Exact Solution of the Ising Model on Group Lattices of Genus $g > 1$

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Abstract

We discuss how to apply the dimer method to Ising models on group lattices having non trivial topological genus $g$. We find that the use of group extension and the existence of both external and internal group isomorphisms greatly reduces the number of distinct Pfaffians and leads to explicit topological formulas for their sign and weight in the expansion of the partition function. The complete solution for the Ising model on the Klein lattice group $L(2,7)$ with $g = 3$ is given.

PACS N. 05.50
I. INTRODUCTION

Among the known approaches to the exact evaluation of the 2D Ising partition function which have followed the celebrated Onsager solution [1], the dimer method [2–4] fully exploits the combinatorial and group theoretical properties of the lattices by relating the partition function $Z$ to the generating function $Z_d$ of close-packed dimer configurations. Though the method is in principle independent from the dimensionality of the lattice, the corresponding analysis in three dimensions has never been developed, due to the difficulties in extending Kasteleyn’s Theorem on lattice orientation. In this paper, we deal with the issue of generalizing the dimer approach to the case of non-abelian 2D lattices of high topological genus which should in fact be equivalent to higher dimensional lattices. Indeed the 3D cubic lattice can be considered as a handlebody 2D lattice of genus $g = N/4$ where $N$ is the number of sites of the lattice. This hints that a non vanishing ratio $g/N$ may be related to an effective dimension $D > 2$ of the lattice. The great difficulty of the problem suggests that a possible concrete way to analyze such lattices is to consider at first graphs possessing the largest possible symmetry group, and the best candidates appear to be the group graphs already well-known in the mathematical literature [5–8]. In this framework we may consider the 2D planar lattice with periodic boundary conditions as an example of abelian lattice group (translational symmetry group) with genus $g = 1$, whereas an example of finite non-abelian lattice with $g = 0$ is given by the fullerene–like lattice studied in ref. [9].

The paper is organized as follows. In Sec.II we outline the basic ideas concerning application of the dimer method to group lattices. In Sec. III we define an extended lattice group $H$ and relate it to the homology and cohomology groups (mod2) $H_1, H^1$ of the original lattice $\Lambda$, define the Grassmann algebra over the decorated lattice $\Lambda^\#$ and the Pfaffians as function on orbits of $H^1$. In Sec.IV we apply the results of Sec. III to discrete groups of particular interest such as the Klein group $L(2,7)$ (of order $N = 168$ and genus $g = 3$) and discuss the role of external automorphisms. In Sec.V we analyze in detail the orbits of
under the action of \( H \), define an invariant duality map \( \varphi : H \to H^1 \) and auxiliary functions of relevant computational interest. Finally, in Sec. VI we construct explicitly the irreps of \( H \), apply them to the harmonic analysis on \( \Lambda^\# \) and derive the expansion for \( Z \). Few preliminary numerical results are also given.

II. GROUP LATTICES AS ISING LATTICES

We recall here key points of the dimer procedure which are specific to group lattices (see also refs. [10]):

(1). A discrete group \( G \) is defined by a presentation given in terms of a set of \( p \) generators \( A_k, \ k = 1, \ldots, p \) and \( t \) relators \( P_i, \ i = 1, \ldots, t \). The \( P_i \) are words in the generators equivalent to the identity. Let \( \mathcal{F} \) be the free group on the \( A_k, k = 1, \ldots, p \). Let \( \mathcal{N} \) be the minimal normal subgroup of \( \mathcal{F} \) containing all relators. Then by definition \( G = \mathcal{F}/\mathcal{N} \).

(2). The Cayley lattice \( \Lambda \) for a group is defined by giving a map \( L : G \to \mathbb{R}^3 \) where \( L(g) \in \mathbb{R}^3 \) is the point of \( \mathbb{R}^3 \) corresponding to the group element \( g \in G \). A bond of color \( k \) is then a line joining \( L(g) \) to \( L(A_k g) \). The genus \( g \) of \( \Lambda \) is that of the surface of minimal genus \( S \subset \mathbb{R}^3 \) on which \( \Lambda \) can be drawn. The Ising Hamiltonian is then defined as

\[
E = - \sum_{h \in G} \sum_{k=1}^{p} J_k \sigma_h \sigma_{A_k h},
\]

where the \( \{ \sigma_h = \pm 1 \} \) are the spin variables and \( \{ J_k \} \) the exchange interactions between connected spins.

(3). Each relator \( P_i \) is then represented on \( \Lambda \) as a closed circuit \( \zeta(P_i) \) made of oriented colored bonds. If \( \zeta(P_i) \) encloses a simply connected region (tile) on \( S \) then \( P_i \) is called local relator. If \( S \) has genus \( g = 0 \) then all relators are local. Of particular interest are models where \( g \) is large.

(4). The group lattice \( \Lambda \) is interesting on its own but cannot be related directly to the partition function of an Ising model and to do this we must consider a decorated lattice \( \Lambda^\# \). This amounts to replace each site of coordination \( q (q > 2) \) of the original lattice by
a sublattice containing 3(q − 2) points and 4q − 9 decorating bonds. The Ising partition function is then related to the dimer covering generating function on the decorated lattice.

(5). In order to compute the generating function we orient \( \Lambda^\# \) according to the Kasteleyn prescription by assigning arrows to each bond inherited from \( \Lambda \) in such a way that for any closed circuit \( \ell \) on \( \Lambda^\# \), the number of bonds of \( \ell \) oriented clockwise is of opposite parity to the number of sites enclosed by \( \ell \). For the decorating bonds see [4] or Sec.V.

(6). The Kasteleyn rules define completely the orientation for lattices of genus \( g = 0 \), whereas for lattices of higher genus we have to deal with further sign fixing for loops which are not homologically trivial, i.e. not the boundary of a union of tiles. The assignment of arrows to \( \Lambda^\# \) (or \( \Lambda \)) is not invariant under the action of \( G \) but rather under an extension \( H \) of \( G \) closely related to the homology \( H_1 \) and cohomology \( H^1 \) groups of \( \Lambda \).

(7). The dimer covering generating function of the lattice can be expressed as a weighted sum of Pfaffians \( Pf(\phi) \), where \( \phi \in H^1 \) and with sign given explicitly by the \( H \)-invariant function \( \theta(\zeta) \), \( \zeta \in H_1 \) defined in Sec.V. Harmonic analysis on \( H \) allows to factorize Pfaffians into determinants of lesser order and external automorphisms induce identifications between Pfaffians.

III. THE EXTENDED LATTICE GROUP

In this section we first discuss the group extension \( H \) of \( G \). Next we show that \( H \) partitiones the homology \( H_1 \) and the cohomology \( H^1 \) groups of \( \Lambda \) into non-intersecting orbits characterized in terms of sign functionals. Their role in the expansion of the dimer generating function is then analyzed.

A. The groups \( H, H_1 \) and \( H^1 \)

The extended lattice group \( H \) can be obtained from \( G \) by replacing the relators \( P_i \) with new relators containing the following elements:
(i) If $P_i, P_j \in G$ are local, then $P_i P_j^{-1}$, $P_i^2$, $P_i A_k P_i^{-1} A_k^{-1}$ are relators in $H$;

(ii) For generic relators $P_i, P_j \in G$ then $P_i^2$, $P_j^2$, $P_i P_j P_i^{-1} P_j^{-1}$ are relators in $H$.

