Limit shape of random convex polygonal lines on $\mathbb{Z}^2$:
Even more universality

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Abstract

The paper is concerned with the limit shape (under some probability measure) of convex polygonal lines on $\mathbb{Z}^2_+$ starting at the origin and with the right endpoint $n = (n_1, n_2) \to \infty$. In the case of the uniform measure, the explicit limit shape $\gamma^*$ was found independently by Vershik, Bárány and Sinai. Bogachev and Zarbaliev recently showed that the limit shape $\gamma^*$ is universal in a certain class of measures analogous to multisets in the theory of decomposable combinatorial structures. In the present work, we extend the universality result to a much wider class of measures, including (but not limited to) analogues of multisets, selections and assemblies. This result is in sharp contrast with the one-dimensional case, where the limit shape of Young diagrams associated with integer partitions heavily depends on the distributional type.

Key words and phrases: Convex lattice polygonal line; Limit shape; Local limit theorem

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1. Introduction

A convex lattice polygonal line $\Gamma$ is a piecewise linear path on the plane, starting at the origin $0 = (0, 0)$, with vertices on the integer lattice $\mathbb{Z}^2_+ := \{(i, j) \in \mathbb{Z}^2 : i, j \geq 0\}$, and such that the inclination of its consecutive edges strictly increases staying between 0 and $\pi/2$. Let $\Pi$ be the set of all convex lattice polygonal lines with finitely many edges, and denote by $\Pi_n \subset \Pi$ the subset of polygonal lines $\Gamma \in \Pi$ whose right endpoint $\xi = \xi_{\Gamma}$ is fixed at $n = (n_1, n_2) \in \mathbb{Z}^2_+$. The limit shape, with respect to a probability measure $P_n$ on $\Pi_n$ as $n \to \infty$, is understood as a planar curve $\gamma^*$ such that, for any $\varepsilon > 0$,

$$
\lim_{n \to \infty} P_n \{ \Gamma \in \Pi_n : d(\hat{\Gamma}_n, \gamma^*) \leq \varepsilon \} = 1,
$$

where $\hat{\Gamma}_n = S_n(\Gamma)$, with a suitable scaling $S_n: \mathbb{R}^2 \to \mathbb{R}^2$, and $d(\cdot, \cdot)$ is some metric on the path space, e.g., induced by the Hausdorff distance between compact sets,

$$
d_H(A, B) := \max \left\{ \max_{x \in A} \min_{y \in B} |x - y|, \max_{y \in B} \min_{x \in A} |x - y| \right\}.
$$

Remark 1.1. By definition, for a polygonal line $\Gamma \in \Pi_n$ the vector sum of its consecutive edges equals $n = (n_1, n_2)$; due to the convexity property, the order of parts in the sum is uniquely determined. Hence, any such $\Gamma$ represents a strict vector partition of $n \in \mathbb{Z}^2_+$ (i.e., without proportional parts; see [19]). For ordinary one-dimensional partitions, the limit shape problem is set out for the associated Young diagrams [20, 22].
Of course, the limit shape and its very existence may depend on the probability law $P_n$. With respect to the uniform distribution on $\Pi_n$, the problem was solved independently by Vershik [19], Bárány [3] and Sinai [16], who showed that, under the scaling $S_n : (x_1, x_2) \mapsto (x_1/n_1, x_2/n_2)$ and with respect to the Hausdorff metric $d_H$, limit (1.1) holds with $\gamma^*$ given by a parabola arc defined by the equation

$$\sqrt{1 - x_1} + \sqrt{x_2} = 1, \quad 0 \leq x_1, x_2 \leq 1.$$  

(1.3)

Recently, Bogachev and Zarbaliev [6, 7] proved that the same limit shape $\gamma^*$ appears for a large class of measures $P_n$ of the form

$$P_n(\Gamma) := \frac{b(\Gamma)}{B_n}, \quad \Gamma \in \Pi_n,$$

(1.4)

with

$$b(\Gamma) := \prod_{e_i \in \Gamma} b_{\ell_i}, \quad B_n := \sum_{\Gamma \in \Pi_n} b(\Gamma),$$

(1.5)

where the product is taken over all edges $e_i$ of $\Gamma \in \Pi_n$, $\ell_i$ is the number of lattice points on the edge $e_i$ except its left endpoint, and

$$b_{\ell_i} := \frac{r(r+1) \cdots (r+\ell-1)}{\ell!}, \quad \ell = 0, 1, 2, \ldots.$$  

(1.6)

This result has provided first evidence in support of a conjecture on the limit shape universality, put forward independently by Vershik [19, p. 20] and Prokhorov [15].

The goal of the present paper is to show that the limit shape $\gamma^*$ given by (1.3) is universal in a much wider class of probability measures of the form (1.4). For instance, along with the uniform measure on $\Pi_n$ this class contains the uniform measure on the subset $\tilde{\Pi}_n \subset \Pi_n$ of polygonal lines that do not have any integer points other than vertices. More generally, measures covered by our method include (but are not limited to) direct analogues of the three classical meta-types of decomposable combinatorial structures — multisets, selections and assemblies [1, 2, 10] (see examples in Section 2.3 below). Let us stress, however, that our universality result is in sharp contrast with the one-dimensional case, where the limit shape of Young diagrams associated with integer partitions heavily depends on the distributional type (see [4, 9, 20, 22]). This suggests that the limit shape of strict vector partitions is a relatively “soft” property as compared to a more demanding case of (one-dimensional) integer partitions.

Let us state our result more precisely. Using the tangential parameterization of convex paths (see [6, §A.1]), let $\tilde{\xi}_n(t)$ denote the right endpoint of part of the scaled polygonal line $\tilde{\Gamma}_n = S_n(\Gamma)$ where the tangent slope (wherever it exists) does not exceed $t \in [0, \infty]$. Similarly, a tangential parameterization of the parabola arc $\gamma^*$ (see (1.3)) is given by

$$g^*(t) = \left( \frac{t^2 + 2t}{(1+t)^2}, \frac{t^2}{(1+t)^2} \right), \quad 0 \leq t \leq \infty.$$  

(1.7)

The tangential distance between $\tilde{\Gamma}_n$ and $\gamma^*$ is defined as

$$d_T(\tilde{\Gamma}_n, \gamma^*) := \sup_{0 \leq t \leq \infty} |\tilde{\xi}_n(t) - g^*(t)|.$$  

(1.8)

It is known [6, §A.1] that the Hausdorff distance $d_H$ (see (1.2)) is dominated by the tangential distance $d_T$.

Our main result is as follows.
Theorem 1.1. Suppose that $0 < c_1 \leq n_2/n_1 \leq c_2 < \infty$, and assume that the coefficients $b_k$ in (1.5) satisfy some mild technical conditions expressed in terms of the power series expansion of the function $y(s) = \ln \left( \sum_k b_k s^k \right)$ (see more details in Section 2.1). Then for any $\varepsilon > 0$

$$\lim_{n \to \infty} P_n \{ \Gamma \in \Pi_n : d_T(\hat{\Gamma}_n, \gamma^*) \leq \varepsilon \} = 1.$$ 

Remark 1.2. Universality of the limit shape $\gamma^*$ has its boundaries: as was shown by Bogachev and Zarbaliev [5, 8], any $C^0$-smooth, strictly convex curve $\gamma$ started at the origin may appear as the limit shape with respect to a suitable probability measure $P_n^\gamma$ on $\Pi_n$, as $n \to \infty$.

Like in [6], our proof employs the elegant probabilistic approach based on randomization and conditioning (see [1, 2]) first used in the polygonal context by Sinai [16]. The idea is to introduce a suitable product measure $Q_\gamma$ on the space $\Pi = \cup_n \Pi_n$ (depending on an auxiliary “free” parameter $z = (z_1, z_2)$), such that the measure $P_n$ on $\Pi_n$ is recovered as the conditional distribution $P_n(\cdot) = Q_\gamma(\cdot \mid \Pi_n)$. Clearly, this device calls for the asymptotics of the probability $Q_\gamma(\Pi_n)$, which is supplied by proving a suitable local limit theorem. Let us also point out that the parameter $z$ is calibrated from the asymptotic equation $E_z(\xi_{\Gamma}) = n \left( 1 + o(1) \right)$, where $\xi_{\Gamma}$ is the right endpoint of the polygonal line $\Gamma \in \Pi$ (so that, e.g., $\Pi_n = \{ \Gamma \in \Pi : \xi_{\Gamma} = n \}$). The main novelty that has allowed us to extend and enhance the argumentation of [6] in a much more general setting considered here is that we choose to work with cumulants rather than moments (see Section 2.1), which proves extremely efficient throughout.

Layout. The rest of the paper is organized as follows. In Section 2 we define the families of measures $Q_\gamma$ and $P_n$. In Section 3 suitable values of the parameter $z = (z_1, z_2)$ are chosen (Theorem 3.2), which implies convergence of “expected” polygonal lines to the limit curve $\gamma^*$ (Theorems 3.3 and 3.4). Refined first-order moment asymptotics are obtained in Section 4 (Theorem 4.4), while higher-order moment sums are analyzed in Section 5. Section 6 is devoted to the proof of the local central limit theorem (Theorem 6.1). Finally, the limit shape result, with respect to both $Q_\gamma$ and $P_n$, is proved in Section 7 (Theorems 7.1 and 7.2).

Some general notations. For a row-vector $x = (x_1, x_2) \in \mathbb{R}^2$, its Euclidean norm (length) is denoted by $|x| := (x_1^2 + x_2^2)^{1/2}$, and $(x, y) := x y^\top = x_1 y_1 + x_2 y_2$ is the corresponding inner product of vectors $x, y \in \mathbb{R}^2$. We denote $\mathbb{Z}_+: = \{ k \in \mathbb{Z} : k \geq 0 \}$, $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and similarly $\mathbb{R}_+: = \{ x \in \mathbb{R} : x \geq 0 \}$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$.

2. Probability measures on spaces of convex polygonal lines

2.1. Global measure $Q_\gamma$ and conditional measure $P_n$

Consider the set

$$\mathcal{X} := \{ x = (x_1, x_2) \in \mathbb{Z}_+^2 : \gcd(x_1, x_2) = 1 \},$$

(2.1)

where “gcd” stands for “greatest common divisor”. Let $\Phi := (\mathbb{Z}_+)^\mathcal{X}$ be the space of functions on $\mathcal{X}$ with nonnegative integer values, and consider the subspace of functions with finite support, $\Phi_0 := \{ \nu \in \Phi : \#(\text{supp } \nu) < \infty \}$, where $\text{supp } \nu := \{ x \in \mathcal{X} : \nu(x) > 0 \}$. It is easy to see that the space $\Phi_0$ is in one-to-one correspondence with the space $\Pi = \bigcup_{n \in \mathbb{Z}_+} \Pi_n$ of all (finite) convex lattice polygonal lines, whereby each $x \in \mathcal{X}$ determines the direction of a potential edge, only utilized if $x \in \text{supp } \nu$, in which case the value $\nu(x) > 0$ specifies the
scaling factor, altogether yielding a vector edge \( x \nu(x) \); finally, assembling all such edges into a polygonal line is uniquely determined by the fixation of the starting point (at the origin) and the convexity property.

Let \( b_0, b_1, b_2, \ldots \) be a sequence of non-negative numbers such that \( b_0 > 0 \) (without loss of generality, we put \( b_0 = 1 \)) and not all \( b_\ell \)'s vanish for \( \ell \geq 1 \), and assume that the function

\[
\beta(s) := 1 + \sum_{\ell=1}^{\infty} b_\ell s^\ell
\]  
(2.2)

is finite for \( |s| < 1 \). Let \( z = (z_1, z_2) \in (0, 1) \times (0, 1) \). Throughout the paper, we shall use the multi-index notation

\[
z^x := z_1^{x_1} z_2^{x_2}, \quad x = (x_1, x_2) \in \mathbb{Z}_+^2.
\]

Let us now define a probability measure \( Q_z \) on the space \( \Phi = \mathbb{Z}_+^X \) as the distribution of a random field \( \nu = \{\nu(x)\}_{x \in X} \) with mutually independent values and marginal distributions

\[
Q_z\{\nu(x) = \ell\} = \frac{b_\ell z^{\ell x}}{\beta(z^x)}, \quad \ell = 0, 1, 2, \ldots \quad (x \in X).
\]  
(2.3)

**Lemma 2.1.** For each \( z \in (0, 1)^2 \), the condition

\[
\tilde{\beta}(z) := \prod_{x \in X} \beta(z^x) < \infty
\]  
(2.4)

is necessary and sufficient in order that \( Q_z(\Phi_0) = 1 \). Furthermore, if \( \beta(s) \) is finite for all \( |s| < 1 \) then condition (2.4) is satisfied.

