Abstract. We provide an explicit formula on the growth rate of ample heights of rational points under iteration of endomorphisms of smooth projective varieties over number fields. As an application, we give a positive answer to a variant of the Dynamical Mordell-Lang conjecture for pairs of étale endomorphisms, which is also a variant of the original one stated by Bell, Ghioca, and Tucker in their monograph.

1. Introduction

Let $X$ be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f: X \to X$ a surjective endomorphism of $X$ over $\overline{\mathbb{Q}}$. Here, an endomorphism simply means a self-morphism. Fix an ample divisor $H$ on $X$ over $\overline{\mathbb{Q}}$ and take a Weil height function $h_H: X(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated with $H$. For a point $P \in X(\overline{\mathbb{Q}})$, the arithmetic degree $\alpha_f(P)$ of $f$ at $P$ is defined by

$$\alpha_f(P) := \lim_{n \to \infty} \max\{1, h_H(f^n(P))\}^{1/n}.$$ 

It is known that the arithmetic degree $\alpha_f(P)$ is well-defined, and independent of the choice of $H$ and $h_H$; see Remark 2.2. By using the arithmetic degree, we can describe the growth rate of the ample heights of rational points under the iteration of the endomorphism as follows.
Theorem 1.1. Let $X$ be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f: X \rightarrow X$ a surjective endomorphism of $X$ over $\overline{\mathbb{Q}}$. Let $H$ be an ample divisor on $X$ over $\overline{\mathbb{Q}}$. Then for any point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) > 1$, there is a non-negative integer $t_f(P) \in \mathbb{Z}_{\geq 0}$, positive real numbers $C_0, C_1 > 0$, and an integer $N_0$ such that

$$C_0 n^{t_f(P)\alpha_f(P)} < h_H(f^n(P)) < C_1 n^{t_f(P)\alpha_f(P)}$$

for all $n \geq N_0$.

As an application of Theorem 1.1 we prove a variant of the Dynamical Mordell-Lang conjecture (see Section 4 for details).

Theorem 1.2. Let $X$ be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f, g: X \rightarrow X$ étale endomorphisms of $X$ over $\overline{\mathbb{Q}}$. Let $P, Q \in X(\overline{\mathbb{Q}})$ be points satisfying the following two conditions:

• $\alpha_f(P)^p = \alpha_g(Q)^q > 1$ for some $p, q \in \mathbb{Z}_{\geq 1}$, and
• $t_f(P) = t_g(Q)$, where $t_f(P)$ and $t_g(Q)$ are as in Theorem 1.1.

Then the set $S_{f,g}(P, Q) := \{(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid f^m(P) = g^n(Q)\}$ is a finite union of the sets of the form

$$\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$.

We now briefly sketch the plan of this paper. In Section 2, we fix some notation. In Section 3, we prove Theorem 1.1. In Section 4, we provide backgrounds of Theorem 1.2 and recall known results related to this theorem. In Section 5, we prove Theorem 1.2. It seems plausible that we can generalize the assertion of Theorem 1.2 further. We give a conjecture (Conjecture 4.6) generalizing Theorem 1.2 and some evidence in Section 6. Furthermore, to see that the results given in Section 6 support our conjecture, we give a definition of the double canonical height in a special case (see the proof of Theorem 6.1).

2. Notation and definitions

Let $X$ be a smooth projective variety, and $f: X \rightarrow X$ a surjective endomorphism of $X$ both over $\overline{\mathbb{Q}}$. We denote the group of divisors, the Picard group, and the Néron-Severi group by $\text{Div}(X), \text{Pic}(X), \text{NS}(X)$, respectively. For a divisor $D$ on $X$, fix a Weil height function $h_D: X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ associated with $D$; see [HS, Theorem 8.3.2]. For a $\mathbb{C}$-divisor $D = \sum_{i=1}^r a_i D_i$ ($a_i \in \mathbb{C}, D_i \in \text{Div}(X)$), we put $h_D := \sum_{i=1}^r a_i h_{D_i}$.

Definition 2.1. Let $H$ be an ample divisor on $X$ over $\overline{\mathbb{Q}}$, and $P \in X(\overline{\mathbb{Q}})$ a point. The arithmetic degree $\alpha_f(P)$ of $f$ at $P$ is defined by

$$\alpha_f(P) := \lim_{n \to \infty} \max\{1, h_H(f^n(P))\}^{1/n}.$$
Remark 2.2. The existence of the arithmetic degree for surjective endomorphisms is proved by Kawaguchi and Silverman (see [KS2, Theorem 3]). They also proved that $\alpha_f(P)$ is independent of the choice of $H$ and $h_H$.

