Vakhitov–Kolokolov and energy vanishing conditions for linear instability of solitary waves in models of classical self-interacting spinor fields

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Abstract
We study the linear stability of localized modes in self-interacting spinor fields, analysing the spectrum of the operator corresponding to linearization at solitary waves. Following the generalization of the Vakhitov–Kolokolov approach, we show that the bifurcation of real eigenvalues from the origin is completely characterized by the Vakhitov–Kolokolov condition \( \frac{dQ}{d\omega} = 0 \) and by the vanishing of the energy functional. We give the numerical data on the linear stability in the generalized Gross–Neveu model and the generalized massive Thirring model in the charge-subcritical, charge-critical and charge-supercritical cases, illustrating the agreement with the Vakhitov–Kolokolov and the energy vanishing conditions.

Keywords: nonlinear Dirac equation, solitary waves, linear stability, Vakhitov–Kolokolov criterion, Gross–Neveu model, massive Thirring model
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1. Introduction
Models of self-interacting spinor fields have been playing a prominent role in physics for a long time starting with [Iva38, FLR51, FFK56, Hei57]. Particularly widely considered are the massive Thirring model (MTM) [Thi58], the Soler model [Sol70] and the massive Gross–Neveu model [GN74, LG75], in which the self-interaction gives rise to nontopological solitons of the
form $\phi_0(x)e^{-i\omega t}$ with $\phi_0$ exponentially localized in space. Such solitary waves have also been found in the Dirac–Maxwell (DM) system [Wak66,Lis95,EGS96,Abe98] and Dirac–Einstein systems [FSY99,Stu10].

We point out that we treat the fermionic field classically, as a $c$-number, completely leaving out the framework of the second quantization. Because of this, we need to mention the role played by such classical solitary wave solutions in physics. In [DHN74,LG75] such classical states were considered from the point of view of classical approximations of hadrons. It was shown in [CZ13] that the (classical) Dirac–Coulomb (DC) and DM systems appear in the quantum field theory where these solitary waves correspond to polarons, formed due to the interaction of fermions with optical phonons or with the gravitational field [LL84] (this reflects the Landau–Pekar approach to the polaron in the conventional nonrelativistic electron theory [Lan33,Pek46]; a similar mechanism is also responsible for the formation of the Cooper pairs in the microscopic mechanism of superconductivity [LL84]). The classical self-interacting spinor field also appears in the Dirac–Hartree–Fock approach in quantum chemistry (see e.g. [QGW04]). Coupled-mode equations in nonlinear optics and the theory of Bose–Einstein condensates could also be treated as a one-dimensional Dirac equation with self-interaction of a particular type [BPZ98,CP06,GW08,PSK04]. The approach given in the present paper is applicable to the stability analysis of all such systems of classical self-interacting spinors.

In view of recent results on stability and instability for the nonlinear Dirac equation [CKMS10,BC12a,BC12b,CGG14], it is becoming clear that the VK criterion is still useful for the spinor models in the nonrelativistic limit. In particular, the ground states (‘smallest energy solitary waves’) in the charge-subcritical nonlinear Dirac equation (with the nonlinearity of order $2k + 1$, with $k < 2/n$) are linearly stable in the nonrelativistic limit $\omega \lesssim m$, which corresponds to solitary waves of small amplitude. The same linear stability is expected to hold for the DM system in the nonrelativistic limit $\omega \gtrsim -m$ [CS12,CZ13].

In the present paper, following [VK73] and [GSS90], we show that in the systems of self-interacting spinor fields the condition

$$E(\phi_0) = 0,$$

along with the VK condition $dQ(\phi_0)/d\omega = 0$, indicates the collision of eigenvalues at the origin, marking a possible border of the stability and instability regions. Above, $E(\phi_0)$ and $Q(\phi_0)$ are the energy and the charge of a corresponding solitary wave.

We then show that our theory applies to the pure-power generalized MTM, with the nonlinearity of order $p = 2k + 1$, $k > 0$. Our numerical results show that in all models with
\(k \neq 1\), the energy functional vanishes at some \(\omega_E(k) \in (-m, 0)\) (with \(\lim_{k \to 1} \omega_E(k) = -m\)). We then compute the eigenvalues of the linearizations at these solitary waves, and show that there is a birth of a pair of positive–negative eigenvalues precisely at the value \(\omega\) which corresponds to solitary waves of zero energy. In figures 1–4 below, we plot the spectra of linearization and the values of the energy for the solitary waves in the generalized MTM with nonlinearity of the order \(2k + 1\), with \(k = 1/2, 1, 2\) and 3. The corresponding plots for the Gross–Neveu model are also given (figures 5 and 6).

