JOINT OPTIMAL PRICING AND INVENTORY MANAGEMENT POLICY AND ITS SENSITIVITY ANALYSIS FOR PERISHABLE PRODUCTS: LOST SALE CASE

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ABSTRACT. In the real world, the demand cannot be depicted exactly because of customer behavior cannot be forecasted without error. In this paper, we study the effect of the error of the estimated price-demand parameters by analyzing the sensitivity of the optimal joint pricing and ordering policy on the price-demand parameters based on a periodic-review, multi-period and lost sale inventory model for perishable products with constant quantity decay rate and price-sensitive demand. Firstly, we formulate the joint pricing and inventory control problem and find the optimal ordering quantity and the optimal price for deterministic price-demand function. The optimal solutions show that the retailer tends to set a lower price in early periods of each ordering cycle in order to reduce the inventory holding costs. Furthermore, the sensitivity of the optimal joint pricing and inventory control system with respect to the price-demand parameters is examined analytically and evaluated numerically. The sensitivity analysis reveals that compared to the optimal ordering quantity, the optimal prices are less sensitive in the demand-price parameters. Finally, according to the findings of the sensitivity analysis, a heuristic method of regulating the estimated demand-price parameters is employed to improve the average profit.

1. Introduction and literature review. There are two main models for describing the deterioration process of perishable products: a) the quality of a product with a fixed lifetime decays[8][17][19]; and b) the quantity of the in-stock products deteriorates in a fixed or random rate[9][11][12][13]. In daily life, the fresh fruits (e.g., kiwis, avocados, strawberries) and the vegetables (e.g. lettuce, celery and cabbage) can be regarded as deteriorating in quantity because (1) as non-standardized products, they do not have a fixed shelf lifetime, i.e., in the end of every period, part of them are totally decayed and cannot be kept in the shelf; (2) moisture in the fresh products in the shelf is lost continuously; (3) some customers’ selection will damage

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the products and leads to the quantity deterioration. Quantity deterioration can result in the profit loss because the sale quantity is less than the order quantity received from the suppliers. In this paper, we focus on this type of perishable products, and assume that a certain amount of them decays at the end of each period. The retailer wants to make an optimal joint pricing and ordering decision for the perishable products whose inventory deteriorates with a fixed decay rate and whose demand is decreasingly linearly in price, i.e., \( d_t = D(p_t) = s - \alpha p_t \), where \( d_t \) is the demand and \( p_t \) is the sales price of the product in period \( t \), the parameter \( s \) is the maximum possible demand in each period, and the parameter \( \alpha \) is the sensitivity of the demand with respect to the price. Hereafter, \( s \) and \( \alpha \) are referred to the maximum demand and the price-demand factor, respectively. The true values of these two fundamental parameters in the demand-price function are basically unknown and have to be estimated. The optimal prices and order quantity are made based on the estimated values of \( s \) and \( \alpha \). Because the estimated parameters will not be completely accurate, the optimal policy obtained based on the estimated parameters may be not optimal for the real system. Therefore, after solving the joint pricing and ordering optimization system, which takes the price-demand parameters (\( s \) and \( \alpha \)) as the inputs and the optimal pricing and ordering decisions (\( \mathbf{F}^* \) and \( u^* \)) as the outputs, we focus on the effect of the errors of the estimated price-demand parameters on the optimization system and give a rule for regulating the estimated parameters. Therefore, in this research, we conduct sensitivity analysis for the effect of estimated parameters on the performance of optimal pricing and ordering decision.

The literature related to this research lie on two aspects: (1) joint ordering and pricing problem, and (2) sensitivity research on inventory control problem.

1.1. Joint ordering and pricing problem. The effect of the price on the demand was well studied in Petruzzi and Dada[20]. Because a low price lead more customers to purchase the items, a lot of research has considered effect of price on demand when make an inventory decision. The joint pricing and ordering problem is firstly considered in 1955 [23]. Research on joint pricing and ordering problem with backlogged demand or without shortage for multi-period case can be found in Chen and Simchi-Levi[5], Yao[24], Chen et al.[4] and Feng et al.[7]. Chen and Simchi-Levi[5] proposed a \((s, S, p)\) policy based on the \( K\)-concave property of the profit-to-go function and proved the optimality of this policy. Yao[24] modeled the price-demand relationship as a Brownian motion in the situation that excess demand is backlogged. Chen et al.[4] studied a joint pricing and inventory control problem for a perishable product with price-dependent demand and find the monotonicity properties of the optimal policy based on the concept of \( L\)-concavity and the accurate inventory-level in every replenishment period. Feng et al.[7] focused the dynamic pricing and ordering policy for the case that no shortage is allowed and solving the problem based on Pontryagin’s maximum principle. These papers give the optimal mathematical solutions because the objective function is concave or convex with respect to the decision variables in the backlog case. Some research focuses on solving the single period joint ordering and pricing problem analytically in which the retailer only order new products at the beginning of the whole sales period[2][6]. Li et al.[16] formulated an optimal joint pricing and inventory control problem for perishable products as a problem of solving a Hamilton-Jacobi-Bellman (HJB) equation. Herbon and Khmelnitsky[10] focused on the ordering and pricing
problem for a perishable product with a demand depending on both price and time since replenishment.

1.2. **Sensitivity research on inventory control problems.** If the uncertainty in demand is taken into consideration, since the demand cannot be estimated exactly, two main methods are often employed: a) finding a method to model or estimate the demand fluctuation mathematically\[15\], and b) finding an acceptable solution by heuristic algorithms\[14\]. Besides these two types of methods, sensitivity analysis is also a very useful technique to deal with uncertainties, which helps to find the most influential parameters on the outputs. Borgonovo and Peccati\[1\] used the Sobol function and variance decomposition method to find the most influential parameters on the output when dealing with the sensitivity analysis of inventory policy with uncertain input parameters. Rabta\[22\] studied the sensitivity of a class of \((s, S)\) inventory models with demand perturbation in terms of ergodicity coefficients and obtained the perturbation bounds. More researchers make sensitivity analysis by numerical experiments based on the results of the deterministic case with assumptions that the demand is backlogged or shortage are not allowed. Research on this aspect can be found in Qin et al.\[21\], Lu et al.\[18\] and Chen et al.\[3\], see Table 1.

| Paper          | Optimization          | Backlog or lost sale or no shortage | Sensitivity analysis objective                                       |
|----------------|-----------------------|------------------------------------|---------------------------------------------------------------------|
| Qin et al.\[21\] | Price, order          | No shortage                        | Decision variables on price-demand parameter and on deterioration function |
| Lu et al.\[18\] | Price, order, quality keeping | No shortage                        | Decision variables on all the system parameters                      |
| Chen et al.\[3\] | Price, order          | backlog                            | Decision variables on decay rate, demand change, holding cost       |

