Quantum description of reality is empirically incomplete

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Empirical falsifiability of the predictions of physical theories is the cornerstone of the scientific method. Physical theories attribute empirically falsifiable operational properties to sets of physical preparations. A theory is said to be empirically complete if such properties allow for a not fine-tuned realist explanation, as properties of underlying probability distributions over states of reality. Such theories satisfy a family of equalities among fundamental operational properties, characterized exclusively by the number of preparations. Quantum preparations deviate from these equalities, and the maximal quantum deviation increases with the number of preparations. These deviations not only signify the incompleteness of the operational quantum formalism, but they simultaneously imply quantum over classical advantage in suitably constrained one-way communication tasks, highlighting the delicate interplay between the two.

Quantum theory is all set to fuel the key technological advances of the 21st century. However, even after almost a century since its conception, there is no consensus about the precise sense in which the structure of reality that quantum theory posits conflicts with classical worldviews. Addressing such questions necessitates formal notions of classicality, and the realist (ontological) framework provides a vital ground for such notions [1–4].

Typically, these notions of classicality ascribe certain operational phenomena a not fine-tuned realist basis by requiring these phenomena to hold intact at the level of potentially inaccessible underlying reality [5]. These phenomena double as empirically falsifiable operational prerequisites for tests of the notions of classicality. For instance, Bell’s local causality [1, 6] ascribes to non-signalling correlations a parameter-independent realist explanation 1, and the no-signaling condition ensured by sufficient spatial separation forms the operational prerequisite of Bell tests. Generalised noncontextuality attributes identical realist counterparts to operationally equivalent physical entities, and the operational indistinguishability of these entities forms the prerequisite of experimental tests of noncontextuality [3]. The recently introduced notion of classicality, bounded ontological distinctness (BOD) explains the distinguishability of operational physical entities by the distinctness of their realist counterparts, and p-distinguishability of the physical entities forms a prerequisite of the experimental tests of BOD [4].

On the other hand, the realist notions of classicality yield empirically falsifiable operational consequences, typically in the form of statistical inequalities. The quantum violation of these inequalities not only highlights the necessity of realist fine-tuning, discarding a large class of realist explanations, but also powers quantum advantage in a plethora of computational, communication and information processing tasks [4, 7–9]. Therefore, empirically falsifiable phenomena feature as the operational prerequisites, as well as the operational consequences of the realist notions of classicality. Moreover, to distill the spectrum of non-classical operational predictions of quantum theory, all empirically falsifiable operational phenomena warrant a not fine-tuned realist basis. Stemming from this unifying perception, in this Letter, we introduce a realist notion of classicality, termed empirical completeness.

Operational theories attribute empirically falsifiable operational properties to sets of preparations. Here, we consider an operationally relevant class of such properties, namely, the maximum success metrics of one-way communication tasks. If these properties allow for a not fine-tuned realist explanation, i.e., if there exists corresponding sets of probability distributions over states of reality (referred to as epistemic states) which exactly and exclusively explain the operational properties, the theory or the fragment thereof is said to be empirically complete.

In contrast to other notions of classicality, we demonstrate that empirically complete theories satisfy a family of equalities among elemental operational properties of set of preparations. These equalities hold irrespective of the particulars of the preparations and are characterised solely by their number. The corresponding properties of sets of quantum preparations deviate from these equalities, signifying the incompleteness of the theory.

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1 Bell’s local-causality entails two distinct realist assumptions. While parameter independence forms the not fine-tuned realist basis for the operational no-signaling condition, outcome independence is an auxiliary assumption of purely causal nature.
Moreover, as classical communication is empirically complete, these deviations imply quantum over classical advantage in suitably constrained one-way communication tasks. The fact that classical preparations adhere to statistical equalities implies that quantum advantage can be obtained with very inefficient detectors, making the experimental demonstrations much more feasible [10]. Finally, employing state of the art semidefinite programming (SDP) techniques we characterise the quantum deviations for the case of three preparations, and provide evidence of increasing maximal quantum deviation with the number of preparations.

