Number of Kummer structures and Moduli spaces of generalized Kummer surfaces

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Abstract
A generalized Kummer surface $X = \text{Km}_3(A, G_A)$ is the minimal resolution of the quotient of a 2-dimensional complex torus by an order 3 symplectic automorphism group $G_A$. A Kummer structure on $X$ is an isomorphism class of pairs $(B, G_B)$ such that $X \cong \text{Km}_3(B, G_B)$. When the surface is algebraic, we obtain that the number of Kummer structures is linked with the number of order 3 elliptic points on some Shimura curve naturally related to $A$. For each $n \in \mathbb{N}$, we obtain generalized Kummer surfaces $X_n$ for which the number of Kummer structures is $2^n$. We then give a classification of the moduli spaces of generalized Kummer surfaces. When the surface is non algebraic, there is only one Kummer structure, but the number of irreducible components of the moduli spaces of such surfaces is large compared to the algebraic case. The endomorphism rings of the complex 2-tori we study are mainly quaternion orders, these orders contain the ring of Eisenstein integers. One can also see this paper as a study of quaternion orders $\mathcal{O}$ over $\mathbb{Q}$ that contain the ring of Eisenstein integers. We obtain that such order is determined up to isomorphism by its discriminant, and when the quaternion algebra is indefinite, the order is principal.

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1 Introduction

A generalized Kummer surface $X = \text{Km}_3(A, G_A)$ (or simply $\text{Km}_3(A)$) is the minimal resolution of the quotient $A/G_A$ of a 2-dimensional complex torus $A$ by an order 3 symplectic automorphism group $G_A$. Let us recall that the quotient surface $A/G_A$ has 9 $A_2$-singularities (9 cusps); the minimal resolution of each cusp is by an $A_2$-configuration, which means two smooth rational curves $C, C'$ in $X$ with intersection $CC' = 1$.

A generalized Kummer structure of $X$ is an isomorphism class of pairs $(B, G_B)$ of complex 2-dimensional torus and order 3 symplectic automorphism group $G_B \subset \text{Aut}(B)$ such that $\text{Km}_3(B, G_B) \cong X$. The generalized Kummer structures on $X$ are in bijective corre-
dence with the orbits of 9A2-configurations on X under Aut(X). Some particular family of
generalized Kummer surfaces is studied in [23], and in [34] some new 9A2-configurations are
constructed, which permit to better understand the automorphism group of X.

In the present paper, we investigate the number $N_{KS}(X)$ of generalized Kummer structures
on X, and the moduli spaces of these surfaces. While for non-algebraic complex tori, it is
easy to see that $N_{KS}(X) = 1$, the situation for abelian surfaces is more complicated. In order
to compute the numbers $N_{KS}(X)$, we use two constructions of abelian surfaces with an order
3 symplectic automorphism group.

The first construction, by Barth [4], is of Hodge-theoretic nature and deals with 2-
dimensional complex tori (for short complex 2-tori) A, algebraic or not. Barth studies the
action of the order 3 automorphism group $G_A$ on the lattice $H^2(A, \mathbb{Z})$. The Picard number
$\rho_X$ of the generalized Kummer surface $X = \text{Km}_3(A)$ is computed to be 18, 19 or 20. If
$\rho_X = 18$, the surface is non-algebraic, if $\rho_X = 20$ the surface is algebraic, but is isolated
in its moduli space. The main case of interest for us will be when X has Picard number 19.
Then the orthogonal complement in the Néron–Severi group of X of the 18 curves above the
9 cusps is generated by a class $L_X$. One has $L^2_X = 0$ or 2 mod 6 and for any integer $\ell$ in $\mathbb{Z}$
congruent to 0 or 2 mod 6, there exists a generalized Kummer surface $X = \text{Km}_3(A)$ with
$L^2_X = \ell$; such a surface X is algebraic if and only if $L^2_X > 0$. We obtain that

**Theorem 1** Consider $X = \text{Km}_3(A)$ and suppose that $\rho_X = 19$. The Néron–Severi group
$\text{NS}(A)$ of the complex 2-torus A is generated by three divisors with intersection matrices

$$
\begin{pmatrix}
\frac{1}{3}(L^2_X - 2) & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
\frac{1}{3}L^2_X & 0 & 0 \\
0 & -2 & 1 \\
0 & 0 & -2
\end{pmatrix}
$$

according if $L^2_X = 2$ mod 6 or $L^2_X = 0$ mod 6. If $L^2_X \neq 0$, the endomorphism ring $\text{End}(A)$
of A is an order in a quaternion algebra over $\mathbb{Q}$.

The above Theorem is proved in Sect. 2 (see Theorems 7 and 20). The ring $\text{End}(A)$ contains the ring of Eisenstein integers (coming from the order 3 automorphism group). The quaternion algebra $\text{End}(A) \otimes \mathbb{Q}$ is indefinite (i.e. there exists an embedding in $M_2(\mathbb{R})$) if and only if $L^2_X > 0$. Let $A$ be the dual complex torus to A. Using the knowledge of $\text{NS}(A)$, we obtain in Theorem 17 the following result:

**Theorem 2** Let $(B, G_B)$ be a generalized Kummer structure on an algebraic generalized
Kummer surface $X = \text{Km}_3(A)$. Then B is isomorphic to A or its dual $\hat{A}$. The abelian surface $A$ admits a principal polarization (and therefore $A \simeq \hat{A}$) if and only if $L^2_X = 2$ mod 6, or
3||$L^2_X$.

Here the symbol $a||n$ means that a divides the integer n and the integers $a, \frac{n}{a}$ are coprime.

The second construction of complex 2-tori, the Shimura construction [38], is of arithmetic
nature and will enable us to compute the number $N_{KS}(X)$ of Kummer structures effectively.
Let A be a complex 2-torus with an order 3 symplectic automorphism $J_A$ and with Picard
number $\rho_A = 3$. The endomorphism ring of $\hat{A}$ is an order in a quaternion algebra $H = \frac{(a,b)}{\mathbb{Q}}$
over $\mathbb{Q}$, where $H$ is generated by $\alpha, \beta$, with $\alpha^2 = a, \beta^2 = b$ and $a\beta = -\beta a$. Since the ring
$\text{End}(A)$ contains the ring $\mathbb{Z}[J_A]$ of Eisenstein integers, there are restrictions on $H$: one can
write $H$ as $H = \frac{(-3, dh)}{\mathbb{Q}}$, where $d_H$ is equal to the discriminant $D_H$ of $H$ or to $\frac{1}{3}D_H$ according
if $3 \nmid D_H$ or not. The integer $d_H$ is a square-free product of primes $p$ with $p = 2$ mod 3.
We obtain in Sect. 3.3 that any quaternion order $\mathcal{O}$ containing the ring of Eisenstein integers
$\mathbb{Z}[j]$ (with $j^2 + j + 1 = 0$) is either isomorphic to a ring of the form

$$
\mathcal{O}_\mu = \mathbb{Z}[j] \oplus \mathbb{Z}[j] \mu \phi,
$$

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for a quaternion $ϕ$ with $jϕ = φj$, such that $φ^2 = dH$, and for some element $μ ∈ Z[j]$, or $𝒪$ contains an order $𝒪_μ$ with index 3, for some $μ ∈ Z[j]$. We obtain moreover that up to conjugation, there is a unique maximal order containing $j$. Conversely, by the results of Shimura and Shimizu, for a given order $𝒪$, there exist complex tori $A$ having quaternionic multiplication by $𝒪$.

To an order $𝒪$ in an indefinite quaternion algebra, it is classically associated the Shimura curve $H/Γ(O^1)$, which is the quotient of the upper half plane by the homorphic action of the group of norm 1 elements of $𝒪$. It is an irreducible component of the coarse moduli space for abelian surfaces with quaternionic multiplication by $𝒪$. The elliptic points of order 3 of $H/Γ(O^1)$ are the ramification points of degree three of the natural quotient map $H → X(𝒪)$. The number $e_3(𝒪)$ of elliptic points of order 3 is the number of orbits of order 3 groups in $𝒪^∗$ under the conjugation by the index $≤ 2$ sub-group $𝒪_1 ⊂ O^∗$ of norm 1 element. We obtain the following results in Theorems 40 and 43:

**Theorem 3** Consider $𝒪 = \text{End}(A)$. Suppose that $𝒪$ is contained in a skew-field. The number $N_{KS}(X)$ of generalized Kummer structures on the algebraic surface $X = Km_3(phis)$ is

$$
\frac{1}{2}e_3(𝒪) \quad \text{if } L_X^2 = 2 \text{ mod } 6 \text{ or } 3|L_X^2 \text{ but } 9 ∤ L_X^2, \\
2e_3(𝒪) \quad \text{if } L_X^2 = 18 \text{ mod } 54, \\
≤ 2e_3(𝒪) \quad \text{if } L_X^2 = 18 \text{ or } 36 \text{ mod } 54.
$$

The ring $𝒪$ is principal: the set of left $𝒪$-ideals modulo the principal ideals has a unique element.

Let us give an example: let $μ ∈ Z[j]$ such that $N = μ\bar{μ}$ is a square-free integer coprime to $D_H$ and to 3, such that the primes dividing $N$ are congruent to 1 mod 3, with $H$ indefinite. Then $𝒪_μ$ is an Eichler order of level $N$, and if we take $A$ such that $\text{End}(A) ≃ O_μ$, we obtain that $N_{KS}(X) = 2^{m+ε}$, where $ε ∈ \{-1, -2\}$ and $m$ is the number of primes dividing $D_HN$ (we suppose here that $D_H ≠ 1$). In particular, the number of Kummer structures can be made arbitrarily large, and in two ways: by varying the indefinite quaternion algebra $H$ or the level $N$. In [12] with D. Cartwright, we compute the number $e_3(𝒪)$ for the orders $𝒪$ such that $3|L_X^2$.

For $ℓ ∈ Z$ an integer such that $ℓ = 0$ or 2 mod 6, let $M_ℓ$ be the moduli space of generalized Kummer surfaces $X = Km_3(phis)$ with the class $L_X ∈ N_{S}(X)$ such that $L_X^2 = ℓ$. The moduli space $M_ℓ$ is one dimensional; when $ℓ$ is positive, it is dominated by some Shimura curve by a map of degree $N_{KS}(X)$, and when $ℓ$ is negative, the space $M_ℓ$ is isomorphic to $\mathbb{P}^1$. The following result is in Theorem 43 and the Section 5.4; it gives an answer to a question of Barth [4, Section 2.4, Problem]:

**Theorem 4** For $ℓ > 0$, the moduli space $M_ℓ$ is irreducible. The number of irreducible components of the moduli space $M_ℓ$ goes to $∞$ when $ℓ$ tends to $−∞$.

The assertion on $M_ℓ$ with $ℓ < 0$, although not difficult to prove, makes an interesting contrast with the case $ℓ > 0$; in fact we obtain much more precise informations on these moduli spaces (see below). For the proof of Theorem 4, we use some results of Miranda and Morrison [26] on embeddings of lattices, and some results of Hosono, Lian, Oguiso and Yau [19, 20].

In [10], Bonfanti and van Geemen also studied abelian surfaces $A$ with an order 3 symplectic automorphism group, in order to obtain concrete examples of Shimura curves. Some of the results on $NS(A)$, $\text{End}(A)$, which we mention here in case $L_X^2 = 0$ mod 6 (for $X = Km_3(phis)$)
algebraic), were previously obtained by them (see also [9], by Bonfanti). In [16], Elkies also studies Shimura curves which parametrize the (classical) Kummer surfaces, associated to abelian surfaces with quaternionic multiplications. Using the moduli space of generalized Kummer surfaces, it seems possible to obtain models of Shimura curves.

The paper is structured as follows. In Sect. 2, we recall the link between generalized Kummer structures on $X = \text{Km}_3(A)$ and Fourier–Mukai partners of $A$, when $A$ is algebraic. If $A$ is not algebraic, the surface $X$ has a unique Kummer structure. For any complex torus $A$ with Picard number 3 with a symplectic order 3 automorphism group $G_A$, we study the class $L_X \in \text{NS}(X)$ which comes from the invariant class $L_A \in \text{NS}(A)$ under the action of $G_A$, and we describe the Néron–Severi group of $A$. We prove that, in the algebraic case, the Néron–Severi group is unique in its genus, and if $(B, G_B)$ is a Kummer structure on $X = \text{Km}_3(A)$, then $B$ is isomorphic to $A$ or its dual $\hat{A}$. For any complex 2-tori $A$ with an order 3 automorphism and Picard number 3, we then give generators of the endomorphism ring of $A$ and we obtain that $\text{End}(A)$ is an order in a quaternion algebra (when $L_X^2 \neq 0$); we express the discriminant of the order $\text{End}(A)$ as a function of $L_X^2$. In Sect. 3, after some preliminaries on quaternion, we describe -up to isomorphisms- all orders of quaternion algebras which contain the ring of the Eisenstein integers. We recall the construction by Shimura of abelian surfaces with quaternionic multiplication, and its analogue for the non-algebraic case. In Sect. 4, we join the results of Sect. 2 and Sect. 3, and obtain that the integer $L_X^2$ determines the quaternion algebra and the endomorphism ring of $A$. We then compute the number $\text{N}_K, S(X)$ of Kummer structures on $X$. In Sect. 5, we study the number of irreducible components of the moduli space $\mathcal{M}_\ell$ of generalized Kummer surfaces $X$ with fixed $\ell = L_X^2$. For each $\ell < 0$, we can describe the irreducible components of $\mathcal{M}_\ell$, using the reflexive rank 4 lattice $U \oplus A_2$ introduced by Barth in [4] for studying generalized Kummer surfaces, and studied by Vinberg in [43].

2 Hodge theory: Néron–Severi, transcendental lattice of $A$

2.1 Notations, conventions

Given two integers $a$, $b$, we write $a|b$ if $a$ divides $b$ and $a$ is coprime to $b/a$.

We take the rather useful convention of Conway and Sloane [13, Chapter 15] that $-1$ is a prime, and $\mathbb{Q}_{-1} = \mathbb{R}$; we will indicate when we use that convention.

For a lattice $\mathbb{L}$ and an integer $n$, $\mathbb{L}(n)$ denotes the group $\mathbb{L}$ with the intersection form of the lattice multiplied by $n$. The lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$) is denoted by $U$ (respectively by $A_2$).

The Néron–Severi group of a surface $X$ is denoted by $\text{NS}(X)$, it is the sub-lattice of classes of divisors in $H^2(X, \mathbb{Z})$. The Picard number of $X$, which is the rank of $\text{NS}(X)$, is denoted by $\rho_X$. The set of numerical equivalence classes is denoted by $\text{Num}(X)$.

If $X$ is a K3 surface, we will often (by abuse) identify a $(-2)$-curve $C$ (i.e. a smooth rational curve contained in $X$ such that $C^2 = -2$) and its class $[C]$ in $\text{NS}(X)$. An $A_2$-configuration of $(-2)$-curves on a K3 surface is the data of two $(-2)$-curves $C, C'$ such that $CC' = 1$.

A standard reference for basics about generalized Kummer surfaces is the article [4] of Barth; one also can refer to [3, 10, 34]. In the following $A$ is a complex 2-torus and $G_A$ is a symplectic automorphism group of order 3, generated by an element $J_A$. Then $X = \text{Km}_3(A, G_A)$ (sometimes also denoted by $X = \text{Km}_3(A, J_A)$) is the associated generalized
Kummer surface. It is a K3 surface which is the minimal resolution of the quotient $A/G_A$. When there is no confusion, we sometimes denote simply $X = \text{Km}_3(A)$.

The surface $X = \text{Km}_3(A)$ has Picard number $\rho_X = 16 + \rho_A \geq 18$. When $\rho_A = 3$, we denote by $L_A$ the generator of the invariant part under $G_A$ of $\text{NS}(A)$, and by $L_X$ the generator of the orthogonal complement of the $18$ $(-2)$-curves of the resolution $X \to A/G_A$. When $A$ is algebraic, we take $L_X, L_A$ to be nef.

