CRYSTAL OF AFFINE TYPE $\hat{A}_{\ell-1}$ AND HECKE ALGEBRAS AT A PRIMITIVE 2\textsuperscript{th} ROOT OF UNITY

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Abstract. Let $\ell \in \mathbb{N}$ with $\ell > 1$. In this paper we give a new realization of the crystal of affine type $\hat{A}_{\ell-1}$ using the modular representation theory of the affine Hecke algebras $\mathcal{H}_n$ of type $A$ and their two level cyclotomic quotients with Hecke parameter being a primitive 2\textsuperscript{th} root of unity. We construct "hat" versions of $i$-induction and $i$-restriction functors on the category $\text{Rep}_I(\mathcal{H}_n)$ of finite dimensional integral modules over $\mathcal{H}_n$, which induce Kashiwara operators on a suitable subgroup of the Grothendieck groups of $\text{Rep}_I(\mathcal{H}_n)$. For any simple module $M \in \text{Rep}_I(\mathcal{H}_n)$, we prove that the simple submodules of $\text{res}_{\mathcal{H}_n^-} M$ which belong to $B(\infty)$ (Definition 5.1) occur with multiplicity two.

The main results generalize the earlier work of Grojnowski and Vazirani on the relations between the crystal of $\hat{A}_\ell$ and the affine Hecke algebras of type $A$ at a primitive $\ell$\textsuperscript{th} root of unity.

1. Introduction

Let $1 < \ell \in \mathbb{N}$. Let $B(\hat{A}_0)$ be the crystal of the integral highest weight module $L(\hat{A}_0)$ of the affine Lie algebra $\hat{\mathfrak{sl}}_\ell$ and $B(A_0 + \Lambda_\ell)$ the crystal of the integral highest weight module $L(A_0 + \Lambda_\ell)$ of the affine Lie algebra $\hat{\mathfrak{sl}}_{2\ell}$, where $\Lambda_0$ and $A_0, \Lambda_\ell$ are the fundamental dominant weights of $\hat{\mathfrak{sl}}_\ell$ and $\hat{\mathfrak{sl}}_{2\ell}$ respectively. By [26] (4.2), there is a natural embedding $\iota : B(\Lambda_0) \cup \{0\} \rightarrow B(A_0 + \Lambda_\ell) \cup \{0\}$ which is defined by

$$\iota(f_{i_1} + \ell z \cdots f_{i_n} + \ell z 1_{\Lambda_0}) = f_{i_1 + 2\ell z} f_{i_2 + \ell + 2\ell z} \cdots f_{i_n + 2\ell z f_{i_1 + \ell + 2\ell z 1_{\Lambda_0 + \Lambda_\ell}}, \forall n \in \mathbb{N}, i_1, \cdots, i_n \in \mathbb{Z},$$

such that $\iota(\{0\}) \subseteq B(A_0 + \Lambda_\ell)$.

The above embedding $\iota$ has some important combinatorial and representation theoretic implication. Recall that $B(\hat{A}_0)$ has a realization in terms of the set $\mathcal{K}_0$ of $\ell$-restricted partitions (or equivalently, Kleshchev partitions), while $B(A_0 + \Lambda_\ell)$ has a realization in terms of the set $\mathcal{K}_{0,\ell}$ of Kleshchev bipartitions with respect to $(\sqrt{\ell} \mathbb{T}, 1, -1)$ ([2] Definition 2.3), [3] Page 605, Definition), where $\sqrt{\ell} \mathbb{T}$ denotes a primitive 2\textsuperscript{th} root of unity. Thus for each $n \in \mathbb{N}$, $\iota$ defines an injection ([22] Corollary 6.9) from the subset $\mathcal{K}_0(n)$ into the subset $\mathcal{K}_{0,\ell}(2n)$, such that if $\emptyset \rightarrow \frac{1}{2} \lambda \rightarrow \frac{1}{2} \lambda \rightarrow \cdots$ $\rightarrow \lambda$ is a path in Kleshchev’s good lattice of $\mathcal{K}_0$ then $\iota(\emptyset) \rightarrow \iota(\frac{1}{2} \lambda) \rightarrow \cdots \rightarrow \iota(\lambda)$ is a path in Kleshchev’s good lattice of $\mathcal{K}_{0,\ell}$, where $\mathcal{K}_0(n)$ (resp., $\mathcal{K}_{0,\ell}(2n)$) denotes the set of $\ell$-restricted partitions of $n$ in $\mathcal{K}_0$ (resp., of Kleshchev bipartitions of $2n$ in $\mathcal{K}_{0,\ell}$). Furthermore, $\iota(B(\hat{A}_0))$ coincides with the fixed point subset of $B(A_0 + \Lambda_\ell)$ under the automorphism “$\iota$” induced by the Dynkin diagram automorphism $i \rightarrow i + \ell + 2\ell \mathbb{Z}$ for all $i \in \mathbb{Z}/2\ell \mathbb{Z}$. The subset $\mathcal{K}_0(n)$ gives a labelling of simple modules for the Iwahori–Hecke algebra of type $A_{n-1}$ (i.e., associated to the symmetric group $\mathfrak{S}_n$) at a primitive $\ell$\textsuperscript{th} root of unity $\sqrt{\ell}$, while the subset $\mathcal{K}_{0,\ell}(2n)$ gives a labelling of simple modules for the Iwahori–Hecke algebra of type $B_{2n}$ at a primitive 2\textsuperscript{th} root of unity.

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References

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2ℓ-th root of unity \( \sqrt[n]{T} \). This gives a first clue on the connection between the modular representations of the Iwahori–Hecke algebras of type \( A \) and of type \( B \) at different roots of unity via (1.1).

The second implication of (1.1) involves the Iwahori–Hecke algebras of type \( D \) at root of unity. Let \( F \) be an algebraically closed field with \( \text{char } F \neq 2 \) and \( 1 \neq q \in F^\times \). Recall that the Iwahori–Hecke algebra \( \mathcal{H}_q(B_n) \) of type \( B_n \) is the unital associative \( F \)-algebra generated by \( T_0, T_1, \ldots, T_{n-1} \) which satisfy the following relations:

\[
\begin{align*}
T_0^2 &= 1, \quad (T_r - q)(T_r + 1) = 0, \quad \forall 1 \leq r < n, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \forall 1 \leq i < n-1, \\
T_i T_j &= T_j T_i, \quad \forall 1 \leq i < j - 1 < n - 1, \\
T_0 T_1 T_0 &= T_1 T_0 T_1.
\end{align*}
\]

The \( F \)-subalgebra of \( \mathcal{H}_q(B_n) \) generated by \( T_0, T_1, T_0, T_1, \ldots, T_{n-1} \) is isomorphic to the Iwahori–Hecke algebra \( \mathcal{H}_q(D_n) \) associated to the Weyl group of type \( D_n \). In a series of earlier works [10], [11], [14], [15], the second author has initiated the study of the modular representations of \( \mathcal{H}_q(D_n) \) using the Clifford theory between \( \mathcal{H}_q(B_n) \) and \( \mathcal{H}_q(D_n) \), with the aim of computing the decomposition numbers of \( \mathcal{H}_q(D_n) \) in terms of the decomposition numbers of \( \mathcal{H}_q(B_n) \). Let \( \mathcal{P}(n) \) and \( \mathcal{P}(n) \) be the set of bipartitions and partitions of \( n \) respectively. Let \( \{ S^λ \mid λ ∈ \mathcal{P}(n) \} \) and \( \{ D^λ \mid \lambda ∈ \mathcal{P}(n) \} \) be the set of Specht modules and simple modules of \( \mathcal{H}_q(B_n) \) respectively, where \( D^λ \) is defined in [6] as certain quotient of \( S^λ \) in the semisimple case. \( S^λ \downarrow_{\mathcal{H}_q(D_n)} \) splits into a direct sum of two simple submodules \( S^λ_+ \oplus S^λ_- \) whenever \( \lambda = (λ, λ) \) for some \( λ ∈ \mathcal{P}(n/2) \). Set \( \mathcal{X}(n) := \{ λ ∈ \mathcal{P}(n) \mid D^λ \neq 0 \} \). By [10], for each \( λ ∈ \mathcal{X}(n) \), \( D^λ \downarrow_{\mathcal{H}_q(D_n)} \) either remains simple, or splits into a direct sum of two simple submodules \( D^λ_+ \oplus D^λ_- \). The most interesting and not well-understood case is when \( n \) is even and the Hecke parameter \( q \) is a primitive \( 2\ell \)-th root of unity. In that case, \( \mathcal{X}(n) = K_{0,2}(n) \), \( D^λ \downarrow_{\mathcal{H}_q(D_n)} \) splits if and only if \( \lambda = \lambda(h(\lambda)), \) and if and only if \( \lambda = \lambda(h(\lambda)) \) for some \( \mu ∈ K_0(n/2) \). Moreover, the set

\[
\{ D^λ \downarrow_{\mathcal{H}_q(D_n)} \mid \lambda ∈ K_{0,2}(n/2) \}
\]

is a complete set of pairwise non-isomorphic simple \( \mathcal{H}_q(D_n) \)-modules, where \( \lambda \sim \mu \) if and only if \( \mu = h(\lambda) \).

A major challenging problem in the understanding of the modular representations of \( \mathcal{H}_q(D_n) \) when \( n \) is even and \( q \) is a primitive \( 2\ell \)-th root of unity is to understand the decomposition numbers \( [S^λ_{\lambda(\mu)} : D^λ_+] \), where \( \lambda \) is a partition of \( n/2 \) and \( \mu \) is an \( \ell \)-restricted partition of \( n/2 \). We suspect that (1.1) reveals not only the bijection between \( K_0(n/2) \) and the set \( \{ \lambda ∈ K_{0,2}(n) \mid D^λ \downarrow_{\mathcal{H}_q(D_n)} \) splits (a fact which was first obtained in [7]), but also indicates some possible connections between the following three (type \( A \), type \( B \) and type \( D \)) decomposition numbers

\[
[S^λ : D^μ], \quad [S^λ_{\lambda(\mu)} : D^λ_+], \quad [S^λ_{\lambda(\mu)} : D^λ_-],
\]

where \( S^λ, D^μ \) denote the Specht module labelled by \( \lambda \) and the simple module labelled by \( \mu \) of \( \mathcal{H}_q(\mathbb{S}_n) \) respectively.

By the celebrated work of Ariki, Lascoux, Leclerc and Thibon [11, 23], the decomposition numbers for the Hecke algebras of type \( A \), type \( B \) or more generally of type \( G(r,1,n) \) when \( \text{char } F = 0 \) can be computed through the calculation of the canonical bases of certain integral highest weight modules over \( \mathfrak{sl}_n \), where \( \ell \) is the multiplicative order of the Hecke parameter \( q \). More recently, using the theory of quiver Hecke algebras, Brundan and Kleshchev [5] show that Ariki-Lascoux-Leclerc-Thibon’s theory can be upgraded into a \( \mathbb{Z} \)-graded setting. In [8] and [9], Grojnowski and Vazirani give a new approach to the modular representations of affine Hecke algebras and their cyclotomic quotients over field of any (possibly positive) characteristic. In their approach a new realization of the crystals of affine type \( A_{\ell-1} \) is obtained using the modular representation theory of affine Hecke algebras and their cyclotomic quotients at a primitive \( \ell \)-th root of unity, where the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) for the crystal are realized as the functors of taking socle of \( i \)-restriction and of taking head of \( i \)-induction. Grojnowski and Vazirani’s approach also successfully applied to other Hecke algebras such as Hecke–Ciffford algebras [4, 29] and KLR algebras [24]. Note that the Dynkin diagram \( A^{(2)}_{2\ell} \) (resp., \( D^{(2)}_{\ell+1} \)) can be obtained by \( A^{(1)}_{2\ell} \) (resp., \( A^{(1)}_{2\ell-1} \)) by some diagram automorphisms [12]. In this paper, motivated by the embedding [13] (which can be viewed as some sort of diagram folding), we give a new realization of the crystal of affine type \( A_{\ell-1} \) using the modular representation theory of the affine Hecke algebras of type \( A \) and their level two cyclotomic quotients (i.e., Iwahori–Hecke algebra of type \( B \)) at a primitive \( 2\ell \)-th root of unity. We realize the Kashiwara operators for the crystal as the functors of taking socle of certain two-steps restriction
and of taking head of certain two-steps induction. For any simple module \( M \in \mathcal{H}_n\)-mod, we prove that the simple submodules of \( \text{res}^{\mathcal{H}_n}_{\mathcal{H}_m} M \) which belong to \( \tilde{B}(\infty) \) (Definition 5.1) occur with multiplicity two. The main tool we use is the theory of real simple modules developed by Kang, Kashiwara, et al. in recent papers [17, 18]. The theory is originally built for the quiver Hecke algebras, but can be transformed into the setting of the affine Hecke algebras of type \( A \) using the categorical equivalence between finite dimensional representations of affine Hecke algebras of type \( A \) and of the quiver Hecke algebras of type \( A \) established in [16].

