Topology of event horizon in axially symmetric spacetime: Classification of Maxwell set with symmetry

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Abstract

The crease set of an event horizon is studied in a spacetime with discrete or continuous symmetry. It determines possible topologies on spatial sections of an event horizon. We thereby investigate the classification of stable topological structure of the crease sets in a spacetime with any symmetry. In practice, we show the classification of the crease set in axially symmetric spacetime. By that we realize the topological structure of axially symmetric event horizons. We will find that many new topological structures become stable which is not stable without symmetry.

1 Introduction

The final state of the black hole is well understood and is simply described by at most three parameters. On the other hand, an early stage of black hole formation is not well understood since there is much variety of their appearance.

In the topological viewpoint, the final state was investigated by many authors and now it is known that under some reasonable conditions such as asymptotic flatness and the weak energy condition, each component of the black hole region is topologically trivial, i.e. simply connected. On the other hand, there were numerical simulations which suggest non-trivial topologies of event horizons in an early stage of black hole formation. There has been some confusion, but it is now well understood that even though the black hole region in the spacetime is simply connected, there are many possible topologies of spatial sections.

In particular, it is revealed that various topologies of the spatial sections of black hole are determined by the endpoint set, or similarly, the crease set, of the event horizon by the author. Therefore the crease set of an event horizon, is an important object which is independent from the choice of time slices and which determines qualitative properties of the event horizon. Thus, in order to restrict the physically possible topologies of the black hole, it is important to restrict the possible (stable) structure of the crease set. Since the crease set can be regarded as a singularity of a distant function (i.e. the Maxwell set) determining the generators of the event horizon. The singularity theory of real maps can classify the stable topological structure of the crease set locally.

These possible topological structures should be studied in numerical experiments of gravitational collapse. Nevertheless, it might be very hard since it requires a huge data set of a geometry in the entirety of numerically generated spacetime, to determine its event horizon. To reduce the cost of the numerical calculation, it may be successful to impose any symmetry on a black hole spacetime. Indeed it
is in axially symmetric gravitational collapse that some remarkable results are reported about non-trivial topologies of the event horizon\cite{2}\cite{3}.

In the symmetric spacetime, however, is the previous result\cite{6} how to be applied to? In a rigorously symmetric spacetime, their perturbation should be also symmetric. Then the stability of the function determining the generators of an event horizon will be altered. We need to study the classifications of the crease set again in this symmetric situation. By this, some new crease sets are added to the classification without symmetry.

This investigation will become relevant to also an almost symmetric realistic spacetime, if the symmetry is dynamically stable and if non-symmetric perturbation is always sufficiently small. In such a situation, the crease set in the almost symmetric spacetime will be approximated by the symmetric classification of the crease set in rigorously symmetric spacetime, by neglecting very fine structures of the crease set (c.f. \cite{9}).

In the present article, we investigate the classification of the Maxwell set under discrete and continuous (especially, reflection and axial) symmetry. Of course, the classification includes the event horizons reported in references \cite{2}\cite{3}.

In the next section, we mention the definition of variational principle to determine the null generators of event horizons in symmetric spacetime. The crease set is mathematically related to the Maxwell set of a potential function in the third section. We investigate the classification of the symmetric Maxwell set in the fourth section. The fifth section gives a concrete classification in reflection symmetric three-dimensional spacetime and axially symmetric four-dimensional spacetime. Then we will discuss the spatial topology of axially symmetric black hole. The final section is devoted to summary and discussions.

2 Fermat potential in symmetric spacetime

An event horizon is generated by null geodesics. A future event horizon $\mathcal{H}$ cannot have future endpoints, but can have past endpoints if it is not eternal. As is pointed out in \cite{4}, the endpoint set $\mathcal{E}$ of a horizon is an arc-wise connected acausal set. Points $u \in \mathcal{E}$ are classified by the multiplicity $m(u)$ of $u$, the number of the generators emanating from $u$:

$$
\mathcal{C} := \{ u \in \mathcal{E} \mid m(u) > 1 \},
\mathcal{D} := \{ u \in \mathcal{E} \mid m(u) = 1 \}. \tag{1}
$$

The set $\mathcal{C}$ is called the crease set of the horizon. The crease set contains the interior of the endpoint set, i.e., the closure of $\mathcal{C}$ contains $\mathcal{E}$\cite{11}. The crease set $\mathcal{C}$ equals the set of points of $\mathcal{E}$ on which the horizon is not differentiable, i.e., the horizon is differentiable at $u \in \mathcal{E}$ if and only if $u \in \mathcal{D}$\cite{11}\cite{12}.

The horizon $\mathcal{H}$ is the envelope of the light cone starting from the crease set $\mathcal{C}$ which is an arc-wise connected acausal subset of $\mathcal{H}$. If the spatial section of the horizon is a topological sphere at late times, the topology of the spatial section of the horizon can be nontrivial only at the crease set and the topology is completely determined by the time slicing of the crease set, which is studied in Ref. \cite{4} in detail. In particular, when the crease set is a single point, each possible spatial section of the horizon topologically is a single sphere. On the contrary, when the crease set has a disk-like structure, the horizon can have toroidal or higher-genus spatial sections. One would see the coalesce of horizons if the crease set has a
line-like structure. Therefore by classifying the structure of the crease set, we will know all possible topologies of the horizons. Here we do not assume that the spatial section of the horizon in the future is a sphere.

The crease set can be determined by Fermat’s principle in simple stationary spacetime. In non-stationary spacetime, we can extend Fermat’s principle and find a variational principle about light paths, imposing some appropriate causality condition such as global hyperbolicity. Here we show an example of the construction of the Fermat potential, and see how the symmetry of spacetime impose a condition to the Fermat potential.

Let us assume that the spacetime $M$ is smooth and is globally hyperbolic from a smooth Cauchy surface $S$ which is diffeomorphic to $\mathbb{R}^n$. Furthermore, we consider spacetime of gravitational collapse, namely, we assume that the event horizon $H$ is in the future of $S$.

By global hyperbolicity, there are always an appropriate smooth global time coordinate $t : M \rightarrow \mathbb{R}$ and a timelike vector field $T$ such that $dt(T) = 1$. The spacetime $M$ is foliated by Cauchy surfaces $S_t = \{q \in M | t(q) = t\}$. The vector field $T = \partial/\partial t$ defines a smooth projection $\pi$ from $M$ into the $S = S_{t_0}$:

$$\pi : M \rightarrow S,$$

$$\pi^{-1}(q) = \{\gamma(t) | \frac{\partial \gamma}{\partial t} = T, \gamma(t_0) = q, t \in \mathbb{R}\}. \quad (2)$$

Conversely, there is a diffeomorphism

$$\phi : \mathbb{R} \times S \rightarrow M,$$

$$t(\phi(t, u)) = t, \quad \pi(\phi(t, u)) = u. \quad (3)$$

Because $H$ is achronal, the restriction of $\pi$ on $H$ is injective and has an inverse, which we denote by $\psi$:

$$\psi : \pi(H) \rightarrow H,$$

$$\psi(u) \in H, \quad \pi(\psi(u)) = u. \quad (4)$$

The map $\psi$ is Lipschitz.

In the present article, we suppose that the spacetime possesses any symmetry. This means that there exists a symmetric Cauchy surfaces $S_{St}$ and there are any isometries $\varphi$ acting on each spatial hypersurface $S_{St}$ isometrically,

$$\varphi : (M, g) \rightarrow (M, g) \quad (5)$$

$$s.t. \varphi(S_{St}) = S_{St} \quad (6)$$

$$\varphi^* g = g, \varphi^* h = h, \quad (7)$$

where $h$ is a spatially induced metric on $S_{St}$. We consider both cases of discrete symmetry and continuous symmetry. For these cases their isometries form discrete group and continuous group, respectively.

Then we adopt a time function of this symmetric timeslicing as a symmetric global time coordinate $t(q)$ and its normal time vector $T = \partial/\partial t \perp S_{St}$ defines symmetric smooth projection $\pi$. By the fact that $\partial/\partial t$ and the Killing vector generating the isometry are orthogonal, $t$ can be a synchronous time,

$$g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = const. \quad (8)$$
We take some (sufficiently large) \( t = t_1 \) and assume that \( S_{St_1} \cap H \) is diffeomorphic to a compact manifold \( M \). We consider \( M \) as a fixed submanifold embedded in \( S \) so that \( S_{St_1} \cap H = \psi(M) \). Consider a neighbourhood \( U \) of \( \pi(H \cap J^- (S_{St_1})) \) in \( S \). For \( x \in M \) and \( u \in U \) we define Fermat potential as follows:

\[
F(x,u) := - \sup \{ t \in \mathbb{R} | \phi(t,u) \in J^- (\psi(x)) \}. \quad (9)
\]

Obviously if there is an event horizon, it should be invariant under the isometry \( \varphi \)

\[
\varphi(H) = H. \quad (10)
\]

Then \( M \) is also invariant under the isometry \( \varphi \)

\[
\varphi(M) = M. \quad (11)
\]

Furthermore we take such \( U \) that it is also invariant subset, \( \varphi(U) = U \).