In summary, existing research is more concerned with how to solve the joint ordering and pricing problem for perishable products under the assumption that the demand is backlogged or the shortage is not allowed. However, in the real market, the price-demand parameters are estimated based on historical sales data. And the effect of the error of estimation, especially for the lost sale case, has not been studied sufficiently in existing research on the joint pricing and inventory control problem for perishable products. In this paper, after solving the optimal joint pricing and ordering optimization problem, we will focus on the sensitivity analysis of the optimal decisions with respect to the errors of the estimated price-demand parameters both analytically and numerically. Firstly, we study the joint pricing and ordering optimization system and obtain the optimal prices \((\vec{p}^*)\) and the optimal ordering quantity \((u^*)\) for the inputs \(s\) and \(\alpha\). Based on the result of the pricing and ordering optimization, the sensitivity of the optimal ordering and pricing decisions (i.e., \(u^*\) and \(\vec{p}^*\)) with respect to the demand-price parameters (i.e., \(s\) and \(\alpha\)) is examined analytically. In addition, numerical experiments are conducted to evaluate the influence of \(s\) and \(\alpha\) on the pricing and ordering optimization process, which reveals significantly different influences of the two parameters on \(u^*\) and
Based on that, heuristic rules are designed to regulate the estimated parameters in the demand-price function so as to improve the average profit.

The rest of this paper is organized as follows: In Section 2, a joint optimal ordering and pricing optimization system with quantity decay of in-stock products is formulated. Based on the zero-inventory condition, an infinite-horizon optimal joint pricing and inventory control model can be simplified into a single-cycle joint pricing and inventory control model. For this single-cycle model, we find the analytical formulation of the optimal ordering quantity, prices and the ordering cycle in Section 3. In Section 4, we examine analytically the sensitivity of the optimal ordering quantity, the optimal prices and the optimal ordering cycle with respect to the price-demand parameters. In Section 5, numerical experiments are conducted to evaluate the sensitivity of the optimal ordering quantity and prices with respect to the demand-price parameters. Furthermore, heuristic rules are designed to regulate the estimated parameters in the demand-price function so as to improve the average profit. Section 6 concludes this paper.

2. Problem formulation. Consider an inventory system of a single type of perishable products with zero lead-time, which has a price-sensitive demand and a constant quantity decay rate. The retailer makes the joint pricing and inventory control decision before placing an order and receives the products without lead-time. The quantity decay happens immediately, and then the demand is satisfied and unsatisfied demand is lost. The remaining products at the end of the period is called the “remaining items” which generates the inventory holding cost that is proportional to its quantity. The optimization objective is to maximize the average profit, which is the difference between the revenue obtained from selling the products and the cost for purchasing and inventory holding. In this paper, we consider the infinite horizon case, and the profit is averaged over time based on the following assumptions:

(1) The demand is a monotonically decreasing linear function of price, i.e., lower price makes more customers to buy products.
(2) The unsatisfied demand is lost (i.e., the lost sale model).
(3) The initial inventory level is zero without loss of generality because the inventory level will always go to zero within a replenishment cycle until a new order is placed.
(4) The lead-time of replenishment is zero.
(5) For simplification, we assume that the quantity decay rate is a constant and all the deteriorated products are discarded.

Based on these assumptions, the mathematical model of the joint optimal pricing and inventory control problem is given by:

(P1)

\[
\text{Maximize} \quad \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T}\{p_t \min[(1-\theta)(x_t + u_t), d_t] - cu_t - K \cdot 1\{u_t\} - hx_{t+1}\}
\]

Subject to:

\[
x_{t+1} = (1-\theta)(x_t + u_t) - \min[(1-\theta)(x_t + u_t), d_t], \quad t = 1, \ldots, T
\]

\[
d_t = D(p_t) = s - \alpha p_t, \quad t = 1, \ldots, T
\]

where

\[T: \text{number of periods;}\]
\( t \): index of period;
\( x_t \): inventory level at the beginning of period \( t \);
\( d_t \): demand quantity in period \( t \);
\( u_t \): ordering quantity in period \( t \);
\( p_t \): selling price in period \( t \);
\( s \): maximum demand (i.e., market capacity);
\( \alpha \): price-demand factor, \( \alpha > 0 \);
\( K \): fixed ordering cost;
\( c \): unit purchasing cost, \( c < p_t \);
\( h \): unit inventory holding cost;
\( \theta \): quantity decay rate, \( 0 < \theta < 1 \);

1\{ \}: indicating function, with \( 1\{x\} = 1 \) if \( x > 0 \) and \( 1\{x\} = 0 \), otherwise.

In Eq. (1), \( p_t \min \left[ (1 - \theta) (x_t + u_t), d_t \right] \) is the revenue in period \( t \), in which \( \min \left[ (1 - \theta) (x_t + u_t), d_t \right] \) is the quantity of sold products and indicates that the problem \( P1 \) is a lost sale model. The inventory cost in period \( t \) consists of three parts: the variable purchasing cost cut, the fixed purchasing cost \( K \cdot 1\{u_t\} \), and the inventory holding cost \( h x_t + 1 \). Eq. (2) is the equation of this inventory system, in which the inventory position at the end of period \( t \) is equal to that at the beginning of period \( t + 1 \). Eq. (3) represents the linear relationship between price and demand.

Before solving the problem \( (P1) \), we give the zero-inventory property for the optimal solution of a finite horizon version of problem \( (P1) \) following Theorem 4.3.1 in the book “Foundations of Inventory Management” by Zipkin[25], in which no backorders or lost sales is allowed for normal products without the quantity decay.

**Property 2.1.** (Zero-Inventory Property). For a finite-horizon (e.g. \( T \)-period) joint optimal pricing and inventory control problem with quantity decay under lost sale case, the optimal joint order quantity and prices \( \{(u_t^*, p_t^*) \mid t = 1, \ldots, T\} \) has the following property: \( u_t^* > 0 \) holds only if \( x_t^* = 0 \); or equivalently, if \( x_t^* > 0 \), then \( u_t^* = 0 \) must hold.

**Proof.** We prove Property 2.1 by contradiction. In the lost sale case, the inventory level is nonnegative because all the unsatisfied demand is lost and no customer waits for a new product if the products are out of stock. Assume that there exists at least one “non-zero” optimal ordering time. For example, we suppose that an order is made at time \( \lambda_i \) when the inventory position is greater than zero, i.e., \( x_{\lambda_i} > 0 \). We define such an ordering time \( (\lambda_i) \) as a “non-zero” ordering time. Assume that in a sequence of ordering times \( \{\lambda_1, \cdots, \lambda_t, t \leq T\} \) under the optimal policy, there exists at least one “non-zero” ordering time. Let \( \lambda_i \) be the earliest “non-zero” ordering time and \( \lambda_{i-1} \) be the ordering time immediately before \( \lambda_i \). \( \lambda_i \) must exist because \( n = 1 \) is not a “non-zero” ordering time (recall that we assume the initial inventory level is zero).

