Surface criticality of antiferromagnetic Potts model

Li-Ru Zhang,1 Chengxiang Ding,1 Youjin Deng,2,3 and Long Zhang4

1School of Science and Engineering of Mathematics and Physics, Anhui University of Technology, Maanshan, Anhui 243002, China
2Department of Modern Physics, University of Science and Technology of China, Hefei, 230027, China
3MinJiang Collaborative Center for Theoretical Physics, College of Physics and Electronic Information Engineering, Minjiang University, Fuzhou 350108, China
4Kashi Institute for Theoretical Sciences and CAS Center for Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100190, China

(Dated: May 3, 2022)

We study the three-state antiferromagnetic Potts model on the simple-cubic lattice, paying attention to the surface critical behaviors. When the nearest neighboring interactions of the surface is tuned, we obtain a phase diagram similar to the XY model, owing to the emergent O(2) symmetry of the bulk critical point. For the ordinary transition, we get \( y_{B1} = 0.780(3), \eta_{R} = 1.44(1) \), and \( \eta_{L} = 0.736(6) \); for the special transition, we get \( y_{S} = 0.59(1), y_{B1} = 1.693(2), \eta_{R} = -0.391(4) \), and \( \eta_{L} = -0.179(5) \); in the extraordinary-log phase, the surface correlation function \( C_{L}(r) \) decays logarithmically, with decaying exponent \( q = 0.60(2) \), however, the correlation \( C_{L}(r) \) still decays algebraically, with critical exponent \( \eta_{L} = -0.442(5) \). If the ferromagnetic next nearest neighboring surface interactions are added, we find two transition points, the first one is a special point between the ordinary phase and the extraordinary-log phase, the second one is a transition between the extraordinary-log phase and the \( Z_{6} \) symmetry-breaking phase, with critical exponent \( y_{S} = 0.41(2) \). The scaling behaviors of the second transition is very interesting, the surface spin correlation function \( C_{||}(r) \) and the surface squared staggered magnetization at this point decays logarithmically, with exponent \( q = 0.37(1) \); however, the surface structure factor with the smallest wave vector and the correlation function \( C_{L}(r) \) satisfy power-law decaying, with critical exponents \( \eta_{||} = -0.69(1) \) and \( \eta_{L} = -0.37(1) \), respectively.

PACS numbers: 03.67.Bg, 03.65.Ud, 05.30.Rt

I. INTRODUCTION

Phase transition and critical phenomena are hot topics in the research of condensed matter and statistical physics. At the critical point of a continuous transition, the system exhibits a variety of singular behaviors characterized by algebraically decaying of correlation functions. Such type of behaviors also appear on the surface of the system and can be different to that of the bulk ones.1–4 Depending on the strength of the surface interactions, the surface critical behavior can be classified as “ordinary transition”, “special transition”, and “extraordinary transition”. Typical examples can be found in the classical O(n) models5,6. Generally, the ordinary transition can be found without tuning the surface interactions \( J_{s} \), i.e., it is the same as the bulk ones, if it is tuned (strengthened), a phase transition may be found, which is the special point \( J_{sp} \), and the phase with \( J_{s} > J_{sp} \) is the extraordinary phase. It is clear that the extraordinary phases of Ising model are ordered, however, whether there is a special transition for the surface of the Heisenberg model and what the extraordinary phase is once controversial7,8. Similar problems also exist in XY model. Until recently, the research interest in surface critical phenomena was renewed by the exotic surface critical behavior in the quantum spin models9,10. A recent theoretical study pointed out that the extraordinary phase on the surface of the Heisenberg model may be an “extraordinary-log” phase characterized by logarithmically decaying of the surface correlation function11, which has been verified numerically.12, similar behavior is also found in the XY model12.