The minimum points \( x \) of \( F \) corresponds to the generator of \( H \) through \( u \). Our definition \( (9) \) is the generalization of the geodesic distance function to the non-static spacetime.

From the above construction of the Fermat potential, it will be understood how the Fermat potential reflects the symmetry of spacetime. Since we adopt a spatially symmetric timeslicing \( S \) and its synchronous normal time vector \( \partial / \partial t \), \( F(x,u) \) is invariant under isometric transformation. In this construction the isometry acts on both state \( X \) and control \( U \) spaces. Then pull back of the Fermat potential satisfies

\[
\varphi^* F(x,u) = F(\varphi(x),\varphi(u)) = F(x,u), \quad (12)
\]

where the isometry \( \varphi \) acts on \( M \) and \( U \) as a invariant subset of \( M \) from \( (10), (11) \) and

\[
\pi \circ \varphi = \varphi \circ \pi. \quad (13)
\]

From \( (9) \), the crease set \( C \) is given by

\[
C = \psi(B_{\text{Maxwell}}(F)), \quad (14)
\]

where \( B_{\text{Maxwell}}(F) \) is the Maxwell set of \( F \) where \( F \) has two or more minimum points. A precise mathematical framework to study its properties is given in \[6\]. By \( (12) \), \( C \) is also invariant \( \varphi(C) = C \).

The goal of our present study is to classify all generally possible structure about the singularities (Maxwell set) of the Fermat potential determining the event horizon. As shown in ref.\[6\], the generic structure will be given by studying singularities of stable Fermat potentials concretely. Then our main task is to give a classification of the stable functions satisfying \( (12) \). When one usually discusses the bifurcation structure, so-called caustics, of a system we analyze the Fermat potential locally in the context of a function germ. However, our main object here is not the bifurcation set but the Maxwell set (the difference is clarified in ref.\[6\]). The problem is not purely local but is rather semi-local. The definition of Maxwell set is local in control (parameter) space \( U \) but is non-local in state (variable) space \( M \). To treat this, one introduces function multigerms and classifies stable multifoldings and their Maxwell sets\[6\]. On the other hand, our Fermat potential in the present study is restricted to a symmetric one \( (12) \). To
give a symmetric multunfoldings, non-local $U$-variable is prepared as an orbit of the isometry and also non-local $M$-variable to include non-isolated critical points of a symmetric Fermat potential. In the next section, we give these concepts as a $G$-orbital multigerm of unfolding.

### 3 Mathematical preparation

The mathematical framework to study the Maxwell set was given in [6]. There the semi-local characteristic of a Fermat potential is investigated in the context of a multigerm. Nevertheless it needs some additional preparation to study the Maxwell set with any symmetry. It is caused mainly by the change of a function space of perturbation. That requires the extension of a multigerm so as to represent an isometry orbit of a multigerm and competence of critical points mapped by the isometries.

First of all, we prepare the same notions as in [6]. A function unfolding $F : M \times U \to \mathbb{R}$ can be considered as a family of functions $f_u : M \to \mathbb{R}$ with $f_u = F(\bullet, u)$. The Maxwell set of a function unfolding is the set of all values of the parameters for which the minimum is attained either at a non-Morse critical point or at two or more critical points.

**Definition 1 (Maxwell set).** For a function unfolding $F : M \times U \to \mathbb{R}$ on a compact manifold $M$, the Maxwell set $B_M(F)$ of $F$ is a subset of $U$ given by

$$B_M(F) := \{u \in U | f_u \text{ has two or more global minimum points}\}.$$  \hspace{1cm} (15)

In the following we sometimes make $x$ as a representative of the state variables and $u$ the control variables. Of course, $F$ is not a global function on $\mathbb{R}^n \times \mathbb{R}^r$ rather a function on manifold $M \times U$, where $M$ is a compact manifold and $U$ is an open subset of $\mathbb{R}^r$.

In the investigation of the Maxwell set we mainly focus on its local structure because the global structure is obtained by the combinations of local ones. Below we extensively use the notion of the germs of objects which provides the best way to characterize their local structure. The rigorous definition of germ is well known and will be given elsewhere [6]. Let $M$, $N$ be $C^\infty$-manifolds. We denote the set of $C^\infty$-maps from $M$ to $N$ by $C^\infty(M, N)$.

The definition of the Maxwell set requires the global information of the function unfolding $F$. A simple but crucial observation is, however, that to determine the local structure of the Maxwell set, i.e., the Maxwell set germs, we only need the local information of $F$ around its global minimum points $p_1, ..., p_k \in M$. We generalize the notion of germs to that of multigerms. Let $M^{(k)}$ be a $k$-tuple of distinct points of $M$, i.e.,

$$M^{(k)} := \{(x_1, ..., x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.$$  \hspace{1cm} (16)

**Definition 2 (Multigerm).** Let $(x_1, ..., x_k) \in M^{(k)}$. A $k$-fold germ $f : (M, (x_1, ..., x_k)) \to N$, or $[f]_{x_1, ..., x_k}$, is the equivalence class of $f \in C^\infty(M, N)$, where two maps are equivalent if they coincide on some open subset of $M$ which contains $x_1, ..., x_k$. A $k$-fold unfolding germ $F : (M \times U, ((x_1, ..., x_k), u_0)) \to N$, or $[F]_{(x_1, ..., x_k), u_0}$, is the equivalence class of $F \in C^\infty(M, \mathbb{R})$ where two functions are equivalent if they coincide on some open subset of $M \times U$ which contains $(x_1, u_0), ..., (x_k, u_0)$. A $k$-fold germ is also called as a multigerm.
A mult germ can be considered as k-tuple of simple germs. For example, a map mult germ \([F]\{x_1,\ldots,x_k\},u_0\) can be considered as k-tuple of function germs \((\{f_1\}_{x_1},\ldots,\{f_k\}_{x_k})\).

To study the Maxwell set under any symmetry, first we prepare a set of mult germs they are generated by a transformation group.

**Definition 3 (G-orbital mult germ of unfolding).** Consider a group of transformation \(G\). Let \((x_1,\ldots,x_k)\in M^{(k)}\). \(G((\{x_1,\ldots,x_k\},u_0)) = \{g((\{x_1,\ldots,x_k\},u_0)|g\in G\} \subset M^{(k)} \times U\) is the orbit of \(G\) through \((\{x_1,\ldots,x_k\},u_0)\). A G-orbital mult germ of unfolding \(F_G : (M \times U, G((\{x_1,\ldots,x_k\},u_0))) \rightarrow N\), or \([F_G]_{G((\{x_1,\ldots,x_k\},u_0))}\), is the equivalence class of \(F \in C^{\infty}(M \times U, \mathbb{R})\) where two functions are equivalent if they coincide on some open subset of \(M^{(k)} \times U\) which contains \(G((\{x_1,\ldots,x_k\},u_0))\).

A G-orbital mult germ can be considered as a suite of mult germs. For example, if \(G\) is a discrete group a G-orbital mult germ \([F_G]_{G((\{x_1,\ldots,x_k\},u_0))}\) can be considered as multituple of k-tuple function germs \((\{f_1|\}_{x_1},\ldots,\{f_k|\}_{x_k})\), \((\{f_1|g(x_1)\}_{x_1},\ldots,\{f_k|g(x_k)\}_{x_k},\ldots)\) where \(g\in G\).

If a representative of \(F_G\) is invariant under actions of \(g\in G\), by definition its G-orbital mult germ of unfolding is invariant under actions of \(G\). Here it should be noticed that discrete \(G\) gives point-wise equivalence similar to usual mult germ otherwise the equivalence is not point-wise. It is also commented that when \(x_i\) or \(u_0\) is a fixed point of \(G\) the equivalence becomes point-wise there even if \(G\) is continuous.

It is required to define not point-wise equivalence of maps to discuss the diffeomorphism class of this G-orbital mult germ of unfolding.

**Definition 4 (Map germ at subset).** Subset \(S \subset M\) is given. Maps \(f,g \in C^{\infty}(M,N)\) are equivalent at subset \(S\) if there is a neighbourhood \(W\) of \(S\) such that \(f|_W = g|_W\). A map germ \(f\) at \(S\), \([f]_S\), is the equivalence class of \(f\). It is also denoted by \(f : (M,S) \rightarrow N\) or \(f : (M,S) \rightarrow (N,f(S))\).

Henceforth we call germ at subset simply germ and sometimes also germ is omitted. Examples of map germs at subset include function germs and diffeomorphism germs.

As easily seen, that G-orbital mult germ of unfolding is not local also in \(U\). When we find the Maxwell set of it, it will be expressed as a non-local equivalence class of set.

