4-REGULAR PARTITIONS AND THE POD FUNCTION

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Abstract. The function $pod(n)$ enumerates the partitions of $n$ wherein odd parts are distinct and even parts are unrestricted. Recently, a number of properties for $pod(n)$ have been established. In this paper, we use connections with 4-regular partitions and, for fixed $k \in \{0, 2\}$, partitions into distinct parts not congruent to $k$ modulo 4 in order to obtain new properties for $pod(n)$. In this context, we derive two new infinite families of linear inequalities involving $pod(n)$. We also obtain new identities of Watson type.

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1. Introduction. A partition $\lambda$ of $n$ is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers (called parts) such that $\sum_{i=1}^{\ell} \lambda_i = n$. We refer to $\lambda_i$ as the $i$th part of $\lambda$, and as usual denote by $p(n)$ the number of partitions of $n$. We note that $p(x) = 0$ if $x \not\in \mathbb{Z}_{\geq 0}$, and since the empty partition $\emptyset$ is the only partition of 0, we have that $p(0) = 1$. The generating function for $p(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_\infty}, \quad (1)$$

where we use the customary $q$-series notation

$$(a; q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0; \end{cases}$$

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

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Moreover, we use the short notation

\[(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty.\]

Because the infinite product \((a; q)_\infty\) diverges when \(a \neq 0\) and \(|q| \geq 1\), whenever \((a; q)_\infty\) appears in a formula, we shall assume \(|q| < 1\).

We denote by \(\text{pod}(n)\) the function which enumerates the partitions of \(n\) with odd parts distinct and even parts unrestricted. Elementary techniques in the theory of partitions give the following equivalent expressions for the generating function for \(\text{pod}(n)\):

\[\sum_{n=0}^{\infty} \text{pod}(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{(q^2; q^4)_\infty}{(q; q)_\infty}.\]  

(2)

The \(\text{pod}(n)\) function has been studied widely. It appears, for example, in the works of K. Alladi [1, 2], G.E. Andrews [3, 4], G.E. Andrews and M. Merca [8], C. Ballantine, M. Merca, D. Passary, A.J. Yee [11], A. Berkovich and F. Garvan [12], S.-P. Cui, W.X. Gu, Z.S. Ma [18], H. Fang, F. Xue, O.X.M. Yao [20], M.D. Hirschhorn and J.A. Sellers [29], S. Radu and J.A. Sellers [50]. In this article, we consider the interpretation of the \(\text{pod}(n)\) function given by the last expression in (2) as the number of partitions of \(n\) into parts not congruent to 2 modulo 4.

**Definition 1.** Let \(n\) be a nonnegative integer. We define \(\text{pod}_e(n)\) (respectively \(\text{pod}_o(n)\)) to be the number of partitions of \(n\) into an even (respectively odd) number of parts which are not congruent to 2 modulo 4.

For example, since the partitions of 8 into parts not congruent to 2 modulo 4 are

\[(8), (7, 1), (5, 3), (5, 1, 1, 1), (4, 4), (4, 3, 1), (4, 1, 1, 1, 1),
(3, 3, 1, 1), (3, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1),\]

we have that \(\text{pod}_e(8) = 7\) and \(\text{pod}_o(8) = 3\).

In this article, we establish connections among partitions enumerated by \(\text{pod}(n)\) and three other classes: 4-regular partitions, and partitions into distinct parts not congruent to \(k\) modulo 4 for \(k \in \{0, 2\}\).

For an integer \(\ell > 1\), a partition is called \(\ell\)-regular if none of its parts is divisible by \(\ell\). In classical representation theory, \(\ell\)-regular partitions of \(n\) parameterize the irreducible \(\ell\)-modular representations of the symmetric group \(S_n\) when \(\ell\) is prime [30]. The arithmetic properties of the number \(b_\ell(n)\) of \(\ell\)-regular partitions of \(n\) have been investigated extensively (see, for example, [13, 16, 19, 22, 28, 35, 45, 46, 58, 59, 55]). The generating function for \(b_\ell(n)\) satisfies the identity

\[\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty}.\]

**Definition 2.** Let \(n\) be a nonnegative integer. We define \(b_{4,e}(n)\) (respectively \(b_{4,o}(n)\)) to be the number of 4-regular partitions of \(n\) into an even (respectively odd) number of parts.
For example, the partitions of 7 into parts that are not multiples of 4 are

\[(7), (6, 1), (5, 2), (5, 1, 1), (3, 3, 1), (3, 2, 2), (3, 2, 1, 1), (3, 1, 1, 1, 1),
    (2, 2, 2, 1), (2, 2, 1, 1, 1), (2, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1),
\]

and so \(b_4(7) = 12\), \(b_{4,e}(7) = 5\) and \(b_{4,o}(7) = 7\). The sequences \(b_{4,e}(n)\) and \(b_{4,o}(n)\) can be found in the On-Line Encyclopedia of Integer Sequences [52, A339406, A339407].

**Definition 3.** Let \(n\) be a nonnegative integer and \(k \in \{0, 2\}\). We define \(Q_k(n)\) to be the number of partitions of \(n\) into distinct parts which are not congruent to \(k\) modulo 4.

For example, \(Q_0(14) = 11\) since the 4-regular partitions of 14 into distinct parts are

\[(14), (13, 1), (11, 3), (11, 2, 1), (10, 3, 1), (9, 5),
    (9, 3, 2), (7, 6, 1), (7, 5, 2), (6, 5, 3), (6, 5, 2, 1),
\]

while \(Q_2(14) = 6\), the relevant partitions being

\[(13, 1), (11, 3), (9, 5), (9, 4, 1), (8, 5, 1), (7, 4, 3).
\]

The generating functions for these sequences satisfy the identities

\[
\sum_{n=0}^{\infty} Q_0(n) q^n = (-q, -q^2, -q^3; q^4)_{\infty}
\]

and

\[
\sum_{n=0}^{\infty} Q_2(n) q^n = (-q, -q^3, -q^4; q^4)_{\infty}.
\]

We remark that \(Q_0(n)\) appears in the On-Line Encyclopedia of Integer Sequences [52, A070048], where we find another combinatorial interpretation for \(Q_0(n)\): the number of partitions of \(n\) into odd parts in which no part appears more than thrice.

Our first main result introduces new combinatorial interpretations for the partition functions \(Q_0(n)\) and \(Q_2(n)\).

**Theorem 1.1.** For \(n \geq 0\) the following hold.

(i) \((-1)^n Q_0(n) = pod_e(n) - pod_o(n)\).

(ii) \((-1)^n Q_2(n) = b_{4,e}(n) - b_{4,o}(n)\).

**Corollary 1.2.** Let \(n \geq 0\).

(i) \(Q_0(n)\) and \(pod(n)\) have the same parity.

(ii) \(Q_2(n)\) and \(b_4(n)\) have the same parity.
We also have the following result relating $Q_0(n)$ and $Q_2(n)$ to $pod(n)$.

**Theorem 1.3.** For $n \geq 0$ the following hold.

(i) $Q_0(n) = pod(n) + 2 \sum_{k=1}^{\infty} (-1)^k pod(n - 4k^2)$.

(ii) $Q_2(n) = \sum_{k=0}^{\infty} (-1)^{k(k+1)/2} pod(n - k(k + 1))$.

In addition, $Q_0(n)$ and $Q_2(n)$ satisfy similar linear recurrence relations involving the triangular numbers. To make these easier to state, we first define

$$\xi_n = \begin{cases} (-1)^m \cdot 2 & \text{if } n = 4m^2 \text{ for some } m > 0, \\ 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\chi_n = \begin{cases} (-1)^{n/2} & \text{if } n = m(m + 1) \text{ for some } m \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 1.4.** For $n \geq 0$ the following hold.

(i) $\sum_{k=0}^{\infty} (-1)^{k(k+1)/2} Q_0(n - k(k + 1)/2) = \xi_n$.

(ii) $\sum_{k=0}^{\infty} (-1)^{k(k+1)/2} Q_2(n - k(k + 1)/2) = \chi_n$.

