Harmonic Crystals in the Half-Space, I. 
Convergence to Equilibrium

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Abstract

We consider the dynamics of a harmonic crystal in the half-space with zero boundary condition. It is assumed that the initial date is a random function with zero mean, finite mean energy density which also satisfies a mixing condition of Rosenblatt or Ibragimov type. We study the distribution $\mu_t$ of the solution at time $t \in \mathbb{R}$. The main result is the convergence of $\mu_t$ to a Gaussian measure as $t \to \infty$ which is time stationary with a covariance inherited from the initial (in general, non-Gaussian) measure.

Key words and phrases: harmonic crystal in the half-space, random initial data, mixing condition, covariance matrices, weak convergence of measures.

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1 Introduction

The paper concerns the problems of the long-time convergence to the equilibrium distribution for the discrete systems. For one-dimensional chains of harmonic oscillators the problem is analyzed in [1, 16]: in [16] for initial measures which have distinct temperatures to the left and to the right, and in [1] for a more general class of initial measures characterized by a mixing condition of Rosenblatt- or Ibragimov-type and which are asymptotically translation-invariant to the left and to the right. For many-dimensional harmonic crystals the convergence has been proved in [13] for initial measures which are absolutely continuous with respect to the canonical Gaussian measure. In [4]–[7] we have started the convergence analysis for partial differential equations of hyperbolic type in $\mathbb{R}^d, d \geq 1$. In [8]–[10] we extended the results to harmonic crystals. In the harmonic approximation the crystal is characterized by the displacements $u(z, t) \in \mathbb{R}^n, z \in \mathbb{Z}^d$, of the crystal atoms from their equilibrium positions.

In the present work the dynamics of the harmonic crystals is studied in the half-space $\mathbb{Z}^d_+, d \geq 1$,

$$\ddot{u}(z, t) = -\sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z}')) u(z', t), \quad z \in \mathbb{Z}^d_+, \quad t \in \mathbb{R},$$

(1.1)

where $\tilde{z} := (-z_1, \tilde{x}), \tilde{x} = (z_2, \ldots, z_d) \in \mathbb{Z}^{d-1}$, with zero boundary condition,

$$u(z, t)|_{z_1=0} = 0,$$

(1.2)

and with the initial data

$$u(z, 0) = u_0(z), \quad \dot{u}(z, 0) = u_1(z), \quad z \in \mathbb{Z}^d_+. \quad \text{(1.3)}$$

Here $\mathbb{Z}^d_+ = \{z \in \mathbb{Z}^d : z_1 > 0\}$, $V(z)$ is the interaction (or force) matrix, $(V_{kl}(z))$, $k, l = 1, \ldots, n$, $u(z, t) = (u_1(z, t), \ldots, u_0(z, t))$, $u_0(z) = (u_{01}(z), \ldots, u_{0n}(z)) \in \mathbb{R}^n$ and correspondingly for $u_1(z)$. To coordinate the boundary and initial conditions we suppose that $u_0(z) = u_1(z) = 0$ for $z_1 = 0$.

Denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t)), \quad Y_0 = (Y^0_0, Y^1_0) \equiv (u_0(\cdot), u_1(\cdot))$. Then (1.1)–(1.3) takes the form of the evolution equation

$$\dot{Y}(t) = \mathcal{A}_+ Y(t), \quad t \in \mathbb{R}, \quad z \in \mathbb{Z}^d_+, \quad Y^0(t)|_{z_1=0} = 0, \quad Y(0) = Y_0. \quad \text{(1.4)}$$

Here $\mathcal{A}_+ = \begin{pmatrix} 0 & 1 \\ -V_+ & 0 \end{pmatrix}$, with $V_+ u(z) := \sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z}')) u(z')$.

It is assumed that the initial state $Y_0$ is given by a random element of the Hilbert space $\mathcal{H}_{a,+}$ of real sequences, see Definition 2.1 below. The distribution of $Y_0$ is a probability measure $\mu_0$ satisfying conditions S1–S4 below. In particular, the initial correlation function $Q_0(z, z')$ is asymptotically translation-invariant as $z_1, z'_1 \to +\infty$ (see Condition S2) and the measure $\mu_0$ has some mixing properties (see Condition S4). Given $t \in \mathbb{R}$, denote by $\mu_t$ the probability measure on $\mathcal{H}_{a,+}$ giving the distribution of the random solution $Y(t)$ to the problem (1.4).
Our main result gives the weak convergence of measures $\mu_t$ on the space $\mathcal{H}_{\alpha,+}$, with $\alpha < -d/2$, to a limit measure $\mu_\infty$, 

$$
\mu_t \overset{\mathcal{H}_{\alpha,+}}{\longrightarrow} \mu_\infty \text{ as } t \to \infty,
$$

(1.5)

where $\mu_\infty$ is an equilibrium Gaussian measure on $\mathcal{H}_{\alpha,+}$. This means the convergence

$$
\int f(Y) \mu_t(dY) \to \int f(Y) \mu_\infty(dY), \quad t \to \infty,
$$

for any bounded continuous functional $f$ on $\mathcal{H}_{\alpha,+}$.

Explicit formulas for the correlation functions of the measure $\mu_0$ are given in (3.3)–(3.5).

The paper is organized as follows. The conditions on the interaction matrix $V$ and the initial measure $\mu_0$ are given in Section 2. The main result is stated in Section 3. Examples of harmonic crystals and the initial measures satisfying all conditions imposed are constructed in Section 4. The convergence of correlation functions of $\mu_t$ is established in Section 5, the compactness of $\mu_t$, $t \geq 0$, and the convergence of characteristic functionals of $\mu_t$ are proved in Sections 6 and 7, respectively.

## 2 Conditions on the system and the initial measure

### 2.1 Dynamics

Let us assume that

$$
V(z) = V(\tilde{z}), \quad \text{where } \tilde{z} := (-z_1, \ldots, \tilde{z}) = (z_2, \ldots, z_d) \in \mathbb{Z}^{d-1}.
$$

(2.1)

Then the solution to the problem (1.1)–(1.3) can be represented as the restriction of the solution to the Cauchy problem with the odd initial date on the half-space. More precisely, consider the following Cauchy problem for the harmonic crystal in the whole space $\mathbb{Z}^d$:

$$
\begin{cases}
\dot{v}(z,t) = -\sum_{z' \in \mathbb{Z}^d} V(z-z')v(z',t), \quad z \in \mathbb{Z}^d, \quad t \in \mathbb{R}, \\
v(z,0) = v_0(z), \quad \dot{v}(z,0) = v_1(z), \quad z \in \mathbb{Z}^d.
\end{cases}
$$

(2.2)

Denote $X(t) = (X^0(t), X^1(t)) \equiv (v(\cdot,t), \dot{v}(\cdot,t))$, $X_0 = (X^0_0, X^1_0) \equiv (v_0(\cdot), v_1(\cdot))$. Then (2.2) has a form

$$
\dot{X}(t) = AX(t), \quad t \in \mathbb{R}, \quad X(0) = X_0.
$$

(2.3)

Here $A = \begin{pmatrix} 0 & 1 \\ -V & 0 \end{pmatrix}$, where $V$ is a convolution operator with the matrix kernel $V$.

Let us assume that the initial date $X_0(z)$ be an odd function w.r.t. $z_1 \in \mathbb{Z}^1$, i.e., $X_0(z) = -X_0(\tilde{z})$. Then the solution $v(z,t)$ of (2.2) is also an odd function w.r.t. $z_1 \in \mathbb{Z}^1$. Let us restrict the solution $v(z,t)$ on the domain $\mathbb{Z}^d_{\geq 1}$ and put $u(z,t) = v(z,t)|_{z_1 \geq 0}$. Then $u(z,t)$ is the solution to the problem (1.4) with the initial date $Y_0(z) = X_0(z)|_{z_1 \geq 0}$.

