The Goldbach conjecture resulting from global-local cuspidal representations and deformations of Galois representations

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Abstract

In the basic general frame of the Langlands global program, a local $p$-adic elliptic semi-module corresponding to a local (left) cuspidal form is constructed from its global equivalent covered by $p^\ell$ roots.

In the same context, global and local bilinear deformations of Galois representations inducing the invariance of their respective residue fields are introduced as well as global and local bilinear quantum deformations leaving invariant the orders of the inertia subgroups. More particularly, the inverse quantum deformation of a closed curve responsible for its splitting directly leads to the Goldbach conjecture.

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1 Introduction

In spite of its apparent simplicity, the Goldbach conjecture asserting that “every integer superior or equal to 4 is the sum of two prime numbers” has resisted since 1742 [Gol] to a convincing proof or disproof.

As for the Wiles demonstration of Fermat’s last theorem [Wil], it it perhaps a sign that these striking problems of number theory cannot be directly solved by means of this unique field but have to be tackled in a more general context for example from the program of Langlands.

The aim of this paper thus consists in approaching the conjecture of Goldbach on the basis of new breakthroughs in the (bilinear) global-local cuspidal representations and in the deformations of Galois representations.

The main results of this work deal with:

a) the generation of a local \( p \)-adic cuspidal form [Lan2] from its global equivalent covered by \( p^\ell \) roots: this leads directly to the Serre-conjecture on \( p \)-adic Galois representations associated with modular forms and to the Shimura-Taniyama-Weil conjecture.

b) global and local bilinear deformations of Galois representations inducing the invariance of their respective residue fields and the introduction of global and local bilinear quantum deformations leaving invariant the orders of the inertia subgroups.

The general basic mathematical frame used in this work is that of the Langlands global program [Pie1] dealing with bijections between the equivalence classes of the \( n^2 \)-dimensional representation of the bilinear global Weil group given by the bilinear algebraic semigroup \( \text{GL}_n(L_\sigma \times L_v) \) and the corresponding conjugacy classes of the cuspidal representation of \( \text{GL}_n(L_\sigma \times L_v) \) where \( L_v \) (resp. \( L_\sigma \)) denotes the complete set of left (resp. right) archimedean pseudo-ramified completions.

Non abelian global class field concepts are reviewed in chapter 2. They are based on a set of increasing finite symmetric splitting semifields characterized by Galois extension degrees which are integers modulo \( N \). The corresponding completions, resulting from isomorphisms of compactification, are infinite archimedean pseudo-ramified completions, defining archimedean pseudo-ramified real places.

Bilinear algebraic semigroups \( \text{GL}_n(L_\sigma \times L_v) = T_n^\sigma(L_\sigma) \times T_n(L_v) \), over the product of the set \( L_v \) of left archimedean pseudo-ramified completions by the symmetric set \( L_\sigma \) of right completions, are expressed according to the Gauss bilinear decomposition, i.e. by means of the product of the group \( T_n^\sigma(L_\sigma) \) of lower triangular matrices with entries in \( L_\sigma \) and referring
to the lower half space by the group $T_n(L_v)$ of upper triangular matrices with entries in $L_v$ and referring to the upper half space.

The algebraic representation space $\text{Repsp}(\text{GL}_n(L_\mathfrak{p} \times L_v))$ of the bilinear algebraic semigroup $\text{GL}_n(L_\mathfrak{p} \times L_v)$ decomposes according to its conjugacy class representatives $g^{(n)}_{R \times L}[i, m_i], 1 \leq i \leq t \leq \infty$, where $m_i$ refers to their multiplicities.

On the toroidal compactification

$$G^{(n)}(L_\mathfrak{p}^T \times L_v^T) \cong \text{Repsp}(\text{GL}_n(L_\mathfrak{p}^T \times L_v^T))$$

of $\text{Repsp}(\text{GL}_n(L_\mathfrak{p} \times L_v))$, the bisemimodule $\Phi(G^{(n)}(L_\mathfrak{p} \times L_v))$ of differentiable smooth bifunctions

$$\phi_{R \times L}(g^{(n)}_{R \times L}[i, m_i]) = \phi(g^{(n)}_{R}[i, m_i]) \times \phi(g^{(n)}_{L}[i, m_i])$$

is defined where $g^{(n)}_{R}[i, m_i]$ (resp. $g^{(n)}_{L}[i, m_i]$) is a toroidal compactified right (resp. left) conjugacy class representative, i.e. a $n$-dimensional semitorus restricted to the lower (resp. upper) half space.

On the set of bisections of $\Phi(G^{(n)}(L_\mathfrak{p}^T \times L_v^T))$, a global elliptic $\Gamma(\Phi(G^{(n)}(L_\mathfrak{p}^T \times L_v^T)))$-bisemimodule $\text{ELLIP}_R(n, i, m_i) \otimes_D \text{ELLIP}_L(n, i, m_i)$ is constructed in such a way that:

a) $\text{ELLIP}_R(n, i, m_i) \otimes_D \text{ELLIP}_L(n, i, m_i)$ covers the corresponding cuspidal form $f_R(z) \otimes_D f_L(z)$ as introduced in [Pie1] and in [Pie2].

b) $$\text{ELLIP}_L(n, i, m_i) = \bigoplus_{i=1}^{t} \lambda^{\frac{i}{2}}(n, i, m_i) e^{2\pi i x} , \quad x \in \mathbb{R}^n$$

(resp. $\text{ELLIP}_R(n, i, m_i) = \bigoplus_{i=1}^{t} \lambda^{\frac{i}{2}}(n, i, m_i) e^{-2\pi i x}$),

is the sum of smooth differentiable functions (resp. cofunctions) on the conjugacy class representatives $g^{(n)}_{L}[i, m_i]$ (resp. $g^{(n)}_{R}[i, m_i]$), where $\lambda^{\frac{i}{2}}(n, i, m_i)$ is the square root of the considered Hecke character.

$\text{ELLIP}_R(n, i, m_i) \otimes_D \text{ELLIP}_L(n, i, m_i)$ then constitutes a cuspidal representation space of the algebraic bilinear semigroup $\text{GL}_n(L_\mathfrak{p} \times L_v)$ as required by the global program of Langlands and recalled in section 3.1.

Let now $L_{[\wp]}$ (resp. $L_{[\wp]}$) denote the truncated set of left (resp. right) archimedean pseudo-ramified completions superior and equal to the $p$-th infinite place, where $p$ is a prime integer.

A $n$-dimensional global elliptic $\Gamma(\Phi(G^{(n)}(L_\mathfrak{p}^T \times L_v^T)))$-bisemimodule $\text{ELLIP}_R(n, i \geq p, m_i) \otimes \text{ELLIP}_L(n, i \geq p, m_i)$ can then be envisaged as well as its covering global elliptic bisemimodule $\text{ELLIP}_R(n, i \geq p, m_i, \mathfrak{p}^{(\ell)}) \otimes \text{ELLIP}_L(n, i \geq p, m_i, \mathfrak{p}^{(\ell)})$ by $p^\ell$ roots in such a way that each term $\text{ellip}_L(n, p+h, m_{p+h}, \mathfrak{p}^{(\ell)}) \simeq \prod_{\ell=1}^{n} r_x(p+h, m_{p+h}, \mathfrak{p}^{(\ell)}) e^{2\pi i x'_c} , \quad x'_c \in \mathbb{R}$,

of $\text{ELLIP}_L(n, i \geq p, m_i, \mathfrak{p}^{(\ell)})$ is the covering by $p^\ell$ roots (“$\ell$” varying from one term to another) of the corresponding term $\text{ellip}_L(n, [p+h], m_{p+h})$ of $\text{ELLIP}_L(n, i \geq p, m_i)$.
It is then proved that every $n$-dimensional semitorus $T^{\ell}_{L_\ell} [p+h, m_{p+h}, \overline{p}^\ell] \simeq \text{ellip}_L(n, [p+h], m_{p+h})$ is a discrete valuation (semi)ring of which uniformizing element is $r(p+h, m_{p+h}, \overline{p}^\ell)^p = \prod_{c=1}^{h} r_c(p+h, m_{p+h}, \overline{p}^\ell)^p$ and units are the invertible elements $e^{2\pi i(p')x'} = \prod_{c=1}^{h} e^{2\pi i(p')x'_c}$, $x' \in \mathbb{R}^n$.

The Kronecker-Weber theorem, expressing that every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension of $\mathbb{Q}$, directly follows from the precedent considerations.

A local $p$-adic elliptic semimodule corresponding to a local left cuspidal form is then constructed in section 3.3 from its global equivalent covered by $p^\ell$ roots.

To this end, it is shown that a set $\{L_{v_{p+h}}\}_h$ of finite increasing global subsemifields “above $p$” can be covered in a etale way by (a) $p$-adic finite extension field(s) leading to a global($\leftrightarrow$)local one-to-one correspondence if the number of global and local elements correspond, i.e. if the number of “global” algebraic points is a power of $p$.

Starting with the two-dimensional global left elliptic $\Gamma(\Phi(G(2)(L^{T}_{[\nu_p]})))$-semimodule $\text{ELLIP}_L(2, \ i \geq p, m_r, \overline{p}^\ell)$, covered by $p^\ell$ roots, the local elliptic left $\text{End}(G(2)(K_p^+))$-semimodule $\text{ELLIP}(2, x, K_p^+) = \oplus \lambda^\ell_p(2, r, m_r)(x) f(\mu^{p-r})$ is constructed in one-to-one correspondence where $x$ is a closed point of the finite Galois extension of the non archimedean $p$-adic left semifield $L_p^+$ introduced in section 3.3.1 in such a way that:

a) the Frobenius substitution $\mu \rightarrow \mu^{q_r}$ on every local Frobenius endomorphism $\mu : x \rightarrow x^p$ be considered where $q_r = \Sigma f_r \cdot e_r$ with $f_r$ the “local” residue degree of the $r$-th prime of the considered Galois extension and $e_r$ the corresponding ramification index.

b) $e_{L_{v_{p+h}}+K_p^+} : \lambda(2, p+h, m_{p+h}) \rightarrow \lambda_p(2, r, m_r)$ be the embedding of the product $\lambda(2, p+h, m_{p+h})$, right by left, of Hecke characters over $L_{[\nu_p]}$ into its equivalent $\lambda_p(2, r, m_r)$ over $K_p^+$.

This condition corresponds to the embedding $i(a_\ell) = \text{trace}(\text{Frob}_\ell)$ into $\mathbb{Q}_p$ of the ring of integers of a finite extension $E_f$ (i.e. the ring of the coefficients of the cuspidal form $f$) of $\mathbb{Q}$, $a_\ell$ being the coefficient of the cuspidal form.

The Serre (Eichler, Deligne, Shimura) conjecture [C-F-T], [D-S], [Swi], asserting that Galois representations $\rho = G(2) \rightarrow \text{GL}_2(\mathbb{Q}_p)$ can be associated to modular forms directly results from this construction of a local $p$-adic cuspidal form.

On this basis, two kinds of explicit deformations of $n$-dimensional representations of Galois or Weil groups given by bilinear algebraic semigroups over complete global and local Noetherian bisemirings are considered in chapter 4.

First, local $p$-adic coefficient semiring homomorphisms are envisaged in such a way that they induce an isomorphism on their residue semifields leading to a base change
in the considered finite Galois extensions. Similarly, global coefficient semiring homomorphisms are defined in such a way that they induce an isomorphism on their global residue semifields.

It is then proved that the inverse image of the homomorphism \( h_{L_p'} : L_{L_p'} \rightarrow L_{L_p} \) between global coefficient semirings \( L_{L_p'} \equiv L_{[v_p]} \) and \( L_{L_p} \equiv L_{[v_p]} \) is isomorphic to the inverse image of the homomorphism \( h_{B_p'} : B_{B_p'} \rightarrow B_{B_p} \) between local coefficient semirings \( B_{B_p'} \equiv B_{[v_p]} \) and \( B_{B_p} \equiv B_{[v_p]} \) if the number of elements of the global kernel \( K(h_{L_p'} - L_{L_p}) \) is equal to the number of elements of the local kernel \( K(h_{B_p'} - B_{B_p}) \), i.e. is a power of \( p \).

A \( n \)-dimensional global bilinear deformation of

\[
\rho_L : \text{Gal}(\hat{L}_{R_p}/k) \times \text{Gal}(\hat{L}_{L_p}/k) \rightarrow \text{GL}_n(L_{R_p} \times L_{L_p})
\]

is then an equivalence class of liftings

\[
\rho_{L_c'} = \rho_L + \delta \rho_{L_c}, \quad 1 \leq c \leq \infty,
\]

where \( \delta \rho_{L_c} \) refers to the kernel of \( \rho_L \), in such a way that the kernels of two deformed algebraic bilinear semigroups \( \text{GL}_n(L_{R_{pc_1}}' \times L_{L_{pc_1}}') \) and \( \text{GL}_n(L_{R_{pc_2}}' \times L_{L_{pc_2}}') \) differ by powers of orders of their inertia bilinear subgroups.

Similarly, a \( n \)-dimensional local \( p \)-adic bilinear deformation of

\[
\rho_K : \text{Gal}(K_p^-/L_p^-) \times \text{Gal}(K_p^+/L_p^+) \rightarrow \text{GL}_n(K_p^- \times K_p^+),
\]

in the sense of Mazur [Maz2], is an equivalence class of liftings

\[
\rho_{K_d'} = \rho_K + \delta \rho_{K_d}, \quad 1 \leq d \leq \infty,
\]

where \( \delta \rho_{K_d} \) refers to the kernel of \( \rho_K \), in such a way that the kernels of the two deformed algebraic bilinear semigroups \( \text{GL}_n(K_p'^- \times K_p'^+) \) and \( \text{GL}_n(K_p'^- \times K_p'^+) \) differ by powers of their ramification indices.

A second type of deformations of Galois representations, called quantum deformations, is envisaged on the basis of global and local coefficient semiring quantum homomorphisms.

A uniform quantum homomorphism

\[
Qh_{L_{Lp+j}} : L_{Lp+j} \rightarrow L_{Lp}
\]

between two global compactified coefficient semirings is such that:

- it induces an isomorphism on their global inertia subgroups;
- it increases the global residue semifield \( L_{L_p} \) by an increment of \( j \) quanta, i.e. \( j \) irreducible closed algebraic subsets of degree \( N \), on every completion.
Similarly, a quantum homomorphism

\[ Qh_{B^+_p \to B^+_r} : B^+_p \longrightarrow B^+_r, \quad t = r + s, \]

between two local coefficient semirings is such that:

- it induces an isomorphism on their “local” inertia subgroups (having thus the same ramification index);
- it increases the residue degrees of the \( r \) residue subsemifields of \( B^+_r \) by a same integer increment.

A \( n \)-dimensional global bilinear quantum deformation of \( \rho_L \), defined above, is then an equivalence class of liftings

\[ \rho_{L_j} = \rho_L + \delta \rho_{L_j}, \quad 1 \leq j \leq \infty, \]

where \( \delta \rho_{L_j} \) refers to the kernel of \( \rho_L \), in such a way that the kernels of the two deformed algebraic bilinear semigroups \( \text{GL}_n(L_{R_p+j_1} \times L_{L_p+j_1}) \) and \( \text{GL}_n(L_{R_p+j_2} \times L_{L_p+j_2}) \) differ by powers of their global residue degrees.

Similarly, a \( n \)-dimensional local bilinear “quantum” deformation of

\[ \rho_{K_{p_r}} : \text{Gal}(K_p^-/L_p^-) \times \text{Gal}(K_p^+/L_p^+) \longrightarrow \text{GL}_n(K_p^- \times K_p^+) \]

in an equivalence class of liftings:

\[ \rho_{K_{p_t}} = \rho_{K_{p_r}} + \delta \rho_{K_{p_t}}, \]

where \( \delta \rho_{K_{p_t}} \) refers to the kernel of \( \rho_{K_{p_r}} \), in such a way that the kernels of the two deformed algebraic bilinear semigroups \( \text{GL}_n(K_{p_1}^- \times K_{p_1}^+) \) and \( \text{GL}_n(K_{p_2}^- \times K_{p_2}^+) \) differ by powers of their local residue degrees.

Taking into account the Langlands global correspondences

\[ \text{ELLIP FREPsp(\text{GL}_n(L_{R \times L_p})) :} \quad \text{GL}_n(L_{R_p} \times L_{L_p}) \longrightarrow \text{ELLIP}_{R \times L}(n, i \geq p, m_i), \]

between the bilinear algebraic semigroup \( \text{GL}_n(L_{R_p} \times L_{L_p}) \) and the \( n \)-dimensional global elliptic bisemimodule \( \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \) introduced above, a \( n \)-dimensional global elliptic bilinear quantum deformation of \( \rho_L^{\text{ELLIP}} = \text{ELLIP FREPsp(\text{GL}_n(L_{R \times L_p}) \circ \rho_L)} \) is an equivalence class of liftings

\[ \rho_{L_j}^{\text{ELLIP}} = \rho_{L}^{\text{ELLIP}} + \delta \rho_{L_j}^{\text{ELLIP}}, \]

inducing the injective morphism

\[ \mathcal{D}^{(p \to (p+j))}_{R \times L}(n) : \quad \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \longrightarrow \text{ELLIP}_{R \times L}(n, i \geq p + j, m_i) \]
which is quantum deformation of \( \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \) increasing the global residue degree "\( i \)" of each left and right term \( i \geq p \) of \( \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \) by an amount of an integer "\( j \)".

The injective morphism

\[
\mathcal{D}^{[p] \rightarrow [p+j]}_{\text{R \times L}}(n) : \quad \text{ellip}_{\text{R \times L}}(n, [p], m_p) \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(n, [p+j], m_{p+j})
\]

restricted to the \((p, m_p)\)-th conjugacy class representative \( \text{ellip}_{\text{R \times L}}(n, [p], m_p) \) of \( \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \), is a quantum equivalence class representative of liftings or an elliptic quantum deformation associated with the exact sequence:

\[
1 \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(n, [j]) \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(n, [p+j], m_{p+j}) \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(n, [p], m_p) \quad \longrightarrow \quad 1.
\]

Let then

\[
\mathcal{D}^{[p+j+k] \rightarrow [p+j]}_{\text{R \times L}}(1) : \quad \text{ellip}_{\text{R \times L}}(1, [p+j+k], m_{p+j+k}) \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(1, [p+j], m_{p+j})
\]

denote the inverse elliptic quantum deformation of a one-dimensional global elliptic subbisme-module of class \([p+j+k]\) towards a class \([p+j]\).

This inverse elliptic quantum deformation corresponds to the endomorphism:

\[
\text{End}_{\text{R \times L}}^{[p+j+k] \rightarrow [p+j]}(1) : \quad \text{ellip}_{\text{R \times L}}(1, [p+j+k], m_{p+j+k}) \quad \longrightarrow \quad \text{ellip}_{\text{R \times L}}(1, [p+j], m_{p+j}) + \text{ellip}_{\text{R \times L}}(1, [k], m_k)
\]

where \( \text{ellip}_{\text{R \times L}}(1, [k], m_k) \) denotes the product, right by left, of a right semicircle at \( k \) quanta, i.e. characterized by a global residue degree \( f_{v_k} = k \) according to section 2.1, and localized in the lower half space by its left equivalent localized in the upper half space.

If we consider now the inverse quantum deformation of a closed curve isomorphic to the left (or right) undoubled semicircle of class \([p'+j'+k']\), we get the following relation, associated with its endomorphism:

\[
f_{v_{p'+j'+k'}} = f_{v_{p'+j'}} + f_{v_{k'}} \quad \text{or} \quad p' + j' + k' = (p' + j') + k'
\]

for the resulting global residue degrees and corresponding to a splitting of the closed curve \( c_2^1[p'+j'+k'] \) of class \([p'+j'+k']\) into two complementary curves of classes \([p'+j']\) and \([k']\).

Taking into account that the global residue degree \( f_{v_{p'+j'+k'}} = p' + j' + k' \) of a closed curve must be an even integer \( G_{\text{even}} \), the relation \( G_{\text{even}} = f_{v_{p'+j'}} + f_{v_{k'}} \) directly leads to the Goldbach conjecture on the basis of the developments of chapter 5 and of [Pie2] dealing with the results of the author on the Riemann hypothesis.
2 New concepts of non abelian global class field theory

2.1 Global class field concepts

Let \( k \) be a global number field of characteristic 0 and let \( \tilde{L} \) denote a finite extension of \( k \).

\( \tilde{L} = \tilde{L}_R \cup \tilde{L}_L \) is assumed to be a symmetric splitting field composed of a right and a left algebraic extension semifields \( \tilde{L}_R \) and \( \tilde{L}_L \) in one-to-one correspondence. \( \tilde{L}_L \) and \( \tilde{L}_R \) are respectively the sets of positive and symmetric negative simple roots of a polynomial ring over \( k \).

\( \tilde{L}_L \) and \( \tilde{L}_R \) are commutative division semirings, i.e. semifields, because they lack for opposite elements with respect to the addition.

Let \( \tilde{L}_{v_1} \subset \cdots \subset \tilde{L}_{v_t} \subset \tilde{L}_L \) (resp. \( \tilde{L}_{\pi_1} \subset \cdots \subset \tilde{L}_{\pi_t} \subset \tilde{L}_L \)) denote the set of increasing subsemifields of \( \tilde{L}_L \) (resp. \( \tilde{L}_R \)).

The completion \( L_{v_i} \) (resp. \( L_{\pi_i} \)) associated with \( \tilde{L}_{v_i} \) (resp. \( \tilde{L}_{\pi_i} \)) is an isomorphism of compactification \( e_{v_i} : L_{v_i} \rightarrow \tilde{L}_{v_i} \) (resp. \( e_{\pi_i} : \tilde{L}_{\pi_i} \rightarrow L_{\pi_i} \)) of \( L_{v_i} \) (resp. \( \tilde{L}_{\pi_i} \)) onto the subsemifield \( L_{v_i} \) (resp. \( L_{\pi_i} \)) which is a closed compact subset of \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_- \)) [Kna], [Ser3].

The equivalence classes of completions of \( \tilde{L}_L \) (resp. \( \tilde{L}_R \)), characterized by they number of elements, are the left (resp. right) infinite real places of \( \tilde{L}_L \) (resp. \( \tilde{L}_R \)).

They are noted \( v = \{ v_1, \ldots, v_t \} \) (resp. \( \pi = \{ \pi_1, \ldots, \pi_t \} \)).

Let \( L_{v_i} \) (resp. \( L_{\pi_i} \)) denote the infinite pseudo-ramified completion proceeding from the subsemifield \( \tilde{L}_{v_i} \) (resp. \( \tilde{L}_{\pi_i} \)):

- it is characterized by an achimedean absolute value in its topology;
- it is generated from an irreducible central \( k \)-semimodule \( L_{v_i}^1 \) (resp. \( L_{\pi_i}^1 \)) of rank \( [L_{v_i}^1 : k] = N \) (resp. \( [L_{\pi_i}^1 : k] = N \));
- it is defined by its rank, i.e. its Galois extension degree \( [L_{v_i} : k] = [L_{\pi_i} : k] = \ast + iN \) which is an integer modulo \( N \), \( N \in \mathbb{N} \), where \( \ast \) denotes an integer inferior to \( N \).

The corresponding pseudo-unramified completion \( L_{v_i}^{nr} \) (resp. \( L_{\pi_i}^{nr} \)) is defined by:

\[
L_{v_i}^{nr} = L_{v_i} / L_{v_i}^1 \quad \text{(resp. } L_{\pi_i}^{nr} = L_{\pi_i} / L_{\pi_i}^1 \text{)}
\]

and is characterized by its global residue degree (in analogy with the local \( p \)-adic treatment) \( f_{v_i} = [L_{v_i}^{nr} : k] = i \) (resp. \( f_{\pi_i} = [L_{\pi_i}^{nr} : k] = i \)).

The infinite pseudo-ramified real place \( v_i \) (resp. \( \pi_i \)) is composed of the basic completion \( L_{v_i} \) (resp. \( L_{\pi_i} \)) and of the set \( \{ L_{v_i,m_i}^{sup(m_i)} \}_{m_i=1}^{sup(m_i)} \) (resp. \( \{ L_{\pi_i,m_i}^{sup(m_i)} \}_{m_i=1}^{sup(m_i)} \)) of equivalent completions characterized by the same rank, where \( sup(m_i) \) denotes the multiplicity of \( v_i \) (resp. \( \pi_i \)).

The set (resp. the direct sum) of the real infinite pseudo-ramified completions is given by \( L_v = \{ L_{v_i,m_i} \}_{i,m_i} \) or \( L_{\pi} = \{ L_{\pi_i,m_i} \}_{i,m_i} \) (resp. \( L_{v_{\emptyset}} = \bigoplus_{i=1}^{t} L_{v_i,m_i} \) or \( L_{\pi_{\emptyset}} = \bigoplus_{i=1}^{t} L_{\pi_i,m_i} \)).
and the product of the primary real infinite pseudo-ramified completions gives rise to the adele semiring:

\[ \mathbb{A}_L^\infty = \prod_{j, p, m} L_{v_j p, m_j p} \quad (\text{resp. } \mathbb{A}_L^\infty = \prod_{j, p, m} L_{\pi_j p, m_j p}) \]

where \( j_p \) denotes the \( j \)-th primary completion.

### 2.2 Galois, inertia and Weil groups

Let \( \tilde{L}_{v_i} \) (resp. \( \tilde{L}_{\pi_i} \)) and \( \tilde{L}_{v_i, m_i} \) (resp. \( \tilde{L}_{\pi_i, m_i} \)) denote respectively the the basic and the equivalent Galois extensions corresponding to the basic completion \( L_{v_i} \) (resp. \( L_{\pi_i} \)) and to the equivalent completion \( L_{v_i, m_i} \) (resp. \( L_{\pi_i, m_i} \)) at the \( v_i \)-th archimedean place.

