Consistent refinement of Bethe strings for spin and electron models and a new non-Bethe solution

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Abstract

Well known Bethe strings in spin chains, δ-electron gas, Hubbard and $t-J$ models are shown to be imprecise, while their consistent refinement along with a new non-Bethe $r$-string are discovered. Connection with the earlier results is established and the string hypothesis problem is discussed in the light of the present findings.

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A new direction in physics, namely the theory of exactly solvable quantum systems opened up following the classic work of Bethe [1]. Though Bethe's objective was to solve exactly the eigenvalue problem of Heisenberg spin chain only, this XXX spin-$\frac{1}{2}$ model proved to be a generic one and almost the same Bethe ansatz or its nested extensions to include particles with more degrees of freedom, e.g. electrons, were found to be applicable to a number of important models in condensed matter physics as well as in quantum field theory [2]. The celebrated Bethe ansatz assumes a specific form for the eigenvectors having parametric dependence on rapidity variables, which in turn are to be determined from specified equations known as the Bethe ansatz equations (BAE). Bethe proposed also a special type of complex solutions for BAE at the thermodynamic limit in the form of a string, known as the Bethe string. He showed further that such strings form a complete set by proving that their total number together with the real roots give exactly the required number of Bethe ansatz states. These Bethe string solutions played important role in all the Bethe ansatz solvable fundamental integrable models like spin chains [3, 4], δ-function electron gas [5, 6], Hubbard model [7, 8], supersymmetric $t-J$ model [9, 10] etc. The string solutions correspond to the bound states and describe excited states of the models. A popular conjecture, known as the string hypothesis, assumes that all complex solutions

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of BAE at the thermodynamic limit must be of the Bethe string type. Though this conjecture was never really proved, it was used successfully to all the above models for calculating important physical quantities like energy, entropy, pressure, magnetization etc. through thermodynamic Bethe ansatz (TBA). However in early eighties some criticisms started appearing in this success story of the Bethe string, when in a series of studies the string hypothesis apparently failed and instead of the Bethe strings non-Bethe complex solutions were reported to appear as close-roots in the form of quartets and 2-strings and as wide-roots with no restriction on their forms [11, 12, 13, 14]. In a subsequent paper [15] an upper bound \( Re(\lambda_l) \ll \sqrt{N} \) on Bethe string roots was identified, beyond which significant deformation of strings was reported [16].

However in spite of the above major criticisms of the string hypothesis, strangely enough no significant attention was given in subsequent years to understand or resolve this problem. Nevertheless, as far as we know, the consistency of the Bethe string form itself has never been questioned before. Therefore, it is even more surprising for us to find that, the well known Bethe string for all the fundamental models mentioned above, widely quoted and used for so many years, is in fact not a precise solution to the BAE even within its validity range and for large \( N \). After noticing this we are able also to obtain here the precise and consistent form of the Bethe string by refining its correction terms, which therefore should replace the accepted string forms presented in a series of celebrated works [1, 3, 4, 5, 6, 7, 8, 9, 10] dealing with the fundamental spin and electron models cited above. Moreover we have discovered a new non-Bethe string with arbitrary \( r \) roots, compatible with the earlier observations. Based on the explicit forms of two kinds of strings one could compare therefore their singularity structures appearing in the BAE, the energies of the states formed by them and hopefully all other thermodynamic quantities, which should contribute toward the solution of the longstanding problem of string hypothesis.

Let us start by checking first the known Bethe string as solutions to the respective BAE at the thermodynamic limit. In all the spin and electron models mentioned above, the same BAE are obtained by both coordinate and the algebraic Bethe ansätze, though in the former they are generated as a consequence of the periodic boundary condition on the wave functions [1], while in the later as the analyticity condition for eigenvalues of the transfer matrix [17]. XXX spin-\( \frac{1}{2} \) chain being a generic model, we intend to examine its solutions in more detail and conclude about the other models by analogy.

