LAPLACIAN COFLOW ON THE 7-DIMENSIONAL HEISENBERG GROUP

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ABSTRACT. We study the Laplacian coflow and the modified Laplacian coflow of $G_2$-structures on the 7-dimensional Heisenberg group. For the Laplacian coflow we show that the solution is always ancient, that is it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$. However, for the modified Laplacian coflow, we prove that in some cases the solution is defined only on a finite interval while in other cases the solution is defined for every positive time.

1. Introduction

A 7-dimensional manifold $M$ carries a $G_2$-structure if $M$ admits a globally defined 3-form $\varphi$, which is called $G_2$ form, and it can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \ldots, e^7\}$ of the local 1-forms on $M$. Here, $e^{127}$ stands for $e^1 \wedge e^2 \wedge e^7$, and so on. Such a 3-form $\varphi$ determines a Riemannian metric $g_\varphi$ and an orientation on $M$. If $\nabla$ denotes the Levi-Civita connection of $g_\varphi$, one can view $\nabla \varphi$ as the torsion of the $G_2$-structure $\varphi$. Thus, if $\nabla \varphi = 0$, which is equivalent to $d \varphi = 0$ and $d^* \varphi = 0$, where $^* \varphi$ is the Hodge star operator with respect to $g_\varphi$, one says that the $G_2$-structure is torsion-free.

The different classes of $G_2$-structures can be described in terms of the exterior derivatives $d \varphi$ and $d^* \varphi$ [2, 7]. If $d \varphi = 0$, then the $G_2$-structure is called closed (or calibrated in the sense of Harvey and Lawson [10]) and if $\varphi$ is coclosed, that is if $^* \varphi$ is closed, then the $G_2$-structure is called coclosed (or cocalibrated [10]).

Since Hamilton introduced the Ricci flow in 1982 [9], geometric flows have been an important tool in studying geometric structures on manifolds. The Laplacian flow for closed $G_2$-structures on a 7-manifold $M$ has been introduced by Bryant in [2], and it is given by

$$\begin{cases}
\frac{\partial}{\partial t} \varphi(t) = \Delta_t \varphi(t), \\
d \varphi(t) = 0, \\
\varphi(0) = \varphi,
\end{cases}$$

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where \( \varphi(t) \) is a closed \( G_2 \) form on \( M \), \( \Delta_t = d d^* + d^* d \) is the Hodge Laplacian operator associated with the metric \( g_{\varphi(t)} \) induced by the 3-form \( \varphi(t) \), and \( \varphi \) is the initial closed \( G_2 \)-structure. A short-time existence and uniqueness for this flow, in the case of compact manifolds, has been proved in \([3]\). Regarding the long-time behavior of the Laplacian flow on compact manifolds \( M \), Lotay and Wei in \([19]\) have proved recently that if the initial closed \( G_2 \) form \( \varphi \) is such that its torsion is sufficiently small (in a suitable sense), then the Laplacian flow of \( \varphi \) will exist for all time and converge to a torsion-free \( G_2 \)-structure. Non-compact examples where the flow converges to a flat \( G_2 \)-structure have been given in \([6]\).

A Shi-type derivative estimates for the Riemann curvature tensor and torsion tensor along the Laplacian flow has been determined in \([18]\), and in \([20]\) it is proved that for each fixed positive time \( t \in (0, T] \), \((M, \varphi(t), g_{\varphi(t)})\) is real analytic. Consequently, any Laplacian soliton is real analytic. Moreover, solitons of the Laplacian flow of \( G_2 \)-structures in the homogeneous case have been studied recently by Lauret in \([?]\) using the bracket flow and the algebraic soliton approach.

Some work has also been done on other related flows of \( G_2 \)-structures - such as the Laplacian coflow, or flow, for coclosed \( G_2 \)-structures. This coflow has been originally proposed by Karigiannis, McKay and Tsui in \([15]\) and, for an initial coclosed \( G_2 \) form \( \varphi \) with \( \psi = \ast \varphi \), it is given by

\[
(1) \quad \frac{\partial}{\partial t} \psi(t) = -\Delta_t \psi(t), \quad d\psi(t) = 0, \quad \psi(0) = \psi,
\]

where \( \psi(t) \) is the Hodge dual 4-form of a \( G_2 \)-structure \( \varphi(t) \), that is \( \psi(t) = \ast_t \varphi(t) \), \( \Delta_t \) is the Hodge Laplacian operator with respect to the Riemannian metric \( g_{\varphi(t)} \). This flow preserves the condition of the \( G_2 \)-structure being coclosed, that is \( \psi(t) \) is closed for any \( t \), and it was studied in \([15]\) for two explicit examples of coclosed \( G_2 \)-structures with symmetry, namely for warped products of an interval, or a circle, with a compact 6-manifold \( N \) which is taken to be either a nearly Kähler manifold or a Calabi-Yau manifold. Nevertheless, in \([8]\) it was shown that the coflow \((1)\) is not even a weakly parabolic flow, and that the symbol of the operator \( \Delta_t \), acting on 4-forms, has a mixed signature. But no general result is known about the short time existence of the coflow \((1)\).

A modified Laplacian coflow was introduced by Grigorian in \([8]\)

\[
(2) \quad \frac{\partial}{\partial t} \psi(t) = \Delta_t \psi(t) + 2d \left( (A - \text{Tr}_t(\tau(t))) \varphi(t) \right), \quad d\psi(t) = 0, \quad \psi(0) = \psi,
\]

where \( \text{Tr}_t(\tau(t)) \) is the trace of the full torsion tensor \( \tau(t) \) of the \( G_2 \)-structure defined by \( \varphi(t) \), and \( A \) is a fixed positive constant (see Section \([3]\) for the details). Moreover, in \([8]\) it is proved that the coflow \((2)\) is weakly parabolic in the direction of closed forms \( \psi(t) \) up to diffeomorphisms and, on compact manifolds, it has a unique solution \( \psi(t) \) for the short time period \( t \in [0, \epsilon] \), for some \( \epsilon > 0 \).
In [1], it is given a classification of 2-step nilpotent Lie groups admitting left invariant coclosed $G_2$-structures. In this paper, we study the coflows (1) and (2) in the case of the 7-dimensional Heisenberg group $H$.

As we mentioned before, it is not known any general result on the short time existence of solution for the coflow (1). Nevertheless, in Theorem 4, we show that the solution of the coflow (1) for any coclosed $G_2$-structure on the Heisenberg group is always ancient, that is it is defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. To our knowledge, these are the first examples of non-compact manifolds having a coclosed $G_2$-structure with long time existence of solution for (1). However, we prove that the solution of the coflow (2) for some coclosed $G_2$-forms on $H$ is defined only on a finite interval (Theorem 9) and, for other coclosed $G_2$ forms, the coflow (2) exists for all time (Theorem 7, part ii)) or it is ancient (Theorem 7, part i), and Theorem 8).

Moreover, considering the coflows (1) and (2) on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$, in a similar way as Lauret did in [16] for the Ricci flow, we show that the underlying metrics $g(t)$ of the solution in Corollary 5 and Theorem 8 converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, as $t$ goes to infinity. Indeed, by [16, Proposition 2.8] the convergence of the metrics in the $C^\infty$ uniformly on compact sets in $\mathbb{R}^7$ is equivalent to the convergence of the nilpotent Lie brackets $\mu(t)$ in the algebraic subset of nilpotent Lie brackets $N \subset (\Lambda^2 \mathbb{R}^7)^* \otimes \mathbb{R}^7$ with the usual vector space topology.

2. Coclosed $G_2$-structures on the Heisenberg group

A 7-dimensional manifold $M$ is said to admit a $G_2$-structure if there is a reduction of the structure group of its frame bundle from $\text{GL}(7, \mathbb{R})$ to the exceptional Lie group $G_2$, which can actually be viewed naturally as a subgroup of $\text{SO}(7)$. Thus, a $G_2$-structure determines a Riemannian metric and an orientation on $M$. In fact, one can prove that the presence of a $G_2$-structure is equivalent to the existence of a differential 3-form $\varphi$ (the $G_2$ form) on $M$, which induces the Riemannian metric $g_\varphi$ given by

\[ g_\varphi(X, Y) \, \text{vol} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi, \]

for any vector fields $X, Y$ on $M$, where $\text{vol}$ is the volume form on $M$, and $\iota_X$ denotes the contraction by $X$. Let $*_\varphi$ be the Hodge star operator determined by $g_\varphi$ and the orientation induced by $\varphi$. We will always write $\psi$ to denote the dual 4-form of a $G_2$-structure $\varphi$, that is

\[ \psi = *_\varphi \varphi. \]

A manifold $M$ has a coclosed (or cocalibrated) $G_2$-structure if there is a $G_2$-structure on $M$ such that the $G_2$ form $\varphi$ is coclosed, that is $d\psi = 0$.

Now, let $G$ be a 7-dimensional simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then, a $G_2$-structure on $G$ is left invariant if and only if the corresponding
3-form \( \varphi \) is left invariant. Thus, a left invariant \( G_2 \)-structure on \( G \) corresponds to an element \( \varphi \) of \( \Lambda^3(g^*) \) that can be written as

\[
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\]

with respect to some orthonormal coframe \( \{e^1, \ldots, e^7\} \) of the dual space \( g^* \), where \( e^{127} \) stands for \( e^1 \wedge e^2 \wedge e^7 \), and so on. So the dual form \( \psi = \star \varphi \varphi \) has the following expression

\[
\psi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}.
\]

Note that in order to recover the left invariant \( G_2 \) form \( \varphi \) from the 4-form \( \star \varphi \varphi \) we need to fix an orientation of \( g \). In fact, the stabilizer of \( \star \varphi \varphi \) in \( \text{GL}(7, \mathbb{R}) \) is \( G_2 \times \mathbb{Z}_2 \) since the matrix \(-Id\) preserves the form \( \star \varphi \varphi \), and so the latter fails to determine the overall orientation.