The content of the paper is organized as follows. In Section 2, we first recall some basic knowledge (like intertwining elements, convolution product, the functor \( \Delta_b \)) about the non-degenerate affine Hecke algebra of type \( A \). Then we introduce in Definitions 2.27 and 2.29 the notion of two-steps induction functors \( \tilde{c}_i \), the generalized Kato modules \( L(\tilde{r}^m) \) and the function \( \varepsilon_i \). We give in Lemmas 2.30, 2.33 a number of results on the functor \( \hat{\Delta}_{\tilde{c}_n} \) and the function \( \varepsilon_i \). In Section 3 we first recall the notion of real simple module and some of their main properties established in [18]. Then we give Lemmas 3.4, 3.7, Corollary 3.10 which provide a number of new real simple modules and their nice properties. In particular, our Corollary 3.11 fixes a gap of [21, 6.3.2] in the case when \( k \) is odd, see [22]. In Section 4 we study the two-steps version \( \tilde{c}_i, \tilde{f}_i \) of the functors \( c_i, f_i \) of taking socle of \( i \)-restriction and taking head of \( i \)-induction. The main result in this section is Proposition 4.3 where we prove that the socle of \( e_i c_i + \epsilon M \) is isomorphic to \( e_i c_i + \epsilon M \) for any finite dimensional simple \( \mathcal{H}_n \)-module \( M \). In addition, we also give generalization in our “hat” setting of some results in [21] for the functors \( \tilde{c}_i, \tilde{f}_i \), \( \Delta_{\tilde{c}_n} \) and the function \( \varepsilon_i \). In Section 5 we give the first two main results Theorems 5.20, 5.23 of this paper, which give a new realization of the crystal of \( U_r(a \tilde{1})^{-} \) using the modular representation theory of the affine Hecke algebras \( \mathcal{H}_n \) of type \( A \) at a primitive \( 2 \)-th root of unity. In Section 6 we give the third main result Theorem 6.20 of this paper, which give a new realization of the crystal of the integral highest weight module \( V(\tilde{\lambda}_0) \) of \( U_r(a \tilde{1}) \) using the modular representation theory of the Iwahori–Hecke algebra of type \( B \) at a primitive \( 2 \)-th root of unity. We also obtain several multiplicity two results in Theorems 6.3, 6.6 and Corollary 6.8 about certain two-steps restrictions and inductions.

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2. Preliminary

Throughout this paper, let \( F \) be an algebraically closed field and \( 1 \neq q \in F^\times \). Let \( \mathcal{H}_n := \mathcal{H}_n(q) \) be the non-degenerate type \( A \) affine Hecke algebra over \( F \) with Hecke parameter \( q \). By definition, \( \mathcal{H}_n \) is the unital associative \( F \)-algebra with generators \( T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1} \) and relations:

\[
(T_i - q(T_i + 1)) = 0, \quad 1 \leq i < n,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i < n,
\]

\[
T_i T_k = T_k T_i, \quad |i - k| > 1,
\]

\[
X_i^{\pm 1} X_k^{\pm 1} = X_k^{\pm 1} X_i^{\pm 1}, \quad 1 \leq i, k \leq n,
\]

\[
X_k X_k^{-1} = 1 = X_k^{-1} X_k, \quad 1 \leq k \leq n,
\]

\[
T_i X_k = X_k T_i, \quad k \neq i, i+1,
\]

\[
X_{i+1} = q^{-1} T_i X_i T_i, \quad 1 \leq i < n.
\]

Note that one can also replace the last relation above with the following:

\[
X_{i+1} T_i = T_i X_i + (q - 1) X_{i+1}, \quad 1 \leq i < n.
\]

Let \( * \) be the anti-isomorphism of \( \mathcal{H}_n \) which is defined on generators by \( T_i^* = T_i, X_i^* = X_j \) for any \( 1 \leq i < n, 1 \leq j \leq n \). For any \( a, b \in \mathbb{N} \) with \( 1 \leq a < b \), we denote by \( \mathcal{H}_{(a, b, a+2, \ldots, b)} \) the affine Hecke algebra which is isomorphic to \( \mathcal{H}_{b-a} \) and whose defining generators and relations are obtained from that of \( \mathcal{H}_{b-a} \) by shifting all the subscript upwards by \( a \). In particular, \( \mathcal{H}_n = \mathcal{H}_{(1, 2, \ldots, n)} \). For later use, we need certain elements of \( \mathcal{H}_n \) which are called intertwining elements.

Definition 2.9. ([31, 4.10], [25], [27]) For each \( 1 \leq k < n \), we define the \( k \)-th intertwining element \( \Phi_k \) to be:

\[
\Phi_k := (1 - X_k X_{k+1}^{-1}) T_k + 1 - q = T_k (1 - X_{k+1} X_k^{-1}) + (q - 1) X_{k+1} X_k^{-1}.
\]
Note that the element $\Phi_k$ defined above is the same as $\Theta_k^*$ in the notation of [5, (4.10)]. These elements have the following nice properties which are mostly easy to check.

**Lemma 2.10.** ([25 Proposition 5.2], [5] (4.11),(4.12),(4.13))

1) For any $1 \leq k < n$,
   $$\Phi_k^2 = (q - X_{k+1}X_k^{-1})(q - X_kX_{k+1}^{-1}).$$
2) For any $1 \leq k < n - 1$, $\Phi_k\Phi_{k+1}\Phi_k = \Phi_{k+1}\Phi_k\Phi_{k+1}$, and $\Phi_k\Phi_j = \Phi_j\Phi_k$ for any $1 \leq j < n$ with $|j - k| > 1$;
3) For any $1 \leq k, j < n$ with $j \neq k$, we have that
   $$\Phi_kX_k = X_{k+1}\Phi_k, \quad \Phi_kX_{k+1} = X_k\Phi_k, \quad \Phi_kX_j = X_j\Phi_k.$$
4) For any $1 \leq k, j < n$ with $j \neq k - 1, k, k + 1$, we have $\Phi_kT_j = T_j\Phi_k$.

Let $S_n$ be the symmetric group on $\{1, 2, \cdots, n\}$. For each $1 \leq k < n$, we define $s_k := (k, k + 1)$. For any $w \in S_n$ and any reduced expression $s_{i_1}\cdots s_{i_m}$ of $w$, we define $\Phi_w := \Phi_{i_1}\cdots\Phi_{i_m}$. Then $\Phi_w$ depends only on $w$ but not on the choice of the reduced expression of $w$ because of the braid relations 2) in Lemma 2.10.

**Corollary 2.11.** Let $w \in S_n$. If $1 \leq k \leq n$ then $\Phi_wX_k = X_{w(k)}\Phi_w$.

*Proof.* This follows from Lemma 2.10 (3).

**Corollary 2.12.** Let $w \in S_n$ and $1 \leq k < n$. Suppose $w(k + 1) = w(k) + 1$. Then $\Phi_wT_k = T_{w(k)}\Phi_w$.

*Proof.* It is clear that $ws_k = s_{w(k)}w$. Since $w(k) < w(k + 1)$, we have $\ell(ws_k) = \ell(w) + 1 = \ell(s_{w(k)}w)$. Therefore, $\Phi_w\Phi_k = \Phi_{w(k)}\Phi_k$ by Lemma 2.10.

By definition, we have
$$\Phi_w\Phi_k = \Phi_w(T_k(1 - X_{k+1}X_k^{-1}) + (q - 1)X_{k+1}X_k^{-1}),$$
$$\Phi_{w(k)}\Phi_k = (T_{w(k)}(1 - X_{w(k+1)}X_{w(k)}^{-1}) + (q - 1)X_{w(k+1)}X_{w(k)}^{-1})\Phi_k$$
$$= T_{w(k)}\Phi_w(1 - X_{k+1}X_k^{-1}) + (q - 1)\Phi_wX_{k+1}X_k^{-1}.$$

Now $\Phi_w\Phi_k = \Phi_{w(k)}\Phi_k$ implies that
$$\Phi_wT_k(1 - X_{k+1}X_k^{-1}) = T_{w(k)}\Phi_w(1 - X_{k+1}X_k^{-1}).$$
Since $1 - X_{k+1}X_k^{-1}$ is not a zero divisor of $H_n$, it follows that $\Phi_wT_k = T_{w(k)}\Phi_w$. \hfill \Box

Let $H_n$ be the category of finite dimensional $H_n$-modules. For any $M \in H_n$-mod, we denote by $\text{hd}(M)$ the head of $M$ (i.e., the maximal semisimple quotient of $M$), and by $\text{soc}M$ the socle of $M$ (i.e., the maximal semisimple submodule of $M$).

**Definition 2.13.** Let $m, n \in \mathbb{N}$. For each $M \in H_m$-mod, $N \in H_n$-mod, we define the convolution product $M \circ N$ of $M$ and $N$ to be:
$$M \circ N := \text{ind}_{m,n}^m M \boxtimes N \in H_{m+n}$$.mod.
Set $M \triangleleft N := \text{hd}(M \circ N)$. For any $k \in \mathbb{N}$, we define $N^\otimes k := \bigotimes_{\binom{k}{k}} N$.

Let $m, n, k \in \mathbb{N}$. It is well-known that for any $M \in H_m$-mod, $N \in H_n$-mod and $K \in H_k$-mod, there is a canonical $H_{m+n+k}$-module isomorphism:
$$M \circ N \circ K \cong M \circ (N \circ K).$$

**Definition 2.15.** Let $\sigma$ and $\tau$ be the automorphism of $H_n$ which is defined on generators as follows:
$$\sigma : \quad T_i \mapsto -qT_{n-i-1}, \quad X_j \mapsto X_{n+1-j},$$
$$\tau : \quad T_i \mapsto T_i, \quad X_j \mapsto -X_j,$$
for all $i = 1, \cdots, n - 1, j = 1, \cdots, n$.

For any composition $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$ of $n$, we define $\nu^\ast := (\nu_r, \cdots, \nu_2, \nu_1)$ and
$$H_{\nu^\ast} := H_{\nu_1} \boxtimes H_{\nu_1+1} \boxtimes \cdots \boxtimes H_{n-\nu_2+1} \cdots \boxtimes H_n,$$
which is parabolic subalgebra of $H_n$. If $M \in H_{\nu^\ast}$-mod, then we can twist the action with $\sigma$ to get a new module $M^\sigma \in H_{\nu^\ast}$-mod.
Lemma 2.16. Let $M \in \mathcal{H}_{m}-\text{mod}$ and $N \in \mathcal{H}_{n}-\text{mod}$. Then

$$(M \circ N)^{q} \cong N^{q} \circ M^{q}.$$  

Let $1 < e \in \mathbb{N}$, $I := \mathbb{Z}/e\mathbb{Z}$, and $q$ is a primitive $e$th root of unity in $F$. When such a $q$ is fixed, we can identify $I$ with $q^{I} \subset F^{\times}$ via $i \mapsto q^{i}$.

Definition 2.17. Let $\text{Rep}_{I} \mathcal{H}_{n}$ be the full subcategory of $\mathcal{H}_{n}-\text{mod}$ consisting of all modules $M$ such that all eigenvalues of $X_{1}, \cdots, X_{n}$ on $M$ belong to $q^{I}$. If $M \in \text{Rep}_{I} \mathcal{H}_{n}$ then we say that the $\mathcal{H}_{n}$-module $M$ is integral.

In [8] and [9], Grojnowski and Vazirani work for both the non-degenerate type $A$ affine Hecke algebras and the degenerate type $A$ affine Hecke algebras. The theory for the non-degenerate case is parallel to the theory for the degenerate case. Kleshchev [21] gives an excellent account and explanation of Grojnowski’s approach in the case of degenerate type $A$ affine Hecke algebras. In most of the time the results and their proof in Kleshchev’s book [21] can be transformed into the case of non-degenerate affine Hecke algebras without any difficulty. In such case we shall simply cite them as “the non-degenerated version” of the corresponding result in [21] whenever we can not find a suitable reference elsewhere.

Let $M \in \text{Rep}_{I} \mathcal{H}_{n}$. For any $a := (a_{1}, \cdots, a_{n}) \in I^{n}$, let

$$M_{a} := \{x \in M \mid (X_{j} - q^{a_{j}})^{N}x = 0 \text{ for any } 1 \leq j \leq n \text{ and } N \gg 0\}.$$  

We define the character of $M$ to be

\begin{equation}
(2.18) \quad \text{ch } M := \sum_{a \in I^{n}} \dim M_{a}[a_{1}, a_{2}, \cdots, a_{n}].
\end{equation}

For any $1 \leq t \leq n$ and $b := (b_{1}, b_{1+1}, \cdots, b_{n}) \in I^{n-t+1}$, we define

\begin{equation}
(2.19) \quad \Delta_{a}M := \bigoplus_{a := (a_{1}, \cdots, a_{n}) \in I^{n}} M_{a}.
\end{equation}

If $t = n$, $b_{n} = b$, then we write $\Delta_{a}M$ instead of $\Delta_{a}M$, see [21] (5.1). In this case, $\Delta_{a}M$ is simply the generalized $q^{a}$-eigenspace of $X_{n}$ on $M$. That is,

\begin{equation}
(2.20) \quad \Delta_{a}M = \bigoplus_{a := (a_{1}, \cdots, a_{n}) \in I^{n}} M_{a} = \{x \in M \mid (X_{n} - q^{b})^{N}x = 0 \text{ for any } N \gg 0\}.
\end{equation}

It is obvious that $\Delta_{a}M$ is $\mathcal{H}_{n-1,1}$-stable. In general, note that though $\Delta_{a}M$ is still $\mathcal{H}_{-1}$-stable, it is not clear whether it is $\mathcal{H}_{-1,n-t+1}$-stable or not unless $b_{1} = \cdots = b_{n-t} = b_{n} = b$. So in general $\Delta_{a}$ does not define a functor from $\mathcal{H}_{n}$-mod to $\mathcal{H}_{-1,n-t+1}$-mod.

Let $1 \leq a \in \mathbb{N}$. Following [8] §8 (see also [8] (5.6)), we define the functor

$$e_{i} := \text{res}_{n-1}^{n-1} \circ \Delta_{i} : \text{Rep}_{I} \mathcal{H}_{n} \rightarrow \text{Rep}_{I} \mathcal{H}_{n-1},$$

The functor $e_{i}$ is denoted by $e_{i}^{*}$ in the notation of [8] §8. Note that if $e_{i}M \neq 0$, then $e_{i}M$ is a self-dual module by the non-degenerate version of [21] Lemma 7.3.1, Remark 8.2.4, Theorem 8.2.5.

