On a Unique Solution of a Class of Stochastic Predator–Prey Models with Two-Choice Behavior of Predator Animals

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Abstract: Simple birth–death phenomena are frequently examined in mathematical modeling and probability theory courses since they serve as an excellent foundation for stochastic modeling. Such mechanisms are inherent stochastic extensions of the deterministic population paradigm for population expansion of a particular species in a habitat with constant resource availability and many other organisms. Most animal behavior research differentiates such circumstances into two different events when it comes to two-choice scenarios. On the other hand, in this kind of research, the reward serves a significant role, because, depending on the chosen side and food placement, such situations may be divided into four groups. This article presents a novel stochastic equation that may be used to describe the vast majority of models discussed in the current studies. It is noteworthy that they are connected to the symmetry of the progression of a solution of stochastic equations. The techniques of fixed point theory are employed to explore the existence, uniqueness, and stability of solutions to the proposed functional equation. Additionally, some examples are offered to emphasize the significance of our findings.

Keywords: stochastic predator–prey model; stability; fixed points

1. Introduction and Preliminaries

The dynamics of populations is a highly contentious issue in biomathematics. The study of the development of diverse habitats has always piqued our attention, beginning with individuals of a single species and progressing to more complex systems, in which many species coexist in the same environment.

Many times, symmetry has appeared in mathematical interpretations, and it has been shown that it is vital for solving issues or progressing studies. It helps to make it possible to find high-quality work that uses significant mathematics and associated topologies to address critical concerns in various domains.

Since Volterra’s groundbreaking work [1], numerous predator–prey models have been developed to comprehend population evolution and dynamics better (see [2,3]). Several variations of the Lotka–Volterra model have been suggested and researched in various fields, including mathematical biology (see [4–6]), ecology (see [7,8]), and economics (see [9,10]), among others (see [11–18] and references therein), in the last half-century. The so-called functional reaction involves the quantity of prey captured per predator and it substantially
influences the kinetic characteristics. It is directly tied to several aspects, including prey density, handling time and attack efficiency (see [1,19]).

The stochastic averaging method helps to explore linear/nonlinear systems triggered by stochastic mechanism. It was originally used on nonlinear systems with Gaussian white noise stimuli (see [20–22]), and later it was extended to nonlinear systems with various forms of stochastic excitation (see [23,24]). It has been used to investigate stationary PDFs (see [25,26] and references therein) and it is the best way to manage species densities in habitats with low self- and stochastic simulations. Furthermore, the average stochastic technique has yet to be used to solve dynamic ecosystem behavior under continuous and random jump excitation (see [27]).

The learning stage in an animal or a human organism, on the other hand, is often seen as a series of options among many possible answers. It is also useful to look for structural changes in the possibilities that indicate variations in the corresponding event probability. The majority of learning research, from this viewpoint, indicates the likelihood of a trial-to-test emergence, which is a hallmark of stochastic processes. As a result, it is not a new concept. In [28,29], the authors established a notion of “reward” based on animals selecting the right side in a two-choice scenario and separated it into four categories: left-reward, right-reward, right-non-reward, and left non-reward. They utilized the following operators to monitor such behavior by relying on four occurrences between a predator and its prey choices:

A few researchers observed the responses of various animals in a two-choice scenario (see [30–36]) using the aforementioned operators (given in Table 1). Recently, in [37], the author used such operators to examine the two-choice behavior of rhesus monkeys in a non-contingent environment. The author focused on the chosen side of the animal rather than the food placement.

Table 1. Some operators and their outcomes presented in [28,29].

| Animal’s Response | Outcome (Left side) | Outcome (Right side) |
|-------------------|---------------------|----------------------|
| Reinforcement     | rx                  | rx + 1 – r           |
| Non-reinforcement | sx + 1 – s          | sx                   |

Operators for habit formation model

| Animal’s Response | Outcome (Left side) | Outcome (Right side) |
|-------------------|---------------------|----------------------|
| Reinforcement     | rx                  | rx + 1 – r           |
| Non-reinforcement | sx                  | sx + 1 – s           |

In contrast to the above work, here, we extend the model by adding two extra compartments discussed in [28,29] to the model with the corresponding probabilities:

\[ \bar{3}(x) = \varphi_1 h_1(x) \bar{3}(q_1(x)) + \varphi_2 h_1(x) \bar{3}(q_2(x)) + \varphi_1 h_2(x) \bar{3}(q_3(x)) + \varphi_2 h_2(x) \bar{3}(q_4(x)), \]  

