BACK-ORDERED INVENTORY MODEL WITH INFLATION IN A CLOUDY-FUZZY ENVIRONMENT

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(Communicated by Harish Garg)

Abstract. In this paper, an Economic Production Quantity model for deteriorating items with time-dependent demand and shortages including partially back-ordered is developed under a cloudy-fuzzy environment. At first, we develop a crisp model by considering linearly time-dependent demand with constant deterioration rate, constant inflation rate and shortages under partially back-ordered, then we fuzzify the model to archive a decision under the cloudy-fuzzy (extension of fuzziness) demand rate, inflation rate, deterioration rate and the partially back-ordered rate which are followed by their practical applications. In this model, we assume ambiances where cloudy normalized triangular fuzzy number is used to handle the uncertainty in information which is coming from the data. The main purpose of our study is to defuzzify the total inventory cost by applying Ranking Index method of fuzzy numbers as well as cloudy-fuzzy numbers and minimize the total inventory cost of crisp, fuzzy, and cloudy-fuzzy model. Finally, a comparative analysis among crisp, fuzzy and cloudy-fuzzy total cost is carried out in this paper. Numerical example, sensitivity analysis, and managerial insights are elaborated to justify the usefulness of the new approach. A comparative inquiry of the numerical result with a new existing paper is also carried out. This paper ends with a conclusion along with advantages and limitations of our solution approach, and an outlook towards possible future studies.

1. Introduction. Inventory is the collection of stock items having an economic value, waiting to be sold. Inventory is listed as a current asset on a company’s balance sheet. Demand is the most important component in a business model.
A person can never start a company without a demand. Demand refers to such products those are waiting to be purchased by a customer. A traditional Economic Production Quantity (EPQ) model was formulated with either constant demand or variable demand. But, in a real-life situation, the demand of various items decreases or increases after a certain time. Demand of some products such as mobile phone, water filter machine, etc., increases as time passes. However, in most of the real-life application, we observe that the demand may have variations from its original value at starting time of a company. Hence, demand rate may be uncertain and for this reason, even the total cost become uncertain. For resolving this type of uncertainty, many researchers paid their attention to different approaches such as fuzzy programming and probabilistic approach, etc. Moradi et al. [25] proposed a multi-objective supply chain model under uncertainty where an interactive fuzzy solution approach was used to solve their proposed multi-objective probabilistic mixed-integer linear programming model. Garai and Garg [8] described a multi-objective fractional inventory model with constraints under generalized intuitionistic fuzzy set environment by minimizing holding cost and considering the selling price as a generalized trapezoidal intuitionistic fuzzy number. Garg [9] developed a fuzzy inventory model for deteriorating items under different types of lead-time by assuming demand rate, holding cost, shortage cost and set-up cost as fuzzy numbers. Shekarian et al. [43] depicted a comprehensive review on a fuzzy inventory model. In another paper, Shekarian et al. [44] analyzed an economic order quantity model by including all input parameters as triangular fuzzy numbers and applied Graded Mean Integration Value for defuzzification. Now we represent the market demand as a cloudy-fuzzy number instead of the constant demand.

Deterioration is defined as decay, damage or spoilage of items and its effect cannot be ignored in an inventory model. We know that some products, such as volatile liquid, medicine, fruit, and various kinds of raw materials, etc., have a certain life period and after this period deterioration occurs. Hence, in an inventory system, deterioration rate of items per unit time have a very important role. In most cases, deterioration is occurred due to damage, evaporation, spoilage, manufacturing problem, obsolescence, that is, the item cannot be applied to its original purpose and do not have any economic value. In our real-life, most of the items like vegetables, fish, fruits, flowers, medicines, food grains, etc., decayed or deteriorated after a schedule time period. Most of the researchers treated in their models as a constant deteriorating items model. But in real the situations, due to evaporation, damage of raw materials, it cannot be assured by us that deterioration rate will be constant through the whole time. So, there are uncertainties of the deterioration rates. Nowadays, various types of uncertainties mixed up in any inventory system; these uncertainties are not always resolved by employing usual probability theory. Because of that, we describe a most suitable process to develop our model by using fuzzy number and cloudy-fuzzy number, instead of probability theory.

In a classical Economic Order Quantity (EOQ) or EPQ model, the case of inflation is commonly avoided by researchers. But, in the real world, it exists, and it is completely soft in nature because the inflation rate of an item is not constant over time. Inflation rate is the rate at which the price of goods and service increase. For example, if the present price of an item is $7 and before two months the price of that item was $5, then we can be conclude that the price of that particular item increased by 40% due to inflation rate. Due to high inflation rate, selling price of items and holding cost increased by balancing minimum total cost and maximum...
profit of a company. Therefore, inflation is an important factor in an inventory model. Generally, inflation rate varies from time to time, province to province as well as country to country. As a result, inflation rate has an uncertainty in nature. In recent years, researchers apply fuzzy number to handle the uncertain information of inflation rate. Jaggi et al. [17] derived an optimal replenishment policy for fuzzy inventory model with fuzzy deterioration rate and shortages under fuzzy inflationary condition. In our proposed model, inflation rate is regarded as a fuzzy number as well as a cloudy-fuzzy number to represent the uncertainties.

A situation in which some more items are needed to fulfil the demand of a customer but it is not available by sufficient amount is known as shortage. In an inventory model, shortage is a vital factor. Shortage of items may occur due to the withdrawal of imperfect items from the stock. The word “back-ordered” refers to an order of goods by customers or retailers that cannot be filled at current time due to shortages of goods, and for which customers or retailers are prepared to wait for sometimes because they do not prefer another source. The back-ordered is of two types, one is partially back-ordered where few customers or retailer are willing to wait, the other is fully back-ordered where all customers or retailers are willing to wait. In our model, fuzzy partially back-ordered model is treated to reflect the real-life situation perfectly.

In a classical EPQ model, it is assumed that holding cost, selling price and ordering cost are known and constant. But, in practical sense, these are highly effected by inflation rate and time. As inflation rate is variable, so holding cost, selling price and ordering cost are also considered as variable in our new model.

Hence, each item described above plays an important role to formulate the proposed model. However, in practical problem and most of the cases many parameters of inventory model depend upon the decision variable and, especially, on the time variable. For handling this type of problems, De and Mahata [4] introduced a new approach towards cloudy-fuzzy environment to solve a fuzzy inventory model with fuzzy back-ordered by addressing a cloudy-fuzzy demand rate. Due to an increase in the system complexity day-by-day, it is very difficult in the real-world problem to handle the fuzziness in the data. For this kind of uncertainties, the concept of fuzzy set was introduced by Zadeh [47]. While this is a well-known technique, the concept of a cloudy-fuzzy number is the extension of a fuzzy set number. The advantages of cloudy-fuzzy number are listed as follows:

1. Cloudy-fuzzy number depends on the decision variable (cycle-time).
2. After infinite time, it converges to a certain value (crisp value).
3. Defuzzification method for this number is easy to understand.
4. It is very convenient for a decision maker to make a decision accordingly.

In an inventory problem, most of the parameters are usually non-crisped values, which also affects the behaviour of the entire system. For example, the value of the inflation rate, demand rate, holding cost, etc., is dependent on cycle-time, i.e., its value is non-crisp. Hence, in our proposed model, we regard these parameters as uncertain and respond to them by cloudy-fuzzy numbers. Here, we first assume a crisp EPQ model; then we express the crisp model as a fuzzy model and as a cloudy-fuzzy model, under the following assumptions:

a. The selling items are deteriorating in nature and deterioration rate is regarded to follow a normal fuzzy and a cloudy-fuzzy number.
b. There is no repairing or replacement of damaged items.
(d) The demand is linearly dependent on time, and it is represented by a normal fuzzy rate, then follows a cloudy-fuzzy rate.
(e) The inflation rate is followed by a normal fuzzy number, then a cloudy-fuzzy number.
(f) The holding cost is dependent on time as well as on inflation.
(g) Ordering cost is a linearly increasing function of time.
(h) Shortages are allowed and these are partially back-ordered.

The main purpose of this investigation is to defuzzify the total inventory cost by using Yager’s [46] Ranking Index Method, to minimize total inventory costs and compare the results of the crisp model, fuzzy model and cloudy-fuzzy model.

The rest of the paper is structured as follows: Survey on related research about our study is elaborated in Section 2. Section 3 describes useful definitions about fuzzy model and cloudy-fuzzy model. Section 4 has two subsections, one is notation and other is assumption. Section 5 represents our crisp mathematical model, while Section 6 gives the formulation of our fuzzy model and of our cloudy-fuzzy model. Sections 7 and 8 present numerical result and discussion of that numerical result, respectively. Section 9 provides an analysis of some graphs and Section 10 contains sensitivity analysis and managerial insights of cloudy-fuzzy model. Finally, Section 11 ends the paper with a conclusion, advantages and limitations of our solution approach, and an outlook towards possible future studies.

2. Survey on related research. In a traditional EPQ model, demand is discussed as constant. An EPQ model with constant demand has been developed by Afshar-Nadjafi and Ghasemi [12]. Khanna et al. [21] worked on credit financing for deteriorating imperfect quality items with allowable shortages under constant demand rate and constant credit period. But in real-life, constant demand rate is not always suitable for many products such as cloths, mobile phone, etc. In recent years, many researchers paid attention to time-dependent demand concept as well as linearly increasing or decreasing demand approach. A production-inventory model for a damageable item was derived by Guchhait et al. [13], where different demand rates were predicted for different type of items. Pervin et al. [27] analyzed an inventory model with shortage by regarding the demand rate as a linear function of time. Later, Pervin et al. [28] studied a two-echelon inventory model with stock-dependent demand. Recently, Shaikh et al. [41] noticed the retailer’s optimal replacement policy of a two-warehouse inventory model for non-instantaneous items with interval-valued inventory costs and stock-dependent demand under inflationary conditions. Once-more, a deteriorating inventory model with preservation technology under price-dependent demand was proposed by Pervin et al. [31], where the production rate was treated as a time-dependent function. Khanna et al. [22] determined order lot size, back-order rate and selling price for an inventory model of imperfect quality items with selling-price-dependent demand under credit financing. Even many researchers interested in an economic model under uncertain demand rate condition. Shekarian et al. [42] presented an EPQ model by seeing back-orders and rework for a single-stage system, triangular and trapezoidal fuzzy numbers were involved to examine the developed fuzzy model. In that paper, graded mean integrations representations (GMIR) is applied for defuzzification. An economic production quantity model involving fuzzy demand rate was proposed by De et al. [6], where the purchase cost was constant. Gautam et al. [11] studied on a strategic defect management for sustainable green supply chain. Roy et al. [34]
developed a two-warehouse inventory model with probabilistic demand and price discount on back-ordered under two levels of trade-credit policy. A deteriorating inventory model was developed by Roy et al. [35] with variable demand under trade-credit policy. Tiwari et al. [45] investigated a two-warehouse model with imperfect trade-credit and inflation on retailer’s ordering policies. A periodic review fuzzy inventory model assuming fuzzy demand was presented by Sarkar and Mahapatra [37]. De and Mahata [4] took a decision on a fuzzy inventory model with fuzzy back-ordered under cloudy-fuzzy demand rate. Thereafter, De and Mahata [5] proposed a cloudy-fuzzy EOQ model for imperfect-quality items with allowable proportionate discounts. In our model, we present the demand by a cloudy-fuzzy number.

