ON FUNCTIONS OF BOUNDED $\Lambda$-VARIATION AND INTEGRAL SMOOTHNESS

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Abstract. We obtain a necessary and sufficient condition for embeddings of integral Lipschitz classes $\text{Lip}(\alpha; p)$ into classes $\Lambda BV$ of functions of bounded $\Lambda$-variation.

1. Introduction

Jordan’s classical concept of bounded variation has been extended in several directions. One well-known generalization is the notion of functions of bounded $p$-variation, due to Wiener.

Waterman [19, 20] extended the class of functions of bounded variation in a different way. Let $f$ be a 1-periodic function on the real line. For any interval $I = [a, b]$, we set $f(I) = f(b) - f(a)$. Denote by $S$ the collection of all positive and nondecreasing sequences $\Lambda = \{\lambda_n\}$ such that $\lambda_n \to \infty$ and

$$
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.
$$

Let $\Lambda = \{\lambda_n\} \in S$, a function $f$ is said to be of bounded $\Lambda$-variation if

$$
v_{\Lambda}(f) = \sup I \sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty,
$$

where the supremum is taken over all sequences $I = \{I_n\}$ of nonoverlapping intervals contained in a period. The class of functions of bounded $\Lambda$-variation is denoted $\Lambda BV$. For a discussion on the origin of classes $\Lambda BV$, see [21]. Observe also that all functions of $\Lambda BV$ are bounded.

Denote by $L^p$ ($1 \leq p < \infty$) the class of all 1-periodic measurable functions $f$ such that

$$
\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p} < \infty.
$$

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For $f \in L^p$ ($1 \leq p < \infty$), the $L^p$-modulus of continuity of $f$ is given by

$$
\omega(f; \delta)_p = \sup_{0 \leq h \leq \delta} \left( \int_0^1 |f(x+h) - f(x)|^p \, dx \right)^{1/p} \quad (0 \leq \delta \leq 1).
$$

Let $\omega$ be any nondecreasing, continuous and subadditive functions defined on $[0, 1]$ with $\omega(0) = 0$. For $1 \leq p < \infty$, set

$$
H^\omega_p = \{ f \in L^p : \omega(f; \delta)_p = O(\omega(\delta)) \}.
$$

In the particular case $\omega(\delta) = \delta^\alpha$, we denote $H^\omega_p = \text{Lip}(\alpha; p)$.

Relations between $H^\omega_p$ and $\Lambda BV$ have attracted some interest in recent years. Kuprikov [8] obtained a sharp estimate of the $L^p$-modulus of continuity ($1 \leq p < \infty$) of a function in terms of its $\Lambda$-variation (for $p = 1$, such estimates were first obtained in [14] and [18]). Afterwards, Goginava [4] proved that Kuprikov’s estimate leads to the necessary and sufficient condition for the embedding

$$
\Lambda BV \subset H^\omega_p \quad (1 \leq p < \infty).
$$

This result was later generalized in [5].

H. Wang [17] studied the reverse embedding, that is

$$
H^\omega_p \subset \Lambda BV \quad (1 \leq p < \infty).
$$

In particular, he observed that a necessary condition for the embedding

$$
\text{Lip}(\alpha; p) \subset \Lambda BV \quad (1 < p < \infty, \ 1/p < \alpha < 1) \quad (1.1)
$$

is

$$
\sum_{n=1}^\infty \left( \frac{1}{\lambda_n} \right)^{1/(1-\alpha)} < \infty. \quad (1.2)
$$

Wang then conjectured that (1.2) is also a sufficient for (1.1) to hold. We remark that the condition $\alpha > 1/p$ in (1.1) is essential; for $\alpha \leq 1/p$, the class $\text{Lip}(\alpha; p)$ contains unbounded functions and (1.1) cannot hold.

The main result of this note is the following.

**Theorem 1.1.** Let $1 < p < \infty$ and $1/p < \alpha < 1$, and set

$$
 r = \frac{1}{\alpha - 1/p} \quad \text{and} \quad r' = \frac{1}{1 + 1/p - \alpha}.
$$

Then the embedding (1.1) holds if and only if

$$
\sum_{n=0}^\infty \left( \sum_{k=2^n}^{2^{n+1}} \left( \frac{1}{k^{\alpha - 1/p} \lambda_k} \right) \right)^{r'/p'} < \infty.
$$
Observe that it follows from Theorem 1.1 that the conjecture of Wang is not true.

