Survival and coexistence for spatial population models with forest fire epidemics.

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Figure: Gypsy moth.
Figure: Egg masses.
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Figure: Configuration at time $t$. Moth living period.
Figure: Growth stage configuration time $t$. Random offspring of mean $\beta$. 
Figure: Growth stage configuration time $t$. Random placement of eggs, uniformly in $V_N$ for each egg.
Figure: Growth stage configuration time \( t \). Moth die and assignation of sites is done.
Figure: Growth stage configuration time $t$. If there is more than one, only one survives (not enough room).
Figure: Epidemic stage configuration time $t + 1/2$. Epidemic attacks with probability $\alpha_N$ each site, independently.
Figure: Epidemic stage configuration time $t + 1/2$. Spreading of epidemic.
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Figure: Epidemic stage configuration time $t + 1/2$. Survivors.
Figure: Configuration time $t + 1$. Moth living period.
Multi-type moth model
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Figure: Epidemic stage configuration time $t + 1/2$. The type is assigned uniformly among all eggs that arrived to each vertex.
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Figure: Configuration time $t + 1$. Moth living period.
\[ \eta_{t+1} = (\eta_{t+1}(v_1), \ldots, \eta_{t+1}(v_{10})) \]
\[ = (0, 0, 0, 0, 0, 0, 0, 1, 0, 2) \]

**Figure:** Configuration time \( t + 1 \). Moth living period.
\[ \rho_{t+1} = (\rho_{t+1}(1), \rho_{t+1}(2)) \]
\[ = (1, 1)/10 \]

*Figure*: Configuration time \( t + 1 \). *Moth living period.*
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\textit{fittest} species (i.e. Biggest $\beta_i$) will always dominate.

\textbf{Spoiler alert!}

Forest fires epidemics \textit{change} this behavior.
Consider a graph $G_N = (V_N, E_N)$ with $N$ vertices and $m \in \mathbb{N}^*$. The MMM is a discrete time Markov process $(\eta_k)_{k \geq 0}$ is defined using an initial configuration $\eta_0 \in \{0, 1, \ldots, m\}^V$ and 2 families of parameters:

1. $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}_+^m$

2. $\vec{\alpha}_N = (\alpha_N(1), \alpha_N(2), \ldots, \alpha_N(m)) \in [0, 1]^m$. 
The dynamics of the process at each time step is divided into two consecutive stages:

- **Growth**: Each individual dies. Before they die, they generate an offspring with mean $\beta_i > 0$ (indep). Each egg is sent to a random uniformly site in $V$ (indep). The type is uniform among the eggs a site received; if none, the type is 0 (indep).

- **Epidemic**: Attack (indep) with probability $\alpha_N(i)$. It spreads to the connected component of the same type.

Density vectors by $(\rho_N^k(1), \rho_N^k(2), ..., \rho_N^k(m))$, $k \geq 0$ defined by

$$
\rho_N^k(i) := \frac{1}{N} \sum_{x \in V} 1\{\eta_N^k(x) = i\},
$$

where $i \in \{1, 2, ..., m\}$.
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Density vectors by $((\rho^N_k(1), \rho^N_k(2), \ldots, \rho^N_k(m)), k \geq 0)$ defined by

$$\rho^N_k(i) := \frac{1}{N} \sum_{x \in V_N} 1\{\eta^N_k(x) = i\}, \quad i \in \{1, 2, \ldots, m\}$$
When $m = 1$ the parameters are no longer vector, so we write $\alpha_N$ and $\beta$.

**Theorem (Durrett & Remenik ’09)**

Suppose $m = 1$ and $(G_N)_{N \in \mathbb{N}}$ a sequence of random uniform 3-regular graphs. Assume that the infection probability satisfies

$$\alpha_N \to 0 \quad \text{and} \quad \alpha_N \log(N) \to \infty, \quad \text{as} \quad N \to \infty$$

and also

$$\rho_0^N \xrightarrow{(d)} p_0 \in [0, 1] \quad \text{as} \quad N \to \infty.$$  

Then the process $(\rho_k^N)_{k \geq 0}$ converges in distribution as $(N \to \infty)$ to the (deterministic) orbits $(p_k)_{k \geq 0}$ of an explicit dynamical system started at $p_0$. 
Figure: Bifurcation diagram in $\beta$ for the dynamical system.
First extension: $m = 1$ and $\alpha_n \to \alpha \in (0, 1)$

