ON NUMBER OF HYPERSURFACES CONTAINING PROJECTIVE CURVES

S.L’vovsky

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Abstract. Generalizing a classical lemma of Castelnuovo, we characterize rational normal curves (resp. linearly normal elliptic curves) as curves $C \subset \mathbb{P}^n$ such that the number of linearly independent hypersurfaces $Z \supset C$ of given degree $m$ is maximal (resp. next to maximal).

1. Introduction

A well-known lemma of Castelnuovo (see, for instance, [GH, Ch. 4, Sec. 3]) says that if $n(n - 1)/2$ linearly independent quadrics pass through $d \geq 2n + 3$ points in general position in $\mathbb{P}^n$, then these points lie on a rational normal curve. As an immediate consequence, one can see that

**Theorem 1.1 (Castelnuovo).** If $n(n - 1)/2$ linearly independent quadrics pass through a non-degenerate curve $C \subset \mathbb{P}^n$, then $C$ is a rational normal curve.

By induction on dimension, one can derive the following result from this theorem:

**Theorem 1.2.** If $X \subset \mathbb{P}^n$ is a non-degenerate irreducible projective variety and $c = \text{codim} X$, then $X$ is contained in at most $c(c + 1)/2$ linearly independent quadrics; the equality is attained iff the $\Delta$-genus of $X$ is 0.

The condition “$\Delta$-genus of $X$ is zero” means that $\deg X = c + 1$; see [H, page 51] for complete classification of such varieties.

In an unpublished paper [Z], F.L. Zak proposed a new proof and a generalization of this fact. The aim of this paper is to give still another proof and a generalization of some results of [Z]. The methods of proof are entirely different.

Theorem 1.2 can be generalized to the case of $\Delta$-genus 1 (cf. [Z]):

**Theorem 1.3.** Suppose that $X \subset \mathbb{P}^n$ is a non-degenerate and non-singular in codimension 1 irreducible projective variety and put $c = \text{codim} X$. If $X$ is contained in $c(c + 1)/2 - 1$ linearly independent quadrics, then $\deg X = \text{codim} X$ and smooth one-dimensional linear sections of $X$ are linearly normal elliptic curves.

The varieties whose one-dimensional linear sections are linearly normal elliptic curves are also completely classified (see, for instance, [F]).
The main step in the proof of these theorems is to establish them for curves (the rest is done by easy induction). If we consider hypersurfaces of degree \( m > 2 \), the induction step does not work, but some results for curves still can be obtained (see section 1.1 for notation and terminology):

**Theorem 1.4.** If \( X \subset \mathbb{P}^n \) is a non-degenerate irreducible projective curve and \( m \geq 2 \), then \( h^0(\mathcal{I}_X(m)) \leq \binom{m+n}{n} - mn - 1 \). The equality is attained iff \( X \) is a rational normal curve.

**Theorem 1.5.** Suppose that \( X \subset \mathbb{P}^n \), \( n \geq 2 \), is a smooth non-degenerate irreducible projective curve and \( m \geq 2 \). If \( h^0(\mathcal{I}_X(m)) < \binom{m+n}{n} - mn - 1 \), then \( h^0(\mathcal{I}_X(m)) \leq \binom{m+n}{n} - mn - 1 \). The equality is attained iff \( \deg X = n+1 \) and the genus of \( X \) is 1.

The proof is based on the study of inflection points of the curve \( X \). It involves two ideas. The first (and very simple) one is that the presence of an inflection point imposes an upper bound on the number of hypersurfaces of degree \( m \) containing \( X \) (Proposition 3.1). The second idea is that if this bound is attained, then, under some additional conditions, \( X \) can be deformed to a certain monomial rational curve. Actually, we prove that Hilbert polynomials of \( X \) and the monomial curve are the same rather than construct an explicit deformation (Proposition 4.11).

Homogeneous rings of monomial curves, which we use in this paper, have been studied by many authors (cf. [B], [Ho], and references therein). Our Proposition 4.1 is contained in Section 4 of [B]. A computation of Hilbert polynomials of monomial curves is contained in [B], too: our Proposition 4.9 agrees with Theorem 4.1 of [B], while methods of proof differ.

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**1.1 Notation and conventions.** We work over an algebraically closed field of arbitrary characteristic.

If \( a \) and \( b \) are integers, then \( (a; b) = \{ n \in \mathbb{Z} : a < n < b \} \) and \([a; b] = \{ n \in \mathbb{Z} : a \leq n \leq b \} \).

For any subset \( X \subset \mathbb{P}^n \), denote by \( \langle X \rangle \) its linear span. A projective subvariety \( X \subset \mathbb{P}^n \) is called non-degenerate iff \( \langle X \rangle = \mathbb{P}^n \).

By \( v_m: \mathbb{P}^n \to \mathbb{P}^{\binom{m+n-1}{n-1}} \) we denote the \( m \)-th Veronese mapping.

If \( 0 < a_1 < \cdots < a_n \) is an increasing sequence of positive integers, then \( a_1, \ldots, a_n \subset \mathbb{Z} \) denotes the semigroup generated by \( 0, a_1, \ldots, a_n \). If a positive integer is not contained in this semigroup, it is called its gap.

**2. Preliminaries on inflection points**

In this section we chiefly recall some well-known definitions (cf. [EH]). Let \( C \) be a smooth projective curve, \( \mathcal{L} \) a line bundle on \( C \), \( V \subset H^0(\mathcal{L}) \) a linear subspace. We will refer to the pair \( (\mathcal{L}, V) \) as a linear system on \( C \).

Put \( \dim V = n + 1 \). For any point \( p \in C \), there exists a basis \( s_0, \ldots, s_n \) of \( V \) such that \( \ord_p(s_0) < \ord_p(s_1) < \cdots < \ord_p(s_n) \). Denote \( \ord_p(s_i) \) by \( a_i(p) \).

The sequence \( a_0(p), a_1(p), \ldots, a_n(p) \) is called the vanishing sequence at \( p \). For any \( s \in V \), the number \( \ord_p(s) \) coincides with one of the \( a_j(p) \)'s. The number \( w(p) = \ldots \)
the following result (cf. [EH, Proposition 1.1]):

\[ \sum_{p \in C} w(p) = (n+1)d + n(n+1)(g-1), \]

where \( d = \deg L \) and \( g \) is the genus of \( C \).

If the characteristic is zero, then almost all points of \( C \) are not inflection points (cf. [GH]).