2.2 The Néron–Severi lattice of $A$

Let $A$ be a complex 2-torus such that there exists an order 3 symplectic automorphism $J_A \in \text{Aut}(A)$. We suppose moreover that $A$ has Picard number 3 and we denote by $L_A$ the generator of $\text{NS}(A)^{J_A}$, the invariant sub-lattice for $J_A$ (we take it ample if $L_A^2 > 0$). Let $X$ be the associated generalized Kummer surface (the desingularization of $A/J_A$) and let $L_X$ be the generator of the orthogonal complement of its natural $9A_2$-configuration (we take $L_X$ nef if $L_X^2 > 0$). Let

$$\pi : A \to A/J_A$$

be the quotient map and $q : X \to A/J_A$ be the minimal resolution. Since $L_A$ is $J_A$-invariant, there exist integers $u_0, v_0$ such that

$$\pi^*q_*L_X = u_0L_A, \quad \pi_*L_A = v_0q_*L_X$$

If $L_A^2 > 0$ these integers are positive since $L_A$ and $L_X$ are nef, otherwise, up to exchanging $L_X$ with $-L_X$, we can suppose that they are also positive. Since moreover

$$\pi_*\pi^*(q_*L_X) = 3q_*L_X,$$

we get $u_0v_0 = 3$, thus $u_0 \in \{1, 3\}$. If $u_0 = 1$, then

$$L_X^2 = \frac{1}{3}L_A^2$$

and if $u_0 = 3$, then

$$L_X^2 = 3L_A^2 = 0 \mod 6.$$

By [4, Section 1.2], the fixed sub-lattice $H_2(A, \mathbb{Z})^{J_A}$ (containing $L_A$) is isomorphic to $U \oplus A_2$ i.e. there exists a basis $\gamma_1, \ldots, \gamma_4$ with Gram matrix:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
$$

(2.1)

Proposition 5 There exist coprime integers $n_1, \ldots, n_4$ such that:

(i) if $L_X^2 = 2 \mod 6$, one has

$$L_A = \pi^*q_*L_X = 3n_1\gamma_1 + 3n_2\gamma_2 + 3n_3\gamma_3 + n_4(\gamma_3 + \gamma_4),$$

with $n_4 \neq 0 \mod 3$ (thus $u_0 = 1$), and then

$$L_X^2 = 6(n_1n_2 + n_3^2 + n_3n_4) + 2n_4^2 = \frac{1}{3}L_A^2,$$
(ii) if \( L_X^2 = 0 \mod 6 \), one has
\[
3L_A = \pi^* q_u L_X = 3(n_1 \gamma_1 + n_2 \gamma_2 + n_3 \gamma_3 + n_4(\gamma_3 + \gamma_4)),
\]
(thus \( u_0 = 3 \)) with \( \gcd(n_1, n_2, n_3, 3) = 1 \), and then
\[
L_X^2 = 6(n_1n_2 + n_3^2 + 3n_3n_4 + 3n_4^2) = 3L_A^2.
\]

**Remark 6** Conversely the choice of such integers \((n_1, \ldots, n_4)\) defines a class \( L \), which is the order of its discriminant group are as follows:

\[
\gcd \quad \text{then}
\]

By [4, Section 1.3, Corollary], and the end of [4, Section 1.2], the primitive class \( L_X \) is given by

\[
L_X = n_1 \xi_1 + \cdots + n_4 \xi_4.
\]

where the integers \( n_1, \ldots, n_4 \) are coprime, and \( \xi_1, \ldots, \xi_4 \) is a basis of the orthogonal in \( H^2(X, \mathbb{Z}) \) of the 18 \((-2)\)-curves above the 9 cusp singularities of \( A/J_A \). The basis is such that \( \pi^* q_u \xi_i = 3\gamma_i \) for \( i \leq 3 \) and \( \pi^* q_u \xi_4 = \gamma_3 + \gamma_4 \), thus:

\[
\pi^* q_u L_X = 3n_1 \gamma_1 + 3n_2 \gamma_2 + 3n_3 \gamma_3 + n_4(\gamma_3 + \gamma_4).
\]

Therefore if \( n_4 \not\equiv 0 \mod 3 \), the integers \( n_1, \ldots, n_4 \) being coprime, the class \( \pi^* q_u L_X \) is primitive, and we have \( L_A = \pi^* q_u L_X \). If \( n_4 = 3n'_4 \) for some \( n'_4 \), we must have \( \gcd(n_1, n_2, n_3, 3) = 1 \) since we suppose that the class \( L_X \) is primitive. Then we have

\[
\pi^* q_u L_X = 3 \left( n_1 \gamma_1 + n_2 \gamma_2 + n_3 \gamma_3 + n'_4(\gamma_3 + \gamma_4) \right),
\]

thus \( L_A = \frac{1}{3} \pi^* q_u L_X \) and the result follows by formally replacing \( n'_4 \) by \( n_4 \) in the notations, since we will not be using again the class \( L_X \) in basis \( \xi_1, \ldots, \xi_4 \).

According to [4, Section 2.4, first Proposition], the Néron–Severi lattice of \( A \) contains the lattice \( M_0 \) generated by the invariant class \( L_A \) and classes \( \delta_1, \delta_2 \), such that \( L_A, \delta_1, \delta_2 \) have Gram matrix

\[
Gr = \begin{pmatrix}
L_A^2 & 0 & 0 \\
0 & -2 & 1 \\
0 & 1 & -2
\end{pmatrix}.
\]

The classes \( \delta_1, \delta_2 \) are non-invariant for the action of \( J_A \), in fact: \( \delta_2 = J_A^* \delta_1 \). Let us prove:

**Theorem 7** According to the cases \( L_X^2 = 2 \) or \( 0 \mod 6 \), the Néron–Severi lattice of \( A \) and the order of its discriminant group are as follows:

| NS(A) | \( L_X^2 \) | \( L_A^2 \) | \( \text{|Disc(NS(A))|} \) |
|-------|----------------|----------------|------------------|
| (i)   | \(
\begin{pmatrix}
2k & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{pmatrix}
\) | \( L_X^2 = 6k + 2 \) | \( L_A^2 = 18k + 6 = 3L_X^2 \) | \( L_X^2 \) |
| (ii)  | \(
\begin{pmatrix}
2k' & 0 & 0 \\
0 & -2 & 1 \\
0 & 1 & -2
\end{pmatrix}
\) | \( L_X^2 = 6k' \) | \( L_A^2 = 2k' = \frac{1}{3}L_X^2 \) | \( L_X^2 \) |

In case i), the lattice NS(A) is generated by \( D_A = \frac{1}{3}(L_A - (2\delta_1 + \delta_2)) \) and \( \delta_1, \delta_2 \), moreover the discriminant group of NS(A) is cyclic of order \( L_X^2 = \frac{1}{3}L_A^2 \). In case ii), the Néron–Severi lattice of \( A \) is generated by \( L_A \) and \( \delta_1, \delta_2 \).
**Remark 8** As observed by Barth, for any integer \( \ell \) with \( \ell = 6k + 2 \) or \( \ell = 6k \) (for some \( k \in \mathbb{Z} \)), there exists a generalized Kummer surface \( X \) with \( L_X^2 = \ell \). Indeed, according if \( L_X^2 = 2 \mod 6 \) or \( L_X^2 = 0 \mod 6 \), one has

\[
L_X^2 = 6(n_1n_2 + n_3^2 + n_3n_4) + 2n_4^2 \text{ or } L_X^2 = 6(n_1n_2 + n_3^2 + 3n_3n_4 + 3n_4^2).
\]

By taking \((n_1, \ldots, n_4) = (k, 1, 0, 1)\) or \((k, 1, 0, 0)\), we get all possible \( L_X^2 \).

**Proof** (of Theorem 7). The discriminant group of the lattice \( M_0 \) is generated by \( \frac{1}{3}L_A, \frac{1}{3}(\delta_1 + 2\delta_2) \); it is isomorphic to \( \mathbb{Z}/L_A^2 \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \). Since \( L_A \) is primitive, the only possibility for having \( \text{NS}(X) \neq M_0 \) is that \( 3|L_A^2 \) and to construct a class in \( \text{NS}(A) \) with the elements \( \frac{1}{3}L_A \) and \( \frac{1}{3}(\delta_1 + 2\delta_2) \) of the discriminant group. In [4, Sections 1.2 & 2.2], Barth expresses the classes \( \delta_1, \delta_2 \) and \( L_A \) in the basis

\[
\alpha_1 \wedge \alpha_2, \alpha_1 \wedge \beta_1, \alpha_1 \wedge \beta_2, \alpha_2 \wedge \beta_1, \alpha_2 \wedge \beta_2, \beta_1 \wedge \beta_2
\]

of \( H^2(A, \mathbb{Z}) \), where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are generators of \( H_1(A, \mathbb{Z}) \), and we have

\[
\delta_1 = \alpha_1 \wedge \alpha_2 - \beta_1 \wedge \beta_2,
\]

\[
\delta_2 = \alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1 + \beta_1 \wedge \beta_2.
\]

Suppose that \( L_X^2 = 2 \mod 6 \). Then \( 3|L_A^2 \) and by Lemma 5:

\[
L_A = 3n_1\gamma_1 + 3n_2\gamma_2 + 3n_3\gamma_3 + n_4(\gamma_3 + \gamma_4)
\]

with \( n_4 \neq 0 \mod 3 \). By [4, Section 1.2], we have

\[
\gamma_1 = -\alpha_1 \wedge \beta_1
\]

\[
\gamma_2 = \alpha_2 \wedge \beta_2
\]

\[
\gamma_3 = \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1
\]

\[
\gamma_4 = \alpha_1 \wedge \alpha_2 + \alpha_1 \wedge \beta_2 + \beta_1 \wedge \beta_2
\]

and therefore in \( H^2(A, \mathbb{Z}/3\mathbb{Z}) \), we have

\[
L_A - (\delta_1 + 2\delta_2) = (n_4 - 1)\alpha_1 \wedge \alpha_2 + (2n_4 - 2)\alpha_1 \wedge \beta_2 + (n_4 + 2)\alpha_2 \wedge \beta_1
\]

\[
+ (n_4 - 1)\beta_1 \wedge \beta_2,
\]

\[
L_A - (2\delta_1 + \delta_2) = (n_4 - 2)\alpha_1 \wedge \alpha_2 + (2n_4 - 1)\alpha_1 \wedge \beta_2 + (n_4 + 1)\alpha_2 \wedge \beta_1
\]

\[
+ (n_4 + 1)\beta_1 \wedge \beta_2.
\]

We conclude that the class \( \frac{1}{3}(L_A - (\delta_1 + 2\delta_2)) \) (respectively \( \frac{1}{3}(L_A - (2\delta_1 + \delta_2)) \)) is in the Néron–Severi lattice if and only if \( n_4 = 1 \mod 3 \) (respectively \( n_4 = 2 \mod 3 \)). We take the freedom to permute the role of \( \delta_1, \delta_2 \) if necessary, so that we have the intersection matrix given in the statement of Theorem 7, for both cases \( n_4 = 1 \) and \( n_4 = 2 \mod 3 \). That proves the result when \( L_X^2 = 2 \mod 6 \). Suppose now that \( L_X^2 = 0 \mod 6 \). If \( 3 \nmid L_X^2 \), since \( L_A \) is primitive, the Néron–Severi group must be generated by \( L_A, \delta_1, \delta_2 \). Suppose \( 3|L_A^2 \) (then \( 9|L_X^2 \)); since by Proposition 5 we have

\[
L_A = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n_4(\gamma_3 + \gamma_4)
\]

and we know the classes \( \gamma_1 \), one can check that the only possibility for having a larger lattice is that \( n_1 = n_2 = n_3 = 0 \mod 3 \) and \( n_4 \neq 0 \mod 3 \). But by Proposition 5, case ii), the integers \( n_1, n_2, n_3 \) must be coprime to 3. \( \square \)
2.3 The genus of the Néron–Severi lattice of $A$

In this section, we suppose that the surfaces $X$, $A$ are algebraic.

For an even lattice $S$, we denote by $A_S$ the discriminant group $S^*/S$ and by $q_{A_S} : A_S \to \mathbb{Q}/2\mathbb{Z}$ the quadratic form on $A_S$ induced by the bilinear form on $S$. The genus $\mathcal{G}(S)$ of $S$ is the set of isomorphism classes of lattices $S'$ of same rank, same signature, and with isometric discriminant group: $(A_S, q_{A_S}) \simeq (A_S', q_{A_S'})$; it is a finite set. Let us denote by $O(A_S)$ the automorphism group of $A_S$ preserving the quadratic form $q_{A_S}$. The orthogonal group $O(S)$ acts on $A_S$ through the natural morphism $O(S) \to O(A_S)$. One has:

**Theorem 9** Let $X = \text{Km}_3(A)$ be an algebraic Kummer surface. The genus of the lattice $\text{NS}(A)$ is $\{\text{NS}(A)\}$ and the natural map $O(\text{NS}(A)) \to O(\text{NS}(A))$ is surjective.

**Proof** Let us recall that, by definition, the length of a finite abelian group $A$ is the minimal number of a generating set of $A$. If $L_X^2 = 2 \mod 6$, the discriminant group of the rank 3 lattice $\text{NS}(A)$ is cyclic, thus it has length $\ell = 1$. If $3|L_X^2$ but $9 \not|L_X^2$, then $L_A^2 = \frac{1}{3}L_X^2$ is coprime to 3 and therefore the discriminant group of $\text{NS}(A)$ (of order $3L_A^2 = L_X^2$) has also length $\ell = 1$. Then in these two cases, the inequality

$$\text{rank}(\text{NS}(A)) = 3 \geq 3 + \ell$$

is satisfied and we can apply [29, Theorem 1.14.2] to conclude that the genus of $\text{NS}(A)$ is $\{\text{NS}(A)\}$ and that the natural map $O(S) \to O(\text{NS}(A))$ is surjective for $S = \text{NS}(A)$.

Suppose now that $9|L_X^2$; the quadratic form associated to $\text{NS}(A)$ is (in the basis $L_A$, $\delta_1$, $\delta_2$):

$$Q(x, y, z) = 2(3kx^2 - y^2 + yz - z^2),$$

(for $k = \frac{1}{9}L_X^2 > 0$) and the discriminant group $\text{NS}(A)$ is (isomorphic to) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3k\mathbb{Z}$. By [25, Chapter VIII, Lemma 7.7(1)], the quadratic form $Q$ is 2-regular, by [25, Chapter VIII, Lemma 7.6(2)] $Q$ is 3-semiregular, and by [25, Chapter VIII, Lemma 7.6(1)] it is $p$-regular for any prime $p \geq 5$. One can therefore apply [25, Chapter VIII, Theorem 7.5] to conclude that also in that case the genus of $\text{NS}(A)$ is $\{\text{NS}(A)\}$ and that the natural map $O(\text{NS}(A)) \to O(\text{NS}(A))$ is surjective. 

\[\square\]

2.4 The transcendental lattice of $A$

Let us recall that Proposition 5 gives the $J_A$-invariant class of $L_A$ in the basis $\gamma_1, \ldots, \gamma_4$ of the sub-lattice $H^2(A, \mathbb{Z})^{J_A} \simeq U \oplus A_2$ which is invariant under the order 3 automorphism $J_A$. The transcendental lattice $T(A) = \text{NS}(X)^{1}$ of $A$ is also the orthogonal complement of $L_A$ in $H^2(A, \mathbb{Z})^{J_A}$. We have:

**Corollary 10** Suppose that $A$ is algebraic. The lattice $T(A)(-1)$ is isomorphic to $\text{NS}(A)$, in particular $T(A)$ is unique in its genus.

**Proof** The lattices $\text{NS}(A)$ and $T(A)$ are orthogonal sub-lattices in $H^2(A, \mathbb{Z})$, which is isomorphic to $U^{\oplus 3}$ and unimodular, thus their discriminant groups $\text{NS}(A)$ and $\text{NS}(A)$ must be isomorphic, and the quadratic forms on $\text{NS}(A)$ and $\text{NS}(A)$ have the opposite sign. Moreover these two lattices are of the same rank, with respective signatures $(1, 2)$, $(2, 1)$. These conditions imply that $T(A)(-1)$ must be in the genus of $\text{NS}(A)$. Since by Theorem 17 the genus contains only $\text{NS}(A)$, the lattice $T(A)(-1)$ is isomorphic to $\text{NS}(A)$. 

\[\square\]
2.5 Fourier–Mukai partners and generalized Kummer structures

2.5.1 Integral Hodge structures, Fourier–Mukai partner, dual abelian surface

Recall that for a K3 surface (resp. a complex torus) $X$, a Fourier–Mukai partner of $X$ is a K3 surface (resp. a complex torus) $Y$ such that there is an isomorphism of Hodge structures

$$(T(Y), \mathbb{C}\omega_Y) \simeq (T(X), \mathbb{C}\omega_X),$$

where $\omega_X$ is a generator of $H^0(X, K_X)$, and $T(X) = \text{NS}(X)^\perp$ is the transcendental lattice. The set of isomorphism classes of Fourier–Mukai partners of $X$ is denoted by $\text{FM}(X)$.

Let $A$ be a complex torus, let $G_A$ be an order 3 automorphism group of $A$ acting symplectically and let $X = \text{Km}_3(A, G_A)$ be the associated generalized Kummer surface. An isomorphism class of pairs $(B, G_B)$ where $B$ is a complex torus (and $G_B$ an order 3 symplectic automorphism group) such that $\text{Km}_3(B, G_B) \simeq X$ is called a generalized Kummer structure on $X$. We denote by $\mathcal{K}(X)$ the set of such isomorphism classes.

Consider the following two assertions:

(I) The surface $B$ is a Fourier–Mukai partner of $A$ (i.e. $B \in \text{FM}(A)$).

(II) The surfaces $\text{Km}_3(B)$ and $\text{Km}_3(A)$ are isomorphic (i.e. $(B, G_B) \in \mathcal{K}(X)$).

The following is the main result of [35]:

**Theorem 11** Let $X = \text{Km}_3(A)$ be an algebraic generalized Kummer surface with Picard number 19.

Suppose that $L_X^2 \neq 0$, $36 \text{ mod } 54$. Then (I) is equivalent to (II).

Suppose that $L_X^2 = 0$ or $36 \text{ mod } 54$. Then (II) implies (I), but (I) does not imply (II) in general.

