FREDHOLM PROPERTIES OF THE \( L^2 \) EXPONENTIAL MAP ON THE SYMPLECTOMORPHISM GROUP

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Abstract. Let \( M \) be a closed symplectic manifold with compatible symplectic form and Riemannian metric \( g \). Here it is shown that the exponential mapping of the weak \( L^2 \) metric on the group of symplectic diffeomorphisms of \( M \) is a non-linear Fredholm map of index zero. The result provides an interesting contrast between the \( L^2 \) metric and Hofer’s metric as well as an intriguing difference between the \( L^2 \) geometry of the symplectic diffeomorphism group and the volume-preserving diffeomorphism group.

1. Introduction. Let \( M \) be a closed symplectic manifold with symplectic form \( \omega \) and Riemannian metric \( g \). We assume \( \omega \) and \( g \) are compatible in the sense that there exists an almost complex structure (an endomorphism of the tangent bundle \( TM \)) satisfying \( J^2 = -I \) and \( \omega(v,Jw) = g(v,w) \). The volume form defined by \( \omega \) coincides with the volume form supplied by the metric \( g \) and we denote this form by \( \mu \). Let \( D^s_\omega(M) \) denote the group of all diffeomorphisms of Sobolev class \( H^s \) preserving the symplectic form \( \omega \) on \( M \). When \( s > \frac{\dim M}{2} + 1 \), \( D^s_\omega(M) \) becomes an infinite dimensional manifold whose tangent space at the identity \( T_eD^s_\omega(M) \) is given by \( H^s \) vector fields on \( M \) satisfying \( L^2v = 0 \). Using right translations, the \( L^2 \) inner-product
\[
(u,v)_{L^2} = \int_M g(u,v) \, d\mu \quad u,v \in T_eD^s_\omega(M)
\]
defines a weak, right-invariant Riemannian metric on \( D^s_\omega \); see the paper of Ebin and Marsden \[5\] for basic facts regarding \( D^s_\omega(M) \).

The action of \( D^s_\omega(M) \) on \( D^s(M) \) by composition on the right is an isometry of \( (1) \) and combined with the Hodge decomposition gives an \( L^2 \) orthogonal splitting of each tangent space
\[
T_\eta D^s = T_\eta D^s_\omega \oplus \omega^\flat (\delta dH^{s+2}(T^*M)) \circ \eta
\]
where \( \omega^\flat : TM \to T^*M \) is an isomorphism defined by \( X \mapsto i_X\omega \) with inverse \( \omega^\sharp : T^*M \to TM \) given by contracting a 1-form with the inverse components of the Symplectic form. The projections onto the first and second summands of \( (2) \) will

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be denoted by $P_\eta$ and $Q_\eta$, respectively, or simply by $P$ and $Q$ if $\eta = e$ the identity. By right-invariance $F_\eta(X) = dR_\eta \circ P \circ dR_\eta^{-1}(X)$, for $X \in T_e D^s$.

Diffeomorphism groups can be realized as the configuration spaces of a number of equations in mathematical physics, and this provides a strong motivation to study their geometry. Perhaps the most famous example is the Euler equations of hydrodynamics, where Arnold, [1], noticed that a curve $\eta(t)$ in the group of smooth volume preserving diffeomorphisms is a geodesic of the $L^2$ metric (1) if and only if the vector field $v$, defined by $\partial_t \eta = v \circ \eta$, solves the Euler equations of hydrodynamics.

Analogously, a curve $\eta(t)$ in $D^s_\omega(M)$ is a geodesic of the $L^2$ metric (1) starting from the identity in the direction $v_\alpha$ if and only if the time dependent vector field

$$v = \dot{\eta} \circ \eta^{-1}$$

on $M$ solves the Symplectic Euler equations

$$\partial_t v + P(\nabla_v v) = 0$$

$$\mathcal{L}_v \omega = 0$$

$$v(0) = v_\alpha,$$

The geodesic equation on $D^s_\omega$, corresponding to the Symplectic Euler equations, is a smooth ODE which can be solved for small values of $t$, cf. [4]. Furthermore, since the solutions depend smoothly on initial data, it follows that the $L^2$ metric has a smooth exponential map

$$\exp_e : T_e D^s_\omega \to D^s_\omega(M)$$

defined, for small $t$, by

$$\exp_e (tv_\alpha) = \eta(t),$$

where $\eta$ is the unique geodesic from the identity with initial velocity $v_\alpha \in T_e D^s_\omega$.

It is well known that if $M$ is a closed surface then solutions to the Euler equations of hydrodynamics exist globally in time. However, when $M$ is three dimensional the existence of global solutions to the Euler equations is a celebrated open problem. In contrast, if $M$ is a closed Symplectic manifold of any dimension then solutions to the Symplectic Euler equations exist globally in time and the map (4) is defined on the whole tangent space $T_e D^s_\omega$, cf. [4].

