Excluding smooth effective one-fixed point actions of finite Oliver groups on low-dimensional spheres

Agnieszka Borowiecka
E-mail: aborowiecka@wp.pl

Piotr Mizerka
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
61-614 Poznań, Poland
E-mail: piotr.mizerka@amu.edu.pl

Abstract

According to [10], a finite group $G$ has a smooth effective one-fixed point action on some sphere if and only if $G$ is an Oliver group. For some finite Oliver groups $G$ of order up to 216, and for $G = A_5 \times C_n$ for $n = 3, 5, 7$, we present a strategy of excluding smooth effective one-fixed point $G$-actions on low-dimensional spheres.

1 Introduction

In this paper, we are interested in excluding smooth effective actions of finite groups on low-dimensional spheres with exactly one fixed point.

Constructions of smooth effective one-fixed point actions of finite groups $G$ on spheres were obtained for the first time by Stein [15] for $G = SL(2, 5)$, then by Petrie [14] for a finite abelian groups of odd order having at least three noncyclic Sylow subgroups, and then by Laitinen, Morimoto and
Pawalowski [11] for any finite nonsolvable group \( G \). Laitinen and Morimoto [10] proved that a finite group \( G \) admits a smooth effective one-fixed point action on some sphere if and only if \( G \) has an algebraic characterization found by Oliver [13] for finite groups admitting smooth fixed point free actions on disks. Now, the groups \( G \) are known as Oliver groups. From the corresponding proofs, however, it is hard to deduce the dimension of the sphere on which a given Oliver group can act effectively with one fixed point. What is the lowest dimension of such a sphere is still unknown for most Oliver groups. Morimoto [12], Furuta [6], Buchdahl, Kwasik and Schultz [3] established this dimension to be at least 6 (in fact, this is the case for the alternating group \( A_5 \)).

In her article [1], Agnieszka Borowiecka developed some methods allowing to exclude one-fixed point actions of \( SL(2,5) \) on sphere of dimension 8. By applying GAP computations, we extend these methods to finite Oliver groups \( G \) of order up to 216, and to \( G = A_5 \times C_n \) for \( n = 3, 5 \) and 7.

The following theorem is the main result of this paper.

**Theorem 1.1.** A finite group \( G \) cannot act smoothly and effectively with one fixed point on \( S^n \) provided:

- \( G = SL(2,5) = SG(120,5) \) and \( n = 6, 8 \) [7],
- \( G = GL(2,3) \times C_3 = SG(144,122) \) and \( n = 6 \),
- \( G = SG(144,124) \) and \( n = 6 \),
- \( G = SG(144,126) \) and \( n = 6 \),
- \( G = SG(144,127) \) and \( n = 6 \),
- \( G = SL(2,3) \times S_3 = SG(144,128) \) and \( n = 6 \),
- \( G = A_4 \times \text{Dic}_3 = SG(144,129) \) and \( n = 6 \),
- \( G = GL(3,2) = SL(3,2) = SG(168,42) \) and \( n = 9, 10 \),
- \( G = SG(168,49) \) and \( n = 6 \),
- \( G = A_5 \times C_3 = GL(2,4) = SG(180,19) \) and \( n = 6, 7, 8, 10 \),
- \( G = A_5 \times C_5 = SG(300,22) \) and \( n = 6, 7, 8, 10 \),
- \( G = A_5 \times C_7 = SG(420,13) \) and \( n = 6, 7, 8, 10 \),
- \( G = SG(216,90) \) and \( n = 6 \),
One-fixed point actions

- $G = \text{SG}(216, 91)$ and $n = 6$,
- $G = \text{SG}(216, 92)$ and $n = 6$,
- $G = \text{SG}(216, 94)$ and $n = 6$,
- $G = \text{SG}(216, 95)$ and $n = 6$,
- $G = \text{SG}(216, 96)$ and $n = 6$,

where SG stands for the SmallGroup in GAP libraries [7].

Throughout the paper, unless otherwise specified, all groups are assumed to be finite. The basic notions from group and representation theory are summarized in appendix.

## 2 Discrete submodules restriction

In this section we present the strategy of excluding smooth effective one-fixed point actions of groups on spheres which concerns the case when the dimensions of the fixed point sets of actions of some particular subgroups of a given group is zero. This justifies the name ”discrete”. This strategy was used in a particular case by Agnieszka Borowiecka in her work [1].

We recall the definition of mod-$p$-cyclic group.

**Definition 2.1.** We call a finite group $G$ a mod-$p$-cyclic group if it contains a normal subgroup $P$ being $p$-group for some prime $p$ and such that the quotient $G/P$ is cyclic.

**Lemma 2.1.** [7] Let $G$ be a mod-$p$-cyclic group acting smoothly on a smooth $\mathbb{Z}_p$-homology sphere $\Sigma$ and $x \in \Sigma^G$. Then

$$\chi(\Sigma^G) = 1 + (-1)^{\dim T_x \Sigma^G}.$$

Using Lemma 2.1 we are able to figure out a strategy of excluding one-fixed point actions, given by the following theorem.

**Theorem 2.1.** Let a finite group $G$ act smoothly on a smooth $\mathbb{Z}$-homology sphere $\Sigma$. Suppose there exist mod-$q$-cyclic subgroups $H_1, H_2 \leq G$ (prime number $q$ can be different for these subgroups) and a $p$-group $P \leq H_1 \cap H_2$ for some prime $p$. Assume $G = \langle H_1 \cup H_2 \rangle$ and $\dim T_x \Sigma^P = 0$ for some $x \in \Sigma^G$. Then $\Sigma^G$ cannot consist of exactly one point.
Proof. From the Smith theory we deduce, that $\Sigma^P$ is a $\mathbb{Z}_p$-homology sphere. Since $\dim T_x \Sigma^P = 0$, we conclude that it consists of two points. As $\Sigma^{H_1}$ and $\Sigma^{H_2}$ are contained in $\Sigma^P$, it follows that $\Sigma^{H_1}$ and $\Sigma^{H_2}$ both consist of at most two points. In particular the Euler characteristics of $\Sigma^{H_1}$ and $\Sigma^{H_2}$ are equal to their cardinalities.