The rest of this paper is organized as follows. In Section 2 we provide analytic and combinatorial proofs of Theorem 1.1. In Section 3 we provide proofs of Theorems 1.3 and 1.4 using generating functions and also give a combinatorial proof of Theorem 1.3 (ii) (a combinatorial proof of Theorem 1.3(i) would be very welcome). In Section 4 we show that the identities of Theorem 1.3 are limiting cases of much more general identities and use the latter to derive two infinite families of linear inequalities involving $pod(n)$. In Section 5 we present several Ramanujan type congruences for $b_4(n)$ modulo 16 and 64. In Section 6 we obtain three identities of Watson type and prove them analytically and combinatorially, and conclude by conjecturing two infinite families of linear inequalities involving $Q_0(n)$ and $Q_2(n)$.

2. Proof of Theorem 1.1.

2.1. Analytic proof. Define

$$F(z, q) := \prod_{k=0}^{\infty} \frac{1}{(1 - zq^{4k+1})(1 - zq^{4k+3})(1 - zq^{4k+4})}.$$
Then, using Euler’s identity $1/(q; q^2)_{\infty} = (-q; q)_{\infty}$ and (3), we have

$$F(-1, q) = \frac{1}{(-q, -q^3, -q^4; q^4)_{\infty}} = \frac{(-q^2; q^4)_{\infty}}{(-q; q)_{\infty}}$$

$$= (q; q^2)_{\infty}(-q^2; q^4)_{\infty} = \sum_{n=0}^{\infty} (-1)^n Q_0(n) q^n.$$ 

On the other hand, the fact that

$$F(z, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} pod(n, m) z^m q^n,$$

where $pod(n, m)$ is equal to the number of partitions of $n$ with $m$ parts none of which are congruent to 2 modulo 4, implies that

$$F(-1, q) = \sum_{n=0}^{\infty} (pod_e(n) - pod_o(n)) q^n,$$

which establishes (i). To prove (ii), we define

$$G(z, q) := \prod_{k=0}^{\infty} \frac{1}{(1-zq^{4k+1})(1-zq^{4k+2})(1-zq^{4k+3})}.$$ 

Then, using Euler’s identity and (4), we have

$$G(-1, q) = \frac{1}{(-q, -q^2, -q^3; q^4)_{\infty}} = \frac{(-q^4; q^4)_{\infty}}{(-q; q)_{\infty}}$$

$$= (q; q^2)_{\infty}(-q^4; q^4)_{\infty} = \sum_{n=0}^{\infty} (-1)^n Q_2(n) q^n.$$ 

Moreover, since

$$G(z, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_4(n, m) z^m q^n,$$

where $b_4(n, m)$ is the number of 4-regular partitions of $n$ with $m$ parts, we also have

$$G(-1, q) = \sum_{n=0}^{\infty} (b_{4,e}(n) - b_{4,o}(n)) q^n,$$

which establishes (ii).

### 2.2. Combinatorial proof.

For the rest of this article we will use calligraphy style capital letters to denote the set of partitions enumerated by the function denoted by the same letters. For example, $\mathcal{P}(n)$ denotes the set of partitions of $n$ and $\mathcal{POD}_e(n)$ is the set of partitions of $n$ into an even number of parts which are...
not congruent to 2 modulo 4. Recalling that \( pod(n) \) has more than one partition theoretic interpretation, in the sequel we assume that \( POD(n) \) represents the set of partitions of \( n \) having distinct odd parts.

If \( \lambda \) is a partition of \( n \), we say that the size of \( \lambda \) is \( n \) and write \( |\lambda| = n \). The length of \( \lambda \), denoted by \( \ell(\lambda) \), is the number of parts in \( \lambda \). We make the convention that \( \lambda_{\ell(\lambda)+1} = 0 \). Given two partitions \( \lambda \) and \( \mu \), we denote by \( \lambda \cup \mu \) the partition whose parts are precisely the parts of \( \lambda \) and \( \mu \) (with multiplicity). If each part of \( \mu \) is also a part of \( \lambda \) with equal or larger multiplicity, we denote by \( \lambda \setminus \mu \) the partition obtained from \( \lambda \) by removing the parts of \( \mu \) (with multiplicity). Finally, we will often write a partition \( \lambda \) as \( (\lambda^e, \lambda^o) \), where \( \lambda^e \) (respectively \( \lambda^o \)) consists of the even (respectively odd) parts of \( \lambda \).

We start by proving the first identity of Theorem 1.1. Denote by \( \tilde{Q}_0(n) \) the set of partitions of \( n \) into odd parts repeated no more than three times. Glaisher’s transformation [24], which iteratively splits each even part of a distinct partition into two equal parts until all parts are odd, yields a bijection between \( Q_0(n) \) and \( \tilde{Q}_0(n) \).

For any partition \( \lambda \in \tilde{Q}_0(n) \), we have \( \ell(\lambda) \equiv n \) \((\text{mod } 2)\). Thus, if \( n \) is even (respectively odd), \( \tilde{Q}_0(n) \) is a subset of \( POD_e(n) \) (respectively \( POD_o(n) \)).

Let \( \mathcal{A}(n) = (POD_e(n) \cup POD_o(n)) \setminus \tilde{Q}_0(n) \). Inspired by [27], we define an involution \( \varphi : \mathcal{A}(n) \to \mathcal{A}(n) \) that reverses the parity of the length of partitions. First, we introduce more useful notation. For any partition \( \lambda \), if \( d \) is a part of \( \lambda \) with multiplicity \( m_d \), we denote by \( s_d \) the nonnegative integer satisfying \( 2^{s_d} \leq m_d < 2^{s_d+1} \). If \( \lambda \in \mathcal{A}(n) \) has an odd part \( d \) with \( m_d \geq 4 \), we let \( r_\lambda \) be the largest value of \( 2^{s_d}d \) among all such parts (else we let \( r_\lambda = 0 \)). If \( \lambda^e = \emptyset \), it follows from the definition of \( \mathcal{A}(n) \) that \( r_\lambda = 0 \). We define \( \varphi(\lambda) \) as follows.

(i) If \( r_\lambda \geq \lambda^e \), merge \( 2^{s_d} \) copies of \( d \) into a new part to obtain \( \varphi(\lambda) \).

(ii) If \( r_\lambda < \lambda^e \), split \( \lambda^e = 2^{k_1}c_1 \), where \( k_1 \geq 1 \) and \( c_1 \) is odd, into \( 2^{k_1} \) parts equal to \( c_1 \) obtain \( \varphi(\lambda) \).

Partitions from case (i) are mapped by \( \varphi \) to partitions from case (ii) and vice-versa. Moreover, \( \varphi \) is its own inverse and reverses the parity of the length of partitions. This finishes the proof of the first identity of Theorem 1.1.

For the second identity, denote by \( \tilde{Q}_2(n) \) the set of partitions \( \lambda = (\lambda^e, \lambda^o) \) of \( n \) such that \( \lambda^o \) has distinct parts and each part of \( \lambda^e \) is congruent to 2 modulo 4 and has even multiplicity. We use the following variant of Glaisher’s transformation to create a bijection between \( Q_2(n) \) and \( \tilde{Q}_2(n) \): given \( \eta \in Q_2(n) \), split each part of \( \eta \) having the form \( 2^kc \), with \( k \geq 2 \) and \( c \) odd, into \( 2^{k-1} \) parts equal to \( 2c \). The inverse of this transformation iteratively merges equal parts of a partition in \( \tilde{Q}_2(n) \) until all parts are distinct. Since even parts of partitions in \( \tilde{Q}_2(n) \) have even multiplicity, all obtained even parts are divisible by 4.

For any partition \( \lambda \in \tilde{Q}_2(n) \) we have \( \ell(\lambda) \equiv n \) \((\text{mod } 2)\). Thus, if \( n \) is even (respectively odd), \( \tilde{Q}_2(n) \) is a subset of \( B_{1,e}(n) \) (respectively \( B_{4,o}(n) \)).

