Synthetic properties of locally compact groups: preservation and transference

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Abstract
Using techniques from TRO equivalence of masa bimodules we prove various transference results: We show that when $\alpha$ is a group homomorphism which pushes forward the Haar measure of $G$ to a measure absolutely continuous with respect to the Haar measure on $H$, then $(\alpha \times \alpha)^{-1}$ preserves sets of compact operator synthesis, and conversely when $\alpha$ is onto. We also prove similar preservation results for operator Ditkin sets and operator M-sets, obtaining preservation results for M-sets as corollaries. Some of these results extend or complement existing results of Ludwig, Shulman, Todorov and Turowska.

Keywords Group homomorphism · Haar measure · Fourier algebras · Spectral synthesis · Operator synthesis · MASA bimodule

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1 Introduction

In this paper we study preservation of synthetic sets, $M$-sets and Ditkin sets by group homomorphisms between locally compact groups both in the classical and in the operator sense.

In the early seventies, Arveson discovered important connections between spectral synthesis and invariant subspace theory and established precise links between notions in harmonic analysis and in operator algebra theory [3].

The failure of spectral synthesis for a locally compact group $G$ can be formulated as the existence of distinct closed ideals of the Fourier algebra of $G$ having the same support.

It was Arveson in his seminal paper [3] who used this phenomenon to produce the first example of two distinct weak-* closed subalgebras on an $L^2(X, \mu)$ space containing the multiplication algebras of $L^\infty(X, \mu)$ which have the same ‘support’, a closed subspace of $X \times X$.

This was later formalised, in the commutative case, by J. Froelich [14] and, in the general locally compact case, by J. Ludwig and L. Turowska [17].

The theory initiated by Arveson has been significantly developed by many authors [2, 12, 14, 17, 19, 21, 24, 26]. Our work is a contribution to this circle of ideas.

Let $G$ and $H$ be locally compact second countable groups with Haar measures $\mu$ and $\nu$ respectively and let $\theta : G \to H$ be a continuous homomorphism. To study preservation properties with respect to $\theta$, we shall use a method based on TRO-equivalence. This notion was introduced in [9] by G.K. Eleftherakis, and used in [10] for the study of masa-bimodules and in [11] for the study of problems of operator synthesis. In Sects. 2, 4 and 6 we prove preservation results for operator synthetic sets, operator Ditkin sets and operator $M$-sets. The main idea of our method is the following: to prove the preservation of a property from $H$ to $G$, we first express the property in terms of appropriate masa-bimodules. Then we show that the masa-bimodules associated to $H$ are TRO-equivalent to the ones associated to $G$, and hence we obtain the preservation of the property we are interested in.

In fact we use this method for more general maps between standard measure spaces. The concept of reduced spectral synthesis as well as its operator theoretic analogue are introduced and studied in [22]. In particular, an inverse image theorem for sets of compact operator synthesis follows from [22, Theorem 4.7] under the assumption that $\theta$ has a special form.

In Sect. 2, we prove Theorem 2.4, which implies an inverse image theorem for sets of compact operator synthesis for general $\theta$ assuming that $\theta_* (\mu)$ is absolutely continuous with respect to $\nu$.

We also show in sect. 4 that if $\theta$ is onto then an $\omega$-closed set $\kappa \subseteq H \times H$ (see below for these notions) is a set of (compact) operator synthesis if and only if $(\theta \times \theta)^{-1}(\kappa)$ is a set of (compact) operator synthesis.

Note that it follows from [19, Theorem 4.7] that if $\theta_* (\mu)$ is absolutely continuous with respect to $\nu$ and $\kappa \subseteq H \times H$ is an $\omega$-closed set of operator synthesis, then $(\theta \times \theta)^{-1}(\kappa)$ is a set of operator synthesis.

In Sect. 5 we prove a preservation result for operator Ditkin sets under the assumption that $\theta$ has open image and compact kernel.
In Sect. 6 we study preservation and transference properties of $M$-sets. These were introduced for general locally compact groups by Bożejko in [6]. Shulman, Todorov and Turowska in [21] introduced the notion of $M_1$ sets in locally compact groups and studied transference and preservation properties of $M$-sets and $M_1$-sets and their operator analogues, which were introduced in [20]. Our results in Sect. 6 complement and improve some of the results of [21].

We show that if $\theta$ is onto, an $\omega$-closed set $\kappa \subseteq H \times H$ is an operator $M$-set (resp. $M_1$-set) if and only if $(\theta \times \theta)^{-1}(\kappa)$ is an operator $M$-set (resp. $M_1$-set). As a corollary we obtain, for a closed set $E \subseteq H$, that $\theta^{-1}(E)$ is an $M$-set (resp. $M_1$-set) if and only if $E$ is an $M$-set (resp. $M_1$-set).

Related results were obtained in [21] under the assumption that $\theta_\ast(\mu)$ has a Radon-Nikodym derivative with respect to $\mu$ which is $\mu$-a.e. finite, which in turn is equivalent to the compactness of $\ker \theta$ (see Corollary 3.6).

We will need some preliminaries. For more details, see for example [2] or [22, Section 4].

Recall that a standard Borel measure space $(X, \mu)$ is a measure space such that $X$ is Borel isomorphic to a Borel subset of a Polish space and $\mu$ is a regular $\sigma$-finite Borel measure on $X$ [15, 12.5].

If $(X, \mu)$ and $(Y, \nu)$ are standard measure spaces, a subset $\kappa \subseteq X \times Y$ is called *marginally null* if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where $M \subseteq X$ and $N \subseteq Y$ are null. A subset $\kappa$ of $X \times Y$ is called *$\omega$-open* if it is marginally equivalent to (i.e. differs by a marginally null set from) the union of a countable set of Borel rectangles. The complements of $\omega$-open sets are called *$\omega$-closed*.

If $\theta : X \to Y$ is a Borel map and $\mu$ is a measure on $X$, we denote by $\theta_\ast(\mu)$ the measure on $Y$ defined by $\theta_\ast(\mu)(E) = \mu(\theta^{-1}(E))$ for every Borel set $E \subseteq Y$.

If $H_1$ and $H_2$ are Hilbert spaces, we write $\mathcal{B}(H_1, H_2)$ for the bounded linear operators from $H_1$ to $H_2$. When $H_1 = L^2(X, \mu)$ and $H_2 = L^2(Y, \nu)$, we will call a subspace $\mathcal{W} \subseteq \mathcal{B}(H_1, H_2)$ a *masa-bimodule* if $M_\theta T M_f \in \mathcal{W}$ for all $T \in \mathcal{W}$, $f \in L^\infty(X, \mu)$ and $g \in L^\infty(Y, \nu)$, where $M_\theta$ and $M_f$ denote the corresponding multiplication operators. The $w^*$-closed masa bimodule generated by a set $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$ will be denoted $\operatorname{Bim}(\mathcal{U})$.

