Reachability-time games on timed automata*

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Abstract

In a reachability-time game, players Min and Max choose moves so that the time to reach a final state in a timed automaton is minimised or maximised, respectively. Asarin and Maler showed decidability of reachability-time games on strongly non-Zeno timed automata using a value iteration algorithm. This paper complements their work by providing a strategy improvement algorithm for the problem. It also generalizes their decidability result because the proposed strategy improvement algorithm solves reachability-time games on all timed automata. The exact computational complexity of solving reachability-time games is also established: the problem is EXPTIME-complete for timed automata with at least two clocks.

1 Introduction

Timed automata [3] are a fundamental formalism for modelling and analysis of real-time systems. They have rich theory, solid modelling and verification tool support [23, 17, 19], and they have been successfully applied to numerous industrial case studies. Timed automata are finite automata augmented by a finite number of continuous real variables which are called clocks because their values increase with time at unit rate. Every clock can be reset to an integer constant when a transition of the automaton is performed, and clock values can be compared to integers to constrain availability of transitions. Adding clocks to finite automata increases their expressive power and the fundamental reachability problem is PSPACE-complete for timed automata [3]. The natural optimization problems of minimizing and maximizing reachability-time in timed automata are also in PSPACE [14].

The reachability (or optimal reachability-time) problems in timed automata are fundamental to the verification of (quantitative timing) properties of systems modeled by timed automata [3]. On the other hand, the problem of control-program synthesis for real-time systems can be cast as a two-player reachability (or optimal reachability-time) games, where the two players, say Min and Max, correspond to the “controller” and the “environment”, respectively, and control-program synthesis corresponds to computing winning (or optimal) strategies for Min. In other words, for control-program synthesis we need to generalize optimization problems to competitive optimization problems. Reachability games [5] and reachability-time games [4] on timed automata are decidable. The former problem is EXPTIME-complete, but the elegant result of Asarin and Maler [4] for reachability-time games is limited to the class of strongly non-Zeno timed automata and no upper complexity bounds are given. A recent result of Henzinger and Prabhu [16] is that values of reachability-time games can be approximated for all timed automata, but computatability of the exact values was left open.

A generalization of timed automata to priced (or weighted) timed automata [7] allows a rich variety of applications, e.g., to scheduling [6, 1, 22, 24]. While the fundamental minimum reachability-price problem is PSPACE-complete [6, 8], the two-player reachability-price games are undecidable on priced timed automata with at least

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three clocks [10]. The reachability-price games are, however, decidable for priced timed automata with one clock [12], and on the class of strongly price-non-Zeno priced timed automata [2][11].

Our contribution. We show that the exact values of reachability-time games on arbitrary timed automata are uniformly computable; here uniformity means that the output of our algorithm allows us, for every starting state, to compute in constant time the value of the game starting from this state. In particular, unlike the paper of Asarin and Maler [4], we do not require timed automata to be strongly non-Zeno. We also establish the exact complexity to compute in constant time the value of the game starting from this state. In particular, unlike the paper of Asarin uniformly computable; here uniformity means that the output of our algorithm allows us, for every starting state, to obtain an elementary and constructive proof of the fundamental determinacy result for reachability-time games, which at the same time yields an efficient algorithm matching the EXPTIME lower bound for the problem. Those techniques were known for finite state systems [21][25] but we are not aware of any earlier algorithmic results based on optimality equations and strategy improvement for real-time systems such as timed automata.

Related and future work. A recent, concurrent, and independent work [13] establishes decidability of slightly different and more challenging reachability-time games “with the element of surprise” [15][16]. In our model of timed games players take turns to take unilateral decisions about the duration and type of subsequent game moves. Games with surprise are more general in two ways: in every round of the game players have a “time race” to be the first to perform a move; moreover, players are forbidden to use strategies which “stop the time”, because such strategies are arguably physically unrealistic and result in Zeno runs.

We conjecture that our principal technique of optimality equations and strategy improvement can be generalized to give an EXPTIME algorithm for reachability-time games with surprise, and we are currently working on it. We also believe that this technique is applicable to many other (competitive) optimization problems on (priced) timed automata and even on restricted classes of hybrid automata; we are currently working on optimality equations and strategy improvement for, e.g., average-time games on timed automata and on o-minimal hybrid systems [9].

2 Reachability-time games

We assume that, wherever appropriate, sets $\mathbb{N}$ of non-negative integers and $\mathbb{R}$ of reals contain a maximum element $\infty$, and we write $\mathbb{N}_{\geq 0}$ for the set of positive integers and $\mathbb{R}_{\geq 0}$ for the set of non-negative reals. For $n \in \mathbb{N}$, we write $[n]_\mathbb{N}$ for the set $\{0, 1, \ldots, n\}$, and $[n]_\mathbb{R}$ for the set $\{r \in \mathbb{R} : 0 \leq r \leq n\}$ of non-negative reals bounded by $n$. For $r \in \mathbb{R}_{\geq 0}$, we write $\lfloor r \rfloor$ for its integer part, and we write $\{r\}$ for its fractional part. For sets $X$ and $Y$, we write $[X \rightarrow Y]$ for the set of functions $F : X \rightarrow Y$, and $[X \rightarrow Y]$ for the set of partial functions $F : X \rightarrow Y$.

Timed automata. Fix a constant $k \in \mathbb{N}$ for the rest of this paper. Let $C$ be a finite set of clocks. A ($k$-bounded) clock valuation is a function $\nu : C \rightarrow [k]_\mathbb{R}$; we write $V$ for the set $[C \rightarrow [k]_\mathbb{R}]$ of clock valuations. If $\nu \in V$ and $t \in \mathbb{R}_{\geq 0}$ then we write $\nu + t$ for the clock valuation defined by $(\nu + t)(c) = \nu(c) + t$, for all $c \in C$. For a set $C' \subseteq C$ of clocks and a clock valuation $\nu : C \rightarrow \mathbb{R}_{\geq 0}$, we define $\text{Reset}(\nu, C')(c) = 0$ if $c \in C'$, and $\text{Reset}(\nu, C')(c) = \nu(c)$ if $c \notin C'$.

The set of clock constraints over the set of clocks $C$ is the set of conjunctions of simple clock constraints, which are constraints of the form $c \gg i$ or $c - c' \gg i$, where $c, c' \in C$, $i \in [k]_\mathbb{N}$, and $\gg \in \{<, >, =, \leq, \geq\}$. Note that there are finitely many simple clock constraints and hence the set of non-equivalent clock constraints is finite. For every clock valuation $\nu \in V$, let $CC(s)$ be the set of simple clock constraints which hold in $\nu \in V$. A clock region is a maximal set $P \subseteq V$, such that for all $\nu, \nu' \in P$, we have $CC(\nu) = CC(\nu')$. In other words, clock
regions are equivalence classes of the equivalence relation relating clock valuations which are indistinguishable by clock constraints. Observe that $\nu$ and $\nu'$ are in the same clock region iff all clocks have the same integer parts in $\nu$ and $\nu'$, and if the partial orders of the clocks determined by their fractional parts in $\nu$ and $\nu'$ are the same. For all $\nu \in V$, we write $[\nu]$ for the clock region of $\nu$.

A clock zone is a convex set of clock valuations which is a union of a set of clock regions. Note that a set of clock valuations is a zone iff it is definable by a clock constraint. For $W \subseteq V$, we write $\overline{W}$ for the closure of the set $W$, i.e., the smallest closed set in $V$ which contains $W$. Observe that for every clock zone $W$, the set $\overline{W}$ is also a clock zone.

Let $L$ be a finite set of locations. A configuration is a pair $(\ell, \nu)$, where $\ell \in L$ is a location and $\nu \in V$ is a clock valuation; we write $Q$ for the set of configurations. If $s = (\ell, \nu) \in Q$ and $c \in C$, then we write $s(c) = \nu(c)$. A region is a pair $(\ell, P)$, where $\ell \in L$ is a location and $P$ is a clock region. If $s = (\ell, \nu)$ is a configuration then we write $[s]$ for the region $(\ell, [\nu])$. We write $R$ for the set of regions. A set $Z \subseteq S$ is a zone if for every $\ell \in L$, there is a clock zone $W_\ell$, such that $Z = \{(\ell, \nu) : \ell \in L \text{ and } \nu \in W_\ell\}$. For a region $R = (\ell, P) \in R$, we write $R$ for the zone $\{(\ell, \nu) : \nu \in P\}$.

A timed automaton $T = (L, C, S, A, E, \delta, \rho, F)$ consists of a finite set of locations $L$, a finite set of clocks $C$, a set of states $S \subseteq Q$, a finite set of actions $A$, an action enabledness function $E : A \rightarrow 2^C$, a transition function $\delta : L \times A \rightarrow L$, a clock reset function $\rho : A \rightarrow 2^C$, and a set of final states $F \subseteq S$. We further require that $T, S, F$, and $E(a)$ for all $a \in A$, are zones.

