Determination of black holes by boundary measurements

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Abstract

For a wave equation with time-independent Lorentzian metric consider an initial-boundary value problem in $\mathbb{R} \times \Omega$, where $x_0 \in \mathbb{R}$ is the time variable and $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let $\Gamma \subset \partial \Omega$ be a subdomain of $\partial \Omega$. We say that the boundary measurements are given on $\mathbb{R} \times \Gamma$ if we know the Dirichlet and Neumann data on $\mathbb{R} \times \Gamma$. The inverse boundary value problem consists of recovery of the metric from the boundary data. In author’s previous works a localized variant of the boundary control method was developed that allows the recovery of the metric locally in a neighborhood of any point of $\Omega$ where the spatial part of the wave operator is elliptic. This allow the recovery of the metric in the exterior of the ergoregion.

Our goal is to recover the black hole. In some cases the ergoregion coincides with the black hole. In the case of two space dimensions we recover the black hole inside the ergoregion assuming that the ergosphere, i.e. the boundary of the ergoregion, is not characteristic at any point of the ergosphere.

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1 Introduction

Let \((x_0, x_1, x_2, ..., x_n) \in \mathbb{R} \times \mathbb{R}^n\), \(x_0 \in \mathbb{R}\) be the time variable, \(x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\). Consider the Lorentzian metric in \(\mathbb{R} \times \mathbb{R}^n\)

\[
\sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k
\]

with the signature \((+1, -1, ..., -1)\). Let \(g(x) = \det[g_{jk}(x)]_{j,k=0}^{n}\) and let \([g^{jk}(x)]_{j,k=0}^{n}\) be the inverse to the metric tensor \([g_{jk}(x)]_{j,k=0}^{n}\). We assume that the metric does not depend on \(x_0 \in \mathbb{R}\).

Let

\[
Lu = \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j} \left( \sqrt{(-1)^n g(x)} g^{jk}(x) \frac{\partial u}{\partial x_k} \right) = 0
\]

be the wave equation corresponding to the metric (1.1).

We assume that

\[
g^{00}(x) > 0.
\]

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\).

We shall consider the following initial-boundary value problem in \(\mathbb{R} \times \Omega\) for the equation (1.2)

\[
u = 0 \quad \text{for} \quad x_0 \ll 0, \; x \in \Omega,
\]

\[
u \bigg|_{\mathbb{R} \times \partial \Omega} = f,
\]

where \(\mathbb{R} \times \partial \Omega\) is not a characteristic surface and \(f\) has a compact support in \(\mathbb{R} \times \partial \Omega\). Since \(\mathbb{R} \times \partial \Omega\) is not characteristic there exists a unique solution of the problem (1.2), (1.4), (1.5) (cf. [14]).

Define the Dirichlet-to-Neumann operator \(\Lambda f\) as

\[
\Lambda f = \sum_{j,k=0}^{n} g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x) \left( \sum_{p,r=0}^{n} g^{pr}(x) \nu_p(x) \nu_r(x) \right)^{-\frac{1}{2}} \bigg|_{\mathbb{R} \times \partial \Omega},
\]

\(u(x_0, x)\) is the solution of (1.2), \(\nu_0 = 0, (\nu_1, ..., \nu_n)\) is the unit outward normal to \(\partial \Omega\).
Let $\Gamma_0$ be a subdomain of $\partial \Omega$. The inverse boundary value problem on $\mathbb{R} \times \Gamma_0$ consists of determining the metric \( (1.1) \) knowing the $\Lambda f$ on $\mathbb{R} \times \Gamma_0$ for all $f$ with the compact support in $\mathbb{R} \times \Gamma_0$.

A powerful boundary control method for solving hyperbolic inverse problems was discovered by M.Belishev (cf. [2]), and further developed in [3], [4], [5], [15], [16], [17]). In author’s works [6], [7], [8], a localized variant of the boundary control method was developed that allows to recover the metric in a neighborhood of any point of $\Omega$ where the spatial part of the wave operator is elliptic. These results are the basis of the present paper.

Let $y = \varphi(x)$ be a diffeomorphism of $\Omega$ on some bounded smooth domain $\Omega_0 \subset \mathbb{R}^n$ and let $a(x) \in C^\infty(\Omega)$, $a(x) = 0$ on $\Gamma_0$.

Consider the map \[
(1.7) \quad (y_0, y) = \Phi(x_0, x) = (\varphi(x), x_0 + a(x))
\]
of $\overline{\Omega} \times \mathbb{R}$ onto $\overline{\Omega}_0 \times \mathbb{R}$ such that $\varphi(x) = x$ and $a(x) = 0$ on $\Gamma_0$.

Note that change of variables $y = \varphi(x), y_0 = x_0 + a(x)$ does not change the DN operator $\Lambda$.

The domain $\Delta \subset \Omega$ is called the ergoregion if (see, for example, [22]),

\[
(1.8) \quad g_{00}(x) \leq 0 \quad \text{on} \quad \Delta.
\]

We assume that $g_{00}(x) > 0$ in the exterior of $\Delta$. Let

\[
(1.9) \quad \Delta(x) = \det[g^{jk}(x)]_{j,k=1}^n.
\]

Then $g_{00}(x) = g^{-1}(x)\Delta(x)$. Thus (1.8) is equivalent to the inequality

\[
(1.10) \quad \Delta(x) \leq 0.
\]

We assume that $\Delta(x) = 0$ is a smooth surface in $\mathbb{R}^n$, $\frac{\partial \Delta(x)}{\partial x} = (\frac{\partial \Delta(x)}{\partial x_1}, \ldots, \frac{\partial \Delta(x)}{\partial x_n}) \neq 0$ when $\Delta(x) = 0$.

Now we shall define the black hole.