(1)

for all \( x \in [\mu, v] \), \( \varphi_1 = (\gamma - \mu)\Delta \delta^{-1} \) and \( \varphi_2 = (v - \gamma)\Delta \delta^{-1} \), where \( 0 \leq \gamma \leq 1, \Delta \delta = v - \mu, \bar{3} : [\mu, v] \to \mathbb{R} \) is an unknown function and \( q_1, q_4 : [\mu, v] \to [\mu, v] \) are given mappings. Moreover, \( h_1, h_2 : [\mu, v] \to \mathbb{R} \) are defined by

\[
\begin{align*}
    \bar{h}_1(x) &= (x - \mu)\Delta \delta^{-1}, \\
    \bar{h}_2(x) &= (v - x)\Delta \delta^{-1},
\end{align*}
\]

(2)

The physical meaning of the parameters/operators are given in Table 2.
Table 2. Physical meaning of the parameters/operators.

| Parameter/Operator | Physical Meaning       |
|--------------------|------------------------|
| $[\mu, \nu]$       | State space            |
| $r, s$             | Learning-rate parameters |
| $\gamma$           | Probability of a chosen side |
| $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ | Transition operators |
| $3$                | Final probability      |

The presented functional Equation (1) with (2) has great importance in mathematical biology and learning theory. Such equations are used to investigate the response of animals in a two-choice situation, and the solution exists when a predator is fixed to one type of prey (see Figure 1).

![Figure 1](image1.png)  
**Figure 1.** (a): A predator with two choices of prey [38]; (b): a predator fixed to one type of prey [39].

On the other hand, the fixed point approach is regarded as a fundamental component and is a very effective method in nonlinear analysis due to its many critical implementations in various fields, including physics, engineering, computer science, biology, economics, and chemistry. This approach is widely used in mathematics to examine game-theoretic models, dynamical systems, statistical models, and differential equations. More specifically, this approach is mainly used to analyze certain integro-differential equations, functional equations, differential and integral equations, and fractional equations, which simplifies the process of obtaining computational solutions to such problems (for the details, see [40–44] and references therein).

In this work, our aim was to use the appropriate fixed point technique to demonstrate the existence of a unique solution to Equation (1) with (2). After that, we considered the stability of solutions to the suggested stochastic Equation (1) under the Hyers–Ulam (HU) and Hyers–Ulam–Rassias type (HUR) stability results. We provide three examples to emphasize the significance of our main conclusions.

Progress of this work requires the accomplishment of the following stated result, which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to find those fixed points.

**Theorem 1** (Banach fixed point theorem). Let $(\mathcal{F}, d)$ be a complete metric space and $\mathcal{F}: \mathcal{F} \to \mathcal{F}$ be a Banach contraction mapping (shortly, BCM) defined by

$$d(3\mu, 3\nu) \leq Yd(\mu, \nu)$$

for some $Y < 1$ and for all $\mu, \nu \in \mathcal{F}$. Then $\mathcal{F}$ has one fixed point. Furthermore, the Picard iteration $\{\mu_n\}$ in $\mathcal{F}$ which is defined by $\mu_n = 3\mu_{n-1}$ for all $n \in \mathbb{N}$, where $\mu_0 \in \mathcal{F}$, converges to the unique fixed point of $\mathcal{F}$.
Proof. For the proof, we refer [45,46]. □

2. Main Results

Let $\mathcal{F} = \{\mu, v\}$ with $\mu < v$, where $\mu, v \in \mathbb{R}$. We denote a class $\mathcal{F}$ having the continuous real-valued functions $\mathcal{F} : \mathcal{F} \to \mathbb{R}$ with $3(\mu) = 0$ and $\sup_{\mathcal{F}} \frac{|3(\xi) - 3(x)|}{|\xi - x|} < \infty$, where

$$\|3\| = \sup_{\xi \neq \chi} \frac{|3(\xi) - 3(x)|}{|\xi - \chi|} \quad (4)$$

for all $3 \in \mathcal{F}$.

We shall use the following conditions to prove the main results:

(A1) There is a nonempty subset $\mathcal{C}$ of $\mathcal{F} := \{3 \in \mathcal{F} \mid 3(v) \leq v\}$ such that $\mathcal{C}$ is a Banach space (for detail, see [34]), where $\|\cdot\|$ is given in (4).