For a configuration $s = (\ell, \nu) \in Q$ and $t \in \mathbb{R}_{\geq 0}$, we define $s + t$ to be the configuration $s' = (\ell, \nu + t)$ if $\nu + t \in V$, and we then write $s \rightarrow_t s'$. We write $s \rightarrow_t s'$ if $s \rightarrow_t s'$ and for all $t' \in [0, t]$, we have $(\ell, s + t') \in S$. For an action $a \in A$, we define $\text{Succ}(s, a)$ to be the configuration $s' = (\ell', \nu')$, where $\ell' = \delta(\ell, a)$ and $\nu' = \text{Reset}(\nu, \rho(a))$, and we then write $s \stackrel{a}{\rightarrow} s'$. We write $s \Rightarrow a s'$ if $s \stackrel{a}{\rightarrow} s'$; $s, s' \in S$; and $s \in E(a)$. For technical convenience and without loss of generality we will assume throughout that timed automata satisfy the requirement that for every $s \in S$, there exists $a \in A$, such that $s \stackrel{a}{\rightarrow} s'$.

For $s, s' \in S$, we say that $s'$ is in the future of $s$, or equivalently, that $s$ is in the past of $s'$, if there is $t \in \mathbb{R}_{\geq 0}$, such that $s \rightarrow_t s'$; we then write $s \rightarrow s'$. For $R, R' \in R$, we say that $R'$ is in the future of $R$, or that $R$ is in the past of $R'$, if there is $s \in R$ and there is $s' \in R'$, such that $s'$ is in the future of $s$; we then write $R \rightarrow R'$. We say that $R'$ is the time successor of $R$ if $R \rightarrow R'$, $R \neq R'$, and for every $R'' \in R$, we have that $R \rightarrow R'' \rightarrow R'$ implies $R'' = R$ or $R'' = R'$; we then write $R \Rightarrow_{+1} R'$ or $R' \Leftarrow_{+1} R$. Similarly, for $R, R' \in R$, we write $R \Rightarrow R'$ if there is $s \in R$, and there is $s' \in R'$, such that $s \stackrel{a}{\rightarrow} s'$.

We say that a region $R \in R$ is thin if for every $s \in R$ and every $\varepsilon > 0$, we have that $[s] \neq [s + \varepsilon]$; other regions are called thick; we write $R_{\text{Thin}}$ and $R_{\text{Thick}}$ for the sets of thin and thick regions, respectively. Note that if $R \in R_{\text{Thick}}$ then for every $s \in R$, there is an $\varepsilon > 0$, such that $[s] = [s + \varepsilon]$. Observe also, that the time successor of a thin region is thick and vice versa.

A timed action is a pair $\tau = (a, t) \in A \times \mathbb{R}_{\geq 0}$. For $s \in Q$, we define $\text{Succ}(s, \tau) = \text{Succ}(s, (a, t))$ to be the configuration $s' = \text{Succ}(s + t, a)$, i.e., such that $s \rightarrow s'' \stackrel{a}{\rightarrow} s'$, and we then write $s \stackrel{a}{\rightarrow} s'$. We write $s \stackrel{a}{\rightarrow} s'$ if $s \rightarrow s'' \stackrel{a}{\rightarrow} s'$. If $\tau = (a, t)$ then we write $s \Rightarrow_{\tau} s'$ instead of $s \stackrel{a}{\rightarrow} s'$, and $s \Leftarrow_{\tau} s'$ instead of $s \stackrel{a}{\rightarrow} s'$.

A finite run of a timed automaton is a sequence $\langle s_0, \tau_1, s_1, \tau_2, \ldots, \tau_n, s_n \rangle \in S \times ((A \times \mathbb{R}_{\geq 0}) \times S)^*$, such that for all $i, 1 \leq i \leq n$, we have $s_{i-1} \Leftarrow_{\tau_i} s_i$. For a finite run $r = \langle s_0, \tau_1, s_1, \tau_2, \ldots, \tau_n, s_n \rangle$, we define $\text{Length}(r) = n$, and we define $\text{Last}(r) = s_n$ to be the state in which the run ends. We write $R_{\text{Runfin}}$ for the set of finite runs. An infinite run of a timed automaton is a sequence $r = \langle s_0, \tau_1, s_1, \tau_2, \ldots \rangle$, such that for all $i \geq 1$, we have $s_{i-1} \Leftarrow_{\tau_i} s_i$. For an infinite run $r$, we define $\text{Length}(r) = \infty$. For a run $r = \langle s_0, \tau_1, s_1, \tau_2, \ldots \rangle$, we define $\text{Stop}(r) = \inf\{i : s_i \in F\}$ and $\text{Time}(r) = \sum_{i=1}^{\text{Length}(r)} t_i$; and we define $\text{RT}(r) = \sum_{i=1}^{\text{Stop}(r)} t_i$ if $\text{Stop}(r) < \infty$, and $\text{RT}(r) = \infty$ if $\text{Stop}(R) = \infty$, where for all $i \geq 1$, we have $t_i = (a_i, t_i)$. 

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Strategies. A reachability-time game $\Gamma$ is a triple $(T, L_{\text{Min}}, L_{\text{Max}})$, where $T = (L, C, S, A, E, \delta, \rho, F)$ is a timed automaton and $(L_{\text{Min}}, L_{\text{Max}})$ is a partition of $L$. We define $Q_{\text{Min}} = \{(l, v) \in Q : l \in L_{\text{Min}}\}$, $Q_{\text{Max}} = Q \setminus Q_{\text{Min}}$, $S_{\text{Min}} = S \cap Q_{\text{Min}}$, $S_{\text{Max}} = S \setminus S_{\text{Min}}$, $R_{\text{Min}} = \{[s] : s \in Q_{\text{Min}}\}$, and $R_{\text{Max}} = R \setminus R_{\text{Min}}$.

A strategy for Min is a function $\mu : R_{\text{Min}}^s \rightarrow A \times \mathbb{R}_{\geq 0}$ such that if $\text{Last}(r) = s \in S_{\text{Min}}$ and $\mu(r) = \tau$ then $s \xrightarrow{\tau} s'$, where $s' = \text{Succ}(s, \tau)$. Similarly, a strategy for Max is a function $\chi : R_{\text{Max}}^s \rightarrow A \times \mathbb{R}_{\geq 0}$ such that if $\text{Last}(r) = s \in S_{\text{Max}}$ and $\chi(r) = \tau$ then $s \xrightarrow{\tau} s'$, where $s' = \text{Succ}(s, \tau)$. We write $\Sigma_{\text{Min}}$ and $\Sigma_{\text{Max}}$ for the sets of strategies for Min and Max, respectively. If players Min and Max use strategies $\mu$ and $\chi$, respectively, then the $(\mu, \chi)$-run from a state $s$ is the unique run $\text{Run}(s, \mu, \chi) = \langle s_0, \tau_1, s_1, \tau_2, \ldots \rangle$, such that $s_0 = s$, and for every $i \geq 1$, if $s_i \in S_{\text{Min}}$ or $s_i \in S_{\text{Max}}$, then $\mu(\text{Run}_i(s, \mu, \chi)) = \tau_{i+1}$, or $\chi(\text{Run}_i(s, \mu, \chi)) = \tau_{i+1}$, respectively, where $\text{Run}_i(s, \mu, \chi) = \langle s_0, \tau_1, s_1, \ldots, s_{i-1}, \tau_i, s_i \rangle$.

We say that a strategy $\mu$ for Min is positional if for all finite runs $r, r' \in R_{\text{Min}}^s$, we have that $\text{Last}(r) = \text{Last}(r')$ implies $\mu(r) = \mu(r')$. A positional strategy for Min can be then represented as a function $\mu : S_{\text{Min}} \rightarrow A \times \mathbb{R}_{\geq 0}$, which uniquely determines the strategy $\mu^\infty \in \Sigma_{\text{Min}}$ as follows: $\mu^\infty(r) = \mu(\text{Last}(r))$, for all finite runs $r \in R_{\text{Min}}^s$. Positional strategies for Max are defined and represented in the analogous way. We write $\Pi_{\text{Min}}$ and $\Pi_{\text{Max}}$ for the sets of positional strategies for Min and for Max, respectively.

