ON THE NONEXISTENCE OF A RELATION BETWEEN 
σ-LEFT-POROSITY AND σ-RIGHT-POROSITY

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ABSTRACT. Given an arbitrarily weak notion of left-⟨f⟩-porosity and an arbitrarily strong notion of right-⟨g⟩-porosity, we construct an example of closed subset of \( \mathbb{R} \) which is not \( \sigma \)-left-⟨f⟩-porous and is right-⟨g⟩-porous. We also briefly summarize the relations between three different definitions of porosity controlled by a function; we then observe that our construction gives the example for any combination of these definitions of left-porosity and right-porosity.

1. Introduction

The topic of this paper originates in the work \([1]\) of R. J. Nárayes and L. Zajíček where the following theorem is proved:

Theorem NZ. There exists a closed set \( F \subseteq \mathbb{R} \) which is right-porous and is not \( \sigma \)-left-porous.

We prove the stronger Theorem 3.1 which shows that, as long as we work with a reasonable notion of upper porosity (i.e. porosity defined by lim sup or some equivalent), there is no connection between \( \sigma \)-left-porosity and right-porosity. The proof is based on the ideas from \([1]\), but is slightly more technical and contains some new concepts. The main difference lies in our usage of the multi-expansions of real numbers (see \([2, 3]\)) as opposed to the ordinary decimal expansion used by Nárayes and Zajíček. When we have chosen a suitable multi-expansion, however, the rest of the proof follows a scheme identical to that from \([1]\).

For our method to conveniently go through, we also need to use the right definition of porosity controlled by a function. The one which suits our purpose the best is \([g]\)-porosity (see Definition 4.4), but that is not a standard notion. For that reason we also give the more common Definitions 4.2 and 4.3, together with Proposition 4.7 which allows us to deduce the validity of our main result in the more standard setting.

Note that porosity controlled by a function can be defined in various ways (see e.g. \([3]\) or \([2]\)) and the notation is not unified. Thus, for example, the notion of \([g]\)-porosity we choose to work with is different from that of \([3]\). However, finding relations between different definitions is usually quite simple and is not our aim in this article.

The proof of Theorem 3.1 uses the so called Foran Lemma (here Lemma 1.2) which is a tool for recognizing non-\(\sigma\)-porous (in various senses) sets developed by L. Zajíček from an original idea of J. Foran. Our proof, however, does not require the most general version, and so we state the definition of Foran system accordingly simplified. The general version for any porosity-like relation in a topologically complete metric space and \(G_\delta\)-sets (instead of closed sets) can be found in the article \([4]\).

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We shall need Notation 4.1 and Definition 4.3 in order to formulate the following definition of Foran System, and the Foran Lemma.

1.1. **Definition.** Let \( g \in G_3 \). We say that \( F \subseteq 2^R \) is a Foran system for left-\([g]-\)porosity if the following conditions hold:

(a) \( F \) is a nonempty family of nonempty closed sets.
(b) For each \( F \in F \) and each open set \( G \subseteq R \) with \( F \cap G \neq \emptyset \), there exists a set \( F^* \in F \) such that \( F^* \subseteq F \cap G \) and \( F \) is left-\([g]-\)porous at no point of \( F^* \).

1.2. **Lemma.** Let \( g \in G_3 \) and let \( F \) be a Foran system for left-\([g]-\)porosity. Then no set from \( F \) is \( \sigma \)-left-\([g]-\)porous.

2. **Preliminaries to the proof**

This section contains the necessary notation and definitions used in the proof of our main theorem, as well as an important lemma which will allow us to choose a suitable multi-expansion.