We can write then $P_j = Q$ ($Q^2 = 1$) for all local relators $P_j$, with $A_k Q = QA_k$, i.e. $Q$ is a central element that we call central signature. Putting $Z_i = P_i Q$ the non local relators will be written as $Z_i^2$, $Z_i Z_j Z_i^{-1} Z_j^{-1}$ where in general $A_k Z_i \neq Z_i A_k$. The $\{Z_i\}$ generate an abelian normal subgroup $HZ \subset H$. Particular examples of this extension will be discussed further on.

Non trivial loops form a chain group $C_1(\Lambda^#, Z_2)$. The equivalence classes of $C_1(\Lambda^#, Z_2)$ modulo boundaries form the homology group $H_1(\Lambda^#, Z_2)$. The class of multiplicative functionals on $H_1(\Lambda^#, Z_2)$ with values $\pm 1$ are then the elements of the cohomology group $H^1(\Lambda^#, Z_2)$. Notice that $H^1(\Lambda^#, Z_2) \sim H^1(\Lambda, Z_2)$ and $H_1(\Lambda^#, Z_2) \sim H_1(\Lambda, Z_2)$ and therefore we denote them briefly by the symbols $H^1, H_1$ respectively. When dealing with elements of $HZ, H_1$ we may use addition instead of the product as composition rule. We write $g \sim g'$ if $g, g' \in H$ define the same site on $\Lambda$.

$H_1$ is isomorphic to $HZ$. To see it consider the closed chain $\ell$ on $\Lambda$ as defined by the sequence $g_0 = g$, $g_p = h_p g_{p-1}$, $p = 1, \ldots, n$ with $h_{n+p} = h_p$. $\zeta(\ell)$ is then defined as

$$\zeta(\ell) = g_0^{-1} h_n h_{n-1} \ldots h_1 g_0 = g_p^{-1} h_{p-1} h_{p-2} \ldots h_1 h_n h_{n-1} \ldots h_{p+1} g_p .$$

Starting from $g_0$ we move $n$ steps across bonds in $\Lambda$ and each bond defines an element $h_i \in H$, where $h_i = A_k^{\pm 1}$, $A_k$ being a generator of $H$. If $g_n \sim g_0$, $\ell$ is closed, $\zeta(\ell) \in HZ$ does not depend on the choice of $g_0$ on $\ell$ and defines a map $\zeta : C_1(\Lambda, Z_2) \rightarrow HZ$. Adding a boundary $\ell_0$ to $\ell$ (i.e. adding the boundary of a union of tiles on the lattice) amounts to replace a sequence of bonds $h_{i_1}, \ldots, h_{i_n}$ by an equivalent one obtained by using local relators only, therefore $\zeta(\ell + \ell_0) = \zeta(\ell)$ and $\zeta$ induces the map $\kappa : H_1 \rightarrow HZ$. Given inversely an element $\zeta \in HZ$, expressed in terms of relators as $\zeta = h_n \ldots h_1$, a corresponding closed chain $\ell(\zeta) \in C_1(\Lambda, Z_2)$, and therefore a cycle in $H_1(\Lambda, Z_2)$, is given by a sequence of $g_i$, $i = 0, \ldots, n$, with $g_0 = 1$ and $h_i g_{i-1} = g_i$. Also if $\zeta = h_n \ldots h_1$, $\zeta' = h_{m+n} \ldots h_{1+n} \in HZ$
and \( g_n \sim g_{m+n} \sim 1 \) the chain \( \ell(\zeta + \zeta') = \ell(\zeta) + \ell(\zeta') \in C_1(\Lambda, \mathbb{Z}_2) \) defined by \( g_0 = 1 \) and \( h_i g_{i-1} = g_i, i = 1...m + n \) corresponds to the element \( \zeta + \zeta' \in HZ \). Hence \( \kappa \) is a group isomorphism.

Right multiplication on \( \Lambda \) by an element \( h \in H \) translates the group lattice and if we replace \( g_i \) with \( g_i h \) in \( \ell \) we obtain a closed path \( \ell_h \) which is the right translation of \( \ell \) by \( h \) and a corresponding element \( \zeta(\ell_h) = h^{-1} \ell h \) which defines the action of \( H \) on \( \ell \in HZ \). In this way \( H \) acts naturally on \( \Lambda^\# \), \( H_1 \) and \( H^1 \) and partitions \( H^1, H_1 \) into non–intersecting orbits. The use of orbits greatly simplifies the computation of \( Z \).

**B. Sign functionals and lattice orientation**

We assign an orientation to \( \Lambda^\# \) according to the Kasteleyn rules and to each site \( h \in H \) a Grassman variable \( a(h) \), with an anticommuting wedge product \( a(h) \wedge a(h') = -a(h') \wedge a(h) \).

Reversing the arrows on all bonds sharing the same site \( i \) corresponds to the change \( a(h) \to -a(h) \).

Let \( \ell \) given by (2). Each \( h_i \) is of the form \( A_{p_i}^{k_i} \) where \( p_i = \pm 1 \) determines the orientation of the arrow in the bond. Closing \( \ell \) means that we must identify \( a(g_0) \) and \( a(g_n) \) as

\[
a(g_0) = p_n a(g_n)
\]  

We define then

\[
\eta(\zeta(\ell)) = -\prod_{i=1}^n p_i
\]  

If \( \zeta \) is trivial then \( \eta(\zeta) = 1 \). All \( \eta(\zeta) \) defined in this way, hereafter called sign functionals, are characterized by a particular recursion relation which can be proved as follows. Following ref. [4] we introduce a reference dimer configuration \( C_0 \) on \( \Lambda^\# \) consisting of all bonds of \( \Lambda^\# \) inherited from \( \Lambda \). By definition, any other dimer configuration \( C \) when superposed on \( C_0 \) generates transition cycles \( \zeta \in H_1 \). In this proof we use transition cycles only. We consider representative chains \( \ell, \ell', \ell'' \in C_1(\Lambda^\#, \mathbb{Z}_2) \) of the pair of intersecting cycles \( \zeta, \zeta' \) and of the
cycle $\zeta + \zeta'$ as in Fig.1. Chains $\ell, \ell'$ run over the sequence of sites $P_i$, $i = 0, 1, 2, 7, 8, 9$ and $i = 5, 6, 7, 2, 3, 4$ respectively, while $\ell''$ runs over $i = 0, 1, 3, 4$ and $i = 5, 6, 8, 9$ but not $i = 2, 7$. We denote with $B_{ij}$ the bonds joining $P_i, P_j$ and with $n(\ell)$ the number of anticlockwise arrows on $\ell$. The key point is that bonds $B_{12}, B_{13}$ are not contained in the intersection of $\zeta$ and $\zeta + \zeta'$ and thus $B_{12}, B_{13} \notin C_0$ whereas $B_{01} \in C_0$. Similarly $B_{34}, B_{27}, B_{56}, B_{89} \in C_0$. By orienting $\Lambda^#$ according to the Kasteleyn rules and counting arrows we see that $n(\ell'') \equiv n(\ell) + n(\ell') + 1 \pmod{2}$. The general case where $B_{ij}$ are replaced by sequences of bonds corresponds to the addition of boundaries to $\ell, \ell', \ell''$ and leads to the same formula. Hence $n(\ell)$ is a function in $H_1$ and $\eta(\zeta + \zeta') = -\eta(\zeta)\eta(\zeta')$ if $\zeta, \zeta'$ intersect and $\eta(\zeta + \zeta') = \eta(\zeta)\eta(\zeta')$ otherwise, i.e.

$$\eta(\zeta + \zeta') = \eta(\zeta)\eta(\zeta')(\omega(\zeta, \zeta') \cdot -1)^\Omega(\zeta, \zeta') \quad ,$$

where $\Omega(\zeta, \zeta') = 0, 1$ is the intersection number (mod2) of $\zeta, \zeta'$. Given a sign functional $\eta(\zeta)$ then $\phi(\zeta)\eta(\zeta), \phi \in H^1$, is another sign functional and so is the $H$–invariant function $\theta(\zeta), \zeta \in H_1$ defined in Sec.V.

