v \leq g_v$. Since $\lambda_{g_v}^2 = 0$ over $\mathcal{M}^{v,\Gamma}$ we may further assume that $h_v < g_v$.

The terms corresponding to $\Gamma$ in $I(n)$ are thus indexed by the set $c(\Gamma)$ of the multi-sets $c = (h_v, m_v)_{v \in V^2(\Gamma)}$ with such that

- $0 \leq h_v < g_v$.
- $0 \leq m_v \leq 2g_v - h_v - 2 - \sum_{i \in I_v} n_i$.

Let $\mathcal{M}^\Gamma$ denote the fixed locus corresponding to $\Gamma$ and $\pi_v : \mathcal{M}^\Gamma \to \mathcal{M}^{v,\Gamma}$ denote the projection map over the product factor corresponding to the vertex $v$. Denote the projection from $\mathcal{M}^\Gamma$ to $\mathcal{M}^\infty_{g,n}$ by $\pi_\infty$. The restriction of $\epsilon$ to $\mathcal{M}^\Gamma$ gives a map from $\mathcal{M}^\Gamma$ to $\mathcal{M}_{g,n+k}$. The contribution corresponding to $\Gamma$ and $c = (h_v, m_v)_{v \in V^2(\Gamma)} \in c(\Gamma)$ takes the form

$$I(n, \Gamma, c) = B(n, \Gamma, c) \epsilon_* \left( \psi(n, \Gamma, c) \cap [\mathcal{M}^\Gamma]^\text{vir} \right)$$
where
\[
\psi(n, \Gamma, c) := \left( \prod_{v \in V^2(\Gamma)} \pi_v^* \left( \lambda_{g_v} \lambda_{h_v} \psi_v^m \psi_v, \Gamma(n) \right) \right) \pi_\infty^* \left( \prod_{v \in V_0(\Gamma)} \frac{1}{1 - p_v \psi_v} \right) e(n, \Gamma, c)
\]

\[
e(n, \Gamma, c) = e - \sum_{v \in V^2(\Gamma)} \left( I_v + 2g_v - 2h_v - m_v - \deg(\psi_v, \Gamma(n)) \right)
\]

and the coefficient \(B(n, \Gamma, c)\) is defined by
\[
\frac{1}{\text{Aut}(\mathfrak{pr})} \left( \prod_{v \in V^{0}(\Gamma)} \frac{p_v^{p_v-1}}{p_v!} \right) \left( \prod_{v \in V^{1}(\Gamma)} \frac{p_v^{p_v-m_v}}{p_v!} \right) \left( \prod_{v \in V^{2}(\Gamma)} \frac{p_v^{p_v+m_v+1}}{p_v!} \right)
\]

Here, for a tautological class \(\psi\), \((\psi)_l\) denotes the part of \(\psi\) corresponding to the cycle dimension \(l\).

For \(\Gamma\) as above and \(v \in V^2(\Gamma)\), set \(J_v = I_v \cap \{1, ..., k\}\) and define
\[
k(\Gamma) = \left| \left\{ n_v \mid v \in V^1(\Gamma) \right\} \cap \{1, ..., k\} \right| + |V^2(\Gamma)|.
\]

Correspondingly, set
\[
\mathcal{N}^{v, \Gamma} = \mathcal{M}_{g_v, 1 + |J_v|}, \quad \forall v \in V^2(\Gamma), \quad \mathcal{N}^{0, \Gamma} = \prod_{v \in V^2(\Gamma)} \mathcal{N}^{0, \Gamma}
\]

\[
\mathcal{N}^{\infty, \Gamma} = \mathcal{M}_{g_\infty, n+k(\Gamma)} \quad \text{and} \quad \mathcal{N}^{\Gamma} = \mathcal{N}^{0, \Gamma} \times \mathcal{N}^{\infty, \Gamma}.
\]

Let \(\pi^{\infty, \Gamma} : \mathcal{M}^{\infty, \Gamma} \to \mathcal{N}^{\infty, \Gamma}\) denote the forgetful map which forgets the markings \(n + k + 1, ..., n + m\). Denote the projection maps from \(\mathcal{N}^{\Gamma}\) to \(\mathcal{N}^{0, \Gamma}\) and \(\mathcal{N}^{\infty, \Gamma}\) by \(q_0\) and \(q_\infty\), respectively, and define \(q_1^* = q_\infty \circ \pi^{\infty, \Gamma}\). The map \(\epsilon : \mathcal{N}^{\Gamma} \to \mathcal{M}_{g, n+k}\) factors through an embedding of \(\mathcal{N}^{\Gamma}\) in \(\mathcal{M}_{g, n+k}\). Thus, there are Chow classes \(\eta_i(n, \Gamma, c) \in R_i(\mathcal{N}^{0, \Gamma})\) with the property that

\[
(1) \quad I(n, \Gamma, c) = j_\Gamma^* \left( \sum_{i=0}^{c} q_0^* \left( \eta_i(n, \Gamma, c) \right) q_1^* \left( \prod_{v \in V_0(\Gamma)} \frac{1}{1 - p_v \psi_v} \right) e_{-i} \right),
\]

where \(j_\Gamma : \mathcal{N}^{\Gamma} \to \mathcal{M}_{g, n+k}\) is the embedding of the quotient of \(\mathcal{N}^{\Gamma}\) by its automorphisms in \(\mathcal{M}_{g, n+k}\).

Associated with the sequence \(n\), we obtain the following relation in the tautological ring of \(\mathcal{M}_{g, n+m}\)

\[
(2) \quad \sum_{\Gamma} \sum_{c \in c(\Gamma)} I(n, \Gamma, c) = 0
\]

where \(I(n, \Gamma, c)\) is given as in (1).
ON THE KAPPA RING OF $\overline{\mathcal{M}}_{g,n}$

3.3. The leading term. Among all $\Gamma$, we distinguish the decorated comb graphs with $V^2(\Gamma) = \emptyset$, and denote the set of such graphs by $\mathcal{J} = \mathcal{J}(g, n, m, d)$. For $\Gamma \in \mathcal{J}$ the set $c(\Gamma)$ is trivial, and the corresponding coefficient is

$$B(n, \Gamma) = \frac{1}{\text{Aut}(\mathcal{P}_\Gamma)} \left( \prod_{v \in V^0(\Gamma)} \frac{p_v^{p_v-1}}{p_v!} \right) \left( \prod_{v \in V^1(\Gamma)} \frac{p_v^{p_v-n_v}}{p_v!} \right)$$

We may thus compute

$$I(n, \Gamma) = B(n, \Gamma) \sum_{c} J(n, \Gamma, c)$$

We call the expression

$$I^{\text{lead}}(n) = \sum_{\Gamma \in \mathcal{J}} I(n, \Gamma)$$

the leading term in the relation of Equation (2).

Associated with a partition in $P(d + g - 1 - e - l; m)$ which is represented by the sequence $n$ there are $m!/\text{Aut}(n)$ different sequences which may be constructed from $n$. For $\sigma \in S_m$, let $\sigma(n)$ denote the sequence obtained by applying the permutation $\sigma$ to $n$. Let

$$J(n, \Gamma, c) := \frac{1}{m!} \sum_{\sigma \in S_m} I(\sigma(n), \Gamma, c) \quad \text{and}$$

$$J(n) := \frac{1}{m!} \sum_{\sigma \in S_m} I(\sigma(n)) = \sum_{\Gamma} \sum_{c \subset c(\Gamma)} J(n, \Gamma, c).$$

Note that $J(n, \Gamma, c)$ and $J(n)$ depend on the partition associated with $n$, and not the sequence $n$ itself.