We say that a measurable subset $\kappa \subseteq X \times Y$ *supports* an operator $T \in \mathcal{B}(H_1, H_2)$ (or that $T$ is *supported by* $\kappa$) if $P(F)T P(E) = 0$ whenever the rectangle $E \times F$ is marginally disjoint from $\kappa$, and write

$$M_{\max}(\kappa) = \{ T \in \mathcal{B}(H_1, H_2) : T \text{ is supported by } \kappa \}.$$  

For any subset $\mathcal{W} \subseteq \mathcal{B}(H_1, H_2)$, there exists a smallest (up to marginal equivalence) $\omega$-closed set $\operatorname{supp}(\mathcal{W}) \subseteq X \times Y$ which supports every operator $T \in \mathcal{W}$ [12].

By [3] and [19], for any $\omega$-closed set $\kappa$, there exists a weak* closed bimodule $M_{\min}(\kappa)$ such that, for every weak* closed masa bimodule $M \subseteq \mathcal{B}(H_1, H_2)$ with support marginally equivalent to $\kappa$ we have that

$$M_{\min}(\kappa) \subseteq M \subseteq M_{\max}(\kappa).$$
2 TRO-equivalence and preservation

The main results of this section are Theorems 2.2 and 2.3, where we use techniques from the TRO equivalence developed in [9]. We use Theorem 2.3 to extend an inverse image theorem for sets of compact operator synthesis due to Shulman, Todorov and Turowska [22, Theorem 4.7]. We will apply Theorem 2.2 to preservation results for group homomorphisms in Sects. 4 and 6.

We begin with a general Lemma which is perhaps of independent interest.

If $U$ is a subspace of operators acting on a Hilbert space we denote by $U \cap K$ its subspace of compact operators.

Let $H, K$ be Hilbert spaces and $\mathcal{M}$ a norm-closed linear subspace of $B(H, K)$. The space $\mathcal{M}$ is called an essential ternary ring of operators (TRO) if $\mathcal{M}\mathcal{M}^* \mathcal{M} \subseteq \mathcal{M}$ (TRO property) and $I_H \in [\mathcal{M}\mathcal{M}^*]^{w*}$ and $I_K \in [\mathcal{M}\mathcal{M}^*]^{w*}$ (essentiality).

Lemma 2.1 Let $H_i, K_i$ be Hilbert spaces and $\mathcal{M}_i \subseteq B(H_i, K_i), i = 1, 2$ be essential ternary rings of operators. Let $U$ be a $w^*$-closed subspace of $B(H_1, H_2)$. We set

$$F(U) = [\mathcal{M}_2 U \mathcal{M}_1^*]^{w*}$$

and

$$F_0(U \cap K) = [\mathcal{M}_2 (U \cap K) \mathcal{M}_1^*]\|\cdot\|.$$

Then

$$F_0(U \cap K) = F(U) \cap K.$$

Proof From [9, Proposition 2.11] it follows that

$$U = [\mathcal{M}_2^* F(U) \mathcal{M}_1^*]^{w*}$$

and hence $\mathcal{M}_2^* (F(U) \cap K) \mathcal{M}_1 \subseteq U \cap K$. Therefore

$$\mathcal{M}_2 \mathcal{M}_2^* (F(U) \cap K) \mathcal{M}_1 \mathcal{M}_1^* \subseteq \mathcal{M}_2 (U \cap K) \mathcal{M}_1^*.$$ (2.1)

Since the $\mathcal{M}_i$ are essential TRO’s, there exist [4, Corollary 8.1.24] nets of the form

$$x_\lambda = \sum_{i=1}^{s_\lambda} z_i^\lambda (z_i^\lambda)^*, \ z_i^\lambda \in \mathcal{M}_2, \ y_\lambda = \sum_{i=1}^{t_\lambda} w_i^\lambda (w_i^\lambda)^*, \ w_i^\lambda \in \mathcal{M}_1, \ \lambda \in \Lambda$$

such that $\|x_\lambda\| \leq 1, \|y_\lambda\| \leq 1$ for all $\lambda$ and

$$\text{SOT- lim } x_\lambda = I_{K_2}, \ \text{SOT- lim } y_\lambda = I_{K_1}.$$

If $x \in F(U) \cap K$ we have from (2.1)

$$x_\lambda x_{\lambda'} y_{\lambda'} \in [\mathcal{M}_2 (U \cap K) \mathcal{M}_1^*], \ \text{for all } \lambda, \lambda' \in \Lambda.$$
Since $x$ is a compact operator the iterated norm limits $\lim_\lambda (\lim_\lambda' x_\lambda x_\lambda') = x$ exist and we have

$$x \in \overline{[M_2(\mathcal{U} \cap \mathcal{K})]^{1\|\|}}.$$ 

We conclude that $F(\mathcal{U}) \cap \mathcal{K} \subseteq \overline{[M_2(\mathcal{U} \cap \mathcal{K})]^{1\|\|}}$ and since $M_2(\mathcal{U} \cap \mathcal{K})M_1^* \subseteq F(\mathcal{U}) \cap \mathcal{K}$ we obtain

$$F(\mathcal{U}) \cap \mathcal{K} = \overline{[M_2(\mathcal{U} \cap \mathcal{K})]^{1\|\|}} = F_0(\mathcal{U} \cap \mathcal{K}).$$

$\square$

For the rest of this section $(X_i, \mu_i), (Y_i, \nu_i), i = 1, 2$ will denote standard Borel measure spaces and $\phi_i : X_i \to Y_i (i = 1, 2)$ will denote measurable maps such that $(\phi_i)_*(\mu_i) \ll \nu_i$.

We define

$$\tilde{\phi}_i : L^\infty(Y_i) \to L^\infty(X_i), \quad \tilde{\phi}_i(f) = f \circ \phi_i, \quad i = 1, 2$$

and the TRO’s

$$\mathcal{N}_i = \{ T \in B(L^2(Y_i), L^2(X_i)) : TP_i(E) = Q_i(\phi_i^{-1}(E))T, \text{ for all } E \subseteq Y_i \text{ Borel} \}$$

(2.2)

(here $P_i(E)$ (resp. $Q_i(\phi_i^{-1}(E))$) is the projection onto $L^2(E)$ (resp. $L^2(\phi_i^{-1}(E))$)).

Fix Borel sets $E_i \subseteq Y_i, i = 1, 2$ such that $\ker \tilde{\phi}_i = L^\infty(Y_i \setminus E_i)$. Then the maps

$$L^\infty(E_i) \to L^\infty(X_i), \quad f|_{E_i} \to f \circ \phi_i, \quad i = 1, 2$$

are 1-1 unital *-homomorphisms. We define, for $i = 1, 2$, the TRO’s

$$\mathcal{M}_i = \{ T : TP_i(E) = Q_i(\phi_i^{-1}(E))T, \text{ for } E \subseteq E_i \text{ Borel} \} \subseteq B(L^2(E_i), L^2(X_i))$$

(2.3)

and we note that $\mathcal{N}_i = \mathcal{M}_i R_i$ where $R_i \in B(L^2(Y_i))$ is the projection onto $L^2(E_i)$, $i = 1, 2$.