Value of reachability-time game and optimality equations $\text{Opt}(\Gamma)$. For every $s \in S$, we define its upper value $\text{Val}^\star(s)$ and its lower value $\text{Val}^\star(s)$ by $\text{Val}^\star(s) = \inf_{\mu \in \Pi_{\text{Min}}} \sup_{\chi \in \Sigma_{\text{Max}}} \text{RT}(\text{Run}(s, \mu, \chi))$ and $\text{Val}^\star(s) = \sup_{\chi \in \Sigma_{\text{Max}}} \inf_{\mu \in \Pi_{\text{Min}}} \text{RT}(\text{Run}(s, \mu, \chi))$. The inequality $\text{Val}^\star(s) \leq \text{Val}^\star(s)$ always holds. A reachability-time game is determined if for every $s \in S$, its lower and upper values are equal to each other; then we say that the value $\text{Val}(s)$ exists and $\text{Val}(s) = \text{Val}^\star(s)$.

For strategies $\mu \in \Sigma_{\text{Min}}$ and $\chi \in \Sigma_{\text{Max}}$, we define $\text{Val}^\mu(s) = \sup_{\chi \in \Sigma_{\text{Min}}} \text{RT}(\text{Run}(s, \mu, \chi))$, and $\text{Val}^\chi(s) = \inf_{\mu \in \Pi_{\text{Min}}} \text{RT}(\text{Run}(s, \mu, \chi))$. For an $\varepsilon > 0$, we say that a strategy $\mu \in \Sigma_{\text{Min}}$ or $\chi \in \Sigma_{\text{Max}}$ is $\varepsilon$-optimal if for every $s \in S$, we have $\text{Val}^\mu(s) \leq \text{Val}(s) + \varepsilon$ or $\text{Val}^\chi(s) \geq \text{Val}(s) - \varepsilon$, respectively. Note that if a game is determined then for every $\varepsilon > 0$, both players have $\varepsilon$-optimal strategies.

We say that a reachability-time game is positionally determined if for every $s \in S$, we have $\text{Val}(s) = \inf_{\mu \in \Pi_{\text{Min}}} \sup_{\chi \in \Sigma_{\text{Max}}} \text{RT}(\text{Run}(s, \mu, \chi))$ and $\text{Val}(s) = \sup_{\chi \in \Sigma_{\text{Max}}} \inf_{\mu \in \Pi_{\text{Min}}} \text{RT}(\text{Run}(s, \mu, \chi))$. Note that if the reachability-time game is positionally determined then for every $\varepsilon > 0$, both players have positional $\varepsilon$-optimal strategies. Our results (Lemma 2, Theorem 6, and Theorem 18) yield a constructive proof of the following fundamental result for reachability-time games.

**Theorem 1 (Positional determinacy).** Reachability-time games are positionally determined.

Let $\Gamma$ be a reachability-time game, and let $T : S \rightarrow \mathbb{R}$ and $D : S \rightarrow \mathbb{N}$. We write $(T, D) \models \text{Opt}_{\text{MinMax}}(\Gamma)$, and we say that $(T, D)$ is a solution of optimality equations $\text{Opt}_{\text{MinMax}}(\Gamma)$, if for all $s \in S$, we have:

- if $D(s) = \infty$ then $T(s) = \infty$; and if $s \in F$ then $(T(s), D(s)) = (0, 0)$;
- if $s \in S_{\text{Min}} \setminus F$ then $T(s) = \inf_{a,t} \{t + T(s') : s \xrightarrow{a} t s'\}$, and $D(s) = \min \{1 + d' : T(s) = \inf_{a,t} \{t + T(s') : s \xrightarrow{a} t s'\} \}$; and
- if $s \in S_{\text{Max}} \setminus F$ then $T(s) = \sup_{a,t} \{t + T(s') : s \xrightarrow{a} t s'\}$, and $D(s) = \max \{1 + d' : T(s) = \sup_{a,t} \{t + T(s') : s \xrightarrow{a} t s'\} \}$. $$

**Lemma 2 ($\varepsilon$-Optimal strategies from optimality equations).** If $(T, D) \models \text{Opt}_{\text{MinMax}}(\Gamma)$, then for all $s \in S$, we have $\text{Val}(s) = T(s)$ and for every $\varepsilon > 0$, both players have positional $\varepsilon$-optimal strategies.
Simple functions and simple timed actions. Let \( X \subseteq Q \). A function \( F : X \to \mathbb{R} \) is simple if either: there is \( e \in \mathbb{Z} \) such that for every \( s \in X \), we have \( F(s) = e \); or there are \( e \in \mathbb{Z} \) and \( c \in C \), such that for every \( s \in X \), we have \( F(s) = e - s(c) \).

Let \( X \subseteq Q \) be convex and let \( F : X \to \mathbb{R} \) be a continuous function. We write \( \overline{F} \) for the unique continuous function \( F' : \overline{X} \to \mathbb{R} \), such that for all \( s \in X \), we have \( F'(s) = F(s) \). Observe that if \( F \) is simple, then \( \overline{F} \) is simple. For functions \( F, F' : X \to \mathbb{R} \) we define functions \( \max(F, F') \), \( \min(F, F') : X \to \mathbb{R} \) by \( \max(F, F')(s) = \max\{ F(s), F'(s) \} \) and \( \min(F, F')(s) = \min\{ F(s), F'(s) \} \), for every \( s \in X \).

**Lemma 3.** Let \( F, F' : R \to \mathbb{R} \) be simple functions defined on a region \( R \subseteq \mathcal{R} \). Then either \( \min(\overline{F}, \overline{F'}) = \overline{F} \) and \( \max(\overline{F}, \overline{F'}) = \overline{F'} \), or \( \min(\overline{F}, \overline{F'}) = \overline{F'} \) and \( \max(\overline{F}, \overline{F'}) = \overline{F} \). In particular, both \( \min(\overline{F}, \overline{F'}) \) and \( \max(\overline{F}, \overline{F'}) \) are simple functions.

Define the finite set of simple timed actions \( \mathcal{A} = A \times \llbracket k \rrbracket_\mathbb{N} \times C \). For \( s \in Q \) and \( \alpha = (a, b, c) \in \mathcal{A} \), we define \( t(s, \alpha) = b - s(c) \) if \( s(c) \leq b \), and \( t(s, \alpha) = 0 \) if \( s(c) > b \); and we define Succ\((s, \alpha)\) to be the state \( s' = \text{Succ}(s, \tau(\alpha)) \), where \( \tau(\alpha) = (a, t(s, \alpha)) \); we then write \( s \overset{\alpha}{\rightarrow} s' \). Note that if \( \alpha \in \mathcal{A} \) and \( s \overset{\alpha}{\rightarrow} s' \) then \([s'] \in \mathcal{R}_{\text{Thin}} \). Observe that for every thin region \( R' \subseteq \mathcal{R}_{\text{Thin}} \), there is a number \( b \in \llbracket k \rrbracket_\mathbb{N} \) and a clock \( c \in C \), such that for every \( R \subseteq \mathcal{R} \) in the past of \( R' \), we have that \( s \in R \) implies \((s + (b - s(c))) \in R'; \) then write \( R \overset{\alpha}{\rightarrow} R' \). For \( \alpha = (a, b, c) \in \mathcal{A} \) and \( R, R' \subseteq \mathcal{R} \), we write \( R \overset{\alpha}{\rightarrow} R' \) if \( R \overset{\alpha}{\rightarrow} R'' \overset{\alpha}{\rightarrow} R' \), for some \( R'' \subseteq \mathcal{R}_{\text{Thin}} \). For \( \alpha \in \mathcal{A} \) and \( R, R' \subseteq \mathcal{R} \), if \( R \overset{\alpha}{\rightarrow} R' \) and \( F : R' \to \mathbb{R} \) then we define the functions \( F^\square_{\alpha} : R \to \mathbb{R} \) and \( F^\square_{\alpha} : R \to \mathbb{R} \) by \( F^\square_{\alpha}(s) = t(s, \alpha) + F(\text{Succ}(s, \alpha)) \) and \( F^\square_{\alpha}(s) = 1 + F(\text{Succ}(s, \alpha)) \), for all \( s \in R \).

**Proposition 4.** Let \( \alpha \in \mathcal{A} \) and \( R, R' \subseteq \mathcal{R} \). If \( R \overset{\alpha}{\rightarrow} R' \) and \( F : R' \to \mathbb{R} \) is simple, then \( F^\square_{\alpha} \) is simple.

For \( \alpha \in \mathcal{A} \) and \( R, R', R'' \subseteq \mathcal{R} \), if \( R \overset{\alpha}{\rightarrow} R'' \), then we define the partial function \( F^\square_{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R} \) by \( F^\square_{\alpha}(t) = t + F(\text{Succ}(s, (a, t))) \), for all \( t \in \mathbb{R}_{\geq 0} \), such that \((s + t) \in R'' \); note that the domain \( \{t \in \mathbb{R}_{\geq 0} : (s + t) \in R'' \} \) of \( F^\square_{\alpha} \) is an interval.