**XXX-spin chain:** The BAE for this model may be given as

\[
V_N^{\lambda}(l) = \prod_{n \neq l} V_1(\lambda_l - \lambda_n), \quad l = 1, \ldots, r, \quad V_\alpha(\lambda) \equiv \frac{\lambda + i\alpha}{\lambda - i\alpha}.
\]  

(1)

The well known Bethe string (BS) solution to the above BAE, valid for large \( N \), is usually given by

\[
\lambda_l^{(bs)} = \lambda_0 + i\eta(l - \frac{r + 1}{2}) + iO(e^{-\alpha N}), \quad l = 1, \ldots, r, \quad \alpha > 0,
\]  

(2)
with \( \eta = 1 \). This form of solution suggests that, exponentially small corrections: \( O(e^{-\alpha N}) \) are needed for large values of \( N \), which presumably are of the same order due to the same \( \alpha \) parameter appearing for all the roots. At \( N \to \infty \) therefore all corrections in \( (3) \) must vanish making it an apparently exact solution \([7, 13]\). Note that conjugate roots are included in \( (3) \) with \( \lambda_1^* = \lambda_{r+1-l}, \ l = 1, \ldots, s \), where \( s = \frac{r}{2} \) for \( r \) even and \( s = \frac{r-1}{2} \) for \( r \) odd.

To follow the idea of this solution we note that, any complex solution with \( Im(\lambda) < 0 \) always makes \( |V_2(\lambda)| < 1 \), which can be proved for the Bethe string with \( l \leq s \) by inserting the explicit form \( (2) \) with arbitrary \( r \):

\[
|V_2(\lambda_l^{(bs)})| = \frac{\lambda_0 + i(l - \frac{r}{2})}{\lambda_0 + i(l - \frac{r+2}{2})} = (1 + \kappa_l)^{-\frac{1}{2}} = e^{-\delta l} < 1,
\]

due to \( \kappa_l = \frac{r+1-2l}{\lambda_0 + i(l - \frac{r}{2})} > 0 \). Consequently, for large values of \( N \) the LHS of BAE \( (1) \): \( V_2^N(\lambda_l^{(bs)}) \sim O(e^{-\eta N}) \) must always vanish exponentially. Therefore for the Bethe string to be a consistent solution, the RHS must also contain an exponentially vanishing factor of the same order of smallness. By direct check one sees easily that in the simplest cases of \( r = 2, 3 \) this condition is fulfilled and hence, the Bethe string satisfies \( (1) \) for large \( N \), as shown explicitly in \([7, 13, 18]\).

However, though at this point it might seem natural to assume that, the same argument must go through for any arbitrary \( r \) \([7, 18]\), we will see by direct insertion that this analogy fails here and the RHS as such becomes inconsistent for the string form \( (2) \) starting from \( r = 4 \). One finds that among the factors in RHS, one with the next higher root gives for the Bethe string \( (2) \): \( V_1(\lambda_l^{(bs)} - \lambda_{l+1}^{(bs)}) = V_1(-i + O(e^{-\alpha N})) \sim O(e^{-\delta N}) \), an exponentially small term. However one can also notice immediately that for \( l > 1 \) and \( r \geq 4 \), another singular factor appears from the adjacent lower root yielding an exponentially large term having the same order: \( V_1(\lambda_l^{(bs)} - \lambda_{l-1}^{(bs)}) = V_1(+i + O(e^{-\alpha N})) \sim O(e^{\alpha N}) \), while the rest of the factors coming from other roots give only finite contributions. Multiplying all these factors we finally get the RHS of \( (1) \) as \( O(e^{-\alpha N})O(e^{\alpha N}) \sim O(1) \), which however has a finite limit contradicting the vanishing LHS. Therefore we see that for all roots \( \lambda_l^{(bs)} \), with \( l = 2, \ldots, r-1 \) and \( r \geq 4 \), having two adjacent neighbors \( \lambda_l^{(bs)} \), the well known string form \( (2) \), as such, is not a consistent solution of BAE \( (1) \), except only for the end-roots \( l = 1, r \) and the real root, which cover also the cases \( r = 2, 3 \). For example for \( r = 4 \), as worked out in our preliminary report \([19]\), the above string form holds for the end-roots with \( l = 1, 4 \), but not for the roots with \( l = 2, 3 \).

\textit{XXZ-spin chain:} The BAE take the same form as in XXX: \( \tilde{V}_2^N(\lambda_l) = \prod_{n \neq l} \tilde{V}_n(\lambda_l - \lambda_n) \), but through a redefined function \( \tilde{V}_\alpha(\lambda) \equiv \frac{\sin(\lambda + i\alpha)}{\sin(\lambda - i\alpha)} \) and its well known string solution for \( \lambda_l^{(bs)} \) can be given again by \( (2) \) \([8]\). Skipping the details we mention only that, since the functions \( \sinh x \) and \( x \) behave similarly at small \( x \), the above reasoning for the XXX-string goes parallelly in this case and one encounters a similar mismatch for its standard string form.