For $i = 1, 2$, let $\mathcal{A}_i \subseteq B(L^2(X_i))$ be the commutant of the commutative von Neumann algebra

$$\{ M_{f \circ \phi_i} : f \in L^\infty(E_i) \}.$$ 

Then it follows from [9, Theorem 3.2] that $[\mathcal{M}_i \mathcal{A}_i \mathcal{M}_i]^\omega = D(E_i)$ and $[\mathcal{M}_i D(E_i) \mathcal{M}_i^*]^\omega = A_i$ for $i = 1, 2$ (where $D(E_i) \subseteq B(L^2(E_i))$ denotes the multiplication algebra of $L^\infty(E_i)$).
If $\mathcal{U}$ is a $D(E_2) - D(E_1)$-bimodule we define

$$F(\mathcal{U}) = [M_2^* XM_1^*]^{w^*},$$

where $M_i, i = 1, 2$ are as in 2.3.

Then, from [9, Proposition 2.11] we obtain that the map

$$\mathcal{U} \rightarrow F(\mathcal{U})$$

maps the family of $D(E_2) - D(E_1)$-bimodules contained in $B(L^2(E_1), L^2(E_2))$ bijectively onto the family of $A_2 - A_1$-bimodules contained in $B(L^2(X_1), L^2(X_2))$. The inverse map of $F$ is given by $F^{-1}(V) = [M_2^* VM_1^*]^{w^*}$.

For each masa bimodule $\mathcal{U} \subseteq B(L^2(Y_1), L^2(Y_2))$ we define

$$F_r(\mathcal{U}) = F(R_2\mathcal{U}R_1).$$

**Theorem 2.2** If $\kappa \subseteq Y_1 \times Y_2$, is an $\omega$-closed set, then

(i) $F_r(M_{\text{max}}(\kappa)) = F(M_{\text{max}}(\kappa \cap (E_1 \times E_2))) = M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa))$ and $F_r(M_{\text{min}}(\kappa)) = F(M_{\text{min}}(\kappa \cap (E_1 \times E_2))) = M_{\text{min}}((\phi_1 \times \phi_2)^{-1}(\kappa))$.

(ii) If the measures $(\phi_i)_{\ast}(\mu_i)$ and $\nu_i$ are equivalent for $i = 1, 2$, we have

$$M_{\text{max}}(\kappa) = [M_2^* M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa)) M_1^*]^{w^*}$$

and $M_{\text{min}}(\kappa) = [M_2^* M_{\text{min}}((\phi_1 \times \phi_2)^{-1}(\kappa)) M_1^*]^{w^*}$.

**Proof** Part (i) follows from the above discussion, using similar arguments as in [11, Theorem 2.4].

For part (ii), note that if the measures are equivalent, then the maps $\tilde{\phi}_i$ are injective, and thus $\nu_i(Y_i \setminus E_i) = 0, \ i = 1, 2$. We conclude that

$$F(M_{\text{max}}(\kappa)) = F_r(M_{\text{max}}(\kappa)) = M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa)),$$

and so

$$M_{\text{max}}(\kappa) = F^{-1}(M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa))).$$

In other words

$$M_{\text{max}}(\kappa) = [M_2^* M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa)) M_1^*]^{w^*}.$$
For every $w^*$-closed subspace $V$ of $B(L^2(E_1), L^2(E_2))$ we write
\[ F_n(V \cap \mathcal{K}) = \overline{N_2(V \cap \mathcal{K}) N_1^\times} \| \cdot \|, \]
where $N_i, i = 1, 2$ are as in 2.2.

**Theorem 2.3** If $\kappa \subseteq Y_1 \times Y_2$ is $\omega$-closed, then
\[ F_n(M_{\text{max}}(\kappa) \cap \mathcal{K}) = F_r(M_{\text{max}}(\kappa)) \cap \mathcal{K} \]
and
\[ F_n(M_{\text{min}}(\kappa) \cap \mathcal{K}) = F_r(M_{\text{min}}(\kappa)) \cap \mathcal{K}. \]

**Proof** We have
\[
F_n(M_{\text{max}}(\kappa) \cap \mathcal{K}) = \overline{N_2(M_{\text{max}}(\kappa) \cap \mathcal{K}) N_1^\times} \| \cdot \|
= \overline{M_2(M_{\text{max}}(\kappa \cap (E_1 \times E_2)) \cap \mathcal{K}) M_1^\times} \| \cdot \|
= F_0(M_{\text{max}}(\kappa \cap (E_1 \times E_2))).
\]
By Lemma 2.1, the last space is equal to
\[ F(M_{\text{max}}(\kappa \cap (E_1 \times E_2))) \cap \mathcal{K} = F_r(M_{\text{max}}(\kappa)) \cap \mathcal{K}. \]
The other equality follows similarly. \(\square\)

In Theorem 2.4 below we improve the inverse image theorem for sets of compact operator synthesis obtained by Shulman, Todorov and Turowska in [22, theorem 4.7]. Recall that an $\omega$-closed set $\kappa \subseteq Y_1 \times Y_2$, is said to be a set of compact operator synthesis if it satisfies $M_{\text{max}}(\kappa) \cap \mathcal{K} = M_{\text{min}}(\kappa) \cap \mathcal{K}$. The sets of compact operator synthesis, as well as their classical counterparts, the sets of reduced synthesis were introduced and studied in [22].

In [22, Theorem 4.7], the authors proved that, if $\kappa$ is an $\omega$-closed set of compact operator synthesis, then $(\phi_1 \times \phi_2)^{-1}(\kappa)$ is a set of compact operator synthesis under the following assumptions: The measures $(\phi_i)_+(\mu_i)$ have a Radon-Nikodym derivative with respect to $\nu_i$ which are finite a.e. for $i = 1, 2$ and the maps $\phi_1$ and $\phi_2$ are of a particular form: the spaces $X_1$ and $X_2$ admit decompositions $X_1 = \tilde{X}_0 \cup \cdots \cup \tilde{X}_m$ and $X_2 = \tilde{Z}_0 \cup \cdots \cup \tilde{Z}_l$ such that $\phi_i$ is 1-1 when restricted to $\tilde{X}_0$ and is constant a.e. on each $\tilde{X}_i, i > 0$, and similarly for $\phi_2$.

We arrive at the same conclusion under the assumption that $\phi_i : X_i \rightarrow Y_i$ are measurable maps such that $(\phi_i)_+(\mu_i) \ll \nu_i$ for $i = 1, 2$.

**Theorem 2.4** If $\kappa \subseteq Y_1 \times Y_2$ is a set of compact operator synthesis then $(\phi_1 \times \phi_2)^{-1}(\kappa)$ is also a set of compact operator synthesis.

**Proof** If $\kappa$ is a set of compact operator synthesis then by Theorem 2.3 we have
\[ F_n(M_{\text{max}}(\kappa) \cap \mathcal{K}) = F_n(M_{\text{min}}(\kappa) \cap \mathcal{K}) \]
hence
\[ F_r(M_{\text{max}}(\kappa)) \cap \mathcal{K} = F_r(M_{\text{min}}(\kappa)) \cap \mathcal{K} \]
and hence \( M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa)) \cap \mathcal{K} = M_{\text{min}}((\phi_1 \times \phi_2)^{-1}(\kappa)) \cap \mathcal{K} \).

\[ \square \]

**Theorem 2.5** Let \( \phi_i : X_i \to Y_i \) be measurable maps such that for \( i = 1, 2 \) the measures \( (\phi_i)_*(\mu_i) \) and \( \nu_i \) are equivalent. Then \( \kappa \subseteq Y_1 \times Y_2 \) is a set of compact operator synthesis if and only if \( (\phi_1 \times \phi_2)^{-1}(\kappa) \) is a set of compact operator synthesis.