Assume that the initial date $Y_0$ of the problem (1.4) belongs to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, defined below.
Definition 2.1 \( \mathcal{H}_{\alpha,+} \) is the Hilbert space of \( \mathbb{R}^n \times \mathbb{R}^n \)-valued functions of \( z \in \mathbb{Z}_+^d \) endowed with the norm

\[
\|Y\|_{\alpha,+}^2 = \sum_{z \in \mathbb{Z}_+^d} |Y(z)|^2 (1 + |z|^2)^\alpha < \infty.
\]

In addition it is assumed that the initial date \( Y_0 = 0 \) if \( z_1 = 0 \).

We impose the following conditions \( \text{E1–E6} \) on the matrix \( V \).

\( \text{E1.} \) There exist positive constants \( C, \gamma \) such that \( \|V(z)\| \leq C e^{-\gamma |z|} \) for \( z \in \mathbb{Z}^d \), \( \|V(z)\| \) denoting the matrix norm.

Let \( \hat{V}(\theta) \) be the Fourier transform of \( V(z) \), with the convention

\[
\hat{V}(\theta) = \sum_{z \in \mathbb{Z}^d} V(z) e^{iz \cdot \theta}, \theta \in \mathbb{T}^d,
\]

where “\( \cdot \)” stands for the scalar product in Euclidean space \( \mathbb{R}^d \) and \( \mathbb{T}^d \) denotes the \( d \)-torus \( \mathbb{R}^d/(2\pi \mathbb{Z})^d \).

\( \text{E2.} \) \( V \) is real and symmetric, i.e., \( V_{kl}(-z) = V_{kl}(z) \in \mathbb{R}, k, l = 1, \ldots, n, z \in \mathbb{Z}^d \).

Both conditions imply that \( \hat{V}(\theta) \) is a real-analytic Hermitian matrix-valued function in \( \theta \in \mathbb{T}^d \).

\( \text{E3.} \) The matrix \( \hat{V}(\theta) \) is non-negative definite for every \( \theta \in \mathbb{T}^d \).

Let us define the Hermitian non-negative definite matrix,

\[
\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0.
\] (2.4)

\( \Omega(\theta) \) has the eigenvalues \( 0 \leq \omega_1(\theta) < \omega_2(\theta) \ldots < \omega_s(\theta) \), \( s \leq n \), and the corresponding spectral projections \( \Pi_\sigma(\theta) \) with multiplicity \( r_\sigma = \text{tr} \Pi_\sigma(\theta) \). \( \theta \mapsto \omega_\sigma(\theta) \) is the \( \sigma \)-th band function. There are special points in \( \mathbb{T}^d \), where the bands cross, which means that \( s \) and \( r_\sigma \) jump to some other value. Away from such crossing points \( s \) and \( r_\sigma \) are independent of \( \theta \). More precisely one has the following lemma.

Lemma 2.2 (see [8, Lemma 2.2]). Let the conditions \( \text{E1 and E2} \) hold. Then there exists a closed subset \( C_\ast \subset \mathbb{T}^d \) such that we have the following:

(i) the Lebesgue measure of \( C_\ast \) is zero.

(ii) For any point \( \Theta \in \mathbb{T}^d \setminus C_\ast \) there exists a neighborhood \( \mathcal{O}(\Theta) \) such that each band function \( \omega_\sigma(\theta) \) can be chosen as the real-analytic function in \( \mathcal{O}(\Theta) \).

(iii) The eigenvalue \( \omega_\sigma(\theta) \) has constant multiplicity in \( \mathbb{T}^d \setminus C_\ast \).

(iv) The spectral decomposition holds,

\[
\Omega(\theta) = \sum_{\sigma=1}^s \omega_\sigma(\theta) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_\ast,
\] (2.5)

where \( \Pi_\sigma(\theta) \) is the orthogonal projection in \( \mathbb{R}^n \). \( \Pi_\sigma \) is a real-analytic function on \( \mathbb{T}^d \setminus C_\ast \).
For $\theta \in T^d \setminus C_*$, we denote by $\operatorname{Hess}(\omega_\alpha)$ the matrix of second partial derivatives. The next condition on $V$ is the following:

**E4.** Let $D_\sigma(\theta) = \det (\operatorname{Hess}(\omega_\alpha(\theta)))$. Then $D_\sigma(\theta)$ does not vanish identically on $T^d \setminus C_*$, $\sigma = 1, \ldots, s$.

Let us denote

$$C_0 = \{ \theta \in T^d : \hat{\dot{V}}(\theta) = 0 \} \text{ and } C_\sigma = \{ \theta \in T^d \setminus C_* : D_\sigma(\theta) = 0 \}, \quad \sigma = 1, \ldots, s. \quad (2.6)$$

Then the Lebesgue measure of $C_\sigma$ vanishes, $\sigma = 0, 1, \ldots, s$ (see [8, Lemma 2.3]).

The last conditions on $V$ are the following:

**E5.** For each $\sigma \neq \sigma'$, the identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_{\pm}$, $\theta \in T^d \setminus C_*$, do not hold with $\text{const}_{\pm} \neq 0$.

This condition holds trivially in the case $n = 1$.

**E6.** $\|\hat{\dot{V}}^{-1}(\theta)\| \in L^1(T^d)$.

If $C_0 = \emptyset$, then $\|\hat{\dot{V}}^{-1}(\theta)\|$ is bounded and **E6** holds trivially.

Denote by $\mathcal{H}_\alpha$ the Hilbert space of $R^n \times R^n$-valued functions of $z \in Z^d$ endowed with the norm

$$\|X\|_{\alpha}^2 = \sum_{z \in Z^d} |X(z)|^2 (1 + |z|^2)^\alpha < \infty.$$

**Proposition 2.3** (see [8, Proposition 2.5]). Let conditions **E1** and **E2** hold, and choose some $\alpha \in \mathbb{R}$. Then (i) for any $X_0 \in H_\alpha$, there exists a unique solution $X(t) \in C(R, \mathcal{H}_\alpha)$ to the Cauchy problem (2.3).

(ii) The operator $U(t) : X_0 \mapsto X(t)$ is continuous in $\mathcal{H}_\alpha$.

**Corollary 2.4** Let conditions (2.1), **E1** and **E2** hold. Then (i) for any $Y_0 \in H_{\alpha,+}$, there exists a unique solution $Y(t) \in C(R, \mathcal{H}_{\alpha,+})$ to the mixed problem (1.4).

(ii) The operator $U_+(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}_{\alpha,+}$.

**Proof.** Corollary 2.4 follows from Proposition 2.3. Indeed, the solution $X(z, t)$ of (2.3) admits the representation

$$X(z, t) = \sum_{z' \in Z^d} G_t(z - z')X_0(z'), \quad (2.7)$$

where the Green function $G_t(z)$ has the Fourier representation

$$G_t(z) := F_{\theta \mapsto z}^{\hat{\dot{V}}^{-1}}(\exp (\hat{\dot{A}}(\theta) t)) = (2\pi)^{-d} \int_{T^d} e^{-iz\theta} \exp (\hat{\dot{A}}(\theta) t) d\theta$$

with

$$\hat{\dot{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{\dot{V}}(\theta) & 0 \end{pmatrix}, \quad \theta \in T^d. \quad (2.8)$$

Therefore, the solution to the problem (1.4) has a form

$$Y(z, t) = \sum_{z' \in Z^d_+} G_{t,+}(z, z')Y_0(z'), \quad z \in Z^d_+, \quad (2.10)$$

where $G_{t,+}(z, z') := G_t(z - z') - G_t(z - \bar{z}')$. Corollary 2.4 follows. 

\[\square\]
2.2 Random initial data and statistical conditions

Denote by $\mu_0$ a Borel probability measure on $\mathcal{H}_{a,+}$ giving the distribution of $Y_0$. Expectation with respect to $\mu_0$ is denoted by $E$.

