Problematic of definition and terminology affecting primary teachers’ mathematical knowledge for teaching geometry

D Susanto
Reformed Liberal Arts, Faculty of Engineering and Science, Calvin Institute of Technology, Indonesia
Corresponding author: dicky.susanto@calvin.ac.id

Abstract. Mathematics is often seen as universal across cultures. However, there exist differences in symbols, conventions, terminology, and definition used in different cultures. In some situations, the distinctions may be mathematically trivial and have no adverse effect on mathematics instructions. However, in other situations, there may be significant consequences on both mathematical understanding and instructions. This paper examines one such instances in the context of learning and teaching geometry in Indonesia, where the terminology, convention, and definition used caused hurdles in primary teachers' knowledge and understanding of the geometry concepts they teach. Some suggestions to ameliorate these challenges are proposed.

1. Introduction
Definitions and terminologies play important roles in the teaching and learning of mathematics. Mathematical definition is a meta-mathematical construct that specifies as precisely and as accurately as possible the meaning of a mathematical term in terms of related well-defined mathematical concepts [1]. In the context of learning and teaching mathematics, Vinner contends that definition creates a serious problem in mathematics learning, where tensions exist between knowledge of mathematics as perceived from the structure of mathematics (concept definition) and knowledge of mathematics from the perspective of learners (concept image), particularly in lower levels [2]. The common notion of the universal and standard nature of the language of mathematics as a is unwarranted [3]. Differences do occur across various cultures, for instance, more than one definition does exist and mathematical objects may be named multiple ways, showing differences or nuances in mathematical meanings [4]. Primary grades curricula and textbooks often are lacking in definitions, or when they are provided, attention is not called to their function in mathematical reasoning, and often fail to build the mathematics from well-defined terms. The challenge, of course, is to provide mathematically rigorous, valid and accurate definitions, while at the same time is useable in the sense that they are suitable and appropriate for primary level students. As a result, at the teacher education level, the complexity of defining terms mathematically while at the same time doing what makes sense pedagogically becomes an issue. Teachers need to deal with these tensions in order to find the best ways to support student learning, without compromising the rigor of mathematics, but at the same time allow the mathematics to be accessible for learners. Teachers’ mathematical knowledge for teaching [5], particularly knowledge of mathematical definitions and terminologies, which is not merely the ability to remembering and reciting textbook definitions, but includes the ability to understand multiple definitions, whether they are congruent or non-congruent [4], and also understanding
definitions that are mathematically accurate yet useful in the context of teaching, especially with learners of different levels.

The purpose of this study is to examine some problems caused by the definitions and terminologies used associated with geometrical shapes, and how they affect teachers’ perceptions on concepts of geometrical shapes. Teachers’ knowledge of mathematical definitions and their concept images have great influence over their instructional choices, such as what mathematical explanations they will use in their teaching, the way they scaffold their instructions for their students, and how they shape the mathematical discourse during discussions [1, 6]. However, their knowledge of equivalent definitions of mathematical concepts or their perceptions of the knowledge of definitions may not match with the actual knowledge that supports their flexibility and fluency in a mathematics classroom. Furthermore, specific knowledge required to do the mathematical task of teaching remains unknown, especially when it is involving definitions such as to inspect definitions given by textbooks, to establish the equivalence of alternative definitions for a given concept, and to choose and develop useable definitions [5]. This study focuses mainly on understanding the problems specifically in the context of the Indonesian language for the mathematical terminology in primary geometry, and how it affects teachers’ knowledge of shapes.

2. Method of Research

This study was conducted in several regions in Indonesia through an initiative to provide professional development in mathematics and science for elementary and middle school teachers. These intensive training programs initiated by several school districts where an independent professional development provider was entrusted with developing and conducting the programs across several regions. For this particular study, a total of 229 elementary and middle school teachers were participants of the science and mathematics professional development programs as mandated by their schools. These teachers’ participation in this study was voluntary. After eliminating data from participants with incomplete responses, the total number of participants dropped to 210 (180 elementary teachers and 30 middle school teachers).