**Proposition 5.** Let \( \alpha \in \mathcal{A} \) and \( R, R', R'' \subseteq \mathcal{R} \). If \( R \overset{\alpha}{\rightarrow} R'' \), \( s \in R \), and \( F : R' \to \mathbb{R} \) is simple, then \( F^\square_{\alpha} : I \to \mathbb{R} \), where \( I = \{t \in \mathbb{R}_{\geq 0} : (s + t) \in R'' \} \), is continuous and nondecreasing.

3 Timed region graph

Timed region graph \( \widehat{\Gamma} \). Let \( \Gamma = (T, \mathcal{L}_{\text{Min}}, \mathcal{L}_{\text{Max}}) \) be a reachability-time game. We define the timed region graph \( \widehat{\Gamma} \) to be the finite edge-labelled graph \((\mathcal{R}, \mathcal{M}) \), where the set \( \mathcal{R} \) of regions of timed automaton \( T \) is the set of vertices, and the labelled edge relation \( \mathcal{M} \subseteq \mathcal{R} \times A \times \mathcal{R} \) is defined in the following way. For \( \alpha = (a, b, c) \in \mathcal{A} \) and \( R, R' \subseteq \mathcal{R} \) we have \((R, \alpha, R') \in \mathcal{M} \), sometimes denoted by \( R \overset{\alpha}{\rightarrow} R' \), if and only if one of the following conditions holds:

- there is an \( R'' \subseteq \mathcal{R} \), such that \( R \overset{\alpha}{\rightarrow} R'' \to R' \); or
- \( R \in \mathcal{R}_{\text{Min}} \), and there are \( R'', R''' \subseteq \mathcal{R} \), such that \( R \overset{\alpha}{\rightarrow} R'' \to_{+1} R''' \overset{\alpha}{\rightarrow} R' \); or
- \( R \in \mathcal{R}_{\text{Max}} \), and there are \( R'', R''' \subseteq \mathcal{R} \), such that \( R \overset{\alpha}{\rightarrow} R'' \to_{+1} R''' \overset{\alpha}{\rightarrow} R' \).

Observe that in all the cases above we have that \( R'' \subseteq \mathcal{R}_{\text{Thin}} \) and \( R''' \subseteq \mathcal{R}_{\text{Thin}} \). The motivation for the second case is the following. Let \( R \overset{\alpha}{\rightarrow} R'' \overset{\alpha}{\rightarrow} R' \), where \( R \in \mathcal{R}_{\text{Min}} \) and \( R''' \subseteq \mathcal{R}_{\text{Thin}} \). One of the main results that we will implicitly establish is that in a state \( s \in R \), among all \( t \in \mathbb{R}_{\geq 0} \), such that \( s + t \in R'' \), the smaller the \( t \), the “better” the timed action \( (a, t) \) is for player Min. Note, however, that the set \( \{t \in \mathbb{R}_{\geq 0} : s + t \in R'' \} \) is an open interval because \( R'' \subseteq \mathcal{R}_{\text{Thin}} \), and hence it does not have the smallest element. Therefore, for every \( s \in R \), we
model the “best” time to wait, when starting from \( s \), before performing an \( a \)-labelled transition from region \( R'' \) to region \( R' \), by taking the infimum of the set \( \{ t \in \mathbb{R}_{\geq 0} : s + t \in R'' \} \). Observe that this infimum is equal to the \( t_{R''} \in \mathbb{R}_{\geq 0} \), such that \( s + t_{R''} \in R'' \), where \( R'' \rightarrow_1 R''' \), and that \( t_{R''} = b - s(c) \), where \( R \rightarrow_{b,c} R'' \). In the timed region graph \( \hat{\Gamma} \), we summarize this model of the “best” timed action from region \( R \) to region \( R' \) via region \( R''' \), by having a move \(( R, \alpha, R' ') \in \mathcal{M} \), where \( \alpha = (a, b, c) \). The motivation for the first and the third cases of the definition of \( \mathcal{M} \) is similar.

**Regional functions and optimality equations** \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \). Recall from Section 2 that a solution of optimality equations \( \text{Opt}_{\text{MinMax}}(\Gamma) \) for a reachability-time game \( \Gamma \) is a pair of functions \(( T, D ) \), such that \( T : S \rightarrow \mathbb{R} \) and \( D : S \rightarrow \mathbb{N} \). Our goal is to define analogous optimality equations \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \) for the timed region graph \( \hat{\Gamma} \).

If \( R \overset{\alpha}{\rightarrow} R' \), where \( R, R' \in \mathcal{R} \) and \( \alpha \in \mathcal{A} \), then \( s \in R \) does not in general imply that \( \text{Succ}(s, \alpha) \in R' \); it is however the case that \( s \in R \) implies \( \text{Succ}(s, \alpha) \in R' \). In order to correctly capture the constraints for successor states which fall out of the “target” region \( R' \) of a move of the form \( R \overset{\alpha}{\rightarrow} R' \), we consider, as solutions of optimality equations \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \), *regional functions* of types \( T : \mathcal{R} \rightarrow [S \rightarrow \mathbb{R}] \) and \( D : \mathcal{R} \rightarrow [S \rightarrow \mathbb{N}] \), where for every \( R \in \mathcal{R} \), the domain of partial functions \( T(R) \) and \( D(R) \) is \( \overline{R} \). Sometimes, when defining a regional function \( F : \mathcal{R} \rightarrow [S \rightarrow \mathbb{R}] \), it will only be natural to define \( F(R) \) for all \( s \in R \), instead of all \( s \in \overline{R} \). This is not a problem, however, because as discussed in Section 2 defining \( F(R) \) on the region \( R \) uniquely determines the continuous extension of \( F(R) \) to \( \overline{R} \). For a function \( F : \mathcal{R} \rightarrow [S \rightarrow \mathbb{R}] \), we define the function \( \hat{F} : S \rightarrow \mathbb{R} \) by \( \hat{F}(s) = F([s])(s) \).

Let \( T : \mathcal{R} \rightarrow [S \rightarrow \mathbb{R}] \) and let \( D : \mathcal{R} \rightarrow [S \rightarrow \mathbb{N}] \). We write \(( T, D ) \models \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \) if for all \( s \in S \), we have the following:

- if \( s \in F \) then \(( \hat{T}(s), \hat{D}(s)) = (0,0) \);
- if \( s \in S_{\text{Min}} \) then \(( \hat{T}(s), \hat{D}(s)) = \min^{\mathsf{lex}}_{m \in \mathcal{M}} \{ (T(R')^{\oplus}_{\alpha}(s), D(R')^{\oplus}_{\alpha}(s)) : m = ([s], \alpha, R') \} \);
- if \( s \in S_{\text{Max}} \) then \(( \hat{T}(s), \hat{D}(s)) = \max^{\mathsf{lex}}_{m \in \mathcal{M}} \{ (T(R')^{\ominus}_{\alpha}(s), D(R')^{\ominus}_{\alpha}(s)) : m = ([s], \alpha, R') \} \).

**Solutions of \( \text{Opt}_{\text{MinMax}}(\Gamma) \) from solutions of \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \).** In this subsection we show that the function \(( T, D ) \rightarrow ( \hat{T}, \hat{D} ) \) translates solutions of reachability-time optimality equations \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \) for the timed region graph \( \hat{\Gamma} \) to solutions of optimality equations \( \text{Opt}_{\text{MinMax}}(\Gamma) \) for the reachability-time game \( \Gamma \). In other words, we establish that the function \( \Gamma \rightarrow \hat{\Gamma} \) is a reduction from the problem of computing values in reachability-time games to the problem of solving optimality equations for timed region graphs. Then in Section 4 we give an algorithm to solve optimality equations for \( \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \).

We say that a function \( F : \mathcal{R} \rightarrow [S \rightarrow \mathbb{R}] \) is *regionally simple* or *regionally constant*, respectively, if for every region \( R \in \mathcal{R} \), the function \( F(R) : \overline{R} \rightarrow \mathbb{R} \) is simple or constant, respectively.

**Theorem 6** (Correctness of reduction to timed region graphs). If \(( T, D ) \models \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \), \( T \) is regionally simple, and \( D \) is regionally constant, then \(( \hat{T}, \hat{D} ) \models \text{Opt}_{\text{MinMax}}(\Gamma) \).

