Bayesian Estimation Analysis of Bernoulli Measurement Error Model for Longitudinal Data

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Abstract: The Bayesian method is used to study the inference of the semi-parametric measurement error model (MEs) with longitudinal data. A semi-parametric Bayesian method combined with fracture prior and Gibbs sampling combined with Metropolis-Hastings (MH) algorithm is applied and applied to the simulation observation from the posterior distribution, and the combined Bayesian statistics of unknown parameters and measurement errors are obtained. We obtained Bayesian estimates of the parameters and covariates of the measurement error model. Under three different priori assumptions, four simulation studies illustrate the effectiveness and utility of the proposed method.

Key words: Bayesian, measurement error, longitudinal data, Bernoulli model.

1. Introduction

Longitudinal data is obtained when the same individual is repeatedly measured at different time points. Longitudinal data is widely found in bio-medicine, epidemiology and labor medicine. For example, biomedical longitudinal samples can generally be obtained through clinical trials and observational cohort studies. Longitudinal data is also widely used in the fields of finance, economy, etc. It is an unbalanced data and is generally processed using a linear hybrid model. Measurement error data and missing data are often encountered for various reasons. When the covariate contains measurement error, You (2006) [1] proposed a profile least squares estimation method for error correction; Zhou (2009) [2] the covariate of measurement error in the research model has statistical inference problem of auxiliary information; Wei (2010) [3] studied the parameter estimation problem of the model when the response variable is missing and the covariate contains the measurement error; Wei (2012) [4] studied the constraint estimation and hypothesis testing of the variable coefficient partial linear measurement error model parameters. There are also Liang, Härdel and Carroll (1999) [5], Ma and Carroll (2006) [6], Liang, Wang and Carroll (2007) [7], Pan, Zeng and Lin (2008) [8] and other literature pairs. Such models have been studied. This paper proposes a hybrid algorithm for generating the observations required for Bayesian inference from the parameter posterior distribution and from the covariates of the ME. The algorithm combines a normal distribution with a mixed normal distribution. Gibbs sampling of a priori and MH algorithms was broken.

2. The Measurement Error Model

For \( i = 1, ..., n \), hypothesis \( Y_i \) is the observation variable, which \( X_i \) is an unobservable covariate vector of order \( r \times 1 \), and \( U_i \) is a covariate vector that can be observed in one order \( p \times 1 \). Let \( Z_i = (X_i^T, U_i^T)^T \), we
assume that the values \( Y_i \) are conditionally independent of each other. For longitudinal data, we consider the following generalized linear measurement error model of structure.

\[
P(Y_i|X_i, \psi) = \exp\left(\frac{Y_i - d_t(i)}{\psi} + c(Y_i, \psi)\right).
\]  

(1)

Here \( Y_i = E(Y_i|X_i) = d_t(i) \) is a divergence parameter, \( d(\cdot) \) and \( c(\cdot, \cdot) \) are specific differentiable functions, and has \( \frac{d(a_t)}{\partial t_i} = \frac{d(a_t)}{\partial t_i} \) and \( \frac{d(a_t)}{\partial t_i^2} = \frac{d^2 a_t}{\partial t_i^2} \). The conditional mean \( Y_i \) satisfies the following equation:

\[
\lambda_i = g(Y_i) = X_i^T \beta_2 + U_i^T \beta_v = Z_i^T \beta.
\]  

(2)

Here \( g(\cdot) \) is a monotonic differentiable link function, which \( \beta = (\beta_2^T, \beta_v^T)^T \) is an unknown regression coefficient with vector \( (r + p) \times 1 \). According to reference [9], for each individual \( i \), we measure \( m \) times for the true value covariate \( X_i \). \( Y_i \) and \( X_i \) are error independent. That is, for each \( j = 1, ..., m \) with following equations, we can’t observe \( X_i \) but we can observe \( W_{ij} \).

\[
W_{ij} = X_i + \eta_{ij}.
\]  

(3)

These measurement error values \( \eta_{ij} \) are subject to unknown distributions, and they are independent of the true values \( X_i \). According to Lachos (2010) [10], our hypothetical distribution \( \eta_{ij} \) is suitable for a mixed model of the Dirichlet Process (DP).