Our proof of Theorem 1.1 makes essential use of estimates of $L^p$-moduli of continuity in terms of moduli of continuity in the space $V_p$ of functions of bounded $p$-variation.

In the final part of the paper, we show that $V_p$ can be expressed in terms of spaces $\Lambda BV$. For $p = 1$, this result was obtained by Perlman [10], who proved that

$$V_1 = \bigcap_{\Lambda \in S} \Lambda BV.$$  

We prove that for $1 < p < \infty$, there holds

$$V_p = \bigcap_{\Lambda \in S_{p'}} \Lambda BV, \quad (1.3)$$

where $\Lambda \in S_{p'}$ means that the sequence $\Lambda \in S$ satisfies

$$\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \right)^{p'} < \infty.$$  

In connection to (1.3), we mention that embeddings between $\Lambda BV$ and other spaces of functions of generalized bounded variation were previously studied in, e.g., [1, 2, 11, 13].

2. Auxiliary results

Let $f$ be a 1-periodic function on the real line and $1 \leq p < \infty$. Below, $\mathcal{I} = \{I_n\}$ will always denote a sequence of nonoverlapping intervals contained in a period. Given any such $\mathcal{I} = \{I_n\}$, set

$$v_p(f; \mathcal{I}) = \left( \sum_{n=1}^{\infty} |f(I_n)|^p \right)^{1/p}.$$  

The function $f$ is said to be of bounded $p$-variation if

$$v_p(f) = \sup_{\mathcal{I}} v_p(f; \mathcal{I}) < \infty,$$

where the supremum is taken over all sequences $\mathcal{I}$. For $p = 1$, this definition was given by Jordan, and for $p > 1$ by Wiener [22].

Given any $\mathcal{I} = \{I_n\}$, we also set $||\mathcal{I}|| = \sup_n |I_n|$, where $|I|$ denotes the length of the interval $I$. Following Terehin [15], we define the modulus of $p$-continuity for $1 < p < \infty$ by

$$\omega_{1-1/p}(f; \delta) = \sup_{||\mathcal{I}|| \leq \delta} v_p(f; \mathcal{I}) \quad (0 < \delta \leq 1),$$
where the supremum is taken over all \( I \) with \( \| I \| \leq \delta \). Note that \( \omega_{1-1/p}(f; 1) = v_p(f) \), the \( p \)-variation of \( f \).

For \( p > 1 \), we may have that \( \lim_{\delta \to 0} \omega_{1-1/p}(f; \delta) = 0 \) for nontrivial functions. Such functions are called \( p \)-continuous. It is not difficult to show that the best order of decay of the modulus of \( p \)-continuity is \( \omega_{1-1/p}(f; \delta) = O(\delta^{1/p'}) \), \( p' = p/(p - 1) \). Moreover, this rate of decay is attained for functions in \( W_1^p \), the class of 1-periodic absolutely continuous functions \( f \) such that \( f' \in L^p \). Indeed, it is a simple application of Hölder’s inequality to show that

\[
\omega_{1-1/p}(f; \delta) \leq \| f' \|_{p'} \delta^{1/p'} \quad (0 \leq \delta \leq 1). \tag{2.1}
\]

Conversely, it was shown in [15] that if \( f \) satisfies \( \omega_{1-1/p}(f; \delta) = O(\delta^{1/p'}) \), then \( f \in W_1^p \).

Let \( 1 < p < \infty \) and \( 0 < \alpha \leq 1 \). Define

\[
\| f \|_{\text{Lip}(\alpha; p)} = \sup_{\delta > 0} \frac{\omega(f; \delta)}{\delta^\alpha}, \tag{2.2}
\]

The next observation will be useful for us. Let \( 1 < p < \infty \) and \( 1/p < \alpha \leq 1 \). Then a function \( f \in \text{Lip}(\alpha; p) \) can be modified on set of measure 0 to be continuous. Moreover, Terehin [16, Corollary 1] proved that there exists a constant \( c_{p, \alpha} > 0 \) such that for the modified function \( \bar{f} \),

\[
c_{p, \alpha} \sup_{\delta > 0} \frac{\omega_{1-1/p}(\bar{f}; \delta)}{\delta^{\alpha-1/p}} \leq \| f \|_{\text{Lip}(\alpha; p)} \leq \sup_{\delta > 0} \frac{\omega_{1-1/p}(\bar{f}; \delta)}{\delta^{\alpha-1/p}}. \tag{2.3}
\]

Thus, if \( p > 1 \), \( 1/p < \alpha \leq 1 \) and \( f \) is a continuous 1-periodic function, then

\[
\omega(f; \delta) = O(\delta^\alpha) \quad \text{if and only if} \quad \omega_{1-1/p}(f; \delta) = O(\delta^{\alpha-1/p}). \tag{2.4}
\]

We shall also use the following construction.