C. Pfaffians and dimer covering generating function

Consider now the form

$$f = \frac{1}{2} \sum_{h, h' \in \Lambda^#} x_{h, h'} a(h) \wedge a(h') \quad ,$$

where $x_{h, h'} = 0$ if $h$ and $h'$ are not connected in $\Lambda^#$ (neighbors). The factor $\frac{1}{2}$ is inserted in order to avoid double counting, the activity is given by $x_b = x_{h, h'} = -x_{h', h} = \coth(\beta J_{h, h'})$ if the bond $b$ is oriented with the arrow from $h$ to $h'$, $\beta$ is the inverse temperature and $J_{h, h'}$ the exchange interaction between spins ($J_k$ in the previous notation) which depends only on the color $k$ of the bond. Under the stated conditions decorating bonds have $x_{h, h'} = 1$. For the sole purpose of safe handling of signs we assume initially $x_b > 0$. All arrows on $\Lambda^#$ and all signs in (3) are then fixed by the sign functional $\eta$. The Pfaffian $Pf(\eta)$ is defined by
where \( M = 3(q - 2)N \), \( N \) is the order of the group (or lattice size) and \( q \) is the lattice coordination. From (7) one derives the well-known relation

\[
\text{Pf}(\eta)^2 = \text{Det}(X(\eta))
\]  

(8)

where \( X(\eta) \) is the \( M \times M \) matrix of elements \( x_{ij} \). In general \( \text{Pf}(\eta) \) is multilinear function of the \( x_b \) considered as independent variables. The partition function \( Z \) is then given by

\[
Z = Z_d 2^N \prod_a \sinh(\beta J_a)^{1/2},
\]

(9)

where \( a \) runs on all the oriented bonds of the undecorated group lattice and

\[
Z_d = 2^{-g} \sum_\eta s_\eta \text{Pf}(\eta) \quad s_\eta = \pm 1.
\]

(10)

To each dimer configuration \( C \) we associate the contribution of \( C \) to \( \text{Pf}(\eta) \), a monomial \( M(C) = \prod_b x_b \) where \( b \) runs over all oriented bonds of \( C \). Because of our conventions \( M(C) \) has sign \( \eta(\zeta) \) where \( \zeta \) is the superposition of \( C \) and \( C_0 \). In general \( \zeta \) is the sum of non-intersecting cycles \( \zeta = \sum_i \zeta_i \) so that \( \eta(\zeta) = \prod_i \eta(\zeta_i) \). The signs \( s_\eta \) must then be chosen in such a way as to set equal to 1 in (10) the coefficient of all \( M(C) \) appearing in \( Z_d \). Once this is done in Sec.V the \( x_b \) can be given any sign.

**IV. 2D ISING LATTICE AND THE L(2,7) LATTICE GROUP**

The above group extension procedure can be naturally applied to a wide class of lattice groups. Here we discuss two examples: the abelian 2D Ising lattice (of genus \( g = 1 \)) and the \( L(2,7) \) Klein group. The latter is non-abelian and of genus \( g = 3 \), and its analysis should hopefully be of interest in the analysis of more general structures of dimension \( D > 2 \).

**A. 2D Ising lattice**

The Onsager solution for the 2D Ising lattice made use of a rectangular \( n \times m \) lattice with sites labelled by \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) (\( n \) even). In this lattice we identify
opposite sides, giving it a toroidal \(( g = 1)\) topology and turning in a group lattice \(G_{nm}\). This last property makes it possible to apply harmonic analysis, i.e. Fourier transform methods, which eventually lead to the final formula. The group \(G_{nm}\) of the lattice is defined by the presentation

\[
ST = TS, \quad S^n = 1, \quad T^m = 1, \quad (11)
\]

for all integers \(n, m\). However in order to satisfy the Kasteleyn rules we must use a central extension \(H_{nm}\) of (11) defined by:

\[
ST = QT S, \quad S^n = Z_1, \quad T^m = Z_2, \quad Q^2 = Z_1^2 = Z_2^2 = 1
\]

\[Q Z_1 = Z_1 Q, \quad Q Z_2 = Z_2 Q, \quad Z_1 Z_2 = Z_2 Z_1. \quad (12)\]

Sites are labeled by elements \(h \in H_{nm}\) with the condition

\[
a(h Q) = -a(h) . \quad (13)
\]

The \(H_{nm}\) lattice is actually a covering of the \(G_{nm}\) lattice with corresponding variables \(a(h)\) identified by the above condition which embodies the Kasteleyn rules.

Besides this conditions we must set

\[
a(h Z_1) = \epsilon_1 a(h), \quad a(h Z_2) = \epsilon_2 a(h), \quad (14)
\]

where \(\epsilon_1\) and \(\epsilon_2\) take the values \(\pm 1\) in all the possible combinations. These correspond to the \(2^{2g} = 4\) possible choices of \(\phi \in H^1\). It is tempting but misleading to label spin sites on the lattice by elements in \(G_{nm}\). In this case the orientation would no longer be invariant under \(G_{nm}\), a clear sign that the true symmetry is that of the extended group \(H_{nm}\).

**B. Non–abelian group lattices of genus \(g > 1\)**

We examine now other types of symmetries which require non–central group extensions and which are of interest for models in \(D > 2\). The natural further step is provided by
a vast array of discrete groups many of which are discussed in detail in the literature (see \[5–8\]). Of particular interest are the Klein groups \(L(2,p)\), where \(p\) is prime. The non-abelian group of genus \(g = 0\) (the fullerene lattice) studied in ref. \[9\] corresponds to \(p = 5\). The next interesting example which is at the same time non abelian and has a non trivial topology is given by the lattice group \(L(2,7)\) (also called \(T(2,3,7)\) in the context of hyperbolic tessellations). The \(L(2,7)\) group, briefly called \(G\), has order 168 and is defined by the presentation

\[
U^7 = 1, \quad V^2 = 1, \quad (UV)^3 = 1, \quad (VU^3)^4 = 1. \tag{15}
\]

The analysis of a lattice possessing this kind of symmetry is one of the chief results of this paper which hopefully opens the way to the investigation of more general and interesting structures. The group lattice given by \((15)\) has genus \(g = 3\) (produced by the non local relator \((VU^3)^4\)) and can be tessellated by 24 heptagons and 56 hexagons, for a total of 168 bonds of type \(U\) and 84 bonds of type \(V\) (see ref. \[11\], pag. 539–549, for more details on the lattice).