**Definition 5 (Set germ at subset).** Subset \(S \subset U\) is given. Subsets \(X, Y\) of \(U\) are equivalent at subset \(S\) if \(X \cap W = Y \cap W\) for each neighbourhood \(W\) of \(S\). A set germ \((X,S)\) of \(X\) at \(S\) is the equivalence class of \(X\). Set germs \((X,S)\) and \((Y,S')\) are diffeomorphic, \((X,S) \simeq (Y,S')\), if there is a diffeomorphism germ \(\phi : (M,S) \rightarrow M\) such that \((\phi(X),\phi(S)) = (Y,S')\).

Next we give a fundamental concepts to investigate the stability of unfolding under symmetry[14][15].

**Definition 6 (Symmetry preserving diffeomorphism).** Let \((M,g)\) be a symmetric space with isometry group \(G\). A diffeomorphism \(\phi\) on \((M,g)\) is a symmetry preserving diffeomorphism (SPD) if it keep the action \(a \in G\) on \((M,g)\) isometry. If a diffeomorphism \(\phi\) is an SPD then \(\phi : a \mapsto \phi \circ a \circ \phi^{-1}\) is an automorphism of \(G\). In the following, \(G\) mainly acts on invariant subset \(M \times U\) as

\[
\varphi : M \times U \rightarrow M \times U
\]

\[
(x,u) \in M \times U, \varphi(x,u) = (\varphi(x),\varphi(u)).
\]

Then SPD is sometimes considered as the diffeomorphism on \(M \times U\) besides \(M\) and \(U\) satisfying above condition.
In the present article we discuss the stability defined by this SPD, since the subset of the function space we treat is invariant under the SPD.

**Definition 7 (Right SPD equivalence).** Function germs \( f : (M, x) \to \mathbb{R} \) and \( g : (M, y) \to \mathbb{R} \) are right equivalent, \( [f]_x \simeq [g]_y \), if there exists a SPD germ \( \phi : (M, x) \to (M, y) \) and \( a \in \mathbb{R} \) such that \( f = g \circ \phi + a \) holds as an equality of function germs at \( x \).

**Definition 8 (Right SPD equivalence at the minimum points).** Unfoldings \( F : (M \times U, M \times \{u_0\}) \to \mathbb{R} \) and \( G : (M \times U, M \times \{v_0\}) \to \mathbb{R} \) are right equivalent at the minimum points if the following conditions hold:

1. The functions \( f_{u_0} = F(\bullet, u_0) \) and \( g_{v_0} = G(\bullet, v_0) \) have the same number of global minimum points, \( p_1, \ldots, p_k \) and \( q_1, \ldots, q_k \), respectively.
2. There exists a SPD multigerm \( \phi : (M \times U, ((p_1, \ldots, p_k), u_0)) \to (M \times U, ((q_1, \ldots, p_k), v_0)) \), a SPD germ \( \psi : (U, u_0) \to (U, v_0) \), and a function germ \( \alpha : (U, u_0) \to \mathbb{R} \) such that

\[
F(x, u) = G(\phi(x, u), \psi(u)) + \alpha(u)
\]

holds with both sides being function multigerms at \( (p_1, \ldots, p_k, u_0) \).

The Maxwell set germ of an unfolding is determined only by the unfolding multigerm at the minimum points:

**Proposition 1.** If unfoldings \( F : (M \times U, M \times \{u_0\}) \to \mathbb{R} \) and \( G : (M \times U, M \times \{v_0\}) \to \mathbb{R} \) are right SPD equivalent at the minimum points, then their Maxwell set germs are equivalent under SPD transformation.

**Proof.** Follows directly from the definitions of right SPD equivalence and of Maxwell set germs.

Below, we will determine the topological structure of \( C^\infty(M, N) \) where all the maps that we treat are included. We define a topology of \( C^\infty(M, N) \) by the \( r \)-jet space \( J^r(M, N) \) below.

**Definition 9 (Jet space).** Let \( f \in C^\infty(M, N) \). The \( r \)-jet \( j^r f(x_0) \) of \( f \) at \( x_0 \) is the equivalence class of \( f \) in \( C^\infty(M, N) \) where two maps are equivalent if all of their \( s \)-th partial derivatives with \( 1 \leq s \leq r \), in some coordinate systems of \( M \) and \( N \), coincide. The \( r \)-jet space of \( C^\infty(M, N) \) is defined by

\[
J^r(M, N) := \{ j^r f(x_0) | f \in C^\infty(M, N) \}.
\]

The space of \( r \)-jets at a point is an \( n \left( \frac{m + r}{r} \right) \)-dimensional manifold, where \( m = \dim M \) and \( n = \dim N \).

Now we endow the space \( C^\infty(M, N) \) with the Whitney \( C^\infty \) topology.

**Definition 10 (Whitney \( C^\infty \) topology).** For an open subset \( O \) of \( J^r(M, N) \), let

\[
W^r(O) := \{ f \in C^\infty(M, N) | j^r f(M) \subset O \}.
\]

The Whitney \( C^\infty \) topology on \( C^\infty(M, N) \) is the topology whose basis is

\[
\bigcup_{r=0}^{\infty} \{ W^r(O) | O \text{ is an open subset of } J^r(M, N) \}.
\]
Hereafter we treat that $C^\infty(M,N)$ as a topological space with the Whitney $C^\infty$ topology. Now we can define stability of the Maxwell set using this topology.

The concepts of the jet space and the Whitney $C^\infty$ topology are used as in the ref.\[12\]. Especially, multiunfolding was investigated by the jet space for $C^\infty(M \times U, \mathbb{R})$. In the present article, however, it is not this $C^\infty(M \times U, \mathbb{R})$ in which we investigate the stability of a $G$-orbital multigerm of unfolding but its subset whose elements are invariant under an isometry group $G$. We call it $G-C^\infty(M \times U, \mathbb{R})$. From \[12\], it can be identified with $C^\infty(\frac{M \times U}{G}, \mathbb{R})$. Here we propose that the topology of the subset is relative topology and equivalent to the topology of $C^\infty(\frac{M \times U}{G}, \mathbb{R})$ in the context of the Whitney topology. $G-C^\infty(M \times U, \mathbb{R})$ is invariant subset under the pull back of SPD.

Now, we have completed the preparation. We will investigate stable structure of the function unfolding and its Maxwell set.

**Definition 11 (G-stable Maxwell set germ).** A $G$-orbital multigerm of unfolding,

$$F_G : (M \times U, G((x_1, \ldots, x_k), u_0)) \to \mathbb{R},$$

is stable with respect to the Maxwell set if for each neighbourhood $W$ of $G(u_0)$ there exists a neighbourhood $U$ of $F_G$ in $G-C^\infty(M \times U, \mathbb{R})$ such that for each $F'_G \in U$ there exists $v_0 \in W$ such that $(B_{\text{Maxwell}}(F_G), G(u_0))$ and $(B_{\text{Maxwell}}(F'_G), G(v_0))$ are symmetry preserving diffeomorphic about $G$.

We call $(B_{\text{Maxwell}}(F), G(u_0))$ a $G$-stable Maxwell set germ.

**Definition 12 (G-stability at the minimum points).** A $G$-orbital multigerm of unfolding $F_G : (M \times U, ((x_1, \ldots, x_k), u_0)) \to \mathbb{R}$ is $G$-stable at the minimum points if for each neighbourhood $W$ of $G(u_0)$ there exists a neighbourhood $U$ of $F_G$ in $G-C^\infty(M \times U, \mathbb{R})$ such that for each $F'_G \in U$ there exists $G(v_0) \in W$ such that $[F]_{G(u_0)}$ and $[F']_{G(v_0)}$ are right SPD equivalent at the minimum points.

From Proposition\[11\] we immediately have the following proposition.

**Proposition 2.** If a $G$-orbital multigerm of unfolding $F_G : (M \times U, M \times G(u_0)) \to \mathbb{R}$ is $G$-stable at the minimum points then $(B_{\text{Maxwell}}(F), G(u_0))$ is a $G$-stable Maxwell set germ.

In this sense, our aim is to classify the $G$-stable Maxwell set germs of the Fermat potential $F$. In the present article, we will not prove this stability in practice. We simply suggest the stability from the result of Ref.\[13\] without any symmetry.

To discuss the stability and classification, we study the orbit of diffeomorphism on some standard functions in $J^r(M, \mathbb{R})$ and its transversality. We will stratify $J^r(M, \mathbb{R})$, i.e., decompose the $J^r(M, \mathbb{R})$ into the union of submanifolds (strata) \[13\].

**Definition 13 (Strata).**

$$A_0 := \{ j^p f(p) \in J^r(M, \mathbb{R}) \mid f \text{ is regular at } p \},$$

$$A_k := \{ j^p f(p) \in J^r(M, \mathbb{R}) \mid [f]_p \simeq [\pm x_1^{k+1}]_0 \}.$$

The following is well discussed \[13\]:
Lemma 1. (1) $A_0$ is an open subset of $J^r(M, \mathbb{R})$ hence is a submanifold of codimension 0.