Let us mention that the solitary waves in the original, cubic MTM (with \(k = 1\)) were recently shown to be orbitally stable in \(L^2\) for all \(\omega \in (-m, m)\) [CPS13], and orbitally stable in \(H^1\) for all \(\omega \in (-\omega_0, \omega_0)\) with some \(\omega_0 \in (0, m)\) [PS14]. These results are based on the complete integrability of the (cubic) MTM.

We also mention situations with coupled-mode equations of the Dirac type which are not Lorentz-invariant, such as that in [BPZ98]. The condition which describes bifurcations of eigenvalues from the origin is formulated in terms of vanishing of the determinant consisting of the derivatives of the conserved quantities at the solitary wave parameters; in the case of coupled-mode equations, the corresponding matrix is not diagonal (see e.g. [KP12]). The present paper shows that for the Lorentz-invariant nonlinear Dirac equations, the corresponding matrix of derivatives is diagonal, with the derivative of the momentum with respect to the speed present paper shows that for the Lorentz-invariant nonlinear Dirac equations, the corresponding matrix is not diagonal (see e.g. [KP12]). The condition which describes bifurcations of eigenvalues from the origin is formulated in terms of vanishing of the determinant consisting of the derivatives of the conserved quantities at the solitary wave parameters; in the case of coupled-mode equations, the corresponding matrix is not diagonal (see e.g. [KP12]).

Let us give an informal outline. After the linearization at a solitary wave, the isolated eigenvalue \(\lambda = 0\) of the linearized equation corresponds to several Jordan blocks related to the symmetries of the system, most importantly the \(\lambda\)-eigenvalue of solitary waves being proportional to the value of the energy of the solitary wave.

We then compute the eigenvalues of the linearizations at these solitary waves, and show that there is a birth of a pair of positive–negative eigenvalues precisely at the value \(\omega\) which corresponds to solitary waves of zero energy. In figures 1–4 below, we plot the spectra of linearization and the values of the energy for the solitary waves in the generalized MTM with nonlinearity of the order \(2k + 1\), with \(k = 1/2, 1, 2\) and 3. The corresponding plots for the Gross–Neveu model are also given (figures 5 and 6).

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Let us mention that similar methods of studying the transition to instability are employed in [GSS90, KKS04].

2. The main results

Let \(\gamma^\mu, 0 \leq \mu \leq n\), be the \(N \times N\) Dirac matrices which satisfy \([\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}I_N\) \((N \in \mathbb{N}\) is even), with \(g^{\mu\nu} = \text{diag}[1, -1, \ldots, -1]\) the Minkowski metric. For \(\psi \in \mathbb{C}^N\), define \(\bar{\psi} = \psi^*\gamma^0\). Consider the Lagrangian density

\[
\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \mathcal{F}(\bar{\psi}, \psi),
\]

with \(m > 0\), \(\psi \in \mathbb{C}^N\) the spinor field, and \(\mathcal{F} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}\), which we assume is sufficiently smooth and satisfies \(|\mathcal{F}(\bar{\psi}, \psi)| = o(|\psi|^2)\) for \(|\psi| \ll 1\). We also assume that \(\mathcal{F}(\bar{\psi}, \psi)\) is \(U(1)\)-invariant:

\[
\mathcal{F}(e^{-i\psi} \bar{\psi}, e^{i\psi} \psi) = \mathcal{F}(\bar{\psi}, \psi), \quad \psi \in \mathbb{C}^N, \quad s \in \mathbb{R}.
\]

The Euler–Lagrange equation obtained by taking the variation of (2) with respect to \(\bar{\psi}\) (considered as independent of \(\psi\)) leads to the equation

\[
i\dot{\psi} = D_m \psi - \beta \nabla_\phi \mathcal{F}.
\]

Above, \(D_m = -i\alpha^j \partial_j + \beta m\) is the Dirac operator, with \(\alpha^j = \gamma^0 \gamma^j\), \(\beta = \gamma^0\) the self-adjoint Dirac matrices. We follow the convention that \(0 \leq \mu, \nu \leq n, 1 \leq j, k \leq n\), and assume that there is a summation with respect to repeated upper–lower indices (unless specified otherwise).
2.1. Conservation laws and the virial identity

