Cluster temperatures and non-extensive thermo-statistics

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Abstract

We propose a novel component to the understanding of the temperature structure of galaxy clusters which does not rely on any heating or cooling mechanism. The new ingredient is the use of non-extensive thermo-statistics which is based on the natural generalization of entropy for systems with long-range interactions. Such interactions include gravity and attraction or repulsion due to charges. We explain that there is growing theoretical indications for the need of this generalization for large cosmological structures. The observed pseudo temperature is generally different from the true thermodynamic temperature, and we clarify the connection between the two. We explain that this distinction is most important in the central part of the cluster where the density profile is most shallow. We show that the observed pseudo temperature may differ up to a factor 2/5 from the true thermodynamic temperature, either larger or smaller. In general the M-T and L-T relations will be affected, and the central DM slope derived through hydrostatic equilibrium may be either more shallow or steeper. We show how the true temperature can be extracted correctly either from the spectrum or from the shape of the Doppler broadening of spectral lines.

Key words:

1 Introduction

Galaxy clusters have been known and studied for many years, and the radial dependence of cluster temperatures is becoming a testing ground for models of structure formation and for our understanding of gas dynamics. The emerging temperature profile is one where the temperature increases from the centre to some maximum temperature, and then decreases again for larger radii.

The central decrement has been much discussed and the possibility of cooling flows has been explained in excellent reviews, see (Fabian 1994, Donahue &
Voit 2004) for references. From an observational point of view this central
temperature decrement is very well established (Allen, Schmidt & Fabian 2001,
Peterson et al. 2003, Kaastra et al. 2004), and numerical simulations are now
beginning to see it too (Motl et al. 2004). The outer temperature decrease
is well established both observationally (Markevitch et al. 1998, Kaastra et
al. 2004) and numerically (Loken et al. 2002, Komatsu & Seljak 2001).

The expected cooling flows are not observed in galaxy clusters, and numerous
explanations have been proposed including various additional heating or lack-
of-cooling mechanisms, see e.g. (Peterson et al. 2003, Donahue & Voit 2004)
for references.

We will here propose a new element in the understanding of cluster temper-
atures. Our solution has galaxy clusters in kinetic equilibrium, but with tem-
peratures defined through non-extensive thermo-statistics which is the natural
generalization of normal statistical mechanics. The need for non-extensive en-
tropy arises when particle interactions are not point-like, and includes gravity
and attraction/repulsion due to charges. These are exactly the conditions for
electrons and protons in galaxy clusters, as we will explain in section 4.

We will structure the paper as follows. First we will consider the theoretical
basis for statistical mechanics with non-extensive entropy. To clarify the signa-
tures of non-extensive statistics, we will apply the results to the Coma cluster.
This is followed by a discussion of reasons for using Tsallis statistics, potential
problems and implications of our findings. Finally we offer our conclusions.

2 Tsallis statistics

Let us first consider the theoretical basis for statistical mechanics with non-
extensive entropy. Statistical mechanics for classical gases can be derived from
the Boltzmann-Gibbs assumption for the entropy, $S_{BG} = -k \sum p_i \cdot \ln p_i$, where
$p_i$ is the probability for a given particle to be in the state $i$, and the sum
is over all states. For normal gases the probability, $p(v)$, coincides with the
velocity distribution function, $f(v)$. This classical statistics can be generalized
to Tsallis (also called non-extensive) statistics (Tsallis 1988), which depends
on the entropic index $q$

$$S_q = -k \sum_i p_i^q \cdot \ln_q p_i. \tag{1}$$

Here the $q$-logarithm is defined by, $\ln_q p = (p^{1-q} - 1)/(1 - q)$, and for $q = 1$ the
normal Boltzmann-Gibbs entropy is recovered, $S_{BG} = S_1$. The probabilities
still obey, $\sum p_i = 1$, while the particle distribution function is now given by
f(v) = p^q(v). Thus, for q < 1 one privileges rare events, whereas q > 1 privileges common events. For a summary of applications see (Tsallis 1999), and for up to date list of references see http://tsallis.cat.cbpf.br/biblio.htm.