Following [8] and [21], for any $i \in I$ and any simple module $M \in \text{Rep}_{I} \mathcal{H}_{n}$, we define

$$\widehat{e}_{i}M := \text{soc } e_{i}M \in \text{Rep}_{I} \mathcal{H}_{n-1}, \quad \widehat{f}_{i}M := \text{hd}(M \circ L(i)) \in \text{Rep}_{I} \mathcal{H}_{n+1},$$

where $L(i)$ is the simple $\mathcal{H}_{0} = F[x_{1}]$-module on which $x_{1}$ acts as multiplication by $q^{i}$. By [8] Theorem 9.4 (or rather the non-degenerate version of [21] Lemma 5.15, Corollary 5.17), we know that

\begin{equation}
(2.21) \quad \widehat{f}_{i}M \text{ is always nonzero and simple, while } \widehat{e}_{i}M \text{ is either 0 or a simple module.}
\end{equation}

Let $1$ denote the trivial simple module of $\mathcal{H}_{0} \cong K$. For any $(i_{1}, \cdots, i_{n}) \in I^{n}$, we define

\begin{equation}
(2.22) \quad L(i_{1}, i_{2}, \cdots, i_{n}) := \widehat{f}_{i_{n}} \cdots \widehat{f}_{i_{2}} \widehat{f}_{i_{1}} 1.
\end{equation}

Then $0 \neq L(i_{1}, \cdots, i_{n}) \in \text{Rep}_{I} \mathcal{H}_{n}$ is a simple module.

Lemma 2.23. Let $i, j \in I$ with $i - j \neq \pm 1$. There is an isomorphism of functors: $e_{i}e_{j} \cong e_{j}e_{i}$.

Proof. Since the case $i = j$ is trivial, we assume that $i \neq j$. For any $M \in \text{Rep}_{I} \mathcal{H}_{n}$, we use $\Phi : M \rightarrow M$ to denote the map given by left multiplication with $\Phi_{n-1}$. By Lemma 2.10 it is clear that $\Phi$ is an $\mathcal{H}_{n-2}$-module homomorphism. We claim that $\Phi(e_{i}e_{j}M) = e_{j}e_{i}M$. 


In fact, for any \( x \in e_i e_j M \), \((X_n - q^i)^k x = 0 = (X_n - q^j)^k x \) for \( k > 0 \). Therefore, using Lemma 2.10 for any \( k > 0 \),
\[
(X_{n-1} - q^i)^k \Phi(x) = (X_{n-1} - q^i)^k \Phi_{n-1}(x) = \Phi_{n-1}(X_{n-1} - q^i)^k(x) = 0,
\]
which implies that \( \Phi(e_i e_j M) \subseteq e_i e_j M \). Similarly, \( \Phi(e_i e_j M) \subseteq e_i e_j M \). To finish the proof, it suffices to show that left multiplication with \( \Phi^{2}_{n-1} \) defines an \( \mathcal{A}_{n-2} \)-module automorphism of \( e_i e_j M \). But this follows from Lemma 2.10(1) and the assumption that \( i - j \not= \pm 1 \) and \( q = \sqrt{\overline{r}} \) for some \( e > 1 \). \( \square \)

Following [8] and [21], for any \( i \in I \) and any simple module \( M \in \mathcal{X}_{n-2} \)-mod, we define
\[ \epsilon_i(M) := \max\{m \geq 0| \Delta_m M \not= 0\} = \max\{m \geq 0| e_i^m M \not= 0\}. \]

**Lemma 2.24.** Let \( i, j \in I \) with \( i - j \not= \pm 1 \). For any simple module \( M \in \mathcal{X}_{n-2} \)-mod, we have that
\[ \tilde{f}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{f}_i M, \quad \tilde{e}_i \tilde{e}_j M \cong \tilde{e}_j \tilde{e}_i M, \quad \tilde{e}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{e}_i M, \quad \epsilon_i(\tilde{f}_j M) = \epsilon_i(M). \]

**Proof.** Since \( i - j \not= \pm 1 \), \( L(i, j) \cong L(i) \circ L(j) \cong L(j) \circ L(i) \cong L(j, i) \) by the non-degenerate version of [21] Theorem 6.1.4. It is easy to see that \( L(i, j) \) is a real simple module (cf. Definition 3.1). Applying Lemma 3.3, we get that
\[ \tilde{f}_i \tilde{f}_j M = (M \triangledown L(j)) \triangledown L(i) \cong M \triangledown (L(j) \circ L(i)) \cong M \triangledown L(j, i) \cong M \triangledown L(i, j) \]
As a result, we see that if \( N = \tilde{e}_i M \not= 0 \) then
\[ \tilde{e}_i \tilde{f}_j M \cong \tilde{e}_j \tilde{f}_i M \cong \tilde{e}_i \tilde{f}_j N \cong \tilde{f}_j \tilde{e}_i N \cong \tilde{f}_j \tilde{e}_i M; \]
If \( \tilde{e}_i M = 0 \), then \( \tilde{f}_j \tilde{e}_i M = 0 \) by the shuffle lemma [9] Lemma 2.4. In a word, we have \( \tilde{e}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{e}_i M \) for any simple module \( M \). Similarly, \( \tilde{e}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{e}_i M \) for any simple module \( M \). As a consequence, \( \epsilon_i(\tilde{f}_j M) = \epsilon_i(M) \).

It remains to show \( \tilde{e}_i \tilde{e}_j M \cong \tilde{e}_j \tilde{e}_i M \). Assume first that \( \tilde{e}_i M \not= 0 \not= \tilde{e}_j M \), then \( \epsilon_i(M) = \epsilon_i(\tilde{e}_j M) > 0 \) by the last paragraph, and hence \( \tilde{e}_i \tilde{e}_j M \not= 0 \). In a similar way we show that \( \tilde{e}_j \tilde{e}_i M \not= 0 \). Thus in this case we have
\[ \tilde{f}_i \tilde{f}_j (\tilde{e}_i \tilde{e}_j M) \cong \tilde{e}_j \tilde{f}_i \tilde{f}_j (\tilde{e}_i \tilde{e}_j M) \cong \tilde{f}_i \tilde{f}_j (\tilde{e}_i \tilde{e}_j M) \]
which implies (by [8] and [21] Corollary 5.2.4) that \( \tilde{e}_i \tilde{e}_j M \cong \tilde{e}_j \tilde{e}_i M \). And if either \( \tilde{e}_i M = 0 \) or \( \tilde{e}_j M = 0 \), then it is easy to see that \( \tilde{e}_i \tilde{e}_j M = 0 = \tilde{e}_j \tilde{e}_i M \) by the shuffle lemma [9] Lemma 2.4. This completes the proof of the lemma. \( \square \)

From now on and until the end of this section, we assume that \( 1 < \ell \in \mathbb{N} \), \( e = 2\ell \), \( I := \mathbb{Z}/2\mathbb{Z} \), and \( q := \xi \) is a primitive 2\( \ell \)th root of unity in \( F \). To simplify notations, for any \( i \in I \) and \( j \in \overline{I} \), we shall often write \( i + j \in I \) instead of \( i + j + 2\mathbb{Z} \in I \).

**Definition 2.25.** Let \( i \in I \) and set \( \widehat{i} := (i, i + \ell) \in \overline{I}^2 \). If \( n \geq 2 \) then we define the two-steps restriction functor
\[ e_i^{\widehat{i}} := e_i e_{i+\ell} : \text{Rep}_I \mathcal{X}_n \rightarrow \text{Rep}_I \mathcal{X}_{n-2}. \]

For any \( \mathcal{G}_n \)-orbit \( \gamma = \mathcal{G}_n \cdot (\xi_1, \cdots, \xi_n) \) of \( \mathcal{G}_n \) on \( (\xi_1)^n \), it determines a central character
\[ \chi_{\gamma} : Z(\mathcal{X}_n) \rightarrow F, \]
\[ f(X_1, \cdots, X_n) \mapsto f(\xi_1, \cdots, \xi_n), \quad \forall f(X_1, \cdots, X_n) \in K[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{\mathcal{G}_n} = Z(\mathcal{X}_n), \]
where \( Z(\mathcal{X}_n) \) denotes the center of \( \mathcal{X}_n \). We use \( (\mathcal{X}_n\text{-mod})[\gamma] \) to denote the block of \( \text{Rep}_I (\mathcal{X}_n) \) corresponding to the central character \( \chi_{\gamma} \) determined by \( \gamma \). This is the full subcategory of \( \text{Rep}_I (\mathcal{X}_n) \) consisting of \( M \) with \( ch M \in \bigoplus (\xi_1, \cdots, \xi_n) \in \gamma \mathbb{Z}[a_1, \cdots, a_n] \), see [21] (4.3)] for the corresponding definition in the degenerate setting.

**Definition 2.26.** For each \( i \in I \) and \( m \in \mathbb{N} \), we define
\[ \widehat{i}^m := (\widehat{i}, \cdots, \widehat{i}) = (i_1, i_1 + \ell, i_1, i_1 + \ell, \cdots, i_1, i_1 + \ell), \]
and
\[ \gamma_{i_1, m} := \mathcal{S}_m \cdot (\xi_1, -\xi_1, \xi_1, -\xi_1, \cdots, \xi_1, -\xi_1). \]
Lemma 2.27. Let $i \in I$ and $n \in \mathbb{N}$. For any permutation $(i_1, \ldots, i_{2n})$ of $\tilde{i}^n$, 

$$L(\tilde{i}^n) \cong L(i_1) \circ L(i_2) \circ \cdots \circ L(i_{2n}) \cong L(i^n) \circ L((i + \ell)^n)$$

is the unique simple module in $\mathcal{H}_{2n}\text{-mod}[\gamma_{i,n}]$.

Proof. Since $\ell > 1$, by the non-degenerate versions of [21] Lemmas 6.1.1, 6.1.2 we know that

$$(2.28) \quad L(i, i + \ell) \cong L(i) \circ L(i + \ell) \cong L(i) \circ L(i + \ell, i),$$

is simple.

By the same reasoning and using (2.28) and the transitivity of induction functors, we can deduce that for any permutation $(i_1, \ldots, i_{2n})$ of $\tilde{i}^n$, 

$$L(i_1) \circ L(i_2) \circ \cdots \circ L(i_{2n}) \cong L(i) \circ L(i + \ell) \circ \cdots \circ L(i) \circ L(i + \ell) = \text{ind}^2_{2n}(\tilde{i}^n).$$

On the other hand,

$$\text{ind}^2_{2n}(\tilde{i}^n) = L(i) \circ L(i + \ell) \circ \cdots \circ L(i) \circ L(i + \ell)$$

$$\cong L(i) \circ L(i) \circ \cdots \circ L(i) \circ L(i + \ell) \circ \cdots \circ L(i + \ell)$$

$$\cong \left( L(i) \circ L(i) \circ \cdots \circ L(i) \right) \circ \left( L(i + \ell) \circ L(i + \ell) \circ \cdots \circ L(i + \ell) \right)$$

$$\cong L(i^n) \circ L((i + \ell)^n),$$

which is simple as $L(i^n)$ and $L((i + \ell)^n)$ are the classical Kato modules ([9, Proposition 3.3]) of $\mathcal{H}_n$. This shows that $L(\tilde{i}^n) \cong L(i_1) \circ L(i_2) \circ \cdots \circ L(i_{2n})$. Finally, the uniqueness follows from [8, Corollary 5.7] and the non-degenerate versions of [21] Lemmas 6.1.4].

Definition 2.29. Suppose $\ell > 1$. Let $i \in I$ and $n \in \mathbb{N}$. We call the $\mathcal{H}_{2n}$-module $L(\tilde{i}^n)$ a generalized Kato module.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a composition of $n$. We define $2\mu := (2\mu_1, 2\mu_2, \ldots, 2\mu_r)$ which is a composition of $2n$. We use $\pi_{\mu,i}$ to denote the projection from $\text{Rep}_I \mathcal{H}_{2\mu}$ onto the block of $\text{Rep}_I \mathcal{H}_{2\mu}$ corresponding to the $\mathfrak{S}_{2\mu}$-orbit of

$$\left( \xi_1, \xi_1, -\xi_1, \xi_1, -\xi_1, \ldots, -\xi_1, \xi_1 \right).$$

Lemma 2.30. Let $i \in I$, $n \in \mathbb{N}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a composition of $n$. Suppose $\ell > 1$. In the Grothendieck group of $\text{Rep}_I \mathcal{H}_{2\mu}$, $[\pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n)] = s[L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r)]$ for some integer $s > 0$, and $\text{soc} \pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n)$ is simple.

Proof. The first equality follows from Lemma 2.27. Thus, any simple submodule of $\pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n)$ must be isomorphic to $L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r)$.

By the transitivity of the induction functor ind, we know that $\text{ind}_{2\mu}^2 L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r) \cong L(\tilde{i}^n)$. Using Frobenius reciprocity, we can deduce that

$$0 < \text{dim} \text{Hom}_{\mathcal{H}_{2\mu}}(L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r), \pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n))$$

$$= \text{dim} \text{Hom}_{\mathcal{H}_{2\mu}}(L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r), \text{res}_{2\mu}^n L(\tilde{i}^n))$$

$$= \text{dim} \text{Hom}_{\mathcal{H}_{2\mu}}(L(\tilde{i}^1) \circ \cdots \circ L(\tilde{i}^r), L(\tilde{i}^n)) = 1.$$ 

It follows that $\text{dim} \text{Hom}_{\mathcal{H}_{2\mu}}(L(\tilde{i}^1) \boxtimes \cdots \boxtimes L(\tilde{i}^r), \pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n)) = 1$. Thus, $\text{soc} \pi_{\mu,i} \text{res}_{2\mu}^n L(\tilde{i}^n)$ is simple.

