Circuit Theory in Projective Space and Homogeneous Circuit Models

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Abstract

This paper presents a general framework for linear circuit analysis based on elementary aspects of projective geometry. We use a flexible approach in which no a priori assignment of an electrical nature to the circuit branches is necessary. Such an assignment is eventually done just by setting certain model parameters, in a way which avoids the need for a distinction between voltage and current sources and, additionally, makes it possible to get rid of voltage- or current-control assumptions on the impedances. This paves the way for a completely general \( m \)-dimensional reduction of any circuit defined by \( m \) two-terminal, uncoupled linear elements, contrary to most classical methods which at one step or another impose certain restrictions on the allowed devices. The reduction has the form

\[
\begin{pmatrix} AP \\ BQ \end{pmatrix} u = \begin{pmatrix} AQ \\ -BP \end{pmatrix} \bar{s}.
\]

Here, \( A \) and \( B \) capture the graph topology, whereas \( P, Q, s \) comprise homogeneous descriptions of all the circuit elements; the unknown \( u \) is an \( m \)-dimensional vector of (say) “seed” variables from which currents and voltages are obtained as \( i = Pu - Qs \), \( v = Qu + Ps \). Computational implementations are straightforward. These models allow for a general characterization of non-degenerate configurations in terms of the multihomogeneous Kirchhoff polynomial, and in this direction we present some results of independent interest involving the matrix-tree theorem. Our approach can be easily combined with classical methods by using homogeneous descriptions only for certain branches, yielding partially homogeneous models. We also indicate how to accommodate controlled sources and coupled devices in the homogeneous framework. Several examples illustrate the results.
1 Introduction

In order to motivate the ideas discussed in this paper, let us consider a seemingly elementary analytical problem. Assume we are given a loop composed of three impedances (see Fig. 2(a); we will return to this example in Section 3) and that we are interested in characterizing the parameter (impedance) values for which the circuit is non-degenerate, i.e. has a unique solution. We allow the real part of the impedances to become zero or negative since otherwise the problem is trivial.

A tentative answer is that the non-degeneracies are defined by the condition \( Z_1 + Z_2 + Z_3 \neq 0 \), because otherwise any non-trivial loop current yields a solution. But it may come as a little surprise that the vanishing of \( Z_1 + Z_2 + Z_3 \) does not describe all possible degeneracies. Indeed, we are presuming here that all three devices admit a description in terms of the impedance parameters \( Z_k \) (\( k = 1, 2, 3 \)), ruling out open-circuits. If we use instead admittance descriptions, via e.g. nodal analysis one can check that the degeneracy condition is defined by the vanishing of the polynomial \( y_{11}y_{21} + y_{11}y_{31} + y_{21}y_{31} \); but short-circuits are now excluded. Combining admittance and impedance descriptions for different devices we would get hybrid polynomials characterizing degenerate configurations in other contexts, but never including all possible scenarios.

One of our goals is to have a single (and simple) description of all possible degenerate parameter sets. This can be achieved by resorting to a homogeneous description of the devices: instead of using the impedance \( Z_k \) or the admittance \( y_k \), we describe each device in terms of a pair of parameters \( (p_k : q_k) \) not vanishing simultaneously. If \( q_k \neq 0 \) then the admittance \( y_k \) is well-defined as the quotient \( p_k/q_k \); conversely, when \( p_k \neq 0 \) the impedance \( z_k \) is \( q_k/p_k \). The pair \( (p_k : q_k) \) now captures all possible cases; this is of interest e.g. when one wishes to include both short- and open-circuits in the same model, for instance in the computation of Thévenin equivalents or in the presence of ideal switches. In the homogeneous setting both \( z_1 + z_2 + z_3 \) and \( y_{11}y_{21} + y_{11}y_{31} + y_{21}y_{31} \) (among other polynomials) will arise as so-called dehomogenizations of a single universal polynomial (cf. (26)), namely \( p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3 \) in our present case, whose zeros describe all degeneracies.

The homogeneous approach provides a key advantage in circuit analysis when we drive the idea further. Specifically, we will describe the characteristic of each device in terms of just one variable \( u_k \) (to be called a homogeneous variable) by means of the parametric form of Ohm’s law defined by the relations \( i_k = p_ku_k, \ v_k = q_ku_k \). Combining this with Kirchhoff laws we avoid the need to assume specific impedance- or admittance-descriptions for the devices, and arrive at a general family of reduced circuit models comprising all possible parameter settings. Sources are easily accommodated, as shown later. The approach can be also naturally combined with classical analysis methods by using homogeneous descriptions only for certain branches.

Using homogeneous descriptions is not new in circuit theory; we can cite, at least, the works [2, 3, 5, 13, 17]. However, from the point of view of the author this formalism has not been systematically exploited or even fully developed. In this direction, the main goal of this paper is to present a comprehensive framework (based on elementary aspects of projective
geometry) accommodating homogeneous descriptions of linear circuits and then, allowed by the broad generality of this approach, to formulate new analysis methods and address certain analytical aspects of circuit theory.

Specifically, we will perform a construction in which any linear circuit, typically in a resistive DC context or in AC sinusoidal steady state (but also in the Laplace domain), can be modelled as a directed graph whose branches are endowed with weights from a certain subset of a projective plane, in a way such that branches need not be classified as sources or impedances/resistors. This will make it possible to get rid of voltage/current control assumptions in the formulation of reduced models. This approach also accounts for the Thévenin/Norton duality when modelling non-ideal sources and, at a higher abstraction level, in the formulation of equivalent circuits. In this setting, from any given digraph a particular linear circuit is defined when choosing a specific configuration, that is, a set of homogeneous parameters for all branches. This framework is presented in Section 2.

The homogeneous formalism allows one to provide a precise characterization of the conditions under which certain families of reduced circuit models are well-defined, in terms of configurations. Actually, we can frame most analysis methods as families of reductions of the circuit equations to certain subspaces of the so-called Kirchhoff and characteristic spaces. By using a suitable parametrization of the characteristic space we introduce a family of homogeneous branch models which (contrary to branch-voltage, branch-current and hybrid methods, but also to nodal and loop analysis models, which at one step or another require certain voltage/current control assumptions) are completely general in the sense that they are well-defined for all possible configurations. This material can be found in Section 3, which is closed with some examples.

In Section 4 we benefit from the homogeneous formalism in the characterization, for any given digraph, of the set of non-degenerate configurations, which are obtained as the complement of the zero set of a universal form of the Kirchhoff or tree-enumerator polynomial. Different dehomogenizations of this universal polynomial yield specific polynomials arising in different analysis methods. Closely related to this are several results of independent interest involving the well-known matrix-tree theorem which are discussed in that section.

We sketch the way in which this approach can be extended to include controlled sources and coupled problems in Section 5. Some potential applications of this formalism in fault isolation problems are discussed in Section 6. Finally, concluding remarks can be found in Section 7.

2 Linear circuits as projectively weighted digraphs

2.1 Background: projective lines and planes

There are many excellent books on projective geometry which may provide a reader with an introduction to this topic, if necessary. We refer him/her in particular to [7, 16, 18]. Note that we will only use elementary aspects of this discipline and all the necessary background is compiled here.
In the sequel we assume that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$: the real case will be used to model resistive circuits with DC sources, whereas the complex case accommodates, via phasors, circuits with AC sources in sinusoidal steady state and, in greater generality, can be used to analyze circuits in the Laplace domain. Let $V$ be a finite-dimensional vector space over $\mathbb{K}$: we will be interested in two- and three-dimensional cases. We define an equivalence relation in the set of non-null vectors $V - \{0\}$ by letting

$$u \sim v \text{ if } u = \mu v \text{ for some } \mu \in \mathbb{K} - \{0\}. \quad (1)$$

The set of equivalence classes (to be denoted as $\mathbb{P}(V)$) is a projective space and each class is a projective point. When $V$ has dimension two or three $\mathbb{P}(V)$ is a projective line or a projective plane, respectively. Note that the dimension of $V$ exceeds by one the dimension of the projective space $\mathbb{P}(V)$. In particular we denote $\mathbb{P}(\mathbb{K}^2)$ by $\mathbb{KP}$ and $\mathbb{P}(\mathbb{K}^3)$ by $\mathbb{KP}^2$.

Let us fix for simplicity $V = \mathbb{K}^2$ in order to focus on the projective line, either with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (be aware of the fact that a complex line has complex dimension one). A point in $\mathbb{KP}$ is an equivalence class defined by the set of non-null vectors in $\mathbb{K}^2$ within any straight line through the origin. If we take any representative $(p, q)$ of this class (with either $p$ or $q$, or both, non-null), we may describe the whole equivalence class, that is, the corresponding point of $\mathbb{KP}$, by writing $(p : q)$, where the notation is aimed to distinguish the representative (namely, $(p, q) \in \mathbb{K}^2 - \{0\}$) from the whole class. In other words, $(p_1 : q_1)$ and $(p_2 : q_2)$ denote the same projective point if (and only if) there is a non-zero constant $\mu$ such that $p_1 = \mu p_2$ and $q_1 = \mu q_2$. Making implicit use of the canonical basis of $\mathbb{K}^2$, we say that $(p : q)$ are homogeneous coordinates for the projective point (we use the same notation $(p : q)$ for homogeneous coordinates arising from the canonical basis and for the projective point as an equivalence class, to avoid the cumbersome $[(p, q)]$ for the latter). Note that homogeneous coordinates are defined only up to a non-vanishing constant; that is, a point in the projective line is not described by a unique coordinate pair but by a whole set of such pairs. As detailed later, homogeneous coordinates in the projective plane $\mathbb{KP}^2$ are defined in an entirely analogous manner; here we choose triads $(p : q : s)$ with at least one non-vanishing entry.

Either in $\mathbb{KP}$ or in $\mathbb{KP}^2$ (in any projective space, actually), the non-vanishing of a homogeneous coordinate will define an affine patch, namely, a space which can be identified either with $\mathbb{K}$ or with $\mathbb{K}^2$, and which only misses from $\mathbb{KP}$ or from $\mathbb{KP}^2$ the point or the line (respectively) at infinity. We will use this in the circuit context to define the current-controlled (impedance) and the voltage-controlled (admittance) patches, characterized by the conditions $p \neq 0$ and $q \neq 0$.