(A2) The mappings $\varphi_1 - \varphi_4 : \mathcal{F} \to \mathcal{F}$ are Banach contraction mappings with contractive coefficients $\varphi_1 - \varphi_4$, respectively, and satisfy the following conditions

$$\begin{align*}
\varphi_1(v) &= v = \varphi_2(v), \quad \text{and} \\
\varphi_3(\mu) &= \mu = \varphi_4(\mu). 
\end{align*}$$

(A3) For a function $\varphi : \mathcal{C} \to [0, \infty)$, we have that for every $3 \in \mathcal{C}$ with $d(3, \mathcal{F}) \leq \varphi(3)$, there is a unique $3^* \in \mathcal{C}$ with $3^* = 3$ and $d(3, 3^*) \leq \varphi(3)$ for some $\varsigma > 0$.

(A4) For $\omega > 0$, we have that for every $3 \in \mathcal{C}$ with $d(3, \mathcal{F}) \leq \omega$, there is a unique $3^* \in \mathcal{C}$ with $3^* = 3$ and $d(3, 3^*) \leq \omega$, for some $\varsigma > 0$.

Now, we begin with the outcome stated below.

Theorem 2. Consider the stochastic functional Equation (1) associated with (2). Suppose that the conditions (A1) and (A2) are satisfied and there exists an $Y < 1$, where

$$Y := |\varphi_1(2\varphi_1 + 2\varphi_3) + \varphi_2(2\varphi_2 + 2\varphi_4)|. \quad (6)$$

Then, for each $3 \in \mathcal{F}$ and for all $x \in \mathcal{F}$, a self-mapping $\mathcal{F}$ from $\mathcal{F}$ to $\mathcal{C}$ is a BCM which is defined by

$$(\mathcal{F}3)(x) = \varphi_1 h_1(x)3(\varphi_1(x)) + \varphi_2 h_2(x)3(\varphi_2(x)) + \varphi_1 h_3(x)3(\varphi_3(x)) + \varphi_2 h_4(x)3(\varphi_4(x)). \quad (7)$$

Proof. Let $3_1, 3_2 \in \mathcal{C}$. For each $\tau_1, \tau_2 \in \mathcal{F}$ such that $\tau_1 \neq \tau_2$, $\Delta \tau = \tau_1 - \tau_2$ and $\Delta 3 = 3_1 - 3_2$, we obtain

$$|\mathcal{F}(\Delta 3)(\tau_1) - \mathcal{F}(\Delta 3)(\tau_2)||\Delta \tau|^{-1}$$

$$= |\Delta \tau ^{-1} \varphi_1 h_1(\tau_1)\Delta 3(\varphi_1(\tau_1)) + \varphi_2 h_2(\tau_1)\Delta 3(\varphi_2(\tau_1)) + \varphi_1 h_3(\tau_1)\Delta 3(\varphi_3(\tau_1)) + \varphi_2 h_4(\tau_1)\Delta 3(\varphi_4(\tau_1))$$

$$+ \varphi_2 h_2(\tau_2)\Delta 3(\varphi_2(\tau_2)) - \varphi_1 h_3(\tau_2)\Delta 3(\varphi_3(\tau_2)) - \varphi_2 h_4(\tau_2)\Delta 3(\varphi_4(\tau_2))$$

$$= \Delta \tau ^{-1} \varphi_1 h_1(\tau_1)\Delta 3(\varphi_1(\tau_1)) - \varphi_1 h_1(\tau_1)\Delta 3(\varphi_1(\tau_2)) + \Delta \tau ^{-1} \varphi_2 h_2(\tau_1)\Delta 3(\varphi_2(\tau_1)) - \varphi_2 h_2(\tau_1)\Delta 3(\varphi_2(\tau_2))$$

$$+ \Delta \tau ^{-1} \varphi_1 h_3(\tau_1)\Delta 3(\varphi_3(\tau_1)) - \varphi_1 h_3(\tau_1)\Delta 3(\varphi_3(\tau_2)) + \Delta \tau ^{-1} \varphi_2 h_4(\tau_1)\Delta 3(\varphi_4(\tau_1)) - \varphi_2 h_4(\tau_1)\Delta 3(\varphi_4(\tau_2))$$

$$+ \Delta \tau ^{-1} \varphi_1 h_2(\tau_1)\Delta 3(\varphi_2(\tau_1)) - \varphi_1 h_2(\tau_1)\Delta 3(\varphi_2(\tau_2)) + \Delta \tau ^{-1} \varphi_2 h_3(\tau_1)\Delta 3(\varphi_3(\tau_1)) - \varphi_2 h_3(\tau_1)\Delta 3(\varphi_3(\tau_2))$$

