Perturbative and Non-Perturbative Aspects
of $\mathcal{N} = 8$ Supergravity

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Abstract

Some aspects of quantum properties of $\mathcal{N} = 8$ supergravity in four dimensions are discussed for non-practitioners.

At perturbative level, they include the Weyl trace anomaly as well as composite duality anomalies, the latter being relevant for perturbative finiteness. At non-perturbative level, we briefly review some facts about extremal black holes, their Bekenstein-Hawking entropy and attractor flows for single- and two-centered solutions.
1 Lecture I

On “Quantum” $\mathcal{N} = 8, d = 4$ Supergravity

$\mathcal{N} = 8, d = 4$ “quantum” supergravity may be defined by starting with the Einstein-Hilbert action, and setting “perturbative” Feynman rules as a bona fide gauge theory of gauge particles of spin 2, the gravitons. In supersymmetric gravity theories with $\mathcal{N}$-extended supersymmetry in $d = 4$ space-time dimensions, the massless particle content is given by

$$\binom{\mathcal{N}}{k} = \frac{\mathcal{N}!}{k!(\mathcal{N} - k)!} \text{ particles of helicity } \lambda = 2 - \frac{k}{2},$$

where $k_{\text{max}} = \mathcal{N}$, and $\mathcal{N} \leq 8$ if $|\lambda| \leq 2$ is requested (namely, no higher spin fields in the massless spectrum).

One possible approach to “quantum” supergravity is to consider it as it comes from $M$-theory restricted to the massless sector. The problem is that this theory, even if preserving maximal $\mathcal{N} = 8$ supersymmetry in $d = 4$ space-time dimensions (corresponding to $32 = 8 \times 4$ supersymmetries), is not uniquely defined, because of the multiple choice of internal compactification manifolds and corresponding duality relations:

I. $M_{11} \rightarrow M_4 \times T_7$ ($GL^+(7, \mathbb{R})$ and $SO(7)$ manifest);

II. $M_{11} \rightarrow AdS_4 \times S^7$ ($SO(8)$ manifest, gauged);

III. $M_{11} \rightarrow M_4 \times T_{7,\mathbb{R}}$ ($SL(8, \mathbb{R})$ and $SO(8)$ manifest),

where $T_7$ is the 7-torus and $S^7$ is the 7-sphere. $T_{7,\mathbb{R}}$ denotes the case in which, according to Cremmer and Julia [1], the dualization of 21 vectors and 7 two-forms makes $SL(8, \mathbb{R})$ (in which $GL^+(7, \mathbb{R})$ is maximally embedded) manifest as maximal non-compact symmetry of the Lagrangian. Note that in case III one can further make $E_{7(7)}$ (and its maximal compact subgroup $SU(8)$) manifest on-shell, by exploiting a Cayley transformation supplemented by a rotation through $SO(8)$ gamma matrices on the vector 2-form field strengths [1, 2]. As we discuss further below, $E_{7(7)}$ can be promoted to a Lagrangian symmetry if one gives up manifest diffeomorphism invariance, as given by treatment in [3], then used in the anomaly study of [4].

It is worth remarking that $\mathcal{N} = 8, d = 4$ gauged supergravity with gauge group $SO(8)$ cannot be used for electroweak and strong interactions model building, because

$$SO(8) \not\supset SU(3) \times SU(2) \times U(1).$$

Furthermore, also the cosmological term problem arises out: the vacuum energy in anti De Sitter space $AdS_4$ is much higher than the vacuum energy in Standard Model of non-gravitational interactions (see e.g. the discussion in [5]). However, by exploiting the $AdS_4/CFT_3$ correspondence, theory II of (1.2) recently found application in $d = 3$ condensed matter physics (see e.g. [6] for a review and list of Refs.). Furthermore, the recently established fluid-gravity correspondence was object of many studies (see e.g. [7] for recent reviews and lists of Refs.).

The fundamental massless fields (and the related number $\sharp$ of degrees of freedom) of $M$-theory in $d = 11$ flat space-time dimensions are

$$g_{\mu\nu} \text{ (graviton)} : \quad \sharp = \frac{(d-1)(d-2)}{2} - 1, \quad \text{in } d = 11 : \sharp = 44;$$

$$\Psi_{\mu\alpha} \text{ (gravitino)} : \quad \sharp = (d - 3)2^{(d - 3)/2}, \quad \text{in } d = 11 : \sharp = 128;$$

$$A_{\mu\nu\rho} \text{ (three-form)} : \quad \sharp = \frac{(d-2)(d-3)(d-4)}{3!}, \quad \text{in } d = 11 : \sharp = 84.$$
Because a \((p+1)\)-form ("Maxwell-like" gauge field) \(A_{p+1}\) couples to \(p\)-dimensional extended objects, and its “magnetic” dual \(B_{d-p-3}\) couples to \((d-p-4)\)-dimensional extended objects, it follows that the fundamental (massive) objects acting as sources of the theory are \(M2\)- and \(M5\)-branes.

In general, a compactification on an \(n\)-torus \(T_n\) has maximal manifest non-compact symmetry \(GL^+(n, \mathbb{R}) \sim \mathbb{R}^\times \times SL(n, \mathbb{R})\). The metric \(g_{IJ}\) of \(T_n\) parametrizes the \(n(n+1)/2\)-dimensional coset \(\mathbb{R}^\times \times SL(n, \mathbb{R})\), whereas the Kaluza-Klein vectors \(g_{\hat{\mu}}\) are in the \(n\)’ irrep. of \(GL^+(n, \mathbb{R})\) itself. By reducing \(M\)-theory on \(T_7\) to \(d = 4\) theory with maximal \((\mathcal{N} = 8)\) local supersymmetry arises. By splitting the \(d = 11\) space-time index \(\mu = 0, 1, ..., 10\) as \(\hat{\mu} = (\hat{\mu}, I)\), where \(\hat{\mu} = 0, 1, ..., 3\) is the \(d = 4\) space-time index, and \(I = 1, ..., 7\) is the internal manifold index, the bosonic degrees of freedom of \(M\)-theory split as follows (below \(1.6\), for simplicity’s sake we will then refrain from hatting the \(d = 4\) curved indices):

