Election in Fully Anonymous Shared Memory Systems:
Tight Space Bounds and Algorithms

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Abstract. This article addresses election in fully anonymous systems made up of
n asynchronous processes that communicate through atomic read-write registers
or atomic read-modify-write registers. Given an integer \( d \in \{1, \ldots, n - 1\} \),
two elections problems are considered: \( d \)-election (at least one and at most \( d \)
processes are elected) and exact \( d \)-election (exactly \( d \) processes are elected). Full
anonymity means that both the processes and the shared registers are anonymous.
Memory anonymity means that the processes may disagree on the names of the
shared registers. That is, the same register name \( A \) can denote different registers
for different processes, and the register name \( A \) used by a process and the register
name \( B \) used by another process can address the same shared register. Let \( n \) be
the number of processes, \( m \) the number of atomic read-modify-write registers,
and let \( M(n, d) = \{k : \forall \ell : 1 < \ell \leq n : \gcd(\ell, k) \leq d\} \). The following
results are presented for solving election in such an adversarial full anonymity
context.

– It is possible to solve \( d \)-election when participation is not required if and
  only if \( m \in M(n, d) \).
– It is possible to solve exact \( d \)-election when participation is required if and
  only if \( \gcd(m, n) \) divides \( d \).
– It is possible to solve \( d \)-election when participation is required if and only if
  \( \gcd(m, n) \leq d \).
– Neither \( d \)-election nor exact \( d \)-election (be participation required or not)
can be solved when the processes communicate through read-write registers
only.

Keywords: Anonymous processes, Anonymous memory, Distributed computabil-
ity, Leader election, Process participation, Read-write register, Read-modify-write
register, Symmetry-breaking, Tight bounds.

1 Introduction

1.1 Leader election

Leader election is a classic basic problem encountered when processes cooperate and
coordinate to solve higher-level distributed computing problems. It consists in design-
ing an algorithm selecting one and only one process from the set of cooperating pro-
cesses. In classical systems where the processes have distinct identities, leader election
algorithms usually amount to electing the process with the smallest (or highest) identity.
Many textbooks describe such algorithms (e.g., [5,14,21,26]).
This article considers two natural generalizations of the election problem in the presence of both process and memory anonymity, where communication is through shared registers. The first one is \emph{$d$-election} in which at least one and at most $d$ processes are elected. The second one is \emph{exact $d$-election} in which exactly $d$ (different) processes are elected.

1.2 System models

\textit{Process anonymity} Process anonymity means that the processes have no identity, have the same code, and have the same initialization of their local variables. Hence, in a process anonymous system, it is impossible to distinguish a process from another process.

Pioneering work on process anonymity in message-passing systems was presented in \cite{3}. Process anonymity has been studied for a long time in asynchronous shared memory systems (e.g., \cite{4}). It has been more recently addressed in the context of crash-prone asynchronous shared memory systems (e.g., \cite{6,11}).

Assuming a system made up of $n$ anonymous asynchronous processes, we use the notation $p_1, \ldots, p_n$ to distinguish the processes. The subscript $i \in \{1, \ldots, n\}$ will also be used to identify the local variables of $p_i$ (identified with names written with lower case letters).

\textit{Shared registers} The processes communicate through a shared memory made up of $m$ atomic registers \cite{15} (identified with names written with upper case letters). Hence, the shared memory appears to the processes as an array of registers denoted $R[1..m]$. \textit{Atomic} means that the operations on a register appear as if they have been executed sequentially, each appearing between its start event and its end event \cite{13}. Moreover, for any $x$, a read of $R[x]$ returns the last value previously written in $R[x]$, where \textit{last} refers to the previous total order on the operations on $R[x]$. (In case $R[x]$ has not been written, the read returns its initial value.) Two communication models are considered in the article.

\begin{itemize}
  \item \textbf{Read-write (RW) model.} This is the basic model in which a register $R[x]$ can be accessed only by a read or a write operation (as the cells of a Turing machine).
  \item \textbf{Read-modify-write (RMW) model.} This model is the RW model enriched with a conditional write operation. This conditional write operation atomically reads the register and (according to the value read) possibly modifies it. This conditional write, denoted \texttt{compare\&\,swap}(R[x], \texttt{old}, \texttt{new}), has three parameters, a shared register, and two values. It returns a Boolean value. If $R[x] = \texttt{old}$, it assigns the value \texttt{new} to $R[x]$ and returns \texttt{true}. If $R[x] \neq \texttt{old}$, $R[x]$ is not modified and the operation returns \texttt{false}. An invocation of \texttt{compare\&\,swap}(R[x], \texttt{old}, \texttt{new}) that returns \texttt{true} is \textit{successful}.
\end{itemize}