Suppose that \( \Delta u_{\lambda_i} \) is the remaining items of the last period in the ordering cycle with the ordering time \( \lambda_{i-1} \), which is also the inventory level at the beginning of the period \( \lambda_i \). Let \( u_{\lambda_{i-1}} \) denote the ordering quantity at the ordering time \( \lambda_{i-1} \), and \( u_{\lambda_i} \) the ordering quantity at the ordering time \( \lambda_i \). As shown in Figure 1, due to the quality decay, we define \( \Delta u_{\lambda_{i-1}} = \Delta u_{\lambda_i} (1 - \theta)^{\lambda_i - \lambda_{i-1}} \), and let \( u_{\lambda_{i-1}} = u_{\lambda_{i-1}} - \Delta u_{\lambda_{i-1}}, u_{\lambda_i} = u_{\lambda_i} + \Delta u_{\lambda_i} \).

Construct a new feasible solution, in which \( \lambda_i \) is no longer a “non-zero” ordering time. Modify the ordering decision by decreasing the ordering quantity at \( \lambda_{i-1} \) by \( \Delta u_{\lambda_{i-1}} \) and increasing the ordering quantity at \( \lambda_i \) by \( \Delta u_{\lambda_i} \). Thus, in the new
solution, the retailer orders $u'_{\lambda_{i-1}}$ products at $\lambda_{i-1}$ and orders $u'_{\lambda_i}$ products at $\lambda_i$ respectively. This change drives the inventory level immediately before $\lambda_i$ to zero. So $\lambda_i$ is not a “non-zero” ordering time any more. Suppose that time $j$ is in the ordering cycle with the ordering time $\lambda_{i-1}$, i.e., $\lambda_{i-1} \leq j \leq \lambda_i - 1$. Therefore, compared to the original solution, the holding cost is reduced by $h\Delta\hat{u}_{\lambda_{i-1}}(1 - \theta)^{j+1-\lambda_{i-1}}$ at time $j$, and the inventory holding cost is reduced by $\lambda_i - 1 \sum_{j=\lambda_{i-1}}^{\lambda_i - 1} \{h\Delta\hat{u}_{\lambda_{i-1}}(1 - \theta)^{j+1-\lambda_{i-1}}\}$ from time $\lambda_{i-1}$ to time $\lambda_i - 1$, during which the retailer will gain the same revenue as before, because it will sell the same quantity of products. In this sense, we say that the “non-zero” ordering time $\lambda_i$ is eliminated. By repeating this procedure, all the remaining “non-zero” ordering times that are greater than $\lambda_i$ can be eliminated, and the average profit will be increased in each step, which is contradict with the assumption we made at the beginning of the proof. Thus, we have proved that in the optimal solution there is no “non-zero” ordering time.

Based on Property 2.1, when $T$ goes to infinity, repetitive ordering and pricing cycles will appear for the infinite horizon case, as depicted in Figure 2. During every ordering cycle, an order is placed only at the beginnin, but the price is optimized in every period. Therefore, we can simplify the long-term joint optimal pricing and inventory control problem to such a problem in a single ordering cycle with zero initial inventory position.

To model the optimal joint pricing and inventory control problem in a single ordering cycle, we define some additional notifications as follows:

- $TP$: length of a single ordering cycle;
- $u$: ordering quantity for a single ordering cycle;
- $\vec{p} = (p_1, \ldots, p_{TP})^T$: selling price vector, $p_t$ is the price in period $t$, $t = 1, ..., TP$;
- $J(\vec{p}, TP)$: average profit.
Considering the stock loss due to quantity decay, the retailer needs to place an order with the quantity of \( D(p_t)/(1-\theta)^t \) at the beginning of the ordering cycle to satisfy the demand in period \( t \), i.e., \( D(p_t) \). Thus, for any \( p_t \) and \( TP \), there exists an optimal ordering quantity \( \tilde{u} = \sum_{t=1}^{TP} D(p_t)/(1-\theta)^t \). Namely, the optimal ordering quantity \( u^* \) is a function of the optimal order cycle \( TP^* \) and the optimal prices \( \vec{p}^* \). Therefore, the objective can be regarded as finding the optimal prices and the optimal cycle length, i.e. \( \{p_t^*, TP^*, t = 1, \cdots, TP^*\} \), to maximize the average profit in an ordering cycle. The model of such a problem is as below:

\[
\text{(P2)}
\]

\[
J(\vec{p}^*, TP^*) = \max J(\vec{p}, TP)
\]

\[
= \max \frac{1}{TP} \left\{ \sum_{t=1}^{TP} p_t D(p_t) - c \sum_{t=1}^{TP} \frac{D(p_t)}{(1-\theta)^t} - h \sum_{t=2}^{TP} \sum_{i=1}^{t-1} \frac{D(p_t)}{(1-\theta)^t} \right\}
\]

Subject to \( D(p_t) = s - \alpha p_t > 0, t = 1, \ldots, TP \).

In Eq. (4), \( D(p_t) \) is the demand determined by the price \( p_t \); \( \sum_{t=1}^{TP} D(p_t)/(1-\theta)^t \) is the ordering quantity; \( \sum_{t=2}^{TP} \sum_{i=1}^{t-1} D(p_t)/(1-\theta)^t \) is the inventory level of the remaining items.

3. Joint Pricing and Inventory Management Optimization. Based on the model of the price and ordering cycle length optimization problem (P2), in this section, we derive the solution to the joint optimal pricing and inventory control problem for the case that the demand is completely determined by the price without any uncertainties. The problem (P2) can be considered equivalent to the original infinite horizon ordering and pricing problem (P1) according to Property 2.1.

