Yangians and their applications*

A. I. Molev

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia

alexm@maths.usyd.edu.au

*To appear in Handbook of Algebra (M. Hazewinkel, Ed.), Vol. 3, Elsevier.
## Contents

1 Introduction \hspace{1cm} 1

2 The Yangian for the general linear Lie algebra \hspace{1cm} 3
   2.1 Algebraic motivations and definitions \hspace{1cm} 3
   2.2 A matrix form of the defining relations \hspace{1cm} 5
   2.3 Automorphisms and anti-automorphisms \hspace{1cm} 6
   2.4 Poincaré–Birkhoff–Witt theorem \hspace{1cm} 7
   2.5 Hopf algebra structure \hspace{1cm} 8
   2.6 Quantum determinant and quantum minors \hspace{1cm} 9
   2.7 The center of \( Y(n) \) \hspace{1cm} 11
   2.8 The Yangian for the special linear Lie algebra \hspace{1cm} 12
   2.9 Two more realizations of the Hopf algebra \( Y(\mathfrak{sl}_2) \) \hspace{1cm} 13
   2.10 Quantum Liouville formula \hspace{1cm} 15
   2.11 Factorization of the quantum determinant \hspace{1cm} 17
   2.12 Quantum Sylvester theorem \hspace{1cm} 18
   2.13 The centralizer construction \hspace{1cm} 19
   2.14 Commutative subalgebras \hspace{1cm} 21

3 The twisted Yangians \hspace{1cm} 24
   3.1 Defining relations \hspace{1cm} 24
   3.2 Embedding into the Yangian \hspace{1cm} 26
   3.3 Sklyanin determinant \hspace{1cm} 27
   3.4 The center of the twisted Yangian \hspace{1cm} 29
   3.5 The special twisted Yangian \hspace{1cm} 30
   3.6 The quantum Liouville formula \hspace{1cm} 30
   3.7 Factorization of the Sklyanin determinant \hspace{1cm} 31
   3.8 The centralizer construction \hspace{1cm} 32
   3.9 Commutative subalgebras \hspace{1cm} 34

4 Applications to classical Lie algebras \hspace{1cm} 35
   4.1 Newton’s formulas \hspace{1cm} 36
   4.2 Cayley–Hamilton theorem \hspace{1cm} 37
   4.3 Graphical constructions of Casimir elements \hspace{1cm} 39
   4.4 Pfaffians and Hafnians \hspace{1cm} 41
1 Introduction

The term Yangian was introduced by Drinfeld (in honor of C. N. Yang) in his fundamental paper [35] (1985). In that paper, besides the Yangians, Drinfeld defined the quantized Kac–Moody algebras which together with the work of Jimbo [64], who introduced these algebras independently, marked the beginning of the era of quantum groups. The Yangians form a remarkable family of quantum groups related to rational solutions of the classical Yang–Baxter equation. For each simple finite-dimensional Lie algebra \( a \) over the field of complex numbers, the corresponding Yangian \( Y(a) \) is defined as a canonical deformation of the universal enveloping algebra \( U(a[x]) \) for the polynomial current Lie algebra \( a[x] \). Importantly, the deformation is considered in the class of Hopf algebras which guarantees its uniqueness under some natural ‘homogeneity’ conditions. An alternative description of the algebra \( Y(a) \) was given later in Drinfeld [37].

Prior to the introduction of the Hopf algebra \( Y(a) \) in [35], the algebra, which is now called the Yangian for the general linear Lie algebra \( gl_n \) and denoted \( Y(gl_n) \), was considered in the work of Faddeev and the St.-Petersburg school in relation with the inverse scattering method; see for instance Takhtajan–Faddeev [161], Kulish–Sklyanin [89], Tarasov [157, 158]. The latter algebra is a deformation of the universal enveloping algebra \( U(gl_n[x]) \).

For any simple Lie algebra \( a \) the Yangian \( Y(a) \) contains the universal enveloping algebra \( U(a) \) as a subalgebra. However, only in the case \( a = sl_n \) does there exist an evaluation homomorphism \( Y(a) \to U(a) \) identical on the subalgebra \( U(a) \); see Drinfeld [35, Theorem 9]. In this chapter we concentrate on this distinguished Yangian which is closely related to \( Y(gl_n) \). For each of the classical Lie algebras \( a = o_{2n+1}, sp_{2n}, o_{2n} \) Olshanski [139] introduced another algebra called the twisted Yangian. Namely, the Lie algebra \( a \) can be regarded as a fixed point subalgebra of an involution \( \sigma \) of the appropriate general linear Lie algebra \( gl_N \). Then the twisted Yangian \( Y(gl_N, \sigma) \) can be defined as a subalgebra of \( Y(gl_N) \) which is a deformation of the universal enveloping algebra for the twisted polynomial current Lie algebra

\[
gl_N[x]^\sigma = \{ A(x) \in gl_N[x] \mid \sigma(A(x)) = A(-x) \}.
\]

For each \( a \) the twisted Yangian contains \( U(a) \) as a subalgebra, and an analog of the evaluation homomorphism \( Y(gl_N, \sigma) \to U(a) \) does exist. Moreover, the twisted Yangian turns out to be a (left) coideal of the Hopf algebra \( Y(gl_N) \).

The defining relations of the Yangian \( Y(gl_n) \) can be written in a form of a single ternary (or RTT) relation on the matrix of generators. This relation has a rich and extensive background. It originates from the quantum inverse scattering theory; see e.g. Takhtajan–Faddeev [161], Kulish–Sklyanin [89], Drinfeld [38]. The Yangians were primarily regarded as a vehicle for producing rational solutions of the
Yang–Baxter equation which plays a key role in the theory integrable models; cf. Drinfeld [35]. Conversely, the ternary relation was used in Reshetikhin–Takhtajan–Faddeev [146] as a tool for studying quantum groups. Moreover, the Hopf algebra structure of the Yangian can also be conveniently described in a matrix form.

Similarly, the twisted Yangians can be equivalently presented by generators and defining relations which can be written as a quaternary (or reflection) equation for the matrix of generators, together with a symmetry relation. Relations of this type appeared for the first time in Cherednik [28] and Sklyanin [152], where integrable systems with boundary conditions were studied.

This remarkable form of the defining relations for the Yangian and twisted Yangians allows special algebraic techniques (the so-called $R$-matrix formalism) to be used to describe the algebraic structure and study representations of these algebras. On the other hand, the evaluation homomorphisms to the corresponding classical enveloping algebras allow one to use these results to better understand the classical Lie algebras themselves. In particular, new constructions of the Casimir elements can be obtained in this way. These include the noncommutative characteristic polynomials for the generator matrices and the Capelli-type determinants. Some other applications include the constructions of Gelfand–Tsetlin bases and commutative subalgebras. Moreover, as was shown by Olshanski [138, 139], the Yangian and the twisted Yangians can be realized as some projective limit subalgebras of a sequence of centralizers in the classical enveloping algebras. This is known as the centralizer construction; see also [120].

The representation theory of the Yangians $Y(\mathfrak{a})$ is a very much nontrivial and fascinating area. Although the finite-dimensional irreducible representations of $Y(\mathfrak{a})$ are completely described by Drinfeld [37], their general structure still remains unknown. This part of the theory of the Yangians will have to be left outside this chapter. We give, however, some references in the bibliography which we hope cover at least some of the most important results in the area.

The Yangians, as well as their super and $q$-analogs, have found a wide variety of applications in physics. This includes the theory of integrable models in statistical mechanics, conformal field theory, quantum gravity. We do not attempt to review all the relevant physics literature but some references are given as a guide to such applications.

At the end of each section we give brief bibliographical comments pointing towards the original articles and to the references where the proofs or further details can be found. The author would like to thank A. N. Kirillov, M. L. Nazarov, G. I. Olshanski and V. O. Tarasov for reading the manuscript and valuable comments.
2 The Yangian for the general linear Lie algebra

As we pointed out in the Introduction, the discovery of the Yangians was motivated by the quantum inverse scattering theory. It is possible, however, to “observe” the Yangian defining relations from purely algebraic viewpoint. We start by showing that they are satisfied by certain natural elements of the universal enveloping algebra $U(\mathfrak{gl}_n)$. Then we show that these relations can be written in a matrix form which provides a starting point for special algebraic techniques to study the Yangian structure.

2.1 Algebraic motivations and definitions

Consider the general linear Lie algebra $\mathfrak{gl}_n$ with its standard basis $E_{ij}$, $i, j = 1, \ldots, n$. The commutation relations are given by

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj},$$

where $\delta_{ij}$ is the Kronecker delta; see e.g. [58]. Introduce the $n \times n$-matrix $E$ whose $ij$-th entry is $E_{ij}$. The traces of powers of the matrix $E$

$$g_s = \text{tr } E^s, \quad s = 1, 2, \ldots$$

are central elements of the universal enveloping algebra $U(\mathfrak{gl}_n)$ known as the Gelfand invariants; see [44]. Moreover, the first $n$ of them are algebraically independent and generate the center. A proof of the centrality of the $g_s$ is easily deduced from the following relations in the enveloping algebra

$$[E_{ij}, (E^s)_{kl}] = \delta_{kj} (E^s)_{il} - \delta_{il} (E^s)_{kj}. \quad (2.1)$$

One could wonder whether a more general closed formula exists for the commutator of the matrix elements of the powers of $E$. The answer to this question turns out to be affirmative and the following generalization of (2.1) can be verified by induction:

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj} (E^s)_{il} - (E^s)_{kj} (E^r)_{il},$$

where $r, s \geq 0$ and $E^0 = 1$ is the identity matrix. We can axiomatize these relations by introducing the following definition.

**Definition 2.1** The Yangian for $\mathfrak{gl}_n$ is a unital associative algebra with countably many generators $t_{ij}^{(1)}$, $t_{ij}^{(2)}$, $\ldots$ where $1 \leq i, j \leq n$, and the defining relations

$$[t_{ij}^{(r+1)}(s), t_{kl}^{(s+1)}(r)] = t_{kj}^{(r)} t_{il}^{(s)} - t_{k}^{(r)} t_{il}^{(s)}, \quad (2.2)$$

where $r, s \geq 0$ and $t_{ij}^{(0)} = \delta_{ij}$. This algebra is denoted by $\mathcal{Y}(\mathfrak{gl}_n)$, or $\mathcal{Y}(n)$ for brevity.
Introducing the generating series, 
\[ t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y(n)[[u^{-1}]], \]
we can write (2.2) in the form
\[
(u - v) [t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u). \tag{2.3}
\]
Divide both sides by \( u - v \) and use the formal expansion 
\[ (u - v)^{-1} = \sum_{r=0}^{\infty} u^{-r-1} v^r \]
to get an equivalent form of the defining relations
\[
[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} (t_{kj}^{(r+s-a)} t_{il}^{(a-1)} - t_{kj}^{(a-1)} t_{il}^{(r+s-a)}). \tag{2.4}
\]
The previous discussion implies that the algebra \( Y(n) \) is nontrivial, as the mapping
\[ t_{ij}^{(r)} \mapsto (E^{r})_{ij} \tag{2.5} \]
defines an algebra homomorphism \( Y(n) \to U(gl_n) \).

Alternatively, the generators of the Yangian can be realized as the “Capelli minors”. Keeping the notation \( E \) for the matrix of the basis elements of \( gl_n \) introduce the Capelli determinant
\[
\det(1 + Eu^{-1}) = \sum_{\sigma \in S_n} \text{sgn} \sigma \cdot (1 + Eu^{-1})_{\sigma(1),1} \cdots (1 + (u - n + 1)^{-1})_{\sigma(n),n}. \tag{2.6}
\]
When multiplied by \( u(u-1) \cdots (u-n+1) \) this determinant becomes a polynomial in \( u \) whose coefficients constitute another family of algebraically independent generators of the center of \( U(gl_n) \). The value of this polynomial at \( u = n - 1 \) is a distinguished central element whose image in a natural representation of \( gl_n \) by differential operators is given by the celebrated Capelli identity \[19\]; see also \[37\]. For a positive integer \( m \leq n \) introduce the subsets of indices \( B_i = \{i, m + 1, m + 2, \ldots, n\} \) and for any \( 1 \leq i, j \leq m \) consider the Capelli minor \( \det(1 + Eu^{-1})_{B_i B_j} \) defined as in (2.6), whose rows and columns are respectively enumerated by \( B_i \) and \( B_j \). These minors turn out to satisfy the Yangian defining relations, i.e., there is an algebra homomorphism
\[
Y(m) \to U(gl_n), \quad t_{ij}(u) \mapsto \det(1 + Eu^{-1})_{B_i B_j}.
\]
These two interpretations of the Yangian defining relations (which will reappear in Sections 2.4, 2.12 and 2.13) indicate a close relationship between the representation theory of the algebra \( Y(n) \) and the conventional representation theory of the general linear Lie algebra. Many applications of the Yangian are based on the following simple observation.
Proposition 2.2 The mapping
\[ \pi : t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1} \] (2.7)
defines an algebra epimorphism \( Y(n) \to U(\mathfrak{gl}_n) \). Moreover,
\[ E_{ij} \mapsto t^{(1)}_{ij} \]
is an embedding \( U(\mathfrak{gl}_n) \hookrightarrow Y(n) \).

In particular, any \( \mathfrak{gl}_n \)-module can be extended to the algebra \( Y(n) \) via (2.7). This plays an important role in the Yangian representation theory.

2.2 A matrix form of the defining relations

Introduce the \( n \times n \) matrix \( T(u) \) whose \( ij \)-th entry is the series \( t_{ij}(u) \). It is convenient to regard it as an element of the algebra \( Y(n)[[u^{-1}]] \otimes \text{End } \mathbb{C}^n \) given by
\[ T(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes e_{ij}, \] (2.8)
where the \( e_{ij} \) denote the standard matrix units. For any positive integer \( m \) we shall be using the algebras of the form
\[ Y(n)[[u^{-1}]] \otimes \text{End } \mathbb{C}^n \otimes \cdots \otimes \text{End } \mathbb{C}^n, \] (2.9)
with \( m \) copies of \( \text{End } \mathbb{C}^n \). For any \( a \in \{1, \ldots, m\} \) we denote by \( T_a(u) \) the matrix \( T(u) \) which acts on the \( a \)-th copy of \( \text{End } \mathbb{C}^n \). That is, \( T_a(u) \) is an element of the algebra (2.9) of the form
\[ T_a(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes 1 \otimes \cdots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \cdots \otimes 1, \] (2.10)
where the \( e_{ij} \) belong to the \( a \)-th copy of \( \text{End } \mathbb{C}^n \) and \( 1 \) is the identity matrix. Similarly, if
\[ C = \sum_{i,j,k,l=1}^n c_{ijkl} e_{ij} \otimes e_{kl} \in \text{End } \mathbb{C}^n \otimes \text{End } \mathbb{C}^n, \]
then for distinct indices \( a, b \in \{1, \ldots, m\} \) we introduce the element \( C_{ab} \) of the algebra (2.9) by
\[ C_{ab} = \sum_{i,j,k,l=1}^n c_{ijkl} 1 \otimes 1 \otimes \cdots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \cdots \otimes 1 \otimes e_{kl} \otimes 1 \otimes \cdots \otimes 1, \] (2.11)
where the $e_{ij}$ and $e_{kl}$ belong to the $a$-th and $b$-th copies of $\text{End} \mathbb{C}^n$, respectively.

Consider now the permutation operator

$$P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji} \in \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n.$$ 

The rational function

$$R(u) = 1 - Pu^{-1} \tag{2.12}$$

with values in $\text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n$ is called the Yang $R$-matrix. (Here and below we write 1 instead of $1 \otimes 1$ for brevity). An easy calculation in the group algebra $\mathbb{C}[S_3]$ shows that the following identity holds in the algebra $(\text{End} \mathbb{C}^n)^{\otimes 3}$

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u). \tag{2.13}$$

This is known as the Yang–Baxter equation. The Yang $R$-matrix is its simplest nontrivial solution. In the following we regard $T_1(u)$ and $T_2(u)$ as elements of the algebra (2.9) with $m = 2$.

**Proposition 2.3** The defining relations (2.2) can be written in the equivalent form

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v). \tag{2.14}$$

The relation (2.14) is known as the ternary or RTT relation.

### 2.3 Automorphisms and anti-automorphisms

Consider an arbitrary formal series which begins with 1,

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]].$$

Also, let $a \in \mathbb{C}$ be an arbitrary element and $B$ an arbitrary nondegenerate complex matrix.

**Proposition 2.4** Each of the mappings

$$T(u) \mapsto f(u)T(u), \tag{2.15}$$

$$T(u) \mapsto T(u + a), \tag{2.16}$$

$$T(u) \mapsto BT(u)B^{-1}$$

defines an automorphism of $Y(n)$.

The matrix $T(u)$ can be regarded as a formal series in $u^{-1}$ whose coefficients are matrices over $Y(n)$. Since this series begins with the identity matrix, $T(u)$ is invertible and we denote by $T^{-1}(u)$ the inverse element. Also, denote by $T^t(u)$ the transposed matrix for $T(u)$. 
Proposition 2.5  Each of the mappings
\[
T(u) \mapsto T(-u),
\]
\[
T(u) \mapsto T'(u),
\]
\[
T(u) \mapsto T^{-1}(u)
\]
defines an anti-automorphism of \(Y(n)\).

2.4 Poincaré–Birkhoff–Witt theorem

Theorem 2.6  Given an arbitrary linear order on the set of the generators \(t_{ij}^{(r)}\), any element of the algebra \(Y(n)\) is uniquely written as a linear combination of ordered monomials in the generators.

Outline of the proof. There are two natural ascending filtrations on the algebra \(Y(n)\). Here we use the one defined by
\[
\text{deg}_1 t_{ij}^{(r)} = r.
\]
(The other filtration will be used in Section 2.7). It is immediate from the defining relations (2.4) that the corresponding graded algebra \(\text{gr}_1 Y(n)\) is commutative. Denote by \(\overline{t}_{ij}^{(r)}\) the image of \(t_{ij}^{(r)}\) in the \(r\)-th component of \(\text{gr}_1 Y(n)\). It will be sufficient to show that the elements \(\overline{t}_{ij}^{(r)}\) are algebraically independent.