Later on we will use the above construction with other vector spaces $V$, specifically the space $L(\mathbb{K}^2, \mathbb{K})$ of linear forms from $\mathbb{K}^2$ to $\mathbb{K}$ and the space of polynomials of degree not greater than one in two indeterminates and with coefficients in $\mathbb{K}$. Note that when a projective space is constructed from an arbitrary vector space $V$, homogeneous coordinates are defined after fixing a basis in $V$ (or, more generally, a projective frame in the projective space $\mathbb{P}(V)$, see again [7]).
2.2 Impedance in the projective line

To get a lighter exposition, from now on our terminology focuses on the complex case; therefore, in most cases we only speak of impedance, admittance, etc. and not of resistance, conductance and so on. In any case, the notation $\mathbb{K}$ recalls that all results apply in both the real and the complex setting.

Elementary circuit theory says that the characteristic of a linear circuit element is governed by Ohm’s law,

$$v = zi,$$

where $z$ represents impedance. This current-controlled description formally excludes an open-circuit. By contrast, the latter (but not a short-circuit) is accommodated in the voltage-controlled or admittance description

$$i = yv.$$  

We may easily accommodate all cases above by means of an homogenization of either (2) or (3): namely, setting $z = q/p$ in (2) and multiplying by $p$, or recasting $y = p/q$ in (3) and then multiplying by $q$, we arrive at the so-called homogeneous form of Ohm’s law,

$$pv - qi = 0,$$

where the parameters $p$ and $q$ cannot vanish simultaneously. This approach essentially makes a projective completion [7] of the complex line where either the impedance or the admittance lies; focusing e.g. on (2), we may rephrase the above approach by recasting the impedance $z$ in homogeneous coordinates as $(1 : z)$ and then allowing the first coordinate to vanish in order to accommodate the infinite impedance case.

But we may do the same from scratch in a way which will admit a natural extension to problems with sources and also to the nonlinear context. Note that what really matters in any form of Ohm’s law is the set of values of $i$ and $v$ which satisfy the device characteristic. All (2), (3) and (4) can be understood to define the kernel of a certain regular (i.e. not identically zero) linear form $f : \mathbb{K}^2 \rightarrow \mathbb{K}$ capturing this characteristic, where the domain $\mathbb{K}^2$ describes the $(i, v)$-space. One may choose many different ways to write this linear form, (2) and (3) being particular instances which are valid under certain (broad but not fully general) hypotheses. By contrast, (4) includes all cases. Moreover, it is obvious that multiplying (4) by any non-null constant $\mu$ we get the same zero set and hence another admissible description of the same relation; equivalently, the kernel of $f$ and of $\mu f$ are the same. This way the projective formalism arises naturally.

Let then $V$ stand for the vector space of linear forms $L(\mathbb{K}^2, \mathbb{K})$, restrict the attention to regular forms ($f \neq 0$) and consider the projective line $\mathbb{P}(V)$ defined by the equivalence classes $[f]$ under the relation (1). This means that, mathematically, we look at an impedance (or at a linear resistor) as a point in this complex (resp. real) projective line, that is, as an equivalence class of regular linear forms. If we let $f$ be a representative in $V$ of the equivalence class of a given impedance, and by defining the basis $f_1(i, v) = v$ and $f_2(i, v) = -i$ of $V$, the impedance is defined by the homogeneous coordinates $(p : q)$ which make $f(i, v) = pf_1(i, v) + qf_2(i, v) = pv - qi = 0$. 
$pv - qi$, as in (4). We will say that $(p : q)$ is a **homogeneous description** of the impedance (or the resistance in the real case).

Equivalently, you can identify the vector spaces $L(\mathbb{K}^2, \mathbb{K})$ and $\mathbb{K}^2$ via the isomorphism $\alpha v + \beta i \rightarrow (\alpha, -\beta)$, and then look at $(p : q)$ as the homogeneous coordinates defined by the canonical basis in $\mathbb{K}^2$. From now on we assume implicitly the use of this isomorphism to denote by $\mathbb{K}P$ the projective line where (homogeneous) impedances lie.

In this framework, the set of projective points defined by homogeneous coordinates $(p : q)$ with $p \neq 0$ will be called the **current-controlled** or **impedance** patch and denoted by $A_z$. Given any point in this patch, the impedance parameter (or the resistance, in the real setting) is uniquely defined from any pair of homogeneous coordinates $(p : q)$ as $z = q/p$. The remaining point in $\mathbb{K}P - A_z$, with homogeneous coordinates $(0 : q)$, can be seen as the point of infinite impedance. Analogously, the **voltage-controlled** or **admittance** patch $A_y$ is defined by the condition $q \neq 0$; in this case the admittance (or the conductance in the real context) is again well defined as $y = p/q$. Note finally that the point of infinite impedance corresponds to the one of zero admittance, and vice-versa.

### 2.3 Sources and the projective plane

The ideas in the previous subsection can be extended to accommodate also active elements (sources) in a natural way. To this end, consider a voltage source $v_s$ in series with an impedance $z$, as in Fig. 1(a).

![Non-ideal voltage and current sources.](image)

With the passive sign convention implicit in that figure (we use this convention to treat impedances and sources in a uniform manner) we get $v = zi + v_s$, that is,

$$v - zi = v_s. \quad (5)$$

With $z = 0$ this amounts to $v = v_s$ and we get an ideal voltage source.

The dual case (Fig. 1(b)) is defined by a current source with an impedance in parallel, typically represented by its admittance parameter $y$. Here the relation reads as

$$yv - i = i_s, \quad (6)$$

with the value $y = 0$ modelling an ideal current source. Note that in both cases we want to encapsulate both the source and the impedance (if present) in a single circuit element.
Both (5) and (6) are particular instances of a general equation of the form

\[ pv - qi = s \]  

(7)

with \( p, q, s \in \mathbb{K} \) and the restriction that \( p, q \) do not vanish simultaneously. In particular, provided that \( p \) does not vanish one gets (5) from (7) by setting \( z = qi/p, \ v_s = s/p \) (analogously, if \( q \neq 0 \) we get (6) with \( y = p/q, \ i_s = s/q \)). If neither \( p \) nor \( q \) do vanish, then both descriptions (5) and (6) are possible and the well-known Thévenin-Norton identities \( v_s = z i_s, \ z = 1/y \) apply. By contrast, the cases \( q = z = 0 \) and \( p = y = 0 \) describe ideal sources and only (5) or (6), respectively, do hold, whereas (7) accommodates both. It is worth mentioning that the formalism above makes also sense (in the real case) for time-dependent sources \( s(t) \).

A key aspect in our approach is that (7) is again defined up to a non-vanishing multiplicative factor, because the zero set of \( pv - qi - s \) does not change if we multiply this polynomial (in \( i, v \)) by a non-zero constant. This makes it natural to set now \( V \) as the (three-dimensional) vector space of polynomials in two indeterminates with coefficients in \( \mathbb{K} \) and degree not greater than one, and assume that the characteristic lies on the projective space \( \mathbb{P}(V) \) resulting from the equivalence relation (1). Additionally, if we denote by \([k]\) the equivalence class defined by all non-zero constants in \( \mathbb{K} \), the requirement that \( p \) and \( q \) do not vanish simultaneously means that the equivalence classes we are interested in belong to the punctured space \( \mathbb{P}(V) - \{[k]\} \). This is the set of equivalence classes corresponding to polynomials of degree (exactly) one. Moreover, it is clear that in the basis defined by the polynomials \( f_1(i, v) = v, \ f_2(i, v) = -i, \ f_3(i, v) = -1, \) the homogeneous coordinates are \((p : q : s)\), with \((p, q) \neq (0, 0)\) by the aforementioned requirement.

Finally, we identify \( \mathbb{P}(V) \) with the projective plane \( \mathbb{K} \mathbb{P}^2 \) by means of the isomorphism \( V \to \mathbb{K}^3 \) defined by \( \alpha v + \beta i + \gamma \to (\alpha, -\beta, -\gamma) \) (and note that, again, the homogeneous coordinates above correspond to the ones obtained in \( \mathbb{K} \mathbb{P}^2 \) from the choice of the canonical basis in \( \mathbb{K}^3 \)). With this isomorphism in mind, we will think of sources as points lying on the projective plane \( \mathbb{K} \mathbb{P}^2 \). More precisely, because of the non-vanishing requirement on (at least) either \( p \) or \( q \), our description of abstract sources is finally bound to lie on the punctured projective plane \( \mathbb{K} \mathbb{P}^2_p = \mathbb{K}^2 \mathbb{P}^2 - \{(0 : 0 : 1)\} \).

2.4 Abstract linear elements

It is not by chance that the discussion in subsection 2.3 generalizes the one in subsection 2.2. Indeed, the relation defined by (7) comprises all possible cases and sets up an abstract linear element whose electrical nature is not determined a priori; it is fixed only after choosing values for the homogeneous parameters \( p, q \) and \( s \). Two natural taxonomies arise.

The first taxonomy depends on the parameter \( s \); the device amounts to a source when an assignment \( s \neq 0 \) is made, or to a linear impedance when \( s = 0 \). In the first case \((s \neq 0)\), it is not necessary to recast the source as a voltage source or as a current source: actually, it is important to understand that (7) may define an abstract source of neither type, even if locally (that is, for specific values of \( p, q \)) it can always be reduced either to a voltage-source...
form (by setting \( p = 1 \)) or to a current-source form (\( q = 1 \)); if both parameters are non-zero then both forms are admissible. But there is no need a priori to do this reduction. Still in the \( s \neq 0 \) patch we may further classify a source as an ideal (resp. a non-ideal) source if either \( p \) or \( q \) vanishes (resp. none of them vanishes). When \( s = 0 \) the circuit element behaves as an impedance and, again, there is no need to specify if it is current- or voltage-controlled: both options exist except in the, say, extremal cases \( p = 0 \) (open-circuit) and \( q = 0 \) (short-circuit).