Let \( \text{Gal}^D(\tilde{L}_{v_i}/k) \) (resp. \( \text{Gal}^D(\tilde{L}_{\pi_i}/k) \)) denote the Galois subgroup of \( \tilde{L}_{v_i} \) (resp. \( \tilde{L}_{\pi_i} \)) and let \( \text{Gal}(\tilde{L}_{v_i, m_i}/k) \) (resp. \( \text{Gal}(\tilde{L}_{\pi_i, m_i}/k) \)) denote the Galois subgroup of the equivalent Galois extension \( \tilde{L}_{v_i, m_i} \) (resp. \( \tilde{L}_{\pi_i, m_i} \)).

So, the Galois subgroup associated with the \( v_i \)-th (resp. \( \pi_i \)-th) infinite pseudo-ramified real place will be given by:

\[ \text{Gal}(\tilde{L}_{v_i}/k) = \text{Gal}^D(\tilde{L}_{v_i}/k) \oplus \text{Gal}(\tilde{L}_{v_i, m_i}/k) \]

(\text{resp. } \text{Gal}(\tilde{L}_{\pi_i}/k) = \text{Gal}^D(\tilde{L}_{\pi_i}/k) \oplus \text{Gal}(\tilde{L}_{\pi_i, m_i}/k)).

For the corresponding pseudo-unramified Galois extensions, we should have:

\[ \text{Gal}(\tilde{L}_{v_i}^{nr}/k) = \text{Gal}^D(\tilde{L}_{v_i}^{nr}/k) \oplus \text{Gal}(\tilde{L}_{v_i, m_i}^{nr}/k) \]

(\text{resp. } \text{Gal}(\tilde{L}_{\pi_i}^{nr}/k) = \text{Gal}^D(\tilde{L}_{\pi_i}^{nr}/k) \oplus \text{Gal}(\tilde{L}_{\pi_i, m_i}^{nr}/k)).

On the other hand, the Galois subgroup of the irreducible central extension \( \tilde{L}_{v_i}^{1} \) (resp. \( \tilde{L}_{\pi_i}^{1} \)), corresponding to the irreducible completion \( L_{v_i}^{1} \) (resp. \( L_{\pi_i}^{1} \)) having a rank \( N \), is obviously the \textbf{global inertia subgroup} \( I_{L_{v_i}} \) (resp. \( I_{L_{\pi_i}} \)) which can be defined by:

\[ I_{L_{v_i}} = \text{Gal}(\tilde{L}_{v_i}^{1}/k) / \text{Gal}(\tilde{L}_{v_i}^{nr}/k) \]

(\text{resp. } \text{Gal}(\tilde{L}_{\pi_i}^{1}/k) / \text{Gal}(\tilde{L}_{\pi_i}^{nr}/k)).

or by the equivalent exact sequence:

\[
1 \longrightarrow I_{L_{v_i}} \longrightarrow \text{Gal}(\tilde{L}_{v_i}/k) \longrightarrow \text{Gal}(\tilde{L}_{v_i}^{nr}/k) \longrightarrow 1 \quad (\text{resp. } 1 \longrightarrow I_{L_{\pi_i}} \longrightarrow \text{Gal}(\tilde{L}_{\pi_i}/k) \longrightarrow \text{Gal}(\tilde{L}_{\pi_i}^{nr}/k) \longrightarrow 1).\]
As the global inertia subgroups are of Galois type, they are all isomorphic:

\[ I_{Lv_1} \simeq \cdots \simeq I_{Lv_i} \simeq \cdots \simeq I_{Lv_t} \]

Let \( \bar{L}_L \) (resp. \( \bar{L}_R \)) denote the union of all finite abelian extensions of \( k \). Then, we have that:

- \( \text{Gal}(\bar{L}_L/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}_{vi}/k) \),
- \( \text{Gal}(\bar{L}_R/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}_{\pi_i}/k) \),
- \( \text{Gal}(\bar{L}^{nr}_L/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}^{nr}_{vi}/k) \),
- \( \text{Gal}(\bar{L}^{nr}_R/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}^{nr}_{\pi_i}/k) \).

In analogy with the \( p \)-adic case where the Weil group is the Galois subgroup of the elements inducing on the residue field an integer power of a Frobenius element, it was assumed [Pie2] that the Weil group in the archimedean case will be the Galois subgroup of the finite pseudo-ramified extensions characterized by extension degrees \( d = 0 \mod N \).

In this respect, if \( \tilde{L}_{vi} \) (and \( \tilde{L}_{\pi_i} \)) denotes a pseudo-ramified real Galois extension of degree \( [\tilde{L}_{vi} : k] = i \cdot N \), the Weil group \( \text{W}_{\tilde{L}_{vi}} \) (resp. \( \text{W}_{\tilde{L}_{\pi_i}} \)), corresponding to the Galois group \( \text{Gal}(\bar{L}_L/k) \) (resp. \( \text{Gal}(\bar{L}_R/k) \)), will be given by:

\[ \text{W}_{\tilde{L}_{vi}} = \bigoplus_{i=1}^t \text{Gal}(\tilde{L}_{vi}/k) \quad (\text{resp.} \quad \text{W}_{\tilde{L}_{\pi_i}} = \bigoplus_{i=1}^t \text{Gal}(\tilde{L}_{\pi_i}/k) ) \]

### 2.3 Non abelian global class field concepts

The set of left (resp. right) real pseudo-ramified completions is, in fact, isomorphic to a one-dimensional left (resp. right) affine scheme \( S^1_L \) (resp. \( S^1_R \)). So, the challenge consists in introducing the \( n \)-dimensional analog of \( S^1_L \) (resp. \( S^1_R \)) which is a \( n \)-dimensional linear algebraic group. But, as the endomorphisms \( \text{End}_k(A) \) of a \( k \)-algebra \( A \) can be handled throughout its enveloping algebra \( A^e = A \otimes_k A^{\text{opp}} \), where \( A^{\text{opp}} \) denotes the opposite algebra of \( A \), because \( A^e \simeq \text{End}_k(A) \), and as fundamental algebras, as the algebra of modular forms, are intrinsically defined in the upper half space, bilinearity instead of linearity will be envisaged [Pie2], [Pie4].

Then, the \( n \)-dimensional equivalent of the product \( S^1_R \times S^1_L \) of the one-dimensional affine schemes \( S^1_R \) and \( S^1_L \) is a \( n^2 \)-dimensional bilinear algebraic semigroup \( G^{(n)}(L_{\pi} \times L_v) \) isomorphic to the product \( GL_n(L_{\pi} \times L_v) \cong T_n(L_{\pi}) \times T_n(L_v) \) of the group \( T_n(L_{\pi}) \) of lower triangular matrices with entries in \( L_{\pi} \) by the group \( T_n(L_v) \) of upper triangular matrices with entries in \( L_v \) where \( L_v = \{L_{v,m_i}\}_{i=1}^t \) and \( L_{\pi} = \{L_{\pi,m_i}\}_{i=1}^t \).
As the algebraic bilinear semigroup \( G^{(n)}(L_{\pi} \times L_{\nu}) \) is constructed over \( L_{\pi} \times L_{\nu} \), it can be decomposed into \( t \) conjugacy classes, \( 1 \leq i \leq t \), having multiplicities \( m^{(i)} = \sup(m_{i}) \), \( m_{i} \in \mathbb{N} \), in such a way that \( m^{(i)} \) denotes the number of equivalent representatives in the \( i \)-th conjugacy class.

The algebraic representation of the algebraic bilinear semigroup \( GL_n(L_{\pi} \times L_{\nu}) \) in the \( G^{(n)}(L_{\pi} \times L_{\nu}) \)-bisemimodule \( M_{R} \otimes M_{L} \) results from the morphism from \( GL_n(L_{\pi} \times L_{\nu}) \) into \( GL(M_{R} \otimes M_{L}) \) where \( GL(M_{R} \otimes M_{L}) \) is the group of automorphisms of \( M_{R} \otimes M_{L} \).

So, \( GL(M_{R} \otimes M_{L}) \) becomes the \( n \)-dimensional equivalent of the product \( W_{L_{\pi \otimes}}^{ab} \times W_{L_{\nu \otimes}}^{ab} \) of the global Weil groups and the \( n \)-dimensional bilinear algebraic semigroup \( G^{(n)}(L_{\pi} \times L_{\nu}) \) is the \( n \)-dimensional representation space of \( W_{L_{\pi \otimes}}^{ab} \times W_{L_{\nu \otimes}}^{ab} \).

Referring to the algebraic bilinear semigroup of matrices \( GL_n(L_{\pi} \times L_{\nu}) \equiv T_n(L_{\pi}) \times T_n(L_{\nu}) \), we see that it is submitted to the following Gauss bilinear decomposition:

\[
GL_n(L_{\pi} \times L_{\nu}) = [D_n(L_{\nu}) \times D_n(L_{\pi})] \times [UT_n(L_{\nu}) \times UT_n(L_{\pi})]
\]

where:

- \( D_n(\cdot) \) is the group of diagonal matrices of order \( n \), also called the \( n \)-split Cartan subgroup;
- \( UT_n(\cdot) \) is the group of upper unitriangular matrices.

In fact, the diagonal bilinear algebraic semigroup of matrices \( D_n(L_{\pi} \times L_{\nu}) \equiv D_n(L_{\pi}) \times D_n(L_{\nu}) \) is more exactly \( D_n(L_{\pi D} \times L_{\nu D}) \) where \( L_{\pi D} \) and \( L_{\nu D} \) are given respectively by \( L_{\pi D} = \{L_{\pi i}\}_{i=1}^{t} \) and \( L_{\nu D} = \{L_{\nu i}\}_{i=1}^{t} \) with \( m^{(i)} = 1 \), \( 1 \leq i \leq t \).

Due to the Gauss bilinear decomposition of the algebraic bilinear semigroup \( GL_n(L_{\pi} \times L_{\nu}) \), its conjugacy classes can be partitioned into:

- diagonal conjugacy classes whose representatives \( g^{(n)}_{R \times L}[i, m^{(i)}] = 1 \) refer to the representation space \( \text{Repsp}(D_n(L_{\pi} \times L_{\nu})) \) of the diagonal bilinear algebraic semigroup \( D_n(L_{\pi} \times L_{\nu}) \);
- off-diagonal conjugacy classes whose representatives \( g^{(n)}_{R \times L}[i, m^{(i)}] > 1 \) are generated from the nilpotent biaction of \( UT_n(L_{\pi}) \times UT_n(L_{\nu}) \) on the diagonal conjugacy classes \( g^{(n)}_{R \times L}[i, m^{(i)}] = 1 \).

So, the conjugacy class representatives \( g^{(n)}_{R \times L}[i, m] \) of the representation space \( \text{Repsp}(GL_n(L_{\pi} \times L_{\nu})) \) are the \( G^{(n)}(L_{\pi D} \times L_{\nu D}) \)-subbisemimodules \( M_{\pi D, m} \otimes M_{\nu D, m} \) of the algebraic representation space of \( GL_n(L_{\pi} \times L_{\nu}) \) given by the \( G^{(n)}(L_{\pi} \times L_{\nu}) \)-bisemimodule \( M_{R} \otimes M_{L} \). So, we have that:

\[
\text{Repsp}(GL_n(L_{\pi \otimes} \times L_{\nu \otimes})) = \bigoplus_{i=1}^{t} \bigoplus_{m} g^{(n)}_{R \times L}[i, m]
\]

such that:
• \( \text{Repsp}(\text{GL}_n(L\varpi \times L_v)) \equiv M_R \otimes M_L ; \)

• \( M_{R_\oplus} \otimes M_{L_\oplus} = \bigoplus_{i=1}^t \bigoplus_{m_i} (M_{\varpi_i,m_i} \otimes M_{v_i,m_i}) ; \)

• \( g^{(n)}_{R \times L}[i,m_i] \equiv M_{\varpi_i,m_i} \otimes M_{v_i,m_i}, 1 \leq i \leq t. \)

Then, we can state the following propositions:

2.4 Proposition

Let \( G^{(n)}(L\varpi \times L_v) \) denote a \( n^{(2)} \)-dimensional bilinear algebraic semigroup isomorphic to the bilinear algebraic semigroup of matrices \( \text{GL}_n(L\varpi_\oplus \times L_{v_\oplus}) \).

Then, \( \text{GL}_n(L\varpi \times L_v) \), having the bilinear Gauss decomposition

\[
\text{GL}_n(L\varpi \times L_v) = [D_n(L\varpi_D \times L_{v_D})] \times [UT^t_n(L\varpi) \times UT_n(L_v)],
\]

is such that its algebraic representation space \( \text{Repsp}(\text{GL}_n(L\varpi_\oplus \times L_{v_\oplus})) \), which is a \( G^{(n)}(L\varpi \times L_v) \)-bisemimodule \( M_{R_\oplus} \otimes M_{L_\oplus} \), decomposes into diagonal and off-diagonal conjugacy class representatives \( g^{(n)}_{R \times L}[i,m_i] \) which are \( G^{(n)}(L\varpi_i,m_i \times L_{v_i,m_i}) \)-subbisemimodules \( M_{\varpi_i,m_i} \otimes M_{v_i,m_i} : \)

\[
M_{R_\oplus} \otimes M_{L_\oplus} = \bigoplus_{i}^t \bigoplus_{m_i} (M_{\varpi_i,m_i} \otimes M_{v_i,m_i})
\]

where:

\[
M_R \otimes M_L \equiv \text{Repsp}(\text{GL}_n(L\varpi \times L_v)).
\]

Proof. results from section 2.3.

2.5 Proposition

Let \( I_{L_{\varpi_i}} \) and \( I_{L_{\varpi_i,m_i}} \) be two global inertia subgroups as introduced in section 2.2 and leading to:

\[
I_{L_v} = \bigoplus_i I_{L_{\varpi_i}} \bigoplus I_{L_{\varpi_i,m_i}} \quad \text{(resp. } I_{L\varpi} = \bigoplus_i I_{L_{\varpi_i}} \bigoplus I_{L_{\varpi_i,m_i}} \text{)}.
\]

Let

\[
\text{Gal}(\bar{L}_L/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}_{\varpi_i}/k) \quad \text{(resp. } \text{Gal}(\bar{L}_R/k) = \bigoplus_{i=1}^t \text{Gal}(\bar{L}_{\varpi_i}/k) \text{)}
\]

be the Galois groups of all finite abelian extensions of \( k \).

Then, we get the explicit \( n \)-dimensional representation spaces:

• \( I_{L\varpi} \times I_{L_v} \longrightarrow P^{(n)}(L_{\varpi}^1 \times L_v^1) \),

• \( \text{Gal}(\bar{L}_R/k) \times \text{Gal}(\bar{L}_L/k) \longrightarrow G^{(n)}(L\varpi \times L_v) \),

where
• $P^{(n)}(L_{\mathfrak{p}}^1 \times L_v^1)$ is the bilinear parabolic subgroup,

• $L_{\mathfrak{p}^1} = \{ L_{\mathfrak{p}_{i,m_i}}^1 \}_{i=1}^l,$

with $L_{\mathfrak{p}_{i,m_i}}$ an irreducible central $k$-subsemimodule of rank $N$.

Proof. 1. The representation

$$\text{Gal}(\tilde{L}_R/k) \times \text{Gal}(\tilde{L}_L/k) \longrightarrow \text{GL}_n(L_{\mathfrak{p}} \times L_v)$$

results from non abelian class field concepts introduced in section 2.3 and leads to the morphism from $\text{GL}_n(L_{\mathfrak{p}} \times L_v)$ into $\text{GL}(M_R \otimes M_L)$.

2. $P^{(n)}(L_{\mathfrak{p}}^1 \times L_v^1)$, being classically defined as the connected component of the identity in $G^{(n)}(L_{\mathfrak{p}} \times L_v)$, constitutes a $n$-dimensional representation of $I_{L_{\mathfrak{p}}} \times I_{L_v}$. ■

2.6 Proposition

Each $G^{(n)}(L_{\mathfrak{p}_{i,m_i}} \times L_{v_i,m_i})$-subbisemimodule $M_{\mathfrak{p}_{i,m_i}} \otimes M_{v_i,m_i}$ is characterized by a rank

$$r_{\mathfrak{p}_{i,v_i}}^{(n)} = i^{n^2} \cdot N^{n^2}.$$

Proof. As $(M_{\mathfrak{p}_{i,m_i}} \otimes M_{v_i,m_i})$ is the $n$-dimensional analog of the one-dimensional bilinear algebraic subsemigroup $L_{\mathfrak{p}_{i,m_i}} \times L_{v_i,m_i}$ having a rank given by $r_{\mathfrak{p}_{i,v_i}}^{(1)} = i^2 \cdot N^2$ according to section 2.1, it is clear, according to the non abelian class field concepts developed in [Pie4], that the rank of $M_{\mathfrak{p}_{i,m_i}} \otimes M_{v_i,m_i}$ is given by:

$$r_{\mathfrak{p}_{i,v_i}}^{(n)} = (f_{\mathfrak{p}_{i}})^n \cdot N^n \cdot (f_{v_i})^n \cdot N^n = i^{n^2} \cdot N^{n^2}.$$ ■

2.7 Lattices and bilattices

Let $B_v$ and $B_{\mathfrak{p}}$ be two division semialgebras of dimension $n$ respectively over $L_v$ and $L_{\mathfrak{p}}$ such that $B_{\mathfrak{p}}$ be the opposite division semialgebra of $B_v$.

If we fix the isomorphisms:

$$B_v \approx T_n(L_v) \quad \text{and} \quad B_{\mathfrak{p}} \approx T_n(L_{\mathfrak{p}}),$$

we have the following isomorphism:

$$B_{\mathfrak{p}} \otimes B_v \approx T_n(L_{\mathfrak{p}}) \times T_n(L_v)$$

for the division bisemialgebra $B_{\mathfrak{p}} \otimes B_v$.

On the other hand, fix the maximal orders $O_{L,v}$ of $L_v$ and $O_{L_{\mathfrak{p}}}$ of $L_{\mathfrak{p}}$.

Then, the maximal orders $\Lambda_v$ and $\Lambda_{\mathfrak{p}}$ respectively in the division semialgebras $B_v$ and $B_{\mathfrak{p}}$ are pseudo-ramified $\mathbb{Z}/N \mathbb{Z}$-lattices in the $B_v$-semimodule $M_L$ and in the $B_{\mathfrak{p}}$-semimodule $M_R$. 12
2.8 Proposition

Let $\Lambda_v$ and $\Lambda_{\tau}$ be pseudo-ramified $\mathbb{Z}/N\mathbb{Z}$-lattices respectively in the division semialgebras $B_v$ and $B_{\tau}$.

Then, the pseudo-ramified bisemilattice $\Lambda_{\tau} \otimes \Lambda_v$ in the $B_{\tau} \otimes B_v$-bisemimodule $M_R \otimes M_L$:

- verifies:
  \[ \Lambda_{\tau} \otimes \Lambda_v \simeq \text{GL}_n(O_{L,\tau} \times O_{L,\nu}) ; \]

- has the decomposition:
  \[ \Lambda_{\tau} \otimes \Lambda_v = \bigoplus_i \bigoplus_m (\Lambda_{v_i,m_i} \otimes \Lambda_{v_i,m_i}) . \]

Proof. 1. As $O_{L,\nu}$ (resp. $O_{L,\tau}$) is a maximal order in $L_v$ (resp. $L_{\tau}$), and as $\Lambda_v$ (resp. $\Lambda_{\tau}$) is a maximal order in the division semialgebra $B_v$ (resp. $B_{\tau}$), we have that:

\[ \Lambda_v \simeq T_n(O_{L,v}) \quad (\text{resp. } \Lambda_{\tau} \simeq T_n(O_{L,\tau}) ) \]

leading to:

\[ \Lambda_{\tau} \otimes \Lambda_v \simeq T_n(O_{L,\tau}) \times T_n(O_{L,v}) . \]

2. As $L_{\tau} \otimes L_v$ can be decomposed into a sum of products of pseudo-ramified completions according to:

\[ L_v \otimes L_v = \bigoplus_i (L_{\tau_i} \otimes L_{v_i}) \oplus (L_{\tau_i,m_i} \otimes L_{v_i,m_i}) \]

(see section 2.1) and, as $\Lambda_i$ and $\Lambda_{\tau_i}$ are supposed to be pseudo-ramified $\mathbb{Z}/N\mathbb{Z}$-lattices respectively in $B_v$ and $B_{\tau}$, we can conclude that $\Lambda_{\tau} \otimes \Lambda_v$ has the decomposition

\[ \Lambda_{\tau} \otimes \Lambda_v = \bigoplus_i \bigoplus_m (\Lambda_{v_i,m_i} \otimes \Lambda_{v_i,m_i}) \]

where

\[ \Lambda_{v_i,m_i} \quad (\text{resp. } \Lambda_{\tau_i,m_i}) \]

is a pseudo-ramified sublattice in the $B_{v_i,m_i}$-subsemimodule $M_{v_i,m_i}$ (resp. $B_{\tau_i,m_i}$-subsemimodule $M_{\tau_i,m_i}$).

\[ \blacksquare \]

2.9 Proposition

Let $\text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2)$ be the general bilinear semigroup of matrices of order $n$ with entries in $(\mathbb{Z}/N\mathbb{Z})^2$.

Then, the Hecke bisemialgebra of dimension $n^2$, $\mathcal{H}_{R \times L}(n^2)$, is generated by all the Hecke bioperators $T_R(n;t) \otimes T_L(n;t)$ having a representation in the subgroup of matrices $\text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2)$.
Proof. 1. A left (resp. right) Hecke operator $T_L(n; t)$ (resp. $T_R(n; t)$) is a left (resp. right) correspondence which associates to the left (resp. right) lattice $\Lambda_v$ (resp. $\Lambda_{\pi}$) the sum of its left (resp. right) sublattices $\Lambda_{v_i,m_i}$ (resp. $\Lambda_{\pi_i,m_i}$) of index $t$ and multiplicities $m(i) = \sup(m_i)$:

$$T_L(n; t) \Lambda_v = \bigoplus_{i=1}^{t} \Lambda_{v_i,m_i}$$

(resp. $T_R(n; t) \Lambda_{\pi} = \bigoplus_{i=1}^{t} \Lambda_{\pi_i,m_i}$).

So, the Hecke bioperator $T_R(n; t) \otimes T_L(n; t)$ is defined by the bicorrespondence:

$$(T_R(n; t) \otimes T_L(n; t)) (\Lambda_{\pi} \otimes \Lambda_v) = \bigoplus_{i,m_i} (\Lambda_{\pi_i,m_i} \otimes \Lambda_{v_i,m_i}) .$$

2. As

$$\Lambda_{\pi_i,m_i} \otimes \Lambda_{v_i,m_i} \simeq g_n(\mathcal{O}_{L_{\pi_i,m_i}} \times \mathcal{O}_{L_{v_i,m_i}})$$

$$\in \text{GL}_n(\mathcal{O}_{L_{\pi}} \times \mathcal{O}_{L_v}) \subset \text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2) ,$$

$g_n(\mathcal{O}_{L_{\pi_i,m_i}} \times \mathcal{O}_{L_{v_i,m_i}})$ can be chosen as a coset representative of the tensor product $T_R(n; t) \otimes T_L(n; t)$ of Hecke operators [Pie1].

2.10 Corollary

There exists an injective morphism:

$$m_{\Lambda_{R \times L} \to M_{R \times L}} : \Lambda_{\pi} \otimes \Lambda_v \longrightarrow M_R \otimes M_L$$

from the pseudo-ramified bisemilattice $\Lambda_{\pi} \otimes \Lambda_v$ to the corresponding $G(n)(L_{\pi} \times L_v)$-bisemimodule $M_R \otimes M_L$.

Proof. Indeed, $\Lambda_{\pi} \otimes \Lambda_v$ is a pseudo-ramified bisemilattice into the $B_{\pi} \otimes B_v$-bisemimodule $M_R \otimes M_L$ according to proposition 2.8. And thus, the decomposition of $M_R \otimes M_L$ into $G(n)(L_{\pi_i,m_i} \times L_{v_i,m_i})$-subbisemimodules $M_{\pi_i,m_i} \otimes M_{v_i,m_i}$, being in one-to-one correspondence with the conjugacy class representatives of $\text{GL}_n(L_{\pi} \times L_v)$, results from the similar decomposition of $\Lambda_{\pi} \otimes \Lambda_v$ into

$$\Lambda_{\pi} \otimes \Lambda_v = \bigoplus_{i,m_i} (\Lambda_{\pi_i,m_i} \otimes \Lambda_{v_i,m_i}) .$$
2.11 Toroidal compactification of lattice bisemispaces

The space \( X = \text{GL}_n(\mathbb{R})/\text{GL}_n(\mathbb{Z}) \) corresponds to the set of lattices of \( \mathbb{R}^n \). In this perspective, a left (resp. right) pseudo-ramified lattice semispace

\[
X_L = T_n(L_L)/T_n(\mathbb{Z}/N\mathbb{Z}) \quad \text{(resp.} \quad X_R = T_n^d(L_R)/T_n^d(\mathbb{Z}/N\mathbb{Z}) \text{)}
\]

where \( L_R \) and \( L_L \) are compactified commutative division semirings corresponding to the semifields \( \bar{L}_R \) and \( \bar{L}_L \) introduced in section 2.1, is introduced in such a way that the cosets of \( X_L \) (resp. \( X_R \)) correspond to the conjugacy classes of \( T_n(L_v) \) (resp. \( T_n^d(L_T) \)).

