BOUNDARY VALUE PROBLEMS FOR HARMONIC FUNCTIONS
ON DOMAINS IN SIERPINSKI GASKETS

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Abstract. We study boundary value problems for harmonic functions on certain domains in the level-$l$ Sierpinski gaskets $S\mathcal{G}_l (l \geq 2)$ whose boundaries are Cantor sets. We give explicit analogues of the Poisson integral formula to recover harmonic functions from their boundary values. Three types of domains, the left half domain of $S\mathcal{G}_l$ and the upper and lower domains generated by horizontal cuts of $S\mathcal{G}_l$ are considered at present. We characterize harmonic functions of finite energy and obtain their energy estimates in terms of their boundary values. This paper settles several open problems raised in previous work.

1. Introduction. A Dirichlet problem is the problem of finding a function which is harmonic in the interior of a given domain that takes continuous prescribed values on the boundary of the domain. The solvability of this problem depends on the geometry of the boundary. For a bounded domain $D$ with sufficiently smooth boundary $\partial D$, the Dirichlet problem is always solvable, and the general solution is given by

$$ u(x) = \int_{\partial D} f(s) \partial_n G(x, s) ds, $$

where $G(x, y)$ is the Green’s function for $D$, $\partial_n G(x, s)$ is the normal derivative of $G(x, y)$ along the boundary and the integration is performed on the boundary. The integral kernel $\partial_n G(x, s)$ is called the Poisson kernel for $D$.

With a well developed theory of Laplacians on post-critically finite (p.c.f.) sets, originated by Kigami [4, 5], it is natural to look for analogous results in the fractal context. Harmonic functions on p.c.f. self-similar sets are of finite dimension. Due to the self-similar construction of the fractal, the Dirichlet problem on the entire fractal always reduces to solving systems of linear equations and multiplying

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matrices. However, for the boundary value problem on bounded subsets of fractals, the knowledge remains far from clear.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Upper and lower domains in $\mathcal{S}\mathcal{G}$ and $\mathcal{S}\mathcal{G}_3$.}
\end{figure}

The study of such problem was initiated in [13] by Strichartz, where the upper domain generated by a horizontal cut of the Sierpinski gasket $\mathcal{S}\mathcal{G}$ was considered. See Figure 1(a). Later it was continued in [11] and [1] to general case. In general, the boundary consists of a Cantor set together with the upper boundary vertex of $\mathcal{S}\mathcal{G}$. An explicit harmonic extension algorithm is given for solving the Dirichlet problem on such domains and the harmonic functions of finite energy are characterized in terms of their boundary values. The main tool is the \textit{Haar series expansion} of the boundary values on the Cantor set with respect to the normalized Hausdorff measure by symmetry consideration. Since the only \textit{generator} of the Haar basis is antisymmetric, one can localize the harmonic extension of this generator to any small scale along the boundary to get other basis harmonic functions. This observation plays a key role in their proof. However, as pointed out in [1], the results could not be extended to other fractals, even for the level-3 Sierpinski gasket $\mathcal{S}\mathcal{G}_3$ on the base of their approach. See Figure 1(b). The reason is that for $\mathcal{S}\mathcal{G}_3$ there exists a generator which is symmetric rather than antisymmetric whose harmonic extension could not be localized to small scales. On the other hand, the problem becomes much harder if we consider the domain lying below the horizontal cut instead. Except the very special case that the domains are made up of $2^m$ adjacent triangles of size $2^{-m}$ lying on the bottom line of $\mathcal{S}\mathcal{G}$ (in this case, the boundary is a finite set), we have little knowledge.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Half domains in $\mathcal{S}\mathcal{G}$ and $\mathcal{S}\mathcal{G}_3$.}
\end{figure}
Recently, there is another natural choice of domain, namely the left half part of $SG$ generated by a vertical cut along one of the symmetry lines of the gasket, becoming be interested. See Figure 2(a). It is the simplest example whose boundary is given as a level set of a harmonic function. In the $SG$ setting, the boundary of the half domain consists of a countably infinite set of points, which makes it possible to study the Dirichlet problem by solving systems of countably infinite linear equations and multiplying infinite matrices. See [10] for a satisfactory discussion on this domain, including an explicit harmonic extension algorithm, the characterization of harmonic functions of finite energy, and an explicit Dirichlet to Neumann map for harmonic functions. However, if we consider the left half domain of level-harmonic functions of finite energy, and an explicit Dirichlet to Neumann map for domain, including an explicit harmonic extension algorithm, the characterization of and multiplying infinite matrices. See [10] for a satisfactory discussion on this study the Dirichlet problem by solving systems of countably infinite linear equations that becomes be interested. See Figure 2(a). It is the simplest example whose boundary of $SG$ becomes a Cantor set together with the single left boundary vertex. See Figure 2(b).

In the following, we will use upper domain, lower domain and half domain to denote the above three types of domains respectively for simplicity. They are probably the simplest domains which should be handled in $SG$. In this paper, we will consider the analogues of them in level-$l$ Sierpinski gasket $SG_l$ for $l \geq 3$ instead, the approach in [10] is not applicable. Compared to the $SG$ case, the essential difference is that the boundary of the left half part of $SG_l$ becomes a Cantor set together with the single left boundary vertex. See Figure 2(b).

Nevertheless, these three types of domains are still the simplest domains in fractals with fractal boundary. We hope our results introduce different ideas and give insight into more general techniques for solving the Dirichlet problem and even other boundary value problems on more general fractal domains.

1.1. Preliminaries and the solvability of Dirichlet problems. Let $l \geq 2$, recall that the level-$l$ Sierpinski gasket $SG_l$ is the unique nonempty compact subset of $\mathbb{R}^2$ satisfying $SG_l = \bigcup_{i=0}^{2^{l+1}-2} F_i SG_l$ with $F_i$'s being contraction mappings defined as $F_i(z) = l^{-1}z + d_{l,i}$ with suitable $d_{l,i} \in \mathbb{R}^2$. The set $V_0$, which consists of the three vertices $q_0, q_1, q_2$ of the smallest triangle containing $SG_l$, is called the boundary. For convenience, we renumber $\{F_i\}_{i=0}^{2^{l+1}-2}$ so that $F_i(q_i) = q_i$ for $i = 0, 1, 2$. $SG_2$ is the standard Sierpinski gasket (denoted by $SG$ for simplicity). For $SG_3$, in addition to $F_0, F_1, F_2$, we denote by $F_3(z) = \frac{1}{2}z + \frac{1}{3}(q_1 + q_2)$, $F_4(z) = \frac{1}{2}z + \frac{1}{3}(q_0 + q_2)$ and $F_5(z) = \frac{1}{2}z + \frac{1}{3}(q_0 + q_1)$ the remaining three mappings, see Figure 3. These fractals have a well-developed theory of Laplacians, which allows us to perform analysis on them. In this paper, We will first describe the situation in more detail in the case of $SG_3$ for half domains and upper domains, and $SG$ for lower domains, then extend the considerations to general $SG_l$ case.
We introduce some necessary notations. Readers can refer to textbooks [6] and [15] for precise definitions and known facts. For $m \geq 1$, let $W_m = \{0, 1, \cdots, \frac{2^m-1}{2}\}^m$ be the collection of words with length $m$ and $W_0 = \{\emptyset\}$. Write $W_* = \bigcup_{m=0}^{\infty} W_m$, and denote the length of $w \in W_*$ by $|w|$. For $w = w_1w_2 \cdots w_m \in W_m$, we define $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ and call $F_wSG_l$ a $m$-cell of $SG_l$. Using the contraction mappings $F_i$, we inductively define sets of vertices $V_m$ by $V_m = \bigcup_{i=0}^{2^m-1} F_iV_{m-1}$ and write $V_* = \bigcup_{m \geq 0} V_m$. Denote $x \sim_m y$ if and only if $x \neq y$ and $x, y \in V_m$ belong to a same $m$-cell. The vertices $V_m$, together with the edge relation $\sim_m$, form a graph $\Gamma_m$ that approximates $SG_l$. See Figure 3 for an illustration for $SG_3$.

For $m \geq 0$, the natural discrete resistance form on $\Gamma_m$ is given by

$$\mathcal{E}_m(u, v) = r^{-m} \sum_{x \sim_m y} (u(x) - u(y))(v(x) - v(y))$$

for $u, v$ being functions defined on $V_m$, where $r = \frac{3}{2}$ for $SG$ and $r = \frac{7}{4}$ for $SG_3$. For a real-valued function $u$ defined on $V_*$, it is easy to check that the graph energies $\mathcal{E}_m(u) := \mathcal{E}_m(u, u)$ is an increasing sequence so that $\lim_{m \to \infty} \mathcal{E}_m(u)$ exists if we allow the value $+\infty$. Define

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u)$$

to be the energy of the function $u$ and say that $u \in \text{dom}\mathcal{E}$ if and only if $\mathcal{E}(u) < \infty$. We regard $\text{dom}\mathcal{E} \subset C(SG_l)$ since each function of finite energy admits a unique continuous extension to $SG_l$. Moreover, $\text{dom}\mathcal{E}$ is dense in $C(SG_l)$. There is a natural resistance form on $SG_l$ defined as

$$\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u, v)$$

for $u, v \in \text{dom}\mathcal{E}$.

Let $\nu$ be the standard(with equal weights) self-similar probability measure on $SG_l$. The standard Laplacian $\Delta$ could be defined using a weak formulation. Suppose $u \in \text{dom}\mathcal{E}$ and $f$ is continuous, say $u \in \text{dom}\Delta$ with $\Delta u = f$ if

$$\mathcal{E}(u, v) = -\int_{SG_l} fv\,d\nu$$

holds for all $v \in \text{dom}\mathcal{E}$ where $\text{dom}\mathcal{E} = \{v : v \in \text{dom}\mathcal{E}, v|_{V_0} = 0\}$.

A function $h$ is harmonic if it minimizes the energy from each level to its next level. All the harmonic functions form a 3-dimensional space, and hence any given values on $V_0$ can uniquely determine a harmonic function on $SG_l$. They are just
the solutions of the equation \( \Delta h = 0 \). In particular, there is an explicit extension algorithm, which determines \( h|_{V_1} \) in terms of \( h|_{V_0} \) and inductively \( h \circ F_w|_{V_1} \) in terms of \( h \circ F_w|_{V_2} \) for any \( w \in W_4 \) in a same manner. See Figure 4 for the exact formula for \( SG_3 \). A harmonic function \( h \) satisfies the mean value property, that is, for each \( m \geq 1, \)
\[
\sum_{y \sim m x} (h(x) - h(y)) = 0, \forall x \in V_m \setminus V_0.
\]

The normal derivative of a function \( u \) at a boundary point \( q_i \in V_0 \) is defined by

\[
\partial_n(q_i) = \lim_{m \to \infty} r^{-m} (2u(q_i) - u(F^m_i q_i) - u(F^m_i q_{i+1}))
\]
(cyclic notation \( q_3 = q_0 \)) providing the limit exists. For harmonic functions, these derivatives can be evaluated without taking limit. We could localize the definition of normal derivative to any vertex in \( V_\star \). Let \( x = F_w q_i \) be a boundary point of a \( m \)-cell \( F_w SG_i \). Define \( \partial_n^w u(x) \) the normal derivative of \( u \) at \( x \) with respect to the cell \( F_w SG_i \) to be \( r^{-m} \partial_n(u \circ F_w)(q_i) \). In this paper, we use the notations \( \partial_n^w, \partial_{n+}^w, \partial_{n-}^w \) to represent the normal derivatives of different directions for simplicity. In particular, \( \partial_n^w u(q_0) = \partial_n u(q_0), \partial_{n-}^w u(q_1) = \partial_n u(q_1), \partial_{n+}^w u(q_2) = \partial_n u(q_2) \). For \( u \in dom \Delta \), the sum of all normal derivatives of \( u \) in different directions must vanish at each \( x \in V_\star \setminus V_0 \). This is called the matching condition.

We have an analogue of Gauss-Green’s formula in the fractal setting. Suppose \( u \in dom \Delta \), then \( \partial_n u(q_i) \) exists for all \( q_i \in V_0 \) and

\[
\mathcal{E}(u, v) = -\int_{SG_i} (\Delta u)vd\nu + \sum_{q_i \in V_0} v(q_i)\partial_n u(q_i), \quad \forall v \in dom \mathcal{E}.
\]

We also have a localized version of this formula,

\[
\mathcal{E}_A(u, v) = -\int_A (\Delta u)vd\nu + \sum_{\partial A} v(x)\partial_n u(x)
\]
for any simple set \( A \), which is defined as a finite union of cells.
Let $\Omega$ be a half, upper or lower domain in $SG_l$. Consider the Dirichlet problem
\begin{equation}
\begin{aligned}
\Delta u &= 0 \text{ in } \Omega, \\
|_{\partial \Omega} u &= f, f \in C(\partial \Omega).
\end{aligned}
\end{equation}

**Proposition 1.** The Dirichlet problem (1) has a unique solution.

**Proof.** First, by Lemma 8.2 in [8], if there exists a function $v \in \text{dom}E$ such that $v|_{\partial \Omega} = f$, then a solution of (1) exists, which minimizes the energy on $\Omega$. For general case, notice that the set $\text{dom}E|_{\partial \Omega} := \{ f \in C(\partial \Omega) \mid \exists v \in \text{dom}E, v|_{\partial \Omega} = f \}$ is dense in $C(\partial \Omega)$, since $\text{dom}E$ is dense in $C(SG_l)$ and $\partial \Omega$ is a closed subset of $SG_l$. Let $\{f_n\}$ be a sequence of functions in $\text{dom}E|_{\partial \Omega}$ converging uniformly to $f$, and $u_n$ be their corresponding solutions of (1). Then $\{u_n\}$ also uniformly converge to a function $u$ with $u|_{\partial \Omega} = f$ by the maximum principle for harmonic functions. It is easy to get that $u$ is harmonic in $\Omega$.

