A RECOGNITION PRINCIPLE FOR THE EXISTENCE OF DESCENT DATA.

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ABSTRACT. Suppose $R \rightarrow S$ is a faithfully flat ring map. The theory of twisted forms lets one compute, given an $R$-module $M$, how many isomorphism classes of $R$-modules $M'$ satisfy $S \otimes_R M \equiv S \otimes_R M'$. This is really a uniqueness problem. But this theory does not help one to solve the corresponding existence problem: given an $S$-module $N$, does there exist some $R$-module $M$ such that $S \otimes_R M \equiv N$? In this paper we work out (in the general language of abelian categories) a criterion for the existence of such an $R$-module $M$, under some reasonable hypotheses on the map $R \rightarrow S$.

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1. INTRODUCTION.

Suppose $R \rightarrow S$ is a faithfully flat map of rings. We will write $\text{Rep}(R), \text{Rep}(S)$ for the representation semirings of $R$ and $S$, that is, the isomorphism classes of finitely generated $R$-modules and $S$-modules, respectively. Here addition is given by direct sum and multiplication by tensor product. We have a base-change ("tensoring-up") map

$\text{Rep}(R) \xrightarrow{f} \text{Rep}(S)$.

This map may fail to be injective, but we have excellent control over its failure to be injective. If $N \in \text{im } f$, then $N \cong f(M)$ for some $M \in \text{Rep}(R)$, and the classical theory of twisted forms tells us that the preimage $f^{-1}(N)$ is in bijection with the cohomology group $H^1(S/R, \text{Aut}(M))$, at least in good cases (for example, when $R, S$ are fields and the map is a Galois extension). What’s happening here is that, since $R \rightarrow S$ is faithfully flat, specifying an $R$-module $M$ such that $f(M) = N$ is equivalent to specifying an $S/R$-descent datum on $N$; and $H^1(S/R, \text{Aut}(M))$ is in bijection with the set of (isomorphism classes of) $S/R$-descent data on $f(M) = S \otimes_R M$. See [5] for a nice exposition of some results of this kind. A modern, very general version is in Mesablishvili’s paper [1].

If one wants to understand the map $\text{Rep}(R) \xrightarrow{f} \text{Rep}(S)$, however, something is missing from this picture: one needs to get some control over the failure of $f$ to be surjective. In other words, we do not know how to recognize which elements of $\text{Rep}(S)$ are indeed in the image of $f$. Another way of putting it is that we want to know, given a finitely generated $S$-module $N$, whether there exists any $S/R$-descent datum on $N$ at all. Equivalently, we

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want to have a simple criterion for determining whether \( N \cong S \otimes_R M \) for some \( R \)-module \( M \). Such a recognition principle, along with the theory of twisted forms as described above, is what one needs to understand the relationship between \( \text{Rep}(R) \) and \( \text{Rep}(S) \), or more generally, to understand how the module theory of a ring changes under faithfully flat extension of that ring.

The purpose of this note is to describe and prove such a recognition principle (Theorem 2.3). Our recognition principle is an abstract statement about abelian categories, and as such, it has sufficient generality to be applied to many nonclassical situations (e.g. \( R, S \) do not have to be commutative, and may have gradings that we insist the modules respect). A short list of the easiest cases to see that this recognition principle applies in is Prop. 3.2, with consequences listed in Cor. 3.3.

The most familiar setting in which our main result applies is the case in which we have an extension

\[
A \to B \to C
\]

of finite-dimensional co-commutative connective Hopf algebras over a field \( k \). (See [4] for the basic theory of extensions of connective Hopf algebras.) Hopf algebra extensions are sufficiently structured and “rigid” that the base-change and restriction-of-scalars functors

\[
\text{Mod}(A) \to \text{Mod}(B) \to \text{Mod}(C)
\]

induced on the module categories have some special, desirable properties. In particular these properties are enough for one to write down a simple criterion for the existence of a \( B/A\)-descent datum on a finitely-generated \( B \)-module \( N \): such a descent datum exists if and only if \( C \otimes_B N \) is a free \( C \)-module. (We prove this as Cor. 3.3 a special case of our Thm. 2.3) In other words: there exists an \( A \)-module \( M \) such that \( B \otimes_A M = N \) if and only if \( C \otimes_B N \) is a free \( C \)-module.

More generally, suppose we have a faithfully flat map \( A \to B \) of finite-dimensional algebras over a field \( k \), such that \( A \) is augmented and the kernel of the surjection \( B \to B \otimes_A k \) is contained in the Jacobson radical of \( B \). Then one has the same criterion: a finitely-generated \( B \)-module \( N \) admits a \( B/A \)-descent datum if and only if \( N \otimes_A k \) is a free \( B \otimes_A k \)-module. In other words: there exists an \( A \)-module \( M \) such that \( B \otimes_A M = N \) if and only if \( N \otimes_A k \) is a free \( B \otimes_A k \)-module.