Let $\Psi_e(d; g, n, k)$ denote the $\mathbb{Q}$-module generated by the subset

$$\left\{ (p)^{g,n,k}_e \mid p \in P(d) \setminus P(d, k - 1) \right\} \subset R_e(\overline{\mathcal{M}}_{g,n+k}),$$

and set $\Psi_e(d; g, n) = \Psi_e(d; g, n, 0)$. The expressions

$$J^{\text{lead}}(n) = \frac{1}{m!} \sum_{\sigma \in S_m} I^{\text{lead}}(\sigma(n))$$

belong to $\Psi_e(d; g, n, k)$.

Suppose that $\Gamma$ is a decorated comb graph which contributes to $I^{\text{lead}}(n)$. Associated with $\Gamma$ is the partition $\mathcal{P}_\Gamma = (p_v)_{v \in V^0(\Gamma)}$ in $P(d)$. The partition associated with $\sigma(\Gamma)$ is $\mathcal{P}_\sigma$ as well. Given $p = (p_1, ..., p_\ell) \in P(d)$, in order to determine the decorated comb graph $\Gamma$ which contributes to $I^{\text{lead}}(n)$ and satisfies $\mathcal{P}_\Gamma = p$ we need to specify an injection $\phi : \{1, ..., m\} \to \{1, ..., \ell\}$,
which determines the $m$ vertices in $V_0(\Gamma) = \{1, \ldots, \ell\}$ which carry the markings $\{n + 1, \ldots, n + m\}$. Let

$$J(n, p) = \sum_{\Gamma \in J: \ p_\Gamma = p} J(n, \Gamma).$$

The above observation implies that

$$J(n, p) = \prod_{i=1}^{\ell} \frac{p_i^{n_i} - 1}{p_i!} \sum_{\phi} \prod_{j=1}^{m} p_{\phi(i)}^{1-n_i} \left( \pi_{g,n,k} \right)_e \left( \frac{1}{(p_i! \sum_{\sigma \in S_i} \prod_{i=1}^{\ell} 1 - p_{\sigma(i)} \psi_{n_i + 1}} \right)_e \sum_{\phi} m \prod_{j=1}^{m} p_{\phi(i)}^{a.m.k}.$$

Let us define

$$\Phi_e(d; g, n, k) := \left\langle f_{\text{lead}}(m^-) \mid m \in P(d) \setminus P(d, \max\{e - g + 1, k - 1\}) \right\rangle_{Q}.$$

**Proposition 3.1.** The $\mathbb{Q}$-module $\Psi_e(d; g, n, k)/\Phi_e(d; g, n, k)$ is generated by the classes in

$$G_e(d; g, n, k) = \left\{ \langle p \rangle_e^{g.n.k} \mid p \in P(d, e - g + 1) \setminus P(d, k - 1) \right\}.$$

**Proof.** The coefficients $C^n_p$ form a matrix $C$ which is a minor of the matrix $M_0(d)$, studied in Proposition 9.1 from [10]. It is shown that there is an upper triangular (with respect to the refinement ordering) square matrix $Y$ whose rows and columns are indexed by the partitions in $P(d)$ such that $M_0(d)Y$ is lower-triangular (with respect to the same order) with non-zero diagonal entries. The minor $C$ of $M_0(d)$ corresponds to the partitions in

$$P(d) \setminus P(d, \max\{e - g + 1, k - 1\}).$$

The matrix $CY$ is thus lower triangular. Note that $CY$ is not a square matrix, but the number of its rows is greater than or equal to the number of its columns, and it makes sense for $CY$ to be lower triangular with non-zero entries on the diagonal. In particular, the classes $\langle p \rangle_e^{g.n.k}$ with $p \in P(d) \setminus P(d, e - g + 1)$ may be expressed in terms of the classes in $\Phi_e(d; g, n, k)$, as well as the classes in $G_e(d; g, n, k)$.

4. THE ASYMPTOTIC BEHAVIOUR OF THE RANK OF THE KAPPA RING

Let $f_1, f_2 : \mathbb{Z}^n \to \mathbb{R}$ be real valued functions. If

$$\limsup_{n \to \infty} \frac{f_1(n)}{f_2(n)} \leq 0,$$

we write $f_1(n) \in o(f_2(n))$. 
Theorem 4.1. Fix the genus $g$, the integer $k \geq 0$, the cycle dimension $e$ and the difference $n - d \in \mathbb{Z}$. Then
\[
\text{rank} \left( \Psi_e(d; g, n, k) \right) - \frac{(n + e + (g + e)k + e)}{(e + 1)!} \in o(n^e).
\]

Proof. First of all, for every $n = m -$ as in Section 3, using (1) we have
\[
j^{\text{lead}}(n) = -\sum_{\Gamma} \sum_{c \in \mathcal{C}(\Gamma)} J(n, \Gamma, c)
\]
\[
= -\frac{1}{m!} \sum_{\sigma \in S_m} \sum_{0 \leq i \leq e} \sum_{\Gamma, c} j^\Gamma_e \left( q_0^\Gamma \left( \eta_i(n), \Gamma, c \right) \frac{1}{1 - p_0 \psi_v} \right) \cap \left( \mathcal{N}^\Gamma \right)^{\text{vir}}
\]
\[
\in \sum_{\Gamma} \sum_{i = 0}^e j^\Gamma_e \left( R_i(\mathcal{N}^0, \Gamma) \otimes \Psi_{e-i}(d; g_\infty(\Gamma), n, k(\Gamma)) \right).
\]

Here, the summation notation is used for summing the subspaces of the tautological ring of $\mathcal{M}_{g,n+k}$ and the sums are over all graphs $\Gamma$ with $g_\infty(\Gamma) < g$.

Proposition 3.1 implies that
\[
\text{rank} \left( \Psi_e(d; g, n, k) \right) - \text{rank} \left( \Phi_e(d; g, n, k) \right) \in o(n^e).
\]

We now use induction on the genus $g$ to prove Theorem 4.1. The induction hypothesis implies that the rank of $R_i(\mathcal{N}^0, \Gamma) \otimes \Psi_{e-i}(d; g_\infty(\Gamma), n, k(\Gamma))$ belongs to $o(n^e)$ unless $i = 0$, since the rank of the ring $R_i(\mathcal{N}^0, \Gamma)$ does not grow with $n$. In other words,
\[
\text{rank} \left( \frac{\Psi_e(d; g, n, k)}{\Psi_e(d; g, n, k) \cap \sum_{\Gamma} j^\Gamma_e \left( \Psi_e(d; g_\infty(\Gamma), n, k(\Gamma)) \right)} \right) \in o(n^e),
\]
where we abuse the notation by setting
\[
j^\Gamma_e \left( \Psi_e(d; g_\infty(\Gamma), n, k(\Gamma)) \right) := j^\Gamma_e \left( R_0(\mathcal{N}^0, \Gamma) \otimes \Psi_e(d; g_\infty(\Gamma), n, k(\Gamma)) \right).
\]

In order to compute the asymptotic rank of $\Psi_e(d; g, n, k)$, we thus restrict ourselves to its subspace which consists of the push-forwards from the strata corresponding to the comb graphs $\Gamma$, and over the corresponding cycle $\mathcal{M}^\Gamma = \mathcal{M}^0, \Gamma \times \mathcal{M}^{\infty, \Gamma}$ we assume that the tautological class is the product of the point class from $\mathcal{M}^0, \Gamma$ and a class in $\Psi_e(d; g_\infty(\Gamma), n, k(\Gamma))$.

We represent the point class corresponding to the factor $\mathcal{M}^{e, \Gamma}$ of $\mathcal{M}^0, \Gamma$ by a nodal pointed curve $C_v$ which is obtained as follows. The curve $C_v$ corresponds to the weighted graph illustrated in Figure 2. Applying the above
inductive scheme to the subspaces $\Psi_e(d; g, n, k(\Gamma))$ we may reduce the genus $g_\infty$ to zero.