2.1. **Notation.**

- Let \( x \in (0,1) \). As usual, we write \( x = 0.a_1a_2 \ldots \) if \( x = \sum_{i=1}^{\infty} a_i10^{-i} \) and \( a_i \in \{0,1,\ldots,9\} \) for each \( i \in \mathbb{N} \). In case \( x \) has two expansions, we only consider the one which ends with zeros and then we denote \( a_i(x) := a_i \).
- For an \( x \in (0,1) \) we put \( l(x) := \sup\{k \in \mathbb{N}; a_k(x) \neq 0\} \).
- We denote by \( \Lambda \) the set of all \( x \in (0,1) \) with \( l(x) < \infty \).

2.2. **Lemma.** Let \( f \in G_3 \). Then there exists a sequence \( \{x_n\}_{n=1}^{\infty} \subseteq (0,\infty) \cap \Lambda \) with the following properties:

(i) The sequence \( \{x_n\}_{n=1}^{\infty} \) is decreasing and \( \lim_{n \to \infty} x_n = 0 \).
(ii) The sequence of natural numbers \( \{l(x_n)\}_{n=1}^{\infty} \) is increasing.
(iii) Let \( M \subseteq R \), \( x \in R \) and let the following proposition be true:

\[
(\exists n_0 \in \mathbb{N})(\forall n > n_0) : x + x_n \in M \lor x + x_{n+1} \in M.
\]  

Then \( M \) is not right-\([f]-\)porous at \( x \).

**Proof.** Let \( f \in G_3 \) be defined on the interval \([0,\delta)\). For \( x \in [0,\delta) \) define \( \alpha(x) := x - f(x) \), and for \( x \in [0,\frac{\delta}{2}) \)

\[
\beta(x) := \min \alpha\left(\left[x, \frac{\delta}{2}\right]\right).
\]

Then \( \beta \) is a non-decreasing continuous function with \( 0 < \beta \leq \alpha \) on \((0,\frac{\delta}{2})\). Let us now define the function \( g : [0,\frac{\delta}{2}) \to (0,\infty) \) by the formula

\[
g(x) := x - \beta^2(x).
\]

Obviously \( \beta(0) = 0 \), so we can find a \( \delta_1 \in \left(0,\frac{\delta}{2}\right)\) such that for each \( x \in (0,\delta_1) \) we have \( \beta(x) < \frac{\delta}{2} \) and thus

\[
\beta^2(x) < \frac{1}{2}\beta(x) \leq \frac{1}{2}\alpha(x).
\]
Recall that \( g \in G_3 \), and hence \( g(x) < x \) for all \( x \in (0, \delta) \). We obtain:

\[
g(g(x)) \leq g(x) - \beta^2(g(x)) \leq x - \beta^2(x) - \beta^2(g(x)) \geq x - 2\beta^2(x) > x - \alpha(x) = f(x).
\]

We shall now construct the sequence \( \{x_n\}_{n=1}^{\infty} \subseteq (0, \infty) \cap A \) by induction. First, choose an arbitrary \( x_1 \in (0, \delta_1) \cap A \). Now assume we have already chosen the number \( x_n \) and take some

\[
y \in \left( \frac{g(x_n), g(x_n) + \frac{x_n - g(x_n)}{n + 1}}{\cap A} \right)
\]

such that \( l(y) > l(x_n) \). We set \( x_{n+1} := y \).

We now have a sequence \( \{x_n\}_{n=1}^{\infty} \) satisfying condition \((ii)\) and it remains to be shown it also satisfies \((i)\) and \((iii)\).

(i): It is obvious that the sequence \( \{x_n\}_{n=1}^{\infty} \) is decreasing and positive; hence, it has a non-negative limit \( c \). To obtain a contradiction, assume that \( c > 0 \). Set

\[
m := \beta^2(c) \quad \text{and} \quad M := \beta^2\left(\frac{c + \delta}{2}\right),
\]

and find an \( n_0 \in \mathbb{N} \) such that

\[
x_{n_0} < \min\left\{ c + m - \frac{M}{n_0 + 1}, \frac{c + \delta}{2} \right\}.
\]