The above results show that the surface critical phenomenon has many interesting characteristics, which are worth studying, and the related research are continuing13–26. In this paper, we study the surface critical properties of the three-state antiferromagnetic Potts model on a simple-cubic lattice27,28. Due to the \( Z_{3} \) symmetry of the spins and the permutation symmetry of two sets of sublattices, the ground state of the system breaks the \( Z_{6} \) symmetry, however, at the bulk critical point, the symmetry of the order parameter is O(2), which is called “emergent O(2) symmetry”29,30, and the corresponding universality class of the phase transition is the same as the XY model. At such “emergent O(2)” bulk critical point, we explore the surface critical behaviors by tuning the nearest neighboring (NN) surface interactions and the next nearest neighboring (NNN) surface interactions, the two phase diagrams we obtained are shown in Fig. 1. In the phase diagram (a), the extraordinary-log phase and the special point are obtained, which is very similar to the XY model27,17. We can also see that the surface can not be ordered only by increasing the strength of the surface NN interactions, this is closely related to the fact that the two-dimensional antiferromagnetic Potts model is not ordered even the temperature is down to zero. However, NNN ferromagnetic interactions can induce \( Z_{6} \) symmetry breaking phase at finite temperature for the two-dimensional antiferromagnetic Potts model31, this inspires us to add NNN ferromagnetic interactions to the surface of the three-dimensional antifer-
The antiferromagnetic Potts model we studied is defined on the simple-cubic lattice
\[ \mathcal{H} = J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j} + J_s \sum_{\langle i,j \rangle'} \delta_{\sigma_i, \sigma_j} - J'_s \sum_{\langle i,j \rangle''} \delta_{\sigma_i, \sigma_j}, \]
where \( J, J_s, \) and \( J'_s \) are the strengths of the NN bulk interactions, the NN surface interactions, and the NNN surface interactions, respectively, noting that the NNN surface interactions are ferromagnetic. The Potts spin \( \sigma_i = 1, 2, 3 \), which can be mapped to a unit vector in the plane
\[ \vec{\sigma}_i = (\cos \theta_i, \sin \theta_i), \]
with \( \theta_i = 2\pi \sigma_i / 3 \). Such mapping shows the symmetry of the spin, the variables we studied are based on such vectors.

The Monte Carlo algorithm we adopt is a combination of the local update (Metropolis algorithm) and global update (cluster algorithms). For the case \( J'_s \neq 0 \), the system has both antiferromagnetic and ferromagnetic interactions, the global update is a mixture of the Swendsen-Wang and WSK algorithms. The high efficiency of the algorithm enables us to perform simulations for the systems with sizes up to \( L = 128 \). In the simulations, the periodic boundary condition is applied along the \( x \) and \( y \) directions, while open boundary condition is applied along the \( z \) direction.

The bulk variables we sampled include the squared (staggered) magnetization \( m^2_s \), the magnetic susceptibility \( \chi_s \), and the Binder Ratio \( Q_s \), which are defined as
\[ m^2_s = \langle M^2_s \rangle, \]
\[ \chi_s = N(\langle M^2_s \rangle - \langle |M_s| \rangle^2), \]
\[ Q = \langle M^2_s \rangle^2 / \langle M^2_s \rangle, \]
where \( T \) is the temperature and \( M_s \) is defined as
\[ M_s = \frac{1}{N} \sum_{\vec{R}} (-1)^{x+y+z} \vec{R}. \]
Here \( \vec{R} = (x, y, z) \) is the coordination, \( N = L^3 \) is the number of sites of the lattice.

We also sample the correlation function \( C(r) \) and correlation length \( \xi \)
\[ C(r) = \langle \vec{\sigma}_i \cdot \vec{\sigma}_{i+r} \rangle, \]
\[ \xi = (m^2_s/F - 1)^{1/2}, \]
where \( F \) is define as
\[ F = \frac{1}{N^2} \left( \left| \sum_{\vec{R}} (-1)^{x+y+z} e^{i\vec{k} \cdot \vec{R}} \vec{R} \right|^2 \right). \]

Generally, in a critical phase, correlation ratio \( \xi/L \) assumes a universal nonzero value in the thermodynamic limit \( L \to \infty \); in a disordered phase, correlation length \( \xi \) is finite and \( \xi/L \) drops to zero, while in an ordered phase, \( \xi/L \) diverges quickly since the structure factor \( F \) vanishes rapidly. Therefore, like the Binder Ratio \( Q_s \), \( \xi/L \) is also very useful in locating the critical point of phase transition.