A toroidal compactification \( \gamma_{X_L^T} \) (resp. \( \gamma_{X_R^T} \)) is envisaged on \( X_L \) (resp. \( X_R \)) in such a way that it corresponds to a projective mapping which can be decomposed into a two step sequence [Pic3]:

1. the points \( \gamma_{a_L[i,m]} \in g_L^{(n)}[i,m] \) (resp. \( \gamma_{a_R[i,m]} \in g_R^{(n)}[i,m] \)) of the conjugacy class representative \( g_L^{(n)}[i,m] \) (resp. \( g_R^{(n)}[i,m] \)) of \( T(n)(L_v) \subset G(n)(L_T \times L_v) \) (resp. \( T(n)(L_T) \)) are mapped onto the origin of \( L_v \) (resp. \( L_T \)).

2. these points \( \gamma_{a_L[i,m]} \) (resp. \( \gamma_{a_R[i,m]} \)) are then projected symmetrically from the origin of \( L_v \) (resp. \( L_T \)) into a connected compact semivariety which is a \( n \)-dimensional real semitorus \( T^n_L[i,m] \) (resp. \( T^n_R[i,m] \)) in \( L_v \) (resp. \( L_T \)) where \( L_v \) (resp. \( L_T \)) is given by:

\[
L_v^T = \{L_{v_i,m_i}^t\}_{i=1}^t \quad \text{(resp.} \quad L_T^T = \{L_{v_i,m_i}^t\}_{i=1}^t \text{)}
\]

with \( L_{v_i,m_i}^t \) (resp. \( L_{v_i,m_i}^t \)) being a left (resp. right) toroidal completion.

The toroidal compactification \( \gamma_{X_L^T} \) (resp. \( \gamma_{X_R^T} \)) of the lattice semispace \( X_L \) (resp. \( X_R \)) is thus the projective mapping:

\[
\gamma_{X_L^T} : \quad X_L \longrightarrow X_L^T = T_n(L_L^T)/T_n(\mathbb{Z}/N\mathbb{Z}) \quad \text{(resp.} \quad \gamma_{X_R^T} : \quad X_R \longrightarrow X_R^T = T_n^d(L_R^T)/T_n^d(\mathbb{Z}/N\mathbb{Z}) \text{)}
\]

sending \( X_L \) (resp. \( X_R \)) into the corresponding toroidal lattice semispace \( X_L^T \) (resp. \( X_R^T \)) such that its cosets correspond to the conjugacy class representatives \( g_T^{(n)}[i,m] \) (resp. \( g_R^{(n)}[i,m] \)) of \( T(n)(L_v) \) (resp. \( T(n)(L_T) \)) which are \( n \)-dimensional real semitori where \( L_R \) (resp. \( L_L \)) is the toroidal equivalent of \( L_R \) (resp. \( L_L \)).

A pseudo-ramified lattice bisemispace

\[
X_{R \times L} = X_R \otimes X_L = \text{GL}_n(L_R \times L_L)/\text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2)
\]

can naturally be generated from the pseudo-ramified lattice semispaces \( X_R \) and \( X_L \).
The toroidal compactification \( \gamma_{X_R^T} \times \gamma_{X_L^T} \) then maps the pseudo-ramified lattice bisemispace \( X_{R\times L} \) into its toroidal equivalent \( X_{R\times L}^T \) according to:

\[
\gamma_{X_R^T} \times \gamma_{X_L^T} : \quad X_{R\times L} \longrightarrow X_{R\times L}^T
\]

where \( X_{R\times L}^T \) is given by:

\[
X_{R\times L}^T = \text{GL}_n(L_R^T \times L_L^T) / \text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2).
\]

2.12 Proposition

Let \( X_{R\times L}^T = \text{GL}_n(L_R^T \times L_L^T) / \text{GL}_n((\mathbb{Z}/N\mathbb{Z})^2) \) be the toroidal pseudo-ramified lattice bisemispace. As \( X_{R\times L}^T \) corresponds to the representation space of \( \text{GL}_n(L_v^T \times L_v^T) \) given by the algebraic bilinear semigroup \( G^{(n)}(L_v^T \times L_v^T) \), it decomposes according to:

\[
X_{R\times L}^T \simeq G^{(n)}(L_v^T \times L_v^T) = \bigoplus_{i=1}^{t} \bigoplus_{m_i} g^{(n)}_{T_{R\times L}} [i, m_i]
\]

where \( g^{(n)}_{T_{R\times L}} \) is a conjugacy class representative given by the product, right by left, of \( n \)-dimensional semitori \( g^{(n)}_{T_R} [i, m_i] \) and \( g^{(n)}_{T_L} [i, m_i] \).

Proof. Indeed, the algebraic bilinear semigroup \( G^{(n)}(L_v^T \times L_v^T) \) results from \( G^{(n)}(L_v^T \times L_v) \) by the toroidal compactification:

\[
\gamma_{X_R^T} \times \gamma_{X_L^T} : \quad G^{(n)}(L_v^T \times L_v) \longrightarrow G^{(n)}(L_v^T \times L_v^T).
\]

As \( G^{(n)}(L_v^T \times L_v) \) decomposes into conjugacy class representatives according to proposition 2.4, it is also the case for the algebraic bilinear semigroup \( G^{(n)}(L_v^T \times L_v^T) \) whose conjugacy class representatives are products, right by left, of \( n \)-dimensional semitori due to the projective morphism \( \gamma_{X_R^T} \times \gamma_{X_L^T} \). ■

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3 Bilinear cuspidal representations

3.1 Bilinear global cuspidal representations on infinite real places

3.1.1 Bisemisheaf of rings on the bilinear algebraic semigroup $G^{(n)}(L_T^E \times L_T^O)$

The set of differentiable smooth functions $\phi(g_{T_L}^{(n)}[i,m_i])$ (resp. $\phi(g_{T_R}^{(n)}[i,m_i])$, 1 ≤ i ≤ t ≤ ∞, on the conjugacy class representatives $g_{T_L}^{(n)}[i,m_i]$ (resp. $g_{T_R}^{(n)}[i,m_i]$) of the algebraic semigroup $G^{(n)}(L_T^E)$ (resp. $G^{(n)}(L_T^O)$) are the sections of a semisheaf of rings $\Phi(G^{(n)}(L_T^E))$ (resp. $\Phi(G^{(n)}(L_T^O))$).

According to proposition 2.12, each section $\phi(g_{T_L}^{(n)}[i,m_i]) \subset \Gamma(\Phi(G^{(n)}(L_T^E)))$ (resp. $\phi(g_{T_R}^{(n)}[i,m_i]) \subset \Gamma(\Phi(G^{(n)}(L_T^O)))$) is a differentiable function on a n-dimensional real semitorus as it will be developed in proposition 3.1.2.

Similarly, the set of differentiable smooth bifunctions $\phi_{R \times L}(g_{T_{R \times L}}^{(n)}[i,m_i]) = \phi(g_{T_L}^{(n)}[i,m_i]) \otimes \phi(g_{T_R}^{(n)}[i,m_i])$ on the conjugacy class representatives $g_{T_{R \times L}}^{(n)}[i,m_i]$ of the algebraic bilinear semigroup $G^{(n)}(L_T^E \times L_T^O)$ is the set $\Gamma(\Phi(G^{(n)}(L_T^E \times L_T^O)))$ of bisections of the bisemisheaf of rings $\Phi(G^{(n)}(L_T^E \times L_T^O))$ [Pie4].

On this set $\Gamma(\Phi(G^{(n)}(L_T^E \times L_T^O)))$ of bisections, a global elliptic $\Gamma(\Phi(G^{(n)}(L_T^E \times L_T^O)))$-bismimodule $ELLIP_R(n,i,m_i) \otimes_D ELLIP_L(n,i,m_i)$ will be explicitly constructed in the following proposition.

3.1.2 Proposition

The functional representation space of the bilinear algebraic semigroup $G^{(n)}(L_T^E \times L_T^O)$ can be given by the product, right by left, of n-dimensional global elliptic semimodules $ELLIP_R(n,i,m_i)$ and $ELLIP_L(n,i,m_i)$ according to [Pie1]:

$$FResp[G^{(n)}(L_T^E \times L_T^O)] = ELLIP_R(n,i,m_i) \otimes ELLIP_L(n,i,m_i)$$

where:

- $ELLIP_L(n,i,m_i) = \bigoplus_{i=1}^{m_i} \lambda^L(n,i,m_i) e^{2\pi i(i)x}$, 1 ≤ i ≤ t ≤ ∞,

- $ELLIP_R(n,i,m_i) = \bigoplus_{i=1}^{m_i} \lambda^R(n,i,m_i) e^{-2\pi i(i)x}$

is a (truncated) Fourier series with

- $x = \sum_{\beta=1}^{n} x_\beta \bar{e}^\beta$ a vector of $\mathbb{R}^n$;

- $\lambda(n,i,m_i) = \prod_{c=1}^{n} \lambda_c(n,i,m_i)$ a product, right by left, of Hecke characters since $\lambda_c(n,i,m_i)$ is an eigenvalue of $g_n(O_{L_T^E,m_i} \times O_{L_T^O,m_i})$;

- each term $\lambda^L(n,i,m_i) e^{2\pi i(i)x}$ (resp. $\lambda^R(n,i,m_i) e^{-2\pi i(i)x}$) of $ELLIP_L(n,i,m_i)$ (resp. $ELLIP_R(n,i,m_i)$) is a section of the semisheaf $\Phi(G^{(n)}(L_T^E))$ (resp. $\Phi(G^{(n)}(L_T^O))$, i.e. a smooth differentiable function on a n-dimensional semitorus $g_{T_L}^{(n)}[i,m_i]$ (resp. $g_{T_R}^{(n)}[i,m_i]$).
Proof. 1. According to proposition 2.12, the representation space $\text{Repsp}(\text{GL}_n(L^n_{\pi_{\circ}} \times L^n_{v_{\circ}}))$ decomposes according to its conjugacy classes representatives:

$$\text{Repsp}(\text{GL}_n(L^n_{\pi_{\circ}} \times L^n_{v_{\circ}})) = G^{(n)}(L^n_{\pi_{\circ}} \times L^n_{v_{\circ}}) = \bigoplus_{i=1}^t \bigoplus_{m_i} g^{(n)}_{T_{R \times L}}[i, m_i]$$

where

$$g^{(n)}_{T_{R \times L}}[i, m_i] = \pi_T[i, m_i] \times g^{(n)}_{T_L}[i, m_i].$$

According to section 2.11, $g^{(n)}_{T_L}[i, m_i]$ (resp. $g^{(n)}_{T_R}[i, m_i]$) is a $n$-dimensional real semitorus localized in the upper (resp. lower) half space.

2. The decomposition of $\text{Repsp}(\text{GL}_n(L^n_T \times L^n_T))$ into conjugacy class representatives $g^{(n)}_{T_R \times L}[i, m_i]$ results from an endomorphism of $\text{Repsp}(\text{GL}_n(L^n_T \times L^n_T))$ into itself generated by the action of Hecke bioperators $T_{R}(n; t) \otimes T_{L}(n; t)$ [Pie1].

3. Each smooth continuous function on the left (resp. right) conjugacy class representative $g^{(n)}_{T_L}[i, m_i]$ (resp. $g^{(n)}_{T_R}[i, m_i]$) is (a function on) a $n$-dimensional real semitorus $T^n_{L}[i, m_i]$ (resp. $T^n_{R}[i, m_i]$) which has the following analytic representation:

$$T^n_{L}[i, m_i] \simeq \lambda^\frac{i}{2}(n, i, m_i) e^{2\pi i(i)x}$$

(resp. $T^n_{R}[i, m_i] \simeq \lambda^\frac{i}{2}(n, i, m_i) e^{-2\pi i(i)x}$).

Indeed, as $g^{(n)}_{T_L}[i, m_i]$ (resp. $g^{(n)}_{T_R}[i, m_i]$) is the non abelian equivalent of the toroidal completion $L^n_{T_{V_i,m_i}}$ (resp. $L^n_{T_{V_i,m_i}}$) according to section 2.3, we have to consider the global Frobenius substitution at the left (resp. right) place $v_i$ (resp. $\overline{v}_i$) given by the mapping:

$$e^{2\pi i x} \mapsto e^{2\pi i(i)x} \quad x \in \mathbb{R}^n,$$

(resp. $e^{-2\pi i x} \mapsto e^{-2\pi i(i)x}$)

$$i = \sqrt{-1},$$

$(i) \in \mathbb{N}$ being the global residue degree of this infinite place $v_i$ (resp. $\overline{v}_i$).

On the other hand, as $T^n_{R \times L}[i, m_i]$ results from an endomorphism of $\text{Repsp}(\text{GL}_n(L^n_T \times L^n_T))$ into itself, the scalar $\lambda(n, i, m_i)$ will correspond to the eigenvalues of the associated coset representative of the product of Hecke operators.

This coset representative of $T_{R}(n; t) \otimes T_{L}(n; t)$ is then given by $g_n(O_{L^n_{T_{V_i,m_i}}} \times O_{L^n_{T_{V_i,m_i}}})$ according to proposition 2.9.

If $\{\lambda_c(n, i, m_i)\}_{c=1}^n$ denotes the set of eigenvalues of $g_n(O_{L^n_{T_{V_i,m_i}}} \times O_{L^n_{T_{V_i,m_i}}})$, then

$$\lambda(n, i, m_i) = \prod_{c=1}^n \lambda_c(n, i, m_i)$$

can be considered as a product, right by left, of Hecke characters and its square root $\lambda^\frac{i}{2}(n, i, m_i)$ can be chosen as the coefficient of $T^n_{L}[i, m_i]$ (resp. $T^n_{R}[i, m_i]$).
3.1.3 Analytic representation of semitori with respect to Hecke characters

Let

\[ \text{FRepsp}[G^{(n)}(L^T_{\mathfrak{p},m_p} \times L^T_{v_p,m_p})] = \text{ellip}_{R \times L}(n, [p], m_p) \]
\[ = \lambda^{\frac{1}{2}}(n, p, m_p) e^{-2\pi i p x} \otimes \lambda^{\frac{1}{2}}(n, p, m_p) e^{2\pi i p x} \]

be the analytic representation of the algebraic bilinear subsemigroup \( G^{(n)}(L^T_{\mathfrak{p},m_p} \times L^T_{v_p,m_p}) \) with respect to the \((p, m_p)\)-th completion of \( R \times L \).

\( \text{FRepsp}[G^{(n)}(L^T_{\mathfrak{p},m_p} \times L^T_{v_p,m_p})] \) has the analytic development \( \text{ellip}_{R \times L}(n, [p], m_p) \) which is in bijection with the product of a right \( n \)-dimensional semitorus by its left equivalent.

Indeed, a left (resp. right) semitorus \( T^1_{\mathfrak{L}}[p, m_p] \) (resp. \( T^1_{R}[p, m_p] \)) is diffeomorphic to

\[ T^1_{\mathfrak{L}}[p, m_p] = T^1_{c_{\mathfrak{L}}}[p, m_p] = r_c(p, m_p) e^{2\pi i p x c} , \quad x_c \in \mathbb{R}^1 , \]

in such a way that the one-dimensional semitorus \( T^1_{c_{\mathfrak{L}}}[p, m_p] \) (resp. \( T^1_{c_{R}}[p, m_p] \)) has the representation given by the following analytic development:

\[ T^1_{c_{\mathfrak{L}}}[p, m_p] = r_c(p, m_p) e^{-2\pi i p x c} , \quad x_c \in \mathbb{R}^1 , \]

whose radius \( r_c(p, m_p) \) can be expressed with respect to \( \lambda_c(p, m_p) \) according to [Pie2]. So, we have that:

\[ \text{ellip}_{\mathfrak{L}}(n, [p], m_p) \approx \prod_{c=1}^{n} r_c(p, m_p) e^{2\pi i p x c} \]

(resp. \( \text{ellip}_{R}(n, [p], m_p) \approx \prod_{c=1}^{n} r_c(p, m_p) e^{-2\pi i p x c} \)).

3.1.4 Proposition

Let \( \text{ellip}_{R \times L}(n, [p], m_p) \in \text{ELLIP}_{R \times L}(n, i, m_i) \) be a global elliptic \((L^T_{\mathfrak{p},m_p} \times L^T_{v_p,m_p})\)-subbisemimodule.

Then,

\[ \text{ellip}_{R \times L}(n, [p], m_p) = \lambda^{\frac{1}{2}}(n, p, m_p) e^{-2\pi i p x} \otimes \lambda^{\frac{1}{2}}(n, p, m_p) e^{2\pi i p x} , \quad x \in \mathbb{R}^n , \]
\[ \approx \prod_{c=1}^{n} T^1_{c_{R}}[p, m_p] \otimes T^1_{c_{\mathfrak{L}}}[p, m_p] \]
\[ = \prod_{c=1}^{n} r_c(p, m_p) e^{-2\pi i p x c} \otimes r_c(p, m_p) e^{2\pi i p x c} \]

is the analytic representation of \( G^{(n)}(L^T_{\mathfrak{p},m_p} \times L^T_{v_p,m_p}) \).
Sketch of proof. In fact, 

\[ \text{ellip}_{R \times L}(n, [p], m_p) = \lambda_1^2(n, p, m_p) e^{-2\pi ip} \otimes \lambda_1^2(n, p, m_p) e^{2\pi ip} \]

is a deformation of the product, right by left, of \( n \)-dimensional semitori:

\[ T_R^n[p, m_p] \otimes T_L^n[p, m_p] \cong \prod_{c=1}^{n} (T^c_{cR}[p, m] \times T^c_{cL}[p, m_p]) \]

resulting from the isomorphism:

\[ I_{EL \rightarrow T} : \text{ellip}_{R \times L}(n, [p], m_p) \sim \prod_{c=1}^{n} r_c(p, m_p) . \]

3.1.5 Proposition (Langlands global correspondence)

The functional representation space of the toroidal compactification of \( G^{(n)}(L_{\overline{\pi}} \otimes L_{\overline{w}}) \) :

\[ \gamma_{G^{(n)}_{TR \times L}} : G^{(n)}(L_{\overline{\pi}} \times L_{\overline{w}}) \rightarrow G^{(n)}(L^T_{\overline{\pi}} \times L^T_{\overline{w}}) , \]

where \( \gamma_{G^{(n)}_{TR \times L}} \) is an isomorphism, leads to the morphism

\[ G^{(n)}(L_{\overline{\pi}} \times L_{\overline{w}}) \rightarrow \text{ELLIP}_{R \times L}(n, i, m_i) \]

which is equivalent to the following Langlands correspondence:

\[ \text{Irr Rep}^{(n)}(W_{L_{\overline{\pi}}} \times W_{L_{\overline{w}}}) \sim \text{Irr cusp(GL}_{n}(L_{\overline{\pi}} \times L_{\overline{w}})) \]

where

- \( \text{Irr Rep}^{(n)}(W_{L_{\overline{\pi}}} \times W_{L_{\overline{w}}}) \) is the sum of products, right by left, of the equivalence classes of the irreducible \( n^2 \)-dimensional representation of the bilinear global Weil group \( (W_{L_{\overline{\pi}}} \times W_{L_{\overline{w}}}) \);

- \( \text{Irr cusp(GL}_{n}(L_{\overline{\pi}} \times L_{\overline{w}})) \) is the sum of the products, right by left, of the conjugacy classes of the irreducible cuspidal representation space of \( \text{GL}_{n}(L_{\overline{\pi}} \times L_{\overline{w}}) \).

Proof. 1. Consider the sequence of morphisms:

\[ G^{(n)}(L_{\overline{\pi}} \times L_{\overline{w}}) \xrightarrow{\gamma_{G^{(n)}_{TR \times L}}} G^{(n)}(L^T_{\overline{\pi}} \times L^T_{\overline{w}}) \xrightarrow{\text{FRepsp}(G^{(n)}_{TR \times L})} \text{ELLIP}_{R \times L}(n, i, m_i) \]

where \( \text{FRepsp}(G^{(n)}_{TR \times L}) \), being the functional representation of the algebraic bilinear semigroup \( G^{(n)}(L^T_{\overline{\pi}} \times L^T_{\overline{w}}) \), is the global elliptic bisemimodule \( \text{ELLIP}_{R \times L}(n, i, m_i) = \text{ELLIP}_R(n, i, m_i) \otimes \text{ELLIP}_L(n, i, m_i) \) according to proposition 3.1.2.

\( G^{(n)}(L_{\overline{\pi}} \times L_{\overline{w}}) \) is then clearly in bijection with \( \text{ELLIP}_{R \times L}(n, i, m_i) \);
2. $G^{(n)}(L_{\mathbb{R}} \times L_{\mathbb{Q}})$ is the $n$-dimensional representation space of $(W_{L_{\mathbb{R}}}^{ab} \times W_{L_{\mathbb{Q}}}^{ab})$ according to section 2.3.

On the other hand, $\text{ELLIP}_{R \times L}(n, i, m_i)$ constitutes a cuspidal representation space of $\text{GL}_n(L_{\mathbb{R}} \times L_{\mathbb{Q}})$ as developed in [Pie1].

3. the searched bijection $\text{Irr Rep}^{(n)}(W_{L_{\mathbb{R}}}^{ab} \times W_{L_{\mathbb{Q}}}^{ab}) \rightarrow \text{Irr cusp}(\text{GL}_n(L_{\mathbb{R}} \times L_{\mathbb{Q}}))$ thus follows.

\section*{3.2 Bilinear global cuspidal representations (on infinite real places) covered by $p^\ell$-th roots}

\subsection*{3.2.1 Proposition (Covering of one-dimensional semitori (i.e. semicircles) by $p^\ell$-th roots)}

Let $p$ be a prime number, $v_p$ (resp. $\overline{v}_p$) the $p$-th primary real infinite place and $v_{p+h}$ (resp. $\overline{v}_{p+h}$) the $h$-th real infinite place above $v_p$ (resp. $\overline{v}_p$).

Then, the left (resp. right) one-dimensional semitorus $T_{cL}^1[p+h, m_{p+h}]$ (resp. $T_{cR}^1[p+h, m_{p+h}]$) can be covered by the $p^\ell$-th complex roots if we introduce the (etale) covering map:

\begin{align*}
R_{L,p^\ell-\rightarrow p+h}^1 : & \rightarrow \quad T_{cL}^1[p+h, m_{p+h}, \overline{p}'] = r_c(p+h, m_{p+h}, p')^p e^{2\pi i (p') x_c} \\
(\text{resp. } R_{R,p^\ell-\rightarrow p+h}^1 : & \rightarrow \quad T_{cR}^1[p+h, m_{p+h}, \overline{p}'] = r_c(p+h, m_{p+h}, p')^p e^{-2\pi i (p') x_c})
\end{align*}

in such a way that:

\begin{align*}
T_{cL}^1[p+h, m_{p+h}] = T_{cL}^1[p+h, m_{p+h}, \overline{p}'] \\
(\text{resp. } T_{cR}^1[p+h, m_{p+h}] = T_{cR}^1[p+h, m_{p+h}, \overline{p}']
\end{align*}

i.e. if:

a) $r_c(p+h, m_{p+h}) = r_c(p+h, m_{p+h}, p')^p$;

b) $p^\ell x' = x_c + 2k\pi$, $0 \leq k \leq (p^\ell - 1)/2$;

c) $p^\ell - 1 / 2 \geq (p + h)$.

\textbf{Proof.} 1. The etale covering map $R_{L,p^\ell-\rightarrow p+h}^1$ (resp. $R_{R,p^\ell-\rightarrow p+h}^1$) is equivalent to finding complex numbers $z'_L$ (resp. $z'_R$) (i.e. $p^\ell$-th complex roots) of which $p^\ell$-th power is equal to $z_L \in T_{cL}^1[p+h, m_{p+h}]$ (resp. $z_R \in T_{cR}^1[p+h, m_{p+h}]$), i.e.

\begin{align*}
(z'_L)^{p^\ell} = z_L & \quad (\text{resp. } (z'_R)^{p^\ell} = z_R)
\end{align*}

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which leads to the conditions a) and b) of this proposition.

2. Condition c) results from the fact that \( x_c \) is a point of order \( \# \text{Nu} \times N \), and, thus, that to each point \( x_c \) correspond \( p^\ell/2 \) roots \( p^\ell \)-th.

So, the number of points on \( T_{c, \ell}^1[p + h, m_{p+h}] \) (or \( T_{c, \ell}^1[p + h, m_{p+h}] \)) is equal to:

\[
n_{T_{c, \ell}^1[p + h]} = \# \text{Nu} \times N \times (p + h),
\]

where \( (p + h) \) is the global residue degree \( f_{v_{p+h}} \), \( \# \text{Nu} \) is the number of nonunits and \( N \) is the degree of an irreducible completion according to section 2.1, while the number of points on \( T_{c, \ell}^1[p + h, m_{p+h}, p] \) (or \( T_{c, \ell}^1[p + h, m_{p+h}, p] \)) is equal to:

\[
n_{T_{c, \ell}^1[p + h, p]} = \# \text{Nu} \times N \times (p^\ell - 1)/2.
\]

The covering map \( R_{L_{p^\ell-\ell p+h}^1} \) (resp. \( R_{R_{p^\ell-\ell p+h}^1} \)) is an isomorphism if \( (p^\ell - 1)/2 = p + h \) and an epimorphism if \( (p^\ell - 1)/2 > p + h \).

\[\blacksquare\]

### 3.2.2 Corollary

To a semitorus \( T_{c, \ell}^1[p, m_p] \) (resp. \( T_{c, \ell}^1[p, m_p] \)) at the \( v_p \)-th \( (\text{resp. } \varpi_p \)-th) real archimedean place corresponds an etale covering map \( R_{L_{p^\ell-\ell p}^1} \) (resp. \( R_{R_{p^\ell-\ell p}^1} \)) by \( p \)-th complex roots.