The following estimate gives a contradiction:

\[
x_{n_0+1} < g(x_{n_0}) + \frac{x_{n_0} - g(x_{n_0})}{n_0 + 1} \leq x_{n_0} - \beta^2(x_{n_0}) + \frac{\beta^2(x_{n_0})}{n_0 + 1} \leq x_{n_0} - \beta^2(x_{n_0}) + \frac{M}{n_0 + 1} \leq x_{n_0} - m + \frac{M}{n_0 + 1} < c.
\]

(iii): Let us have a set \( M \subseteq \mathbb{R} \) and a point \( x \in \mathbb{R} \) such that \((i)\) is true; let us take some \( n_0 \) from \((i)\). Then we have \( x + x_{n_0+1} \in M \) or \( x + x_{n_0+2} \in M \). Now choose an arbitrary \( y \in (x, x + x_{n_0+2}) \) and find the largest \( n_1 \in \mathbb{N} \) such that \( y \leq x + x_{n_1} \). Now:

\[
f(y - x) < g(g(y - x)) \leq g(g(x_{n_1})) \leq x_{n_1+2} < x_{n_1+1} < y - x.
\]

Setting \( I := (x + f(y - x), y) \) we obtain that \( x + x_{n_1+1} \in M \cap I \) or \( x + x_{n_1+2} \in M \cap I \) and hence, by the definition, \( M \) is not right-[\( f \)]-porous at \( x \).

\( \square \)

2.3. Remark. It is obvious that the sequence \( \{x_n\}_{n=1}^{\infty} \) with \((i)\), \((ii)\), \((iii)\) from the previous lemma also has the following property.

(iv) Let \( M \subseteq \mathbb{R}, x \in \mathbb{R} \) and let the following proposition be true:

\[
(\exists n_0 \in \mathbb{N}) (\forall n > n_0) : x - x_n \in M \ \vee \ x - x_{n+1} \in M.
\]

Then \( M \) is not left-[\( f \)]-porous at \( x \).

2.4. Definition. Let \( \{d_n\}_{n=1}^{\infty} \subseteq \mathbb{N} \) be a sequence. We denote \( D_0 := 0 \) and then for a number \( n \in \mathbb{N} \) we set

\[
D_n := \sum_{k=1}^{n} d_k.
\]
For \( x \in (0, 1) \) we define the multi-digit expansion of \( x \) with respect to \( \{d_n\}_{n=1}^{\infty} \) as the sequence \( \{b_n(x)\}_{n=1}^{\infty} \), where for \( n \in \mathbb{N} \),

\[
b_n(x) := \sum_{k=1+D_{n-1}}^{D_n} 10^{D_n-k} a_k(x).
\]

The sequence \( \{b_n\}_{n=1}^{\infty} \) of functions defined on \( (0, 1) \) by (8) is called the multi-digit expansion with respect to \( \{d_n\}_{n=1}^{\infty} \). We will need the symbols

\[
C(x, n) := \# \left\{ k \in \mathbb{N}; \ n^2 < k \leq (n+1)^2, b_k(x) = 10^{d_k} - 1 \right\},
\]

\[
E(x, n) := \# \left\{ k \in \mathbb{N}; \ n^2 < k \leq (n+1)^2, b_k(x) \neq 10^{d_k} - 1 \right\}.
\]

2.5. Remark. Let us have a multi-digit expansion \( \{b_n\}_{n=1}^{\infty} \) with respect to some sequence \( \{d_n\}_{n=1}^{\infty} \subseteq \mathbb{N} \) and a real number \( x \in (0, 1) \). Then for each \( n \in \mathbb{N} \) we have \( b_n(x) \in \{0, 1, \ldots, 10^{d_n} - 1\} \). We can also observe that \( x \in (0, 1) \) can be written in the form

\[
x = \sum_{n=1}^{\infty} b_n(x) \cdot 10^{-D_n}.
\]

Note that we still respect the convention from the first point of 2.1 hence \( b_n(x) \neq 10^{d_n} - 1 \) for infinitely many \( n \in \mathbb{N} \).

