Variational approximations for stationary states of Ising–like models

A. Pelizzola

1 Dipartimento di Scienza Applicata e Tecnologia, CNISM and Center for Computational Studies, Politecnico di Torino, Corso Duca degli Abruzzi 24, I–10129 Torino, Italy
2 INFN, Sezione di Torino, via Pietro Giuria 1, I-10125 Torino, Italy
3 Human Genetics Foundation, HuGeF, Via Nizza 52, I-10126 Torino, Italy

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Abstract – We introduce a new variational approach to the stationary state of kinetic Ising–like models. The approach is based on the cluster expansion of the entropy term appearing in a functional which is minimized by the system history. We rederive a known mean–field theory and propose a new method, here called diamond approximation, which turns out to be more accurate and faster than other methods of comparable computational complexity.

Introduction. – In equilibrium statistical mechanics, the exact solution of a model with a large number of interacting variables is most often analytically unfeasible and computationally intractable. For this reason, many approximate methods have been developed in the last century to deal with this problem. Among these, mean–field–like techniques play a fundamental role and are still the subject of a research activity aimed to improve accuracy and speed and to refine theoretical foundations. The importance of these techniques can be understood by considering that they are simple tools, easily applicable to many different models, and usually much faster than Monte Carlo simulations. Moreover, they can provide exact results in special cases.

Nonequilibrium statistical mechanics, although not as well developed as its equilibrium counterpart, poses similar problems and mean–field–like techniques have been and are being adapted for application to out of equilibrium models. In this context, a typical and well defined problem, which actually goes far beyond the boundaries of statistical mechanics, is finding the stationary state of a Markov process [1].

In developing mean–field–like techniques for this problem, statistical physicists often focused on specific examples, like kinetic Ising and Ising–like models, or epidemic processes. In trying to go beyond the simple mean–field theory [2], several approaches were developed. Here we cannot make an exhaustive review, but we shall try to briefly recall the approaches which are the most relevant for the present proposal.

One line of approach, which sometimes goes under the name of local equilibrium approximation [2–5], is based on the assumption that in the stationary state, the probability distribution of a model with many variables factors into a suitable product of marginals, each involving a small number of variables. In the extreme case, when one assumes factoring into a product of single–variable marginals, a mean–field theory is obtained.

Another technique is the path probability method (PPM) [6–9] (the dynamical version of the cluster variation method [10–12]), where the kinetic problem is written in terms of a 2–times variational problem, and the entropy terms appearing in the kinetic functional to be minimized are approximated by means of a cluster expansion. This technique is not unrelated to the local equilibrium approximation, and in specific cases equivalence has been rigorously proved [13].

In addition, a very recent proposal is the so–called dynamic cavity method [14–18], an interesting generalization of the well–known and widely applied cavity method [10–12] to kinetic problems. This is a message–passing algorithm, which has been shown to be efficient on large systems and to perform better than mean–field theories [18].

In the present letter we try to make a step forward in
this line of research by proposing a new variational method for the stationary state problem. We retain the cluster expansion idea on which the PPM is based, but instead of applying it to a 2–times functional, we start with the functional which describes the full system history. In this way, we have more freedom in the choice of the clusters which enter the expansion. While in PPM the main clusters are obtained by selecting a suitable set of interacting variables and using the same variables for two consecutive times, our main clusters can involve more than 2 consecutive times and need not be time invariant, that is different variables can be involved at different times. We illustrate our idea by two examples: the first one, termed star approximation, reduces to an already known mean–field theory; in the second one, termed diamond approximation, the choice of the main clusters is inspired by the dynamic cavity method. The result is a new method, more general than dynamic cavity, which, compared with other techniques of comparable complexity, will prove to be more accurate and faster.

**Variational approximations.** We consider a model with discrete variables $s_i$, associated to the nodes $i = 1, 2, \ldots, N$ of an undirected graph $G = (V, E)$, where $V$ denotes the set of nodes and $E$ the set of edges. The neighbourhood of a node $i$ is denoted by $\partial i = \{ j \in V : (i, j) \in E \}$, and its cardinality $d_i = |\partial i|$ is called the degree of node $i$.