Average values are calculated through the particle distribution function, and one e.g. has the mean energy (Tsallis, Mendes & Plastino 1998)

\[ U_q = \frac{\sum p_i^q E_i}{c_p}, \tag{2} \]

where \( c_p = \sum p_i^q \), and \( E_i \) are the energy eigenvalues. Optimization of the entropy in eq. (1) under the constraints leads to the probability (Silva, Plastino & Lima 1998, Tsallis, Prato & Plastino 2003)

\[ p_i = \frac{[1 - (1 - q)\beta_q (E_i - U_q)]^{1/(1-q)}}{Z_q}, \tag{3} \]

where \( Z_q \) normalizes the probabilities, \( \beta_q = \beta/c_p \), and \( \beta \) is the optimization Lagrange multiplier associated with the average energy. Adding a constant energy, \( \epsilon_0 \), to all the energy eigenvalues leads to \( U_q \to U_q + \epsilon_0 \), which leaves all the probabilities, \( p_i \), invariant (Tsallis, Mendes & Plastino 1998). By defining

\[ \alpha = 1 + (1 - q)\beta_q U_q, \]

eq. (3) can be written as

\[ p_i = \frac{(1 - (1 - q)(\beta_q/\alpha) E_i)^{1/(1-q)}}{Z'_q}, \tag{5} \]

and we see that the probabilities have the shape of q-exponential functions

\[ p_i = \frac{\exp_q (-\beta'_q E_i)}{Z'_q}, \tag{6} \]

where

\[ \beta'_q = \frac{\beta_q}{\alpha}. \tag{7} \]

For q = 1 one recovers the standard Maxwell distribution with \( p_i \sim \exp(-\beta E_i) \).

We now note a very important detail, namely that the distribution function contains the ‘pseudo-inverse temperature’, \( \beta'_q \), which differs from the real temperature \( \beta_q = \partial S_q/\partial U_q, \) as described in eq. (7). If one was blindly to fit a given
spectrum with the shape of eq. (5), then one gets the observed \( \beta'_q \), which is not the true (inverse) temperature. Instead, the true (inverse) temperature is \( \beta_q = \beta'_q \cdot \alpha \) (for a discussion on temperature in non-extensive statistics see e.g. (Rama 2000, Martinez, Pennini & Plastino 2001)).

In section 4 we will discuss in more detail why and where one should expect Tsallis statistics to be important for cluster physics.

A brief summary of our findings so far is the following. For particles with long-range interactions (e.g. charged particles or particles with gravitational interactions only) the velocity distribution function may in general be different from the normal Maxwell distribution, \( f(v) \sim \exp(-\beta E) \), and will instead follow the shape given in eq. (5) with \( f(v) = p^q(v) \), for systems in kinetic equilibrium. It is important to keep in mind that these distributions are in kinetic equilibrium, and they are therefore stable even though the equilibration timescale for electrons and protons is much smaller than the age of the cluster. The observed pseudo temperature is generally different from the true thermodynamic temperature, and differs by the value of \( \alpha \) in eq. (4).

### 3 An example: Coma

To show the signatures of non-extensive statistics, we now wish to consider observed X-ray data of a cluster structure, and we consider the central part of NGC 4874, which is near the center of the Coma cluster (Arnaud et al. 2001). This is just to exemplify how one can extract the true cluster temperature. In figure 1 we plot the continuum part of the spectrum for two radial bins near the center of NGC 4874, namely the center-most bin, where the temperature was observed to drop to approximately 6.6 keV, and also the third bin, where the temperature plateau is already reached with \( T \approx 8.5 \) keV (Arnaud et al. 2001). By ‘continuum part’ we simply consider the energy ranges (in keV) 0.2–0.5, 3.5–3.7, 4.25–6.3, 7.2–7.5, and 8.7–10. The data is binned to have signal to noise ratio of 10 sigma, and other binnings leave the spectral shape intact. The horizontal error-bars represent the bins in energy. The plotted data is background subtracted and convolved with the instrument response matrix. For this example we are going to assume (incorrectly) that the response matrix is unity, and can therefore compare theoretical curves directly with the data-points. We emphasize that this is only an example, since the detector response drops at high energies. As is clear from the (green) triangles on the figure, the outer radial bin with particles in the temperature plateau are well fitted with a normal exponential, corresponding to \( q = 1 \), that is, the continuum is well fitted with a single exponential (straight dashed line) which implies that the normal temperature concept is correct. This is contrasted with the inner radial bin (red circles), where an exponential (straight dashed
Fig. 1. The continuum part of the spectra for two radial bins near NGC 4874. The (green) triangles from the non-central region is well fitted with an exponential, corresponding to $q = 1$. The (red) circles are not well fitted with an exponential in the high energy part. Instead, using formula (5) we find agreement when using $q = 0.87$. This corresponds to the true temperature being overestimated by approximately 20%. This figures serves only as an illustration of the signatures of non-extensive statistics, and the fitted $q = 0.87$ serves only as an example since the detector response is not included.