**Proof.** According to (2.3), \( Q_z\{\nu(x) > 0\} = 1 - \beta(z^x)^{-1} \) (\( x \in X \)). Hence, Borel–Cantelli’s lemma implies that \( Q_z\{\nu \in \Phi_0\} = 1 \) if and only if \( \sum_{x \in X} (1 - \beta(z^x)^{-1}) < \infty \). In turn, the latter inequality is equivalent to (2.4).

To prove the second statement, observe using (2.2) that

\[
\ln \tilde{\beta}(z) = \sum_{x \in X} \ln \beta(z^x) \leq \sum_{x \in X} (\beta(z^x) - 1) = \sum_{\ell=1}^{\infty} b_\ell \sum_{x \in X} z^{\ell x}.
\]  
(2.5)

Furthermore, for any \( \ell \geq 1 \)

\[
\sum_{x \in X} z^{\ell x} \leq \sum_{x_1=1}^{\infty} z_1^{\ell x_1} + \sum_{x_1=0}^{\infty} z_1^{\ell x_1} \sum_{x_2=1}^{\infty} z_2^{\ell x_2} = \frac{z_1^\ell}{1 - z_1} + \frac{z_2^\ell}{1 - z_2} \leq \frac{z_1^\ell}{1 - z_1} + \frac{z_2^\ell}{(1 - z_1)(1 - z_2)}.
\]

Substituting this into (2.5) and recalling (2.2), we obtain

\[
\ln \tilde{\beta}(z^x) \leq \frac{\beta(z_1)}{1 - z_1} + \frac{\beta(z_2)}{(1 - z_1)(1 - z_2)} < \infty,
\]

which implies (2.4).
Lemma 2.1 ensures that a sample configuration of the random field $\nu(\cdot)$ belongs ($Q_z$-a.s.) to the space $\Phi_0$ and therefore determines a (random) finite polygonal line $\Gamma \in \Pi$. By the mutual independence of the values $\nu(x)$, the corresponding $Q_z$-probability is given by

$$Q_z(\Gamma) = \prod_{x \in X} \frac{b_{\nu(x)} z^{x_\nu(x)}}{\beta(z)} = \frac{b(\Gamma) z^\xi}{\beta(z)}, \quad \Gamma \in \Pi; \quad (2.6)$$

where $\xi = \sum_{x \in X} x \nu(x)$ is the right endpoint of $\Gamma$, and

$$b(\Gamma) := \prod_{x \in X} b_{\nu(x)} < \infty, \quad \Gamma \in \Pi. \quad (2.7)$$

Remark 2.1. The infinite product in (2.7) contains only finitely many terms different from 1 (since $b_{\nu(x)} = b_0 = 1$ for $x \notin \text{supp} \nu$); hence, (2.7) can be rewritten in an intrinsic form (1.5).

In particular, for the trivial polygonal line $\Gamma_0 \leftrightarrow \nu \equiv 0$ formula (2.6) yields

$$Q_z(\Gamma_0) = \tilde{\beta}(z)^{-1} > 0.$$ 

Note, however, that $Q_z(\Gamma_0) < 1$, since $\beta(s) > \beta(0) = 1$ for $s > 0$ and hence, according to definition (2.4), $\tilde{\beta}(z) > 1$.

On the subspace $\Pi_n \subset \Pi$ of polygonal lines with the right endpoint fixed at $n = (n_1, n_2)$, the measure $Q_z$ induces the conditional distribution

$$P_n(\Gamma) := Q_z(\Gamma | \Pi_n) = \frac{Q_z(\Gamma)}{Q_z(\Pi_n)}, \quad \Gamma \in \Pi_n. \quad (2.8)$$

Formula (2.8) is well defined as long as $Q_z(\Pi_n) > 0$, that is, there is at least one polygonal line $\Gamma \in \Pi_n$ with $b(\Gamma) > 0$ (see (2.6) and (2.7)). A simple sufficient condition is as follows.

Lemma 2.2. Suppose that $b_1 > 0$. Then $Q_z(\Pi_n) > 0$ for all $n \in \mathbb{Z}_+^2$ such that $n_1, n_2 > 0$.

Proof. Observe that $n = (n_1, n_2) \in \mathbb{Z}_+^2$ (with $n_1, n_2 \geq 1$) can be represented as

$$(n_1, n_2) = (n_1 - 1, 1) + (1, n_2 - 1), \quad (2.9)$$

where both points $x^{(1)} = (n_1 - 1, 1)$ and $x^{(2)} = (1, n_2 - 1)$ belong to the set $X$. Moreover, $x^{(1)} \neq x^{(2)}$ unless $n_1 = n_2 = 2$, in which case instead of (2.9) we can write $(2, 2) = (1, 0) + (1, 2)$, where again $x^{(1)} = (1, 0) \in X$, $x^{(2)} = (1, 2) \in X$. If $\Gamma^* \in \Pi_n$ is a polygonal line with two edges determined by the values $\nu(x^{(1)}) = 1$, $\nu(x^{(2)}) = 1$ (and $\nu(x) = 0$ otherwise), then, according to definition (2.6), $Q_z(\Pi_n) \geq Q_z(\Gamma^*) = b_1^2 z^{n_1} \tilde{\beta}(z)^{-1} > 0$. □

The parameter $z$ may be dropped in notation (2.8) due to the following key fact.

Lemma 2.3. The measure $P_n$ in (2.8) does not depend on $z$.

Proof. If $\Pi_n \ni \Gamma \leftrightarrow \nu \in \Phi_0$ then $\xi = n$ and hence formula (2.6) is reduced to

$$Q_z(\Gamma) = \frac{b(\Gamma) z^n}{\beta(z)}, \quad \Gamma \in \Pi_n.$$ 

Accordingly, using (2.4) and (2.8) we get the expression

$$P_n(\Gamma) = \frac{b(\Gamma)}{\sum_{\Gamma \in \Pi_n} b(\Gamma')}, \quad \Gamma \in \Pi_n, \quad (2.10)$$

which is $z$-free. □
2.2. A class of measures \(Q_z\)

Recalling expansion (2.2) for the generating function \(\beta(s)\) (with \(\beta(0) = b_0 = 1\)), consider the corresponding expansion of its logarithm,

\[
\ln \beta(s) = \sum_{k=1}^{\infty} a_k s^k, \quad |s| < 1.
\] (2.11)

**Remark 2.2.** Substituting expansion (2.2) into (2.11) it is clear that \(a_1 = b_1\); more generally, if \(\ell^* := \min\{\ell \geq 1 : b_\ell > 0\}\) and \(k^* := \min\{k \geq 1 : a_k \neq 0\}\) then \(\ell^* = k^*\) and \(b_{\ell^*} = a_{k^*}\).

Under the measure \(Q_z\) defined in (2.3), the probability generating function \(\phi_{\nu}(s; x) := E_z[s^{\nu(x)}]\) of \(\nu(x)\) is given by the ratio

\[
\phi_{\nu}(s; x) = \frac{\beta(sz)}{\beta(z)}, \quad |s| \leq 1
\] (2.12)

(2.13)

Likewise, the characteristic function \(\varphi_{\nu}(t; x) := E_z[e^{it\nu(x)}]\) is given by

\[
\varphi_{\nu}(t; x) = \frac{\beta(e^{it}z)}{\beta(z)}, \quad t \in \mathbb{R}.
\] (2.14)

and the principal branch of its logarithm (corresponding to \(\ln \varphi_{\nu}(0; x) = 0\)) is represented as

\[
\ln \varphi_{\nu}(t; x) = \sum_{k=1}^{\infty} a_k(e^{ikt} - 1) z^{kx}, \quad t \in \mathbb{R}.
\] (2.15)

For \(q \in \mathbb{N}\), denote by \(m_q = m_q(x) := E_z[\nu(x)^q]\) the moments of \(\nu(x)\), and let \(\kappa_q = \kappa_q(x)\) be the cumulants of \(\nu(x)\), with the exponential generating function

\[
\ln \phi_{\nu}(e^t; x) = \sum_{q=1}^{\infty} \kappa_q(x) \frac{t^q}{q!}.
\] (2.16)

Substituting (2.13) into (2.16) and Taylor expanding the exponential function, we get

\[
\ln \phi_{\nu}(e^t; x) = \sum_{k=1}^{\infty} a_k(e^{kt} - 1) z^{kx} = \sum_{q=1}^{\infty} \frac{t^q}{q!} \sum_{k=1}^{\infty} k^q a_k z^{kx},
\]

and by a comparison with (2.16) it follows that

\[
\kappa_q(x) = \sum_{k=1}^{\infty} k^q a_k z^{kx}, \quad q \in \mathbb{N}.
\] (2.17)
In particular, from (2.17) we obtain the mean and variance of \( \nu(x) \),

\[
E_z[\nu(x)] = m_1(x) = \kappa_1(x) = \sum_{k=1}^{\infty} k a_k x^k,
\]

\[
\text{Var}[\nu(x)] = m_2(x) - m_1(x)^2 = \kappa_2(x) = \sum_{k=1}^{\infty} k^2 a_k x^k.
\]

More generally, using a well-known recursion between the cumulants and moments (see, e.g., [14, §3.14])

\[
m_q = \kappa_q + \sum_{i=1}^{q-1} \left( q - i \right) \kappa_i m_{q-i}
\]

it is easy to see by a simple induction that the moments \( m_q \) (\( q \in \mathbb{N} \)) are expressed as linear combinations of the cumulants \( \kappa_1, \ldots, \kappa_q \) with positive (in fact, integer) coefficients, which gives, in view of (2.17),

\[
m_q(x) = \kappa_q(x) + \sum_{i=1}^{q-1} C_{i,q} \kappa_i(x) + \sum_{i=1}^{q-1} C_{i,q} \sum_{k=1}^{\infty} k^i a_k x^k,
\]

with \( C_{i,q} > 0 \) (\( i = 1, \ldots, q-1 \)). Furthermore, using a rescaling relation \( \kappa_q[cX] = c^q \kappa_q[X] \) and the additive property of cumulants for independent summands, we obtain the cumulants of the random variables \( \xi_j = \sum_{x \in X} x_x \nu(x) \) (\( j = 1, 2 \)),

\[
\kappa_q[\xi_j] = \sum_{x \in X} x_j^q \kappa_q(x) = \sum_{x \in X} x_j^q \sum_{k=1}^{\infty} k^q a_k x^k,
\]

and, similarly to (2.20), the corresponding moments

\[
E_z(\xi_j^q) = \kappa_q[\xi_j] + \sum_{i=1}^{q-1} C_{i,q} \sum_{x \in X} x_j^i \sum_{k=1}^{\infty} k^i a_k x^k.
\]

For \( s \in \mathbb{C} \) such that \( \sigma := \Re s > 0 \), denote

\[
A(s) := \sum_{k=1}^{\infty} \frac{a_k}{k^s}, \quad A^+(\sigma) := \sum_{k=1}^{\infty} \frac{|a_k|}{k^{\sigma}} \leq \infty.
\]

Most of our results are valid under the condition \( A^+(2) < \infty \), or sometimes \( A^+(1) < \infty \) (in particular, in Theorem 4.1). However, for a local limit theorem (see Theorem 6.1) we require an additional technical condition on the generating function \( \beta(s) \).

**Assumption 2.1.** The coefficients \( (a_k) \) in expansion (2.22) of \( \ln \beta(s) \) are such that \( a_1 > 0 \) and, for any \( \theta \in (0, 1) \) and all \( t \in \mathbb{R} \), the following inequality holds, with some constant \( C_1 > 0 \),

\[
\sum_{k=1}^{\infty} a_k \theta^k (1 - \cos kt) \geq C_1 a_1 \theta (1 - \cos t).
\]

**Remark 2.3.** Assumption 2.1 is obviously satisfied (with \( C_1 = 1 \)) when all \( a_k \) are positive.
Due to Remark 2.2, the condition $a_1 > 0$ is equivalent to $b_1 > 0$. Moreover, from (2.14) and (2.15) we note that

$$\ln |\varphi_\nu(t; x)| = \frac{1}{2} \ln \frac{\beta(z^e u^t) \beta(z^e u^{-t})}{\beta(z)^2} = - \sum_{k=1}^{\infty} a_k z^k (1 - \cos kt);$$  \hspace{1cm} (2.25)

hence, condition (2.24) can be equivalently rewritten (for any $\theta \in (0, 1)$ and all $t \in \mathbb{R}$) as

$$\frac{1}{2} \ln \frac{\beta(\theta u^t) \beta(\theta u^{-t})}{\beta(\theta)^2} \leq - C_1 b_1 \theta (1 - \cos t).$$  \hspace{1cm} (2.26)

2.3. Examples

Let us now consider a few illustrative examples. The first three have direct analogues in the theory of (one-dimensional) decomposable combinatorial structures, corresponding, respectively, to the three well-known meta-classes: multisets, selections and assemblies (see [1, 2, 10]). To the best of our knowledge, Example 2.4 was first considered in [4] in the context of integer partitions.