The variables $s_i$ take values in a finite set, which is usually (but does not need to) the same for all nodes $i$. Typical examples are the Ising model, where $s_i \in \{ +1, -1 \}$, the Potts model, where $s_i \in \{ 0, 1, \ldots, q \}$, epidemic models, where $s_i \in \{ Susceptible, Infectected, Recovered \}$ or variants thereof. The value of $s_i$ at time $t$ is denoted by $s^t_i$, and the state of the system at time $t$ by $s^t = \{ s_1^t, s_2^t, \ldots, s_N^t \}$. The kinetics we consider is formulated in terms of a discrete time Markov process,

\[
P(s^{t+1}) = \sum_{s^t} P(s^t) W(s^t \to s^{t+1}),
\]

specified through the transition matrix

\[
W(s^t \to s^{t+1}) = P(s^{t+1} | s^t),
\]

that is the conditional probability of the state at time $t+1$, given the state at time $t$. The transition matrix must obey the normalization condition

\[
\sum_{\sigma} W(s \to \sigma) = 1, \quad \forall \sigma.
\]

In principle, one is interested in finding the whole history of the system, that is $P(s_0^t, s_1^t, \ldots, s^t, \ldots)$, given an initial condition $P(s^0)$. For many purposes, however, a knowledge of the long–time behaviour is sufficient. In most cases of physical interest, this can be described by a stationary state $\pi(s)$, defined as the state which satisfies

\[
\pi(\sigma) = \sum_{s} \pi(s) W(s \to \sigma)
\]
to be minimized subject to the marginalization constraints

\[ P(s^\tau) = \sum_{s^{\tau-1}} P(s^{\tau-1}, s^\tau), \quad \tau = 1, \cdots t \tag{9} \]

\[ P(s^\tau) = \sum_{s^{\tau-1}} P(s^{\tau-1}, s^\tau), \quad \tau = 1, \cdots t. \tag{10} \]

We could introduce suitable Lagrange multipliers to enforce the constraints, it is however simpler to use eq. 9 as a definition of \( P(s^{\tau-1}) \). Then, minimizing \( \mathcal{F} \) with respect to the 2–times probabilities we obtain

\[ P(s^{\tau-1}, s^\tau) = W(s^{\tau-1} \to s^\tau)P(s^{\tau-1}), \tag{11} \]

and eq. 11 reduces to eq. 10 showing that this variational formulation is indeed equivalent to the original kinetic problem.

In order to obtain a variational formulation for the stationary state problem it is now sufficient to observe that, assuming that the long–time kinetics converges to a stationary state, the 1–time and 2–times marginals become time–independent and the density (per unit time) corresponding to our functional is

\[ f = \sum_{s,\sigma} P(s, \sigma) [-\ln W(s \to \sigma) + \ln P(s, \sigma)] - \sum_{s} P(s) \ln P(s), \tag{12} \]

to be minimized with the constraint \( P(\sigma) = \sum_s P(s, \sigma) \).

The variational functionals in eqs. 8 and 12 (for the stationary state) are used as starting points to develop variational approximations in the PPM [7,9]: for a given graph \( G \), a set \( R \) of clusters (subsets of \( V \)) is selected according to the principles of the cluster variation method (CVM) [10,12], the 2–times and 1–time entropies are expanded into a sum of contributions associated to such clusters, and the resulting functional is minimized with respect to the 1–time and 2–times probability distributions \( P(s^\alpha) \) and \( P(s^{\tau-1}, s^\tau) \) of each cluster \( \alpha \in R \), with the appropriate marginalization constraints. The constrained variational problem can then be solved by means of simple generalizations of message–passing algorithms like those developed in [20]. The PPM has been shown, at least in one case [13], to be equivalent to a technique sometimes called local equilibrium approach, based on the assumption of a suitable factorization of the stationary state [2,11,15].

Here, however, we would like to take a slightly different route with respect to PPM, by applying the cluster expansion of the entropy directly to the functional \( \mathcal{F} \) in eq. 9. The variables \( s^\tau_i \) will be regarded as associated to the nodes, labeled by the pair \((i, t)\), of an extended graph \( G_T \), obtained by time translation of \( G \).