By Nöther’s theorem, due to the \( U(1) \)-invariance of the Hamiltonian, there is a charge functional

\[
Q(\psi) = \int \psi^*(x, t) \psi(x, t) \, dx
\]

(4)

whose value is conserved along the trajectories. (Here and below, each integral is over \( \mathbb{R}^n \) unless stated otherwise.) The local law of charge conservation has the form

\[
\partial_\mu J^\mu = 0,
\]

(5)

where

\[
J^\mu = \bar{\psi} \gamma^\mu \psi
\]

(6)

is the 4-vector of the charge-current density.

By \([BD65]\), the density of the energy–momentum tensor is given by

\[
T_{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \psi)} g_{\nu\rho} \partial^\rho \psi - g_{\mu\nu} L.
\]

With

\[
L = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi + \mathcal{F}(\bar{\psi}, \psi)
\]

from (2), we have

\[
\mathcal{H} = T_{00} = -\bar{\psi} (i \gamma^j \partial_j) \psi + m \bar{\psi} \psi - \mathcal{F}(\bar{\psi}, \psi).
\]

(7)

The components of the energy–momentum tensor \( T_{\mu\nu} = \int T_{\mu\nu} \, dx \) are given by

\[
T_{00} = E = \int \mathcal{H} \, dx,
\]

\[
T_{0k} = T_{k0} = i \int \bar{\psi} \gamma^0 g^{k\rho} \partial_\rho \psi \, dx = ig^{k\rho} (\psi, \partial_\rho \psi),
\]

\[
T_{jk} = \langle \psi, i \alpha_j \partial_\rho \psi \rangle g^{\rho k} - g^{jk} L,
\]

(8)

where \( L = \int \mathcal{L} \, dx \). Note that \( T_{\mu\nu} \) is Hermitian.

Now let us consider a solitary wave solution

\[
\psi_\omega(x, t) = \phi_\omega(x) e^{-i \omega t},
\]

(9)

with \( \phi_\omega(x) \in \mathbb{C}^N \) of Schwartz class in \( x \). Comparing (2) and (7), we obtain

\[
L(\psi_\omega) = -E(\psi_\omega) + \omega Q(\psi_\omega).
\]

(10)

By the Stokes theorem, the local form of the charge conservation (5) leads to

\[
0 = \partial_t \int J_0 x_k \, dx = -\int (\partial_j J^j x_k) \, dx = \int J_k^* \, dx.
\]

(11)

Therefore, for a solitary wave (9), one has

\[
J^k := \int J^k \, dx = 0, \quad 1 \leq k \leq n.
\]

(12)

Similarly, since the energy–momentum tensor is the conserved Nöther current associated with space–time translations, for any fixed \( 0 \leq \nu \leq n \), there is the identity \( \partial_\mu \mathcal{T}^{\mu \nu} = 0 \) which follows from the Euler–Lagrange equations. This leads to

\[
T^{ij} = T^{ij} := \int \mathcal{T}^{ij} \, dx = 0, \quad 1 \leq j \leq n.
\]

(13)

We decompose the Hamiltonian functional into

\[
E(\psi) = K(\psi) + M(\psi) + V(\psi),
\]

with

\[
K(\psi) = \int \psi^* (-i \alpha \cdot \nabla) \psi \, dx,
\]

\[
M(\psi) = m \int \psi^* \beta \psi \, dx,
\]

\[
V(\psi) = -\int \mathcal{F}(\bar{\psi}, \psi) \, dx.
\]
Combining (8) and (12), we conclude that

\[ i \int \phi^* \alpha^j \partial_j \phi \, dx = i \int \psi^* \alpha^j \partial_j \psi \, dx = \delta^j_k L(\psi). \tag{14} \]

Taking the trace of (14), we obtain the virial identity

\[ K(\phi) = K(\psi) = -nL(\psi), \tag{15} \]

where \( K(\psi) \) is defined in (13).

Note that for a solitary wave \( \psi(x, t) = \phi(x) e^{-i \omega t} \), one has \( E(\psi) = E(\phi) \), since the Hamiltonian density (7) does not contain the time derivatives; similarly, the values of \( K, M \) and \( V \) are the same on \( \psi \) and \( \phi \).

