NORMALITY AND MONTEL’S THEOREM

GOPAL DATT AND SANJAY KUMAR

Abstract. In this article, we prove a normality criterion for a family of meromorphic functions having zeros with some multiplicity which involves sharing of a holomorphic function by the members of the family. Our result generalizes Montel’s normality test in a certain sense.

1. Introduction and main results

The notion of normal families has played a key role in the progress of function theory. The convergence of a family of functions always has far reaching consequences. The concept of local convergence of a sequence of functions was introduced by Montel who later gave the notion of normal family. He gave a result on the convergence of the sequence of holomorphic functions which says that a sequence of uniformly bounded holomorphic functions has a subsequence that is locally uniformly convergent. Let us recall the definition: A family of meromorphic (holomorphic) functions defined on a domain $D \subset \mathbb{C}$ is said to be normal in the domain, if every sequence in the family has a subsequence which converges spherically uniformly on compact subsets of $D$ to a meromorphic (holomorphic) function or to $\infty$. [1, 4, 5, 10].

The most celebrated result in the theory of normal families is Montel’s Critère Fondamental (Fundamental Normality Test), which says that: A family $\mathcal{F}$ of meromorphic functions in a domain $D \subset \mathbb{C}$, which omits three distinct complex numbers, is normal in the domain $D$. This result supports Bloch’s heuristic principle which says that a family $\mathcal{F}$ of meromorphic functions endowed with a property $P$ is normal if condition $P$ reduces a meromorphic function to a constant in the plane. Although this principle is not true in general, many researchers gave normality criteria for families of meromorphic functions supporting Bloch’s heuristic principle. Inspired by Bloch’s principle, Schwick discovered a connection between shared values and normality [6]. Since then many researchers proved normality criteria concerning shared values [2, 3, 7, 9]. Let us recall the meaning of shared values. Let $f$ be a meromorphic function of a domain $D \subset \mathbb{C}$. For $p \in \mathbb{C}$, let

$$E_f(p) = \{ z \in D : f(z) = p \}$$

and let

$$E_f(\infty) = \text{poles of } f \text{ in } D.$$ 

For $p \in \mathbb{C} \cup \{ \infty \}$, two meromorphic functions $f$ and $g$ of $D$ share the value $p$ if $E_f(p) = E_g(p)$.

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Improving the fundamental normality test Sun [7] proved the following theorem.

**Theorem A.** [7] Let \( \mathcal{F} \) be a family of functions meromorphic in a plane domain \( D \). If each pair of functions \( f \) and \( g \) share \( 0, 1, \infty \), then \( \mathcal{F} \) is normal in \( D \).

This result of Sun was further improved by Xu [9] as follows.

**Theorem B.** [9] Let \( \mathcal{F} \) be a family of functions meromorphic in a plane domain \( D \). Suppose that
1. \( f \) and \( g \) share \( 0 \) in \( D \), for each pair \( f \) and \( g \) in \( \mathcal{F} \),
2. all zeros of \( f - 1 \) are of multiplicity at least 3 (or 2), for each \( f \in \mathcal{F} \),
3. all poles of \( f \) are of multiplicity at least 2 (or 3), for each \( f \in \mathcal{F} \),
then \( \mathcal{F} \) is normal in \( D \).

In the same paper Xu also proved the following normality criterion.

**Theorem C.** [9] Let \( \mathcal{F} \) be a family of functions meromorphic in a plane domain \( D \) and let \( \psi \not\equiv 0, \infty \) be a meromorphic function in \( D \). Suppose that
1. \( f \) and \( g \) share \( 0, \infty, \psi(z) \) in \( D \), for each pair \( f \) and \( g \) in \( \mathcal{F} \),
2. the multiplicity of \( f \in \mathcal{F} \) is larger than that of \( \psi(z) \) at the common zeros or poles of \( f \) and \( \psi(z) \) in \( D \),
then \( \mathcal{F} \) is normal in \( D \).

**Observation.** We observe that we can not assure normality in case each pair \( f, g \) of \( \mathcal{F} \) shares \( 0 \) and \( \infty \). We have the following example supporting this observation.