\textit{Memory anonymity} The notion of an anonymous memory has been introduced in \cite{27}. In a non-anonymous memory, the address $R[x]$ denotes the same register whatever the process that invokes $R[x]$; there is an a priori agreement on the name of each register. In an anonymous memory, there is no such agreement on the names of the shared registers. While the name $R[x]$ used by a given process $p_i$ always denotes the very same register,
the same name $R[x]$ used by different processes $p_i$, $p_j$, $p_k$ ..., may refer to different registers. More precisely, an anonymous memory system is such that:

- For each process $p_i$, an adversary defined a permutation $f_i(\cdot)$ over the set $\{1, 2, ..., n\}$ such that, when $p_i$ uses the name $R[x]$, it actually accesses $R[f_i(x)]$.
- No process knows the permutations,
- All the registers are initialized to the same default value.

In an anonymous memory system, ALL the registers are anonymous. Moreover, the size of the anonymous memory is not under the control of the programmer, and it is imposed on her/him. As shown in \[2\] (for non-anonymous processes and anonymous memory), and in this article (for fully anonymous systems) the size of the anonymous memory is a crucial parameter when one has to characterize the pairs $\langle n, m \rangle$ for which election can be solved in fully anonymous $n$-process systems.

Process participation As in previous works on election in anonymous or non-anonymous memory systems \[10\], this article considers two types of assumptions on the behavior of the processes: (1) algorithms that require the participation of all the processes to compete to be leaders, and (2) algorithms that do not (i.e., an arbitrary subset of processes may participate but not necessarily all the processes).

Symmetric algorithm Considering a system in which the processes have distinct identifiers, a symmetric algorithm is an algorithm where the processes can only compare their identities with equality \[25\]. So there is no notion of smaller/greater on process identities, and those cannot be used to index entries of arrays, etc. This notion of symmetry associated with process identities is the “last step” before their anonymity. In this article, we will consider algorithms in which the processes (and memories) are anonymous but will also mention symmetric algorithms.

1.3 Related works on anonymous memories Since its introduction, several problems have been addressed in the context of memory anonymity: mutual exclusion, election, consensus, set-agreement and renaming. We discuss below work on the first two problems that are more related to our work.

Mutual exclusion First, we observe that no shared memory-based mutual exclusion algorithm requires the participation of the processes. Let $M(n) = \{k : \forall \ell : 1 < \ell \leq n : \gcd(\ell, k) = 1\}$ (all the integers $2, ..., n$ are relatively prime with $k$). The following results have been recently established.

- There is a deadlock-free symmetric mutual exclusion algorithm in the RMW (resp. RW) model made up of $m$ anonymous registers if and only if $m \in M(n)$ (resp. $m \in M(n) \setminus \{1\}$) \[2\].
- There is a deadlock-free mutual exclusion algorithm in the process anonymous and memory anonymous RMW model made up of $m$ registers if and only if $m \in M(n)$. Moreover, there is no such algorithm in the fully anonymous RW communication model \[23\].
The conditions relating $m$ and $n$ can be seen as the seed needed to break symmetry despite anonymous memory, and symmetric or anonymous processes, thereby allowing mutual exclusion and election to be solved. A single leader election can be considered one-shot mutual exclusion, where the first process to enter its critical section is elected.

**Election in the symmetric model** Considering the symmetric process model in which all the processes (unlike in the mutual exclusion problem) are required to participate. The following results are presented in [10] for such a model.

- There is a $d$-election symmetric algorithm in the memory anonymous RW and RMW communication models made up of $m$ registers if and only if $\gcd(m, n) \leq d$.
- There is an exact $d$-election symmetric algorithm in the memory anonymous RW and RMW communication models made up of $m$ registers if and only if $\gcd(m, n)$ divides $d$.

We emphasize that the above results for $d$-election assume that the processes are symmetric and not that they are anonymous, as done in this article. Finally, fully anonymous agreement problems are investigated in [24].