Example 2.1 (multisets). For $r \in (0, \infty)$, $\rho \in (0, 1]$, let $Q_z$ be a measure determined by formula (2.3) with coefficients (1.6). A particular case with $\rho = 1$ was considered in [6]. Note that $b_0 = 1$, in accordance with our convention in Section 2.1, and $b_1 = r \rho > 0$. By the binomial expansion formula, the generating function of sequence (1.6) is given by

$$\beta(s) = (1 - \rho s)^{-r}, \quad |s| < \rho^{-1};$$  \hspace{1cm} (2.27)

and formula (2.3) specializes to

$$Q_z\{\nu(x) = \ell\} = \binom{r + \ell - 1}{\ell} \rho^{\ell} z^{\ell x} (1 - \rho z^x)^r, \quad \ell \in \mathbb{Z}_+,$$  \hspace{1cm} (2.28)

which is a negative binomial distribution with parameters $r$ and $p = 1 - \rho z^x$.

If $r = 1$ then $b_1 = \rho^\ell$, $\beta(s) = (1 - \rho s)^{-1}$ and, according to (2.28),

$$Q_z\{\nu(x) = \ell\} = \rho^\ell z^{\ell x} (1 - \rho z^x), \quad \ell \in \mathbb{Z}_+.$$  \hspace{1cm} (2.29)

In turn, from formulas (1.5) and (2.10) we get

$$P_n(\Gamma) = \frac{\rho^{N_{\Gamma}}}{\sum_{\Gamma' \in \Pi_n} \rho^{N_{\Gamma'}}}, \quad \Gamma \in \Pi_n,$$  \hspace{1cm} (2.29)

where $N_{\Gamma} := \sum_{x \in \chi} \nu(x)$ is the total number of integer points on $\Gamma \setminus \{0\}$. Furthermore, if also $\rho = 1$ then (2.29) is reduced to the uniform distribution on $\Pi_n$ (see (2.10),

$$P_n(\Gamma) = \frac{1}{\#(\Pi_n)}, \quad \Gamma \in \Pi_n.$$  \hspace{1cm} (2.29)

In the general case, using (2.27) we note that

$$\ln \beta(s) = -r \ln(1 - \rho s) = r \sum_{k=1}^{\infty} \rho^k s^k.$$  \hspace{1cm} (2.27)
and so the coefficients \((a_k)\) in expansion (2.11) are given by

\[ a_k = \frac{r \rho^k}{k} > 0, \quad k \in \mathbb{N}. \]

As pointed out in Remark 2.3, this implies that Assumption 2.1 is satisfied; also, it readily follows that \(A^+(\sigma) < \infty\) for any \(\sigma > 0\).

**Example 2.2 (selections).** For \(r \in \mathbb{N}, \ \rho \in (0, 1]\), consider the generating function

\[ \beta(s) = (1 + \rho s)^r, \quad |s| < \rho^{-1}, \quad (2.30) \]

with the coefficients in expansion (2.2) given by

\[ b_\ell = \binom{r}{\ell} \rho^\ell = \frac{r(r-1) \cdots (r-\ell+1)}{\ell!} \rho^\ell, \quad \ell = 0, 1, \ldots, r. \quad (2.31) \]

In particular, \(b_0 = 1, b_1 = r \rho > 0\). Accordingly, formula (2.3) gives a binomial distribution

\[ Q_z\{\nu(x) = \ell\} = \binom{r}{\ell} \rho^\ell z^\ell (1 + \rho z^x)^{r-\ell}, \quad \ell = 0, 1, \ldots, r, \quad (2.32) \]

with parameters \(r\) and \(p = \rho z^x (1 + \rho z^x)^{-1}\). From (2.30) we obtain

\[ \ln \beta(s) = r \ln(1 + \rho s) = r \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \rho^k}{k} s^k, \]

hence the coefficients \((a_k)\) in expansion (2.11) are given by

\[ a_k = \frac{r (-1)^{k-1} \rho^k}{k}, \quad k \in \mathbb{N}, \]

and in particular \(a_1 = r \rho > 0\). Note that \(A^+(\sigma) < \infty\) for any \(\sigma > 0\).

In the special case \(r = 1\), the measure \(Q_z\) is concentrated on the subspace \(\tilde{\Pi}\) of polygonal lines with “simple” edges, that is, containing no lattice points between the adjacent vertices. Here we have \(b_0 = 1, b_1 = \rho\) and \(b_\ell = 0 \ (\ell \geq 2)\), so that (2.32) is reduced to

\[ Q_z\{\nu(x) = \ell\} = \frac{\rho^\ell}{1 + \rho z^x}, \quad \ell = 0, 1 \quad (x \in \mathcal{X}), \]

Accordingly, formula (2.10) specifies on the corresponding subspace \(\tilde{\Pi}_n\) the distribution

\[ P_n(\Gamma) = \sum_{\Gamma' \in \tilde{\Pi}_n} \rho^{N_{\Gamma'}} \rho^{N_{\Gamma'}}, \quad \Gamma \in \tilde{\Pi}_n, \quad (2.33) \]

where the number of integer points \(N_{\Gamma'}\) coincides here with the number of vertices on \(\Gamma \setminus \{0\}\). Furthermore, if also \(\rho = 1\) then (2.33) is reduced to the uniform distribution on \(\tilde{\Pi}_n\),

\[ P_n(\Gamma) = \frac{1}{\#(\tilde{\Pi}_n)}, \quad \Gamma \in \tilde{\Pi}_n. \]
Finally, let us check that Assumption 2.1 holds (with $C_1 = (1 + \rho)^{-2}$). It is more convenient to use version (2.26). Substituting (2.30) and recalling that $b_1 = r \rho > 0$, we obtain

$$
\frac{1}{2} \ln \left( \frac{\beta(\theta e^{it}) \beta(\theta e^{-it})}{\beta(\theta)^2} \right) = \frac{r}{2} \ln \left( \frac{1 + 2 \rho \theta \cos t + \rho^2 \theta^2}{(1 + \rho \theta)^2} \right) \leq \frac{r}{2} \left( \frac{1 + 2 \rho \theta \cos t + \rho^2 \theta^2}{(1 + \rho \theta)^2} - 1 \right) \leq - \frac{b_1}{b_1} \frac{(1 - \cos t)}{(1 + \rho \theta)^2}.
$$

Example 2.3 (assemblies). For $r \in (0, \infty)$, $\rho \in [0, 1]$, consider the generating function

$$
\beta(s) = \exp \left( \frac{rs}{1 - \rho s} \right) = \exp \left( \sum_{k=1}^{\infty} s^k \rho^{k-1} \right), \quad |s| < \rho^{-1}. \tag{2.34}
$$

Clearly, the corresponding coefficients $b_\ell$ in expansion (2.2) are positive, with $b_0 = 1$, $b_1 = r$, $b_2 = \frac{1}{2} r^2 + r \rho$, etc.; more systematically, one can use the well-known Faà di Bruno’s formula generalizing the chain rule to higher derivatives (see, e.g., [13, Ch. I, §12, p. 34]) to obtain

$$
b_\ell = \rho^\ell \sum_{m=1}^\ell \left( \frac{r}{\rho} \right)^m \sum_{(j_1, \ldots, j_\ell) \in J_m} \frac{1}{j_1! \cdots j_\ell!}, \quad \ell \in \mathbb{N}, \tag{2.35}
$$

where $J_m$ is the set of all non-negative integer $\ell$-tuples $(j_1, \ldots, j_\ell)$ such that $j_1 + \cdots + j_\ell = m$ and $1 \cdot j_1 + 2 \cdot j_2 + \cdots + \ell \cdot j_\ell = \ell$.

Remark 2.4. Note that the $\ell$-tuples $(j_1, \ldots, j_\ell) \in J_m$ are in a one-to-one correspondence with partitions of $\ell$ involving precisely $m$ different integers as parts, where an element $j_i$ has the meaning of the multiplicity of $i \in \mathbb{N}$ (i.e., the number of times $i$ is used in a partition of $\ell$).

Taking the logarithm of (2.34), we see that the coefficients $a_k$ in (2.11) are given by

$$
a_k = r \rho^{k-1} > 0, \quad k \in \mathbb{N}. \tag{2.36}
$$

Therefore, Assumption 2.1 is automatic; moreover, $A^+(\sigma) < \infty$ for any $\sigma > 0$, except for the case $\rho = 1$ where $A^+(\sigma) < \infty$ only for $\sigma > 1$.

In the particular case $\rho = 0$, we have $\beta(s) = e^{rs}$ and so expression (2.35) is replaced by $b_\ell = r^\ell / \ell!$, whereas (2.36) simplifies to $a_1 = r$ and $a_k = 0$ for $k \geq 2$. The random variables $\nu(x)$ have a Poisson distribution with parameter $r z^x$,

$$
Q_z \{ \nu(x) = \ell \} = \frac{r^\ell z^\ell}{\ell!} e^{-rz}, \quad \ell \in \mathbb{Z}_+,
$$

which leads, according to (2.10), to the following distribution on $\Pi_n$

$$
P_n(\Gamma) = \left( \sum_{\{\ell_x'\} \in \Pi_n} \prod_{x \in X} \frac{r^{\ell_x'} x!}{\ell_x' x!} \right)^{-1} \prod_{x \in X} \frac{r^{\ell_x} x!}{\ell_x x!}, \quad \Gamma \leftrightarrow \{\ell_x\} \in \Pi_n.
$$

Example 2.4. Let $r \in (0, \infty)$, $\rho \in (0, 1]$, and consider the generating function

$$
\beta(s) = \left( - \frac{\ln(1 - \rho s)}{\rho s} \right)^r = \left( 1 + \sum_{\ell=1}^{\infty} \frac{r^\ell s^\ell}{\ell + 1} \right)^r =: f(s)^r. \tag{2.37}
$$
From (2.37) it is clear that \(b_0 = 1\), \(b_1 = \frac{1}{r} \rho > 0\) and, more generally, all \(b_k > 0\). Let us analyze the coefficients \((a_k)\) in the power series expansion of \(\ln \beta(s) = r \ln f(s)\) (see (2.11)). Differentiation of this identity with respect to \(s\) gives

\[
rf'(s) = f(s) \sum_{k=1}^{\infty} k a_k s^{k-1}.
\] (2.38)

Differentiating (2.38) further \(m\) times \((m \geq 0)\), by the Leibniz rule we obtain

\[
f^{(m+1)}(s) = \frac{1}{r} \sum_{j=0}^{m} \binom{m}{j} f^{(m-j)}(s) \sum_{k=j+1}^{\infty} \frac{k!}{(k-j-1)!} a_k s^{k-j-1},
\]

and in particular

\[
f^{(m+1)}(0) = \frac{1}{r} \sum_{j=0}^{m} \binom{m}{j} f^{(m-j)}(0)(m+1)! a_{m+1}.
\] (2.39)

But we know from (2.37) that \(f^{(j)}(0) = \rho^j j!/(j + 1)\), so (2.39) specializes to the equation

\[
\frac{\rho^{m+1}(m+1)!}{m+2} = \frac{1}{r} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} \cdot \frac{\rho^{m-j}(m-j)!}{m-j+1} (j+1)! a_{j+1},
\]

or, after some cancellations,

\[
\frac{m+1}{m+2} = \frac{1}{r} \sum_{j=0}^{m} \frac{\rho^{-j-1}(j+1)}{m-j+1} a_{j+1}.
\] (2.40)

Denoting for short \(\tilde{a}_j := r^{-1} \rho^{-j} j a_j\), equation (2.40) simplifies to

\[
\frac{m+1}{m+2} = \tilde{a}_1 + \frac{\tilde{a}_2}{m+1} + \cdots + \frac{\tilde{a}_m}{m} + \tilde{a}_{m+1}.
\] (2.41)

Setting here \(m = 0, 1, 2, 3, \ldots\) we can in principle find successively all \(\tilde{a}_m\),

\[
\tilde{a}_1 = \frac{1}{2}, \quad \tilde{a}_2 = \frac{5}{12}, \quad \tilde{a}_3 = \frac{3}{8}, \quad \tilde{a}_4 = \frac{251}{720}, \ldots,
\]

but the fractions quickly become quite cumbersome. However, it is not hard to obtain suitable estimates of \(\tilde{a}_m\). Observe that (2.41) implies

\[
\frac{m+1}{m+2} \leq \frac{\tilde{a}_1}{m} + \frac{\tilde{a}_2}{m-1} + \cdots + \tilde{a}_m + \tilde{a}_{m+1} = \frac{m}{m+1} + \tilde{a}_{m+1},
\]

and it follows that

\[
\tilde{a}_{m+1} \geq \frac{m+1}{m+2} - \frac{m}{m+1} = \frac{1}{(m+1)(m+2)} > 0,
\]

or explicitly

\[
a_{m+1} \geq \frac{\rho^{m+1}}{(m+1)^2(m+2)} > 0.
\] (2.42)
On the other hand, from (2.41) we get
\[ \tilde{a}_{m+1} = \frac{m + 1}{m + 2} - \frac{\tilde{a}_1}{m + 1} - \frac{\tilde{a}_2}{m} - \cdots - \frac{\tilde{a}_m}{m + 2} \leq \frac{m + 1}{m + 2} - \frac{\tilde{a}_1}{m + 1}, \]
hence
\[ \tilde{a}_{m+1} \leq \frac{m + 1}{m + 2} - \frac{1/2}{m + 1} = \frac{2m^2 + 3m}{2(m + 1)(m + 2)} \]
and therefore
\[ a_{m+1} \leq r \rho^{m+1}(2m^2 + 3m) \]
\[ 2(m + 1)(m + 2). \]
(2.43)

As a result, combining (2.42) and (2.43) we obtain, for all \( k \in \mathbb{N} \),
\[ \frac{r \rho^{k}}{k^2(k + 1)} \leq a_{k} \leq \frac{r \rho^{k}(2k^2 - k - 1)}{2k^2(k + 1)} \leq \frac{r \rho^{k}}{k + 1}. \]
In particular, this implies that \( A^+(\sigma) < \infty \) for any \( \sigma > 0 \); furthermore, since all \( a_{k} > 0 \) it follows that Assumption 2.1 is automatically satisfied.

