THE SANDPILE MODEL ON THE COMPLETE SPLIT GRAPH, COMBINATORIAL NECKLACES, AND TIERED PARKING FUNCTIONS

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Abstract. In this paper we perform a classification of the recurrent states of the Abelian sandpile model (ASM) on the complete split graph. There are two distinct cases to be considered that depend upon the location of the sink vertex in the complete split graph. We define and use a new toppling order, called the mascoi toppling order, to perform this classification. This mascoi toppling order allows us to give a bijection between the decreasing recurrent states of the ASM on the complete split graph and two classes of tri-coloured combinatorial necklaces. This characterisation of decreasing recurrent states is then used to provide a characterisation of the general recurrent states. We also give a characterisation of the recurrent states in terms of a new type of parking function that we call a tiered parking function. These parking functions are characterised by assigning a tier (or colour) to each of the cars, and specifying how many cars of a lower-tier one wishes to have parked before them. We also enumerate the different sets of recurrent configurations studied in this paper, and in doing so derive a formula for the number of spanning trees of the complete split graph. This paper lays the foundations for a study into statistics on these recurrent configurations, as was done in the case of the complete bipartite graph in the author’s papers [7] and [1].

1. Introduction

The complete split graph is a bipartite graph consisting of two distinct parts, a clique part in which all distinct pairs of vertices are connected by a single edge, and an independent part in which no two vertices are connected by an edge. There is precisely one edge between every vertex in the clique part and every vertex in the independent part. We denote the complete split graph which has \( m \) vertices in the clique part and \( n \) vertices in the independent part by \( S_{m,n} \). The graph \( S_{5,3} \) is illustrated in Figure 1.

![Figure 1. The complete split graph \( S_{5,3} \)](image)

The graph \( S_{m,n} \) contains the complete graph \( K_m \) as a subgraph, but is also a bipartite graph in its own right, and it is this dual feature that we find interesting to examine in terms of the sandpile model. The sandpile model has been studied on several classes of graphs, and rich connections to other combinatorial structures have been established in each of the cases. For the case of the sandpile model on the complete graph \( K_{n+1} \) in which one designated vertex is a sink, it was shown by Cori and Rossin [4] that the set of recurrent states of the sandpile model on this graph are in one-to-one correspondence with parking functions of order \( n \).

In the case of the sandpile model on the complete bipartite graph \( K_{m,n} \) in which a designated vertex is the sink, it was shown by the author (in collaboration with Le Borgne [7]) that the set of recurrent states admit...
a description in terms of planar animals called parallelogram polyominoes. Parallelogram polyominoes are also known as staircase polyominoes in the literature. Cori and Poulalhon in [3] showed that the recurrent states of the sandpile model on the complete tri-partite graph $K_{1,p,q}$, in which the solitary vertex is the sink, admits a description in terms of a parking function for cars of two different colours.

In this paper we will classify the recurrent states of the Abelian sandpile model on the complete split graph. There are two distinct cases to be considered that depend on whether the sink is a clique vertex or an independent vertex in the graph. To accomplish this classification, we define and use a new toppling order that we called the mascoi toppling order; the name is an abbreviation of its constituent rules. This mascoi toppling order allows us to give a bijection between the decreasing recurrent states of the ASM on the complete split graph and two classes of tri-coloured combinatorial necklaces, thereby providing an alternative combinatorial characterisation of the general recurrent states. Further to this, we give a second characterisation of the recurrent states in terms of a new type of parking function that we call a tiered parking function. These parking functions are characterised by assigning a tier (or colour) to each of the cars, and specifying how many cars of a lower-tier one wishes to have parked before them in a one-way street. We also enumerate the different sets of recurrent configurations studied in this paper, and in doing so derive a formula for the number of spanning trees of the complete split graph.

This paper lays the foundations for a study into statistics on these recurrent configurations, as was done in the case of the complete bipartite graph in [7] and [1]. We posit that there are many interesting correspondences to be uncovered by studying Dhar’s burning algorithm in the context of bijective combinatorics. This paper bolsters this proposition by establishing some new and surprising correspondences, and we believe this is the first time combinatorial necklaces have featured prominently in a physical model.

In Section 2 we give a description of the sandpile model on a general graph, and define the recurrent states of the model. We then define our graph of interest and sets of configurations on this graph that will be used throughout the paper. We also introduce mascoi topplings, a toppling order that will play a prominent role in our proofs. In Section 3 we perform a classification of recurrent states of the sandpile model for when the sink is in the clique part of the complete split graph and provide several examples verifying this classification. In Section 4 we perform a classification of recurrent states of the sandpile model for when the sink is in the independent part of the complete split graph along with several examples verifying the classification. In Section 5 we introduce combinatorial necklaces, and prove that two classes of combinatorial necklaces are in one-to-one correspondence with the weakly decreasing recurrent states that were characterised in Sections 3 and 4. In Section 6 we introduce tiered parking functions. These new parking functions allow for a new and novel classification of the recurrent states studied in this paper. In Section 7 we enumerate the different sets of recurrent configurations that were studied in this paper. We use Kirchhoff’s matrix tree theorem to derive a formula for the number of spanning trees of the complete split graph.

2. Recurrent States of the Sandpile Model on a Graph and the Mascoi Toppling Order

Let us first describe the sandpile model on a general graph and define what it means for a configuration or state on such a graph to be recurrent. In Sections 3 and 4 we will consider the two different cases of the sandpile model on the complete split graph. The different cases occur due to the location of the sink: it may be a vertex in the clique part, or in the independent part. The labelling of the vertices and choice of sink have been chosen so as to accommodate the cleanest classification with respect to the notation. The graph remains the same for both cases in that the complete split graph will have $m$ clique-part vertices and $n$ independent-part vertices.

The Abelian sandpile model (ASM) may be defined on any undirected graph $G$ with a designated vertex $s$ called the sink. A configuration on $G$ is an assignment of non-negative integers to the non-sink vertices of $G$:

$$c : V(G) \setminus \{s\} \rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}. $$
The number \( c(v) \) is sometimes referred to as the number of grains at vertex \( v \), or as the height of \( v \). Given a configuration \( c \), a vertex \( v \) is said to be stable if the number of grains at \( v \) is strictly smaller than the threshold of that vertex, which is the degree of \( v \), denoted \( \text{deg}(v) \). Otherwise \( v \) is unstable. A configuration is called stable if all non-sink vertices are stable.

If a vertex is unstable then it may topple, which means the vertex donates one grain to each of its neighbors. The sink vertex has no height associated with it and it only absorbs grains, thereby modelling grains exiting the system. Given this, it is possible to show that starting from any configuration \( c \) and toppling unstable vertices, one eventually reaches a stable configuration \( c' \). Moreover, \( c' \) does not depend on the order in which vertices are toppled in this sequence. We call \( c' \) the stabilisation of \( c \). We use the notation ChipFiring\((G, s)\) to indicate that we are considering the ASM as described here on the graph \( G \) with sink \( s \). We will find it useful to use the notation \( \mathcal{V}(\text{ChipFiring}(G, s)) = \mathcal{V}(G) \setminus \{s\} \) for the non-sink vertices.

Starting from the empty configuration, one may indefinitely add any number of grains to any vertices in \( G \) and topple vertices should they become unstable. Certain stable configurations will appear again and again, that is, they recur, while other stable configurations will never appear again. These recurrent configurations are the ones that appear in the long term limit of the system. Determining the set of recurrent configurations for ChipFiring\((G, s)\) is not a straightforward task. In [8], Section 6, Dhar describes the so-called burning algorithm, which establishes in linear time whether a given stable configuration is recurrent. We recall the result here.

**Proposition 2.1** ([8], Section 6.1). Let \( G \) be a graph with sink \( s \), and let \( c \) be a stable configuration on \( G \). Then \( c \) is recurrent if and only if there exists an ordering \( v_0 = s, v_1, \ldots, v_n \) of the vertices of \( G \) such that, starting from \( c \), for any \( i \geq 1 \), toppling the vertices \( v_0, \ldots, v_{i-1} \) causes the vertex \( v_i \) to become unstable. Moreover, if such a sequence exists, then toppling \( v_0, \ldots, v_n \) returns the initial configuration \( c \).

This algorithm has seen several equivalent formulations over the years, and we refer the interested reader to two recent books on the topic that discuss these in a clear and insightful way; Corry & Perkinson [5] §7.5 and Klivans [11] §2.6.7.

Let us define the complete split graph \( S_{m,n} \) to have vertex set \( \mathcal{V}(S_{m,n}) := \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \) and edge set
\[
E(S_{m,n}) := \{(v_i, v_j) : 1 \leq i < j \leq m\} \cup \{(v_i, w_j) : i \in [1, m] \text{ and } j \in [1, n]\}.
\]
Here, and throughout the paper, we use the notation \([a, b] = \{a, a+1, a+2, \ldots, b\}\). In the literature one typically refers to the part of the graph on vertices \( V = \{v_1, \ldots, v_m\} \) as the clique-part, and that part of the graph on vertices \( W = \{w_1, \ldots, w_n\} \) as the independent-part. We will find it useful to name the sets \( V' := V \setminus \{v_m\} \) and \( W' := W \setminus \{w_n\} \). By a clique vertex, we will mean a vertex in \( V \), and similarly for an independent vertex.

A configuration on ChipFiring\((S_{m,n}, s)\) is a vector \( c = (c_1, \ldots, c_{m+n-1}) \) whereby \( c_i \) represents the number of chips/grains at the \( i \)th vertex in the set
\[
\{v_1, \ldots, v_m, w_1, \ldots, w_n\} \setminus \{s\},
\]
with respect to the following total order \( \mathcal{O} = (\mathcal{O}, \prec) \) on \( \mathcal{V}(S_{m,n}) \):
\[
\mathcal{O} : v_1 \prec v_2 \prec \ldots \prec v_m \prec w_1 \prec \ldots \prec w_n.
\]
Given \( u \in \mathcal{V}(S_{m,n}) \) and a configuration \( c \) on ChipFiring\((S_{m,n}, s)\), let us define the surplus of vertex \( u \) as
\[
\text{surplus}_c(u) := c(u) - \text{deg}(u).
\]
Let us denote the recurrent states of the model by \( \text{Rec}(\text{ChipFiring}(S_{m,n}, s)) \). Due to the high number of symmetries in \( S_{m,n} \), we will restrict our analysis to those configurations that are weakly decreasing with respect to vertex labels. This restriction will allow us to focus our analysis to consider characterizing all those ‘different’ configurations, and from which we can generate all configurations through permutations. Furthermore, for the two distinct cases of where the sink may be (in the clique part or the independent part),
for the sake of notation we will consider the sink to be \( s = v_m \) in the first case, and \( s = w_n \) in the second case.

We denote the set of weakly decreasing recurrent states on \( \text{ChipFiring}(S_{m,n}, v_m) \) by

\[
\text{DecRec}(\text{ChipFiring}(S_{m,n}, v_m)) := \{(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \text{Rec}(\text{ChipFiring}(S_{m,n}, v_m)) : a_1 \geq \cdots \geq a_{m-1} \text{ and } b_1 \geq \cdots \geq b_n\}
\]

and the set of weakly decreasing recurrent states on \( \text{ChipFiring}(S_{m,n}, w_n) \) by

\[
\text{DecRec}(\text{ChipFiring}(S_{m,n}, w_n)) := \{(a_1, \ldots, a_m, b_1, \ldots, b_{n-1}) \in \text{Rec}(\text{ChipFiring}(S_{m,n}, w_n)) : a_1 \geq \cdots \geq a_m \text{ and } b_1 \geq \cdots \geq b_{n-1}\}.
\]

We will sometimes find it convenient to replace the comma that separates the clique and independent parts in a configuration with a semi-colon.

Example 2.2. \( \text{DecRec}(\text{ChipFiring}(S_{2,2}, v_2)) = \{(0; 1, 1), (1; 1, 0), (1; 1, 1), (2; 0, 0), (1; 2, 1), (2; 1, 0)\} \). Note that the numbers \((a_1; b_1, b_2)\) correspond to the number of chips at the vertices \((v_1; w_1, w_2)\).

The graph for \( \text{ChipFiring}(S_{5,3}, v_5) \) is illustrated in Figure 2.

Figure 2. The complete split graph \( S_{5,3} \) with sink at vertex \( v_5 \)

Let us now introduce a new toppling order for configurations of the ASM on the complete split graph. This new toppling order is certainly different to the parallel toppling order that has proven successful classifying recurrent states of the ASM on other graphs.

Definition 2.3. Suppose \( c \) is a configuration on \( H = \text{ChipFiring}(S_{m,n}, s) \). Let us define the \textit{max-superl & clique-over-independent (mascoi) toppling order} on \( c \) to be the outcome of the following procedure:

M0. Set \( t \leftarrow 1 \) and let \( c^{(t)} \leftarrow c \).

M1. Let \( X_t \) be the set of unstable vertices \( u \) for which \( \text{surplus}_{c^{(t)}}(u) \) is currently largest, i.e.

\[
X_t = \{ u \in V(H) : \text{surplus}_{c^{(t)}}(u) = m^* \} \text{ where } m^* = \max_{u \in V(H)} \text{surplus}_{c^{(t)}}(u).
\]

M2. Choose the vertex \( u^* \) in \( X_t \) that is smallest with respect to the total order \( \mathcal{O} \).

M3. Topple \( u^* \) and let \( c^{(t+1)} \) be the resulting configuration.

M4. If \( c^{(t+1)} \) is an unstable configuration, then set \( t \leftarrow t + 1 \) and go to step M1.

Let the outcome of this process be \( \text{mascoi}_H(c) := (u_1, \ldots, u_k) \) where \( \{u_1, \ldots, u_k\} \subseteq V(H) \). For any vertex \( u \in V(H) \), let us define \( h_t(u) := c^{(t)}(u) \), the height of vertex \( u \) at time \( t \).

