Physical Examples of the Heun-to-Hypergeometric Reduction *

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Abstract

The Heun’s equation having four regular singularities emerge in many applications in physics. Therefore relating this equation with the well-known hypergeometric equation may provide a complete understanding of the mathematics of the Heun’s equation as well as the behavior of the physical system. We studied the Heun-to-hypergeometric reductions for three physical problems: The Schrödinger’s equation which was written in terms of the Heun’s general equation for the Coulomb problem on a 3-sphere, the s-wave bound state equation in the problem of the attractive inverse square potential and the equation for the limit density function for the discrete-time quantum walk. We applied the limiting cases and tried to figure out if a reduction is possible. While some problems permit these reductions, some others does not because of the restrictions on the physical parameters. A SAGE code is given in the appendix for the calculation of some Heun identities.

Keywords: Heun’s equation, Heun to hypergeometric reductions

1 Introduction

The need of understanding the mathematical aspects of the Heun’s equation becomes more important as new physical applications giving them as solutions emerge frequently. A comprehensive updated literature on the recent physical applications can be found in Hortaçsu’s work [1]. The number of studies related to the mathematical analysis of the Heun’s equation is very few regarding the number of occurrences of this equation in the modern applications. Among these limited number of mathematical works, articles on reducing the Heun’s equation to the hypergeometric equations have a particular importance since constituting a relationship with probably the most well-studied equations in the literature gives a deep insight about the less known Heun’s equation and its confluent types [2]-[11]. The literature lacks the physical applications of this reduction, among very few articles we can cite Kwon et al [12] and Cunha et al.’s [13] works.

In this paper, we will try to give examples of Heun-to-hypergeometric reductions using three physical problems with some limiting cases given by Maier [11]. The first one is the quantum mechanical Coulomb problem on a 3-sphere analyzed by Bellucci et al. [14]. They studied the Schrodinger’s equation in the generalized parabolic coordinates and found that the equation can be written in the form of the Heun’s general equation. The second one is the s-wave bound state equation in the problem of the attractive inverse square potential studied by Bouaziz and Bawin which is again given in terms of the Heun’s equation [15]. The third example is the equation for the limit density function for the discrete-time quantum walk as studied by Konno et al. where they found a relation between the limit density function and the Heun’s equation [16].

In the next section, we will summarize the identities and the limiting cases which are needed for the Heun-to-hypergeometric reductions. The subsequent sections contain the examples in which the limiting cases are studied in detail. A symbolic SAGE code is given in the appendix for the calculation of the identities that we will cover in the next section [17].

* Dedicated to Prof. Mahmut Hortaçsu on the Occasion of His Seventy-first Birthday
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2 Heun’s equation and its reductions

The general Heun’s equation can be given by

\[ H'' + \left( \frac{\Gamma}{z} + \frac{\Delta}{z-1} + \frac{\epsilon}{z-d} \right) H' + \frac{abz - q}{z(z-1)(z-d)} H = 0, \]  

(1)

where the prime denotes differentiation with respect to \( z \) and \( a + b + 1 = \Gamma + \Delta + \epsilon \) should hold. The equation has four regular singularities at \( \{0, 1, d, \infty\} \), \( \{d \neq 0, 1\} \). The solution of this equation is Heun’s function and it is denoted by \( H(d; q; a, b, \Gamma; z) \) \[18\].

There are mainly five cases that this equation reduces to a hypergeometric equation:

1. \( \epsilon = 0 \) and \( q = abd \)
2. \( \Delta = 0 \) and \( q = ab \)
3. \( \Gamma = 0 \) and \( q = 0 \)
4. \( ab = 0 \) and \( q = 0 \) (the trivial case)
5. Nontrivial cases

In the first four cases, the equation loses one singular point explicitly. The fourth case implies the absence of the singularity at infinity as given by the Definition (2.1) of Maier \[11\]. One can study the nontrivial reductions when the singular points satisfy the harmonic (equally spaced collinear points) or equianharmonic (equilateral triangle) conditions as given by the Theorem (3.1) of Maier \[11\].

We will restrict ourselves with the cases exhibiting harmonic behavior among the singular points (i.e. the real case). There are mainly two identities that we can use to change the \( d \) values to \( \frac{d}{d-1} \) or \( \frac{1}{d} \).