(2) $A_k$ is a submanifold of $J^r(M, \mathbb{R})$ of codimension $k$.

(3) $\Sigma := J^r(M, \mathbb{R}) - \bigcup_{k=0}^{3} A_k$ is the union of a finite number of submanifolds of codimension 4 or greater:

$$\Sigma = W_1 \cup ... \cup W_s,$$

(26)
where the codimension is determined for the case $M$ is one-dimensional. It is different from that of Ref. [2].

Definition 14 (Natural stratification). The natural stratification of $J^r(M, \mathbb{R})$, where $r > 3$, is the one given by

$$S(J^r(M, \mathbb{R})) := \{A_0, A_1, ..., A_3, W_1, ..., W_s\}.$$  (27)

This natural stratification corresponds to the classification of the stable unfolding under the aid of well-known transversality theorem. To discuss the Maxwell set we naturally extend this concept to the one at several minimum points. For that, we extend the jet space to multijet space.

Definition 15 (Multijet space). The $k$-fold $r$-jet, or simply, $r$-multijet, $kj^r f$ of $f : M \to \mathbb{R}$ at $(x_1, ..., x_k) \in M^{(k)}$ is

$$kj^r f(x_1, ..., x_k) := (j^r f(x_1), ..., j^r f(x_k)).$$  (28)

The $r$-multijet space $kJ^r(M, N)$ is given by

$$kJ^r(M, N) := \{kj^r f(x_1, ..., x_k) \in (J^r(M, N))^k : (x_1, ..., x_k) \in M^{(k)}\}.  $$  (29)

The map $kj^r f : M^{(k)} \to kJ^r(M, N)$ is called an $r$-multijet section.

Definition 16 (Natural stratification of $kJ^r(M, \mathbb{R})$). The natural stratification of $kJ^r(M, \mathbb{R})$ is the one given by

$$S(kJ^r(M, \mathbb{R})) := \{\Delta_k \cap X_1 \times ... \times X_k | X_1, ..., X_k \in S(J^r(M, \mathbb{R}))\},$$

(30)
where

$$\Delta_k := \{(j^r f_1(p_1), ..., j^r f_k(p_k)) \in kJ^r(M, \mathbb{R}) | f_1(p_1) = ... = f_k(p_k)\}.  $$  (31)

Hence we find that the stable multiunfolding germ is given by multuple of well known single stable unfolding under the aid of Mather’s multitransversality theorem[17]. This does not change in the case with discrete symmetry since the multifolding is point-wise. With continuous symmetry we cannot use the multitransversality theorem directly since a family of uncountable unfoldings is considered.

Finally we give a minimum function to investigate the concrete Maxwell set. If the minimum function $\mu_F$ has a singularity at the point $u_0$, then the function $f_{u_0}$ has several global minimal points on the manifold $M$. This will give just the Maxwell set even if the minimum point is not isolated in the case of continuous symmetry.
Definition 17 (Minimum function). The minimum function $\mu_F : U \to \mathbb{R}$ of an unfolding $F : M \times U \to \mathbb{R}$ is given by

$$
\mu_F(u) := \min \{F(x,u) | x \in M\}.
$$

(32)

There are two cases when the minimum function $\mu_F$ has a singularity at $u$:

1. the function $f_u$ has several minimum points in $M$, or
2. the number of minimum points changes there.

In the case (1) $u$ is a point of the Maxwell set. In the case (2) $u$ is not a point of the Maxwell set but corresponds to a point of the endpoint set $\mathcal{E}$.

4 classification of the symmetric Maxwell set

In the paper[6], the stable Maxwell set of multiunfolding $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$, where $M$ is two dimensional manifold and $U$ is three-dimensional open subset, were mathematically classified. There, based on well established local investigation of the unfolding[10], semi-local classification of the universal multiunfolding is made and it gives the classification of the Maxwell set. Then using multitransversality theorem, it is shown that they are stable.

On the other hand, some revisions are required for the present study in symmetric spacetime. With any symmetry, we should impose a condition for the symmetry to the list of the standard functions which are representatives of classification. Then diffeomorphism equivalence of the classification would be restricted to SPD equivalence under the request of the symmetry. We realize the restriction by using some global coordinates related to the symmetry instead of the local coordinate of neighborhoods.

That simply results from imposing a symmetry to the Maxwell set but revisions are not only this. With any symmetry, we should change the concept of the stability of the unfoldings from general ones to symmetric ones (Def.11, Def.12), since the definitions of perturbation and the stable multiunfolding are altered by the SPD and the symmetric subset $G-C^\infty(M \times U, \mathbb{R})$. Consequently, it is expected that a new class of multiunfolding will be added to the classification as a stable one. This seriously embarrasses the problem. Especially for the case of continuous symmetry, the function space where unfolding and perturbation are defined changes its dimensions (strictly speaking, the dimensions of its jet space) by infinite dimensions. This implies that an excluded function germ because of its infinite codimensions (which means the function germ cannot have universal unfolding) in function space $C^\infty(M \times U, \mathbb{R})$ is possible to be with finite codimensions and provide a stable unfolding in $G-C^\infty(M \times U, \mathbb{R})$. To find this directly, we should take a systematic survey of all functions in $G-C^\infty(M \times U, \mathbb{R}) \sim C^\infty(M \times U/G, \mathbb{R})$, and is not easy because $U/G$ has a boundary at the fixed points of $G$. In addition to it, because of such continuous symmetry, the concept of the usual point-wise germ will fail since the function germ orbits along the isometry. Especially at the fixed point of the isometry in $U$, their minimum points become non-isolated.

In the previous section, a $G$-orbital multigerm have been defined to resolve the latter problem (Def.3). To avoid the former trouble, we first consider a discrete symmetry $G_d$ in low-dimensional spacetime $M', U'$ whose catastrophe is essentially equivalent to the continuous symmetry $G_c$ because of $U/G_c \sim U'/G_d$. For example, the reflection symmetry in $(n-1)$-dimensional spacetime and axial symmetry in $n$-dimensional
spacetime are so. Since the change from the general function space \( C^\infty(M' \times U', \mathbb{R}) \) to the discrete symmetric function space \( G_d C^\infty(M' \times U', \mathbb{R}) \) is exceedingly small so that one can systematically pick up multunfolding newly added to the stable classes.

First, we recall a symmetry condition about the unfolding. We suppose the spacetime \((M, g)\) is symmetric and there exists a group of isometry \(G \ni \varphi, \varphi^* g = g\) (eq. (7)). As stated in the section 2, \(M\) and \(U\) are invariant subspace by the construction,

\[
\varphi(M) = M, \quad \varphi(U) = U, \quad \varphi(M \times U) = M \times U.
\] (33)

As the discussion in the section 2, the unfolding is pulled back by \(\varphi\), and should be invariant under the isometric transformation,

\[
F(x, u) = \varphi^*(F)(x, u) = F(\varphi(x), \varphi(u)).
\] (34)

Therefore, we confirm that the Maxwell set should be invariant by the isometry group. To find a classification of the Maxwell set in this symmetric spacetime, we will list the \(G\)-stable multunfolding. It should be a \(G\)-orbital multigerm of unfolding (Def. 3).

4.1 discrete symmetry

In the case of discrete symmetry investigation is easy to perform. Here, we suppose the isometry group is a discrete group \(G_d \ni \varphi_d\).

4.1.1 Class A multunfolding

If the symmetry is discrete, as long as there is no fixed point, the classification of stable multunfolding is essentially same as the case without symmetry [6]. Without fixed points, locally \(U_M \times U_U \sim \frac{U_M}{G} \times \frac{U_U}{G}\), where \(U_M \subset M\) and \(U_U \subset U\) is an open subset of \(M\) and \(U\), respectively. Then the identification

\[
k^* J^r \left( \frac{U_M \times U_U}{G}, \mathbb{R} \right) \sim k^* J^r \left( \frac{U_M}{G} \times \frac{U_U}{G}, \mathbb{R} \right),
\] (35)

implies that the catastrophe of this case is equivalent to the catastrophe without symmetry.

By the action of the isometry \(\varphi_d\) on an unfolding \(F\), we generate a \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding from a multigerm of unfolding since \(G_d\)-orbital multigerm is point-wise germ. We write this \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding as \(G_d[F]\).

\[
G_d[F] = [G_d[F]]\left(\{(x_1, \ldots, x_k, u_0), (\phi_d(x_1), \ldots, \phi_d(x_k), \phi_d(u_0)) \} : \phi_d \in G_d\right)
\] (36)

\[
= [F]\left(\{x_1, \ldots, x_k, u_0\}, [\phi_d^* F]_{(\phi_d(x_1), \ldots, \phi_d(x_k), \phi_d(u_0))} \right) = [F, \phi_d^* F],
\] (37)

\(\phi_d \in G_d\).