The composition of the automorphism \(T(u) \mapsto T^{-1}(-u)\) of \(Y(n)\) and the homomorphism (2.7) yields another homomorphism \(Y(n) \rightarrow U(\mathfrak{gl}_n)\) such that
\[
T(u) \mapsto (1 - Eu^{-1})^{-1}.
\]
The image of the generator \(t_{ij}^{(r)}\) is given by (2.3). For any nonnegative integer \(m\) consider the Lie algebra \(\mathfrak{gl}_{n+m}\) and now let \(E\) denote the corresponding matrix formed by its basis elements \(E_{ij}\). Then formula (2.3) still defines a homomorphism
\[
Y(n) \rightarrow U(\mathfrak{gl}_{n+m}).
\]
Consider the canonical filtration on the universal enveloping algebra \(U(\mathfrak{gl}_{n+m})\) and observe that the homomorphism (2.18) is filtration-preserving. So, it defines a homomorphism of the corresponding graded algebras
\[
\text{gr}_1 Y(n) \rightarrow S(\mathfrak{gl}_{n+m}),
\]
where \(S(\mathfrak{gl}_{n+m})\) is the symmetric algebra of \(\mathfrak{gl}_{n+m}\). One can show that for any finite family of elements \(\overline{t}_{ij}^{(r)}\) there exists a sufficiently large parameter \(m\) such that their images under (2.19) are algebraically independent. \(\Box\)

Corollary 2.7  \(\text{gr}_1 Y(n)\) is the algebra of polynomials in the variables \(\overline{t}_{ij}^{(r)}\).
2.5 Hopf algebra structure

A Hopf algebra $A$ (over $\mathbb{C}$) is an associative algebra equipped with a coproduct (or comultiplication) $\Delta : A \to A \otimes A$, an antipode $S : A \to A$ and a counit $\epsilon : A \to \mathbb{C}$ such that $\Delta$ and $\epsilon$ are algebra homomorphisms, $S$ is an anti-automorphism and some other axioms are satisfied. These can be found in any textbook on the subject. In [25, Chapter 4] the Hopf algebra axioms are discussed in the context of quantum groups.

Theorem 2.8 The Yangian $Y(n)$ is a Hopf algebra with the coproduct

$$\Delta : t_{ij}(u) \mapsto \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u),$$

the antipode

$$S : T(u) \mapsto T^{-1}(u),$$

and the counit

$$\epsilon : T(u) \mapsto 1.$$

Proof. We only verify the most nontrivial axiom that $\Delta : Y(n) \to Y(n) \otimes Y(n)$ is an algebra homomorphism. The remaining axioms follow directly from the definitions. We slightly generalize the notation used in Section 2.2. Let $p$ and $m$ be positive integers. Introduce the algebra

$$(Y(n)[[u^{-1}]])^\otimes p \otimes (\text{End} \mathbb{C}^n)^\otimes m$$

and for all $a \in \{1, \ldots, m\}$ and $b \in \{1, \ldots, p\}$ consider its elements

$$T_{[b]a}(u) = \sum_{i,j=1}^{n} (1^\otimes b-1 \otimes t_{ij}(u) \otimes 1^\otimes p-b) \otimes (1^\otimes a-1 \otimes e_{ij} \otimes 1^\otimes m-a).$$

The definition of $\Delta$ can now be written in a matrix form,

$$\Delta : T(u) \mapsto T_{[1]}(u) T_{[2]}(u),$$

where $T_{[b]}(u)$ is an abbreviation for $T_{[b]1}(u)$. We need to show that that $\Delta(T(u))$ satisfies the ternary relation $$(2.14),$$ i.e.,

$$R(u - v)T_{[1]1}(u) T_{[2]1}(u) T_{[1]2}(v) T_{[2]2}(v) = T_{[1]2}(v) T_{[2]2}(v) T_{[1]1}(u) T_{[2]1}(u) R(u - v).$$

However, this is implied by the ternary relation (2.14) and the observation that $T_{[2]1}(u)$ and $T_{[1]2}(v)$, as well as $T_{[1]1}(u)$ and $T_{[2]2}(v)$, commute. \qed
2.6 Quantum determinant and quantum minors

Consider the rational function $R(u_1, \ldots, u_m)$ with values in the algebra $(\text{End} \mathbb{C}^n)^{\otimes m}$ defined by

$$R(u_1, \ldots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \cdots (R_{1m} \cdots R_{12}), \quad (2.20)$$

where we abbreviate $R_{ij} = R_{ij}(u_i - u_j)$. Applying the Yang–Baxter equation (2.13) and the fact that $R_{ij}$ and $R_{kl}$ commute if the indices are distinct, we can write (2.20) in a different form. In particular,

$$R(u_1, \ldots, u_m) = (R_{12} \cdots R_{1m}) \cdots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}).$$

As before, we use the notation $T_a(u_a)$ for the matrix $T(u_a)$ of the Yangian generators corresponding to the $a$-th copy of End $\mathbb{C}^n$.

**Proposition 2.9** We have the relation

$$R(u_1, \ldots, u_m) T_1(u_1) \cdots T_m(u_m) = T_m(u_m) \cdots T_1(u_1) R(u_1, \ldots, u_m).$$

We let the $e_i$, $i = 1, \ldots, n$ denote the canonical basis of $\mathbb{C}^n$, and $A_m$ the antisymmetrizer in $(\mathbb{C}^n)^{\otimes m}$ given by

$$A_m(e_1 \otimes \cdots \otimes e_m) = \sum_{\sigma \in S_m} \text{sgn} \sigma \cdot e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}}. \quad (2.21)$$

Note that this operator satisfies $A_m^2 = m! A_m$.

**Proposition 2.10** If $u_i - u_{i+1} = 1$ for all $i = 1, \ldots, m - 1$ then

$$R(u_1, \ldots, u_m) = A_m.$$

By Propositions 2.9 and 2.10, we have the identity

$$A_m T_1(u) \cdots T_m(u - m + 1) = T_m(u - m + 1) \cdots T_1(u) A_m. \quad (2.22)$$

Suppose now that $m = n$. Then the antisymmetrizer is a one-dimensional operator in $(\mathbb{C}^n)^{\otimes n}$. Therefore, the element (2.22) equals $A_n$ times a scalar series with coefficients in $Y(n)$ which prompts the following definition.

**Definition 2.11** The quantum determinant is the formal series

$$\text{qdet} T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots \in Y(n)[[u^{-1}]] \quad (2.23)$$

such that the element (2.22) (with $m = n$) equals $A_n \text{qdet} T(u)$. 


Proposition 2.12 For any permutation $\rho \in S_n$ we have
\[
\text{qdet } T(u) = \text{sgn } \rho \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot t_{\sigma(1), \rho(1)}(u) \cdots t_{\sigma(n), \rho(n)}(u - n + 1) \tag{2.24}
\]
\[
= \text{sgn } \rho \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot t_{\rho(1), \sigma(1)}(u - n + 1) \cdots t_{\rho(n), \sigma(n)}(u). \tag{2.25}
\]

Example 2.13 In the case $n = 2$ we have
\[
\text{qdet } T(u) = t_{11}(u) t_{22}(u - 1) - t_{21}(u) t_{12}(u - 1)
= t_{22}(u) t_{11}(u - 1) - t_{12}(u) t_{21}(u - 1)
= t_{11}(u - 1) t_{22}(u) - t_{12}(u - 1) t_{21}(u)
= t_{22}(u - 1) t_{11}(u) - t_{21}(u - 1) t_{12}(u).
\]

More generally, assuming that $m \leq n$ is arbitrary, we can define $m \times m$ quantum minors as the matrix elements of the operator \((2.22)\). Namely, the operator \((2.22)\) can be written as
\[
\sum_{c_1, \ldots, c_m} t^{c_1 \cdots c_m}_{d_1 \cdots d_m}(u) \otimes \epsilon_{c_1 d_1} \otimes \cdots \otimes \epsilon_{c_m d_m},
\]
summed over the indices $c_i, d_i \in \{1, \ldots, n\}$, where $t^{c_1 \cdots c_m}_{d_1 \cdots d_m}(u) \in \mathbb{Y}(n)[[u^{-1}]]$. We call these elements the quantum minors of the matrix $T(u)$. The following formulas are obvious generalizations of \((2.24)\) and \((2.25)\),
\[
t^{c_1 \cdots c_m}_{d_1 \cdots d_m}(u) = \sum_{\sigma \in S_m} \text{sgn } \sigma \cdot t_{c_{\sigma(1)} d_1}(u) \cdots t_{c_{\sigma(m)} d_m}(u - m + 1)
= \sum_{\sigma \in S_m} \text{sgn } \sigma \cdot t_{c_1 d_{\sigma(1)}}(u - m + 1) \cdots t_{c_m d_{\sigma(m)}}(u).
\]

It is clear from the definition that the quantum minors are skew-symmetric with respect to permutations of the upper indices and of the lower indices.

Proposition 2.14 The images of the quantum minors under the coproduct are given by
\[
\Delta(t^{c_1 \cdots c_m}_{d_1 \cdots d_m}(u)) = \sum_{a_1 \leq \cdots \leq a_m} t^{c_1 \cdots c_m}_{a_1 \cdots a_m}(u) \otimes t^{a_1 \cdots a_m}_{d_1 \cdots d_m}(u), \tag{2.26}
\]
summed over all subsets of indices $\{a_1, \ldots, a_m\}$ from $\{1, \ldots, n\}$.

Proof. Using the notation of Section 2.5 we can write the image of the left hand side of \((2.22)\) under the coproduct $\Delta$ as
\[
A_m T^{[1]}_1(u) T^{[2]}_1(u) \cdots T^{[1]}_m(u - m + 1) T^{[2]}_m(u - m + 1).
\]

Since $m! A_m = A_m^2$, by \((2.22)\) this coincides with
\[
\frac{1}{m!} A_m T^{[1]}_1(u) \cdots T^{[1]}_m(u - m + 1) A_m T^{[2]}_1(u) \cdots T^{[2]}_m(u - m + 1).
\]
Taking here the matrix elements and using the skew-symmetry of the quantum minors we come to (2.26).

Corollary 2.15 We have

\[ \Delta : \text{qdet } T(u) \mapsto \text{qdet } T(u) \otimes \text{qdet } T(u). \]

2.7 The center of \( \mathcal{Y}(n) \)

Proposition 2.16 We have the relations

\[ [\tau_{ab}(u), \tau_{c_1 \cdots c_m}(v)] = \frac{1}{u - v} \left( \sum_{i=1}^{m} \tau_{c_i b}(u) \tau_{d_1 \cdots d_m}(v) - \sum_{i=1}^{m} \tau_{d_1 \cdots d_m}(v) \tau_{a d_i}(u) \right), \]

where the indices \( a \) and \( b \) in the quantum minors replace \( c_i \) and \( d_i \), respectively.

Corollary 2.17 For any indices \( i, j \) we have

\[ [\tau_{c_i d_j}(u), \tau_{c_1 \cdots c_m}(v)] = 0. \]

Recall the elements \( d_i \in \mathcal{Y}(n) \) are defined by (2.23).

Theorem 2.18 The coefficients \( d_1, d_2, \ldots \) of the series \( \text{qdet } T(u) \) belong to the center of the algebra \( \mathcal{Y}(n) \). Moreover, these elements are algebraically independent and generate the center of \( \mathcal{Y}(n) \).

Outline of the proof. The first claim follows from Corollary 2.17. To prove the second claim introduce a filtration on \( \mathcal{Y}(n) \) by setting \( \text{deg}_2 \tau_{ij}^{(r)} = r - 1 \).

The corresponding graded algebra \( \text{gr}_2 \mathcal{Y}(n) \) is isomorphic to the universal enveloping algebra \( U(\mathfrak{gl}_n[x]) \) where \( \mathfrak{gl}_n[x] \) is the Lie algebra of polynomials in an indeterminate \( x \) with coefficients in \( \mathfrak{gl}_n \). Indeed, denote by \( \tau_{ij}^{(r)} \) the image of \( \tau_{ij}^{(r)} \) in the \( (r - 1) \)-th component of \( \text{gr}_2 \mathcal{Y}(n) \). Then \( E_{ij} x^{r-1} \mapsto \tau_{ij}^{(r)} \) is an algebra homomorphism \( U(\mathfrak{gl}_n[x]) \to \text{gr}_2 \mathcal{Y}(n) \). Its kernel is trivial by Theorem 2.6.

We observe now from (2.24) (with \( \rho = 1 \)) that the coefficient \( d_r \) of \( \text{qdet } T(u) \) has the form

\[ d_r = \tau_{11}^{(r)} + \cdots + \tau_{nn}^{(r)} + \text{terms of degree } < r - 1. \]

This implies that the elements \( d_r, r \geq 1 \) are algebraically independent. Furthermore, the image of \( d_r \) in the \( (r - 1) \)-th component of \( \text{gr}_2 \mathcal{Y}(n) \) coincides with \( Z x^{r-1} \) where \( Z = E_{11} + \cdots + E_{nn} \). It remains to note that the elements \( Z x^{r-1} \) with \( r \geq 1 \) generate the center of \( U(\mathfrak{gl}_n[x]) \). The latter follows from the fact that the center of \( U(\mathfrak{sl}_n[x]) \) is trivial [22]; see also [14, Proposition 2.12].
2.8 The Yangian for the special linear Lie algebra

For any simple Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ the corresponding Yangian $Y(\mathfrak{a})$ is a deformation of the universal enveloping algebra $U(\mathfrak{a}[x])$ in the class of Hopf algebras. Two different presentations of $Y(\mathfrak{a})$ are given in [35] and [37]. The type $A$ Yangian $Y(\mathfrak{sl}_n)$ can also be realized as a Hopf subalgebra of $Y(n)$, as well as a quotient of $Y(n)$.

For any series $f(u) \in 1 + u^{-1}\mathbb{C[[u^{-1}]]}$ denote by $\mu_f$ the automorphism $(2.15)$ of $Y(n)$.

**Definition 2.19** The Yangian for $\mathfrak{sl}_n$ is the subalgebra $Y(\mathfrak{sl}_n)$ of $Y(n)$ which consists of the elements stable under all automorphisms $\mu_f$.

We let $Z(n)$ denote the center of $Y(n)$.

**Theorem 2.20** The subalgebra $Y(\mathfrak{sl}_n)$ is a Hopf algebra whose coproduct, antipode and counit are obtained by restricting those from $Y(n)$. Moreover, $Y(n)$ is isomorphic to the tensor product of its subalgebras

$$Y(n) = Z(n) \otimes Y(\mathfrak{sl}_n).$$

**Outline of the proof.** It is easy to verify that there exists a unique formal power series $\tilde{d}(u)$ in $u^{-1}$ with coefficients in $Z(n)$ which begins with $1$ and satisfies

$$\tilde{d}(u) \tilde{d}(u-1) \cdots \tilde{d}(u-n+1) = q\det T(u).$$

Then by Proposition 2.12, the image of $\tilde{d}(u)$ under $\mu_f$ is given by

$$\mu_f : \tilde{d}(u) \mapsto f(u) \tilde{d}(u).$$

This implies that all coefficients of the series

$$\tau_{ij}(u) = \tilde{d}(u)^{-1} t_{ij}(u)$$

belong to the subalgebra $Y(\mathfrak{sl}_n)$. In fact, they generate this subalgebra. Furthermore, the coefficients of the series $\tilde{d}(u)$ are algebraically independent over $Y(\mathfrak{sl}_n)$ which gives $(2.27)$. Corollary 2.13 implies that

$$\Delta : \tilde{d}(u) \mapsto \tilde{d}(u) \otimes \tilde{d}(u)$$

and so the image of $Y(\mathfrak{sl}_n)$ under the coproduct is contained in $Y(\mathfrak{sl}_n) \otimes Y(\mathfrak{sl}_n)$. Using Definition 2.11, we find that the image of $q\det T(u)$ under the antipode $S$ is $(q\det T(u))^{-1}$ and so,

$$S : \tilde{d}(u)^{-1} T(u) \mapsto \tilde{d}(u) T^{-1}(u).$$

Due to $(2.28)$, $Y(\mathfrak{sl}_n)$ is stable under $S$. \square

**Corollary 2.21** The algebra $Y(\mathfrak{sl}_n)$ is isomorphic to the quotient of $Y(n)$ by the ideal generated by the center, i.e.,

$$Y(\mathfrak{sl}_n) \cong Y(n)/(q\det T(u) = 1).$$
2.9 Two more realizations of the Hopf algebra $Y(\mathfrak{sl}_2)$

**Definition 2.22** Let $A$ denote the associative unital algebra with six generators $e, f, h, J(e), J(f), J(h)$ and the defining relations

\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f, \\
[x, J(y)] = J([x, y]), \quad J(ax) = aJ(x),
\]

where $x, y \in \{e, f, h\}$, $a \in \mathbb{C}$, and

\[
[[J(e), J(f)], J(h)] = (J(e)f - eJ(f))h.
\]

The Hopf algebra structure on $A$ is defined by

\[
\Delta : x \mapsto x \otimes 1 + 1 \otimes x, \\
\Delta : J(x) \mapsto J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2} [x \otimes 1, t], \\
S : x \mapsto -x, \quad J(x) \mapsto -J(x) + x, \\
\epsilon : x \mapsto 0, \quad J(x) \mapsto 0,
\]

where $t = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$.

This definition generalizes to any simple Lie algebra $\mathfrak{a}$. Given a basis $e_1, \ldots, e_n$ of $\mathfrak{a}$, the corresponding Yangian $Y(\mathfrak{a})$ is a Hopf algebra generated by $2n$ elements $e_i, J(e_i), i = 1, \ldots, n$ as originally defined by Drinfeld [35].

**Theorem 2.23** The mapping

\[
e \mapsto t_{12}^{(1)}, \quad f \mapsto t_{21}^{(1)}, \quad h \mapsto t_{11}^{(1)} - t_{22}^{(1)},
\]

\[
J(e) \mapsto t_{12}^{(2)} - \frac{1}{2}(t_{11}^{(1)} + t_{22}^{(1)} - 1)t_{12}^{(1)}, \\
J(f) \mapsto t_{21}^{(2)} - \frac{1}{2}(t_{11}^{(1)} + t_{22}^{(1)} - 1)t_{21}^{(1)}, \\
J(h) \mapsto t_{11}^{(2)} - t_{22}^{(2)} - \frac{1}{2}(t_{11}^{(1)} + t_{22}^{(1)} - 1)(t_{11}^{(1)} - t_{22}^{(1)}),
\]

defines a Hopf algebra isomorphism $A \to Y(\mathfrak{sl}_2)$.