The definition of the surface variables are very similar to the bulk ones, but the spins in Eqs. (6), (7), and (9) should be restricted to the surface ones. For clarity, we add a subscript \( 1 \) to the surface variables, i.e., the surface squared magnetization, the surface magnetic susceptibility, the surface Binder Ratio, the surface structure factor, and the surface correlation length are written as \( m^2_{s1}, \chi_{s1}, Q_{s1}, F_1, \) and \( \xi_1 \), respectively, except the surface correlation functions, which are written as \( C_{\parallel}(r) \) and \( C_{\perp}(r) \); both \( C_{\parallel}(r) \) and \( C_{\perp}(r) \) are calculated as Eq. (11), for \( C_{\parallel}(r) \), both site \( i \) and \( i + r \) are on the surface; for \( C_{\perp}(r) \), site \( i \) is on the surface but site \( i + r \) is in the bulk, and the line from \( i \) to \( i + r \) is perpendicular to the surface.

III. RESULTS

A. Refine the bulk critical point

The simple-cubic antiferromagnetic Potts model has ever been studied by Monte Carlo simulations in Refs. [28] and [29], whose critical exponents coincide with the XY universality class, however, the accuracy is not very high. A more
accurate numerical results about the critical exponents can be found in Ref. [31], while the interactions studied there are antiferromagnetic along the x and y directions but ferromagnetic along the z direction, such mixed interactions does not change the universality class of the phase transition but lead to different (bulk) critical point. In current paper, before studying the surface critical behaviors, we refine the bulk critical point for the simple-cubic antiferromagnetic model (with full antiferromagnetic interactions).

\[ m^2_{s1}L^2 = c + L^{2y_{h1}-2}(a + bL^{y_1}), \]  
\[ C_\parallel(L/2) = L^{-1-\eta_\parallel}(a + bL^{y_1}), \]  
\[ F_1L^2 = c + L^{2y_{h1}-2}(a + bL^{y_1}), \]  
\[ C_\perp(L/2) = L^{-1-\eta_\perp}(a + bL^{y_1}), \]

where \( y_{h1}, \eta_\parallel, \) and \( \eta_\perp \) are critical exponent; \( y_1 < 0 \) is the correction-to-scaling exponents; \( a, b, \) and \( c \) are unknown parameters, with \( c \) the analytical part of \( m^2_{s1}L^2 \) or \( F_1 L^2 \), originating from the contribution of short-term correlations. Although mathematically Eqs. (11) and (13) are equivalent to

\[ m^2_{s1} = c \cdot L^{-2} + L^{2y_{h1}-4}(a + bL^{y_1}), \]  
\[ F_1 = c \cdot L^{-2} + L^{2y_{h1}-4}(a + bL^{y_1}), \]

respectively, technically, in the fitting of \( m^2_{s1} \), the RHS of (11) or (13) will be dominated by the first term and the fitting result of \( y_{h1} \) is often unreliable if the value of \( y_{h1} \) is smaller than 1, which is exactly the case in ordinary transition.

The data fitting, with \( y_1 = -1 \), gives \( y_{h1} = 0.780(3) \) (from \( m^2_{s1} \)), \( y_{h1} = 0.782(4) \) (from \( F_1 \)), \( \eta_\parallel = 1.44(1) \), and \( \eta_\perp = 0.736(6) \), these values coincide with those of the XY model[12] and satisfy the scaling laws

\[ \eta_\parallel = d - 2y_{h1}, \]  
\[ 2\eta_\perp = \eta_\parallel + \eta, \]

with \( d = 3 \) the dimension of the system and \( \eta = 0.0385 \) the exponent of the bulk correlation.

