On Time-Variant Distortions in Multicarrier Transmission with Application to Frequency Offsets and Phase Noise

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Abstract—Phase noise and frequency offsets are due to their time-variant behavior one of the most limiting disturbances in practical OFDM designs and therefore intensively studied by many authors. In this paper we present a generalized framework for the prediction of uncoded system performance in the presence of time-variant distortions including the transmitter and receiver pulse shapes as well as the channel. Therefore, unlike existing studies, our approach can be employed for more general multicarrier schemes. To show the usefulness of our approach, we apply the results to OFDM in the context of frequency offset and Wiener phase noise, yielding improved bounds on the uncoded performance. In particular, we obtain exact formulas for the averaged performance in AWGN and time-invariant multipath channels.

Index Terms—Multicarrier transmission, OFDM, Gabor theory, frequency offset, phase noise

I. INTRODUCTION

Multicarrier (MC) transmission is a promising concept for future mobile communications. In particular, the popular OFDM scheme has already been implemented in several standards such as digital video and audio broadcasting (DAB and DVB-T), WLAN (IEEE 802.11a) and is proposed for next generation mobile communication networks. The main advantage of OFDM is its easy implementation and simple equalization which is based on the orthogonality of the subcarriers in time-invariant channels. However, time variances due to Doppler shifts, carrier frequency mismatching or phase noise destroy this orthogonality and causes intercarrier interference leading to serious performance degradation in OFDM transmission [1].

Most authors [2], [3], [4], [5], [6] study offsets and phase noise in the context of OFDM. Unfortunately their approaches are difficult to apply to many recently proposed MC schemes, which are optimized in terms of pulse adaption and bandwidth efficiency [7], [8], [9], [10]. Even in the OFDM case exact results are not known. However, exact results are crucial for next generation systems, that are expected to operate at high mobility as well as at high data rates. Moreover, to overcome the disadvantages of low-cost hardware at the mobile side normally some kind of tracking is used to provide a minimum signal quality. On the other hand tracking increases receiver complexity. Hence in performance evaluation this has to be taken into account which is provided by this contribution.

Another goal of the paper is to establish a new approach for general MC schemes without the restriction to “textbook” OFDM. The main idea is, that for many practical schemes the average total power received on a single subcarrier is preserved, giving bounds on the expected interference power. Hence, interference analysis reduces to the investigation of separated subcarriers.

The paper is organized as follows. First the cyclic prefix based OFDM transmission model which is relevant for most applications is introduced. Next the transition to generalized MC transmission is performed together with some introductory notes on the underlying Gabor theory. For this system model we study in Sec.III the effective mapping that results from time-invariant channel and a linear distortion. The evaluations lead to a theorem on the resulting interference. In Sec.IV and Sec.V the results are applied to time-frequency offsets and phase noise.

II. SYSTEM DESCRIPTION

A. OFDM signaling

The cyclic prefix based OFDM (cp-OFDM) baseband transmitted signal is

\[ s(t) = \sum_{(mn)\in\mathcal{I}} x_{mn} e^{j2\pi nt/T_u} \gamma(t - n(T_u + T_{cp})) \]

where \( i \) is the imaginary unit and \( \gamma(\cdot) \) is the rectangular pulse:

\[ \gamma(t) = \frac{1}{\sqrt{T_u + T_{cp}}} \chi_{[-T_{cp}, T_u]}(t + t_0). \]

The function \( \chi_{[-T_{cp}, T_u]} \) is the characteristic function of the interval \([-T_{cp}, T_u]\), where \( T_u \) denotes the length of the useful part of the signal and \( T_{cp} \) the length of the cyclic prefix. Without loss of generality we set the time origin to \( t_0 = 0 \), but note that this has influence on several common phase errors given in this paper. The subcarrier spacing is \( F = 1/T_u \) and \( x_{mn} \) are the complex data symbols at time instant \( n \) and subcarrier index \( m \). The indices \((mn)\) range over the doubly-countable index set \( \mathcal{I} \), referring to the data burst to
be transmitted. Note that in practice only a finite number of subcarriers is considered. However for theoretical reasons it is beneficial to consider an infinite set. We will denote the synchronization defects by the random linear operator $\mathcal{S}$, the linear time-invariant channel by $\mathcal{H}$ and the additive white Gaussian noise process (AWGN) by $n(t)$. The received signal is then

$$r(t) = (\mathcal{S}\mathcal{H}s)(t) + (\mathcal{S}n)(t) = (\mathcal{S}(s * h))(t) + (\mathcal{S}n)(t)$$

with $h$ being a realization of the (causal) channel impulse response of finite maximum delay spread $\tau_d$. The standard OFDM receiver estimates the complex symbol as

$$\hat{x}_{kl} = \int e^{-i2\pi kl/T_c} g(t - l(T_u + T_{cp})) r(t) dt$$

using the rectangular pulse $g(t) = \frac{1}{T_c} \chi_{[0,T_c]}(t + l)$ which removes the cyclic prefix. If we assume that the receiver has perfect channel knowledge (given by $h$) zero forcing equalization of the form $\hat{x}_{kl}^0 = h(k/T_c)^{-1} \hat{x}_{kl}$ (or alternatively MMSE equalization if the noise variance is known) is performed where $h$ is the transfer function of the channel (Fourier transform of $h$).

B. Gabor Multicarrier signaling

The standard OFDM setup presented in the previous section can be embedded into a generalization of MC signaling as already proposed by several authors. Because our analysis is based on this more abstract formulation, we will give a brief introduction. Furthermore we include some remarks on facts from Gabor theory that are important for our investigations. Hence we focus on a multicarrier system where the transmitter modulates data symbols $x_{mn} \in \mathbb{C}$ on transmitter waveforms $\gamma_{mn}$ with

$$\gamma_{mn}(t) = (S_{nT,mF}\gamma)(t) = (t - nT)e^{i2\pi mFt}$$

being time- and frequency-shifted versions of one transmit prototype pulse $\gamma \in L_2(\mathbb{R})$. Those sets of functions, denoted here as Gabor sets, are generated by unitary representations on $L_2(\mathbb{R})$ of the so called Weyl-Heisenberg group [11], namely the time-frequency-shift operators. Their definition is obviously not unique, but (2) is a valid choice. Most of the calculations later on can be done by using

$$S_{a,b}^* = e^{-i2\pi ab}S_{-a,-b}$$
$$S_{a,b}S_{c,d} = e^{-i2\pi ab}S_{a+c,b+d}$$
$$S_{a,b}S_{c,d} = e^{-i2\pi (ad-bc)}S_{c,d}S_{a,b}$$