**Example.** Let \( D = \{ z : |z| < 1 \} \) and \( \mathcal{F} = \{ nz : n \in \mathbb{N} \} \). Clearly each pair \( f, g \) of \( \mathcal{F} \) shares \( 0, \infty \) but \( \mathcal{F} \) is not normal in \( D \).

This example also confirms that normality will no longer be assured if each pair \( f, g \) of \( \mathcal{F} \) shares a holomorphic function which is identically 0 in \( D \).

It is natural to ask if we can assure normality after removing the condition of sharing \( 0 \) and \( \infty \) in Theorem C? In this paper we discuss this problem and propose a normality criterion where a holomorphic function is shared by each pair of functions of the family. Let us recall the definition of shared function. We say two functions \( f \) and \( g \) share a function \( h \) IM in a domain \( D \), if \( \{ z \in D : f(z) = h(z) \} = \{ z \in D : g(z) = h(z) \} \). We obtain the following result which clearly generalizes Theorem B.

**Theorem 1.1.** Let \( \mathcal{F} \) be a family of meromorphic functions defined on a domain \( D \subset \mathbb{C} \) and let \( \psi \not\equiv 0 \) be a holomorphic function in \( D \) such that zeros of \( \psi(z) \) are of multiplicity at most \( m \). Suppose that
1. all poles of \( f \) are of multiplicity at least \( 3(m + 1) \) (or \( 2(m + 1) \)),
2. all zeros of \( f \) are of multiplicity at least \( 2(m + 1) \) (resp. \( 3(m + 1) \)),
3. each pair \( f \) and \( g \) of \( \mathcal{F} \) shares \( \psi \) IM in \( D \),
then \( \mathcal{F} \) is normal in \( D \).

**Corollary 1.2.** Let \( \mathcal{F} \) be a family of meromorphic functions defined on a domain \( D \subset \mathbb{C} \). Suppose that
1. all poles of \( f \) are of multiplicity at least 3 (or 2),
(2) all zeros of $f$ are of multiplicity at least 2 (resp. 3),
(3) each pair $f$ and $g$ of $\mathcal{F}$ shares 1 IM in $D$,
then $\mathcal{F}$ is normal in $D$.

Corollary 1.3. Let $\mathcal{F}$ be a family of holomorphic functions defined on a domain $D \subset \mathbb{C}$ and let $\psi \not\equiv 0$ be a holomorphic function in $D$ such that zeros of $\psi(z)$ are of multiplicity at most $m$. Suppose that

(1) all zeros of $f$ are of multiplicity at least $2(m + 1)$,
(2) each pair $f$ and $g$ of $\mathcal{F}$ shares $\psi$ IM in $D$,
then $\mathcal{F}$ is normal in $D$.

Remark 1.4. It is easy to see that Corollary 1.2 can also be obtained from Theorem B, by considering the family $\mathcal{F}_1 = \{1 - f : f \in \mathcal{F}\}$.

The following example shows that the condition on the multiplicities of zeros in Corollary 1.2 (and hence also in Theorem 1.1) is necessary.

Example 1.5. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{nz + 1 : n \in \mathbb{N}\}$. Let $\psi(z) \equiv 1$. Then $m = 0$ and each pair $f$, $g$ of $\mathcal{F}$ shares $\psi$ but $\mathcal{F}$ is not normal in $D$.

We thank the referee for suggesting that by using a result of Xu (cf. Theorem D below) and Corollary 1.2 one can relax condition (1) in Theorem 1.1 to multiplicity at least 3. We could improve this further. We state the improved result as follows.

Theorem 1.6. Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain $D \subset \mathbb{C}$ and let $\psi \not\equiv 0, \infty$ be a meromorphic function in $D$. Suppose that

(1) every zero of $f$ has multiplicity at least 2,
(2) every pole of $f$ has multiplicity at least 3,
(3) at the common zeros or poles of $f$ and $\psi$, the multiplicity of $f$ is larger than that of $\psi$,
(4) $f$ and $g$ share $\psi$, for each pair $f$ and $g$ in $\mathcal{F}$,
then $\mathcal{F}$ is normal in $D$.