We shall illustrate the idea with two examples: a simple one, reducing to a mean–field theory, and a more advanced one, leading to a new and powerful technique. To fix ideas we shall restrict our discussion to kinetic Ising–like models with parallel (or synchronous) update, where the transition matrix has the simple form

\[ W(s \to \sigma) = \prod_{i \in G} W_i(\sigma_i | s_{\partial i}). \tag{13} \]

In the case of Ising variables, a frequently adopted choice for the transition matrix is the Glauber one, specified by

\[ W_i(\sigma_i | s_{\partial i}) = \frac{\exp[\sigma_i(h_i + \sum_{j \in \partial i} J_{ij}s_j)]}{2\cosh(h_i + \sum_{j \in \partial i} J_{ij}s_j)}. \tag{14} \]

where \( h_i \) is a local field, \( J_{ij} \) is a coupling (in general \( J_{ij} \neq J_{ji} \)) and temperature has been absorbed into fields and couplings.

In order to perform a cluster expansion in eq. 9 a key observation is that if one does not want to introduce additional approximations, the set \( R \) of clusters used in the entropy expansion should contain the clusters involved in the specification of \( W_i \); in the present case the star–like clusters \( A_{i,t} = \{(i, t)\} \cup \{(j, t-1), j \in \partial i\} \) with \((i, t) \in G_T \) and \( t > 0 \). The simplest possible choice is then to take \( A_{i,t} \) as maximal clusters and expand the entropy term in eq. 9 according to the rules of the CVM [11,12] (we shall call star approximation the resulting method). For simplicity, we shall consider a locally tree–like graph, without short loops. In such a case the only intersections of our star clusters are single nodes of \( G_T \), and each node \((i, t)\) appears in \( d_i + 1 \) star clusters (only \( d_i \) if \( t = 0 \)). The cluster expansion of the entropy in eq. 9 is then

\[ \mathcal{F} \approx \sum_{(i,t>0)} \sum_{s^\tau_i, s^{\tau-1}_{\partial i}} P(s^\tau_i, s^{\tau-1}_{\partial i}) [-\ln W_i(s^\tau_i | s^{\tau-1}_{\partial i}) + \ln P(s^\tau_i | s^{\tau-1}_{\partial i})] - \sum_{(i,t>0)} d_i \sum_{s^\tau_i} P(s^\tau_i) \ln P(s^\tau_i) - \sum_{i} (d_i - 1) \sum_{s^\tau_i} P(s^\tau_i) \ln P(s^\tau_i) \tag{15} \]

where \( s^{\tau-1}_{\partial i} = \{s^{\tau-1}_j, j \in \partial i\} \). If the graph \( G \) contains short loops additional terms may enter the expansion, but the following results can still be used as a low–order approximation. The above functional must be minimized with respect to the star cluster and single node probability distributions, subject to the marginalization constraints

\[ P(s^{\tau-1}_j | s^{\tau-1}_{\partial i}) = \sum_{s^\tau_i} P(s^\tau_i, s^{\tau-1}_{\partial i}), \tag{16} \]

\[ P(s^\tau_i) = \sum_{s^{\tau-1}_{\partial i}} P(s^\tau_i, s^{\tau-1}_{\partial i}). \tag{17} \]

Using eq. 16 as a definition for the single–node probabilities and minimizing \( \mathcal{F} \) in eq. 15 with respect to the star cluster probabilities we obtain

\[ P(s^\tau_i, s^{\tau-1}_{\partial i}) = W_i(s^\tau_i | s^{\tau-1}_{\partial i}) \prod_{j \notin \partial i} P(s^{\tau-1}_j), \tag{18} \]
while eq. (18) becomes

\[ P(s^t_i) = \sum_{s^t_{\partial i}} W_i(s^t_i|s^t_{\partial i}) \prod_{j \in \partial i} P(s^t_j), \]  

which in the stationary limit reduces to

\[ P(\sigma) = \sum_{s_{\partial i}} W_i(\sigma|s_{\partial i}) \prod_{j \in \partial i} P(s_j), \]

and can be used as a basis for an iterative solution. Our star approximation is then a mean-field-like approximation, structurally similar to the hard-spin mean-field theory [24] [22] for the equilibrium problem, where the stationary state is assumed to factor into a product of single node probabilities, as discussed for example in [4] [5].