In order to calculate the previously set covariate measurement error model, we also need to define a real covariate model. The true covariate model for \( X_{ki} \) \((k = 1, ..., r)\) can be defined as

\[
X_{ki} = \alpha_{k0} + \alpha_{kv}^k U_i + \xi_{ki}, \quad \xi_{ki} \sim N(0, \sigma_k^2).
\]  

(4)

Here \( \alpha_{k0} \) is an intercept, \( \alpha_{kv} = (\alpha_{k1}, ..., \alpha_{kp})^T \) is an order of \( p \times 1 \) unknown regression coefficient vectors. Let \( Y = \{Y_1, ..., Y_n\} \), \( X = \{X_1, ..., X_n\} \), \( U = \{U_1, ..., U_n\} \), \( \eta = \{\eta_1, ..., \eta_n\} \) and \( W = \{W_1, ..., W_n\} \), \( X_i = (X_{i1}, ..., X_{ir})^T \), \( \eta_i = (\eta_{i1}, ..., \eta_{ir})^T \) and \( W_i = (W_{i1}, ..., W_{ir})^T \), for each \( i = 1, ..., n \). Suppose \( \varepsilon_y = (\beta, \psi) \), \( \varepsilon_a = \{\alpha_{10}, ..., \alpha_{rv}, \alpha_{1v}, ..., \alpha_{rv}, \sigma_k^2\} \) and \( \varepsilon = \{\varepsilon_y, \varepsilon_a, \varepsilon_\eta\} \), \( \varepsilon_\eta \) is the parameter of equation (3). The joint probability density function for \( \{Y, W, \theta, X\} \) representation

\[
P(Y, \psi, \theta, X|\psi, \varepsilon) = \prod_{i=1}^n P(Y_i|X_i, U_i; \varepsilon_y) P(W_i|X_i; \varepsilon_a) P(X_i|U_i; \varepsilon_a).
\]  

(5)

We set these parameters \( \beta, \psi, \alpha = (\alpha_{k0}, \alpha_{kv})^T \) for \( k = 1, ..., r \) and \( \sigma_k^2 \) with a priori obey the following distribution

\[
\beta|\psi, \beta^0, H_\beta^0 \sim N_{r+p}(\beta^0, \psi^{-1}H_\beta^0), \quad \psi^{-1}|b_1, b_2 \sim \Gamma(b_1, b_2), \quad \sigma_k^{-2}|d_1, d_2 \sim \Gamma(d_1, d_2).
\]

These \( b_1, b_2, \beta^0, H_\beta^0, \alpha_{kv}, H_{vk}^k, \sigma_k^2 \) and \( \alpha_{k0} \) are hyperparameters, and assume that their values are given by a priori information. According to the joint probability density function given above and their priors, we can use the Bayesian method to make statistical inferences on the parameters \( \psi = \{\varepsilon_y, \varepsilon_a, \varepsilon_\eta\} \). In addition, we use Gibbs sampling and Metropolis-Hastings algorithm to analyze the measurement error model with longitudinal data. We get the posterior distribution of the interested parameters

\[
P(\sigma_k^{-2}|X, U, \alpha) \sim \Gamma(d_1 + 0.5n, d_2 + 0.5 \sum_{k=1}^p \sum_{i=1}^n (X_{ki} - \alpha_{k0} - \alpha_{kv}^k U_i)^2); \]

\[
P(\beta|\psi, Y, X, U) \propto \exp(\psi^{-1} \sum_{i=1}^n (Y_i - d_t(i))^2), \quad 0.5\psi^{-1}(\beta - \beta^0)^T(H_\beta^0)^{-1}(\beta - \beta^0)P(\alpha_k|X, U, \sigma_k^2) \sim N(\mu_{ak}, \Omega_{ak})
\]

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where

\[
\mu_{\omega_i}^* = \Omega_{\omega_i}^* \left( \sum_{j=1}^{n} U_i^* X_i \sigma_j^{-2} + (H_{0k}^0)^{-1} \alpha_k^0 \right) \Omega_{\omega_k}^* = \left( \sum_{i=1}^{n} U_i^* U_i^T \sigma_z^{-2} + (H_{0k}^0)^{-1} \right)^{-1}, \quad U_i^* = (1, U_i^T)^T.
\]

