Online Posted Pricing with Unknown Time-Discounted Valuations

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Abstract
We study the problem of designing posted-price mechanisms in order to sell a single unit of a single item within a finite period of time. Motivated by real-world problems, such as, e.g., long-term rental of rooms and apartments, we assume that customers arrive online according to a Poisson process, and their valuations are drawn from an unknown distribution and discounted over time. We evaluate our mechanisms in terms of competitive ratio, measuring the worst-case ratio between their revenue and that of an optimal mechanism that knows the distribution of valuations. First, we focus on the identical valuation setting, where all the customers value the item for the same amount. In this setting, we provide a mechanism $\mathcal{M}_c$ that achieves the best possible competitive ratio, discussing its dependency on the parameters in the case of linear discount. Then, we switch to the random valuation setting. We show that, if we restrict the attention to distributions of valuations with a monotone hazard rate, then the competitive ratio of $\mathcal{M}_c$ is lower bounded by a strictly positive constant that does not depend on the distribution. Moreover, we provide another mechanism, called $\mathcal{M}_{pc}$, which is defined by a piecewise constant pricing strategy and reaches performances comparable to those obtained with $\mathcal{M}_c$. This mechanism is useful when the seller cannot change the posted price too often. Finally, we empirically evaluate the performances of our mechanisms in a number of experimental settings.

Introduction
Posted-price mechanisms try to sell an item by proposing a take-it-or-leave-it price to each arriving agent, who then decides whether to buy the item or not (Chawla et al. 2010). If an agent opts for purchasing the item, then the mechanism terminates; otherwise, the agent leaves without any further possibility of buying the item, and the mechanism goes on by proposing prices to upcoming agents. Over the last years, growing attention has been devoted to the analysis of posted-price mechanisms, both in the classical economic literature (Seifert 2006) and in computer science (Babaioff et al. 2015, 2017, Adamczyk et al. 2017, Correa et al. 2017), within artificial intelligence and machine learning in particular (Kleinberg and Leighton 2003, Shah, Johari, and Blanche 2019). This is mainly motivated by the overwhelming number of online economic transactions carried out by posted-price mechanisms. This happens, for example, in online travel agencies (e.g., Expedia), accommodation websites (e.g., Booking.com), and e-commerce platforms (e.g., Amazon, eBay). As studied by Einav et al. (2018), an increasing number of eBay users prefer buying goods via posted prices rather than participating in auctions.

Posted-price mechanisms provide many advantages over traditional auction-style mechanisms. From the designer’s perspective, posting prices requires a much lower effort than running an auction, since it avoids the burden of first eliciting information (the bids) from the agents, and then collecting the payments. At the same time, posted-price mechanisms retain most of the desirable properties of classical auctions, such as truthfulness. Indeed, even though the agents are not required to report their valuations for the item, they are always better off deciding whether to buy the item or not on the basis of their true valuations, without acting strategically (Babaioff et al. 2017). From the agents’ perspective, participating in a posted-price mechanism is preferable over competing in an auction, for several reasons. For instance, agents may prefer revealing minimal information about their true preferences if they plan to participate in similar markets in the future. Moreover, in some real-world settings, requiring the agents to figure out their true valuations for the item might need some additional efforts on their behalf, while answering a take-it-or-leave-it offer is usually much easier.

In this work, we study posted-price mechanisms for selling a single unit of a single item within a finite period of time, when the value of the item is discounted over time according to an arbitrary continuous and non-increasing discount function. Discounting is common in many real-world applications and widely studied for a number of economic situations, such, e.g., bargaining (Rubinstein 1982, Gatti, Di Giunta, and Marino 2008), and auctions (Mao et al. 2018). We tackle settings in which agents arrive sequentially—a common assumptions in online mechanism design (Lavi and Nisan 2004, Parkes 2007)—and the number of agents is unknown a priori. In particular, following a mainstream approach in economics (see, e.g., Mason and Valimäki 2011, Rosenthal 2011), we assume that agents’ arrivals are governed by a Poisson process. Remarkably, posted pricing with Poisson arrivals has been previously investigated by Wang (1993) and Rong, Qin, and An (2018) for undiscounted settings, though without providing any theoretical result.
We assume that each agent arriving at the mechanism has a different initial (i.e., undiscounted) valuation for the item, which is independently drawn according to a common probability distribution. This leads to a fundamental trade-off between setting high prices so as to achieve high revenue and, on the other side, progressively lowering posted prices so as to increase the probability of selling the item. Our assumption is that the mechanism is only aware of the range of valuations, while it does not know anything about the shape of the distribution. This is reasonable since, differently from the actual distribution, the range of valuations can be estimated from previous data or market surveys. Lavi and Nisan (2004) and Babaioff et al. (2017) provide the main state-of-the-art results on posted-price mechanisms for single-item single-unit scenarios. However, their models do not fit to our setting, since the agents’ valuations are not discounted over time and the number of agents is known a priori. As a result, these models do not embed an explicit time representation and the proposed pricing strategies are only driven by the number of agents arrived.

Our model encompasses many real-world scenarios, such as, e.g., long-term rental of rooms and apartments. Think of a website renting rooms to students for fixed periods of one year. The value of a room naturally decreases over time, reflecting the fact that a future tenant will benefit from the room for a period shorter than one year. Moreover, the potential customers arrive at the renting website according to a stochastic process, which can be reasonably modeled by a Poisson process whose rate parameter can be easily estimated by looking at traffic logs of the website.

**Original Contributions** We adopt the perspective of competitive analysis (Borodin and El-Yaniv 2005) and evaluate our mechanisms in terms of competitive ratio, measuring the worst-case ratio between their revenue and that of an optimal mechanism that knows the distribution of valuations. As it is customary in the literature (see, e.g., Babaioff et al. (2017) and Kleinberg and Leighton (2003)), we first focus on the identical valuation setting in which all the agents share the same initial valuation for the item. Then, we extend our results to the random valuation setting where the agents’ valuations are drawn i.i.d. from the same distribution satisfying the monotone hazard rate condition (when the distributions of valuations are unrestricted, Lavi and Nisan (2004) and Babaioff et al. (2017) show that then there is no algorithm with good performances). In the identical valuation setting, we design a posted-price mechanism $M_C$ and prove that it is optimal, i.e., it provides the best possible competitive ratio. In order to derive the ratio, we first identify two crucial properties that characterize optimal mechanisms: their undiscounted price is non-increasing in time and they always guarantee the same fraction of the expected revenue of an optimal mechanism that knows the agents’ valuation, independently of its actual value. For the specific case of linear discount, we discuss how the competitive ratio depends on the parameters. In the random valuation setting, we first show that mechanism $M_{PC}$ still provides good performances by proving that its competitive ratio is lower bounded by a constant, which does not depend on the distribution of agents’ valuations. Then, motivated by real-world scenarios in which the seller is constrained not to change the posted prices too often, we propose a new mechanism $M_{PC}$ defined by a piecewise constant pricing strategy and prove that its performances in terms of competitive ratio are comparable with those obtained by $M_C$. In conclusion, we empirically compare $M_C$ with a natural adaption of the mechanism proposed by Babaioff et al. (2017) to our setting, showing that the latter is inefficient even without time discounting. We also empirically evaluate the performances of $M_C$ and $M_{PC}$ as the frequency with which prices are allowed to change decreases, showing that, when this is not too low, then the performances of $M_{PC}$ and $M_C$ are comparable.

**Other Related Works** As showed by Hajighayi, Kleinberg, and Sandholm (2007) for single-item settings, posted pricing is strictly related to the secretary problem and prophet inequalities; see also the work by Babaioff et al. (2009) for single-item settings and that of Lucier (2017) for multi-item scenarios. Differently from our model, these works assume that the mechanism knows the probability distribution of agents’ valuations and that the agents reveal their actual valuation for the item upon arrival. When multiple units of the same item are available, learning approaches based on bandit techniques are customarily adopted. In particular, Kleinberg and Leighton (2003) study an unlimited-supply setting where the number of buyers is fixed, and derive upper bounds on the regret. Several recent works extend the results in (Kleinberg and Leighton 2003). Shah, Johari, and Blanchet (2019) study a contextual setting, providing a semi-parametric model that learns from the observation of a binary outcome which stands for acceptance or rejection of the offered price. Mohri and Munoz (2014) study revenue-maximizing learning algorithms for posted pricing with strategic buyers. They consider a repeated game in which, at each round, the seller offers the item at a certain price and a strategic buyer accepts or rejects it. In that work, the goal is to learn the buyers’ valuation for the item by minimizing the strategic-regret of the algorithm.

**Preliminarries** We study a model in which a seller is interested in selling a single unit of an item within a finite time period of length $T$. The seller implements a posted-price mechanism by setting a take-it-or-leave-it price at each time $t \in [0, T]$. We denote by $p: [0, T] \rightarrow \mathbb{R}_+$ the pricing strategy adopted by the seller, with $p(t)$ being the price offered at time $t \in [0, T]$. The agents (i.e., the buyers) arrive sequentially over time, according to a Poisson process with rate parameter $\lambda > 0$.

We label agents according to their order of arrival (i.e., agent $i$ is the $i$-th agent arriving in $[0, T]$). Each agent $i$ has a private valuation $V_i$ for the item, drawn from a distribution $F$ with finite support $[v_{\text{min}}, v_{\text{max}}]$, where $v_{\text{max}} > v_{\text{min}} > 0$ denote the maximum and minimum valuation, respectively. In the following, for the ease of presentation, we normalize agents’ valuations in the range $[1, h]$, where we define $h := \frac{v_{\text{max}}}{v_{\text{min}}}$. Accordingly, we scale the support of $F$ to $[1, h]$. Then,
we denote by $f$ the probability density function of $F$.

The value of the item for sale decreases over time. In particular, $V_i$ is agent $i$’s initial valuation at time $t = 0$. We model decreasing values by introducing a continuous non-increasing discount function $\xi : [0, T] \to [0, 1]$ such that $\xi(0) = 1$ and $\xi(T) = 0$. By letting $W_i$ be the random variable representing the arrival time of agent $i$, we define the agent $i$’s discounted valuation as $D_i := V_i \xi(W_i)$, which represents how much agent $i$ is willing to pay upon her arrival. As a result, whenever agent $i$ arrives, she buys the item if and only if $D_i \geq p(W_i)$, i.e., her discounted valuation is at least as large as the price offered by the mechanism.

We introduce the following additional notation. We denote by $\int_{s \tau} := [s, s + \tau] \subseteq [0, T]$ the time interval of length $\tau \in [0, T]$ starting from time $s \in [0, T - \tau]$. The number of agents arriving in $\int_{s \tau}$ is a random variable denoted by $N_{s \tau}$. Given $\tau \in [0, T]$, the random variables $N_{s \tau}$ are equally distributed for all $s \in [0, T - \tau]$, as the arrivals are generated by a Poisson process. For the sake of presentation, we omit $s$ in $N_{s \tau}$, denoting by $N_{\tau}$ the random variable of the total number of agents arriving in the overall time period. In the following, we sometimes focus on the linear discount function, denoted as $\xi_{\text{lin}} : [0, T] \to [0, 1]$ with $\xi_{\text{lin}}(t) := 1 - \frac{t}{T}$. In this case, each agent $i$’s discounted valuation is $D_i := V_i (1 - \frac{W_i}{T})$.

Performances of Posted-Price Mechanisms

Given a deterministic posted-price mechanism $\mathcal{M}$ defined by a price function $p_{\mathcal{M}} : [0, T] \to \mathbb{R}_+$, we denote by $E_F[\mathcal{R}(\mathcal{M})]$ the expected revenue that the mechanism provides to the seller. The expectation is calculated with respect to both the Poisson arrivals and the distribution $F$ of agents’ initial valuations. We made explicit the dependence on $F$, as we will frequently refer to it along the paper.

We adopt the perspective of competitive analysis and measure the performances of a mechanism $\mathcal{M}$ by comparing the seller’s expected revenue with that of a benchmark mechanism $\mathcal{M}^\star$, which is optimal having knowledge of the distribution $F$. Notice that the benchmark has no information on the actual realizations of agents’ initial valuations, whereas the mechanisms we propose operate having knowledge of their range only.

Our goal is to bound the performances of our mechanisms w.r.t. those of the benchmark $\mathcal{M}^\star$ by looking at the worst case over the set $F$ of possible distributions $F$, i.e., all those with support $[1, h]$. This is captured by the following:

**Definition 1.** The competitive ratio of a deterministic posted-price mechanism $\mathcal{M}$ is defined as:

$$\rho(\mathcal{M}) := \min_{F \in \mathcal{F}} \rho_F(\mathcal{M}), \quad \text{where} \quad \rho_F(\mathcal{M}) := \frac{E_F[\mathcal{R}(\mathcal{M})]}{E_F[\mathcal{R}(\mathcal{M}^\star)]}.$$ 

Moreover, we say that a mechanism is optimal when its competitive ratio is the highest possible among all the deterministic posted-price mechanisms.

Notice that $\rho(\mathcal{M}) \in [0, 1]$ and, for every possible distribution $F \in \mathcal{F}$, we are guaranteed that the seller’s expected revenue $E_F[\mathcal{R}(\mathcal{M})]$ provided by mechanisms $\mathcal{M}$ is at least a fraction $\rho(\mathcal{M})$ of that achieved by $\mathcal{M}^\star$, i.e., it holds $E_F[\mathcal{R}(\mathcal{M})] \geq \rho(\mathcal{M}) E_F[\mathcal{R}(\mathcal{M}^\star)]$.

As previously showed by Babaioff et al. (2017) in similar settings, we can safely restrict our analysis to mechanisms maintaining the bottom price for a non-negligible period of time. Indeed, in the case in which $F$ places all the probability mass on 1, any mechanism providing a non-null seller’s expected revenue must set the minimum price during some time interval, otherwise no agent would buy the item.