The subgroup of Hamiltonian diffeomorphisms plays a role in plasma dynamics analogous to the role played by the volume preserving diffeomorphism group in incompressible hydrodynamics, [2], [7], [16], [10], [11], [20]. Given a Symplectic manifold $M$ of dimension $2n$, the Vlasov Hamiltonian is defined by

$$H[f] = \frac{1}{2} \int \int f(x) G(x, y) f(y) \, d^n x \, d^n y = \frac{1}{2} \| f \|^2_G,$$

where the kernel $G$ is taken so that it defines an appropriate norm on the space of densities of $M$. When $G = (-\Delta)^{-1}$, the geodesic Vlasov equation, obtained via $H$, is given by the Symplectic Euler equations (3) rewritten on the space of Hamiltonian functions:

$$\partial_t \Delta F + \{ F, \Delta F \} = 0,$$

where the time-dependent Hamiltonian $F$ is related to the solution $v$ of the Symplectic Euler equations by $v = J \nabla F$, and $\{ \cdot, \cdot \}$ is the Poisson bracket defined by $\omega$. We note that this is also the 2D Helmholtz equation of incompressible fluids, cf. [23], [7].

A bounded linear operator $S$ between Banach spaces is said to be Fredholm if it has closed range, finite dimensional Kernel and finite dimensional co-kernel. $S$
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is said to be semi-Fredholm if it has closed range and at least one of the other two conditions holds. The index of a semi-Fredholm operator is defined as

$$\text{ind } S = \dim \ker S - \dim \text{coker } S,$$

and is a continuous function on the set of Fredholm operators into $\mathbb{Z} \cup \{\pm \infty\}$. A smooth map $f : M \to N$ between Banach manifolds is called a Fredholm map if its Frechet derivative $df(p)$ is a Fredholm operator for each $p$. If the domain of $f$ is connected then the index of the operator $df(p)$ is independent of $p$ and by definition is the index of $f$, cf. Smale [22].

Let $\eta(t)$ be a geodesic of the $L^2$ metric (1) in $D^s_\omega$, emanating from the identity $e$ in the direction $v_0 \in T_e D^s_\omega$. The point $\eta(t^*)$, $t^* > 0$, is said to be conjugate to $\eta(0)$ if the linear operator

$$D\exp_e(t^*v_0) : T_e D^s_\omega \to T_{\eta(t^*)} D^s_\omega$$

fails to be an isomorphism. If $\dim \ker D\exp_e(t^*v_0) = k$, $k$ is called the multiplicity of the conjugate point.

A linear operator between Hilbert spaces with empty kernel need not be an isomorphism. Therefore, $\eta(t^*)$ may be a conjugate point even if $D\exp_e(t^*v_0)$ has empty kernel. A point $\eta(t^*)$ is monoconjugate to $e$ if $D\exp_e(t^*v_0)$ fails to be injective and a point $\eta(t^*)$ is epiconjugate to $e$ if $D\exp_e(t^*v_0)$ fails to be surjective.

It was shown in [6] that the exponential mapping $\exp_{\mu,ex}e$ is a non-linear Fredholm map of index zero, in the sense of Smale [23]. In particular, conjugate points in $D^s_{\mu,ex}(M^2)$ are isolated, of finite multiplicity, and the two types of conjugacies coincide.

In this paper we extend the result of [6] and prove

**Theorem 1.1.** Let $M$ be a closed Symplectic manifold of dimension $2n$ endowed with a fixed symplectic form $\omega$ and Riemannian metric $g$ which are compatible. Then the exponential map (4) of the $L^2$ metric on $D^s_\omega(M)$ is a nonlinear Fredholm map of index zero.

We remark that Theorem 1.1 provides a distinction between symplectic diffeomorphisms and Volume preserving diffeomorphisms, when equipped with the $L^2$ metric. Theorem 1.1 holds for any symplectic manifold of dimension $2n$, while Theorem 1.1 fails for the Volume-preserving diffeomorphism group of manifolds of dimension 3 and higher, cf. [6] and [18]. The relationship between Theorem 1.1 and known classifications (e.g. $C^0$ closure, Gromov’s non-squeezing Theorem) of symplectic diffeomorphisms is unclear. The result also provides an interesting contrast to the Hofer metric which is a bi-invariant metric on the group of Symplectomorphisms whose conjugate points do not have such a nice description. See Hofer-Zehnder [8] or Ustilovsky [25].

In a forthcoming paper we show how Theorem 1.1 yields a new characterization of conjugate points in terms of coadjoint orbits. We also show that the notions of Eulerian and Lagrangian stability of the symplectic Euler equations coincide, which completes the results of Preston [17] in that we make no assumptions on the topology of $M$, nor on the qualitative behaviour of solutions to the symplectic Euler equations.

Denote by $T_e D^s_{\text{Ham}}$ the Lie subalgebra of $T_e D^s_\omega$ consisting of globally Hamiltonian vector fields. The algebra $T_e D^s_{\text{Ham}}$ consists of vector fields of the form

$$v_F = J \nabla F$$
where \( F \in H^{s+1}_0(M) = \{ F : M \to \mathbb{R} : F \in H^{s+1}(M), \int_M F \, d\mu = 0 \} \). To each element in \( T_c D^s_\omega \) there corresponds a unique element of \( H^{s+1}_0(M) \). Also, each element of \( H^{s+1}_0 \) uniquely determines an element of \( T_c D^s_\omega \). Since \( T_c D^s_\omega \) is of finite codimension in \( T_c D^s_\omega \), it suffices to prove Theorem 1.1 for the exponential mapping restricted to \( T_c D^s_\omega \). Then on \( T_c D^s_\omega \) the exponential mapping will still be Fredholm.