Let us begin by revisiting the requisite preliminaries. Operational physical theories such as quantum theory serve a two fold purpose, (i) they prescribe mathematical objects to experimental procedures, and (ii) predict the consequent observations. Here we consider simple experiments entailing a preparation $P \in \mathcal{P}_O$ of a physical system, followed by a $K$-outcome measurement $M \in \mathcal{M}_O$, where $\mathcal{P}_O$ and $\mathcal{M}_O$ are sets of preparations and measurements featuring in an operational theory. The theory yields predictions of the form $\mathbb{P}(k|P,M)\equiv \mathbb{P}(k|M)$ specifying the probability of observing an outcome $k \in [K]$ (where $[a] \equiv \{1, \ldots, a\}$) as a result of performing a measurement $M$ on a preparation $P$.

In quantum theory preparations are described by density operators $\rho_P \in \{\rho \in \mathcal{B}_+(\mathcal{H}) | \text{Tr} \rho = 1\}$, and measurements are defined by POVMs $\{M_k \in \mathcal{B}_+(\mathcal{H})\}_k$ such that $\sum_k M_k = \mathbb{I}$, where $\mathcal{B}_+(\mathcal{H})$ is the set of bounded positive semidefinite operators on some Hilbert space $\mathcal{H}$ which is assumed to be finite dimensional, and $\mathbb{I}$ is the identity operator on $\mathcal{H}$. The predictions of quantum theory are specified by the Born rule $\mathbb{P}(k|P,M) = \text{Tr}(\rho_P M_k)$. Yet another relevant class of operational models is that of $d$-levelled classical communication. In such theories an operational preparation $P$ corresponds to an encoding scheme entailing a probability distribution $e_P$ over a $d$-levelled classical message $\omega \in [d]$, and a $K$-outcome measurement $M$ is described by a decoding scheme specifying the probability $d_M(k|\omega)$ of producing an outcome $k \in [K]$ upon receiving the message $\omega$. The predictions of such models are given by $\mathbb{P}(k|P,M) = \sum_\omega e_P(\omega)d_M(k|\omega)$.

Owing to the ubiquitous distinction between preparation and measurement procedures, operational theories implicitly assign empirically falsifiable properties to sets of physical preparations. In this Letter, we consider an operationally relevant subclass of such properties defined as follows,

**Definition 1** (Operational properties). Associated to a given set of $n$ preparations $P \equiv \{P_x \in \mathcal{P}_O\}_{x=1}^n$, an empirically falsifiable operational property has the generic form,

\[
S_n^{(O)}(P) = \max_{M \in \mathcal{M}_O} \left\{ \sum_{x,k} c_{x,k} \mathbb{P}(k|P_x,M) \right\},
\]

where $\{c_{x,k} \in \mathbb{R}\}_{x,k}$ are real coefficients and the maximisation is over the set of all measurements $\mathcal{M}_O$ allowed in the operational theory.

The operational properties (1) are closely linked to one-way communication tasks, wherein the sender encodes the classical input $x \in [n]$ onto a physical preparation $P_x \in \mathcal{P}$, and receiver decodes the message by performing some measurement $M \in \mathcal{M}_O$ to produce an outcome $k$. We gauge the performance of these resources using a success metric $S_n^{(O)}(P,M) = \sum_{x,k} c_{x,k} \mathbb{P}(k|P_x,M)$. For a given set of preparations $P$, each operational property (1) corresponds to the maximal attainable success metric, i.e., $S_n^{(O)}(P) = \max_{M \in \mathcal{M}_O} \{S_n^{(O)}(P,M)\}$. The maximization relieves $S_n^{(O)}(P)$ of its dependence on the measurements featuring in an operational theory deeming it to be an exclusive operational property of the given set of preparations $P$.

For a given set of preparations $P$, if one can experimentally demonstrate higher success than $S_n^{(O)}(P)$ in the associated communication task, then the operational theory and its prescriptions are falsified. For a given set of quantum preparations $\rho \equiv \{\rho_x\}_{x=1}^n$, evaluating an operational property $S_n^{(O)}(\rho) = \max_{M \in \mathcal{M}_O} \{\sum_{x,k} c_{x,k} \text{Tr}(\rho_x M_k)\}$, invariably constitutes a semidefinite program, which owing to its strong duality yields the precisely the maximum success metric along with the optimal POVM $\{M_k\}$ [11]. Whereas evaluating an operational property $S_n^{(O)}(e)$ for a given set of encoding schemes $e \equiv \{e_x\}_{x=1}^n$ constitutes a linear program.