Example 2.4. Consider the unstable configuration \( c = c^{(1)} = (7, 6, 4, 3; 4, 3, 1) \) on \( H = \text{ChipFiring}(S_{5,3}, v_5) \). A clique vertex is unstable if its height is 7 or greater and an independent vertex is unstable if its height is 5 or greater. There is only one unstable vertex at time 1, so in step M1, we have \( X_1 = \{v_1\} \). Topple this vertex to get \( c^{(2)} = (0, 7, 5, 4; 5, 4, 2) \) and return to M1. For step M1 we now have \( X_2 = \{v_2, w_1\} \) and \( \text{surplus}(v_2) = 0 = \text{surplus}(w_1) \). As \( v_2 \prec w_1 \), step M3 tells us to topple \( v_2 \) to get \( c^{(3)} = (1, 0, 6, 5; 6, 5, 3) \).
For step M1 we now have $X_3 = \{w_1, w_2\}$. Step M2 selects $w^* = w_1$ and step M3 tells us to topple $w_1$ to get $c^{(4)} = (2, 1, 7, 6; 1, 5, 3)$. Now $X_4 = \{v_3, w_2\}$ and surplus$(v_3) = \text{surplus}(w_2) = 0$. Step M2 tells us to topple $v_3$. This gives $c^{(5)} = (3, 2, 0, 7; 2, 6, 4)$ and $X_5 = \{w_2\}$, so we topple $w_2$ to get $c^{(6)} = (4, 3, 1, 8; 2, 1, 4)$. Now we have $X_6 = \{v_4\}$ so we topple $v_4$ to get $c^{(7)} = (5, 4, 2, 1; 3, 2, 5)$. Finally $X_7 = \{v_3\}$ and we topple it go get $c^{(8)} = (6, 5, 3, 2; 3, 2, 0)$, which is stable. We record the sequence of topplings as \( \text{mascoi}_H(c) = (v_1, v_2, w_1, v_3, w_2, v_4, w_3) \).

We will consider the characterisations of recurrent states using the mascoi toppling order for both the clique and the independent case. The rationale behind this new toppling rule is that it is at the heart of the correspondence with combinatorial necklaces that we will discuss in Section 5.

**Proposition 2.5.** A weakly decreasing stable configuration $c$ on $F = \text{ChipFiring}(S_{m,n}, v_m)$ is in DecRec($F$) iff there exists an ordering $z = (z_1, \ldots, z_{m+n-1})$ of the vertices $\{v_1, \ldots, v_{m-1}, w_1, \ldots, w_n\}$ such that $z = \text{mascoi}_F(c)$ and the following conditions hold:

(i) \( h_t(z_t) \geq \deg(z_t) \) for all $t \in [1, m+n-1]$, 
(ii) \( h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}) \) for all $t \in [1, m+n-1]$, 
(iii) If $z_t$ is independent and $z_{t+1}$ is clique then $h_t(z_t) - m > h_t(z_{t+1}) - (m+n-1)$.

**Proof.** Proposition 2.3 gives a necessary and sufficient condition for checking that a stable configuration on ChipFiring($S_{m,n}, v_m$) is recurrent. We will use the mascoi toppling order (Definition 2.5) in conjunction with Proposition 2.1. Let $F = \text{ChipFiring}(S_{m,n}, v_m)$. The sequence $\delta_F = (l_1, \ldots, l_{m+n-1})$ where $l_i$ is the number of edges from the sink $v_m$ to the $i$th vertex of $V(F)$ with respect to the order $O$, is simply the sequence of all ones. This is the vector that represents the contributions to vertices on toppling the sink of $F$.

A weakly decreasing stable configuration $c$ on $F$ is in DecRec($F$) iff $\text{mascoi}_F(c + \delta_F) = (z_1, \ldots, z_{m+n-1})$ where $\{z_1, \ldots, z_{m+n-1}\} = V(F)$. The threshold of vertex $z_i$ in $S_{m,n}$ is

\[
\deg(z_i) = \begin{cases} 
  m + n - 1 & \text{if } z_i \in V \\
  m & \text{if } z_i \in W.
\end{cases}
\]

Let us consider the mascoi toppling process on a weakly decreasing stable configuration $c$. In order to do this, let $h_t(z) := (h_t(z_1), \ldots, h_t(z_{m+n-1}))$ for all times $t = 0, \ldots, m+n-1$, where $h_t$ is defined in Definition 2.3. Let $h_0(z)$ be the sequence $(c(z_1), c(z_2), \ldots, c(z_{m+n-1}))$. This will be a permutation of $c$ whereby the heights of the clique vertices and the independent vertices respect their original order. On toppling the sink we have

\[
h_1(z) = (1 + h_0(z_1), \ldots, 1 + h_0(z_{m+n-1})).
\]

The toppling criterion for a recurrent sequence means that all vertices topple exactly once, so it must be the case that $1 + h_0(z_1)$ is at least as large as the threshold $\deg(z_1)$.

The mascoi toppling rule in conjunction with Proposition 2.3 give the following: At time $t$, all vertices in the set $\{z_1, \ldots, z_{t-1}\}$ have toppled and vertex $z_t$ is unstable. (Note that it could be that $z_t$ was unstable before this.) This means

\[
h_t(z_t) \geq \deg(z_t) \text{ for all } t \in [1, m+n-1].
\]

Vertex $z_t$ will topple between times $t$ and $t+1$: We must also be careful that the toppling order rules hold true with respect to the mascoi toppling order. This means that at time $t$, the height of unstable vertex $z_t$ above its threshold is greater than the height vertex $z_{t+1}$ above its threshold at time $t$.

\[
h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}).
\]

The order $O$ in M2 indicates a preference for a clique vertex to topple before an independent vertex, should both have a surplus of the same size. Thus should they have the same surplus over their respective thresholds.
at the same time, M2 is equivalent to the following: if \( z_t \) is an independent vertex that is unstable, and \( z_{t+1} \) is a clique vertex that is unstable, then 
\[
    h_t(z_t) - \deg(z_t) = h_t(z_t) - m > h_t(z_{t+1}) - \deg(z_{t+1}) = h_t(z_{t+1}) - (m + n - 1).
\]
This inequality together with the two inequalities in (2) and (3) yield: a weakly decreasing stable configuration \( c \) on \( F \) is in DecRec(\text{ChipFiring}(S_{m,n}, w_n)) if and there exists an ordering \( z = (z_1, \ldots, z_{m+n-1}) \) of the vertices \( \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \) such that \( z = \text{mascoi}_F(c) \) and the following conditions hold:

(i) \( h_t(z_t) \geq \deg(z_t) \) for all \( t \in [1, m + n - 1] \),
(ii) \( h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}) \) for all \( t \in [1, m + n - 1] \),
(iii) If \( z_t \) is independent and \( z_{t+1} \) is clique then \( h_t(z_t) - m > h_t(z_{t+1}) - (m + n - 1) \).

The characterisation of these states when the sink is changed to be an independent vertex contains several details that are different. Consequently this new case warrants a presentation as a separate proposition.

![Figure 3. The complete split graph \( S_{5,4} \) with sink at vertex \( w_4 \)](image)

**Proposition 2.6.** A weakly decreasing stable configuration \( c \) on \( F = \text{ChipFiring}(S_{m,n}, w_n) \) is in DecRec(\( F \)) iff there exists an ordering \( z = (z_1, \ldots, z_{m+n-1}) \) of the vertices \( \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \) such that \( z = \text{mascoi}_F(c) \) and the following conditions hold:

(i) \( z_1 \) is clique with \( c(z_1) = m + n - 2 \), and \( h_t(z_t) \geq \deg(z_t) \) for all \( t \in [2, m + n - 1] \),
(ii) \( h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}) \) for all \( t \in [1, m + n - 1] \),
(iii) If \( z_t \) is independent and \( z_{t+1} \) is clique then \( h_t(z_t) - m > h_t(z_{t+1}) - (m + n - 1) \).

**Proof.** Again, we use Proposition 2.1 in conjunction with Definition 2.3 to characterise DecRec(\text{ChipFiring}(S_{m,n}, w_n)). Let \( F = \text{ChipFiring}(S_{m,n}, w_n) \). The sequence \( \delta_F = (t_1, \ldots, t_{m+n-1}) \) where \( t_i \) is the number of edges from the sink \( w_n \) to the \( i \)th vertex of \( V(F) \) with respect to the order \( O \), is simply the sequence \((1, \ldots, 1, 0, \ldots, 0)\) where there are \( m \) ones and \( n - 1 \) zeros.

As before, a weakly decreasing stable configuration \( c \) on \( F \) is in DecRec(\( F \)) iff mascoi(\( c+\delta_F = (z_1, \ldots, z_{m+n-1}) \)) where \( \{z_1, \ldots, z_{m+n-1}\} = V(F) \). Let \( h_0(z) \) be the sequence \( (c(z_1), c(z_2), \ldots, c(z_{m+n-1})) \). This will be a permutation of \( c \) whereby the heights of the clique vertices and the independent vertices respect their original order (since the configuration heights are weakly decreasing in their respective parts). On toppling the sink we now have
\[
    h_1(z) = (h_0(z) + 1 \lceil z_i \text{ clique} \rceil)_{i=1}^{m+n-1}.
\]

The toppling criterion for a recurrent sequence means that all vertices topple exactly once, so it must be the case that \( h_0(z_1) + 1 \lceil z_1 \text{ clique} \rceil \) is at least as large as the threshold \( \deg(z_1) \). However, if \( z_1 \) is an independent vertex, then \( 1 \lceil z_1 \text{ clique} \rceil = 0 \), which means it cannot reach the threshold. Thus it must be that \( z_1 \) is a clique vertex and \( h_0(z_1) + 1 = \deg(z_1) = m + n - 1 \), i.e. \( h_0(z_1) = m + n - 2 \).
With this initial observation noted, our analysis is essentially the same as in the proof of Prop [2.5].

The mascoi toppling rule in conjunction with Proposition [2.1] give the following: At time \( t \), all vertices in the set \( \{ z_1, \ldots, z_{t-1} \} \) have topped and vertex \( z_t \) is unstable. This means

\[
 h_t(z_t) \geq \deg(z_t) \quad \text{for all } t \in [1, m + n - 1].
\]

The case \( t = 1 \) has already been discussed above. Vertex \( z_t \) will topple between times \( t \) and \( t + 1 \): we must be careful that the toppling order rules hold true with respect to the mascoi toppling order. This means that at time \( t \), the height of unstable vertex \( z_t \) above its threshold is greater than the height vertex \( z_{t+1} \) above its threshold at time \( t \).

\[
 h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}).
\]

The order \( O \) in M2 indicates a preference for a clique vertex to topple before an independent vertex, should both have a surplus of the same size. Thus should they have the same surplus over their respective thresholds at the same time, M2 is equivalent to the following: if \( z_t \) is an independent vertex that is unstable, and \( z_{t+1} \) is a clique vertex that is unstable, then

\[
 h_t(z_t) - \deg(z_t) = h_t(z_t) - m > h_t(z_{t+1}) - \deg(z_{t+1}) = h_t(z_{t+1}) - (m + n - 1).
\]

This inequality together with the inequalities above yield: a weakly decreasing stable configuration \( c \) on \( F \) is in \( \text{DecRec}(\text{ChipFiring}(S_{m,n}, v_n)) \) iff there exists an ordering \( z = (z_1, \ldots, z_{m+n-1}) \) of the vertices \( \{v_1, \ldots, v_m, w_1, \ldots, w_{n-1} \} \) such that \( z = \text{mascoi}_F(c) \) and the following conditions hold:

- (i) \( z_1 \) is clique with \( c(z_1) = m + n - 2 \), and \( h_t(z_t) \geq \deg(z_t) \) for all \( t \in [2, m + n - 1] \),
- (ii) \( h_t(z_t) - \deg(z_t) \geq h_t(z_{t+1}) - \deg(z_{t+1}) \) for all \( t \in [1, m + n - 1] \),
- (iii) If \( z_t \) is independent and \( z_{t+1} \) is clique then \( h_t(z_t) - m > h_t(z_{t+1}) - (m + n - 1) \).

\[ \square \]

3. Recurrent states for the case of a clique-sink

In this section we will classify recurrent states of the sandpile model on the complete split graph in the case of a clique vertex acting as the sink. This will be done using Proposition [2.5] to given a novel sequence encoding of the recurrent states. In terms of the dynamics of the model in this particular case, the effect of toppling the sink is to increase the height of every non-sink vertex by one.

**Example 3.1.** The first few sets of weakly decreasing recurrent configurations for small values of \( m \) and \( n \) are:

\[
\begin{align*}
\text{DecRec}(\text{ChipFiring}(S_{2,1}, v_2)) &= \{(0;1),(1;0),(1;1)\} \\
\text{DecRec}(\text{ChipFiring}(S_{2,2}, v_2)) &= \{(0;1,1),(1;1,0),(1;1,1),(2;0,0),(2;1,0),(2;1,1)\}.
\end{align*}
\]

and

\[
\begin{align*}
\text{DecRec}(\text{ChipFiring}(S_{3,2}, v_3)) &= \{(1,0;2,2),(1,1;2,2),(2,0;2,1),(2,0;2,2),(2,1;2,0),(2,1;2,1),(2,1;2,2),(2,2;2,0),(2,2;2,1),
(2,2;2,2),(3,0;1,1),(3,0;2,1),(3,0;2,2),(3,1;1,0),(3,1;1,1),(3,1;2,0),(3,1;2,1),(3,1;2,2),
(3,2;0,0),(3,2;1,0),(3,2;1,1),(3,2;2,0),(3,2;2,1),(3,2;2,2),(3,3;0,0),(3,3;1,0),(3,3;1,1),
(3,3;2,0),(3,3;2,1),(3,3;2,2)\}
\end{align*}
\]

**Theorem 3.2.** A weakly decreasing stable configuration \( c = (a_1, \ldots, a_{m-1}; b_1, \ldots, b_n) \) on \( \text{ChipFiring}(S_{m,n}, v_m) \) is recurrent iff there exists a sequence \( x \) of \( m-1 \) zeros, \( m-1 \) ones, and \( n \) zeros with the following property:

- (i) For all \( i \in [1,2m+n-2] \):
  \[ |\{ j \in [1,i] : x_j = 0 \}| \geq |\{ j \in [1,i] : x_j = 1 \}|. \]
- (ii) The number of ones to the right of the \( j \)th two in \( x \) is \( b_j \).
(iii) The number of ones and twos to the right of the $j^{th}$ zero in $x$ is $a_j + 1$.