The following example shows that the condition on common zeros is necessary in Theorem 1.6.

Example 1.7. Let $k, m$ be two integers such that $2 \leq k \leq m$, let $D := \{z : |z| < 1\}$, $\psi(z) = z^m$ and
$$\mathcal{F} = \{f_n(z) = (n + 2)z^k : z \in D, n \in \mathbb{N}\}.$$ For each $f_n \in \mathcal{F}$, we have

(1) $f_n$ has zeros of multiplicity $k \geq 2$,
(2) $f$ has no pole,
(3) for each $i, j$, $f_i$ and $f_j$ share $\psi$ in $D$,
(4) at the common zero of $f_n$ and $\psi$ the multiplicity of $\psi$ is larger than or equal to the multiplicity of $f_n$.
But $\mathcal{F}$ is not normal in $D$.

The following example shows that the condition on common poles is necessary in Theorem 1.6.
Example 1.8. Let \( k, m \) be two integers such that \( 3 \leq k \leq m \), let \( D := \{ z : |z| < 1 \} \), \( \psi(z) = \frac{1}{z^m} \) and
\[
F = \left\{ f_n(z) = \frac{1}{(n+2)z^k} : z \in D, n \in \mathbb{N} \right\}.
\]

For each \( f_n \in F \), we have
1. \( f_n \) has poles of multiplicity \( k \geq 3 \),
2. \( f \) has no zero,
3. for each \( i, j, f_i \) and \( f_j \) share \( \psi \) in \( D \),
4. at the common pole of \( f_n \) and \( \psi \) the multiplicity of \( \psi \) is larger than or equal to that of \( f_n \).

But \( F \) is not normal in \( D \).

The following result was proved by Xu [8].

Theorem D. [8] Let \( F \) be a family of meromorphic functions defined in a domain \( D \subset \mathbb{C} \) and let \( \psi(\neq 0) \) be a meromorphic function in \( D \). For every \( f \in F \), if
1. \( f \) has only multiple zeros,
2. the poles of \( f \) have multiplicity at least 3,
3. at the common poles of \( f \) and \( \psi \), the multiplicity of \( f \) does not equal the multiplicity of \( \psi \),
4. \( f(z) \neq \psi(z) \),
then \( F \) is normal in \( D \).

We find that Theorem D is not true. Example 1.8 shows that the analysis was not completed in Theorem D. Theorem D can be stated as follows in its correct formulation. It can be seen easily that now the proof of Xu [8] works smoothly.

Theorem D’. Let \( F \) be a family of meromorphic functions defined in a domain \( D \). Let \( \psi (\neq 0, \infty) \) be a function meromorphic in \( D \). For every function \( f \in F \), if
1. every zero of \( f \) has multiplicity at least 2,
2. every pole of \( f \) has multiplicity at least 3,
3. at the common poles of \( f \) and \( \psi \), the multiplicity of \( f \) is larger than that of \( \psi \),
4. \( f(z) \neq \psi(z) \),
then \( F \) is normal in \( D \).

In [8] the meaning of \( f(z) \neq \psi(z) \) is not defined. But from condition (iii) in Theorem D it is clear that it is meant to mean that the meromorphic function \( f - \psi \) has no zeros. As \( \psi \) and \( f \) might have poles, this is a weaker condition than \( f(z_0) \neq \psi(z_0) \) for all \( z_0 \in \mathbb{C} \). In particular, Theorem D’ is not an immediate corollary to Theorem 1.6 (apart from the fact that we are using Theorem D’ to prove Theorem 1.6).

2. Proof of Main Theorems

We need some preparation for proving our main result. Zalcman proved a striking result that studies consequence of non-normality [11]. Roughly speaking, it says that in
an infinitesimal scaling the family gives a non-constant entire function under the compact-open topology. We state this renormalization result which has now come to be known as Zalcman’s Lemma.