**Remark** While addressing a different problem in a different context, it is worth mentioning the work presented in [8] that addresses the exploration of an $m$-size anonymous not-oriented ring by a team of $n$ identical, oblivious, asynchronous mobile robots that can view the environment but cannot communicate. Among other results, the authors have shown that there are initial placements of the robots for which gathering is impossible when $n$ and $m$ are not co-prime, i.e., when $\gcd(n, m) \neq 1$. They also show that the problem is always solvable for $\gcd(n, m) = 1$ when $n \geq 17$.

### 1.4 Motivation and content

**Motivation** The main motivation of this work is theoretical. It investigates a fundamental symmetry-breaking problem (election) in the worst adversarial context, namely asynchronous and fully anonymous systems. Knowing what is possible/impossible, stating computability and complexity lower/upper bounds are at the core of algorithmics [1,12], and trying to find solutions “as simple as possible” is a key if one wants to be able to master the complexity of future applications [11]. This article aims to increase our knowledge of what can/cannot be done in the full anonymity context, providing associated necessary and sufficient conditions which enrich our knowledge on the system assumptions under which fundamental problems such as election can be solved.

When one has to solve a symmetry-breaking problem, the main issue consists in finding the “as weak as possible” initial seed from which the initial system symmetry

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1 On an application-oriented side, it has been shown in [16,20] that the process of genome-wide epigenetic modifications (which allows cells to utilize the DNA) can be modeled as a fully anonymous shared memory system where, in addition to the shared memory, also the processes (that is proteins modifiers) are anonymous. Hence fully anonymous systems can be useful in the context of biologically-inspired distributed systems [17,20].
can be broken. So, considering FULL anonymity, this article complements previously known results on anonymous systems (which were on non-anonymous processes and anonymous memory\footnote{10}). In all cases, the seed that allows breaking the very strong (adversary) symmetry context defined by FULL anonymity is captured by necessary and sufficient conditions relating the number of anonymous processes and the size of the anonymous memory.

Content of the article Let $m,n$ and $d$ be the number of registers, the number of processes and the number of leaders, respectively, and (as previously defined) $M(n,d) = \{k : \forall \ell : 1 < \ell \leq n : \gcd(\ell,k) \leq d\}$. Table 1 summarizes the four main results.

| Problem                  | Register type | Participation | Necessary & sufficient condition on $(m,n,d)$ | Necessary | Sufficient | Section |
|--------------------------|---------------|---------------|-----------------------------------------------|-----------|-----------|---------|
| $d$-election             | RMW           | not required  | $m \in M(n,d)$                                | Thm 1     | Thm 3     | 2       |
| Exact $d$-election       | RMW           | required      | $\gcd(m,n)$ divides $d$                       | Follows from [10] | Thm 5 | 3       |
| $d$-election             | RMW           | required      | $\gcd(m,n) \leq d$                           | Follows from [10] | Thm 7 | 4       |
| $d$-election and Exact $d$-election | RW | required or not required | Impossible | Corollary 2 | | 5       |

Table 1. Election in the fully anonymous shared memory systems.

1. A $d$-election algorithm in the RMW communication model, which does not require participation of all the processes. It is also shown that the condition $m \in M(n,d)$ is necessary and sufficient for such an algorithm. Notice that $M(n,1)$ is the set $M(n)$ that appears in the results for fully anonymous mutual exclusion discussed earlier.

2. An exact $d$-election algorithm for the RMW communication model in which all the processes are required to participate. It is also shown that the necessary and sufficient condition for such an algorithm is $\gcd(m,n)$ divides $d$.

3. A $d$-election algorithm (which is based on the previous result) for the RMW communication model in which all the processes are required to participate. It is also shown that $\gcd(m,n) \leq d$ is a necessary and sufficient condition for such an algorithm. (The short algorithm appears in the proof of Theorem 7.)

4. An impossibility result that regardless whether participation is required or not, there is neither $d$-election nor exact $d$-election algorithm in the anonymous RW communication model.