In future work on Hopf algebroids and algebraic stacks we plan to use the same recognition principle on comodule categories over Hopf algebroids, equivalently quasicoherent module categories over certain Artin stacks.

We use these results in our work on stable representation theory and stable algebraic \( G \)-theory, [3].

2. The main definition and the main result.

In this section we offer our main definition, Def. 2.2, and our main theorem, Thm. 2.3.

First we need to define an extension of abelian categories. The idea here is to recognize and isolate the most important structures and properties one has on \( \text{Mod}(A), \text{Mod}(B), \) and \( \text{Mod}(C) \) which come from a connective Hopf algebra extension \( A \to B \to C \). When one has three abelian categories equipped with appropriate functors between them which have exactly these kinds of properties and structures, we say that we have an extension of abelian categories. We break this definition into two parts. First we describe the structure we need, a composable pair of abelian categories:
Definition 2.1. Let $\kappa$ be a semisimple abelian category. By an composable pair of abelian categories over $\kappa$ we mean a 6-tuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, F_{\mathcal{A}/\kappa}, F_{\mathcal{B}/\kappa}, F_{\mathcal{C}/\kappa})$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories and

\[
\begin{array}{ccc}
\kappa & \xrightarrow{F_{\mathcal{A}/\kappa}} & \mathcal{A} \\
& & \xrightarrow{F_{\mathcal{B}/\kappa}} \mathcal{B} \\
& & \xrightarrow{F_{\mathcal{C}/\kappa}} \mathcal{C}
\end{array}
\]

are additive functors with right adjoints.

Suppose $(\mathcal{A}, \mathcal{B}, \mathcal{C}, F_{\mathcal{A}/\kappa}, F_{\mathcal{B}/\kappa}, F_{\mathcal{C}/\kappa})$ is a composable pair. In order to ease the weight of the notations, we will write $F_{\mathcal{B}/\kappa}, F_{\mathcal{C}/\kappa}, F_{\mathcal{C}/\mathcal{A}}$ for the composites

\[
\begin{align*}
F_{\mathcal{B}/\kappa} &= F_{\mathcal{B}/\mathcal{A}} \circ F_{\mathcal{A}/\kappa} \\
F_{\mathcal{C}/\kappa} &= F_{\mathcal{C}/\mathcal{B}} \circ F_{\mathcal{B}/\kappa} \circ F_{\mathcal{A}/\kappa} \\
F_{\mathcal{C}/\mathcal{A}} &= F_{\mathcal{C}/\mathcal{B}} \circ F_{\mathcal{B}/\mathcal{A}} \\
\end{align*}
\]

We will write $G_{\mathcal{B}/\kappa}$ for the right adjoint of $F_{\mathcal{A}/\kappa}$, and we will write $G_{\mathcal{B}/\kappa}$ for the right adjoint of $F_{\mathcal{B}/\kappa}$, etc.

Now we describe the axioms we require a composable pair of abelian categories to satisfy in order to be called an extension of abelian categories.

Definition 2.2. Let $\kappa$ be a semisimple abelian category. By an extension of abelian categories over $\kappa$ we mean a composable pair $(\mathcal{A}, \mathcal{B}, \mathcal{C}, F_{\mathcal{A}/\kappa}, F_{\mathcal{B}/\kappa}, F_{\mathcal{C}/\kappa})$ of abelian categories over $\kappa$ satisfying the following axioms:

1. The functors $G_{\mathcal{B}/\kappa}$ and $G_{\mathcal{C}/\kappa}$ preserve epimorphisms (equivalently, are exact), and $G_{\mathcal{B}/\mathcal{A}}$ is faithful.
2. For any object $X$ of $\mathcal{B}$, if $G_{\mathcal{C}/\mathcal{B}}F_{\mathcal{C}/\kappa}X \cong 0$ in $\mathcal{C}$, then $X \cong 0$.
3. ("Exactness of $0 \to \mathcal{A} \to \mathcal{B}"$) The functor $F_{\mathcal{B}/\mathcal{A}}$ is comonadic, that is, if we write $G_{\mathcal{B}/\mathcal{A}}$ for a right adjoint to $F_{\mathcal{B}/\mathcal{A}}$, then the comparison functor $\mathcal{A} \to \mathcal{B}F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}$, from $\mathcal{A}$ to coalgebras over the comonad $F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}$, is an equivalence of categories.
4. ("Exactness of $\mathcal{B} \to \mathcal{C} \to 0"$) The unit map $X \to G_{\mathcal{C}/\mathcal{B}}F_{\mathcal{C}/\kappa}X$ is an epimorphism in $\mathcal{B}$ for all objects $X$ of $\mathcal{B}$.
5. ("Exactness of $\mathcal{A} \to \mathcal{B} \to \mathcal{C}"$) Suppose $Y$ is an object of $\kappa$ and $M$ an object of $\mathcal{B}$ and $g : F_{\mathcal{B}/\kappa}Y \to M$ a map such that $F_{\mathcal{C}/\kappa}g : F_{\mathcal{C}/\mathcal{B}}F_{\mathcal{B}/\kappa}Y \to F_{\mathcal{C}/\kappa}M$ is an isomorphism. Write $i : \ker g \to F_{\mathcal{B}/\kappa}Y$ for the inclusion of the kernel of $g$. Then the map $G_{\mathcal{B}/\mathcal{A}}i : G_{\mathcal{B}/\mathcal{A}}\ker g \to G_{\mathcal{B}/\mathcal{A}}F_{\mathcal{B}/\kappa}Y$ factors through the unit map $F_{\mathcal{B}/\kappa}Y \to G_{\mathcal{B}/\mathcal{A}}F_{\mathcal{B}/\mathcal{A}}F_{\mathcal{B}/\kappa}Y$ of the adjunction of $F_{\mathcal{B}/\mathcal{A}}, G_{\mathcal{B}/\mathcal{A}}$.

When $(\mathcal{A}, \mathcal{B}, \mathcal{C}, F_{\mathcal{A}/\kappa}, F_{\mathcal{B}/\kappa}, F_{\mathcal{C}/\kappa})$ is an extension of abelian categories we will sometimes write, as shorthand, that

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F_{\mathcal{A}/\kappa}} & \mathcal{B} \\
& & \xrightarrow{F_{\mathcal{C}/\kappa}} \mathcal{C}
\end{array}
\]

is an extension of abelian categories.

Now we are ready for the main theorem of this note: when one has an extension

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F_{\mathcal{A}/\kappa}} & \mathcal{B} \\
& & \xrightarrow{F_{\mathcal{C}/\kappa}} \mathcal{C}
\end{array}
\]

of abelian categories, an object $N$ of $\mathcal{B}$ is in the essential image of $F_{\mathcal{B}/\mathcal{A}}$ if and only if $F_{\mathcal{C}/\mathcal{B}}(N)$ is in the essential image of $F_{\mathcal{C}/\kappa}$. In the case of an extension

$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$

of finite-dimensional co-commutative connective Hopf algebras over a field, this is saying that a $\mathcal{B}$-module $N$ is of the form $B \otimes_A M$ if and only if $C \otimes_B N$ is a free $C$-module. Here is the theorem and its proof:
Theorem 2.3. Let $\mathcal{K}$ be a semisimple abelian category and let $(\mathbb{A}, \mathbb{B}, \mathbb{C}, F_{\mathcal{A}/\mathcal{K}}, F_{\mathcal{B}/\mathcal{A}}, F_{\mathcal{C}/\mathcal{B}})$ be an extension of abelian categories over $\mathcal{K}$. Suppose $M$ is an object of $\mathbb{B}$ and $Y$ is an object of $\mathcal{K}$ and there exists an isomorphism $F_{\mathcal{C}/\mathbb{B}} M \cong F_{\mathcal{C}/\mathcal{K}} Y$ in $\mathbb{C}$. Then there exists an object $N$ of $\mathbb{A}$ such that $F_{\mathcal{B}/\mathcal{A}} N \cong M$ in $\mathbb{B}$.

Proof. This proof is a little long but the basic idea is that we are going to use the isomorphism $F_{\mathcal{C}/\mathbb{B}} M \cong F_{\mathcal{C}/\mathcal{K}} Y$ to produce a descent datum, that is, an $F_{\mathcal{B}/\mathcal{A}} G_{\mathcal{B}/\mathcal{A}}$-coalgebra structure map, on $M$. Then comonadicity of $F_{\mathcal{B}/\mathcal{A}}$, axiom 5 implies that there exists $N$ such that $F_{\mathcal{B}/\mathcal{A}} N$ is isomorphic to $M$.

