Approximability of word maps by homomorphisms

Alexander Bors

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Abstract

Generalizing a recent result of Mann, we show that there is an explicit function $f : (0, 1] \to (0, 1]$ such that for every reduced word $w$, say in $d$ variables, there is an explicit reduced word $v$ in at most $3d$ variables (nontrivial if the length of $w$ is at least 2) such that for all $\rho \in (0, 1]$, the following holds: If $G$ is any finite group for which the word map $w_G : G^d \to G$ agrees with some fixed homomorphism $G^d \to G$ on at least $\rho|G|^d$ many arguments, then the word map $v_G : G^{3d} \to G$ has a fiber of size at least $f(\rho)|G|^{3d}$. We also discuss some applications of this result.

1 Introduction

Various authors have studied finite groups $G$ with an automorphism mapping a certain minimum proportion of elements of $G$ to their $e$-th power, for $e \in \{-1, 2, 3\}$ fixed. Some notable results in this context are the following: If $G$ has an automorphism inverting (resp. squaring, cubing) more than $\frac{3}{4}|G|$ (resp. $\frac{1}{2}|G|$, $\frac{3}{4}|G|$) many elements of $G$, then $G$ is abelian, see [9] (resp. [6, Theorem 3.5], [7, Theorem 4.1]), and if $G$ has an automorphism inverting (resp. squaring, cubing) more than $\frac{1}{16}|G|$ (resp. $\frac{1}{2}|G|$, $\frac{1}{8}|G|$) many elements of $G$, then $G$ is solvable, see [10, Corollary 3.2] (resp. [3, Theorem C], [4, Theorem 4.1]). In [1], the author studied finite groups with an automorphism inverting, squaring or cubing at least $\rho|G|$ many elements, for a fixed, but arbitrary $\rho \in (0, 1]$, and recently, in [8], Mann proved the following, yielding an approach that works under the weaker assumption where the word “automorphism” is replaced by “endomorphism” and consists in rewriting the assumption into a nontrivial lower bound on the proportion of solutions to a certain word equation in three variables over $G$:

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*University of Salzburg, Mathematics Department, Hellbrunner Straße 34, 5020 Salzburg, Austria. E-mail: alexander.bors@sbg.ac.at

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Theorem 1.1. (Mann, [8, Theorem 9]) There is a function \( f : (0, 1] \rightarrow (0, 1] \) such that for all \( e \in \mathbb{Z} \), all \( \rho \in (0, 1] \) and all finite groups \( G \): If \( G \) has an endomorphism \( \phi \) such that \( \phi(x) = x^e \) for at least \( \rho|G| \) many \( x \in G \), then the word equation \((xyz)^e = x^ey^ez^e\) has at least \( f(\rho)|G|^3 \) many solutions over \( G \).

The aim of this note is to generalize Mann’s approach, allowing for the use of results on word equations to study a large class of problems; we will also give an exemplary application of this more general method. Recall that to every (reduced) word \( w \in F(X_1, \ldots, X_d) \) and every group \( G \), there is associated a word map \( w_G : G^d \rightarrow G \), induced by substitution.

Theorem 1.2. There is a function \( f : (0, 1] \rightarrow (0, 1] \) such that for all words \( w \in F(X_1, \ldots, X_d) \), all \( \rho \in (0, 1] \) and all finite groups \( G \): If there is a homomorphism \( \phi : G^d \rightarrow G \) such that \( |\{g \in G^d \mid \phi(g) = w(g)\}| \geq \rho|G|^d \), then the following word equation in \( 3d \) pairwise distinct variables \( x_i, y_i, z_i, i = 1, \ldots, d \), has at least \( f(\rho)|G|^{3d} \) many solutions over \( G \):

\[
w(x_1^{-1}y_1z_1, \ldots, x_d^{-1}y_dz_d) = w(x_1, \ldots, x_d)^{-1}w(y_1, \ldots, y_d)w(z_1, \ldots, z_d).
\]

Mann’s Theorem 1.1 is the special case where \( w = X_1^e \) for some \( e \in \mathbb{Z} \) (note that the inversion from Theorem 1.2 can be removed in that special case through the substitution \( x_1 \mapsto x_1^{-1} \)). Using Theorem 1.2 and recent results on fibers of word maps (see [8]), one gets:

Corollary 1.3. Let \( w \in F(X_1, \ldots, X_d) \) be a reduced word of length at least 2. Then for all \( \rho \in (0, 1] \), the orders of the nonabelian composition factors of a finite group \( G \) such that \( w_G \) agrees with some homomorphism \( G^d \rightarrow G \) on at least \( \rho|G|^d \) many arguments are bounded in terms of \( w \) and \( \rho \).