$$+ \Delta \tau ^{-1} \varphi_1 h_4(\tau_1)\Delta 3(\varphi_4(\tau_1)) - \varphi_1 h_4(\tau_1)\Delta 3(\varphi_4(\tau_2)).$$
From the above equation, we can write
\[
|\mathcal{F}(\Delta \mathcal{A})_1 - \mathcal{F}(\Delta \mathcal{A})_2|/|\Delta \tau|^{-1} \\
\leq |\Delta \mathcal{F}|^{-1} |\varphi_1 h_1(\mathcal{A}_1) + \varphi_1 h_1(\mathcal{A}_2)| + |\Delta \mathcal{F} - \varphi_1 h_1(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_1(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_2(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_2(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_3(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_3(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_4(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_4(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_5(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_5(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_6(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_6(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_7(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_7(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_8(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_8(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_9(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_9(\mathcal{A}_2)| \\
+ |\Delta \mathcal{F} - \varphi_1 h_{10}(\mathcal{A}_1)| + |\Delta \mathcal{F} - \varphi_1 h_{10}(\mathcal{A}_2)|.
\]

As \(e_1 - e_4 : \mathcal{F} \to \mathcal{F}\) are Banach contraction mappings, i.e.,
\[
|e_1(\mathcal{A}_1) - e_1(\mathcal{A}_2)| \leq \omega_1 |\mathcal{A}_1 - \mathcal{A}_2|,
\]
\[
|e_2(\mathcal{A}_1) - e_2(\mathcal{A}_2)| \leq \omega_2 |\mathcal{A}_1 - \mathcal{A}_2|,
\]
\[
|e_3(\mathcal{A}_1) - e_3(\mathcal{A}_2)| \leq \omega_3 |\mathcal{A}_1 - \mathcal{A}_2|,
\]
\[
|e_4(\mathcal{A}_1) - e_4(\mathcal{A}_2)| \leq \omega_4 |\mathcal{A}_1 - \mathcal{A}_2|
\]
where \(\omega_1 - \omega_4\) are contractive coefficients, respectively. Thus, by using the above relation with the definition of norm (4), we have
\[
|\mathcal{F}(\Delta \mathcal{A})_1 - \mathcal{F}(\Delta \mathcal{A})_2|/|\Delta \tau|^{-1} \leq \mathcal{Y}/|\Delta \mathcal{A}|,
\]
where \(\mathcal{Y}\) is given in (6). This gives that
\[
d(\mathcal{F}_1, \mathcal{F}_2) = \|\mathcal{F}_1 - \mathcal{F}_2\| \leq \mathcal{Y} \|\mathcal{A}_1 - \mathcal{A}_2\| = \mathcal{Y}d(\mathcal{A}_1, \mathcal{A}_2).
\]

It follows from \(0 < \mathcal{Y} < 1\) that \(\mathcal{F}\) is a BCM. This completes the proof. \(\square\)

**Theorem 3.** Consider the Equation (1) with (2). Suppose that the conditions \((A_1)\) and \((A_2)\) are satisfied and there exists an \(\mathcal{Y} \leq 1\), where \(\mathcal{Y}\) is given in (6). The mapping \(\mathcal{F} : \mathcal{X} \to \mathcal{X}\) is a BCM, which is defined in (7). Thus, the proposed problem (1) associated with (2) has a unique solution in \(\mathcal{X}\). Moreover, the iteration \(\mathcal{F}_n \in \mathcal{X}\) \((\forall n \in \mathbb{N} \text{ and } \mathcal{F}_0 \in \mathcal{X})\) given by
\[
(\mathcal{F}_n)(x) = \varphi_1 h_1(\mathcal{A}_1) + \varphi_2 h_1(\mathcal{A}_2) + \varphi_3 h_1(\mathcal{A}_3) + \varphi_4 h_1(\mathcal{A}_4) + \cdots + \varphi_n h_1(\mathcal{A}_n)
\]
converges to the unique solution of (1).

**Proof.** As \(\mathcal{F} : \mathcal{X} \to \mathcal{X}\) is a BCM, we obtain the outcome of this result by combining Theorem 2 with the Banach fixed point theorem. \(\square\)

Here, we shall look at different conditions. If \(e_1 - e_4 : \mathcal{F} \to \mathcal{F}\) are Banach contraction mappings with contractive coefficients \(\omega_1 \leq \omega_2 \leq \omega_3 \leq \omega_4\), respectively, then by Theorems 2 and 3, the outcomes are as follows.