Choose an isomorphism $a : F_{\mathcal{C}/\mathbb{B}} M \xrightarrow{\cong} F_{\mathcal{C}/\mathcal{K}} Y$ in $\mathbb{C}$. Then we let $\sigma : F_{\mathcal{B}/\mathcal{K}} Y \to G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} M$ be the composite

\[
\begin{array}{ccc}
F_{\mathcal{B}/\mathcal{K}} Y & \xrightarrow{\eta F_{\mathcal{B}/\mathcal{K}} Y} & G_{\mathcal{C}/\mathbb{B}} F_{\mathcal{C}/\mathbb{B}} M \\
\downarrow \sigma & & \downarrow \sigma \\
\mathbb{Y} & \xrightarrow{\eta M} & G_{\mathcal{C}/\mathbb{B}} F_{\mathcal{C}/\mathbb{B}} M
\end{array}
\]

where $\eta$ is the unit map of the adjunction of $F_{\mathcal{C}/\mathbb{B}}, G_{\mathcal{C}/\mathbb{B}}$. We claim that there exists a map $\tilde{\sigma} : F_{\mathcal{B}/\mathcal{K}} Y \to M$ making the above diagram commute. Indeed, $\sigma \in \text{hom}_{\mathbb{B}}(F_{\mathcal{B}/\mathcal{K}} Y, G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} M)$, so a map $\tilde{\sigma}$ as desired exists if the map of abelian groups

\[
\text{hom}_{\mathbb{A}}(F_{\mathcal{B}/\mathcal{K}} Y, M) \to \text{hom}_{\mathbb{B}}(F_{\mathcal{B}/\mathcal{K}} Y, G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} M),
\]

induced by the unit map $\eta M : M \to G_{\mathcal{C}/\mathbb{B}} F_{\mathcal{C}/\mathbb{B}} M$, is a surjection. However, the map 2.3 fits into the commutative diagram

\[
\begin{array}{ccc}
\text{hom}_{\mathbb{A}}(F_{\mathcal{B}/\mathcal{K}} Y, M) & \to & \text{hom}_{\mathbb{B}}(F_{\mathcal{B}/\mathcal{K}} Y, G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} M) \\
\downarrow \equiv & & \downarrow \equiv \\
\text{hom}_{\mathcal{K}}(Y, G_{\mathcal{B}/\mathcal{K}} M) & \to & \text{hom}_{\mathcal{K}}(Y, G_{\mathcal{C}/\mathcal{K}} F_{\mathcal{C}/\mathcal{B}} M).
\end{array}
\]

By axiom 4 $\eta M$ is an epimorphism, hence by axiom 1 $G_{\mathcal{B}/\mathcal{K}} \eta M : G_{\mathcal{B}/\mathcal{K}} M \to G_{\mathcal{C}/\mathcal{K}} F_{\mathcal{B}/\mathcal{B}} M$ is an epimorphism in $\mathcal{K}$, hence a split epimorphism since $\mathcal{K}$ is assumed semisimple. So by the commutativity of diagram 2.4 the map 2.3 is a surjection. Hence $\tilde{\sigma}$ exists making diagram 2.2 commute.

We now observe that our map $\tilde{\sigma}$ has the property that

\[
G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} \tilde{\sigma} = G_{\mathcal{C}/\mathbb{A}} a.
\]

This follows from the commutativity of diagram 2.2 together with basic monad theory giving us the equations

\[
\sigma = G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} \tilde{\sigma} \circ \eta F_{\mathcal{C}/\mathbb{B}} Y = G_{\mathcal{C}/\mathbb{A}} a \circ \eta F_{\mathcal{C}/\mathbb{B}} Y.
\]

Now, by axiom 4 $\eta F_{\mathcal{C}/\mathbb{B}} Y$ is epic, i.e., right-cancellable; so $G_{\mathcal{C}/\mathbb{A}} F_{\mathcal{C}/\mathbb{B}} \tilde{\sigma} = G_{\mathcal{C}/\mathbb{A}} a$. 

We now check that $\bar{\sigma}$ is epic. We have the commutative diagram with exact rows

\[
\begin{array}{c}
F_{B/k}Y \\
\downarrow \sigma \\
G_{C/g}F_{B/k}Y \\
\downarrow G_{C/g}\psi \\
G_{C/g}F_{C/g}M
\end{array} \quad \xrightarrow{\sigma} \quad \begin{array}{c}
M \\
\downarrow \sigma \\
coker \bar{\sigma} \\
\downarrow \sigma \\
0
\end{array}
\]

Exactness of the second row is due to axiom 1 implying that $G_{C/g}$ preserves cokernels, and $F_{C/s}$ preserving cokernels by virtue of being a left adjoint. By equation 2.5, $G_{C/g}F_{C/g}\bar{\sigma} = G_{C/g}a$ is an isomorphism (since $a$ is), so its cokernel is trivial. Now axiom 2 implies that $\text{coker} \bar{\sigma} \cong 0$, so $\bar{\sigma}$ is epic.