In section 2 we study the exponential map and its derivative in terms of solutions to the Jacobi equation, which is simply the linearization of the symplectic Euler equation and the flow equation \( \dot{\eta} = v \circ \eta \), where \( v \) solves the symplectic Euler equation (3). Using a convenient decomposition of the solution operator to the Jacobi equation we show, in section 3, that the derivative of the exponential mapping is the sum of an invertible linear operator and a compact operator and is therefore Fredholm.

2. The Jacobi equation. Let \( \eta \) be the geodesic of (1) starting from the identity in the direction \( v_\circ \in T_c D^s_\omega \). In order to study Fredholmness of the exponential map it is convenient to express its derivative at \( tv_\circ \) in terms of solutions to the Jacobi equation

\[
\nabla_\eta^\omega Y + R^\omega(Y, \eta) \dot{\eta} = 0
\]

along \( \eta(t) = \exp_t(tv_\circ) \) with initial conditions

\[
Y(0) = 0, \quad Y'(0) = w_\circ.
\]

Here, \( \nabla^\omega \) is the right-invariant Levi-Civita connection of (1) given along a geodesic \( \eta(t) \) by

\[
\nabla_\eta^\omega X = P \left( \frac{\partial u}{\partial t} + \nabla_v u \right) \circ \eta(t),
\]

for \( X = u \circ \eta, u \in T_c D^s_\omega \) and \( \nabla \) the Levi-Civita connection on \( M \), cf. [3]. \( R^\omega \) is the right-invariant Riemann curvature tensor of (1), given by

\[
R^\omega(X, Y)Z = \nabla_\eta^\omega \nabla^\omega_\eta Z - \nabla^\omega_\eta \nabla_\eta^\omega Z - \nabla^\omega[Z, X]Y
\]

\[
= (P \circ \nabla_u P \circ \nabla_v) \circ \eta - (P \circ \nabla_v P \circ \nabla_u w) \circ \eta - (P \circ \nabla[w, v]w) \circ \eta
\]

for \( X = u \circ \eta, Y = v \circ \eta, Z = w \circ \eta \), with \( u, v, w \in T_c D^s_\omega \). The curvature tensor \( R^\omega \) is a bounded multi-linear operator in the \( H^s \) topology. To see this, let \( u, v \in T_c D^s_\omega \) and define

\[
(u, v)_{H^s} = (u, v)_{L^2} + (u, \Delta^s v)_{L^2},
\]

where \( \Delta \) denotes the Laplace-Beltrami operator. Extending \( (u, v)_{H^s} \) to \( T \mathcal{D}_\omega \) by right-invariance gives a smooth invariant Riemannian metric on \( \mathcal{D}_\omega \) whose induced topology is equivalent to the underlying topology on \( \mathcal{D}_\omega \). Let \( u, v, w \in T_c D^s_\omega \) and \( z \in C^\infty(TM) \) with \( \nabla \omega = 0 \). Since \( s > \frac{\dim M}{2} + 1 \) and \( P \) is an orthogonal projection onto \( T_c D^s_\omega \) and \( \nabla \omega \) is a weak Riemannian connection, we have

\[
(R^\omega(u, v)w, z)_{L^2}
\]

\[
= - (P(\nabla_v w), \nabla u, z)_{L^2} + (P(\nabla_u w), \nabla v z)_{L^2} - (P(\nabla[w, v]w), z)_{L^2}
\]

\[
\leq C \|u\|_{H^s} \|v\|_{H^s} \|w\|_{H^s} \|z\|_{H^s}.
\]

From the tensorial character of \( R^\omega \) and the fact that the map \( \eta \mapsto P_\eta \) is continuously differentiable we similarly obtain

\[
(R^\omega(u, v)w, \Delta^s z)_{L^2} \leq C \|u\|_{H^s} \|v\|_{H^s} \|w\|_{H^s} \|z\|_{H^s},
\]
so that
\[
\|R^\omega(u,v)w\|_{H^s} = \sup \{ \langle R^\omega(u,v)w, z \rangle_{H^s} : z \in C^\infty(TM), L_z \omega = 0, \|z\|_{H^s} \leq 1 \}
\leq C \|u\|_{H^s} \|v\|_{H^s} \|w\|_{H^s}
\]
where \(C\) only depends on \(M\). The general case follows from right invariance. Consequently, Jacobi fields exist, are unique and global in time along the geodesics in \(D^s_\omega\). If \(K\) is the Jacobi field along \(\eta\) with initial conditions (6), then
\[
\Phi(t)w_o := D \exp_v(tw_o)tw_o = Y(t)
\]
defines a family \(\Phi(t)\) of bounded linear operators from \(T_\eta T^*\) to \(T^*_\eta\).