2.6. Convention. In the following we shall often say briefly multi-expansion instead of multi-digit expansion with respect to \( \{d_n\}_{n=1}^{\infty} \), as we will only work with a single sequence \( \{d_n\}_{n=1}^{\infty} \). The functions \( b_n \) will also be called multi-digits.

3. Main result

3.1. Theorem. Let \( f, g \in G_3 \). Then there exists a closed set \( F \subseteq \mathbb{R} \) which is right-[\( g \)]-porous and is not \( \sigma \)-left-[\( f \)]-porous.

Proof. Let the two functions \( f, g \in G_3 \) be given. First we need to choose the multi-expansion we will work with in the rest of the proof. That is, we need to find a suitable (according to the functions \( f \) and \( g \)) sequence \( \{d_n\}_{n=1}^{\infty} \subseteq \mathbb{N} \). Using the multi-expansion we will then define a Foran system \( \mathcal{F} \) for left-[\( f \)]-porosity with right-[\( g \)]-porous elements. According to Lemma 2.2 any set \( F \in \mathcal{F} \) will have the desired properties.

**Multi-expansion:** Take the sequence \( \{x_n\}_{n=1}^{\infty} \) from Lemma 2.2 for \( f \); this sequence will remain fixed till the end of the proof. We shall now define the sequence \( \{d_n\}_{n=1}^{\infty} \subseteq \mathbb{N} \) by induction, respecting the notation from Definition 2.3. At the same time we also inductively define the multi-expansion \( \{b_n\}_{n=1}^{\infty} \) by the formula (8).

Set \( d_1 := \min\{k \in \mathbb{N}; a_k(x_1) \neq 0\} \). Now suppose we have already chosen the numbers \( d_k \in \mathbb{N} \) for all \( k \leq n \). Set \( k_0 := \max\{k \in \mathbb{N}; b_n(x_k) \neq 0\} \) and choose a natural number \( d_{n+1} \) such that the following conditions hold:

- \( d_{n+1} > d_n \),
- \( g \left(10^{-D_n}\right) > 10^{-D_{n+1}} \),
- \( D_{n+1} = \sum_{k=1}^{n+1} d_k > \max\{l(x_k); b_n(x_k) \neq 0\}, l(x_{k_0+1})\).

These conditions can be met, since \( x_n \in \mathcal{A} \) for all \( n \) and the sequence tends to zero.

Hence, we have a fixed multi-expansion \( \{b_n\}_{n=1}^{\infty} \). Observe that each member of the sequence \( \{x_n\}_{n=1}^{\infty} \) either has only one non-zero multi-digit or it has two consecutive non-zero
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The sets: Fix an arbitrary $\max$satisfying (9). We already know that satisfy the following conditions (see Definition 2.4):

$$\{\text{closed. To that end, take a convergent sequence} \ b_n \text{it is now easy to see that} (10) \text{gives}$$

$$\text{It is easy to see that such an} \ M$$

$$\text{Denote by} \ F$$

$$\text{Multi-digits. We will also use the easy fact that, for each} \ n \in \mathbb{N}, \ \max\{k; b_k(x_{n+1}) \neq 0\} \leq \max\{k; b_k(x_n) \neq 0\} + 1.$$  

The sets are closed: We will show that each set of the form $A := A(B_1, \ldots, B_{N^2}, \varepsilon)$ is closed. To that end, take a convergent sequence $\{y_m\}_{m=1}^\infty \subseteq A$ with limit $y$. We want to check that $y \in A$.

Condition (A2) clearly implies the existence of an $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$,

$$C(x, n) \neq 0 \quad \text{and} \quad E(x, n) \neq 0. \quad (10)$$

Choose an arbitrary $n \geq n_0$ and find some $m(n) \in \mathbb{N}$ such that

$$|y_{m(n)} - y| < 10^{-D} \quad \text{where} \quad D := D_{(n+1)^2+1}.$$  

It is now easy to see that (10) gives $b_k(y_{m(n)}) = b_k(y)$ for each $k \leq n^2$. But since the number $n$ was chosen arbitrarily and the corresponding $y_{m(n)}$ is an element of $A$, it follows from the definition of $A$ that $y \in A$.

