A coupled KPZ equation, its two types of approximations and existence of global solutions

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March 13, 2018

Abstract

This paper concerns the multi-component coupled Kardar-Parisi-Zhang (KPZ) equation and its two types of approximations. One approximation is obtained as a simple replacement of the noise term by a smeared noise with a proper renormalization, while the other one introduced in [6] is suitable for studying the invariant measures. By applying the paracontrolled calculus introduced by Gubinelli et al. [8, 9], we show that two approximations have the common limit under the properly adjusted choice of renormalization factors for each of these approximations. In particular, if the coupling constants of the nonlinear term of the coupled KPZ equation satisfy the so-called “trilinear” condition, the renormalization factors can be taken the same in two approximations and the difference of the limits of two approximations are explicitly computed. Moreover, under the trilinear condition, the Wiener measure twisted by the diffusion matrix becomes stationary for the limit and we show that the solution of the limit equation exists globally in time when the initial value is sampled from the stationary measure. This is shown for the associated tilt process. Combined with the strong Feller property shown by Hairer and Mattingly [12], this result can be extended for all initial values.

1 Introduction and main results

1.1 Coupled KPZ equation

We consider the following $\mathbb{R}^d$-valued coupled KPZ equation for $h(t, x) = (h^\alpha(t, x))_{\alpha=1}^d$ defined on the one dimensional torus $T = \mathbb{R}/\mathbb{Z} = [0, 1)$:

\begin{equation}
\partial_t h^\alpha = \frac{1}{2} \partial_x^2 h^\alpha + \frac{1}{4} \Gamma_{\beta\gamma} \partial_x h^\beta \partial_x h^\gamma + \sigma_{\beta\xi}^\alpha \xi^\beta, \quad x \in T,
\end{equation}

where $\Gamma_{\beta\gamma}$ and $\sigma_{\beta\xi}^\alpha$ are the diffusion matrix and the noise matrix, respectively.

Keywords: Stochastic partial differential equation, KPZ equation, Paracontrolled calculus, Renormalization.

2010 MSC: 60H15, 82C28.

The first author is supported in part by the JSPS KAKENHI Grant Numbers (S) 24224004, (S) 16H06338, (B) 26287014 and 26610019. The second author is supported by JSPS KAKENHI, Grant-in-Aid for JSPS Fellows, 16J03010.
for $1 \leq \alpha \leq d$. Here summation symbols $\sum$ over $\beta$ and $\gamma$ are omitted by Einstein’s convention. $(\sigma_\beta^\alpha)_{1 \leq \alpha, \beta \leq d}$ and $(\Gamma^\alpha_{\beta \gamma})_{1 \leq \alpha, \beta, \gamma \leq d}$ are given constants, and $\xi(t,x) = (\xi^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise. In particular, it has the covariance structure

$$E[\xi^\alpha(t,x)\xi^\beta(s,y)] = \delta^{\alpha \beta} \delta(x-y)\delta(t-s),$$

where $\delta^{\alpha \beta}$ denotes Kronecker’s $\delta$. We always assume that the coupling constants $\Gamma^\alpha_{\beta \gamma}$ satisfy

$$\Gamma^\alpha_{\beta \gamma} = \Gamma^\alpha_{\gamma \beta}$$

for all $\alpha, \beta, \gamma$, and the diffusion matrix $\sigma = (\sigma_\beta^\alpha)_{1 \leq \alpha, \beta \leq d}$ is invertible. The symmetry or bilinearity (1.2) of $\Gamma^\alpha = (\Gamma^\alpha_{\beta \gamma})_{\beta \gamma}$ for each $\alpha$ is natural due to the form of the equation (1.1).

One of the motivations to study the coupled KPZ equation (1.1) comes from the nonlinear fluctuating hydrodynamics recently discussed by Spohn and others [5, 18, 19], whose origin goes back to Landau. From microscopic systems with random evolutions, in a proper space-time scaling, one can derive certain nonlinear partial differential equations (PDEs) as a result of a local average due to the local ergodicity. This procedure is called the hydrodynamic limit. If the system has $d$ (local) conserved quantities, we have a system of $d$ coupled nonlinear PDEs in the limit. The noises in the microscopic systems are averaged out and disappear in the macroscopic limit equations. However, if we consider a linearization of this system around a global equilibrium, the noise terms survive in a proper scaling and we obtain linear stochastic PDEs (SPDEs) in the limit. At least heuristically, if the system involves a weak asymmetry and if we expand the equation to the second order, one can expect to obtain the coupled KPZ equations in the limit in a proper scaling. If some of $\Gamma^\alpha_{\beta \gamma}$ are degenerate, then the solution involves different scalings such as diffusive, KPZ or (anomalous) Lévy type scalings.

### 1.2 Two approximating equations

The coupled KPZ equation (1.1) itself is ill-posed, so that we need to introduce its approximations; see [7] for a scalar-valued KPZ equation. A simple approximation of (1.1) is defined as follows. Let $\eta \in C_0^\infty(\mathbb{R})$ be a function satisfying $\eta(x) = \eta(-x)$ and $\int_\mathbb{R} \eta(x)\,dx = 1$; note that $\eta$ may not be non-negative. We set $\eta^\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon > 0$ and consider the $\mathbb{R}^d$-valued KPZ approximating equation for $h = h^\varepsilon(t,x) \equiv (h^\varepsilon,\alpha(t,x))_{\alpha=1}^d$ with a smeared noise and a proper renormalization:

$$\partial_t h^\varepsilon,\alpha = \frac{1}{2} \partial^2_x h^\varepsilon,\alpha + \frac{1}{2} \Gamma^\alpha_{\beta \gamma} (\partial_x h^\varepsilon,\beta \partial_x h^\varepsilon,\gamma - c^\varepsilon A^{\beta \gamma} - B^{\varepsilon,\beta \gamma}) + \sigma_\beta^\alpha \sigma_\delta^\varepsilon \delta^\varepsilon_{\beta \gamma} \eta^\varepsilon,$$

for $1 \leq \alpha \leq d$, where $A^{\beta \gamma} = \sum_{\delta=1}^d \sigma_\beta^\delta \sigma_\delta^\gamma$, $c^\varepsilon = \frac{1}{\varepsilon} \|\eta\|_{L^2(\mathbb{R})}$ and $B^{\varepsilon,\beta \gamma}$ is a renormalization factor defined in Section 4, which diverges as $O(\log \varepsilon^{-1})$ as $\varepsilon \downarrow 0$ in general. We consider $\varepsilon > 0$ small enough, so that the support of $\eta^\varepsilon$ is in the interval $(-1/2, 1/2)$.

Another approximation of (1.1) suitable for studying invariant measures is introduced as follows. Let $\eta_2(x) = \eta \ast \eta(x)$, $\eta_3^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$ and consider the following $\mathbb{R}^d$-valued
equation for $\tilde{h} = \tilde{h}^\varepsilon(t, x) \equiv (\tilde{h}^{\varepsilon, \alpha}(t, x))_{\alpha=1}^{d}$ with a smeared noise and a proper renormalization:

$$
\partial_t \tilde{h}^{\varepsilon, \alpha} = \frac{1}{2} \partial^2_{\xi} \tilde{h}^{\varepsilon, \alpha} + \frac{1}{2} \Gamma^\alpha_{\beta \gamma} (\partial_\beta \tilde{h}^{\varepsilon, \beta} \partial_\gamma \tilde{h}^{\varepsilon, \gamma} - c^\varepsilon A^{\alpha \beta \gamma} - \tilde{B}^{\varepsilon, \beta \gamma}) * \eta^\varepsilon + \sigma^\varepsilon \xi^{\beta} * \eta^\varepsilon,
$$

for $1 \leq \alpha \leq d$, where $\tilde{B}^{\varepsilon, \beta \gamma}$ is a renormalization factor defined in Section [3] which diverges as $O(\log \varepsilon^{-1})$ as $\varepsilon \downarrow 0$ in general. We assume that the support of $\eta^\varepsilon$ is in $(-1/2, 1/2)$. The difference of (1.4) from (1.3) is that it has a convolution factor $* \eta^\varepsilon$ in the nonlinear term.

In [6], assuming that $\sigma$ is an identity matrix $I$, under the additional assumption, which we call the trilinear condition, on $\Gamma$:

$$
\Gamma^\alpha_{\beta \gamma} = \Gamma^\alpha_{\gamma \beta} = \Gamma^\beta_{\gamma \alpha},
$$

for all $\alpha, \beta, \gamma$, the infinitesimal invariance of the smeared Wiener measure for the tilt process $u^\varepsilon = \partial_\varepsilon \tilde{h}^\varepsilon$ of the solution $\tilde{h}^\varepsilon$ of (1.4) with $\tilde{B}^{\varepsilon, \beta \gamma} = 0$ is shown (actually on $\mathbb{R}$ instead of $T$). Namely, let $(B_x)_{x \in T} = ((B^d_{\xi})_{\xi=1}^d)_{x \in T}$ be the $d$-dimensional periodic Brownian motion such that $B_0 = B_1 = 0$ a.s. Then the distribution of $\partial_\varepsilon (B * \eta^\varepsilon)$ is infinitesimally invariant for $u^\varepsilon = \partial_\varepsilon \tilde{h}^\varepsilon$ determined from (1.4) with $\sigma = I$ and $\tilde{B}^{\varepsilon, \beta \gamma} = 0$.

This result can be easily extended to our general setting with $\sigma$. Indeed, let $\tilde{h}^\varepsilon = (\tilde{h}^{\varepsilon, \alpha})$ be the solution of (1.4) and set $\tilde{h}^{\varepsilon, \alpha} := \tau^\beta_{\alpha} \tilde{h}^{\varepsilon, \beta}$, where $\tau = (\tau^\beta_{\alpha})$ is the inverse matrix of $\sigma$. Then, we easily see that $\tilde{h}^\varepsilon = (\tilde{h}^{\varepsilon, \alpha})$ is a solution of (1.4) with $(\sigma^\varepsilon \xi^{\beta} * \eta^\varepsilon, A^{\beta \gamma}, \tilde{B}^{\varepsilon, \beta \gamma}, \Gamma^\beta_{\gamma \beta})$, replaced by $(\tau^\beta_{\alpha} \xi^{\beta}, \delta^{\beta \gamma}, \tau^\beta_{\alpha} \gamma \tau^\gamma_{\beta \gamma} \tilde{B}^{\varepsilon, \beta \gamma}, \Gamma^\beta_{\gamma \beta})$, where

$$
\hat{\Gamma}^\alpha_{\beta \gamma} = \tau^\alpha_{\beta \gamma} \hat{\Gamma}^\alpha_{\beta \gamma} \sigma^\beta_{\gamma \delta} \sigma^\gamma_{\delta} = \tau^\beta_{\alpha} \tau^\beta_{\gamma} \sigma^\beta_{\alpha \gamma} \sigma^\gamma_{\delta} \delta^{\beta \gamma} \sigma^\beta_{\gamma \delta},
$$

which arises from $\Gamma$ under the change of variables $\hat{h} = \tau \tilde{h}$. In fact, $\Gamma$ is a tensor of type (1,2) and $\hat{\Gamma}$ defined by (1.6) is its transform under the change of basis. Note that the bilinearity (1.2): $\hat{\Gamma}^\alpha_{\beta \gamma} = \Gamma^\alpha_{\beta \gamma}$ automatically holds. Therefore, if $\hat{\Gamma}$ determined from $\Gamma$ as in (1.6) satisfies the trilinear condition:

$$
\hat{\Gamma}^\alpha_{\beta \gamma} = \hat{\Gamma}^\alpha_{\beta \gamma} = \hat{\Gamma}^\beta_{\gamma \alpha},
$$

for all $\alpha, \beta, \gamma$, then the distribution of the derivative of the $d$-dimensional periodic and smeared Brownian motion $(\partial_x (\eta^\varepsilon))_{x \in T} = ((\partial_x (\sigma^\varepsilon \xi^\beta  * \eta^\varepsilon(x))_{\xi=1}^d)_{x \in T}$ multiplied by $\sigma$ is infinitesimally invariant for the tilt process $u = \partial_\varepsilon \tilde{h}$ of the solution $\tilde{h}$ of (1.4) with $\tilde{B}^{\varepsilon, \beta \gamma} = 0$.

When $d = 1$ and $\Gamma^\alpha_{\beta \gamma} = \sigma^\beta_{\gamma \delta}$ for simplicity, the approximating equations (1.3) with $\tilde{B}^{\varepsilon, \beta \gamma} = 0$ and (1.4) with $\tilde{B}^{\varepsilon, \beta \gamma} = 0$ have the forms:

$$
\partial_t h = \frac{1}{2} \partial^2_{\xi} h + \frac{1}{2} \left( (\partial_\varepsilon h)^2 - c^\varepsilon \right) * \eta^\varepsilon + \xi * \eta^\varepsilon,
$$

and

$$
\partial_t \tilde{h} = \frac{1}{2} \partial^2_{\xi} \tilde{h} + \frac{1}{2} \left( (\partial_\varepsilon \tilde{h})^2 - c^\varepsilon \right) * \eta^\varepsilon + \xi * \eta^\varepsilon,
$$

respectively. It is shown that the solution of (1.8) converges as $\varepsilon \downarrow 0$ to the so-called Cole-Hopf solution $h_{\text{CH}}(t, x)$ of the KPZ equation [10, 11], while the solution of (1.9) converges to $h_{\text{CH}}(t, x) + \frac{1}{2} t$ under the equilibrium setting [7] and the non-equilibrium setting for a maximal solution [14]. The method of [7] is based on the Cole-Hopf transform, which is not available for our multi-component coupled equation in general.
1.3 Main results

Our first goal is to study the limits of the solutions of two types of approximating equations (1.3) and (1.4) as \( \varepsilon \downarrow 0 \) based on the paracontrolled calculus introduced by Gubinelli et al. [8, 9] as in [14] for \( d = 1 \). Especially, we study the difference between these two limits, which extends the results for the scalar-valued KPZ equation mentioned above. For \( \kappa \in \mathbb{R} \) and \( r \in \mathbb{N} \), \( (C^\kappa)^r := B_{\infty, \infty}^\kappa((\mathbb{T}; \mathbb{R}^r)) \) denotes the \( \mathbb{R}^r \)-valued Besov space on \( \mathbb{T} \). Our first two main theorems are formulated as follows.

**Theorem 1.1.** (1) Let \( 0 < \delta < \delta' < \frac{1}{2} \) be fixed. For every \( h(0) \in (C^\delta)^d \), there exists a unique solution \( h^\varepsilon \) of the KPZ approximating equation (1.3) up to the survival time \( T_{\text{sur}}^\varepsilon \in (0, \infty) \) (i.e. \( T_{\text{sur}}^\varepsilon = \infty \) or \( \lim_{T \uparrow T_{\text{sur}}^\varepsilon} \|h^\varepsilon\|_{C([0,T],[C^\delta]^d)} = \infty \)). With a proper choice of \( B^\varepsilon, B^\beta \), there exists a random time \( T_{\text{sur}}^\varepsilon \in (0, \infty) \) such that \( T_{\text{sur}}^\varepsilon \leq \liminf_{\varepsilon \downarrow 0} T_{\text{sur}}^\varepsilon \) in probability and \( h^\varepsilon \) converges to some \( h \in C([0,T],[C^\delta]^d) \cap C((0,T),(C^\delta)^d) \) in probability for every \( 0 < T < T_{\text{sur}}^\varepsilon \). This \( T_{\text{sur}}^\varepsilon \) can be chosen maximal in the sense that \( T_{\text{sur}}^\varepsilon = \infty \) or \( \lim_{T \uparrow T_{\text{sur}}^\varepsilon} \|h^\varepsilon\|_{C([0,T],[C^\delta]^d)} = \infty \). The survival time \( T_{\text{sur}}^\varepsilon \) depends on the initial value \( h(0) \) and driving processes introduced in Section 3.2.

(2) A similar result holds for the solution \( h_{\hat{\varepsilon}}^\varepsilon \) of the KPZ approximating equation (1.4) with some limit \( h_{\hat{\varepsilon}}^\varepsilon \) under a proper choice of \( \hat{B}^\varepsilon, \hat{B}^\beta \). Moreover, under a well-adjusted choice of the renormalization factors \( B^\varepsilon, B^\beta \) and \( \hat{B}^\varepsilon, \hat{B}^\beta \) as in Section 4, we can make \( h \equiv h_{\hat{\varepsilon}}^\varepsilon \).

**Remark 1.1.** Precisely, the convergence \( h^\varepsilon \to h \) considered here means that

\[
P(\|h^\varepsilon - h\|_{C([0,T],[C^\delta]^d)} < \lambda, T < T_{\text{sur}} \wedge T_{\text{sur}}^\varepsilon) + P(T_{\text{sur}}^\varepsilon \leq T_{\text{sur}} - \lambda, T_{\text{sur}} < \infty) + P(T_{\text{sur}}^\varepsilon \leq T, \ T_{\text{sur}} = \infty) \to 0
\]

for every \( 0 < t < T \) and \( \lambda > 0 \). The convergence \( h^\varepsilon \to h \) is similarly understood.

**Theorem 1.2.** All components of the renormalization matrices \( B^\varepsilon \) and \( \hat{B}^\varepsilon \) defined in Section 2 behave as \( O(1) \) if and only if the trilinear condition (1.7) holds. In particular, when (1.7) holds, we can choose \( B^\varepsilon = B^\beta = 0 \) in the approximating equations (1.3) and (1.4), and the corresponding solutions \( h^\varepsilon_{B=0} \) and \( h_{\hat{\varepsilon}}^\varepsilon_{B=0} \) converge to \( h_{B=0} \) and \( h_{\hat{\varepsilon}}^\varepsilon_{B=0} \) respectively, as \( \varepsilon \downarrow 0 \). In the limit, we have

\[
h^\varepsilon_{B=0}(t,x) = h_{B=0}(t,x) + c^\alpha t, \quad 1 \leq \alpha \leq d,
\]

where

\[
c^\alpha = \frac{1}{24} \sum_{\beta_1, \beta_2} \sigma^\beta \hat{\Gamma}_{\alpha_1, \alpha_2}^\beta \hat{\Gamma}_{\beta_1, \beta_2}^{\alpha_1} \hat{\Gamma}_{\beta_1, \beta_2}^{\alpha_2},
\]

**Remark 1.2.** For the equation (1.3) with \( d = 1 \) (then the condition (1.7) is trivial), Hairer [10] first obtained that the two logarithmic renormalization factors (i.e., \( O(\log \varepsilon^{-1}) \) terms) cancel with each other and the constant \( \frac{1}{24} \) arises from the difference of these two terms, see also Section 4.