Assume that $\Sigma^G = \text{pt}$. The Euler characteristics of $\Sigma^{H_1}$ and $\Sigma^{H_2}$ both are greater or equal 1. Moreover, from Lemma 2.1, we deduce $\chi(\Sigma^{H_1}), \chi(\Sigma^{H_2}) \in \{0, 2\}$. Since these characteristics are positive, we conclude that

$$\chi(\Sigma^{H_1}) = \chi(\Sigma^{H_2}) = 2.$$ 

This means $\Sigma^{H_1}, \Sigma^{H_2}$ and $\Sigma^P$ consist of two points, which in connection with $\Sigma^{H_1}, \Sigma^{H_2} \subseteq \Sigma^P$, gives

$$\Sigma^P = \Sigma^{H_1} = \Sigma^{H_2}.$$ 

However, since $\langle H_1 \cup H_2 \rangle = G$

$$\text{pt} = \Sigma^G = \Sigma^{H_1} \cap \Sigma^{H_2} = \Sigma^P,$$

which is a contradiction, for $\Sigma^P$ is a two-point set.

The assumptions of Theorem 2.1 are easy to check in GAP language.

3 Intersection number restriction

Following [5], suppose $M$ is a smooth compact manifold of dimension $m$ and $A, B$ are its smooth, compact, connected and oriented submanifolds of dimensions $a$ and $b$ respectively such that $m = a + b$. Moreover, assume $A$ and $B$ are transverse in $M$. Choose a point $x \in A \cap B$. Since $A$ and $B$ are transverse and of complementary dimensions in $M$, we can look at the tangent space to $M$ at the point $x$ in two ways:

- as $T_x M$ with basis induced from the orientation of $M$,
- as $T_x A \oplus T_x B$ with basis given by the bases of $T_x A$ and $T_x B$ induced by the orientations of $A$ and $B$.

Denote by $\eta_x$ the sign of change-of-base matrix from $T_x M$ to $T_x A \oplus T_x B$. Having all this, we can introduce the following definition.

Definition 3.1. [5] The intersection number of $A$ and $B$ in $M$ is the value

$$A \cdot B = \sum_{p \in A \cap B} \eta_x$$
Now, consider the fundamental classes \([A] \in H_a(A)\) and \([B] \in H_b(B)\). They induce elements \((i_A)_*[A] \in H_a(M)\) and \((i_B)_*[B] \in H_b(M)\), where \(i_A : A \subset M\) and \(i_B : B \subset M\) are inclusions. The intersection number reveals a very interesting algebraic property which we shall use later.

**Theorem 3.1.** \([5]\) For \(A, B\) and \(M\) are as above, we have

\[
A \cdot B = \langle \alpha \cup \beta, [M] \rangle,
\]

where \(\alpha, \beta\) are Poincare duals to \((i_A)_*[A]\) and \((i_B)_*[B]\) respectively, \([M]\) is the fundamental class of \(M\) and \(\langle , \rangle\) is the Kronecker pairing.

We will focus now on ensuring the conditions which allow us to define the intersection number for our particular purposes.

**Lemma 3.1.** Assume a finite group \(G\) acts smoothly on a smooth manifold \(M\) with \(M^G\) connected. Suppose there exist subgroups \(H_1, H_2 \leq G\) and \(H \leq H_1 \cap H_2\) such that the following conditions hold:

1. \(\langle H_1 \cup H_2 \rangle = G\),

2. \[
\dim C(H_1) + \dim C(H_2) - \dim M^G = \dim C(H),
\]

where, for a given subgroup \(K \leq G\), \(C(K)\) stands for the connected component of \(M^K\) containing \(M^G\).

Then \(M^{H_1}\) and \(M^{H_2}\) are transverse in \(M^H\).

**Proof.** Pick \(x \in M^{H_1} \cap M^{H_2}\) \((H_1 \cup H_2) = G\) \(M^G\). It is enough to show that

\[
(3.1) \quad \dim T_x M^{H_1} + \dim T_x M^{H_2} - \dim (T_x M^{H_1} \cap T_x M^{H_2}) = \dim T_x M^H
\]

From the dimension assumption we have

\[
\dim T_x M^{H_1} + \dim T_x M^{H_2} - \dim T_x (M^{H_1} \cap M^{H_2}) = \dim T_x M^H
\]

If we prove that

\[
(3.2) \quad T_x (M^{H_1} \cap M^{H_2}) = T_x M^{H_1} \cap T_x M^{H_2}
\]

we get \((3.1)\).

From the Slice Theorem there exists an open neighborhood \(U\) of \(x\) and a \(G\)-diffeomorphism \(f : U \rightarrow T_x M\).
Let $K \leq G$. We show that
\begin{equation}
(3.3) \quad f(U^K) = T_x M^K
\end{equation}
Since $f$, as a $G$-diffeomorphism, preserves fixed point sets, we have $f(U^K) = (T_x M)^K$. As the action of $K$ on $T_x M$ is linear, it follows that $(T_x M)^K$ is a linear subspace of $T_x M$ and therefore it is connected. Hence $U^K$ is connected. One can consider then the dimensions of $U^K$ and $(T_x M)^K$. The former if equal to $\dim C(K)$. As $f$ is a $G$-diffeomorphism, it preserves the dimension, hence
\[\dim(T_x M)^K = \dim f(U^K) = \dim U^K = \dim C(K).\]
On the other hand, $\dim T_x M^K = \dim C(K)$. Therefore we see that $T_x M^K$ and $(T_x M)^K$ are linear subspaces of $T_x M$ of the same dimension. So, to prove $T_x M^K = (T_x M)^K$, it suffices to show $T_x M^K \subseteq (T_x M)^K$.