First of all, we rewrite Eq. (4) as below

\[
\text{(P2)}
\]

\[
J(\vec{p}^*, TP^*) = \max J(\vec{p}, TP)
\]
\[
\begin{align*}
&= \max_{\vec{p}, TP} \left\{ \frac{1}{TP} \left\{ \sum_{t=1}^{TP} \left( p_t D(p_t) - \frac{c D(p_t)}{1 - \theta} \right) \right. \\
&\quad + \left. \frac{1}{TP} \sum_{t=2}^{TP} \left( p_t - \frac{c}{1 - \theta} - \frac{h}{\theta(1 - \theta)^{t-1}} \left[ 1 - (1 - \theta)^{t-1} \right] \right) D(p_t) - K \right\} \\
&\text{For the convenience of mathematical manipulation, we introduce a new variable} \\
&\kappa_t = \frac{c}{(1 - \theta)^t} + \frac{h}{\theta(1 - \theta)^{t-1}} = \left( \frac{c}{1 - \theta} + \frac{h}{\theta} \right) \frac{1}{(1 - \theta)^{t-1}} - \frac{h}{\theta} \quad (6)
\end{align*}
\]

for \( t = 2, \ldots, TP \). Especially, we define \( \kappa_1 = c_1 - \theta \), which is also consistent with Eq. (5) for \( t = 1 \). Therefore, \( \kappa_t > 0 \) for all the \( t \geq 1 \). Then Eq. (4) can be rewritten as

\[
J(\vec{p}^*, TP^*) = \max_{\vec{p}, TP} \left\{ \frac{1}{TP} \sum_{t=1}^{TP} \left( p_t - \kappa_t \right) D(p_t) - K \right\} \quad (7)
\]

The Hessian matrix of the average profit \( J(\vec{p}, TP) \) with respect to \( \vec{p} = (p_1, \ldots, p_{TP})^T \) is \( \nabla^2 J(\vec{p}, TP) = \begin{bmatrix} -2\alpha & \ddots \\
\vdots & -2\alpha \end{bmatrix} \).

Since \( \alpha > 0 \), \( \nabla^2 J(\vec{p}, TP) \) is negative definite. Hence, \( J(\vec{p}, TP) \) has a unique maximum that makes \( \frac{\partial J(\vec{p}, TP)}{\partial p_t} = s + \alpha \kappa_t - 2\alpha p_t = 0 \). By solving \( \frac{\partial J(\vec{p}, TP)}{\partial p_t} = s + \alpha \kappa_t - 2\alpha p_t = 0 \), we obtain the optimal prices as below

\[
p_t^* = \frac{s}{2\alpha} + \frac{\kappa_t}{2}, \quad \text{for} \quad t = 1, \ldots, TP \quad (8)
\]

The corresponding demands under the optimal prices is

\[
D(p_t^*) = \frac{s}{2} - \frac{\alpha \kappa_t}{2}, \quad \text{for} \quad t = 1, \ldots, TP \quad (9)
\]

Because \( \kappa_t \) increases monotonically with respect to \( t \), we have \( p_1^* < \cdots < p_{TP}^* \), and \( D(p_1^*) > \cdots > D(p_{TP}^*) \) holds accordingly. Thus, we have the following property about the optimal prices.

**Property 3.1.** For the joint pricing and inventory control problem for a single ordering cycle (P2), the optimal prices increase over time, i.e., \( p_1^* < \cdots < p_{TP}^* \), where \( TP \) is the ordering cycle.

Considering Eqs. (5)-(6), we can get

\[
\begin{align*}
J(\vec{p}^*, TP^*) &= \max_{\vec{p}, TP} \left\{ \frac{1}{TP} \sum_{t=1}^{TP} \left( p_t^* - \kappa_t \right) D(p_t^*) - K \right\} \\
&= \max_{\vec{p}, TP} \left\{ \frac{1}{TP} \sum_{t=1}^{TP} \left( \frac{s}{2\alpha} \right) \frac{\kappa_t}{2} D(p_t^*) - K \right\} \\
&= \max_{\vec{p}, TP} \left\{ \frac{s}{2} - \frac{\alpha \kappa_t}{2} \right\} - K \\
&\text{Because} \quad D(p_1^*) > \cdots > D(p_{TP}^*) = \frac{s}{2} - \frac{\alpha \kappa_{TP}}{2} > 0, \quad \text{we can get the upper bound of} \quad TP, \quad \text{i.e.,} \quad TP < 1 + \ln \frac{s \alpha + \alpha h}{s \theta + \alpha h} / \ln (1 - \theta), \quad \text{with} \quad \alpha \theta + \alpha h < s \theta + \alpha h \quad \text{because of}
\end{align*}
\]
s > αc, which comes from s − αpt > 0 and pt > c. Hence, ln \( \frac{αθ + αh}{αθ + αh} \) \( \ln (1 – θ) > 0 \) and 1 ≤ TP < 1 + ln \( \frac{αθ + αh}{αθ + αh} \) \( \ln (1 – θ) \) holds. Therefore, we have

\[
J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) = \frac{1}{αTP(TP + 1)} \left\{ \sum_{t=1}^{TP} \left[ D^2\left(p_{t,T+1}^*\right) - D^2\left(p_t^*\right) \right] + αK \right\}
\]

(11)

Because the demands decrease over time, i.e., \( D\left(p_{T+1}^*\right) < D\left(p_t^*\right), t = 1, \cdots, TP \), we have \( D^2\left(p_{T+1}^*\right) - D^2\left(p_t^*\right) < 0 \). Therefore, there exists an unique optimal ordering cycle \( TP^* \in \left[1, 1 + ln \frac{αθ + αh}{αθ + αh} \right] \) such that \( J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) > 0 \) holds if \( TP < TP^* \) and \( J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) \leq 0 \) holds if \( TP ≥ TP^* \). Especially, if there exists \( J\left(\vec{p}^*, TP + 1\right) = J\left(\vec{p}^*, TP\right) \), let \( TP^* = TP \).

Therefore, we have the following property for optimal ordering cycle:

**Property 3.2.** (Unimodality of with respect to TP) For the joint pricing and inventory control problem for a single ordering cycle (P2), the profit \( J\left(\vec{p}^*, TP\right) \) under the optimal prices is unimodal in the ordering cycle \( TP \).

It is not difficult to find the optimal ordering cycle \( TP^* \) by enumeration from \( TP = 1 \). Actually,

\[
TP^* \in \begin{cases} 
TP | J\left(\vec{p}^*, TP - 1\right) - J\left(\vec{p}^*, TP\right) < 0 \\
\text{and} J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) \leq 0 
\end{cases}
\]

(12)

Especially, if \( J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) \leq 0 \) for all the \( TP \), then \( TP^* = 1 \), this is a relatively simple case. In the following work, if \( TP^* = 1 \), the referred relationship of period \( TP - 1 \) and the period \( TP \) will not be taken into consideration. According to the optimal price and the optimal ordering cycle, the optimal ordering quantity is given by

\[
u^* = \sum_{t=1}^{TP^*} D\left(p_t^*\right) \left\{ (1 - θ)^t \right\} = \frac{1}{2} \sum_{t=1}^{TP^*} \left( s - ακt \right) \left\{ (1 - θ)^t \right\}
\]

(13)

To summarize, the optimal joint pricing and inventory control policy to maximize \( J\left(\vec{p}, TP\right) \) is

\[
u^* = \frac{1}{2} \sum_{t=1}^{TP^*} \left( s - ακt \right) \left\{ (1 - θ)^t \right\}
\]

\[
p_t^* = \frac{s}{2α} + \frac{κt}{2}, t = 1, \cdots, TP^*
\]