The second taxonomy classifies linear elements according to the values of \( p, q \), disregarding \( s \). The patch \( A_z \), defined by the condition \( p \neq 0 \), accommodates the set of current-controlled elements, including in particular short-circuits and also voltage sources (more precisely, abstract sources that admit a description as voltage sources, either non-ideal or ideal). Assignments with \( q \neq 0 \) (defining the patch \( A_y \)) correspond to voltage-controlled elements, including open-circuits and sources that admit a description as current sources.

### 2.5 Linear circuits as digraphs with projective weights. Configurations

In order to lift the projective formalism of previous subsections to the circuit level we only need to combine this approach with the classical description of circuits in terms of directed graphs. The reader is referred to [1, 4, 5, 15] for background in this regard.

A linear circuit is simply modelled in our framework as a directed graph \( G \) endowed with a map \( \gamma : E(G) \to \mathbb{KP}^2 \), that is, a map assigning to each branch \( e \in E(G) \) a point in the punctured projective plane \( \mathbb{KP}^2_* \) defined above. In terms of homogeneous coordinates, we assign to every branch \( e \in E(G) \) a triad of (homogeneous) parameters \( (p : q : s) = \gamma(e) \), with \( (p, q) \neq (0, 0) \). We call \( \gamma \) a configuration map.

A source-free configuration is a map \( \hat{\gamma} : E(G) \to \mathbb{KP} \) assigning to every branch \( e \) a projective point with homogeneous coordinates \( (p : q) = \hat{\gamma}(e) \). Via the projection map

\[
\pi : \mathbb{KP}^2_* \to \mathbb{KP} \\
(p : q : s) \to (p : q),
\]

we associate to any configuration \( \gamma \) a source-free configuration via the relation \( \hat{\gamma} = \pi \circ \gamma \). In practice, this amounts to setting \( s = 0 \) for all branches or, in classical terms, to replacing voltage (resp. current) sources by short (resp. open) circuits. Note that (8) is well-defined because \( (0 : 0 : s) \notin \mathbb{KP}^2_* \).

We may further assume that the set of branches \( E(G) \) just amounts to \( \{1, \ldots, m\} \). This way, the whole set of parameter values is described as a point in the space \( \mathbb{KP}^2_* \times \binom{m}{2} \times \mathbb{KP}^2 \). We will call this Cartesian product the configuration space or the parameter space and, for simplicity in the notation, from now on we skip the symbol \( m \) in such products. In the absence of sources, the parameter set takes values in the source-free configuration space \( \mathbb{KP} \times \ldots \times \mathbb{KP} \). This is often called a multiprojective space and, as a cautionary remark, the reader should be aware of the fact that it is not equivalent (if \( m > 1 \)) to \( \mathbb{KP}^m \) (e.g. \( \mathbb{KP} \times \mathbb{KP} \) is not the projective plane \( \mathbb{KP}^2 \)).
3 Homogeneous circuit models

The formalism presented in Section 2 makes it possible to introduce a comprehensive framework where different circuit model families can be properly placed and analyzed. We undertake this task in the present section, where in particular we formulate an $m$-dimensional reduction of linear circuits holding without restrictions (cf. (18)).

3.1 General homogeneous model. Characteristic space

We refer the reader to [1, 4, 5, 6, 15] for background on the topics here discussed. We will assume for simplicity that the digraphs/circuits involved in all results are connected. Denote by $n$ and $m$ the number of nodes and the number of branches in the circuit.

We assume that the reader is familiar with the full incidence matrix $A \in \mathbb{R}^{n \times m}$ of a directed graph, with entries defined as $a_{ij} = +1$ (resp. $-1$) if the $j$-th branch leaves (resp. enters) the $i$-th node, and 0 otherwise. Denote by $I = \ker A \subseteq \mathbb{R}^m$ the so-called cycle space, which is known to comprise all current vectors solving Kirchhoff’s current law. Analogously, the cut space $V = \text{im} A^T \subseteq \mathbb{R}^m$ defines the set of voltage vectors solving Kirchhoff’s voltage law. Both spaces are orthogonal to each other, according to Tellegen’s identity. In a connected digraph, we have $\dim I = m - n + 1$, $\dim V = n - 1$.

Now, let $A \in \mathbb{R}^{(n-1)\times m}$ and $B \in \mathbb{R}^{(m-n+1)\times m}$ be maximal rank matrices with entries in $\{\pm 1, 0\}$ which satisfy $\ker A = I$, $\ker B = V$, so that Kirchhoff laws can be written as the independent sets of linear equations $Ai = 0$, $Bv = 0$. We call $A$ a cut matrix and $B$ a cycle matrix. Typically, $A$ is a reduced incidence matrix or a reduced cutset matrix of the digraph and $B$ can be chosen as a reduced loop matrix.

In order to join together the different linear elements defining the circuit, we assume that $(p_k : q_k : s_k)$ are homogeneous coordinates describing the $k$-th circuit element and let $P$ and $Q$ be diagonal matrices of order $m$ with $p_k$ and $q_k$, respectively, in the $k$-th diagonal position. We write as $s$ the vector defined by the excitation terms $(s_1, \ldots, s_m)$ (set also $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$). Recall that all these parameters may take values either in $\mathbb{R}$ (for resistive circuits with DC sources) or in $\mathbb{C}$ (AC circuits in sinusoidal steady state, or circuits modelled in the Laplace domain), and that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. In the light of subsection 2.5, the set of homogeneous parameters $((p_1 : q_1 : s_1), \ldots, (p_m : q_m : s_m))$ defines a point in the multiprojective space $\mathbb{KP}^2_1 \times \ldots \times \mathbb{KP}^2_2$.

The fact that $p_k$ and $q_k$ cannot vanish simultaneously is reformulated here by requiring the $m \times 2m$ matrix $(P \quad Q)$ to have maximal rank. For later use, the fact that each homogeneous triad $(p_k : q_k : s_k)$ is defined up to a non-null constant (say $d_k \in \mathbb{K} - \{0\}$) is recast by saying that $P, Q, s$ are defined up to premultiplication by a non-singular, diagonal, order $m$ matrix $D$, with $d_k$ in the $k$-th diagonal position. Be also aware of the fact that diagonal matrices commute, so that $DP = PD, DQ = QD$ and also $PQ =QP$.

In these terms, the equations of an arbitrary (uncoupled, time-invariant) linear circuit
can be written as
\[
\begin{pmatrix}
A & 0 \\
0 & B \\
-Q & P
\end{pmatrix}
\begin{pmatrix}
\mathbf{i} \\
\mathbf{v}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\] (9)

Let us define \( \mathcal{M} \) as the \( m \)-dimensional affine subspace of \( \mathbb{K}^{2m} \) defined by the set of current-voltage pairs which satisfy the characteristic equations of all circuit branches, that is,
\[
\mathcal{M} = \{(i, v) \in \mathbb{K}^{2m} / -Qi + Pv = s\}.
\] (10)

We will call this the characteristic space. Clearly, the solutions of the circuit equations (9) are the current-voltage vectors which belong to the intersection space \((I \times V) \cap \mathcal{M}\).

The circuit equations (9) have \( 2m \) unknowns and an order reduction is advisable. Many (if not all) families of circuit analysis models may be understood as a reduction of the circuit equations (9) to one of the spaces above (that is, the “Kirchhoff space” \( I \times V \) or the characteristic space \( \mathcal{M} \)), or to a certain subspace of one of them, after a suitable parametrization is given. Traditionally, in the literature such reductions are performed under certain assumptions on the controlling variables (an example of such an assumption would be “all devices are voltage-controlled”, what in a classical framework means that voltage sources are excluded and that impedances have an admittance description, to get a reduction in terms of voltages; precise details are given later). Our goal is either to perform such reductions in total generality, something that will be achieved by an appropriate reduction to the characteristic space \( \mathcal{M} \) or, in other cases, to give a precise characterization of the configurations allowing for whatever reduction. This framework can be also used in the characterization of nodal- and loop-analysis models.

3.2 Classical reductions

Focus on the subspace of the multiprojective configuration space \( \mathbb{K}P^2_z \times \ldots \times \mathbb{K}P^2_z \) defined by the condition that all \( p_k \) coordinates are non-null or, equivalently, that the matrix \( P \) defined above is non-singular (this defines the current-controlled patch \( A_z \); we use the boldface symbol \( z \) to distinguish it from the corresponding “one-element” patch \( A_z \subseteq \mathbb{K}P^2_z \), with \( A_z = A_z \times \ldots \times A_z \)). In this patch, all sources admit a classical description as voltage sources, with excitation vector \( v_s = P^{-1}s \in \mathbb{K}^m \); branches for which \( s_k = 0 \), which correspond to impedances, contribute a null entry to the vector \( v_s \). Additionally, the impedance parameter of all branches, including sources, is well-defined and can be described by means of the impedance matrix \( Z = P^{-1}Q \): this applies in particular to non-ideal source branches (in these cases \( z_k \) is the series impedance of Fig. 1(a)) and also to ideal voltage sources (for which \( z_k = 0 \)). It is easy to check that both
\[
Z = P^{-1}Q, \quad v_s = P^{-1}s
\] (11)
are independent of the choice of homogeneous coordinates.
In essence, what we are using is a parametrization of the characteristic space $\mathbb{M}$ by the current variable $i$, namely through the relation

$$v = Z_i + v_s,$$

(12)

with $Z$ and $v_s$ given by (11). This yields the so-called *branch-current model*

$$A_i = 0$$

(13a)

$$BZ_i = -Bv_s.$$  

(13b)

The key idea is that both the parametrization (12) and the branch-current model (13) are well-defined only for parameter sets lying on the patch $A_z$. Incidentally, splitting the matrices $A, B$ by columns (according to the source/impedance nature of branches), and redefining $v_s$ accordingly, one gets the classical form of the branch-current model, cf. [4]. But in our formalism we do not need to make this splitting; the entries of $Z$ and $v_s$ implicitly perform this task.