Definition 2.3. For a column vector $v = t(x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$, we set
$$
\|v\| := \max_{0 \leq i \leq N} \{|x_i|\}.
$$
For a square matrix $A = (a_{i,j}) \in M_{N+1}(\mathbb{C})$, we similarly set
$$
\|A\| := \max_{0 \leq i,j \leq N} \{|a_{i,j}|\}.
$$
We frequently use the following inequality
$$
\|Av\| \leq (N + 1)\|A\| \cdot \|v\|.
$$

Definition 2.4. For sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ of positive real numbers, we write $a_n \preceq b_n$ if there is a positive real number $C > 0$ and an integer $N_0$ such that the inequality $a_n \leq C \cdot b_n$ holds for all $n \geq N_0$. If both $a_n \preceq b_n$ and $b_n \preceq a_n$ hold, we write $a_n \asymp b_n$.

3. Proof of Theorem 1.1

First, we give some lemmata on linear algebra. Then, we shall prove Theorem 1.1.

For a non-negative integer $\ell \in \mathbb{Z}_{\geq 0}$ and a complex number $\lambda \in \mathbb{C}$, let
$$
\Lambda := \begin{pmatrix}
\lambda & 1 & \lambda & O \\
1 & \lambda & \ddots & \ddots \\
O & \ddots & \lambda & 1 \\
\end{pmatrix}
$$
be the Jordan block matrix of the size $(\ell + 1) \times (\ell + 1)$. We put
$$
N := \Lambda - \lambda I.
$$

Lemma 3.1. (a) When $|\lambda| \geq 1$, we have $\|\Lambda^n\| \asymp n^\ell |\lambda|^n$.
(b) When $|\lambda| < 1$, we have $\|\Lambda^n\| \leq n^\ell |\lambda|^{n-\ell}$.

Proof. Note that $\binom{n}{k} \asymp n^k$ and $\binom{n}{k} \leq n^k$. Then both assertions follow from the following equalities:
$$
\|\Lambda^n\| = \|(\Lambda + N)^n\|
$$
$$
= \left\| \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} N^k \right\|
$$
$$
= \left\| \sum_{k=0}^{\ell} \binom{n}{k} \lambda^{n-k} N^k \right\| \quad \text{since } N^{\ell+1} = 0
$$
$$
= \max_{0 \leq k \leq \ell} \binom{n}{k} |\lambda|^{n-k}.
$$
Lemma 3.2. Assume $|\lambda| > 1$. For a non-zero column vector $v = (x_0, \ldots, x_\ell) \in \mathbb{C}^{\ell+1} \setminus \{0\}$, we have

$$\|\Lambda^n v\| \asymp n^t |\lambda|^n,$$

where we put

$$t := \ell - \min\{i \mid 0 \leq i \leq \ell, \ x_i \neq 0\}.$$ 

Proof. We may assume $t = \ell$, so $x_0 \neq 0$. For a negative integer $j < 0$, we set $x_j := 0$. The asymptotic inequality $\|\Lambda^n v\| \preceq n^\ell |\lambda|^n$ follows from the following inequalities.

$$|(\Lambda^n v)_j| = \left| \sum_{k=0}^\ell \binom{n}{k} \lambda^{n-k}(\Lambda^k v)_j \right|$$

$$= \left| \sum_{k=0}^\ell \binom{n}{k} \lambda^{n-k}x_{j-k} \right|$$

$$\leq \sum_{k=0}^j n^k \cdot |\lambda|^n \cdot |x_{j-k}|$$

$$\leq (\ell + 1) \cdot n^\ell \cdot |\lambda|^n \cdot \|v\|.$$ 

The converse asymptotic inequality $\|\Lambda^n v\| \succeq n^\ell |\lambda|^n$ follows from the following asymptotic inequalities.

$$|(\Lambda^n v)_\ell| = \left| \sum_{k=0}^\ell \binom{n}{k} \lambda^{n-k}x_{\ell-k} \right|$$

$$\geq \left( \frac{n}{\ell} \right)|\lambda^{\ell}x_0| - \sum_{k=0}^{\ell-1} \left( \binom{n}{k} \lambda^{n-k}x_{\ell-k} \right)$$

$$\geq n^\ell |\lambda|^n - n^{\ell-1} |\lambda|^n$$

$$\geq n^\ell |\lambda|^n.$$ 

Hence we conclude $\|\Lambda^n v\| \asymp n^\ell |\lambda|^n$. 