**Corollary 1.** Consider the stochastic Equation (1) associated with (2). Assume that the condition \((A_1)\) is satisfied and there exists an \(\mathcal{Y} := 4\omega_4 < 1\). Then, for each \(\mathcal{F} \in \mathcal{X}\) and for all \(x \in \mathcal{X}\), a self-mapping \(\mathcal{F}\) from \(\mathcal{X}\) to \(\mathcal{X}\) is a BCM, which is defined by
\[
(\mathcal{F}_n)(x) = \varphi_1 h_1(\mathcal{A}_1) + \varphi_2 h_1(\mathcal{A}_2) + \varphi_3 h_1(\mathcal{A}_3) + \varphi_4 h_1(\mathcal{A}_4).
\]

**Corollary 2.** Consider the Equations (1) and (2). Assume that the condition \((A_1)\) is satisfied and there exists an \(\mathcal{Y} := 4\omega_4 < 1\). The mapping \(\mathcal{F} : \mathcal{X} \to \mathcal{X}\) is a BCM, which is defined in (9). Thus, the proposed problem (1) associated with (2) has a unique solution in \(\mathcal{X}\). Additionally, the iteration \(\mathcal{F}_n \in \mathcal{X}\) \((\forall n \in \mathbb{N} \text{ and } \mathcal{F}_0 \in \mathcal{X})\) is defined as
\[(3_n)(x) = \varphi_1 h_1(x) 3_{n-1}(e_1(x)) + \varphi_2 h_1(x) 3_{n-1}(e_2(x)) + \varphi_1 h_2(x) 3_{n-1}(e_3(x)) + \varphi_2 h_2(x) 3_{n-1}(e_4(x)) \quad (10)\]

converges to the unique solution of (1).

**Remark 1.** The suggested stochastic Equation (1) is a generalization of the equations discussed in [30–34].

3. Stability Analysis

Here, we shall discuss the stability of the solution to the Equation (1) (see [47–51] for the details).

**Theorem 4.** In light of Theorem 2’s assumptions, the equation \( F \bar{3} = 3 \), where \( F : \mathcal{C} \rightarrow \mathcal{C} \) is defined as

\[(F \bar{3})(x) = \varphi_1 h_1(x) 3(e_1(x)) + \varphi_2 h_1(x) 3(e_2(x)) + \varphi_1 h_2(x) 3(e_3(x)) + \varphi_2 h_2(x) 3(e_4(x)), \quad (11)\]

for all \( \bar{3} \in \mathcal{C} \) and \( x \in \mathcal{J} \), has HUI stability (defined in (A_3)).

**Proof.** Let \( \bar{3} \in \mathcal{C} \) such that \( d(F \bar{3}, 3) \leq \varphi(3) \). By using Theorem 2, we have a unique \( \bar{3}^* \in \mathcal{C} \), such that \( F \bar{3}^* = 3^* \). Thus, we obtain

\[d(\bar{3}, \bar{3}^*) \leq d(\bar{3}, 3) + d(3, \bar{3}^*) \leq \varphi(3) + d(\bar{3}, 3^*) \leq \varphi(3) + Y d(\bar{3}, 3^*) \]

where \( Y \) is defined in (6), and so by \( \varsigma := \frac{1}{1 - Y} \), we have

\[d(\bar{3}, \bar{3}^*) \leq \varsigma \varphi(3) \]

\( \Box \)

We gain the following conclusion about the HU stability from the aforementioned investigation.

**Corollary 3.** In light of Theorem 2’s assumptions, the equation \( F \bar{3} = 3 \), where \( F : \mathcal{C} \rightarrow \mathcal{C} \) is given by

\[(F \bar{3})(x) = \varphi_1 h_1(x) 3(e_1(x)) + \varphi_2 h_1(x) 3(e_2(x)) + \varphi_1 h_2(x) 3(e_3(x)) + \varphi_2 h_2(x) 3(e_4(x)), \quad (12)\]

for all \( \bar{3} \in \mathcal{C} \) and \( x \in \mathcal{J} \), has HU stability (defined in (A_4)).

4. Some Illustrative Examples

Here, we provide the following examples to justify our findings.