Now let $\psi_{std} : F_{B/k}Y \rightarrow F_{B/k}Y$ denote the standard $F_{B/k}$-coalgebra map on $F_{B/k}Y$, i.e., $\psi_{std} = G_{B/k}\eta F_{B/k}Y$. We write $i$ for the inclusion $i : \ker \bar{\sigma} \rightarrow F_{B/k}Y$ of the kernel of $\bar{\sigma}$ into $F_{B/k}Y$. We have the commutative diagram with exact column and exact row

\[
\begin{array}{c}
0 \\
\downarrow \end{array} \quad \begin{array}{c}
G_{B/k} \ker \bar{\sigma} \\
\downarrow j \\
0 \\
\downarrow \end{array} \quad \begin{array}{c}
F_{B/k}Y \\
\downarrow \eta F_{B/k}Y \\
G_{B/k}F_{B/k}Y \\
\downarrow G_{B/k}\psi_{std} \\
G_{B/k}M \\
\downarrow G_{B/k}F_{B/k}M
\end{array} \quad \xrightarrow{G_{B/k}\psi_{std} - \eta G_{B/k}F_{B/k}Y} \quad \begin{array}{c}
G_{B/k}F_{B/k}Y \\
\downarrow G_{B/k}\sigma \\
G_{B/k}F_{B/k}Y \\
\downarrow G_{B/k}\sigma
\end{array}
\]

with exactness of the row due to axiom 3 and existence of the map $j$ making the diagram commute being due to axiom 5. So we have

\[
0 = (G_{B/k}\psi_{std} - \eta G_{B/k}F_{B/k}Y) \circ G_{B/k}i,
\]

and hence we have

\[
G_{B/k}F_{B/k}G_{B/k} \circ G_{B/k}\psi_{std} \circ G_{B/k}i = G_{B/k}F_{B/k}G_{B/k} \circ \eta G_{B/k}F_{B/k}Y \circ G_{B/k}i
\]

\[
= \eta G_{B/k}M \circ G_{B/k} \circ G_{B/k}i
\]

\[
= \eta G_{B/k}M \circ G_{B/k}(\bar{\sigma} \circ i)
\]

\[
= \eta G_{B/k}M \circ G_{B/k} \circ 0
\]

\[
= 0.
\]

Since $G_{B/k}(F_{B/k}G_{B/k} \circ \psi_{std} \circ i) = 0$ and $G_{B/k}$ is faithful by axiom 1, we have that the composite $F_{B/k}G_{B/k} \circ \psi_{std} \circ i = 0$. Hence we have a canonical map $\bar{\phi}$ filling in the
We claim that the map $\tilde{\psi} : M \to F_{B/A}G_{B/A}M$ is a $F_{B/A}G_{B/A}$-coalgebra structure map on $M$. We must check that $\tilde{\psi}$ is counital and coassociative. We check counitality first: we have the commutative diagram with exact columns

\[
\begin{array}{ccc}
0 & \xrightarrow{\id} & F_{B/A}G_{B/A}M \\
\ker \tilde{\sigma} & \xrightarrow{i} & F_{B/A}G_{B/A}M \\
F_{B/A}F_{A/Y}Y & \xrightarrow{\psi_{\text{std}}} & F_{B/A}G_{B/A}F_{A/Y}Y \\
\tilde{\sigma} & \xrightarrow{\tilde{\psi}} & F_{B/A}G_{B/A} \tilde{\sigma} \\
M & \xrightarrow{\epsilon} & F_{B/A}G_{B/A}M \\
0 & \xrightarrow{0} & M
\end{array}
\]

where $\epsilon$ is the counit natural transformation of the adjunction of $F_{B/A}, G_{B/A}$. We get the identity map across the top of the diagram because of $\psi_{\text{std}}$ being itself an $F_{B/A}G_{B/A}$-coalgebra structure map, hence itself counital. From the commutativity of this diagram we get the equality

\[
\epsilon M \circ \tilde{\psi} \circ \tilde{\sigma} = \tilde{\sigma} \circ \epsilon F_{B/A}F_{A/Y}Y \circ \psi_{\text{std}}
\]

Now since $\tilde{\sigma}$ is epic, i.e., right-cancellable, we have that $\epsilon M \circ \tilde{\psi} = \id$, which is precisely the statement of counitality for $\tilde{\psi}$.

