INEQUALITIES FOR PLANE PARTITIONS

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Abstract. Inequalities are important features in the context of sequences of numbers and polynomials. The Bessenrodt–Ono inequality for partition numbers and Nekrasov–Okounkov polynomials has only recently been discovered. In this paper we study the log-concavity (Turán inequality) and Bessenrodt–Ono inequality for plane partitions and their polynomization.

1. Introduction and Main Results

In this paper we address inequalities for plane partitions and their polynomization. Plane partitions are, according to Stanley, fascinating generalizations of partitions of integers ([St99], Section 7.20). Andrews [An98] gave an excellent introduction of plane partitions in the context of higher-dimensional partitions. We also refer to Krattenthaler’s survey on plane partitions in the work of Stanley and his school [Kr16].

A plane partition \( \pi \) of \( n \) is an array \( \pi = (\pi_{ij})_{i,j \geq 1} \) of non-negative integers \( \pi_{ij} \) with finite sum \( |\pi| := \sum_{i,j=1}^{\infty} \pi_{ij} = n \), which is weakly decreasing in rows and columns. It can be considered as the filling of a Ferrers diagram with weakly decreasing rows and columns, where the sum of all these numbers is equal to \( n \). Let the numbers in the filling represent the heights for stacks of blocks placed on each cell of the diagram (Figure 1). This is a natural generalization of the concept of classical partitions [An98 On03].

\begin{center}
\begin{tabular}{cccccc}
5 & 4 & 4 & 3 & 3 & 2 & 1 \\
4 & 3 & 2 \\
2 & 1 \\
\end{tabular}
\end{center}

\[ \rightarrow \]

Figure 1. Representation of plane partitions.

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Let $p(n)$ denote the number of partitions of $n$ and $pp(n)$ the number of plane partitions of $n$. As usual, $p(0) := 1$ and $pp(0) := 1$. In Table 1 we have listed the first values of the partition and plane partition function.

| $n$  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|------|----|----|----|----|----|----|----|----|----|----|----|
| $p(n)$ | 1  | 1  | 2  | 3  | 5  | 7  | 11 | 15 | 22 | 30 | 42 |
| $pp(n)$ | 1  | 1  | 3  | 6  | 13 | 24 | 48 | 86 | 160| 282| 500|

Table 1. Values for $0 \leq n \leq 10$.

We investigate the log-concavity [Br89, St89] and the Bessenrodt–Ono inequality [BO16, HNT20] of plane partitions. A sequence of real, non-negative numbers $\{\alpha(n)\}_{n=0}^{\infty}$ is log-concave if for all $n \in \mathbb{N}$:

\[
\alpha(n)^2 > \alpha(n-1)\alpha(n+1).
\]

We say log-concave at $n$ if (1.1) is satisfied for a specific $n$. A sequence of real non-negative numbers $\{\alpha(n)\}_{n=0}^{\infty}$ satisfies the Bessenrodt–Ono inequality for a set $S \subset \mathbb{N} \times \mathbb{N}$, if for all $(a, b) \in S$:

\[
\alpha(a)\alpha(b) > \alpha(a+b).
\]

1.1. Related Work and Recent Results. We recall that Nicolas [Ni78] proved that the partition function is log-concave for $n > 25$:

\[
p(n)^2 > p(n-1)p(n+1).
\]

It is valid for all $n$ even and fails for $1 \leq n \leq 25$ odd. In the course of proving several conjectures of Chen and Sun, DeSalvo and Pak [DP15] reproved this result. They remarked that due to the Hardy–Ramanujan asymptotic formula

\[
p(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2}{3}n}} \quad \text{as } n \to \infty,
\]

there is no way of knowing precisely when the asymptotic formula dominates the calculation. Their proof is based on Rademacher type estimates [Ra37] by Lehmer (e. g. [Le38, Le39]), which provide the demanded, explicit, guaranteed error estimate. This proves the log-concavity for $n \geq 2,600$ [DP15]. More generally, let $p_k(n)$ be the $k$-colored partition function, obtained for every $k \in \mathbb{N}$ by the generating function

\[
\prod_{n=1}^{\infty} (1 - q^n)^{-k} = \sum_{n=0}^{\infty} p_k(n) q^n,
\]
which is essentially the $k$th power of the reciprocal of the Dedekind eta function \cite{Ono03}. Then \cite[(1.3)]{Ono03} was extended by Chern–Fu–Tang \cite{CFT18} to an interesting conjecture: Suppose $k, \ell, n \in \mathbb{N}$ with $k \geq 2$ and $n > \ell$. Let $(k, n, \ell) \neq (2, 6, 4)$, then
\[ p_k(n-1)p_k(\ell+1) \geq p_k(n)p_k(\ell). \]
The conjecture was extended to $k \in \mathbb{R} \geq 2$ \cite{HN21A}, which involves the so-called D’Arcais polynomials or Nekrasov–Okounkov polynomials \cite{NO06, Ha10}. The Conjecture by Chern–Fu–Tang and some portion of the Conjecture by Heim–Neuhauser was recently proven by Bringmann, Kane, Rolen, and Tripp \cite{BKRT21}. The proof is based on Rademacher type formulas for the coefficients of powers of the Dedekind eta function, utilizing the weak modularity property.

Bessenrodt and Ono \cite{BO16} discovered a beautiful and simple inequality for partition numbers. Let $a, b \in \mathbb{N}$. Suppose $a, b \geq 2$ and $a + b \geq 10$, then
\begin{equation}
(1.4) \quad p(a)p(b) > p(a+b).
\end{equation}
The inequality is symmetric and always fails for $a = 1$ or $b = 1$. Let $2 \leq b \leq a$. There is equality for the pairs $(4, 3), (6, 2), (7, 2)$ and the opposite inequality of (1.4) is exactly true for the pairs $(2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (5, 3)$. Bessenrodt and Ono’s proof is based on an analytic result of Lehmer \cite{Le39} of Rademacher type, similar to the proof of the log-concavity of $p(n)$ \cite{Ni78, DP15}. Shortly after the result was published, Alanazi, Gangola III, and Munagi \cite{AGM17} came up with a subtle combinatorial proof. Chern, Fu, and Tang \cite{CFT18} generalized and proved the Bessenrodt–Ono inequality to $k$-colored partitions. In \cite{HN19} this was extended to $k$ real, again involving polynomials. The proof was given by induction and involving derivatives. Further, the work of Bessenrodt and Ono triggered the results of Beckwith and Bessenrodt \cite{BB16} on $k$-regular partitions, Hou and Jagadeesan \cite{HJ18} on the numbers of partitions with ranks in a given residue class modulo 3 and Males \cite{Ma21} for general $t$, and Heim and Neuhauser \cite{HN19}, and Dawsey and Masri \cite{DM19} for the Andrews spt-function.