(38)

In \([X_1, X_2, \ldots]\), \([\ ]\) means that each element \(X_i\) does not share \(u_0\) and is not competent to others, differently from \((X_1, X_2, \ldots)\).

There is no need to care the change of the function space \(\mathbb{R}\). To get a representative function of the classification we simply use a function \(F\) in the case of the classification without symmetry.

11
Though $u_0$ is mapped to $\varphi_d(u_0)$ in $(38)$, there is no more degrees of freedom for control parameter space $U$ of $G_d[F]$ than those of the original multiunfolding $F$. Of course it is obviously $G_d$-stable at the minimum points if the original $F$ is stable at the minimum points. The reason is because perturbation is considered also in $kJ^r(UG, R)$. We call this a Class A multiunfolding in the following.

4.1.2 Class B multiunfolding

Next we consider the case with fixed points of the isometry on $M$ or $U$ in the neighborhood of minimum points $G_d((p_1, p_2, \ldots, p_k), u_0)$. Though this point in $M^{(k)}$ or $U$ is the fixed point of only several elements $g_f \in G$ in general, considering the subgroup $(g_f) = G_f \subseteq G$ generated by them, we suppose the fixed point is of all elements of the isometry in the present article. General case would be developed by combining the discussion of both cases with and without fixed points.

If a minimum point $p_i$ for $u_0$ is a fixed point of $\varphi_d$, $u_0$ should also be on a fixed point of $\varphi_d$ in our discussion. This is required in the case of an event horizon. A couple of $u_0$ and a minimum point $p_i$ determines a generator $(u_0, p_i)$ of the event horizon. Nevertheless, from the symmetry also $(\varphi(u_0), \varphi(p_i) = p_i)$ is a generator of the same event horizon. This means two generators pass a one point $p_i$ on $S_{St1} \cap \mathcal{H} = \psi(M)$. Nevertheless this is not allowed since there is no future endpoint on the event horizon. Then we suppose $u_0$ is always a fixed point of the isometry here.

In this case, we must pay attention to the changes of the function space from $C^\infty(M \times U, \mathbb{R})$ to $G_d-C^\infty(M \times U, \mathbb{R})$, where unfolding and perturbation is given since the essential control space $U/G_d$ has boundary at the fixed point. To care this problem, it is convenient also to represent a standard function by not local coordinates but global coordinates to indicate the symmetry. By the action of the isometry to the global coordinate, we observe that the codimension of the stratum for multijet space (Def.15) is possible to be diminished since $u_0$ is common among multituple of multiunfolding for the $G_d$-orbital multigerm of unfolding. When we list the strata for the multijet space, in advance, we should take extra strata assuming the decreasing of the control parameter. By this, there is possibility that a new multiunfolding becomes stable.

We find $G_d$-stable multiunfoldings by the following steps.

1. One prepares strata for a single-jet space which possess minimum points from Def.13 and give its universal unfolding, so that the number of control parameter does not exceed the dimension of $U$. This is well understood and is known to be stable by Thom’s theorem in the context of elementary catastrophe. They are written in local coordinates.

2. List the possible multitupeles of unfoldings and sum the numbers of their control parameters. By the isometric transformation of unfolding (34), some of control parameters are identified in the global coordinate for the symmetry. Then we manage the list so that also the sum of the number of control parameters does not exceed the dimension of $U$ after such identifications.

3. Imposing the symmetry relation (34) on a possible globalization of coordinate, we will find a $G_d$-invariant $G_d$-orbital multigerm of unfolding. Of course any of the multitupeles may not generate $G_d$-invariant $G_d$-orbital multigerm of unfolding.
4. In the same logic as in \([6]\), it can be shown that the \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding constructed above, is the \(G_d\)-stable at the minimum points, since each unfolding at a minimum points is \(G_d\)-stable unfolding even after imposing the symmetry. This is guaranteed by the fact the jet section \(j^*F : M \times U \mapsto J^r(M \times U, \mathbb{R})\) transversely crosses the strata also in \(G_d\)-\(C^\infty(M \times U, \mathbb{R})\) as long as the projection \(\rho : C^\infty(M \times U, \mathbb{R}) \mapsto G_d\)-\(C^\infty(M \times U, \mathbb{R})\) maps neither the stratum nor the jet section degenerate into a point. It is easily checked that the condition (35) never does so in most of significant cases.

Since the \(G_d\)-orbital multigerm of unfolding found in this way is \(G_d\)-invariant, it should be provided as the one generated by the isometry group \(G_d\). Nevertheless it includes competence at the fixed point \(u_0\) among multifoldings generated by each elements \(\varphi_d \in G_d\), and is not \(G_d[F]\). Then we write such a \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding as \(G_d(F)\):

\[
G_d(F) = [G_d(F)](\{(x_1,...,x_k),u_0,((\varphi_d(x_1),...,\varphi_d(x_k)),u_0)\}...\}\phi_d \in G_d) \tag{39}
\]

\[
= ([F](\{(x_1,...,x_k),u_0\}),[\phi^*_d F](\varphi_d(x_1),...,\varphi_d(x_k),u_0,...)) = (F,\phi^*_d F,...) \tag{40}
\]

\[
\phi_d \in G_d. \tag{41}
\]

\(F,\phi^*_d F,...\) share \(u_0\) and compete to each other. We call this a Class B multiumfolding.

4.2 continuous symmetry

Now we write continuous isometry group as \(\varphi_c \in G_c\). To avoid some difficulties caused by the continuous symmetry, here we suppose that the catastrophe of the case of this continuous symmetry is equivalent to a catastrophe with any discrete symmetry. Here an essential control parameter space is not \(U\) rather \(U/G\) since the isometry does not allow the unfolding in the direction of the isometry. Then essentially the classification of stable multiumfolding with the continuous symmetry is already completed, if such a corresponding discrete symmetry really exists and if their catastrophe is equivalent, since in the previous subsection, we have investigated the procedure to classify the \(G_d\)-stable \(G_d\)-orbital multigerm of unfolding.

In the next section, it will be shown that there is identification between the control space for a reflection symmetry in three-dimensional spacetime and that for an axial symmetry in four-dimensional spacetime. Since their catastrophes are equivalent, we only have to translate the classification of the multiumfolding with discrete symmetry into that with continuous symmetry.

4.2.1 Class C multiunfolding

To find a \(G_c\)-invariant \(G_c\)-orbital multigerm of unfolding for the continuous symmetry, it is an easy way to generate it by the isometry from the unfolding of discrete symmetry as an orbit of the continuous isometry. When there is no fixed point in the neighborhood of \((\{p_1,...,p_k\},u_0)\), it is possible to generate \(G_c\)-invariant \(G_c\)-orbital multigerm of unfolding from a \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding by straightforward correspondence.

Here one should be careful to start from a multiumfolding in Class A. To find a \(G_d\)-invariant \(G_d\)-orbital multigerm of unfolding in Class A, it was generated by the isometry group as \(G_d[F]\). Nevertheless, in the continuous symmetry, \(G_c\)-orbital multigerm suffers overlap of neighborhoods \(U_1\) and \(U_2\) generated by an isometry \(\varphi_c(U_1) = U_2\). Therefore the multigerm \(F(\varphi_c(x),\varphi_c(u))\) may be inconsistent to \(F(x,u)\) on
\( \mathcal{U}_1 \cap \mathcal{U}_2 \). To resolve this inconsistency it is sufficient that the original unfolding \( F \) is invariant along the continuous isometry in \( \mathcal{U}_1 \). This also is realized by the global coordinate of the continuous symmetry.

To represent the equivalence class of \( G_d \)-orbital multigerm, multituple notation was possible in the discrete symmetry. On the other hand, in this case of continuous symmetry, we write it as a family of uncountable members,

\[
G_c[F] = \{ (\varphi_a(x_1), \ldots, \varphi_c(x_k)) | \varphi_c \in G_c \} = \{ \varphi_c^* F | \varphi_c \in G_c \}.
\]

Since original \( F \) is invariant under the isometry \( \varphi_c^* F = F \) in global coordinate it is sufficient to write as \( [F|G_c] \).

Now we need to revive a dummy variable \( u_t \) of \( U \), which is omitted by the quotient \( U/G_c \), to decide the Maxwell set in \( U \). Since the generator of isometry \( \partial/\partial t \) should not effect the catastrophe, that can be achieved by adding a Morse function from the splitting lemma\(^{[16]}\). Therefore it is sufficient to add a \( \delta t^2 \) with \( \varphi_c(\delta t) = \delta t \) in order to find \( G_c \)-invariant \( F \). The dummy variable \( u_t \) is included in this as \( \delta t = \delta t(u_t) \). This case is named as a Class C multiunfolding.