**Proof.** We need to show that for every \( s \in S_{\text{Min}} \setminus F \), we have: (a) \( \hat{T}(s) = \inf_{t \in \mathbb{N}} \{ t + \hat{T}(s') : s \overset{\alpha}{\rightarrow} t s' \} \); and (b) \( \hat{D}(s) = \min_{d' \in \mathbb{N}} \{ 1 + d' : \hat{T}(s) = \inf_{t \in \mathbb{N}} \{ t + \hat{T}(s') : s \overset{\alpha}{\rightarrow} t s' \text{ and } \hat{D}(s') = d' \} \} \). The proof of the
corresponding equalities for states $s \in S_{\text{Max}} \setminus F$ is similar and omitted. We prove the equality (a) here.

$$
\tilde{T}(s) = \min_{m \in \mathcal{M}} \{ T(R')^\oplus_\alpha(s) : m = ([s], \alpha, R') \}
= \min \left\{ \min_{R'' \alpha R'} \{ T(R')^\oplus_{s, a}(b - s(c)) : [s] \to_{b, c} R'' \to R' \}, \min_{R'' \alpha R'} \{ T(R')^\oplus_{s, a}(b - s(c)) : [s] \to_{b, c} R'' \to_{+1} R'' \to R' \} \right\}
= \min_{R'' \alpha R'} \left\{ \inf_t \{ T(R')^\oplus_{s, a}(t) : [s + t] = R'' \} : [s] \to_s R'' \to R' \right\}
= \inf_{a, t} \{ t + \tilde{T}(s) : s \to_a t \}
$$

The first equality holds by the assumption that $T \models \text{Opt}^\text{MinMax}(\bar{\Gamma})$. The second equality holds by the definition of the move relation $\mathcal{M}$ of the timed graph $\bar{\Gamma}$, and because if $\alpha = (a, b, c)$ then

$$
T(R')^\oplus_\alpha(s) = b - s(c) + T(R')(\text{Succ}(s, (a, b - s(c)))) = T(R')^\oplus_{s, a}(b - s(c)).
$$

For the third equality we invoke simplicity of $T$ which by Proposition 5 implies that the function $T(R')^\oplus_{s, a}$ is continuous and nondecreasing. If either $[s] \to_{b, c} R'' \to R'$, or $[s] \to_{b, c} R'' \to_{+1} R'' \to R'$, then we have that $\inf_t \{ t + [s + t] = R'' \} = b - s(c)$, and hence

$$
\inf_t \{ T(R')^\oplus_{s, a}(t) : [s + t] = R'' \} = T(R')^\oplus_{s, a}(b - s(c)),
$$

because $T(R')^\oplus_{s, a}$ is continuous and nondecreasing. The fourth equality holds because $[s + t] = R''$ and $R'' \to R'$ imply that $\text{Succ}(s, (a, t)) = R'$, and hence $T(R')\text{Succ}(s, (a, t)) = \tilde{T}(\text{Succ}(s, (a, t)))$. 

\end{proof}

4 Solving optimality equations by strategy improvement

Positional strategies. A positional strategy for player Max in a timed region graph $\bar{\Gamma}$ is a function $\chi : S_{\text{Max}} \to \mathcal{M}$, such that for every $s \in S_{\text{Max}}$, we have $\chi(s) = ([s], \alpha, R)$, for some $\alpha \in \mathcal{A}$ and $R \in \mathcal{R}$. A strategy $\chi : S_{\text{Max}} \to \mathcal{M}$ is regionally constant if for all $s, s' \in S_{\text{Max}}$, we have that $[s] = [s']$ implies $\chi(s) = \chi(s')$; we can then write $\chi([s])$ for $\chi(s)$. Positional strategies for player Min are defined analogously. We write $\Delta_{\text{Max}}$ and $\Delta_{\text{Min}}$ for the sets of positional strategies for players Max and Min, respectively.

If $\chi \in \Delta_{\text{Max}}$ is regionally constant then we define the strategy subgraph $\bar{\Gamma}|\chi$ to be the subgraph $(\mathcal{R}, \mathcal{M}_\chi)$ where $\mathcal{M}_\chi \subseteq \mathcal{M}$ consists of: all moves $(R, \alpha, R') \in \mathcal{M}$, such that $R \in \mathcal{R}_{\text{Min}}$; and of all moves $m = (R, \alpha, R')$, such that $R \in \mathcal{R}_{\text{Max}}$ and $\chi(R) = m$. The strategy subgraph $\bar{\Gamma}|\mu$ for a regionally constant positional strategy $\mu \in \Delta_{\text{Min}}$ for player Min is defined analogously. We say that $R \in \mathcal{R}$ is choiceless in a timed region graph $\bar{\Gamma}$ if $R$ has a unique successor in $\bar{\Gamma}$. We say that $\bar{\Gamma}$ is 0-player if all $R \in \mathcal{R}$ are choiceless in $\bar{\Gamma}$; we say that $\bar{\Gamma}$ is 1-player if either all $R \in \mathcal{R}_{\text{Min}}$ or all $R \in \mathcal{R}_{\text{Max}}$ are choiceless in $\bar{\Gamma}$; every timed region graph $\bar{\Gamma}$ is 2-player. Note that if $\chi$ and $\mu$ are positional strategies in $\bar{\Gamma}$ for players Max and Min, respectively, then $\bar{\Gamma}|\chi$ and $\bar{\Gamma}|\mu$ are 1-player and $(\bar{\Gamma}|\chi)|\mu$ is 0-player.

For functions $T : \mathcal{R} \to [S \to R]$ and $D : \mathcal{R} \to [S \to R]$, and $s \in S_{\text{Max}}$, we define sets $M^*(s, (T, D))$ and $M_s(s, (T, D))$, respectively, of moves enabled in $s$ which are (lexicographically) $(T, D)$-optimal for player Max and Min, respectively:

$$
M^*(s, (T, D)) = \arg\max_{m \in \mathcal{M}} \{ (T(R')^\oplus_\alpha(s), D(R')^\oplus_\alpha(s)) : m = ([s], \alpha, R') \},
$$

$$
M_s(s, (T, D)) = \arg\min_{m \in \mathcal{M}} \{ (T(R')^\oplus_\alpha(s), D(R')^\oplus_\alpha(s)) : m = ([s], \alpha, R') \}.
$$
Let Choose : \(2^M \to \mathcal{M}\) be a function such that for every non-empty set of moves \(M \subseteq \mathcal{M}\), we have Choose\((M) \in M\). For regional functions \(T : \mathcal{R} \to [S \to \mathbb{R}]\) and \(D : \mathcal{R} \to [S \to \mathbb{N}]\), the canonical \((T, D)\)-optimal strategies \(\chi_{(T, D)}\) and \(\mu_{(T, D)}\) for player Max and Min, respectively, are defined by: \(\chi_{(T, D)}(s) = \text{Choose}(M^*(s, (T, D)))\), for every \(s \in S_{\text{Max}}\); and \(\mu_{(T, D)}(s) = \text{Choose}(M^*(s, (T, D)))\), for every \(s \in S_{\text{Min}}\).

**Optimality equations** \(\text{Opt}(\tilde{\Gamma}), \text{Opt}_{\text{Max}}(\tilde{\Gamma}), \text{Opt}_{\text{Min}}(\tilde{\Gamma}), \text{Opt}_{\geq}(\tilde{\Gamma})\) and \(\text{Opt}_{\leq}(\tilde{\Gamma})\). Let \(T : \mathcal{R} \to [S \to \mathbb{R}]\) and \(D : \mathcal{R} \to [S \to \mathbb{N}]\). We write \((T, D) \models \text{Opt}_{\text{Max}}(\tilde{\Gamma})\) or \((T, D) \models \text{Opt}_{\text{Min}}(\tilde{\Gamma})\), respectively, if for all \(s \in F\), we have \((\tilde{T}(s), \tilde{D}(s)) = (0, 0)\), and for all \(s \in S \setminus F\), we have, respectively:

\[
(\tilde{T}(s), \tilde{D}(s)) = \max_{m \in \mathcal{M}} \{(T(R'_{\alpha}^m(s), D(R'_{\alpha}^m(s)) : m = ([s], \alpha, R')\}, \text{ or}\]

\[
(\tilde{T}(s), \tilde{D}(s)) = \min_{m \in \mathcal{M}} \{(T(R'_{\alpha}^m(s), D(R'_{\alpha}^m(s)) : m = ([s], \alpha, R')\}.
\]

If \(\tilde{\Gamma}\) is 0-player then \(\text{Opt}_{\text{Max}}(\tilde{\Gamma})\) and \(\text{Opt}_{\text{Min}}(\tilde{\Gamma})\) are equivalent to each other and denoted by \(\text{Opt}(\tilde{\Gamma})\).