\footnote{Both the algorithms described in the paper are simple. Their early versions were far from being simple, and simplicity is a first class property. As said by Y. Perlis (the recipient of first Turing Award) “Simplicity does not precede complexity, but follows it”\footnote{13}.}
Let us notice that, due to the very nature of the anonymous process model, no process can know the “identity” of elected processes. So, at the end of an election algorithm in the anonymous process model, a process only knows if it is or not a leader. We point out that the leader election problem has several variants, and the most general one, where a process only knows if it is or not a leader is a very common variant [5][22][26].

2 d-Election in the RMW Model
Where Participation is Not Required

Throughout this section, it is assumed that communication is through RMW anonymous registers and that the processes are not required to participate.

2.1 A necessary condition for d-election

In this subsection, it is further assumed that processes have identities that can only be compared (symmetry constraint). As they are weaker models, it follows that the necessary condition proved below still holds in RMW model where both the processes and the memory are anonymous, and in the model where communication is through anonymous RW registers.

**Theorem 1.** There is no symmetric d-election algorithm in the RMW communication model for \( n \geq 2 \) processes using \( m \) anonymous registers if \( m \not\in M(n, d) \).

**Proof** Let \( k \) be an arbitrary positive number such that \( 1 \leq k \leq n \). Below we examine what must be the relation between \( k, m \) and \( d \), when assuming the existence of a symmetric \( d \)-election algorithm for \( n \) processes using \( m \geq 1 \) anonymous RMW registers. To simplify the modulo notation, the processes are denoted \( p_0, ..., p_{n-1} \).

Let \( \gcd(m, k) = \delta \), for some positive number \( \delta \). We will construct a run in which exactly \( k \) processes participate. Let us partition these \( k \) processes into \( \delta \geq 1 \) disjoint sets, denoted \( P_0, ..., P_{\delta - 1} \), such that there are exactly \( k/\delta \) processes in each set. This partitioning is achieved by assigning process \( p_i \) (where \( i \in \{0, ..., k-1\} \)) to the set \( P_{i \mod \delta} \). For example, when \( k = 6 \) and \( \delta = 3 \), \( P_0 = \{p_0, p_3\}, P_1 = \{p_1, p_4\}, \) and \( P_2 = \{p_2, p_5\} \) (top of Figure 1). Such a division is possible since, by definition, \( \gcd(m, k) = \delta \).

![Fig. 1. Illustration of the runs for \( k = 6 \) and \( \delta = 3 \)](image-url)
Let us arrange the $m$ registers on a ring with $m$ nodes where each register is placed on a different node. To each one of the $\delta$ sets of processes $P_i$ (where $i \in \{0, ..., \delta - 1\}$), let us assign an initial register (namely, the first register that each process in that set accesses) such that for every two sets $P_i$ and its ring successor $P_{i+1}$ mod $\delta$ the distance between their assigned initial registers is exactly $\delta$ when walking on the ring in a clockwise direction. This is possible since $\gcd(m, k) = \delta$.

The lack of global names for the RMW anonymous registers allows us to assign, for each one of the $k$ processes, an initial register and an ordering which determines how the process scans the registers. An execution in which the $k$ processes are running in lock-steps, is an execution where we let each process take one step (in the order $p_0, ..., p_{k-1}$), and then let each process take another step, and so on. For a given $d$-election algorithm $A$, let us call this execution, in which the processes run in lock-steps, $\rho_A$. For simplicity, we will omit the subscript $A$ and simply write $\rho$.

For process $p_i$ and integer $j$, let $\text{order}(p_i, j)$ denotes the $j$th new (i.e., not yet assigned) register that $p_i$ accesses during the execution $\rho$, and assume that we arrange that $\text{order}(p_i, j)$ is the register whose distance from $p_i$’s initial register is exactly $(j - 1)$, when walking on the ring in a clockwise direction.

We notice that $\text{order}(p_i, 1)$ is $p_i$’s initial register, $\text{order}(p_i, 2)$ is the next new register that $p_i$ accesses and so on. That is, $p_i$ does not access $\text{order}(p_i, j + 1)$ before accessing $\text{order}(p_i, j)$ at least once, but for every $j' \leq j$, $p_i$ may access $\text{order}(p_i, j')$ several times before accessing $\text{order}(p_i, j + 1)$ for the first time. Since the memory is anonymous, when a process accesses a register for the first time, say register $\text{REG}[x]$, we may map $x$ to any (physical) register that it hasn’t accessed yet. However, when it accesses $\text{REG}[x]$ again, it must access the same register it has accessed before when referring to $x$.