The uniqueness of the solution is an immediate consequence of the maximum principle. \hfill \Box

1.2. The organization of the paper. Throughout this paper, although in different situations, we always use the same symbol $\Omega$ to denote the domain and $X$ to denote the Cantor set boundary without causing any confusion.

In Section 2, we solve the Dirichlet problem for the half domain in $SG_3$. An explicit harmonic extension algorithm is provided. Let $f$ be the prescribed value on $\partial \Omega$. We only need to find the explicit formula for the values of the extended harmonic function $u$ on $V_1 \cap \Omega$, since if we do so, then the value of $u$ in the 1-cells contained in $\bar{\Omega}$ is determined by the harmonic extension algorithm, and then the problem of finding values of $u$ in the remaining region is essentially the same by dilation. An interesting phenomenon is that the solution could be expressed explicitly in terms of only a countable set of points which is dense in $\partial \Omega$. We also characterize the energy estimate of solutions of finite energy in terms of their boundary values.

We consider the Dirichlet problem for the upper domain in $SG_3$ in Section 3. Basing on the same reason, we find the explicit formula for a finite number of crucial points, and then use dilation to continue. For the energy estimate, we still use the technique of Haar series expansion. But now we expand the boundary values with respect to a more natural probability measure rather than the normalized Hausdorff measure.

In Section 4, we deal with the lower domain in $SG$. Essentially the method is the same as before, but the situation is more complicated. We still obtain the explicit harmonic extension algorithm. However, it is unclear how to work out the energy estimate in term of the boundary values.

Finally, we show that our methods on the above three types of domains are still valid for general $SG_l$ and briefly state the outcomes. We present an intriguing correspondence between the normal derivatives and the boundary values of harmonic functions along boundaries on the half domain of $SG$, although we have no idea on how to extend it to general cases.

At the end of this section, we list some previous work on related topics. See [1, 2, 3, 7, 9, 10, 11, 16] and the references therein. In particular, in [9], some extension problems on $SG$, which aim at finding a function minimizing Sobolev types of norms with certain prescribed data such as values and derivatives at a finite set, are studied. It is interesting to consider analogous problems on domains...
considered in this paper. We leave these as open problems for future research. The energy estimates considered in this paper characterize the restriction to the Cantor set boundary $X$ of functions of finite energy on $\Omega$. It is also interesting to characterize the traces on $X$ of functions in some other Sobolev spaces, such as $\text{dom}_{L^2}(\Delta^k)$ defined as $\{u \in L^2(\mathcal{S}_l): \Delta u \in L^2(\mathcal{S}_l), \forall j \leq k\}$. One may also consider how to extend a function of finite energy defined on $\Omega$ to a function of finite energy on the whole $\mathcal{S}_l$, and analogous problems for other Sobolev spaces. Related problems are discussed in [10, 1]. The above mentioned Sobolev spaces are easily characterized in terms of expansions in eigenfunctions of the Laplacian, see [14]. For half domains, as pointed out in [10] in the $\mathcal{S}_l$ setting, essentially, there are no new eigenfunctions. A complete theory of the eigenspaces of the Laplacian on the upper domain in $\mathcal{S}_l$ with $X$ equal to the bottom line segment is given in [12].

2. Dirichlet problem on the half domain of $\mathcal{S}_3$. In this section, we focus on solving the Dirichlet problem on the half domain of $\mathcal{S}_3$. We will first give an extension algorithm for harmonic functions with continuous prescribed boundary values, then estimate the energies of them in terms of their boundary values.

2.1. Extension algorithm. The domain $\Omega$ can be defined by a level set of an antisymmetric harmonic function, denoted by $h_a$, with boundary values $h_a|_{\{q_0,q_1,q_2\}} = (0,1,-1)$. Define $\Omega = \{x \in \mathcal{S}_3 \setminus V_0 : h_a(x) > 0\}$, and the boundary $\partial \Omega = \{q_1\} \cup X$, with $X = \{x \in \mathcal{S}_3 : h_a(x) = 0\}$.

Let $\tilde{\Omega}$ denote the closure of $\Omega$. It is easy to check that $\tilde{\Omega} = F_1\mathcal{S}_3 \cup F_5\mathcal{S}_3 \cup F_0\tilde{\Omega} \cup F_3\tilde{\Omega}$.

As shown in Section 1, to solve the Dirichlet problem (1), we only need to find the explicit algorithm for the values of the harmonic function $u$ on $V_1 \cap \Omega$. For convenience, we use $x_0, y_0, z_0$ to represent the three “crucial” vertices in $V_1 \cap \Omega$ with $x_0 = F_1q_2, y_0 = F_1q_0, z_0 = F_0q_1$.

We also denote $p_0 = F_3q_0$.

For $m \geq 0$, write $\tilde{W}_m = \{0,3\}^m$ and $\tilde{W}_* = \bigcup_{m=0}^{\infty} \tilde{W}_m$.

Obviously, $\tilde{W}_m \subset W_m$ and $\tilde{W}_* \subset W_*$. Denote $x_w = F_w x_0, y_w = F_w y_0, z_w = F_w z_0, p_w = F_w p_0$ for $w \in \tilde{W}_*$. Obviously, $\{x_w, y_w, z_w\}_{w \in \tilde{W}_*} \subset V_* \cap \Omega$ and $\{p_w\}_{w \in \tilde{W}_*} = V_* \cap X \setminus \{q_0\}$.

Now, we proceed to show how to determine the values of the harmonic function $u$ on $V_1 \cap \Omega$ in terms of the boundary function $f$. From the matching condition at
each vertex in $V_1 \cap \Omega$, we have
\begin{align*}
\begin{cases}
\frac{7}{15} \partial_n^- u(x_0) + 2u(x_0) - u(y_0) - f(q_1) = 0, \\
4u(y_0) - u(x_0) - u(z_0) - f(q_1) - f(p_0) = 0, \\
\frac{7}{15} \partial_n^- u(z_0) + 2u(z_0) - u(y_0) - f(p_0) = 0.
\end{cases}
\end{align*}

To make the equations (2) enough to determine the unknowns, we need to represent the normal derivatives at $x_0$ and $z_0$ in terms of \{\{u(x_0), u(y_0), u(z_0)\}\} and the boundary data $f$.

We will prove that there exists a signed measure on the boundary $\partial \Omega$ such that the normal derivative of $u$ at $q_1$ could be evaluated as the integral of $f$ with respect to this measure. This signed measure is determined by the normal derivative of the antisymmetric harmonic function $h_a$ along the boundary $\partial \Omega$. See Figure 5 for the values of $h_a$ on $V_1 \cap \Omega$.

\[ \text{Figure 5. The values of } h_a \text{ on } V_1 \cap \Omega. \]

**Theorem 2.1.** Let $u$ be a solution of the Dirichlet problem (1). Then
\[ \partial_n^- u(q_1) = 3f(q_1) - \sum_{w \in W} \frac{6}{7} \mu_w f(p_w), \]
where $\mu_w = \mu_{w_1} \mu_{w_2} \cdots \mu_{w_{|w|}}$ with $\mu_0 = \frac{1}{7}$ and $\mu_3 = \frac{4}{7}$. In addition, if $u \in \text{dom} E_{\Omega}$, we have
\[ E_{\Omega}(h_a, u) = \partial_n^- u(q_1). \]

**Proof.** Set $O_1 = F_1 SG_3 \cup F_2 SG_3$, and $O_m = \bigcup_{w \in W_m, |w| \leq m-1} F_w O_1$ for $m \geq 2$. Obviously, $\bar{\Omega}$ equals the closure of $\bigcup_{m \geq 1} O_m$. See Figure 6 for $O_1$ and $O_2$.

Applying the local Gauss-Green’s formula on $O_m$, we get
\[ E_{O_m}(h_a, u) = \partial_n^- h_a(q_1) f(q_1) + \sum_{w \in W_m, |w| \leq m-1} \partial_n^- h_a(p_w) f(p_w) \]
\[ + \sum_{w \in W_{m-1}} \left( \partial_n^- h_a(x_w) u(x_w) + \partial_n^+ h_a(z_w) u(z_w) \right). \]
It is easy to calculate the normal derivatives of $h_a$ at $p_w, x_w, z_w$, 
\[
\partial_n^+h_a(p_w) = -\frac{6}{7}\mu_w, \quad \partial_n^-h_a(x_w) = -\frac{12}{7}\mu_w, \quad \partial_n^\uparrow h_a(z_w) = -\frac{3}{7}\mu_w.
\]
So we have the estimate that 
\[
\left| \mathcal{E}_{O_m}(h_a, u) - (3f(q_1) - \sum_{w\in W_*} \frac{6}{7}\mu_w f(p_w)) \right| 
\leq \left| \sum_{w\in W_{m-1}} (\partial_n^+h_a(x_w) + \partial_n^\uparrow h_a(z_w)) + \sum_{w\in W_{m-1}, |w|\geq m} \partial_n^+h_a(p_w) \right| \cdot \| f \|_{\infty} 
= \frac{30}{7} \left( \frac{5}{7} \right)^{m-1} \| f \|_{\infty}.
\]
Thus if $u \in \text{dom}\mathcal{E}_{\Omega}$, we have $\mathcal{E}_{\Omega}(h_a, u) = 3f(q_1) - \sum_{w\in W_*} \frac{6}{7}\mu_w f(p_w)$ by taking the limit.

For the rest part of the theorem, we introduce a sequence of harmonic functions \( \{u_n\}_{n\geq 0} \) which are piecewise constant on $X$, defined as $u_n|_{\bar{f}_n} = f(p_{\tau}), \forall \tau \in \bar{W}_n$. The existence of such functions is ensured by Proposition 1. Moreover, it is easy to check that $u_n$ uniformly converges to $u$ by the maximum principle for harmonic functions. Applying Gauss-Green’s formula, we have 
\[
\mathcal{E}_{O_m}(h_a, u_n) = \partial_n^-u_n(q_1)h_a(q_1) + \sum_{w\in W_{m-1}, |w|\leq m-1} \partial_n^-u_n(p_w)h_a(p_w) 
+ \sum_{w\in W_{m-1}} (\partial_n^-u_n(x_w)h_a(x_w) + \partial_n^\uparrow u_n(z_w)h_a(z_w)) 
= \partial_n^-u_n(q_1) + \sum_{w\in W_{m-1}} (\partial_n^-u_n(x_w)h_a(x_w) + \partial_n^\uparrow u_n(z_w)h_a(z_w)).
\]

For fixed $n$ and $\tau \in \bar{W}_n$, we have that $\partial_n^-u_n(x_{\tau w}), \partial_n^\uparrow u_n(z_{\tau w})$ take the same sign, and $\sum_{w\in W_{m-1}, |w|=m} (\partial_n^-u_n(x_{\tau w}) + \partial_n^\uparrow u_n(z_{\tau w}))$ are uniformly bounded, as $u_n \circ
\[ F_r = c_1 + c_2 h_a \] for some constants \( c_1, c_2 \). In addition, \( h_a(x_w) \) and \( h_a(z_w) \) converge uniformly to 0 as \( |w| \to \infty \). Thus, letting \( m \to \infty \), we get

\[
\mathcal{E}_\Omega(h_a, u_n) = \lim_{m \to \infty} \left( \partial_n^- u_n(q_1) + \sum_{\tau \in W_n} \sum_{w \in W_{na}} \left( \partial_n^- u(x_{tw})h_a(x_{tw}) + \partial_n^+ u(z_{tw})h_a(z_{tw}) \right) \right)
\]

\[ = \partial_n^- u_n(q_1). \]

Combining this equality with the first part of the proof, we then have

\[
\partial_n^- u_n(q_1) = 3u_n(q_1) - \sum_{w \in W_x} \frac{6}{7} \mu_w u_n(p_w).
\]

Taking \( n \to \infty \), we get (3). \( \square \)

**Remark 1.** One can regard the signed measure \( 3\delta_{q_1} - \sum_{w \in W_x} \frac{6}{7} \mu_w \delta_{p_w} \) as the normal derivative of \( h_a \) on \( \partial \Omega \). In this opinion, Theorem 2.1 is just a result of the extended “Guass-Green’s formula” acting on \( h_a \) and \( u \).

In the following, we denote \( \mu \) the probability measure \( \sum_{w \in W_x} \frac{2}{7} \mu_w \delta_{p_w} \) on \( X \). Thus we could write

\[
\partial_n^- u(q_1) = 3f(q_1) - 3 \int_X f d\mu.
\]

(4)

Now, we have enough information to calculate the values \( u(x_0), u(y_0), u(z_0) \).

**Theorem 2.2** (Extension algorithm). There exists a unique solution of the Dirichlet problem (1). In addition, we have

\[
u(x_0) = \frac{4}{15} f(q_1) + \frac{4}{15} f(p_0) + \frac{1}{15} \int_X f \circ F_0 d\mu + \frac{19}{30} \int_X f \circ F_3 d\mu,
\]

(5)

\[
u(y_0) = \frac{1}{3} f(q_1) + \frac{1}{3} f(p_0) + \frac{1}{6} \int_X f \circ F_0 d\mu + \frac{1}{15} \int_X f \circ F_3 d\mu,
\]

(6)

\[
u(z_0) = \frac{1}{15} f(q_1) + \frac{1}{15} f(p_0) + \frac{19}{30} \int_X f \circ F_0 d\mu + \frac{1}{30} \int_X f \circ F_3 d\mu.
\]

(7)

**Proof.** The existence and uniqueness of a solution of (1) has been shown in Proposition 1. Taking \( \partial_n^- u(x_0) = \frac{12}{7} \partial_n^- (u \circ F_3)(q_1) = \frac{15}{7} (3u(x_0) - 3 \int_X f \circ F_3 d\mu) \) and \( \partial_n^- u(z_0) = \frac{15}{7} \partial_n^- (u \circ F_0)(q_1) = \frac{15}{7} (3u(z_0) - 3 \int_X f \circ F_0 d\mu) \) into (2), and solving the system of linear equations, we get (5), (6) and (7). \( \square \)

### 2.2. Energy estimate

In Theorem 2.2, we have shown that the harmonic function \( u \) could be explicitly determined by its values at only countably infinite vertices \( \{q_1\} \cup \{p_w\}_{w \in W_x} \). It is natural to hope that the energy estimate of \( u \) also depends on the same values as well.