Let notation be the same as in Theorem 1.1. Let $V_H$ be the $\mathbb{Q}$-vector subspace of $\text{Pic}(X)_{\mathbb{Q}} := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by the set $\{(f^n)^*H \mid n \geq 0\}$, and $V_H$ the image of $V_H$ in $\text{NS}(X)_{\mathbb{Q}} := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is known that $V_H$ is a finite dimensional $\mathbb{Q}$-vector space (see the proof of [KS2, Theorem 3]). We decompose the $\mathbb{C}$-vector space $(V_H)_{\mathbb{C}} := V_H \otimes_{\mathbb{Z}} \mathbb{C}$ into Jordan blocks with respect to the $\mathbb{C}$-linear map $f^* : (V_H)_{\mathbb{C}} \to (V_H)_{\mathbb{C}}$:

$$(V_H)_{\mathbb{C}} = \bigoplus_{i=1}^\tau V_i.$$
For each $1 \leq i \leq \tau$, let $\lambda_i \in \mathbb{C}$ be the eigenvalue of $f^*_i|_{V_i}$. By changing the order of the Jordan blocks if necessary, we may assume

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_\sigma| > 1 \geq |\lambda_{\sigma+1}| \geq \cdots \geq |\lambda_\tau|$$

for some $0 \leq \sigma \leq \tau$. We put $\ell_i := \dim_{\mathbb{C}} V_i - 1$.

We take a $\mathbb{C}$-basis $\{D_{i,j}\}_{0 \leq j \leq \ell_i}$ of $V_i$ satisfying the following linear equivalences:

$$(3.1) \quad f^* D_{i,j} \sim D_{i,j} - 1 + \lambda_i D_{i,j} \quad (0 \leq j \leq \ell_i)$$

where we set $D_{i,-1} := 0$. For each $1 \leq i \leq \sigma$, let

$$\hat{h}_{D_{i,j}} : X(\overline{Q}) \to \mathbb{C} \quad (0 \leq j \leq \ell_i)$$

be unique functions satisfying the normalization condition

$$\hat{h}_{D_{i,j}} = h_{D_{i,j}} + O(1) \quad (0 \leq j \leq \ell_i)$$

and the functional equation

$$\hat{h}_{D_{i,j}} \circ f = \hat{h}_{D_{i,j-1}} + \lambda_i \hat{h}_{D_{i,j}} \quad (0 \leq j \leq \ell_i)$$

(see [KS2, Theorem 5] for the existence of such functions).

For each $1 \leq i \leq \tau$, we set

$$h_{D_{i}} := (h_{D_{i,0}}, h_{D_{i,1}}, \ldots, h_{D_{i,\ell_i}}).$$

For each $1 \leq i \leq \sigma$, we set

$$\hat{h}_{D_{i}} := (\hat{h}_{D_{i,0}}, \hat{h}_{D_{i,1}}, \ldots, \hat{h}_{D_{i,\ell_i}}).$$

Let $\Lambda_i$ be the Jordan block matrix of size $(\ell_i + 1) \times (\ell_i + 1)$ associated with the eigenvalue $\lambda_i$.

**Lemma 3.3.** For each $\sigma + 1 \leq i \leq \tau$ and a point $P \in X(\overline{Q})$, we have

$$\|h_{D_i}(f^n(P))\| \leq n^{\ell_i + 1}.$$

**Proof.** By (3.1), there is a positive real number $C_0 > 0$ such that the inequality

$$\|h_{D_i} \circ f - \Lambda_i \cdot h_{D_i}\| \leq C_0$$

is satisfied.
holds on \( X(\overline{\mathbb{Q}}) \). Therefore, there is a positive real number \( C_1 > 0 \) such that for every point \( P \in X(\overline{\mathbb{Q}}) \), the following inequalities hold:

\[
\| h_{D_i} \circ f^n(P) \| - \| A^n_i \cdot h_{D_i}(P) \| \\
\leq \| h_{D_i} \circ f^n(P) - A^n_i \cdot h_{D_i}(P) \| \\
\leq \sum_{k=0}^{n-1} \| A_k^{\ell_i} \cdot (h_{D_i} \circ f^{n-k}(P)) - A_k^{\ell_i+1} \cdot (h_{D_i} \circ f^{n-k-1}(P)) \| \\
\leq \sum_{k=0}^{n-1} (\ell_i + 1) \| A_k^{\ell_i} \| \cdot \| h_{D_i}(f^{n-k}(P)) - A_i \cdot h_{D_i}(f^{n-k-1}(P)) \| \\
\leq \sum_{k=0}^{n-1} (\ell_i + 1) \| A_k^{\ell_i} \| C_0 \\
\leq \sum_{k=0}^{n-1} (\ell_i + 1) \cdot n^{\ell_i} \cdot |\lambda_i|^{k-\ell_i} \cdot C_0 \\
\leq C_1 n^{\ell_i+1} \]

by Lemma 3.1 because \( |\lambda_i| \leq 1 \).

Furthermore, we have

\[
\| A^n_i \cdot h_{D_i}(P) \| \leq (\ell_i + 1) \cdot \| A^n_i \| \cdot \| h_{D_i}(P) \| \\
\leq (\ell_i + 1) \cdot n^{\ell_i} \cdot |\lambda_i|^{n-\ell_i} \cdot \| h_{D_i}(P) \| \quad \text{by Lemma 3.1} \\
\leq (\ell_i + 1) \cdot n^{\ell_i} \cdot \| h_{D_i}(P) \| \quad \text{because } |\lambda_i| \leq 1.
\]

Combining these inequalities, the assertion is proved. □

**Lemma 3.4 (see [KS2, Lemma 18]).** For a \( \mathbb{C} \)-divisor \( D \) on \( X \) and a point \( P \in X(\overline{\mathbb{Q}}) \), we have

\[
\max\{1, h_H(f^n(P))\} \geq |h_D(f^n(P))|.
\]

**Proof.** The assertion is obviously true when the forward \( f \)-orbit of \( P \) is a finite set. Hence, we may assume the forward \( f \)-orbit of \( P \) is an infinite set. Thus we may assume

\[
(3.2) \quad h_H(f^n(P)) \to \infty \quad (\text{as } n \to \infty).
\]

Write \( D = D_r + \sqrt{-1}D_c \), where \( D_r \) and \( D_c \) are \( \mathbb{R} \)-divisors on \( X \). By the triangle inequality, it is enough to prove the assertion for \( D_r \) and \( D_c \). Thus we may assume \( D \) is an \( \mathbb{R} \)-divisor.
Take a sufficiently large positive real number $C > 0$ such that $CH \pm D$ are ample. The function $C \cdot h_H - |h_D|$ is bounded below on $X(\overline{Q})$. There is a (not necessarily positive) real number $C' \in \mathbb{R}$ satisfying
\begin{equation}
(3.3) \quad C h_H(f^n(P)) - |h_D(f^n(P))| \geq C'
\end{equation}
for all $P \in X(\overline{Q})$. By (3.2) and (3.3), the assertion follows.

**Proof of Theorem 1.1.** Write $H$ in terms of the $\mathbb{C}$-linear bases of $(V_H)_C$:
\[ H = \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} D_{i,j} \quad (c_{i,j} \in \mathbb{C}). \]

Let $P \in X(\overline{Q})$ be a point with $\alpha_f(P) > 1$. If $\hat{h}_{D_i}(P) = 0$ for all $1 \leq i \leq \sigma$, we have the following inequalities:
\begin{align*}
&h_H(f^n(P)) \\
\leq & \left| \sum_{i=1}^{\sigma} \sum_{j=0}^{\ell_i} c_{i,j} \hat{h}_{D_{i,j}}(f^n(P)) \right| + \left| \sum_{i=\sigma+1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1) \\
= & \left| \sum_{i=\sigma+1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1) \\
\leq & n^{\lambda_{\max, \ell_i+1}} \quad \text{by Lemma 3.3.}
\end{align*}

Thus we get
\[ \alpha_f(P) \leq \lim_{n \to \infty} n^{(\max, \ell_i+1)/n} = 1. \]

But this contradicts $\alpha_f(P) > 1$. Hence we get $\hat{h}_{D_i}(P) \neq 0$ for some $1 \leq i \leq \sigma$.