Let us now consider another division of the $k$ processes into sets. We divide the $k$ processes into $k/\delta$ disjoint sets, denoted $Q_0, ..., Q_{k/\delta - 1}$, such that there are exactly $\delta$ processes in each set. This partitioning is achieved by assigning process $p_i$ (where $i \in \{0, ..., k-1\}$) to the set $Q_{i/\delta}$. For example, when $k = 6$ and $\delta = 3$, $Q_0 = \{p_0, p_1, p_2\}$, and $Q_1 = \{p_3, p_4, p_5\}$. Again, such a partitioning is possible since $\gcd(m, k) = \delta$ (bottom of Figure[1]).

We notice that $Q_0$ includes the first process to take a step in the execution $\rho$, in each one of the $\delta$ sets, $P_0, ..., P_{\delta-1}$. Similarly, $Q_1$ includes the second process to take a step in the execution $\rho$, in each one of the $\delta$ sets, $P_0, ..., P_{\delta-1}$, and so on.

Since only comparisons for equality are allowed, and all registers are initialized to the same value—which (to preserve anonymity) is not a process identity—in the execution $\rho$, for each $i \in \{0, ..., n/\delta - 1\}$, all the processes in the set $Q_i$ that take the same number of steps must be at the same state. (This is because all the processes in $Q_i$ are located at the same distance around the ring. At each lockstep, they invoke the Read/Modify/Write operation into different locations, so because of the symmetry assumption, it is not possible to break the symmetry between them, (either all or none are elected.) Thus, in the run $\rho$, it is not possible to break symmetry within a set $Q_i$ ($i \in \{0, ..., k/\delta - 1\}$), which implies that either all the $\delta$ processes in the set $Q_i$ will be elected, or no process in $Q_i$ will be elected.
Thus, the number of elected leaders in \( \rho \) equals \( \delta \) times the number of \( Q_i \) sets \((i \in \{0, \ldots, k/\delta - 1\})\) that all their members were elected, and (by definition of \( d \)-election) it must be a positive number. That is, the number of elected leaders in \( \rho \) equals \( a\delta \) for some integer \( a \in \{1, \ldots, k/\delta\} \).

Since in a \( d \)-election algorithm at most \( d \) leaders are elected in run \( \rho \), it follows from the fact that for some positive integer \( a \), it must be the case that \( a\delta \leq d \). Thus, it must be the case that \( \text{gcd}(m, k) = \delta \leq d \). Since \( k \) was chosen arbitrarily from \( \{1, \ldots, n\} \), it follows that a necessary requirement for a symmetric \( d \)-election algorithm for \( n \geq 2 \) processes using \( m \) anonymous RMW registers is that, for every \( 1 \leq k \leq n \), \( \text{gcd}(m, k) \leq d \).

\[ \Box \text{Theorem} \]  

2.2 A \( d \)-election algorithm in RMW fully anonymous systems

Anonymous memory The anonymous memory is made up of \( m \) RMW registers \( R[1..m] \), each initialized to the default value 0. It is assumed that \( m \in M(n,d) \) (recall that \( M(n,d) = \{k : \forall \ell : 1 < \ell \leq n : \text{gcd}(\ell, k) \leq d\} \)).

Local variables at each process \( p_i \) Each process \( p_i \) manages the following set of local variables.

- \( \text{counter}_i \): used to store the number of RMW registers owned by \( p_i \). A process owns a register when it is the last process that wrote a positive value into this register.
- \( \text{myview}_i[1..n] \): array of Boolean values, each initialized to \( \text{false} \). When \( \text{myview}_i[j] \) is equal to \( \text{true} \), \( p_i \) owns the register \( R_i[j] \).
- \( \text{round}_i \) (initialized to 0): round number (rung number in the ladder metaphor, see below) currently attained by \( p_i \) in its competition to be a leader. When \( \text{round}_i = n - d + 1 \), \( p_i \) becomes a leader.
- \( \text{competitors}_i \): maximal number of processes that compete with \( p_i \) when it executes a round.

Participation and output Any number of processes can invoke the election algorithm. A process exits the algorithm when it invokes \( \text{return(\text{res})} \) where \( \text{res} \) is leader or not leader.

