Research article

Blow-up criteria for the full compressible Navier-Stokes equations involving temperature in Vishik Spaces

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Abstract: In this paper, we consider the conditional regularity for the 3D incompressible Navier-Stokes equations in Vishik spaces. These results will be regarded an improvement of the results given by Huang-Li-Xin, (SIAM J. Math. Anal., 2011) and Jiu-Wang-Ye,(J. Evol. Equ., 2021).

Keywords: full compressible Navier-Stokes equations; strong solutions; blow-up criteria

Mathematics Subject Classification: 35B65, 35D30, 76D05

1. Introduction

We study the following system of Newton heat-conducting compressible fluid in three-dimensional space

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho, \theta) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u &= 0, \\
c_v[\rho \theta_t + \rho u \cdot \nabla \theta] + P \text{div} u - \kappa \Delta \theta &= \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2, \\
(\rho, u, \theta)|_{t=0} &= (\rho_0, u_0, \theta_0).
\end{aligned}
\] (1.1)

Here, $\rho$, $u$, $\theta$ stand for the flow density, velocity and the absolute temperature, respectively. The scalar function $P$ represents the pressure, the state equation of which is determined by

\[ P = R \rho \theta, \quad R > 0, \] (1.2)

and $\kappa$ is a positive constant and two constants $\mu$ and $\lambda$ are the coefficients of viscosity satisfying the physical restrictions $\mu > 0$, $2\mu + 3\lambda \geq 0$. The initial conditions satisfy

\[ \rho(x, t) \to 0, \quad u(x, t) \to 0, \quad \theta(x, t) \to 0, \quad \text{as} \ |x| \to \infty, \quad \text{for} \ t \geq 0. \] (1.3)
Let $\gamma > 0$. For all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, we consider the following scaled functions:

$$
\rho_\lambda = \rho(\lambda^2 t, \lambda x), \quad u_\lambda = \lambda u(\lambda^2 t, \lambda x), \quad \theta_\lambda = \lambda^2 \theta(\lambda^2 t, \lambda x).
$$

(1.4)

There are huge literatures on the study of the existence of solutions to compressible Navier-Stokes equations, we only give a brief survey for blow-up criteria rather than the existence of solutions. When the initial data contain vacuums, after Xin’s blow-up works [21, 22], the various result for blow up criteria for strong solutions to the system (1.1) is investigated. In present paper, in particular, we focus on the Serrin type criteria (e.g. [6–9]) as

$$
\lim \sup_{T \to T^*} \left( \|\text{div} \, u\|_{L^1(0,T;L^{\infty}(\mathbb{R}^3))} + \|u\|_{L^p(0,T;L^q(\mathbb{R}^3))} \right) = \infty, \quad \frac{2}{p} + \frac{3}{q} = 1, \quad q > 3,
$$

or

$$
\lim \sup_{T \to T^*} \left( \|\rho\|_{L^{\infty}(0,T;L^{q}(\mathbb{R}^3))} + \|u\|_{L^p(0,T;L^q(\mathbb{R}^3))} \right) = \infty, \quad \frac{2}{p} + \frac{3}{q} = 1, \quad q > 3
$$

and it is aimed to expand them into Vishik space motivated by the results of two recent papers Kanamaru [10] and Wu [20] (see also [2–5, 11–16, 18, 19] for other criteria containing Beale-Kato-Majda blow-up mechanism).

We remind the local well-posedness of strong solutions to the equations (1.1) (see [1]).

**Theorem 1.1.** Let $\lambda < 3\mu$. Suppose $u_0, \theta_0 \in (D^1 \cap D^2)(\mathbb{R}^3)$ and $\rho_0 \in (W^{1,q} \cap H^1 \cap L^1)(\mathbb{R}^3)$ for some $q \in (3, 6)$. If $\rho_0$ is nonnegative and the initial data satisfy the compatibility condition

$$
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} \, u_0 + \nabla P(\rho_0, \theta_0) = \sqrt{\rho_0} g_1
$$

$$
\Delta \theta_0 + \frac{\mu}{2} |\nabla u_0| + (\nabla u_0)^2 + \lambda (\text{div} \, u_0)^2 = \sqrt{\rho_0} g_2
$$