Recall that the seven dimensional Heisenberg group \( H \) is the simply connected nilpotent Lie group whose Lie algebra \( h \) is defined by

\[
h = \left(0, 0, 0, 0, 0, \frac{\sqrt{6}}{6} (e^{12} + e^{34} + e^{56})\right).
\]

This notation means that the dual space \( h^* \) is spanned by \( \{e^1, \ldots, e^7\} \) satisfying

\[
de^i = 0, \quad 1 \leq i \leq 6, \quad de^7 = \frac{\sqrt{6}}{6} (e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6).
\]

3. On the coflows of coclosed \( G_2 \)-structures

Here we show the expression of each one of the coflows \([11]\) and \([2]\) in terms of the intrinsic torsion forms of a coclosed \( G_2 \)-structure \([2, 8]\).

Let \( M \) be a 7-dimensional manifold with a \( G_2 \)-structure defined by a 3-form \( \varphi \). Denote by \( \psi \) the 4-form \( \psi = \star \varphi \varphi \), where \( \star \varphi \) is the Hodge star operator of the metric \( g \varphi \) induced by \( \varphi \). Let \( (\Omega^*(M), d) \) be the de Rham complex of differential forms on \( M \). Then, Bryant in \([2]\) proved that the forms \( d\varphi \) and \( d\psi \) are such that

\[
\begin{align*}
d\varphi &= \tau_0 \psi + 3 \tau_1 \wedge \varphi + \star \varphi \tau_3, \\
d\psi &= 4\tau_1 \wedge \psi - \star \varphi \tau_2,
\end{align*}
\]

where \( \tau_0 \in \Omega^0(M) \), \( \tau_1 \in \Omega^1(M) \), \( \tau_2 \in \Omega^2(M) \) and \( \tau_3 \in \Omega^3(M) \). Here \( \Omega^2(M) \) and \( \Omega^3(M) \) are the spaces

\[
\Omega^2(M) = \{\alpha \in \Omega^2(M) \mid \alpha \wedge \varphi = -\star \varphi \alpha\},
\]

\[
\Omega^3(M) = \{\beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0 = \beta \wedge \star \varphi \varphi\}.
\]
The differential forms $\tau_i$ ($i = 0, 1, 2, 3$) that appear in (7), are called the **intrinsic torsion forms** of $\varphi$. According to Grigorian [8] the **full torsion tensor** $\tau$ of $\varphi$ is the tensor field on $M$ given by

$$
\tau = \frac{1}{4}\tau_0 g_\varphi - \iota_{\tau_1} \varphi - \frac{1}{3} j_\varphi(\tau_3) + \frac{1}{2} \tau_2,
$$

where $\iota_{\tau_1}$ denotes the contraction by $\tau_1$ using the metric $g_\varphi$ induced by $\varphi$ (that is, if $U$ is the vector field on $M$ such that $\tau_1 = \iota_U g_\varphi$, then $\iota_{\tau_1} \varphi = \iota_U \varphi$) and $j_\varphi : \Omega^3(M) \to S^2(M)$ is the map defined by

$$
j_\varphi(\gamma)(X,Y) = \star_\varphi \left( (i_X \varphi) \wedge (i_Y \varphi) \wedge \gamma \right),
$$

where $\gamma \in \Omega^3(M)$, and $X,Y$ are vector fields on $M$ [2]. In particular, by [2] $j_\varphi$ is an isomorphism between the space $\Omega^3_{27}(M)$ and the space $S^2_0(M)$ of trace-free symmetric 2-tensors on $M$.

Recall that $\varphi$ defines a coclosed $G_2$-structure on $M$ if $\psi$ is closed, that is $d\psi = 0$. In this case, (7) implies that the forms $\tau_1$ and $\tau_2$ vanish, and so the full torsion tensor $\tau$ has the following expression

$$
\tau = \frac{1}{4}\tau_0 g_\varphi - \frac{1}{3} j_\varphi(\tau_3).
$$

Since $\tau_3 \in \Omega^3_{27}(M)$, the trace of $j_\varphi(\tau_3)$ vanishes. Therefore, $\text{Tr}(\tau)$ of $\tau$ is given by

$$
\text{Tr}(\tau) = \frac{1}{4}\tau_0 \text{Tr}(g_\varphi) = \frac{7}{4}\tau_0.
$$

**Lemma 1.** Let $M$ be a 7-dimensional manifold with a coclosed $G_2$ form $\varphi$. Denote by $\tau_0$ and $\tau_3$ the torsion forms of $\varphi$. Then, the torsion forms $\tilde{\tau}_0$ and $\tilde{\tau}_3$ of $-\varphi$ satisfy

$$
\tilde{\tau}_0 = -\tau_0, \quad \tilde{\tau}_3 = \tau_3.
$$

**Proof.** Using (7), we see that $\tilde{\tau}_0 = -\tau_0$ and $\tilde{\tau}_3 = \tau_3$ since $\star_{-\varphi} = -\star_\varphi$. \hfill \Box

**Proposition 2.** Let $M$ be a 7-dimensional manifold with a coclosed $G_2$ form $\varphi$. Then, the coflow (1) for $\varphi$ has the following expression

$$
\frac{\partial}{\partial t} \psi(t) = -d(\tau_0(t)) \wedge \varphi(t) - \left( \tau_0(t) \right)^2 \psi(t) - \tau_0(t) * t \tau_3(t) - d\tau_3(t),
$$

$\psi(t) = 0, \quad \varphi(0) = \varphi,$

and the coflow (2) is expressed as

$$
\frac{\partial}{\partial t} \psi(t) = \tau_0(t) \left( 2A - \frac{5}{2} \tau_0(t) \right) \psi(t) + \left( 2A - \frac{5}{2} \tau_0(t) \right) * t \tau_3(t) + d\tau_3(t)
$$

$$
+ \frac{5}{2} \varphi(t) \wedge d\tau_0(t),
$$

$\psi(t) = 0, \quad \varphi(0) = \varphi,$
where $\tau_0(t)$ and $\tau_3(t)$ are the torsion forms of $\varphi(t)$ (according with (7)), $\star_t$ is the Hodge star operator with respect to the Riemannian metric $g_{\varphi(t)}$ induced by $\varphi(t)$ and $A$ is a fixed positive constant.

**Proof.** The coflows (1) and (2) preserve both the closeness of $\psi(t) = \star_t \varphi(t)$. Therefore, by (7) and the vanishing of the torsion forms $\tau_1(t)$ and $\tau_2(t)$ of $\varphi(t)$,

$$d\varphi(t) = \tau_0(t) \psi(t) + \star_t \tau_3(t).$$

Hence,

$$\Delta_t \psi(t) = d d^* \psi(t) = d \star_t d \varphi(t) = d \star_t \left( \tau_0(t) \psi(t) + \star_t \tau_3(t) \right)$$

and

$$2d \left( (A - \text{Tr}(\tau(t))) \varphi(t) \right) = 2d \left( (A - \frac{7}{4} \tau_0(t)) \varphi(t) \right)$$

$$= -\frac{7}{2} d(\tau_0(t)) \wedge \varphi(t) + \left( 2A - \frac{7}{2} \tau_0(t) \right) (\tau_0(t) \psi(t) + \star_t \tau_3(t)).$$

Thus,

$$\Delta_t \psi + 2d \left( (A - \text{Tr}(\tau(t))) \varphi(t) \right) = \frac{5}{2} d(\tau_0(t)) \wedge \varphi(t) + \frac{5}{2} \left( 2A - \frac{5}{2} \tau_0(t) \right) \psi(t)$$

$$+ \left( 2A - \frac{5}{2} \tau_0(t) \right) \star_t \tau_3(t) + d\tau_3(t),$$

and the Proposition follows. \(\square\)

**Remark 1.** Note that (9) and Proposition 2 imply that the solution of the coflow (G) for $\varphi$ (if such a solution exists) changes when the initial coclosed $G_2$ form is $-\varphi$ instead of $\varphi$ (see Theorem 7 and Theorem 8). However, the study of the coflow (C) is independent of whether the initial condition is $\varphi$ or $-\varphi$.

**Remark 2.** By [8], since $\text{Tr}(\tau(t)) = \frac{2}{7} \tau_0(t)$, as long as the condition $0 \leq \frac{2}{7} \tau_0(t) \leq \frac{4}{3} A$ holds for the time of existence, we have the following inequality for the volume

$$A \int_M \frac{7}{4} \tau_0(t) \text{vol} \geq \int_M \frac{3}{4} \left( \frac{7}{4} \tau_0(t) \right)^2 \text{vol}.$$ 

4. **Explicit solutions for the Laplacian coflow**

In this section we study the Laplacian coflow on the seven dimensional Heisenberg Lie group $H$ with structure equations (6).

Let $\varphi_0$ be a left invariant coclosed $G_2$-structure on $H$. Denote by $g_0$ the underling metric and by $\psi_0 = \star_0 \varphi_0$ its Hodge dual.
Let $\eta = \|e^7\|_0^{-1}e^7$. Clearly $\|\eta\|_0 = 1$ and $d\eta \in \Lambda^2 \text{Ker}(\eta)^*$ is a non-degenerate two-form on $\text{Ker}(\eta)$. If we identify $\mathfrak{h}$ with $\text{Ker}(\eta) \oplus \mathfrak{z}$, being $\mathfrak{z} = [\mathfrak{h}, \mathfrak{h}] = \text{Span}(e_7)$ the commutator of $\mathfrak{h}$, then every four-form $\psi \in \Lambda^4 \mathfrak{h}$ has a unique decomposition as
\[
\psi = \psi^{(4)} + \psi^{(3)} \wedge \eta,
\]
where $\psi^{(i)} \in \Lambda^i \text{Ker}(\eta)^*$, $i = 3, 4$ are closed forms.