**Definition 2.1.** Let \( I = [a, b] \subset [0, 1] \) be an interval, \( N \in \mathbb{N} \) and \( \mathbf{H} = (H_0, H_1, ..., H_{N-1}) \in \mathbb{R}^N \) be a vector with \( H_j \geq 0 \) for \( 0 \leq j \leq N-1 \). Set \( h = (b-a)/N \), \( \xi_j = a + jh \ (j = 0, 1, ..., N) \) and \( \xi_j^* = a + (j+1/2)h \ (j = 0, 1, ..., N-1) \). The function \( F(x) = F(I, N, \mathbf{H}; x) \) is defined to be the continuous 1-periodic function such that \( F(x) = 0 \) for \( x \in [0, 1] \setminus I \), \( F(\xi_j) = 0 \ (j = 0, 1, ..., N) \), \( F(\xi_j^*) = H_j \ (j = 0, 1, ..., N-1) \), and \( F \) is linear on each of the intervals \([\xi_j, \xi_j^*]\) and \([\xi_j^*, \xi_{j+1}]\) \( (j = 0, 1, ..., N-1) \).
Thus, the graph of $F$ consists of $N$ isosceles triangles of heights $H_j$ ($j = 0, ..., N - 1$) and bases $h$. It is easy to see that

$$v_p(F) = 2^{1/p} \left( \sum_{j=0}^{N-1} H_j^p \right) \quad (1 \leq p < \infty), \quad (2.5)$$

and

$$\|F'\|_p = 2h^{-1/p'} \left( \sum_{j=0}^{N-1} H_j^p \right)^{1/p} \quad (1 \leq p < \infty). \quad (2.6)$$

The next lemma is of a known type (cf. [12]). In particular, it can be proved in the same way as Lemma 2.4 in [7].

**Lemma 2.2.** Let $\{\alpha_k\} \in l^1$ be a sequence of non-negative numbers and let $\theta > 1$ and $\gamma > 0$. There exists a sequence $\{\beta_k\}$ of positive numbers such that

$$\alpha_k \leq \beta_k, \quad k \in \mathbb{N},$$

$$\sum_{k=1}^{\infty} \beta_k \leq \frac{\theta^{1+\gamma}}{(\theta - 1)(\theta^\gamma - 1)} \sum_{k=1}^{\infty} \alpha_k,$$

and

$$\theta^{-\gamma} \leq \frac{\beta_{k+1}}{\beta_k} \leq \theta, \quad k \in \mathbb{N}.$$

We shall also use the following Hardy-type inequality (see [9]).

**Lemma 2.3.** Let $\beta > 0$ and $1 < r < \infty$ be fixed. Let $\{a_k\}$ be a sequence of nonnegative real numbers, and $\{\nu_n\}$ an increasing sequence of positive real numbers with $\nu_0 = 1$. Then there exists a constant $c_{\beta,r} > 0$ such that

$$\sum_{n=0}^{\infty} 2^{-n\beta} \left( \sum_{1 \leq k \leq \nu_n} a_k \right)^{1/r} \leq c_{\beta,r} \sum_{n=1}^{\infty} 2^{-n\beta} \left( \sum_{\nu_{n-1} \leq k \leq \nu_n} a_k \right)^{1/r}. \quad (2.7)$$

Finally, we formulate the next well-known result (see, e.g., [3, Ch.6]).

**Lemma 2.4.** Let $1 < p < \infty$. Then $\{x_n\} \in l^p$ if and only if

$$\sum_{n=1}^{\infty} \alpha_n x_n < \infty,$$

for all $\{\alpha_n\} \in l^{p'}$. Moreover,

$$\sup_{\|\{\alpha_n\}\|_{p'} \leq 1} \sum_{n=1}^{\infty} \alpha_n x_n = \|\{x_n\}\|_p.$$
3. EMBEDDING OF LIPSCHITZ CLASSES

We shall now prove our main results. Recall that \( \|f\|_{\text{Lip}(\alpha;p)} \) is given by (2.2).