The group \(G_{nm}\) given by \((11)\) is abelian and has a non trivial genus \(g = 1\). On a lattice of genus \(g\) we expect that the Kasteleyn rule determines the orientation up to \(2^g\) signs with a total of \(2^{2g}\) configurations. Each close path on the lattice \(\Lambda\) can be considered as an element of the homology group \(H_1\) and an assignment of all remaining signs as an element of the cohomology group \(H^1\). It is therefore sufficient to assign signs on a suitable basis of \(2g\) elements of \(H_1\). In the toroidal case this was done by giving \(\epsilon_1, \epsilon_2\). In fact \(Z_1, Z_2\) are central in the group \(G_{nm}\).

The group \(G\), as defined by \((13)\), is non abelian and has a non trivial genus, in particular the \(Z_i\) are no longer central elements and we deal with the discussed non central extension \(H\) of \(G\) defined by :

\[
U^7 = Q, \quad V^2 = Q, \quad (UV)^3 = Q, \quad (VU^3)^4 = Q, \quad (VU^3)^4 = Z(1), \tag{16}
\]
and additional relators which can be expressed in terms of the auxiliary elements,

\[ Z(h) = h^{-1}Wh, \quad h \in H, \quad (17) \]

classified by

\[ Z(h_1)Z(h_2) = Z(h_2)Z(h_1), \quad (Z(h))^2 = 1, \quad \forall h_1, h_2, h \in H. \quad (18) \]

The \( Z(h) \) are not all independent and can be expressed in terms of the subset of \( 2g = 6 \) elements

\[ Z_n \equiv Z(U^n) = U^{-n}Z(1)U^n. \quad (19) \]

Clearly \( Z_n = Z_{n,(\text{mod }7)} \) and \( Z_0 = Z(1) = Z_7 \). The following identities equivalent under (18)

\[ Z_5Z_1Z_4Z_0Z_5Z_6Z_2 = 1, \quad Z_1Z_3Z_5Z_0Z_2Z_4Z_6 = 1, \quad (20) \]

reduces to 6 the number of independent elements \( Z_i \) and can be proved by repeated application of relators (16) but not of (18).

A generic \( \zeta \in H_1 \) can be always written in the additive form \( \sum n_iZ_i \) where \( n_i = 0, 1 \). Because of \( \sum_{i=0}^6 Z_i = 0 \) we can always choose the \( n_i \) in such a way as to have \( \sum_i n_i = 0 \) or 1. All elements \( Z(h), h \in H \), can be obtained by repeated conjugation of the \( Z_n \) by \( U \) and \( V \). Conjugation by \( U \) is trivial, i.e. \( U^{-1}Z_nU = Z_{n+1} \). Conjugation by \( V \) is less obvious. By using (16) and not (18) we find that

\[ V^{-1}Z_0V = Z_4, \quad V^{-1}Z_1V = Z_3^{-1}Z_0^{-1}, \quad V^{-1}Z_2V = Z_4^{-1}Z_2^{-1}Z_0^{-1}, \]
\[ V^{-1}Z_3V = Z_4^{-1}Z_4^{-1}, V^{-1}Z_4V = Z_0, \quad V^{-1}Z_5V = Z_5^{-1}, \quad V^{-1}Z_6V = Z_6^{-1}, \quad (21) \]

which can be further simplified by using (18). In the additive form (21) and (19) can we written in full generality as:

\[ h^{-1}Z_ih = \sum_{k=0}^6 P_{ik}(h)Z_k, \quad h \in H \quad (22) \]

where \( P(h) \) is a representation mod 2 of \( H \).
Moreover the group $H$ has an external automorphism $\nu$ given by:

\[
\begin{align*}
\nu(V) &= V^{-1} = -V \\
\nu(U) &= U^{-1} \\
\nu(Z_p) &= Z_{4-p}^{-1} = Z_{4-p}
\end{align*}
\] (23)

**V. ORBITS OF HOMOLOGY AND COHOMOLOGY GROUPS**

As anticipated, the analysis of orbits and of external automorphisms plays a central role in the computation of $Z_d$. We thus give here the explicit construction of such orbits in $H_1$ and $H_1$, together with their duality map.

A functional $\phi \in H^1$ can be defined by the equivalent conditions:

\[
a(gZ_i) = \epsilon_i a(g) , \quad \phi(Z_i) = \epsilon_i \quad , \quad i = 0, \ldots , 6 \quad ; \quad \prod_{i=0}^{6} \epsilon_i = 1
\] (24)

$\phi$ is then identified by $(\epsilon_0, \ldots , \epsilon_6)$. Let $\zeta \in H_1$ and let $h^{-1}\zeta h$ be the translated cycle. Then the translated functional $\phi_h$ is defined by

\[
\phi_h(h^{-1}\zeta h) = \phi(\zeta).
\] (25)

We have then:

\[
\phi_U(U^{-1}Z_iU) = \phi(Z_i) = \epsilon_i = \phi_U(Z_{i+1}) ,
\] (26)

and hence

\[
\phi_U(Z_i) = \epsilon_{i-1} , \quad \phi_{U^{-1}}(Z_i) = \epsilon_{i+1} .
\] (27)

Similarly we find

\[
\phi_V(V^{-1}Z_0V) = \phi(Z_0) = \epsilon_0 = \phi_V(Z_4) ,
\] (28)

and
\[ \phi_V(Z_0) = \epsilon_1, \quad \phi_V(Z_3) = \epsilon_1 \epsilon_4, \quad \phi_V(Z_2) = \epsilon_0 \epsilon_4 \epsilon_2, \]
\[ \phi_V(Z_1) = \epsilon_3 \epsilon_0, \quad \phi_V(Z_5) = \epsilon_1, \quad \phi_V(Z_6) = \epsilon_6. \]  
(29)

There are \( 2^{2g} = 64 \) (\( g = 3 \) denoting the genus of \( \Lambda \)) different elements in \( H^1 \) which fall in 5 different orbits containing 1, 7, 7, 21, 28 elements. We list these orbits with only one signature out of each cyclically permuted septet:

\[ A : \ (1, 1, 1, 1, 1, 1, 1) \quad \text{(trivial)} \]
\[ B : \ (1, 1, -1, 1, -1, -1, -1) \]
\[ C : \ (-1, -1, -1, -1, -1, 1, 1) \]
\[ D : \ (-1, -1, -1, -1, 1, -1, -1), (-1, -1, 1, 1, 1, -1, -1), (1, 1, -1, 1, -1, 1, 1) \]
\[ E : \ (-1, 1, -1, 1, -1, -1, -1), (1, -1, -1, 1, -1, -1, 1), (-1, 1, 1, 1, 1, 1, -1), \]
\[ (1, -1, 1, 1, 1, -1, 1) \]  
(30)

The orbits \( B, C \) are mapped into each other by the action of \( \nu \). This proves incidentally that \( \nu \) is external. We also list in similar fashion the dual orbits in \( H_1 \):

\[ A : \ (0, 0, 0, 0, 0, 0, 0) \quad \text{(trivial)} \]
\[ B : \ (1, 1, 0, 0, 1, 0, 1) \]
\[ C : \ (1, 0, 1, 0, 0, 1, 1) \]
\[ D : \ (0, 0, 1, 0, 1, 0, 0), (0, 0, 1, 0, 1, 1, 0), (0, 1, 0, 0, 0, 1, 0) \]
\[ E : \ (1, 0, 0, 0, 0, 1, 1), (1, 1, 0, 1, 1, 1, 1), (1, 1, 0, 0, 0, 1, 1), \]
\[ (1, 0, 1, 0, 1, 0, 1) \]  
(31)

as it can be checked by using (28)–(29).