Let \( \mathcal{B}(n) = (B_{4,e}(n) \cup B_{4,o}(n)) \setminus \tilde{Q}_2(n) \). Notice that for every partition \( (\lambda^e, \lambda^o) \in \mathcal{B}(n) \), if \( \lambda^o \) has distinct parts, then there is at least one part in \( \lambda^e \) with odd multiplicity. We define a transformation \( \varepsilon : \mathcal{B}(n) \to \mathcal{B}(n) \) as follows.
(i) If $\lambda$ has a part $4k + 2$ with odd multiplicity with the property that all parts less than $2k + 1$ in $\lambda^o$ have multiplicity one, let $4a + 2$ be the smallest such part and define
\[ \varepsilon(\lambda) = (\lambda^e \setminus (4a + 2), \lambda^o \cup (2a + 1, 2a + 1)). \]

(ii) Else, let $2c + 1$ be the smallest repeated part in $\lambda^o$ and define
\[ \varepsilon(\lambda) = (\lambda^e \cup (4c + 2), \lambda^o \setminus (2c + 1, 2c + 1)). \]

Then $\varepsilon$ is an involution on $B(n)$ that maps a partition from case (i) to a partition from case (ii) and vice-versa, and changes the length of a partition by one. This finishes the proof of the second identity of Theorem 1.1.

3. Proof of Theorems 1.3 and 1.4.

3.1. Analytic proofs. The following theta identities are often attributed to Gauss [5, p. 23, Equations (2.2.12), (2.2.13)]:
\[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}} \]  
(5)

and
\[ \sum_{n=0}^{\infty} (-1)^{n(n+1)/2} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}}. \]  
(6)

The Jacobi triple product identity (cf. [23, Equation (1.6.1)]) states that
\[ (z; q)_{\infty}(q/z; q)_{\infty}(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2}. \]  
(7)

Using (5) we have
\[ (-q, -q^2, -q^3; q^4)_{\infty} \cdot \frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = (-q^2; q^4)_{\infty}(q^2; q^2)_{\infty} \]
\[ = (q^4; q^8)_{\infty}(q^4; q^4)_{\infty} \]
\[ = \frac{(q^4; q^4)_{\infty}}{(-q^4; q^4)_{\infty}} \]
\[ = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2}, \]  
(8)

while invoking (7) with $q$ replaced by $q^8$ and $z$ replaced by $q^2$ allows us to write
\[ (-q, -q^2, -q^3; q^4)_{\infty} \cdot \frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = (-q^4; q^4)_{\infty}(q^2; q^2)_{\infty} \]
\[
\begin{align*}
&= \frac{(q^2; q^2)_\infty}{(q^4; q^8)_\infty} \\
&= (q^2, q^6; q^8)_\infty \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n(2n-1)} \\
&= \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)}.
\end{align*}
\]

Thus, we deduce that
\[
\sum_{n=0}^{\infty} Q_0(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2} \right) \\
= \left( \sum_{n=0}^{\infty} \text{pod}(n) q^n \right) \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2} \right) \\
= \sum_{n=0}^{\infty} \left( \text{pod}(n) + 2 \sum_{k=1}^{\infty} (-1)^k \text{pod}(n - 4k^2) \right) q^n,
\]

and
\[
\sum_{n=0}^{\infty} Q_2(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)} \\
= \left( \sum_{n=0}^{\infty} \text{pod}(n) q^n \right) \left( \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} (-1)^{(k+1)/2} \text{pod}(n - k(k+1)) \right) q^n,
\]
from which Theorem 1.3 follows.

Using (6), the relations (8) and (9) can be written as
\[
\left( \sum_{n=0}^{\infty} Q_0(n) q^n \right) \left( \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)/2} \right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2}
\]
and
\[
\left( \sum_{n=0}^{\infty} Q_2(n) q^n \right) \left( \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)/2} \right) = \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)}.
\]

Theorem 1.4 follows from rewriting these identities as
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} (-1)^{(k+1)/2} Q_0(n - k(k+1)/2) \right) q^n = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2}
\]
and
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} (-1)^{(k+1)/2} Q_2(n - k(k+1)/2) \right) q^n = \sum_{n=0}^{\infty} (-1)^{(n+1)/2} q^{n(n+1)}.
\]
3.2. Combinatorial proof of Theorem 1.3 (ii). The Ferrers diagram of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) is an array of left justified boxes such that the \( i \)th row from the top contains \( \lambda_i \) boxes.

If \( n \) and \( k \) are nonnegative integers, we define a transformation on \( \POD(n - k(k + 1)) \) as follows. Start with \( \lambda = (\lambda^e, \lambda^o) \in \POD(n - k(k + 1)) \). The 2-modular Ferrers diagram of \( \lambda^e \) is obtained from the ordinary Ferrers diagram of \( \lambda^e \) by replacing two boxes at a time in each row with a single box filled with a 2. To this diagram we append at the top the rotated Ferrers diagram of the staircase of length \( k \) with each box filled with 2. Next, starting in the upper left corner of the resulting diagram, we draw a zig-zag line beginning with a right step and continuing with pairs of alternating down and right steps for as long as both step segments border boxes of the diagram.

For example, if \( \lambda^e = (14, 14, 12, 12, 8, 4) \) and \( k = 3 \), we obtain

\[
\begin{array}{c}
\begin{array}{cccccc}
1 & & & & & \\
2 & 2 & & & & \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & \\
2 & 2 & 2 & 2 & \\
2 & 2 & 2 & \\
2 & 2 & \\
2 & \\
\end{array}
\end{array}
\]

The diagram defines two partitions into distinct even parts: the partition \( \alpha \) whose 2-modular Ferrers diagram is made up of the columns below the zig-zag line, and the partition \( \beta \) whose 2-modular Ferrers diagram is formed by the rows to the right of the zig-zag line. In the example above, \( \alpha = (18, 16, 12, 10, 6, 4) \) and \( \beta = (6, 4) \).

The triple \((\lambda^o, \alpha, \beta)\) is completely determined by \( \lambda \) and \( k \). Moreover \(|\lambda^o| + |\alpha| + |\beta| = n\) and \( k \leq \ell(\alpha) - \ell(\beta) \leq k + 1 \). Denote by \( A_k(n) \) the set of triples of partitions \((\lambda^o, \alpha, \beta)\), where \( \lambda^o \) is a partition into distinct odd parts, \( \alpha \) and \( \beta \) are partitions into distinct even parts, \(|\lambda^o| + |\alpha| + |\beta| = n\), and \( k \leq \ell(\alpha) - \ell(\beta) \leq k + 1 \). Then the transformation described above is a bijection between \( \POD(n - k(k + 1)) \) and \( A_k(n) \) (for the inverse transformation we refer the reader to [10, Section 2.1], where a similar transformation for ordinary Ferrers diagrams is defined). Thus, it remains to show that

\[
Q_2(n) = \sum_{k=0}^{\infty} (-1)^{k(k+1)/2} |A_k(n)|. \tag{10}
\]

Let \( A(n) \) be the set of triples \((\lambda^o, \alpha, \beta)\), where \( \lambda^o \) is a partition into distinct odd parts and \( \alpha \) and \( \beta \) are partitions into distinct even parts such that \(|\lambda^o| + |\alpha| + |\beta| = n\) and \( \ell(\alpha) - \ell(\beta) \geq 0 \).