**Theorem 3.1.** Let \( \Lambda \in \mathcal{S} \) be given and \( 1 < p < \infty, \ 1/p < \alpha < 1 \). Set
\[
   r = \frac{1}{\alpha - 1/p} \quad \text{and} \quad r' = \frac{1}{1 + 1/p - \alpha}.
\]  
(3.1)

There exists a constant \( c_{p,\alpha} > 0 \) depending only on \( \alpha \) and \( p \) such that for any \( f \in \text{Lip}(\alpha;p) \),
\[
   v_\Lambda(f) \leq c_{p,\alpha} \|f\|_{\text{Lip}(\alpha;p)} \left( \sum_{n=0}^{\infty} \left( \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^{\alpha-1/p} \lambda_k} \right)^{r'/p'} \right)^{1/r'}.
\]  
(3.2)

**Proof.** In light of (2.3), we may without loss of generality assume that
\[
   \sup_{\delta > 0} \frac{\omega_{1-1/p}(f;\delta)}{\delta^{\alpha-1/p}} = 1.
\]  
(3.3)

Take an arbitrary sequence \( \mathcal{I} = \{I_j\} \) of nonoverlapping intervals contained in a period. Denote
\[
   \sigma_k(\mathcal{I}) = \{j : 2^{-k-1} < |I_j| \leq 2^{-k}\} \quad (k \geq 0).
\]

Then we have
\[
   V = \sum_{j=1}^{\infty} \frac{|f(I_j)|}{\lambda_j} = \sum_{k=0}^{\infty} \sum_{j \in \sigma_k(\mathcal{I})} \frac{|f(I_j)|}{\lambda_j}.
\]

We shall estimate \( V \). By Hölder’s inequality, we have
\[
   V \leq \sum_{k=0}^{\infty} \left( \sum_{j \in \sigma_k(\mathcal{I})} |f(I_j)|^p \right)^{1/p} \left( \sum_{j \in \sigma_k(\mathcal{I})} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'} \leq \sum_{k=0}^{\infty} \omega_{1-1/p}(f;2^{-k}) \left( \sum_{j \in \sigma_k(\mathcal{I})} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'}.
\]  
(3.4)

Thus, by (3.3) and (3.4)
\[
   V \leq \sum_{k=0}^{\infty} 2^{-k(\alpha-1/p)} \left( \sum_{j \in \sigma_k(\mathcal{I})} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'}.
\]  
(3.5)
Let the sequence \( \{\delta_n\} \) be defined by
\[
\text{card} \left( \bigcup_{k=0}^{n} \sigma_k(I) \right) = 2^n \delta_n,
\]
where, \( \text{card}(A) \) denotes the number of elements of the finite set \( A \). Set also \( \delta_{-1} = 0 \). There exists an \( n_0 \geq 0 \) such that \( \delta_n > 0 \) for all \( n \geq n_0 \), and we may assume \( n_0 = 0 \). We observe that \( \|\{\delta_n\}\|_{l^1} \leq 4 \). Indeed, first note that
\[
\infty \sum_{k=0}^\infty 2^{-k} \text{card}(\sigma_k(I)) \leq 2 \sum_{j=0}^\infty |I_j| \leq 2.
\]
On the other hand, for \( n \geq 0 \), we have
\[
2^{-n} \text{card}(\sigma_n(I)) = \delta_n - \delta_{n-1}/2.
\]
Whence, for any \( N \in \mathbb{N} \), we have
\[
\sum_{n=0}^{N+1} (\delta_n - \delta_{n-1}/2) = \delta_{N+1} + \frac{1}{2} \sum_{n=0}^N \delta_n \geq \frac{1}{2} \sum_{n=0}^N \delta_n,
\]
and consequently, \( \|\{\delta_n\}\|_{l^1} \leq 4 \).