**Proposition 1.** Every deterministic posted-price mechanism $\mathcal{M}$ such that $\rho(\mathcal{M}) > 0$ must set the minimum price $p_{\mathcal{M}}(t) = \xi(t)$ for every $t$ in a time interval of length $\tau > 0$.

**Identical Valuation Setting**

We start studying the identical valuation (IV) setting, where all the agents share the same initial valuation $v \in [1, h]$ for the item, i.e., it holds $V_i = v$ and $D_i = v \xi(W_i)$ for every agent $i$. The IV setting is a special case of the general random valuation model where one restricts the attention to distributions $F$ placing all the probability mass on a single valuation in $[1, h]$. In the following, we adjust notation for expected revenues and competitive ratios accordingly, writing $E_v[\mathcal{R}(\mathcal{M})]$ and $\rho_v(\mathcal{M})$ instead of $E_F[\mathcal{R}(\mathcal{M})]$ and $\rho_F(\mathcal{M})$.

Our main result (Theorem 1) is to provide a deterministic posted-price mechanism, called $\mathcal{M}_v$, which is optimal for the IV setting for every discount function $\xi$. We also study the specific case of a linear discount function $\xi_{\text{lin}}$, where we design an optimal mechanism $\mathcal{M}_{v, \text{lin}}$ (Theorem 2), that enjoys an easily interpretable analytical description. All the proofs are in the Appendix.

First, we describe the shape of the benchmark mechanism $\mathcal{M}^\star$ for the IV setting. Indeed, since $\mathcal{M}^\star$ knows the actual initial valuation $v$, its price function $p_{\mathcal{M}^\star} : [0, T] \to \mathbb{R}_+$ is such that $p_{\mathcal{M}^\star}(t) = v \xi(t)$ for $t \in [0, T]$. Therefore, we can compute the expected revenue of $\mathcal{M}^\star$ as follows:

$$E_v[\mathcal{R}(\mathcal{M}^\star)] := \int_0^T p_{\mathcal{M}^\star}(t) \lambda e^{-\lambda t} dt = \int_0^T v \xi(t) \lambda e^{-\lambda t} dt = v k^\star,$$

where $k^\star := \int_0^T \xi(t) \lambda e^{-\lambda t} dt$ does not depend on $v$, but only on the problem parameters $T$, $\lambda$, and the discount function $\xi$. Let us remark that the expected revenue of the benchmark $\mathcal{M}^\star$ defined in Equation (1) is expressed as a linear function of $v$.

**Optimal Mechanism for a General Discount**

We start proving two lemmas that highlight two crucial properties which characterize optimal posted-price mechanisms for the IV setting. Lemma 1 implies that the pricing strategy of an optimal mechanism must be such that the discounted price defined as $p(t) / \xi(t)$ is non-increasing in $t$, whereas Lemma 2 shows that any mechanism which always provides
a constant fraction of the expected revenue of the benchmark, independently of the agents’ initial valuation $v$, is an optimal mechanism.

**Lemma 1.** *In the IV setting, given any deterministic posted-price mechanism $\mathcal{M}$, there always exists a deterministic posted-price mechanism $\mathcal{M}'$ with undiscounted price $p_{\mathcal{M}'}(t)$ non-increasing in $t$ such that $E_v[R(\mathcal{M})] \leq E_v[R(\mathcal{M}')]$ for every possible agents’ initial valuation $v \in [1, h]$.*

Notice that, since $\xi$ is continuous and non-increasing by definition, Lemma 1 also shows that there is always an optimal mechanism whose pricing strategy is non-increasing. Moreover, by recalling Proposition 1, we can conclude that any optimal mechanism must set the minimum price at the end of the overall time period, i.e., during a time interval $[t_0, T] \subseteq [0, T]$ defined for some $t_0 \in [0, T]$. This result is exploited to prove the following lemma.

**Lemma 2.** *In the IV setting, let $\mathcal{M}$ be a deterministic posted-price mechanism whose pricing strategy $p_{\mathcal{M}}$ satisfies $p_{\mathcal{M}}(t) = \xi(t)$ for $t \in [t_0, T]$ with $t_0 \in [0, T]$. If the ratio $\rho_{\mathcal{M}}(\mathcal{M}) = E_v[R(\mathcal{M})]$ for $\mathcal{M}$ does not depend on the agents’ initial valuation $v$, then $\mathcal{M}$ is an optimal mechanism.*

By Lemma 2 in order to find an optimal mechanism for the IV setting, we can restrict the attention to mechanisms $\mathcal{M}$ whose ratios $\rho_{\mathcal{M}}(\mathcal{M})$ do not depend on the initial valuation $v$. Therefore, since the expected revenue of the benchmark $\mathcal{M}^*$ is a linear function of $v$ (see Equation (1)), we can search for an optimal mechanism among those having an expected revenue which linearly depends on $v$. This crucial observation allows us to design the optimal mechanism $\mathcal{M}_{c,lin}$ in Theorem 1 by leveraging the condition $E_v[R(\mathcal{M}_{c,lin})] = kv$ for every $v \in [1, h]$ with $k$ being a suitably defined constant independent of $v$. The key insight that allows us to derive an expression for $\mathcal{M}_{c,lin}$ is that we can always find the desired pricing strategy $p_{c,lin}$ among the continuous price functions such that $\frac{\xi(t)}{\sqrt{t}}$ is non-increasing in $t \in [0, T]$.

**Theorem 1.** *In the IV setting, there exists an optimal deterministic posted-price mechanism $\mathcal{M}_{c,lin}$ whose pricing strategy $p_{c,lin}$ is defined as follows:

$$p_{c,lin}(t) := \begin{cases} a e^{ \lambda t} b(t) \xi(t) dt & \text{if } t \in [0, t_0) \\ \xi(t) & \text{if } t \in [t_0, T] \end{cases},$$

where $b$ is a function such that $b(t) = \lambda - \frac{\zeta(t)}{\xi(t)}$ with $\xi(t) = \frac{1}{\sqrt{t}}$, whereas $a, k, t_0$ are suitably defined constants that do not depend on the agents’ initial valuation $v$.*

As a byproduct of the proof of Theorem 1, we also get an expression for the competitive ratio of the mechanism $\mathcal{M}_{c,lin}$, as stated by the following corollary.

**Corollary 1.** *In the IV setting, mechanism $\mathcal{M}_{c,lin}$ achieves:

$$\rho(\mathcal{M}_{c,lin}) = \frac{\int_0^t \xi(t) \lambda e^{-\lambda t} dt}{\int_0^t \xi(t) \lambda e^{-\lambda t} dt}.$$*

**Optimal Mechanism for a Linear Discount**

The pricing strategy $p_{\mathcal{M}_{c,lin}}$ of the optimal mechanism defined in Theorem 1 still depends on some parameters, namely $a$, $k$, and $t_0$, which do not admit an easy analytical formula for a general discount function $\xi$. Nevertheless, they can be expressed analytically if we restrict the attention to functions $\xi$ having a particular shape. In the following Theorem 2 and Corollary 2, we analyze the case of a linear discount function $\xi_{lin}$, defining an optimal mechanism $\mathcal{M}_{c,lin}$ for such setting.

**Theorem 2.** *In the IV setting with linear discount function $\xi_{lin}$, there exists an optimal deterministic posted-price mechanism $\mathcal{M}_{c,lin}$ whose pricing strategy $p_{c,lin}$ is defined as:

$$p_{c,lin}(t) := \begin{cases} h (1 - \frac{\xi(t)}{\sqrt{t}}) e^{(1 - \frac{1}{b})t + \frac{\lambda t}{\sqrt{t}}} & \text{if } t \in [0, t_0) \\ 1 - \frac{\xi(t)}{\sqrt{t}} & \text{if } t \in [t_0, T] \end{cases},$$

where $k := \lambda t_0 - \frac{2T t_0}{\sqrt{\pi}}$ and the time $t_0 \in [0, T]$ is defined as the unique positive real root of the following equation:

$$1 - \frac{\lambda t_0}{\sqrt{\pi}} = \lambda t_0 e^{-\lambda T}.$$

The prices posted by $\mathcal{M}_{c,lin}$ decrease as a linearly discounted exponential function until $t = t_0$, starting, at time $t = 0$, by setting the price equal to the maximum agents’ initial valuations $h$. Then, during the time interval $[t_0, T]$, the price function linearly decreases and equals zero in $t = T$. The asymptotic values of $t_0$ and $\rho(\mathcal{M}_{c,lin})$ as $T, \lambda, h$ go to infinity are in Table 1 (see the Appendix for more details). In particular, $\rho(\mathcal{M}_{c,lin})$ goes asymptotically to 1 as $\lambda$ or $T$ increases. This corresponds to having an infinite number of agents and, thus, selling the item with certainty. Instead, $\rho(\mathcal{M}_{c,lin})$ decreases as $h$ increases, going asymptotically to 0 as $\frac{1}{\log h}$. The range $[1, h]$ represents the degree of uncertainty that the mechanism has on the agents’ valuation. Therefore, $\rho(\mathcal{M}_{c,lin})$ decreases as the uncertainty increases and it cannot be lower bounded by any strictly positive constant if no finite upper bound on $h$ is known (i.e., when $h \to +\infty$). However, the dependency of $\rho(\mathcal{M}_{c,lin})$ on the degree of uncertainty is logarithmic. Instead, notice that a trivial mechanism setting the price equal to $\xi_{lin}(t)$ for $t \in [0, T]$ would have a competitive ration of $\frac{1}{\sqrt{t}}$, which decreases linearly on the degree of uncertainty.

| $\rho(\mathcal{M}_{c,lin})$ | $T \to \infty$ | $\lambda \to \infty$ | $h \to \infty$ |
|--------------------------|----------------|----------------|---------------|
| $\frac{\Theta(T)}{\sqrt{\pi}}$ | $\frac{\Theta(T)}{\sqrt{\pi}}$ | $\frac{\Theta(T)}{\sqrt{\pi}}$ | $\frac{\Theta(T)}{\sqrt{\pi}}$ |
Random Valuation Setting

We now switch to the random valuation (RV) setting, where agents’ initial valuations $V_i$ are i.i.d. random variables defined by a cumulative distribution function $F$ with support $[1, h]$. We focus on distributions $F$ satisfying the monotone hazard rate (MHR) condition. Formally, a distribution $F$ is MHR if the hazard rate $H(x) := \frac{f(x)}{1-F(x)}$ is non-decreasing in $x$. This assumption is common when studying posted-price mechanisms that operate without knowing the shape of the distribution of valuations (see Babaioff et al. [2015, 2017]), and many distributions used in practice satisfy it (such as, e.g., uniform, normal, and exponential distributions). Moreover, the MHR condition is necessary for proving our main results (Theorems 3 and 4). Indeed, when the family of possible distributions is unrestricted, one can’t proving our main results (Theorems 3 and 4). Indeed, when the family of possible distributions is unrestricted, one cannot design posted-price mechanisms guaranteeing a constant fraction of the revenue of $M^*$ independently of the distribution $F$, as shown by Babaioff et al. [2017], for the easier setting in which agents do not arrive stochastically. All the proofs are provided in the Appendix.

Auxiliary Definitions and Results

We introduce the random variable $X_{\lambda \tau}$, as the maximum initial valuation of agents arriving in an interval of length $\tau \in (0, T]$. Formally:

$$X_{\lambda \tau} := \max_{i \in \{1, \ldots, N_{\tau}\}} V_i.$$

$X_{\lambda \tau}$ is the first order statistic of $N_{\tau}$ samples drawn from $F$ and, since agents’ arrivals are governed by a Poisson process, its cumulative distribution function $F_{X_{\lambda \tau}}$ is defined as:

$$F_{X_{\lambda \tau}}(x) := \sum_{j=1}^{\infty} \mathbb{P}\{N_{\tau} = j\} F_{X_{\lambda \tau}|N_{\tau}=j}(x) = \sum_{j=1}^{\infty} \frac{(\lambda \tau)^j e^{-\lambda \tau}}{j!} [F(x)]^j = e^{-\lambda \tau (1-F(x))}.$$

We also define $Y_{s, \lambda \tau}$ as the random variable representing the maximum discounted valuation of agents arriving in an interval $I_{s, \lambda \tau}$ of length $\tau \in (0, T]$ starting at $s \in [0, T - \tau]$:

$$Y_{s, \lambda \tau} := \max_{i \in \{1, \ldots, N_{\tau}\}} D_i.$$

The cumulative distribution function $F_{Y_{s, \lambda \tau}}$ of $Y_{s, \lambda \tau}$ is:

$$F_{Y_{s, \lambda \tau}}(x) := \sum_{j=1}^{\infty} \mathbb{P}\{N_{\tau} = j\} F_{Y_{s, \lambda \tau}|N_{\tau}=j}(x) = \sum_{j=1}^{\infty} \frac{(\lambda \tau)^j e^{-\lambda \tau}}{j!} F_{Y_{s, \lambda \tau}|N_{\tau}=j}(x),$$

where $F_{Y_{s, \lambda \tau}|N_{\tau}=j}$ is the cumulative distribution function of $Y_{s, \lambda \tau}$ conditioned on the event $N_{\tau} = j$. Let us remark that, by definition, $F_{Y_{s, \lambda \tau}}$ depends on distribution $F$. In the following, we also let $Y_{\lambda \tau} := Y_{0, \lambda \tau}$ be the random variable representing the maximum discounted valuation of agents arriving in the overall time period $[0, T]$. In the Supplemental Material, for the specific case of a linear discount function, we show how to exploit some useful properties of Poison processes so as to find an analytical expression for $F_{Y_{\lambda \tau}}$.