**Zalcman’s Lemma.** [11] A family $\mathcal{F}$ of functions meromorphic (holomorphic) on the unit disc $\Delta$ is not normal if and only if there exist

(a) a number $r$, $0 < r < 1$

(b) points $z_j, |z_j| < r$

(c) functions $\{f_j\} \subseteq \mathcal{F}$

(d) numbers $\rho_j \to 0^+$

such that

$$f_j(z_j + \rho_j \zeta) \to g(\zeta)$$

spherically uniformly (uniformly) on compact subsets of $\mathbb{C}$, where $g$ is a non-constant meromorphic (entire) function on $\mathbb{C}$.

Before proving Theorem 1.1 we prove some auxiliary results.

**Lemma 2.1.** Let $f$ be a transcendental meromorphic function and let $p \not\equiv 0$ be a polynomial. Suppose that every zero of $f$ has multiplicity at least $2$ (or $3$) and every pole of $f$ has multiplicity at least $3$ (resp. $2$). Then $f - p$ has infinitely many zeros.

**Proof.** Clearly $p(z)$ satisfies $T(r, p(z)) = o\{T(r, f(z))\}$. Suppose that $f(z) - p(z)$ has only finitely many zeros. Then by invoking the second fundamental theorem of Nevanlinna for three small functions $a_1(z) = 0, a_2(z) = \infty$ and $a_3(z) = p(z)$, we get

$$(1 + o(1))T(r, f) \leq \mathcal{N}(r, f) + \mathcal{N}\left(r, \frac{1}{f}ight) + \mathcal{N}\left(r, \frac{1}{f - p}\right) + S(r, f)$$

$$= \mathcal{N}(r, f) + \mathcal{N}\left(r, \frac{1}{f}\right) + S(r, f)$$

$$\leq \frac{N(r, f)}{3} + \frac{N\left(r, \frac{1}{f}\right)}{2} + S(r, f)$$

$$\leq \frac{5}{6}T(r, f) + S(r, f),$$

which is a contradiction. \(\square\)

**Lemma 2.2.** Let $f$ be a non-constant rational function and let $p \not\equiv 0$ be a polynomial of degree at most $m$, where $m$ is a fixed positive integer. Suppose that every zero of $f$ has multiplicity at least $2(m + 1)$ (or $3(m + 1)$) and every pole of $f$ has multiplicity at least $3(m + 1)$ (resp. $2(m + 1)$). Then $f - p$ has at least two distinct zeros.

**Proof.** For the sake of convenience, we fix the degree of polynomial $p$ as $m$ ($\deg p = m$). For $\deg p < m$, this proof works verbatim. Now we discuss the following cases.

**Case 1.** Suppose $f$ is a non-constant polynomial, then we write

$$f(z) = A(z - \alpha_1)^{m_1} \cdots (z - \alpha_s)^{m_s},$$

where $A$ is a non-zero constant, $m_i \geq 2(m + 1)$ are integers. Then by the fundamental theorem of algebra $f - p$ has zeros. Assume $f - p$ has exactly one zero at $z_0$ and thus we
can write
\[ (2.2) \quad f(z) - p(z) = B(z - z_0)^l, \quad l \geq 2(m + 1). \]

Now differentiating (2.1) and (2.2) \( m \) times, we get
\[ (2.3) \quad f^{(m)}(z) = (z - \alpha_1)^{m_1-m} \ldots (z - \alpha_s)^{m_s-m} g(z), \]
where \( g(z) \) is a polynomial with \( \deg g(z) \leq m(s - 1) \). And
\[ (2.4) \quad f^{(m)}(z) - C = B_1(z - z_0)^{l-m}, \]
where \( C \) and \( B_1 \) are non-zero constants. It is easy to see that \( z_0 \neq \alpha_i \) for any \( i \in \{1, \ldots s\} \), otherwise \( C = 0 \). Again differentiating (2.3) and (2.4), we get
\[ (2.5) \quad f^{(m+1)}(z) = (z - \alpha_1)^{m_1-m-1} \ldots (z - \alpha_s)^{m_s-m-1} g_1(z), \]
where \( g_1(z) \) is a polynomial with \( \deg g_1(z) \leq (m + 1)(s - 1) \). And
\[ (2.6) \quad f^{(m+1)}(z) = B_2(z - z_0)^{l-m-1}, \]
where \( B_2 \) is a non-zero constant. From (2.6), we see that \( f^{(m+1)}(\alpha_i) \neq 0 \), for \( i = 1, \ldots, s \). This shows that the multiplicity of zeros of \( f \) is at most \( m \), which is a contradiction.