### C. Special transition and extraordinary phase by tuning the NN surface interaction \( J_s \)

Exactly at the bulk critical point \( T_{\text{bulk}}^{\text{bulk}} = 1.22603 \), we perform simulations with open boundary condition along the z direction and periodic boundary conditions along the x and y directions, the surface interactions are set as \( J_s = J = 1 \) and \( J_s = 0 \). In this case, the surface critical behaviors originates from the bulk correlations, which is called the “ordinary transition”. The data of the surface squared magnetization \( m^2_{s1} \), the surface correlation function \( C_\parallel \), the surface structure factor \( F_1 \), and the surface correlation \( C_\perp \) are fit according to the scaling formulas
point satisfy the scaling formula [10], with $T$ replaced by $J_s$ and the critical exponent $y_1$ replaced by $y_s$; the fitting, with $y_1 = -1$, gives the critical point $J_{sc} = 2.04119(40)$ and the critical exponent $y_s = 0.59(1)$. We can see that the value of $y_s$ is consistent with that of the XY model.

Exactly at the special transition point $J_{sc} = 2.04119$, we investigate the scaling behaviors of the surface squared magnetization $m^2_{s1}$, the surface correlation function $C_{\parallel}$, the surface structure factor $F_1$, and the surface correlation $C_{\perp}$, they also satisfy the FSS formulas [11], [12], [13], and [14], respectively. Alternatively, $m^2_{s1}$ and $F_1$ can also be fit by Eqs. [15] and [16], respectively, because in current case $y_{h1} > 1.5$, the RHS of the equation is dominated by the second term, while the first one behaves like a correction-to-scaling term. The data fitting gives $y_{h1} = 1.693(2)$ (from $m^2_{s1}$), $y_{h1} = 1.694(4)$ (from $F_1$), $\eta_{\parallel} = -0.391(4)$, and $\eta_{\perp} = -0.179(5)$, these results coincide with those of the XY model and satisfy the scaling laws [17] and [18].

For $J_s > J_{sc}$, we find that the system is in the extraordinary-log phase, characterizing by the logarithmically decaying of correlation function $C_{\parallel}(r)$ and related variables.  

\begin{align}
  m^2_{s1} &= a \cdot [\ln(L/L_0)]^{-q}, \\
  C_{\parallel}(L/2) &= a \cdot [\ln(L/L_0)]^{-q'}, \\
  F_1 &= a \cdot [\ln(L/L_0)]^{-q'},
\end{align}

where $q$ and $q'$ are the critical exponents, $a$ and $L_0$ are nonuniversal parameters. From the fitting of $C_{\parallel}$, we get $q = 0.60(2)$ and $L_0 = 1.54(9)$; from the fitting of $m^2_{s1}$ and $F_1$, we get $q = 0.61(1)$ and $q' = 1.61(1)$, respectively. We can see that the values of $q$ and $q'$ are consistent with those of the XY model [17], and satisfy the relation $q' = 1 + q$. It should be noted that in the fitting of $m^2_{s1}$ and $F_1$, we have set the value of $L_0$ fixed to 1.54, such trick is the same as that in Ref. [17] for XY model. For the surface correlation function $C_{\perp}(L)$, we find that it still decaying algebraically, as shown in Fig. [18]b), therefore we fit it according to Eq. [14] with $y_1 = -1$, we get $\eta_{\perp} = -0.442(5)$.

We also study the case of $J_s = 10$, and get $q = 0.63(2)$, $q' = 1.64(2)$, $L_0 = 1.2(1)$, and $\eta_{\perp} = -0.444(7)$. The values of $q$ and $q'$ coincide with the cases of $J_s = 5$ and the XY model [17], which means the critical behaviors of the extraordinary-log phase of the O(2) surface are universal.