In particular, by letting $Z := V U$, where $V$ and $U$ are independent random variables distributed according to $F$ and $U((0, 1))$, respectively, we obtain:

$$F_{Y_{\lambda \tau}}(x) := \sum_{j=1}^{\infty} \frac{(\lambda \tau)^j e^{-\lambda \tau}}{j!} [F_Z(x)]^j,$$

where

$$F_Z(x) := \left\{ \begin{array}{ll}
\frac{1}{e} (1 + x)^{-1} f(x) df & \text{if } x \in [0, 1) \\
F(x) + \frac{1}{e} e^{-1} f(x) df & \text{if } x \in [1, h].
\end{array} \right.$$
Lemma 3. In the RV setting with agents’ initial valuations drawn from a distribution $F$, given $\tau \in (0, T]$ and $0 < \epsilon < 1$, there exists at least an interval $I_{s, \tau}$ of length $\tau \in (0, T]$ starting at $s \in [0, T-\tau]$ such that the prices $p_{M_{C}}(t)$ posted by mechanism $M_{C}$ during the time instants $t \in I_{s, \tau}$ lie in the range $[E[X_{\tau}](s+\tau)(1-\epsilon)]$, $E[X_{\tau}]\xi(s+\tau)(1-\epsilon)$. The following two lemmas are the final pieces that we need to prove Theorem 4. Lemma 4. $X_{\tau}$ has non-decreasing monotone hazard rate. Lemma 5. For every $\tau, \tau' \in (0, T]$ with $\tau' \leq \tau$, it holds: $E[X_{\tau}'] \geq \ln (\tau')$. Theorem 3. Consider the RV setting with $\lambda_T = (\lambda T)^{1-\epsilon} \geq 1 - \ln(e-1)$ for some $\tau \in (0, T]$ and $0 < \epsilon < 1$. Then, restricted to the set $F$ of distributions $F$ satisfying the MHR condition, mechanism $M_{C}$ has a competitive ratio that can be lower bounded as follows: $\rho(M_{C}) \geq \frac{\xi((t_0 + T^{1-\epsilon} \lambda^{-\epsilon})(1-\epsilon))}{\kappa' e}$. The idea of the proof is to use $\rho_F(M_{C}) \geq \frac{E_F[R(M_{C})]}{E_F[Y_{\tau}]}$, following from the fact that $E_F[R(M^*)]$ cannot be larger than $E[Y_{\tau}]$, which is the expected revenue achieved by an optimal mechanism that knows the realization of agents’ initial valuations and arrivals. Then, $E_F[R(M_{C})]$ is lower bounded by the revenue that $M_{C}$ achieves in a suitably defined interval $I_{s, \tau}$, whose existence is guaranteed by Lemma 3. Moreover, Lemmas 3, 4 and 5 together with the properties of MHR distributions, allow us to write $E_F[R(M_{C})] \geq E[X_{\tau}](s_{\tau} + \tau)(1-\epsilon)$, giving the result as $E[Y_{\tau}] \leq E[X_{\tau}]$. A Mechanism with a Piecewise Constant Price We introduce a new mechanism $M_{PC}$ whose pricing strategy $p_{M_{PC}}$ is a piecewise constant function. This turns out to be useful in all the situations in which the seller is constrained not to change the posted price too often, e.g., when the mechanism is required to set prices for time intervals having a given minimum length. Our main result (Theorem 4) is a lower bound on the competitive ratio of $M_{PC}$ in the RV setting, which is comparable to that obtained for $M_{C}$ in Theorem 3. Thus, we show that, even in presence of constraints on the allowed pricing strategies, we are still able to design mechanisms with good performances in terms of competitive ratio. Clearly, $M_{PC}$ depends on the minimum length requirement, which influences the resulting lower bound. In particular, $M_{PC}$ is tuned by a parameter $\delta$ related to the number of time intervals in which the price must be constant. Mechanism $M_{PC}$ works by evenly partitioning the time interval $[0, t_0]$ into $[\log h]$ sub-intervals of length $\tau$, where $\delta \in (1, h)$ and $t_0 \in [0, T]$ are suitably defined parameters. Then, the remaining time $[t_0, T]$ is organized in other sub-intervals of length $\tau$. As a result, $[0, T]$ is partitioned into $[\frac{t}{\tau}]$ sub-intervals, which, overloading notation, we denote by $I_i := [(i-1)\tau, \min{[i, \tau, T]}]$ for $i = 1, \ldots, [\frac{t}{\tau}]$. Notice that $\tau = \frac{t_0}{\log h}$, and, thus, parameters $t_0$ and $\delta$ can be tuned to match the required minimum length $\tau$. The pricing strategy $p_{M_{PC}}$ of $M_{PC}$ is defined in such a way that the price is constant in each interval $I_i$. By letting $p_{M_{PC}}(I_i)$ be the price posted during $I_i$, we define the function $p_{M_{PC}}$ as follows: $p_{M_{PC}}(I_i) := \begin{cases} \frac{1}{\delta} \xi(i\tau) & \text{if } i = 1, \ldots, [\log h] \\ \xi(i\tau) & \text{if } i = [\log h] + 1, \ldots, [\frac{t}{\tau}] - 1. \end{cases}$ We compare in Figure 1 the prices of $M_{C,lin}$ and $M_{PC,lin}$ (i.e., $M_{PC}$ with a linear discount) in a specific setting for two values of $\tau$. Notice that $M_{PC}$ can be thought of as an extension of the Equal-Sample-of-Every-Scale (ESoES) mechanism by Babaioff et al. (2017) to the more general setting in which agents arrive stochastically according to a Poisson process and agents’ valuations are discounted over time. Before proving our main result, we need the following lemma, which is the analogous of Lemma 3 working for mechanism $M_{PC}$ instead of $M_{C}$. Lemma 6. In the RV setting with agents’ initial valuations drawn from a distribution $F$, given $0 < \epsilon < 1$, there exists $i = 1, [\log h]$ such that the price $p_{M_{PC}}(I_i)$ posted by $M_{PC}$ during the interval $I_i$ lies in the range $[\frac{1}{\delta} \xi((i+1)\tau), \nu \xi(i\tau)]$, where $\nu := \max\{1, E[X_{\tau}]\}$. Now, we provide our main result. The idea behind its proof is similar to the one used for Theorem 5. Theorem 4. Consider the RV setting with $\lambda_T = (\lambda T)^{1-\epsilon} \geq 1 - \ln(e-1)$ for some $\tau \in (0, T]$ and $0 < \epsilon < 1$. Then, restricted to the set $F$ of distributions $F$ satisfying the MHR condition, mechanism $M_{PC}$ has a competitive ratio that can be lower bounded as follows: $\rho(M_{PC}) \geq \frac{\xi((\log h) + 1)T^{1-\epsilon} \lambda^{-\epsilon})(1-\epsilon)}{\delta e}$. Whenever $T$ is not divisible by $\tau$, then the last time interval is shorter than $\tau$. Thus, in order to satisfy the minimum length constraint, we set its price equal to the one in the preceding interval.
Empirical Evaluation

We evaluate mechanisms $\mathcal{M}_C$, $\mathcal{M}_{PC}$, and a natural adaptation of the ESoES mechanism by Babaioff et al. (2017) to stochastic settings with no time discounting (called ESoES-SS). The pricing strategy of ESoES-SS is defined as follows. First, we compute the prices of ESoES by setting the number of agents equal to the expected number $\lambda T$ of agents arriving in $[0, T]$ according to a Poisson process of parameter $\lambda$. Then, ESoES-SS proposes the price that ESoES would propose to the $i$-th agent arrived if $i \leq \lambda T$ and 1 otherwise.

We use the following parameters values for the experiments: $\lambda \in \{1, \ldots, 20\}$, $T \in \{10, 20, 50, 100\}$, and $h \in \{2, \ldots, 20\}$. The following results do not consider time discounting so as to have a fair comparison between our mechanisms and ESoES-SS. Further results with a linear discount function are provided in the Appendix.

Result #1 We study a RV setting with a uniform probability distribution over $[1, h]$. For every combination of values of $\lambda$, $T$, $h$, we run 1000 Monte Carlo simulations, evaluating the revenue provided by mechanisms ESoES-SS, $\mathcal{M}_C$, and $\mathcal{M}_{PC}$. In particular, we analyze some variants of mechanisms $\mathcal{M}_{PC}$ differing for the number of subintervals (i.e., $N_{sub}$) in which $[0, t_0]$ is partitioned. Furthermore, we normalize the revenue provided by the mechanisms in each simulation with respect to $h$. We report the results in Figure 2 for $T = 10$ and $T = 50$, when $h = 10$. The results obtained for different values of $h$ are similar. $\mathcal{M}_C$ and $\mathcal{M}_{PC}$ with $N_{sub} = 232$ have overlapping performances that beat those of the other mechanisms. $\mathcal{M}_{PC}$ with $N_{sub} = 13$ has a performance close to that of the previous two mechanisms, showing that mechanism $\mathcal{M}_{PC}$ provides good performances even with few subintervals. $\mathcal{M}_{PC}$ with $N_{sub} = 4$ and ESoES-SS have almost overlapping performances, showing that very few subintervals are sufficient to $\mathcal{M}_{PC}$ to match the performances of ESoES-SS. The worst mechanism is $\mathcal{M}_{PC}$ with $N_{sub} = 2$. The loss of ESoES-SS w.r.t. $\mathcal{M}_C$ averaged over the values of $\lambda$ is about $0.3 h$ when $T = 10$, and $0.4 h$ when $T = 50$. Surprisingly, the performances of ESoES-SS seem to do not strictly depend on $\lambda$ and $T$.

Result #2 We study an IV setting. For every combination of values of $\lambda$, $T$, $h$, and for every $v \in \{1.0, 1.5, 2.0, \ldots, 5\}$, we run 1000 Monte Carlo simulations, evaluating the normalized revenue provided by mechanisms ESoES-SS and $\mathcal{M}_C$. For every combination of values of $\lambda$, $T$, $h$, we calculate $\max_v \frac{E_v[R(\mathcal{M}_C)] - E_v[R(\text{ESoES-SS})]}{E_v[R(\text{ESoES-SS})]}$ corresponding to the maximum normalized loss of ESoES-SS w.r.t. $\mathcal{M}_C$ over all valuations $v$, and $\max_v \frac{E_v[R(\text{ESoES-SS})] - E_v[R(\mathcal{M}_C)]}{E_v[R(\text{ESoES-SS})]}$ corresponding to the maximum normalized loss of $\mathcal{M}_C$ w.r.t. ESoES-SS over all valuations $v$. These two indexes are shown in Figure 3 for $T = 10$ and $T = 50$, when $h = 10$. The results obtained for different values of $h$ are similar. The loss of ESoES-SS w.r.t. $\mathcal{M}_C$ is always larger than $0.5 h$ except when both $\lambda$ and $T$ assume small values, while the loss of $\mathcal{M}_C$ w.r.t. ESoES-SS is negligible. Furthermore, the two losses converge to two constants as $\lambda$ and $T$ increase. This shows that, even if there are some special settings where ESoES-SS performs better than $\mathcal{M}_C$, the improvement is negligible. Instead, mechanism $\mathcal{M}_C$, which is designed to deal with stochastic arrivals, provides a very significant improvement. In particular, we observe that the difference between the revenue provided by ESoES-SS and that provided by $\mathcal{M}_C$ is maximized for small values of $v$ close to 1, while between $\mathcal{M}_C$ and ESoES-SS for large values of $v$ close to $h$.

Conclusion and Future Works

We study distribution-free posted-price mechanisms in order to sell a unique item within a finite time period. In our model, the agents arrive online according to a Poisson process, and their valuations for the item are discounted over time. Following a worst-case competitive analysis, we design a mechanism $\mathcal{M}_C$ providing an optimal competitive ratio in the identical valuation setting. Then, as for the random valuation setting, we analyze the performances of $\mathcal{M}_C$ and of a new mechanism $\mathcal{M}_{PC}$ that is constrained to set constant prices during time intervals having a given minimum length. We prove that both mechanisms achieve a competitive ratio that is constant with respect to the actual valuation when the distribution of the valuations has a monotone hazard rate. This shows that our mechanisms are robust even in nonstationary markets subject to arbitrary distribution changes preserving the same support.

In future, we will investigate hybrid settings in which our robust mechanisms can be combined with machine learning tools. For instance, data could be used to learn a class of distributions, and we could design a mechanism robust with respect to all the distributions of that class.
Ethical Impact

Posted-price mechanisms are widely adopted in real-world economic transactions, thanks to their simplicity: a seller posts prices and buyers arrive sequentially, deciding whether to accept the offer or not. Nowadays, most e-commerce websites implement this form of interaction with their users. Our mechanisms apply to concrete scenarios where the probability distribution of buyers’ valuations is unknown, the value of item for sale may decrease over time, and buyers’ arrivals are stochastic. In these settings, our mechanisms can make economic transactions more efficient and robust, allowing agents (buyers and sellers) to find better economic agreements. As we argued in the paper, our mechanisms provide theoretical guarantees in terms of online worst-case performance. This could have an arguably positive societal impact when applied to real-world economic problems. However, further research in this direction is required to prevent scenarios with an unbalanced reward structure, where agreements may just award one side (buyers or sellers) with the largest utilities at the expense of the others.

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Appendix

Omitted Proofs for the IV Setting

Lemma 1. In the IV setting, given any deterministic posted-price mechanism \( \mathcal{M} \), there always exists a deterministic posted-price mechanism \( \mathcal{M}' \) with undiscounted price \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} \) non-increasing in \( t \) such that \( E_v[\mathcal{R}(\mathcal{M})] \leq E_v[\mathcal{R}(\mathcal{M}')] \) for every possible agents’ initial valuation \( v \in [1, h] \).