Proof. Proposition 2.5 tells us that a weakly decreasing stable configuration $F$ gives values $v$ such that the size of the set in the second case is the number of clique and independent vertices to the right of $z_t$ and $z_{t+1}$ is in $W$. We can now reformulate cases (Ai)–(Aiii) as follows.

Let us observe that since $z_t$ is the only vertex that topples at time $t$, we can write down an expression for $h_t(z_t)$:

$$h_t(z_t) = 1 + h_0(z_t) + \begin{cases} (t - 1) & \text{if } t \leq i \text{ and } z_t \in V' \\ (t - 1) - \deg(z_t) & \text{if } t > i \text{ and } z_t \in V' \\ \{|j| [1, t - 1] : z_j \in V'\} & \text{if } t \leq i \text{ and } z_t \in W \\ \{|j| [1, t - 1] : z_j \in V'\} - \deg(z_t) & \text{if } t > i \text{ and } z_t \in W. \end{cases}$$

The values $h_t(z_t)$ and $h_t(z_{t+1})$ feature in conditions (Ai)–(Aiii) above, and using this expression for $h_t(z_t)$ gives

$$h_t(z_t) = 1 + h_0(z_t) + \begin{cases} t - 1 & \text{if } z_t \in V' \\ \{|j| [1, t - 1] : z_j \in V'\} & \text{if } z_t \in W. \end{cases}$$

The term $(t - 1)$ in the first case above is the number of vs and ws to the left of $z_t$ (which is in $W$) in $z$. The size of the set in the second case is the number of vs to the left of $z_t$ (which is in $W$) in $z$. In the remainder of the proof, we will use observations such as these to formulate equivalent inequalities that admit a slightly simpler description. Using the same reasoning, we also have

$$h_t(z_{t+1}) = 1 + h_0(z_{t+1}) + \begin{cases} t - 1 & \text{if } z_{t+1} \in V' \\ \{|j| [1, t - 1] : z_j \in V'\} & \text{if } z_{t+1} \in W. \end{cases}$$

We can now reformulate cases (Ai)–(Aiii) as follows.

**Case (Ai):** If $z_t$ is a clique vertex then $h_0(z_t) \geq m + n - 1 - t$. The quantity $m + n - 1 - t$ is simply the number of clique and independent vertices to the right of $z_t$ in $z$. This inequality may be replaced with

$$h_0(z_t) \geq \{|j| [t + 1, m + n - 1] : z_j \text{ is clique or independent}\} \quad \text{for all clique vertices } z_t.$$

Likewise, if $z_t$ is an independent vertex then $h_0(z_t) \geq m - 1 - \{|j| [1, t - 1] : z_j \in V'\}$. Since there are $m - 1$ clique vertices in $V'$ and $z_t$ is not a clique vertex, the terms $\{|j| [1, t - 1] : z_j \in V'\}$ and $\{|j| [t + 1, m + n - 1] : z_j \in V\}$ must sum to $m - 1$. The inequality for this may be replaced with

$$h_0(z_t) \geq \{|j| [t + 1, m + n - 1] : z_j \text{ is clique}\} \quad \text{for all independent vertices } z_t.$$

The reformation of (i) is now:

(Bi) $h_0(z_t) \geq \{|j| [t + 1, m + n - 1] : z_j \text{ is clique or independent}\} \quad \text{for all clique vertices } z_t.$

(Bii) $h_0(z_t) \geq \{|j| [t + 1, m + n - 1] : z_j \text{ is clique}\} \quad \text{for all independent vertices } z_t.$

**Case (Aii):** This case partitions into four sub-cases depending on the types of the vertices $z_t$ and $z_{t+1}$:

(Bii) If $z_t$ and $z_{t+1}$ are both clique vertices then $h_0(z_t) \geq h_0(z_{t+1})$.

(Biv) If $z_t$ and $z_{t+1}$ are both independent vertices then $h_0(z_t) \geq h_0(z_{t+1})$. 


More descriptively, each of these points can be rephrased as follows:

(Bv) If \(z_t\) is a clique vertex and \(z_{t+1}\) is an independent vertex, then we have \(h_t(z_t) = 1 + h_0(z_t) + t - 1 = t + h_0(z_t)\) and \(h_t(z_{t+1}) = 1 + h_0(z_{t+1}) + |\{j \in [1, t-1] : z_j \in V'\}|. The inequality in (Aii) holds iff \(h_t(z_t) - (m + n - 1) \geq h_t(z_{t+1}) - m\). Using the expressions for \(h_t\) we find this inequality happens iff \(h_0(z_t) \geq h_0(z_{t+1}) + |\{j \in [1, t-1] : z_j \in V'\}| + n - t\). As \(\{j \in [1, t-1] : z_j \in V'\} = t - 1 - |\{j \in [1, t-1] : z_j \in V'\}|\), the expression \(|\{j \in [1, t-1] : z_j \in V'\}| + n - t\) becomes \(n - t + t - 1 - |\{j \in [1, t-1] : z_j \in W\}| = n - 1 - |\{j \in [1, t-1] : z_j \in W\}|\). There are precisely \(n\) elements in \(z\) that are in \(W\), thus \(n - |\{j \in [1, t-1] : z_j \in W\}| = |\{j \in [t, m + n - 1] : z_j \in W\}|\), and this latter expression may be replaced with \(|\{j \in [t + 1, m + n - 1] : z_j \in W\}|\) since, by assumption, \(z_t\) is in \(V'\). The inequality above is thus equivalent to

\[h_0(z_t) \geq h_0(z_{t+1}) + |\{j \in [t + 1, m + n - 1] : z_j \in W\}| - 1.\]

(Bvi') If \(z_t\) is an independent vertex and \(z_{t+1}\) is a clique vertex, then by precisely the same considerations employed in (Bv) above, we have

\[h_0(z_t) + |\{j \in [t + 1, m + n - 1] : z_j \in W\}| \geq h_0(z_{t+1}).\]

Case (Aiii): This case translates into: If \(z_t\) is independent and \(z_{t+1}\) is a clique vertex, then \(1 + h_0(z_t) + |\{j \in [1, t-1] : z_j \in V'\}| - m \geq 1 + h_0(z_{t+1}) + t - 1 - (m + n - 1). This is equivalent to

\[h_0(z_t) > h_0(z_{t+1}) + 1 + (t - 1 - |\{j \in [1, t-1] : z_j \in V'\}|) - n\]
\[\iff h_0(z_t) > h_0(z_{t+1}) + 1 + |\{j \in [1, t-1] : z_j \in W\}| - n\]
\[\iff h_0(z_t) > h_0(z_{t+1}) + |\{j \in [1, t] : z_j \in W\}| - n\]
\[\iff h_0(z_t) > h_0(z_{t+1}) - |\{j \in [t+1, m + n - 1] : z_j \in W\}|.\]

Thus we have

(Bvi) If \(z_t\) is independent and \(z_{t+1}\) is a clique vertex, then

\[h_0(z_t) > h_0(z_{t+1}) - |\{j \in [t + 1, m + n - 1] : z_j \in W\}|.\]

Notice that condition (Bvi) implies condition (Bvi'), so we may replace (Bvi') with (Bvi). Another observation of note is that since the configuration \(c\) we are considering is weakly decreasing to begin with, we may assume that in \(z\), the vertices \(\{v_1, \ldots, v_{m-1}\}\) appear from left to right in that order, and the vertices \(\{w_1, \ldots, w_n\}\) appear from left to right in that same order. (In other words \(z\) is an interleaving of the sequences \(\{v_1, \ldots, v_{m-1}\}\) and \(\{w_1, \ldots, w_n\}\).)

Thus we have a weakly decreasing configuration \(c\) on \(F\) is in \(\text{DecRec}(\text{ChipFiring}(S_{n,m}, v_m))\) iff there exists an interleaving \(z = (z_1, \ldots, z_{m+n-1})\) of the sequences of clique vertices \(\{v_1, \ldots, v_{m-1}\}\) and independent vertices \(\{w_1, \ldots, w_n\}\) such that:

(Ci) \(h_0(z_t) \geq |\{j \in [t + 1, m + n - 1] : z_j \text{ is clique or independent}\}|\) for all clique vertices \(z_t\).

(Cii) \(h_0(z_t) \geq |\{j \in [t + 1, m + n - 1] : z_j \text{ is clique}\}|\) for all independent vertices \(z_t\).

(Ciii) If \(z_t\) and \(z_{t+1}\) are both clique vertices then \(h_0(z_t) \geq h_0(z_{t+1})\).

(Civ) If \(z_t\) and \(z_{t+1}\) are both independent vertices then \(h_0(z_t) \geq h_0(z_{t+1})\).

(Cv) If \(z_t\) is a clique vertex and \(z_{t+1}\) is an independent vertex, then

\[h_0(z_t) \geq h_0(z_{t+1}) + |\{j \in [t + 1, m + n - 1] : z_j \text{ is independent}\}| - 1.\]

(Cvi) If \(z_t\) is an independent vertex and \(z_{t+1}\) is a clique vertex, then

\[h_0(z_t) + |\{j \in [t + 1, m + n - 1] : z_j \text{ is independent}\}| > h_0(z_{t+1}).\]

More descriptively, each of these points can be rephrased as follows:

(Ci) The height of a clique vertex in \(z\) is at least the number of vertices to its right in \(z\).

(Cii) The height of an independent vertex in \(z\) is at least the number of clique vertices to its right.

(Ciii) The heights of clique vertices in \(z\) from left to right is weakly decreasing.
(Civ) The heights of independent vertices in \(z\) from left to right is weakly decreasing.

(Cv) If a clique vertex is immediately followed by an independent vertex in \(z\), then the height of this clique vertex is at least the height of the independent vertex plus the number of independent vertices to the right of the clique vertex, less one.

(Cvi) If an independent vertex is immediately followed by a clique vertex in \(z\), then the height of the clique vertex is strictly less than the height of the independent vertex plus the number of independent vertices to the right of the clique vertex.

These conditions give bounds on the heights of vertices in the sequence \(z\) based on the number of vertices of a particular type relative to a vertex. We will now show that we can use an integer sequence to encode the 'deficit' between these lower bounds and the heights of the vertices. It will transpire that these six inequalities (Ci)–(Cvi) work harmoniously to give a particularly nice and simple characterisation. To do this we require two steps, the first step will show how we can encode this deficit in the sequence for just the independent vertices. Following this, we will show how we can, in a sense, partition the deficits of the first step amongst clique vertices.

First let us consider only the independent vertices of \(z\). These vertices are \(w_i = z_{k_i}\) for \(i \in [1, n]\) and some strictly increasing sequence \(1 \leq k_1 < \ldots < k_n \leq m + n - 1\).

\[
z = z_{k_1} \cdots z_{k_2} \cdots z_{k_n} \cdots .
\]

Let us insert non-negative integer values that we will call primary values (for lack of a better phrase) between all adjacent pairs of these independent vertices, and as a suffix to them. Let us write \(s_j\) for the primary value between \(z_{k_j}\) and \(z_{k_{j+1}}\) and \(s_n\) is the primary value after \(z_{k_n}\). We will also allow for a primary value \(s_0\) before \(z_{k_1}\) that, while not used just now, will become useful a little later. Define these primary values be \(s_n := h_0(z_{k_n})\) and \(s_j := h_0(z_{k_j}) - h_0(z_{k_{j+1}})\) for all \(j \in [1, n - 1]\).

\[
z = z_{k_1} s_1 \cdots z_{k_2} s_2 \cdots z_{k_3} \cdots z_{k_{n-1}} s_{n-1} \cdots z_{k_n} s_n \cdots .
\]

The height of an independent vertex in \(z\) is the sum of the primary values to its right. In terms of this construction, condition (Cii) simply translates into the values \(s_i\) being non-negative integers. Furthermore, since all vertices in \(z\) are stable, vertex \(z_{k_1}\) has height \(s_1 + \ldots + s_n\) and this can be at most \(m - 1\). Condition (Cii) is equivalent to the sum of the primary values to the right of an independent vertex in \(z\) being greater than or equal to the number of clique vertices to its right in \(z\), i.e.

\[
s_i + s_{i+1} + \ldots + s_n \geq |\{j \in [1 + k_i, m + n - 1] \mid z_j \in V'\}| \text{ for all } i \in [1, n]. \tag{6}
\]

Now let us introduce secondary values. These will be a sequence of non-negative integers \((g_1, \ldots, g_{m+n-1})\).