In the dual case, under the assumption that $Q$ is non-singular, so that the admittance matrix $Y = PQ^{-1}$ is well defined and sources admit a description as current sources with $i_s = Q^{-1}s$, one gets the *branch-voltage model* (cf. again [4] in the classical setting)

$$AYv = Ai_s$$

(14a)

$$Bv = 0.$$  

(14b)

Again, this can be seen as a reduction of the circuit model to the characteristic space $\mathbb{M}$, in this case by means of the parametrization $i = Yv - i_s$. This reduction is only valid on the voltage-controlled patch $A_y \subseteq \mathbb{K}P_2 \times \ldots \times \mathbb{K}P_2$.

Finally, even if we omit details for the sake of brevity, hybrid models (see [8] in this regard), which combine current- and voltage-controlled descriptions, can be also described as a reduction of (9) under appropriate parametrizations of the characteristic space $\mathbb{M}$. But also in this case an *a priori* assignment of a control variable to all branches is still necessary, so that neither these models can be used on the whole configuration space $\mathbb{K}P_2 \times \ldots \times \mathbb{K}P_2$.

### 3.3 Homogeneous reductions

Contrary to the models above, the homogeneous formalism makes it possible to perform an $m$-dimensional reduction without any working assumption on controlling variables, and accommodating all possible cases (even degenerate ones; cf. Section 4), that is, applying on the whole configuration space $\mathbb{K}P_2 \times \ldots \times \mathbb{K}P_2$. To do so we introduce abstract variables $u \in \mathbb{K}^m$, to be termed *homogeneous variables* (cf. subsection 3.5 below), which parametrize the characteristic space $\mathbb{M}$ in the form

$$i = Pu + i_0$$

(15a)

$$v = Qu + v_0.$$  

(15b)
For these relations to actually parametrize $M$, the vector $(i_0, v_0)$ needs to solve the equation $-Qi + Pv = s$. This means that $(i_0, v_0)$ is itself a point of $M$. We will call it the origin of the parametrization (the terminology just reflects that (15) sets up an affine system of coordinates for $M$) and will elaborate on possible choices later on.

By inserting (15) into (9) we get a general homogeneous branch model of the form

$$\begin{pmatrix} AP \\ BQ \end{pmatrix} u = \begin{pmatrix} -Ai_0 \\ -Bv_0 \end{pmatrix},$$

which is defined without the need to impose any restriction on $P, Q$. Currents and voltages are simply obtained from the solutions $u$ of (16) as $i = Pu + i_0, v = Qu + v_0$ (cf. (15)).

The models of subsection 3.2 can be derived as particular instances of (16). Focus for example on the branch-current model (13), which holds under the assumption that $P$ is non-singular. The latter means that the choice $\hat{P} = I$ is admissible for the homogeneous description of circuit elements; the remaining parameters then read as $\hat{Q} = Z, \hat{s} = v_s$ (cf. (11) and (12)). Together with $i_0 = 0, v_0 = v_s$, one can easily check that (15) reads as $i = u, v = Zu + v_s$ and that (16) amounts to the branch-current model (13). Similar remarks apply to the branch-voltage model (14), which can be derived analogously from (16) under appropriate assumptions.

### 3.4 Symmetric form

The origin $(i_0, v_0)$ can be always chosen, without any further assumptions on $P, Q$, as

$$i_0 = -Q(P^2 + Q^2)^{-1}s,$$

$$v_0 = P(P^2 + Q^2)^{-1}s,$$

where we use the fact that the (diagonal) matrix $P^2 + Q^2$ is invertible because $(p_k, q_k) \neq (0, 0)$ for all circuit branches. With the choice (17), the homogeneous branch model takes the (say) symmetric form

$$\begin{pmatrix} AP \\ BQ \end{pmatrix} u = \begin{pmatrix} AQ \\ -BP \end{pmatrix} (P^2 + Q^2)^{-1}s. \tag{18}$$

We will sometimes write $s = (P^2 + Q^2)^{-1}s$ for notational simplicity. We emphasize that this reduced model is well-defined for any linear circuit, even for degenerate ones; that is, no restriction in the the circuit parameters is necessary for the model (18) to hold. This makes it useful for analytical purposes and suitable for computational implementation. Note that the inverse involved is that of a diagonal matrix and hence poses no computational difficulties. The branch currents and voltages are recovered from the solutions of (18) from the relations depicted in (15) and (17), which altogether yield

$$\begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} u + \begin{pmatrix} -Q \\ P \end{pmatrix} (P^2 + Q^2)^{-1}s. \tag{19}$$
3.5 Homogeneous variables

Equations (16) and (18) provide completely general models for an m-dimensional description of linear circuits, not requiring any assumptions on the existence of current-controlled or voltage-controlled descriptions of the circuit elements. The price, of course, is the lack of a physical meaning on the unknowns, namely, the $u$ variables, in contrast to the current $i$ and the voltage $v$; the variables $u$ can be thought of as a “seed” from which both the current and the voltage stem by means of the relations (15) or (19). More prosaically, we will say that $u$ is a vector of homogeneous variables, borrowing the term from the homogeneous nature of the parameters $P$, $Q$, and $s$. There is a terminological abuse here; see however the remarks in the following paragraph.

The variables $u$ are not uniquely defined, because they arise as the unknowns in (16) or (18) and the actual form of both systems (in other words, the coordinate system parametrizing $M$) depends on the choice of $P$, $Q$, $s$ and $(i_0, v_0)$. There is no problem with this because $u$ is always accompanied by $P$, $Q$ when recovering $i$, $v$ via (15) or (19): in practice, the simpler choice of $P$, $Q$, the better. In any case, two different choices of these parameters (say $P$, $Q$, $i_0$, $v_0$ and $\tilde{P}$, $\tilde{Q}$, $\tilde{i}_0$, $\tilde{v}_0$) yield two sets of homogeneous variables which result from a diagonal rescaling of one another. Indeed, the identities $\tilde{P} = PD$, $\tilde{Q} = QD$ must hold for some scaling matrix $D$, as indicated in subsection 3.1, and one can show that this implies that under the symmetric choice (19) the relation $u = D\tilde{u}$ holds. This means that each scalar variable $U_k$ is defined up to a non-vanishing constant, exactly as the parameters $p_k$, $q_k$, $s_k$ are: this (informally) gives the $u$ variables a homogeneous flavor. For the sake of completeness, be aware that in general (more precisely, when the choice of parameters yields a difference vector $(\tilde{i}_0 - i_0, \tilde{v}_0 - v_0)$ not belonging to ker $(P \; Q)$) the equation relating $u$ and $\tilde{u}$ can be proved to have the form

$$u = D\tilde{u} + (P^2 + Q^2)^{-1} (P \; Q) \begin{pmatrix} \tilde{i}_0 - i_0 \\ \tilde{v}_0 - v_0 \end{pmatrix}. \quad (20)$$

3.6 Elementary examples. Partially homogeneous models

We illustrate the ideas above by means of a very simple example, depicted in Fig. 2(a).

![Figure 2](image_url)

Figure 2: (a) A 3-loop. (b) Assume the first branch to be a current source.

We show here how to set up the models by hand (whereas the examples in Section 6...
make systematic use of the digraph matrices A, B. Direct all branches counterclockwise, and write Kirchhoff laws as \( i_1 = i_2 = i_3 \) and \( v_1 + v_2 + v_3 = 0 \). We assume that branches 2 and 3 do not accommodate sources (i.e. \( s_2 = s_3 = 0 \)) but that branch 1 does. The relations (19) read as

\[
\begin{align*}
    i_1 &= p_1u_1 - \frac{q_1}{p_1^2 + q_1^2}s_1, \\
v_1 &= q_1u_1 + \frac{p_1}{p_1^2 + q_1^2}s_1,
\end{align*}
\]

for the first branch, and \( i_k = p_ku_k, v_k = q_ku_k \) for \( k = 2, 3 \). Inserting all of them in Kirchhoff’s relations we get the symmetric model (18), which here has the form

\[
\begin{align*}
P_1u_1 - P_2u_2 &= \frac{q_1}{p_1^2 + q_1^2}s_1, \\
P_2u_2 - P_3u_3 &= 0, \\
q_1u_1 + q_2u_2 + q_3u_3 &= -\frac{p_1}{p_1^2 + q_1^2}s_1.
\end{align*}
\]

The determinant of the coefficient matrix (which corresponds to the one on the left-hand-side of (18)) is

\[
P_1P_2q_3 + P_1q_2P_3 + q_1P_2P_3,
\]

which already arose in the Introduction (find a detailed discussion on the form of this polynomial in Section 4). Note that for the moment we are not assuming either a voltage- or a current-controlled form for any device, hence the multihomogeneous form of the polynomial above.

In most practical cases there would be no loss of generality in giving the source a classical form, e.g. as a (non-ideal) current source, as in Fig. 2(b). In this setting there is no advantage in keeping a homogeneous variable \( u_1 \) in the model; it is more convenient to describe the source simply by means of the relation \( i_1 = y_1v_1 - i_s \) and use \( v_1 \) as a model variable. Retaining the homogeneous form for the other two branches (to include simultaneously all possible cases, including short- and open-circuits) we get a partially homogeneous model

\[
\begin{align*}
y_1v_1 - p_2u_2 &= i_s, \\
p_2u_2 - p_3u_3 &= 0, \\
v_1 + q_2u_2 + q_3u_3 &= 0,
\end{align*}
\]

with unknowns \( (v_1, u_2, u_3) \).