In standard Lie group notation, the group adjoint operator on \(T_e D^s_\omega\) is \(Ad_\eta = dR_\eta \circ dL_\eta\), where \(\eta \in D^s_\omega\) and \(R_\eta\) and \(L_\eta\) are the right and left translations on \(D^s_\omega\) given by the composition with \(H^s\) diffeomorphisms on the right, respectively the left. Consequently
\[
Ad_\eta(X) = D\eta \cdot X \circ \eta^{-1}.
\]

See [2] for derivations of formulas (8), (9).

The group coadjoint \(\text{Ad}^*_{\eta} : T^*_e D^s_\omega \rightarrow T^*_e D^s_\omega\) is defined so that
\[
\langle \text{Ad}^*_{\eta} v, w \rangle_{L^2} = \langle v, \text{Ad}_\eta w \rangle_{L^2} \quad w \in T^*_e D^s_\omega
\]
and the Lie algebra coadjoint \(\text{ad}^*_{\eta} : T_e D^s_\omega \rightarrow T_e D^s_\omega\) is defined so that
\[
\langle \text{ad}^*_{\eta} v, w \rangle_{L^2} = \langle v, \text{ad}_\eta w \rangle_{L^2} \quad u, w \in T_e D^s_\omega
\]

**Lemma 2.1.** For \(v_F\) and \(v_H\) in \(T_e D^s_{\text{Ham}}\) and any \(\eta \in D^s_\omega\)
\[
\text{ad}^*_{v_F} v_H = P (\triangle H \cdot \nabla F)
\]

**Proof.** Let \(v_G\) be any vector in \(T_e D^s_{\text{Ham}}\). Using (9)
\[
\langle \text{ad}^*_{v_F} v_H, v_G \rangle_{L^2} = \langle v_H, \text{ad}_{v_F} v_G \rangle_{L^2}
\]
\[
= \langle v_H, J\nabla \omega(v_F, v_G) \rangle = -\int_M g(\nabla H, \nabla \omega(v_F, v_G)) \, d\mu
\]
\[
= \int_M \omega(v_G, v_F) \cdot \triangle H \, d\mu = \int_M g(v_G, \nabla F) \cdot \triangle H \, d\mu
\]
consequently
\[
\text{ad}^*_{v_F} v_H = P (\triangle H \cdot \nabla F)
\]

For \(\sigma \geq 0\), let \(T_e D^s_{\text{Ham}}\) denote the closure of the space of globally Hamiltonian vector fields in the \(H^\sigma\) norm. By the Hodge decomposition this is a closed subspace in the space of all \(H^\sigma\) vector fields ([15]). For \(\sigma > \frac{\dim M}{2} + 1\) this coincides with the actual tangent space to \(D^s_{\text{Ham}}\). However, for smaller \(\sigma\), \(D^s_{\text{Ham}}\) is not necessarily a smooth manifold.

We have the following decomposition of the solution operator to the Jacobi equation whose proof can be found in [6]. Notice that the decomposition loses one derivative; the equations are only defined on \(H^\sigma\) if the initial velocity defining the geodesic is in \(H^{\sigma+1}\). However, we will be able to compensate for this by using a
smoother geodesic and applying a density argument to obtain the result on $H^s$, $s > \dim M/2 + 1, s \geq \sigma + 1$.

**Proposition 2.1.** If $\eta$ is a geodesic in $\mathcal{D}_s^\omega(M)$, with $s > \dim M/2 + 1, s \geq \sigma + 1$ then the map $\Phi(t)$ defined in (5) extends to a continuous linear operator from $T_e\mathcal{D}_0^\omega$ to $T_\eta(t)\mathcal{D}_\omega^\sigma$. In addition, we have the formula

$$\Phi(t) = D\eta(t)(\Omega(t) - \Gamma(t))$$

(13)

where

$$\Omega(t) = \int_0^t \Ad_{\eta(\tau)^{-1}}\Ad_{\eta(\tau)^{-1}} d\tau$$

(14)

$$\Gamma(t) = \int_0^t \Ad_{\eta(\tau)^{-1}}K_v(\tau)dR_{\eta^{-1}(\tau)}\Phi(\tau) d\tau$$

(15)

and $K_v(\cdot) = \text{ad}_{\tau}^* v$ and $v$ solves the Euler equation.

**Proof.** [6]

We will also make use of the following convenient formulas for the Hodge projections $P$ and $Q$.