Now we check coassociativity. Due to basic properties of adjunctions and their comonads and also coassociativity of $\psi_{\text{std}}$ since it itself is a $F_{B/A}G_{B/A}$-coalgebra structure map,

\[
\epsilon M \circ \tilde{\psi} \circ \tilde{\sigma} = \tilde{\sigma} \circ \epsilon F_{B/A}F_{A/Y}Y \circ \psi_{\text{std}}
\]

Now since $\tilde{\sigma}$ is epic, i.e., right-cancellable, we have that $\epsilon M \circ \tilde{\psi} = \id$, which is precisely the statement of counitality for $\tilde{\psi}$.
we have the equalities
\[
F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\psi} \circ \tilde{\sigma} = F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\psi} \circ F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\sigma} \circ \psi_{std}
\]
\[
= F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}(\tilde{\psi} \circ \tilde{\sigma}) \circ \psi_{std}
\]
\[
= F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}(F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\psi} \circ \psi_{std}) \circ \psi_{std}
\]
\[
= F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}(F_{\mathcal{B}/\mathcal{A}}\eta G_{\mathcal{B}/\mathcal{A}}F_{\mathcal{B}/\mathcal{A}}\mathcal{F} \mathcal{A} Y) \circ \psi_{std}
\]
\[
= F_{\mathcal{B}/\mathcal{A}}\eta G_{\mathcal{B}/\mathcal{A}}M \circ F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\sigma} \circ \psi_{std}
\]
\[
= F_{\mathcal{B}/\mathcal{A}}\eta G_{\mathcal{B}/\mathcal{A}}M \circ \tilde{\psi} \circ \tilde{\sigma},
\]
and since \( \tilde{\sigma} \) is epic, i.e., right-cancellable, this tells us that
\[
F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}}\tilde{\psi} \circ \tilde{\phi} = F_{\mathcal{B}/\mathcal{A}}\eta G_{\mathcal{B}/\mathcal{A}}M \circ \tilde{\psi},
\]
which is precisely the statement that \( \tilde{\psi} \) is coassociative.

Hence \( \tilde{\psi} \) is a \( F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}} \)-coalgebra structure map. Hence, by axiom 3 since \( M \) admits the structure of a \( F_{\mathcal{B}/\mathcal{A}}G_{\mathcal{B}/\mathcal{A}} \)-coalgebra, it is itself in the essential image of the functor \( F_{\mathcal{B}/\mathcal{A}} \). □

3. Special cases and applications.

We introduce a quick definition of a certain class of monoids which are suitable for being the monoids of grading for graded objects, e.g. \( \mathbb{N} \) and \( \mathbb{Z} \).

**Definition 3.1.** We will say that a commutative monoid \( \mathbb{M} \) is finitely-generated and weakly free, or FGWF for short, if \( \mathbb{M} \) is isomorphic to a finite Cartesian product of copies of \( \mathbb{N} \) and \( \mathbb{Z} \).

Here are some examples of extensions of abelian categories. Thm. 2.3 applies to each case.

**Proposition 3.2.** Suppose \( k \) is a field. The following are examples of extensions of abelian categories:

- Suppose \( A \) is an augmented algebra over \( k \), and \( f : A \to B \) is a faithfully flat map of \( k \)-algebras. Suppose both \( A \) and \( B \) are finite-dimensional as \( k \)-vector spaces. Let \( C \) be the algebra \( B \otimes_A k \) and suppose that the kernel of the surjection \( B \to C \) is contained in the Jacobson radical of \( B \). Then
  \[
  \text{fgMod}(A) \to \text{fgMod}(B) \to \text{fgMod}(C)
  \]
is an extension of abelian categories over \( \text{fgMod}(k) \). Here \( \text{fgMod}(A) \) is the category of finitely generated \( A \)-modules, \( \text{fgMod}(B) \) is the category of finitely generated \( B \)-modules, etc.

- The previous example works as well in the graded setting. Suppose \( \mathbb{M} \) is an FGWF monoid (e.g. \( \mathbb{M} = \mathbb{N} \) or \( \mathbb{M} = \mathbb{Z} \)) and suppose that \( A \) is an \( \mathbb{M} \)-graded augmented algebra over \( k \) and suppose \( f : A \to B \) is an \( \mathbb{M} \)-grading-preserving faithfully flat map of \( \mathbb{M} \)-graded \( k \)-algebras. Suppose both \( A \) and \( B \) are finite-dimensional as \( k \)-vector spaces. Let \( C \) be the \( \mathbb{M} \)-graded algebra \( B \otimes_A k \) and suppose that the kernel of the surjection \( B \to C \) is contained in the Jacobson radical of \( B \). Then
  \[
  \text{gr fgMod}(A) \to \text{gr fgMod}(B) \to \text{gr fgMod}(C)
  \]
is an extension of abelian categories over \( \text{fgMod}(k) \). Here \( \text{gr fgMod}(A) \) is the category of finitely generated \( \mathbb{M} \)-graded \( A \)-modules and grading-preserving module maps, etc.
Suppose \( A \to B \to C \) is an extension of connected co-commutative Hopf algebras over \( k \). In other words, \( A, B, C \) can each be given \( \mathbb{N} \)-gradings such that the degree zero summand of each of \( A, B, C \) consists of the image of the unit map from \( k \); and \( B \cong A \otimes_k C \) both as an \( A \)-module and as a \( C \)-comodule. (See [4] for extensions of graded connected Hopf algebras.) Suppose each of \( A, B, C \) is finite-dimensional as a \( k \)-vector space. Then

\[
\text{fgMod}(A) \to \text{fgMod}(B) \to \text{fgMod}(C)
\]
is an extension of abelian categories over \( \text{fgMod}(k) \).

