AN IMPROVED ERROR TERM FOR MINIMUM $H$-DECOMPOSITIONS OF GRAPHS

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Abstract. We consider partitions of the edge set of a graph $G$ into copies of a fixed graph $H$ and single edges. Let $\phi_H(n)$ denote the minimum number $p$ such that any $n$-vertex $G$ admits such a partition with at most $p$ parts. We show that $\phi_H(n) = \text{ex}(n, K_r) + \Theta(\text{biex}(n, H))$ for $\chi(H) \geq 3$, where $\text{biex}(n, H)$ is the extremal number of the decomposition family of $H$. Since $\text{biex}(n, H) = O(n^2)$ for some $\gamma > 0$ this improves on the bound $\phi_H(n) = \text{ex}(n, H) + o(n^2)$ by Pikhurko and Sousa [J. Combin. Theory Ser. B 97 (2007), 1041–1055]. In addition it extends a result of Özkahya and Person [J. Combin. Theory Ser. B, to appear].

1. Introduction

We study edge decompositions of a graph $G$ into disjoint copies of another graph $H$ and single edges. More formally, an $H$-decomposition of $G$ is a decomposition $E(G) = \bigcup_{i \in [t]} E(G_i)$ of its edge set, such that for all $i \in [t]$ either $|E(G_i)| = 1$ or $G_i$ is isomorphic to $H$. Let $\phi_H(G)$ denote the minimum $t$ such there is a decomposition $E(G) = \bigcup_{i \in [t]} E(G_i)$ of this form, and let $\phi_H(n) := \max_{v(G) = n} \phi_H(G)$.

The function $\phi_H(n)$ was first studied in the seventies by Erdős, Goodman and Pósa [3], who showed that the minimal number $k(n)$ such that every $n$-vertex graph admits an edge decomposition into $k(n)$ cliques equals $\phi_{K_3}(n)$. They also proved that $\phi_{K_3}(n) = \text{ex}(n, K_3)$, where $\text{ex}(n, H)$ is the maximum number of edges in an $H$-free graph on $n$ vertices. A decade later this result was extended to $K_r$ for arbitrary $r$ by Bollobás [1] who showed that $\phi_{K_r}(n) = \text{ex}(n, K_r)$ for all $n \geq r \geq 3$.

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General graphs $H$ were considered only recently by Pikhurko and Sousa [6], who proved the following upper bound for $\phi_H(n)$.

**Theorem 1** (Theorem 1.1 from [6]). If $\chi(H) = r \geq 3$ then

$$\phi_H(n) = \text{ex}(n, K_r) + o(n^2).$$

Pikhurko and Sousa also conjectured that if $\chi(H) \geq 3$ and if $n$ is sufficiently large, then the correct value is the extremal number of $H$.

**Conjecture 2.** For any graph $H$ with chromatic number at least $3$, there is an $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$ for all $n \geq n_0$.

We remark that the function $\text{ex}(n, H)$ is known precisely only for some graphs $H$, which renders Conjecture 2 difficult. However, $\text{ex}(n, H)$ is known for the family of edge-critical graphs $H$, that is, graphs with $\chi(H) > \chi(H - e)$ for some edge $e$. And in fact, after Sousa [9, 7, 8] proved Conjecture 2 for a few special edge-critical graphs, Özkahya and Person [5] verified it for all of them.

Our contribution is an extension of the result of Özkahya and Person to arbitrary graphs $H$, which also improves on Theorem 1. We need the following definition. Given a graph $H$ with $\chi(H) = r$, the decomposition family $\mathcal{F}_H$ of $H$ is the set of bipartite graphs which are obtained from $H$ by deleting $r - 2$ colour classes in some $r$-colouring of $H$. Observe that $\mathcal{F}_H$ may contain graphs which are disconnected, or even have isolated vertices. Let $\mathcal{F}'_H$ be a minimal subfamily of $\mathcal{F}_H$ such that for any $F \in \mathcal{F}_H$, there exists $F' \in \mathcal{F}'_H$ with $F' \subseteq F$. We define

$$\text{biex}(n, H) := \text{ex}(n, \mathcal{F}_H) = \text{ex}(n, \mathcal{F}'_H).$$

Our main result states that the $o(n^2)$ error term in Theorem 1 can be replaced by $O(\text{biex}(n, H))$, which is $O(n^2 - \gamma)$ for some $\gamma > 0$ by the result of Kövari, Turán and Sós [4]. Furthermore, we show that our error term is of the correct order of magnitude.

