Spectra of disc operator for twisted acceleration-enlarged Newton-Hooke space-times

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Abstract

The time-dependent spectra of disc area operator for twisted acceleration-enlarged Newton-Hooke space-times are derived. It is demonstrated that the corresponding area quanta are expanding or oscillating in time.
Recently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [1]-[4]) as well as with field theoretical models (see e.g. [5], [6]), in which the quantum space-time is employed. The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [7]. Recently, there were also found formal arguments based mainly on Quantum Gravity [8] and String Theory models [9], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. On the other side, the main reason for such considerations follows from the suggestion that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [10], [11].

Presently, it is well known, that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries, one can distinguish three basic types of quantum spaces [12], [13]:

1) Canonical ($\theta^{\mu\nu}$-deformed) space-time

\[
[x_\mu, x_\nu] = i\theta^{\mu\nu} ; \quad \theta^{\mu\nu} = \text{const},
\]

introduced in [14], [15] in the case of Poincare quantum group and in [16] for its Galilean counterpart.

2) Lie-algebraic modification of classical space

\[
[x_\mu, x_\nu] = i\theta^{\rho}_{\mu\nu}x_\rho,
\]

with particularly chosen coefficients $\theta^{\rho}_{\mu\nu}$ being constants. This type of noncommutativity has been obtained as the representations of the $\kappa$-Poincare [17] and $\kappa$-Galilei [18] as well as the twisted relativistic [19] and nonrelativistic [16] symmetries, respectively.

3) Quadratic deformation of Minkowski and Galilei space

\[
[x_\mu, x_\nu] = i\theta^{\rho\tau}_{\mu\nu}x_\rho x_\tau,
\]

with coefficients $\theta^{\rho\tau}_{\mu\nu}$ being constants. This kind of deformation has been proposed in [20], [21], [19] at relativistic and in [16] at nonrelativistic level.

Besides, it has been demonstrated in [22], that in the case of so-called acceleration-enlarged Newton-Hooke Hopf algebras $U_0(\hat{NH}_\pm)$ the (twist) deformation provides the new space-time noncommutativity, which is expanding ($U_0(\hat{NH}_+)$) or periodic ($U_0(\hat{NH}_-)$) in time, i.e. it takes the form $^4$

\[
[t, x_i] = 0 , \quad [x_i, x_j] = if_{\pm}\left(\frac{t}{\tau}\right)\theta_{ij}(x) ,
\]

$^1$The $U_0(\hat{NH}_\pm)$ acceleration-enlarged Newton-Hooke Hopf structures are obtained by adding to the $\hat{NH}_\pm$ algebras (algebraic parts) the trivial coproduct $\Delta_0(a) = a \otimes 1 + 1 \otimes a$.

$^2$x_0 = ct.
with time-dependent functions

\[ f_+ \left( \frac{t}{\tau} \right) = f \left( \sinh \left( \frac{t}{\tau} \right), \cosh \left( \frac{t}{\tau} \right) \right) \quad , \quad f_- \left( \frac{t}{\tau} \right) = f \left( \sin \left( \frac{t}{\tau} \right), \cos \left( \frac{t}{\tau} \right) \right) , \]

and \( \theta_{ij}(x) \sim \theta_{ij} = \text{const} \) or \( \theta_{ij}(x) \sim \theta_{ij} x_i \). Such a kind of noncommutativity follows from the presence in acceleration-enlarged Newton-Hooke symmetries \( \mathcal{U}_0(\hat{NH}_\pm) \) of the time scale parameter (cosmological constant) \( \tau \). As it was demonstrated in \[22\] that just this parameter is responsible for oscillation or expansion of space-time noncommutativity.

It should be noted that both Hopf structures \( \mathcal{U}_0(\hat{NH}_\pm) \) contain, apart from rotation \( (M_{ij}) \), boost \( (K_i) \) and space-time translation \( (P_i, H) \) generators, the additional ones denoted by \( F_i \), responsible for constant acceleration. Consequently, if all generators \( F_i \) are equal zero we obtain the twisted Newton-Hooke quantum space-times \[23\], while for time parameter \( \tau \) running to infinity we get the acceleration-enlarged twisted Galilei Hopf structures proposed in \[22\]. Besides, for \( F_i \rightarrow 0 \) and \( \tau \rightarrow \infty \) we reproduce the canonical \[11\], Lie-algebraic \[12\] and quadratic \[13\] (twisted) Galilei spaces provided in \[16\]. Finally, it should be noted, that all mentioned above noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see \[24\], \[25\], \[14\], \[15\]), for quantum acceleration-enlarged Newton-Hooke Hopf algebras, respectively.