In most of the inventory models, it is assumed that the deterioration rate is constant with time. Pervin et al. [28] presented a two-echelon inventory model by assuming constant deterioration. A fuzzy inventory model for a deteriorating item with price-dependent demand has been published by Saha [36] by assuming an uncertain cycle-time. But, in practice, deterioration rate is dynamic with time. Sarkar and Chakrabarti [38] developed an EPQ model, where the deterioration rate follows a Weibull distribution. Dutta and Kumar [7] expressed a fuzzy inventory model, where the deterioration rate regarded as a fuzzy number. Fuzzy inventory model for deteriorating items with time-varying demand and shortages was developed by Jaggi et al. [16], where deterioration rate and demand were reflected as a triangular fuzzy number. In our model, we represent the deterioration rate by a cloudy-fuzzy number.

In the last few years, inflation has been introduced in many inventory models. The demand as well as holding cost and selling price are influenced by inflation rate. Hence, in an inventory system, inflation plays a vital role. In the traditional inventory system, the inflation rate was assumed as constant. Researchers, such as Muniappan et al. [26] developed an EOQ model with constant inflation and time-value of money by addressing time-dependent deteriorating rate. Pervin et al. [32] formulated an integrated vendor-buyer model for deteriorating items including quadratic demand rate. Shaboni et al. [39] developed an inventory model with fuzzy deterioration and fully backlogged shortages under inflation. De and Goswami [3] presented an inventory model under fuzzy inflation rate and fuzzy deterioration rate including delay in payment. A fuzzy imperfect production and repair inventory model with time-dependent demand, production and repair rates under inflationary conditions was developed by Jain et al. [18], where all costs parameters were represented as triangular fuzzy numbers. Partially back-ordered shortages have received high attention during the past few years. An inventory model with generalized type demand, deterioration including decreasing back-ordered rate was analysed by Hung [14]. Kazemi et al. [20] connected fuzzy lot-sizing problem to human learning effect with back-orders. Pervin et al. [29] proposed a model by treating constant partially back-ordered rate. Furthermore, an imperfect production quantity inventory model with carbon emission for an integrated supply chain has been developed by Gautam and Khanna [10], where shortages were fully backlogged. Kumar et al. [24] described a fuzzy inventory model for deteriorating items with time-dependent demand and partial backlogging, where backlogging rate was treated as a fuzzy number. A fuzzy inventory model for deteriorating items with shortages under fully backlogging condition was constructed by Indrajitsingha et al. [15]. Later,
Shaikh et al. [40] discussed a fuzzy inventory model for a deteriorating item under variable demand, permissible delay in payments and partial backlogging under shortage follows inventory policy by employing quantum-behaved particle swarm optimization technique to solve the optimization problems. Pervin et al. [30] discussed an integrated model by taking variable holding cost. A multi-item inventory model was studied by Pervin et al. [33], where holding cost of manufacturer as well as retailer was dependent on purchasing cost per item per unit time. Let us refer to Table 1 which represents core contributions by different authors related to this field.

3. Useful definitions. Fuzzy set was introduced by Zadeh [47]. According to Zadeh [47], a fuzzy set \( \tilde{B} \) of a universal set \( X \) is a set of ordered pairs \( \{(x, \mu_{\tilde{B}}(x)) : x \in X\} \), where \( \mu_{\tilde{B}}(x) : x \mapsto y \in [0, 1] \) which assigns a real number \( y \) in \([0, 1]\) to each element \( x \in X \). The function \( \mu_{\tilde{B}}(x) \) is called the membership function of the fuzzy set \( \tilde{B} \), which is a continuous mapping from \( X \) to the closed interval \([0, 1]\), and the set \( \tilde{B} \) is a convex set. The \( \alpha \)-cut of \( \tilde{B} \) is denoted by \( B_\alpha \) which is a set of real numbers and it is expressed as:

\[
B_\alpha = \{x : \mu_{\tilde{B}}(x) \geq \alpha : 0 \leq \alpha \leq 1\}.
\]

**Normalized General Triangular Fuzzy Number (NGTFN):** A NGTFN \( \tilde{A} = (a, b, c) \) is represented along with membership function \( \mu_{\tilde{A}}(x) \) as follows:

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & \text{if } x < a \text{ and } x > c, \\
\frac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\
\frac{c-x}{c-b}, & \text{if } b \leq x \leq c.
\end{cases}
\]

Moreover, the right and left \( \alpha \)-cuts of \( \tilde{A} \) are defined as \( A_R(\alpha) = c - (c - b)\alpha \) and \( A_L(\alpha) = a + (b - a)\alpha \). Observe that the measures of fuzziness can be calculated by applying the following index theory.

**Index Theory:** If we denote left and right \( \alpha \)-cuts of the fuzzy number \( \tilde{A} \) as \( A_L(\alpha) \) and \( A_R(\alpha) \), respectively, then the defuzzification rule depends on the Yager’s [46] Index Theorem is

\[
I(\tilde{A}) = \int_0^1 \frac{A_L(\alpha) + A_R(\alpha)}{2} d\alpha = \frac{a + 2b + c}{4}.
\]

The degree of fuzziness can be obtained by the formula, \( D_f = (u_b - l_b) / (2M) \), where \( u_b \) and \( l_b \) are upper bound and lower bound of the fuzzy numbers, respectively, and \( M \) is their respective mode.

**Cloudy Normalized Triangular Fuzzy Number (CNTFN):** A triangular fuzzy number \( \tilde{D} = (a, b, c) \) is said to be a CNTFN if the set itself converges to a crisp singleton set after an infinite time; i.e., if time \( t \to \infty \), then both \( a \to b \) and \( c \to b \).

Based on the above definition, we consider a cloudy triangular fuzzy number as follows:

\[
\tilde{D} = \langle b(1 - \rho \frac{\gamma}{1+\gamma}), b(1 + \frac{\gamma}{1+\gamma}) \rangle, \text{ where } 0 < \gamma, \rho < 1.
\]

Here, \( \lim_{t \to \infty} b(1 - \rho \frac{\gamma}{1+\gamma}) = b \) and \( \lim_{t \to \infty} b(1 + \frac{\gamma}{1+\gamma}) = b \), and, therefore, \( \tilde{D} \to \{b\} \).

Now, the membership function of the cloudy-fuzzy number \( \tilde{D} \), which is a function of two variables, is introduced as follows:
\[ \mu_{\tilde{D}}(x, t) = \begin{cases} 
0, & \text{if } x < b(1 - \frac{\rho}{1+t}) \text{ and } x > b(1 + \frac{\gamma}{1+t}), \\
\frac{x-b(1 - \frac{\rho}{1+t})}{\frac{\rho}{1+t}}, & \text{if } b(1 - \frac{\rho}{1+t}) \leq x \leq b, \\
\frac{b(1 + \frac{\gamma}{1+t}) - x}{\frac{\gamma}{1+t}}, & \text{if } b \leq x \leq b(1 + \frac{\gamma}{1+t}). 
\end{cases} \]

(2)

Ranking Index method on Cloudy Normalized Triangular Fuzzy Number:

According to the extension of De and Beg’s [2] Ranking Index on CNTFN, if we take left and right \( \alpha \)-cuts of \( \mu_{\tilde{D}}(x, t) \) from Eqn. 2, denoted by \( D_L(\alpha, t) \) and \( D_R(\alpha, t) \), respectively, then the Defuzzification Rule is

\[ I(\tilde{D}) = \frac{1}{T} \int_{t=0}^{t=T} \int_{\alpha=0}^{\alpha=1} \frac{D_L(\alpha, t) + D_R(\alpha, t)}{2} d\alpha dt, \]

(3)

where \( \alpha \) and \( t \) are independent variables.

Now the left and right \( \alpha \)-cuts of \( \mu_{\tilde{D}}(x, t) \) of the CNTFN stated above are, respectively: \( D_L(\alpha, t) = b(1 - \frac{\rho}{1+t}) + \frac{\rho}{1+t} \alpha \) and \( D_R(\alpha, t) = b(1 + \frac{\gamma}{1+t}) - \frac{\gamma}{1+t} \alpha \); hence, by using the formula 3, we obtain \( I(\tilde{D}) = \frac{b}{T} \left\{ T + \frac{\gamma-\rho}{4} \log(1+T) \right\} \), which can be written as \( I(\tilde{D}) = b \left\{ 1 + \frac{\gamma-\rho}{4} \frac{\log(1+T)}{T} \right\} \). Obviously, \( \lim_{T \to \infty} \frac{\log(1+T)}{T} = 0 \), so \( I(\tilde{D}) \to b \) as \( T \to \infty \).

The factor \( \frac{\log(1+T)}{T} \) is known as Cloud Index. In general, the time horizon \( T \) can never be infinite, and hence, the cloud index \( \frac{\log(1+T)}{T} \) can not be 0 and the index value of function can never go back to its original value. The cloud index lies in the open interval \((0, 1)\).

The significance of the cloud index is that diminishing cloud index increases the stability of the function; i.e., the uncertainty of the function decreases.

4. Notations and assumptions. The notations and assumptions which are used throughout this paper are given in two subsections.