**Remark 12** By [11, Proposition 5.3], the number $|\text{FM}(A)|$ of Fourier–Mukai partners of an abelian surface $A$ is finite.

Let $A$ be an abelian surface with Picard number 3 that possesses an order 3 symplectic automorphism $J_A$. We denote by $\hat{J}_A$ the symplectic order 3 automorphism of the dual complex abelian surface $\hat{A} = \text{Pic}^0(A)$, induced by the action $\mathcal{L} \to J_A^*\mathcal{L}$. Let $X = \text{Km}_3(A)$ be the associated generalized Kummer surface.

**Corollary 13** Suppose that $L_X^2 \neq 0$, $36 \text{ mod } 54$. Then $\text{Km}_3(A)$ and $\text{Km}_3(\hat{A})$ are isomorphic: $(\hat{A}, \hat{J}_A) \in \mathcal{K}(X)$.

**Proof** By Shioda’s Torelli Theorem [39] recalled in Sect. 5, there exists an isomorphism of Hodge structures

$$(T(\hat{A}), \mathbb{C}\omega_{\hat{A}}) \simeq (T(A), \mathbb{C}\omega_A).$$

Therefore $A$ and $\hat{A}$ are Fourier–Mukai partners and, using the hypothesis on $L_X^2$, we can apply Theorem 11 to the pairs $(A, J_A)$, $(\hat{A}, \hat{J}_A)$ to conclude that $(\hat{A}, \hat{J}_A) \in \mathcal{K}(X).$  

**Remark 14** As we will see below, the abelian surface $A$ has a principal polarization if and only if $L_X^2 = 2 \text{ mod } 6$ or $L_X^2 = 6k$ with $k \neq 0 \text{ mod } 3$. In that case, let $\phi$ be an isomorphism between $A$ and $\hat{A}$. The pairs $(\hat{A}, \hat{J}_A)$ and $(A, J'_A)$ with $J'_A = \phi^{-1}\hat{J}_A \phi$ define the same Kummer structure, but we do not know if $(A, J'_A)$ define the same Kummer structure as $(A, J_A)$.
2.5.2 The generalized Kummer structures come from Fourier–Mukai partners

Let $A$ be an abelian surface; for a lattice $S$, let us denote by $FM(A, S)$ the set of Fourier–Mukai partners $B$ of $A$ such that $NS(B) \simeq S$. The following result is a consequence of Theorems 11 and 7:

**Corollary 15** Let $X = Km_3(A)$ be an algebraic generalized Kummer surface with Picard number 19. Let $(B, G_B)$ be a generalized Kummer structure of $X$. Then $B \in FM(A, NS(A))$.

**Proof** By Theorem 11, the surface $B$ is a Fourier–Mukai partner of $A$. The rank 18 orthogonal complement of $L_X^2$ in $NS(X)$ is -up to isomorphism- independent of $L_X^2$ (see [34]). That implies that $L_X^2 = L_Y^2$ for $Y = Km_3(B) \simeq Km_3(A)$. By Theorem 7, the Néron–Severi group of $A$ (respectively $B$) is uniquely determined by the integer $L_X^2$ (respectively $L_Y^2$), and therefore $NS(A) \simeq NS(B)$. □

For a 2-dimensional complex torus $A$, let us denote by $G_{Ho}$ the group of Hodge isometries of the transcendental lattice $T(A)$. It acts on $O(A_S)$ through the natural isomorphism $(A_S, q_{A_S}) \simeq (A_T, q_{A_T})$, where $S = NS(A)$ and $T = T(A) = NS(A)^\perp$ are contained in $H^2(A, \mathbb{Z}) \simeq U^\oplus$. Related to Corollary 15, let us recall the following result:

**Theorem 16** ([20, Theorem 2.3] and [19, Section 2, Claim]) Let $\mathcal{P}_A$ be the double coset $\mathcal{P}_A = O(NS(A)) \setminus O(NS(A))/G_{Ho}$. There exists a map

$$\xi : FM(A, NS(A)) \rightarrow \mathcal{P}_A,$$

which is onto and such that $\xi^{-1}(\xi([B])) = [B, \hat{B}]$.

Let $\hat{A}$ be the dual torus of $A$. Define $X = Km_3(A)$.

**Theorem 17** We have $FM(A) = FM(A, NS(A)) = \{A, \hat{A}\}$, in particular if $(B, G_B)$ is a generalized Kummer structure, then either $B \simeq A$ or $B \simeq \hat{A}$.

The surface $\hat{A}$ has a principal polarization if and only if $L_X^2 = 2 \text{ mod } 6$ or $3 || L_X^2$. Suppose that $L_X^2 = 6k$ with $k = 3 \text{ mod } 9$. Then $(A, J_A)$ and $(\hat{A}, \hat{J}_A)$ are two distinct Kummer structures on $Km_3(A)$.

**Proof** By Theorem 9, the map $O(NS(A)) \rightarrow O(NS(A))$ is onto, therefore $|\mathcal{P}_A| = 1$ and by Theorem 16, $FM(A, NS(A)) = \{A, \hat{A}\}$. If $B$ is a Fourier–Mukai partner of $A$, then $T(B) \simeq T(A)$. By Corollary 10, one has $T(A)(-1) \simeq NS(A)$ and $T(B)(-1) \simeq NS(B)$, thus $B \in FM(A, NS(A))$. We thus obtain that

$$FM(A) = FM(A, NS(A)) = \{A, \hat{A}\}.$$  

By Corollary 15, if $(B, G_B)$ is a generalized Kummer structure one has $B \in FM(A)$, thus either $B \simeq A$ or $B \simeq \hat{A}$.

Suppose that $L_X^2 = 2 \text{ mod } 6$ or $3 || L_X^2$. If the quadratic forms

$$(xD_A + y\delta_1 + z\delta_2)^2 \text{ or } (xL_A + y\delta_1 + z\delta_2)^2 \tag{2.2}$$

(according to the cases) associated to the lattice $NS(A)$ represents 2, then the surface $A$ has a principal polarization, and therefore $A$ is isomorphic to its dual $\hat{A}$. We know that this quadratic form is unique in its genus, therefore one can apply the Hasse-Minkowsky principle: if there exists a local solution for each prime and over $\mathbb{R}$, then there exists an integral (global)
solution. That can be checked using [36, Chapitre 4]. We leave it to the reader; a complete proof may be found on arXiv version 1 of this paper.

Suppose now that $9 \lfloor L_X^3 \rfloor$ and let $k' \in \mathbb{N}$ be such that $L_A^2 = 3k'$. The integers represented by the quadratic form $y^2 - yz + z^2 = 0$ or $1 \mod 3$ thus the equation $3k'x^2 - (y^2 - yz + z^2) = 1$ has no solution modulo 3, and therefore, there is no solution to equation $(xL_A^2 + y\delta_1 + z\delta_2)^2 = 2$; $A$ has no principal polarization.

Suppose that $L_X^2 = 6k$ with $k = 3 \mod 9$. By Shioda’s Torelli Theorem [39], $A$ and $\hat{A}$ are Fourier–Mukai partners. Then by implication (I) $\Rightarrow$ (II) of Theorem 11, the pair $(\hat{A}, \hat{J}_A)$ is a generalized Kummer structure of $X = \text{Km}(A, J_A)$. Since $A$ and $\hat{A}$ are not isomorphic, the generalized Kummer structures $(A, J_A)$ and $(\hat{A}, \hat{J}_A)$ are distinct. □

**Remark 18** In case $k = 0 \mod 9$ or $k = 6 \mod 9$, we do not know if $(\hat{A}, \hat{J}_A)$ is a generalized Kummer structure of $X = \text{Km}(A, J_A)$.

### 2.6 Non-algebraic case: a unique generalized Kummer structure

Let $(A, G_A), (B, G_B)$ be two non-algebraic complex 2-tori with an order 3 automorphism group.

**Proposition 19** Suppose that $\text{Km}_3(A) \simeq \text{Km}_3(B)$. Then there exists an isomorphism $\tau : A \to B$ such that $\tau^{-1}G_B\tau = G_A$. In particular $\text{Km}_3(A)$ has a unique generalized Kummer structure.

**Proof** The Néron–Severi group of $X = \text{Km}_3(A)$ has rank $\rho_X = 18$ or 19. The lattice $\text{NS}(X)$ is described in [34, Section 2], (using [6, Proof of Proposition 1.3]), where this is stated for abelian surfaces, but it is valid more generally for complex 2-tori since the proof only uses algebra. With the notations of [34], in case $\rho_X = 18$, one has $\text{NS}(X) = K_3$, otherwise $\text{NS}(X)$ is given in [34, Theorem 2].

Suppose that $\text{NS}(X)$ is negative definite (this is equivalent to suppose that the map $\text{NS}(A) \to \text{Num}(A)$ has trivial kernel). By computing the roots (the elements of square $-2$) of $\text{NS}(X)$, one finds that in both cases $\rho_X = 18$ or 19, the only irreducible curves on $X$ are the 18 ($-2$)-curves coming from the desingularization of $A/G_A$. Since $H_1(X, \mathbb{Z}) = \{0\}$ on K3 surfaces, the cyclic triple cover $\eta_A$ of $X$ branched over the 29 curves in the exceptional locus of the minimal resolution $X \to A/G_A$ is unique (see [5, Chapter I, Lemma 17.1]), and from that triple cover we may recover $A$. Since there are only 18 curves on $\text{Km}_3(A)$ and $\text{Km}_3(B)$, the isomorphism $\phi : \text{Km}_3(B) \to \text{Km}_3(A)$ must send the 18 ($-2$)-curves of $\text{Km}_3(B)$ to the 18 ($-2$)-curves of $X$. The maps $\phi \circ \eta_B$ and $\eta_A$ are triple covers with same branch locus, therefore by the unicity of the triple cover, the surface $B$ is isomorphic to $A$ and we obtain that $\text{Km}_3(A)$ possesses a unique generalized Kummer structure.

Suppose that the map $\text{NS}(A) \to \text{Num}(A)$ has a non-trivial kernel. Let $F$ be a generator of that kernel. One has $F^2 = 0$, therefore there exists a fibration $\varphi : A \to E$ of the complex torus onto an elliptic curve. The fibration $\varphi$ induces an elliptic fibration $\varphi' : X = \text{Km}_3(A) \to \mathbb{P}^1$ and the 9 fixed points of $G_A$ (which are 3-torsion points of $A$) induce singular fibers on $X$. Using Kodaira’s Table [5, Chapter V, Section 7], the singular fibers $F$ of an elliptic fibration have Euler number $e(F)$ as follows:
The fibers $I_1, I_2, II, III$ do not contain a $A_2$-configuration i.e. two $(-2)$-curves $C, C'$ such that $CC' = 1$ (see Kodaira’s Table) and the Euler number of a fiber containing (at least) a $A_2$ configuration is $\geq 3$. By [5, Chapter III, Proposition 11.4], since the Euler number of a K3 surface $X$ is 24, one has

$$24 = \sum_{s \in \mathbb{P}^1} e(F_s)$$

where $F_s$ is the fiber over $s$ of an elliptic fibration $X \to \mathbb{P}^1$ (one has $e(F_s) = 0$ if $F_s$ is smooth). Therefore, for example, the nine $A_2$-configurations cannot be spread into 9 distinct fibers of $\varphi'$. Also from the Kodaira Table, the 9 fixed points of $G$ cannot belong to the same fiber of $\varphi$. Using that, and the fact that the image of a 3-torsion group by $\varphi$ must be a 3-torsion group, we obtain that the unique possibility for the 9 fixed points of $G$ is that they are distributed by 3 on 3 fibers of $\mathbb{A}$, and that the elliptic fibration $\varphi'$ has 3 singular fibers of type $\tilde{E}$ (each such fiber has 7 irreducible components), and $\varphi'$ has no other singular fibers.

There cannot be curves that are multi-sections, otherwise there would exist a divisor of square $> 0$. Thus the surface $X$ contains exactly 21 $(-2)$-curves. Moreover, using Kodaira’s Table, among these 21 curves, there is a unique set of 18 curves that form 9 disjoint $A_2$-configurations. Then we can use the same argument as in case when $\text{NS}(X)$ is negative definite to conclude that $\text{Km}_3(A)$ has also a unique generalized Kummer structure.

2.7 The endomorphism ring of $A$

Let $(A, J_A)$ be a complex torus and an order 3 automorphism. Let $X = \text{Km}_3(A)$ be the associated generalized Kummer surface. For $a, b \in \mathbb{Q}, ab \neq 0$, let us denote by $\mathbb{Q}(a,b)$ the quaternion algebra generated by $r, \phi$ such that $r^2 = a, \phi^2 = b, r\phi = -\phi r$. Let us prove the following result

**Theorem 20** (i) Suppose that $L_X^2 = 2 \mod 6$. The endomorphism ring of $A$ contains elements $j = J_A, r, \phi, \psi$, such that

$$r = 1 + 2j, \quad r^2 = -3, \quad \phi^2 = \frac{1}{2}L_X^2, \quad r\phi = -\phi r, \quad \psi = \frac{r}{3}(\phi - 1),$$

with $\psi^2 = \frac{1}{6}(L_X^2 - 2) \in \text{End}(A)$. Moreover for general $A$, we have

$$\text{End}(A) = \mathbb{Z}[j, \psi], \quad \text{End}_\mathbb{Q}(A) = \frac{(-3, \frac{1}{2}L_X^2)}{\mathbb{Q}}.$$

(ii) Suppose that $L_X^2 = 0 \mod 6$. The endomorphism ring of $A$ contains elements $j = J_A, r, \phi$, such that

$$r = 1 + 2j, \quad r^2 = -3, \quad \phi^2 = \frac{1}{6}L_X^2, \quad r\phi = -\phi r.$$
Moreover for general A, we have

\[ \text{End}(A) = \mathbb{Z}[j, \phi], \text{ and if } L_X^2 \neq 0 \text{ then } \text{End}_{\mathbb{Q}}(A) = \left(\frac{-3, \frac{1}{6}L_X^2}{\mathbb{Q}}\right). \]

In both cases (i) and (ii) with \( L_X^2 \neq 0 \), the discriminant of the quaternion order \( \text{End}(A) \) is equal to \( \frac{1}{2}L_X^2 \).

**Example 21** Suppose that \( L_A^2 = L_X^2 = 0 \) (for example take \( L_A = \gamma_2 \)). Since \( L_A^2 = 0, L_A\delta_1 = L_A\delta_2 = 0, \) the map \( \text{NS}(X) \to \text{Num}(X) \) (where \( \text{Num}(X) \) is the group of numerical equivalence classes) has a one dimensional kernel generated by \( L_A \). Since \( \text{Num}(X) \) is negative definite, the torus \( A \) is not algebraic. The class \( L_A \) is the class of an elliptic curve \( E \), and there is an extension

\[ 0 \to E \to A \to E' \to 0, \]

where \( E' \) is also an elliptic curve; this is a so-called Shafarevich extension, see [7, Proposition 7.1]. The endomorphism \( \phi \) is non-zero and such that \( \phi^2 = 0 \) and \( r\phi = -\phi r \).

**Remark 22** In his thesis [9] Bonfanti, and in [10], Bonfanti and van Geemen obtain the construction of some \((1, d)\)-polarized abelian surfaces with an order 3 symplectic action \( J_A \). These surfaces \((A, J_A)\) have a deformation to \( E_j \times E_j \), where \( E_j = \mathbb{C}/\mathbb{Z}[j] \) for \( j^2 + j + 1 = 0 \), and \( J_{E_j \times E_j} = \text{Diag}(j, j^2) \). They also prove that the abelian surfaces they study have endomorphism ring \( \text{End}(A) \) generated by \( J_A, \psi \) where \( \psi^2 = [d] \), see [10, Theorem 1.2]. The generalized Kummer surface \( X \) associated to such \( A \) is such that \( L_X^2 = 0 \) mod 6. In fact one can check that the \((1, d)\)-polarization they obtain has class

\[ L_A = d\gamma_1 + \gamma_2, \]

with \( L_A^2 = 2d = \frac{1}{3}L_X^2 \), and these surfaces form one of the two families defined in Remark 8.

**Proof** (Of Theorem 20). We proceed as in the proof of [10, Theorem 1.2]: the elements \( D \) in the Néron–Severi group of \( A \) are identified with alternate forms on \( H_1(A, \mathbb{Z}) \) (and concretely with alternate matrices in the dual basis of basis \( \alpha_1, \beta_1, \alpha_2, \beta_2 \)).

We know from Theorem 7 the generators \( \delta_1, \delta_2 \in \text{NS}(A) \) (which are independent of \( n_1, n_2, n_3, n_4 \) and \( D_A \) or \( L_A \)) according if \( L_X^2 = 2 \) or 0 mod 6. By [8, Proposition 5.2.1a], if \( D \in \text{NS}(A) \) is such that \( D^2 > 0 \), then (as an alternate matrix) it is invertible and for any element \( D' \in \text{NS}(A) \), the matrix \( D^{-1}D' \) is the rational representation of an element of \( \text{End}_{\mathbb{Q}}(A) \) in basis \( \alpha_1, \beta_1, \alpha_2, \beta_2 \). It is an element of \( \text{End}(A) \) if and only if the coefficients are integers. We remark that if \( D' \in \text{NS}(A) \) is also invertible (as a matrix), then \( D'^{-1}D \) is also the rational representation of an endomorphism. We remark moreover that \( \delta_1, \delta_2 \) have determinant 1 and \( \delta_1^{-1}\delta_2 = j \) (the transpose of the action of \( J_A \) on \( H_1(A, \mathbb{Z}) \), since we are in the dual basis), we define \( r = 1 + 2j, \phi = r^{-1}\delta_1^{-1}L_A, \) and \( \psi = \frac{1}{j}\delta_1^{-1}(L_A - \delta_1 - 2\delta_2) = \frac{\ell}{j}(\phi - 1) \).