\textit{Repulsive \( \delta \)-function electron gas:} For this and all other electron models considered below an ad-
ditional set of rapidity variables is needed. The BAE is therefore extended to include another set of equations \( \prod_{j}^{N_e} V_{\pm}^{(l)}(\lambda_l - k_j) = \prod_{n\neq l} V_{\pm}(\lambda_l - \lambda_n) \), though its structure is very similar to (1) and the string solution for \( \lambda_l \) is given in the same form (2) with \( \eta = c > 0 \) and \( \{k_j\} \) real \( \mathbb{R} \). Since here \( k_j \)'s are real, for complex \( \lambda_l^{(bs)} \) with \( l \leq s \), the LHS becomes a product of terms each being \( \beta \). Therefore for the string solutions at large \( N_e \), one gets a vanishing LHS, while the RHS having exactly the same form as in (3) remains finite as argued above.

**Hubbard model:** The additional BAE exhibit very similar structure to the above electron model giving \( \prod_{j}^{N_e} V_{\pm}^{(l)}(\lambda_l - \sin k_j) = \prod_{n\neq l} V_{\pm}^{(l)}(\lambda_l - \lambda_n) \). However in this case along with the string solutions for \( \{\lambda_l, \lambda_n\} \) in the form (2) with \( \eta = \frac{\pi}{2} \), \( M' \)-pairs of solutions from \( \{k_j\} \) can also be of string type satisfying \( k_n^+ - \lambda_n^+ = \mp i\frac{\pi}{4}, n = 1, \ldots, M' \). Note that despite of \( k_n^\pm \) being complex, due to the presence of their conjugates also in the factors of LHS, any complex string root for \( \lambda_l, l \leq s \) will make the factors \( \beta \). Therefore for \( N_e \to \infty \) the LHS \( \to 0 \), while the RHS due to its same form as in the above discussed cases, gives finite contribution for \( r \geq 4 \).

**Supersymmetric \( t-J \) model:** For a particular (BBF) type of excitations [10], one set of BAE takes the form \( V_{\pm}^{(l)}(\lambda_l - \gamma_{\beta}) = \prod_{n\neq l} V_{\pm}(\lambda_l - \lambda_n) \). The string solutions for \( \lambda_l \) are again given by (2) with all \( \gamma_{\beta} \)'s real [10]. Noticing the first factor in the LHS to be the same as in XXX case and \( \{\gamma_{\beta}\} \) being real, we conclude as before that LHS \( \to 0 \) for \( N \to \infty \). The RHS however being same again as in XXX gives non-vanishing terms. starting from \( r \geq 4 \).

As we see from the above arguments, the mismatch of the well known Bethe string form (3) is due to its imprecise correction terms. We therefore propose the consistent and precise form of the Bethe string (PBS) as

\[
\lambda_l^{(pbs)} = \lambda_0 + i(l - \frac{r+1}{2}) + iO(e^{-\alpha_l N}), \quad l = 1, \ldots, s \quad \text{and} \quad \lambda_{r+1-l} = \lambda_l^+, \tag{4}
\]

provided the exponential orders in its correction terms are fine-tuned as a strictly growing sequence

\[
0 < \alpha_1 < \cdots < \alpha_l < \alpha_{l+1} < \cdots < \alpha_s, \quad \text{with} \quad \alpha_l - \alpha_{l-1} = v_l = \frac{1}{2} \ln(1 + \kappa_l) > 0, \tag{5}
\]

with \( \kappa_l \) as defined in (3). Note that the term \( O(e^{-\alpha_l N}) \) in (4) stands for the terms like \( c_l e^{-\alpha_l N} \) with \( l \)-dependent multiplicative constants \( c_l \). The essential point in proving the validity of this refined Bethe string is that unlike the known form (3) the adjacent roots contribute now zeros and poles of different orders of smallness in the RHS of BAE, since \( V_1(\lambda_l^{(pbs)} - \lambda_{l+1}^{(pbs)}) = O(e^{-\alpha_l N}) - O(e^{-\alpha_{l+1} N}) \approx O(e^{-\alpha_l N}), \) while \( V_1(\lambda_l^{(pbs)} - \lambda_l^{(pbs)}) = \left( O(e^{-\alpha_l N}) - O(e^{-\alpha_{l-1}} N) \right)^{-1} \approx O(e^{\alpha_{l-1} N}), \) using the strict inequality (3). Therefore the RHS becomes \( O(e^{-\alpha_l N})O(e^{\alpha_{l-1} N}) \approx O(e^{-\eta N}), \) i.e. consistent with the LHS, which has the same vanishing limit as before. It is important to note that, contrary to its known form (3), the correction terms present in the proper Bethe string (4) are rather complicated with simultaneous involvement of small terms of all different orders and none of them can be neglected from the beginning, even at \( N \to \infty \).
We emphasize again that, XXX-spin chain being a generic case, the same consistent Bethe string \( \mathbf{1} \) with refinement \( \mathbf{2} \), will be equally valid for all fundamental integrable models discussed above and should therefore replace the corresponding well known and widely used Bethe strings appearing in related works cited above. Curiously however this seems not to affect string based physical results obtained through TBA. The TBA method \( \mathbf{3}, \mathbf{4} \) needs apparently not the solution of individual BAE, but the solution of a product of several BAEs with different roots for the same string: 