Partially homogeneous models can be also used in the context of other circuit analysis techniques, as illustrated in the sequel. Assume for instance that all impedances in Fig. 2(b) are known to have an admittance description (i.e. we work in the patch \( A_y \)), and that we want to compute the Thévenin equivalent across the third branch. Our goal is to set up only one model allowing us to compute both the Thévenin (open-circuit) voltage and the Norton (short-circuit) current. To this end, we simply model a virtual load in parallel with the third
branch in homogeneous terms, and use a classical description for the remaining branches. The nodal analysis model can be checked to read as

\[ y_1 e_1 + y_2(e_1 - e_2) = i_s \]  \hspace{1cm} (25a)

\[ y_2(e_2 - e_1) + y_3 e_2 - p_i u_i = 0 \]  \hspace{1cm} (25b)

\[ e_2 + q_i u_i = 0 \]  \hspace{1cm} (25c)

with unknowns \((e_1, e_2, u_i)\); \(e_1\) and \(e_2\) are the potentials at the NW and NE nodes and the subscript \(i\) is used for the (virtual) load. Solving this system with \(p_i = 0\) yields the Thévenin voltage as \(q_i u_i\), whereas the Norton (short-circuit) current is obtained as \(p_i u_i\) with \(q_i = 0\).

More conclusions can be derived from the latter model. The determinant is now \((y_1 y_2 + y_1 y_3 + y_2 y_3) q_i + (y_1 + y_2) p_i\). When \(p_i = 0\) (so that \(q_i \neq 0\)), the non-vanishing of the first term requires \(y_1 y_2 + y_1 y_3 + y_2 y_3 \neq 0\), which characterizes the set of configurations in \(A_y\) for which the Thévenin voltage (and the Thévenin equivalent circuit) is well-defined; analogously, when \(q_i = 0\) (and then \(p_i \neq 0\)) the non-vanishing of the above polynomial requires \(y_1 + y_2 \neq 0\), a condition which characterizes the parameter values (again in \(A_y\)) for which the Norton current (and the Norton equivalent) are well-defined.

4 Non-degeneracy. The matrix-tree theorem

The homogeneous formalism makes it possible to address in general the non-degeneracy problem, that is, the characterization of the conditions under which a linear circuit has a unique solution. This is e.g. of interest in non-passive problems, in which the real part of some impedances (or the resistance in a DC context) may become negative. These properties have been typically examined in circuit theory in terms of certain reduced models which, as discussed earlier, are not completely general. In particular, in nodal or loop analysis the results are formulated in terms of the nodal admittance matrix or the loop-impedance matrix, respectively [4]. In the nodal or the loop analysis context, the degeneracy of a circuit is characterized by the zeros of a tree- or a cotree-enumerator polynomial, respectively. However, such polynomials provide no explicit information about what happens when the assumptions supporting such reductions do not hold.

In this section we perform this analysis in full generality, using a projective version of the aforementioned matrices (cf. (27)) and the multihomogeneous version (26) of the Kirchhoff or tree-enumerator polynomial: find related results regarding the latter in a matroid context in [3, 17]. This approach makes it possible to obtain smoothly previous polynomials as dehomogenizations of this universal polynomial. In other (closely related) language, these results provide essentially a projectively-weighted version of the matrix-tree theorem (cf. (43)), extending the results of [11].

To present our results we need some additional background on the cut and cycle matrices introduced in 3.1 (see [1] for further background). As before, we focus on connected problems for simplicity. It is well known that a square submatrix \(A_T\) of \(A\) of order \(n - 1\) has a non-null determinant if and only if the branches defining the columns of \(A_T\) form a spanning tree.
Moreover, for a given $A$ all such submatrices are known to have the same determinant in absolute value; based on this we assign to $A$ a constant $k_A$, namely the positive integer for which the identities $\det A_T = \pm k_A$ hold. We have $k_A = 1$ for so-called totally unimodular choices of $A$ (e.g. for a reduced incidence matrix). The same applies to maximal square submatrices of a fixed cycle matrix $B$: non-singular ones have a determinant of the form $\pm k_B$ for a positive integer $k_B$, and this happens iff the chosen columns specify a co-tree (the complement of a spanning tree), the determinant being zero otherwise. Totally unimodular choices of $B$ are always possible.

With this background, the matrix-tree theorem in the unweighted context is simply expressed by the identity $\det(AA^T) = \tau k_A^2$, where $\tau$ is the total number of spanning trees. If $A$ is totally unimodular (in particular if it is a reduced incidence matrix), then the identity amounts to $\det(AA^T) = \tau$, which is probably the most popular form of the matrix-tree theorem and which, in essence, can be traced back to the work of Kirchhoff and Maxwell [9, 12]. Dual results hold for $B$, namely, $\det(BB^T) = \tau k_B^2$ holds true and, in particular, $\det(BB^T) = \tau$ for totally unimodular choices of $B$. We will elaborate on these results (and their weighted counterparts) in subsection 4.3.

4.1 Multihomogeneous Kirchhoff polynomial

We assume in the sequel that the branch set $E(G)$ is simply $\mathbb{N}_m = \{1, \ldots, m\}$, so that the value assigned by the source-free configuration map $\gamma$ to the $i$-th branch is $(p_i : q_i)$. Both $p_i$ and $q_i$ will be indeterminates in the polynomial (26); similarly, $y_i = p_i/q_i$ and $z_i = q_i/p_i$ will arise as indeterminates in certain dehomogenizations of this multihomogeneous polynomial. Additionally, a spanning tree will be represented by its set of branches; with the convention above, a spanning tree is unambiguously defined as a subset $T$ of $\mathbb{N}_m$ with $n - 1$ elements. A cotree is denoted as $\overline{T}$, meaning $\mathbb{N}_m - T$. By $\mathcal{T}$ we denote the set of spanning trees of the digraph.

With this notation, the multihomogeneous Kirchhoff polynomial of a connected (di)graph (see [3, 17] and references therein) is defined as

$$K(p, q) = \sum_{T \in \mathcal{T}} \left( \prod_{j \in T} p_j \prod_{k \notin T} q_k \right),$$

(26)

that is, every spanning tree $T$ in the graph sets up a monomial in $K(p, q)$, which includes $p_i$ (resp. $q_i$) as a factor if the $i$-th branch belongs to $T$ (resp. to $\overline{T}$). This polynomial is homogeneous (of degree one) in each pair of variables $(p_i, q_i)$, because necessarily either $p_i$ or $q_i$, but not both, appears in each monomial; hence the “multihomogeneous” label.

4.2 Non-degenerate configurations

**Definition 1.** A non-degenerate configuration on a connected digraph $G$ is a configuration map $\gamma : E(G) \to \mathbb{K}P^2$ for which the circuit equations (9) have a unique solution.
Remark. The non-degenerate character of a configuration does not depend on the excitation vector \( s \) (which only matters for the location of the solution). We may hence study the non-degeneracy of a configuration by examining the associated source-free configuration \( \tilde{\gamma}: \mathcal{E}(G) \rightarrow \mathbb{KP} \) defined in subsection 2.5 as \( \pi \circ \gamma \). This amounts in practice to the well-known fact that e.g. the voltage of a voltage source does not matter for the existence and uniqueness of solutions, and its voltage can be hence fixed at zero. This is consistent with the fact that Theorem 1 below can be stated in terms of source-free configurations. The other way round, the reader may understand that Theorem 1 applies to configurations with sources by assuming that the polynomial \( K \) depends vacuously on \( s \): that is, if a pair of vectors \( p, q \) annihilates the polynomial, then so it does the triad \( (p, q, s) \) for any \( s \).

**Theorem 1.** The set of (source-free) degenerate configurations of a digraph is the zero set of the multihomogeneous Kirchhoff polynomial (26).

The proof of Theorem 1 is based on several auxiliary results that we state in advance. The first of them essentially says that the non-singularity of the coefficient matrix of the general model (9) is equivalent to the one of the coefficient matrices of the reduced homogeneous models (16) and (18), namely

\[
M = \begin{pmatrix} AP \\ BQ \end{pmatrix}.
\]  

(27)

**Lemma 1.** Let \( P \) and \( Q \) be diagonal matrices in \( \mathbb{K}^{m \times m} \) with \( (P \ Q) \) of maximal rank. If \( A \) and \( B \) are arbitrary matrices in \( \mathbb{K}^{r \times m} \) and \( \mathbb{K}^{(m-r) \times m} \), respectively, then

\[
\det\begin{pmatrix} A & 0 \\ 0 & B \\ -Q & P \end{pmatrix} = \det\begin{pmatrix} AP \\ BQ \end{pmatrix}.
\]  

(28)

Proof. Write

\[
\begin{pmatrix} A & 0 \\ 0 & B \\ -Q & P \end{pmatrix} \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} = \begin{pmatrix} AP & -AQ \\ BQ & BP \\ 0 & P^2 + Q^2 \end{pmatrix}
\]  

(29)

because \( P \) and \( Q \) commute, and use the fact that

\[
\det\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} = \det(P^2 + Q^2).
\]  

(30)

The latter is very simple to check, since an obvious (determinant-preserving) permutation of rows and columns drives the matrix in the left-hand side to block-diagonal form, with blocks of the form

\[
\begin{pmatrix} p_k & -q_k \\ q_k & p_k \end{pmatrix}.
\]

This makes it clear that the first determinant of (30) amounts to

\[
\prod_{k=1}^{m}(p_k^2 + q_k^2) = \det(P^2 + Q^2).
\]
Finally, the maximal rank condition on \((P\ Q)\) means that for each \(k\) at least one of the two parameters \(p_k, q_k\) does not vanish and this makes both determinants in (30) non-null. The result then follows from the identity (29). \(\square\)

Proposition 1 below makes systematic use of certain results from linear algebra which are compiled here. Recall first that the signature of a permutation is \((-1)^q\), where \(q\) is the number of transpositions in any decomposition of the permutation as a product of transpositions. We are interested in certain classes of permutations of \(m\) elements, namely those in which \(N_m = \{1, \ldots, m\}\) can be partitioned in two subsets \(\sigma_1 = \{j_1, \ldots, j_r\}\), \(\overline{\sigma_1} = N_m - \sigma_1 = \{j_{r+1}, \ldots, j_m\}\) (where we assume \(j_1 < j_2 < \ldots < j_r\) and \(j_{r+1} < \ldots < j_m\)) in a way such that the restrictions of the permutation to both \(\sigma_1\) and \(\overline{\sigma_1}\) are order-preserving. That is, if \(k_i\) is the image of \(j_i\) (for \(i = 1, \ldots, m\)), then both \(k_1 < \ldots < k_r\) and \(k_{r+1} < \ldots < k_m\) hold. Letting \(\sigma_2 = \{k_1, \ldots, k_r\}\) and \(\overline{\sigma_2} = \{k_{r+1}, \ldots, k_m\}\), we will denote the permutation by \(\mathcal{P}_m(\sigma_1, \sigma_2)\), allowed by the fact that it is completely defined by \(\sigma_1\) and \(\sigma_2\); indeed, the elements of the former are mapped into those of the latter in increasing order, and the same happens with \(\overline{\sigma_1}\) and \(\overline{\sigma_2}\). In this context, the signature of the permutation can be computed as follows (a detailed proof can be found e.g. in [19]):

\[
\text{sgn}\left(\mathcal{P}_m(\sigma_1, \sigma_2)\right) = (-1)^{\sum_{j \in \sigma_1} j + \sum_{k \in \sigma_2} k}.
\]

