BREAKDOWN OF THE EQUIVALENCE BETWEEN GRAVITATIONAL MASS AND ENERGY FOR A QUANTUM BODY: THEORY AND SUGGESTED EXPERIMENTS

Andrei G. Lebed*

Department of Physics, University of Arizona, 1118 E. 4-th Street,
Tucson, Arizona 85721, USA; lebed@physics.arizona.edu

We review recent theoretical results, obtained for the equivalence between gravitational mass and energy of a composite quantum body as well as for its breakdown at macroscopic and microscopic levels. In particular, we discuss that the expectation values of passive and active gravitational masses operators are equivalent to the expectation value of energy for electron stationary quantum states in a hydrogen atom. On the other hand, for superpositions of the stationary quantum states, inequivalence between the gravitational masses and energy appears at a macroscopic level. It reveals itself as time-dependent oscillations of the expectation values of passive and active gravitational masses, which can be, in principle, experimentally measured. Inequivalence between passive gravitational mass and energy at a microscopic level can be experimentally observed as unusual electromagnetic radiation, emitted by a macroscopic ensemble of the atoms. We propose the corresponding experiment, which can be done on the Earth's orbit, using small spacecraft. If such experiment is done it would be the first direct observation of quantum effects in general relativity.

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1. INTRODUCTION

Unification theory of all interactions in nature has been considered as one of the major problems in physics for many years. The idea to unify all existing forces is behind such successful theories as the Maxwell’s electromagnetism, electroweak theory of Salam, Weinberg, and Glashow and finally - the so-called standard theory of elementary particles. What is obviously missing here is gravitational force, which has not been unified with quantum mechanics so far. Note that the current situation with possible unification of gravitation theory and quantum mechanics is very unclear. This is partially due to the fact that the fundamentals of these two theories are very different from each other and partially due to the lack of the corresponding experimental data. Normally, one can expect that quantum gravity reveals itself in full at the so-called Planck energy of the order of $10^{28}eV$, which is much higher than the energy range accessible to experimentalists. On the other hand, if one is interested in quantum effects due to quantization of matter only, they can appear at accessible energies (see, for example, neutron experiments [1,2]). Note that, despite of their significant importance, the experiments [1,2] have an obvious drawback: they study quantum mechanical effects in Newtonian gravity, where all general relativity effects are negligible. Below, we review two recently suggested quantum phenomena in general relativity [3-6], which have to be accessible at the existing energies, as well as two corresponding idealized experiments to discover these phenomena. It is also important that the above mentioned quantum effects in general relativity significantly break the textbook equivalence between gravitational mass of a composite quantum body and its energy, which also can be experimentally tested.

It is known that the notion of gravitational mass of a composite classical body is not a trivial one. Moreover, it is related to the following interesting paradoxes. Indeed, if we apply the so-called Tolman’s formula for active gravitational mass [7],

$$m^g_a = \frac{1}{c^2} \int [T^0_0(r) - T^1_1(r) - T^2_2(r) - T^3_3(r)] d^3r,$$

(1)

to a free photon with energy $E$, we obtain $m^g_a = 2E/c^2$ (i.e., two times bigger value than the expected one)[8]. Let us now consider the photon in a box with mirror walls (i.e., a composite body at rest). Then, as shown by Misner and Putnam [8], the Einstein’s equation, $m^g_a = E/c^2$, restores, if we take into account a negative contribution to active gravitational mass from stress in the box walls. So, in the example above, both kinetic and potential energies make contributions to gravitational mass and the Einstein’s equation (what is important) is restored only after averaging over time. Nordtvedt [9] and Carlip [10] considered more realistic example - classical model of a hydrogen atom (i.e., positive and negative charges bound by the Coulomb interaction). They showed that kinetic energy, $K$, and potential energy, $U$, are coupled to external gravitational field in different ways. More specifically, they demonstrated [9,10] that the following combination, $3K + 2U$, is coupled with the gravitational field, which is rather unexpected.
Nevertheless, the Einstein’s equation can be restored for this classical model of a hydrogen atom after averaging over time [9,10]. Indeed, due to the classical virial theorem, the following time averaging is zero:

\[ <2K + U>_t = 0. \]  

Therefore, if we consider averaged over time active and passive gravitational masses of classical model of a hydrogen atom, we obtain:

\[ <m_{g,a,p}>_t = m_e + m_p + \frac{<3K + 2U>_t}{c^2} = \frac{E}{c^2}, \]  

where \( m_e \) and \( m_p \) are the bare electron and proton masses, respectively.

2. GOAL

The goal of our review is to describe in detail the recent results [3-6] related to passive and active gravitational masses of a composite quantum body. As the simplest example of such a body, we consider a hydrogen atom in external gravitational field. In section 3, we reproduce the results of Refs. [9,10], related to passive gravitational mass of classical model of a hydrogen atom, using different method. Section 4 of the review is devoted to the following quantum results: the expectation values of passive and active gravitational masses are equivalent to the expectation value of energy for stationary quantum states [3-6]. In section 5, we review the results that demonstrate that the above mentioned equivalence is broken for the corresponding expectation values for superpositions of stationary quantum states [3,5]. This breakdown happens at macroscopic level and reveals itself as time dependent oscillations of the expectation values of passive and active gravitational masses. In section 6, we review suggested in Ref. [5] an idealized experiment to discover these oscillations. In sections 7 and 8, we discuss breakdown of the equivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states and the suggested idealized experiment to discover this effect [3-6]. In section 9, we summarize our results and discuss some still unresolved problems. Note that, if one of the suggested in Refs. [3-6] and described in detail in the review experiments is done, it would be the first direct experiment to detect quantum effects in general relativity.

3. GRAVITATIONAL MASS IN CLASSICAL PHYSICS

In this section, we derive the Lagrangian and Hamiltonian for classical model of a hydrogen atom in the Earth’s gravitational field [9,3]. Below, we account for terms only of the order of \( 1/c^2 \) and, thus, take into account only couplings of non-relativistic kinetic and the Coulomb potential energies with weak gravitational field. To be more specific, we disregard electromagnetic and gravitational waves, magnetic force, spin-orbital interaction, spin-spin interaction, etc. It is very important that we disregard also all tidal effects. To derive the corresponding Lagrangian, let us express the interval in the Earth’s gravitational field by means of the so-called weak field approximation [11]:

\[ ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)(cdt)^2 + \left(1 - 2\frac{\phi}{c^2}\right)(dx^2 + dy^2 + dz^2), \phi = -\frac{GM}{R}, \]  

where \( c \) is the velocity of light, \( G \) is the gravitational constant, \( M \) is mass of the Earth, \( R \) is distance between center of the Earth and center of mass of a hydrogen atom (in our approximation - proton). Note that, to calculate the Lagrangian and Hamiltonian in a linear with respect to small parameter, \( |\phi(R)|/c^2 \ll 1 \), approximation, we don’t need to keep in Eq.(4) the terms of the order \( |\phi(R)|/c^2 \), unlike the classical problem about perihelion of the Mercury calculations [11].