Let $\mathcal{G}$ denote the set of all stable weighted graph with a distinguished vertex $v_\infty$ such that $\theta(v_\infty) = (0, n + k_\infty)$ for some $0 \leq k_\infty = k_\infty(G) \leq k$, and for all $v \in V(G) \setminus \{v_\infty\}, \theta(v) = (0, k_v)$ with $k_v + d_v = 3$ (where $d_v$ denotes the degree of the vertex $v$). Moreover, we assume that $k_\infty + \sum_v k_v = k$, that $G$ has $g$ self-edges and that by deleting these self-edges $G$ becomes a connected tree. For $G \in \mathcal{G}$ we get an embedding

$$i^G : C_G \simeq \mathcal{M}_{0,k_\infty+d_\infty} \to \mathcal{M}_{g,n+k}.$$  

An inductive use of (3) implies that

$$\text{rank} \left( \frac{\Psi_e(d; g, n, k)}{\sum_{G \in \mathcal{G}} i^G_* (\Psi_e(d; 0, n, k_\infty(G)))} \right) \in o(n^e).$$  

Finally, note that all embeddings $i^G$ factor through the embedding

$$i^{g,k} : \mathcal{C}_{g,k} = \mathcal{C}_{g_\infty,k_\infty} \to \mathcal{M}_{g,n+k}$$

which corresponds to the stable weighted graph $G_{g,k}$ with vertices $v_\infty, 1, 2, \ldots, g$, such that for every $i = 1, \ldots, g$ $G_{g,k}$ contains an edge connecting the vertex $i$ to $v_\infty$ together with a self edge from $i$ to itself and $\theta(i) = (0, 0)$. Moreover, $\theta(v_\infty) = (0, k)$. As a result of this observation, from (4) we obtain

$$\text{rank} \left( \frac{\Psi_e(d; g, n, k)}{\left( \Psi_e(d; g, n, k) \cap i^{g,k}_* (R_e(C_{g,k})) \right)} \right) \in o(n^e).$$
It is thus enough to prove that
\[
\text{rank} \left( \Psi_e(d; g, n, k) \cap i_*^{g,k} (R_e(C_{g,k})) \right) - \frac{(n+\epsilon)(g+k+\epsilon)}{(e+1)!} \in \mathfrak{o}(n^e)
\]

The tautological ring of $C_{g,k} \simeq \overline{\mathcal{M}}_{0,n+g+k}$ is generated by combinatorial cycles. Let $S_{n,g,k}$ denote the set of permutations in $S_{n+g+k}$ which preserve the sets $\{1, \ldots, n\}, \{n+1, \ldots, n+g\}$ and $\{n+g+1, \ldots, n+g+k\}$. If $D \subset \overline{\mathcal{M}}_{0,g+n+k}$ is a combinatorial cycle (i.e. one of the boundary strata in $\overline{\mathcal{M}}_{0,g+n+k}$), every $\sigma \in S_{n,g,k}$ acts on $D$ by permuting the markings on the curves in $D$ to give a corresponding combinatorial cycle $\sigma(D)$.

Suppose that the push-forward $\beta = i_*^{g,k}(\alpha)$ belongs to $\Psi_e(d; g, n, k)$ for $\alpha = \sum_{i=1}^{N} D_i$ with $D_i \in R_e(C_{g,k})$. Since $C_{g,k} \simeq \overline{\mathcal{M}}_{0,n+g+k}$, we may set
\[
\sigma(\alpha) = \sum_{i=1}^{N} \sigma(D_i) \quad \forall \, \sigma \in S_{n+g+k}.
\]

Note that
\[
\beta = i_*^{g,k}(\alpha) = \sigma(\beta) = i_*^{g,k}(\sigma(\alpha)) \quad \forall \, \sigma \in S_{n,g,k}
\]
\[
\Rightarrow \beta = i_*^{g,k} \left( \frac{1}{n! \times g! \times k!} \sum_{\sigma \in S_{n,g,k}} \sigma(\alpha) \right) =: i_*^{g,k}(\tau^\alpha).
\]

The class $\tau^\alpha \in R_e(\overline{\mathcal{M}}_{0,n+g+k})$ is determined by its integrals over combinatorial cycles. The $\mathbb{Q}$-module generated by the combinatorial cycles in codimension $e$ is the same as $R_e = R_{n+g+k-3-e}(\overline{\mathcal{M}}_{0,n+g+k})$. If $D$ is a combinatorial cycle of codimension $e$ in $R_e$ we have
\[
\langle \tau^\alpha, \sigma(D) \rangle = \langle \sigma^{-1}(\tau^\alpha), D \rangle = \langle \tau^\alpha, D \rangle \quad \forall \, \sigma \in S_{n,g,k}.
\]

Integration against $\tau$ thus gives a map
\[
\int_{\tau}: R_{n+g+k-3-e}(\overline{\mathcal{M}}_{0,n+g+k}) \to \mathbb{Q}
\]
which determines $\tau^\alpha$.

Every combinatorial cycle $D$ as above determines a combinatorial cycle in $R_{n+3g+k-3-e}(\overline{\mathcal{M}}_{g,n+k})$ as follows. Suppose that $D$ is associated with a stable weighted graph $G$ and that
\[
\epsilon_G : V(G) \to \mathbb{Z}_{\geq 0} \times 2^{\{1, \ldots, n+g+k\}}
\]
is the corresponding weight function. For every vertex $v \in V(G)$, $\epsilon_G(v) = (0, I_v)$ with $I_v$ disjoint subsets of $\{1, \ldots, n+g+k\}$ which give a partition of it. Let $\pi(G)$ denote the stable weighted graph with the same underlying graph $G$ and the weight function defined by
\[
\epsilon_{\pi(G)}(v) := (|I_v \cap \{n+1, \ldots, n+g\}|, I_v \setminus \{n+1, \ldots, n+g\}) \quad \forall \, v \in V(G).
\]
Let $\pi(D)$ denote the combinatorial cycle associated with $\pi(G)$. If $\pi(D) = \pi(D')$ then $i^{g,k}_{*}(D) = i^{g,k}_{*}(D')$. Moreover, the intersection of $\pi(D)$ with $C_{g,k}$ is transverse and

$$\pi(D) \cap i^{g,k}_{*}(\overline{M}_{0,n+g+k}) = \# \{D' \mid \pi(D') = \pi(D)\} i^{g,k}_{*}(D).$$

From here we obtain

$$\langle i^{g,k}_{*}(\alpha), \pi(D) \rangle = \# \{D' \mid \pi(D') = \pi(D)\} \langle \alpha, D \rangle.$$

In other words, the map $\int_{\beta}$ is determined by the evaluation $\int_{\beta}(D) = \frac{1}{\# \{D' \mid \pi(D') = \pi(D)\}} \langle \beta, \pi(D) \rangle$.

Since $\beta \in \Psi_{e}(d; g, n, k)$ the evaluation $\int_{\beta}(D)$ only depends on the modified weight function associated with $D$, and not the underlying graph $G$. In other words, in order to determine $\int_{\beta}(D)$ one needs to specify

- The dimensions $(d_0, d_1, \ldots, d_e)$ of each one of the $e+1$ components of $D$, with the property that $\sum d_i = n + g + k - 3 - e$.
- The number of markings from $\{n+1, \ldots, n+g\}$ on each one of the $e+1$ components of $D$.
- The number of markings from $\{n+g+1, \ldots, n+g+k\}$ on each one of the $e+1$ components of $D$.

Asymptotically, the number of ways this can be done is equal to

$$\frac{(n+e)(g+e)(k+e)}{(e+1)!}.$$

This completes the proof of the theorem.