for vector fields $g_1, g_2 \in L^2(\mathbb{R}^3)$. Then there exist a time $T \in (0, \infty)$ and unique solution to the equations (1.1)–(1.3), satisfying

$$(\rho, u, \theta) \in C([0,T); (L^1 \cap H^1 \cap W^{1,q})(\mathbb{R}^3) \times C([0,T); (D^1 \cap D^2)(\mathbb{R}^3)) \times L^2([0,T); D^{2,q}(\mathbb{R}^3)),$$

$$(\rho_t, u_t, \theta_t) \in C([0,T); (L^2 \cap L^q)(\mathbb{R}^3)) \times L^2([0,T); D^1(\mathbb{R}^3)) \times L^2([0,T); D^1(\mathbb{R}^3)),$$

$$(\rho^{1/2} u_t, \rho^{1/2} \theta_t) \in L^\infty([0,T); L^2(\mathbb{R}^3) \times L^\infty([0,T); L^2(\mathbb{R}^3)).$$

If the maximal existence time $T^*$ is finite, then there holds

$$
\lim \sup_{T \to T^*} \left( \|\rho\|_{L^{\infty}(0,T;L^{\infty}(\mathbb{R}^3))} + \|\theta\|_{L^{\infty}(0,T;L^{p}(\mathbb{R}^3))} \right) = \infty, \quad q > \frac{3}{2},
$$

(1.5)

where $\sigma \in [1, \infty], \theta \in [1, \sigma]$.

**Remark 1.1.** In the light of the arguments in [7, 8], we observe that (1.5) be replaced by

$$
\lim \sup_{T \to T^*} \left( \|\text{div} \, u\|_{L^1(0,T;L^{\infty}(\mathbb{R}^3))} + \|\theta\|_{L^p(0,T;L^{q}(\mathbb{R}^3))} \right) = \infty.
$$

We note that the condition (1.5) is in scaling invariant norm in the sense of (1.4) for the temperature.
Remark 1.2. Without the restriction $\lambda < 3\mu$, in the case away from vacuum, through the argument in [9] and our proof, we obtain the similar results [9, Theorem 1.3] of what the authors in [9] says in Vishik space.

Next, we consider the full compressible Navier-Stokes equations without temperature.

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla (\text{div} u) + \nabla P(\rho) &= 0, \\
(\rho, u)(x, 0) &= (\rho_0, u_0)(x),
\end{align*}
\]

(1.6)

where $\rho$, $u$, and $P$ are the density, velocity and pressure respectively. The equation of state is given by

\[ P(\rho) = a \rho^\gamma, \quad (a > 0, \gamma > 1). \]

(1.7)

The constants $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions: $\mu > 0$, $\lambda + 2\mu \geq 0$.

Through a similar scheme in Theorem 1.1, we also obtain the following result for the equations (1.6).

**Theorem 1.2.** Let $(\rho, u)$ be a strong solution to the Cauchy problem (1.6)–(1.7) with the initial data $(\rho_0, u_0)$ satisfy

\[ 0 \leq \rho_0 \in (L^1 \cap H^1 \cap W^{1,r})(\mathbb{R}^3), \quad u_0 \in (D^1 \cap D^2)(\mathbb{R}^3), \]

for some $r \in (3, \infty)$ and the compatibility condition:

\[-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g \quad \text{for some } g \in L^2(\mathbb{R}^3).\]

If $T^* < \infty$ is the maximal time of existence, then both

\[
\lim_{T \to T^*} \left( \|\text{div} u\|_{L^1(0, T; L^\infty(\mathbb{R}^3))} + \|u\|_{L^p(0, T; V_{\alpha,1}^0(\mathbb{R}^3))}^{2p} \right) = \infty,
\]

and

\[
\lim_{T \to T^*} \left( \|\rho\|_{L^p(0, T; L^\infty(\mathbb{R}^3))} + \|u\|_{L^p(0, T; V_{\alpha,1}^0(\mathbb{R}^3))}^{2p} \right) = \infty, \quad 3 < p \leq \infty,
\]

where $\alpha \in [1, \infty], \theta \in [1, \sigma]$.