The equation (3.4.11) of Arscott \[19\] is the first identity that we will be using,

\[ H(d_1, q_1; a_1, b_1, \Gamma_1, \Delta_1; z_1) = H(d_2, q_2; a_2, b_2, \Gamma_2, \Delta_2; z_2), \]

(2)

where

\[
\begin{align*}
d_2 &= \frac{d_1}{d_1 - 1}, \\
q_2 &= \frac{q_1 + a_1 d_1 \Gamma_1}{d_1 - 1}, \\
a_2 &= a_1, \\
b_2 &= a_1 - \Delta_1 + 1, \\
\Gamma_2 &= \Gamma_1, \\
\Delta_2 &= a_1 + b_1 + 1, \\
z_2 &= \frac{z_1}{z_1 - 1},
\end{align*}
\]

(3) \hfill (4) \hfill (5) \hfill (6) \hfill (7) \hfill (8) \hfill (9)

and the second one is the equation (3.4.3) of the same article, namely

\[ H(d_1, q_1; a_1, b_1, \Gamma_1, \Delta_1; z_1) = H(d_3, q_3; a_3, b_3, \Gamma_3, \Delta_3; z_3), \]

(10)

where

\[
\begin{align*}
d_3 &= \frac{1}{d_1}, \\
q_3 &= \frac{q_1}{d_3}, \\
a_3 &= a_1, \\
b_3 &= b_1, \\
\Gamma_3 &= \Gamma_1, \\
\Delta_3 &= a_1 + b_1 - \Gamma_1 - \Delta_1 + 1, \\
z_3 &= \frac{z_1}{d_1}.
\end{align*}
\]

(11) \hfill (12) \hfill (13) \hfill (14) \hfill (15) \hfill (16) \hfill (17)
A short SAGE code is given in the appendix to apply these identities easily [17].

Then, according to the Theorem 3.7 of Maier [11], the Heun’s equation is nontrivial if $ab \neq 0$ and $q \neq 0$ and the solution can be reduced to a hypergeometric function by a formula of the type $H(t) = F(R(t))$, with a rational function $R$ only if the parameters satisfy $q = abp$ with $(a, p)$ equal to one of the 23 pairs stated in the theorem with a given degree for $R(t)$. For example, by using the identities stated above we can have $\{d = \{-1, 2, \frac{1}{2}\}\}$ which corresponds to a harmonic cross ratio orbit according to Theorem 3.1 (1a) of Maier [11]. The related pairs are $(d, p) = (-1, 0), (\frac{1}{2}, \frac{1}{2}), (2, 1)$ and they come with degree 2 or 4 for the function $R$.

3 Quantum mechanical Coulomb problem on a 3-sphere

The equation that we are going to study was given by Bellucci and Yeghikyan in [14] as

$$H'' + \left( \frac{\Gamma}{z} + \frac{\Delta}{z-1} + \frac{\epsilon}{z+1} \right) H' + \frac{abz - q}{z(z-1)(z+1)} H = 0,$$

where

$$\Gamma = 1 - \sqrt{1 + E + i\gamma},$$
$$\Delta = \epsilon = |m| + 1,$$
$$q = \frac{i\beta}{2},$$
$$a = 1 + |m| + \frac{\sqrt{1 + E - i\gamma} - \sqrt{1 + E + i\gamma}}{2},$$
$$b = 1 + |m| - \frac{\sqrt{1 + E + i\gamma} + \sqrt{1 + E - i\gamma}}{2}.$$

$\beta$ is a separation constant, $m$ is an exponential associated with the axial symmetry, $\gamma$ is a parameter defined in the suggested solution and $E$ is the energy. We have the third regular singular point at $d = -1$ in our case. From now on, we will use $m$ instead of its absolute value $|m|$.