Each triple \((\lambda^o, \alpha, \beta) \in A(n)\) with \( \ell(\alpha) > \ell(\beta) \) appears in exactly two of the sets \( A_k(n) \), namely those with \( k = \ell(\alpha) - \ell(\beta) \) and \( k = \ell(\alpha) - \ell(\beta) - 1 \). Triples \((\lambda^o, \alpha, \beta) \in A(n)\) with \( \ell(\alpha) = \ell(\beta) \) appear only in \( A_0(n) \). Thus, from the parity of triangular numbers, the contribution of \((\lambda^o, \alpha, \beta)\) with \(|\lambda^o| + |\alpha| + |\beta| = n\) to the
righthand side of (10) is

\[
\begin{cases}
1 & \text{if } \ell(\alpha) = \ell(\beta), \\
0 & \text{if } \ell(\alpha) - \ell(\beta) \equiv 1 \pmod{2}, \\
2 & \text{if } \ell(\alpha) - \ell(\beta) \equiv 0 \pmod{4}, \\
-2 & \text{if } \ell(\alpha) - \ell(\beta) \equiv 2 \pmod{4}.
\end{cases}
\]

Denote by \( \mathcal{MA}_0(n) \) be the multiset of triples \((\lambda^o, \alpha, \beta) \in \mathcal{A}(n) \) satisfying \( \alpha \neq \beta \) and \( \ell(\alpha) - \ell(\beta) \equiv 0 \pmod{4} \), with the multiplicity of the triple defined to be 2 if \( \ell(\alpha) - \ell(\beta) > 0 \) and 1 if \( \ell(\alpha) - \ell(\beta) = 0 \). Similarly, we denote by \( \mathcal{MA}_2(n) \) be the multiset of triples \((\lambda^o, \alpha, \beta) \in \mathcal{A}(n) \) with \( \ell(\alpha) - \ell(\beta) \equiv 2 \pmod{4} \) where each triple has multiplicity 2.

For any triple \((\lambda^o, \alpha, \beta) \in \mathcal{A}(n) \) let \( i \) be the smallest positive integer such that \( \alpha_i \neq \beta_i \).

We now define a map from \( \mathcal{MA}_0(n) \) to \( \mathcal{MA}_2(n) \) as follows. Start with \((\lambda^o, \alpha, \beta) \in \mathcal{MA}_0(n) \) and suppose \( \ell(\alpha) - \ell(\beta) = k \).

Case 1: \( \alpha_i < \beta_i \). Let \( \tilde{\alpha} = \alpha \cup (\beta_i) \) and \( \tilde{\beta} = \beta \setminus (\beta_i) \). Then, \( \ell(\tilde{\alpha}) - \ell(\tilde{\beta}) = k + 2 \), \( \tilde{\alpha}_i > \beta_i \), and the first \( i-1 \) parts of \( \tilde{\alpha} \) and \( \tilde{\beta} \) are equal. Moreover, the triple \((\lambda^o, \tilde{\alpha}, \tilde{\beta}) \) lies in \( \mathcal{MA}_2(n) \).

Case 2: \( \beta_i < \alpha_i \). Let \( \tilde{\alpha} = \alpha \setminus (\alpha_i) \) and \( \tilde{\beta} = \beta \cup (\alpha_i) \). Then, \( \ell(\tilde{\alpha}) - \ell(\tilde{\beta}) = k - 2 \), \( \tilde{\beta}_i > \tilde{\alpha}_i \), and the first \( i-1 \) parts of \( \tilde{\alpha} \) and \( \tilde{\beta} \) are equal. If \( k > 0 \), the triple \((\lambda^o, \tilde{\alpha}, \tilde{\beta}) \) lies in \( \mathcal{MA}_2(n) \). If \( k = 0 \), then since \( \ell(\tilde{\beta}) - \ell(\tilde{\alpha}) = 2 \), the triple \((\lambda^o, \tilde{\beta}, \tilde{\alpha}) \) lies in \( \mathcal{MA}_2(n) \). With regard to multiplicities, note that this triple is also obtained from the Case 1 triple \((\lambda^o, \beta, \alpha) \).

For the inverse of the transformation, start with \((\lambda^o, \tilde{\alpha}, \tilde{\beta}) \in \mathcal{MA}_2(n) \) and suppose \( \ell(\tilde{\alpha}) - \ell(\tilde{\beta}) = k \).

Case I: \( \tilde{\beta}_i < \tilde{\alpha}_i \). Let \( \alpha = \tilde{\alpha} \cup (\tilde{\beta}_i) \) and \( \beta = \tilde{\beta} \setminus (\tilde{\beta}_i) \). Then \( \ell(\alpha) - \ell(\beta) = k + 2 \) and \((\lambda^o, \alpha, \beta) \in \mathcal{MA}_0(n) \).

Case II: \( \tilde{\alpha}_i < \tilde{\beta}_i \). Let \( \alpha = \tilde{\alpha} \setminus (\tilde{\alpha}_i) \) and \( \beta = \tilde{\beta} \cup (\tilde{\alpha}_i) \). Then \( \ell(\alpha) - \ell(\beta) = k - 2 \) and \((\lambda^o, \alpha, \beta) \in \mathcal{MA}_0(n) \). If \( k = 2 \), one copy of \((\lambda^o, \tilde{\alpha}, \tilde{\beta}) \) is mapped to \((\lambda^o, \alpha, \beta) \) and the other to \((\lambda^o, \beta, \alpha) \).

Therefore, the transformation defined above is a bijection between \( \mathcal{MA}_0(n) \) and \( \mathcal{MA}_2(n) \). We complete our proof by noting that the righthand side of (10) is equal to the number of triples \((\lambda^o, \alpha, \beta) \in \mathcal{A}(n) \) with \( \alpha = \beta \) and that these triples are in bijection with the partitions in \( \mathcal{Q}_2(n) \) via the mapping that takes \((\lambda^o, \alpha, \alpha) \) to \( \lambda^o \cup 2\alpha \), where \( 2\alpha \) is the partition obtained from \( \alpha \) by doubling each of its parts.

4. Linear inequalities involving \( pod(n) \). Linear inequalities involving partition functions have been studied extensively [7, 8, 26, 31, 37, 38, 39, 40, 41]. For example, V.J.W. Guo and J. Zeng [26] proved that

\[
(-1)^{k-1} \sum_{j=0}^{2k-1} (-1)^{(j+1)/2} pod(n - j(j+1)/2) \geq 0
\]
for all $n, k > 0$. Recently, G.E. Andrews and M. Merca [8, Corollary 10] established a partition theoretic interpretation of this inequality by showing that

$$(-1)^{k-1} \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} \operatorname{pod}(n - j(j + 1)/2) = MP_k(n),$$

where $MP_k(n)$ is the number of partitions of $n$ in which the smallest part larger than $2k - 1$ is odd and appears exactly $k$ times and all other odd parts appear at most once. For example, $MP_2(19) = 10$, and the partitions in question are

$$(9, 9, 1), (9, 5, 5), (8, 5, 5, 1), (7, 7, 3, 2), (7, 7, 2, 2, 1), (7, 5, 5, 2), (6, 5, 5, 3), (6, 5, 5, 2, 1), (5, 5, 3, 2, 2), (5, 5, 2, 2, 2, 2, 1).$$

Shortly after that, C. Ballantine, M. Merca, D. Passary and A.J. Yee [11] gave combinatorial proofs of this interpretation.

In this section, inspired by Theorem 1.3, we obtain new infinite families of linear inequalities for $\operatorname{pod}(n)$. To this end, we recall that an overpartition of $n$ is a nonincreasing sequence of positive integers whose sum is $n$ in which the first occurrence of a number may be overlined [15]. For example, the eight overpartitions of 3 are

$$(3), (\overline{3}), (2, 1), (2, \overline{1}), (\overline{2}, 1), (2, \overline{1}), (1, 1, 1), (\overline{1}, 1, 1).$$

G.E. Andrews and M. Merca [8] introduced the function $\overline{M}_k(n)$ which counts the number of overpartitions of $n$ in which the first part larger than $k$ appears at least $k + 1$ times. For example, $\overline{M}_2(12) = 16$, with the relevant overpartitions being

$$(4, 4, 4), (\overline{4}, 4, 4), (3, 3, 3, 3), (\overline{3}, 3, 3, 3), (3, 3, 3, 2, 1), (3, 3, 3, \overline{2}, 1), (3, 3, 3, 2, \overline{1}), (3, 3, 3, 2, \overline{1}), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1), (\overline{3}, 3, 3, 2, 1).$$

We now prove an identity that has Theorem 1.3.(i) as its limiting case when $k \to \infty$.