Remark 2.5. Specific choices of the coefficients \((b_{\ell})\) in Examples 2.1–2.4 above can be used in the context of integer partitions (see, e.g., [10, 20, 22] and also a recent preprint [4]). More specifically, Example 2.1 corresponds to the ensemble of weighted partitions including the case of all unrestricted partitions under the uniform distribution; Example 2.2 leads to (weighted) partitions with bounds on the multiplicities of parts, including the case of uniform partitions with distinct parts; Example 2.3 corresponds to partitions representing the cycle structure of permutations; finally, Example 2.4 introduced in [4] defines a new ensemble of random partitions. Note that the limit shapes of partitions (or rather their Young diagrams) in the first three cases are known to exist, at least under some technical conditions on the coefficients (see [4, 9, 20, 22], but they are all drastically different from each other, as opposed to the case of lattice polygonal lines representing strict vector partitions, for which the limit shape is universal in all four examples.

3. Asymptotics of the expectation

In what follows, the asymptotic notation of the form \( x_n \asymp y_n \) with \( n = (n_1, n_2) \) means that
\[ 0 < \lim \inf_{n_1, n_2 \to \infty} \frac{x_n}{y_n} \leq \lim \sup_{n_1, n_2 \to \infty} \frac{x_n}{y_n} < \infty. \]
We also use the standard notation \( x_n \sim y_n \) for \( x_n/y_n \to 1 \) as \( n_1, n_2 \to \infty \).

Throughout the paper, we adopt the following convention about the limit \( n \to \infty \).

Assumption 3.1. The notation \( n \to \infty \) signifies that \( n_1, n_2 \to \infty \) in such a way that \( n_1 \asymp n_2 \).
In particular, this implies that \( n_1 \asymp |n|, n_2 \asymp |n|, \) where \( |n| = (n_1^2 + n_2^2)^{1/2} \to \infty \).

3.1. Calibration of the parameter \( z \)

We want to find the parameter \( z = (z_1, z_2) \) from the asymptotic conditions
\[ E_z(\xi_1) \sim n_1, \quad E_z(\xi_2) \sim n_2 \quad (n \to \infty), \]
(3.1)
where \( \xi_j = \sum_{x \in X} x_j \nu(x) \) and \( E_z \) denotes expectation with respect to \( Q_z \). Set

\[
z_j = e^{-\alpha_j}, \quad \alpha_j = \delta_j n_j^{-1/3} \quad (j = 1, 2),
\]

(3.2)

where the quantities \( \delta_1, \delta_2 > 0 \) (possibly depending on the ratio \( n_2/n_1 \)) are presumed to be bounded from above and separated from zero. Hence, recalling formula (2.18), we get

\[
E_z(\xi) = \sum_{k=1}^{\infty} k a_k \sum_{x \in X} x e^{-k\langle \alpha, x \rangle}.
\]

(3.3)

To deal with sums over the set \( X \), the following lemma will be instrumental. Recall that the M"obius function \( \mu(m) \) \((m \in \mathbb{N})\) is defined as follows: \( \mu(1) := 1 \), \( \mu(m) := (-1)^d \) if \( m \) is a product of \( d \) different prime numbers, and \( \mu(m) := 0 \) otherwise (see [11, §16.3, p. 234]); in particular, \( |\mu(m)| \leq 1 \) for all \( m \in \mathbb{N} \).

**Lemma 3.1.** Suppose that a function \( f : \mathbb{R}_+^2 \to \mathbb{R} \) is such that \( f(0, 0) = 0 \) and, for any \( h > 0 \),

\[
\sum_{k=1}^{\infty} \sum_{x \in \mathbb{Z}_+^2} |f(h k x)| < \infty.
\]

(3.4)

For \( h > 0 \), consider the functions

\[
F^\sharp(h) := \sum_{x \in X} f(hx), \quad (3.5)
\]

\[
F(h) := \sum_{m=1}^{\infty} F^\sharp(hm) = \sum_{m=1}^{\infty} \sum_{x \in X} f(hmx).
\]

(3.6)

Then the following identities hold for all \( h > 0 \)

\[
F(h) = \sum_{x \in \mathbb{Z}_+^2} f(hx), \quad (3.7)
\]

\[
F^\sharp(h) = \sum_{m=1}^{\infty} \mu(m) F(hm), \quad (3.8)
\]

where \( \mu(m) \) is the M"obius function.

**Proof.** Recalling definition (2.1) of the set \( X \), observe that \( \mathbb{Z}_+^2 = \bigcup_{m=0}^{\infty} mX \); hence, (3.6) is reduced to (3.7). Then representation (3.8) follows from the M"obius inversion formula (see [11, Theorem 270, p. 237]), provided that \( \sum_{k,m} |F^\sharp(hkm)| < \infty \). To verify the last condition, using (3.5) we obtain (cf. (3.6) and (3.7))

\[
\sum_{k,m=1}^{\infty} |F^\sharp(kmh)| \leq \sum_{k=1}^{\infty} \left( \sum_{m=1}^{\infty} \sum_{x \in X} |f(hkx)| \right) = \sum_{k=1}^{\infty} \sum_{x \in \mathbb{Z}_+^2} |f(hkx)| < \infty,
\]

according to (3.4). This completes the proof. \( \square \)
Theorem 3.2. Suppose that $A^+(2) < \infty$ (see (2.23)), and choose $\delta_1, \delta_2$ in (3.2) as follows,

\[
\delta_1 = \kappa (n_2/n_1)^{1/3}, \quad \delta_2 = \kappa (n_1/n_2)^{1/3},
\]

where

\[
\kappa := \left( \frac{A(2)}{\zeta(2)} \right)^{1/3}
\]

and $\zeta(2) := \sum_{k=1}^{\infty} k^{-2} = \pi^2/6$. Then conditions (3.1) are satisfied.

Remark 3.1. Observe that (3.2) and (3.9) imply the scaling relations

\[
\alpha_1^2 \alpha_2 n_1 = \alpha_1 \alpha_2^2 n_2 = \kappa^3, \quad \alpha_2 n_2 = \alpha_1 n_1.
\]

Proof of Theorem 3.2. Let us prove (3.1) for $\xi_1$ (the proof for $\xi_2$ is similar). Setting

\[
f(x) := x_1 e^{-\langle \alpha, x \rangle}, \quad x \in \mathbb{R}_+^2,
\]

and following notations (3.5) and (3.6) of Lemma 3.1, a projection of equation (3.3) to the first coordinate takes the form

\[
E_z(\xi_1) = \sum_{k=1}^{\infty} a_k F^\sharp(k).
\]

Note that

\[
F(h) = h \sum_{x_1=1}^{\infty} x_1 e^{-\alpha_1 x_1} \sum_{x_2=0}^{\infty} e^{-\alpha_2 x_2} = \frac{h e^{-\alpha_1}}{(1 - e^{-\alpha_1})^2 (1 - e^{-\alpha_2})},
\]

and it easily follows that condition (3.4) of Lemma 3.1 is satisfied. Hence, using (3.8) and (3.14), we can rewrite (3.13) as

\[
E_z(\xi_1) = \sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} \mu(m) F(km) = \sum_{k=1}^{\infty} ka_k \sum_{m=1}^{\infty} \frac{m \mu(m) e^{-km\alpha_1}}{(1 - e^{-km\alpha_1})^2 (1 - e^{-km\alpha_2})},
\]

or, recalling relations (3.11),

\[
n_1^{-1} E_z(\xi_1) = \frac{1}{\kappa^3} \sum_{k,m=1}^{\infty} ka_k \mu(m) \frac{\alpha_1^2 \alpha_2 e^{-km\alpha_1}}{(1 - e^{-km\alpha_1})^2 (1 - e^{-km\alpha_2})}.
\]

Note that for any $b > 0$, $\theta > 0$, there is a global bound

\[
\frac{e^{-\theta t}}{(1 - e^{-t})^b} \leq C t^{-b}, \quad t > 0,
\]

with some constant $C = C(b, \theta) > 0$. This gives, uniformly in $k$ and $m$,

\[
\frac{e^{-km\alpha_1/2}}{(1 - e^{-km\alpha_1})^2} = O(1), \quad \frac{e^{-(n_2/n_1)km\alpha_2/2}}{1 - e^{-km\alpha_2}} = O(1),
\]
where in the second estimate we used Assumption 3.1. Therefore, the summand in (3.16) is bounded by \( O(\{a_k\} k^{-2} m^{-2}) \), which is a term of a convergent series due to the assumption \( A^+(2) < \infty \). Hence, by Lebesgue’s dominated convergence theorem we obtain

\[
\lim_{n \to \infty} n^{-1} E_z(\xi_i) = \frac{1}{\kappa^3} \sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{A(2)}{\kappa^3 \zeta(2)} = 1, \tag{3.19}
\]

according to (3.10); we also used the identity

\[
\sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} = \frac{1}{\zeta(s)}, \tag{3.20}
\]

which readily follows by the Möbius inversion formula (3.8) applied to \( F^s(h) = h^{-s}, F(h) = \sum_{m=1}^{\infty} (hm)^{-s} = h^{-s} \zeta(s) \) (cf. [11, Theorem 287, p. 250]).

**Assumption 3.2.** Throughout the rest of the paper, we assume that the parameters \( z_1, z_2 \) are chosen according to formulas (3.2), (3.9). In particular, the measure \( Q_2 \) becomes dependent on \( n = (n_1, n_2) \), as well as the \( Q_2 \)-probabilities and the corresponding expected values.

### 3.2. Asymptotics of the mean polygonal lines

For \( \Gamma \in \Pi \), denote by \( \Gamma(t) (t \in [0, \infty]) \) the part of \( \Gamma \) where the slope does not exceed \( tn_2/n_1 \). Consider the set

\[
\mathcal{X}(t) := \{ x \in \mathcal{X} : x_2/x_1 \leq tn_2/n_1 \}, \quad t \in [0, \infty]. \tag{3.21}
\]

According to the association \( \Pi \ni \Gamma \leftrightarrow \nu \in \Phi_0 \) described in Section 2.1, for each \( t \in [0, \infty] \) the polygonal line \( \Gamma(t) \) is determined by a truncated configuration \( \{\nu(x), x \in \mathcal{X}(t)\} \), hence its right endpoint \( \xi(t) = (\xi_1(t), \xi_2(t)) \) is given by

\[
\xi(t) = \sum_{x \in \mathcal{X}(t)} x \nu(x), \quad t \in [0, \infty]. \tag{3.22}
\]

In particular, \( \mathcal{X}(\infty) = \mathcal{X}, \xi(\infty) = \xi \). Similarly to (3.3),

\[
E_z[\xi(t)] = \sum_{k=1}^{\infty} ka_k \sum_{x \in \mathcal{X}(t)} x e^{-k(\alpha,x)}, \quad t \in [0, \infty]. \tag{3.23}
\]

Recall that the function \( g^s(t) = (g_1^s(t), g_2^s(t)) \) is defined in (1.7).