**Theorem 3.** For every integer $r \geq 3$ and every graph $H$ with $\chi(H) = r$ there are constants $c = c(H) > 0$ and $C = C(H)$ and an integer $n_0$ such that for all $n \geq n_0$ we have

$$\text{ex}(n, K_r) + c \cdot \text{biex}(n, H) \leq \phi_H(n) \leq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H).$$

Since for every edge-critical $H$ and every $n$ we have $\text{biex}(n, H) = 0$, this is indeed an extension of the result of Özkahya and Person.

2. **OUTLINE OF THE PROOF AND AUXILIARY LEMMAS**

The lower bound of Theorem 3 is obtained as follows. We let $F$ be an $n$-vertex $\mathcal{F}'_H$-free graph with $\text{biex}(n, H)$ edges, and let $c = (r - 1)^2$. There is an $n/(r - 1)$-vertex subgraph $F'$ of $F$ with at least $c \cdot e(F)$ edges. We let $G$ be obtained from the complete balanced $(r - 1)$-partite graph on $n$ vertices by inserting $F'$ into the largest part. Clearly, we
have $e(G) \geq \text{ex}(n, K_r) + c \cdot \text{biex}(n, H)$, and by definition of $\mathcal{F}_H$, the graph $G$ is $H$-free, and therefore satisfies $\phi_H(G) = e(G) \geq \text{ex}(n, K_r) + c \cdot \text{biex}(n, H)$.

The upper bound of Theorem 3 is an immediate consequence of the following result.

**Theorem 4.** For every integer $r \geq 3$ and every graph $H$ with $\chi(H) = r$ there is a constant $C = C(H)$ and an integer $n_0$ such that the following holds. Every graph $G$ on $n \geq n_0$ vertices and with
\[ e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H) \]
satisfies $\phi_H(G) \leq \text{ex}(n, K_r)$.

The proof of this theorem (see Section 3) uses the auxiliary lemmas collected in this section and roughly proceeds as follows. We start with a graph $G = (V, E)$ on $n$ vertices with $e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H)$. For contradiction we assume that $\phi_H(G) > \text{ex}(n, K_r)$. This allows us to use a stability-type result (Lemma 5), which supplies us with a partition $V = V_1, \ldots, V_{r-1}$ with parts of roughly the same size and with few edges inside each part. Since $e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H)$ we also know that between two parts only few edges are missing. Next, in each part $V_i$ we identify the (small) set $X_i$ of those vertices with many edges to $V_i$ and set $V'_i := V_i \setminus X_i$ and $X := \bigcup X_i$.

Then we consider the graph $G[V \setminus X]$, and identify a copy of some $F$ in the decomposition family of $H$ in any $G[V'_i]$, which we then complete to a copy of $H$ using the classes $V'_j$ with $j \neq i$ (see Lemma 6). We delete this copy of $H$ from $G$ and repeat this process. We shall show that this is possible until the number of edges in all $V_i \setminus X_i$ drops below $\text{biex}(n, H)$, and thus this gives many edge-disjoint copies of $H$ in $G$.

Finally, we find edge-disjoint copies of $H$ each of which has one of its colour classes in $X$ and the other $(r - 1)$ colour classes in $V'_1, \ldots, V'_{r-1}$ (see Lemma 7). It is possible to find many copies of $H$ in this way, because every vertex in $X$ has many neighbours in every $V_i$.

In total these steps will allow us to find enough $H$-copies to obtain a contradiction.