Assume that the initial measure $\mu_0$ has the following properties.

S1. $Y_0(z)$ has zero expectation value, $E_0(Y_0(z)) = 0$, $z \in \mathbb{Z}_+^d$.

For $a, b, c \in \mathbb{C}^n$, denote by $a \otimes b$ the linear operator $(a \otimes b) c = a \sum_{j=1}^n b_j c_j$.

S2. The correlation matrices of the measure $\mu_0$ have a form

$$Q^{ij}_0(z, z') = E_0(Y^{i}_0(z) \otimes Y^{j}_0(z')) = q^{ij}_0(z_1, z'_1, \bar{z} - \bar{z}'), \ z, z' \in \mathbb{Z}_+^d, \ i, j = 0, 1. \quad (2.11)$$

where

(i) $q^{ij}_0(z_1, z'_1, \bar{z}) = 0$ for $z_1 = 0$ or $z'_1 = 0$,

(ii) $\lim_{y \to +\infty} q^{ij}_0(z_1 + y, y, \bar{z}) = q^{ij}_0(z), \ z = (z_1, \bar{z}) \in \mathbb{Z}_+^d$. Here $q^{ij}_0(z)$ are correlation functions of some translation invariant measure $\nu_0$ with zero mean value in $\mathcal{H}_\alpha$.

Definition 2.5 A measure $\nu$ is called translation invariant if $\nu(T_h B) = \nu(B), \ B \in \mathcal{B}(\mathcal{H}_\alpha), \ h \in \mathbb{Z}_+^d$, where $T_h X(z) = X(z - h), \ z \in \mathbb{Z}_+^d$.

S3. The measure $\mu_0$ has a finite variance and finite mean energy density,

$$e_0(z) = E_0(|Y_0^0(z)|^2 + |Y_0^1(z)|^2) = \text{tr} \left[ Q^{00}_0(z, z) + Q^{11}_0(z, z) \right] \leq e_0 < \infty, \ z \in \mathbb{Z}_+^d. \quad (2.12)$$

Finally, it is assumed that the measure $\mu_0$ satisfies a mixing condition. To formulate this condition, let us denote by $\sigma(\mathcal{A}), \mathcal{A} \subset \mathbb{Z}_+^d$, the $\sigma$-algebra in $\mathcal{H}_{a,+}$ generated by $Y_0(z)$ with $z \in \mathcal{A}$. Define the Ibragimov mixing coefficient of the probability measure $\mu_0$ on $\mathcal{H}_{a,+}$ by the rule (cf. [12, Definition 17.2.2])

$$\varphi(r) = \sup_{\mathcal{A}, B \subset \mathbb{Z}_+^d} \sup_{\text{dist}(\mathcal{A}, B) \geq r} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}. \quad (2.13)$$

Definition 2.6 A measure $\mu_0$ satisfies the strong uniform Ibragimov mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

S4. The measure $\mu_0$ satisfies the strong uniform Ibragimov mixing condition with

$$\int_0^\infty r^{d-1}\varphi^{1/2}(r) \, dr < \infty. \quad (2.14)$$

This condition can be considerably weakened (see Remarks 3.4 (i), (ii)).

3 Main results

Definition 3.1 (i) We define $\mu_t$ as the Borel probability measure on $\mathcal{H}_{a,+}$ which gives the distribution of the random solution $Y(t)$,

$$\mu_t(B) = \mu_0(U_+(-t)B), \ B \in \mathcal{B}(\mathcal{H}_{a,+}), \ t \in \mathbb{R}.$$
(ii) The correlation functions of the measure $\mu_t$ are defined by

$$Q^{ij}_t(z, z') = E\left(Y^i(z, t) \otimes Y^j(z', t)\right), \quad i, j = 0, 1, \quad z, z' \in \mathbb{Z}^d. \quad (3.1)$$

Here $Y^i(z, t)$ are the components of the random solution $Y(t) = (Y^0(\cdot, t), Y^1(\cdot, t))$ to the problem (1.4).

The main result of the paper is the following theorem.

**Theorem A** Let $d, n \geq 1$, $\alpha < -d/2$, and assume that the conditions (2.1), E1–E6 and S1–S4 hold. Then

(i) the convergence in (1.5) holds.

(ii) The limit measure $\mu_\infty$ is a Gaussian measure on $\mathcal{H}_{\alpha, +}$.

(iii) The correlation matrices of the measures $\mu_t$ converge to a limit, for $i, j = 0, 1$,

$$Q^{ij}_t(z, z') = \int (Y^i(z) \otimes Y^j(z')) \mu_t(dY) \to Q^{ij}_\infty(z, z'), \quad t \to \infty, \quad z, z' \in \mathbb{Z}^d. \quad (3.2)$$

The correlation matrix $Q^{ij}_\infty(z, z') = (Q^{ij}_\infty(z, z'))_{i, j = 0}^1$ of the limit measure $\mu_\infty$ has a form

$$Q_\infty(z, z') = q_\infty(z - z') - q_\infty(z - z') - q_\infty(\bar{z} - \bar{z}') + q_\infty(\bar{z} - \bar{z}'), \quad z, z' \in \mathbb{Z}^d. \quad (3.3)$$

Here $q_\infty(z) = q^+_{\infty}(z) + q^-_{\infty}(z)$, where in the Fourier transform we have

\begin{align*}
\hat{q}_+^{\infty}(\theta) &= \frac{1}{4} \sum_{\sigma=1}^s \Pi_{\sigma}(\theta) (\hat{q}_0(\theta) + C(\theta)\hat{q}_0(\theta)C(\theta)^*) \Pi_{\sigma}(\theta), \quad (3.4) \\
\hat{q}_-^{\infty}(\theta) &= \frac{i}{4} \sum_{\sigma=1}^s \text{sign} (\partial_{\theta} \omega_{\sigma}(\theta)) \Pi_{\sigma}(\theta) (C(\theta)\hat{q}_0(\theta) - \hat{q}_0(\theta)C(\theta)^*) \Pi_{\sigma}(\theta), \quad \theta \in T^d \setminus C_{*, +} \quad (3.5)
\end{align*}

$\Pi_{\sigma}(\theta)$ is the spectral projection from Lemma 2.2 (iv) and

$$C(\theta) = \begin{pmatrix} 0 & \Omega(\theta)^{-1} \\ -\Omega(\theta) & 0 \end{pmatrix}, \quad C(\theta)^* = \begin{pmatrix} 0 & -\Omega(\theta)^{-1} \\ \Omega(\theta)^{-1} & 0 \end{pmatrix}. \quad (3.6)$$

(iv) The measure $\mu_\infty$ is time stationary, i.e., $[U_+(t)]^* \mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.

(v) The group $U_+(t)$ is mixing with respect to the measure $\mu_\infty$, i.e.,

$$\lim_{t \to \infty} \int f(U_+(t)Y)g(Y) \mu_\infty(dY) = \int f(Y) \mu_\infty(dY) \int g(Y) \mu_\infty(dY)$$

for any $f, g \in L^2(\mathcal{H}_{\alpha, +}, \mu_\infty)$.

The assertions (i), (ii) of Theorem A can be deduced from Propositions 3.2 and 3.3 below.