We performed several numerical experiments and are convinced that some of the recorded results can be transferred to plane partitions $pp(n)$ and its generalization. MacMahon \cite{Ma97, Ma99, Ma60} proved the following non-trivial result, which took him several years. The generating function of the plane partition is given by
\[ \prod_{n=1}^{\infty} (1 - q^n)^{-n} = \sum_{n=0}^{\infty} pp(n) q^n. \]
Since this generating function is not related to a weakly modular form, in contrast to the partition numbers, we have only an asymptotic formula provided by Wright...
based on the circle and saddle point method for plane partition numbers. Wright proved the following asymptotic behavior as \( n \) goes to infinity:

\[
pp(n) \sim \frac{\zeta(3)^{7/36}}{\sqrt{12\pi}} \left( \frac{2^{2/3}}{n} \right)^{2/3} \exp \left( 3 \zeta(3) \frac{1}{3} \left( \frac{n}{2} \right)^{2/3} + \zeta'(-1) \right).
\]

Here \( \exp(z) = e^z \) and \( \zeta(s) \) denotes the Riemann zeta function. Thus, we are in a similar situation as described before by DeSalvo and Pak \cite{DP15} for partition numbers.

Recently, we invented a new proof method \cite{HN21B} and reproved some known results related to the Bessenrodt–Ono inequality for the partition function, the \( k \)-colored partitions and extension to the D’Arcais polynomials.

1.2. Main Results: Plane Partitions. We first start with the Bessenrodt–Ono inequality \cite{BessenrodtOno}.

**Theorem 1.1** (Bessenrodt–Ono inequality). Let \( a \) and \( b \) be positive integers. Let \( a, b \geq 2 \) and \( a + b \geq 12 \). Then

\[ pp(a) \cdot pp(b) > pp(a + b). \]

Equality is never satisfied.

Due to symmetry, let us assume that \( 2 \leq b \leq a \). Then \( pp(a) \cdot pp(b) < pp(a + b) \) for \( (a, b) \in \{(a, 2) : 2 \leq a \leq 9\} \cup \{(a, 3) : 2 \leq a \leq 5\} \). Note that \( pp(n) < pp(n + 1) \), similar to \( p(n) < p(n + 1) \). We have

\[ (pp(3))^2 < pp(3 + 3) \text{ and } (pp(4))^2 > pp(4 + 4). \]

Based on our investigations we state the following

**Conjecture 1.** Let \( n \geq 12 \). Then the sequence \( \{pp(n)\}_n \) of plane partitions is log-concave.

\[
pp(n)^2 > pp(n - 1) \cdot pp(n + 1).
\]

We can show with Wright’s formula \cite{Wright}, that \( (1.6) \) is true for large \( n \), and it seems that this is already true for all \( n \) even and for all \( n \geq 12 \).

**Theorem 1.2** (Log-Concavity). Let \( 12 \leq n \leq 10^5 \). Then Conjecture 1 is true. It is further true for all \( n \) even and false for all odd \( n \) below 12. Furthermore there is an \( N \) such that it is true for all \( n > N \).

It would be very interesting to determine such a \( N \) of reasonable size and to finally prove the conjecture.
1.3. Main Results: Polynomization. It is possible to consider \( \{pp(n)\}_n \) as special values of a family of polynomials \( \{P_n(x)\}_n \). We will have \( pp(n) = P_n(1) \). This makes it possible to generalize the Bessenrodt–Ono inequality and the log-concavity. We view the inequalities as a property of the largest positive real zeros of new polynomials associated with \( \{P_n(x)\}_n \).

**Definition.** Let \( \sigma_2(n) := \sum_{d|n} d^2 \). Let \( P_0(x) := 1 \) and

\[
(1.7) \quad P_n(x) := \frac{x}{n} \sum_{k=1}^{n} \sigma_2(k) P_{n-k}(x).
\]

We have listed the first polynomials in Table 2.

| \( n \) | \( P_n(x) \) |
|---|---|
| 1 | \( x \) |
| 2 | \( \frac{1}{2}x^2 + \frac{5}{2}x \) |
| 3 | \( \frac{1}{6}x^3 + \frac{5}{2}x^2 + \frac{10}{3}x \) |
| 4 | \( \frac{1}{24}x^4 + \frac{5}{4}x^3 + \frac{155}{24}x^2 + \frac{21}{4}x \) |
| 5 | \( \frac{1}{120}x^5 + \frac{5}{12}x^4 + \frac{115}{24}x^3 + \frac{163}{12}x^2 + \frac{26}{5}x \) |

**Table 2.** Polynomials \( P_n(x) \) for \( n \in \{1, 2, 3, 4, 5\} \).

Let \( q, z \in \mathbb{C} \) and \( |q| < 1 \). It is a standard procedure to show that

\[
\sum_{n=0}^{\infty} P_n(z) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-n^2} = \exp \left( z \sum_{n=1}^{\infty} \frac{\sigma_2(n) q^n}{n} \right).
\]

Thus, \( n \ pp(n) = \sum_{k=1}^{n} \sigma_2(k) \ pp(n-k) \), applying MacMahon’s discovery. Let \( k \) be a positive integer. We would like to call \( P_n(k) \) the \( k \)-colored plane partitions (compare \[BBPT19\]), but at the moment there is no combinatorial interpretation available, as in the case of partitions \[HNT20, BKRT21\]. The topic is quite complicated, since MacMahon’s result is already non-trivial and \( pp(n) \) can also be identified with the number of all partitions of \( n \), where each part \( n_j \) is allowed to have \( n_j \) colors.

