Stochastic Noise as a Source of Decoherence in a Solid State Quantum Computer

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We examine a stochastic noise process that has a decohering effect on the average evolution of qubits in the quantum register of the solid state quantum computer proposed by Kane [4]. We consider the effects of this process on the single qubit operations necessary to perform quantum logical gates and derive an expression for the fidelity of these gates in this system. We then calculate an upper bound on the level of this stochastic noise tolerable in a workable quantum computer.
I. INTRODUCTION

The process of computation by quantum logic, so called quantum computation, has recently been shown to be far more powerful for solving certain classes of problem than is classical computation [1–3]. The superiority stems from the ability of the quantum analogue of bits, qubits, to maintain coherence between different classical states. This allows the quantum computer to perform computations over many classical input states at once, giving a quantum computer an exponential increase in speed for solving certain problems. In order to exploit this advantage in quantum computation it is vital that the evolution of the qubits be not only coherent, but precisely known to the operator. In this paper we examine the evolution of qubits under the influence of a stochastic noise process, the effect of which is to make the exact evolution of the system uncertain. We thus consider an ensemble of qubits and calculate their average evolution. We find that the effect of the noise is to produce a decay of the average phase coherence of the qubits in the quantum register and to depolarize qubits undergoing single qubit operations. This ensemble decoherence manifests itself as a decay of the fidelities of the quantum operations the qubits are undergoing. We calculate this fidelity and use it to determine an upper bound on the level of stochastic noise that the computer can tolerate yet still operate successfully within the limits set by current error correcting codes.

II. THE KANE SOLID STATE QUANTUM COMPUTER

Throughout this paper we will be considering a solid state quantum computer (QC), as proposed by Kane [4]. In this system the qubits are simply spin $^{31}\text{P}$ nuclei, in a silicon substrate. The system is subject to a background magnetic field oriented in the $z$ direction, $B_z$. At low energies the effective Hamiltonian for the nucleus-electron system is given by [4–6]
\[ H_{n+e} = \mu_B B_z \sigma^z_e - g_n \mu_n B_z \sigma^z_n + A \vec{\sigma}_e \cdot \vec{\sigma}_n. \]  

(1)

\[ A = \frac{8}{3} \pi \mu_B g_n \mu_n |\psi(0)|^2 \] and \( |\psi(0)|^2 \) is the probability density of the electron wave function evaluated at the position of the nucleus. We can alter A by applying a voltage to a so-called “A-gate” situated above the nucleus. This applied voltage shifts the electron wave function away from the nucleus and thus reduces A. If \( A << 2B_z \mu_B \) the electron spin states are separated in energy by a factor of approximately \( \mu_B/(g_n \mu_n) = 1633.8 \) greater than the nuclear states. Thus the nuclei can be manipulated without significantly altering the electron’s state. We therefore consider as our quantum computing basis a sub-space of the entire Hilbert space spanned by the Hamiltonian Eq(1), namely the states \( \ket{\downarrow 0}, \ket{\downarrow 1} \). This corresponds to the basis of nuclear states with the electron in its ground state. The effective Hamiltonian in this sub-basis to first order in \( A/\mu_B B_z \), up to a constant, is given by

\[ H = B_z \gamma \sigma^z. \]  

(2)

For convenience we have omitted the \( n \) subscript on the Pauli operator as we shall do for the rest of the paper. We can tune the Lamour frequency of the nucleus via the A-gate bias;

\[ \gamma = -g_n \mu_B - \frac{A_0 - \eta V}{B_z}, \]  

(3)

where \( V \) is the applied A-gate voltage, \( \eta = 5\pi \times 10^7 \text{Hz/V} \) and \( A_0 \) is the value of \( A \) when \( V = 0 \).