Denote by $\star_0$ and $\star$ the Hodge operators on $\mathfrak{h}$ and $\text{Ker}(\eta)$, respectively. Note that the $G_2$-structure $\varphi_0$ defines an SU(3)-structure $(\omega_0, \rho_0)$ on $\text{Ker}(\eta)$. Using this fact, the four-form $\psi_0 = \star_0 \varphi_0$ on $\mathfrak{h}$ can be written as
\[
\psi_0 = \frac{1}{2} \omega_0^2 + \hat{\rho}_0 \wedge \frac{1}{\varepsilon_0} \eta,
\]
where $\hat{\rho}_0 = J_0 \rho_0$, being $J_0$ the almost complex structure induced by $(\omega_0, \rho_0)$, and $\varepsilon_0$ is the norm of $\eta$ with respect to $g_0$. Indeed, if $x_0 \in \mathfrak{h}$ is the vector defined by
\[
g_0(x_0, y) = \eta(y),
\]
for every $y \in \mathfrak{h}$, then
\[
\text{Ker}(\eta) = \{y \in \mathfrak{h} \mid g_0(x_0, y) = 0\} = \text{Span}(x_0)^{\perp_0},
\]
and we can apply Proposition 4.5 in [22] to define the SU(3)-structure $(\omega_0, \rho_0)$.

For a general SU(3)-structure on a real vector space we have the following result.

**Lemma 3.** Let $(\omega, \rho)$ be a linear SU(3)-structure on $\mathbb{R}^6$, and let $\alpha \in \Lambda^2(\mathbb{R}^6)^*$. Then the following inequalities hold
\[
1. \|\alpha\|^2 + \|\frac{1}{2}\omega^2 \wedge \alpha\|^2 = \|\alpha \wedge \omega\|^2 \leq 4\|\alpha\|^2;
2. \|\alpha^2\|^2 \leq 6\|\alpha\|^6, \text{ where } \|\cdot\| \text{ is the norm induced by the scalar product defined by the SU(3)-structure } (\omega, \rho).
\]

**Proof.** Let us fix an orthonormal basis $\{e^1, \ldots, e^6\}$ of $(\mathbb{R}^6)^*$ so that $\omega = e^{12} + e^{34} + e^{56}$, and write $\alpha = \sum_{1 \leq h < k \leq 6} a_{hk} e^h \wedge e^k$. Then,
\[
\|\alpha\|^2 = \sum_{1 \leq h < k \leq 6} a_{hk}^2.
\]

On the other hand,
\[
\omega \wedge \alpha = e^{12} \wedge (a_{34} e^{34} + a_{35} e^{35} + a_{36} e^{36} + a_{45} e^{45} + a_{46} e^{46} + a_{56} e^{56}) + e^{34} \wedge (a_{12} e^{12} + a_{13} e^{13} + a_{14} e^{14} + a_{23} e^{23} + a_{24} e^{24} + a_{34} e^{34}) + e^{56} \wedge (a_{12} e^{12} + a_{13} e^{13} + a_{14} e^{14} + a_{23} e^{23} + a_{24} e^{24} + a_{34} e^{34}).
\]

Thus,
\[
\|\omega \wedge \alpha\|^2 = \|\alpha\|^2 + (a_{12} + a_{34} + a_{56})^2 = \|\alpha\|^2 + \left\|\frac{1}{2} \omega^2 \wedge \alpha\right\|^2.
\]
Moreover,

\[ \left\| \frac{1}{2} \omega^2 \wedge \alpha \right\|^2 = \left\| \star (\omega) \wedge \alpha \right\|^2 = (\omega | \alpha)^2 \leq ||\omega||^2 ||\alpha||^2 = 3 ||\alpha||^2. \]

This equality together with \((11)\) and \((12)\) imply the first part of the Lemma.

To prove 2. note that the spectral theorem guarantees the existence of an orthonormal basis of 1-forms \(\{f^1, \ldots, f^6\}\) such that \(\alpha = \lambda_1 f^{12} + \lambda_2 f^{34} + \lambda_3 f^{56}\), for some real numbers \(\lambda_i\) with \(i = 1, 2, 3\). Indeed, any real skew-symmetric matrix can be diagonalized by a unitary matrix. Since the eigenvalues of a real skew-symmetric matrix are imaginary, it is possible to transform it to a block diagonal form by an orthogonal transformation. Therefore,

\[ ||\alpha||^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \]

and

\[ \alpha^3 = 6 \lambda_1 \lambda_2 \lambda_3 f^{123456}. \]

Thus,

\[ ||\alpha^3||^2 = 36 \lambda_1^2 \lambda_2^2 \lambda_3^2, \]

and 2. follows. \(\square\)

**Theorem 4.** Let \(H\) be the seven dimensional Heisenberg group whose Lie algebra is defined by \((\mathfrak{g})\). Then, for any left invariant coclosed \(G_2\) form \(\varphi_0\), the solution \(\phi_t\) of the Laplacian coflow \((\mathfrak{g})\) with initial condition \(\psi_0 = \star_0 \varphi_0\) is given by

\[
\psi(t) = \frac{1}{2} \omega(t)^2 + \hat{\rho}(t) \wedge \frac{1}{6} \eta,
\]

where \(6 \varepsilon_t^2 = \star_0 (\omega(t)^3)\), and \(\omega(t)\) and \(\hat{\rho}(t)\) are forms on \(\text{Ker}(\eta)\), given respectively by

\[
\omega(t) = \lambda_1(t) f^{12} + \lambda_2(t) f^{34} + \lambda_3(t) f^{56},
\]

\[
\hat{\rho}(t) = \sqrt{\lambda_1(t) \lambda_2(t) \lambda_3(t)} \left( -f^{246} + f^{136} + f^{145} + f^{235} \right),
\]

with respect to some \(g_0\)-orthonormal frame \(\{f_1, \ldots, f_6\}\) of \(\text{Ker}(\eta)\), and the functions \(\lambda_i(t), i = 1, 2, 3\), satisfy

\[
\begin{align*}
\lambda'_1(t) &= -\frac{\lambda_2(t) \lambda_1(t) + n_2 n_3 \lambda_3'(t)}{\lambda_1(t) \lambda_2(t) \lambda_3(t)}, \\
\lambda'_2(t) &= -\frac{\lambda_1(t) \lambda_3(t) + n_1 n_3 \lambda_2'(t)}{\lambda_1(t) \lambda_2(t) \lambda_3(t)}, \\
\lambda'_3(t) &= -\frac{\lambda_1(t) \lambda_2(t) + n_1 n_2 \lambda_3'(t)}{\lambda_1(t) \lambda_2(t) \lambda_3(t)}, \\
\lambda_1(0) &= \omega(0)(f_1, f_2), \quad \lambda_2(0) = \omega(0)(f_3, f_4), \quad \lambda_3(0) = \omega(0)(f_5, f_6),
\end{align*}
\]

for \(n_j \in \{1, -1\}\). In particular, the solution is ancient with singular time \(0 < T < \frac{1}{\sqrt{6 ||\alpha||_0^2}}\).
Proof. We are going first to show that the system (11) implies the conditions (13). Let \( \psi(t) \) be the solution of \( (11) \) with initial condition \( \psi_0 = *_0 \varphi_0 \), and let \( \varepsilon_t \) be the norm \( ||\eta||_t \) of \( \eta \) with respect to the metric induced by \( \psi(t) \). Then we can write

\[
\psi(t) = \frac{1}{2} \omega(t)^2 + \tilde{\rho}(t) \wedge \frac{1}{\varepsilon_t} \eta,
\]

where the pair \( (\omega(t), \rho(t)) \) defines an SU(3)–structure on \( \text{Ker}(\eta) \) and \( \tilde{\rho}(t) = J_t \rho(t) \).

In fact, if \( x_t \in \mathfrak{h} \) is the vector defined by \( g_t(x_t, y) = \eta(y) \), for any \( y \in \mathfrak{h} \), then \( \text{Ker}(\eta) = \{ y \in \mathfrak{h} | g_t(x_t, y) = 0 \} \) is the orthogonal complement of the span of \( x_t \) with respect to \( g_t \). Thus we can apply Proposition 4.5 in [22].

With respect to the decomposition \( (10) \) we have

\[
\psi(t) = \psi^{(4)}(t) + \psi^{(3)}(t) \wedge \eta,
\]

and so,

\[
\psi^{(4)}(t) = \frac{1}{2} \omega(t)^2, \quad \psi^{(3)}(t) = \frac{1}{\varepsilon_t} \tilde{\rho}(t).
\]

Moreover, the forms \( \omega(t) \in \Lambda^2 \text{Ker}(\eta)^* \) and \( \tilde{\rho}(t) \in \Lambda^3 \text{Ker}(\eta)^* \) are closed. Since the cohomology class of \( \psi(t) \) is fixed by the flow, it turns out that \( \frac{d}{dt} \psi(t) \) is exact, and hence

\[
\frac{d}{dt} \psi(t) = \frac{d}{dt} \psi^{(4)}(t) + \frac{d}{dt} \psi^{(3)}(t) \wedge \eta \in d\Lambda^3 \mathfrak{h}^* \subseteq \Lambda^4 \text{Ker}(\eta)^*.
\]

Therefore, \( \frac{d}{dt} \psi^{(3)}(t) = 0 \) and

\[
\tilde{\rho}(t) = \varepsilon_t \psi^{(3)}(0) = \varepsilon_t \tilde{\rho}_0.
\]

Consequently, the almost complex structure \( J_t \) defined by \( (\omega(t), \rho(t)) \) does not change along the flow, i.e. \( J_t \equiv J_0 \), where \( J_0 \) is the almost complex structure defined by \( (\omega_0, \rho_0) \). Thus \( \rho(t) = -J_0 \tilde{\rho}(t) = \varepsilon_t \rho_0 \) and

\[
\frac{1}{6} \omega(t)^2 = \frac{1}{4} \rho(t) \wedge \tilde{\rho}(t) = \varepsilon_t^2 \ast_0 (1).
\]

Now let us compute the Laplacian of \( \psi(t) \) with respect to the metric \( g_t \):

\[
\begin{align*}
  d \ast_t d \ast_t \psi(t) &= d \ast_t d \ast_t \left( \frac{1}{2} \omega(t)^2 + \tilde{\rho}(t) \wedge \varepsilon_t^{-1} \eta \right) \\
  &= d \ast_t d(\omega(t) \wedge \varepsilon_t^{-1} \eta + \rho(t)) \\
  &= d \ast_t (\varepsilon_t^{-1} \omega(t) \wedge d\eta) \\
  &= d (\varepsilon_t^{-2} \ast_t (\omega(t) \wedge d\eta) \wedge \eta) \\
  &= \varepsilon_t^{-2} \ast_t (\omega(t) \wedge d\eta) \wedge d\eta.
\end{align*}
\]

On the other hand we have

\[
\frac{d}{dt} \psi_t = \frac{d}{dt} \left( \frac{1}{2} \omega(t)^2 \right).
\]
Thus, by $\frac{d}{dt} \psi(t) = -\Delta_t \psi(t)$ we obtain

$$\frac{1}{2} \left( \frac{d}{dt} \omega(t)^2 \right) = -\varepsilon_t^{-2} (\omega(t) \wedge d\eta) \wedge d\eta.$$  

We observe that

$$0 = \varepsilon_t d\psi(t) = \hat{\rho}(t) \wedge d\eta$$

and so $d\eta$ is $J_0$-invariant, i.e. $J_0(d\eta) = d\eta$.