The Plücker formula implies the following well-known

**Proposition 2.1.** Suppose that no point of \( C \) is an inflection point with respect to the linear system \((C, L, V)\). Then this linear system defines an isomorphism of \( C \) onto a rational normal curve in \( \mathbb{P}^n \).

### 3. Abstract vanishing sequences

For a non-negative integer \( n \), denote by \( A_n \) the set of strictly increasing sequences of \( n + 1 \) non-negative integers:

\[ A_n = \{ (a_0, \ldots, a_n) \mid 0 \leq a_0 < \cdots < a_n \}. \]

Elements of \( A_n \) will be called abstract vanishing sequences of length \( n \).

For any integer \( k \geq 1 \) and any \( A = (a_0, \ldots, a_n) \in A_n \), put

\[ v_k(A) = \{ a_{i_1} + \cdots + a_{i_k} \mid 0 \leq i_1 < \cdots < i_k \leq n \}. \]

Sometimes we will assume that \( v_0(a) = \{0\} \). By \( k \)-span of \( A \) we will mean the cardinality of \( v_k(A) \). We will denote \( k \)-span of \( A \) by \( \text{Span}_k(A) \).

The following easy observation plays the key role in the sequel.

**Proposition 3.1.** If \((L, V)\) is a linear system on a smooth curve \( C \), where \( \dim L = n + 1 \), and if \( A = (a_0, \ldots, a_n) \) is the vanishing sequence at a point \( p \in C \), then

\[ \dim \text{Im} \left( \text{Sym}^m V \to H^0(L^m) \right) \geq \text{Span}_m(A) \text{ for any } m \geq 1. \]

**Proof.** Let \((s_0, \ldots, s_n)\) be an adapted to \( p \) basis of \( V \). Among \( m \)-fold products of \( s_j \)'s there are exactly \( \text{Span}_m(A) \) that have different orders of vanishing at \( p \). It is clear that all these sections of \( L^m \) are linearly independent, whence the result.

**Corollary 3.2.** If \( X \subset \mathbb{P}^n \) is an irreducible non-degenerate curve such that \( X = \phi(C) \), where \( C \) is a smooth curve and \( \phi \) is defined by a linear system \((L, V)\) on \( C \), and if \( p \in C \) is a point with vanishing sequence \((a_0, \ldots, a_n)\), then \( \dim \langle v_m(X) \rangle \geq \text{Span}_m(A) - 1 \).

Now let us find out in what cases \( m \)-span of a sequence is small.
Proposition 3.3. Suppose that \( A \in \mathcal{A}_n \). Then

(i) \( \text{Span}_m(A) \geq mn + 1 \);
(ii) \( \text{Span}_m(A) = mn + 1 \) if and only if \( A \) is an arithmetic progression.
(iii) If \( \text{Span}_m(A) > mn + 1 \), then \( \text{Span}_m(A) \geq m(n+1) \).
(iv) Suppose that \( n \geq 3 \). Then \( \text{Span}_m(A) = m(n+1) \) if and only if either
\[ a_n - a_{n-1} = 2(a_j - a_{j-1}) \text{ for all } j \in [1; n-1] \text{ or } a_1 - a_0 = 2(a_j - a_{j-1}) \text{ for all } j \in [2; n]. \]

Proof. For \( 1 \leq i, j \leq n \), put \( b_{ij} = (m - i)a_0 + a_j + (i-1)a_n \in v_m(A) \). It is clear that the following chain of inequalities holds:

\[
\begin{align*}
ma_0 < b_{11} < b_{12} < & \cdots < b_{1n} < \\
b_{21} < b_{22} < & \cdots < b_{2n} < \\
& \cdots \cdots \cdots \\
b_{m1} < b_{m2} < & \cdots < b_{mn}
\end{align*}
\]

(3.1)

Since this chain contains \( mn + 1 \) elements of \( v_m(A) \), assertion (i) follows. Suppose now that \( A \) is an arithmetic progression with step \( d \). Then the difference of any two elements of \( v_m(A) \) is a multiple of \( d \), and any two consecutive members of the chain (3.1) differ by \( d \). Since \( ma_0 \) and \( b_{mn} = ma_n \) are the least and the greatest element of the set \( v_m(A) \), we see that there are no elements in \( v_m(A) \) except for members of (3.1); this proves the “if” part of assertion (ii).

To prove the “if” part of assertion (iv), assume that \( a_1 - a_0 = 2d \) and \( a_j - a_{j-1} = d \) for all \( j > 1 \) (in the other case the proof is analogous). Then the difference of any two elements of \( v_m(A) \) is again a multiple of \( d \), and any two consecutive members of the chain (3.1) differ by \( d \), except that

\[ b_{21} - b_{1n} = b_{31} - b_{2n} = \cdots = b_{m1} - b_{m-1,n} = 2d. \]

It is clear that

\[ b_{jn} < (m - j - 1)a_0 + a_1 + a_{n-1} + (j-1)a_n < b_{j+1,1} \quad \text{for all } j < m. \]

If we insert the \( m - 1 \) numbers \( (m - j - 1)a_0 + a_1 + a_{n-1} \) for \( j \in [1; m-1] \) into the chain (3.1), we obtain a strictly increasing sequence of length \( m(n+1) \) such that the difference of any pair of its consecutive elements equals \( d \). Since the difference of any two elements of \( v_m(A) \) is divisible by \( d \), no extra element can be inserted into this sequence, whence the cardinality of \( v_m(A) \) equals \( m(n+1) \).

To proceed, we need the following

Lemma 3.4.

(i) If \( (b_{k,n-1}; b_{k+1,1}) \cap v_m(A) = \{b_{kn}\} \) for some \( k \in [1; m-1] \), then \( a_n - a_{n-1} = a_1 - a_0 \).
(ii) If \( a_{j+1} - a_j = a_1 - a_0 \) for some \( j \in [2; n-1] \) and \( (b_{k,j-1}; b_{k,j+1}) \cap v_m(A) = \{b_{kj}\} \) for some \( k \in [1; m-1] \), then \( a_j - a_{j-1} = a_1 - a_0 \).

Proof of the lemma. To prove (i), observe that the number \( x = (m - k - 1)a_0 + a_1 + a_{n-1} + (k-1)a_n \in v_m(A) \) belongs to the interval \( (b_{k,n-1}; b_{k+1,1}) \); thus \( x = b_{kn} \), whence \( a_n - a_{n-1} = a_1 - a_0 \).
To prove (ii), observe that the hypothesis implies the equality
\[ b_{k,j+1} = (m - k - 1)a_0 + a_1 + a_j + (k - 1)a_n. \]
Hence, the number \( x = (m - k - 1)a_0 + a_1 + a_{j-1} + (k - 1)a_n \) belongs to the interval \((b_{k,j-1}; b_{k,j+1})\); thus \( x = bk_j \), whence \( a_j - a_{j-1} = a_1 - a_0 \).