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One should be very careful when trying to directly compare to previous temperature measurements. First of all the cluster temperatures are normally found by a full spectral analysis, and not trivially with fit-by-eye of the (approximately) exponential tail. The exponential tail of the photon counts does come from the exponential tail of the distribution of the plasma particles (Rybicki & Lightman 1979), and for a generalized q-exponential the photon count will follow the generalized shape of eq. (6). The same statement holds true for non-thermal bremsstrahlung, and also in our case must the Gaunt factors be recalculated. Also the spectral lines are expected to be
different in non-extensive thermo-statistics (Silva, Plastino & Lima 1998). A proper implementation and comparison is beyond the scope of this short article, so we will leave that for a future analysis. From figure 1 we see that the non-extensive curve (solid line) is slightly above the exponential continuum curve for low energies, $E < 5$ keV, and it will be interesting to study in detail if this is connected to the observed soft excess.

4 Reasons for using Tsallis statistics

An important question to ask is naturally if there is any reason for considering the non-extensive thermo-statistics, that is, why fit the continuum with eq. (5) and not just with a simple exponential? We will now explain that the answer to this question, from a theoretical point of view, is a definite yes.

Systems with particles having point interaction generally have extensive entropy, however, systems with long-range interactions do not have extensive entropy, e.g. dark matter experiences long-range gravitational interaction (only), and do therefore not obey the rule of extensivity.

The electrons and protons in clusters of galaxies experience two forces, namely the attraction and repulsion from charges when the particles are nearby each other, and the gravitational attraction when the particles are charge screened. Both these forces are long-range, and we will consider them in turn.

Charged particles have long-range repulsion and attraction, and it is known that a pure electron plasma in 2 dimensions is correctly described by Tsallis statistics and has entropic index $q = 1/2$ (Huang & Driscoll 1994, Boghosian 1996). Laboratory experiments of inverse bremsstrahlung absorption in plasma certainly exhibit non-Maxwellian form (Liu et al. 1994), and are in fact well described by the generalized form in eq. (5) (Tsallis & de Souza 1997). Generally, the Tsallis statistics applies very well to a variety of astrophysical plasma environments, see e.g. (Leubner 2002) for references. One should therefore expect that charged electrons and protons, as in galaxy clusters, may have entropic index different from unity when the interactions are purely due to the charges, i.e. for distances shorter than the distances on which the charges are neutralized.

On larger distances the only force is the gravitational attraction. It is well-known that the velocity distribution for particles in an equilibrated structure with only gravitational interactions do not follow a Maxwell distribution. In fact, for the simplest structures, polytropes, the distribution function is exactly the Tsallis distribution of eq. (6) (Plastino & Plastino 1993), and not
just any random non-thermal distribution function\(^1\). Infact, a simple connection between the density slope, \(\gamma = d\ln \rho / d\ln r\) and the entropic index, \(q\), can be found (Hansen et al. 2004), \(q = (10 + 3\gamma)/(6 + \gamma)\). Here one sees directly that structures with density slope of \(\rho \sim r^{-2}\) have \(q \approx 1\), and hence normal extensive thermo-statistics provides a correct description. Hence, when the density slope is about \(-2\) (as in the outer part of a beta-model), then normal thermo-statistics applies. Further, if the particles are not fully ionized then particle interactions are again point like, and hence one should expect normal extensive thermodynamics to give a correct description in the outer cluster region.

However, the inner part of a cluster may have a density slope more shallow than \(-2\), and one should therefore expect that the entropic index should differ from unity in the central cluster region.

One may wonder if centrally disturbing objects (like a cD galaxy) may be related to a different entropic index. Observationally the very central region near the cD galaxy M87 is known to not follow a single phase structure locally (Matsushita et al. 2002). This could come about since the timescale for reaching kinetic equilibrium (correctly described by non-extensive thermodynamics) is much shorter than the timescale for reaching thermal equilibrium. The thermalization of energetic particles in a cold gas was in fact shown numerically (Waldeer & Urbassek 1991) to lead to power-law distributions, which eq. (5) indeed is (Tsallis 1999). This question of timescales can be addressed by considering the spectra from shocks of merging clusters which are still dynamically young. If our interpretation with rapidly achieved kinetic equilibrium is correct, then such dynamically young structures may have entropic indices different from unity. Another approach would be a numerical study of the equilibration of electron-proton plasma in an external gravitational field, however, that is a numerically very hard challenge.

To briefly summarize this section, then there are growing theoretical indications that the central region of clusters very well may have an entropic index different from unity. Since the forces involved are gravity and electromagnetic attraction and repulsion, which both are known (experimentally, numerically and theoretically) to have distribution functions of the Tsallis type, then it is very likely that the electron distribution in fact does follow the shape in eq. (6).