2. Notations and some auxiliary lemmas

We follow the notation of [6] and [9]. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^3)$ represents the usual Lebesgue space. The classical Sobolev space $W^{k,p}(\mathbb{R}^3)$ is equipped with the norm $\|f\|_{W^{k,p}(\mathbb{R}^3)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^3)}$. A function $f$ belongs to the homogeneous Sobolev spaces $D^{k,l}$ if $u \in L^1_{t,x}(\mathbb{R}^3) : \|\nabla^k u\|_{L^l} < \infty$. $C > 0$ is an absolute constant which may be different from line to line unless otherwise stated in this paper. We also now introduce a Banach space $\dot{V}^s_{\mu,\lambda,0}(\mathbb{R}^3)$ which is larger than the homogeneous Besov space; see [10, 17].
Definition 2.1. Let \( s \in \mathbb{R}, p, \sigma \in [1, \infty], \theta \in [1, \sigma] \), the Vishik space \( \dot{V}^s_{p,\sigma,\theta} \) is defined by

\[
\dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3) := \{ f \in \mathcal{D}'(\mathbb{R}^3) : \|f\|_{\dot{V}^s_{p,\sigma,\theta}} < \infty \},
\]

with the norm

\[
\|f\|_{\dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3)} := \left\{ \begin{array}{l}
sup_{N=1,2,\ldots} \left( \sum_{k \in \mathbb{N}^3} \|\Delta^k f\|_{L^p}^p \right)^{\frac{1}{p}}, \quad \theta \neq \infty, \\
\|f\|_{B^s_{p,\infty}(\mathbb{R}^3)}, \quad \theta = \infty.
\end{array} \right.
\]

Here \( \mathcal{D}'(\mathbb{R}^3) \) is the dual space of \( \mathcal{D}(\mathbb{R}^3) = \{ f \in \mathcal{S}(\mathbb{R}^3) ; D^s f(0) = 0, \forall \sigma \in \mathbb{N}^3 \} \). As mentioned in [20], we remind that the following continuous embeddings hold:

\[
\dot{B}^s_{p,\sigma}(\mathbb{R}^3) = \dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,\theta_1}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,\theta_2}(\mathbb{R}^3) \subset \dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3)
\]

for \( s \in \mathbb{R}, p, \sigma \in [1, \infty] \) and \( \theta_1, \theta_2 \in [1, \sigma] \) with \( \theta_1 \geq \theta_2 \).

In what follows, for simplicity, we write

\[
L^p = L^p(\mathbb{R}^3), \quad H^k = W^{k,2}(\mathbb{R}^3), \quad D^k = D^{k,2}(\mathbb{R}^3), \quad \dot{V}^s_{p,\sigma,\theta} := \dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3).
\]

3. Proof of Theorem 1.1

We will prove Theorem 1.1 by a contradiction argument. Therefore, we assume that

\[
\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^{\infty}(0,T;\dot{V}^0_{p,1}(\mathbb{R}^3))} \leq C, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad q > \frac{3}{2}, \quad (3.1)
\]

Lemma 3.1. Suppose that (3.1) is valid and \( \lambda < 3\mu \), then there holds

\[
\sup_{0 \leq s \leq T} \int_{\mathbb{R}^3} \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \frac{1}{2\mu + \lambda} p^2 - 2P \text{div } u + \frac{C}{2} \rho \theta^2 + \frac{C + 1}{2\mu} |\rho| |u|^4 \right] + \int_0^T [\kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} |\rho| |u|^2 + |u|^2 |\nabla u|^2] \, dt \leq C.
\]

Proof. From Lemma 2.3 and Lemma 3.1 in [9], we know that

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \frac{1}{2\mu + \lambda} p^2 - 2P \text{div } u + \frac{C}{2} \rho \theta^2 \right] + \kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\rho| |u|^2 \leq C \int \rho |\theta|^3 + C \int \rho |u|^2 |\theta|^2 + C \int |u|^2 |\nabla u|^2, \quad (3.2)
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + \int_{\mathbb{R}^3 \cap |u| > 0} |u|^2 |\nabla u|^2 \leq C \int \rho |u|^2 |\theta|^2. \quad (3.3)
\]