1.3.1. Bessenrodt–Ono Inequalities.

**Theorem 1.3.** Let \( x \in \mathbb{R} \) and \( x > 5 \). Then

\[
P_a(x) P_b(x) > P_{a+b}(x)
\]

for all positive integers \( a \) and \( b \).
It is also possible to get results for \( x = 1, 2, 3, 4, 5 \). This leads to restrictions on \( a \) and \( b \), reflected in Table 3 where we have recorded the largest real zero of
\[
P_{a,b}(x) := P_a(x) P_b(x) - P_{a+b}(x).
\]
Thus, studying the polynomials \( P_n(x) \) and \( P_{a,b}(x) \) and their leading coefficients and zeros, provides the big picture and reveals information on the original task, studying properties of plane partitions \( \text{pp}(n) = P_n(1) \). Note that \( P_{a,b}(x) \) goes to infinity for \( a, b \geq 1 \), as \( x \) goes to infinity.

**Theorem 1.4.** Let \( x \in \mathbb{R} \) and \( x \geq 2 \). Then
\[
P_a(x) P_b(x) > P_{a+b}(x)
\]
for all positive integers \( a \) and \( b \) satisfying \( a + b \geq 12 \).

It would be interesting to search for a combinatorial proof for the plane partition numbers (Theorem 1.1) and their generalization to \( k \)-colored plane partitions (Theorems 1.3 and 1.4).

| \( a \) \( \backslash \) \( b \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|
| 1       | 5.0 | 3.2 | 3.0 | 2.6 | 2.5 | 2.3 | 2.2 | 2.1 | 2.0 | 2.0 | 1.9 |
| 2       | 3.2 | 1.9 | 1.6 | 1.4 | 1.3 | 1.2 | 1.1 | 1.1 | 1.0 | 1.0 | 1.0 | 0.9 |
| 3       | 3.0 | 1.6 | 1.5 | 1.2 | 1.1 | 1.0 | 1.0 | 0.9 | 0.9 | 0.8 | 0.8 | 0.8 |
| 4       | 2.6 | 1.4 | 1.2 | 0.9 | 0.9 | 0.8 | 0.7 | 0.7 | 0.6 | 0.6 | 0.6 | 0.6 |
| 5       | 2.5 | 1.3 | 1.1 | 0.9 | 0.9 | 0.7 | 0.7 | 0.6 | 0.6 | 0.6 | 0.6 | 0.5 |
| 6       | 2.3 | 1.2 | 1.0 | 0.8 | 0.7 | 0.6 | 0.6 | 0.5 | 0.5 | 0.5 | 0.5 | 0.4 |
| 7       | 2.2 | 1.1 | 1.0 | 0.7 | 0.7 | 0.6 | 0.6 | 0.5 | 0.5 | 0.5 | 0.5 | 0.4 |
| 8       | 2.1 | 1.1 | 0.9 | 0.7 | 0.6 | 0.5 | 0.5 | 0.5 | 0.4 | 0.4 | 0.4 | 0.4 |
| 9       | 2.1 | 1.0 | 0.9 | 0.6 | 0.6 | 0.5 | 0.5 | 0.4 | 0.4 | 0.4 | 0.4 | 0.3 |
| 10      | 2.0 | 1.0 | 0.8 | 0.6 | 0.6 | 0.5 | 0.5 | 0.4 | 0.4 | 0.4 | 0.3 | 0.3 |
| 11      | 2.0 | 1.0 | 0.8 | 0.6 | 0.6 | 0.5 | 0.5 | 0.4 | 0.4 | 0.4 | 0.3 | 0.3 |
| 12      | 1.9 | 0.9 | 0.8 | 0.6 | 0.5 | 0.4 | 0.4 | 0.4 | 0.3 | 0.3 | 0.3 | 0.3 |

**Table 3.** Approximative largest real zeros of \( P_{a,b}(x) \) for \( 1 \leq a, b \leq 12 \).

### 1.3.2. Log-Concavity.

In the spirit of the Chern–Fu–Tang Conjecture [CFT18] on \( k \)-colored partitions and their polynomization [HN21A] (see also [BKRT21]), we consider the polynomials
\[
\Delta_{a,b}(x) := P_{a-1}(x) P_{b+1}(x) - P_a(x) P_b(x).
\]
Note that $\Delta_{a+1,a-1}(x) = P_a(x)^2 - P_{a-1}(x) P_{a+1}(x)$. Then $\Delta_{a+1,a-1}(1) > 0$ is the log-concavity condition for $pp(a)$.

This leads to the following conjecture on the log-concavity of the polynomials $P_n(x)$.

**Conjecture 2 (Turán inequality).** Let $a$ be an integer with $a \geq 12$ and let $x$ be a real number with $x \geq 1$. Then

$$P_a(x)^2 > P_{a-1}(x) P_{a+1}(x).$$

This is a natural extension of Conjecture 1 on the log-concavity of plane partitions.

It would also be interesting to study the hyperbolicity of the associated Jensen polynomials [GORZ19] and higher Turán inequalities [CJW19]. Let $\alpha(m) := pp(m)$ or more general, $\alpha(m) := P_m(x)$, $x \in \mathbb{R}_{>0}$. The Jensen polynomial of degree $d$ and shift $n$ attached to the sequence $\{\alpha(0), \alpha(1), \alpha(2), \ldots\}$ of non-negative real numbers is the polynomial

$$J_{\alpha}^{d,n}(X) := \sum_{k=0}^{d} \binom{d}{k} \alpha(n+k) X^k.$$ 

It follows from the zero distribution of the polynomial $\Delta_{a+1,a-1}(x) := P_a(x)^2 - P_{a-1}(x) P_{a+1}(x)$ (see Figure 2) that Conjecture 2 is true for $12 \leq a \leq 100$, since the polynomial goes to infinity, as $x$ goes to infinity. Let $a$ be even and $2 \leq a \leq 1000$, then the Turán inequality in Conjecture 2 is already valid for $x > 0$. Actually, in this case all coefficients of $\Delta_{a+1,a-1}(x)$ are non-negative.