4.1. Notations.

- \( D \) rate of demand per unit time;
- \( p \) production quantity per unit time;
- \( \theta \) deterioration rate per item per unit time;
- \( \delta \) fraction of demand that will be back-ordered, \( 0 < \delta < 1 \);
- \( c_1 \) shortage cost per unit item per unit time;
- \( s \) lost sale cost per unit item per unit time;
- \( N \) per unit time inflation rate of money in market;
- \( H(t) \) holding cost per unit item which is time dependent as well as inflation dependent, defined by \( H(t) = b(1 + Nt) \), where \( b \) is the unit holding cost at zero time, \( b > 0 \);
- \( f \) fraction of deteriorating production rate per unit time;
- \( p(1-f) \) per unit time actual production;
- \( S(t) \) selling price function per unit item per unit time, which is \( S(t) = a_1 + Nt, a_1, N \geq 0 \);
- \( T \) cycle-time;
- \( T_1 \) production time per cycle;
- \( T_2 \) time for which shortages are allowed;
Table 1: Contribution of various authors related to inventory models.

| Author(s)               | Variable demand Rate | Fuzzy demand Rate | Cloudy Demand Rate | Fuzzy Deterioration | Cloudy Deterioration | Fuzzy Inflation | Cloudy Inflation | Variable Holding cost | Fuzzy Back order Rate | Cloudy Back order Rate |
|-------------------------|----------------------|-------------------|--------------------|---------------------|----------------------|-------------------|-------------------|------------------------|------------------------|------------------------|
| De et al. (2003)        | √                    |                   |                    |                     |                      |                   |                   |                        |                        |                        |
| De and Goswami (2006)   |                      | √                 |                    |                     |                      |                   |                   |                        |                        |                        |
| Jaggi et al. (2012)     | √                    |                   |                    |                     |                      |                   |                   |                        |                        |                        |
| Dutta and Kumar (2013)  | √                    |                   |                    |                     |                      |                   |                   |                        |                        |                        |
| Nadjfi (2013)           |                      |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Shaboni et al. (2014)   |                      |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Pervin et al. (2015)    | √                    |                   |                    |                     |                      |                   |                   |                        |                        |                        |
| Kumar and Rajput (2015) | √                    |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Pervin et al. (2016)    | √                    |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| De and Mahata (2017)    | √                    | √                 |                    |                     |                      |                   |                   |                        |                        | √                      |
| Parvin et al. (2017)    | √                    |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Kumar and Kumar (2017)  |                      |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Pervin et al. (2018)    | √                    |                   |                    |                     |                      |                   |                   |                        |                        | √                      |
| Our paper               | √                    | √                 | √                  | √                   | √                    | √                 | √                 | √                      | √                      | √                      |
$I_1(t)$ inventory level that changes with time $t$ during production period;

$I_2(t)$ inventory level that changes with time $t$ during non-production period;

$I_3(t)$ inventory level which changed with time $t$ during shortage period;

$A(t)$ time dependent ordering cost of row material per order, which is $a + ct$; $a, c > 0$;

$B$ set-up cost per order which is constant;

$Z(T, T_1, T_2)$ total inventory cost.

4.2. Assumptions.

i) Ordering cost is a linearly increasing function of time.

ii) Shortage are allowed and partially back-ordered. Thus, the production rate is less than the demand rate.

iii) Demand is time-dependent and fuzzy and it is defined by $D(t) = \beta + \alpha t$, where $\beta > 0$; $\beta$ and $\alpha$ are fuzzy numbers, and cloudy-fuzzy numbers also, respectively.

iv) Lead time is considered as 0.

v) There is no repair or replacement of deteriorating items during the cycle-time.

vi) Deterioration rate and partially back-ordered rate follow fuzzy and cloudy-fuzzy number, respectively.

viii) The inflation rate follows a general fuzzy and a cloudy-fuzzy number.

5. Mathematical formulation. Based on our assumptions, the inventory system can be presented as follows:

At the beginning (i.e., at time $t = 0$), the cycle starts with production quantity $p$. The production continues up to time $T_1$. During the time period $[0, T_1]$, inventory piles up by adjusting the demand in the market. This accumulated inventory level at time $T_1$ gradually diminishes during the period $[T_1, T_2]$ and ultimately falls to 0 at time $t = T_2$. The time interval $[T_2, T]$ is the shortage interval. During the shortage interval, the unsatisfied demand is back-ordered at a constant rate $\delta$. After the scheduling period $T$, the cycle repeat itself. Figure 1 shows a graphical presentation of our developed model.

Now, the following differential equations involve the instantaneous states of the inventory level in the interval $[0, T]$:

$$\frac{dI_1(t)}{dt} + \theta(t)I_1(t) = p(1 - f) - D(t), \quad 0 \leq t \leq T_1,$$

with $I_1(0) = 0$;

$$\frac{dI_2(t)}{dt} + \theta(t)I_2(t) = -D(t), \quad T_1 \leq t \leq T_2,$$

with $I_2(T_2) = 0$, and $I_1(T_1) = I_2(T_1)$;

$$\frac{dI_3(t)}{dt} = -\delta D(t), \quad T_2 \leq t \leq T,$$

with $I_3(T_2) = 0$. The solutions of the differential Eqns. 4, 5 and 6 are respectively represented as follows:

$$I_1(t) = (p(1 - f) - \beta)(t - \frac{\theta t^2}{2}) - \frac{\alpha t^2}{2}, \quad 0 \leq t \leq T_1,$$

$$I_2(t) = \frac{\alpha t^2}{2}, \quad T_1 \leq t \leq T_2,$$

$$I_3(t) = -\delta \int_{T_2}^{t} D(u) du, \quad T_2 \leq t \leq T.$$
and

\[ I_2(t) = (\beta + \alpha t)(T_2 - t) + \frac{\beta \theta + \alpha \theta T_2 - \alpha}{2}(T_2 - t)^2, \quad T_1 \leq t \leq T_2, \quad (8) \]

with

\[ I_3(t) = (T_2 - t)(\delta \beta + \frac{\delta \alpha}{2}(T_2 + t)), \quad T_2 < t < T. \quad (9) \]

For finding the total inventory cost, we have to calculate the subsequent costs:

\( a. \) The total holding cost \((THC)\) during the cycle-time \([0, T]\) is given by:

\[
THC = \int_0^{T_1} H(t) I_1 dt + \int_{T_1}^{T_2} H(t) I_2 dt \\
= \int_0^{T_1} b(1 + Nt)[(p(1 - f) - \beta)(t - \frac{\theta t^2}{2}) - \frac{\alpha t^2}{2}] dt \\
+ \int_{T_1}^{T_2} b(1 + Nt)[(\beta + \alpha t)(T_2 - t) + \frac{\beta \theta + \alpha \theta T_2 - \alpha}{2}(T_2 - t)^2] dt.
\]

After some fundamental manipulation, we obtain:

\[
THC = bp(1 - f)\left(\frac{T_1^2}{2} - \frac{\theta T_1^3}{6}\right) + bn_p(1 - f)\left(\frac{T_1^3}{3} - \frac{\theta T_1^4}{8}\right) + \frac{b \beta T_2^2}{2} - b \beta T_2 T_1 \\
+ \frac{1}{2}(b \alpha + \frac{b \beta \theta}{3} + \frac{b \beta \theta}{2})T_2^3 - \frac{1}{2}(b \alpha + b \theta)T_2^2 T_1 + \frac{(b \beta \theta - \frac{b \beta \theta}{2} - b \beta \theta)}{2} T_2^2 T_1^2 \\
+ \left(\frac{b \alpha \theta}{8} + \frac{b \beta \theta}{24}\right)T_2^4 - \frac{b \alpha \theta}{2} T_2^3 T_1 + \frac{(b \alpha \theta - \frac{b \beta \theta}{4} - b \beta \theta)}{2} T_2^2 T_1^2 \\
+ \left(\frac{b \beta \theta}{3} - \frac{b \beta \theta}{6}\right)T_2 T_1^3 + \frac{b \beta \theta}{24} T_2^3 - \frac{b \beta \theta}{4} T_2^3 T_1^2 + \frac{b \beta \theta}{3} T_2^2 T_1^3 - \frac{b \beta \theta}{8} T_2 T_1^4.
\]

\( b. \) Since \( f \) is a fraction of defective items due to faulty machine in production time, so there arise a cost which must be added to the total inventory cost. This
additional cost $L_p$ is as follows:

$$L_p = \int_0^{T_1} A(t)S(t)p \, dt$$

$$= \int_0^{T_1} (a + ct)(a_1 + N_t)p \, dt$$

$$= p f[a_1 a T_1 + \frac{a N}{2} T_1^2 + \frac{c a_1}{2} T_1^2 + \frac{c N}{3} T_1^3].$$

c. Due to the deterioration per cycle $[0, T)$, the deteriorating cost $DC$ is:

$$DC = \int_0^{T_1} \theta(t)I_1(t)A(t) \, dt + \int_{T_1}^{T_2} \theta(t)A(t)I_2(t) \, dt$$

$$= \int_0^{T_1} \theta(a + ct)[(p(1 - f) - \beta)(t - \frac{\theta t^2}{2}) - \frac{\alpha t^2}{2}] \, dt$$

$$+ \int_{T_1}^{T_2} \theta(a + ct)[(\beta + \alpha T_2)(T_2 - t) + \frac{\beta \theta + \alpha \theta T_2 - \alpha}{2} (T_2 - t)^2] \, dt.$$

After some calculation, we obtain the following result:

$$DC = \theta a p(1 - f)(\frac{T_1^2}{2} - \frac{\theta T_1^3}{6}) + \theta a p(1 - f)\frac{T_1^3}{3} - \frac{\theta T_1^4}{8} + \frac{\theta \beta a T_2^2}{2} - \frac{\theta \beta a T_2 T_1}{2} - (\frac{\theta a \alpha}{2} + \frac{\theta^2 a \beta}{2}) T_2 T_1^2 + \frac{1}{2} \theta^2 a \beta T_2 T_1^2 + \frac{\theta^2 a a \alpha}{6} + \frac{\theta \alpha a}{8} + \frac{\theta^2 a \beta}{24} T_2^4$$

$$+ \frac{\theta^2 a a}{2} - \frac{\theta \alpha a}{4} - \frac{\theta^2 a \beta}{4} T_2^2 T_1^2 - (\frac{\theta^2 a a}{6} - \frac{\theta \alpha a}{3}) T_2 T_1^3 + \frac{\theta^2 a a}{24} T_2^3 - \frac{\theta \alpha a}{4} T_2^3 T_1^2$$

$$- \frac{\theta^2 a a}{8} T_2 T_1^4 + (\frac{1}{2} \theta a a + \frac{\theta^2 a \beta}{6}) T_2^3 - \frac{\theta^2 a a}{2} T_2^3 T_1^2 + \frac{\theta^2 a a}{3} T_2^2 T_1^3.$$

d. The shortage cost $SC$ during the interval $[T_2, T]$ is indicated as:

$$SC = c_1 \int_{T_2}^{T} I_3(t) \, dt$$

$$= c_1 \int_{T_2}^{T} (T_2 - t)(\delta \beta + \frac{\delta a}{2}(T_2 + t)) \, dt$$

$$= c_1[\delta \beta T T_2 - \frac{\delta \beta T^2}{2} + \frac{\delta a}{2} T_2^2 T - \frac{1}{6} \delta a T^3 - \frac{\delta \beta}{2} T_2^2 - \frac{\delta a}{3} T_2^3].$$

e. Due to shortage, many customer are not interested to wait for buying their product, which may cause a loss in profit. Hence, this lost sale cost $LSC$ is formulated as:

$$LSC = s \int_{T_2}^{T} (1 - \delta) D(t) \, dt$$

$$= s(1 - \delta) \int_{T_2}^{T} (\beta + \alpha t) \, dt$$

$$= \beta s(1 - \delta)(T - T_2) + \alpha s(1 - \delta) \frac{T^2 - T_2^2}{2}.$$
f. Purchase cost $PC$ during the interval $[T_2, T]$ is stated as:

$$PC = \int_{T_2}^{T} A(t)\delta D(t)dt$$

$$= \int_{T_2}^{T} (a + ct)\delta(\beta + ct)dt$$

$$= \delta[a\beta(T - T_2) + \frac{a\alpha + c\beta}{2}(T^2 - T_2^2) + \frac{c\alpha}{3}(T^3 - T_2^3)].$$