Applying Lemma 2.2 with \( \theta = 2 \) and \( \gamma = 1/2 \) to \( \{\delta_k\} \) yields a sequence \( \{\beta_k\} \) such that \( \delta_k \leq \beta_k \),
\[
2^{-1/2} \leq \frac{\beta_{k+1}}{\beta_k} \leq 2 \quad (k \in \mathbb{N}) \quad \text{and} \quad \|\{\beta_k\}\|_{l^1} \leq 64. \tag{3.6}
\]
Set \( \nu_k = 2^k \beta_k \). By the first relation of (3.6), we have
\[
2^k \nu_k \leq \nu_{k+2} \leq 16 \nu_k \quad (k \in \mathbb{N}). \tag{3.7}
\]
Since \( \text{card}(\sigma_k(I)) \leq 2^k \delta_k = \nu_k \), and \( \{\lambda_j\} \) is increasing, we have by (3.5)
\[
V \leq \sum_{k=0}^{\infty} 2^{-k(\alpha-1/p)} \left( \sum_{1 \leq j \leq \nu_k} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'}.
\]
Applying (2.7) to the right-hand side of the previous inequality, we get
\[
V \leq c_{p,\alpha} \sum_{k=1}^{\infty} 2^{-k(\alpha-1/p)} \left( \sum_{\nu_{k-1} \leq j \leq \nu_k} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'},
\]
for some constant \( c_{p,\alpha} > 0 \). Since \( 2^{-k} = \beta_k/\nu_k \), we have
\[
V \leq c_{p,\alpha} \sum_{k=1}^{\infty} \left( \frac{\beta_k}{\nu_k} \right)^{\alpha-1/p} \left( \sum_{\nu_{k-1} \leq j \leq \nu_k} \left( \frac{1}{\lambda_j} \right)^{p'} \right)^{1/p'}. \tag{3.8}
\]
By using Hölder’s inequality with exponents \( r \) and \( r' \), and the second inequality of (3.6), we estimate the right-hand side of (3.8)

\[
V \leq c_{p,\alpha} \| \{ \beta_k^{1/r} \} \|_r \left( \sum_{k=1}^{\infty} \nu_k^{-r'(\alpha-1/p)} \left( \sum_{\nu_{k-1} \leq j \leq \nu_k} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'/p'} \right)^{1/r'} \right)
\leq 64c_{p,\alpha} \left( \sum_{k=1}^{\infty} \left( \sum_{\nu_{k-1} \leq j \leq \nu_k} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'/p'} \right)^{1/r'} \right)^{1/r'}.
\]  

(3.9)

By collecting the terms of the sum at the right-hand side of (3.9) in pairs, and using that \( a^q + b^q \leq 2(a + b)^q \) for any \( q \geq 0 \) and \( a, b \geq 0 \), we get

\[
V \leq 128c_{p,\alpha} \left( \sum_{k=0}^{\infty} \left( \sum_{\nu_{2k-1} \leq j \leq \nu_{2k}+1} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'/p'} \right)^{1/r'} \right)^{1/r'}.
\]  

(3.10)

For \( k \geq 0 \), we define \( m_k \geq 0 \) as the greatest integer \( m \) such that

\[
2^m < \nu_{2k}.
\]

By (3.7), we have \( 2\nu_{2k} \leq \nu_{2k+2} \), and thus,

\[
2^{m_k+1} < \nu_{2k+2}.
\]

Consequently,

\[
m_{k+1} \geq m_k + 1 \quad (k \geq 0).
\]  

(3.11)

Further, by (3.7), we have \( \nu_{2k+2} \leq 16\nu_{2k} \). Therefore, for all \( k \geq 0 \),

\[
[\nu_{2k}, \nu_{2k+2}] \subset [2^{m_k}, 2^{m_k+5}].
\]

Whence,

\[
\sum_{\nu_{2k} \leq j \leq \nu_{2k+2}} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'} \leq \sum_{j=2^{m_k}}^{2^{m_k+5}} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'}.
\]

Since the terms of the previous sum decrease, it follows that

\[
\sum_{j=2^{m_k}}^{2^{m_k+5}} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'} \leq 40 \sum_{j=2^{m_k}}^{2^{m_k+1}} \left( \frac{1}{j^{\alpha-1/p}} \right)^{r'}.
\]
Consequently, by the previous inequality and (3.10),

$$V \leq c'_{p,\alpha} \left( \sum_{n=1}^{\infty} \left( \sum_{k=2^n}^{2^{n+1}-1} \left( \frac{1}{k^{\alpha-1/p} \lambda_k} \right)^{p'} \right)^{1/p'} \right)^{1/r'}$$

for some $c'_{p,\alpha} > 0$. By (3.11), for each $k \geq 0$, the intersection of $[2^m, 2^{m+1}]$ and $[2^{m+1}, 2^{m+1}+1]$ consists of at most one point. Hence,

$$V \leq c'_{p,\alpha} \left( \sum_{n=0}^{\infty} \left( \sum_{j=2^n}^{2^{n+1}} \left( \frac{1}{j^{\alpha-1/p} \lambda_j} \right)^{p'} \right)^{1/r'} \right)^{1/r'}$$

This proves (3.2). $\square$

The estimate (3.2) is sharp in a sense. Namely, we have the following result.