**Lemma 2.2.** The projections $P$ and $Q$, given by orthogonal projection onto the first and second summands of (2), respectively, are given by

$$P = -J\nabla\triangle^{-1}\text{div}J$$

$$Q = \omega^2\delta\triangle^{-1}\text{div}J.$$

**Proof.** For any $v \in T_e\mathcal{D}_s^\omega$, we have $\omega^2(v) = d\delta\beta + \sigma = d\triangle^{-1}\delta\omega^2(v) + \sigma$ for some $\beta \in H^{s+2}(T^*M)$ and $\sigma \in \mathcal{H}$. Hence the orthogonal projection $P : T_e\mathcal{D}_s^\omega \to T_e\mathcal{D}_s^\omega$ can be written as

$$v \mapsto P(v) = \omega^2d\delta\beta + \omega^2(\sigma) = \omega^2d\triangle^{-1}\delta\omega^2(P(v) + \omega^2(P^\mathcal{H}(v)))$$

where $P^\mathcal{H}$ denotes the projection onto the finite-dimensional space of harmonic forms which we henceforth neglect. Using the compatible structure $J$ we may further write

$$P(v) = \omega^2g^b g^d\delta\omega^2 - \text{div}J(v) = -J\nabla\triangle^{-1}\text{div}J(v).$$

$Q$ is computed similarly. [\square]

3. **Proof of Fredholmness.** We are now in a position to prove theorem 1.1 by showing that the operator $\Omega(t)$ defined in (14) is invertible and the operator $\Gamma(t)$ defined by (15) is compact. As observed in the Introduction, $T_e\mathcal{D}_s^\sigma_{\text{Ham}}$ is of finite codimension in $T_e\mathcal{D}_s^\omega$. Thus, it suffices to prove that $\Phi(t)$ of Proposition 2.2 is Fredholm on $T_e\mathcal{D}_s^\sigma_{\text{Ham}}$.

**Lemma 3.1.** For $s > \dim M/2 + 1$ the operator $\Omega(t)$ defined by (14) is an invertible linear operator on $T_e\mathcal{D}_s^\sigma_{\text{Ham}}$ with

$$\|\Omega(t)v_H\|_{L^2} \geq C(t)\|v_H\|_{L^2}$$

(16)

$$C(t) = \int_0^t \frac{1}{\|D\eta(\tau)\|_{L^\infty}\|D\eta(\tau)^2\|_{L^\infty}} d\tau$$
Proof. We compute
\[
\langle v_H, \Omega(t)v_H \rangle = \int_0^t \langle v_H, \text{Ad}_{\eta^{-1}}\text{Ad}_{\eta^{-1}}v_H \rangle \, d\tau
\]
\[
= \int_0^t \|\text{Ad}_{\eta^{-1}}v_H\|_{L^2}^2 \, d\tau \geq \int_0^t \frac{1}{\|\text{Ad}_{\eta(t)}\|_{L^2}} \|v_H\|_{L^2}^2 \, d\tau
\]
\[
\geq \int_0^t \|D\eta(t)D\eta(t)^T\|_{L^\infty} \, d\tau \|v_H\|_{L^2}^2,
\]

since
\[
\|\text{Ad}_{\eta(t)}\|_{L^2}^2 = \|\text{Ad}_{\eta(\tau)}\|_{L^2}^2 \leq \|D\eta(\tau)^T D\eta(\tau)\|_{L^\infty}.
\]

Here we have used right-invariance of the \(L^2\) metric and the \(L^\infty\) denotes the maximum on \(M\) of the largest eigenvalue of the symmetric matrix \(D\eta(\tau)^T D\eta(\tau)\), which is well defined since \(\eta\) is \(C^1\). By the Schwartz inequality,
\[
\|D(t)\|_{L^2} \leq \|\Omega(t)v_H\|_{L^2}
\]

So \(\Omega(t)\) has empty kernel and closed range. Since \(\Omega(t)\) is also self-adjoint it has empty cokernel and is therefore invertible on \(T_x D^0_{\text{Ham}}\). \(\square\)

Let \(O\) be a coordinate patch on \(M\) and \(H \in H^s(O)\). The Sobolev topology can be defined locally by
\[
\|H \circ \eta\|_{H^s(O)} = \sum_{|\alpha| \leq \sigma} \|\partial^\alpha H(\eta)\|_{L^2(O)},
\]
where \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is a multi-index with \(|\alpha| = \sum_{j=1}^n \alpha_j\) and \(\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}\).

Lemma 3.2. Suppose \(M\) is a compact symplectic manifold of dimension \(2n\), without boundary, and \(O\) a coordinate patch in \(M\). Suppose \(\eta \in D^s_{\text{Ham}}\), with \(s \geq \sigma + 1\) and \(s > \frac{\dim M}{2} + 1\). Then, for any multi-index \(\alpha\) with \(|\alpha| \leq \sigma\) and any \(w \in H^s\), we have the estimate
\[
\|[\partial^\alpha, P_\eta] w\|_{L^2(O)} \leq B_\alpha \|w\|_{H^{s-1}(O)}
\]
for some constant \(B_\alpha\).

Proof. Let \(w \in H^s\). Then we can write \(w = v + J g^l \delta \alpha\), for some \(H^{s+1}\) 1-form \(\alpha\) and \(v = J \nabla F \in T_x D^s_{\text{Ham}}\) for some \(H^{s+1}\) function \(F\). We prove the estimate by induction.