**Remark 1.3.** Kupiainen and Marcozzi [13] studied another approximation of the equation (1.1) with \( \sigma = 1 \) and obtained the cancellation of the logarithmic renormalization factors under the trilinear condition (1.10).
Our second goal is to show the global-in-time existence of the limit process $h$ under the condition (1.7). Let $\mu_A$ be the Gaussian measure on the space $(C^{d-1}_0)^d := \{u \in (C^{d-1}_0)^d : \int_T u = 0\}$, $\delta > 0$, under which $u = (u^\alpha)^a_{\alpha=1} \in (C^{d-1}_0)^d$ has the covariance

$$E[u^\alpha(x)u^\beta(y)] = A^{\alpha\beta}\delta(x - y).$$

Note that $\mu_A$ is the distribution of $(\partial_x(\sigma B)_{x \in \mathbb{T}}$, which is the limit in law of that of $(\partial_x(\sigma B * \eta^\varepsilon))_{x \in \mathbb{T}}$ as $\varepsilon \downarrow 0$. When $\sigma = I$, $\mu_A$ is called an $\mathbb{R}^d$-valued spatial white noise on $\mathbb{T}$.

**Theorem 1.3.** Let $0 < \delta < \delta' < \frac{1}{2}$ and assume the trilinear condition (1.7). Then there exists a subset $H \subset (C^{d-1}_0)^d$ such that $\mu_A(H) = 1$, and if $\partial_x h(0) \in H$, the convergence to the limit process $h$ as above holds on whole $[0, \infty)$ (i.e., $h^\varepsilon$ and $\hat{h}^\varepsilon$ exist in the space $C([0, \infty), (C^\delta)^d)$) almost surely, and both of them converge to the same $h$ in the space $C([0, T], (C^\delta)^d \cap C([t, T], (C^\delta)^d)$ for every $0 < t < T$ in probability. Moreover, the spatial derivative $u = \partial_x h$ of the limit process $h$ is a Markov process on $(C^{d-1}_0)^d$ which admits $\mu_A$ as an invariant measure.

**Remark 1.4.** Proposition 5.4 of Hairer and Mattingly [12] (combined with Theorem 1.3) shows that the limit process $h$ exists on $[0, \infty)$ almost surely for all initial values $h(0) \in (C^\delta)^d$, since the measure $\mu_A$ has a dense support in $(C^{d-1}_0)^d$.

Finally in this subsection, we note that the Cole-Hopf transform works for the coupled KPZ equation (1.1) in special cases. For example, Ertaş and Kardar [4] considered the $\mathbb{R}^2$-valued coupled equations

\[
\begin{align*}
\partial_t h^1 &= \frac{1}{2}\partial_x^2 h^1 + \frac{1}{2}\{(\lambda_1\partial_x h^1)^2 + \lambda_2(\partial_x h^2)^2\} + \sigma_1 \xi^1, \\
\partial_t h^2 &= \frac{1}{2}\partial_x^2 h^2 + \lambda_1 \partial_x h^1 \partial_x h^2 + \sigma_2 \xi^2
\end{align*}
\]

as a linearizable case. In general, if we assume that there exists an invertible matrix $s = (s^{\alpha}_{\alpha'})_{1 \leq \alpha, \alpha' \leq d}$ ($s$ may be complex valued) such that

\[
\Gamma^{\alpha}_{\beta\gamma} = \sum_{\alpha''}\left(s^{-1}\right)^{\alpha}_{\alpha''} s^{\alpha'}_{\beta'} s^{\alpha''}_{\gamma},
\]

then $\hat{h}^\alpha = s^{\alpha}_{\beta} h^\beta$ defined from the solution $h$ of (1.11) satisfies

\[
\begin{align*}
\partial_t \hat{h}^\alpha &= \frac{1}{2}\partial_x^2 \hat{h}^\alpha + \frac{1}{2}s^{\alpha'}_{\beta'} \Gamma^{\alpha'}_{\beta\gamma} \partial_x h^\beta \partial_x \hat{h}^\gamma + s^{\alpha}_{\beta} \sigma^{\beta}_{\gamma} \xi^\gamma \\
&= \frac{1}{2}\partial_x^2 \hat{h}^\alpha + \frac{1}{2}(\partial_x \hat{h}^\alpha)^2 + s^{\alpha}_{\beta} \sigma^{\beta}_{\gamma} \xi^\gamma.
\end{align*}
\]

In this way, the nonlinear term is decoupled. Hence the Cole-Hopf transform $Z^\alpha = \exp \hat{h}^\alpha$ linearizes (1.1), so that the argument in [9] yields the global existence of $h$. In fact, the equation (1.10) satisfies the condition (1.11) with $s = (\lambda_1 \lambda_1, -\lambda_1 \lambda_2)^{1/2}$. Meanwhile, even if $d = 2$, the matrices $\Gamma^1 = \left(\begin{smallmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{smallmatrix}\right)$ and $\Gamma^2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ satisfy the trilinear condition (1.7), but not (1.11).

As for the invariant measure, the tilt process $\partial_x \hat{h}^\alpha$ of each component $\hat{h}^\alpha$ of the transformed process has the distribution $\mu_\alpha$ of $(\sqrt{\sum_{\gamma}(s^x_{\beta} \sigma_{\gamma}^2)^2} \partial_x B)_{x \in \mathbb{T}}$ as its invariant measure,
where $B$ is the 1-dimensional periodic Brownian motion. This is seen by applying the result of [7] or Theorem [1.3] stated above for each component noting that $s^\alpha_0 \sigma_{\gamma} \xi^\gamma$ in (1.12) is the scalar-valued space-time white noise with covariance $\sum_{\gamma}(s^\alpha_\gamma \sigma^\gamma)^2$. In particular, if (1.11) holds, we see that the tilt process $u = \partial_s h$ of the solution $h = (h^\alpha)$ of (1.1) in the limit with a suitable renormalization has an invariant measure, whose marginals under the transform $h = sh$ is given by $\mu_\alpha$ for each $\alpha$. Indeed, with the help of Rellich type theorem, one can easily show the tightness on the space $(C^{\delta-1}_0)^d$ of the Cesàro mean $\nu_T = \frac{1}{T} \int_0^T \mu(t)dt$ over $[0,T]$ of the distributions $\mu(t)$ of $\partial_s h(t)$ having an initial distribution $\otimes \mu_\alpha$, so that the limit distribution of $\nu_T$ as $T \to \infty$ is the invariant measure. However, the joint distribution of such invariant measure is unclear.

**Remark 1.5.** As we stated in Theorem [1.2] and will see in Lemma [4.1] below, the trilinear condition (1.7) is equivalent to the condition “$F = G$ as matrices”, which is also equivalent to that the logarithmic renormalization factors $B^\alpha_{\beta,\gamma}$ and $\tilde{B}^\alpha_{\beta,\gamma}$ behave as $O(1)$; indeed, $\tilde{B}^\alpha_{\beta,\gamma} = 0$ under this condition. However, the necessary and sufficient condition for the logarithmic renormalization factors staying bounded in the KPZ approximating equations (1.3) and (1.4) is that the quantities $\Gamma_{\beta,\gamma}^\alpha B^\alpha_{\rho,\tau}$ or $\Gamma_{\beta,\gamma}^\alpha \tilde{B}^\alpha_{\rho,\tau}$, rather than $B^\alpha_{\beta,\gamma}$ or $\tilde{B}^\alpha_{\beta,\gamma}$ themselves, are bounded for every $\alpha$, respectively. From the expressions given in Lemma [4.1], this is equivalent to the identity

$$
\hat{\Gamma}_{\alpha_1\alpha_2} \hat{\Gamma}_{\alpha_3\alpha_4} \hat{\Gamma}_{\alpha_5\alpha_6} = \hat{\Gamma}_{\alpha_1\alpha_2} \hat{\Gamma}_{\alpha_3\alpha_4} \hat{\Gamma}_{\alpha_5\alpha_6},
$$

where the sums $\sum$ over $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are omitted, holds for every $\alpha$. Both (1.7) and (1.11) are sufficient conditions of (1.13), but neither of them are necessary conditions. In particular, the logarithmic renormalization factor does not appear in the equation (1.12).

### 1.4 Notations and organization of the paper

The Fourier transform on $\mathbb{R}$ is denoted by $\varphi = \mathcal{F}\eta \in \mathcal{S}(\mathbb{R})$, i.e. $\varphi(\theta) = \int_{\mathbb{R}} e^{-2\pi i x \theta} \eta(x)dx$, $\theta \in \mathbb{R}$. When $\eta$ is even and satisfies $\int_{\mathbb{R}} \eta(x)dx = 1$, then $\varphi$ is real-valued and satisfies $\varphi(0) = 1$ and $\varphi(-\theta) = \varphi(\theta)$. The convolution operators $* \eta^\epsilon$ and $* \eta^\epsilon_\tau$ are represented by the Fourier multipliers $\varphi(\epsilon D)$ and $\varphi^\epsilon(\epsilon D)$, where $\varphi(D)u := \mathcal{F}^{-1}(\varphi \mathcal{F} u)$.

We also consider the Fourier transform on $\mathbb{T}$ and use the same notation $\mathcal{F}$ and $\mathcal{F}^{-1}$:

$$
\mathcal{F}u(k) = \hat{u}(k) = \int_{\mathbb{T}} e^{-2\pi ikx} u(x)dx, \quad k \in \mathbb{Z},
$$

$$
\mathcal{F}^{-1}u(x) = \sum_k e^{2\pi ikx} \hat{u}(k), \quad x \in \mathbb{R}.
$$

Then, the heat kernel associated with $\partial_t - \frac{1}{2} \partial_x^2$ is given by

$$
p(t, x) = \sum_k e^{2\pi ikx} e^{-2\pi^2 k^2 t} = \mathcal{F}^{-1}(e^{-2\pi^2 k^2 t})(x), \quad t > 0, \ x \in \mathbb{T}.
$$

The noises $\xi^\beta(t, x)$ are transformed into complex-valued white noises $\xi^\beta, k(s) = (\mathcal{F} \xi^\beta(s))(k)$ such that $\xi^\beta, k(s) = \xi^\beta, -k(s)$ and

$$
E[\xi^\beta, k_1(t) \xi^\gamma, k_2(s)] = \delta^{\beta\gamma} \delta(t-s) 1_{\{k_1+k_2=0\}}.
$$
In fact, the left hand side of (1.14) is given by
\[ E \left[ \int_T e^{-2\pi ik_1 x} \xi^\beta(t,x)dx \right] \int_T e^{-2\pi ik_2 y} \xi^\gamma(s,y)dy = \delta^{\beta\gamma} \delta(t-s) \int_T e^{-2\pi ik_1 x} e^{-2\pi ik_2 x} dx. \]
The smeared noise is defined by \( \xi * \eta^\varepsilon = \varphi(\varepsilon D)\xi \), where \( \varphi(D)u = F^{-1}(\varphi F u) \) as we mentioned above.

This paper is organized as follows. In Section 2, following [14] we formulate a fixed point problem associated with (1.1) and solve it by constructing a deterministic solution map from the initial value and deterministic driving terms. In Section 3, we prove the probabilistic part of Theorem 1.1, i.e., we give controls of stochastic drivers and calculations of renormalization factors. In Section 4, we prove Theorem 1.2 under the trilinear condition (1.7). In Section 5, we repeat the same arguments as in Section 2 for the stochastic Burgers equation:
\[ \partial_t u^\alpha = \frac{1}{2} \partial_x^2 u^\alpha + \frac{1}{2} \Gamma^\alpha_{\beta\gamma} \partial_x u^\beta \partial_x u^\gamma + \sigma^\alpha_{\beta} \partial_x \xi^\beta, \]
and construct a well-defined solution map. We show the invariance of \( \mu_A \) under (1.1) at the Burgers level and prove Theorem 1.3. At last, we touch the global well-posedness of the approximating equations (1.3) and (1.4) at the Burgers level.

2 Formal expansion and solving the coupled KPZ equation

2.1 Preliminary consideration due to formal expansion

In the coupled KPZ equation (1.1), we think of the noise as the leading term and the nonlinear term as its perturbation. Although we eventually take \( a = 1 \), we put \( a > 0 \) in front of the nonlinear term:
\[ \mathcal{L}h^\alpha = \frac{a}{2} \Gamma^\alpha_{\beta\gamma} \partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_{\beta} \xi^\beta, \]
where \( \mathcal{L} = \partial_t - \frac{1}{2} \partial_x^2 \). Then, at least formally, one can expand the solution \( h \) in \( a \):
\[ h^\alpha = \sum_{k=0}^\infty a^k h^\alpha_k. \]
Indeed, by inserting (2.2) to (2.1), we have that
\[ \sum_{k=0}^\infty a^k \mathcal{L}h^\alpha_k = \sigma^\alpha_{\beta} \xi^\beta + \frac{a}{2} \sum_{k_1, k_2=0}^\infty a^{k_1+k_2} \Gamma^\alpha_{\beta\gamma} \partial_x h^{\beta}_{k_1} \partial_x h^{\gamma}_{k_2}. \]

Thus, comparing the terms of order \( a^0, a^1, a^2, a^3 \) in both sides and noting the condition (1.2), we obtain the following identities:
\[ \begin{align*}
\mathcal{L}h^\alpha_0 &= \sigma^\alpha_{\beta} \xi^\beta, \\
\mathcal{L}h^\alpha_1 &= \frac{1}{2} \Gamma^\alpha_{\beta\gamma} \partial_x h^\beta_0 \partial_x h^\gamma_0, \\
\mathcal{L}h^\alpha_2 &= \Gamma^\alpha_{\beta\gamma} \partial_x h^\beta_1 \partial_x h^\gamma_0, \\
\mathcal{L}h^\alpha_3 &= \frac{1}{2} \Gamma^\alpha_{\beta\gamma} \partial_x h^\beta_1 \partial_x h^\gamma_1 + \Gamma^\alpha_{\beta\gamma} \partial_x h^\beta_2 \partial_x h^\gamma_0.
\end{align*} \]
The first equation determines $h_0^\alpha$, which is actually the Ornstein-Uhlenbeck process, and $h_0^\alpha \in C^{1/2-} := \int_{t>0} C^{1/2-}$ in $x$. Therefore, the product $\partial_x h_0^\alpha \partial_x h_0^\beta$ in the second equation is not definable in a usual sense. When $\xi^\beta$ is replaced by the smeared noise $\xi^\beta := \xi^\beta * \eta^\varepsilon$, this product makes sense, since $h_0^\alpha \in C^\infty$ for such case. However, as we will see later in (2.16), for $h_0^\alpha$ to converge, we need to introduce a renormalization. At this moment, we just assume $h_1^\alpha \in C^{1-}$ (note $-1/2 - 1/2 + 2 = 2$) and then $h_2^\alpha \in C^{3/2-}$ (note $-1/2 + 2 = 2$) are defined in some sense. We denote $h_0^\alpha, h_1^\alpha, h_2^\alpha$ with stationary initial values by $H_1^\alpha, H_2^\alpha, H_3^\alpha$, respectively, see Section 3 for details.

After defining $H_1^\alpha, H_2^\alpha, H_3^\alpha$, the KPZ equation (with $a = 1$) for $h^\alpha = H_1^\alpha + H_2^\alpha + H_3^\alpha + h_{\geq 3}$ can be rewritten into an equation for the remainder term $h_{\geq 3}$:

$$L h_{\geq 3} = \Phi^\alpha + L h_3^\alpha,$$

where $\Phi^\alpha = \Phi^\alpha(H_1, H_2, H_3, h_{\geq 3})$ is given by

$$\Phi^\alpha = \Gamma_{\beta\gamma}(\partial_x h_\geq^\beta, \partial_x H_1^\beta, \partial_x H_2^\gamma) + \Gamma_{\beta\gamma}(\partial_x H_2^\beta, \partial_x h_\geq^\beta, \partial_x H_1^\gamma) + \Gamma_{\beta\gamma}(\partial_x H_2^\beta, \partial_x h_\geq^\beta, \partial_x h_\geq^\gamma).$$

This is easily obtained from (1.1) and (2.2) by computing $\mathcal{L} h - \mathcal{L} H_1 - \mathcal{L} H_2 - \mathcal{L} H_3$ and expanding $\mathcal{L} h - \mathcal{L} H_1 = \frac{1}{2} \Gamma_{\beta\gamma}(\partial_x H_1^\beta + H_2^\beta + H_3^\beta + h_\geq^\beta) \partial_x (H_1^\gamma + H_2^\gamma + H_3^\gamma + h_\geq^\gamma)$. Note that $h_3^\alpha = h_3^\alpha(H_1, H_2, H_3)$ in (2.3) is defined through the last identity in (2.3).

We now recall that Section 2 of [14] briefly summarizes definitions and known results on Besov spaces, Bony’s paraproducts $u \odot v$, $u \odot v$ of $u$ and $v$, mollifier estimates, Schauder estimates, commutator estimates and others; see [8, 9] for details. To define $h_{\geq 3}$, we need to introduce four more objects as driving terms:

$$H_\gamma^\beta = \frac{1}{2} \partial_x H_2^\beta \partial_x H_1^\gamma, \quad H_\gamma^\gamma = \partial_x H_2^\beta \odot \partial_x H_1^\gamma,$$

$$H_\gamma^\gamma = \text{"stationary solution of } \mathcal{L} H_2^\gamma = 0\text{",} \quad H_\gamma^\gamma = \partial_x H_2^\beta \odot \partial_x H_1^\gamma.$$

Indeed, to solve the equation (2.4), we divide $h_{\geq 3}^\alpha$ into the sum of two parts $f^\alpha$ and $g^\alpha$: $h_{\geq 3}^\alpha = f^\alpha + g^\alpha$, which solve

$$\mathcal{L} f^\alpha = \Gamma_{\beta\gamma}(\partial_x H_2^\beta + \partial_x f^\beta + \partial_x g^\beta) \odot \partial_x H_1^\gamma,$$

$$\mathcal{L} g^\alpha = \Gamma_{\beta\gamma}(\partial_x H_2^\beta + \partial_x f^\beta + \partial_x g^\beta)(\odot + \odot) \partial_x H_1^\gamma + \text{other terms},$$

respectively. Here, the implicitly written “other terms” contain nonlinear operators of sufficiently regular functions, so that they are well-defined if we can define $H_\gamma^\beta, H_\gamma^\gamma \in C^0$. To define the term $\partial_x f^\beta \odot \partial_x H_1^\gamma$ in the right hand side of $\mathcal{L} g^\alpha$, we first introduce $H_\xi^\alpha$ as a solution of $\mathcal{L} H_\xi^\alpha = \partial_x H_1^\alpha$. Then by definition, $f^\alpha$ has the form

$$f^\alpha = P P^0(0) + \Gamma_{\beta\gamma}(\partial_x H_2^\beta + \partial_x h_{\geq 3}^\beta, H_\xi^\gamma, P H_\xi^\gamma(0)) + C_1^\alpha(H_\xi, h_{\geq 3}, H_1),$$

with a term $C_1^\alpha(H_\xi, h_{\geq 3}, H_1)$ sufficiently regular in the sense that the resonant $\partial_x C_1^\beta \odot \partial_x H_1^\gamma$ is well-defined, see Lemma 3.1 of [14]. Here $P \equiv P_1$ is the heat semigroup defined by
$P_t u = \int_T f(t, \cdot - y) u(y) dy$. By the commutator estimate, e.g., see Lemma 2.4 of [8] or Proposition 2.12 of [14], the term $\partial_x f^3 \circ \partial_x H^3_\theta$ is defined if $H^3_{\psi} \in C^{0-}$ is given a priori.