The single qubit operations are implemented by the application of an oscillating transverse magnetic field. This field can be brought into resonance by tuning \( \gamma \) to satisfy the resonance condition \( \omega = 2B_z \gamma / \hbar \). In this way specific qubits can be operated on without affecting the rest of the qubits in the register. The Hamiltonian for single qubit rotations then becomes \( H_{sq} = H + H(t) \), where
\[ H(t) = -B_{ac}g_n\mu_n(\cos(\omega t + \phi)\sigma^y - \sin(\omega t + \phi)\sigma^x). \] \hspace{1cm} (4)

Here the phase factor \( \phi \) determines the axis of the rotation. Without loss of generality we set \( \phi = 0 \) and consider only \( y \) rotations. We convert to a frame rotating at this resonance frequency by transforming to the interaction picture, and find

\[ \tilde{H}_{sqr} = -B_{ac}g_n\mu_n\sigma^y. \] \hspace{1cm} (5)

III. DECOHERENCE OF THE QUANTUM REGISTER

We now consider the effect of a stochastic white noise in the applied A-gate voltage, that is we write the voltage signal

\[ V(t) = V_0(1 + \Delta(t)), \] \hspace{1cm} (6)

where \( \Delta(t) \) describes a white noise process [6]. We can thus write

\[ \Delta(t)dt = \sqrt{\lambda}dW(t), \] \hspace{1cm} (7)

where \( dW(t) \) is the Wiener increment, and \( \sqrt{\lambda} \) scales the noise. We can calculate \( \lambda \) by integrating Eq(6) over the duration of the voltage pulse, \( \tau \), to give the pulse area:

\[ \Gamma(\tau) = V_0\tau + \Delta\Gamma(\tau). \] \hspace{1cm} (8)

Here \( \Delta\Gamma(\tau) \) is a Gaussian random variable with a mean of zero and a variance \( V_0^2\lambda\tau \). The ratio of the rms value of the fluctuations in the pulse area to average pulse area is given by:

\[ \frac{\Delta\Gamma(\tau)_{rms}}{\Gamma(\tau)} = \sqrt{\frac{\lambda}{\tau}}. \] \hspace{1cm} (9)

We find then, that the Hamiltonian for a qubit in the quantum register, that is a qubit not undergoing an operation \( (B_{ac} = 0) \), in the presence of this noise is given by
\[ H = B_z (\gamma + \xi(t)) \sigma^z. \] (10)

Here \( \xi(t) \) gives the stochastic fluctuations in the Lamour frequency of the qubit caused by the noise, it is related to the noise in the voltage signal by

\[ \xi(t) = \frac{\eta h V_0}{B_z} \Delta(t). \] (11)

We define \( \xi(t) dt = \sqrt{\epsilon} dW(t) \), where

\[ \epsilon = (\frac{\eta h V_0}{B_z})^2 \lambda. \] (12)

Let us transform into a frame rotating at the Lamour frequency, that is transform to the interaction picture as we have already done for the Hamiltonian Eq(5). In this picture a qubit in a noiseless quantum register does not evolve at all, the Hamiltonian is zero. However if we include the white noise, the evolution is generated by the Hamiltonian

\[ \tilde{H} = B_z \xi(t) \sigma^z. \] (13)

From this we can form the Ito stochastic differential equation for the density operator in the interaction picture

\[
\begin{align*}
    d\tilde{\rho}(t) &= \frac{B_z}{i\hbar} \sqrt{\epsilon} dW(t)[\sigma^z, \tilde{\rho}(t)] \nonumber \\
    &- \frac{\epsilon B_z^2}{2\hbar^2} [\sigma^z, [\sigma^z, \tilde{\rho}(t)]] dt.
\end{align*}
\] (14)

Because we do not know the precise history of the evolution, we take an average over noise histories and thus calculate the evolution of the average density operator. Using \( \langle dW(t) \rangle = 0 \) we find

\[
\frac{d\tilde{\rho}_{av}(t)}{dt} = -\frac{\epsilon B_z^2}{2\hbar^2} [\sigma^z, [\sigma^z, \tilde{\rho}_{av}(t)]].
\] (15)

Master equations with this nested commutator structure have been studied extensively in the field of quantum optics \[7\,8\]. Let us define polarization vector by
\[ \tilde{\rho}_{av}(t) = \frac{1}{2}(1 + \vec{P}(t) \cdot \vec{\sigma}). \]