**Definition 2.24** Let $B$ be the associative algebra with generators $e_k, f_k, h_k$ where $k = 0, 1, 2, \ldots$ and the defining relations given in terms of the generating series

\[
e(u) = \sum_{k=0}^{\infty} e_k u^{-k-1}, \\
f(u) = \sum_{k=0}^{\infty} f_k u^{-k-1}, \\
h(u) = 1 + \sum_{k=0}^{\infty} h_k u^{-k-1}
\]

13
as follows

\[ [h(u), h(v)] = 0, \quad [e(u), f(v)] = -\frac{h(u) - h(v)}{u - v}, \]

\[ [e(u), e(v)] = -\frac{(e(u) - e(v))^2}{u - v}, \]

\[ [f(u), f(v)] = \frac{(f(u) - f(v))^2}{u - v}, \]

\[ [h(u), e(v)] = -\frac{\{h(u), e(u) - e(v)\}}{u - v}, \]

\[ [h(u), f(v)] = \frac{\{h(u), f(u) - f(v)\}}{u - v}, \]

where we have used the notation \{a, b\} = ab + ba. The Hopf algebra structure on \( \mathcal{B} \) is defined by the coproduct

\[ \Delta : e(u) \mapsto e(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k f(u + 1)^k h(u) \otimes e(u)^{k+1}, \]

\[ \Delta : f(u) \mapsto 1 \otimes f(u) + \sum_{k=0}^{\infty} (-1)^k f(u)^{k+1} \otimes h(u) \otimes e(u + 1)^k, \]

\[ \Delta : h(u) \mapsto \sum_{k=0}^{\infty} (-1)^k (k + 1) f(u + 1)^k h(u) \otimes h(u) \otimes e(u + 1)^k, \]

the antipode

\[ S : e(u) \mapsto -(h(u) + f(u + 1)e(u))^{-1} e(u), \]

\[ S : f(u) \mapsto -f(u) (h(u) + f(u)e(u + 1))^{-1}, \]

\[ S : h(u) \mapsto (h(u) + f(u + 1)e(u))^{-1}(1 - f(u + 1)(h(u) + f(u + 1)e(u))^{-1} e(u)), \]

and the counit

\[ \epsilon : e(u) \mapsto 0, \quad f(u) \mapsto 0, \quad h(u) \mapsto 1. \]

Explicitly, the defining relations of \( \mathcal{B} \) can be written in the form

\[ [h_k, h_l] = 0, \quad [e_k, f_l] = h_{k+l}, \quad [h_0, e_k] = 2e_k, \quad [h_0, f_k] = -2f_k, \]

\[ [e_{k+1}, e_l] - [e_k, e_{l+1}] = e_k e_l + e_l e_k, \]

\[ [f_{k+1}, f_l] - [f_k, f_{l+1}] = -f_k f_l - f_l f_k, \]

\[ [h_{k+1}, e_l] - [h_k, e_{l+1}] = h_k e_l + e_l h_k, \]

\[ [h_{k+1}, f_l] - [h_k, f_{l+1}] = -h_k f_l - f_l h_k. \]

Such a realization exists for an arbitrary Yangian \( Y(\mathfrak{a}) \). Some authors call it the new realization following the title of Drinfeld’s paper [37] where it was introduced.
This presentation of the Yangian is most convenient to describe its finite-dimensional irreducible representations \[37\]. However, in the case of an arbitrary simple Lie algebra \(a\) no explicit formulas for the coproduct and antipode are known.

**Theorem 2.25** The mapping

\[
\begin{align*}
    e(u) &\mapsto t_{22}(u)^{-1} t_{12}(u), \\
    f(u) &\mapsto t_{21}(u) t_{22}(u)^{-1}, \\
    h(u) &\mapsto t_{11}(u) t_{22}(u)^{-1} - t_{21}(u) t_{22}(u)^{-1} t_{12}(u) t_{22}(u)^{-1},
\end{align*}
\]

defines a Hopf algebra isomorphism \(B \rightarrow Y(\mathfrak{sl}_2)\).

Combining the two above theorems we obtain a Hopf algebra isomorphism \(A \rightarrow B\) given by

\[
\begin{align*}
    e &\mapsto e_0, \\
    f &\mapsto f_0, \\
    h &\mapsto h_0,
\end{align*}
\]

and

\[
\begin{align*}
    J(e) &\mapsto e_1 - \frac{1}{4} (e_0 h_0 + h_0 e_0), \\
    J(f) &\mapsto f_1 - \frac{1}{4} (f_0 h_0 + h_0 f_0), \\
    J(h) &\mapsto h_1 + \frac{1}{2} (e_0 f_0 + f_0 e_0 - h_0^2).
\end{align*}
\]

As we have seen in the proof of Theorem 2.18, the graded algebra \(\text{gr}_2 Y(\mathfrak{sl}_2)\) is isomorphic to the universal enveloping algebra \(U(\mathfrak{sl}_2[x])\). The images of the generators of the algebra \(A\) in the graded algebra clearly correspond to \(e, f, h, ex, fx, hx\) while the images of the generators \(e_k, f_k, h_k\) of \(B\) correspond to \(ex^k, fx^k, hx^k\).

### 2.10 Quantum Liouville formula

Here we give another family of generators of the center of \(Y(n)\). Introduce the series \(z(u)\) with coefficients from \(Y(n)\) by the formula

\[
    z(u)^{-1} = \frac{1}{n} \text{tr} \left( T(u) T^{-1}(u - n) \right),
\]

so that

\[
    z(u) = 1 + z_2 u^{-2} + z_3 u^{-3} + \cdots, \quad z_i \in Y(n).
\]

**Definition 2.26** The quantum comatrix \(\hat{T}(u)\) is defined by

\[
\hat{T}(u) T(u - n + 1) = \text{qdet} T(u).
\]
Proposition 2.27 The matrix elements $\hat{t}_{ij}(u)$ of the matrix $\hat{T}(u)$ are given by

$$\hat{t}_{ij}(u) = (-1)^{i+j} t^{1\cdots\hat{j}\cdots\hat{i}\cdots n}(u),$$

(2.31)

where the hats on the right hand side indicate the indices to be omitted. Moreover, we have the relation

$$\hat{T}^t(u - 1) T^t(u) = q\text{det } T(u).$$

(2.32)

Proof. Using Definition 2.11 we derive from (2.30)

$$A_n T_1(u) \cdots T_{n-1}(u - n + 2) = A_n \hat{T}_n(u).$$

Taking the matrix elements we come to (2.31). Further, consider the automorphism $\varphi : T(u) \mapsto T^t(-u)$ of $Y(n)$; see Proposition 2.23. Using Proposition 2.12 we find

$$\varphi : q\text{det } T(u) \mapsto q\text{det } T(-u + n - 1), \quad \varphi : \hat{T}(u) \mapsto \hat{T}^t(-u + n - 2).$$

Now applying $\varphi$ to (2.30) and replacing $-u + n - 1$ by $u$ we get (2.32). \qed

The following is a ‘quantum’ analog of the classical Liouville formula; see [119, Remark 5.8] for more comment.

Theorem 2.28 We have the relation

$$z(u) = \frac{q\text{det } T(u - 1)}{q\text{det } T(u)}.$$  

(2.33)

Proof. From (2.29) and (2.30) we find

$$z(u)^{-1} = \frac{1}{n} \text{tr} \left( T(u) \hat{T}(u - 1) (q\text{det } T(u - 1))^{-1} \right).$$

Using the centrality of $q\text{det } T(u)$ and (2.32) we get (2.33). \qed

Corollary 2.29 The coefficients $z_2, z_3, \ldots$ of $z(u)$ are algebraically independent generators of the center of $Y(n)$.

Proposition 2.30 The square of the antipode $S$ is the automorphism of $Y(n)$ given by

$$S^2 : T(u) \mapsto z(u + n) T(u + n).$$

In particular, $q\text{det } T(u)$ is stable under $S^2$. 

16
Outline of the proof. The series \( z(u) \) can be defined equivalently in a way similar to the quantum determinant; cf. Definition 2.11. Multiply the ternary relation (2.14) by \( T_2^{-1}(v) \) from both sides and take the transposition with respect to the second copy of \( \text{End} \mathbb{C}^n \). We obtain the relation

\[
R^t(u - v) \tilde{T}_2(v) T_1(u) = T_1(u) \tilde{T}_2(v) R^t(u - v),
\]

where \( \tilde{T}(v) = (T^{-1}(v))^t \) and

\[
R^t(u) = 1 - Qu^{-1}, \quad Q = \sum_{i,j=1}^n e_{ij} \otimes e_{ij}.
\]

Observe that \( Q \) is a one-dimensional operator satisfying \( Q^2 = nQ \). Therefore, \( R^t(u)^{-1} \) has a simple pole at \( u = n \) with residue \( Q \). Relation (2.34) now implies

\[
Q T_1(u) \tilde{T}_2(u - n) = \tilde{T}_2(u - n) T_1(u) Q,
\]

and this element equals \( Q z(u)^{-1} \). It now suffices to apply the antipode \( S \) to both sides of the identity \( T(u) T^{-1}(u) = 1 \). The second claim follows from Theorem 2.28.

\[
\square
\]

2.11 Factorization of the quantum determinant

Definition 2.31 Let \( X \) be a square matrix over a ring with 1. Suppose that there exists the inverse matrix \( X^{-1} \) and its \( ji \)-th entry \( (X^{-1})_{ji} \) is an invertible element of the ring. Then the \( ij \)-th quasi-determinant of \( X \) is defined by the formula

\[
|X|_{ij} = ((X^{-1})_{ji})^{-1}.
\]

For any \( 1 \leq m \leq n \) denote by \( T^{(m)}(u) \) the submatrix of \( T(u) \) corresponding to the first \( m \) rows and columns.

Theorem 2.32 The quantum determinant \( \text{qdet} T(u) \) admits the factorization in the algebra \( Y(n)[[u^{-1}]] \)

\[
\text{qdet} T(u) = t_{11}(u) |T^{(2)}(u - 1)|_{22} \cdots |T^{(n)}(u - n + 1)|_{nn}.
\]

Moreover, the factors on the right hand side are permutable.

Proof. By Definition 2.26 we have

\[
\hat{T}(u) = \text{qdet} T(u) T^{-1}(u - n + 1).
\]
Taking the \(nn\)-th entry we come to

\[ q\det T(u) \left(T^{-1}(u-n+1)\right)_{nn} = \hat{t}_{nn}(u). \]

Proposition 2.27 gives

\[ q\det T(u) = q\det T^{(n-1)}(u) |T^{(n)}(u-n+1)|_{nn}. \]

Note that the factors here commute by the centrality of the quantum determinant. An obvious induction completes the proof.

There is a generalization of this result providing a block factorization of \(q\det T(u)\).

For subsets \(P\) and \(Q\) in \(\{1,\ldots,n\}\) and an \(n\times n\)-matrix \(X\) we shall denote by \(X_{PQ}\) the submatrix of \(X\) whose rows and columns are enumerated by \(P\) and \(Q\) respectively.

Fix an integer \(0 \leq m \leq n\) and set \(A = \{1,\ldots,m\}\) and \(B = \{m+1,\ldots,n\}\). We let \(T^*(u)\) denote the matrix \(T^{-1}(-u)\).

**Theorem 2.33** We have the identity

\[ q\det T(u) q\det T^*(-u+n-1)_{AA} = q\det T(u)_{BB}. \]

We keep the notation \(t'_{ij}(u)\) for the matrix elements of the matrix \(T^{-1}(u)\).

**Proposition 2.34** We have the relations

\[ [t_{ij}(u), t_{kl}'(v)] = \frac{1}{u-v} \left( \delta_{kj} \sum_{a=1}^{n} t_{ia}(u) t_{al}'(v) - \delta_{il} \sum_{a=1}^{n} t_{ka}'(v) t_{aj}(u) \right). \]

In particular, the matrix elements of the matrices \(T(u)_{AA}\) and \(T^*(v)_{BB}\) commute with each other.

**Proof.** It suffices to multiply the ternary relation (2.14) by \(T^{-1}_2(v)\) from both sides and equate the matrix elements. \(\square\)

### 2.12 Quantum Sylvester theorem

The following commutation relations between the quantum minors generalize Proposition 2.16.

**Proposition 2.35** We have the relations

\[
[t_{b_1 \cdots b_k}(u), t_{d_1 \cdots d_l}'(v)] = \sum_{p=1}^{\min\{k,l\}} \frac{(-1)^{p-1} p!}{(u-v-k+1) \cdots (u-v-k+p)} \sum_{i_1 < \cdots < i_p \atop j_1 < \cdots < j_p} \left( t_{b_1 \cdots b_k}(u) t_{c_1 \cdots c_l}(v) - t_{d_1 \cdots d_l}(v) t_{a_1 \cdots a_k}(u) \right).
\]
Here the \( p \)-tuples of upper indices \((a_{i_1}, \ldots, a_{i_p})\) and \((c_{j_1}, \ldots, c_{j_p})\) are respectively interchanged in the first summand on the right hand side while the \( p \)-tuples of lower indices \((b_{i_1}, \ldots, b_{i_p})\) and \((d_{j_1}, \ldots, d_{j_p})\) are interchanged in the second summand.

As in the previous section, we fix an integer \( m \) satisfying \( 1 \leq m \leq n \). For any indices \( 1 \leq i, j \leq m \) introduce the following series with coefficients in \( Y(n) \)

\[
\tilde{t}_{ij}(u) = t^{i,m+1 \cdots n}_{j,m+1 \cdots n}(u)
\]

and combine them into the matrix \( \tilde{T}(u) = (\tilde{t}_{ij}(u)) \). The following is an analog of the classical Sylvester theorem. As before, we denote \( \mathcal{B} = \{m+1, \ldots, n\} \).

**Theorem 2.36** The mapping

\[
t_{ij}(u) \mapsto \tilde{t}_{ij}(u), \quad 1 \leq i, j \leq m,
\]

defines an algebra homomorphism \( Y(m) \to Y(n) \). Moreover, one has the identity

\[
q\det \tilde{T}(u) = q\det T(u) q\det T(u - 1)_{BB} \cdots q\det T(u - m + 1)_{BB}.
\]

(2.37)

**Outline of the proof.** Using Proposition 2.35 we check that the series \( \tilde{t}_{ij}(u) \) satisfy the Yangian defining relations (2.3) which proves the first claim. The identity (2.37) is derived by induction on \( m \) from the relation

\[
\hat{T}(u)_{AA} \tilde{T}(u - m + 1) = q\det T(u) q\det T(u - m + 1)_{BB},
\]

where \( \hat{T}(u) \) is the quantum comatrix; see Definition 2.26.

\( \square \)

### 2.13 The centralizer construction

Fix a nonnegative integer \( m \) and for any \( n \geq m \) denote by \( \mathfrak{g}_m(n) \) the subalgebra in \( \mathfrak{gl}_n \) spanned by the basis elements \( E_{ij} \) with \( m + 1 \leq i, j \leq n \). The subalgebra \( \mathfrak{g}_m(n) \) is isomorphic to \( \mathfrak{gl}_{n-m} \). Let \( A_m(n) \) denote the centralizer of \( \mathfrak{g}_m(n) \) in the universal enveloping algebra \( A(n) = U(\mathfrak{gl}_n) \). Let \( A(n)^0 \) denote the centralizer of \( E_{nn} \) in \( A(n) \) and let \( I(n) \) be the left ideal in \( A(n) \) generated by the elements \( E_{in}, i = 1, \ldots, n \). Then \( I(n)^0 = I(n) \cap A(n)^0 \) is a two-sided ideal in \( A(n)^0 \) and one has a vector space decomposition

\[
A(n)^0 = I(n)^0 \oplus A(n - 1).
\]

Therefore the projection of \( A(n)^0 \) onto \( A(n - 1) \) with the kernel \( I(n)^0 \) is an algebra homomorphism. Its restriction to the subalgebra \( A_m(n) \) defines a filtration preserving homomorphism

\[
\pi_n : A_m(n) \to A_m(n - 1)
\]

(2.38)
so that one can define the algebra \( A_m \) as the projective limit with respect to this sequence of homomorphisms in the category of filtered algebras.

By the Harish-Chandra isomorphism [34, Section 7.4], the center \( A_0(n) \) of \( U(\mathfrak{gl}_n) \) is naturally isomorphic to the algebra \( \Lambda^*(n) \) of polynomials in \( n \) variables \( \lambda_1, \ldots, \lambda_n \) which are symmetric in the shifted variables \( \lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1 \). So, in the case \( m = 0 \) the homomorphisms \( \pi_n \) are interpreted as the specialization homomorphisms \( \pi_n : \Lambda^*(n) \to \Lambda^*(n - 1) \) such that

\[
\pi_n : f(\lambda_1, \ldots, \lambda_n) \mapsto f(\lambda_1, \ldots, \lambda_{n-1}, 0).
\]

(2.39)

The corresponding projective limit in the category of filtered algebras is called the algebra of shifted symmetric functions and denoted by \( \Lambda^* \). The elements of \( \Lambda^* \) are well-defined functions on the set of all sequences \( \lambda = (\lambda_1, \lambda_2, \ldots) \) which contain only a finite number of nonzero terms. The following families of elements of \( \Lambda^* \) are analogs of power sums, elementary symmetric functions and complete symmetric functions:

\[
p_m(\lambda) = \sum_{k=1}^{\infty} ((\lambda_k - k)^m - (-k)^m), \quad m = 1, 2, \ldots,
\]

\[
1 + \sum_{m=1}^{\infty} e_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 + (\lambda_k - k)t}{1 - kt},
\]

\[
1 + \sum_{m=1}^{\infty} h_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 + kt}{1 - (\lambda_k - k)t}.
\]

Each of the families \( \{p_m\}, \{e_m\}, \{h_m\} \) can be taken as a system of algebraically independent generators of the algebra \( \Lambda^* \). To summarize, we have the following.

**Proposition 2.37** The algebra \( A_0 \) is isomorphic to the algebra of shifted symmetric functions \( \Lambda^* \).

Now consider the homomorphism (2.17) and take its composition with an automorphism of \( Y(n) \) given by (2.16) to yield another homomorphism \( \varphi_n : Y(n) \to A(n) \) such that

\[
\varphi_n : T(u) \mapsto \left(1 - \frac{E}{u + n}\right)^{-1}.
\]

It follows from the defining relations (2.3) that the image of the restriction of \( \varphi_n \) to the subalgebra \( Y(m) \) is contained in the centralizer \( A_m(n) \) thus yielding a homomorphism \( \varphi_n : Y(m) \to A_m(n) \).

**Theorem 2.38** For any fixed \( m \geq 1 \) the sequence \( (\varphi_n \mid n \geq m) \) defines an algebra embedding \( \varphi : Y(m) \hookrightarrow A_m \). Moreover, one has an isomorphism

\[
A_m = A_0 \otimes Y(m),
\]

(2.40)

where \( Y(m) \) is identified with its image under the embedding \( \varphi \).
Outline of the proof. First we verify that the family \( (\varphi_n \mid n \geq m) \) is compatible with the chain of homomorphisms (2.38). Further, to prove the injectivity of \( \varphi \) we consider the corresponding commutative picture replacing \( A(n) \) with its graded algebra \( \text{gr} A(n) \cong S(\mathfrak{gl}_n) \). This reduces the task to the description of the invariants in the symmetric algebra \( S(\mathfrak{gl}_n) \) with respect to the action of the group \( GL(n - m) \) corresponding to the Lie algebra \( \mathfrak{g}_m(n) \). This can be done with the use of the classical invariant theory [172] which also implies the decomposition (2.40).