\[
g_{\mu\nu} \rightarrow \begin{cases} 
g_{\hat{\mu}\hat{\nu}} (d = 4 \text{ graviton}), & 1 + 1; 
g_I^\mu (\text{vectors}), & 7'; 
g_{IJ} (\text{scalars}), & 28; 
\end{cases} 
\]

\[
A_{\mu\nu\rho} \rightarrow \begin{cases} 
A_{\hat{\mu}\hat{\nu}\hat{\rho}} (d = 4 \text{ domain wall}), & 7; 
A_{I\hat{\mu}\hat{\nu}} (\text{antisymmetric tensors : strings}), & 21; 
A_{I{\hat{\mu}J}} (\text{vectors}), & 35; 
A_{IJK} (\text{scalars}), & 70;
\end{cases}
\]

where the indicated irreps. pertain to the maximal manifest non-compact symmetry \(GL^+(7, \mathbb{R})\), whose maximal compact subgroup is \(SO(7)\). The 28 scalars \(g_{IJ}\) (metric of \(T_7\)) parametrize the coset \(\mathbb{R}^\times \times \frac{SL(7, \mathbb{R})}{SO(7)}\).

By switching to formulation \(III\) of \(1.12\) \(1.1\), the 7 antisymmetric rank-2 tensors \(A_{\hat{\mu}\hat{\nu}I}\) (sitting in the 7 of \(GL^+(7, \mathbb{R})\) can be dualized to scalars \(\phi^I\) (in the 7' of \(GL^+(7, \mathbb{R})\)), and therefore one obtains \(35 + 28 + 7 = 70\) scalar fields. It is worth remarking that in Cremmer and Julia’s \(1\) theory the gravitinos \(\psi_I\) and the gauginos \(\chi_{IJK}\) respectively have the following group theoretical assignment (\(I\) in 8 of \(SU(8)\)):

\[
\text{theory } III : \begin{cases} 
\psi_I : SO(7) \subset SO(8) \subset SU(8); 
\chi_{IJK} : SO(7) \subset SO(8) \subset SU(8). 
\end{cases} 
\]

Thus, in this theory the 70 scalars arrange as

\[
\text{theory } III : s = 0 \text{ dofs : } SO(7) \subset SO(8) \subset SU(8), 
\]

\(1 + 7 + 21 + 35 + 35 + 35 = 70\)

where \(70\) is the rank-4 completely antisymmetric irrep. of \(SU(8)\), the maximal compact subgroup of the \(U\)-duality group \(E_7(7)\) (also called \(\mathcal{R}\)-symmetry).

\(1\) As evident from \(1.7\), we use a different convention with respect to \(3\), (see e.g. Table 36 therein). Indeed, we denote as \(8_s, \) of \(SO(8)\) the irrep. which decomposes into \(7 + 1\) of \(SO(7)\), whereas the two spinorial irreps. \(8_s\) and \(8_c\) both decompose into \(8\) of \(SO(7)\). The same change of notation holds for \(35\) and \(56\) irreps..
On the other hand, also the vector fields $A_{\mu I}$ (sitting in the $21$ of $GL^+(7, \mathbb{R})$) can be dualized to $A^I_{\mu}$ (sitting in the $21'$ of $GL^+(7, \mathbb{R})$). Together with $g^{IJ}_\mu$, the “electric” and “magnetic” vector degrees of freedom can thus be arranged as follows:

$$s = 1 \text{ dofs} :$$

$$\begin{align*}
&\begin{cases}
G^+ (7, \mathbb{R}) \subset SL (8, \mathbb{R}) \subset E_{7(7)}; \\
7^+ + 21^1 + 21 \quad 28^+ + 28 \quad 56
\end{cases} \\
&\begin{cases}
SO (7) \subset SO (8) \subset SU (8), \\
7^+ + 21^1 + 21 \quad 28^+ + 28 \quad 28 + 28
\end{cases}
\end{align*}$$

(1.9)

The counting of degrees of freedom is completely different in the gauged maximal supergravity theory $\mathbf{II}$ of [12], based on the $AdS_4 \times S^7$ solution of $d = 11, \mathcal{N} = 1$ $M$-theory field equations; in this framework, rather than using torus indices as in theories $\mathbf{I}$ and $\mathbf{III}$ of [12], Killing vector/spinor techniques are used (for a discussion, see e.g. [8], and the lectures [9], and Refs. therein). However, the 70 scalars still decompose as $35_v + 35_c$ of $SO (8)$ but, with respect to the chain of branchings (1.8), they lack of any $SO (7)$ interpretation. It is worth recalling that a formulation of this theory directly in $d = 4$ yields to the de Wit and Nicolai’s $\mathcal{N} = 8, d = 4$ gauged supergravity [10].

Since the 70 scalar fields fit into an unique irrep. of $SU (8)$, it follows that they parameterize a non-compact coset manifold $G \overline{\mathcal{H}}_{SU (8)}$. Indeed, the $SU (8)$ under which both the scalar fields and the fermion fields transform is the “local” $SU (8)$, namely the stabilizer of the scalar manifold.

On the other hand, the $SU (8)$ appearing in the second line of (1.9) is the “global” $SU (8)$ ($R$-symmetry group). Roughly speaking, the physically relevant group $SU (8)$ is the diagonal one in the product $SU_{\text{local}} (8) \times SU_{\text{global}} (8)$ (see also discussion below).

Remarkably, there exists an unique simple, non-compact Lie group with real dimension $70 + 63 = 133$ and which embeds $SU (8)$ as its maximal compact subgroup: this is the real, non-compact split form $E_{7(7)}$ of the exceptional Lie group $E_7$, thus giving rise to the symmetric, rank-7 coset space

$$\frac{E_{7(7)}}{SU (8)/\mathbb{Z}_2},$$

(1.10)

which is the scalar manifold of $\mathcal{N} = 8$, $d = 4$ supergravity ($\mathbb{Z}_2$ is the kernel of the $SU (8)$-representations of even rank; in general, spinors transform according to the double cover of the stabilizer of the scalar manifold; see e.g. [11], [12]).