In order to go beyond this mean-field approximation, one should at least take into account correlations in \( s^t_{\partial i} \). If the graph \( G \) does not contain short loops, these correlations will be primarily due to the interactions that variables in \( s^t_{\partial i} \) have with \( s^t_i \). This observation naturally leads to introduce a new approximation, by choosing as maximal clusters in our entropy expansion the diamond-like clusters \( B_{i,t} = \{(i,t)\} \cup \{(j,t-1), j \in \partial i\} \cup \{(i,t-2)\) \} with \((i,t) \in G_T \) and \( t > 1 \) (we shall call diamond approximation the resulting method). The choice of these clusters, besides being quite natural, is inspired by the dynamic cavity method, whose recursive equations involves the same sets of variables [18], but the resulting method will be more general and more powerful. In a graph without short loops, the cluster expansion of eq. (5) based on our diamond-like clusters contains also terms corresponding to the following clusters: the pairs \( \{(i,t), (j,t-1)\), with \( j \in \partial i \) (whose entropy terms will have a coefficient -1, since they are subclusters of 2 different diamond clusters); the single nodes \( (i,t) \) (with coefficient \( d_i - 1 \), since they are subclusters of \( d_i + 2 \) diamond clusters and \( 2d_i \) pair clusters). We obtain

\[ \mathcal{F} \approx \sum_{(i,t) s^t_{\partial i}} \sum_{s^t_{\partial i}, s^t_{\partial i}, s^t_{\partial i}, s^t_{\partial i}} P(s^t_i | s^t_{\partial i}, s^t_{\partial i}, s^t_{\partial i}) \times \]

\[ \times \left[ -\ln W_i(s^t_i | s^t_{\partial i}) + \ln P(s^t_i | s^t_{\partial i}, s^t_{\partial i}) \right] - \sum_{(i,t-2) j \in \partial i} \sum_{s^t_{\partial i}, s^t_{\partial i}} P(s^t_{j,t-2} | s^t_{\partial i}, s^t_{\partial i}) \ln P(s^t_{j,t-2} | s^t_{\partial i}), \]

\[ \sum_{(i,t-2)} (d_i - 1) \sum_{s^t_{\partial i}} P(s^t_{\partial i} | s^t_{\partial i}) \ln P(s^t_{\partial i} | s^t_{\partial i}) \]

\[ + \text{boundary terms} \]

(there is no need to specify boundary terms since our main interest is the stationary state), with the following pair-node

\[ P(s^t_{i,t-2}) = \sum_{s^t_{j,t-2}} P(s^t_{j,t-2} | s^t_{i,t-2}) \]

and diamond-pair

\[ P(s^t_{j,t-2}, s^t_{i,t-2}) = \sum_{s^t_{j,t-2}, s^t_{i,t-2}} P(s^t_{j,t-2} | s^t_{i,t-2}) \]

and diamond-pair

\[ P(s^t_{j,t-2}, s^t_{i,t-2}) = \sum_{s^t_{j,t-2}, s^t_{i,t-2}} P(s^t_{j,t-2} | s^t_{i,t-2}) \]

and diamond-pair

\[ P(s^t_{j,t-2}, s^t_{i,t-2}) = \sum_{s^t_{j,t-2}, s^t_{i,t-2}} P(s^t_{j,t-2} | s^t_{i,t-2}) \]

marginalization constraints. Proceeding as before, we define \( P(s^t_{i,t-2}) \) and \( P(s^t_{j,t-2}, s^t_{i,t-2}) \) as marginals of the diamond cluster probabilities using eqs. (22) and (24). Minimizing \( \mathcal{F} \) in eq. (21) with respect to the diamond cluster probabilities we then obtain

\[ P(s^t_{i,t-2}, s^t_{j,t-2}) = W_i(s^t_{i,t-2}) \times \]

\[ \times [P(s^t_{i,t-2})]^{d_i - 1} P(s^t_{j,t-2}) \prod_{j \in \partial i} P(s^t_{j,t-2} | s^t_{i,t-2}). \]

or equivalently, in terms of conditional pair probabilities,

\[ P(s^t_{i,t-2}, s^t_{j,t-2}) = W_i(s^t_{i,t-2}) P(s^t_{j,t-2} | s^t_{i,t-2}) \prod_{j \in \partial i} P(s^t_{j,t-2} | s^t_{i,t-2}). \]