Proposition (F-Linker-Remenik '18)

*Under the same hypothesis of Durrett & Remenik’s theorem, the convergence to an explicit dynamical system is also true when $\alpha_n \to \alpha \in (0, 1)$.***
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**Proposition (F-Linker-Remenik ’18)**

*Under the same hypothesis of Durrett & Remenik’s theorem, the convergence to an explicit dynamical system is also true when \( \alpha_N \to \alpha \in (0, 1) \).*

**Figure:** Left: Bifurcation diagram in \( \beta \) for the dynamical system with fixed \( \alpha = 0.1 \). Right: stochastic process simulations densities for \( \alpha = 0.1 \) and different \( \beta \)’s.
Define the **effective offspring parameter**

\[
\phi(\alpha_N, \beta) = \beta(1 - \alpha),
\]

and **the extinction time**

\[
\tau_N = \inf\{k \geq 0 : \rho_k^N = 0\}.
\]

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**Theorem (F-Linker-Remenik ’18)**

- **Extinction:** *When \( \phi(\alpha_N, \beta) < 1 \) there is \( C > 0 \) independent of \( N \) such that*

  \[
  \mathbb{E}(\tau_N) \leq C \log(N). \tag{1}
  \]

- **Survival:** *If \( \phi(\alpha_N, \beta) > 1 \) and \( \rho_0^N \) the initial density is bounded away from 0, then there exists \( c > 0 \) (depending only on \( \rho_0^N \) and \( \alpha_N \)) such that*

  \[
  \mathbb{E}(\tau_N) \geq cN. \tag{2}
  \]
Vectors again!

**Theorem (F-Linker-Remenik ’18)**

Consider $m \geq 2$ and $\vec{\alpha} \in [0, 1]^m$ (epidemic parameters). If

\[ \vec{\alpha}_N \to \vec{\alpha} \quad \text{and} \quad \alpha_N(i) \log(N) \to \infty \quad \text{as} \quad N \to \infty, \; \forall i \in \{1, \ldots, m\}, \]

and also

\[ \vec{\rho}_N^0 \xrightarrow{(d)} \vec{p}_0 \in [0, 1] \quad \text{as} \quad N \to \infty. \]

Then, the sequence of density vectors $(\vec{\rho}_k, k \geq 0)$ converges for the product topology to the orbits

\[ (\vec{p}_k, k \geq 0) \]

of an explicit dynamical system depending on $\vec{\beta}$ and $\vec{\alpha}$.
Proposition (F-Linker-Remenik ’18)

There are explicit regions of the parameter space giving either domination (black/white regions) or coexistence (gray region) for the dynamical system.

Figure: Left $\alpha(1) = \alpha(2) = 0$ and right $\alpha(1) = \alpha(2) = 0.1$. 
Define $\bar{\alpha} := \min\{\alpha(1), \alpha(2)\}$.

**Theorem (F-Linker-Remenik ’18)**

For $m = 2$, assume that $\vec{\rho}_0^N \to \vec{\rho}_0$. Then, under some technical condition:

1. **In the domination regime** (of the dynamical system):
   - The weakest type dies out in time of order $\log(N)$.
   - The strongest one survives for at least order
     $$\begin{cases} e^{\sqrt{\log(N)}} & \text{if } \bar{\alpha} = 0 \\ N^{\bar{\alpha}/5} & \text{if } \bar{\alpha} > 0. \end{cases}$$

2. **In the coexistence regime** (of the dynamical system):
   - Both types survive for at least order
     $$\begin{cases} e^{\sqrt{\log(N)}} & \text{if } \bar{\alpha} = 0 \\ N^{\bar{\alpha}/5} & \text{if } \bar{\alpha} > 0. \end{cases}$$
In words:

1. We proved the results when each egg is placed uniformly in $\mathcal{N}_N(x) = B(x, r_N)$ with $r_N \to \infty$ at a certain rate.

2. We can work with $d$-regular graphs instead. **Difficulties:** the explicit dynamical system turns out to be ugly and the regimes of **domination** and **coexistence** cannot be treated at once for a generic $d$.

3. We think that for $m = 1$ the survival regime satisfies an expected absorption time of exponential order.
Thanks for your attention!