If we disregard all tidal effect, we can introduce the local proper spacetime coordinates,

\[ x' = \left(1 - \frac{\phi}{c^2}\right)x, \quad y' = \left(1 - \frac{\phi}{c^2}\right)y, \quad z' = \left(1 - \frac{\phi}{c^2}\right)z, \quad t' = \left(1 + \frac{\phi}{c^2}\right)t, \]  

and write the Lagrangian and action for classical model of a hydrogen atom in the following standard forms:

\[ L' = -m_pc^2 - m_ec^2 + \frac{1}{2}m_e(v')^2 + \frac{e^2}{r'}, \quad S' = \int L'dt', \]
where \( e \) is the electron charge, \( r' \) is distance between electron and proton, and \( v' \) is electron velocity. [Note that, in this review, we use inequality \( m_p \gg m_e \). Therefore, we disregard kinetic energy of proton and consider its position as center of mass of a hydrogen atom.] It is easy to show that the Lagrangian \( (6) \) can be expressed in inertial coordinates \( (x, y, z, t) \) as

\[
L = -m_p e^2 - m_e e^2 + \frac{1}{2} m_e v^2 + \frac{e^2}{r} - m_e \phi - \left( 3m_e \frac{v^2}{2} - 2 \frac{e^2}{r} \right) \frac{\phi}{c^2} .
\]  

(7)

To calculate the Hamiltonian of classic model of a hydrogen atom in the Earth’s gravitational potential, we use the standard procedure: \( \hat{H}(\mathbf{p}, \mathbf{r}) = \mathbf{p}v - L(\mathbf{v}, \mathbf{r}) \), where \( \mathbf{p} = \partial L(\mathbf{v}, \mathbf{r}) / \partial \mathbf{v} \). After simple calculations, we obtain:

\[
\hat{H} = m_p c^2 + m_e c^2 + \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + m_p \phi + m_e \phi + \left( \frac{3}{2m_e} - \frac{2}{r} \right) \frac{\phi}{c^2} .
\]  

(8)

where the canonical momentum in gravitational field is \( \mathbf{p} = m_e \mathbf{v}(1 - 3\phi/c^2) \). [It is important that, in the review, we disregard all tidal effects. This means that we consider a hydrogen atom as a point-like body and do not differentiate gravitational potential with respect to the coordinates \( \mathbf{r} \) and \( \mathbf{r}' \), corresponding to electron position in the center of mass coordinate system. It is possible to show that in such a way we disregard tidal effects of the order of \( |\phi/c^2|/(r_B/R) \sim 10^{-26} \), where \( r_B \) is the so-called Bohr radius (i.e., the typical size of a hydrogen atom).] Let us introduce averaged over time electron passive gravitational mass, \( \langle m^g_p \rangle \). From Eq.(8), it follows that

\[
\langle m^g_p \rangle = m_e + \left( \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} \right) \frac{1}{c^2} + \left( \frac{3}{2m_e} - \frac{2}{r} \right) \frac{1}{c^2} = m_e + \frac{E}{c^2} ,
\]  

(9)

where \( E = \mathbf{p}^2/2m_e - e^2/r \) is electron energy. We pay attention to the fact that the third term in Eq.(9) is equal to zero due to the classical virial theorem. Therefore, we reproduce, using Hamiltonian approach, the results of Refs.[9,10]. Indeed, as directly seen from Eq.(9) the averaged over time passive gravitational mass of a composite body in classical physics is related to its energy by the Einstein’s equation.

\section{4. EQUVALENCE OF THE EXPECTATION VALUES OF GRAVITATIONAL MASS AND ENERGY FOR STATIONARY QUANTUM STATES}

In this section, we review some of the general results of Refs.[3-6].

\subsection{4.1. Operator of passive gravitational mass}

Here, we start to consider a quantum problem of gravitational mass of a composite body \([3-6]\). Let us quantize the Hamiltonian \( (8) \) by substituting momentum operator, \( \hat{p} = -\hbar \partial / \partial \mathbf{r} \), instead of canonical momentum, \( \mathbf{p} \). For convenience, we represent the quantized Hamiltonian in the following form:

\[
\hat{H} = m_p c^2 + m_e c^2 + \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} + m_p \phi + \hat{m}^g_p \phi ,
\]  

(10)

where we define electron passive gravitational mass operator as operator, which is proportional to electron weight operator in the weak gravitational field \([3-6]\),

\[
\hat{m}^g_p = m_e + \left( \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) \frac{1}{c^2} + \left( \frac{3}{2m_e} - \frac{2}{r} \right) \frac{1}{c^2} .
\]  

(11)

We pay attention to the fact that the first term in Eq.(11) is the bare electron mass, the second term is the expected kinetic and potential energies contributions to electron mass, whereas the third term is the so-called virial one. It is important that the virial operator in Eq.(11) does not commute with electron energy operator, taken in the absence of gravitational field. Note that Eq.(11) can be obtained directly from the Dirac equation in curved spacetime in a weak gravitational field. Indeed, it is possible to show that Eq.(11) can be derived from the Hamiltonian \((3.24)\) of Ref.[12], if we omit all tidal terms. Note that, in Ref.[12], a completely different physical problem is considered.
4.2. Equivalence of the expectation values of passive gravitational mass and energy for stationary quantum states

4.2.1. Non-relativistic case

Let us discuss an important consequence of Eq.(11) for electron passive gravitational mass operator. Suppose that we have a macroscopic ensemble of hydrogen atoms each of them being in a ground state with energy $E_1$,

$$\Psi_1(r, t) = \Psi_1(r) \exp(-iE_1t/\hbar).$$

(12)

Then, as directly seen from Eq.(11), the expectation value of the electron gravitational mass operator is

$$\langle \hat{m}_p^g \rangle = m_e + \frac{E_1}{c^2} + \left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) \frac{1}{c^2} = m_e + \frac{E_1}{c^2},$$

(13)

where the third term in Eq.(13) is zero due to the quantum virial theorem [13]. It is trivial to extend the obtained result (13) to any stationary quantum state in a hydrogen atom. Thus, we can conclude that the expectation values of passive gravitational mass and energy are equivalent to each other for stationary quantum states [3-6].

4.2.2 Relativistic corrections

In this subsection, we study a more general case [5], where we include the so-called relativistic corrections to electron motion in a hydrogen atom. If we disregard all tidal effects, we can write the Schrödinger equations in external gravitational field in proper spacetime coordinates in the following way:

$$i\hbar \frac{\partial \Psi(r', t')}{\partial t'} = \hat{H}(\hat{p}', \hat{r}') \Psi(r', t').$$

(14)

It is important that the Hamiltonian (14) now contains both non-relativistic and relativistic parts and can be written as [14]

$$\hat{H}(\hat{p}', \hat{r}') = \hat{H}_0(\hat{p}', \hat{r}') + \hat{H}_1(\hat{p}', \hat{r}'),$$

(15)

where

$$\hat{H}_0(\hat{p}', \hat{r}') = m_e c^2 + \frac{\hat{p}'^2}{2m_e} - \frac{e^2}{r'},$$

(16)

and

$$\hat{H}_1(\hat{p}', \hat{r}') = \alpha \hat{p}'^4 + \beta \delta^3(\hat{r}') + \gamma \frac{\hat{S} \cdot \hat{L}'}{(r')^3},$$