3. Simulation and Bayesian Estimation

In order to test the feasibility of the Bayesian method in the case where our previously assumed model obeys a large number of different distributions in the measurement error \( \eta_{ij} \), for the sample size \( n = 200, m = 5 \) in the generalized linear measurement error model, 100 sets of data sets \( \{(Y_i, U_i, W_i, X_i): i = 1, \ldots, n\} \) are repeatedly generated from the probability density function with (Bernoulli) distribution for simulation studies

\[
Y_i \sim B(1, p_i).
\]

Here \( \lambda_i = \log \{ p_i/(1 - p_i) \} = X_i^T \beta_x + U_i^T \beta_v = Z_i^T \beta \). Assumptions \( U_i \sim N(0, 0.25I_3), X_{i1} \) and \( X_{2i} \) are derived from the data generated according to equation (4). Under this distribution, the value \( \psi \) is known to be a constant according to equation (1). For \( k = 1 \) and \( 2 \), the true values \( \beta_x, \beta_v, \alpha_k \) and \( \sigma_z^2 \) are taken as

\[
\beta_x = (0.8, 0.9)^T, \beta_v = (0.5, 0.5, 0.5)^T, \alpha_k = (0.2, 0.2, 0.2, 0.5)^T \quad \text{and} \quad \sigma_z^2 = 1.\]

To test the validity of the prior metric measurement error \( \eta_{ij} \), for the sample size \( n = 200, m = 5 \) in the generalized linear measurement error model, 100 sets of data sets \( \{(Y_i, U_i, W_i, X_i): i = 1, \ldots, n\} \) are repeatedly generated from the probability density function with (Bernoulli) distribution for simulation studies

Simulation 1: We assume \( \eta_{ijk} \) that it is from \( N(0, 1.2^2) \).

Simulation 2: We assume \( \eta_{ijk} \) that it is from

\[
\eta_{ijk} \sim 0.6N(-0.4, 0.2^2) + 0.4N(0.6, 0.2^2).
\]

Simulation 3: We assume \( \eta_{ijk} \) that it is from

\[
\eta_{ijk} \sim 0.3N(0.5, 0.1) + 0.2N(3.0, 1) + 0.5N(-1.5, 0.1)
\]

Simulation 4: We assume \( \eta_{ijk} \) that it is from

\[
\eta_{ijk} \sim 0.3N(0.5, 0.1) + 0.2N(3.0, 1) + 0.1N(-3.5, 0.1) + 0.4N(-1.0, 1).
\]

To study the sensitivity of a priori to Bayesian estimation, we consider the following three priori assumptions for the parameters \( \beta \) and \( \alpha_k \).

**Type A.** About a priori hyperparameters \( \beta \) and \( \alpha_k \) are chosen to be \( \beta_0 = 1.5 \times (0.8, 0.9, 0.5, 0.5, 0.5)^T \), \( H_{\beta}^0 = 0.75I_5 \), \( H_{0k}^0 = 0.75I_4 \) and \( \alpha_k = 1.5 \times (0.2, 0.2, 0.2, 0.5)^T \). This ensures a weak priori information in the simulation test.

**Type B.** About a priori hyperparameters \( \beta \) and \( \alpha_k \) are chosen to be \( \beta_0 = 0 \times (0.8, 0.9, 0.5, 0.5, 0.5)^T \), \( H_{\beta}^0 = 10I_5 \), \( \alpha_k = 0 \times (0.2, 0.2, 0.2, 0.5)^T \) and \( H_{0k}^0 = 10I_4 \). This ensures noninformative priori information in the simulation test.

The simulation results are listed in Table 1-4. We drop the first 5000 iterations of all parameters and collect 5000 data after 5000th to generate 100 sets of data from the posterior distribution of the full data through Marko. Bayesian Monte Carol (MCMC) sampling was used to evaluate Bayesian estimates. The results of the above two hypotheses and their three different a priori designs are given in Tables 1-4, where 'Bias' is the absolute value of the difference between the true value and the parameter mean of the 100 sets of replicates;
and 'RMS' It is the mean square error of the parameter estimates and true values for 100 replicates. We also have plotted densities of $\eta_{ijk}$ and $\hat{\eta}_{ijk}$ for Simulation 4 under Type C prior inputs in Fig. 1.