To prove assertion (iii) of Proposition 3.3 together with the “only if” part of (ii), it suffices to prove that \( A \) is an arithmetic progression whenever \( \text{Span}_m(A) < m(n+1) \). Indeed, in this case at most \( m - 2 \) elements of \( v_m(A) \) are not contained in the chain (3.1). Since for \( k \in [1; m - 1] \) the intervals \((b_{k,n-1}; b_{k+1,1})\) are disjoint, it follows that \((b_{k,n-1}; b_{k+1,1}) \cap v_m(A) = \{b_{kn}\} \) for some \( k \in [1; m - 1] \), whence \( a_n - a_{n-1} = a_1 - a_0 \) by virtue of Lemma 3.4(i). The same argument shows that for each \( i \in [2; n - 1] \) there exists a number \( k_i \in [1; n] \) such that \((b_{k_i,1}; b_{k_i+1}) \cap v_m(A) = \{b_{ki}\} \). Applying Lemma 3.4(ii) \( n - 2 \) times, we see that \( a_{n-1} - a_{n-2} = \cdots = a_1 - a_0 \), i.e. \( A \) is an arithmetic progression.

It remains to prove the “only if” part of (iv). First we do it for \( m = 2 \):

**Lemma 3.5.** Suppose that \( A \in A_n \), where \( n \geq 3 \), and \( \text{Span}_2(A) = 2n + 2 \). Then either \( a_0 - a_{n-1} = 2(a_j - a_{j-1}) \) for all \( j \in [1; n - 1] \) or \( a_1 - a_0 = 2(a_j - a_{j-1}) \) for all \( j \in [2; n] \).

**Proof of the lemma.** For any \( A = (a_0, \ldots, a_n) \in A_n \), put \( A' = (a_0, \ldots, a_{n-1}) \in A_{n-1} \) and \( A'' = (a_1, \ldots, a_n) \in A_{n-1} \). First observe that
\[ (3.2) \quad \text{Span}_2(A) \geq \text{Span}_2(A') + 2 \quad \text{and} \quad \text{Span}_2(A) \geq \text{Span}_2(A'') + 2. \]

Indeed, since \( A \) is a strictly increasing sequence, we see that \( a_{n-1} - a_n \neq 2a_n \), and none of these numbers equals \( a_i + a_j \), where \( 0 \leq i \leq j < n \). Hence, \( v_2(A) \setminus v_2(A') \supset \{a_{n-1} + a_n, 2a_n\} \), whence the desired inequality. The proof of the second inequality is analogous.

Now we proceed by induction on \( n \). For \( n = 3 \), the proof is straightforward. Suppose now that the lemma is proved for all \( A \in A_k \), where \( k < n \), and that \( n > 3 \). If \( A \in A_n \) and \( \text{Span}_2(A) = 2n + 2 \), then (3.2) implies that \( \text{Span}_2(A') \leq 2n \) and \( \text{Span}_2(A'') \leq 2n \). Moreover, we see that one of these inequalities is actually an inequality (this is not the case, then part (ii) of Proposition 3.3, which we have already proved, implies that \( A' \) and \( A'' \) are arithmetic progressions, whence \( A \) is an arithmetic progression and \( \text{Span}_2(A) = 2n + 1 \), contrary to the assumption). From now on we will assume that \( \text{Span}_2(A') = 2n \) (the argument for the case when \( \text{Span}_2(A'') = 2n \) is the same if one mirrors the indices with respect to \( n/2 \)). Observe that \( \{a_{n-1} + a_n, 2a_n\} \cap v_2(A') = \emptyset \); it follows from our assumptions that \( v_2(A) = v_2(A') \cup \{a_{n-1} + a_n, 2a_n\} \), whence \( a_{n-2} + a_n \in v_2(A') \). The only element of \( v_2(A') \) that is a priori not less than \( a_{n-2} + a_n \), is \( 2a_{n-1} \). Hence, \( a_{n-2} + a_n = 2a_{n-1} \).

If we apply the induction hypothesis to \( A' \), we see that either \( a_1, \ldots, a_{n-1} \) is an arithmetic progression with step \( d \) and \( a_1 - a_0 = 2d \), or \( a_0, \ldots, a_{n-2} \) is an arithmetic progression with step \( d \) and \( a_{n-1} - a_{n-2} = 2d \). In the first case it is clear that \( a_1, \ldots, a_n \) is an arithmetic progression with the same step \( d \), and we are done. In the second case it is easy to show that \( \text{Span}_2(A) = 2n + 3 \), which contradicts the hypothesis. This completes the proof of the lemma.

To prove (iv) in full generality, we may assume that \( m > 2 \). Denote by \( E \) the set of elements of \( v_m(A) \) that do not belong to the chain (3.1). Suppose that \( v_m(A) = \cdots \).
m(n + 1), i.e. \( \#E = m - 1 \). I claim that each of the \( m - 1 \) intervals \((b_{k,n-1}; b_{k+1,1})\) contains exactly one element of \( E \). Indeed, assume the converse; then Lemma 3.4(i) implies that \( a_n = a_{n-1} = a_0 \). Moreover, arguing as in the proof of assertion (iii) above, we see that for each \( i \in [2; n - 1] \) there exists a number \( k_i \in [1; n] \) such that \((b_{k_i,i-1}; b_{k_i,i+1}) \cap v_m(A) = \{b_{k_i,i}\}\); applying Lemma 3.4(ii) \( n - 2 \) times we see that \( A \) is an arithmetic progression—a contradiction.

Hence, for each \( k \in [1; m - 1] \) there exists a number \( x_k \in (b_{k,n-1}; b_{k+1,1}) \), \( x_j \neq b_{kn} \), and each of the intervals \((b_{i-1,1}; b_{i+1,1}), i \in [2; n - 1] \) does not contain elements of \( v_m(A) \). Now Lemma 3.4(ii) implies that \( A \) is an arithmetic progression whenever \( a_n - a_{n-1} = a_1 - a_0 \). Hence, \( a_n - a_{n-1} \neq a_1 - a_0 \), and for any \( k \in [1; n - 1] \) the number \((m - k - 1)a_0 + a_1 + a_{n-1} + (k - 1)a_n \in v_m(A)\) belongs to the interval \((b_{k,n-1}; b_{k+1,1})\) and does not coincide with \( b_{kn} \). This shows that \( x_k = (m - k - 1)a_0 + a_1 + a_{n-1} + (k - 1)a_n \). If is clear that either \( b_{k,n-1} < x_k < b_{kn} \) for all \( k \in [1; n - 1] \) (this is the case if \( a_1 - a_0 < a_n - a_{n-1} \)), or \( b_{kn} < x_k < b_{k+1,1} \) (this is the case if \( a_1 - a_0 > a_n - a_{n-1} \)).