**Example 1.** Consider the stochastic functional equation given below

\[3(x) = \varphi_1 h_1(x) 3 \left( v(k - \mu) \Delta \vartheta^{-1} + (v - k) \Delta \vartheta^{-1} x \right) + \varphi_2 h_1(x) 3 \left( v(v - k) \Delta \vartheta^{-1} + (k - \mu) \Delta \vartheta^{-1} x \right) + \varphi_1 h_2(x) 3 \left( (v - \ell)(x - \mu) \Delta \vartheta^{-1} + \mu \right) \]
for all \( x \in \mathcal{J} \) with \( \mu < k, \ell < v \) and \( \mathfrak{J} \in \mathcal{J} \). If we set the mappings \( q_1 - q_4 : \mathcal{J} \to \mathcal{J} \) by
\[
\begin{align*}
q_1(x) &= \upsilon(k - \mu)\Delta \delta^{-1} + (v - k)\Delta \delta^{-1}x, \\
q_2(x) &= \upsilon(v - k)\Delta \delta^{-1} + (k - \mu)\Delta \delta^{-1}x, \\
q_3(x) &= (v - \ell)(x - \mu)\Delta \delta^{-1} + \mu, \\
q_4(x) &= (\ell - \mu)(x - \mu)\Delta \delta^{-1} + \mu,
\end{align*}
\]
for all \( x \in \mathcal{J} \), so (1) decreases to the Equation (13). It is easy to see that the mappings \( q_1 - q_4 \) satisfy (A2), i.e.,
\[
\begin{align*}
|q_1(x) - q_1(y)| &\leq (v - k)\Delta \delta^{-1}|x - y|, \\
|q_2(x) - q_2(y)| &\leq (k - \mu)\Delta \delta^{-1}|x - y|, \\
|q_3(x) - q_3(y)| &\leq (v - \ell)\Delta \delta^{-1}|x - y|, \\
|q_4(x) - q_4(y)| &\leq (\ell - \mu)\Delta \delta^{-1}|x - y|,
\end{align*}
\]
for all \( x, y \in \mathcal{J} \), where \( a_1 = (v - k)\Delta \delta^{-1} \), \( a_2 = (k - \mu)\Delta \delta^{-1} \), \( a_3 = (v - \ell)\Delta \delta^{-1} \), \( a_4 = (\ell - \mu)\Delta \delta^{-1} \) are contractive coefficients, respectively. Here, if (A3) is satisfied with
\[
Y := \left|q_1(4v - 2k - 2\ell)\Delta \delta^{-2} + q_2(2k + 2\ell - 4\mu)\Delta \delta^{-2}\right| < 1,
\]
then the mapping \( F : \mathcal{X} \to \mathcal{X} \) defined on (13) is a BCM. Thus, all conditions of Theorem 2 are fulfilled and, therefore, we obtain the existence of a solution to the functional Equation (13).

For a unique solution of (13), we define \( \mathfrak{J}_0 = \mathbf{1} \in \mathcal{X} \) as a starting approximation (whereas \( \mathbf{1} \) is an identity function), then by Theorem 3, we obtain the convergence of the following iteration process
\[
\begin{align*}
\mathfrak{J}_1(x) &= \Delta \delta^{-3} \left[\left(\frac{2\ell v - 2\gamma k + k\mu - \ell\mu + k\upsilon - \ell\upsilon)x^2}{-3\mu^2 v - 2k\mu v + 2\gamma k\upsilon - 2\ell v \mu + 2\ell v \mu - \ell^2 v}\right) + (k\mu v^2 - \ell \mu^2 v + k\upsilon^2 v) - 2\gamma k\mu v + 2\ell v \mu - \ell^2 v) \right], \\
\mathfrak{J}_2(x) &= \varphi_1 h_1(x)\mathfrak{J}_1(\upsilon(k - \mu)\Delta \delta^{-1} + (v - k)\Delta \delta^{-1}x) \\
&\quad + \varphi_2 h_1(x)\mathfrak{J}_1(\upsilon(v - k)\Delta \delta^{-1} + (k - \mu)\Delta \delta^{-1}x) \\
&\quad + \varphi_1 h_2(x)\mathfrak{J}_1((v - \ell)(x - \mu)\Delta \delta^{-1} + \mu) \\
&\quad + \varphi_2 h_2(x)\mathfrak{J}_1((\ell - \mu)(x - \mu)\Delta \delta^{-1} + \mu), \\
&\vdots \\
\mathfrak{J}_n(x) &= \varphi_1 h_1(x)\mathfrak{J}_{n-1}(\upsilon(k - \mu)\Delta \delta^{-1} + (v - k)\Delta \delta^{-1}x) \\
&\quad + \varphi_2 h_1(x)\mathfrak{J}_{n-1}(\upsilon(v - k)\Delta \delta^{-1} + (k - \mu)\Delta \delta^{-1}x) \\
&\quad + \varphi_1 h_2(x)\mathfrak{J}_{n-1}((v - \ell)(x - \mu)\Delta \delta^{-1} + \mu) \\
&\quad + \varphi_2 h_2(x)\mathfrak{J}_{n-1}((\ell - \mu)(x - \mu)\Delta \delta^{-1} + \mu),
\end{align*}
\]
for all \( n \in \mathbb{N} \).