**Case 2.** Suppose \( f \) is a non-polynomial rational function, then we set
\[ (2.7) \quad f(z) = A \frac{(z - \alpha_1)^{m_1} \ldots (z - \alpha_s)^{m_s}}{(z - \beta_1)^{n_1} \ldots (z - \beta_t)^{n_t}}, \]
where \( A \) is a non-zero constant, \( m_i \geq 2(m+1) \) \( (i = 1, 2, \ldots, s) \) and \( n_j \geq 3(m+1) \) \( (j = 1, 2, \ldots, t) \).

Let us define
\[ (2.8) \quad \sum_{i=1}^{s} m_i = M \geq 2(m + 1)s \quad \text{and} \quad \sum_{j=1}^{t} n_j = N \geq 3(m + 1)t. \]

On differentiating (2.7) \( m + 1 \) times, we get
\[ (2.9) \quad f^{(m+1)}(z) = A_1 \frac{(z - \alpha_1)^{m_1-m-1} \ldots (z - \alpha_s)^{m_s-m-1} h(z)}{(z - \beta_1)^{n_1+m+1} \ldots (z - \beta_t)^{n_t+m+1}}, \]
where \( A_1 \) is a non-zero constant and \( h(z) \) is a polynomial with \( \deg h(z) \leq (m + 1)(s+t-1) \).

Now we discuss the following cases.

**Case 2a.** If \( f - p \) has no zeros, then we can write
\[ (2.10) \quad f(z) = p(z) + \frac{A_2}{(z - \beta_1)^{n_1} \ldots (z - \beta_t)^{n_t}}, \]
where \( A_2 \) is a non-zero constant. We notice that (2.7) and (2.10) together gives \( M \geq N \).

On differentiating (2.10) \( m + 1 \) times, we get
\[ (2.11) \quad f^{(m+1)}(z) = \frac{h_1(z)}{(z - \beta_1)^{n_1+m+1} \ldots (z - \beta_t)^{n_t+m+1}}. \]
where $h_1(z)$ is a polynomial with $\deg h_1(z) \leq (m + 1)t$.

Also, by \((2.9)\) and \((2.11)\), we have $M - (m + 1)s \leq (m + 1)t$, which gives that $M \leq (m + 1)(s + t)$ and combining this with \((2.8)\) we get

$$M \leq (m + 1)(s + t) \leq \frac{5}{6}M < M,$$

which is a contradiction.

Here we note that when the multiplicity of poles of $f$ is $\geq 2(m + 1)$ and the multiplicity of zeros of $f$ is $\geq 3(m + 1)$, the proof is exactly the same.

**Case 2b.** If $f - p$ has exactly one zero at $z_0$, then we can write

\[
(2.12) \quad f(z) = p(z) + \frac{C_1(z - z_0)^l}{(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_t}},
\]

where $C_1$ is a non-zero constant and $l$ is a positive integer. On differentiating \((2.12)\) $m + 1$ times, we get

\[
(2.13) \quad f^{(m+1)}(z) = \frac{(z - z_0)^{l-m-1}h_2(z)}{(z - \beta_1)^{n_1+m+1} \cdots (z - \beta_t)^{n_t+m+1}},
\]

where $h_2(z)$ is a polynomial with $\deg h_2(z) \leq (m + 1)t$. Note that \((2.13)\) also holds in the case $l \leq m$. In this case

$$f^{(m+1)}(z) = \frac{h_2(z)}{(z - \beta_1)^{n_1+m+1} \cdots (z - \beta_t)^{n_t+m+1}}.$$

Now, we claim that $z_0 \neq \alpha_i$ for any $i \in \{1, \ldots, s\}$. Suppose that $z_0 = \alpha_i$ for some $i \in \{1, \ldots, s\}$. If $l \geq m + 1$, this would mean that $z_0$ is a zero of order at least $m + 1$ of $p$. And if $l \leq m$, then from \((2.12)\), $z_0$ is a zero of $p$ with multiplicity at least $l$. Now from \((2.12)\) we have

\[
(2.14) \quad f_1(z) = p_1(z) + \frac{C_2}{(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_t}},
\]

where $C_2$ is a constant, $f_1 = f/(z - z_0)^l$ and $p_1 = p/(z - z_0)^l$. Now proceed as in the Case 2a and get a contradiction. Hence, we have $z_0 \neq \alpha_i$ for any $i \in \{1, \ldots, s\}$.