In the low temperature, the order parameter of the antiferromagnetic Potts model on the simple-cubic lattice breaks the $Z_6$ symmetry [32], therefore one may ask whether the surface can be ordered with such symmetry breaking when the surface interactions are strengthened. However, by performing the simulations with much larger $J_s$, we find that the surface is always in the extraordinary-log phase, it can not be ordered by the strengthen of $J_s$. This may be related to the fact that in the limit of $J_s \to \infty$, the decoupled two dimensional antiferromagnetic Potts model can not be ordered even the temperature is zero [32]. However, as we will show in the next subsection, the $Z_6$ symmetry breaking surface can be reached by adding the NNN ferromagnetic interactions $J_s'$ in the surface.

D. New special transition induced by next nearest interactions $J_s''$

When the NNN ferromagnetic interactions $J_s'$ is added to the surface, we can find two phase transitions, as shown by the behaviors of the surface susceptibility $\chi_{s1}$ in Fig. [19]a). In the intermediate region $J_{s1c1} < J_s' < J_{s2c2}$, with $J_s' \approx 0.41$ and $J_{s2c2} \approx 0.85$, the scaling behaviors of $\chi_{s1}$ is very different from that in region $J_s' < J_{s1c1}$ or $J_s' > J_{s2c2}$. As we will shown later, in this region, the system is in the extraordinary-log phase. Therefore, the peak of $\chi_{s1}$ at $J_{s1c1} \approx 0.41$ indicates a special transition point like that studied in above subsection.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{(a) Log-log plot of $m^2_{s1}$, $C_{\parallel}(L/2)$, and $F_1$ versus $\ln(L/L_0)$, for the antiferromagnetic Potts model with $T = T^{bulk} = 1.22603$, $J_s = 5$, and $J_s' = 0$; $L_0 = 1.54$ is taken for all the three variables. (b) Log-log plot of the surface correlation $C_{\perp}(L)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Surface critical behaviors of the antiferromagnetic Potts model with $T = T_c = 1.22603$ and $J_s = 1$: (a) surface susceptibility $\chi_{s1}$, (b) surface squared magnetization $m^2_{s1}$, (c) surface structure factor $F_1$, (d) surface Binder Ratio $Q_{s1}$.}
\end{figure}

The second transition at $J_{s2c2}$, which we call “new special point”, is revealed not only by the surface susceptibility...
\( \chi_{s1} \), but also by the surface squared magnetization \( m_{s1}^2 \) and surface structure factor \( F_1 \), as shown in Fig. 3(b) and (c), respectively; however, the signature of the Binder Ratio \( Q_{s1} \) for this transition is not obvious, as shown in Fig. 3(d).

We can see that the second peak of \( \chi_{s1} \) diverges much slower than the first one, namely the singularity of such transition is very weak. The transition point \( J_{sc2} \approx 0.85 \) is obtained by the position of such peak of the largest system with size \( L = 128 \); however, considering the finite-size effect, the critical point in the thermodynamic limit should be a little smaller than this one. A more accurate estimation of the critical point can be obtained from the scaling behavior of the structure factor \( F_1 \), which satisfies the logarithmic decaying in the region \( J_{sc1} < J_{c} < J_{sc2} \) but a power-law scaling at the transition point \( J_{sc2} = 0.815 \); this is demonstrated in Fig. 6.

In the region \( J_{sc1} < J_{c} < J_{sc2} \), we check the FFS behaviors of \( F_1 \) for several cases, the results of data fitting, according to Eq. (21), are listed in Table I. In this region, \( m_{s1} \) and \( C_{||} \) also satisfy the logarithmic decaying, the fitting results, according to Eq. (19) and (20), are also listed in Table I. These results claim that in this region the system is in an extraordinary-log phase; the critical exponents \( q \) and \( q' \) in this phase are universal and satisfy the relation \( q' = 1 + q \), as that in XY model. In this phase, the correlation \( C_{\perp} \) still satisfy the power-law scaling formula (12), the fitting results of \( \eta_{\perp} \) for several cases are also listed in Table I. In this range, \( J_{c} > J_{sc2} \), the surface is in a long-range ordered phase, the structure factor \( F_1 \) decays much faster, which satisfies neither a logarithmically decaying formula nor a simple pow-law formula, this is also demonstrated in Fig. 6.