We will also use in Proposition 1 the general Schur complement (i.e. the Schur complement of a non-principal submatrix), cf. [20]. Given \(M \in \mathbb{K}^{n \times n}\) and two non-empty subsets \(\alpha, \omega\) of \(N_m = \{1, \ldots, m\}\), denote by \(M[\alpha, \omega]\) the submatrix of \(M\) defined by the rows and columns specified by the index sets \(\alpha\) and \(\omega\), respectively. If \(\alpha\) and \(\omega\) have the same number of elements and \(M[\alpha, \omega]\) is non-singular, the Schur complement of \(M[\alpha, \omega]\) in \(M\) is defined as \(M/M[\alpha, \omega] = M[\alpha, \omega] - M[\alpha, \omega](M[\alpha, \omega])^{-1}M[\alpha, \omega]\), where again \(\alpha, \omega\) stand for \(N_m - \alpha, N_m - \omega\). In this setting, we have the identity

\[
det M = \text{sgn}\left(\mathcal{P}_m(\alpha, \omega)\right) \det(M[\alpha, \omega]) \det(M/M[\alpha, \omega]).
\]

**Proposition 1.** Let \(A\) and \(B\) be two arbitrary cut and cycle matrices of a given connected digraph, with their columns arranged according to the same order of branches. Assume that \(T_1\) and \(T_2\) specify two spanning trees, and let \(\overline{T_1}\) and \(\overline{T_2}\) represent the corresponding cotrees. Then

\[
det A_{T_1} \det B_{T_1} = \text{sgn}\left(\mathcal{P}_m(T_1, T_2)\right) \det A_{T_2} \det B_{T_2}.
\]

**Proof.** Let \(\alpha = \{1, \ldots, n - 1\}\). For notational brevity, denote by \(A_{T_i}\) the submatrix of \(A\) defined by the columns indexed by \(T_i\) (that is, \(A_{T_i} = A[\alpha, T_i]\)); analogously, \(B_{T_i}\) is the submatrix of \(B\) defined by the columns indexed by \(T_i\). By writing

\[
det \begin{pmatrix} A \\ B \end{pmatrix}
\]

in terms of \(\det A_{T_1}\) and \(\det A_{T_2}\) (using (32)), we get

\[
\text{sgn}(\mathcal{P}_m(\alpha, T_1)) \det A_{T_1} \det(B_{T_1} - B_{T_1}A_{T_1}^{-1}A_{T_1}) = \text{sgn}(\mathcal{P}_m(\alpha, T_2)) \det A_{T_2} \det(B_{T_2} - B_{T_2}A_{T_2}^{-1}A_{T_2})
\]
and therefore

\[
\det A_T \det(B_{T_1} - B_T A_{T_1}^{-1} A_{T_1}) = \text{sgn} (\mathcal{P}_m(T_1, T_2)) \det A_T \det(B_{T_2} - B_T A_{T_2}^{-1} A_{T_2})
\] (35)

since \(\text{sgn} (\mathcal{P}_m(\alpha, T_1)) \text{sgn} (\mathcal{P}_m(T_1, T_2)) = \text{sgn} (\mathcal{P}_m(\alpha, T_2))\).

Making use of the fact that \(\det B_{T_i} \det B_{T_i}^T = (\det B_{T_i})^2 = k_B^2\) does not depend on \(i\), multiply the left-hand side of (35) by \(\det B_{T_1}^T\) and the right-hand side by \(\det B_{T_2}^T\) to recast this identity as

\[
\det A_T \det(B_{T_1}^T - B_T A_{T_1}^{-1} A_{T_1} B_{T_1}) = \\
\text{sgn} (\mathcal{P}_m(T_1, T_2)) \det A_T \det(B_{T_2}^T - B_T A_{T_2}^{-1} A_{T_2} B_{T_2}).
\] (36)

By writing the orthogonality of the cycle and cut spaces as \(AB^T = 0\) one can derive the identity \(-A_{T_1}^{-1} A_{T_1} B_{T_1}^T = B_{T_1}^T\). This allows us to rewrite the left-hand side of (36) as

\[
\det A_T \det(B_{T_1}^T + B_T B_{T_1}^T) = \det A_T \det(B_{T_1}^T) - \det A_T \det(B_T)
\]

and, analogously, the right-hand side as

\[
\text{sgn} (\mathcal{P}_m(T_1, T_2)) \det A_T \det(B_{T_2}^T). 
\]

The identity (33) then follows from the fact that \(\det(B_B) = \tau k_B^2\) does not vanish.

The key idea in our approach is that there is no chance to determine whether the signs of \(\det A_T\) and \(\det A_T\) are the same in purely combinatorial terms, that is, by just looking at the permutation \(\mathcal{P}_m(T_1, T_2)\). The same applies to \(\det B_{T_1}\) and \(\det B_{T_2}\). But it is indeed possible to do it when we look at the products \(\det A_T \det B_{T_i}\), as shown above. Another way to express the same is to say, allowed by Proposition 1, that, for any pair of cut/cycle matrices \(A, B\), the quantity

\[
k_{AB} = (-1)^{n(n-1)/2} + \sum_{\varepsilon \in T^J} \det A_T \det B_T 
\] (37)

is a (non-null) invariant, that is, it does not depend on the actual choice of the spanning tree \(T\). To prove this just use (31), (33) and the fact that \((-1)^{n(n-1)/2}\) is a quantity which does not depend on the tree \(T\).

Accordingly, we will say that \(A\) and \(B\) are positively matched if \(k_{AB}\) is positive and, in particular, that they are well-matched if \(k_{AB} = 1\). Since \(\det A_T\) and \(\det B_{T_i}\) are integers, \(A\) and \(B\) are well-matched if and only if they are positively matched and both of them are totally unimodular. An important consequence (that the author could not find in the literature) is that the fundamental matrices

\[
A = (I \ A_T), \quad B = (B_T \ I), \quad \text{with} \quad B_T = -A_T^T,
\] (38)

defined by any spanning tree \(T\) are well-matched since, on the one hand, \(T = \{1, \ldots, n-1\}\), so that the exponent in (37) is even, and on the other \(\det A_T = \det B_T = +1\) by construction.
(both $A_T$ and $B_T$ are identity matrices). We will elaborate on the implications of this later (cf. Corollary 2).

**Proof of Theorem 1.** The matrix in the left-hand side of (28) is the coefficient matrix of (9) and therefore the characterization of non-degenerate configurations amounts to characterizing the non-singularity of this matrix. To do this, we compute this determinant using a generalized Laplace expansion along the first $n - 1$ rows of the matrix (27), arising in the right-hand side of (28). Generalized Laplace expansions are explained in many linear algebra texts; see e.g. [10]. With $\alpha = \{1, \ldots, n - 1\}$, such expansion of $\det M$ reads as

$$(-1)^{n(n-1)/2} \sum_{|\omega| = n-1} (-1)^{\Sigma_j \omega_j} \det(M[\alpha, \omega]) \det(M[\overline{\alpha}, \overline{\omega}]),$$

the exponent $n(n - 1)/2$ being the sum of the indices of $\alpha$.

Because of the diagonal structure of $P$ and $Q$ and the properties of the digraph matrices $A$ and $B$ stated at the beginning of this section, the non-null determinants of the submatrices arising in this expansion come from spanning tree/cotree pairs and have the form $\det A_T \det P_T$ (where $P_T$ is the submatrix of $P$ defined by the rows and columns indexed by $T$) and $\det B_T \det Q_T$ (with the same convention for $Q_T$; find the notation for $A_T$, $B_T$ in the proof of Proposition 1). Using these remarks, we may recast the expansion above as

$$(-1)^{n(n-1)/2} \sum_{T \in T} (-1)^{\Sigma_{j \in T} j} \det A_T \det P_T \det B_T \det Q_T.$$

Now, the key step in the proof is the fact that

$$(-1)^{n(n-1)/2} (-1)^{\Sigma_{j \in T} j} \det A_T \det B_T$$

does not depend on the choice of the tree $T$ (this is the constant $k_{AB}$ in (37)). Hence

$$\det \begin{pmatrix} AP \\ BQ \end{pmatrix} = k_{AB} \sum_{T \in T} \det P_T \det Q_T = k_{AB} \sum_{T \in T} \left( \prod_{i \in T} p_i \prod_{j \in T} q_j \right),$$

that is,

$$\det \begin{pmatrix} AP \\ BQ \end{pmatrix} = k_{AB} K(p, q),$$

and the claim follows from the fact that $k_{AB} \neq 0$. \qed

The following is essentially the unweighted version of Theorem 1, which is obtained by setting $P = Q = I$ in (40); just note that $K(1, 1) = \tau$, where $1 = (1, \ldots, 1)$.