Now, we discuss the following two subcases.

**Subcase 2b.1.** If $M \geq N$. Since $z_0 \neq \alpha_i$ for any $i \in \{1, \ldots, s\}$, therefore by \((2.9)\) and \((2.13)\), we have $M - (m + 1)s \leq (m + 1)t$. Which further implies that $M \leq (m + 1)(s + t)$ and combining this with \((2.8)\) we get

$$M \leq (m + 1)(s + t) \leq \frac{5}{6}M < M,$$

which is a contradiction.
Suppose \( 2.2 \). If \( M < N \), then by (2.9) and (2.13), we deduce that \( l - m - 1 \leq (m + 1)(s + t - 1) \), which gives that

\[
l \leq (m + 1)(s + t) \leq \frac{5}{6}N < N.
\]

But by (2.7) and (2.12), we have \( M \leq \max \{N + \deg p, l\} \), with equality if \( N + \deg p \neq l \). So \( l < N \) leads to the contradiction \( M > N \). \( \square \)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Without loss of generality, we may assume that \( D = \{z \in \mathbb{C} : |z| < 1\} \). Suppose on the contrary that \( \mathcal{F} \) is not normal at \( z_0 = 0 \). Now we have two cases to consider.

**Case 1.** Suppose \( \psi(0) \neq 0 \). Then by Zalcman’s Lemma there exist

1. a sequence of complex numbers \( z_j \to z_0 = 0 \), \( |z_j| < r < 1 \),
2. a sequence of functions \( f_j \in \mathcal{F} \),
3. a sequence of positive numbers \( \rho_j \to 0 \),

such that \( g_j(\xi) = f_j(z_j + \rho_j \xi) \) converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function \( g(\xi) \). It is evident from Hurwitz’s theorem that \( g \) satisfies the following properties:

- (a) every zero of \( g \) has multiplicity at least \( 2(m + 1) \) (or \( 3(m + 1) \)),
- (b) every pole of \( g \) has multiplicity at least \( 3(m + 1) \) (resp. \( 2(m + 1) \)).

Also on every compact subsets of \( \mathbb{C} \), not containing poles of \( g \), we get that

\[
f_j(z_j + \rho_j \xi) - \psi(z_j + \rho_j \xi) = g_j(\xi) - \psi(z_j + \rho_j \xi) \to g(\xi) - \psi(0).
\]

Clearly, \( g(\xi) - \psi(0) \neq 0 \). Therefore by Lemma 2.1 and Lemma 2.2, we know that \( g(\xi) - \psi(0) \) has at least two distinct zeros. Let \( w_1 \) and \( w_2 \) be two distinct zeros of \( g(\xi) - \psi(0) \). We can find two disjoint neighborhoods \( N_{\delta_1}(w_1) = \{z : |z - w_1| < \delta_1\} \) and \( N_{\delta_2}(w_2) = \{z : |z - w_2| < \delta_2\} \) such that \( N_{\delta_1}(w_1) \cup N_{\delta_2}(w_2) \) contains no zero of \( g(\xi) - \psi(0) \) other than \( w_1 \) and \( w_2 \). By Hurwitz’s theorem, there exist two sequences \( \{w_{1j}\} \subset N_{\delta_1}(w_1), \{w_{2j}\} \subset N_{\delta_2}(w_2) \) converging to \( w_1, w_2 \) respectively and for sufficiently large \( j \), we have

\[
f_j(z_j + \rho_j w_{1j}) - \psi(z_j + \rho_j w_{1j}) = 0,
f_j(z_j + \rho_j w_{2j}) - \psi(z_j + \rho_j w_{2j}) = 0.
\]