In order to further explore the critical behaviors of the new special point, we plot \( F_1 L^{-1-\eta_1} = F_1 L^{0.31} \) versus \( J_{c} \) in Fig. 7(a), we can see that \( F_1 L^{-1-\eta_1} \) plays the role of a dimensionless variable like the Binder Ratio; Fig. 7(b) is the data collapses of \( F_1 L^{0.31} \) versus \( (J_{c} - 0.815) L^{y_s} \), with critical exponent \( y_s = 0.41 \). More rigorously, one can fit the data of \( F_1 \) in the vicinity of \( J_{c} = 0.815 \) by the scaling formula

\[
F_1 = L^{-1-\eta_1} \left[ a_0 + \sum_{k=1}^{k_{max}} a_k (J_{c} - J_{sc2})^k L^{y_s} + b L^{y_1} \right],
\]

which gives \( \eta_{\perp} = -0.69(1) \), \( J_{sc2} = 0.815(2) \), and \( y_s = 0.41(2) \), these results are consistent with those applied in the data collap in Fig. 7(b), thus give a self-consistent check.

In summary, for the new special transition, it is mainly characterized by the the following critical behaviors,

\[
J_{sc2} = 0.815(2), \\
y_s = 0.41(2), \\
F_1 \sim a L^{-1-\eta_1}, \text{ with } \eta_{\perp} = -0.69(1), \\
m_{s1}^2 \sim a \cdot \left[ \ln(L/L_0) \right]^q, \text{ with } q = 0.37(1), \\
C_{||} \sim a \cdot \left[ \ln(L/L_0) \right]^{-q}, \text{ with } q = 0.38(1), \\
C_{\perp} \sim a L^{-1-\eta_\perp}, \text{ with } \eta = -0.37(1).
\]
IV. CONCLUSION AND DISCUSSION

In summary, we have studied the surface critical behaviors of the three-state antiferromagnetic Potts model on the simple-cubic lattice, we obtain a phase diagram similar to the XY model by tuning the NN surface interactions, the universality classes of the special transition and extraordinary-log phase are the same as the XY model. A long range-order surface which breaks the $Z_6$ symmetry and a new type of special transition between such ordered phase and the extraordinary-log phase is obtained by tuning the NNN ferromagnetic surface interactions. At such new special transition point, the scaling behaviors are very interesting; the surface squared magnetization $m_{\perp}^2$ and the surface correlation function $C_{\parallel}$ satisfy the logarithmic decaying, with exponent $q = 0.37(1)$, but the structure factor $F_1$ and the correlation function $C_{\perp}$ still satisfy the power-law decaying, with critical exponent $\eta_\parallel = -0.69(1)$ and $\eta_\perp = -0.37(1)$, respectively. We also visualized the symmetries of different phases of the surface, where the extraordinary-log phase is shown to conserve $O(2)$ symmetry.

We have to stress that all our conclusions about the new special transition are based on the assumption that it is a second-order phase transition, which does not rule out the possibility of other types of phase transitions, such as the Berezinskii-Kosterlitz-Thouless transition.

In classical XY model, although the symmetry of the bulk critical point is also $O(2)$, the surface can not be ordered because of Mermin-Wagner-Hohenberg theorem, therefore such new type of special transition can not appear in classical XY model; discrete Hamiltonian symmetry and emergent $O(n)$ symmetry of bulk criticality seems a necessary condition for such new type of special transition, such as the clock model. In addition to the emergent $O(2)$ critical point, other emergent $O(n)$ critical point have also been found in systems with discrete symmetry of Hamiltonian. It is obviously an important and interesting question to study such new special transition and related critical properties in these models.