Let us study the endomorphism ring of abelian surfaces \( A \) with polarization

\[ L_A = 3n_1\gamma_1 + 3n_2\gamma_2 + 3n_3\gamma_3 + n_4(\gamma_3 + \gamma_4), \]
so that \( L_X^2 = 6(n_1n_2 + n_3^2 + n_3n_4) + 2n_4^2 = \frac{1}{3}L_A^2 = 2 \text{ mod } 6. \) We already know \( j \) and one has

\[
\phi = \begin{pmatrix}
2n_3 + n_4 & n_3 + n_4 & 2n_2 & n_2 \\
-n_3 & -2n_3 - n_4 & -n_2 & -2n_2 \\
2n_1 & n_1 & -2n_3 - n_4 & -n_3 \\
-n_1 & -2n_1 & n_3 + n_4 & 2n_3 + n_4
\end{pmatrix},
\]

\[
\psi = \frac{1}{3} \begin{pmatrix}
n_4 - 1 & -3n_3 - n_4 - 2 & 0 & -3n_2 \\
-3n_3 - 2n_4 + 2 & -n_4 + 1 & -3n_2 & 0 \\
0 & -3n_1 & n_4 - 1 & 3n_3 + 2n_4 - 2 \\
-3n_1 & 0 & 3n_3 + n_4 + 2 & -n_4 + 1
\end{pmatrix},
\]

both have trace 0. These formulas are for the elements \( \delta_1, \delta_2 \) specified in the proof of Theorem 7 when \( n_4 = 1 \text{ mod } 3 \) (the coefficients of \( \psi \) are integers). Similar formulas exist for \( n_4 = 2 \text{ mod } 3 \). It is easy to check that in both cases \( r\phi = -\phi r \) and \( \phi^2 = \frac{1}{2}L_X^2 \), therefore the ring \( \mathbb{Q}[j, \phi] \) is the quaternion algebra \((-\frac{1}{4}, \frac{L_X^2}{2})\).

In order to understand the generators of the endomorphism ring \( \text{End}(A) \), let us take 4 variables \( a_1, \ldots, a_4 \in \mathbb{Q} \) and search if the matrix

\[ a_1 + a_1j + a_2\phi + a_4\psi \]

has integral coefficients. That gives 16 equations for the 4 unknowns \( a_1, \ldots, a_4 \). The ideal generated by all the 4 by 4 minors of that system is \( 2(n_1, \ldots, n_4)^2 \), which is equal to \( 2\mathbb{Z} \) since by hypothesis the polarization is primitive, and therefore the \( n_j \) are coprime. Thus if \( p \) is a non-trivial denominator of one of the \( a_k \), then \( p = 2 \). By reducing the equation modulo 2, one finds exactly one more class (modulo the lattice generated by 1, \( j, \phi, \psi \)), which is \( \psi' = \frac{1}{2}(1 - \phi - \psi) \), but since one can check that \( \psi' = j\psi \), we obtain that \( j, \psi \) generate the ring \( \text{End}(A) \) and the elements \( \beta_1 = 1, \beta_2 = j, \beta_3 = \psi, \beta_4 = j\psi \) form a basis of the free \( \mathbb{Z} \)-module \( \text{End}(A) \). The reduced trace of an element \( \beta \in \text{End}(A) \) is \( \frac{1}{2} \) of the trace of \( \beta \) seen as an element of \( M_4(\mathbb{Q}) \) (as an element of a quaternion algebra, it is equal to \( \beta + \bar{\beta} \)). One can then compute the reduced discriminant \( D_{\text{End}(A)} = \det(\text{Tr}_{\text{red}}((\gamma_1\gamma_2)_{1\leq s, t \leq 4})) \) of \( \text{End}(A) \) and one obtains that \( D_{\text{End}(A)} = \frac{1}{2}L_X^2 \).

Suppose that \( L_X^2 = 0 \text{ mod } 6 \). Then

\[ L_A = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n_4(\gamma_3 + \gamma_4), \]

with \( L_X^2 = 6(n_1n_2 + n_3^2 + 3n_3n_4 + 3n_4^2) = 3L_A^2 \). The endomorphism \( \phi = \delta_1^{-1}L_A \) is

\[
\phi = \begin{pmatrix}
n_4 & -n_3 - n_4 & 0 & -n_2 \\
n_3 - 2n_4 & -n_4 & -n_2 & 0 \\
0 & -n_1 & n_4 & n_3 + 2n_4 \\
-n_1 & 0 & n_3 + n_4 & -n_4
\end{pmatrix},
\]

which has square \( \phi^2 = \frac{1}{2}L_X^2 \). Moreover \( r\phi = -\phi r \). Using the fact that \( n_1, \ldots, n_4 \) are coprime, one computes that either \( \text{End}(A) \) is the ring generated by \( j, \phi \) (with reduced discriminant \( D_{\text{End}(A)} = \frac{1}{2}L_X^2 \)), or one can find another class if \( n_1 = n_2 = n_3 = 0 \text{ mod } 3 \) and \( n_4 \neq 0 \text{ mod } 3 \). But by Proposition 5 the \( \gcd \) of \( n_1, n_2, n_3 \) must be coprime to 3.

A direct consequence of Theorem 20 is

**Corollary 23** Two complex tori \( A, B \) with order 3 automorphism such that \( L_X^2 = L_Y^2 \), where \( X = \text{Km}_3(A), Y = \text{Km}_3(B) \) have isomorphic endomorphism ring.
3 Abelian surfaces with quaternionic multiplication

3.1 Complex torus with an order 3 symplectic automorphism

Let $J_A$ be an order 3 symplectic automorphism of $A$. One has:

$$J_A^2 + J_A + I_A = 0$$

in $\text{End}(A)$, where $I_A$ is the identity of $A$. The Néron–Severi group of $A$ has rank $\rho_A = 2, 3$ or 4. From Theorem 20, we obtain:

**Proposition 24** Suppose that $\rho_A = 3$ and $L_A^2 \neq 0$. The surface $A$ has quaternionic multiplication: there exists a quaternion algebra $H$ over $\mathbb{Q}$ and an order $O$ in $H$ such that $O \cong \text{End}(A)$. Since $J_A^2 + J_A + I_A = 0$, the ring of Eisenstein integers is contained in $O$.

The quaternion algebra $H$ is indefinite (i.e. $H \otimes \mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R})$) if and only if $L_A^2 > 0$.

Let us therefore study quaternion algebras and their orders $O$ which contain the ring $\mathbb{Z}[j]$, with $j^2 + j + 1 = 0$. As we will see, the Shimura construction gives a converse: if $O$ is an order in a quaternion algebra such that $\mathbb{Z}[j] \hookrightarrow O$, then there exists a one dimensional family of abelian surfaces $A$ with $O \cong \text{End}(A)$ and Picard number 3 (generically). Let us start by recalling the definitions for quaternion algebras and by fixing the notations.

3.2 Quaternion algebras and their discriminant

More details on the definitions related to quaternion algebras are given in the long version v1 of this paper on arXiv.

3.2.1 First definitions and notations

For $a, b \in \mathbb{Q}$, we denote by $H = \frac{(a, b)}{\mathbb{Q}}$ the quaternion algebra generated by $\alpha, \beta$ with $\alpha^2 = a$, $\beta^2 = b$ and $\alpha\beta = -\beta\alpha$. The reduced discriminant $D_H$ of $H$ is the product of primes $p$ at which there is a ramification i.e. such that $H_p = H \otimes \mathbb{Q} \mathbb{Q}_p$ is division algebra (by convention we count $-1$ as a prime and $\mathbb{Q}_{-1} = \mathbb{R}$). There are always an even number of such primes, and two quaternion algebras are isomorphic if and only if they are ramified at the same places, which is equivalent to ask that their reduced discriminants are equal. The quaternion algebra $H$ is a matrix algebra ($H \cong M_2(\mathbb{Q})$) if and only if $D_H = 1$.

If the field $\mathbb{Q}(\sqrt{d})$ splits $H$ (i.e. $H \otimes \mathbb{Q}(\sqrt{d})$ is isomorphic to the matrix algebra $M_2(\mathbb{Q}(\sqrt{d}))$), then there exists $c \in \mathbb{Q}^*$ such that $H = \frac{(c, d)}{\mathbb{Q}}$ and an embedding $\Phi : H \hookrightarrow M_2(\mathbb{Q}(\sqrt{d}))$ is given by

$$\Phi(x + y\alpha + z\beta + t\alpha\beta) = \begin{pmatrix} x + y\sqrt{d} & z + t\sqrt{d} \\ c(z - t\sqrt{d}) & x - y\sqrt{d} \end{pmatrix}.$$  (3.1)

The reduced norm and trace of $w \in H$ are then equal to the determinant and trace of $\Phi(w)$.

3.2.2 Eichler orders, ideals and class numbers

An Eichler order is an order which is the intersection of two maximal orders. If $D_H$ is the discriminant of $H$, the level of $O$ is the integer $N$ such that $D_O = ND_H$, where $D_O$ is the reduced discriminant of $O$. By [1, Proposition 1.54], the level of an Eichler order is coprime to $D_H$, moreover:
Proposition 25 [1, Corollary 1.58] Let \( N \) be an integer coprime to \( D_H \). Up to conjugation, there exists an unique Eichler order of level \( N \). [1, Proposition 1.54] Let \( \mathcal{O} \) be an order. Suppose \( D_{\mathcal{O}} = ND_H \) with \( N \) a square-free and prime to \( D_H \) integer. Then \( \mathcal{O} \) is an Eichler order.

Two ideals \( I, J \) are said equivalent on the right if \( I = Jh \) for some \( h \in H^* \). Let \( \mathcal{O} \) be an order. The classes of equivalent (on the right) \( \mathcal{O} \)-ideals are called the right classes of \( \mathcal{O} \). There is the same notion for the left. The class number \( h(\mathcal{O}) \) of \( \mathcal{O} \)-ideals is the number of classes of left \( \mathcal{O} \)-ideals; that number is also equal to the number of classes of right \( \mathcal{O} \)-ideals.

3.2.3 Legendre, Kronecker and Hilbert symbol

For the notions of Legendre and Kronecker symbols \((-\cdot)\), and the quadratic reciprocity law, we refer to [1, Section 1.1.2] or [36]. For a prime \( p \), the Hilbert symbol is the function \((-,-)_p : \mathbb{Q}_p^* \times \mathbb{Q}_p^* \to \{-1,1\} \) defined by

\[
(a, b)_p = \begin{cases} 
1 & \text{if } z^2 = ax^2 + by^2 \text{ has a non-trivial solution} \\
-1 & \text{otherwise.}
\end{cases}
\]

For the computation of the Hilbert symbol, see [36], or the version 1 of that paper on arXiv. One has:

Proposition 26 The quaternion algebra \( \frac{(a,b)}{\mathbb{Q}_p} \) is isomorphic to \( M_2(\mathbb{Q}_p) \) if and only if \( (a, b)_p = 1 \).

Two quaternion algebras \( \frac{(a,b)}{\mathbb{Q}}, \frac{(c,d)}{\mathbb{Q}} \) are isomorphic if and only if they have the same Hilbert symbols: \( (a, b)_p = (c, d)_p \) for all primes \( p \), including \(-1\).

3.2.4 The discriminant of the quaternion algebras \( \frac{(-3,d)}{\mathbb{Q}} \).

We recall that, following Conway and Sloane, we take the convention that \(-1\) is a prime. Let us study the quaternion algebras \( H \) over \( \mathbb{Q} \) which contain an element \( r \) with \( r^2 = -3 \), as in Proposition 24. Let \( d \in \mathbb{Q} \) such that \( H = \frac{(-3,d)}{\mathbb{Q}} \). Since for any integer \( n \neq 0 \), we have \( \frac{(-3,d)}{\mathbb{Q}} = \frac{(-3,n^2d)}{\mathbb{Q}} \), we can suppose that \( d = p_1 \cdots p_m \) is a square free integer. Let us denote by \( r, \phi_o \) the generators of \( H \) with \( r^2 = -3, \phi_o^2 = d \) and \( r\phi_o = -\phi_or \), so that \( 1, r, \phi_o, r\phi_o \) is a \( \mathbb{Q} \)-basis of \( H \). If \( 3 \) is among the primes \( p_1, \ldots, p_m \), then \( \frac{1}{2}r\phi_o^2 = \frac{d}{3} \in \mathbb{Z} \), and \( r\left(\frac{1}{2}r\phi_o\right) = -\phi_o \), thus \( \frac{(-3,d)}{\mathbb{Q}} \) is isomorphic to \( \frac{(-3,\frac{d}{3})}{\mathbb{Q}} \), and we can suppose that \( 3 \mid d \). Let us define \( j = -\frac{1}{2}r^2 \), so that \( j^2 + j + 1 = 0, r = 1+2j \in \mathbb{Z}[j] \), and \( j\phi_o = \phi_o j^2 = \phi_o j \) (since we know that \( r\phi_o = -\phi_or \)). Let \( p \) be a prime dividing \( d \) and such that \( p = 1 \mod 3 \). Then \( p \) is a square modulo 3 and there exists \( u \in \mathbb{Z}[j] \) with \( uu^* = p \). By writing \( u = a + bj \) with \( a, b \in \mathbb{Z} \), since \( \phi_o j = j\phi_o \), we obtain the relation \( \phi_o u = uu^* \phi_o \), and

\[
\left( \frac{u}{\phi_o} \right)^2 = \frac{uu^*}{p^2} \phi_o^2 = \frac{p}{p^2} = \frac{d}{p^2} \in \mathbb{Z}, \quad r\left( \frac{u}{p} \phi_o \right) = -\left( \frac{u}{p^2} \phi_o \right) r,
\]

thus the quaternion algebra \( H \) is isomorphic to \( \frac{(-3,\frac{d}{p^2})}{\mathbb{Q}} \). We can therefore suppose that all prime divisors of \( d \) are congruent to 2 mod 3 and \( d \) is square free. Let us define \( d_H = d \): this is a square-free integer such that the primes dividing it are congruent to 2 mod 3.
Lemma 27  Under the above hypotheses on $d_H$, the discriminant of $H = \frac{(-3,d_H)}{\mathbb{Q}}$ is

$$D_H = \begin{cases} d_H & \text{if } m \text{ is even} \\ 3d_H & \text{if } m \text{ is odd} \end{cases}$$

where $d_H = p_1 \cdots p_m$ is the decomposition of $d_H$ as a product of primes (including eventually $-1$) $p_k$ congruent to $2 \mod 3$.

**Proof** The possible ramification is over primes $3$, $a$ natural prime. By Sect. 3.2.3, one has $H$ thus

$$\text{Lemma 27}$$

Remark 28 One has $H$ if and only if $-1 \in \{p_1, \ldots, p_m\}$. We have

$$(-3, p)_p = \left(\frac{-1}{p}\right) \left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = -1,$$

thus $H$ is ramified at $p$. If $p = 2$, we also obtain that $(-3, 2)_2 = -1$ and $H$ is ramified at 2. The quaternion algebra $H$ is definite if and only if $-1 \in \{p_1, \ldots, p_m\}$. We have

$$(-3, d)_3 = \prod_{k=1}^m (-3, p_k)_3 = \prod_{k=1}^m \left(\frac{p_k}{3}\right) = (-1)^m,$$

therefore $3$ is ramified in $H$ if and only if $m$ is odd. \hfill $\Box$

**Remark 28** One has $D_H = 3d_H$ if and only if $d_H = 2 \mod 3$, and $D_H = d_H$ if and only if $d_H = 1 \mod 3$.

### 3.3 Orders containing the Eisenstein integers

Let us recall that $r, \phi_o$ are the generators of $H = \frac{(-3,d_H)}{\mathbb{Q}}$ such that $r^2 = -3$, $\phi_o^2 = d_H$, $r\phi_o = -\phi_or$. We also defined $j = \frac{1}{2}(-1+r)$, which satisfies $j^2 + j + 1 = 0$, $j\phi_o = \phi_0 j = \phi_0 j^2$. If $D_H = 1 \mod 3$, let us define $\theta = \frac{\zeta}{3}(\phi_o - 1)$ (then $\theta^2 = \frac{d_H-1}{3}$) and if $D_H = 0 \mod 3$, let us define $\theta = \phi_o$.

**Proposition 29** The quaternions $1, j, \theta, j\theta \in H$ are generators of a maximal order $\mathcal{O}_m$. Suppose that $H$ is a definite quaternion algebra and let $j' \in \mathcal{O}_m$ be such that $j'^2 + j' + 1 = 0$. Then $j' = j$ or $j^2$.