**Theorem 3.2.** Let $\Lambda \in S$ be given, $1 < p < \infty$, $1/p < \alpha < 1$ and $r, r'$ be defined by (2.7). Then there exists a function $g$ and constants $c'_{p,\alpha}, c''_{p,\alpha} > 0$ depending only on $\alpha$ and $p$ such that

$$\omega_{1-1/p}(g; \delta) \leq c'_{p,\alpha} \delta^{\alpha-1/p} \quad (0 < \delta \leq 1),$$

and

$$v_\Lambda(g) \geq c''_{p,\alpha} \left( \sum_{n=1}^{\infty} \left( \sum_{k=2^n}^{2^{n+1}-1} \left( \frac{1}{k^{\alpha-1/p} \lambda_k} \right)^{p'/r'} \right)^{1/r'} \right)^{1/r}.$$ (3.13)

**Proof.** Let $\{\delta_n\} \in l^1$ be a fixed but arbitrary positive sequence with $\|\{\delta_n\}\|_l \leq 1$. Applying Lemma 2.2 with $\gamma = 1$ and $\theta = 3/2$ (the value of $\gamma$ does not matter, it is only important that $1 < \theta < 2$) to the sequence $\{\delta_n\}$, we obtain a positive sequence $\{\beta_n\}$ such that $\delta_n \leq \beta_n$ ($n \in \mathbb{N}$),

$$\frac{2}{3} \leq \frac{\beta_{n+1}}{\beta_n} \leq \frac{3}{2} \quad (n \in \mathbb{N}) \quad \text{and} \quad L = \|\{\beta_n\}\|_l \leq 9. \quad (3.14)$$

Subdivide the interval $[0, 1]$ into non-overlapping intervals $J_n$ ($n \in \mathbb{N}$) with $|J_n| = \beta_n/L$. For $n \in \mathbb{N}$, denote

$$S_n = \left( \sum_{k=2^n}^{2^{n+1}-1} \left( \frac{1}{\lambda_k} \right)^{p'/r'} \right)^{1/r'}$$

and

$$H_k^{(n)} = (2^{-n}\beta_n)^{\alpha-1/p} \lambda_k^{-1/(p-1)} S_n^{p'/p} \quad \text{for} \quad 2^n \leq k \leq 2^{n+1} - 1.$$
Let also $H_n = (H_{2^n}, H_{2^{n+1}}, ..., H_{2^{n+1}-1}) \in \mathbb{R}^{2^n}$. Put $F_n(x) = F(J_n, 2^n, H_n; x)$ (see Definition 2.1), and

$$g(x) = \sum_{n=1}^{\infty} F_n(x).$$

It is clear that

$$v_\Lambda(g) \geq 2 \sum_{n=1}^{\infty} \frac{2^{n+1}-1}{2^n} \sum_{k=2^n}^{2^{n+1}-1} \frac{H_k(n)}{\lambda_k}.$$ (3.15)

On the other hand,

$$\sum_{k=2^n}^{2^{n+1}-1} \frac{H_k(n)}{\lambda_k} = (2^{-n} \beta_n)^{\alpha-1/p} S_n.$$ (3.16)

Thus, since $\delta_k \leq \beta_k$ for $k \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \frac{H_k(n)}{\lambda_k} = \sum_{n=1}^{\infty} \frac{\delta^{\alpha-1/p}}{2^{n+1}-1} \sum_{k=2^n}^{2^{n+1}-1} \left( \frac{1}{k^{\alpha-1/p} \lambda_k} \right)^{1/p}.$$ (3.17)

We proceed to estimate $\omega_{1-1/p}(g; \delta)$. By the first relation of (3.14), we have

$$\frac{1}{3} \leq \frac{2^{-n} \beta_{n+1}}{2^{-n} \beta_n} \leq \frac{3}{4} < 1.$$ (3.18)

In particular, the sequence $\{2^{-n} \beta_n\}$ is strictly decreasing and $2^{-n} \beta_n \to 0$ as $n \to \infty$. Fix $0 < \delta < 1$. If $\delta > 2^{-1} \beta_1$, then we set $m = 0$. Otherwise, define $m \in \mathbb{N}$ to be the unique natural number such that

$$2^{-m} \beta_{m+1} < \delta \leq 2^{-m} \beta_m.$$ (3.19)