(17)

with the following values of the parameters:

$$\alpha = -\frac{1}{8m_e^2c^2}, \quad \beta = \frac{\pi e^2 \hbar^2}{2m_e^2c^2}, \quad \gamma = \frac{e^2}{2m_e^2c^2}.$$  

(18)

[Here, $\delta^3(\hat{r})$ is the three-dimensional Dirac’s delta-function, $\hat{L}' = -i\hbar [\hat{r}' \times \partial / \partial \hat{r}']$ is electron angular momentum operator.] We point out that the first term in Eq.(17) is a relativistic correction to electron kinetic energy, the second term (with delta function) is the so-called Darwin’s term, and the third one is the spin-orbital interaction. Using the coordinate transformation (5), we can rewrite the Hamiltonian (15)-(18) in inertial coordinate system ($x, y, z, t$) in the following way:

$$\hat{H}(\hat{p}, \hat{r}, \phi) = [\hat{H}_0(\hat{p}, \hat{r}) + \hat{H}_1(\hat{p}, \hat{r})] \left(1 + \frac{\phi}{c} \right) + \left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(\hat{r}) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{(r^3)}\right) \frac{\phi}{c^2}.$$  

(19)

From Eq.(19) it follows that passive gravitational electron mass operator with relativistic corrections can be written in a more complicated than Eq.(11) form:

$$\hat{m}_p^g = m_e + \left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + \alpha \hat{p}^4 + \beta \delta^3(\hat{r}) + \gamma \frac{\hat{S} \cdot \hat{L}}{r^3}\right) / c^2 + \left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(\hat{r}) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3}\right) / c^2.$$  

(20)
Suppose that we consider again a macroscopic ensemble of hydrogen atoms with each of them being in ground quantum state with energy $E_1'$,

$$\Psi_1(r,t) = \Psi_1(r) \exp(-iE_1't/\hbar),$$  \hspace{1cm} (21)

where $E_1'$ is calculated in the presence of relativistic corrections (17) to electron energy in a hydrogen atom. In this case, the expectation value of electron passive gravitational mass can be expressed as

$$\langle \hat{m}_p \rangle = m_e + E_1'/c^2 + \left\langle 2 \frac{\hat{p}^2}{2m_e} - \frac{\alpha^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(r) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3} \right\rangle/c^2.$$ \hspace{1cm} (22)

Below, we demonstrate that the expectation value of the third term in Eq.(22) is zero and, thus, the equivalence between passive gravitational mass and energy survives for stationary quantum states in the presence of relativistic corrections [5]. To this end, we define the so-called virial operator [13],

$$\hat{G} = \frac{1}{2}(\hat{p} \hat{r} + \hat{r} \hat{p}),$$ \hspace{1cm} (23)

and write a standard equation for motion of its expectation value,

$$\frac{d}{dt} \left\langle \hat{G} \right\rangle = \frac{i}{\hbar} \left\langle [\hat{H}_0(\hat{p}, \hat{r}) + H_1(\hat{p}, \hat{r}), \hat{G}] \right\rangle,$$ \hspace{1cm} (24)

where $[\hat{A}, \hat{B}]$ is, as usual, a commutator of two operators, $\hat{A}$ and $\hat{B}$. Note that, for the stationary quantum state (21), the derivative $d \langle \hat{G} \rangle /dt = 0$ and, therefore,

$$\left\langle [\hat{H}_0(\hat{p}, \hat{r}) + H_1(\hat{p}, \hat{r}), \hat{G}] \right\rangle = 0,$$ \hspace{1cm} (25)

where the Hamiltonians in Eqs.(24),(25) are defined by Eqs. (16)-(18). Using straightforward but lengthy calculations, it is possible to show that

$$\frac{[\hat{H}_0(\hat{p}, \hat{r}), \hat{G}]}{-i\hbar} = 2 \frac{\hat{p}^2}{2m_e} - \frac{\alpha^2}{r}, \quad \frac{[\alpha \hat{p}^4, \hat{G}]}{-i\hbar} = 4\alpha \hat{p}^4, \quad \frac{[\beta \delta^3(r), \hat{G}]}{-i\hbar} = 3\beta \delta^3(r), \quad \frac{1}{-i\hbar} \left[ \frac{\hat{S} \cdot \hat{L}}{r^3}, \hat{G} \right] = 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3},$$ \hspace{1cm} (26)

where we use the following property of the Dirac’s delta function: $x[d\delta(x)/dx] = -\delta(x)$. As directly seen from Eqs.(25),(26),

$$\left\langle 2 \frac{\hat{p}^2}{2m_e} - \frac{\alpha^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(r) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3} \right\rangle = 0,$$ \hspace{1cm} (27)

and, thus, Eq.(22) satisfies the Einstein’s equation:

$$\langle \hat{m}_p \rangle = m_e + E_1'/c^2.$$ \hspace{1cm} (28)

It is important that Eq.(27) extends the so-called relativistic quantum virial theorem [15] to the case, where the particles have spin $1/2$.

Note that in this subsection we have established the equivalence between the expectation values of passive gravitational mass of a hydrogen atom and its energy in the presence of the relativistic corrections to electron energy [5]. Here, we put forward a hypothesis that such equivalence between the expectations values of passive gravitational mass of a quantum system and its energy always exists for stationary quantum states.

4.3. Equivalence of the expectation values of active gravitational mass and energy for stationary quantum states

In this subsection, we review some general results, obtained in Ref.[5].
4.3.1. Active gravitational mass in classical physics

Here, we introduce active gravitational mass for classical model of a hydrogen atom, where we have light negatively charged particle (i.e., electron) exhibiting a bound motion in the Coulomb electrostatic field of heavy positively charged particle (i.e., proton). At large distances from the atom, $R \gg r_B$, gravitational potential in the first approximation is

$$\phi(R) = -G \frac{m_p + m_e}{R}, \quad (29)$$

where so far we have not taken into account electron kinetic and potential energies contributions. Since $m_p \gg m_e$, as usual, we disregard kinetic energy of proton and consider its position as center of mass of the atom. Now, we are in a position to calculate how kinetic and potential energies of electron contribute to electron active gravitational mass. To this end, we define active gravitational mass of the atom by evaluating gravitational potential, which act in a position to calculate how kinetic and potential energies of electron contribute to electron active gravitational mass. As it follows from general theory of a weak gravitational field [7,11], gravitational potential for our case can be written as

$$\phi(R, t) = -G \frac{m_p + m_e}{R} - G \int \frac{\Delta T^{\text{kin}}_{\alpha\beta}(t, \mathbf{r}) + \Delta T^{\text{pot}}_{\alpha\beta}(t, \mathbf{r})}{c^2 R} d^3 \mathbf{r}, \quad (30)$$

where $\Delta T^{\text{kin}}_{\alpha\beta}(t, \mathbf{r})$ and $\Delta T^{\text{pot}}_{\alpha\beta}(t, \mathbf{r})$ are changes of stress-energy tensor component $T_{\alpha\beta}(t, \mathbf{r})$ due to kinetic and the Coulomb potential energies, correspondingly. We note that in Eq.(30) only terms of the order of $1/c^2$ are taken into account, which, for example, means that we disregard all retardation effects. Therefore, in this approximation, electron active gravitational mass is equal to

$$m_a^g = m_e + \frac{1}{c^2} \int [\Delta T^{\text{kin}}_{\alpha\beta}(t, \mathbf{r}) + \Delta T^{\text{pot}}_{\alpha\beta}(t, \mathbf{r})] d^3 \mathbf{r}. \quad (31)$$