### 2.2 Drivers

Fix $\kappa \in (\frac{1}{3}, \frac{1}{2})$. The driver of the coupled KPZ equation is the element $\mathbb{H}$ of the form

$$\mathbb{H} := (H_1, H_1', H_\phi, H_{\phi'}, H_\chi, H_{\chi'}, H_\psi, H_{\psi'})$$

$$\in C([0, T], (C^\kappa)^d) \times C([0, T], (C^{2\kappa})^d) \times \{C([0, T], (C^{\kappa+1})^d) \cap C^{1/4}([0, T], (C^{\kappa+1/2})^d)\}$$

$$\times C([0, T], (C^{2\kappa-1})^d) \times C([0, T], (C^{\kappa+1})^d) \times C([0, T], (C^{2\kappa-1})^d),$$

which satisfies $LH_\psi = \partial_x H_1$. Note that, for $H_{\phi'}$ and others, the Besov space is $\mathbb{R}^d \otimes \mathbb{R}^d$, valued. We denote by $H^\mu_{\text{KPZ}}$ the class of all drivers. We write $\|\mathbb{H}\|_T$ for the product norm of $\mathbb{H}$ on the above space. Due to the preliminary and heuristic consideration, these terms should be defined a priori in some sense.

### 2.3 Deterministic result

Fix $\lambda \in (\frac{1}{3}, \kappa)$ and $\mu \in (-\lambda, \lambda]$. For a $\mathcal{D}'(\mathbb{T}, \mathbb{R}^d)$-valued functions $f = (f^\alpha)_{\alpha=1}^d$ and $g = (g^\alpha)_{\alpha=1}^d$ on $[0, T]$, we write $(f, g) \in D^{\lambda, \mu}_{\text{KPZ}}([0, T])$ if

$$\|(f, g)\|_{D^{\lambda, \mu}_{\text{KPZ}}([0, T])} :=$$

$$\sup_{t \in [0, T]} t^{\lambda + \mu} \|f(t)\|_{(C^{\lambda+1})^d} + \sup_{t \in [0, 1]} \|f(t)\|_{(C^{\mu+1})^d} + \sup_{s \in [0, T]} s^{\lambda - \mu} \frac{\|f(t) - f(s)\|_{(C^{\lambda+1/2})^d}}{|t - s|^{1/4}}$$

$$+ \sup_{t \in [0, T]} t^{\lambda - \mu} \|g(t)\|_{(C^{2\lambda+1})^d} + \sup_{t \in [0, 1]} \|g(t)\|_{(C^{2\mu+1})^d} + \sup_{s \in [0, T]} s^{\lambda - \mu} \frac{\|g(t) - g(s)\|_{(C^{2\lambda+1/2})^d}}{|t - s|^{1/4}}$$

is finite.

The following theorem is due to the paracontrolled calculus and fixed point theorem. For the detailed proof, see Section 3 of [14].

**Theorem 2.1** (Theorem 3.3 and Lemma 3.5 of [14]). (1) Let $T > 0$ and $\mathbb{H} \in H^\mu_{\text{KPZ}}$ be given. Then, for every initial value $(f(0), g(0)) \in (C^{\mu+1})^d \times (C^{2\mu+1})^d$ the system (2.1) admits a unique solution in $D^{\lambda, \mu}_{\text{KPZ}}([0, T_*])$ up to the time

$$T_* = C(1 + \|f(0)\|_{(C^{\mu+1})^d} + \|g(0)\|_{(C^{2\mu+1})^d} + \|\mathbb{H}\|_{T}^3)^{-\frac{2}{\kappa - \lambda} \wedge T},$$

where $C$ is a universal constant depending only on $\kappa, \lambda, \mu$ and $T$. The solution satisfies

$$\|(f, g)\|_{D^{\lambda, \mu}_{\text{KPZ}}([0, T_*])} \leq C'(1 + \|f(0)\|_{(C^{\mu+1})^d} + \|g(0)\|_{(C^{2\mu+1})^d} + \|\mathbb{H}\|_{T}^3),$$

with a universal constant $C'$.

(2) Let $T_{\text{sup}} \leq T$ be the maximal time such that the existence and uniqueness of the solution...
hold on \([0, T_{\text{sur}}]\). The map \((f(0), g(0), \mathbb{H}) \mapsto T_{\text{sur}}\) is lower semi-continuous. If \(T_{\text{sur}} < T\), then
\[
\lim_{t \uparrow T_{\text{sur}}} \|h\|_{\mathcal{C}([0,t] \cup \{t\}^+)} = \infty,
\]
where \(h = H_1 + H_Y + H_\Psi + f + g\). The map \(S_{\text{KPZ}}\) from \((f(0), g(0), \mathbb{H}) \in (\mathcal{C}^{\mu+1})^d \times (\mathcal{C}^{2\mu+1})^d \times \mathcal{H}_{\text{KPZ}}^\epsilon\) to the maximal solution \(h \in \mathcal{C}([0, T_{\text{sur}}], (\mathcal{C}^{\mu+1})^d \times (\mathcal{C}^{2\mu+1})^d)\) is continuous.

We denote by \(h = H_1 + H_Y + H_\Psi + f + g \equiv S_{\text{KPZ}}(f(0), g(0), \mathbb{H})\) the maximal solution.

### 2.4 Renormalization

By replacing \(\xi^\beta\) by \(\xi^{\epsilon,\beta} = \xi^\beta \ast \eta^\epsilon\) in (1.1) and introducing the renormalization factors \(-c\epsilon A^{\beta\gamma}, C^{\epsilon,\beta\gamma}\) and \(D^{\epsilon,\beta\gamma}\), we have the following identities for the formal expansion \(h^{\epsilon,\alpha} = \sum_{k=0}^\infty a^k h_k^{\epsilon,\alpha}\) of the solution of the approximating equation (1.3):

\[
\mathcal{L} h_0^{\epsilon,\alpha} = \sigma_0^{\epsilon,\alpha}, \\
\mathcal{L} h_1^{\epsilon,\alpha} = \frac{1}{2} \Gamma^\alpha_{\beta\gamma} (\partial_x h_0^{\epsilon,\beta} \partial_x h_0^{\epsilon,\gamma} - c_\epsilon A^{\beta\gamma}), \\
\mathcal{L} h_2^{\epsilon,\alpha} = \Gamma^\alpha_{\beta\gamma} \partial_x h_1^{\epsilon,\beta} \partial_x h_0^{\epsilon,\gamma}, \\
\mathcal{L} h_3^{\epsilon,\alpha} = \frac{1}{2} \Gamma^\alpha_{\beta\gamma} (\partial_x h_1^{\epsilon,\beta} \partial_x h_0^{\epsilon,\gamma} - C^{\epsilon,\beta\gamma}) + \Gamma^\alpha_{\beta\gamma} (\partial_x h_2^{\epsilon,\beta} \partial_x h_0^{\epsilon,\gamma} - D^{\epsilon,\beta\gamma}).
\]

Then we obtain the renormalized driver \(\mathbb{H}^{\epsilon}\) corresponding to \(\xi^{\epsilon}\), which is defined by the solutions of
\[
\mathcal{L} H^{\epsilon,\alpha} = \sigma_\beta^{\epsilon,\alpha} \partial_x H^{\epsilon,\beta}, \\
\mathcal{L} H_1^{\epsilon,\alpha} = \frac{1}{2} \Gamma^\alpha_{\beta\gamma} (\partial_x H_1^{\epsilon,\beta} \partial_x H_1^{\epsilon,\gamma} - c_\epsilon A^{\beta\gamma}), \\
\mathcal{L} H_2^{\epsilon,\alpha} = \Gamma^\alpha_{\beta\gamma} \partial_x H_1^{\epsilon,\beta} \partial_x H_1^{\epsilon,\gamma}, \\
\mathcal{L} H_3^{\epsilon,\alpha} = \partial_x H_1^{\epsilon,\alpha}
\]

with stationary initial values, and products
\[
H^{\epsilon,\beta\gamma}_Y = \frac{1}{2} (\partial_x H^{\epsilon,\beta} \partial_x H^{\epsilon,\gamma} - C^{\epsilon,\beta\gamma}), \\
H^{\epsilon,\beta\gamma}_\Psi = \partial_x H^{\epsilon,\beta}_Y \partial_x H^{\epsilon,\gamma} - D^{\epsilon,\beta\gamma}, \\
H^{\epsilon,\beta\gamma}_\Psi = \partial_x H^{\epsilon,\beta}_Y \partial_x H^{\epsilon,\gamma}.
\]

From this, we see that \(h^{\epsilon} := S_{\text{KPZ}}(f(0), g(0), \mathbb{H}^{\epsilon})\) solves (1.3) with
\[
B^{\epsilon,\beta\gamma} = C^{\epsilon,\beta\gamma} + 2 D^{\epsilon,\beta\gamma}.
\]

Theorem 2.1 combined with the convergence of drivers \(\mathbb{H}^{\epsilon}\) to \(\mathbb{H}\) shown in Theorem 3.2 below proves Theorem 1.1(1).

We do similar arguments for the equation with \(*\eta^\epsilon = \varphi^2(\epsilon D)\) for the nonlinear term:
\[
\partial_t h^{\alpha} = \frac{1}{2} \partial_x^2 h^{\alpha} + \frac{1}{2} \Gamma^\alpha_{\beta\gamma} \varphi^2(\epsilon D)(\partial_x h^{\beta} \partial_x h^{\gamma}) + \sigma_\beta^{\epsilon,\alpha} \xi^\beta.
\]
Then for the formal expansion $\tilde{h}^\alpha = \sum_{k=0}^\infty a^k \tilde{h}_k^\alpha$, we have

\[
\mathcal{L} \tilde{h}_0^\alpha = \sigma_0^2 \xi^\alpha,
\]
\[
\mathcal{L} \tilde{h}_1^\alpha = \frac{1}{2} \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_0^\alpha \partial_x \tilde{h}_0^\gamma),
\]
\[
\mathcal{L} \tilde{h}_2^\alpha = \frac{1}{2} \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_1^\alpha \partial_x \tilde{h}_1^\gamma),
\]
\[
\mathcal{L} \tilde{h}_3^\alpha = \frac{1}{2} \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_2^\alpha \partial_x \tilde{h}_2^\gamma) + \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_2^\alpha \partial_x \tilde{h}_0^\gamma).
\]

We can construct a solution map $h = S^\varepsilon_{KPZ}(f(0), g(0), \mathbb{H})$ corresponding to the mollified equation (2.7), though the driver $C$ in we again obtain the renormalized driver $\tilde{h}$ and $H$ equation (2.7), though the driver $C$ in we again obtain the renormalized driver $\tilde{h}$ and $H$

**Theorem 2.2** combined with the convergence of drivers $\tilde{h}$.

Theorem 2.2 combined with the convergence of drivers $\tilde{h}$.

By replacing $\xi^\alpha$ by $\xi^\varepsilon, \beta$ in (2.7) and introducing the renormalization factors $-c^\varepsilon A^{\beta\gamma}$, $\tilde{C}^\varepsilon, \beta, \gamma$, $\tilde{D}^\varepsilon, \beta, \gamma$, we again obtain the renormalized driver $\tilde{h}^\varepsilon$ corresponding to the approximating equation (1.4), which is defined by the solutions of

\[
\mathcal{L} \tilde{h}_{i,\alpha}^\varepsilon = \sigma_0^2 \partial_x \xi^\varepsilon, \beta, 
\]
\[
\mathcal{L} \tilde{h}_{Y,\alpha}^\varepsilon = \frac{1}{2} \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_1^\alpha \partial_x \tilde{h}_1^\gamma) - \varepsilon^\alpha A^{\beta\gamma},
\]
\[
\mathcal{L} \tilde{h}_{Y,\alpha}^\varepsilon = \Gamma_{\beta\gamma}^\alpha \varphi^2(\varepsilon D)(\partial_x \tilde{h}_1^\alpha \partial_x \tilde{h}_1^\gamma),
\]
\[
\mathcal{L} \tilde{h}_{\xi,\alpha}^\varepsilon = \partial_x \varphi^2(\varepsilon D)\tilde{h}_{i,\alpha}^\varepsilon
\]

with stationary initial values, and products

\[
\tilde{H}_{\chi,\gamma}^\varepsilon = \frac{1}{2} (\partial_x \tilde{h}_{i,\beta}^\varepsilon \partial_x \tilde{h}_{i,\gamma}^\varepsilon - \tilde{C}^\varepsilon, \beta, \gamma),
\]
\[
\tilde{H}_{\chi,\gamma}^\varepsilon = \partial_x \tilde{h}_{Y,\beta}^\varepsilon \partial_x \tilde{h}_{i,\gamma}^\varepsilon - \tilde{D}^\varepsilon, \beta, \gamma,
\]
\[
\tilde{H}_{\chi,\gamma}^\varepsilon = \partial_x \tilde{h}_{\xi,\beta}^\varepsilon \partial_x \tilde{h}_{i,\gamma}^\varepsilon.
\]

From this, we see that $\tilde{h}^\varepsilon := S^\varepsilon_{KPZ}(f(0), g(0), \mathbb{H})$ solves (1.4) with

\[
\tilde{B}^\varepsilon, \beta, \gamma = \tilde{C}^\varepsilon, \beta, \gamma + 2 \tilde{D}^\varepsilon, \beta, \gamma.
\]

Theorem 2.2 combined with the convergence of drivers $\tilde{h}$ to $\mathbb{H}$ shown in Theorem 3.3 proves Theorem 1.1(2).

### 3 Computation of renormalization factors

#### 3.1 Product formula

We first prepare the product formula to compute the Wiener chaos expansions of the products of two multiple Wiener-Itô integrals.
Let \( \{\xi_{\beta,k}^\varepsilon(s)\}_{\beta \in \{1, \ldots, d\}, k \in \mathbb{Z}} \) be complex-valued Gaussian white noises on \( \mathbb{R} \) which satisfy \( \xi^\varepsilon_{\beta,k}(s) = \xi^\varepsilon_{\beta,-k}(s) \) and have the covariance structure (1.14), which are realized on a probability space \((\Omega, \mathcal{F}, P)\), where \( \mathcal{F} \) is a \( \sigma \)-field generated by \( \{\xi_{\beta,k}(1(s))\}_{\beta,k,s < t} \). The Hilbert space \( \mathcal{H} = L^2(\Omega, \mathcal{F}, P) \) can be decomposed into the direct sum \( \oplus_n \mathcal{H}_n \), where \( n = (n_{\beta,k}) \in \mathbb{Z}^{\{1, \ldots, d\} \times \mathbb{Z}} \) satisfies \( n = \sum n_{\beta,k} < \infty \) and \( \mathbb{Z}_{\leq 0} = \mathbb{N} \cup \{0\} \). For \( f = ((s_{i}^{\beta,k})_{i=1}^{n_{\beta,k}})_{\beta,k} \in \mathcal{H}_n := L^2(\mathbb{R}^{|n|}, ds_n) \) with \( ds_n = \prod_{\beta,k} \prod_{i=1}^{n_{\beta,k}} ds_i^{\beta,k} \), we define multiple Wiener-Itô integrals

\[
I_n(f) = \int f \left( (s_{i}^{\beta,k})_{i=1}^{n_{\beta,k}} \right) \prod_{\beta,k} \prod_{i=1}^{n_{\beta,k}} \xi^{\beta,k}(ds_i^{\beta,k}).
\]

Note that we don’t divide by \( n! = \prod_{\beta,k} n_{\beta,k}! \) in the definition of \( I_n(f) \) compared with (1.10).

For given \( n = (n_{\beta,k}) \) and \( m = (m_{\beta,k}) \), a diagram \( \lambda \subset \prod_{\beta,k} \{1, \ldots, n_{\beta,k}\} \times \prod_{\beta,k} \{1, \ldots, m_{\beta,k}\} \) consists of disjoint pairs \((i_{\beta,k}, j_{\beta,k})\) connecting \( i_{\beta,k} \in \{1, \ldots, n_{\beta,k}\} \) and \( j_{\beta,k} \in \{1, \ldots, m_{\beta,k}\} \). Note that \( \beta \) and \( k \) have opposite signs in these pairs. The set of all possible diagrams \( \{\lambda\} \) is denoted by \( \Gamma(n,m) \). For \( \lambda \in \Gamma(n,m) \), we denote \( \lambda = (\beta, k) \), where \( \bar{\lambda} = (\beta, k) \) for each \( \beta, k \). Then \( n + m - \lambda \) is defined componentwisely by \( (n + m - \lambda)_{\beta,k} = n_{\beta,k} + m_{\beta,k} - \bar{\lambda}_{\beta,k} \). For \( f_1, f_2 \in \mathcal{H}_n \) and \( f_3 \in \mathcal{H}_m \), we define \( f_\lambda \in \mathcal{H}_{n+m-\lambda} \) by

\[
f_\lambda(s_n \cup s_m \setminus s_\lambda) = \int f_1(s_n) f_2(s_m) \prod_{i_{\beta,k} \in \lambda} ds_i^{\beta,k},
\]

where \( s_n \cup s_m \setminus s_\lambda = ((s_{i_{\beta,k}}^{\beta,k})_{i_{\beta,k} \in \lambda} \cup (s_{j_{\beta,k}}^{\beta,k})_{j_{\beta,k} \notin \lambda})_{\beta,k} \), and \( s_m \) is defined by \( s_m \) with \( s_{j_{\beta,k}} \) replaced by \( s_{i_{\beta,k}} \) when the pair \((i_{\beta,k}, j_{\beta,k})\) appears in \( \lambda \). Then we have the following product formula; see Theorem 5.3 of [16] with \( m = 2 \) shown in a slightly different setting from ours.

**Proposition 3.1.** For \( f_1, f_2 \in \mathcal{H}_n \), we have

\[
I_n(f_1)I_m(f_2) = \sum_{\lambda \in \Gamma(n,m)} I_{n+m-\lambda}(f_\lambda).
\]

### 3.2 Definition of driving processes

Here we precisely define the components of \( \mathbb{H}_\varepsilon \) and \( \mathbb{H}_\varepsilon^\varepsilon \). Now we write \( \mathcal{H}_n = \oplus n=\varepsilon I_n(\mathcal{H}_n) \).