The rules can be easily verified and are essentially the Weyl–Heisenberg group operation. The bandwidth efficiency $\epsilon$ (in symbols) of the signaling given in (2) is $\epsilon \equiv (TF)^{-1}$. The synthesis of the baseband transmit signal corresponding to the transmit symbol sequence $x = (x_{mn}, \ldots)^T$ is performed via the Gabor synthesis operator $\Gamma$ related to the pulse $\gamma$. This operator is defined as

$$\Gamma x \equiv \sum_{mn} x_{mn} \gamma_{mn} = (\sum_{mn} x_{mn} S_{nT,mF})$$

We will call a point $(mn) \in \mathcal{I}$ from now on TF-slot. It represents in analogy to OFDM the $nth$ subcarrier of the $n$th multicarrier symbol. The index set $\mathcal{I} \subset \mathbb{Z}^2$ itself again formally refers to the subset of the rectangular lattice $F^2 \mathbb{Z} \times T\mathbb{Z}$ on the time–frequency plane used for transmission. Without loss of generality we can embed each $x \in \mathbb{C}^2$ into $\mathbb{C}^2$ by setting $x_{mn} = 0$ for $(mn) \in \mathbb{Z}^2 \setminus \mathcal{I}$. Hence the transmit signal is given as

$$s(t) = (\Gamma x)(t)$$

which essentially represents a transmitter side filterbank operation. The signal at the receiver, after passing through the channel $\mathcal{H}$ and the time–variant distortion $\mathcal{S}$, is

$$r(t) = (\mathcal{S}\mathcal{H}\Gamma x)(t) + (\mathcal{S}n)(t)$$

A linear MC receiver projects the received signal onto the Gabor set $\{y_{mn}\}_{(mn)\in \mathcal{I}}$ to give the sequence $\hat{x}$. This can be formally written by employing the Gabor analysis operator $G^*$ that corresponds to the pulse $g$, i.e.

$$\hat{x} = G^* r \equiv (\ldots, (g_{mn}, r), \ldots)^T$$

where $(x, y) = x^* y$ is here the standard inner product on $L_2(\mathbb{R})$ and the operation $^*$ means conjugate transpose. It is easy to see that $G^*$ is the adjoint of $G$ (same for $\Gamma^*$ and $\Gamma$). Thus, the operator $G^*$ implements the receiver side filterbank.

Now the overall transmission chain is given as follows

$$\hat{x} = G^* [S\mathcal{H}\Gamma x + \mathcal{S}n] = G^* S\mathcal{H}\Gamma x + G^* \mathcal{S}n$$

Some important properties can directly be stated in terms of synthesis and analysis operators: (1) perfect reconstruction of the data symbols is expressed as $G^* \Gamma = \mathbb{I}$ (biorthogonality), (2) orthogonal transmitter waveform design means $\Gamma^* \Gamma = \mathbb{I}$. The matrix $\Gamma^* \Gamma$ is called the Gram–matrix of the transmitter pulses. Several properties can also be related to the operator $\Gamma^* \Gamma$. For example in an orthogonal transmitter waveform design $\Gamma^* \Gamma$ is an orthogonal projector on the transmitters signal space. If the operator norm of $\Gamma^* \Gamma$ respectively $\Gamma^* \Gamma$ is finite the set $\{\gamma_{mn}\}_{(mn)\in \mathbb{Z}^2}$ is a Bessel sequence and $B_{\gamma}$ is called the corresponding Bessel bound. If in particular there holds $0 < A_{\gamma} \equiv \inf_{f \in L_2(\mathbb{R})} \|\Gamma^* f\|^2 / \|f\|^2$ the set establishes a frame [12] (in our definition it is a frame for $L_2(\mathbb{R})$) and $\Gamma^* \Gamma$ is called the frame operator. Furthermore it is a Gabor- or Weyl–Heisenberg frame due to the underlying group structure (for an introduction see for example [13]). An important case arise if $A_{\gamma} = B_{\gamma}$, i.e. $\Gamma^* \Gamma = B_{\Gamma}$. Then this establishes a tight frame, which can be understood as a generalization of orthogonal bases to overcomplete expansions.

Finally we adopt the following normalization of the pulses. The normalization of $g$ will have no effect on the later used system performance measures. The normalization of $\gamma$ is determined by the transmit power constraint. In our system model this can be absorbed into noise scale, thus we assume $g$ and $\gamma$ to be normalized to one. Furthermore we assume a noise power of $\sigma^2$ per component for the projected noise vector $\hat{n}$ and $E\{\hat{x} \hat{x}^*\} = \mathbb{I}$. 

III. INTERFERENCE ANALYSIS

For this section we provide a rather generic approach for the evaluation of the desired performance measures. The reason is, that some drawbacks of pure textbook OFDM can be overcome with optimized MC transmission schemes. A lot of OFDM evolutions have been proposed, where namely pulse shaping and different time-frequency densities and constellations are considered. Motivated by this observation we start with a generic theorem, followed by a corollary related to our channel model. Both handle a large class of linear distortions in multicarrier transmission. Only in the last step we restrict ourselves to OFDM.

General interference analysis: Writing the received complex symbol \( \tilde{x}_{kl} \) in the absence of AWGN yields

\[
\tilde{x}_{kl} = \frac{\Delta_{kl}}{\mathbf{H}_{kl}} x_{kl} + \sum_{(mn) \neq (kl)} H_{kl,mn} x_{mn} \tag{5}
\]

where we defined the following expectation value \( \mathbf{T}_{kl} \equiv \mathbb{E}_S \{ H_{kl,kl} \} \). Further we define the second moment with respect to the statistics of \( S \) as \( P_{kl} \equiv \mathbb{E}_S \{ |H_{kl,kl}|^2 \} \). Thus the transmitted symbol \( x_{kl} \) will be multiplied by a constant and disturbed by two zero mean random variables (RV). The first RV \( \Delta_{kl} \) represents a distortion which comes from the randomness of \( S \). This part can be understood as a noise contribution if the receiver does not know \( H_{kl,kl} \). But with proper tracking of \( H_{kl,kl} \) the receiver can “move” the power of \( \Delta_{kl} \), given as \( D_{kl} \equiv P_{kl} - |\mathbf{T}_{kl}|^2 \), to the desired signal contribution yielding an improved performance. Examples are the correction of the common phase errors and Wiener phase noise tracking. For many applications tracking is mandatory to ensure an overall system performance, where interference cancellation has less priority due to the steeply increasing complexity. Therefore the RV ICI (interference from other TF-slots, thus intercarrier and intersymbol interference) remains and gives a noisy contribution of power \( I_{kl} \).

Using this argumentation we establish two performance measures important for uncoded communication. With the separation in (5) we define the signal-to-interference-and-noise-ratio in the TF-slot \( (kl) \) for the case where \( H_{kl,kl} \) is exactly known at the receiver - namely SINR\(_{kl} \). This measure represents the signal quality if the receiver performs ideal tracking of the single TF-slots. If performing no tracking we will represent this with a remaining sinr\(_{kl} < \) SINR\(_{kl} \). These definitions and the corresponding bounds are summarized in the following theorem.

**Theorem 1** If the realizations \( H_{kl,kl} \) are known to the receiver, the SINR\(_{kl} \) of the TF-slot \( (kl) \) is lower bounded by

\[
\text{SINR}_{kl} \equiv \frac{P_{kl}}{\sigma^2 + I_{kl}} \geq \frac{P_{kl}}{\sigma^2 + B_\gamma \beta_{kl} - P_{kl}} \tag{6}
\]

If it is only possible to track the mean \( \bar{H}_{kl} \), the remaining sinr\(_{kl} \leq \) SINR\(_{kl} \) is lower bounded by

\[
\text{SINR}_{kl} \equiv \frac{|\bar{H}_{kl}|^2}{\sigma^2 + I_{kl} + D_{kl}} \geq \frac{|H_{kl}|^2}{\sigma^2 + B_\gamma \beta_{kl} - |\bar{H}_{kl}|^2} \tag{7}
\]

where \( \beta_{kl} = \mathbb{E}_S \{ |\mathbf{H}^* S g_{kl}||^2 \} \) denotes here the channel bound and \( B_\gamma \) is the Bessel bound of \( \{ \gamma_{mn} \} \). Equality is given if the set \( \{ \gamma_{mn} \} \) is a tight frame.