The generalization of the former Chern–Fu–Tang conjecture of $k$-colored partitions and its polynomization [HN21A] leads to:
Conjecture 3. Let $a$ and $b$ be integers. Suppose $a - 1 > b \geq 0$ and $(a, b) \notin \{(4, 2), (6, 4)\}$. Then for all real numbers $x \geq 2$:

$$\Delta_{a,b}(x) := P_{a-1}(x) P_{b+1}(x) - P_a(x) P_b(x) > 0.$$ 

We refer to Table 8, where we recorded the largest real zero for $\Delta_{a,b}$ for all pairs $(a, b)$ with $a - 1 > b \geq 0$ and $2 \leq a \leq 19$, which shows that the Conjecture 3 is valid for all admissible pairs $(a, b)$ in this range.

2. Basic Properties of $\sigma_2(n)$ and $P_n(x)$

For proof of Theorem 1.1 and Theorem 1.3 we need some elementary properties of $\sigma_2(n)$ and $P_n(x)$. As a further evidence that Conjecture 2 most likely to be true for even arguments $n$, we prove that $\sigma_2(n)/n$ is log-concave for $n$ even. Note that this is false for $\sum_{d|n} d$, where $n$ is even (which is related to the Chern–Fu–Tang conjecture for $k$-colored partitions).

Proposition 2.1. Let $n$ be an even, positive integer. Then

$$\left(\frac{\sigma_2(n)}{n}\right)^2 > \frac{\sigma_2(n-1)}{n-1} \frac{\sigma_2(n+1)}{n+1}.$$ 

Proof. For $n$ odd we estimate

$$\sigma_2(n) = \sum_{t|n} t^2 = \sum_{t|n} \left(\frac{n}{t}\right)^2 = n^2 \sum_{t|n} t^{-2} \leq n^2 \left(1 - \frac{1}{4}\right) \sum t^{-2} \leq \frac{\pi^2}{8} n^2 < \frac{5}{4} n^2.$$ 

Now, let $n \geq 6$ be even. Then $\sigma_2(n) \geq n^2 + \left(\frac{n}{2}\right)^2 + 4 + 1 = \frac{5}{4} (n^2 + 4)$. Therefore,

$$\frac{\sigma_2(n-1) \sigma_2(n+1)}{n^2 - 1} < \frac{25}{16} (n^2 - 1) < \left(\frac{5 n^2 + 4}{4 n}\right)^2 \leq \left(\frac{\sigma_2(n)}{n}\right)^2.$$ 

The inequality holds also for even $n \leq 4$ (Table 4).

| $n$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $\sigma_2(n)$ | 1  | 5  | 10 | 21 | 26 |

Table 4. Values of $\sigma_2(n)$ for $n \in \{1, 2, 3, 4, 5\}$.

Note, that $\sigma_2(n) < \tilde{\sigma}_2(n) := 2n^2$ for all $n \in \mathbb{N}$. This follows from

$$\sum_{t|n} t^2 = n^2 \sum_{t|n} \frac{1}{t^2} \leq n^2 \left(1 + \int_1^n t^{-2} \, dt\right) < 2n^2.$$
The functions $P_n(x)$ are polynomials of degree $n$. This can be deduced directly from the recurrence formula (1.7). We have $P_n(x) = \frac{1}{n!} \sum_{m=1}^{n} A_{n,m} x^m$, where $A_{n,m} \in \mathbb{N}$ for $n \geq 1$. We later use the fact that $A_{n,1} = n! \frac{\sigma_2(n)}{n}$. Further, we have the following properties.

**Proposition 2.2.** Let $n,m$ be natural numbers and $x \geq 1$ real. Then

\[
(2.1) \quad P'_n(x) = \sum_{k=1}^{n} \sigma_2(k) \frac{k}{k} P_{n-k}(x),
\]

\[
(2.2) \quad P_{n+1}(x) > P_n(x),
\]

\[
(2.3) \quad P_n(x) > \sum_{\ell=1}^{m} \frac{1}{\ell!} \left( \frac{n+\ell-1}{2\ell-1} \right) x^\ell \text{ for } n > 1.
\]

**Proof.** The formula for the derivative $P'_n(x)$ is given similarly as that obtained for the polynomials attached to $\sigma(n) = \sum_{d|n} d$ [HN18]. We prove (2.2) by induction.

Let $\Delta_n(x) := P_n(x) - P_{n-1}(x)$.

Let $n = 1$. Then we have $\Delta_1(x) > 0$ for all $x > 1$. Suppose $\Delta_m(x) > 0$ for all $1 \leq m \leq n - 1$. We obtain for $P'_{n+1}(x)$ the strict lower bound

\[
\sum_{k=1}^{n} \frac{\sigma_2(k)}{k} P_{n+1-k}(x) \geq \sum_{k=1}^{n} \frac{\sigma_2(k)}{k} P_{n-k}(x).
\]

Thus, $P'_{n+1}(x) > P'_n(x)$. The plane partition function is strictly increasing:

$$\text{pp}(n) < \text{pp}(n+1)$$

for all $n \in \mathbb{N}$, since every plane partition of $n$ can be lifted to a plane partition of $n + 1$. This provides

$$P_{n+1}(1) = \text{pp}(n+1) > \text{pp}(n) = P_n(1).$$

Thus, the claim is proven. The lower bound given in (2.3) is obtained in the following way. We have $n^2 < \sigma_2(n)$ for $n > 1$ and $\sigma_2(1) = 1$. Thus, the coefficients of the polynomial $S_n(x)$, defined by $S_0(x) := 1$ and $S_n(x) = \frac{1}{n} \sum_{k=1}^{n} k^2 S_{n-k}(x)$, are smaller than the coefficients of $P_n(x)$ for $n > 1$. The $m$th coefficient is given by $\frac{1}{m!} \binom{n+m-1}{2m-1}$. This can be deduced from [HLN19], Section 4. □