### 2.1. Notation.

Let $G$ be a graph and $V(G) = V_1 \cup \cdots \cup V_s$ a partition of its vertex set. We write $e(V_i)$ for the number of edges of $G$ with both ends in $V_i$ and $e(V_i, V_j)$ for the number of edges of $G$ with one end in $V_i$ and one end in $V_j$. Moreover, for $v \in V$ we let $\deg_G(v, V_i) = \deg(v, V_i)$ denote the number of neighbours of $v$ in $V_i$. An edge of $G$ is called *crossing* (for $V_1 \cup \cdots \cup V_s$) if its ends lie in different classes of this partition. A subgraph $H$ is called crossing if all of its edges are crossing, and *non-crossing* if none of its edges is crossing. The *chromatic excess* $\sigma(H)$ of $H$ denotes the smallest size of a colour class in a proper $\chi(H)$-colouring of $H$. 
2.2. Auxiliary lemmas. The proof of Theorem 4 relies on the following three lemmas. Firstly, we use a stability-type result which was observed in [5].

Lemma 5 (stability lemma [5]). For every $\gamma > 0$, every integer $r \geq 3$, and every graph $H \neq K_r$, with $\chi(H) = r$ there is an integer $n_0$ such that the following holds. If $G = (V, E)$ has $n \geq n_0$ vertices and satisfies $\phi_H(G) \geq \text{ex}(n, K_r)$, then there is a partition $V = V_1 \cup \cdots \cup V_{r-1}$ such that

(a) $\deg(v, V_i) \leq \deg(v, V_j)$ for all $v \in V_i$ and all $i, j \in [r - 1]$,
(b) $\sum_i e(V_i) < \gamma n^2$, and
(c) $\frac{n}{r-1} - 2\sqrt{\gamma}n \leq |V_i| \leq \frac{n}{r-1} + 2r\sqrt{\gamma}n$. \hfill $\square$

We remark that this lemma is stated in [5] only with assertion (b). However, we can certainly assume that the partition obtained is a maximal $(r - 1)$-cut, which implies (a), and for (c) see Claim 8 in [5].

The following lemma allows us to find many $H$-copies in a graph $G$ with a partition such that each vertex has few neighbours inside its own partition class.

Lemma 6. For every integer $r \geq 3$, every graph $H$ with $\chi(H) = r$ and every positive $\beta \leq 1/(100e(H)^4)$ there is an integer $n_0$ such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices, with a partition $V = V_1 \cup \cdots \cup V_{r-1}$ such that for all $i, j \in [r - 1]$ with $i \neq j$

(i) $\deg(v, V_j) \geq (\frac{1}{r-1} - \beta) n$ for every $v \in V_i$,
(ii) $\sum_{i=1}^{r-1} e(V_i) \leq \beta^2 n^2/e(H)$ and $\Delta(V_i) \leq 2\beta n$.

Then we can consecutively delete edge-disjoint copies of $H$ from $G$, until $e(V_i) \leq \text{biex}(n, H)$ for all $i \in [r - 1]$. Moreover, these $H$-copies can be chosen such that each of them contains a non-crossing $F \in \mathcal{F}_H^r$ and all edges in $E(H) \setminus E(F)$ are crossing.

Proof. Let $G = (V, E)$ be a graph and $V = V_1 \cup \cdots \cup V_{r-1}$ be a partition satisfying the conditions of the lemma. We proceed by selecting copies of $H$ in $G$ and deleting them, one at a time, in the following way. First we find a copy of some $F \in \mathcal{F}_H^r$ in $G[V_i]$ for some partition class $V_i$. Then we extend this $F$ to a copy of $H$, using only vertices $v$ of $G$ for $H \setminus F$ which have at least $(\frac{1}{r-1} - 2\beta) n$ neighbours in every partition class other than their own. We say that such vertices $v$ are $\beta$-active.

We need to show that this deletion process can be performed until $e(V_i) \leq \text{biex}(n, H)$ for all $i \in [r - 1]$. Clearly, while $e(V_i) > \text{biex}(n, H)$ for some $i$, we find some $F \in \mathcal{F}_H^r$ in $G[V_i]$. Let such a copy of $F$ be fixed in the following and assume without loss of generality that $V(F) \subseteq V_{r-1}$. It remains to show that $F$ can be extended to a copy of $H$.