For generic elements \( \zeta, \zeta' \in H_1 \) we use the additive form:

\[ \zeta = \sum_{i=0}^{6} n_i Z_i, \quad \zeta' = \sum_{i=0}^{6} m_i Z_i, \]  
(32)

and define the intersection number (mod2)
\[ \tau(\zeta, \zeta') = \tau(\zeta', \zeta) = (-1)^{\Omega(\xi, \xi')} . \] (33)

\[ \Omega(\xi, \xi') = \sum_{i,k=0}^{6} \chi(i-k) n_i m_k \] (34)

with \( \chi(i) = 1 \) if \( i \equiv 1, 2, 5, 6 \pmod{7} \) and \( \chi(i) = 0 \) otherwise, as it can be verified on the graph \( \Lambda^\# \). \( \tau(\zeta, \zeta') \) is invariant under conjugations by \( H \), see (21), and is multiplicative, i.e.

\[ \tau(\zeta, \zeta' + \zeta'') = \tau(\zeta, \zeta') \tau(\zeta, \zeta'') . \] (35)

We define also:

\[ \theta(\zeta) = (-1)^{\frac{1}{2}} \sum_{i,k=0}^{6} \chi(i-k) n_i n_k , \] (36)

so that \( \theta(Z_i) = \theta(1) = 1 \) and \( \theta(\zeta) \) is a sign functional as \( \eta \) in (6):

\[ \theta(\zeta + \zeta') = \theta(\zeta) \theta(\zeta') \tau(\zeta, \zeta') . \] (37)

All these definitions can be extended naturally to a wide class of lattice groups.

Let then :

\[ m_i = \sum_{k=0}^{6} \chi(i-k) n_k \] (38)

and define the duality map \( \varphi : H_1 \rightarrow H^1 \):

\[ \varphi \left( \sum_{i=0}^{6} n_i Z_i \right) = ((-1)^{m_0}, ..., (-1)^{m_6}) \] (39)

The \( 2^{2g} \) elements of \( H_1, H^1 \) are labeled as \( \zeta_I, \phi_I \), where \( \varphi(\zeta_I) = \phi_I \), by an index \( I = 1, ..., 2^{2g} \) with \( \zeta_1, \phi_1 \) the trivial elements and sorted in such a way that in (51), (50) \( \varphi \) maps corresponding orbits in \( H_1, H^1 \). Clearly then \( \phi_I(\zeta_K) = \tau(\zeta_I, \zeta_K) \). We set \( \eta_I(\zeta) = \phi_I(\zeta) \theta(\zeta) \), \( s_I = s_{\eta_I} \) so that we may label Pfaffians equally well with elements \( \phi \in H^1 \) and rewrite (10) as:

\[ Z_d = 2^{-g} \sum_I s_I Pf(\phi_I) . \] (40)
The importance of orbits should be clear once we realize that in (40) for elements \(\phi_I, \phi_J\) in the same orbit we have \(Pf(\phi_I) = Pf(\phi_J)\). In this way the effective number of different terms in \(Z_d\) reduces to the number of orbits in \(H^1\).

Further reductions arise from external automorphisms. When omitted, as in (40), we assume summation ranges on \(I, K\) to be \(1...2^{2g}\). Since \(\varphi\) commutes with the group operations it maps orbits in \(H_1\) into dual orbits in \(H^1\) and we use the same label \(A...E\) for pairs of dual orbits. Moreover the \(2^{2g} \times 2^{2g}\) matrix \(\Phi = \Phi^T\) of elements \(\Phi_{KI} = 2^{-g} \tau(\zeta_I, \zeta_K)\) is orthogonal.

We have first of all:

\[
\sum_I \Phi_{KI}^2 = 2^{-2g} \sum_I 1 = 1 .
\]  

(41)

We have next:

\[
p(K, K') = \sum_I \Phi_{KI} \Phi_{K'I} = 2^{-2g} \sum_I \phi_K(\zeta_I) \phi_{K'}(\zeta_I) = 2^{-2g} \sum_I \phi_K(\zeta_I + \xi) \phi_{K'}(\zeta_I + \xi)
\]

(42)

therefore \(p(K, K') = 0\) if \(\exists \xi : \phi_K(\xi) \phi_{K'}(\xi) \neq 1\). But if \(\forall \xi : \phi_K(\xi) \phi_{K'}(\xi) = 1\) then \(K = K'\) therefore \(p(K, K') = 0\) if \(K \neq K'\). This proves that \(\Phi\) is orthogonal and that \(\Phi = \Phi^{-1}\).

We conclude this section with some rather technical formulas whose rôle will be crucial in what follows (sect. VIII).

The sum \(\sum_K \theta(\zeta_K)\) can be readily evaluated by using a standard basis for \(H_1\) given by \(X_i, Y_i, \ i = 1...g\) such that

\[
\theta(\sum_i (p_i X_i + q_i Y_i)) = (-1)^{\sum_i p_i q_i}
\]

(43)

where for instance:

\[
X_1 = Z_0, Y_1 = Z_1, X_2 = Z_1 + Z_5, Y_2 = Z_0 + Z_2 + Z_5, X_3 = Z_0 + Z_3, Y_3 = Z_1 + Z_3 + Z_6
\]

(44)

In this form \(\sum_K \theta(\zeta_K)\) factors into \(g\) independent and equal sums each yielding a factor 2 and hence \(\sum_K \theta(\zeta_K) = 2^g\).

Furthermore:
\[ 2^{-g} \sum_K \tau(\zeta_I, \zeta_K) \theta(\zeta_K) = 2^{-g} \theta(\zeta_I) \sum_K \theta(\zeta_K + \zeta_I) = \theta(\zeta_I). \] (45)

since \( \zeta_K + \zeta_I \) runs over the whole group \( H_1 \) taking every element once just as \( \zeta_K \) and therefore \( \sum_K \theta(\zeta_K + \zeta_I) = \sum_K \theta(\zeta_K) = 2^g \).

VI. IRREPS OF \( H \)

The group \( H \) has \( 168 \times 64 = 10752 \) elements and in order to perform harmonic analysis and obtain partial block diagonalization of \( X(\phi) \) and of Pfaffians we must find the unitary irreducible representations (irreps) of \( H \).

The trivial functional \((1,1,1,1,1,1,1)\) must be dealt with separately and requires the construction of the irreps of the factor group \( H_0 \) of \( H \) and central extension of \( G \) defined by

\[ U^7 = Q, \quad V^2 = Q, \quad (UV)^3 = Q, \quad (VU^3)^4 = Q, \] (46)

(i.e. \( Z_i = 1 \)) of order \( 168 \times 2 = 336 \). Therefore we have the relations :

\[ a(gZ_i) = a(g), \quad a(gQ) = -a(g). \] (47)

Because of (47), only a subset of the unitary irreps of \( H_0 \) is actually used in the harmonic analysis. In order to see it let us write such irreps as \( D^J_{\alpha\beta}(g) \) with \( J \) a convenient label and \( \alpha, \beta = 1, \ldots, d_J \) where \( d_J \) is the dimension of the irrep. The matrix elements satisfy the orthogonality relations:

\[ \sum_{g \in H_0} D^J_{\alpha\beta}(g)^* D^{J'}_{\alpha'\beta'}(g) = \delta^{JJ'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \] (48)

Let us define

\[ a^J_{\alpha\beta} = \sum_{g \in H_0} a(g) D^J_{\alpha\beta}(g). \] (49)