By (2.1), we have

$$\omega_{1-1/p}(g; \delta) \leq \delta^{1/p} \sum_{n=1}^{m} \| F_n \|_p + \sum_{n=m+1}^{\infty} v_p(F_n).$$ (3.17)
(The first sum is taken as zero if \( m = 0 \)). We shall estimate the terms at the right-hand side of (3.17). It follows from (2.5) and the definition of \( H_k^{(n)} \) that
\[
v_p(F_n) = 2^{1/p} \left( \sum_{k=2^n}^{2^{n+1}-1} (H_k^{(n)})^p \right)^{1/p} = 2^{1/p} (2^{-n} \beta_n)^{\alpha-1/p}.
\]
(3.18)

Further, by (2.6),
\[
\|F_n'\|_p = 2^{1/p'} \left( \frac{\beta_n L}{2^n} \right)^{-1/p'} \left( \sum_{k=2^n}^{2^{n+1}-1} (H_k^{(n)})^p \right)^{1/p'} = 2 L^{1/p'} (2^{-n} \beta_n)^{\alpha-1}.
\]
(3.19)

By the estimate \( L \leq 9 \), (3.17), (3.18) and (3.19),
\[
\omega_{1-1/p}(g; \delta) \leq 18 \delta^{1/p'} \sum_{n=1}^{m} (2^{-n} \beta_n)^{\alpha-1/p} + 2^{1/p} \sum_{n=m+1}^{\infty} (2^{-n} \beta_n)^{\alpha-1/p}.
\]
(3.20)

Since
\[
\left( \frac{2^{-n+1} \beta_{n-1}}{2^{-n} \beta_n} \right)^{\alpha-1} = \left( \frac{2 \beta_{n-1}}{\beta_n} \right)^{\alpha-1} = \left( \frac{\beta_n}{2 \beta_{n-1}} \right)^{1-\alpha} \leq \left( \frac{3}{4} \right)^{1-\alpha} < 1,
\]
we get
\[
\sum_{n=1}^{m} (2^{-n} \beta_n)^{\alpha-1} \leq (2^{-m} \beta_m)^{\alpha-1} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^{n(1-\alpha)} = c_\alpha (2^{-m} \beta_m)^{\alpha-1} \leq c_\alpha \delta^{\alpha-1}.
\]
(3.21)

Similarly,
\[
\sum_{n=m+1}^{\infty} (2^{-n} \beta_n)^{\alpha-1/p} \leq (2^{-m-1} \beta_{m+1})^{\alpha-1/p} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^{n(\alpha-1/p)} \leq c_{p,\alpha} \delta^{\alpha-1/p}.
\]
(3.22)

Thus, by (3.17), (3.21) and (3.22),
\[
\omega_{1-1/p}(g; \delta) \leq c'_{p,\alpha} \delta^{\alpha-1/p} \quad (0 < \delta \leq 1).
\]
(3.23)

Denote
\[
L_n = \left( \sum_{k=2^n}^{2^{n+1}-1} \left( \frac{1}{k^{\alpha-1/p} \lambda_k} \right)^{p'} \right)^{1/p'}.
\]
Clearly, \( \{\delta_n\} \in l^1 \) is equivalent to \( \{\delta_n^{\alpha-1/p}\} \in l^{r'} \). By Lemma 2.4, we can choose \( \{\delta_n\} \in l^1 \) such that

\[
\sum_{n=1}^{\infty} \delta_n^{\alpha-1/p} L_n \geq \frac{1}{2} \left( \sum_{n=1}^{\infty} L_n^{r'} \right)^{1/r'}.
\] (3.24)

If \( \{L_n\} \notin l^{r'} \), then we must interpret (3.24) in the sense that we may choose \( \{\delta_n\} \in l^1 \) such that the left-hand side of (3.24) is infinite. In any case, the function \( g \) constructed above with this choice of \( \{\delta_n\} \) satisfies (3.12) and (3.13), by (3.23), (3.16) and (3.24). \( \square \)

Clearly, it follows from Theorem 3.2 that the conjecture of Wang is not true (that is, (1.2) is not sufficient for the embedding (1.1)). Combining Theorems 3.1 and 3.2, we obtain Theorem 1.1.

**Remark 3.3.** For \( 1 \leq p < \infty, \alpha = 1 \), we have \( \text{Lip}(1;p) = W^{1}_p \). It is easy to show that the embedding \( W^{1}_p \subset \Lambda BV \) holds for all sequences \( \Lambda \in \mathcal{S} \).

4. A Perlman-type theorem

As mentioned in the introduction, Perlman [10] showed that

\[
V_1 = \bigcap_{\Lambda \in \mathcal{S}} \Lambda BV.
\]

Recall that \( \mathcal{S}_{p'} \) denotes the class of all sequences \( \Lambda = \{\lambda_n\} \in \mathcal{S} \) such that

\[
\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \right)^{p'} < \infty.
\]

In this section, we prove the following complement to Perlman’s theorem.