The realist framework seeks to explain the predictions of an operational theory, whilst assuming the existence of observer independent attributes associated with physical systems. The complete specification of each attribute is referred to as the state of reality $\lambda$ of a physical system. A realist model for a prepare and measure fragment of an operational theory consists of the following three elements: (i) a measurable space, $\Lambda$, known as the **realist state space**, (ii) a probability measure $\mu_P$ on $\Lambda$ describing the epistemic state of the system for each preparation $P \in \mathcal{P}_O$, and (iii) for every state of reality $\lambda \in \Lambda$ and measurement $M \in \mathcal{M}_O$, a probability distribution $\xi_M(\cdot|\lambda)$ over the possible outcomes of $M$, referred to as a response scheme. A realist model explains the operational predictions if $\mathbb{P}(k|P,M) = \int_{\Lambda} d\lambda \mu_P(\lambda)\xi_M(k|\lambda)$.

A realist explanation of an operational property (1) has the form $S_n^{(O)}(P) = \max_{\{\mu_x\}_{x=1}^n} \{\sum_{x,k} c_{x,k} \int_{\Lambda} d\lambda \mu_x(\lambda)\xi_M(k|\lambda)\}$, where the maximization is over the possibly fine-tuned set of operationally accessible response schemes. To distill the necessity of such fine-tuning, we define not fine-tuned exclusive properties of sets of epistemic states as,

**Definition 2** (Not fine-tuned realist properties). Associated to a given set of $n$ epistemic states $\mu \equiv \{\mu_x\}_{x=1}^n$
a not fine-tuned realist property has the generic form,

\[ S_n^{(A)}(\mu) = \max_{\{\xi(k|\lambda)\}} \left\{ \sum_{x,k} c_k^x \int \lambda \mu_x(\lambda)|\xi(k|\lambda)\rangle \right\}, \]  

(2)

where \( c_k^x \in \mathbb{R} \) are real coefficients and the maximization is over the set of all valid response schemes which satisfy positivity (\( \forall \lambda \in \Lambda, k \in [K] : \xi(k|\lambda) \geq 0 \)) and completeness (\( \forall \lambda \in \Lambda : \sum_k \xi(k|\lambda) = 1 \)).

As the set of response schemes constrained by only positivity and completeness forms a convex polytope with deterministic response functions as extremal points, we can solve the maximization in (2) by selecting the response functions that for each \( \lambda \), yield the outcome \( k \) which maximises the function \( \sum_x c_k^x \mu_x(\lambda) \) such that,

\[ S_n^{(A)}(\mu) = \int \lambda \max_k \left\{ \sum_x c_k^x \mu_x(\lambda) \right\}. \]  

(3)

This expression further substantiates the fact that the maximization over response schemes relieves \( S_n^{(A)}(\mu) \) from its dependence on response schemes, deeming it to be an exclusive property of the set of epistemic states \( \mu \).

Now we have all the necessary ingredients to formally introduce empirical completeness as a characterising feature of operation theories,

**Definition 3 (Empirically complete theories).** An operational theory or a fragment thereof is said to be empirically complete (EC) if for all sets of preparations \( P \equiv \{ P_x \in \mathcal{P}_O \}_{x=1}^n \) and all associated empirically falsifiable operational properties \( S_n^{(O)}(P) \) (1), there exists underlying sets of epistemic states \( \mu \equiv \{ \mu_x \}_{x=1}^n \) with no fine-tuned realist properties \( S_n^{(A)}(\mu) \) (2) such that,

\[ S_n^{(A)}(\mu) = S_n^{(O)}(P). \]  

(4)

As an example, \( d \)-levelled classical communication models are empirically complete, owing to the fact that the \( d \)-levelled classical message \( \omega \) forms an operationally accessible sufficient statistic for the underlying state of reality \( \lambda \). This implies that for all empirically falsifiable operational properties \( S_n^{(C)}(e) \) associated with a set of encoding schemes \( e \equiv \{ e_x \}_{x=1}^n \), we can obtain not fine tuned realist explanations by taking the message itself to be the state of reality, i.e., \( \lambda = \omega \), and the encoding schemes to be the epistemic states, i.e., \( e \equiv \mu \), such that \( S_n^{(A)}(\mu) = S_n^{(C)}(e) \).

