HOW TO MAKE RSA AND SOME OTHER ENCRYPTIONS PROBABILISTIC

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Abstract. A new scheme of probabilistic subgroup-related encryption is introduced. Some applications of this scheme based on the RSA, Diffie-Hellman and ElGamal encryption algorithms are described. Security assumptions and main advantages of this scheme are discussed. We outline that this scheme is potentially semantically secure under reasonable cryptographic assumptions.

1. Some algorithmic problems for finite fields and modular rings.

In [1], [2], we proposed a novel probabilistic public-key encryption, based on the RSA cryptosystem. Its security is based on intractability of the membership and exponent problems for modular rings and usual security assumptions for the RSA encryption (see exact definitions and explanations below). Also these problems can be considered for finite fields. There are many reasons to estimate these two problems as hard mathematical problems. We note in this connection that the well-known Quadratic residuosity problem is a partial case of the membership problem.

1.1. The membership, order and exponent problems for finite fields and modular rings. We consider multiplicative groups of finite fields and modular rings, specified by a list of generators, or by some other effective way. The two most basic questions about such groups are membership in and the order of the group. It is not known how to answer these questions without solving hard number theoretic problems (factoring and discrete log).

Let \( \mathbb{Z}_n \) be a product of two different primes \( p \) and \( q \). Let \( \mathbb{Z}_n^{*} \) be the corresponding modular ring. Let \( Q_n \) be the subgroup of \( G_n = \mathbb{Z}_n^{*} \) consisting of all quadratic residues. The following problem is one of the most known decision problems in number theory and cryptography.

The Quadratic residuosity problem (QRP). Given an element \( f \in G_n \), determine if \( f \in Q_n \).

We assume that this determination should be effective. This problem is considered by many authors as intractable. A number of cryptographic schemes are based on this intractability, and the famous Goldwasser-Micali cryptosystem is one of them. It is important to note that the semantic security property of the Goldwasser-Micali cryptosystem is based on the
intractability of the QRP. The QRP is a particular case of the following decision problem.

The Membership Problem (MP_H). Let G be a group, and H be a subgroup of G. Given an element f ∈ G, determine if f ∈ H.

It seems that the following problem was previously considered only for matrix groups.

The order problem (OP). Given an element g (subgroup H) of a group G, determine order |g| (|H|).

Lemma 1. Let p and q be two different odd primes such that p,q ≡ 3(mod 4), and n = pq. Then solvability of OP for G_n = Z*_n implies solvability of QRP for G_n.

Proof. Let p = 4k + 3 and q = 4l + 3. Then |G_n| = (p − 1)(q − 1) = 4(2k + 1)(2l + 1). As index of Q_n in G_n is 4, then Q_n has odd order m = (2k + 1)(2l + 1). An element g lies in Q_n if and only if |g| is odd. Hence, if we can determine |g| for each g ∈ G_n, we can solve QRP for G_n.

Let F_q, q = p^r, be the finite field of order q and characteristic p. If the primality decomposition of q − 1 is known then there is a polynomial algorithm solving OP for elements of F_q^* (see [3]).

The exponent problem (EP). Given a subgroup H of a group G, determine exponent e(H), i.e., the minimal positive integer e such that h^e = 1 for every element h ∈ H.

Obviously, the solvability EP implies solvability OP, and by Lemma 1 solvability QRP in the case p,q ≡ 3(mod 4).

1.2. Constructing finite fields and modular rings that contain subgroups of prescribed exponent. Let r be a positive integer, and one party, say Bob, wants to construct a field F of a sufficiently large size that contains an element g ∈ F^* such that |g| = r. Then he seeks to find a prime p in the form p = 1 + 2rx where x is taken randomly. He checks the primality of p with some of the known primality tests. When it turns out p is prime, there is an element g ∈ F_p^* such that |g| = r. It can be found by ordinary effective procedure (see [3]). As r divides p^k − 1 for any k, he can take field F_q for q = p^k for suitable k, and to find element g ∈ F_q^* such that |g| = r. If he wants to get a couple g_1,...,g_t of elements such that |g_i| = r_i, for prescribed positive integers r_i (i = 1,...,t) he seeks to find a prime p in the form p = 1 + 2rx, where r = \prod_{i=1}^r r_i, and x is taken randomly. Then he gets desired elements g_1,...,g_t of a field F_q, q = p^k for a suitable k as above.

Let e be a positive integer, and Bob wants to construct a field F of a sufficiently large size such that the multiplicative group F^* contains subgroup H of exponent e. Then he generates a couple of positive integers r_1,...,r_t such that lcm(r_1,...,r_k) = e, then constructs a field F_q, q = p^k, for a suitable k, and finds elements g_1,...,g_t ∈ F_q^* such that |g_i| = r_i for i = 1,...,t, as explained above, and then succeeds in determining H = gp(g_1,...,g_t).