\[
\prod_\alpha V_1^N(\lambda_\alpha^{(1)}(bs)) = \prod_{\alpha<j} V_1(\lambda_\alpha^{(1)}(bs) - \lambda_j^{(2)}(bs)).
\]

Note that since the complex conjugate of each root is also present in the product, its LHS remains finite even for large \( N \) and therefore the standard string solutions hold for such product-BAE. Therefore inspite of being imprecise in form for the individual BAE, the known Bethe string fortunately is capable of producing correct TBA results.

Nevertheless the precise form of the Bethe string we find here is important not only as a consistent form of solution for the BAE but also for its possible comparison with a non-Bethe solution. For comparison one needs also an explicit non-Bethe string with arbitrary form of solution for the BAE but also for its possible comparison with a non-Bethe solution. For the \( \prod \) individual BAE, the known Bethe string fortunately is capable of producing correct TBA results.

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\]
$O(e^{-\sum_a^M \phi_{ia}})$, where \( \phi_{ia} = \frac{1}{2} \ln(1 + \kappa_{ia}) \), with \( \kappa_{ia} = \frac{1}{D}(r + 1 - 2l) \), \( D = \frac{1}{r^2}((\lambda_0 - \nu_a)^2 + \frac{1}{r^2}(2l - 1)^2) \).

Therefore for large \( M \) there appears now a possibility to compensate for the singular term arising in the LHS of the product BAE mentioned above and thus to resolve the controversy for the existence of strings like \( \blacksquare \). The exact values of \( \nu_a \) should in fact be determined self-consistently from the BAE. However this is difficult to achieve in practice for large values of \( M, N \), even numerically and such strings can exist only for large \( N, M \). Therefore we make here only some estimates to show the right trend. Since the product of the BAE’s for the \( l \)-th root with its partner leading to a contradiction for the isolated NBS is our main concern, we observe first that each of the additional terms in its RHS, e.g. \( e^{-\nu_a - \phi_{(i+s)a}} \) exhibits diminishing (increasing) trend also for \( l < l* \) (\( l > l* \)), matching exactly with the behaviour of the LHS established above. This shows at least qualitatively that for the NBS the coupling with real roots can indeed yield necessary singular terms in the RHS that might match such terms appearing in the LHS. For having further quantitative estimates we have checked numerically the ratios of the singular terms arising in both the sides and sought for the real solutions of \( \nu_a \) when they match. We find in particular that for various values of \( r \) and \( t \) there always exists a pair of real solutions for \( \nu_a \) to the relation: \( e^{-2(|\nu_{i} - \nu_{i+s}|)} \approx e^{-\nu_a - \phi_{(i+s)a}} \) taking one positive and another negative values. For \( \lambda_0 = 0 \) for example, we get \( \nu_a \) placed symmetrically around the origin, which gives the total momentum of the system: \( P = 2\pi \cdot \text{Integer} \), in agreement with the translational invariance of the system. Therefore for \( M = \frac{N}{2} \) and a distribution of \( \nu_a \)'s having the same order in magnitude one is likely to match the singularities in both sides of the BAE allowing the NBS \( \blacksquare \) to exist. Note that in the thermodynamic limit this distribution of real roots should correspond to the AFGS and the NBS should correspond to the earlier non-Bethe string results observed also over the AFGS \( [11, 12, 13, 14] \). Now we focus on to some interesting properties of NBS \( \blacksquare \) and show its more resemblance with earlier observations. Firstly we notice that due to the reduced inter-root distance \( \Delta = \frac{1}{s} \), its length \( L(r) = \frac{i(r-l)}{s} \) is bounded as \( 1 \geq |L(r)| \geq 2 \) and in contrast to the Bethe string exhibits therefore a close-root form, which is in accordance with the earlier observations. It is also evident that together with each root \( (\lambda_l) \) its partner \( (\tilde{\lambda}_l) \), its conjugate \( (\lambda_l^*) \) and the partner of its conjugate \( (\tilde{\lambda}_l^*) \) form a closed unit of four and being decisive contributors to the equation at large \( N \), become almost independent entries at the thermodynamic limit. These groups of four may also get reduced to form a doublet or a triplet due to possible degeneracies (see fig.1). However this can occur only once and that also when \( r \) is not divisible by 4. Thus the close-root string \( \blacksquare \) may split up into units of four, three and two at the thermodynamic limit, reproducing again the non-Bethe structures observed earlier. Moreover, we see that for each of the roots with \( l < l* \), which corresponds to \( |Im(\lambda_l)| > \frac{1}{2} \), its partner \( \tilde{\lambda}_l \) is always with \( |Im(\tilde{\lambda}_l)| < \frac{1}{2} \). This fact also mimics amazingly the observation of \( [13] \) stating that the close root of type I must have their partners from among the close roots of type II. Due to such striking agreement with earlier studies we hope that the more general non-Bethe string structure \( \blacksquare \) should survive also in the thermodynamic limit and should be consistent with the integral equations derived earlier for the complex roots in the anti-ferromagnetic case. The precise and explicit forms of the general Bethe (PBS) \( \blacksquare \) and non-Bethe (NBS) \( \blacksquare \) for strings found here should also be useful for
comparing their corresponding properties. As we have noticed, their correction terms induce different nature of singularity structures in the BAE at large $N$. Generically roots of the Bethe-string (PBS) \( (1) \) produce zeros and poles of different orders of smallness in the BAE, arranged in a growing sequence and the roots are located mostly in the wide-root region: \( |\text{Im}(\lambda_l)| > 1 \). The non-Bethe string (NBS) \( (3) \) on the other hand, allows the singularities of its roots to be of the same order of smallness and the roots themselves are concentrated only in the close-root region. Therefore, if one looks for the complex roots of the BAE at the AFGS with their singularities having the same order in the whole complex plane, then among the string solutions only the close-root NBS are likely to appear, which supports the earlier findings. Similarly the energies of the states created by Bethe and non-Bethe $r$-strings can also be compared using their explicit forms giving 