Proof. We only need to prove the result for mechanisms \( \mathcal{M} \) whose undiscounted price \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} \) is not non-increasing in \( t \), otherwise the statement of the lemma is trivially true. The main idea of the proof is to let the time period \( [0, T] \) be evenly partitioned into time intervals of length \( \tau \) such that the undiscounted price function of \( \mathcal{M} \) is constant in each interval. This is w.l.o.g. if we take \( \tau \to 0 \). Then, there must be two consecutive time intervals, namely \( I_1 := I_s, \tau \) and \( I_2 := I_{s+\tau}, \tau \) for some starting time \( s \in [0, T - \tau] \), such that there exist \( p_1 < p_2 \in [1, h] \) with \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} = p_1 \) and \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} = p_2 \) during \( I_1 \) and \( I_2 \), respectively (otherwise the undiscounted price would be non-increasing). Now, let us define a mechanism \( \mathcal{M}' \) whose undiscounted price function is the same as that of \( \mathcal{M} \), except for the fact that \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} = p_2 \) during \( I_1 \) and \( \frac{p_{\mathcal{M}}(t)}{\xi(t)} = p_1 \) during \( I_2 \) (i.e., intuitively, we exchange the values in the two intervals so as to make the undiscounted price non-increasing in that window of time).

We show that the expected revenue provided by \( \mathcal{M}' \) is always greater than or equal to that achieved by \( \mathcal{M} \), as long as \( \tau \to 0 \). In order to compare the expected revenues of the two mechanisms, it is sufficient to focus on the window of time \( I_1 \cup I_2 \), where their price functions differ. Given \( p_1 \) and \( p_2 \), we can partition the agents’ valuations \( v \in [1, h] \) into three different subsets, as follows:

- \( v < p_1 \), implying that \( v \xi(t) < p_\mathcal{M}(t) \) and \( v \xi(t) < p_\mathcal{M}'(t) \) for every time instant \( t \in I_1 \cup I_2 \);
- \( p_1 \leq v \leq p_2 \), implying that \( p_\mathcal{M}(t) \leq v \xi(t) \leq p_\mathcal{M}'(t) \) for every time instant \( t \in I_1 \) and \( p_\mathcal{M}'(t) \leq v \xi(t) \leq p_\mathcal{M}(t) \) for every time instant \( t \in I_2 \);
- \( v > p_2 \), implying that \( v \xi(t) > p_\mathcal{M}'(t) \) and \( v \xi(t) > p_\mathcal{M}(t) \) for every time instant \( t \in I_1 \cup I_2 \).

In the first case, \( E_v[\mathcal{R}(\mathcal{M})] - E_v[\mathcal{R}(\mathcal{M}')] = 0 \), since both \( \mathcal{M} \) and \( \mathcal{M}' \) achieve an expected revenue equal to 0 during the time window \( I_1 \cup I_2 \), given that the item is never sold in that window (as both \( p_\mathcal{M}(t) \) and \( p_\mathcal{M}'(t) \) are always higher than the agents’ discounted valuation \( v \xi(t) \)). As for the second case, let us assume \( p_1 < v < p_2 \) (since the cases \( v = p_1 \) and \( v = p_2 \) are analogous). Then, \( \mathcal{M} \) can sell the item only during the interval \( I_1 \), while \( \mathcal{M}' \) can sell the item only during the other interval \( I_2 \). Thus, we have the following:

\[
E_v[\mathcal{R}(\mathcal{M})] - E_v[\mathcal{R}(\mathcal{M}')] = \int_s^{s+\tau} p_1 \xi(t) e^{-\lambda t} dt - \int_{s+\tau}^{s+2\tau} p_1 \xi(t) e^{-\lambda t} dt,
\]

which goes to 0 as \( \tau \to 0 \), given that \( \xi \) is continuous. Finally, in the third case, we can compute the difference between the expected revenues of the two mechanisms as follows:

\[
E_v[\mathcal{R}(\mathcal{M})] - E_v[\mathcal{R}(\mathcal{M}')] = \int_s^{s+\tau} p_1 \xi(t) e^{-\lambda t} dt + \int_{s+\tau}^{s+2\tau} p_2 \xi(t) e^{-\lambda t} dt - \int_s^{s+\tau} p_2 \xi(t) e^{-\lambda t} dt - \int_{s+\tau}^{s+2\tau} p_1 \xi(t) e^{-\lambda t} dt
= (p_1 - p_2) \int_s^{s+\tau} \xi(t) e^{-\lambda t} dt - (p_1 - p_2) \int_{s+\tau}^{s+2\tau} \xi(t) e^{-\lambda t} dt
= (p_1 - p_2) \left[ \int_s^{s+\tau} \xi(t) e^{-\lambda t} dt - \int_{s+\tau}^{s+2\tau} \xi(t) e^{-\lambda t} dt \right],
\]

which is less than or equal to 0 as \( \tau \to 0 \), by continuity of \( \xi \).

By re-iterating the procedure on all the pairs of consecutive infinitesimal intervals (since \( \tau \to 0 \)) defined as \( I_1 \) and \( I_2 \) (each time using the last mechanism \( \mathcal{M}' \) as the new \( \mathcal{M} \)), we can render the undiscounted price function non-increasing, obtaining a final mechanism \( \mathcal{M}' \) such that \( E_v[\mathcal{R}(\mathcal{M})] \leq E_v[\mathcal{R}(\mathcal{M}')] \) for every possible agents’ valuation \( v \in [1, h] \).

Lemma 2. In the IV setting, let \( \mathcal{M} \) be a deterministic posted-price mechanism whose pricing strategy \( p_{\mathcal{M}} \) satisfies \( p_{\mathcal{M}}(t) = \xi(t) \) for \( t \in [t_0, T] \) with \( t_0 \in [0, T) \). If the ratio \( p_v(\mathcal{M}) = \frac{E_v[\mathcal{R}(\mathcal{M})]}{E_v[\mathcal{R}(\mathcal{M}')]} \) for \( \mathcal{M} \) does not depend on the agents’ initial valuation \( v \), then \( \mathcal{M} \) is an optimal mechanism.
Proof. By contradiction, suppose that $\mathcal{M}$ is not optimal, i.e., there exists another deterministic posted-price mechanism $\mathcal{M}'$ such that $\rho(\mathcal{M}') > \rho(\mathcal{M})$. According to Proposition 1 and Lemma 1, $\mathcal{M}'$ must be defined by a pricing strategy $p_{\mathcal{M}'}$ such that the undiscounted price $p_{\mathcal{M}'}(t)$ is non-increasing in $t$ and the minimum price is selected for a time interval $[t'_0, T] \subseteq [0, T]$ having non-zero length (recall that $\rho(\mathcal{M}) > 0$ does not depend on $v$ and $\rho(\mathcal{M}) = \min_{v \in [1, h]} \rho_v(\mathcal{M}))$.

Case $t'_0 \geq t_0$. Let us consider the valuation $v = 1$. Then, we have that the expected revenue of mechanism $\mathcal{M}$ is $E[v][\mathcal{R}(\mathcal{M})] = \int_{t_0}^{T} \xi(t) \lambda e^{-\lambda t} dt$ (accounting for the case in which an agent arrives at $t \geq t_0$ and buys the item at price $\xi(t)$), which is greater than or equal to the expected revenue of mechanism $\mathcal{M}'$, defined as $E[v][\mathcal{R}(\mathcal{M}')] = \int_{t'_0}^{T} \xi(t) \lambda e^{-\lambda t} dt$. Intuitively, $E[v][\mathcal{R}(\mathcal{M})] \geq E[v][\mathcal{R}(\mathcal{M}')]$ since $\mathcal{M}'$ posts the minimum price for a period of time shorter than that of $\mathcal{M}$. Therefore, it holds $\rho(\mathcal{M}') \leq \rho_v(\mathcal{M}') \leq \rho(\mathcal{M})$, which is a contradiction.

Case $t'_0 < t_0$. First, suppose that there exists a time instant $t' \in [0, t'_0]$ defined as $t' = \sup \{ t \in [0, t'_0] \mid p_{\mathcal{M}}(t) < p_{\mathcal{M}'}(t) \}$, i.e., the last time instant in which $p_{\mathcal{M}}(t)$ changes from being less than $p_{\mathcal{M}'}(t)$ to being larger than or equal to $p_{\mathcal{M}'}(t)$. Clearly, it holds $p_{\mathcal{M}}(t) \geq p_{\mathcal{M}'}(t)$ for every $t \in [0, T] : t > t'$. Moreover, let us consider the agents’ valuation $v \in [1, h]$ such that $v \xi(t') = p_{\mathcal{M}}(t')$ and focus on the case in which $p_{\mathcal{M}}(t) = p_{\mathcal{M}'}(t)$ (as the other cases are analogous). Notice that, for every time instant $t \leq t'$, mechanism $\mathcal{M}'$ cannot sell the item, since, by using Lemma 1 we get:

$$v \xi(t) < vp_{\mathcal{M}}(t) - \frac{\xi(t')}{p_{\mathcal{M}'}(t')} = v p_{\mathcal{M}'}(t) - \frac{\xi(t')}{p_{\mathcal{M}'}(t')} \leq v p_{\mathcal{M}'}(t) - \frac{\xi(t')}{v \xi(t')} \leq p_{\mathcal{M}'}(t).$$

Additionally, with an analogous reasoning we can show that, for all the times $t \in [0, T] : t > t'$, both mechanisms may sell the item, but the price posted by $\mathcal{M}'$ is always less than or equal to that chosen by $\mathcal{M}$, with a non-empty time interval in which the former is strictly less than the latter (as $t'_0 < t_0$). Thus, in this case, it holds $\rho_v(\mathcal{M}) > \rho_v(\mathcal{M}')$, which implies that $\rho(\mathcal{M}') < \rho(\mathcal{M})$, a contradiction. Finally, it remains to analyze the case in which a time instant $t'$ defined above does not exist. Since the undiscounted price functions are non-increasing by Lemma 2 and $t'_0 < t_0$, it must be the case that there is no intersection point between the two functions. Hence, it must be $p_{\mathcal{M}}(t) > p_{\mathcal{M}'}(t)$ for all $t \in [0, t_0]$, which implies that $\rho(\mathcal{M}') < \rho(\mathcal{M})$ by taking $v = h$. This leads to a contradiction. \qed

**Theorem 1.** In the IV setting, there exists an optimal deterministic posted-price mechanism $\mathcal{M}_C$ whose pricing strategy $p_{\mathcal{M}_C}$ is defined as follows:

$$p_{\mathcal{M}_C}(t) := \begin{cases} a e^b \frac{b(t)dt}{\xi(t)} & \text{if } t \in [0, t_0) \\ \frac{1}{\xi(t)} & \text{if } t \in [t_0, T], \end{cases}$$

where $b$ is a function such that $b(t) := \lambda - \frac{1}{\xi(t)} - \frac{\xi'(t)}{\xi(t)}$ with $\xi(t) := \frac{1}{\xi(T)}$, whereas $a, k, t_0$ are suitably defined constants that do not depend on the agents’ initial valuation $v$.

Proof. By Lemma 2 and using $\rho_v(\mathcal{M}_C) = E[v][\mathcal{R}(\mathcal{M}_C)]$, it is sufficient to search for an optimal mechanism $\mathcal{M}_C$ whose pricing strategy $p_{\mathcal{M}_C}$ is such that the expected revenue of the mechanism is linearly dependent in $v$, i.e., for every valuation $v \in [1, h]$, it must be the case that:

$$E[v][\mathcal{R}(\mathcal{M}_C)] = kv,$$

where $k > 0$ is a suitably defined constant that does depend on $v$. In the following, for the case of presentation, we omit the index $\mathcal{M}_C$ from $p_{\mathcal{M}_C}$ as the mechanism is clear from the context.

From Proposition 1 there must be a $t_0 \in [0, T]$ such that $p(t) = \xi(t)$ for every $t \in [t_0, T]$, otherwise $\rho_v(\mathcal{M}_C) = 0$ for the valuation $v = 1$. Thus, it remains to define $p(t)$ for $t \in [0, t_0]$.