A different embedding \( Y(m) \hookrightarrow A_m \) can be constructed with the use of the quantum Sylvester theorem; see Section 2.12. Consider the homomorphism \( Y(m) \to Y(n) \) provided by Theorem 2.36 and take its composition with (2.7). We obtain an algebra homomorphism \( \psi_n : Y(m) \to A(n) \) given by
\[
\psi_n : t_{ij}(u) \mapsto q\det (1 + Eu^{-1})_{B_i B_j},
\]
where \( B_i \) denotes the set \( \{i, m + 1, \ldots, n\} \). Corollary 2.17 implies that this image commutes with the elements of the subalgebra \( \mathfrak{g}_m(n) \) and so, (2.41) defines a homomorphism \( \psi_n : Y(m) \to A_m(n) \). Furthermore, the family of homomorphisms \( (\psi_n \mid n \geq m) \) is obviously compatible with the projections (2.38) and thus defines an algebra homomorphism \( \psi : Y(m) \to A_m \). Denote by \( \tilde{A}_0 \) the projective limit of the sequence of the centers of the universal enveloping algebras \( U(\mathfrak{g}_m(n)) \), where \( n = m, m + 1, \ldots \), defined by the corresponding homomorphisms (2.39). By Proposition 2.37, \( \tilde{A}_0 \) is isomorphic to the algebra of shifted symmetric functions in the variables \( \lambda_{m+1}, \lambda_{m+2}, \ldots \). The following is an analog of Theorem 2.38.

**Theorem 2.39** The homomorphism \( \psi : Y(m) \to A_m \) is injective. Moreover, one has an isomorphism
\[
A_m = \tilde{A}_0 \otimes Y(m),
\]
where \( Y(m) \) is identified with its image under the embedding \( \psi \).

### 2.14 Commutative subalgebras

Here we use the notation introduced in Section 2.2. Consider the algebra (2.9) with \( m = n \). Fix an \( n \times n \) matrix \( C \) with entries in \( \mathbb{C} \) and for any \( 1 \leq k \leq n \) introduce the series \( \tau_k(u, C) \) with coefficients in \( Y(n) \) by
\[
\tau_k(u, C) = \text{tr} A_n T_1(u) \cdots T_k(u - k + 1) C_{k+1} \cdots C_n,
\]
where \( A_n \) is the antisymmetrizer defined by (2.21) and the trace is taken over all \( n \) copies of \( \text{End} \mathbb{C}^n \).
Theorem 2.40 All the coefficients of the series $\tau_1(u,C), \ldots, \tau_n(u,C)$ commute with each other. Moreover, if the matrix $C$ has simple spectrum then the coefficients at $u^{-1}, u^{-2}, \ldots$ of these series are algebraically independent and generate a maximal commutative subalgebra of the Yangian $Y(n)$.

Consider the epimorphism $\pi: Y(n) \to U(\mathfrak{gl}_n)$ defined in (2.7). Clearly, the coefficients of the images of the series $\tau_k(u,C), k = 1, \ldots, n$ under $\pi$ form a commutative subalgebra $C \subseteq U(\mathfrak{gl}_n)$.

Theorem 2.41 If the matrix $C$ has simple spectrum then the subalgebra $C$ of $U(\mathfrak{gl}_n)$ is maximal commutative.

Bibliographical notes

2.2. For the origins of the RTT relation and associated $R$-matrix formalism see for instance the papers Takhtajan–Faddeev [161], Kulish–Sklyanin [89], Reshetikhin–Takhtajan–Faddeev [149]. The statistical mechanics background of quantum groups is also explained in the book by Chari and Pressley [25, Chapter 7].

2.4. The Poincaré–Birkhoff–Witt theorem for general Yangians is due to Drinfeld (unpublished). Another proof was given by Levendorskiı̆ [97]. The details of the proof outlined here can be found in [119]. It follows the approach of Olshanskiı̆ [138].

2.6. The definition of the quantum determinant $q\det T(u)$ (in the case $n = 2$) originally appeared in Izergin–Korepin [63]. The basic ideas and formulas associated with the quantum determinant for an arbitrary $n$ are contained in Kulish–Sklyanin’s survey paper [89]. Detailed proofs are given in [119]. Proposition 2.14 is contained e.g. in Iohara [59] and Nazarov–Tarasov [130].

2.8. By the general approach of Drinfeld [38], the Yangian for $\mathfrak{sl}_n$ should be defined as a quotient algebra of $Y(n)$. The fact that it can also be realized as a (Hopf) subalgebra of $Y(n)$ was observed by Olshanskiı̆ [119].

2.9. For any simple Lie algebra $\mathfrak{a}$ the Yangian $Y(\mathfrak{a})$ was defined by Drinfeld [35, 37]. The two definitions given here are particular cases for $\mathfrak{a} = \mathfrak{sl}_2$. The formulas for the coproduct and antipode in Definition 2.24 are due to the author; see e.g. Khoroshkin–Tolstoy [70]. These formulas were employed in [70] in the construction of the double of the Yangian. These results were generalized to the Yangian $Y(\mathfrak{sl}_3)$ by Soloviev [154], and to $Y(\mathfrak{sl}_n)$ by Iohara [59].

2.10. The series $z(u)$ was introduced by Nazarov [124]. The quantum Liouville formula is also due to him. The argument given in Section 2.10 is a simplified version of his $R$-matrix proof [119].

2.11. A general theory of quasi-determinants of matrices over noncommutative rings is developed by Gelfand and Retakh [14, 15]. Various analogs of the classical theorems for such determinants are given. Quasi-determinant factorizations of quantum
determinants for the quantized algebra $GL_q(n)$ are obtained in those papers; see also Krob and Leclerc [84]. In a more general context of Hopf algebras such factorizations are constructed by Etingof and Retakh [41].

2.12. An analog of Sylvester’s theorem for the algebra $GL_q(n)$ was given by Krob and Leclerc [84] with the use of the quasi-determinant version of this theorem due to Gelfand and Retakh [17]. The approach of [84] works for the Yangians as well. The proof outlined here follows [117] where a proof of Proposition 2.35 is given. The latter result and some other quantum minor relations are known to specialists as ‘folklore theorems’. Some more quantum analogs of the classical minor relations are collected in Iohara [59].

2.13. The centralizer construction is due to Olshanski [136, 138]. The modified version (Theorem 2.39) based on the quantum Sylvester theorem is given in [117]. The algebra $\Lambda^*$ of shifted symmetric functions is studied in detail by Okounkov and Olshanski [134].

2.14. The commutative subalgebras in the Yangian originate from the integrable models in statistical mechanics (specifically, from the six vertex or XXX model); see Baxter [9]. The common eigenvectors of such a commutative subalgebra in certain standard Yangian modules can be constructed by a special procedure called the algebraic Bethe ansatz; see Faddeev [42], Kulish–Sklyanin [89], Kulish–Reshetikhin [85], Kirillov–Reshetikhin [75, 76]. Theorems 2.40 and 2.41 are proved by Nazarov and Olshanski [128].

Finite-dimensional irreducible representations of the Yangians were classified by Drinfeld [37] with the use of the particular case of $Y(gl_2)$ considered earlier by Tarasov [157, 158]. A detailed exposition of these results for $Y(gl_n)$ is contained in [112]. Cherednik [29, 30] used the Yangians to ‘materialize’ the second Weyl character formula. Nazarov [124, 126] and Okounkov [133] employed the Yangian techniques to obtain remarkable immanant analogs of the classical Capelli identity [19, 20]. An explicit construction of all representations of $Y(gl_2)$ as tensor products of the evaluation modules is given by Tarasov [158] and Chari–Pressley [23, 26]. In particular, this provides an irreducibility criterion of tensor products of the $Y(gl_2)$ evaluation modules. A generalization of this criterion to $Y(gl_n)$ with an arbitrary $n$ was given in [116]; see also Leclerc–Nazarov–Thibon [96]. An important part of the criterion is the binary property established by Nazarov and Tarasov [132]; see also Kitanine, Maillet and Terras [68, 103]. A character formula for an arbitrary finite-dimensional irreducible representation of $Y(gl_n)$ is given by Arakawa [3] with the use of the Drinfeld functor; see Drinfeld [36]. This formula is also implied by the earlier results of Ginzburg–Vasserot [19] combined with the work of Lusztig; see Nakajima [122], Varagnolo [171]. The irreducible characters are expressed in terms of those for the ‘standard tensor product modules’ via the Kazhdan–Lusztig polynomials. Bases of Gelfand–Tsetlin type for ‘generic’ representations of $Y(gl_n)$
are constructed in [107]. More general class of ‘tame’ Yangian modules was introduced and explicitly constructed by Nazarov and Tarasov [130]. The earlier works of Nazarov and Tarasov [129] and the author [107] provide different constructions of the well-known Gelfand–Tsetlin basis for representations of $\mathfrak{g}l_n$. A surprising connection of the Yangian $Y(\mathfrak{g}l_n)$ with the finite $W$-algebras was discovered by Ragoucy and Sorba [145, 146]; see also Briot and Ragoucy [18]. The Yangian actions on certain modules over the affine Lie algebras were constructed by Uglov [167].

3 The twisted Yangians

Here we describe the structure of the twisted Yangians corresponding to the orthogonal and symplectic Lie algebras $\mathfrak{o}_N$ and $\mathfrak{sp}_N$. We consider both cases simultaneously, unless otherwise stated.

3.1 Defining relations

Given a positive integer $N$, we number the rows and columns of $N \times N$ matrices by the indices $\{-n, \ldots, -1, 0, 1, \ldots n\}$ if $N = 2n + 1$, and by $\{-n, \ldots, -1, 1, \ldots n\}$ if $N = 2n$. Similarly, in the latter case the range of indices $-n \leq i, j \leq n$ will exclude 0. It will be convenient to use the symbol $\theta_{ij}$ which is defined by

$$\theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case,} \\ \text{sgn} i \cdot \text{sgn} j & \text{in the symplectic case.} \end{cases}$$

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the orthogonal case and the lower sign to the symplectic case. By $X \mapsto X^t$ we will denote the matrix transposition such that for the matrix units we have

$$(e_{ij})^t = \theta_{ij} e_{-j,-i}. \quad (3.1)$$

Introduce the following elements of the Lie algebra $\mathfrak{gl}_N$:

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i}, \quad -n \leq i, j \leq n.$$

The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij}$ is a realization of a simple Lie algebra $\mathfrak{g}_n$ of rank $n$ (see e.g. [58]). In the orthogonal case $\mathfrak{g}_n$ is of type $D_n$ or $B_n$ if $N = 2n$ or $N = 2n + 1$, respectively. In the symplectic case, $N = 2n$ and $\mathfrak{g}_n$ is of type $C_n$. Thus,

$$\mathfrak{g}_n = \mathfrak{o}_{2n}, \mathfrak{o}_{2n+1} \text{ or } \mathfrak{sp}_{2n}.$$ 

**Definition 3.1** Each of the twisted Yangians $Y^+(2n)$, $Y^+(2n+1)$ and $Y^-(2n)$ corresponding to the Lie algebras $\mathfrak{o}_{2n}$, $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$, respectively, is a unital associative
algebra with generators $s_{ij}^{(1)}$, $s_{ij}^{(2)}$, ... where $-n \leq i, j \leq n$, and the defining relations written in terms of the generating series

$$s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots$$

as follows

$$(u^2 - v^2) [s_{ij}(u), s_{kl}(v)] = (u + v) \left( s_{kj}(u) s_{il}(v) - s_{kl}(v) s_{ij}(u) \right) - (u - v) \left( \theta_{k,-j} s_{i,-k}(u) s_{j,-l}(v) - \theta_{i,-l} s_{k,-i}(v) s_{-l,j}(u) \right) + \theta_{i,-j} \left( s_{k,-i}(u) s_{-j,l}(v) - s_{k,-i}(v) s_{-j,l}(u) \right)$$

and

$$\theta_{ij} s_{-j,-i}(-u) = s_{ij}(u) \pm \frac{s_{ij}(u) - s_{ij}(-u)}{2u}.$$ (3.2)

These relations can be conveniently presented in an equivalent matrix form analogous to (2.14). For this introduce the transposed $R^t(u)$ for the Yang $R$-matrix (2.12) (correcting the indices for the matrix elements appropriately) by

$$R^t(u) = 1 - Qu^{-1}, \quad Q = \sum_{i,j=-n}^n e^t_{ij} \otimes e_{ji}.$$ 

Furthermore, denote by $S(u)$ the $N \times N$ matrix whose $ij$-th entry is $s_{ij}(u)$. As in (2.8) we regard $S(u)$ as an element of the algebra $Y^\pm(N) \otimes \text{End} \mathbb{C}^N$ given by

$$S(u) = \sum_{i,j=-n}^n s_{ij}(u) \otimes e_{ij}.$$ 

**Proposition 3.2** The defining relations for the twisted Yangian $Y^\pm(N)$ are equivalent to the quaternary relation

$$R(u - v) S_1(u) R^t(-u - v) S_2(v) = S_2(v) R^t(-u - v) S_1(u) R(u - v)$$ (3.3)

and the symmetry relation

$$S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}.$$ (3.4)

The following relation between the twisted Yangians and the corresponding classical Lie algebras plays a key role in many applications; cf. Proposition 2.2.

**Proposition 3.3** The mapping

$$\pi : s_{ij}(u) \mapsto \delta_{ij} + F_{ij} \left( u \pm \frac{1}{2} \right)^{-1}$$ (3.5)

defines an algebra epimorphism $Y^\pm(N) \to U(\mathfrak{g}_n)$. Moreover,

$$F_{ij} \mapsto s_{ij}^{(1)}$$

is an embedding $U(\mathfrak{g}_n) \hookrightarrow Y^\pm(N)$. 

25
3.2 Embedding into the Yangian

We keep the notation \( t^{(r)}_{ij} \) for the generators of the Yangian \( Y(N) \). However, in accordance with the above, we now let the indices \( i, j \) run over the set \( \{-n, \ldots, n\} \). Also, the matrix transposition is now understood in the sense (3.1).

**Theorem 3.4** The mapping

\[
S(u) \mapsto T(u) T^t(-u)
\]

defines an embedding \( Y\pm(N) \hookrightarrow Y(N) \).

**Outline of the proof.** It is straightforward to verify that the matrix \( T(u) T^t(-u) \) satisfies both relations (3.3) and (3.4). To show that the homomorphism (3.6) is injective we use the corresponding homomorphism of the graded algebras \( \text{gr}_1 Y\pm(N) \to \text{gr}_1 Y(N) \), where \( \text{gr}_1 Y\pm(N) \) is defined by setting \( \text{deg}_1 s^{(r)}_{ij} = 1 \). Then apply Corollary 2.7.

This result allows us to regard the twisted Yangian as a subalgebra of \( Y(N) \). The following is an analog of the Poincaré–Birkhoff–Witt theorem for the algebra \( Y\pm(N) \).

**Corollary 3.5** Given an arbitrary linear order on the set of the generators

\[
s^{(2k)}_{ij}, \quad i + j \leq 0; \quad s^{(2k-1)}_{ij}, \quad i + j < 0; \quad k = 1, 2, \ldots,
\]
in the case of \( Y^+(N) \), and the set of the generators

\[
s^{(2k)}_{ij}, \quad i + j < 0; \quad s^{(2k-1)}_{ij}, \quad i + j \leq 0; \quad k = 1, 2, \ldots,
\]
in the case of \( Y^-(N) \), any element of the algebra \( Y\pm(N) \) is uniquely written as a linear combination of ordered monomials in the generators.

**Proposition 3.6** The subalgebra \( Y\pm(N) \) is a left coideal of \( Y(N) \),

\[
\Delta(Y\pm(N)) \subseteq Y(N) \otimes Y\pm(N)
\]

**Proof.** This follows from the explicit formula

\[
\Delta : s_{ij}(u) \mapsto \sum_{a,b=-n}^n \theta_{bj} t_{ia}(u) t_{-j,-b}(-u) \otimes s_{ab}(u).
\]

The restrictions of some of the automorphisms and anti-automorphisms of \( Y(N) \) described in Section 2.3 preserve the subalgebra \( Y\pm(N) \) and so we have the following.

**Proposition 3.7** The mapping

\[
S(u) \mapsto S^t(u)
\]
defines an anti-automorphism of \( Y\pm(N) \). For any formal series \( g(u) \in 1 + u^{-2} \mathbb{C}[[u^{-2}]] \) the mapping

\[
S(u) \mapsto g(u) S(u)
\]
defines an automorphism of \( Y\pm(N) \).
3.3 Sklyanin determinant

Here we use the notation of Section 2.6 with the usual convention on the matrix element indices. Define $R(u_1, \ldots, u_m)$ by (2.20) and set

$$S_i = S_i(u_i), \quad 1 \leq i \leq m \quad \text{and} \quad R_{ij}^t = R_{ji}^t = R_{ij}^t(-u_i - u_j), \quad 1 \leq i < j \leq m,$$

where $S_a(u)$ and $R_{ab}^t(u)$ are defined by an obvious analogy with (2.10) and (2.11).

For an arbitrary permutation $(p_1, \ldots, p_m)$ of the numbers $1, \ldots, m$, we abbreviate

$$\langle S_{p_1}, \ldots, S_{p_m} \rangle = S_{p_1}(R_{p_1p_2}^t \cdots R_{p_1p_m}^t)S_{p_2}(R_{p_2p_3}^t \cdots R_{p_2p_m}^t) \cdots S_{p_m}.$$

Proposition 3.8 We have the identity

$$R(u_1, \ldots, u_m) \langle S_1, \ldots, S_m \rangle = \langle S_m, \ldots, S_1 \rangle R(u_1, \ldots, u_m).$$

Now take $m = N$ and specialize the variables $u_i$ as

$$u_i = u - i + 1, \quad i = 1, \ldots, N.$$

By Propositions 2.10 and 3.8 we have

$$A_N \langle S_1, \ldots, S_N \rangle = \langle S_N, \ldots, S_1 \rangle A_N. \quad (3.8)$$

Definition 3.9 The Sklyanin determinant is the formal series

$$\text{sdet } S(u) = 1 + c_1 u^{-1} + c_2 u^{-2} + \cdots \in Y^\pm(N)[[u^{-1}]]$$

such that the element (3.8) equals $A_N \text{sdet } S(u)$.

In the next theorem we regard $Y^\pm(N)$ as a subalgebra of $Y(N)$.