$E_{7(7)}$ acts as electric-magnetic duality symmetry group [13], and its maximal compact subgroup $SU (8)$ has a chiral action on fermionic as well as on (the vector part of the) bosonic fields. While the chiral action of $SU (8)$ on fermions directly follows from the chirality (complex nature) of the relevant irreps. of $SU (8)$ (as given by Eq. (1.7)), the chiral action on vectors is a crucial consequence of the electric-magnetic duality in $d = 4$ space-time dimensions. Indeed, this latter allows for “self-dual/anti-self-dual” complex combinations of the field strengths, which can then fit into complex irreps. of the stabilizer $H$ of the coset scalar manifold $G/H$ itself. For the case of maximal $\mathcal{N} = 8$ supergravity, the relevant chiral complex irrep. of $H = SU (8)$ is the rank-2 antisymmetric $28$, as given by Eq. (1.9).

Note that if one restricts to the $SL (8, \mathbb{R})$-covariant sector, the chirality of the action of electric-magnetic duality is spoiled, because the maximal compact subgroup of $SL (8, \mathbb{R})$, namely $SO (8)$, has not chiral irreps.

\footnote{There are three distinct 35-dimensional $SO (8)$ irreps., usually denoted as $35_v$, $35_c$, and $35_e$, obeying the relations:

$$(ab) \leftrightarrow [ABCD]_+, \quad [abcd]_2 \leftrightarrow [ABCD]_-, \quad [abcd]_3 \leftrightarrow (AB),$$

where $a, b = 1, \ldots, 8$ are in $8_v$, $A, B, C, D = 1, \ldots, 8$ are in $8_c$ (or in $8_e$), and “$\pm$” denotes self-dual/anti-self-dual irreps.. For a discussion, see e.g. [11] and [9].}
Composite (sigma model $G/H$) anomalies can arise in theories in which $G$ has a maximal compact subgroup with a chiral action on bosons and/or fermions (see e.g. [14, 15, 11]). Surprising cancellations among the various contributions to the composite anomaly can occur as well. An example is provided by $N = 8$, $d = 4$ supergravity itself, in which standard anomaly formulæ yield the remarkable result [15, 14]

$$3 \text{Tr}_8 X^3 - 2 \text{Tr}_{28} X^3 + \text{Tr}_{56} X^3 = (3 - 8 + 5) \text{Tr}_8 X^3 = 0,$$

where $X$ is any generator of the Lie algebra $\mathfrak{su}(8)$ of the rigid ($i.e.$ global) $SU(8)$ group ($R$-symmetry). In light of the previous considerations, the first and third contributions to (1.11) are due to fermions: the 8 gravitinos $\psi_A$ and the 56 spin-$\frac{1}{2}$ fermions $\chi_{ABC}$, respectively, whereas the second contribution is due to the 28 chiral vectors. Note that, for the very same reason, the local $SU(8)$ (stabilizer of the non-linear sigma-model of scalar fields), under which only fermions do transform, would be anomalous [14]. In an analogous way, in [15] it was discovered that $N = 6$ and $N = 5$ “pure” supergravities are composite anomaly-free, whereas $N \leq 4$ theories are not.

A crucial equivalence holds at the homotopical level:

$$E_7(7) \cong (SU(8)/\mathbb{Z}_2) \times \mathbb{R}^{70},$$

implying that the two group manifolds have the same De Rham cohomology. This is a key result, recently used in [24] to show that the aforementioned absence of $SU(8)$ current anomalies yield to the absence of anomalies for the non-linearly realized $E_7(7)$ symmetry, thus implying that the $E_7(7)$ continuous symmetry of classical $N = 8$, $d = 4$ supergravity is preserved at all orders in perturbation theory (see e.g. [16, 17, 18, 19, 20, 21, 22, 23]). This implies the perturbative finiteness of supergravity at least up to seven loops; Bern, Dixon et al. explicitly checked the finiteness up to four loops included [16] (computations at five loops, which might be conclusive, are currently in progress; for a recent review, see e.g. [24]).

In order to achieve the aforementioned result on the anomalies of $E_7(7)$, in [24] the manifestly $E_7(7)$-covariant Lagrangian formulation of $N = 8$, $d = 4$ supergravity [3] was exploited, by using the ADM decomposition of the $d = 4$ metric, namely:

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right),$$

with lapse $N$ and shift $N^i$ ($h_{ij}$ is the metric on the spatial slice). Within this approach [23, 8], the diffeomorphism symmetry is not realized in the standard way on the vector fields: the 28 vector fields $A_\mu^a$ of the original formulation [11, 10] are replaced by 56 vector fields $A_i^B$ ($B = 1, \ldots, 56$) with only spatial components, which recover the number of physical degrees of freedom by switching to an Hamiltonian formulation. Besides the $56 \times 56$ symplectic metric $\Omega$:

$$\Omega^T = -\Omega, \quad \Omega^2 = -I,$$

a crucial quantity is the scalar field-dependent $56 \times 56$ symmetric matrix $M$ (see Eq. (2.20) below), which is symplectic (see e.g. [26]):

$$M \Omega M = \Omega,$$

Also scalar fields transform under local $SU(8)$, but they do not contribute to the composite anomaly, because they sit in the self-real (and thus non-chiral) rank-4 antisymmetric irrep. 70 of $SU(8)$.

We use units in which the Newton gravitational constant $G$ and the speed of light in vacuum $c$ are all put equal to 1.
and negative definite due to the positivity of the vector kinetic terms (see also discussion below). $M$ allows for the introduction of a symplectic, scalar field-dependent complex structure:

$$J ≡ MΩ ⇔ J^2 = MΩMΩ = −I.$$  

(1.16)

Thus, the equations of motion of the 56 vector fields $A^B_i$ can be expressed as a twisted self-duality condition\(^5\) for their super-covariant field strengths $\hat{F}^B_{\mu\nu}$, namely (see \([3, 4]\) for further elucidation)

$$\hat{F}^B_{\mu\nu} = −\frac{1}{2\sqrt{|g|}} \epsilon^{\rho\sigma} J^A_{\alpha} \hat{F}^A_{\mu\nu},$$  

(1.17)

Although the time components $A^B_0$ do not enter the Lagrangian, they appear when solving the equations of motion for the spatial components $A^B_i$, and diffeomorphism covariance is recovered on the solutions of the equations of motion \([3, 4]\).