Suppose \( A \to B \to C \) is an extension of \( \mathbb{N} \)-graded connected co-commutative Hopf algebras over \( k \). In other words, \( A, B, C \) are each equipped with \( \mathbb{N} \)-gradings such that the degree zero summand of each of \( A, B, C \) consists of the image of the unit map from \( k \); and \( B \cong A \otimes_k C \) both as an \( \mathbb{N} \)-graded \( A \)-module and as an \( \mathbb{N} \)-graded \( C \)-comodule. Suppose each of \( A, B, C \) is finite-dimensional as a \( k \)-vector space. Then

\[
\mathbb{N} - \text{gr} \ \text{fgMod}(A) \to \mathbb{N} - \text{gr} \ \text{fgMod}(B) \to \mathbb{N} - \text{gr} \ \text{fgMod}(C)
\]
is an extension of abelian categories over \( \mathbb{N} - \text{gr} \ \text{fgMod}(k) \).

**Proof.**

- Let \( A \) be an augmented algebra over \( k \), \( f : A \to B \) a faithfully flat map of \( k \)-algebras, \( C \) the tensor product algebra \( B \otimes_A k \). We write \( g \) for the surjection \( B \to C \). We check the axioms in order:
  - Axiom 1 follows immediately from restriction-of-scalars functors induced by ring homomorphisms being always faithful and exact.
  - Axiom 2 follows from the following observation: in our setting, the functor \( F_{C/k} \) is the base-change functor
    \[
    M \mapsto M \otimes_B (B \otimes_A k) \cong M \otimes_A k \cong M/(\ker g).
    \]
    If \( M/(\ker g) \cong 0 \), then the inclusion \( M/(\ker g) \hookrightarrow M \) is an isomorphism.
    Nakayama’s Lemma, in its noncommutative form, now applies: since \( (\ker g) \) is contained in the Jacobson radical of \( B \), \( (\ker g)M = M \) and \( M \) finitely generated together imply that \( M = 0 \).
  - Axiom 3 is immediately implied by the assumption that \( f \) is faithfully flat, by classical faithfully flat descent.
  - Axiom 4 follows from the counit map \( M \to G_{C/k} F_{C/k} M \) being precisely the quotient module map \( M \to M/(\ker g) \).
  - Axiom 5 follows from the observation that, if \( M \) is an \( A \)-module, then
    \[
    F_{C/k} F_{B/A} M \cong M \otimes_A B \otimes_B (B \otimes_A k) \cong (M \otimes_A k) \otimes_k (B \otimes_A k)
    \]
as a \( B \otimes_A k \)-module, i.e., \( F_{C/k} F_{B/A} M \) is the free \( B \otimes_A k \)-module generated by the \( k \)-modules \( M \otimes_A k \), i.e.,

\[
F_{C/k} F_{B/A} M \cong F_{C/k}(M \otimes_A k),
\]
as desired.

- In the \( \mathbb{M} \)-graded setting, axioms 1, 2, 4, and 5 all follow immediately from the ungraded case, case 3.2. Axiom 3 does as well, once one observes that, for an \( \mathbb{M} \)-graded faithfully flat ring extension \( A \to B \), any \( \mathbb{M} \)-graded descent datum (i.e., \( \mathbb{M} \)-graded \( F_{B/A} G_{B/A} \)-coalgebra structure map) is effective and descends to an \( \mathbb{M} \)-graded \( A \)-module.
• The case of an extension of connected co-commutative Hopf algebras is actually a special case of case 3.2. Since $B \cong A \otimes_k C$, we know that $B$ is free, hence faithfully flat as an $A$-module. Being connected and finite $k$-dimensional forces the augmentation ideals of each of $A, B, C$ to be nilpotent ideals. The kernel of $B \to B \otimes_A k$ is contained in the augmentation ideal, hence nilpotent, hence contained in the nilradical, hence contained in the Jacobson radical.