**Theorem 2.3.** Let \( f \in C(\partial \Omega) \), and write

\[
Q(f) = (f(q_1) - f(p_0))^2 + \sum_{w \in W_x} \left( \frac{15}{7} \right)^{|w|} \left( (f(p_w) - f(p_{w0}))^2 + (f(p_w) - f(p_{w3}))^2 \right).
\]

Then we have the energy estimate that

\[
C_1 Q(f) \leq \mathcal{E}(u) \leq C_2 Q(f),
\]

where \( C_1, C_2 \) are two positive constants independent of \( f \).
Proof. Notice that
\[ E_{O_1}(u) = \frac{15}{7} \left( (f(q_1) - u(x_0))^2 + (f(q_1) - u(y_0))^2 + (u(x_0) - u(y_0))^2 \right. \\
\left. + (f(p_0) - u(y_0))^2 + (f(p_0) - u(z_0))^2 + (u(y_0) - u(z_0))^2 \right) \]
\[ \geq \frac{45}{28} (f(q_1) - f(p_0))^2, \]
where the equality holds when \( u(x_0) = \frac{3}{4} f(q_1) + \frac{1}{4} f(p_0), u(y_0) = \frac{1}{2} f(q_1) + \frac{1}{2} f(p_0) \) and \( u(z_0) = \frac{1}{4} f(q_1) + \frac{3}{4} f(p_0) \). Similarly, for any \( w \in W_* \), we also have
\[ E_{F_wO_1}(u) + E_{F_wO_1}(u) \geq c_1 \left( \frac{15}{7} |w| (f(p_w) - f(p_{w0}))^2 \right), \]
and
\[ E_{F_wO_1}(u) + E_{F_wO_1}(u) \geq c_2 \left( \frac{15}{7} |w| (f(p_w) - f(p_{w3}))^2 \right), \]
where \( c_1, c_2 \) are suitable positive constants. Thus, we have
\[ E_{O}(u) = \sum_{w \in W_*} E_{F_wO}(u) = E_{O_1}(u) + \sum_{w \in W_*} \left( E_{F_wO_1}(u) + E_{F_wO_1}(u) \right) \]
\[ = \frac{1}{3} E_{O_1}(u) + \frac{1}{3} \sum_{w \in W_*} \left( 2E_{F_wO_1}(u) + E_{F_wO_1}(u) + E_{F_wO_1}(u) \right) \]
\[ \geq \frac{1}{3} \min\{c_1, c_2, \frac{45}{28} \} \cdot Q(f). \]

Conversely, we assume without loss of generality that \( Q(f) < \infty \), otherwise there is nothing to prove. Consider a piecewise harmonic function \( v \) defined on \( \Omega \), which is harmonic in \( F_wO_1, \forall w \in W_* \) with values \( v|_{\partial \Omega} = f \) and \( v(x_w) = v(z_w) = f(p_w), \forall w \in W_* \). See Figure 7 for the value of this function.
It is easy to calculate the energy of $v$,
\[
\mathcal{E}_\Omega(v) = \sum_{w \in \tilde{W}} \mathcal{E}_{F_w, O_1}(v) \\
= \frac{15}{4} (f(q_1) - f(p_0))^2 \\
+ \sum_{w \in \tilde{W}} \frac{7}{4} (\frac{15}{4} |w| + 2) \left( (f(p_w) - f(p_{w0}))^2 + (f(p_w) - f(p_{w3}))^2 \right) \\
\leq \frac{225}{28} Q(f).
\]

On the other hand, $\mathcal{E}_\Omega(v) \geq \mathcal{E}_\Omega(u)$, as harmonic functions minimize the energy.

3. **Dirichlet problem on upper domains of $SG_3$.** In this section, we deal with the Dirichlet problem on upper domains of $SG_3$. Prescribe that the boundary vertices $q_0, q_1, q_2 \in \mathbb{R}^2$ take the following coordinates,
\[
q_0 = \left( \frac{1}{\sqrt{3}}, 1 \right), \quad q_1 = (0, 0), \quad q_2 = \left( \frac{2}{\sqrt{3}}, 0 \right).
\]
Then for each $0 < \lambda < 1$, define the upper domain
\[
\Omega_\lambda = \{(x, y) \in SG_3 \setminus V_0 | y > 1 - \lambda \},
\]

together with the boundary
\[
\partial \Omega_\lambda = \{q_0\} \cup X_\lambda, \quad \text{with} \quad X_\lambda = \{(x, y) \in SG_3 | y = 1 - \lambda \}.
\]
See Figure 8 for an illustration. Denote $\bar{\Omega}_\lambda = \Omega_\lambda \cup \partial \Omega_\lambda$ the closure of $\Omega_\lambda$. In the following context, we write $X$ instead of $X_\lambda$ when there is no confusion.

![Figure 8. The upper domain.](image)

For $0 < \lambda \leq 1$, there is a unique representation
\[
\lambda = \sum_{k=1}^{\infty} t_k 3^{-m_k}, \quad (8)
\]
with an integer sequence $0 < m_1 < m_2 < \cdots$, and $t_k = 1$ or 2. Denote
\[
R\lambda = \sum_{k=2}^{\infty} t_k 3^{-(m_k - m_1)}.
\]
Inductively, write
\[ \lambda_n = R^n \lambda = \sum_{k=n+1}^{\infty} t_k 3^{-(m_k - m_n)}. \]

Set \( \lambda_0 = \lambda \) and \( m_0 = 0 \).

It is easy to check the following relationship between \( \Omega_{\lambda_n} \) and \( \Omega_{\lambda_{n+1}} \),

\[ \bar{\Omega}_{\lambda_n} = \begin{cases} F_{m_{n+1} - m_n}^{m_{n+1} - m_n - 1} (F_0SG_3 \cup F_4\bar{\Omega}_{\lambda_{n+1}} \cup F_5\bar{\Omega}_{\lambda_{n+1}}) & \text{if } \iota_{n+1} = 1, \\ F_{m_{n+1} - m_n}^{m_{n+1} - m_n - 1} (F_0SG_3 \cup F_4SG_3 \cup F_5SG_3 \cup F_1\bar{\Omega}_{\lambda_{n+1}} \cup F_2\bar{\Omega}_{\lambda_{n+1}} \cup F_3\bar{\Omega}_{\lambda_{n+1}}) & \text{if } \iota_{n+1} = 2. \end{cases} \]  

(9)

For \( 1 \leq i \leq 5 \), write \( p_i^\lambda = F_{m_1 - 1} F_{w_1} \cdots F_{m_n - m_{n-1}}^{-1} F_{w_n} \), for \( w \in W_\lambda^n \).

Then \( \forall n \geq 0, \)

\[ X = \bigcup_{w \in W_\lambda^n} X_w \]

where \( X_w = F_w^\lambda SG_3 \cap X \). It is easy to see that for \( \lambda \) not a triadic rational, \( X \) is homeomorphic to the space \( \Sigma^{\lambda} = \prod_{k=1}^{\infty} S_{\iota_k} \) equipped with the product topology. Otherwise, \( X \) is a union of finite segments.

Here we give an example to help readers to get familiar with the notations.

**Example 1.** We plot the area \( \Omega_{\lambda} \) for \( \lambda = 0.39 = 3^{-1} + 3^{-2} + 2 \cdot 3^{-3} + \cdots \). See Figure 10. In this case, \( \iota_1 = 1, \iota_2 = 1, \iota_3 = 2 \) and

\[ W_1^\lambda = \{4, 5\}, W_2^\lambda = \{4, 5\}^2 = \{44, 45, 54, 55\}, W_3^\lambda = \{4, 5\}^2 \times \{1, 2, 3\}. \]
Figure 10. The upper domain $\Omega_{0.39}$. The shaded regions are $F^3_5\Omega_{\lambda_1}$, $F^3_5\Omega_{\lambda_2}$, $F^3_5\Omega_{\lambda_3}$.

3.1. Extension algorithm. We still use $f$ to denote the boundary data on $\partial \Omega_\lambda$ and $u$ the harmonic solution of the Dirichlet problem (1). We only need to find an explicit algorithm for the values of $u$ at $p_i$'s since if we do so, the problem of finding values of $u$ in the remaining region is essentially the same after dilation.

From the matching condition at each vertex $p_i$, we have the following system of equations.

Case 1($\iota_1 = 1$):
\[
\begin{align*}
\partial_\nu u(p_5) + \left(\frac{15}{7}\right)^{m_1} \left(2u(p_5) - u(p_4) - f(q_0)\right) &= 0, \\
\partial_\nu u(p_4) + \left(\frac{15}{7}\right)^{m_1} \left(2u(p_4) - u(p_5) - f(q_0)\right) &= 0.
\end{align*}
\] (10)

Case 2($\iota_1 = 2$):
\[
\begin{align*}
4u(p_5) - u(p_1) - u(p_3) - u(p_4) - f(q_0) &= 0, \\
4u(p_4) - u(p_2) - u(p_3) - u(p_5) - f(q_0) &= 0, \\
\partial_\nu u(p_1) + \left(\frac{15}{7}\right)^{m_1} \left(2u(p_1) - u(p_4) - u(p_5)\right) &= 0, \\
\partial_\nu u(p_2) + \left(\frac{15}{7}\right)^{m_1} \left(2u(p_2) - u(p_3) - u(p_4)\right) &= 0, \\
\partial_\nu u(p_3) + \left(\frac{15}{7}\right)^{m_1} \left(4u(p_3) - u(p_1) - u(p_2) - u(p_4) - u(p_5)\right) &= 0.
\end{align*}
\] (11)

Due to the same consideration in Section 2, we need to express the normal derivatives $\partial_\nu u(p_i)$'s in the above equations in terms of $u(p_i)$'s and the boundary data $f$. Thus we turn to find the explicit representation of $\partial_\nu u(q_0)$ in terms of the boundary values.

For this purpose, we need to look at the normal derivative along the boundary of a special harmonic function on $\Omega_\lambda$, denoted by $h_0^\lambda$, assuming value 1 at $q_0$ and 0 along $X$. We will write $h_0 = h_0^\lambda$ for simplicity if there is no confusion.

Lemma 3.1. Denote $\alpha(\lambda) = h_0(p_4)$ ($= h_0(p_5)$ by symmetry) and $\eta(\lambda) = \partial_\nu h_0(q_0) = 2\left(\frac{15}{7}\right)^{m_1} \left(1 - \alpha(\lambda)\right)$. Then
\[
\alpha(\lambda) = \begin{cases} 
\frac{1}{1 + \eta(R\lambda)} & \text{if } \iota_1 = 1, \\
\frac{6 + 6\eta(R\lambda) + \eta(R\lambda)^2}{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2} & \text{if } \iota_1 = 2.
\end{cases}
\] (12)
Denote

\[
T^\lambda(x) = \begin{cases}
    1 & \text{if } \iota_1 = 1, \\
    1 + 2\left(\frac{15}{7}\right)^{m_2-m_1}(1-x) & \text{if } \iota_1 = 2,
\end{cases}
\]

then \(\alpha(\lambda) = T^\lambda(\alpha(R\lambda))\). In addition,

\[
\alpha(\lambda) = \lim_{n \to \infty} T^\lambda \circ T^{\lambda_1} \circ \cdots \circ T^{\lambda_n}(0).
\]

**Proof.** By (9) and the definition of \(h_0\), we have \(h_0 \circ F_1^\lambda = h_0(p_i)h_i^{R\lambda}, \forall i \in W_1^\lambda\). Taking \(u = h_0\) and \(\partial_n^i h_0(p_i) = (\frac{15}{7})^{m_1} h_0(p_i) \eta(R\lambda)\) into (10) or (11), noticing that \(h_0\) is symmetric in \(\Omega_\lambda\), we have,

\[
\eta(R\lambda) \alpha(\lambda) + (2\alpha(\lambda) - \alpha(\lambda) - 1) = 0, \text{ if } \iota_1 = 1,
\]

or

\[
\begin{cases}
    4\alpha(\lambda) - \alpha(\lambda) - 1 - h_0(p_1) - h_0(p_3) = 0, \\
    \eta(R\lambda) h_0(p_1) + (2h_0(p_1) - \alpha(\lambda) - h_0(p_3)) = 0, \text{ if } \iota_1 = 2. \\
    \eta(R\lambda) h_0(p_1) + (4u(p_3) - 2\alpha(\lambda) - 2h_0(p_1)) = 0,
\end{cases}
\]

Solving the above equations, we get (12) and

\[
\begin{cases}
    h_0(p_1) = h_0(p_2) = \frac{6 + \eta(R\lambda)}{6 + 2\eta(R\lambda)}, \\
    h_0(p_3) = \frac{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2}{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2},
\end{cases}
\]

in case of \(\iota_1 = 2\). (13)

Moreover, substituting \(\eta(R\lambda) = 2\left(\frac{15}{7}\right)^{m_2-m_1}(1 - \alpha(R\lambda))\) into (12), we have \(\alpha(\lambda) = T^\lambda(\alpha(R\lambda))\). Inductively,

\[
\alpha(\lambda) = T^\lambda \circ T^{\lambda_1} \circ \cdots \circ T^{\lambda_{n-1}}(\alpha(\lambda_n)).
\]

For the rest part of the theorem, we introduce a sequence of functions \(u_n^\lambda\) which assume values \(u_n^\lambda(F_w^{\lambda_n}W_4^\lambda) = u_n^\lambda(F_w^{\lambda_n}P_5^\lambda) = 0, \forall w \in W_1^\lambda\) and \(u_n^\lambda|_{\partial \Omega_\lambda} = h_0|_{\partial \Omega_\lambda}\), and take harmonic extension in the remaining region. A similar discussion yields that

\[
u_n^\lambda(p_4) = T^\lambda \circ T^{\lambda_1} \circ \cdots \circ T^{\lambda_{n-1}}(0).
\]

In fact, \(\forall i \in W_1^\lambda\), we have \(u_n^\lambda \circ F_1^\lambda = u_n^\lambda(p_i)u_i^{R\lambda} \circ V_{-1}.\) Taking \(u = u_n^\lambda\) and \(\partial_n^i u_n^\lambda(p_i) = (\frac{15}{7})^{m_1} u_n^\lambda(p_i) \partial_n^i u_i^{R\lambda} \circ V_{-1}(q_0)\) into (10) or (11), after a similar calculation we have \(u_n^\lambda(p_4) = T^\lambda(u_{n-1}^\lambda(p_4))\).