We set
\[ \lambda := \max\{|\lambda_i| \mid 1 \leq i \leq \sigma, \hat{h}_{D_i}(P) \neq 0\}. \]
If $\hat{h}_{D_{i,j}}(P) = 0$, we set $t_{f,i}(P) := -\infty$. Otherwise, we set
\[ t_{f,i}(P) := \ell_i - \min\{j \mid 0 \leq j \leq \ell_i, \hat{h}_{D_{i,j}}(P) \neq 0\}. \]

Finally, we set
\[ t_{f}(P) := \max\{t_{f,i}(P) \mid \lambda = |\lambda_i|\}. \]
Since $\hat{h}_{D_i}(P) \neq 0$ holds for some $1 \leq i \leq \sigma$, we get $t_{f}(P) \neq -\infty$. It is enough to prove
\[ h_H(f^n(P)) \asymp n^{t_{f}(P)} \lambda^n. \]
Note that \( \lambda = \alpha f(P) \) follows from this asymptotic inequality. We have (3.4)

\[
h_H(f^n(P)) \leq \left| \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1)
\]

\[
\leq \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} |c_{i,j}| \cdot |h_{D_{i,j}}(f^n(P))| + O(1)
\]

\[
\leq \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} |c_{i,j}| \cdot h_H(f^n(P))
\]

by Lemma 3.3

\[
\leq h_H(f^n(P)).
\]

By the equality \( \hat{h}_{D_i}(f(P)) = \Lambda_i \hat{h}_{D_i}(P) \) and Lemma 3.2 we get

(3.5) \[ \| \hat{h}_{D_i}(f^n(P)) \| \asymp n^{\ell_i} \lambda_i^n \quad (1 \leq i \leq \sigma). \]

Combining Lemma 3.3 and (3.5), we conclude

\[
\sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} |c_{i,j}| \cdot |h_{D_{i,j}}(f^n(P))| \asymp n^{\ell_i} \lambda^n.
\]

The assertion follows from this asymptotic equality and (3.4). \( \square \)

4. Backgrounds and general conjectures

Theorem 1.2 gives a positive answer to a variant of the Dynamical Mordell-Lang conjecture for pairs of étale endomorphisms, which is a variant of the original one stated by Bell, Ghioca, and Tucker (see \cite[Question 5.11.0.4]{BGT2}). In \cite{GTZ1} and \cite{GTZ2}, Ghioca, Tucker, and Zieve studied similar problems for polynomial maps and got deeper results. Moreover, they introduced some of reductions including Lemma 5.2 we use. In \cite{GN}, Ghioca and Nguyen also studied similar problems for self-maps of semi-abelian varieties.

First, we recall a version of the Dynamical Mordell-Lang conjecture. Note that there are several variants of the Dynamical Mordell-Lang conjecture. Many results are obtained in various situations (see \cite{BGT2} for details).

**Conjecture 4.1** (the Dynamical Mordell-Lang conjecture \cite[Conjecture 1.7]{GT}). Let \( X \) be a quasi-projective variety over \( \mathbb{C} \). Let \( f : X \rightarrow X \) be an endomorphism of \( X \) over \( \mathbb{C} \). For any \( \mathbb{C} \)-rational point \( P \in X(\mathbb{C}) \) and any closed subvariety \( Y \subset X \), the set

\[
S_f(P,Y) := \{ n \in \mathbb{Z}_{\geq 0} \mid f^n(P) \in Y(\mathbb{C}) \}
\]

is a union of finitely many sets of the form

\[
\{a_i + b_i m \mid m \in \mathbb{Z}_{\geq 0}\}
\]

for some non-negative integers \( a_i, b_i \in \mathbb{Z}_{\geq 0} \).
In Section 5, we shall use the following result proved by Bell, Ghioca, and Tucker, which is a special case of the Dynamical Mordell-Lang conjecture.

**Theorem 4.2 (BGT1 Theorem 1.3).** If \( f : X \to X \) is an étale endomorphism, Conjecture 4.1 holds.

Furthermore, in [BGT2 Question 5.11.0.4], the following conjecture is stated.