2.2. Linearization at a solitary wave

We assume that there are solitary wave solutions to (3) of the form (9) with \( \omega \in \Omega_1 \), where \( \Omega_1 \) is an open set. Many quantities appearing below will depend on \( \omega \), which we will indicate with the subscript \( \omega \); sometimes the subscript will be omitted to shorten the notation.

To study the linear stability of the solitary waves (9), we consider the solution \( \psi \) in the form

\[ e^{-i \omega t} (\phi(x) + \rho(x, t)). \]

The linearized equation is not \( C \)-linear in \( \rho \). To apply the linear operator theory, we write the linearized equation on \( \rho \) in the \( C \)-linear form

\[ \dot{\rho} = JL(\omega)\rho, \quad \rho = \begin{bmatrix} \text{Re} \rho \\ \text{Im} \rho \end{bmatrix}, \]

with

\[ J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}, \quad L(\omega) = D_m + V, \]

where \( D_m = J\alpha^j \partial_k + \beta m \) and

\[ \alpha^j = \begin{bmatrix} \text{Re} \alpha^j & -\text{Im} \alpha^j \\ \text{Im} \alpha^j & \text{Re} \alpha^j \end{bmatrix}, \quad \beta = \begin{bmatrix} \text{Re} \beta & -\text{Im} \beta \\ \text{Im} \beta & \text{Re} \beta \end{bmatrix}. \]

The matrix-valued function \( V \) is self-adjoint and of Schwartz class in \( x \); its dependence on \( \omega \) is via \( \phi_\omega \).

2.3. The structure of the null space

Due to the \( U(1) \)-invariance of the equation, the perturbation \( \rho(x, t) \) that corresponds to infinitesimal multiplication of the solitary wave by a constant unitary phase is in the kernel of the linearization \( JL \). Similarly, the translational invariance and the rotational symmetry result in vectors in the kernel of the linearized operator. As a result,

\[ J\phi_\omega, \partial_j \phi_\omega \in \ker JL(\omega), \tag{16} \]

where \( \phi_\omega = \begin{bmatrix} \text{Re} \phi_\omega \\ \text{Im} \phi_\omega \end{bmatrix} \). These inclusions follow from taking the derivatives in \( \omega \) and \( x^j \) of the relation \( E'(\phi_\omega) = \omega Q'(\phi_\omega) \). One can check by direct computation that there is a Jordan block corresponding to each of these eigenvectors:

\[ JL\partial_\omega \phi_\omega = J\phi_\omega, \quad JL\xi_j = \partial_j \phi_\omega, \tag{17} \]
where
\[ \xi_j = \omega x_j \cdot J \phi_{\omega} - \frac{1}{2} \alpha J \phi_{\omega}. \] (18)

By (17), there are Jordan blocks of size at least 2 corresponding to each of the vectors \( J \phi_{\omega}, \partial_j \phi_{\omega} \) from the null space. When two (or more) eigenvalues collide at \( \lambda = 0 \), at a particular value of \( \omega \), they can instantaneously join one of these two Jordan block types permanently residing at 0. We now consider these two events.

2.4. \( U(1) \)-invariance and the VK criterion

Let us revisit the VK criterion from the point of view of the size of a particular Jordan block at \( \lambda = 0 \). By (17), the Jordan block of \( JL \) corresponding to the unitary invariance is of size at least 2. The size of this Jordan block jumps up when we can solve the generalized eigenvector equation \( JL \phi = \partial \omega \phi \). Since \( L \) is Fredholm (this follows from \( L \) being self-adjoint and \( 0 \notin \sigma_{\text{ess}}(L) = \mathbb{R}(m, -m) \); see e.g. [EE87]), such a \( \phi \) exists if \( \partial \omega \phi \) is orthogonal to the null space of \( JL^* = -LJ \). The generalized eigenvector \( \partial \omega \phi \) is always orthogonal to \( J \cdot \partial^k \phi \in \ker LJ, 1 \leq k \leq n \). Indeed, we have
\[ \langle \partial \omega \phi, \partial^k \phi \rangle = \langle \phi, -\frac{1}{2} \alpha \cdot \partial^k \phi \rangle - \omega \langle \phi, x^k J \phi \rangle, \] (19)
where we used (17) and self-adjointness of \( J \). By (11), the first term in the right-hand side is zero. The second term in the right-hand side is zero due to the skew-symmetry of \( J \). Thus,
\[ \langle \partial \omega \phi, J \partial^k \phi \rangle = 0, \quad 1 \leq k \leq n. \] (20)