For any valuation $v \in [1, h]$, by letting $t^* := \sup \{ t \in [0, t_0] | p(t) > v \xi(t) \}$, we can express the expected revenue of the mechanism $\mathcal{M}_C$ as a function of $t^*$. First, notice that, it holds $p(t^*) = v \xi(t^*)$. Moreover, by using Lemma 1 it must be the case that $p(t) > v \xi(t)$ for every $t < t^*$, since:

$$v \xi(t) < v p(t) \frac{\xi(t^*)}{p(t^*)} = v p(t) \frac{\xi(t^*)}{v \xi(t^*)} = p(t).$$

As a result, the item is never sold before time $t^*$, which allows us to write the following:

$$E[v][\mathcal{R}(\mathcal{M}_C)] = e^{\lambda t^*} \int_{t^*}^{t_0} p(t) \lambda e^{-\lambda t} dt + e^{\lambda t^*} \int_{t_0}^{T} \xi(t) \lambda e^{-\lambda t} dt.$$
By deriving the left-hand side of Equation (2) with respect to $t^*$, we get:

$$\frac{d\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)]}{dt^*} = e^{\lambda t^*} \frac{dG(t^*)}{dt^*} + \lambda \left[ e^{\lambda t^*} \int_{t^*}^{T} p(t) \lambda e^{-\lambda t} dt + \lambda e^{\lambda t^*} \int_{t^0}^{T} \xi(t) \lambda e^{-\lambda t} dt \right]$$

$$= -\lambda p(t^*) + \lambda k\zeta(t^*)p(t^*)$$

where $G(t^*) := \int_{t^*}^{T} p(t) \lambda e^{-\lambda t} dt = \int_{t^*}^{T} -p(t) \lambda e^{-\lambda t} dt = \int_{t^*}^{T} g(t) dt$, with $g(t) := -p(t) \lambda e^{-\lambda t}$. By applying the fundamental theorem of calculus, we have that $\frac{dG(t^*)}{dt^*} = g(t^*) = -p(t^*) \lambda e^{-\lambda t^*}$. Thus, the last equality is readily obtained by noticing that the term in the squared brackets is exactly equal to the expected revenue $\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)]$, which, in turn, must be equal to $k\zeta(t^*)p(t^*)$. Furthermore, by deriving the right-hand side of Equation (2) with respect to $t^*$, we get:

$$\frac{d}{dt^*} [k\zeta(t^*)p(t^*)] = k\zeta'(t^*)p(t^*) + k\zeta(t^*)p'(t^*)$$

By equating the derivatives of the two sides of Equation (2), we get the following differential equation:

$$p'(t^*) = \left[ \lambda - \frac{\lambda}{k\zeta(t^*)} - \frac{\zeta'(t^*)}{\zeta(t^*)} \right] p(t^*)$$

(3)

By solving Equation (3) for $p(t)$, we obtain the function:

$$p(t) = a e^{\left[ \lambda - \frac{\lambda}{k\zeta(t)} - \frac{\zeta'(t)}{\zeta(t)} \right] t}$$

and, from the boundaries conditions $p(0) = h$ and $p(t_0) = \xi(t_0)$, we can derive constants $a$ and $k$. Notice that the condition $p(0) = h$ can be derived from the fact that, if $p(0) < h$, then the expected revenue $\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)]$ is the same for all the valuations $v \in [1, h]$ such that $p(0) \leq v \leq h$, which is not possible since we want that $\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)]$ linearly depends on $v$.

We recall that $\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)] = kv$ for all $v \in [1, h]$. Thus, we can use this in order to find $t_0$ as a function of the problem parameters $\lambda$, $T$, $h$, and function $\xi$. Using $v = 1$, we get:

$$\int_{t^0}^{T-t_0} \xi(t) \lambda e^{-\lambda t} dt = k,$$

(4)

which gives $t_0$ after replacing $k$ with the expression we got from the boundaries conditions.

**Corollary 1.** *In the IV setting, mechanism $\mathcal{M}_c$ achieves:

$$\rho(\mathcal{M}_c) = \frac{\int_{t^0}^{T} \xi(t) \lambda e^{-\lambda t} dt}{\int_{t^0}^{T} \xi(t) \lambda e^{-\lambda t} dt}.$$*

**Proof.** Let us recall that, from the proof of Theorem [1], $\mathcal{M}_c$ is characterized by the same ratio $\rho_v(\mathcal{M}_c)$ for all $v \in [1, h]$. Hence, we can calculate the competitive ratio by taking $v = 1$:

$$\rho(\mathcal{M}_c) = \frac{\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c)]}{\mathbb{E}_v[\mathcal{R}(\mathcal{M}_c^*)]} = k = \frac{\int_{t^0}^{T-t_0} \xi(t) \lambda e^{-\lambda t} dt}{\int_{t^0}^{T} \xi(t) \lambda e^{-\lambda t} dt},$$

(5)

where we used Equation (1) and Equation (4) from the proof of Theorem [1].

**Theorem 2.** *In the IV setting with linear discount function $\xi_{lin}$, there exists an optimal deterministic posted-price mechanism $\mathcal{M}_{c,lin}$ whose pricing strategy $p_{\mathcal{M}_{c,lin}}$ is defined as:

$$p_{\mathcal{M}_{c,lin}}(t) := \begin{cases} h \left( 1 - \frac{t}{T} \right) e^{\lambda(1 - \frac{1}{T})t + \frac{\lambda}{2T^2} t^2} & \text{if } t \in [0, t_0) \\ 1 - \frac{1}{\lambda} & \text{if } t \in [t_0, T] \end{cases}$$

where $k := \frac{\lambda t_0}{2(T - t_0)}$, and the time $t_0 \in [0, T]$ is defined as the unique positive real root of the following equation: $1 - \frac{1}{\lambda} \left( 1 + \lambda t_0 - e^{-\lambda(T - t_0)} \right) = k$.*

**Proof.** We follow the line of the proof of Theorem [1] i.e., we look for a mechanism $\mathcal{M}_{c,lin}$ such that $\mathbb{E}_v[\mathcal{R}(\mathcal{M}_{c,lin})] = kv$ for every $v \in [1, h]$, where $k > 0$ is suitably defined constant that does not depend on $v$. For the ease of presentation, we omit the subscript $\mathcal{M}_{c,lin}$ from the pricing strategy $p_{\mathcal{M}_{c,lin}}$. 

Let us fix $v \in [1, h]$. By defining $t^*$ as in the proof of Theorem 1 since in this case the discount is $\xi_{\text{lin}}(t) = 1 - \frac{t}{T}$, for $t \in [0, T]$, we have $p(t^*) = v \left(1 - \frac{t^*}{T}\right)$, which allows us to write the following:

$$e^{\lambda t^*} \int_{t^*}^{T} p(t) e^{-\lambda t} \, dt + e^{\lambda T} \int_{0}^{T} \left(1 - \frac{t}{T}\right) \lambda e^{-\lambda t} \, dt = k \frac{T}{T - t^*} p(t^*),$$

where the left-hand side is the expected revenue $E_v[\mathcal{R}(\mathcal{M}_{\text{lin}})]$ and the right-hand side is $kv$. By deriving with respect to $t^*$ the left-hand side of the Equation (6), we get:

$$\frac{d}{dt^*} \left(\int_{t^*}^{T} p(t) e^{-\lambda t} \, dt + e^{\lambda T} \int_{0}^{T} \left(1 - \frac{t}{T}\right) \lambda e^{-\lambda t} \, dt\right) = -\lambda p(t^*) + \lambda e^{\lambda T} \frac{T}{T - t^*} p(t^*),$$

where $G(t^*)$ is defined as in the proof of Theorem 1. Now, we derive the right-hand side of Equation (6) with respect to $t^*$:

$$\frac{d}{dt^*} \left(\frac{kT}{T - t^*} p(t^*)\right) = \frac{kT}{T - t^*} p'(t^*) + \frac{kT}{(T - t^*)^2} p(t^*).$$

By equating the derivatives of the two sides of Equation (6), we get the following differential equation:

$$p'(t^*) = \left[\lambda - \frac{\lambda (T - t^*)}{kT} - \frac{1}{T - t^*}\right] p(t^*).$$

(7)

After solving Equation (7) for $p(t)$, we obtain the general solution:

$$p(t) = a e^{\int \left[\lambda - \frac{\lambda (T - t)}{kT} - \frac{1}{T - t}\right] \, dt} = a e^{\lambda (1 - \frac{t}{T}) + \frac{\lambda t}{kT} T + \ln(T - t)},$$

where, using boundary conditions $p(0) = h$ and $p(t_0) = 1 - \frac{t_0}{T}$, we can derive the expressions $a := \frac{h}{T}$ and $k := \lambda e^{\lambda T} \frac{2T - t_0}{2T T (\lambda t_0 + \ln(h))}$.

Since $E_v[\mathcal{R}(\mathcal{M}_{\text{lin}})] = kv$ for all $v \in [1, h]$, we can use the equation in order to define $t_0$ with respect to the problem parameters $\lambda$, $T$ and $h$. For $v = 1$:

$$\int_{0}^{T - t_0} \left(1 - \frac{t}{T}\right) \lambda e^{-\lambda t} \, dt = 1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda (T - t_0)}\right) = k,$$

(8)

and, by replacing $k$ with the expression we got from the boundary conditions, we obtain:

$$1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda (T - t_0)}\right) = \lambda t_0 \frac{2T - t_0}{2T (\lambda t_0 + \ln(h))}$$

(9)

Finally, we can define $t_0$ as the unique positive real root of Equation (9). In particular, it is easy to show that Equation (9) always admits a positive real root in the range $(0, T)$. Indeed, we call $q(x) = \lambda x e^{\lambda x (T - x)} - 1 + \frac{1}{\lambda T} (1 + \lambda x - e^{-\lambda (T - x)})$.

We observe that $q(x)$ is continuous on the interval $[0, T]$ and that $q(T) > 0$ and $q(0) < 0$, therefore, for Bolzano’s theorem, there exists at least a $t_0 \in (0, T)$ such that $q(t_0) = 0$. The uniqueness can be derived as a consequence of Lemma 2.

**Corollary 2.** In the IV setting with linear discount function $\xi_{\text{lin}}$, $\mathcal{M}_{\text{lin}}$ achieves a competitive ratio:

$$\rho(\mathcal{M}_{\text{lin}}) = \frac{1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda (T - t_0)}\right)}{1 - \frac{1}{\lambda T} \left(1 - e^{-\lambda T}\right)}.$$

**Proof.** We can calculate it by taking $v = 1$:

$$\rho(\mathcal{M}_{\text{lin}}) = \frac{E_v[\mathcal{R}(\mathcal{M}_{\text{lin}})]}{E_v[\mathcal{R}(\mathcal{M}^*)]} = \frac{k^*}{k^*} = \frac{\int_{0}^{T - t_0} \left(1 - \frac{t}{T}\right) \lambda e^{-\lambda t} \, dt}{\int_{0}^{T} \left(1 - \frac{t}{T}\right) \lambda e^{-\lambda t} \, dt} = \frac{1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda (T - t_0)}\right)}{1 - \frac{1}{\lambda T} \left(1 - e^{-\lambda T}\right)},$$

where we used Equation (1) and Equation (5) from the proof of Theorem 2. \□
Examples of Mechanisms $\mathcal{M}_{c,\text{lin}}$ and $\mathcal{M}_{\text{PC,lin}}$ and Competitive Ratio Analysis

In order to ease the reader in the understanding of our mechanisms, we provide their graphical representation for the case of a linear discount function $\xi_{\text{lin}}(t) := 1 - \frac{t}{T}$. In particular, we focus on mechanisms $\mathcal{M}_{c,\text{lin}}$ and $\mathcal{M}_{\text{PC,lin}}$, where the latter is the linear-discount version of the general-discount mechanism $\mathcal{M}_{\text{PC}}$. The price function $p_{\mathcal{M}_{c,\text{lin}}}$ of $\mathcal{M}_{\text{PC,lin}}$ can be easily obtained from that of $\mathcal{M}_{\text{PC}}$ by using the specific definition of the discount function. We report it below for completeness.

$$p_{\mathcal{M}_{\text{PC,lin}}}(I_i) := \begin{cases} \frac{h}{t^i} (1 - \frac{t^i}{T}) & \text{if } i = 1, \ldots, \lceil \log_{\delta} h \rceil \\ 1 - \frac{1}{T} & \text{if } i = \lceil \log_{\delta} h \rceil, \ldots, \lceil T/\tau \rceil - 1 \\ 1 - \frac{(i-1)\tau}{T} & \text{if } i = \left\lfloor \frac{T}{\tau} \right\rfloor \end{cases}$$

We tune the parameters $h$, $\lambda$, and $T$ so as to simulate real-world scenarios representing the long-term rental of a single room. In particular, we fix the parameter values by analyzing data from a real-world co-living company operating on the web, counting over 7000 rooms. In this scenario, the goal is to rent a single room to students for a fixed period of one year. We set $T = 12$, assuming that each time interval of length 1 corresponds to a period of one month, and we fix the starting time $t = 0$ as the time in which the contract of the previous tenant ends. Therefore, the room value is discounted over time as an effect of the ever shorter period of stay of the future tenant. We also set $h = 2.8$, which means that the highest valuation for the room is around three times the lowest one.

Figure 4 shows how the shape of mechanism $\mathcal{M}_{c,\text{lin}}$ changes by varying the arrival rate $\lambda$, which is the expected number of agents arriving in a time interval of one month. We observe that the price function decreases as a linearly discounted exponential function in the time interval $[0, t_0]$, and, then, as a linear function in $[t_0, T]$. Notice that, by comparing Figure 4(a) and Figure 4(b), it is easy to see that the time instant $t_0$ gets closer to zero as the arrival rate $\lambda$ increases.

This can be explained by recalling that the mechanism has to deal with the trade-off between setting high prices so as to achieve high revenues and posting lower prices in order to increase the probability of selling the item. In the first period of time, the seller posts high prices hoping for the arrival of an agent having an high valuation. This phase cannot be too long, otherwise the item risks to remain unsold, and, on the other hand, it cannot even be too short, otherwise the probability of encountering such an high-valuation agent becomes too small. Therefore, when the arrival rate decreases, the high-price phase must be enlarged in order to still have some chance of concluding the purchase for an high price (Figure 4(b)), while, if $\lambda$ increases, it suffices to post high prices for a shorter time period (Figure 4(a)).

Figure 5 represents the behavior of mechanism $\mathcal{M}_{\text{PC,lin}}$ when we impose different constraints on the minimum time in which the price must be constant. In particular, Figure 5a and Figure 5b show the shape of $\mathcal{M}_{\text{PC,lin}}$ when the posted price does not change for time intervals of length $\tau$ equal to one month (i.e., $\tau = 1$) and one week (i.e., $\tau = 0.25$), respectively.