(14)

where \( TP^* \in \begin{cases} 
TP | J\left(\vec{p}^*, TP - 1\right) - J\left(\vec{p}^*, TP\right) < 0 \\
\text{and} J\left(\vec{p}^*, TP + 1\right) - J\left(\vec{p}^*, TP\right) \leq 0 
\end{cases}
\]

Consequently, the maximum average profit is

\[
J\left(\vec{p}^*, TP^*\right) = \frac{s^2}{4α} - \frac{1}{TP^*} \left\{ \sum_{t=1}^{TP^*} \left( \frac{sκt}{2} - \frac{ακt^2}{4} \right) + K \right\}
\]

(15)
4. Sensitivity analysis. In Section 3, we give the optimal ordering quantity $u^*$ and the optimal prices $\vec{p}^*$ (outputs) that depend on the price-demand parameters $s$ and $\alpha$ (inputs) for the joint ordering and pricing optimization system. However, the true values of the two parameters may deviate from their estimated values. Therefore, it is necessary to analyze the sensitivity of $u^*$ and $\vec{p}^*$ (outputs) on $s$ and $\alpha$ (inputs). We will first analyze the impact of the estimated error of $J$ to-go function $\kappa$ namely, change from the optimal ordering cycle, and then on the optimal ordering quantity and prices.

The optimal joint pricing and inventory control policy to maximize the profit-to-go function $J(\vec{p}, TP)$ is given by Eq. (14). When the price-demand parameters change from $(s, \alpha)$ to $(s + \Delta s, \alpha + \Delta \alpha)$, where $\Delta s$ and $\Delta \alpha$ are the deviations of the two parameters, respectively, the optimal ordering quantity and the optimal price become

$$
\begin{align*}
&u^*(\Delta s, \Delta \alpha) = \frac{1}{2} \sum_{t=1}^{TP_{\text{new}}^*} \frac{(s + \Delta s) - (\alpha + \Delta \alpha) \kappa_t}{(1 - \theta)^t}, \\
&p_t^*(\Delta s, \Delta \alpha) = \frac{s + \Delta s}{2(\alpha + \Delta \alpha)} + \frac{\kappa_t}{2},
\end{align*}
$$

(16)

where $TP_{\text{new}}^*$ is the optimal ordering cycle of the inventory system under parameters $(s + \Delta s, \alpha + \Delta \alpha)$, which can be obtained by enumerating from $TP_{\text{new}}^* = 1$ through

$$
TP_{\text{new}}^* \in \left\{ TP | J\left(\vec{p}^*(\Delta s, \Delta \alpha), TP - 1\right) - J\left(\vec{p}^*(\Delta s, \Delta \alpha), TP\right) < 0 \right\}
$$

and $J\left(\vec{p}^*(\Delta s, \Delta \alpha), TP + 1\right) - J\left(\vec{p}^*(\Delta s, \Delta \alpha), TP\right) \leq 0$

The new optimal pricing policy is $\vec{p}_{\text{new}}^* = (p_t^*(\Delta s, \Delta \alpha), t = 1, ..., TP_{\text{new}}^*)$.

4.1. Sensitivity analysis of the optimal ordering cycle. The fluctuation of $s$ or $\alpha$ will probably change the optimal ordering cycle. Based on Eq. (12) and the analysis in Section 3, the optimal ordering cycle has the following property

$$
\begin{align*}
&J(\vec{p}^*, TP^*) - J(\vec{p}^*, TP^* - 1) \geq 0, \\
&J(\vec{p}^*, TP^* + 1) - J(\vec{p}^*, TP^*) \geq 0
\end{align*}
$$

(17)

which is the necessary and sufficient condition for the optimal ordering cycle $TP^*$. Combing Eq. (10) with Eq. (17), we obtain the following inequalities after simplification

$$
\frac{\frac{1}{2} \left( TP^* \kappa_{TP^*+1} - \sum_{t=1}^{TP^*} \kappa_t \right) + \frac{1}{2} \left( \sum_{t=1}^{TP^*} \kappa_t^2 - TP^* \kappa_{TP^*+1}^2 \right) - K}{TP^* (TP^* + 1)} > 0
$$

(18)

$$
\frac{\frac{1}{2} \left( \sum_{t=1}^{TP^*-1} \kappa_t - (TP^* - 1) \kappa_{TP^*} \right) + \frac{1}{2} \left( (TP^* - 1) \kappa_{TP^*}^2 - \sum_{t=1}^{TP^*-1} \kappa_t^2 \right) + K}{TP^* (TP^* - 1)} > 0
$$

(19)

Since $\kappa_t = \left( \frac{c}{1-\sigma} + \frac{h}{\theta} \right) \frac{1}{(1-\theta)^t} - \frac{h}{\theta} > 0$ and $\theta$ is a constant, $\kappa_t$ increases in $t$, namely, $\kappa_1 < \kappa_2 < ... < \kappa_{TP^*-1} < \kappa_{TP^*} < \kappa_{TP^*+1}$. Hence, $TP^* \kappa_{TP^*+1} - \sum_{t=1}^{TP^*} \kappa_t > 0$, $\sum_{t=1}^{TP^*} \kappa_t^2 - TP^* \kappa_{TP^*+1}^2 < 0$, $\sum_{t=1}^{TP^*} \kappa_t - (TP^* - 1) \kappa_{TP^*} < 0$ and $(TP^* - 1) \kappa_{TP^*}^2 -
\[ \sum_{t=1}^{TP^*-1} \kappa_t^2 > 0. \] After examining the coefficients of terms with \( s \) and \( \alpha \) in Eqs. (18) and (19), we can conclude that Eq. (18) holds when \( s \) increases or \( \alpha \) decreases, and Eq. (19) holds when \( s \) decreases or \( \alpha \) increases. To explain the trend of \( TP^* \), we analyze a typical case that \( s \) increases but \( \alpha \) is fixed (\( \Delta \alpha = 0 \)) and give the results of other cases directly in Table 2.

According to the analysis in Section 3, \( J(\vec{p}^*, TP^* + 1) - J(\vec{p}^*, TP) > 0 \) holds if \( TP < TP^* \) and \( J(\vec{p}^*, TP + 1) - J(\vec{p}^*, TP) < 0 \) holds if \( TP \geq TP^* \). When \( s \) increases, for any \( TP \geq TP^* \), \( J(\vec{p}^*, TP^* + 1) - J(\vec{p}^*, TP) < 0 \) holds because Eq. (18) holds. However, Eq. (19) may not hold because the maximum demand \( s \) is positive but the coefficient of the term with \( \alpha \) is negative. Therefore, \( TP^*_{\text{new}} \leq TP^* \), the optimal ordering cycle tends to be smaller. The new optimal ordering cycle \( TP^*_{\text{new}} = TP^* \) if Eq. (19) holds and \( TP^*_{\text{new}} \geq TP^* \) if Eq. (19) does not hold.