Theorem 3.10 We have

$$\text{sdet } S(u) = \gamma_n(u) \text{qdet } T(u) \text{qdet } T(-u + N - 1),$$

where

$$\gamma_n(u) = \begin{cases} 1 & \text{for } Y^+(N), \\ 2u + 1 & \text{for } Y^-(2n). \\ 2u - 2n + 1 & \end{cases}$$

There is an explicit formula for the Sklyanin determinant in terms of the generators $s_{ij}(u)$. It uses a special map of the symmetric groups

$$\pi_N : S_N \rightarrow S_N, \quad p \mapsto p'.$$
defined by an inductive procedure. First of all, \( p'(N) = N \) so that \( p' \) can be regarded as an element of \( \mathfrak{S}_{N-1} \). Given a set of positive integers \( \omega_1 < \cdots < \omega_N \) we regard \( \mathfrak{S}_N \) as the group of their permutations. If \( N = 2 \) we define \( \pi_2 \) as the only map \( \mathfrak{S}_2 \to \mathfrak{S}_1 \). For \( N > 2 \) define a map from the set of ordered pairs \((\omega_k, \omega_l)\) with \( k \neq l \) into itself by the rule

\[
\begin{align*}
(\omega_k, \omega_l) & \mapsto (\omega_l, \omega_k), & k, l < N, \\
(\omega_k, \omega_N) & \mapsto (\omega_{N-1}, \omega_k), & k < N-1, \\
(\omega_N, \omega_k) & \mapsto (\omega_k, \omega_{N-1}), & k < N-1, \\
(\omega_{N-1}, \omega_N) & \mapsto (\omega_{N-1}, \omega_{N-2}), \\
(\omega_N, \omega_{N-1}) & \mapsto (\omega_{N-1}, \omega_{N-2}).
\end{align*}
\tag{3.10}
\]

Let \( p = (p_1, \ldots, p_N) \) be a permutation of the indices \( \omega_1, \ldots, \omega_N \). Define its image \( p' = (q_1, \ldots, q_{N-1}) \) under the map \( \pi_N \) as follows. First take \( (q_1, q_{N-1}) \) as the image of the ordered pair \((p_1, p_N)\) under the map \([3.10]\). Then define \((q_2, \ldots, q_{N-2})\) as the image of \((p_2, \ldots, p_{N-1})\) under \( \pi_{N-2} \) where \( \mathfrak{S}_{N-2} \) is regarded as the group of permutations of the family of indices obtained from \((\omega_1, \ldots, \omega_N)\) by deleting \( p_1 \) and \( p_N \). To describe the combinatorial properties of the map \( \pi_N \) introduce the signless Stirling numbers of the first kind \( c(N, k) \) by

\[
\sum_{k=1}^{N} c(N, k) x^k = x(x+1) \cdots (x+N-1).
\]

We also need the Boolean posets \( B_n \). The elements of \( B_n \) are subsets of \( \{1, \ldots, n\} \) with the usual set inclusion as the partial ordering. Equip the symmetric group \( \mathfrak{S}_N \) with the standard Bruhat order.

**Theorem 3.11** Each fiber of the map \( \pi_N : \mathfrak{S}_N \to \mathfrak{S}_N \) is an interval in \( \mathfrak{S}_N \) with respect to the Bruhat order, isomorphic to the Boolean poset \( B_k \) for some \( k \). Moreover, for any \( k \) the number of intervals isomorphic to \( B_k \) is the Stirling number \( c(N-1, k) \).

Below we denote the matrix elements of the transposed matrix \( S^t(u) \) by \( s_{ij}^t(u) \). For any permutation \( p \in \mathfrak{S}_N \) we denote by \( p' \) its image under the map \( \pi_N \).

**Theorem 3.12** Let \((a_1, \ldots, a_N)\) be an arbitrary permutation of the set of indices \((-n, -n+1, \ldots, n)\). Then

\[
\text{sdet} S(u) = (-1)^n \gamma_n(u) \sum_{p \in \mathfrak{S}_N} \text{sgn} pp' \cdot s_{-a_{p(1)}, a_{p(1)}}^t (-u) \cdots s_{-a_{p(n)}, a_{p(n)}}^t (-u+n-1) \\
\times s_{-a_{p(n+1)}, a_{p(n+1)}} (u-n) \cdots s_{-a_{p(N)}, a_{p(N)}} (u-N+1)
\]

and also

\[
\begin{align*}
&= (-1)^n \gamma_n(u) \sum_{p \in \mathfrak{S}_N} \text{sgn} pp' \cdot s_{-a_{p(1)}, a_{p(1)}} (u-N+1) \cdots s_{-a_{p(n)}, a_{p(n)}} (u-N+n) \\
&\quad \times s_{-a_{p(n+1)}, a_{p(n+1)}} (-u+N-n-1) \cdots s_{-a_{p(N)}, a_{p(N)}} (-u)\end{align*}
\]

p. 28
Outline of the proof. The key idea is to use the symmetry relation (3.4) in order to eliminate the intermediate factors $R_{ij}^t$ in the expression (3.8) which defines the Sklyanin determinant.

Example 3.13 In the case $N = 2$ we have

$$\text{sdet } S(u) = \frac{2u + 1}{2u \pm 1} \left( s_{-1,-1}(u-1)s_{-1,-1}(-u) \mp s_{1,1}(u-1) + s_{1,1}(-u) \right).$$

3.4 The center of the twisted Yangian

Theorem 3.10 implies the following relation between the coefficients of the Sklyanin determinant

$$\gamma_n(u) \text{sdet } S(-u + N - 1) = \gamma_n(-u + N - 1) \text{sdet } S(u).$$

In particular, the odd coefficients of sdet $S(u)$ can be expressed in terms of the even.

Theorem 3.14 All the coefficients of the series sdet $S(u)$ belong to the center of the algebra $Y^{\pm}(N)$. Moreover, the even coefficients $c_2, c_4, \ldots$ are algebraically independent and generate the center of $Y^{\pm}(N)$.

Outline of the proof. The first assertion is immediate from Theorems 2.18 and 3.10. Alternatively, one can also prove the centrality of sdet $S(u)$ by using the matrix form of the defining relations of $Y^{\pm}(N)$. To prove the second assertion introduce a filtration on $Y^{\pm}(N)$ by setting $\text{deg}_2 s_{ij}^{(r)} = r - 1$. To describe the corresponding graded algebra $\text{gr}_2 Y^{\pm}(N)$ consider the involution $\sigma$ on the Lie algebra $\mathfrak{gl}_N$ defined by $\sigma : A \mapsto -A^t$. Then the fixed point subalgebra is the classical Lie algebra $\mathfrak{g}_n$. Introduce the twisted polynomial current Lie algebra $\mathfrak{gl}_N[x]^{\sigma}$ by (1.1). Then we have an algebra isomorphism

$$\text{gr}_2 Y^{\pm}(N) \cong U(\mathfrak{gl}_N[x]^{\sigma}).$$

Identifying these algebras we find that the image of the coefficient $c_{2m}$ in the $(2m - 1)$-st component of $\text{gr}_2 Y^{\pm}(N)$ coincides with $Z x^{2m-1}$ where

$$Z = E_{-n,-n} + E_{-n+1,-n+1} + \cdots + E_{n,n}.$$

To complete the proof of the theorem we use the fact that the center of the algebra $U(\mathfrak{gl}_N[x]^{\sigma})$ is generated by the elements $Z x^{2m-1}$ with $m \geq 1$. ☐
3.5 The special twisted Yangian

In the next definition we regard $Y^\pm(N)$ as a subalgebra of the Yangian $Y(N)$. Recall the definition of the Yangian $Y(sl_N)$ from Section 2.8.

**Definition 3.15** The special twisted Yangian $SY^\pm(N)$ is the subalgebra of $Y(N)$ defined by

$$SY^\pm(N) = Y(sl_N) \cap Y^\pm(N).$$

Equivalently, $SY^\pm(N)$ is the subalgebra of $Y^\pm(N)$ which consists of the elements stable under all automorphisms of the form (3.7). The following result is implied by Theorem 2.20.

**Theorem 3.16** The algebra $Y^\pm(N)$ is isomorphic to the tensor product of its center $Z^\pm(N)$ and the subalgebra $SY^\pm(N)$,

$$Y^\pm(N) = Z^\pm(N) \otimes SY^\pm(N).$$

In particular, the center of $SY^\pm(N)$ is trivial.

**Corollary 3.17** The subalgebra $SY^\pm(N)$ of $Y(sl_N)$ is a left coideal.

3.6 The quantum Liouville formula

Define the series $\zeta(u)$ with coefficients in $Y^\pm(N)$ by

$$\zeta(u) - 1 = \frac{1}{N} \text{tr} \left\{ \left( \frac{2u - N}{2u - N + 1} S^t(-u) \pm \frac{1}{2u - N + 1} S(-u) \right) S^{-1}(u - N) \right\}.$$

**Theorem 3.18** We have the relation

$$\zeta(u) = \varepsilon_n(u) \frac{s\text{det} S(u - 1)}{s\text{det} S(u)},$$

where $\varepsilon_n(u) = \gamma_n(u) \gamma_n(u - 1)^{-1}$.

**Outline of the proof.** The quaternary relation (3.3) implies

$$QS_i^{-1}(-u) R(2u - N) S_2(u - N) = S_2(u - N) R(2u - N) S_i^{-1}(-u) Q.$$

It is deduced from the definition of $\zeta(u)$ that this expression coincides with $\zeta(u)Q$ up to a scalar function. Combining this with the matrix definition (2.33) of $z(u)$ we obtain

$$\zeta(u) = z(u) z(-u + N)^{-1}.$$

Now the claim follows from Theorems 2.28 and 3.10.

**Corollary 3.19** The coefficients of the series $\zeta(u)$ generate the center of $Y^\pm(N)$. 
3.7 Factorization of the Sklyanin determinant

Let $1 \leq m \leq n$. Denote by $S^{(m)}(u)$ the submatrix of $S(u)$ corresponding the rows and columns enumerated by $-m, -m + 1,\ldots,m$, and by $\tilde{S}^{(m)}(u)$ the submatrix of $S^{(m)}(u)$ obtained by removing the row and column enumerated by $-m$. Set

$$c(u) = \frac{1}{\gamma_n(u + N/2 - 1/2)} \operatorname{sdet} S(u + N/2 - 1/2).$$

Then by Theorem 3.10, $c(u)$ is an even formal series in $u^{-1}$, with coefficients in the center of the $Y^{\pm}(N)$. We shall use the quasi-determinants introduced in (2.31).

**Theorem 3.20** If $N = 2n$ then

$$c(u) = |\tilde{S}^{(1)}(-u - 1/2)|_{11} \cdot |S^{(1)}(u - 1/2)|_{11} \cdot \cdots |\tilde{S}^{(n)}(-u - n + 1/2)|_{nn} \cdot |S^{(n)}(u - n + 1/2)|_{nn}.$$

If $N = 2n + 1$ then

$$c(u) = s_{00}(u) \cdot |\tilde{S}^{(1)}(-u - 1)|_{11} \cdot |S^{(1)}(u - 1)|_{11} \cdot |\tilde{S}^{(n)}(-u - n)|_{nn} \cdot |S^{(n)}(u - n)|_{nn}.$$

Moreover, the factors on the right side of each expression are permutable.

**Outline of the proof.** Define the Sklyanin comatrix $\tilde{S}(u) = (\tilde{s}_{ij}(u))$ by the formula

$$\tilde{S}(u) S(u - N + 1) = \operatorname{sdet} S(u). \quad (3.11)$$

Taking the $nn$-th entry gives

$$\operatorname{sdet} S(u) = \tilde{s}_{nn}(u) |S(u - N + 1)|_{nn}.$$

Then proceed by induction with the use of the formula

$$\tilde{s}_{nn}(u) = \frac{2u + 1}{2u + 1} |\tilde{S}^{(n)}(-u)|_{nn} \operatorname{sdet} S^{(n-1)}(u - 1),$$

which is deduced from the definition of $\operatorname{sdet} S(u)$.

**Proposition 3.21** The mapping

$$S(u) \mapsto \gamma_N(u) \tilde{S}(-u + \frac{N}{2} - 1),$$

defines an automorphism of the algebra $Y^{\pm}(N)$.
Consider the algebra $\tilde{Y}^\pm(N)$ which is defined exactly as the twisted Yangian $Y^\pm(N)$ (Definition 3.11) but with the symmetry relation (3.2) dropped. We denote the generators of $\tilde{Y}^\pm(N)$ by the same symbols $s_{ij}^{(r)}$. Note that the definition of the Sklyanin determinant does not use the symmetry relation; see Definition 3.9. Therefore, we can define $\text{sdet} S(u)$ in the same way for $\tilde{Y}^\pm(N)$. One can show that all coefficients of $\text{sdet} S(u)$ belong to the center of this algebra; cf. Theorem 3.14. The matrix $S^*(u) = S^{-1}(-u - N/2)$ satisfies the quaternary relation (3.3) and so, its Sklyanin determinant $\text{sdet} S^*(u)$ is well-defined.

Fix a nonnegative integer $M \leq N$ such that $N - M$ is even and put $m = \lfloor M/2 \rfloor$. Set

$$A_m = \{-n, \ldots, -m - 1, m + 1, \ldots, n\}$$

and

$$B_m = \{-m, \ldots, m\}$$

and use the notation of Section 2.11 for submatrices of $S(u)$.

**Theorem 3.22** In the algebra $\tilde{Y}^\pm(N)$ we have

$$\text{sdet} S(u) \text{sdet} S^*(-u + N/2 - 1)_{AA} = \text{sdet} S(u)_{BB}.$$

We shall use the notation $s'_{ij}(u)$ for the matrix elements of the matrix $S^{-1}(u)$.

**Proposition 3.23** In the algebra $\tilde{Y}^\pm(N)$ the matrix elements of the matrices $S(u)_{AA}$ and $S^{-1}(v)_{BB}$ commute with each other.

### 3.8 The centralizer construction

Let $\mathfrak{g}_n \subset \mathfrak{g}_{1,n}$ be the classical Lie algebra of type $B_n$, $C_n$ or $D_n$, as defined in Section 3.11. Fix an integer $m$ satisfying $0 \leq m \leq n$ if $N = 2n$, and $-1 \leq m \leq n$ if $N = 2n + 1$. Denote by $\mathfrak{g}_m(n)$ the subalgebra of $\mathfrak{g}_n$ spanned by the elements $F_{ij}$ subject to the condition $m + 1 \leq |i|, |j| \leq n$. Let $A_m(n)$ denote the centralizer of $\mathfrak{g}_m(n)$ in the universal enveloping algebra $A(n) = U(\mathfrak{g}_n)$. In particular, $A_0(n)$ (respectively, $A_{-1}(n)$) is the center of $A(n)$. Let $A(n)^0$ denote the centralizer of $F_{nn}$ in $A(n)$ and let $I(n)$ be the left ideal in $A(n)$ generated by the elements $F_{in}, i = -n, \ldots, n$. Then $I(n)^0 = I(n) \cap A(n)^0$ is a two-sided ideal in $A(n)^0$ and one has a vector space decomposition

$$A(n)^0 = I(n)^0 \oplus A(n - 1).$$

Therefore the projection of $A(n)^0$ onto $A(n - 1)$ with the kernel $I(n)^0$ is an algebra homomorphism. Its restriction to the subalgebra $A_m(n)$ defines a filtration preserving homomorphism

$$\pi_n : A_m(n) \to A_m(n - 1)$$

so that one can define the algebra $A_m$ as the projective limit with respect to this sequence of homomorphisms in the category of filtered algebras.
We denote by $\mathfrak{h}_n$ the diagonal Cartan subalgebra of $\mathfrak{g}_n$, and by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the subalgebras spanned by the upper triangular and lower triangular matrices, respectively. We identify $U(\mathfrak{h}_n)$ with the algebra of polynomial functions on $\mathfrak{h}_n^*$ and let $\lambda_i$ denote the function which corresponds to $F_{ii}$. For $i = 1, \ldots, n$ denote

$$\rho_{-i} = -\rho_i = \begin{cases} i - 1 & \text{for } \mathfrak{g}_n = \mathfrak{o}_{2n}, \\ i - \frac{1}{2} & \text{for } \mathfrak{g}_n = \mathfrak{o}_{2n+1}, \\ i & \text{for } \mathfrak{g}_n = \mathfrak{sp}_{2n}, \end{cases} \quad (3.12)$$

and set $l_i = -l_{-i} = \lambda_i + \rho_i$. We also set $\rho_0 = 1/2$ in the case of $\mathfrak{g}_n = \mathfrak{o}_{2n+1}$. Recall [34, Section 7.4] that the image $\chi(z) \in U(\mathfrak{h}_n)$ of an element $z$ of the center of $A(n)$ under Harish-Chandra isomorphism $\chi$ is uniquely determined by the condition

$$z - \chi(z) \in (\mathfrak{n}^- A(n) + A(n) \mathfrak{n}^+).$$

If we identify $U(\mathfrak{h}_n)$ with the algebra of polynomials in the variables $l_1, \ldots, l_n$ then $\chi(z)$ belongs to the subalgebra $M^*(n)$ of those polynomials $f = f(l_1, \ldots, l_n)$ which are invariant under the shifted action of the Weyl group. More precisely, $f$ must be invariant under all permutations of the variables and all transformations $l_i \mapsto \pm l_i$, where in the case of $\mathfrak{g}_n = \mathfrak{o}_{2n}$ the number of ‘$-$’ has to be even. In the case of the minimum value of $m$ ($m = 0$ or $m = -1$, respectively) the homomorphisms $\pi_n$ are interpreted as the specialization homomorphisms $\pi_n : M^*(n) \to M^*(n - 1)$ such that

$$\pi_n : f(\lambda_1, \ldots, \lambda_n) \mapsto f(\lambda_1, \ldots, \lambda_{n-1}, 0).$$

The corresponding projective limit in the category of filtered algebras is an analog of the algebra of shifted symmetric functions which is denoted by $M^*$. The elements of $M^*$ are well-defined functions on the set of all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ which contain only a finite number of nonzero terms. The following families of elements of $M^*$ are analogs of power sums, elementary symmetric functions, and complete symmetric functions:

$$p_m(\lambda) = \sum_{k=1}^{\infty} (t_k^{2m} - \rho_k^{2m}), \quad m = 1, 2, \ldots;$$

$$1 + \sum_{m=1}^{\infty} e_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 + l_k^2 t}{1 + \rho_k^2 t};$$

$$1 + \sum_{m=1}^{\infty} h_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 - \rho_k^2 t}{1 - l_k^2 t}.$$

Each of the families $\{p_m\}$, $\{e_m\}$, $\{h_m\}$ can be taken as a system of algebraically independent generators of the algebra $M^*$.  

33
Proposition 3.24 The algebra $A_0$ or $A_{-1}$ in the case $N = 2n$ or $N = 2n + 1$, respectively, is isomorphic to the algebra of shifted symmetric functions $M^*$.

Consider the homomorphism (3.5) and take its composition with the automorphism of $Y^+(N)$ given by Proposition 3.21 to yield another homomorphism $\varphi_n : Y^+(N) \to U(\mathfrak{g}_n)$. Set $M = 2m$ or $M = 2m + 1$ depending on whether $N = 2n$ or $N = 2n + 1$. The image of the restriction of $\varphi_n$ to the subalgebra $Y^+(M)$ is contained in the centralizer $A_m(N)$.

Theorem 3.25 The sequence of homomorphisms $(\varphi_n | n \geq m)$ defines an algebra embedding $\varphi : Y^+(M) \hookrightarrow A_m$. Moreover, one has an isomorphism

$$A_m = M^* \otimes Y^+(M),$$

where $Y^+(M)$ is identified with its image under the embedding $\varphi$.