Description of the of the algorithm The code of each anonymous process \( p_i \) appears in Figure 2. When the process \( p_i \) invokes \( \text{elect()} \), it enters a repeat loop that it will exit at line 11 if it is not elected, and at line 13 if it is elected.

Once in a new round, \( p_i \) first writes its new round number in all the registers it owns, those are the registers \( R_i[j] \) such that \( \text{myview}_i[j] = \text{true} \) (line 4). Then, it strives to own as many registers as possible (without compromising liveness). To this end, it considers all the registers \( R_i[j] \) such that \( R_i[j] < \text{round}_i \) (line 6). If such a register is equal to 0 (i.e., is not currently owned by another process), \( p_i \) invoke \( \text{compare\&swap}(R_i[j], 0, \text{round}_i) \) to own it (line 7). If it is the case, it accordingly increases \( \text{counter}_i \) (line 8).

Then \( p_i \) computes the maximal number of processes that, at round \( \text{round}_i \), can compete with them (variable \( \text{competitors}_i \) at line 9). There are then two cases. If it owns
fewer registers than the average number \(m/\text{competitors}\) (division on real numbers), \(p_i\) resets the registers it owns to their initial value (line 11), and withdraws from the leader competition (line 12). Otherwise, if \(\text{round}_i < n - d + 1\), \(p_i\) re-enters the repeat loop to progress to the next round. If \(\text{round}_i = n - d + 1\), \(p_i\) is one of the at most \(d\) leaders (line 14).

Let us note that a (successful) assignment of a round number to \(R_i[j]\) by a process \(p_i\) at line 7 has \(R_i[j] = 0\) as pre-condition and \(R_i[j] > 0\) as post-condition. Moreover, both the assignment of \(R_i[j]\) at lines 4 and 11 have \(R_i[j] > 0\) as pre-condition. It follows that, between the lines 3 and 9 \(\text{counter}_i\) counts the number of registers owned by \(p_i\).

\begin{algorithm}
\textbf{Algorithm 1: code of a process }p_i\textbf{ in the fully anonymous RMW model (non mandatory participation)}

The initial value of all the shared registers is 0.

\begin{verbatim}
algorithm elect() is
    counter_i ← 0; round_i ← 0; for each j ∈ {1,...,m} do myview_i[j] ← false end do
repeat
    round_i ← round_i + 1 // progress to the next round
    for each j ∈ {1,...,m} do if myview_i[j] then R_i[j] ← round_i fi end do // owned
    for each j ∈ {1,...,m} do if R_i[j] < round_i do // R[j] < round_i implies myview[j] = false
        myview_i[j] ← compare&swap(R_i[j],0,round_i) // try to own R_i[j]
        if myview_i[j] then counter_i ← counter_i + 1 fi end do // own + 1
    end if
    competitors_i ← n - round_i + 1 // max # of competing processes
    if counter_i < m/competitors_i then // too many competitors
        for each j ∈ {1,...,m} do if myview_i[j] then R_i[j] ← 0 fi end do // free owned
    fi end if
until round_i = n - d + 1 end repeat // p_i is elected
end algorithm
\end{verbatim}

Fig. 2. \(d\)-election for \(n\) anonymous processes and \(m \in \mathcal{M}(n,d)\) anonymous RMW registers

\[2.3 \textbf{Proof of Algorithm 1}\]

Let us say that “process \(p_i\) executes round \(r\)” when its local variable \(\text{round}_i = r\).

Reminder: \(m \in \mathcal{M}(n,d)\) where \(\mathcal{M}(n,d) = \{k : \forall \ell : 1 < \ell \leq n : \gcd(\ell,k) \leq d\}\).

\textbf{Lemma 1.} For every \(r \in \{1,...,n - d + 1\}\), at most \(n - r + 1\) processes may execute round \(r\). In particular, at most \(d\) processes may execute round \(n - d + 1\).

\textbf{Proof} The proof is by induction on the number of rounds. The induction base is simple since at most \(n\) processes may execute round \(r = 1\). Let us assume (induction hypothesis) that the lemma holds for round \(r < n - d + 1\) and prove that the lemma also holds
(induction step) for round \( r + 1 \). That is, we need to show that at most \( n - r \) processes execute round \( r + 1 \).

Let \( P_r \) be the set of processes that execute round \( r \). If \(|P_r| < n - r + 1\) then we are done, so let us assume that \(|P_r| = n - r + 1\). Notice that, since \( r < n - d + 1 \), \(|P_r| > d \).