We define \( I(\zeta) \equiv I(\zeta)(t, x) \) for a noise \( \zeta = \{\zeta(s, y); s \in \mathbb{R}, y \in \mathbb{T}\} \) by

\[
I(\zeta)(t, x) = \int_{-\infty}^t \int_{\mathbb{T}} p(t - s, x, y) \Pi_0^T \zeta(s, y) dy ds + \int_0^t \Pi_0 \zeta(s) ds.
\]

Here \( \Pi_0 \) is the orthogonal projection of \( L^2(\mathbb{T}) \) onto the set of constant functions, and \( \Pi_0^T = 1 - \Pi_0 \). \( I(\zeta) \) is a solution of \( L I(\zeta) = \zeta \) with stationary initial value for non-zero Fourier modes but zero initial value for the zero mode. Note that \( \partial_x I(\zeta) \) is a stationary process since

\[
\partial_x I(\zeta)(t, x) = \int_{-\infty}^t \int_{\mathbb{T}} \partial_x p(t - s, x, y) \zeta(s, y) dy ds.
\]

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By taking the smeared noise $\xi^{\varepsilon,\beta}$, which is extended for $t \in \mathbb{R}$, set

\begin{equation}
H_1^{\varepsilon,\alpha} = \tilde{H}_1^{\varepsilon,\alpha} := \sigma_0^\beta I(\xi^{\varepsilon,\beta}) \in \mathcal{H}_1.
\end{equation}

Note that this solves the first equations of (2.6) and (2.8). This converges to the process $H_1^\alpha = \sigma_0^\beta I(\xi^\beta)$ as $\varepsilon \downarrow 0$.

We define

\begin{equation}
H_Y^{\varepsilon,\alpha} := \frac{1}{2} \Gamma_{\beta\gamma}^\alpha I(\partial_x H_1^{\varepsilon,\beta} \partial_x H_1^{\varepsilon,\gamma} - c^\varepsilon A^{\beta\gamma}) \in \mathcal{H}_2,
\end{equation}

\begin{equation}
\tilde{H}_Y^{\varepsilon,\alpha} := \frac{1}{2} \Gamma_{\beta\gamma}^\alpha \varphi^2 (\varepsilon D) I(\partial_x H_1^{\varepsilon,\beta} \partial_x H_1^{\varepsilon,\gamma} - c^\varepsilon A^{\beta\gamma}) \in \mathcal{H}_2.
\end{equation}

These solve the second equations of (2.6) and (2.8), respectively. Note that $\partial_x I(\xi^{\varepsilon,\beta})$ is stationary in $t$, so that the $\mathcal{H}_0$-components of $H_Y^{\varepsilon,\alpha}$ and $\tilde{H}_Y^{\varepsilon,\alpha}$ are compensated by $c^\varepsilon A^{\beta\gamma}$.

We define

\begin{equation}
H_{\phi}^{\varepsilon,\alpha} := \Gamma_{\beta\gamma}^\alpha I(\partial_x H_Y^{\varepsilon,\beta} \partial_x H_1^{\varepsilon,\gamma}) \in \mathcal{H}_3 \oplus \mathcal{H}_1,
\end{equation}

\begin{equation}
\tilde{H}_{\phi}^{\varepsilon,\alpha} := \Gamma_{\beta\gamma}^\alpha \varphi^2 (\varepsilon D) I(\partial_x H_Y^{\varepsilon,\beta} \partial_x H_1^{\varepsilon,\gamma}) \in \mathcal{H}_3 \oplus \mathcal{H}_1.
\end{equation}

These solve the third equations of (2.6) and (2.8), respectively. We can also define

\begin{align*}
H_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} &= \frac{1}{2} (\partial_x H_Y^{\varepsilon,\beta} \partial_x H_Y^{\varepsilon,\gamma} - C^{\varepsilon,\beta\gamma}) \in \mathcal{H}_4 \oplus \mathcal{H}_2, \\
\tilde{H}_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} &= \frac{1}{2} (\partial_x \tilde{H}_Y^{\varepsilon,\beta} \partial_x \tilde{H}_Y^{\varepsilon,\gamma} - \tilde{C}^{\varepsilon,\beta\gamma}) \in \mathcal{H}_4 \oplus \mathcal{H}_2, \\
H_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} &= \partial_x H_Y^{\varepsilon,\beta} \odot \partial_x H_1^{\varepsilon,\gamma} - D^{\varepsilon,\beta\gamma} \in \mathcal{H}_4 \oplus \mathcal{H}_2, \\
\tilde{H}_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} &= \partial_x \tilde{H}_Y^{\varepsilon,\beta} \odot \partial_x H_1^{\varepsilon,\gamma} - \tilde{D}^{\varepsilon,\beta\gamma} \in \mathcal{H}_4 \oplus \mathcal{H}_2,
\end{align*}

by subtracting the corresponding $\mathcal{H}_0$-components.

We further define

\begin{align*}
H_{\phi}^{\varepsilon,\alpha} := I(\partial_x H_1^{\varepsilon,\alpha}) \in \mathcal{H}_1, \\
\tilde{H}_{\phi}^{\varepsilon,\alpha} := \varphi^2 (\varepsilon D) I(\partial_x H_1^{\varepsilon,\alpha}) \in \mathcal{H}_1,
\end{align*}

and

\begin{align*}
H_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} := \partial_x H_Y^{\varepsilon,\beta} \odot \partial_x H_1^{\varepsilon,\gamma} \in \mathcal{H}_2, \\
\tilde{H}_{\phi_{\varepsilon,\gamma}}^{\varepsilon,\beta\gamma} := \partial_x \tilde{H}_Y^{\varepsilon,\beta} \odot \partial_x H_1^{\varepsilon,\gamma} \in \mathcal{H}_2.
\end{align*}

Note that the $\mathcal{H}_0$-components vanish because the function $\varphi = \mathcal{F}(\eta)$ is even.

The following result is shown in a similar way to Theorem 5.1 of [14].

**Theorem 3.2.** There exists an $\mathcal{H}_\text{KPZ}$-valued random variable $\mathbb{H}$ such that, for every $T > 0$ and $p \geq 1$,

\[ E \| \mathbb{H} \|_T^p < \infty, \quad \lim_{\varepsilon \downarrow 0} E \| \mathbb{H}^{\varepsilon} - \mathbb{H} \|_T^p = \lim_{\varepsilon \downarrow 0} E \| \tilde{\mathbb{H}}^{\varepsilon} - \mathbb{H} \|_T^p = 0. \]

In particular, both $h^{\varepsilon} = S_{\text{KPZ}}(f(0), g(0), \mathbb{H}^{\varepsilon})$ and $\tilde{h}^{\varepsilon} = S_{\text{KPZ}}(f(0), g(0), \tilde{\mathbb{H}}^{\varepsilon})$ converge to $h = S_{\text{KPZ}}(f(0), g(0), \mathbb{H})$ in probability as $\varepsilon \downarrow 0$ in $C([0, T_\text{sur}], (\mathcal{C}(\mathbb{M}, \Z)^p) \odot (\mathbb{M}^{\varepsilon})^\varepsilon).$
3.3 Derivation of \(c^e A\)

In Sections 3.3-3.5 we compute the precise values of renormalization factors. First we consider the \(H_0\)-component of the product \(\partial_x H_1 \partial_x H_1\).

Recall that \(H_1^e = \tilde{H}_1^e\) is given by (3.1). Since \(p(t, x, y) = \mathcal{F}^{-1}(e^{-2\pi^2 k^2 t})(x - y)\), we have

\[
\Pi_0^e H_1^{e, \alpha}(t, x) = \sigma_\beta^\alpha \sum_{k \neq 0} \int_{-\infty}^t e^{2\pi i k \xi} e^{-2\pi^2 k^2 (t - s)} \varphi(\varepsilon k) \xi^{\beta, k}(ds).
\]

Therefore,

\[
\partial_x H_1^{e, \alpha}(t, x) = \sum_{k \neq 0} \int K_{1}^{e, \alpha}(t, x)_{\beta, k}(s) \xi^{\beta, k}(ds) \in \mathcal{H}_1,
\]

where

\[
K_{1}^{e, \alpha}(t, x)_{\beta, k}(s) = \sigma_\beta^\alpha e^{2\pi i k \xi} \varphi(\varepsilon k) (2\pi ik) 1_{t \geq s} e^{-2\pi^2 k^2 (t - s)}.
\]

By Proposition 3.1 the \(H_2\)-component of \((\partial_x H_1^{e, \alpha_1} \partial_x H_1^{e, \alpha_2})(t, x)\) is given by Wiener-Itô integral with the kernel

\[
K_{1}^{e, \alpha_1 \alpha_2}(t, x)_{(\beta_1, k_1), (\beta_2, k_2)}(s_1, s_2) := K_{1}^{e, \alpha_1}(t, x)_{\beta_1, k_1}(s_1) K_{1}^{e, \alpha_2}(t, x)_{\beta_2, k_2}(s_2)
\]

\[
= \sigma_{\beta_1}^{\alpha_1} \sigma_{\beta_2}^{\alpha_2} e^{2\pi i (k_1 + k_2) x} \varphi(\varepsilon k_1) \varphi(\varepsilon k_2)
\]

\[
\times (2\pi i k_1) 1_{t \geq s_1} e^{-2\pi^2 k_1^2 (t - s_1)} (2\pi i k_2) 1_{t \geq s_2} e^{-2\pi^2 k_2^2 (t - s_2)},
\]

while its \(H_0\)-component is given by

\[
\int_T K_{1}^{e, \alpha_1}(t, x)_{\beta, k}(s) K_{1}^{e, \alpha_2}(t, x)_{\beta, -k}(s) ds
\]

\[
= \sum_{\beta=1}^d \sigma_{\beta}^{\alpha_1} \sigma_{\beta}^{\alpha_2} \sum_{k \neq 0} \varphi^2(\varepsilon k) \int (2\pi ik)(-2\pi ik)(1_{t \geq s}) e^{-2\pi^2 k^2 (t - s)} 2 \, ds
\]

\[
= A^{\alpha_1 \alpha_2} \sum_{k \neq 0} \varphi^2(\varepsilon k) \int_0^\infty 4\pi^2 k^2 e^{-4\pi^2 k^2 s} ds
\]

\[
= A^{\alpha_1 \alpha_2} \sum_{k \neq 0} \varphi^2(\varepsilon k) = c^e A^{\alpha_1 \alpha_2},
\]

note that \(\varphi(\varepsilon k) = \varphi(-\varepsilon k)\).

3.4 Derivation of \(C^e\)

Recall that \(H_1^e\) and \(\tilde{H}_1^e\) are given by (3.2). Now we introduce the kernels

\[
H_t(k) := 1_{(k \neq 0, t > 0)} e^{-2\pi^2 k^2 t}, \quad h_t(k) := (2\pi ik) H_t(k).
\]
Then for the function $F(t, x) = e^{2\pi ikx} f(t)$, we have the formula
\[ \partial_x I(F) = e^{2\pi ikx} \int_R h_{t-u}(k) f(u) du. \]

The following convolution formula is useful, cf. the proof of Lemma 6.11 of [13].

**Lemma 3.3.**
\[ \int_R h_{t-u}(k) h_{s-u}(-k) du = H_{|t-s|}(k). \]

**Proof.** Since the integral is over $u < t \wedge s$, the left hand side is rewritten as
\[ (2\pi ik)(-2\pi ik) \int_{-\infty}^{t \wedge s} e^{-2\pi^2 k^2 \{(t-u)+(s-u)\}} du = e^{-2\pi^2 k^2 (t+s-2u)} \int_{u=-\infty}^{t \wedge s} e^{-2\pi^2 k^2 |t-s|} = H_{|t-s|}(k). \]

Note that $t + s - 2(t \wedge s) = |t - s|$. \( \square \)

The kernels of $\partial_x H^\varepsilon,\alpha_Y, \partial_x \tilde{H}^\varepsilon,\alpha_Y \in \mathcal{H}_2$ are given by
\[ K^\varepsilon,\alpha_Y(t, x)(\beta_1, k_1), (\beta_2, k_2)(s_1, s_2) = \frac{i}{\Gamma^\varepsilon,\gamma_1, \gamma_2}(\beta_1, k_1), (\beta_2, k_2)(s_1, s_2)\{\beta_1, k_1\}, (\beta_2, k_2\}(s_1, s_2)) = \int e^{2\pi i (k_1+k_2)x} \varphi(\varepsilon k_1) \varphi(\varepsilon k_2) \int h_{t-u}(k_1 + k_2) h_{u-s_1}(k_1) h_{u-s_2}(k_2) du,
\]
and
\[ \tilde{K}^\varepsilon,\alpha_Y(t, x)(\beta_1, k_1), (\beta_2, k_2)(s_1, s_2) = \frac{i}{\Gamma^\varepsilon,\gamma_1, \gamma_2}(\beta_1, k_1), (\beta_2, k_2)(s_1, s_2)\{\beta_1, k_1\}, (\beta_2, k_2\}(s_1, s_2)) = \int e^{2\pi i (k_1+k_2)x} \varphi(\varepsilon k_1) \varphi(\varepsilon k_2) \varphi^2(\varepsilon(k_1 + k_2)) \times \int h_{t-u}(k_1 + k_2) h_{u-s_1}(k_1) h_{u-s_2}(k_2) du,
\]
respectively. Here $E^\alpha_{\beta_1, \beta_2} := \Gamma^\alpha_{\gamma_1, \gamma_2} \sigma^\gamma_{\beta_1, \beta_2}.$

Now we compute two expectations
\[ C^\varepsilon,\alpha_1, \alpha_2 = E[\{\partial_x H^\varepsilon,\alpha_1 Y, \partial_x H^\varepsilon,\alpha_2 Y\}(t, x)], \quad \tilde{C}^\varepsilon,\alpha_1, \alpha_2 = E[\{\partial_x \tilde{H}^\varepsilon,\alpha_1 Y, \partial_x \tilde{H}^\varepsilon,\alpha_2 Y\}(t, x)]. \]

By Proposition 3.1
\[ C^\varepsilon,\alpha_1, \alpha_2 = 2 \sum_{\beta_1, \beta_2} \sum_{k_1, k_2} \int \int K^\varepsilon,\alpha_1 Y(t, x)(\beta_1, k_1), (\beta_2, k_2)(s_1, s_2) \times K^\varepsilon,\alpha_2 Y(t, x)(\beta_1, -k_1), (\beta_2, -k_2)(s_1, s_2) ds_1 ds_2,
\]
and $\tilde{C}^\varepsilon,\alpha_1, \alpha_2$ is obtained by replacing $K^\varepsilon_Y$ by $\tilde{K}^\varepsilon_Y$. The factor 2 comes from the symmetry of the kernels.
Lemma 3.4. We have

\[ C^{\epsilon, \beta, \gamma} = \frac{F^{\beta, \gamma}}{4\pi^2} \sum_{k_1, k_2} \frac{\varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2)}{k_1^2 + k_1 k_2 + k_2^2}, \]

\[ \tilde{C}^{\epsilon, \beta, \gamma} = \frac{F^{\beta, \gamma}}{4\pi^2} \sum_{k_1, k_2} \frac{\varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2) \varphi^4(\varepsilon(k_1 + k_2))}{k_1^2 + k_1 k_2 + k_2^2}, \]

where

\[ F^{\beta, \gamma} = \sum_{\gamma_1, \gamma_2} E^{\beta}_{\gamma_1, \gamma_2} E^{\gamma}_{\gamma_1, \gamma_2} = \sum_{\gamma_1, \gamma_2} \Gamma_{\beta, \delta_2}^{\gamma} \Gamma_{\delta_3, \delta_4}^{\gamma} \sigma_{\gamma_1}^{\delta_1} \sigma_{\gamma_2}^{\delta_2} \sigma_{\gamma_1}^{\delta_3} \sigma_{\gamma_2}^{\delta_4}. \]

Here \( \sum^* \) means the sum over \( k_1, k_2 \) such that \( k_1, k_2, k_1 + k_2 \neq 0 \).

**Remark 3.1.** When \( d = 1 \), \( C^{\epsilon, \beta, \gamma} \) and \( \tilde{C}^{\epsilon, \beta, \gamma} \) coincide with \( C^{\epsilon, \gamma} \) and \( \tilde{C}^{\epsilon, \gamma} \), respectively, in [14] with \( E^{\beta, \gamma}_{\gamma_1, \gamma_2} = 1 \).