In the first case, there are exactly \( 2n + 2 \) elements of \( v_m(A) \) in the interval \([b_{m-2,n}; b_{mn}]\). Since a number \( x \in v_m(A) \) belongs to \([b_{m-2,n}; b_{mn}]\) whenever \( x = (m - 2)a_n + y \) with \( y \in v_2(A) \), we see that \( \text{Span}_2(A) = 2n + 2 \), and the conclusion about \( A \) follows from Lemma 3.5.

In the second case the argument is analogous if we consider \( v_m(A) \cap [ma_0; b_{1n}] \).

This completes the proof.

Remark 3.6. For a given \( A \in A_n \) and all \( m \gg 0 \), the number \( \text{Span}_m(A) \) is easily computable. An explicit formula is contained in [B, Theorem 4.1] (see also Proposition 4.10 below).

By way of an amusing application, we show how Proposition 3.3(iv) yields a proof of the following well-known

**Proposition 3.7.** If \( C \subset \mathbb{P}^n \) is an elliptic curve of degree \( n + 1 \), then \( C \) has exactly \((n + 1)^2\) distinct inflection points, and the weight of each of these points equals 1 (or, equivalently, the vanishing sequence is of the form \((1, \ldots, n - 1, n + 1)\)).

**Proof.** It follows immediately from the Plücker formula that \( \sum_{p \in C} w(p) = (n + 1)^2 \); the point is that there are \((n + 1)^2\) distinct inflection points. To prove this it suffices to show that \( w(p) = 1 \) for any inflection point \( p \in C \). To that end, denote by \( A \) the vanishing sequence at \( p \); since \( C \) is contained in \( n(n - 1)/2 - 1 \) linearly independent quadrics, we see that \( \dim \text{Im}(H^0(O_{\mathbb{P}^n}(2)) \to H^0(O_C(2))) \leq 2n + 2 \), whence \( \text{Span}_2(A) \leq 2n + 2 \) by virtue of Proposition 3.1. Since \( a_0 = 0 \) and \( a_1 = 1 \), we see, by virtue of Proposition 3.3(iv), that \( A = (0, 1, \ldots, n - 1, n + 1) \) and \( w(p) = 1 \).

**Remark 3.8.** Of course, this fact can be established directly by Riemann-Roch. The \((n + 1)^2\) inflection points are points of order \( n + 1 \) on the elliptic curve.

4. ADAPTedin Bases and Filtrations

If \( R = k[X_0, \ldots, X_n] \) is the ring of polynomials over a field \( k \), then \( R_m \subset R \) denotes the space of homogeneous polynomials of degree \( m \). Let \( A = (a_0, \ldots, a_n) \) be an abstract vanishing sequence. Suppose that \( \xi = X_0^{a_0} \cdots X_n^{a_n} \in R \) is a monomial. Then the number \( a_0k_0 + a_1k_1 + \cdots + a_nk_n \) is called weight of \( \xi \) and is denoted by \( w^A(\xi) \) (or simply \( w(\xi) \)). Denote by \( R[j] \subset R \) the \( k \)-subspace spanned by the monomials of weight \( j \). Then \( R[j] \supseteq R[j+1] \) and \( R = \bigoplus R[j] \). Thus, the \( R[j] \)'s
define a (quasihomogeneous) grading of $R$; we will say that elements of $R^{[j]}$ are quasihomogeneous of weight $j$. If $R^{[j]}_m = R_m \cap R^{[j]}$, then $R = \bigoplus R^{[j]}_m$; we will sometimes refer to elements of $R^{[j]}_m$ as bihomogeneous of bidegree $(m,j)$.

For $j, m \geq 0$ put

$$(J^A_m)^{[j]} = \left\{ f \in R^{[j]}_m \mid f(1,1,\ldots,1) = 0 \right\}.$$ 

Put $J^A_a = \bigoplus_s (J^A_m)^{[j]}$, $(J^A)^{[j]} = \bigoplus_s (J^A_m)^{[j]}$, and $J^A = \bigoplus J^A_m = \bigoplus (J^A)^{[j]}$. Then $J^A \subseteq R$ is an ideal that is both homogeneous and quasihomogeneous. Put $S^A = R/J^A$; the ring $S^A$ inherits both homogeneous and quasihomogeneous grading from $R$. Homogeneous (resp. quasihomogeneous) components of $S^A$ will be denoted by $S^A_m$ (resp. $(S^A)^{[j]}$). Dimensions of homogeneous components of $S^A$ are easily computed (cf. [B, Section 4]):

**Proposition 4.1.** $\dim_k(S^A_m) = \text{Span}_m(A)$.

**Proof.** It is clear from the definition that $\dim(J^A_m)^{[j]} = \max(0, \dim R^{[j]}_m - 1)$. Hence, $\dim S^A_m = \sum_s \dim(S^A)^{[j]} = \sum_s (\dim R^{[j]}_m - \max(0, \dim R^{[j]}_m - 1)) = \# \left\{ j \mid \dim R^{[j]}_m \neq 0 \right\}$. The latter number equals $\text{Span}_m(A)$, which completes the proof.

Some simple properties of rings $S^A$ are gathered in the following

**Proposition 4.2.** Let $A = (a_0,\ldots,a_n)$ and $B = (b_0,\ldots,b_n)$ be abstract vanishing sequences.

(i) If there exists a constant $c$ such that $b_i = a_i + c$ for all $i$, then $S^A \cong S^B$.

(ii) If there exists a constant $d$ such that $b_i = a_i \cdot d$ for all $i$, then $S^A \cong S^B$.

(iii) If $a_i = b_{n-i}$ for all $n$, then $S^A \cong S^B$.

In all three cases we mean isomorphism of graded rings with respect to homogeneous grading.