Consider \([\partial^\alpha, P_\eta]\) (where \(P_\eta = dR_\eta \circ P \circ dR_{\eta^{-1}}\)):
\[
[\partial^\alpha, P_\eta] w = \partial^\alpha (v \circ \eta) - P(\partial^\alpha (w \circ \eta \circ \eta^{-1}) \circ \eta).
\]

We will prove
\[
\|[\partial^\alpha, P_\eta] w \circ \eta\|_{L^2} \leq b_1 \|w\|_{L^2},
\]
with \(N^j_l = \frac{\partial g^j}{\partial x^l} \circ \eta^{-1}\), \(J^i_j\) the components of the almost complex structure, \(g^{lm}\) the components of the inverse metric, \((\delta \alpha)_m\) the components of the 1-form \(\delta \alpha\), and \(i, j\) denoting a derivative in the \(x^j\) variable,
\[
\partial^\alpha (w^k \circ \eta) \circ \eta^{-1} \partial_k = (N^j_l w^k_{i,j}) \partial_k
\]
\[
= \left(N^j_l v^k_{i,j} + N^j_l J^k_{l,j} g^{lm} (\delta \alpha)_m + N^j_l J^k_{l,j} g^{lm} (\delta \alpha)_m + N^j_l J^k_{l,j} g^{lm} (\delta \alpha)_{m,j}\right) \partial_k.
\]
Now $(\delta \alpha)_{m,j} = (\delta \alpha, j)_{m}$ (since $\delta = \star d \star$, with $\star$ the Hodge star operator) and $N^j \star d \star \alpha = \star N^j d \star \alpha = \star d \star (N^j \alpha) - \star \left[ d(N^j) \wedge \star \alpha \right] = \delta(N^j \alpha) - i \gamma_{N^j} \alpha_j$ (which follows from the formula $\delta(f \cdot \gamma) = f \delta \gamma + i \gamma \delta \gamma$ for any function $f$ and any $k$-form $\gamma$). Consequently

$$\partial_x (w^k \circ \eta) \circ \eta^{-1}(\partial_k) = (N^j v^l_j + N^j j^l j^m (\delta \alpha)_m + N^j j^l j^m (\delta \alpha)_m - j^l g^l m (i \gamma_{N^j} \alpha_j)_m) \partial_k + J g^l m (i \gamma_{N^j} \alpha_j)_m) \partial_k \circ \eta.$$

Since $P = J \nabla \triangle - \text{div} J$ and $J^2 = -I$, the last term projects to zero. Therefore

$$\partial_x (w^k \circ \eta) = Q(N^j v^l_j \partial_k) \circ \eta.$$

The $L^2$ norm of the second term is bounded by the $L^2$ norm of the first derivatives of $\alpha$ which are, in turn, bounded by the $L^2$ norm of $w$. It suffices to bound the first term by the $L^2$ norm of $w$.

For any vector field $w$

$$Q(w) = \omega^k \delta \triangle^{-1} d \omega^k (w),$$

by Lemma 2.3. Using the Leibniz rule and the fact that $d \omega^k (v) = 0$, the expression $Q(N^j v^l_j \partial_k) \circ \eta$ only involves first derivatives of $v$. Hence

$$\| Q(v) \|_{L^2} \leq \| \Delta^{-1} d \omega^k (v) \|_{H^1} \leq C \| v \|_{L^2} \leq C \| w \|_{L^2}$$

and we obtain (18). Inductively, the estimate (18) implies that for any multi-index $\alpha$, with no more than $\sigma$ terms, we will have (17).

**Lemma 3.3.** For $s > \frac{\text{dim} M}{2} + 1$, $s \geq \sigma + 1$ the operator $\Omega(t)$ defined by (14) is an invertible linear operator on $T_{e} D_{E}^{s} \mathbb{R}$ with

$$C(t) \| v_h \|_{H^s} \leq \| \Omega(t) v_h \|_{H^s} + K \| v_h \|_{H^{s-1}} \quad (19)$$

$$C(t) = \int_0^t \frac{1}{\| D\eta(\tau) \|_{L^\infty} \| D\eta(\tau) \|^T_{L^\infty}} d\tau.$$

In particular, $\Omega(t)$ is invertible on $T_{e} D_{E}^{s-1} \mathbb{R}$.

**Proof.** We have

$$\| \Omega(t) v_h \|_{H^s} = \sum \| \partial^\alpha \Omega(t) v_h \|_{L^2} \geq \sum \| \Omega(t) \partial^\alpha v_h \|_{L^2} - \sum \| [\partial^\alpha, \Omega(t)] v_h \|_{L^2} \geq C(t) \| v_h \|_{H^s} - \sum \| [\partial^\alpha, \Omega(t)] v_h \|_{L^2},$$

where we have used Lemma 3.1 in the last inequality.