**Corollary 4.2.** Fix the codimension $e$ and the genus $g$ and let the number $n$ of the markings grow large. Then the rank of $\kappa_{e}(\overline{M}_{g,n})$ as a module over $\mathbb{Q}$ is asymptotic to

$$\frac{(n+e)(g+e)}{(e+1)!}.$$

**Proof.** By Theorem 4.1 we know that

$$\text{rank } (\kappa_{e}(\overline{M}_{g,n})) - \frac{(n+e)(g+e)}{(e+1)!} \in o(n^e).$$

Theorem 5 from [2] implies that

$$\frac{(n+e)(g+e)}{(e+1)!} - \text{rank } (\kappa_{e}(\overline{M}_{g,n})) \in o(n^e).$$

These two observations complete the proof of the corollary. \qed
5. The kappa ring of $\mathcal{M}_{1,n}$

5.1. The kappa ring and the combinatorial kappa ring. We would now like to focus on the study of $\kappa^d(\mathcal{M}_{1,n}) = \Psi_{n-d}(d; 1, n)$. As before, let

$$\Phi_{n-d}(d; 1, n) = \left< J^{\text{lead}}(\mathbf{m}^-) \mid \mathbf{m} \in \mathcal{P}(d) \setminus \mathcal{P}(d, n-d) \right>.$$  

Other than the comb graphs in $\mathcal{J} = \mathcal{J}(1, n, m, d)$, the only possible comb graphs are the comb graphs $\Gamma$ with a distinguished vertex $v_0 \in V_0(\Gamma)$ with associated genus $g_0 = 1$, and with $g_v = 0$ for all other vertices of $\Gamma$. The image of the corresponding components of the fixed locus under the forgetful map

$$\epsilon : \mathcal{M}_{1,n+2}(\mathcal{P}(V), d) \to \mathcal{M}_{1,n}$$

coincides with the image of

$$\imath = \imath^{1,0} : \mathcal{M}_{0,n+2} \simeq \{\text{pt}\} \times \mathcal{M}_{0,n+1} \subset \mathcal{M}_{1,1} \times \mathcal{M}_{0,n+1} \to \mathcal{M}_{1,n}.$$  

In particular, for every $\mathbf{m} \in \mathcal{P}(d) \setminus \mathcal{P}(d, n-d)$ we have

$$J^{\text{lead}}(\mathbf{m}^-) = \imath_*(\kappa(\mathbf{m})) \quad \kappa(\mathbf{m}) \in \Psi_{n-d}(d; 0, n, 1).$$

**Theorem 5.1.** The quotient map from $\kappa^*(\mathcal{M}_{1,n})$ to $\kappa_c^*(\mathcal{M}_{1,n})$ is an isomorphism.

**Proof.** It is enough to show that if a kappa class $\kappa \in \kappa_c(\mathcal{M}_{g,n})$ is combinatorially trivial then it is zero. Suppose that $\kappa = <\phi>_{1,n}$ for some $\phi \in \Psi(d)$ (we refer to Section 2 for the definitions). There is a homomorphism

$$J : \mathcal{M}_{0,n+2} \to \mathcal{M}_{1,n},$$

which gives an embedding of $\mathcal{M}_{0,n+2}(\mathbb{Z}/2\mathbb{Z})$ into $\mathcal{M}_{1,n}$. The integral of $\kappa$ over all combinatorial cycles of the form $J_*(D)$ is trivial, since $\kappa$ is combinatorially trivial. However, this implies that

$$\int_D <\phi>_{0,n+2} = \int_{J_*(D)} <\phi>_{1,n} = 0 \quad \forall \ D,$$

i.e. $<\phi>_{0,n+2} = 0$. By Remark 2.4

$$<\phi>_{0,n+2} = 0 \Rightarrow \phi = \sum_{\mathbf{m} \in \mathcal{P}(d) \setminus \mathcal{P}(d, n-d)} a_{\mathbf{m}} \left( \sum_{\mathcal{P}(d)} C_{\mathbf{m}^{-}}^{D} \right),$$

for some rational coefficients $a_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{P}(d) \setminus \mathcal{P}(d, n-d)$, i.e. $\kappa$ is a linear combination of the kappa classes $J^{\text{lead}}(\mathbf{m}^-)$. Thus, there is a tautological class $\psi \in \mathcal{R}_{d-2}(\mathcal{M}_{0,n+1})$ such that $\kappa = \imath_*(\psi)$. In particular

$$<\psi, D> = \frac{1}{\# \{ D' \mid \pi(D) = \pi(D') \}} (\kappa, \pi(D)) = 0 \quad \forall \ D,$$

$$\Rightarrow \psi = 0 \Rightarrow \kappa = 0.$$  

This completes the proof. \qed
5.2. The $\kappa$-trivial combinatorial cycles. We call a formal linear combination
\[ a_1 q_1 + a_2 q_2 + \ldots + a_k q_k \quad a_i \in \mathbb{Q}, \quad q_i \in Q(d; g, n) \]
a $\kappa$-trivial cycle if for every kappa class $\kappa \in \kappa^d(\mathcal{M}_{g,n})$
\[ a_1 \langle \kappa, q_1 \rangle + a_2 \langle \kappa, q_2 \rangle + \ldots + a_k \langle \kappa, q_k \rangle = 0. \]
The space of $\kappa$-trivial cycles is a subspace of the vector space $\langle Q(d; g, n) \rangle_{\mathbb{Q}}$ freely generated by the elements of $Q(d; g, n)$, and we denote its rank by $r(d; g, n)$. The quotient $V(d; g, n)$ of $\langle Q(d; g, n) \rangle_{\mathbb{Q}}$ by the space of $\kappa$-trivial cycles is a vector space isomorphic to $\kappa^d(\mathcal{M}_{g,n})$. Thus, the rank of the combinatorial kappa ring $\kappa^* (\mathcal{M}_{g,n})$ in degree $d$ may be computed as
\[ \text{rank} \left( \kappa^d_c(\mathcal{M}_{g,n}) \right) = |Q(d; g, n)| - r(d; g, n). \]

**Proposition 5.2.** The rank of $\kappa^d_c(\mathcal{M}_{1,n})$ is at most $|P_1(d, n - d)|$.

**Proof.** Theorem 3 from \cite{2} implies that
\[ \frac{1}{24} q_0(n) - \sum_{i=1}^{n-1} \binom{n-2}{i-1} q_i(n - i) \]
is $\kappa$-trivial in $\langle Q(n + d - k - 1; 1, n + d) \rangle_{\mathbb{Q}}$. Thus, for every $\mathbf{n} \neq (1, 1, ..., 1)$ in $P(n)$, $q_0(\mathbf{n}) \in V(n - |\mathbf{n}|; 1, n)$ is equal to a linear combination of the cycles $q_l(\mathbf{m})$ for $l \geq 1$ and $\mathbf{m} \in P(n - l; n)$. In other words, $V(d; 1, n)$ is generated by $q_l(\mathbf{n})$ for $l \geq 1$ and $\mathbf{n} \in P(n - l; n - d)$.

For $\mathbf{n} = (a, b) \in P(n)$ with $a > b$, Theorem 3 from \cite{2} gives the following two equations in $V(n - 2; 1, n)$:
\[ \frac{1}{24} q_0(\mathbf{n}) = \sum_{i=1}^{a-1} \binom{a-2}{i-1} q_i(a - i, b) \]
\[ = \sum_{i=1}^{b-1} \binom{b-2}{i-1} q_i(a, b - i). \]

Thus,
\[ q_{b-1}(a, 1, n_1, ..., n_k) \in V \left( a + b - 2 - k + \sum_i n_i; 1, a + b + \sum_i n_i \right) \]
may be expressed as a linear combination of
\[ q_i(a - i, b, n_1, ..., n_k), \quad i = 1, ..., a - 1 \]
and
\[ q_j(a, b - j, n_1, ..., n_k), \quad j = 1, ..., b - 2. \]
This observation implies that $V(d; 1, n)$ is generated by the following elements of $Q(d; 1, n)$ (with $l \geq 1$):
\begin{itemize}
\item $q_1(n_1 \leq \ldots \leq n_{n-d})$ with $\sum_{i=1}^{n-d} n_i = n - l$, and $n_1 \geq 2$
\item $q_2(1 \leq n_2 \leq \ldots \leq n_{n-d})$ with $\sum_{i=2}^{n-d} n_i = n - l - 1$ and $n_{n-d} \leq l + 1$.
\end{itemize}