**Proof.** Referring to proposition 3.2.1, we see that this case corresponds to the condition c) with \( h = 0 \) and thus \( \ell = 1 \). The covering map \( R_{L_{p^\ell-\ell p+h}^1} \) (and \( R_{R_{p^\ell-\ell p+h}^1} \)) of this proposition occurs then at the conditions:

a) \( r_c(p, m_p) = r_c(p, m_p, p) \);

b) \( px_c = x_c + 2k\pi, \ 0 \leq k \leq (p - 1)/2 \);

c) \( (p^\ell - 1)/2 = p + h \) with \( \ell = 1 \) and \( h = 0 \).

\[\blacksquare\]

### 3.2.3 Corollary

The semitori \( T_{c, \ell}^1[j, m_j] \) (resp. \( T_{c, \ell}^1[j, m_j] \)) at the \( v_j \)-th \( (\text{resp. } \varpi_j \)-th) real archimedean places below the \( v_p \)-th \( (\text{resp. } \varpi_p \)-th) place, i.e. for \( j < p \), cannot be covered by \( p \)-th complex roots.

**Proof.** Indeed, if \( v_j < v_p \) \( (\text{resp. } \varpi_j < \varpi_p) \), then there are points of \( T_{c, \ell}^1[j, m_j] \) (resp. \( T_{c, \ell}^1[j, m_j] \)) which are not covered by \( p \)-th complex roots.

\[\blacksquare\]
3.2.4 Proposition

The \( n \)-dimensional semitorus \( T^n_L[p + h, m_{p+h}] \) (resp. \( T^n_R[p + h, m_{p+h}] \)), at the \( v_{p+h} \)-th (resp. \( \varpi_{p+h} \)-th) archimedean place, will be covered by \( p^\ell \)-th complex roots according to:

\[
R^n_{L,p^\ell\rightarrow p+h}: \quad T^n_L[p + h, m_{p+h}, \overline{p}'] \simeq \prod_{c=1}^{n} r_c(p + h, m_{p+h}, \overline{p}') p^{\ell} e^{2\pi i(p') x'_c} \\
\longrightarrow T^n_R[p + h, m_{p+h}] \simeq \prod_{c=1}^{n} r_c(p + h, m_{p+h}) e^{2\pi i(p) x_c}
\]

(resp. \( R^n_{R,p^\ell\rightarrow p+h} \):

\[
T^n_R[p + h, m_{p+h}, \overline{p}'] \simeq \prod_{c=1}^{n} r_c(p + h, m_{p+h}, \overline{p}') p^{\ell} e^{-2\pi i(p') x'_c} \\
\longrightarrow T^n_R[p + h, m_{p+h}] \simeq \prod_{c=1}^{n} r_c(p + h, m_{p+h}) e^{-2\pi i(p) x_c}
\]

leading to the same conditions as these considered in proposition 3.2.1.

Proof. This is evident since this proposition is the \( n \)-dimensional generalization of proposition 3.2.1.

3.2.5 Proposition (Base change under covering by \( p^\ell \)-th complex roots)

Let the covering map \( R^n_{L,p^\ell\rightarrow p+h} \) (resp. \( R^n_{R,p^\ell\rightarrow p+h} \)) of the semitorus \( T^n_L[p + h, m_{p+h}] \) (resp. \( T^n_R[p + h, m_{p+h}] \)) on the \( v_{p+h} \)-th (resp. \( \varpi_{p+h} \)-th) place by the semitorus \( T^n_L[p + h, m_{p+h}, \overline{p}'] \) (resp. \( T^n_R[p + h, m_{p+h}, \overline{p}'] \)) be an isomorphism.

Then, this covering map by \( p^\ell \)-th complex roots corresponds to an equivariant base change from a base of dimension \( (p + h)^n \) to a covering base of dimension \( ((p^\ell - 1)/2)^n \).

Proof. This results directly from proposition 3.2.1, condition c), and proposition 3.2.4.

3.2.6 Corollary

Only the \( n \)-dimensional global elliptic semimodules

\[
\text{ELLIP}_L(n, p \leq i, m_i) = \bigoplus_{i=p}^{m_i} \lambda_L^{i\overline{i}}(n, p \leq i, m_i) e^{2\pi i x}, \\
\text{ELLIP}_R(n, p \leq i, m_i) = \bigoplus_{i=p}^{m_i} \lambda_L^{i\overline{i}}(n, p \leq i, m_i) e^{-2\pi i x}, 
\]

\( x \in \mathbb{R}^n, i \equiv p + h, h \) running from 0 to \( \infty, p \leq i \leq t \leq \infty \),

with terms \( i \geq p \) can be covered by \( p^\ell \)-th complex roots, \( \ell \) varying.

Proof. Indeed, according to proposition 3.2.1 and corollary 3.2.3, only \( n \)-dimensional semitori \( T^n_L[p + h, m_{p+h}] \) (resp. \( T^n_R[p + h, m_{p+h}] \)) having a rank \( (p + h)^n \) can be covered by \( p^\ell \)-th complex roots. They thus correspond to terms \( i = p + h, 0 \leq h \leq \infty \).

\[\blacksquare\]
3.2.7 \textbf{Semisheaves associated with ELLIP}_L(n, p \leq i, m_i) \textbf{ and ELLIP}_R(n, p \leq i, m_i)

Let $\Phi(G^{(n)}(L_T^v))$ (resp. $\Phi(G^{(n)}(L_T^h))$) be the semisheaf on the algebraic semigroup $G^{(n)}(L_T^v)$ (resp. $G^{(n)}(L_T^h)$) and let ELLIP$_L(n, i, m_i)$ (resp. ELLIP$_R(n, i, m_i)$) be the associated global elliptic semimodule $\Gamma(\Phi(G^{(n)}(L_T^v)))$ (resp. $\Gamma(\Phi(G^{(n)}(L_T^h)))$) semimodule in such a way that to each section $\phi(g^{(n)}_T[i, m_i])$ (resp. $\phi(g^{(n)}_T[i, m_i])$) of $\Phi(G^{(n)}(L_T^v))$ (resp. $\Phi(G^{(n)}(L_T^h))$) corresponds one term $\text{ellip}_L(n, [i], m_i) = \lambda^2(n, i, m_i) e^{2\pi i x} \eta$ (resp. $\text{ellip}_R(n, [i], m_i) = \lambda^2(n, i, m_i) e^{-2\pi i x}$) of ELLIP$_L(n, i, m_i)$ (resp. ELLIP$_R(n, i, m_i)$).

Let then $\Phi(G^{(n)}(L_T^v))$ (resp. $\Phi(G^{(n)}(L_T^h))$) denote the semisheaf on the algebraic semigroup over the set of completions $[v_p] = \{v_p, v_{p+1}, \ldots, v_1\}$ (resp. $[\overline{v}_p] = \{\overline{v}_p, \overline{v}_{p+1}, \ldots, \overline{v}_1\}$) superior and equal to $p$ and let ELLIP$_L(n, p \leq i, m_i)$ (resp. ELLIP$_R(n, p \leq i, m_i)$) be the associated global elliptic semimodule as introduced in corollary 3.2.6.

Then, there exists the epimorphism

\[
\text{em}^{(n)}_{v-[v_p]} : \Phi(G^{(n)}(L_T^v)) \longrightarrow \Phi(G^{(n)}(L_T^v)) \quad (\text{resp. } \text{em}^{(n)}_{\overline{v}-[\overline{v}_p]} : \Phi(G^{(n)}(L_T^h)) \longrightarrow \Phi(G^{(n)}(L_T^h)))
\]

of which kernel $\ker(\text{em}^{(n)}_{v-[v_p]}) = \Phi(G^{(n)}(L_T^v-[v_p]))$ (resp. $\ker(\text{em}^{(n)}_{\overline{v}-[\overline{v}_p]}) = \Phi(G^{(n)}(L_T^h-[\overline{v}_p]))$ is the complementary semisheaf $\Phi(G^{(n)}(L_T^v-[v_p]))$ (resp. $\Phi(G^{(n)}(L_T^h-[\overline{v}_p]))$) on the algebraic semigroup $G^{(n)}(L_T^v-[v_p])$ (resp. $G^{(n)}(L_T^h-[\overline{v}_p])$) over the set of completions $v-[v_p] = \{v_1, \ldots, v_{p-1}\}$ (resp. $\overline{v}=[\overline{v}_p] = \{\overline{v}_1, \ldots, \overline{v}_{p-1}\}$).

3.2.8 \textbf{Proposition}

\textit{Each semitorus} $T^1_{cl}[p+h, m_{p+h}, \overline{p}^0]$ (resp. $T^1_{cr}[p+h, m_{p+h}, \overline{p}^0]$) \textit{covering the global elliptic subsemimodule} $\text{ellip}_L(1, [p+h], m_{p+h}) = \lambda^2(1, [p+h], m_{p+h}) e^{2\pi i x c}$ (resp. $\text{ellip}_R(1, [p+h], m_{p+h}) = \lambda^2(1, [p+h], m_{p+h}) e^{-2\pi i x c}$) \textit{is a discrete valuation semiring of which:}

\begin{enumerate}
  \item the uniformizing element is $r_c(p+h, m_{p+h}, \overline{p}^0) p$
  \item the units are the invertible elements $e^{2\pi i x c}$ (resp. $e^{-2\pi i x c}$) each $x_c$ verifying $p^c x_c = x_c + 2k\pi$, $x_c \in \mathbb{R}$.
\end{enumerate}

\textbf{Proof.} 1. Referring to proposition 3.2.1, we see that the semitorus $T^1_{cl}[p+h, m_{p+h}, \overline{p}^0] = r_c(p+h, m_{p+h}, \overline{p}^0) p e^{2\pi i x c}$ (resp. $T^1_{cr}[p+h, m_{p+h}, \overline{p}^0] = r_c(p+h, m_{p+h}, \overline{p}^0) p e^{-2\pi i x c}$) covering the global elliptic subsemimodule $\text{ellip}_L(1, [p+h], m_{p+h})$ (resp. $\text{ellip}_R(1, [p+h], m_{p+h})$) must be a discrete valuation semiring because it is a principal ideal domain having a unique nonzero prime ideal given by:

\[
T^1_{cl}[p+h, m_{p+h}, \overline{p}] = r_c(p+h, m_{p+h}, \overline{p}) p e^{2\pi i x c} \quad (\text{resp. } T^1_{cr}[p+h, m_{p+h}, \overline{p}] = r_c(p+h, m_{p+h}, \overline{p}) p e^{-2\pi i x c}).
\]
Each element of this discrete valuation semiring $T_{cL}^1[p + h, m_{p+h}, \nu]$ (resp. $T_{cR}^1[p + h, m_{p+h}, \nu]$) then writes as the product of the $\ell$ power of the uniformizing element $r_c(p + h, m_{p+h}, \nu^p)$ by a unit $e^{2\pi i (p^\ell)x_c}$ (resp. $e^{-2\pi i (p^\ell)x_c}$).

The valuation of this element is the integer $\ell$.

3.2.9 Corollary

The etale covering of the global elliptic subsemimodules $\text{ellip}_L(1, [p+h], m_{p+h})$ (resp. $\text{ellip}_R(1, [p+h], m_{p+h})$) by the semitori $T_{cL}^1[p + h, m_{p+h}, \nu]$ (resp. $T_{cR}^1[p + h, m_{p+h}, \nu]$) [Pie2] can give rise to other discrete valuation semirings of which

a) the uniformizing element is $r_c(p + h, m_{p+h})$;

b) the units are the invertible elements $e^{2\pi i k}$, $0 \leq k \leq (p^\ell - 1)$, in one-to-one correspondence with the $p^\ell$-th complex roots $e^{2\pi i k/p^\ell}$ of unity;

c) the unique valuation is the integer 1.

Proof. Indeed, the semitors $T_{cL}^1[p + h, m_{p+h}, \nu] = r_c(p + h, m_{p+h}, \nu)^{p^\ell} e^{2\pi i (p^\ell)x_c}$ (resp. $T_{cR}^1[p + h, m_{p+h}, \nu] = r_c(p + h, m_{p+h}, \nu)^{p^\ell} e^{-2\pi i (p^\ell)x_c}$) can also be written according to:

$$T_{cL}^1[p + h, m_{p+h}] = r_c(p + h, m_{p+h}) e^{2\pi i (p+h)x_c} e^{2\pi i k}$$

(resp. $T_{cR}^1[p + h, m_{p+h}] = r_c(p + h, m_{p+h}) e^{-2\pi i (p+h)x_c} e^{-2\pi i k}$)

since $p^\ell x_c = x_c + 2k\pi$, $i \leq k \leq (p^\ell - 1)/2$ according to proposition 3.2.1.

Consequently, it is a principal ideal domain of which the unique prime ideal is $T_{cL}^1[p + h, m_{p+h}] = r_c(p + h, m_{p+h}) e^{2\pi i (p+h)x_c}$ (resp. $T_{cR}^1[p + h, m_{p+h}] = r_c(p + h, m_{p+h}) e^{-2\pi i (p+h)x_c}$).

The uniformizing element is $r_c(p + h, m_{p+h})$ or $r_c(p + h, m_{p+h}) e^{2\pi i (p+h)x_c}$ with $x_c = 0$ and the units are $e^{2\pi i}$.

Consequently, the $(p^\ell - 1)/2 \times N \times N$ points of $T_{cL}^1[p + h, m_{p+h}]$ (resp. $T_{cR}^1[p + h, m_{p+h}]$) can be expressed from the product of the uniformizing element $r_c(p + h, m_{p+h}) e^{2\pi i (p+h)x_c}$ (resp. $r_c(p + h, m_{p+h}) e^{-2\pi i (p+h)x_c}$) at $x_c = 0$, by the $p^\ell$ units $e^{2\pi i k}$ (resp. $e^{-2\pi i k}$), $0 \leq k \leq (p^\ell - 1)/2$.

3.2.10 Corollary

The Kronecker-Weber theorem follows directly from the existence of discrete valuation semirings $T_{cL}^1[p + h, m_{p+h}]$ (resp. $T_{cR}^1[p + h, m_{p+h}]$), $0 \leq h \leq \infty$, $h \in \mathbb{N}$.
Proof. Corollary 3.2.8 shows that the points of the valuation semirings \( T_{cL}^1[p + h, m_{p+h}] \) (resp. \( T_{cR}^1[p + h, m_{p+h}] \)) can be expressed by multiplying the uniformizing element \( r_c(p + h, m_{p+h}) e^{2\pi i(x+h)c} \) (resp. \( r_c(p + h, m_{p+h}) e^{-2\pi i(x+h)c} \)) at \( x = 0 \) by the units \( e^{2\pi ik} \), \( 0 \leq k \leq (p^\ell - 1)/2 \) which are in one-to-one correspondence with the \( p^\ell \)-th roots of unity \( e^{2\pi ik/p^\ell} \), solution of the equation \( x^{p^\ell} - 1 = 0 \). (These roots form a cyclic group having as generator \( e^{2\pi i/p^\ell} \)).

Note that there is an inflation map:

\[
\text{INF}_L : \quad e^{2\pi ik} \longrightarrow r_c(p + h, m_{p+h}) e^{2\pi ix_h} \cdot e^{2\pi ik} \\
\text{(resp. INF}_R : \quad e^{-2\pi ik} \longrightarrow r_c(p + h, m_{p+h}) e^{-2\pi ix_h} \cdot e^{-2\pi ik}
\]

from the units, points of a circle having a radius equal to 1, to the complex numbers \( r_c(p + h, m_{p+h}) e^{2\pi ix_h} \cdot e^{2\pi ik} \) (resp. \( r_c(p + h, m_{p+h}) e^{-2\pi ix_h} \cdot e^{-2\pi ik} \)), points of a circle having a radius equal to \( r_c(p + h, m_{p+h}) \).

When the covering map \( R^1_{\ell p^\ell \rightarrow p+h} \) (resp. \( R^1_{R^\ell p^\ell \rightarrow p+h} \)) is an epimorphism (see proposition 3.2.1), i.e. the case where \( (p^\ell - 1)/2 > p + h \), the finite abelian extension of \( k = \mathbb{Q} \), characterized by a global residue degree \( f_{p+h} = p + h \), is thus related to the cyclotomic field of \( p^\ell \)-th roots of unity which corresponds to a Galois cyclotomic extension of order \( (p^\ell - 1)/2 \). The Kronecker-Weber theorem, expressing that each finite abelian extension of \( \mathbb{Q} \) is contained in a cyclotomic extension of \( \mathbb{Q} \), is thus reached here. ■

3.2.11 Proposition

Each \( n \)-dimensional semitorus \( T_{cL}^n[p + h, m_{p+h}, \mathbb{P}^\ell] \) (resp. \( T_{cR}^n[p + h, m_{p+h}, \mathbb{P}^\ell] \)) covering the global elliptic subsemimodule

\[
\text{ell}_{L}(n,[p + h], m_{p+h}) = \lambda^{\frac{1}{2}}(n, p + h, m_{p+h}) e^{2\pi i(p+h)x} \quad x \in \mathbb{R}^n ,
\]

(resp. \( \text{ell}_{R}(n, [p + h], m_{p+h}) = \lambda^{\frac{1}{2}}(n, p + h, m_{p+h}) e^{-2\pi i(p+h)x} \))

is a discrete valuation semiring of which:

a) the uniformizing element is \( r(p + h, m_{p+h}, \mathbb{P}^\ell)^p = \prod_{c=1}^n r_c(p + h, m_{p+h}, \mathbb{P}^\ell)^p \);

b) the units are the invertible elements \( e^{2\pi i(p^\ell)x'} = \prod_{c=1}^n e^{2\pi i(p^\ell)x'_c} \) (resp. \( e^{-2\pi i(p^\ell)x'_c} \)), \( x' \in \mathbb{R}^n \).

Proof. This proposition is the \( n \)-dimensional generalization of proposition 3.2.5, taking into account corollary 3.2.6. ■
3.2.12 Global elliptic semimodules covered by $p^\ell$-roots

Let $\text{ELLIP}_L(n, i \geq p, m_i)$ (resp. $\text{ELLIP}_R(n, i \geq p, m_i)$) be the $\Gamma(\Phi(G^{(n)}(L^T_{[\nu_p]})))$-semimodule (resp. $\Gamma(\Phi(G^{(n)}(L^T_{[\nu_p]})))$-semimodule) with terms $i \geq p$.

The $n$-dimensional global elliptic semimodule

$$\text{ELLIP}_L(n, i \geq p, m_i, \overline{p}^{(\ell)}) = \bigoplus_{h=0}^{w} \oplus_{p+h} \text{ellip}_L(n, p+h, m_{p+h}, \overline{p}^{(\ell)})$$

$$0 \leq h \leq w \leq \infty ,$$

(resp. $\text{ELLIP}_R(n, i \geq p, m_i, \overline{p}^{(\ell)}) = \bigoplus_{h=0}^{w} \oplus_{p+h} \text{ellip}_R(n, p+h, m_{p+h}, \overline{p}^{(\ell)})$)

covers $\text{ELLIP}_L(n, i \geq p, m_i)$ (resp. $\text{ELLIP}_R(n, i \geq p, m_i)$) in the sense that each term

$$\text{ellip}_L(n, p+h, m_{p+h}, \overline{p}) \simeq \prod_{c=1}^{n} r_c(p+h, m_{p+h}, \overline{p})^{p^\ell} e^{2\pi i(p^\ell)x_c'}$$

(resp. $\text{ellip}_R(n, p+h, m_{p+h}, \overline{p}) \simeq \prod_{c=1}^{n} r_c(p+h, m_{p+h}, \overline{p})^{p^\ell} e^{-2\pi i(p^\ell)x_c'}$)

of $\text{ELLIP}_L(n, i \geq p, m_i, \overline{p}^{(\ell)})$ (resp. $\text{ELLIP}_R(n, i \geq p, m_i, \overline{p}^{(\ell)})$) is the covering by $p^\ell$ roots ($\ell$ varying from one term to another) of each term $\text{ellip}_L(n, [p+h], m_{p+h})$ (resp. $\text{ellip}_R(n, [p+h], m_{p+h})$) of $\text{ELLIP}_L(n, i \geq p, m_i)$ (resp. $\text{ELLIP}_R(n, i \geq p, m_i)$).

A semisheaf $\Phi(G^{(n)}(L^T_{[\nu_p]}))$ (resp. $\Phi(G^{(n)}(L^T_{[\nu_p]}))$), of which sections are the functions $\text{ellip}_L(n, p+h, m_{p+h}, \overline{p})$ (resp. $\text{ellip}_R(n, p+h, m_{p+h}, \overline{p})$), is associated with the $n$-dimensional global elliptic semimodule $\text{ELLIP}_L(n, i \geq p, m_i, \overline{p}^{(\ell)})$ (resp. $\text{ELLIP}_R(n, i \geq p, m_i, \overline{p}^{(\ell)})$) as introduced precedently.

3.3 Bilinear local cuspidal representations associated with their global correspondents covered by $p^\ell$-th roots

3.3.1 Non-archimedean local fields

Let $K_p^+/L_p^+$ (resp. $K_p^-/L_p^-$) denote a finite Galois extension of a non-archimedean $p$-adic left (resp. right) semifield $L_p^+$ (resp. $L_p^-$) which is finite extension of $\mathbb{Q}_p^+ = \mathbb{Z}_p \otimes \mathbb{Q}_+$ (resp. $\mathbb{Q}_p^- = \mathbb{Z}_p \otimes \mathbb{Q}_-$) where $\mathbb{Z}_p = \lim \mathbb{Z}/(p^n)$ and where $\mathbb{Q}_p^+$ (resp. $\mathbb{Q}_p^-$) is the completion of $\mathbb{Q}_+$ (the positive rational numbers) (resp. $\mathbb{Q}_-$ (the negative rational numbers)) in the $p$-adic metric.

Let $[K_p^+ : L_p^+] = q$ denote the degree of this extension, $v_p$ a discrete valuation of $L_p^+$ with semiring $A_p^+$ and $\omega_r$ the different prolongations of $v_p$ to $K_p^+$ [Ser3].

Let $B_p^+$ be the integral closure of $A_p^+$ into $K_p^+$ in such a way that the $A_p^+$-semimodule $B_p^+$ be finitely generated and that its field of fractions be $K_p^+$.

Let $\beta_p^+ \subset \beta_{p_2}^+ \subset \cdots \subset \beta_{p_r}^+$ be a chain of distinct prime ideals of $B_p^+$ and let $m(A_p^+) = \beta_{p_r}^+ \cap A_p^+$ define the division of $\beta_{p_r}^+$ by the maximal ideal $m(A_p^+)$, noted $\beta_{p_r}^+ | m(A_p^+)$ ($\beta_{p_r}^+$ contains
For each prime ideal \( \beta \)

the exponent \( e_{K_p} \) is the “local” residue degree

\[
\frac{f_{\beta_p}}{r} = [B_p^+/\beta_p^+ : A_p^+/m(A_p^+)]
\]

of \( \beta_p \) in the extension \( K_p^+/L_p^+ \).

The exponent \( e_{\beta_p}^+ \) of \( \beta_p^+ \) in the decomposition of \( m(A_p^+)B_p^+ \) into prime ideal is the ramification index of \( \beta_p^+ \) in the extension \( K_p^+/L_p^+ \).

The semiring \( B_p^+/m(A_p^+) \) is an \( A_p^+/m(A_p^+) \)-semialgebra of degree \( q = \Sigma f_{\beta_p}^+ e_{\beta_p}^+ \) where \( \beta_p | m(A_p^+) \).

Remark that the ramification indices \( e_{\beta_p}^+ \), referring to \( m(A_p^+) \) are all equal to \( e_{\beta_p}^+ \) in Galois extensions [Ser3].

To each prime ideal \( \beta_p^+ \) corresponds a residue semifield \( k_{K_p^+/\beta_p^+} = \mathcal{O}_{K_p^+}/\beta_p^+ \) where \( \mathcal{O}_{K_p^+}/\beta_p^+ \) is the semiring of integers of \( K_p^+ \) restricted to \( \beta_p^+ \).

According to J.P. Serre, the semiring \( B_p^+ \) is a discrete valuation semiring [Ser3]. Taking into account that the different valuations \( w_r \), corresponding to the prime ideal \( \beta_p^+ \), define each one a norm on \( K_p^+ \) making \( K_p^+ \) a Hausdorff topological vector semispace over \( L_p^+ \). As \( L_p^+ \) is assumed to be complete and as the topology \( F_r \) defined by \( w_r \) is a product topology on \( K_p^+ \) not depending on the index \( r \), there is only one \( w_r \) which is relevant.

Let \( \tilde{\omega}_{K_p^+} \) denote the uniformizer (i.e. a prime element) in \( \mathcal{O}_{K_p^+} \). \( \mathcal{O}_{K_p^+} \) is then the inverse limit of \( \mathcal{O}_{K_p^+}/(\tilde{\omega}_{K_p^+}) \).

The number of elements in \( K_p^+ \) is thus \( p^q = \Sigma f_{r \cdot e_r} \equiv p^r \) [Kna], where \( f_{\beta_p^+} \) and \( e_{\beta_p^+} \) have been written in condensed form respectively as \( f_r \) and \( e_r \).

Remark that the right case can be handled similarly with the evident following notations: semifields \( L_p^- \) and \( K_p^- \), semiring \( A_p^- \), \( A_p^- \)-semimodule \( B_p^- \), prime ideals \( \beta_p^- \), semiring of integers \( \mathcal{O}_{K_p^-} \), residue semifield \( k_{K_p^-} \) and so on.