Defined in this way, empirical completeness underlies other well-known notions of classicality. As the distinguishability of a set of preparations forms an empirically falsifiable operational property, empirical completeness directly implies (symmetric) maximal \( \psi \)-epistemology [12], bounded ontological distinctness of preparations [4], and preparation noncontextuality [3]. Equipped with certain quantum theory dependent assumptions, empirical completeness can also be shown to imply [4, 13, 14] generalized noncontextuality [3], Kochen-Specker noncontextuality [2] and Bell local-causality [1]. Consequently, the quantum violation of the operational consequences of all other well-known notions of classicality which typically constitute statistical inequalities, imply the operational incompleteness of quantum theory.

To bring forth the characteristic empirically falsifiable operational consequences of empirical completeness we consider the following elemental operational properties associated with a set of \( n \geq 2 \) operational preparations \( P \equiv \{ P_x \}_{x=1}^n \) and its two member subsets, respectively,

(i.) **Average set distinguishability** \( (\bar{D}_n^{(O)}) \) is the average maximum success probability of correctly guessing which non-trivial \( m \)-member subset a given preparation \( P_x \) belongs to,

\[ \bar{D}_n^{(O)}(P) = \frac{1}{n-1} \sum_{m=1}^{n-1} D_{n,m}^{(O)}(P), \]  

(5)

where \( D_{n,m}^{(O)}(P) = \frac{1}{n} \max_{\{P_x\}} \sum_{i_1 < \ldots < i_m} \sum_{x \in \{i_1, \ldots, i_m\}} \{ p(k = \{i_1, \ldots, i_m\}|P_x, M) \} \) forms an operational property of entire set of preparations \( P \equiv \{ P_x \}_{x=1}^n \), where the first summation is over all distinct \( m \)-member subsets \( \{i_1, \ldots, i_m\} \subset [n] \). We note here that \( D_{n,1}^{(O)}(P) \) corresponds to the distinguishability of the set of preparations \( P \), whereas \( D_{n,n-1}^{(O)}(P) \) is equivalent to their anti-distinguishability [13, 15].

(ii.) **Average pair-wise distinguishability** \( (\bar{D}_n^{(O)}) \) is the average of maximum success probability of perfectly distinguishing distinct pairs of preparations \( \{ P_i, P_j \in \{ P_x \} \} \),

\[ \bar{D}_n^{(O)}(P) = \frac{1}{\binom{n}{2}} \sum_{i<j} \bar{D}_{2,1}^{(O)}(\{ P_i, P_j \}), \]  

(6)

where \( \bar{D}_{2,1}^{(O)}(\{ P_i, P_j \}) = \frac{1}{n} \max_{M \in \mathcal{M}_O} \{ \sum_{x \in \{i,j\}} \{ p(k = x|P_x, M) \} \} \) forms an operational property of the pair \( \{ P_i, P_j \} \), commonly referred to as their distinguishability.

Now we are prepared to present the operational consequences of empirical completeness, namely, a family of empirically falsifiable statistical inequalities,

**Theorem 1.** For any empirically complete theory (EC), for any given set of \( n \) preparations \( P \equiv \{ P_x \}_{x=1}^n \), the average set distinguishability is exactly equal to average pair-wise distinguishability, i.e.,

\[ \bar{D}_n^{(EC)}(P) = \bar{D}_n^{(EC)}(P). \]  

(7)

The proof has been deferred to the supplementary material for brevity. These equalities imply that for any empirically complete theory, and any set of \( n \) preparations \( P \), if there exists measurements that achieve an average pair-wise distinguishability \( \bar{D}_n^{(EC)}(P) = p \), there must exist measurements that attain an average set distinguishability with exactly the same efficiency, i.e., \( \bar{D}_n^{(EC)}(P) = p \),
and vice versa. On the contrary, if either average pairwise distinguishability or average set distinguishability falls short of the other, i.e., there is non-zero deviation from the equalities $\Delta_n^{(c)}(P) = \tilde{\Delta}_n^{(c)}(P) \neq 0$, then this signifies the lack of operational measurements which would saturate the equalities. The deviation $\Delta_n^{(c)}(P)$ forms a measure of incompleteness of the operational theory with respect to the set of preparations $P$. Moreover, it is easy to see that any realist model underlying such an operational theory must fine-tune the set of operationally accessible response schemes to explain the operational deviation.