### 4.2.2 Class D multiunfolding

At the fixed point, also the \( G_c \)-invariant \( G_c \)-orbital multigerm of unfolding is generated by the continuous isometry since the fixed point of \( G_c \) is same as that of \( G_d \). Using global coordinates, it is possible to rewrite such multiunfoldings in symmetric form,

\[
G_c(F) = \{ (\varphi_a(x_1), \ldots, \varphi_c(x_k), u_0) | \varphi_c \in G_c \} = \{ \varphi_c^* F | \varphi_c \in G_c \}.
\]

Since original \( F \) is \( G_c \)-invariant just like the Class C multiunfolding, it is sufficient to write \( (F) \). Here \( (F) \) should be written as a standard function using a global coordinate in the direction of the continuous isometry to clarify that its minimum points make an orbit along the isometry. This will be found by discussion in a following concrete situation. Instead of the Morse function, a global function which is locally right SPD equivalent to a Morse function are added to revive a dummy variable (see next section).

By the splitting lemma\(^{[16]}\), there is no need to discuss the transversality and stability, since it is supposed that the essential control parameter space \( U/G_c \) for the continuous symmetry is identical to that of the discrete symmetry \( U/G_d \) in low-dimensional spacetime where the stability have been studied.

These are class D multiunfoldings.

### 5 concrete classification: reflection symmetry and axial symmetry

#### 5.1 reflection symmetry in three-dimensional spacetime

The main purpose of the present article is to classify the stable crease set of an event horizon in an axially symmetric spacetime. To do so, we study Maxwell sets in reflection symmetric three-dimensional
spacetime where \( U \) is two dimensional and \( M \) is one-dimensional, in advance.

A reflection isometry form a \( \mathbb{Z}_2 \) group, that is
\[
\varphi_r : M \mapsto M, U \mapsto U,
\]
\[
G_r = \{id, \varphi_r | \varphi_r \circ \varphi_r = id \} \simeq \mathbb{Z}_2,
\]
where \( id \) is the identical map.

### 5.1.1 Class A multiunfolding with reflection symmetry

First, we consider a reflection symmetric \( G_r \)-orbital multigerm of unfolding \( F_{G_r} \) without a fixed point (a Class A multiunfolding). As discussed in the previous section, a \( G_r \)-invariant \( G_r \)-orbital multigerm of unfolding \( G_r[F] \) is generated by the reflection isometry \( \varphi_r \) from a usual multiunfolding \( F \) and is \( G_r \)-stable at the minimum points if the original multiunfolding \( F \) is stable at the minimum points.

We put the classification of multiunfolding without symmetry, that is the case of \( \text{codim} \leq 4 \) from \( \mathbb{R}^2 \), so that they are also stable in the case with two-dimensional \( U \). A stable multiunfolding \( F \) is right equivalent to one of the following multiunfolding germ at their minimum points.

- \( A_3 = x^4 + u_2 x^2 + u_1 x, \) at \((0,0) \in \mathbb{R} \times \mathbb{R}^2\).
- \((A_1, A_1) = (x_r^2 + u_1, x_2^2), \) at \((0_1,0_2), 0 \in \mathbb{R}^2 \times \mathbb{R}^2 \) where \( 0_1 \) is the origin of local coordinate \((x_i, y_i)\).
- \((A_1, A_1, A_1) = (x_r^3 + u_1 x_2 + u_2, x_3^2), \) at \((0_1,0_2,0_3), 0 \in \mathbb{R}^3 \times \mathbb{R}^2\).

where \( A_i \) means a stable unfolding of the stratum \( A_i \). The unfolding \( A_1 \) is omitted since its Maxwell set is empty set. Then, we give reflection symmetric \( G_r \)-orbital multigerm of unfolding as \( G_r(F) \).

**Class A multiunfolding**

A \( G_r \)-invariant \( G_r \)-stable multiunfolding in Class A is right SPD-equivalent to one of the following \( G_r \)-orbital multigerm of unfolding at the minimum points.

- \( G_r[A_3] = [A_3, \varphi_r^* A_3] = [A_3(x,u), A_3(\varphi_r(x), \varphi_r(u))]\).
- \( G_r[(A_1, A_1)] = [(A_1, A_1), \varphi_r^*(A_1, A_1)] = [(A_1, A_1), (A_1(\varphi_r(x), \varphi_r(u)), A_1(\varphi_r(x), \varphi_r(u)))]\).
- \( G_r[(A_1, A_1, A_1)] = [(A_1, A_1, A_1), \varphi_r^*(A_1, A_1, A_1)] = [(A_1, A_1, A_1), (A_1(\varphi_r(x), \varphi_r(u)), A_1(\varphi_r(x), \varphi_r(u)), A_1(\varphi_r(x), \varphi_r(u)))]\).

Since the function space of perturbation is same as that of without symmetry in the neighborhood of \((p_1, ..., p_k, u_0)\), these are all of the stable multiunfolding in this class. Their Maxwell sets are given by the Maxwell sets in \( \mathbb{R} \) and their reflectional images. They are illustrated in Fig.1.

### 5.1.2 Class B multiunfolding with reflection symmetry

Next we consider Class B multiunfoldings, where \( u_0 \) is on a fixed point of \( \varphi_r \). For convenience, we introduce global coordinates \((x, u_x, u_z) \in M \times U \) by which the reflection symmetry can be represented. The reflection isometry is given by \((x, u_x, u_z) \mapsto (-x, -u_x, u_z)\) in our convention of the coordinates.

We will find the list of the stable multiunfolding in a procedure developed in the previous section.
1. Since we treat only global minimums and $M$ is one- and $U$ is two-dimensional, only $A_1$ and $A_3$ are considered among the list of the strata (Def.13), whose codimension is less than four and the number of control parameter is not more than two ($= \text{dim}(U)$). Their stable unfoldings are $A_1 = x^2$, and $A_3 = x^4 + u_2x^2 + u_1x$.

2. Since even after one imposes the reflection symmetry the codimension of the strata for multijet space $\Delta_k \cap (A_i × \ldots)$ does not decrease more than $\lfloor \text{codim}/2 \rfloor$ ($\lfloor x \rfloor$ is the Gauss’s symbol), it is sufficient to consider the case in which codimension of $\Delta_k \cap (A_i × \ldots)$ is less than five (the number of control parameters is less than four), where the minimum value is fixed by the reflection symmetry. The list of the $k$-tuple of the stable unfoldings is

$$
(A_1, A_1) = (x_1^2 + u_1, x_2^2) \quad (48)
$$

$$
(A_1, A_1, A_1) = (x_1^2 + u_1, x_2^2 + u_2, x_3^2) \quad (49)
$$

$$
(A_1, A_1, A_1, A_1) = (x_1^2 + u_1, x_2^2 + u_2, x_3^2 + u_3, x_4^2) \quad (50)
$$

$$
(A_3) = (x_4^4 + u_2x_1^2 + u_1x_1) \quad (51)
$$

$$
(A_3, A_1) = (x_4^4 + u_2x_1^2 + u_1x_1, x_2^2 + u_3) \quad (52)
$$

3. Imposing the symmetry relation (34) by the global coordinate, the above $k$-tuples of unfoldings become the following $G_r$-invariant $G_r$-orbital multigerms of unfolding if the number of their control parameters $u_i$ finally is not exceeding $\text{dim}(U)$. $(A_3, A_1)$ is not allowed since the number of parameter is three even after imposing symmetry.

4. We can concretely confirm that projection $\rho : C^\infty(M × U, \mathbb{R}) \rightarrow G_r$-$C^\infty(M × U, \mathbb{R})$ maps neither the strata nor jet section into a point. Therefore the following $G_r$-invariant $G_r$-orbital multigerms of unfolding $G_r(F)$ are $G_r$-stable at the minimum points, since their multijet section transversally crosses the strata $\Delta_k \cap (A_i × \ldots)$ in the jet space of $G_r$-$C^\infty(M × U, \mathbb{R})$. This will be proved in the same logic as in [6].

**Class B multiunfolding**

A $G_r$-invariant $G_r$-stable multiunfolding in Class B is right SPD-equivalent to one of the following $G_r$-orbital multigerms of unfolding at the minimum points.

- $(A_1, A_1) = (x_{r_1}^2 + u_z, x_{r_2}^2)$.

To find the Maxwell set of it, we determine the minimum function (Def.17) of the $G_r$-orbital multigerm of unfolding,

$$
\min(A_1, A_1) = \begin{cases} 
  u_z & u_z < 0 \\
  0 & u_z \geq 0
\end{cases}.
$$

Then the Maxwell set that is the singular point of the minimum function is

$$
B_{\text{Maxwell}}(A_1, A_1) = \{(u_x, u_z)|u_z = 0\}.
$$

- $G_r(A_1) = (A_1, \varphi^*_r A_1) = ((x_r - x_0)^2 + u_x, (x_r + x_0)^2 - u_x) \sim (x_1^2 + u_x, x_2^2 - u_x)$,

where $x_1$ and $x_2$ are local coordinates.
The reflection symmetry. Especially, it is noticeable that (\(A\)).