**Proof** If \( \kappa \) is a set of compact operator synthesis then, by Theorem 2.4, \( (\phi_1 \times \phi_2)^{-1}(\kappa) \) is also a set of compact operator synthesis. Conversely, assume that \( (\phi_1 \times \phi_2)^{-1}(\kappa) \) is a set of compact operator synthesis. By Theorem 2.2, (ii) and Lemma 2.1

\[
M_{\text{max}}(\kappa) \cap \mathcal{K} = \frac{[M_2^*(M_{\text{max}}((\phi_1 \times \phi_2)^{-1}(\kappa)) \cap \mathcal{K} \cap \mathcal{M}_1)]}{\| \cdot \|},
\]

\[
M_{\text{min}}(\kappa) \cap \mathcal{K} = \frac{[M_2^*(M_{\text{min}}((\phi_1 \times \phi_2)^{-1}(\kappa)) \cap \mathcal{K} \cap \mathcal{M}_1)]}{\| \cdot \|}.
\]

It follows that \( \kappa \) is a set of compact operator synthesis. \[ \square \]

## 3 Strict group morphisms and measures

We will need the following results:

Let \( G \) be a locally compact group, \( N \) a closed normal subgroup of \( G \); let \( G/N \) be the quotient group and \( \pi \) the quotient map \( G \to G/N \). Choose Haar measures \( \mu, \lambda \) and \( \nu \) on \( G, N \) and \( G/N \) respectively such that, writing \( \dot{x} = \pi(x) \), we have

\[
\int_G f(x) d\mu(x) = \int_{G/N} \left( \int_N f(xh) d\lambda(h) \right) d\nu(\dot{x})
\]

for every continuous, compactly supported function \( f \) on \( G \) [5, Ch. VII, Proposition 10].

Then from [5, Ch. VII, §2, p.57 b)] we have the following

**Lemma 3.1** Let \( G \) be a second countable locally compact group and \( N \) a closed normal subgroup of \( G \). If \( \mu, \lambda \) and \( \nu \) are Haar measures on \( G, N \) and \( G/N \) chosen as above, then

\[
\mu(\pi^{-1}(E)) = \lambda(N)\nu(E)
\]

for every Borel set \( E \subseteq G/N \).

**Remark 3.2** In particular \( \pi_*(\mu) \) and \( \nu \) are equivalent. Note that if \( \lambda(N) = +\infty \) and \( \nu(E) = 0 \), second countability ensures that we have \( \mu(\pi^{-1}(E)) = 0 \).

If \( G, H \) are locally compact groups, \( \theta : G \to H \) is a homomorphism and \( \mu \) is a measure on \( G \), we recall that we denote by \( \theta_*(\mu) \) the measure on \( H \) defined by \( \theta_*(\mu)(E) = \mu(\theta^{-1}(E)) \) for every Borel set \( E \subseteq H \).

Recall that \( \theta \) is said to be a strict morphism if \( \theta(G) \), with the relative topology induced by \( H \), is homeomorphic to \( G/\ker \theta \).
Lemma 3.3 Let $G, H$ be locally compact groups and $\theta : G \to H$ be a strict morphism. Let $\mu, \nu$ and $\lambda$ be Haar measures on $G, \theta(G)$ and $\ker \theta$ respectively. Then there exists $0 < c < +\infty$ such that
\[
\mu(\theta^{-1}(E)) = c\lambda(\ker \theta)\nu(E)
\]
for every Borel set $E \subseteq \theta(G)$.

Proof Since $\theta$ is strict, $G/\ker \theta$ is topologically isomorphic to $\theta(G)$. The assertion follows from Lemma 3.1, by uniqueness of Haar measure on $\ker \theta$. □

Lemma 3.4 Let $G, H$ be locally compact second countable groups and $\theta : G \to H$ a continuous homomorphism. Assume that $\theta(G)$ is an open subgroup of $H$. Then

(i) $\theta$ is a strict morphism
(ii) $\theta$ is an open map.

Proof An open subgroup of a topological group is also closed. Hence, it follows from [8, Proposition 6] that $\theta$ is a strict morphism. The quotient map $G \to G/\ker \theta$ is always open and since $\theta$ is a strict morphism, the induced map $G/\ker \theta \to H$ is a homeomorphism. So, $\theta$ is an open map. □

Theorem 3.5 Let $G, H$ be locally compact second countable groups with Haar measures $\mu$ and $\nu$ respectively, and $\theta : G \to H$ be a continuous homomorphism. The following are equivalent:

(i) $\theta^\ast(\mu) \ll \nu$
(ii) $\theta(G)$ is an open subgroup of $H$.

Proof (i) $\Rightarrow$ (ii) If $\nu(\theta(G)) = 0$ then $\theta^\ast(\mu)(\theta(G)) = \mu(G) = 0$ which is absurd. Thus $\nu(\theta(G)) > 0$, which by Weil’s theorem [27, p. 50] implies that $\theta(G)$ contains an open subset, hence is an open subgroup of $H$.

(ii) $\Rightarrow$ (i) Let $v_0$ be the restriction of $\nu$ to $\theta(G)$. Since $\theta(G)$ is an open subgroup of $H$, $v_0$ is a Haar measure on $\theta(G)$. By Lemma 3.4 $\theta$ is a strict morphism. By Lemma 3.3, for some $c > 0$ we have
\[
\theta^\ast(\mu)(E) = \theta^\ast(\mu)(E \cap \theta(G)) = c\lambda(\ker \theta)v_0(E \cap \theta(G))
\]
for every Borel set $E \subseteq H$. Therefore
\[
\theta^\ast(\mu)(E) \leq c\lambda(\ker \theta)\nu(E)
\]
and hence $\theta^\ast(\mu) \ll \nu$. □

Corollary 3.6 Let $G, H$ be locally compact second countable groups with Haar measures $\mu$ and $\nu$ respectively, and let $\theta : G \to H$ be a continuous homomorphism such that $\theta^\ast(\mu) \ll \nu$. Then

(i) If $\ker \theta$ is compact then $\theta^\ast(\mu)$ is a Haar measure for $\theta(G)$. 

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(ii) If \( \ker \theta \) is not compact then \( \theta_*(\mu)(E) \in \{0, +\infty\} \) for every Borel set \( E \subseteq \theta(G) \).