**Proposition 3.2** The family of measures $\{\mu_t, t \in \mathbb{R}\}$ is weakly compact on the space $\mathcal{H}_{\alpha, +}$ for any $\alpha < -d/2$, and the bounds $\sup_{t \geq 0} E\|U_+(t)Y_0\|_{\alpha, +}^2 < \infty$ hold.
Set \( S = [S(Z^d_+) \otimes \mathbb{R}^n]^2 \), where \( S(Z^d_+) \) stands for the space of rapidly decreasing real sequences. Denote \( \langle Y, \Psi \rangle_+ = \langle Y^0, \Psi^0 \rangle_+ + \langle Y^1, \Psi^1 \rangle_+ \) for \( Y = (Y^0, Y^1) \in \mathcal{H}_{\alpha,+} \) and \( \Psi = (\Psi^0, \Psi^1) \in S \), where

\[
\langle Y^i, \Psi^i \rangle_+ = \sum_{z \in Z^d_+} Y^i(z) \cdot \Psi^i(z), \quad i = 0, 1.
\]

**Proposition 3.3** For every \( \Psi \in S \), the characteristic functionals converge to a Gaussian one,

\[
\hat{\mu}_t(\Psi) := \int e^{i \langle Y, \Psi \rangle + \mu_t(dY)} \to \exp \left\{ -\frac{1}{2} Q_\infty(\Psi, \Psi) \right\}, \quad t \to \infty, \tag{3.7}
\]

where \( Q_\infty \) is the quadratic form defined as

\[
Q_\infty(\Psi, \Psi) = \sum_{i,j=0}^1 \sum_{z,z' \in Z^d_+} \left( Q^{ij}_\infty(z, z'), \Psi^i(z) \otimes \Psi^j(z') \right).
\]

Proposition 3.2 ensures the existence of the limit measures of the family \( \{ \mu_t, t \in \mathbb{R} \} \), while Proposition 3.3 provides the uniqueness. They are proved in Sections 6 and 7, respectively. The assertion (iii) of Theorem A is proved in Section 5, item (iv) follows from (1.5) and item (v) can be proved using a method of [11].

**Remarks 3.4** (i) The uniform Rosenblatt mixing condition [15] also suffices, together with a higher power \( > 2 \) in the bound (2.12): there exists \( \delta > 0 \) such that

\[
E \left( |Y^0(z)|^{2+\delta} + |Y^1(z)|^{2+\delta} \right) \leq C < \infty, \quad z \in Z^d_+. \tag{3.8}
\]

Then (2.14) requires a modification:

\[
\int_0^{+\infty} r^{d-1} \alpha^p(r) dr < \infty, \quad \text{with} \quad p = \min(\delta/(2+\delta), 1/2).
\]

Here \( \alpha(r) \) is the Rosenblatt mixing coefficient defined as in (2.13) but without \( \mu_0(B) \) in the denominator:

\[
\alpha(r) = \sup \{ \alpha_Y(\mathcal{A}, \mathcal{B}) : \mathcal{A}, \mathcal{B} \subset Z^d_+, \text{dist}(\mathcal{A}, \mathcal{B}) \geq r \},
\]

where

\[
\alpha_Y(\mathcal{A}, \mathcal{B}) = \sup \{|\mu_0(\mathcal{A} \cap \mathcal{B}) - \mu_0(\mathcal{A})\mu_0(\mathcal{B})| : \mathcal{A} \in \sigma(\mathcal{A}), \mathcal{B} \in \sigma(\mathcal{B})\}.
\]

Under these modifications, the statements of Theorem A and their proofs remain essentially unchanged.

(ii) The uniform Rosenblatt mixing condition also could be weakened. Let \( K(z, s) = \prod_{i=1}^d [z_i - s, z_i + s] \), where \( s > 0, z \in Z^d \), stand for the cube in \( Z^d \), \( \bar{K}_r = Z^d \setminus K(z, s + r) \). Let us define the mixing coefficient \( \alpha_I(r) \) by the rule

\[
\alpha_I(r) = \sup \left\{ \alpha_Y \left( K(z, s) \cap Z^d_+, \bar{K}_r \cap Z^d_+ \right) : z \in Z^d_+, 0 \leq s \leq l \right\}.
\]

To prove Theorem A it suffices to assume, together with (3.8), that

\[
\alpha_I(r) \leq \frac{C l^k}{(1 + r)^{k'}}.
\]

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with some constants $C, \kappa, \kappa' > 0$. See [2, 3] for a more detailed discussion about the different mixing conditions.

(iii) The condition $E5$ could be considerably weakened. Namely, it suffices to assume the following condition: 

**E5'** If for some $\sigma \neq \sigma'$ one has $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm$ with $\text{const}_\pm \neq 0$, then

$$
\begin{aligned}
\{ p_{\sigma\sigma'}^{11}(\theta) \mp \omega_\sigma(\theta)\omega_{\sigma'}(\theta)p_{\sigma\sigma'}^{00}(\theta) \equiv 0, \\
\omega_\sigma(\theta)p_{\sigma\sigma'}^{01}(\theta) \pm \omega_{\sigma'}(\theta)p_{\sigma\sigma'}^{10}(\theta) \equiv 0. 
\end{aligned}
$$

(3.9)

Here $p_{\sigma\sigma'}^{ij}(\theta)$ stand for the matrices

$$
p_{\sigma\sigma'}^{ij}(\theta) := \Pi_\sigma(\theta)\hat{q}_{ij}^{\sigma}(\theta)\Pi_{\sigma'}(\theta), \quad \theta \in \mathbb{T}^d, \quad \sigma, \sigma' = 1, \ldots, s, \quad i, j = 0, 1,
$$

where $\hat{q}_{ij}^{\sigma}(\theta)$ are Fourier transforms of the correlation functions $q_{ij}^{\sigma}(z)$.

Note that the condition $E5'$ is fulfilled, for instance, if $q_{00}(z)$ is a covariance matrix of a Gibbs measure $\nu_0$ on $\mathcal{H}_\alpha$, $\alpha < -d/2$. Formally the Gibbs measure $\nu_0$ is

$$
\nu_0(dv_0, dv_1) = \frac{1}{Z_\beta} e^{-\frac{\beta}{2} \sum_{z \in \mathbb{Z}^d} (|v_1(z)|^2 + \mathcal{V}v_0(z) \cdot v_0(z))},
$$

where $Z_\beta$ is the normalization factor, $\beta = T^{-1}$, $T > 0$ is an absolute temperature. Let us introduce the Gibbs measure $\nu_0$ as the Gaussian measure with the correlation matrices defined by their Fourier transform as

$$
\hat{q}_{00}^{\sigma}(\theta) = T\hat{V}^{-1}(\theta), \quad \hat{q}_{11}^{\sigma}(\theta) = T(\delta_{kl})_{k,l=1}^n, \quad \hat{q}_{01}^{\sigma}(\theta) = \hat{q}_{10}^{\sigma}(\theta) = 0.
$$

Then $p_{\sigma\sigma'}^{ij} = 0$ for $\sigma \neq \sigma'$ and (3.9) holds.

### 4 Examples

Let us give the examples of the equations (1.1) and measures $\mu_0$ which satisfy all conditions $E1$–$E6$ and $S1$–$S4$, respectively.