\[
E_r^{(ba)} = \frac{r}{\lambda_0 + \frac{r}{2}} \quad \text{and} \quad E_r^{(nbs)} = \sum_{j=1}^{s} \frac{2g_j}{\lambda_0 + g_j},
\]

respectively, where the factor \( g_j = g_1 - \frac{1}{2}(j - 1) \), with \( g_1 = \frac{3}{2} \) for odd and \( g_1 = \frac{3}{2} - \frac{1}{2s} \) for even $r$. One can show analytically for $r \to \infty$ and numerically for finite $r$ (see fig. 2) that $E_r^{(ba)}$ is always lower than $E_r^{(nbs)}$, while both have lower values than that of the $r$-free magnons: 

\[
E_r^{(free)} = \frac{r}{\lambda_0 + \frac{r}{4}}.
\]

Therefore one concludes that, though both Bethe and non-Bethe strings may give bound states, the non-Bethe ones must be more loosely bound with higher energies. However at this stage conclusive statements are still difficult to make regarding the string hypothesis problem. One perhaps should consider the TBA analysis using the general $r$-NBS over the AFGS, which we leave as a future problem.

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Figure 1: Arrangement of roots in string solutions with $r = 7$. a) Non-Bethe string with length $L(7) = 6\Delta = 2i$ represents a close-root form and breaks up into a quartet with \{\(\lambda_2, \tilde{\lambda}_2(= \lambda_5), \lambda_2'(= \lambda_0), \tilde{\lambda}_2'(= \lambda_3)\}\} and a triplet with \{\(\lambda_1, \tilde{\lambda}_1(= \lambda_4) = \lambda_1^*, \lambda_1'(= \lambda_7)\}\}. b) Bethe string with length $L(7) = 6\Delta = 6i$ in the wide-root region cannot break up into smaller units since each root has two partners, each of which in turn has a different partner.
Figure 2: Comparison of energies corresponding to $r$-Bethe string: $E(bs)$, $r$-non-Bethe string: $E(nbs)$ and $r$ free magnons: $E(free)$ for $r = [1 : 10]$, showing both strings as bound states with $E(free) \geq E(nbs) \geq E(bs)$. 