![Figure 4: Mechanism $\mathcal{M}_{c,\text{lin}}$ with different rate parameters $\lambda$.](image)

### Analysis of the Competitive Ratio $\rho(\mathcal{M}_{c,\text{lin}})$

From Corollary 2 we know that, in the IV setting with linear discount function $\xi_{\text{lin}}$, mechanism $\mathcal{M}_{c,\text{lin}}$ achieves a competitive ratio:

$$\rho(\mathcal{M}_{c,\text{lin}}) = \frac{1 - \frac{1}{\lambda T} \left( 1 + \lambda t_0 - e^{-\lambda(T-t_0)} \right)}{1 - \frac{1}{\lambda T} (1 - e^{-\lambda T})},$$

where $t_0 \in [0, T]$ is defined in Theorem 2 as the unique positive real root of the equation:

$$\lambda t_0 = \frac{2 T - t_0}{2 T (\lambda t_0 + \ln h)} = 1 - \frac{1}{\lambda T} \left( 1 + \lambda t_0 - e^{-\lambda(T-t_0)} \right).$$

\(1\) We cannot disclose the name of the company for privacy reasons.
Indeed, the numerator of $\rho$ we analyze how revenue of an optimal mechanism. There are no guarantees for valuation functions with unbounded support. In the following, having a valuation function with finite support is fundamental in order to achieve a certain fraction of the expected results in the following table:

|                | $T \to \infty$ | $\lambda \to \infty$ | $h \to \infty$ |
|----------------|-----------------|-----------------------|----------------|
| $t_0$          | $\Theta(\sqrt{T})$ | $\Theta(\sqrt{T/\lambda})$ | $\Theta(T)$ |
| $\lim \rho(M_{c,\text{lin}})$ | $\Theta \left(1 - \frac{1}{\sqrt{T}} \right)$ | $\Theta \left(1 - \frac{1}{\sqrt{\lambda}} \right)$ | $\Theta \left(\frac{1}{\log^2(h)} \right)$ |

By using Equation (10), we can conclude that $t_0$ is asymptotically equivalent to $\sqrt{T}$ when $T \to \infty$. Then, the limit of the competitive ratio is:

$$
\lim_{T \to \infty} \rho(M_{c,\text{lin}}) = \lim_{T \to \infty} 1 - \frac{t_0}{T} = \lim_{T \to \infty} 1 - \frac{1}{\sqrt{T}} = 1.
$$

Similarly, $t_0$ is asymptotically equivalent to $\sqrt{\frac{\lambda}{x}}$ when $\lambda \to \infty$. Thus:

$$
\lim_{\lambda \to \infty} \rho(M_{c,\text{lin}}) = \lim_{\lambda \to \infty} 1 - \frac{t_0}{T} = \lim_{\lambda \to \infty} 1 - \frac{1}{\sqrt{T \lambda}} = 1.
$$

Moreover, it is easy to see that $t_0 \to T$ when $h \to \infty$. Indeed, in this case we have that $t_0$ is the unique positive real root of the equation:

$$
1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda(T-t_0)} \right) = 0.
$$

Since $t_0 \to T$, the limit of the competitive ratio is:

$$
\lim_{h \to \infty} \rho(M_{c,\text{lin}}) = \frac{1 - \frac{1}{\lambda T} \left(1 + \lambda T - e^{-\lambda(T-T_0)} \right)}{1 - \frac{1}{\lambda T} \left(1 - e^{-\lambda T} \right)} = 0.
$$

Therefore, having a valuation function with finite support is fundamental in order to achieve a certain fraction of the expected revenue of an optimal mechanism. There are no guarantees for valuation functions with unbounded support. In the following we analyze how $\rho(M_{c,\text{lin}})$ goes to 0. We first observe that $\rho(M_{c,\text{lin}})$ is proportional to $\frac{\lambda(T-t_0)^2}{2T} + o(T-t_0)^2$ when $h \to \infty$. Indeed, the numerator of $\rho(M_{c,\text{lin}})$ depends on $h$ through $t_0$:

$$
1 - \frac{1}{\lambda T} \left(1 + \lambda t_0 - e^{-\lambda(T-t_0)} \right) = \frac{z}{T} - \frac{1}{\lambda T} \left(1 - e^{-\lambda z} \right) = \frac{\lambda(T-t_0)^2}{2T} + o(T-t_0)^2,
$$

where $z := T - t_0 \to 0$ as $h \to \infty$, and the Taylor series $e^{-\lambda z} = 1 - \lambda z + \frac{\lambda^2 z^2}{2} + o(z^2)$ is used to expand function $e^{-\lambda z}$ at $t_0 = T$. Notice that the competitive ratio is decreasing in $t_0$. We compute $t_0$, which is an upper bound for $t_0$, by solving Equation (10) with the exponential term $e^{-\lambda(T-t_0)}$ substituted by parameter $a \varepsilon$. We impose $\varepsilon$ and $e^{-\lambda(T-t_0)}$ to have the same domain, hence $\varepsilon \in (0, 1)$. Thus, we obtain the following equation:

$$
\lambda x - \frac{2T - x}{2T (\lambda x + \ln h)} = 1 - \frac{1}{\lambda T} \left(1 + \lambda x - \varepsilon \right), \quad (11)
$$

Figure 5: Mechanism $M_{c,\text{lin}}$ with different constraints on the minimum time in which the price must be constant.
The solution
\[ x = \frac{\sqrt{2\lambda T \ln(h) + \ln^2 h + \varepsilon^2 - 2\varepsilon + 1 - \ln h + \varepsilon - 1}}{\lambda} \]
is increasing in \( \varepsilon \), being its first partial derivative positive for all \( \varepsilon \in (0, 1) \):
\[ \frac{\partial x}{\partial \varepsilon} = \frac{\varepsilon - 1}{\lambda \sqrt{2\lambda T \ln(h) + \ln^2 h + (\varepsilon - 1)^2}} + \frac{1}{\lambda}. \]

Hence, by setting \( \varepsilon = 1 \), we get the following upper bound on \( t_0 \):
\[ \bar{t}_0 := \frac{-\ln h + \sqrt{2\lambda T \ln(h) + \ln^2 h}}{\lambda} = \frac{2T \ln(h)}{\sqrt{2\lambda T \ln(h) + \ln^2 h + \ln h}}. \]

Notice that \( T - \bar{t}_0 \) is a lower bound for \( T - t_0 \). By asymptotic analysis, as \( h \to \infty \) we have:
\[ T - \bar{t}_0 = \frac{2\lambda T^2 \ln h}{2 \ln^2 h + 2\lambda T \ln h + 2 \ln h \sqrt{2\lambda T \ln h + \ln^2 h}} \sim C_1 \frac{1}{\ln(h)}, \]
where \( C_1 \) is constant with respect to \( h \) and depends on parameters \( \lambda \) and \( T \). Hence, as \( h \to \infty \):
\[ \rho(M_{c,\text{lin}}) \sim \frac{\lambda(T - t_0)^2}{2T - \frac{2}{\lambda}(1 - e^{-\lambda T})} \geq \frac{\lambda(T - \bar{t}_0)^2}{2T - \frac{2}{\lambda}(1 - e^{-\lambda T})} \sim C_2 \frac{1}{\ln^2(h)}, \]
where \( C_2 \) is a constant with respect to \( h \) and depends on parameters \( \lambda \) and \( T \). We conclude that, as \( h \to \infty \), the competitive ratio \( \rho(M_{c,\text{lin}}) \) converges to 0 slower than or the same as the function \( \frac{1}{\log^2(h)} \).

**Omitted Proofs for the RV Setting**

**Lemma 3.** In the RV setting with agents’ initial valuations drawn from a distribution \( F \), given \( \tau \in (0, T) \) and \( 0 < \varepsilon < 1 \), there exists at least an interval \( I_{s,\tau} \) of length \( \tau \in (0, T] \) starting at \( s \in [0, T - \tau] \) such that the prices \( p_{M_c}(t) \) posted by mechanism \( M_c \) during the time instants \( t \in I_{s,\tau} \) lie in the range \( \left[ \frac{\mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon)}{\kappa_\tau(s)}, \mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon) \right] \).

**Proof.** Given how the function \( p_{M_c} \) is defined, we can always define a time interval \( I_{s,\tau} \) as desired by selecting its starting time \( s \in [0, T - \tau] \) in such a way that \( p_{M_c}(s) = \mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon) \). From Definition\([5]\) we know that \( \kappa_\tau(s) \leq \kappa_\tau \). Hence,
\[ p_{M_c}(s + \tau) = \frac{p_{M_c}(s)}{\kappa_\tau(s)} = \frac{\mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon)}{\kappa_\tau(s)} \geq \frac{\mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon)}{\kappa_\tau}. \]

Since \( p_{M_c} \) is non-increasing by Lemma\([1]\) for every \( t \in I_{s,\tau} \) we have:
\[ p_{M_c}(t) \in [p_{M_c}(s), p_{M_c}(s + \tau)] \subseteq \left[ \frac{\mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon)}{\kappa_\tau}, \mathbb{E}[X_{\lambda T}]\xi(s + \tau)(1 - \varepsilon) \right]. \]

Notice that the inequality involving \( p_{M_c}(s + \tau) \) holds with equality if \( s \in \arg\max_{s \in [0, T - \tau]} \kappa_\tau(s) \). If this is the case, then there exists a unique interval verifying the statement. \( \square \)

**Lemma 4.** \( F_{X_{\lambda T}} \) has non-decreasing monotone hazard rate.

**Proof.** Let us recall that the cumulative distribution function of \( X_{\lambda T} \) is such that:
\[ F_{X_{\lambda T}}(x) = e^{-\lambda T(1-F(x))}. \]

We compute the hazard rate of \( F_{X_{\lambda T}} \) and show it is non-decreasing, as follows:
\[ H_{X_{\lambda T}}(x) = \frac{f_{X_{\lambda T}}(x)}{1 - F_{X_{\lambda T}}(x)} = \frac{\frac{\partial}{\partial x} F_{X_{\lambda T}}(x)}{1 - F_{X_{\lambda T}}(x)} = \frac{\frac{\lambda T f(x)e^{-\lambda T(1-F(x))}}{e^{\lambda T(1-F(x))} - 1}}{\lambda T - f(x)} = \frac{\lambda T - f(x)}{1 - F(x) e^{\lambda T(1-F(x))} - 1}. \]
Since \( F \) is MHR, the hazard rate \( H(x) \) is non-decreasing. Notice that \( F(x) \) is non-decreasing, and, thus, \( 1 - F(x) \) is non-increasing. As a result, proving that \( \frac{1 - F(x)}{x} \) is non-decreasing in \( x \) is equivalent to show that \( g(y) := \frac{y}{e^{\lambda y} - 1} \) is non-increasing in \( y \). We study the first derivative of \( g(y) \):

\[
\frac{d}{dy} g(y) = \frac{e^{\lambda y} (1 - \lambda y) - 1}{(e^{\lambda y} - 1)^2} \leq 0 \quad \text{for all } y \in [0, 1].
\]

This implies that \( g(y) \) is non-increasing in \( y \); hence, \( \frac{1 - F(x)}{x} \) is non-decreasing in \( x \). We conclude that \( H_{X,\lambda}(x) \) is monotone non-decreasing. \( \square \)

In order to prove Lemma 5, we first state the following variant of the Chebyshev inequality [Mitrovic, Pecaric, and Fink 2013], where the adopted notation is specific for the proposition.

**Proposition 2** ([Mitrovic, Pecaric, and Fink 2013]). Suppose function \( h(x) \) is positive and non-decreasing on \([a, b]\), function \( g(x) \) is non-decreasing on \([a, b]\), and function \( f(x) \) is continuous on \([a, b]\), then the following inequality holds:

\[
\int_a^b h(x) f(x) g(x) dx \geq \int_a^b f(x) g(x) dx.
\]

**Lemma 5.** For every \( \tau, \tau' \in (0, T) \) with \( \tau \leq \tau' \), it holds:

\[
\frac{\mathbb{E}[X_{\lambda\tau}]}{\mathbb{E}[X_{\lambda\tau'}]} \geq \frac{\ln (\lambda \tau)}{\ln (\lambda \tau')},
\]

**Proof.** Recall that \( F_{X,\lambda}(x) = e^{-\lambda \tau (1 - F(x))} \). Then, we can write the following:

\[
\mathbb{E}[X_{\lambda\tau}] = \int_0^\infty x f_{X,\lambda\tau}(x) dx = \int_0^\infty 1 - F_{X,\lambda\tau}(x) dx = \int_0^\infty 1 - e^{-\lambda \tau (1 - F(x))} dx
\]

\[
= \int_0^\infty \frac{1}{f(x)} \frac{1 - e^{-\lambda \tau (1 - F(x))}}{1 - F(x)} dF(x)
\]

\[
= \int_0^\infty \frac{1}{H(F^{-1}(1 - \eta))} \frac{1 - e^{-\lambda \tau \eta}}{\eta} d\eta.
\]

Now, we apply Lemma 2 \( F \) having non-decreasing monotone hazard rate implies that \( h(\eta) := \frac{1}{H(F^{-1}(1 - \eta))} \) is a non-decreasing function of \( \eta \). Hence, \( h(k) \) is non-decreasing and positive on \([0, 1]\). \( g(\eta) := \frac{1 - e^{-\lambda \tau \eta}}{1 - e^{-\lambda \tau}} \) is non-decreasing on \([0, 1]\) and \( f(\eta) := \frac{1 - e^{-\lambda \tau \eta}}{\eta} \) is continuous on \([0, 1]\). Thus,

\[
\mathbb{E}[X_{\lambda\tau}] = \int_0^1 \frac{1}{H(F^{-1}(1 - \eta))} \frac{1 - e^{-\lambda \tau \eta}}{\eta} d\eta
\]

\[
= \int_0^1 \frac{1 - e^{-\lambda \tau \eta}}{\eta} d\eta = \int_0^\tau Ei(-\lambda \tau) + \frac{\ln (\lambda \tau)}{\ln (\lambda \tau')} \geq \frac{\ln (\lambda \tau)}{\ln (\lambda \tau')},
\]

where \( Ei(x) := \int_0^x e^{t} \frac{dt}{t} \) is the entire exponential integral function, \( Ei(x) := \int_x^{\infty} e^{t} \frac{dt}{t} \) is the exponential integral function, and \( \gamma \approx 0.577 \) is the Euler’s constant. \( \square \)