Applying (47) to (49) we get

\[ \sum_{g \in H_0} a(gQ) D^J_{\alpha\beta}(g) = -a^J_{\alpha\gamma} = a^J_{\alpha\gamma} D^J_{\gamma\beta}(Q). \] (50)
But $Q$ is a central element and $Q^2 = 1$. Therefore by Schur lemma $D^J_{\gamma\beta}(Q)$ must be proportional to the identity, i.e.

$$D^J_{\gamma\beta}(Q) = \kappa \delta_{\gamma\beta}, \kappa = \pm 1,$$

therefore from (50)

$$\kappa a^J_{a\gamma} = -a^J_{a\gamma},$$

i.e. $\kappa = -1$. It follows that only irreps having $\kappa = -1$ actually contribute to the Fourier expansion of $a(g)$. Those irreps which are also irreps of the original group $G$ (i.e. those with $\kappa = 1$) are absent from the expansion. There are only 5 irreps of $H_0$ with $\kappa = -1$ of dimension 4, 4, 6, 6, 8, satisfying separately the Burnside condition $4^2 + 4^2 + 6^2 + 6^2 + 8^2 = 168$. The detailed form is listed in Appendix A. The matrix elements of these irreps are polynomials in $K = \exp(i\pi/7)$ with $K^7 = -1$. By abuse of language we may write

$$Q = -1, \quad U^7 = -1, \quad V^2 = -1, \quad (UV)^3 = -1, \quad (VU^3)^4 = -1,$$

instead of (46).

As for the remaining irreps of $H$ the most convenient way is to obtain them as induced representations on the cosets of the subgroup $L$ of $H$ generated by:

$$V, \quad U^{-1}VU^2, \quad U^{-6}VU^3, \quad U^{-4}VU^4, \quad U^{-5}VU^5,$$

as well as the cosets of the group $\nu(L)$ generated by:

$$V^{-1}, \quad UV^{-1}U^{-2}, \quad U^6V^{-1}U^{-3}, \quad U^4V^{-1}U^{-4}, \quad U^5V^{-1}U^{-5}.$$ (55)

All generators and their inverses are of the form $U^{a(b)}V^b$, $b = 0, \ldots, 6$. Let set briefly $v = V$ and $t = U^{-1}VU^2$; we find then

$$U^{-6}VU^3 = vt, \quad U^{-4}VU^4 = Z_1(tv)^2 = (tv)^2Z_4$$

$$U^{-5}VU^5 = vt(vt^{-1})^2Z_1 = Z_2Z_6vt(vt^{-1})^2$$

$$(vt)^4 = QZ_3, \quad (tv)^4 = QZ_1Z_4.$$

(56)
It can be verified that \{v, t, Z_i, Q\} generate the whole subgroup L. Let suppose that a representation \( \lambda : L \to \text{Hom}(L) \) is given on a linear space \( L \) of dimension \( n \). Then \( \lambda \) can be extended to the induced representation \( \mu : H \to \text{Hom}(H) \) where \( H = \bigoplus_{p=0}^{6} L_p \) and \( L_p \) are isomorphic to \( L \). Let \( \psi_i \) be a basis on \( L \), \( \psi_{i,p} \) a basis on \( L_p \) and \( \psi_{i,p+7} = -\psi_{i,p} \). The action of \( U \) on \( H \) is then defined by

\[
U \psi_{i,p} = \psi_{i,p+1} , \quad (57)
\]
hence \( U^7 = -1 \). Moreover we set

\[
w \psi_{i,0} = \psi_{k,p} \lambda_{k,i}(w) , \quad w \in L . \quad (58)
\]

From (56)–(58) derive the representation of \( V \) on \( H \) as follows:

\[
V \psi_{i,b} = VU^b \psi_{i,0} = U^a(b)U^{-a(b)}VU^b \psi_{i,0} =
U^a(b) \psi_{k,0} \lambda(U^{-a(b)}VU^b)_{k,i} = \psi_{k,a(b)} \lambda(U^{-a(b)}VU^b)_{k,i} . \quad (59)
\]

Since \( b \) can take all values 0, \ldots, 6 we deduce the complete representation of \( V \) on \( H \). The problem reduces now to that of finding a suitable representation of \( L \) on \( L \). Should we set \( Q = Z_i = 1 \), \( L \) becomes the octahedral group \( T(2,3,4) \) of order 24. We are interested in irreps of \( L \) in which all \( \lambda(Z_i) \) are diagonal and \( \lambda(Z_i^2) = 1 \). Each \( \lambda(Z_i) \) is then a diagonal block matrix. In fact we have

\[
Z_0 \psi_{i,p} = Z_0 U^p \psi_{i,0} = U^p Z_p \psi_{i,0} = U^p \lambda(Z_p) \psi_{i,0} = \lambda(Z_p) \psi_{i,p} . \quad (60)
\]

It is then clear that \( \mu( Z_s ) \) is the diagonal matrix with entries \( \lambda(Z_s), \lambda(Z_{s+1}), \ldots, \lambda(Z_{s+6}) \), where \( \lambda(Z_s) \) has \( n \) eigenvalues \( \xi_{i,s} = \pm 1, i = 1, \ldots, n \), \( s = 0, \ldots, 6 \). It follows also that each vector \( \psi_{i,p} \) has eigenvalues

\[
Z_s \psi_{i,p} = \xi_{i,s+p} \psi_{i,p} , \quad (61)
\]

which define the element \( \Xi \in H^1 : \)

\[
\Xi = (\xi_{i,p}, \xi_{i,p+1}, \ldots, \xi_{i,p+6}) \quad (62)
\]
For the group $\nu(L)$ the corresponding formulae are obtained by setting $v' = V^{-1}$, $t' = UV^{-1}U^{-2}$. We have then

$$U^0V^{-1}U^{-3} = v't', \quad U^4V^{-1}U^{-4} = Z_3(t'v')^2 = (t'v')^2 Z_0,$$

(63)

$$\quad (v't')^4 = Q Z_0^{-1}, \quad (t'v')^4 = Q Z_0^{-1} Z_3^{-1}.$$

(64)

If an irrep of $H$ contains a signature $\Xi$ it contains also the whole orbit including its cyclical permutations. Non trivial functionals have 7 distinct cyclical permutation of any $\Xi$ corresponding to $p = 0, \ldots , 6$ in (62). The irrep $\mu$ of $H$ generated by $\lambda$ has dimension $7n$ and as many signatures. If it contains different orbits then is reducible. Therefore irreps can be grouped in disjoint subsets characterized by orbits. Irreps belonging to orbits $A, D, E$ can be obtained either from $L$ or $\nu(L)$ whereas $B$ and $C$ can be obtained only from $L$ and $\nu(L)$ respectively. We list in appendix B the explicit representations $\lambda$ of the group $L$ corresponding to each orbit by giving $\lambda(v)$, $\lambda(t)$ and $\lambda(Z_1)$. In this way we obtain the irreps of the orbits $A, B, D, E$. Irreps of orbit $C$ can be obtained by applying $\nu$ to irreps of $B$. Clearly orbits may appear in some irreps with multiplicities $m = 1, 2, 4$ where $n/m$ is the number of inequivalent signatures under cyclical permutations contained in the orbit. The method applied to orbits A yields representations which reduce into the irreps already discussed. The dimensions of the irreps satisfy a separate Burnside condition

$$\sum_{J \in O} d_J^2 = n(O)N$$

(65)

where $n(O)$ is the number of signatures belonging to a given orbit $O$.