#### 4.1 Harmonic crystals

The conditions $E1$–$E6$ in particular are fulfilled in the case of the nearest neighbor crystal, i.e., when the interaction matrix $V(z) = (V_{kl}(z))_{k,l=1}^n$ has a form:

$$
V_{kl}(z) =
\begin{cases}
0 & \text{for } k \neq l, \\
-\gamma_k & \text{for } |z| = 1, \\
2\gamma_k + m_k^2 & \text{for } z = 0, \\
0 & \text{for } |z| \geq 2,
\end{cases}
$$

(4.1)

with $\gamma_k > 0$, $m_k \geq 0$. Then the equation (1.1) becomes

$$
\ddot{u}_k(z,t) = (\gamma_k \Delta_L - m_k^2)u_k(z,t), \quad k = 1, \ldots, n.
$$
Here $\Delta_L$ stands for the discrete Laplace operator on the lattice $\mathbb{Z}^d$,

$$\Delta_L u(z) := \sum_{e, |e|=1} (u(z + e) - u(z)).$$

Then the eigenvalues of $\hat{V}(\theta)$ are

$$\omega_k(\theta) = \sqrt{2\gamma_1(1 - \cos \theta_1) + \ldots + 2\gamma_d(1 - \cos \theta_d) + m_k^2}, \quad k = 1, \ldots, n.$$ 

These eigenvalues have to be labelled as follows

$$\omega_1(\theta) \equiv \omega_1(\theta), \quad \omega_{r_1}(\theta) \equiv \omega_{r_1}(\theta), \quad \omega_{r_{s-1}+1}(\theta) \equiv \omega_{r_s}(\theta), \quad \omega_2(\theta) < \omega_3(\theta) < \ldots < \omega_s(\theta) \equiv \omega_{s}(\theta).$$

Clearly conditions E1–E5 hold with $\mathcal{C}_s = \emptyset$. In the case all $m_k > 0$ the set $\mathcal{C}_0$ is empty and condition E6 holds automatically. Otherwise, if $m_k = 0$ for some $k$, $\mathcal{C}_0 = \{0\}$. Then E6 is equivalent to the condition $\omega_{s}^{-2}(\theta) \in L^1(\mathbb{T}^d)$, which holds if $d \geq 3$. Therefore, the conditions E1–E6 hold provided either (i) $d \geq 3$, or (ii) $d = 1, 2$ and all $m_k > 0$.

In the case (4.1) formulas (3.4) and (3.5) can be rewritten as follows. Denote

$$M_{kl}^{ij} = \frac{1}{4} \sum_{\sigma=1}^{s} \chi_k(\sigma) M_{kl}^{ij}, \quad i, j = 0, 1, \quad k, l = 1, \ldots, n,$$

where

$$M_{kl}^{11} = \omega_s^2 M_{kl}^{00} + [\omega_s^2 q_0^{10} - q_0^{01}]_{kl},$$

$$M_{kl}^{01} = -M_{kl}^{10} = \left[q_0^{01} - q_0^{10} + \frac{\mathrm{sign}(\sin \theta_1)}{\omega_1(\theta)} \left(\omega_s^2 q_0^{00} + q_0^{11}\right)\right]_{kl}.$$

### 4.2 Gaussian measures

We consider $n = 1$ and construct Gaussian initial measures $\mu_0$ satisfying S1–S4. Let us define $\nu_0$ in $\mathcal{H}_\alpha$ by the correlation functions $\tilde{q}_0^{ij}(z - z')$ which are zero for $i \neq j$, while for $i = 0, 1$,

$$\tilde{q}_0^{ii}(\theta) := F_{z \to \theta}[\nu_0^{ii}(z)] \in L^1(\mathbb{T}^d), \quad \tilde{q}_0^{ii}(\theta) \geq 0. \quad \text{(4.2)}$$

Then by the Minlos theorem there exists a unique Borel Gaussian measure $\nu_0$ on $\mathcal{H}_\alpha$, $\alpha < -d/2$, with the correlation functions $q_0^{ij}(z - z')$, because

$$\int \|X\|^2_\alpha d\nu_0(dX) = \sum_{z \in \mathbb{Z}^d} (1 + |z|^2)^\alpha (\text{tr} q_0^{00}(0) + \text{tr} q_0^{11}(0)) = C(\alpha, d) \int_{\mathbb{T}^d} \text{tr}(\hat{q}_0^{00}(\theta) + \hat{q}_0^{11}(\theta)) d\theta < \infty.$$ 

The measure $\nu_0$ satisfies S1 and S3. Let us take a function $\zeta \in C(\mathbb{Z})$ such that

$$\zeta(s) = \begin{cases} 1, & \text{for } s > a, \\ 0, & \text{for } s \leq 0 \end{cases} \quad \text{with an } a > 0.$$
Let us introduce \( X(z) \) as a random function in probability space \( (\mathcal{H}_\alpha, \nu_0) \). Define a Borel probability measure \( \mu_0 \) on \( \mathcal{H}_{\alpha,+} \) as a distribution of the random function \( Y_0(z) = \zeta(z_1)X(z), \quad z \in \mathbb{Z}_+^d \). Then correlation functions of \( \mu_0 \) are

\[
Q_{ij}^0(z, z') = q_{ij}^0(z - z')\zeta(z_1)\zeta(z'_1), \quad i, j = 0, 1,
\]

where \( z = (z_1, \ldots, z_d), \ z' = (z'_1, \ldots, z'_d) \in \mathbb{Z}_+^d \), and \( q_{ij}^0 \) are the correlation functions of the measure \( \nu_0 \). The measure \( \mu_0 \) satisfies S1–S3. Further, let us provide, in addition to (4.2), that

\[
q_{ii}^0(z) = 0, \quad |z| \geq r_0. \quad (4.3)
\]

Then the mixing condition S4 follows with \( \varphi(r) = 0, \ r \geq r_0 \). For instance, (4.2) and (4.3) hold if we set \( q_{ii}^0(z) = f(z_1)f(z_2)\ldots f(z_d) \), where \( f(z) = N_0 - |z| \) for \( |z| \leq N_0 \) and \( f(z) = 0 \) for \( |z| > N_0 \) with \( N_0 := \lfloor r_0/\sqrt{d} \rfloor \) (the integer part). Then \( f(\theta) = (1 - \cos N_0\theta)/(1 - \cos \theta) \), \( \theta \in T^1 \), and (4.2) holds.

## 5 Convergence of correlation functions

### 5.1 Bounds for initial covariance

**Definition 5.1** By \( l^p \equiv l^p(\mathbb{Z}_+^d) \otimes \mathbb{R}^n \) (\( (l_+^p) \equiv l^p(\mathbb{Z}_+^d) \otimes \mathbb{R}^n \)), \( p \geq 1, \ n \geq 1 \), we denote the space of sequences \( f(z) = (f_1(z), \ldots, f_n(z)) \) endowed with norm \( \|f\|_p = \left( \sum_{z \in \mathbb{Z}_+^d} |f(z)|^p \right)^{1/p} \) (\( \|f\|_{l_+^p} := \left( \sum_{z \in \mathbb{Z}_+^d} |f(z)|^p \right)^{1/p} \), resp.).

The next Proposition reflects the mixing property of initial correlation functions.

**Proposition 5.2** (i) Let conditions S1–S4 hold. Then for \( i, j = 0, 1 \), the following bounds hold

\[
\sum_{z' \in \mathbb{Z}_+^d} |Q_{ij}^0(z, z')| \leq C < \infty \quad \text{for all } z \in \mathbb{Z}_+^d, \quad (5.1)
\]

\[
\sum_{z \in \mathbb{Z}_+^d} |Q_{ij}^0(z, z')| \leq C < \infty \quad \text{for all } z' \in \mathbb{Z}_+^d. \quad (5.2)
\]

Here the constant \( C \) does not depend on \( z, z' \in \mathbb{Z}_+^d \).

(ii) \( \hat{q}_{ij}^0 \in C(T^d) \), \( i, j = 0, 1 \).

**Proof.** (i) By [12, Lemma 17.2.3], conditions S1, S3 and S4 imply

\[
|Q_{ij}^0(z, z')| \leq C e_0 \varphi^{1/2}(|z - z'|), \quad z, z' \in \mathbb{Z}_+^d. \quad (5.3)
\]

Hence, condition (2.14) implies (5.1),

\[
\sum_{z \in \mathbb{Z}_+^d} |Q_{ij}^0(z, z')| \leq C e_0 \sum_{z \in \mathbb{Z}_+^d} \varphi^{1/2}(|z|) < \infty. \quad (5.4)
\]
(ii) The bound (5.3) and condition S2 imply the following bound:

$$|q_0^ij(z)| \leq C_{t0} \varphi^{1/2}(z), \quad z \in \mathbb{Z}^d. \quad (5.5)$$

Hence, from (2.14) it follows that $q_0^ij(z) \in l^1$, what implies $q_0^ij \in C(T^d)$.