By condition (i), at the beginning of the deletion process every vertex is $\beta$-active, and every vertex which gets inactive has lost at least
\( \beta n \) neighbours in some partition class other than its own. Further, by condition \((ii)\) we can find at most \( \beta^2 n^2/e(H) \) copies of \( H \) in this way. Hence we conclude that even after the very last deletion step, the number of vertices which are not \( \beta \)-active is at most

\[
\frac{\beta^2 n^2}{e(H)} e(H) \cdot \frac{1}{\beta n} = \beta n.
\]

In addition, by condition \((ii)\) we have \( \Delta(V_{r-1}) \leq 2\beta n \) at the beginning of the deletion process. Recall moreover that, in this process, we use inactive vertices only in copies of some graph in \( \mathcal{F}_H \) (and not to complete such a copy to an \( H \)-copy). Hence, throughout the process, we have for all \( j \in [r - 1] \) and all \( v \in V \setminus V_j \) that

\[
(1) \ \deg(v, V_j) \geq \left(\frac{1}{r-1} - 2\beta\right) n - e(H) - e(H) \cdot 2\beta n \geq \frac{n}{r-1} - 5e(H)\beta n.
\]

By condition \((i)\) each partition class \( V_j \) has size at least \( \left(\frac{1}{r-1} - 2\beta\right)n \), and thus size at most \( \frac{n}{r-1} + 2r\beta n \). Moreover, by \((1)\) each vertex \( v \in V_j \) has at most

\[
\frac{n}{r-1} + 2r\beta n - \left(\frac{n}{r-1} - 6e(H)\beta n\right) \leq 8e(H)\beta n
\]

non-neighbours in each \( V_j \) with \( j \neq j' \). Hence, any set \( S \subseteq V \setminus V_j \) with \(|S| \leq r \cdot v(H)\) has at least \( |V_j| - 8r \cdot v(H)e(H)\beta n \) common neighbours in \( V_j \). In particular, \( S \) has at least

\[
\left(\frac{1}{r-1} - 2\beta\right)n - \beta n - 8r \cdot v(H)e(H)\beta n \geq \frac{n}{r-1} - 11e(H)^3 \beta n > \beta n \geq v(H)
\]

common neighbours in \( V_j \) which are \( \beta \)-active, where we used the condition \( \beta \leq 1/(100e(H)^4) \) in the second inequality, and in the last inequality that \( n \) is sufficiently large.

When \( F \) gets selected in the deletion process, we use the above observation to construct within the \( \beta \)-active common neighbours of \( F \) a copy of the complete \((r-2)\)-partite graph with \( v(H) \) vertices in each part, as follows. We inductively find sets \( S_i \subseteq V_i \) of size \( v(H) \) which form the parts of this complete \((r-2)\)-partite graph. For each \( 1 \leq i \leq r-2 \) in turn, we note that \( v(F) + (i - 1)v(H) \leq r \cdot v(H) \), and therefore the set \( v(F) \cup S_1 \cup \cdots \cup S_{i-1} \) has at least \( \beta n \geq v(H) \) common neighbours in \( V_i \) which are \( \beta \)-active. We let \( S_i \) be any set of \( v(H) \) of these \( \beta \)-active common neighbours. Thus we can extend \( F \) to a copy of \( H \) in \( G \). \( \square \)

With the help of the next lemma we will find \( H \)-copies using those vertices which have many neighbours in their own partition class.

**Lemma 7.** For every integer \( r \geq 3 \), every graph \( H \) with \( \chi(H) = r \), and every positive \( \beta \leq 1/(2e(H)^2) \) there are integers \( K \) and \( n_0 \) such that the following holds. Let \( G = (V, E) \) be a graph on \( n \geq n_0 \) vertices, with a partition \( V = X \cup V'_1 \cup \cdots \cup V'_{r-1} \) such that

\[
(i) \ \ e(V_i', V_j') > |V_i'||V_j'| - \beta^6 n^2 \text{ for each } i, j \in [r-1] \text{ with } i \neq j,
\]

\[
(ii) \ |X| \leq \beta^6 n.
\]
Then we can consecutively delete edge-disjoint copies of \( H \) from \( G \), until for all but at most \( K(\sigma(H) - 1) \) vertices \( x \in X \) there is an \( i \in [r - 1] \) such that \( \deg(x, V'_i) \leq \beta^2 n \). Moreover, these \( H \)-copies can be chosen such that they are crossing for the partition \( X \cup V'_i \cup \cdots \cup V'_{r-1} \) and each of them uses exactly exactly \( \sigma(H) \) vertices of \( X \).