Let r be a positive integer, and Bob wants to construct a modular ring Z_n of a sufficiently large size n = pq, where p and q are two different odd
prime numbers, that contains an element \( g \in \mathbb{Z}_n^* \) such that \( |g| = r \). Then he seeks to find primes \( p \) and \( q \) in the forms \( p = 1+2r_1 x \) and \( q = 1+r_2 y \), where \( r_1 \) and \( r_2 \) are positive integers such that \( r = \text{lcm}(r_1, r_2) \), and \( x, y \) are taken randomly. He checks the primality of \( p \) and \( q \) with some known primality test. Then he finds elements \( g_1 \in \mathbb{F}_p^* \) and \( g_2 \in \mathbb{F}_q^* \) such that \( |g_1| = r_1 \) and \( |g_2| = r_2 \), as above. By the Chinese Remainder Theorem, he gets a solution \( g \) of the set of equations \( g = g_1 \mod p \), \( z = g_2 \mod q \). It follows that \( |g| = r \).

Hence Bob can construct a couple \( g_1, ..., g_t \) of elements such that \( |g_i| = r_i \), for prescribed positive integers \( r_i \) \((i = 1, ..., t)\), and to construct a modular ring \( \mathbb{Z}_n, n = pq \), where \( p \) and \( q \) are different odd primes, that contains a subgroup \( H \leq \mathbb{Z}_n^* \) of prescribed exponent \( e \).

When Bob publics the field \( \mathbb{F}_q, q = p^k \), or modular ring \( \mathbb{Z}_n, n = pq \), and the subgroup \( H = \text{gp}(g_1, ..., g_t) \), constructed as above, he doesn’t public the primality decompositions of \( p - 1 \) in the field case, and keeps secret the factors of \( n \) in the modular ring case. In both cases the exponent \( e = e(H) \) is private.

Oscar, an opponent, who wants to crack, modify, substitute, or replay messages, cannot compute \( e \) without solving hard number theoretic primality factoring problem with respect to \( p - 1 \) in the field case. In the modular ring case, he needs not only in knowing of the factors of \( n \), but also he needs in primality factorings of \( p - 1 \) and \( q - 1 \). Hence, in both the cases, the exponent problem with respect to \( H \) can be considered as intractable.

2. Basic scheme of probabilistic subgroup-related encryption founded on the intractability of EP.

Suppose two parties, say Alice and Bob, want to establish a secure transport connection through a non-secure channel. They agree to use the multiplicative group \( K^* \) as a platform, where \( K \) is a finite field \( \mathbb{F}_q, q = p^k \), or a modular ring \( \mathbb{Z}_n, n = pq \), where \( p \) and \( q \) are different sufficiently large primes. Also they agree that Bob will choose all parameters and then he will send all public parameters to Alice. The encryption by Alice, and the decryption by Bob will be done as follows.

Field version.

(1) Bob creates a probabilistic cryptographic system. Namely, he chooses a field \( \mathbb{F}_q, q = p^k \) and two subgroups \( H \) and \( U \) of \( \mathbb{F}_q^* \) of coprime orders \( r \) and \( s \) respectively. Elements of \( U \) encode all possible messages, and elements of \( H \) play role of masks. Also Bob computes \( t = r^{-1}(mod s) \).

(2) Bob sends the public parameters, namely: \( p, k, H, U \), to Alice. The subgroups \( H \) and \( U \) are specified by their generating elements, or by some other effective method that does not reveal the secret parameters \( r \) and \( s \).

(3) To transport a message \( u \in U \), Alice chooses \( h \in H \) randomly, then she sends \( g = hu \) to Bob through non-secure channel.
(4) Bob receives $g$. Then he succeeds to reveal the message $u$ as follows:
\[ g^{rt} = (h^r)^t(u^{rt}) = u. \]

**Remark 1.**
- In particular, the subgroups $H$ and $U$ can be chosen such that $F_q^* = HU$. Then every element $g \in F_q^*$ can be uniquely written in the form $g = hu$, where $h \in H$ and $u \in U$.
- The orders $|H|$ and $|U|$ have to be sufficiently large. The presentations of $H$ and $U$ should be chosen in a way that doesn’t give a possibility to find the decomposition $n = pq$.