**Theorem 4.1.** For $n, k > 0$, we have

$$(-1)^k \left( \operatorname{pod}(n) + 2 \sum_{j=1}^{k} (-1)^j \operatorname{pod}(n - 4j^2) - Q_0(n) \right) = \sum_{j=0}^{\lfloor n/4 \rfloor} Q_0(n - 4j) \overline{M}_k(j).$$

**Proof.** According to G.E. Andrews and M. Merca [8, Theorem 7], we have the following truncated version of (5):

$$\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{j^2} \right) = 1 + (-1)^k \frac{(-q; q)_k}{(q; q)_k} \sum_{j=k+1}^{\infty} \frac{2 q^{j(k+1)}}{1 - q^j} \cdot \left( -q^{j+1}; q \right)_{\infty},$$

(11)
As explained in [8, Proof of Corollary 8], the series on the right hand side of this identity is the generating function for $M_k(n)$, i.e.,

$$
\sum_{n=0}^{\infty} M_k(n) q^n = \frac{(-q; q)_k}{(q; q)_k} \sum_{j=k+1}^{\infty} \frac{2 q^{j(k+1)}}{1 - q^j} \cdot \frac{(-q^{j+1}; q)_\infty}{(q^{j+1}; q)_\infty}.
$$

Therefore, replacing $q$ by $q^4$ in (11) yields

$$
\frac{(-q^4; q^4)_\infty}{(q^4; q^4)_\infty} \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{4j^2} \right) = 1 + (-1)^k \sum_{n=0}^{\infty} M_k(n) q^{4n}.
$$

Multiplying both sides of this identity by $(-q, -q^2, -q^3; q^4)_\infty$ we obtain

$$
\frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{4j^2} \right) - (-q, -q^2, -q^3; q^4)_\infty
$$

$$
= (-1)^k (-q, -q^2, -q^3; q^4)_\infty \sum_{n=0}^{\infty} M_k(n) q^{4n},
$$

and using (2) and (3), the identity becomes

$$
\left( \sum_{n=0}^{\infty} pod(n) q^n \right) \left( 1 + 2 \sum_{j=1}^{k} (-1)^j q^{4j^2} \right) - \sum_{n=0}^{\infty} Q_0(n) q^n
$$

$$
= (-1)^k \left( \sum_{n=0}^{\infty} Q_0(n) q^n \right) \left( \sum_{n=0}^{\infty} M_k(n) q^{4n} \right).
$$

The theorem follows by comparing coefficients of $q^n$ on the two sides of this equation. \qed

As a consequence of Theorem 4.1 we obtain the following infinite family of linear inequalities involving $pod(n)$.

COROLLARY 4.2. For $n, k > 0$, we have

$$
(-1)^k \left( pod(n) + 2 \sum_{j=1}^{k} (-1)^j pod(n - 4j^2) - Q_0(n) \right) \geq 0,
$$

with strict inequality if and only if $n \geq 4(k + 1)^2$.

REMARK. For example, Corollary 4.2 implies

$$
pod(n) - 2pod(n - 4) \leq Q_0(n),
$$

$$
pod(n) - 2pod(n - 4) + 2pod(n - 16) \geq Q_0(n),
$$
\( \text{pod}(n) - 2\text{pod}(n - 4) + 2\text{pod}(n - 16) - 2\text{pod}(n - 36) \leq Q_0(n), \)

\( \text{pod}(n) - 2\text{pod}(n - 4) + 2\text{pod}(n - 16) - 2\text{pod}(n - 36) + 2\text{pod}(n - 64) \geq Q_0(n). \)

Our next result has Theorem 1.3.(ii) as its limiting case when \( k \to \infty \).

**Theorem 4.3.** For \( n, k > 0 \), we have

\[
(-1)^k \left( Q_2(n) - \sum_{j=0}^{2k-1} (-1)^{j+1/2} \text{pod}(n - j(j + 1)) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} Q_2(n - 2j) MP_k(j).
\]

**Proof.** According to G.E. Andrews and M. Merca [8, Theorem 9], we have the following truncated version of (6):

\[
\frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=0}^{2k-1} (-q)^{j(j+1)/2} = 1 + (-1)^{k-1} \left( \frac{(-q; q^2)_k}{(q^2; q^2)_{k-1}} \sum_{j=k}^\infty \frac{q^{k(2j+1)} (-q^{2j+3}; q^2)_\infty}{(q^{2j+2}; q^2)_\infty} \right). \tag{12}
\]

As explained in [8, Proof of Corollary 10], the series on the right hand side of this identity is the generating function for \( MP_k(n) \), i.e.,

\[
\sum_{n=0}^\infty MP_k(n) q^n = \frac{(-q; q^2)_k}{(q^2; q^2)_{k-1}} \sum_{j=k}^\infty \frac{q^{k(2j+1)} (-q^{2j+3}; q^2)_\infty}{(q^{2j+2}; q^2)_\infty}.
\]

Therefore, replacing \( q \) by \( q^2 \) in (12) yields

\[
\frac{(-q^2; q^4)_\infty}{(q^4; q^4)_\infty} \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} q^{j(j+1)} = 1 + (-1)^{k-1} \sum_{n=0}^\infty MP_k(n) q^{2n}.
\]

Multiplying both sides of this identity by \( (-q, -q^3, -q^4; q^4)_\infty \) we obtain

\[
\frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} q^{j(j+1)} = (-q, -q^3, -q^4; q^4)_\infty \sum_{n=0}^\infty MP_k(n) q^{2n},
\]

and using (2) and (4), the identity becomes

\[
\left( \sum_{n=0}^\infty \text{pod}(n) q^n \right) \left( \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} q^{j(j+1)} \right) - \sum_{n=0}^\infty Q_2(n) q^n
\]

\[
= (-1)^{k-1} \left( \sum_{n=0}^\infty Q_2(n) q^n \right) \left( \sum_{n=0}^\infty MP_k(n) q^{2n} \right).
\]
The theorem follows by comparing coefficients of $q^n$ on the two sides of this equation.

As a consequence of Theorem 4.3 we obtain the following infinite family of linear inequalities involving $\text{pod}(n)$.

**Corollary 4.4.** For $n, k > 0$, we have

$$(-1)^k \left( Q_2(n) - \sum_{j=0}^{2k-1} (-1)^{(j+1)/2} \text{pod}(n-j(j+1)) \right) \geq 0,$$

with strict inequality if and only if $n \geq 2k(2k+1)$.

**Remark.** For example, Corollary 4.4 implies

$$\text{pod}(n) - \text{pod}(n-2) \geq Q_2(n),$$

$$\text{pod}(n) - \text{pod}(n-2) - \text{pod}(n-6) + \text{pod}(n-12) \leq Q_2(n),$$

$$\text{pod}(n) - \text{pod}(n-2) - \text{pod}(n-6) + \text{pod}(n-12) + \text{pod}(n-20) - \text{pod}(n-30) \geq Q_2(n),$$

$$\text{pod}(n) - \text{pod}(n-2) - \text{pod}(n-6) + \text{pod}(n-12) + \text{pod}(n-20) - \text{pod}(n-30) - \text{pod}(42) + \text{pod}(56) \leq Q_2(n).$$

5. **Ramanujan type congruences.** In recent years, many congruences for the number of $\ell$-regular partitions have been discovered by G.E. Andrews, M.D. Hirschhorn and J.A. Sellers [6], S.C. Chen [14], S.-P. Cui and N.S.S. Gu [16, 17], B. Dandurand and D. Penniston [19], D. Furcy and D. Penniston [22], B. Gordon and K. Ono [25], W.J. Keith [32], B.L.S. Lin and A.Y.Z. Wang [34], J. Lovejoy and D. Penniston [35], D. Penniston [45, 46], E.X.W. Xia [57, 58], E.X.W. Xia and O.X.M. Yao [59, 60], O.X.M. Yao [61], and J.J. Webb [55].