The third singular point is $d = -1$ but we can change it to $d = 2$ or $d = \frac{1}{2}$ using the identities given in the previous section. We can write the solution as

$$H(-1, q; a, b, \Gamma, \Delta; z) = H\left(\frac{1}{2}, \frac{a\Gamma - q}{2}; a, a - \Delta + 1, \Gamma, a + b + 1; \frac{z}{z-1}\right),$$

and

$$H(-1, q; a, b, \Gamma, \Delta; z) = H(-1, -q; a, b, \Gamma, a + b - \Gamma - \Delta + 1; -z),$$

or

$$H\left(\frac{1}{2}, \frac{a\Gamma - q}{2}; a, a - \Delta + 1, \Gamma, a + b + 1; \frac{z}{z-1}\right) = H\left(2, a\Gamma - q; a, a - \Delta + 1, \Gamma, a - b - \Delta - \Gamma + 1; \frac{2z}{z-1}\right).$$

Now let us study the special cases where the equation reduces to a hypergeometric equation.

The cases with $\{\epsilon = 0\}$ and $q = abd$ and $\{\Delta = 0\}$ and $q = ab$ are not possible for our problem as $\Delta = \epsilon = m + 1$ and this value is restricted to have nonzero values by $m$ which is actually the absolute value of this parameter.

3.1 The case with $\Gamma = 0$ and $q = 0$

$\Gamma$ and $q$ being zero yields $E = -i\gamma$ and $\beta = 0$. This reduces our equation to the Legendre equation with three regular singularities at $\{-1, 1, \infty\}$ as

$$H = (z^2 - 1)^{-m/2} \left[ P \left( \frac{\sqrt{1 - 2i\gamma} - 1}{2}, m; z \right) + Q (...) \right],$$

which is a special form of the hypergeometric equation.
3.2 The trivial case \( \{ab = 0 \text{ and } q = 0 \} \)

We can restrict the energy using the multiplication of \( a \) and \( b \) being zero as

\[
E = \frac{m(m^3 + 4m^2 + 5m + 2)}{(1 + m)^2} - \frac{\gamma^2}{4(1 + m)^2},
\]

and \( \beta = 0 \) to satisfy \( q = 0 \).

The solution then reduces to the hypergeometric function

\[
H = F \left( \frac{m + 1}{2} - \sqrt{C_1 - \frac{c + 8i\gamma}{8}}, \frac{m + 1}{2} + \sqrt{C_1 + \frac{c + 8i\gamma}{8}}; 1 - \sqrt{\frac{C_1}{4}}; z^2 \right)
\]

where

\[
C_1 = 4(E + i\gamma + 1).
\]

3.3 Nontrivial cases

The easiest approach would be using the equation (3.8) of Maier \[11\] with \( (d, p) = (-1, 0) \), namely

\[
H \left(-1, 0; a, b, \Gamma, \frac{a + b - \Gamma + 1}{2}; t \right) = F \left( \frac{a}{2}, \frac{b}{2}, \frac{\Gamma + 1}{2}; t^2 \right),
\]

as \( d = -1 \) originally. \( q = 0 \) (with \( \beta = 0 \)) and the condition \( \Delta = \frac{a + b - \Gamma + 1}{2} = m + 1 \) is automatically satisfied using the physical parameters.

The solution of the equation for this case can also be written as

\[
H(-1, q; a, b, \Gamma, \Delta; z) = H \left( \frac{1}{2} - \frac{q}{2}; a, a - \Delta + 1, \Gamma, a + b + 1; \frac{z}{z - 1} \right),
\]

after applying the identities to the original solution and \( (d, p) = (2, 1) \) yields \( a\Gamma - q = a(a - \Delta + 1) \) by the condition \( q = abp \). Using this, we can specify the value of the corresponding energy as follows

\[
E = -\gamma^4 + 4\beta\gamma^3 + \sqrt{-6\beta^2 + 4m(m + 2)(1 + m)^2}\gamma^2 + 8\beta \left[ \frac{\beta^2}{4} + (1 + m)^2 \right] \gamma - 4\beta^2 \left[ \frac{\beta^2}{4} + (1 + m)^2 \right].
\]

We can then try the equation (3.5a) of Maier \[11\], namely

\[
H(2, ab; a, b, \Gamma, a + b - 2\Gamma + 1; t) = F \left( \frac{a}{2}, \frac{b}{2}; \Gamma; t(2 - t) \right).
\]