Denote the above two sets of generators by $A_1(d; 1, n)$ and $A_2(d; 1, n)$ respectively, and set

$$A(d; 1, n) = A_1(d; 1, n) \cup A_2(d; 1, n).$$

Every element of $A_1(d; 1, n)$ corresponds to the partition

$$(n_1 - 1, \ldots, n_{n-d} - 1) \in P(d - l; n - d).$$

The size $|A_1(d; 1, n)|$ is thus equal to $\sum_{l=1}^{2d-n} |P(d - l; n - d)|$. Every partition in $A_2(d; 1, n)$ gives the partition

$$((n_2 - 1) \leq (n_3 - 1) \leq \ldots \leq (n_{n-d} - 1) \leq l) \in P(d, n - d).$$

Thus, $V(d; 1, n)$ is generated by a set of size

$$|P(d, n - d)| + \sum_{l=1}^{2d-n} |P(d - l; n - d)|.$$

Sending the partition $n = (n_1 \leq \ldots \leq n_{n-d}) \in P(d - l; n - d)$ to

$$n[l] := (n_1, \ldots, n_{n-d}, 1, \ldots, 1) \in P(d)$$

gives a bijection (extending the inclusion $P(d, n - d) \subset P_1(d, n - d)$)

$$P(d, n - d) \cup \prod_{l=1}^{2d-n} P(d - l; n - d) \longrightarrow P_1(d, n - d).$$

This completes the proof of Proposition \ref{prop:injective}. \hfill \Box

\subsection{Independence of the generators.}

\textbf{Theorem 5.3.} The rank of $\kappa^d_1(\overline{\mathcal{M}}_{g, n})$ is equal to $|P_1(d, n - d)|$.

\textbf{Proof.} By Proposition \ref{prop:injective} it is enough to show that the elements of $A(d; 1, n)$ are linearly independent in $V(d; 1, n)$.

If $n \in P(d, n - d)$, the integral of $\psi(n) \in \kappa^d_1(\overline{\mathcal{M}}_{1, n})$ against every $q \in A_1(d; 1, n)$ is zero, since the length of $p(q)$ is $n - d + 1$, while the length of $n$ is at most $n - d$ (thus $n$ does not refine $p(q)$). Meanwhile, the map $p : Q(d; 1, n) \rightarrow P(d, n - d + 1)$ gives an injection

$$p : A_2(d; 1, n) \rightarrow P(d, n - d).$$

With respect to the refinement ordering on $P(d, n - d)$ the matrix

$$\left( \langle \psi(p(q)), q' \rangle \right)_{q, q' \in A_2(d; 1, n)}$$
is triangular with non-zero diagonal entries, and is thus full-rank. The above two observations reduce the proof of Theorem 5.3 to showing that the elements of $A_1(d;1,n)$ are linearly independent in $V(d;1,n)$.

For every $n \in \mathbb{P}(n-l; n-d)$, every $p \in \mathbb{P}(d)$, and every integer $N \geq 0$

$$\langle \psi(p), q_i(n) \rangle_{1,n} = \langle \psi(p), q_i(n[N]) \rangle_{1,n+N}.$$  

In order to prove the independence of the elements of $A_1(d;1,n)$, it is thus enough to prove the independence of the elements of $A_1^N(d;1,n) \subset A_1(d;1,n + N)$ consisting of $q_i(n[N])$ with $q_i(n) \in A_1(d;1,n)$.

For $n = (n_1, ..., n_k) \in \mathbb{P}(n)$ and $m = (m_1, ..., m_p) \in \mathbb{P}(m)$ define

- $\hat{n} := (n_1 + 1, ..., n_k + 1, 1, ..., 1) \in \mathbb{P}(2n; n)$ and
- $n \cup m := (n_1, ..., n_k, m_1, ..., m_p) \in \mathbb{P}(m + n)$

Let $\tilde{\mathbb{P}}(d)$ denote the set of all sequences $n = (n_1, ..., n_k)$ of positive integers such that $n_1 + ... + n_k = d$. There is a natural map from $\tilde{\mathbb{P}}(d)$ to $\mathbb{P}(d)$, which is implicitly used below to think of the elements of $\tilde{\mathbb{P}}(d)$ as partitions.

**Lemma 5.4.** For every positive integer $l$ and every $n \in \mathbb{P}(n)$ the cycle

$$q_{i}(n[l]) + \frac{1}{24} \sum_{m=(m_1,...,m_k)\in\tilde{\mathbb{P}}(l)} \binom{(-1)|m| m_1}{d(m)} \binom{d(m)}{m} q_0(\hat{m} \cup n)$$

is $\kappa$-trivial.

**Proof.** We use induction on $l$. For $l = 1$, Lemma 5.4 follows directly from (6). Suppose now that the claim is proved for $1, 2, ..., l - 1$. Using (6), for every $n \in \mathbb{P}(n)$ we make the following computation in $V(n + l; 1, k + l)$:

$$q_i(n[l]) = \frac{1}{24} q_0(\{l + 1 \cup n[l - 1]\}) - \sum_{i=1}^{l-1} \binom{l-1}{i} q_{l-i}(\{i + 1 \cup n[l - 1]\})$$

$$= \frac{1}{24} q_0(\{\hat{l} \cup n\}) - \sum_{i=1,...,l-1}^{m_1} \frac{m_1}{l-i} \binom{l-1}{i} (l-i) \binom{-1}{m} \frac{(l-k)^k}{24} q_0\left(\{\{i\} \cup m\} \cup n\right)$$

$$= -\frac{1}{24} \sum_{m=(m_1,...,m_k)\in\tilde{\mathbb{P}}(l)} \binom{(-1)|m| m_1}{d(m)} \binom{d(m)}{m} q_0(\hat{m} \cup n)$$

This completes the proof of Lemma 5.4. \qed
In particular, every element of \( A_{1}^{2d-n}(d; 1, n) \subset A(d; 1, 2d) \) is a linear combination (in \( V(d; 1, 2d) \)) of the cycles of the form \( q_0(n) \) with \( n \in P(2d; d) \) having at least \( n - d + 1 \) terms greater than or equal to 2. Such \( n \)'s are determined by
\[
m = n^{-} \in P(d) - P(d, n - d).
\]
Define \( q(m) = q_0(n) \).

Let us denote the matrix expressing the elements of \( A_{1}^{2d-n}(d; 1, n) \) in terms of \( q(m) \) with \( m \in P(d) - P(d, n - d) \) by \( M(d; 1, n) \). The rows of \( M(d; 1, n) \) are thus indexed by the elements of \( P(d) - P(d, n - d) \) and its columns are indexed by the elements of \( A_1(d; 1, n) \). In particular,
\[
q_{(m)}(n) = A_1(d; 1, n) \Rightarrow n[l] \in P(d) - P(d, n - d),
\]
and the \((q_{(m)}, n[l])\) component of \( M(d; 1, n) \) is equal to \((-1)^{l-1}(l-1)!\). Moreover, if \( m \in P(2d, d) \) corresponds to some non-zero entry of \( M(d; 1, n) \) in the column corresponding to \( q_{(m)}(n) \) then \( m^{-} \) refines \( n \). In other words, the square sub-matrix of \( M(d; 1, n) \) corresponding to the rows indexed by \( n[l] \) with \( q_{(m)}(n) \in A_1(d; 1, n) \) is triangular with non-zero elements on the diagonal (if we use the refinement ordering on the partitions). Hence \( M(d; 1, n) \) is a matrix of full rank equal to \( |A_1(d; 1, n)| \).