We now need to check whether \( \partial \omega \phi \) is orthogonal to \( \phi_{\omega} \in \ker LJ \). The orthogonality condition takes the form
\[ \langle \partial \omega \phi, \phi_{\omega} \rangle = \frac{1}{2} \partial \omega Q(\phi_{\omega}) = 0. \] (21)

This is in agreement with the VK criterion \( \frac{d}{d \omega} Q(\phi_{\omega}) < 0 \) derived in the context of the nonlinear Schrödinger equation and more abstract Hamiltonian systems with \( U(1) \)-invariance [VK73, GSS87].

2.5. Translation invariance and the energy criterion for linear instability

Let us find the condition for the increase in size of the Jordan block corresponding to translational invariance. This happens if there is a \( \xi \) such that \( JL \xi = \xi \), where \( \xi = \sum_{j=1}^n c_j \xi_j \neq 0 \) is some nontrivial linear combination of generalized eigenvectors. Since \( L \) is Fredholm, the sufficient condition is that \( \xi \) is orthogonal to vectors from \( \ker LJ \). By (19) and (20), one always has
\[ \left( \alpha \cdot J \phi_{\omega} - \frac{1}{2} \alpha J \phi_{\omega}, \phi_{\omega} \right) = 0, \] (22)
ensuring orthogonality of \( \xi \) to \( \phi_{\omega} \in \ker LJ \).

Now we need to ensure orthogonality to all of \( J^{-1} \partial \phi_{\omega} \in \ker LJ, 1 \leq k \leq n \). We may write this condition in the form
\[ \det C_{jk}(\omega) = 0, \quad C_{jk}(\omega) := -2 \langle \xi_j, J \partial_k \phi_{\omega} \rangle. \] (23)

Substituting \( \xi_j \) in from (18), we have
\[ C_{jk}(\omega) = \langle \alpha \cdot J \phi_{\omega} - 2 \alpha \cdot J \phi_{\omega}, J \partial_k \phi_{\omega} \rangle. \] (24)
Since
\[
\langle 2x^j J\phi_\omega, J\partial_k \phi_\omega \rangle = \int x^j \partial_k (\phi_\omega^* \phi_\omega) \, dx = -\delta^j_k Q(\phi_\omega),
\]
we rewrite (24) as
\[
C_{jk}(\omega) = \langle \alpha^j \phi_\omega, J\partial_k \phi_\omega \rangle + \omega \delta^j_k Q(\omega). \tag{25}
\]
Using (10), (14) and (25), we get
\[
C_{jk}(\omega) = E(\omega) \delta^j_k.
\]
Thus, the condition (23) for the increase of the size of the Jordan block in the nonlinear Dirac equation is equivalent to
\[
E(\phi_\omega) = 0. \tag{26}
\]

**Lemma 1.** Let \( J(\tilde{\psi}, \psi) \) be homogeneous of degree \( k+1 \) in \( \tilde{\psi}, \psi \), and assume that \( J(\tilde{\psi}, \psi) > 0 \) for \( \psi \neq 0 \). Then one has
\[
E(\phi_\omega) > 0 \quad \text{for} \quad \omega \geq 0, \tag{27}
\]
for an exponentially localized solitary wave profile \( \phi_\omega \neq 0 \).

**Proof.** Substituting into \( \partial_\lambda |_{\lambda = 1} E(\phi_\lambda) = \omega \partial_\lambda |_{\lambda = 1} Q(\phi_\lambda) \) the family \( \phi_\lambda(x) = \lambda \phi(x) \), we show that quantities (13) satisfy the relation
\[
\omega Q = K + M + (k+1)V.
\]
With \( V := -\int J(\tilde{\psi}, \psi) \, dx < 0 \), for \( \omega \geq 0 \) we arrive at \( E = K + M + V = \omega Q - kV > 0 \). \( \square \)

### 2.6. Rotational symmetry

In the \((3 + 1)D\) case, the kernel of the linearized operator contains the eigenvectors due to the rotational symmetry. For \( 1 \leq j \leq 3 \), define
\[
\Sigma_j = \text{diag}[\sigma_j, \sigma_j], \quad \Sigma_j = \begin{bmatrix} \text{Re} \Sigma_j & -\text{Im} \Sigma_j \\ \text{Im} \Sigma_j & \text{Re} \Sigma_j \end{bmatrix}.
\]
Then
\[
\Theta_j = -J \Sigma_j \phi + 2\epsilon_{jkl} x^j \partial_k \phi_\omega \in \ker JL(\omega) \tag{28}
\]
are the eigenvectors from the null space which correspond to infinitesimal rotations. Above, \( \epsilon_{jkl} \) are the Levi–Civita symbols.