With the definition of $\Omega(t)$ we write

$$\Omega(t) = \int_0^t \Delta \eta^{-1} d R \eta P \eta^{-1} (D \eta^{-1})^T d \tau$$

and use this to show that $\| [\partial^\alpha, \Omega(t)] v_h \|_{L^2}$ is bounded above by the $H^{s-1}$ norm of $v_h$. It is enough to show that, for each $\tau$,

$$\sum \| [\partial^\alpha, D \eta^{-1} P \eta (D \eta^{-1})^T] v_h \|_{L^2} \leq K \| v_h \|_{H^{s-1}}, \quad (20)$$

where $d R \eta P \eta^{-1} = P \eta$. Now

$$\sum \| [\partial^\alpha, D \eta^{-1} P \eta (D \eta^{-1})^T] v_h \|_{L^2} \leq \sum \| [\partial^\alpha, D \eta^{-1}] P \eta (D \eta^{-1})^T v_h \|_{L^2}$$
smooth vector fields $v$ since the projection of a gradient field vanishes. Then, for any limit of compact operators is compact it suffices to show that $K$ operator is compact. By Lemma 3.4 we can approximate $F$ with zero mean and observe that $H$ order differentiation operator and a multiplication operator is an operator of order $\sigma - 1$, by the Leibniz rule. Therefore, the first and third terms above are bounded above by $\|v_H\|_{H^{\sigma - 1}}$. Applying Lemma 3.2 to the second term we obtain (20) and therefore (19).

The estimate (19) shows that $\Omega(t)$ has empty kernel (since $\Omega(t)$ has empty kernel in $L^2$) and closed range on $T_eD_{\text{Ham}}^{\sigma}$. Choose any $v_H \in T_eD_{\text{Ham}}^{\sigma}$. Since $\Omega(t)$ is invertible on $T_eD_{\text{Ham}}^{\sigma}$ we can find a $v_F \in T_eD_{\text{Ham}}^{\sigma}$ such that $\Omega(t)v_F = v_H$. But now the estimate (19) shows that $v_F$ is in $H^\sigma$ as well and $\Omega(t)$ is therefore invertible on $T_eD_{\text{Ham}}^{\sigma}$.

We now proceed to prove compactness of the operator $\Gamma(t)$.

Let $\mathcal{H} = \cap_{s \geq 0} H_0^{s+1}(M)$ be the subspace consisting of smooth functions on $M$ with zero mean and observe that $\mathcal{H}$ is dense in $H_0^{s+1}(M)$. Given $F \in H_0^{s+1}(M)$, let $F_k$ be a sequence of smooth functions in $\mathcal{H}$ approximating $F$ in the $H^{s+1}$ norm. Consequently, $v_{F_k}$ is a smooth sequence of $H^\sigma$ vector fields approximating $v_F$ in the $H^\sigma$ norm.

**Lemma 3.4.** Let $s > \frac{\dim M}{2} + 1$, $s \geq \sigma + 1$ and let $F$ and $\{F_k\}_{k \in \mathbb{N}}$ be as above. Then, $K_{v_{F_k}} \to K_{v_F}$ in the $H^\sigma$ norm, where $K_X$ is the operator defined in Proposition 2.2.

**Proof.** Let $v_H \in T_eD_{\text{Ham}}^{\sigma}$. We estimate

$$\left\|K_{v_{F_k}}v_H - K_{v_{F_k}}v_H\right\|_{H^\sigma} = \left\|P_k(\nabla F \cdot \nabla H - \nabla F_k \cdot \nabla H)\right\|_{H^\sigma}$$

$$\leq C \left\|\nabla F - \nabla F_k\right\|_{H^s} \cdot \|v_H\|_{H^\sigma}$$

$$\leq C \left\|F - F_k\right\|_{H^{s+2}} \cdot \|v_H\|_{H^\sigma}$$

$$\leq C \left\|F - F_k\right\|_{H^{s+1}} \cdot \|v_H\|_{H^\sigma}$$

and the Lemma follows.

**Lemma 3.5.** Let $s > \frac{\dim M}{2} + 1$, $s \geq \sigma + 1$. For any vector field $v_F \in T_eD_{\text{Ham}}^{\sigma}$ the operator $K_{v_F}$ defined in Proposition 2.2 is compact on $T_eD_{\text{Ham}}^{\sigma}$.

**Proof.** By Lemma 3.4 we can approximate $v_F$ in the $H^\sigma$ norm by a sequence of smooth vector fields $v_{F_k}$ such that $K_{v_{F_k}} \to K_{v_F}$ in the $H^\sigma$ operator norm. Since a limit of compact operators is compact it suffices to show that $K_{v_F}$ is compact when $v_F$ is smooth.

By Proposition 2.2 and Lemma 2.1, the operator $K_{v_F}$ may be written as

$$K_{v_F}(v_H) = P(\nabla F \cdot \nabla H) = -P(H \cdot \nabla \Delta F)$$

since the projection of a gradient field vanishes. Then, for any $v_H \in T_eD_{\text{Ham}}^{\sigma}$,

$$\|K_{v_F}(v_H)\|_{H^{s+1}} = \left\|P(H \cdot \nabla \Delta F)\right\|_{H^{s+1}}$$

$$\leq C \|H \cdot \nabla \Delta F\|_{H^{s+1}}$$

$$\leq C' \|H\|_{H^{s+1}}$$

$$\leq C' \|v_H\|_{H^\sigma}$$

Therefore the map $v_H \mapsto K_{v_F}(v_H)$, as a map from $H^\sigma$ vector fields to $H^{\sigma+1}$ vector fields, is compact by the Rellich embedding Theorem.
Corollary 3.1. Suppose $M$ is a closed symplectic manifold and let $s > \frac{\dim M}{2} + 1$, $s \geq \sigma + 1$. Let $v_0 \in T_e D\gamma_0$ and $\eta(t) = \exp_e(tv_0)$. Then, the operator $\Gamma(t)$ defined by (15) is compact on $T_e D\gamma_0$.