**Proof.** Without loss of generality we assume that there are only crossing edges in \( G \) (otherwise delete the non-crossing edges). We proceed as follows. In the beginning we set \( X' := X \). Then we identify \( \sigma(H) \) vertices in \( X' \) which are completely joined to a complete \((r - 1)\)-partite graph \( K_{r-1}(v(H)) \) in \( V \setminus X \), with \( v(H) \) vertices in each part. The subgraph of \( G \) identified in this way clearly contains a copy of \( H \) with the desired properties, whose edges we delete from \( G \). Next we delete those vertices \( x \) from \( X' \) with \( \deg(x, V'_i) \leq \beta^2 n \) for some \( i \in [r - 1] \). Then we continue with the next copy of \( H \).

We need to show that this process can be repeated until \( X' \leq K(\sigma(H) - 1) \). Indeed, assume that we still have \( X' > K(\sigma(H) - 1) \). Observe that since \( \sum_{x \in X} \deg(x) < |X|n \) we can find less than \( |X|n \leq \beta^6 n^2 \) copies of \( H \) with the desired properties in total, where we used condition \((ii)\). Hence, throughout the process at most \( \epsilon(H) \beta^6 n^2 \) edges are deleted from \( G \). In addition, for each \( x \in X' \) we have by definition \( \deg(x, V'_i) > \beta^2 n \) for all \( i \in [r - 1] \). Hence we can choose for each \( i \) a set \( S_i \subseteq N_{V'_i}(x) \) of size \( \beta^2 n \). By condition \((i)\) the graph \( G[\cup S_i] \) has density at least

\[
\frac{r^{-1}}{2} \left( \frac{\beta^2 n^2 - \beta^6 n^2 - \epsilon(H) \beta^6 n^2}{r^{-1} \beta^2 n^2} \right) \geq \frac{r - 2}{r - 1} (1 - (e(H) + 1)\beta^2) > \frac{r - 3}{r - 2},
\]

where we used \( \beta \leq 1/(2e(H)^2) \) in the last inequality. Thus, since \( n \) is sufficiently large, we can apply the supersaturation theorem of Erdős and Simonovits \([2]\), to conclude that the graph \( G[\cup S_i] \) contains at least \( \delta n^{(r - 1)\epsilon(H)} \) copies of \( K_{r-1}(v(H)) \), where \( \delta > 0 \) depends only on \( \beta \) and \( \epsilon(H) \). Choosing \( K := 1/\delta \), we can then use the pigeonhole principle and the fact that \( |X'| > K(\sigma(H) - 1) \) to infer that there are \( \sigma(H) \) vertices in \( X' \) which are all adjacent to the vertices of one specific copy of \( K_{r-1}(v(H)) \) in \( G[\cup S_i] \) as desired. \( \square \)

In addition we shall use the following easy fact about \( \text{biex}(n, H) \).

**Fact 8.** Let \( H \) be an \( r \)-chromatic graph, \( r \geq 3 \). If \( \text{biex}(n, H) < n - 1 \) then \( \sigma(H) = 1 \).

**Proof.** If \( \sigma(H) \geq 2 \), then each \( F \) from \( \mathcal{F}_H \) contains a matching of size 2. Thus \( \text{biex}(n, H) \geq n - 1 \) since the star \( K_{1, n-1} \) does not contain two disjoint edges. \( \square \)

3. **Proof of Theorem 4**

In this section we show how Lemmas 5, 6 and 7 imply Theorem 4.
Proof of Theorem 4. Let $r$ and $H$ with $\chi(H) = r \geq 3$ be given. If $H = K_r$, then the result of Bollobás [1] applies, hence we can assume that $H \neq K_r$. We choose

\[(2) \quad \beta := \frac{1}{1000e(H)^4} \quad \text{and} \quad \gamma := \frac{\beta^{12}}{1000e(H)^4}.
\]

Let $K'$ be the constant from Lemma 7 and choose

\[(3) \quad C := K' \cdot v(H)^\beta^{-1}.
\]

Finally let $n_0$ be sufficiently large for Lemmas 5, 6 and 7.