We write \((T, D) \models \text{Opt}_{\geq}(\tilde{\Gamma})\) or \((T, D) \models \text{Opt}_{\leq}(\tilde{\Gamma})\), resp., if for all \(s \in F\), we have \((\tilde{T}(s), \tilde{D}(s)) \geq_{\text{lex}} (0, 0)\) or \((\tilde{T}(s), \tilde{D}(s)) \leq_{\text{lex}} (0, 0)\), respectively; and for all \(s \in S \setminus F\), we have, respectively:

\[
(\tilde{T}(s), \tilde{D}(s)) \geq_{\text{lex}} \max_{m \in \mathcal{M}} \{(T(R'_{\alpha}^m(s), D(R'_{\alpha}^m(s)) : m = ([s], \alpha, R')\}, \text{ or}\]

\[
(\tilde{T}(s), \tilde{D}(s)) \leq_{\text{lex}} \min_{m \in \mathcal{M}} \{(T(R'_{\alpha}^m(s), D(R'_{\alpha}^m(s)) : m = ([s], \alpha, R')\}.
\]

**Proposition 7** (Relaxations of optimality equations). If \((T, D) \models \text{Opt}_{\text{Max}}(\tilde{\Gamma})\) then \((T, D) \models \text{Opt}_{\geq}(\tilde{\Gamma})\), and if \((T, D) \models \text{Opt}_{\text{Min}}(\tilde{\Gamma})\) then \((T, D) \models \text{Opt}_{\leq}(\tilde{\Gamma})\).

**Lemma 8** (Solution of \(\text{Opt}(\tilde{\Gamma})\) is regionally simple). Let \(\tilde{\Gamma}\) be a 0-player timed region graph. If \((T, D) \models \text{Opt}(\tilde{\Gamma})\) then \(T\) is regionally simple and \(D\) is regionally constant.

**Solving 1-player maximum reachability-time optimality equations** \(\text{Opt}_{\text{Max}}(\tilde{\Gamma})\). In this section we give a strategy improvement algorithm for solving maximum reachability-time optimality equations \(\text{Opt}_{\text{Max}}(\tilde{\Gamma})\) for a 1-player timed region graph \(\tilde{\Gamma}\).

We define the following strategy improvement operator \(\text{Improve}_{\text{Max}}\):

\[
\text{Improve}_{\text{Max}}(\chi, (T, D))(s) = \begin{cases} 
\chi(s) & \text{if } \chi(s) \in M^*(s, (T, D)), \\
\text{Choose}(M^*(s, T)) & \text{if } \chi(s) \notin M^*(s, (T, D)).
\end{cases}
\]

Note that \(\text{Improve}_{\text{Max}}(\chi, (T, D))(s)\) may differ from the canonical \((T, D)\)-optimal choice \(\chi_{(T, D)}(s)\) only if \(\chi(s)\) is itself \((T, D)\)-optimal in state \(s\), i.e., if \(\chi(s) \in M^*(s, (T, D))\).

**Lemma 9** (Improvement preserves regional constancy of strategies). If \(\chi \in \Delta_{\text{Max}}\) is regionally constant, \(T : \mathcal{R} \to [S \to \mathbb{R}]\) is regionally simple, and \(D : \mathcal{R} \to [S \to \mathbb{N}]\) is regionally constant, then \(\text{Improve}_{\text{Max}}(\chi, (T, D))\) is regionally constant.

**Algorithm 1. Strategy improvement algorithm for** \(\text{Opt}_{\text{Max}}(\tilde{\Gamma})\).

1. (Initialisation) Choose a regionally constant positional strategy \(\chi_0\) for player Max in \(\tilde{\Gamma}\); set \(i := 0\).
2. (Value computation) Compute the solution \((T_i, D_i)\) of \(\text{Opt}(\tilde{\Gamma} | \chi_i)\).
3. (Strategy improvement) If \(\text{Improve}_{\text{Max}}(\chi_i, (T_i, D_i)) = \chi_i\), then return \((T_i, D_i)\).
   Otherwise, set \(\chi_{i+1} := \text{Improve}_{\text{Max}}(\chi_i, (T_i, D_i))\); set \(i := i + 1\); and goto step 2.
Solving 2-player reachability-time optimality equations. In this section we give a strategy improvement algorithm for solving optimality equations \( \text{Opt}_{\text{Max}}(\hat{\Gamma}) \) for a 2-player timed region graph \( \hat{\Gamma} \). The structure of the algorithm is very similar to that of Algorithm 1. The only difference is that in step 2. of every iteration we solve 1-player optimality equations \( \text{Opt}_{\text{Max}}(\hat{\Gamma}|\mu) \) instead of 0-player optimality equations \( \text{Opt}(\hat{\Gamma}|\chi) \). Note that we can perform step 2. of Algorithm 1 below by using Algorithm 1.

We define the following strategy improvement operator \( \text{Improve}_{\text{Min}} \):

\[
\text{Improve}_{\text{Min}}(\mu, (T, D))(s) = \begin{cases} 
\mu(s) & \text{if } \mu(s) \in M_*(s, (T, D)), \\
\text{Choose}(M_*(s, (T, D))) & \text{if } \mu(s) \notin M_*(s, (T, D)).
\end{cases}
\]
Corollary 20. The problem of solving reachability-time games is in EXPTIME.
Courcoubetis and Yannakakis proved that the reachability problem for timed automata with at least three clocks is PSPACE-complete \[14\]. We complement their result by showing that solving 2-player reachability games on timed automata with at least two clocks is EXPTIME-complete. Note that the best currently known lower bound for the reachability problem for timed automata with two clocks is NP-hardness \[20\].

**Theorem 21** (Complexity of reachability games on timed automata). *The problem of solving reachability games is EXPTIME-complete on timed automata with at least two clocks.*

**Theorem 22** (Complexity of reachability-time games on timed automata). *The problem of solving reachability-time games is EXPTIME-complete on timed automata with at least two clocks.*

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Observe that for every state $s \in S$ there is a positional strategy $\mu_\varepsilon : S_{\text{Min}} \to A \times \mathbb{R}_{\geq 0}$ for player Min, such that for every strategy $\chi$ for player Max, if $s \in S$ is such that $D(s) < \infty$, then we have $RT(\text{Run}(s, \mu_\varepsilon, \chi)) \leq T(s) + \varepsilon$. The proof, that for every $\varepsilon > 0$, there exists a positional strategy $\chi_\varepsilon : S_{\text{Max}} \to A \times \mathbb{R}_{\geq 0}$ for player Max, such that for every strategy $\mu$ for player Min, if $s \in S$ is such that $D(s) < \infty$ then we have $RT(\text{Run}(s, \mu, \chi_\varepsilon)) \geq T(s) - \varepsilon$, is similar and omitted. The proof, that if $D(s) = \infty$ then player Max has a strategy to prevent ever reaching a final state, is routine and omitted as well.

Together, these facts imply that $T$ is equal to the value function of the reachability-time game, and the positional strategies $\mu_\varepsilon$ and $\chi_\varepsilon$, defined in the proof below for all $\varepsilon > 0$, are $\varepsilon$-optimal.

For $\varepsilon > 0$, $T : S \to \mathbb{R}$, and $s \in S_{\text{Min}} \setminus F$, we say that a timed action $(a, t) \in A \times \mathbb{R}_{\geq 0}$ is $\varepsilon'$-optimal for $(T, D)$ in $s$ if $s \xrightarrow{a, t} s'$, and

$$D(s') \leq D(s) - 1 \quad \text{and} \quad t + T(s') \leq T(s) + \varepsilon'. \quad (2)$$

Observe that for every state $s \in S_{\text{Min}}$ and for every $\varepsilon' > 0$, there is a $\varepsilon'$-optimal timed action for $(T, D)$ in $s$ because $(T, D) \models \text{Opt}_{\text{MinMax}}(\Gamma)$. Moreover, again by $(T, D) \models \text{Opt}_{\text{MinMax}}(\Gamma)$ we have that for every $s \in S_{\text{Max}} \setminus F$ and timed action $(a, t)$, such that $s \xrightarrow{a, t} s'$, we have

$$D(s') \leq D(s) - 1 \quad \text{and} \quad t + T(s') \leq T(s). \quad (3)$$

Let $\varepsilon > 0$; we define $\mu_\varepsilon : S_{\text{Min}} \to A \times \mathbb{R}_{\geq 0}$ by setting $\mu_\varepsilon(s)$, for every $s \in S_{\text{Min}}$, to be a timed action which is $\varepsilon'(s)$-optimal for $(T, D)$ in $s$, where $\varepsilon'(s) > 0$ is sufficiently small (to be determined later). Let $\chi$ be an arbitrary strategy for player Max and let $r = \text{Run}(s, \mu_\varepsilon, \chi) = (s_0, (a_1, t_1), s_1, (a_2, t_2), \ldots)$. Let $N = \text{Stop}(r)$. Our goal is to prove that $RT(r) \leq T(s) + \varepsilon$, i.e., that $T(s) \geq \sum_{k=1}^{N} t_k - \varepsilon$.