**Modular ring version.**
(1) Bob creates a probabilistic cryptographic system. Namely, he chooses a modular ring $\mathbb{Z}_n$, $n = pq$, $p$ and $q$ different primes, and two subgroups $H$ and $U$ of $G_n = \mathbb{Z}_n^*$ of coprime orders $r$ and $s$ respectively. Elements of $U$ encode all possible messages, and elements of $H$ play role of masks. Also Bob computes $t = r^{-1}(mod s)$.
(2) Bob sends the public parameters, namely: $n, H, U$, to Alice. The subgroups $H$ and $U$ are specified by their generating elements, or by some other effective method that does not reveal the secret parameters $r$ and $s$.
(3) To transport a message $u \in U$, Alice chooses $h \in H$ randomly, then she sends $g = hu$ to Bob through non-secure channel.
(4) Bob receives $g$. Then he succeeds to reveal the message as follows.
\[ g^{rt} = (h^r)^t(u^{rt}) = u. \]

**Remark 2.**
- In particular, the subgroups $H$ and $U$ can be chosen such that $F_q^* = HU$. Then every element $g \in F_q^*$ can be uniquely written in the form $g = hu$, where $h \in H$ and $u \in U$.
- The orders $|H|$ and $|U|$ have to be sufficiently large. The presentations of $H$ and $U$ should be chosen in a way that doesn’t give a possibility to find the decomposition $n = pq$.

### 3. Some applications.

#### 3.1. Probabilistic encryption based on RSA encryption.

The following encrypting has been proposed in [1] and [2]. The described above basic scheme is combined with the standard RSA algorithm.

Let $p$ and $q$ be two different odd primes. Bob chooses subgroups $H$ and $U$ of the multiplicative group $G_n = \mathbb{Z}_n^*$ of the modular ring $\mathbb{Z}_n$ such that their respective orders $t$ and $r$ are coprime. He presents these subgroups by their generating elements as: $U = gp(u_1, ..., u_m)$ and $H = gp(h_1, ..., h_s)$, or in a different effective way. We suppose that $U$ is the message space, i.e., each message $u$ is presented as an element of $U$, and vice versa. Now we fix public and secret data that are established off-line as follows.
Public data: $n$ (and therefore $\mathbb{Z}_n$ and $G_n = \mathbb{Z}_n^*$), $u_1, ..., u_m$ (and therefore $U$), $h_1, ..., h_s$ (and therefore $H$).

Secret data: $p, q, \varphi(n) = (p - 1)(q - 1), r, t$.

Alice chooses public key $e \in \mathbb{Z}$ such that $\gcd(e, r) = 1$. Then she computes the secret key $d = td_1$ such that $(te)d_1 = 1 \pmod{r}$. This is possible because $\gcd(t, r) = \gcd(e, r) = 1$ by our assumption. Then $ted = 1 + rk$ for some integer $k$. Thus, Bob has the following keys:

Public key: $e$.
Secret key: $d$.

To send a message $u \in U$ to Bob, the other party, Alice, chooses a random element $h \in H$ (secret session key) and acts as follows:

Encryption: $u \rightarrow c = (hu)^e \pmod{n}$.

Bob recovers $u$ as follows:

Decryption: $c \rightarrow u = c^d \pmod{n}$.

Correctness: $c^d = (h^t)^{ed} u (u^r)^k = u \pmod{n}$.

Security assumption. If the adversary can to find the factors $p$ and $q$ of $n$ he can not reveal message $u$ without knowing of primality factorings of $p - 1$ and $q - 1$. Hence, a security of the algorithm is based on intractability of the full primality decomposition problem for positive integers. A priori, the primality decomposition problem is harder than the decomposition problem for positive integers of the form $n = pq$.

3.2. Probabilistic encryption based on the ElGamal key exchange protocol. The ElGamal algorithm \cite{4} is an asymmetric encryption algorithm for public key cryptography, based on hardness of the Discrete logarithm problem. ElGamal is semantically secure under reasonable assumptions, and is probabilistic \cite{5}. The original ElGamal encryption scheme \cite{4} operates as follows. First, Bob fixes a large prime $p$, a generating element $g$ of the multiplicative group $\mathbb{F}_p^*$, and a positive integer $a$ in the range $1 < a < p$. Bob’s private key is taken to be $a$, while his public key is the tuple $(g, y = g^a, p)$. Alice performs encryption by segmenting the plaintext and encoding it as a sequence of integers in the range $0 < u < p$. Alice chooses a temporary secret in the form of an auxiliary random integer $r$ and encrypts a plaintext as $c = u \cdot y^r = u \cdot g^{ar}$. Along with this encrypted message $c$, Alice includes the header $g^r$.

Bob performs decryption by first manipulating the header $(g^r)^a = g^{ar}$. It follows that the original message can be recovered by noting that $c \cdot (g^{ar})^{-1} = u$.

The ElGamal cipher leverages the purported difficulty of computing the discrete logarithm.

We propose a new version of the ElGamal encryption that does not use header. As the original, our version is probabilistic, but uses a very different idea. This encryption scheme operates as follows. First, Bob chooses two positive coprime integers $r$ and $s$. Then he fixes a large prime $p$ such that $p - 1$ divides to $rs$, i.e., $p - 1 = rsk$ for some $k$. He sets $a = sk$ and $b = rk$. 
Bob’s private key is taken to be the tuple \((r, s, a)\) while his public key is the tuple \((g, y = g^a, g^b, p)\).