Fixing a frame $(x_1, \ldots, x_6)$ of $\text{Ker}(\eta)$, we get

$$\ast_t(d\eta \wedge \omega(t)) = \sum_{1 \leq i < j \leq 6} \ast_t((d\eta)_{ij} x^j \wedge \omega(t))$$

where $(\cdot, \cdot)_t$ denotes the scalar product induced by $g_t$ and $\ast$ is the contraction.

Therefore we can reformulate (14) as

$$\frac{d}{dt} \left( \frac{1}{2} \omega(t)^2 \right) = -\varepsilon_t^{-2} [(d\eta, \omega(t))_t \omega(t) \wedge d\eta - d\eta \wedge d\eta],$$

with $\omega(t) \in \Lambda^2 \text{Ker}(\eta)^*$. Define on $\text{Ker}(\eta)$ the following bilinear form

$$h(x, y) = d\eta(x, J_0 y), \quad x, y \in \text{Ker}(\eta).$$

Since $J_0(d\eta) = d\eta$, we have that $h$ is symmetric. If we consider $\text{Ker}(\eta)$ as complex vector space through $J_0$, then both $g_0$ and $h$ define on $\text{Ker}(\eta)$ sesquilinear forms $g^c$ and $h^c$, respectively. Clearly $g^c$ is positive definite while $h^c$ is non-degenerate with mixed signature. By the spectral theorem there exists a complex $g^c$-orthonormal basis $\{k_1, k_2, k_3\}$ of the complex vector space $\text{Ker}(\eta)$ such that

$$h^c(k_i, k_j) = \delta_{ij} \frac{n_i}{l_i}, \quad n_i \in \{1, -1\}, \quad l_i > 0.$$  

Therefore, if $\{k^1, k^2, k^3\}$ is the dual basis of $\{k_1, k_2, k_3\}$, putting $f^i = \sqrt{l_i} k^i$, for $i = 1, 2, 3$, we get

$$d\eta = -\sum_{i=1}^3 n_i f^i \wedge J_0 f^i, \quad \omega_0 = -\sum_{i=1}^3 l_i f^i \wedge J_0 f^i.$$
We claim that there exist positive functions \( \lambda_1(t), \lambda_2(t) \) and \( \lambda_3(t) \) such that

(16) \[ \omega(t) = -3 \sum_{i=1}^{3} \lambda_i(t) f^i \wedge J_0 f^i \]

and \( \lambda_j(0) = I_j \). Indeed, suppose that \( \omega(t) \) is given by (16), then \( \omega(t) \) is a non-degenerate 2-form of type \((1,1)\) with respect to \( J_0 \), and

\[
\frac{d}{dt} \left( \frac{1}{2} \omega(t)^2 \right) = \sum_{i<j} \{ \lambda_i'(t) \lambda_j(t) + \lambda_i(t) \lambda_j'(t) \} f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j.
\]

Now, from the equation (15) and using

\[
\varepsilon_t^2 = \frac{1}{6} \ast_0 \omega(t)^3 = \lambda_1(t) \lambda_2(t) \lambda_3(t),
\]

we get

\[
\sum_{i<j} \{ \lambda_i'(t) \lambda_j(t) + \lambda_i(t) \lambda_j'(t) \} f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j =
\]

\[
- \frac{1}{\varepsilon_t^2} \sum_{i<j} \left[ \left( \frac{n_1}{\lambda_1(t)} + \frac{n_2}{\lambda_2(t)} + \frac{n_3}{\lambda_3(t)} \right) (n_i \lambda_j(t) + n_j \lambda_i(t)) - 2n_i n_j \right] f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j.
\]

This is the system of ordinary differential equations given by (13). Clearly the system (13) has an unique solution and therefore, by the unicity of the solution of the Laplacian coflow, we obtain that the solution of (11) is

\[
\psi(t) = \frac{1}{2} \omega(t)^2 + \tilde{\rho}(t) + \frac{1}{\varepsilon_t} \eta,
\]

where \( \tilde{\rho}(t) = \varepsilon_t \tilde{\rho}(0) \) and \( \omega(t) \) is given by (16) with the functions \( \lambda_i(t), \ i = 1, 2, 3, \) satisfying (13).

Let \( (\tau, T) \) be the maximal interval of existence of \( \psi(t) \), where \( -\tau, T \in (0, +\infty) \). We want to prove that \( T < +\infty \) and \( \tau = -\infty \). First observe that the condition \( 6 \varepsilon_t^2 = \ast_0(\omega(t)^3) \) implies

\[
12 \varepsilon_t \varepsilon_t' = 3 \ast_0 \left( \frac{d}{dt} \omega(t) \wedge \omega(t)^2 \right).
\]

Then,

\[
\varepsilon_t' = -\frac{\varepsilon_t^2}{4\varepsilon_t} \ast_0 \left[ \ast_0 (\omega(t) \wedge d\eta) \wedge \omega(t) t \wedge d\eta \right] = -\frac{\varepsilon_t^2}{4\varepsilon_t} \ast_0 (||\omega(t) \wedge d\eta||_t^2 \ast_t (1)) = -\frac{1}{4\varepsilon_t} \ast_0 (||\omega(t) \wedge d\eta||_t^2 \varepsilon_t^2 \ast_0 (1)) = -\frac{1}{4\varepsilon_t} ||\omega(t) \wedge d\eta||_t^2.
\]
This implies that $\varepsilon'_t < 0$ and also the existence of $\lim_{t \to T} \varepsilon_t = \varepsilon_T \geq 0$. Observe that

$$\tfrac{1}{\varepsilon_t} \eta = \tfrac{1}{\varepsilon_t}(1) = \varepsilon_t^2 \ast_0 (1) = \frac{1}{6} \delta_0 (d\eta)^3,$$

where

$$\delta_0 = \frac{1}{6} \|(d\eta)^3\|_0.$$ 

Thus,

$$\frac{1}{6} \|(d\eta)^3\|^2_t = \frac{\delta_0^2}{\varepsilon_t^4}.$$ 

Now by Lemma 3 we have

$$\|d\eta\|^6_t \geq 6 \|1/6(d\eta)^3\|^2_t = \frac{6\delta_0^2}{\varepsilon_t^4},$$

and

$$\|\omega(t) \wedge d\eta\|^2_t \geq \|d\eta\|^2_t.$$ 

Therefore, we can estimate $\varepsilon'_t$ as follows

$$\varepsilon'_t = -\frac{1}{4\varepsilon_t} \|\omega(t) \wedge d\eta\|^2_t \leq -\frac{1}{4\varepsilon_t} \|d\eta\|^2_t \leq -\frac{1}{4\varepsilon_t} \left(6 \|1/6(d\eta)^3\|^2_t\right)^{\frac{1}{3}} = -\frac{\sqrt[3]{6 \delta_0^2}}{4\varepsilon_t \sqrt[3]{\varepsilon_t}} = -\frac{C}{\varepsilon_t^{1+\frac{1}{3}}}.$$ 

As a consequence, if $t \in (0, T)$, we get

$$\varepsilon_t - 1 = \int_0^t \varepsilon'_s ds \leq -C \int_0^t \frac{1}{\varepsilon_s^{1+\frac{1}{3}}} ds \Rightarrow t = \int_0^t \frac{1}{\varepsilon_s^{1+\frac{1}{3}}} ds \leq \frac{1 - \varepsilon_t}{C},$$

where we have used that $\varepsilon_s < 1$ if $s \in (0, t)$. So

$$T \leq \frac{1 - \varepsilon_T}{C} = 4 \frac{1 - \varepsilon_T}{\sqrt[3]{6 \delta_0^2}} \leq \frac{4}{\sqrt[3]{6 \delta_0^2}}.$$ 

It remains to show that $\tau = -\infty$. Firstly, we prove that it is true if $h$ is positive or negative definite. Note that in this case $n_i n_j = 1$, for every $i, j = 1, 2, 3$. So $\lambda_i'(t) < 0$, for any $i = 1, 2, 3$. Define

$$f(t) = \lambda_1(t) + \lambda_2(t) + \lambda_3(t).$$

Then it is clear that the solution exists until $f(t) < +\infty$ and, consequently, $f(\tau) = +\infty$. Now observe that by (13),

$$f''(t) = -\frac{a}{dt} \left( \sum_{a, b, c} \frac{\lambda_a^3 + \lambda_b \lambda_c (t)}{\lambda_a^3 + \lambda_b^2 + \lambda_c^2} \right)$$

$$= -\sum_{a, b, c} \frac{2\lambda_a \lambda_b^3 \lambda_c^3 + 2\lambda_a \lambda_b \lambda_c^3 - 2\sum_{i, j, k} (\lambda_i \lambda_j \lambda_k) \lambda_i^2 \lambda_j^2 \lambda_k^2}{\lambda_a^3 + \lambda_b^2 + \lambda_c^2},$$

where $(a, b, c)$ and $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. But, from (13), for any $(a, b, c)$, it follows

$$2\lambda_a \lambda_b \lambda_c \lambda_1^2 \lambda_2^2 \lambda_3^2 - 2(\lambda_a \lambda_b^2 \lambda_c^2) \lambda_a^2 = 0,$$
and
\[ \lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_2^2 - 2(\lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_3) \lambda_0 \lambda_2 = \lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_3 (\lambda_3 - 2\lambda_2) > 0, \]
\[ \lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_2^2 = 2(\lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_3) \lambda_0 \lambda_2 = \lambda_0 \lambda_2 \lambda_1^2 \lambda_1^2 \lambda_3 (\lambda_3 - 2\lambda_2) > 0. \]

Therefore \( f''(t) < 0 \), for \( t \in (\tau, T) \). But, for \( t \in (\tau, 0) \),
\[ f(0) - f(t) = \int_t^0 f'(s)ds \geq \int_t^0 f'(0)ds = -tf'(0). \]

Thus,
\[ f(t) \leq f(0) + tf'(0), \]
which means \( \tau = -\infty \).