3.3.2 Proposition (Global ↔ local correspondences between extension (semi)fields)

The set of intermediate subsemifields \( \{ \tilde{L}_{v_{p+h, m+p+h}} \}_{h=0}^{\infty} \) (resp. \( \{ \tilde{L}_{v_{p+h, m_p+h}} \}_{h=0}^{\infty} \)) of \( \tilde{L}_L \) (resp. \( \tilde{L}_R \)), extensions of a numberfield \( k \) of char 0 as introduced in section 2.1, or equivalently the set of corresponding archimedean completions \( \{ L_{v_{p+h, m+p+h}} \}_{h=0}^{\infty} \) (resp. \( \{ L_{v_{p+h, m_p+h}} \}_{h=0}^{\infty} \)), can be covered in an etale way by a (set of) \( p \)-adic finite extension semifield(s) leading to a global ↔ local isomorphism if

\[
p^\Sigma f_{r \cdot e_r} = \# Nu \times \Sigma f_{v_{p+h, m_p+h}} \text{.}
\]
i.e. if their numbers of elements correspond, where:

- \( \# \text{Nu} \) is the number of global nonunits;
- \( N \in \mathbb{N} \) is the Galois extension degree of a quantum, i.e. an irreducible subsemifield \( \tilde{L}_{v_1} \) or \( \tilde{L}_{\pi_1} \);
- \( f_{v_p+h,m_p+h} = p + h \in \mathbb{N} \) is the global residue degree of \( \tilde{L}_{v_p+h,m_p+h} \);
- \( \sum_r f_r \cdot e_r \) is the number of elements in the finite \( p \)-adic Galois extension(s) as developed in section 3.3.1.

Proof. 1. According to section 2.1, the Galois extension degree of the pseudo-ramified completion \( L_{v_p+h,m_p+h} \) (or \( L_{\pi_p+h,m_p+h} \)) at the infinite real place \( v_p+h \) (or \( \pi_p+h \)) is

\[
[L_{v_p+h,m_p+h} : k] \equiv [L_{\pi_p+h,m_p+h} : k] = (p + h) \cdot N
\]

(in the residue class zero of the integers modulo \( N \)).

Consequently, the number of elements in \( L_{v_p+h,m_p+h} \) or in the subsemifield \( \tilde{L}_{v_p+h,m_p+h} \) is equal to:

\[
\left| \tilde{L}_{v_p+h,m_p+h} \right| = \# \text{Nu} \times N \times f_{v_p+h}.
\]

2. Referring to section 3.3.1, if the number of elements \( \left| \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \right| = \# \text{Nu} \times N \times \sum_h f_{v_p+h,m_p+h} \) of the set of subsemifields \( \tilde{L}_{v_p+h,m_p+h} \) is a power of \( p \), i.e. if \( \left| \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \right| = p^q \), where \( q = \sum_r f_r \cdot e_r \), then \( \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \) is isomorphic to the \( p \)-adic extension semifield \( K_p^+ \) of dimension \( q \).

Similarly, if \( \left| \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \right| < p^q \), \( \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \) is monomorphic to \( K_p^+ \), and if \( \left| \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \right| > p^q \), \( \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \) is epimorphic to \( K_p^+ \).

3.3.3 Corollary

The set of completions \( \{ \tilde{L}_{v_p+h,m_p+h} \}_{h} \) (resp. \( \{ \tilde{L}_{\pi_p+h,m_p+h} \}_{h} \)) is a \( p \)-adic semifield if:

a) its number of elements is a power of \( p \), i.e. if

\[
\left| \{ L_{v_p+h,m_p+h} \}_{h} \right| = \left| \{ L_{\pi_p+h,m_p+h} \}_{h} \right| = p^q;
\]

b) they are defined as completion(s) of \( K_p^+ \) (resp. \( K_p^- \)) for the topology defined by its \( p \)-adic absolute value.
Proof. a) According to proposition 3.3.2, if the number of elements of \( \{ L_{v^p+h,m^p+h} \} \) is a power of \( q \), this set of infinite completions is isomorphic to a finite \( p \)-adic extension semifield.

b) As each infinite completion \( L_{v^p+h,m^p+h} \) results from an isomorphism of compactification into a closed compact subset of \( \mathbb{R}_+ \), \( \{ L_{v^p+h,m^p+h} \} \) will define a completion of a subset of \( K^+_p \) if the considered topology refers to an ultrametric \( p \)-adic absolute value.

Similarly, \( \{ L_{\pi^p+h,m^p+h} \} \) can generate a completion of a subset of \( K^-_p \).

3.3.4 Proposition (Local elliptic semimodule)

Let \( \Phi(G^{(2)}(L^T_{[v^p]})) \) (resp. \( \Phi(G^{(2)}(L^T_{[p^m]})) \)) be the two-dimensional global semimodule to which is associated the global elliptic semimodule \( \text{ELLIP}_L(2, i \geq p, m_i) \) (resp. \( \text{ELLIP}_R(2, i \geq p, m_i) \)) and let \( \Phi(G^{(2)}_{[p^m]}(L^T_{[v^p]})) \) (resp. \( \Phi(G^{(2)}_{[p^m]}(L^T_{[p^m]})) \)) be its etale covering global semimodule by \( p^i \) roots to which is associated the global elliptic semimodule \( \text{ELLIP}_L(2, i \geq p, m_i, L^i) \) (resp. \( \text{ELLIP}_R(2, i \geq p, m_i, L^i) \)).

Then, \( \Phi(G^{(2)}_{[p^m]}(L^T_{[v^p]})) \) (resp. \( \Phi(G^{(2)}_{[p^m]}(L^T_{[p^m]})) \)) is covered by a \( p \)-adic local semigroup \( G^{(2)}(K^+_p) \equiv T_2(K^+_p) \) (resp. \( G^{(2)}(K^-_p) \equiv T_2(K^-_p) \)) on a semischeme \( S^+ \) (resp. \( S^- \)) [Mes] which is a flat \( O_{K^+_p} \)-semimodule (resp. \( O_{K^-_p} \)-semimodule) if:

a) \( p^m = \# \text{Nu} \times N \), i.e. if the number of elements \( \# \text{Nu} \times N \) in a global quantum is a power of \( p \);

b) \( p^{2q} = \left( \sum_{i}(p^i - 1) / 2 \times m_{(p^i-1)/2} \times p^m \right)^2 \) (case \( n = 2 \), two-dimensional case), where \( m_{(p^i-1)/2} \) denotes the multiplicity of the covering sections by \( p^i \) roots, or

\[
p^{2q} = \left( \sum_{h}(p + h) \times m_{p+h} \times p^m \right)^2 ;
\]

c) there are Frobenius endomorphisms with generator \( x \to x^{p^{r'}} \) (resp. \( -x \to -x^{p^{r'}} \)) resulting from the cyclicity of the Galois semigroup \( \text{Gal}(k^+_p / k^+_L) \) (resp. \( \text{Gal}(k^-_p / k^-_L) \)) where \( k^+_p \) (resp. \( k^-_p \)) and \( k^+_L \) (resp. \( k^-_L \)) are respectively residue semifields of \( K^+_p \) (resp. \( K^-_p \)) and of \( L^+_p \) (resp. \( L^-_p \)).

On the other hand, the image of a member of \( \text{Gal}(K^+_p / L^+_p) \) (resp. \( \text{Gal}(K^-_p / L^-_p) \)) is of the form \( x \to x^{p^{r'}} = \mu^r \) (resp. \( -x \to -x^{p^{r'}} = -\mu^r \));

d) there are embeddings

\[
e_{L_{[v^p]} \to K^+_p} : \quad \lambda(2, p + h, m_{p+h}) \quad \longrightarrow \quad \lambda_p(2, r, m_r)
\]

(resp. \( e_{L_{[p^m]} \to K^-_p} : \quad \lambda(2, p + h, m_{p+h}) \quad \longrightarrow \quad \lambda_p(2, r, m_r) \))

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of the product, right by left, of Hecke characters $\lambda(2, p + h, m_{p+h})$ over $L_{[p]}$ (resp. $L_{[\overline{p}]}$) into their equivalents $\lambda_p(2, r, m_r)$ over $K_r^+$ (resp. $K_r^-$),

in such a way that the global elliptic semimodule $\text{ELLIP}_L(2, p \leq i, m_i)$ (resp. $\text{ELLIP}_R(2, p \leq i, m_i)$) and its covering by $p^{(2)}$ roots $\text{ELLIP}_L(2, p \leq i, m_i, \overline{p}^{(2)})$ (resp. $\text{ELLIP}_R(2, p \leq i, m_i, \overline{p}^{(2)})$) be covered by the local elliptic $\text{End}(G^{(2)}(K_r^+))$-semimodule (resp. $\text{End}(G^{(2)}(K_r^-))$-semimodule) referring to Drinfeld [Drin] and Anderson [And]:

$$\text{ELLIP}(2, x, K_r^+) = \bigoplus \left( \lambda_p^2(2, r, m_r)(x) f(\mu^{q-r}) \right)$$

(resp. $\text{ELLIP}(2, -x, K_r^-) = \bigoplus \left( \lambda_p^2(2, r, m_r)(-x) f(-\mu^{q-r}) \right)$)

for every closed point $x$ (resp. $-x$) of $K_r^+$ (resp. $K_r^-$) where $f(\mu^{q-r})$ (resp. $f(-\mu^{q-r})$) is a function of the Frobenius endomorphism.

Proof. It is thus asserted that there exists an isomorphism

$$\Phi(G^{(2)}_{\overline{p}(p)}(L_{[v_p]}^T)) \rightarrow G^{(2)}(K_r^+)$$

(resp. $\Phi(G^{(2)}_{\overline{p}(p)}(L_{[\overline{p}_p]}^T)) \rightarrow G^{(2)}(K_r^-)$)

from the global semisheaf $\Phi(G^{(2)}_{\overline{p}(p)}(L_{[v_p]}^T))$ (resp. $\Phi(G^{(2)}_{\overline{p}(p)}(L_{[\overline{p}_p]}^T))$) to the local semigroup $G^{(2)}(K_r^+)$ (resp. $G^{(2)}(K_r^-)$) in such a way that the diagram

$$\Phi(G^{(2)}_{\overline{p}(p)}(L_{[v_p]}^T)) \rightarrow G^{(2)}(K_r^+)$$

be commutative.

This can be achieved if the number of points of the local semigroup is equal to the number of points of the global semisheaf.

The number of points of the local semigroup $G^{(2)}(K_r^+)$ (resp. $G^{(2)}(K_r^-)$) is $p^{2q}$ according to proposition 3.3.2, the factor “2” resulting from the dimension $n = 2$.

The number of points of the global semisheaf $\Phi(G^{(2)}_{\overline{p}(p)}(L_{[v_p]}^T))$, covered by $G^{(2)}(K_r^+)$, is

$$\left( \Sigma_{q}((p^{q} - 1)/2 \times m_{p^{q}/2} \times p^{m}) \right)^2$$

since its sections of type $T_{cL}^{11}[p + h, m_{p+h}, \overline{p}]$, according to proposition 3.2.1, have a number of points

$$n_{T_{cL}[p + h, \overline{p}]} = \# Nu \times N \times (p^{q} - 1)/2.$$
From proposition 3.2.5, it results that the covering map \( R_{p^\ell}^2 \rightarrow \mathbb{Z}/2\mathbb{Z} \) is an isomorphism if \( ((p^\ell - 1)/2)^2 = (p + h)^2 \), which explains that
\[
p^{2q} = \left( \sum_{h} (p + h) \times m_{(p+h)} \times p^m \right)^2.
\]
By this way, each global point of the algebraic semigroup \( G_2(\ell)(LT_{[v_p]}) \) is in one-to-one correspondence with a local closed point of the local group \( G(2)(K^+_p) \); this results from the conditions a) and b) of this proposition.

In order that the local \( p \)-adic elliptic \( \text{End}(G(2)(K^+_p)) \)-semimodule \( \text{ELLIP}(2, x, K^+_p) \) (resp. \( \text{End}(G(2)(K^-_p)) \)-semimodule \( \text{ELLIP}(2, -x, K^-_p) \)) corresponds to a local cuspidal left (resp. right) form, the conditions c) and d) must be fulfilled in analogy with the global case considered in proposition 3.1.2, i.e.

1. a Frobenius substitution:
\[
\mu \quad \longrightarrow \quad \mu^{q_r} \quad \quad \text{(resp.} \quad -\mu \quad \longrightarrow \quad -\mu^{q_r})
\]
on every local Frobenius endomorphism
\[
\mu : \quad x \quad \longrightarrow \quad x^p \quad \quad \text{(resp.} \quad -x \quad \longrightarrow \quad -x^p);
\]
2. an embedding
\[
e_{L_\mu \rightarrow K^+_p} : \quad \lambda(2, p + h, m_{p+h}) \quad \longrightarrow \quad \lambda_p(2, r, m_r)
\]
(resp. \( e_{L_\mu \rightarrow K^-_p} : \quad \lambda(2, p + h, m_{p+h}) \quad \longrightarrow \quad \lambda_p(2, r, m_r) \))
of the Hecke character \( \lambda(2, p + h, m_{p+h}) \) into \( \lambda_p(2, r, m_r) \) which is the square of the coefficient for every local point \( x \) (resp. \( -x \)) of the local elliptic \( \text{End}(G(2)(K^+_p)) \)-semimodule \( \text{ELLIP}(2, x, K^+_p) \) (resp. \( \text{End}(G(2)(K^-_p)) \)-semimodule \( \text{ELLIP}(2, -x, K^-_p) \)).

This condition corresponds to the embedding \( i(a_\ell) = \text{trace}(\text{Frob}_\ell) \) into \( \mathbb{Q}_p \) of the ring of the integers of a finite extension \( E_f \) (i.e. the ring of the coefficients of the cuspidal form \( f \)) of \( \mathbb{Q} \), \( a_\ell \) being the coefficient of the cuspidal form, as introduced by Deligne [Del] and Serre [Ser4] and mentioned in [Win].

### 3.3.5 Corollary

There is an epimorphism
\[
e_{\Phi(G(2)(K^+_p)) \rightarrow \Phi(G(2)(K^-_p))} : \quad G(2)(K^+_p) \quad \longrightarrow \quad \Phi(G(2)(K^-_p)(LT_{[v_p]}))
\]
(resp. \( e_{\Phi(G(2)(K^-_p)) \rightarrow \Phi(G(2)(K^+_p))} : \quad G(2)(K^-_p) \quad \longrightarrow \quad \Phi(G(2)(K^+_p)(LT_{[v_p]})) \))
from the local semigroup $G^{(2)}(K_p^+)$ (resp. $G^{(2)}(K_p^-)$) into the global semisheaf $\Phi(G^{(2)}_{p\ell}(L^T_{[v_p]}))$ (resp. $\Phi(G^{(2)}_{p\ell}(L^T_{[v_p]}))$) such that ELLIP$(2, x, K_p^+)$ (resp. ELLIP$(2, -x, K_p^-)$) be projected onto ELLIP$_L(2, i \geq p, m_i, \overline{p}(l))$ (resp. ELLIP$_R(2, i \geq p, m_i, \overline{p}(l))$) if the number of “local” points is superior to the number of “global points”, i.e. if

$$p^{2q} > \left(\sum_{i} (p^{\ell}/2) \times m^{\ell}/2 \times p^m\right)^2.$$

Proof. This directly results from proposition 3.3.4 and, more particularly, from condition b), the other conditions a), b) and d) being unchanged. 

3.3.6 Proposition

The local $p$-adic elliptic $\text{End}(G^{(2)}(K_p^+))$-semimodule ELLIP$(2, x, K_p^+)$ (resp. $\text{End}(G^{(2)}(K_p^-))$-semimodule ELLIP$(2, -x, K_p^-)$), corresponding to a local $p$-adic left (resp. right) cuspidal form, results from the left (resp. right) semisheaf $\Phi(G^{(2)}(L^T_v))$ (resp. $\Phi(G^{(2)}(L^T_{[v_p]}))$) on the global algebraic semigroup $G^{(2)}(L^T_v)$ (resp. $G^{(2)}(L^T_{[v_p]}))$ by the commutative diagram (the left case being only considered here):

\[
\begin{array}{cccccc}
\Phi(G^{(2)}(L^T_v)) & \xrightarrow{\text{em}^{(2)}_{v \rightarrow [v_p]}} & \Phi(G^{(2)}(L^T_{[v_p]})) & \xrightarrow{\sim} & \Phi(G^{(2)}_{p\ell}(L^T_{[v_p]})) & \xrightarrow{i_{p\ell}^{(2)}} & G^{(2)}(K_p^+) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{ELLIP}_L(2, i, m_i) & \xrightarrow{\sim} & \text{ELLIP}_L(2, p \leq i, m_i) & \xrightarrow{\sim} & \text{ELLIP}_L(2, p \leq i, m_i, \overline{p}(l)) & \xrightarrow{i_{p\ell}^{(2)}} & \text{ELLIP}_L(2, x, K_p^+) \\
\end{array}
\]

Proof. This is a consequence of corollary 3.2.6, sections 3.2.7 and 3.2.12 as well as proposition 3.3.4.

That is to say that:

a) the epimorphism $\text{em}^{(2)}_{v \rightarrow [v_p]}$ sends the semisheaf $\Phi(G^{(2)}(L^T_v))$ on the algebraic semigroup $G^{(2)}(L^T_v)$ over “$t$” sets of toroidal archimedean completions, $1 \leq i \leq t \leq \infty$, into the semisheaf $\Phi(G^{(2)}(L^T_{[v_p]}))$ on the algebraic semigroup $G^{(2)}(L^T_{[v_p]})$ restricted to toroidal completions above $v_p$.

b) this allows to find the covering semisheaf $\Phi(G^{(2)}_{p\ell}(L^T_{[v_p]}))$ by $p^{(\ell)}$ roots of the semisheaf $\Phi(G^{(2)}(L^T_{[v_p]}))$ and the $p$-adic local semigroup $G^{(2)}(K_p^+)$. 

The local $p$-adic elliptic $\text{End}(G^{(2)}(K_p^+))$-semimodule ELLIP$(2, x, K_p^+)$ then results from the global elliptic $\Gamma(\Phi(G^{(2)}(L^T_v)))$-semimodule ELLIP$_L(2, i, m_i)$. 

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3.3.7 Proposition

The Serre (Eichler, Deligne, Shimura) conjecture, asserting that Galois representations \( \rho : G_\mathbb{Q} \to \text{GL}_2(\overline{\mathbb{Q}}_p) \) can be associated to modular forms, directly results from proposition 3.3.6, and more particularly, from the epimorphism:

\[
\begin{align*}
\text{em}^{(2)}_{L^T_v \to K_p^+} : \quad & \text{GL}_2(L^T_v) \longrightarrow \text{GL}_2(K_p^+) , \quad \text{with } T^2(L^T_v) \equiv \text{GL}_2(L^T_v) , \\
\text{(resp. } \text{em}^{(2)}_{L^T_v \to K_p^-} : \quad & \text{GL}_2(L^T_v) \longrightarrow \text{GL}_2(K_p^-) ) ,
\end{align*}
\]

sending the algebraic semigroup \( \text{GL}_2(L^T_v) \) (resp. \( \text{GL}_2(L^T_v) \)) over the set \( L^T_v \) (resp. \( L^T_w \)) of archimedean completions into the algebraic semigroup \( \text{GL}_2((K_p^+)) \) (resp. \( \text{GL}_2((K_p^-)) \)) at the following conditions:

1. \( L^T_v \) (resp. \( L^T_w \)) is extended to \( \overline{\mathbb{Q}}_p \) (resp. \( \overline{\mathbb{Q}}_p^- \)), \( K^+_p \) to \( \mathbb{Q}_p^+ \) and \( K^-_p \) to \( \mathbb{Q}_p^- \);

2. there is an epimorphism

\[
\begin{align*}
\text{em}^{(2)}_{L^T_v \to L^T_{v[p]}}, \quad & \text{GL}_2(L^T_v) \longrightarrow \text{GL}_2(L^T_{v[p]}) \\
\text{(resp. } \text{em}^{(2)}_{L^T_w \to L^T_{w[p]}}, \quad & \text{GL}_2(L^T_w) \longrightarrow \text{GL}_2(L^T_{w[p]}) )
\end{align*}
\]

from \( \text{GL}_2(L^T_v) \) (resp. \( \text{GL}_2(L^T_w) \)) into the algebraic semigroup \( \text{GL}_2(L^T_{v[p]}) \) (resp. \( \text{GL}_2(L^T_{w[p]}) \)) over the set of completions \( L^T_{v[p]} \) (resp. \( L^T_{w[p]} \)) superior or equal to \( v_p \) (resp. \( w_p \));

3. \( p^m = \# \text{Nu} \times N \);

4. \( p^{2q} = \sum_h (p + h) \times m_{(p+h)} \times p^m \)^2 ;

5. there are Frobenius endomorphisms \( \mu : x \to x^{p^{fr \times er}} = \mu^{v} \) (resp. \( -\mu : -x \to -x^{p^{fr \times er}} = -\mu^{v} \));

6. there are embeddings \( e_{L_v \to K^+_p} : \lambda(2,i,m_i) \to \lambda_p(2,r,m_r) \) (resp. \( e_{L_v \to K^-_p} : \lambda(2,i,m_i) \to \lambda_p(2,r,m_r) \)) of the product, right by left, of Hecke characters \( \lambda(2,i,m_i) \) over \( L_v \) (resp. \( L_w \)) into their equivalents \( \lambda_p(2,r,m_r) \) over \( K^+_p \) (resp. \( K^-_p \)).

Proof. (for the left case, the right case being handled similarly)

1. First, we have to consider the mapping of \( G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), or, more restrictively, of \( \text{Gal}(\overline{L}_L/k) \) or of \( W_{\overline{L}_v} \), defined in section 2.4, into the set (or the sum) of the equivalence classes of the representations space \( \text{Irr Rep}^{(2)}(W_{\overline{L}_v}) \) of the global Weil group \( W_{\overline{L}_v} \) in such a way that:

\[
\text{Irr Rep}^{(2)}(W_{\overline{L}_v}) = G^{(2)}(L_v)
\]

as developed in proposition 3.1.5.
Then, the algebraic semigroup (of matrices) $GL_2(L_T)$, isomorphic to the algebraic semigroup $G^{(2)}(L_v)$ according to the preceding developments of this paper, is sent by the epimorphism $em^{(2)}_{L_T 	o L_T[v_p]}$ into the algebraic semigroup $GL_2(L_T[v_p])$ with coefficients on completions superior or equal to $v_p$.

Finally, $GL_2(L_T[v_p])$ is sent by the isomorphism $im^{(2)}_{L_T[v_p] 	o K_p^+}$ into $GL_2(K_p^+)$ at the conditions a), b), c) and d) of proposition 3.3.4.

It then results that the Galois representation

$$\rho : \quad G_\mathbb{Q} \longrightarrow GL_2(\overline{\mathbb{Q}_p})$$

is in fact an epimorphism corresponding mainly to the compositions of morphisms:

$$im^{(2)}_{L_T[v_p] \to K_p^+} \circ em^{(2)}_{L_T^L \to L_T[v_p]} \circ \text{Irr Rep}^{(2)}(W^b_{L_T}) ;$$

2. The fact that the Galois representation $\rho : G_\mathbb{Q} \to GL_2(\overline{\mathbb{Q}_p})$ can be associated to a modular form results from the commutative diagram of proposition 3.3.6, taking into account that the global elliptic semimodule $\text{ELLIP}_L(2, i, m_i)$ constitutes a cuspidal representation of $G^{(2)}(L_v)$ according to proposition 3.1.5 and is in one-to-one correspondence with a cuspidal form as developed in [Pie2].

**3.3.8 Corollary**

**The Shimura-Taniyama-Weil conjecture,** associated with the action of $G_\mathbb{Q}$ on the elliptic curve $E[p]$ leading to a continuous representation

$$\rho_{E,p} : \quad G_\mathbb{Q} \longrightarrow GL_2(F_p)$$

in such a way that:

$$\text{trace}(\rho_{E,p}(\text{Frob}_p)) = p + 1 - \#E(F_p) \pmod{p},$$

also directly results from propositions 3.3.4 and 3.3.6.

**Proof.** In the new context proposed here, the Shimura-Taniyama-Weil conjecture is a special case of the Serre conjecture since, referring to proposition 3.3.6 and to the extension of the residue field $k_{K_p}$ to $\mathbb{F}_p$, it results from a global elliptic semimodule $\text{ELLIP}_L(2, p \leq i, m_i)$ (resp. $\text{ELLIP}_R(2, p \leq i, m_i)$) restricted to the $i$-th $p$-th terms, i.e. to the case where $h = 0$ and $\ell = 1$, according to corrolaries 3.2.2 and 3.2.6.

Remark that the Shimura-Taniyama-Weil-conjecture was specifically studied in [Pie2] from this new point of view. 

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4 Deformations of Galois representations

Two kinds of deformations of \( n \)-dimensional representations of Galois or Weil groups give by bilinear algebraic semigroups over complete global and local noetherian bisemirings, in reference with the work of B. Mazur [Maz1], are envisaged:

1. global and local bilinear deformations inducing the invariance of their respective global and local bilinear residue (semi)fields.

2. global and local bilinear “quantum” deformations leaving invariant the orders of the inertia subgroups.

4.1 Local and global coefficient semiring homomorphisms

4.1.1 Local coefficient semiring homomorphisms (left case)

A coefficient semiring \( B_p^+ \) is, according to B. Mazur [Maz2], a complete noetherian local semiring with finite residue semifield \( k_{K_p^+} \). It is characterized by a profinite topology given by a base of prime ideals \( \tilde{\omega}_{K_p^+} B_p^+ \) in such a way that:

\[
\mathcal{B}_p^+ = \lim_{r \to \infty} \mathcal{B}_p^+ / \tilde{\omega}_{K_p^+} B_p^+
\]

where \( \tilde{\omega}_{K_p^+} B_p^+ \) is the maximal ideal of \( \mathcal{B}_p^+ \).