Eq. (26) (or 27), together with eqs. (23) and (25) provides the solution to our problem. Given the single-node probabilities at time \( t = 2 \), and the pair probabilities at times \( (t-2, t-1) \), we can use these equations to find the same probabilities one time step later. We have therefore an iterative scheme which (if convergent) provides, in the long time limit, an approximation to the stationary state.

The accuracy of the method will be tested numerically, in specific cases, in the next section. Here it is interesting to qualitatively compare our approximation with the recently proposed dynamic cavity method [14] [18]. The latter method, which needs the so-called time-factorization (or one-time) approximation [16] [18] to make it tractable, is a message-passing algorithm where messages (which can be thought of as a suitable parametrization of cavity marginals) are exchanged between neighbouring nodes. In order to understand the difference between our method and dynamic cavity, it is useful to rewrite our eq. (28) using eqs. (23) and (24) with the following result:

\[ P(s^t_i) = \sum_{s^t_{\partial i}} W_i(s^t_i | s^t_{\partial i}) P(s^t_{\partial i} | s^t_{\partial i}) \prod_{j \in \partial i} P(s^t_{j,t-2} | s^t_{i,t-2}). \]

This is structurally similar to the recursive equation for single node marginals found in [17]. The latter however contains, in place of our conditional pair probabilities, certain quantities (messages) which in the case of Ising variables are parametrized as

\[ \mu(s^t_{i,t-2} | s^t_i) = \frac{\exp[s^t_{i,t-2}(u_{j \to i} + J_{ij}s^t_{i,t-2})]}{2 \cosh(u_{j \to i} + J_{ij}s^t_{i,t-2}),} \]

where \( u_{j \to i} \) is determined recursively. It has to be noticed here that our conditional pair probabilities can be written
in a similar form, but the corresponding effective interaction between $s_i^{k-2}$ and $s_i^{k-1}$ is not constrained to $J_{ij}$. In a general problem it can take any value, although it reduces to the true coupling $J_{ij}$ in the fully symmetric case $(J_{ij} = J_{ji}, \forall (i,j) \in E)$, when the two methods become equivalent (and exact if $G$ is a tree): see the Appendix for a proof of this equivalence. As a consequence, our method has more parameters than dynamic cavity, and needs more equations. Indeed we have to solve equations for both single-node and pair probabilities, while dynamic cavity with the time-factorization approximation is expressed through recursive equations on single-node (cavity) marginals only \[15\]. One might expect this to affect the relative performance of the two methods, making ours slower by a factor of $q$ for $q$-state variables, but in the next section we shall see that this is not the case.

Results. – Here we compare the star- and diamond-cluster approximations we have derived in the previous section with approximations of comparable complexity: the dynamic cavity method in the time-factorization approximation \[17, 18\]; a 3-times mean-field approximation proposed in \[5\], which involves a summation over the second neighbourhood of a node; the so-called naive mean field (see e.g. \[15\]). We also considered the pair approximation \[17, 23\], based on assuming a factorization of the stationary state probability at the level of equal-time neighbouring pairs, however we do not report the corresponding results since they are indistinguishable from those of our star approximation on the scale of the graphs. For reference, exact or Monte Carlo results will be used, depending on the size of the graph. We shall consider Ising-like models, with the transition matrices defined in eqs. \[13\] and \[14\] with random, independently drawn, fields and couplings. $h_i$ will be taken from a uniform distribution in $(-1/2, 1/2)$, while $J_{ij} \neq J_{ji}$ will be taken from a uniform distribution in $(-J_0, J_0)$. The quantity $\delta m = \sqrt{\frac{1}{N} \sum (m_i - m_i^{\text{exact}})^2}$ is used as a measure of the performance of the methods, where $m_i$ is the estimate of $\langle s_i \rangle$ in the stationary state provided by an approximate method and $m_i^{\text{exact}}$ is the corresponding exact result.