Let $[\gamma] \in T_x M^K$, where $\gamma : I \rightarrow M^K$ is a curve in $M^K$. Pick $k \in K$. We have for $t \in I$
\[k\gamma(t) = \gamma(t)\]
since $\gamma$ is invariant under the action of $K$. Hence $k\gamma = \gamma$ and therefore $k[\gamma] = [k\gamma] = [\gamma]$. We conclude then that $[\gamma] \in (T_x M)^K$ and $T_x M^K \subseteq (T_x M)^K$ and, as a consequence, $T_x M^K = (T_x M)^K$, and, indeed
\[f(U^K) = T_x M^K.\]
Using (3.3) and the fact that $f$ is injective, we have
\begin{align*}
T_x M^G &= f(U^G) = f(M^G \cap U) = f(M^{H_1} \cap M^{H_2} \cap U) \\
&= f((M^{H_1} \cap U) \cap (M^{H_2} \cap U)) = f(U^{H_1} \cap U^{H_2}) \\
&= f(U^{H_1}) \cap f(U^{H_2}) = T_x M^{H_1} \cap T_x M^{H_2}.
\end{align*}
On the other hand
\[T_x M^G = T_x (M^{H_1} \cap M^{H_2}),\]
which completes the proof of (3.2).

Let us care about the orientability assumption.

**Lemma 3.2.** Any $\mathbb{Z}_p$-homology sphere is orientable.

**Proof.** Suppose $\Sigma$ is a $\mathbb{Z}_p$-homology sphere which is not orientable. This means $H_{n}(\Sigma) = 0$, where $n = \dim \Sigma$. Using the Universal Coefficient Theorem, we have for $k = 1, \ldots, n$ the following exact sequence
\[0 \rightarrow H_k(\Sigma) \otimes \mathbb{Z}_p \rightarrow H_k(\Sigma; \mathbb{Z}_p) \rightarrow \text{Tor}(H_{k-1}(\Sigma), \mathbb{Z}_p) \rightarrow 0\]
Since $H_k(\Sigma; \mathbb{Z}_p)$ vanish for $k = 1, \ldots, n - 1$, we have

\[ H_k(\Sigma) \otimes \mathbb{Z}_p = \text{Tor}(H_{k-1}(\Sigma), \mathbb{Z}_p) = 0 \]

and consequently

\[ H_k(\Sigma) \cong \mathbb{Z}_{q_k,1} \oplus \ldots \oplus \mathbb{Z}_{q_k,i_k} \]

where $\gcd(q_k,i,p) = 1$.

For $k = n$, we have $H_n(\Sigma) = 0$ by our assumption. Hence, the Universal Coefficient Theorem gives in this case the exactness of

\[ 0 \longrightarrow H_n(\Sigma; \mathbb{Z}_p) \longrightarrow \text{Tor}(H_{n-1}(\Sigma), \mathbb{Z}_p) \longrightarrow 0 \]

Since $H_n(\Sigma; \mathbb{Z}_p) \cong \mathbb{Z}_p$, we conclude that

\[ \text{Tor}(H_{n-1}(\Sigma), \mathbb{Z}_p) \cong \mathbb{Z}_p \]

On the other hand, by (3.4),

\[
\text{Tor}(H_{n-1}(\Sigma), \mathbb{Z}_p) \cong \text{Tor}(\mathbb{Z}_{q_{n-1},1} \oplus \ldots \oplus \mathbb{Z}_{q_{n-1},i_{n-1}}, \mathbb{Z}_p) \\
\cong \text{Tor}(\mathbb{Z}_{q_{n-1},1}, \mathbb{Z}_p) \oplus \ldots \oplus \text{Tor}(\mathbb{Z}_{q_{n-1},i_{n-1}}, \mathbb{Z}_p) = 0.
\]

The following lemma will play the key role in excluding one-fixed point actions for our reasoning.

Lemma 3.3. If $M$ is a smooth manifold with finite $H^k(M)$, then the Kronecker pairing,

\[ \langle , \rangle : H^k(M) \times H_k(M) \longrightarrow \mathbb{Z} \]

is trivial.

Proof. Suppose $\langle [\varphi], [\Sigma] \rangle = a \neq 0$ for some nonzero $\varphi \in C^k(M)$ and $\Sigma \in C_k(M)$. Since $|H^k(M)| < \infty$, so $0 < ||\varphi|| = d < \infty$. From linearity of $\langle , \rangle$ in the first coordinate, we get

\[ 0 = \langle 0, [\Sigma] \rangle = \langle d[\varphi], [\Sigma] \rangle = d\langle [\varphi], [\Sigma] \rangle = da \neq 0, \]

a contradiction. Hence $\langle , \rangle$ is trivial.