**Corollary 1.** For any pair of cut/cycle matrices $A, B$, we have

$$\det \begin{pmatrix} A \\ B \end{pmatrix} = \tau k_{AB},$$

where $\tau$ is the number of spanning trees and $k_{AB}$ is the constant arising in (37).
At this point it is worth comparing (41) with previous (and closely related) results in this direction, a full account of which can be found in [4, 11]. Chen’s approach obtains for the determinant (41) the expression $\pm \tau k_A k_B$: this is, specifically, (2.144b) in [4] (essentially the same holds in [11], cf. Theorem 4' there), but an ambiguity in the sign remains. It is clear that $k_{AB} = \pm k_A k_B$ but we emphasize that with our approach $CAB$ captures the $\pm$ sign, which can be computed from any tree-cotree pair using (37). Actually, the difference between both approaches is a subtle one: Chen uses an indirect approach to compute up to a sign the determinant (41), and then derives a result (Lemma 2.8 in [4]) which can be understood as an alternative statement of Proposition 1 above. By contrast, our approach captures the essential combinatorial property expressed by (33) in Proposition 1, to derive from it the expression (41) with a well-defined sign.

**Corollary 2.** If $A$ and $B$ are well-matched cut/cycle matrices (in particular, if they are the fundamental matrices (38) defined by a spanning tree), then

$$\det \begin{pmatrix} A \\ B \end{pmatrix} = \tau. \quad (42)$$

This result, which provides an alternative form of the matrix-tree theorem giving the number of spanning trees in terms of (well-matched) cut/cycle matrices, is just (41) with $k_{AB} = 1$.

### 4.3 Dehomogenization. Classical forms of the Kirchhoff polynomial and the matrix-tree theorem

For simplicity, assume in what follows $k_{AB} = 1$, namely, that $A$ and $B$ are well-matched. Then (40) reads as

$$\det \begin{pmatrix} AP \\ BQ \end{pmatrix} = K(p, q). \quad (43)$$

From this expression, which can be understood as a projectively-weighted version of the matrix-tree theorem, we may derive certain known forms of this theorem in a weighted setting, involving classical forms of the Kirchhoff polynomial which are shown below to arise as dehomogenizations of (26).

Let us focus the attention on the $A_\Phi$ affine patch defined in $\mathbb{KP} \times \ldots \times \mathbb{KP}$ by the conditions $q_i \neq 0$ for $i = 1, \ldots, m$. This is the patch where the admittance matrix $Y$ is well-defined (and equals $PQ^{-1}$ for arbitrary choices of $P, Q$, as far of course as $Q$ is non-singular). In this patch the choice $Q = I$ is always possible; this yields $P = Y$. With these parameters, the matrix in (43) amounts to the one in the branch-voltage system (14), and the identity (43) becomes

$$\det \begin{pmatrix} AY \\ B \end{pmatrix} = K(y, 1), \quad (44)$$

(cf. [11] in this regard). The polynomial in the right-hand side of (44) is a dehomogenization of $K(p, q)$, which amounts to the so-called Maxwell’s form or tree-based form of the Kirchhoff polynomial.
polynomial, namely,

\[ K_0(y) = \sum_{T \in \mathcal{T}} \prod_{j \in T} y_j, \]  

(45)

which is set up simply by inserting the admittance parameter \( y_j \) in the monomial corresponding to the spanning tree \( T \) if branch \( j \) belongs to \( T \).

The vectors of admittances which do not annihilate the polynomial (45) define the set of non-degenerate configurations in the voltage-controlled patch \( \mathbb{A}_y \); equivalently, these are the admittances for which the branch-voltage system (14) (which is defined only on \( \mathbb{A}_y \)) has a unique solution. Moreover, from (44) it is not difficult to derive also the identity

\[ \det (AYA^T) = K_0(y), \]  

(46)

which is the usual form of the weighted matrix-tree theorem (note, in particular, that \( A \) may be a reduced incidence matrix). This shows that the non-vanishing of \( K_0(y) \) also characterizes the set of admittances where the classical nodal equations are well-defined. Note finally that by setting \( Y = I \) in (46) we get the unweighted version of the matrix-tree theorem in its classical form, namely \( \det (AA^T) = \tau \): here \( A \) typically denotes a reduced incidence matrix.

We leave it to the reader to check that the expression for the polynomial which characterizes non-degenerate configurations in the patch \( \mathbb{A}_y \subset \mathbb{K}^P \times \ldots \times \mathbb{K}^P \) is

\[ K_1(z) = \sum_{T \in \mathcal{T}} \prod_{k \in T} z_k, \]  

(47)

arising as the dehomogenization \( K(1, z) \). The zeros of this polynomial characterize the degeneracies of the branch-current system (13) and also of the loop analysis equations. In this case the polynomial is constructed by including, in the monomial corresponding to a given tree, the impedance parameter \( z_k \) iff the \( k \)-th branch belongs to the cotree. And even if we omit it for brevity, it is possible to derive analogously the polynomial characterizing non-degenerate configurations in so-called hybrid models, mixing admittance and impedance descriptions (find a detailed discussion in this regard in [8]).

The form of the different polynomials arising above can be easily illustrated in terms of the graph in Fig. 2(a). It is obvious that there are three spanning trees, each one excluding one of the branches. The universal Kirchhoff polynomial (26) has in this case the expression depicted in (23): here the term \( p_1p_2q_3 \) comes from the tree defined by branches 1 and 2, which are responsible for the \( p_1p_2 \) factor, whereas branch 3 defines the co-tree and yields the \( q_3 \) factor. The other terms are obtained analogously. In turn, the form (45) reads here \( y_1y_2 + y_1y_3 + y_2y_3 \), each term coming from one of the spanning trees, and is only valid if all branches admit an admittance description (namely, this holds in the patch \( \mathbb{A}_y \)). The dual case is given by (47) and reads \( z_1 + z_2 + z_3 \) (cf. Section 1), where each term arises in this case as the impedance in each co-tree; this expression is valid only in the patch \( \mathbb{A}_z \).
5 Controlled sources

The framework developed in previous sections can be extended to accommodate controlled sources, as discussed below. A salient advantage of our approach in this context is that it avoids the need for the classical distinction among the four types of sources (depending on the controlling/controlled variables); independent sources and switches can be also included in a comprehensive manner. For the sake of brevity we only sketch the results.

Disregarding excitation terms, an abstract controlled source is described by a pair of equations of the form

\[
\begin{align*}
    p_1 v_1 - q_1 i_1 &= 0, \\
    p_2 v_2 - q_2 i_2 + q_3 i_1 &= 0,
\end{align*}
\]

with \((p_k, q_k) \neq (0, 0)\) for \(k = 1, 2\). The first (resp. second) equation describes the controlling (resp. controlled) device. Ideal VCVS’s (voltage-controlled voltage sources), CCVS’s, VCCS’s and CCCS’s (with the same convention in the acronyms) are described by the parameter values \(q_2 = 0, \alpha \neq 0, \beta = 0; q_2 = 0, \alpha = 0, \beta \neq 0; p_2 = 0, \alpha \neq 0, \beta = 0; \) and \(p_2 = 0, \alpha = 0, \beta \neq 0\), respectively. Non-ideal cases, displaying an impedance in series or parallel with an ideal voltage/current source, are included above with \(p_2 \neq 0 \neq q_2\), with the Thévenin/Norton equivalence holding again. The addition of excitation terms \(s_1\) and \(s_2\) in the right-hand side would easily accommodate independent sources in the same setting. Note that the formalism above may also account for ideal switches, since the parameter values \(q_2 = 0\) and \(p_2 = 0\), respectively, model a closed switch (short-circuit) and an open switch, with \(\alpha = \beta = 0\) in both cases; this might of interest, for instance, in modelling saturation/cut-off regimes in transistors.

![Figure 3: Small-signal II-model of a transistor.](image-url)

The interest of this formulation relies on the fact that a unique analysis can be performed for all four types of controlled sources and accounting also for all possible open- and short-circuits in the same model. We illustrate this idea by considering an abstract II-model of the small-signal equivalent of a transistor depicted in Fig. 3. Here, the abstract controlled source (which is controlled by the first branch) would amount to a CCCS for a bipolar junction transistor and to a VCCS for a MOSFET, but we do not need to make such a distinction. We join together the parallel impedance (2b in the figure) with the controlled source in a...
single circuit element. To be specific, we focus on the problem of the existence of a two-port, 
Z-parameter description.

Using \( v_1 = v_{\text{in}} \) and \( v_2 = v_{\text{out}} \) the circuit equations can be written as

\[
\begin{align*}
    i_{\text{in}} &= i_1 + i_3 \\
    i_{\text{out}} &= i_2 - i_3 \\
    v_1 &= v_2 + v_3 \\
    p_1 v_1 - q_1 i_1 &= 0 \\
    p_2 v_2 - q_2 i_2 + \alpha v_1 + \beta i_1 &= 0 \\
    p_3 v_3 - q_3 i_3 &= 0,
\end{align*}
\]

and the existence of a Z-parameter description relies on the non-singularity of the coefficient
matrix of the variables \( v_1 = v_{\text{in}}, \, v_2 = v_{\text{out}}, \, v_3, \, i_1, \, i_2 \) and \( i_3 \), since this makes it possible to
write \( v_{\text{in}} \) and \( v_{\text{out}} \) just in terms of \( i_{\text{in}} \) and \( i_{\text{out}} \), as intended. An easy computation shows that the
determinant of this coefficient matrix is

\[
p_1 p_2 p_3 + p_1 q_2 p_3 + q_1 p_2 p_3 + \alpha q_1 p_3 + \beta p_1 p_3,
\]

the non-vanishing of which characterizes the existence of a Z-parameter description. From
the general expression (50) one may draw conclusions in many different settings. Just to give
a glimpse, let us assume that the controlled source is an ideal current source, so that \( p_2 = 0 \),
and that the bridge admittance does not vanish (that is, \( p_3 \neq 0 \)), in order to examine the
dependence of the non-degeneracy expression above on the controlling branch. Under these
assumptions, the vanishing of (50) amounts to that of \( p_1 q_2 + \alpha q_1 + \beta p_1 \). In the CCCS context
(\( \alpha = 0 \); the BJT case) this further amounts to \( p_1 q_2 + \beta p_1 \), whereas in the VCCS one (\( \beta = 0 \);
the MOSFET case) the expression reads as \( p_1 q_2 + \alpha q_1 \). Should the controlling branch be a
(say) regular impedance (\( p_1 \neq 0 \neq q_1 \)) then both contexts are essentially the same, since
one can easily resort from the voltage-controlled to the current-controlled framework and
vice-versa, just setting the gains in a way such that \( \alpha q_1 = \beta p_1 \). More can be derived from
the homogeneous formalism, though: indeed, when the controlling branch is open-circuited
(\( p_1 = 0 \)) then the BJT case always degenerates and no Z-description holds; by contrast,
the assumption \( p_1 = 0 \) poses no problem for the MOSFET, since there is a non-null extra
term of the form \( \alpha q_1 \). We emphasize the fact that these conclusions can be derived from a
single model; needless to say, they can be also obtained from classical circuit analysis but
one would need to set up different models in order to cover the variety of scenarios.