Similarly, we can get the range of the new optimal ordering cycle \( TP^*_{\text{new}} \) for other cases of the change of \( s \) and \( \alpha \), which is shown in Table 2.

**Table 2. Sensitivity analysis on perishability problem**

| \( \Delta s \) | \( \Delta \alpha \) | \( TP^*_{\text{new}} \) |
|-----------------|-----------------|-------------------|
| \( \Delta s > 0 \) | \( \Delta \alpha = 0 \) | \( TP^* \leq TP^*_{\text{new}} \) |
| \( \Delta s < 0 \) | \( \Delta \alpha = 0 \) | \( TP^* \geq TP^*_{\text{new}} \) |
| \( \Delta s = 0 \) | \( \Delta \alpha > 0 \) | \( TP^* \geq TP^*_{\text{new}} \) |
| \( \Delta s = 0 \) | \( \Delta \alpha < 0 \) | \( TP^* \leq TP^*_{\text{new}} \) |

### 4.2. Sensitivity analysis of optimal ordering quantity and optimal prices.

Sensitivity analysis of the optimal ordering quantity and the optimal price consists of two cases: A) sensitivity analysis on \( s \) with \( \Delta \alpha = 0 \) and B) sensitivity analysis on \( \alpha \) with \( \Delta s = 0 \). Define \( SC_{\Omega, \sigma} = \frac{\Delta \Omega(\Delta \sigma)}{\Delta \sigma} \) as the sensitivity coefficient, which is the ratio between the change rate of parameter \( \sigma \) (i.e., \( s \) or \( \alpha \)) and the change rate of the optimal ordering or pricing decision \( \Omega \) (i.e., \( u^* \) or \( p_t^* \)). Hereafter, if \( |SC_{\Omega, \sigma}| < 1 \), the relative change of \( \Omega \) is smaller than that of \( \sigma \), we say \( \Omega \) is robust with respect to \( \sigma \).

**Case A.** \( \Delta s \neq 0 \) and \( \Delta \alpha = 0 \).

According to Eq. (16), the sensitivity coefficients of the optimal ordering quantity and the optimal prices with respect to parameter \( s \) are given by

\[
SC_{u^*, s} = \frac{\sum_{t=1}^{TP^*_{\text{new}}} \frac{s + \Delta s - \alpha \kappa_t}{(1 - \theta)^t} - \sum_{t=1}^{TP^*} \frac{s - \alpha \kappa_t}{(1 - \theta)^t}}{\Delta s} \quad \frac{TP^*_{\text{new}}}{s}
\]

\[
SC_{p_t^*, s} = \frac{s}{s + \alpha \kappa_t}
\]

From Eq. (20), we know that

1. \( SC_{p_t^*, s} \) is a constant for each period \( t \). Because of \( s > 0, \alpha > 0 \) and \( \kappa_t > 0 \), \( |s/(s + \alpha \kappa_t)| < 1 \) holds. Therefore, the relative change of the optimal price is smaller than that of the maximum demand, i.e., \( p_t^* \) is robust with respect to \( s \).
2. Due to \( s/(s + \alpha_k t) > 0 \), the optimal price \( p^*_t \) is monotonically increasing in the maximum demand \( s \).

3. \( SC^*_p,s \) is independent of \( TP^*_{\text{new}} \). Hence, the change of the optimal ordering cycle has no effect on \( SC^*_p,s \) for every \( t \).

4. The change of the optimal ordering cycle affects \( SC^*_u,s \):

\[
\begin{align*}
SC^*_{u,s} &= \begin{cases} 
\frac{TP^*_{\text{new}}}{s(1-\theta)} - \frac{TP^*}{s(1-\theta)} & \text{if } TP^*_{\text{new}} < TP^* \\
\frac{TP^*_{\text{new}}}{s(1-\theta)} - \frac{TP^*}{s(1-\theta)} & \text{if } TP^*_{\text{new}} = TP^* \\
\frac{TP^*_{\text{new}}}{s(1-\theta)} + \frac{TP^*}{s(1-\theta)} & \text{if } TP^*_{\text{new}} > TP^*
\end{cases}
\end{align*}
\]

From Eq. (21), if \( TP^*_{\text{new}} = TP^* \), then \( SC^*_u,s \) is a constant. For the case of \( TP^*_{\text{new}} \neq TP^* \), \( SC^*_u,s \) depends on the value of \( TP^*_{\text{new}} \). Moreover, \((s - \alpha_k t)/(1 - \theta)^t > 0 \) in Eq. (21) holds because \( D(p^*_t) = (s - \alpha_k t)/2 > 0 \). As a result, \( SC^*_u,s \) is concave in \( \Delta s/s \) for every \( TP^*_{\text{new}} \), i.e., \( SC^*_u,s \) is piecewise concave.

**Case B.** \( \Delta s = 0 \) and \( \Delta \alpha \neq 0 \).

Similarly, the sensitivity coefficients of the optimal ordering quantity and the optimal prices with respect to parameter \( \alpha \) are given by

\[
\begin{align*}
SC^*_{u,\alpha} &= \frac{TP^*_{\text{new}}}{s(1-\theta)} - \frac{TP^*}{s(1-\theta)} \left/ \frac{\Delta \alpha}{\alpha} \right. \\
SC^*_{p,\alpha} &= \frac{-s}{(s + \alpha_k t)(1 + \Delta \alpha/\alpha)}
\end{align*}
\]

From Eq. (22), we can find that

1. \( SC^*_{p,\alpha} \) is increasing and concave in \( \Delta \alpha/\alpha \) for each period \( t \) and the change of the optimal ordering cycle has no effect on the optimal price.

2. Because of \(-s/[(s + \alpha_k t)(1 + \Delta \alpha/\alpha)] < 0 \), the optimal price \( p^*_t \) is monotonically decreasing in \( \alpha \).
3. The change of the optimal ordering cycle affects $SC_{u^*,a}$:

$$
SC_{u^*,a} = \begin{cases} 
\frac{TP^*}{TP^*} \sum_{i=1}^{\frac{s-\alpha\kappa_t}{1-\theta}} - \frac{TP^*}{TP^*} \sum_{i=1}^{\frac{s-\alpha\kappa_t}{1-\theta}} \frac{1}{\Delta s}, & \text{if } TP^*_\text{new} < TP^* \\
\frac{TP^*}{TP^*} \sum_{i=1}^{\frac{s-\alpha\kappa_t}{1-\theta}} + \frac{TP^*}{TP^*} \sum_{i=1}^{\frac{s-\alpha\kappa_t}{1-\theta}} \frac{1}{\Delta s}, & \text{if } TP^*_\text{new} > TP^*
\end{cases}
$$

From Eq. (23), if $TP^*_\text{new} = TP^*$, $SC_{u^*,a}$ is a constant. If $TP^*_\text{new} \neq TP^*$, $SC_{u^*,a}$ is concave for every $TP^*_\text{new}$, because $(s - \alpha\kappa_t)/(1-\theta)$ holds due to $D(p^*_t) = (s - \alpha\kappa_t)/2 > 0$.