**VII. THE DECORATED LATTICE AND THE PARTITION FUNCTION**

We need now a more explicit description of the decorated lattice $\Lambda^\#$ along lines already discussed in ref. [9]. Each element $h \in H$ identifies uniquely a site of $\Lambda$ which we also label with $h$, being intended that $hQ \sim hZ_i \sim Z_i h$ identify the same site of $\Lambda$. $h$ is connected to other 3 sites $Vh, Uh, U^{-1}h$ and is replaced in $\Lambda^\#$ by 3 sites $h_a, h_b, h_c$ to which we associate Grassmann variables $a(h), b(h), c(h)$. The original oriented bonds $h \rightarrow Vh, h \rightarrow Uh$
are replaced by oriented bonds $h_a \rightarrow Vh_a, h_b \rightarrow Uh_c$ (with exchange interaction $J_V, J_U$ respectively) together with the additional decorating bonds $h_a \rightarrow h_c, h_c \rightarrow h_b, h_b \rightarrow h_a$. For each Pfaffian we need boundary conditions of the kind discussed above:

$$a(hZ_i) = \epsilon_ia(h), \ a(Qh) = a(hQ) = -a(h)$$

and the corresponding conditions obtained by replacing $a$ with $b,c$. Under these assumptions the 2-form (6) can be rewritten explicitly as:

$$f = \sum_{h \in \Lambda} \left( a(h) \wedge c(h) + c(h) \wedge b(h) + b(h) \wedge a(h) + \frac{y}{2} a(h) \wedge (Vh) + x b(h) \wedge (Uh) \right)$$

where $y = \coth(\beta J_V), x = \coth(\beta J_U)$.

Harmonic analysis on $\Lambda$ can be performed by recalling the Fourier components of $a(h), b(h), c(h)$ given by $a(h) = \sum_{\alpha \beta} D_{\alpha\beta}^J(h)^* a_{\alpha\beta}^J$, etc. To this purpose we consider the matrices $a^J, b^J, c^J$ of elements $a_{\alpha\beta}^J, b_{\alpha\beta}^J, c_{\alpha\beta}^J$ where $\alpha, \beta = 1, \ldots, d_J$. By using the orthonormality relations (68) satisfied by the matrices $D_{\alpha\beta}^J(h)$ we rewrite $f = \sum_J f^J$ where

$$f^J = \text{Tr} \left( a^{J^\dagger} \wedge c^{J^\dagger} + c^{J^\dagger} \wedge b^{J^\dagger} + b^{J^\dagger} \wedge a^{J^\dagger} + \frac{y}{2} a^{J^\dagger} \wedge (Vh)^J a^J + x b^{J^\dagger} \wedge (Uh)^J c^J \right)$$

and $J^\dagger$ labels the complex conjugate irrep of $J$. The key point is that different pairs $J, J^\dagger$ lead to disjoint sets of Grassman variables, moreover $f^J + f^{J^\dagger}$ separates into the sum of partial forms $\sum_a (f^J)_a + c.c.$ Therefore the final Pfaffian is the product of $d_J$ partial (and identical) Pfaffians, which must be computed explicitly by use of (5). Once this is done we must determine the coefficients $s_I$ appearing in (10). Since all the $s_I$ and Pfaffians $Pf(\phi_I)$ sharing the same orbit $O$ of $\phi_I$ are equal we can set $S_O = n(O) s_I, Pf(\phi_I) = pf(O), \forall \phi_I \in O$ and (10) reduces to:

$$Z_d = 2^{-g} \sum_O S_O pf(O)$$

The sign in $Pf(\phi_I)$ of a term associated with the path $\zeta$ is given by $\eta_I(\zeta) = \tau(\zeta_I, \zeta) \theta(\zeta)$. The $s_I$ must be chosen in such a way as to set the final coefficient of $\zeta_K$ in the expansion (10):
\[ 2^{-g} \sum_I s_I \tau(\zeta_I, \zeta_K) \theta(\zeta_K) \] (70)
equal to 1. Taking into account (45) we see that the condition

\[ \forall I : 2^{-g} \sum_I s_I \tau(\zeta_I, \zeta_K) = \sum_I s_I \Phi_{IK} = \theta(\zeta_K) \] (71)

has the simple solution:

\[ s_I = \theta(\zeta_I) , \] (72)

and thus \( S_A = 1, S_B = S_C = 7, S_D = 21, S_E = -28 \) and \( S_A + S_B + S_C + S_D + S_E = 8 = 2^g \).

VIII. PFAFFIANS AND HARMONIC ANALYSIS ON THE LATTICE

The boundary conditions (66) on the Grassmann variables \( a(h), b(h), c(h) \) imply strong restrictions on the Fourier components \( a^J_{\alpha\beta}, b^J_{\alpha\beta}, c^J_{\alpha\beta} \) similar to those already derived for \( Q \). Consider now Pf \((\phi)\) where \( \phi = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6) \). From (66) we get:

\[ \sum_{h \in H} a(hZ_i)D^J_{\alpha\beta}(h) = \epsilon_i a^J_{\alpha\gamma} = a^J_{\alpha\gamma}D^J_{\gamma\beta}(Z_i) \] (73)

In all the explicit representation of \( H \) listed in the Appendix B the matrices \( D^J_{\gamma\beta}(Z_i) \) are diagonal and

\[ D^J_{\gamma\beta}(Z_i) = \epsilon^J_i(\gamma) \delta_{\gamma\beta} . \] (74)

Clearly \( a^J_{\alpha\gamma}, b^J_{\alpha\gamma}, c^J_{\alpha\gamma} = 0 \) unless \( \epsilon^J_i(\gamma) = \epsilon_i, \ i = 0...6 \). The orbit \( O \) of \( \phi \) identifies a subset of irreps and in (68) only a rectangular submatrix of \( a^J, b^J, c^J \) of dimension \( d_J \times d_J / n(O) \) survives where \( d_J \) is the dimension of the irrep and \( d_J / n(O) \) the number of times a given signature in \( O \) appears in the irrep. The total number of surviving components of \( a^J_{\alpha\gamma} \) among all irreps is in any case equal to the order \( N \) of the group \( G \), i.e. 168.

Each \( a^J, b^J, c^J \) is therefore partitioned into \( d_J / n(O) \) rectangular submatrices labeled by the signatures in \( O \). Only one of these matrices needs to be used because they all yield the same final expression for the Pfaffian. Since in (68) all partial forms \( f_{Ja} \) yield identical
Pfaffians we may consider just one contribution and drop the index \( \alpha \). In this case the square of the partial Pfaffian can be written as the determinant of the block matrix \( \Delta_J \)

\[
\Delta_J = \begin{bmatrix}
  yD^J(V) & 1 & -1 \\
  -1 & 0 & 1 - xD^J(U) \\
  1 & -1 + xD^J(U^{-1}) & 0
\end{bmatrix}
\]  

(75)

Notice that \( \Delta_J^\dagger = -\Delta_J \). Upon multiplication of the last group of columns by \( (1 - xD^J(U))^{-1} \) and the last group of rows by \( (1 - xD^J(U^{-1}))^{-1} \) we find as in [9] :

\[
\text{Det} \ (\Delta_J) = \text{Det} \left[ y \left(1 - xD^J(U)\right)D^J(V) \left(1 - xD^J(U^{-1})\right) + x \left(D^J(U) - D^J(U^{-1})\right) \right]
\]

(76)

and therefore the effective dimension of the final determinant reduces to \( d_J \). The dimer generating function \( Z_d \) is then:

\[
Z_d = 2^{-g} \sum_O S_O pf(O) = 2^{-g} \sum_O S_O \prod_{J \in O} Pf_{d_J/n(O)}
\]

(77)

The degree of \( Pf_J \) in \( x, y \) is \( d_J, d_J/2 \) and therefore the total degree of \( pf(O) \) in \( x \) is \( \sum_{J \in O} d_J^2/n(O) = N \) because of the Burnside condition and half as much for \( y \). Not all \( \text{Det} \ (\Delta_J) \) are distinct, besides \( \text{Det} \ (\Delta_J) = \text{Det} \ (\Delta_{J'}) \) the external automorphism \( \nu \) maps \( pf(B) \) into \( pf(C) \) and vice versa thus \( pf(C) = pf(B) \). The number of independent Pfaffians is therefore further reduced to four (64 \( \rightarrow \) 5 \( \rightarrow \) 4) by the existence of external automorphisms.