- Suppose there exist some clique vertices after the final independent vertex of \(z\), i.e. \(k_n < m + n - 1\), with \(z_{k_n}\) an independent vertex and \(z_{k_n+1}, \ldots, z_{m+n-1}\) are all clique vertices. Condition (Cvi) equates to \(h_0(z_{k_n}) + |\{j \in [k_n + 1, m + n - 1] \mid z_j \text{ is independent}\}| > h_0(z_{k_n+1}).\) Now \(h_0(z_{k_n}) = s_n\) and \(|\{j \in [k_n + 1, m + n - 1] \mid z_j \text{ is independent}\}| = 0\) since \(z_{k_n}\) is the rightmost independent vertex in \(z\). Thus \(h_0(z_{k_n+1}) < s_n\) and so \(h_0(z_{k_n+1}) \in [0, s_n - 1]\). Condition (Cii) corresponds to

\[
s_n - 1 \geq h_0(z_{k_{n+1}}) \geq h_0(z_{k_{n+2}}) \geq \ldots \geq h_0(z_{m+n-1}) \geq 0.
\]

We can represent this by determining the weak integer composition \((g_{k_n}, \ldots, g_{m+n-1}) \models s_n\) such that

\[
h_0(z_{k_{n+j}}) = g_{k_n+j} + g_{k_n+j+1} + \ldots + g_{m+n-1} - 1 \text{ for all } 1 \leq j \leq m + n - 1 - k_n.
\]

and

\[
g_{k_n} := s_n - (g_{k_n+1} + \ldots + g_{m+n-1}).
\]

This can be realised systematically by beginning with \(j = m + n - 1 - k_n\), and then work backwards to \(j = 1\). After these have been formed, set

\[
g_{k_n} := s_n - (g_{k_n+1} + \ldots + g_{m+n-1}).
\]
We think of value $g_j$ as being a value between the two entries $z_j$ and $z_{j+1}$ in $z$.

$$z = \cdots z_{k_n}^{s_n} \cdots z_{k_i}^{s_i} \cdots z_{k_1}^{s_1} \cdots$$

Next let us suppose that there are clique vertices between $z_{k_i}$ and $z_{k_{i+1}}$, the $i$th and $i+1$th independent vertices of $z$ for some $i \in [1, n-1]$. Then $k_{i+1} > k_i + 1$ and $z_{k_i}, \ldots, z_{k_{i+1} - 1}$ are clique vertices (there are $k_{i+1} - 1 - (k_i + 1) + 1 = k_{i+1} - k_i - 1$ of these). Furthermore, $z_{k_i}$ is an independent vertex and $z_{k_{i+1}}$ is a clique vertex, so (Cv) tells us that

$$h_0(z_{k_i}) + |\{j \in [k_i + 1, m + n - 1] : z_j \text{ is independent}\}| > h_0(z_{k_{i+1}}).$$

In this expression, we have $h_0(z_{k_i}) = s_i + \cdots + s_n$ and $|\{j \in [k_i + 1, m + n - 1] : z_j \text{ is independent}\}| = n - i$, thus

$$s_i + \cdots + s_n + n - i > h_0(z_{k_{i+1}}).$$

Since $z_{k_{i+1} - 1}$ is a clique vertex and $z_{k_{i+1}}$ is an independent vertex, we have from (Cv):

$$h_0(z_{k_{i+1} - 1}) \geq h_0(z_{k_{i+1}}) + |\{j \in [k_{i+1}, m + n - 1] : z_j \text{ is independent}\}| - 1.$$

In this expression, we have $h_0(z_{k_{i+1}}) = s_{i+1} + \cdots + s_n$ and $|\{j \in [k_{i+1}, m + n - 1] : z_j \text{ is independent}\}| = n - i$, giving

$$h_0(z_{k_{i+1} - 1}) \geq s_{i+1} + \cdots + s_n + n - i - 1.$$

Combining both of these and using (Ciii) we have

$$s_i + \cdots + s_n + n - i > h_0(z_{k_{i+1}}) \geq \cdots \geq h_0(z_{k_{i+1} - 1}) \geq s_{i+1} + \cdots + s_n + n - i - 1.$$

We may represent this in a unique way by forming a secondary set of values $(g_k, \ldots, g_{k_{i+1} - 1})$ of non-negative integers that are a weak integer composition of $s_i$ such that:

$$h_0(z_{k_{j+1}}) = g_{k_i + j} + g_{k_i + j + 1} + \cdots + g_{k_{i+1}} + \cdots + g_{m + n - 1} + 1 - n - i$$

for all $j \in [1, k_{i+1} - k_i - 1]$. Again, we can do this systematically by starting with the largest such $j$ and working backwards. The outstanding value is $g_{k_i} := s_i - (g_{k_i + 1} + \cdots + g_{k_{i+1} - 1})$

and we visualise this transition from primary to secondary values as follows:

$$z = \cdots z_{k_i}^{g_i} \cdots z_{k_1}^{g_1} \cdots$$

Let us suppose that there is a clique vertex before the independent vertex $z_{k_i}$ in $z$. Then $k_1 > 1$ and $z_1, \ldots, z_{k_i} - 1$ are clique vertices. As $z_{k_i}$ is a clique vertex and $z_{k_1}$ is an independent vertex, condition (Cv) is equivalent to:

$$h_0(z_{k_{i+1} - 1}) \geq h_0(z_{k_i}) + |\{j \in [k_i, m + n - 1] : z_j \text{ is independent}\}| - 1.$$

In this expression, we have $h_0(z_{k_i}) = s_1 + \cdots + s_n$ and $|\{j \in [k_i, m + n - 1] : z_j \text{ is independent}\}| = n$, giving

$$h_0(z_{k_{i+1} - 1}) \geq s_1 + \cdots + s_n + n - 1.$$

Combining this with (Ciii) we have

$$m + n - 1 > h_0(z_1) \geq \cdots \geq h_0(z_{k_{i+1} - 1}) \geq s_1 + \cdots + s_n + n - 1.$$

We may represent this is a unique way by choosing the secondary set of values $(g_1, \ldots, g_{k_{i+1} - 1})$ that form a weak integer composition of $s_0$ where

$$s_0 := m - 1 - (s_1 + \cdots + s_n).$$

These values are such that:

$$h_0(z_j) = g_j + g_{j+1} + \cdots + g_{k_i - 1} + \cdots + g_{m + n - 1} + 1 + n.$$
Thus we can write an expression for the height of a general vertex in such a sequence $z$ in terms of the secondary entries and the number of independent entries:

$$ h_0(z_t) = \sum_{j=t}^{m+n-1} g_j + \begin{cases} 0 & \text{if } z_t \text{ independent} \\ |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1 & \text{if } z_t \text{ clique.} \end{cases} \quad (7) $$

The only condition yet to be translated is (Ci). This has the following equivalent formulation in terms of these secondary values. By Equation 7 above, for a clique vertex $z_t$ in $z$:

$$ h_0(z_t) = \sum_{j=t}^{m+n-1} g_j + |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1. $$

The expression in (Ci) becomes

$$ \sum_{j=t}^{m+n-1} g_j + |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1 \geq m + n - 1 - t, $$

which is equivalent to:

$$ \sum_{j=t}^{m+n-1} g_j - 1 \geq |\{j \in [t+1, m+n-1] : z_j \text{ clique}\}|, $$

which, again, is equivalent to:

$$ \sum_{j=t}^{m+n-1} g_j \geq |\{j \in [t, m+n-1] : z_j \text{ clique}\}|. \quad (8) $$

In other words: the sum of the secondary values to the right of a clique vertex is $\geq$ the number of clique vertices weakly to the right of that clique vertex (and this includes counting the vertex $z_t$ itself).

To recap in light of the reformulation of conditions (Ci)–(Cvi) in terms of the secondary values: a weakly decreasing stable configuration $c = (a_1, \ldots, a_{m-1}; b_1, \ldots, b_n)$ on $F$ is in DecRec(ChipFiring$(S_{m,n}, v_m)$) iff there exists an interleaving $z = (z_1, \ldots, z_{m+n-1})$ of the sequences of clique vertices $(v_1, \ldots, v_{m-1})$ and independent vertices $(w_1, \ldots, w_n)$ and a weak integer composition $(g_1, \ldots, g_{m+n-1})$ of $m-1$ such that:

(C1) The sum of the secondary values to the right of the $j^{th}$ independent vertex in $z$ is $b_j$.

(C2) The sum of the secondary values to the right of an independent vertex is $\geq$ the number of clique vertices to its right (Equation 8).

(C3) The sum of the secondary values to its right of the $j^{th}$ clique vertex $+ \text{ the number of independent vertices to the right of the } j^{th} \text{ clique vertex, less one, in } z \text{ is } a_j$.

(C4) The sum of the secondary values to the right of a clique vertex is $\geq$ the number of clique vertices weakly to the right (equivalently, not left) of that clique vertex (Equation 8).

Statements (C2) and (C4) may be replaced with a single statement for all $t \in [1, m+n-1]$,

$$ g_t + \ldots + g_{m+n-1} \geq |\{j \in [t, m+n-1] : z_j \text{ is clique}\}|. $$

Thus we have the list of conditions:

(C1') The sum of the secondary values to the right of the $j^{th}$ independent vertex in $z$ is $b_j$.  

Statements (C2) and (C4) may be replaced with a single statement for all $t \in [1, m+n-1]$,

$$ g_t + \ldots + g_{m+n-1} \geq |\{j \in [t, m+n-1] : z_j \text{ is clique}\}|. $$

Thus we have the list of conditions:

(C1') The sum of the secondary values to the right of the $j^{th}$ independent vertex in $z$ is $b_j$.  

Statements (C2) and (C4) may be replaced with a single statement for all $t \in [1, m+n-1]$,
Proposition 3.5. A stable configuration with label in the graph, we have the following characterisation of general recurrent states on $m$.

Example 3.4. The configuration $c = (a_1, ..., a_9; b_1, ..., b_{12}) = (20, 17, 13, 12, 12, 6, 5, 2, 1; 9, 9, 9, 8, 8, 7, 7, 6, 4, 4, 2, 2)$ is in $\text{DecRec}(\text{ChipFiring}(S_{10,12}, v_{10}))$. This is evidenced by the sequence $x = (0, 2, 2, 2, 0, 1, 2, 2, 1, 0, 2, 0, 0, 2, 1, 2, 1, 2, 0, 2, 0, 1, 1, 2, 0, 2, 0, 1, 1)$. Theorem 3.2 characterises all these weakly decreasing states. In light of this construction of words on the alphabet $\{0, 1, 2\}$, and by noting that the $i$th zero or two in a sequence $x$ does not depend on the precise vertex label in the graph, we have the following characterisation of general recurrent states on $\text{ChipFiring}(S_{m,n}, v_m)$.

Proposition 3.5. A stable configuration $c$ on $\text{ChipFiring}(S_{m,n}, v_m)$ is recurrent iff there exists a sequence $x$ of $m - 1$ zeros, $m - 1$ ones, and $n$ twos with the following property:

(i) For all $i \in [1, 2m + n - 2]$:

\[ |\{j \in [1, i] : x_j = 0\}| \geq |\{j \in [1, i] : x_j = 1\}|. \]

(ii) The number of ones to the right of the $j$th two in $x$ is the height of the $j$th highest independent vertex.

(iii) The number of ones and twos to the right of the $j$th zero in $x$ is 1 plus the height of the $j$th highest clique vertex.

Note that the vertex having greatest height is the ‘1st highest vertex’, etc.
4. Recurrent states for the case of an independent-sink

We will now consider the other case whereby an independent vertex is the sink. This change certainly affects the dynamics of the system since only clique vertices are connected to the sink, so all chips have to go through a clique vertex before exiting the system. The analysis of this case is certainly similar in spirit to that of Section 3, but the details do differ.

**Example 4.1.** The first few sets are:

\[
\text{DecRec}(\text{ChipFiring}(S_{2,2}, w_2)) = \{(2, 2; 1), (2, 2; 0), (2, 1; 0), (2, 0; 1), (2, 1; 1)\}
\]

\[
\text{DecRec}(\text{ChipFiring}(S_{3,3}, w_3))
\]

\[
= \{(3, 3; 1, 1), (3, 3; 0, 0), (3, 2; 0, 0), (3, 2; 1, 0), (3, 1; 1, 0), (3, 1; 1, 1), (3, 0; 1, 1), (3, 3; 1, 0), (3, 2; 1, 1)\}
\]

\[
\text{DecRec}(\text{ChipFiring}(S_{3,2}, w_2))
\]

\[
= \left\{(3, 3; 2), (3, 3; 3; 2), (3, 3; 2; 1), (3, 2; 1, 1), (3, 2; 1, 0), (3, 2; 0; 1), (3, 2; 0; 2), (3, 3; 0; 1), (3, 3; 1; 1), (3, 2; 0; 2), (3, 2; 0; 1), (3, 2; 1; 0), (3, 2; 2), (3, 2; 2; 1), (3, 2; 2; 2)\right\}.
\]

**Theorem 4.2.** A weakly decreasing stable configuration \(c = (a_1, \ldots, a_m; b_1, \ldots, b_{m-1})\) on \(\text{ChipFiring}(S_{m,n}, w_n)\) is recurrent iff there exists a sequence \(x\) of \(m\) zeros, \(m\) ones, and \(n - 1\) twos with the following property:

(i) For all \(i \in [1, 2m + n - 1]\):

\[\{|j \in [1, i] : x_j = 0\}| \geq \{|j \in [1, i] : x_j = 1\}|.\]

(ii) The number of ones to the right of the \(j^\text{th}\) two in \(x\) is \(b_j\).

(iii) The number of ones and twos to the right of the \(j^\text{th}\) zero in \(x\) is \(a_j + 1\).

(iv) The first two in \(x\) appears after the first one.

**Proof.** The proof of this closely mirrors the proof of Theorem 3.2. The main difference arises from the fact that the heights of the vertices \(h_1(z_t)\) are different to the previous case on toppling the sink. In the case of a clique sink, when it is toppled all other vertices will have their height increased by one. However, in this case it is only the clique vertices that increase in height by one when toppling the sink. We use Proposition 2.0 in conjunction with observations regarding height change as in the previous proof. Notice that for this case, at time \(t\):

\[
h_t(z_i) = \begin{cases} 
1 + h_0(z_i) + (t - 1) & \text{if } t \leq i \text{ and } z_i \in V \\
1 + h_0(z_i) + (t - 1) - \deg(z_i) & \text{if } t > i \text{ and } z_i \in V \\
h_0(z_i) + \{|j \in [1, t - 1] : z_j \in V\} & \text{if } t \leq i \text{ and } z_i \in W \\
h_0(z_i) + \{|j \in [1, t - 1] : z_j \in V\} - \deg(z_i) & \text{if } t > i \text{ and } z_i \in W.
\end{cases}
\]

Using these, the values \(h_t(z_t)\) and \(h_t(z_{t+1})\) that feature in Proposition 2.0 become

\[
h_t(z_t) = \begin{cases} 
h_0(z_t) + t & \text{if } z_t \in V \\
h_0(z_t) + \{|j \in [1, t - 1] : z_j \in V\} & \text{if } z_t \in W',
\end{cases}
\]

and

\[
h_t(z_{t+1}) = \begin{cases} 
h_0(z_{t+1}) + t & \text{if } z_{t+1} \in V \\
h_0(z_{t+1}) + \{|j \in [1, t - 1] : z_j \in V\} & \text{if } z_{t+1} \in W'.
\end{cases}
\]

We can now reformulate the conditions in Prop. 2.0 as follows:

**Case (i):** If \(z_t\) is a clique vertex then \(h_0(z_t) \geq m + n - 1 - t\). If \(z_t\) is an independent vertex then \(h_0(z_t) \geq m - \{|j \in [1, t - 1] : z_j \in V\} = \{|j \in [t + 1, m + n - 1] : z_j \in V\}|\), just as in Thm. 3.2. Hence we have the same reformulation as before:

(Di) \(h_0(z_t) \geq \{|j \in [t + 1, m + n - 1] : z_j \text{ is clique or independent}\}\) for all clique vertices \(z_t\).