The discrete valuation semiring \( B_p^+ \), introduced in section 3.3.1 as the integral closure of \( A_p^+ \) in the finite Galois extension \( K_p^+ / L_p^+ \) of the \( p \)-adic semifield \( L_p^+ \), is a noetherian local semiring if the chain \( \beta_{p_1}^+ \subset \beta_{p_2}^+ \subset \cdots \subset \beta_{p_r}^+ \) of prime ideals of \( B_p^+ \) tends to \( \infty \), i.e. if \( r \to \infty \).

It will then be assumed in this chapter that \( B_p^+ \) is a noetherian local semiring on the completion of \( K_p^+ \).

Let \( B_p^{'+} \) be another coefficient semiring being the integral closure of \( A_p^+ \) in another finite Galois extension \( B_p^{'+} / L_p^+ \) and let \( k_{K_p^{'+}} = \{ \mathcal{O}_{K_p^{'+}} / \beta_{p_r}^+ \}_r \) be its residue semifield defined on the set of residue subsemifields \( \mathcal{O}_{K_p^{'+}} / \beta_{p_r}^+ / m(A_p^+) \) restricted to the prime ideals \( \beta_{p_r}^+ \) and \( B_p^{'+} \), referring to section 3.3.1.

The semiring \( B_p^{'+} / m(A_p^+) B_p^{'+} \) is also an \( A_p^+ / m(A_p^+) \)-semialgebra of degree \( q' = \sum \beta_{p_r}^+ \beta_{p_r}^+ e_{\beta_{p_r}^+} = [K_p^{'+} : L_p^+] \) where \( f_{\beta_{p_r}^+} \) is the residue degree of \( \beta_{p_r}^+ \) in the extension \( K_p^{'+} / L_p^+ \) and \( e_{\beta_{p_r}^+} \) is the corresponding ramification index.

A coefficient semiring homomorphism [Maz2]

\[
h_{B_p^{'+} \to B_p^+} : B_p^{'+} \longrightarrow B_p^+
\]

sending \( B_p^+ \) into \( B_p^{'+} \) is such that:
a) the inverse image of the maximal ideal \( \bar{\omega}_K + B_p^+ \) of \( B_p^+ \) is the maximal ideal \( \bar{\omega}_K + B_p' \) of \( B_p' \);

b) the induced homomorphism

\[
h_{k_{K_p'} \rightarrow k_{K_p}} : k_{K_p'} \sim \rightarrow k_{K_p}
\]

on the residue semifields is an isomorphism leading to the evident condition

\[
\sum f_{\beta_{pr}'} = \sum f_{\beta_{pr}}
\]
on the residue degrees.

### 4.1.2 Proposition

The kernel of the coefficient semiring homomorphism \( h_{B_p' \rightarrow B_p} : B_p' \rightarrow B_p \) is characterized by a degree of extension:

\[
[K_p' : L_p] - [K_p : L_p] = \sum f_{\beta_{pr}} e_{\beta_{pr}}' - \sum f_{\beta_{pr}} e_{\beta_{pr}}
\]

\[
= (e_{\beta_{pr}}' - e_{\beta_{pr}}) \left( \sum f_{\beta_{pr}} \right).
\]

**Proof.** This is a consequence of section 4.1.1 leading generally to the inequality

\[
\sum f_{\beta_{pr}} e_{\beta_{pr}}' > \sum f_{\beta_{pr}} e_{\beta_{pr}}.
\]

On the other hand, as \( \sum f_{\beta_{pr}} = \sum f_{\beta_{pr}}' \) and as the ramification indices \( e_{\beta_{pr}}' \) are equal to \( e_{\beta_{pr}} \) and the \( e_{\beta_{pr}} \) are equal to \( e_{\beta_{pr}}' \) according to section 3.3.1, the preceding inequality results from the fact that \( e_{\beta_{pr}}' > e_{\beta_{pr}} \).

### 4.1.3 Corollary

The coefficient semiring homomorphism \( h_{B_p' \rightarrow B_p} : B_p' \rightarrow B_p \) corresponds to a base change from \( K_p' \) to \( K_p' \).

**Proof.** The degree of this change of basis is thus:

\[
[K_p' : L_p] - [K_p : L_p] = q' - q
\]

if \( q = e_{\beta_{pr}} \sum f_{\beta_{pr}} \) and \( q' = e_{\beta_{pr}}' \sum f_{\beta_{pr}}' \).
4.1.4 Global coefficient semiring homomorphisms

A global coefficient semiring $\tilde{L}_{L_p}$ (resp. $\tilde{L}_{R_p}$) is, according to section 2.1, a complete noetherian global semiring characterized by a set of embedded subsemifields above "p":

$$\tilde{L}_{L_p} \subset \cdots \subset \tilde{L}_{L_p+h} \subset \cdots$$ (resp. $\tilde{L}_{\tilde{L}_p} \subset \cdots \subset \tilde{L}_{\tilde{L}_p+h} \subset \cdots$)

which, being compactified, give rise to the corresponding infinite pseudo-ramified completions:

$$L_{L_p} \subset \cdots \subset L_{L_p+h} \subset \cdots \in L_{L_p} \equiv L_{[v_p]} \ , \quad 1 \leq h \leq \infty,$$

(resp. $\tilde{L}_{\tilde{L}_p} \subset \cdots \subset \tilde{L}_{\tilde{L}_p+h} \subset \cdots \in L_{\tilde{L}_p} \equiv L_{[\tilde{L}_p]}$).

Referring to proposition 3.3.2, the number of elements of $L_{L_p}$ (resp. $L_{R_p}$), a more manageable notation than $L_{[v_p]}$ (resp. $L_{[\tilde{L}_p]}$), is:

$$|L_{L_p}| = |L_{R_p}| = \# \text{Nu} \times N \times \sum_h \sum f_{v_p+h,m_{p+h}}$$

while the number of elements of the corresponding global unramified compactified coefficient semiring $L_{L_p}^{nr}$ (resp. $L_{R_p}^{nr}$) is:

$$|L_{L_p}^{nr}| = |L_{R_p}^{nr}| = \# (\text{Nu})' \times N' \times \sum_h \sum f_{v_p+h,m_{p+h}}'$$

where $N$ is the order of the global inertia subgroup(s).

Let $L'_{L_p}$ (resp. $L'_{R_p}$) denote another global compactified coefficient semiring characterized by the same set of embedded infinite pseudo-ramified completions

$$L'_{L_p} \subset \cdots \subset L'_{L_p+h} \subset \cdots \in L'_{L_p} \quad \text{(resp. } L'_{\tilde{L}_p} \subset \cdots \subset L'_{\tilde{L}_p+h} \subset \cdots \in L'_{R_p})$$

of which number of elements is

$$|L'_{L_p}| = |L'_{R_p}| = \# (\text{Nu})' \times N' \times \sum_h \sum f'_{v_p+h,m_{p+h}}'.$$

$L'_{L_p}$ (resp. $L'_{R_p}$) then differs from $L_{L_p}$ (resp. $L_{R_p}$) by the number of non units $(\text{Nu})'$ and by the order $N'$ of the inertia subgroup, the (unramified) maximal orders being by hypothesis the same in $L_{L_p}$ (resp. $L_{R_p}$) and in $L'_{L_p}$ (resp. $L'_{R_p}$) and characterized essentially by the global residue degrees $f_{v_p+h,m_{p+h}}'$ as developed subsequently.

A coefficient semiring homomorphism (isomorphism)

$$h_{L'_{L_p} \rightarrow L_{L_p}} : L'_{L_p} \longrightarrow L_{L_p}$$

induces a homomorphism (resp. an isomorphism)

$$h_{L'_{L_p}^{nr} \rightarrow L_{L_p}^{nr}} : L'_{L_p}^{nr} \longrightarrow L_{L_p}^{nr}$$ (resp. $i_{L'_{L_p}^{nr} \rightarrow L_{L_p}^{nr}} : L'_{L_p}^{nr} \sim \longrightarrow L_{L_p}^{nr}$)
on the global unramified compactified left coefficient semi-rings \( L^\prime_{L_p} \) and \( L^\prime_{L_p} \) (which corresponds in characteristic 0 to (global) residue semi-fields) at the condition that:

\[
\sum_h \sum_{m_{p+h}} f^\prime_{v_{p+h},m_{p+h}} = \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}}
\]

(resp. \( \#(\text{Nu})' \times \sum_h \sum_{m_{p+h}} f^\prime_{v_{p+h},m_{p+h}} = \# \text{Nu} \times \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}} \)).

4.1.5 Proposition (left case)

The kernel of the coefficient semi-ring homomorphism:

\[
h_{L_{L_p}} : L'_{L_p} \rightarrow L_{L_p},
\]

inducing the homomorphism \( h_{L^\prime_{L_p} \rightarrow L'_{L_p}} \) (resp. the isomorphism \( i_{L^\prime_{L_p} \rightarrow L'_{L_p}} \)) on their global residue semi-fields, is characterized by an extension degree:

\[
[L'_{L_p} : k] - [L_{L_p} : k] = (N' - N) \times \left( \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}} \right)
\]

and a number of elements:

\[
|L'_{L_p}| - |L_{L_p}| = \left( \#(\text{Nu})' \times N' \right) - \left( \# \text{Nu} \times N \right) \times \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}}
\]

(resp. \( |L'_{L_p}| - |L_{L_p}| = (N' - N) \times \# \text{Nu} \times \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}} \)).

Proof. This is a consequence of section 4.1.4 restricted to the left case, the right case being handled similarly.

4.1.6 Corollary

The coefficient semi-ring homomorphism

\[
h_{L'_{L_p} \rightarrow L_{L_p}} : L'_{L_p} \rightarrow L_{L_p}
\]

corresponds to a base change from \( L_{L_p} \) into \( L'_{L_p} \) of which degree is:

\[
[L'_{L_p} : k] - [L_{L_p} : k] = (N' - N) \times \sum_h \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}}
\]
4.1.7 Proposition (left case)

The inverse image of the homomorphism \( h_{L_{Lp}'} \rightarrow L_{Lp} \) between global coefficient semirings is isomorphic to the inverse image of the homomorphism \( h_{B_p^+} : B_p^+ \rightarrow B_p^+ \) between local coefficient semirings if:

1. the number of elements of the kernel \( K(h_{L_{Lp}'} \rightarrow L_{Lp}) \) is equal to the number of elements of the kernel \( K(h_{B_p^+} : B_p^+ \rightarrow B_p^+) \), i.e. if

\[
|K(h_{L_{Lp}'} \rightarrow L_{Lp})| = |K(h_{B_p^+} : B_p^+ \rightarrow B_p^+)|
\]

given by:

\[
(N' - N) \times \# Nu \times \sum_{h} \sum_{m_{p+h}} f_{v_{p+h},m_{p+h}} = p^{q'} - q
\]

where \( q = e_{p^{\beta_p}} \sum_{r} f_{p^{\beta_p}} \) and \( q' = e_{p^{\beta_p}} \sum_{r} f_{p^{\beta_p}} \).

2. \( B_p^+ \) covers isomorphically \( L_{Lp} \).

Proof. In order that the inverse image of \( h_{L_{Lp}'} \rightarrow L_{Lp} \) be isomorphic to the inverse image of \( h_{B_p^+} : B_p^+ \rightarrow B_p^+ \), it is necessary that the induced homomorphism on the respective residue semifields be an isomorphism as it was seen in sections 4.1.1 and 4.1.4 where \( h_{L_{Lp}^{nr}} \rightarrow L_{Lp}^{nr} \) must be the isomorphism \( i_{L_{Lp}^{nr} \rightarrow L_{Lp}^{nr}} \).

On the other hand, the inverse image of the homomorphism \( h_{L_{Lp}'} \rightarrow L_{Lp} \) is given by:

\[
L_{Lp}' = L_{Lp} + K\left(h_{L_{Lp}'} \rightarrow L_{Lp}\right)
\]

with respect to its kernel \( K\left(h_{L_{Lp}'} \rightarrow L_{Lp}\right) \) and the inverse image of the homomorphism \( h_{B_p^+} : B_p^+ \rightarrow B_p^+ \) is similarly given by:

\[
B_p' = B_p^+ + K\left(h_{B_p^+} : B_p^+ \rightarrow B_p^+\right).
\]

4.1.8 Proposition (left case)

1. To each global coefficient semiring \( L_{Lp} \) corresponds the category \( \mathcal{C}(L_{Lp}') \), \( 1 \leq c \leq \infty \), associated with the set \( \{h_{L_{Lp}'} : L_{Lp}\}N_c \) of coefficient semiring homomorphisms in such a way that:

   (a) the extension degree \([L_{Lpc}^{'}, L_{Lpc}] : k\) of \( L_{Lpc}^{'}, L_{Lpc} \) differ from the extension degree of \( L_{Lp} \) by the orders “\( N_c' \)” of the inertia subgroups, \( N_c' \neq N \).

   (b) the set \( \{f_{v_{p+h},m_{p+h}}\}_{h} \) of global residue degrees is an invariant in \( L_{Lpc} \) and in the set \( \{L_{Lpc}'\}N_c \).
2. Similarly, to each local coefficient semiring $B_p^+$ corresponds the category $\mathcal{C}(B_p^+)$, $1 \leq d \leq \infty$, associated with the set $\{h_{B_p^+} \to B_p^+\}_{e_{p_d}^+}$ of coefficient semiring homomorphisms in such a way that:

(a) the extension degrees $[K_{p_d}^+ : L_p^+]$ of $K_{p_d}^+$ differ from the extension degree of $K_p^+$ by the ramification indices $e_{p_d}^+$ different from $e_{p}^+$.

(b) the set $\{f_{p_d}^+\}_r$ of residue degrees is an invariant in $K_p^+$ and in the set $\{K_{p_d}^+\}_d$.

Proof. a) The existence of the category $\mathcal{C}(L_{p_c}^+)$ whose objects are global coefficient semirings $L_{p_c}^+$ and whose morphisms are the homomorphisms $h_{L_{p_c}^+} \to L_{p_c}^+$ results from the orders $N_c^r$ of the inertia subgroups of $L_{p_c}^+$, $c \in \mathbb{N}$, $1 \leq c \leq \infty$.

b) In the same way, the existence of the category $\mathcal{C}(B_{p_d}^+)$, $1 \leq d \leq \infty$, whose objects are local coefficient semirings $B_{p_d}^+$ and whose morphisms are homomorphisms $h_{B_{p_d}^+} \to B_{p_d}^+$ results from the ramification indices $e_{p_d}^+$ of $B_{p_d}^+$ different from the ramification index $e_{p}^+$ of $B_p^+$.

4.2 Deformations of Galois representations over local and global noetherian bisemirings

4.2.1 Definition (Global bilinear deformation representative)

A global bilinear deformation representative resulting from a global bilinear coefficient semiring homomorphism

$$h_{L_{p_c}^+} : L_{p_c}^+ \to L_{p_c}^+ : L_{p_c}^+ \times L_{p_c}^+ \to L_{p_c}^+ \times L_{p_c}^+,$$

inducing the isomorphism $i_{L_{p_c}^+} : L_{p_c}^+ \times L_{p_c}^+ \to L_{p_c}^+ \times L_{p_c}^+$ on their global bilinear residue semifields is an equivalence class representative $\rho_{L_p}^+$ of lifting

$$\text{Gal}(\hat{L}_{L_p}^+ / k) \times \text{Gal}(\hat{L}_{L_p}^+ / k) \xrightarrow{h_{L_p}^+ - \text{Gal}(\hat{L}_{L_p}^+ / k) \times \text{Gal}(\hat{L}_{L_p}^+ / k)} \text{GL}_n(L_{p_c}^+ \times L_{p_c}^+).$$

with the evident bilinear notations introduced in sections 4.1.4 and 4.1.5 for the right and left global semiring homomorphisms, in section 2.4 for the Galois and Weil groups and in section 2.5 for the algebraic bilinear semigroups $\text{GL}_n(L_{p_c}^+ \times L_{p_c}^+)$ and $\text{GL}_n(L_{p_c}^+ \times L_{p_c}^+)$.
4.2.2 Proposition (Global bilinear deformation)

Let \( K(h_{L_{pc}' \times L_{pc}' - L_{R_{pc}} \times L_{L_{pc}'}}) \) be the kernel of the bihomomorphism \( h_{L_{pc}' \times L_{pc}' - L_{R_{pc}} \times L_{L_{pc}'}} \), where \( L_{R_{pc}} \times L_{L_{pc}'} \) belongs to the bicategory \( C(L_{R_{pc}} \times L_{L_{pc}'}) \), \( 1 \leq c \leq \infty \), defined similarly as for the left case in proposition 4.1.8.

Let\[
\text{Gal}(\hat{\delta L}_{R_{pc}'}/k) \times \text{Gal}(\hat{\delta L}_{L_{pc}'}/k) = \left[ \text{Gal}(\hat{L}_{R_{pc}'}/k) - \text{Gal}(\hat{L}_{R_{p}}/k) \right] \times \left[ \text{Gal}(\hat{L}_{L_{pc}'}/k) - \text{Gal}(\hat{L}_{L_{p}}/k) \right]
\]
be the Weil subgroup associated with this kernel \( K(h_{L_{R_{pc}}' \times L_{L_{pc}'} - L_{R_{pc}} \times L_{L_{pc}'}}) \).

Then, a \( n \)-dimensional global bilinear deformation of \( \rho_{L} \) is an equivalence class of liftings \( \{\rho_{L_{c}}\} \), \( 1 \leq c \leq \infty \), described by the following commutative exact sequence

\[
\begin{array}{cccc}
1 & \rightarrow & \text{Gal}(\hat{\delta L}_{R_{pc}'}/k) & \rightarrow \text{Gal}(\hat{L}_{R_{pc}'}/k) \times \text{Gal}(\hat{L}_{L_{pc}'}/k) \rightarrow \text{Gal}(\hat{L}_{R_{p}}/k) \rightarrow 1 \\
& & \downarrow \delta \rho_{L_{c}} & \downarrow \rho_{L_{c}} & \downarrow \rho_{L} \\
1 & \rightarrow & \text{GL}_{n}(\delta L_{R_{pc}}' \times \delta L_{L_{pc}}') & \rightarrow \text{GL}_{n}(L_{R_{pc}}' \times L_{L_{pc}'}) & \rightarrow \text{GL}_{n}(L_{R_{p}} \times L_{L_{p}}) & \rightarrow 1
\end{array}
\]

of which “Weil kernel” is \( \text{Gal}(\hat{\delta L}_{R_{pc}'}/k) \times \text{Gal}(\hat{\delta L}_{L_{pc}'}/k) \) and “\( \text{GL}_{n}(\bullet \times \bullet) \) kernel” is \( \text{GL}_{n}(\delta L_{R_{pc}}' \times \delta L_{L_{pc}}') \) where (\( \delta L_{R_{pc}}' \times \delta L_{L_{pc}}' \)) is given by

\[
(\delta L_{R_{pc}}' \times \delta L_{L_{pc}}') = \left[ (L_{R_{pc}}' - L_{R_{p}}) \times (L_{L_{pc}}' - L_{L_{p}}) \right].
\]

Proof. A \( n \)-dimensional global bilinear deformation of \( \rho_{L} \) is thus an equivalence class of liftings

\[
\rho_{L_{c}} = \rho_{L} + \delta \rho_{L_{c}} \quad \forall c, \quad 1 \leq c \leq \infty,
\]

where two liftings \( \rho_{L_{c_{1}}} \) and \( \rho_{L_{c_{2}}} \) are strictly equivalent if they can be transformed one into another by conjugation by bielements of \( \text{GL}_{n}(L_{R_{pc}}' \times L_{L_{pc}'}) \) in the kernel \( \text{GL}_{n}(\delta L_{R_{pc}}' \times \delta L_{L_{pc}'}) \) of \( h_{G_{c_{1}}} \): this is the worked out condition of (bilinear) deformations proposed by B. Mazur for example in [Maz2].

In other words, the lifting \( \rho_{L_{c_{1}}} \) generates the bilinear algebraic semigroup \( \text{GL}_{n}(L_{R_{pc_{1}}}' \times L_{L_{pc_{1}}}') \) having a (presumed) rank

\[
r_{G_{n}}(L_{R_{pc_{1}}}' \times L_{L_{pc_{1}}}') = \left( (N_{c_{1}}') \cdot f_{v}^{n} \right) \times \left( (N_{c_{1}}') \cdot f_{v}^{n} \right)
\]

where:

- \( N_{c_{1}}' \) is the order of the global inertia subgroup according to section 4.1.4;
• \( f_v \ (\equiv f_\pi) \) is a condensed notation for \( \sum_{h} \sum_{m_p+h} f_{v_{p+h,m_{p+h}}} \).

And, the lifting \( \rho_{Lc_2} \) generates the algebraic bilinear semigroup \( \text{GL}_n(L'_{R_{pc_2}} \times L'_{L_{pc_2}}) \) of which rank

\[
r_{G_n(L'_{R_{c_2}} \times L'_{L_{c_2}})} = ((N'_{c_2})^n \cdot f_{\pi}^n) \times ((N'_{c_2})^n \cdot f_{\pi}^n)
\]
differs from the rank \( r_{G_n(L'_{R_{c_1}} \times L'_{L_{c_1}})} \) of \( \text{GL}_n(L'_{R_{c_1}} \times L'_{L_{c_1}}) \) by

\[
\delta r_{G_n(2-1)} = [(N'_{c_2})^n - (N'_{c_1})^n](f_{\pi})^{n^2}
\]
taking into account that \( f_v = f_\pi \) if we refer to proposition 4.1.5.

As a consequence, the deformed algebraic bilinear semigroups \( \text{GL}_n(L'_{R_{pc_1}} \times L'_{L_{pc_1}}) \) and \( \text{GL}_n(L'_{R_{pc_2}} \times L'_{L_{pc_2}}) \) differ from the algebraic bilinear semigroup \( \text{GL}_n(L'_{R_{p}} \times L'_{L_{p}}) \) by their respective kernels \( \text{GL}_n(\delta L'_{R_{pc_1}} \times \delta L'_{L_{pc_1}}) \) and \( \text{GL}_n(\delta L'_{R_{pc_2}} \times \delta L'_{L_{pc_2}}) \) in such a way that \( \text{GL}_n(L'_{R_{pc_1}} \times L'_{L_{pc_1}}) \) can be transformed into \( \text{GL}_n(L'_{R_{pc_2}} \times L'_{L_{pc_2}}) \) by conjugation of bielements in the first kernel \( \text{GL}_n(\delta L'_{R_{pc_1}} \times \delta L'_{L_{pc_1}}) \) bringing it into the second kernel \( \text{GL}_n(\delta L'_{R_{pc_2}} \times \delta L'_{L_{pc_2}}) \) and inversely.

4.2.3 Corollary (The transformation of kernels)

\[
\text{GL}_n(\delta L'_{R_{pc_1}} \times \delta L'_{L_{pc_1}}) \longrightarrow \text{GL}_n(\delta L'_{R_{pc_2}} \times \delta L'_{L_{pc_2}})
\]
corresponds to a base change of \( \text{GL}_n(L'_{R_{pc_1}} \times L'_{L_{pc_1}}) \) into \( \text{GL}_n(L'_{R_{pc_2}} \times L'_{L_{pc_2}}) \) whose dimension is given by the difference of ranks

\[
\delta r_{G_n(2-1)} = [(N'_{c_2})^n - (N'_{c_1})^n](f_{\pi})^{n^2}.
\]

Proof. This results directly from proposition 4.2.2.

As the local semirings of category \( \mathcal{C}(B_{p_{c_2}^+}) \) differ between themselves by their ramification indices and as the global semirings of the category \( \mathcal{C}(L_{pc}^+) \) differ between themselves by their orders of inertia subgroups, the local (bilinear) deformations can be described similarly as it was done for the global bilinear deformations.

4.2.4 Definition (Local bilinear deformation representative)

A local bilinear deformation representative resulting from a local bilinear coefficient semiring homomorphism:

\[
h_{B_{p^-}^+ \times B_{p^-}^+} : \quad B_{p^-}^+ \times B_{p^-}^+ \longrightarrow B_{p^-}^+ \times B_{p^-}^+,
\]

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inducing an isomorphism on their residue (bi)semifields, is an equivalence class representative \( \rho_{K'} \) of lifting

\[
\begin{array}{ccc}
\text{Gal}(K_p^-/L_p^-) \times \text{Gal}(K_p^+/L_p^+) & \xrightarrow{h_{K' \to K}} & \text{Gal}(K_p^-/L_p^-) \times \text{Gal}(K_p^+/L_p^+) \\
\downarrow \rho_{K'} & & \downarrow \rho_K \\
\text{GL}_n(K_p' \times K_p') & \xrightarrow{h_{GL' \to GL}} & \text{GL}_n(K_p' \times K_p')
\end{array}
\]

where:

- \( K_p^+ \) (resp. \( K_p^- \)) is a finite left (resp. right) \( p \)-adic Galois extension of the left (resp. right) \( p \)-adic semifield \( L_p^+ \) (resp. \( L_p^- \)) according to section 3.3.1 in such a way that \( B_{pd}^+ \) (resp. \( B_{pd}^- \)) be the (valuation) coefficient semiring in \( K_p^+ \) (resp. \( K_p^- \)) referring to section 4.1.1.

- \( K_p' \) (resp. \( K_p'^- \)) is another finite left (resp. right) \( p \)-adic Galois extension of \( L_p^+ \) (resp. \( L_p^- \)) in such a way that the extension degree \( q' \) of \( K_p' \) (resp. \( K_p'^- \)) is superior to the extension degree \( q \) of \( K_p^+ \) (resp. \( K_p^- \)) referring to section 4.1.1 and proposition 4.1.2.