Table 1. First Simulated Parameter Estimation

| Parameter | True value | Type A | Type B | Type C |
|-----------|------------|--------|--------|--------|
| $\alpha_{10}$ | 0.2 | 0.0032 | 0.0731 | 0.0731 |
| $\alpha_{11}$ | 0.2 | 0.0439 | 0.1536 | 0.1536 |
| $\alpha_{12}$ | 0.2 | 0.0076 | 0.1594 | 0.1594 |
| $\alpha_{13}$ | 0.5 | 0.0061 | 0.1079 | 0.1079 |
| $\alpha_{20}$ | 0.2 | 0.0138 | 0.0669 | 0.0669 |
| $\alpha_{21}$ | 0.2 | 0.0128 | 0.1711 | 0.1711 |
| $\alpha_{22}$ | 0.2 | 0.0078 | 0.1316 | 0.1316 |
| $\alpha_{23}$ | 0.5 | 0.0269 | 0.1443 | 0.1443 |
| $\beta_0$ | 0.8 | 0.0173 | 0.0858 | 0.0858 |
| $\beta_1$ | 0.9 | 0.0034 | 0.0763 | 0.0763 |
| $\beta_2$ | 0.5 | 0.0374 | 0.1391 | 0.1391 |
| $\beta_3$ | 0.5 | 0.0356 | 0.1330 | 0.1330 |
| $\beta_4$ | 0.5 | 0.0132 | 0.1458 | 0.1458 |
| $\sigma^2_{\eta}$ | 1.0 | 0.0073 | 0.0805 | 0.0805 |

Table 2. Second Simulated Parameter Estimation

| Parameter | True value | Type A | Type B | Type C |
|-----------|------------|--------|--------|--------|
| $\alpha_{10}$ | 0.2 | 0.0186 | 0.0352 | 0.0352 |
| $\alpha_{11}$ | 0.2 | 0.0150 | 0.1308 | 0.1308 |
| $\alpha_{12}$ | 0.2 | 0.0185 | 0.1991 | 0.1991 |
| $\alpha_{13}$ | 0.5 | 0.0008 | 0.1293 | 0.1293 |
| $\alpha_{20}$ | 0.2 | 0.0064 | 0.0919 | 0.0919 |
| $\alpha_{21}$ | 0.2 | 0.0003 | 0.1638 | 0.1638 |
| $\alpha_{22}$ | 0.2 | 0.0388 | 0.1313 | 0.1313 |
| $\alpha_{23}$ | 0.5 | 0.0089 | 0.1357 | 0.1357 |
| $\beta_0$ | 0.8 | 0.0046 | 0.0833 | 0.0833 |
| $\beta_1$ | 0.9 | 0.0062 | 0.0546 | 0.0546 |
| $\beta_2$ | 0.5 | 0.0137 | 0.1252 | 0.1252 |
| $\beta_3$ | 0.5 | 0.0601 | 0.1327 | 0.1327 |
| $\beta_4$ | 0.5 | 0.0160 | 0.1882 | 0.1882 |
| $\sigma^2_{\eta}$ | 1.0 | 0.0061 | 0.0624 | 0.0624 |