As it was already mentioned, the impact of quantum spaces on the formal structure of physical systems has been discussed in different context in \[1\]-\[6\]. Particularly, it has been demonstrated that the different kinds of noncommutativity produce additional interaction terms in the Hamiltonian of dynamical systems, in particular, it has been shown, that such quantum spaces generate nonlocality. Besides, there were also performed the studies on the fractal structure of quite popular \( \kappa \)-Minkowski space-time as well as the quantum sphere \[26\]. More precisely, there has been shown that both spaces have a scale dependent fractal dimension, which deviates from its classical value at short scales. It can be added, that the formal (algebraic) properties of \( \kappa \)-Poincare algebra gave also a mathematical tool for such theoretical constructions as Double Special Relativity (see e.g. \[27\]), which postulates two observer-independent scales, of velocity, describing the speed of light, and of mass, which can be identify with \( \kappa \)-parameter - the fundamental Planck mass.

Recently, an interesting property of (non-)relativistic (three-dimensional) canonically deformed space-time \[11\]

\[ [t, x_{1,2}] = 0 \quad , \quad [x_1, x_2] = i\theta , \]

has been studied in the article \[28\]. It has been demonstrated that the spectra of disc area operator

\[ S \equiv \pi R^2 = \pi \left( [x_1]^2 + [x_2]^2 \right) , \]

takes the discrete values

\[ s_n = 2\pi \theta (n + 1/2) \quad ; \quad n = 0, 1, 2, \ldots , \]

i.e. it has been shown that there appear, in such a case, proportional to the deformation parameter \( \theta \), quanta of area \( s_\theta = 2\pi \theta \). This result appears to be quite interesting due to
similarity to Quantum Gravity area operator \[29\]. In other words, such a result provides the link between canonical space \[14\], \[15\], and the quantum gravitational considerations performed in the framework of spin foam model \[30\] (more precisely, it provides the similar to (7) area spectrum \[31\]

\[ s(j_k) = \gamma l_p^2 (j_k + 1/2) ; \quad j_k = 0, 1/2, 1, 3/2, 2, \ldots, \] (8)

with Planck length \( l_p = (\hbar G/c)^{1/2} = 10^{-33} \) cm and constant \( \gamma \) analogous to the Immirzi parameter).

Another interesting result has been obtained in \[32\] where the Bekenstein spectrum of black hole horizon area \[33\]

\[ s_n = \tilde{\gamma} l_p^2 n ; \quad n = 0, 1, 2, \ldots ; \tilde{\gamma} \text{ dimensionless constant} , \] (9)

has been rederived as a spectrum of disc area operator in quantum Snyder space \[7\].

In this article we find, following the algorithm used in \[28\], \[32\], the disc spectra of area operator for all (with classical time and quantum space directions) three-dimensional acceleration-enlarged Newton-Hooke quantum space-times provided in \[22\]. They can be written as follows

\[ \left[ t, x_{1,2} \right] = 0 , \quad \left[ x_1, x_2 \right] = if(t) , \] (10)

where function \( f(t) \) is given by \[3\]

\[ f(t) = f_{\kappa_1}(t) = f_{\pm,\kappa_1} \left( \frac{t}{\tau} \right) = \kappa_1 C_\pm \left( \frac{t}{\tau} \right) , \] (11)

\[ f(t) = f_{\kappa_2}(t) = f_{\pm,\kappa_2} \left( \frac{t}{\tau} \right) = \kappa_2 \tau C_\pm \left( \frac{t}{\tau} \right) S_\pm \left( \frac{t}{\tau} \right) , \] (12)

\[ f(t) = f_{\kappa_3}(t) = f_{\pm,\kappa_3} \left( \frac{t}{\tau} \right) = \kappa_3 \tau^2 S_\pm^2 \left( \frac{t}{\tau} \right) , \] (13)

\[ f(t) = f_{\kappa_4}(t) = f_{\pm,\kappa_4} \left( \frac{t}{\tau} \right) = 4\kappa_4 \tau^4 \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right)^2 , \] (14)

\[ f(t) = f_{\kappa_5}(t) = f_{\pm,\kappa_5} \left( \frac{t}{\tau} \right) = \pm \kappa_5 \tau^2 \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right) C_\pm \left( \frac{t}{\tau} \right) , \] (15)

\[ f(t) = f_{\kappa_6}(t) = f_{\pm,\kappa_6} \left( \frac{t}{\tau} \right) = \pm \kappa_6 \tau^3 \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right) S_\pm \left( \frac{t}{\tau} \right) , \] (16)

for six types of acceleration-enlarged Newton-Hooke Hopf algebras \( U_{\kappa_1}(\hat{\mathcal{N}}H_\pm) \), \( U_{\kappa_2}(\hat{\mathcal{N}}H_\pm) \), \( U_{\kappa_3}(\hat{\mathcal{N}}H_\pm) \), \( U_{\kappa_4}(\hat{\mathcal{N}}H_\pm) \), \( U_{\kappa_5}(\hat{\mathcal{N}}H_\pm) \) and \( U_{\kappa_6}(\hat{\mathcal{N}}H_\pm) \) respectively, with

\[ C_{+/-} \left( \frac{t}{\tau} \right) = \cosh / \cos \left( \frac{t}{\tau} \right) \quad \text{and} \quad S_{+/-} \left( \frac{t}{\tau} \right) = \sinh / \sin \left( \frac{t}{\tau} \right) \]