We now demonstrate that such instances of incompleteness of an operational theory fuel advantage over classical communication in one-way communication tasks. To demonstrate this, we cast set distinguishibility as an one way communication task, wherein the sender (Alice) encodes her classical input $x \in \{n\}$ onto an operational preparation $P_x \in \mathcal{P} \equiv \{P_x \in \mathcal{P}\}_{x=1}^n$, and transmits it to the receiver (Bob). Bob upon receiving the transmission performs a $(n)$-outcome measurement $M^y \in \mathcal{M} \equiv \{M^y \in \mathcal{M}_c\}$ based on her input $y \in \{n-1\}$. They aim to maximize the success metric $\tilde{\Delta}_n^{(c)}(P, M)$. Unlike communication complexity problems, here the amount of communication remains unconstrained, instead Alice’s preparations are constrained such that their average pair-wise distinguishability $\tilde{\Delta}_n^{(c)}(P)$ is at most $p$. Now as classical communication is empirically complete, (7) implies that the success metric is bounded by $p$, i.e., $\tilde{\Delta}_n^{(c)} = \max_{\{e,d\}} \tilde{\Delta}_n^{(c)}(e,d) \leq p$. Similarly, average pair-wise distinguishability can also be cast as a promised communication task with the success metric $\tilde{\Delta}_n^{(c)}(P, M)$ and the channel constraint $\tilde{\Delta}_n^{(c)}(P) \leq p$, such that for such tasks $\tilde{\Delta}_n^{(c)} \leq p$. If a set of operational preparations $P$, have $\Delta_n^{(c)}(P) < 0$ they fuel advantage in the former task, and when $\Delta_n^{(c)}(P) > 0$ an advantage is obtained in the latter.

Consider the set of three qubit preparations $P_\Omega \equiv \{\rho_\Omega = \frac{1+n_x}{3} \mathbb{1} \}_{x=1}^3$ which form an equilateral triangle on the Bloch sphere, specifically with Bloch vectors $\mathbf{n}_1 = [1,0,0]^T, \mathbf{n}_2 = [\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}, 0]^T$ and $\mathbf{n}_3 = [\cos \frac{4\pi}{3}, \sin \frac{4\pi}{3}, 0]^T$. While their average set distinguishability turns out to be $\tilde{\Delta}_3^{(c)}(P_\Omega) = \frac{\sqrt{3}}{6} \approx 0.5$ their average pairwise distinguishibility is $\Delta_3^{(c)}(P_\Omega) = \frac{\sqrt{3}}{2} (1 + \frac{\sqrt{3}}{2}) \approx 0.933$, such that the deviation from equality is $\Delta_3^{(c)}(P_\Omega) = \frac{\sqrt{3}}{2} (1 + \frac{\sqrt{3}}{2}) \approx 0.933$. The set of four qubit preparations $P_{\Omega_2}$ which form a regular tetrahedron on the Bloch sphere attains a higher deviation from the four preparation equality, i.e., $\Delta_4^{(c)}(P_{\Omega_2}) \approx 0.1453$.