Multiplying the inequality (3.3) by \( (C + 1) \) and adding the result to the inequality (3.2), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \frac{1}{2\mu + \lambda} p^2 - 2P \text{div } u + \frac{C}{2} \rho \theta^2 + \frac{C + 1}{2\mu} |\rho| |u|^4 \right]
\]
\[ + \kappa \int |\nabla \theta|^2 + \frac{1}{2} \int \rho|\dot{u}|^2 + \int |u|^2 |\nabla u|^2 \leq C \int \rho|\theta|^3 + C \int \rho|u|^2 |\theta|^2. \] (3.4)

For the second term in the right hand side of (3.4), we note that

\[ \int \rho|u|^2 |\theta|^2 = \int \rho^{\frac{1}{2}} |\theta| \rho^{\frac{1}{2}} |u|^2 |\sum_{j \leq -N} \Delta_j \theta| + \int \rho^{\frac{1}{2}} |\theta| \rho^{\frac{1}{2}} |u|^2 |\sum_{j = N}^{j = N} \Delta_j \theta| \] (3.5)

\[ + \int \rho^{\frac{1}{2}} |\theta| \rho^{\frac{1}{2}} |u|^2 |\sum_{j > N} \Delta_j \theta| := I + II + III. \]

Now, let's control each term sequentially by Hölder's inequality, interpolation inequality (for the term II below), Sobolev embedding theorem, Berstein's inequality and Young's inequalities:

(The term (I)):

\[ I \leq \| \sum_{j \leq -N} \Delta_j \theta \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \leq C \| \sum_{j \leq -N} 2^{j/2} \| \theta \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \]

\[ \leq C 2^{-3N} (\| \rho^{\frac{1}{2}} \theta \|_{L^2}^2 + \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2}^2). \]

(The term (II)):

\[ II \leq \sum_{j = N}^{j = N} \| \Delta_j \theta \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \]

\[ \leq \sum_{j = N}^{j = N} \| \Delta_j \theta \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \]

\[ \leq C N^{-1} \sup_{N = 1, 2, \ldots} \| \Delta_j \theta \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \]

\[ \leq C \| \theta \|_{L^3}^{\frac{2}{3}} \rho^{1/2} \| \rho^{1/2} |u|^2 \|_{L^2} + \frac{1}{16} \| \nabla \theta \|_{L^2}^2 + \frac{1}{16} \| \rho^{1/2} |u|^2 \|_{L^2} \]

\[ \leq C \| \theta \|_{L^3}^{\frac{2}{3}} (\| \rho^{1/2} \theta \|_{L^2}^2 + \| \rho^{1/2} |u|^2 \|_{L^2}^2) + \frac{1}{16} \| \nabla \theta \|_{L^2}^2 + \frac{1}{16} \| \nabla |u|^2 \|_{L^2}^2. \]

(The term (III)):

\[ III \leq \sum_{j \geq N} \| \Delta_j \theta \|_{L^2} \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \| \rho^{\frac{1}{2}} \theta \|_{L^2} \leq C \| \rho^{\frac{1}{2}} |u|^2 \|_{L^2} \sum_{j \geq N} 2^{j/2} \| \theta \|_{L^2} \| \rho^{1/2} \theta \|_{L^2} \]

\[ \leq C 2^{-N} \| \theta \|_{L^2} \| \nabla |u|^2 \|_{L^2} \| \rho^{1/2} \theta \|_{L^2} \leq 2^{-N} \| \theta \|_{L^2}^2 \| \rho^{1/2} \theta \|_{L^2}^2 + \frac{1}{32} \| \rho^{1/2} \|_{L^2}^2. \]