For example, for $\alpha \geq 1$ and $n \geq 0$, from [6] we have

$$b_4 \left( 3^{2\alpha+1}n + \frac{17 \cdot 3^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{2},$$

$$b_4 \left( 3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8} \right) \equiv 0 \pmod{2},$$

$$b_4 \left( 3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8} \right) \equiv 0 \pmod{2},$$

and for $r \in \{13, 21, 29, 37\}$, from [14] we have

$$b_4 \left( 5^{2\alpha+2}n + \frac{r \cdot 5^{2\alpha+1} - 1}{8} \right) \equiv 0 \pmod{4}.$$  

From [57], for $\alpha \geq 0$, $n \geq 0$ and $r \in \{11, 19\}$, we have

$$b_4 \left( 3^{4\alpha+4}n + \frac{r \cdot 3^{4\alpha+3} - 1}{8} \right) \equiv 0 \pmod{8}.$$
To facilitate the study of partition functions, S. Radu [49, 48] considered a class of functions $a(n)$ defined by

$$
\sum_{n=0}^{\infty} a(n) q^n = \prod_{\delta|M} (q^\delta; q^\delta)^{r_\delta},
$$

(13)

where the product is over the positive divisors of $M > 0$ and $r_\delta \in \mathbb{Z}$. Using the ideas of H. Rademacher [47], M. Newman [42, 43] and O. Kolberg [33], Radu [49] discovered a method for establishing congruences of the form

$$
a(mn + j) \equiv 0 \pmod{u}
$$

for fixed $m, j$ and $u$, and developed the so-called Ramanujan-Kolberg algorithm [48] for deriving identities involving the generating functions of $a(mn + j)$ and modular functions on $\Gamma_0(N)$. Very recently, N.A. Smoot [53] provided a successful Mathematica implementation of Radu’s algorithm. The package is called RaduRK and requires 4ti2, a software package for algebraic, geometric and combinatorial problems on linear spaces. Instructions for the proper installation of these packages can be found in [53].

The procedure

$$
\text{RK}[N,M,r,m,j]
$$

take as input an integer $N \geq 2$, a generating function defined by $M$ and $r = (r_\delta)_{\delta|M}$ as in (13), and an arithmetic progression $mn + j$ with $0 \leq j < m$. The algorithm decides if there exists an identity of the form

$$
f(q) \cdot \prod_{j' \in P} \left( \sum_{n=0}^{\infty} a(mn + j') q^n \right) = \sum_{g \in A} g \cdot p_g(t),
$$

where $j \in P$, $A$ is a finite set of modular functions for $\Gamma_0(N)$ containing $1$, $f$ and $t$ are a modular function on $\Gamma_0(N)$, and $p_g$ are polynomials. For the definition of these notions and a general introduction to RaduRK algorithm, see [44]. For the correctness proof and details of the algorithm, see [53].

In this section, we use the implemented RaduRK algorithm to prove several congruences modulo 16 for $b_4(n)$. Note that letting $M = 4$ and $r_1 = -1, r_2 = 0, r_4 = 1$ in (13) yields the generating function for $b_4(n)$.

**Theorem 5.1.** Let $\alpha \in \{8, 13, 18, 23\}$. For all $n \geq 0$, we have

$$
b_4(25n + \alpha) \equiv 0 \pmod{16}.
$$

Since, $b_4(8) = 16$, $b_4(25 + 13) = 8528$, $b_4(18) = 208$ and $b_4(23) = 592$, we have that

$$
\sum_{n=0}^{\infty} b_4(25n + \alpha) q^n \neq 0 \pmod{32},
$$

for all $\alpha \in \{8, 13, 18, 23\}$. Thus, Theorem 5.1 follows directly from the following two lemmas.
Lemma 5.2.
\[
\left( \sum_{n=0}^{\infty} b_4(25n + 8) q^n \right) \left( \sum_{n=0}^{\infty} b_4(25n + 23) q^n \right) \equiv 0 \pmod{256}.
\]

Proof. We consider the RaduRK program with input

\[
\text{RK}[20, 4, \{-1, 0, 1\}, 25, 8],
\]
in which case it returns the following:

\[
P = \{8, 23\}
\]

\[
f(q) = \frac{(q; q)^{32}_\infty (q^4; q^4)^{12}_\infty (q^{10}; q^{10})^{32}_\infty}{q^{31} (q^2; q^2)^{32}_\infty (q^5; q^5)^{10}_\infty (q^{20}; q^{20})^{54}_\infty}
\]

\[
t = \frac{(q^4; q^4)^4_\infty (q^{10}; q^{10})^2_\infty}{q^2 (q^2; q^2)^2_\infty (q^{20}; q^{20})^4_\infty}
\]

\[
A = \left\{ 1, \frac{(q^4; q^4)^\infty_\infty (q^5; q^5)^5_\infty}{q^3 (q; q)^\infty_\infty (q^{20}; q^{20})^5_\infty} - \frac{(q^4; q^4)^\infty_\infty (q^{10}; q^{10})^2_\infty}{q^2 (q^2; q^2)^2_\infty (q^{20}; q^{20})^4_\infty} \right\}
\]

\[
\{ p_g(t) \}_{g \in A} = \left\{ -387500000000 t^2 - 4722300000000 t^3 + 1975524000000 t^4
\right.
\]

\[
- 13492968000000 t^5 - 2890299600000 t^6 + 5172328240000 t^7
\]

\[
- 27746680960000 t^8 - 697717120000 t^9 + 7036326368000 t^{10}
\]

\[
- 2875996422400 t^{11} + 195120171520 t^{12} + 113806525952 t^{13}
\]

\[
+ 2380696832 t^{14} + 2340096 t^{15},
\]

\[
12500000000 t + 262500000000 t^2 + 1449800000000 t^3
\]

\[
- 9119240000000 t^4 + 13179468000000 t^5 - 2546388000000 t^6
\]

\[
- 9465344000000 t^7 + 8751301760000 t^8 - 2237298720000 t^{9}
\]

\[
- 628733856000 t^{10} + 318550950400 t^{11} + 22754516480 t^{12}
\]

\[
+ 119739648 t^{13} + 9472 t^{14} \right\}.
\]

Since 256 is the common factor of the coefficients of the polynomials \( p_g(t) \), we deduce the identity

\[
\frac{(q; q)^{32}_\infty (q^4; q^4)^{12}_\infty (q^{10}; q^{10})^{32}_\infty}{q^{31} (q^2; q^2)^{32}_\infty (q^5; q^5)^{10}_\infty (q^{20}; q^{20})^{54}_\infty} \left( \sum_{n=0}^{\infty} b_4(25n + 8) q^n \right) \left( \sum_{n=0}^{\infty} b_4(25n + 23) q^n \right)
\]

\[
= 256 Y_1 + 256 Y_2 \left( \frac{(q^4; q^4)^\infty_\infty (q^5; q^5)^5_\infty}{q^3 (q; q)^\infty_\infty (q^{20}; q^{20})^5_\infty} - \frac{(q^4; q^4)^\infty_\infty (q^{10}; q^{10})^2_\infty}{q^2 (q^2; q^2)^2_\infty (q^{20}; q^{20})^4_\infty} \right),
\]

where

\[
Y_1 = -1513671875 t^2 - 18446484375 t^3 + 77168906250 t^4
\]

\[
- 52706906250 t^5 - 112902328125 t^6 + 202044071875 t^7
\]

\[
- 108385472500 t^8 - 2725457500 t^9 + 2748569875 t^{10}
\]
in which case it returns the following:

\[ Y_2 = 48828125 t + 1025390625 t^2 + 5663281250 t^3 - 35622031250 t^4 \\
+ 51482296875 t^5 - 9946828125 t^6 - 36973962500 t^7 \\
+ 34184772500 t^8 - 873948125 t^9 - 245591625 t^{10} \\
+ 1244339650 t^{11} + 88884830 t^{12} + 467733 t^{13} + 37 t^{14}. \]

Lemma 5.2 follows immediately. \( \square \)

**Lemma 5.3.**

\[
\left( \sum_{n=0}^{\infty} b_4(25n + 13) q^n \right) \left( \sum_{n=0}^{\infty} b_4(25n + 18) q^n \right) \equiv 0 \pmod{256}.
\]