Here we consider axial symmetry in a four-dimensional spacetime. In this case, its isometry group is added to the classification. These Maxwell sets are illustrated in Fig. 2.

The minimum function is
\[
\min(A_1, \varphi^*_r A_1) = \begin{cases} 
-u_x & u_x < 0 \\
u_x & u_x \geq 0 
\end{cases}.
\]  
(55)

The Maxwell set that is the singular point of the minimum function is
\[
B_{\text{Maxwell}}(A_1, \varphi^*_r A_1) = \{(u_x, u_z)|u_x = 0\}.
\]  
(56)

- \((G_r(A_1), A_1) = ((A_1, \varphi^*_r A_1), A_1) = (x_1^2 + u_x + u_z, x_2^2 - u_x + u_z, x_3, x_4).\)

The minimum function is
\[
\min((A_1, \varphi^*_r A_1), A_1) = \min(-|u_x| + u_z, 0) = \begin{cases} 
|u_x| & |u_x| > u_z \\
0 & |u_x| \leq u_z 
\end{cases}.
\]  
(57)

The Maxwell set that is the singular point of the minimum function is
\[
B_{\text{Maxwell}}((A_1, \varphi^*_r A_1), A_1) = \{(u_x, u_z)|u_x = 0, u_z < 0\}
\cup \{(u_x, u_z)|u_x = u_z\}.
\]  
(58)

- \(G_r((A_1, A_1)) = (G_r(A_1), G_r(A_1)) = ((A_1, \varphi^*_r A_1), (A_1, \varphi^*_r A_1))
\]
\[(x_1^2 + u_x, x_2^2 - u_x), (x_3^2 + u_x + u_z, x_4^2 - u_x + u_z)),\]

where \(x_1, x_2, x_3, x_4\) are local coordinates.

The minimum function is
\[
\min((A_1, \varphi^*_r A_1), (A_1, \varphi^*_r A_1)) = \min(-|u_x|, u_z - |u_x|) = \begin{cases} 
-|u_x| + u_z & u_z < 0 \\
-u_x & |u_z| \geq 0 
\end{cases}.
\]  
(59)

The Maxwell set that is the singular point of minimum function is
\[
B_{\text{Maxwell}}((A_1, \varphi^*_r A_1), (A_1, \varphi^*_r A_1)) = \{(u_x, u_z)|u_x = 0\}
\cup \{(u_x, u_z)|u_z = 0\}.
\]  
(60)

- \(G_r(A_3) = A_3 = x_1^2 + u_z x_2^2 + u_x x_r.\)

The Maxwell set is same as that of \(A_3\) without symmetry.
\[
B_{\text{Maxwell}}(A_3) = \{(u_x, u_z)|u_x < 0, u_z = 0\}.\]  
(61)

It should be noted that some multiunfoldings decreases the number of control parameter after imposing the reflection symmetry. Especially, it is notable that \(((A_1, \varphi^*_r A_1), (A_1, \varphi^*_r A_1))\) is not stable without the reflection symmetry but stable with the reflection symmetry. Then the entirely new Maxwell set have been added to the classification. These Maxwell sets are illustrated in Fig. 2.

### 5.2 Axial symmetry in four-dimensional spacetime

Here we consider axial symmetry in a four-dimensional spacetime. In this case, its isometry group \(G_a\) is continuous group and isomorphic to \(U(1),\)
\[
\varphi_\theta : M \mapsto M, U \mapsto U, \varphi_{\theta_0} = \exp \left[ \theta_0 \left( \frac{\partial}{\partial \theta} \right) \right]
\]
\[G_a = \{\varphi_\theta|0 \leq \theta \leq 2\pi, \varphi_{2\pi} = \text{id}\} \simeq U(1),\]  
(62)
5.2.1 Class C multiunfolding with axial symmetry

For the reflection symmetry, the function space was restricted by only finite codimensions and there does not appear an essentially new unfolding without a fixed point. As the axial symmetry, however, is continuous, our standard investigation will fail by two reasons. Since the function space (strictly speaking, its jet space) is restricted into infinite codimensional subspace, a new function which never provides stable unfolding owing to its infinite codimension can provide a stable unfolding, e.g. \((x^2 + y^2)^2\) see [5]. To find all of these new functions it is required to survey the whole right SPD equivalence class of the restricted function space systematically and it is embarrassing.

Another is that infinite number of stationary points can be generated by the axial isometry as \(\varphi_a^*F\). Then we cannot use semi-local discussion where usual function germ defined. Nevertheless, the essential control space of this axial symmetry in four dimensional spacetime \(U/G_\alpha\) is identical to that of reflection symmetry in three-dimensional spacetime \(U'/G_r\). Omitting the angular direction of the state space \(M\) and control space \(U\), we treat the remaining as a state space and control space \(U/G_\alpha\). We can study it in semi-local formalism.

To determine a correspondence between \(U_r = (U' \sim \mathbb{R}^2)/G_r\) and \(U_a = (U \sim \mathbb{R}^3)/G_\alpha\), it is convenient to use coordinate systems \((u'_x, u'_y, (u_r, u_\theta, u_z))\). By identifications \(u'_x = u_r\) and \(u'_z = u_z\), a correspondence \(\mathbb{R}^2/G_r \mapsto \mathbb{R}^3/G_\alpha\) is given, since the conditions \(0 \leq u'_x < \infty, -\infty < u'_z < \infty\) are consistent to \(0 \leq u_r < \infty, -\infty < u_z < \infty\). This correspondence simply gives the correspondence between Class A multiunfolding and the Class C multiunfolding. We also use local coordinate \((x_i, u_i)\) to show local diffeomorphism equivalence. \(x_i\) and \(u_i\) are local coordinate in \(r-z\) or \(r-u_z\) section of \(M\) and \(U\), respectively.

To determine a Maxwell set we should give a unfolding in \(U\) not in \(U/G_\alpha\). From the splitting lemma[16], we understand that it is sufficient to add a Morse function by a dummy variable \(u_\theta\) of angular direction, which does not concern the structure of the minimum point, as \(\delta \theta^2 = (\theta - u_\theta)^2\) where \(\theta - u_\theta\) is considered to be small\(^3\).

**Class C multiunfolding**

A \(G_\alpha\)-invariant \(G_\alpha\)-stable multiunfolding in Class C is right SPD-equivalent to one of the following \(G_\alpha\)-orbital multigerm of the unfolding at the minimum points.

- \(G_\alpha[(A_1, A_1)] = (x_1^2 + y_1^2 + u_1 + (\theta_1 - u_\theta)^2, x_2^2 + y_2^2 + (\theta_2 - u_\theta)^2)\),
  \(0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi, 0 \leq u_\theta \leq 2\pi\).

For \(G_\alpha[(A_1, A_1)]\), the minimum function is

\[
\min(G_\alpha[(A_1, A_1)]) = \begin{cases} u_1 & u_1 < 0 \\ 0 & u_1 \geq 0 \end{cases}.
\]

Then its Maxwell set is given by

\[
B_{\text{Maxwell}}(G_\alpha[(A_1, A_1)]) = \{(u_1, u_2, u_\theta)|u_1 = 0\}.
\]

- \(G_\alpha[(A_1, A_1, A_1)] = (x_1^2 + y_1^2 + u_1 + (\theta_1 - u_\theta)^2, x_2^2 + y_2^2 + u_2 + (\theta_2 - u_\theta)^2, x_3^2 + y_3^2 + (\theta_3 - u_\theta)^2)\),
  \(0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi, 0 \leq \theta_3 \leq 2\pi, 0 \leq u_\theta \leq 2\pi\).

\(^3\)One should not translate this as the generator of an event horizon does not change its angular direction. To write this form, we may change the origin or scale of the angular coordinate \(\theta\).
For $G_a[(A_1, A_1, A_1)]$, the minimum function is

$$\min(G_a[(A_1, A_1, A_1)]) = \begin{cases} 
  u_1 & u_1 < u_2, u_1 < 0 \\
  u_2 & u_2 < u_1, u_2 < 0 \\
  0 & u_1 > 0, u_2 > 0 
\end{cases} \quad (65)$$

This is singular at $u_1 = u_2 < 0, u_1 = 0, u_2 > 0$ and $u_2 = 0, u_1 > 0$. Then its Maxwell set is given by

$$B_{\text{Maxwell}}(G_a[(A_1, A_1, A_1)]) = \{(u_1, u_2, u_\theta)|u_1 = u_2 < 0\} \cup \{(u_1, u_2, u_\theta)|u_1 = 0, u_2 > 0\} \cup \{(u_1, u_2, u_\theta)|u_2 = 0, u_1 > 0\} \quad (66)$$

- $G_a[A_3] = x_1^4 + u_2 x_1^3 + u_1 x_1 + (\theta_1 - u_\theta)^2$, 
  $0 \leq \theta_1 \leq 2\pi, 0 \leq u_\theta \leq 2\pi$.