Summing up the estimates above, we have

\[ \int \rho|u|^2 |\theta|^2 \leq C 2^{-3N} \| \theta \|_{L^2}^2 (\| \rho^{1/2} \theta \|_{L^2}^2 + \| \rho^{1/2} |u|^2 \|_{L^2}^2) + C \| \theta \|_{L^3}^{\frac{2}{3}} (\| \rho^{1/2} \theta \|_{L^2}^2 + \| \rho^{1/2} |u|^2 \|_{L^2}^2) \]

\[ + \frac{1}{16} \| \nabla \theta \|_{L^2}^2 + \frac{1}{16} \| \nabla |u|^2 \|_{L^2}^2 + 2^{-N} \| \theta \|_{L^2}^2 \| \rho^{1/2} \theta \|_{L^2}^2. \] (3.7)
By similar above arguments, we get
\[
\int \rho |\theta|^3 = \int \rho^2 |\theta|_{L^2}^2 \leq C 2^{-3N^2} \|\theta\|_{L^2}^2 |\rho|^2_{L^2} + C \|\theta\|_{L^2}^{2\alpha} \|\rho\|_{L^2}^{\beta} + \frac{1}{16} \|\nabla \theta\|_{L^2}^2 + \frac{1}{16} \|\nabla u\|_{L^2}^2 + 2^{-N^2} |\theta|^2_{L^2} |\rho|^2_{L^2}.
\] (3.8)

Substituting (3.7) and (3.8) into (3.4), we obtain
\[
\begin{align*}
\frac{d}{dt} \int \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \text{div} u + C \frac{1}{2\mu} |\rho|^4 \right] \\
&+ \frac{k}{2} \int |\nabla \theta|^2 + \frac{1}{2} \int \rho |\theta|^2 + \frac{1}{2} \int_{\mathbb{R}^3 \cap \{u > 0\}} |u|^2 |\nabla u|^2 \\
&\leq C 2^{-3N^2} \|\theta\|_{L^2}^2 \left( |\rho|^2_{L^2} + |\rho|^2_{L^2} \right) + C \|\theta\|_{L^2}^{2\alpha} \left( |\rho|^2_{L^2} + |\rho|^2_{L^2} \right) \\
&+ \frac{1}{16} \|\nabla \theta\|_{L^2}^2 + \frac{1}{16} \|\nabla u\|_{L^2}^2 + 2^{-N^2} |\theta|^2_{L^2} |\rho|^2_{L^2} \\
&\leq C \left( C 2^{-3N^2} + 2^{-N^2} |\theta|^2_{L^2} + |\theta|^{2\alpha}_{L^2} \right) \int \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \text{div} u \\
&+ C \frac{1}{2\mu} |\rho|^4 \right] + \frac{1}{16} \|\nabla \theta\|_{L^2}^2 + \frac{1}{16} \|\nabla u\|_{L^2}^2,
\end{align*}
\] (3.9)

where we used the fact that
\[
\int \left[ (\mu + \lambda)(\text{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \text{div} u + C \frac{1}{2\mu} |\rho|^4 \right] \geq \int \rho \theta^2,
\]
for a sufficiently large constant $C > 0$. Now, choosing $N > 0$ sufficiently large such that $C 2^{-N^2} \|\theta\|_{L^2}^2 \leq \frac{1}{128}$, the estimate (3.9) becomes
\[
\frac{d}{dt} \int \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \text{div} u + C \frac{1}{2\mu} |\rho|^4 \right] \\
+ \frac{k}{2} \int |\nabla \theta|^2 + \frac{1}{2} \int \rho |\theta|^2 + \frac{1}{2} \int_{\mathbb{R}^3 \cap \{u > 0\}} |u|^2 |\nabla u|^2 \\
\leq C \int \rho \theta^3 + C \int \rho |u|^2 |\theta|^2 \leq CN \|u\|^{2\alpha}_{L^2} \|\nabla u\|_{L^2}^2.
\] (3.10)

Then, Grönwall’s inequality and (3.10) enables us to obtain the desired results.

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \left[ \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2 + \frac{1}{2\mu + \lambda} P^2 - 2P \text{div} u + C \frac{1}{2\mu} |\rho|^4 \right] \\
+ \int_0^T \left[ k \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \rho |\theta|^2 + |u|^2 |\nabla u|^2 \right] dt \leq C.
\]

\( \square \)

**Proof of Theorem 1.1.** In the proof in Theorem 1.1 in [9], as long as Lemma 3.2 in [9] is only replaced by Lemma 3.1 in present paper, the proof is completed.