### 4.2.5 Proposition (Local bilinear deformation)

Let \( K(h_{B_{pd}^- \times B_{pd}^+} - B_p^+ \times B_p^-) \) be the kernel of the bihomomorphism \( h_{B_{pd}^- \times B_{pd}^+} - B_p^+ \times B_p^- \) where \( B_{pd}^- \times B_{pd}^+ \) belongs to the bicategory \( C(B_{pd}^- \times B_{pd}^+) \), \( 1 \leq d \leq \infty \), defined similarly as for the left case in proposition 4.1.8.

Let

\[
\text{Gal}(\delta K_{pd}^-/L_p^-) \times \text{Gal}(\delta K_{pd}^+/L_p^+) = [\text{Gal}(K_{pd}^-/L_p^-) - \text{Gal}(K_p^-/L_p^-)] \times [\text{Gal}(K_{pd}^+/L_p^+) - \text{Gal}(K_p^+/L_p^+)]
\]

be the Weil subgroup corresponding to this kernel \( K(h_{B_{pd}^- \times B_{pd}^+} - B_p^+ \times B_p^-) \).

Then, a \( n \)-dimensional local bilinear deformation of \( \rho_K \) is an equivalence class of liftings \( \{\rho_{K_d'}\} \) described by the following commutative exact sequence:

\[
\begin{array}{ccc}
1 & \to & \text{Gal}(\delta K_{pd}^-/L_p^-) \\
& \times & \text{Gal}(\delta K_{pd}^+/L_p^+) \\
\downarrow \rho_{K_d} & & \downarrow \rho_{K_d} \\
1 & \to & \text{GL}_n(\delta K_{pd}^- \times \delta K_{pd}^+) \\
& \to & \text{GL}_n(K_{pd}^- \times K_{pd}^+) \\
& \to & \text{GL}_n(K_p^- \times K_p^+) & \to 1
\end{array}
\]

of which “Weil kernel” is \( \text{Gal}(\delta K_{pd}^-/L_p^-) \times \text{Gal}(\delta K_{pd}^+/L_p^+) \) and “\( \text{GL}_n(\bullet \times \bullet) \) kernel” is \( \text{GL}_n(\delta K_{pd}^- \times \delta K_{pd}^+) \) where

\[
(\delta K_{pd}^- \times \delta K_{pd}^+) = [(K_{pd}^- - K_p^-) \times (\delta K_{pd}^+/K_p^+)]
\]
Proof. A \( n \)-dimensional local bilinear deformation of \( \rho_k \) is an equivalence class of liftings

\[
\rho_{K_d'} = \rho_K + \delta \rho_{K_d'}, \quad \forall \ d, \ 1 \leq d \leq \infty,
\]

where two local liftings \( \rho_{K_{d_1}'} \) and \( \rho_{K_{d_2}'} \) are strictly equivalent if they can be transformed one into another by conjugation of bielements of \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) in the local kernel \( \text{GL}_n(\delta K_{p_{d_1}}^- \times \delta K_{p_{d_1}}^+) \) of \( h_{\text{GL}_d'\text{GL}} \) \cite{Maz2}.

In other words, the lifting \( \rho_{K_{d_1}'} \) generates the bilinear algebraic semigroup \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) having a rank

\[
\tau_{\text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+)} = ((e_{\beta_{p_{d_1}}^-} \times (f_{\beta_{p_{d_1}}^+}))^n) = (e_{\beta_{p_{d_1}}'}^n) \times (f_{\beta_{p_{d_1}}^+})^n
\]
where:

- \( e_{\beta_{p_{d_1}}^-} = e_{\beta_{p_{d_1}}^+} \) is the ramification index of \( K_{p_{d_1}}^- \) or of \( K_{p_{d_1}}^+ \).
- \( f_{\beta_{p_{d_1}}^-} = f_{\beta_{p_{d_1}}^+} = \sum_r f_{\beta_{p_{d_1}}^r} \) is the residue degree of \( K_{p_{d_1}}^- \) or of \( K_{p_{d_1}}^+ \).

And, the lifting \( \rho_{K_{d_2}'} \) generates the algebraic bilinear semigroup \( \text{GL}_n(K_{p_{d_2}}^- \times K_{p_{d_2}}^+) \) of which rank

\[
\tau_{\text{GL}_n(K_{p_{d_2}}^- \times K_{p_{d_2}}^+)} = (e_{\beta_{p_{d_2}}'}^n) \times (f_{\beta_{p_{d_2}}^+})^n
\]
differs from the rank \( \tau_{\text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+)} \) of \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) by

\[
\delta \tau_{\text{GL}_n(2-1)} = [(e_{\beta_{p_{d_2}}'}^n)^2 - (e_{\beta_{p_{d_1}}'}^n)^2](f_{\beta_{p_{d_2}}^+})^n)]
\]
because all the residue degrees of \( f_{\beta_{p_{d_2}}^r} \) are equal since the homomorphisms on the envisaged residue semifields are isomorphisms according to section 4.1.1. Consequently, the deformed algebraic bilinear semigroups \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) and \( \text{GL}_n(K_{p_{d_2}}^- \times K_{p_{d_2}}^+) \) differ from the algebraic bilinear semigroup \( \text{GL}_n(K_{p}^- \times K_{p}^+) \) by their respective kernels \( \text{GL}_n(\delta K_{p_{d_1}}^- \times \delta K_{p_{d_1}}^+) \) and \( \text{GL}_n(\delta K_{p_{d_2}}^- \times \delta K_{p_{d_2}}^+) \) in such a way that \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) can be transformed into \( \text{GL}_n(K_{p_{d_2}}^- \times K_{p_{d_2}}^+) \) by conjugation of bielements in the first kernel bringing it to the second kernel and inversely.

\[\Box\]

4.2.6 Corollary (The transformation of kernels)

\[
\text{GL}_n(\delta K_{p_{d_1}}^- \times \delta K_{p_{d_1}}^+) \longrightarrow \text{GL}_n(\delta K_{p_{d_2}}^- \times \delta K_{p_{d_2}}^+)
\]
corresponds to a base change of \( \text{GL}_n(K_{p_{d_1}}^- \times K_{p_{d_1}}^+) \) into \( \text{GL}_n(K_{p_{d_2}}^- \times K_{p_{d_2}}^+) \) of which dimension is given by the difference of ranks

\[
\delta \tau_{\text{GL}_n(2-1)} = [(e_{\beta_{p_{d_2}}'}^n)^2 - (e_{\beta_{p_{d_1}}'}^n)^2](f_{\beta_{p_{d_2}}^+})^n]
\]

Proof. This results from proposition 4.2.5 similarly as for the global case. \[\Box\]

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4.3 Global and local coefficient semiring quantum homomorphisms

4.3.1 Global coefficient semiring quantum homomorphisms (left case)

Let $L_{L_p}$ (resp. $L_{R_p}$) denote a global compactified coefficient semiring characterized by a set of embedded infinite pseudo-ramified completions above “$p$”:

$$L_{v_p} \subset \cdots \subset L_{v_{p+h}} \subset \ldots \quad (\text{resp.} \quad L_{\pi_p} \subset \cdots \subset L_{\pi_{p+h}} \subset \ldots) \quad 1 \leq h \leq \infty.$$  

Let $L_{L_{p+j}}$ (resp. $L_{R_{p+j}}$) denote another global compactified coefficient semiring composed of the same number of corresponding embedded pseudo-ramified completions above “$p+j$”:

$$L_{v_{p+j}} \subset \cdots \subset L_{v_{p+j+h}} \subset \ldots \quad (\text{resp.} \quad L_{\pi_{p+j}} \subset \cdots \subset L_{\pi_{p+j+h}} \subset \ldots)$$

where:

- $q = p + j$ is assumed to be a prime number;
- the global residue degree of $L_{v_{p+j+h}}$ (and $L_{\pi_{p+j+h}}$) in $L_{L_{p+j}}$ (resp. $L_{R_{p+j}}$) is given by:
  $$f_{v_{p+j+h}} = f_{v_{p+h}} + j = p + h + j$$
  with respect to its correspondent $L_{v_{p+h}}$ in $L_{L_p}$ (resp. $L_{R_p}$);
- the number of non units and the degree of the inertia subgroup are the same in $L_{L_p}$ (resp. $L_{R_p}$) and in $L_{L_q}$ (resp. $L_{R_q}$).

Let

$$Qh_{L_{L_{p+j}} \rightarrow L_{L_p}} : L_{L_{p+j}} \longrightarrow L_{L_p}$$

be a uniform quantum homomorphism between global compactified coefficient semirings inducing an isomorphism on their global inertia subgroups having the same degree $N$.

4.3.2 Proposition

The kernel $K(Qh_{L_{L_{p+j}} \rightarrow L_{L_p}})$ of the uniform quantum homomorphism

$$Qh_{L_{L_{p+j}} \rightarrow L_{L_p}} : L_{L_{p+j}} \longrightarrow L_{L_p}$$

inducing an isomorphism on their global inertia subgroups, is characterized by an extension degree:

$$[L_{L_{p+j}} : k] - [L_{L_p} : k] = N \times j \times \sum_{h} m_{p+h}$$

if $m_{p+h+j} = m_{p+h}$. 

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Proof. The extension degree \([L_{Lp+j} : k]\) is given by:

\[
[L_{Lp+j} : k] = N \times \sum_h \sum m_{p+h+j} f_{v_{p+h+j}},
\]

\(j\) being fixed, where \(f_{v_{p+h+j}}\) is the global residue degree of the completion \(L_{v_{p+h+j}}\), and the extension degree \([L_{Lp} : k]\) is given by:

\[
[L_{Lp} : k] = N \times \sum_h m_{p+h} f_{v_{p+h}}, \quad 0 \leq h \leq \infty.
\]

So, referring to section 4.3.1, we have that:

\[
[L_{Lp+j} : k] - [L_{Lp} : k] = N \times \left( \sum_h (m_{p+h+j}) \times (p+h+j) \right) - N \times \left( \sum_h (m_{p+h}) \times (p+h) \right)
\]

\[
= N \times j \times \sum_h m_{p+h},
\]

where \(m_{p+h}\) is the multiplicity of the completion \(L_{v_{p+h}}\), if \(m_{p+h+j} = m_{p+h}\), condition resulting from the homomorphisms \(Qh_{L_{p+j} \to L_{Lp}}\).

4.3.3 Corollary

The uniform quantum homomorphim:

\[
Qh_{L_{Lp+j} \to L_{Lp}} : \quad L_{Lp+j} \longrightarrow L_{Lp}
\]

corresponds to a base change from \(L_{Lp}\) into \(L_{Lp+j}\) of which extension degree

\[
[L_{Lp+j} : k] - [L_{Lp} : k] = N \times j \times \sum_h m_{p+h}
\]

means an increment of \(j\) quanta of degree \(N\) on each completion of the global compactified coefficient semiring \(L_{Lp}\).

Proof. 1. From the preceding developments, it appears that the kernel \(K(Qh_{L_{Lp+j} \to L_{Lp}})\) of the uniform quantum homomorphism \(Qh_{L_{Lp+j} \to L_{Lp}}\) measures the extent of the base change from \(L_{Lp}\) into \(L_{Lp+j}\).

2. This base change corresponds to an increment of \(j\) quanta on each completion of \(L_{Lp}\), taking into account that a quantum was defined [Pie3] as a (compact) closed-irreducible algebraic subset of degree \(N\).
4.3.4 Remarks

1. The quantum homomorphism \( Qh_{L_{p+}(j)} \rightarrow L_{L_p} \) will be said non uniform if the completions of \( L_{L_p} \) are increased by different numbers of quanta, the integer “\( j \)” then varying from one completion of \( L_{p+}(j) \) to another.

2. The uniform quantum homomorphism \( Qh_{L_{p+} \rightarrow L_{L_p}} \) is in fact induced by a quantum homomorphism

\[
Qh_{L_{p+}^n \rightarrow L_{L_p}^n} : L_{L_p}^n \longrightarrow L_{L_p}^n
\]

between the corresponding unramified global compactified coefficient semirings \( L_{L_p}^n \) and \( L_{L_p}^n \).

4.3.5 “Quantum” homomorphisms between local coefficient semirings (left case)

Let \( B_{p^t}^{+} \) denote a noetherian local left coefficient semiring on the completion of a finite Galois extension \( K_{p^t}^{+} \) of the \( p \)-adic semifield \( L_{p}^{+} \).

\( B_{p^t}^{+} \) is a discrete valuation semiring, being the integral closure of \( A_{p^t}^{+} \) in \( K_{p^t}^{+} / L_{p}^{+} \). It is characterised by a chain \( \beta_{p^t}^{+} \subset \cdots \subset \beta_{p^e}^{+} \) of \( r \) prime ideals and has a finite residue semifield \( k_{K_{p^t}^{+}} \) according to section 4.1.1.

Let \( B_{p^t}^{+} \) be another local left coefficient semiring, with \( t = r + s, r \leq t \leq s + r \), in such a way that \( B_{p^t}^{+} \) be the integral closure of \( A_{p^t}^{+} \) in the finite Galois extension \( K_{p^t}^{+} / L_{p}^{+} \) and that

\[
k_{K_{p^t}^{+}} = \{ O_{K_{p^t}^{+} | \beta_{p^t}^{+}} \}_i^{r=0}
\]

be its residue semifield defined on the set of \( r \) residue semifields \( O_{K_{p^t}^{+} | \beta_{p^t}^{+}} / m(A_{p^t}^{+}) \).

The semiring \( B_{p^t}^{+} / m(A_{p^t}^{+}) B_{p^t}^{+} \) is also an \( A_{p^t}^{+} / m(A_{p^t}^{+}) \)-semialgebra of degree

\[
d_t = [K_{p^t}^{+} : L_{p}^{+}] = \sum_{\beta_{p^t}^{+} \beta_{p^t}^{+}} f_{\beta_{p^t}^{+} e_{\beta_{p^t}^{+}}}
\]

where \( f_{\beta_{p^t}^{+}} \) is the residue degree of \( \beta_{p^t}^{+} \) in the extension \( K_{p^t}^{+} / L_{p}^{+} \) and \( e_{\beta_{p^t}^{+}} \) is the ramification index in the extensions \( K_{p^t}^{+} / L_{p}^{+} \) and \( K_{p^t}^{+} / L_{p}^{+} \).

A coefficient semiring quantum homomorphism

\[
Qh_{B_{p^t}^{+} \rightarrow B_{p^t}^{+}} : B_{p^t}^{+} \longrightarrow B_{p^t}^{+}
\]

sending \( B_{p^t}^{+} \) into \( B_{p^t}^{+} \) is such that:

a) the inverse image of the maximal ideal \( \tilde{\omega}_{K_{p^t}^{+}} \) of \( B_{p^t}^{+} \) is the maximal ideal \( \tilde{\omega}_{K_{p^t}^{+}} B_{p^t}^{+} \) of \( B_{p^t}^{+} \).

b) the induced homomorphism

\[
Qh_{k_{K_{p^t}^{+}} \rightarrow k_{K_{p^t}^{+}}} : k_{K_{p^t}^{+}} \longrightarrow k_{K_{p^t}^{+}}
\]

on the left residue semifields is an isomorphism.
c) there is an isomorphism between the inertia subgroups $K_{p_r}^+$ and $K_{p_t}^+$ having the same ramification index $e_{\beta_p^+}$.

4.3.6 Proposition

The kernel of the coefficient semiring quantum homomorphism

$$Qh_{B_{p_t}^+ \rightarrow B_{p_r}^+} : B_{p_t}^+ \rightarrow B_{p_r}^+$$

is characterized by a degree

$$[K_{p_t}^+ : L_{p}^+] - [K_{p_r}^+ : L_{p}^+] = e_{\beta_p^+} \times \left[ \sum_{t=r}^{r+s} f_{\beta_p^+}^t - \sum_{r}^{f_{\beta_p^+}^r} \right]$$

and measures the extent of the base change from $K_{p_r}^+$ to $K_{p_t}^+$.

Proof. This is evident from the preceding developments.

4.3.7 Categories associated with quantum homomorphisms (left case)

1. To each global coefficient semiring $L_{L_p}$ corresponds the category $\mathcal{C}(L_{L_{p+j}}), 1 \leq j \leq \infty$, associated with the set $\{Qh_{L_{p+j} \rightarrow L_{L_p}}\}_{j=1}^{\infty}$ of uniform quantum homomorphisms in such a way that:

   (a) the extension degrees $[L_{L_{p+j}} : k]$ of $L_{L_{p+j}}$ differ from the extension degree of $L_{L_p}$ by the numbers of $j$ quanta of degree $N$, $j > 0$.

   (b) the order $N$ of the inertia subgroups is the same in $L_{L_{p+j}}$ and in $L_{L_p}$; it is thus an invariant of $\mathcal{C}(L_{L_{p+j}})$.

2. To each local coefficient semiring $B_{p_r}^+$ corresponds the category $\mathcal{C}(B_{p_t}^+), r \leq t \leq r+s$, associated with the set $\{Qh_{B_{p_t}^+ \rightarrow B_{p_r}^+}\}$ of uniform quantum homomorphisms in such a way that:

   (a) the extension degrees $[K_{p_t}^+ : L_{p}^+]$ of $K_{p_t}^+$ differ from the extension degree $[K_{p_r}^+ : L_{p}^+]$ of $K_{p_r}^+$ by the differences $[f_{K_{p_t}^+} - f_{K_{p_r}^+}]$ in their (local) residue degrees according to proposition 4.3.6.

   (b) the ramification index $e_{\beta_p^+}$ is the same in $K_{p_t}^+$ and in $K_{p_r}^+$; it is thus an invariant of $Qh_{B_{p_t}^+ \rightarrow B_{p_r}^+}$. 
4.4 Quantum deformations of Galois representations over local and global noetherian bisemirings

4.4.1 Definition (Global bilinear quantum deformation representative)

A global bilinear quantum deformation representative, resulting from a global bilinear coefficient semiring quantum homomorphism:

\[ Q_{h_{R_{p+j}} \times L_{p+j}}^{L_{p+j} \times L_{p+j}} : L_{R_{p+j}} \times L_{p+j} \to L_{R_p} \times L_{L_p} , \]

characterized by global inertia subgroups having the same degree \( N \), is an equivalence class representative \( \rho_{L_j} \) of lifting

\[
\text{Gal}(\hat{\delta}_{R_{p+j}/k}) \times \text{Gal}(\hat{\delta}_{L_{p+j}/k}) \xrightarrow{Q_{h_{L_j} \to L}} \text{Gal}(\hat{\delta}_{L_{R_p}/k}) \times \text{Gal}(\hat{\delta}_{L_{L_p}/k})
\]

\[
\text{GL}_n(L_{R_{p+j}} \times L_{p+j}) \xrightarrow{Q_{h_{G_j} \to G}} \text{GL}_n(L_{R_p} \times L_{L_p})
\]

with the evident bilinear notations.

4.4.2 Proposition (Global bilinear quantum deformation)

Let \( K(Q_{h_{L_{R_{p+j}} \times L_{p+j} \to L_{R_p} \times L_{L_p}}}) \) be the kernel of the uniform quantum bihomomorphism \( Q_{h_{L_{R_{p+j}} \times L_{p+j} \to L_{R_p} \times L_{L_p}}} \) where \( L_{R_{p+j}} \times L_{L_{p+j}} \) belongs to the bicategory \( \mathcal{C}(L_{R_{p+j}} \times L_{L_{p+j}}), 1 \leq j \leq \infty \).

Let

\[
\text{Gal}(\hat{\delta}_{R_{p+j}/k}) \times \text{Gal}(\hat{\delta}_{L_{p+j}/k}) = [\text{Gal}(\hat{\delta}_{R_{p+j}/k}) - \text{Gal}(\hat{\delta}_{L_{R_p}/k})] \times [\text{Gal}(\hat{\delta}_{L_{p+j}/k}) - \text{Gal}(\hat{\delta}_{L_{L_p}/k})]
\]

be the Weil subgroup associated with this kernel.

Then, a \( n \)-dimensional global bilinear quantum deformation of \( \rho_L \) is an equivalence class of liftings \( \{\rho_{L_j}\}, 1 \leq j \leq \infty \), described by the following commutative exact sequence:

\[
1 \to \text{Gal}(\hat{\delta}_{L_{R_{p+j}}/k}) \times \text{Gal}(\hat{\delta}_{L_{p+j}/k}) \xrightarrow{Q_{h_{L_j} \to L}} \text{Gal}(\hat{\delta}_{L_{R_p}/k}) \times \text{Gal}(\hat{\delta}_{L_{L_p}/k}) \to 1
\]

\[
\text{GL}_n(\delta_{L_{R_{p+j}} \times L_{L_{p+j}}}) \to \text{GL}_n(L_{R_{p+j}} \times L_{p+j}) \xrightarrow{Q_{h_{G_j} \to G}} \text{GL}_n(L_{R_p} \times L_{L_p}) \to 1
\]

of which “Weil kernel” is \( \text{Gal}(\hat{\delta}_{L_{R_{p+j}}/k}) \times \text{Gal}(\hat{\delta}_{L_{p+j}/k}) \) and “ \( \text{GL}_n(\bullet \times \bullet) \) kernel” is \( \text{GL}_n(\delta_{L_{R_{p+j}} \times \delta_{L_{p+j}}}) \).
Proof. The proof is similar to that of proposition 4.2.2.
A $n$-dimensional global bilinear quantum deformation of $\rho_L$ is an equivalence class of liftings:

$$\rho_{L_j} = \rho_L + \delta\rho_{L_j}$$

where two liftings $\rho_{L_{j_1}}$ and $\rho_{L_{j_2}}$ are strictly equivalent if they can be transformed one into another by conjugation by bielements of $GL_n(L_{R_{p+j}} \times L_{L_{p+j}})$ in the kernel of $Qh_{G_j \rightarrow G}$.

The lifting $\rho_{L_{j_1}}$ generates the algebraic bilinear semigroup $GL_n(L_{R_{p+j}} \times L_{L_{p+j}})$ having a rank

$$r_{G_n(L_{R_{j_1}} \times L_{L_{j_1}})} = ((N)^n \cdot f^n_{\underline{v}(+j_1)}) \times ((N)^n \cdot f^n_{\underline{v}(+j_1)})$$

where

$$f^n_{\underline{v}(+j_1)} = \sum_{h} m_{p+h+j_1}(p + h + j_1)$$

$$= \sum_{h} \sum_{m+h+j_1} f_{p+h+j_1,m+h+j_1}.$$ 

Proceeding similarly for the lifting $\rho_{L_{j_2}}$, we find that the difference of ranks between $GL_n(L_{R_{p+j}} \times L_{L_{p+j}})$ and $GL_n(L_{R_{p+j_2}} \times L_{L_{p+j_2}})$ is

$$\delta r_{G_n(j_2-j_1)} = N^{n^2} \left( f^n_{\underline{v}(+j_2)} - f^n_{\underline{v}(+j_1)} \right).$$

The deformed algebraic bilinear semigroup $GL_n(L_{R_{p+j}} \times L_{L_{p+j}})$ can be transformed into $GL_n(L_{R_{p+j_2}} \times L_{L_{p+j_2}})$ by conjugation of bielements in the first kernel $GL_n(\delta L_{R_{p+j_1}} \times \delta L_{L_{p+j_1}})$ bringing it into the second kernel $GL_n(\delta L_{R_{p+j_2}} \times \delta L_{L_{p+j_2}}).$

4.4.3 Corollary

The transformation of kernels

$$GL_n(\delta L_{R_{p+j_1}} \times \delta L_{L_{p+j_1}}) \longrightarrow GL_n(\delta L_{R_{p+j_2}} \times \delta L_{L_{p+j_2}})$$

corresponds to a base change of $GL_n(L_{R_{p+j_1}} \times L_{L_{p+j_1}})$ into $GL_n(L_{R_{p+j_2}} \times L_{L_{p+j_2}})$ of which dimension is given by the difference of ranks

$$\delta r_{G_n(j_2-j_1)} = N^{n^2} \left( f^n_{\underline{v}(+j_2)} - f^n_{\underline{v}(+j_1)} \right).$$

4.4.4 Definition (Local bilinear “quantum” deformation representative)

A local bilinear “quantum” deformation representative resulting from a local bilinear coefficient semiring “quantum” homomorphism:

$$Qh_{B_{p_1}^+ \times B_{p_1}^+ \rightarrow B_{p_1}^+ \times B_{p_1}^+} : B_{p_1}^- \times B_{p_1}^+ \longrightarrow B_{p_1}^- \times B_{p_1}^+,$$
inducing an isomorphism on their inertia subgroups having the same ramification index $e_{\beta^+}$, is an equivalence class representative $\rho_{K_{pr}}$ of lifting

\[
\begin{array}{ccc}
\text{Gal}(K_{p_t}/L_p^-) \times \text{Gal}(K_{p_t}^+/L_p^+) & \xrightarrow{Qh_{K_{pt}^- \rightarrow K_{pt}^+}} & \text{Gal}(K_{p_t}^-/L_p^-) \times \text{Gal}(K_{p_t}^+/L_p^+) \\
\downarrow \rho_{K_{pt}} & & \downarrow \rho_{K_{pt}} \\
\text{GL}_n(K_{p_t}^- \times K_{p_t}^+) & \xrightarrow{Gh_{\text{GL}(t) \rightarrow \text{GL}(r)}} & \text{GL}_n(K_{p_t}^- \times K_{p_t}^+)
\end{array}
\]

of which notations refer to section 4.3.5.

4.4.5 Proposition (Local bilinear “quantum” deformation)

Let $K(Qh_{B_{p_t}^- \times B_{p_t}^+ \rightarrow B_{p_t}^- \times B_{p_t}^+})$ be the kernel of the bihomomorphism $Qh_{B_{p_t}^- \times B_{p_t}^+ \rightarrow B_{p_t}^- \times B_{p_t}^+}$ where $B_{p_t}^- \times B_{p_t}^+$ belongs to the bicategory $C(B_{p_t} \times B_{p_t}^+), \ r \leq t \leq r + s$.