**Proof** From the proof of Theorem 3.5 it follows that there exists \( 0 < c < +\infty \) such that

\[
\mu(G) = \theta_*(\mu)(\theta(G)) \leq c \lambda(\ker \theta) \nu(\theta(G))
\]

and hence \( \lambda(\ker \theta) > 0 \). Therefore, using (3.1),

\[
\theta_*(\mu)(E) = c \lambda(\ker \theta) \nu(E)
\]

for every Borel set \( E \subseteq \theta(G) \). This fact implies the conclusion. \( \square \)

### 4 Synthesis and compact operator synthesis

Recall that the Fourier-Stieltjes algebra of \( G \) is the set of all coefficient functions \( s \mapsto \langle \pi(s)\xi, \eta \rangle, (\xi, \eta \in H_\pi) \) defined by unitary representations \((\pi, H_\pi)\) of \( G \), while the Fourier algebra \( A(G) \) of \( G \) consists of the coefficients of the left regular representation \( s \mapsto \lambda_s \) on \( L^2(G) \), given by \( \lambda_s(t) := \xi(s^{-1}t) \).

We study preservation and transference properties of continuous group homomorphisms between locally compact groups. These homomorphisms preserve important objects of Harmonic Analysis. For example, if \( \theta : G \to H \) is a continuous homomorphism, then the map \( u \mapsto u \circ \theta \) is a contraction from the Fourier algebra \( A(H) \) into the Fourier-Stieltjes algebra \( B(G) \) of \( G \) [13, Théorème 2.20(1)].

If \( J \subseteq A(G) \) is an ideal, we denote by \( Z(J) \) the set of all points of \( G \) where all \( u \in J \) vanish. If \( E \) is a closed subset of \( G \), we set

\[
I_G(E) = I(E) = \{ u \in A(G) : u(s) = 0, s \in E \}, \quad J_G(E) = J(E) = \{ u \in A(G) : u \text{ has compact support disjoint from } E \}.
\]

Then

\[
Z(J(E)) = Z(I(E)) = E
\]

and, if \( J \subseteq A(G) \) is a closed ideal with \( Z(J) = E \), then \( J(E) \subseteq J \subseteq I(E) \).

A closed subset \( E \subseteq G \) is called a set of **spectral synthesis** if \( I(E) = J(E) \). Shulman, Todorov and Turowska introduce and study in [22] the notions of reduced spectral synthesis and reduced local spectral synthesis for closed subsets of a locally compact group \( G \). A closed set \( E \) is a set of **reduced spectral synthesis** if \( C^*_r(G) \cap I(E) = C^*_r(G) \cap J(E) \). It is a set of **reduced local spectral synthesis** if \( C^*_r(G) \cap I_c(E) \cap J(E) = C^*_r(G) \cap J(E) \), where \( I_c(E) \) denotes the functions in \( I(E) \) of compact support. Here \( C^*_r(G) \) is the reduced \( C^* \)-algebra of \( G \), that is, the \( C^* \)-subalgebra of \( B(L^2(G)) \) generated by all operators \( \lambda(f), f \in L^1(G) \), where \( \lambda(f)(h) = f \ast h, h \in L^2(G) \). For motivation, examples and relevant discussion about these concepts, see [22, Section 3].
In what follows, for a subset $E \subseteq G$ we write

$$E^* = \{(s, t) \in G \times G : ts^{-1} \in E\}.$$  

**Theorem 4.1** Let $G$, $H$ be locally compact second countable groups with Haar measures $\mu$ and $\nu$ respectively and let $\theta : G \to H$ be a continuous homomorphism such that $\theta(G)$ is an open set in $H$. If $E \subseteq H$ is a set of reduced local spectral synthesis then $\theta^{-1}(E)$ is also a set of reduced local spectral synthesis.

**Proof** If $E \subseteq H$ is a set of reduced local spectral synthesis then [22, Theorem 5.1] implies that $E^*$ is a set of compact operator synthesis. By Theorem 2.4, $(\theta \times \theta)^{-1}(E^*)$ is also a set of compact operator synthesis. Since $\theta^{-1}(E^*) = (\theta \times \theta)^{-1}(E^*)$, using [22, Theorem 5.1] again we conclude that $\theta^{-1}(E)$ is a set of reduced local spectral synthesis. $\square$

Under the assumptions of Theorem 4.1, observe that the measures $\theta_*^*(\mu)$ and $\nu$ are equivalent if and only if $\theta$ is onto. Indeed, if $\theta$ is onto, by Theorem 3.5 and Corollary 3.6 we have that

$$\mu(\theta^{-1}(A)) = c\nu(A)$$

for every Borel set $A \subseteq H$, where $c$ is a positive constant, perhaps infinity if $\ker \theta$ is not compact. Thus $\theta_*^*(\mu)$ and $\nu$ are equivalent. Conversely, assume that $\theta_*^*(\mu)$ and $\nu$ are equivalent. Then, since $\theta_*^*(\mu)(H \setminus \theta(G)) = 0$ we have $\nu(H \setminus \theta(G)) = 0$. But every open subgroup is also closed, so $H \setminus \theta(G)$ is open; since it is $\nu$-null, it must be empty. Thus $\theta(G) = H$.

Under the above assumptions the map

$$\tilde{\theta} : L^\infty(H) \to L^\infty(G), \ f \mapsto f \circ \theta$$

is an injective $\ast$-homomorphism. Indeed assume that $\tilde{\theta}$ is not injective then $\ker \tilde{\theta} = L^\infty(A)$ for a Borel set $A \subseteq H$ with positive measure $\nu(A)$. Thus, $\mu(\theta^{-1}(A))$ is positive. This implies that $\tilde{\theta}(\chi_A) \neq 0$ which is a contradiction.

In the following result, the ‘only if’ direction is due to Shulman and Turowska [19, Theorem 4.7]. Recall that a subset $\kappa \subseteq H \times H$ is $\omega$-closed if its complement is marginally equivalent to a countable union of Borel rectangles.

**Theorem 4.2** Let $G$, $H$ be locally compact second countable groups with Haar measures $\mu$ and $\nu$ respectively, $\theta$ be a continuous onto homomorphism from $G$ to $H$, and $\kappa \subseteq H \times H$ be an $\omega$-closed set. Then the set $\kappa$ is a set of operator synthesis if and only if $(\theta \times \theta)^{-1}(\kappa)$ is a set of operator synthesis.

**Proof** We define the TRO

$$\mathcal{M} = \{T \in B(L^2(H), L^2(G)) : TP(E) = Q(\theta^{-1}(E))T, \text{ for all } E \subseteq H \text{ Borel}\}.$$
where \( P(E) \) is the projection onto \( L^2(E, \nu) \) and \( Q(\theta^{-1}(E)) \) is the projection onto \( L^2(\theta^{-1}(E), \mu) \). Since the map \( \tilde{\theta} : L^\infty(H) \to L^\infty(G) : f \to f \circ \theta \) is an injective \(*\)-homomorphism, from Theorem 2.2 we have that

\[
M_{\text{max}}((\theta \times \theta)^{-1}(\kappa)) = \left[ M\mathcal{M}_{\text{max}}(\kappa) \mathcal{M}^* \right]_{\nu^*},
\]

\[
M_{\text{max}}(\kappa) = \left[ \mathcal{M}^*(M_{\text{max}}(\theta \times \theta)^{-1}(\kappa)) \mathcal{M} \right]_{\nu^*},
\]

\[
M_{\text{min}}((\theta \times \theta)^{-1}(\kappa)) = \left[ M\mathcal{M}_{\text{min}}(\kappa) \mathcal{M}^* \right]_{\nu^*},
\]

\[
M_{\text{min}}(\kappa) = \left[ \mathcal{M}^*(M_{\text{min}}(\theta \times \theta)^{-1}(\kappa)) \mathcal{M} \right]_{\nu^*}.
\]

By the definition of operator synthesis, the result follows. \( \square \)

In [17, Theorems 4.3 and 4.11] the authors show that a closed set \( E \subseteq G \) is a set of local synthesis if and only if \( E^* \subseteq G \times G \) is a set of operator synthesis. Therefore, using the above theorem, we obtain the following result which is proved by Lohoué in [16].