\( \square \)
4. Proof of Theorem 1.2

Let \((\rho, u)\) be a strong solution to the problem (1.6)-(1.7) as described in Theorem 1.2. Then the standard energy estimate yields

\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2} u(t)\|_{L^2}^2 + \|\rho\|_{L^1} + \|\rho\|_{L^2}^2 \right) + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C, \quad 0 \leq T < T^*.
\] (4.1)

We first prove Theorem 1.2 by a contradiction argument. Otherwise, there exists some constant \(M_0 > 0\) such that

\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^2\cap L^p(\mathbb{R}^3)} \right) \leq M_0.
\] (4.2)

The first key estimate on \(\nabla u\) will be given in the following lemma.

**Lemma 4.1.** Under the condition (4.2), it holds that

\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho u_t^2 dx dt \leq C, \quad 0 \leq T < T^*.
\] (4.3)

**Proof.** It follows from the momentum equations in (1.6) that

\[
\triangle G = \text{div}(\rho \dot{u}), \quad \mu \triangle \omega = \nabla \times (\rho \dot{u}),
\]

where \(\dot{v} := v_t + u \cdot \nabla v, \quad G := (2\mu + \lambda)\text{div}u - P(\rho), \quad \omega := \nabla \times u\) are the material derivative of \(f\), the effective viscous flux \(G\) and the vorticity \(\omega\), respectively. In particular, for the effective viscous flux, it is well-known that

\[
\|\nabla G\|_{L^p} \leq \|\rho \dot{u}\|_{L^p}, \quad \forall p \in (1, +\infty),
\]

and

\[
\|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C(\|\rho u\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2}).
\] (4.4)

Multiplying the momentum equation (1.6) by \(u\) and integrating the resulting equation over \(\mathbb{R}^3\) gives

\[
\frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u|^2 + (\lambda + \mu) (\text{div}u)^2) dx + \int \rho u_t^2 dx = \int P \text{div}u dx - \int \rho u \cdot \nabla u \cdot u dx.
\] (4.5)

From (1.6), we note that

\[
P_t + \text{div}(P u) + (y - 1) P \text{div}u = 0.
\]

For the first term in the right hand side of (4.5), one has

\[
\int P \text{div}u dx \leq \frac{d}{dt} \int P \text{div}u dx + \frac{1}{8} \|\nabla G\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C.
\] (4.6)

Substituting (4.6) into (4.5), we have

\[
\frac{d}{dt} \int \left( \frac{\mu}{2} |\nabla u|^2 + \frac{\lambda + \mu}{2} (\text{div}u)^2 - P \text{div}u \right) dx + \frac{1}{2} \int \rho u_t^2 dx \leq C \|\nabla u\|_{L^2}^2 + \int \rho |u \cdot \nabla u \cdot u| dx + C.
\]
For the second term in the right hand side of (4.5), we have

$$\int |\rho^{1/2} u \cdot \nabla u - \rho^{1/2} u| dx \leq \int |\rho^{1/2} \sum_{j=-N}^{j=N} \Delta_j u \nabla u| dx$$

$$+ \int |\rho^{1/2} \sum_{j=-N}^{j=N} \Delta_j u \nabla u| dx + \int |\rho^{1/2} \sum_{j=-N}^{j=N} \Delta_j u \nabla u| dx : = I + II + III.$$  

In a similar way to (3.5), we let control each term sequentially.

(The term (I)):

$$I \leq C 2^{-3N^2} ||u||_{L^2}^2 ||\nabla u||_{L^2}^2 + \frac{1}{32} ||\rho^{1/2} u||_{L^2}^2.$$  

(The term (II)):

$$II \leq C N ||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^2 + \frac{1}{32} (||\rho^{1/2} u||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2).$$  

(The term (III)):

$$III \leq C 2^{-3N^2} ||u||_{L^2}^2 ||\nabla u||_{L^2}^2 + C 2^{-N^2} ||u||_{L^2}^2 ||\nabla^2 u||_{L^2}^3.$$  