**Corollary 2.3.** Let $n \in \mathbb{N}$ and $x \geq 1$. Then $\Delta'_n(x) > 0$.

3. **Proof Strategy for the Bessenrodt–Ono Inequality for Plane Partitions**

In this section we lay out a general strategy for proving Bessenrodt–Ono type inequalities. This also makes the appearance of exceptions transparent.
3.1. Input Data and Proof by Induction. Let $A, B, N_0 \in \mathbb{N}$ be given with $2 \leq B < N_0$ and $x_0 \in \mathbb{R}_{\geq 1}$. Let $n \geq B$. Let $S(n)$ be the following mathematical statement for one of the three cases: $x \in \mathbb{R}$ with $x = x_0$, $x > x_0$, or $x \geq x_0$. For all $A \leq b \leq a$ with $a + b = n$ and all $x$ with fixed case, we have

$$P_{a,b}(x) := P_a(x) P_b(x) - P_{a+b}(x) > 0.$$ 

Note that $P_{a,b}(x_0) = P_{b,a}(x_0)$ and that $\lim_{x \to \infty} P_{a,b}(x) = +\infty$.

Given input data $A, B, x_0$ by induction we prove $S(n)$ for one of the given cases. We choose $N_0$ and prove first manually or by numerical calculation (utilizing PARI/GP) that $S(\ell)$ is true for all $B \leq \ell \leq N_0$. In the case of $x \geq x_0$ or $x > x_0$ we study the real zeros of the polynomial $P_{a,b}(x)$ for all $A \leq b \leq a$ with $a + b = \ell$. Let $n \geq N_0$. Then we prove $S(n)$ by assuming that $S(m)$ is true for all $B \leq m \leq n - 1$, the induction hypothesis.

3.2. Basic Decomposition. Let $A, B \in \mathbb{N}$ be given with $B \geq 2$. Further, let $a, b \in \mathbb{N}$ satisfy $A \leq b \leq a$ and $a + b \geq B$. We define

$$k_0 := a - \max\{B - b, A\} + 1,$$

$$f_k(a, b, x) := \sigma_2(k) \left( \frac{P_{a-k}(x) P_b(x)}{a} - \frac{P_{a+b-k}(x)}{a + b} \right).$$

We consider the decomposition $P_{a,b}(x) = L_{a,b}(x) + R_{a,b}(x)$, where

$$L = L_{a,b}(x) := - \sum_{k=1}^{b} \frac{\sigma_2(k + a)}{a + b} P_{b-k}(x),$$

$$R = R_{a,b}(x) := \sum_{k=1}^{a} f_k(a, b, x).$$

Utilizing (2.2) leads immediately to

$$L > -4b a P_b(x).$$

Further, let $R = R_1 + R_2 + R_3$, where

$$R_1 := f_1(a, b, x), \ R_2 := \sum_{k=2}^{k_0-1} f_k(a, b, x), \ R_3 := \sum_{k=k_0}^{a} f_k(a, b, x).$$

Suppose $P_{a+b-k}(x) < P_{a-k}(x) P_b(x)$ for $1 \leq k \leq k_0 - 1$, then

$$R_1 > \frac{b}{2a^2} P_{a-1}(x) P_b(x) \text{ and } R_2 > 0.$$
Further, we put $R_3 = R_{31} + R_{32} + R_{33}$, where

$$R_{31} := \sum_{k=k_0}^{a-A} f_k(a,b,x), \quad R_{32} := \sum_{k=a-A+1}^{a-1} f_k(a,b,x), \quad R_{33} := f_a(a,b,x).$$

Then $R_{33} > 0$ and $R_{32} = 0$ if $A = 1$. Moreover, let $B - b \leq A$, then $R_{31} = 0$. Thus, $R_3$ can only have a negative contribution to $R$ if $A \geq 2$ (by $R_{32}$) and if $B - b > A$ (by $R_{31}$). Note that $n^2 \leq \sigma_2(n) < \tilde{\sigma}_2(n) := 2n^2$. We obtain by straight-forward estimations the following.

**Lemma 3.1.** Let $B - b > A$. Then

$$R_{31} > \frac{a-A-k_0+1}{a} \left( k_0^2 P_A(x) P_b(x) - \tilde{\sigma}_2(a-A) P_{a-b-k_0}(x) \right).$$

If we know that some $P_k(x) P_b(x) - P_{b+k}(x)$ is negative we can use the estimate

$$R_{31} > \frac{\tilde{\sigma}_2(a)}{a} \sum_{k=A}^{B-b} (P_k(x) P_b(x) - P_{k+b}(x))) -$$

where $v_- = v$ if $v < 0$, otherwise $v_- = 0$.

### 4. Proof of Theorem 1.3 and Theorem 1.4

**Proof of Theorem 1.3.** Let $x > 5$. We prove that $P_{a,b}(x) > 0$ for all $a, b \geq 1$. Due to symmetry, we can assume $b \leq a$. We follow the strategy presented in Section 3. Let $A = 1$ and $B = 2$. Let $x_0 = 5$ and $x > x_0$. Note that $x = x_0$ does not work, since $P_{1,1}(5) = 0$. Let $N_0 = 12$. Then the mathematical statement $S(n)$ is true for all $A \leq a, b \leq N_0$ (see Table 3). Let $n > N_0$, we assume that $S(m)$ is true for all $B \leq m \leq n - 1$. Let $a + b = n$ with $A \leq b \leq a$. Then $P_{a,b}(x) > L + R_1$. Note that $R_2, R_3 \geq 0$. With (2.2) we have

$$L > -4 a b P_b(x),$$

$$R > \frac{b}{2a^2} P_{a-1}(x) P_b(x).$$

**Final step.** Putting things together and estimating $P_{a-1}(x)$ from below by (2.3), we obtain

$$P_{a,b}(x) > \frac{b P_b(x)}{2a^2} \left( -8 a^3 + \sum_{\ell=1}^{5} \left( \frac{a+\ell-2}{2\ell-1} \right) \frac{5^\ell}{\ell!} \right).$$