It turns out that \( J\phi \in \ker JL \) is a linear combination of the \( \Theta_j \), \( 1 \leq j \leq 3 \), so these three eigenvectors only contribute 2 to the dimension of \( \ker JL \).

One can check that the condition for the generalized eigenvector \( \partial_\omega \phi \) to be orthogonal to \( J\Theta_j \), \( 1 \leq k \leq 3 \), is given by the VK condition
\[
\langle \partial_\omega \phi, J\Theta_k \rangle = \langle \partial_\omega \phi_\omega, \phi_\omega \rangle.
\]
One can also check that the generalized eigenvectors \( \xi_j \), \( 1 \leq j \leq 3 \), are always orthogonal to \( \Theta_k \), \( 1 \leq k \leq 3 \):
\[
\left\{ \omega x^j J\phi - \frac{1}{2} \alpha^j \phi, J\Theta_k \right\} = 0.
\]
Therefore, the presence of these eigenvectors in the kernel of \( JL \) in the \((3 + 1)D\) case does not affect the size of the Jordan blocks associated with unitary and translational invariance; these sizes are completely characterized by the conditions (21) and (26).
There are no new Jordan blocks associated with $\Theta_j$. For example, for the standard ansatz
\[
\phi_\omega(x) = \begin{bmatrix} g_\omega(r) & 1 \\ i f_\omega(r) & 0 \end{bmatrix} \begin{bmatrix} \cos \theta \\ e^{i \omega \sin \theta} \end{bmatrix},
\]
(29)
one has $\Theta_3 = -J\phi$, $\Theta_1 = J\Theta_2$, so
\[
\langle \Theta_1, J\Theta_2 \rangle = \langle \Theta_1, \Theta_1 \rangle > 0.
\]
As a consequence, the Jordan block corresponding to $\Theta_3$ is the same as the one corresponding to the unitary invariance (whose size is controlled by the VK condition (21)), and there are no Jordan blocks corresponding to $\Theta_1, \Theta_2$ since neither is orthogonal to $\ker (JL)^* \ni J^{-1}\Theta_k$.

Remark 2. In the $(2 + 1)$D case, the story is similar: the eigenvector from the null space which corresponds to the infinitesimal rotation coincides with $J\phi$, the same eigenvector as corresponds to the unitary symmetry. The size of the corresponding Jordan block jumps (indicating collision of eigenvalues at the origin) if and only if the VK condition $\partial_\omega Q(\phi_\omega) = 0$ is satisfied.

3. Applications

3.1. The generalized MTM

The (generalized) MTM in $(1 + 1)$D is characterized by the Lagrangian
\[
\mathcal{L}_{\text{MTM}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{\|\bar{\psi}\gamma^\mu \psi\|_g^{1+k}}{1+k},
\]
(30)
where $\|\cdot\|_g$ is the length in the Minkowski metric,
\[
\|\xi\|_g^2 = g_{\mu\nu} \xi^\mu \xi^\nu, \quad \xi \in \mathbb{R}^{1+1},
\]
with $g = \text{diag}[1, -1]$ the Minkowski tensor. We notice that $\|\bar{\psi}\gamma^\mu \psi\|_g^2 = (\psi^*\psi)^2 - (\psi^*a^1\psi)^2 \geq 0$.

The choice $k = 1$ leads to the nonlinear Dirac equations with cubic nonlinearities originally considered in [Thi58]. In the nonrelativistic limit $\omega \lesssim m$, for $k \in (0, 2)$, one has spectral stability according to [BC12b]; for $k > 2$, there is linear instability by [CGG14].