**Theorem 3.** Consider the RV setting with \( \lambda \tau = (\lambda T)^{1 - \epsilon} \geq 1 - \ln(\epsilon - 1) \) for some \( \tau \in (0, T] \) and \( 0 < \epsilon < 1 \). Then, restricted to the set \( \mathcal{F} \) of distributions \( F \) satisfying the MHR condition, mechanism \( M_c \) has a competitive ratio that can be lower bounded as follows:

\[
\rho(M_c) \geq \frac{\xi(t_0 + T^{1 - \epsilon} \lambda^{-\epsilon})(1 - \epsilon)}{\kappa_{\tau \epsilon}}.
\]

**Proof.** By hypothesis, we have \( \lambda \tau = (\lambda T)^{1 - \epsilon} \) for some \( \tau \in (0, T] \) and \( 0 < \epsilon < 1 \), which implies that \( 1 - \epsilon = \frac{\ln(\lambda \tau)}{\ln(\lambda T)} \). Moreover, let us fix a distribution \( F \) satisfying the MHR condition. From Lemma 5 there exists a time interval \( I_{s, \tau} \) with starting time
s ∈ [0, T − τ] such that \( p_{M_e}(t) \) ∈ \( \left[ \frac{\mathbb{E}[X_{\lambda_T} \xi(s + \tau)(1-\epsilon)]}{\kappa}, \mathbb{E}[X_{\lambda_T}] \xi(s + \tau)(1-\epsilon) \right] \) for every \( t \in I_{s,\tau} \). We distinguish two cases, depending on whether the starting time of the interval is before or after the time \( t_0 \) characterizing mechanism \( M_e \) (as defined in Theorem 1).

**Case s < t_0.** By using the fact that the seller’s expected revenue for the overall time period is at least that achieved during the interval \( I_{s,\tau} \), we have:

\[
\mathbb{E}_F[\mathcal{R}(M_e)] \geq p_{M_e}(s + \tau) \mathbb{P}\{Y_{\lambda_T} \geq p_{M_e}(s)\} \\
\geq p_{M_e}(s + \tau) \mathbb{P}\{X_{\lambda_T} \xi(s + \tau) \geq \mathbb{E}[X_{\lambda_T}] \xi(s + \tau)(1-\epsilon)\} \\
= p_{M_e}(s + \tau) \mathbb{P}\{X_{\lambda_T} \geq \mathbb{E}[X_{\lambda_T}](1-\epsilon)\} \\
= p_{M_e}(s + \tau) \mathbb{P}\left\{X_{\lambda_T} \geq \mathbb{E}[X_{\lambda_T}] \frac{\ln(\tau)}{\ln(\lambda_T)}\right\} \\
\geq p_{M_e}(s + \tau) \mathbb{P}\{X_{\lambda_T} \geq \mathbb{E}[X_{\lambda_T}]\} \\
\geq p_{M_e}(s + \tau) \mathbb{E}[X_{\lambda_T}] \xi(s + \tau)(1-\epsilon) \\
\geq \frac{\mathbb{E}[X_{\lambda_T}] \xi(s + \tau)(1-\epsilon)}{\kappa \tau e}.
\]

Equation (12) holds since \( X_{\lambda_T} \xi(s + \tau) \) is a random variable representing the maximum initial valuation of agents arriving in a time interval of length \( \tau \) weighted by the maximum possible discount, and, thus, it is always smaller than or equal to \( Y_{\lambda_T} \).

Equation (13) follows from Lemma 5. Equation (14) follows from a result by Barlow and Marshall (1964), which implies that, for any MHR distribution, the probability of exceeding its expectation is at least \( \frac{1}{\tau e} \).

**Case s ≥ t_0.** In this case, we can lower bound the seller’s expected revenue for the overall time period with that obtained during the interval \( I_{t_0,\tau} \), as follows:

\[
\mathbb{E}_F[\mathcal{R}(M_e)] \geq p_{M_e}(t_0 + \tau) (1 - e^{-\lambda\tau}) \\
\geq \xi(t_0 + \tau) \frac{1}{e} \\
\geq \frac{\mathbb{E}[X_{\lambda_T}] \xi(t_0 + \tau)(1-\epsilon)}{\kappa \tau e},
\]

where for the first inequality we used the fact that the expected revenue in \( I_{t_0,\tau} \) is at least the lowest price posted during the interval times the probability that at least one agent arrives in \( I_{t_0,\tau} \), the second inequality holds since \( (1 - e^{-\lambda\tau}) \geq \frac{1}{e} \) when \( \lambda\tau \geq 1 - \ln(e - 1) \simeq 0.46 \), while the last inequality follows from the fact that \( s \geq t_0 \). Indeed, by Lemma 3, we can write the following:

\[
p_{M_e}(s + \tau) = \xi(s + \tau) \geq \frac{\mathbb{E}[X_{\lambda_T}] \xi(s + \tau)(1-\epsilon)}{\kappa \tau e},
\]

which implies that \( \frac{\mathbb{E}[X_{\lambda_T}] (1-\epsilon)}{\kappa \tau e} \leq 1 \).

We can now compute a lower bound on the ratio \( \rho_F(M_e) \) of mechanism \( M_e \), as follows:

\[
\rho_F(M_e) = \frac{\mathbb{E}_F[\mathcal{R}(M_e)]}{\mathbb{E}_F[\mathcal{R}(M^*)]} \\
\geq \frac{\mathbb{E}_F[\mathcal{R}(M_e)]}{\mathbb{E}_F[\mathcal{R}(Y_{\lambda_T})]} \\
\geq \frac{\mathbb{E}[X_{\lambda_T}] \xi(t_0 + \tau)(1-\epsilon)}{\mathbb{E}[Y_{\lambda_T}]} \frac{\kappa \tau e}{\kappa \tau e} \\
\geq \frac{\xi(t_0 + \tau)(1-\epsilon)}{\kappa \tau e}
\]

where the first inequality holds since \( \mathbb{E}[Y_{\lambda_T}] \) is the expected revenue of a mechanism that knows the actual realization of agents’ initial valuations and arrival times, i.e., the realization of variable \( Y_{\lambda_T} \). This mechanism achieves an expected revenue greater than or equal to that obtained by the benchmark \( M^* \), since the latter only knows the distribution of valuations. As for the second inequality, it is easy to see that \( \frac{\mathbb{E}[X_{\lambda_T}]}{\mathbb{E}[Y_{\lambda_T}]} \geq 1 \). Finally, by recalling the condition \( \lambda\tau = (\lambda T)^{1-\epsilon} \), we have \( \tau = T^{1-\epsilon} \lambda^{-\epsilon} \), which
allows us to write the following bound:
\[
\rho(M_{pc}) \geq \frac{\xi(t_0 + T^{1-\epsilon}\lambda^{-\epsilon})(1-\epsilon)}{\kappa \epsilon}.
\]
This concludes the proof.

Lemma 6. In the RV setting with agents’ initial valuations drawn from a distribution \( F \), given \( 0 < \epsilon < 1 \), there exists \( i = 1, \ldots, [\log_3 h] \) such that the price \( p_{M_{pc}}(I_i) \) posted by \( M_{pc} \) during the interval \( I_i \) lies in the range \( \left[ \frac{\nu}{3} \xi(\tau), \nu \xi(\tau) \right] \), where \( \nu := \max\{1, \mathbb{E}[X_{\lambda T}] (1-\epsilon)\} \).

Proof. In the following, for the ease of presentation, we let \( \tilde{I}_i := \left[ \frac{\nu}{3} \xi(\tau), \nu \xi(\tau) \right] \) for any \( i = 1, \ldots, [\log_3 h] \). By contradiction, suppose that there is no \( i = 1, \ldots, [\log_3 h] \) such that \( p_{M_{pc}}(I_i) \in \tilde{I}_i \). Notice that \( \nu \) is a lower bound on \( \mathbb{E}[X_{\lambda T}] \) and belongs to the range \( \in [1, h) \).

We reach a contradiction by employing an iterated reasoning. As a first step, we observe that either \( \nu \in \left( \frac{h}{3}, h \right) \) or \( \nu \in \left[ 1, \frac{h}{3} \right] \).

If \( \nu \in \left( \frac{h}{3}, h \right) \), then \( p_{M_{pc}}(I_1) = \frac{2}{3} \xi(\tau) \) is in the range \( \tilde{I}_1 = \left[ \frac{\nu}{3} \xi(\tau), \nu \xi(\tau) \right] \). Hence, it must hold \( \nu \in \left[ 1, \frac{h}{3} \right] \). Then, as a second step, we can conclude that either \( \nu \in \left( \frac{h}{3}, \frac{h}{5} \right) \) or \( \nu \in \left[ 1, \frac{h}{3} \right] \). If \( \nu \in \left( \frac{h}{3}, \frac{h}{5} \right) \), then \( p_{M_{pc}}(I_2) = \frac{\nu}{3} \xi(2\tau) \) is in the range \( \tilde{I}_2 = \left[ \frac{\nu}{3} \xi(2\tau), \nu \xi(2\tau) \right] \). Hence, it must hold \( \nu \in \left[ 1, \frac{h}{5} \right] \). By iterating the reasoning until the \( \lfloor \log_3 h \rfloor \)-th step, we obtain that either \( \nu \in \left( \frac{h}{\lfloor \log_3 h \rfloor}, \frac{h}{\lfloor \log_3 h \rfloor - 1} \right) \) or \( \nu \in \left[ 1, \frac{h}{\lfloor \log_3 h \rfloor} \right] \).

Let us first consider the case in which it holds \( \lfloor \log_3 h \rfloor \neq \lfloor \log_3 h \rfloor \). If \( \nu \in \left( \frac{h}{\lfloor \log_3 h \rfloor}, \frac{h}{\lfloor \log_3 h \rfloor - 1} \right) \), then \( p_{M_{pc}}(I_{\lfloor \log_3 h \rfloor}) \in \tilde{I}_{\lfloor \log_3 h \rfloor} \) since:
\[
\frac{h}{\lfloor \log_3 h \rfloor} \xi(\lfloor \log_3 h \rfloor \tau) \in \left[ \frac{\nu}{3} \xi(\lfloor \log_3 h \rfloor \tau), \nu \xi(\lfloor \log_3 h \rfloor \tau) \right].
\]
Hence, it must hold \( \nu \in \left[ 1, \frac{h}{\lfloor \log_3 h \rfloor} \right] \). Then, \( p_{M_{pc}}(I_{\lfloor \log_3 h \rfloor}) = \xi(\lfloor \log_3 h \rfloor \tau) \) belongs to the range \( \tilde{I}_{\lfloor \log_3 h \rfloor} = \left[ \frac{\nu}{3} \xi(\lfloor \log_3 h \rfloor \tau), \nu \xi(\lfloor \log_3 h \rfloor \tau) \right] \), which leads to a contradiction.

Now, suppose that \( \lfloor \log_3 h \rfloor = \lfloor \log_3 h \rfloor = \log_3 h \). Then, in the \( \lfloor \log_3 h \rfloor \)-th step of the iterated reasoning, we can conclude that \( \nu \in \left[ 1, \frac{h}{\lfloor \log_3 h \rfloor} \right] \) and \( p_{M_{pc}}(I_{\lfloor \log_3 h \rfloor}) = \xi(\lfloor \log_3 h \rfloor \tau) \) is in the range \( \tilde{I}_{\lfloor \log_3 h \rfloor} = \left[ \frac{\nu}{3} \xi(\lfloor \log_3 h \rfloor \tau), \nu \xi(\lfloor \log_3 h \rfloor \tau) \right] \), which leads to the final contradiction.

Theorem 4. Consider the RV setting with \( \lambda \tau = (\lambda T)^{1-\epsilon} \geq 1 - \ln(e-1) \) for some \( \tau \in (0, T] \) and \( 0 < \epsilon < 1 \). Then, restricted to the set \( F \) of distributions \( F \) satisfying the MHR condition, mechanism \( M_{pc} \) has a competitive ratio that can be lower bounded as follows:
\[
\rho(M_{pc}) \geq \frac{\xi(\lfloor \log_3 h \rfloor + 1)T^{1-\epsilon}\lambda^{-\epsilon}(1-\epsilon)}{\delta e}.
\]

Proof. By hypothesis we have \( \lambda \tau = (\lambda T)^{\tau} \) for \( \tau = \frac{\log \ln(e-1)}{\log h} \in (0, T] \). We distinguish two cases, depending on whether \( \mathbb{E}[X_{\lambda T}] (1-\epsilon) \) is greater or lower than one. Note that \( \mathbb{E}[X_{\lambda T}] (1-\epsilon) \) is a lower bound for \( \mathbb{E}[X_{\lambda T}] \) and that one is the minimum value that \( \mathbb{E}[X_{\lambda T}] \) can assume. In particular \( \mathbb{E}[X_{\lambda T}] = 1 \) when \( F \) is the point distribution such that \( P(V_i \leq 1) = P(V_i = 1) = 1 \).