The Geometry Scale of the MKT measures was used as the instrument to evaluate the effectiveness of a professional development program. These measures were designed to measure teachers’ content knowledge of geometry in grades 3 through 8 and consisted of 35 items. Two parallel forms were used as a pretest (Form A - item stems, 19 items) and posttest (Form B - 8 item stems, 23 items). There were seven items common in the two forms for linking purposes. The appropriateness of the translated and adapted MKT measures as assessments to evaluate Indonesian elementary teachers’ mathematical knowledge for teaching geometry is reported elsewhere [7]. In this study, the focus is on the qualitative analysis of items where the participants scored particularly low (more than 2 standard deviation units). In addition, excerpts taken from the professional development transcripts were included to exemplify and complement the results from the analysis.

To gauge teachers’ perceptions and understanding of mathematical definitions and their role in teaching and learning, this paper draws from parts of research studies conducted in Indonesia [8] that examined the adaptability of the U.S. developed MKT measures. Qualitative analyses of focus group interviews with teachers in this country and quantitative results based on psychometric analyses of teachers’ responses on the MKT measures were conducted to examine their perceptions and understanding of mathematical definitions.

3. Results and Discussion

From the total of 35 distinct items in Form A and Form B of the MKT geometry measures, participants scored particularly low (two or more standard deviation units) on items that ask teachers to relate shapes and to evaluate definitions of shapes. Due to the confidentiality nature of the instruments, the specific items are not reported here.
3.1. Relationships of Shapes
As reported elsewhere, participants in this study found questions that require them to understand the inclusive definitions of quadrilaterals to be challenging [8, 9]. For instance, one of the questions stems from the MKT measures asks teachers to determine whether a statement is always true, sometimes true, or never true. The statements presented in the instrument were: “A rectangle is a square,” “A square is a rectangle,” and “The diagonals of a parallelogram intersect at right angles.” Majority of the participants (about 94%) responded “never true” for these statements. Another question asks teachers to evaluate whether a statement is possible or impossible. The statements presented in the instrument were: “A rectangle that is a parallelogram,” “A parallelogram with diagonals of equal lengths,” and “A trapezoid with no lines of symmetry.” Majority of the teachers (about 90%) answered “impossible” to these statements. Responses to the first set of questions are discussed below. Similar reasoning can be applied to the second set of questions.

For the first statement “A rectangle is a square” the answer is correct. However, the reason for the correct answer stems from an exclusive definitions of quadrilaterals, where each of the five quadrilaterals has their own distinct definitions without any particular relationships other than they all have four straight sides [10]. Their incorrect responses to the statements “A square is a rectangle” and “The diagonals of a parallelogram intersect at right angles” showed that they do not consider the possibility that a square could be a special kind of rectangle where all the sides are congruent or that a rectangle could be a special kind of parallelogram where all the angles are congruent. Therefore, there is strong evidence that they adopt an exclusive definition of the shapes.

In the 2016 revised version of the 2013 curriculum, students are expected to know three-dimensional dan two-dimensional shapes in grade 1, to explain two-dimensional and three-dimensional shapes based on their characteristics in grade 2, to analyze various two-dimensional shapes based on their properties in grade 3, and to analyze characteristics of regular and irregular polygons in grade 4. Although there is an expectation to analyze shapes in grade 3, it is unclear or there is no specific guidelines to relate these shapes. In comparison, in the U.S. curriculum, students are expected to “…compare, and analyze attributes of two- and three-dimensional shapes and develop the vocabulary to describe the attributes; and classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids” [11, p. 164].

The terminology in the Indonesian language for rectangle (literally “long square”) has a strong effect on teachers’ concept image. For them, a square can never be a rectangle since a rectangle has to have a pair of parallel sides “longer” than the other pair of sides, which is incongruent with the inclusive definitions, where a square can be a rectangle because it satisfies all the properties of a rectangle — thus, a square is a special kind of rectangle. Consequently, the teachers in this study found it to be extremely difficult to relate the two shapes in a way that one shape is the subset of another. This result confirms other studies that have shown that language plays a significant role in promoting mathematical understanding [12], and extend beyond the topic of understanding of place value [13] and fractions [14] to geometry.