There is an interesting behaviour for $\omega$ away from the nonrelativistic limit. It turns out that for any $k \neq 1$, there is the following phenomenon: there is $\omega_E = \omega_E(k) \in (-m, 0)$ such that $E(\phi_\omega)|_{\omega = \omega_E} = 0$. According to our theory, at $\omega = \omega_E$, two purely imaginary eigenvalues collide at the origin, turning into a pair of two real (one positive, one negative) eigenvalues for $\omega \in (-m, \omega_E)$, guaranteeing linear instability in this region of frequencies. In figures 1, 2, 3 and 4 we plot the results of the numerical analysis for the cases $k = 1/2, 1, 2$ and 3, giving both the values of the energy and charge functionals (as functions of $\omega \in (-m, m)$; we take $m = 1$) and the spectrum of the operator corresponding to the linearization at a solitary wave. We show that the collision of eigenvalues at the origin does indeed correspond to either $\partial_\omega Q(\phi_\omega) = 0$ or $E(\phi_\omega) = 0$.

3.2. The Gross–Neveu model

The Soler model [Sol70] has the Lagrange density
\[
\mathcal{L}_{\text{Soler}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + F(\bar{\psi} \psi).
\]
(31)
Figure 1. MTM with $k = 1/2$. Top: energy (solid line) and charge (dotted line) as functions of $\omega \in (-1, 1)$. Bottom: the spectrum of the linearization at a solitary wave on the upper half of the imaginary axis. Solid vertical lines symbolize the (upper half of the) essential spectrum. Dotted and solid curves denote ‘even’ and ‘odd’ eigenvalues (of the same as $\phi$ and of the opposite parity, respectively). The solid line eigenvalue collides with its opposite at the origin when $\omega = \omega_E \approx -0.6276$ (at $\omega_E$, the ‘energy condition’ $E(\omega) = 0$ is satisfied). For $\omega \in (-1, \omega_E)$, the spectrum contains one positive eigenvalue and one negative eigenvalue (not shown).
Figure 2. MTM with $k = 1$. Top: energy (solid line) and charge (dotted line) as functions of $\omega \in (-1, 1)$. Bottom: the spectrum of the linearization at a solitary wave on the upper half of the imaginary axis. Note the absence of nonzero eigenvalues in the case of the completely integrable model.
Figure 3. MTM with $k = 2$. Top: energy (solid line) and charge (dotted line) as functions of $\omega \in (-1, 1)$. Bottom: the spectrum of the linearization at a solitary wave on the upper half of the imaginary axis. For $\omega \lesssim 1$, note the presence of a purely imaginary eigenvalue near $\lambda = 0$ (dotted line) whose trajectory is tangent to the horizontal axis; this is due to quintic NLS being charge-critical in one spatial dimension. For $\omega \in (\omega_E, 0)$, there is a purely imaginary eigenvalue near the threshold $\lambda = i(1 - |\omega|)$. It collides with its opposite at the origin when $\omega = \omega_E \gtrsim -1$, with $\omega_E$ corresponding to a solitary wave with zero energy. For $\omega \in (-1, \omega_E)$, there is one positive eigenvalue and one negative eigenvalue in the spectrum (not shown).
Figure 4. MTM with $k = 3$. Top: energy (solid line) and charge (dotted line) as functions of $\omega \in (-1, 1)$. Bottom: the spectrum of the linearization at a solitary wave on the upper half of the imaginary axis. This case is charge-supercritical; for $\omega \lesssim 1$, there is a positive eigenvalue (its trajectory is shown by triangles in the plot). At $\omega = \omega_{VK}$ (when the VK condition $dQ/d\omega = 0$ is satisfied), this real positive eigenvalue collides at the origin with its opposite, producing a pair of purely imaginary eigenvalues (the one with positive imaginary part is given by the dotted line in the plot). For $\omega \in (\omega_E, 0)$, there is a purely imaginary eigenvalue near the threshold $\lambda = i(1 - |\omega|)$. It collides with its opposite at the origin when $\omega = \omega_E \gtrsim -1$, where $\omega_E$ corresponds to a solitary wave of zero energy. For $\omega \in (-1, \omega_E)$, there is one positive eigenvalue and one negative eigenvalue in the spectrum (not shown).
The corresponding equation is
\[ i \dot{\psi} = (-i \alpha \cdot \nabla + m \beta) \psi - f(\psi^* \beta \psi) \beta \psi, \]  
with \( f(s) = F'(s) \). It follows that if \( \phi_\omega(x) e^{-i \omega t} \) is a solitary wave solution to the nonlinear Dirac equation (32), then \( \phi_\omega(x) e^{i \omega t} \) is a solitary wave solution to the nonlinear Dirac equation
\[ i \dot{\psi} = (-i \hat{\alpha} \cdot \nabla + m \hat{\beta}) \psi - \hat{f}(\psi^* \hat{\beta} \psi) \hat{\beta} \psi, \]  
with \( \hat{\alpha} = -\alpha, \hat{\beta} = -\beta, \) and \( \hat{f}(s) = f(-s) \). Thus, one can always rewrite a solitary wave solution with \( \omega \in (-m, 0) \) as a solitary wave with \( \omega \in (0, m) \) (with the same stability properties). Therefore, we conclude from lemma 1 that for the Soler model for pure power nonlinearities \( F(s) = \text{sign}(s)|s|^k/(k + 1), k > 0, \) the condition \( E(\phi_\omega) = 0 \) is not triggered; the collisions of eigenvalues at the origin are described solely by the VK condition \( \frac{d}{d \omega} Q(\phi_\omega) = 0 \). At the same time, for the nonlinearities \( F(s) = |s|^k/(k + 1), k > 0 \) according to the construction in [BC12a], in one dimension there are no solitary waves with \( \omega \leq 0 \). We provide the corresponding plots in figures 5 (\( k = 1/2 \) and \( k = 1 \)) and 6 (\( k = 2, 3 \)). One can see that the collision of eigenvalues at the origin is indeed completely described by the VK condition \( dQ/d\omega = 0 \), in agreement with lemma 1.
Figure 6. Gross–Neveu model. Left: \( k = 2 \); right: \( k = 3 \). Top row: charge (dotted line) and energy (solid line) of the solitary waves as functions of \( \omega \in (0, 1) \). Bottom row: the spectrum on the upper half of the imaginary axis. The case \( k = 2 \) is charge-critical; note the purely imaginary eigenvalue whose trajectory is tangent to \( \lambda = 0 \) for \( \omega \lesssim 1 \). The case \( k = 3 \) is charge-supercritical; there is a real eigenvalue (plotted with triangles) born from the origin at \( \omega = 1 \) and persisting for \( \omega \lesssim 1 \). At \( \omega = \omega_{VK} \) (when the VK condition \( dQ/d\omega = 0 \) is satisfied), this eigenvalue collides with its opposite at the origin, producing a pair of purely imaginary eigenvalues (one with the positive imaginary part is plotted with dots).