Now let $G$ be a graph with $n \geq n_0$ vertices and

\[(4) \quad e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H),
\]

and assume for contradiction that

\[(5) \quad \phi_H(G) \geq \text{ex}(n, K_r).
\]

Observe first that we may assume without loss of generality that

\[(6) \quad \delta(G) \geq \delta(T_{r-1}(n)) - 1.
\]

Indeed, if this is not the case, we can consecutively delete vertices of minimum degree until we arrive at a graph $G_{n^*}$ on $n^*$ vertices with $\delta(G_{n^*}) \geq \delta(T_{r-1}(n^*))$. Denote the sequence of graphs obtained in this way by $G_n := G, G_{n-1}, \ldots, G_{n^*}$. We have

\[
\text{ex}(n, K_r) \leq \phi_H(G) \leq \phi_H(G_{n-1}) + \delta(T_{r-1}(n)) - 1
\]

and thus $\phi_H(G_{n-1}) \geq \text{ex}(n - 1, K_r) + 1$. Similarly $\phi_H(G_{n-i}) \geq \text{ex}(n - i, K_r) + i$. Since $n$ is sufficiently large there is an $i^*$ such that $n - i^* \geq n_0$ and $i^* \geq \binom{n - i^*}{2} + 1$. Hence $n^* > n - i^* \geq n_0$, since otherwise $\phi_H(G_{n^*}) \geq \text{ex}(n^*, K_r) + \binom{n^*}{2} + 1$, a contradiction. Thus we may assume (6).

Next, by (5), we can apply Lemma 5, which provides us with a partition $V_1 \cup \ldots \cup V_{r-1}$ of $V(G)$ such that assertions (a), (b) and (c) in Lemma 5 are satisfied. Let $m := \sum_{i=1}^{r-1} e(V_i)$. Equation (4) and Lemma 5(b) imply

\[(7) \quad C \cdot \text{biex}(n, H) \leq m \leq \gamma n^2 \leq \beta^2 n^2/e(H).
\]

Further, by the definition of $m$ we clearly have $e(G) \leq \text{ex}(n, K_r) + m$. Hence it will suffice to find $m/e(H) - 1 + 1$ edge-disjoint copies of $H$ in $G$, since this would imply

\[
\phi_H(G) \leq \text{ex}(n, K_r) + m - \left(\frac{m}{e(H) - 1} + 1\right) (e(H) - 1) < \text{ex}(n, K_r),
\]

contradicting (5). So this will be our goal in the following, which we shall achieve by first applying Lemma 6 and then Lemma 7.

We prepare these applications by identifying for every $i \in [r - 1]$ the set $X_i$ of vertices in $V_i$ with high degree to its own class, that is,

\[
X_i := \{v \in V_i : \deg(v, V_i) \geq \frac{1}{2} \beta n\}.
\]
Let $X := \bigcup_{i \in [r-1]} X_i$. This implies
\begin{equation}
|X| \leq \frac{2m}{\beta n} \leq \frac{2\gamma n^2}{\beta n} \leq \sqrt{\gamma} n \leq \beta^6 n.
\end{equation}

In addition we set $V'_i := V_i \setminus X_i$ for all $i \in [r-1]$, $n' := |V \setminus X|$, $m' := \sum_{i=1}^{r-1} e(V'_i \setminus X_i)$ and $m_X := m - m' = e(X) + \sum_{i=1}^{r-1} e(X_i, V'_i)$.

**Step 1.** We want to apply Lemma 6 to the graph $G[V \setminus X]$ and the partition $V'_1 \cup \cdots \cup V'_{r-1}$. We first need to check that the conditions are satisfied. By Lemma 5(c) and (8) we have for each for each $i, j \in [r-1]$ with $i \neq j$ that $|V \setminus (V_i \cup V_j)| \leq (\frac{r-3}{r-1} + 6\sqrt{\gamma})n$. Moreover, by (8) we clearly have $n' \geq n/2$. Hence, by the definition of $X$, for each $v \in V'_i$ we have
\begin{equation}
\deg(v, V'_j) \geq \delta(T_{r-1}(n)) - |V \setminus (V_i \cup V_j)| - \frac{1}{2} \beta n
\end{equation}
and thus condition (i) of Lemma 6 is satisfied. Condition (ii) of Lemma 6 holds by (7) and the definition of $X$. Therefore we can apply Lemma 6.