If \( d_J/n(O) \) is odd the sign of \( Pf_J = \pm \sqrt{\text{Det} \ (\Delta_J)} \) must be fixed unambiguously by computing it for instance in the limit \( \beta \rightarrow 0 \). \( Z_d \) becomes then a polynomial in \( x, y \) of degree \( N, N/2 \) [12]. In Fig.2 we give the plot of the specific heat corresponding to a ferromagnetic choice of the exchange interactions, \( (J_U = J_V = 1) \). The specific heat presents a very sharp peak which prefigures the transition form and ordered ferromagnetic phase to a disordered paramagnetic one. When \( J_U, J_V \) have different signs or are both antiferromagnetic (i.e. negative) the model is totally frustrated and the specific heat becomes smoother. Moreover, different finite values (> \( \log 2/N \)) of the ground state entropy are found, signaling an exponential degeneracy of the ground state.
IX. CONCLUSIONS

In this paper we have developed a formalism capable of dealing efficiently with Ising models defined on group lattices of non trivial genus $g$. In particular we have applied the method to the Ising model on the Klein group $L(2, 7)$ having $g = 3$ and $N = 168$. We found that the computation of the partition function $Z$ is greatly simplified by use of symmetries of an extended group, both internal and external to the group, which reduce the number of and provide explicit formulas and topological interpretation for the sign and weight of Pfaffians in the expansion of $Z$. We plan to apply this method to other lattices where $N$ is large and $g$ is comparable to $N$.

X. ACKNOWLEDGMENTS.

Credit for ideas in this paper should be shared with Mario Rasetti to whom we are indebted for many illuminating discussions and suggestions.
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[12] Not all determinants can be computed symbolically but all of them can be evaluated numerically by very fast algorithms.

**Appendix A**
Fermionic irreps of $H_0$:

$U^{(1)} = \text{diag}(-1, K, -K^2, -K^4)$, $U^{(2)} = (U^{(1)})^*$, $U^{(3)} = \text{diag}(K, -K^2, -K^4, -K^6, K^5, K^3)$, $U^{(4)} = (U^{(3)})^*$, $U^{(5)} = \text{diag}(K, -K^2, -K^4, -1, -K^6, K^5, K^3, -1)$.

$$V^{(1)} = \frac{2i}{\sqrt{7}} \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & c_1 & c_3 \\ 1/\sqrt{2} & c_3 & c_2 \\ 1/\sqrt{2} & c_2 & c_1 \end{pmatrix},$$

with $c_k = \cos\left(\frac{2k\pi}{7}\right), k = 1, 2, 3$. $V^{(2)} = (V^{(1)})^*$.

$$V^{(3)} = \frac{2i}{7} \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

where

$$A = \begin{pmatrix} s_2 - \sqrt{2}s_3 & s_1 - \sqrt{2}s_2 & s_3 - \sqrt{2}s_1 \\ s_1 - \sqrt{2}s_2 & s_3 - \sqrt{2}s_1 & s_2 - \sqrt{2}s_3 \\ s_3 - \sqrt{2}s_1 & s_2 - \sqrt{2}s_3 & s_1 - \sqrt{2}s_2 \end{pmatrix},$$

$$B = r \begin{pmatrix} s_3 + (\sqrt{2} - 1)s_1 & s_2 + (\sqrt{2} - 1)s_3 & s_3 + (\sqrt{2} - 1)s_2 \\ s_2 + (\sqrt{2} - 1)s_3 & s_1 + (\sqrt{2} - 1)s_2 & s_3 + (\sqrt{2} - 1)s_1 \\ s_1 + (\sqrt{2} - 1)s_2 & s_3 + (\sqrt{2} - 1)s_1 & s_2 + (\sqrt{2} - 1)s_3 \end{pmatrix},$$

with $s_1 = \sin\left(\frac{\pi}{7}\right), s_2 = -\sin\left(\frac{2\pi}{7}\right), s_3 = -\sin\left(\frac{4\pi}{7}\right)$ and $r = \sqrt{2 + \sqrt{2}}$.

Finally $V^{(4)} = (V^{(3)})^*$ and

$$V^{(5)} = \frac{i}{7} \begin{pmatrix} C & D \\ D & -C \end{pmatrix},$$

where
All these irreps are changed into equivalent and conjugate expressions by replacing $K$ with $-K^2$.

**Appendix B**

Putting $\lambda(x) = x_O$ for the representations $\lambda$ belonging to the orbit $O$ we have

Case $B$: 

$$v_B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad t_B = \begin{pmatrix} \frac{1+i\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{(1-i\sqrt{2})}{2} \end{pmatrix}, \quad Z_{1,B} = I_2;$$

Case $C$: $v_C = v_B^{-1}$, $t_C = t_B^{-1}$, $Z_{1,C} = Z_{1,B};$

Case $D$: 

$$v_D^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & -i \\ 0 & 0 & 0 & -i & i \end{pmatrix}, \quad t_D^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & -i \\ 0 & 0 & 0 & 0 & -i & i \\ -i & -i & 0 & 0 & 0 & 0 \\ -i & i & 0 & 0 & 0 & 0 \end{pmatrix},$$

$Z_{1,D}^{(1)} = \text{diag}(1, 1, 1, 1, -1, -1),$
\[ v_D^{(2)} = (v_D^{(1)})^*, \quad t_D^{(2)} = (t_D^{(1)})^*, \quad Z_{1,D}^{(2)} = Z_{1,D}^{(1)}; \]

Case E:

\[
v_E^{(1)} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad t_E^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -i & 0 & 0 \end{pmatrix},
\]

\[ Z_{1,E}^{(1)} = \text{diag}(1, -1, 1, -1), \quad v_E^{(2)} = (v_E^{(1)})^*, \quad t_E^{(2)} = (t_E^{(1)})^*, \quad Z_{1,E}^{(2)} = -Z_{1,E}^{(1)}; \]

\[
v_E^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix},
\]

\[
t_E^{(3)} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[ Z_{1,E}^{(3)} = \text{diag}(-1, -1, 1, 1, -1, -1, 1, 1). \]
FIG. 1. Intersection between $\ell$ and $\ell'$ and schematic representation of $\zeta, \zeta'$ and $\zeta + \zeta'$ (or $\ell, \ell'$ and $\ell''$).

FIG. 2. Specific heat versus temperature for $J_U = J_V = 1$. 