**Proof** Suppose that the discriminant of $H = \frac{(-3,d_H)}{\mathbb{Q}}$ is $d_H$ i.e. $D_H = d_H = 1 \mod 3$. Let $k \in \mathbb{N}$ such that $d_H = 3k + 1$. The quaternions $j, \phi_o$ and $\theta$ are integral over $\mathbb{Z}$ and $\mathcal{O}_m$ is a ring. Consider $v_1 = 1$, $v_2 = j$, $v_3 = \theta$, $v_4 = j\theta$. The matrix $(\text{Tr}(v_s v_t))_{1 \leq s, t \leq 4}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 1 & 2k & -k \\ 1 & -1 & -k & 2k + 1 \end{pmatrix}$$

which has determinant $-d^2 = -D_H^2$, thus $\mathcal{O}_m$ is maximal by [1, Proposition 1.32]. Suppose that the discriminant of $H = \frac{(-3,d)}{\mathbb{Q}}$ is $D_H = 3d_H$. Then $d_H = p_1 \cdots p_m$ is a product of an odd number of different primes (eventually including $-1$) congruent to $2 \mod 3$. The elements $j$ and $\phi_o$ are integral over $\mathbb{Z}$. Consider $v_1 = 1$, $v_2 = j$, $v_3 = \phi_o$, $v_4 = j\phi_o$. The matrix $(\text{Tr}(v_s v_t))_{1 \leq s, t \leq 4}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 2d & -d \\ 0 & 0 & -d & 2d \end{pmatrix}$$
which has determinant \(-9d^2 = -D_H^2\), thus \(O_m\) is maximal by \([1, \text{Proposition 1.32}]\).

For the second assertion, we only give a sketch of the proof (see the long version of that paper on arXiv for a complete proof). The idea is to write \(j' = c_1 + c_2j + c_3\theta + c_4j\theta\) for integers \(c_i\) and to remark that \(j'^2 = \bar{j}'\), so that \(1 = j'^3 = \bar{j}'j'\) is expressed as a quadratic form in the \(c_i\). The hypothesis on \(H\) (which is in fact \(d_H < 0\)) insures that this quadratic form is definite positive, and finally the only solution for the relation \(j'^2 + j' + 1 = 0\) to hold is that \(j' = j\) or \(j' = j^2\).

Let us define the order

\[ O_\mu = \mathbb{Z}[j] + \mathbb{Z}[j]\mu \theta \subset O_m, \]

where \(\theta\) equals \(\frac{1}{2}(\phi_o - 1)\) or \(\phi_o\) according if \(D_H = 1\) or \(0\mod3\). Let \(O\) be an order contained in \(O_m\).

**Proposition 30** Suppose that \(j \in O\). There exists \(\mu \in \mathbb{Z}[j]\) such that \(O = O_\mu\) and the discriminant of \(O_\mu\) is \(\mu \bar{\mu} D_H\).

**Proof** Since \(O\) contains \(\mathbb{Z}[j]\), this is a rank 2 \(\mathbb{Z}[j]\)-module and since \(\mathbb{Z}[j]\) is principal, we can write \(O = \mathbb{Z}[j] \oplus \mathbb{Z}[j]\tau\) for some \(\tau\) in the order \(O_m\), which has basis \(1, j, \theta, j\theta\), and the result follows. The assertion on the discriminant follows from a direct computation. \(\square\)

Let \(\mu, \mu' \in \mathbb{Z}[j]\) be such that \(\mu \bar{\mu} = \mu' \bar{\mu}'\).

**Proposition 31** The order \(O_\mu\) is isomorphic to \(O_{\mu'}\).

**Proof** The only trouble is with the primes \(p_1, \ldots, p_r\) dividing \(\mu \bar{\mu}\) which are split in \(\mathbb{Z}[j]\). Let \(p_1, \bar{p}_1, \ldots, p_r, \bar{p}_r\) be the primes in \(\mathbb{Z}[j]\) over these primes \(p_k\). There is a unique factorization (up to invertible elements):

\[ \mu = r^{a_1}p_1^{a_1} \bar{p}_1^{b_1} \cdots p_r^{a_r} \bar{p}_r^{b_r} m_2, \]

for some integers \(a, a_k, b_k\), and \(m_2 \in \mathbb{Z}\), a product of primes congruent to 2 mod 3. Consider \(\delta = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}\) and define

\[ \mu'' = r^{a_1}p_1^{a_1+b_1} p_2^{a_2+b_2} \cdots p_r^{a_r+b_r} m_2. \]

One can check that \(\delta(\delta\phi_o)\delta^{-1} = \delta\phi_o\) (thus \(\delta(\mu\phi_o)\delta^{-1} = \mu'\phi_o\)), that \(\delta\delta\theta\delta^{-1} = \delta\theta + \gamma\) for some \(\gamma \in \mathbb{Z}[j]\) (thus \(\delta(\mu\theta)\delta^{-1} \in O_{\mu''}\)), and that therefore the conjugation map \(x \mapsto \delta x \delta^{-1}\) is an isomorphism \(O_\mu \mapsto O_{\mu''}\). We obtain similarly that \(O_{\mu'}\) is isomorphic to \(O_{\mu''}\), since \(\mu' = (-j)^{b_1} r^{a_1} p_1^{a_1} p_1^{b_1} \cdots p_r^{a_r} p_r^{b_r} m_2\) for some integers \(b, a', b'\) such that \(a' + b' = a_k + b_k\), and the result follows. \(\square\)

Suppose that the quaternion algebra \(H/\mathbb{Q}\) is indefinite. Then all the maximal orders are conjugated to \(O_m\). Therefore:

**Corollary 32** An order of \(H\) containing \(\mathbb{Z}[j]\) is determined up to isomorphism by its discriminant.

Still in the indefinite case, let \(O \subset H\) be an order containing an element \(j'\) with \(j'^2 + j' + 1 = 0\). Up to conjugation by an element of \(H\), we can suppose that \(O \subset O_m\). The group of Atkin–Lehner involutions of \(O_m\) acts transitively on the orbits of order 3 sub-groups of \(O_m^*\). Therefore, up to conjugation and replacing \(j'\) by \(j'^2\), we can suppose that up to isomorphism \(j' = j\). We thus obtained up to isomorphism all the possible endomorphism rings of abelian surfaces which have Picard number 3 and an order 3 symplectic automorphism.
Remark 33  Note that using Corollary 23, one obtains more generally that for any endomorphism ring of a complex torus $A$ with an order 3 symplectic group, there exists a quaternion algebra $H$ and $\mu \in \mathbb{Z}[j]$ such that End($A$) = $O_\mu$. Indeed we will see below that each ring $O_\mu$ determines such a complex torus $B$, and the values of $L^2_Y$ (for $Y = \text{Km}_3(B)$) when $H$ and $\mu$ varies form the set \( \{ \ell \in \mathbb{Z} \mid \ell = 0 \text{ or } 2 \text{ mod } 6 \} \).

### 3.4 Shimura curves

Let us denote by $\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ the Poincaré upper-half plane and by $\bar{\mathcal{H}}$ its conjugate. The group $SL_2(\mathbb{R}) \mod \pm I_2$ acts on $\mathcal{H}$ (and also on $\bar{\mathcal{H}}$) by homographic transformations:

$$\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), z \in SL_2(\mathbb{R}) \times \mathcal{H} \rightarrow \gamma(z) = \frac{az + b}{cz + d} \in \mathcal{H}.$$  

A transformation $\gamma$ is called **elliptic** if it has a fixed point $z \in \mathcal{H}$. This is equivalent to require $|\text{Tr}(\gamma)| < 2$, in particular $\gamma \neq \pm I_2$.

Let $\Gamma$ be a discrete (for the usual topology) subgroup of $SL_2(\mathbb{R})$. A point $z \in \mathcal{H}$ is called elliptic with respect to $\Gamma$ if there exists an elliptic transformation $\gamma \in \Gamma$ such that $\gamma(z) = z$.

The isotropy group of a point $z \in \mathcal{H}$ is the group $\Gamma_z = \{ \gamma \in \Gamma \mid \gamma(z) = z \}$. Since $\Gamma$ is discrete, the isotropy group of an elliptic point is finite and cyclic. By definition, the *order* of an elliptic point $z$ is $|\Gamma_z|$ if $-I_2 \notin \Gamma$, and $\frac{1}{2} |\Gamma_z|$ if $-I_2 \in \Gamma$. If $z \in \mathcal{H}$ is an elliptic point, then for any $\gamma \in \Gamma$, the element $\gamma(z) \in \mathcal{H}$ is an elliptic point and their isotropy groups are conjugate: $\Gamma_{\gamma(z)} = \gamma \Gamma_z \gamma^{-1}$.

Let $H = \frac{(c,d)}{Q}$ be an indefinite quaternion algebra over $\mathbb{Q}$, with $d > 0$. Let $I \subset H$ be a $\mathbb{Z}$-ideal and let $O$ be the left order of $I$. We denote by $O^*$ the group of invertible elements and by $O^1 \subset O^*$ the sub-group of reduced norm 1 elements. The group $O^1$ has index 2 or 1 in $O^*$ according if there exists an element of $O^*$ of norm $-1$ or not. The group $\Gamma(O^1) := \Phi(O^1)$ (respectively $\Gamma(O^*) := \Phi(O^*)$) is a sub-group of $SL_2(\mathbb{Q}(\sqrt{d}))$, which is discrete in $SL_2(\mathbb{R})$ (where the map $\Phi$ is defined in Sect. 3.2.1). The elements of $\Gamma(O^*)$ are called **quaternionic transformations**. Since $H$ is defined over $\mathbb{Q}$, the elliptic transformations of $\Gamma(O^*)$ have order 2 or 3 (see [1, Lemma 2.10]).

The group $\Gamma(O^*)$ (respectively $\Gamma(O^1)$) acts on $\mathcal{H} \cup \bar{\mathcal{H}}$ (respectively $\mathcal{H}$ and $\bar{\mathcal{H}}$). If it exists, an element $\gamma \in O^*$ of norm $-1$ exchanges the upper and lower half-planes $\mathcal{H}$ and $\bar{\mathcal{H}}$. The quotient $\mathcal{H} / \Gamma(O^1)$ is a complex Riemann surface. The quotient $X(O) = (\mathcal{H} \cup \bar{\mathcal{H}}) / \Gamma(O^*)$ is isomorphic to $\mathcal{H} / \Gamma(O^1)$ if and only if $-1$ is a norm. If $-1$ is not a norm, then $X(O)$ is the disjoint union of the two curves $\mathcal{H} / \Gamma(O^1)$, $\bar{\mathcal{H}} / \Gamma(O^1)$. The compactification of $X(O)$ is a Shimura curve. The curve $\mathcal{H} / \Gamma(O^1)$ is compact if and only if $H$ is a division algebra; we will mainly suppose that this is the case in the following. A point $p$ on the curve $\mathcal{H} / \Gamma(O^1)$ is called elliptic if this is the image of some elliptic point $z$ for $\Gamma(O^1)$ by the quotient map $\mathcal{H} \rightarrow \mathcal{H} / \Gamma(O^1)$; by definition, its order is the order of the elliptic point $z$. We remark that $\Phi(-1) = -I_2 \in \Gamma(O^1)$ and we obtain immediately:

**Proposition 34** There is a bijection between the set of elliptic points of order $k$ on $\mathcal{H} / \Gamma(O^1)$ and the set of orbits under conjugation by $\Gamma(O^1)$ of order $2k$ subgroups of $\Gamma(O^1)$.

The curve $X(O) = (\mathcal{H} \cup \bar{\mathcal{H}}) / \Gamma(O^*)$ is a coarse moduli space for abelian surfaces with quaternionic multiplication by $O$.

Let $e_k(O)$ be the number of elliptic points of order $k$ ($k \in \{2, 3\}$) of the group $\Gamma(O^1)$. Suppose that $O$ is an Eichler order of level $N$ (necessarily coprime to $D_H$) or is maximal ($N = 1$), then
Proposition 35 [1, Proposition 2.29] The number of order 3 elliptic points of \( \mathcal{H}/\Gamma(O^1) \) is
\[
e_3(O) = \begin{cases} 
\prod_{p|D_H}(1 - \left(\frac{-3}{p}\right)) \prod_{p|N}(1 + \left(\frac{-3}{p}\right)) & \text{if } 9 \nmid N, \\
0 & \text{if } 9|N,
\end{cases}
\]
where \( D_H \) is the discriminant of \( H \).

This number is 0 or a power of 2. One has \( e_3(O) = 1 \) if and only if \( D_H = 1 \) and \( N = 1 \) or 3. We remark that as soon as the level \( N \) is divisible by a prime \( p \) such that \( p = 2 \mod 3 \), one has \( e_3(O) = 0 \), which means that there are no injective ring homomorphism \( \mathbb{Z}[j] \rightarrow O \), (where \( j^2 + j + 1 = 0 \)).

3.5 Complex 2-tori with QM and generalized Kummer surfaces

Let \( H = \frac{(c,d)}{\mathbb{Q}} \) be a quaternion algebra over \( \mathbb{Q} \) (definite or not) and let \( O \subset H \) be an order. One says that a complex 2-torus \( A \) admits a quaternionic multiplication by \( H \) if there is an embedding
\[
\iota : H \hookrightarrow \text{End}(A) \otimes \mathbb{Q},
\]
moreover, \( A \) admits a quaternionic multiplication by the order \( O \) if
\[
\iota(H) \cap \text{End}(A) = \iota(O)
\]
in \( \text{End}(A) \otimes \mathbb{Q} \) (then, for the general torus, one has \( \text{End}(A) = \iota(O) \)). Let \( \Phi : H \hookrightarrow M_2(\mathbb{C}) \) or \( M_2(\mathbb{R}) \) be an embedding (see Sect. 3.2.1), according if \( H \) is definite or not. The complex 2-tori with quaternionic multiplication by \( O \) are those of the form
\[
A_{h,v} = \mathbb{C}^2/\Phi(I) \left( \frac{z}{z'} \right),
\]
where \( I \subset O \) is a left \( O \)-ideal, for \( z, z' \in \mathbb{C} \) with \( zz' \neq 0 \) if \( H \) is definite, and if \( H \) is indefinite, \( z' = 1 \) and \( z \) is in \( H \cup \bar{H} \); in that case we denote simply: \( A_z = A_{h,v} \). These results are due to Shimura [38] for \( H \) indefinite, and for \( H \) definite, we refer to the paper of Shimizu [37, Section 4]. Note that in the definite case, if \( \lambda \in \mathbb{C}^* \), one has \( A_{\lambda h, \lambda v} \simeq A_{h,v} \) and the complex 2-tori with quaternionic multiplication by \( O \) are therefore parametrized by \( \mathbb{P}^1 \).

When \( H \) is indefinite, one has \( A_z \simeq A_w \) if and only if \( z = \gamma(w) \) for some \( \gamma \in O^* \), where \( O^* = \{ \gamma \in O \mid \text{Nr}(\gamma) = \pm 1 \} \) is the group of invertible elements of \( O \).

Two left \( O \)-ideals \( I, J \) give the same family of complex 2-tori with endomorphism ring \( O \) if and only if there exists a principal ideal \( P \) such that \( I = JP \). The automorphism group of the general complex 2-tori \( A \) is isomorphic to \( O^* \).

4 Junction between Shimura and Barth constructions

4.1 The polarization on the generalized Kummer surface and its isogeny classes

As in Sect. 3.5, let \( d_H \) be a square free integer divisible only by primes congruent to 2 mod 3 and let \( H \) be the quaternion algebra \( H = \frac{(-3,d_H)}{\mathbb{Q}} \), we denote by \( r, \phi_0 \), the generators such that \( r^2 = -3, \phi_0^2 = d_H, \phi_0 r = -r \phi_0 \). The discriminant of \( H \) satisfies \( D_H = d_H \) if \( d_H = 1 \mod 3 \) and \( D_H = 3d_H \) if \( d_H = 2 \mod 3 \). Let us also define \( j = \frac{1}{2}(-1 + r) \). We will suppose in this section that \( H \) is indefinite.
Definition 36 For an integer $c$, we denote by $\text{rad}_2(c)$ the product of the primes congruent to $2 \mod 3$ that divide $c$ to an odd power (for example $\text{rad}_2(12) = 1$).

Let us say that two generalized Kummer surfaces $X = \text{Km}_3(A)$, $Y = \text{Km}_3(B)$ are in the same isogeny class if there exists an isogeny of complex tori $A \to B$. If this is the case, there exists a natural dominant rational map $X \dashrightarrow Y$.

Let $A$ be an abelian surface with an order 3 automorphism $J_A$, with Picard number 3 and quaternionic multiplication by $H$. The aim of this section is to prove the following result:

Theorem 37 There exists $\mu \in \mathbb{Z}[j]$ such that endomorphism ring of $A$ is (isomorphic to)

$$
\mathcal{O}_\mu = \mathbb{Z}[j] + \mathbb{Z}[j] \mu \theta,
$$

where $\theta = \frac{j}{2}(\phi_0 - 1)$ or $\theta = \phi_0$, according if $D_H = 1$ or $0 \mod 3$. The generalized Kummer surface $X$ is polarized by $L_X$ with $L_X^2 = 2\mu^2 D_H$. The integer $L_X^2$ determines $\text{End}(A)$. Let $X = \text{Km}_3(A)$, $Y = \text{Km}_3(B)$ be two generalized Kummer surfaces. If the surfaces $X$ and $Y$ are in the same isogeny class then $\text{rad}_2(\frac{1}{2}L_X^2) = \text{rad}_2(\frac{1}{2}L_Y^2)$.