This lemma asserts that we can consecutively delete copies of $H$ from $G[V \setminus X]$, each containing a non-crossing $F \in \mathcal{F}_H$ and crossing edges otherwise, until $e(V'_i) \leq \text{biex}(n', H)$. Denote the graph obtained after these deletions by $G_1$.

We have $\max_{F \in \mathcal{F}_H} e(F) \leq e(H) - 2$, since $\chi(H) \geq 3$. Hence each copy of $H$ deleted in this way uses at most $e(H) - 2$ non-crossing edges, and so this gives at least
\begin{equation}
\frac{m' - (r - 1) \text{biex}(n, H)}{e(H) - 2} \geq \frac{m' - r \text{biex}(n, H)}{e(H) - 2}
\end{equation}
edge-disjoint copies of $H$ in $G[V \setminus X]$.

By assertion (b) of Lemma 5 and the assumption $e(G) \geq \text{ex}(n, K_r)$, we have that $\epsilon_G(V_i, V_j) \geq |V_i||V_j| - \gamma n^2$, and thus $\epsilon_G(V'_i, V'_j) \geq |V'_i||V'_j| - \gamma n^2$. Again by assertion (b) of Lemma 5, in obtaining $G_1$ as described above we delete at most $\gamma n^2$ copies of $H$, so we have
\begin{equation}
\epsilon_{G_1}(V'_i, V'_j) \geq |V'_i||V'_j| - \gamma n^2 - (e(H) - 1)\gamma n^2
\end{equation}
\begin{equation}
= |V'_i||V'_j| - e(H)\gamma n^2 \geq |V'_i||V'_j| - \beta^6 n^2.
\end{equation}

**Step 2.** Next we want to apply Lemma 7 to $G_1$ and the partition $X \cup V'_1 \cup \cdots \cup V'_{r-1}$. Note that condition (i) of Lemma 7 is satisfied by (11) and condition (ii) by (8). Hence Lemma 7 allows us to delete crossing copies of $H$ from $G$ until all vertices $x$ of a subset $X_0 \subseteq X$ with $|X| - |X_0| \leq K(\sigma(H) - 1)$ have $\deg(x, V'_{i(x)}) \leq \beta^2 n$ for some $i(x) \in [r-1]$. Denote the graph obtained after these deletions by $G_2$. 

Now let \( x \in X_j \) for some \( j \in [r - 1] \) be arbitrary. We set \( m_x := \deg_G(x, V_j \setminus X) \). Since no edges adjacent to \( x \) were deleted in step 1, if \( x \in X_0 \) then the number of edges adjacent to \( x \) deleted in step 2 is at least \( \deg_G(x, V_i \setminus X) - \beta^2 n \geq m_x - 2\beta^2 n \), where we used assertion (a) of Lemma 5 and (8) in the inequality. Hence, since \( m_X = \sum_{x \in X} m_x + e(X) \), in total at least
\[
m_X - K'n - |X_0|\beta^2 n - e(X) \geq m_X - K'n - 2\beta^2 n|X|
\]
edges adjacent to \( X \) were deleted in step 2. By Fact 8 we have \( K' = K'(\sigma(H) - 1) = 0 \) if \( \text{biex}(n, H) < n - 1 \). If \( \text{biex} \geq n - 1 \geq n/2 \) on the other hand, then \( m \geq Cn/2 \) by (7) and thus \( K'n \leq 2K'm/C \). Observe moreover that, because \( H \neq \overline{K}_3 \), each \( H \)-copy deleted in this step uses at least 2 edges which are not adjacent to \( X \). We conclude that at least
\[
m_X - K'n - |X_0|\beta^2 n - e(X) \geq m_X - K'n - 8\beta m
\]
edge-disjoint copies of \( H \) were deleted from \( G_1 \) in step 2.

Combining (10) and (12) reveals that \( G \) contains
\[
\frac{m' - r \text{biex}(n, H)}{e(H) - 2} + \frac{m_X - 9\beta m}{e(H) - 2} \geq \frac{m - \frac{r}{2}m - 9\beta m}{e(H) - 2} \geq \frac{m - 10\beta m}{e(H) - 2} \geq \frac{m}{e(H) - 1} + 1
\]
e edge-disjoint copies of \( H \), which gives the desired contradiction. \( \square \)

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