In order to prove that \( \tau = -\infty \) if \( h \) is indefinite, we proceed by contradiction as follows. Since that \( h \) is indefinite, we can assume that \( n_1 n_2 = n_1 n_3 = -n_2 n_3 = -1 \). Moreover, if \( \tau > -\infty \) and \( \lambda_1(t), \lambda_2(t) \) and \( \lambda_3(t) \) are all bounded near \( \tau \) then, by maximality, at least one of them must go to zero; but this is not possible because \( \lambda_1(t) \lambda_2(t) \lambda_3(t) = 1/6 \ast_0 (\omega_2^*) = \mathcal{E}^2_2 \) decreases. Therefore, if \( \tau > -\infty \), there is at least one \( \lambda_i(t) \) \( (i = 1, 2, 3) \) which should be unbounded. Suppose now that \( \lambda_2(t) \) is unbounded. Then, choosing a sequence of negative times \( \{t_n\} \) converging to \( \tau \) and such that \( \lambda_2(t_n) - \lambda_2(t_{n-1}) \) diverges, it follows that
\[ \lambda_2(t_n) - \lambda_2(t_{n-1}) = -\int_{t_{n-1}}^{t_n} \frac{\lambda_1(s) \lambda_3(s) - \lambda_2(s)}{\lambda_1(s)^2 \lambda_2(s)^2}ds \]
\[ = -\int_{t_{n-1}}^{t_n} \left( \frac{1}{\lambda_1(s)^2 \lambda_3(s)^2} - \frac{1}{\lambda_1(s)^2 \lambda_3(s)^2} \right)ds \]
\[ = \left( -\frac{1}{\lambda_1(t_n) \lambda_2(t_n)} + \frac{1}{\lambda_1(t_{n-1}) \lambda_2(t_{n-1})} \right) \to +\infty, \]
where \( \bar{t}_n \in (t_n, n-1) \). Hence \( \lambda_1(\bar{t}_n) \lambda_3(\bar{t}_n) \to 0 \). Indeed \( \lambda(\bar{t}_n)^2 \lambda_2(\bar{t}_n) \) stays away from zero. But for \( n \) large
\[ 0 > \frac{\lambda_2(t_n) - \lambda_2(t_{n-1})}{t_n - t_{n-1}} = -\frac{\lambda_1(t_n) \lambda_3(t_n) - \lambda_2(t_n)}{\lambda_1(t_n)^2 \lambda_2(t_n)^2} \]
\[ > 0. \]

Thus \( \lambda_2(t) \) must be bounded. The same argument shows that \( \lambda_3(t) \) must be bounded as well. So, the only possibility is that \( \lambda_1(t) \) is unbounded but both \( \lambda_2(t) \) and \( \lambda_3(t) \) are bounded. As done previously choose \( \{t_n\} \) so that \( t_n \to \tau \) and
\[ \lambda_1(t_n) - \lambda_1(t_{n-1}) = -\int_{t_{n-1}}^{t_n} \frac{\lambda_2(s) \lambda_3(s) + \lambda_1(s)^2}{\lambda_1(s)^2 \lambda_2(s)^2 \lambda_3(s)^2}ds \to +\infty. \]

Therefore, there exists \( \bar{t}_n \in (t_n, t_{n-1}) \) such that \( \lambda_2(\bar{t}_n) \lambda_3(\bar{t}_n) \to 0 \). We can certainly assume that \( \lambda_3(t_n) \lambda_3(\bar{t}_n) \) decreases in \( n \) by choosing a suitable subsequence which we will still denote by \( t_n \). Then,
\[ 0 > \lambda_2(t_n) \lambda_3(t_n) - \lambda_2(t_{n-1} \lambda_3(t_{n-1})) = \left( \frac{d}{dt}(\lambda_2 \lambda_3) \right)(s_n)(t_n - t_{n-1}), \]
for some $s_n \in (t_n, t_{n-1})$. On the other hand, by (13), it turns out that
\[
\frac{d}{dt} (\lambda_2(t)\lambda_3(t)) = -\frac{\lambda_1(t)(\lambda_2(t)^2 + \lambda_3(t)^2) - \lambda_2(t)\lambda_3(t)(\lambda_2(t) + \lambda_3(t))}{\lambda_1(t)^2\lambda_2(t)^2\lambda_3(t)^2}.
\]
Since
\[
\lambda_1(t_n) \to +\infty, \quad \frac{\lambda_2(t_n)\lambda_3(t_n)(\lambda_2(t_n) + \lambda_3(t_n))}{\lambda_2(t_n)^2} \to 0,
\]
we obtain that $\frac{d}{dt} (\lambda_2\lambda_3)(s_n) < 0$, for $n$ large. Then we get the following contradiction:
\[
0 > \lambda_2\lambda_3(t_n) - \lambda_2\lambda_3(t_{n-1}) = \left(\frac{d}{dt}\lambda_2\lambda_3\right)(s_n)(t_n - t_{n-1}) > 0.
\]
Thus, $\lambda_1(t)$ must be bounded, which completes the proof.

We now solve the coflow (1) on the 7-dimensional Heinsenber group when the initial coclosed $G_2$ form is equal to $\varphi_i$ ($i = 1, 2$), where $\varphi_1$ and $\varphi_2$ are defined by
\[
(17) \quad \varphi_1 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\]
and
\[
(18) \quad \varphi_2 = e^{127} - e^{347} - e^{567} + e^{135} - e^{146} + e^{236} + e^{245},
\]
respectively. We will show in the next section that the behaviour of the solution for the modified co-clow is different.

**Corollary 5.** The solution of the Laplacian coflow (1) on $H$ with the initial coclosed $G_2$ form $\varphi_1$, defined by (17), is given by
\[
(19) \quad \varphi(t) = \frac{1}{y(t)}(e^{127} + e^{347} + e^{567}) + y(t)^3 (e^{135} - e^{146} - e^{236} - e^{245}), \quad t \in \left(-\infty, \frac{3}{5}\right),
\]
where $y = y(t)$ is the positive function
\[
(20) \quad y(t) = \sqrt[10]{1 - \frac{5}{3}t}.
\]
The underlying metrics $g_t$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $H$ as $t$ goes to $-\infty$.

**Proof.** For each $t \in (-\infty, \frac{3}{5})$, we consider the basis $\{f^1(t), \ldots, f^7(t)\}$ of left invariant 1-forms on $H$ defined by
\[
f^i = f^i(t) = y(t) e^i, \quad 1 \leq i \leq 6,
\]
\[
f^7 = f^7(t) = y(t)^{-3} e^7,
\]
where the function $y = y(t)$ is given by (20). Then, $f^i(0) = e^i$, for $i \in \{1, \cdots, 7\}$, and the structure equations of $H$, with respect to the basis $\{f^1(t), \ldots, f^7(t)\}$, are

$$
df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y(t)^{-5} (f^{12} + f^{34} + f^{56}).$$

Now, for any $t$, the 3-form $\varphi(t)$ defined by (19) has the following expression

$$\varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}.$$  

Note that $\varphi(0) = \varphi_1$ and, for any $t$, the 3-form $\varphi(t)$ on $H$ induces the metric $g_t$ such that the coframe $\{f^1(t), \ldots, f^7(t)\}$ of $\mathfrak{h}^*$ is orthonormal. Denote by $\star_t$ the Hodge star operator determined by $g_t$. Using (11), (15) and (21), we have $d \star_t \varphi(t) = 0$, where the 4-form

$$\star_t \varphi(t) = f^{1234} + f^{2356} + f^{1367} + f^{1457} - f^{2357} - f^{2467} + f^{3456},$$

So, in terms of the coframe $\{e^1, \ldots, e^7\}$ of $\mathfrak{h}^*$, $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4 (e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$ 

Thus,

$$\frac{d}{dt} (\star_t \varphi(t)) = 4y(t)^3 y'(t) (e^{1234} + e^{1256} + e^{3456}).$$

Moreover, using (21) and (22), we have

$$-\Delta_t \star_t \varphi(t) = -d \star_t d \varphi(t) = -\frac{\sqrt{6}}{3} y(t)^{-5} d \star_t (f^{1234} + f^{1256} + f^{3456})$$

$$= -\frac{2}{3} y(t)^{-10} (f^{1234} + f^{1256} + f^{3456}),$$

or, equivalently,

$$-\Delta_t \star_t \varphi(t) = -\frac{2}{3} y(t)^{-6} (e^{1234} + e^{1256} + e^{3456}).$$

The last equality and (23) prove that (19) is the solution of the coflow (11) when the function $y = y(t)$ is given by (20).