From power counting arguments in quantum gravity, an $n$-loop counterterm contains $2n + 2$ derivatives, arranged such that it does not vanish on-shell. In $N = 8$ supergravity the first (non-BPS) full superspace integral which is $E_{7(7)}$-invariant is the super-Vielbein superdeterminant, which may contain as last component a term $\sim \partial^8 R^4$ (see e.g. \([28]\), and also \([29]\)), then possibly contributing to a divergence in the four-graviton amplitude. However, in \([22]\) R. Kallosh argued that, by exploiting the light-cone formulation, candidate counterterms can be written in chiral, but not in real, light-cone superspace. This would then imply the ultraviolet finiteness of $N = 8$, $d = 4$ supergravity, if supersymmetry and $E_{7(7)}$ symmetry are non-anomalous. Recently, in \([30]\) the latter symmetry was advocated by the same author to imply ultraviolet finiteness of the theory to all orders in perturbation theory.

A puzzling aspect of these arguments is that string theory certainly violates continuous $E_{7(7)}$ symmetry at the perturbative level, as it can be easily realized by considering the dilaton dependence of loop amplitudes (see e.g. \([23]\)). However, this is not the case for $N = 8$ supergravity. From this perspective, two (perturbatively finite) theories of quantum gravity would exist, with 32 local supersymmetries; expectedly, they would differ at least in their non-perturbative sectors, probed e.g. by black hole solutions. String theorists \([31, 32, 33]\) claim that $N = 8$, $d = 4$ supergravity theory is probably not consistent at the non-perturbative level. From a purely $d = 4$ point of view, their arguments could be overcome by excluding from the spectrum, as suggested in \([13]\), black hole states which turn out to be singular or ill defined if interpreted as purely four-dimensional gravitational objects. Inclusion of such singular states (such as $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS black holes) would then open up extra dimensions, with the meaning that a non-perturbative completion of $N = 8$ supergravity would lead to string theory \([31]\). Extremal black holes with a consistent $d = 4$ interpretation may be defined as having a Bertotti-Robinson \([34]\) $AdS_2 \times S^2$ near-horizon geometry, with a non-vanishing area of the event horizon. In $N = 8$ supergravity, these black holes are $\frac{1}{8}$-BPS or non-BPS (for a recent review and a list of Refs., see e.g. \([34]\)). The existence of such states would in any case break the $E_{7(7)}(\mathbb{R})$ continuous symmetry, because of Dirac-Schwinger-Zwanziger dyonic charge quantization conditions. The breaking of $E_{7(7)}(\mathbb{R})$ into an arithmetic subgroup $E_{7(7)}(\mathbb{Z})$ would then manifest only in exponentially suppressed contributions to perturbative amplitudes (see e.g. the discussion in \([4]\), and Refs. therein), in a similar way to instanton effects in non-Abelian gauge theories.

The composite anomaly concerns the gauge-scalar sector of the supergravity theories. Another anomaly, originated in the gravitational part of the action is the so-called gravitational anomaly, which only counts the basic degrees of freedom associated to the field content of the

\(^5\) For interesting recent developments on twisted self-duality, see \([27]\).

\(^6\) We also remark that these are the only black holes for which the Freudenthal duality \([30, 37]\) is well defined.
theory itself [38, 39] (see also [40] for a review):

\[ g_{\mu\nu} \langle T^{\mu\nu} \rangle_{1\text{-loop}} = \frac{\mathcal{A}}{32\pi^2} \int d^4 x \sqrt{|g|} \left( R_{\mu\nu\lambda\rho}^2 - 4 R_{\mu\nu}^2 + R^2 \right), \]  

(1.18)

where \( \langle T^{\mu\nu} \rangle_{1\text{-loop}} \) is the 1-loop vev of the gravitational stress-energy tensor. In general, this trace anomaly is a total derivative and therefore it can be non-vanishing only on topologically non-trivial \( d = 4 \) backgrounds. Furthermore, as found long time ago by Faddeev and Popov [41], \((p + 1)\)-form gauge fields have a complicated quantization procedure, due to the presence of ghosts; thus, their contribution to the parameter \( \mathcal{A} \) appearing in the formula (1.18) vary greatly depending on the field under consideration. This is because at the quantum level different field representations are generally inequivalent [38]. Consequently, one may expect that different formulations of \( \mathcal{N} = 8, d = 4 \) supergravity (1.2), give rise to different gravitational anomalies. This is actually what happens:

- in the formulation \textbf{III} of (1.2) [1], with maximal manifest compact symmetry \( SO(8) \), the antisymmetric tensors \( A_{\mu\nu I} \) are dualized to scalars, and \( \mathcal{A} \neq 0 \).
- in the formulation \textbf{I} of (1.2) [1], with maximal manifest compact symmetry \( SO(7) \), obtained by compactifying \( d = 11 \) M-theory on \( T_7 \), the antisymmetric tensors \( A_{\mu\nu I} \) are not dualized, and, as found some time ago in [38], the gravitational anomaly vanishes: \( \mathcal{A} = 0 \).
- Recently, a wide class of models has been shown to have \( \mathcal{A} = 0 \), by exploiting \textit{generalized mirror symmetry} for seven-manifolds [42].
- in the formulation \textbf{II} of (1.2) [10, 39] (see also [8, 9] and the discussion above), with maximal manifest compact \textit{gauged} symmetry \( SO(8) \), the gravitational anomaly is the sum of two contributions: one given by (1.18), and another one related to the non-vanishing cosmological constant \( \Lambda \), given by

\[ B \int d^4 x \sqrt{|g|} \Lambda^2, \]  

(1.19)

where \( B \), through the relation \( \Lambda \sim -e^2 \) [39], vanishes whenever the charge \( e \) normalization beta function [43]

\[ \beta_e (s) = \frac{\hbar}{90\pi^2} e^3 C_s \left( 1 - 12s^2 \right) (-1)^{2s} \]  