**Proof.** We consider the RaduRK program with input

\[ \text{RK}[20, 4, \{-1, 0, 1\}, 25, 13], \]

in which case it returns the following:

\[ P = \{13, 18\} \]

\[ f(q) = \frac{(q; q)_{52} (q^4; q^4)_{12} (q^{10}; q^{10})_{32}}{q^{31} (q^2; q^2)_{32} (q^5; q^5)_{10} (q^{20}; q^{20})_{54}} \]

\[ t = \frac{(q^4; q^4)_{10} (q^{10}; q^{10})_{2}}{q^{2} (q^2; q^2)_{2} (q^{20}; q^{20})_{4}} \]

\[ A = \left\{ 1, \frac{(q^4; q^4)_{5} (q^5; q^5)_{5}}{q^{2} (q^2; q^2)_{2} (q^{20}; q^{20})_{4}} - \frac{(q^4; q^4)_{5} (q^{10}; q^{10})_{2}}{q^{2} (q^2; q^2)_{2} (q^{20}; q^{20})_{4}} \right\} \]

\[ \{p_9(t)\}_{g \in A} = \{62500000000 t - 650000000000 t^2 - 430980000000 t^3 \\
+ 1946374000000 t^4 - 1337296000000 t^5 - 2904996000000 t^6 \\
+ 519499944000000 t^7 - 279415849600000 t^8 - 5956625600000 t^9 \\
+ 70025450880000 t^{10} - 28697110528000 t^{11} + 194775499520 t^{12} \\
+ 113785213952 t^{13} + 2380097792 t^{14} + 2273536 t^{15}, \\
- 50000000000 t + 587500000000 t^2 + 71230000000 t^3 \\
- 814774000000 t^4 + 12342468000000 t^5 - 2047288000000 t^6 \\
- 96691664000000 t^7 + 88012137600000 t^8 - 2240988480000 t^9 \\
- 6299088960000 t^{10} + 3187359168000 t^{11} + 22753946880 t^{12} \\
+ 120139008 t^{13} + 13312 t^{14}\} . \]

As in the proof of Lemma 5.2, we deduce the identity

\[
\frac{(q; q)_{52} (q^4; q^4)_{12} (q^{10}; q^{10})_{32}}{q^{31} (q^2; q^2)_{32} (q^5; q^5)_{10} (q^{20}; q^{20})_{54}} \left( \sum_{n=0}^{\infty} b_4(25n + 13) q^n \right) \left( \sum_{n=0}^{\infty} b_4(25n + 18) q^n \right)
\]
= 256 Y_1 + 256 Y_2 \left( \frac{(q^4; q^4)_\infty (q^5; q^5)_\infty}{(q^3; q)_\infty (q^{20}; q^{20})_\infty} - \frac{(q^4; q^4)_\infty (q^{10}; q^{10})_\infty}{q^2 (q^2; q^2)_\infty (q^{20}; q^{20})_\infty} \right),

where

\begin{align*}
Y_1 &= 244140625 \ t - 2539062500 \ t^2 - 16835156250 \ t^3 + 76030234375 \ t^4 - 52238156250 \ t^5 - 113476546875 \ t^6 \\
&+ 202929665625 \ t^7 - 109146816250 \ t^8 - 2326806875 \ t^9 \\
&+ 27353691750 \ t^{10} - 112098088000 \ t^{11} + 76084179516250 \ t^{12} \\
&+ 444473492 \ t^{13} + 9297257 \ t^{14} + 8881 \ t^{15},
\end{align*}

\begin{align*}
Y_2 &= -195312500 \ t + 2294921875 \ t^2 + 2782421875 \ t^3 - 31827109375 \ t^4 \\
&+ 48212765625 \ t^5 - 7997218750 \ t^6 - 37770181250 \ t^7 \\
&+ 34379741250 \ t^8 - 8753861250 \ t^9 - 24605816250 \ t^{10} \\
&+ 12450621750 \ t^{11} + 88882605 \ t^{12} + 469293 \ t^{13} + 52 \ t^{14}.
\end{align*}

Lemma 5.3 follows immediately. \hfill \Box

The RaduRK algorithm can also be used to introduce congruences modulo 64 for \( b_4(n) \).

**Theorem 5.4.** Let \( \alpha \in \{13, 20, 27, 34, 41, 48\} \). For all \( n \geq 0 \), we have

\[ b_4(49n + \alpha) \equiv 0 \pmod{64}. \]

Theorem 5.4 follows directly from the following lemma whose proof we omit beyond noting that one can use the RaduRK program as above with

\[ \text{RK}[28, 4, \{-1, 0, 1\}, 49, 13] \]

and

\[ \text{RK}[28, 4, \{-1, 0, 1\}, 49, 27]. \]

**Lemma 5.5.** The following congruences hold.

\begin{enumerate}
  \item \[ \prod_{\alpha \in \{13, 20, 34\}} \sum_{n=0}^{\infty} b_4(49n + \alpha) q^n \equiv 0 \pmod{64^3}. \]
  \item \[ \prod_{\alpha \in \{27, 41, 48\}} \sum_{n=0}^{\infty} b_4(49n + \alpha) q^n \equiv 0 \pmod{64^3}. \]
\end{enumerate}

We end this section by noting that, while we were unable to find more congruences due to the running time of the RaduRK program, there likely exist infinite families of congruences modulo 16 or modulo 64 for \( b_4(n) \).
6. Open problems and concluding remarks. In this paper, inspired by the decompositions of \( Q_0(n) \) and \( Q_2(n) \) in terms of the partition function \( \text{pod}(n) \) given by Theorem 1.3, we derived new infinite families of linear inequalities involving \( \text{pod}(n) \). Here we show that there is another way to decompose \( Q_0(n) \) using \( \text{pod}(n) \), and also establish decompositions of \( Q_2(n) \) and \( b_4(n) \) in terms of the function \( \overline{p}(n) \) which enumerates the overpartitions of \( n \) (see [15]).

**Theorem 6.1.** For all \( n \geq 0 \) we have

\[
Q_0(n) = \sum_{k=0}^{\infty} \text{pod} \left( \frac{n-k(k+1)/2}{2} \right). \tag{14}
\]

**Proof.** (Analytic proof) By (3) we have

\[
\sum_{n=0}^{\infty} Q_0(n) \ q^n = (-q, -q^2, -q^3; q^4) = (-q, -q^3, q^4; q^4) \cdot \frac{(-q^2; q^4) \_}{(q^4; q^4) \_}.
\]

Next, using (2) and replacing \( q \) by \( q^4 \) and \( z \) by \( -q \) in (7), we find

\[
\sum_{n=0}^{\infty} Q_0(n) \ q^n = \left( \sum_{n=-\infty}^{\infty} q^{2n^2-n} \right) \left( \sum_{n=0}^{\infty} \text{pod}(n) \ q^{2n} \right)
\]

\[
= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} \text{pod}(n) \ q^{2n} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \text{pod} \left( \frac{n-k(k+1)/2}{2} \right) \right) q^n.
\]

□

**Proof.** (Combinatorial proof) We begin by recalling a result due to G.N. Watson [54]. If we denote by \( Q_{\text{odd}}(n) \) the number of partitions of \( n \) into distinct odd parts, then

\[
Q_{\text{odd}}(n) = \sum_{k=0}^{\infty} p \left( \frac{n-k(k+1)/2}{4} \right). \tag{15}
\]

A combinatorial proof of (15) is given in [56] using abacus displays, which were first introduced in [30]. For the convenience of the reader, we briefly describe the construction here, using the language of [51, section 2].