**Corollary 5.3** Proposition 5.2 (i) implies, by the Shur lemma, that for any $\Phi, \Psi \in l^2_+^d$ the following bound holds:

$$|\langle Q_0(z, z'), \Phi(z) \otimes \Psi(z') \rangle_+| \leq C \|\Phi\|_2^1 \|\Psi\|_2^1.$$

### 5.2 Proof of the convergence (3.2)

From condition (2.1), formulas (2.8) and (2.9) it follows that $G_t(z) = G_t(\tilde{z})$ with $\tilde{z} = (-z_1, z_2, \ldots, z_d)$. Then, by the explicit representation (2.10), the covariance $Q_t(z, z')$ can be decomposed into a sum of four terms:

$$Q_t(z, z') = \sum_{y, y' \in \mathbb{Z}^d} G_{t, +}(z, y)Q_0(y, y')G^T_{t, +}(z', y') = R_t(z, z') - R_t(z, \tilde{z}') - R_t(\tilde{z}, z') + R_t(\tilde{z}, \tilde{z}'),$$

where

$$R_t(z, z') := \sum_{y, y' \in \mathbb{Z}^d} G_t(z - y)Q_0(y, y')G^T_t(z' - y').$$

Therefore, (3.2) follows from the following convergence

$$R_t(z, z') \to q_\infty(z - z') \quad \text{as} \quad t \to \infty, \quad z, z' \in \mathbb{Z}^d. \quad (5.6)$$

To prove (5.6) let us define

$$Q_\ast(z, z') = \left\{ \begin{array}{ll} Q_0(z, z') & \text{for } z, z' \in \mathbb{Z}^d_+, \\
0 & \text{otherwise}.
\end{array} \right. $$

First we split the function $Q_\ast(z, z')$ into the following three matrices

$$Q^+(z, z') := \frac{1}{2}q_0(z - z'), \quad (5.7)$$

$$Q^-(z, z') := \frac{1}{2}q_0(z - z') \, \text{sign}(z'_1), \quad (5.8)$$

$$Q^r(z, z') := Q_\ast(z, z') - Q^+(z, z') - Q^-(z, z'). \quad (5.9)$$

Next introduce the matrices

$$R^a_t(z, z') := \sum_{y, y' \in \mathbb{Z}^d} \left( G_t(z - y)Q^a(y, y')G^T_t(z' - y') \right), \quad z, z' \in \mathbb{Z}^d, \quad t > 0, \quad (5.10)$$

for each $a = \{+, -, r\}$, and split $R_t(z, z')$ into three terms: $R_t(z, z') = R^+_t(z, z') + R^-_t(z, z') + R^r_t(z, z')$. Then the convergence (5.6) follows from the following lemma.

**Lemma 5.4** (i) $\lim_{t \to \infty} R^+_t(z, z') = q^+_\infty(z - z'), \quad z, z' \in \mathbb{Z}^d$, with the matrix $q^+_\infty$ defined in (3.4),

(ii) $\lim_{t \to \infty} R^-_t(z, z') = q^-\infty(z - z'), \quad z, z' \in \mathbb{Z}^d$, with the matrix $q^-\infty$ defined in (3.5),

(iii) $\lim_{t \to \infty} R^r_t(z, z') = 0, \quad z, z' \in \mathbb{Z}^d$.

This lemma can be proved using the technique of [9, Proposition 7.1]. To justify the main idea of the proof we sketch the proof of Lemma 5.4 (i) in Appendix.
6 Compactness of measures family

Proposition 3.2 follows from the bound (6.1) by the Prokhorov compactness theorem [17, Lemma II.3.1] by a method used in [17, Theorem XII.5.2], since the embedding $H_{\alpha,+} \subset H_{\beta,+}$ is compact if $\alpha > \beta$.

**Lemma 6.1** Let conditions S1, S3, S4 hold and $\alpha < -d/2$. Then the following bounds hold

$$\sup_{t \geq 0} E \|U_+(t)Y_0\|_{\alpha,+}^2 < \infty. \tag{6.1}$$

**Proof.** Definition 2.1 implies

$$E \|Y(\cdot, t)\|_{\alpha,+}^2 = \sum_{z \in \mathbb{Z}_+^d} (1 + |z|^2)^\alpha \left( \text{tr} Q^{00}_t(z, z) + \text{tr} Q^{11}_t(z, z) \right) < \infty.$$

Since $\alpha < -d/2$, it remains to prove that

$$\sup_{t \in \mathbb{R}} \sup_{z, z' \in \mathbb{Z}_+^d} \|Q_t(z, z')\| \leq C < \infty.$$

The representation (2.10) gives

$$Q^{ij}_t(z, z') = E \left( Y^i(z, t) \otimes Y^j(z', t) \right) = \sum_{y, y' \in \mathbb{Z}_+^d} \sum_{k, l=0,1} G^{ij}_{t,+}(z, y) Q^{kl}_t(y, y') G^{ij}_{t,+}(z', y')$$

$$= \langle Q_0(y, y'), \Phi^i_{z}(y, t) \otimes \Phi^j_{z'}(y', t) \rangle_+,$$

where

$$\Phi^i_{z}(y, t) := \left( G^{00}_{t,+}(z, y), G^{11}_{t,+}(z, y) \right)$$

$$= \left( G^{00}_t(z - y) - G^{00}_t(z - \tilde{y}), G^{11}_t(z - y) - G^{11}_t(z - \tilde{y}) \right), \quad i = 0, 1.$$

Note that the Parseval identity, formula (8.2) and condition E6 imply

$$\|\Phi^i_{z}(\cdot, t)\|_{l_2}^2 = (2\pi)^{-d} \int_{T^d} |\hat{\Phi}^i_{z}(\theta, t)|^2 d\theta \leq C \int_{T^d} \left( |\hat{G}^{00}_t(\theta)|^2 + |\hat{G}^{11}_t(\theta)|^2 \right) d\theta \leq C_0 < \infty.$$

Then Corollary 5.3 gives

$$|Q^{ij}_t(z, z')| = |\langle Q_0(y, y'), \Phi^i_{z}(y, t) \otimes \Phi^j_{z'}(y', t) \rangle_+| \leq C \|\Phi^i_{z}(\cdot, t)\|_{l_2} \|\Phi^j_{z'}(\cdot, t)\|_{l_2} \leq C_1 < \infty,$$

where the constant $C_1$ does not depend on $z, z' \in \mathbb{Z}_+^d$, $t \in \mathbb{R}$.

7 Convergence of characteristic functionals

We derive (3.7) by using the explicit representation (2.10) of the solution $Y(t)$, the Bernstein ‘room - corridor’ technique and a method of [4]–[9]. The method gives a representation of $\langle Y(t), \Psi \rangle_+$ as a sum of weakly dependent random variables (see formula (7.6) below). Then (3.7) follows from the central limit theorem under a Lindeberg-type condition. The similar technique of the proof is applied in [9, Sections 9, 10]. Then we remark only the main steps of the proof.
7.1 Asymptotics of \( U'_+(t) \Psi \)

At first, let us evaluate of scalar product \( \langle Y(t), \Psi \rangle \). Let us introduce a function \( \Psi_*(z) \) as

\[
\Psi_*(z) = \begin{cases} 
\Psi(z), & \text{if } z_1 > 0, \\
0, & \text{if } z_1 = 0, \\
-\Psi(\tilde{z}), & \text{if } z_1 < 0.
\end{cases}
\]

Therefore

\[
\langle Y(z, t), \Psi(z) \rangle = \langle Y(z, t), \Psi_*(z) \rangle = \langle Y_0(z'), \Phi(z', t) \rangle,
\]

where

\[
\Phi(z', t) := U'_+(t) \Psi_*(z') = \sum_{z \in \mathbb{Z}^d} G^T_{t, z'}(z, z') \Psi(z) = \sum_{z \in \mathbb{Z}^d} G^T_t(z - z') \Psi_*(z)
\]

\[
= (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-iz' \cdot \theta} \hat{G}_t(\theta) \hat{\Psi}_*(\theta) d\theta.
\]

Definition 7.1

(i) The critical set \( C := C_0 \cup C_* \cup (\cup_{j} C_{\sigma}) \) (see E4).