We then consider the evolution of the average polarization vector in the quantum computation basis. In this notation the information about coherence is contained in the \( x \) and \( y \) components of the polarization vector. The magnitude of the polarization vector gives the purity of the state; if the state is completely pure then \(|\vec{P}| = 1\), in a completely mixed state the magnitude is 0 and partially pure states have a magnitude between these two extremes. The evolution leads to an exponential decay of the \( x \) and \( y \) components, indicating a loss of phase coherence, while the \( z \) component remains unchanged:

\[
\frac{dP_z(t)}{dt} = 0, \tag{16}
\]
\[
\frac{dP_{x,y}(t)}{dt} = -\frac{2\epsilon B^2}{\hbar} P_{x,y}(t). \tag{17}
\]

Thus phase coherence is destroyed but population probabilities are conserved. These equations have the solution

\[
P_z(t) = P_z(0),
\]
\[
P_{x,y}(t) = \exp\left[-\frac{2B^2 \epsilon t}{\hbar^2}\right] P_{x,y}(0), \tag{18}
\]

in which we can explicitly see the exponential decay of phase coherence. We can see how this decoherence of the average density operator effects the operation of the QC by calculating the average fidelity of the operation. This gives the probability that the state evolves as we expect it to if there were no noise, and can be obtained by calculating the trace of the product of the evolved density operator and the density operator we would expect from a noisless evolution, in this case the zero time density operator.

\[
F = \text{Tr}[\tilde{\rho}_{av}(t) \tilde{\rho}_{av}(0)],
\]
\[
= \frac{1}{2}(1 + P_z(0)^2 + (P_x(0)^2 + P_y(0)^2))\exp\left[-\frac{2B^2 \epsilon t}{\hbar^2}\right]. \tag{19}
\]

We can see that the fidelity depends on the initial state of the system, specifically on how much phase coherence the initial state possessed. If the qubit is initially in a
classical state the system cannot decohere and we get a fidelity of 1, a perfect gate. Generally however, the initial state will be some kind of superposition of classical states and coherence will be destroyed, leading to a loss of fidelity. The worst case is when the initial state is a maximum superposition, the system decoheres, and the fidelity is given by

$$F = \frac{1}{2}(1 + \exp[-\frac{2B_z^2\epsilon t}{\hbar^2}])$$  \hspace{1cm} (20)

In this case the fidelity eventually decays to a limiting value of $$\frac{1}{2}$$. Because the process is only phase destroying, a measurement of $$\sigma^z$$, that is a projection onto the basis of classical states will yield the same result as in the noiseless case. In this sense we can say that the classical information has been retained, the qubit has been converted to a classical bit.

**IV. FIDELITY OF SINGLE QUBIT OPERATIONS**

To calculate the effect of the stochastic fluctuations in the A-gate voltage bias on a qubit undergoing a single particle operation we consider the example of a rotation around the $$y$$-axis. The inclusion of white noise transforms the Hamiltonian Eq(5) to

$$\tilde{\mathcal{H}}_{sqr} = \xi(t)B_x\sigma^z - B_{ac}g_n\mu_n\sigma^y.$$  \hspace{1cm} (21)

We can then form the Ito stochastic differential equation

$$d\tilde{\rho}(t) = -\frac{B_{ac}g_n\mu_n}{i\hbar}[\sigma^y, \tilde{\rho}(t)]dt + \frac{B_x\sqrt{\mathcal{W}(t)}}{i\hbar}[\sigma^z, \tilde{\rho}(t)]$$

$$- \frac{\epsilon B_z^2}{2\hbar^2}[\sigma^z, [\sigma^z, \tilde{\rho}(t)]].$$  \hspace{1cm} (22)