(Dii) \(h_0(z_t) \geq \{|j \in [t + 1, m + n - 1] : z_j \text{ is clique}\}\) for all independent vertices \(z_t\).
More descriptively, each of these points can be rephrased as follows:

Case (ii): There are four sub-cases depending on the types of the vertices $z_i$ and $z_{i+1}$:

- (Diii) If $z_i$ and $z_{i+1}$ are both clique vertices then $h_0(z_i) \geq h_0(z_{i+1})$, as before.
- (Div) If $z_i$ and $z_{i+1}$ are both independent vertices then $h_0(z_i) \geq h_0(z_{i+1})$, as before.
- (Dv) If $z_i$ is a clique vertex and $z_{i+1}$ is an independent vertex, then we have $h_0(z_i) = t + h_0(z_{i+1})$ and $h_0(z_{i+1}) = h_0(z_{i+1}) + |\{j \in [1, t-1]: z_j \in V\}|$. Using the same reasoning as before, we find this to be equivalent to:
  \[ h_0(z_i) \geq h_0(z_{i+1}) + |\{j \in [t+1, m+n-1]: z_j \text{ is independent}\}| - 1. \]
- (Dvi') If $z_i$ is an independent vertex and $z_{i+1}$ is a clique vertex, then by precisely the same considerations employed in (Dv) above, we have
  \[ h_0(z_i) + |\{j \in [t+1, m+n-1]: z_j \text{ is independent}\}| \geq h_0(z_{i+1}). \]

Case (iii): If $z_i$ is an independent vertex and $z_{i+1}$ is a clique vertex, then $1 + h_0(z_i) + |\{j \in [1, t-1]: z_j \in V\}| - m > h_0(z_{i+1}) + t - (m+n-1)$. This is equivalent to

\[ h_0(z_i) > h_0(z_{i+1}) + 1 + (t-1 - |\{j \in [1, t-1]: z_j \in V\}|) - n \]
\[ \iff h_0(z_i) > h_0(z_{i+1}) + 1 + |\{j \in [1, t-1]: z_j \in W\}| - n \]
\[ \iff h_0(z_i) > h_0(z_{i+1}) + |\{j \in [1, t]: z_j \in W\}| - n \]
\[ \iff h_0(z_i) + |\{j \in [t+1, m+n-1]: z_j \in W\}| > h_0(z_{i+1}). \]

Thus we have

- (Dvi) If $z_i$ is independent and $z_{i+1}$ is a clique vertex, then
  \[ h_0(z_i) + |\{j \in [t+1, m+n-1]: z_j \in W\}| > h_0(z_{i+1}). \]

As before, (Dvi) implies (Dvi'), so we may replace (Dvi’) with (Dvi). Another observation of note is that since the configuration $c$ we are considering is weakly decreasing to begin with, we may assume that in $z$, the vertices $\{v_1, \ldots, v_{m-1}\}$ appear from left to right in that order, and the vertices $\{w_1, \ldots, w_n\}$ appear from left to right in that same order. (In other words $z$ is an interleaving of the sequences $\{v_1, \ldots, v_{m-1}\}$ and $\{w_1, \ldots, w_n\}$.) Since an independent vertex is the sink, we have that $z_1$ must be a clique vertex.

Thus we have a weakly decreasing configuration $c$ on $F$ is in $\text{DecRec}(\text{ChipFiring}(S_{m,n}^m, w_n))$ iff there exists an interleaving $z = (z_1, \ldots, z_{m+n-1})$ of the sequences of clique vertices $(v_1, \ldots, v_{m-1})$ and independent vertices $(w_1, \ldots, w_n)$ such that:

- (Ei) \[ h_0(z_i) \geq |\{j \in [t+1, m+n-1]: z_j \text{ is clique or independent}\}| \text{ for all clique vertices } z_i. \]
- (Eii) \[ h_0(z_i) \geq |\{j \in [t+1, m+n-1]: z_j \text{ is clique}\}| \text{ for all independent vertices } z_i. \]
- (Eiii) If $z_i$ and $z_{i+1}$ are both clique vertices then $h_0(z_i) \geq h_0(z_{i+1})$.
- (Eiv) If $z_i$ and $z_{i+1}$ are both independent vertices then $h_0(z_i) \geq h_0(z_{i+1})$.
- (Ev) If $z_i$ is a clique vertex and $z_{i+1}$ is an independent vertex, then
  \[ h_0(z_i) \geq h_0(z_{i+1}) + |\{j \in [t+1, m+n-1]: z_j \text{ is independent}\}| - 1. \]
- (Evi) If $z_i$ is an independent vertex and $z_{i+1}$ is a clique vertex, then
  \[ h_0(z_i) + |\{j \in [t+1, m+n-1]: z_j \text{ is independent}\}| > h_0(z_{i+1}). \]
- (Evii) $z_1 = v_1$.

More descriptively, each of these points can be rephrased as follows:

- (Ei) The height of a clique vertex in $z$ is at least the number of vertices to its right in $z$.
- (Eii) The height of an independent vertex in $z$ is at least the number of clique vertices to its right.
- (Eiii) The heights of clique vertices in $z$ from left to right is weakly decreasing.
- (Eiv) The heights of independent vertices in $z$ from left to right is weakly decreasing.
(Ev) If a clique vertex is immediately followed by an independent vertex in \( z \), then the height of this clique vertex is at least the height of the independent vertex plus the number of independent vertices to the right of the clique vertex, less one.

(Evi) If an independent vertex is immediately followed by a clique vertex in \( z \), then the height of the clique vertex is strictly less than the height of the independent vertex plus the number of independent vertices to the right of the clique vertex.

(Evii) The first vertex to topple is \( v_1 \), i.e. \( h_0(v_1) = m + n - 2 \).

First let us consider only the independent vertices of \( z \). These vertices are \( w_i = z_{k_i} \) for \( i \in [1, n - 1] \) and some strictly increasing sequence \( 2 \leq k_1 < \ldots < k_{n-1} \leq m + n - 1 \).

As before, let us insert non-negative integer values that we will call \textit{primary values} between all adjacent pairs of these independent vertices, and as a suffix to them. Let us write \( s_j \) for the primary value between \( z_{k_j} \) and \( z_{k_{j+1}} \) and \( s_{n-1} \) is the primary value after \( z_{k_{n-1}} \). We will also allow for a primary value \( s_0 \) before \( z_{k_1} \) to be used later. Define these primary values be \( s_{n-1} := h_0(z_{k_{n-1}}) \) and \( s_j := h_0(z_{k_j}) - h_0(z_{k_{j+1}}) \) for all \( j \in [1, n - 2] \).

The height of an independent vertex in \( z \) is the sum of the primary values to its right. In terms of this construction, condition (Eiv) simply translates into the values \( s_i \) being non-negative integers. Furthermore, since all vertices in \( z \) are stable, vertex \( z_{k_i} \) has height \( s_1 + \ldots + s_n \) and this can be at most \( m - 1 \). This means \((s_0, s_1, \ldots, s_n)\) must be a weak integer composition of \( m \) with the property that \( s_0 > 0 \), this latter fact being a consequence of (Evii). Condition (Eii) tells us the sum of the primary values to the right of an independent vertex in \( z \) being greater than or equal to the number of clique vertices to its right in \( z \), i.e.

\[
s_i + s_{i+1} + \ldots + s_{n-1} \geq |\{ j \in [1 + k_i, m + n - 1] : z_j \in V \}| \quad \text{for all } i \in [1, n - 1].
\]

Now let us introduce secondary values as before. These will be a sequence of non-negative integers \((g_1, \ldots, g_{m+n-1})\) that form a weak integer composition of \( m \).

- Suppose there exist some clique vertices after the final independent vertex of \( z \), i.e. \( k_{n-1} < m + n - 1 \), with \( z_{k_{n-1}} \) an independent vertex and \( z_1, \ldots, z_{m+n-1} \) are all clique vertices. Condition (Evi) tells us that \( h_0(z_{k_{n-1}}) + |\{ j \in [k_{n-1} + 1, m + n - 1] : z_j \text{ is independent} \}| > h_0(z_{k_{n-1}}) \). Now \( h_0(z_{k_{n-1}}) = s_{n-1} \) and \( |\{ j \in [k_{n-1} + 1, m + n - 1] : z_j \text{ is independent} \}| = 0 \) since \( z_{k_{n-1}} \) is the rightmost independent vertex in \( z \). Thus \( h_0(z_{k_{n-1}+1}) < s_{n-1} \) and so \( h_0(z_{k_{n-1}+1}) \in [0, s_{n-1} - 1] \). Condition (Eiiii) tells us that

\[
s_{n-1} - 1 \geq h_0(z_{k_{n-1}+1}) \geq h_0(z_{k_{n-1}+2}) \geq \ldots \geq h_0(z_{m+n-1}) \geq 0.
\]

We may represent this is a unique way, as before, by choosing the weak integer composition

\[
(g_{k_{n-1}}, g_{k_{n-1}+1}, \ldots, g_{m+n-1}) \models s_{n-1}
\]

such that

\[
h_0(z_{k_{n-1}+j}) = g_{k_{n-1}+j} + g_{k_{n-1}+j+1} + \ldots + g_{m+n-1} - 1 \quad \text{for all } 1 \leq j \leq m + n - 1 - k_{n-1},
\]

and \( g_{k_{n-1}} := s_{n-1} - (g_{k_{n-1}+1} + \ldots + g_{m+n-1}) \).

- Next let us suppose that there are clique vertices between \( z_{k_i} \) and \( z_{k_{i+1}} \), the \( i \)th and \( i + 1 \)th independent vertices of \( z \) for some \( i \in [1, n - 2] \). Then \( k_{i+1} > k_i + 1 \) and \( z_{k_{i+1}}, \ldots, z_{k_{i+1}+1} \) are clique vertices (there are \( k_{i+1} - 1 - (k_i + 1) + 1 = k_{i+1} - k_i - 1 \) of these). Furthermore, \( z_{k_i} \) is an independent vertex and \( z_{k_{i+1}} \) is a clique vertex, so (Evi) tells us that

\[
h_0(z_{k_i}) + |\{ j \in [k_i + 1, m + n - 1] : z_j \text{ is independent} \}| > h_0(z_{k_{i+1}}).
\]

In this expression, we have \( h_0(z_{k_i}) = s_i + \ldots + s_{n-1} \) and \( |\{ j \in [k_i + 1, m + n - 1] : z_j \text{ is independent} \}| = n - 1 - i \), thus

\[
s_i + \ldots + s_{n-1} + n - 1 - i > h_0(z_{k_{i+1}}).
\]
Thus we can write an expression for the height of a general vertex in such a sequence which is equivalent to:

\[ h_0(z_{k_{i+1}-1}) \geq h_0(z_{k_{i+1}}) + |\{j \in [k_{i+1}, m+n-1] : z_j \text{ is independent}\}| - 1. \]

In this expression, we have \( h_0(z_{k_{i+1}}) = s_{i+1} + \ldots + s_{n-1} \) and \( |\{j \in [k_{i+1}, m+n-1] : z_j \text{ is independent}\}| = n - 1 - i \), giving

\[ h_0(z_{k_{i+1}-1}) \geq s_{i+1} + \ldots + s_{n-1} + n - 1 - i. \]

Combining both of these and using (Eii) we have

\( s_i + \ldots + s_{n-1} + n - 1 - i > h_0(z_{k_{i+1}}) \geq \ldots \geq h_0(z_{k_{i+1}-1}) \geq s_{i+1} + \ldots + s_{n-1} + n - 1 - i. \)

We may represent this in a unique way by forming a secondary set of values \((g_k, \ldots , g_{k_{i+1}-1})\) of non-negative integers that sum to \(s_i\) such that:

\[ h_0(z_{k_{i+1}+j}) = g_{k_{i+j}} + g_{k_{i+j}+1} + \ldots + g_{k_{i+1}} + \ldots + g_{m+n-1} + n - 1 - i \]

for all \( j \in [1, k_{i+1} - k_{i}]. \) Again, we can do this systematically by starting with the largest such \( j \) and working backwards. The outstanding value is \( g_{k_{i}} := s_i - (g_{k_{i+1}} + \ldots + g_{k_{i+1}-1}). \)

- For this case, there is a clique vertex before the independent vertex \( z_{k_j} \) in \( z, \) as per condition (Eii). Then \( k_{i+1} > 1 \) and \( z_{1}, \ldots , z_{k_{i+1}-1} \) are clique vertices. As \( z_{k_{i+1}-1} \) is a clique vertex and \( z_{k_{i+1}} \) is an independent vertex, condition (Ev) is equivalent to:

\[ h_0(z_{k_{i+1}-1}) \geq h_0(z_{k_{i+1}}) + |\{j \in [k_{i+1}, m+n-1] : z_j \text{ is independent}\}| - 1. \]

Note that since the toppling of the sink does not cause any other independent vertices to topple, we must have that the independent vertex \( w_{1} \) has height \( m - 1 \) to begin with. In this expression, we have \( h_0(z_{k_{i+1}}) = s_1 + \ldots + s_{n-1} = m - 1 \) and \( |\{j \in [k_{1}, m+n-1] : z_j \text{ is independent}\}| = n - 1, \) giving

\[ h_0(z_{k_{i+1}-1}) \geq m - 1 + n - 1. \]