Let us note that all above nonrelativistic space-times are equipped with the classical time and quantum spatial directions. Of course, as it was already mentioned above, for

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\[3\] See formulas (2) and (4) respectively.
generators $F_i$ approaching zero (and) parameter $\tau$ running to infinity the space-times (11)-(16) become the same as twisted Newton-Hooke (and) Galilei spaces proposed in [23] (and) [16] respectively. In the case $\tau \to \infty$ we get the acceleration-enlarged Galilei quantum space-times proposed in [22].

In a first step of our construction we define two time-dependent operators ($i = 1, \ldots, 6$)

$$a_{\kappa_i}(t) = \frac{1}{\sqrt{2f_{\kappa_i}(t)}} (x_1 + ix_2) ,$$

$$a_{\kappa_i}^\dagger(t) = \frac{1}{\sqrt{2f_{\kappa_i}(t)}} (x_1 - ix_2) .$$

One can check that they satisfy the following (standard) commutation relations

$$[a_{\kappa_i}(t), a_{\kappa_i}^\dagger(t)] = 1 .$$

Next, we build in a standard way the number operator

$$N_{\kappa_i}(t) = a_{\kappa_i}^\dagger(t)a_{\kappa_i}(t) ,$$

which satisfies

$$[N_{\kappa_i}(t), a_{\kappa_i}^\dagger(t)] = a_{\kappa_i}^\dagger(t) , \quad [N_{\kappa_i}(t), a_{\kappa_i}(t)] = -a_{\kappa_i}(t) .$$

Hence, one can identify $a_{\kappa_i}^\dagger(t)$ and $a_{\kappa_i}(t)$ objects with creation and annihilation operators respectively; more precisely, as we shall see for a moment, they create and annihilate the time-dependent disc area quanta of space-times (10).

It is easy to see that the number operator (20) can be rewritten in terms of coordinate variables as follows

$$N_{\kappa_i}(t) = \frac{1}{2f_{\kappa_i}(t)} (\left[ x_1 \right]^2 + \left[ x_2 \right]^2 - f_{\kappa_i}(t)) .$$

Then, the disc area operator

$$S = \pi (\left[ x_1 \right]^2 + \left[ x_2 \right]^2) ,$$

takes the form

$$S_{\kappa_i}(t) = 2\pi f_{\kappa_i}(t)(N_{\kappa_i}(t) + 1/2) .$$

One can find (see e.g. [34]), that the operator $S_{\kappa_i}(t)$ is quantized with levels equally spaced by the interval (quanta) $s_{\kappa_i}(t) = 2\pi f_{\kappa_i}(t)$, with the following eigenvalues

$$S_{\kappa_i}(t)\varphi_{n,\kappa_i} = s_{n,\kappa_i}(t)\varphi_{n,\kappa_i} = 2f_{\kappa_i}(t)(n + 1/2) \varphi_{n,\kappa_i} ; \quad n = 0, 1, 2, \ldots ,$$

and the eigenstates given by

$$\varphi_{n,\kappa_i}(t) = \frac{1}{\sqrt{n!}} (a_{\kappa_i}^\dagger(t))^n \varphi_{0,\kappa_i} ; \quad a_{\kappa_i}(t)\varphi_{0,\kappa_i} = 0 .$$
We see, therefore, that objects $a^\dagger_{\kappa_1}(t)$ and $a_{\kappa_1}(t)$ are (in fact) responsible for creation and annihilation of area quanta $s_{\kappa_1}(t)$ respectively.

Finally, let us study the basic properties of the above spectra. First of all, it should be noted that for fixed time parameter $t$, all (twelve) time-dependent area quanta

\[
s_{\kappa_1}(t) = s_{\pm,\kappa_1} \left( \frac{t}{\tau} \right) = 2\pi\kappa_1 C_{\pm} \left( \frac{t}{\tau} \right),
\]

\[
s_{\kappa_2}(t) = s_{\pm,\kappa_2} \left( \frac{t}{\tau} \right) = 2\pi\kappa_2 C_{\pm} \frac{t}{\tau} S_{\pm} \left( \frac{t}{\tau} \right),
\]

\[
s_{\kappa_3}(t) = s_{\pm,\kappa_3} \left( \frac{t}{\tau} \right) = 2\pi\kappa_3 \tau \sqrt{C_{\pm} \left( \frac{t}{\tau} \right) - 1} S_{\pm} \left( \frac{t}{\tau} \right),
\]