**Proof.** Left to the reader.

**Corollary 4.3.** Let $A = (a_0,\ldots,a_n)$ and $B = (b_0,\ldots,b_n)$ be abstract vanishing sequences.

(i) If there exists a constant $c$ such that $b_i = a_i + c$ for all $i$, then $\text{Span}_m(A) = \text{Span}_m(B)$ for all $m$.

(ii) If there exists a constant $d$ such that $b_i = a_i \cdot d$ for all $i$, then $\text{Span}_m(A) = \text{Span}_m(B)$ for all $m$.

(iii) If $a_i = b_{n-i}$ for all $n$, then $\text{Span}_m(A) = \text{Span}_m(B)$ for all $m$.

Let $(O, m, k)$ be a discrete valuation ring. We will always assume that $k$ is embedded into $O$. Choose a generator $\pi \in m$. Suppose that there exists an increasing sequence of integers $a_0 < a_1 < \cdots < a_n$ and a sequence $s_0, s_1,\ldots, s_n \in O$ such that $\text{ord}(s_j) = a_j$ and $s_j \equiv \pi^{a_j} \mod m^{a_j+1}$. Put $I = \{ f \in R \mid f(s_0,\ldots,s_n) = 0 \}$. Denote by $I \subseteq \mathcal{I}$ the largest homogeneous ideal contained in $I$. Put $S = R/I$. The quasihomogeneous grading induces a descending filtration on $R$:

$$(4.1) F^i R = \sum_{j \geq i} R^{[j]}.$$ 

Put $F^i I = F^i R \cap I$. The following proposition provides an upper bound for dimensions of homogeneous components of $I$. 

Proposition 4.4. \( \dim (F^j I_m/F^{j+1} I_m) \leq \dim (J^A_m)^{[j]} \).

Proof. Let us construct an injection of \( (F^j I_m/F^{j+1} I_m) \) into \( (J^A_m)^{[j]} \). To that end, associate to any \( f \in F^j I_m \) its quasihomogeneous component of weight \( j \). We are to check two things:

(i) The kernel of this morphism coincides with \( F^{j+1} I_m \);
(ii) The image of this morphism lies in \( (J^A_m)^{[j]} \).

Assertion (i) is obvious; to prove (ii), suppose that \( f + g \in F^j I_m \), where \( f \) is quasihomogeneous of weight \( (m, j) \) and \( g \in F^{j+1} R \). Then \( f(s_0, \ldots, s_n) \equiv f(1, \ldots, 1) \pi_j \mod m^{j+1} \). On the other hand, the equality \( f(s_0, \ldots, s_n) + g(s_0, \ldots, s_n) = 0 \) implies that \( f(s_0, \ldots, s_n) \in m^{j+1} \). This proves (ii) and the proposition.

Corollary 4.5. \( \dim S_m \leq \dim S^A_m = \text{Span}_m(A) \).

This corollary suggests the following

Definition 4.1. A system \( (s_0, \ldots, s_n) \) with vanishing sequence \( A \) is called \( m \)-maximal if \( \dim S_m = \dim S^A_m \).

Here’s how this definition works:

Proposition 4.6. If a system \( (s_0, \ldots, s_n) \) is \( m \)-maximal and if \( \sum_{t \geq m} J^A_t = R J^A_m \), then the system \( (s_0, \ldots, s_n) \) is \( t \)-maximal for all \( t \geq m \).

Proof. Observe that the system \( (s_0, \ldots, s_n) \) is \( m \)-maximal if and only if the homomorphisms \( (F^j I_m/F^{j+1} I_m) \to (J^A_m)^{[j]} \) are isomorphisms for all \( j \). Denote by \( q(f) \in R_m \) the sum of quasihomogeneous components of the lowest degree for any \( f \in R_m \).

Consider now an element \( f \in J^A_t \), where \( t > m \); we are to prove that \( f = q(f) \), where \( f \in I_t \). To that end, observe that \( f = \sum g_i h_i \), where \( h_i \in J^A_m \), by hypothesis; on the other hand, \( h_i = q(h_i) \), where \( h_i \in I_m \), because the system \( (s_0, \ldots, s_n) \) is \( m \)-maximal. It is clear that one can put \( f = \sum g_i h_i \).

Another application of the notion of \( m \)-maximality is the following

Proposition 4.7. If a system \( (s_0, \ldots, s_n) \) is \( m \)-maximal and \( \sum_{t \geq m} J^A_t = R J^A_m \), then \( \sum_{t \geq m} I_t = R I_m \).

Proof. Suppose that \( f \in F^j I_k \setminus F^{j+1} I_t \), where \( t \geq m \). I claim that \( f = \varphi + h \), where \( \varphi \in R I_m \) and \( h \in F^{j+1} I_t \).

Indeed, it follows from the proof of Proposition 4.4 that \( q(f) \in (J^A_m)^{[j]} \), whence \( q(f) = \sum g_i h_i \), where \( h_i \in J^A_m \). The observation at the beginning of the proof of Proposition 4.6 shows that \( h_i = q(u_i) \), where \( u_i \in I_m \). If \( \varphi = \sum g_i u_i \), then it is clear that \( \varphi \in R I_m \) and \( h = f - \varphi \in F^{j+1} I_m \). This proves our claim.

The proposition follows by induction from what we have proved, because quasihomogeneous weights of elements of \( R \) are bounded.

Let us apply our results to linear systems on curves. Let \( (\mathcal{L}, V) \) be a linear system on a smooth curve \( C \). Fix a point \( p \in C \); put \( (\mathcal{O}, m) = (\mathcal{O}_p, m_p) \) and fix an isomorphism \( \mathcal{L}_p \to \mathcal{O} \). If \( (s_0, \ldots, s_n) \) is an adapted basis of \( V \), then we can treat \( s_i \)’s as elements of \( \mathcal{O} \) and apply the previous results to the system \( (s_0, \ldots, s_n) \). Observe that \( S \) is the homogeneous coordinate ring of the curve \( \phi(C) \subset \mathbb{P}^n \), where \( \phi \) is the mapping defined by the linear system \( (\mathcal{L}, V) \).
We will say that a linear system $(\mathcal{L}, V)$ is $m$-maximal at the point $p$ whenever $(s_0, \ldots, s_n)$ is $m$-maximal in the sense of Definition 4.1.

Besides the curve $C$, we will consider a certain rational curve $C^A$, which depends only on the vanishing sequence $A$. By definition, $C^A = \text{Proj}(S^A) \subset \mathbb{P}^n = \text{Proj} R$ (we mean Proj with respect to homogeneous grading).