On the other hand, as \( 0 \leq Y < 1 \), we obtain
\[
\varsigma := \frac{1}{1 - Y} > 0.
\]
If a function \( \mathcal{F} \in \mathcal{C} \) satisfies the inequality
\[
d(\mathcal{F} \mathcal{F}, \mathcal{F}) \leq \varphi(\mathcal{F}), \quad \text{for some } \varphi(\mathcal{F}) > 0,
\]
then Theorem 4 implies that there exists a unique \( \mathcal{F}^* \in \mathcal{C} \), such that
\[
\mathcal{F} \mathcal{F}^* = \mathcal{F}^* \quad \text{and} \quad d(\mathcal{F}, \mathcal{F}^*) \leq c \varphi(\mathcal{F}).
\]

**Example 2.** Consider the functional equation given below
\[
\mathcal{F}(x) = \varphi_1 \varphi_2 + \varphi_3 + \varphi_4,
\]
where \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \) are contractive coefficients, respectively. Here, if \( (A_4) \) is satisfied with
\[
\mathcal{F} = 4 \left| (p - \mu) \Delta \theta^{-1} \right| < 1,
\]
then the mapping \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{C} \) defined on (14) is a BCM. Thus, it fulfills all the conditions of Corollary 1 and, therefore, we obtain the results related to the existence of a solution to the functional Equation (14).

If we define \( \mathcal{F}_0 = I \in \mathcal{C} \) as a starting approximation (whereas \( I \) is an identity function), then by Corollary 2, we have a unique solution of (14) followed by the iteration process stated below:
\[ 3(x) = \Delta^{3} \left[ (p\gamma - vp - \gamma m + \ell v - \ell \gamma + \gamma k + m\mu - k\mu)x^2 + (\gamma v^2 + v^2p + nmv - \ell v^2 + v^3 + \ell \gamma v - \gamma k\mu - \gamma m\mu + m\nu\mu + m\nu^2 + m\mu^2 + 3m^2v + k\mu^2 - m^3) x + (-m\mu\nu - \ell \mu v + \gamma \mu v + k\mu v - 3\mu v^2 - \gamma k\mu - m\mu^2 + 3\mu^2v + k\mu^2 - m^3) x \right], \]

\[ 3(x) = \varphi_1 h_1(x) 3_1 \left( (k - \mu)\Delta x + (1 - (k - \mu)\Delta x) \right) + \varphi_2 h_2(x) 3_1 \left( (\ell - \mu)\Delta x + (1 - (\ell - \mu)\Delta x) \right) + \varphi_1 h_2(x) 3_1 \left( (m - \mu)\Delta x + (v - m)\Delta x \right) + \varphi_2 h_2(x) 3_1 \left( (p - \mu)\Delta x + (v - p)\Delta x \right), \]

for all \( n \in \mathbb{N} \).

As \( 0 \leq \tilde{Y} < 1 \), we have

\[ \xi := \frac{1}{1 - \tilde{Y}} > 0. \]

If a function \( 3 \in \mathcal{G} \) satisfies the inequality

\[ d(\mathcal{F} 3, 3) \leq \varphi(3), \quad \text{for some } \varphi(3) > 0, \]

then Theorem 4 implies that there exists a unique \( 3^* \in \mathcal{G} \), such that

\[ \mathcal{F} 3^* = 3^* \quad \text{and} \quad d(3, 3^*) \leq \xi \varphi(3). \]