**Remark 3.2.** The expression of the factor \( F^{\beta, \gamma} \) can be obtained by the following graphic rules. Each leaf of the graph “\( \epsilon, \gamma \)” correspond to a label of a noise. When two noises are contracted, the two labels are equal. Each edge attached to a noise corresponds to the factor \( \sigma_{\beta}^2 \). Each vertex with the shape “\( \gamma \)” corresponds to the factor \( \Gamma_{\beta, \gamma}^2 \). Indeed, we have

![Diagram](image)

\[ = \sum_{\gamma_1, \gamma_2} \Gamma_{\delta_1, \delta_2}^{\beta} \Gamma_{\delta_3, \delta_4}^{\gamma} \sigma_{\gamma_1}^{\delta_1} \sigma_{\gamma_2}^{\delta_2} \sigma_{\gamma_1}^{\delta_3} \sigma_{\gamma_2}^{\delta_4}. \]

**Proof.** From Lemma 3.3

\[ C^{\epsilon, \alpha_1, \alpha_2} = \frac{1}{2} \sum_{\beta_1, \beta_2, \gamma_1, \gamma_2} E_{\beta_1, \gamma_1}^{\alpha_1} E_{\beta_2, \gamma_2}^{\alpha_2} \varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2) \]

\[ \times \int \int \int \int h_{t-u_1}(k_1 + k_2) h_{u_1-s_1}(k_1) h_{u_2-s_2}(k_2) \]

\[ \times h_{t-u_2}(-k_1 - k_2) h_{u_2-s_1}(-k_1) h_{u_2-s_2}(-k_2) du_1 du_2 ds_1 ds_2 \]

\[ = \frac{1}{2} F^{\alpha_1, \alpha_2} \sum_{k_1, k_2 \neq 0} \varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2) \]

\[ \times 4\pi^2(k_1 + k_2)^2 \int \int H_{t-u_1}(k_1 + k_2) H_{t-u_2}(k_1 + k_2) \]

\[ \times H_{u_1-u_2}(k_1) H_{u_1-u_2}(k_2) du_1 du_2. \]

Note that the dependence in \( x \) cancels. Changing the variables as \( t - u_1 = r_1, t - u_2 = r_2 \), the integral can be rewritten as

\[ \int_0^\infty dr_1 \int_0^\infty dr_2 e^{-2\pi^2(k_1 + k_2)^2(r_1 + r_2)} e^{-2\pi^2(k_1^2 + k_2^2)|r_1 - r_2|} \]

\[ = \int_0^\infty dr_2 \int_0^{r_2} dr_1 e^{-2\pi^2(k_1 + k_2)^2(r_1 + r_2)} e^{-2\pi^2(k_1^2 + k_2^2)(r_2 - r_1)} \]

\[ + \text{(a similar term with } k_1 \leftrightarrow k_2) \].
Since $k_1, k_2 \neq 0$, the first integral is equal to
\[
\int_0^\infty e^{-2\pi^2((k_1+k_2)^2+(k_1^2+k_2^2))} dr_2 \int_0^{r_2} e^{-2\pi^2((k_1+k_2)^2-(k_1^2+k_2^2))} dr_1
\]
\[
= \frac{1}{2\pi^2((k_1+k_2)^2-(k_1^2+k_2^2))} 
\times \int_0^\infty e^{-2\pi^2((k_1+k_2)^2+(k_1^2+k_2^2))} dr_2 (1 - e^{-2\pi^2((k_1+k_2)^2-(k_1^2+k_2^2))} ) dr_2
\]
\[
= \frac{1}{4\pi^2 k_1 k_2} \int_0^\infty \left( \frac{1}{2\pi^2((k_1+k_2)^2+(k_1^2+k_2^2))} - \frac{1}{2\pi^2((k_1+k_2)^2-(k_1^2+k_2^2))} \right) dr_2
\]
\[
= \frac{1}{4\pi^2 k_1 k_2} \frac{1}{2\pi^2(2(k_1+k_2)^2+(k_1^2+k_2^2))(k_1+k_2)^2} = \frac{1}{16\pi^4(k_1^2+k_2^2)(k_1+k_2)^2}.
\]

By the symmetry under $k_1 \leftrightarrow k_2$, (3.5) is equal to
\[
\frac{1}{8\pi^4(k_1^2+k_2^2)(k_1+k_2)^2}.
\]

One can compute $\tilde{\gamma}^\varepsilon,\beta \gamma$ similarly by noting that the kernel $\tilde{K}_Y^\varepsilon$ has an extra factor $\varphi^2(\varepsilon(k_1+k_2))$ compared with $K_Y^\varepsilon$. This leads to the conclusion. 

3.5 Derivation of $D^\varepsilon$

For the $H_1$-component of $(\partial_x H_Y^{\varepsilon,\alpha_1} \partial_x H_1^{\alpha_2})(t, x) \in H_3 \oplus H_1$, by Proposition 3.1 the kernel is given by
\[
K_{Y}^{\varepsilon,\alpha_1\alpha_2}(t, x)_{\beta, k}(s)
\]
\[
= 2 \sum_{\beta'} \sum_{k' \neq 0} K_Y^{\varepsilon,\alpha_1}(t, x)_{(\beta, k), (\beta', k')}(s, s') K_Y^{\varepsilon,\alpha_2}(t, x)_{\beta', -k'} ds'
\]
\[
= e^{2\pi i k x} \sum_{\beta'} \int h_{-u}(k+k') h_{u-s}(k) h_{u-s'}(k') h_{-u'}(-k') du ds'
\]
\[
= e^{2\pi i k x} \sum_{\beta'} \sum_{k' \neq 0} E_{\beta}\sigma_{\beta'}^{\alpha_1} \sum_{k' \neq 0} \varphi^2(\varepsilon k')
\]
\[
\times \int h_{-u}(k+k') h_{u-s}(k) h_{u-s'}(k') h_{-u'}(-k') du ds'
\]
\[
= e^{2\pi i k x} \sum_{\beta'} \int h_{-u}(k+k') h_{u-s}(k) H_{-u}(k') du.
\]

Note that 2 in the first line comes from the symmetry of $K_{Y}^{\varepsilon,\alpha}(t, x)_{(\beta_1, k_1), (\beta_2, k_2)}(s_1, s_2)$ in $(\beta_1, k_1, s_1)$ and $(\beta_2, k_2, s_2)$. We use Lemma 3.3 to have the last equality. Similarly, the
The $H_1$-component of $(\partial_x \tilde{H}_Y^{\varepsilon,\alpha_1} \partial_x H_1^{\varepsilon,\alpha_2})(t, x)$ is given by the kernel

\[
\tilde{K}_Y^{\varepsilon,\alpha_1\alpha_2}(t, x)_{\beta,k}(s) = e^{2\pi ikx} \varphi(z_k) \sum_{\beta' = 1}^{\beta} E^{\alpha_1}_{\beta' \beta} \delta^{\alpha_2}_{\beta'} \sum_{k' \neq 0} \varphi^2(\varepsilon k') \varphi^2(\varepsilon (k + k')) \\
\times \int h_{t-u}(k + k') h_{u-s}(k) H_{t-u}(k') du.
\]

Recall that $H_Y^\varepsilon$ and $\tilde{H}_Y^\varepsilon$ are given by (3,3). For the $H_1$-component of $\partial_x H_Y^{\varepsilon,\alpha}$ and $\partial_x \tilde{H}_Y^{\varepsilon,\alpha}$, the kernels are respectively given by

\[
K_Y^{\varepsilon,\alpha}(t, x)_{\beta,k}(s) = \Gamma_{\gamma_1\gamma_2}^{\alpha} \partial_x I(K_{\gamma_1\gamma_2}^{\varepsilon,\alpha}(\cdot, \cdot)_{\beta,k}(s))(t, x) = e^{2\pi ikx} \sum_{\beta' = 1}^{d} \Gamma^{\alpha}_{\gamma_1\gamma_2} E^{\gamma_1}_{\beta' \beta} \sigma^{\gamma_2}_{\beta'} \varphi(z_k) \sum_{k' \neq 0} \varphi^2(\varepsilon k') \\
\times \int h_{t-u}(k) h_{v-u}(k + k') h_{u-s}(k) H_{v-u}(k') du dv,
\]

and

\[
\tilde{K}_Y^{\varepsilon,\alpha}(t, x)_{\beta,k}(s) = e^{2\pi ikx} \sum_{\beta' = 1}^{d} \Gamma^{\alpha}_{\gamma_1\gamma_2} E^{\gamma_1}_{\beta' \beta} \sigma^{\gamma_2}_{\beta'} \varphi(z_k) \sum_{k' \neq 0} \varphi^2(\varepsilon k') \varphi^2(\varepsilon (k + k')) \\
\times \int h_{t-u}(k) h_{v-u}(k + k') h_{u-s}(k) H_{v-u}(k') du dv.
\]

Now we start to compute the expectations

\[
D^{\varepsilon,\alpha_1\alpha_2} = E[(\partial_x H_Y^{\varepsilon,\alpha_1} \partial_x H_1^{\varepsilon,\alpha_2})(t, x)],
\]

\[
\tilde{D}^{\varepsilon,\alpha_1\alpha_2} = E[(\partial_x \tilde{H}_Y^{\varepsilon,\alpha_1} \partial_x H_1^{\varepsilon,\alpha_2})(t, x)].
\]

By Proposition 3.1 we have

\[
D^{\varepsilon,\alpha_1\alpha_2} = \sum_{\beta = 1}^{d} \sum_{k \neq 0} \int K_Y^{\varepsilon,\alpha_1}(t, x)_{\beta,k}(s) K_1^{\varepsilon,\alpha_2}(t, x)_{\beta,-k}(s) ds,
\]

and $\tilde{D}^{\varepsilon,\alpha_1\alpha_2}$ is obtained by replacing $K_Y^{\varepsilon,\alpha}$ by $\tilde{K}_Y^{\varepsilon,\alpha}$.

**Lemma 3.5.** We have

\[
D^{\varepsilon,\beta_1\beta_2} = -\frac{G^{\beta_1\beta_2}}{4\pi^2} \sum_{k_1, k_2} \frac{(k_1 + k_2) \varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2)}{k_1^2 + k_1 k_2 + k_2^2},
\]

\[
\tilde{D}^{\varepsilon,\beta_1\beta_2} = -\frac{G^{\beta_1\beta_2}}{4\pi^2} \sum_{k_1, k_2} \frac{(k_1 + k_2) \varphi^2(\varepsilon k_1) \varphi^2(\varepsilon k_2) \varphi^4(\varepsilon (k_1 + k_2))}{k_1^2 + k_1 k_2 + k_2^2},
\]

where

\[
G^{\beta_1\beta_2} = \sum_{\beta_1, \beta_2} \Gamma^{\beta_1\beta_2}_{\gamma_1\gamma_2} \sigma^{\gamma_1}_{\beta_1} \sigma^{\gamma_2}_{\beta_2} = \sum_{\beta_1, \beta_2} \Gamma^{\beta_1\beta_2}_{\gamma_1\gamma_2} \Gamma^{\gamma_1}_{\gamma_2} \sigma^{\gamma_1}_{\beta_1} \sigma^{\gamma_2}_{\beta_2} \delta^{\gamma_1}_{\beta_1} \delta^{\gamma_2}_{\beta_2}.
\]
Remark 3.3. When \( d = 1 \), \( D^{ε,γ} \) and \( \tilde{D}^{ε,γ} \) coincide with \( e^{εX} \) and \( \bar{e}^{εX} \), respectively, in \( \mathcal{L}^γ \) with \( G^{γ} = 1 \).

Remark 3.4. Similarly to Remark 3.2, the expression of the factor \( G^{γ} \) can be obtained by the following graphic rules as follows.

\[
\sum_{β_1, β_2} \Gamma_1^β γ_1 γ_2 δ_1 δ_2 σ_β_1 σ_β_2 σ_β_3 σ_β_4 γ_1 γ_2 γ_3 γ_4.
\]

Proof. From Lemma 3.3,

\[
\begin{align*}
D^{ε,α_1 α_2} &= \sum_{β_1, β_2, k, k' \neq 0} \Gamma_1^{α_1} γ_1 γ_2 δ_1 δ_2 σ_β_1 σ_β_2 \varphi^2(εk) \varphi^2(εk') \\
&\times \int \int \int h_{t-v}(k) h_{u-v}(k + k') h_{u-s}(k) H_{v-u}(k') h_{t-s}(-k) \, du \, dv \, ds \\
&= \sum_{k_1, k_2 \neq 0} G^{α_1 α_2} \varphi^2(εk_1) \varphi^2(εk_2) (2πi(k_1 + k_2)) \{2πi(k_1 + k_2)\} \\
&\times \int_{-∞}^t du \int_{u}^{t} dv \int H_{t-v}(k_1) H_{v-u}(k_1 + k_2) H_{v-u}(k_2) H_{t-u}(k_1).
\end{align*}
\]

Changing the variables as \( t - u = u', t - v = v' \), the integral is computed as

\[
\begin{align*}
\int_0^∞ du' \int_0^{u'} dv' &\ e^{-2π^2 k_1^2 u'} e^{-2π^2 ((k_1 + k_2)^2 + k_2^2)} (u' - v') e^{-2π^2 k_1^2 u'} \\
&= \frac{1}{2π^2 \{(k_1 + k_2)^2 + k_2^2\}} \int_0^∞ dv' e^{-2π^2 ((k_1 + k_2)^2 + k_2^2 + k_1^2 + k_1^2)} (u' - v') e^{-2π^2 ((k_1 + k_2)^2 + k_2^2 + k_1^2 + k_1^2)} - 1 \\
&= \frac{1}{2π^2 \{(k_1 + k_2)^2 + k_2^2 + k_1^2\}} \int_0^∞ (e^{-4π^2 k_1^2 u'} - e^{-2π^2 ((k_1 + k_2)^2 + k_2^2 + k_1^2 + k_1^2)}) \, du' \\
&= \frac{1}{2π^2 \{(k_1 + k_2)^2 + k_2^2 + k_1^2\}} \left( \frac{1}{4π^2 k_1^2} - \frac{1}{2π^2 \{(k_1 + k_2)^2 + k_1^2 + k_1^2\}} \right) \\
&= \frac{1}{2π^2 \{(k_1 + k_2)^2 + k_2^2 + k_1^2\}} \frac{1}{2π^2 2k_1^2 \{(k_1 + k_2)^2 + k_1^2 + k_1^2\}} \\
&= \frac{1}{16π^4 k_1^2(k_2^2 + k_1^2 + k_2^2)}.
\end{align*}
\]

The computation of \( \tilde{D}^{ε,γ} \) is similar with two extra factors \( \varphi^2(ε(k_1 + k_2)) \). This leads to the conclusion. \( \square \)
4 Renormalization factors under the trilinear condition (1.7)

In Lemmas 3.4 and 3.5, we have already computed the renormalization factors
\[ B^{\varepsilon,\beta\gamma} = C^{\varepsilon,\beta\gamma} + 2D^{\varepsilon,\beta\gamma}, \quad \tilde{B}^{\varepsilon,\beta\gamma} = \tilde{C}^{\varepsilon,\beta\gamma} + 2\tilde{D}^{\varepsilon,\beta\gamma} \]

with four renormalization factors given by
\[ C^{\varepsilon,\beta\gamma} = F^{\beta\gamma} C^{\varepsilon}, \quad D^{\varepsilon,\beta\gamma} = G^{\beta\gamma} D^{\varepsilon}, \quad \tilde{C}^{\varepsilon,\beta\gamma} = F^{\beta\gamma} \tilde{C}^{\varepsilon}, \quad \tilde{D}^{\varepsilon,\beta\gamma} = G^{\beta\gamma} \tilde{D}^{\varepsilon}, \]

where \( C^{\varepsilon}, \tilde{C}^{\varepsilon}, D^{\varepsilon}, \tilde{D}^{\varepsilon} \) depend only on \( \varphi \) and \( \varepsilon \), and diverges as \( O(-\log \varepsilon) \), while \( F \) and \( G \) are matrices determined from \( \sigma \) and \( \Gamma \). In this way, the renormalization factors are completely factorized into the products of two terms, one determined from the scalar-valued KPZ equation as is pointed out in Remarks 3.1 and 3.3 and the other from \( \sigma \) and \( \Gamma \).

Lemma 4.1. The constants \( G^{\beta\gamma} \) and \( F^{\beta\gamma} \) are rewritten as
\[ G^{\beta\gamma} = \sigma_{\alpha_1}^{\beta,\gamma} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} \tilde{\Gamma}_{\alpha_2}^{\beta_1,\beta_2}, \quad F^{\beta\gamma} = \sigma_{\alpha_1}^{\beta,\gamma} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} \tilde{\Gamma}_{\alpha_2}^{\beta_1,\beta_2}, \]
respectively. Here the sums \( \sum \) over \( (\alpha_1,\alpha_2,\beta_1,\beta_2) \) are omitted. Moreover, the equality \( F^{\beta\gamma} = G^{\beta\gamma} \) holds for every \((\beta,\gamma)\) if and only if the trilinear condition (1.7) holds.

Proof. In what follows, summation symbols \( \sum \) over the repeated indices are omitted. Noting \( \Gamma^{\gamma}_{\gamma_1\gamma_2} \sigma^{\gamma_1}_{\beta_1} \sigma^{\gamma_2}_{\beta_2} = \sigma^{\gamma}_{\beta} \tilde{\Gamma}^{\gamma}_{\beta_1\beta_2} \), these constants are easily computed as
\[ G^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2} \sigma^{\gamma_1}_{\alpha_1} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} \tilde{\Gamma}_{\alpha_2}^{\beta_1,\beta_2}, \qquad F^{\beta\gamma} = \sigma^{\beta}_{\alpha_1} \sigma^{\gamma}_{\alpha_2} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} \tilde{\Gamma}_{\alpha_2}^{\beta_1,\beta_2}. \]
Hence if we assume the trilinear condition (1.7), we have
\[ F^{\beta\gamma} = G^{\beta\gamma} = \sigma^{\beta}_{\alpha_1} \sigma^{\gamma}_{\alpha_2} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} \tilde{\Gamma}_{\alpha_2}^{\beta_1,\beta_2}. \]

Conversely,
\[ G^{\beta\gamma} - F^{\beta\gamma} = \sigma^{\beta}_{\alpha_1} \sigma^{\gamma}_{\alpha_2} \tilde{\Gamma}_{\alpha_1}^{\beta_1,\beta_2} (\tilde{\Gamma}^{\beta}_{\alpha_2} - \tilde{\Gamma}^{\alpha_2}_{\beta_1}) = 0 \]
is equivalent to
\[ \tilde{\Gamma}^{\beta}_{\beta_1\beta_2} (\tilde{\Gamma}^{\beta}_{\beta_1\beta_2} - \tilde{\Gamma}^{\beta}_{\beta_1\beta_2}) = 0, \quad 1 \leq \beta, \gamma \leq d, \]
because \( \sigma \) is invertible. Taking \( \beta = \gamma \) and summing them over \( \beta \), we have
\[ (4.1) \]
\[ \sum_{\beta_1,\beta_2} \tilde{\Gamma}^{\beta}_{\beta_1\beta_2} (\tilde{\Gamma}^{\beta}_{\beta_1\beta_2} - \tilde{\Gamma}^{\beta}_{\beta_1\beta_2}) = 0. \]
Replacing the role of variables \( \beta \) and \( \beta_1 \) in (4.1), we have
\[ (4.2) \]
\[ \sum_{\beta,\beta_1,\beta_2} \tilde{\Gamma}^{\beta}_{\beta_1\beta_2} (\tilde{\Gamma}^{\beta}_{\beta_1\beta_2} - \tilde{\Gamma}^{\beta}_{\beta_1\beta_2}) = 0. \]
Taking the difference of (4.1) and (4.2), we have
\[ \sum_{\beta,\beta_1,\beta_2} (\tilde{\Gamma}^{\beta}_{\beta_1\beta_2} - \tilde{\Gamma}^{\beta}_{\beta_1\beta_2})^2 = 0, \]
which yields \( \tilde{\Gamma}^{\beta}_{\beta_1\beta_2} = \tilde{\Gamma}^{\beta}_{\beta_1\beta_2} \) for every \((\beta,\beta_1,\beta_2)\).
Proof of Theorem 1.2. A computation for the scalar-valued case made in Proposition 5.32 of [14] shows

\[
\tilde{C}^\varepsilon + 2\tilde{D}^\varepsilon = 0, \quad C^\varepsilon + 2D^\varepsilon = -\frac{1}{12} + O(\varepsilon)
\]

(see also Lemma 6.5 of [10]), and this implies that all components \(B^{\varepsilon, \beta\gamma}\) and \(\tilde{B}^{\varepsilon, \beta\gamma}\) behave as \(O(1)\) if and only if \(F = G\) as matrices, which is equivalent to the condition (1.7) by Lemma 4.1.