**Conjecture 4.3 (BGT2 Question 5.11.0.4).** Let \( X \) be a projective variety over \( \mathbb{Q} \). Let \( H \) be an ample \( \mathbb{R} \)-divisor on \( X \) over \( \overline{\mathbb{Q}} \). Let \( f, g : X \to X \) be étale endomorphisms of \( X \) over \( \mathbb{Q} \) such that \( f^*H \equiv \delta_f H \) and \( g^*H \equiv \delta_g H \) hold in \( \text{NS}(X)_{\mathbb{R}} \) for some \( \delta_f, \delta_g \in \mathbb{R}_{>1} \). Then for any points \( P, Q \in X(\overline{\mathbb{Q}}) \), the set

\[
S_{f,g}(P, Q) := \{ (m,n) \mid f^m(P) = g^n(Q) \}
\]

is a union of finitely many sets of the form

\[
\{ (a_i + b_i \ell, c_i + d_i \ell) \mid \ell \in \mathbb{Z}_{\geq 0} \}
\]

for some non-negative integers \( a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0} \).

Bell, Ghioca, and Tucker proved a special case of Conjecture 4.3.

**Theorem 4.4 (BGT2 Theorem 5.11.0.1).** Conjecture 4.3 holds if \( \delta_f = \delta_g \).

**Remark 4.5** (see [KS1 Theorem 2 (a), Proposition 7] for details). If an ample \( \mathbb{R} \)-divisor \( H \) satisfies \( f^*H \equiv dH \) in \( \text{NS}(X)_{\mathbb{R}} \) for some \( d \in \mathbb{R}_{>1} \), the limit

\[
\hat{h}_{f,H}(P) := \lim_{n \to \infty} \frac{h_H(f^n(P))}{d^n}
\]

converges for all \( P \in X(\overline{\mathbb{Q}}) \) and satisfies

\[
\hat{h}_{f,H}(f(P)) = d\hat{h}_{f,H}(P)
\]

and

\[
(4.1) \quad \hat{h}_{f,H} - h_H = O(\sqrt{h_H}).
\]

The function \( \hat{h}_{f,H} \) is called the canonical height. Furthermore, the following conditions are equivalent to each other:

- \( \alpha_f(P) > 1 \),
- \( \alpha_f(P) = d \),
- \( \hat{h}_{f,H}(P) \neq 0 \), and
- the forward \( f \)-orbit of \( P \) is an infinite set.

In the setting of Theorem 4.4, when the forward \( f \)-orbit of \( P \) or the forward \( g \)-orbit of \( Q \) is finite, the assertion of Theorem 4.3 is obviously true. By Remark 4.5, we have

\[
\alpha_f(P) = \delta_f = \delta_g = \alpha_g(Q) > 1
\]
in the remaining case. Thus, when \( X \) is smooth, Theorem 1.2 is a generalization of Theorem 4.4. The assumption of Conjecture 4.3 seems too strong. We propose a more general conjecture as follows.

**Conjecture 4.6.** Let \( X \) be a smooth projective variety over \( \overline{\mathbb{Q}} \), and let \( f, g: X \to X \) be \( \acute{e}tale \) endomorphisms of \( X \) over \( \mathbb{Q} \). For points \( P, Q \in X(\overline{\mathbb{Q}}) \) with \( \alpha_f(P) > 1 \) and \( \alpha_g(Q) > 1 \), the following statements hold.

(a) The set \( S_{f,g}(P, Q) \) is a finite union of the sets of the form

\[
\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}
\]

for some non-negative integers \( a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0} \).

(b) If \( \log \alpha_f(P) \alpha_g(Q) \) is irrational or \( t_f(P) \neq t_g(Q) \), the set \( S_{f,g}(P, Q) \) is finite.

The part (b) asserts that the hypotheses from Theorem 1.2 regarding \( \delta_f(P), \delta_g(Q), t_f(P), t_g(Q) \) must met if the set \( S_{f,g}(P, Q) \) were infinite. In Section 6, we give some examples of endomorphisms for which Conjecture 4.6 (b) hold.

5. **Proof of Theorem 1.2**

In this section we prove Theorem 1.2.