2 Proof of Theorem 1.2

Our proof of Theorem 1.2 actually yields an explicit example of such a function \( f \):

One can take \( f := f_1, f_2 \), where \( f_1, f_2 : (0, 1] \rightarrow (0, 1] \) are given by Notation 2.1 below.

We note that in principle, Mann’s counting argument in [8, proof of Theorem 9] gives an explicit example of a function \( f \) as in Theorem 1.1 and that essentially the same argument can be applied to give an explicit proof of Theorem 1.2. Still, we will give a slightly different argument, for two reasons: Firstly, Mann’s argument simplifies the situation a bit by replacing two quantities by (asymptotic) approximations, and to get a correct value for \( f(\rho) \) (not just an approximation to a correct value) one would have to go through some, as Mann says himself, “tedious” computations. Secondly, our alternative approach allows us to highlight Lemma 2.2 below, which we think is, in spite of its elementarity, an interesting result in its own right.

Let us first introduce the two functions \( f_1 \) and \( f_2 \) that make \( f \) from Theorem 1.2 explicit:

Notation 2.1. We introduce the following functions \( f_1, f_2 : (0, 1] \rightarrow (0, 1] \):

\[
\begin{align*}
f_1(x) &= \max \left\{ \frac{\log(x)}{\log(1/\rho)} \mid \rho \in (0, 1] \right\}, \\
f_2(x) &= \max \left\{ \frac{\log(x)}{\log(1/\rho)} \mid \rho \in (0, 1] \right\}.
\end{align*}
\]
1. \( f_1(\rho) := \min\{\rho^2/(12[2\rho^{-1}]), \rho^3/(4[2\rho^{-1}])\} \), for each \( \rho \in (0,1] \).
2. \( f_2(\rho) := \rho/([2\rho^{-1}] \cdot ([2\rho^{-1}] + 1)) \), for each \( \rho \in (0,1] \).

Our proof of Theorem 1.2 relies on the following combinatorial lemma, already mentioned above and an extension of \([1, \text{Lemma 2.1.2}]\):

**Lemma 2.2.** For all \( \rho \in (0,1] \), all finite sets \( X \) and all families \( (M_i)_{i \in I} \) of subsets of \( X \) such that \( |M_i| \geq \rho |X| \) for all \( i \in I \) and \( |I| \geq \rho |X| \), there are at least \( f_1(\rho)|X|^2 \) many pairs \((i_1,i_2) \in I^2\) such that \( |M_{i_1} \cap M_{i_2}| \geq f_2(\rho)|X| \).

**Proof.** We make a case distinction.

- **Case 1:** \( |X| < 4[2\rho^{-1}]\rho^{-1} \). Then, using that for all \( i \in I \),

\[
|M_i \cap M_i| = |M_i| \geq \rho|X| \geq f_2(\rho)|X|,
\]

we see that for at least

\[
|I| = \frac{1}{|I|}|I|^2 \geq \frac{1}{4[2\rho^{-1}]\rho^{-1}} \cdot \rho^2|X|^2 \geq f_1(\rho)|X|^2
\]

many pairs \((i_1,i_2) \in I^2\), we have \( |M_{i_1} \cap M_{i_2}| \geq f_2(\rho)|X| \), as required.