3.9 Commutative subalgebras

Fix an $N \times N$ matrix $C$ with entries in $\mathbb{C}$ such that $C^t = C$ or $C^t = -C$, where the transposition is defined in (3.1). Consider the algebra $Y^+(N)[[u^{-1}]] \otimes (\text{End} \mathbb{C}^N)^{\otimes N}$ and for any $1 \leq k \leq N$ introduce its element

$$S(u, k) = \langle S_1, \ldots, S_k \rangle$$

as in Section 3.3 with the variables $u_i$ specialized to $u_i = u - i + 1$ for $i = 1, \ldots, k$. Similarly, define the element $C(u, k)$ by

$$C(u, k) = C_{k+1} \tilde{R}^t_{k+1,k+2} \cdots \tilde{R}^t_{k+1,N} C_{k+2} \tilde{R}^t_{k+2,k+3} \cdots \tilde{R}^t_{k+2,N} \cdots C_{N-1} \tilde{R}^t_{N-1,N} C_{N},$$

where we abbreviate $\tilde{R}^t_{ij} = R^t_{ij}(-2u - N + i + j + 2)$. Introduce the series $\sigma_k(u, C)$ with coefficients in $Y^+(N)$ by

$$\sigma_k(u, C) = \text{tr} A_N S(u, k) \left( \prod_{i=1,\ldots,k} \tilde{R}^t_{ij} \prod_{j=k+1,\ldots,N} R^t_{ij} \right) C(u, k),$$

where $R^t_{ij} = R^t_{ij}(-2u + i + j + 2)$, $A_N$ is the antisymmetrizer defined by (2.21) and the trace is taken over all $N$ copies of End $\mathbb{C}^N$.

Theorem 3.26 All the coefficients of the series $\sigma_1(u, C), \ldots, \sigma_N(u, C)$ commute with each other. Moreover, if the matrix $C$ has simple spectrum and satisfies $C^t = -C$ then these coefficients generate a maximal commutative subalgebra of the twisted Yangian $Y^+(N)$.
Consider the epimorphism \( \pi : Y^\pm(N) \to U(\mathfrak{g}_n) \) defined in (3.3) (recall that \( \mathfrak{g}_n \) denotes the classical Lie algebra \( \mathfrak{o}_{2n}, \mathfrak{o}_{2n+1} \) or \( \mathfrak{sp}_{2n} \)). Clearly, the coefficients of the images of the series \( \sigma_k(u,C) \), \( k = 1, \ldots, N \) under \( \pi \) form a commutative subalgebra \( \mathcal{C} \subseteq U(\mathfrak{g}_n) \).

**Theorem 3.27** If the matrix \( C \) has simple spectrum and satisfies \( C^t = -C \) then the subalgebra \( \mathcal{C} \) of \( U(\mathfrak{g}_n) \) is maximal commutative.

**Bibliographical notes**

The twisted Yangians were introduced by Olshanski in [139] where he also outlined their basic properties. A detailed exposition of the most of the results presented here can be found in [119].

3.3. Theorem 3.12 is proved in [108]. Theorem 3.11 was conjectured by Lascoux and proved by the author in [111].

3.7. Quasi-determinant factorization of the Sklyanin determinant is given in [109].

3.8. The centralizer construction originates from Olshanski [139]. A detailed proof of Theorem 3.25 is given in [120].

3.9. The commutative subalgebras in the twisted Yangians and the classical enveloping algebras were constructed in Nazarov–Olshanski [128].

Finite-dimensional irreducible representations of the twisted Yangians are classified in [112] with the symplectic case done earlier in [106]. Explicit constructions of all representations of \( Y^\pm(2) \) are also given in [112]. These results were used in [113, 114, 115] to construct weight bases of Gelfand–Tsetlin type for the classical Lie algebras. Ragoucy [144] discovered a relationship between the twisted Yangians and folded \( W \)-algebras. A family of algebras defined by a quaternary type relation (or reflection equation) is defined by Sklyanin [152]. He also constructed commutative subalgebras and some representations of these algebras. They were also studied in the physics literature in connection with the integrable models with boundary conditions and the nonlinear Schrödinger equation; see e.g. Kulish–Sklyanin [90], Kulish–Sasaki–Schwiebert [11], Kuznetsov–Jørgensen–Christiansen [94], Liguori–Mintchev–Zhao [100], Mintchev–Ragoucy–Sorba–Zaugg [104, 105]. The action of such algebras on hypergeometric functions was studied by Koornwinder and Kuznetsov [82].

4 Applications to classical Lie algebras

Here we give constructions of families of Casimir elements for the classical Lie algebras implied by the results discussed in the previous sections. All of these constructions (including some well known) are related with the quantum determinant for the Yangian \( Y(n) \) or the Sklyanin determinant for the twisted Yangian \( Y^\pm(N) \). We keep the
notation \( \mathfrak{g}_n \) for the classical Lie algebra of type \( B_n, C_n \) or \( D_n \) as in Section 3.1. For any element \( z \) of the center of the universal enveloping algebra \( U(\mathfrak{gl}_n) \) or \( U(\mathfrak{g}_n) \) we shall denote by \( \chi(z) \) its Harish-Chandra image; see Sections 2.13 and 3.8 respectively. In the case of \( \mathfrak{g}_n \) we keep using the parameters \( \rho_i \) defined in (3.12).

4.1 Newton’s formulas

Consider the case of \( \mathfrak{gl}_n \) first. As in Section 2.1 we denote by \( E \) the \( n \times n \)-matrix whose \( ij \)-th entry is \( E_{ij} \). Denote by \( C(u) \) the Capelli determinant

\[
C(u) = \sum_{p \in S_n} \text{sgn } p \cdot (u + E)_{p(1),1} \cdots (u + E - n + 1)_{p(n),n}.
\]

This is a polynomial in \( u \) whose coefficients belong to the center of \( U(\mathfrak{gl}_n) \). The image of \( C(u) \) under the Harish-Chandra isomorphism is clearly given by

\[
\chi : C(u) \mapsto (u + l_1) \cdots (u + l_n),
\]

where \( l_i = \lambda_i - i + 1 \). Another family of central elements is provided by the Gelfand invariants \( \text{tr} E^k \); see Section 2.1.

The following can be regarded as a noncommutative analog of the classical Newton formula which relates the elementary and power sums symmetric functions; see e.g. Macdonald [101].

**Theorem 4.1** We have the formula

\[
1 + \sum_{k=0}^{\infty} \frac{(-1)^k \text{tr} E^k}{(u - n + 1)^{k+1}} = \frac{C(u + 1)}{C(u)}.
\]

**Proof.** Apply the homomorphism (2.7) to the quantum Liouville formula; Section 2.10.

**Corollary 4.2** The images of the Gelfand invariants under the Harish-Chandra isomorphism are given by

\[
1 + \sum_{k=0}^{\infty} \frac{(-1)^k \chi(\text{tr} E^k)}{(u - n + 1)^{k+1}} = \prod_{i=1}^{n} \left( 1 + \frac{1}{u + l_i} \right).
\]

In the case of \( \mathfrak{g}_n \) we denote by \( F \) the \( N \times N \)-matrix whose \( ij \)-th entry is \( F_{ij} \). Introduce the Capelli-type determinant

\[
C(u) = (-1)^n \sum_{p \in S_N} \text{sgn } pp' \cdot (u + \rho_{-n} + F)_{a_{p(1)},a'_{p'(1)}} \cdots (u + \rho_{n} + F)_{a_{p(N)},a'_{p'(N)}}.
\]

36
where \((a_1, \ldots, a_N)\) is any permutation of the indices \((-n, \ldots, n)\) and \(p'\) is the image of \(p\) under the map (3.9). Using Theorem 3.12 and applying the homomorphism (3.5) we deduce that all coefficients of the polynomial \(C(u)\) belong to the center of \(U(g_n)\). If we take \((a_1, \ldots, a_N) = (-n, \ldots, n)\) then the image of \(C(u)\) under the Harish-Chandra isomorphism is easily found. For \(N = 2n\) we get

\[
\chi : C(u) \mapsto \prod_{i=1}^{n} (u^2 - l_i^2),
\]

and for \(N = 2n + 1\)

\[
\chi : C(u) \mapsto (u + \frac{1}{2}) \prod_{i=1}^{n} (u^2 - l_i^2).
\]

**Theorem 4.3** If \(N = 2n\) we have

\[
1 + \frac{2u + 1}{2u + 1 + 1} \sum_{k=0}^{\infty} (-1)^k \text{tr} F^k (u + \rho_n)^{k+1} = \frac{C(u + 1)}{C(u)},
\]

where the upper sign is taken in the orthogonal case and the lower sign in the symplectic case. If \(N = 2n + 1\) then the same formula holds with \(C(u)\) replaced by

\[
\overline{C}(u) = \frac{2u}{2u + 1} C(u).
\]

**Proof.** Apply the homomorphism (3.5) to the quantum Liouville formula; Section 3.6.

**Corollary 4.4** The images of the Gelfand invariants under the Harish-Chandra isomorphism are given by

\[
1 + \frac{2u + 1}{2u + 1 + 1} \sum_{k=0}^{\infty} (-1)^k \chi \text{tr} F^k (u + \rho_n)^{k+1} = \prod_{i=-n}^{n} (1 + \frac{1}{u + l_i}),
\]

where the zero index is skipped in the product if \(N = 2n\), while for \(N = 2n + 1\) one should set \(l_0 = 0\).

### 4.2 Cayley–Hamilton theorem

The polynomials \(C(u)\) turn out to be the noncommutative characteristic polynomials for the matrices \(E\) and \(F\). Consider the case of \(gl_n\) first.

**Theorem 4.5** We have the identities

\[
C(-E + n - 1) = 0 \quad \text{and} \quad C(-E^t) = 0.
\]
Proof. Applying the homomorphism (2.7) to the relations (2.30) and (2.32) and multiplying by the denominators we get

\[ C(u) = \tilde{C}(u)(u + E - n + 1) \quad \text{and} \quad C(u) = \tilde{C}^t(u - 1)(u + E^t), \]

where \( \tilde{C}(u) \) is polynomial in \( u \) with coefficients in \( U(\mathfrak{gl}_n) \otimes \text{End } \mathbb{C}^n \).

Taking the images of the identities (4.4) in a highest weight representation \( L \) of \( \mathfrak{gl}_n \) with the highest weight \( (\lambda_1, \ldots, \lambda_n) \) we derive the characteristic identities.

**Corollary 4.6** The image of the matrix \( E \) in \( L \) satisfies the identities

\[ \prod_{i=1}^{n} (E - l_i - n + 1) = 0 \quad \text{and} \quad \prod_{i=1}^{n} (E^t - l_i) = 0, \]

where \( l_i = \lambda_i - i + 1 \).

Now turn to the case of the Lie algebras \( \mathfrak{g}_n \). As before, we consider the three cases simultaneously.

**Theorem 4.7** We have the identity

\[ C(-F - \rho_n) = 0. \quad (4.5) \]

**Outline of the proof.** Apply (3.5) to the relation (3.14) and multiply by the denominators.

Let \( L \) be a highest weight representation of \( \mathfrak{g}_n \) with the highest weight \( (\lambda_1, \ldots, \lambda_n) \) with respect to the basis elements \( F_{11}, \ldots, F_{nn} \) of the Cartan subalgebra \( \mathfrak{h}_n \). The following are the characteristic identities for \( \mathfrak{g}_n \) which are obtained by taking the image of (4.5) in \( L \).

**Corollary 4.8** The image of the matrix \( F \) in \( L \) satisfies the identities

\[ \prod_{i=-n}^{n} (F - l_i + \rho_n) = 0, \]

where \( l_i = \lambda_i + \rho_i \). The zero index is skipped in the product if \( N = 2n \), while for \( N = 2n + 1 \) one should set \( l_0 = \frac{1}{2} \).
4.3 Graphical constructions of Casimir elements

For $1 \leq m \leq n$ denote by $E^{(m)}$ the $m \times m$-matrix with the entries $E_{ij}$, where $i, j = 1, \ldots, m$. Let $\mathcal{E}^{(m)}$ denote the complete oriented graph with the vertices $\{1, \ldots, m\}$, the arrow from $i$ to $j$ is labelled by the $ij$-th matrix element of the matrix $E^{(m)} - m + 1$. Then every path in the graph defines a monomial in the matrix elements in a natural way. A path from $i$ to $j$ is called simple if it does not pass through the vertices $i$ and $j$ except for the beginning and the end of the path. Using this graph introduce the elements $\Lambda_k^{(m)}$, $S_k^{(m)}$, $\Psi_k^{(m)}$ and $\Phi_k^{(m)}$ of the universal enveloping algebra $U(\mathfrak{gl}_n)$ as follows. For $k \geq 1$

$(-1)^{k-1}\Lambda_k^{(m)}$ is the sum of all monomials labelling simple paths in $\mathcal{E}^{(m)}$ of length $k$ going from $m$ to $m$;

$S_k^{(m)}$ is the sum of all monomials labelling paths in $\mathcal{E}^{(m)}$ of length $k$ going from $m$ to $m$;

$\Psi_k^{(m)}$ is the sum of all monomials labelling paths in $\mathcal{E}^{(m)}$ of length $k$ going from $m$ to $m$, the coefficient of each monomial being the length of the first return to $m$;

$\Phi_k^{(m)}$ is the sum of all monomials labelling paths in $\mathcal{E}^{(m)}$ of length $k$ going from $m$ to $m$, the coefficient of each monomial being the ratio of $k$ to the number of returns to $m$.

**Theorem 4.9** The center of the algebra $U(\mathfrak{gl}_n)$ is generated by the scalars and each of the following families of elements

$$\Lambda_k = \sum_{i_1 + \cdots + i_n = k} \Lambda_{i_1}^{(1)} \cdots \Lambda_{i_n}^{(n)},$$

$$S_k = \sum_{i_1 + \cdots + i_n = k} S_{i_1}^{(1)} \cdots S_{i_n}^{(n)},$$

$$\Psi_k = \sum_{m=1}^n \Psi_k^{(m)},$$

$$\Phi_k = \sum_{m=1}^n \Phi_k^{(m)},$$

where $k = 1, 2, \ldots, n$. Moreover, $\Psi_k = \Phi_k$ for any $k$, and the images of $\Lambda_k$, $S_k$ and $\Psi_k$ under the Harish-Chandra isomorphism are, respectively, the elementary, complete and power sums symmetric functions of degree $k$ in the variables $l_1, \ldots, l_n$.

**Proof.** Consider the polynomial $C(u)$ introduced in Section 4.1 and set $\tilde{C}(t) = t^n C(t^{-1})$. Applying the homomorphism (2.7) to the decomposition (2.36) we obtain a decomposition in the algebra of formal series with coefficients in $U(\mathfrak{gl}_n)$,

$$\tilde{C}(t) = |1 + tE^{(1)}|_{11} \cdots |1 + t(E^{(n)} - n + 1)|_{nn}.$$
and the factors are permutable. By [40, Proposition 7.20], the elements introduced above are now interpreted as follows

\[
1 + \sum_{k=1}^{\infty} \Lambda_k^{(m)} t^k = |1 + t(E^{(m)} - m + 1)|_{mm},
\]

\[
1 + \sum_{k=1}^{\infty} S_k^{(m)} t^k = |1 - t(E^{(m)} - m + 1)|_{mm}^{-1},
\]

\[
\sum_{k=1}^{\infty} \Psi_k^{(m)} t^{k-1} = |1 - t(E^{(m)} - m + 1)|_{mm} d \frac{dt}{|1 - t(E^{(m)} - m + 1)|_{mm}^{-1}},
\]

\[
\sum_{k=1}^{\infty} \Phi_k^{(m)} t^{k-1} = -\frac{d}{dt} \log(|1 - t(E^{(m)} - m + 1)|_{mm}).
\]

Due to the relations between the classical symmetric functions [101], the second part of the theorem follows from (4.1).

Similarly, in the case of \( g_n \) for any \( 1 \leq m \leq n \) denote by \( F^{(m)} \) the matrix with the entries \( F_{ij} \), where \( i, j = -m, -m + 1, \ldots, m \) (the index 0 is skipped if \( N = 2n \)).

Consider the complete oriented graph \( F_m \) with the vertices \( \{-m, -m + 1, \ldots, m\} \), the arrow from \( i \) to \( j \) is labelled by the \( ij \)-th matrix element of the matrix \( F^{(m)} + \rho_m \).

Introduce the elements \( \Lambda_k^{(m)}, \tilde{\Lambda}_k^{(m)}, S_k^{(m)}, \tilde{S}_k^{(m)}, \Phi_k^{(m)}, \tilde{\Phi}_k^{(m)} \) of the universal enveloping algebra \( U(g_n) \) as follows: for \( k \geq 1 \)

\((-1)^{k-1} \Lambda_k^{(m)} \) (resp. \( -\tilde{\Lambda}_k^{(m)} \)) is the sum of all monomials labelling simple paths in \( F^{(m)} \) (resp. simple paths that do not pass through \(-m\)) of length \( k \) going from \( m \) to \( m \);

\( S_k^{(m)} \) (resp. \( (-1)^k \tilde{S}_k^{(m)} \)) is the sum of all monomials labelling paths in \( F^{(m)} \) (resp. paths that do not pass through \(-m\)) of length \( k \) going from \( m \) to \( m \);

\( \Phi_k^{(m)} \) (resp. \( (-1)^k \tilde{\Phi}_k^{(m)} \)) is the sum of all monomials labelling paths in \( F^{(m)} \) (resp. paths that do not pass through \(-m\)) of length \( k \) going from \( m \) to \( m \), the coefficient of each monomial being the ratio of \( k \) to the number of returns to \( m \).

**Theorem 4.10** Each of the following families of elements is contained in the center of the algebra \( U(g_n) \):

\[
\Lambda_{2k} = \sum_{i_1 + \cdots + i_{2n} = 2k} \tilde{\Lambda}_{i_1}^{(1)} \Lambda_{i_2}^{(1)} \cdots \tilde{\Lambda}_{i_{2n-1}}^{(n)} \Lambda_{i_{2n}}^{(n)},
\]

\[
S_{2k} = \sum_{i_1 + \cdots + i_{2n} = 2k} \tilde{S}_{i_1}^{(1)} S_{i_2}^{(1)} \cdots \tilde{S}_{i_{2n-1}}^{(n)} S_{i_{2n}}^{(n)},
\]

\[
\Phi_{2k} = \sum_{m=1}^{n} (\tilde{\Phi}_{2k}^{(m)} + \Phi_{2k}^{(m)}),
\]
where \( k = 1, 2, \ldots \). Moreover, the images of \((-1)^k \Lambda_{2k}, S_{2k} \) and \( \Phi_{2k}/2 \) under the Harish-Chandra isomorphism are, respectively, the elementary, complete and power sums symmetric functions of degree \( k \) in the variables \( l_1^2, \ldots, l_n^2 \).