Definition 2.31. Let $i \in I$ and $m \in \mathbb{N}$. We define

$$\Delta_{\tilde{i}^m} := \bigoplus_{\mu = (\mu_1, \ldots, \mu_{2m}) \in \Pi_m} \Delta_{\mu}.$$ 

It is clear that for any $M \in \mathcal{H}_m\text{-mod}$, $\Delta_{\tilde{i}^m} M$ is an $\mathcal{H}_{-2m,2m}\text{-mod}$-submodule of $\text{res}_{-2m,2m}^n M$. So $\Delta_{\tilde{i}^m}$ does define a functor $\text{Rep}_I \mathcal{H}_m \to \text{Rep}_I \mathcal{H}_{-2m,2m}$. We have a functorial isomorphism:

$$(2.32) \quad \text{Hom}_{\mathcal{H}_{-2m,2m}}(N \boxtimes L(\tilde{i}^m), \Delta_{\tilde{i}^m} M) \cong \text{Hom}_{\mathcal{H}_m}(N \circ L(\tilde{i}^m), M).$$
Lemma 2.33. Let $i \in I$, $m \in \mathbb{N}$ with $1 \leq m \leq n/2$. Let $M \in \mathcal{H}_n$-mod. Then $\hat{\Delta}_{im} M$ is the largest submodule of $\text{res}_{im-2m}^m M$ such that all of its composition factors are of the form $N \boxtimes L(i^m)$ for some simple module $N \in \mathcal{H}_{n-2m}$-mod.

Proof. This follows from Lemma 2.27.

Let $M \in \text{Rep}_I \mathcal{H}_n$ and $0 \leq m \leq n/2$. We write $\text{ch } M = \sum_{\alpha \in I^m} r_\alpha [a_1, \cdots, a_n]$. Then

$$\text{ch } \hat{\Delta}_{im} M = \sum_{\alpha = (a_1, \cdots, a_n) \in I^m} r_\alpha [a_1, \cdots, a_n],$$

for some $\hat{\alpha} = (a_1, \cdots, a_n) \in I_{\hat{\eta},m}$.

Definition 2.34. Let $M$ be a simple module in $\text{Rep}_I \mathcal{H}_n$. Let $i \in I$. We define

$$\varepsilon_i(M) := \max \{m \geq 0 | \hat{\Delta}_{im} M \neq 0 \}.$$

Corollary 2.35. Let $M$ be a simple module in $\text{Rep}_I \mathcal{H}_n$. Let $i \in I$. Then

$$\varepsilon_i(M) = \max \{m \geq 0 | e_{i}^{m} M \neq 0 \}.$$

Proof. If $e_{i}^{m} M \neq 0$ then it is clear that $\hat{\Delta}_{im} M \neq 0$. Conversely, if $\hat{\Delta}_{im} M \neq 0$, then it follows from Lemma 2.33 and Lemma 2.27 that $e_{i}^{m} M \neq 0$. Hence the corollary follows.

Lemma 2.36. Let $M$ be a simple module in $\text{Rep}_I \mathcal{H}_n$. Let $i \in I$, $\varepsilon := \varepsilon_i(M)$. If $N \boxtimes L(i^m)$ is a simple submodule of $\hat{\Delta}_{im} M$ for some $0 \leq m \leq \varepsilon$, then $\varepsilon_i(N) = \varepsilon - m$.

Proof. By the definition of $\varepsilon_i$ we see that $\varepsilon_i(N) \leq \varepsilon - m$. By (2.32) we get a nonzero (and hence surjective) homomorphism $N \circ L(i^m) \to M$. So, by characters consideration and the Shuffle Lemma [9, Lemma 2.4], we can deduce that $\varepsilon_i(N) + m \geq \varepsilon$. Hence $\varepsilon_i(N) = \varepsilon - m$.

3. SOME REAL SIMPLE MODULES

Throughout this section, unless otherwise specified, we assume that $2 < e \in \mathbb{N}$, $I := \mathbb{Z}/e\mathbb{Z}$, and $q$ is a primitive $e$th root of unity in $F$.

In [13], Kang, Kashiwara et al. develop the theory of real simple modules for quantum affine algebras and for quiver Hecke algebras which is proved to be a very powerful tool. Rouquier has first presented an isomorphism in [28, Proposition 3.18] between certain localized forms of affine Hecke algebras and of quiver Hecke algebras. It follows from the isomorphism that there is an equivalence ([28, Theorem 3.19]) between the category of finite dimensional integral representations of the affine Hecke algebras of type $A$ and the category of finite dimensional representations of the quiver Hecke algebras of type $A$.

Later, using a lift of Brundan-Kleshchev’s isomorphism [3] between cyclotomic Hecke algebras of type $A$ and cyclotomic quiver Hecke algebras of type $A$ to their affine counterparts, the second author of this paper and Fang Li proved in [16, §5] an isomorphism between some generalized Ore localizations of some modified affine Hecke algebras and of the quiver Hecke algebras of type $A$. The equivalence between these two finite dimensional modules categories induced by the isomorphisms is compatible with the convolution products on both sides.

We mimic [18] to give the definition of real simple module in the category of finite dimensional modules over $\mathcal{H}_n$.

Definition 3.1. Let $M \in \text{Rep}_I \mathcal{H}_n$ be a simple module. If $M \circ M$ is simple $\mathcal{H}_n$-module, then we call $M$ a real simple $\mathcal{H}_n$-module.

Example 3.2. For each $i \in I$ and $n \in \mathbb{N}$, the Kato module $L(i^n)$ is a real simple $\mathcal{H}_n$-module. In fact, this is clear because $L(i^n) \circ L(i^n) \cong L(i^{2n})$ is again a simple (Kato) module.

Lemma 3.3. Let $M \in \text{Rep}_I \mathcal{H}_n$ be a real simple module, $N \in \text{Rep}_I \mathcal{H}_m$ be a simple module.

1) $M \circ N$ has a simple head and a simple socle. Similarly, $N \circ M$ has a simple head and a simple socle;

2) if $M \circ N \cong N \circ M$, then $M \circ N$ is a simple $\mathcal{H}_{m+n}$-module. Conversely, if $M \circ N$ is simple, then $M \circ N \cong N \circ M$;

3) if $N$ is a real simple module too and $M \circ N \cong N \circ M$, then $M \circ N$ is a real simple $\mathcal{H}_{m+n}$-module. In particular, for any $k \geq 1$, $M^\circ k$ is a real simple $\mathcal{H}_k$-module;

4) if $M \circ N \cong N \circ M$, then $M \circ N$ is simple.
\begin{proof}
For 1) and 2), they follow from \cite[Theorem 3.2, Corollaries 3.3, 3.4]{18} and \cite[(5.10)]{16} follows from \cite[Corollary 3.9]{18} because the socle of $M \circ N$ is isomorphic to the head of $(M \circ N)^* \cong N^* \circ M^* \cong N \circ M$. Finally, 3) is a consequence of 2).
\end{proof}

We remark that the notion of real simple modules can also be defined for the category of finite dimensional modules over the degenerate affine Hecke algebra of type $A$ in a similar way, and the above lemma also holds in the context of the degenerate affine Hecke algebra of type $A$ as the main results in \cite{16} work for both the non-degenerate and the degenerate affine Hecke algebras of type $A$.

For the rest of this section, we fix $i, j \in I$.

\textbf{Lemma 3.4.} The module $L(i, j)$ is a real simple $\mathcal{H}_2$-module and $L(i, j) \circ L(i, j) \cong L(i^2, j^2)$.

\textbf{Proof.} Suppose $i - j \notin \{\pm 1\}$ then by the non-degenerate version of \cite[Theorem 6.1.4]{21}, $L(i) \circ L(j) \cong L(j) \circ L(i)$ is simple. Since $L(i)$ and $L(j)$ are real simple modules, it follows from Lemma \cite[3.6]{9} that both $L(i) \circ L(j)$ and $L(j) \circ L(i)$ have unique simple heads and hence $L(i, j) \cong L(i) \circ L(j) \cong L(j) \circ L(i) \cong L(i, j)$, from which the lemma follows at once.

Now suppose $i - j \in \{\pm 1\}$. Set $L := L(i, j) \circ L(i, j)$. Recall that $\text{ch}(L(i, j)) = [i, j]$. By the Shuffle Lemma \cite[Lemma 2.4]{9},

\begin{equation}
\text{ch}(L) = 4[i^2, j^2] + 2[i, j, i, j].
\end{equation}

We set $\gamma := \mathfrak{S}_4 \cdot (i, j, i, j)$. By the non-degenerate version of \cite[Lemma 6.2.2]{21},

$$\text{ch}(L) \cong \text{ch}(L(i, j)) \cong 2\text{ch}(i, j, i, j).$$

It follows from \cite[Corollary 3.6]{9} that $L(i^2, j^2) \cong L(i, j, i, j)$.

The block subcategory $(\mathcal{H}_2\text{-mod})[\gamma]$ has at most 4 distinct isoclasses of simple modules. Namely,

$$[L(i^2, j^2)] = [L(i, j, i, j)], \quad [L(j^2, i^2)] = [L(j, i, j, i)], \quad [L(i, j, i, j)], \quad [L(j, i, j, i)].$$

By the non-degenerate version of \cite[Lemma 6.1.1]{21}, $[i_1, i_2, i_3, i_4]$ always appears nonzero coefficient in the character of $\text{ch}(L(i_1, i_2, i_3, i_4))$. By characters consideration and the Shuffle Lemma \cite[Lemma 2.4]{9}, we see that $L(i, j, i, j) \cong L(i^2, j^2)$ is the only composition factor of $L$. Furthermore, $[L : L(i, j, i, j)] \in \{1, 2\}$.

Suppose that $[L : L(i, j, i, j)] = 2$. Then we have a short exact sequence

$$0 \to L(i^2, j^2) \to L \to L(i^2, j^2) \to 0.$$

Applying $e_j$ on the above exact sequence, we get a new exact sequence

$$0 \to e_jL(i^2, j^2) \to e_jL \to e_jL(i^2, j^2) \to 0.$$

It follows that

$$\text{ch}(e_jL) = 4[i^2, j] + 2[i, j, i] = 2\text{ch}(e_jL(i^2, j^2)).$$

On the other hand, by \cite[Theorem 9.13]{9},

$$\text{ch}(e_jL(i^2, j^2)) = 2\text{ch}(e_jL(i^2, j^2)) + \sum_{\alpha} c_{\alpha} \text{ch}(M_{\alpha}) = 2\text{ch}(L(i^2, j^2)) + \sum_{\alpha} c_{\alpha} \text{ch}(M_{\alpha}),$$

where for each $\alpha$, $M_{\alpha} \in \mathcal{H}_2\text{-mod}$ is simple, $c_{\alpha} \in \mathbb{N}$ and $e_j(M_{\alpha}) = 0$. However, by the above calculation, \cite[55]{55} and \cite[336]{336}, we have that

$$\text{ch}(e_jL(i^2, j^2)) = \frac{1}{2} \text{ch}(e_jL) = 2[i^2, j] + [i, j, i],$$

$$2\text{ch}(L(i^2, j^2)) = 2(2[i^2, j] + [i, j, i]) = 4[i^2, j] + 2[i, j, i].$$

We get a contradiction! This proves that $L \cong L(i^2, j^2)$ is simple. Hence $L(i, j)$ is a real simple module.
\end{proof}

\textbf{Lemma 3.7.} For any $m, k \in \mathbb{N}$, $L(i, j)^{ok} \circ L(i^m) \cong L(i^m) \circ L(i, j)^{ok}$ is simple, and $L(j, i)^{ok} \circ L(i^m) \cong L(i^m) \circ L(j, i)^{ok}$ is simple.

\textbf{Proof.} If $i - j \notin \{\pm 1\}$ then the lemma clearly holds because $L(i^m) \cong L(i)^{om}$ and in that case $L(i) \circ L(j) \cong L(j) \circ L(i)$ is simple by the non-degenerate version of \cite[Theorem 6.1.4]{21}. It remains to consider the case when $i - j \in \{\pm 1\}$.

In this case, by Lemma \cite[3.3, 3.4]{55} and Lemma \cite[3.4]{334}, it suffices to prove the lemma for $k = 1 = m$ as $L(i^m) \cong L(i)^{om}$ and both $L(i)$ and $L(i, j)$ are real simple modules. By the non-degenerated version of \cite[Lemma 6.2.2]{21}, we can get that

$$L(i, j) \cong L(i) \circ L(i, j), \quad L(j, i^2) \cong L(j) \circ L(i) \cong L(i) \circ L(j, i).$$
which are both simple by Lemma 3.3. This proves the lemma. □

**Lemma 3.8.** Set \( \gamma_n := \mathcal{G}_n \cdot (\xi^1, \ldots, \xi^n, \xi^2) \), where \( 2 \leq n \in \mathbb{N} \). Suppose \( i - j \in \{\pm 1\} \). The block \( n - 1 \)-copy subcategory \( \left( \mathcal{H}_n \cdot \text{-mod} \right)[\gamma_n] \) has only two distinct isoclasses of simple modules, namely, \([L(i, j, i^{n-2})]\) and \([L(j, i^{n-1})]\). Moreover, \( L(i^r, j, i^s) \cong L(i, j, i^{r+s-1}) \) for any \( r \geq 1, s \geq 0 \).

**Proof.** By [9] Proposition 3.3] and the transitivity of induction functors,
\[
L(i, j) \circ L(i^{n-2}) \cong L(i, j) \circ L(i) \circ \cdots \circ L(i) \quad \text{\textmd{\( n - 2 \) copies}}
\]

By Lemma 3.7, \( L(i, j) \circ L(i^k) \) is simple for any \( k \geq 0 \). It follows that
\[
L(i, j) \circ L(i^{n-2}) \cong L(i, j) \circ L(i)^{o_{n-2}} \cong \tilde{f}_i^{n-2}L(i, j) = L(i, j, i^{n-2}).
\]

Similarly, we have \( L(j, i) \circ L(i^{n-2}) \cong L(j, i^{n-1}) \).