Again we average over the noise histories and find

$$\frac{d\tilde{\rho}_{av}(t)}{dt} = -\frac{B_{ac}g_n\mu_n}{i\hbar}[\sigma^y, \tilde{\rho}_{av}(t)] - \frac{\epsilon B_z^2}{2\hbar^2}[\sigma^z, [\sigma^z, \tilde{\rho}_{av}(t)]].$$  \hspace{1cm} (23)
This gives a set of coupled differential equations for the polarization vector components of the average density operator

\[
\frac{dP_x(t)}{dt} = -\frac{2B^2_{\epsilon}}{\hbar^2}P_x(t) - \frac{2B_{ac}g_n\mu_n}{\hbar}P_z(t),
\]
\[
\frac{dP_y(t)}{dt} = -\frac{2B^2_{\epsilon}}{\hbar^2}P_y(t),
\]
\[
\frac{dP_z(t)}{dt} = \frac{2B_{ac}g_n\mu_n}{\hbar}P_x(t),
\]

(24)

that can be solved to give

\[
P_x(t) = \exp\left[\frac{-B^2_{\epsilon}t}{\hbar^2}\right]\{\cosh\left(\frac{\alpha t}{\hbar^2}\right) - \frac{B^2_{\epsilon}}{\alpha}\sinh\left(\frac{\alpha t}{\hbar^2}\right)\}P_x(0) - \frac{2B_{ac}g_n\mu_n\hbar}{\alpha}\sinh\left(\frac{\alpha t}{\hbar^2}\right)P_z(0),
\]
\[
P_y(t) = \exp\left[\frac{-2B^2_{\epsilon}t}{\hbar^2}\right]P_y(0),
\]
\[
P_z(t) = \exp\left[\frac{-B^2_{\epsilon}t}{\hbar^2}\right]\{\cosh\left(\frac{\alpha t}{\hbar^2}\right) + \frac{B^2_{\epsilon}}{\alpha}\sinh\left(\frac{\alpha t}{\hbar^2}\right)\}P_z(0) + \frac{2B_{ac}g_n\mu_n\hbar}{\alpha}\sinh\left(\frac{\alpha t}{\hbar^2}\right)P_x(0).
\]

(25)

Here we have defined \(\alpha = \sqrt{B^4_{\epsilon} - 4(B_{ac}g_n\mu_n\hbar)^2}\). In this case we see that all the components of the polarization vector decay to zero, thus leaving a totally mixed state regardless of the initial state. This is known as a depolarizing process, not only is the phase coherence lost, but the population probabilities become uniform and the qubit is equally likely to be in the classical \(|0\rangle\) or \(|1\rangle\) state. The action of the single qubit gate is to mix the \(P_x\) and \(P_z\) components of the polarization vector.

The dephasing process causes the decay of the \(P_x\) and \(P_y\) components and these two processes combine to cause the depolarization. The two processes define two time scales in the system, the first \(\tau_{op} = \frac{\hbar}{4B_{ac}g_n\mu_n}\) is the time it takes to perform a typical single qubit logic operation, a Hadamard gate. The time scale of the dephasing process is given by \(\tau_{dec} = \frac{\hbar^2}{2B^2_{\epsilon}}\). Obviously a functioning quantum computer requires \(\tau_{op}/\tau_{dec} << 1\). To zeroth order in this ratio we find that Eqs(23) give

\[
P_x(t) = \exp\left[\frac{-B^2_{\epsilon}t}{\hbar^2}\right]\{\cos\left(\frac{2B_{ac}g_n\mu_n t}{\hbar}\right)P_x(0) + \sin\left(\frac{2B_{ac}g_n\mu_n t}{\hbar}\right)P_z(0)\},
\]
\[
P_y(t) = \exp\left[\frac{-2B^2_{\epsilon}t}{\hbar^2}\right]P_y(0),
\]

(26)
\[ P_z(t) = \exp\left(-\frac{B^2_{ac}g_n\mu_n t}{\hbar}\right) \{ \cos\left(\frac{-2B_{ac}g_n\mu_n t}{\hbar}\right)P_z(0) - \sin\left(\frac{-2B_{ac}g_n\mu_n t}{\hbar}\right)P_x(0) \}. \]