To calculate $\Delta T^{\text{kin}}_{\alpha\beta}(t, \mathbf{r})$, we use the standard expression for stress-energy tensor of a moving relativistic point mass [7,11]:

$$T^{\alpha\beta}(\mathbf{r}, t) = \frac{mv^\alpha(t)v^\beta(t)}{\sqrt{1 - v^2(t)/c^2}} \delta^3[\mathbf{r} - \mathbf{r}_e(t)], \quad (32)$$

where $v^\alpha$ is a four velocity and $\mathbf{r}_e$ is electron trajectory in three dimensional space. From Eq.(32), it directly follows that for low enough velocities, $v \ll c$,

$$\Delta T^{\text{kin}}_{\alpha\beta}(t) = \int \Delta T^{\text{kin}}_{\alpha\beta}(t, \mathbf{r}) d^3 \mathbf{r} = \frac{3}{2} \frac{m_e v^2(t)}{c^2}. \quad (33)$$

Now, let us write the standard formula for stress energy tensor of electromagnetic field,

$$T^{\mu\nu}_{\text{em}} = \frac{1}{4\pi} [F_\mu^\alpha F_\nu^\alpha - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}], \quad (34)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric in the absence of gravitational field, $F_{\alpha\beta}$ is the so-called tensor of electromagnetic field [7]. We stress that, in this subsection, we take into account only the Coulomb electric field without relativistic corrections. In this case, Eq.(34) can be significantly simplified and, as a result, we obtain the following expression for change of the electromagnetic stress energy tensor:

$$\Delta T^{\text{pot}}_{\alpha\beta}(t) = \int \Delta T^{\text{pot}}_{\alpha\beta}(t, \mathbf{r}) d^3 \mathbf{r} = -2 \frac{e^2}{r(t)} \quad (35)$$

From Eqs.(33),(35), it follows that active electron gravitational mass can be written in a similar way as the passive one:

$$m_a^g = m_e + \left(\frac{m_e v^2}{2} - \frac{e^2}{c^2 r(t)}\right)/c^2 + \left(2 \frac{m_e v^2}{2} - \frac{e^2}{c^2 r(t)}\right)/c^2. \quad (36)$$
Note that the last term in Eq.(36) is the virial one, which depends on time. Therefore, we can conclude that active gravitational mass of a classical model of a hydrogen atom depends on time too. Nevertheless, it is possible to restore the Einstein’s equation for the equivalence between the mass and energy. To this end, we introduce active gravitational mass of electron, averaged over time [9,10]:

\[ \langle m^g_a \rangle_t = m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 = m_e + E / c^2, \]  

where the averaged over time virial term is zero due to the classical virial theorem.

### 4.3.2. Active gravitational mass in quantum physics

Let us first rewrite Eq.(36) for active electron gravitational mass using momentum, instead of velocity,

\[ m^g_a = m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2, \]  

and let us quantize the expression (38):

\[ \hat{m}^g_a = m_e + \left( \hat{\mathbf{p}}^2 \right) / c^2 + \left( 2 \hat{\mathbf{p}}^2 \right) / c^2. \]  

In this subsection, we make use of semi-classical approach to quantum general relativity [16], where gravitational field is not quantized but matter is quantized,

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \langle \hat{T}_{\mu\nu} \rangle, \]  

where \( \langle \hat{T}_{\mu\nu} \rangle \) stands for the expectation value of quantum operator, corresponding to the stress energy tensor. From Eqs.(39),(40), it follows that the expectation value of electron active gravitational mass is

\[ \langle \hat{m}^g_a \rangle = m_e + \left( \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) / c^2, \]  

where, as usual, the last term is the virial one. Now, let us suppose that we have a macroscopic ensemble of hydrogen atoms with each of them being in ground state (12). Then, the expectation value of electron active gravitational mass operator is equal to

\[ \langle \hat{m}^g_a \rangle = m_e + \frac{E}{c^2}, \]  

where we take into account that the expectation value of the virial term in Eq.(41) is equal to zero in stationary quantum states due to the quantum virial theorem [13]. Therefore, we can make the statement that, in stationary quantum states, active gravitational mass of a composite quantum body is equivalent to its energy [5].

### 5. INEQUIVALENCE BETWEEN GRAVITATIONAL MASS AND ENERGY FOR SUPERPOSITION OF STATIONARY QUANTUM STATES

In this section, we review the results of Refs.[3,5] that, in superpositions of stationary quantum states, where the expectation values of energy do not depend on time, the expectation values of passive and active gravitational masses exhibit time dependent quantum oscillations. These oscillations represent novel quantum phenomenon. Moreover, their observation would be the first observation of quantum effects in general relativity. In section 6, we suggest the corresponding idealized experiment.
5.1. Inequivalence between passive gravitational mass and energy

Let us consider the simplest superposition of stationary quantum states in a hydrogen atom - the superposition of the ground and first s-wave excited states (i.e., 1S and 2S electron orbits),

\[ \Psi_{1,2}(r,t) = \frac{1}{\sqrt{2}} [\Psi_1(r) \exp(-iE_1t) + \Psi_2(r) \exp(-iE_2t)]. \quad (43) \]

It is easy to prove that superposition (43) is characterized by the following constant expectation value of energy:

\[ \langle E \rangle = \frac{E_1 + E_2}{2}. \quad (44) \]

On the other hand, the expectation value of passive gravitational mass operator (11) for quantum state (43) is not constant and, as possible to show, oscillates with time:

\[ \langle \hat{m}_g \rangle = m_e + \frac{E_1 + E_2}{2c^2} + \frac{V_{1,2}}{c^2} \cos \left[ \frac{(E_1 - E_2)t}{\hbar} \right], \quad (45) \]

where we introduce matrix elements of the virial operator by the following equation:

\[ V_{1,2} = \int \Psi_1^*(r) \left( \frac{2\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) \Psi_2(r) d^3r. \quad (46) \]

Note that the mathematical origin of this result is that the quantum virial theorem is valid for the expectation value of the virial operator only in a stationary quantum state. From Eqs.(45),(46), we can make a conclusion that the time-dependent oscillations of the expectation value of passive gravitational mass directly demonstrate breakdown of the equivalence between the expectation values of passive gravitational mass and energy for the superpositions of stationary quantum states. On the other hand, it is easy to prove that the Einstein’s equation is fulfilled for averaged over time expectation values of passive gravitational mass and energy. Indeed, from Eq.(45), it directly follows that:

\[ \langle \langle \hat{m}_g \rangle \rangle_t = m_e + \frac{E_1 + E_2}{2c^2} = \langle \frac{E}{c^2} \rangle. \quad (47) \]