Combining this with (Eiii) we have

\[ m + n - 1 > h_0(z_1) \geq \ldots \geq h_0(z_{k_{i+1}-1}) \geq m + n - 2. \]

This means \( h_0(z_1) = \ldots = h_0(z_{k_{i+1}-1}) = m + n - 2 \) and in representing this in the same unique way by forming the secondary set of values \((g_1, \ldots, g_{k_{i+1}-1})\) of non-negative integers that sum to \( s_0, \) all of these values are forced to be \( 0. \) Regardless, we have

\[ h_0(z_j) = g_j + g_{j+1} + \ldots + g_{k_{i+1}-1} + \ldots + g_{m+n-1} + n - 1 + n \]

for all \( j \in [1, k_{i+1}-1]. \)

Thus we can write an expression for the height of a general vertex in such a sequence \( z \) in terms of the secondary entries and number of independent entries:

\[ h_0(z_t) = \sum_{j=t}^{m+n-1} g_j + \begin{cases} 0 & \text{if } z_t \text{ independent} \\ |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1 & \text{if } z_t \text{ clique.} \end{cases} \tag{10} \]

Condition (Ei) has the following equivalent formulation in terms of these secondary values. By Equation \( 10 \) above, for a clique vertex \( z_t \) in \( z: \)

\[ h_0(z_t) = \sum_{j=t}^{m+n-1} g_j + |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1. \]

The expression in (Ei) becomes

\[ \sum_{j=t}^{m+n-1} g_j + |\{j \in [t+1, m+n-1] : z_j \text{ independent}\}| - 1 \geq m + n - 1 - t, \]

which is equivalent to:

\[ \sum_{j=t}^{m+n-1} g_j - 1 \geq |\{j \in [t+1, m+n-1] : z_j \text{ clique}\}|, \]

\[ 17 \]
which, again, is equivalent to:

$$\sum_{j=t}^{m+n-1} g_j \geq |\{j \in [t, m + n - 1] : z_j \text{ clique}\}|.$$  \hspace{1cm} (11)

In other words: the sum of the secondary values to the right of a clique vertex is \(\geq\) the number of clique vertices weakly to the right of that clique vertex.

To recap in light of the reformulation of conditions (Ei)–(Evii) in terms of secondary values: a weakly decreasing stable configuration \(c = (a_1, \ldots, a_m; b_1, \ldots, b_{n-1})\) on \(F\) is in DecRec(ChipFiring(\(S_{m,n}, w_n\))) iff there exists an interleaving \(z = (z_1, \ldots, z_{m+n-1})\) of the sequences of clique vertices \((v_1, \ldots, v_m)\) and independent vertices \((w_1, \ldots, w_{n-1})\) and a sequence of non-negative integers \((g_1, \ldots, g_{m+n-1})\) which sum to \(m\) such that:

- (E1) The sum of the secondary values to the right of the \(j^{th}\) independent vertex in \(z\) is \(b_j\).
- (E2) The sum of the secondary values to the right of the \(j^{th}\) clique vertex is \(\geq\) the number of clique vertices to its right (Equation (11)).
- (E3) The sum of the secondary values to the right of the \(j^{th}\) clique vertex + the number of independent vertices to the right of the \(j^{th}\) clique vertex, in \(z\) is \(a_j + 1\).
- (E4) The sum of the secondary values to the right of a clique vertex is \(\geq\) the number of clique vertices to the right of that clique vertex (Equation (11)).
- (E5) The secondary values before the first independent vertex in \(z\) are all 0.

The two statements containing inequalities above can be replaced with a single conditions: the sum of the secondary values to the right of a vertex is \(\geq\) the number of clique vertices to its right. Thus we have the list of conditions:

- (E1') The sum of the secondary values to the right of the \(j^{th}\) independent vertex in \(z\) is \(b_j\).
- (E2') The sum of the secondary values to the right of the \(j^{th}\) clique vertex + the number of independent vertices to the right of the \(j^{th}\) clique vertex, in \(z\) is \(a_j + 1\).
- (E3') The sum of the secondary values to the right of a vertex is \(\geq\) the number of clique vertices to its right.
- (E4') The secondary values before the first independent vertex in \(z\) are all 0.

This allows us to translate this into an interpretation in terms of words on the alphabet \(\{0, 1, 2\}\) as before. Transform \(z\) into a sequence \(x\) in the following way: in \(z\) from left to right, replace every clique vertex with a 0, and replace every independent vertex with a 2. Between positions \(t\) and \(t + 1\) in \(z\), insert precisely \(g_t\) 1s. The result is a sequence \(x\) consisting of \(m\) 0s, \(m\) 1s, and \(n - 1\) 2s that satisfies the following conditions:

- (i) For all \(i \in [1, 2m + n - 1]\):
  \[
  |\{j \in [1, i] : x_j = 0\}| \geq |\{j \in [1, i] : x_j = 1\}|.
  \]
- (ii) The number of ones to the right of the \(j^{th}\) two in \(x\) is \(b_j\).
- (iii) The number of ones and twos to the right of the \(j^{th}\) zero in \(x\) is \(a_j + 1\).
- (iv) There are no 2s before the first 1 in \(x\).

\(\square\)

**Example 4.3.**

(a) The stable and weakly decreasing configuration \(c = (a_1, a_2, a_3; b_1, b_2) = (4, 3, 1; 2, 1)\) is a member of \(\text{DecRec}(\text{ChipFiring}(S_{3,3}, w_3))\). The is evidenced by the sequence \(x = (01021021)\).

(b) The stable and weakly decreasing configuration \(c = (a_1, a_2, a_3; b_1, b_2) = (4, 3, 0; 3, 1)\) is not in \(\text{DecRec}(\text{ChipFiring}(S_{3,3}, w_3))\). This would be represented by the sequence \(x = (02011201)\), but this violates the final condition of Theorem 4.2 since a 2 appears before the first 1.
Example 4.4. The configuration
c = (a_1, \ldots, a_{11}; b_1, \ldots, b_9) = (19, 17, 16, 15, 14, 12, 8, 3, 3; 10, 10, 8, 7, 7, 3, 3, 3)
is in DecRec(ChipFiring(S_{11, 10}, w_{10})). This is evidenced by the sequence
\[ x = (0, 1, 2, 0, 0, 2, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 2, 1, 0, 1, 1, 2, 2, 0, 0, 2, 1, 1, 1). \]

Theorem 4.2 can be rephrased in the following way so that the set of all recurrent configurations is more explicit:

Proposition 4.5. A stable configuration \( c \) on ChipFiring(\( S_{m,n}, w_n \)) is recurrent iff there exists a sequence \( x \) of \( m \) zeros, \( m \) ones, and \( n - 1 \) twos with the following property:

1. For all \( i \in [1, 2m + n - 1] \):
   \[ |\{ j \in [1, i] : x_j = 0 \}| \geq |\{ j \in [1, i] : x_j = 1 \}|. \]
2. The number of ones to the right of the \( j \)th two in \( x \) is the height of the \( j \)th highest independent vertex.
3. The number of ones and twos to the right of the \( j \)th zero in \( x \) is 1 plus the height of the \( j \)th highest clique vertex.
4. The first two in \( x \) appears before the first one.

5. Recurrent states and combinatorial necklaces

In this section we will present an interpretation of the recurrent states classified in the previous sections in terms of combinatorial necklaces. As far as we are aware this is the first explicit use of combinatorial necklaces representing critical states of a dynamical system. First we will introduce combinatorial necklaces and then we will define the two particular classes of combinatorial necklaces pertinent to this paper. Following this will show how Theorems 3.2 and 4.2 provide a bijections between the two representations of weakly decreasing recurrent states in this paper and the two classes of combinatorial necklaces that we have defined.

A combinatorial necklace is an arrangement of coloured beads on a circle. Mathematically this is represented by assigning a number to each colour and then listing the sequence of numbers in a clockwise direction around the circle. The same necklace can be represented in numerous ways depending on the ‘beginning’ bead. The usual convention is to use the first \( k \) non-negative numbers for necklaces consisting of \( k \) distinct colours.

More formally, let \( \Sigma_k^n \) be the set of all length-\( n \) strings over the alphabet \( \Sigma_k = \{0, 1, \ldots, k - 1\} \). Define two strings \( \alpha = \alpha_1 \ldots \alpha_n \) and \( \beta = \beta_1 \ldots \beta_n \) in \( \Sigma_k^n \) to be equivalent \( \alpha \sim \beta \) if there is a cyclic rotation \( \alpha_i \ldots \alpha_n \alpha_1 \ldots \alpha_i^{-1} \) of the sequence \( \alpha \) that equals \( \beta \). A combinatorial necklace is an equivalence class \( [\alpha]_\sim \) on \( \Sigma_k^n \), i.e. \( [\alpha]_\sim \) consists of all those sequences that correspond to the same coloured necklace, and the set of all necklaces is the quotient set \( \Sigma_k^n / \sim \).

Example 5.1. There are precisely three necklaces consisting of 4 beads for which two beads have colour 0, one bead has colour 1, and one bead has colour 2. These are
\[ \{ (0, 1, 0, 2) \sim (1, 0, 2, 0) \sim (0, 2, 0, 1) \sim (2, 0, 1, 0), \ (0, 0, 2, 1) \sim (0, 2, 1, 0) \sim (2, 1, 0, 0) \sim (1, 0, 0, 2), \ (0, 0, 1, 2) \sim (0, 1, 2, 0) \sim (1, 2, 0, 0) \sim (2, 0, 0, 1) \} \]
where we have listed the lexicographically smallest sequences of each equivalence class on the left. Thus we can write
\[ \text{Necklaces}(2, 1, 1) = \{ [(0, 1, 0, 2)]_\sim, [(0, 0, 2, 1)]_\sim, [(0, 0, 1, 2)]_\sim \}, \]
and illustrate as follows (the first entry is at the highest position in each, and we read the other entries in a clockwise manner):
In the literature, a typical way to list necklaces is by listing those sequences that are lexicographically smallest in their equivalence classes. In this paper, we will find it advantageous to adopt a different method of systematically writing representatives from each of the equivalence classes. We will be particularly interested in the set \( \text{Necklaces}(a, a-1, b) \), the set of necklaces having \( a \) beads of colour 0, \( a-1 \) beads of colour 1, and \( b \) beads of colour 2. Moreover, we will be interested in specific representatives for each of the necklaces in \( \text{Necklaces}(a, a-1, b) \) which we define next. Let \( \text{Dyck}_n \) be the set of Dyck words of length \( n \) on the alphabet \( \{0, 1\} \): those words with the property that in every left prefix the number of 0s is at least as large as the number of 1s. E.g. \( \text{Dyck}_3 = \{000111, 001011, 001101, 010011, 010101\} \).

**Definition 5.2.** Given \([c] \sim \in \text{Necklaces}(a, a-1, b)\), let \( c^* \) be that member of \([c] \sim \) whose subword consisting of only 0s and 1s is lexicographically smallest. Let \( \text{Necklaces}^*(a, a-1, b) \) be the set of sequences \( c^* \) with the above property. More explicitly, \( \text{Necklaces}^*(a, a-1, b) \) is the set of sequences \((c_1, \ldots, c_{2a+b-1})\) consisting of \( a \) zeros, \( a-1 \) ones, \( b \) twos with the following properties: \( c_1 = 0 \), and the number of 0s in every left prefix is always at least one more than the number of ones in that left prefix. Note that such a sequence \( c^* \) without the 2s and the leading 0 is a Dyck word in \( \text{Dyck}_{a-1} \).

**Example 5.3.** In Example 5.1 there are three equivalence classes. For the first sequence \( c = (0, 1, 0, 2) \) is lexicographically smallest and begins with a 0, but is not the sequence identified by Definition 5.2. With the 2s removed it becomes the sequence \((0, 1, 0)\). Consider the third sequence in that equivalence class \((0, 2, 0, 1)\): with the 2s removed it becomes \((0, 0, 1)\) which is lexicographically smaller than \((0, 1, 0)\). For this equivalence class we have \( c^* = (0, 2, 0, 1) \). For the other two equivalence classes, the \( c^* \) will coincide with the lexicographically smallest, so the other two entries will be \( c^* = (0, 0, 2, 1) \) and \( c^* = (0, 0, 1, 2) \). Thus

\[
\text{Necklaces}^*(2, 1, 1) = \{(0, 2, 0, 1), (0, 0, 2, 1), (0, 0, 1, 2)\}.
\]

**Example 5.4.** The following table gives the members of \( \text{Necklaces}^*(3, 2, 2) \). We also find it appropriate to list the associated binary necklace \( D \) that is achieved by removing the 2s from \( c^* \), and the set \( T \) listing the indices of necklace entries in \( c^* \) that are 2. Note that \( |D| = 2a-1 \) and \( T \subseteq \{2, \ldots, 7 = 2a-1+b\} \) has size \( |T| = b = 2 \).

| \( c^* \in \text{Necklaces}^*(3, 2, 2) \) | \( (D(c^*), T(c^*)) \) | \( c^* \in \text{Necklaces}^*(3, 2, 2) \) | \( (D(c^*), T(c^*)) \) |
|---|---|---|---|
| 0220011 | (000111, [2, 3]) | 0220101 | (001011, [2, 3]) |
| 0202011 | (000111, [2, 4]) | 0202101 | (001011, [2, 4]) |
| 0200211 | (000111, [2, 5]) | 0201201 | (001011, [2, 5]) |
| 0200121 | (000111, [2, 6]) | 0201021 | (001011, [2, 6]) |
| 0200112 | (000111, [2, 7]) | 0201012 | (001011, [2, 7]) |
| 0022011 | (000111, [3, 4]) | 0022101 | (001011, [3, 4]) |
| 0020211 | (000111, [3, 5]) | 0021201 | (001011, [3, 5]) |
| 0020121 | (000111, [3, 6]) | 0021021 | (001011, [3, 6]) |
| 0020112 | (000111, [3, 7]) | 0021012 | (001011, [3, 7]) |
| 0002211 | (000111, [4, 5]) | 0012201 | (001011, [4, 5]) |
| 0002121 | (000111, [4, 6]) | 0012021 | (001011, [4, 6]) |
| 0002112 | (000111, [4, 7]) | 0012012 | (001011, [4, 7]) |
| 0001221 | (000111, [5, 6]) | 0010221 | (001011, [5, 6]) |
| 0001212 | (000111, [5, 7]) | 0010212 | (001011, [5, 7]) |
| 0001122 | (000111, [6, 7]) | 0010122 | (001011, [6, 7]) |
5.1. Necklaces for the clique-sink case. Let us now introduce a construction that relates the combinatorial necklaces we have defined above to stable configurations on ChipFiring\((S_{m,n}, v_m)\).