The sets are nonempty: Condition (A2) from the definition of $A(B_1, \ldots, B_{N^2}, \varepsilon)$ is equivalent to the statement

$$C(x, n) \in [(1 - \frac{\varepsilon}{n^\alpha})(2n + 1), 2n + 1] =: I_n, \quad \text{for each} \ n \geq N.$$  

Assume $n \geq N$; then (9) gives the following estimates:

$$\left(1 - \frac{\varepsilon}{n^\alpha}\right)(2n + 1) > \left(1 - \frac{\varepsilon}{1 + \varepsilon}\right)(2(1 + \varepsilon)^{\frac{1}{\alpha}})^{(0 < \alpha < 1)} > 2,$$

$$(2n + 1) - \left(1 - \frac{\varepsilon}{n^\alpha}\right)(2n + 1) = \frac{\varepsilon}{n^\alpha}(2n + 1) > 2\varepsilon n^{1 - \alpha} > 2.$$  

These two inequalities imply that the interval $I_n$ contains some natural number. Hence, the set $A(B_1, \ldots, B_{N^2}, \varepsilon)$ is nonempty, whenever $N$ fulfills condition (9).

Foran system: Denote by $F$ the system of all sets of the form $A(B_1, \ldots, B_{N^2}, \varepsilon)$ with $N$ satisfying (9). We already know that $F$ satisfies the first condition from the definition of Foran system [12] We claim that condition (b) from [12] is also true for $F$.

To prove it, take an arbitrary set $F := A(B_1, \ldots, B_{N^2}, \varepsilon) \in F$ and an open set $G \subseteq \mathbb{R}$ with $F \cap G \neq \emptyset$. Now choose any point $y \in F \cap G$ and a natural number $M$ such that

$$M > \max\left\{N, \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\alpha - 1}}\right\} \quad \text{and} \quad (12)$$

$$F^* := A\left(B_1, \ldots, B_{N^2}, b_{N^2+1}(y), \ldots, b_{M^2}(y), \frac{\varepsilon}{2}\right) \subseteq F \cap G.$$  

It is easy to see that such an $M$, indeed, exists. Hence, we have the set $F^*$ which clearly belongs to $F$. We now need to show that the set $F$ is left-$[f]$-porous at no point of $F^*$. 

Let us fix an arbitrary point $z \in F^*$. As the sequence $\{x_n\}_{n=1}^\infty$ has the property (iv) from Remark 2.3, it is sufficient to find a $p_0 \in \mathbb{N}$ such that, for each $p > p_0$, at least one of the points $z_p^− := z - x_p$ and $z_{p+1}^− := z - x_{p+1}$ belongs to $F$. Find a $p_0 \in \mathbb{N}$ such that, for each $p > p_0$,

$$x_p < 10^{-D} \quad \text{where} \quad D := D_{(M+1)^2+1}^2. \quad (13)$$

Now choose any $p > p_0$ and $x \in \{z_p^−, z_{p+1}^−\}$. The estimate (13) gives that $b_k(x_p) = 0$ for all $k \leq (M+1)^2$. We also know that each $x_n$ has at most two non-zero multi-digits and it follows that, for all $n \in \mathbb{N}$,

$$C(x, n) \geq C(z, n) - 2. \quad (14)$$

Further, since $z \in F^*$, we have $C(z, M) > 0$ (we even know that $C(z, M) > 2$) and this yields that, for each $k \leq M^2$, $b_k(x) = b_k(z)$. In particular, $x$ satisfies the condition (A1) from the definition of $F = A(B_1, \ldots, B_{N^2}, \varepsilon)$.