(ii) \( S^0 := \{ \Psi \in S = [S(\mathbb{Z}^d) \otimes \mathbb{R}^n]^2 : \hat{\Psi}(0) = 0 \text{ in a neighborhood of } C \} \).

Note that \( \text{mes } C = 0 \) (see [9, lemma 7.3]) and it suffices to prove (3.7) for \( \Psi_* \in S^0 \) only.

For the function \( \Phi(z, t) \) the following lemma holds.

Lemma 7.2 (cf Lemma 9.1 from [9]) Let conditions E1–E4 and E6 hold. Then for any fixed \( \Psi_* \in S^0 \), the following bounds hold:

(i) \( \sup_{z \in \mathbb{Z}^d} |\Phi(z, t)| \leq C t^{-d/2} \).

(ii) For any \( p > 0 \) there exist \( C_p, \gamma > 0 \) such that

\[
|\Phi(z, t)| \leq C_p (1 + |z| + |t|)^{-p}, \quad |z| \geq \gamma t.
\]

(7.3)

This lemma follows from (7.2), (8.2), Definition 7.1 and the standard stationary phase method.

7.2 Bernstein’s argument

Let us introduce a ‘room - corridor’ partition of the half-ball \( \{ z \in \mathbb{Z}^d_+ : |z| \leq \gamma t \} \), with \( \gamma \) from (7.3). For \( t > 0 \) we choose \( \Delta_t \) and \( \rho_t \in \mathbb{N} \). Let us choose 0 < \( \delta < 1 \) and

\[
\rho_t \sim t^{1-\delta}, \quad \Delta_t \sim \frac{t}{\log t}, \quad t \to \infty.
\]

(7.4)

Let us set \( h_t = \Delta_t + \rho_t \) and

\[
a^j = jh_t, \quad b^j = a^j + \Delta_t, \quad j = 0, 1, 2, \ldots, N_t = [(\gamma t)/h_t].
\]

We call the slabs \( R^j_t = \{ z \in \mathbb{Z}^d_+ : |z| \leq N_t h_t, a^j \leq z_1 < b^j \} \) the ‘rooms’, \( C^j_t = \{ z \in \mathbb{Z}^d_+ : |z| \leq N_t h_t, b^j \leq z_1 < a^{j+1} \} \) the ‘corridors’ and \( L_t = \{ z \in \mathbb{Z}^d_+ : |z| > N_t h_t \} \) the ‘tail’. Here
Let \( z = (z_1, \ldots, z_d) \), \( \Delta_t \) is the width of a room, and \( \rho_t \) of a corridor. Let us denote by \( \chi^j_t \) the indicator of the room \( R^j_t \), \( \xi^j_t \) that of the corridor \( C^j_t \), and \( \eta_t \) that of the tail \( L_t \). Then

\[
\sum_t [\chi^j_t(z) + \xi^j_t(z)] + \eta_t(z) = 1, \quad z \in \mathbb{Z}^d,
\]

where the sum \( \sum_t \) stands for \( \sum_{N_t-1}^{N_t} \). Hence, we get the following Bernstein’s type representation:

\[
\langle Y_0, \Phi(\cdot, t) \rangle_+ = \sum_t \left[ \langle Y_0, \chi^j_t \Phi(\cdot, t) \rangle_+ + \langle Y_0, \xi^j_t \Phi(\cdot, t) \rangle_+ \right] + \langle Y_0, \eta_t \Phi(\cdot, t) \rangle_+. \tag{7.5}
\]

Let us define the random variables \( r^j_t, c^j_t, l_t \) by

\[
r^j_t = \langle Y_0, \chi^j_t \Phi(\cdot, t) \rangle_+, \quad c^j_t = \langle Y_0, \xi^j_t \Phi(\cdot, t) \rangle_+, \quad l_t = \langle Y_0, \eta_t \Phi(\cdot, t) \rangle_+.
\]

Therefore, from (7.1) and (7.5) it follows that

\[
\langle Y(t), \Psi \rangle_+ = \langle Y_0, \Phi(\cdot, t) \rangle_+ = \sum_t (r^j_t + c^j_t) + l_t. \tag{7.6}
\]

**Lemma 7.3** Let \( S_1 - S_4 \) hold and \( \Psi_* \in S^0 \). The following bounds hold for \( t > 1 \):

\[
E|r^j_t|^2 \leq C(\Psi) \Delta_t/t, \quad \forall j,
E|c^j_t|^2 \leq C(\Psi) \rho_t/t, \quad \forall j,
E|l_t|^2 \leq C_p(\Psi) t^{-p}, \quad \forall p > 0.
\]

The proof is based on Lemma 7.2 and Proposition 5.2 (i) (see [9, Lemma 9.2]).

Further, to prove (3.7) we use a version of the central limit theorem developed by Ibragimov and Linnik. If \( Q_{\infty}(\Psi, \Psi) = 0 \), the convergence (3.7) follows from (3.2). Thus, we may assume that for a given \( \Psi_* \in S^0 \),

\[
Q_{\infty}(\Psi, \Psi) \neq 0. \tag{7.7}
\]

At first, we obtain

\[
|E \exp\{i\langle Y_0, \Phi(\cdot, t) \rangle_+ \} - \hat{\mu}_\infty(\Psi)| = \left| E \exp\left\{ i\sum_i r^i_t \right\} - \exp\left\{ -\frac{1}{2} \sum_t E|r^i_t|^2 \right\} \right| + o(1), \quad t \to \infty.
\]

This fact follows from Lemma 7.3, convergence (3.2), condition \( S_4 \) and (7.4) (cf [9, p.1073-1075]).

Secondly, by the mixing condition \( S_4 \), we derive that

\[
\left| E \exp\left\{ i\sum_i r^i_t \right\} - \prod_{0}^{N_t-1} E \exp\left\{ i r^i_t \right\} \right| \leq C N_t \varphi(\rho_t) \to 0, \quad t \to \infty.
\]

Hence, it remains to check that

\[
\left| \prod_{0}^{N_t-1} E \exp\left\{ i r^j_t \right\} - \exp\left\{ -\frac{1}{2} \sum_t E|r^j_t|^2 \right\} \right| \to 0, \quad t \to \infty.
\]
According to the standard statement of the central limit theorem (see, e.g. [14, Theorem 4.7]), it suffices to verify the Lindeberg condition: \( \forall \delta > 0, \)

\[
\frac{1}{\sigma_t} \sum_t E_{\delta} |r_t|^2 \to 0, \quad t \to \infty.
\]

Here \( \sigma_t \equiv \sum_t E |r_t|^2 \), and \( E_{\delta} f \equiv E(X_{\delta} f) \), where \( X_{\delta} \) is the indicator of the event \( |f| > \varepsilon^2 \). Note that (3.2) and (7.7) imply that \( \sigma_t \to Q_\infty(\Psi, \Psi) \neq 0, \quad t \to \infty \). Hence it remains to verify that

\[
\sum_t E_{\varepsilon} |r_t|^2 \to 0, \quad t \to \infty, \quad \text{for any} \quad \varepsilon > 0.
\]

This condition is checked using the technique from [9, section 10].

8 Appendix. Outline of the proof of Lemma 5.4 (i)

Obviously, the assertion of Lemma 5.4 (i) is equivalent to the next proposition.