Outline of the proof. The proof is the same as for Theorem 4.9 with the use of Theorem 3.20 and the homomorphism (3.3).

\[ \square \]

4.4 Pfaffians and Hafnians

Suppose first that \( \mathfrak{g}_n \) is the orthogonal Lie algebra \( \mathfrak{o}_{2n} \) or \( \mathfrak{o}_{2n+1} \). For any \( 1 \leq k \leq n \) consider a subset \( I \subseteq \{-n, \ldots, n\} \) of cardinality \( 2k \) so that the elements of \( I \) are \( i_1 < \cdots < i_{2k} \). The matrix \([F_{i_p, -i_q}]\) is skew-symmetric. Introduce the corresponding Pfaffian \( \text{Pf} F^I \) by

\[
\text{Pf} F^I = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \text{sgn} \sigma \cdot F_{i_{\sigma(1)}, -i_{\sigma(2)}} \cdots F_{i_{\sigma(2k-1)}, -i_{\sigma(2k)}}.
\]

Given a subset \( I \) set \( I^* = \{-i_{2k}, \ldots, -i_1\} \) and denote

\[
c_k = (-1)^k \sum_{|I|=2k} \text{Pf} F^I \text{Pf} F^{I^*}, \quad k \geq 1,
\]

and \( c_0 = 1 \). Consider the Capelli-type determinant \( C(u) \) introduced in Section 4.1.

**Theorem 4.11** The elements \( c_k \) belong to the center of \( \mathfrak{U}(\mathfrak{g}_n) \) and one has a decomposition

\[
C(u) = \sum_{k=0}^n c_k (u^2 - \rho^2_1) \cdots (u^2 - \rho^2_{n-k})
\]

if \( N = 2n \), and

\[
C(u) = (u + \frac{1}{2}) \sum_{k=0}^n c_k (u^2 - \rho^2_1) \cdots (u^2 - \rho^2_{n-k})
\]

if \( N = 2n + 1 \). Moreover, the image of \( c_k \) under the Harish-Chandra isomorphism is given by

\[
\chi(c_k) = (-1)^k \sum_{i_1 < \cdots < i_k} (l^2_{i_1} - \rho^2_{i_1}) \cdots (l^2_{i_k} - \rho^2_{i_k-k+1}).
\]

Using the definition of the Sklyanin determinant one can write a similar expression for the Capelli-type determinant in any realization of the orthogonal or symplectic Lie algebra. Consider the realization of \( \mathfrak{o}_N \) corresponding to the canonical symmetric form so that the elements of \( \mathfrak{o}_N \) are skew-symmetric matrices with respect to the usual
transposition. For this discussion only, we use more standard numbering 1, . . . , N of the rows and columns of such matrices. Here F will denote the N × N-matrix whose ij entry is \( F_{ij} = E_{ij} - E_{ji} \). The elements \( c_k \) are now given by

\[
c_k = \sum_{|I|=2k} (\text{Pf } F^I)^2,
\]

where \( I = \{i_1, \ldots, i_{2k}\} \) is a subset of \( \{1, \ldots, N\} \) and \( \text{Pf } F^I \) is the Pfaffian of the skew-symmetric matrix \( [F_{i_p,i_q}] \). The Capelli-type determinant is given by

\[
C(u) = \sum_{\rho \in \mathfrak{S}_N} \text{sgn } \rho \cdot \prod_{k=1}^n (u + F + \sigma_1)_{\rho(1),\rho'(1)} \cdots (u + F + \sigma_N)_{\rho(N),\rho'(N)},
\]

where \( \sigma_i = N/2 - i \) for \( i \leq n \) and \( \sigma_i = N/2 - i + 1 \) for \( i > n \). Theorem [4.11] holds in the same form with \( \rho_i \) replaced by \( \sigma_{n-i+1} \) for every \( i = 1, \ldots, n \). Introduce also more standard determinant

\[
D(u) = \sum_{\rho \in \mathfrak{S}_N} \text{sgn } \rho \cdot \prod_{k=1}^n (u + F + m)_{\rho(1),1} (u + F + m-1)_{\rho(2),2} \cdots (u - m+1)_{\rho(N),N}, \tag{4.6}
\]

where \( m = N/2 \).

**Theorem 4.12** The coefficients of the polynomial \( D(u) \) are central in \( U(\mathfrak{o}_N) \) and the following decomposition holds

\[
D(u) = \sum_{k=0}^n c_k (u + m - k)(u + m - k - 1) \cdots (u - m + k + 1). \tag{4.7}
\]

In particular, we get the following two analogs of the relation \((\text{Pf } A)^2 = \det A\) which holds for a numerical skew-symmetric \( 2n \times 2n \)-matrix \( A \).

**Corollary 4.13** If \( N = 2n \) then

\[
(\text{Pf } F)^2 = C(0) = D(0).
\]

Now consider the symplectic Lie algebra \( \mathfrak{sp}_{2n} \) and return to our standard notation. For any \( k \geq 1 \) consider a sequence \( I \) of indices from \( \{-n, \ldots, n\} \) of cardinality \( 2k \) so that the elements of \( I \) are \( i_1 \leq \cdots \leq i_{2k} \). Set \( \bar{F}_{ij} = \text{sgn } i \cdot F_{ij} \). Then we have \( \bar{F}_{i,-j} = \bar{F}_{j,-i} \). Introduce the corresponding **Hafnian** \( \text{Hf } F^I \) by

\[
\text{Hf } F^I = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \bar{F}_{i_{\sigma(1)},-i_{\sigma(2)}} \cdots \bar{F}_{i_{\sigma(2k-1)},-i_{\sigma(2k)}}.
\]

For each \( I \) let \( f_{\pm1}, \ldots, f_{\pm n} \) be the multiplicities of the indices \( \pm1, \ldots, \pm n \) in \( I \), and let

\[
\text{sgn } I = (-1)^{f_{-1} + \cdots + f_{-n}}.
\]

42
Set \( I^* = \{-i_2, \ldots, -i_1\} \) and denote
\[
d_k = (-1)^k \sum_{|I|=2k} \frac{\text{sgn} I \cdot Hf^I \cdot Hf^{I^*}}{f_1!f_{-1}! \cdots f_n!f_{-n}!}.
\]

Consider the Capelli-type determinant \( C(u) \) introduced in Section 4.1 and introduce the series \( c(u) \) by
\[
c(u) = \frac{C(u)}{(u^2 - \rho_1^2) \cdots (u^2 - \rho_n^2)}.
\]

**Theorem 4.14** The elements \( d_k \) belong to the center of \( U(\mathfrak{sp}_{2n}) \) and one has a decomposition
\[
c(u)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{d_k}{(u^2 - (n + 1)^2) \cdots (u^2 - (n + k)^2)}.
\]

Moreover, the image of \( d_k \) under the Harish-Chandra isomorphism is given by
\[
\chi(d_k) = \sum_{i_1 \leq \cdots \leq i_k} \left( l_{i_1}^2 - i_1^2 \right) \cdots \left( l_{i_k}^2 - (i_k + k - 1)^2 \right).
\]

**Bibliographical notes**

4.1. The images of the Gelfand invariants under the Harish-Chandra isomorphism were first found by Perelomov and Popov \[140, 141, 142\]; see also Želobenko \[173\]. The observation that the formulas (4.2) and (4.3) are related with the theory of Yangians is due to Cherednik \[30\]. A derivation of the Perelomov–Popov formulas from the Liouville formula is contained in \[110\]. Of course, given the Harish-Chandra images of the polynomials \( C(u) \), Theorems 4.1 and 4.3 follow from Corollaries 4.2 and 4.4. Different proofs of Newton’s formulas without using the Yangians are given by Itoh and Umeda \[61, 61, 170\] for the general linear and orthogonal Lie algebras.

4.2. Theorem 4.5 is due to Nazarov and Tarasov \[129\]. Theorem 4.7 is proved in \[108\]. An independent proof is given by Nazarov (unpublished) without using the map (3.9). One more proof in the orthogonal case is given by Itoh \[61\] with the use of a determinant of type (4.6) instead of \( C(u) \). Corollaries 4.6 and 4.8 are the remarkable characteristic identities which are due to Bracken and Green \[17, 52\]. More general identities are obtained by Gould \[50\].

4.3. The elements \( \Lambda_k, S_k, \Psi_k, \Phi_k \) are the noncommutative symmetric functions associated with a matrix; see Gelfand et al \[40\]. Theorem 4.9 is proved in \[40\]. Section 7.5. Theorem 4.10 is contained in \[109\]. A different version for the orthogonal Lie algebra is given in \[111\].

4.4. Most of these results are contained in \[118\]; see also \[111\]. The centrality of the determinant (4.6) was first proved by Howe and Umeda \[57\]. Its relationship with the
Pfaffians (formula (4.7)) was established by Itoh and Umeda [62]. The first relation in Corollary 4.13 is proved in [108]. Both Pfaffians and Hafnians are key ingredients in the analogs of the celebrated Capelli identity [19, 20] for the classical Lie algebras given in [118]; see also Weyl [172], Howe [56], Howe–Umeda [57] for the role of the Capelli identity in the classical invariant theory. The polynomials $\chi(c_k)$ and $\chi(d_k)$ are respectively the elementary and complete factorial symmetric functions; see e.g. Okounkov–Olshanski [134, 135].

References

[1] C. Ahn and W. M. Koo, $gl(n|m)$ color Calogero-Sutherland models and Super Yangian Algebra, Phys. Lett. B 365 (1996), 105–112.

[2] C. Ahn and S. Nam, Yangian symmetries in the $SU(N)_1$ WZW model and the Calogero-Sutherland model, Phys. Lett. B 378 (1996), 107–112.

[3] T. Arakawa, Drinfeld functor and finite-dimensional representations of Yangian, Comm. Math. Phys. 205 (1999), 1–18.

[4] D. Arnaudon, J. Avan, L. Frappat, E. Ragoucy and M. Rossi, On the quasi-Hopf structure of deformed double Yangians, Lett. Math. Phys. 51 (2000), 193–204.

[5] D. Arnaudon, J. Avan, L. Frappat and E. Ragoucy, Yangian and quantum universal solutions of Gervais–Neveu–Felder equations, preprint math.QA/0104181.

[6] J. Avan, A. Jevicki and J. Lee, Yangian-invariant field theory of matrix-vector models, Nucl. Phys. B 486 (1997), 650–672.

[7] B. Basu-Mallick, P. Ramadevi and R. Jagannathan, Multiparametric and coloured extensions of the quantum group $GL_q(N)$ and the Yangian algebra $Y(gl_N)$ through a symmetry transformation of the Yang-Baxter equation, Int. J. Mod. Phys. A 12 (1997), 945–962.

[8] B. Basu-Mallick and A. Kundu, Multi-parameter deformed and nonstandard $Y(gl_M)$ Yangian symmetry in a novel class of spin Calogero-Sutherland models, Nucl. Phys. B 509 (1998), 705–728.

[9] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press, New York, 1982.

[10] D. Bernard, An Introduction to Yangian Symmetries, Int. J. Mod. Phys. B 7 (1993), 3517–3530.

[11] D. Bernard and A. LeClair, Quantum group symmetries and non-local currents in 2D QFT, Commun. Math. Phys. 142, 99–138 (1991).

[12] D. Bernard, Z. Maassarani and P. Mathieu, Logarithmic Yangians in WZW models, Mod. Phys. Lett. A 12 (1997), 535–544.

[13] P. Bouwknegt, A. W. W. Ludwig and K. Schoutens, Spinon bases, Yangian symmetry and fermionic representations of Virasoro characters in conformal field theory, Phys. Lett. B 338 (1994), 448–456.
[14] P. Bouwknegt and K. Schoutens, The SU(n)1 WZW Models: Spinon Decomposition and Yangian Structure, Nucl. Phys. B 482 (1996), 345–372.

[15] P. Bouwknegt and K. Schoutens, Spinon decomposition and Yangian structure of \( \hat{\mathfrak{sl}}_n \) modules, Geometric analysis and Lie theory in mathematics and physics, pp. 105–131, Austral. Math. Soc. Lect. Ser., 11, Cambridge Univ. Press, Cambridge, 1998.

[16] S. I. Boyarchenko and S. Z. Levendorski˘ı, On affine Yangians, Lett. Math. Phys. 32 (1994), 269–274.

[17] A. J. Bracken and H. S. Green, Vector operators and a polynomial identity for SO(n), J. Math. Phys. 12 (1971), 2099–2106.

[18] C. Briot and E. Ragoucy, RTT presentation of finite W-algebras, preprint math.QA/0005111.

[19] A. Capelli, Über die Zurückführung der Cayley’schen Operation \( \Omega \) auf gewöhnliche Polar-Operationen, Math. Ann. 29 (1887), 331–338.

[20] A. Capelli, Sur les opérations dans la théorie des formes algébriques, Math. Ann. 37 (1890), 1–37.

[21] V. Chari, On the fermionic formula and the Kirillov-Reshetikhin conjecture, preprint math.QA/0006093.

[22] V. Chari and S. Ilangovan, On the Harish-Chandra homomorphism for infinite-dimensional Lie algebras, J. Algebra 90 (1984), 476–494.

[23] V. Chari and A. Pressley, Yangians and R-matrices, L’Enseign. Math. 36 (1990), 267–302.

[24] V. Chari and A. Pressley, Fundamental representations of Yangians and rational R-matrices, J. Reine Angew. Math. 417 (1991), 87–128.

[25] V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, 1994.

[26] V. Chari and A. Pressley, Yangians: their representations and characters, Acta Appl. Math. 44, 39–58 (1996).

[27] V. Chari and A. Pressley, Yangians, integrable quantum systems and Dorey’s rule, Comm. Math. Phys. 181 (1996), 265–302.

[28] I. V. Cherednik, Factorized particles on the half-line and root systems, Theor. Math. Phys. 61 (1984), no. 1, 35–44.

[29] I. V. Cherednik, A new interpretation of Gelfand–Tzetlin bases, Duke Math. J. 54 (1987), 563–577.

[30] I. V. Cherednik, Quantum groups as hidden symmetries of classic representation theory, in ‘Differential Geometric Methods in Physics (A. I. Solomon, Ed.)’, World Scientific, Singapore, 1989, pp. 47–54.

[31] T. Curtright and C. Zachos, Supersymmetry and the Nonlocal Yangian Deformation Symmetry, Nucl. Phys. B 402 (1993), 604–612.
[32] X. M. Ding, Bo-Yu Hou and L. Zhao, \( h \) (Yangian) deformation of the Miura map and Virasoro algebra, Internat. J. Modern Phys. A 13 (1998), 1129–1144.

[33] X. M. Ding and L. Zhao, Free Boson Representation of \( DY_h(\mathfrak{sl}_2)_k \) and the Deformation of the Feigin-Fuchs, Commun. Theor. Phys. 32 (1999), 103–108.

[34] J. Dixmier, Algèbres Enveloppantes, Gauthier-Villars, Paris, 1974.

[35] V. G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Soviet Math. Dokl. 32 (1985), 254–258.

[36] V. G. Drinfeld, Degenerate affine Hecke algebras and Yangians, Funct. Anal. Appl. 20 (1986), 56–58.

[37] V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212–216.

[38] V. G. Drinfeld, Quantum Groups, in ‘Proc. Int. Congress Math., Berkeley, 1986’, AMS, Providence RI, 1987, pp. 798–820.

[39] B. Enriquez and G. Felder, A construction of Hopf algebra cocycles for the Yangian double \( DY(\mathfrak{sl}_2) \), J. Phys. A 31 (1998), 2401–2413.

[40] B. Enriquez and G. Felder, Coinvariants for Yangian doubles and quantum Knizhnik-Zamolodchikov equations, Internat. Math. Res. Notices 1999, no. 2, 81–104.

[41] P. Etingof and V. Retakh, Quantum determinants and quasideterminants, Asian J. Math. 3 (1999), 345–352.

[42] L. D. Faddeev, Integrable models in \((1+1)\)-dimensional quantum field theory, in “Recent advances in field theory and statistical mechanics, Les Houches Lectures 1982”, pp. 561–608, North-Holland, Amsterdam 1984.

[43] L. D. Faddeev and L. A. Takhtajan, Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, J. Soviet Math. 24 (1984), 241–267.

[44] I. M. Gelfand, The center of an infinitesimal group ring, Mat. Sbornik N.S. 26 (68) (1950), 103–112 (Russian). English transl. in: I. M. Gelfand, “Collected papers”. Vol II, Berlin: Springer-Verlag 1988.

[45] I. M. Gelfand and M. L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Dokl. Akad. Nauk SSSR 71 (1950), 825–828 (Russian). English transl. in: I. M. Gelfand, “Collected papers”. Vol II, Berlin: Springer-Verlag 1988.

[46] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh and J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995), 218–348.

[47] I. M. Gelfand and V. S. Retakh, Determinants of matrices over noncommutative rings, Funct. Anal. Appl. 25 (1991), 91-102.

[48] I. M. Gelfand and V. S. Retakh, A theory of noncommutative determinants and characteristic functions of graphs, Funct. Anal. Appl. 26 (1992), 1-20; Publ. LACIM, UQAM, Montreal, 14, 1-26.
[49] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type $A_n$*, Internat. Math. Res. Not. (1993) No. 3, 67–85.

[50] M. D. Gould, *Characteristic identities for semi-simple Lie algebras*, J. Austral. Math. Soc. B 26 (1985), 257–283.

[51] M. D. Gould and Y.-Z. Zhang, *On super-RS algebra and Drinfeld realization of quantum affine superalgebras*, Lett. Math. Phys. 44 (1998), 291–308.

[52] H. S. Green, *Characteristic identities for generators of $GL(n), O(n)$ and $Sp(n)$*, J. Math. Phys. 12 (1971), 2106–2113.

[53] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard and V. Pasquier, *Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory*, Phys. Rev. Lett. 69 (1992), 2021–2025.

[54] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, Contemporary Math. 248 (1999), 243–291.

[55] T. Hauer, *Systematic proof of the existence of Yangian symmetry in chiral Gross-Neveu models*, Phys. Lett. B 417 (1998), 297–302.

[56] R. Howe, *Remarks on classical invariant theory*, Trans. AMS 313 (1989), 539–570.

[57] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. 290 (1991), 569–619.

[58] J. E. Humphreys, *Introduction to Lie algebras and Representation Theory*, Graduate Texts in Mathematics 9, Springer, New York, 1972.

[59] K. Iohara, *Bosonic representations of Yangian Double $DY_\hbar(g)$ with $g = gl_N, sl_N$*, J. Phys. A 29 (1996), 4593–4621.

[60] M. Itoh, *Explicit Newton’s formulas for $gl_n$*, J. Algebra 208 (1998), 687–697.

[61] M. Itoh, *Capelli elements for the orthogonal Lie algebras*, J. Lie Theory 10 (2000), 463–489.

[62] M. Itoh and T. Umeda, *On central elements in the universal enveloping algebras of the orthogonal Lie algebras*, Compos. Math., to appear.

[63] A. G. Izergin and V. E. Korepin, *A lattice model related to the nonlinear Schrödinger equation*, Sov. Phys. Dokl. 26 (1981) 653–654.