Essentially we used the principle of energy (power) conservation here to upper bound the received power in each TF-slot.

**Proof:** Let \( I \subset \mathbb{Z}^2 \). First observe the following upper bound

\[
\mathbb{E}_S \{ \sum_{(mn) \in \mathbb{Z}^2} |H_{kl,mn}|^2 \} = \mathbb{E}_S \{ (g_{kl}, \mathbf{S} \mathbf{H^*} \mathbf{G} \mathbf{H^*} g_{kl}) \}
\]

\[
\leq B_\gamma \mathbb{E}_S \{ |\mathbf{S}^* \mathbf{H^*} g_{kl}|^2 \} \tag{8}
\]

where \( B_\gamma \) is the Bessel bound of the Gabor set \( \{ \gamma_{mn} \}_{(mn) \in \mathbb{Z}^2} \). Equality in (8) is achieved if \( I = \mathbb{Z}^2 \) and the Gabor set establishes a tight frame. Inequality occurs first due to the possible incompleteness of the set \( \{ \gamma_{mn} \}_{(mn) \in \mathbb{Z}^2} \). Then the adjoint channel could map \( g_{kl} \) into the orthogonal complement of span \( \{ \gamma_{mn} \}_{(mn) \in \mathbb{Z}^2} \). Moreover (8) can be seen as “uniformity” property (related to the condition number) of the mapping \( \mathbf{G} \). Furthermore there could be a sampling loss if \( I \neq \mathbb{Z}^2 \).

We can now write the total amount of received signal power in the TF-slot \( (kl) \) as

\[
\mathbb{E}_{x,n,S} \{ |\tilde{x}_{kl}|^2 \} = \sum_{(mn) \in I} \mathbb{E}_S \{ |H_{kl,mn}|^2 \} + \sigma^2 \tag{9}
\]

\[
\leq B_\gamma \beta_{kl} \cdot |R_{kl}(\mathbf{S}^* \mathbf{H})|^2 + \sigma^2 \leq B_\gamma \beta_{kl} + \sigma^2
\]

The second inequality is caused by the limitation to \( I \subset \mathbb{Z}^2 \). The associated contribution, given by

\[
|R_{kl}(\mathbf{S}^* \mathbf{H})|^2 \cdot \mathbb{E}_S \{ \sum_{(mn) \notin I} |H_{kl,mn}|^2 \} \geq \mathbb{E}_S \{ |H_{kl,mn}|^2 \} \tag{10}
\]

is the interference from non-existing TF-slots that were counted in the infinite sum in (8). Its computation would improve (9) in particular at the boundaries of \( I \) but in the following we will neglect this term. Finally, based only on the “gain” \( P_{kl} \) and the channel bound \( \beta_{kl} \), we arrive at the following bound to the ICI-power \( I_{kl} \)

\[
I_{kl} \leq B_\gamma \beta_{kl} - P_{kl} \tag{11}
\]

which then straightforward leads to the bounds on SINR\(_{kl} \) and sinr\(_{kl} \).

The latter theorem is a selection of the worst case scenario where all symbol energy sent by the transmitter is uniformly collected at the receiver. The calculation of the lower bounds is in most cases simpler than a direct study of the interferer.

The importance of Theorem 1 relies in the fact that it is of very general type in the sense that concepts like (bi-) orthogonality and completeness of neither the transmit sequences, the receiver sequences nor jointly are required. So it is well suited for studying distortions that can not be formulated.
within orthogonality of the subcarriers. Moreover, it provides a tool for performance evaluations for general non-orthogonal multicarrier schemes.

Using these bounds requires the computation of the Bessel bound $B_{\gamma}$, which is independent of $\mathcal{H}$ and $\mathcal{S}$ and only related to the fixed transmitter setup. For example if $\{\gamma_{mn}\}$ is an orthogonal basis for its span it follows that $\mathcal{P}^*$ is the orthogonal projector on span$\{\gamma_{mn}\}_{(mn)\in \mathbb{Z}^2}$, i.e. $B_{\gamma} = 1$ is the minimal achievable Bessel bound for $\{\gamma_{mn}\}$ being all normalized. For overcomplete sets the minimal Bessel bound, achieved by tight frames, is given by the redundancy introduced by the normalized $\{\gamma_{mn}\}$. For Gabor sets we have

$$B_{\gamma} = \max\{1, 1/TF\} \quad (12)$$

Given $B_{\gamma}$, it remains to compute $\overline{H}_{kl}$ (or $P_{kl}$) and $\beta_{kl}$ for the $\mathcal{H}, \mathcal{S}$. By observing that $B_{\gamma}$ is related to the transmitter only, we have the desired separation between the system setup and distortion. Note that distortions like frequency offset and phase noise are obviously not completely known neither at the receiver nor the transmitter, so in practice we have to study $\sinr_{kl}$ given by (7), hence assuming that at least $\overline{H}_{kl}$ is known. This implies that the receiver corrects the phase so that the "full power" $|\overline{H}_{kl}|^2$ can be used for signal reception. When we perform an average over the channel we indirectly use (6) because we have to assume the channel must be ideally known for equalization. Moreover the asymptotic performance for ideal tracking of $\mathcal{S}$ can be obtained from (6), as shown later on for Wiener phase noise. An explicit use of (6) is given in (14). Finally, if the distortion $\mathcal{S}$ is not random, it is $\sinr_{kl} = \sinr_{kl}$.

**Incorporating the time-invariant channel:** In the aim of using (7) of Theorem 1 for the case where $\mathcal{H}$ is time-invariant, known to the receiver and $\mathcal{S}$ represents the time-invariant distortion, we have to specify $\beta_{kl}$ and $\overline{H}_{kl}$. Moreover it is straightforward to extent Theorem 1 and carry out the average over the channel. Hence, let $\mathcal{H}$ given as

$$\mathcal{H} = \int_0^{T_d} h(\tau)S_{\tau,0}d\tau.$$  

(13)

where $h$ is a realization of the (causal) channel impulse response $h$ with finite maximum delay spread $T_d$. A commonly statistical model for a time-invariant channel is $E[h(t_1)h(t_2)] = p_h(t_1)\delta(t_1 - t_2)$ where $p_h$ is the power delay profile and $||p_h||_1$ is the overall channel power (path loss). For this scenario an evaluation of $\beta_{kl}$ (Appendix B) yields

$$\beta_{kl} \text{ def } E_{\mathcal{H}}\{||\mathcal{H}^*S^*g_{kl}\|_2^2\} \leq \|\hat{h}\|_2^2 E_{\mathcal{H}}\{||S^*g_{kl}\|_2^2\} = \tau_d \|\hat{h}\|_2^2 \quad (14)$$

and averaging over the channel

$$E_{\mathcal{H}}\{\beta_{kl}\} = E_{\mathcal{H},S}\{||\mathcal{H}^*S^*g_{kl}\|_2^2\}$$

$$\int_0^{T_d} p_h(\tau)E_{\mathcal{H}}\{||S_{\tau,0}^*S^*g_{kl}\|_2^2\}d\tau$$

$$= ||p_h||_1 E_{\mathcal{H}}\{||S^*g_{kl}\|_2^2\} = ||p_h||_1 $$

which is independent of $(kl)$. Here we have assumed that $E_{\mathcal{H}}\{||Sf\|_2^2\} = ||f\|_2^2$ which covers the frequency and timing offset as well as phase noise. The effective channel matrix for a fixed realization $h$, given as

$$H_{kl,mn} = \int_0^{T_d} h(\tau)\langle g_{kl}, SS_{\tau,0}\gamma_{mn} \rangle d\tau$$