Lemma 3.4. If $\Sigma$ is a $\mathbb{Z}_p$-homological sphere with dimension $n$, then the Kronecker pairing,

\[ \langle , \rangle : H^K(\Sigma) \times H_k(\Sigma) \longrightarrow \mathbb{Z} \]

is trivial for $k = 1, \ldots, n - 1$.  

\[\boxed{\text{Lemma 3.3.}}\]
Proof. Let \( k \in \{1, \ldots, n - 1\} \). From the proof of Lemma 3.2 we conclude that \( H_{n-k}(\Sigma) \) has only the torsion part. By Lemma 3.2, \( \Sigma \) is orientable, hence \( H^k(\Sigma) \cong H_{n-k}(\Sigma) \). Therefore \( H^k(\Sigma) \) consists only of the torsion part as well. The Lemma’s assertion follows now directly from Lemma 3.3.

We are ready now to establish the intersection number strategy of excluding smooth effective one-fixed point group actions on spheres.

Theorem 3.2. Let a finite group \( G \) act smoothly on a smooth, compact, \( \mathbb{Z} \)-homology sphere \( \Sigma \) with \( \Sigma^G \) connected. Assume there exist subgroups \( H_1, H_2 \leq G \) with \( \langle H_1 \cup H_2 \rangle = G \). Suppose there exists a \( p \)-group \( P \leq H_1 \cap H_2 \) such that

\[
\dim C(H_1) + \dim C(H_2) = \dim \Sigma^P.
\]

and one of the following is true:

1. the orders of \( H_1 \) and \( H_2 \) are odd,
2. \( P \) is normal in \( H_1 \) and \( H_2 \) and the orders of quotient groups \( H_1/P \) and \( H_2/P \) are odd.

Then \( \Sigma^G \) is not a one-point set.

Proof. Assume \( \Sigma^G \) consists of one point. We conclude from Lemma 3.1 that \( \Sigma^{H_1} \) and \( \Sigma^{H_2} \) are transverse in \( \Sigma^P \). Now, we show that \( \Sigma^P \) is an orientable manifold. We know from Smith theory, that it is a \( \mathbb{Z}_p \)-homology sphere. Hence, by Lemma 3.2 we conclude that \( \Sigma^P \) is orientable. Having this, notice that \( \Sigma^{H_1} \) and \( \Sigma^{H_2} \) are orientable too. Indeed, if \( H_1 \) and \( H_2 \) are of odd order, this follows directly from [2][2.1 Theorem]. While the second case holds and the orders of the quotient groups \( H_1/P, H_2/P \) are odd, we again apply this theorem to state that \( \Sigma^{H_1} \) and \( \Sigma^{H_2} \) are orientable, since

\[
\Sigma^{H_i} = \left( \Sigma^P \right)^{H_i/P}.
\]

Therefore we have a well-defined intersection number of \( C(H_1) \) and \( C(H_2) \) in \( \Sigma^P \). Since \( \Sigma^P \) is a \( \mathbb{Z}_p \)-homology sphere, we derive that \( H^k(\Sigma^P; \mathbb{Z}_p) = 0 \) for \( k = 1, \ldots, \dim \Sigma^P - 1 \). Hence, from Theorem 3.1 we have

\[
C(H_1) \cdot C(H_2) = 0.
\]

On the other hand, by our supposition, \( |\Sigma^G| = 1 \), and therefore the intersection number of \( C(H_1) \) and \( C(H_2) \) with coefficients in \( \mathbb{Z}_p \) is either 1 or \( p - 1 \), a contradiction. \( \square \)
4 The Exclusion Algorithm

In this paragraph we present the exclusion algorithm combining the discussed strategies.

\textit{Input:}

\begin{itemize}
    \item group $G$,
    \item $n$ - the dimension of a sphere on which we want to exclude a one-fixed point action of $G$
\end{itemize}

Throughout, we assume that the postulated fixed point is $x \in S^n$. In order to apply the exclusion algorithm, one should perform the following steps.

1. \textit{Determine all faithful real characters of $G$ of dimension $n$}

   Since only faithful $\mathbb{R}G$-modules of dimension $n$ can occur as the $\mathbb{R}G$-module structure of $T_xS^n$ (Theorem \textbf{A3}), we can restrict to faithful real characters of $G$ of dimension $n$. Moreover we can rule out the trivial character since we have an isolated fixed point. We use a combinatoric approach to generate these characters. First, prepare the partitions of $n$ into summands being the dimensions of real nontrivial irreducibles. Next, for any partition

   
   \[ n = a_1 d_1 + \ldots + a_m d_m, \]

   $d_1 < \ldots < d_m$, we arrange all choices of real irreducibles $X_{i,1}, \ldots, X_{i,a_i}$ for $i = 1, \ldots, m$ such that

   \[ X_{i,j}(1) = d_i \]

   for $j = 1, \ldots, a_i$. Now, it only remains to verify whether the obtained character, \( \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq a_i} X_{i,j} \), is faithful. For this purpose we use Lemma \textbf{A1}.

2. \textit{Determine good subgroups triples}

   In this step we look for subgroups of $G$, $H_1, H_2$ and $P \leq H_1 \cap H_2$ satisfying the conditions of either of the strategies - Theorem \textbf{2.1} or Theorem \textbf{3.2} not concerning the dimension (they shall be checked later since these conditions depend on the module - character):

   \begin{itemize}
       \item for discrete submodules strategy:
           \[ \langle H_1, H_2 \rangle = G, \]
   \end{itemize}
- $H_1$ and $H_2$ are mod-$p$-cyclic (possibly for different $p$’s)
- $P$ is a $p$ group for some prime $p$,

- for intersection number strategy:
  - $\langle H_1, H_2 \rangle = G$,
  - $P$ is a $p$ group for some prime $p$,
  - one of the following is true:
    * the orders of $H_1$ and $H_2$ are odd,
    * $P$ is normal in both $H_1$ and $H_2$ and the orders of the quotient groups $H_i/P$ are odd for $i = 1, 2$.