Given a partition \( \lambda \), fill the boxes of its Ferrers diagram with alternating 0s and 1s starting with 0 in the upper left corner. Drawing horizontal lines through rows ending in 0 and vertical lines through columns ending in 1, the boxes at the intersection of these lines (pushed toward the northwest) form the Ferrers diagram of a partition \( \alpha \). Repeating this process with horizontal lines through rows ending in 1 and vertical lines through columns ending in 0 yields a partition \( \beta \). The pair \( (\alpha, \beta) \) is referred to as the 2-quotient of \( \lambda \). The 2-core of \( \lambda \) is the staircase partition \( \delta \) whose Ferrers diagram is obtained from the Ferrers diagram of \( \lambda \) by repeatedly removing pairs of adjacent boxes (2-hooks) such that at ever step the resulting diagram is the Ferrers diagram of a partition. Then

\[
|\lambda| = |\delta| + 2|\alpha| + 2|\beta|.
\]
It is shown in [30, Theorem 2.7.30] that the triple \((\delta, \alpha, \beta)\) completely determines \(\lambda\). If \(\lambda\) is self-conjugate, then clearly \(\alpha = \beta\) and \((\delta, \alpha)\) determines \(\lambda\).

Given a partition \(\mu \in Q_{odd}(n)\), consider the partition \(\mu^{sc}\) whose Ferrers diagram is obtained from that of \(\mu\) by straightening the hooks nested along the diagonal (this is the classical bijection proving that the number of partitions of \(n\) into distinct odd parts equals the number of self-conjugate partitions of \(n\)). If we define \(\xi(\mu)\) to be the partition \(\alpha\) in the 2-quotient \((\alpha, \alpha)\) of \(\mu^{sc}\), then the transformation

\[
\xi : Q_{odd}(n) \to \bigcup_{k=0}^{\infty} \mathcal{P} \left( \frac{n - k(k+1)/2}{4} \right)
\]

is a bijection.

To prove (14), start with a partition \(\lambda = (\lambda^e, \lambda^o) \in Q_0(n)\) and let \(\tilde{\lambda}^e\) be the partition whose parts are the parts of \(\lambda^e\) divided by 2. Since the parts of \(\lambda^e\) are congruent to 2 modulo 4, \(\tilde{\lambda}^e\) is a partition with distinct odd parts. Denoting by \(\tilde{\lambda}^o\) the partition whose parts are the parts of \(\xi(\lambda^o)\) multiplied by 2 and defining \(\psi(\lambda) = \tilde{\lambda}^o \cup \tilde{\lambda}^e\), the map

\[
\psi : Q_0(n) \to \bigcup_{k=0}^{\infty} \mathcal{POD} \left( \frac{n - k(k+1)/2}{2} \right)
\]

is a bijection. \(\square\)

In a similar way, \(Q_2(n)\) and \(b_4(n)\) can be expressed in terms of \(p(n)\).

**Theorem 6.2.** For all \(n \geq 0\) we have

\[
Q_2(n) = \sum_{k=0}^{\infty} p \left( \frac{n - k(k+1)/2}{4} \right)
\]

and

\[
b_4(n) = \sum_{k=0}^{\infty} \overline{p} \left( \frac{n - k(k+1)/2}{2} \right).
\]

**Proof.** (Analytic proof) Arguing as in the proof of Theorem 6.1, we have

\[
\sum_{n=0}^{\infty} Q_2(n) q^n = (-q, -q^3, q^4; q^4)^\infty \cdot (-q^4, q^4; q^4)^\infty \cdot (q^4; q^4)^\infty
\]

\[
= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} p(n) q^{4n} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \overline{p} \left( \frac{n - k(k+1)/2}{4} \right) \right) q^n
\]
and
\[
\sum_{n=0}^{\infty} b_4(n) q^n = \frac{1}{(q; q^2)_{\infty} (q^2; q^4)_{\infty}} = (-q; q)_{\infty} (-q^2; q^2)_{\infty} \\
= (-q, -q^3; q^4)_{\infty} \cdot \frac{(-q^2; q^2)_{\infty}}{(q^2; q^4)_{\infty}} \\
= (-q, -q^3, q^4)_{\infty} \cdot \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\
= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} p(n) q^{2n} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \overline{p} \left( \frac{n - k(k + 1)/2}{2} \right) \right) q^n.
\]

\[
\begin{align*}
\sum_{n=0}^{\infty} b_4(n) q^n &= (-q; q)_{\infty} (-q^2; q^2)_{\infty} \\
&= (-q, -q^3, q^4)_{\infty} \cdot (-q^2; q^2)_{\infty} \\
&= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} p(n) q^{2n} \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \overline{p} \left( \frac{n - k(k + 1)/2}{2} \right) \right) q^n.
\end{align*}
\]

Proof. (Combinatorial proof) Let \( \lambda \in Q_2(n) \). If we denote by \( \bar{\lambda}^e \) the overpartition whose parts are the parts of \( \lambda^e \) divided by 4 and each part is overlined, and define \( \rho(\lambda) = \xi(\lambda^o) \cup \bar{\lambda}^e \), then
\[
\rho : Q_2(n) \to \bigcup_{k=0}^{\infty} \overline{P} \left( \frac{n - k(k + 1)/2}{4} \right)
\]
is a bijection.

To prove the second identity we make use of the Fu-Tang combinatorial proof [21] of another identity due to G.N. Watson [54]. If we denote by \( Q(n) \) the number of partitions of \( n \) into distinct parts, then the Fu-Tang bijection
\[
\zeta : Q(n) \to \bigcup_{k=0}^{\infty} P \left( \frac{n - k(k + 1)/2}{2} \right)
\]
(see Section 2 of [9] for a succinct description of \( \zeta \)) yields
\[
Q(n) = \sum_{k=0}^{\infty} p \left( \frac{n - k(k + 1)/2}{2} \right).
\]

(16)

Let \( \lambda \in B_4(n) \). Dividing each part of \( \lambda^e \) by 2 yields a partition \( \tilde{\lambda}^e \) into odd parts. We use Glaisher’s bijection to transform \( \tilde{\lambda}^e \) into a partition with distinct parts whose parts we overline to obtain an overpartition \( \bar{\lambda}^e \). Next, we apply Glaisher’s bijection to \( \lambda^o \) to obtain a partition \( \tilde{\lambda}^o \) into distinct parts and let \( \chi(\lambda) = \zeta(\tilde{\lambda}^o) \cup \bar{\lambda}^e \). Then the function
\[
\chi : B_4(n) \to \bigcup_{k=0}^{\infty} \overline{P} \left( \frac{n - k(k + 1)/2}{2} \right)
\]
is a bijection. \( \square \)
We note that Theorems 6.1 and 6.2 provide new identities of Watson type (more on such identities can be found in [9]).

Numerical evidence suggests that $Q_0(n)$ and $Q_2(n)$ satisfy the following linear homogeneous inequalities analogous to those given by Corollaries 4.2 and 4.4.

**Conjecture 1.** For all $n, k \geq 0$ we have

$$( -1 )^{k-1} \left( \sum_{j=0}^{2k-1} ( -1 )^{j(j+1)/2} Q_0( n - j(j+1)/2 ) - \xi_n \right) \geq 0,$$

with strict inequality if and only if $n \geq k(2k + 1)$.

**Remark.** For example, Conjecture 1 implies

$$Q_0(n) - Q_0(n-1) \geq \xi_n,$$
$$Q_0(n) - Q_0(n-1) - Q_0(n-3) + Q_0(n-6) \leq \xi_n,$$
$$Q_0(n) - Q_0(n-1) - Q_0(n-3) + Q_0(n-6) + Q_0(n-10) - Q_0(n-15) \geq \xi_n.$$

**Conjecture 2.** For all $n, k \geq 0$ we have

$$( -1 )^{k-1} \left( \sum_{j=0}^{2k-1} ( -1 )^{j(j+1)/2} Q_2( n - j(j+1)/2 ) - \chi_n \right) \geq 0,$$

with strict inequality if and only if $n \geq k(2k + 1)$.

**Remark.** For example, Conjecture 2 implies

$$Q_2(n) - Q_2(n-1) \geq \chi_n,$$
$$Q_2(n) - Q_2(n-1) - Q_2(n-3) + Q_2(n-6) \leq \chi_n,$$
$$Q_2(n) - Q_2(n-1) - Q_2(n-3) + Q_2(n-6) + Q_2(n-10) - Q_2(n-15) \geq \chi_n.$$

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