**Corollary 4.3** Let \( G, H \) be locally compact second countable groups, \( \theta \) be a continuous onto homomorphism from \( G \) to \( H \), and \( E \subseteq H \) be a closed set. Then \( E \) is a set of local synthesis if and only if \( \theta^{-1}(E) \) is a set of local synthesis.

**Remark 4.4** Note that, conversely, one could prove that the conclusion of theorem 4.2 holds for subsets of \( H \times H \) of the form \( E^* \), where \( E \) is a closed subset of \( H \), using the result of Lohoué and the above mentioned results of [17].

**Theorem 4.5** Let \( G, H \) be locally compact second countable groups with Haar measures \( \mu \) and \( \nu \) respectively, \( \theta \) be a continuous onto homomorphism from \( G \) to \( H \), and \( \kappa \subseteq H \times H \) be an \( \omega \)-closed set. The set \( \kappa \) is a set of compact operator synthesis if and only if \( (\theta \times \theta)^{-1}(\kappa) \) is a set of compact operator synthesis.

**Proof** In this case the measures \( \theta^*(\mu) \) and \( \nu \) are equivalent and the conclusion follows from Theorem 2.5. \( \square \)

### 5 Ditkin sets

Operator Ditkin sets were first defined and studied by Shulman and Turowska in [19]. Ludwig and Turowska introduced the notion of strong operator Ditkin sets in [17]. In this section we prove an inverse image theorem for strong operator Ditkin sets.

Recall that, for a locally compact group \( G \), the predual of \( B(L^2(G)) \) may be identified with the space \( T(G) \) of (equivalence classes of marginally a.e equal) functions \( h : G \times G \to \mathbb{C} \) for which there exist sequences \( (f_n) \) and \( (g_n) \) in \( L^2(G) \) such that

\[
h(x, y) = \sum_{n=1}^{\infty} f_n(x)g_n(y), \quad \sum_{n=1}^{\infty} \|f_n\|_2\|g_n\|_2 < +\infty.
\]
The norm of such a function is
\[ \|h\|_{T(G)} = \inf \sum_{n=1}^{\infty} f_n(x)g_n(y) \]
where the infimum is taken over all such sequences for which
\[ h(x, y) = \sum_{n=1}^{\infty} f_n(x)g_n(y) \]
marginally a.e.. The duality between \( T(G) \) and \( B(L^2(G)) \) is given by the pairing
\[ \langle T, h \rangle := \sum_{n=1}^{\infty} \langle Tf_n, \tilde{g}_n \rangle \]
[3, p. 494].

Let \( \mathfrak{S}(G) \) be the multiplier algebra of \( T(G) \); by definition, a measurable function \( w : G \times G \to \mathbb{C} \) belongs to \( \mathfrak{S}(G) \) if the map \( m_w : h \mapsto wh \) leaves \( T(G) \) invariant (that is, if \( wh \) is marginally equivalent to a function from \( T(G) \), for every \( h \in T(G) \)) and defines a bounded map on \( T(G) \) [23] (see also [25, Proposition 4.9] where it is shown that the continuity of \( m_w \) is automatic). The elements of \( \mathfrak{S}(G) \) are called (measurable) Schur multipliers.

Fix locally compact second countable groups \( G \) and \( H \) with Haar measures \( \mu \) and \( \nu \) respectively.

We will assume that the Fourier algebra \( A(G) \) satisfies Ditkin’s condition at infinity (or \( D_\infty \)), namely, that every \( u \in A(G) \) belongs to \( \overline{uA(G)} \). See Remark 5.2 below.

We also assume that \( \theta : G \to H \) is a continuous homomorphism such that \( \ker \theta \subseteq G \) is compact and \( \theta(G) \subseteq H \) is open. Since \( \theta(G) \) is an open subgroup, by Corollary 3.6, \( \theta_*(\mu) \) is a Haar measure for \( \theta(G) \), so (multiplying \( \nu \) by a positive constant if necessary) we may assume that \( \nu|_{\theta(G)} = \theta_*(\mu) \). Observe that the map \( L^2(H) \to L^2(G) : f \mapsto f \circ \theta \) is contractive; hence the map
\[ T(H) \to T(G) : h \mapsto h \circ (\theta \times \theta) \]
is a contraction.

If \( \kappa \subseteq G \times G \) is an \( \omega \)-closed set, we write
\[ \Psi(\kappa) = \{ h \in T(G) : h\chi_\kappa = 0 \text{ marginally a.e.} \} \, . \]

(This denoted is by \( \Phi_0(\kappa) \) in [19].) By [19, Theorem 4.3],
\[ M_{\min}(\kappa) = \Psi(\kappa)_{\perp} \]
(5.1)

Recall (see [17]) that an \( \omega \)-closed set \( \kappa \subseteq G \times G \) is called a strong operator Ditkin set if there exists a sequence \( (w_n) \) in \( \mathfrak{S}(G) \) such that each \( w_n \) vanishes in an
\(\omega\)-neighbourhood of \(\kappa\) and

\[ \| w_n h - h \|_{T(G)} \to 0, \quad \text{for all} \quad h \in \Psi(\kappa). \]

Let \(E \subseteq H\) be a closed set such that \(E^*\) is a strong operator Ditkin set. It follows from [17, Theorem 5.4] that \(E\) is a set of local spectral synthesis. Recall that a closed subset \(E \subseteq G\) is called a set of local spectral synthesis if \(J(E)\) contains every compactly supported function of \(I(E)\).