**Definition 5.5** (Necklaces from configurations). Given a configuration \(c \in \text{DecRec}(\text{ChipFiring}(S_{m,n}, v_m))\) with \(c = (a_1, \ldots, a_{m-1}, b_1, \ldots, b_n)\), let us construct a sequence \(f_{m,n}^{(CN)}(c)\) as follows: let \(s\) be the sequence of \(n\) twos:

\[
2 \; 2 \; \cdots \; 2 \; 2.
\]

For all \(1 \leq j < n\), insert precisely \(b_{j+1} - b_j\) ones between the \(j\)th and \(j + 1\)st twos of \(s\). Insert \(b_n\) ones after the final two of \(s\):

\[
\begin{array}{cccccc}
\text{b}_1 - \text{b}_2 & \text{b}_2 - \text{b}_3 & \cdots & \text{b}_{n-1} - \text{b}_n & \text{b}_n \\
2 \; 1 \; 1 \; \cdots \; 1 \; 2 \; 1 \; 1 \; \cdots \; 1.
\end{array}
\]

Next, insert \(m - 1\) zeros into \(s\) so that the number of 1s/2s to the right of the \(i\)th 0 is \(a_i + 1\):

\[
\begin{array}{cccccc}
\text{a}_1 - \text{a}_2 & \text{a}_2 - \text{a}_3 & \cdots & \text{a}_{m-2} - \text{a}_{m-1} & \text{a}_{m-1} + 1 \\
0 \; 2 \; 1 \; 2 \; \cdots \; 1 \; 2 \; 1 \; \cdots \; 1.
\end{array}
\]

Finally, prepend a 0 to the resulting sequence:

\[
f_{m,n}^{(CN)}(c) := \begin{array}{cccccc}
\text{a}_1 - \text{a}_2 & \text{a}_2 - \text{a}_3 & \cdots & \text{a}_{m-2} - \text{a}_{m-1} & \text{a}_{m-1} + 1 \\
0 \; 2 \; 1 \; 2 \; \cdots \; 1 \; 2 \; 1 \; \cdots \; 1.
\end{array}
\]

**Example 5.6.** Consider \(c\) from Example 5.4

\[
c = (a_1, \ldots, a_9; b_1, \ldots, b_{12}) = (20, 17, 13, 12, 12, 6, 5, 2, 1; 9, 9, 9, 8, 8, 7, 7, 6, 4, 4, 2, 2)
\in \text{DecRec}(\text{ChipFiring}(S_{10,12}, v_{10})).
\]

To construct \(f_{10,12}^{(CN)}(c)\) we form the sequence consisting of 12 twos:

\[
222222222222
\]

Next place the appropriate number of ones between each and at the end so that the number of 1s to the right of the \(i\)th 2 is the \(i\)th entry in the sequence \((9, 9, 9, 8, 8, 7, 7, 6, 4, 4, 2, 2)\):

\[
222122122121122112211
\]

Next insert the zeros so that the number of 1s and 2s after the \(i\)th 0 is the \(i\)th entry of the sequence \((a_1 + 1, \ldots, a_9 + 1) = (21, 18, 14, 13, 13, 7, 6, 3, 2)\):

\[
022220122102002122112202011202011
\]

Finally prepend a 0 to this sequence to get

\[
f_{10,12}^{(CN)}(c) = 022220122102002122112202011202011.
\]
The construction in Definition 5.5, while detailed, is straightforward to unravel in the opposite direction. In fact, it admits the following short construction:

**Definition 5.7 (Configurations from necklaces).** Given $c \in \text{Necklaces}^*(m, m - 1, n)$, let us define

$$f^{(NC)}_{m,n}(c) := (\mu_1, \ldots, \mu_{m-1}; \nu_1, \ldots, \nu_n)$$

where $\mu_i$ is the number of 1s and 2s to the right of the $(i + 1)$th 0 in $c$, less one, and $\nu_j$ be the number of 1s to the right of the $j$th 2 in $c$.

**Example 5.8.** Consider $c = 0022201221020021211202011202011$. For the second 0 of $c$, the number of 1s and 2s to its right is 21, so $\mu_1 = 21 - 1 = 20$. For the third 0 of $c$, the number of 1s and 2s to its right is 18, so $\mu_2 = 18 - 1 = 17$. For the tenth 0 of $c$, the number of 1s and 2s to its right is 2, so $\mu_9 = 2 - 1 = 1$.

For the first 2 of $c$, the number of 1s to its right is 9, so $\nu_1 = 9$. The second and third 2s of $c$ are together with the first 2, so the number is exactly the same and $\nu_2 = \nu_3 = 9$. The fourth 2 of $c$ has 8 1s to its right, so $\nu_4 = 8$. From this, we have

$$f^{(NC)}_{m,n}(c) := (20, 17, 13, 12, 6, 5, 2, 1; 9, 9, 8, 8, 7, 6, 4, 4, 2, 2).$$

We now state the main result of this section, which shows that combinatorial necklaces in $\text{Necklaces}^*(m, m - 1, n)$ are in one-to-one correspondence with weakly decreasing recurrent configurations of the sandpile model $\text{ChipFiring}(S_{m,n}, v_m)$.

**Theorem 5.9.** A sequence $c \in \text{Necklaces}^*(m, m - 1, n)$ iff $f^{(NC)}_{m,n}(c) \in \text{DecRec}(\text{ChipFiring}(S_{m,n}, v_m))$.

**Proof.** A sequence $c \in \text{Necklaces}^*(m, m - 1, n)$ iff it satisfies the properties stated in Definition 5.2. By convention $c$ begins with two 0s, so if we remove the first 0 of $c$ then we have a sequence $x$ consisting of $m - 1$ 0s, $m - 1$ 1s, and $n$ 2s such that the subsequence of 0s and 1s is a Dyck word in $\text{Dyck}_{m-1}$. Combining this fact with the construction for $f^{(NC)}_{m,n}(c)$ in Definition 5.7 and comparing with the statement of Theorem 3.2 yields the stated result. 

\[\square\]
Example 5.10. The set \( \text{Necklaces}^\dagger(2, 1, 2) = \{00122, 00212, 02012, 00221, 02021, 02201\} \), and from Example 3.1 we have

\[
\text{DecRec}(\text{ChipFiring}(S_{2, 2}, v_2)) = \{(0; 1, 1), (1; 1, 0), (1; 1, 1), (2; 0, 0), (2; 1, 0), (2; 1, 1)\}.
\]

We can see that these sets are in one-to-one correspondence as

\[
\begin{align*}
\text{f}^{(NC)}(00122) &= (2; 0, 0) & \text{f}^{(NC)}(00212) &= (2; 1, 0) \\
\text{f}^{(NC)}(02012) &= (1; 1, 0) & \text{f}^{(NC)}(00221) &= (2; 1, 1) \\
\text{f}^{(NC)}(02021) &= (1; 1, 1) & \text{f}^{(NC)}(02201) &= (0; 1, 1).
\end{align*}
\]

5.2. Necklaces for the independent-sink case. The results of Theorem 5.2 and 5.4 that characterise the two different sets of weakly decreasing recurrent configurations are quite similar. Aside from the technical detail that concerns states on \( m - 1 \) clique vertices and \( n \) independent vertices changing to states on \( m \) clique and \( n - 1 \) independent vertices, the key difference lies in the additional detail stated in Theorem 5.2(iv) detailing an initial-constraint. When translated through Definition 5.5 we identify the following subset of ‘our’ necklaces.

Definition 5.11. Let \( \text{Necklaces}^\dagger(m, m - 1, n) \) be the collection of necklaces \( c \in \text{Necklaces}^\dagger(m, m - 1, n) \) with the following property: the leftmost 2 appears to the right of the leftmost 1.

In terms of necklace diagrams, these are those necklaces in \( \text{Necklaces}^\dagger(m, m - 1, n) \) for which the first appearances of colours 0, 1, and 2 are in that same order when starting from the top of the necklace and going in a clockwise direction.

Example 5.12. In Example 5.4 we listed the members of \( \text{Necklaces}^\dagger(3, 2, 2) \). We also find it appropriate to list the associated binary necklace \( D \) that is achieved by removing the 2s from \( c^* \), and the set \( T \) listing the indexes of necklace entries in \( c^* \) that are 2. Note that \(|D| = 2a - 1\) and \( T \subseteq \{2, \ldots, 7 = 2a - 1 + b\} \) has size \(|T| = b = 2\).

| \( c \) | Necklace | \((D(c^*), T(c^*))\) |
|---|---|---|
| 0001221 | ![Necklace](image1) | \((00011, [5, 6])\) |
| 0001212 | ![Necklace](image2) | \((00011, [5, 7])\) |
| 0001122 | ![Necklace](image3) | \((00011, [6, 7])\) |
| 0012201 | ![Necklace](image4) | \((00101, [4, 5])\) |
| 0012021 | ![Necklace](image5) | \((00101, [4, 6])\) |

Theorem 5.13. A sequence \( c \in \text{Necklaces}^\dagger(m + 1, m, n - 1) \) iff \( \text{f}^{(NC)}(m,n)(c) \in \text{DecRec}(\text{ChipFiring}(S_{m,n}, w_n)) \).

Proof. A sequence \( c \in \text{Necklaces}^\dagger(m + 1, m, n - 1) \) iff it satisfies the properties stated in Definition 5.11. By convention \( c \) begins with two 0s, so if we remove the first 0 of \( c \) then we have a sequence \( x \) consisting of \( m \) 0s, \( m \) 1s, and \( n - 1 \) 2s such that the subsequence of 0s and 1s is a Dyck word in \( \text{Dyck}_{m-1} \) and the first one appears before the first two. Combining this fact with the construction for \( \text{f}^{(NC)}(m,n)(c) \) in Definition 5.7 and comparing with the statement of Theorem 5.2 yields the stated result. \( \square \)
Example 5.14. From Example 5.12 above we have

Necklaces$^\dagger(3, 2, 2) = \{0001221, 0001212, 0001122, 0012201, 0012021, 0012012, 0010221, 0010212, 0010122\}$. In Example 4.11 we have

DecRec(ChipFiring($S_{2,3}, w_3$))

$= \{(3, 3; 1, 1), (3, 3; 0, 0), (3, 2; 1, 0), (3, 1; 1, 0), (3, 1; 1, 1), (3, 0; 1, 1), (3, 3; 1, 0), (3, 2; 1, 1)\}$. We can see that these sets are in one-to-one correspondence as

\[
\begin{align*}
f^{(NC)}(0001221) &= (3, 3; 1, 1) & f^{(NC)}(0001212) &= (3, 3; 1, 0) \\
f^{(NC)}(0001122) &= (3, 3; 0, 0) & f^{(NC)}(0012201) &= (3, 0; 1, 1) \\
f^{(NC)}(0012021) &= (3, 1; 1, 1) & f^{(NC)}(0012012) &= (3, 1; 1, 0) \\
f^{(NC)}(0010221) &= (3, 2; 1, 1) & f^{(NC)}(0010212) &= (3, 2; 1, 0) \\
f^{(NC)}(0010122) &= (3, 2; 0, 0).
\end{align*}
\]

6. Tiered parking functions

In light of the classification of weakly decreasing recurrent states as combinatorial necklaces, we may also offer the following equivalent definition of such configurations as a new type of parking function that we will call a tiered parking function. Parking functions were mentioned earlier in the paper in relation to the recurrent states of the sandpile model on the complete graph. Moreover, the $G$-parking functions of Postnikov and Shapiro [12] provide a useful language in which an alternative description of recurrent states of the sandpile model on a general graph. The application of $G$-parking functions to the complete split graph is different to what we present in this section. Our aim is to provide a new ‘type’ of parking function, and provide a setting in which recurrence is quite easily established in this new context. We refer the interested reader to Yan [15] for a discussion of $G$-parking functions and their relation to the ASM. Our definition is inspired by Cori and Poulalhon’s [3] concept of $(p, q)$-parking functions.

Definition 6.1 (k-tiered parking function). Let $m_1, \ldots, m_k$ be a sequence of positive integers with $m_1 + \ldots + m_k = M$. Suppose that there are $m_i$ cars of colour/tier $i$ and there are $M$ parking spaces. We will call a sequence $P = (m_1; P_2, \ldots, P_k)$ of sequences $P_i = (p^{(i)}_1, \ldots, p^{(i)}_{m_i})$ a $k$-tiered parking function of order $(m_1, \ldots, m_k)$ if there exists a parking configuration of the $M$ cars that satisfies the following preferences for all drivers:

for the driver of the $j$th car having colour $i > 1$ asks that
there be at least $p^{(i)}_{j-1}$ cars of colours $\{1, \ldots, i-1\}$ parked before him.

Drivers of cars having colour 1 have no preferences with regard to other coloured cars, which is why we only list their number $m_1$ in $P$.

Example 6.2. The sequence $P = (4; (2, 1, 0, 4, 2), (8, 2, 1, 2), (4, 10, 8))$ is a 4-tiered parking function of order $(4, 5, 4, 3)$. This is realised by the following parking configuration where the leftmost entry represents the first parking spot, and the colour of the parked car is indicated.

\[
\rightarrow \text{direction of traffic} \rightarrow \\
\begin{array}{cccccccccc}
2 & 3 & 1 & 3 & 4 & 3 & 2 & 1 & 2 & 4 & 2 & 1 & 4 & 1 & 3 & 2
\end{array}
\]

Example 6.3. The sequence $P = (9; (8, 2, 9, 4, 6, 9, 7, 8, 2, 7, 9, 4), (18, 2, 14, 6, 21, 13, 7, 13, 3))$ is a 3-tiered parking function of order $(9, 12, 9)$. This is realised by the following parking configuration:

\[
\rightarrow \text{direction of traffic} \rightarrow \\
\begin{array}{cccccccccc}
1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 1 & 2 & 2 & 1 & 3 & 2 & 2 & 2 & 3
\end{array}
\]
We will gather in this section some enumerative results related to the objects discussed in this paper.