Total inventory cost per unit time is:

$$Z(T, T_1, T_2) = \frac{1}{T}(THC + DC + L_v + SC + LSC + PC + B)$$

$$= \frac{1}{T}(R_1 + R_2 + R_3 + B).$$

Here, $R_1$, $R_2$ and $R_3$ are defined as:

$$R_1 = p(1 - f)(\frac{T_2^2}{2} - \frac{\theta T_1^3}{8})(b + a\theta) + p(1 - f)(\frac{T_1^3}{3} - \frac{\theta T_1^4}{8})(bN + c\theta) + (\frac{b\beta}{2})$$

$$+ \frac{a\beta}{2})T_2^2 - \beta(b + a\theta)T_2T_1 + (\frac{ba}{2} + b\beta\delta + bN\beta + \frac{\alpha}{2} + \theta^2 a\beta + \frac{c\beta}{6})T_2^3$$

$$- \frac{1}{2}(ba + b\beta\alpha + \theta a\alpha + \theta^2 a\beta T_2^2 T_1 + (\frac{b\beta}{2} - bN\beta - \frac{\alpha}{2} - \theta^2 a\beta - \frac{c\beta}{6})T_2T_1^2 - (\frac{b\alpha}{2} + bN\alpha + bN\beta + \frac{\alpha}{8} + \theta a\alpha + \theta\alpha)$$

$$+ \frac{\alpha^2}{8})T_2T_1^4 - (\frac{b\alpha}{2} + \frac{a\alpha^2}{2})T_2^3 T_1 + (\frac{b\alpha}{6} + bN\alpha + bN\beta + \frac{\alpha}{24} + \frac{\alpha^2}{6} + \theta a\alpha + \theta\alpha)$$

$$+ \frac{\alpha^2}{24} T_2^4 + \frac{b\alpha}{2} + \frac{\alpha^2}{24})T_2^3 T_1^3 - (\frac{b\alpha}{4} + \frac{a\alpha^2}{4})T_2^3 T_1^3 - (\frac{b\alpha}{2} + bN\alpha + bN\beta + \frac{\alpha}{4} + \frac{\alpha^2}{4})T_2^3 T_1^3$$

$$- (\frac{b\alpha}{2} + bN\alpha + \frac{\alpha}{6} - \frac{\alpha^2}{3})T_2T_1^5,$$

$$R_2 = p[(a_1 aT_1 + \frac{aN}{2} T_1^2 + \frac{c\alpha_1}{2} T_1^2 + \frac{cN}{3} T_1^3] + c_1[\delta \beta TT_2 + \frac{\delta\alpha}{2} T_2^2 - \frac{1}{6} \delta\alpha T^3$$

$$- \frac{\delta\beta}{2} (T^2 + T_2^2) - \frac{\delta\alpha}{3} T_2^3],$$

$$R_3 = (T - T_2)(\beta s - \beta \delta s + \delta \alpha a) + \frac{T^2 - T_2^2}{2} [\alpha s(1 - \delta) + a a + c \beta] + \frac{\alpha c}{3}(T^3 - T_2^3).$$

6. Fuzzy and cloudy-fuzzy model. Let us consider the demand, deterioration rate, inflation rate and back-ordered rate at first follow the general fuzzy numbers, and then follow the cloudy-fuzzy numbers over the inventory run time. Since demand is $\beta + \alpha t$, so we consider $\beta$ and $\alpha$ as fuzzy and cloudy-fuzzy numbers. Hence, consider $N$, $\alpha$, $\beta$, $\theta$, $\delta$ as follows:

$$\tilde{N} = \begin{cases} \langle N_1, N_2, N_3 \rangle, & \text{for NGTFN}, \\ \langle N(1 - \frac{\theta}{T_T}), N(1 + \frac{\theta}{T_T}) \rangle, & \text{for CNTFN}, \end{cases}$$

for $0 < \rho$, $\gamma < 1$ and $T > 0$,

$$\tilde{\alpha} = \begin{cases} \langle \alpha_1, \alpha_2, \alpha_3 \rangle, & \text{for NGTFN}, \\ \langle \alpha(1 - \frac{\theta}{T_T}), \alpha(1 + \frac{\theta}{T_T}) \rangle, & \text{for CNTFN}, \end{cases}$$

for $0 < \rho$, $\gamma < 1$ and $T > 0$,

$$\tilde{\beta} = \begin{cases} \langle \beta_1, \beta_2, \beta_3 \rangle, & \text{for NGTFN}, \\ \langle \beta(1 - \frac{\theta}{T_T}), \beta(1 + \frac{\theta}{T_T}) \rangle, & \text{for CNTFN}, \end{cases}$$

for $0 < \rho$, $\gamma < 1$ and $T > 0$. 


\[ \ddot{Z} = \begin{cases} (\delta_1, \delta_2, \delta_3), & \text{for NGTFN,} \\ (\delta(1 - \frac{\rho}{1 + \tau}), \delta, \delta(1 + \frac{\gamma}{1 + \tau})), & \text{for CNTFN, where } 0 < \rho, \gamma < 1 \text{ and } T > 0, \end{cases} \]

\[ \ddot{Z} = \begin{cases} (\beta_1, \beta_2, \beta_3), & \text{for NGTFN,} \\ (\beta(1 - \frac{\rho}{1 + \tau}), \beta, \beta(1 + \frac{\gamma}{1 + \tau})), & \text{for CNTFN, where } 0 < \rho, \gamma < 1 \text{ and } T > 0. \end{cases} \]

The corresponding total fuzzy and cloudy-fuzzy inventory cost per unit time are as follows:

\[ \ddot{Z} = \begin{cases} (Z_1, Z_2, Z_3), & \text{for NGTFN,} \\ (Z_1^c, Z_2^c, Z_3^c), & \text{for CNTFN,} \end{cases} \]

and \( Z_1, Z_2, Z_3, \) and \( Z_1^c, Z_2^c, Z_3^c, \) are defined as:

\[
\begin{align*}
Z_1 &= \frac{1}{T}(R_{11} + R_{21} + R_{31} + B), \\
Z_2 &= \frac{1}{T}(R_{12} + R_{22} + R_{32} + B), \\
Z_3 &= \frac{1}{T}(R_{13} + R_{23} + R_{33} + B), \\
Z_1^c &= \frac{1}{T}(R_{11}^c + R_{21}^c + R_{31}^c + B), \\
Z_2^c &= \frac{1}{T}(R_{12}^c + R_{22}^c + R_{32}^c + B), \\
Z_3^c &= \frac{1}{T}(R_{13}^c + R_{23}^c + R_{33}^c + B),
\end{align*}
\]

where

\[
R_{11} = p(1-f)(\frac{T_1^2}{2} - \theta_1T_1^3)(b + a\theta_1) + p(1-f)(\frac{T_1^3}{3} - \frac{\theta_1T_1^4}{8})(bN_1 + c\theta_1) + \frac{b\beta_1}{2}
\]
\[
+ \frac{\theta_1a\beta_1}{2}T_2^2 - \beta_1(b + a\theta_1)T_2^2 + \frac{b\alpha_1a\beta_1}{6} + b\alpha_1\beta_1 + \frac{\theta_1a\alpha_1}{2} + \frac{\theta_1a\beta_1}{6}
\]
\[
+ \frac{\theta_1c\beta_1}{6}T_2^2 - \frac{1}{2}(b\alpha_1 + b\beta_1\theta_1 + \theta_1a\alpha_1 + \theta_1^2a\beta_1)T_2^2 + \frac{b\beta_1\beta_1}{2} - \frac{b\beta_1\beta_1}{2} + \frac{\theta_1^2a\beta_1}{2}
\]
\[
- \frac{\theta_1c\beta_1}{8}T_2T_1^2 - \frac{b\alpha_1\alpha_1\theta_1}{8} + \frac{\theta_1^2c\alpha_1}{8}T_2T_1^2 - \frac{b\alpha_1\alpha_1\theta_1}{2} + \frac{\theta_1^2\alpha_1\alpha_1}{2}T_1^2 + \frac{b\alpha_1\alpha_1\theta_1}{24}
\]
\[
+ \frac{\theta_1c\alpha_1}{24}T_2T_1^4 - \frac{b\alpha_1\alpha_1\theta_1}{24} + \frac{\theta_1^2c\alpha_1}{24}T_2T_1^4 + \frac{b\alpha_1\alpha_1\theta_1}{4} - \frac{b\alpha_1\alpha_1\theta_1}{4} + \frac{\theta_1^2\alpha_1\alpha_1}{4}T_1^4 + \frac{b\alpha_1\alpha_1\theta_1}{4}
\]
\[
- \frac{\theta_1^2c\alpha_1}{4}T_2^2T_1^2 - \frac{b\alpha_1\alpha_1\theta_1}{6} - \frac{b\alpha_1\alpha_1\theta_1}{6} + \frac{\theta_1^2c\alpha_1}{6}T_2T_1^2 + \frac{b\alpha_1\alpha_1\theta_1}{3} - \frac{\theta_1^2\alpha_1\alpha_1}{3}
\]
\[
+ \frac{\theta_1^2c\alpha_1}{3}T_1^2T_3.
\]

Similarly, \( R_{12} \) and \( R_{13} \) are obtained by putting \( N = N_2, \alpha = \alpha_2, \beta = \beta_2, \theta = \theta_2, \delta = \delta_2, \) and \( N = N_3, \alpha = \alpha_3, \beta = \beta_3, \theta = \theta_3, \delta = \delta_3 \) in \( R_1, \) respectively; furthermore,

\[
R_{21} = p[f(a_1T_3) + \frac{a_1N_1}{2}T_1^2 + \frac{a_1}{2}T_1^2 + \frac{c_1N_1}{3}T_1^3] + c_1[\delta_1\beta_1TT_2 + \frac{\delta_1\alpha_1}{2}TT_2^2 - \frac{1}{6}\delta_1\alpha_1T_3^3]
\]
\[
- \frac{\delta_1\alpha_1}{2}(T_2^2 + T_2^3) - \frac{\delta_1\alpha_1}{3}T_2^3].
\]