Let $S(x) = 0$ be a closed surface in $\mathbb{R}^n$ and $\Omega_{int}$ be the interior of the surface $S(x) = 0$. We call the region $\Omega_{int}$ a black hole if no signal (disturbance) inside $S(x) = 0$ can reach the exterior of $S(x) = 0$. Let $S(x) = 0$ be a characteristic surface for the equation \( (1.2) \), i.e.

\[
(1.11) \quad \sum_{j,k=0}^n g^{jk}(x)S_{x_j}(x)S_{x_k}(x) = 0 \quad \text{when} \quad S(x) = 0.
\]
We have (cf. [9]) that \( S(x) = 0 \) in a boundary of a black hole if \( S(x) = 0 \) is a characteristic surface and

\[
(1.12) \quad \sum_{j=1}^{n} g^{0j}(x)S_{x_j}(x) < 0 \quad \text{when} \quad S(x) = 0.
\]

The boundary \( S(x) = 0 \) of the black hole is called the black hole event horizon.

When the metric (1.1) is not a solution of the Einstein equation, i.e. the metric (1.1) is not related to the general theory of relativity (cf. [24]), the black hole \( \Omega_{int} \) is called an analogue black hole. In physical applications the analogue black holes appear when one studies the propagation of waves in a moving medium (cf. [1], [20]). An example of the analogue metric is the following acoustic metric (cf. [21], [22]):

Consider a fluid flow in a vortex with the velocity field

\[
(1.13) \quad v = (v^1, v^2) = \frac{A}{r} \hat{r} + \frac{B}{r} \hat{\theta},
\]

where \( r = |x| \), \( \hat{r} = \left( \frac{x_1}{|x|}, \frac{x_2}{|x|} \right) \), \( \hat{\theta} = \left( -\frac{x_2}{|x|}, \frac{x_1}{|x|} \right) \), \( A \) and \( B \) are constants, \( A < 0 \). When \( B \neq 0 \) (1.13) is a rotating flow.

The inverse metric tensor \([g^{jk}]_{j,k=1}^2\) has the form

\[
(1.14) \quad g^{00} = \frac{1}{\rho c}, \quad g^{0j} = g^{j0} = \frac{1}{\rho c} v^j, \quad 1 \leq j \leq 2,
\]

\[
\quad g^{jk} = \frac{1}{\rho c} (-c^2 \delta_{ij} + v^j v^k), \quad 1 \leq j, k \leq 2.
\]

Here \( c \) is the sound speed, \( \rho \) is the density. We shall assume, for the simplicity, that \( \rho = 1, \ c = 1 \). It was shown in [22] that \( \{ r \leq \sqrt{A^2 + B^2} \} \) is the ergoregion and \( \{ r < |A| \} \) is the black hole.

We conclude the introduction by a brief description of the content of this paper.

In §2 we show that the boundary data given on any subdomain of \( \mathbb{R} \times \partial \Omega \) allow to recover the metric (1.1) outside the ergoregion.

We specify the particular case of the inverse boundary value problem when the ergoregion coincides with the black hole.

We consider another example of analogue metric, the Gordon metric ([18], [19]), that arise when one studies the propagation of light in a moving dielectric medium:
Let \( w = (w_1(x), w_2(x), w_3(x)) \) be the velocity of the flow. The Gordon metric has the form

\[
\sum_{j,k=0}^{3} g_{jk}(x) dx_j dx_k,
\]

where \( g_{jk}(x) = \eta_{jk} + (n^{-2}(x) - 1)v_j v_k \), \( n(x) \) is the index of the refraction,

\[
v_0 = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \quad \text{and} \quad v_j(x) = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} w_j(x) / c, \quad 1 \leq j \leq 3,
\]

\( \eta_{jk} \) is the Lorentz metric.

In §3 for \( n = 2 \) we describe the black hole inside the ergosphere.

In §4 we show that knowing the metric on the ergosphere one can recover the black hole inside.

In §5 we consider the axisymmetric case in three space dimensions.

### 2 Recovery of the ergosphere from the boundary measurements

Let \( \Gamma' \) be any small subset of \( \partial \Omega \) and \( P_0 \in \Gamma' \).

It was proven in [6], Theorem 2.3, (see also [7], [8]), that knowing boundary data on \([0, +\infty) \times \Gamma'\) one can recover, modulo change of variables (1.7), the metric in \([0, +\infty) \times V(P_0)\) where \( V(P_0) \) is a neighborhood of \( P_0 \) in \( \overline{\Omega} \). The key condition for the validity of Theorem 2.3 (condition 2.3 in [6]) is that the spatial part of the equation (1.2) is elliptic in \( V(P_0) \), i.e. \( V(P_0) \) is outside the ergosphere.

Next taking \( P_1 \in V(P_0) \) one can recover (1.1) in \([0, \infty) \times V(P_1)\) where \( V(P_1) \subset \overline{\Omega} \) is a neighborhood of \( P_1 \). Repeating this argument infinitely many times we can recover the metric outside the ergosphere \( \Delta(x) = 0 \) when \( \Delta(x) \) is the same as in (1.9). Taking the limit we can recover the metric on \( \Delta(x) = 0 \) too. Thus we have the following theorem:

**Theorem 2.1.** Knowing the DN operator (1.6) on \( \mathbb{R} \times \Gamma_0 \) we can determinate the ergosphere \( \Delta(x) = 0 \), and the metric (1.1) on \( \Delta(x) = 0 \).