We collect the triples of such subgroups $(H_1, H_2, P)$ into two parts corresponding to a given strategy.

Note: We can limit ourselves to conjugacy classes of subgroups since, by Corollary [A1], the choice of their representatives doesn’t influence fixed point dimensions. We must be careful however, whether there exists representatives of these classes satisfying necessary conditions. Nevertheless, the restriction to conjugacy classes increases the efficiency.

3. Check the dimension conditions for characters

For any faithful character generated in the first step, we consider every good subgroup triple for either the strategy. Once we fix such character $X$ and triple $(H_1, H_2, P)$, we check the dimension conditions. If they are satisfied for at least one of the strategies, we conclude that are an obstacle to one-fixed point action. If these obstacles are found for any character to be considered, we can exclude the one-fixed point action.

If the conditions of Theorem 2.1 or Theorem 3.2 are satisfied, this algorithm provides the proof that there does not exist the postulated action. The algorithm has been implemented in GAP language.

5 Results

In the introduction, we presented particular Oliver groups and dimensions for which the exclusion strategy ruled out one-fixed point actions on spheres of these dimensions. Those results were obtained by GAP computations. The strategy was tested for all Oliver groups of order up to 216, as well as for $A_5 \times C_5$ and $A_5 \times C_7$. 
Let us see how the strategy works for the particular example of $G = A_5 \times C_3$ for $n = 6, 7, 8$.

$G$ can be presented as the subgroup of $S_8$ generated by $(1\ 5\ 2\ 4\ 3)(6\ 8\ 7)$ and $(1\ 4\ 2\ 5\ 3)(6\ 7\ 8)$. $G$ has the following conjugacy classes:

- $c_1 = (\text{id})$,
- $c_2 = ( (2\ 3\ 4\ 5))$ which elements are of order 2,
- $c_{3,1} = ( (6\ 7\ 8))$, $c_{3,2} = ( (6\ 8\ 7))$, $c_{3,3} = ( (3\ 4\ 5))$, $c_{3,4} = ( (3\ 4\ 5)(6\ 7\ 8))$, $c_{3,5} = ( (3\ 4\ 5)(6\ 8\ 7))$ which elements are of order 3,
- $c_{5,1} = ( (1\ 2\ 3\ 4\ 5))$, $c_{5,2} = ( (1\ 2\ 3\ 5\ 4))$ which elements are of order 5,
- $c_{6,1} = ( (2\ 3\ 4\ 5)(6\ 7\ 8))$, $c_{6,2} = ( (2\ 3\ 4\ 5)(6\ 8\ 7))$ which elements are of order 6,
- $c_{15,1} = ( (1\ 2\ 3\ 4\ 5)(6\ 7\ 8))$, $c_{15,2} = ( (1\ 2\ 3\ 4\ 5)(6\ 8\ 7))$, $c_{15,3} = ( (1\ 2\ 3\ 5\ 4)(6\ 7\ 8))$, $c_{15,4} = ( (1\ 2\ 3\ 5\ 4)(6\ 8\ 7))$ which elements are of order 15.

We compute the real nontrivial irreducible characters of $G$ (we put $\xi_5 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$),

| $X_i$  | $c_1$ | $c_{3,1}$ | $c_{3,2}$ | $c_{3,3}$ | $c_{3,4}$ | $c_{3,5}$ | $c_{6,1}$ | $c_{6,2}$ |
|-------|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $X_2$ | 2     | -1        | -1        | 2         | -1        | 2         | -1        | -1        |
| $X_{3,1}$ | 3     | 3         | 3         | 0         | 0         | -1        | -1        | -1        |
| $X_{3,2}$ | 3     | 3         | 3         | 0         | 0         | -1        | -1        | -1        |
| $X_{6,1}$ | 6     | -3        | -3        | 0         | 0         | -2        | 1         | 1         |
| $X_{6,2}$ | 6     | -3        | -3        | 0         | 0         | -2        | 1         | 1         |
| $X_4$ | 4     | 4         | 4         | 1         | 1         | 0         | 0         | 0         |
| $X_8$ | 8     | -4        | -4        | 2         | -1        | -1        | 0         | 0         |
| $X_5$ | 5     | 5         | 5         | -1        | -1        | -1        | 1         | 1         |
| $X_{10}$ | 10    | -5        | -5        | -2        | 1         | 1         | 2         | -1        |
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
 & $c_{5,1}$ & $c_{15,1}$ & $c_{15,2}$ & $c_{5,2}$ & $c_{15,3}$ & $c_{15,4}$ \\
\hline
$X_2$ & 2 & -1 & -1 & 2 & -1 & -1 \\
$X_{3,1}$ & $-\xi_5 - \xi_5^4$ & $-\xi_5 - \xi_5^4$ & $-\xi_5 - \xi_5^4$ & $-\xi_5^2 - \xi_5^3$ & $-\xi_5^2 - \xi_5^3$ & $-\xi_5^2 - \xi_5^3$ \\
$X_{3,2}$ & $-\xi_5 - \xi_5^4$ & $-\xi_5^2 - \xi_5^3$ & $-\xi_5^2 - \xi_5^3$ & $-\xi_5 - \xi_5^4$ & $-\xi_5 - \xi_5^4$ & $-\xi_5 - \xi_5^4$ \\
$X_{6,1}$ & $-2\xi_5 - 2\xi_5^4$ & $\xi_5 + \xi_5^4$ & $\xi_5 + \xi_5^4$ & $-2\xi_5^2 - 2\xi_5^3$ & $\xi_5^2 + \xi_5^3$ & $\xi_5^2 + \xi_5^3$ \\
$X_{6,2}$ & $-2\xi_5^2 - 2\xi_5^3$ & $\xi_5^2 + \xi_5^3$ & $\xi_5^2 + \xi_5^3$ & $-2\xi_5 - 2\xi_5^4$ & $\xi_5 + \xi_5^4$ & $\xi_5 + \xi_5^4$ \\
$X_1$ & -1 & -1 & -1 & -1 & -1 & -1 \\
$X_5$ & 0 & 0 & 0 & 0 & 0 & 0 \\
$X_{10}$ & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\end{table}