For every state $s \in S$, such that $D(s) < \infty$, define $\varepsilon'(s) = \varepsilon \cdot 2^{-D(s)}$. Note that if we add left- and right-hand sides of the inequalities (3) or (5), respectively, for all states $s_i$, and $\varepsilon'(s_i)$-optimal timed actions $\mu_\varepsilon(s_i)$ if $s_i \in S_{\text{Min}}$, where $i = 0, 1, \ldots, N - 1$, then we get

$$T(s) = T(s_0) \geq \sum_{k=1}^{N} t_k - \sum_{k=0}^{N-1} \varepsilon'(s_k) \geq \sum_{k=0}^{N-1} t_k - \varepsilon.$$
The first inequality holds by $T(s_N) = T(s_{\text{Stop}(r)}) = 0$, and the second inequality holds because

$$\sum_{k=0}^{N-1} e'(s_k) = \sum_{k=0}^{N-1} (\varepsilon \cdot 2^{-D(s_k)}) \leq \varepsilon \cdot \sum_{d=1}^{\infty} 2^{-d} \leq \varepsilon,$$

where the first inequality follows by (2) and (4).

It may be worth noting that if the finite values of the function $D$ are bounded, i.e., if $B < \infty$, where $B = \sup_{s \in S} \{D(s) : D(s) < \infty\}$, then in the above proof it is sufficient to define $e'(s) = \varepsilon/B$, for all $s \in S$, which gives arguably more realistically “physically implementable” $\varepsilon$-optimal strategies.

**Proof of Proposition 5.** We prove the lemma for functions $\min(F, F')$ and $\max(F, F')$ instead of $\min(F, F')$ and $\max(F, F')$, respectively. Extending the result to the unique continuous extensions to $\overline{X}$ is routine. The case when both $F$ and $F'$ are constant functions is straightforward. Hence it suffices to consider the following two cases.

Case 1. Let $F(s) = e - s(c)$ and let $F'(s) = e'$, for some $e, e' \in \mathbb{Z}$ and a clock $c \in C$. Note that for every state $s \in R$, we have $[F'(s) - F(s)] = (e' - e) + [s(c)]$ and hence $[F' - F]$ is a constant function in region $R$. Therefore either $F'(s) - F(s) \geq 0$ for all $s \in R$, or $F'(s) - F(s) \leq 0$ for all $s \in R$, i.e., either $\min(F, F') = F$ and $\max(F, F') = F'$, or $\min(F, F') = F'$ and $\max(F, F') = F$.

Case 2. Let $F(s) = e - s(c)$ and $F'(s) = e' - s(c')$, for some $e, e' \in \mathbb{Z}$ and clocks $c, c' \in C$. Note that for every state $s \in R$, we have $[F'(s) - F(s)] = (e' - e) + [s(c') - s(c)]$ and

$$[s(c') - s(c)] = \begin{cases} [s(c')] - [s(c)] & \text{if } [s(c')] \geq [s(c)], \\ [s(c') - [s(c)] - 1 & \text{if } [s(c')] < [s(c)]. \end{cases}$$

In particular, as in the previous case we have that $[F' - F]$ is a constant function in region $R$ and hence one of the functions $F$ or $F'$ is equal to $\max(F, F')$ and the other is equal to $\min(F, F')$.

**Proof of Proposition 4.** Let $\alpha = (a, b, c)$. If $F$ is a constant function, i.e., if there is some $e \in \mathbb{Z}$, such that for all $s' \in R'$, we have $F(s') = e$, then $F^\alpha_{s}(s) = t(s, a) + e$. If $s(c) > b$ for all $s \in R$, then $t(s, \alpha) = 0$ for all $s \in R$, and hence $F^\alpha_{s}(s) = e$ and $F^\alpha_{s}$ is simple. If instead $s(c) \leq b$ for all $s \in R$, then $F^\alpha_{s}(s) = (b - s(c)) + e = (b + e) - s(c)$ and hence it is a simple function.

The other case is when $F$ is not a constant function, i.e., if there are a constant $e \in \mathbb{Z}$ and a clock $c' \in C$, such that for all $s' \in R'$, we have $F(s') = e - s'(c')$. We consider two subcases.

If $c' \in \rho(a)$ then $F^\alpha_{s}(s) = t(s, a) + e - s'(c') = t(s, a) + e$, because by the assumption that $c' \in \rho(a)$ we have that $s'(c') = 0$. If $s(c) > b$ for all $s \in R$, then $t(s, \alpha) = 0$ for all $s \in R$, and hence $F^\alpha_{s}(s) = e$ which is a simple function. If instead $s(c) \leq b$ for all $s \in R$, then $F^\alpha_{s}(s) = (b + e) - s(c)$ which is also a simple function.

If instead $c' \notin \rho(a)$ then $F^\alpha_{s}(s) = t(s, \alpha) + (e - s'(c') + t(s, \alpha)) = e - s'(c')$, because by the assumption that $c' \notin \rho(a)$ we have that $s'(c') = s(c') + t(s, \alpha)$, and hence $F^\alpha_{s}(s)$ is a simple function.

**Proof of Proposition 5.** We consider two cases. If $F$ is a constant function, i.e., if there is $e \in \mathbb{Z}$, such that for all $s' \in R'$ we have $F(s') = e$, then $F^\alpha_{s}(t) = t + F(\text{Succ}(s, (a, t))) = t + e$, which is a continuous and nondecreasing function of $t$.

The other case is when $F$ is not a constant function, i.e., if there are a constant $e \in \mathbb{Z}$ and a clock $c' \in C$, such that for all $s' \in R'$, we have $F(s') = e - s'(c')$. We consider two subcases. If $c' \in \rho(a)$ then $F^\alpha_{s}(t) = t + e$ which is continuous and nondecreasing. If instead $c' \notin \rho(a)$ then $F^\alpha_{s}(t) = t + (e - (s + t)(c')) = t + e - (s(c') + t) = e - s(c')$, i.e., $F^\alpha_{s,a}$ is a constant function and hence continuous and nondecreasing.
Proofs from Section 3

Proof of Theorem 6 (Correctness of reduction to timed region graphs). Now we prove the equality (b).

\[ \tilde{D}(s) = \min_{m \in M} \{ D(R')_{\alpha}^{\oplus}(s) : \tilde{T}(s) = T(R')_{\alpha}^{\oplus}(s) \text{ and } m = ([s], \alpha, R') \} \]

\[ = \min_{d' \in N} \{ 1 + d' : \tilde{T}(s) = T(R')_{\alpha}^{\oplus}(s) \text{ and } ([s], \alpha, R') \in M \text{ and } D(R') \equiv d' \} \]

\[ = \min_{d' \in N} \{ 1 + d' : \tilde{T}(s) = \inf_{a,t} \{ t + \tilde{T}(s') : s \xrightarrow{\alpha, t} s' \text{ and } \tilde{D}(s') = d' \} \} \]

The first equality holds by the assumption that \( (T, D) \models \text{Opt}_{\text{MinMax}}(\hat{\Gamma}) \). The second equality holds because of the assumption that \( D \) is regionally constant, and we write \( D(R') \equiv d' \), where \( d' \in N \), to express that for all \( s \in R' \), we have \( D(R')(s) = d' \). Finally, to establish the third equality it is sufficient to perform a calculation analogous to the above proof of (a), in order to show that

\[ \tilde{T}(s) = T(R')_{\alpha}^{\oplus}(s) \text{ and } ([s], \alpha, R') \in M \text{ and } D(R') \equiv d' \]

if and only if

\[ \tilde{T}(s) = \inf_{a,t} \{ t + \tilde{T}(s') : s \xrightarrow{\alpha, t} s' \text{ and } \tilde{D}(s') = d' \} \].

Proofs from Section 4

Proof of Lemma 8 (Solution of \( \text{Opt}(\hat{\Gamma}) \) is regionally simple). In a 0-player timed region graph \( \hat{\Gamma} \), for every region \( R \), there is at most one outgoing labelled edge \( (R, \alpha, R') \in M \), and hence for every region \( R \), there is a unique \( M \)-path from \( R \) in \( \hat{\Gamma} \). For every region \( R \in \mathcal{R} \), we define the distance \( d(R) \in N \) to be the smallest number of edges in the unique \( M \)-path from \( R \), that one needs to reach a final region. It is easy to show that for every state \( s \in S \), we have that \( D([s])(s) = d([s]) \), and hence \( D \) is regionally constant.

We prove that for every region \( R \in \mathcal{R} \), the function \( T(R) : \overline{R} \to \mathbb{R} \) is simple, by induction on \( d(R) \). If \( d(R) = 0 \) then \( T(R)(s) = 0 \) for all \( s \in \overline{R} \), and hence \( T(R) \) is simple on \( \overline{R} \).