- **Case 2:** \( |X| \geq 4[2\rho^{-1}]\rho^{-1} \). Set \( \epsilon := \rho/2 \); we will be using the fact (see \([1, \text{Lemma 2.1.2}(3)]\)) and note that the necessary inequality \( |I'| \geq 2[\epsilon^{-1}] = 2[2\rho^{-1}] \) holds by our case assumption on \(|X|\) and the assumption on \(|I'|\) below) that for all subsets \( I' \subseteq I \) such that \( |I'| \geq \epsilon |X| = \frac{\epsilon}{2} |X| \), there is an \( i' \in I' \) such that for at least \( \frac{\epsilon}{2[\epsilon^{-1}]} |I'| \geq \frac{\epsilon}{4[2\rho^{-1}]\rho^{-1}} |X| \) many \( j' \in I' \setminus \{i'\} \), we have

\[
|M_{i'} \cap M_{j'}| \geq \frac{2\epsilon}{|\epsilon^{-1}| \cdot ([\epsilon^{-1}] + 1)} |X| = f_2(\rho)|X|.
\]

Set \( I_0 := I \). Assume that we have already defined, for some \( t \leq \frac{\epsilon}{2} |X| + 1 \), a strictly decreasing chain of index sets \( I_0 \supset I_1 \supset \cdots \supset I_t \) and indices \( i_0,\ldots,i_{t-1} \in I \) such that for \( k = 1,\ldots,t \), \( I_k = I_{k-1} \setminus \{i_{k-1}\} \), and such that for \( k = 0,\ldots,t-1 \), there are at least \( \frac{\rho}{4[2\rho^{-1}]} |X| \) many \( j \in I_k \) with \( |M_{i_{k-1}} \cap M_j| \geq f_2(\rho)|X| \). Then since

\[
|I_t| = |I| - t \geq \rho |X| - \left(\frac{\rho}{3} |X| + 1\right) \geq \rho |X| - \frac{\rho}{2} |X| = \epsilon |X|;
\]

we conclude that there is also \( i_t \in I_t \) such that for at least \( \frac{\rho}{4[2\rho^{-1}]} |X| \) many \( j \in I_t \setminus \{i_t\} \), we have \( |M_{i_t} \cap M_j| \geq f_2(\rho)|X| \). Hence we can set \( I_{t+1} := I_t \setminus \{i_t\} \) to proceed with the recursion. Altogether, this yields at least

\[
\frac{\rho}{3} |X| \cdot \frac{\rho}{4[2\rho^{-1}]} |X| = \frac{\rho^2}{12[2\rho^{-1}]} |X|^2 \geq f_1(\rho)|X|^2
\]

many pairs \((i_1,i_2) \in I^2\) such that \( |M_{i_1} \cap M_{i_2}| \geq f_2(\rho)|X| \) in this case as well.
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\[ \vec{a} \]

and so, writing \((G, \vec{a})\) and \((\vec{g}, \vec{d})\) independently uniformly randomly chosen elements of many arguments has commuting probability at least \(f \vec{\phi}\). Let Corollary 3.1.

Proof. Set \(\alpha = (\vec{g}, \vec{d})\) for \(\alpha \in S^2\) such that

\[ |S \cap \vec{S} \vec{t} | = |S \cap \vec{S} \vec{r} | \geq f_2(\rho)|G|^d. \]

Hence for at least \(f_1(\rho)f_2(\rho)|G|^{3d}\) many triples \((\vec{s}, \vec{r}, \vec{u}) \in S^3\), we have \(\vec{s}^{-1}\vec{t}\vec{u} \in S\), and so, writing \(\vec{d} = (a_1, \ldots, a_d)\) for \(a \in \{s, t, u\},

\begin{align*}
  w_G(s_1^{-1}t_1u_1, \ldots, s_d^{-1}t_du_d) &= w_G(\vec{s}^{-1}\vec{t}\vec{u}) = \varphi(\vec{s})^{-1}\varphi(\vec{t})\varphi(\vec{u}) = w_G(\vec{s})^{-1}w_G(\vec{t})w_G(\vec{u}) \\
  &= w_G(s_1, \ldots, s_d)^{-1}w_G(t_1, \ldots, t_d)w_G(u_1, \ldots, u_d).
\end{align*}

\[ \square \]

3 Applications of Theorem 1.2

We give two applications of Theorem 1.2 in this note. The first is Corollary 1.3, the proof of which is easy now:

Proof of Corollary 1.3. By our assumption that \(w\) is reduced of length at least 2, the word equation \(w(x_1^{-1}y_1z_1, \ldots, x_d^{-1}y_dz_d) = w(x_1, \ldots, x_d)^{-1}w(y_1, \ldots, y_d)w(z_1, \ldots, z_d)\) is nontrivial (when moving the symbols from the right-hand side to the left-hand side by corresponding right multiplications, the last factor \((x_i^{-1}y_iz_i)\) on the left-hand side does not fully cancel). Hence the assertion follows from [3, Theorem 1.1] (more precisely, from the fact, implied by [3, Theorem 1.1], that the orders of the nonabelian composition factors of a finite group \(G\) such that, for a given nontrivial reduced word \(v \in F(X_1, \ldots, X_t)\), the word map \(v_G\) has a fiber of size at least \(\rho|G|^d\), are bounded in terms of \(v\) and \(\rho\).)

\[ \square \]

The second application illustrates the use of Theorem 1.2 for studying a concrete example where one can say more about the structure of \(G\) than Corollary 1.3 does.

The commuting probability of a finite group \(G\) is the probability that two independently uniformly randomly chosen elements of \(G\) commute (in other words, it is the number \(|\{ (g, h) \in G^2 \mid gh = hg \}|/|G|^2|\):

Corollary 3.1. Let \(\rho \in (0, 1]\). A finite group \(G\) whose group multiplication, viewed as a function \(G^2 \to G\), agrees with some homomorphism \(G^2 \to G\) on at least \(\rho|G|^2\) many arguments has commuting probability at least \(f_1(\rho)f_2(\rho)|G|^d/2\).

Proof. Set \(\epsilon := f_1(\rho)f_2(\rho)|G|^d/2\). We make a case distinction.

- Case 1: \(|G| < \sqrt{2/\epsilon}\). Then since each element of \(G\) commutes with itself, the commuting probability of \(G\) is at least \(|G|^{-1} > \sqrt{\epsilon}/2\), which is bounded from below by \(\epsilon/(2 - \epsilon)\) since by the definitions of \(f_1\) and \(f_2\), we certainly have \(\epsilon < \sqrt{1/2}\).
Case 2: $|G| \geq \sqrt[4]{2/\epsilon}$. By our proof of Theorem 1.2 applied with $w := X_1X_2$, we conclude that for at least $\epsilon|G|^6$ many sextuples $(s_1, s_2, t_1, t_2, u_1, u_2) \in G^6$, the equation

$$s_1^{-1}t_1u_1s_2^{-1}t_2u_2 = s_2^{-1}s_1^{-1}t_1t_2u_1u_2$$

holds, which is equivalent to

$$s_1s_2s_1^{-1} = t_1t_2u_1^{-1}s_2u_2^{-1}t_1^{-1}. \quad (1)$$

We claim that there are at least $\epsilon^2|G|^4$ many quadruples $(t_1, t_2, u_1, u_2) \in G^4$ each having the property that for at least $\frac{\epsilon}{2}|G|^2$ many pairs $(s_1, s_2) \in G^2$, Equation (1) holds. Indeed, otherwise, we would get that the number of sextuples satisfying Equation (1) is strictly smaller than $\left( \frac{\epsilon}{2} + (1 - \frac{\epsilon}{2}) \cdot \frac{\epsilon}{2} \right)|G|^6 = \epsilon|G|^6$, a contradiction. Now by our case assumption, we have that $\frac{\epsilon}{2}|G|^4 \geq 1$, so we can fix a quadruple $(t_1, t_2, u_1, u_2)$ with the described property, and then the set $M$ consisting of the at least $\frac{\epsilon}{2}|G|^2$ matching pairs $(s_1, s_2)$ has the following property: For each $s_2 \in G$, the set $N_{s_2}$ of all $s_1 \in G$ such that $(s_1, s_2) \in M$ is contained in a single coset of the centralizer $C_G(s_2)$, so $|N_{s_2}| \leq |C_G(s_2)|$. From this, we infer that

$$|\{(g, h) \in G^2 \mid gh = hg\}| = \sum_{h \in G} |C_G(h)| \geq \sum_{h \in G} |N_h| = |M| \geq \frac{\epsilon}{2 - 2\epsilon}|G|^2,$$

as required.

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