Theorem 5.3 in [2] implies that

\[ M_{\min}(\theta^{-1}(E)^*) = Bim(I_G(\theta^{-1}(E)^*)) \]

and Corollary 2.5 in [11] implies that \(\theta^{-1}(E)^*\) is a set of operator synthesis. Thus,

\[ M_{\max}(\theta^{-1}(E)^*) = M_{\min}(\theta^{-1}(E)^*). \]

Now, if \(\rho(u) = u \circ \theta\) for all \(u \in A(H)\), from Theorem 3.4 in [11] we have that

\[ M_{\max}(\theta^{-1}(E)^*) = Bim(\langle \rho(I_H(E)) \rangle^\perp), \]

where \(\langle \rho(I_H(E)) \rangle\) is the closed ideal in \(A(G)\) generated by \(\rho(I_H(E))\). Thus

\[ Bim(I_G(\theta^{-1}(E)^*)) = Bim(\langle \rho(I_H(E)) \rangle^\perp). \]

Since \(A(G)\) satisfies \(D_{\infty}\), Lemma 4.5 in [2] implies that

\[ I_G(\theta^{-1}(E)) = \langle \rho(I_H(E)) \rangle. \]

Thus

\[ \Psi((\theta \times \theta)^{-1}(E^*))^\perp = Bim(\langle \rho(I_H(E)) \rangle^\perp). \]

**Lemma 5.1** The space

\[ \Psi((\theta \times \theta)^{-1}(E^*)) \]

is a subspace of the masa-bimodule generated in \(T(G)\) by the set

\[ \{ h \circ (\theta \times \theta) : h \in \Psi(E^*) \}. \]

**Proof** We saw that the space \(\Psi((\theta \times \theta)^{-1}(E^*))\) is the preannihilator of \(Bim(\langle \rho(I_H(E)) \rangle^\perp)\). By [2, Theorem 3.2] this preannihilator is

\[ \text{span}\{ N(\rho(u))h : u \in I_H(E), \quad h \in T(G) \}. \]

where \(N(f)(s, t) := f(ts^{-1})\) (see [2, Proposition 3.1]).

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Let \( u \in I_H(E) \). The restriction \( u|_{\theta(G)} \) of \( u \) to \( \theta(G) \) belongs to \( A(\theta(G)) \) [13, Proposition 3.21(2)] and the function \( u|_{\theta(G)} \circ \theta \) belongs to \( A(G) \) [13, Proposition 3.25(1)]. Since \( u|_{\theta(G)} \circ \theta = u \circ \theta, \rho(u) = u \circ \theta \) is in \( A(G) \) and therefore \( N(u) \circ (\theta \times \theta) \) vanishes m.a.e. on \( (\theta \times \theta)^{-1}(E^*) \). It follows that for all \( h \in T(G) \), the function \( (N(u) \circ (\theta \times \theta))h \) belongs to the masa-bimodule generated by the set

\[ \{ k \circ (\theta \times \theta) : k \in \Psi(E^*) \}. \]

\[ \square \]

**Remark 5.2** Note that Lemma 4.5 of [2], which we have used, remains true if instead of assuming that \( A(G) \) has an approximate identity, we assume that \( A(G) \) satisfies the formally weaker Ditkin’s condition at infinity. This follows immediately from its proof. Ditkin’s condition at infinity is a ‘local’ condition, closer in spirit to reflexivity (as defined by Loginov-Shulman) rather than the ‘global’ condition of an approximate identity. However it is unknown if there are any locally compact groups failing either condition. We note that a characterization of Ditkin’s condition at infinity related to our discussion is given by Andreou [1].

**Theorem 5.3** Let \( G \) and \( H \) be locally compact second countable groups such that \( A(G) \) satisfies \( D_\infty \) and let \( \theta : G \to H \) be a continuous homomorphism such that \( \ker \theta \subseteq G \) is compact and \( \theta(G) \subseteq H \) is open. Let \( E \subseteq H \) be a closed subset. If \( E^* \) is a strong operator Ditkin set, then \((\theta^{-1}(E))^*\) is also a strong operator Ditkin set.

**Proof** Writing \( H_1 = \theta(G) \) for brevity, we can easily see that

\[ \Psi((E \cap \theta(G))^*) = \zeta_0 \Psi(E^*), \]

where \( \zeta_0 = \chi_{H_1 \times H_1} \). Since \( E^* \) is an operator Ditkin set, there exists a sequence \( (w_n) \subseteq \mathcal{S}(H) \) such that \( w_n \) vanishes in an \( \omega \)-neighbourhood of \( E^* \) and

\[ \|w_n h - h\|_{T(H)} \to 0, \quad \text{for all} \quad h \in \Psi(E^*). \]

Thus

\[ \|w_n h \zeta_0 - h \zeta_0\|_{T(H_1)} \to 0, \quad \text{for all} \quad h \in \Psi(E^*) \]

and so

\[ \|w_n h_1 - h_1\|_{T(H_1)} \to 0, \quad \text{for all} \quad h_1 \in \Psi((E \cap H_1)^*). \]

Since the map

\[ T(H_1) \to T(G) : h_1 \to h_1 \circ (\theta \times \theta) \]

is a contraction we have

\[ \| (w_n \circ (\theta \times \theta)(h_1 \circ (\theta \times \theta)) - h_1 \circ (\theta \times \theta)) \|_{T(G)} \to 0, \quad \text{for all} \quad h_1 \in \Psi((E \cap H_1)^*) \]
and, a fortiori,
\[ \|(w_n \circ (\theta \times \theta)(h_1 \circ (\theta \times \theta))w - h_1 \circ (\theta \times \theta))w\|_{T(G)} \to 0, \]
for all \( h_1 \in \Psi((E \cap H_1)^*) \) and \( w \in \mathcal{S}(G) \). But now if \( h_2 \in \Psi((E \cap H)^*) = \Psi((\theta \times \theta)^{-1}(E^*)) \) then, by Lemma 5.1, \( h_2 \) is in the masa bimodule \( \{ h \circ (\theta \times \theta)w : h \in \Psi(E^*), w \in \mathcal{S}(G) \} \) (since masa bimodules are invariant under Schur multipliers) and so we have
\[ \|(w_n \circ (\theta \times \theta)h_2 - h_2\|_{T(H)} \to 0, \]
for all \( h_2 \in \Psi((\theta \times \theta)^{-1}(E^*) \).
Since \( w_n \circ (\theta \times \theta) \) vanishes in an \( \omega \)-neighbourhood of \( (\theta \times \theta)^{-1}(E^*) \), we conclude that \( (\theta^{-1}(E))^* \) is a strong operator Ditkin set. \( \square \)

**Remark 5.4** Note, for comparison, that Parthasarathy and Kumar have proved in [18] that if \( G \) is a compact group and \( H \) a normal subgroup of \( G \), then a closed subset \( E \) of \( G/H \) is a strong Ditkin set if and only if \( \pi^{-1}(E) \) is a strong Ditkin set, where \( \pi : G \to G/H \) is the quotient map.

**Remark 5.5** The above theorem together with Theorem 5.4 in [17] implies the following: Let \( G \) and \( H \) be locally compact second countable groups such that \( A(G) \) satisfies \( D_{\infty} \) and let \( \theta : G \to H \) be a continuous homomorphism such that \( \ker \theta \subseteq G \) is compact and \( \theta(G) \subseteq H \) is open. Let \( E \subseteq H \) be a closed subset. If \( E^* \) is a strong operator Ditkin set then \( \theta^{-1}(E) \) is a local Ditkin set.

### 6 M-sets

The concept of \( M \)-sets for general locally compact groups was introduced by Bożejko in [6]. Shulman, Todorov and Turowska introduced in [21] the notion of \( M_1 \)-sets in locally compact groups and studied transference and preservation properties of \( M \)-sets and \( M_1 \)-sets and their operator analogues, which had been introduced in [20].