Looking at the region bounded by \(\{q_0\} \cup \{F_w^{\lambda_n}W_4^\lambda, F_w^{\lambda_n}P_5^\lambda\}_{w \in W_1^\lambda}\), applying the maximum principle for harmonic functions, we get

\[
0 \leq (h_0 - u_n^\lambda)(p_4) \leq \max_{w \in W_1^\lambda, 4,5} (h_0 - u_n^\lambda)(F_w^{\lambda_n}p_5^\lambda).
\]

Since for \(i = 4, 5\), \(u_n^\lambda(F_w^{\lambda_n}P_5^\lambda) = 0, \forall w \in W_1^\lambda\), and \(h_0(F_w^{\lambda_n}P_5^\lambda)\) converges uniformly to 0 as \(|w| \to \infty\), we then have \(u_n^\lambda(p_4) \to h_0(p_4)\) as \(n \to \infty\).

**Remark 2.** In fact, we have

\[
\alpha(\lambda) = \lim_{n \to \infty} T^\lambda \circ T^{\lambda_1} \circ \cdots \circ T^{\lambda_n}(c),
\]

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Definition 3.2. Let $0 < \lambda\leq 1$, denote
\[
\mu_i^\lambda = \begin{cases}
\frac{1}{2} & \text{for } i = 4, 5, \text{ if } \iota_1 = 1, \\
\frac{6 + \eta(R\lambda)}{18 + 4\eta(R\lambda)} & \text{for } i = 1, 2, \\
\frac{9 + 2\eta(R\lambda)}{18 + 4\eta(R\lambda)} & \text{for } i = 3,
\end{cases}
\]
or
\[
\mu_i^\lambda = \begin{cases}
\frac{6 + \eta(R\lambda)}{18 + 4\eta(R\lambda)} & \text{for } i = 1, 2, \\
\frac{9 + 2\eta(R\lambda)}{18 + 4\eta(R\lambda)} & \text{for } i = 3,
\end{cases}
\]
where $\eta(R\lambda)$ is an increasing function of $\lambda$ on $(\frac{1}{3}, 1]$, and $h_0^\lambda|_{\bar{\Omega}_{\lambda_a}} \geq h_0^{\lambda_a}$ if $\lambda_a \geq \lambda_a$. For $0 < \lambda \leq \frac{1}{3}$, we have $\alpha(\lambda) = \alpha(3^{\frac{1}{m_1}-1}\lambda)$ by dilation. Thus
\[
0 < \alpha(\lambda) \leq \alpha(1) = \frac{75 - \sqrt{2353}}{60} \approx 0.441538,
\]
where $\alpha(1)$ is the root of $x = \frac{3 + 6 + \frac{15}{2}(1-x)+2(\frac{1}{2})^2(1-x)^2}{3 + 15 + 6(\frac{1}{2})^2(1-x)^2}$.

Theorem 3.4. Let $u$ be a solution of the Dirichlet problem (1). Then
\[
\partial_n^\lambda u(q_0) = \eta(\lambda)(f(q_0) - \int_X f d\mu^\lambda).
\]
In addition, if $u \in \text{dom} \mathcal{E}_{\Omega_{\lambda}}$, we have
\[
\mathcal{E}_{\Omega_{\lambda}}(h_0, u) = \partial_n^\lambda u(q_0) = \eta(\lambda)(f(q_0) - \int_X f d\mu^\lambda).
\]
Proof. For \( n \geq 1 \), let \( O_{\lambda,n} \) be the simple set with boundary vertices \( \{q_0\} \cup \{P_{\lambda}q_0\}_{w \in W_{\lambda}} \). Noticing that \( \{O_{\lambda,n}\}_{n \geq 1} \) is expanding with \( \bigcup_{n \geq 1} O_{\lambda,n} = \Omega_{\lambda} \), using a similar proof as that of Theorem 2.1, the theorem follows.

**Corollary 3.5.** \( \mathcal{E}_{\Omega_{\lambda}}(h_0) = \eta(\lambda) \).

**Proof.** We only need to show that \( h_0 \in \text{dom}\mathcal{E}_{\Omega_{\lambda}} \). This is obvious since for any \( n \geq 1 \), by Gauss-Green’s formula, we always have

\[
\mathcal{E}_{O_{\lambda,n}}(h_0) = \partial_n^i h_0(q_0) - \sum_{w \in W_{\lambda}} \partial_n^i h_0(P_{\lambda}q_0)h_0(P_{\lambda}q_0) \leq \partial_n^i h_0(q_0).
\]

By (14) and the fact that \( \eta(\lambda) = 2\left(\frac{\alpha(\lambda)}{d}\right) (1 - \alpha(\lambda)) \), the energy of \( h_0 \) is estimated by

\[
2(1 - \alpha(1)) \left(\frac{15}{7}\right)^{m_1} \approx 1.116924\left(\frac{15}{7}\right)^{m_1} \leq \eta(\lambda) = \mathcal{E}_{\Omega_{\lambda}}(h_0) < 2\left(\frac{15}{7}\right)^{m_1}.
\]

This result will be helpful in the energy estimate for general functions.

Combining Theorem 3.4 with equations (10) and (11), we could calculate the values of the solution \( u \) at the “crucial” points \( p_i \)’s for \( i \in W_{\lambda}^1 \), which are sufficient to recover \( u \) by induction.

**Theorem 3.6 (Extension Algorithm).** There exists a unique solution of the Dirichlet problem (1.1). In addition, we have the following formulas for \( u(p_i), i \in W_{\lambda}^1 \).

**Case 1 (\( i_1 = 1 \)):**

\[
u(p_1) = \frac{1}{1 + \eta(R\lambda)} f(q_0) + \frac{2\eta(R\lambda) + \eta(R\lambda)^2}{3 + 4\eta(R\lambda) + \eta(R\lambda)^2} \int_{X_{R\lambda}} f \circ F_{\lambda}^1 d\mu_{R\lambda} + \frac{\eta(R\lambda)}{3 + 4\eta(R\lambda) + \eta(R\lambda)^2} \int_{X_{R\lambda}} f \circ F_{\lambda}^2 d\mu_{R\lambda}.
\]

**Case 2 (\( i_1 = 2 \)):**

\[\begin{align*}
u(p_2) &= \frac{1}{54 + 165\eta(R\lambda) + 102\eta(R\lambda)^2 + 15\eta(R\lambda)^3} \left( (54 + 39\eta(R\lambda) + 5\eta(R\lambda)^2) f(q_0) \ight. \left. + (60\eta(R\lambda) + 76\eta(R\lambda)^2 + 15\eta(R\lambda)^3) \int_{X_{R\lambda}} f \circ F_{\lambda}^1 d\mu_{R\lambda} + \right. \\
&\quad \left. + (30\eta(R\lambda) + \eta(R\lambda)^2) \int_{X_{R\lambda}} f \circ F_{\lambda}^2 d\mu_{R\lambda} + \ight. \\
&\quad \left. + (36\eta(R\lambda) + 20\eta(R\lambda)^2) \int_{X_{R\lambda}} f \circ F_{\lambda}^3 d\mu_{R\lambda} \right),
\]

\[\begin{align*}
u(p_3) &= \frac{6 + 2\eta(R\lambda)}{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2} f(q_0) + \frac{5\eta(R\lambda) + 3\eta(R\lambda)^2}{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2} \int_{X_{R\lambda}} f \circ F_{\lambda}^1 d\mu_{R\lambda} + \\
&\quad + \frac{4\eta(R\lambda)}{6 + 15\eta(R\lambda) + 3\eta(R\lambda)^2} \left( \int_{X_{R\lambda}} f \circ F_{\lambda}^2 d\mu_{R\lambda} + \int_{X_{R\lambda}} f \circ F_{\lambda}^3 d\mu_{R\lambda} \right).
\]
\[
\begin{align*}
\psi_{(1),\lambda}^3(1) & = \frac{1}{54 + 165\eta(R\lambda) + 102\eta(R\lambda)^2 + 15\eta(R\lambda)^3} \\
& \quad \times \left((54 + 84\eta(R\lambda) + 39\eta(R\lambda)^2 + 5\eta(R\lambda)^3) f(q_0) \right) \\
& \quad + (24\eta(R\lambda) + 12\eta(R\lambda)^2 + \eta(R\lambda)^3) \int_{X_{RA}} f \circ F_3^\lambda d\mu^\lambda \\\n& \quad + (30\eta(R\lambda) + 27\eta(R\lambda)^2 + 4\eta(R\lambda)^3) \int_{X_{RA}} f \circ F_2^\lambda d\mu^\lambda \\
& \quad + (27\eta(R\lambda) + 24\eta(R\lambda)^2 + 5\eta(R\lambda)^3) \int_{X_{RA}} f \circ F_1^\lambda d\mu^\lambda.
\end{align*}
\]

The formulas for \(u(p_4), u(p_5)\) can be obtained symmetrically.

**Proof.** See Proposition 1 for the existence and uniqueness of the solution. Substituting \(\partial_1^w u(p_i) = (\frac{15}{\eta})^{\mu-\eta} f(R\lambda)(u(p_i) - \int_{X_{RA}} f \circ F_i^\lambda d\mu^\lambda)\) for \(i \in W_3^\lambda\) into (10) or (11), after solving linear equations, we get the result. \(\square\)

### 3.2. Haar series expansion and energy estimate

Now we consider the energy estimate for the harmonic functions in terms of their boundary values. For a harmonic function \(u\) with boundary value \(u|_\Omega = f\) in \(L^2(X, \mu^\lambda)\), we will give an estimation of \(\mathcal{E}_{\Omega\lambda}(u)\) in terms of the Fourier coefficients of \(f\) with respect to a Haar basis.

**Definition 3.7.** (1) Assume \(0 < \lambda \leq 1\). Define \(\psi^{(1),\lambda}\) and \(\psi^{(2),\lambda}\) to be piecewise constant functions on \(X\lambda\) such that,

\[
\psi^{(1),\lambda}|_{X_3} = 1, \quad \psi^{(1),\lambda}|_{X_4} = -1, \quad \text{if } \iota_1 = 1, \quad (18)
\]

\[
\begin{cases}
\psi^{(1),\lambda}|_{X_1} = 1, \quad \psi^{(1),\lambda}|_{X_2} = -1, \quad \psi^{(1),\lambda}|_{X_3} = 0, \\
\psi^{(2),\lambda}|_{X_1} = \psi^{(2),\lambda}|_{X_2} = \mu_3^\lambda, \quad \psi^{(2),\lambda}|_{X_3} = -2\mu_1^\lambda,
\end{cases} \quad \text{if } \iota_1 = 2, \quad (19)
\]

and there is no \(\psi^{(2),\lambda}\) for \(\iota_1 = 1\). In addition, for \(w \in W_n^\lambda\), define \(\psi^{(1),\lambda}_w\) supported in \(X_w\) by

\[
\psi^{(1),\lambda}_w = \psi^{(1),\lambda}_n \circ (F_w^\lambda)^{-1},
\]

and similarly

\[
\psi^{(2),\lambda}_w = \psi^{(2),\lambda}_n \circ (F_w^\lambda)^{-1}, \quad \text{if } \iota_{w+1} = 2.
\]

(2) For \(w \in W_n^\lambda\) and \(j \leq \iota_{w+1}\), define a series of harmonic functions on \(\Omega\lambda\) by

\[
h^{(j),\lambda}_w(q_0) = 0, \quad \text{and } h^{(j),\lambda}|_X = \psi^{(j),\lambda}_w.
\]

We will write \(\psi^{(j)}_w\) and \(h^{(j)}_w\) instead of \(\psi^{(j),\lambda}_w\) and \(h^{(j),\lambda}_w\) when it causes no confusion. Set \(\psi^{(1),\lambda}_n = \psi^{(1),\lambda}_0\) and \(h^{(1),\lambda}_n = h^{(1),\lambda}_0\).

It is easy to check that for \(w \in W_n^\lambda\) and \(j \leq \iota_{w+1}\), \(h^{(j)}_w\) is supported in \(F_w^\lambda \Omega_n\lambda\) with \(h^{(j)}_w = h^{(j),\lambda}_n \circ (F_w^\lambda)^{-1}\), noticing that \(\partial_{q_0}^n h^{(j)}_w(q_0) = 0\) by Theorem 3.4. Moreover, \(\{\psi^{(j)}_w\}_{w \in W_n^\lambda, j \leq \iota_{w+1}} \cup \{1\}\) form an orthogonal basis of \(L^2(X, \mu^\lambda)\). See Figure 11 for an illustration of \(h^{(1)}\) and \(h^{(2)}\).

Before performing the energy estimate, we list two basic lemmas.
Lemma 3.8. There exist two positive constants $C_1$ and $C_2$, such that

$$C_1 \left( \frac{15}{7} \right)^{m|w|+1} \leq \mathcal{E}_{\Omega,\lambda}(h_{(j)}^\lambda) \leq C_2 \left( \frac{15}{7} \right)^{m|w|+1},$$

for all $0 < \lambda \leq 1$ and $w \in W^\lambda$, $j \leq t_{|w|+1}$.

Proof. We only need to prove that $\mathcal{E}_{\Omega,\lambda}(h_{(j)}^\lambda)$ is bounded above and below by multiples of $(\frac{15}{7})^{m_1}$, as $h_{(j)}^\lambda = h_{(j)}^{\lambda,\lambda_1} \circ (F_{(j)}^\lambda)^{-1}$. In particular, we will restrict our consideration to $\frac{1}{7} < \lambda \leq 1$, since for general $0 < \lambda \leq 1$, $h_{(j)}^\lambda$ is supported in $F_0^{m_1-1}(SG_3)$ with $h_{(j)}^\lambda = h_{(j)}^{\lambda,3^{m_1-1}} \circ F_0^{-m_1+1}$. So it is sufficient to assume $m_1 = 1$ and prove that $\mathcal{E}_{\Omega,\lambda}(h_{(j)}^\lambda)$ is bounded above and below by two positive constants.