Similarly, \( R_{22} \) and \( R_{23} \) are received by putting \( N = N_2, \alpha = \alpha_2, \beta = \beta_2, \delta = \delta_2, \) and \( N = N_3, \alpha = \alpha_3, \beta = \beta_3, \delta = \delta_3 \) in \( R_2, \) respectively; moreover,

\[
R_{31} = (T - T_2)(\beta_1s - \beta_1\delta_1s + \delta_1\beta_1\alpha) + \frac{T^2T_2^2}{2}[\alpha_1s(1 - \delta_1) + a_1 + c_3] + \frac{c_3}{3}(T^3 - T_2^2).
\]

We continue and we get \( R_{32} \) and \( R_{33} \) by putting \( \alpha = \alpha_2, \beta = \beta_2, \delta = \delta_2, \) and \( \alpha = \alpha_3, \beta = \beta_3, \delta = \delta_3 \) in \( R_3, \) respectively.
To compute the values of $Z_1^c$, $Z_2^c$ and $Z_3^c$ which are defined in Eqn. 11, we must calculate the following terms:

$$R_{11}^c = p(1 - f)(T_2^2 - \frac{\theta(1 - \frac{\rho}{1 + T})}{6}b + a\theta(1 - \frac{\rho}{1 + T}))$$

$$+ p(1 - f)(\frac{T_1^3}{3} - \frac{\theta(1 - \frac{\rho}{1 + T})}{8}b) + a\theta(1 - \frac{\rho}{1 + T}) + \frac{b\beta(1 - \frac{\rho}{1 + T})}{2}$$

$$+ \frac{\theta c(1 - \frac{\rho}{1 + T})}{6}T_2^3 - \frac{b\alpha\theta(1 - \frac{\rho}{1 + T})}{2} + a\alpha(1 - \frac{\rho}{1 + T})$$

Similarly, $R_{31}^c$ is obtained by putting $\alpha = \alpha(1 + \frac{\gamma}{1 + T})$, $\beta = \beta(1 + \frac{\gamma}{1 + T})$, $\theta = \theta(1 + \frac{\gamma}{1 + T})$, $\delta = \delta(1 + \frac{\gamma}{1 + T})$, and $N = N(1 + \frac{\gamma}{1 + T})$ in $R_1$ and $R_{12}^c = R_1$. Furthermore, we get:

$$R_{21}^c = pf[a_1 a T_1 + \frac{a N(1 - \frac{\rho}{1 + T})}{2} T_1^2 + \frac{c a_1}{2} T_1^2 + \frac{c N(1 - \frac{\rho}{1 + T})}{3} T_1^3]$$

$$+ c_1[\delta(1 - \frac{2\rho}{1 + T})TT_2 + \frac{\delta\alpha(1 - \frac{2\rho}{1 + T})}{2} TT_2^2 - \frac{1}{6}\delta\alpha(1 - \frac{2\rho}{1 + T})T^3$$

$$- \frac{\delta\beta(1 - \frac{2\rho}{1 + T})}{2} (T^2 + T_2^2) - \frac{\delta\alpha(1 - \frac{2\rho}{1 + T})}{3} T_2^3].$$
We go on with $R_{22}^c = R_2$, $R_{32}^c = R_3$, and putting $\alpha = \alpha(1 + \frac{\gamma}{T})$, $\beta = \beta(1 + \frac{\gamma}{T})$, $N = N(1 + \frac{\gamma}{T})$, and $\delta = \delta(1 + \frac{\gamma}{T})$ in $R_2$ and $R_3$, we get $R_{23}^c$ and $R_{33}^c$, respectively.

From the definition of a membership function of a fuzzy number, we formulate the membership function of fuzzy inventory cost as follows:

$$
\mu(Z) = \begin{cases} 
0 & \text{if } Z < Z_1 \text{ and } Z > Z_2, \\
\frac{Z - Z_1}{Z_2 - Z_1} & \text{if } Z_1 \leq Z \leq Z_2, \\
\frac{Z - Z_1}{Z_3 - Z_2} & \text{if } Z_1 \leq Z \leq Z_3,
\end{cases}
$$

where $Z_1$, $Z_2$ and $Z_3$ are given in Eqn. 11.

By using Eqn. 1, we obtain the fuzzy index value of fuzzy inventory cost as follows:

$$
I(\tilde{Z}) = \frac{Z_1}{4} + 2Z_2 + Z_3
$$

$$
= \frac{1}{4T}[R_{11} + R_{21} + R_{31} + B + 2(R_{12} + R_{22} + R_{32} + B) + R_{13} + R_{23} + R_{33} + B]
$$

$$
= \frac{1}{4T}[(R_{11} + 2R_{12} + R_{12}) + (R_{21} + 2R_{22} + R_{23}) + (R_{31} + 2R_{32} + R_{33}) + 4B]
$$

$$
= \frac{1}{4T}(X + \frac{Y}{2} + Z + \frac{1}{2} + B).
$$

(12)

Now, $X, Y$ and $Z$ are stated subsequently:

$$
X = p(1 - f)\frac{T^2}{2}(b + aS) - p(1 - f)\frac{T^3}{2}(bS_0 + aS_{a2}) + p(1 - f)\frac{T^3}{3}(bS_N + cS_N)
$$

$$
- p(1 - f)\frac{T^3}{3}(bS_{a2} + aS_{a2}) + (\frac{bS_0}{2} + \frac{aS_{a2}}{2})T^2 - (bS_0 + aS_0)T^2 + (\frac{bS_0}{2})
$$

$$
+ \frac{bS_0}{6} + \frac{bS_{a2}}{6} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a}T^3 - \frac{1}{2}(bS_0 + aS_0 + aS_{a2})T^2 - (\frac{bS_0}{2})T^2
$$

$$
+ \frac{aS_{a2}}{2}T^3 + \frac{bS_0}{6} + \frac{bS_{a2}}{6} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a}T^3 - \frac{1}{2}(bS_0 + aS_0 + aS_{a2})T^2 - (\frac{bS_0}{2})T^2
$$

$$
+ \frac{aS_{a2}}{24}T^5 - (\frac{bS_0}{24} + \frac{bS_{a2}}{24} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a})T^3 - (\frac{bS_0}{24} + \frac{bS_{a2}}{24} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a})T^3
$$

$$
+ \frac{aS_{a2}}{24}T^3T^2 + (\frac{bS_0}{24} + \frac{bS_{a2}}{24} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a})T^3 - (\frac{bS_0}{24} + \frac{bS_{a2}}{24} + \frac{S_{a2}}{a} + \frac{aS_0}{6} + \frac{cS_0}{a})T^3
$$

$$
= pf[^{4a_1}aT_1 + \frac{aS_N}{2}T^2 + \frac{4a_1}{2}T^2 + \frac{cS_N}{3}T^3] + c_1[S_{b2}TT^2 + \frac{S_{a2}}{2}TT^2 - \frac{1}{6}S_{a2}T^3
$$

$$
- S_{\beta}T^2 - T^2 - \frac{S_{a2}}{3}T^3],
$$

$$
Z = (T - T_2)(sS_\beta - sS_{a2}) + \frac{T^2}{2}(s + a)S_a + cS_\beta - sS_{a2} + \frac{cS_0}{3}(T^3 - T^2).
$$

Here, $S_a = \alpha_1 + 2\alpha_2 + \alpha_3$, $S_\beta = \beta_1 + 2\beta_2 + \beta_3$, $S_\theta = \theta_1 + 2\theta_2 + \theta_3$, $S_5 = \delta_1 + 2\delta_2 + \delta_3$, $S_N = N_1 + 2N_2 + N_3$, $S_{a2} = \alpha_1\alpha_2 + \beta_2\alpha_2 + \beta_3\alpha_3$, $S_{a\beta} = \alpha_1\beta_2 + 2\beta_2\beta_3 + \alpha_3\beta_3$, $S_{a\alpha} = N_1\alpha_1 + 2N_2\alpha_2 + \alpha_3N_3$, $S_{ab\alpha} = \alpha_1^2\alpha_2 + 2\alpha_2^2\alpha_3 + \alpha_3^2\alpha_3$, etc.

Particular cases (for fuzzy model):

(i) If $\alpha_1 \rightarrow \alpha_2$ and $\alpha_3 \rightarrow \alpha$, $\beta_1 \rightarrow \beta_2$ and $\beta_3 \rightarrow \beta$, $\theta_1 \rightarrow \theta_2$ and $\theta_3 \rightarrow \theta_2$, $\delta_1 \rightarrow \delta_2$ and $\delta_3 \rightarrow \delta$, $N_1 \rightarrow N_2$ and $N_3 \rightarrow N_2 \rightarrow N$, then $X \rightarrow 4R_1$, $Y \rightarrow 4R_2$ and $Z \rightarrow 4R_3$, so $I(\tilde{Z}) = \frac{1}{2}(R_1 + R_2 + R_3 + B)$, which is equal to $Z(T, T_1, T_2)$. This is a partially back-ordered EPQ model under constant inflation, constant deterioration rate and linearly increasing demand with time.
(ii) If $T_2 \to T$, then:
$I(\tilde{Z}) \to \frac{1}{T}[R_1 + pf(a_1 a T_1 + a_2 N T_1^2 + c_1 T_1^3 + c_2 N T_1^3)].$

This is an EPQ Model with constant inflation rate, constant deterioration rate and
time-dependent demand rate.