**Proposition 4.8.** If $k$ is algebraically closed, then $C^A = \Phi^A(\mathbb{A}^1) \subset \mathbb{P}^n$, where $\Phi^A: \mathbb{A}^1 \to \mathbb{P}^n$ acts by the formula $t \mapsto (t^{a_0} : t^{a_1} : \ldots : t^{a_n})$.

**Proof.** Obvious.

Degree an arithmetic genus of the rational curve $C^A$ are easily computable. Indeed, let $A \in \mathbb{A}_n$ be an abstract vanishing sequence. Observe that, in view of Proposition 4.2, one may assume that $a_0 = 0$ and g.c.d. of $a_j$’s equals 1.

**Proposition 4.9.** Under the above assumptions on $A$, degree of $C^A$ equals $a_n$ and arithmetic genus of $C^A$ equals $L_0 + L_\infty$, where $L_0$ is the number of gaps of the semigroup $\langle a_1, \ldots, a_n \rangle$ and $L_\infty$ is the number of gaps of the semigroup $\langle a_n - a_{n-1}, \ldots, a_n - a_1, a_n \rangle$.

**Proof.** It follows from the hypothesis that there exist integers $c_0, \ldots, c_n$ such that $\sum c_i a_i = 1$. Put $\psi(x_1, \ldots, x_n) = \prod x_i^{c_i}$. Then the rational map $\psi: C^A \to \mathbb{A}^1$ is a birational inverse to $\Phi^A$. Hence, $\text{deg} C^A = a_n$ and $p_a(C^A) = \text{length}(\Phi_*, \mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_{C^A})$, where $\Phi$ is the extension of $\Phi^A$ to $\mathcal{O}_{\mathbb{P}^1}$. Since $\psi$ is regular on $C^A$ outside $\{0, \infty\}$, this length is the sum of the lengths of stalks of $\Phi_*, \mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_{C^A}$ at $\Phi(0)$ and $\Phi(\infty)$. The stalk of $\Phi_*, \mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_{C^A}$ at $\Phi(0)$ is isomorphic to $k[[t]]/k[[t^{a_1}, \ldots, t^{a_n}]]$, and its length clearly equals $L_0$. Ditto for $\Phi(\infty)$.

Now we can compute $\text{Span}_m(A)$ for a given $A \in \mathbb{A}_n$ and $m \gg 0$. The following proposition is contained in [B, Theorem 4.1].

**Proposition 4.10.** Let $A \in \mathbb{A}_n$ be an abstract vanishing sequence. Suppose that $a_0 = 0$ and g.c.d. of $a_j$’s equals 1. Then for all $m \gg 0$ one has $\text{Span}_m(A) = a_n \cdot m + 1 - L_0 - L_\infty$, where $L_0$ and $L_\infty$ are as in Proposition 4.9.

**Proof.** Proposition 4.1 implies that, for $m \gg 0$, $\text{Span}_m(A) = P(m)$, where $P$ is the Hilbert polynomial of the curve $C^A$. Now Proposition 4.9 applies.

Proposition 4.6 yields the following

**Proposition 4.11.** If $(\mathcal{L}, V)$ is $m$-maximal for some $m$ and $\sum_{j \geq m} J^A_m = RJ^A_m$, then the degree and arithmetic genus of $X = \phi(C)$ are equal to those of $C^A$.

**Proof.** Proposition 4.6 shows that $(\mathcal{L}, V)$ is $t$-maximal for all $t \geq m$. Hence, Hilbert polynomials of $X$ and $C^A$ coincide.

5. On equations defining certain monomial curves

To apply Proposition 4.11, one should know the degrees of generators of the homogeneous ideal $J^A$, or, equivalently, of the homogeneous ideal of the curve $C^A$. In general, it is not clear how these degrees depend on $A$. The following proposition shows that in the simplest cases $J^A$ is generated by quadrics.
Proposition 5.1. Let $A \in \mathcal{A}_n$ be an abstract vanishing sequence. Suppose that one of the following conditions holds:

(i) $n \geq 2$ and $A$ is an arithmetic progression;
(ii) $n \geq 3$ and either $a_n - a_{n-1} = 2(a_j - a_{j-1})$ for all $j \in [1; n-1]$ or $a_1 - a_0 = 2(a_j - a_{j-1})$ for all $j \in [2; n]$.

Then $J^A = R_2J^A$.

Corollary 5.2. If $A \in \mathcal{A}_n$ is as in Proposition 5.1, then $\sum_{j\geq m} J^A_m = RJ^A_m$ for all $m \geq 2$.

Remark 5.3. If $n = 2$ in the case (ii), then $C^A$ is the plane cuspidal cubic and is not defined by quadratic equations.

Remark 5.4. The curve $C^A \subset \mathbb{P}^n$ is a rational normal curve in the case (i) and a linearly normal curve of arithmetic genus 1 in the case (ii). It is well known that the homogeneous ideals of such curves are generated by its components of degree 2 (this is especially true for the case (i)). However, we give a proof since no suitable reference for the case (ii) is known to the author.

The rest of this section is devoted to the proof of Proposition 5.1. We start with some general observations. Consider an arbitrary $A = (a_0, \ldots, a_n)$. It is clear from the definitions that $J^A_m \subset k[X_0, \ldots, X_n]$ is generated, as a linear space, by differences $\xi - \eta$, where $\xi$ and $\eta$ are monomials of degree $m$ and equal weight (recall that weight of a monomial $\xi = X^d_0 \cdots X^d_n$ is the number $\text{wt}_A(\xi) = \sum a_i d_i$). For an integer $t \geq 2$, we will say that monomials $\xi = X^{d_0}_0 \cdots X^{d_n}_n$ and $\eta = X^{e_0}_0 \cdots X^{e_n}_n$ are $t$-neighbors whenever $\deg \xi = \deg \eta$, $\text{wt}_A(\xi) = \text{wt}_A(\eta)$, and $\# \{ i \in [0; n] \mid d_i \neq e_i \} = t$. We will say that the monomials $\xi$ and $\xi'$ are $t$-equivalent whenever there exists a chain of monomials $\xi = \xi_0, \xi_1, \ldots, \xi_l = \xi'$ such that for all $j \in [1; l]$ the monomials $\xi_j$ and $\xi_{j-1}$ are $t$-neighbors. The following propositions follow immediately from the above definitions.

Proposition 5.5. $J^A_m = R_{m-1}J^A$ if and only if any two monomials $\xi$ and $\eta$ of degree $m$ and equal weight are $t$-equivalent.