First, we consider $\frac{2}{3} < \lambda \leq 1$. In this case, $t_1 = 2$. Let $c_1, c_2, c_3$ be some selected constants independent of $\lambda$. For each $\frac{2}{3} < \lambda \leq 1$, write $v^\lambda$ the harmonic function on $\Omega_\lambda$ which assumes 0 at $q_0$ and takes constant $c_i$ along $X_i$ for $i = 1, 2, 3$. We claim that

$$\mathcal{E}_{\Omega_\lambda}(v_{\lambda_1}^\lambda) \geq \mathcal{E}_{\Omega_\lambda}(v_{\lambda_2}^\lambda), \text{ if } \lambda_1 \leq \lambda_2.$$

To prove the claim, we construct another function $\bar{v}$ on $\Omega_{\lambda_2}$ such that $\bar{v}|_{\partial \Omega_{\lambda_2}} = v_{\lambda_1}^\lambda|_{\partial \Omega_{\lambda_2}}, \bar{v}(p_i) = v_{\lambda_1}^\lambda(p_i)$ for each $1 \leq i \leq 5$ and $\bar{v}$ is harmonic in remaining region. So $\bar{v}|_A = v_{\lambda_1}^\lambda|_A$, for $A = F_0 SG_3 \cup F_4 SG_3 \cup F_5 SG_3$, which says that

$$\mathcal{E}_{\Omega_\lambda}(\bar{v}) \leq \mathcal{E}_{\Omega_{\lambda_2}}(v_{\lambda_1}^\lambda),$$

by using Corollary 3.5 and the fact that $\eta(R\lambda)$ is decreasing on $\frac{2}{3} < \lambda \leq 1$. In addition, we have $\mathcal{E}_{\Omega_{\lambda_2}}(\bar{v}) \geq \mathcal{E}_{\Omega_{\lambda_2}}(v_{\lambda_1}^\lambda)$, since harmonic functions minimize the energy. Combining the two inequalities, we obtain the claim.

Thus, $\mathcal{E}_{\Omega_{\lambda}}(v_{\lambda}^\lambda) \geq \mathcal{E}_{\Omega_{\lambda}}(v_{\lambda_1}^\lambda)$ (more precisely, $\mathcal{E}_{\Omega_{\lambda}}(v_{\lambda}^\lambda) \geq \lim_{\varepsilon \to 0} \mathcal{E}_{\Omega_{\lambda}}(v_{\lambda_1}^\lambda)$) which provides a lower bound of $\{\mathcal{E}_{\Omega_{\lambda}}(v_{\lambda}^\lambda)\}$. On the other hand, to find an upper bound of $\{\mathcal{E}_{\Omega_{\lambda}}(v_{\lambda}^\lambda)\}$, consider the function $\bar{v} \in C(A)$ which is harmonic in $A$ and assumes 0 at $q_0, c_i$ at $p_i$ for $i = 1, 2, 3$. It is easy to find that $\mathcal{E}_{\Omega_{\lambda}}(v_{\lambda}^\lambda) \leq \mathcal{E}_{\Omega}(\bar{v})$ by extending $\bar{v}$ to $\Omega_{\lambda}$ with $\bar{v}|_{F_0^\lambda \Omega_{\lambda}} = c_i$.

The energy estimate of $h_{(1)}$ is a special case in the above discussion. To estimate the energy of $h_{(2)}$, observe that the boundary values $(h_{(2)}^{\lambda,1}|_{X_1}, h_{(2)}^{\lambda,2}|_{X_2}, h_{(2)}^{\lambda,3}|_{X_3})$ vary within a compact set, denoted by $C$, since we always have $\frac{1}{7} < \mu_1 < \mu_2 < \frac{1}{5}$ and
$\frac{1}{3} < \mu_1^\lambda < \frac{1}{2}$. We then have

$$0 < \inf \{ \mathcal{E}_{\Omega_\lambda}(v^1) : (v^1|_{X_{1,1}}, v^1|_{X_{1,2}}, v^1|_{X_{1,3}}) \in C \}$$

$$\leq \varepsilon_{\Omega_\lambda}(h^{(j)}) \leq \sup \{ \mathcal{E}_{\lambda}(v) : (v(p_1), v(p_2), v(p_3)) \in C \} < \infty.$$ 

Thus we have proved that $\mathcal{E}_{\Omega_\lambda}(h^{(j)})$ is bounded above and below by two positive constants, when $\frac{2}{3} < \lambda \leq 1$.

A similar discussion is valid for $\frac{1}{3} < \lambda \leq \frac{2}{3}$. \hfill \qed

The next lemma shows that all the basis functions $\{h^{(j)}_w\}$ are pairwise orthogonal in energy.

**Lemma 3.9.** Assume $0 < \lambda \leq 1$ and $w, w' \in W^\lambda_\lambda$. Then $\mathcal{E}_{\Omega_\lambda}(h^{(j)}_w, h^{(j')}_{w'}) \neq 0$ if and only if $w = w'$ and $j = j'$.

**Proof.** We discuss in different cases.

If $X_w \cap X_{w'} = \emptyset$, then obviously $\mathcal{E}_{\Omega_\lambda}(h^{(j)}_w, h^{(j')}_{w'}) = 0$, as $h^{(j)}_w$ and $h^{(j')}_{w'}$ support in disjoint regions.

If $X_w \subseteq X_{w'}$, then

$$\mathcal{E}_{\Omega_\lambda}(h^{(j)}_w, h^{(j')}_{w'}) = \left( \frac{15}{7} \right)^{m_\lambda} \left( \partial_\lambda^{(j)} (h^{(j)}_w \circ F^\lambda_w)(q_0) \right) \left( h^{(j')}_{w'}(F^\lambda_{w'} q_0) - h^{(j')}_{w'} \right) |_{X_w} = 0,$$

by Theorem 3.4 and the fact that $\partial_\lambda^{(j)} (h^{(j)}_w \circ F^\lambda_w)(q_0) = 0$.

If $w = w'$ and $j \neq j'$, then $h^{(j)}_w$ and $h^{(j')}_{w'}$ assume different symmetries. So we also have $\mathcal{E}(h^{(j)}_w, h^{(j')}_{w'}) = 0$. \hfill \qed

**Remark 3.** The proof of Lemma 3.9 also implies that $\mathcal{E}_{\Omega_\lambda}(h^{(j)}_w, h_0) = 0$ for each $w \in W^\lambda_\lambda$, $j \leq \ell_{|w|+1}$.

Now for the harmonic solution $u$ of the Dirichlet problem (1), we have the following estimate in energy in terms of its boundary data $f$.

**Theorem 3.10.** Let $u$ be the harmonic function in $\Omega_\lambda$ with boundary values $u(q_0) = a$ and $u|_{\partial \Omega_\lambda} = f$, where

$$f = b + \sum_{w \in W^\lambda_\lambda} \sum_{j \leq \ell_{|w|+1}} c^{(j)}_w \psi^{(j)}_w$$

with

$$c^{(j)}_w = \left( \int_{X_\lambda} \left( \psi^{(j)}_w \right)^2 d\mu_\lambda \right)^{-1} \int_{X_\lambda} f \psi^{(j)}_w d\mu_\lambda.$$

Then $\mathcal{E}_{\Omega_\lambda}(u)$ is bounded above and below by multiples of

$$\left( \frac{15}{7} \right)^{m_\lambda} (a - b)^2 + \sum_{n=0}^{\infty} \sum_{w \in W^\lambda_\lambda} \sum_{j=1}^{\ell_{n+1}} \left( \frac{15}{7} \right)^{m_{n+1}} |c^{(j)}_w|^2.$$ 

(20)

In particular, $u$ has finite energy if and only if (20) is finite.

**Proof.** We have

$$u = b + (a - b)h_0 + \sum_{n=0}^{\infty} \sum_{w \in W^\lambda_\lambda} \sum_{j=1}^{\ell_{n+1}} c^{(j)}_w h^{(j)}_w.$$
Since we have shown in Lemma 3.9 that the functions $h_0 \bigcup \{h^{(j)}_w\}$ are orthogonal in energy,

$$E_{\Omega_{\lambda}}(u) = (a - b)^2 E_{\Omega_{\lambda}}(h_0) + \sum_{n=0}^{\infty} \sum_{w \in W_{\lambda}} \sum_{j=1}^{\ell_{n+1}} |c^{(j)}_w|^2 E_{\Omega_{\lambda}}(h^{(j)}_w).$$

Then (3.14) follows from Lemma 3.8 and Corollary 3.5.

4. Dirichlet problems on lower domains of $SG$. In this section, we consider the Dirichlet problem on lower domains of $SG$. Similar to the last section, we assume $SG$ is contained in $\mathbb{R}^2$ with boundary vertices $q_0 = (\frac{1}{\sqrt{3}}, 0)$, $q_1 = (0, 0)$, $q_2 = (\frac{2}{\sqrt{3}}, 0)$.

Let $0 \leq \lambda < 1$, the lower domain of $SG$, denoted by $\Omega_{\lambda}^-$, is defined as $\Omega_{\lambda}^- = \{(x, y) \in SG \setminus V_0 | y < 1 - \lambda\}$.

The boundary of $\Omega_{\lambda}^-$ is

$$\partial \Omega_{\lambda}^- = X_{\lambda}^- \cup \{q_1, q_2\},$$

with

$$X_{\lambda}^- = \begin{cases} V_{d(\lambda)} \cap X_{\lambda}, & \text{if } \lambda \text{ is a dyadic rational}, \\ X_{\lambda}, & \text{if } \lambda \text{ is not a dyadic rational}, \end{cases}$$

where $d(\lambda)$ is the smallest integer such that $\lambda$ is a multiple of $2^{-d(\lambda)}$ and $X_{\lambda} = \{(x, y) \in SG | y = 1 - \lambda\}$. We still abbreviate $X_{\lambda}^-$ to $X$ throughout this section for convenience. Write $\bar{\Omega}_{\lambda}^- = \Omega_{\lambda}^- \cup \partial \Omega_{\lambda}^-$ the closure of $\Omega_{\lambda}^-$. See Figure 12 for two typical domains.

![Figure 12. Two typical domain $\Omega_{\lambda}^-$'s. $\lambda$ is (or not) a dyadic rational.](image)

For $0 \leq \lambda < 1$, we write $\lambda$ in its binary expansion,

$$\lambda = \sum_{k=1}^{\infty} e_k(\lambda)2^{-k}, e_k(\lambda) = 0, 1 \text{ for } k \geq 1.$$ 

We forbid infinitely consecutive 1's to make the expansion unique. Denote

$$S\lambda = \sum_{k=1}^{\infty} e_{k+1}(\lambda)2^{-k}.$$ 

It is easy to check the relationship between $\Omega_{\lambda}^-$ and $\Omega_{S\lambda}^-$ as following,

$$\bar{\Omega}_{\lambda}^- = \begin{cases} F_0 \Omega_{S\lambda}^- \bigcup F_1 SG \bigcup F_2 SG, & \text{if } e_1(\lambda) = 0, \\ F_1 \Omega_{S\lambda}^- \bigcup F_2 \Omega_{S\lambda}, & \text{if } e_1(\lambda) = 1. \end{cases}$$ (21)
See Figure 13 for an illustration.

\[ \text{Figure 13. The relationship between } \Omega^-_\lambda \text{ and } \Omega^-_\lambda'. e_1(\lambda) = 0 \text{ in the left one and } e_1(\lambda) = 1 \text{ in the right one.} \]

It is natural to introduce the following sets of words

\[ \tilde{W}_m^\lambda = \{ w \in W_m | w_k = 0 \text{ if } e_k(\lambda) = 0, \text{ and } w_k = 1 \text{ or } 2 \text{ if } e_k(\lambda) = 1, \forall 1 \leq k \leq m \}, \]

so that

\[ X = \bigcup_{w \in \tilde{W}_m^\lambda} X_w, \forall m \geq 0, \]

where \( X_w = F_w S \mathcal{G} \cap X \). Write \( \tilde{W}_*^\lambda = \bigcup_{m \geq 0} \tilde{W}_m^\lambda \). In addition, denote \( A_{\lambda,m} \) to be the closure of \( \Omega^-_\lambda \setminus \bigcup_{w \in \tilde{W}_m^\lambda} F_w S \mathcal{G}^\lambda = m \), which is the union of all \( m \)-cells contained in \( \Omega^-_\lambda \). For example, \( A_{\lambda,1} = F_1 S \mathcal{G} \cup F_2 S \mathcal{G} \) if \( e_1(\lambda) = 0 \), and \( A_{\lambda,1} = \emptyset \) if \( e_1(\lambda) = 1 \).

4.1. **Extension algorithm.** The Dirichlet problem on \( \Omega^-_\lambda \) is stated as

\[
\begin{align*}
\Delta u &= 0 \text{ in } \Omega^-_\lambda, \\
|\partial_\Omega^- u| &= f, f \in C(\partial \Omega^-_\lambda).
\end{align*}
\]  

(22)

Analogous to the previous two sections, we only need to find an explicit algorithm for \( u_{|\Omega^-_\lambda \setminus V_1} \) in terms of the boundary data \( f \), since then the problem of finding values of \( u \) elsewhere in \( \Omega^-_\lambda \) is essentially the same after dilation.