We can now calculate the fidelity of these noisy operations

\[ F = \text{Tr}[\hat{\rho}_{av}(t)\rho(t)] = \frac{1}{2}(1 + \exp\left[\frac{-2B^2_{ac}g_n\mu_n t}{\hbar^2}\right]P_y(0)^2 + \exp\left[\frac{B^2_{ac}g_n\mu_n t}{\hbar^2}\right](P_x(0)^2 + P_z(0)^2)), \quad (27) \]

where \( \rho(t) \) is the output density operator for a noiseless operation. In this case the fidelity decays to \( \frac{1}{2} \) regardless of the input state, however the rate of decay is dependent of the initial state. We see that the rate of fidelity loss is not faster than is the case for a qubit in the quantum register and in the worst case they are equal. The worst case occurs, for \( y \) rotations when the qubit is initially in a \( \sigma^y \) eigenstate, this means that the gate cannot rotate the state out of the basis in which it decoheres.

**V. TOLERANCE TO NOISE**

To calculate the level of noise in the voltage signal that the quantum computer can tolerate in performing a typical single particle operation, the Hadamard gate. There is still some debate as to how much error a QC can tolerate and still function usefully, even with error correcting codes. The estimates range between an error probability of \( \delta = 10^{-6} - 10^{-4} \) per qubit per operation \[ [10,11] \]. For the purposes of this calculation we will hedge our bets and use the limit \( \delta = 10^{-5} \). We use the operating parameters prescribed by Kane \[ 4 \]; \( B_z = 2T \) and \( B_{ac} = 0.001T \) and an A-gate bias of 1V, a value at the top of the bias range. We then find that using the worst case fidelity function given by eq(20) the error probability is given by

\[ \delta = 1 - F = \frac{1}{2}(1 - \exp\left[\frac{-\tau_{op}}{\tau_{dec}}\right]), \quad (28) \]
which substituting in our limit for $\delta$ gives

$$\frac{\tau_{\text{op}}}{\tau_{\text{dec}}} < 2 \times 10^{-5}. \quad (29)$$

This limit justifies our zeroth order approximation in obtaining eqs(26,27). We now require

$$\epsilon < \frac{B_{ac} g_n \mu_n \hbar}{\pi B_z^2} \times 4 \times 10^{-5}. \quad (30)$$

Using eq(12) we find

$$\lambda = \frac{B_{ac} g_n \mu_n \hbar}{\pi \eta^2 V_0^2 \hbar} \times 4 \times 10^{-5}, \quad (31)$$

and so we get a limit on the acceptable noise in the voltage signal. To implement a Hadamard gate requires a pulse of duration $t = \tau_{\text{op}}$, thus we find that the ratio of the rms fluctuations in the pulse area to the mean pulse area is restricted by

$$\frac{\Delta \Gamma(t_{\text{Had}})_{\text{rms}}}{\Gamma(t_{\text{Had}})} < \frac{B_{ac} \gamma'}{\pi \eta V_0 \hbar} \times 1.3 \times 10^{-2}, \quad (32)$$

which under the operating conditions of the Kane computer gives

$$\frac{\Delta \Gamma(t_{\text{Had}})_{\text{rms}}}{\Gamma(t_{\text{Had}})} < 1.4 \times 10^{-6}. \quad (33)$$

**VI. CONCLUSIONS**

We have found that the stochastic white noise process which causes dephasing of qubits in the quantum register becomes a depolarizing process for qubits undergoing rotations in the quantum computer. Under similar operating conditions the fidelity of these qubit rotations decays more slowly for certain highly coherent input states, than does the fidelity of the register. We find that in order to satisfy currently accepted limits of on error probability per qubit operation we must keep the ratio of the rms fluctuations in the pulse area to the intended pulse area $\frac{\Delta \Gamma(t_{\text{Had}})_{\text{rms}}}{\Gamma(t_{\text{Had}})} < 1.4 \times 10^{-6}$. 
VII. ACKNOWLEDGMENTS

CJW would like to acknowledge the support of an Australian Postgraduate Award, a Melbourne University Postgraduate Abroad Scholarship and the Max-Planck-Institut für Kernphysik. CJW would also like to acknowledge valuable discussions with B.H.J.McKellar, and the theory group at the University of Queensland. LCLH wishes to acknowledge the support of the Alexander von Humboldt foundation and the Max-Planck-Institut für Kernphysik.

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