**Proposition 7.1.**

For a slightly different formulation can be given in Definition 6.1 whereby cars of colour 1 are ‘less than’ another colour be considered. 

**Remark 6.5.** A slightly different formulation can be given in Definition 6.1 whereby cars of colour 1 are ‘less than’ another colour be considered.

**Question 6.6.** How many 2-tiered parking functions are there? How many 3-tiered parking functions of order (a, b, c)?

7. Enumeration

We will gather in this section some enumerative results related to the objects discussed in this paper.

**Proposition 7.1.** \(|\text{Necklaces}^\ast(m, m-1, n)| = \frac{1}{m} \binom{2(m-1)}{m-1} \binom{2(m-1)+n}{n} = |\text{DecRec}(\text{ChipFiring}(S_{m,n}, v_m))|.

**Proof.** Definition 5.2 tells us that \(\text{Necklaces}^\ast(m, m-1, n)\) can be decomposed as a pair \((D, T)\) where \(D\) is a Dyck word in \(\text{Dyck}_{m-1}\) and \(T\) is a subset of \(\{2, \ldots, 2m-1+n\}\) of size \(n\). The number of Dyck words in \(\text{Dyck}_{m-1}\) is \(\frac{1}{m}(2^{m-1})\). The number of ways to choose \(T\), or equivalently to embed this Dyck word into a sequence of length \(2(m-1)+n\), is \(\binom{2(m-1)+n}{n}\), with the understanding the empty positions are filled with 2s. \(\square\)

**Proposition 7.2.** The second set of interest \(\text{Necklaces}^\dagger(m+1, m, n-1)\) has

\[|\text{Necklaces}^\dagger(m+1, m, n-1)| = \frac{1}{m} \binom{n+m-2}{m-1} \binom{n+2m-1}{m-1} = |\text{DecRec}(\text{ChipFiring}(S_{m,n}, w_n))|.

**Proof.** To enumerate this set, we consider a refinement of the enumeration in the proposition above. Every necklace in this set is a sequence \(x\) of length \(2m+n-1\) consisting of \(m\) 0s, \(m\) 1s, and \((n-1)\) 2s. The first two must come after the first one. Thus every such sequence may be uniquely expressed as a pair \((D, T)\) where \(D\) is a Dyck word in \(\text{Dyck}_m\), and \(T\) is an \((n-1)\)-subset of \(\{k(D)+1, \ldots, 2m+n-1\}\) where \(k(D)\) is the index of the first one of \(D\) in \(x\). The number of ways to choose the latter is \(\binom{2m+n-1-k}{n-1}\).

Let \(\text{Dyck}_m(k)\) be the set of Dyck words in \(\text{Dyck}_m\) such that the first one is in position \(k \in [2, m]\). The number of these words is

\[|\text{Dyck}_m(k)| = \binom{2m-k}{m-1} - \binom{2m-k}{m} = \binom{2m-k+1}{m} \frac{k-1}{2m-k+1} = \frac{k-1}{m} \binom{2m-k}{m-1}.

Theorem 6.4.

(a) (Clique-sink case) A configuration \(c = (a_1, \ldots, a_{m-1}; b_1, \ldots, b_n) \in \text{ChipFiring}(S_{m,n}, v_m)\) iff there exists a 3-tiered parking function of order \((m-1, n, m-1)\) with

\[P = ((b_1, \ldots, b_n), (a_1 + 1, \ldots, a_{m-1} + 1)).\]

(b) (Independent-sink case) A configuration \(c = (a_1, \ldots, a_m; b_1, \ldots, b_{n-1}) \in \text{ChipFiring}(S_{m,n}, w_n)\) iff there exists a 3-tiered parking function of order \((m, n-1, m)\) with

\[P = ((b_1, \ldots, b_{n-1}), (a_1 + 1, \ldots, a_m + 1)),\]

with the added restriction that there is no colour 2 car that is parked after all of the \(n-1\) colour 1 cars.

**Remark 6.5.** A slightly different formulation can be given in Definition 6.1 whereby cars of colour 1 are treated as empty parking spaces, and the number of empty places along with the number of cars of colour ‘less than’ another colour be considered.

**Question 6.6.** How many 2-tiered parking functions are there? How many 3-tiered parking functions of order \((a, b, c)\)?
and we have

\[
|\text{Necklaces}^1(m+1,m,n-1)| = \sum_{k=2}^{m+1} |\text{Dyck}_m(k)| \binom{2m+n-1-k}{n-1} \\
= \frac{1}{m} \sum_{k=2}^{m+1} (k-1) \binom{2m-k}{m-1} \binom{2m+n-1-k}{n-1} \\
= \frac{1}{m} \sum_{\ell=0}^{m-1} ((m-1) + 1 - \ell) \binom{m-1 + \ell}{m-1} \binom{m-1 + n - 1 + \ell}{n-1}.
\]

The sum can now be evaluated by using Lemma 7.3 (which follows this proof) in which \(m\) is replaced with \(m-1\) and \(n\) is replaced with \(n-1\). This gives:

\[
|\text{Necklaces}^1(m+1,m,n-1)| = \frac{1}{m} \binom{(n-1) + (m-1)}{m-1} \binom{(n-1) + 2(m-1) + 2}{m-1} \\
= \frac{1}{m} \binom{n + m - 2}{m-1} \binom{n + 2m - 1}{m-1}.
\]

**Lemma 7.3.** \(\sum_{\ell=0}^{m} (m - \ell + 1) \binom{m + n + \ell}{m + \ell} \binom{m + \ell}{m} = \binom{n + 2m + 2}{n + m + 2} \binom{n + m}{m}.
\)

**Proof.** Let us consider the set

\(Z_{m,n} := \{(A, B) : B \subseteq \{1, \ldots, n+2m+2\} \text{ with } |B| = n+m+2 \text{ and } A \subseteq B \setminus \{B_{\min}, B_{\min}^{\prime}\} \text{ with } |A| = m\}.
\)

Here we use the notation \(B_{\min}\) for the smallest element of the set \(B\), and \(B_{\min}^{\prime}\) for the second smallest element of the set \(B\). We can enumerate \(Z_{m,n}\) in various way, the most straightforward of which is the following. The set \(B\) can be chosen in \(\binom{n+2m+2}{n+m+2}\) ways. Once \(B\) has been chosen, we may select \(A\) from \(B\) by choosing an \(m\)-element subset that does not contain the two smallest entries of \(B\). The number of ways of doing this is \(\binom{(n+m+2)-2}{m} = \binom{n+m}{m}\). We have

\[
|Z_{m,n}| = \binom{n + 2m + 2}{n + m + 2} \binom{n + m}{m}.
\]

(12)

Let us now enumerate \(Z_{m,n}\) in a slightly different way:

- first choose the value of \(B_{\min}\), the second smallest element of the set \(B\)
- then select the smallest value \(B_{\min}^{\prime}\)
- next select those elements of \(B \setminus \{B_{\min}, B_{\min}^{\prime}\}\) that are not in \(A\)
- and finally select the elements of \(A\).

The two extreme instances of \(B\) are \(B = \{1, 2, \ldots, m+n+2\}\) and \(B = \{m+1, m+2, \ldots, n+2m+2\}\), and from these we see the value \(B_{\min}\) takes values in the set \(\{2, \ldots, m+2\}\). Let \(\ell := m+2 - B_{\min}\) so that \(\ell\) takes values in the set \(\{0, \ldots, m\}\).

If we fix \(\ell\) (which is equivalent to selecting a value for \(B_{\min}\), then \(B_{\min}\) must be in the set \(\{1, \ldots, B_{\min} - 1\} = \{1, \ldots, m + 1 - \ell\}\), and so there are \((m+1-\ell)\) choices for \(B_{\min}\).

With \(B_{\min}\) and \(B_{\min}\) now chosen, we must choose \(B \setminus \{B_{\min}, B_{\min}^{\prime}\}\) from the set \(\{B_{\min} + 1, \ldots, n+2m+2\}\) (a set of size \(n+m+\ell\)), and from that set we must choose an \(m\)-element subset \(A\). We can do this by first selecting the elements of \(B' := B \setminus (A \cup \{B_{\min}, B_{\min}^{\prime}\})\), of which there are \(n\), from the \(n+m+\ell\) elements \(B_{\min} + 1, \ldots, n+2m+2\). There are \(\binom{n+m+\ell}{n}\) ways to do this.
Next we select an \( m \)-element subset \( A \) from the set \( \{B_{\text{min}} + 1, \ldots, n + 2m + 2\} \setminus B' \) which is a set of size \( n + m + \ell - n = m + \ell \). This may be done in \( \binom{m + \ell}{m} \) ways. Combining these, we have

\[
|Z_{m,n}| = \sum_{\ell=0}^{m} (m+1-\ell) \binom{m+n+\ell}{m} \binom{m+\ell}{n}.
\]

(13)

**Proposition 7.4.** \(|\text{Rec}(\text{ChipFiring}(S_{m,n}, v_m))| = |\text{Rec}(\text{ChipFiring}(S_{m,n}, w_n))| = m^{n-1}(m+n)^{m-1}.

**Proof.** For the case of general recurrent configurations, we know that the number of recurrent configurations is in one-to-one correspondence with the number of spanning trees of the underlying graph. Let \( \text{Spanning}(S_{m,n}) \) be the set of spanning trees of \( S_{m,n} \). This number is independent of the choice of sink in the graph and so \(|\text{Rec}(\text{ChipFiring}(S_{m,n}, v_m))| = |\text{Rec}(\text{ChipFiring}(S_{m,n}, w_n))| = |\text{Spanning}(S_{m,n})|\). We may derive a formula for \( |\text{Spanning}(S_{m,n})| \) using Kirchhoff’s matrix tree theorem as follows. The Laplacian of the graph \( S_{m,n} \) is the following \( (m+n) \times (m+n) \) matrix:

\[
Q = \begin{pmatrix}
(m+n-1) & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & (m+n-1) & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & (m+n-1) & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & (m+n-1) & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & -1 & m & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & m & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots m
\end{pmatrix}
\]

The labels of the rows and columns is the order given by \( \mathcal{O} : \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \). If we remove the final row and column, then we have the \( (m+n-1) \times (m+n-1) \) matrix:

\[
Q^* = \begin{pmatrix}
(m+n-1) & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & (m+n-1) & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & (m+n-1) & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & (m+n-1) & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & -1 & m & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & m & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots m
\end{pmatrix}
\]

The number of spanning trees of \( S_{m,n} \) is the determinant of \( Q^* \). Let us perform some elementary row operations on \( Q^* \) and comment at each step how the operations change the determinant. For \( Q^* \), subtract the \( (m+1) \)th row from all other rows to get:

\[
Q' = \begin{pmatrix}
(m+n) & 0 & 0 & \cdots & 0 & -1-m & -1 & \cdots & -1 \\
0 & (m+n) & 0 & \cdots & 0 & -1-m & -1 & \cdots & -1 \\
0 & 0 & (m+n) & \cdots & 0 & -1-m & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & m+n & -1-m & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & -1 & m & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & m & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -m & m & \cdots
\end{pmatrix}
\]

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We note that $\det(Q^*) = \det(Q')$. Next perform the elementary row operation $r_{m+1} \leftarrow r_{m+1} + \frac{1}{m+n} (r_1 + \ldots + r_m)$:

$$Q'' = \begin{pmatrix}
m + n & 0 & 0 & \ldots & 0 & -1 - m & -1 & \ldots & -1 \\
0 & m + n & 0 & \ldots & 0 & -1 - m & -1 & \ldots & -1 \\
0 & 0 & m + n & \ldots & 0 & -1 - m & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \frac{m(n-1)}{m+n} & \frac{-m}{m+n} & \ldots & \frac{-m}{m+n} \\
0 & 0 & 0 & \ldots & 0 & -m & m & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & -m & 0 & \ldots & m
\end{pmatrix}.$$  

We now have that $\det(Q'') = (m + n)^n \det(Q''')$ where $Q'''$ is the $(n-1) \times (n-1)$ matrix that forms the bottom right hand corner of $Q''$:

$$Q''' = \begin{pmatrix}
\frac{m(n-1)}{m+n} & -m & -m & \ldots & -m \\
-m & m & 0 & \ldots & 0 \\
-m & 0 & m & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-m & 0 & 0 & \ldots & m
\end{pmatrix}.$$  

We may take the factor $\frac{1}{m+n}$ out of the first row to find $\det(Q''') = \frac{1}{m+n} \det(Q''''')$ where

$$Q''''' = \begin{pmatrix}
m & -m & -m & \ldots & -m \\
0 & m & 0 & \ldots & 0 \\
0 & 0 & m & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & m
\end{pmatrix},$$

Finally, on $Q'''''$ perform the elementary row operation $c_1 \leftarrow c_1 + c_2 + \ldots + c_{n-1}$ to get the upper triangular matrix:

$$Q'''''' = \begin{pmatrix}
m & -m & -m & \ldots & -m \\
0 & m & 0 & \ldots & 0 \\
0 & 0 & m & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & m
\end{pmatrix},$$

which has $\det(Q'''''') = m^{n-1}$. Combining these relations we have

$$|\text{Spanning}(S_{m,n})| = \det(Q^*) = (m + n)^m \frac{1}{m+n} m^{n-1} = (m + n)^{m-1} m^{n-1}.$$  

The form of the enumeration of the number of spanning trees suggests the following question:

**Question 7.5.** Is it possible to prove an enumeration of the number of spanning trees of the graph $S_{m,n}$ using Prüfer codes?

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