3.2. Definitions of Shapes
Two questions ask teachers to evaluate definitions. The first item asks teachers to evaluate two different definitions of trapezoid — a quadrilateral with exactly one pair of parallel sides versus a quadrilateral with at least one pair of parallel sides — and how the choice of definition may impact the relationship of a rectangle and a trapezoid. The second question asks teachers to analyze the relationship between prisms and pyramids based on their definitions. A prism is defined as a solid figure that has two congruent, polygon-shaped bases, where all other faces are rectangles. Whereas a pyramid is defined as a solid figure with a polygon base where all other faces are triangles that meet at a single vertex. The results show that the teachers in this study struggle to evaluate the impact of the definitions on the relationships of the shapes. Again, we argue that their concept images strongly influence their understanding of the shapes as described in the previous section. In addition, in the
context of Indonesia, textbooks often are treated as a blueprint for content coverage and instructional sequence. Therefore, teachers perceive that definitions are typically given in the textbook; it does not warrant a need to scrutinize definitions and question their validity.

3.3. Knowledge of Terminology: Length of Rectangle
In addition to the quantitative data from the MKT measures that were analyzed qualitatively, we also include data from the professional development sessions. The following excerpt is taken from transcripts of the professional development session with the teachers, discussing which side of a rectangle is called the length and which is the width.

I: So, which one is the length (panjang) and which one is the width (lebar)?
T1: The longer side is the length and the shorter side is the width.
I: Does everyone agree with that?
(Majority of the teachers nod the heads agreeing with that statement).
I: Ok. Why does the length have to be longer than the width?
T2: Because the word panjang (length) means it has to be longer.
I: I see. So, everyone agrees with that idea?

The teachers were quite content with this way of assigning the dimension of a rectangle. It is important to note that in the Indonesian language, the word for length, panjang, can also be translated as “long”. Thus, the concept image that the length has to be the longer side of the rectangle has a strong effect on their concept definition.

One teacher, however, disagreed with this notion that the length has to be the longer side, but rather emphasized the visual aspect.

T3: I disagree. I don’t think the length has to be longer than the width. I think the length is the one that is horizontal. (The teacher draws a rectangle with horizontal base and points to it as the length and the vertical side to the width).
I: Oh, why is that?
T3: I don’t know. It’s just the way the book (textbook) shows it.
I: So even if the base is shorter as long as it is horizontal then it is the length?
T3: Hmm, I am not sure. Usually, the base is drawn longer.

This teacher was strongly influenced by the visual aspect of the prototypical rectangles drawn horizontally and failed to provide any reasoning for the choice of dimension assignment. When pushed further for reasoning by introducing a scenario where the rectangle is tilted, the teacher reverted back to the previous notion that the longer side is the length.

I: I see. But, do you have to draw rectangles with horizontal bases? What if I draw a tilted rectangle? Is this ok?
T3: No, I don’t think only the horizontal rectangle. Hmm, I guess if it is tilted, it’s the longer side that is the length? I don’t know.
I: Ok. So I think we’re back to the agreement that the length is the side that is longer.

Another teacher proposed a different idea to assign the length of a rectangle to account for tilted rectangles.

T4: The length is the side that is closest to you.
I: Oh what do you mean?
T4: (Draws a rectangle with the horizontal base that is longer) For this one, the side closest to me (points to the horizontal base) is the length and the other one is the width. (Draws another rectangle
with the horizontal base that is shorter) And this one (points to the shorter horizontal base) is the length because it is the closest to me.

I: Interesting. What are other people’s thought on this idea?

The teachers were not sure and no one responded to this idea. The instructor steered the discussion to ask the teachers to think about the consequences of such definition.

I: Ok, let’s think about the consequences of adopting one definition over another. If the length is defined as the side of a rectangle closest to you, what do you think is the consequence? (Silence)

I: Well, what happens if I rotate the paper or I move my position?

The teachers started to realize that the side that is assigned as the length changed. They agreed that the assignment of the length became very subjective and different students would assign the length differently, causing more confusion. The instructor further asked them to evaluate the consequence of adopting the earlier definition that the longer side is the length.

I: Ok, now let’s think about the consequence of adopting the definition that the length is the longer side. Let’s look at a typical problem in Algebra:

“A farmer wants to build a rectangular pen for his goats using a fence of 24 meters long. What would be the dimension of the pen to fit the most goats?”