(15)

could still be a RV. To separate the channel from the distortion in the evaluation of $\overline{H}_{kl}$ as much as possible let us define

$$E_{\mathcal{S}}\{\langle g_{kl}, SS_{\tau,0}\gamma_{kl} \rangle\} = e^{-i2\pi kF T} E_{\mathcal{S}}\{\langle g, S_{IT,kF}^*S_{IT,kF}S_{\tau,0}\gamma \rangle\}$$

(16)

$$= e^{-i2\pi kF T} (g, E_{\mathcal{S}}\{S_{IT,kF}^*S_{IT,kF}S_{\tau,0}\gamma \})$$

$$= e^{-i2\pi kF T} S_{\gamma_{kl}}(\tau) = (S_{0,kF}\gamma_{kl})(\tau)$$

(17)

which essentially contains the distortion of the $\tau$th path contribution in terms of the pulses conjugated by $S_{IT,kF}$, i.e. "shifted" to $(IT,kF)$ in the time-frequency plane. The mean diagonal (with respect to $S$) is then given as

$$\overline{H}_{kl} = \int_0^{T_d} h(\tau)E_{\mathcal{S}}\{\langle g_{kl}, SS_{\tau,0}\gamma_{kl} \rangle\} d\tau$$

$$= (S_{0,kF}, h, s_{kl})$$

For the channel average we need the second moment of (17) with respect to $\mathcal{H}$ as already intended in Theorem 1 because the channel is known, thus

$$E_{\mathcal{H}}\{||\overline{H}_{kl}|^2\} = E_{\mathcal{H}}\{E_{\mathcal{S}}\{\langle g_{kl}, SS_{\tau,0}\gamma_{kl} \rangle\}^2\}$$

$$= \int_0^{T_d} p_h(\tau)E_{\mathcal{S}}\{\langle g_{kl}, SS_{\tau,0}\gamma_{kl} \rangle\}^2d\tau$$

$$= \int_0^{T_d} p_h(\tau)||s_{kl}(\tau)||^2d\tau = ||p_h||_1 ||s_{kl}||_2$$

(18)

The disturbances are bounded now as

$$I_{kl} + D_{kl} \leq B_{\gamma}||s_{kl}||_2 - ||S_{0,kF}h, s_{kl}||_2^2$$

$$E_{\mathcal{H}}\{I_{kl} + D_{kl}\} \leq B_{\gamma}||p_h||_1 - ||p_h, ||_1^2$$

Let us summarize the refinement of Theorem 1 in the following corollary

**Corollary 1** If the channel realizations of $\mathcal{H}$ are given as a convolution with the impulse response $h$ known to the receiver and the distortion $\mathcal{S}$ is only known in the mean, the $\sinr_{kl}$ is lower bounded as

$$\sinr_{kl} = \frac{\overline{H}_{kl}}{\sigma^2 + I_{kl}} \geq \frac{||S_{0,kF}h, s_{kl}||_2^2}{\sigma^2 + B_{\gamma}||s_{kl}||_2^2}$$

(19)

Furthermore in the average over the channel statistics it becomes

$$\sinr_{kl} = \frac{E_{\mathcal{H}}\{||\overline{H}_{kl}||_2^2\}}{\sigma^2 + E_{\mathcal{H}}\{I_{kl}\}} \geq \frac{\langle p_h, ||s_{kl}||_2^2 \rangle}{\sigma^2 + B_{\gamma}||p_h||_1 - \langle p_h, ||s_{kl}||_2^2 \rangle}$$

(20)

where $p_h$ is the power delay profile of the channel.

So it remains to compute $s_{kl}(\tau)$ (independent of $h$) and $\beta_{kl}$ (depending on $h$) to get a "worst-case" $\sinr_{kl}$-bound. We will mainly concentrate on the channel average where it remains to compute $||s_{kl}(\tau)||_2^2$ only. The phase of $s_{kl}(\tau)$ is also important
to get a view of what the receiver has to correct – separately or within the channel equalization.

**cp-OFDM specifics:** Before proceeding by applying Corollary 1 to the problem of time–frequency offsets and phase noise, we will introduce a slight modification for cp-OFDM. The OFDM transmitter does not exploit an orthogonal set when using a cyclic prefix. In the Appendix A it is shown that the Gram matrix $\Gamma^T \Gamma$ is block-Toeplitz with the maximal eigenvalue given as twice the bandwidth efficiency $\epsilon$, thus $B_\gamma = 2\epsilon$. By using this value Corollary 1 covers an arbitrary linear distortion. However it can be shown, that for distortion considered in the paper $\epsilon B_\gamma$ instead of $B_\gamma$ can be used, where $B_\gamma = 1$ is the Bessel bound of the orthogonal receiver set. The latter is related to the cyclic structure inserted by the special choice of $g$ and $\gamma$ in cp-OFDM. This improves the prediction in (11), hence we define for our application now $B_{\text{offset}} \eqdef \epsilon B_\gamma = \epsilon$.

**IV. Timing and Frequency Offsets**

Performance evaluation of communication systems under timing and carrier frequency offset is of fundamental importance. In particular OFDM systems suffer from a mismatch of local oscillator frequency at the receiver with respect to the carrier frequency. That means that the decoupling of the subcarriers in time-invariant channels achieved with the cyclic prefix OFDM is destroyed. First let us start to establish formulas for the general offset problem and then refine them to OFDM.