Then we need

\[
H \left( 2, a\Gamma - q; a, a - \Delta + 1, \Gamma, a - b - \Delta - \Gamma + 1; \frac{2z}{z - 1} \right) = H \left( 2, ab; a, b, \Gamma, a + b - 2\Gamma + 1; t \right),
\]

\[
= H(2, a(a - \Delta + 1);
\]

\[
a, a - \Delta + 1,
\]

\[
\Gamma, 2a - \Delta - 2\Gamma + 2; t)
\]

In order to satisfy \( a - b - \Delta - \Gamma + 1 = 2a - \Delta - 2\Gamma + 2 \) we need \( m = -1 \) which is not possible as the parameter \( m \) is restricted to have only positive values.

Now, let us try the equation (3.5c) of Maier \[11\], as given by

\[
H(2, ab; a, b, \frac{a + b + 2}{4}, \frac{a + b}{2}; t) = F \left( \frac{a}{4}, \frac{b}{4}, \frac{a + b + 2}{4}, \frac{a + b}{2}; 1 - 4 \left[ t(2 - t) - \frac{1}{2} \right]^2 \right)
\]
Then we need to have

\[
H \left( 2, a\Gamma - q; a, a - \Delta + 1, \Gamma, a - b - \Delta - \Gamma + 1; \frac{2z}{z-1} \right) = H(2, ab; a, b, \frac{a+b+2}{4}, \frac{a+b}{2}; t),
\]

and thus \( \Gamma = \frac{2a-\Delta+3}{4} \) and \( a - b - \Delta - \Gamma + 1 = \frac{2a-\Delta+1}{2} \) should be satisfied. Solving these equations we can find that the positive \( m \) values can be obtained only when we set a lower bound for the energy value. We need \( E > \frac{11}{9} \) either for \( m = -\frac{22}{13} + \frac{3}{13} \sqrt{22 + 26E} \) or for \( m = -\frac{38}{37} + \frac{3}{37} \sqrt{70 + 74E} \), the only solutions giving positive \( m \) values. Then we can write

\[
H(-1, q; a, b, \Gamma, \Delta; z) = H \left( 2, a\Gamma - q; a, a - \Delta + 1, \Gamma, a - b - \Delta - \Gamma + 1; \frac{2z}{z-1} \right)
\]

as the solution of our main equation reduced into the hypergeometric type.

4 s-wave bound state equation in the problem of the attractive inverse square potential

The equation which Bouaziz and Bawin obtained for the wave function can be rewritten as

\[
H'' + \left( \frac{\Gamma}{z} + \frac{\Delta}{z-1} + \frac{\epsilon}{z-d} \right) H' + \frac{abz - q}{z(z-1)(z-d)} H = 0,
\]

where

\[
\begin{align*}
\Gamma & = \frac{3}{2}, \\
\Delta & = \frac{1}{2} - \omega_4, \\
\epsilon & = 2, \\
q & = \frac{3}{2} + \frac{\kappa}{1 - 2\omega}, \\
a & = \frac{1}{2} (3 - \omega_4 - \nu), \\
b & = \frac{1}{2} (3 - \omega_4 + \nu), \\
\tilde{\nu} & = \sqrt{(\omega_4 - 1)^2 - \frac{4\kappa}{1 - 2\omega}}, \\
d & = \frac{2\omega}{2\omega - 1}.
\end{align*}
\]

Note that we changed to sign of the accessory parameter \( q \) in the original paper to the standard form we used in the second section. Here, \( \omega_4 \) and \( \kappa \) depends on \( \beta ' s \) coming from the modified commutation relations between the position and momentum operators, \([\hat{X}, \hat{P} j] = i\hbar \left[ 1 + \beta \hat{P}^2 \right] \delta_{ij} + \beta' \hat{P} i \hat{P} j \] and \( (\beta, \beta') > 0 \). The relations can be given as \( \omega_4 = \frac{\beta'}{\beta + \beta'} \) and \( \kappa = \frac{m\alpha}{2\hbar} \) where \( \alpha = \gamma - \beta' (\frac{\beta + 1}{\beta + \beta'})^2 > 0 \) and \( \gamma \) is an arbitrary constant \([15] \).