5.2. Inequivalence between active gravitational mass and energy

Let us study the question if the expectation value of active gravitational mass is equivalent to the expectation value of energy for superpositions of stationary quantum states. For this purpose, we again consider the simplest superposition of stationary quantum states in a hydrogen atom, given by Eq.(43). As we discussed in the previous subsection, the expectation value of energy for such superposition (44) does not depend on time. Nevertheless, it is easy to show that the expectation value of electron active mass operator (39) oscillates with time:

\[ \langle \hat{m}_a \rangle = m_e + \frac{E_1 + E_2}{2c^2} + \frac{V_{1,2}}{c^2} \cos \left[ \frac{(E_1 - E_2)t}{\hbar} \right], \quad (48) \]

where matrix element \( V_{1,2} \) is given by Eq.(46). As seen from Eq.(48), these time dependent oscillations are similar to that for the expectation value of passive electron gravitational mass (45). Eq.(48) directly demonstrates inequivalence between the expectation values of active gravitational mass and energy for superpositions of stationary quantum states. As we discussed before such oscillations have a pure quantum mechanical origin and do not have classical analogs. To simplify the situation, in the same way as in the previous subsection, we can introduce the averaged over time expectation value of active gravitational mass, which obeys the Einstein’s equation:

\[ \langle \langle \hat{m}_a \rangle \rangle_t = m_e + \frac{E_1 + E_2}{2c^2} = \langle \frac{E}{c^2} \rangle. \quad (49) \]

6. FIRST SUGGESTED IDEALIZED EXPERIMENT

Here, we suggest idealized experiment how to observe oscillations of the expectation value of active gravitational mass of electrons. By using laser, it is possible to create a macroscopic ensemble of coherent superpositions of electron
stationary states in some gas [see Eq(43)]. They will be characterized by a feature that each molecule has the same phase difference between two wave function components, $\Psi_1(r)$ and $\Psi_2(r)$. In this case, the macroscopic ensemble of atoms generates gravitational field, which oscillates in time. In the case of hydrogen atoms, these oscillations correspond to the following oscillations of the active gravitational mass

$$M_n^2 = n m_e + n \frac{E_1 + E_2}{2c^2} + n \frac{V_{1,2}}{c^2} \cos \left( \frac{(E_1 - E_2)t}{\hbar} \right),$$

(50)

where $n$ is the number of coherent atoms. The oscillating gravitational field can be, in principle, measured by a probing mass.

Let us discuss some characteristic features of oscillations (50), including their relative magnitude. If we use actual numbers for a hydrogen atom, we can obtain the following value for the matrix element (46): $V_{1,2} \simeq 5.7 \, eV$. If we take into account that $m_e c^2 \simeq 0.5 \, MeV$ and $m_p \simeq 1800 \, m_e$, we can estimate relative strength of the oscillations (50): $\delta m_e/m_e \sim 10^{-5}$ and $\delta m/m \sim 10^{-8}$. From these estimations, we can conclude that the oscillations (50) are small but not negligible. For the corresponding experiment it may be important that they correspond to the following frequency: $\nu = 2.5 \times 10^{15} \, Hz$. Here, we discuss what kind of molecules have to be used for the above described experiment. As an idealized example, we in the review always use a hydrogen atom or a macroscopic ensemble of such atoms. In reality, as known, free hydrogen atoms exist only in the Universe far from the Earth. Therefore, more realistic examples are helium atoms or hydrogen molecules. The corresponding calculations for a helium atom (see Ref.[17]) show that all qualitative results of the review become unchanged. Moreover, all estimations of the possible experimental characteristics of oscillations (50), obtained in this section, are valid. Nevertheless, there exists extra important for the possible experiment point. Indeed, for creation of superpositions of stationary quantum states by laser, we need that the corresponding dipole matrix element will not be zero. On the other hand, for the existence of the oscillations (50) we need non-zero matrix element (46). Simple analysis shows that these are possible only for molecules without center of symmetry [17]. We postpone the detail discussions of the realistic experiments until the corresponding calculations are complete. To summarize this section, we hope that the oscillations of active gravitational mass are experimentally found, despite the fact that they correspond to quasi-stationary quantum states (43). If such oscillations are experimentally observed this would be the first direct observation of quantum effects in general relativity.

7. BREAKDOWN OF THE EQUIVALENCE BETWEEN PASSIVE GRAVITATIONAL MASS AND ENERGY FOR STATIONARY QUANTUM STATES

In section 4, we have discussed that the expectation values of both passive and active gravitational masses are equivalent to energy for stationary quantum states. Nevertheless, as known, quantum mechanics is not science about average values, but it is science about an individual measurement. Therefore, in this section, we review question about an individual measurement of passive gravitational mass for a hydrogen atom. In particular, we discuss in detail how Eqs.(10),(11) break the equivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states [3-6]. First, let us pay attention that passive gravitational mass operator for electron in a hydrogen atom (11) does not commute with energy operator taken in the absence of gravitational field. From physical point of view, this means that quantum state of a hydrogen atom with definite energy does not correspond to quantum state with definite passive gravitational mass. In other words, measurements of gravitational mass in such state can give different values which, as shown below, are quantized [3-6]. In this section, we illustrate the inequivalence between energy and passive gravitational mass by means of two thought experiments.

7.1. First thought experiment

Here, we discuss the inequivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states by considering the following thought experiment [4,5]. Suppose, there exists quantum state of a hydrogen atom with definite energy in the absence of gravitational field (i.e, at $t \to -\infty$),

$$\Psi_1(r,t) = \Psi_1(r) \exp(-im_e c^2 t/\hbar - iE_1 t/\hbar).$$

(51)
Then, we switch on adiabatically gravitational field (4) and, therefore, in the presence of gravitational field (i.e., near \( t = 0 \)), a hydrogen atom is characterized by the following wave function:

\[
\Psi(r, t) = \sum_{n=1}^{\infty} a_n(t) \Psi_n(r) \exp\left(-im_e c^2 t / \hbar - iE_n t / \hbar \right) .
\]  

(52)

[Note that, due to special symmetry of our problem, we need to keep in Eq.(52) only normalized wave functions, \( \Psi_n(r) \), corresponding to isotropic \( nS \) wave functions of the \( n \)-th energy level in a hydrogen atom.]