While these form lower bounds specific to the set of states $\rho$, finding out the maximal quantum deviation from the equalities (7) $|\Delta_n^{(Q)}| = \max_\rho \{|\Delta_n^{(Q)}(\rho)|\}$ boils down to finding the maximal quantum values of a $n$ preparation success metric, i.e., $S_n^{(Q)} = \max_\rho \{S_n^{(Q)}(\rho)\}$ given an upper bound on another generic operational property, i.e., $T_n^{(Q)}(\rho) \leq p$, where $S_n^{(Q)}(\rho)$, and $T_n^{(Q)}(\rho)$ are arbitrary operational properties of $n$ density operators of the form (1). Such optimization problems are very arduous to solve as the dimension of the quantum systems remains unconstrained. To this end, in a forthcoming article [16], we devise a hierarchy of SDP relaxations for such problems, to obtain efficient dimension independent tightening upper bounds $S_n^{(Q)}$, where $\mathcal{L} \subset \mathbb{N}$ is the level of the relaxation such that $\forall \mathcal{L} \in \mathbb{N}_+: S_n^{(Q)} \geq S_n^{(Q_{\mathcal{L}+1})} \geq S_n^{(Q)}$. We also devise a see-saw SDP algorithm to obtain dimension dependent efficient lower bounds $S_n^{(Q_{\mathcal{L}+1})} \leq S_n^{(Q_{\mathcal{L}})}$. Whenever the upper bounds from the relaxations coincide with the dimension dependent lower bounds, we obtain the maximal quantum value $S_n^{(Q)}$ (upto machine precision).

We employed the hierarchy of SDP relaxations for the optimization problem $\tilde{\Delta}_n^{(Q)} = \max_\rho \tilde{\Delta}_n^{(Q)}(\rho)$ such that $\tilde{\Delta}_n^{(Q)}(\rho) \leq p$. Remarkably, the second level of the hierarchy yields a maximum deviation from the three preparation equality saturated by $\rho_\Delta$, i.e., $\Delta_3^{(Q)}(\rho_\Delta) = 3\frac{\sqrt{3}-1}{2}$ when $p = \tilde{\Delta}_3^{(Q)}(\rho_\Delta) = \frac{\sqrt{3}}{5}$. Quantum preparations can also violate the inequality the other way. We employed the SDP hierarchy and the see-saw technique to obtain upper and lower bounds for the solution of the inverse optimization problem, $\tilde{\Delta}_n^{(Q)} = \max_\rho \tilde{\Delta}_n^{(Q)}(\rho)$ such that $\tilde{\Delta}_n^{(Q)}(\rho) \leq p$. We find that the third level of the hierarchy yields the maximum deviation $-\Delta_3^{(Q)} = \Delta_3^{(Q)}(\rho_\Delta) \approx 0.0277$ when $p = \tilde{\Delta}_3^{(Q)}(\rho) \approx 0.8214$ saturated by a triplet of qudits $\rho$. This implies that the maximal quantum deviation from the three preparation equality is $|\Delta_4^{(Q)}| = \Delta_3^{(Q)}(\rho_\Delta) = 3\frac{\sqrt{3}-1}{2}$. Using these SDP techniques, we characterized the quantum deviations from the three preparation equality (see FIG. 1). Moreover, the second level of the hierarchy yields a higher maximum deviation from the four preparation equality $|\Delta_4^{(Q)}| = \Delta_4^{(Q)}(\rho_{\Omega_2}) \approx 0.1453$. In fact, we present evidence that the maximal quantum deviation from the equalities increases with the number of preparations in FIG. 2.

In summary, we introduced a realist notion of classicality, termed empirical completeness, which requires empirically falsifiable operational properties associated with sets of preparations, to have a not fine-tuned realist explanation, as properties of stochastic distributions over states of reality. However, unlike the other notions, the

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2 Similar to the Navascués–Pironio–Acín hierarchy for nonlocal quantum correlations [17], and [18, 19] for generalized contextuality scenarios, [20] for informationally restricted correlations.
distinguishing experimentally falsifiable operational consequence of empirical completeness constitute a family of equalities among elemental operational properties of sets of preparations. Quantum preparations violate these equalities both ways, implying the theory’s incompleteness. In general, operational violation of empirical incompleteness highlights the operational properties which require realist fine-tuning, and fuels advantage over classical communication in communication tasks wherein the sender’s preparations are constrained such that they attain bounded success in other communication tasks.