The case with \( \{ \epsilon = 0 \text{ and } q = abd \} \) is impossible to apply as the parameter \( \epsilon \) is fixed as \( \epsilon = 2 \). The case with \( \{ \Gamma = 0 \text{ and } q = 0 \} \) is also not applicable as we have \( \Gamma = \frac{3}{2} \).
The case with $\Delta = 0$ and $q = ab$

$\Delta = 0$ yields $\omega_4 = \frac{1}{2}$ and also $q = ab$ yields the same condition. Therefore $\omega_4 = \frac{1}{2}$ is the sufficient condition to have this case to transform from Heun to hypergeometric equation as

$$H = F \left( \frac{5 \sqrt{2} \omega - 1 - \sqrt{2} \omega - 1 + 16 \kappa}{4 \sqrt{2} \omega - 1}, \frac{5 \sqrt{2} \omega - 1 + \sqrt{2} \omega - 1 + 16 \kappa}{4 \sqrt{2} \omega - 1}, \frac{3}{2}, \frac{(2 \omega - 1)}{2\omega}^{-z} \right). \tag{56}$$

The trivial case \{\text{\textit{ab}} = 0 \text{ and } q = 0\}

$q = 0$ yields $\kappa = 3\omega - \frac{3}{2}$ and $ab = 0$ yields $\omega_4 = \frac{1}{2}$ and the solution simplifies dramatically as

$$H = C_1 + C_2 \left[ \arctan \left( \frac{\sqrt{2z} (2 \omega - 1)}{2 \sqrt{\omega} (2 \omega - 1)} \right) \frac{3 \sqrt{2} (2 \omega - 1)}{8 \omega^2 \sqrt{2} (2 \omega - 1)} - \frac{3z(2 \omega - 1) - 4\omega}{4\omega^2 \sqrt{2} |z(2 \omega - 1) - 2\omega|} \right], \tag{57}$$

where $C_1$ and $C_2$ are constants.

Nontrivial cases

As we have $d = \frac{2\omega}{2\omega - 1}$, we may seem to be free to choose a $(d, p)$ pair among the 23 pairs stated in the Theorem 3.7 of Maier [11] but picking $(d, p) = (2, 1)$ results in $\omega_4 = \frac{1}{2}$ which was also found in the case with $\{\Delta = 0 \text{ and } q = ab\}$, so this should be trivial. The pair $(d, p) = (-1, 0)$ to have $\omega = \frac{1}{2}$ and $\kappa = -\frac{3}{2}$ is also prohibited as $\kappa$ is restricted to have positive values ($\alpha > 0$). In general, the pairs with $p = 1$ yields $\omega_4 = \frac{1}{2}$ (trivial) and the pairs with $p = 0$ brings negative $\kappa$ values for the permitted $d$ values, namely $d \in \{-1, -3, -\frac{3}{2}\}$. The other pairs also result in nonpositive $\kappa$'s as $\omega_4$ should also be positive. The $\kappa$ values in these cases depend on $\omega_4$'s such that we always need a negative $\omega_4$ to have a positive $\kappa$ value. Therefore, we can conclude that this problem cannot be reduced nontrivially via harmonic singular points.

The equation for the limit density function for the discrete-time quantum walk

The relation between the limit density function and the Heun’s equation found by Konno et.al. can be stated as

$$H'' + \left( \frac{\Gamma}{z} + \frac{\Delta}{z - 1} + \frac{\epsilon}{z - a} \right) H' + \frac{abz - q}{z(z - 1)(z - d)} H = 0, \tag{58}$$

where

$$\Gamma = \frac{1}{2}, \Delta = 2, \epsilon = \frac{3}{2}, a = b = \frac{3}{2},$$

$$q = \frac{2d + 1}{4}, \tag{59}$$

and $d > 0$ [16].

We do not have the trivial cases as the parameters have fixed values and they do not satisfy the conditions needed for the them.