We have to show that at least one process in \( P_r \) will not proceed to round \( r + 1 \), i.e., to show that at least one process in \( P_r \) will withdraw at line 12. This amounts to show that for at least one process \( p_i \in P_r \) the predicate \( \text{counter}_i < m/\text{competitors}_i \) is evaluated to true when \( p_i \) executes line 10 during round \( r \).

Assume by contradiction that the predicate \( \text{counter}_i < m/\text{competitors}_i \) in line 13 is evaluated to false for each process \( p_i \in P_r \). For each \( 1 \leq i \leq |P_r| \), let \( \text{counter}(i) \) denotes the value of \( \text{counter}_i \) at that time (when the predicate is evaluated to false). Thus, for all \( 1 \leq i \leq |P_r| \), \( \text{counter}(i) = m/(n - r + 1) \). Hence, it follows from the following facts,

1. \( \text{counter}(1) + \cdots + \text{counter}(|P_r|) = m \),
2. \( \forall 1 \leq i \leq |P_r| : \text{counter}(i) \geq m/(n - r + 1) \), and
3. \( |P_r| = n - r + 1 \),

that \( \forall 1 \leq i \leq |P_r| : \text{counter}(i) = m/(n - r + 1) \). Moreover, as

1. \( \text{counter}(i) \) is a positive integer, we have \( \gcd(n - r + 1, m) = n - r + 1 \),
2. \( r < n - d + 1 \) it follows that \( n - r + 1 > d \),

from which follows that \( \gcd(n - r + 1, m) > d \), which contradicts the assumption that \( m \in M(n, d) \). \( \square \)

**Theorem 1**

**Lemma 2.** At most \( d \) processes are elected.

**Proof** The proof is an immediate consequence of Lemma 1, which states that at most \( d \) processes may execute round \( n - d + 1 \). If they do not withdraw from the competition, each of these processes exits the algorithm at line 14, and considers it is a leader. \( \square \)

**Lemma 3.** For every \( r \in \{1, \ldots, n - d + 1\} \), at least one process executes round \( r + 1 \). In particular, at least one process executes round \( n - d + 1 \) at the end of which it claims it is a leader.

**Proof** Considering the (worst) case where the \( n \) processes execute round \( r = 1 \), we show that at least one process attains round 2. To this end, let us assume by contradiction that no process attains round 2. This means that all the processes executed line 10 and found the predicate equal to true (they all withdrew) hence each process \( p_i \) is such that \( \text{counter}_i < m/(n - r + 1) = m/n \). Using the notations and the observations of Lemma 1 we have

1. \( |P_r| = n \),
2. \( \text{counter}(1) + \cdots + \text{counter}(n) = m \),
3. \( \forall 1 \leq i \leq n : \text{counter}(i) < m/(n - r + 1) = m/n \).
If then follows from the last item that $\text{counter}(1) + \cdots + \text{counter}(n) < n \times m/n = m$
which contradicts the second item. It follows from this contradiction that there is
at least one process for which the predicate of line 10 is false at the end of round 1, and conse-
sequently this process progresses to round $r = 2$.

Assuming now by induction that at most $(n - r + 1)$ processes execute round $r$, we
show that at least one process progresses to round $r+1$. The proof follows from the three
previous items where $|P_r| = n - r + 1$ (item 1), $\text{counter}(1) + \cdots + \text{counter}(|P_r|) = m$
(item 2), and $\forall 1 \leq i \leq |P_r| : \text{counter}(i) < m/(n - r + 1)$ (item 3), from which we
conclude $\text{counter}(1) + \cdots + \text{counter}(|P_r|) < n \times (m - r + 1)/(m/(m - r + 1))$, i.e.
$m < m$, a contradiction. It follows that at least one process executes the round $r + 1$
during which it finds the predicate of line 10 false and consequently progresses to the
next round if $r < n - d + 1$. If $r = n - d + 1$, the process executes line 14 and becomes
a leader.

Theorem 2. Let $m$, $n$ and $d$ be such that $m \in M(n,d)$, and assume at least one
process invokes elect(). Algorithm 1 (Fig. 2) solves $d$-election in a fully anonymous
system where communication is through RMW registers.