Note that the matrix \( [g^{jk}(x)]_{j,k=1}^{n} \) has rank \( n - 1 \) when \( \Delta(x) = \det[g^{jk}(x)]_{j,k=1}^{n} = 0 \). Therefore there exists a null-vector
\[
e(x) = (e_1(x), \ldots, e_n(x)) \text{ of the matrix } [g^{jk}(x)]_{j,k=1}^n \text{ smoothly dependent on } x \in \Delta(x)
\]

\[
(2.1) \quad \sum_{k=1}^n g^{jk}(x)e_k(x) \equiv 0, \quad 1 \leq j \leq n, \quad \Delta(x) = 0.
\]

We will consider separately two cases:

a) \(e(x)\) is orthogonal to the surface \(\Delta(x) = 0\) for all \(x\), i.e. \(\Delta(x) = 0\) is characteristic at any \(x \in \Delta\).

b) \(e(x)\) is not orthogonal to the surface \(\Delta(x) = 0\) for any \(x\), i.e. is not characteristic for any \(x \in \Delta\).

Note that there are many other cases when \(e(x)\) is orthogonal to \(\Delta(x) = 0\) only on a part of \(\Delta(x) = 0\) but we will not consider them.

In the case a) we have that \(e(x)\) is collinear to the gradient \(\frac{\partial \Delta(x)}{\partial x}\), \(\Delta(x) = 0\), for all \(x\). Therefore it follows from (2.1) that

\[
(2.2) \quad \sum_{k=1}^n g^{jk}(x)\Delta_{x_k}(x) = 0, \quad 1 \leq j \leq n, \quad \Delta(x) = 0.
\]

Multiplying (2.2) by \(\Delta_{x_j}\) and summing in \(j\) we get

\[
(2.3) \quad \sum_{j,k=1}^n g^{jk}(x)\Delta_{x_j}\Delta_{x_k} = 0 \text{ when } \Delta(x) = 0,
\]

i.e. \(\Delta(x) = 0\) is a characteristic surface. Therefore \(\Delta(x) = 0\) is a boundary either of black hole or a white hole (cf. [9]).

It follows from [9] that \(\Delta(x) = 0\) is the boundary of a black hole if

\[
(2.4) \quad \sum_{j=1}^n g^{0j}(x)\frac{\partial \Delta}{\partial x_j} < 0 \text{ when } \Delta(x) = 0.
\]

Consider, for example, a Schwartzchild metric. It has the following form in Cartesian coordinates (cf. [23]):

\[
(2.5) \quad ds^2 = \left(1 - \frac{2m}{R}\right)dt^2 - dx^2 - dy^2 - dz^2 - \frac{4m}{R}dt dR - \frac{2m}{R}(dR)^2,
\]

where \(R = \sqrt{x^2 + y^2 + z^2}\).
It is easy to see (cf. [23]) that
\[(2.6) \quad 1 - \frac{2m}{R} = 0\]
is simultaneously an ergosphere and a characteristic surface. Thus \[\{1 - \frac{2m}{R} < 0\}\] is an ergoregion and a black hole.

We shall call the metric such that the ergoregion coincide with the black hole the Schwartzschield type metric.

Another example of a Schwartzscheld metric is an acoustic metric \(1.13\) when \(B = 0, \ A < 0\).

Therefore the problem of recovery of the black hole by the boundary measurements is solved for the Schwartzshield type metrics since it consists of the recovery of the ergosphere.

Further example of Schwartzschield type black hole appears in the study of the Gordon equation corresponding to the metric \(1.16\).

The condition \(\Delta(x) = 0\) is equivalent to \(g_{00} = 0\).

Using \(1.16\) we get
\[(2.7) \quad g_{00} = 1 + (n^{-2} - 1)v_0^2 = 1 + \frac{(n^{-2} - 1)}{1 - |w|^2c^2} = 0\]

Therefore
\[(2.8) \quad |w(x)|^2 = \frac{c^2}{n^2(x)}\]
is the ergosphere for the Gordon equation.

In the case of the Gordon metric we have, from \(1.16\) and \(2.3\):
\[(2.9) \quad |\Delta_x|^2 = \frac{n^2 - 1}{c^2\left(1 - \frac{|w|^2}{c^2}\right)}\left(\sum_{j=1}^n w_j(x)\Delta_{x_j}\right)^2.\]

Since \(|w(x)|^2 = \frac{c^2}{n^2(x)}\) we get \(\Delta_x \cdot w|^2 = |\Delta_x|^2|w|^2\). Therefore
\[(2.10) \quad \Delta_x(x) = \alpha(x)w(x) \quad \text{when} \quad \Delta(x) = 0.\]

Since \(\Delta_x\) is outward normal we have that \(\alpha(x) < 0\) when \(w(x)\) is pointed inside \(\Delta(x) = 0\).
The condition (2.4) has the following form for the Gordon metric

\[
\sum_{j=1}^{n} \frac{(n^2 - 1) w_j}{1 - |w|_c^2} \Delta x_j < 0.
\]

It follows from (2.10) that (2.11) holds when \( w(x) \) is pointed inside \( \Delta(x) = 0 \), and hence \( \Delta(x) = 0 \) is a boundary of a black hole.

The case of not Schwarzschild type metrics is more difficult. We shall study only the case of two dimensions and a metrics such that any point of the ergosphere is non-characteristic (i.e. the case b).

First we describe the black hole for this case.

3 Description of the black hole inside the ergosphere in the case of two space dimensions

Let \( \Delta(x) = 0, n = 2, \) be the ergosphere. Assume that the normal to that ergosphere is not characteristic for any \( x \in \{ \Delta(x) = 0 \} \)

\[
\sum_{j,k=1}^{2} g^{jk}(x) \nu_j \nu_k \neq 0 \quad \text{for all} \quad x \in \{ \Delta(x) = 0 \},
\]

where \((\nu_1, \nu_2)\) is the unit normal to \( \Delta = \{ \Delta(x) = 0 \} \). We assume that the ergosphere \( \Delta(x) = 0 \) is smooth, i.e. \( \frac{\partial \Delta}{\partial x} = \left( \frac{\partial \Delta}{\partial x_1}, \frac{\partial \Delta}{\partial x_2} \right) \neq 0 \) when \( \Delta(x) = 0 \).