Using Eqs.(10),(11), the adiabatically switched gravitational field can be written as the following time-dependent perturbation to quantum Hamiltonian in the absence of the field:

\[
\hat{U}(R, t) = \phi(R) \left[ m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 + \left( 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) / c^2 \right] \exp(\lambda t),
\]  

(53)

where \( \lambda \to 0 \). [It is important that our choice of the perturbation in the adiabatical form (53) allows to avoid in the Hamiltonian some extra terms related to velocities (see e.g., the Lagrangian in Ref.[9]).] Now, let us perform the standard calculations by means of the quantum mechanical time dependent perturbation theory. As a result, we obtain for amplitudes in Eq.(52) the following expressions:

\[
a_1(t) = \exp\left[-\frac{i\phi(R)m_e c^2 t + i\phi(R)E_1 t}{c^2 \hbar}\right],
\]  

(54)

\[
a_n(0) = -\frac{\phi(R)}{c^2} \frac{V_{1,n}}{E_n - E_1}, \quad n \neq 1,
\]  

(55)

where

\[
V_{1,n} = \int \Psi^*_n(r) \left( \frac{2 \hat{p}^2}{2m_e} - \frac{e^2}{r} \right) \Psi_n(r) d^3 r.
\]  

(56)

[We pay attention to the fact that perturbation (53) has the following selection rule: electrons from 1S ground state of a hydrogen atom can be excited only into \( nS \) excited level.] Here, we discuss in detail Eqs.(54)-(56). We note that Eq.(54) represents the well known red shift of atomic ground state energy level in gravitational field. It is important that amplitudes (55) correspond to some novel effect. Indeed, they show that there is non-zero probability,

\[
P_n = |a_n(0)|^2 = \left[ \frac{\phi(R)}{c^2} \right]^2 \left( \frac{V_{1,n}}{E_n - E_1} \right)^2, \quad n \neq 1,
\]  

(57)

that, in the presence of gravitational field (i.e., at \( t = 0 \)), electron occupies \( nS \) energy level in a hydrogen atom. In other words, this demonstrates that quantum measurements of passive electron gravitational mass (11) in quantum state with definite energy (52) can result in the following quantized values:

\[
m_p^q(n) = m_e + E_n / c^2,
\]  

(58)

which correspond to probabilities (57). Note that probabilities (57) are of the second order with respect to the small parameter, \( |\phi(R)| / c^2 \ll 1 \), and, therefore, they are rather small. Nevertheless, we pay attention that the corresponding changes of electron passive gravitational mass are comparably large. Indeed, they do not contain small parameter, \( |\phi(R)| / c^2 \ll 1 \), and are of the order of \( 5m_p^q / m_p^0 \sim \alpha^2 \sim 10^{-4} \), where \( \alpha \) is the fine structure constant. It is important that quantization law (58) can be measured indirectly. The point is that the all excited atomic levels are quasi-stationary and, thus, the quantization law (58) can be discovered by measuring electromagnetic radiation from macroscopic ensemble of the atoms. This procedure is discussed in more detail in section (8).

### 7.2. Second thought experiment

To further study phenomenon of the inequivalence between passive gravitational mass and energy for stationary quantum states at the microscopic level, let us consider the second possible thought experiment [3,5]. Suppose that, at \( t = 0 \), we create, in gravitational field (4), electron stationary quantum state, corresponding to ground state of a
hydrogen atom in the absence of gravitational field [see Eq.(51)]. Then, we take into account gravitational field and find that the wave function (51) is not anymore a ground state of the Hamiltonian (10),(11). It is important that for proper local observer, in accordance with (5) and main principles of general relativity, a general solution of the Schrödinger equation in the presence of the gravitational field (4) is

$$
\Psi(r, t) = (1 - \phi/c^2)^{3/2} \sum_{n=1}^{\infty} a_n \Psi_n[(1 - \phi/c^2)r] \exp[-im_n c^2(1 + \phi/c^2)t/\hbar] \exp[-iE_n(1 + \phi/c^2)t/\hbar].
$$

(59)

It is easy to show that wave function (59) is a series of eigenfunctions of passive gravitational mass operator (11) if we restrict ourselves only linear terms with respect to the parameter \(\phi/c^2\). We note that, in Eq.(59), factor \((1 - \phi/c^2)\) in a hydrogen atom wave functions, \(\Psi_n[(1 - \phi/c^2)r]\), is due to a curvature of space and the total wave function (59) is normalized. Another factor \((1 + \phi/c^2)\) corresponds to the famous red shift of electron energy levels in the gravitational field (4).

According to the main principles of quantum mechanics, probability that electrons with wave function (51) occupies excited quantum state with energy \(m_e c^2(1 + \phi/c^2) + E_n(1 + \phi/c^2)\) can be written as

$$
P_n = |a_n|^2, \quad a_n = \int \Psi_1^*(r)\Psi_n[(1 - \phi/c^2)r]d^3r \approx -(\phi/c^2) \int \Psi_1^*(r)r\Psi_n'(r)d^3r.
$$

(60)

[It is important that the calculations show that for \(a_1\) in Eq.(60) the linear term with respect to gravitational potential \(\phi\) is zero. This corresponds to the statement of section 4 about the equivalence of the expectation values of passive gravitational mass and energy and is a consequence of the quantum virial theorem.] If we take into account the fact that the Hamiltonian is a Hermitian operator, we can show that, for \(n \neq 1\),

$$
\int \Psi_1^*(r)r\Psi_n'(r)d^3r = V_{1,n}/(E_n - E_1),
$$

(61)

where the matrix element of the virial operator is given by Eq.(56). Here, we discuss physical meaning of Eqs.(60),(61). We point out that they directly demonstrate that there exist a non-zero probability,

$$
P_n = |a_n|^2 = \left[\frac{\phi(R)}{c^2}\right]^2 \left(\frac{V_{n1}}{E_n - E_1}\right)^2, \quad n \neq 1,
$$

(62)

that, in the presence of gravitational field, electron occupies aS atomic orbital, which directly breaks the expected Einstein’s equation, \(m_e^p = m_e + E_1/c^2\). Let us discuss this question in more detail. Indeed, non-zero probabilities (62) demonstrate that there are situations, where quantum measurements of electron passive gravitational mass in state with definite energy give the quantized values (58). Note that, as it has to be, the probabilities (57) and (62) are equal, which show the equivalence of the first and second thought experiments. We pay attention that the probabilities (57),(62) are quadratic with respect to small parameter, \(|\phi(R)|/c^2 \ll 1\). It is easy to make sure that, to calculate the probabilities (62) with such accuracy for \(n \neq 1\), we need to calculate wave functions (52),(54),(55) in linear approximation with respect to the above mentioned small parameter. Moreover, analysis shows that calculation of wave function (59) with accuracy of \(|\phi(R)|^2/c^4\) will change electron passive gravitational mass of the order of \(\delta m^p_\phi \sim (\phi/c^2)m_e \sim 10^{-5}m_e\). This value is much smaller than the characteristic difference between the quantized values in Eq.(58), \(\delta m^p_\phi \sim \alpha^2 m_e \sim 10^{-4}m_e\). We also would like to stress that the existence of the small probabilities (57),(62), \(P_n \sim 10^{-18}\), to break the Einstein’s equation for passive gravitational mass and energy do not contradict to the existing Eötvös type experimental measurements [11], confirming the equivalence principle with accuracy of the order of \(\delta m/m \sim 10^{-13}\). As mentioned in the previous subsection, an important role plays the fact that all excited levels of a hydrogen atom are not strictly stationary. Therefore, quantization law (58) can be measured by optical methods, which are much more sensitive that the Eötvös type experiments.