**A. The General Offset Problem**

As a simple application of section III we will now study synchronization errors that consist of a non–random time-frequency shift, i.e. $S = S_{d,\nu}$. Because $S$ is non-random and unitary it follows that $\|S^* g_{kl}\|_2 = \|g_{kl}\|_2 = 1$. We have to evaluate (16), thus

$$s_{kl}(\tau) = e^{i2\pi[\nu(T-d\tau)+d\nu]} (g, S_{\tau+d,\nu} \gamma)$$

which is related to the cross ambiguity function $A_{\gamma\gamma}(\cdot, \cdot)$ of the pulses $g$ and $\gamma$

$$A_{\gamma\gamma}(d, \nu) = (g, S_{d,\nu} \gamma)$$

Before proceeding further, we like to state that in the absence of a channel we already have

$$\overline{P}_{kl} = s_{kl}(0) = e^{i2\pi(\nu d)/\epsilon} A_{\gamma\gamma}(d, \nu)$$

$$\beta_{kl} = 1$$

$$l_{kl} \leq B_\gamma - |A_{\gamma\gamma}(d, \nu)|^2$$

for arbitrary $d$ and $\nu$. Using this in (7) yields

$$\sin r \geq \frac{|A_{\gamma\gamma}(d, \nu)|^2}{\sigma^2 + B_\gamma - |A_{\gamma\gamma}(d, \nu)|^2}$$

which is achievable if the receiver ideally corrects the common phase error, hence the phase of $\overline{P}_{kl}$. Then it turns out that $\text{SINR}_{kl} = \text{SINR}_{kl}$ since this distortion is non-random ($D_{kl} = 0$).

Let us consider the case where a channel is present. Computing

$$\langle S_{0,kF}, s_{kl} \rangle^2 = \int_0^{T_d} h(\tau) e^{-i2\pi\nu kF} A_{\gamma\gamma}(d + \tau, \nu) d\tau$$

$$\langle p_h, s_{kl} \rangle^2 = \int_0^{T_d} p_h(\tau) |A_{\gamma\gamma}(\tau + d, \nu)|^2$$

and using Corollary 1 gives first the bound on $\text{SINR}_{kl}$ for a fixed channel. The second equation gives the bound on $\sin r$ in the channel average, which turns out to be independent of $(kl)$. Both hold for general (Gabor-based) MC schemes given by its pulses $g$ and $\gamma$. To establish an improved channel bound $\beta_{kl}$ with respect to the pure bound already given in (14) is quite difficult for this general constellation. However for the frequency offset alone we get (given in Appendix C)

$$\beta_{kl} \leq \|H^* g_{kl}\|_2 + 4\pi T_\nu |\hat{h}|_\infty (1 + T_\nu |\nu|)$$

providing a separation of the frequency offset and the channel. Another approximation (not a strict bound) was already presented in [15]. In the following we will refine to the problem of the frequency offset in OFDM which is of more practical importance.

**B. Frequency Offset in OFDM**

Let us consider now a cyclic prefix based OFDM transmission (instead of $B_\gamma$ we use now $B_{\text{offset}} = \epsilon$) distorted by constant unknown offsets $\nu$ and $d$ under ideal channel knowledge. The cross ambiguity function for $\gamma$ and $g$ as introduced in (22) can be compactly written by

$$\langle \cdot, \cdot \rangle_{\text{cp}} : \tau \rightarrow \langle \cdot, \cdot \rangle_{\text{cp}} = \begin{cases} \tau & \tau \leq 0 \\ 0 & 0 < \tau < T_{\text{cp}} \\ \tau - T_{\text{cp}} & \tau \geq T_{\text{cp}} \end{cases}$$

as

$$A_{\gamma\gamma}(\tau, \nu) = \sqrt{\epsilon} \sin \nu T_u - \langle \rho \rangle_{\text{cp}} \frac{\pi \nu T_u}{\pi \nu T_u + \epsilon}$$

The phase $\phi_0 = \pi \nu T_u$ is related to our choice of time origin $t_0 = 0$ in (1).

The signal quality in the presence of time- and frequency shifts can now be directly obtained from (28). Apart from $\langle \cdot \rangle_{\text{cp}}$ and $\sqrt{\epsilon}$ (the loss in mean signal amplitude due to the cyclic prefix) (28) agrees with the well known auto ambiguity function for rectangular pulses $g = \gamma$ of width $T_u$. If the system exhibits a time offset $d$ only and $|d + T_{\text{cp}}| = 0$, the time dependency in the cross ambiguity function cancels, thus

$$s_{kl}(\tau) = e^{-i2\pi dk/\epsilon} A_{\gamma\gamma}(d, 0) = \sqrt{\epsilon} e^{i\phi_0 - 2\pi dk/\epsilon}$$

and only phase rotations occur (normally corrected by channel estimation and equalization). Contrary to this, time offsets with $|d + T_{\text{cp}}| \neq 0$ causes interference. For frequency offsets interference occurs immediately as seen from Fig.1. Going
back to the case $[d + \tau_d]_{cp} = 0$

$$\mathcal{T}_{kl} = \langle S_{0,kF}h, s_{kl} \rangle = e^{i2\pi(\hat{\nu}-d\hat{k})/\nu} \sqrt{1-\sin^2 \hat{\nu}/(\pi\nu)^2} H(t/T)$$

(29)

holds. Obviously $\hat{\nu}$ ($\hat{d}$) induces a rotating phase over the time slots $l$ (frequency slots $k$) as we would expect. With (20), (26) and (29) we get in the channel average

$$\sin \geq \frac{\sin^2 \pi \hat{\nu}/(\pi \nu)^2}{\sigma^2/(\epsilon\|p_h\|_1) + (1-\sin^2 \pi \hat{\nu}/(\pi \nu)^2)}$$

(30)

where we again used the fact that $|A_{g\gamma}(\tau+d, \nu)|^2 = |A_{g\gamma}(0, \nu)|^2$ for $[\tau_d + d]_{cp} = 0$. This result is consistent with a lower bound presented in [3]. Restricting to $\hat{\nu} \leq \frac{1}{\pi}$ (half the subcarrier spacing) as has been done in [3] we immediately obtain their result together with an analytical expression for their numerically estimated bound on the interference. It can be found by observing that $\text{INT} \leq (1-\frac{1}{\pi^2}) \sin^2 \pi \nu = 0$, 5947 $\sin^2 \pi \nu$, thus

$$\sin \geq (30) \geq \frac{\|p_h\|_1/\sigma^2}{1 + 0, 5947 \sin^2 \pi \nu \cdot \|p_h\|_1/\sigma^2}$$

(31)

Note that this bound holds only for $\hat{\nu} \leq 0.5$ and is less tight than (30) (see Fig.2).

V. PHASE NOISE

The continuous time system model we consider in this section is

$$r_0(t) = (S\mathcal{H}s)(t) + (S\mathcal{N})(t) \triangleq \theta(t)(\mathcal{H}s)(t) + \theta(t)n(t).$$

Obviously the multiplication by the phase noise process $\theta(t) = e^{i\phi(t)}$ fulfills $E_S\{\|S^*f\|_2\} = \|f\|_2$ and does not change the noise statistics. To derive $\sin^{kl}$ we start with (17), respectively (16), which gives

$$s_{kl}(t) = (g, S_{\gamma}^{*}T^{-1,kF}S_{\gamma}T, kF S_{\gamma}S_{\gamma}) = (g, S_{\gamma}^{*}T^{-1,kF}S_{\gamma}S_{\gamma})$$

(32)

where the first moment is defined as $\overline{\theta}(t) \triangleq E_{\phi(t)}\{\theta(t)\}$. The second step follows because $\overline{\theta}$ is a pointwise multiplication, so that the frequency shifts will cancel out.