We study the behavior of the underlying metric $g_t$ of the solution $\varphi(t)$ in the limit for $t \to -\infty$. The limit can be computed fixing the $G_2$-structure and changing the Lie bracket as in [17]. If we evolve the Lie brackets $\mu(t)$ instead of the 3-form defining the $G_2$-structure, the corresponding bracket flow has a solution for every $t$. Indeed, if we fix on $\mathbb{R}^7$ the 3-form $f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}$, then the basis $(f_1(t), \ldots, f_7(t))$ defines, for every $t < 3/5$, a nilpotent Lie algebra with bracket $\mu(t)$ such that $\mu(0)$ is the Lie bracket of $\mathfrak{h}$. Moreover, the solution converges to the null bracket corresponding to the abelian Lie algebra. □

In a similar way we can prove the following
Corollary 6. The solution of the Laplacian coflow (1) on H with initial coclosed G\(_2\) form \(\phi_2\), defined by (18), is ancient and it is given by

\[
\phi(t) = \frac{y(t)}{z(t)^2} e^{127} - \frac{1}{y(t)} e^{347} - \frac{1}{y(t)} e^{567} + y(t)z(t)^2 \left( e^{135} - e^{146} + e^{236} + e^{245} \right),
\]

where the functions \(y = y(t)\) and \(z = z(t)\) satisfy

\[
\begin{cases}
\frac{d}{dt} y(t) = -\frac{1}{12} \frac{y(t)^4 + z(t)^4}{y(t)z(t)^8}, & \frac{d}{dt} z(t) = \frac{1}{12} \frac{z(t)^2 - y(t)^2}{y(t)z(t)^8}, \\
y(0) = 1, & z(0) = 1.
\end{cases}
\]

5. Explicit solutions for the modified Laplacian coflow

We study the modified Laplacian coflow (2) for each of the coclosed G\(_2\) forms \(\phi_i\), \(i = 1, 2\), defined respectively by (17) and (18), on the 7-dimensional Heisenberg group. In particular, we prove that the solution of (2) for \(\phi_1\) is ancient only if the positive constant \(A\), that appears in (2), take values in a certain open interval, while the solution of (2) for \(-\phi_1\) is ancient for any \(A\). However, we prove that the solution of (2) for \(\phi_2\) is never ancient.

Theorem 7. The solution of the modified Laplacian coflow (2) for the coclosed G\(_2\) form \(\phi_1\), defined by (17), is given by

\[
\varphi(t) = \frac{1}{y(t)} \left( e^{127} + e^{347} + e^{567} \right) + y(t)^3 \left( e^{135} - e^{146} - e^{236} - e^{245} \right),
\]

where the function \(y = y(t)\) satisfies

\[
\begin{cases}
\frac{d}{dt} y(t) = \frac{2A\sqrt{6}y(t)^5 - 1}{12y(t)^8}, \\
y(0) = 1.
\end{cases}
\]

Moreover,

i) if \(0 < A < \frac{1}{2\sqrt{6}}\), then \(t \in (-\infty, T)\), with \(T = -\frac{1}{10 A^2} \left( 2\sqrt{6} A + \log \left( 1 - 2\sqrt{6} A \right) \right) > 0\). Therefore, in this case, the solution (26) is ancient;

ii) if \(A \geq \frac{1}{2\sqrt{6}}\), then \(t \in (-\infty, +\infty)\).

Proof. By Cauchy Theorem, there exist a maximal open interval \(I\), containing 0, and a smooth function \(y : I \to (0, +\infty)\), which is the unique solution of (27).

To prove that (26) is the solution to the coflow (2) for \(\varphi_1\), we proceed as follows. As in the proof of Theorem 5, for each \(t \in I\), we consider the basis \(\{f^1(t), \ldots, f^7(t)\}\) of left invariant 1-forms on \(H\) defined by

\[
\begin{align*}
\hat{f}^i &= f^i(t) = y(t) e^i, & i &= 1, \ldots, 6, \\
\hat{f}^7 &= f^7(t) = y(t)^{-3} e^7,
\end{align*}
\]
where the function \( y = y(t) \) satisfies now (27). Then, \( f^i(0) = e^i \), for \( i \in \{1, \ldots, 7\} \), and the structure equations of \( H \), with respect to the basis \( \{f^1(t), \ldots, f^7(t)\} \), are

\[
df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y^{-5}(t)(f^{12} + f^{34} + f^{56}).
\]

Moreover, for any \( t \in I \), the 3-form \( \varphi(t) \) defined by (26) has the following expression

\[
\varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}.
\]

So, \( \varphi(0) = \varphi_1 \) and, for any \( t \in I \), the 3-form \( \varphi(t) \) on \( H \) induces the metric \( g_t \) such that the coframe \( \{f^1(t), \ldots, f^7(t)\} \) of \( \mathfrak{h}^* \) is orthonormal. Denote by \( \star_t \) the Hodge star operator determined by \( g_t \). Using (1), (5) and (28), we have \( d \star_t \varphi(t) = 0 \), where \( \star_t \varphi(t) \) is given by

\[
\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.
\]

Thus, in terms of the coframe \( \{e^1, \ldots, e^7\} \) of \( \mathfrak{h}^* \), the 4-form \( \star_t \varphi(t) \) has the following expression

\[
\star_t \varphi(t) = y(t)^4(e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.
\]

This implies

\[
\frac{d}{dt} \star_t \varphi(t) = 4 y(t)^3 y'(t)(e^{1234} + e^{1256} + e^{3456}),
\]

that is

\[
\frac{d}{dt} \star_t \varphi(t) = \frac{2A\sqrt{6} y(t)^5 - 1}{3 y(t)^6}(e^{1234} + e^{1256} + e^{3456}),
\]

since the function \( y = y(t) \) satisfies (27).

On the other hand, by (7) we know that the torsion forms \( \tau_i(t) \) \( (i = 0, 1, 2, 3) \) of \( \varphi(t) \) are such that \( \tau_1(t) = 0 = \tau_2(t) \) since \( d(\star_t \varphi(t)) = 0 \). Then, from (28), (29) and (7), we have

\[
d\varphi(t) = \frac{\sqrt{6}}{3 y(t)^5}(f^{1234} + f^{1256} + f^{3456}) = \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t),
\]

where

\[
\tau_3(t) = \frac{\sqrt{6}}{y(t)^5}(-f^{135} + f^{146} + f^{236} + f^{245} + \frac{4\sqrt{6}}{21 y(t)^5}(f^{127} + f^{347} + f^{567}),
\]

\[
\star_t \tau_3(t) = \frac{\sqrt{6}}{y(t)^5}(-f^{1367} - f^{1457} - f^{2357} + f^{2467} + \frac{4\sqrt{6}}{21 y(t)^5}(f^{1234} + f^{1256} + f^{3456}),
\]

and

\[
\tau_0(t) = \frac{\sqrt{6}}{7 y(t)^5}.
\]
So, according with the first equality of (31),

\[
\Delta_t \ast_t \phi(t) + 2d\left((A - \frac{7}{4} \tau_0)\phi(t)\right) = d \ast_t d(\phi(t)) + 2(A - \frac{7}{4} \tau_0)d\phi(t) = \frac{2A\sqrt{6} y(t)^5 - 1}{3 y(t)^5} (f^{1234} + \phi^{1256} + f^{3456}) ,
\]

that is

\[
\Delta_t \ast_t \phi(t) + 2d\left((A - \frac{7}{4} \tau_0)\phi(t)\right) = \frac{2A\sqrt{6} y(t)^5 - 1}{3 y(t)^5} (e^{1234} + e^{1256} + e^{3456}) .
\]

The last equality, together with (8) and (30), show that (26) solves the modified Laplacian coflow (2) for \( \phi(t) \).

In order to show that the solution \( \phi(t) \), given by (26), is ancient, we analyse the behaviour of the function \( y = y(t) \) according with the values of the positive constant \( A \). If \( A = \frac{1}{2\sqrt{6}} \), then (27) implies that \( y'(t) = 0 \). Thus, \( y(t) \equiv 1 \), for all \( t \in (-\infty, +\infty) \), since \( y(0) = 1 \). Assume \( A \neq \frac{1}{2\sqrt{6}} \). In this case, we first observe that \( y = y(t) \) is monotone. Indeed, the constant function \( \hat{y}(t) \equiv (2\sqrt{6}A)^{-1/5} \) satisfies the differential equation that appears in (27), which is autonomous, and so their solutions are monotone functions. Hence, the solution \( y = y(t) \) of the system (27) is monotone and it must satisfy either \( y(t) > \hat{y}(t) \) or \( y(t) < \hat{y}(t) \) for any \( t \in I \), according to the value of \( A \) and the fact \( y(0) = 1 \), that is if \( 2\sqrt{6}A < 1 \), then \( y(t) < \hat{y}(t) \), but if \( 2\sqrt{6}A > 1 \), then \( y(t) > \hat{y}(t) \).

Now, we rewrite the differential equation that appears in (27) as

\[
\left(\frac{\sqrt{6}}{A} y(t)^4 + \frac{\sqrt{6}}{A} \frac{y(t)^4}{2\sqrt{6}Ay(t)^5 - 1}\right) y'(t) = 1 .
\]

Integrating this equation from 0 to \( t \), we have

\[
(32) t = \frac{\sqrt{6}}{5A} (y(t)^5 - 1) + \frac{1}{10A^2} \log \left| \frac{1 - 2\sqrt{6}Ay(t)^5}{1 - 2\sqrt{6}A} \right| .
\]

Therefore, taking into account (32), if \( 2\sqrt{6}A < 1 \), then \( y = y(t) \) decreases from \((2\sqrt{6}A)^{-1/5}\) to 0 as \( t \) goes from \(-\infty\) to \(-\frac{2A\sqrt{6}+\log(1-2\sqrt{6}A)}{10A^2}\). Otherwise, if \( 2\sqrt{6}A > 1 \), then \( y = y(t) \) increases from \((2\sqrt{6}A)^{-1/5}\) to \(+\infty\) as \( t \) goes from \(-\infty\) to \(+\infty\). In particular, we have that the definition interval \( I \) of the function \( y = y(t) \) is

\[
I = (-\infty, -\frac{2\sqrt{6}A+\log(1-2\sqrt{6}A)}{10A^2}), \quad \text{if } A < \frac{1}{2\sqrt{6}} ,
\]

and

\[
I = (-\infty, +\infty), \quad \text{if } A \geq \frac{1}{2\sqrt{6}} .
\]

□
Remark 3. In a similar way as in the proof of Theorem 3, one can check that the Riemannian curvature $R(g_t)$ of the metric $g_t$ induced by (26) is such that
$$\|R(g_t)\|_{g_t}^2 = \frac{23}{48}y(t)^{-20},$$
and so, in the case iii) (corresponding to $A > \frac{1}{2\sqrt{6}}$) $\lim_{t \to +\infty} R(g_t) = 0$.