Let \( d(R) = n + 1 \) and let \( (R, \alpha, R') \in M \) be the unique edge going out of \( R \) in \( \hat{\Gamma} \). Observe that \( T(R) = T(R')_{\alpha}^{\oplus} \) because for every \( s \in R \), we have \( T(R)(s) = T([s])(s) = T(R')_{\alpha}^{\oplus}(s) \), where the second equality follows from \( (T, D) \models \text{Opt}(\hat{\Gamma}) \). Moreover, by the induction hypothesis the function \( T(R') : \overline{R'} \to \mathbb{R} \) is simple, and hence by Proposition 4 we get that \( T(R')_{\alpha}^{\oplus} = T(R) \) is simple.

If \( d(R) = \infty \), i.e., if the unique \( M \)-path from \( R \) in \( \hat{\Gamma} \) never reaches a final region, then we set \( T(R')(s) = \infty \), for all \( s \in \overline{R} \). Therefore \( T(R') : \overline{R} \to \mathbb{R} \) is a constant function and hence it is simple.

Proof of Lemma 9 (Improvement preserves regional constancy of strategies). We need to prove that for \( s, s' \in S \), if \( [s] = [s'] \) then \( \chi'(s) = \chi'(s') \), where \( \chi' = \text{Improve}_{\text{MinMax}}(\chi, (T, D)) \). By regionality of \( \chi \) it is sufficient to prove that \( M^*(s, (T, D)) = M^*(s', (T, D)) \). By regional simplicity of \( T \), and by Proposition 4 we have that functions \( T(R)_{\alpha}^{\oplus} : [s] \to \mathbb{R} \), for all \( m = ([s], \alpha, R) \in M \), are simple. Then we have

\[ M^*(s, (T, D)) = \arg\max_{m \in M} \{ \langle T(R)_{\alpha}^{\oplus}(s), D(R)_{\alpha}^{\oplus}(s) \rangle : m = ([s], \alpha, R) \} \]

\[ = \arg\max_{m \in M} \{ \langle T(R)_{\alpha}^{\oplus}(s'), D(R)_{\alpha}^{\oplus}(s') \rangle : m = ([s'], \alpha, R) \} \]

\[ = M^*(s', (T, D)) \]

where the second equality follows from \( [s] = [s'] \), regional constancy of \( D \), and by Lemma 8 applied to the (finite) set of functions \( \{ T(R)_{\alpha}^{\oplus} : ([s], \alpha, R) \in M \} \).
Proof of Lemma 12 (Strict strategy improvement for Max). First we argue that \((T, D) \models \text{Opt}_\leq (\tilde{\Gamma} \mid \chi')\) which by Proposition 11 implies that \((T, D) \preceq (T', D')\). Indeed, for every \(s \in S \setminus F\), if \(\chi(s) = ([n, \alpha, R])\) and \(\chi'(s) = ([n, \alpha', R])\), then we have
\[
(\tilde{T}(s), \tilde{D}(s)) = (T(R)\circ_\alpha(s), D(R)\circ_\alpha(s)) \preceq \text{lex} (T(R')\circ_\alpha(s), D(R')\circ_\alpha(s)),
\]
where the equality follows from \((T, D) \models \text{Opt}_\leq (\tilde{\Gamma} \mid \chi)\), and the inequality follows from the definition of \text{Improve}_{\text{Max}}. Moreover, if \(\chi \neq \chi'\) then there is \(s \in S_{\text{Max}} \setminus F\) for which the above inequality is strict. Then \((T, D) \not\models \text{Opt}_\leq (\tilde{\Gamma} \mid \chi')\) because every vertex in \(\tilde{\Gamma} \mid \chi'\) has a unique successor, and hence again by Proposition 11 we conclude that \((T, D) \not\models (T', D')\). \qed

Proofs from Section 5

Proof of Lemma 19 (Complexity of strategy improvement). An \(O(|R|)\) algorithm for solving \(\text{Opt}(\tilde{\Gamma}_0)\) is implicit in the proof of Lemma 8.

Let \((T, D) \models \text{Opt}_{\text{Max}}(\tilde{\Gamma}_1)\); and for all \(i \geq 0\), let \(\chi_i \in \Delta_{\text{Max}}\) be the strategy in the \(i\)-th iteration of Algorithm 1 and let \((T_i, D_i) \models \text{Opt}(\tilde{\Gamma}_1 \mid \chi_i)\). We claim that for every \(i \geq 0\), if \(D(R) \equiv i\) then for all \(j \geq i\), we have \((T_j(R), D_j(R)) = (T(R), D(R))\). This can be established by a routine induction on the values of the regionally constant function \(D\). Observe that the finite values of the function \(D\) are bounded by \(|R|\), because in the proof of Lemma 8 they are set to be the length of a simple path in a timed region graph. Algorithm 1 must therefore terminate no later than after \(|R| + 1\) iterations, because for every \(i \geq 0\), in the \(i\)-th iteration there must be \(R \in R\) whose value \(D(R)\) is set to \(i\).

An analogous routine proof by induction on the value of \(D\) can be used to prove that Algorithm 2 terminates in \(O(|R|)\) iterations. \qed

Proof of Theorem 21 (Complexity of reachability games on timed automata). In order to solve a reachability game on a timed automaton it is sufficient to solve the reachability game on the finite region graph of the automaton. Observe that every region, and hence also every configuration of the game, can be written down in polynomial space, and that every move of the game can be simulated in polynomial time. Therefore, the winner in the game can be determined by a straightforward alternating PSPACE algorithm, and hence the problem is in EXPTIME because \(\text{APSPACE} = \text{EXPTIME}\).

In order to prove EXPTIME-hardness of solving reachability games on timed automata with two clocks, we reduce the EXPTIME-complete problem of solving countdown games to it. Let \(G = (N, M, \pi, n_0, B_0)\) be a countdown game, where \(N\) is a finite set of nodes, \(M \subseteq N \times N\) is a set of moves, \(\pi : M \rightarrow \mathbb{N}_{>0}\) assigns a positive integer number to every move, and \((n_0, B_0) \in N \times \mathbb{N}_{>0}\) is the initial configuration. In every move of the game from a configuration \((n, B) \in N \times \mathbb{N}_{>0}\), first player 1 chooses a number \(p \in \mathbb{N}_{>0}\) such that \(p \leq B\) and \(\pi(n, n') = p\) for some move \((n, n') \in M\), and then player 2 chooses a move \((n, n''') \in M\), such that \(\pi(n, n'''') = p\); the new configuration is then \((n'', B - p)\). Player 1 wins a play of the game when a configuration \((n, 0)\) is reached, and he loses (i.e., player 2 wins) when a configuration \((n, B)\) is reached in which player 1 is stuck, i.e., for all moves \((n, n') \in M\), we have \(\pi(n, n') > B\).

We define the timed automaton \(T_G = (L, C, S, A, E, \delta, \rho, F)\) by setting \(C = \{b, c\}\); \(S = L \times ([B_0]_{\mathbb{R}})^2\);
\[ A = \{ * \} \cup P \cup M, \text{ where } P = \pi(M), \text{ the image of the function } \pi : M \rightarrow \mathbb{N}_{>0}; \]

\[ L = \{ * \} \cup N \cup \{(n, p) : \text{ there is } (n, n') \in M, \text{ s.t. } \pi(n, n') = p\}; \]

\[ E(a) = \begin{cases} \{(n, \nu) : n \in N \text{ and } \nu(b) = B_0\} & \text{if } a = *, \\ \{(n, \nu) : \text{ there is } (n, n') \in M, \text{ s.t. } \pi(n, n') = p \text{ and } \nu(c) = 0\} & \text{if } a = p \in P, \\ \{(n, p, \nu) : \pi(n, n') = p \text{ and } \nu(c) = p\} & \text{if } a = (n, n') \in M, \end{cases} \]

\[ \delta(\ell, a) = \begin{cases} * & \text{if } \ell = n \in N \text{ and } a = *, \\ (n, p) & \text{if } \ell = n \in N \text{ and } a = p \in P, \\ n' & \text{if } \ell = (n, p) \in N \times P \text{ and } a = (n, n') \in M; \end{cases} \]

\[ \rho(a) = \{ c \}, \text{ for every } a \in A; \text{ and } F = \{ * \} \times V. \text{ Note that the timed automaton } T_G \text{ has only two clocks and that the clock } b \text{ is never reset.} \]

Finally, we define the reachability game \( \Gamma_G = (T_G, L_1, L_2) \) by setting \( L_1 = N \) and \( L_2 = L \setminus L_1 \). It is routine to verify that player 1 has a winning strategy from state \((n_0, (0, 0))\) in the reachability game \( \Gamma_G \) if and only if player 1 has a winning strategy (from the initial configuration \((n_0, B_0))\) in the countdown game \( G \). \( \square \)