\(^1\) The Tsallis shape of the distribution function has also been seen directly from cosmological N-body simulations, and the \(\alpha\) from eq. (4) may in fact be crucial for the understanding of the observed relation between the anisotropy and the density slope of structures of purely gravitating particles (Hansen & Moore 2004).
5 Implications

We saw above that the observed pseudo temperature, $\beta'_q$, differs from the true thermodynamic temperature, $\beta_q$. How big can this difference be? It is known (Lima, Silva & Plastino 2001) that $q$ is bounded from below by $q > 0$, and if we adopt $U_q\beta_q = 3/2$ (Boghosian 1999, Hansen et al. 2004), then we see that $\alpha$ is bounded from above by $\alpha = 5/2$, and hence the true temperature may be a factor $5/2$ smaller than the observed temperature. Similarly, to assure positiveness of thermal conductivity Boghosian (Boghosian 1999) found for an ideal gas that $q$ is bounded from above by $q < 7/5$, which implies a lower bound of $\alpha > 2/5$. E.g. cooling flow clusters have been observed to have a central temperature which is at most a factor 3 below the the peak temperature (Kaastra et al. 2004, Peterson et al. 2003), however, the true central temperature may thus be different by a factor of $2/5$ from the observed value. This may be either larger or smaller depending on the value of $q$, and may thus reduce or increase the cooling flow problem.

If our proposed component to the understanding of cluster temperatures indeed turns out to be correct, then it will have numerous implications. All quantities that depend on the velocity distribution function of electrons will be different, this includes in particular the cooling function, heat conduction coefficient, Gaunt factors, and the Sunyaev-Zeldovich effect. The temperature of the Coma cluster was estimated to be $T_{SZ} = 6.6 \pm 13$ keV, using purely the SZ observations from the Coma cluster (Hansen 2004). Worse yet, the normal connections between thermodynamical quantities (e.g. how temperature is related to pressure and density) will have to be reconsidered, since they were derived under the normal assumption of extensive entropy. Clearly this may affect the cooling flow problem, however, we leave a detailed study of this aspect for the future. Since the derived temperature may differ from the true thermodynamic temperature, then the M-T and L-T relations will be affected. One could imagine that the smaller structure will have a relatively larger fraction of the cluster which should be described by non-extensive thermodynamics, in which case the lower temperature clusters have a true temperature which is lower than the observed one. This could push the low-T tail of the L-T relation towards the theoretically expected values. Also the tests of DM structure will be different (Lewis, Buote & Stocke 2003, Pointecouteau et al. 2004) where e.g. a lower central temperature (as exemplified above in NGC 4874) would lead to a shallower DM profile.

Let us finalize by reconsidering the spectral lines (Silva, Plastino & Lima 1998). The temperature determination from the 'exponential' tail is very accurate, but it is always nice with an alternative consistency check. The temperature can in principle be measured directly from the Doppler broadening of the spectral lines (Rybicki & Lightman 1979), e.g. through the Doppler width.
\[ \Delta \nu_D = \nu / c \sqrt{2 \ln 2 / m\beta}, \]
where \( \beta \) is the inverse temperature. In the generalized case this becomes

\[ \Delta \nu_D = \nu / c \sqrt{-2\alpha / m\beta q \ln_q (1/2)^{1/q}}. \]  

(8)

This method requires somewhat better energy resolution than what we have today. This method requires that the line shape is sufficiently accurately determined that one can separate turbulent gas motion from the microphysical distribution function. Other methods of temperature determination, e.g. through line ratios is more complicated to calculate for the generalized statistics, but may be interesting to consider given that we already have sufficient resolution to use this (Nevalainen et al. 2003).

6 Conclusions

We propose to use non-extensive thermo-statistics to analyse galaxy clusters. We explain that there is growing theoretical indications that this Tsallis statistics is the correct statistics to consider in the central part of cosmological structures.

The use of such generalized thermo-statistics leads to a difference between the observed and the true temperature as explained through eqs. (4, 7). We show that this difference may be as big as a factor 5/2. Thus, the observed low central cluster temperature may be different from what we believe today, both higher or lower. In general the M-T and L-T relations will be affected, as well as the central DM slope derived through hydrostatic equilibrium may be either more shallow or steeper.

We show how a correct temperature determination can be done either through the use of the quasi-exponential continuum spectrum, or through the Doppler broadening of the spectral lines.

We have raised more questions than given answers, and it will be very interesting in the near future to test our proposal on dynamically young structures, such as ongoing mergers and shocks in galaxy clusters, which are only kinetically equilibrated.
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