**Membership function of objective function and index value under cloudy-fuzzy environment:**
Using the definition of membership function of cloudy-fuzzy number, the membership function for the fuzzy total inventory cost per unit time under the cloudy-fuzzy model is given by

$$C_p(Z) = \begin{cases} 
0, & \text{if } Z < Z_1^c \text{ and } Z > Z_3^c, \\
\frac{Z - Z_1^c}{Z_3^c - Z_1^c}, & \text{if } Z_1^c \leq Z \leq Z_2^c, \\
\frac{Z_3^c - Z}{Z_3^c - Z_2^c}, & \text{if } Z_2^c \leq Z \leq Z_3^c,
\end{cases}$$

where $Z_1^c$, $Z_2^c$ and $Z_3^c$ are stated in Eqn. 11. Now the index value of cloudy-fuzzy total inventory cost per unit time from Eqn. 3 is given below:

$$CI(\tilde{Z}) = \frac{1}{\psi} \int_{T=0}^{\psi} \left(\frac{R_1}{T} + 2R_{12}^c + R_{13}^c\right) dT$$

$$= \frac{1}{4\psi} \int_{T=0}^{\psi} \left[R_1 + 2R_{12}^c + R_{13}^c\right] dT + \int_{T=0}^{\psi} \left[R_2 + 2R_{22}^c + R_{23}^c\right] dT$$

$$+ \int_{T=0}^{\psi} \left[R_{31} + 2R_{32}^c + R_{33}^c\right] dT + \int_{T=0}^{\psi} 4B d\psi$$

$$= \frac{1}{4\psi} \left[4R_1 - 4(T_1(s\beta - s\delta + a\beta) + \frac{T_2^2}{2}(\alpha s - \alpha s\delta + a\alpha - c\beta) + \frac{cN}{3} T_1^3)\right]$$

$$+ 4pf\left(a_1 a T_1 + \frac{aN}{2} T_1^2 + \frac{cN}{2} T_1^3\right) - 4c_1(\frac{\delta\beta}{2} T_2^2 + \frac{\delta\alpha}{3} T_3^3) + 4B \log \frac{\psi}{\epsilon}$$

$$+ \frac{1}{4\psi} \left[4(s\beta - s\delta + a\beta) + 4c_1(\delta\beta T_2 + \frac{\delta\alpha}{2} T_2^2)\right] \frac{\psi}{\epsilon} + \frac{1}{4\psi} \left[4(\alpha s - \alpha s\delta + a\alpha + c\beta) - \frac{\delta\beta}{4} \frac{\psi^2}{\epsilon} + \frac{1}{4\psi} \left[\frac{4\alpha a}{3} - 4c_1 \frac{\delta\beta}{6} \frac{\psi^2}{\epsilon} + \frac{\psi}{\epsilon} (\gamma - \rho)(A_2 - T_2(s\beta - 2s\delta + 2a\beta))\right] - \frac{T_2}{2} \left(\alpha s - 2s\alpha\delta + a\alpha + c\beta\right) - \frac{cN}{3} T_1^3 + pf\left(\frac{aN}{2} T_1^2 + \frac{cN}{3} T_1^3\right)\right] \log \frac{\psi}{\epsilon(1 + \psi)}$$

$$((s\beta - s\delta + 2a\beta) - \frac{\alpha a}{2} - \alpha a\delta + \frac{c\beta}{2} + \frac{\alpha a}{2} + \frac{c\beta}{2} + \frac{\alpha a}{2}) + c_1(2\beta T_2 + \delta\alpha T_2^2)$$

$$- \frac{\delta\alpha}{3} - \delta\beta T_2^2 - \frac{2c\beta}{3} T_3^3)) \log(1 + \psi) + \frac{\alpha a}{3} - \frac{\delta\alpha}{3} \frac{\psi^2}{\epsilon} + \frac{\delta\alpha}{3} - \delta\beta + \frac{\alpha a}{2} - s\alpha\delta + \frac{c\beta}{2} - \frac{\alpha a}{2})$$

$$\frac{\psi}{\epsilon(1 + \psi)}.$$
$T_2^2T_1 + \frac{1}{2}(2b\beta \theta - 2bN\beta - 3\theta^2a\alpha - 2\theta c\beta)T_2T_1^2 + \left(\frac{b\alpha \theta}{3} + \frac{bN\alpha}{4} + \frac{bN\beta \theta}{8} + \frac{\theta^2a\alpha}{2}\right)
+ \left(\frac{\theta c\alpha}{4} + \frac{\theta^2c\beta}{8}\right)T_2^4 + \left(\frac{bN\alpha \theta}{8} + \frac{\theta c\alpha}{8}\right)T_2^5 - \frac{3}{4}(bN\alpha \theta + \theta c\alpha)T_2T_1^2 + (bN\alpha \theta + \theta^2c\alpha)$

$T_2^2T_1^2 - \frac{3}{8}(bN\alpha \theta + \theta^2c\alpha)T_2T_1^4 - \frac{1}{2}(2b\alpha \theta + 3\theta^2a\alpha)T_2^2T_1 + (b\alpha \theta - \frac{bN\alpha}{2} - \frac{3bN\beta \theta}{4})$

$+ \left(\frac{3\theta^2a\alpha}{2} - \theta c\alpha - \frac{3\theta^2c\beta}{2}\right)T_2^2T_1^2 - (b\beta + 2a\beta \theta)T_2T_1$

$- \left(\frac{b\alpha \theta}{3} - bN\beta \theta + \frac{\theta c\alpha}{3} - \theta c\beta\right)T_2T_1^3$.

**Stability analysis and particular cases (cloudy-fuzzy model):**

(i) If $\gamma - \rho \to 0$, then $CI(Z) \to \frac{1}{4\psi}(4R_1 - 4(T_2(s\beta - s\beta\delta + a\delta\beta) + \frac{b\alpha \theta}{4})\psi$.

(ii) If $\gamma \to 0$, $\rho \to 0$, then our model reduces to (i); hence, we choose $\epsilon$ in such way that the above reduces to the partially back-order EPQ model with constant inflation, constant deterioration rate and time-dependent demand, i.e.,

$\frac{1}{4\psi}(4R_1 - 4(T_2(s\beta - s\beta\delta + a\delta\beta) + \frac{b\alpha \theta}{2})\psi$.

Now, comparing left hand side and right hand side of above expression, we get:

$\frac{\log \frac{2}{\psi}}{\psi} = \frac{1}{4}$ and $\frac{2}{\psi} = T$ which gives $\psi = 2T$.

If we take $T = 1$, then $\psi = 2$, which implies that $\epsilon \to 2e^{-2} < 1$. Thus the model is stable and in this case we write $\psi = 2T$.

**Application of cloudy-fuzzy number to the parameters of inventory model:**

Measuring of fuzziness always depends on the question about which quantities are going to be measured. In an inventory system, the cycle-time is a crucial decision variable and most of the parameters directly or indirectly depend on the cycle-time. Usually, in an inventory system, the uncertainties are viewed as high with respect to time increases. So, it becomes necessary for a system manager to remove all types of ambiguities from the system.

A decision maker (DM) usually has no idea how the inflation rate will affect the selling price or purchasing cost of the raw material for an inventory system. Thus, ambiguity of the inflation rate occurs from the beginning. But, as time increases, the DM starts to acquire knowledge about the nature of the given inflation. As time passes, DM learn whether the inflation value is below the crisp value or above the crisp value or approximately the same as the crisp value. Therefore, from Section 3 we conclude that the application of cloudy-fuzzy number to the parameters, specially, the inflation rate, has a justification because a cloudy-fuzzy number depend upon time.
7. **Numerical example.** We now consider an inventory system with the following parametric values.

**Crisp model:** Let us assume, $B = $500 per cycle; $p = 1000$ units item per cycle; $b = $5; $N = $0.7 per week; $s = $30 per unit item per week; $a = $1 per unit item per order; $a_1 = $3 per unit item per week; $c_1 = $19 per unit item week; $f = 0.006$ units item per week; $\alpha = 0.7$ units item per week; $\beta = 3$ units item per week; $\delta = 0.005$ units item per week; $\theta = 0.06$ units item per week; $c = $0.5 per unit item per order. By using Mathematica software, we have the optimal solution of crisp model; the optimal solution is $Z^* = $206.931 and $T^* = 4.21999$ weeks, $T_1^* = 0.0101996$ weeks and $T_2^* = 3.16691$ weeks. Figure 2 shows the convexity of the total cost of our proposed crisp model.

**Fuzzy model:** Let us consider the demand rate $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle 0.63, 0.7, 0.84 \rangle$, $\langle \beta_1, \beta_2, \beta_3 \rangle = \langle 2.7, 3, 3.6 \rangle$, the inflation rate $\langle N_1, N_2, N_3 \rangle = \langle 0.63, 0.7, 0.84 \rangle$, and the deterioration rate $\langle \theta_1, \theta_2, \theta_3 \rangle = \langle 0.058, 0.06, 0.072 \rangle$ and partially back-ordered rate $\langle \delta_1, \delta_2, \delta_3 \rangle = \langle 0.0045, 0.005, 0.006 \rangle$, and while the other parameters remain same as the crisp model’s parameters. By using Mathematica software, we get the optimal solution of our proposed fuzzy model as: the solution is $Z^* = $213.634 and $T^* = 4.84194$ weeks, $T_1^* = 0.00915884$ weeks and $T_2^* = 2.80802$ weeks. Figure 4 shows the convexity of the total cost of our proposed fuzzy model.
Now, for computing the degree of fuzziness, we need to find the mode of every fuzzy parameters. Mean of \( \langle 0.63, 0.7, 0.84 \rangle = 0.72 \), Mean of \( \langle 2.7, 3.3, 6 \rangle = 3.1 \), Mean of \( \langle 0.63, 0.7, 0.84 \rangle = 0.72 \), Mean of \( \langle 0.058, 0.06, 0.072 \rangle = 0.06 \), and Mean of \( \langle 0.0045, 0.005, 0.006 \rangle = 0.0052 \).

Hence, Mode \( \alpha_m = 3 \times \text{Median} - 2 \times \text{Mean} = 0.66 \), Mode \( \beta_m = 2.8 \), Mode \( N_m = 0.66 \), Mode \( \theta_m = 0.054 \) and Mode \( \delta_m = 0.0046 \), where \( \alpha_m, \beta_m, N_m \) and \( \theta_m, \delta_m \) are respective modes of the fuzzy parameters \( \tilde{\alpha}, \tilde{\beta}, \tilde{N} \) and \( \tilde{\theta}, \tilde{\delta} \). We apply the formula for finding the degree of fuzziness: 
\[
d_f = \frac{(u_b - l_b)}{(2M)},
\]
where \( u_b, l_b \) and \( M \) are upper bound, lower bound and mode of corresponding fuzzy parameter, respectively.

\[\text{Figure 3. Graphical representation of the convexity of total cost of our model in fuzzy environment. The figure represents the total cost } I(\tilde{Z}), T_1 \text{ and } T_2, \text{ along the axis of blue colour, the axis of red colour and the axis of green colour, respectively.}\]

Cloudy-fuzzy model: Let us consider \( \gamma = 0.17, \rho = 0.15, \) and \( \epsilon = 0.2 \) and the other parameters remain same as the crisp model. Therefore, from Eqn. (6.12), we get the results as: the total cost \( Z^* = $169.752 \) with optimal cycle-time \( T^* = 5.72418 \) weeks, optimal production time \( T_1^* = 0.0101397 \) weeks and optimal non-production time with out shortages duration, \( T_2^* = 3.15275 \) weeks.

8. Discussion of Tables 2, 3 and 4. From Table 2, we observe that in a crisp environment, the optimal inventory cost per unit time of our proposed model is $206.931 with the approximate cycle-time of 5 weeks, and while the optimal inventory cost in a fuzzy environment is $213.634. It is surprising that when we regard the cloudy-fuzzy environment, then the total inventory cost per unit time reduces to
$169.901$ by increasing the cycle-time from $4.21999$ weeks to $5.72418$ weeks, whereas at same time it is observed that the degree of fuzziness has been reduced from $0.16$ to $0.14$.