Let \(h^\varepsilon_{B=0} = 0\) and \(\tilde{h}^\varepsilon_{B=0} = 0\) be the solutions of two KPZ approximating equations (1.3) and (1.4) with \(B^{\varepsilon, \beta\gamma}, \tilde{B}^{\varepsilon, \beta\gamma} = 0\), which are actually the shifts

\[
h^{\varepsilon, \sigma}_{B=0} = h^{\varepsilon, \sigma} + \frac{1}{2} \Gamma^{\alpha}_{\beta\gamma} B^{\varepsilon, \beta\gamma}, \quad \tilde{h}^{\varepsilon, \sigma}_{B=0} = \tilde{h}^{\varepsilon, \sigma} + \frac{1}{2} \Gamma^{\alpha}_{\beta\gamma} \tilde{B}^{\varepsilon, \beta\gamma}
\]

of the solutions \(h^\varepsilon\) and \(\tilde{h}^\varepsilon\) of (1.3) and (1.4), respectively. Both of them converge because \(B^{\varepsilon, \beta\gamma}, \tilde{B}^{\varepsilon, \beta\gamma} = O(1)\) when (1.7) holds. Let \(h^\alpha_{B=0} = 0\) and \(\tilde{h}^\alpha_{B=0} = 0\) be the respective limits. The difference

\[
\tilde{h}^\alpha_{B=0} - h^\alpha_{B=0} = (\tilde{h}^{\varepsilon, \alpha} - h^{\varepsilon, \alpha}) + \frac{1}{2} \Gamma^{\alpha}_{\beta\gamma} (\tilde{B}^{\varepsilon, \beta\gamma} - B^{\varepsilon, \beta\gamma}),
\]

converges because \(h^{\varepsilon, \alpha} - h^{\varepsilon, \alpha} \to 0\) by Theorem 1.1-(2). Furthermore, noting (4.3), we have in the limit

\[
\tilde{h}^\alpha_{B=0}(t, x) = h^\alpha_{B=0}(t, x) + c^\alpha t,
\]

where

\[
c^\alpha := \frac{1}{24} \Gamma^{\alpha}_{\beta\gamma} F^{\beta\gamma} = \frac{1}{24} \sum_{\beta_1, \beta_2} \Gamma^{\beta}_{\gamma \alpha} \sigma_{\alpha 1} \sigma_{\alpha 2} \tilde{\Gamma}^{\alpha 1}_{\beta_1 \beta_2} \tilde{\Gamma}^{\alpha 2}_{\beta_1 \beta_2} = \frac{1}{24} \sum_{\beta_1, \beta_2} \sigma_{\beta}^{\alpha} \tilde{\Gamma}^{\beta}_{\gamma 1} \tilde{\Gamma}^{\alpha 1}_{\beta_1 \beta_2} \tilde{\Gamma}^{\alpha 2}_{\beta_1 \beta_2}.
\]

\[\square\]

5 Global existence for a.e.-initial values under the stationary measure

When \(d = 1\), the global-in-time existence of the solution of the KPZ equation was obtained by Gubinelli and Perkowski [9], using the Cole-Hopf transform. In the multi-component case, however, such transform does not work in general, so that the global existence is non-trivial. In this section, by similar arguments to Da Prato and Debussche [2], we show the global existence for initial values sampled from the invariant measure of (1.1), under the trilinear condition (1.7).

Precisely, the process which has the invariant measure is the derivative \(u = \partial_x h\), which solves the coupled stochastic Burgers equation

\[
\partial_t u^\alpha = \frac{1}{2} \partial^2_x u^\alpha + \frac{1}{2} \Gamma^{\alpha}_{\beta\gamma} (u^\beta u^\gamma) + \sigma^{\alpha}_{\beta} \partial_x \xi^\beta.
\]

(5.1)

We can apply the paracontrolled calculus to (5.1) and construct a well-posed solution map similarly to the coupled KPZ equation. Indeed, these two schemes are equivalent. If \(h\) solves (1.1), then \(u = \partial_x h\) solves (5.1). Conversely, the solution \(\hat{h}\) of

\[
\partial_t \hat{h}^\alpha = \frac{1}{2} \partial^2_x \hat{h}^\alpha + \frac{1}{2} \Gamma^{\alpha}_{\beta\gamma} (\hat{u}^\beta u^\gamma) + \sigma^{\alpha}_{\beta} \xi^\beta
\]

(5.2)
coincides with the original $h$. Hence the global existence of $u$ is equivalent to that of $h$. The equation (5.1) has the Gaussian invariant measure $\mu_A$, under the condition (1.7). As we will see, this implies the global existence of $u$ starting from a.e.-initial values under the invariant measure. We will justify the above arguments in this section.

From now we consider the space of zero mean functions denoted by
\[ C_0^\alpha = \{ u \in C^\alpha ; \int_T u(x)dx = 0 \}. \]

### 5.1 Relation between KPZ equation and stochastic Burgers equation

Construction of the solution map for (5.1) is parallel to that for the coupled KPZ equation. As in Section 2, although we eventually take $a = 1$, considering the formal expansion $u^\alpha = \sum_{k=0}^{\infty} a^k u_k^\alpha$ of the solution of
\[ L u^\alpha = \frac{a}{2} \Gamma_{\beta\gamma} \partial_x (u^\beta u^\gamma) + \sigma_{\beta\gamma} \partial_x \xi^\beta, \]

we obtain the identities
\[ L u_0^\alpha = \sigma_{\beta\gamma} \partial_x \xi^\beta, \]
\[ L u_1^\alpha = \frac{1}{2} \Gamma_{\beta\gamma} \partial_x (u_0^\beta u_0^\gamma), \]
\[ L u_2^\alpha = \Gamma_{\beta\gamma} \partial_x (u_1^\beta u_0^\gamma), \]
\[ L u_3^\alpha = \frac{1}{2} \Gamma_{\beta\gamma} \partial_x (u_1^\beta u_1^\gamma) + \Gamma_{\beta\gamma} \partial_x (u_2^\beta u_0^\gamma). \]

We denote $u_0, u_1, u_2$ with stationary initial values by $U_0, U_1, U_2$, respectively. To define $L u_3$, we introduce
\[ U_0^\beta = \frac{1}{2} \partial_x (U_1^\beta U_1^\gamma), \quad U_1^\gamma = \partial_x (U_0^\beta \odot U_1^\gamma). \]

After defining these objects, (5.1) for $u = U_1 + U_2 + U_3 + u_{\geq 3}$ can be rewritten as
\[ L u_{\geq 3}^\alpha = \Psi^\alpha + L u_3^\alpha, \]

where
\[ \Psi^\alpha = \Gamma_{\beta\gamma} \partial_x (u_{\geq 3}^\beta U_1^\gamma) + \Gamma_{\beta\gamma} \partial_x (U_1^\beta u_{\geq 3}^\gamma) U_1^\gamma + \frac{1}{2} \Gamma_{\beta\gamma} \partial_x (U_0^\beta u_{\geq 3}^\gamma) (U_1^\gamma + u_{\geq 3}^\gamma). \]

The term $u_{\geq 3}^\beta U_1^\gamma$ is still ill-posed. To make sense, we divide $u_{\geq 3} = v + w$, which solve
\[ L v^\alpha = \Gamma_{\beta\gamma} \partial_x \{ ((U_0^\beta + v^\beta + w^\beta) \odot U_1^\gamma) \}, \]
\[ L w^\alpha = \Gamma_{\beta\gamma} \partial_x \{ ((U_0^\beta + v^\beta + w^\beta) (\odot + \odot) U_1^\gamma) \} + \text{other terms}, \]

respectively. The only remaining problem is to give the definition of $v^\beta \odot U_1^\gamma$. Introducing $U_\xi^\alpha$ as a stationary solution of $L U_\xi^\alpha = \partial_x U_\xi^\alpha$, $v^\alpha$ has the form
\[ v^\alpha = \Gamma_{\beta\gamma} (U_0^\beta + v^\beta + w^\beta) \odot U_\xi^\gamma + \text{regular terms}. \]
Thus, if $U_\xi = U_\xi^\beta \otimes U_1^\gamma$ is given a priori, one can define the term $v^\beta \otimes U_1^\gamma$.

We summarize these arguments. Fix $\kappa \in \left(\frac{1}{3}, \frac{1}{2}\right)$. The driver of the coupled stochastic Burgers equation \([\text{CSB}]\) is the element $U$ of the form

$$
U := (U_1, U_Y, U_Y, U_Y, U_\xi, U_\xi)
$$

\(\in C([0, T], (C_{0}^{\kappa-1})^d) \times C([0, T], (C_{0}^{\kappa-1})^d) \times \{C([0, T], (C_{0}^{\kappa})^d) \cap C^{1/4}([0, T], (C_{0}^{\kappa-1/2})^d)\} \times C([0, T], (C_{0}^{\kappa-2})^d) \times C([0, T], (C_{0}^{\kappa})^d) \times C([0, T], (C_{0}^{\kappa-2})^d),
$$

which satisfies $LU_\xi = \partial_x U_1$. We denote by $U_{\text{CSB}}^{\kappa}$ the class of all drivers. We write $\|U\|_T$ for the product norm on the above space. Comparing with $H_{\text{KPZ}}^\kappa$, note that

\[ (5.3) \]

\[ U_\sigma = \partial_x H_\sigma, \quad (\sigma = 1, Y, Y, Y, \xi, \xi), \]

\[ U_\xi = H_\xi. \]

Now we can prove a similar result to Theorem 2.1. Fix $\lambda \in \left(\frac{1}{3}, \kappa\right)$ and $\mu \in (-\lambda, \lambda)$. For a $\mathcal{D}(\mathbb{T}, \mathbb{R}^d)$-valued functions $v = (v^\alpha)_{\alpha=1}^d, w = (w^\alpha)_{\alpha=1}^d$ on $[0, T]$, we write $(v, w) \in \mathcal{D}_{\text{CSB}}^{\lambda, \mu}(0, T)$ if

\[
\| (v, w) \|_{\mathcal{D}_{\text{CSB}}^{\lambda, \mu}(0, T)} :=
\sup_{t \in [0, T]} t^{\lambda - \mu} \| v(t) \|_{(C_{0}^{\lambda})^d} + \sup_{t \in [0, 1]} \| v(t) \|_{(C_{0}^{\mu})^d} + \sup_{s < t \in [0, T]} \frac{s^{\lambda - \mu} \| v(t) - v(s) \|_{(C_{0}^{\lambda-1/2})^d}}{|t - s|^{1/4}}
\]

\[ + \sup_{t \in [0, T]} t^{\lambda - \mu} \| w(t) \|_{(C_{0}^{\lambda})^d} + \sup_{t \in [0, 1]} \| w(t) \|_{(C_{0}^{\mu})^d} + \sup_{s < t \in [0, T]} \frac{s^{\lambda - \mu} \| w(t) - w(s) \|_{(C_{0}^{\lambda+1/2})^d}}{|t - s|^{1/4}} \]

is finite. For every initial value $(v(0), w(0)) \in (C_{0}^{\mu})^d \times (C_{0}^{\mu})^d$, the system (5.2) admits a unique local solution $(v, w) \in \mathcal{D}_{\text{CSB}}^{\lambda, \mu}$. Denote by $u = U_1 + U_Y + U_\xi + v + w \equiv S_{\text{CSB}}(v(0), w(0), U)$ the maximal solution up to the survival time.

From the constructions, we see that

\[ \partial_x S_{\text{KPZ}}(f(0), g(0), U) = S_{\text{CSB}}(\partial_x f(0), \partial_x g(0), U), \]

where $U$ satisfies (5.3). The problem is to restore the solution map $S_{\text{KPZ}}$ from $S_{\text{CSB}}$. Since the right hand sides of (2.5) depend only on the derivatives of $f$ and $g$, we can write

\[ (5.4) \]

\[ \mathcal{L} f^\alpha = \Gamma_{\beta \gamma}^\alpha (U_\gamma^\beta + v^\beta + w^\beta) \otimes U_1^\gamma, \]

\[ \mathcal{L} g^\alpha = \Gamma_{\beta \gamma}^\alpha (U_\gamma^\beta + v^\beta + w^\beta)(\otimes + \otimes) U_1^\gamma + \text{other terms}. \]

Conversely let $f, g$ be the solutions of (5.4) with initial values $f(0), g(0)$. Then $f, g$ should satisfy $\partial_x f = v$ and $\partial_x g = w$ by uniqueness of the solution of (5.2). Inserting these relations into (5.4), we see that $f, g$ satisfy (2.5). Hence $(f, g)$ is the solution of the original KPZ equation. In this way, $S_{\text{KPZ}}$ can be recovered from $S_{\text{CSB}}$. To sum up, we have the following equivalence.
Theorem 5.1. Assume that $\mathbb{H}$ and $\mathbb{U}$ are related by (5.3). Let $T_{\text{sur}}^{\text{KPZ}}$ and $T_{\text{sur}}^{\text{CSB}}$ be the survival times of the solutions of (2.3) and (5.2), respectively. Then we have $T_{\text{sur}}^{\text{KPZ}} = T_{\text{sur}}^{\text{CSB}}$ and

$$\partial_x S_{\text{KPZ}}(f(0), g(0), \mathbb{H}) = S_{\text{CSB}}(\partial_x f(0), \partial_x g(0), \mathbb{U}).$$

We constructed an $\mathcal{H}_{\text{KPZ}}^\kappa$-valued random variable $\mathbb{H}$ from space-time white noise $\xi$, in Section 3. The relation (5.3) determines a $\mathcal{U}_{\text{CSB}}^\kappa$-valued random variable $\mathbb{U}$. Note that renormalization factors vanish because we take the derivative $\partial_x$. In the following sections, we study the probabilistic properties of the solution $u = S_{\text{CSB}}(v(0), w(0), \mathbb{U})$.

5.2 Gaussian stationary measure of the OU process

Let $\{u_{\alpha,k}\}_{\alpha\in\{1,\ldots,d\}, k\neq 0}$ be the family of centered complex Gaussian variables such that $u_{\alpha,-k} = u_{\alpha,k}$ and has covariance

$$E[u_{\alpha,k}u_{\beta,l}] = A_{\alpha\beta}1_{\{k+l=0\}}.$$ 

Denote the distribution on $D'(\mathbb{T}, \mathbb{R}^d)$ of $u_{\alpha}(x) = \sum_k u_{\alpha,k}e^{2\pi ikx}$ by $\mu_A$. Indeed, $\mu_A$ has a support on $(C_{-1/2})^d$.

Lemma 5.2. Let $\kappa < \frac{1}{2}$. For $\mu_A$-a.e. $u \in D'$, we have $u \in (C_{0}^{\kappa-1})^d$.

Proof. Let $\zeta \in \mathbb{R}$ and $p \geq 1$. Computing $L^{2p}(\Omega, P)$-norm of $\|u_{\alpha}\|_{B_{2p}^{2\zeta}}$ as

$$E\|u_{\alpha}\|_{B_{2p}^{2\zeta}}^{2p} = \sum_j 2^{j\zeta}2^{2p}E\|\Delta_j u_{\alpha}\|_{L^{2p}(\mathbb{T})}^{2p} = \sum_j 2^{j\zeta}2^{2p}\int_T E|\Delta_j u_{\alpha}(x)|^{2p}dx,$$

where $\Delta_j = \rho(2^{-j}D)$ is a Littlewood-Paley projection (see Section 2.2 of [1]), from the hypercontractivity of Wiener chaos, we have

$$E|\Delta_j u_{\alpha}(x)|^{2p} \lesssim (E|\Delta_j u_{\alpha}(x)|^{2})^p.$$

Since we have

$$E|\Delta_j u_{\alpha}(x)|^{2} = \sum_k \rho_j(k)^2 E[u_{\alpha,k}u_{\alpha,-k}] = A_{\alpha\alpha} \sum_k \rho_j(k)^2 \lesssim 2^j,$$

independently of $x$, for every $\zeta < -\frac{1}{2}$

$$E\|u_{\alpha}\|_{B_{2p}^{2\zeta}}^{2p} \lesssim \sum_j 2^{(2\zeta+1)jp} < \infty.$$

Since $B_{2p}^{2\zeta} \subset B_{\infty,\infty}^{\zeta-1/(2p)}$ by Besov embedding, see Theorem 2.71 of [1] or Proposition 2.2 of [1], we have the conclusion for sufficiently large $p$. □

Though it is well-known, we show that the Ornstein-Uhlenbeck process $u$ determined by

\begin{equation}
(5.5) \qquad \mathcal{L}u_{\alpha} = \sigma_{\beta}^\alpha \partial_x \xi^\beta
\end{equation}

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has an invariant measure $\mu_A$. Taking Fourier transform $u_0^{\alpha,k} = \hat{u}_0^{\alpha}(k)$, we can solve it as

$$u^{\alpha,k}(t) = e^{-2\pi^2t^2}u^{\alpha,k}(0) - \sigma^\alpha_\beta(2\pi ik)\int_0^t e^{-2\pi^2k^2(t-s)}\xi^{\beta,k}(s)ds.$$  

For a given $u(0)$, $\{u^{\alpha,k}(t)\}_{\alpha,k}$ is a Gaussian family. If $u(0) \sim \mu_A$, i.e., $u(0)$ is distributed under $\mu_A$, and if $u(0)$ is independent of $\xi$, we see that $u^{\alpha,k}$ has mean zero and covariance

$$E[u^{\alpha,k}(t)u^{\beta,k}(0)] = e^{-2\pi^2(k^2+l^2)t}E[u^{\alpha,k}(0)u^{\beta,k}(0)] + \sigma^\alpha_\beta(2\pi ik)(2\pi il)\int_0^t e^{-4\pi^2k^2(t-s)}\delta^{\gamma_1\gamma_2}1_{\{k+l=0\}}ds$$

$$= A^{\alpha\beta}1_{\{k+l=0\}}\left(e^{-4\pi^2k^2t} + 4\pi^2k^2\int_0^t e^{-4\pi^2k^2(t-s)}ds\right)$$

$$= A^{\alpha\beta}1_{\{k+l=0\}}.$$  

Hence $u(t) \sim \mu_A$ for every $t > 0$.

### 5.3 Galerkin approximation

For $N \in \mathbb{N}$, we consider the approximation

$$(5.6) \quad \partial_t u^{N,\alpha} = \frac{1}{2}\partial_x^2 u^{N,\alpha} + F^{\alpha}_N(u^N) + \sigma_\beta^0 \partial_x \xi^{\beta},$$

of the equation (5.1), where

$$(5.7) \quad F^{\alpha}_N(u^N) = \frac{1}{2}\Gamma^\beta\gamma P_N(\partial_x u^{N,\beta}u^N)$$

and $P_N = \psi(N^{-1}D)$ is the Fourier multiplier defined by an even cut-off function $\psi \in C^\infty_0(\mathbb{R})$ taking values in $[0, 1]$ and supported in the interval $[-1, 1]$. Since $F_N$ depends on finitely many Fourier components of $u^N$, the equation (5.6) is well-posed.