In the following theorem we study the modified Laplacian coflow (2) when the initial coclosed $G_2$ form on the 7-dimensional Heisenberg group is equal to $-\varphi_1$, where $\varphi_1$ is defined by (17).

**Theorem 8.** The solution of the modified Laplacian coflow (2) with initial coclosed $G_2$ form $-\varphi_1$ is ancient and it is given by
$$\varphi(t) = -\frac{1}{y(t)} \left( e^{127} + e^{347} + e^{567} \right) - y(t)^3 \left( e^{135} - e^{146} - e^{236} - e^{245} \right),$$
where $t \in (-\infty, T)$, with $T = \frac{\sqrt{6}}{5A} \left( 1 - (2A\sqrt{6})^{-1} \log(2A\sqrt{6} + 1) \right)$, and the function $y = y(t)$ satisfies
$$\begin{cases}
\frac{d}{dt}y(t) = -\frac{2A\sqrt{6}y(t)^{5+1}}{12y(t)^9}, \\
y(0) = 1.
\end{cases}$$

The underlying metrics $g_t$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $H$ as $t$ goes to $-\infty$.

**Proof.** By Cauchy’s Theorem there exist a maximal open interval $I$, containing 0, and a smooth function $y : I \to (0, +\infty)$, which is the unique solution of (34).

To prove that (33) is the solution of the coflow (2) for $-\varphi_1$, we proceed as follows. As in the proof of Theorem 7, for each $t \in I$, we consider the basis $\{f^1(t), \ldots, f^7(t)\}$ of left invariant 1-forms on $H$ defined by
$$f^i = f^i(t) = y(t) \, e^i, \quad i = 1, \ldots, 6$$
$$f^7 = f^7(t) = y(t)^{-3} \, e^7,$$
where the function $y = y(t)$ satisfies now (34). Then, $f^i(0) = e^i$, for $i \in \{1, \ldots, 7\}$, and the structure equations of $H$, with respect to the basis $\{f^1(t), \ldots, f^7(t)\}$, are
$$df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y(t)^{-5} \left( f^{12} + f^{34} + f^{56} \right).$$

Now, for any $t \in I$, the 3-form $\varphi(t)$ defined by (33) has the following expression
$$\varphi(t) = -(f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}).$$
So, \( \varphi(0) = -\varphi_1 \) and, for any \( t \in I \), the metric \( g_t \) induced by \( \varphi(t) \) is such that the coframe \( \{f^1(t), \ldots, f^7(t)\} \) of \( h^* \) is orthonormal. Denote by \( *_t \) the Hodge star operator determined by \( g_t \). Using (33), we have \( d *_t \varphi(t) = 0 \), where \( *_t \varphi(t) \) is given by

\[
*_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.
\]

Then, in terms of the coframe \( \{e^1, \ldots, e^7\} \) of \( h^* \), the 4-form \( *_t \varphi(t) \) has the following expression

\[
*_t \varphi(t) = y(t)^4(e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.
\]

Therefore,

\[
\frac{d}{dt} *_t \varphi(t) = 4y(t)^3 y'(t)(e^{1234} + e^{1256} + e^{3456}),
\]

that is,

\[
\frac{d}{dt} *_t \varphi(t) = -\frac{2A \sqrt{y(t)}^5}{3y(t)^5} (e^{1234} + e^{1256} + e^{3456}),
\]

since the function \( y = y(t) \) satisfies (34).

On the other hand, by (7) we know that the torsion forms \( \tau_i(t) \) (\( i = 0, 1, 2, 3 \)) of \( \varphi(t) \) are such that \( \tau_1(t) = 0 = \tau_2(t) \) since \( d(*_t \varphi(t)) = 0 \). Then, from (35), (36) and using again (7), we have

\[
d\varphi(t) = -\frac{\sqrt{6}}{3y(t)^5}(f^{1234} + f^{1256} + f^{3456}) = \tau_0(t) *_t \varphi(t) + *_t \tau_3(t),
\]

where

\[
\tau_3(t) = \frac{\sqrt{6}}{3y(t)^5}(-f^{135} + f^{146} + f^{236} + f^{245}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{127} + f^{347} + f^{567}),
\]

\[
*_t \tau_3(t) = \frac{\sqrt{6}}{3y(t)^5}(-f^{135} - f^{146} - f^{236} + f^{245}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{127} + f^{347} + f^{567}),
\]

and

\[
\tau_0(t) = -\frac{\sqrt{6}}{7y(t)^5}.
\]

Then, according with the first equality of (38),

\[
\Delta_t *_t \varphi(t) + 2d\left((A - \frac{7}{4} \tau_0) \varphi(t)\right) = d *_t d(\varphi(t)) + 2(A - \frac{7}{4} \tau_0) d\varphi(t)
\]

\[
= -\frac{2A \sqrt{y(t)}^5}{3y(t)^5} (e^{1234} + e^{1256} + e^{3456}),
\]

or, equivalently,

\[
\Delta_t *_t \varphi(t) + 2\left((A - \frac{7}{4} \tau_0) \varphi(t)\right) = -\frac{2A \sqrt{y(t)}^5}{3y(t)^5} (e^{1234+1256} + e^{3456}).
\]

The last equality, together with (8) and (37), show that (38) solves the modified Laplacian flow (2) for \(-\varphi_1\).
To show that the solution $\varphi(t)$, given by (33), is ancient, we study the behaviour of the function $y = y(t)$. To this end, we rewrite the differential equation that appears in (34) as

$$-\frac{12 y(t)^3}{2A\sqrt{6} y(t)^5 + 1} y' = 1.$$ 

Integrating this equation from $0$ to $t$ we obtain

$$\frac{\sqrt{6}}{5A} (1 - y^5(t)) + \frac{1}{10A^2} \log \left( \frac{2A\sqrt{6} y^5(t) + 1}{2A\sqrt{6} + 1} \right) = t. \quad (39)$$

Clearly $y'(t) < 0$ since the function $y = y(t)$ satisfies the differential equation that appears in (34). Then, (39) implies that the function $y = y(t)$ decreases from $+\infty$ to $0$ as $t$ goes from $-\infty$ to $\frac{\sqrt{6}}{5A} \left( 1 - \frac{1}{2A\sqrt{6}} \log(2A\sqrt{6} + 1) \right)$.

To study the behaviour of the underlying metric $g_t$ of the solution (33) for $t \to -\infty$, we proceed in a similar way as in the proof of Theorem 5.

Concerning the modified Laplacian coflow (2) for the coclosed $G_2$ form $\varphi_2$ on the 7-dimensional Heisenberg group $H$ we have the following.

**Theorem 9.** The solution of the modified Laplacian coflow (2) with initial coclosed $G_2$-structure $\varphi_2$ is defined on a bounded interval, and it is given by

$$\varphi(t) = \frac{y(t)}{z(t)^2} e^{127 - y(t)^{-1} \left( e^{347} + e^{567} \right)} + y(t)z(t)^2 \left( e^{135} - e^{146} + e^{236} + e^{245} \right), \quad (40)$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy

$$\begin{cases} 
\frac{d}{dt} y(t) = \frac{2A\sqrt{6} y(t)z(t)^6 + 2z(t)^2 + y(t)^2}{12 y(t)^2 z(t)^8}, \\
\frac{d}{dt} z(t) = -\frac{2A\sqrt{6} y(t)z(t)^4 + 1}{12 y(t)^2 z(t)^3}, \\
y(0) = 1, \\
z(0) = 1. 
\end{cases} \quad (41)$$

**Proof.** By Cauchy’s Theorem, there exist a maximal open interval $I$, containing 0, and two smooth functions $y, z : I \to (0, +\infty)$, which are the unique solution of (41).