(1.20)

vanishes, namely in \( \mathcal{N} > 4 \) supergravities (compare e.g. Table II of [39] with Table 1 of [43]). The contribution to the coefficients \( \mathcal{A} \) and \( B \) of (1.18) and (1.19) depends on the spin \( s \) of the massless particle, but also, as mentioned above, on the its field representation ( [39]; see also e.g. Table 1 of [42]):

\[
\begin{array}{cccccc}
\mathcal{s} & 0 & 0 & \frac{1}{2} & 1 & 1 & \frac{3}{2}
\end{array}
\begin{array}{cccccc}
\mathcal{A} & 4 & -720 & 7 & -52 & 364 & -233
\end{array}
\begin{array}{cccccc}
\mathcal{B} & -1 & 0 & -3 & -12 & 0 & 137
\end{array}

(1.21)

\[ C_s \] is the appropriate (positive) quadratic invariant for the gauge group representation in which the particle of spin \( s \) sits (see e.g. Table 1 of [43], and Refs. therein).
2 Lecture II
(Multi-Center) Black Holes and Attractors

If $E_{7(7)}$ is a continuous non-anomalous symmetry of $\mathcal{N} = 8$ supergravity, then it is likely that non-perturbative effects are exponentially suppressed in perturbative amplitudes.

Black holes (BHs) are examples of non-perturbative states which, in presence of Dirac-Schwinger-Zwanziger dyonic charge quantization, would break $E_{7(7)}(\mathbb{R})$ to a suitable (not unique) arithmetic subgroup of $E_{7(7)}(\mathbb{Z})$ (see e.g. [44] 38, 19, 35, and Refs. therein).

Here we confine ourselves to recall some very basic facts on extremal BHs (for further detail, see e.g. [45], and Refs. therein), and then we will mention some recent developments on multi-center solutions.

For simplicity’s sake, we consider the particular class of extremal BH solutions constituted by static, asymptotically flat, spherically symmetric solitonic objects with dyonic charge vector $Q$ and scalars $\phi$ describing trajectories (in the radial evolution parameter $r$) with fixed points determined by the Attractor Mechanism [46]:

\[
\begin{align*}
\lim_{r \to r_H^+} \phi(r) &= \phi_H(Q); \\
\lim_{r \to r_H^-} \frac{d\phi(r)}{dr} &= 0.
\end{align*}
\] (2.1)

At the horizon, the scalars lose memory of the initial conditions (i.e. of the asymptotic values $\phi_\infty \equiv \lim_{r \to \infty} \phi(r)$), and the fixed (attractor) point $\phi_H^*(Q)$ only depends on the BH charges $Q$.

In the supergravity limit, for $\mathcal{N} > 2$ supersymmetry, the attractor behavior of such BHs is now completely classified (see e.g. [17], [48] for a review and list of Refs.).

The classical BH entropy is given by the Bekenstein-Hawking entropy-area formula [49]

\[ S(Q) = \frac{A_H(Q)}{4} = \pi V_{BH}(\phi_H(Q), Q) = \pi \sqrt{|I_4(Q)|}. \] (2.2)

where $V_{BH}$ is the effective BH potential [50] (see Eq. (2.2) below).

The last step of (2.2) holds for those theories admitting a quartic polynomial invariant $I_4$ in the (symplectic) representation of the electric-magnetic duality group in which $Q$ sits. This is the case at least for the “groups of type $E_7$” [51], which are the electric-magnetic duality groups of supergravity theories in $d = 4$ with symmetric scalar manifolds (see e.g. [37] for recent developments, and a list of Refs.). These include all $\mathcal{N} \geq 3$ supergravities as well as a broad class of $\mathcal{N} = 2$ theories in which the vector multiplets’ scalar manifold is a special Kähler symmetric space (see e.g. [52], [53], [54], [68], and Refs. therein). In the D-brane picture of type IIA supergravity compactified on Calabi-Yau threefolds $CY_3$, charges can be denoted by $q_0$ (D0), $q_a$ (D2), $p^a$ (D4) and $p^b$ (D6), and the quartic invariant polynomial $I_4$ is given by [55]

\[ I_4 = - (p^0 q_0 + p^a q_a)^2 + 4 \left( -p^0 I_3(q) + q_0 I_3(p) + \frac{\partial I_3(p) \partial I_3(q)}{\partial p^a} \frac{\partial p^a}{\partial q_a} \right); \] (2.3)

\[ I_3(p) \equiv \frac{1}{3!} d_{abc} p^a p^b p^c; \quad I_3(q) \equiv \frac{1}{3!} d^{abc} q_a q_b q_c, \] (2.4)

where $d_{abc}$ and $d^{abc}$ are completely symmetric rank-3 invariant tensors of the relevant electric and magnetic charge irreps. of the $U$-duality group in $d = 5$. A typical (single-center) BPS

\[ \text{charge vector} Q \]

\[ \phi \]

\[ I_4 \]

\[ I_3 \]

\[ d_{abc} \]

\[ d^{abc} \]
configuration is \((q_0, p^0)\), with all charges positive (implying \(I_4 > 0\)), while a typical non-BPS configuration is \((p^0, q_0)\) (implying \(I_4 < 0\)), see \textit{e.g.} the discussion in [56] (other charge configurations can be chosen as well). In the dressed charge basis, manifestly covariant with respect to the \(\mathcal{R}\)-symmetry group, the charges arrange into a complex skew-symmetric central charge matrix \(Z_{AB}\). This latter can be skew-diagonalized to the form [57]

\[ Z_{AB} = \text{diag}(z_1, z_2, z_3, z_4) \otimes \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \]  

(2.5)

and the quartic invariant can be recast in the following form [58]:

\[ I_4 = 4 \sum_{i=1}^{4} |z_i|^4 - 2 \sum_{i<j=1}^{4} |z_i|^2 |z_j|^2 + 4 \prod_{i=1}^{4} z_i + 4 \prod_{i=1}^{4} \bar{z}_i. \]  

(2.6)

In such a basis, a typical BPS configuration is the one pertaining to the Reissner-Nördstrom BH, with charges \(z_1 = (q + i p)\) and \(z_2 = z_3 = z_4 = 0\) (implying \(I_4 = (q^2 + p^2)^2 > 0\)), whereas a typical non-BPS configuration has (at the event horizon) \(z_i = \rho e^{i \pi / 4} \forall i = 1, ..., 4\) (implying \(I_4 = -16 \rho^4 < 0\)); see \textit{e.g.} the discussion in [59, 60, 61].