Nontrivial cases

The singular point $d$ is restricted to have positive values. Therefore we exclude the $(d, p)$ pairs with negative $d$ values. Another restriction is that the a and b parameters have fixed values to give $2d + 1 = 9p$ using $q = abp$. Only the pair $(d, p) = (4, 1)$ satisfies this condition. We have

$$H(4, ab; a, b, \frac{1}{2}, \frac{2(a + b)}{3}; z) = F \left[ \frac{a}{3}, \frac{b}{3}, \frac{1}{2}, 1 - (z - 1)^2 \left( 1 - \frac{z}{4} \right) \right], \tag{61}$$

as the equation (3.5b) in [11] and thus we have

$$H(4, \frac{9}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2; z) = F \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - (z - 1)^2 \left( 1 - \frac{z}{4} \right) \right], \tag{62}$$

as the reduction to the hypergeometric function. We should note that the pair $(4, 1)$ corresponds to degree 3 for the function $R$ as can be seen from the result.
6 Conclusion

We tried to give physical examples to the limiting cases in which the Heun equation reduces to the hypergeometric or related equations. We firstly analyzed the main equation obtained by studying the Coulomb problem on a 3-sphere and tried to reduce the solution which was originally given by the Heun’s functions to hypergeometric functions in some limiting cases. The second example was the s-wave bound state equation in the problem of the attractive inverse square potential, and we saw that the nontrivial cases were inapplicable in this problem but the trivial cases simplified the problem to a great extent. The last example was the equation for the limit density function for the discrete-time quantum walk and the trivial cases were impossible to study in this problem whereas in the nontrivial case we successfully reduced the equation using a polynomial transformation of degree 3.

The Heun-to-hypergeometric reductions are important mainly for two reasons: The first reason is that the hypergeometric equations have an enormous literature written on them and the second one is that the occurrence of the hypergeometric equations can be related to some symmetries of systems such as SL(2,R) symmetry [20]. Heun’s equations frequently emerge in modern applications in physics and their importance increases, whereas hypergeometric equations are much more studied and very well-known in detail. Therefore studying the reductions are important for understanding Heun’s equation and related physical systems better.

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8 Appendix: The SAGE code for calculating the identities

The code given in this appendix can be used to apply the identities mentioned in the text. The code is also accessible via GitHub [22].

```
reset()
# Define the variables:
d1,q1,a1,b1,Gamma1,Delta1,z1=var(’d1,q1,a1,b1,Gamma1,Delta1,z1’)
q,a,b,Gamma,Delta,z=var(’q,a,b,Gamma,Delta,z’)

# The values:
d1,q1,a1,b1,Gamma1,Delta1,z1 = -1,q,a,b,Gamma,Delta,z
oldvars=[d1,q1,a1,b1,Gamma1,Delta1,z1]

def iden1(oldvars):
    newvars=[]
    newvars.append(oldvars[0]/(oldvars[0]-1))
    newvars.append((oldvars[1]+oldvars[2]*oldvars[0]*oldvars[4])/(oldvars[0]-1))
    newvars.append(oldvars[2])
    newvars.append(oldvars[2]-oldvars[5]+1)
    newvars.append(oldvars[4])
    newvars.append(oldvars[2]+oldvars[3]+1)
    newvars.append(oldvars[6]/(oldvars[6]-1).simplify_full())
```
def iden2(oldvars):
    newvars=[]
    newvars.append(1/oldvars[0])
    newvars.append(oldvars[1]/oldvars[0])
    newvars.append(oldvars[2])
    newvars.append(oldvars[3])
    newvars.append(oldvars[4])
    newvars.append(oldvars[2]+oldvars[3]-oldvars[4]-oldvars[5]+1)
    newvars.append((oldvars[6]/oldvars[0]).simplify_full())
    return newvars

# Do the calculation:

# This is how you apply the first identity:
newvars=iden1(oldvars)

# This is how you apply the second identity:
newvars=iden2(oldvars)

# You can also apply the identities successively:
newvars=iden2(iden1(oldvars))

# Print the result:
writeresult='H( %s, %s; %s, %s, %s, %s; %s) \n=H( %s, %s; %s, %s, %s, %s; %s)''%(latex(d1),latex(q1),latex(a1),latex(b1)
    ,latex(Gamma1),latex(Delta1),latex(z1)
    ,latex(newvars[0]),latex(newvars[1])
    ,latex(newvars[2]),latex(newvars[3])
    ,latex(newvars[4]),latex(newvars[5]),latex(newvars[6]))
show(writeresult)

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