Suppose that $L_X^2 = L_Y^2$. Then $\text{End}(A) \simeq \text{End}(B)$.

Let us recall the following result of Theorem 20 according to the cases $L_X^2 = 2$ or $0 \mod 6$:

| Case | $L_X^2$ | $\phi^2$ | $\psi$ | $\mathcal{O}_\mu$ | $D_{\text{End}(A)}$ |
|------|---------|---------|-------|----------------|------------------|
| (i)  | $2 \mod 6$ | $\frac{1}{2}L_X^2 = \frac{1}{6}L_A^2$ | $\psi = \frac{j}{2}(\phi - 1)$ | $\mathbb{Z}[j, \psi]$ | $\frac{1}{2}L_X^2 = \frac{1}{6}L_A^2$ |
| (ii) | $0 \mod 6$ | $\phi^2 = \frac{1}{6}L_X^2 = \frac{1}{2}L_A^2$ | $\mathbb{Z}[j, \phi]$ | $\frac{1}{2}L_X^2 = \frac{1}{2}L_A^2$ |

where $D_{\text{End}(A)}$ is the discriminant of $\text{End}(A)$. For proving Theorem 37, we need the following result:

Lemma 38 One has $\text{rad}_2(D_{\text{End}(A)}) = \text{rad}_2(\frac{1}{2}L_X^2) = d_H$.

Proof By Theorem 20, the discriminant of $\text{End}(A)$ equals $\frac{1}{2}L_X^2$. According to Proposition 30, there exists $\mu \in \mathbb{Z}[j]$ such that $\text{End}(A) \simeq \mathcal{O}_\mu$; the discriminant of $\mathcal{O}_\mu$ is $\mu^2 D_H$. Let $p \in \mathbb{Z}$ be a prime. If $p = 1 \mod 3$, then $p$ is split in $\mathbb{Z}[j]$: there exists a prime $p$ in $\mathbb{Z}[j]$ such that $p = \mathfrak{p} \mathfrak{p}$. If $p = 2 \mod 3$, then $p$ is inert: it is still a prime in $\mathbb{Z}[j]$. Finally $3$ is ramified: $3 = -r^2 = r \tilde{r}$. Thus for any $\mu \in \mathbb{Z}[j]$, there exists uniquely determined integers $a, m_1, m_2$ such that $\mu \tilde{\mu} = 3a m_1 m_2^2$, where $a \geq 0$ and the integer $m_i$ ($i \in \{1, 2\}$) is a product of primes congruent to $i \mod 3$. Therefore $\text{rad}_2(\mu \tilde{\mu} D_H) = d_H$ since $d_H$ is square free and the primes dividing $d_H$ are congruent to $2 \mod 3$.

Proof (Of Theorem 37). Let $X = \text{Km}_3(A)$, $Y = \text{Km}_3(B)$ be two generalized algebraic Kummer surfaces. There exists $\mu, \mu' \in \mathbb{Z}[j]$ such that the discriminants of the endomorphism rings of $A, B$ are:

$$
D_{\text{End}(A)} = \mu \tilde{\mu} D_H, \quad D_{\text{End}(B)} = \mu' \tilde{\mu}' D_{H'}.
$$

Suppose that $L_X^2 = L_Y^2$. Then according to Theorem 20,

$$
\mu \tilde{\mu} D_H = \mu' \tilde{\mu}' D_{H'}.
$$
thus by Lemma 38, \( d_H = d_{H'} \) which implies \( H \cong H' \), and moreover \( \mu \bar{\mu} = \mu' \bar{\mu}' \). The orders \( \text{End}(A), \text{End}(B) \) being of the same discriminant and of the form \( \mathcal{O}_\mu \) for some \( \mu \in \mathbb{Z}[j] \), they are isomorphic by Proposition 32.

Finally, we remark that if the surfaces \( X \) and \( Y \) are isogeneous, the abelian surfaces \( A \) and \( B \) have quaternionic multiplication by the same quaternion algebra \( H \), and therefore \( \text{rad}_2(\frac{1}{2}L_X^2) = \text{rad}_2(\frac{1}{2}L_Y^2) \).

\[ \square \]

\section{4.2 Order that contains an element of norm \(-1\)}

Let \( \mathcal{O} \) be the order of a quaternion algebra containing the Eisenstein integers. Let \( A \) be a general abelian surface with quaternionic multiplication by \( \mathcal{O} \), and let \( J \) be an order 3 symplectic automorphism of \( A \). We denote by \( X = \text{Km}_3(A, J) \) the associated order 3 Kummer surface. For future use in the proof of Proposition 40, let us prove the following result:

\textbf{Lemma 39} \ The group \( \mathcal{O}^* \) contains an element of norm \(-1\) if and only if \( L_X^2 = 2 \mod 6 \) or \( 3 \| L_X^2 \).

\textbf{Proof} \ We follow the proof of [22, Lemma 3.3] given for Eichler orders. Let \( Q \) be quadratic form \( Q(q) = \text{Tr}(q)^2 - \text{Nr}(q) \) for \( q \in \mathcal{O} = \mathcal{O}_\mu \). It is degenerate of rank 3 and induces a non-degenerate ternary quadratic form (also denoted by \( Q \)) on \( \mathcal{O}_\mu / \mathbb{Z} \), called the discriminant form. For any \( q \in \mathcal{O}_\mu \), the integer \( Q(q) \) is the discriminant of the quadratic subring \( \mathbb{Z}[q] \). It suffices to show that the discriminant form represents a prime \( p \) congruent to 1 mod 4, since then \( \mathcal{O} = \mathcal{O}_\mu \) contains a real quadratic order of discriminant \( p \), whose fundamental unit has norm \(-1\).

Suppose that \( d_H = 2 \mod 3 \) i.e. \( D_H = 3d_H \). A basis of \( \mathcal{O}_\mu \) is \( 1, j, \mu \phi, \mu j \phi \) with \( \phi^2 = d_H \). Consider \( d = \mu \bar{\mu}d_H = (\mu \phi)^2 \). The discriminant form is

\[ Q(xj + yj\mu \phi + zj\mu \phi) = -3x^2 + 4d(y^2 + yz + z^2). \]

Suppose that \( d_H = 1 \mod 3 \) i.e. \( D_H = d_H \) is coprime to 3. A basis of \( \mathcal{O}_\mu \) is \( 1, j, \mu \psi, \mu j \psi \) with \( \psi = \frac{1}{3}(\phi - 1) \), and \( \phi^2 = d_H \). Consider \( d = \mu \bar{\mu}d_H = (\mu \phi)^2 \). The discriminant form is

\[ Q(xj + yj\mu \psi + zj\mu \psi) = -3x^2 + 4xy - 2xz + z^2 + 4d(y^2 + yz + z^2). \]

In both cases, this is an indefinite quadratic form. Suppose that \( \mu \) is coprime to 3. Thus \( d \) is coprime to 3, the form \( Q \) is not 0 modulo any prime and we conclude as in [22, Lemma 3.3] that the quadratic form represents prime numbers \( p \) congruent to 1 mod 4, and therefore there exists a unit of norm \(-1\).

Suppose that \( 9 \| L_X^2 \) and consider \( \tau = x + yj + u\phi + vj\phi \). The equation

\[ \tau \bar{\tau} = x^2 - xy + y^2 - 3d'(u^2 - uv + v^2) = -1 \]

has no solution \( \tau = x + yj + u\phi + vj\phi \) modulo 3 for any \( d' \), thus the equation \( \tau \bar{\tau} = -1 \) has no solution: there is no unit of norm \(-1\).

\[ \square \]

\section{4.3 Number of generalized Kummer structures}

In this section, we consider an indefinite quaternion algebra \( H = \frac{(-3,d_H)}{\mathcal{O}} \), where \( d_H > 0 \) is a square-free product of primes congruent to 2 mod 3, so that the discriminant of \( H \) equals to
$d_H$ or $3d_H$. The generators are $r, \phi_0$ with $r^2 = -3, \phi_0^2 = d_H$ and $r\phi_0 = -\phi_0r$. As before, let $\theta$ be the quaternion $\phi_0$ or $\frac{1}{2}r(\phi_0 - 1)$ according if $3 \mid D_H$ or not. Let $\mathcal{O}$ be an order containing the sub-ring $\mathbb{Z}[j]$. By Proposition 30, we can suppose that $\mathcal{O}$ is contained in the maximal order $\mathcal{O}_m$, and there exists $\mu \in \mathbb{Z}[j]$ such that $\mathcal{O}$ is the ring

$$\mathcal{O}_\mu = \mathbb{Z}[j] \oplus \mathbb{Z}[j]\mu\theta.$$  

Let $A = \mathbb{C}^2/\Phi(I) \{ z, 1 \}$ be an abelian surface with quaternionic multiplication by $\mathcal{O} = \mathcal{O}_\mu$ and Picard number 3 ($I$ is an $\mathcal{O}$-ideal). Let $J_A$ be the order 3 automorphism acting on $A$ corresponding to $j \in \mathcal{O}$. Let

$$X = \text{Km}_3(A) = \text{Km}_3(A, J_A)$$

be the associated generalized Kummer surface: it is the minimal resolution of $A$ by the group generated by $J_A$.

Let $\mathcal{K}(X)$ denote the set of generalized Kummer structures on $X$, which is the set of isomorphisms classes of pairs $(B, G)$ such that $X \cong \text{Km}_3(B, G)$, where the pair $(B, G)$ is said isomorphic to $(B', G')$ if there exists an isomorphism $\phi : B \to B'$ of abelian surfaces such that $G = \phi G' \phi^{-1}$ (see [23]).

Let $\mathcal{C}_A$ be the set of orbits (under conjugation by $\text{Aut}(A)$) of order 3 symplectic automorphism sub-groups $G \subset \text{Aut}(A) = \mathcal{O}^*$.  

Let us recall that we proved in Theorem 17 that

$$\text{FM}(A) = \text{FM}(A, \text{NS}(A)) = \{ A, \hat{A} \}$$

and that by Theorem 11, if $(B, G_B) \in \mathcal{K}(X)$, then $B \in \text{FM}(A)$.

We also recall that when $L_X^2 \neq 0$, 36 mod 54, by Theorem 11, the generalized Kummer surface $\text{Km}(A, G)$ associated to any group $G \in \mathcal{C}_A$ is isomorphic to $X = \text{Km}_3(A, J_A)$. Hence in that case, there is a well-defined map

$$\Delta_A : \mathcal{C}_A \to \mathcal{K}(X), \ G \to (A, G),$$

which is moreover injective. Also from the discussion $\mathcal{K}(X) = \Delta_A(\mathcal{C}_A) \cup \Delta_{\hat{A}}(\mathcal{C}_{\hat{A}})$, with $\Delta_A(\mathcal{C}_A) = \Delta_{\hat{A}}(\mathcal{C}_{\hat{A}})$ if $A$ has a principal polarization and with $\Delta_A(\mathcal{C}_A) \cap \Delta_{\hat{A}}(\mathcal{C}_{\hat{A}}) = \emptyset$ otherwise.

If $L_X^2 = 0$ or 36 mod 54, then $\mathcal{K}(X)$ is bounded from above by the order of $\mathcal{C}_A \cup \mathcal{C}_{\hat{A}}$, and it can happen that two pairs $(A, G), (A, G')$ for $G, G' \in \mathcal{C}_A$ give non-isomorphic K3 surfaces [35, Corollary 24].

Let us recall that we denoted by $e_3(\mathcal{O})$ the number of elliptic points of order 3 on the Shimura curve $\mathcal{H}/\Gamma(\mathcal{O}^1)$. This is also the number of orbits of order 3 symplectic automorphism sub-groups $G \subset \text{Aut}(A)$ under conjugation by $\mathcal{O}^1 \subset \text{Aut}(A)$. We have

**Theorem 40** Suppose that $D_H = 1$ or that $L_X^2 = 0$ or 36 mod 54. Then

- $|\mathcal{K}(X)| \leq e_3(\mathcal{O})$ if $L_X^2 = 2 \text{ mod } 6$ or $3||L_X^2$,
- $|\mathcal{K}(X)| \leq 2e_3(\mathcal{O})$ otherwise.

Suppose that $D_H \neq 1$ and $L_X^2 \neq 0$, 36 mod 54. Then

- $|\mathcal{K}(X)| = \frac{1}{2}e_3(\mathcal{O})$ if $L_X^2 = 2 \text{ mod } 6$ or $3||L_X^2$,
- $|\mathcal{K}(X)| = 2e_3(\mathcal{O})$ otherwise.

**Proof** Let $(B, G) \in \mathcal{K}(X)$ be a Kummer structure on $X$. By Theorem 11, we can suppose that $B = A$ or $B = \hat{A}$. 

\[ \hat{A} \]
The surface $A$ is isomorphic to $\hat{A}$ if and only if $A$ is principally polarized, which is equivalent by 17 to

$$L_X^2 = 2 \mod 6 \text{ or } 3||L_X^2.$$  

By Lemma 39, this is also equivalent to the condition that group $O^*$ contains an element of norm $-1$, which is equivalent to suppose that $\Gamma(O^1)$ has index $2$ in $\Gamma(O^*)$.

* Case $L_X^2 = 2 \mod 6$ or $3||L_X^2$. Let us suppose that $L_X^2 = 2 \mod 6$ or $3||L_X^2$. Then the elements of norm $-1$ exchange the half planes $\mathcal{H}$ and $\bar{\mathcal{H}}$ and one has

$$\mathcal{H} \cup \bar{\mathcal{H}} / \Gamma(O^*) = \mathcal{H} / \Gamma(O^1).$$

In any case, that proves that $K(X)$ has order $\leq e_3(O)$. Suppose now that $H$ is a skew-field i.e. $D_H \neq 1$. Let $\gamma \in O^*$ be a norm $-1$ element. The matrix $\Phi(\gamma)$ has determinant $-1$ and acts anti-holomorphically on the Poincaré upper-plane by $z \rightarrow (a\bar{z} + b)/(c\bar{z} + d)$. That action induces a real structure on the Shimura curve $\mathcal{H}/\Gamma(O^1)$. It is known that this action has no fixed-point since $H$ is a skew field (see [15, Section 2.2]; a Shimura curve has no real points), since we suppose $D_H \neq 1$. Therefore there are no fixed points among the $e_3(O)$ order 3 elliptic points on the Shimura curve $\mathcal{H}/\Gamma(O^1)$. At the level of orbits in $\mathcal{H}/G$, if $G \subseteq O^1$ has order 3 and fixes $z \in \mathcal{H}$, then the point $\gamma z \in \mathcal{H}$ is fixed by $\gamma G\gamma^{-1} \subseteq O^1$, and is not in the orbit of $z$ under $O^1$. Therefore $\gamma$ maps the orbit $O(G) = \{\tau G \tau^{-1} | \tau \in O^1\}$ to the orbit $O(\gamma G\gamma^{-1})$, which is also equal to $\gamma O(G)\gamma^{-1}$. The action of the conjugation by $\gamma$ on the set of $e_3(O)$ orbits $O(G) = \{\tau G \tau^{-1} | \tau \in O^1\}$ has therefore no fixed element, hence the number of conjugacy classes of order 3 groups contained in $O^*$ under the action by conjugation of $O^*$ is $\frac{1}{2}e_3(O)$.

* Suppose that $L_X^2 = 18 \mod 54$. From Theorems 11 and 17, the Kummer structures on $X$ are the pairs $(A, G), (\hat{A}, \hat{G})$ with $G, \hat{G}$ symplectic of order 3. One has $O^* = O^1$. The abelian surface $A$ has $e_3(O)$ orbits of symplectic order 3 automorphism groups, and so has the surface $\hat{A} \neq A$. Thus the number of Kummer structures on $X$ is $2e_3(O)$.

* Suppose that $L_X^2 = 6k$ with $k = 0$ or $6 \mod 9$. Since all Kummer structures on $X = \text{Km}_3(G)$ are of the form $(A, G)$ or $(\hat{A}, \hat{G})$ with $G$ of order 3, we obtain the upper-bound $2e_3(O)$. This is only an upper bound since it can happen that there is a group $G$ such that $(A, G)$ is not a Kummer structure of $X$. 

Suppose that $\mu = q_1 \ldots q_n \in \mathbb{Z}[j]$ is a product of primes $q_1, \ldots, q_n$ in $\mathbb{Z}[j]$, with norm some natural primes $q_1, \ldots, q_n$ such that $q_j = 1 \mod 3$ and the $q_j$ are distinct. Let $A$ be a general abelian surface with quaternionic multiplication by $O_\mu$ and let $X = \text{Km}_3(A)$ be the associated generalized Kummer surface. A consequence of Propositions 35 is

**Corollary 41** The order $O_\mu = \mathbb{Z}[j] \oplus \mathbb{Z}[j] \mu \theta$ is an Eichler order of level $\mu \tilde{\mu}$. The number of Kummer structures on $X$ is equal to $2^{m+\varepsilon}$, where $m$ is the number of primes dividing $\frac{1}{2}L_X^2 = \mu \tilde{\mu}D_H$, and where $\varepsilon = -2$ if $3|D_H$, $\varepsilon = -1$ otherwise.