A. GAUSSIAN PHASE NOISE

A typical model which occurs in phase synchronization loops is $\phi(t) \sim N(0, S_{\phi})$ with $E\{\phi(t)\phi(t + \tau)\} = C_\phi(t) = \int S_{\phi}(\tau) e^{i2\pi f^\tau} df$. It was already observed in [2] that $\sin^{kl}$ is independent of the phase noise spectrum $S_{\phi}(\tau)$, where the authors only considered an classical OFDM system based on rectangular pulses (without cyclic prefix and additional channel). For the general bounds presented in this work this is a direct consequence from the fact that the bounds depend only on the first moments. Thus, with the mean $\overline{\theta}(t) = e^{-\frac{S_{\phi}}{2}}$ for the Gaussian case we get

$$s_{kl}(t) = e^{-\frac{S_{\phi}}{2}} (g, S_{\gamma},\nu, \gamma) = e^{-\frac{S_{\phi}}{2}} A_{g\gamma}(\tau, 0)$$

and we continue (except of the constant factor) as in Sec.IV. That is

$$\mathcal{T}_{kl} = e^{-\frac{S_{\phi}}{2}} \int \overline{\theta}(\tau) e^{-i2\pi kF\tau} A_{g\gamma}(\tau, 0) d\tau$$

If no channel is present we obtain for general (Gabor-based) MC schemes with $A_{g\gamma}(0, 0) = (g, \gamma)$

$$\sin \geq \frac{|(g, \gamma)|^2}{e^{S_{\phi}}(\sigma^2 + B_{\gamma}) - |(g, \gamma)|^2}$$

and for OFDM (using $B_{\text{oddm}} = \epsilon$)

$$\sin \geq \frac{1}{e^{S_{\phi}}(\sigma^2/\epsilon + 1) - 1}$$

Fig. 1. Ambiguity function for cyclic prefix OFDM - The ambiguity function $A_{g\gamma}(\tau, \nu)$ describes the behavior of the pulse shaped system with respect to single time-frequency shifts, hence natural arise in the offset problematic. It illustrates the cp-OFDM fundamentals that the magnitude stays constant for time offsets $\tau$ with $|\tau|_{cp} = 0$, where it is rapidly decreasing in $\nu$ yielding an increased interference.
For the channel average we get similar to (25)

$$\langle p_h, |s_{kl}|^2 \rangle = e^{-S_\phi} \int_0^{T_d} p_h(\tau)|A_{g\gamma}(\tau, 0)|^2 d\tau$$

determining sinr (see Corollary 1) for general (Gabor-based) MC schemes. For OFDM transmission (using $B_{\text{ofdm}} = \epsilon$ and (28)) this gives

$$\sinr_{kl} \geq e^{S_\phi} (\sigma^2/\epsilon) + \beta_{kl} - |\hat{h}(k/T_u)|^2$$

if the channel delay spread does not exceed the cyclic prefix. While in the latter the calculation of $\beta_{kl}$ is still left open for practical applications, we obtain in the channel average directly

$$\sinr \geq e^{S_\phi} (\sigma^2/\epsilon) + 1 - 1$$

B. Wiener Phase Noise

A widely used model in frequency synchronization is

$$\phi(t) = \int_0^t \dot{\phi} d\tau$$

with the instantaneous frequency $\frac{1}{2\pi} \dot{\phi}(\tau)$. The power density spectrum (PDS) of the signal $r(t)$ corrupted by phase noise is given by

$$S_\phi(\omega) = \langle S_\phi \ast S_{\text{fa}}(\omega) \rangle$$

with $S_\phi(\omega) = \int_{-\infty}^{\infty} e^{-i2\pi\omega t}C_\phi(\tau)$ and $C_\phi$ being the autocorrelation of the phase noise process. With the application of the Wiener-Khintchine-theorem the autocorrelation can be expressed using $S_\phi$ (the PDS of the instantaneous frequencies)

$$C_\phi(\tau) = e^{-\frac{\sigma^2}{2}} \int_0^\infty S_\phi(\omega) \sin^2 \left( \frac{\omega \tau}{2} \right)$$

For $S_\phi(\omega) = S_\phi = \text{const}$ the autocorrelation of the process is given by

$$C_\phi(\tau) = e^{-\frac{\sigma^2}{2} |\tau|}$$

with the typical Lorentzian PDS

$$S_\phi(\omega) = \frac{4S_\phi}{\omega^2 + (2\pi\omega)^2}$$

In the presence of Wiener phase noise communication via coherent detection is not possible due to the infinite distortions of the phase. A common approach is to correct the phase from period to period, therefore we use

$$\theta(t) = e^{i\phi(t-t_{\text{sync}})}$$

with $\phi(\cdot)$ being a realization of the Wiener process $\phi(t) = \int_0^t \dot{\phi} d\tau$ on $[0, \infty)$ and $t_{\text{sync}}$ denotes the time of the last phase synchronization. For simplicity let us assume that $t_{\text{sync}} = t_{\text{sync}} T$, i.e. a multiple of the symbol time shift. The mean of $\theta(t)$ (defined on $[t_{\text{sync}}, \infty)$) is

$$\mathbb{E}[\theta(t)] = e^{-\frac{\sigma^2}{2}\frac{T}{T_u}} e^{i\phi(t-t_{\text{sync}})}$$

where $\mathbb{E}[\theta(t)]$ is a phase noise process defined on $[0, \infty)$.

We will need the time-frequency shifted version of $\mathbb{E}[\theta]$ in (32). Thus with the commutation relation $S_{-IT,0} \mathbb{E}[\theta] = e^\frac{\sigma^2}{2} \mathbb{E}[\theta] S_{-IT,0}$ this gives

$$s_{kl}(\tau) = e^{\frac{\sigma^2}{2} \text{sync}} \langle g, S_{-IT,0} \mathbb{E}[\theta], S_{\tau,0} \rangle$$

which depends obviously on the pulse shapes $g$ and $\gamma$. The last step is correct in a rough sense only. The reason is that the pulse shapes could be much longer than $T$. So one has to assure that the domain of $\mathbb{E}[\theta]$ is not/originally marginally violated (this is obviously not relevant for cyclic prefix OFDM). We fix the time origin now so that $t_{\text{sync}} = 0$ and normalize the phase noise power with respect to the subcarrier spacing $\rho = S_\phi/F$, i.e.

$$s_{kl}(\tau) = e^{-\frac{\sigma^2}{2} \text{sync}} \langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle$$

Now, using (33) we can directly establish the following: if no channel is present a bound on $\sinr_{kl}$ is

$$\sinr_{kl} \geq \frac{|\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle|^2}{e^{\rho^2/\epsilon} (\sigma^2 + B_\gamma) - |\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle|^2}$$

and in the channel average

$$\sinr \geq \frac{\int_0^\infty p_h(\tau)|\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle|^2 d\tau}{e^{\rho^2/\epsilon} (\sigma^2 + B_\gamma) - \int_0^\infty p_h(\tau)|\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle|^2 d\tau}$$

Both formulas hold again for general (Gabor-based) MC schemes. And - from (34) and (35) one can see that there is an inherent exponential degradation of the signal quality. To obtain closed formulas it is left to calculate $\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle$. For OFDM we get

$$\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle = \frac{2\sqrt{\rho}}{S_\phi T_u} (1 - e^{-\frac{\sigma^2}{2} \text{sync}}) \chi_{[-T_u,T_u]}(\tau)$$