According to the theoretical analysis, since the change of the optimal ordering cycle only affects the optimal ordering quantity, the retailer is suggested to get the knowledge of how the ordering cycle changes when making decision on ordering optimization. Moreover, $p^*_t$ is robust with respect to $s$, which is a crucial property in practice, which means that the relative error of $p^*_t$ (i.e., $[p^*_t (\Delta s) - p^*_t]/p^*_t$) is smaller than the relative estimation error of $s$ (i.e., $\Delta s/s$).

5. Numerical experiments. In this section, numerical experiments are conducted to verify the sensitivity analysis results we have obtained in subsection 3.2 and the range of the sensitivity coefficients. We only consider the case of $TP^* > 0$ and $TP^*_\text{new} > 0$ without loss of generality.

We conduct experiments on 100 instances of the inventory system with their parameters randomly generated from normal distributions satisfying $c \in [10, 20]$, $h \in [0.05c, 0.1c]$, $\theta \in [0.05, 0.1]$, $K \in [100c, 500c]$, $\alpha \in [2, 8]$ and $s \in [50s, 150s]$. For each of the 100 instances, sensitivity analysis is carried out in the following two ways: (1) sensitivity analysis of the joint pricing and inventory control optimization system with respect to parameter $s$ when $\Delta s = 0$; and (2) sensitivity analysis of the system with respect to parameter $\alpha$ when $\Delta \alpha = 0$.

5.1. Sensitivity analysis of the maximum demand $s$ with fixed $\alpha$ ($\Delta s \neq 0$, $\Delta \alpha = 0$). For each randomly generated instance of the inventory system, we call the case of $\Delta s = 0$ and $\Delta \alpha = 0$ its “nominal system”. For each instance, we increase $\Delta s/s$ from −50% to 50% with a step size of 10% excluding the case of $\Delta s/s = 0$.

The change of the optimal ordering cycle $TP^*_\text{new} - TP^*$ is depicted in Figure 3, in which the dots indicate $TP^*_\text{new} - TP^*$ at different values of $\Delta s/s$ and the solid line is the detailed change of $TP^*_\text{new} - TP^*$ for one of the 100 instances. Figure 3 confirms the trends of the optimal ordering cycle we have given in Table 2 in subsection 4.1, i.e., $TP^*_\text{new} - TP^* \geq 0$ when $\Delta s/s < 0$ and $TP^*_\text{new} - TP^* \leq 0$ when $\Delta s/s > 0$. If $\Delta s/s$ is between −0.1 and 0.1, the change of the optimal ordering cycle will be 0 or 1.
The results of $SC_{u^*,s}$ and $SC_{p^*_1,s}$ are depicted in Figure 4 and Figure 5, respectively. In both figures, each vertical dotted line illustrates the distribution of the sensitivity coefficient $SC_{u^*,s}$ or $SC_{p^*_1,s}$ at one possible value of $\Delta s/s$, and the solid line or curve shows the detailed variation of the sensitivity coefficient $SC_{u^*,s}$ or $SC_{p^*_1,s}$ for one of the 100 instances. From Figure 4 and Figure 5, we can observe that:

1) $SC_{u^*,s}$ is a constant for the case of $TP^*_{\text{new}} = TP^*$ and is piecewise concave for the case of $TP^*_{\text{new}} \neq TP^*$. Moreover, $SC_{p^*_1,s}$ is always a constant, which is consistent with the analysis in subsection 4.2;

2) for $TP^*_{\text{new}} > 0$, the range of $SC_{u^*,s}$ becomes narrower when $|\Delta s/s|$ increases. For example, $SC_{u^*,s} \in (-4.2)$ for $\Delta s/s = -0.1$ and $SC_{u^*,s} \in (0,1)$ for $\Delta s/s = -0.5$;

3) the range of $SC_{p^*_1,s}$ is from 0.74 to 0.93, which is less than 1.

![Figure 3. Change of the optimal ordering cycle](image)

![Figure 4. Sensitivity coefficient of the optimal ordering quantity](image)
This implies that the optimal price is robust with respect to $s$, in the sense that the magnitude of the relative change of the optimal price is less than that of the relative estimated error of the maximum demand ($s$).

5.2. **Sensitivity analysis of the price-demand parameter $\alpha$ with a fixed $s$ ($\Delta s = 0$, $\Delta \alpha \neq 0$).** For each instance, we increase $\Delta \alpha/\alpha$ from $-50\%$ to $50\%$ with a step size of $10\%$ excluding the case of $\Delta \alpha/\alpha = 0$. The change of the optimal ordering cycle $TP^*_\text{new} - TP^*$ is depicted in Figure 6 with dots and a solid line. Figure 6 confirms the analysis we have given in Table 2 in Section 4.1, i.e., $TP^*_\text{new} - TP^* \leq 0$ when $\Delta \alpha/\alpha < 0$ and $TP^*_\text{new} - TP^* \geq 0$ when $\Delta \alpha/\alpha > 0$. If $\Delta \alpha/\alpha$ is between $-0.1$ and $0.1$, the change of optimal ordering cycle will be 0 or 1.

The values of $SC_{u^*,\alpha}$ and $SC_{p^*,\alpha}$ for the 100 instances are depicted in Figure 7 and Figure 8 with vertical dotted lines and a solid line respectively. From Figure 7 and Figure 8, we know that:
1) $SC_{u^*,\alpha}$ is a constant for the case of $TP_{\text{new}}^* = TP^*$ and is piecewise convex for the case of $TP_{\text{new}}^* \neq TP^*$, and $SC_{p^*,\alpha}$ increases and is concave in $\Delta \alpha/\alpha$, which confirms our analysis in subsection 4.2.

2) Similar to the analysis in subsection 5.1, the range of $SC_{u^*,\alpha}$ becomes narrower when $|\Delta \alpha/\alpha|$ increases, for example, $SC_{u^*,\alpha} \in (-1, 3.5)$ for $\Delta \alpha/\alpha = -0.1$ and $SC_{u^*,\alpha} \in (-0.5, 0.5)$ for $\Delta \alpha/\alpha = -0.5$.

3) The absolute value of $SC_{p^*,\alpha}$ is less than 1 for every $\Delta \alpha/\alpha$ and its range becomes narrower as $\Delta \alpha/\alpha$ increases. Furthermore, $|SC_{p^*,\alpha}| < 1$ if $\Delta \alpha/\alpha > 0$, which implies that the optimal price is robust with respect to $\alpha$ when $\Delta \alpha/\alpha > 0$.