In fig. 1 a random graph of regular degree 3 ($d_i = 3, \forall i \in G$) and $N = 14$ nodes was considered, and $\delta m$ is reported as a function of $J_0$ for the various methods. It is seen that the star approximation outperforms the other mean-field techniques and is practically equivalent to the dynamic cavity method, while the diamond approximation outperforms (by almost an order of magnitude for large $J_0$) also the dynamic cavity method.

Fig. 2 contains a similar plot in the case of a graph with $N = 10^3$ nodes, and Monte Carlo results were used in place of exact ones. For each data point, Monte Carlo results are obtained by averaging over $10^8$ time steps, after waiting $10^5$ time steps for reaching the stationary state. The relative performance of the various methods is the same as in the case of the small graph. The plateau in the bottom-left part of the figure means that for small enough $J_0$ some approximations are more accurate than the Monte Carlo simulations.

As a further check, we considered a square lattice with $N = 30^2$ nodes and periodic boundary conditions. Corresponding results, using the same parameters as in previous cases, are reported in Fig. 3. Here, the 3-times mean-field approximation was not used, since it becomes too slow (much slower than Monte Carlo simulations) due to the sum over the second neighbourhood. Moreover, the naive mean-field theory does not converge for $J_0 > 1.2$. The performance of the other approximations is reduced, but the general picture is qualitatively the same as before. As far as the diamond approximation is concerned, we must stress that here we have used eqs. \[20\] and \[21\] which were developed in the previous section for a graph without short loops. Taking into account short loops in the entropy expansion should result in a more accurate (although slower) algorithm.

It is also interesting to observe that the star and diamond approximations are much faster than the dynamic cavity method, see Fig. 4. At $J_0 = 0.1$ the diamond approximation is 12 times faster than dynamic cavity, and this figure increases with $J_0$. Since a single iteration of the dynamic cavity is computationally comparable to a single iteration of the diamond approximation (and becomes simpler for large $q$), this means that the latter requires a smaller number of iterations.

Finally, in order to further explore the behaviour of our approach in the case of a graph with many short loops, we repeated the above tests for a simple cubic lattice with $N = 10^3$ nodes and periodic boundary conditions. Results are reported in Figs. 5 ($\delta m$) and 6 (CPU times).

The picture is similar to the square lattice case: naive mean-field is unreliable, while the other approximations perform better than Monte Carlo at $J_0 \leq 0.2$ and show an
Discussion. – We have introduced a new variational approach to the stationary states of kinetic Ising–like models. The approximation is based on the cluster expansion of the entropy term appearing in a variational functional which is minimized by the system history. The new feature is that the cluster expansion is performed on the full functional (eq. 5) and not on a 2–times functional (eqs. 8 and 12) as in the path probability method. This leads us to use as main clusters subsets of nodes which are more general than those obtained by simple time translation of the typical clusters used in equilibrium approximations. The approximation, in its present formulation, is limited to discrete time kinetics.

We have illustrated our idea with two examples. At the lowest level we have obtained a well–known mean–field theory, here called the star approximation. Adding only one node to the main clusters we have obtained a new method, here called diamond approximation. This was tested against other methods of comparable computational complexity on models defined on random graphs and on graphs with many short loops (a square lattice and a simple cubic lattice) with random fields and random interactions of varying strength. The diamond approximation turned out to be more accurate and faster than other methods of similar complexity, including the ordinary pair approximation and the recently proposed dynamic cavity method.