8. SECOND SUGGESTED IDEALIZED EXPERIMENT

The second suggested idealized experiment [3-5] is designed to observe the passive gravitation mass quantization law (58) by measuring radiation, emitted by a macroscopic ensemble of some atoms supported and moved in the Earth’s gravitational field. All calculations below are done for hydrogen atoms, although in practice helium atoms or hydrogen molecules can be used [17]. We cannot except that some kind of solids may be more convenient to use in such kind of experiments, but this discussion has to be postponed until more detailed calculations are done. As mentioned before, in this review, we restrict ourself by the following idealized experimental conditions. There is a small spacecraft, which carries a tank with hydrogen atoms. The atoms do not interact with each other and are supported in the Earth’s gravitational field and moved with constant velocity in the field.
8.1. Lagrangian and Hamiltonian

Here, we review in detail derivation of the Hamiltonian [3], corresponding to the suggested idealized experiment [3-6]. To this end, we first consider the Lagrangian for the following three bodies: electron and proton, bound by the Coulomb electrostatic field, and the Earth moving from the atom with small constant velocity, \( u \ll c \). This case for classical model of a hydrogen atom is considered in Ref.[9] and, therefore, we can use the result [9] that the above mentioned three-body Lagrangian can be written as sum

\[
L = L_{\text{kin}} + L_e + L_G + L_{e,G},
\]

(63)

where \( L_{\text{kin}}, L_e, L_G, \) and \( L_{e,G} \) are kinetic, electric, gravitational, and electro-gravitational parts of the Lagrangian, correspondingly. We recall that, in our approximation, we omit in the Hamiltonian and Lagrangian all terms of the order of \( \phi^2/c^4 \) and \((v/c)^4\). In other words, we keep only classical kinetic energy and the electrostatic Coulomb energy interactions with external gravitational field. In this case four contributions to the Lagrangian (63) can be written in the following simplified forms:

\[
L_{\text{kin}} + L_e = -Mc^2 - mp^2 - m_ec^2 + M\frac{u^2}{2} + m_e\frac{v^2}{2} + \frac{e^2}{r} ,
\]

(64)

\[
L_G = G \frac{m_pM}{(R + ut)} + G \frac{m_eM}{(R + ut)} \left\{ 1 - \frac{1}{2}[v \cdot u + (v \cdot \hat{R})(u \cdot \hat{R})]/c^2 \right\} + \frac{3}{2}G \frac{m_eM}{(R + ut)} \frac{v^2}{c^2} ,
\]

(65)

\[
L_{e,G} = -2G \frac{M}{(R + ut)c^2} \frac{e^2}{r} ,
\]

(66)

where \( \hat{R} \) is a unit vector directed along \( R \) and where we use the inequality \( m_e \ll m_p \).

For further development, it is natural to suppose that the spacecraft velocity, \( u \), is small enough comparable to the typical electron velocity in atoms and molecules, \( v \sim \alpha c \):

\[
u \ll v \sim \alpha c .
\]

(67)

Using Eq.(67), we simplify the Lagrangian (63)-(66) in the following way:

\[
L = m_e\frac{v^2}{2} + \frac{e^2}{r} - \phi(R + ut) \left[ m_e + 3m_e\frac{v^2}{2} - 2\frac{e^2}{r} \right] .
\]

(68)

[We pay attention that, in the Lagrangian (68), we disregard the fact that the center of mass of a hydrogen is not exactly coincides with proton. It is possible to show that, doing so, we disregard terms of the order of \(|\phi(R + ut)|m_evu/c^2 \ll |\phi(R + ut)|m_ev^2/c^2\). We also use electron mass, instead of the so-called reduced mass, which is possible for \( m_e \ll m_p \).]

From Eq.(68), it is easy to find the corresponding Hamiltonian:

\[
H = \frac{p^2}{2m_e} - \frac{e^2}{r} + \phi(R + ut) \left[ m_e + 3m_e\frac{p^2}{2m_e} - 2\frac{e^2}{r} \right] ,
\]

(69)

which can be easily quantized:

\[
\hat{H} = \frac{\hat{p}^2}{2m_e} - \frac{\hat{e}^2}{r} + \hat{\phi}(R + ut) \left[ m_e + 3m_e\frac{\hat{p}^2}{2m_e} - 2\frac{\hat{e}^2}{r} \right] .
\]

(70)

8.2. Mass quantization law and emitted photons

Once the quantum Hamiltonian (70) is known, we can consider theory of the second suggested idealized experiment. We assume that at initial moment of time, \( t = 0 \), all hydrogen atoms are in a ground state and located in small spacecraft, which is placed at distance \( R' \) from the Earth’s center. The wave function of each hydrogen atom in this case can be written in the following form [see Eq.(59)]:

\[
\hat{\Psi}_1(r, t) = (1 - \phi/c^2)^{3/2}\Psi_1[(1 - \phi/c^2)r] \times \exp[-im_e\phi^2(1 + \phi/c^2)t/\hbar] \exp[-iE_1(1 + \phi/c^2)t/\hbar] ,
\]

(71)
where $\phi' = \phi(R')$. Since we consider non-interacting hydrogen atoms, here we discuss optical properties of a single atom and later will take into account the fact that we have a macroscopic ensemble to the atoms. We recall that the atom is supported in gravitational field and moved from the Earth with constant velocity, $u \ll \alpha c$ [see Eq.(67)], by spacecraft. From Eq.(70), it follows that electron wave function at latter moments of time in the inertial coordinate system, related to the spacecraft, can be expressed as

$$\Psi(r, t) = (1 - \phi'/c^2)^{3/2} \sum_{n=1}^{\infty} \tilde{a}_n(t) \Psi_n[(1 - \phi'/c^2)r] \exp[-im_ee^2(1 + \phi'/c^2)t/\hbar] \exp[-iE_n(1 + \phi'/c^2)t/\hbar],$$

(72)

where the perturbation for "free" Hamiltonian of a hydrogen atom is

$$\hat{U}(r, t) = \frac{\phi(R' + ut) - \phi(R')} {c^2} \left( 3 \frac{\hat{p}^2} {2m_e} - \frac{2e^2} {r} \right).$$

(73)

(It is important to stress that in the spacecraft, moving with constant velocity, each hydrogen atom and electron experience gravitational force. In this section, we suggest the experiment, where this force is compensated by some forces of non-gravitational origin. It is possible to show that this causes small changes of the atomic energy levels, which are not important for the further calculations. Since the gravitational forces are compensated, the atom does not feel acceleration, $g$, but rather feels the time dependent gravitational potential, $\phi(R' + ut)$ [see Eq.(73)].)