**Theorem 4.1.** Let \( 1 < p < \infty \). Then

\[
V_p = \bigcap_{\Lambda \in \mathcal{S}_{p'}} \Lambda BV.
\]

**Proof.** Let \( f \) be a given function and \( \{I_n\} \) an arbitrary sequence of nonoverlapping intervals contained in a period. Applying Hölder’s inequality, we have

\[
\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} \leq v_p(f) \left( \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \right)^{p'} \right)^{1/p'}.
\]
Thus, if $\Lambda \in S_{p'}$, then $V_{p} \subset \Lambda BV$. Whence,

$$V_{p} \subset \bigcap_{\Lambda \in S_{p'}} \Lambda BV.$$ 

Let now $f$ be a bounded function with $f \notin V_{p}$. Then there exists a sequence $\{J_{n}\}$ of nonoverlapping intervals contained in a period such that

$$\sum_{n=1}^{\infty} |f(J_{n})|^p = \infty.$$ 

Since $\{|f(J_{n})|\} \notin l^p$, there exists $\{\alpha_n\} \in l^p$ such that

$$\sum_{n=1}^{\infty} \alpha_n |f(J_{n})| = \infty,$$ 

by Lemma 2.4. We may assume that $\alpha_n > 0$ for all $n \in \mathbb{N}$ and that $\{|f(J_{n})|\}$ is ordered nonincreasingly. Let $\{\alpha_n^*\}$ be the nonincreasing rearrangement of $\{\alpha_n\}$, set $\lambda_n = 1/\alpha_n^*$ and $\Lambda = \{\lambda_n\}$. Since $\{|f(J_{n})|\}$ is nonincreasing, we have

$$\sum_{n=1}^{\infty} \frac{|f(J_{n})|}{\lambda_n} = \sum_{n=1}^{\infty} \alpha_n^* |f(J_{n})| \geq \sum_{n=1}^{\infty} \alpha_n |f(J_{n})| = \infty, \quad (4.1)$$

whence $f \notin \Lambda BV$. It remains to show that $\Lambda \in S_{p'}$. Clearly $\Lambda$ is a positive and nondecreasing sequence. Moreover, $|f(I)| \leq 2\|f\|_{\infty}$ for any interval. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \geq \frac{1}{2\|f\|_{\infty}} \sum_{n=1}^{\infty} \frac{|f(J_{n})|}{\lambda_n} = \infty,$$ 

by (4.1). Whence, $\Lambda \in S$. Furthermore, since $\{\alpha_n\} \in l^{p'}$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^{p'} = \sum_{n=1}^{\infty} (\alpha_n^*)^{p'} = \sum_{n=1}^{\infty} \alpha_n^{p'} < \infty.$$ 

Thus, $\{\lambda_n\} \in S_{p'}$. $\square$

**Remark 4.2.** As a generalization of the class $V_{p}$, L.C. Young [23] introduced the class $V_{\Phi}$ of functions of bounded $\Phi$-variation. A result similar to Theorem 4.1 can be proved also for $V_{\Phi}$.

**Remark 4.3.** We can apply Theorem 4.1 to prove that there is a sequence $\Lambda \in S$ that satisfies (1.2) but still $\text{Lip}(\alpha; p) \notin \Lambda BV$.

Note first that $1 < 1/\alpha < p < \infty$. It was proved in [6] that there exists a function $f$ such that $\omega_{1-1/p}(f; \delta) = O(\delta^{\alpha-1/p})$, and at the same
time $f \notin V_{1/\alpha}$. In light of (2.4), this means exactly that there is a function $f \in \text{Lip}(\alpha; p)$ such that $f \notin V_{1/\alpha}$. Theorem 4.1 states that

$$V_{1/\alpha} = \bigcap_{\Lambda \in S_{1/(1-\alpha)}} \Lambda BV. \quad (4.2)$$

Observe that $S_{1/(1-\alpha)}$ is the collection of all sequences in $S$ that satisfies (1.2). Since $f \notin V_{1/\alpha}$, (4.2) implies that for some $\Lambda \in S_{1/(1-\alpha)}$, we have $f \notin \Lambda BV$. But since $f \in \text{Lip}(\alpha; p)$, we have shown that there exists a $\Lambda$ that satisfies (1.2) while the embedding (1.1) does not hold.

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