As in [11] introduce coordinates \((\rho, \theta)\) where \( \rho = 0 \) is the equation of \( \Delta(x) = 0, \rho = -\Delta(x) \) near \( \rho = 0 \). For the convenience we extend \( \theta \in [0, 2\pi] \) to \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). We have \( 0 \leq \rho \leq \rho_0(\theta) \) where \( \rho = \rho_0(\theta) \) is the black hole event horizon, \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \).

Since the set \( \{ \Delta(x) = \text{det}[g^{jk}(x)] < 0 \} \) is inside the ergosphere, there are two characteristics \( S^\pm(x) \) such that

\[
\sum_{j,k=1}^{2} g^{jk}(x) S^\pm_{x_j} S^\pm_{x_k} = 0, \quad \Delta(x) < 0,
\]

or, in \((\rho, \theta)\) coordinates,

\[
\hat{g}^{\rho\rho} (\hat{S}^\pm_{\rho})^2 + 2\hat{g}^{\rho\theta} \hat{S}^\pm_{\rho} \hat{S}^\pm_{\theta} + \hat{g}^{\theta\theta} (\hat{S}^\pm_{\theta})^2 = 0,
\]
where \( \left[ \hat{g}^{\rho \rho} \hat{g}^{\rho \theta} \hat{g}^{\theta \rho} \right] \) is the matrix \( \left[ g^{11} \ g^{12} \ g^{21} \ g^{22} \right] \) in \((\rho, \theta)\) coordinates.

We assume that \( \hat{S}^\pm (\rho, \theta) \) satisfy the following boundary conditions
\[(3.3) \quad \hat{S}^\pm (0, \theta) = \theta \quad \text{for any} \quad \theta \in \mathbb{R}/2\pi \mathbb{Z}. \]

Solving the quadratic equation (3.2) we get
\[(3.4) \quad \hat{S}^\pm_\rho (\rho, \theta) = - \hat{g}^{\rho \theta} \pm \sqrt{- \Delta \hat{g}^{\rho \rho}} \hat{g}_\theta (\rho, \theta), \]
where \( \Delta = \hat{g}^{\rho \rho} \hat{g}^{\theta \theta} - (\hat{g}^{\rho \theta})^2. \)

Consider the equations for the null-bicharacteristics (null geodesics). When we use the time variable as a parameter we have (cf. [9]) two families of null-geodesics:
the \((+)\) null-geodesics \((\rho^+(x_0), \theta^+(x_0))\), \(\rho^+(0) = 0, \ \theta^+(0) = \theta_0, \ x_0 \geq 0, \) and the \((-)\) null-geodesics \((\rho^-(x_0), \theta^-(x_0))\), \(\rho^-(0) = 0, \ \theta^-(0) = \theta_0, \ x_0 \leq 0. \)

Note that condition (3.1) is equivalent that both \((+)\) and \((-)\) null-geodesics are not tangent to \(\Delta (x) = 0. \)

Assume that the ergoregion \(\{\Delta (x) \leq 0\} \) contains a trapped region \(O_\varepsilon\), i.e. a region that both \((+)\) and \((-)\) null-geodesics reach when \(x_0 \to +\infty \) and stay there. One of the examples of trapped region is a neighborhood of a singularity of the metric described in [11] (cf. (1.3), (1.4) in [11]).

**Remark 3.1.** The trapped region described above leads to the existence of the black hole. If we change the definition of the trapped region requiring that all \((+)\) and \((-)\) null-geodesics tends to \(O_\varepsilon\) when \(x_0 \to -\infty \) we get that there exists a white black hole (cf. [9]).

When \(x_0 \to +\infty \ (\rho^+(x_0), \theta^+(x_0)) \) reaches the trapped region \(O_\varepsilon\) and remains there for all large \(x_0. \)

The null-geodesics \(\gamma_- = (\rho^-(x_0), \theta^-(x_0))\), \(x_0 < 0\), ends at \(((0, \theta_0)\) when \(x_0 = 0\) and \((\rho^-(x_0), \theta^-(x_0))\) cannot reach \(O_\varepsilon\) when \(x_0 \to -\infty. \) Thus \((\rho^-(x_0), \theta^-(x_0))\) has no limit points in \(\{\Delta (x) \leq 0\} \setminus O_\varepsilon. \)

Therefore the limiting set of the trajectories \(\{(\rho^-(x), \theta^-(x_0)), x_0 < 0\}\) is inside \(\Delta \setminus O_\varepsilon. \) By the Poincare-Bendixson theorem (cf. [13]) there exists a limit cycle, i.e. closed periodic solution \(\gamma_0 = \{(\rho^0_0(x_0), \theta^0_0(x_0))\}\) that is a black hole event horizon. The solution \(\gamma_-\) approaches \(\gamma_0\) spiraling around \(\gamma_0. \) All other \((-)\) null-geodesics also approach \(\gamma_0\) spiraling.

Denote by \(\Pi\) the infinite strip
\[(3.5) \quad \Pi = \{0 \leq \rho < \rho_0(\theta), \ -\infty < \theta < +\infty\}, \]
where \( \{ \rho = \rho_0(\theta), \theta \in \mathbb{R}/2\pi \mathbb{Z} \} \) is the equation of the black hole event horizon \( \gamma_0 \). Therefore \( \Pi \) is traced by all \((-\) null-)geodesics, and \( \gamma_0 \) is the boundary of \( \Pi \): \( \gamma_0 \subset \partial \Pi \). Thus the following theorem holds:

**Theorem 3.1.** Let (3.1) holds, i.e. the ergosphere \( \Delta(x) = 0 \) is not characteristic for all \( x \in \{ \Delta(x) = 0 \} \). Suppose the ergoregion \( \{ \Delta(x) \leq 0 \} \) contains a trapped region \( O_\varepsilon \). Then there exists a black hole \( \Pi = \{ 0 \leq \rho \leq \rho_0(\theta), -\infty < \theta < +\infty \} \) and \( \{ \rho = \rho_0(\theta), \theta \in \mathbb{R}/2\pi \mathbb{Z} \} \) is the black hole event horizon.