**Proof** The order $O_\mu$ is generated by $\gamma_1 = 1, \gamma_2 = j, \gamma_3 = \mu \theta, \gamma_4 = j \mu \theta$, and one computes that the reduced discriminant of $O_\mu$ satisfies

$$D_{O_\mu}^2 = \det(\text{Tr}(\gamma_i \gamma_j)) = (\mu \tilde{\mu})^2 D_H^2.$$  

By the hypothesis on $\mu$, the integer $\mu \tilde{\mu}$ is square free, thus by [1, Proposition 1.54], $O_\mu$ is an Eichler order of level $\mu \tilde{\mu}$.

The formula in Proposition 35 gives that $e_3(O_\mu) = 2^m$ if $D_H$ is coprime to 3 and $e_3(O_\mu) = 2^{m-1}$ if $3|D_H$. With the hypothesis on $H$ and $\mu$, either $L_X^2 = 2 \mod 6$ or $3||L_X^2$ and Theorem 40, gives that the number of Kummer structures equals to $2^{m+\varepsilon}$. 

\[\Box\]
Remark 42 From Corollary 41 and the above examples of Eichler orders, we see that in the same isogeny class of $X$, the number of generalized Kummer structures can be arbitrarily large. Moreover letting varying $H$, and taking the unique maximal order (up to conjugation), we also obtain generalized Kummer surfaces with an arbitrarily large number of Kummer structures.

Using the formulas in [12], the following Table gives the number of Kummer structures for some low values of $L^2_X$:

| $L^2_X$ | 6  | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 | 90 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $|\mathcal{K}(X)|$ | 1  | 1  | 1  | 1  | 1  | 4* | 1  | 1  | 3* | 2  | 1  | 2  | 4* |

where for $L^2_X = 0$ or 36 mod 54, the exponent * indicates that this is only an upper bound.

5 Moduli spaces of generalized Kummer surfaces

5.1 The algebraic case: irreducibility of the moduli space of generalized Kummer surfaces

For each integer $\ell > 0$ such that $\ell = 0$ or 2 mod 6, let $\mathcal{N}S_\ell$ be the Néron–Severi lattice of the generalized Kummer surface $X$ with Picard number 19 and the divisor $L_X$ such that $L^2_X = \ell$. We denote by $\mathcal{M}_\ell$ the one dimensional moduli space of algebraic K3 surfaces $X$ polarized by $\mathcal{N}S_\ell$, as defined in [14]. Let us recall that we obtain in Sect. 2.2 a unique rank 3 lattice $\text{NS}_\ell$ such that, for $X = \text{Km}_3(A)$ with $L^2_X = \ell$, one has $\text{NS}(A) \simeq \text{NS}_\ell$. Let us denote by $\mathcal{O} = \mathcal{O}_\mu$ the unique quaternion order with discriminant $\bar{\mu}D_H = \frac{1}{2}\ell$, so that the abelian surface $A$ has quaternionic multiplication by $\mathcal{O}$.

In [4] Barth asks how many irreducible components has the moduli space $\mathcal{M}_\ell$. Such a question on algebraic lattice polarized K3 surfaces has been studied for example in [14]. However, since the Néron–Severi group of the generalized Kummer surfaces is large, the results of [14] do not apply in our situation. We obtain:

**Theorem 43** The order $\mathcal{O}$ is principal. The moduli space of abelian surfaces with quaternionic multiplication by $\mathcal{O}$ has one irreducible component if $\ell = 2$ mod 6 or $3 || \ell$, and 2 irreducible components otherwise. The moduli space $\mathcal{M}_\ell$ of generalized Kummer surfaces is irreducible for any $\ell > 0$ (with $\ell = 0$ or 2 mod 6).

The remaining of this section is devoted to the proof of Theorem 43, during which we give the definition of what we mean by the moduli space of abelian surfaces with quaternionic multiplication by $\mathcal{O}$.

A Hodge structure of weight 2 and K3 type on the lattice $U^{\oplus 3}$ is the data of spaces $V^{p,q} \subset U^{\oplus 3} \otimes \mathbb{C}$ for $p + q = 2$, $p \geq 0$ such that $V^{p,q} = V^{q,p}$ and $\text{dim}_\mathbb{C} V^{2,0} = 1$, with a generator $\omega$ of $V^{2,0}$ satisfying $\omega \bar{\omega} = 0$, $\omega, \bar{\omega} > 0$, where the intersection form is induced from the intersection form on $U^{\oplus 3}$. For a lattice $\text{NS}_\ell \subset U^{\oplus 3}$ of rank 3 and signature $(1,2)$, the family of Hodge structures $(V^{p,q})_{p+q=2}$ such that $V^{1,1} \cap U^{\oplus 3} = \text{NS}_\ell$ is one-dimensional (it is biholomorphic to $\mathcal{H} \cup \bar{\mathcal{H}}$, where $\mathcal{H}$ is the upper half plane, see e.g. [4, Section 2.3] or [45, Section 7.2.3]).
Let $\NS$ be an even rank 3 lattice of signature $(1, 2)$ or $(0, 3)$. The families of complex tori $A$ with $\NS(A) \simeq \NS$ are in one-to-one correspondence with the embeddings $\NS \hookrightarrow U^{\oplus 3}$ up to isometry. By [21, Chapter 14, Proposition 1.8], such an embedding always exists. The orthogonal complement $T$ of $\NS$ in $U^{\oplus 3}$ is such that $T(-1)$ is in the same genus as $\NS$. Conversely any lattice $T$ in the genus of $\NS(-1)$ can be glued to $\NS$ using a glue map $\varphi : A_{\NS} \to A_T$ to form a rank 6 even unimodular lattice $\NS \oplus \varphi T$, which therefore is isomorphic to $U^{\oplus 3}$ (for glueing theory of lattices, we refer to the detailed Sect. 2 of [24]).

Let us denote by $T_\ell$ the transcendental lattice of an abelian surface $A$ with $\NS(A) \simeq \NS_\ell$. Since $\ell > 0$, by Corollary 10, the lattice $T_\ell$ is isomorphic to $\NS_\ell(-1)$. By Theorem 17, the co-set $\mathcal{P} = O(T_\ell)\backslash O(A_{T_\ell})$ has a unique element. Thus (see [20, Definition 1]), if $t_0, t_1 : T_\ell \hookrightarrow U^{\oplus 3}$ are any two embeddings, there exist $g \in O(T_\ell), \Phi \in O(U^{\oplus 3})$ such that $t_1 = \Phi \circ t_0 \circ g$.

Let us prove that the glueing of $\NS_\ell$ and $T_\ell$ is unique (one could refer to the general result of [26, Theorem], but we prefer to give a proof here). Let $\varphi_1, \varphi_2 : A_{T_\ell} \to A_{\NS_\ell}$ be two glueing maps. One obtains two glue-ups $T_\ell \oplus \varphi_1 \NS_\ell, T_\ell \oplus \varphi_2 \NS_\ell$ and after identification of these glue-ups with $U^{\oplus 3}$ by using suitable isomorphisms, we get two embeddings $t_0, t_1 : T_\ell \hookrightarrow U^{\oplus 3}$ and $t_0, t_1 : \NS_\ell \hookrightarrow U^{\oplus 3}$.

Let $g \in O(T_\ell), \Phi \in O(U^{\oplus 3})$ be such that $t_1 = \Phi \circ t_0 \circ g$. Since $\Phi$ is an element of the orthogonal group, $\Phi(\tau_0(\NS_\ell)) = \tau_1(\NS_\ell)$. Therefore the glueing of $\NS_\ell$ and $T_\ell$ is unique up to isomorphism (when $\ell < 0$ we will see examples of non-unique glue-ups).

Let $A$ be an abelian surface with $T(A) \simeq T_\ell$. The Hodge structure on $H^2(A, \mathbb{Z})$ is isomorphic to a Hodge structure $(\nu^p,q)_{p+q=2}$ on $U^{\oplus 3}$ such that $\nu^{1,1} \cap U^{\oplus 3} = T^\perp_\ell$. We thus obtain that there exists a unique 1-dimensional family of such Hodge structures.

**Definition 44** We call the corresponding one dimensional family of abelian surfaces $A$ with these Hodge structures the moduli space of abelian surfaces with quaternionic multiplication by $\mathcal{O}$.

It is the curve $(\mathcal{H} \cup \mathcal{H}) / \Gamma(\mathcal{O}^*)$, which has 2 irreducible components if and only if $9|L_X^2$ (see Sect. 3.4) and one irreducible component otherwise (here the surfaces $A$ are such that $L_X^2 = \ell$, for $X = \Km_3(A)$).

That implies that the set of $\End(A)$-ideals modulo principal ideals has a unique class, since $(\mathcal{H} \cup \mathcal{H}) / \Gamma(\mathcal{O}^*)$ is the unique family of abelian surfaces with multiplication by $\End(A) \simeq \mathcal{O}$.

Take any $\ell = 0$ or $2 \mod 6$. For a generic element $X = \Km_3(A)$ in $\mathcal{M}_\ell$, let $\mathcal{O}$ be the endomorphism ring of $A$. Up to isomorphism, the ring $\mathcal{O}$ is isomorphic to the order $\mathcal{O}_\mu = \mathbb{Z}[j] \oplus \mathbb{Z}[j] \mu \theta$ defined in Sect. 3, with $\frac{1}{2}L_X^2 = \mu \mu D_h$. An element of the curve $\mathcal{X}(\mathcal{O}) = (\mathcal{H} \cup \mathcal{H}) / \Gamma(\mathcal{O}^*)$ is a pair $(A, \iota_A)$, where $\iota_A : \mathcal{O}_\mu \hookrightarrow \End(A)$ is an embedding. To such a pair, one can associate the pair $(A, G_A)$, where $G_A$ is the order 3 symplectic group $G_A = \langle \iota_A(j) \rangle$. Conversely, if $G_A = \langle j_A \rangle$ is an order 3 symplectic automorphism group on an abelian surface $A$, by Sect. 2.7 and Corollary 32 there is an isomorphism $\iota_B$ from an order $\mathcal{O}_\mu$ (which is determined up to isomorphism by $L_X^2$) to the endomorphism ring $\End(B)$ such that $(A, G_A) \simeq (B, \langle \iota_B(j) \rangle)$.

The degree of the forgetful map

$$\Psi : \mathcal{X}(\mathcal{O}) = (\mathcal{H} \cup \mathcal{H}) / \Gamma(\mathcal{O}^*) \to \mathcal{M}_\ell$$

which associates to $(A, \iota_A)$ the generalized Kummer surface $X = \Km_3(A, G_A)$, where $G_A = \langle \iota_A(j) \rangle$, is therefore the number of Kummer structures on a generic $X \in \mathcal{M}_\ell$. If $\mathcal{O}$ is an Eichler order (then $\ell = 2 \mod 6$ or $3|\ell$), the map $\Psi$ is the quotient of $\mathcal{X}(\mathcal{O})$ (which is irreducible in that case) by the Atkin–Lehner involutions.
When $9|\ell$, the conjugation map $c_\phi : \gamma \to \phi \gamma \phi^{-1}$ by $\phi \in \mathcal{O}_\mu = \mathcal{O}$ still preserves $\mathcal{O}$ (thus $\mathcal{O}^*$). Since the norm of $\phi$ is negative, the action of $\phi$ on $\mathcal{H} \cup \overline{\mathcal{H}}$ exchanges the two half planes $\mathcal{H}$, $\overline{\mathcal{H}}$. The homomorphism $i : \mathcal{O} \to \text{End}(A)$ is not equivalent to $i \circ c_\phi$, and the map $\Psi$ factors through the quotient of $\mathcal{X}(\mathcal{O})$ by the involution on $\mathcal{X}(\mathcal{O})$ induced by $\phi$, which exchanges the two irreducible components of $\mathcal{X}(\mathcal{O})$. The moduli space $M_\ell$ is therefore irreducible.

### 5.2 Preliminaries on the lattice $U \oplus A_2$

In order to study the irreducible components of the moduli space of non-algebraic generalized Kummer surfaces, we need some preliminary results on the lattice $U \oplus A_2$ introduced in Sect. 2.2. By results of Vinberg [43], the lattice $(U \oplus A_2)(-1)$ is the Néron–Severi group of some K3 surfaces with a finite number of automorphisms, which have been studied in [2]. The following elements $c_1, \ldots, c_4$ of $U \oplus A_2$

$$c_1 = (0, 1, -1, 1), \ c_2 = (-1, -1, 0, 0), \ c_3 = (1, 0, 0, -1), \ c_4 = (1, 0, 1, 0)$$

(in basis $\gamma_1, \ldots, \gamma_4$ of $U \oplus A_2$) have square $c_j^2 = 2$; their intersection matrix is

$$\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{pmatrix}.$$

Let us denote by $r_1, \ldots, r_4$ the order 2 reflection

$$x \in U \oplus A_2 \rightarrow x - (c_k x) c_k \in U \oplus A_2,$$

where $(, )$ denotes the intersection form between elements of the lattice $U \oplus A_2$. The element $c_5 = (0, 0, 1, 1) \in U \oplus A_2$ has square $c_5^2 = 6$ and the reflection

$$r_5 : x \rightarrow x - \frac{1}{6} (c_5 x) c_5$$

preserves the lattice $U \oplus A_2$. In [43], Vinberg proves that lattice $U \oplus A_2$ (of signature $(3, 1)$) has the remarkable property that the group $W$ generated by the reflections through the roots of square 2 has finite index in the orthogonal group, and that in fact $W$ is generated by $r_1, \ldots, r_4$. Since the discriminant of the lattice $U \oplus A_2$ is 3, only reflections through elements of square 2 and 6 can preserve $U \oplus A_2$. There is, up-to the action of $W$, a unique element of square 6 and the orthogonal group $O(U \oplus A_2)$ is generated by $r_1, \ldots, r_5$. We denote by $SO(U \oplus A_2)$ the index 2 subgroup of $O(U \oplus A_2)$ of elements with determinant 1; it is generated by the elements $r_i r_j$, $1 \leq i, j \leq 5$.

### 5.3 The Barth moduli space of complex tori with order 3 symplectic groups

The following construction is due to Barth and we take the same notations as [4, Section 2.1]. Let $J_A$ be the generator of the order 3 symplectic group $G_A$ acting on $H_1(A, \mathbb{Z})$. In the basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $H_1(A, \mathbb{Z})$, the action of the symplectic order 3 automorphism $J_A$ is by the matrix

$$J = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1
\end{pmatrix}.$$
Knowing the complex structure on $A$ amounts to fixing some $\mathbb{R}$-linear maps $z_1, z_2 : H_1(A, \mathbb{R}) \to \mathbb{C}$ inducing an isomorphism $H_1(A, \mathbb{R}) \to \mathbb{C}^2$ of real vector spaces. The coordinates $(z_1, z_2)$ determine a period $\omega = z_1 \wedge z_2 \in H^2(A, \mathbb{C})$. The period $\omega$ is determined by the complex structure on $A$ up to multiplication by a scalar, and $\omega \wedge \omega = 0$, $\omega \wedge \bar{\omega}$ is positively oriented, so that $(H^2(A, \mathbb{C}), \mathbb{C}\omega)$ is a weight 2 Hodge structure. For the converse, let us recall Shioda’s Torelli Theorem:

**Theorem 45** [39] Let $A$ and $B$ be two 2-tori. There is an isomorphism of Hodge structures

$$ (H^2(B, \mathbb{Z}), \mathbb{C}\omega_B) \simeq (H^2(A, \mathbb{Z}), \mathbb{C}\omega_A) $$

if and only if $B \simeq A$ or $B \simeq \hat{A}$, where $\hat{A}$ is the dual of $A$. For any given weight 2 Hodge structure $(U^{\oplus 3}, \mathbb{C}\omega)$, there exists a complex 2-torus $A$ with an Hodge isometry

$$ (H^2(A, \mathbb{Z}), \mathbb{C}\omega_A) \simeq (U^{\oplus 3}, \mathbb{C}\omega). $$

Thus a space $\mathbb{C}\omega \subset H^2(A, \mathbb{C})$ with $\omega \wedge \omega = 0$ and $\omega \wedge \bar{\omega}$ positively oriented determines (at most) two complex structures (in [4, Section 2.1], Barth did not address that problem). Moreover if the map $J$ preserves $\omega$, i.e. if $J^*\omega = \omega$, then $J$ is $\mathbb{C}$-linear for the complex structure defined by $\mathbb{C}\omega$ and $J_A$ is holomorphic. The group $G_A = (J_A)$ then acts on $\wedge^2 H^1(A, \mathbb{Z}) = H^2(A, \mathbb{Z}) \simeq U^{\oplus 3}$, and the invariant part is the lattice $U \oplus A_2$. In [4, Section 2.1], Barth considers the natural domain

$$ \Omega = \{ \omega \in \mathbb{P}(U \oplus A_2) \otimes \mathbb{C}) \mid \omega \wedge \omega = 0, \ \omega \wedge \bar{\omega} > 0 \}, $$

which he calls a period domain for pairs $(A, G_A)$ of a complex 2-torus $A$ with an order 3 automorphism group $G_A$. There is a group $\Gamma \subset O(U \oplus A_2)$ (described below) acting on $\Omega$, such that the quotient $\mathcal{M}_B = \Omega/\Gamma$ (of dimension 2; the action of $\Gamma$ is not discrete) is a moduli space for these pairs $(A, G_A)$. This is done by associating for each such $A$, with Hodge structure $(H^2(A, \mathbb{Z}), \omega_A)$ and a marking $\iota : H^2(A, \mathbb{Z}) \xrightarrow{\iota} U^{\oplus 3}$, the period $[\iota(\omega_A)] \in \Omega \subset \mathbb{P}(U \oplus A_2) \otimes \mathbb{C})$; taking the period $\iota(\omega_A)$ modulo the group $\Gamma$ makes the map independent of the choice of the marking $\iota$.