With the normalized phase noise power $\rho = S_\phi T_u$

$$\langle g, \mathbb{E}[\theta], S_{\tau,0} \rangle = \frac{2\sqrt{\rho}}{\rho} (1 - e^{-\frac{\sigma^2}{2} \text{sync}})$$

follows as long as the channel delay spread does not exceed the cyclic prefix. If no channel is present (34) yields

$$\sinr_{kl} \geq \frac{1}{e^{\rho^2/\epsilon} (\sigma^2 + B_\gamma) - 1}$$

The bound for the channel average given by (35) reads now

$$\sinr \geq \frac{1}{e^{\rho^2/\epsilon} (\sigma^2 + B_\gamma) - 1}$$

And finally for a particular channel realization $h$ we obtain

$$\sinr_{kl} \geq \frac{|\hat{h}(k/T_u)|^2}{e^{\rho^2/\epsilon} (\sigma^2 + B_\gamma) - |\hat{h}(k/T_u)|^2}$$

In the last estimates we used again that $B_{\text{ofdm}} = \epsilon$. Note the exponential decay in (38), thus tracking for Wiener phase noise is crucial. One can directly obtain the tracking gain,
hence $l = 0$ ($l = 1$) means that phase synchronization was performed in the current (previous) OFDM symbol. It is interesting to see which asymptotic performance results if the receiver ideally removes the phase noise on each subcarrier but does no further interference cancellation. We can give an answer to this question by using formula (6) from Theorem 1, namely the bound on $\text{SINR}_{kl}$, which is shown in the Appendix D to be

$$\text{SINR}_{kl} \geq \frac{1}{4k(\rho - 2e^{\rho/2} - \rho^2)}(\sigma^2/\epsilon + 1) - 1 \geq \sinr_{kl}$$

The graphical summary, i.e. the comparison of the latter to (36), is shown in Fig.3.

![Graphical summary](image)

**Fig. 3.** Wiener phase noise tracking in OFDM systems - The $\sinr_l$ for $l = 0, 1, 2$ is shown over the normalized Wiener phase noise power $\rho$ and compared to the asymptotic performance given by SINR. The latter corresponds to a perfect tracking of phase noise in each subcarrier, but without a cancellation of the induced interference.

**VI. PERFORMANCE EVALUATIONS FOR OFDM**

**A. Frequency Offset**

To demonstrate the prediction of the degradation due to a frequency offset we present in Fig.4 the theoretical and simulated symbol estimation error $\text{MSE} \equiv \|\hat{x}_{kl} - x_{kl}\|^2$ over the normalized frequency offset for a fixed sample channel. With the approximation in (26) for the channel bound $\beta_{kl}$ the theoretical prediction agrees nicely with our simulation. An unknown frequency offset has significant impact on the MSE especially if a (known and equalized) channel is present.

![MSE due to Frequency Offset](image)

**Fig. 4.** MSE due to Frequency Offset in OFDM systems - the impact of the frequency offset and its prediction on the MSE over the normalized offset with and without a LTI channel.

large enough so that the main degradation is $I_{kl}$ with many contributions (central limit theorem).

**B. Phase Noise**

As an example of the evaluation in section V-B we present here the symbol error rate (SER) of a cyclic prefix OFDM system in the presence of Wiener phase noise. For the SER-prediction the interference due to phase noise is assumed to be Gaussian (see Fig.5). Then it can be treated as additional noise. We like to state that this is only appropriate for $S_{\hat{\phi}}$

![BPSK symbol error rate](image)

**Fig. 5.** BPSK symbol error rate - the simulated impact of Wiener phase noise on the BPSK performance and its prediction is shown. The phase is synchronized at each OFDM symbol.

**VII. CONCLUSIONS**

We derived a framework for the evaluation of bounds on uncoded system performance of linear distorted (Gabor-based) multicarrier schemes with inclusion of a time-invariant channel. We identified the dominating terms determining interference levels for the case where the receiver has perfect knowledge or only mean knowledge on the distortion. Our contribution provides analytical insights into the interrelation of pulse shapes and time-frequency density of Gabor systems to practical problems of imperfect, hence distorted radio frontends. The bounds apply without requiring (bi-)orthogonality of the subcarriers commonly needed for those evaluations. Our study was motivated by the impact of time-variant distortions on the OFDM performance caused by imperfect receiver structures. Therefore we applied the theoretical framework
on time-frequency offsets and phase noise, both being effects limiting the performance of current OFDM implementations. Finally we verified our theoretical predictions with computer simulations.

**APPENDIX**

A. **Bessel bound for cyclic prefix OFDM**

The Bessel bound is given by the norm of $\mathbf{G^*}$ which is equal to the largest eigenvalue of the Gram matrix $\mathbf{G^*}$. Computing this for $\gamma(t) = \frac{1}{\sqrt{T_u + T_c}} \chi(-T_c, T_u)(t)$ is

$$(\mathbf{G^*})_{k,m} = \delta_{kn} e^{-i\frac{\pi}{\nu}(m-k)} \frac{\sin \frac{\pi}{\nu}(m-k)}{\pi(n-m)}$$

where $\epsilon = T_u/(T_u + T_c)$. This is a Toeplitz matrix in the frequency slots $k$ and $m$ generated by the symbol

$$\phi(\omega) = \sum_{n=-\infty}^{\infty} e^{i\pi(\omega - \frac{1}{\nu})n} \sin \frac{\pi}{\nu}n = 1 + \frac{2\epsilon}{\pi} \sum_{n=1}^{\infty} \cos \pi(\omega - \frac{1}{\nu})n \cdot \sin \frac{\pi}{\nu}n$$

$$= 1 + \epsilon - \epsilon [(1/\epsilon - \omega) \mod 1 + \omega] = \epsilon (1 - \epsilon - \omega) + 1$$

(39)

where in the last step we restrict $\omega \in (0, 1)$. Its known that the spectrum of the infinite Toeplitz operator $\mathbf{G^*}$ is dense in the image of $\phi$. Thus the Bessel bound is given as $B_g = \|\phi\|_{\infty}$. For cyclic Toeplitz matrices already for finite dimension $N$ the $k$th eigenvalue is given as $\phi(k/N)$. As seen from (39) $\phi(\omega)$ is a step-like function taking on $[0, 1]$ only the two values $\epsilon([\frac{1}{\epsilon}] + 1)$ and $\epsilon([\frac{1}{\epsilon} - 1] + 1)$. A special case is $\epsilon = 0.5$ where the first value is not anymore in $[0, 1)$. In that case and also for $\epsilon = 1$ the spectrum is constant $\phi(\omega) = 1$, hence the set $\{\gamma_{mn}\}$ forms an orthonormal basis for its span. For $\epsilon > 0.5$, relevant for the application in cp-OFDM, $\phi(\omega)$ is a step-function with the functions $\epsilon$ and $2\epsilon$, thus in general we have to use $B_g = 2\epsilon$. Note here that normally (but not in this paper) the transmit pulse is normalized for transmit power $1$, thus $\|\phi\|_{\infty} = 1/\epsilon$ and then follows $B_g = 2$. Using $B_g = 2\epsilon$ in (18) gives the most general bound on the disturbances. In particular this is needed if $S$ represents non-causal operations, like $S_{t,T}$ with $t > 0$, occurring in time-offset correction. For the distortions considered in this paper one can show, that one can equivalently use $\epsilon B_g$ where $B_g$ is the Bessel bound of the receiver set $\{g_{mn}\}_{(mn)\in \mathbb{Z}^2}$. Its easy to verify that $\{g_{mn}\}_{(mn)\in \mathbb{Z}^2}$ is an orthonormal set, i.e. $\mathbf{G^*}$ an orthogonal projector onto its span, hence $B_g = 1$.