Let

\[
\text{Gal}(\delta K_{p_t}^-/L_p^-) \times \text{Gal}(\delta K_{p_t}^+/L_p^+) = [\text{Gal}(K_{p_t}^-/L_p^-) - \text{Gal}(K_{p_t}^-/L_p^-)] \times [\text{Gal}(K_{p_t}^+/L_p^+) - \text{Gal}(K_{p_t}^+/L_p^+)]
\]

be the Weil subgroup corresponding to this kernel.

Then, a $n$-dimensional local bilinear “quantum” deformation of $\rho_{K_{pr}}$ is an equivalence class of liftings $\{\rho_{K_{pt}}\}$ described by the following exact sequence:

\[
\begin{array}{cccccc}
1 & \rightarrow & \text{Gal}(\delta K_{p_t}^-/L_p^-) & \xrightarrow{Qh_{K_{pt}^- \rightarrow K_{pt}^+}} & \text{Gal}(K_{p_t}^-/L_p^-) & \rightarrow 1 \\
\times \text{Gal}(\delta K_{p_t}^+/L_p^+) & \times \text{Gal}(K_{p_t}^+/L_p^+) & \times \text{Gal}(K_{p_t}^+/L_p^+) & \rightarrow & 1 \\
\downarrow \delta \rho_{K_{pt}} & \downarrow \rho_{K_{pt}} & \downarrow \rho_{K_{pt}} & & \\
1 & \rightarrow & \text{GL}_n(\delta K_{p_t}^- \times K_{p_t}^+) & \xrightarrow{Qh_{\text{GL}(t) \rightarrow \text{GL}(r)}} & \text{GL}_n(K_{p_t}^- \times K_{p_t}^+) & \rightarrow 1
\end{array}
\]

of which “Weil kernel” is $\text{Gal}(\delta K_{p_t}^-/L_p^-) \times \text{Gal}(\delta K_{p_t}^+/L_p^+)$ and “$\text{GL}_n(\bullet \times \bullet)$ kernel” is $\text{GL}_n(\delta K_{p_t}^- \times \delta K_{p_t}^+)$. 

Proof. A $n$-dimensional local bilinear “quantum” deformation of $\rho_{K_{pr}}$ is an equivalence class of liftings

\[
\rho_{K_{pt}} = \rho_{K_{pt}} + \delta \rho_{K_{pt}}
\]

where two local liftings $\rho_{K_{p_{t1}}}$ and $\rho_{K_{p_{t2}}}$ are strictly equivalent if they can be transformed one into another by conjugation of bielements of $\text{GL}_n(\delta K_{p_{t1}}^- \times K_{p_{t1}}^+)$ in the kernel $\text{GL}_n(\delta K_{p_{t1}}^- \times \delta K_{p_{t1}}^+)$ of $Qh_{\text{GL}(t_1) \rightarrow \text{GL}(r)}$. 

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In other words, the difference of ranks of the two bilinear algebraic semigroups $GL_n(K_p \times K_{p_2}^+)$ and $GL_n(K_{p_1}^- \times K_{p_2}^+)$ is

$$\delta r_{GL_n(t_2-t_1)} = e_n^{\beta_p} \left( f_{K_{p_2}}^{n^2} - f_{K_{p_1}}^{n^2} \right),$$

as it results from proposition 4.3.6.

Thus, the deformed algebraic bilinear semigroups $GL_n(K_{p_1}^- \times K_{p_2}^+)$ and $GL_n(K_{p_1}^- \times K_{p_2}^+)$ differ from the algebraic bilinear semigroup $GL_n(K_{p_1}^- \times K_{p_2}^+)$ by their respective kernels in such a way that $GL_n(K_{p_1}^- \times K_{p_2}^+)$ can be transformed into $GL_n(K_{p_1}^- \times K_{p_2}^+)$ by conjugation of bielements in their first kernel bringing it into the second kernel and inversely.

4.4.6 Corollary

The transformation of kernels

$$GL_n(\delta K_{p_1}^- \times \delta K_{p_1}^+) \longrightarrow GL_n(\delta K_{p_2}^- \times \delta K_{p_2}^+)$$

corresponds to a base change of $GL_n(K_{p_1}^- \times K_{p_2}^+)$ into $GL_n(K_{p_2}^- \times K_{p_2}^+)$ of which dimension is given by the difference of ranks $\delta r_{GL_n(t_2-t_1)}$. 
5 Inverse quantum lifts and the Goldbach conjecture

5.1 Global bilinear elliptic quantum deformations

5.1.1 Definition (n-dimensional global elliptic bilinear quantum deformation)

According to proposition 4.4.2, a n-dimensional global bilinear quantum deformation of

\[ \rho_L : \text{Gal}(\hat{L}_{R_p}/k) \times \text{Gal}(\hat{L}_{L_p}/k) \rightarrow \text{GL}_n(L_{R_p} \times L_{L_p}) \]

is an equivalence class of liftings

\[ \rho_{L_j} = \rho_L + \delta \rho_{L_j} \]

where \( \rho_{L_j} \) is the morphism:

\[ \rho_{L_j} : \text{Gal}(\hat{L}_{R_{p+j}}/k) \times \text{Gal}(\hat{L}_{L_{p+j}}/k) \rightarrow \text{GL}_n(L_{R_{p+j}} \times L_{L_{p+j}}) . \]

Taking into account the Langlands global correspondence recalled in proposition 3.1.5, we can define a n-dimensional global elliptic bilinear quantum deformation of

\[ \rho^\text{ELLIP}_L = \text{ELLIP FREPsp}(\text{GL}_n(L_{R \times L_p}) \circ \rho_L) \]

where

\[ \text{ELLIP FREPsp}(\text{GL}_n(L_{R \times L_p})) : \text{GL}_n(L_{R_p} \times L_{L_p}) \rightarrow \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \]

is the epimorphism from the bilinear algebraic semigroup \( \text{GL}_n(L_{R} \times L_p) \) into the n-dimensional global elliptic bisemimodule

\[ \text{ELLIP}_{R \times L}(n, i \geq p, m_i) = \left( \sum_{i=p}^{t} \sum_{m_i} \lambda^2(n, i \geq p, m_i) e^{-2\pi i (i)x} \right) \otimes_D \left( \sum_{i=p}^{t} \sum_{m_i} \lambda^4(n, i \geq p, m_i) e^{2\pi i (i)x} \right) , \]

\[ x \in \mathbb{R}^n , \ i = p + h , \ p \leq i \leq t \leq \infty , \ 0 \leq h \leq \infty , \]

introduced in corollary 3.2.6.

This n-dimensional global elliptic bilinear quantum deformation of \( \rho^\text{ELLIP}_L \) is an equivalence class of liftings:

\[ \rho^\text{ELLIP}_{L_j} = \rho^\text{ELLIP}_L + \delta \rho^\text{ELLIP}_{L_j} \]

in such a way that \( \rho^\text{ELLIP}_{L_j} \) be given by:

\[ \rho^\text{ELLIP}_{L_j} = \text{ELLIP FREPsp}(\text{GL}_n(L_{R \times L_{(p+j)}}) \circ \rho_{L_j}) \]

where

\[ \text{ELLIP FREPsp}(\text{GL}_n(L_{R \times L_{(p+j)}})) : \text{GL}_n(L_{R_{p+j}} \times L_{L_{p+j}}) \rightarrow \text{ELLIP}_{R \times L}(n, i \geq p + j, m_i) \]
is the epimorphism from $\text{GL}_n(L_{R+p+j} \times L_{p+j})$ into the $n$-dimensional global elliptic bisemimodule

$$\text{ELLIP}_{R \times L}(n, i \geq p + j, m_i) = \left( \frac{1}{i_p \cdot j} \sum_{i=p+j}^{m_i} \lambda^i(n, i \geq p + j, m_i) e^{-2\pi i(i)x} \right) \otimes D \left( \frac{1}{i_p \cdot j} \sum_{i=p+j}^{m_i} \lambda^i(n, i \geq p + j, m_i) e^{2\pi i(i)x} \right).$$

5.1.2 Proposition

The $n$-dimensional elliptic global bilinear quantum deformation $\rho_{L_j}^{\text{ELLIP}}$ of $\rho_L^{\text{ELLIP}}$ results from the $n$-dimensional global bilinear quantum deformation $\rho_L$ by the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gal}(L_{R+p+j}/k) \times \text{Gal}(L_{L+p+j}/k) & \xrightarrow{Q^{h_{L_j}}_{L_j \rightarrow L}} & \text{Gal}(L_{R}/k) \times \text{Gal}(L_{L_p}/k) \\
\rho_{L_j}^{\text{ELLIP}} \downarrow \rho_{L_j} & & \rho_{L_j} \downarrow \rho_{L} \\
\text{GL}_n(L_{R+p+j} \times L_{p+j}) & \xrightarrow{Q^{h_{C_j \rightarrow C}}_{C_j \rightarrow C}} & \text{GL}_n(L_{R} \times L_{L_p}) \\
\text{ELLIP} \downarrow \text{FResp}(\text{GL}_n(L_{R \times L(p+j)})) & & \text{ELLIP} \downarrow \text{FResp}(\text{GL}_n(L_{R \times L_p})) \\
\text{ELLIP}_{R \times L}(n, i \geq p + j, m_i) & \xrightarrow{Q^{h_{E_{L_j} \rightarrow E\text{L}}}_{E_{L_j} \rightarrow E\text{L}}} & \text{ELLIP}_{R \times L}(n, i \geq p, m_i)
\end{array}
\]

in such a way that the injective morphism $D_{R \times L}^{[p]}(n) = Q^{h_{E_{L_j} \rightarrow E\text{L}}}_{E_{L_j} \rightarrow E\text{L}}^{-1}$

$$D_{R \times L}^{[p]}(n) : \text{ELLIP}_{R \times L}(n, i \geq p, m_i) \longrightarrow \text{ELLIP}_{R \times L}(n, i \geq p + j, m_i)$$

be a quantum deformation of the $n$-dimensional global elliptic bisemimodule $\text{ELLIP}_{R \times L}(n, i \geq p, m_i)$.

Proof. This results immediately from definition 5.1.1.

5.1.3 Proposition

Let

$$D_{R \times L}^{[p]}(n) : \text{ellip}_{R \times L}(n, [p], m_p) \longrightarrow \text{ellip}_{R \times L}(n, [p + j], m_{p+j})$$

denote a quantum equivalence class representative of liftings called in a more condensed form a quantum deformation of the $(p, m_p)$-th conjugacy class representative $\text{ellip}_{R \times L}(n, i \geq p, m_i)$ of $\text{ELLIP}_{R \times L}(n, i \geq p, m_i)$, i.e. the $(p, m_p)$-th term of $\text{ellip}_{R \times L}(n, i \geq p, m_i)$.

This elliptic quantum deformation is then associated with the exact sequence:

$$1 \longrightarrow \text{ellip}_{R \times L}(n, [j]) \longrightarrow \text{ellip}_{R \times L}(n, [p + j], m_{p+j}) \longrightarrow \text{ellip}_{R \times L}(n, [p], m_p) \longrightarrow 1.$$
Proof. Indeed, this exact sequence results from the commutative diagram:

\[
\begin{array}{cccccc}
1 & \rightarrow & \text{Gal}(\hat{L}_{\pi_j}/k) & \rightarrow & \text{Gal}(\hat{L}_{p+j}/k) & \rightarrow & \text{Gal}(\hat{L}_{p}/k) & \rightarrow & 1 \\
 & & \times \text{Gal}(\hat{L}_{v_j}/k) & \times \text{Gal}(\hat{L}_{v_{p+j}}/k) & \times \text{Gal}(\hat{L}_{v_p}/k) & & & \\
1 & \rightarrow & \text{gl}_n(L_{\pi_j} \times L_{v_j}) & \rightarrow & \text{gl}_n(L_{\pi_{p+j}} \times L_{v_{p+j}}) & \rightarrow & \text{gl}_n(L_{\pi_p} \times L_{v_p}) & \rightarrow & 1 \\
& & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
1 & \rightarrow & \text{ellip}_{R \times L}(n, [j]) & \rightarrow & \text{ellip}_{R \times L}(n, [p + j], m_{p+j}) & \rightarrow & \text{ellip}_{R \times L}(n, [p], m_p) & \rightarrow & 1 \\
\end{array}
\]

where:

- \(\text{gl}_n(L_{\pi_{p+j}} \times L_{v_{p+j}}) \in \text{GL}_n(L_{R_{p+j}} \times L_{L_{p+j}})\) is the \((p + j)\)-th conjugacy class of the bilinear algebraic semigroup \(\text{GL}_n(L_{R_{p+j}} \times L_{L_{p+j}})\);

- \(\text{gl}_n(L_{\pi_j} \times L_{v_j})\) is the kernel at \(\text{“} j^n \text{”} \) biquanta of the exact sequence

\[
\begin{array}{cccccc}
1 & \rightarrow & \text{gl}_n(L_{\pi_j} \times L_{v_j}) & \rightarrow & \text{gl}_n(L_{\pi_{p+j}} \times L_{v_{p+j}}) & \rightarrow & \text{gl}_n(L_{\pi_p} \times L_{v_p}) & \rightarrow & 1 \\
\end{array}
\]

if it is referred to corollary 4.3.3.

\[\blacksquare\]

### 5.2 Inverse elliptic quantum deformations and the Goldbach conjecture

#### 5.2.1 Definition (Inverse elliptic quantum deformations)

Let \(\mathcal{D}_{R \times L}[p+j],[p+j+k](n)\) denote the equivalence class representative of lifting of the global elliptic subsemimodule \(\text{ellip}_{R \times L}(n, [p + j], m_{p+j})\), of class \([p + j]\), towards the global elliptic subsemimodule \(\text{ellip}_{R \times L}(n, [p + j + k], m_{p+j+k})\) of class \([p + j + k]\) as considered in proposition 5.1.3.

Then, the inverse deformation \(\mathcal{D}_{R \times L}[p+j+k],[p+j](n)\) can be introduced by the surjective mapping:

\[
\mathcal{D}_{R \times L}[p+j+k],[p+j](n) : \text{ellip}_{R \times L}(n, [p + j + k], m_{p+j+k}) \rightarrow \text{ellip}_{R \times L}(n, [p + j], m_{p+j})
\]

which is associated with the exact sequence:

\[
1 \rightarrow \text{ellip}_{R \times L}(n, [k]) \rightarrow \text{ellip}_{R \times L}(n, [p + j + k], m_{p+j+k}) \rightarrow \text{ellip}_{R \times L}(n, [p + j], m_{p+j}) \rightarrow 1
\]

in such a way that:

\[
\text{ellip}_{R \times L}(n, [p + j], m_{p+j}) \simeq \text{ellip}_{R \times L}(n, [p + j + k], m_{p+j+k}) - \text{ellip}_{R \times L}(n, [k])
\]
corresponds to the endomorphism:
\[
\text{End}_{R\times L}^{[p+j+k]-[p+j]}(n) : \ellip_{R\times L}(n, [p + j], m_{p+j+k})
\]
\[\longrightarrow \ellip_{R\times L}(n, [p + j], m_{p+j}) + \ellip_{R\times L}(n, [k]) .\]

5.2.2 Proposition

The set of inverse quantum deformations \( \{\ellip_{R\times L}^{[p+j+k]-[p+j]}(n)\}_{k=1}^{k_{up}} \) areartinian deformations.

Proof. Indeed, the set of class representatives of decreasing global elliptic subbisemimodules:

\[
\ellip_{R\times L}(n, [p + j + k_{up}], m_{p+j+k_{up}}) \subset \cdots \subset \ellip_{R\times L}(n, [p + j + k], m_{p+j+k})
\]
\[\subset \cdots \subset \ellip_{R\times L}(n, [p + j], m_{p+j}), \quad 1 \leq k \leq k_{up} \leq \infty ,\]
generated under the set of inverse quantum deformations \( \{\ellip_{R\times L}^{[p+j+k]-[p+j]}(n)\}_{k=1}^{k_{up}} \), forms an artinian sequence.

5.2.3 Inverse quantum deformations of one-dimensional tori

Consider the inverse quantum deformation:

\[
\ellip_{R\times L}^{[p+j+k]-[p+j]}(1) : \ellip_{R\times L}(1, [p + j + k], m_{p+j+k})
\]
\[\longrightarrow \ellip_{R\times L}(1, [p + j], m_{p+j})
\]
of a global elliptic subbisemimodule \( \ellip_{R\times L}(1, [p + j + k], m_{p+j+k}) \) from a class \([p + j + k]\) to a class \([p + j]\),

where

- \( \ellip_{R\times L}(1, [p + j], m_{p+j}) \sim T^1_{R}[p + j, m_{p+j}] \times T^1_{L}[p + j, m_{p+j}] \) is in bijection with the product, right by left, of the analytic developments of semitori of class \([p + j]\), i.e. having a rank \( r^{(1)}_{p+j} = (p + j) \cdot N \) such that \((p + j)\) is a global residue degree;

- \( \ellip_{R\times L}(1, [p + j + k], m_{p+j+k}) \sim T^1_{R}[p + j + k, m_{p+j+k}] \times T^1_{L}[p + j + k, m_{p+j+k}] \) of which semitori have a rank \( r^{(1)}_{p+j+k} = (p + j + k) \cdot N \).

The semitorus (or semicircle) \( T^1_{L}[p + j, m_{p+j}] \) is defined in the upper half plane. Consequently, if we have to consider an analytic continuation of it to the whole plane, we have to undouble it according to:

\[
T^1_{L}[p + j, m_{p+j}] \longrightarrow T^1_{2L}[2(p + j), m_{2(p+j)}] \equiv T^1_{2L}[p' + j', m_{p'+j'}]
\]
such that:
• $T_{2L}^1[\cdot]$ has a rank $r_{v_2(p+j)}^{(1)} = 2 \cdot (p+j) \cdot N$;

• $T_{2L}^1[p' + j', m_{p' + j'}]$ has been defined with respect to a new prime number $p' \neq p$.

So, a general class representative of inverse lifting corresponding to an inverse quantum deformation of a $1D$-torus of class $[p' + j' + k']$ will be defined by the projective mapping:

$$\mathcal{D}_L^{[p' + j' + k'] - [p' + j']}(1): \quad T_{2L}^1[p' + j' + k', m_{p' + j' + k'}] \longrightarrow T_{2L}^1[p' + j', m_{p' + j'}]$$

associated with the endomorphism:

$$\text{End}_L^{[p' + j' + k'] - [p' + j']}(1): \quad T_{2L}^1[p' + j' + k', m_{p' + j' + k'}] \longrightarrow T_{2L}^1[p' + j', m_{p' + j'}] + T_{2L}^1[k', m_k]$$

splitting the circle of class $[p' + j' + k']$ into two complementary portions of circles of classes $[p' + j']$ and $[k']$.

If the global residue degrees $f_{v_{p' + j'}} = p' + j'$ and $f_{v_{k'}} = k'$ are not even integers, then $T_{2L}^1[p' + j', m_{p' + j'}]$ and $T_{2L}^1[k', m_k]$ are not “closed” circles.

But, in any case, we have the following equality between the global residue degrees associated with the endomorphism $\text{End}_L^{[p' + j' + k'] - [p' + j']}(1)$ of the circle $T_{2L}^1[p' + j' + k', m_{p' + j' + k'}]$:

$$f_{v_{p' + j' + k'}} = f_{v_{p' + j'}} + f_{v_{k'}} \quad \Rightarrow \quad p' + j' + k' = (p' + j') + (k')$$

where $f_{v_{p' + j' + k'}} = p' + j' + k'$ is an even integer since it is assumed to be the global residue degree of a one-dimensional torus undoubled from the corresponding semicircle.

### 5.2.4 Proposition (Goldbach conjecture)

Let $c_{2L}^1[p' + j' + k', m_{p' + j' + k'}]$ denote a closed curve isomorphic to the one-dimensional torus $T_{2L}^1[p' + j' + k', m_{p' + j' + k'}]$ of class $(p' + j' + k')$ generated from the corresponding semitorus localized in the upper half plane.

Let

$$\mathcal{D}_L^{[p' + j' + k'] - [p' + j']}(1): \quad c_{2L}^1[p' + j' + k', m_{p' + j' + k'}] \longrightarrow c_{2L}^1[p' + j', m_{p' + j'}]$$

be the projective inverse quantum deformation of the closed curve $c_{2L}^1[p' + j' + k', m_{p' + j' + k'}]$ associated with the endomorphism

$$\text{End}_L^{[p' + j' + k'] - [p' + j']}(1): \quad c_{2L}^1[p' + j' + k', m_{p' + j' + k'}] \longrightarrow c_{2L}^1[p' + j', m_{p' + j'}] + c_{2L}^1[k', m_k]$$

splitting the closed curve $c_{2L}^1[p' + j' + k', m_{p' + j' + k'}]$ into two complementary portions, in such a way that:
• the curves \( c_{2L}'[p' + j', m_{p'+j'}] \) and \( c_{2L}'[k', m_{k'}] \) are not necessarily closed;

• the global residue degrees of these curves verify:

\[
f_{v_{p'+j'+k'}} = f_{v_{p'+j'}} + f_{v_{k'}} \quad \Rightarrow \quad p' + j' + k' = (p' + j') + (k')
\]

where \( G_{\text{even}} = 2G = p' + j' + k' \) is an even integer.

Then, to the even integer \( G_{\text{even}} \), there corresponds at least one basic class representative of inverse lifting corresponding to the inverse deformation \( D_{L}^{[p'+j'+k']-[p'+j']}(1) \) of the closed curve \( c_{2L}'[p' + j' + k', m_{p'+j'+k'}] \) such that the global residue degrees associated to \( D_{L}^{[p'+j'+k']-[p'+j']}(1) \) verify:

\[
G_{\text{even}} = f_{v_{p'+j'}} + f_{v_{k'}} \quad , \quad G_{\text{even}} \leq \infty ,
\]

where:

\[
f_{v_{p'+j'}} = p' + j' \quad \text{and} \quad f_{v_{k'}} = q' + n' = k'
\]

in such a way that the even integer \( F_{\text{even}} = G_{\text{even}} - j' - n' = p' + q' \), is the sum of two prime numbers \( p' \) and \( q' \), \( 4 \leq F_{\text{even}} \leq \infty \).

Proof:

1. To the set of one-dimensional tori \( \{T_{2L}'[p' + j' + k', m_{p'+j'+k'}]\}_{j',k'} \) is associated a classical Riemann \( \zeta \)-function whose trivial zeros are negative even integers which are proved to be in one-to-one correspondence with the global residue degrees

\[
f_{v_{p'+j'+k'}} = p' + j' + k' = 2G
\]

as developed in [Pie2].

It was also seen in [Pie2] that the trivial zeros and the nontrivial zeros of the classical \( \zeta \)-function all on the line \( \sigma = \frac{1}{2} \) are in one-to-one correspondence under the action of the Lie algebra of the decomposition group.

So, if we admit that the Riemann hypothesis is verified, then, the prime number theorem follows:

\[
\Pi(x) \sim \frac{x}{\ln x} \quad \text{when} \quad x \to \infty
\]

[B-K], [Ing], [K-S],

where \( \Pi(x) \) is the number of primes not exceeding \( x \).

2. To every even integer \( 2G \), there corresponds a pair of nontrivial zeros of the Riemann zeta function:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}
\]

and a set of prime numbers inferior to \( 2G \).
So, according to the prime number theorem fixing the density of the prime numbers, it is likely that there exists at least a pair \( \{ p', q' \} \) of prime numbers whose sum is an even integer.

But, the question is now the following: does there always exist a pair \( \{ p', q' \} \) of prime numbers whose sum is an even integer? The response is affirmative.

Indeed, the integers \( G_{\text{even}} \), \( f_{v_{p',j'}} \), and \( f_{v_{k'}} \), are interpreted as global residue degrees associated with the inverse quantum deformation \( D_L^{[p'+j'+k']-[p'+j']}(1) \) of the closed curve \( c_2[p' + j' + k', m_{p'+j'+k'}] \). Now, the global residue degree \( G_{\text{even}} = p' + j' + k' \) was defined in 2.1 and in [Pie1] by:

\[
[L_{v_{p'+j'+k'}}^{nr} : k] = f_{v_{p'+j'+k'}} = hp' + j'' + k'' = p' + j' + k',
\]

where \( j'' \) and \( k'' \) are integers, referring to congruence classes modulo \( p' \), and where \( p' \) is a prime number.

Similarly, we have that

\[
[L_{v_{p'+j'}}^{nr} : k] = f_{v_{p'+j'}} = \ell p' + j'' = p' + j'
\]

and

\[
[L_{v_{k'}}^{nr} : k] = f_{v_{k'}} = mq' + n = q' + n' = k'
\]

where \( q' \) is a prime number.

Indeed, the global residue degrees can be defined with respect to prime numbers in order that the corresponding local fields be able to be handled by classical \( p' \) (resp. \( q' \))-adic methods as it was developed in section 3.3.

We thus have that

\[
F_{\text{even}} = G_{\text{even}} - j' - n' = p' + q'
\]

where \( j' \) and \( n' \) are both either even or odd integers if \( F_{\text{even}} \) is an even integer.

Now, it is always possible to find pairs of even or odd integers \( j' \) and \( n' \) in order that \( F_{\text{even}} \) be an even integer as it results from section 3.2.1.

Finally, if \( F_{\text{even}} \) could not be defined as the sum \( (p' + q') \) of two primes \( p' \) and \( q' \), that should mean that \( G_{\text{even}} \) could not be equal to the sum \( (f_{v_{p'+j'}} + f_{v_{k'}}) \) of two global residue degrees and, thus, that it could not be possible to define all class representatives of inverse liftings of every closed curve, which is absurd.

As a result, we have the

5.2.5 Goldbach’s conjecture

Every even integer superior or equal to 4 is the sum of two prime numbers.
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