Several remarks are in order regarding the extension of the results of Sections 3 and 4,
even if a detailed analysis is left for future work. Note that we may account for coupling
parameters in the \( P \) and \( Q \) matrices simply by writing the blocks corresponding to (48) as

\[
P = \begin{pmatrix} p_1 & 0 \\ \alpha & p_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & 0 \\ -\beta & q_2 \end{pmatrix}.
\]

One can easily check that the matrix \( (P \ \ Q) \) (or, equivalently, \( (-Q \ \ P) \)) still has maximal
rank and this means that for each controlling-controlled pair the characteristic equation
\(-Qi + Pv = 0\) (that is, \((48)\)) can be again described for all possible parameter values just in terms of two homogeneous variables \(u_1, u_2\): \such a parametrization now reads as \(i = Pu\), \(v = Qu\), with \(u = (u_1, u_2)\) and blocks of the form

\[
\hat{P} = \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} q_1 & 0 \\ \delta & q_2 \end{pmatrix},
\]

where \(\gamma = \frac{q_2}{p_2 + q_2} (\alpha q_1 + \beta p_1)\), \(\delta = -\frac{p_2}{p_2 + q_2} (\alpha q_1 + \beta p_1)\). This is possibly of interest in order to include in different circuit models all possible types of controlled sources in terms of a single (and the same) pair of variables \(u_1, u_2\). Moreover, and even if we state the following without proof, an identity such as \((28)\) within the reduction process arising in Lemma 1 still holds in this context when \(P\) and \(Q\) in the right-hand side of \((28)\) are replaced by \(\hat{P}\) and \(\hat{Q}\) as defined above. We also note that the terms involving control parameters (i.e. \(\alpha, \beta\)) in determinantal expansions such as \((50)\) may be addressed in terms of certain pairs of spanning trees, accounting for off-diagonal terms in the \(P\) and \(Q\) matrices. Finally, fully coupled problems are also in the scope of future research. More difficulties are likely to show up in this context, including not only analytical but also computational aspects; worth mentioning in this regard is the term cancellation problem: cf. [14].

6 Example: fault isolation in a bridge circuit

We illustrate in this section, by means of a simple example, how the above approach can be used in practice. To this end, consider the Wien bridge circuit depicted in Fig. 4(a).

In a classical approach, we may assume that all linear elements are impedances (exception made of the 0-th branch, which will correspond to an ideal voltage source with voltage \(v_0\))
and that they are characterized by their impedance parameter $z_k$. To fix simple values, let the bridge be balanced (namely, let $z_1/z_2 = z_3/z_4$, with $z_3$ and $z_4$ denoting the parallel/series connection of $z_{3a}$ and $z_{3b}$ and of $z_{4a}$ and $z_{4b}$) by taking e.g. $z_1 = 1$, $z_2 = 2$, $z_{3a} = z_{4a} = 1$, $z_{3b} = z_{4b} = -j$; additionally, in order to model a substantially larger impedance across the bridge set e.g. $z_5 = 100$. Assume if you want that all impedances are in kΩ's. The balance condition yields $v_5 = 0$ regardless of the values of $z_5$ and $v_0$.

Assume now that we want to simulate bridging faults in this circuit, by checking the actual values displayed by $v_5$ in different (and possibly faulty) scenarios. Bridging faults arise e.g. in integrated circuits from spurious connections between metal interconnects, and are typically modelled as short-circuits. In practice (in larger scale circuits) one may identify critical pairs of nodes which are sensitive to this type of faults and then simulate, for later use, the behavior of the faulty circuits resulting from such bridging effects. According to this strategy, the goal in our context would be to set up a table of expected values for $v_5$ under different faulty conditions, making it possible to identify later the spurious connection which would eventually be responsible for a failure.

To perform such simulations, in a classical framework (say e.g. in terms of the branch-current model (13)) one would need to reformulate the topology of the circuit and hence the model for each fault scenario, by adding a (short-circuit) virtual branch, as in Fig. 4(b). Note that an additional current variable $i_k$ must be included in the equations when modelling a fault in the $k$-th branch. Be aware of the fact that the no-faults scenario cannot be accommodated as a particular case of any of the resulting faulty models, since both cases correspond to the extremal values $y_k = 0$ (no fault) and $z_k = 0$ (bridging fault) and each model involves a different set of variables: specifically, the $i_k$ variable in the faulty scenario for branch $k$ would not be present in the original model.

By contrast, if we resort to the homogeneous framework we may include all faulty circuits and the no-fault original one in single model, even accommodating simultaneous faults. To this end, we may simply characterize the virtual branches of Fig. 4(b) by using homogeneous impedances $(p_k : q_k)$ for $k = 6, 7, 8$ and homogeneous variables $u_6$, $u_7$ and $u_8$ (branches 6 and 7 are the NW-SE and SW-NE diagonals in the figure, and branch 8 is the one on the right). Currents and voltages will be computed as $i_k = p_k u_k$, $v_k = q_k u_k$ for $k = 6, 7, 8$. The resulting model reads as

To model the no-fault case just set all $p_k$'s to zero (and, if you want, $q_k = 1$ for simplicity, although this is not strictly necessary), whereas the different bridging faults are obtained just by resetting the corresponding parameter $q_k$ to 0 (and, optionally, $p_k = 1$). Here we need to set up one model, instead of four, one for each individual fault plus the original one, in the branch-current setting. Worth emphasizing is the partially homogeneous nature of (52), where some branches are modelled in classical terms as impedances (or as a source), with their current variables entering the model, and only certain branches are given a homogeneous description.

The same idea may be further exploited, e.g. to include faults also in the original impedances (namely, 1 to 4b; we do not model faults in the source or in $z_5$). Short-circuit
faults at these branches can be naturally framed in the branch-current model by setting
\( z_k = 0 \), but this is not the case with open-circuit faults, resulting e.g. from wire breaks,
an excess of insulating material, etc. They should be modelled by the conditions \( y^k = 0 \)
and again this would require another model; be also aware that resorting to an admittance
description in the original model would exclude short-circuit faults.

Instead, by using homogeneous variables for all branches we accommodate all scenarios
in a single model, just setting \( q_k \) or \( p_k \) to zero to model short-circuit and open-circuit faults.
In this case we avoid defining up to ten models in a classical framework by setting up only
one model in the homogeneous context, namely Note that the expressions of the \( A \) and \( B \)
matrices used in the model can be easily derived from the first four and last seven rows of
the matrix in (53); just set \( p_k = q_k = 1 \) there (i.e. set \( P = Q = I \) in (27)).

Here \( s_0 = s_0/(p_0^2 + q_0^2) \). We emphasize the fact that all the variables are now homogeneous,
and that the model takes the symmetric form (18). Note only that excitation terms are ruled
out for all branches except for the 0-th one; that is, we are assuming that there are no sources in the remaining branches by implicitly setting $s_1 = \ldots = s_8 = 0$. If desired, you can further simplify the model by setting $p_0 = 1$, $q_0 = 0$ (capturing the assumption that the source is an ideal voltage source), so that $s_0 = s_0 = v_0$ and $u_0$ can be replaced by the current variable $i_0$; additionally, you can set $p_5 = 1$ and $q_5 = z_5$, and replace $u_5$ by the current variable $i_5$, since we are not modelling open-circuits in the fifth branch. Again, this would lead to a partially homogeneous model.

For completeness, we can check that the values of $v_5$, computed as $q_5 u_5$ from the solutions of the homogeneous model (53) with $v_0 = 1$, are indeed able to isolate up to thirteen different faults. Namely, bridging faults (captured in the vanishing of $q_6$, $q_7$, $q_8$) yield, respectively, the values $-0.178 - 0.230j$, $0.662$ and $-0.330 - 0.001j$ for $v_5$. Analogously, the presence of other short-circuit faults, defined by the respective vanishing of $q_1$, $q_2$, $q_{3a}$ (or, equivalently, of $q_{3b}$), $q_{4a}$ or $q_{4b}$, lead to the values $0.332 + 0.001j$, $-0.664 - 0.002j$, $-0.331$, $0.065 + 0.199j$ and $0.066 - 0.198j$. Finally, open-circuits in the original impedances, captured in the vanishing of $p_1$, $p_2$, $p_{3a}$, $p_{3b}$ or $p_{4a}$ (or $p_{4b}$), respectively, yield for $v_5$ the values $-0.651 - 0.002j$, $0.329 + 0.001j$, $0.067 - 0.198j$, $0.065 + 0.198j$ and $-0.329 - 0.002j$. Recall that in the no-fault case we have $v_5 = 0$. In any case, it is worth emphasizing that the interest of the simulation does not rely on the actual values met by $v_5$, but on the fact that all computations are performed in terms of one and the same model (53), as indicated above. Even if constructed (for illustrative purposes) at a small scale, our example suggests that this approach may be fruitful in the definition of fault simulation strategies in larger scale circuits.

7 Concluding remarks

The homogeneous formalism introduced in this paper seems to be of interest in the study of other analytical aspects of linear circuits. Further reductions related to nodal analysis, as well as partially homogeneous models and computational aspects, are the subject of ongoing research. Additionally, a nonlinear version of this approach has proved feasible and is currently in preparation. Regarding applications, homogeneous and partially homogeneous models are probably worth being examined further for fault isolation, computation of Thévenin equivalents, analysis of circuits with ideal switches, etc. The inclusion of fully coupled and multiterminal devices is in the scope of future research.

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