B. **General Channel bounds**

$\mathbf{H^*}$ is a convolution with $h^R(t) \overset{\text{def}}{=} h(-t)$. With $(h^R)^* = \bar{h}$

this gives for the channel bound $\beta_{kl}$

$$\beta_{kl} = \mathbb{E}_S \{ \| \mathbf{S}^* g_{kl} \|^2 \} = \mathbb{E}_S \{ \| \bar{h} \cdot (\mathbf{S}^* g_{kl}) \|^2 \}$$

$$\leq \| \| \bar{h} \|^2 \| \cdot \mathbb{E}_S \{ \| \| \mathbf{S}^* g_{kl} \|^2 \| \cdot \| \}$$

The last step is for $1 = 1/p + 1/q$ (Hölder’s inequality). From practical point of view the essential support of $\bar{h}$ is often less then $g$. Therefore using $p = \infty$ and $q = 1$

$$\beta_{kl} \leq \| \bar{h} \|_{\infty} \cdot \mathbb{E}_S \{ \| \mathbf{S}^* g_{kl} \| \} = \| \bar{h} \|_{\infty}$$

(40)

with $\mathbb{E}_S \{ \| \mathbf{S}^* f \| \} = \| f \|$ and $\| g_{kl} \|^2 = 1$.

C. **Channel bound for the frequency offset**

Let $S = S_{0,\nu}$ and $\mu = \hat{h}^R$, then

$$\mathbb{E}_S \{ \| \mathbf{H^* S}^* g_{kl} \|_{\frac{\nu}{2}}^2 \} = \int_{\omega} \| \mu(\omega) \|_{\nu}^2 |\hat{g}_{kl}(\omega - \nu)|^2 d\omega$$

We can express the first part of the integrand as

$$|\hat{\mu}(\omega + \nu)|^2 = |\hat{\mu}(\omega)|^2 + \int_{\omega}^{\omega + \nu} \hat{\mu}'(f) df$$

(a)

$$= 2 \Re \{ \hat{\mu}(\omega) \int_{\omega}^{\omega + \nu} \hat{\mu}'(f) df \}$$

(b)

By using

$$|\hat{\mu}'(f)|^2 = \frac{\partial}{\partial f} \int_{\tau_d}^{\tau_d + T} d\tau \mu(\tau)e^{i2\pi f \tau}$$

$$\leq 2\pi \int_{\tau_d}^{\tau_d + T} d\tau \tau \mu(\tau)e^{i2\pi f \tau}$$

$$\leq (2\pi)^2 \tau_d^2 |\hat{\mu}(f)|^2$$

we upper bound term (a) as (using Jensen’s integral inequality and the measure $df/|\nu|$)

(a) $$\| \nu \|^2 \int_{\omega}^{\omega + \nu} \frac{df}{|\nu|} |\hat{\mu}'(f)|^2$$

$$\leq \| \nu \|^2 \int_{\omega}^{\omega + |\nu|} \frac{df}{|\nu|} |\hat{\mu}'(f)|^2$$

$$\leq \| \nu \|^2 (2\pi)^2 \tau_d^2 \| \hat{\mu} \|_{\infty}^2$$

Further for the term (b) follows

$$\leq 2|\hat{\mu}(\omega)\int_{\omega}^{\omega + \nu} df| |\hat{\mu}'(f)|$$

$$\leq 2|\hat{\mu}(\omega)| \int_{\omega}^{\omega + |\nu|} df| |\hat{\mu}'(f)| \leq 4\pi \| \mu \|_{\infty} |\hat{\mu}(\omega)| |\tau_d| |\nu|$$

$$\leq 4\pi \| \hat{\mu} \|_{\infty}^2 \tau_d |\nu|$$

Putting both together and using that $\| \hat{\mu} \|_{\infty}^2 \leq \tau_d |\mu|_{2}^2$ gives

$$|\hat{\mu}(\omega + \nu)|^2 \leq |\hat{\mu}(\omega)|^2 + 4\pi \tau_d |\nu||\hat{\mu}|_{\infty}^2 (1 + \tau_d |\nu|)$$

$$\leq |\hat{\mu}(\omega)|^2 + 4\pi \tau_d |\nu|(1 + \tau_d |\nu| \nu)$$
D. Wiener Phase noise - tracking asymptotic

Let $S = \theta(t)$ be the pointwise multiplication with the phase noise process and $C_p(t)$ the phase noise autocorrelation. Moreover, for simplification no additional channel should be present, i.e. $\mathcal{H} = I$. Let assume further that the receiver can ideally track and correct the phase noise on each subcarrier, but does no interference cancellation. The performance of this asymptotic situation can be obtained from Theorem 1. The SINR in formula (6) is determined only by $P_{kl}$. The channel bound is $b_{kl} = 1$ (because $\mathcal{H} = I$) and $B_{\text{oldm}} = \epsilon$ (Appendix A). Then

$$P_{kl} = \mathbb{E}_S \{|H_{kl,kl}|^2\}$$

$$= \int \int \frac{g(t_1)g(t_2)C_p(t_1 - t_2)}{T_s} \gamma(t_1) \gamma(t_2) d^2 t$$

$$= \int \int f(t_1)C_p(t_1 - t_2) f(t_2) d^2 t$$

$$= \int |\hat{f}(\omega)|^2 S_p(\omega) d\omega$$

with $f(t) = \frac{g(t)}{\sqrt{T_s} \sqrt[4]{\mathbb{E}_f}}$. For OFDM follows $f(t) = \sum_{k=-\infty}^{\infty} \frac{\sin^2(\pi \omega T_u)}{(\pi \omega)^2}$, For Wiener phase noise follows

$$P_{\text{kl}} = \epsilon \int \frac{s^2 \pi^2 T_u}{(\pi T_u)^2 \gamma} \frac{4S_{\phi}}{S_{\phi}^2 + 4(2\pi\omega)^2} d\omega$$

$$= \frac{4\epsilon e^{-\pi^2 T_u}}{(S_{\phi} T_u)^2} \left(S_{\phi}^2 T_u - 2 + 2e^{-\pi^2 T_u}\right)$$

$$= \frac{4\epsilon}{\rho^2} (\rho - 2 + 2e^{-\pi^2 T_u})$$

where $\rho = S_{\phi} T_u$. Due to “permanent” ideal tracking the $(kl)$ dependence disappeared. The SINR bound is now

$$\text{SINR} \geq \frac{1}{u(\rho^2 - 2 - 2\rho / \rho^2)} (\sigma^2 / \epsilon + 1) - 1$$

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