The right hand side of (4.1) is polynomial in $a$ of degree 9 with a positive leading coefficient. Calculating the largest real zero shows that the right hand side is positive for $a \geq 7$. $\square$
Proof of Theorem 1.4. Let $x \geq 2$. We put $A = 1$ and $B = 12$. Let $N_0 = 58$. Then by induction, as before, we obtain
\[ P_{a,b}(x) > L_1 + R_{31}, \]
where for $b \geq 12$ we can apply the induction hypothesis and obtain $R_{31} > 0$. In case $1 \leq b \leq 11$ we find from Table 3 that $P_{a-k,b}(2) > 0$ for $b = 1$ and $a-k \notin \{1,2,\ldots,11\}$ or $k = a-1$ and $b \notin \{1,2,\ldots,11\}$. Therefore,
\[ R_{31} > -\sum_{k=a-11}^{a-1} \sigma(k) P_{a-k,1}(x) / a. \]

It can be checked that the polynomials $P_{k,1}(x)$, $1 \leq k \leq 10$, are monotonically increasing for $x \geq 2$. Thus,
\[ R > R_{31} > -641 \cdot 2a \geq -641 a b P_b(x) \]

from (3.1) and Table 5.

Final step: Putting everything together leads to
\[ P_{a,b}(x) > b P_b(x) - \frac{1290a^3}{2a^2} \left( a + \ell - 2 \right) \frac{2^\ell}{\ell!}. \]

In the last step we used the property (1.7) and that $x \geq 2$. We obtain that the expression (4.2) is positive for all $a \geq 27$. Since the leading coefficient of $P_{a,b}(x)$ is positive we only have to determine the largest real zero of all remaining $P_{a,b}(x)$. We checked this for $12 \leq b + a \leq 52$ with PARI/GP (compare Table 3). \[ \square \]

| $b$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
|     | -641 | -4 | -11 | -16 | -38 | -52 | -101 | -126 | -180 | -110 |

Table 5. Values of $\sum_{k=1}^{11-b} (P_k(2)P_b(2) - P_{k+b}(2))_-$ for $b \leq 10$.

5. Proof of the Bessenrodt–Ono Inequality: Theorem 1.1

We start with the following auxiliary result.

**Lemma 5.1.** We have $pp(2) = 3 pp(1)$ and for $n \neq 1$:
\[ pp(n + 1) < 3 pp(n). \]

**Proof.** The proof is by mathematical induction. We checked with PARI/GP that (5.1) holds for $n \leq N_0 = 24$. \[ \square \]
Now let \( n > N_0 \). Then

\[
3 \text{pp} (n) - \text{pp} (n + 1) = \frac{3}{n} \sum_{k=1}^{n} \sigma_2 (k) \text{pp} (n - k) - \frac{1}{n+1} \sum_{k=1}^{n+1} \sigma_2 (k) \text{pp} (n + 1 - k)
\]

\[
= -\frac{\sigma_2 (n + 1)}{n + 1} + \sum_{k=1}^{n} \sigma_2 (k) \left( \frac{3}{n} \text{pp} (n - k) - \frac{1}{n+1} \text{pp} (n - k + 1) \right)
\]

\[
> -2 \left( \frac{n + 1}{n + 1} \right)^2 + \left( \frac{3}{n} - \frac{3}{n+1} \right) \text{pp} (n - 1)
\]

\[
\geq -2 (n + 1) + \frac{3}{(n + 1)^2} \sum_{\ell=1}^{3} \left( \frac{n + \ell - 2}{2\ell - 1} \right) \frac{1}{\ell!} > 0.
\]

We simplified (5.2) by the induction hypothesis. 

\[\square\]

**Proof of Theorem 1.1.** The proof is again by induction. We follow the proof strategy stated in Section 3. See also Section 4.

Let \( A = 2, \ B = 12 \) and \( x = 1 \). Then \( k_0 = a - \max \{B - b, A\} + 1 = a - \max \{11 - b, 1\} \). Furthermore, \( L > -4ab \text{pp} (b), R_1 > \frac{b}{2a^2} \text{pp} (a - 1) \text{pp} (b), \) and \( R_2 > 0 \). For \( R_{33} \) we obtain the lower bound 0 and for \( R_{32} \) we obtain

\[
\sigma_2 (a - 1) \left( \frac{\text{pp} (b)}{a} - \frac{\text{pp} (b + 1)}{a + b} \right) \geq -\sigma_2 (a - 1) \frac{2}{a} \text{pp} (b) > -4a \text{pp} (b).
\]

Finally,

\[
R_{31} > \sum_{k=k_0}^{a-A} \frac{\sigma (k)}{a} (\text{pp} (a - k) \text{pp} (b) - \text{pp} (a + b - k))
\]

\[
\geq -106 \cdot 2a = -212a
\]

(Table 6). Therefore, note \( R_3 > -4a \text{pp} (b) - 212a > -38ab \text{pp} (b) \). Putting everything together leads to

\[
\text{pp} (a) \text{pp} (b) - \text{pp} (a + b) > \frac{b}{2a^2} \left( -76a^3 + \text{pp} (a - 1) \right)
\]

\[
> \frac{b}{2a^2} \left( -76a^3 + \sum_{\ell=1}^{3} \frac{1}{\ell!} \left( a - \ell - 2 \right) \right)
\]

using (1.7). This is positive for \( a \geq 237 \). We have checked with PARI/GP that \( \text{pp} (a) \text{pp} (b) - \text{pp} (a + b) > 0 \) for \( B \leq a + b \leq N_0 = 472 \). 

\[\square\]
Table 6. Values of $\sum_{k=2}^{11-b} (\text{pp}(k) \text{pp}(b) - \text{pp}(k+b))$ for $2 \leq b \leq 9$.

6. Proof of Theorem 1.2

Lemma 6.1. Let $s \in \mathbb{R}$. There are $C_0, N > 0$ such that

\begin{equation}
2n^s - (n+1)^s - (n-1)^s = (1-s)sn^{s-2} + E_n n^{s-3}
\end{equation}

for $n > N$ with $|E_n| < C_0$.