In order to finish the proof, it is enough to show that the matrix
\[
N(d; 1, g) = (\langle \psi(p), q(p') \rangle)_{p, p' \in P(d) - P(d, n - d)}
\]
is invertible. This is true since the matrix is upper triangular with non-zero diagonal elements with respect to the refinement ordering over the partitions. This completes the proof of Theorem 5.3.

6. The Kappa Ring of \( \mathcal{M}_{2,n} \)

Let us now consider the case \( g = 2 \). Fix a partition
\[
n \in P(d, 2d - 2 - n - l), \quad l > 0.
\]
If the contribution of \( \Gamma \) to \( J(n) \) is non-trivial \( \mathcal{N}^{0, \Gamma} \) is either trivial, or one of \( \mathcal{M}_{1,1}, \mathcal{M}_{2,1} \) or \( \mathcal{M}_{1,1} \times \mathcal{M}_{1,1} \).

Correspondingly, the map \( j^{\Gamma} : \mathcal{N}^{\Gamma} \to \mathcal{M}_{2,n} \) is one of the four maps
\[
\begin{align*}
J^0 &= 1d : \mathcal{M}_{2,n} \to \mathcal{M}_{2,n} \\
J^1 &= \mathcal{M}_{1,1} \times \mathcal{M}_{1,1} \times \mathcal{M}_{0,n+2} \to \mathcal{M}_{2,n} \\
J^{2,n} &= \mathcal{M}_{2,1} \times \mathcal{M}_{0,n+1} \to \mathcal{M}_{2,n} \\
J^{3,n} &= \mathcal{M}_{1,1} \times \mathcal{M}_{1,n+1} \to \mathcal{M}_{2,n}.
\end{align*}
\]
The comb graphs which correspond to \( J^0 \) form the leading term \( J^{lead}(n) \) as their contribution to \( J(n) \). Since a factor \( \lambda_1 \) appears over either of the two \( \mathcal{M}_{1,1} \) components in the domain of \( J^1 \), the contribution of the comb graphs which correspond to \( J^1 \) is a class of the form \( i^{2,n}_{2,n}(\kappa(n)) \) for some tautological
class \( \kappa(n) \in R^d-4(\mathcal{M}_{0,n+2}) \). With a similar reasoning, the comb graphs corresponding to \( j^3 \) contribute via a class of the form

\[
j^2_s(\pi^*_1(\lambda_1)\pi^*_2(\psi(n))) \, , \quad \psi(n) \in \Psi_{n+3-d}(d; 1, n, 1),
\]

where (abusing the notation) \( \pi_i \) denotes the projection map from the domain of either of \( j^i \) over the \( i \)-th product factor, for \( j = 0, 1, 2, 3 \) and \( i = 1, 2, 3 \).

Let \( \delta \) denote the divisor

\[
\mathcal{M}_{2,1} \setminus \mathcal{M}_{2,1} = \frac{[\mathcal{M}_{1,3}]}{\mathbb{Z}/2\mathbb{Z}} + [\mathcal{M}_{1,1} \times \mathcal{M}_{1,2}] = \delta_0 + \delta_1.
\]

By the argument of Section 8 from [8] over \( \mathcal{M}_{2,1} \) we get

\[
\begin{align*}
\kappa_1 &= 2\lambda_1 + \frac{1}{2}\delta_1 + \psi_1 \quad \text{and} \quad \lambda_1 = \frac{1}{12}(\kappa_1 + \delta - \psi_1) \\
\Rightarrow \lambda_2\kappa_1 &= \frac{4}{3}\lambda_2\lambda_1 + \lambda_2\psi_1 \quad \text{and} \quad \lambda_2\delta_1 = \frac{20}{3}\lambda_2\lambda_1.
\end{align*}
\]

Thus, the Chow factor over the component \( \mathcal{M}_{2,1} \) in the domain of \( j^2 \) is a linear combination of \( \lambda_2, \lambda_2\psi_1, \lambda_2\lambda_1 \) and the point class.

For every tautological class \( \psi \in R^*(\mathcal{M}_{0,n+1}) \) note that

\[
j^2_s(\pi^*_1(\psi)) \in \operatorname{Im}(i^2_s(n)).
\]

Consequently, the total contribution corresponding to the comb graphs \( \Gamma \) with \( j^3 = j^2 \) and \( c \subset c(\Gamma) \) corresponding to a multiple of the point class over \( \mathcal{M}_{2,1} \) is of the form \( i^2_s(n)(\kappa'(n)) \) for some \( \kappa'(n) \in R^d-4(\mathcal{M}_{0,n+2}) \). We thus obtain relations of the form

\[
\begin{align*}
j^{\text{lead}}(n) &= i^2_s(\kappa(n) + \kappa'(n)) + j^3_s(\pi^*_1(\lambda_1)\pi^*_2(\psi(n))) \\
&\quad + j^2_s\left[\pi^*_1(\lambda_2)\pi^*_2(\beta(n)) + \pi^*_1(\lambda_2\lambda_1)\pi^*_2(\gamma_1(n)) + \pi^*_1(\lambda_2\psi_1)\pi^*_2(\gamma_2(n))\right]
\end{align*}
\]

where \( \beta(n) \in R^{d-3}(\mathcal{M}_{0,n+1}) \) and \( \gamma_i(n) \in R^{d-4}(\mathcal{M}_{0,n+1}) \), \( i = 1, 2 \).

Let us now assume that \( \kappa \in \kappa^d(\mathcal{M}_{2,n}) \) is combinatorially trivial. For every stable weighted graph \( H \) with the property that the combinatorial cycle associated with \( H \) is of dimension \( d \) and lives in \( \mathcal{M}_{2,n} \) we get \( \int_{|H|}\kappa = 0 \). Applying the above assumption to the stable weighted graphs with the zero genus associated with all vertices we find that \( \kappa \) is a linear combination of the classes \( j^{\text{lead}}(n) \) by Remark 2.4. The above observation implies that there are classes

\[
\begin{align*}
\alpha &\in R^{d-4}(\mathcal{M}_{0,n+2}), \quad \gamma_i \in R^{d-4}(\mathcal{M}_{0,n+1}) \quad i = 1, 2 \\
\beta &\in R^{d-3}(\mathcal{M}_{0,n+1}) \quad \text{and} \quad \psi \in \Psi_{n+3-d}(d; 1, n, 1)
\end{align*}
\]
such that
\[
\kappa = j_2^3[\pi_1^*(\lambda_1)\pi_2^*(\psi)] + \lambda_2 j_2^4[\pi_3^*(\alpha)]
\]
\[
+ \lambda_2 j_2^2[\pi_2^*(\beta) + \pi_1^*(\lambda_1)\pi_2^*(\gamma_1) + \pi_1^*(\psi_1)\pi_2^*(\gamma_2)]
\]

In getting rid of \(j_2^{2,n}\) and replacing it with \(j_2^4\) we are using the fact that
\[
\frac{1}{24} j_2^{2,n}(a) = j_2^4(\pi_1^*(\lambda_1)\pi_2^*(\lambda_1)\pi_3^*(a)) \quad \forall \ a \in R^*(\hat{M}_{0,n+2}).
\]

Moreover, since the restriction of \(\lambda_2\) to
\[
\hat{M}_{1,1} \times \hat{M}_{1,1} \times \hat{M}_{0,n+2} \subset \hat{M}_{2,n}
\]
gives a factor of \(\lambda_1\) over either of the product factors \(\hat{M}_{1,1}\),
\[
j_2^4(\pi_1^*(\lambda_1)\pi_2^*(\lambda_1)\pi_3^*(a)) = \lambda_2 j_2^4(\pi_3^*(a)).
\]

Consider a stable weighted graph \(G\) which determines a combinatorial cycle over \(\hat{M}_{n+1}\) with cycle dimension \(d - 2\) (i.e. of codimension \(n + 3 - d\)). Let \(v\) denote the vertex of \(G\) which carries the special marking \(n + 1\). As in Section 4 let \(\pi(G)\) denote the stable weighted graph obtained from \(G\) by removing the special marking from \(v\) and increasing the genus \(g_v\) by 1. Note that \(\pi(G)\) determines a combinatorial cycle of dimension \(d\) in \(\hat{M}_{2,n}\). Let us assume that the genus associated with all vertices of \(G\) is zero. Since \(\pi(G)\) contains a loop, the restriction of \(\lambda_2\) to \([\pi(G)]\) is trivial. We thus find
\[
0 = \int_{[\pi(G)]} \kappa = \int_{[\pi(G)]} j_2^3(\pi_1^*(\lambda_1)\pi_2^*(\psi)) = \frac{1}{24} \int_{[G]} \psi.
\]

In particular, the integral of \(\psi \in \Psi_{n+3-d}(d; 1, n, 1)\) over all combinatorial cycles consisting only of genus zero components is trivial.