V. ACKNOWLEDGMENT

We thank Jian-Ping Lv for valuable discussions, and Cenke Xu for the advice of visualizing the symmetries of the suface by histograms. C.D. is supported by the Na-
K. Binder and P. C. Hohenberg, Surface effects on magnetic phase transitions, Phys. Rev. B 9, 2194 (1974).
2. K. Binder, in Phase Transitions and Critical Phenomena, Vol. 8, edited by C. Domb and J. L. Lebowitz (Academic Press, London, England, 1983).
3. T. C. Lubensky and M. H. Rubin, Critical phenomena in semi-infinite systems. I. expansion for positive extrapolation length, Phys. Rev. B 11, 4533 (1975).
4. T. C. Lubensky and M. H. Rubin, Critical phenomena in semi-infinite systems. II. Mean-field theory, Phys. Rev. B 12, 3885 (1975).
5. M. N. Barber, Scaling Relations for Critical Exponents of Surface Properties of Magnets, Phys. Rev. B 8, 407 (1973).
6. M. Hasenbusch, Monte Carlo study of surface critical phenomena: the special point, Physical Review B 84, 134405 (2011).
7. Y. Deng, H. W. J. Blote, and M. P. Nightingale, Surface and bulk transitions in three-dimensional O(n) models, Phys. Rev. E 72, 016128 (2005).
8. Y. Deng, Bulk and surface phase transitions in the three-dimensional O(4) spin model, Phys. Rev. E 73, 056116 (2006).
9. C. S. Arnold and D. P. Pappas, Gd(0001): A Semi-Infinite Three-Dimensional Heisenberg Ferromagnet with Ordinary Surface Transition, Phys. Rev. Lett. 85, 5202 (2000).
10. M. Krech, Surface scaling behavior of isotropic Heisenberg systems: Critical exponents, structure factor, and profiles, Phys. Rev. B 62, 6360 (2000).
11. L. Zhang and F. Wang, Unconventional surface critical behavior induced by a quantum phase transition from the two-dimensional Affleck-Kennedy-Lieb-Tasaki phase to a Néel-ordered phase, Phys. Rev. Lett. 118, 087201 (2017).
12. C. Ding, L. Zhang, and W. Guo, Engineering Surface Critical Behavior of (2 + 1)-Dimensional O(3) Quantum Critical Points, Phys. Rev. Lett. 120, 235701 (2018).
13. L. Weber, F. Parisen Toldin, and S. Wessel, Nonordinary edge criticality of two-dimensional quantum critical magnets, Phys. Rev. B 98, 140403(R) (2018).
14. L. Weber and S. Wessel, Nonordinary criticality at the edges of planar spin-1 Heisenberg antiferromagnets, Phys. Rev. B 100, 054437 (2019).
15. M. A. Metlitski, Boundary criticality of the O(N) model in d = 3 critically revisited, arXiv:2009.05119 (2020).
16. F. P. Toldin, Boundary Critical Behavior of the Three-Dimensional Heisenberg Universality Class, Phys. Rev. Lett. 126, 135701 (2021).
17. M. Hu, Y. Deng, and J.-P. Lv, Extraordinary-Log Surface Phase Transition in the Three-Dimensional XY Model, Phys. Rev. Lett. 127, 120603 (2021).
18. W. Zhu, C. Ding, L. Zhang, and W. Guo, Surface critical behavior of coupled Haldane chains, Phys. Rev. B 103, 024412 (2021).
19. L. Weber and S. Wessel, Spin versus bond correlations along dangling edges of quantum critical magnets, Phys. Rev. B 103, L020406 (2021).
20. C.-M. Jian, Y. Xu, X.-C. Wu, and C. Xu, Continuous Néel-VBS quantum phase transition in non-local one-dimensional systems with SO(3) symmetry, SciPost Phys. 10, 033 (2021).
21. F. P. Toldin, M. A. Metlitski, Boundary criticality of the 3d O(N) model: from normal to extraordinary, arXiv:2111.03613 (2021).