In the examples to verify that the requirement for the existence of the black hole inside the ergosphere are satisfied, one need to check that \( \Delta \) is not characteristic when \( \Delta(x) = 0 \), i.e.

\[
(3.6) \quad \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial \Delta}{\partial x_j} \frac{\partial \Delta}{\partial x_k} \neq 0 \quad \text{when} \quad \Delta(x) = 0.
\]

In the case of the Gordon metric we have that \( \Delta(x) = 0 \) is equivalent to (cf. (2.8))

\[
(3.7) \quad |w(x)|^2 = \frac{c^2}{n^2(x)}.
\]

Equation (3.6) in the case of Gordon metric gives

\[
(3.8) \quad -|\Delta| \neq \left( \frac{|w|^2}{c^2} \right)^{-1} \left( \sum_{i=1}^{2} \Delta_{x_i} w_j c \right)^2 \neq 0,
\]

where we used again (3.1). Thus, if (3.8) holds, there exists a black hole inside the ergosphere.

Now we shall verify the conditions for the existence of the black hole in the case of the acoustic metric.

Introduce polar coordinates \((r, \varphi)\). Then the Hamiltonian in polar coordinates has the form

\[
(3.9) \quad H = \left( \xi_0 + \frac{A}{r} \xi_r + \frac{B \xi_\varphi}{r} \right)^2 - \left( \xi_\varphi + \frac{1}{r^2} \xi_\varphi \right)^2.
\]

Therefore \( g^{rr} = \left( \frac{A^2}{r^2} - 1 \right), \ g^{r\varphi} = \frac{AB}{r^2}, \ g^{\varphi\varphi} = \frac{B^2}{r^2} - \frac{1}{r^2} \).
The ergosphere is

\[ \Delta_0 = \left( \frac{A^2}{r^2} - 1 \right) \left( \frac{B}{r^4} - \frac{1}{r^2} \right) - \frac{A^2 B^2}{r^6} = -\frac{A^2}{r^4} - \frac{B^2}{r^4} + \frac{1}{r^2} = 0. \]

It is easy to check that \( \Delta_0 \) is not characteristic. Therefore a black hole exists inside the ergosphere.

One can show (cf. [9]) that \( \{ r < |A| \} \) when \( A < 0 \) is a black hole.

**Remark 3.2.** Consider the case \( B = 0 \), i.e. non-rotating fluid. Then the ergosphere is \( \Delta_0(x) = \frac{1}{r^2}(1 - \frac{A^4}{r^2}) = 0 \). It was shown in [9] that \( \{ r < |A| \} \) is a black hole. Thus we have another example of the Schwartzschield type black hole.

### 4 Recovery of the black hole knowing the boundary data on the ergosphere

Consider the characteristic equations \( \Delta \). Note that

\[ \tilde{\Delta}(\rho, \theta) = g^{\rho \rho}g^{\theta \theta} - (g^{\rho \theta})^2 \equiv -C_1 \rho, \ C_1 > 0. \]

Let \( \rho = \rho^+(x_0), \ \theta = \theta^+(x), \ x_0 \geq 0 \), be the null-geodesics (cf. §3). Then

\[ S^+(\rho^+(x_0), \theta^+(x_0)) \equiv S^+(\rho(0), \theta(0)), \rho(0) = 0, \ \theta(0) = \theta_0. \]

We have that \( (\rho^+(x_0), \theta^+(x_0)) \) crosses the black hole horizon \( \gamma_0 = \{ \rho = \rho_0(\theta), \theta \in \mathbb{R} \} \) at some point \( x = x_0 \) and remains inside the black hole. The null-geodesics \( (\rho^-(x_0), \theta^-(x_0)) \) approach the black hole event horizon \( \gamma_0 \) when \( x_0 \to -\infty \). Thus \( \gamma_0 \) is the limit set of \( (\rho^-(x_0), \theta^-(x_0)) \), \( x_0 < 0 \).

Periodically extending \( \theta \in [0, 2\pi] \) to \( \theta \in (-\infty, +\infty) \) we have that the \( - \) null-geodesics cover the strip \( \Pi = \{ 0 \leq \rho < \rho_0(\theta), -\infty < \theta < +\infty \} \) when \( 0 \leq \rho(x_0) < \rho_0, \ \theta(x_0) = \theta, \ -\infty < \theta < +\infty \).

Let

\[ \sigma = S^+(\rho, \theta), \ \tau = S^-(\rho, \theta) \]

when \( (\rho, \theta) \in \Pi, \ \Pi = \{ 0 \leq \rho < \rho_0(\theta), \theta \in \mathbb{R}^1 \}, \rho = \rho_0(\theta) \) is the black hole event horizon.
Note that $Lu = 0$ has the form
\begin{equation}
\frac{\partial^2 u}{\partial \sigma \partial \tau} = 0
\end{equation}
in coordinates $(\sigma, \tau)$.

Make a new change of variables
\begin{equation}
y_1 = \frac{\sigma + \tau}{2} = \frac{S^+(\rho, \theta) + S^-(\rho, \theta)}{2},
\end{equation}
\begin{equation}
y_2 = \frac{\sigma - \tau}{2}, \quad y_1 \big|_{\rho=0} = \theta, \quad y_2 \big|_{\rho=0} = 0.
\end{equation}

We have that $(y_1, y_2) = \Phi(\sigma, \tau)$ is the map of $\Pi$ onto the half-plane $\mathbb{R}^2_+ = \{(y_1, y_2), y_1 \geq 0, y_2 \in (-\infty, +\infty)\}$. Note that $\Phi$ is a homeomorphism of $\Pi$ onto $\mathbb{R}^2_+$, $\Phi$ is the identity on $\rho = 0$.