Proof. Since the operators $\Phi(t)$, $dR\eta^{-1}$ and $A\eta$ are all continuous and $K_{\exp}$ is compact, the composition appearing under the integral in (15) is compact. Thus the integral, as a limit of Riemann sums of compact operators, is compact.

Proof of Theorem 1.1. Since $\Omega(t)$ is invertible and $\Gamma(t)$ is compact on $T_e D\gamma_0$, the sum $\Omega(t) - \Gamma(t)$ is Fredholm on $T_e D\gamma_0$. Since $D\eta(t)$ is continuous and invertible on $T_e D\gamma_0$, the operator $\Phi(t) = D\eta(\Omega(t) - \Gamma(t))$ is Fredholm on $T_e D\gamma_0$ and hence $d\exp_{\gamma}(tv_F)$ is Fredholm as well.

In order to prove the result in $H^s$ we need to approximate $\eta$ by a smoother $\tilde{\eta}$ since if $\eta$ is in $H^s$ the decomposition (13) only works in $H^{s-1}$ due to derivatives of $\eta$. We prove that $\Phi(t)$ has closed range and finite dimensional kernel in $H^s$. Then $\Phi(t)$ is semi-Fredholm and we can compute its index. Since the index is continuous on the space of Fredholm operators we have the same index for all time. Now $\Phi(0)$ is the identity with index zero so we can conclude that $\Phi(t)$ is Fredholm of index zero for all time.

Finite dimensionality of the kernel in $H^s$ follows from finite dimensionality of the kernel in $L^2$ since the former is a subset of the latter. In order to prove closed range it will suffice to establish an estimate of the form

$$A \|u_0\|_{H^s} \leq \|\Phi(t)u_0\|_{H^s} + \|\kappa(t)u_0\|_{H^s}$$

(21)

for some positive constant $A$ and a compact operator $\kappa(t)$ on $H^s$.

Choose a $C^\infty$ vector field $v_{F_0}$ close to $v_{F_0}$ in the $H^s$ topology, in a sense to specified shortly. Then the geodesic $\tilde{\eta}(t)$ defined by $\tilde{v}_{F_0}$ is smooth and is defined for all time. Then, from the decomposition (13) we have

$$\tilde{\Phi}(t) = D\tilde{\eta}[\tilde{\Omega}(t) - \tilde{\Gamma}(t)]$$

which is valid on $T_e D\gamma_0$ as well.

Since the geodesic and Jacobi equations are smooth on $D\gamma_0$, solutions depend continuously on the initial conditions. Therefore $\tilde{\Phi}$ is close to $\Phi$ in the $H^s$ operator norm and $\tilde{\eta}$ is close to $\eta$ in the $H^s$ norm. Then

$$\|\Phi(t)u_0\| \geq \|\tilde{\Phi}(t)u_0\|_{H^s} - \|\Phi(t) - \tilde{\Phi}(t)\|_{H^s} \|u_0\|_{H^s}$$

$$\geq \|D\tilde{\eta}(t) \cdot \tilde{\Omega}(t)u_0\|_{H^s} - \|D\tilde{\eta}(t) \cdot \tilde{\Gamma}(t)u_0\|_{H^s} - \|\Phi(t) - \tilde{\Phi}(t)\|_{H^s} \|u_0\|_{H^s}$$

$$\geq \frac{C(t)}{\|D\tilde{\eta}\|_{L^\infty}} \|u_0\|_{H^s} - \|D\tilde{\eta}(t) \cdot \tilde{\Gamma}(t)u_0\|_{H^s} - \|\Phi(t) - \tilde{\Phi}(t)\|_{H^s} \|u_0\|_{H^s}$$

where we have used the estimate (19) of Lemma 3.1. Therefore, with $\kappa(t) = D\tilde{\eta}(t) \cdot \tilde{\Gamma}(t)$ we obtain the estimate

$$A \|u_0\|_{H^s} \leq \|\Phi(t)u_0\|_{H^s} + \|\kappa(t)u_0\|_{H^s}$$

with

$$A = \frac{C(t)}{\|D\tilde{\eta}\|_{L^\infty}} - \left(\frac{C(t)}{\|D\tilde{\eta}\|_{L^\infty}} - \frac{\tilde{C}(t)}{\|D\tilde{\eta}\|_{L^\infty}}\right) - \left\|\Phi(t) - \tilde{\Phi}(t)\right\|_{H^s}.$$
is close to $\|D\eta\|_{L^\infty}$. Also, $\left\| \Phi(t) - \tilde{\Phi}(t) \right\|_{H^s}$ is close to zero so that $A$ can be made positive and the estimate (21) is satisfied. Thus $\Phi(t)$, and hence $d\exp(tvF_0)$, has closed range in the $H^s$ topology.

This completes the proof of Theorem 1.1.

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