**Lemma 5.1.** To prove Theorem 1.2 we may assume \( p = q = 1 \).

**Proof.** We see that

\[
S_{f,g}(P, Q) = \bigcup_{0 \leq i \leq p-1} \{ (i + pm, j + qn) \mid (m, n) \in S_{f^p,g^q}(f^i(P), g^j(Q)) \}.
\]

Thus to prove Theorem 1.2, it is enough to prove it for \( f^p \) and \( g^q \). By using Theorem 1.1 twice, we have

\[
n^{t_f(P)\alpha_f(P)n} \asymp h_H(f^{pn}(P)) \asymp (pn)^{t_f(P)\alpha_f(P)^m} \asymp n^{t_f(P)\alpha_f(P)^m}.
\]

Similarly, we obtain

\[
n^{t_g(Q)\alpha_g(Q)n} \asymp n^{t_g(Q)\alpha_g(Q)^qn}.
\]

Hence combining with the assumption of Theorem 1.2 for \( f \) and \( g \), we get

\[
t_{f^p}(P) = t_f(P) = t_g(Q) = t_{g^q}(Q),
\]

\[
\alpha_{f^p}(P) = \alpha_f(P)^p = \alpha_g(Q)^q = \alpha_{g^q}(Q).
\]

Hence our assertion follows. \( \square \)
Lemma 5.2. To prove Theorem 1.2 in the case $p = q = 1$, it is enough to prove

$$\sup_{(m,n) \in S_{f,g}(P,Q)} |m - n| < \infty.$$ 

Proof. We set

$$M := \sup_{(m,n) \in S_{f,g}(P,Q)} |m - n|.$$ 

Then we have

$$S_{f,g}(P, Q) = \bigcup_{0 \leq k \leq M} \{(m, m + k) \in S_{f,g}(P, Q) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \bigcup_{1 \leq k \leq M} \{(n + k, n) \in S_{f,g}(P, Q) \mid n \in \mathbb{Z}_{\geq 0}\}.$$ 

Let $f \times g: X \times X \to X \times X$ be the product of the endomorphisms $f, g$. Let $\Delta \subset X \times X$ be the diagonal. Then we have

$$(m, m + k) \in S_{f,g}(P, Q) \iff f^m(P) = g^{m+k}(Q)$$

$$\iff (f \times g)^m(P, g^k(Q)) \in \Delta.$$ 

Since $f \times g$ is étale, the Dynamical Mordell-Lang conjecture is true for $f \times g$ by Theorem 4.2. Hence the set

$$\{(m, m + k) \in S_{f,g}(P, Q) \mid m \in \mathbb{Z}_{\geq 0}\}$$

is a finite union of the sets of the form

$$\{(a_i + b_i \ell, a_i + b_i \ell + k) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i \in \mathbb{Z}_{\geq 0}$. Similarly, the set

$$\{(n + k, n) \in S_{f,g}(P, Q) \mid n \in \mathbb{Z}_{\geq 0}\}$$

is a finite union of the sets of the form

$$\{(c_i + d_i \ell + k, c_i + d_i \ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $c_i, d_i \in \mathbb{Z}_{\geq 0}$. Thus the assertion follows. $\square$

Remark 5.3. Lemma 5.2 is the only part where the assumption of the étaleness of $f, g$ is used. So Theorem 1.2 is true if the Dynamical Mordell-Lang conjecture (Conjecture 4.1) is true for the endomorphism $f^p \times g^q: X \times X \to X \times X$ and the diagonal $\Delta \subset X \times X$.

Proof of Theorem 1.2. By Lemma 5.1, we may assume $p = q = 1$. By Theorem 1.1 there are positive real numbers $C_0, C_1, C_2, C_3 > 0$ such that the inequalities

$$C_0 m^{r_f(P)} \alpha_f(P)^m \leq h_H(f^m(P)) \leq C_1 m^{r_f(P)} \alpha_f(P)^m,$$

$$C_2 n^{r_g(Q)} \alpha_g(Q)^n \leq h_H(g^n(Q)) \leq C_3 n^{r_g(Q)} \alpha_g(Q)^n$$
hold except for finitely many \( m \) and \( n \). Suppose \( f^m(P) = g^n(Q) \) with \( m \geq n \). Then we get

\[
(m/n)^{t_f(P)} \alpha_f(P)^{m-n} \leq C_3/C_0
\]

because \( t_f(P) = t_g(Q) \) and \( \alpha_f(P) = \alpha_g(Q) \) by assumption. Hence we get

\[
\alpha_f(P)^{m-n} \leq C_3/C_0.
\]

(5.1)

Since \( \alpha_f(P) > 1 \), the inequality (5.1) holds only for finitely many values of \( m - n \). By the same argument for the case \( m < n \), we conclude

\[
\sup_{(m,n) \in S_{f,g}(P,Q)} |m - n| < \infty.
\]

Hence by Lemma 5.2 the proof of Theorem 1.2 is complete. \( \square \)

6. Some evidence for Conjecture 4.6

In this section, we prove the following theorem which gives some evidence for Conjecture 4.6 (b).