By Lemma 3.7, \( L := L(i^{n-2}) \circ L(i, j) \) is simple, so by Shuffle Lemma [9] Lemma 2.4] \( \epsilon_j(L) = 1 \) and hence \( \tilde{c}_jL \neq 0 \) is forced to be isomorphic to \( L(i^{n-1}) \). Thus \( L \cong L(i^{n-1}, j) \) by [9] Corollary 3.6. Applying Lemma 3.7, \( L \cong L(i, j, i^{n-2}) \).

In general, for any \( r \geq 1, s \geq 0 \), \( L(i^r, j, i^s) \cong \text{hd}(L(i^r, j) \circ L(i^s)) \) by [9] Corollary 3.6(i). Therefore, by Lemma 3.7 and the result we obtained in the last paragraph,
\[
L(i^r, j) \circ L(i^s) \cong L(i^{r-1}) \circ L(i, j) \circ L(i^s) \cong L(i, j) \circ L(i^{r-1}) \circ L(i^s) \cong L(i, j) \circ L(i^{r+s-1})
\]

This proves the second part of the lemma. Finally, note that
\[
\tilde{c}_i^{n-1}L(i, j, i^{n-1}) \cong L(j) \neq 0, \quad \tilde{c}_i^{n-1}L(i, j, i^{n-2}) = \tilde{c}_iL(i, j) = 0.
\]

We see that \( L(j, i^{n-1}) \neq L(i, j, i^{n-2}) \). Now the first part of the lemma follows from the second part and the result we obtained in the last paragraph. □

**Corollary 3.9.** Let \( m \in \mathbb{N} \). Then \( L(i^2, j^2) \circ L(i^m) \cong L(i^m) \circ L(i^2, j^2) \) is simple, and \( L(j^2, i^2) \circ L(i^m) \cong L(i^m) \circ L(j^2, i^2) \) is simple.

**Proof.** This follows from Lemma 3.7 and Lemma 3.4 □

**Corollary 3.10.** Let \( m, k \in \mathbb{N} \). Then both \( L(i^m) \circ L(i, j)^{ok} \) and \( L(i, j)^{ok} \circ L(i^m) \) are real simple \( \mathcal{H}_{m+2k} \)-modules.

**Proof.** By Lemma 3.7, \( L(i^m) \circ L(i, j)^{ok} \cong L(i, j)^{ok} \circ L(i^m) \) and \( L(i^m) \circ L(j, i)^{ok} \cong L(j, i)^{ok} \circ L(i^m) \). Since both \( L(i^m) \cong L(i)^{om}, L(i, j)^{ok} \) and \( L(j, i)^{ok} \) are real simple modules, it follows from Lemma 3.3

**4. The Operators \( \tilde{c}_i, \tilde{f}_i \) and Their Properties**

In this section, we assume that \( 1 < \ell \in \mathbb{N}, q := \xi \) is a primitive \( 2\ell \text{th} \) root of unity in \( F \). In particular, this implies that \( \text{char } F \neq 2 \).

**Lemma 4.1.** Let \( M \) be a simple module in \( \text{Rep}_F \mathcal{H}_n \) and \( i \in I \). Then \( \tilde{c}_i \tilde{c}_{i+\ell}M \cong \tilde{c}_{i+\ell} \tilde{c}_iM \) and \( \tilde{c}_i \tilde{c}_{i+\ell}M \neq 0 \) if and only if \( \epsilon_i \neq 0 \).
Proof. Since \( \ell > 1 \), it follows from Lemma \([2, 22]\) that \( \bar{e}_i \bar{e}_{i+\ell} M \cong \bar{e}_{i+\ell} \bar{e}_i M \).

Suppose that \( e_i M \neq 0 \). By Lemma \([2, 22]\), we have that \( e_i M \neq 0 \neq e_{i+\ell} M \). Hence \( \bar{e}_i M \neq 0 \neq \bar{e}_{i+\ell} M \).

This implies that \( \epsilon_i(M) \geq 1 \geq \epsilon_{i+\ell}(M) \). We set \( N := \bar{e}_{i+\ell} M \neq 0 \). Applying Lemma \([2, 22]\), we can deduce that \( \epsilon_i(N) = \epsilon_{i+\ell}(M) \geq 1 \). Thus \( \bar{e}_i N = \bar{e}_i \bar{e}_{i+\ell} M \neq 0 \) as required.

Conversely, if \( \bar{e}_i \bar{e}_{i+\ell} M \neq 0 \) then \( e_i e_{i+\ell} M \neq 0 \), and hence \( e_i M = e_i e_{i+\ell} M \neq 0 \). This completes the proof of the lemma. \( \square \)

Recall from Definition \([2, 24]\) and Corollary \([2, 33]\) that

\[
\epsilon_i(M) = \max \{ m \geq 0 : \Delta_m, M \neq 0 \} = \max \{ m \geq 0 : e_i^m M \neq 0 \}.
\]

**Lemma 4.2.** Let \( m \in \mathbb{N}, i \in I \) and \( M \in \text{Rep}_I \mathcal{H}_n^\ell \) be a simple module. Then \( \text{hd}(M \circ L(i^m)) \cong (\bar{f}_i \bar{f}_{i+\ell})^m M \).

**Proof.** Since \( L(i) \cong L(i) \circ L(i + \ell) \cong L(i + \ell) \circ L(i) \), it is easy to see that \( (\bar{f}_i \bar{f}_{i+\ell})^m M \) is a quotient of \( M \circ L(i^m) \). Using Lemma \([3, 3] \), we know that \( L(i^m) \) is a real simple module. Now the lemma follows from Lemma \([3, 3] \). \( \square \)

**Proposition 4.3.** Let \( M \) be a simple module in \( \text{Rep}_I \mathcal{H}_n^\ell \). Let \( i \in I \). If \( e_i M \neq 0 \) then \( \text{soc} e_i M \) is simple. Furthermore, \( \text{soc} e_i M \cong \bar{e}_i \bar{e}_{i+\ell} M \) and \( \epsilon_i(\text{soc} e_i M) = \epsilon_i(M) - 1 \).

**Proof.** Being a center element of \( \mathcal{H}_n^\ell \), \( X_1 + \cdots + X_n \) acts as a scalar \( c \) on \( M \). Similarly, the center element \( \sum_{1 \leq i < j \leq n} X_i X_j \) acts as a scalar \( c' \) on \( M \).

Assume that \( e_i M \neq 0 \). Let \( L \subseteq e_i M \) be any simple \( \mathcal{H}_{n-2} \)-submodule. Then the center element \( X_1 + \cdots + X_{n-2} \) acts as the scalar \( c = q^i = q^{i+\ell} \) on \( L \). Similarly, the center element \( \sum_{1 \leq i < j \leq n-2} X_i X_j \) acts as a scalar on \( L \).

Since

\[
X_{n-1} X_n = (X_1 + \cdots + X_n) - (X_1 + \cdots + X_{n-2}) = \sum_{1 \leq i < j \leq n} X_i X_j - \sum_{1 \leq i < j \leq n-2} X_i X_j - (X_1 + \cdots + X_{n-2})(X_{n-1} + X_n),
\]

it follows that both \( X_{n-1} + X_n \) and \( X_{n-1} X_n \) act as scalars on \( L \) and these scalars are invariant when \( L \) varies by block consideration which implies that both \( X_{n-1} + X_n \) and \( X_{n-1} X_n \) act as scalars on \( \text{soc} e_i M \). Note that both \( X_{n-1} + X_n \) and \( X_n \) stabilize \( \text{soc} e_i M \), and \( q^i = q^{i+\ell} \) is the only eigenvalue of \( X_{n-1} \) on \( \text{soc} e_i M \). If either \( A \) or \( B \) is a diagonal matrix, then it is immediate that both \( A \) and \( B \) are diagonal matrices and our claim follows. Suppose that this is not the case. Since \( A \) is a non-diagonal Jordan matrix, \( B \) is upper-triangular and \( A + B = 0 \), it is easy to see that if \( A(i, i+1) = 1 \) then \( B(i, i+1) = -1 \) and hence \( AB = -q^{2i} \). This proves our claim.

Therefore, \( X_{n-1} \) acts as \( q^i \) on \( \text{soc} e_i M \), and \( X_n \) act as \( -q^i \) on \( \text{soc} e_i M \). In particular, any constituent \( L \) of \( \text{soc} e_i M \) contributes a simple submodule of \( \text{res}_{n-2,1,1}^n M \) which is isomorphic to \( L \times L(i) \times L(i + \ell) \).

By Frobenius reciprocity, we have a surjective homomorphism

\[
\text{ind}_{n-2,1,1}^n L \otimes L(i, i + \ell) \cong \text{ind}_{n-2,1,1}^n L \otimes L(i) \otimes L(i + \ell) \twoheadrightarrow M.
\]

Since \( \ell > 1 \), \( L(i, i + \ell) \) is a real simple module by Lemma \([5, 24]\). It follows from Lemma \([8, 33] \) that \( L \circ L(i, i + \ell) \) has a unique simple head. On the other hand, by definition \([21, 22] \), there is a natural surjection:

\[
\text{ind}_{n-2,1,1}^n L \otimes L(i) \otimes L(i + \ell) \twoheadrightarrow \bar{f}_{i+\ell} \bar{f}_i L \neq 0.
\]

It follows that \( M \cong \text{hd}(\text{ind}_{n-2,1,1}^n L \otimes L(i, i + \ell)) \cong \bar{f}_{i+\ell} \bar{f}_i L \). Hence by \([8] \), the non-degenerate version of \([21, 5.2.3] \) and Lemma \([4, 11] \), \( L \cong \bar{e}_i \bar{e}_{i+\ell} M \cong \bar{e}_{i+\ell} \bar{e}_i M \).
Applying Frobenius reciprocity together with the proof in the above two paragraphs, we get that
\[
\dim \text{Hom}_{\mathcal{H}_{n-2}}(\tilde{e}_i \tilde{e}_{i+\ell} M, e_i e_{i+\ell} M) = \dim \text{Hom}_{\mathcal{H}_{n-2,1,1}}(\tilde{e}_i \tilde{e}_{i+\ell} M \boxtimes L(i) \boxtimes L(i + \ell), \text{res}_{n-2,1,1}^{n} M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n}}(\text{ind}_{n-2,1,1}^{n} \tilde{e}_i \tilde{e}_{i+\ell} M \boxtimes L(i) \boxtimes L(i + \ell), M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n}}(\text{ind}_{n-2,2}^{n} \tilde{e}_i \tilde{e}_{i+\ell} M \boxtimes L(\hat{i}), M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n}}(\text{ind}_{n-2,2}^{n} L \boxtimes L(\hat{i}), M) = 1.
\]
Thus \(\text{soc} \ e_i M \cong \tilde{e}_i \tilde{e}_{i+\ell} M\) is simple.

Apply Lemma 2.36 to the case \(m = 1\), we get \(e_i(\text{soc} \ e_i M) = e_i(M) - 1\).

**Definition 4.4.** Let \(i \in I\) and \(M\) a simple module in \(\text{Rep}_I \mathcal{H}_n\). We define
\[
\tilde{e}_i(M) := \text{soc} \circ \epsilon_i(M), \quad \check{f}_i(M) := \text{hd}(\text{ind}_{n-2,2}^{n} M \boxtimes L(i, i + \ell)).
\]

**Corollary 4.5.** Let \(i \in I\) and \(M\) a simple module in \(\text{Rep}_I \mathcal{H}_n\). Then
\[
\tilde{e}_i(M) \cong \tilde{e}_i \tilde{e}_{i+\ell} M \cong \tilde{e}_i e_i M, \quad \check{f}_i(M) \cong \check{f}_i \check{f}_{i+\ell} M \cong \check{f}_{i+\ell} \check{f}_i M,
\]
\[
\epsilon_i(M) = \max \{m \geq 0 | \tilde{e}_i^m M \neq 0\} = \max \{m \geq 0 | (\tilde{e}_i \tilde{e}_{i+\ell})^m M \neq 0\} = \min \{\epsilon_i(M), e_{i+\ell}(M)\}
\]

Furthermore, if \(N \in \text{Rep}_I \mathcal{H}_{n+2}\) is a simple module, then \(\check{f}_i M \cong N\) if and only if \(\tilde{e}_i N \cong M\).

By Corollary 4.5 it is clear that for any \(n \geq 1\) and any simple module \(M \in \text{Rep}_I \mathcal{H}_n\),
\[
0 \neq \tilde{e}_i M \text{ is simple if and only if } \epsilon_i(M) > 0.
\]

The following lemma is a “hat” analogue of [21] Lemma 5.1.4, Theorem 5.1.6 and Lemma 5.2.1. However, our proof is different. In fact, we remark that [21] Theorem 5.1.6 and Lemma 5.2.1 and their proof cannot be transformed directly into our “hat” set-up because the naive “hat” version of [21] Lemma 5.1.4 does not hold unless we have the extra assumption that \(\epsilon_i(M) = e_{i+\ell}(M)\).

**Lemma 4.7.** Let \(M\) be a simple module in \(\text{Rep}_I \mathcal{H}_n\). Let \(i \in I\), \(\epsilon := \epsilon_i(M)\).

1) For any \(0 \leq m \leq \epsilon\), \(\text{soc} \tilde{\Delta}_n M \cong (\tilde{e}_i \tilde{e}_{i+\ell})^m M \boxtimes L(\hat{m})\) with \(\epsilon_i((\tilde{e}_i \tilde{e}_{i+\ell})^m M) = \epsilon - m\).