**Theorem 3.3.** For each \( t \in [0, \infty] \),

\[
\lim_{n \to \infty} n_j^{-1} E_z[\xi_j(t)] = g_j^s(t) \quad (j = 1, 2). \tag{3.24}
\]

**Proof.** Theorem 3.2 implies that (3.24) holds for \( t = \infty \). Assume that \( t < \infty \) and let \( j = 1 \) (the case \( j = 2 \) is considered in a similar manner). Setting for brevity \( c_n := n_2/n_1 \) and arguing as in the proof of Theorem 3.2 (see (3.3), (3.13) and (3.16)), from (3.23) we obtain

\[
E_z[\xi_1(t)] = \sum_{k,m=1}^{\infty} ka_k m \mu(m) \sum_{x_1=1}^{\infty} x_1 e^{-k\alpha_1 x_1} \sum_{x_2=0}^{X_2} e^{-k\alpha_2 x_2} \quad = \sum_{k,m=1}^{\infty} ka_k m \mu(m) \sum_{x_1=1}^{\infty} x_1 e^{-k\alpha_1 x_1} \frac{1 - e^{-k\alpha_2 (x_2+1)}}{1 - e^{-k\alpha_2}}, \tag{3.25}
\]

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where $\hat{x}_2 = \hat{x}_2(t)$ denotes the integer part of $tc_n x_1$, so that
\begin{equation}
0 \leq tc_n x_1 - \hat{x}_2 < 1. \tag{3.26}
\end{equation}

Aiming to replace $\hat{x}_2 + 1$ by $tc_n x_1$ in (3.25), we recall (3.11) and rewrite the sum over $x_1$ as
\begin{equation}
\sum_{x_1=1}^{\infty} x_1 e^{-kma_1 x_1} \left(1 - e^{-kma_1 x_1} \right) + \Delta_{k,m}(t, \alpha), \tag{3.27}
\end{equation}
where
\[ \Delta_{k,m}(t, \alpha) := \sum_{x_1=1}^{\infty} x_1 e^{-kma_1 x_1 (1+t)} \left(1 - e^{-kma_2(\hat{x}_2 + 1 - tc_n x_1)} \right). \]

Using that $0 < \hat{x}_2 + 1 - tc_n x_1 \leq 1$ (see (3.26)) and applying estimate (3.17), we obtain, uniformly in $k, m \geq 1$ and $t \in [0, \infty]$,
\[ 0 < \frac{\Delta_{k,m}(t, \alpha)}{1 - e^{-kma_2}} \leq \sum_{x_1=1}^{\infty} x_1 e^{-kma_1 x_1} = \frac{e^{-kma_1}}{(1 - e^{-kma_1})^2} = O(1) \frac{e^{-ma_1/2}}{(kma_1)^2}. \]

Substituting this estimate into (3.25) and using the condition $A^+(2) < \infty$, we see that the error resulting from the replacement of $\hat{x}_2 + 1$ by $tc_n x_1$ is dominated by
\[ O(\alpha^{-2}) \sum_{k=1}^{\infty} \frac{|a_k|}{k} \sum_{m=1}^{\infty} \frac{e^{-ma_1/2}}{m} = O(\alpha^{-2}) \ln \left(1 - e^{-a_1/2}\right) = O(\alpha^{-2} \ln \alpha_1). \]

Returning to representation (3.25) and evaluating the sum in (3.27), we find
\begin{equation}
E_z[\xi_1(t)] = \sum_{k,m=1}^{\infty} ka_k \frac{m \mu(m)}{1 - e^{-kma_2}} \cdot \frac{e^{-kma_1 y}}{(1 - e^{-kma_1 y})^2} \bigg|_{y=1}^{y=1+t} + O(\alpha^{-2} \ln \alpha_1). \tag{3.28}
\end{equation}

Then, passing to the limit by Lebesgue’s dominated convergence theorem, similarly to the proof of Theorem 3.2 (cf. (3.19)) we get, as $n \to \infty$,
\[ n_1^{-1} E_z[\xi_1(t)] \to \frac{1}{\kappa^3} \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \left(1 - \frac{1}{(1 + t)^2}\right) = \frac{t^2 + 2t}{(1 + t)^2}, \]
which coincides with $g_1^*(t)$, as claimed. \hfill \Box

**Theorem 3.4.** Convergence in (3.24) is uniform in $t \in [0, \infty]$, that is,
\[ \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left| n_j^{-1} E_z[\xi_j(t)] - g_j^*(t) \right| = 0 \quad (j = 1, 2). \]

For the proof, we shall use the following simple criterion of uniform convergence proved in [6, Lemma 4.3].
Lemma 3.5. Let \( \{f_n(t)\} \) be a sequence of nondecreasing functions on a finite interval \([a, b]\), such that, for each \( t \in [a, b] \), \( \lim_{n \to \infty} f_n(t) = f(t) \), where \( f(t) \) is a continuous (nondecreasing) function on \([a, b]\). Then the convergence \( f_n(t) \to f(t) \) as \( n \to \infty \) is uniform on \([a, b]\).

Proof of Theorem 3.4 Suppose that \( j = 1 \) (the case \( j = 2 \) is handled similarly). Note that for each \( n \) the function
\[
f_n(t) := n_1^{-1} E_z[\xi_1(t)] = \frac{1}{n_1} \sum_{x \in A(t)} x_1 E_z[\nu(x)]
\]
is nondecreasing in \( t \). Therefore, by Lemma 3.5 the convergence in (3.24) is uniform on any interval \([0, t^*] \) \( (t^* < \infty) \). Since \( n_1^{-1} E_z[\xi_1(\infty)] \to g_1^*(\infty) \) and the function \( g_1^*(t) \) is continuous at infinity (see (1.7)), it remains to show that for any \( \varepsilon > 0 \) there is \( t^* \) such that, for all large enough \( n_1, n_2 \) and all \( t \geq t^* \),
\[
n_1^{-1} E_z[\xi_1(\infty) - \xi_1(t)] \leq \varepsilon. \tag{3.29}
\]

To this end, on account of (3.28) we have
\[
E_z[\xi_1(\infty) - \xi_1(t)] = \sum_{k,m=1}^{\infty} k\alpha_k \frac{m\mu(m)}{1 - e^{-km\alpha_2}} \cdot \frac{e^{-km\alpha_1(t)}(1+t)}{(1 - e^{-km\alpha_1(t)})(1+t)^2} + O(\alpha_1^{-2} \ln \alpha_1). \tag{3.30}
\]

Note that by inequality (3.17), uniformly in \( k, m \geq 1 \),
\[
\frac{e^{-km\alpha_2}}{1 - e^{-km\alpha_2}} \cdot \frac{e^{-km\alpha_1(t)}(1+t)}{(1 - e^{-km\alpha_1(t)})(1+t)^2} = \frac{O(1)}{\alpha_1^2 \alpha_2(km)^3(1+t)^2}.
\]

Returning to (3.30) and using the condition \( A^+(2) < \infty \), we obtain, uniformly in \( t \geq t^* \),
\[
\alpha_2^2 \alpha_2 E_z[\xi_1(\infty) - \xi_1(t)] = \frac{O(1)}{(1 + t)^2} \sum_{k=1}^{\infty} \frac{|a_k|}{k^2} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{O(1)}{(1 + t^*)^2},
\]
whence by (3.2) we get (3.29).

\[ \square \]

4. Refined asymptotics of the expectation

We need to sharpen the asymptotic estimate \( E_z(\xi) - n = o(|n|) \) provided by Theorem 3.2.

Theorem 4.1. Under the condition \( A^+(1) < \infty \), we have \( E_z(\xi) - n = O(|n|^{2/3}) \) as \( n \to \infty \).

For the proof of Theorem 4.1 some preparations are required.

4.1. Integral approximation of sums

Let a function \( f : \mathbb{R}_+^2 \to \mathbb{R} \) be continuous and absolutely integrable on \( \mathbb{R}_+^2 \), together with its partial derivatives up to the second order. Set
\[
F(h) := \sum_{x \in \mathbb{Z}_+^2} f(hx), \quad h > 0 \tag{4.1}
\]
(as one can verify, the above conditions on \( f \) ensure that the series in (4.1) is absolutely convergent \([6, \text{p. 21}]\), and assume that for some \( \beta > 2 \)
\[
F(h) = O\left(h^{-\beta}\right), \quad h \to \infty.
\] (4.2)

Consider the Mellin transform of \( F(h) \) (see, e.g., \([21, \text{Ch. VI, §9}]\)),
\[
\hat{F}(s) := \int_0^\infty h^{s-1} F(h) \, dh,
\] (4.3)
and set
\[
\Delta_f(h) := F(h) - \frac{1}{h^2} \int_{\mathbb{R}^2_+} f(x) \, dx, \quad h > 0.
\] (4.4)

The following general lemma can be proved using the well-known Euler–Maclaurin summation formula (see details in \([6, \text{Lemmas 5.1 and 5.3, pp. 20–22}]\)).

**Lemma 4.2.** The function \( \hat{F}(s) \) is meromorphic in the strip \( 1 < \Re s < \beta \), with a single (simple) pole at \( s = 2 \). Moreover, \( \hat{F}(s) \) satisfies the identities
\[
\hat{F}(s) = \int_0^\infty h^{s-1} \Delta_f(h) \, dh, \quad 1 < \Re s < 2,
\] (4.5)
\[
\Delta_f(h) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-s} \hat{F}(s) \, ds, \quad 1 < c < 2.
\] (4.6)

**4.2. Proof of Theorem 4.1**

Our argumentation follows the same lines as in a similar result in \([6, \text{pp. 22–27}]\) for distributions determined by coefficients \((1.6)\) (with \( \rho = 1 \)). For the reader’s convenience, we repeat all the steps but skip some word-by-word repetitions, giving specific references to \([6]\).

Let us consider \( \xi_1 \) (for \( \xi_2 \) the proof is similar). Recalling the notations \( f(x) \) and \( F(h) \) introduced in Section 3.2 (see (3.12) and (3.14), respectively), we have, according to (3.15),
\[
E_z(\xi_1) = \sum_{k,m=1}^{\infty} a_k \mu(m) F(km),
\] (4.7)
where
\[
F(h) = \sum_{x \in \mathbb{Z}^2_+} f(hx) = \frac{h e^{-\alpha_1 h}}{(1 - e^{-\alpha_1 h})^2 (1 - e^{-\alpha_2 h})}, \quad h > 0,
\]
\[
f(x) = x_1 e^{-\langle \alpha, x \rangle}, \quad x \in \mathbb{R}^2_+.
\]

Note that
\[
\int_{\mathbb{R}^2_+} f(x) \, dx = \int_0^\infty x_1 e^{-\alpha_1 x_1} \, dx_1 \int_0^\infty e^{-\alpha_2 x_2} \, dx_2 = \frac{1}{\alpha_1^2 \alpha_2}.
\]
Moreover, using (3.10) and (3.11) we have (cf. (3.19))
\[
\sum_{k,m=1}^{\infty} \frac{a_k \mu(m)}{(km)^2 \alpha_1^2 \alpha_2} = \frac{n_1}{k^3} \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \equiv n_1.
\] (4.8)
Subtracting (4.8) from (4.7), we obtain the representation

\[ E_z(\xi_1) - n_1 = \sum_{k,m=1}^{\infty} a_k \mu(m) \Delta_f(km), \]  

where \( \Delta_f(h) \) is defined in (4.4). Clearly, the functions \( f \) and \( F \) satisfy the hypotheses of Lemma 4.2 (with \( \beta = \infty \)). Setting \( c_n := n_2/n_1 \) and using (3.11), the Mellin transform of \( F(h) \) defined by (4.3) can be represented as

\[ \hat{F}(s) = \alpha_1^{-s-1} \tilde{F}(s), \]  

where

\[ \tilde{F}(s) := \int_0^\infty \frac{y^s e^{-y}}{(1 - e^{-y})^2 (1 - e^{-y/c_n})} \, dy, \quad \Re s > 2. \]  

It is easy to verify (see [6, p. 23] for details) that the analytic continuation of expression (4.11) into domain \( 1 < \Re s < 2 \) is explicitly given by

\[ \tilde{F}(s) = J(s) + c_n \zeta(s - 1) \Gamma(s) + \frac{1}{2} \zeta(s) \Gamma(s + 1), \]  

where \( \Gamma(s) = \int_0^\infty u^{s-1} e^{-u} \, du \) is the gamma function, \( \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \) is the Riemann zeta function and

\[ J(s) := \int_0^\infty \frac{y^s e^{-y}}{(1 - e^{-y})^2} \left( \frac{1}{1 - e^{-y/c_n}} - \frac{c_n}{y} - \frac{1}{2} \right) \, dy. \]  

Note that for \( \Re s > 0 \) the integral in (4.13) is absolutely convergent and therefore \( J(s) \) is regular. Furthermore, it is well known that \( \Gamma(s) \) is analytic for \( \Re s > 0 \) [17 §4.41, p. 148], whereas \( \zeta(s) \) has a single pole at point \( s = 1 \) [17 §4.43, p. 152]. Thus, the right-hand side of (4.12) is meromorphic in the half-plane \( \Re s > 0 \), with simple poles at \( s = 1 \) and \( s = 2 \).