Note that \(\psi\) may be represented as the sum of an element
\[
\psi' \in \Phi_{n+3-d}(d; 1, n, 1)
\]
and a linear combination of the classes in \(G_{n+3-d}(d; 1, n, 1)\). Every linear combination of the classes in \(G_{n+3-d}(d; 1, n, 1)\) is a linear combination of the classes of the form \(\psi^{p-1}_{n+1}(p)\) with \(p\) a partition in \(P(d - 2 - p, n + 2 - d)\) by Proposition 3.1. Let \(\hat{P}(d)\) denote the set of marked partitions \((p; p)\) of \(d\), i.e. the set of pairs \((p; p)\) such that \(p \leq d\) is a positive integer and \(p \in P(d - p)\). Let \(\hat{P}(d, k)\) denote the subset of \(\hat{P}(d)\) which consists of the marked partitions \((p; p)\) with \(|p| < k\). The above observation implies that associated with every \((p; p) \in \hat{P}(d - 2, n + 3 - d)\) there is a rational coefficient \(A_{(p; p)}\) such that
\[
\psi = \psi' + \sum_{(p; p) \in \hat{P}(d - 2, n + 3 - d)} A_{(p; p)} \psi^{p-1}_{n+1}(p).
\]
Since the integral of both $\psi$ and $\psi'$ over all combinatorial cycles which only consist of genus zero components is zero, we conclude that for every such combinatorial cycle $D$ we have

$$ \sum_{(p;\mathbf{p})\in \hat{P}(d-2,n+3-d)} A_{(p;\mathbf{p})} \int_D \psi_{n+1}^{p-1} \psi(\mathbf{p}) = 0. $$

For a combinatorial cycle $D$ as above which consists only of genus zero components let $p_D-1$ denote the dimension of the component containing the $(n+1)$-th marking. Let $\mathbf{p}_D$ denote the partition of $d-2-p_D$ determined by the dimensions of the rest of the components. Note that the marked partition $(\mathbf{p}_D,\mathbf{p}_D)$ of $D$ consists of at most $n-(d-3)$ components. The integral of $\psi_{n+1}^{p-1} \psi(\mathbf{p})$ against the combinatorial cycle $D$ only depends on $(\mathbf{p}_D;\mathbf{p}_D) \in P(d-2,n+3-d)$.

The $|\hat{P}(d-2,n+3-d)| \times |\hat{P}(d-2,n+3-d)|$ matrix containing all possible integrals $\langle \psi_{n+1}^{p-1} \psi(\mathbf{p}), (\mathbf{p}_D;\mathbf{p}_D) \rangle$ is triangular with respect to the refinement ordering, the equations of (9) implies that $A_{(p;\mathbf{p})} = 0$ for all $(p;\mathbf{p}) \in \hat{P}(d-2,n+3-d)$. In particular, $\psi = \psi'$ belongs to $\Phi_{n+3-d}(d;1,n,1)$.

Since $\psi \in \Phi_{n+3-d}(d;1,n,1)$ it may be expressed in terms of the tautological classes pushed forward using $i^1: \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,n+1} \rightarrow \overline{\mathcal{M}}_{1,n+1}$ and $i^2: \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+2} \rightarrow \overline{\mathcal{M}}_{1,n+1}$.

Repeating the argument which was employed at the beginning of this subsection we may write

$$ \psi = i^1_\ast [\pi_1^\ast (\lambda_1) \pi_2^\ast (\gamma_3)] + \iota^2_\ast [\pi_1^\ast (\lambda_1) \pi_2^\ast (\alpha')] $$

$$ \Rightarrow j_2^\ast [\pi_1^\ast (\lambda_1) \pi_2^\ast (\psi)] = \lambda_2 \left[ j_2^\ast (\pi_3^\ast (\alpha')) + j_2^\ast (\pi_1^\ast (\delta) \pi_2^\ast (\gamma_3)) \right] $$

$$ \Rightarrow \kappa = \lambda_2 \left[ j_2^\ast (\pi_3^\ast (\alpha + \alpha')) + j_2^\ast (\pi_2^\ast (\beta)) \right] $$

$$ + \lambda_2 j_2^\ast [\pi_1^\ast (\lambda_1) \pi_2^\ast (\gamma_1)] + \pi_1^\ast (\psi_1) \pi_2^\ast (\gamma_2) + \pi_1^\ast (\delta_1) \pi_2^\ast (\gamma_3)]. $$

The second equality follows since the restriction of $\lambda_2$ to either of $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,n+1}, \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \overline{\mathcal{M}}_{0,n+2} \subset \overline{\mathcal{M}}_{2,n}$ gives a factor of 1 = $\lambda_0$ over every product factor $\overline{\mathcal{M}}_{0,*}$, and a factor of $\lambda_1$ over every product factor $\overline{\mathcal{M}}_{1,*}$.

The above considerations imply the following lemma.

**Lemma 6.1.** If the integral of $\kappa \in \kappa^d(\overline{\mathcal{M}}_{2,n})$ over all combinatorial cycles in $\overline{\mathcal{M}}_{2,n}$ with the sum of the genera of the components less than 2 is trivial then there are tautological classes $a \in R^{d-4}(\overline{\mathcal{M}}_{0,n+2})$, $b \in R^{d-3}(\overline{\mathcal{M}}_{0,n+1})$ and $c, c' \in R^{d-4}(\overline{\mathcal{M}}_{0,n+1})$ such that

$$ \kappa = \iota^1_\ast n(a) + \lambda_2 j_2^\ast n \left( \pi_2^\ast (b) + \pi_1^\ast (\lambda_1) \pi_2^\ast (c) + \pi_1^\ast (\psi_1) \pi_2^\ast (c') \right). $$
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**Theorem 6.2.** The quotient map from $\kappa^d(\overline{M}_{2,n}) \to \kappa^d_c(\overline{M}_{2,n})$ is an isomorphism.