2) If \(\epsilon_i(M) = e_{i+\ell}(M)\), then \(\tilde{\Delta}_n M = \text{soc} \tilde{\Delta}_n M\).

**Proof.** 1) Let \(N \boxtimes L(\hat{m})\) be a simple submodule of \(\tilde{\Delta}_n M\), then by Frobenius reciprocity (2.32) and Lemma 4.2, \(N \cong (\tilde{e}_i \tilde{e}_{i+\ell})^m \epsilon_i M\). Since
\[
\dim \text{Hom}_{\mathcal{H}_{n-2m,2m}}((\tilde{e}_i \tilde{e}_{i+\ell})^m \epsilon_i M \boxtimes L(\hat{m}), \tilde{\Delta}_n M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n-2m,2m}}((\tilde{e}_i \tilde{e}_{i+\ell})^m \epsilon_i M \boxtimes L(\hat{m}), \text{res}_{n-2m,2m}^{n} M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n}}((\tilde{e}_i \tilde{e}_{i+\ell})^m \epsilon_i M \boxtimes L(\hat{m}), M)
\]
\[
= \dim \text{Hom}_{\mathcal{H}_{n}}((\tilde{e}_i \tilde{e}_{i+\ell})^m \epsilon_i M \boxtimes L(\hat{m}), \check{f}_i M) = 1. \quad \text{(as } L(\hat{m}) \text{ is real by Lemma 3.3)}
\]
This proves that \(\text{soc} \tilde{\Delta}_n M\) is simple. The second part of 1) follows from Lemma 2.36.

2) Suppose \(\epsilon_i(M) = e_{i+\ell}(M)\). By Lemma 2.36, we can assume \(N \boxtimes L(\hat{\epsilon})\) is a simple submodule of \(\tilde{\Delta}_n M\), where \(N \in \text{Rep}_I(\mathcal{H}_{n-2})\) is simple with \(\epsilon_i(N) = 0\). Applying (2.32) and Lemma 4.2, we can deduce that \(M\) is the unique simple head of \(N \circ L(\hat{\epsilon})\) and \(\epsilon_i(N) = \epsilon_i(M) - \epsilon = e_{i+\ell}(M) - \epsilon \equiv e_{i+\ell}(N)\). In particular, \(\epsilon_i(N) = 0 = e_{i+\ell}(N)\) by Corollary 4.5 and \(\tilde{\Delta}_n M\) is a quotient of \(\tilde{\Delta}_n N \circ L(\hat{\epsilon})\). Using the shuffle Lemma 9 and the equality \(\epsilon_i(N) = 0 = e_{i+\ell}(N)\), we get that \(\tilde{\Delta}_n (N \circ L(\hat{\epsilon})) \cong N \boxtimes L(\hat{\epsilon})\). Thus \(\tilde{\Delta}_n M \cong N \boxtimes L(\hat{\epsilon})\) is simple and \(N \cong (\tilde{e}_i \tilde{e}_{i+\ell})^m M\) by the result proved in 1). This proves 2).

**Corollary 4.8.** Let \(i, j \in I\) and \(M\) be a simple module in \(\text{Rep}_I \mathcal{H}_n\). Suppose \(i \not\in \{j \pm 1, j + \ell \pm 1\}\). Then
\[
\tilde{e}_i \tilde{e}_j M \cong \tilde{e}_{j+1} \tilde{e}_i M, \quad \check{f}_i \check{f}_j M \cong \check{f}_i \check{f}_{j+1} M, \quad \tilde{e}_i \tilde{e}_{j+1} M \cong \tilde{e}_{j+1} \tilde{e}_i M, \quad \epsilon_i(\check{f}_j M) = \epsilon_i(M).
\]

**Proof.** The first three isomorphisms follows from Lemma 2.21 and Corollary 4.5. The fourth equality follows from the fourth equality in Lemma 2.24 by noting that \(\epsilon_i(M) = \min(\epsilon_i(M), e_{i+\ell}(M))\).
Lemma 4.9. Let $i \in I, 0 \leq m \leq n/2$. Suppose that $N \in \text{Rep}_I \mathcal{H}_{n-2m}$ is simple with $\varepsilon = \varepsilon_i(N)$. Set $M := N \circ L(\hat{m})$. Then

1) $K := \text{hd}(M)$ is simple with $\varepsilon(K) := m + \varepsilon$;
2) If $\varepsilon_i(N) = \varepsilon_{i+\ell}(N)$, then all the other composition factors $L$ of $M$ have $\varepsilon_i(L) < m + \varepsilon$.

Proof. 1) By Lemma 3.4, $L(\hat{m})$ is a real simple module. Applying Lemma 3.3 we deduce that $K := \text{hd}(M)$ is simple. Using Lemma 1.2 and Corollary 4.5 we see that $K \cong M/\text{hd}(M)$.

2) By Corollary 4.6 and assumption, we have $\varepsilon_i(N) = \varepsilon_{i+\ell}(N) = \varepsilon_i(N) = \varepsilon$. Applying Lemma 3.3 we get a natural surjection

$$M \cong N \circ L(\hat{m}) \circ L((i + \ell)m) \to (N \triangleright L(\hat{m})) \triangleright L((i + \ell)m) \cong \text{hd}(M)$$

as $M$ has a unique simple head.

Applying Lemma 3.5, we can deduce that $\varepsilon_i(N \triangleright L(\hat{m})) = \varepsilon_i(N \triangleright M) = \varepsilon + m$, while all the other composition factors $L_1$ of $N \circ L(\hat{m})$ have $\varepsilon_i(L_1) < \varepsilon + m$. Hence for those $L_1$, $\varepsilon_i(L') \leq \varepsilon_i(L') < m + \varepsilon$ where $L'$ is any composition factor of $L_1 \circ L((i + \ell)m)$. Now by Lemma 2.24, $\varepsilon_i(N \triangleright L(\hat{m})) = \varepsilon_{i+\ell}(N) = \varepsilon$. Applying Lemma 3.3 again, we see that if $L$ is a composition factor of $(N \triangleright L(\hat{m})) \circ L((i + \ell)m)$ which is not equal to $(N \triangleright L(\hat{m})) \triangleright L((i + \ell)m)$, then $\varepsilon_i \leq \varepsilon_{i+\ell}(L) < m + \varepsilon$. This completes the proof of 2.

5. The crystal $\hat{B}(\infty)$

In this section we shall give the main results Theorem 5.20 and Theorem 5.23 of this paper. Throughout we assume that $\ell \geq 2, q := \xi \in F$ is a primitive $2\text{nd}$ root of unity in $F$, $I := \mathbb{Z}/2\mathbb{Z}$.

We fix an embedding $\theta : \mathbb{Z}/\ell\mathbb{Z} \hookrightarrow I, i + \ell\mathbb{Z} \mapsto i + 2\mathbb{Z}$ for $i = 0, 1, 2, \ldots, \ell - 1$. By some abuse of notations, for any $i \in \mathbb{Z}/\ell\mathbb{Z}$, we set $\tilde{i} := \theta(i)$.

We define

$$L(\tilde{i}_1, \ldots, \tilde{i}_n) := L(\theta(i_1), \theta(i_1) + \ell, \ldots, \theta(i_n)).$$

As a special case of (2.22), we have $L(\tilde{i}_1, \ldots, \tilde{i}_n) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_n}.1$.

Definition 5.1. We define

$$\hat{B}(\infty) := \{ L(\tilde{i}_1, \ldots, \tilde{i}_n) \mid n \in \mathbb{N}, i_1, \ldots, i_n \in \mathbb{Z}/\ell\mathbb{Z} \}.$$

For each $n \in \mathbb{N}$, we define $\hat{B}(n) := \{ L(\tilde{i}_1, \ldots, \tilde{i}_n) \mid i_1, \ldots, i_n \in \mathbb{Z}/\ell\mathbb{Z} \}$. In particular, $\hat{B}(\infty) = \bigsqcup_{n \in \mathbb{N}} \hat{B}(n)$.

Recall the definition of the automorphism $\tau$ of $\mathcal{H}_n$ in Definition 2.15.

Proposition 5.2. $\hat{B}(n)$ consists of the iso-classes of those simple modules $L$ in $\text{Rep}_I \mathcal{H}_{2n}$ such that $L^\tau \simeq L$.

Proof. We prove this by induction on $n$. For $n = 0$, it is trivial. Assume that $n > 0$ and the result holds for $n - 1$. By (2.22) and characters consideration we can deduce that $L(i, i + \ell)^\tau \simeq L(i, i + \ell)$. For $M \in \hat{B}(n)$, there exists $N \in \hat{B}(n-1)$, s.t. $M = N \triangleright L(i, i + \ell)$ for some $i$ by definition. As $N \simeq N^\tau$ by induction hypothesis, $(N \circ L(i, i + \ell))^\tau \simeq N \circ L(i, i + \ell)$. This implies that $M^\tau \simeq M$.

Conversely, if $M \in \text{Rep}_I \mathcal{H}_{2n}$ is s.t. $M^\tau \simeq M$, then $(e_iM)^\tau \simeq e_{i+\ell}M$ and therefore $\varepsilon_i(M) \neq 0 \iff \varepsilon_{i+\ell}(M) = 0$. For such an $i$, $(e_i e_{i+\ell} M)^\tau \simeq e_{i+\ell}e_i M \neq 0$, hence $\tilde{e}_iM \in \hat{B}(n-1)$ by induction hypothesis, and $M \in \hat{B}(n)$.

The following result is a key step in the proof of our main results Theorem 5.20 and Theorem 5.23 of this paper.

Theorem 5.3. Let $M \in \hat{B}(n)$. Then

$$\tilde{e}_iM = \varepsilon_i(M) = \varepsilon_{i+\ell}(M),$$

and

$$\tilde{f}_iM \in \hat{B}(n+1), \quad \tilde{e}_iM \in \hat{B}(n-1) \cup \{0\}.$$

Furthermore, if $n \geq 1$ then $\tilde{e}_iM \in \hat{B}(n-1)$ if and only if $\varepsilon_i(M) > 0$. 

Proof. \( \tilde{c}_i M \in \bar{B}(n - 1) \cup \{0\} \) has been proved in the last proposition and \( \tilde{f}_i M \in \bar{B}(n + 1) \) follows from definition. To show the equalities \( c_i^*(M) \), as \( c_i^*(M) = \min\{c_i(M), c_{i+\ell}(M)\} \), it suffices to show the second one. This follows from the fact that \( (c_i^m M)^\sigma \simeq c_{i+\ell}^m(M^\sigma) \).

\[ \]

**Definition 5.6.** Let \( M \in \operatorname{Rep}_I \mathcal{H}_n \) be a simple module, \( i \in I \) and \( 0 \leq m \leq n \). We define

\[
\begin{align*}
\tilde{c}_i^m M &:= (\tilde{c}_i(M^\sigma))^\sigma, \\
\tilde{f}_i^m M &:= (\tilde{f}_i(M^\sigma))^\sigma \cong \operatorname{hd}(L(i) \circ M), \\
\epsilon_i^*(M) &:= c_i(M^\sigma) := \max\{m \geq 0 | (\tilde{c}_i^m M \neq 0)\}, \\
\tilde{c}_i^0 M &:= \tilde{c}_i \tilde{c}_{i+\ell}^n M, \\
\tilde{f}_i^0 M &:= \tilde{f}_i \tilde{f}_{i+\ell}^n M, \\
\epsilon_i^0(M) &:= \max\{m \geq 0 | (\tilde{c}_i^m M \neq 0)\}.
\end{align*}
\]

Let \( i \in I \) and \( M \in \operatorname{Rep}_I \mathcal{H}_n \) be a simple module. By the non-degenerate version of [21] Lemma 10.1.3, we have that \( \epsilon_i^*(\tilde{f}_i M) \in \{\epsilon_i^*(M), \epsilon_i^*(M) + 1\} \), and for any \( i \neq j \in I \), \( \epsilon_i^*(\tilde{f}_j M) = \epsilon_i^*(M) \).

**Lemma 5.7.** Let \( i \in I \) and \( M \in \operatorname{Rep}_I \mathcal{H}_n \) be a simple module. The following statements are equivalent:

1. \( \epsilon_i(\tilde{f}_i^2 M) = \epsilon_i(M) + 1 \);
2. \( M \circ L(i) \cong L(i) \circ M \);
3. \( M \circ L(i) \cong L(i) \circ M \) is simple;
4. \( \tilde{f}_i M \cong \tilde{f}_i^2 M \).

Proof. We shall prove that \((4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4)\). If \( \tilde{f}_i M \cong \tilde{f}_i^2 M \), then \( M \cong L(i) \circ M \). Note that \( L(i) \) is a real simple module. Applying Lemma 5.3, we can deduce that \( M \circ L(i) \cong L(i) \circ M \) is simple. In particular,

\[
\epsilon_i(\tilde{f}_i^2 M) = \epsilon_i(\operatorname{hd}(L(i) \circ M)) = \epsilon_i(\operatorname{hd}(M \circ L(i))) = \epsilon_i(\tilde{f}_i M) = \epsilon_i(M) + 1.
\]

Conversely, assume that \( \epsilon_i(\tilde{f}_i^2 M) = \epsilon_i(\tilde{f}_i M) = \epsilon_i(M) + 1 \). Applying [21] Lemma 10.1.6, \( \tilde{c}_i \tilde{f}_i^2 M \cong M \), which implies that \( \tilde{f}_i M \cong \tilde{f}_i^2 M \).

**Lemma 5.8.** Let \( i \in I \) and \( M \in \bar{B}(n) \). The following statements are equivalent:

1. \( M \circ L(i) \cong M \circ L(i) \);
2. \( M \circ L(i) \cong L(i) \circ M \);
3. \( \epsilon_i(\tilde{f}_i^2 M) = \epsilon_i(M) + 1 \);
4. \( \epsilon_{i+\ell}(\tilde{f}_{i+\ell}^2 M) = \epsilon_{i+\ell}(M) + 1 \).