The closure $\overline{\Pi} = \{0 \leq \rho \leq \rho_0(\theta), \theta \in \mathbb{R}^1\}$ has the form $\overline{\Pi} = \Pi \cup \gamma_0$. Thus the black hole event horizon $\gamma_0$ belongs to the closure of $\Pi$.

Suppose we have another metric $g_1$ in $\Omega$ such that $\Lambda_1 f \big|_{\mathbb{R} \times \Gamma_0} = \Lambda f \big|_{\mathbb{R} \times \Gamma_0}$ for all $f \in \mathbb{R} \times \Gamma_0$ where $\Lambda_1, \Lambda$ are two DN operators for $g_1$ and $g$, respectively. Then, as in §2, $g = g_1$ modulo of the change of variables (1.7) outside of the ergosphere.

Therefore, without loss of generality, we can assume the ergosphere $\Delta = 0$ for metrics $g, g_1$ is the same and the metrics $g, g_1$ are equal on $\Delta = 0$.

Let $\varphi^\pm$ be a solution of characteristic equation of the form (3.2) with $[\hat{g}^{\rho^j_\theta}]_{j, k=1}^2$ replaced by $[g_1^{\rho^j_\theta}]_{j, k=1}^2$.

We assume, as in (4.2), that
\begin{equation}
\varphi^\pm(\rho^+_1(x_0), \theta^+_1(x_0)) = \varphi^\pm(\rho_1(0), \theta_1(0)), \quad \theta_1(0) = \hat{\theta}_1.
\end{equation}

Make change of variables as in (4.3)
\begin{equation}
\sigma' = \varphi^+(\rho, \theta), \quad \tau' = \varphi^-(\rho, \theta),
\end{equation}
where
\begin{equation}
(\rho', \theta') \in \Pi', \quad \Pi' = \{0 \leq \rho' < \rho_0(\theta'), \theta' \in \mathbb{R}^1\},
\end{equation}
$\rho = \rho_0(\theta'), \theta' \in \mathbb{R}$ is the black hole event horizon for the metric $g_1$. Note that $L'u' = 0$ has the form $\frac{\partial^2 u'}{\partial \sigma' \partial \tau'} = 0$ in $(\sigma', \tau')$ coordinates.
As in (4.5), (4.6), make a change of variables

\begin{equation}
(4.10) \quad y_1' = \frac{\sigma' + \tau'}{2} = \frac{\varphi^+ (\rho, \theta) + \varphi^- (\rho, \theta)}{2},
\end{equation}

\begin{equation}
(4.11) \quad y_2' = \frac{\sigma' - \tau'}{2} = \frac{\varphi^+ - \varphi^-}{2},
\end{equation}

\[ y_1'|_{\rho=0} = \theta, \quad y_2'|_{\rho=0} = 0. \]

Note that \( L'u' = 0 \) has the form \((\partial^2 \partial y_1^2 - \partial^2 \partial y_2^2)u = 0, (y_1, y_2) \in \mathbb{R}^2_+ \).

Let \( \Phi_1 \) be the map (4.10), \( \Phi_1 \) is a homeomorphism of \( \Pi' \) onto \( \mathbb{R}^2_+ \). Therefore, \( \Phi_0 = \Phi^{-1} \Phi_1 \) is a homeomorphism of \( \Pi' \) onto \( \Pi \). Note that \( \gamma_0' = \partial \Pi' \) is the black hole event horizon for the metric \( g_1 \). Since the closure \( \overline{\Phi_0} \) maps \( \overline{\Pi} \) onto \( \overline{\Pi} \) we have that \( \gamma_0' \) is mapped onto \( \gamma_0 \). Thus the event horizon \( \gamma_0 \) can be recovered up to a change of variables. Therefore the following theorem holds:

**Theorem 4.1.** Suppose we have two wave equations \( Lu = 0, L'u' = 0 \) such that corresponding DN operators are equal on \( \mathbb{R} \times \Gamma_0 \). Suppose the ergosphere \( \Delta(x) = 0 \) of \( Lu = 0 \) is not characteristic for any \( \Delta(x) = 0 \) and the trapped region \( \Omega_\epsilon \subset \Delta \) exists. Then the ergosphere \( \Delta'(x) = 0 \) of \( L'u' = 0 \) is also non-characteristic for all \( \Delta'(x') = 0 \) and has a trapped region \( \Omega_\epsilon' \). Moreover, if \( \overline{\Pi} \) is the black hole of \( Lu = 0 \) and \( \overline{\Pi'} \) is the black hole of \( L'u' = 0 \), then \( \overline{\Pi} \) and \( \overline{\Pi'} \) are homeomorphic.

**Remark 4.1** Since the kernel of \([g^{jk}]_{j,k=1}^2 \) is one-dimensional for all \( x \) belonging to the ergosphere \( \Delta \), there exists a vector \( e(x) \in \ker [g^{jk}]_{j,k=1}^2 \) smoothly depending on \( x \in \Delta \). Above we considered two cases when \( e(x) \) is normal to \( \Delta(x) = 0 \) for all \( x \in \Delta \) and when \( e(x) \) is not normal to \( \Delta \) for all \( x \in \Delta \). The last condition is equivalent to the conditions that the null-geodesics on \( \Delta \) are not tangent to \( \Delta \).

There is also a case 3) when \( e(x) \) is normal to \( \Delta = 0 \) only on some subset of \( \Delta = 0 \). Black holes in the case 3) were studied in [11] and we shall explain the ideas behind these studies.