4. Discussion and conclusions

We have shown that generically the condition \( E(\phi_\omega) = 0 \) indicates the birth of a pair of a positive and negative eigenvalues, thus possibly marking the border of the linear instability region. This condition is auxiliary to the Vakhitov–Kolokolov condition \( \partial_\omega Q(\phi_\omega) = 0 \) (see [Com11] on its application in the context of nonlinear Dirac equations). Together, these two conditions describe the birth of real eigenvalues from the point \( \lambda = 0 \), as a result of a collision of a pair of purely imaginary eigenvalues.

The real eigenvalues produced after the collision of eigenvalues at \( \lambda = 0 \), when \( E(\phi_\omega) = 0 \), correspond to vectors which are essentially parallel to \( \partial_\omega \phi_\omega \). Thus, the unstable behaviour develops from a slight push, after which a solitary wave starts accelerating and loses its shape.

The condition of the energy vanishing is of general type and is applicable to any classical model of self-interacting spinors. We have shown that in the (generalized) massive Thirring model in \((1 + 1)D\), the energy vanishing correctly predicts the eigenvalue collision, and also have shown that this condition is not triggered in the (generalized) Gross–Neveu model.
It can also be shown that for the NLS or Klein–Gordon equations, the analogous criterion is never triggered (each Jordan block corresponding to translations is of size exactly 2), so the collision of eigenvalues at the origin is completely described by the VK condition \( \partial_\omega Q(\phi_\omega) = 0 \). As a result, in the fermionic models, the energy vanishing condition may take place away from the nonrelativistic limit. Indeed, we demonstrated that in the pure power case the energy vanishing is only possible for solitary waves with \( \omega < 0 \).

It is important to mention that this new criterion and the Vakhitov–Kolokolov criterion do not exhaust all scenarios of instability. In particular, two pairs of purely imaginary eigenvalues may collide away from the origin and turn into a quadruplet of complex eigenvalues with nonzero real part. It is also possible that the instability takes over due to eigenvalues bifurcating from the essential spectrum. We hope to address these scenarios in forthcoming research.

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