Proposition 5.6. If the monomials $\xi$ and $\eta$ are $t$-equivalent, then $\lambda \xi$ is $t$-equivalent to $\lambda \eta$ for any monomial $\lambda$.

Suppose that $\xi = X^{d_0}_0 \cdots X^{d_n}_n$ is a monomial; let us say that the support of $\xi$ is the set $\text{supp}(\xi) = \{ i \in [0; n] \mid d_i \neq 0 \}$. We will say that two subsets $U, V$ of a segment $[k; l]$ are interlaced iff either there exist numbers $p', p'' \in U$ and $q \in V$ such that $p' < q < p''$ or there exist numbers $p \in U$ and $q', q'' \in V$ such that $q' < p < q''$.

Lemma 5.7. If $\xi$ and $\eta$ are monomials of equal weight, then $\text{supp}(\xi)$ and $\text{supp}(\eta)$ are interlaced.

Proof. Assume the converse. If $\xi = X^{d_0}_0 \cdots X^{d_n}_n$ and $\eta = X^{e_0}_0 \cdots X^{e_n}_n$, then we may suppose without loss of generality that $d_i < e_i$ for all $i$. Hence, $\sum a_i d_i < \sum a_i e_i$, which contradicts the hypothesis.

To proceed with the proofs, it will be convenient to consider a certain “mathematical game.” This game is played on a board that is a rank of $n + 1$ squares numbered by integers $0, 1, \ldots, n$ from left to right. Any number of pieces can stand on each square. If we assign a position with $d_i$ pieces on the $i$-th square to each
monomial \( \xi = X_0^{d_0} \cdots X_n^{d_n} \), we get a 1-1 correspondence between the set of monomials in \( X_0, \ldots, X_n \) of degree \( m \) and the set of positions in this game with \( m \) pieces on the board.

The pieces can be moved according to the following rule: If \( a_i + a_j = a_{i'} + a_{j'} \) and there are pieces on the \( i \)-th and \( j \)-th squares, then both these pieces can be moved simultaneously to the \( i' \)-th and \( j' \)-th square respectively. This rule does not exclude the cases \( i = j \) or \( i' = j' \) (of course, if \( i = j \), then there should be at least two pieces on the \( i \)-th square).

Using this language, one can restate the definition of 2-equivalence as follows:

**Proposition 5.8.** Two monomials of equal degree are 2-equivalent if and only if the corresponding positions in the game can be joined by a sequence of moves.

Now we can resume the proofs.

**Lemma 5.9.** Suppose that there exist integers \( k \) and \( l \) such that \( 0 \leq k < l \leq n \) and the sequence \( (a_k, a_{k+1}, \ldots, a_l) \) is an arithmetic progression. If \( \xi \) and \( \eta \) are two monomials such that the sets \( \text{supp}(\xi) \cap [k, l] \) and \( \text{supp}(\eta) \cap [k, l] \) are interlaced, then there exist monomials \( \xi' \) and \( \eta' \) such that \( \xi' \) is 2-equivalent to \( \xi \), \( \eta' \) is 2-equivalent to \( \eta \), and \( \text{supp}(\xi') \cap \text{supp}(\eta') \neq \emptyset \).

**Proof.** Without loss of generality we may assume that there exist integers \( p', p'' \in \text{supp}(\xi) \) and \( q \in \text{supp}(\eta) \) such that \( k \leq p' < q < p'' \leq l \). Consider a position in our game that corresponds to the monomial \( \xi \). There are pieces on the squares \( p' \) and \( p'' \); since the restriction of the sequence \( (a_0, \ldots, a_n) \) to the segment \( [p'; p''] \) is an arithmetic progression, we may start moving both these pieces towards each other, to one square each at a time, until they are on the same or adjacent square(s).

En route, one of the pieces will step on the \( q \)-th square; if \( \xi' \) is the monomial corresponding to the position at that moment, then \( \xi' \) is 2-equivalent to \( \xi \) and \( \text{supp}(\xi') \cap \text{supp}(\eta') \neq \emptyset \).

**Proof of Proposition 5.1(i).** In view of Proposition 5.5 it suffices to prove that any two monomials of degree \( m \) and equal weight are 2-equivalent. Let us do it by induction on \( m \).

If \( m = 2 \), there is nothing to prove. Suppose that our assertion is proved for all \( m' < m \), and let \( \xi \) and \( \eta \) be two monomials of degree \( m \) and equal weight. Lemma 5.7 implies that \( \text{supp}(\xi) \) and \( \text{supp}(\eta) \) are interlaced; since \( A \) is an arithmetic progression, Lemma 5.9 implies that there exist monomials \( \xi' \) and \( \eta' \) such that \( \xi' \) is 2-equivalent to \( \xi \), \( \eta' \) is 2-equivalent to \( \eta \), and \( \text{supp}(\xi') \cap \text{supp}(\eta') \neq \emptyset \). Hence, there exist monomials \( \xi \), \( \eta \) and \( \lambda \) such that \( \xi' = \lambda \xi \), \( \eta' = \lambda \eta \), and \( \deg \lambda > 0 \). Since weights of \( \xi \) and \( \eta \) are equal, the induction hypothesis implies that \( \xi \) and \( \eta \) are 2-equivalent, whence \( \xi \) and \( \eta \) are 2-equivalent by virtue of Proposition 5.6. This completes the proof.

**Proof of Proposition 5.1(ii).** Proposition 4.2 shows that we may assume that \( A = (0, 1, \ldots, n - 1, n + 1) \), which we will do further on. Proceeding by induction as in the previous proof, one can see that it suffices to prove the following claim:

If \( \xi \) and \( \eta \) are two monomials of degree \( m > 2 \) and equal weight, then there exist monomials \( \xi' \) and \( \eta' \) such that \( \xi' \) is 2-equivalent to \( \xi \), \( \eta' \) is 2-equivalent to \( \eta \), and \( \text{supp}(\xi') \cap \text{supp}(\eta') \neq \emptyset \).
To prove this claim, observe that \( \mathrm{supp}(\xi) \) and \( \mathrm{supp}(\eta) \) are interlaced by virtue of Lemma 5.7. If \( n \notin \mathrm{supp}(\xi) \cup \mathrm{supp}(\eta) \), then we are done by Lemma 5.9. Hence, we may assume that \( n \in \mathrm{supp}(\xi) \) and \( \mathrm{supp}(\xi) \cap \mathrm{supp}(\eta) = \emptyset \) (in particular, \( \mathrm{supp}(\eta) \not\supset n \)).