The underlying idea in analyzing the black holes in two space dimensions is the following:

Consider the Hamiltonian

\begin{equation}
(4.12) \quad H(x_1, x_2, \xi_0, \xi_1, \xi_2) = \sum_{j,k=0}^2 g^{jk}(x)\xi_j\xi_k.
\end{equation}
Let

\[
\begin{align*}
\frac{dx_k}{ds} &= \frac{\partial H(x, \xi)}{\partial \xi_k}, \quad \frac{d\xi_k}{ds} = -\frac{\partial H}{\partial x_k}, \quad 0 \leq k \leq 3, \\
x_k(0) &= y_k, \quad \xi_k(0) = \eta_k
\end{align*}
\]

be the equation of null-bicharacteristics. Thus

\[
H(x_1(s), x_2(s), \xi_0(s), \xi_1(s), \xi_2(s)) = 0 \quad \text{for all } s.
\]

Since \(H\) is independent of \(x_0\) we have that \(\frac{dx_0}{ds} = 0\), i.e. \(\xi_0(s) = \eta_0\) is a constant. We choose \(\xi_0 = 0\), and we shall call the null-bicharacteristics with \(\xi_0 = 0\) the zero energy null-bicharacteristics. The projection of zero energy null-bicharacteristics on the \((x_1, x_2)\)-space is called the zero-energy null-geodesics. Therefore we have

\[
\sum_{j,k=1}^{2} g^{jk}(x(s)) \xi_j(s) \xi_k(s) \equiv 0, \quad \forall s.
\]

Equation (4.15) is a quadratic equation in \(\xi_j(s), \ 1 \leq j \leq 2\), and therefore we have two families of solutions

\[
\xi_j^\pm(s) = p_j^\mp(x(s)), \quad x = (x_1, x_2), \quad j = 1, 2.
\]

If substitute \(\xi_j^\pm\) in (4.13) and choose \(x_0\) as a parameter instead of \(s\) we obtain two \(2 \times 2\) system of differential equations in \((x_1, x_2)\):

\[
\begin{align*}
\frac{dx_j^\pm}{dx_0} &= \frac{g^{j1}(x^\pm)p_{j1}^\pm(x^\pm) + g^{j2}(x^\pm)p_{j2}^\pm(x^\pm)}{g^{01}(x^\pm)p_{1}^\pm(x^\pm) + g^{02}(x^\pm)p_{2}^\pm(x^\pm)}, \quad j = 1, 2.
\end{align*}
\]

Therefore the solution of \(4 \times 4\) system of null-bicharacteristics (4.13) is reduced to the solution of \(2 \times 2\) system (4.17). This reduction substantially simplifies the study of black holes. In the case when the boundary of the ergoregion is not characteristic it was done in §3.

In the case 3) black holes also exist and the boundary of the black hole consists of segments of “plus” or “minus” zero energy null-geodesics. In some cases the boundary of black hole may have corners when “plus” null-geodesics and “minus” null-geodesics intersect.

We will not consider the inverse problems for the black holes in the case 3).
5 The case of the axisymmetric metrics in three space dimensions

Consider the wave equation (1.2) when \( n = 3 \) and the metric \( g = \sum_{j,k=0}^{3} g_{jk} dx_j dx_k \) is axisymmetric.

Let \( (\rho, \varphi, z) \) be the cylindrical coordinates \( x = \rho \cos \varphi, y = \rho \sin \varphi, z = z \).

Since metric is axisymmetric the Hamiltonian of the spatial part of (1.2) has the form

\[
H = g_{\rho\rho}(\rho, z) \xi_{\rho}^2 + 2g_{\rho z}(\rho, z) \xi_{\rho} \xi_{z} + g_{zz}(\rho, z) \xi_{z}^2 + g_{\rho\varphi}(\rho, z) \xi_{\rho} \xi_{\varphi} + g_{z\varphi}(\rho, z) \xi_{z} \xi_{\varphi} + g_{\varphi\varphi}(\rho, z) \xi_{\varphi}^2, \tag{5.1}
\]

where the coefficients of (5.1) are independent of \( \varphi \). Therefore the dual variable \( \xi_{\varphi} \) is constant. We choose \( \xi_{\varphi} = 0 \), i.e. instead of (5.1) we have

\[
H_0 = g_{\rho\rho}(\rho, z) \xi_{\rho}^2 + g_{\rho z}(\rho, z) \xi_{\rho} \xi_{z} + g_{zz}(\rho, z) \xi_{z}^2. \tag{5.2}
\]

We shall call (5.2) the restricted axisymmetric Hamiltonian. The wave equation corresponding to the restricted axisymmetric Hamiltonian is obtained from (1.2) by dropping all derivatives in \( \varphi \).

Factoring \( H_0 = 0 \) we get

\[
\xi_{\rho} = \frac{-g_{\rho z} \pm \sqrt{(g_{\rho z})^2 - g_{\rho\rho} g_{zz}}}{g_{\rho\rho}} \xi_{z}. \tag{5.3}
\]

Note that

\[
\Delta_1(\rho, z) = g_{\rho\rho} g_{zz} - (g_{\rho z})^2 \leq 0 \tag{5.4}
\]

is the ergoregion for the restricted axisymmetric equation. As in §3 there exists a smooth vector \( e_1(\rho, z) \) on \( \Delta_1(\rho, z) = 0 \) such that

\[
\begin{bmatrix}
  g_{\rho\rho} & g_{\rho z} \\
  g_{\rho z} & g_{zz}
\end{bmatrix}
\begin{bmatrix}
  e_{1}(\rho, z)
\end{bmatrix} = 0 \text{ for all } (\rho, z) \in \Delta_1 = 0.
\tag{5.5}
\]