$G$ has 21 conjugacy classes of subgroups:

- $d_1$ with representative $\{id\}$ isomorphic to the trivial group,
- $d_2$ with representative $\langle (2 \ 5) (3 \ 4) \rangle$ isomorphic to $C_2$,
- $d_{3,1}, d_{3,2}, d_{3,3}$ with representatives $\langle (6 \ 8 \ 7) \rangle, \langle (2 \ 4 \ 5) \rangle$ and $\langle (2 \ 4 \ 5) (6 \ 8 \ 7) \rangle$ respectively, isomorphic to $C_3$,
- $d_4$ with representative $\langle (2 \ 4) (3 \ 5) \rangle, \langle (2 \ 4) (3 \ 4) \rangle$ isomorphic to $C_2 \times C_2$,
- $d_5$ with representative $\langle (1 \ 3 \ 4 \ 5 \ 2) \rangle$ isomorphic to $C_5$,
- $d_{6,1}$ with representative $\langle (1 \ 3) (2 \ 5), (2 \ 5 \ 4) \rangle$ isomorphic to $D_6$,
- $d_{6,2}$ with representative $\langle (2 \ 5) (3 \ 4), (6 \ 8 \ 7) \rangle$ isomorphic to $C_6$,
- $d_9$ with representative $\langle (2 \ 4 \ 5), (6 \ 8 \ 7) \rangle$ isomorphic to $C_5 \times C_3$,
- $d_{10}$ with representative $\langle (2 \ 3) (4 \ 5), (1 \ 2 \ 5 \ 4 \ 3) \rangle$ isomorphic to $D_{10}$,
- $d_{12,1}, d_{12,2}, d_{12,3}$ with representatives $\langle (2 \ 4 \ 5) (2 \ 3) (4 \ 5), (2 \ 5) (3 \ 4) \rangle$, $\langle (2 \ 4 \ 5) (6 \ 8 \ 7), (2 \ 3) (4 \ 5), (2 \ 5) (3 \ 4) \rangle$ and $\langle (2 \ 4 \ 5) (6 \ 7 \ 8), (2 \ 3) (4 \ 5), (2 \ 5) (3 \ 4) \rangle$ respectively, isomorphic to $A_4$,
- $d_{12,4}$ with representative $\langle (2 \ 4) (3 \ 5), (2 \ 5) (3 \ 4), (2 \ 4 \ 5) (6 \ 8 \ 7) \rangle$ isomorphic to $C_6 \times C_2$,
- $d_{15}$ with representative $\langle (1 \ 3 \ 4 \ 5 \ 2), (6 \ 8 \ 7) \rangle$ isomorphic to $C_{15}$,
• $d_{18}$ with representative $\langle (1 \ 3 \ 4 \ 5) , (2 \ 5 \ 4) , (6 \ 8 \ 7) \rangle$ isomorphic to $D_6 \times C_3$,

• $d_{30}$ with representative $\langle (1 \ 5 \ 3 \ 4 \ 6 \ 8 \ 7) , (1 \ 4 \ 2 \ 5 \ 6 \ 8 \ 7) \rangle$ isomorphic to $D_{10} \times C_3$,

• $d_{36}$ with representative $\langle (2 \ 4 \ 5) , (2 \ 4) (3 \ 5) , (6 \ 8 \ 7) \rangle$ isomorphic to $A_4 \times C_3$,

• $d_{60}$ with representative $\langle (1 \ 3 \ 4 \ 5 \ 2) , (1 \ 4 \ 3 \ 5 \ 2) \rangle$ isomorphic to $A_5$,

• $d_{180}$ with the only representative isomorphic to $G$.

The following table shows the fixed point dimensions of the characters for all subgroups of $G$:

|       | $d_1$ | $d_2$ | $d_{3,1}$ | $d_{3,2}$ | $d_{3,3}$ | $d_4$ | $d_5$ | $d_{6,1}$ | $d_{6,2}$ | $d_9$ | $d_{10}$ |
|-------|-------|-------|-----------|-----------|-----------|-------|-------|-----------|-----------|-------|---------|
| $X_2$ | 2     | 2     | 0         | 2         | 2         | 2     | 2     | 0         | 0         | 0     | 2       |
| $X_{3,1}$ | 3     | 1     | 3         | 1         | 1         | 0     | 1     | 0         | 1         | 1     | 0       |
| $X_{3,2}$ | 3     | 1     | 3         | 1         | 1         | 0     | 1     | 0         | 1         | 1     | 0       |
| $X_{6,1}$ | 6     | 2     | 0         | 2         | 2         | 0     | 2     | 0         | 0         | 0     | 0       |
| $X_{6,2}$ | 6     | 2     | 0         | 2         | 2         | 0     | 2     | 0         | 0         | 0     | 0       |
| $X_4$ | 4     | 2     | 4         | 2         | 2         | 1     | 0     | 1         | 2         | 2     | 0       |
| $X_5$ | 8     | 4     | 0         | 4         | 2         | 2     | 0     | 2         | 0         | 0     | 0       |
| $X_6$ | 5     | 3     | 5         | 1         | 1         | 2     | 1     | 1         | 3         | 1     | 1       |
| $X_{10}$ | 10    | 6     | 0         | 2         | 4         | 4     | 2     | 2         | 0         | 0     | 2       |