Summing up the estimates $I-III$, it is bounded by

$$C 2^{-3N^2} ||u||_{L^2}^2 ||\nabla u||_{L^2}^2 + C N ||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^2 + C 2^{-N^2} ||u||_{L^2}^2 ||\nabla^2 u||_{L^2}^2 + C N ||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^2$$

$$\leq \frac{1}{16} (||\rho^{1/2} u||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2).$$  

On the other hand, due to (4.4), we note that

$$||\nabla^2 u||_{L^2}^2 \leq C (||\sqrt{\rho} u||_{L^2}^2 + ||\rho u \cdot \nabla u||_{L^2}^2)$$

$$\leq C ||\sqrt{\rho} u||_{L^2}^2 + C 2^{-3N^2} ||u||_{L^2}^2 ||\nabla u||_{L^2}^2 + C 2^{-N^2} ||u||_{L^2}^2 ||\nabla^2 u||_{L^2}^2 + C N ||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^2.$$  

Collecting (4.7) and (4.8), we have

$$\frac{d}{dt} \int_\Omega \left( \frac{\mu}{2} ||\nabla u||^2 + \frac{\lambda}{2}(\text{div} u)^2 - P\text{div} u \right) dx + \frac{1}{4} \int \rho u_t^2 dx + \int ||\nabla G||^2 dx$$

$$\leq C ||\sqrt{\rho} u||_{L^2}^2 + C 2^{-3N^2} ||u||_{L^2}^2 ||\nabla u||_{L^2}^2 + C 2^{-N^2} ||u||_{L^2}^2 ||\nabla^2 u||_{L^2}^2 + C N ||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^2.$$  

Now, choosing $N > 0$ sufficiently large such that $C 2^{-N^2} ||u||_{L^2}^2 \leq \frac{1}{128}$, (indeed, the constant $C > 0$ is also depending on $||\rho^{1/2} u||_{L^2}^2$) the estimate (4.9) becomes

$$\frac{d}{dt} \int_\Omega \left( \frac{\mu}{2} ||\nabla u||^2 + \frac{\lambda}{2}(\text{div} u)^2 - P\text{div} u + \rho ||u||^2 + \rho + \rho^2 \right) dx$$

$$+ \int_\Omega \left( ||\nabla G||^2 + ||\nabla u||^2 + \frac{1}{4} \rho ||u||^2 \right) dx \leq C N (||u||_{L^2}^{2\alpha} + 1)(||\nabla u||_{L^2}^2 + 1),$$

which, together with (4.2) and Grönwall’s inequality, gives (4.3). The proof of Lemma 4.1 is completed.

Proof of Theorem 1.2. In the proof in Theorem 1.1 in [6], as long as Lemma 3.1 in [6] is only replaced by Lemma 4.1 in our paper, the proof is completed.  

\[\square\]
5. Appendix

For the convenience of the reader, we give the proof for (4.6), given in [6].

\[
\int P \text{div} u dx = \frac{d}{dt} \int P \text{div} u dx - \int P(\text{div} u)^2 dx \\
= \frac{d}{dt} \int P \text{div} u dx + \int \text{div}(Pu) \text{div} u dx + (\gamma - 1) \int P(\text{div} u)^2 dx \\
= \frac{d}{dt} \int P \text{div} u dx - \int (Pu) \cdot \nabla \text{div} u dx + (\gamma - 1) \int P(\text{div} u)^2 dx \\
= \frac{d}{dt} \int P \text{div} u dx - \frac{1}{\mu + \lambda} \int Pu \cdot \nabla G dx - \frac{1}{2(\mu + \lambda)} \int Pu \cdot \nabla G dx \\
+ (\gamma - 1) \int P(\text{div} u)^2 dx \\
\leq \frac{d}{dt} \int P \text{div} u dx + \frac{1}{8} \|\nabla G\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C,
\]

(5.1)

6. Discussion

Our result is focused on the full compressible Navier-Stokes equations. However, it is believed that our results can be expanded in various ways for the coupled equations or system. In this regard, we think of it as a future study and intend to produce more meaningful results.

7. Conclusions

The current paper results are Blow-up criteria for solutions in Vishik Space which is a weaker space to Besov space and Lebesgue space. It seems to be a meaningful result in this regard, and it is new.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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