From the matching condition at each vertex in \( V_1 \cup \Omega^-_\lambda \), there exist the equations,

\[
\begin{cases}
4u(F_1 q_2) - f(q_1) - f(q_2) - u(F_0 q_1) - u(F_0 q_2) = 0, \\
3 \partial^-_n u(F_0 q_1) + (2u(F_0 q_1) - u(F_1 q_2) - f(q_1)) = 0, & \text{if } e_1(\lambda) = 0, \\
3 \partial^-_n u(F_0 q_2) + (2u(F_0 q_2) - u(F_1 q_2) - f(q_2)) = 0,
\end{cases}
\]

and

\[
\partial^-_n u(F_1 q_2) + \partial^-_n u(F_1 q_2) = 0, \quad \text{if } e_1(\lambda) = 1.
\]  

(23)

(24)

We need to express the involved normal derivatives in terms of \( u(F_0 q_1), u(F_0 q_2), u(F_1 q_2) \) and the boundary data \( f \). For this purpose, we introduce two important coefficients

\[ \eta_1(\lambda) = \partial^-_n h_1^\lambda(q_1), \eta_2(\lambda) = -\partial^-_n h_1^\lambda(q_2), \]

(25)
where \( h_1^\lambda \) is the harmonic function with values \( h_1^\lambda(q_1) = 1, h_1^\lambda(q_2) = 0 \) and \( h_1^\lambda|_X = 0 \). Symmetrically, we have
\[
\eta_2(\lambda) = -\partial_n^- h_2^\lambda(q_1), \eta_1(\lambda) = \partial_n^+ h_2^\lambda(q_2),
\]
where \( h_2^\lambda \) is the harmonic function with values \( h_2^\lambda(q_1) = 0, h_2^\lambda(q_2) = 1 \) and \( h_2^\lambda|_X = 0 \).

We omit the superscript \( \lambda \) of \( h_1^\lambda \) when there is no confusion caused. We will discuss on how to calculate these coefficients in the second part of this section. Here we only mention the following property.

**Lemma 4.1.** For \( 0 \leq \lambda < 1 \), we have \( \eta_1(\lambda) \geq 2, 0 \leq \eta_2(\lambda) \leq 1 \).

**Proof.** By using the maximum principle for harmonic functions, it is easy to see that \( \eta_1 \) is an increasing function of \( \lambda \), and \( \eta_2 \) is a decreasing function of \( \lambda \). Thus we have
\[
\eta_1(\lambda) \geq \eta_1(0) = 2, \quad \eta_2(\lambda) \leq \eta_2(0) = 1.
\]
Obviously, \( \eta_2(\lambda) = -\partial_n^- h_1(q_2) \) is nonnegative. \( \square \)

**Remark 4.** More precisely, we have \( \eta_1(\lambda) + \eta_2(\lambda) \geq 3 \). In fact, we just need to consider the antisymmetric harmonic function \( h_1 - h_2 \) whose normal derivative at \( q_1 \) is \( \eta_1(\lambda) \) + \( \eta_2(\lambda) \). Using the maximum principle on the left half part of \( \Omega^\lambda_1 \), one can check that \( \eta_1 + \eta_2 \) is an increasing function of \( \lambda \).

Using the coefficients \( \eta_1(\lambda), \eta_2(\lambda) \), we have the following two lemmas.

**Lemma 4.2.** Assume \( h = h(q_1)h_1 + h(q_2)h_2 \). If \( e_1(\lambda) = 0 \), then
\[
\begin{pmatrix}
\partial_n^- h(F_0q_1) \\
\partial_n^- h(F_0q_2)
\end{pmatrix}
= M_{\lambda}^0
\begin{pmatrix}
\partial_n^- h(q_1) \\
\partial_n^- h(q_2)
\end{pmatrix}.
\]
If \( e_1(\lambda) = 1 \), then
\[
\begin{pmatrix}
\partial_n^- h(F_1q_1) \\
\partial_n^- h(F_1q_2)
\end{pmatrix}
= M_{\lambda}^1
\begin{pmatrix}
\partial_n^- h(q_1) \\
\partial_n^- h(q_2)
\end{pmatrix}, \quad \text{and} \quad
\begin{pmatrix}
\partial_n^- h(F_2q_1) \\
\partial_n^- h(F_2q_2)
\end{pmatrix}
= M_{\lambda}^2
\begin{pmatrix}
\partial_n^- h(q_1) \\
\partial_n^- h(q_2)
\end{pmatrix}.
\]
Here the matrices \( M_{\lambda}^\alpha \) are
\[
M_{\lambda}^0 = \begin{pmatrix}
3 + 3\eta_1(S\lambda) + 3\eta_2(S\lambda) & 3 + \eta_1(S\lambda) + \eta_2(S\lambda) \\
6 + 4\eta_1(S\lambda) + 4\eta_2(S\lambda) & 6 + 4\eta_1(S\lambda) + 4\eta_2(S\lambda)
\end{pmatrix},
M_{\lambda}^1 = \begin{pmatrix}
\frac{1}{\eta_2(S\lambda)} & 0 \\
0 & \frac{1}{\eta_1(S\lambda)}
\end{pmatrix}, \quad \text{and} \quad M_{\lambda}^2 = \begin{pmatrix}
\frac{\eta_2(S\lambda)}{2\eta_1(S\lambda)} & \frac{\eta_2(S\lambda)}{2\eta_1(S\lambda)} \\
0 & \frac{\eta_2(S\lambda)}{2\eta_1(S\lambda)}
\end{pmatrix}.
\]

**Proof.** It follows by direct computation. We omit it. \( \square \)

By iteratively using the above matrices, we have
\[
\begin{pmatrix}
\partial_n^- h(F_wq_1) \\
\partial_n^- h(F_wq_2)
\end{pmatrix}
= M_{\lambda}^w
\begin{pmatrix}
\partial_n^- h(q_1) \\
\partial_n^- h(q_2)
\end{pmatrix},
\]
for \( w \in \hat{W}_a^\lambda \) with \( M_{\lambda}^w = M_{w_1}^{\lambda - 1} \cdots M_{w_2}^1 M_{w_1}^\lambda \). Moreover,

**Lemma 4.3.** Assume \( h = h(q_1)h_1 + h(q_2)h_2 \). Then \( \forall m \geq 0, \)
\[
\sum_{w \in \hat{W}_m^\lambda} (\partial_n^- h(F_wq_1) + \partial_n^+ h(F_wq_2)) = \partial_n^- h(q_1) + \partial_n^+ h(q_2),
\]
(28)
and $\forall w \in W^{\lambda}_s$, 
\[ \partial_n^+ h(F_w q_1) + \partial_n^- h(F_w q_2) \geq 0, \quad \text{if } h(q_1) \geq 0 \text{ and } h(q_2) \geq 0. \]  \hspace{1cm} (29)

**Proof.** By using the local Gauss-Green’s formula on $A_{\lambda,m}$ we get (28) holds for each $m \geq 0$. Notice that for $w \in W^{\lambda}_s$, 
\[ \partial_n^+ h(F_w q_1) + \partial_n^- h(F_w q_2) = \left( \frac{5}{3} \right)^{|w|} (h(F_w q_1) + h(F_w q_2)) (\eta_1(S^{[w]} \lambda) - \eta_2(S^{[w]} \lambda)). \]

By using Lemma 4.1 and the maximum principle for harmonic functions, we have both $\eta_1(S^{[w]} \lambda) - \eta_2(S^{[w]} \lambda) \geq 0$ and $h(F_w q_1) + h(F_w q_2) \geq 0$, in case of $h(q_1) \geq 0$ and $h(q_2) \geq 0$, which gives (29). \hfill \Box

According to Lemma 4.2 and 4.3, we introduce two measures on $X$. 

**Definition 4.4.** Define $\mu^\lambda_i$ to be the unique probability measure on $X$ satisfying 
\[ \mu^\lambda_i(X_w) = \frac{1}{\eta_1(\lambda) - \eta_2(\lambda)} \left( \frac{\eta_1(\lambda)}{-\eta_2(\lambda)} \right), \quad \forall w \in W^{\lambda}_s. \]

Symmetrically, define $\mu^\lambda_i$ by 
\[ \mu^\lambda_i(X_w) = \frac{1}{\eta_1(\lambda) - \eta_2(\lambda)} \left( \frac{-\eta_2(\lambda)}{\eta_1(\lambda)} \right), \quad \forall w \in W^{\lambda}_s. \]

Note that for $i = 1, 2$, we have $\partial_n^+ h_1(F_w q_1) + \partial_n^- h_1(F_w q_2) = (\eta_1(\lambda) - \eta_2(\lambda)) \mu^\lambda_i(X_w)$. 

**Theorem 4.5.** Let $u$ be a solution of the Dirichlet problem (22). Then 
\[ \partial_n^+ u(q_1) = \eta_1(\lambda)f(q_1) - \eta_2(\lambda)f(q_2) - (\eta_1(\lambda) - \eta_2(\lambda)) \int_X f d\mu^\lambda_1, \]  

\[ \partial_n^+ u(q_2) = \eta_1(\lambda)f(q_2) - \eta_2(\lambda)f(q_1) - (\eta_1(\lambda) - \eta_2(\lambda)) \int_X f d\mu^\lambda_2. \]  \hspace{1cm} (30) \hspace{1cm} (31)

In addition, if $E_{X}^{-\lambda}(u) < \infty$, then 
\[ E_{X}^{-\lambda}(h_1, u) = \partial_n^+ u(q_1), \quad E_{X}^{-\lambda}(h_2, u) = \partial_n^- u(q_2). \]  \hspace{1cm} (32)

**Proof.** Using the local Gauss-Green’s formula on $A_{\lambda,m}$, we have 
\[ E_{\lambda,m}(h_1, u) = \partial_n^+ h_1(F_w q_1) f(q_1) + \partial_n^- h_1(F_w q_2) f(q_2) \]
\[ = \sum_{w \in W^{\lambda}_m} (\partial_n^+ h_1(F_w q_1) u(F_w q_1) + \partial_n^- h_1(F_w q_2) u(F_w q_2)) \]
\[ = \eta_1(\lambda) f(q_1) - \eta_2(\lambda) f(q_2) \]
\[ - \sum_{w \in W^{\lambda}_m} u(F_w q_1) + u(F_w q_2) \]
\[ = \sum_{w \in W^{\lambda}_m} u(F_w q_1) - u(F_w q_2) \]
\[ = \sum_{w \in W^{\lambda}_m} \frac{u(F_w q_1) + u(F_w q_2)}{2} \cdot (\partial_n^+ h_1(F_w q_1) + \partial_n^- h_1(F_w q_2)) \]
\[ = \frac{u(F_w q_1) + u(F_w q_2)}{2} \cdot (\partial_n^+ h_1(F_w q_1) + \partial_n^- h_1(F_w q_2)) \]
\[ \lim_{m \to \infty} \sum_{w \in W^{\lambda}_m} \frac{u(F_w q_1) + u(F_w q_2)}{2} \cdot (\partial_n^+ h_1(F_w q_1) + \partial_n^- h_1(F_w q_2)) \]
\[ = (\eta_1(\lambda) - \eta_2(\lambda)) \int_X f d\mu_i^\lambda. \]
On the other hand, by Lemma 4.1, we have
\[
|\frac{\partial}{\partial n} h_1(Fwq_1) - \frac{\partial}{\partial n} h_1(Fwq_2)| = \left| \frac{5}{3} m(h_1(Fwq_1) - h_1(Fwq_2)) + \frac{1}{3} \left( \frac{5}{3} m h_1(Fwq_1) + h_1(Fwq_2) \right) \left( \eta_1(S^m \lambda) + \eta_2(S^m \lambda) \right) \right| \\
\leq 3 \left| \frac{5}{3} m h_1(Fwq_1) + h_1(Fwq_2) \right| \left( \eta_1(S^m \lambda) - \eta_2(S^m \lambda) \right) \\
= 3 \left( \frac{\partial}{\partial n} h_1(Fwq_1) + \frac{\partial}{\partial n} h_1(Fwq_2) \right).
\]
So by Lemma 4.3, \( \sum_{w \in \mathcal{W}_{\lambda}} |\frac{\partial}{\partial n} h_1(Fwq_1) - \frac{\partial}{\partial n} h_1(Fwq_2)| \leq 3(\eta_1(\lambda) - \eta_2(\lambda)). \) Noticing that \( u \) is uniformly continuous on \( \bar{\Omega}_{\lambda} \), we have
\[
\lim_{m \to \infty} \sum_{w \in \mathcal{W}_{\lambda}} \frac{u(Fwq_1) - u(Fwq_2)}{2} \cdot \left( \frac{\partial}{\partial n} h_1(Fwq_1) - \frac{\partial}{\partial n} h_1(Fwq_2) \right) = 0.
\]
Combining the above facts, we get
\[
\mathcal{E}_{\Omega_{\lambda}^1}(h_1, u) = \eta_1(\lambda)f(q_1) - \eta_2(\lambda)f(q_2) - (\eta_1(\lambda) - \eta_2(\lambda)) \int_X f d\mu_1^\lambda. \tag{33}
\]
Similarly,
\[
\mathcal{E}_{\Omega_{\lambda}^1}(h_2, u) = \eta_1(\lambda)f(q_2) - \eta_2(\lambda)f(q_1) - (\eta_1(\lambda) - \eta_2(\lambda)) \int_X f d\mu_2^\lambda. \tag{34}
\]
The rest part of the proof is similar to that of Theorem 2.1. \( \square \)

Combining Theorem 4.5 with (23) and (24), after an easy calculation, we finally get the following extension algorithm.

**Theorem 4.6 (Extension Algorithm).** There exists a unique solution of the Dirichlet problem (22). In addition, we have the following formulas for \( u|_{\Omega_1 \cap \Omega_{\lambda}^1} \).