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FIG. 1. Characterising quantum deviation from the three preparation equality: The average pair-wise distinguishability \( D_3^{(Q)}(\cdot) \) is plotted against the average set distinguishability \( \overline{\rho}_3^{(Q)}(\cdot) \) for randomly sampled (from an Haar uniform distribution) triplets of density operators \( \rho \) (green crosses), pure states \( \psi \) (orange solid circles). The solid black line represents the three preparation equality \( \overline{\rho}_3^{(EC)}(\cdot) = \overline{\rho}_3^{(EC)}(\cdot) \) satisfied by all empirically complete (EC) theories (7). All random samples deviated from the equality with over 99% exhibiting a deviation \( |\Delta_3^{(Q)}(\cdot)| \geq 0.0005 \). The red solid line represents the dimension independent upper bounds \( \overline{\rho}_3^{(Q)}(\cdot) \) such that \( \overline{\rho}_3^{(Q)}(\cdot) = \overline{\rho}_3^{(Q)}(\cdot) \) when \( \overline{\rho}_3^{(Q)}(\cdot) = \overline{\rho}_3^{(Q)}(\cdot) \) from the third level of the hierarchy of SDP relaxations. The blue circles represent lower bounds \( \tilde{\rho}_3^{(Q,\text{LB})} \) from the see-saw technique. The triplet of qubit states \( \rho_\Delta \) which form an equilateral triangle on the Bloch sphere attain the maximal quantum deviation from the equality \( |\Delta_3^{(Q)}(\cdot)| = \overline{\rho}_3^{(Q)}(\cdot) - \overline{\rho}_3^{(Q)}(\cdot) = \frac{\sqrt{3} - 1}{2} \approx 0.0997 \). Whereas, a triplet of qutrit states \( \rho \) attain the maximal quantum difference \( \Delta_3^{(Q)}(\cdot) - \overline{\rho}_3^{(Q)}(\cdot) \approx 0.0277 \).

FIG. 2. Increasing quantum deviation: This graphic provides evidence for increasing maximum quantum deviation \( |\Delta_3^{(Q)}(\cdot)| = \Delta_3^{(Q)}(\cdot) \) from the equalities (7) with the number of preparations \( n \). The larger blue bars represent upper bounds \( \Delta_3^{(Q)}(\cdot) \) obtained from the first level of the hierarchy of SDP relaxations, while the smaller orange bars represent lower bounds \( \Delta_3^{(Q,\text{LB})} \) obtained via see-saw SDP technique. For \( n = \{3, 4, 5\} \) the yellow circles represent upper bounds from the second level \( \Delta_n^{(Q)} \) of the hierarchy of SDP relaxations which saturate the lower bounds for maximal quantum deviation \( |\Delta_3^{(Q)}(\cdot)| = \Delta_4^{(Q)}(\cdot) = \Delta_5^{(Q)}(\cdot) = \Delta_3^{(Q,\text{LB})} \), and hence constitute the maximal quantum deviations \( |\Delta_3^{(Q)}(\cdot)| \).

\[ \Delta_3^{(Q)}(\cdot) = \Delta_3^{(Q)}(\cdot) \]

\[ \Delta_4^{(Q)}(\cdot) = \Delta_4^{(Q)}(\cdot) \]

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SUPPLEMENTAL MATERIAL

Proof of Theorem 1

Theorem 1 states that for any empirically complete theory, for any given set of \( n \) preparations \( \mathbf{P} = \{P_x\}_{x=1}^{n} \), the average set distinguishability \( \bar{D}_n^{\mathcal{EC}}(\mathbf{P}) \) is exactly equal to average pair-wise distinguishability \( \bar{D}_n^{(\mathcal{EC})}(\mathbf{P}) \), i.e. \( \bar{D}_n^{\mathcal{EC}}(\mathbf{P}) = \bar{D}_n^{(\mathcal{EC})}(\mathbf{P}) \). To prove this thesis, first, for a set of epistemic states \( \mu = \{\mu_x\}_{x=1}^{n} \), we obtain expressions for the not fine-tuned realist counterparts of average set distinguishability \( \bar{D}_n^{(\mathcal{E})}(\mu) \) and average pair-wise distinguishability \( \bar{D}_n^{(\mathcal{E})}(\mu) \),

\[
\bar{D}_n^{(\mathcal{E})}(\mu) = \frac{1}{n(n-1)} \sum_{i<j} \int_{\Lambda} d\lambda \max\{\mu_i(\lambda), \mu_j(\lambda)\},
\]

\[
\bar{D}_n^{(\mathcal{E})}(\mu) = \frac{1}{n-1} \sum_{m=1}^{n-1} \int_{\Lambda} d\lambda \max\{\mu_x(\lambda)\}_{x \in \{i_1, \ldots, i_m\}} \sum_{x \in \{i_1, \ldots, i_m\}} \mu_x(\lambda),
\]

where the second and fourth equality follows from \( \bar{D}_n^{(\mathcal{E})} = \frac{1}{n} \int_{\Lambda} d\lambda \max\{\mu_x(\lambda)\}_{x \in \{i_1, \ldots, i_m\}} \sum_{x \in \{i_1, \ldots, i_m\}} \mu_x(\lambda) \) which in turn follows from (3). Now we introduce following elementary identity [21] which forms the key ingredient of our proof technique,