Using (4.6) and (4.10), and recalling formulas (2.23) and (3.20), we can rewrite (4.9) as

\[ E_z(\xi_1) - n_1 = \frac{1}{2\pi i} \sum_{k,m=1}^{\infty} a_k \mu(m) \int_{c-i\infty}^{c+i\infty} \frac{\tilde{F}(s)}{\alpha_1^{s+1}(km)^s} \, ds \]
\[ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A(s)}{\alpha_1^{s+1} \zeta(s)} \tilde{F}(s) \, ds \quad (1 < c < 2). \]  

Noting that \( \zeta(s) \neq 0 \) for \( \Re s \geq 1 \), let us show that the integration contour \( \Re s = c \) in (4.14) can be moved to \( \Re s = 1 \). By the Cauchy theorem, it suffices to check that

\[ \lim_{t \to \infty} \int_{1-it}^{c+it} \frac{A(s)}{\alpha_1^{s+1} \zeta(s)} \tilde{F}(s) \, ds = 0, \quad (1 < c < 2). \]  

To this end, note that for \( s = \sigma + it \) with \( 1 \leq \sigma \leq c < 2 \)

\[ |A(s)| \leq A^+(1) < \infty, \quad |\alpha_1^{-s-1}| \leq \alpha_1^{-c-1}, \]  

whereas integration by parts in (4.13) yields a uniform estimate

\[ J(s) = O(t^{-2}), \quad t \to \infty. \]  

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Furthermore, we have the following asymptotic estimates as \( t \to \infty \), uniform in the strip \( \Re s \in [1, c] \) (see \cite{17} §4.42, p. 151) for (4.18), \cite{12} Theorem 1.9, p. 25] for (4.19) and \cite{18} Eq. (3.11.8), p. 60) for (4.20)

\[
\Gamma(s) = O(|t|^{s-1/2} e^{-\pi|t|/2}), \quad \Gamma(s + 1) = O(|t|^{s+1/2} e^{-\pi|t|/2}), \quad (4.18)
\]

\[
\zeta(s) = O(|t|), \quad \zeta(s - 1) = O(t^{1-s/2} \ln |t|), \quad (4.19)
\]

\[
\zeta(s)^{-1} = O(\ln |t|), \quad (4.20)
\]

Substituting estimates (4.18) and (4.19) into (4.12), we get \( \tilde{F}(s) = O(t^{-2}) \), and on account of (4.16) and (4.17) we see that (4.15) follows. Hence, representation (4.14) takes the form

\[
E_z(\xi_1) - n_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A(1 + it)}{\zeta(1 + it)} \tilde{F}(1 + it) \, dt = O(\alpha^{-2}) = O(|n|^{2/3}),
\]

according to (3.2). The proof of Theorem 4.1 is complete.

Remark 4.1. If condition \( A^+(\sigma) < \infty \) is satisfied with some \( \sigma \in (0, 1) \), then the statement of Theorem 4.1 can be enhanced to \( E_z(\xi) - n = o(|n|^{2/3}) \) (cf. \cite{6} p. 26). This is the case for all examples in Section 2.3.

5. Asymptotics of higher-order moments

In this section, we again assume that \( A^+(2) < \infty \).

5.1. Second-order moments

As before, denote \( a_z := E_z(\xi) \), and let \( K_z := \text{Cov}(\xi, \xi) = E_z(\xi - a_z)(\xi - a_z)^\top \) be the covariance matrix of the random vector \( \xi = \sum x \nu(x) \). Recalling that the random variables \( \nu(x) \) are independent for different \( x \in X \) and using (2.19), we see that the elements \( K_z(i, j) = \text{Cov}(\xi_i, \xi_j) \) \((i, j \in \{1, 2\})\) of the matrix \( K_z \) are given by

\[
K_z(i, j) = \sum_{x \in X} x_i x_j \text{Var}[\nu(x)] = \sum_{x \in X} x_i x_j \sum_{k=1}^{\infty} k^2 a_k z^{kx}. \quad (5.1)
\]

Theorem 5.1. As \( n \to \infty \),

\[
K_z(i, j) \sim B_{ij} (n_1 n_2)^{2/3}, \quad i, j \in \{1, 2\}, \quad (5.2)
\]

where the matrix \( B := (B_{ij}) \) is given by

\[
B = \kappa^{-1} \begin{pmatrix} 2n_1/n_2 & 1 \\ 1 & 2n_2/n_1 \end{pmatrix}. \quad (5.3)
\]

Proof. Let us consider \( K_z(1, 1) \) (the other elements of \( K_z \) are analyzed in a similar manner). Substituting (5.2) into (5.1), we obtain

\[
K_z(1, 1) = \sum_{x \in X} x_1^2 \sum_{k=1}^{\infty} k^2 a_k e^{-k(\alpha, x)}.
\]

(5.4)
Using the Möbius inversion formula (3.8), similarly to (3.16) the double sum in (5.4) can be rewritten in the form

\[
K_z(1, 1) = \sum_{m=1}^{\infty} m^2 \mu(m) \sum_{k=1}^{\infty} k^2 a_k \sum_{x \in \mathbb{Z}^2} x_1^2 e^{-km(\alpha, x)}
\]

\[
= \sum_{k,m=1}^{\infty} m^2 \mu(m) k^2 a_k \sum_{x_1=1}^{\infty} x_1^2 e^{-kma_1x_1} \sum_{x_2=0}^{\infty} e^{-kma_2x_2}
\]

\[
= \sum_{k,m=1}^{\infty} m^2 \mu(m) k^2 a_k \frac{e^{-kma_1(1 + e^{-kma_1})}}{(1 - e^{-kma_1})^3(1 - e^{-kma_2})}.
\] (5.5)

By estimate (3.17), the general term in series (5.5) is bounded by \(\alpha_1^{-3} \alpha_2^{-1} O(|a_k| k^{-2} m^{-2})\), uniformly in \(k, m\), and furthermore (see (2.23))

\[
\sum_{k,m=1}^{\infty} \frac{|a_k|}{k^2 m^2} = A^+(2) \zeta(2) < \infty.
\]

Therefore, Lebesgue’s dominated convergence theorem yields

\[
\alpha_1^3 \alpha_2 K_z(1, 1) \to 2 \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{2A(2)}{\zeta(2)}, \quad \alpha_1, \alpha_2 \to 0.
\] (5.6)

Hence, using (3.2), (3.9) and (3.10), from (5.6) we get, as \(n \to \infty\),

\[
K_z(1, 1) \sim \frac{2A(2)}{\zeta(2)} k^3 (n_1/n_2) (n_1 n_2)^{2/3} = B_{11} (n_1 n_2)^{2/3},
\]

as required (cf. (5.2), (5.3)).

The next lemma is a direct corollary of Theorem 5.1.

**Lemma 5.2.** As \(n \to \infty\),

\[
\det K_z \sim 3k^{-2} (n_1 n_2)^{4/3}.
\]

Lemma 5.2 implies that the matrix \(K_z\) is non-degenerate, at least asymptotically as \(n \to \infty\). In fact, from (5.1) it is easy to see (e.g., using the Cauchy–Schwarz inequality together with the characterization of the equality case) that \(K_z\) is positive definite; in particular, \(\det K_z > 0\) and hence \(K_z\) is invertible. Let \(V_z = K_z^{-1/2}\) be the (unique) square root of the matrix \(K_z^{-1}\), that is, a symmetric, positive definite matrix such that \(V_z^2 = K_z^{-1}\).

Recall that the matrix norm induced by the Euclidean vector norm \(| \cdot |\) is defined by \(\|A\| := \sup_{|x|=1} |xA|\). We need some general facts about this matrix norm (see [6, §7.2, pp. 33–34] for simple proofs and bibliographic comments).

**Lemma 5.3.** If \(A\) is a real matrix then \(\|A^T A\| = \|A\|^2\).

**Lemma 5.4.** If \(A = (a_{ij})\) is a real \(d \times d\) matrix, then

\[
\frac{1}{d} \sum_{i,j=1}^{d} a_{ij}^2 \leq \|A\|^2 \leq \sum_{i,j=1}^{d} a_{ij}^2.
\] (5.7)
Lemma 5.5. Let $A$ be a symmetric $2 \times 2$ matrix with $\det A \neq 0$. Then

$$\|A^{-1}\| = \|A\| |\det A|^{-1}.$$  \hspace{1cm} (5.8)

Let us now estimate the norms of the matrices $K_z$ and $V_z = K_z^{-1/2}$.

Lemma 5.6. As $n \to \infty$, one has $\|K_z\| \asymp |n|^{4/3}$.

Proof. Lemma 5.4 and Theorem 5.1 imply

$$\|K_z\|^2 \asymp \sum_{i,j=1}^{n_1 n_2} K_z(i,j)^2 \asymp (n_1 n_2)^{4/3} \asymp |n|^{8/3} \quad (n \to \infty),$$

and the required estimate follows. \hfill \square

Lemma 5.7. For the matrix $V_z = K_z^{-1/2}$, one has $\|V_z\| \asymp |n|^{-2/3}$ as $n \to \infty$.

Proof. Using Lemmas 5.3 and 5.5 we have

$$\|V_z\|^2 = \|V_z^2\| = \|K_z^{-1}\| = \|K_z\| |\det K_z|^{-1},$$

and an application of Lemmas 5.2 and 5.6 completes the proof. \hfill \square

5.2. Auxiliary estimates

Denote

$$\nu_0(x) := \nu(x) - E_z[\nu(x)], \quad x \in X,$$ \hspace{1cm} (5.9)

and consider the moments of order $q \in \mathbb{N}$

$$m_q(x) := E_z[\nu(x)^q], \quad \mu_q(x) := E_z[|\nu_0(x)^q|]$$ \hspace{1cm} (5.10)

(for simplicity, we suppress the dependence on $z$).

Let us note a simple general inequality (cf. \cite{6} Lemma 6.2]).

Lemma 5.8. For each $q \geq 1$ and all $x \in X$,

$$\mu_q(x) \leq 2^q m_q(x).$$ \hspace{1cm} (5.11)

Proof. Using the elementary inequality $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for any $a, b > 0$ and $q \geq 1$ (which follows from Hölder’s inequality for the function $y = x^q$), we obtain

$$\mu_q(x) \leq E_z[(\nu(x) + m_1(x))^q] \leq 2^{q-1}(m_q(x) + m_1(x)^q) \leq 2^q m_q(x),$$

where we used Lyapunov’s inequality $m_1(x)^q \leq m_q(x)$. \hfill \square

The following two lemmas are useful for estimation of higher-order moment sums.
Lemma 5.9. For \( q \in \mathbb{N} \), the function

\[
S_q(\theta) := \sum_{x=1}^{\infty} x^{q-1} e^{-\theta x}, \quad \theta > 0, \tag{5.12}
\]

admits a representation

\[
S_q(\theta) = \sum_{j=1}^{q} c_{j,q} \frac{e^{-\theta j}}{(1 - e^{-\theta})^j}, \quad \theta > 0, \tag{5.13}
\]

with some constants \( c_{j,q} > 0 \) \( (j = 1, \ldots, q) \); in particular, \( c_{q,q} = (q - 1)! \).

Proof. In the case \( q = 1 \), expression (5.12) is reduced to a geometric series

\[
S_1(\theta) = \sum_{x=1}^{\infty} e^{-\theta x} = \frac{e^{-\theta}}{1 - e^{-\theta}},
\]

which is a particular case of (5.13) with \( c_{1,1} := 1 \). Assume now that (5.13) is valid for some \( q \geq 1 \). Then, differentiating identities (5.12) and (5.13) with respect to \( \theta \), we obtain

\[
S_{q+1}(\theta) = -\frac{d}{d\theta} S_q(\theta) = \sum_{j=1}^{q} c_{j,q} \left( \frac{j e^{-\theta j}}{(1 - e^{-\theta})^j} + \frac{j e^{-\theta(j+1)}}{(1 - e^{-\theta})^{j+1}} \right)
\]
\[
= \sum_{j=1}^{q+1} c_{j,q+1} \frac{e^{-\theta j}}{(1 - e^{-\theta})^j},
\]

where we set

\[
c_{j,q+1} := \begin{cases} 
  c_{1,q}, & j = 1, \\
  j c_{j,q} + (j - 1) c_{j-1,q}, & 2 \leq j \leq q, \\
  q c_{q,q}, & j = q+1.
\end{cases}
\]

In particular, \( c_{q+1,q+1} = q c_{q,q} = q(q - 1)! = q! \). Thus, formula (5.13) holds for \( q + 1 \) and hence, by induction, for all \( q \geq 1 \).