|       | $d_{12,1}$ | $d_{12,4}$ | $d_{12,2}$ | $d_{12,3}$ | $d_{15}$ | $d_{18}$ | $d_{30}$ | $d_{36}$ | $d_{60}$ | $d_{180}$ |
|-------|------------|------------|------------|------------|----------|----------|----------|----------|----------|-----------|
| $X_2$ | 2          | 0          | 0          | 0          | 0        | 0        | 0        | 2        | 0        | 0         |
| $X_{3,1}$ | 0          | 0          | 0          | 0          | 1        | 0        | 0        | 0        | 0        | 0         |
| $X_{3,2}$ | 0          | 0          | 0          | 0          | 1        | 0        | 0        | 0        | 0        | 0         |
| $X_{6,1}$ | 0          | 0          | 0          | 0          | 0        | 0        | 0        | 0        | 0        | 0         |
| $X_{6,2}$ | 0          | 0          | 0          | 0          | 0        | 0        | 0        | 0        | 0        | 0         |
| $X_4$ | 1          | 1          | 1          | 1          | 0        | 1        | 0        | 1        | 0        | 0         |
| $X_5$ | 2          | 0          | 0          | 0          | 0        | 0        | 0        | 0        | 0        | 0         |
| $X_{10}$ | 0          | 2          | 0          | 0          | 1        | 1        | 1        | 0        | 0        | 0         |

Now, we analyze all the possible characters which can occur as characters of tangent module to $S^n$ at the fixed point and exclude the one-fixed point actions for them.

• $n = 6$; we have the following characters:
\( X = X_{6, i} \) for \( i = 1, 2 \); in this case take

\[
\begin{align*}
H_1 &= \langle (2 \ 5 \ (3 \ 4), (6 \ 8 \ 7) \rangle \in d_{6,2}, \\
H_2 &= \langle (1 \ 5 \ 4), (6 \ 8 \ 7) \rangle \in d_9, \\
P &= \langle (6 \ 8 \ 7) \rangle \in d_{4,1}
\end{align*}
\]

We see that \( P \) is contained in \( H_1 \cap H_2 \). From the fixed dimension table, we see that \( \dim X^P = 0 \) and we exclude the effective one-fixed point action by the discrete submodules restriction.

\( X = X_2 + X_4 \); we take

\[
\begin{align*}
H_1 &= \langle (6 \ 8 \ 7) \rangle \in d_{3,1}, \\
H_2 &= \langle (1 \ 3 \ 4 \ 5 \ 2), (1 \ 4 \ 3 \ 5 \ 2) \rangle \in d_{60}, \\
P &= \{id\} \in d_1
\end{align*}
\]

and apply the intersection number restriction, since \( \dim X^{H_1} + \dim X^{H_2} = 4 + 2 = 6 = \dim X^P \).

- \( n = 7 \); The possible characters are:

  - \( X = 2X_2 + X_3, i = 1, 2 \); the action is ruled out by the intersection number strategy for

    \[
    \begin{align*}
    H_1 &= \langle (6 \ 8 \ 7) \rangle \in d_{3,1}, \\
    H_2 &= \langle (1 \ 3 \ 4 \ 5 \ 2), (1 \ 4 \ 3 \ 5 \ 2) \rangle \in d_{60}, \\
    P &= \{id\} \in d_1.
    \end{align*}
    \]

  - \( X = X_2 + X_5 \); the subgroups and the strategy are the same as in the previous case.

- \( n = 8 \); The possible characters are:

  - \( X \in \{X_2 + 2X_{3,1}, X_2 + 2X_{3,2}, X_2 + X_{3,1} + X_{3,2}, 2X_2 + X_4\} \); the intersection number strategy for the same subgroups as in case \( n = 7 \).

  - \( X \in \{X_2 + X_{6,1}, X_2 + X_{6,2}, X_8\} \); the discrete submodules strategy for the same subgroups as in two first characters for \( n = 6 \).
6 Conclusions

The exclusion algorithm works well for the case where there are relatively few faithful $\mathbb{R}G$-modules of a given dimension. It is often the case that the big amount of such modules generates situations which do not fit for either of the strategies. However, for low dimensions, such as from 6 up to 10, there are few faithful modules and in many cases, we are able to exclude smooth effective one-fixed point actions. Even this is very interesting, since for the current state of knowledge we don’t know whether a particular Oliver group can act on some sphere of low dimensions varying from 6 to 10. If one looks closer for the case of $A_5 \times C_p$ for $p = 3, 5, 7$ one can observe that we could exclude quite a lot of dimensions. Since the dimension 9 was not ruled out, one could try to construct the smooth effective one fixed point action on $S^9$ for these groups. If it was the case, we would know the minimal dimension of a sphere on which $A_5 \times C_p$ can act smoothly and effectively with one fixed point for $p = 3, 5, 7$. It would be an essential step concerning the following more general open question.

**Question.** For a given Oliver group $G$, what is the least $n$ for which $G$ can act smoothly and effectively with one fixed point on $S^n$?

**Appendix**

Let a group $G$ act smoothly on a smooth manifold $M$ with fixed point $x \in M$. Then the action of $G$ induces its linear action on the tangent space $T_xM$. Therefore $T_xM$ has the structure of $\mathbb{R}G$-module.

**Lemma A1.** [3][13.11 Theorem] An $\mathbb{R}G$-module $V$ is faithful iff the only element $g \in G$ with $\chi_V(g) = \chi_V(1)$ is $g = 1$.