Proof. We have

\[
2n^s - (n+1)^s - (n-1)^s \\
= \left( 2 - \left( 1 - \frac{1}{n} \right)^s - \left( 1 + \frac{1}{n} \right)^s \right) n^s \\
= \left( 2 - \left( 1 - \frac{s}{n} \right) - \left( 1 + \frac{s}{n} \right) \frac{D_1,n}{n^3} \right) \left( 1 + \frac{s}{n} - \left( 1 - \frac{s}{n} \right) \frac{D_2,n}{n^3} \right) n^s \\
= (1-s)sn^{s-2} - E_n n^{s-3}
\]

with $|D_1,n|, |D_2,n|, |E_n| < C_0$ for some $C_0, N > 0$ and all $n > N$. \qed

Corollary 6.2. Let $C_1 > 0$. There is $N > 0$ such that

\begin{equation}
1 + \frac{C_1}{9} n^{-4/3} < \exp \left( 2C_1 n^{2/3} - C_1 (n+1)^{2/3} - C_1 (n-1)^{2/3} \right) < 1 + \frac{4C_1}{9} n^{-4/3}
\end{equation}

for all $n > N$.

Proof. For example for the lower bound we obtain from (6.1)

\[
2n^{2/3} - (n+1)^{2/3} - (n-1)^{2/3} \geq \frac{2}{9} n^{-4/3} - 2C_0 n^{-7/3} > \frac{1}{9} n^{-4/3}
\]

for some $C_0 > 0$ and all $n > N$ for some $N$. Therefore

\[
\exp \left( 2C_1 n^{2/3} - C_1 (n-1)^{2/3} - C_1 (n+1)^{2/3} \right) > \exp \left( \frac{C_1}{9} n^{-4/3} \right) \\
> 1 + \frac{C_1}{9} n^{-4/3}.
\]

\qed
**Theorem 6.3.** Let $C_1, C_2, C_3 > 0$, $r, \gamma_1, \gamma_2 \in \mathbb{R}$, $\beta(n) = C_2 n^r e^{C_1 n^{2/3}}$, and
\[
\left| \frac{\alpha(n)}{\beta(n)} - 1 - \gamma_1 n^{-2/3} - \gamma_2 n^{-4/3} \right| \leq C_3 n^{-2}
\]
for $n > N_0$ for some $N_0$. There is an $N \geq N_0$ such that $(\alpha(n))_{n \geq N_0}$ is log-concave for $n > N$.

**Proof.** Let $f_{\pm}(n) = \gamma_1 n^{-2/3} + \gamma_2 n^{-4/3} \pm C_3 n^{-2}$. Then there is an $N_1 \geq N_0$ such that
\[
(1 + f_-(n)) \beta(n) < \alpha(n) < (1 + f_+(n)) \beta(n)
\]
for all $n > N_1$. Therefore
\[
(\alpha(n))^2 - \alpha(n-1) \alpha(n+1) > ((1 + f_-(n)) \beta(n))^2 - (1 + f_+(n-1)) (1 + f_+(n+1)) \beta(n-1) \beta(n+1)
\]
\[
= (1 + f_-(n))^2 \beta(n+1) \beta(n-1) \cdot \left( \exp \left( 2Bn^{2/3} - B(n-1)^{2/3} - B(n+1)^{2/3} \right) - \frac{(1 + f_+(n-1))(1 + f_+(n+1))}{(1 + f_-(n))^2} \left( \frac{n^2 - 1}{n^2} \right)^r \right).
\]
Now
\[
\frac{(1 + f_+(n-1))(1 + f_+(n+1))}{(1 + f_-(n))^2}
\]
\[
= 1 + \frac{f_+(n-1) + f_+(n+1) - 2f_-(n)}{1 + f_-(n)} + \frac{(f_+(n-1) - f_-(n))(f_+(n+1) - f_-(n))}{(1 + f_-(n))^2}.
\]

With (6.1) we obtain
\[
|f_+(n-1) + f_+(n+1) - 2f_-(n)| = \left| -\frac{10}{9} \gamma_1 n^{-8/3} + E_n n^{-10/3} \right| < C_5 n^{-8/3}
\]
with $|E_n| < C_4$, $C_4, C_5 > 0$, and
\[
|(n+v)^u - n^u| = \left| 1 + \frac{uv}{n} + U_n \frac{u}{n^2} - 1 \right| n^u = \left| uv n^u - 1 + U_n n^{u-2} \right| < C_6 n^{-5/3}
\]
with $|U_n| < C_7$, $C_6, C_7 > 0$, and $u \leq -2/3$. Therefore
\[
\left| \frac{(1 + f_+(n-1))(1 + f_+(n+1))}{(1 + f_-(n))^2} \right| < 1 + C_8 n^{-8/3}
\]
for some $C_8 > C_5$ which implies with (6.2)

$$\exp \left(2C_1 n^{2/3} - C_1 (n - 1)^{2/3} - C_1 (n + 1)^{2/3}\right) -$$

$$\frac{(1 + f_+ (n - 1)) (1 + f_+ (n + 1))}{(1 + f_-(n))^2} \left(1 + \frac{1}{n^2 - 1}\right)^{-r}$$

$$> 1 + \frac{C_1}{9} n^{-4/3} - (1 + C_8 n^{-8/3}) \left(1 + \frac{C_9}{n^2}\right) > 0$$

for some $C_9 > 0$. This is positive for $n > N$ and $N$ sufficiently large.

\[\square\]

**Proof of Theorem 1.2.** Wright’s formula ([Wr31], Formula (2.21)) tells us that there are $C_1, C_2, C_3 > 0, r = -25/36, \text{ and } \gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\text{pp} (n) = C_2 n^{-\frac{25}{36}} e^{C_1 n^{2/3}} \left(1 + \frac{\gamma_1 n^2}{n^3} + \frac{\gamma_2 n^{4/3} + G_n}{n^4}\right)$$

with $|G_n| < C_3$ for all $n > N_0, N_0$ sufficiently large.

\[\square\]

7. **Conjecture 2 and Conjecture 3**

In this section we provide evidence for the Conjectures by information on the zeros of the underlying polynomials. Here we also use the crucial property that the leading coefficient of these polynomials always has positive sign.