For $G_a[A_3]$, the Maxwell set is same as that of $A_3$ on the section $u_\theta = 0$. Then its Maxwell set is given by

$$B_{\text{Maxwell}}(G_a[A_3]) = \{(u_1, u_2, u_\theta)|u_2 > 0, u_1 = 0\} \quad (67)$$

These Maxwell sets are illustrated in Fig.3.

5.2.2 Class D multiunfolding with axial symmetry

Similarly to the Class C multiunfolding, the correspondence between $U_r = \mathbb{R}^2/G_r$ and $U_a = \mathbb{R}^3/G_a$ also gives correspondence of multiunfolding between Class B and Class D. In addition that, it is necessary to show that the minimum is not isolated but make an orbit of the isometry. This is realized by global coordinates of the symmetry so that the Morse function $\delta \theta^2$, which is added to revive a dummy variable $u_\theta$, is extended to a global function of $\delta \theta$. It is achieved by $\cos \delta \theta \sim 1 - \delta \theta^2/2 + ...$.

**Class D multiunfolding**

A $G_a$-invariant $G_a$-stable multiunfolding in Class D is right SPD-equivalent to one of the following $G_a$-orbital multigerm of unfolding at the minimum points.

- $(A_1, A_1) = (r_1^2 + u_z, r_2^2)$.

For $(A_1, A_1)$, the minimum function is

$$\min(A_1, A_1) = \begin{cases} 
  u_z & u_z < 0 \\
  0 & u_z \geq 0 
\end{cases} \quad (68)$$

The Maxwell set that is the singular point of minimum function is

$$B_{\text{Maxwell}}(\varphi_\theta A_1) = \{(u_r, u_z, u_\theta)|u_z = 0\} \quad (69)$$

The toroidal event horizon reported in ref.2 is caused by the crease set which is embedding of this Maxwell set.

- $G_a(A_1) = z^2 + u_r \cos(\theta - u_\theta)$, 
  $0 \leq \theta \leq 2\pi, 0 \leq u_\theta \leq 2\pi$. 

19
For $G_a(A_1)$, the minimum function is

$$\min(G_a(A_1)) = u_r.$$  \hspace{2cm} (70)

This is singular at $u_r = 0$,

$$B_{\text{Maxwell}}(G_a(A_1)) = \{(u_r, u_z, u_\theta)|u_r = 0\}. \hspace{2cm} (71)$$

This Maxwell set is reported in the collision of the two black holes $G$. 

- $(G_a(A_1), A_1) = (z^2 + u_r \cos(\theta - u_\theta), r_2^2 + u_z), 0 \leq \theta \leq 2\pi, 0 \leq u_\theta \leq 2\pi$

The minimum function of $(G_a(A_1), A_1)$ is

$$\min(G_a(A_1), A_1) = \min(u_r, u_z) = \begin{cases} 
  u_z & u_r > u_z \\
  u_r & u_r \leq u_z 
\end{cases}. \hspace{2cm} (72)$$

This is singular at $u_r = 0, u_z > 0$ and $u_z = u_r, u_z < 0$;

$$B_{\text{Maxwell}}(G_a(A_1)) = \{(u_r, u_z, u_\theta)|u_r = 0, u_z > 0\} \cup \{(u_r, u_z, u_\theta)|u_z < 0, u_z = u_r\}. \hspace{2cm} (73)$$

- $(G_a(A_1), G_a(A_1)) = (z^2 + u_r \cos(\theta_1 - u_\theta) + u_z, z^2 + u_r \cos(\theta_2 - u_\theta)), 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi, 0 \leq u_\theta \leq 2\pi$

The minimum function of $(G_a(A_1), G_a(A_1))$ is

$$\min((G_a(A_1), G_a(A_1))) = \min(u_r + u_z, u_r) = \begin{cases} 
  u_r & u_z > 0 \\
  u_r + u_z & u_z \leq u_r \leq 0 
\end{cases}. \hspace{2cm} (74)$$

This is singular at $u_r = 0$ and $u_z = 0$:

$$B_{\text{Maxwell}}((G_a(A_1), G_a(A_1))) = \{(u_r, u_z, u_\theta)|u_r = 0\} \cup \{(u_r, u_z, u_\theta)|u_z = 0\}. \hspace{2cm} (75)$$

- $G_a(A_3) = A_3 = r^4 + u_z r^2 + u_r r$

where $r$ and $u$ is the position vector to $(r, \theta)$ and $(u_r, u_\theta)$ surface of $M$ and $U$, respectively.

For $G_a(A_3)$, the Maxwell set is same as that of $A_3$

$$B_{\text{Maxwell}}(G_a(A_3)) = \{(u_r, u_z, u_\theta)|u_r = 0, u_z < 0\}. \hspace{2cm} (76)$$

Their Maxwell sets are illustrated in Fig 4

Here it should be noted that the Maxwell set stably possesses one-dimensional component $G_a(A_1)$, $(G_a(A_1), A_1), (G_a(A_1), G_a(A_1))$ or $G_a(A_3)$, while only two-dimensional structure is stable without symmetry $[5][6]$.

From the above classifications, we are able to see the crease set. As stated in $[6][10]$, the crease set is an acausal embedding of the Maxwell set into the spacetime so that the crease set is homeomorphic to a
ball and approaches to null at its boundary. We give a figure of an example of axially symmetric Maxwell set in Fig.5.

In the example in Fig.5, by an appropriate timeslicing, it is observed that toroidal black hole and spherical black hole coalesce. At the lower spatial hypersurface, the section of the crease set is a ring and a point. They would be a crease of a toroidal event horizon and a cusp on a spherical event horizon. As time goes by they approach each other and the ring shrinks, and then they unite and disappear. This implies the coalescence of toroidal black hole and a spherical black hole.

To find an example in Fig.5, we have combined the structures of the Maxwell set so that a total Maxwell set is homeomorphic to a ball, since it is required by the assumption that the event horizon finally becomes a single sphere. Nevertheless it might be important to consider topologically non-trivial final state of the event horizon, which becomes meaningful in studies of higher-dimensional black hole. For example, for a toroidal final event horizon, we compose the Maxwell set so that it is homeomorphic to a ring.

6 conclusion and discussions

We have investigated the Maxwell set under a discrete symmetry and a continuous symmetry. In the context of singularity theory, the stable Maxwell set is classified. Then we concretely showed the classification of the stable Maxwell set with reflection symmetry in three dimensional spacetime and axial symmetry in four-dimensional spacetime. Remarkable fact is that by the axial symmetry one-dimensional segments of the crease set is possible to become stable while it is not stable in four-dimensional spacetime without symmetry.

It might be worth to comment that the present result also is relevant in the case of almost symmetric where the symmetry is dynamically stable, if a very fine structure is neglected. Some cases will be almost axially symmetric in realistic gravitational collapse. Nevertheless it is not mathematically clear how can we neglect the fine structure. Under the present situation, our result only suggests the structure of the crease set in almost symmetric spacetime.

By the way, one may be doubtful that peculiar crease sets for example like in Fig.5 can be formed. However, it will be realistic in the collision of collapsing stars. In an oblate spheroidal event horizon, it is probable that the crease set is naturally two dimensional. If sufficiently oblate spheroidal collapsing star and spherical collapsing star are sufficiently away in direction of their symmetry axis and if they coalesce, the crease set like Fig.5 will be formed. Furthermore, when several black holes coalesce, their crease set will become fairly complicated.

Since these collapses are axially symmetric, it will be possible to simulate them by numerical calculation of the gravitational collapse. Not only the fairly simple case in refs.2,3 (these cases are also reflection symmetric about x-y plane) but also some considerably complex cases should be studied. With that the various topological black hole will be observed simultaneously. That will be thought-provoking for the gravitational collapse and phenomena concerning the event horizon, e.g. quasi-normal mode of gravitational radiation.
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Figure 1: The Maxwell sets in two-dimensional reflection symmetric spacetime not on its fixed points. They are the Maxwell sets of Class A multifoldings.

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Figure 2: The Maxwell sets in two-dimensional reflection symmetric spacetime on its fixed points. They are the Maxwell sets of Class B multiumfoldings.
Figure 3: The Maxwell sets in three-dimensional axially symmetric spacetime not on its fixed points. They are the Maxwell sets of Class C multiumfoldings.
Figure 4: The Maxwell sets in three-dimensional axially symmetric spacetime on its fixed points. They are the Maxwell sets of Class D multiumfoldings.
Figure 5: An example of the axially symmetric Maxwell set is shown. That is composed of $G_a(A_3)$, $(G_a(A_1), A_1)$ and $G_a[A_3]$, which are connected by $G_a(A_1)$ and $G_a[A_1, A_1]$. Next figure is the crease set which is given by embedding the Maxwell set into a spacetime as a spatial set. The bottom figure indicate the change of the topology of an event horizon with that crease set.