After teachers were given sufficient time to work on the problem, the instructor asked them to think about the implication of adopting the definition that the length of a rectangle is the longer side to this problem. Some teachers began to see that the answer was a square with dimension of 6 meters, but debated whether it was allowed since the question asked for a rectangular shape. Others who used algebraic equations or table of values realized that using the variables $l$ and $w$ to depict the length and width of the rectangle might end up with $w$ longer than $l$. Eventually, discussions around the idea that the two dimensions were arbitrary, that length was not necessarily longer than width, were conducted.

4. Conclusion

Teachers’ mathematical knowledge for teaching includes knowledge of mathematical definitions and terminologies, not only knowing “standard” definitions stated in the textbook, but being able to evaluate the mathematical accuracy yet pedagogically useful. In this study, the Indonesian teachers’ mathematical knowledge for teaching geometry are very much affected by their concept images of two dimensional shapes such as square and rectangle as seen from the extremely low scores in items that ask them to relate shapes and to evaluate definitions of shapes. The reason for these concept images is due to the strong influence of the language and terminology used. In addition, the standards in the curriculum, which are then translated into the textbooks, do not explicitly or particularly emphasize on relating shapes. Therefore, definitions and terminologies are treated as given, and teachers and students are not required to think more critically about them and how they may affect in their understanding of the concepts. In this study, we show the importance for teachers to develop a critical stance when it comes to definitions and terminologies, and be able to evaluate the usefulness and consequences of adopting certain definitions, especially when the textbooks do not provide such guidance. Particularly, the problems of using the term panjang for length and the persegi panjang for rectangle warrant serious discussions among mathematicians, mathematics educators, and other stakeholders to reach a consensus on how to eradicate or at least ameliorate the situations without compromising the mathematical integrity nor causing disservice to learners. Furthermore, considerations of whether to adopt an exclusive or an inclusive definition of shapes should also be part of an on-going conversation, especially from the perspectives of the learners.
5. Significance
In this paper, we argue for the importance for mathematics teacher educators to emphasize on mathematical definitions and terminologies and their impact on the teaching and learning of mathematics in order to support a robust understanding of students. Furthermore, prospective teachers need to be engaged in not only teaching definitions, but also to teach to define and to evaluate the mathematical accuracy and usability of definitions [15]. The consensus among mathematics educators on appropriate definitions and terminologies that impact understanding needs to be reached to revise current versions. Textbook authors need to consider various definitions, and when choosing particular definitions, they need to provide a rationale and impact for that choice.

References
[1] Leikin R & Zazkis R 2010 Int. J. Math. Educ. Sci. Technol. 41 (4) 451
[2] Vinner S 1991 The role of definitions in the teaching and learning of mathematics. In Tall, D.(Ed). Advanced Mathematical Thinking (Dordrecht: Kluwer)
[3] Morgan, C 2005 Lang. Educ. 19 (2) 102
[4] Ng, D, Mosvold R & Fauskanger J 2012 Math. Enthusiast 9 (1) 149
[5] Ball D L, Thames M H & Phelps G 2008 J. Teach. Educ. 59 (5) 389
[6] Zazkis R & Leikin R 2008 Educ. Stud. Math. 69 (2) 131
[7] Ng D 2012 ZDM 44 (3) 401
[8] Ng D 2011 Asia-Pacific J. Teach. Educ. 39 (2) 151
[9] Fujita T & Jones K 2007 Res. Math. Educ. 9 (1) 3
[10] Usiskin Z 2008 The Classification of Quadrilaterals: A Study of Definition (North Carolina: IAP)
[11] National Council of Teachers of Mathematics 2000 Principles and standards for school mathematics (Ed.)
[12] Miura I T, Kim C C, Chang C M, & Okamoto Y 1988 Child Dev. 1445
[13] Miura I T, Okamoto Y, Kim C C, Steere M & Fayol M 1993 J. Educ. Psychol. 85 (1) 24
[14] Miura I T, Okamoto Y, Vlahovic-Stetic V, Kim C C & Han J H 1999 J. Exp. Child Psychol. 74 (4) 356
[15] Edwards B S & Ward M B 2004 Am. Math. Mon. 111 (5) 411