From Figure 3 to Figure 8 and all the above analysis, we can find that the optimal prices are partially robust. That is to say, the optimal prices are always robust with respect to $s$, but it is robust with respect to $\alpha$ only when $\Delta \alpha/\alpha > 0$. However,
the optimal ordering quantity is neither robust with respect to \( s \) nor robust with respect to \( \alpha \).

5.3. Heuristic regulation of price-demand parameters. Based on all the analytical and experimental results, we propose a heuristic method for regulating the price-demand parameters to reduce the negative effect of the estimation inaccuracy, which is called “heuristic regulation method” hereafter.

We consider a long-term joint pricing and inventory control problem (the total periods \( T > 1000 \)), in which we use the sine functions to simulate the situation that the price-demand parameters (i.e., \( s \) and \( \alpha \)) change over time around their mean values periodically with the variable amplitudes \( |\Delta s_{\text{max}}| \) or \( |\Delta \alpha_{\text{max}}| \) and the frequency \( f \). Besides, in the price-demand function exists a normally distributed additive error \( \varepsilon_t \), whose mean value is zero. The price-demand parameters (i.e., \( s \) and \( \alpha \)) are regulated according to the difference between the real sale quantities and the estimated demands, which are calculated from the optimal prices, and the sale quantities of the previous ordering cycle. We only consider the “valid periods” in the previous ordering cycle. For the lost sale case, if the inventory level at the end of a period is greater than zero, the real demand is equal to the sale quantity in this period. We call such a period a “valid period” hereafter.

The experiments are carried out on two aspects: 1) \( s \) changes over time for a given \( \alpha \); and 2) \( \alpha \) changes over time for a given \( s \). Denote the amplitudes of the change of \( s \) and \( \alpha \) by \( |\Delta s_{\text{max}}| \) and \( |\Delta \alpha_{\text{max}}| \) respectively. Then the maximum average profits under the joint pricing and inventory control decision based on the regulated price-demand parameters are denoted by \( JR(\Delta s) \) and \( JR(\Delta \alpha) \), respectively. Moreover, we denote the maximum average profit under the joint pricing and inventory control decision based on the original values of \( s \) and \( \alpha \) by \( JC \). Let \( s'_t \) and \( \alpha'_t \) denote the real values of the price-demand parameters in period \( t \). Then the real demand in period \( t \) is \( d'_t = s'_t - \alpha'_t p_t + \varepsilon_t \), and the estimated demand in period \( t \) is \( d_t = s - \alpha p_t \). In all the valid periods, \( d'_t \) is equal to the sales quantity, and therefore \( d'_t \) is observable. The heuristic regulation rules are as follows:

Rule 1 (Regulating \( s \) for a given \( \alpha \), i.e., \( \alpha'_t = \alpha \)). For all the valid periods in the previous ordering cycle, we have \( d'_t - d_t = s'_t - s + \varepsilon_t \), and then \( s'_t = (d'_t - d_t) + s - \varepsilon_t \) holds. Therefore, the parameter \( s \) in the current period can be regulated to \( \tilde{s} = \frac{1}{|V|} \sum_{t \in V} [(d'_t - d_t) + s] \), where \( V \) is the set of all the valid periods in the previous ordering cycle, and \( |V| \) is the number of the elements in the set \( V \).

Rule 2 (Regulating \( \alpha \) for a given \( s \), i.e., \( s'_t = s \)). For all the valid periods in the previous ordering cycle, we have \( d'_t - d_t = -\alpha'_t p_t + \alpha p_t + \varepsilon_t \), and then \( \alpha'_t = (\alpha p_t + \varepsilon_t + d_t - d'_t)/p_t \) holds. Therefore, the parameter \( \alpha \) in the current period can be regulated to \( \tilde{\alpha} = \frac{1}{|V|} \sum_{t \in V} (\alpha p_t + \varepsilon_t + d_t - d'_t)/p_t \) where \( V \) is the set of all the valid periods in the previous ordering cycle, and \( |V| \) is the number of the elements in the set \( V \). The performance of the heuristic regulation method is depicted in Figure 9 and Figure 10.

From Figure 9, we can find that if the change frequency of \( s \) is high, the regulation does not have any advantage comparing to the decision made based on the unchanged \( s \) (see the line of \( f = 1 \)). However, as the change frequency of \( s \) becomes low, the heuristic regulation method has better performance as \( |\Delta s_{\text{max}}| \) increases. The performance of the heuristic regulation method becomes better when
the change frequency of $s$ becomes lower. Figure 10 shows that the heuristic regulation method brings more profit for the retailer significantly under the high frequency of the change of $\alpha$. On the contrary, if the change frequency of $\alpha$ becomes lower, the heuristic regulation method does not have obvious better performance for different change frequencies of $\alpha$ although it still increases the profit based on the original parameter estimation.

In summary, in the low change frequency case, the heuristic regulation method has an obvious good performance on $\alpha$. Under the high change frequency case, the heuristic regulation method performs better on $s$. However, for all the cases, the heuristic regulation method has a good regulating function to bring more profit for the retailer, and it performs better when $|\Delta s_{\text{max}}|$ or $|\Delta \alpha_{\text{max}}|$ increases.

**Figure 9.** Performance of the heuristic regulation method on $s$

**Figure 10.** Performance of the heuristic regulating method on $\alpha$

**6. Conclusion.** In this paper, we firstly investigate the joint pricing and ordering problem of perishable products with a price-sensitive demand. Then analytical expression of the optimal prices, which are increasing over time, are obtained; and an efficient method is developed to find the optimal ordering cycle based on the
unimodality of the maximum profit with respect to the ordering cycle. We further examine analytically the impact of the variation of the price-demand parameters on the optimal ordering quantity and the prices by sensitivity analysis. Finally, numerical experiments are conducted to validate the theoretical analysis results. We find that the optimal prices are always robust with respect to $s$, but it is robust with respect to $\alpha$ only when $\Delta \alpha / \alpha > 0$. Besides, the optimal ordering quantity is not robust with respect to the same parameters. Hence, although it is difficult for a retailer to find a deterministic policy with high robustness and good performance at the same time, we suggested that the retailer has a tendency to underestimate the parameter $\alpha$ and he should pay more attention to the optimal ordering quantity when estimate the parameters. Because of this finding, a heuristic method based on historical data of sale quantity is proposed to regulate the price-demand parameters. In future research, we will develop a joint pricing and inventory control system that has better performances on both optimality and robustness for perishable products. Besides, in this work, we only consider the quantity deterioration of the perishable products. Actually, many products decay not only in quantity but also in quality. Therefore, the perishable products with both quality and quantity deteriorations will be considered in the future research.

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