These results suggest that several extensions and improvements might be worth considering. In the case of graphs with many short loops, like Euclidean lattices in 2 or 3 dimensions, one could try to get further improvements by introducing additional terms in the entropy expansion. In this case, message–passing algorithms like the generalized belief propagation [20] may be needed for the minimization. In case of convergence problems, provably convergent algorithms [12,24] may be used instead. Moreover, the use of larger clusters can be considered. The simplest example in this direction is including node \((i,t–1)\) in the diamond–like cluster \(B_{i,t}\): this will allow to introduce self–interaction terms like \(J_{ii}s_{i,t}^{-1}s_{i,t}^{1}\) in the transition matrix, which is relevant e.g. for models of epidemic processes. Another line of investigation could involve the use of different kinds of transition matrices, the simplest examples being asynchronous updates and exchange dynamics: the latter, in particular, may need the introduction of larger clusters. A further important question to address is related to the applicability of these methods to an approximate description of the transient, although it was shown in [16, 18] that these mean–field like techniques are more appropriate for the stationary state. Work is in progress along these lines.

Appendix. – In the case of symmetric interactions \((J_{ji} = J_{ij}, \forall (i,j) \in E)\), it was shown in [17] that a fixed point of ordinary belief propagation (BP) corresponds to a stationary state of the dynamic cavity method in the time–factorization approximation. In this Appendix we shall show that this property is shared by our diamond approximation, thereby establishing an equivalence between the dynamic cavity method and the diamond approximation. As shown in the previous sections, for non–symmetric interactions the two methods give different results, so the equivalence is limited to the case of symmetric interactions.

Let us consider the stationary state problem defined by the transition matrix eq. 11 with \(J_{ij} = J_{ji}\), and the corresponding equilibrium problem, defined by the Ising Hamil-
marginals given by above equilibrium problem, with the single–node and pair
from node \( k \) transitions corresponds to a stationary state of our diamond
marginalization constraints yield (up to normalization)

In the above equations, \( H = - \sum_{i \in G} h_i s_i - \sum_{(i,j) \in E} J_{ij} s_i s_j \). (30)

In the following, we will use a symmetrized form of the above Hamiltonian, where \( H = \sum_{(i,j) \in E} H_{ij}(s_i, s_j) \) and

\[
H_{ij}(s_i, s_j) = -J_{ij} s_i s_j - \frac{h_i}{a_i} s_i - \frac{h_j}{a_j} s_j. \quad (31)
\]

Moreover, we shall define \( \psi_{ij}(s_i, s_j) = \exp[-H_{ij}(s_i, s_j)] \).

BP \([12, 19]\) provides an approximate solution to the above equilibrium problem, with the single–node and pair marginals given by

\[
P(s_i) = \frac{1}{Z_i} \prod_{k \in \partial i} m_{k \rightarrow i}(s_i), \quad (32)
\]

\[
P(s_i, s_j) = \frac{\psi_{ij}(s_i, s_j)}{Z_{ij}} \prod_{k \in \partial j \setminus i} m_{k \rightarrow i}(s_i) \times 
\prod_{l \in \partial j \setminus i} m_{l \rightarrow j}(s_j). \quad (33)
\]

In the above equations, \( m_{k \rightarrow i}(s_i) \) is called the message from node \( k \) to node \( i \), while \( Z_i \) and \( Z_{ij} \) are normalization constants. Messages are determined by imposing the marginalization constraints \( P(s_i) = \sum_{s_j} P(s_i, s_j) \), which yield (up to normalization)

\[
m_{j \rightarrow i}(s_i) \propto \sum_{s_j} \psi_{ij}(s_i, s_j) \prod_{l \in \partial j \setminus i} m_{l \rightarrow j}(s_j). \quad (34)
\]

usually solved by iteration to a fixed point.

We can now show that a fixed point of the BP equations corresponds to a stationary state of our diamond approximation. In eqs. (26) and (27), let us suppose that the transition matrix is given by eq. (14) and rewrite it as

\[
W(s_i \! \! \! \! \! \! \! \rightarrow s_i'; d_i) = \frac{\prod_{k \in \partial i} \psi_{ki}(s_i'; s_k)}{\sum_{s'_{i}} \prod_{k \in \partial i} \psi_{ki}(s_i', s_k)}. \quad (35)
\]
Fig. 6: Same as fig. 4 for a cubic lattice with $N = 10^3$ nodes.

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