The simplest example of BH metric is the Schwarzschild BH:

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \]  

(2.7)

where \(M\) is the ADM mass [62], and \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\psi^2\). This BH has no \textit{naked singularity}, i.e. the singularity at \(r = 0\) is \textit{covered} by the event horizon at \(r_H = 2M\).

The metric (2.7) can be seen as the neutral \(q, p \to 0\) limit of the Reissner-Nordström (RN) BH:

\[ ds^2_{RN} = - \left( 1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2. \]  

(2.8)

Such a metric exhibits two horizons, with radii

\[ r_{\pm} = M \pm \sqrt{M^2 - q^2 - p^2}. \]  

(2.9)

In the \textit{extremal} case \(r_{+} = r_{-}\), and it holds that

\[ M^2 = q^2 + p^2, \]  

(2.10)

thus a unique event horizon exists at \(r_H = M\). Notice that for RN BHs the extremality condition coincides with the saturation of the \textit{BPS bound} [63]

\[ M^2 \geq q^2 + p^2. \]  

(2.11)

By defining \(\rho \equiv r - M = r - r_H\), the extremal RN metric acquires the general static Papapetrou-Majumdar [64] form

\[ ds^2_{RN, extr} = - \left( 1 + \frac{M}{\rho} \right)^2 dt^2 + \left( 1 + \frac{M}{\rho} \right)^2 (d\rho^2 + \rho^2 d\Omega^2) = - e^{2U} dt^2 + e^{-2U} d\bar{x}^2, \]  

(2.12)

where \(U = U(\vec{x})\) is an harmonic function satisfying the \(d = 3\) Laplace equation

\[ \Delta e^{-U(\vec{x})} = 0. \]  

(2.13)
In order to determine the near-horizon geometry of an extremal RN BH, let us define a new radial coordinate as $\tau = -\frac{1}{\rho} = \frac{1}{r_H - r}$. Thus, after a further rescaling $\tau \to M^2 \tau$, the near-horizon limit $\rho \to 0^+$ of extremal metric $[24,22]$ reads

$$\lim_{\rho \to 0^+} ds^2_{RN,extr} = \frac{M^2}{\tau^2} (-dt^2 + d\tau^2 + \tau^2 d\Omega^2),$$

which is nothing but the $AdS_2 \times S^2$ Bertotti-Robinson metric $[34]$, both flat and conformally flat.

In presence of scalar fields coupled to the BH background, the BPS bound gets modified, and in general extremality does not coincide with the saturation of BPS bound (and thus with supersymmetry preservation) any more. Roughly speaking, the charges $Q$ gets “dressed” with scalar fields $\phi$ into the central extension of the local $N$-extended supersymmetry algebra, which is an antisymmetric complex matrix $Z_{AB}(\phi, Q)$, named central charge matrix $(A,B = 1,\ldots,N)$:

$$\begin{align*}
\{Q_\alpha A, Q_\beta B\} &= \delta^B_A \sigma^\mu_{\alpha\beta} P^\mu; \\
\{Q_\alpha A, Q_{\beta B}\} &= \epsilon_{\alpha\beta} Z_{AB}(\phi, Q). 
\end{align*}\tag{2.15}$$

In general

$$Z_{AB}(\phi, Q) = L^A_{AB}(\phi) Q_B,$$

where $L^A_{AB}(\phi)$ are the scalar field-dependent symplectic sections of the corresponding (generalized) special geometry (see e.g. $[26, 59, 37]$, and Refs. therein).

In the BH background under consideration, the general Ansätze for the vector 2-form field strengths $F^\Lambda_{\mu\nu}$ of the $n_V$ vector fields $(\Lambda = 1,\ldots,n_V)$ and their duals $G^\Lambda_{\mu\nu} = \frac{\delta L}{\delta F^\Lambda_{\mu\nu}}$ are given by $[50]

$$
\begin{align*}
F_1 &= e^{2U} C M(\phi) Q dt \wedge d\tau + Q \sin \theta d\theta \wedge d\psi; \\
F_2 &= \left( \begin{array}{c}
F^\Lambda_{\mu\nu} \\
G^\Lambda_{\mu\nu}
\end{array} \right) \frac{dx^\mu dx^\nu}{2},
\end{align*}\tag{2.17,2.18}$$

and electric and magnetic charges $Q \equiv (p^\Lambda, q^\Lambda)^T$ are defined by

$$q^\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} G^\Lambda, \quad p^\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} F^\Lambda,$$

where $S^2_\infty$ is the 2-sphere at infinity. $M(\phi)$, already discussed in Sec. 1, is a $2n_V \times 2n_V$ real symmetric $Sp(2n_V, \mathbb{R})$ matrix (see Eq. 1.15) whose explicit form reads $[26]$ $[50]$

$$M(\phi) = \begin{pmatrix}
I + R I^{-1} R & -R I^{-1} \\
-R I^{-1} R & I^{-1}
\end{pmatrix},$$

with $I \equiv \text{Im} \mathcal{N}_{\Lambda\Sigma}$ and $R \equiv \text{Re} \mathcal{N}_{\Lambda\Sigma}$, where $\mathcal{N}_{\Lambda\Sigma}$ is the (scalar field dependent) kinetic vector matrix entering the $d = 4$ Lagrangian density

$$\mathcal{L} = -\frac{1}{2} R \partial_{ij}(\phi) \partial^i \phi \partial^j \phi + I_{\Lambda\Sigma} F^\Lambda \wedge^* F^\Sigma + R_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma.$$

(2.21)

The black hole effective potential $[46]$ is given by

$$V_{BH}(\phi, Q) = -\frac{1}{2} Q M(\phi) Q,$$

(2.22)
This is the effective potential which arises upon reducing the general \( d \geq 4 \) Lagrangian on the BH background to the \( d = 1 \) almost geodesic action describing the radial evolution of the \( n_V + 1 \) scalar fields \( (U(\tau), \phi^i(\tau)) \) \[65\]:

\[
S = \int \mathcal{L} d\tau = \int (\ddot{U} + g_{ij} \dot{\phi}^i \dot{\phi}^j + e^{2U} V_{BH}(\phi(\tau), p, q)) d\tau.
\] (2.23)