Now, let us apply the standard time-dependent quantum mechanical perturbation theory to determine amplitudes $\tilde{a}_n(t)$ in Eq.(72):

$$\tilde{a}_n(t) = -\frac{V_{1,n}} {\hbar \omega_{1,1} c^2} \left\{ \phi(R' + ut) - \phi(R') \right\} \exp(i\omega_{1,1} t) - \frac{u} {i\omega_{1,1}} \int_0^t \frac{d\phi(R' + ut)} {dR'} d(\exp(i\omega_{1,1} t)) \right\}, \ n \neq 1 ,$$

(74)

where

$$\omega_{1,1} = (E_n - E_1)/\hbar$$

(75)

and $V_{1,n}$ is given by Eq.(56). We point out that, in the suggested experiment, the following condition is valid:

$$u \ll \omega_{1,1} R \sim \alpha c(R/R_B) \sim 10^{13} c.$$  

(76)

It is easy to prove that Eq.(76) means that we can omit the second term in Eq.(74):

$$\tilde{a}_n(t) = -\frac{V_{1,n}} {\hbar \omega_{1,1} c^2} \left(\phi(R' + ut) - \phi(R')\right) \exp(i\omega_{1,1} t) , \ n \neq 1 .$$

(77)

Let us discuss the main consequences of Eq.(77). As directly seen from (77), if the excited levels of the atom were strictly stationary, then there would exist possibilities to find the quantized values of passive gravitational mass with $n \neq 1$ in Eq.(58). They correspond to the following probabilities:

$$\hat{P}_n(t) = \left( \frac{V_{1,n}} {\hbar \omega_{1,1}} \right)^2 \frac{[\phi(R' + ut) - \phi(R')]^2} {c^4} = \left( \frac{V_{1,n}} {\hbar \omega_{1,1}} \right)^2 \frac{[\phi(R'') - \phi(R'')]^2} {c^4} , \ n \neq 1 ,$$

(78)

where $R'' = R'+ut$. In reality, the excited levels of a hydrogen atom are quasi-stationary, therefore, they spontaneously decay with time. In this case, it is possible to measure the breakdown of the Einstein’s equation indirectly by measuring electromagnetic radiation, emitted by a macroscopic ensemble of the atoms. Then, Eq.(78) corresponds to probability that a hydrogen atom emits a photon with frequency (75) during the time interval $t$. [We would like to stress here that the dipole matrix element for quantum transition $nS \rightarrow 1S$ is zero. In this situation, we can expect quadrupole transition $2S \rightarrow 1S$ and dipole transitions like $3S \rightarrow 2P$, etc.]

Important fact is that the probabilities (78) can be written in the form, where they depend only on gravitational potentials at the initial and final points of spacecraft trajectories, $\phi' = \phi(R')$ and $\phi'' = \phi(R'')$. This allow to determine their physical meaning. Below, we consider a general solution of the Hamiltonian (10),(11) at final position of spacecraft, where it is located at distance $R'$ from center of the Earth:

$$\Psi(r,t) = (1 - \phi''/c^2)^{3/2} \sum_{n=1}^{\infty} \tilde{a}_n \Psi_n[(1 - \phi''/c^2)r] \exp[-im_ee^2(1 + \phi''/c^2)t/\hbar] \exp[-iE_n(1 + \phi''/c^2)t/\hbar].$$

(79)
We pay attention that wave function (79) is an infinite series of the eigenfunctions corresponding to a definite weight (i.e., passive gravitational mass) at distance $R''$ from the Earth’s center. If we take into account that the wavefunction (71) corresponds to quantum state with definite energy at distance $R''$ from the Earth’s center, then, according to the main principles of quantum mechanics, the expression,

$$a_n = \int \Psi_1^*[(1 - \phi'/c^2)r]\Psi_n[(1 - \phi''/c^2)r]d^3r \approx -[(\phi'' - \phi')/c^2] \int \Psi_1^*(r)r\Psi_n'(r)d^3r,$$

$$P_n = |a_n|^2 = (\phi'' - \phi')^2 \left| \Psi_1^*(r)r\Psi_n'(r)d^3r \right|^2,$$

represents the probability that electron with definite energy at position $R''$ has the quantized passive gravitational mass (58) values with $n \neq 1$. Now, let us use Eq.(61) and rewrite probabilities (80) in a familiar way,

$$P_n = \left( \frac{V_{n,1}}{\hbar \omega_{n,1}} \right)^2 \frac{(\phi(R'') - \phi(R'))^2}{c^2}, \quad n \neq 1,$$

coinciding with Eq.(78). Here, we can make a conclusion that all photons, emitted during spacecraft motion between positions $R'$ and $R''$, are due to breakdown of the Einstein’s equation for passive gravitational mass and energy (i.e., they are due to quantization of gravitational mass [see Eq.(58)]). Below, we estimate the probabilities (78). It is easy to see that, by using spacecraft, we can reach the condition $|\phi(R'')| \ll |\phi(R')|$. In this case Eq.(78) coincides with Eqs.(57),(62) and can be written as

$$\tilde{P}_n = \left( \frac{V_{n,1}}{E_n - E_1} \right)^2 \frac{\phi^2(R')}{c^4} \simeq 0.49 \times 10^{-18} \left( \frac{V_{n,1}}{E_n - E_1} \right)^2,$$

where we make use of the following values of the Earth’s radius and mass, $R_0 \simeq 6.36 \times 10^6m$ and $M \simeq 6 \times 10^{24}kg$, respectively. Characteristic feature of probability (82) is that it is quadratic with respect to small parameter, $|\phi(R'')| \ll 1$, and, thus, small, $P_n \sim 10^{-18}$. Nevertheless, the number of the emitted photons can be very large since $\frac{V_{n,1}}{E_n - E_1} \sim 1$ and the Avogadro number (i.e., in our case, the number of atoms in one mole of atomic hydrogen) is $N_A = 6 \times 10^{23}$. The calculations show that, for 1000 moles of the atoms, the number of emitted photons is estimated as

$$N_{n,1} = 2.95 \times 10^8 \left( \frac{V_{n,1}}{E_n - E_1} \right)^2, \quad N_{2,1} = 0.9 \times 10^8.$$

Presumably such large amount of photons can be experimentally detected, where $N_{n,1}$ determines the number of photons, emitted with frequency (75).

### 9. SUMMARY

In the review, we have discussed in detail the question about active and passive gravitational masses of a composite quantum body [3-6], using a hydrogen atom in the Earth’s gravitational field as the simplest example. In particular, we have shown that the expectation values of active and passive gravitational masses are equivalent to the expectation value of energy for stationary quantum states. On the other hand, this equivalence has been discussed to be broken for non-stationary quantum states, where the expectation values of energy are constant, but time-dependent oscillations of the expectation values of the masses appears. We have suggested idealized experiment how to detect these time-dependent oscillations of active gravitational mass. Here, we pay attention to the following difficulty to be resolved before such experiment is done. The point is that the non-stationary quantum states decay with time. Therefore, it is necessary to distinguish experimentally weak oscillating gravitational signal from strong electromagnetic radiation effects. For stationary quantum states, we have also discussed breakdown of the equivalence between passive gravitational mass and energy at a microscopic level. We have illustrated this phenomenon by discussing two thought experiments and one idealized. Our idealized experiment also needs an improvements and some additional calculations. First, the existing calculations [3-6] have to be extended to the cases of helium and molecular hydrogen, since large amounts of atomic hydrogen exist only in the Universe very far from the Earth. Second, effects of atomic (molecular) interactions have to be taken into account. We cannot even exclude that it maybe more experimentally effective to use some solid state bodies, instead of a tank of a gas, suggested above and in Refs.[3-6]. We hope that all the above mentioned and possible other experimental difficulties are solved and the discussed above two experiments are done. They would be the first direct experimental observations of quantum effects in general relativity.
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∗Also at: L.D. Landau Institute for Theoretical Physics, RAS, 2 Kosygina Street, Moscow 117334, Russia.

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