Case \( \mathbb{E}[X_{\lambda T}] (1-\epsilon) \geq 1 \). For Lemma 6 there exists an \( i \in \{1, \ldots, \lfloor \log_3 h \rfloor \} \) such that the price \( p_i^* = p_{M_{pc}}(I_i) \) lies in the range \( \tilde{I}_i = \left[ \mathbb{E}[X_{\lambda T}] \frac{\xi(\tau)(1-\epsilon)}{\delta}, \mathbb{E}[X_{\lambda T}] \xi(\tau)(1-\epsilon) \right] \). By using the fact that the seller’s expected revenue for the overall time
period is at least that achieved during the interval $I_i$, we have:

$$
E[\mathcal{R}(\mathcal{M}_{pc})] \geq p^{*}_{PC}(Y_{X,T} \geq p^{*})
$$

$$
= p^{*}_{PC}(X_{X,T} \geq E[X_{X,T}])(1 - \epsilon)
$$

$$
= p^{*}_{PC}(X_{X,T} \geq E[X_{X,T}] ln(\lambda)/ln(\lambda T))
$$

$$
\geq p^{*}_{PC}(X_{X,T} \geq E[X_{X,T}])
$$

$$
= \frac{p^{*}_{PC}}{\epsilon}
$$

$$
\geq \frac{E[X_{X,T}]\xi(i\tau)(1 - \epsilon)}{\epsilon}
$$

$$
\geq \frac{E[X_{X,T}]\xi(\lfloor \log_{\delta} h \rfloor + 1)\tau(1 - \epsilon)}{\epsilon}
$$

Equation (15) holds since $X_{X,T}\xi(i\tau)$ is a random variable representing the maximum initial valuation of agents arriving in a time interval of length $\tau$ weighted by the maximum possible discount, thus it is always smaller than or equal to $Y_{X,T,i}$. Equation (16) follows from Lemma 5. Equation (17) follows from a result by Barlow and Marshall (1964), which implies that, for any MHR distribution, the probability of exceeding its expectation is at least $\frac{1}{\epsilon}$.

**Case** $E[X_{X,T}](1 - \epsilon) < 1$. In this case we can lower bound the seller’s expected revenue for the overall time period with that obtained during the interval $I_{\lfloor \log_{\delta} h \rfloor + 1}$, as follows:

$$
E_F[\mathcal{R}(\mathcal{M}_{pc})] \geq p^{*}_{PC}(1 - e^{-\lambda T})
$$

$$
\geq \xi(\lfloor \log_{\delta} h \rfloor + 1)\tau\frac{1}{\epsilon}
$$

$$
\geq \frac{E[X_{X,T}]\xi(\lfloor \log_{\delta} h \rfloor + 1)\tau(1 - \epsilon)}{\epsilon}
$$

where for the first inequality we used the fact that the expected revenue in $I_{\lfloor \log_{\delta} h \rfloor + 1}$ is the price posted during the interval times the probability that at least one agent arrives in $I_{\lfloor \log_{\delta} h \rfloor + 1}$, the second inequality holds since $(1 - e^{-\lambda T}) \geq \frac{1}{\epsilon}$ when $\lambda T \geq 1 - \ln(e - 1) \approx 0.46$, while the last inequality follows from the fact that $E[X_{X,T}](1 - \epsilon) < 1$ and $\delta \geq 1$.

We can now compute a lower bound on the ratio of the mechanism $\rho_F(\mathcal{M}_{pc})$, as follows:

$$
\rho_F(\mathcal{M}_{pc}) = \frac{E_F[\mathcal{R}(\mathcal{M}_{pc})]}{E_F[\mathcal{R}(\mathcal{M}^*)]}
$$

$$
\geq E_F[\mathcal{R}(\mathcal{M}_{pc})]
$$

$$
\geq \frac{E[X_{X,T}]\xi(\lfloor \log_{\delta} h \rfloor + 1)\tau(1 - \epsilon)}{\epsilon}
$$

$$
\geq \frac{E[X_{X,T}]\xi(\lfloor \log_{\delta} h \rfloor + 1)\tau(1 - \epsilon)}{\epsilon}
$$

where it is easy to see that $\frac{E[X_{X,T}]}{E[Y_{X,T}]} \geq 1$. By recalling the condition $\lambda T = (\lambda T)^{1-\epsilon}$, we have $\tau = T^{1-\epsilon}\lambda^{-\epsilon}$, which allows us to write the following bound:

$$
\rho_F(\mathcal{M}_{pc}) \geq \frac{\xi(\lfloor \log_{\delta} h \rfloor + 1)T^{1-\epsilon}\lambda^{-\epsilon}(1 - \epsilon)}{\epsilon}
$$

This concludes the proof. □

**An Analytical Expression of $F_{Y_{X,T}}$ for the RV Setting with Linear Discount**

We study the cumulative distribution function of the random variable $Y_{X,T}$ so as to unveil its dependence on $F$. We perform our analysis for the specific case of a linear discount function; thus:

$$
Y_{X,T} = \max_{i \in \{1, \ldots, N_T \}} V_i \left( 1 - \frac{W_i}{T} \right).
$$
The results presented in the following crucially rely on some properties of Poisson processes.

First, we introduce some auxiliary definitions and results.

**Proposition 3** (Ross et al. (1996)). The random variable $W_i$ representing the arrival time of agent $i$ has a Gamma distribution $\Gamma(i, \lambda)$, with shape parameter $i > 0$ and rate parameter $\lambda > 0$, whose probability density function is defined as follows:

$$f_{W_i}(w) := \frac{\lambda^i w^{i-1}}{(i-1)!} e^{-\lambda w}, \quad \text{for every } w \in [0, T].$$

**Theorem 5** (Pinsky and Karlin (2010)). Let $W_1, W_2, \ldots$ be random variables representing the arrival times in a Poisson process with rate parameter $\lambda > 0$. Conditioned on the event $N_T = n$, the variables $W_1, \ldots, W_n$ have a joint probability density function defined as follows:

$$f_{W_1, \ldots, W_n|N_T=n}(w_1, \ldots, w_n) = n! T^{-n}, \quad \text{for } 0 < w_1 < \ldots < w_n \leq T.$$

Intuitively, as discussed in (Ross et al. 1996), a consequence of Theorem 5 is that, conditioned on the event $N_T = n$, the times $W_1, \ldots, W_n$ at which the $n$ arrivals occur, considered as unordered random variables, are distributed uniformly and independently in the interval $[0, T]$. This is the crucial observation that allows to derive the following theorem.

**Theorem 6.** The random variable representing the maximum discounted valuation of agents arriving in the overall time period $[0, T]$ conditioned on the event that $N_T = n$ is defined as follows:

$$Y_{\lambda T|N_T=n} := \max_{i \in \{1, \ldots, n\}} V_i U_i, \quad \text{where } U_i \sim \mathcal{U}(0, 1).$$

**Proof.** Given the symmetry of the functional $\max_{i \in \{1, \ldots, N_T\}} V_i \left(1 - \frac{W_i}{T}\right)$ and Theorem 5, we can write the following:

$$\mathbb{P} \{ Y_{\lambda T} = y \mid N_T = n \} = \mathbb{P} \left\{ \max_{i \in \{1, \ldots, N_T\}} V_i \left(1 - \frac{W_i}{T}\right) = y \mid N_T = n \right\}$$

$$= \mathbb{P} \left\{ \max_{i \in \{1, \ldots, n\}} V_i \left(1 - \frac{\hat{U}_i}{T}\right) = y \right\}$$

where $\hat{U}_i$ is a random variable distributed according to $\mathcal{U}(0, T)$, which is a continuous uniform distribution with support $[0, T]$. Letting $U_i := \left(1 - \frac{\hat{U}_i}{T}\right)$, it is easy to show that $U_i \sim \mathcal{U}(0, 1)$. Formally, for every $x \in [0, 1]$, the cumulative distribution function $F_{U_i}$ of $U_i$ is defined as follows:

$$F_{U_i}(x) := \mathbb{P} \{ U_i \leq x \} = \mathbb{P} \left\{ \left(1 - \frac{\hat{U}_i}{T}\right) \leq x \right\} = \mathbb{P} \left\{ T(1 - x) \leq \hat{U}_i \right\}$$

$$= 1 - \mathbb{P} \left\{ \hat{U}_i \leq T(1 - x) \right\} = 1 - \frac{T(1-x)}{T} = x$$

Moreover, for $x < 0$ it holds $F_{U_i}(x) = 0$, while for $x > 1$ it holds $F_{U_i}(x) = 1$. Thus, $F_{U_i}$ is the cumulative distribution function of a random variable drawn from a uniform distribution with support $[0, 1]$.

In the following, we denote by $Z$ a product variable $V U$, where $V$ and $U$ are random variables distributed according to $F$ and $\mathcal{U}(0, 1)$, respectively. Moreover, we let $Z_i := V_i U_i$ be the variable $Z$ referred to a specific agent $i$. Theorem 6 allows us to express $F_{Y_{\lambda T}|N_T=j}$ as follows:

$$F_{Y_{\lambda T}|N_T=j}(x) = F_{\max_{i \in \{1, \ldots, j\}} Z_i}(x) = \mathbb{P} \left\{ \bigcap_{i=1}^{j} Z_i \leq x \right\} = \prod_{i=1}^{j} \mathbb{P} \{ Z_i \leq x \} = [F_Z(x)]^j.$$

Hence, we can write $F_{Y_{\lambda T}}$ as:

$$F_{Y_{\lambda T}}(x) = \sum_{j=1}^{\infty} \frac{(\lambda T)^j e^{-\lambda T}}{j!} [F_Z(x)]^j,$$

where

$$F_Z(x) = \begin{cases} xf_{1}^h \frac{1}{h}f(v)dv & \text{if } x \in [0, 1) \\ F(x) + x f_{1}^h \frac{1}{h}f(v)dv & \text{if } x \in [1, h] \end{cases} \quad (19)$$

We denote by $\mathcal{U}(a, b)$ a continuous uniform distribution over the interval $[a, b]$. 
Thus, it is easy to see that $F_{Y_\lambda}$ depends on $F$ and $f$, which are the cumulative distribution function and the probability density function of agents’ initial valuations, respectively.

It remains to show how to derive the expression of $F_Z$ in Equation 19. Notice that, since $U \sim U(0, 1)$, the probability density function of $U$ is defined as $f_U(u) = 1_{[0,1]}(u)$, while its cumulative distribution function is $F_U(u) = u 1_{[0,1]}(u)$. The support of $Z$ is $[0, h]$, being $V$ defined on $[1, h]$. Therefore,

$$F_Z(z) = P\{ V U \leq z \} = P\{ U \leq \frac{z}{V} \} =$$

$$= 1_{[0,1]}(z) \int_{D'} f(v) f_U(u) du + 1_{[1,h]}(z) \int_{D''} f(v) f_U(u) du =$$

$$= 1_{[0,1]}(z) \int_0^h f(v) f_U(u) du +$$

$$+ 1_{[1,h]}(z) \left( \int_1^z f(v) dv + \frac{1}{v} \int_0^1 f_U(u) du + \int_z^h f(v) \frac{v}{z} f_U(u) du \right)$$

where the domains of integration $D'$ and $D''$ are defined as:

$$D' := \{(u, v) : 0 \leq u \leq \frac{z}{v}, 1 \leq v \leq h\}$$

$$D'' := D''_1 \cup D''_2 := \{(u, v) : 0 \leq u \leq 1, 1 \leq v \leq z\} \cup \{(u, v) : 0 \leq u \leq \frac{z}{v}, z < v \leq h\}$$

See also Figure 6 for a graphical representation of the domains.

![Graphical representation of the domain of integration $D'$ and $D''$.](image)

Figure 6: Graphical representation of the domain of integration $D'$ and $D''$.

### Additional Experiments

We provide other empirical evaluations of our mechanisms in the RV setting. We compare $\mathcal{M}_C$, $\mathcal{M}_{PC}$, and ES0ES-SS when the distribution of the agents’ valuations is not MHR. Then, we show the performances of $\mathcal{M}_C$ and $\mathcal{M}_{PC}$ when valuations are linearly discounted over time.

**Result #3** We perform an experiment similar to that of Result #1. Here, agents’ valuations are drawn from a truncated normal distribution with $\mu = \frac{h-1}{2}$, $\sigma^2 = 2$, and support $[1, h]$. Figure 7 is similar to Figure 2. Observe that, in this setting, the performances of $\mathcal{M}_{PC}$ with $N_{sub} = 13$ and $\mathcal{M}_{PC}$ with $N_{sub} = 232$ are analogous. This means that, tuning the parameters in a suitable way, we can impose a time constraint with almost no loss in the normalized mean revenue. Moreover, the truncated normal distribution is not MHR, hence, all the bounds on the competitive ratio of the mechanism do not hold. Despite this fact, we see that, in this scenario, the behavior of the mean normalized revenue is comparable to that of Result #1. In particular, the loss of ES0ES-SS w.r.t. $\mathcal{M}_C$ averaged over the values of $\lambda$ is about 0.2 h when $T = 10$, and slightly larger when $T = 50$. 


Result #4  We analyze mechanisms $\mathcal{M}_C$ and $\mathcal{M}_{PC}$ with different $\text{Nsub}$ values when the valuations of the agents are linearly discounted. For every $\lambda$ we run 1000 Monte Carlo simulations, with $h = 10$. Given parameters $\lambda$ and $h$, we simulated the arrivals of agents drawn from a uniform distribution with support $[1, h]$ and we computed the revenue of the mechanisms. We normalized the results by $h$ and, for each value of $\lambda$, we average by the simulations. Then, for each value of $\lambda$, we plot in Figure 8 the normalized mean revenues of the mechanisms, for $T = 10$ and $T = 50$. We observe that $\mathcal{M}_C$ is no longer the best mechanism in terms of normalized mean revenue. The interesting fact is that, a suitably tuned mechanism $\mathcal{M}_{PC}$ can reach a better average revenue than $\mathcal{M}_C$ in some IV scenarios.