In this case, we have \( \tilde{f}_i^2 M \cong \tilde{f}_i M \in \bar{B}(n + 1) \).

Proof. By Lemma 5.7, it suffices to prove \((1) \Rightarrow (2)\). Suppose that \( L(i) \circ M \cong M \circ L(i) \). Let \( c_{q^f} \) be the automorphism of \( \mathcal{H}_n \) which is defined on the generators by

\[ c_{q^f} : T_i \mapsto T_i, \quad X_j \mapsto q^f X_j, \]

for \( i = 1, \ldots, n - 1, j = 1, \ldots, n \). The automorphism \( c_{q^f} \) induces a covariant automorphism of categories:

\[ c_{q^f} : \operatorname{Rep}_I \mathcal{H}_n \cong \operatorname{Rep}_I \mathcal{H}_n \text{ such that } c_{q^f}(L(i_1, \ldots, i_n)) \cong L(i_1 + \ell, \ldots, i_n + \ell) \text{ for any } i_1, \ldots, i_n \in I. \]

Since \( \ell > 1 \), it follows from Lemma 2.2.24 that \( c_{q^f}(L(\tilde{i}_1, \ldots, \tilde{i}_n)) \cong L(\tilde{i}_1, \ldots, \tilde{i}_n) \) for any \( L(\tilde{i}_1, \ldots, \tilde{i}_n) \in \bar{B}(n) \). In particular, \( c_{q^f}(M) \cong M \). Therefore,

\[
L(i + \ell) \circ M \cong c_{q^f}(L(i)) \circ M \cong c_{q^f}(L(i) \circ M) \cong c_{q^f}(M \circ L(i)) = M \circ L(i + \ell).
\]

In this case, by Lemma 5.3

\[
\tilde{f}_i^2 M = \tilde{f}_i \tilde{f}_i^2 M \cong L(i) \circ L(i + \ell) \circ M \\
\cong M \circ L(i + \ell) \circ L(i) \\
= \tilde{f}_i \tilde{f}_{i+\ell} M \\
\cong \tilde{f}_i M \in \bar{B}(n + 1).
\]

**Corollary 5.9.** Let \( i \in I \) and \( M \in \bar{B}(n) \). The following statements are equivalent:

1. \( \epsilon_i(\tilde{f}_i M) = \epsilon_i^*(M) + 1 \);
2. \( M \circ L(i) \cong L(i) \circ M \) is simple;
3. \( M \circ L(i + \ell) \cong L(i + \ell) \circ M \).
Theorem 5.10. Let \( n \in \mathbb{N} \). Then
\[
\hat{B}(n) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_n} \mid i_1, \ldots, i_n \in \mathbb{Z}/(\mathbb{Z}) \}.
\]
For any \( i \in I \) and \( M \in \hat{B}(n) \), \( \tilde{f}_i M = \tilde{B}(n+1) \) and \( \tilde{c}_i M = \tilde{B}(n-1) \cup \{0\} \). Furthermore, \( \varepsilon_i^+(M) = \varepsilon_i^+(M) = \varepsilon_i^+(M) \), and \( \tilde{c}_i M = \tilde{B}(n-1) \) if and only if \( \varepsilon_i^+(M) > 0 \).

Using Theorems 5.3, 5.10 and the non-degenerated version of [21, 10.1.1–10.1.7] we can immediately get the “hat” version of all the results in [21 §10.1].

Lemma 5.11. Let \( M \in \hat{B}(n) \).
1) For any \( i \in \mathbb{Z}/(\mathbb{Z}) \), \( \varepsilon_i(\tilde{f}_i M) = \varepsilon_i(M) + 1 \).
2) For any \( i, j \in \mathbb{Z}/(\mathbb{Z}) \) with \( i \neq j \), we have \( \varepsilon_i(\tilde{f}_j M) = \varepsilon_i(M) \).

Lemma 5.12. Let \( M \in \hat{B}(n) \) and \( i, j \in \mathbb{Z}/(\mathbb{Z}) \). Assume that \( \varepsilon_i(\tilde{f}_j M) = \varepsilon_i(M) \), \( \varepsilon := \varepsilon_i(M) \). Then \( \tilde{c}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{c}_i M \).

Lemma 5.13. Let \( M \in \hat{B}(n) \).
1) For any \( i \in \mathbb{Z}/(\mathbb{Z}) \), \( \varepsilon_i(\tilde{f}_i M) = \varepsilon_i(M) + 1 \).
2) For any \( i, j \in \mathbb{Z}/(\mathbb{Z}) \) with \( i \neq j \), we have \( \varepsilon_i(\tilde{f}_j M) = \varepsilon_i(M) \).

Lemma 5.14. Let \( M \in \hat{B}(n) \) and \( i, j \in \mathbb{Z}/(\mathbb{Z}) \). Assume that \( \varepsilon_i^+(\tilde{f}_j M) = \varepsilon_i^+(M) \), \( a := \varepsilon_i^+(M) \). Then \( \tilde{c}_i \tilde{f}_j M \cong \tilde{f}_j \tilde{c}_i M \).

Corollary 5.15. Let \( M \in \hat{B}(n) \) and \( i \in \mathbb{Z}/(\mathbb{Z}) \). Let \( M_1 := \varepsilon_i^+(M) \), \( M_2 := \tilde{c}_i \varepsilon_i^+(M) \). Then \( \varepsilon_i^+(M) = \varepsilon_i(M_1) \) if and only if \( \varepsilon_i(M) = \varepsilon_i(M_2) \).

Lemma 5.16. Let \( M \in \hat{B}(n) \) and \( i \in \mathbb{Z}/(\mathbb{Z}) \) satisfying \( \varepsilon_i(\tilde{f}_j M) = \varepsilon_i(M) + 1 \). Then \( \tilde{c}_i \tilde{f}_j M \cong M \).

Lemma 5.17. Let \( M \in \hat{B}(n) \) and \( i \in \mathbb{Z}/(\mathbb{Z}) \) satisfying \( \varepsilon_i^+(\tilde{f}_j M) = \varepsilon_i^+(M) + 1 \). Then \( \tilde{c}_i \tilde{f}_j M \cong M \).

Let \( \hat{\mathfrak{g}}_{2\ell} \) be the affine Lie algebra of type \( \hat{A}_{2\ell-1} \). Let \( \{ \alpha_i \mid i \in I \} \) (resp., \( \{ h_i \mid i \in I \} \)) be the set of simple roots (resp., coroots) of \( \hat{\mathfrak{g}}_{2\ell} \). We define
\[
B(\infty) := \{ L(i_1, \ldots, i_n) \mid n \in \mathbb{N}, i_1, \ldots, i_n \in I \}.
\]
For any \( M = L(i_1, \ldots, i_n) \in B(\infty) \), where \( i_1, \ldots, i_n \in I \), we define the weight function “wt” by \( \text{wt}(M) := -\gamma \) such that
\[
\gamma = \sum_{i \in I} \gamma_i \alpha_i, \quad \text{where } \gamma_i = \# \{ 1 \leq j \leq n \mid i_j = i \}, \quad \forall i \in I.
\]

For each \( i \in I \), we define \( \varphi_i(M) := \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle \). Let \( U_o(\hat{\mathfrak{g}}_{2\ell})^- \) be the negative part of the quantized enveloping algebra \( U_o(\hat{\mathfrak{g}}_{2\ell}) \) of the affine Lie algebra \( \hat{\mathfrak{g}}_{2\ell} \). Let \( P, Q \) be the weight lattice and root lattice of \( \mathfrak{g}_{\ell} \) respectively. Let \( \{ \hat{\alpha}_i \mid i \in \mathbb{Z}/(\mathbb{Z}) \} \) be the set of fundamental dominant weights of \( \hat{\mathfrak{g}}_{\ell} \).
Theorem 5.20. The set $\hat{B}(\infty)$, the functions $\varepsilon_i, \varphi_i, \hat{w}t$, together with the operators $\tilde{c}_i, \tilde{f}_i$ form a crystal in the sense of Kashiwara [19, §7.2].

Proof. This follows from Theorems 5.3 and Corollary 4.5 and the definitions of the functions $\varepsilon_i, \varphi_i, \hat{w}t$. □

Following [19], for each $i \in \mathbb{Z}/\ell\mathbb{Z}$, we have the crystal $B_i = \{b_i(n) | n \in \mathbb{Z}\}$ with functions

$$
\varepsilon_j(b_i(n)) := \begin{cases} 
-n, & \text{if } j = i, \\
-\infty, & \text{if } j \neq i;
\end{cases}
\varphi_j(b_i(n)) := \begin{cases} 
n, & \text{if } j = i, \\
-\infty, & \text{if } j \neq i;
\end{cases}
$$

and $\text{wt}(b_i(n)) := n\alpha_i$, and operators

$$
\tilde{c}_j(b_i(n)) := \begin{cases} 
b_i(n+1), & \text{if } j = i, \\
0, & \text{otherwise};
\end{cases}
\tilde{f}_j(b_i(n)) := \begin{cases} 
b_i(n-1), & \text{if } j = i, \\
0, & \text{otherwise};
\end{cases}
$$

We set $b_i := b_i(0)$. We define a map $\Psi_i : \hat{B}(\infty) \rightarrow \hat{B}(\infty) \otimes B_i$ which sends each $[M] \in \hat{B}(\infty)$ to $[\tilde{c}_i^n M] \otimes \tilde{f}_i^n b_i$, where $a := \varepsilon_i^n(M)$.

The proof of the following two lemmas is completely the same as the proof of [21, Lemma 10.3.1, Lemma 10.3.2] by using Lemmas 5.11, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17. We leave the details to the readers.

Lemma 5.21. Let $M \in \hat{B}(\infty)$ and $i, j \in \mathbb{Z}/\ell\mathbb{Z}$ with $i \neq j$. We set $a := \varepsilon_i^n(M)$.

(i) $\varepsilon_i(M) = \varepsilon_i^n([\tilde{c}_i^n M])$;
(ii) If $\varepsilon_i(M) > 0$, then $\varepsilon_i^n(\tilde{c}_j M) = \varepsilon_i^n(M)$ and $\tilde{c}_i^n \tilde{c}_j M \cong \tilde{c}_i^n M$.

Lemma 5.22. Let $M \in \hat{B}(\infty), i \in \mathbb{Z}/\ell\mathbb{Z}$. Set $a := \varepsilon_i^n(M)$ and $L := (\tilde{c}_i^n) M$.

(i) $\varepsilon_i(M) = \max\{\varepsilon_i(L), a - \langle h_i, \text{wt}(L) \rangle\}$;
(ii) If $\varepsilon_i(M) > 0$, then

$$
\varepsilon_i^n(\tilde{c}_i M) = \begin{cases} 
a, & \text{if } \varepsilon_i(L) \geq a - \langle h_i, \text{wt}(L) \rangle; \\
a - 1, & \text{otherwise}.
\end{cases}
$$

(iii) If $\varepsilon_i(M) > 0$, then

$$
(\tilde{c}_i^n) M \cong \begin{cases} 
\tilde{c}_i L, & \text{if } \varepsilon_i(L) \geq a - \langle h_i, \text{wt}(L) \rangle; \\
L, & \text{otherwise},
\end{cases}
$$

where $b := \varepsilon_i^n(\tilde{c}_i M)$.

Theorem 5.23. The crystal $\hat{B}(\infty)$ is isomorphic to Kashiwara’s crystal associated to the crystal base of $U_q(\mathfrak{gl}_\ell^-)$.

Proof. This follows from a similar argument used in the proof of [21, Theorem 10.3.4] via using Lemma 5.21 and Lemma 5.22 and [20, Proposition 3.2.3]. □
6. The crystal $\hat{B}(\Lambda_0 + \Lambda_\ell)$ and the Multiplicity Two Theorem

Throughout we assume that $\text{char } F \neq 2$, $\ell \geq 2$, $q := \xi \in F$ is a primitive $2\ell$th root of unity in $F$, $I := \mathbb{Z}/2\mathbb{Z}$.

Recall that $\hat{a}_{2\mathbb{Z}}$ is the affine Lie algebra of type $\hat{A}_{2\ell-1}$, $\{a_i | i \in I\}$ (resp., $\{h_i | i \in I\}$) is the set of simple roots (resp., coroots) of $\hat{a}_{2\mathbb{Z}}$, and $P, Q$ are the weight lattice and root lattice of $\hat{a}_{2\mathbb{Z}}$ respectively. Let $Q^+_n := \{\alpha = \sum_{i \in I} \gamma_i a_i \in Q | \sum_{i \in I} \gamma_i = n, \gamma_i \in \mathbb{N}, \forall i \in I\}$. Let $\{\Lambda_i | i \in I\}$ be the set of fundamental dominant weights of $\hat{a}_{2\mathbb{Z}}$. Let $P^+ := \sum_{i \in I} \Lambda_i$ be the set of integral dominant weights.

For any $\Lambda \in P^+$, we define $J_\Lambda$ to be the two-sided ideal of $\mathcal{H}_n$ generated by $\prod_{i \in I}(X_1 - \xi^{(h_i, \Lambda)})$. We define the non-degenerate cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ to be the quotient

$$\mathcal{H}_n^\Lambda := \mathcal{H}_n/J_\Lambda.$$  

In this section, we are mostly interested in the special level two case, i.e., when $\Lambda = \lambda := \Lambda_0 + \Lambda_\ell$.

**Definition 6.1.** The Iwahori–Hecke algebra $\mathcal{H}_n^\lambda$ of type $B_n$ is defined to be the quotient $\mathcal{H}_n^\lambda := \mathcal{H}_n/J_\lambda$, where $J_\lambda$ is the two-sided ideal of $\mathcal{H}_n$ generated by $(X_1 - 1)(X_1 + 1) \in \mathcal{H}_n$.