Any $\mathbb{R}G$-module can be expressed as a direct sum of irreducible $\mathbb{R}G$-modules, which can be extracted from irreducible $\mathbb{C}G$-modules.

**Definition A1.** Let $V$ be a complex $\mathbb{C}G$-module. A **Frobenius-Schur indicator** of $V$ is the value

$$\iota(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2).$$

The precise way of reading off the characters of real irreducible representations from complex ones gives the following theorem.
**Theorem A1.** \[\text{\cite{3}}\] If $U$ is an irreducible $\mathbb{R}G$-module, then its character has one of the following forms:

1. $\chi_U$, and $U$ is an irreducible $\mathbb{C}G$-module,

2. $2 \Re \chi_W$, for some irreducible $\mathbb{C}G$-module $W$.

Moreover, for any irreducible $\mathbb{C}G$ module $V$, one of the following is true:

1. if $\iota(V) = 1$, then $V$ is an irreducible real representation,

2. if $\iota(V) \neq 1$, then $2 \Re \chi_V$ is a character of some irreducible real representation.

The above theorem allows us to construct all the real irreducible characters from the table of complex irreducible characters.

Since any $\mathbb{R}G$-module $U$, by its associated representation, induces a $\mathbb{C}G$-module $V$, the dimension (real) of $U$ is equal to the complex dimension of $V$. Thus we can obtain the dimensions of the fixed point sets using the following result.

**Theorem A2.** \[\text{\cite{4}}\] Let $V$ be a $\mathbb{C}G$-module and $H$ be a subgroup of $G$. Then the dimension of the fixed point set of the action of $H$ on $V$ is given by the formula

\[\dim V^H = \frac{1}{|H|} \sum_{h \in H} \chi_V(h).\]

**Corollary A1.** Let $V$ be a $\mathbb{C}G$-module and $H_1, H_2$ be conjugate subgroups of $G$. Then

\[\dim V^{H_1} = \dim V^{H_2}.\]

**Proof.** Suppose $H_2 = gH_1g^{-1}$ for some $g \in G$. Hence for any $h_2 \in H_2$ there exists a unique $h_1 \in H_1$ such that $h_2 = gh_1g^{-1}$. Using the formula from Theorem A2 we have

\[\dim V^{H_2} = \frac{1}{|H_2|} \sum_{h_2 \in H_2} \chi_V(h_2) = \frac{1}{|H_1|} \sum_{h_1 \in H_1} \chi_V(gh_1g^{-1}) = \sum_{h_1 \in H_1} \chi_V(h_1) = \dim V^{H_1}.\]

\[\square\]

We recall now a very useful property:

**Theorem A3.** If a group $G$ acts smoothly and effectively on a smooth closed manifold $M$ with a fixed point $x \in M$, then $T_xM$ is a faithful $\mathbb{R}G$-module.
Proof. One uses the fact that there exists a $G$-invariant Riemannian metric on $M$ [9][Theorem 2.].

Definition A2. [13] An Oliver group is a group $G$ not of prime power order for which there does not exist a sequence of normal subgroups, $P \trianglelefteq H \trianglelefteq G$ such that $P$ is a $p$-group, $G/H$ is a $q$-group for (possibly the same) primes $p$ and $q$ and the group $H/P$ is cyclic.

Theorem A4. [13][11][10] A group $G$ can act smoothly and effectively with one fixed point on some sphere if and only if $G$ is an Oliver group.

Acknowledgements

The authors would like to thank Prof. Krzysztof Pawałowski for the support and remarks which substantially improved the presentation of this paper. We are very thankful to all the participants of the algebraic topology seminar held at Adam Mickiewicz University. They listened to our talks concerning this paper and gave us essential remarks.

References

[1] Agnieszka Borowiecka. SL(2, 5) has no smooth effective one-fixed-point action on $S^8$. *Bull. Pol. Acad. Sci. Math.*, 64(1):85–94, 2016.

[2] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[3] N. P. Buchdahl, Sławomir Kwasik, and Reinhard Schultz. One fixed point actions on low-dimensional spheres. *Invent. Math.*, 102(3):633–662, 1990.

[4] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original.

[5] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[6] Mikio Furuta. A remark on a fixed point of finite group action on $S^4$. *Topology*, 28(1):35–38, 1989.
[7] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.8.10*, 2018.

[8] Gordon James and Martin Liebeck. *Representations and characters of groups*. Cambridge University Press, New York, second edition, 2001.

[9] J.-L. Koszul. *Lectures on groups of transformations*. Notes by R. R. Simha and R. Sridharan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 32. Tata Institute of Fundamental Research, Bombay, 1965.

[10] Erkki Laitinen and Masaharu Morimoto. Finite groups with smooth one fixed point actions on spheres. *Forum Math.*, 10(4):479–520, 1998.

[11] Erkki Laitinen, Masaharu Morimoto, and Krzysztof Pawałowski. Deleting-inserting theorem for smooth actions of finite nonsolvable groups on spheres. *Comment. Math. Helv.*, 70(1):10–38, 1995.

[12] Masaharu Morimoto. On one fixed point actions on spheres. *Proc. Japan Acad. Ser. A Math. Sci.*, 63(4):95–97, 1987.

[13] Bob Oliver. Fixed point sets and tangent bundles of actions on disks and Euclidean spaces. *Topology*, 35(3):583–615, 1996.

[14] Ted Petrie. One fixed point actions on spheres. I, II. *Adv. in Math.*, 46(1):3–14, 15–70, 1982.

[15] Elliott Stein. Surgery on products with finite fundamental group. *Topology*, 16(4):473–493, 1977.