**Theorem 6.1.** Let \( X \) be a smooth projective variety over \( \overline{\mathbb{Q}} \). Let \( f, g: X \to X \) be surjective endomorphisms on \( X \) over \( \overline{\mathbb{Q}} \). Assume that \( f \) commutes with \( g \), and there is an ample \( \mathbb{R} \)-divisor \( H \) on \( X \) over \( \overline{\mathbb{Q}} \) such that \( f^*H \equiv dH \) in \( \text{NS}(X)_{\mathbb{R}} \) for some \( d \in \mathbb{R}_{>1} \). Let \( P, Q \in X(\overline{\mathbb{Q}}) \) be points with \( \alpha_f(P) > 1 \) and \( \alpha_g(Q) > 1 \). Assume that \( \log \alpha_f(P) \alpha_g(Q) \) is an irrational real number. Then the set \( S_{f,g}(P,Q) \) is finite.

**Proof of Theorem 6.1.** It is enough to prove that once \( f^{m_0}(P) = g^{n_0}(Q) \) is satisfied for some \( m_0 \) and \( n_0 \), we have \( f^{m+m_0}(P) \neq g^{n+n_0}(Q) \) for all \( (m, n) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0,0)\} \). Therefore, it is enough to prove that for every point \( R \in X(\overline{\mathbb{Q}}) \), we have \( f^m(R) \neq g^n(R) \) for all \( (m, n) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0,0)\} \).

Fix a Weil height function \( h_H \) associated with \( H \) so that \( h_H \geq 1 \).

By Theorem 1.1, there are positive real numbers \( C_0 > 0 \) and \( C_1 > 0 \) satisfying

\[
C_0 n^{t_s(R)\alpha_g(R)} \leq h_H(g^n(R)) \leq C_1 n^{t_s(R)\alpha_g(R)}
\]

for all \( n \in \mathbb{Z}_{\geq 1} \). We set

\[
\hat{h}_{f,g,H}(R) := \liminf_{n \to \infty} \frac{h_f(g^n(R))}{n^{t_s(R)\alpha_g(R)}}.
\]

From (1.1) and (6.1), we have

\[
\hat{h}_{f,g,H}(R) = \liminf_{n \to \infty} \frac{h_H(g^n(R))}{n^{t_s(R)\alpha_g(R)}} \geq C_0 > 0.
\]

Since the asymptotic behavior of \( h_H(g^n(R)) \) does not depend on the choice of an ample divisor \( H \) and a height function \( h_H \), one can see
that
\[
  h_H(g^n \circ f(R)) = h_H(f \circ g^n(R)) \\
  = h_{f^* H}(g^n(R)) + O(1) \\
  \asymp n^{\log(R) \alpha_g(R)},
\]
where the first equality follows from the commutativity of \( f \) and \( g \). This asymptotic equality means that we have \( t_g(f(R)) = t_g(R) \) and \( \alpha_g(f(R)) = \alpha_g(R) \). Hence the functional equations
\[
  \hat{h}_{f,g,H}(f(R)) = \alpha_f(R) \hat{h}_{f,g,H}(R), \\
  \hat{h}_{f,g,H}(g(R)) = \alpha_g(R) \hat{h}_{f,g,H}(R)
\]
hold (see Remark 4.5). Thus, if we have \( f^m(R) = g^n(R) \), we get
\[
  \alpha_f(R)^m \hat{h}_{f,g,H}(R) = \hat{h}_{f,g,H}(f^m(R)) \\
  = \hat{h}_{f,g,H}(g^n(R)) \\
  = \alpha_g(R)^n \hat{h}_{f,g,H}(R).
\]
Hence \( \alpha_f(R)^m = \alpha_g(R)^n \). This equality can hold only when \( m = n = 0 \) since we are assuming \( \log_{\alpha_f(R)} \alpha_g(R) \) is an irrational real number. \( \Box \)

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: ksano@math.kyoto-u.ac.jp