Following [21], we have two natural functors

$$\text{pr}^\lambda : \text{Rep}_I \mathcal{H}_n \to \mathcal{H}_n^\lambda\text{-mod}, \quad \text{infl}^\lambda : \mathcal{H}_n^\lambda\text{-mod} \to \text{Rep}_I \mathcal{H}_n,$$

where $\text{pr}^\lambda M := M/J_\lambda M$, and $\text{infl}^\lambda$ is the natural inflation along the epimorphism $\pi_\lambda : \mathcal{H}_n \to \mathcal{H}_n^\lambda$. The functor $\text{infl}^\lambda$ is right adjoint to $\text{pr}^\lambda$.

Following [8] and [21], for each $i \in I$ and each simple module $M \in \mathcal{H}_n^\lambda\text{-mod}$, we define the action of cyclotomic crystal operators:

$$\bar{e}_i^\lambda M := \text{pr}^\lambda \circ \bar{e}_i \circ \text{infl}^\lambda M, \quad \bar{f}_i^\lambda M := \text{pr}^\lambda \circ \bar{f}_i \circ \text{infl}^\lambda M.$$

By [8], we know that both $\bar{e}_i^\lambda$ and $\bar{f}_i^\lambda$ define a map $B(\lambda) \to B(\lambda) \cup \{0\}$.

Recall that the set of $\mathfrak{S}_n$-orbits $\{\mathfrak{S}_n \cdot (0^n, \ldots , (2\ell - 1)^{2\ell-1}) \mid \sum_{j=0}^{2\ell-1} \gamma_j = n\}$ is in one-to-one correspondence with the set $\{\sum_{i \in I} \gamma_i a_i \in Q^+_n | \gamma_i \in \mathbb{N}, \forall i \in I\}$. Thus we can also use $Q^+_n$ to label the blocks of $\text{Rep}_I \mathcal{H}_n$. There is a decomposition $\mathcal{H}_n^\lambda\text{-mod} \cong \bigoplus_{\gamma \in Q^+_n} (\mathcal{H}_n^\lambda\text{-mod})[\gamma]$, where

$$(\mathcal{H}_n^\lambda\text{-mod})[\gamma] := \{M \in \mathcal{H}_n^\lambda\text{-mod} | \text{infl}^\lambda M \in (\mathcal{H}_n\text{-mod})[\gamma]\}.$$  

If $M \in (\mathcal{H}_n\text{-mod})[\gamma]$ for some $\gamma = \sum_{i \in I} \gamma_i a_i$, then we define

$$e_i^\lambda M := \begin{cases} (\text{res}_{\mathcal{H}_n^{\lambda-1}}^{\mathcal{H}_n^\lambda} M)[\gamma - a_i], & \text{if } \gamma_i > 0; \\ 0, & \text{if } \gamma_i = 0. \end{cases}$$

while for any simple module $M \in \mathcal{H}_n^\lambda\text{-mod}[\gamma]$ with $\gamma = \sum_{i \in I} \gamma_i a_i$, we define

$$f_i^\lambda M := (\text{infl}_{\mathcal{H}_n^{\lambda+1}}^{\mathcal{H}_n^\lambda} M)[\gamma + a_i].$$

The functors $e_i^\lambda, f_i^\lambda$ are both left and right adjoint to each other, and hence are exact and send projectives to projectives.

**Definition 6.2.** Let $n \in \mathbb{N}$ and $M \in \text{Rep}_I (\mathcal{H}_n)$. We define

$$
\text{soc}_B M := \sum_{M \supseteq L \in \mathcal{B}(\mathfrak{S}_n)} L.
$$

It is clear that $\text{soc}_B M$ is a direct summand of $\text{soc} M$.

**Theorem 6.3.** (Multiplicity Two Theorem) Let $i \in \mathbb{Z}/2\mathbb{Z}$ and $M \in \text{Rep}_I \mathcal{H}_n$ be a simple module, where $n \geq 1$. Then $e_i^\lambda M$ is either 0 or a self-dual indecomposable module with simple socle $\bar{e}_i^\lambda M \cong \bar{e}_{i+1}^\lambda M$. Furthermore, if $n \geq 2$, then $\text{soc}_B \text{res}_{\mathcal{H}_n^{\lambda-2}}^{\mathcal{H}_n^\lambda} M \neq 0$ only if $n$ is even and $M \in \mathcal{B}(n/2)$. In that case, each simple module in $\text{soc}_B \text{res}_{\mathcal{H}_n^{\lambda-2}}^{\mathcal{H}_n^\lambda} M$ occurs with multiplicity two.

**Proof.** By [8] Corollary 9.14, we can find a $\Lambda \in P^+$ such that $\text{pr}^\lambda M = M$. In other words, $M \in \mathcal{H}_n^\lambda\text{-mod}$. It follows that $e_i M \in \mathcal{H}_n^{\lambda-1}\text{-mod}, e_{i+1} M \in \mathcal{H}_n^{\lambda-1}\text{-mod}, e_i e_{i+1} M \in \mathcal{H}_n^{\lambda-2}\text{-mod}$. Therefore, $e_i M \cong \text{infl}^\lambda e_i^\lambda e_i^\lambda M$. Now the self-duality property of $e_i M$ follows from the non-degenerate version of [21] Lemma 8.2.2. Since $\ell > 2$ and $\text{char } F \neq 2$, it follows from Proposition [13] that $e_i M$ is either 0 or has
simple socle $\tilde{e}_i M \cong \tilde{e}_i e_{i+t} M$. Furthermore, if $\soc B M \neq 0$ then clearly $n$ is even and $M \in B(n/2)$. In that case, since $\res B_{\ell}\mathcal{H}_n M \cong \bigoplus_{i \in I} e_i M \cong \bigoplus_{i,j \in J} \epsilon_i e_j M$, it follows that each simple module in $\soc B M$ occurs with multiplicity two (as both $\tilde{e}_i e_{i+t} M$ and $\tilde{e}_{i+t} e_i M$ occur).

Let $1_\lambda \cong F$ be the trivial simple $\mathcal{H}^\lambda$-module. Let $i \in \mathbb{Z}/\ell \mathbb{Z}$ and $M \in \mathcal{H}^\lambda$-mod be a simple module. Recall from [3] and [21] §8.4 that

$$
\varepsilon_i^\lambda(M) := \max\{ m \geq 0 | (\tilde{e}_i^\lambda)^m M \neq 0 \} = \varepsilon_i(\inf^\lambda M),
$$

$$
\varphi_i^\lambda(M) := \max\{ m \geq 0 | (\tilde{f}_i^\lambda)^m M \neq 0 \}.
$$

In particular, $\varepsilon_i(1_\lambda) = 0$ and $\varphi_i(1_\lambda) = \langle h_i, \lambda \rangle$. If furthermore, $M \in \mathcal{H}^\lambda$-mod[\gamma] for some $\gamma = \sum_{i \in I} \gamma_i \alpha_i$, then we define the weight function $\wt^\lambda(M) := \lambda - \gamma$. Then $\varphi_i^\lambda(M) = \varepsilon_i^\lambda(M) + \langle h_i, \lambda - \gamma \rangle$. We define

$$
B(\lambda) := \{ M | M \in \mathcal{H}^\lambda$-mod is simple, $n \in \mathbb{N} \}.
$$

**Lemma 6.4.** ([3]) The set $B(\lambda)$, the functions $\varepsilon_i^\lambda, \varphi_i^\lambda, \wt^\lambda$, together with the operators $\tilde{e}_i^\lambda, \tilde{f}_i^\lambda$ form a crystal in the sense of Kashiwara [19] §7.2. Moreover, it is isomorphic to Kashiwara’s crystal associated to the crystal base of the integral highest weight module $V(\lambda)$ over $U_q(\mathfrak{sl}_2)$.

**Definition 6.5.** For each $i \in \mathbb{Z}/\ell \mathbb{Z}$ and each simple module $M \in \mathcal{H}^\lambda$-mod, define

$$
e_i^{\lambda_0} M := e_i e_{i+t} M, \quad f_i^{\lambda_0} M := f_i f_{i+t} M.
$$

$$
\tilde{e}_i^{\lambda_0} M := \varepsilon_i e_{i+t} M, \quad \tilde{f}_i^{\lambda_0} M := \varphi_i f_{i+t} M.
$$

**Theorem 6.6.** Let $i \in \mathbb{Z}/\ell \mathbb{Z}$ and $M \in \mathcal{H}^\lambda$-mod be a simple module. Then

1) $\soc e_i^{\lambda_0} M \cong \tilde{e}_i^{\lambda_0} M$. Furthermore, $e_i^{\lambda_0} M \neq 0$ if and only if $\tilde{e}_i^{\lambda_0} M \neq 0$, in which case $e_i^{\lambda_0} M$ is a self-dual indecomposable module with simple socle and head isomorphic to $\tilde{e}_i^{\lambda_0} M$;

2) $\soc f_i^{\lambda_0} M \cong \tilde{f}_i^{\lambda_0} M$. Furthermore, $f_i^{\lambda_0} M \neq 0$ if and only if $\tilde{f}_i^{\lambda_0} M \neq 0$, in which case $f_i^{\lambda_0} M$ is a self-dual indecomposable module with simple socle and head isomorphic to $\tilde{f}_i^{\lambda_0} M$.

**Proof.** 1) follows from Theorem 5.3 2) follows from 1), Lemma 6.4 and the fact that $\varepsilon_i, f_i$ are both left and right adjoint to each other.

**Definition 6.7.** We define $\tilde{B}(\lambda_0) := \{ M = \tilde{f}_i^{\lambda_0} \cdots \tilde{f}_i^{\lambda_0} 1_\lambda \neq 0 | n \in \mathbb{N}, i_1, \cdots, i_n \in \mathbb{Z}/\ell \mathbb{Z} \}$. If $M = \tilde{f}_i^{\lambda_0} \cdots \tilde{f}_i^{\lambda_0} 1_\lambda$ is non-zero, where $i_1, \cdots, i_n \in \mathbb{Z}/\ell \mathbb{Z}$, and

$$
\wt^\lambda(M) = \lambda - \sum_{i \in \mathbb{Z}/\ell \mathbb{Z}} \gamma_i (\alpha_{\theta(i)} + \alpha_{\theta(i)+\ell}),
$$

then we define

$$
\varepsilon_i^{\lambda_0}(M) := \varepsilon_i^\lambda(M) = \varepsilon_{i+t}^\lambda(M), \quad \wt^{\lambda_0}(M) := \lambda_0 - \sum_{i \in \mathbb{Z}/\ell \mathbb{Z}} \gamma_i \alpha_i, \quad \varphi_i^{\lambda_0}(M) = \varepsilon_i^{\lambda_0}(M) + \langle h_i, \lambda_0 - \sum_{i \in \mathbb{Z}/\ell \mathbb{Z}} \gamma_i \alpha_i \rangle.
$$

Note that the above $\varepsilon_i^{\lambda_0}(M)$ is well-defined by Theorem 5.3

Let $M \in \mathcal{H}^\lambda$-mod be a simple module. We define

$$
\soc B M := \sum_{M \supseteq L \in B(\lambda)} L, \quad \hd B M := \sum_{\hd(M) \supseteq L \in B(\lambda)} L.
$$

**Corollary 6.8.** Let $i \in \mathbb{Z}/\ell \mathbb{Z}$ and $M \in \mathcal{H}^\lambda$-mod be a simple module.

1) if $n \geq 2$ then $\soc B \res \mathcal{H}^\lambda_{\ell - 2} M \neq 0$ only if $n$ is even and $M \in \tilde{B}(\lambda_0)$. In that case, each simple module in $\soc B \res \mathcal{H}^\lambda_{\ell - 2} M$ occurs with multiplicity two.

2) $\hd B \ind \mathcal{H}^\lambda_{\ell - 2} M \neq 0$ only if $n$ is even and $M \in \tilde{B}(\lambda_0)$. In that case, each simple module in $\hd B \ind \mathcal{H}^\lambda_{\ell - 2} M$ occurs with multiplicity two.

**Proof.** This follows from Theorem 6.6 and the non-degenerate version of [21] Lemma 8.2.2. \(\square\)
Theorem 6.9. (8) The set $\widehat{B}(\Lambda_0)$, the functions $e_{i}^{\Lambda_0}$, $f_{i}^{\Lambda_0}$, $\text{wt}^{\Lambda_0}$, together with the operators $e_{i}$, $f_{i}$ form a crystal in the sense of Kashiwara [19] 7.2. Moreover, it is isomorphic to Kashiwara’s crystal associated to the crystal base of the integral highest weight module $V(\Lambda_0)$ over $U_q(\mathfrak{sl}_n)$.

Proof. Note that by definition it is easy to check that $\varphi_{i}^{\Lambda_0}(M) = \varphi_{i}^{\lambda}(M) = \varphi_{i+\epsilon}^{\lambda}(M)$ for any $M \in \widehat{B}(\Lambda_0)$. Now the theorem follows from a similar argument as that was used in the proof of [21, Lemma 10.2.1, Theorem 10.3.4] by using Theorem 5.3, Theorem 5.23 and [19, Propositions 8.1, 8.2, Theorem 8.2]. □

Finally, we remark that although we chose a level two cyclotomic quotient for $\lambda = \Lambda_0 + \Lambda_\ell$ and consider the Hecke algebras of types $D_n$ and $B_n$ in this paper, the construction and the main results of this paper should be able to be generalized to the cases of the cyclotomic Hecke algebras of type $G(pd, p, n)$ and some cyclotomic Hecke algebra of type $G(pd, 1, n)$, and the dominant weight $\lambda$ should be replaced with some h-symmetric dominant weights in $P^+$ in a suitable sense, where h is the map defined in [14] Theorem 4.2.

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