We remark that although a period $[\omega] \in \mathbb{P}(U \oplus A_2) \otimes \mathbb{C})$ lifts to a unique period $\omega$ in $\mathbb{P}(U^{\oplus 3} \otimes \mathbb{C})$ (we recall that there is a natural inclusion $U \oplus A_2 \subset U^{\oplus 3}$), according to Shioda’s Torelli Theorem, there are (at most) two surfaces $A, \hat{A}$ which have the same period $\omega$. Of course when $A$ is a principally polarized abelian surface, one has $A \simeq \hat{A}$, and there is no confusion, but in general the moduli space $\mathcal{M}_B$ parametrizes pairs $(A, G_A)$, $(\hat{A}, \hat{G}_A)$ of two dimensional complex tori with an order 3 automorphism group.

The groups $O(U \oplus A_2)$, $SO(U \oplus A_2)$ defined in Sect.5.2 are related to the group $\Gamma$ as follows. As defined in [4, Section 2.1], let $\Gamma'$ be the group of invertible matrices in basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ preserving the orientation of $H_1(A, \mathbb{Z})$ and commuting with the action of $J$. This is the group of elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrices $A, B, C, D \in GL_2(\mathbb{Z})$ that are commuting with $T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. The group $\Gamma$ is $\Gamma = \rho(\Gamma')$, the image by the representation $\rho : \Gamma' \to O(U \oplus A_2)$ of the group $\Gamma'$ acting on $U \oplus A_2 \subset U^{\oplus 3} = \wedge^2 H^1(A, \mathbb{Z})$. From that description, we obtain the following result:

**Proposition 46** We have $GL_2(\mathbb{Z}[j]) \simeq \Gamma'$ and the kernel of $\rho$ is $\langle -j I_2 \rangle$, where $j^2 + j + 1 = 0$. The group $SO(U \oplus A_2)$ is equal to the group $\Gamma$ and is isomorphic to $GL_2(\mathbb{Z}[j])/\langle -j I_2 \rangle$.

So as expected by Barth, the group $\Gamma$ is close to be the orthogonal group $O(U \oplus A_2)$.
**Proof** (Of Proposition 46). The fact that \( GL_2(\mathbb{Z}[j]) \simeq \Gamma' \) is obtained readily from the above description, by interpreting the \( 2 \times 2 \) matrices as block matrices, by substituting \( T = -\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \) to \( j \), and the size 2 identity matrix to 1. It is not difficult to check that the kernel of \( \rho \) contains the matrix \(-\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}\). The morphism \( \rho : GL_2(\mathbb{Z}[j]) \to O(U \oplus A_2) \) extends to a morphism \( \rho_Q : GL_2(\mathbb{Q}[j]) \to GL((U \oplus A_2) \otimes \mathbb{Q}) \). The kernel of \( \rho_Q \) is a distinguished sub-group. The distinguished sub-groups of \( GL_2(\mathbb{Q}[j]) \) are classified: these are either the group of homotheties, \( SL_2(\mathbb{Q}[j]) \) or some subgroups containing \( SL_2(\mathbb{Q}[j]) \).

Only the sub-group \( \langle -jI_2 \rangle \) of the group of homotheties is in the kernel of \( \rho \). The group \( SL_2(\mathbb{Q}[j]) \) is not contained in the kernel of \( \rho_Q \), thus neither are his over-groups. We conclude that \( \text{Ker}(\rho) = \langle -jI_2 \rangle \) and \( \Gamma \simeq GL_2(\mathbb{Z}[j])/\langle -jI_2 \rangle \).

The following matrices

\[
\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

are generators of \( GL_2(\mathbb{Z}[j]) \) (see e.g. [41]). A direct computation gives that their action on \( U \oplus A_2 \) is by the determinant 1 matrices

\[
\begin{align*}
\tau_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \\
\tau_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\tau_4 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{align*}
\]

in basis \( \gamma_1, \ldots, \gamma_4 \) of \( U \oplus A_2 \), therefore \( \tau_1, \ldots, \tau_4 \) are generators of \( \Gamma \). One can check that

\[
\begin{align*}
\tau_1 &= (r_1r_3r_2)^2r_4r_2, \\
\tau_2 &= r_1r_2r_4r_2r_1r_3r_2r_3, \\
\tau_3 &= r_3r_1r_2r_5r_1r_4, \\
\tau_4 &= r_1r_3r_2r_3r_1r_2,
\end{align*}
\]

which implies that \( \Gamma \subset SO(U \oplus A_2) \). Conversely, the products \( r_ir_j \) with \( 1 \leq i < j \leq 5 \) satisfy

\[
\begin{align*}
r_1r_2 &= \tau_3^2\tau_2\tau_4, \\
r_1r_3 &= \tau_1^2\tau_2\tau_4\tau_2, \\
r_1r_4 &= \tau_2^2\tau_2\tau_1\tau_4, \\
r_1r_5 &= \tau_3^2\tau_2\tau_3 \\
r_2r_3 &= \tau_4\tau_3\tau_3\tau_4\tau_2, \\
r_2r_4 &= \tau_3\tau_4\tau_3\tau_4\tau_4, \\
r_2r_5 &= \tau_3\tau_4\tau_2^2, \\
r_3r_4 &= \tau_4\tau_3^2\tau_2\tau_3\tau_4, \\
r_3r_5 &= \tau_4\tau_3(\tau_4\tau_4)^2, \\
r_4r_5 &= \tau_3\tau_4\tau_3\tau_2\tau_3\tau_4,
\end{align*}
\]

and since \( r_1r_j = (r_jr_i)^{-1} \) (because \( r_k^2 = 1 \)), one also knows the products \( r_ir_j \) with \( i > j \). Therefore \( SO(U \oplus A_2) \subset \Gamma \) and the two groups are equal. \( \square \)

### 5.4 The irreducible components of the moduli space of non-algebraic generalized Kummer surfaces

In this section, we study the moduli spaces \( \mathcal{M}_\ell \) of generalized Kummer surfaces \( X = \text{Km}_3(A) \) with non-zero invariant class \( L_X \in \text{NS}(X) \) such that \( L_X^2 = \ell \leq 0 \). We give an algorithm to compute a complete set of all embeddings classes of the lattice \( \text{NS}_h \) into \( U^{\oplus 3} \) for any \( \ell \leq 0 \) with \( \ell = 0 \) or 2 mod 6. Contrary to the case with \( \ell > 0 \), such an embedding is, in general, far from being unique up to isometry. Moreover, the number of elements in the genus of \( \text{NS}_h \) grows to infinity with \( \ell \). Indeed, for any integer \( k \), by the Smith–Minkowski–Siegel mass formula, there is only a finite number of definite quadratic forms of rank \( \geq 3 \) that have less than \( k \) elements in their genus.
Consider \( P = (5, -4, 3, -3) \in U \oplus A_2 \): it has square \(-22\) and \( c_j P = -1 \) for \( j \in \{1, \ldots, 4\} \) (see Sect. 5.2 for the definition of \( c_j \)). Let us define the negative cone \( N \) as

\[
N = \{ x \in U \oplus A_2 \mid x^2 \leq 0, \ P x \leq 0 \}
\]

(\( P \) realizes a choice between the two connected components of \( \{ x^2 \leq 0 \} \setminus \{0\} \)). By [43, Section 6], the polyhedral cone

\[
\Pi = \{ x \in U \oplus A_2 \mid x c_j \leq 0, \ j = 1, \ldots, 4 \} \cap N
\]

is a fundamental domain for the action of the reflection group \( W \) on the cone \( N \). Thus the cone

\[
\Pi' = \{ x \in U \oplus A_2 \mid x c_j \leq 0, \ j = 1, \ldots, 5 \} \cap N
\]

is a fundamental domain for the action of \( O(U \oplus A_2) \) on \( N \). Since the group \( SO = SO(U \oplus A_2) \) is an index 2 sub-group of \( O(U \oplus A_2) \), any reflection \( r_j \) is such that the cosets of \( O(U \oplus A_2)/SO \) are \( SO, r_j SO \). We thus get that \( \Pi' + r_j \Pi' \) is a fundamental domain for \( SO \), in particular:

**Proposition 47** The cone \( \Pi = \Pi' + r_5 \Pi' \) is also a fundamental domain for the action of \( SO(U \oplus A_2) = \Gamma \).

The Hilbert basis of \( \Pi \) has 5 elements

\[
w_1 = (1, -1, 1, -1), \ w_2 = (2, -1, 1, -1), \ w_3 = (2, -2, 1, -1), \ w_4 = (3, -3, 2, -1), \ w_5 = (3, -3, 1, -2),
\]

which means that every element of the cone \( \Pi \) is linear combination with positive or zero integral coefficients of the vectors \( w_k, k = 1, \ldots, 5 \). The extremal rays of \( \Pi \) are \( w_1, w_2, w_4, w_5 \). The intersection matrix of the vectors \( w_k, k = 1, \ldots, 5 \) is

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 3 \\
1 & 2 & 4 & 6 & 6 \\
2 & 4 & 6 & 9 & 9 \\
3 & 6 & 9 & 12 & 15 \\
3 & 6 & 9 & 15 & 12
\end{pmatrix}
\]

For \( j \in \{1, 2, 4, 5\} \), let \( F_j \) be the cone generated by the rays \( w_k, k \in \{1, 2, 4, 5\} \setminus \{j\} \). The cones \( F_j \) with \( j \in \{1, 2, 4, 5\} \) are the 4 facets of \( \Pi \); one can check that the facets \( F_4, F_5 \) are exchanged under the group \( \Gamma \). By the above description, one can compute all representatives of the polarizations \( L_A \) (with \( L_A^2 \leq 0 \)) of the moduli spaces of complex tori in the polyhedral cone \( \Pi \). We have to take into account that one must avoid repetition from the facets \( F_4, F_5 \), which means (for example) to choose to exclude solutions that are in \( F_5 \setminus F_4 \). Moreover, by Proposition 2.2, if we are looking for surfaces \( X \) with \( L_X^2 = 0 \mod 6 \), we must take only the solutions \( L_A = (n_1, n_2, n_3 + n_4, n_4) \) in basis \( \gamma_1, \ldots, \gamma_4 \) with \( \gcd(n_1, n_2, n_3, n_4) = 1 \) and \( \gcd(n_1, n_2, n_3, 3) = 1 \). If we search for surfaces with \( L_X^2 = 2 \mod 6 \), we must search solution of the form \( (3n_1, 3n_2, 3n_3 + n_4, n_4) \) with \( \gcd(3n_1, 3n_2, 3n_3, n_4) = 1 \).

For example, the element \( w_1 \) of the Hilbert basis is the unique element of the cone \( \Pi \) of square 0, therefore

**Corollary 48** The moduli space of non-algebraic generalized Kummer surfaces \( X \) such that \( L_X^2 = 0 \) is irreducible.
Let us give another example. The polarizations

\[ L_1 = (6, -5, 4, -4), \quad L_2 = (15, -14, 14, -14), \quad L_3 = (9, -7, 7, -7) \]

are in \( \Pi \) and are the representatives of all polarizations \( L_A \) of square \(-28\) modulo the action of \( \Gamma \). Therefore the moduli space \( \mathcal{M}_{-84} \) has exactly 3 irreducible components. Corresponding to \( L_1, L_2, L_3 \), there are three non-isomorphic embeddings of the lattice \( \text{NS}(A) \) in the even unimodular lattice \( U^{\oplus 3} \). Knowing the classes \( L_j \), one can compute these embeddings, and the transcendental lattice \( T_j \) of the complex tori \( A \) with invariant class \((\text{under } G_A) L_j, j \in \{1, 2, 3\} \). The lattices \( T_2, T_3 \) are isomorphic to \( \text{NS}(A)(-1) \), and the lattice \( T_1 \), which has a basis with Gram matrix

\[
\begin{pmatrix}
4 & 2 & -2 \\
2 & 6 & -3 \\
-2 & -3 & 6
\end{pmatrix},
\]

is not isomorphic to \( \text{NS}(A)(-1) \) (the minimum of the lattice \( T_1 \) is 4, against 2 for \( T_2, T_3 \)), but of course it is in the same genus.

Using the above algorithm, for an integer \( k \), the following table gives in line \#\( L_A \) the number of irreducible components in the moduli space \( \mathcal{M}_{-6k} \) of generalized Kummer surfaces \( X \) such that \( 3L_X^2 = L_X^2 = -6k \) (which is also the number of inequivalent polarizations \( L_A \) under \( \Gamma \)), and in the line \#\( G \) the number of elements in the genus of \( \text{NS}(A) \):

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \#\( L_A \) | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 4 | 2 | 4 | 5 |
| \#\( G \) | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 3 | 1 | 3 | 2 | 2 | 2 | 4 | 1 |

The following table gives the same information for \( k \) such that \( L_X^2 = -6k + 2 \):

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \#\( L_A \) | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 2 | 5 | 4 | 5 | 3 | 7 | 3 | 5 |
| \#\( G \) | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 4 | 2 | 5 | 2 | 4 |

Let us remark that the set of Hodge structures on \( T(A) \) for a complex torus \( A \) with \( \text{NS}(A) \cong \text{NS}_{\ell} \) is \( \mathbb{P}^1 \) for \( \ell < 0 \):

**Lemma 49** Let be \( L \in U \oplus A_2 \) such that \( L^2 < 0 \). The domain \( \Omega_L = \Omega \cap L^\perp \) is isomorphic to \( \mathbb{P}^1 \).

**Proof** The lattice \( L^\perp \subset U \oplus A_2 \) is positive definite, so there are real coordinates \( x_1, x_2, x_3 \) of \( L^\perp \otimes \mathbb{R} \) in which the intersection form is \( x_1^2 + x_2^2 + x_3^2 = 0 \). Let \( z_1, z_2, z_3 \) be the corresponding coordinates on \( L^\perp \otimes \mathbb{C} \); each \( \omega \in \Omega_L \) is such that \( \omega.\omega = 0 \) i.e. \( z_1^2 + z_2^2 + z_3^2 = 0 \), and the condition \( \omega.\bar{\omega} = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 > 0 \) is empty since \( \omega \in \mathbb{P}((U \oplus A_2) \otimes \mathbb{C}) \). Therefore \( \Omega_L \) is a smooth quadric in \( \mathbb{P}^2 \) and \( \Omega_L \cong \mathbb{P}^1 \). \( \Box \)

The moduli space \( \mathcal{M}_L \) of complex tori \( A \) such that \( L \) is in the Néron–Severi group of \( A \) is the quotient of \( \mathbb{P}^1 \) by \( \Gamma_L \), the stabilizer of \( L \) in \( \Gamma \). Since \( \Gamma_L \) must preserve the positive definite
lattice $\text{NS}(A)^{\perp}$, such a group is finite (the automorphism group of $\text{NS}(A)^{\perp}$ is $(\mathbb{Z}/2\mathbb{Z})^2 \times S_3$ or the dihedral group $D_6$ of order 12 according if $L_X^2 = 0 \mod 6$ or $L_X^2 = 2 \mod 6$). If $L$ is in the interior of the fundamental domain $\Pi$, the group $\Gamma_L$ is trivial.

### 5.5 Concluding remark

**Computing the number of polarizations up to automorphisms:** By a theorem of Narasimhan–Nori, there are always - up to automorphisms- a finite number of polarizations of any given square on an abelian variety $A$. The computation of that number for principal polarizations has been done for several classes of abelian varieties by various authors (Lange, Rotger...). Let now $A$ be an abelian surface with an order 3 symplectic automorphism group, and suppose (to simplify things) that $A$ has a principal polarization. The generalized Kummer structures on $X = \text{Km}_3(A)$ are the conjugacy classes of such groups $G_A$ on $A \cong \hat{A}$. To $G_A$, we associate the polarization $L_A \in \text{NS}(A)$, which generates the invariant part of $\text{NS}(A)$ under $G_A$ and the number $L_X^2$ does not depend on the group $G_A$. If $L \in \text{NS}(A)$ is a polarization of square $L^2 = L_A^2$, its orthogonal complement in $\text{NS}(A)$ is a definite lattice of rank 2 and discriminant 3, thus is the lattice $A_2$. Therefore by [39, Section 4, Theorem], $L$ corresponds to the fixed part of an order 3 automorphism. The integer $\frac{1}{2}e_3(O_L)$ is therefore also the number of orbits under $\text{Aut}(A)$ of polarization $L \in \text{NS}(A)$ of square $L^2 = L_A^2$.

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