Table 3. Third Simulated Parameter Estimation

| Parameter | True value | Type A | Type B | Type C |
|-----------|------------|--------|--------|--------|
| $\alpha_{10}$ | 0.2 | 0.0002 | 0.0482 | 0.0482 |
| $\alpha_{11}$ | 0.2 | 0.0112 | 0.1152 | 0.1152 |
| $\alpha_{12}$ | 0.2 | 0.0117 | 0.1221 | 0.1221 |
| $\alpha_{13}$ | 0.5 | 0.0018 | 0.1372 | 0.1372 |
| $\alpha_{20}$ | 0.2 | 0.0035 | 0.0856 | 0.0856 |
| $\alpha_{21}$ | 0.2 | 0.0045 | 0.1108 | 0.1108 |
| $\alpha_{22}$ | 0.2 | 0.0243 | 0.1030 | 0.1030 |
| $\alpha_{23}$ | 0.5 | 0.0035 | 0.1138 | 0.1138 |
| $\beta_0$ | 0.8 | 0.0046 | 0.0324 | 0.0324 |
| $\beta_1$ | 0.9 | 0.0035 | 0.0603 | 0.0603 |
| $\beta_2$ | 0.5 | 0.0125 | 0.1340 | 0.1340 |
| $\beta_3$ | 0.5 | 0.0105 | 0.1243 | 0.1243 |
| $\beta_4$ | 0.5 | 0.0206 | 0.1338 | 0.1338 |
| $\sigma^2_{\eta}$ | 1.0 | 0.0135 | 0.0562 | 0.0562 |
Table 4. Fourth Simulated Parameter Estimation

| Parameter | True value | Type Bias | A | Type Bias | B | Type Bias | C |
|-----------|------------|-----------|---|-----------|---|-----------|---|
| $\alpha_{10}$ | 0.2 | 0.0162 | 0.0562 | 0.0142 | 0.0766 | 0.0191 | 0.0864 |
| $\alpha_{11}$ | 0.2 | 0.0060 | 0.1386 | 0.0092 | 0.1685 | 0.0125 | 0.1574 |
| $\alpha_{12}$ | 0.2 | 0.0292 | 0.1133 | 0.0024 | 0.1142 | 0.0209 | 0.1476 |
| $\alpha_{13}$ | 0.5 | 0.0018 | 0.1814 | 0.0241 | 0.1792 | 0.0361 | 0.1549 |
| $\alpha_{20}$ | 0.2 | 0.0086 | 0.0411 | 0.0103 | 0.0614 | 0.0224 | 0.0881 |
| $\alpha_{21}$ | 0.2 | 0.0013 | 0.1192 | 0.0405 | 0.1543 | 0.0264 | 0.1753 |
| $\alpha_{22}$ | 0.2 | 0.0195 | 0.1320 | 0.0321 | 0.1286 | 0.0027 | 0.1405 |
| $\alpha_{23}$ | 0.5 | 0.0062 | 0.1116 | 0.0143 | 0.1236 | 0.0216 | 0.1330 |
| $\beta_0$ | 0.8 | 0.0144 | 0.0162 | 0.0112 | 0.0619 | 0.0256 | 0.0662 |
| $\beta_1$ | 0.9 | 0.0156 | 0.0703 | 0.0163 | 0.0551 | 0.0134 | 0.0844 |
| $\beta_2$ | 0.5 | 0.0241 | 0.1526 | 0.0442 | 0.1895 | 0.0327 | 0.1124 |
| $\beta_3$ | 0.5 | 0.0160 | 0.1249 | 0.0254 | 0.1399 | 0.0216 | 0.1768 |
| $\beta_4$ | 0.5 | 0.0204 | 0.1135 | 0.0406 | 0.1183 | 0.0102 | 0.1821 |
| $\sigma^2_z$ | 1.0 | 0.0079 | 0.0853 | 0.0041 | 0.0897 | 0.0052 | 0.0752 |

Fig. 1. Estimated versus true densities of $\eta_{ij1}$ and $\eta_{ij2}$ as assumption under Type C prior inputs.

4. Conclusion

According to Table 1-4 and Fig. 1, we know that (i) the model uses Bayesian estimation to be reasonable and correct, regardless of the distribution $\eta_{ijk}$ and a priori assumptions, because the unknown parameters produce Bias values less than 0.1 and RMS values less than 0.2. (ii) Dirichlet priori is generally sufficient to capture the characteristics of the various distribution hypotheses of the measurement error model. (iii) The results show that the proposed method is a good estimate of the distribution $\eta_{ijk}$.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Meilan Qiu and Dewang Li conducted the research; Meilan Qiu and Zhongyi Ke analyzed the data; Dewang Li and Zhongyi Ke wrote the paper; all authors had approved the final version.

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