We now turn our attention to condition (A2). The following estimate holds for $n \geq M$:

$$\frac{2}{2n+1} = \varepsilon \left(\frac{\varepsilon}{2n+1}\right)^{1-\alpha} \leq \frac{\varepsilon n^{1-\alpha}}{2n+1} < \frac{\varepsilon}{2n^\alpha} \quad (15)$$

Thus, for $n \geq M$, we obtain

$$\frac{C(x, n)}{2n+1} \geq \frac{C(z, n) - 2}{2n+1} \geq 1 - \frac{\varepsilon}{2n^\alpha} - \frac{2}{2n+1} \quad (15)$$

It remains to be shown that, for $x = z_p^−$ or $x = z_{p+1}^−$,

$$E(x, n) \neq 0 \quad \text{for each} \quad n \geq M.$$  

To that end, assume that, for some $n \geq M$, $E(z_p^−, n) = 0$. Set $k_1 := \max\{m \in \mathbb{N}; b_m(x_p) \neq 0\}$ and $k_2 := \max\{m \in \mathbb{N}; b_m(x_{p+1}) \neq 0\}$. The way the multi-expansion was constructed implies (as we have observed) that $k_2 \in \{k_1, k_1 + 1\}$. We also noted that $b_m(x_p) = 0$ for all $m \in \mathbb{N} \setminus \{k_1, k_1 - 1\}$. Taking into account that for each $m \geq M$ we have $C(z, m) > 2$, it is easy to see that

$$k_1 = n^2 + i \quad \text{where} \quad i \in \{1, \ldots, 2n-1\}, \quad (16)$$

$$b_m(z) = 10^{d_m} - 1 \quad \text{for each} \quad m \in \{n^2 + i + 1, \ldots, (n+1)^2\}. \quad (17)$$

We distinguish two cases. In case $k_2 = k_1$, we use the property (ii) of the sequence $\{x_n\}_{n=1}^\infty$ which gives that $l(x_{p+1}) > l(x_p)$. It follows that $b_{k_1}(x_{p+1}) \neq b_{k_1}(x_p)$, and thus $b_{k_1}(z_{p+1}) \neq b_{k_1}(z_p^-) = 10^{d_{k_1} - 1}$ which means that $E(z_{p+1}^−) \neq 0$. If, on the other hand, $k_2 = k_1 + 1$, then from (16) and (17) it follows that $b_{k_1+1}(z_{p+1}^-) \neq 10^{d_{k_1+1} - 1}$. Again, the conclusion is that $E(z_{p+1}^−) \neq 0$. Hence, $\mathcal{F}$ is a Foran system for left-[f]-porosity.

**Right-[g]-porosity:** To conclude the proof of the theorem we shall choose any set $F := A(B_1, \ldots, B_{N^2}, \varepsilon) \in \mathcal{F}$ and prove it is right-[g]-porous.

Fix a point $x \in F$. For each $n \in \mathbb{N}$, let $m_n$ be the maximum of those $i \in \mathbb{N}$ for which there exist natural numbers $u$ and $v$ such that $v - u = i$,

$$n^2 \leq u < v \leq (n+1)^2 \quad \text{and} \quad b_s(x) = 10^{d_s} - 1 \quad \text{for each} \quad u < s \leq v. \quad (18)$$

The following estimate is obvious:

$$2n+1 - E(x, n) = C(x, n) \leq m_n(E(x, n) + 1) \quad (19)$$
Moreover, condition (A2) from the definition of $F$ implies that, for each $n \geq N$,

$$E(x, n) \leq \frac{\varepsilon(2n + 1)}{n^\alpha}.$$  \hfill (20)

Hence, for all $n \geq N$ the following estimate holds:

$$m_n \geq \frac{2n + 1 - E(x, n)}{E(x, n) + 1} \geq \frac{(2n + 1)(1 - \frac{\varepsilon}{n^\alpha})}{\varepsilon(2n + 1) + n^\alpha} \geq$$

$$\geq \frac{n^\alpha(2n + 1)\frac{1}{1 + \varepsilon}}{\varepsilon(2n + 1) + (2n + 1)} = \frac{n^\alpha}{1 + \varepsilon} = \frac{n^\alpha}{(1 + \varepsilon)^2} =: cn^\alpha.$$  \hfill (21)

Now, for each $n$, choose natural numbers $u_n$ and $v_n$ such that

$$v_n - u_n = m_n$$ and (18) holds for $u = u_n$, $v = v_n$.