Since each pair \( f_a, f_b \) of \( \mathcal{F} \) shares \( \psi \) in \( D \), therefore for any positive integer \( m \) we have

\[
f_m(z_j + \rho_j w_{1j}) - \psi(z_j + \rho_j w_{1j}) = 0,
f_m(z_j + \rho_j w_{2j}) - \psi(z_j + \rho_j w_{2j}) = 0.
\]

Fixing \( m \) and taking \( j \to \infty \), we see that \( z_j + \rho_j w_{1j} \to 0, z_j + \rho_j w_{2j} \to 0 \) and \( f_m(0) - \psi(0) = 0 \). Since the zero set is discrete, for large values of \( j \) we have

\[
z_j + \rho_j w_{1j} = 0 = z_j + \rho_j w_{2j},
\]

hence

\[
w_{1j} = -\frac{z_j}{\rho_j} = w_{2j}.
\]
This contradicts the fact that \( N_{\delta_1}(w_1) \cap N_{\delta_2}(w_2) = \emptyset \).

**Case 2.** Let \( \psi(0) = 0 \). We can write \( \psi(z) = z^t \phi(z) \), where \( t(\leq m) \) is a positive integer and \( \phi(z) \) is a holomorphic function in \( D \) such that \( \phi(0) \neq 0 \). Now we consider the following subcases:

**Subcase 2.1.** If \( f(0) \neq \psi(0) \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( f(z) \neq 0 \) and \( f(z) \neq \psi(z) \), for all \( z \in N_r(0) \) and for all \( f \in \mathcal{F} \). Then, normality is confirmed by Theorem D'.

**Subcase 2.2.** If \( f(0) = \psi(0) \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( f(z) \neq \psi(z) \), for all \( z \in N_r'(0) = \{ z : 0 < |z| < r \} \) and for all \( f \in \mathcal{F} \). Now consider the family \( \mathcal{G} = \{ f(z)/z^t : f \in \mathcal{F} \} \) and the function \( \phi(z) \). On \( N_r(0) \), \( \mathcal{G} \) satisfies Subcase 2.1, therefore \( \mathcal{G} \) is normal in \( N_r(0) \). Now, we show that \( \mathcal{F} \) is normal in \( N_r(0) \). Clearly, \( \mathcal{F} \) is normal in \( N_r'(0) \) and \( g(0) = 0 \), for all \( g \in \mathcal{G} \). So there exists \( 0 < \delta < r \) such that if \( g \in \mathcal{G} \), \( |g(z)| \leq 1 \), for \( z \in N_\delta(0) \). On the boundary of \( N_\delta(0) \), \( |f(z)| = |z|^t |g(z)| \leq \delta^t \). Thus, by the maximum principle \( |f(z)| \leq \delta^t \) on \( N_\delta(0) \), for all \( f \in \mathcal{F} \). So \( \mathcal{F} \) is normal on \( N_\delta(0) \) and hence on \( N_r(0) \).

Now we give the proof of Theorem 1.6.

**Proof of Theorem 1.6.** Since normality is a local property, it is sufficient to show that \( \mathcal{F} \) is normal at each point of \( D \). Now we consider the following cases to check the normality at an arbitrarily chosen point \( z_0 \in D \).

**Case 1.** If \( f(z_0) \neq \psi(z_0) \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( f(z) \neq \psi(z) \), for all \( z \in N_r(z_0) \) and for all \( f \in \mathcal{F} \). Then \( \mathcal{F} \) is normal at \( z_0 \), by Theorem D'.

**Case 2a.** If \( f(z_0) = \psi(z_0) \neq 0, \infty \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( f(z) \neq \psi(z) \) and \( f(z)/\psi(z) \neq 0, \infty \), in \( N_r'(z_0) \). Now consider the family \( \mathcal{F}_1 = \{ f(z)/\psi(z) : f \in \mathcal{F} \} \). By Corollary 1.2, \( \mathcal{F}_1 \) is normal at \( z_0 \). Since each \( f/\psi \in \mathcal{F}_1 \) is holomorphic and \( \mathcal{F}_1 \) is normal at \( z_0 \), we get \( \mathcal{F} \) is normal at \( z_0 \).