7.1. **Conjecture 2.** Figure 2 indicates that it is most likely that if one considers the sequence of real parts of the zero of $\{\Delta_{a+1,a-1}(x)\}$ with the largest real part, that these numbers tend in the limit to zero. Another aspect is given by coefficients of these polynomials (Table 7).

| $a$ | $\Delta_{a+1,a-1}(x)$ |
|-----|----------------------|
| 2   | $\frac{1}{13} x^4 + \frac{32}{13} x^2$ |
| 3   | $\frac{144}{141} x^8 + \frac{48}{45} x^6 + \frac{145}{101} x^4 - \frac{101}{144} x^3 - \frac{145}{21} x^2$ |
| 4   | $\frac{1}{2880} x^{10} + \frac{1}{1152} x^8 + \frac{5}{192} x^6 + \frac{941}{2880} x^4 - \frac{1373}{144} x^3 + \frac{491}{48} x^2$ |
| 5   | $\frac{1}{86400} x^{12} + \frac{1}{1152} x^8 + \frac{5}{192} x^6 + \frac{941}{2880} x^4 - \frac{1373}{144} x^3 + \frac{491}{48} x^2$ |

| Table 7. Polynomials $\Delta_{a+1,a-1}(x)$ for $a \in \{2, 3, 4, 5\}$. |

We observe that for $a$ odd there are coefficients, which are negative. In the case $a$ even, as already mentioned in the introduction, we have calculated the polynomials
for $2 \leq a \leq 1000$ and observed that the coefficients are all non-negative. Let

$$\Delta_{a+1,a-1}(x) = \sum_{k=2}^{2a} B_{2a,k} x^k,$$

then we deduce from Proposition 2.1 that $B_{2a,2} > 0$.

7.2. **Conjecture 3.** Theorem 1.4 implies that $\Delta_{a,0}(x) = P_{a-1}(x) - P_a(x) > 0$ for $a \geq 12$ and $x \geq 2$. Since $\Delta_{a,0}(x) > 0$ also for $3 \leq a \leq 11$ (Table 8), we obtain:

**Corollary 7.1.** Let $x \geq 2$. Then Conjecture 3 is valid for all $a \geq 3$ and $b = 0$.

Next, we prove Conjecture 3 for $x \geq 1$ and all pairs $(a,1)$ with $a \geq 3$. We refer also to Figure 3 for the pairs $(a,b)$ with $2 \leq b \leq 4$. Note, for $(a,1)$ and $3 \leq a \leq 100$, there are no zeros with a positive real part.

![Figure 3. Zeros with the largest real part of $\Delta_{a,b}(x)$ for $2 \leq b \leq 4$, blue = real zero, red = imaginary zero.](image)

**Proposition 7.2.** Let $x \geq 1$ and $a \geq 3$. Then $\Delta_{a,1}(x) > 0$.

**Proof.** We deduce from Table 8 that $\Delta_{a,1}(x) > 0$ for $x > 0$ and $a \leq 20$. Moreover, $\Delta_{a,1}(1) = 3 \, \text{pp}(a - 1) - \text{pp}(a) > 0$ for $a \geq 3$. This follows from Lemma 5.1. Let $\Delta_{a,1}(x) = x \, F_a(x)$. We prove that $F_a(x) > 0$ for $x \geq 1$ and $a \geq 3$ by induction on $a$. Let $a \geq 6$ and $F_m(x) > 0$ for $3 \leq m < a$ and $x \geq 1$. We show that $F_a'(x) > 0$, which completes the proof, since $F_a(1) > 1$, for $a \geq 3$. Recall Formula (2.1). Then
the derivative $F'_a(x)$ is equal to
\[
\frac{x + 5}{2} \sum_{k=1}^{a-1} \frac{\sigma_2(k)}{k} P_{a-1-k}(x) - \sum_{k=1}^{a} \frac{\sigma_2(k)}{k} P_{a-k}(x) + \frac{P_{a-1}(x)}{2}.
\]
By the induction hypothesis we obtain
\[
\frac{x + 5}{2} \sum_{k=1}^{a-1} \frac{\sigma_2(k)}{k} P_{a-1-k}(x) > \frac{x + 5 \sigma_2(a-1)}{2} - \frac{a-1}{a} + \sum_{k=1}^{a-2} \frac{\sigma_2(k)}{k} P_{a-k}(x).
\]
This leads to
\[
F'_a(x) > \frac{x + 5 \sigma_2(a-1)}{2} - \sum_{k=a-1}^{a} \frac{\sigma_2(k)}{k} P_{a-k}(x) + \frac{P_{a-1}(x)}{2} = \frac{P_{a-1}(x)}{2} - \frac{x \sigma_2(a-1)}{2(a-1)} + \frac{5 \sigma_2(a-1)}{2(a-1)} - \frac{\sigma_2(a)}{a}.
\]
Moreover, we have
\[
2F'_a(x) > \left(P_{a-1}(x) - \frac{\sigma_2(a-1)}{a} x\right) + a - 5,
\]
since $a^2 < \sigma_2(a) < 2a^2$. We observe that $P_{a}(x) - \frac{\sigma_2(a)}{a} x$ has non-negative coefficients. Finally, this implies that $F'_a(x) > 0$ for all $x > 1$. Thus, $F_a(x) > 0$ since $F_a(1) > 0$. \qed

**Corollary 7.3.** Let $x \geq 1$ and $a \geq 3$. Then $\Delta'_{a,1}(x) > 0$.

**Remarks.**
a) The real part of the zeros of $\Delta_{a,1}(x)/x$ is negative for $3 \leq a \leq 100$.
b) The method of the proof is similar to the one outlined in [HN21A].
Table 8. Approximative largest positive real zeros of $\Delta_{a,b}(x)$ for $0 \leq b < a - 1 \leq 19$.

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