In order to have the same equations of motion of the original theory, the action must be complemented with the Hamiltonian constraint, which in the extremal case reads \[50\]

\[
\dot{U}^2 + g_{ij} \dot{\phi}^i \dot{\phi}^j - e^{2U} V_{BH}(\phi(\tau), p, q) = 0.
\] (2.24)

The black hole effective potential \( V_{BH} \) can generally be written in terms of the superpotential \( W(\phi) \) as

\[
V_{BH} = W^2 + 2g^{ij} \partial_i W \partial_j W.
\] (2.25)

This formula can be viewed as a differential equation defining \( W \) for a given \( V_{BH} \), and it can lead to multiple choices, one corresponding to BPS solutions, and the others associated to non-BPS ones. \( W \) allows to rewrite the ordinary second order supergravity equations of motion

\[
\ddot{U} = e^{2U} V_{BH};
\]
\[
\ddot{\phi}^i = g^{ij} \frac{\partial V_{BH}}{\partial \phi_j} e^{2U},
\] (2.26) (2.27)

as first order flow equations, defining the radial evolution of the scalar fields \( \phi^i \) and the warp factor \( U \) from asymptotic (radial) infinity towards the black hole horizon \[66\] :

\[
\dot{U} = -e^{U} W, \\
\dot{\phi}^i = -2e^{U} g^{ij} \partial_j W.
\] (2.28)

At the prize of finding a suitable “fake” first order superpotential \( W \), one only has to deal with these first order flow equations even for non-supersymmetric solutions, where one does not have Killing spinor equations \[66, 67\].

For \( \frac{1}{2} \) \( \frac{1}{2} \)-BPS supersymmetric BHs in \( \mathcal{N} \geq 2 \) supergravity theories (with central charge matrix \( Z_{AB} \)), \( W \) is given by the square root \[10\] of the largest of the eigenvalues of \( Z_{AB} Z_{BC} \) \[66, 67\]. Furthermore, \( W \) has a known analytical expression for all \( \mathcal{N} \geq 2 \) charge configurations with \( I_A > 0 \) (for \( \mathcal{N} = 2 \), this applies to special Kähler geometry based on symmetric spaces, see e.g. \[68\] \[69\] \[70\] \[71\]). For \( I_A < 0 \), \( W^2 \) has an analytical expression for rank-1 and rank-2 cosets \[69, 70, 71\], while it is known to exist in general as a solution of a sixth order algebraic equation \[70, 74, 72\].

The Bekenstein-Hawking BH entropy \[49\] \[2.2\] can be written in terms of \( W \) as follows:

\[
S(Q) = \pi W^2 \bigg|_{\partial W = 0},
\] (2.29)

where the critical points of the suitable \( W \) reproduce a class of critical points of \( V \) itself. It is worth remarking that the value of the superpotential \( W \) at radial infinity also encodes other basic properties of the extremal black hole, namely its \( \text{ADM mass} \) \[62\], given by \( (\phi_\infty^i \equiv \lim_{r \rightarrow \infty} \phi^i(r)) \)

\[
M_{\text{ADM}}(\phi_\infty, Q) = \dot{U}(\tau = 0) = W(\phi_\infty, Q),
\] (2.30)

and the scalar charges

\[
\Sigma^i(\phi_\infty, Q) = 2g^{ij}(\phi_\infty) \frac{\partial W}{\partial \phi^j}(\phi_\infty, Q).
\] (2.31)

\[10\] The subscript “h” stands for “the highest”. 
Multi-center BHs are a natural extension of single-center BHs, and they play an important role in the dynamics of quantum theories of gravity, such as superstrings and M-theory.

In fact, interesting multi-center solutions have been found for BPS BHs in $d = 4$ theories with $\mathcal{N} = 2$ supersymmetry, in which the Attractor Mechanism \cite{16, 50} is generalized by the so-called split attractor flow \cite{73}. This name comes from the existence, for 2-center solutions, of a co-dimension one region (named marginal stability (MS) wall) in the scalar manifold, where in fact a stable 2-center BH configuration may decay into two single-center constituents, whose scalar flows then separately evolve according to the corresponding attractor dynamics.

The study of these phenomena has recently progressed in many directions. By combining properties of $\mathcal{N} = 2$ supergravity and superstring theory, a number of interesting phenomena, such as split flow tree, entropy enigma, bound state recombination walls, and microstate counting have been investigated (see e.g. \cite{74, 75, 76, 77}).

The MS wall is defined by the condition of stability for a marginal decay of a 2-center BH compound solution with charge $Q = Q_1 + Q_2$ into two single-center BHs (respectively with charges $Q_1$ and $Q_2$):

$$M (\phi_\infty, Q_1 + Q_2) = M (\phi_\infty, Q_1) + M (\phi_\infty, Q_2).$$

(2.32)

As mentioned, after crossing the MS wall each flow evolves towards its corresponding attractor point, and the classical entropy of each BH constituent follows the Bekenstein-Hawking formula \cite{2.2}. It should be noted that the entropy of the original compound (conceived as a single-center BH with total charge $Q = Q_1 + Q_2$) can be smaller, equal, or larger than the sum of the entropies of its constituents:

$$S (Q_1 + Q_2) \gtrless S (Q_1) + S (Q_2).$$

(2.33)

For $\mathcal{N} = 2$ BPS compound and constituents in $\mathcal{N} = 2$, $d = 4$ supergravity (in which $Z_{AB} = \epsilon_{ABZ}$), \cite{2.2} can be recast as a condition on the central charge ($Z_i \equiv M (\phi_\infty, Q_i)$, $i = 1, 2$, and $Z_{1+2} \equiv Z (\phi_\infty, Q_1 + Q_2) = Z_1 + Z_2$):

$$|Z_1 + Z_2| = |Z_1| + |Z_2|. \quad (2.34)$$

Furthermore, before crossing the MS wall, the relative distance $|\vec{x}_1 - \vec{x}_2|$ of the two BH constituents with mutually non-local charges $(Q_1, Q_2) \neq 0$ is given by \cite{74}

$$|\vec{x}_1 - \vec{x}_2| = \frac{1}{2} \frac{|Q_1, Q_2| |Z_1 + Z_2|}{\text{Im} (Z_1 Z_2^*)}, \quad (2.35)$$

where

$$2 |\text{Im} (Z_1 Z_2^*)| = \sqrt{4 |Z_1|^2 |Z_2|^2 - \left( |Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2 \right)^2}. \quad (2.36)$$