• The case of a graded connected extension of co-commutative Hopf algebras follows from case 3.2 together with the argument given above for case 3.2. □

There is another important class of cases, that of an extension of (commutative) Hopf algebroids; see Appendix A of [2] for basic definitions. Describing those cases depends on first having a well-developed theory of base-change and restriction-of-scalars functors induced by a map of Hopf algebroids on their comodule categories, something which we plan to address in a (hopefully near) future paper. Furthermore, since affine covers of Artin stacks are specified by Hopf algebroids, those cases really are about algebraic stacks as well as Hopf algebroids.

Now, finally, we write down some explicit consequence of Thm. 2.3 in some of the special cases listed in Prop. 3.2.

**Corollary 3.3.** Suppose that $A \to B$ is a faithfully flat map of algebras over a field $k$, such that:

- $A$ is augmented,
- $A, B$ are finite-dimensional as $k$-vector spaces, and
- the kernel of the surjection $B \to B \otimes_A k$ is contained in the Jacobson radical of $B$.

(For example, all these conditions are satisfied if $A \to B$ is an injective map of connected co-commutative Hopf algebras over $k$ which are finite-dimensional as $k$-vector spaces.)

Suppose $N$ is a finitely-generated $B$-module. Then there exists an $A$-module $M$ such that $B \otimes_A M \cong N$ if and only if $N \otimes_A k$ is a free $B \otimes_A k$-module.

**Proof.** From Thm. 2.3 and Prop. 3.2, we know that, in each of these cases, if $M$ is a right $B$-module such that $M \otimes_A k$ is a free $B \otimes_A k$-module, then $M$ admits an $F_{\text{Mod}(B)}/\text{fgMod}(A)G_{\text{Mod}(B)}$ coalgebra structure map, i.e., a $B/\text{A}$-descent datum. By classical faithfully flat descent (e.g. as in [5]), every $B/\text{A}$-descent datum is effective, hence gives an $A$-module $N$ such that $N \otimes_A B \equiv M$.

The opposite inclusion is much easier to prove: if $M$ is a right $B$-module such that that $M \equiv N \otimes_A B$ for some right $A$-module $N$, then

$$M \otimes_B (B \otimes_A k) \equiv (N \otimes_A B) \otimes_B (B \otimes_A k) \equiv (N \otimes_A k) \otimes_A (B \otimes_A k),$$

clearly a free right $B \otimes_A k$-module. □

**Corollary 3.4.** Suppose, for any ring $R$, we write $\text{Rep}(R)$ for the commutative monoid of isomorphism classes of finitely-generated right $R$-modules, with addition given by direct sum. Suppose, $A, B, k$ are as in Cor. 3.3. Then the image of the base-change (“tensoring up”) map of monoids $\text{Rep}(A) \to \text{Rep}(B)$ consists of exactly the isomorphism classes of $B$-modules $M$ such that $M \otimes_A k$ is a free $B \otimes_A k$-module.
Corollary 3.5. Suppose, $A$, $B$, $k$ are as in Cor. 3.3. Write $\text{StableRep}(A)$ (resp. $\text{StableRep}(B)$) for the stable representation monoid of $A$ (resp. $B$), that is, the monoid of stable equivalence classes of finitely generated $A$-modules (resp. finitely generated $B$-modules). Suppose every finitely generated projective $B \otimes_A k$-module is free. Then the sequence of monoid maps

$$\text{StableRep}(A) \to \text{StableRep}(B) \to \text{StableRep}(B \otimes_A k) \to 0$$

is exact.

The proof of Cor. 3.3 also holds, without significant change, in a graded setting:

Corollary 3.6. Now suppose that $\mathcal{M}$ is an FGWF monoid (e.g. $\mathcal{M} = \mathbb{N}$ or $\mathcal{M} = \mathbb{Z}$), and suppose that, for any $\mathcal{M}$-graded ring $R$, we write $\text{Rep}(R)$ for the commutative monoid of isomorphism classes of $\mathcal{M}$-graded finitely-generated right $R$-modules, with addition given by direct sum. Suppose that

$$A \to B$$

is a faithfully flat map of $\mathcal{M}$-graded algebras over a field $k$, such that:

- $A$ is augmented,
- $A$, $B$ are finite-dimensional as $k$-vector spaces, and
- the kernel of the surjection $B \to B \otimes_A k$ is contained in the Jacobson radical of $B$.

(For example, all these conditions are satisfied if $A \to B$ is an injective map of $\mathcal{M}$-graded connected co-commutative Hopf algebras over $k$ which are finite-dimensional as $k$-vector spaces.) Then the image of the base-change ("tensoring up") map $\text{Rep}(A) \to \text{Rep}(B)$, Then the image of the base-change ("tensoring up") map $\text{Rep}(A) \to \text{Rep}(B)$ consists of exactly the isomorphism classes of $B$-modules $M$ such that $M \otimes_A k$ is a free $\mathcal{M}$-graded $B \otimes_A k$-module.

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