22. C. Ding, W. Zhu, W.-A. Guo, L. Zhang, Special Transition and Extraordinary Phase on the Surface of a (2+1)-Dimensional Quantum Heisenberg Antiferromagnet, arXiv:2110.04762 (2021).
23. X.-J. Yu, R.-Z. Huang, H.-H. Song, L. Xu, C. Ding, L. Zhang, Conformal Boundary Conditions of Symmetry-Enriched Quantum Critical Spin Chains, arXiv:2111.10945 (2021).
24. W. Zhu, C. Ding, L. Zhang, W. Guo, Exotic surface behaviors induced by geometrical settings of two-dimensional dimerized quantum XXZ model, arXiv:2111.12336 (2021).
25. F. P. Toldin, M. A. Metlitski, Boundary criticality of the 3d O(N) model: from normal to extraordinary, arXiv:2111.03613 (2021).
26. J. Padayasi, A. Krishnan, M. A. Metlitski, I. A. Gruzuerg, M. Meineri, The extraordinary boundary transition in the 3d O(N) model via conformal bootstrap, arXiv:2111.03071 (2021).
27. J. R. Banavar, G. S. Grest, and D. Jasnow, Ordering and phase Transitions in antiferromagnetic potts models, Phys. Rev. Lett. 45, 1424 (1980).
28. J.-S. Wang, R. H. Swendsen, and R. K. Kotecky, Three-state antiferromagnetic Potts models: A monte Carlo study, 42, 2465 (1990).
29. C. Yamaguchi and Y. Okabe, Three-dimensional antiferromagnetic q-state Potts models: application of the Wang-Landau algorithm, J. Phys. A: Math. Gen. 34, 8781-8794 (2001).
30. C. Ding, W. Guo, and Y. Deng, Reentrance of Berezinskii-Kosterlitz-Thouless-like transitions in a three-state Potts antiferromagnetic thin film, Phys. Rev. B 90, 134420 (2014).
31. Chengxiang Ding, Henk W. J. Blote, and Youjin Deng, Emergent O(n) symmetry in a series of three-dimensional Potts models, Phys. Rev. B 94, 104402 (2016).
32. J. Salas and A. D. Sokal, The three-state square-lattice Potts antiferromagnet at zero temperature, J. Stat. Phys. 92, 729 (1998).
33. M. P. M. den Nijs, M. P. Nightingale, and M. Schick, Critical fans in the antiferromagnetic three-state Potts model, Phys. Rev. B 26, 2490 (1982).
34. Robert H. Swendsen and Jian-Sheng Wang, Nonuniversal critical dynamics in Monte Carlo simulations, Phys. Rev. Lett. 58, 86 (1987).
M. P. Nightingale, in *Finite-Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privaman (World Scientific, Singapore, 1990).

M. N. Barber, in *Phase Transitions and Critical Phenomena*, Vol. 8, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1983).

Q. Liu, Y. Deng, T. M. Garoni, and H. W. J. Blöte, The $O(n)$ loop model on a three-dimensional lattice, Nucl. Phys. B 859, 107 (2012).

W. Xu, Y. Sun, J.-P. Lv, and Y. Deng, High-precision Monte Carlo study of several models in the three-dimensional $U(1)$ universality class, Phys. Rev. B 100, 064525 (2019).

N. D. Mermin and H. Wagner, Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models, Phys. Rev. Lett. 17, 1133 (1966).

P. C. Hohenberg, Existence of long-range order in one and two dimensions, Phys. Rev. 158, 383 (1967).

F. Léonard and B. Delamotte, Critical Exponents Can Be Different on the Two Sides of a Transition: A Generic Mechanism, Phys. Rev. Lett. 115, 200601

X. Zou, S. Liu, W. Guo, Surface critical properties of the three-dimensional clock model, arXiv:2204.13612.