Set $L_n := D_{v_n-1}$, $K_n := D_{v_n}$, and define

$$y_n := x + 10^{-K_n} \quad \text{and} \quad z_n := x + 10^{-L_n}.$$  \hfill (22)

For each $t \in (10^{-K_n}, 10^{-L_n})$, min\{k \in \mathbb{N}; b_k(t) \neq 0\} = v_n$. It is now easy to see that, for each $t \in (y_n, z_n)$ and each $u_n < k \leq v_n - 1$, we have $b_k(t) = 0$. Hence, for each $t \in (y_n, z_n)$,

$$C(t, n) \leq 2n + 1 - (m_n - 1).$$  \hfill (23)

Since $\alpha > 1 - \alpha$, we can find an $n_0 \in \mathbb{N}$ such that, for each natural $n > n_0$,

$$2n + 2 - cn^\alpha < 2n + 1 - 2\varepsilon n^{1-\alpha} - \varepsilon n^{-\alpha} = \left(1 - \frac{\varepsilon}{n^\alpha}\right)(2n + 1).$$  \hfill (24)

Now, for each $n > n_0$, we obtain the estimate

$$C(t, n) \leq 2n + 1 - (m_n - 1) \leq 2n + 2 - cn^\alpha < \left(1 - \frac{\varepsilon}{n^\alpha}\right)(2n + 1),$$

which implies that $t \notin F = A(B_1, \ldots, B_{n^2}, \varepsilon, \alpha)$. Thus, for each $n > n_0$, $(y_n, z_n) \cap F = \emptyset$.

It only remains to recall the second condition from the construction of our multi-expansion. It says that

$$g(z_n - x) = g(10^{-L_n}) > 10^{-K_n} = y_n - x,$$

and hence we obtain the inclusion $(y_n, z_n) \supseteq (x + g(z_n - x), z_n)$. It follows that the set $F$ is right-$[g]$-porous which concludes the proof. \hfill \Box

4. Several Definitions of Porosity and Their Relations

In this last section we briefly discuss the connection between the notion of porosity used in the proof of Theorem 3.1 and two of the more standard definitions of porosity controlled by a function.

4.1. Notation. The different notions of porosity use the following sets of control functions:

- $G$ is the set of all increasing continuous functions $f : [0, \delta) \to (0, \infty)$ (where $\delta > 0$) with $f(0) = 0$.
- $G_1$ is the set of all $f \in G$ such that \(\lim_{x \to 0^+} \frac{f(x)}{x} > 0\).
- $G_2$ is the set of all $f \in G$ such that $f(x) > x$ for each $x \neq 0$ from the domain of $f$.
- $G_3$ is the set of all $f \in G$ such that $f(x) < x$ for each $x \neq 0$ from the domain of $f$. 
4.2. **Definition** ((g)-porosity). Let \( M \subseteq \mathbb{R} \) be a set and let \( I \subseteq \mathbb{R} \) be an interval. We denote by \( \lambda(M, I) \) the length of the largest open subinterval of \( I \) which is disjoint from \( M \). For \( x \in \mathbb{R} \) and \( g \in G_1 \) we set
\[
 p^+_g(M, x) := \limsup_{h \to 0_+} \frac{g(\lambda(M, (x, x + h)))}{h},
\]
\[
 p^-_g(M, x) := \limsup_{h \to 0_+} \frac{g(\lambda(M, (x - h, x)))}{h}.
\]
We say \( M \) is **right-(g)-porous at** \( x \) if \( p^+_g(M, x) > 0 \).