[64] M. Jimbo, *A $q$-difference analogue of $U(g)$ and the Yang–Baxter equation*, Lett. Math. Phys. 10 (1985), 63–69.

[65] Guo-xing Ju, Shi-kun Wang and Ke Wu, *The Algebraic Structure of the $gl(n|m)$ Color Calogero-Sutherland Models*, J. Math. Phys. 39 (1998), 2813–2820.

[66] S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, *Combinatorics, the Bethe ansatz and representations of the symmetric group*, J. Sov. Math. 41 (1988), 916–924.

[67] S. Khoroshkin, D. Lebedev and S. Pakuliak, *Traces of intertwining operators for the Yangian double*, Lett. Math. Phys. 41 (1997), 31–47.
[68] S. M. Khoroshkin, A. A. Stolin and V. N. Tolstoy, *Rational solutions of Yang-Baxter equation and deformation of Yangians*, in ‘From field theory to quantum groups’, pp. 53–75, World Sci. Publishing, River Edge, NJ, 1996.

[69] S. M. Khoroshkin, A. A. Stolin and V. N. Tolstoy, *Deformation of Yangian Y(sl2)*, Comm. Algebra 26 (1998), 1041–1055.

[70] S. M. Khoroshkin and V. N. Tolstoy, *Yangian double*, Lett. Math. Phys. 36 (1996), 373–402.

[71] A. N. Kirillov, *Combinatorial identities and completeness of states for the Heisenberg magnet*, J. Sov. Math. 30 (1985), 2298–3310.

[72] A. N. Kirillov, *Completeness of states of the generalized Heisenberg magnet*, J. Sov. Math. 36 (1987), 115–128.

[73] A. N. Kirillov, A. Kuniba and T. Nakanishi, *Skew Young diagram method in spectral decomposition of integrable lattice models*, Comm. Math. Phys. 185 (1997), 441–465.

[74] A. N. Kirillov, A. Kuniba and T. Nakanishi, *Skew Young diagram method in spectral decomposition of integrable lattice models. II. Higher levels*, Nuclear Phys. B 529 (1998), 611–638.

[75] A. N. Kirillov and N. Yu. Reshetikhin, *Yangians, Bethe ansatz and combinatorics*, Lett. Math. Phys. 12, (1986) 199–208.

[76] A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, J. Soviet Math. 41 (1988), 925–955.

[77] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, J. Soviet Math. 52 (1990), 3156–3164.

[78] N. Kitanine, J.-M. Maillet and V. Terras, *Form factors of the XXZ Heisenberg spin-1/2 finite chain*, Nucl. Phys. B 554 (1999), 647–678.

[79] M. Kleber, *Combinatorial structure of finite-dimensional representations of Yangians: the simply-laced case*, Int. Math. Res. Not. 7 (1997), 187–201.

[80] H. Knight, *Spectra of tensor products of finite-dimensional representations of Yangians*, J. Algebra 174 (1995), 187–196.

[81] H. Konno, *Free Field Representation of Level-k Yangian Double DY(sl2)k and Deformation of Wakimoto*, Lett. Math. Phys. 40 (1997), 321–336.

[82] T. H. Koornwinder and V. B. Kuznetsov, *Gauss hypergeometric function and quadratic R-matrix algebras*, St. Petersburg Math. J. 6 (1994), 161–184.

[83] D. Korotkin and H. Samtleben, *Yangian Symmetry in Integrable Quantum Gravity*, Nucl. Phys. B 527 (1998), 657–689.

[84] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. 169 (1995), 1–23.
[85] P. P. Kulish and N. Yu. Reshetikhin, *Diagonalisation of GL(N) invariant transfer matrices and quantum N-wave system (Lee model)*, J. Phys. A 16 (1983), L591–L596.

[86] P. P. Kulish and N. Yu. Reshetikhin, *GL_{3}*-invariant solutions of the Yang-Baxter equation, J. Soviet Math. 34 (1986), 1948–1971.

[87] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Yang–Baxter equation and representation theory*, Lett. Math. Phys. 5 (1981), 393–403.

[88] P. P. Kulish and E. K. Sklyanin, *On the solutions of the Yang–Baxter equation*, J. Soviet Math. 19 (1982), 1596–1620.

[89] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method: recent developments*, in ‘Integrable Quantum Field Theories’, Lecture Notes in Phys. 151 Springer, Berlin-Heidelberg, 1982, pp. 61–119.

[90] P. P. Kulish and E. K. Sklyanin, *Algebraic structures related to reflection equations*, J. Phys. A 25 (1992), 5963–5975.

[91] P. P. Kulish, R. Sasaki and G. Schwiebert, *Constant solutions of reflection equations and quantum groups*, J. Math. Phys. 34 (1993), 286–304.

[92] P. P. Kulish and A. A. Stolin, *Deformed Yangians and integrable models*, Czechoslovak J. Phys. 47 (1997), 1207–1212.

[93] A. Kuniba and J. Suzuki, *Analytic Bethe Ansatz for Fundamental Representations of Yangians*, Commun. Math. Phys. 173 (1995), 225–264.

[94] V. B. Kuznetsov, M. F. Jørgensen and P. L. Christiansen, *New boundary conditions for integrable lattices*, J. Phys. A 28 (1995), 4639–4654.

[95] A. Leclair and F. Smirnov, *Infinite Quantum Group Symmetry of Fields in Massive 2D Quantum Field Theory*, Int. J. Mod. Phys. A 7 (1992), 2997–3022.

[96] B. Leclerc, M. Nazarov and J.-Y. Thibon, *Induced representations of affine Hecke algebras and the canonical bases for quantum groups*, preprint math.QA/0011074.

[97] S. Z. Levendorski˘ı, *On PBW bases for Yangians*, Lett. Math. Phys. 27 (1993), 37–42.

[98] S. Z. Levendorski˘ı, *On generators and defining relations of Yangians*, J. Geom. Phys. 12 (1993), 1–11.

[99] S. Z. Levendorski˘ı and A. Sudbery, *Yangian construction of the Virasoro algebra*, Lett. Math. Phys. 37 (1996), 243–247.

[100] A. Liguori, M. Mintchev and L. Zhao, *Boundary exchange algebras and scattering on the half line*, Comm. Math. Phys. 194 (1998), 569–589.

[101] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 2nd edition 1995.

[102] J. M. Maillet and J. Sanchez de Santos, *Drinfeld twists and algebraic Bethe ansatz*, in ‘L. D. Faddeev’s Seminar on Mathematical Physics’, pp. 137–178, Amer. Math. Soc. Transl. Ser. 2, 201, Amer. Math. Soc., Providence, RI, 2000.
[103] J. M. Maillet and V. Terras, *On the quantum inverse scattering problem*, Nucl. Phys. B 575 (2000), 627–644.

[104] M. Mintchev, E. Ragoucy, P. Sorba and Ph. Zaugg, *Yangian symmetry in the nonlinear Schrödinger hierarchy*, J. Phys. A 32 (1999), 5885–5900.

[105] M. Mintchev, E. Ragoucy and P. Sorba, *Spontaneous symmetry breaking in the gl(N)-NLS hierarchy on the half line*, preprint hep-th/0104079.

[106] A. I. Molev, *Representations of twisted Yangians*, Lett. Math. Phys. 26 (1992), 211–218.

[107] A. I. Molev, *Gelfand–Tsetlin basis for representations of Yangians*, Lett. Math. Phys. 30 (1994), 53–60.

[108] A. I. Molev, *Sklyanin determinant, Laplace operators and characteristic identities for classical Lie algebras*, J. Math. Phys. 36 (1995), 923–943.

[109] A. I. Molev, *Noncommutative symmetric functions and Laplace operators for classical Lie algebras*, Lett. Math. Phys. 35 (1995), 135-143.

[110] A. I. Molev, *Yangians and Laplace operators for classical Lie algebras*, in “Confronting the Infinite”, Proceedings of the Conference in Celebration of the 70th Years of H. S. Green and C. A. Hurst, pp. 239–245. World Scientific, Singapore 1995.

[111] A. I. Molev, *Stirling partitions of the symmetric group and Laplace operators for the orthogonal Lie algebra*, Discrete Math. 180 (1998), 281–300.

[112] A. I. Molev, *Finite-dimensional irreducible representations of twisted Yangians*, J. Math. Phys. 39 (1998), 5559–5600.

[113] A. I. Molev, *A basis for representations of symplectic Lie algebras*, Comm. Math. Phys. 201 (1999), 591–618.

[114] A. I. Molev, *A weight basis for representations of even orthogonal Lie algebras*, in “Combinatorial Methods in Representation Theory”, Adv. Studies in Pure Math., 28 (2000), 223–242.

[115] A. I. Molev, *Weight bases of Gelfand–Tsetlin type for representations of classical Lie algebras*, J. Phys. A: Math. Gen., 33 (2000), 4143–4168.

[116] A. I. Molev, *Irreducibility criterion for tensor products of Yangian evaluation modules*, Duke Math. Journal, to appear.

[117] A. I. Molev, *Yangians and transvector algebras*, Discrete Math., to appear.

[118] A. Molev and M. Nazarov, *Capelli identities for classical Lie algebras*, Math. Annalen 313 (1999), 315–357.

[119] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Surveys 51:2 (1996), 205–282.

[120] A. Molev and G. Olshanski, *Centralizer construction for twisted Yangians*, Selecta Math., N. S., 6 (2000), 269–317.
[121] S. Murakami and F. Göhmann, *Yangian Symmetry and Quantum Inverse Scattering Method for the One-Dimensional Hubbard Model*, Phys. Lett. A 227 (1997), 216–226.

[122] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. 14 (2001), 145–238.

[123] T. Nakanishi, *Fusion, mass, and representation theory of the Yangian algebra*, Nucl. Phys. B 439 (1995), 441–460.

[124] M. L. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. 21 (1991), 123–131.

[125] M. L. Nazarov, *Yangians of the ‘strange’ Lie superalgebras*, in ‘Quantum Groups (P. P. Kulish, Ed.)’, Lecture Notes in Math. 1510, Springer, Berlin-Heidelberg, 1992, pp. 90–97.

[126] M. Nazarov, *Yangians and Capelli identities*, in ‘Kirillov’s Seminar on Representation Theory’, (G. I. Olshanski, Ed.) Amer. Math. Soc. Transl. 181, AMS, Providence, 1998, pp. 139–163.

[127] M. Nazarov, *Yangian of the queer Lie superalgebra*, Comm. Math. Phys. 208 (1999), 195–223.

[128] M. Nazarov and G. Olshanski, *Bethe subalgebras in twisted Yangians*, Comm. Math. Phys. 178 (1996), 483–506.

[129] M. Nazarov and V. Tarasov, *Yangians and Gelfand–Zetlin bases*, Publ. RIMS, Kyoto Univ. 30 (1994), 459–478.

[130] M. Nazarov and V. Tarasov, *Representations of Yangians with Gelfand–Zetlin bases*, J. Reine Angew. Math. 496 (1998), 181–212.

[131] M. Nazarov and V. Tarasov, *On irreducibility of tensor products of Yangian modules*, Internat. Math. Research Notices (1998), 125–150.

[132] M. Nazarov and V. Tarasov, *On irreducibility of tensor products of Yangian modules associated with skew Young diagrams*, Duke Math. Journal, to appear.

[133] A. Okounkov, *Quantum immanants and higher Capelli identities*, Transform. Groups 1 (1996), 99–126.

[134] A. Okounkov and G. Olshanski, *Shifted Schur functions*, St. Petersburg Math. J. 9 (1998), 239–300.

[135] A. Okounkov and G. Olshanski, *Shifted Schur functions II. Binomial formula for characters of classical groups and applications*, in ‘Kirillov’s Seminar on Representation Theory’, (G. I. Olshanski, Ed.) Amer. Math. Soc. Transl. 181, AMS, Providence, 1998, pp. 245–271.

[136] G. I. Olshanski, *Extension of the algebra U(g) for infinite-dimensional classical Lie algebras g, and the Yangians Y(g(m))*. Soviet Math. Dokl. 36 (1988), 569–573.

[137] G. I. Olshanski, *Yangians and universal enveloping algebras*. J. Soviet Math. 47 (1989), 2466–2473.
[138] G. I. Olshanski, *Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians*, in ‘Topics in Representation Theory’ (A. A. Kirillov, Ed.), Advances in Soviet Math. 2, AMS, Providence RI, 1991, pp. 1–66.

[139] G. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in ‘Quantum Groups’ (P. P. Kulish, Ed.), Lecture Notes in Math. 1510, Springer, Berlin-Heidelberg, 1992, pp. 103–120.

[140] A. M. Perelomov and V. S. Popov, *Casimir operators for U(n) and SU(n)*, Soviet J. Nucl. Phys. 3 (1966), 676–680.

[141] A. M. Perelomov and V. S. Popov, *Casimir operators for the orthogonal and symplectic groups*, Soviet J. Nucl. Phys. 3 (1966), 819–824.

[142] A. M. Perelomov and V. S. Popov, *Casimir operators for semisimple Lie algebras*, Isv. AN SSSR 32 (1968), 1368–1390.

[143] H. Pfeiffer, *Factorizing twists and the universal R-matrix of the Yangian Y(sl_2)*, J. Phys. A: Math. Gen. 33 (2000), 8929–8951.

[144] E. Ragoucy, *Twisted Yangians and folded W-algebras*, preprint [math.QA/0012182](http://arxiv.org/abs/math.QA/0012182).

[145] E. Ragoucy and P. Sorba, *Yangians and finite W-algebras*, Quantum groups and integrable systems (Prague, 1998). Czechoslovak J. Phys. 48 (1998), 1483–1487.

[146] E. Ragoucy and P. Sorba, *Yangian realisations from finite W-algebras*, Comm. Math. Phys. 203 (1999), 551–572.

[147] N. Yu. Reshetikhin, *Integrable models of quantum one-dimensional magnets with O(n) and Sp(2k)-symmetry*, Theor. Math. Phys. 63 (1985), no. 3, 347–366.

[148] N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, *Central extensions of quantum current groups*, Lett. Math. Phys., 19 (1990), 133–142.

[149] N. Yu. Reshetikhin, L. A. Takhtajan and L. D. Faddeev, *Quantization of Lie Groups and Lie algebras*, Leningrad Math. J. 1 (1990), 193–225.

[150] K. Schoutens, *Yangian Symmetry in Conformal Field Theory*, Phys. Lett. B 331 (1994), 335–341.

[151] E. K. Sklyanin, *Quantum version of the method of inverse scattering problem*, J. Soviet. Math. 19 (1982), 1546–1596.

[152] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A21 (1988), 2375–2389.

[153] E. K. Sklyanin, *Separation of variables in the quantum integrable models related to the Yangian Y[sl(3)]*, J. Math. Sci. 80 (1996), 1861–1871.

[154] A. Soloviev, *Cartan-Weyl Basis for Yangian Double DY(sl_3)*, Theoret. and Math. Phys. 111 (1997), no. 3, 731–743.
[155] A. Stolin and P. Kulish, *New rational solutions of Yang-Baxter equation and deformed Yangians*, Czechoslovak J. Phys. 47 (1997), 123–129.

[156] K. Takemura, *The Yangian symmetry in the spin Calogero model and its applications*, J. Phys. A 30 (1997), 6185–6204.

[157] V. O. Tarasov, *Structure of quantum L-operators for the R-matrix of of the XXZ-model*, Theor. Math. Phys. 61 (1984), 1065–1071.

[158] V. O. Tarasov, *Irreducible monodromy matrices for the R-matrix of the XXZ-model and lattice local quantum Hamiltonians*, Theor. Math. Phys. 63 (1985), 440–454.

[159] V. O. Tarasov, *Cyclic monodromy matrices for sl(n) trigonometric R-matrices*, Commun. Math. Phys. 158 (1993), 459–484.

[160] K. Takemura and D. Uglov, *The orthogonal eigenbasis and norms of eigenvectors in the Spin Calogero-Sutherland Model*, J. Phys. A 30 (1997), 3685–3717.

[161] L. A. Takhtajan and L.D. Faddeev, *Quantum inverse scattering method and the Heisenberg XYZ-model*, Russian Math. Surv. 34 (1979), no. 5, 11–68.

[162] V. N. Tolstoy, *Connection between Yangians and quantum affine algebras*, in ‘New symmetries in the theories of fundamental interactions’ (Karpacz, 1996), pp. 99–117, PWN, Warsaw, 1997.

[163] V. N. Tolstoy, *q-deformation of Yangian Y(sl2)*, in ‘Lie theory and its applications in physics’ (Clausthal, 1995), pp. 179–186, World Sci. Publishing, River Edge, NJ, 1996.

[164] V. N. Tolstoy, *Drinfeldians*, in "Lie Theory and Its Applications in Physics II" (Clausthal, Germany, 1997).

[165] D. B. Uglov, *Symmetric functions and the Yangian decomposition of the Fock and basic modules of the affine Lie algebra slN*, Quantum many-body problems and representation theory, pp. 183–241, MSJ Mem., 1 (1998), Math. Soc. Japan.

[166] D. B. Uglov, *Yangian Gelfand-Zetlin bases, glN-Jack polynomials and computation of dynamical correlation functions in the spin Calogero-Sutherland model*, Comm. Math. Phys. 191 (1998), 663–696.

[167] D. B. Uglov, *Skew Schur functions and Yangian actions on irreducible integrable modules of slN*, Conference on Combinatorics and Physics (Los Alamos, NM, 1998). Ann. Comb. 4 (2000), 383–400.

[168] D. B. Uglov, *Yangian actions on higher level irreducible integrable modules of affine gl(N)*, preprint math.QA/9802048.

[169] D. B. Uglov and V. E. Korepin, *The Yangian symmetry of the Hubbard model*, Phys. Lett. A 190, 238–242 (1994).

[170] T. Umeda, *Newton’s formula for glN*, Proc. Amer. Math. Soc. 126 (1998), 3169–3175.

[171] M. Varagnolo, *Quiver varieties and Yangians*, Lett. Math. Phys. 53 (2000), 273–283.
[172] H. Weyl, *Classical Groups, their Invariants and Representations*, Princeton Univ. Press, Princeton NJ, 1946.

[173] D. P. Želobenko, *Compact Lie groups and their representations*. Transl. of Math. Monographs 40 AMS, Providence RI, 1973.

[174] R. B. Zhang, *Representations of super Yangian*, J. Math. Phys. 36 (1995), 3854–3865.

[175] R. B. Zhang, *The quantum super-Yangian and Casimir operators of U_q(gl(M|N))*, Lett. Math. Phys. 33 (1995), 263–272.

[176] R. B. Zhang, *The gl(M|N) super Yangian and its finite-dimensional representations*, Lett. Math. Phys. 37 (1996), 419–434.

[177] Y.-Z. Zhang, *Super-Yangian double and its central extension*, Phys. Lett. A 234 (1997), 20–26.