A closed subset \( E \subseteq G \) is called a set of multiplicity (or an \( M \)-set) if \( C^*_e(G) \cap J(E)^\perp \neq \{0\} \). The set \( E \) is called an \( M_1 \)-set if \( C^*_e(G) \cap I(E)^\perp \neq \{0\} \).

An \( \omega \)-closed set \( \kappa \subseteq G \times G \) is called an operator \( M \)-set if \( M_{\text{max}}(\kappa) \) contains nonzero compact operators, and is called an operator \( M_1 \)-set if \( M_{\text{min}}(\kappa) \) contains nonzero compact operators.

In the following theorem we show that a set \( \kappa \) is an operator \( M \)-set if and only if \( (\theta \times \theta)^{-1}(\kappa) \) is an operator \( M \)-set. The ‘only if’ direction of this result is proved in [21] under the assumption that \( \theta_*(\mu) \) has a Radon-Nikodym derivative which is \( \mu \)-a.e. finite, which in turn is equivalent to the compactness of \( \ker \theta \) (see Corollary 3.6).

**Theorem 6.1** Let \( G, H \) be locally compact second countable groups with Haar measures \( \mu \) and \( \nu \) respectively, \( \theta \) be a continuous onto homomorphism from \( G \) to \( H \), and \( \kappa \subseteq H \times H \) be an \( \omega \)-closed set.

Then the set \( \kappa \) is an operator \( M \)-set (resp. \( M_1 \)-set) if and only if \( (\theta \times \theta)^{-1}(\kappa) \) is an operator \( M \)-set (resp. \( M_1 \)-set).
**Proof** As we noted in sect. 4, the condition \( \theta(G) = H \) is equivalent to the requirement that the measures \( \theta_*(\mu) \) and \( \nu \) are equivalent, even if \( \theta_*(\mu) \) is not \( \sigma \)-finite.

We define the TRO

\[
\mathcal{M} = \{ x : xP(E) = Q(\theta^{-1}(E))x, \text{ for all } E \subseteq H \text{ Borel} \},
\]

where \( P(E) \) is the projection onto \( L^2(E, \nu) \) and \( Q(\theta^{-1}(E)) \) is the projection onto \( L^2(\theta^{-1}(E), \mu) \). From Theorem 2.2 and Lemma 2.1 we have that

\[
\begin{align*}
M_{\max}((\theta \times \theta)^{-1}(\kappa)) \cap \mathcal{K} &= \overline{[\mathcal{M}(\mathcal{M}_{\max}(\kappa) \cap \mathcal{K})\mathcal{M}^*]_{||\cdot||}}, \\
M_{\max}(\kappa) \cap \mathcal{K} &= \overline{[\mathcal{M}^*(\mathcal{M}_{\max}((\theta \times \theta)^{-1}(\kappa)) \cap \mathcal{K})\mathcal{M}]_{||\cdot||}}, \\
M_{\min}((\theta \times \theta)^{-1}(\kappa)) \cap \mathcal{K} &= \overline{[\mathcal{M}(\mathcal{M}_{\min}(\kappa) \cap \mathcal{K})\mathcal{M}^*]_{||\cdot||}}, \\
M_{\min}(\kappa) \cap \mathcal{K} &= \overline{[\mathcal{M}^*(\mathcal{M}_{\min}((\theta \times \theta)^{-1}(\kappa)) \cap \mathcal{K})\mathcal{M}]_{||\cdot||}}.
\end{align*}
\]

If \( \kappa \) is an operator \( M \)-set then \( M_{\max}(\kappa) \) contains a non-zero compact operator \( x \). We claim that \( \mathcal{M}x\mathcal{M}^* \neq \{0\} \). Indeed: assume to the contrary that \( \mathcal{M}x\mathcal{M}^* = \{0\} \). Then \( \overline{[\mathcal{M}^*\mathcal{M}]_{\omega^*}x[\mathcal{M}^*\mathcal{M}]_{\omega^*}} = \{0\} \). By [9, Lemma 3.1] the algebra \( \overline{[\mathcal{M}^*\mathcal{M}]_{\omega^*}} \) contains the identity operator and thus \( x = 0 \). This contradiction shows that \( \mathcal{M}x\mathcal{M}^* \neq \{0\} \) which implies that \( M_{\max}((\theta \times \theta)^{-1}(\kappa)) \cap \mathcal{K} \neq \{0\} \). Therefore the set \( (\theta \times \theta)^{-1}(\kappa) \) is an operator \( M \)-set.

Reversing the roles of \( \mathcal{M} \) and \( \mathcal{M}^* \), we show that if \( (\theta \times \theta)^{-1}(\kappa) \) is an operator \( M \)-set then \( \kappa \) is an operator \( M \)-set.

Similarly we can prove that \( \kappa \) is an operator \( M_1 \)-set if and only if \( (\theta \times \theta)^{-1}(\kappa) \) is an operator \( M_1 \)-set. \( \square \)

In [21], the authors showed that if \( \theta : G \to H \) is a continuous homomorphism onto \( H \) and \( E \) is a closed subset of \( H \) which is an \( M \)-set (resp. an \( M_1 \)-set) then \( \theta^{-1}(E) \) is also an \( M \)-set (resp. an \( M_1 \)-set) under the assumption that \( \theta_*(\mu) \) has a Radon-Nikodym derivative which is \( \mu \)-a.e. finite, which in turn is equivalent to the compactness of \( \ker \theta \) (see Corollary 3.6).

In [11] it was proved that if \( \theta : G \to H \) is an open continuous homomorphism and \( \theta^{-1}(E) \) is an \( M \)-set (resp. an \( M_1 \)-set) then \( E \) is also an \( M \)-set (resp. an \( M_1 \)-set).

It follows from [21] that a closed set \( E \subseteq G \) is an \( M \)-set (resp. an \( M_1 \)-set) if and only if \( E^* \subseteq G \times G \) is an operator \( M \)-set (resp. an operator \( M_1 \)-set). Hence Theorem 6.1 implies Corollary 6.2 below, which for \( M \)-sets is due to Derighetti [7]. We note that conversely, one could prove that the conclusion of theorem 6.1 holds for subsets of \( H \times H \) of the form \( E^* \), where \( E \) is a closed subset of \( H \), using the result of Derighetti and the above mentioned result of [21].

**Corollary 6.2** Let \( G, H \) be locally compact second countable groups, \( \theta \) be a continuous onto homomorphism from \( G \) to \( H \), and \( E \subseteq H \) be a closed set. Then \( E \) is an \( M \)-set (resp. an \( M_1 \)-set) if and only if \( \theta^{-1}(E) \) is an \( M \)-set (resp. an \( M_1 \)-set).
Remark 6.3 In case $\theta$ is not onto, Corollary 6.2 does not necessarily hold even if $\theta$ is an open map with compact kernel. For example take $G = \mathbb{Z}_2$ and $\theta: G \to G$, $\theta(x) = 0$, for all $x \in G$.

Consider the set $E = \{1\}$. This is an $M$-set and an $M_1$-set, but $\theta^{-1}(E) = \emptyset$ is not an $M$-set or an $M_1$-set.

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