As in §3 there are three choices:

\[
e_1(\rho, z) = C \left( \frac{\partial \Delta_1}{\partial \rho}, \frac{\partial \Delta_1}{\partial z} \right) \text{ for all } (\rho, z) \in \{ \Delta_1 = 0 \}, \tag{5.6}
\]

\[
a) \quad e_1(\rho, z) \neq C \left( \frac{\partial \Delta_1}{\partial \rho}, \frac{\partial \Delta_1}{\partial z} \right) \text{ for all } (\rho, z) \in \{ \Delta_1 = 0 \} \quad \text{and}
\]

\[
b) \quad e_1(\rho, z) = C \left( \frac{\partial \Delta_1}{\partial \rho}, \frac{\partial \Delta_1}{\partial z} \right) \text{ only for a subset of } (\rho, z) \in \{ \Delta_1 = 0 \}.\]
In the case a) $\Delta_1(\rho, z) = 0$ is a characteristic curve and we have a Schwarzschild type black hole. In the case b) there exists a black hole inside $\{\Delta_1(\rho, z) < 0\}$. The case c) will not be considered in this paper.

Consider the case of Kerr metric - the most celebrated example of the axisymmetric metric (cf. [18], [23]). The restricted Kerr Hamiltonian is (cf. (5.2)):

$$H_0(\rho, z, \xi_\rho, \xi_z) = (-\xi_\rho^2 - \xi_z^2) + K(b_\rho \xi_\rho + b_z \xi_z)^2,$$

where

$$K = \frac{2mr^3}{r^4 + a^2 z^2}, \quad b_\rho = \frac{\rho^2}{r^2 + a^2}, \quad b_z = \frac{z}{r},$$

where $r$ is defined by the relation

$$\frac{\rho^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

We have

$$g^{\rho\rho} = -1 + Kb_\rho^2, \quad g^{zz} = -1 + Kb_z^2, \quad g^{\rho z} = 2Kb_\rho b_z,$$

where $g^{\rho\rho}, g^{\rho z}, g^{zz}$ are the same as in (5.2).

Thus

$$\mathcal{\hat{\Delta}}_1(\rho, z) = \det \begin{bmatrix} g^{\rho\rho} & g^{\rho z} \\ g^{\rho z} & g^{zz} \end{bmatrix} = Kb_\rho^2 + Kb_z^2 - 1.$$

It was shown in [10] that

$$\mathcal{\hat{\Delta}}_1(\rho, z) = C_\pm (r - r_\pm),$$

where

$$C_\pm \neq 0, \quad r_\pm = m \pm \sqrt{m^2 - a^2}.$$

Thus there are two ergospheres: the outer ergosphere $\Delta_1^+(\rho, z) = C_0^+(r - r_+)$ and the inner ergosphere $\Delta_1^- = C_0^-(r - r_-)$ for the restricted Kerr metric. It is known (cf. [23]) that

$$\frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} = 1$$

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are characteristic surfaces for the Kerr equation.

Note that

\begin{align}
(5.14) \quad & r - r_+ = C_1^+ \left( \frac{\rho^2}{r_+^2 + a^2} + \frac{z^2}{r_+^2} - 1 \right), \\
(5.15) \quad & r - r_- = C_1^- \left( \frac{\rho^2}{r_-^2 + a^2} + \frac{z^2}{r_-^2} - 1 \right),
\end{align}

where \( C_1^+ \) and \( C_1^- \) \( \neq 0 \).

Therefore \( \Delta_1 = \hat{\Delta}_1^+ \hat{\Delta}_1^- \).

\begin{align}
(5.16) \quad & \hat{\Delta}_1^\pm (\rho, z) = C_3^\pm \left( \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 \right), \\
\end{align}

Denote

\begin{align}
(5.17) \quad & S^\pm = \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1.
\end{align}

Then

\begin{align}
(5.18) \quad & \frac{\partial}{\partial \rho} \hat{\Delta}_1^\pm = \left( \frac{\partial}{\partial \rho} C_3^\pm \right) \left( \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 \right) + C_3^\pm \frac{\partial}{\partial \rho} \left( \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 \right), \\
(5.19) \quad & \frac{\partial}{\partial z} \hat{\Delta}_1^\pm = \left( \frac{\partial}{\partial z} C_3^\pm \right) \left( \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 \right) + C_3^\pm \frac{\partial}{\partial z} \left( \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 \right).
\end{align}

When \( S^\pm = \frac{\rho^2}{r_\pm^2 + a^2} + \frac{z^2}{r_\pm^2} - 1 = 0 \) we got that the first terms in (5.18), (5.19) vanish.

Therefore

\begin{align}
& g^{\rho\rho} \left( \frac{\partial}{\partial \rho} \hat{\Delta}_1^\pm \right)^2 + 2 g^{\rho\theta} \frac{\partial}{\partial \rho} \hat{\Delta}_1^\pm \frac{\partial}{\partial \theta} \hat{\Delta}_1^\pm + g^{\theta\theta} \left( \frac{\partial \hat{\Delta}_1^\pm}{\partial \theta} \right)^2 \\
= & g^{\rho\rho} (S_\rho^\pm)^2 + 2 g^{\rho\theta} \frac{\partial S^\pm}{\partial \rho} \frac{\partial S^\pm}{\partial \theta} + g^{\theta\theta} \left( \frac{\partial S^\pm}{\partial \theta} \right)^2 = 0
\end{align}

since \( S^\pm = 0 \) and \( S^\pm \) are characteristic surfaces.

Therefore the following theorem holds:

**Theorem 5.1.** We have that \( \hat{\Delta}_1^+ = 0 \) is an outer ergosphere and an outer black hole horizon and \( \hat{\Delta}_1^- = 0 \) is an inner ergosphere and an inner black hole horizon for the restricted Kerr equation.'
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