If \( \lambda(1) = 0 \), then
\[
u(F_0q_1) = \frac{9 + 5\eta_1(S\lambda) + \eta_2(S\lambda)}{4\eta_1(S\lambda)^2 + 14\eta_1(S\lambda) - 2\eta_2(S\lambda) - 4\eta_2(S\lambda)^2 + 12} f(q_1) \\
+ \frac{3 + \eta_1(S\lambda) + 5\eta_2(S\lambda)}{4\eta_1(S\lambda)^2 + 14\eta_1(S\lambda) - 2\eta_2(S\lambda) - 4\eta_2(S\lambda)^2 + 12} f(q_2) \\
+ \frac{1}{4\eta_1(S\lambda)^2 + 14\eta_1(S\lambda) - 2\eta_2(S\lambda) - 4\eta_2(S\lambda)^2 + 12} \int_{X_{S\lambda}^1} f \circ F_0 d\mu_1^{S\lambda} \\
+ \frac{1}{4\eta_1(S\lambda)^2 + 14\eta_1(S\lambda) - 2\eta_2(S\lambda) - 4\eta_2(S\lambda)^2 + 12} \int_{X_{S\lambda}^1} f \circ F_0 d\mu_2^{S\lambda},
\]
and \( u(F_1q_2) = \frac{1}{2} \left( u(F_0q_1) + u(F_0q_2) + f(q_1) + f(q_2) \right) \). The formula of \( u(F_0q_2) \) is symmetrical to that of \( u(F_0q_1) \).

If \( \lambda(2) = 1 \), then
\[
u(F_1q_2) = \frac{\eta_2(S\lambda)}{2\eta_1(S\lambda)} \left( f(q_1) + f(q_2) \right) \\
+ \frac{\eta_1(S\lambda) - \eta_2(S\lambda)}{2\eta_1(S\lambda)} \left( \int_{X_{S\lambda}^1} f \circ F_1 d\mu_1^{S\lambda} + \int_{X_{S\lambda}^1} f \circ F_2 d\mu_1^{S\lambda} \right).
\]
4.2. The calculation of $\eta$. In this section, we focus on the calculation of $\eta_1(\lambda)$, $\eta_2(\lambda)$. In particular, we will prove the following theorem.

**Theorem 4.7.** (a) $\eta_1(\lambda)$ is increasing in $[0, 1)$, $\eta_2(\lambda)$ is decreasing in $[0, 1)$.

(b) For $0 \leq \lambda < 1$, $(\eta_1(\lambda), \eta_2(\lambda)) = T_{e_1}(\lambda)(\eta_1(S\lambda), \eta_2(S\lambda))$ with

$$ T_0(x, y) = \left( \frac{5}{6} \cdot \frac{3 + 2x + 2y}{2 + x + y} + \frac{5}{2} \cdot \frac{x - y}{3 + 2x - 2y}, \frac{5}{6} \cdot \frac{3 + 2x + 2y}{2 + x + y} - \frac{5}{2} \cdot \frac{x - y}{3 + 2x - 2y} \right), \quad (35) $$

and

$$ T_1(x, y) = \left( \frac{5}{3} \cdot \frac{x - y^2}{2x}, \frac{5y^2}{6x} \right). \quad (36) $$

In addition, for any fixed positive numbers $c_1 > c_2$, we have

$$ (\eta_1(\lambda), \eta_2(\lambda)) = \lim_{m \to \infty} T_{e_1}(\lambda) \circ T_{e_2}(\lambda) \circ \cdots \circ T_{e_m}(\lambda)(c_1, c_2). \quad (37) $$

**Proof.** By using maximum principle for harmonic functions, (a) follows easily. So we only need to prove (b).

Looking at the functions $h_1 + h_2$ and $h_1 - h_2$ pictured in Figure 14, by computing their normal derivatives at $q_1$, we get

$$ \begin{align*}
\begin{cases}
\eta_1(\lambda) + \eta_2(\lambda) &= \frac{5}{3} \cdot \frac{3 + 2\eta_1(S\lambda) + 2\eta_2(S\lambda)}{3 + 2\eta_1(S\lambda) - 2\eta_2(S\lambda)}, \\
\eta_1(\lambda) - \eta_2(\lambda) &= \frac{5}{3} \cdot \frac{3\eta_1(S\lambda) - 3\eta_2(S\lambda)}{3 + 2\eta_1(S\lambda) - 2\eta_2(S\lambda)},
\end{cases}
\end{align*} $$

if $e_1(\lambda) = 0$,
The above equations lead to
\[
\begin{cases}
\eta_1(\lambda) + \eta_2(\lambda) = \frac{5}{3}\eta_1(S\lambda), \\
\eta_1(\lambda) - \eta_2(\lambda) = \frac{5}{3}(\eta_1(S\lambda) - \frac{\eta_2(S\lambda)}{\eta_1(S\lambda)}),
\end{cases}
\]
if \(\xi_1(\lambda) = 1\).

The above equations lead to
\[
(\eta_1(\lambda), \eta_2(\lambda)) = T_{\xi_1(\lambda)}(\eta_1(S\lambda), \eta_2(S\lambda)).
\] (38)

For the rest proof of Theorem 4.7(b), we need some claims and notations.

Claim 1. For \(0 \leq \lambda < 1\), we have

\[
h_1(F_w q_i) \leq \left(\frac{3}{5}\right)^{|w|} \frac{\eta_1(\lambda) - \eta_2(\lambda)}{\eta_1(S\lambda) - \eta_2(S\lambda)}, \forall w \in W^* \lambda, \forall i = 1, 2.
\]

Proof of Claim 1. Let \(m = \frac{|w|}{5}\). Noticing that \(h_1(F_w q_i) \geq 0\) by the maximum principle for harmonic functions, by Lemma 4.1, we have

\[
\partial_n^\alpha h_1(F_w q_i) + \partial_n^\beta h_1(F_w q_i) = \left(\frac{5}{3}\right)^m (\eta_1(S^m \lambda) - \eta_2(S^m \lambda)) (h_1(F_w q_i) + h_1(F_w q_2)) \geq 0.
\]

On the other hand, by Lemma 4.3,

\[
\partial_n^\alpha h_1(F_w q_1) + \partial_n^\beta h_1(F_w q_2) \leq \partial_n^\alpha h_1(q_1) + \partial_n^\beta h_1(q_2) = \eta_1(\lambda) - \eta_2(\lambda).
\]

Combining the above two inequalities, we get the desired result. \(\square\)

Notation. (a) For \(0 \leq \lambda < 1\) and fixed positive numbers \(c_1 > c_2\), define a sequence of resistance forms \(E_{m}(c_1, c_2)\) on \(V_m^\lambda = (V_m \cap \Omega^\lambda_\lambda) \cup \{F_w q_0\}_{w \in W^* \lambda}\) with the conductances

\[
c_{\lambda, m}^{x, y}(c_1, c_2) = \begin{cases}
0, & \text{if } x \not\sim_m y, \\
\left(\frac{5}{3}\right)^m(c_1 - c_2), & \text{if } \{x, y\} = \{F_w q_0, F_w q_1\} \text{ or } \{x, y\} = \{F_w q_0, F_w q_2\} \\
\frac{5}{3}c_2, & \text{if } \{x, y\} = \{F_w q_1, F_w q_2\} \text{ for some } w \in \tilde{W}_m^\lambda, \\
\frac{5}{3}m, & \text{Otherwise}.
\end{cases}
\]

(b) Let \(h_{1, m}^{\lambda, (c_1, c_2)}\) be a sequence of functions on \(V_m^\lambda\) harmonic with respect to \(E_{m}(c_1, c_2)\), assuming boundary values

\[
h_{1, m}^{\lambda, (c_1, c_2)}(q_1) = 1, h_{1, m}^{\lambda, (c_1, c_2)}(q_2) = 0, h_{1, m}^{\lambda, (c_1, c_2)}(F_w q_0) = 0, \forall w \in \tilde{W}_m^\lambda.
\]

Still denote by \(h_{1, m}^{\lambda, (c_1, c_2)}\) the piecewise harmonic function which assumes the same value on \(V_m^\lambda\) and takes harmonic extension elsewhere in \(\Omega^\lambda_m \cup \{F_w S\tilde{G}\}_{w \in \tilde{W}_m^\lambda}\).

In Figure 15, we give an example of \(V_m^\lambda\) together with some conductances. We abbreviate \(h_{1, m}^{\lambda, (c_1, c_2)}\) to \(h_{1, m}\) when there is no confusion caused. By Theorem 4.5, one can easily check that \(h_{1, m}(V_m \cap \Omega^\lambda_\lambda) = h_1|_{V_m \cap \Omega^\lambda_\lambda}\), when \((c_1, c_2) = (\eta_1(S^m \lambda), \eta_2(S^m \lambda))\).

Without loss of generality, we assume that \(1 - \lambda > 2^{-j}\) for some integer \(j\).

Claim 2. For \(c_1 > c_2 > 0\), we have

\[
\left(\frac{5}{3}\right)^j (2 - h_{1, m}(F_1^j q_0) - h_{1, m}(F_1^j q_2)) \cdot \left(\frac{5}{3}\right)^j (h_{1, m}(F_2^j q_0) + h_{1, m}(F_2^j q_1))
\]

\[
= T_{\xi_1(\lambda)} \circ T_{\xi_2(\lambda)} \circ \cdots \circ T_{\xi_m(\lambda)}(c_1, c_2).
\]
Proof of Claim 2. It is easy to see the claim holds by inductively using (38) when \((c_1, c_2) = (\eta_1(S^m\lambda), \eta_2(S^m\lambda))\), since then \(h_{1,m}|_{V_m \cap \Omega_\lambda} = h_1|_{V_m \cap \Omega_\lambda}\). For general \(c_1, c_2\), the claim still holds in a completely similar way. \(\square\)

**Claim 3.** Write \((c_1, m, c_2, m) = T_{e_1(\lambda)} \circ T_{e_2(\lambda)} \circ \cdots \circ T_{e_m(\lambda)}(c_1, c_2)\) for short. Then

\[ h_{1,m}(F_{w_qi}) \leq \left(\frac{3}{5}\right)^m \frac{c_{1,m} - c_{2,m}}{c_1 - c_2} \leq 4\left(\frac{3}{5}\right)^{m-j} \frac{1}{c_1 - c_2}, \quad \forall w \in \tilde{W}_m^\lambda, \forall i = 1, 2. \]

**Proof of Claim 3.** The first inequality is obtained analogously to the proof of Claim 1. The second inequality follows from Claim 2 and the fact \(\|h_{1,m}\|_\infty \leq 1. \square\)

Notice that \(h_{1,m}^{(c_1, c_2)}\) and \(h_{1,m}^{(\eta_1(S^m\lambda), \eta_2(S^m\lambda))}\) satisfy the same mean value equations on \(V_m^\lambda \setminus (\bigcup_{w \in \tilde{W}_m^\lambda} F_w \cap V_0)\). By Claim 3, we have the following estimate due to the maximum principle for harmonic functions,

\[ |(h_{1,m}^{(c_1, c_2)} - h_{1,m}^{(\eta_1(S^m\lambda), \eta_2(S^m\lambda))})(F_{i}^j q_{i'})| \leq 4\left(\frac{3}{5}\right)^{m-j} \frac{1}{c_1 - c_2} + 1, \]

for \((i, i') \in \{(1, 0), (1, 2), (2, 0), (2, 1)\}\), where we use the fact that \(\eta_1(S^m\lambda) - \eta_2(S^m\lambda) \geq 1\).

Thus, by Claim 2, we have

\[
\begin{align*}
&\left\| T_{e_1(\lambda)} \circ T_{e_2(\lambda)} \circ \cdots \circ T_{e_m(\lambda)}(c_1, c_2) - (\eta_1(\lambda), \eta_2(\lambda)) \right\| \\
= &\left\| T_{e_1(\lambda)} \circ T_{e_2(\lambda)} \circ \cdots \circ T_{e_m(\lambda)}(c_1, c_2) - T_{e_1(\lambda)} \circ T_{e_2(\lambda)} \circ \cdots \circ T_{e_m(\lambda)} \\
&\quad (\eta_1(S^m\lambda), \eta_2(S^m\lambda)) \right\| \\
\leq &8\left(\frac{3}{5}\right)^{m-2j} \frac{1}{c_1 - c_2} + 1 
\end{align*}
\]

with \(\|(a, b)\| = \max\{|a|, |b|\}\), which yields (37) as \(m \to \infty\) for \(0 \leq \lambda < 1 - 2^{-j}\). Noticing that \(j\) is arbitrary, we complete the proof of Theorem 4.7(b). \(\square\)

## 4.3. Some calculations on \(M_w^\lambda\).

In this subsection, we consider two special cases, \(e_1(\lambda) = e_2(\lambda) = \cdots = e_m(\lambda) = 0\) or \(e_1(\lambda) = e_2(\lambda) = \cdots = e_m(\lambda) = 1\) for some \(m\). It

![Figure 15. \(V_m^\lambda\) and some conductances. \((\frac{5}{8} < \lambda < \frac{3}{4}, m = 3)\)
would be convenient to get a direct expression for $M^\lambda_w, w \in \hat{W}^\lambda_m$ and $T_0 \circ T_0 \circ \cdots \circ T_0$

or $T_1 \circ T_1 \cdots \circ T_1$.

**Proposition 2.** Assume $e_1(\lambda) = e_2(\lambda) = \cdots = e_m(\lambda) = 0$. Then we have

(a) $\eta_1(\lambda) + \eta_2(\lambda) = 3 + \frac{14(\eta_1(S^m\lambda) + \eta_2(S^m\lambda) - 3)}{3(15^m - 1)(\eta_1(S^m\lambda) + \eta_2(S^m\lambda) - 3) + 14 \cdot 15^m},$

(b) $M^\lambda_{0^m} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a + b = 1$ and

$m \cdot \eta_1(S^m\lambda) + \eta_2(S^m\lambda))\
(9 \cdot 15^m + 5)(\eta_1(S^m\lambda) + \eta_2(S^m\lambda)) + 15(15^m - 1),$

where we use the notation $0^m$ to represent the word $00 \cdots 0$ of length $m$.