**Proposition 8.1** Let conditions E1–E6 and S1–S4 hold. Then for any \( \Psi \in S \),

\[
\lim_{t \to \infty} \langle R^+_t(z, z'), \Psi(z) \otimes \Psi(z') \rangle = \langle q^+_\infty(z - z'), \Psi(z) \otimes \Psi(z') \rangle. \tag{8.1}
\]

**Proof.** It suffices to prove (8.1) for \( \Psi \in S^0 \) only. It can be proved similarly as in [9, Lemma 7.6].

At first, let us apply the Fourier transform to the matrix \( R^+_t(z, z') \) defined by (5.10): \( \hat{R}^+_t(\theta, \theta') := F_{z - \theta} R^+_t(z, z') = \hat{G}_t(\theta) \hat{Q}^+(\theta, \theta') \hat{G}^*_t(\theta'), \) where \( \hat{Q}^+(\theta, \theta') := F_{z' - \theta'} Q^+(z, z') \). From (5.7) it follows that \( \hat{Q}^+(\theta, \theta') = \delta(\theta + \theta') (2\pi)^d \hat{q}_0(\theta)/2 \). Hence,

\[
\hat{R}^+_t(\theta, \theta') = (2\pi)^d \frac{1}{2} \delta(\theta + \theta') \hat{G}_t(\theta) \hat{q}_0(\theta) \hat{G}^*_t(-\theta).
\]

Secondly, \( \hat{G}_t(\theta) \) has a form

\[
\hat{G}_t(\theta) = \begin{pmatrix} \cos \Omega t & \sin \Omega t \Omega^{-1} \\ -\sin \Omega t & \cos \Omega t \end{pmatrix}, \tag{8.2}
\]

where \( \Omega = \Omega(\theta) \) is the Hermitian matrix defined by (2.4). Let \( C(\theta) \) be defined by (3.6) and \( I \) be the identity matrix. Then

\[
\hat{G}_t(\theta) = \cos \Omega t I + \sin \Omega t C(\theta). \tag{8.3}
\]

Moreover, by condition E2, \( \hat{G}^*_t(-\theta) = \hat{G}^*_t(\theta) = \cos \Omega t I + \sin \Omega t C(\theta)^* \). Therefore,

\[
\langle R^+_t(z, z'), \Psi(z) \otimes \Psi(z') \rangle = (2\pi)^{-2d} \langle \hat{R}^+_t(\theta, \theta'), \hat{\Psi}(\theta) \otimes \hat{\Psi}(\theta') \rangle
\]

\[
= \frac{1}{2(2\pi)^d} \langle \hat{G}_t(\theta) \hat{q}_0(\theta) \hat{G}^*_t(\theta), \hat{\Psi}(\theta) \otimes \overline{\hat{\Psi}(\theta)} \rangle. \tag{8.4}
\]
Further, let us choose certain smooth branches of the functions \( \Pi_\sigma(\theta) \) and \( \omega_\sigma(\theta) \) to apply the stationary phase arguments which require a smoothness in \( \theta \). We choose a finite partition of unity
\[
\sum_{m=1}^{M} g_m(\theta) = 1, \quad \theta \in \text{supp} \Psi,
\]
where \( g_m \) are nonnegative functions from \( C^\infty_0(\mathbb{T}^d) \) and vanish in a neighborhood of the set \( \mathcal{C} \) defined in Definition 7.1, (i). Further, using (8.5) we rewrite the RHS of (8.4). Applying formula (8.3) for \( \hat{G}_t(\theta) \), one obtains
\[
\langle R_t^+(z, z'), \Psi(z) \otimes \Psi(z') \rangle = \frac{1}{2(2\pi)^d} \sum_{m=1}^{s} \int_{\mathbb{T}^d} g_m(\theta) \left( \Pi_\sigma(\theta) R_{t,\sigma^{\prime}}(\theta) \Pi_{\sigma^{\prime}}(\theta), \hat{\Psi}(\theta) \otimes \overline{\Psi}(\theta) \right) d\theta,
\]
where \( R_{t,\sigma^{\prime}}(\theta) \) stands for the \( 2n \times 2n \) matrix,
\[
R_{t,\sigma^{\prime}}(\theta) = \frac{1}{2} \sum_{\epsilon = \pm} \left\{ \cos(\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)) t \left[ \hat{q}_0(\theta) \mp C(\theta) \hat{q}_0(\theta) C(\theta)^* \right] \right. + \sin(\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)) t \left[ C(\theta) \hat{q}_0(\theta) \mp \hat{q}_0(\theta) C(\theta)^* \right] \}
\]
(8.6)

If \( \sigma = \sigma' \), then
\[
R_{t,\sigma}(\theta) = \frac{1}{2} \left[ \hat{q}_0(\theta) + C(\theta) \hat{q}_0(\theta) C(\theta)^* \right] + \frac{1}{2} \cos(2\omega_\sigma(\theta) t) \left[ \hat{q}_0(\theta) - C(\theta) \hat{q}_0(\theta) C(\theta)^* \right] + \frac{1}{2} \sin(2\omega_\sigma(\theta) t) \left[ C(\theta) \hat{q}_0(\theta) + \hat{q}_0(\theta) C(\theta)^* \right].
\]
(8.7)

By Lemma 2.2 and the compactness arguments, we choose the eigenvalues \( \omega_\sigma(\theta) \) and the matrix \( \Pi_\sigma(\theta) \) as real-analytic functions inside the \( \text{supp} g_m \) for every \( m \): we do not mark the functions by the index \( m \) to not overburden the notations. Let us analyze the Fourier integrals with \( g_m \).

At first, note that the identities \( \omega_\sigma(\theta) + \omega_{\sigma'}(\theta) \equiv \text{const}_+ \) or \( \omega_\sigma(\theta) - \omega_{\sigma'}(\theta) \equiv \text{const}_- \) with the \( \text{const}_\pm \neq 0 \) are impossible by condition \( \text{E5} \). Furthermore, the oscillatory integrals with \( \omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \neq \text{const} \) vanish as \( t \to \infty \). Hence, only the integrals with \( \omega_\sigma(\theta) - \omega_{\sigma'}(\theta) \equiv 0 \) contribute to the limit, since \( \omega_\sigma(\theta) + \omega_{\sigma'}(\theta) \equiv 0 \) would imply \( \omega_\sigma(\theta) \equiv \omega_{\sigma'}(\theta) \equiv 0 \) which is impossible by \( \text{E4} \). By formulas (8.6) and (8.7), one obtains
\[
\langle R_t^+(z, z'), \Psi(z) \otimes \Psi(z') \rangle = (2\pi)^{-d} \sum_{m=1}^{s} \frac{1}{4} \int_{\mathbb{T}^d} g_m(\theta) \left( \Pi_\sigma(\theta) \left[ \hat{q}_0(\theta) + C(\theta) \hat{q}_0(\theta) C(\theta)^* \right] \Pi_\sigma(\theta) + \ldots, \hat{\Psi}(\theta) \otimes \overline{\Psi}(\theta) \right) d\theta
\]

where ”...” stands for the oscillatory integrals which contain \( \cos(\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)) t \) and \( \sin(\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)) t \) with \( \omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \neq \text{const} \). The oscillatory integrals converge to zero by the Lebesgue-Riemann theorem since all the integrands in ”...” are summable, and \( \nabla (\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)) \neq 0 \) for all \( \theta \) in the support of \( g_m \).
\( \omega_\sigma'(\theta) = 0 \) only on the set of the Lebesgue measure zero. The summability follows from Proposition 5.2 (ii) and E6 (if \( C_0 \neq \emptyset \)) since the matrices \( \Pi_\sigma(\theta) \) are bounded. The zero measure follows similarly to Lemma 2.2 (i) since \( \omega_\sigma(\theta) \pm \omega_\sigma'(\theta) \neq \text{const} \). Lemma 5.4 (i) is proved.

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