**Proof.** Every combinatorially trivial kappa class $\kappa$ has a representation of the form $\int [\pi(G)]$. Consider a stable weighted graph $G$ which corresponds to a combinatorial cycle in $\overline{M}_{0,n+2}$. Treating the last two markings in $\overline{M}_{0,n+2}$ as the special markings, $\pi(G)$ may be defined as a stable weighted graph determining a combinatorial cycle in $\overline{M}_{2,n}$. If the intersection of $\pi(G)$ with the image of $j^{2,n}$ is non-empty then the markings $p = n + 1$ and $q = n + 2$ lie over the same vertex of $G$. In this latter case, the intersection of $[\pi(G)]$ with the image of $j^{2,n}$ is transverse, unless the vertex of $G$ containing $p, q$ is a vertex $v$ with $d(v) = 1$ and $\epsilon(v) = (0, \{p, q\})$. However, if the intersection of $j^{2,n}$ with $[\pi(G)]$ is transverse then

$$\int [\pi(G)] \lambda_2 j^{1,n} \left[ \pi^*(b) + \pi^*(\lambda_1) \pi^*(c) + \pi^*(\psi_1) \pi^*(c') \right]$$

$$= \int [\text{Im}(j^{2,n})] \lambda_2 \left[ \pi^*(b) + \pi^*(\lambda_1) \pi^*(c) + \pi^*(\psi_1) \pi^*(c') \right].$$

In this case, the intersection includes a factor $\overline{M}_{2,1}$, and since the degree of the Chow class over this factor is at most 3 the above integral is trivial. We conclude that unless $G$ contains a vertex $v$ with $\epsilon(v) = (0, \{p, q\})$ and $d(v) = 1$

$$\int_{[\pi(G)]} a = 2 \times 24^2 \times \#\{D | \pi(D) = [\pi(G)]\} \times \int_{[\pi(G)]} \kappa = 0.$$

Let us now assume that the stable weighted graph $G$ has a vertex $v$ with $d(v) = 1$ and $\epsilon(v) = (0, \{p, q\})$. Let $w$ be the vertex of $G$ which is connected to $v$ by an edge $e$. The vertex $v$ corresponds to a product factor $\overline{M}_{0,3}$ where the three markings are labelled by $\{p, q, e\}$. The vertex $w$ corresponds to a factor $\overline{M}_{0,k+1}$ where the markings are denoted by $\{e, p_1, ..., p_k\}$, and with $e$ denoting the marking which corresponds to the edge $e$. For every subset $A \subset \{p_1, ..., p_k, p, q\} = B$ with $2 \leq |A| \leq k$ let $G_A$ denote the stable weighted graph obtained as follows. Delete the edges of $G$ which are adjacent to either of $v$ and $w$, except for $e$, to obtain a sub-graph $H$ of $G$ with $V(H) = V(G)$. If $e'$ denotes a deleted edge of $G$ which connects some vertex $u$ of $G$ to $w$ then $e'$ corresponds to one of the markings $p_i \in \{p_1, ..., p_k\}$. If $p_i \in A$ then add an edge to $H$ which connects $u$ to $v$. Otherwise, add an edge to $H$ which connects $u$ to $w$. This gives a graph $G_A$. Let $\epsilon(w) = (0, I_w)$. Define the weight function over the vertices of $G_A$ by

$$\epsilon_A(u) = \begin{cases} (0, A \cap (I_w \cup \{p, q\})) & \text{if } u = v \\ (0, (B \setminus A) \cap (I_w \cup \{p, q\})) & \text{if } u = w \\ \epsilon(u) & \text{otherwise} \end{cases}$$

**Proof.** Set $a = \alpha + \alpha'$, $b = \beta$, $c = \gamma_1 + \frac{22}{3} \gamma_3$ and $c' = \gamma_2$. \qed
In particular, $G_{\{p,q\}} = G$ as stable weighted graphs.

Keel’s Theorem [7] implies that
\[
\sum_{A: p,q \in A \atop p_1,p_2 \in B \setminus A} [G_A] = \sum_{A: p,p_1 \in A \atop q,p_2 \in B \setminus A} [G_A]
\]

In particular, we obtain
\[
\int_{[G]} a = \sum_{A: p,p_1 \in A \atop q,p_2 \in B \setminus A} \int_{[G_A]} a - \sum_{A: p,q \in A \atop p_1,p_2 \in B \setminus A} [G_A] a = 0.
\]

The above discussion implies that the integral of $a$ over all combinatorial cycles is trivial, and thus $a = 0$.

On the other hand, if $G$ is a stable weighted graph containing a vertex $v$ with $d(v) = 1$ and $\epsilon(v) = (0, \{p,q\})$, the combinatorial cycle $[\pi(G)]$ is included in the image of the map $j^{2,n}$. Every such stable weighted graph $G$ corresponds to another stable weighted graph $G^*$ obtained from $G$ by removing the vertex $v$ from $G$. If $w$ is the unique vertex adjacent to $v$ by the edge $e$, we define
\[
\epsilon_{G^*}(u) = \begin{cases} (0, I_w \cup \{e\}) & \text{if } u = w \\ \epsilon(u) & \text{otherwise} \end{cases}
\]

The stable weighted graph $G^*$ determines a combinatorial cycle in $\overline{M}_{0,n+1}$, where $e$ corresponds to the last marking. Conversely, every combinatorial cycle in $\overline{M}_{0,n+1}$ is of the form $G^*$. For every $H = G^*$ we obtain
\[
0 = \int_{[\pi(G)]} \kappa = \frac{1}{2 \times 24^2} \int_{[G]} a + \left( \int_{[\overline{M}_{2,1}]} \lambda_2 \lambda_1 \psi_1 \right) \int_{[H]} c + \left( \int_{[\overline{M}_{2,1}]} \lambda_2 \psi_1^2 \right) \int_{[H]} c'
\]
\[
= \frac{1}{2 \times 24^2} \int_{[H]} (c + \frac{9}{4} c').
\]

Thus $4c + 9c' = 0$. Next, let $G$ be a stable weighted graph which corresponds to a cycle of dimension $d - 4$ in $\overline{M}_{0,n+1}$. Let $v$ denote the vertex of $G$ which contains the marking $n+1$, and let $\epsilon(v) = (0, I_v)$. Let $\tilde{G}$ denote the stable weighted graph obtained from $G$ as follows. We add a vertex $w$ to $G$ and connect it to $v$ by a single edge. Then we set
\[
\epsilon_{\tilde{G}}(u) = \begin{cases} (1, \emptyset) & \text{if } u = w \\ (1, I_v \setminus \{n+1\}) & \text{if } u = v \\ \epsilon(u) & \text{otherwise} \end{cases}
\]
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The intersection of the combinatorial cycle $[\tilde{G}]$ with the image of $j^{2,n}$ is always transverse, and they cut each other in $[\delta_1] \times [G] \subset \overline{M}_{2,1} \times \overline{M}_{0,n+1}$.

For every stable weighted graph $G$ as above we thus find

$$0 = \int_{[\tilde{G}]} \kappa = \left( \int_{[\delta_1]} \lambda_2 \lambda_1 \right) \int_{[G]} c + \left( \int_{[\delta_1]} \lambda_2 \psi_1 \right) \int_{[G]} c' = \frac{1}{24^2} \int_{[G]} c'.$$

Thus $c = c' = 0$ and $\kappa = \lambda_2 j^{2,n}_* (\pi_*^2(b))$.

Finally, let $G$ be a stable weighted graph which corresponds to a cycle of dimension $d - 3$ in $\overline{M}_{0,n+1}$. Let $v$ denote the vertex of $G$ which contains the marking $n+1$, and let $\epsilon(v) = (0, I_v)$. Let $\overline{G}$ denote the stable weighted graph obtained from $G$ as follows. The graph $\overline{G}$ is obtained from $G$ by adding a pair of vertices $w_1$ and $w_2$ to $G$, which are connected by the edges $e_1$ and $e_2$ to $v$. We define the corresponding weight function by

$$\epsilon_{\overline{G}}(u) = \begin{cases} (1, \emptyset) & \text{if } u = w_i, \quad i = 1, 2 \\ (1, I_v \setminus \{n + 1\}) & \text{if } u = v \\ \epsilon(u) & \text{otherwise} \end{cases}.$$ 

The combinatorial cycle $[\overline{G}]$ cuts the image of $j^{2,n}$ transversely in $$(\overline{M}_{1,1} \times \overline{M}_{1,1} \times \overline{M}_{0,3}) \times [G] \subset \overline{M}_{2,1} \times \overline{M}_{0,n+1}.$$ 

For every stable weighted graph $G$ as above we thus obtain

$$0 = \int_{[\overline{G}]} \kappa = \left( \int_{\overline{M}_{1,1}} \lambda_1 \right)^2 \int_{[G]} b = \frac{1}{24^2} \int_{[G]} b.$$ 

Thus $b = 0$ and $\kappa$ is trivial.

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