The equation (5.6) is formally equivalent to the (spatial derivative of the) approximating equation (5.3). Indeed, for the solution $\tilde{u}^\epsilon$ of such equation, we would have that $\tilde{u}^\epsilon := \varphi^{-1}(\epsilon D)\tilde{u}^\epsilon \equiv \tilde{u}^\epsilon(\epsilon^\gamma)\varphi^{-1}$ solves

$$\partial_t \tilde{u}^\epsilon = \frac{1}{2}\partial_x^2 \tilde{u}^\epsilon + \frac{1}{2}\Gamma^{\beta\gamma} \partial_x \tilde{u}^\epsilon(\tilde{u}^{\gamma,\gamma} \star \eta^\epsilon) \star \eta^\epsilon + \sigma_\beta^0 \partial_x \xi^{\beta}.$$  

Here $\varphi^{-1}(\epsilon D)$ is an inverse operator of the convolution $\ast \eta^\epsilon = \varphi(\epsilon D)$ defined in a finite dimensional subspace of $D'(\mathbb{T})$. Then (5.6) is obtained by setting $u^N = \tilde{u}^\epsilon$ and $P_N = \ast \eta^\epsilon$. Since $\tilde{u}^\epsilon$ has an invariant measure $\mu^{\epsilon}_A$, which is the distribution of the derivative of the $d$-dimensional periodic and smeared Brownian motion $(\partial_x \sigma B \ast \eta^\epsilon)_{x \in \mathbb{T}}$, $\tilde{u}^\epsilon$ should admit $\mu_A$ as an invariant measure.

Unlike the usual Galerkin approximation, we use the operator $P_N$ rather than Fourier cut-off $\Pi_N = 1_{[-N,N]}(D)$. This is because $P_N$ has the approximating properties

$$\|P_Nu\|_{C^\sigma} \lesssim \|u\|_{C^\sigma}, \quad \|P_Nu - u\|_{C^{\alpha-s}} \lesssim N^{-\delta}\|u\|_{C^\sigma},$$

$$\|P_Nu\|_{C^\sigma} \lesssim \|u\|_{C^\sigma}$$
for $\alpha \in \mathbb{R}$ and $\delta \in [0, 2]$; see Lemma A.5 of [8], or Lemmas 2.4 and 2.5 of [14]. We can construct the solution map $u^N = S_{\text{CSB}}^N(v(0), w(0), U^N)$ corresponding to (5.6), where $U^N$ is defined by the stationary solutions of

$$
\mathcal{L} U_1^{N,\alpha} = \sigma_2^2 \partial_x \xi^2,
$$
(5.8)

$$
\mathcal{L} U_Y^{N,\alpha} = \frac{1}{4} \Gamma_\alpha \partial_x P_N(P_N U_Y^{N,\beta} P_N U_Y^{N,\gamma}),
$$

$$
\mathcal{L} U_\xi^{N,\alpha} = \Gamma_\alpha \partial_x P_N(P_N U_Y^{N,\beta} P_N U_Y^{N,\gamma}),
$$

$$
\mathcal{L} U_\zeta^{N,\alpha} = \partial_x P_N U_1^{N,\alpha},
$$

and products

$$
U_{X,Y}^{N,\beta \gamma} = \frac{1}{2} \partial_x (P_N U_Y^{N,\beta} P_N U_Y^{N,\gamma}),
$$

$$
U_{X,Y}^{N,\beta \gamma} = \partial_x (P_N U_Y^{N,\beta} \circ P_N U_1^{N,\gamma}),
$$

$$
U_{X,Y}^{N,\beta \gamma} = P_N U_Y^{N,\beta} \circ P_N U_1^{N,\gamma}.
$$

(5.9)

From the approximating properties of $P_N$, we have that $S_{\text{CSB}}^N \to S_{\text{CSB}}$ similarly to the approximation $S_{\text{KPZ}}^N$.

**Theorem 5.3.** If $(v^N(0), w^N(0)) \to (v(0), w(0))$ in $(C_0^m)^d \times (C_0^m)^d$ and $U^N \to U$ in $U_{\text{CSB}}$, then we have the convergence $S_{\text{CSB}}^N(v(0), w(0), U^N) \to S_{\text{CSB}}(v(0), w(0), U)$ in $C([0, T_{su}), (C_0^{(k-1)\mu, \mu, 2\mu})^d)$.

If we define $U^N$ by (5.8)–(5.9) from the space-time white noise $\xi$, then $S_{\text{CSB}}^N(v(0), w(0), U^N)$ coincides with the strong solution of (5.6) with initial value $(U_1^N + U_Y^N + U_\xi^N + u + w)(0)$.

Our goal is to show that $\mu_A$ is invariant under $(u^N)$ if the trilinear condition (1.7) holds; see Proposition 5.5 below. Let $u_{OU}$ be the solution of (5.6) with initial value $u^N(0)$. Obviously, the solution of (5.6) is given by $u^N = \Pi_N u_{\text{OU}} + U^N$, where $\Pi_N := 1 - \Pi_N$ and $U^N$ solves the finite dimensional SDE

$$
\partial_t u^{N,\alpha} = \frac{1}{2} \sigma_2^2 u^{N,\alpha} + F_N(u^N) + \sigma_2^3 \partial_x \Pi_N \xi^\beta
$$

with $u^N(0) = \Pi_N u^N(0)$. If $u^N(0)$ is independent of $\xi$, then $\Pi_N u^N(0) \sim (\Pi_N)^{-1} \mu_A$ is independent of $\Pi_N u^N(0) \sim (\Pi_N)^{-1} \mu_A$. Since $\mu_A$ is invariant under $u_{\text{OU}}$, we have $\Pi_N u^N(t) \sim (\Pi_N)^{-1} \mu_A$ for all $t$. Thus we need the following lemma to complete the proof of the invariance of $\mu_A$ under $(u^N)$.

**Lemma 5.4.** If the trilinear condition (1.7) holds, then the solution $U^N$ of (5.10) exists globally in time, and admits $\mu_A^N := (\Pi_N)^{-1} \mu_A$ as an invariant measure.

**Proof.** Note that if we define $\tilde{\Gamma}_{\alpha \beta \gamma} := (A^{-1})_{\alpha \alpha'} \Gamma_{\beta \gamma}^{\alpha'}$, then the condition (1.7) is equivalent to

$$
\tilde{\Gamma}_{\alpha \beta \gamma} = \tilde{\Gamma}_{\alpha \gamma \beta} = \tilde{\Gamma}_{\beta \alpha \gamma}.
$$

(5.11)
Indeed, since \((A^{-1})_{\alpha\alpha'} = \sum_{\gamma} \tau_{\alpha}^{\gamma} \tau_{\alpha'}^{\gamma}\) and \(\tau_{\alpha}^{\gamma} \Gamma_{\beta\gamma}^{\alpha} = \tau_{\beta}^{\gamma} \Gamma_{\beta\gamma}^{\alpha}\),
\[
\Gamma_{\alpha\beta\gamma} = \sum_{\gamma'} \tau_{\alpha}^{\gamma'} \tau_{\alpha'}^{\gamma'} \Gamma_{\beta\gamma'}^{\alpha} = \sum_{\gamma'} \tau_{\alpha}^{\gamma'} \tau_{\beta}^{\gamma'} \Gamma_{\beta\gamma'}^{\alpha}\).
\]
Thus \(\hat{\Gamma}_{\gamma'}^{\beta\gamma'} = \hat{\Gamma}_{\gamma'}^{\beta\gamma'}\) leads to \(\hat{\Gamma}_{\alpha\beta\gamma} = \hat{\Gamma}_{\beta\alpha\gamma}\). From (5.11), we have the identity
\[
(\alpha^{-1})_{\alpha\beta} \langle F_{\alpha}^N(u), u^\beta \rangle_{L^2(\mathbb{T})} = -\frac{1}{2} (\alpha^{-1})_{\alpha\beta} \Gamma_{\beta\gamma'}^{\alpha} \langle P_N u^\beta, P_N u^\gamma, \partial_\xi P_N u^\beta \rangle_{L^2(\mathbb{T})}
\]
\[
= -\frac{1}{6} \hat{\Gamma}_{\beta\gamma'} \int_{\mathbb{T}} \partial_\xi \langle P_N u^\beta, P_N u^\gamma, u^\beta \rangle(x)dx = 0.
\]
We show the global existence of \(U^N\). For this, we set
\[
H_t^N := \sum_{\alpha} \| r_{\alpha} U^N(t) \|^2_{L^2(\mathbb{T})} = (\alpha^{-1})_{\alpha\beta} \langle U^{N,\alpha}(t), U^{N,\beta}(t) \rangle.
\]
Then, by Itô’s formula, we have
\[
dH_t^N = 2 (\alpha^{-1})_{\alpha\beta} \langle U^{N,\alpha}(t), dU^{N,\beta}(t) \rangle + (\alpha^{-1})_{\alpha\beta} \langle dU^{N,\alpha}(t), dU^{N,\beta}(t) \rangle
\]
\[
= -(\alpha^{-1})_{\alpha\beta} \langle \partial_\xi U^{N,\alpha}(t), \partial_\xi U^{N,\beta}(t) \rangle dt + (\alpha^{-1})_{\alpha\beta} \langle U^{N,\alpha}(t), F_N^\beta(U^N)(t) \rangle dt
\]
\[
+ (\alpha^{-1})_{\alpha\beta} \langle \sigma^\alpha \partial_\xi \Pi_N \xi^\gamma(dt), \sigma^\beta \partial_\xi \Pi_N \xi^\delta(dt) \rangle + 2 (\alpha^{-1})_{\alpha\beta} \langle U^{N,\alpha}(t), \sigma^\alpha \partial_\xi \Pi_N \xi^\gamma(dt) \rangle
\]
\[
\leq C_N dt + dM_t^N,
\]
where \(C_N = d \sum_{|k| \leq N} 4 \pi^2 k^2\) and \(M^N\) is a local martingale with the quadratic variation
\[
d[M^N]_t = 4 (\alpha^{-1})_{\alpha\beta} \langle \partial_\xi U^{N,\alpha}(t), \partial_\xi U^{N,\beta}(t) \rangle dt
\]
\[
= 4 (\alpha^{-1})_{\alpha\beta} \sum_{|k| \leq N} (2 \pi i k) \langle U^{N,\alpha}(t), e^{-2 \pi i k} \rangle \langle U^{N,\beta}(t), e^{-2 \pi i k} \rangle
\]
\[
\leq 4 (2 \pi N)^2 H_t^N dt.
\]
By Doob’s inequality, we have
\[
E \left[ \sup_{0 \leq t \leq T} (M_t^N)^2 \right] \leq 4 E[(M_t^N)^2] \leq 64 \pi^2 N^2 \int_0^T E(H_t^N) dt.
\]
Thus, we obtain
\[
E \left[ \sup_{0 \leq t \leq T} H_t^N \right] \leq H_0^N + C_N T + E \left[ \sup_{0 \leq t \leq T} |M_t^N| \right]
\]
\[
\leq H_0^N + C_N T + 1 + E \left[ \sup_{0 \leq t \leq T} (M_t^N)^2 \right]
\]
\[
\leq H_0^N + C_N T + 1 + 64 \pi^2 N^2 \int_0^T E(H_t^N) dt.
\]
Applying Gronwall’s inequality, we obtain
\[ E \left[ \sup_{0 \leq t \leq T} H^N_t \right] \leq (H^N_0 + C_N T + 1)e^{64\pi^2 N^2 T} < \infty. \]

This implies that the process \((U^N(t))_{0 \leq t < \infty}\) does not explode because \(A^{-1}\) is non-degenerate.

Next we show the invariance of \(\mu^N_A\). For the sake of simplicity, we consider the orthonormal basis
\[ e_k(x) = \begin{cases} \sqrt{2} \cos(2\pi kx), & k > 0, \\ \sqrt{2} \sin(2\pi kx), & k < 0, \end{cases} \]
of \(H = \{f \in L^2(\mathbb{T}) : \int_\mathbb{T} f(x) dx = 0\}\). We write \(u^{\alpha,k} = \langle u^\alpha, e_k \rangle\) for \(u = (u^\alpha)_{\alpha=1}^{d} \in H^d\).

For every smooth function \(\Phi : \mathbb{R}^{2Nd}(\simeq (\Pi_N H)^d) \to \mathbb{R}\), which has bounded derivatives, by Itô’s formula,
\[ d\Phi(U^N)(t) = \partial_{(\alpha,k)}\Phi(U^N) dU^{N,\alpha,k}(t) + \frac{1}{2} \partial^2_{(\alpha,k)(\beta,l)}\Phi(U^N) dU^{N,\alpha,k}dU^{N,\beta,l}(t) \]
\[ = - \sum_{|k| \leq N} 2\pi^2 k^2 (U^{N,\alpha,k}) \partial_{(\alpha,k)} - A^{\alpha\beta} \partial^2_{(\alpha,k)(\beta,l)}\Phi(U^N) dt \]
\[ + F^{\alpha,k}_N(U) \partial_{(\alpha,k)}\Phi(U^N) dt + \text{martingale} \]
\[ =: L^N_1 \Phi(U^N) dt + L^N_2 \Phi(U^N) dt + \text{martingale}. \]

By Echeverría’s criterion [3], to complete the proof of the invariance of \(\mu^N_A\), it suffices to show \(\int (L^N_1 + L^N_2) \Phi(u) \mu^N_A(du) = 0\). Since \(L_1\) is the generator of \(\Pi_N u_{OU}\), under which \(\mu^N_A\) is invariant, we have
\[ \int L^N_1 \Phi(u) \mu^N_A(du) = 0. \]

For \(L_2\), note that under \(\mu^N_A\) the \(\mathbb{R}^d\)-valued random variables \(\{(u^{\alpha,k}) \}_{k}\) are independent and each of them has the distribution \(\mathcal{N}(0, A)\). Since \(\mathcal{N}(0, A)\) has a density function \(\gamma_A(u^k), u^k \in \mathbb{R}^d\), satisfying \(\partial_{\beta} \gamma_A(u^k) = -(A^{-1})_{\alpha\beta} u^{\alpha,k} \gamma_A(u^k)\), by the integration by parts, we have
\[ \int L^N_2 \Phi(u) \mu^N_A(du) = \sum_k \int_{\mathbb{R}^{2Nd}} \partial_{(\alpha,k)} \Phi(u) F^{\alpha,k}_N(u) u^{\beta,k} \prod_{0 < |l| \leq N} \gamma_A(u^l) du^l \]
\[ = \sum_k \int_{\mathbb{R}^{2Nd}} \Phi(u) F^{\alpha,k}_N(u)(A^{-1})_{\alpha\beta} u^{\beta,k} \prod_{0 < |l| \leq N} \gamma_A(u^l) du^l \]
\[ = \int_{(\Pi_N H)^d} \Phi(u)(A^{-1})_{\alpha\beta} (F^\alpha_N(u), u^\beta)_{L^2(\mathbb{T})} \mu^N_A(du) = 0 \]
from (5.12). Here we have used \(\sum_k \partial_{(\alpha,k)} F^{\alpha,k}_N(u) = 0\), which is shown as follows:
\[ \sum_k \partial_{(\alpha,k)} F^{\alpha,k}_N(u) = \frac{1}{2} \sum_k \langle \partial_{(\alpha,k)} \partial_x \Gamma^\alpha_{\beta\gamma} P_N (P_N u^\beta P_N u^\gamma), e_k \rangle \]
\[ = \sum_k \langle \partial_x \Gamma^\alpha_{\beta\gamma} P_N (P_N u^\beta P_N e_k), e_k \rangle. \]

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Theorem 1.3. When \( \partial T \) and initial values \( (5.2) \), we assume the trilinear condition \( cf., \text{Theorem 2.1-(2)} \). Our main result of this section is formulated as follows.

(5.14) is finite then it vanishes after applying \( \partial_x \).

If \( \Pi_N u^N(t) \sim (\Pi_N)^{-1} \mu_A \) and \( \Pi_N u^N(t) \sim \mu_A^N \), then \( u^N(t) \sim \mu_A \) by definition. As a consequence, we have the invariance of \( \mu_A \) under \( (u^N) \).

Proposition 5.5. If the trilinear condition \((1.7)\) holds, the solution \( u^N \) of \((5.6)\) exists globally in time, and admits \( \mu_A \) as an invariant measure.

5.4 Global existence for a.e.-initial values

Let \( U, U^N \) be the \( U^N_{CSB} \)-random variables defined from the space-time white noise \( \xi \), corresponding to the equations \((5.1)\) and \((5.6)\), respectively. The following result is obtained similarly to Theorem 5.1 of [2]. Note that if \( T > 0 \) and \( p \geq 1 \),

\[
\lim_{N \to \infty} E\|U^N - U\|^p_T = 0.
\]

In particular, \( u^N = S^N_{CSB}(v(0), w(0), U^N) \) converges to \( u = S_{CSB}(v(0), w(0), U) \) in probability as \( N \to \infty \) in \( C([0, T_{\text{sur}}], (C_0^{\kappa-1})^d) \).

From now we set \( \mu = \frac{\kappa - 1}{2} \), so that \( (\kappa - 1) \land \mu \land 2\mu = \kappa - 1 \). We consider initial values \( u(0) \in (C_0^{\kappa-1})^d \) and set

\[
(5.13) \quad v(0) = 0, \quad w(0) = u(0) - U_1(0) - U_2(0) - U_3(0).
\]

We can prove the following result in a similar way to Theorem 5.1 of [2]. Note that if \( T_{\text{sur}} \) is finite then

\[
(5.14) \quad \lim_{t \to T_{\text{sur}}} \|u\|_{C([0,t], (C_0^{\kappa-1})^d)} = \infty,
\]

cf., Theorem 2.1 (2). Our main result of this section is formulated as follows.

Theorem 5.7. We assume the trilinear condition \((1.7)\). Then, for every \( T > 0 \) and \( \mu_A \)-a.e. \( u(0) \in (C_0^{\kappa-1})^d \), there exists a unique solution \( (v, w) \in D^\mu_{CSB}([0, T]) \) of the system \((5.2)\) with initial values \((5.13)\), which satisfies for every \( p \geq 1 \),

\[
E\|S_{CSB}(v(0), w(0), U)\|^p_{C([0,T], (C_0^{\kappa-1})^d)} < \infty.
\]

In particular, \( T_{\text{sur}} = \infty \) a.s. Furthermore, \( u = S_{CSB}(v(0), w(0), U) \) is a Markov process on \( (C_0^{\kappa-1})^d \) which admits \( \mu_A \) as an invariant measure.