Now the game comes to help. Place \( m \) white pieces and \( m \) black pieces on the board in such a way that the position of white (resp. black) pieces corresponds to the monomial \( \xi \) (resp. \( \eta \)). We are to show that one can move white and/or black pieces according to the rules of the game in such a way that one of the pieces steps on a square occupied by a piece of the opposite color.

If the sets \( \mathrm{supp}(\xi) \cap [0; n-1] \) and \( \mathrm{supp}(\eta) \cap [0; n-1] \) are interlaced, then we can make white and black pieces meet using the method from the proof of Lemma 5.9. Hence, in the sequel we may and will assume that \( \mathrm{supp}(\xi) \cap [0; n-1] \) and \( \mathrm{supp}(\eta) \cap [0; n-1] \) are not interlaced. It follows from this assumption that for any \( i \in \mathrm{supp}(\xi) \setminus \{n\} \) and \( j \in \mathrm{supp}(\eta) \) we have \( i < j \).

If \( \mathrm{supp}(\xi) \supset \{n, j\} \), where \( j \leq n - 3 \), we can make the following move: a white piece jumps from \( j \) to \( j + 2 \), and another white piece jumps from \( n \) to \( n - 1 \). Let us repeat such a move (with various \( j \)'s maybe) while it is possible. When this process is finished, one of the following assertions holds:

1. There are no white pieces left on the \( n \)-th square.
2. There are no white pieces left on the squares whose number is less than \( n - 2 \), and there is a white piece on the \( n \)-th square.

Denote the monomial corresponding to the new position of white pieces by \( \xi' \). In the first case, we can finish off the proof by applying Lemma 5.9 to \( \xi' \) and \( \eta \). In the second case, all white pieces are concentrated on the \( (n - 2) \)-th and \( n \)-th squares. If \( \mathrm{supp}(\xi') \cap [0; n-1] \) is interlaced with \( \mathrm{supp}(\eta) \), we can again make white and black piece meet by the same method as in the proof of Lemma 5.9. Hence, we may assume that these two sets are not interlaced and disjoint. This is possible only if all black pieces are concentrated on the \( (n - 1) \)-th square. Since \( n \geq 3 \) by the hypothesis, we can make the following move: one black piece jumps from \( n - 1 \) to \( n \), and another jumps from \( n - 1 \) to \( n - 3 \). After this move, black and white pieces will meet on the \( n \)-th square. This completes the proof.

6. PROOFS OF MAIN RESULTS

Proof of Theorem 1.4. If \( X \subset \mathbb{P}^n \) is an irreducible non-degenerate curve, then \( X \) is the image of a birational morphism \( f: C \to \mathbb{P}^n \) defined by a linear system \( (\mathcal{L}, \mathcal{V}) \), where \( C \) is a smooth curve. If \( p \in C \) is any point with the vanishing sequence \( A = (a_0, \ldots, a_n) \), then Proposition 3.3(i) shows that \( \text{Span}_m(A) \geq mn + 1 \), whence \( \dim \text{Im}(\text{Sym}^m(V) \to H^0(\mathcal{L}^m)) \geq mn + 1 \) by Proposition 3.1. The first part of the theorem follows immediately from this inequality.

The fact that the bound is attained for rational normal curves is trivial. Hence, we can assume for the sequel that \( h^0(\mathcal{I}_X(m)) = \binom{m+n}{n} - mn - 1 \), or, equivalently, that \( \dim \text{Im}(\text{Sym}^m(V) \to H^0(\mathcal{L}^m)) = mn + 1 \), and we are to prove that \( X \) is a rational normal curve.

To that end, pick a point \( p \in C \) and denote by \( A = (a_0, \ldots, a_n) \) its vanishing sequence at \( p \). It follows from Proposition 3.1 that \( \text{Span}_m(A) \leq mn + 1 \). Now Proposition 3.3(i,ii) shows that \( A \) is an arithmetic progression and that \( (\mathcal{L}, \mathcal{V}) \) is \( m \)-maximal at \( p \).
Hence, $C^A$ is the normal rational curve of degree $n$ in $\mathbb{P}^n$. Proposition 4.11 and Corollary 5.2 show that $X$ has degree $n$ and arithmetic genus 0 as well, whence $X$ is rational normal curve.

Proof of Theorem 1.5. Consider the linear system $(\mathcal{L}, V)$, where $\mathcal{L} = \mathcal{O}_X(1)$ and $V = \text{Im}(H^0(\mathcal{O}_{P^n}(1) \to H^0(\mathcal{O}_X(1)))$. If this linear system has no inflection points, then $X$ is a normal rational curve by Proposition 2.1, whence $h^0(\mathcal{I}_X(m)) = \binom{m+n}{n} - mn - 1$, which contradicts the hypothesis. Hence, we may assume that there is an inflection point $p \in X$. Denote its vanishing sequence by $A = (a_0, \ldots, a_n)$. It is clear that $a_0 = 0$ and $a_1 = 1$, whence $A$ is not an arithmetic progression.

Proposition 3.3(iii) implies that $\text{Span}_m(A) \geq m(n+1)$, and Proposition 3.1 implies that $\dim \text{Im} (\text{Sym}^m V \to H^0(\mathcal{L}^\otimes m)) \geq m(n+1)$, whence the first assertion of the theorem.

To prove the second assertion, suppose that $h^0(\mathcal{I}_X(m)) = \binom{m+n}{n} - mn - 1$. Propositions 3.3(iv) and 3.1 imply that $A$ is of the form $(0, 1, \ldots, n-1, n+1)$ and the linear system $(\mathcal{L}, V)$ is $2$-maximal at $p$. Corollary 5.2 shows that Proposition 4.11 applies, whence the degree and arithmetic genus of $X$ equal those of $C^A$. Since $C^A$ is a cuspidal rational curve of degree $n+1$ and arithmetic genus 1, we are done.

Proof of Theorems 1.2 and 1.3. We will proceed by induction on $\dim X$. If $\dim X = 1$, then our theorem are just the $m = 2$ case of Theorems 1.4 and 1.5. If $\dim X > 1$, consider a generic hyperplane section $Y \subset X$. There is an exact sequence

$$0 \to \mathcal{I}_X(1) \to \mathcal{I}_X(2) \to \mathcal{I}_Y(2) \to 0.$$ 

Since $X$ is non-degenerate, this sequence yields the inequality $h^0(\mathcal{I}_X(2) \leq h^0(\mathcal{I}_Y(2))$, and the theorems follow from the induction hypothesis.

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