**Case 2b.** If \( f(z_0) = \psi(z_0) = 0 \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( \psi(z) \neq 0, \infty \) and \( f(z) \neq \psi(z) \) in \( N_r'(z_0) \). Let \( m \) be the multiplicity of the zero of \( \psi \) at \( z = z_0 \). Consider the family \( \mathcal{G} = \{ f/(z-z_0)^m : f \in \mathcal{F} \} \) and the function \( \psi(z)/(z-z_0)^m \). On \( N_r(z_0) \), \( \mathcal{G} \) satisfies case 1. Therefore \( \mathcal{G} \) is normal in \( N_r(z_0) \). Now, we show that \( \mathcal{F} \) is normal in \( N_r(z_0) \). Clearly, \( \mathcal{F} \) is normal in \( N_r'(z_0) \). By condition (3) of the theorem, \( g(z_0) = 0 \), for all \( g \in \mathcal{G} \). So there exists \( 0 < \delta < r \) such that if \( g \in \mathcal{G} \), \( |g(z)| \leq 1 \), for \( z \in N_\delta(z_0) \). On the boundary of \( N_\delta(z_0) \), \( |f(z)| = |z-z_0|^m |g(z)| \leq \delta^m \). Thus, by the maximum principle \( |f(z)| \leq \delta^m \) on \( N_\delta(z_0) \), for all \( f \in \mathcal{F} \). So \( \mathcal{F} \) is normal on \( N_\delta(z_0) \) and hence on \( N_r(z_0) \).

**Case 2c.** If \( f(z_0) = \psi(z_0) = \infty \), for some \( f \in \mathcal{F} \). Then, there exists \( r > 0 \) such that \( f(z) \neq \psi(z) \) in \( N_r'(z_0) \). Let \( k \) be the multiplicity of the pole of \( \psi(z) \) at \( z = z_0 \). Consider the family \( \mathcal{H} = \{ (z-z_0)^k f : f \in \mathcal{F} \} \) and the function \( (z-z_0)^k \psi(z) \). Then \( \mathcal{H} \) satisfies case 1, so \( \mathcal{H} \) is normal in \( N_r(z_0) \). Now, we prove that \( \mathcal{F} \) is normal at \( N_r(z_0) \). Clearly, \( \mathcal{F} \) is normal
in $N'_r(z_0)$. Also, by condition (3) of the theorem, $h(z_0) = \infty$, for all $h \in \mathcal{H}$. So there exists $0 < \delta(< r)$ such that $|h(z)| \geq 1$, for all $h \in \mathcal{H}$ and $z \in N_\delta(z_0)$. It follows that, $f(z) \neq 0$ in $N_\delta(z_0)$, for all $f \in \mathcal{F}$. Since $\mathcal{F}$ is normal in $N'_r(z_0)$, then the family $1/\mathcal{F} = \{1/f : f \in \mathcal{F}\}$ is holomorphic in $N_\delta(z_0)$ and normal in $N'_\delta(z_0)$, but it is not normal at $z = z_0$. Thus, there exists a sequence $\{1/f_n\} \subset 1/\mathcal{F}$ which converges locally uniformly in $N'_\delta(z_0)$, but no subsequence of $\{1/f_n\}$ converges uniformly in a neighborhood of $z_0$. The maximum modulus principle implies that $1/f_n \to \infty$ on compact subsets in $N'_\delta(z_0)$. Hence, $f_n \to 0$ uniformly on compact subsets of $N'_\delta(z_0)$ and this shows that $h_n \to 0$ uniformly on compact subsets of $N'_\delta(z_0)$. Which is a contradiction to the fact that $|h_n(z)| \geq 1$ in $N_\delta(z_0)$. □

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Department of Mathematics, University of Delhi, Delhi–110 007, India
E-mail address: ggopal.datt@gmail.com, gdatt1@maths.du.ac.in

Department of Mathematics, Deen Dayal Upadhyaya College, University of Delhi, Delhi–110 015, India
E-mail address: sanjpant@gmail.com