**Lemma 1.** For any set of \( n \) real numbers \( \{u_x \in \mathbb{R}\}_{x=1}^{n} \), the following identity holds,

\[
\sum_{i<j} \max\{u_i, u_j\} = \sum_{m=1}^{n-1} \max_{i_1 < \ldots < i_m \in [n]} \sum_{x \in \{i_1, \ldots, i_m\}} \{u_x\}. \tag{9}
\]

**Proof.** To prove the above thesis, we consider an ordered list \( \{a_x\}_{x=1}^{n} \) such that \( a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq a_n \) associated to a given set of real numbers \( \{u_x\}_{x=1}^{n} \). Consequently, we re-express \( \sum_{i<j} \max\{u_i, u_j\} \) in terms of members of the ordered list \( \{a_x\}_{x=1}^{n} \) as,

\[
\sum_{i<j} \max\{u_i, u_j\} = \sum_{x=1}^{n-1} (n-x)a_x \tag{10}
\]
Notice that the right hand side of the above equation can be regrouped into sums of \( m \leq n - 1 \) largest numbers such that,

\[
\sum_{i<j} \max\{u_i, u_j\} = (n - 1)a_1 + (n - 2)a_2 + \ldots + 2a_{n-2} + a_{n-1} = \sum_{m=1}^{n-1} \sum_{x=1}^{m} a_x, \tag{11}
\]

Finally, observe that for any \( m \leq n \),

\[
\max_{i_1 \ldots i_m} \sum_{x \in \{i_1, \ldots, i_m\}} \{u_x\} = \sum_{x=1}^{m} a_x, \tag{12}
\]

which when plugged back into (11) yields the desired thesis (9).

As any set of epistemic states \( \mu \equiv \{\mu_x\}_{x=1}^{n} \) yields a set of positive numbers \( \{\mu_x \lambda\}_{x=1}^{n} \) for each ontic state \( \lambda \in \Lambda \), Lemma 1 implies,

\[
\sum_{i<j} \max\{\mu_i(\lambda), \mu_j(\lambda)\} = \sum_{m=1}^{n-1} \max_{i_1 \ldots i_m, \lambda} \sum_{x \in \{i_1, \ldots, i_m\}} \{\mu_x(\lambda)\}, \quad \forall \lambda \in \Lambda. \tag{13}
\]

Integrating both sides of (13) over the ontic-state space, and multiplying by \( \frac{1}{n(n-1)} \) yields,

\[
\bar{D}_n^{(\Lambda)}(\mu) = \frac{1}{n(n-1)} \int_{\Lambda} d\lambda \sum_{i<j} \sum_{x \in \{i_1, \ldots, i_m\}} \{\mu_x(\lambda)\} = \frac{1}{n(n-1)} \int_{\Lambda} d\lambda \sum_{m=1}^{n-1} \max_{i_1 \ldots i_m, \lambda} \sum_{x \in \{i_1, \ldots, i_m\}} \{\mu_x(\lambda)\} = \bar{D}_n^{(\Lambda)}(\mu), \tag{14}
\]

where we have used the relations (8). Finally, as for any empirically complete theory, for any set of preparations \( P \equiv \{P_x\}_{x=1}^{n} \) there exists an underlying set of epistemic states \( \mu \equiv \{\mu_x\}_{x=1}^{n} \), such all that operational properties have a not fine-tuned realist explanation, i.e., \( S_n^{(EC)} = S_n^{(A)} \). In particular, empirical completeness allows us to port the relation (14) to the operational level such that \( \bar{D}_n^{(EC)}(P) = \bar{D}_n^{(EC)}(P) \), which concludes the proof.