4.3. **Definition** ((g)-porosity). Let \( M \subseteq \mathbb{R} \) be a set, \( r, \delta \in \mathbb{R} \) be such that \( 0 < r < \delta \) and let \( g \in G_2 \) be defined on \([0, \delta)\). Denote
\[
 S^+(g, r, M) := \bigcup \{ (y - g(\sigma), y) ; y \in \mathbb{R}, \sigma \in (0, r), (y - \sigma, y) \cap M = \emptyset \}.
\]
We say the set \( M \) is **right-(g)-porous at** \( x \) if
\[
 x \in \bigcap_{0 < r < \delta} S^+(g, r, M).
\]

4.4. **Definition** ([g]-porosity). Let \( M \subseteq \mathbb{R}, x \in \mathbb{R} \) and \( g \in G_3 \). We say the set \( M \) is **right-[g]-porous at** \( x \) if there exists a sequence \( \{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) such that the following conditions are satisfied:

(i) The sequence \( \{\alpha_n\}_{n=1}^{\infty} \) is decreasing and it tends to \( 0 \).
(ii) For each \( n \in \mathbb{N} \) we have \((x + g(\alpha_n), x + \alpha_n) \cap M = \emptyset\).

Of course, definitions 4.2, 4.3, and 4.4 can be stated “for the left side” in the obvious symmetrical way.

4.5. **Definition.** Let \( M \subseteq \mathbb{R}, x \in \mathbb{R} \), and assume \( V \) is one of the symbols \((g), \langle g \rangle, [g]\). We say the set \( M \) is

- **V-porous at** \( x \) if it is left-V-porous at \( x \) or right-V-porous at \( x \),
- **V-porous** if it is V-porous at each of its points,
- **right-V-porous** if it is right-V-porous at each of its points,
- **\( \sigma \)-V-porous** if it is a countable union of V-porous sets,
- **\( \sigma \)-right-V-porous** if it is a countable union of right-V-porous sets.

The following proposition deals (to some extent) with the relation of the previously defined notions of porosity. In order to formulate it in a briefer form, we need a special notation:

4.6. **Notation.**

- If \( g \in G_1 \), we write \(^1g\) instead of \( (g) \).
- If \( g \in G_2 \), we write \(^2g\) instead of \( \langle g \rangle \).
- If \( g \in G_3 \), we write \(^3g\) instead of \([g]\).

4.7. **Proposition.** Let \( i, j \in \{1, 2, 3\} \) and let \( f \in G_i \). Then:

(i) If \( j \neq 1 \), then there is a function \( g \in G_j \) such that whenever \( M \subseteq \mathbb{R} \) is right-\(^i\)g-porous (resp. left-\(^i\)g-porous), then \( M \) is right-\(^i\)f-porous (resp. left-\(^i\)f-porous).
(ii) There exists a function \( h \in G_j \) such that whenever \( M \subseteq \mathbb{R} \) is right-\(^i\)f-porous (resp. left-\(^i\)f-porous), then \( M \) is right-\(^i\)h-porous (resp. left-\(^i\)h-porous).
We omit the proof as it is completely straightforward but quite long. The sole purpose of this proposition in the article is to show that Theorem 3.1, which is formulated with the non-standard \([f]\)-porosity, is also true with other kinds of porosity controlled by a function. (In fact, to see this, we only need to consider the case \(j = 3\).)

4.8. **Remark.** The part (i) of Proposition 4.7 says that, for any given type of porosity controlled by a given function, a function \(g \in G_j\) can be found such that the notion of right-\(j\)-porosity “is stronger”. We can not include the case \(j = 1\), because the \(1\)-porosity is the strongest possible for \(g(x) = x\), and that is the ordinary upper porosity. For the other two cases, however, we can obtain (in some sense) arbitrarily strong notions of porosity.

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