We first prove that (40) is the solution of the coflow (2) for $\varphi_2$. As in the proof of Theorem 6 for each $t \in I$, we consider the basis $\{f^1(t), \ldots, f^7(t)\}$ of left invariant 1-forms on $H$ defined by

$$f^i = f^i(t) = y(t) e^i, \quad i = 1, 2,$$

$$f^i = f^i(t) = z(t) e^i, \quad i = 3, \ldots, 6,$$

$$f^7 = f^7(t) = y(t)^{-1} z(t)^{-2} e^7,$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy now (41). Then, $f^i(0) = e^i$, for $i \in \{1, \ldots, 7\}$, and the structure equations of $H$, with respect to the basis
\( \left\{ f^1(t), \ldots, f^7(t) \right\} \), are

\[
\begin{align*}
&df^i = 0, \quad 1 \leq i \leq 6, \\
&df^7 = \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \left( y(t)^{-2} f^1 z^{-2} f^{12} + z(t)^{-2} f^{34} + z(t)^{-2} f^{56} \right).
\end{align*}
\]

Moreover, for any \( t \in I \), the 3-form \( \varphi(t) \) defined by (40) has the following expression

\[
\varphi(t) = f^{127} - f^{347} - f^{567} + f^{135} - f^{146} + f^{236} + f^{245}.
\]

So \( \varphi(0) = \varphi_2 \) and, for any \( t \in I \), the 3-form \( \varphi(t) \) on \( H \) induces the metric \( g_t \) such that \( \left\{ f^{1}(t), \ldots, f^{7}(t) \right\} \) of \( \mathfrak{h}^* \) is an orthonormal basis of \( \mathfrak{h}^* \). Denote by \( \ast_t \) the Hodge operator determined by \( g_t \). Using (4), (5) and (42), we have \( d \ast_t \varphi(t) = 0 \), where \( \ast_t \varphi(t) \) is given by

\[
\ast_t \varphi(t) = -f^{1234} - f^{1256} - f^{1367} - f^{1457} + f^{2357} - f^{2467} + f^{3456}.
\]

Thus, in terms of the coframe \( \left\{ e^1, \ldots, e^7 \right\} \) of \( \mathfrak{h}^* \), the 4-form \( \ast_t \varphi(t) \) has the following expression

\[
\ast_t \varphi(t) = y(t)^2 z(t)^2 (-e^{1234} - e^{1256}) - e^{1367} - e^{1457} + e^{2357} - e^{2467} + z(t)^4 e^{3456}.
\]

Therefore,

\[
\frac{d}{dt} (\ast_t \varphi(t)) = 2 \left( y(t) z(t)^2 y'(t) + y(t)^2 z(t) z'(t) \right) (-e^{1234} - e^{1256}) + 4z(t)^3 z'(t) e^{3456},
\]

that is

\[
\begin{align*}
\frac{d}{dt} \ast_t \varphi(t) &= A \sqrt{6} (y(t)^3 z(t)^2 - y(t)z(t)^4) - \frac{1}{3 y(t)^2 z(t)^4} (e^{1234} + e^{1256}) \\
&\quad - \frac{2 A \sqrt{6} y(t) z(t)^4 + 1}{3 y(t)^2 z(t)^4} e^{3456},
\end{align*}
\]

since the functions \( y = y(t) \) and \( z = z(t) \) satisfy (11).

On the other hand, let us consider the torsion forms \( \tau_i(t) \) \( (i = 0, 1, 2, 3) \) of \( \varphi(t) \). By (7), \( \tau_1(t) = 0 = \tau_2(t) \) since \( d(\ast_t \varphi(t)) = 0 \). Then, from (12), (13) and using again (7), we have

\[
\begin{align*}
d \varphi(t) &= \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \left( (z(t)^{-2} - y(t)^{-2}) (f^{1234} + f^{1256}) - 2z(t)^{-2} f^{3456} \right) \\
&= \tau_0(t) \ast_t \varphi(t) + \ast_t \tau_3(t),
\end{align*}
\]

where

\[
\begin{align*}
\tau_3(t) &= -\frac{\sqrt{6} \left( 5 y(t)^2 + z(t)^2 \right)}{21 y(t)^3 z(t)^4} f^{127} + \frac{\sqrt{6} \left( 3 y(t)^2 - 5 z(t)^2 \right)}{42 y(t)^3 z(t)^4} (f^{347} + f^{567}) \\
&\quad + \frac{\sqrt{6} \left( 2 y(t)^2 - z(t)^2 \right)}{21 y(t)^3 z(t)^4} (f^{135} - f^{146} + f^{236} + f^{245}).
\end{align*}
\]
\[ \tau_t \tau_3(t) = -\frac{\sqrt{\pi} (5y(t)^2 + z(t)^2)}{21 y(t)^3 z(t)^4} f_{3456} + \frac{\sqrt{\pi} (3y(t)^2 - 5z(t)^2)}{42 y(t)^3 z(t)^4} (f_{1234} + f_{1256}) \\
+ \frac{\sqrt{6} (2y(t) - z(t))^2}{21 y(t)^3 z(t)^4} (-f_{1367} - f_{1457} + f_{2357} - f_{2467}), \]
and
\[ \tau_0(t) = -\frac{\sqrt{6}}{21 y(t)^3 z(t)^4} (2y(t)^2 - z(t)^2). \]

Then, according with the first equality of (45),
\[ \Delta_t \tau_\tau \varphi(t) + 2d \left( (A - \frac{z}{4} \tau_0) \varphi(t) \right) = d \tau_t d(\varphi(t)) + 2(A - \frac{z}{4} \tau_0) d \varphi(t) \]
\[ = \frac{A \sqrt{\pi} (y(t)^3 z(t)^2 - y(t) z(t)^4)}{3 y(t)^2 z(t)^4} \left( f_{1234} + f_{1256} \right) \]
or, equivalently,
\[ \Delta_t \tau_\tau \varphi(t) + 2d \left( (A - \frac{z}{4} \tau_0) \varphi(t) \right) = \frac{A \sqrt{6} (y(t)^3 z(t)^2 - y(t) z(t)^4)}{3 y(t)^2 z(t)^4} \left( e_{1234} + e_{1256} \right) \]
\[ - \frac{2A \sqrt{6} y(t) z(t)^4 + f_{3456}}{3 y(t)^2 z(t)^4} c_{3456}. \]

The last equality, together with (8) and (14), show that (40) solves the modified Laplacian flow (2) for \( \varphi_2 \).

To prove that (40) is defined on a bounded interval, we will show that \( t_+ = \sup(I) < +\infty \) and \( t_- = \inf(I) > -\infty \). On the one hand, we know that the functions \( y = y(t) \) and \( z = z(t) \) are positive. Then, the system (41) implies that \( z'(t) < 0 \) for \( t \in I \), for any \( t \in I \). Therefore, the function \( z = z(t) \) is decreasing, and \( y = y(t) \) is increasing. Thus, there exist
\[ \lim_{t \to t_-} y(t) = y_0 \in [0, 1] \quad \text{and} \quad \lim_{t \to t_+} z(t) = z_0 \in [0, 1]. \]

Now, using (41), it is straightforward to verify that the function \( z'' = z''(t) \) satisfies
\[ z'' = -\frac{1}{244 y z^2 \sqrt{6}} \left( 24A^2 (3y^4 z^2 - y^2 z^{10}) + 2A \sqrt{6} (9y^3 z^2 - 4yz^6) + 5y^2 - 4z^2 \right), \]
for any \( t \in I \). Note that in the last equality, the functions \( 3y^4 z^2 - y^2 z^{10} \), \( 9y^3 z^2 - 4yz^6 \), \( 5y^2 - 4z^2 \) are positive functions in \((0, t_+).\) Indeed, their values at \( t = 0 \) are positive, and \( z = z(t) \) decreases while \( y = y(t) \) increases in \((0, t_+).\) Therefore, \( z''(t) < 0 \), for \( t \in (0, t_+).\) Thus, \( z'(t) < z'(0) < 0 \), for any \( t \in (0, t_+).\) Now, we choose a sequence \( \{t_n\} \subset I \) of positive times converging to \( t_+.\) Then,
\[ z(t_n) - 1 = \int_0^{t_n} z'(t) dt < \int_0^{t_n} z'(0) dt < z'(0) t_n. \]
So, \( t_n < \frac{z(t_n)-1}{z(0)} \) and, consequently, \( t_+ \leq \frac{z-1}{z(0)} < +\infty \).

Using again (41), we have

\[
-144y^7z^{16}y'' = 48A^2(z^{12}y^2 - z^{10}y^4) + 2A\sqrt{6}(10z^8y - 11z^6y^3 - 8z^4y^5)
+ 12z^4 - 4z^2y^2 - 7y^4.
\]

Then, it is possible to show that \( y''(t) < 0 \) in some neighbourhood of \( t_- \). Indeed, the functions \( z^{12}y^2 - z^{10}y^4 \) and

\[
(12z^4 - 4z^2y^2 - 7y^4) = 4z^2(z^2 - y^2) + (8z^4 - 7y^4)
\]

are both positive on \((t_-, 0)\), since the functions \( z^2 - y^2 \) and \( 8z^4 - 7y^4 \) are both decreasing. Moreover, the solution is maximal for \( t \) going to \( t_- \). Therefore, the limits \( \lim_{t\to t_-} z(t) = z_- \) and \( \lim_{t\to t_-} y(t) = y_- \) cannot be both finite and different from zero, otherwise we can restart the flow. As a consequence, since \( y'(t) > 0 \) and \( z'(t) < 0 \), for any \( t \in I \), we get that either \( z_- = +\infty \) (and consequently \( y_- = 0 \)) or \( z_- = +\infty \).

In the first case, the leading term (as polynomial in \( z \)) of the right side of (46) is \( 12z^4 \), so it must be positive in a neighbourhood of \( t_- \). On the other hand \( -144y^7z^{16} < 0 \), so \( y''(t) < 0 \) in some neighbourhood of \( t_- \). In the other case (i.e. when \( z_- = +\infty \)),

\[
\lim_{t\to t_-} (10z^8 - 11z^6y^2 - 8z^4y^4) = +\infty
\]

since \( z_- = +\infty \) and \( y \) is bounded. Therefore \( y(10z^8 - 11z^6y^2 - 8z^4y^4) \) is positive in some neighbourhood of \( t_- \). Hence, in both cases, it follows that \( y'' < 0 \) for \( t \in (\bar{t}, t_-) \), for some \( \bar{t} \in (t_-, 0) \), i.e. that \( y'(t) > y'(\bar{t}) \), for \( t \in (t_-, \bar{t}) \). Now, we choose a sequence of negative times \( \{t_n\} \subset (t_-, \bar{t}) \) converging to \( t_- \). Then,

\[
y(\bar{t}) - y(t_n) = \int_{t_n}^{\bar{t}} y'(t) \, dt > \int_{t_n}^{\bar{t}} y'(\bar{t}) \, dt = (\bar{t} - t_n) y'(\bar{t}).
\]

It follows that \( t_n > \frac{y(\bar{t}) - y(t_n)}{y'(\bar{t})} + \bar{t} \). So, \( t_- \geq \frac{y_- - y(\bar{t})}{y'(\bar{t})} + \bar{t} > -\infty \).

\[\square\]

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