**Example 3.** Consider the functional equation given below

\[ 3(x) = \gamma x 3 \left( \frac{x + 2}{3} \right) + (1 - \gamma) x 3 \left( \frac{x + 6}{7} \right) + \gamma (1 - x) 3 \left( \frac{x}{11} \right) + (1 - \gamma)(1 - x) 3 \left( \frac{2x}{17} \right), \quad (15) \]

for all \( x \in \mathcal{F} = [0, 1] \) and \( 3 \in \mathcal{P} \). If we set the mappings \( \varrho_1 - \varrho_4 : \mathcal{F} \rightarrow \mathcal{F} \) by

\[ \varrho_1(x) = \frac{x + 2}{3}, \quad \varrho_2(x) = \frac{x + 6}{7}, \quad \varrho_3(x) = \frac{x}{11}, \quad \varrho_4(x) = \frac{2x}{17}, \]

for all \( x \in \mathcal{F} \), so (1) decreases to the Equation (15). It is easy to see that the mappings \( \varrho_1 - \varrho_4 \) satisfy \((A_2)\), i.e.,

\[
\begin{align*}
|\varrho_1(x) - \varrho_1(y)| &\leq \frac{1}{3}|x - y|, \\
|\varrho_2(x) - \varrho_2(y)| &\leq \frac{1}{7}|x - y|, \\
|\varrho_3(x) - \varrho_3(y)| &\leq \frac{1}{11}|x - y|, \\
|\varrho_4(x) - \varrho_4(y)| &\leq \frac{2}{17}|x - y|,
\end{align*}
\]
for all \( x, y \in J \), where \( \omega_1 = \frac{1}{5}, \ \omega_2 = \frac{1}{7}, \ \omega_3 = \frac{1}{11}, \ \omega_4 = \frac{2}{17} \) are contractive coefficients, respectively. Here, if \((A_1)\) is satisfied with

\[
Y = \frac{1286}{3927} x + \frac{62}{119} < 1,
\]

then the mapping \( F : C \to C \) defined on (15) is a BCM. Thus it fulfills all the conditions of Theorem 2 and, therefore, we obtain the results related to the existence of a solution to the functional Equation (15).

If we define \( Z_0 = I \in C \) as a starting approximation (whereas \( I \) is an identity function), then by Theorem 2, we have a unique solution of (15) followed by the iteration process stated below:

\[
3_1(x) = \frac{1}{3927} \left[ 853 \gamma x^2 - 853 \gamma x + 99 x^2 + 3828 x \right],
\]

\[
3_2(x) = \frac{1}{4943637468} \left[ \begin{array}{c}
1265337942 \gamma x^2 + 95473731 \gamma^2 x^3 - 94357154 \gamma^2 x^2 \\
-209916476 \gamma^2 x + 26046658 \gamma x^3 - 193075796 \gamma x \\
+279494028 x^2 + 1736955 x^3 + 5510579580 x
\end{array} \right],
\]

\[
3_n(x) = \gamma x 3_{n-1} \left(\frac{x + 2}{3}\right) + (1 - \gamma) x 3_{n-1} \left(\frac{x + 6}{7}\right) + \gamma (1 - x) 3_{n-1} \left(\frac{x}{11}\right)
\]

\[
+ (1 - \gamma)(1 - x) 3_{n-1} \left(\frac{2x}{17}\right),
\]

for all \( n \in \mathbb{N} \).

As \( 0 \leq Y < 1 \), we have

\[
\varsigma := \frac{1}{1 - Y} = \frac{3927}{1891 - 1286 \gamma} > 0.
\]

If a function \( Z \in C \) satisfies the inequality

\[
d(F Z, Z) \leq \varphi(Z), \quad \text{for some } \varphi(Z) > 0,
\]

then Theorem 4 implies that there exists a unique \( Z^* \in C \), such that

\[
F Z^* = Z^* \quad \text{and} \quad d(Z, Z^*) \leq \left( \frac{3927}{1891 - 1286 \gamma} \right) \varphi(Z).
\]

5. Conclusions

In population biology, predator–prey or host–parasite relationships are arguably the most often simulated phenomena. According to such models, a predator has two prey options, and the solution is determined when the predator is drawn to a certain prey type. In this research, we presented a generic functional equation that can cover numerous learning theory models in the current study. Additionally, we analyzed the solution to the proposed stochastic equation for its existence, uniqueness, and stability. To demonstrate the significance of our findings, we presented two examples. Our method is novel and can be applied to many mathematical models associated with mathematical psychology and learning theory.

Finally, we present the following open problems for those who are interested in this research.

**Question 1:** what would happen if a predator does not approach any prey and remains stuck to its original position?

**Question 2:** is there another way to establish the conclusions from Theorems 2 and 3?
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