By the equivalence of \( S_{KPZ} \) and \( S_{CSB} \) shown in Theorem 5.1, we have \( T_{\text{sur}}^{KPZ} = \infty \) a.s. when \( \partial_x h(0) \) is taken from \( \mu_A \)-full set of \( (C_0^{\kappa-1})^d \). This together with Theorem 5.7 implies Theorem 1.3.
Proof of Theorem 5.7. We denote by $u^N(\cdot, u(0))$ the solution of (5.6) with initial value $u(0)$. The local existence result like Theorem 2.1(1) holds even for the stochastic Burgers equation, which implies that there exist $C, C' > 0$ independent of $N$ such that, for given $M > 0$, if

$$\|U^N\|_{\mathcal{F}}^3 \leq CM, \quad \|u(0)\|_{(C_0^{-1})^d} \leq CM,$$

then

$$\sup_{t \in [0, t^*_M]} \|u^N(t, u(0))\|_{(C_0^{-1})^d} \leq M, \quad t^*_M = C'M^{-\frac{2}{d}} \wedge T.$$

This yields that for every $u(0) \in (C_0^{-1})^d$,

$$P(\sup_{t \in [0, T]} \|u^N(t, u(0))\|_{(C_0^{-1})^d} > M) \leq \sum_{k=0}^{[T/t^*_M]} P(\sup_{t \in [kt^*_M, (k+1)t^*_M]} \|u^N(t, u(0))\|_{(C_0^{-1})^d} > M) \leq \sum_{k=0}^{[T/t^*_M]} P(\|u(kt^*_M, u(0))\|_{(C_0^{-1})^d} > CM) + ([T/t^*_M] + 1)P(\|U^N\|_{\mathcal{F}}^3 > CM).$$

Since $\sup_N E[\|U^N\|_{\mathcal{F}}^p] < \infty$ for every $p \geq 1$, the second term is bounded as

$$([T/t^*_M] + 1)P(\|U^N\|_{\mathcal{F}}^3 > CM) \lesssim_p M^{-p}.$$

For the first term, since $\mu_A$ is invariant under $u^N$ and $\mu_A$ is Gaussian, we have

$$\int_{(C_0^{-1})^d} P(\|u(kt^*_M, u(0))\|_{(C_0^{-1})^d} > CM)\mu_A(du(0)) = \mu_A(\|u(0)\|_{(C_0^{-1})^d} > CM) \lesssim_p M^{-p}.$$

These are summarized into the bound

$$\int_{(C_0^{-1})^d} P(\sup_{t \in [0, T]} \|u^N(t, u(0))\|_{(C_0^{-1})^d} > M)\mu_A(du(0)) \lesssim_p M^{-p},$$

for every $p \geq 1$, which leads

$$\int_{(C_0^{-1})^d} E[\sup_{t \in [0, T]} \|u^N(t, u(0))\|_{(C_0^{-1})^d}^p]\mu_A(du(0)) \lesssim_p 1.$$

Since bounded set in $L^p(\Omega \times (C_0^{-1})^d, \mu_A)$ is weak* compact for $p > 1$, there exists a subsequence $\{N_k\}$ and $M \in L^p(\Omega \times (C_0^{-1})^d, \mu_A)$ such that

$$\sup_{t \in [0, T]} \|u^{N_k}(t, u(0))\|_{(C_0^{-1})^d} \rightarrow M$$

as $k \rightarrow \infty$ in weak* topology. On the other hand, for every $u(0) \in (C_0^{-1})^d$ and $m \in \mathbb{N}$, $u^{N_k}$ converges to $u = S_{CSB}(v(0), w(0), U)$ in probability in the space $C([0, (m^{-1}T_{su}) \wedge}$

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\( \tau \)

\( \{ \) element

\( \}

by

and the distance

and, by defining

\( u \)

\( C \)

This implies that for \( \mu \) \( m \) \( N \) \( k \)

\( \rightarrow \infty \) in weak* topology for every \( m \in \mathbb{N} \). From the uniqueness of the limit, we have

\( \frac{||u||_{C([0, \frac{1}{m} T_{\text{sur}} \land T], (C_0^{\kappa - 1})^d)}}{m} \rightarrow M^m \)

as \( k_m \rightarrow \infty \).

\( \mu \)

\( \in \mathbb{N} \). From the uniqueness of the limit, we have

\( \frac{||u||_{C([0, \frac{1}{m} T_{\text{sur}} \land T], (C_0^{\kappa - 1})^d)}}{m} \rightarrow M^m \), \( P \times \mu_{\mathcal{A}} \) a.e.

Taking the limit \( m \rightarrow \infty \), we have

\( \frac{||u||_{C([0, T_{\text{sur}} \land T], (C_0^{\kappa - 1})^d)}}{M} \leq P \times \mu_{\mathcal{A}} \) a.e.

Markov property of \( u \) is obtained as follows. Let \( u = u(\cdot, 0, ) \in C([0, T_{\text{sur}}], (C_0^{\kappa - 1})^d) \) be the solution starting at \( u(0) \) and driven by \( \mathcal{G} \). Now we introduce the space \((C_0^{\kappa - 1})^d \cup \{ \Delta \})\) and the distance

\( d(u, \Delta) = (1 + ||u||_{(C_0^{\kappa - 1})^d})^{-1} \),

and, by defining \( u(t) = \Delta \) for \( t \geq T_{\text{sur}} \), we regard \( u \) as a random variable taking values in \((C_0, (C_0^{\kappa - 1})^d \cup \{ \Delta \})\). By the uniqueness result, we have the identity

\( u(t, u(s, u(0), U), \tau_s U) = u(s + t, u(0), U), \quad s, t \geq 0, \)

where \( \tau_s \) is a shift operator defined by \( \tau_s U(\cdot) = U(\cdot + s) \). Let \( \mathcal{F}_1 \) be the \( \sigma \)-algebra generated by \( \{ 1 \in [r, s] \} \) \(-- \infty < r < s \leq T \). Then from the construction of the solution, for deterministic element \( v, u(s, v, U) \in \mathcal{F}_s \)-measurable. Moreover, \( u(t, v, \tau_s U) \) is independent of \( \mathcal{F}_s \) because it coincides with the solution of (the approximation of)

\begin{align*}
\begin{aligned}
\partial_t u^\alpha &= \frac{1}{2} \partial_x^2 u^\alpha + \frac{1}{2} \Gamma_{\beta \gamma}^\delta \partial_x (u^\beta u^\gamma) + \sigma_{\beta \gamma}^\delta \partial_x (\tau_s \xi^\beta) \quad t > 0, \\
u(0, \cdot) &= v.
\end{aligned}
\end{align*}

As a consequence, for every bounded measurable function \( \Phi \) on \((C_0^{\kappa - 1})^d \cup \{ \Delta \})\), we have

\[ E[\Phi(u(s + t, u(0), U)) | \mathcal{F}_s] = P_{st}(u(s, u(0), U)), \]

where \( P_{st}(v) := E[\Phi(u(t, v, \tau_s U))] \). In particular, when \( u(0) \sim \mu_A \) independently of \( \xi \), since \( T_{\text{sur}} = \infty \) almost surely, \( u \) is a Markov process on \((C_0^{\kappa - 1})^d\). The invariance of \( \mu_A \) under \( (u(t))_{t \geq 0} \) is an immediate consequence of the convergence \( u^\xi(t) \rightarrow u(t) \) and Proposition 5.5

\subsection{5.5 Global existence for two coupled KPZ approximating equations}

The derivatives \( u = \partial_x h^\varepsilon \) and \( \tilde{u} = \partial_x \tilde{h}^\varepsilon \) of the solutions \( h^\varepsilon \) and \( \tilde{h}^\varepsilon \) of the coupled KPZ approximating equations (1.3) and (1.3) solve the following coupled stochastic Burgers equations

\begin{equation}
\begin{aligned}
\partial_t u^\alpha &= \frac{1}{2} \partial_x^2 u^\alpha + \frac{1}{2} \Gamma_{\beta \gamma}^\delta \partial_x (u^\beta u^\gamma) + \sigma_{\beta \gamma}^\delta \partial_x \xi^\beta \ast \eta^\varepsilon, \\
\end{aligned}
\end{equation}
and
\[ \frac{\partial_t \tilde{u}_\alpha}{2} = \frac{1}{2} \partial^2 u^\alpha + \frac{1}{2} \Gamma_{\beta \gamma}^\alpha \partial_x \{ (\tilde{u} \tilde{u}) \ast \eta_k^2 \} + \sigma_{\beta \gamma} \partial_x \xi^\beta \ast \eta^\alpha, \]
respectively. We show the global existence of the solutions of these two equations. Since \( \varepsilon > 0 \) is fixed, we drop the superscripts \( \varepsilon \) from \( u^\varepsilon \) and \( \tilde{u}^\varepsilon \) and simply write \( u \) and \( \tilde{u} \), respectively.

### 5.5.1 The equation \( (5.15) \)

For the equation \( (5.15) \), we apply the classical method of the energy inequality to show the global well-posedness.

**Lemma 5.8.** Assume \( E[\|u(0)\|_{L^2(T,R^d)}] < \infty \). Then, for every \( T > 0 \), we have
\[ E\left[\sup_{0 \leq t \leq T} \|u(t)\|_{L^2(T,R^d)}^2\right] < \infty. \]
In particular, \( T_{\text{sur}} = \infty \) a.s.

**Proof.** We use a similar argument given in Section 5.3. We consider the approximation
\[
\begin{align*}
\partial_t u^N_{\alpha} &= \frac{1}{2} \partial^2 u^N_{\alpha} + F^N_\alpha(u^N) + \sigma_{\gamma} \partial_x \xi^\gamma \ast \eta^\alpha, \\
\alpha &= N(0) = \Pi_N u(0),
\end{align*}
\]
where \( F^N_\alpha \) is the operator defined by \( (5.7) \). We can use the same argument as in the proof of Lemma 5.4 and show the global existence of \( (u^N(t))_{0 \leq t < \infty} \). Indeed, by applying Itô’s formula to \( H^N := \sum_{\alpha} \|\tau^\alpha u^N\|_{L^2(T)}^2 \) we have
\[ dH^N_t \leq C_N dt + dM^N_t, \]
where \( C_N = d \sum_{|k| \leq N} 4\pi^2 k^2 \varphi(ek)^2 \) and \( M^N \) is a local martingale with the quadratic variation
\[ d[M^N]_t = 4(A^{-1})_{\alpha \beta} \partial_x \eta^\alpha(t) \partial_x \eta^\beta(t) dt \]
\[ = 4 \sum_{\alpha} \|\tau^\alpha u^N \ast \partial_x \eta^\alpha(t)\|_{L^2(T)}^2 dt \leq 4C'H^N_t dt, \]
where \( C' = \|\partial_x \eta\|_{L^2(T)}^2 \). The same argument as Lemma 5.4 shows that
\[ E\left[\sup_{0 \leq t \leq T} \sum_{\alpha} \|\tau^\alpha u^N(t)\|_{L^2(T)}^2\right] \leq (E[\sup_{t} \|\tau^\alpha \Pi_N u(0)\|_{L^2(T)}^2] + C_N T + 1) e^{16C'T}. \]
Since \( \tau \) is invertible, a similar estimate holds for \( \sum_{\alpha} \|u^N_{\alpha}(t)\|_{L^2(T)}^2 = \|u^N(t)\|_{L^2(T,R^d)}^2 \). Since \( \|\Pi_N u(0)\|_{L^2(T,R^d)} \leq \|u(0)\|_{L^2(T,R^d)} \) and \( C_N \leq d \sum_k 4\pi^2 k^2 \varphi(ek)^2 < \infty \), we have the uniform estimate
\[ \sup_N E\left[\sup_{0 \leq t \leq T} \|u^N(t)\|_{L^2(T,R^d)}^2\right] < \infty. \]
It is not difficult to prove that \((u^N)_N\) converges to the solution \(u\) of (5.15) up to time \(T_{\text{sur}}\), as an application of the paracontrolled calculus. From this convergence, in a similar way to the proof of Theorem 5.7, we can show that
\[
E[\|u\|_{C([0,T_{\text{sur}}\wedge T],L^2(T,R^d))}^2] < \infty
\]
and this implies \(T_{\text{sur}} = \infty\) a.s. \(\square\)

Next we consider the case that \(u_0 \in (C^{\delta - 1}_0)^d\). We fix \(T > 0\). By Theorem 2.1, for every \(K > 0\) there exists a (deterministic) \(t = t(u_0,K) \in (0,T]\) such that
\[
u^K_t = \begin{cases} u_t, & \|H^\epsilon \|_t \leq K, \\ 0, & \text{otherwise} \end{cases}
\]
satisfies \(\|\nu^K_t\|_{L^2(T,R^d)} \lesssim 1 + \|u_0\|_{C^{\delta - 1}_0} + K^3\), so that \(E[\|\nu^K_t\|_{L^2(T,R^d)}^2] < \infty\). Since the solution of (5.15) with initial value \(\nu^K_t\) exists globally, we have
\[
P(u \in C([0,T],(C^{\delta - 1}_0)^d)) \geq P(\|H^\epsilon \|_t \leq K) \geq P(\|H^\epsilon \|_T \leq K).
\]
By letting \(K \to \infty\), we see that \(u\) exists up to the time \(T\) almost surely. Since \(T > 0\) is arbitrary, we have the global existence of \(u\).

5.5.2 The equation (5.16)

For the equation (5.16), in a similar way to Sections 5.3-5.4, one can first show the global existence for a.e.-initial values under the stationary measure, and extend it to all initial values by combining it with the strong Feller property.

Precisely, we first use the approximation
\[
\begin{align*}
\partial_t \tilde{u}^N,\alpha &= \frac{1}{2}\tilde{\sigma}^{\alpha 2} \tilde{u}^N,\alpha + F^\alpha_N(\tilde{u}^N) \star \eta^2_{\tilde{\epsilon}} + \sigma^{\alpha 2} \tilde{x}^\beta \star \eta^\beta_{\tilde{\epsilon}}, \\
\tilde{u}^N(0) &= \tilde{u}(0) \in (C^{\delta - 1}_0)^d.
\end{align*}
\]
where \(F^\alpha_N\) is the operator defined by (5.7). Without loss of generality, we may assume \(\varphi(\epsilon k) \neq 0\) for all \(k \in \mathbb{Z}\). We can see that this equation has a unique global solution \(\tilde{u}^N\) by applying Itô’s formula to
\[
H^N := \sum_\alpha \|\tau^\alpha \varphi^{-1}(\epsilon D)\Pi_N \tilde{u}^N\|_{L^2(T)}^2 = (A^{-1})_{\alpha \beta} \langle \Pi_N \tilde{u}^{N,\alpha}, \varphi^{-2}(\epsilon D) \Pi_N \tilde{u}^{N,\beta}\rangle.
\]
Moreover, \(\tilde{u}^N\) admits the measure \(\mu^\epsilon_A\) as an invariant measure, where \(\mu^\epsilon_A\) is the distribution on \(\mathcal{D}'(T,R^d)\) of \(u^\alpha(x) = \sum_k \varphi(\epsilon k)u^\alpha,k \epsilon^{2\pi ikx}\) with the family \(\{u^\alpha,k\}\) of centered complex Gaussian variables such that \(u^{\alpha,-k} = \overline{u^\alpha,k}\) and covariance
\[
E[u^{\alpha,k}u^{\beta,l}] = A^{\alpha \beta} \{k+l=0\},
\]
recall Lemma 5.4. Then the similar argument to the proof of Theorem 5.7 shows that the solution \(\tilde{u}\) of the equation (5.16) exists globally for \(\mu^\epsilon_A\)-a.e. initial values.
Acknowledgements

The authors thank Cédric Bernardin for bringing the paper [4] into their attention, Xue-Mei Li for her interest in this paper, especially, the discussion on the trilinear conditions (1.5) and (1.7), and Hao Shen for his significant remarks on the logarithmic renormalization factors. They also thank anonymous referees for their helpful remarks, in particular, suggestions of the condition (1.13) and Remarks 3.2 and 3.4.

References

[1] H. Bahouri, J-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Springer, 2011.

[2] G. Da Prato and A. Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal., 196 (2002), 180–210.

[3] P. Echeverría, *A criterion for invariant measures of Markov processes*, Z. Wahrsch. Verw. Gebiete, 61 (1982), 1–16.

[4] D. Ertas and M. Kardar, *Dynamic roughening of directed lines*, Phys. Rev. Lett., 69 (1992), 929–932.

[5] P.L. Ferrari, T. Sasamoto and H. Spohn, *Coupled Kardar-Parisi-Zhang equations in one dimension*, J. Stat. Phys., 153 (2013), 377–399.

[6] T. Funaki, *Infinitesimal invariance for the coupled KPZ equations*, Memoriam Marc Yor – Séminaire de Probabilités XLVII, Lect. Notes Math., 2137, 37–47, Springer, 2015.

[7] T. Funaki and J. Quastel, *KPZ equation, its renormalization and invariant measures*, Stoch. PDE: Anal. Comp., 3 (2015), 159–220.

[8] M. Gubinelli, P. Imkeller and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math., Pi, 3 (2015) 75pp.

[9] M. Gubinelli and N. Perkowski, *KPZ reloaded*, arXiv:1508.03877.

[10] M. Hairer, *Solving the KPZ equation*, Ann. Math, 178 (2013), 559–664.

[11] M. Hairer, *A theory of regularity structures*, Invent. Math., 198 (2014), 269–504.

[12] M. Hairer and J. Mattingly, *The strong Feller property for singular stochastic PDEs*, arXiv:1610.03415.

[13] M. Hairer and J. Quastel, *A class of growth models rescaling to KPZ*, arXiv:1512.07845.

[14] M. Hoshino, *Paracontrolled calculus and Funaki-Quastel approximation for the KPZ equation*, arXiv:1605.02624.
[15] A. Kupiainen and M. Marcozzi, *Renormalization of Generalized KPZ Equation*, J. Stat. Phys. **166** (2017), no. 3-4, 876-902.

[16] P. Major, *Multiple Wiener-Itô integrals, with applications to limit theorems*, Lect. Notes Math., **849**, 2nd Edition, Springer, 2014.

[17] J.-C. Mourrat and H. Weber, *Global well-posedness of the dynamic \( \Phi^4_3 \) model on the torus*, arXiv 1601.01234.

[18] H. Spohn, *Nonlinear fluctuating hydrodynamics for anharmonic chains*, J. Stat. Phys., **154** (2014), 1191–1227.

[19] H. Spohn and G. Stoltz, *Nonlinear fluctuating hydrodynamics in one dimension: the case of two conserved fields*, J. Stat. Phys., **160** (2015), 861–884.