From Tables 3 - 4 we further learn that within the time gap of cycle-time $2$ to $10$ weeks, the total inventory cost function per unit time of the crisp, fuzzy and cloudy-fuzzy model follows a convex curve at over $5$, $4$ and $6$ weeks, respectively. Figure 5 shows that, in a cloudy-fuzzy environment, the total inventory cost per unit time has a global minimum value at cycle-time of $6$ weeks. Although, if we investigate the tendency of changing the values of $T_1$ and $T_2$ with cycle-time, then we see that their tendency is to grow in all environments.

The advantages of the cloudy results are reflected subsequently:

(i) The cloudy-fuzzy model always gives the average total cost of the model.

(ii) Less fuzziness does not mean more profit by the model.

(iii) Diminishing the cloud index decreases the uncertainty of the total cost function as well as parameters.

(iv) For DM, a cloudy-fuzzy environment is quite easy to understand, and it is convenient to make a decision accordingly.

(v) Cloudy-fuzzy environment is a perfect choice to solve an inventory system under uncertainty.
Table 2: Optimal solutions of our models.

| Model       | $T^\ast$(week) | $T_1^\ast$(week) | $T_2^\ast$(week) | Minimum cost $Z^\ast($($) | $d_f = \frac{u_k-l_k}{2M}$ | $CI= \frac{1}{ln(1+T)}$ |
|-------------|----------------|------------------|------------------|--------------------------|--------------------------|--------------------------|
| Crisp       | 4.21999        | 0.0101996        | 3.16691          | 206.931                  | —                        | —                        |
| Fuzzy       | 4.84194        | 0.00915884       | 2.80802          | 213.634                  | 0.16                     | —                        |
| Cloudy fuzzy| 5.72418        | 0.0101397        | 3.15275          | 169.752                  | —                        | 0.14                     |

Table 3: Cycle-time variation in crisp model and fuzzy model.

| Cycle time | Crisp model | Fuzzy model |
|------------|-------------|-------------|
|            | $T_1$ | $T_2$ | $Z^\ast$ | $T_1$ | $T_2$ | $Z^\ast$ |
| 2          | 0.0102881 | 3.18061 | 253.543 | 0.00921317 | 2.82304 | 265.894 |
| 3          | 0.0102483 | 3.17444 | 218.974 | 0.00919408 | 2.81776 | 228.391 |
| 4          | 0.0102084 | 3.16827 | 207.867 | 0.00917496 | 2.81248 | 215.966 |
| 5          | **0.0101665** | 3.16208 | **206.271** | **0.00915582** | **2.80719** | **213.7** |
| 6          | 0.0101286 | 3.15589 | 209.536 | 0.00913665 | 2.80188 | 216.623 |
| 7          | 0.0100887 | 3.14968 | 215.667 | 0.00911746 | 2.79657 | 220.391 |
| 8          | 0.0100488 | 3.14347 | 223.674 | 0.00909825 | 2.79125 | 226.574 |
| 9          | 0.0100089 | 3.13725 | 232.997 | 0.00907901 | 2.78591 | 230.544 |
| 10         | 0.009997 | 3.13101 | 243.307 | 0.00905975 | 2.78057 | 250.359 |

* Bold represents optimal solution.

9. Analysis of Figures 5 and 6. From Figure 6 we see that there is a larger difference between the total inventory costs of the fuzzy model as well as crisp model, when compared with the cloudy-fuzzy model. Furthermore, we notice that the highest inventory costs obtained in fuzzy model and crisp model, respectively, and cloudy-fuzzy model have the lowest value of the inventory cost with respect to all time. Hence, the cloudy-fuzzy environment is an excellent choice to model and solve an inventory system under uncertainty. From Figure 5 it is seen that the cloudy objective function gives an approximate alphabet letter U-tern shape over cycle-time of 6 weeks; thus, it is convex.

9.1. Comparison the result obtained from proposed approach with Karbassi Yazdi et al. [19] model. A comparative study on EPQ model under space constraint with different kinds of data was derived by Karbassi Yazdi et al. [19]. In the paper [19], the concept of fuzzy number and grey number were employed to handle the uncertain information on demand rate of a multi-item economic production quantity model. According to the paper [19], the grey number of a crisp number $x$ is denoted by $\otimes x$, and defined by $\otimes x = [x_l, x_u]$, where $x_l$ and $x_u$ are lower and upper bounds, respectively, of the grey number.

To change a grey number to a crisp number, the following formula is used:

$$x = \alpha x_l + (1 - \alpha) x_u, \alpha \in [0, 1].$$

Now, we treat the uncertain parameters, such as $\alpha, \beta, \theta, \delta, \text{ and } N$ of our proposed model as grey numbers, i.e., $\otimes \alpha = [0.63, 0.84]$, $\otimes \beta = [2.7, 3.6]$, $\otimes \theta = [0.058, 0.072]$, $\otimes \delta = [0.0045, 0.006]$, and $\otimes N = [0.63, 0.84]$, and putting these parameters in Eqn. 10, then by applying the above formula of transferring grey number to crisp number, we get the corresponding results: the minimum total cost $Z^\ast$ = $\$212.267$ with
optimal cycle-time $T^* = 4.57666$ weeks, optimal production time $T_1^* = 0.010634$ weeks and optimal non-production time without shortages duration, $T_2^* = 3.09962$ weeks.

From this result and Table 2, we conclude that a cloudy-fuzzy environment gives a better result on an inventory system. Furthermore, a cloudy-fuzzy number converges to its crisp value, but in grey environment it does not do. Hence, a cloudy-fuzzy environment is the best choice, when compared with a fuzzy number and a grey number to handle uncertain information of parameters.

Table 4: Cycle-time variation in cloudy-fuzzy model.

| Cycle time | $T_1$  | $T_2$  | $Z^*$  |
|------------|--------|--------|--------|
| 2          | 0.0102881 | 3.18061 | 233.896 |
| 3          | 0.0101824 | 3.15873 | 195.051 |
| 4          | 0.0101661 | 3.1565  | 177.841 |
| 5          | 0.0101501 | 3.15431 | 170.946 |
| 6          | 0.0101357 | 3.15216 | 169.901 |
| 7          | 0.0101212 | 3.15005 | 172.574 |
| 8          | 0.010107  | 3.14797 | 177.808 |
| 9          | 0.0100932 | 3.14592 | 184.919 |
| 10         | 0.0100796 | 3.1439  | 193.475 |

* Bold represents optimal solution.

Figure 5. Cost variation with respect to cycle-time variation for cloudy-fuzzy model.

10. Sensitivity analysis of cloudy fuzzy model. Let us now investigate the effects of changes of the parameters $B$, $c_1$, $\gamma$, $\rho$, $\alpha$, $\beta$, $\theta$, $s$, $\delta$, $N$ and $\epsilon$ by $+50\%$, $+30\%$, $+15\%$, $-15\%$, $-30\%$ and $-50\%$ on the optimal solution of the model, i.e., $Z^*$, $T^*$, $T_1^*$ and $T_2^*$, which are shown in Tables 5 - 6.

The following observation are obtained on the basis of Table 5.

- The total inventory cost $Z^*$ expanded by 45\% as well as the cycle-time $T^*$ highly increases with the change at $+50\%$ of the value of set-up cost $B$, and hence, we suggest that to decrease the total cost, we reduce the set-up cost.
- Due to growth of deterioration rate $\theta$ from $-50\%$ to $+50\%$, the total optimal cost increases gradually from $-1.35\%$ to $+1.05\%$, furthermore, the cycle-time increases excessively.
- The solution $Z^*$, $T^*$, $T_1^*$ and $T_2^*$ are more sensitive with regard to changes of the parameters $\alpha$ and $\beta$, in fact when $\alpha$ increases up to $+50\%$, the total cost
increases by +12.61% whereas by the increasing of \( \beta \) up to +50%, the total cost as well as cycle-time decrease.

- When changing shortage cost, lost-sale cost and partially back-ordered rate at +50%, the total inventory cost per cycle can reduce by \(-0.33\%\), \(-58.24\%\) and \(-0.35\%\), respectively.

- Total cost is moderately sensitive with respect to changes of inflation rate of money per unit item, \( N \). This indicates that, if the inflation rate increases, then the total cost as well as the cycle-time increase, but the production time decrease. A company always tries to diminish the total cost; thus, a smaller inflation rate of raw materials is more beneficial to a company.

- Regarding a variation of the fuzzy variable \( \gamma \) from \(-50\%\) to +50%, the optimal total cost \( Z^* \) slightly grows from \(-0.4\%\) to +0.17\%, whereas under a variation of the fuzzy parameter \( \rho \), from \(-50\%\) to +50\%, the optimal total cost reduces moderately.

Ultimately, we may say that with alternation of the values of parameters by +50\%, +30\%, +15\%, -15\%, -30\% and -50\%, the total cost increases by 0.02\% to 45\%.

The sensitivity investigation of Table 5 also explores that for any kind of parametric variation, the range of the cycle time is being limited to 1 to 10 weeks.

10.1. Managerial Insights. From analytic models and numerical example demonstration, we conclude the following managerial insights for practitioners:

- Table 2 and Figure 6 reveal that the optimal total inventory cost for the cloudy-fuzzy model with crisp decision variable \( T \) is less than that for the fuzzy model or the crisp model. A cloudy-fuzzy model always gives the average total cost of the model. Less fuzziness does not mean more profit by the model.

- For any DM cloudy-fuzzy environment is quite easy to understand and it is convenient to make a decision accordingly.

- Table 5 and Figure 7 reflect that with increasing of set-up cost \( B \), total inventory cost as well as cycle-time increase, whereas non-production time \( T_2 \) decreases. So the manager or manufacturer should try to ensure for the company an as low as possible set-up cost.
In Table 5, we represent the different optimum values for different values of $\rho$ and $\gamma$, by keeping the values of $N$, $\alpha$, $\theta$, $\delta$ and $\beta$ fixed. We observe that for fixed $\rho$, when $\gamma$ increases, the total cost $Z$ as well as optimal cycle-time $T^*$ increase, whereas for fixed $\gamma$, when $\rho$ increases, the total cost and optimum cycle-time decreases. Thus, the manager of the company should treat the parameters of the inventory management as a cloudy-fuzzy parameter with less $\gamma$ and more $\rho$.

From Figure 8, it can be concluded that $N$ proportionally varies with total cost $Z$ as well as cycle-time $T$.

From Subsection 9.1 it is shown that cloudy-fuzzy solution concept is a better approach for solving inventory model under uncertainty.