Correspondingly, the 2-center BH has an intrinsic (orbital) angular momentum, given by \cite{74}

$$\vec{J} = \frac{1}{2} (Q_1, Q_2) \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}. \quad (2.37)$$

Note that when the charge vectors $Q_1$ and $Q_2$ are mutually local (i.e., $(Q_1, Q_2) = 0$), $|\vec{x}_1 - \vec{x}_2|$ is not constrained at all, and $J = 0$. Actually, this is always the case for the scalarless case of extremal Reissner-Nördstrom double-center BH solutions in $\mathcal{N} = 2$ pure supergravity. Indeed, in this case the central charge simply reads (see also discussion above)

$$Z_{RN} (p, q) = q + ip, \quad (2.38)$$

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and it is immediate to check that the marginal stability condition \((2.34)\) implies \(\langle Q_1, Q_2 \rangle = q_1 p_2 - p_1 q_2 = 0\).

It is here worth observing that \(\text{Im}(Z_1 Z_2) = 0\) both describes marginal and anti-marginal stability \([76]\). Marginal stability further requires

\[
\text{Re} (Z_1 Z_2) > 0 \iff |Z_1 + Z_2|^2 > |Z_1|^2 + |Z_2|^2. \tag{2.39}
\]

The other (unphysical) branch, namely

\[
\text{Re} (Z_1 Z_2) < 0 \iff |Z_1 + Z_2|^2 < |Z_1|^2 + |Z_2|^2, \tag{2.40}
\]

pertains to anti-marginal stability, reached for \(|Z_1 + Z_2| = |Z_1| - |Z_2|\).

Eq. \((2.35)\) implies the stability region for the 2-center BH solution to occur for

\[
\langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) > 0, \tag{2.41}
\]

while it is forbidden for \(\langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) < 0\). The scalar flow is directed from the stability region towards the instability region, crossing the MS wall at \(\langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) = 0\). This implies that the stability region is placed \emph{beyond} the MS wall, \textit{and on the opposite side of the split attractor flows.}

By using the fundamental identities of \(\mathcal{N} = 2\) special Kähler geometry in presence of two (mutually non-local) symplectic charge vectors \(Q_1\) and \(Q_2\) (see \textit{e.g.} \([73, 78, 59]\)), one can compute that at BPS attractor points of the centers 1 or 2:

\[
\langle Q_1, Q_2 \rangle = -2 \text{Im}(Z_1 Z_2) \Rightarrow 2 \langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) = -\langle Q_1, Q_2 \rangle^2 < 0. \tag{2.42}
\]

By using \((2.35)\) and \((2.42)\), one obtains \(|\vec{x}_1 - \vec{x}_2| < 0\): this means that, as expected, the BPS attractor points of the centers 1 or 2 do not belong to the stability region of the 2-center BH solution. Furthermore, the result \((2.42)\) also consistently implies:

\[
\text{stability region :}
\langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) = |\langle Q_1, Q_2 \rangle| \sqrt{4 |Z_1|^2 |Z_2|^2 - \left(|Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2\right)^2} > 0; \tag{2.43}
\]

\[
\text{instability region :}
\langle Q_1, Q_2 \rangle \text{Im}(Z_1 Z_2) = -|\langle Q_1, Q_2 \rangle| \sqrt{4 |Z_1|^2 |Z_2|^2 - \left(|Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2\right)^2} < 0, \tag{2.44}
\]

where a particular case of \((2.41)\), holding at the attractor points, is given by \((2.42)\).

As shown in \([77]\), by exploiting the theory of matrix norms, all above results can be extended \textit{at least} to \(\mathcal{N} = 2\) non-BPS states with \(Z_4 > 0\), as well as to BPS states in \(\mathcal{N} > 2\) supergravity.

For two-center BHs, by replacing \(|Z|\) with \(\sqrt{\lambda_{\hat{r}}}\), the generalization of \((2.35)\) \textit{e.g.} to \(\mathcal{N} = 8\) maximal supergravity reads

\[
|\vec{x}_1 - \vec{x}_2| = \frac{|\langle Q_1, Q_2 \rangle| \sqrt{\lambda_{1+2,h}}}{\sqrt{4 \lambda_{1,h} \lambda_{2,h} - (\lambda_{1+2,h} - \lambda_{1,h} - \lambda_{2,h})^2}}, \tag{2.45}
\]

where \(\lambda_{1+2,h} \equiv \lambda_h (\phi, Q_1 + Q_2)\) and \(\lambda_{i,h} \equiv \lambda_h (\phi, Q_i)\).
Analogously, also result (2.42) can be generalized e.g. to suitable states in $\mathcal{N} = 8$ supergravity. Indeed, by exploiting the $\mathcal{N} = 8$ generalized special geometry identities \[ Z_i \equiv Z_{AB}(\phi_\infty, Q_i) \]

\[ \langle Q_1, Q_2 \rangle = - \text{Im} \left( \text{Tr} \left( Z_1 Z_2^\dagger \right) \right), \]  

(2.46)

one can compute that at the $\frac{1}{8}$-BPS attractor points of the centers 1 or 2 it holds

\[ |\langle Q_1, Q_2 \rangle| = \sqrt{4\lambda_{h,1}\lambda_{h,2} - (\lambda_{1,h} + \lambda_{2,h} - \lambda_{1+2,h})^2}. \]  

(2.47)

Analogously to the $\mathcal{N} = 2$ case treated above, note that $\frac{1}{8}$-BPS attractor points of the centers 1 or 2 do not belong to the stability region of the two-center BH solution, but instead they are placed, with respect to the stability region, on the opposite side of the MS wall.

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