Lemma 5.10. For each \( q \in \mathbb{N} \), there exists a positive constant \( C_q \) such that, for all \( \theta > 0 \),

\[
0 < S_q(\theta) \leq \frac{C_q e^{-\theta}}{(1 - e^{-\theta})^q}, \tag{5.14}
\]

Proof. Observe that for \( j = 1, \ldots, q \) and all \( \theta > 0 \)

\[
\frac{e^{-\theta j}}{(1 - e^{-\theta})^j} \leq \frac{e^{-\theta}}{(1 - e^{-\theta})^q}.
\]

Substituting these inequalities into (5.13) and recalling that the coefficients \( c_{j,q} \) are positive, we obtain (5.14) with \( C_q := \sum_{j=1}^{q} c_{j,q} \).

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5.3. Asymptotics of moment sums

According to (2.21) and (3.2), the cumulants of $\xi_j = \sum_{x \in \mathcal{X}} x_j \nu(x)$ ($j = 1, 2$) are given by

$$\kappa_q[\xi_j] = \sum_{x \in \mathcal{X}} x_j^q \kappa_q(x) = \sum_{x \in \mathcal{X}} x_j^q \sum_{k=1}^{\infty} k^q a_k e^{-k(\alpha,x)}, \quad q \in \mathbb{N}. \tag{5.15}$$

**Lemma 5.11.** For each $q \in \mathbb{N}$ and $j = 1, 2$,

$$\kappa_q[\xi_j] \asymp |n|^{(q+2)/3}, \quad n \to \infty. \tag{5.16}$$

**Proof.** Let $j = 1$ (the case $j = 2$ is treated in a similar fashion). Using the Möbius inversion formula (8.8), similarly to (3.16) the right-hand side of (5.15) (with $j = 1$) can be rewritten as

$$\kappa_q[\xi_1] = \sum_{k, m=1}^{\infty} m^q \mu(m) k^q a_k \sum_{x \in \mathcal{X}_1} x_j^q e^{-km(\alpha,x)}$$

$$= \sum_{k, m=1}^{\infty} m^q \mu(m) k^q a_k \sum_{x_1=1}^{\infty} x_j^q e^{-km\alpha_1 x_1} \sum_{x_2=0}^{\infty} e^{-km\alpha_2 x_2}$$

$$= \sum_{k, m=1}^{\infty} m^q \mu(m) k^q a_k S_{q+1}(km\alpha_1) (1 - e^{-km\alpha_2})^{-1}, \tag{5.17}$$

where in the last line we used notation (5.12). Lemma 5.10 and inequality (3.17) show that the general term in series (5.17) is bounded in absolute value, uniformly in $k$ and $m$, as follows

$$m^q k^q |a_k| \frac{C_{q+1} e^{-km\alpha_1}}{(1 - e^{-km\alpha_1})^{q+1}(1 - e^{-km\alpha_2})} = O(1) |a_k| k^2 m^2 \alpha_1^{q+1} \alpha_2$$

(cf. (3.18)). Hence, expanding $S_{q+1}(km\alpha_1)$ by Lemma 5.9 we can pass to the limit in (5.17) as $\alpha_1, \alpha_2 \to 0$ to obtain

$$\alpha_1^{q+1} \alpha_2 \kappa_q[\xi_1] = \sum_{j=1}^{q+1} c_{j,q+1} \sum_{k, m=1}^{\infty} m^q \mu(m) k^q a_k \frac{\alpha_1^{q+1} \alpha_2 e^{-km\alpha_1}}{(1 - e^{-km\alpha_1})^{j} (1 - e^{-km\alpha_2})}$$

$$\to c_{q+1,q+1} \sum_{k, m=1}^{\infty} \frac{\mu(m) a_k}{k^2 m^2} = q! \frac{A(2)}{\zeta(2)} = q! \kappa^3. \tag{5.18}$$

Finally, according to (3.2) we have $\alpha_1^{q+1} \alpha_2 \asymp |n|^{-(q+2)/3}$, and hence (5.18) implies (5.16). \qed

There is a similar upper asymptotic bound for the mixed cumulants.

**Lemma 5.12.** For each $q \in \mathbb{N}$ and any $t_1, t_2 \in \mathbb{R}$,

$$\kappa_q[t_1 \xi_1 + t_2 \xi_2] = O(|n|^{(q+2)/3}), \quad n \to \infty. \tag{5.19}$$

**Proof.** Similarly to representation (5.15), we have

$$\kappa_q[t_1 \xi_1 + t_2 \xi_2] = \sum_{x \in \mathcal{X}} (t_1 x_1 + t_2 x_2)^q \kappa_q(x).$$
Hence, by the inequality \((a + b)^q \leq 2^{q-1}(a^q + b^q)\) (already used in the proof of Lemma 5.8), we obtain
\[
|\kappa_q[t_1 \xi_1 + t_2 \xi_2]| \leq \sum_{x \in \mathcal{X}} |t_1 x_1 + t_2 x_2|^q |\kappa_q(x)| \\
\leq \sum_{x \in \mathcal{X}} \left( |t_1|^q x_1^q + |t_2|^q x_2^q \right) \sum_{k=1}^\infty k^q |a_k| e^{-k(\alpha, x)}. \tag{5.19}
\]

Repeating the arguments used in the proof of Lemma 5.11 we see that the right-hand side of (5.19) admits an asymptotic bound \(O(|n|^{(q+2)/3})\), and the lemma is proved. \(\square\)

In view of relation (2.22), Lemma 5.11 immediately yields the following corollary.

**Lemma 5.13.** For each \(q \in \mathbb{N}\) and \(j = 1, 2\),
\[
E_z(\xi_j^q) \asymp |n|^{(q+2)/3}, \quad n \to \infty.
\]

We also have a similar upper estimate for the centered moments.

**Lemma 5.14.** For each \(q \in \mathbb{N}\) and \(j = 1, 2\),
\[
E_z(\xi_j - E_z(\xi_j))^q = O(|n|^{(q+2)/3}), \quad n \to \infty.
\]

**Proof.** Applying an inequality similar to (5.11), we obtain
\[
E_z(\xi_j - E_z(\xi_j))^q \leq 2^{q-1} E_z(\xi_j^q) \asymp |n|^{(q+2)/3},
\]
according to Lemma 5.13. \(\square\)

**Lemma 5.15.** For each \(q \in \mathbb{N}\),
\[
\sum_{x \in \mathcal{X}} |x|^q m_q(x) = O(|n|^{(q+2)/3}), \quad n \to \infty. \tag{5.20}
\]

**Proof.** Using the elementary inequalities \(|x|^q \leq (x_1 + x_2)^q \leq 2^{q-1}(x_1^q + x_2^q)\) and recalling definition (5.10), observe that
\[
\sum_{x \in \mathcal{X}} |x|^q m_q(x) \leq 2^{q-1} \left( E_z(\xi_1^q) + E_z(\xi_2^q) \right),
\]
whence (5.20) readily follows by Lemma 5.13. \(\square\)

**Lemma 5.16.** As \(n \to \infty\),
\[
\sum_{x \in \mathcal{X}} |x|^3 \mu_3(x) \asymp |n|^{5/3},
\]
where \(\mu_3(x) := E_z|\nu_0(x)|^3\) (see (5.10)).

**Proof.** An upper bound \(O(|n|^{5/3})\) follows from inequality (5.11) and Lemma 5.15. On the other hand, we have
\[
\mu_3(x) = E_z|\nu(x) - m_1(x)|^3 \geq E_z(\nu(x) - m_1(x))^3 = \kappa_3(x),
\]
using that the third-order centered moment coincides with the third-order cumulant. Hence, on account of formula (5.15),
\[
\sum_{x \in \mathcal{X}} |x|^3 \mu_3(x) \geq \sum_{x \in \mathcal{X}} x_1^3 \kappa_3(x) = \kappa_3[x_1] \asymp |n|^{5/3},
\]
according to Lemma 5.11 (with \(q = 3\)). \(\square\)
Let us introduce the Lyapunov coefficient
\[ L_z := \|V_z\|_3 \sum_{x \in X} |x|^3 \mu_3(x). \] (5.21)

The next asymptotic estimate is an immediate consequence of Lemmas 5.7 and 5.16.

**Lemma 5.17.** As \( n \to \infty \), one has \( L_z \asymp |n|^{-1/3} \).

### 6. Local limit theorem

The role of a local limit theorem in our approach is to yield the asymptotics of the probability \( Q_z\{\xi = n\} \equiv Q_z(\Pi_n) \) appearing in the representation of the measure \( P_n \) as a conditional distribution, \( P_n(\cdot) = Q_z(\cdot | \Pi_n) = Q_z(\cdot)/Q_z(\Pi_n) \).

#### 6.1. Statement of the theorem

As before, we denote \( a_z := E_z(\xi) \), \( K_z := \text{Cov}(\xi, \xi) \), \( V_z := K_z^{-1/2} \) (see Section 5.1). Consider the probability density function of a two-dimensional normal distribution \( \mathcal{N}(a_z, K_z) \) (with mean \( a_z \) and covariance matrix \( K_z \)), given by
\[ f_{a_z, K_z}(x) = \frac{1}{2\pi \sqrt{\det K_z}} \exp \left( -\frac{1}{2} |(x - a_z)V_z|^2 \right), \quad x \in \mathbb{R}^2. \] (6.1)

**Theorem 6.1.** Assume that \( A^+(2) < \infty \) and suppose that Assumption 2.1 holds. Then, uniformly in \( m \in \mathbb{Z}^2_+ \),
\[ Q_z\{\xi = m\} = f_{a_z, K_z}(m) + O\left(|n|^{-5/3}\right), \quad n \to \infty. \] (6.2)

**Corollary 6.2.** Under the conditions of Theorem 6.1
\[ Q_z\{\xi = n\} \asymp (n_1n_2)^{-2/3}, \quad n \to \infty. \] (6.3)

Let us point out that the cumulant asymptotics obtained in Section 5.3 (see Lemma 5.12), together with the asymptotics of the first two moments of \( \xi \) (Theorems 3.2 and 5.1) immediately lead to a central limit theorem consistent with Theorem 6.1.

**Theorem 6.3** (CLT). The distribution of the random vector \( (\xi - a_z)V_z \) converges weakly, as \( n \to \infty \), to the standard two-dimensional normal distribution \( \mathcal{N}(0, I) \).

#### 6.2. Estimates of the characteristic functions

Before proving Theorem 6.1, we have to make some technical preparations. Recall from Section 2.1 that, with respect to the measure \( Q_z \), the random variables \( \{\nu(x)\}_{x \in X} \) are independent and have characteristic functions \( \varphi_\nu(\lambda) \). Hence, the characteristic function \( \varphi_\xi(\lambda) := E_z(e^{i\langle \lambda, \xi \rangle}) \) of the vector sum \( \xi = \sum_{x \in X} x \nu(x) \) is given by
\[ \varphi_\xi(\lambda) = \prod_{x \in X} \varphi_\nu(\langle \lambda, x \rangle; x) = \prod_{x \in X} \frac{\beta(z^x e^{i\langle \lambda, x \rangle})}{\beta(z^x)}, \quad \lambda \in \mathbb{R}^2. \] (6.4)

Let us start with a general absolute estimate for the characteristic function of a centered random variable (for a proof, see [6, Lemma 7.10]).
Lemma 6.4. Let \( \varphi_{v_0}(t; x) := E_z(e^{i t v_0(x)}) \) be the characteristic function of the random variable \( v_0(x) := \nu(x) - E_z[\nu(x)] \). Then

\[
|\varphi_{v_0}(t; x)| \leq \exp\left\{-\frac{1}{2}\mu_2(x)t^2 + \frac{1}{3}\mu_3(x)|t|^3\right\}, \quad t \in \mathbb{R},
\]

where \( \mu_q(x) := E_z[|v_0(x)|^q] \).

The next lemma provides two estimates (proved in [6 Lemmas 7.11 and 7.12]) for the characteristic function \( \varphi_{\xi_0}(\lambda) := E_z(e^{i\lambda \xi_0}) \) of the centered vector \( \xi_0 := \xi - a_z = \sum_{x \in \mathcal{X}} x v_0(x) \). Recall that the Lyapunov coefficient \( L_z \) is defined in (5.21), and \( V_z := K_z^{-1/2} \).

Lemma 6.5. (a) For all \( \lambda \in \mathbb{R}^2 \),

\[
|\varphi_{\xi_0}(\lambda V_z)| \leq \exp\left\{-\frac{1}{2}|\lambda|^2 + \frac{1}{3}L_z|\lambda|^3\right\}.
\]

(b) If \( |\lambda| \leq L_z^{-1} \) then

\[
|\varphi_{\xi_0}(\lambda V_z) - e^{-|\lambda|^2/2}| \leq 16L_z|\lambda|^3 e^{-|\lambda|^2/6}.
\]

Let us also prove the following global bound (cf. [6 Lemma 7.13]).