Table 5: Sensitivity Analysis of the cloudy-fuzzy model.

| Parameter | % change | New value of parameter | $T$ | $T_1$ | $T_2$ | $Z$ | $\frac{\text{change}}{Z} \times 100\%$ |
|-----------|----------|------------------------|-----|-------|-------|-----|-------------------------------------|
| $B$       | +50      | 750                    | 8.10737 | 0.010155 | 3.14775 | 246.259 | 45                                    |
|           | +30      | 650                    | 7.24503 | 0.01017 | 3.14954 | 217.988 | 28.3                                 |
|           | +15      | 575                    | 6.52651 | 0.010128 | 3.15104 | 194.942 | 14.74                                |
|           | -15      | 425                    | 4.80246 | 0.010153 | 3.15474 | 141.317 | -0.17                                |
|           | -30      | 350                    | 3.68948 | 0.010171 | 3.15718 | 108.416 | -36.19                               |
|           | -50      | 250                    | 1.45517 | 0.010210 | 3.16223 | 45.2895 | -73.34                               |
| $c_1$     | +50      | 70.265                 | 5.75603 | 0.010158 | 3.15869 | 169.34 | -0.33                                |
|           | +30      | 24.7                   | 5.73997 | 0.010143 | 3.15332 | 169.505 | -0.23                                |
|           | +15      | 21.85                  | 5.73204 | 0.010146 | 3.15304 | 169.628 | -0.16                                |
|           | -30      | 13.3                   | 5.70862 | 0.010135 | 3.15216 | 169.997 | 0.06                                 |
|           | -50      | 9.5                    | 5.69837 | 0.010133 | 3.15175 | 170.161 | 0.15                                 |
| $\gamma$  | +50      | 70.265                 | 5.73720 | 0.010175 | 3.13834 | 170.195 | 0.17                                 |
|           | +30      | 650                    | 5.73593 | 0.010161 | 3.14403 | 170.018 | 0.07                                 |
|           | +15      | 625                    | 5.73007 | 0.010196 | 3.14347 | 169.906 | 0.03                                 |
|           | -15      | 144.5                 | 5.73236 | 0.010129 | 3.15716 | 169.617 | -0.17                                |
|           | -30      | 119                    | 5.72331 | 0.010187 | 3.16161 | 169.302 | -0.4                                 |
|           | -50      | 95                     | 5.72138 | 0.010133 | 3.15175 | 170.161 | 0.15                                 |
| $\rho$    | +50      | 650                    | 5.72139 | 0.010109 | 3.16084 | 169.345 | -0.32                                |
|           | +30      | 195                    | 5.72254 | 0.010121 | 3.16056 | 169.514 | -0.23                                |
|           | +15      | 172.5                 | 5.72337 | 0.010134 | 3.15664 | 169.633 | -0.16                                |
|           | -15      | 127.5                 | 5.72496 | 0.010149 | 3.14888 | 169.87 | -0.02                                |
|           | -30      | 105                    | 5.72573 | 0.010158 | 3.14505 | 169.987 | 0.05                                 |
|           | -50      | 72                     | 5.72672 | 0.010173 | 3.13998 | 170.143 | 0.14                                 |
| $\alpha$  | +50      | 650                    | 7.41414 | 0.010304 | 3.11271 | 191.321 | 12.65                                |
|           | +30      | 650                    | 5.74547 | 0.011206 | 3.12682 | 183.808 | 8.19                                 |
|           | +15      | 5.8547                 | 0.011213 | 3.12682 | 183.808 | 8.19                                 |
|           | -15      | 5.74547                 | 0.011206 | 3.12682 | 183.808 | 8.19                                 |
|           | -30      | 5.70432                 | 0.009690 | 3.16852 | 161.137 | -5.16                                |
|           | -50      | 5.68432                 | 0.009690 | 3.16852 | 161.137 | -5.16                                |
| $\beta$   | +50      | 650                    | 4.51458 | 0.015765 | 3.19932 | 136.373 | -19.13                               |
|           | +30      | 650                    | 5.02358 | 0.014956 | 3.18289 | 150.54 | -11.4                                |
|           | +15      | 3.45                   | 5.38207 | 0.011816 | 3.16882 | 160.41 | -5.58                                |
|           | -15      | 2.55                   | 6.05227 | 0.008456 | 3.13411 | 178.618 | 5.13                                 |
|           | -30      | 2.1                    | 6.36819 | 0.006794 | 3.11212 | 187.067 | 10.1                                 |
|           | -50      | 1.5                    | 6.77288 | 0.000696 | 3.07548 | 197.757 | 16.4                                 |
| $\theta$  | +50      | 650                    | 5.18734 | 0.015680 | 3.07474 | 171.88 | 1.05                                 |
|           | +30      | 650                    | 5.62698 | 0.010257 | 3.10254 | 170.932 | 0.61                                 |
|           | +15      | 650                    | 5.74389 | 0.010215 | 3.12725 | 170.351 | 0.61                                 |
|           | -15      | 5.70382                 | 0.010061 | 3.17908 | 169.133 | -0.45                                |
|           | -30      | 6.8276                  | 0.009799 | 3.20629 | 168.493 | -0.83                                |
|           | -50      | 5.65355                 | 0.0098641 | 3.24406 | 167.606 | -1.35                                |

Continuation of Table 5.
## Back-Ordered Inventory Model with Inflation

| Parameter | % change | New value of parameter | $T_1$ | $T_2$ | $Z_*$ | $\frac{Z_*/Z^*}{2} \times 100\%$ |
|-----------|----------|------------------------|-------|-------|-------|-------------------------------|
| $s$       | +50      | 45                     | 1.60044 | 0.0162324 | 4.01898 | 70.9432 | -58.24 |
|           | -30      | 39                     | 3.51682 | 0.0138168 | 3.69291 | 129.89 | -23.55 |
|           | +15      | 34.5                   | 4.6588 | 0.0119896 | 3.43165 | 154.538 | -9.04 |
|           | -15      | 24.5                   | 7.00539 | 0.007835 | 2.78187 | 179.177 | 5.46 |
|           | -30      | 21                     | 7.83848 | 0.00634301 | 2.52404 | 180.747 | 6.38 |
|           | -50      | 15                     | 9.35126 | 0.00367255 | 2.02744 | 176.399 | 3.82 |
| $\delta$  | +50      | 0.0075                 | 5.76822 | 0.0101149 | 3.1489 | 169.312 | -0.35 |
|           | +30      | 0.0065                 | 5.75046 | 0.0101249 | 3.14046 | 169.489 | -0.24 |
|           | +15      | 0.00575                | 5.73726 | 0.0101324 | 3.15161 | 169.621 | -0.16 |
|           | -15      | 0.00425                | 5.71112 | 0.010147 | 3.15388 | 169.882 | -0.01 |
|           | -30      | 0.0035                 | 5.69833 | 0.0101543 | 3.155 | 170.012 | 0.07 |
|           | -50      | 0.0025                 | 5.68133 | 0.0101638 | 3.15647 | 170.183 | 0.17 |
| $N$       | +50      | 1.05                   | 6.0393 | 0.00775952 | 2.7678 | 179.705 | 5.77 |
|           | +30      | 0.91                   | 5.93784 | 0.008571 | 2.90253 | 176.282 | 3.75 |
|           | +15      | 0.805                  | 5.83929 | 0.00929355 | 3.01911 | 173.268 | 0.02 |
|           | -15      | 0.595                  | 5.58678 | 0.011551 | 3.30832 | 165.573 | -2.55 |
|           | -30      | 0.49                   | 5.41893 | 0.0124013 | 3.49299 | 160.49 | -5.54 |
|           | -50      | 0.35                   | 5.12323 | 0.0146078 | 3.8047 | 151.594 | -10.77 |
| $\epsilon$| +50      | 0.3                    | 5.27247 | 0.0101332 | 3.15255 | 159.446 | -6.15 |
|           | +30      | 0.26                   | 5.43884 | 0.0101356 | 3.15261 | 163.181 | -3.95 |
|           | +15      | 0.23                   | 5.57568 | 0.0101376 | 3.15267 | 166.293 | -2.12 |
|           | -15      | 0.17                   | 5.89021 | 0.0101421 | 3.15284 | 173.664 | 2.21 |
|           | -30      | 0.14                   | 6.07952 | 0.0101448 | 3.15295 | 178.199 | 4.88 |
|           | -50      | 0.1                    | 6.38786 | 0.010149 | 3.15313 | 185.743 | 9.32 |

* Bold shows the most sensitive total inventory cost.

**Figure 7.** Graph of Set up cost vs inventory total cost.

### Conclusion and direction of future research

In this paper, an EPQ model is developed under a cloudy-fuzzy environment, where deterioration of items and inflation rate of money are considered. Furthermore, shortages are allowed with partial back-ordered. From literature, we have seen that most of the EPQ models
are developed either in crisp, general fuzzy or in stochastic environment. But, to establish and solve model with uncertainties, a cloudy-fuzzy environment leads to a best solutions. In this paper, a crisp model and a fuzzy model are developed first, then the cloudy-fuzzy environment is applied. We have observed that the minimum inventory costs of crisp and fuzzy model are higher than the minimum inventory cost of the corresponding cloudy-fuzzy model. Not only that, the total cycle-time in crisp and fuzzy environment takes a higher value than the cloudy-fuzzy value. The degree of fuzziness in cloudy-fuzzy model is less than the degree of fuzziness in fuzzy model and cloudy minimization problem provides smaller values. This result is realistic in application and it is the main aim of our model. If no information is available to a company about how many customers or retailers are interested to purchase their products or items and what will be the back-ordered rate of products during shortage time or how will be high the inflation rate, then we confirm that our new model is more realistic for those companies. From comparison, the result obtained by our proposed approach with an existing paper Karabassi Yazdi et al. [19], we conclude that cloudy-fuzzy environment is a better way to handle the uncertainties in this respect.

However, in scientific research, every work has its merits and demerits; therefore, our model has few advantages and limitations. The main advantages of our model is that a cloudy-fuzzy number basically depends on the time variable such as cycle-time, and this type of fuzzy variation is generally occurs due to learning effects in decision making problem. For this reason, for any DM it is convenient to understand and to make a decision accordingly. The limitation is that the calculations of fuzzification and defuzzification are rather lengthy. This approach is not applicable for strictly increasing or strictly decreasing parameters.

In future research, our novel model can be extended in several ways. One can further enhance this model under a “dense fuzzy number” environment [1]. Graded
Mean Integration representation method can be employed for defuzzification from cloudy fuzzification. Carbon emission may be introduced into our advanced model as well. In the future, we can apply the “sign distance” method [23] in our model to defuzzify the fuzzy objective function. Eventually, our model could become the basis of an attractive research direction, where researchers can replace the demand rate by carbon-concerned demand rate, and they can consider imperfect quality deteriorating items, too.

Acknowledgments. The authors are very much thankful to the Editor-in-Chief and the anonymous referee for his or her valuable comments.

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Received June 2019; revised October 2019.

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