\[
s_{\kappa_4}(t) = s_{\pm,\kappa_4} \left( \frac{t}{\tau} \right) = 8\pi\kappa_4 \tau^4 \left( C_{\pm} \left( \frac{t}{\tau} \right) - 1 \right)^2,
\]

\[
s_{\kappa_5}(t) = s_{\pm,\kappa_5} \left( \frac{t}{\tau} \right) = \pm 2\pi\kappa_5 \tau^3 \left( C_{\pm} \left( \frac{t}{\tau} \right) - 1 \right) S_{\pm} \left( \frac{t}{\tau} \right),
\]

and

\[
s_{\kappa_6}(t) = s_{\pm,\kappa_6} \left( \frac{t}{\tau} \right) = \pm 2\pi\kappa_6 \tau^3 \left( C_{\pm} \left( \frac{t}{\tau} \right) - 1 \right) S_{\pm} \left( \frac{t}{\tau} \right),
\]

become the same as mentioned above "canonical" quanta $s_\theta = 2\pi\theta$. Such a situation appears for

\[
t_{\kappa_1} = t_{\pm,-,\kappa_1} = -\tau \arccosh/\arccos \left( -\sqrt{\frac{\theta}{\kappa_1}} \right),
\]

\[
t_{\kappa_2} = t_{\pm,-,\kappa_2} = \frac{\tau}{2} \arcsinh/\arcsin \left( \frac{2\theta}{\tau\kappa_2} \right),
\]

\[
t_{\kappa_3} = t_{\pm,-,\kappa_3} = -\tau \arcsinh/\arcsin \left( \sqrt{\frac{\theta}{\tau^2\kappa_3}} \right),
\]

\[
t_{\kappa_4} = t_{\pm,-,\kappa_4} = -\tau \arccosh/\arccos \left( -\sqrt{\frac{\theta}{4\tau^4\kappa_4}} + 1 \right),
\]

\[
t_{\kappa_5} = t_{\pm,-,\kappa_5} = -\tau \arccosh/\arccos \left( 1 + / - \sqrt{1 + / - \frac{4\theta}{\tau^2\kappa_5}} \right),
\]

\[
t_{\kappa_6} = t_{\pm,-,\kappa_6} = -\tau \arccosh/\arccos \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - / + A_{\pm,-} + B_{\pm,-}} + \frac{1}{2} \sqrt{2 + / - A_{\pm,-} - B_{\pm,-}} - \frac{2}{\sqrt{1 - / + A_{\pm,-} + B_{\pm,-}}} \right),
\]
in the case of acceleration-enlarged Newton-Hooke Hopf structures $U_{\kappa_1}(\hat{NH}_\pm)$, $U_{\kappa_2}(\hat{NH}_\pm)$, $U_{\kappa_3}(\hat{NH}_\pm)$, $U_{\kappa_4}(\hat{NH}_\pm)$, $U_{\kappa_5}(\hat{NH}_\pm)$ and $U_{\kappa_6}(\hat{NH}_\pm)$ respectively. Besides, one can also observe that first, third, fourth and fifth acceleration-enlarged Newton-Hooke spectrum is invariant with respect time reflection $t \to -t$, whereas such a symmetry is broken in the case of second and sixth quanta $s_{\kappa_2}(t)$ and $s_{\kappa_6}(t)$. The last interesting property concerns the "duality" of functions $s_{\kappa_1}(t)$, $s_{\kappa_2}(t)$ and $s_{\kappa_3}(t)$ respectively, i.e. it should be noted that the first of them passes into the last one (and vice-versa) after substitution

$$C_{\pm} \left( \frac{t}{\tau} \right) \leftrightarrow \tau S_{\pm} \left( \frac{t}{\tau} \right), \quad (39)$$

while the second quanta $s_{\kappa_2}(t)$ passes into itself\(^4\). Obviously, for parameters $\kappa_i$ approaching infinity all deformations disappear and the area spectra become classical.

In summary. In this short article we derive the spectra of disc area operator defined on acceleration-enlarged Newton-Hooke space-times, equipped with quantum spatial directions $x_i$ and classical time $t$. We demonstrate that such obtained spectra are quantized with level equally spaced by time-dependent area quanta, proportional to the "value of noncommutativity", i.e. to the function $f(t)$. Besides, it should be noted, that the above quanta are covariant with respect the action of corresponding twist deformed acceleration-enlarged Newton-Hooke Hopf symmetries, respectively. Such a property obviously follows from the fact, that considered quantum space-times are defined as Hopf modules (representation spaces) of proper quantum groups. The more physical applications for obtained results are under investigation.

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\(^4\)In other words it is "self-dual" with respect "duality" transformation (39).
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