**Proposition 3.** Assume $e_1(\lambda) = e_2(\lambda) = \cdots = e_m(\lambda) = 1$. Denote $0 < x < 1$ the solution of $\frac{\eta_1(S^m\lambda)}{\eta_2(S^m\lambda)} = \frac{x + x^{-1}}{2}$. Then we have

(a) $\eta_1(\lambda) = \frac{5}{3} x - x^{-1} \frac{x^{2m} + x^{-2m}}{x^{2m} - x^{-2m}} \eta_1(S^m\lambda)$ and $\eta_2(\lambda) = \frac{5}{3} x - x^{-1} \frac{x^{2m} - x^{-2m}}{x^{2m} - x^{-2m}} \eta_2(S^m\lambda)$. (b) For $w \in \hat{W}^\lambda_m$, the matrix $M^\lambda_w$ is given by

$M^\lambda_w = \begin{pmatrix} x^j & -x^{2m-j} \\ -x^{j+1} & x^{2m-j-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -x^{2m} \\ -x^{2m} & 1 \end{pmatrix}^{-1},$

with $j$ being the integer such that $0 \leq j \leq 2^m - 1$ satisfying

$$j = \sum_{k=1}^{m} (w_k - 1) \cdot 2^{m-k}.$$ (39)

The proofs of Proposition 2 and 3 directly follow from elementary computations. We omit them. Proposition 3 is motivated by Theorem 5.4 of [S1], which solves the special case that $\lambda = 1 - 2^{-m}$.

5. **Extension to $\mathcal{S}G_l$.** In this section, we will briefly discuss how to extend previous results to level-$l$ Sierpinski gasket $\mathcal{S}G_l$.

5.1. **Half domains.** We still use $\Omega$ to denote the half domain and $X$ its Cantor set boundary. As shown in Section 2, to solve the Dirichlet problem on the half domain of $\mathcal{S}G_l$, it suffices to obtain the extension algorithm for $u|_{V_1 \cup \Omega}$ in terms of the boundary data $f$. We summarize it into following two steps.

**Step 1.** Find a formula for $\partial_n^\nu u(q_1)$, in the form of

$$\partial_n^\nu u(q_1) = 3f(q_1) - 3 \int_X f d\mu.$$
In Figure 16 (a), we label the contraction mappings of Example 2.

\[ W_m = \{ i : \#(F_i S G_1 \cap X) = \infty \} \text{ and } \hat{W}_m = \bigcup_{m=0}^{\infty} W_m. \]

For \( w \in \hat{W}_m, 1 \leq j \leq \left[ \frac{1}{2} \right], \) denote \( p_{j, w} = F_w p_{j, \emptyset}. \) Obviously, \( \{ p_{j, w} \}_{1 \leq j \leq \left[ \frac{1}{2} \right], w \in \hat{W}_m} = V_{\hat{W}} \cap X \setminus \{ q_0 \} \) is dense in \( X. \) See the following example for an illustration.

**Example 2.** In Figure 16 (a), we label the contraction mappings of \( S G_4. \) So we have \( \hat{W}_m = \{ 0, 6 \}^m. \) The vertices \( \{ p_{j, \emptyset} \}_{j=1}^2 \) are plotted in Figure 16 (b).

![Figure 16. The half domain of \( S G_4. \)](attachment:figure16.png)

With the above notations, introduce the pure atomic probability measure \( \mu \) on \( X \) satisfying

\[ \mu(\{ p_{j, w} \}) = -\frac{1}{3} \partial^{-1}_n h_\alpha(p_{j, w}), \text{ for } w \in \hat{W}_m \text{ and } 1 \leq j \leq \left[ \frac{1}{2} \right]. \] (40)

For \( i \in \hat{W}_1, \) denote \( \mu_i = r^{-1} h_\alpha(F_i q_1) \) and write \( \mu_w = \mu_{w_1} \mu_{w_2} \cdots \mu_{w_{\# W}}, \) where \( r \) is the renormalization factor of the energy on \( S G_i. \) It is easy to check that

\[ \partial^{-1} n h_\alpha(p_{j, w}) = \mu_w \partial^{-1} n h_\alpha(p_{j, \emptyset}), \text{ so that } \mu(\{ p_{j, w} \}) = -\frac{\mu_w}{3} \partial^{-1} n h_\alpha(p_{j, \emptyset}). \]

**Step 2.** Solve linear equations determined by the matching conditions of normal derivatives on \( V_1 \cap \Omega. \)

That is

\[
\begin{align*}
\left\{ r \partial^{-1} n u(F_i q_1) + \sum_{y \sim x, F_i q_1, y \in F_i V_0} (u(F_i q_1) - u(y)) = 0, \quad &i \in \hat{W}_1, \\
\sum_{y \sim x} (u(x) - u(y)) = 0, \quad &\text{for other } x \in V_1 \cap \Omega.
\end{align*}
\]

(41)

Taking \( \partial_n^{-1} u(F_i q_1) = 3r^{-1}(u(F_i q_1) - F_i f) \) into (41), the remaining problem is solving the linear equations. However, even for the values of \( h_\alpha, \) the calculation becomes much more complicated, so we could not provide a general solution of (41).

Let’s look at the simplest case.

**Example 3.** Consider the half domain of \( S G. \) In this case, \( \hat{W}_m = \{ \emptyset, 0, 00, \ldots \} \) and \( p_{1, \emptyset} = F_1 q_2. \) For convenience, we write \( p_k = F_0^k p_{1, \emptyset}. \) See Figure 17 below for the notations.
The measure $\mu$ is given by

$$\mu(\{p_k\}) = 2 \cdot 3^{-k-1}.$$ 

There is only one equation in (41),

$$2u(F_0 q_1) - f(q_1) - f(p_0) + 3 \cdot (u(F_0 q_1) - \int_X f \circ F_0 d\mu) = 0.$$ 

So

$$u(F_0 q_1) = \frac 1 5 f(q_1) + \frac 1 5 f(p_0) + \frac 3 5 \int_X f \circ F_0 d\mu,$$  \hspace{1cm} (42)

and thus $u(F_0 q_1) = \frac 1 5 f(q_1) + \frac 1 5 f(p_0) + \frac 6 5 \sum_{k=1}^{\infty} 3^{-k} f(p_k)$, which is Corollary 2.5 in [10].

We mention that the Dirichlet to Neumann map was studied in [10], and it was shown that $\|(\frac{3}{5})^{k+1} \partial_n^- u(p_k)\|_\infty < \infty$ if $\{f(p_k)\}_{k=0}^{\infty} \in \ell^\infty$. Here, we present another interesting observation, which describes where $\{\partial_n^- u(p_k)\}_{k \geq 0}$ live in when $f \in C(\partial \Omega)$.

**Theorem 5.1.** Let $u$ be the solution of the Dirichlet problem (1) on the half domain of $SG$, then the series $\sum_{k=0}^{\infty} (\frac{3}{5})^{k+1} \partial_n^- u(p_k)$ converges.

Conversely, given a convergence series $\{\eta_k\}_{k=-1}^{\infty}$, there exists a unique $f \in C(\partial \Omega)$ such that $f(q_1) = 0$, $\partial_n^- u(q_1) = \eta_{-1}$ and $\partial_n^- u(p_k) = (\frac{3}{5})^{k+1} \eta_k$ for $k \geq 0$, where $u$ is the harmonic function with boundary data $f$.

**Proof.** By using (42) and the fact that $\int_X f \circ F_0^k d\mu = \frac{3}{5} f(p_k) + \frac{3}{5} \int_X f \circ F_0^{k+1} d\mu$, we have

$$(\frac{3}{5})^{k+1} \partial_n^- u(p_k) = 2 f(p_k) - u(F_0^k q_1) - u(F_0^{k+1} q_1)$$

$$= 2 f(p_k) - u(F_0^k q_1) - u(F_0^{k+1} q_1) + \frac{5}{2} (u(F_0^{k+1} q_1) - \frac{1}{5} u(F_0^k q_1))$$

$$= - \frac{1}{5} u(p_k) - \frac{3}{5} \int_X f \circ F_0^{k+1} d\mu$$

$$= \frac{3}{2} (u(F_0^{k+1} q_1) - u(F_0^k q_1)) + \frac{3}{2} f(p_k) - \frac{3}{2} \int_X f \circ F_0^{k+1} d\mu$$

$$= \frac{3}{2} (u(F_0^{k+1} q_1) - u(F_0^k q_1)) + \frac{9}{4} \left( \int_X f \circ F_0^k d\mu - \int_X f \circ F_0^{k+1} d\mu \right).$$

Thus we have $\sum_{k=0}^{\infty} (\frac{3}{5})^{k+1} \partial_n^- u(p_k)$ converges to $\frac{9}{4} \int_X f d\mu - \frac{3}{2} f(q_0) - \frac{3}{2} f(q_1)$. 

\[\square\]
Conversely, suppose there exists such a $f \in C(\partial \Omega)$, it must satisfies
\[
\begin{cases}
2f(p_k) - u(F_0^k q_1) - u(F_0^{k+1} q_1) = \eta_k, \\
f(p_k) + u(F_0^k q_1) - 2u(F_0^{k+1} q_1) = \sum_{i=1}^{k} (\frac{3}{2})^{k-i} \eta_i,
\end{cases}
\] for some positive constants $\eta_k$. 
which arise from the definition of normal derivatives at $p_k$ and $F_0^{k+1} q_1$, using the fact that $-\partial_n^+ u(F_0^{k+1} q_1) = \partial_n^+ u(q_1) + \sum_{i=0}^{k} \partial_n^+ u(p_i)$. The solution of the above equations is
\[
\begin{cases}
u(F_0^{k+1} q_1) = -\eta_1 - \frac{4}{3} \sum_{i=0}^{k} \eta_i + \sum_{i=1}^{k} (\frac{3}{2})^{k-i} \eta_i, \\
v(p_k) = -\eta_1 - \frac{4}{3} \sum_{i=0}^{k} \eta_i + \frac{4}{3} \sum_{i=1}^{k} (\frac{3}{2})^{k-i} \eta_i + \frac{1}{3} \eta_k,
\end{cases}
\] for all $k \geq 0$, where $f(p_k)$ and $u(F_0^k q_1)$ converge to $f(q_0) = -\eta_1 - \frac{4}{3} \sum_{i=0}^{\infty} \eta_i$. Thus we find a unique function $f$ which satisfies the prescribed conditions.

However, it is not clear where $\{\partial_n^+ u(p_j, w)\}_{w \in \tilde{W}, 1 \leq j \leq \frac{l}{2}}$ live in for general cases, even for $\mathcal{S}_\Omega^3$.

For the energy estimate part, for $\mathcal{S}_\Omega^1$, we have
\[
C_1 Q(f) \leq E_\Omega(u) \leq C_2 Q(f)
\] for some positive constants $C_1, C_2$, with
\[
Q(f) = \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} \left( f(q_1) - f(p_j, \theta) \right)^2 + \sum_{w \in \tilde{W}} \sum_{i \in \tilde{W}_1} \sum_{j', j''} r^{-|w|} \left( f(p_j, w) - f(p_{j'}, w) \right)^2.
\] The method is essentially the same as that for $\mathcal{S}_\Omega^3$ case.

5.2. Upper or lower domains. The Dirichlet problem on the upper and lower domains in general $\mathcal{S}_\Omega^1$ are much more complicated.

For the upper domains, we use the infinite expansion
\[
\lambda = \sum_{k=1}^{\infty} \epsilon_k \cdot l^{-m_k}
\]
to characterize $\Omega_\lambda$, where $\{m_k\}_{k \geq 1}$ is an increasing sequence of positive integers and $\epsilon_k$ take values from $\{1, 2, \cdots, l-1\}$. The number $\epsilon_k$ decides the relationship between $\Omega_{R^{k-1}}$ and $\Omega_{R^k \lambda}$, where $R \lambda = \sum_{k=2}^{\infty} \epsilon_k \cdot l^{-(m_k - m_1)}$, and there are $l - 1$ choices in the $\mathcal{S}_\Omega^1$ setting.

As for the lower domains, we refer to a different expansion
\[
\lambda = \sum_{k=1}^{\infty} \epsilon_k(\lambda) l^{-k},
\]
with $\epsilon_k(\lambda)$ taking values from $\{0, 1, \cdots, l-1\}$. We forbid infinitely consecutive $(l-1)$'s to make the expansion unique. Similarly, different $\epsilon_k(\lambda)$ determines different type of relationships between $\Omega_{S^{k-1} \lambda}$ and $\Omega_{S^k \lambda}$, where $S \lambda = \sum_{k=1}^{\infty} \epsilon_k+1(\lambda) l^{-k}$.

The approaches in Section 3 and Section 4 to solve the Dirichlet problem on upper or lower domains still work, although the calculations involved turn to be rather complicated. We list the main steps.

**Step 1.** Denote $\eta(\lambda) = \partial_n^+ h_0(q_0)$ (or $\eta_1(\lambda) = \partial_n^+ h_1(q_1), \eta_2(\lambda) = -\partial_n^+ h_1(q_2)$). Represent $\eta(\lambda)$ in terms of $\eta(R \lambda)$ with the relationship between $\Omega_\lambda$ and $\Omega_{R \lambda}$ (or
represent $\eta_1(\lambda), \eta_2(\lambda)$ in terms of $\eta_1(S\lambda)$ and $\eta_2(S\lambda))$. Use the above representations iteratively to approximate $\eta(\lambda)$ (or $\eta_1(\lambda), \eta_2(\lambda)$), and the proof is essentially the same as Lemma 3.1 (or Theorem 4.7).

**Step 2.** Calculate the normal derivatives of $h_0$ (or $h_1, h_2$) along the Cantor set $X$, using the crucial coefficients $\eta(\lambda)$ (or $\eta_1(\lambda), \eta_2(\lambda)$). The normal derivatives of $h_0$ (or $h_1, h_2$) hold the key to the representation of $\partial^+_n u(q_0)$ (or $\partial^+_n u(q_1), \partial^+_n u(q_2)$) in terms of the boundary data $f$.

**Step 3.** Solve the linear equations determined by the matching conditions of normal derivatives on the crucial points.

Lastly, the Haar series expansion used in the energy estimate still works in general $SG_i$ cases. The key observation is that we can still use the analogue of Theorem 3.4 to show that we can decompose the harmonic solution associated with a square integrable boundary value data (with respect to a suitable choice of measure), into a summation of countably infinite, pairwise orthogonal in energy, locally supported harmonic functions with suitable piecewise constant boundary values.

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