He announces that the message space is subgroup \(gp(g^b)\) which order is \(s\). Then he computes \(t = r^{-1}(mod\ s)\).

Alice chooses a temporary secret in the form of an auxiliary random integer \(l\) and encrypts a plaintext \(u = (g^b)^x = g^{bx}\) as \(c = u \cdot y^l = u \cdot g^{al}\).

Bob performs decryption by first manipulating \(c' = u^r \cdot (g^{al})^r = u^r \cdot (g^{rs})^l = u^r \cdot (g^{p-1})^l = u^r\). It follows that the original message can be recovered by noting that \((u^r)^l = u\).

Let \(u_1, u_2 \in \text{gp}(g^b)\) be two different messages, and \(c_i = u_i \cdot g^{al} (i \in \{0, 1\}\) be the cipher text of one of them. The Oscar, opponent, has to guess about correct value of \(i\). Suppose that he can give correct answer with probability significantly greater than \(1/2\). As \(u_j^{-1} \cdot c_i\) lies in \(\text{gp}(g^b)\) if and only if \(i = j\), Oscar can to solve the membership problem for \(\text{gp}(g^a)\) with probability significantly greater than \(1/2\). Assuming the latter problem is hard when we take elements from \(\text{gp}(g^a) \cdot \text{gp}(g^b)\), we conclude that this version of ElGamal encryption is semantically secure. Note that we can take \(a\) and \(b\) such that \(\text{gp}(g^a) \cdot \text{gp}(g^b) = F_p^a\).

3.3. Probabilistic encryption based on the Diffie-Hellman key exchange protocol. The Diffie-Hellman algorithm \([6]\) is an asymmetric encryption algorithm for public key cryptography, based on hardness of the Discrete logarithm problem. The original Diffie-Hellman encryption scheme \([6]\) operates as follows. First, Alice and Bob agree upon a large prime \(p\), number \(q = p^l\) and a generating element \(g\) of the multiplicative group \(F_q^\ast\).

Bob’s private key is taken to be a number \(b\) while his public key is \(g^b\). Alice’s private key is a number \(a\) and her public key is \(g^a\). The secret key they exchange is then \(g^{ab}\). Both users can compute this key. An unauthorized third party will be unable to determine the key if it is computationally infeasible to compute \(g^{ab}\) knowing only \(g^a\) and \(g^b\).

We propose a new version of the Diffie-Hellman encryption that is probabilistic. This encryption scheme operates as follows. First, Bob chooses a private positive integer \(r\), and Alice chooses a private positive integer \(s\). Then Bob takes randomly number \(x\) and publics \(r_1 = rx\). Alice takes randomly \(y\) and publics \(s_1 = sy\). Then they agree upon prime of the form \(p = 1 + 2r_1s_1z\). They use field \(F_p\) as platform and agree upon some fixed public element \(g \in F_p^\ast\). Ideally, \(g\) should be a generator of \(F_p^\ast\); however, this is not absolutely necessary. Bob finds a subgroup \(H\) of \(F_p^\ast\) of exponent \(r\) and publics its generators; Alice finds a subgroup \(U\) of exponent \(s\) and publics its generators.

To exchange secret key Bob chooses randomly private number \(b\), element \(u \in U\), and Alice chooses randomly private number \(a\) and element \(h \in H\). Bob sends \(u g^{br}\) to Alice, Alice sends \(h g^{as}\) to Bob.

Bob computes \((hg^{as})^rb = g^{abrs}\). Alice computes \((ug^{br})^sa = g^{abrs}\), that is the exchanged secret key.
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REFERENCES

[1] Roman’kov V. A semantically secure public-key cryptosystem based on RSA // Prikl. Discr. Mat. — 2015. — No. 3. P. 32-40 (In Russian).
[2] Roman’kov V. New probabilistic public-key cryptosystem based on the RSA cryptosystem // Groups, Complexity, Cryptology. — 2015. — V. 7. No. 2. — P. 153-156.
[3] Menezes A.J., Oorschot P.C. van, Vanstone S.A. Handbook of Applied Cryptography. CRC Press. Taylor & Francis Group. Boca Raton, 1996.
[4] ElGamal T. A public-key cryptosystem and a signature scheme based on discrete logarithms // IEEE Trans. Inform. Theory. — 1985. — V. 31. No. 4. P. 469-472.
[5] Goldwasser S., Micali S. Probabilistic encryption // J. of Computer and System Sciences. — 28 (1984). — V. 28. — P. 270-299.
[6] Diffie W., Hellman M. E. New directions in cryptography // IEEE Trans. Inform. Theory. — 1976. — V. 22. No 6. — P. 644–654.

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