Lemma 6.6. As in Theorem 6.1, suppose that Assumption 2.1 is satisfied. Then

\[
|\varphi_{\xi_0}(\lambda)| \leq \exp\{-C_0 J_\alpha(\lambda)\}, \quad \lambda \in \mathbb{R}^2,
\]

where \( C_0 \) is a positive constant and

\[
J_\alpha(\lambda) := \sum_{x \in \mathcal{X}} e^{-\langle \alpha, x \rangle}(1 - \cos(\lambda, x)) \geq 0, \quad \lambda \in \mathbb{R}^2.
\]

Proof. From (6.4) we have

\[
|\varphi_{\xi_0}(\lambda)| = |\varphi_\xi(\lambda)| = \exp\left\{\sum_{x \in \mathcal{X}} \ln|\varphi_\nu((\lambda, x); x)|\right\}.
\]

Recall that under Assumption 2.1 we have, according to (2.25) and (2.26),

\[
\ln|\varphi_\nu(t; x)| = \frac{1}{2} \ln \frac{\beta(z^\tau e^{i t})\beta(z^\tau e^{-i t})}{\beta(z^\tau)^2} \leq -C_1 b_1 z^\tau (1 - \cos t),
\]

with \( C_1 > 0 \) and \( b_1 > 0 \). Utilizing this estimate under the sum in (6.10) (with \( t = \langle \lambda, x \rangle \)) and recalling notation (3.2), we arrive at (6.8) with \( C_0 := C_1 b_1 > 0 \).

6.3. Proof of Theorem 6.1 and Corollary 6.2

Let us first deduce the corollary from the theorem.

Proof of Corollary 6.2 According to Theorem 4.1, \( a_z = E_z(\xi) = n + O(\|n\|^{2/3}) \). Together with Lemma 5.7 this implies

\[
|(n - a_z) V_z| \leq |n - a_z| \cdot \|V_z\| = O(1).
\]

Hence, by Lemma 5.2 we get

\[
f_{a_z, F_z}(n) = \frac{1}{2\pi \sqrt{\det K_z}} e^{-|(n - a_z) V_z|^2/2} \asymp (n_1 n_2)^{-2/3},
\]

and (6.3) now readily follows from (6.2).
Proof of Theorem 6.1. By definition, the characteristic function of the random vector $\xi_0 = \xi - a_x$ is given by the Fourier series

$$\varphi_{\xi_0}(\lambda) = \sum_{m \in \mathbb{Z}_+^2} Q_z \{\xi = m\} e^{i(\lambda, m - a_x)}, \quad \lambda \in \mathbb{R}^2,$$

hence the Fourier coefficients are expressed as

$$Q_z \{\xi = m\} = \frac{1}{4\pi^2} \int_{T^2} e^{-i(\lambda, m - a_x)} \varphi_{\xi_0}(\lambda) \, d\lambda, \quad m \in \mathbb{Z}_+^2,$$  \hspace{1cm} (6.11)

where $T^2 := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 : |\lambda_1| \leq \pi, |\lambda_2| \leq \pi\}$. On the other hand, the characteristic function corresponding to the normal probability density $f_{a_x, K_x}(x)$ (see (6.1)) is given by

$$\varphi_{a_x, K_x}(\lambda) = e^{i(\lambda, a_x) - |\lambda V_z^{-1}|^2/2}, \quad \lambda \in \mathbb{R}^2,$$

so by the Fourier inversion formula

$$f_{a_x, K_x}(m) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i(\lambda, m - a_x) - |\lambda V_z^{-1}|^2/2} \, d\lambda, \quad m \in \mathbb{Z}_+^2.$$  \hspace{1cm} (6.12)

Note that if $|\lambda V_z^{-1}| \leq L_z^{-1}$ then, according to Lemmas 5.7 and 5.17

$$|\lambda| \leq |\lambda V_z^{-1}| \cdot \|V_z\| \leq L_z^{-1} \|V_z\| = O(|n|^{-1/3}) = o(1),$$

which of course implies that $\lambda \in T^2$. Using this observation and subtracting (6.12) from (6.11), we get, uniformly in $m \in \mathbb{Z}_+^2$,

$$\left| Q_z \{\xi = m\} - f_{a_x, K_x}(m) \right| \leq I_1 + I_2 + I_3,$$  \hspace{1cm} (6.13)

where

$$I_1 := \frac{1}{4\pi^2} \int_{\{\lambda : |\lambda V_z^{-1}| \leq L_z^{-1}\}} \left| \varphi_{\xi_0}(\lambda) - e^{-|\lambda V_z^{-1}|^2/2} \right| \, d\lambda,$$

$$I_2 := \frac{1}{4\pi^2} \int_{\{\lambda : |\lambda V_z^{-1}| > L_z^{-1}\}} e^{-|\lambda V_z^{-1}|^2/2} \, d\lambda,$$

$$I_3 := \frac{1}{4\pi^2} \int_{T^2 \cap \{\lambda : |\lambda V_z^{-1}| > L_z^{-1}\}} |\varphi_{\xi_0}(\lambda)| \, d\lambda.$$

By the substitution $\lambda = y V_z$, the integral $I_1$ is reduced to

$$I_1 = \left| \frac{\det V_z}{4\pi^2} \int_{|y| \leq L_z^{-1}} \left| \varphi_{\xi_0}(y V_z) - e^{-|y|^2/2} \right| \, dy \right|$$

$$= O(1) \left( \det K_x \right)^{-1/2} L_z \int_{\mathbb{R}^2} |y|^{3} e^{-|y|^2/6} \, dy = O(|n|^{-5/3}),$$  \hspace{1cm} (6.14)

on account of Lemmas 5.2, 5.17 and 6.5 b). Similarly, again putting $\lambda = y V_z$ and passing to the polar coordinates, we get, due to Lemmas 5.2 and 5.17

$$I_2 = \frac{|\det V_z|}{2\pi} \int_{L_z^{-1}}^{\infty} r e^{-r^2/2} \, dr = O\left( |n|^{-4/3} \right) e^{-L_z^{-2}/2} = o\left( |n|^{-5/3} \right).$$  \hspace{1cm} (6.15)

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Finally, let us turn to $I_3$. Using Lemma 6.6, we obtain
\begin{equation}
I_3 = O(1) \int_{T^2 \cap \{|\lambda V^{-1}_z| > L_z^{-1}\}} e^{-C_0 J_\alpha(\lambda)} \, d\lambda,
\end{equation}
where $J_\alpha(\lambda)$ is given by (6.9). $|\lambda V^{-1}_z| > L_z^{-1}$ then $|\lambda| > \eta|\alpha|$ for a suitable (small enough) constant $\eta > 0$, which implies that $\max\{|\lambda_1|/\alpha_1, |\lambda_2|/\alpha_2\} > \eta$, for otherwise from (3.2) and Lemmas 5.6 and 5.17 it would follow
\[1 < L_z |\lambda V^{-1}_z| \leq L_z \eta |\alpha| \cdot \|K_z\|^{1/2} = O(\eta) \to 0 \quad \text{as} \quad \eta \downarrow 0,
\]
which is a contradiction. Hence, estimate (6.16) is reduced to
\begin{equation}
I_3 = O(1) \left( \int_{|\lambda_1| > \eta \alpha_1} + \int_{|\lambda_2| > \eta \alpha_2} \right) e^{-C_0 J_\alpha(\lambda)} \, d\lambda.
\end{equation}

To estimate the first integral in (6.17), by keeping in summation (6.9) only pairs of the form $x = (x_1, 1)$, $x_1 \in \mathbb{Z}_+$, we obtain
\begin{align*}
J_\alpha(\lambda) &\geq \sum_{x_1=0}^{\infty} e^{-\alpha x_1} \left( 1 - \Re e^{i(\lambda_1 x_1 + \lambda_2)} \right) = \frac{1}{1 - e^{-\alpha_1}} - \Re \left( \frac{e^{i\lambda_2}}{1 - e^{-\alpha_1 + i\lambda_1}} \right) \\
&\geq \frac{1}{1 - e^{-\alpha_1}} - \frac{1}{|1 - e^{-\alpha_1 + i\lambda_1}|},
\end{align*}
because $\Re u \leq |u|$ for any $u \in \mathbb{C}$. Since $\eta \alpha_1 \leq |\lambda_1| \leq \pi$, we have
\[|1 - e^{-\alpha_1 + i\lambda_1}| \geq |1 - e^{-\alpha_1 + i\eta \alpha_1}| \sim \alpha_1 (1 + \eta^2)^{1/2} \quad (\alpha_1 \to 0).
\]
Substituting this estimate into (6.18), we conclude that $J_\alpha(\lambda)$ is asymptotically bounded from below by $C(\eta) \alpha_1^{-1} \sim |n|^{1/3}$ (with some constant $C(\eta) > 0$), uniformly in $\lambda$ such that $\eta \alpha_1 \leq |\lambda_1| \leq \pi$. Thus, the first integral in (6.17) is bounded by
\[O(1) \exp(-\text{const} \cdot |n|^{1/3}) = o(|n|^{-5/3}).
\]

Similarly, the second integral in (6.17) is estimated by reducing the summation in (6.9) to that over $x = (1, x_2)$ only. As a result, $I_2 = o(|n|^{-5/3})$. Substituting this estimate, together with (6.14) and (6.15), into (6.13) we get (6.2), and so the theorem is proved.

7. Proof of the limit shape results

Let us first establish the universality of the limit shape under the measure $Q_z$.

Theorem 7.1. For each $\varepsilon > 0$,
\[\lim_{n \to \infty} Q_z \left\{ \sup_{0 \leq t \leq \infty} |n_j^{-1} \xi_j(t) - g^*_j(t)| \leq \varepsilon \right\} = 1 \quad (j = 1, 2).
\]
Proof. By Theorems 3.3 and 3.4 the expectation of the random process \( n_j^{-1} \xi_j(t) \) uniformly converges to \( g_j^*(t) \) as \( n \to \infty \). Therefore, we only need to check that, for each \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} Q \left\{ \sup_{0 \leq t \leq \infty} n_j^{-1} \xi_j(t) - E_z[\xi_j(t)] > \varepsilon \right\} = 0.
\]

Note that the random process \( \xi_{0j}(t) := \xi_j(t) - E_z[\xi_j(t)] \) is a martingale with respect to the filtration \( \mathcal{F}_t := \sigma\{\nu(x), x \in X(t)\}, t \in [0, \infty] \). From the definition of \( \xi_j(t) \) (see (3.22)), it is also clear that \( \xi_{0j}(t) \) is a càdlàg process (i.e., its paths are everywhere right-continuous and have left limits). Therefore, applying the Kolmogorov–Doob submartingale inequality (see, e.g., [23, Corollary 2.1, p. 14]) and using Theorem 5.1, we obtain

\[
Q \left\{ \sup_{0 \leq t \leq \infty} |\xi_{0j}(t)| > n_j \varepsilon \right\} \leq \frac{\text{Var}(\xi_j)}{(\varepsilon n_j)^2} = O(|n|^{-2/3}) \to 0,
\]

and the theorem is proved.

We are finally ready to prove our main result about the universality of the limit shape under the measures \( P_n \) (cf. Theorem 1.1).

**Theorem 7.2.** For any \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} P_n \left\{ \sup_{0 \leq t \leq \infty} n_j^{-1} \xi_j(t) - g_j^*(t) \leq \varepsilon \right\} = 1 \quad (j = 1, 2).
\]

**Proof.** Like in the proof of Theorem 7.1, the claim is reduced to the limit

\[
\lim_{n \to \infty} P_n \left\{ \sup_{0 \leq t \leq \infty} |\xi_{0j}(t)| > \varepsilon n_j \right\} = 0, \quad (7.1)
\]

where \( \xi_{0j}(t) := \xi_j(t) - E_z[\xi_j(t)] \). Using (2.8) we get

\[
P_n \left\{ \sup_{0 \leq t \leq \infty} |\xi_{0j}(t)| > \varepsilon n_j \right\} \leq \frac{Q \left\{ \sup_{0 \leq t \leq \infty} |\xi_{0j}(t)| > \varepsilon n_j \right\}}{Q \{ \xi = n \}}. \quad (7.2)
\]

By the Kolmogorov–Doob submartingale inequality and Lemma 5.14 (with \( q = 4 \)), we have

\[
Q \left\{ \sup_{0 \leq t \leq \infty} |\xi_{0j}(t)| > \varepsilon n_j \right\} \leq \frac{E_z[\xi_j - E_z[\xi_j]]^4}{(\varepsilon n_j)^4} = O(|n|^{-2}). \quad (7.3)
\]

On the other hand, by Corollary 6.2

\[
Q \{ \xi = n \} \asymp (n_1 n_2)^{-2/3} \asymp |n|^{-4/3}. \quad (7.4)
\]

Combining (7.3) and (7.4), we conclude that the right-hand side of (7.2) is dominated by a quantity of order of \( O(|n|^{-2/3}) \to 0 \), and so the limit in (7.1) follows.

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