Proof The proof follows directly from the lemmas 2 and 3.

3 Exact $d$-election in the RMW Model
Where Participation is Required

This section considers the fully anonymous RMW model in which all the processes are
required to participate. In such a context, it presents an exact $d$-election algorithm that
assumes that $d$ is a multiple of $\gcd(m,n)$. It also shows that this condition is necessary
for exact $d$-election in such a system model.

3.1 A necessary condition for exact $d$-election

The following theorem, which considers anonymous memory and non-anonymous pro-
cesses with the symmetry constraint, has been stated and proved in [10].

Theorem 3 (See [10]). There is no symmetric exact $d$-election algorithm in the RMW
communication model for $n \geq 2$ processes using $m$ anonymous registers if $\gcd(m,n)$
does not divide $d$.

As in Section 2.1 this impossibility still holds in the RMW model where both the
processes and the memory are anonymous, and in the model where communication
is through anonymous RW registers.

3.2 An exact $d$-election algorithm

Anonymous memory All the registers of the anonymous memory $R[1..m]$ are RMW
registers initialized to 0. Moreover, the size $m$ of the memory is such that $\gcd(m,n)$
divides $d$. 

\[ \text{counter}(1) + \cdots + \text{counter}(n) < n \times m/n = m \]
An anonymous register $R[x]$ will successively contain the values 1, 2, ... where the increases by 1 are produced by successful executions of compare\&swap($R[x]$, $val$, $val+1$) issued by the processes (lines 6 and 8 in Figure 3). The fact that a process can increase the value of a register to $val+1$ only if its current value is $val$ is the key of the algorithm.

Underlying principle The key idea that governs the algorithm is Bezout’s identity, a Diophantine equation that relates any pair of positive integers according to their Greatest Common Divisor.

**Theorem 4 (Bezout, 1730-1783).** Let $m$ and $n$ be two positive integers and let $d = \gcd(m, n)$. There are two positive integers $u$ and $v$ such that $u \times m = v \times n + d$.

Consider a rectangle made up of $u \times m$ squares. On one side, this means that $u$ squares are associated with each of the $m$ anonymous registers. On another side, each of the $n$ processes progresses until it has “captured” $v$ squares (from an operational point of view, the capture of a square is a successful invocation of compare\&swap($R[x]$, $val$, $val+1$).

Then, when $v \times n$ squares have been captured by the processes, each process competes to capture one more square. As it remains only $d = u \times m - v \times n$ squares, the processes that succeed in capturing one more square are the $d$ leaders.

Local variables at each process $p_i$
- $won_i$ (initialized to 0): number of squares captured by $p_i$.
- $sum_i$ (initialized to 0): local view of the numbers of squares captured by all the processes.
- $myview_i[1..m]$: local copy (non-atomically obtained) of the anonymous memory $R[1..m]$.

Description of the algorithm Assuming $d$ is a multiple of $\gcd(m, n)$ and all the processes participate, Algorithm 2 (described in Figure 3) solves exact $d$-election for $n$ anonymous processes and $m$ RMW anonymous registers.

When it invokes elect(), a process $p_i$ enters a repeat loop lines 1-10. Each time it enters the loop, $p_i$ asynchronously reads the anonymous memory non-atomically (line 2) and then counts in $sum_i$ the number of squares that have been captured by all processes as indicated by the previous asynchronously reads (line 3).

If $p_i$ sees a register $R[x]$ that has been captured less than $u$ times (line 4), there are two cases.
- If $won_i < v$, $p_i$ tries to capture one of the $u$ squares of $R[x]$. To this end $p_i$ uses the RMW operation: it invokes compare\&swap($R[x]$, $myview_i[x]$, $myview_i[x]+1$).
  If it is successful, it increases $won_i$, the number of squares it has captured so far (line 6).

3 This principle has already been used in [10] to solve exact $d$-election with a symmetric algorithm in a system where the (non-anonymous) processes cooperate through an anonymous RW registers.

4 The pair $(u, v)$ is not unique. Euclid’s $\gcd(m, n)$ algorithm can be used to compute such pairs.
Algorithm 2: code of a process $p_i$ in the fully anonymous RMW model (mandatory participation)

$u$ and $v$: smallest positive integers such that $u \times m = v \times n + d$

The initial value of all the shared registers is 0.

\begin{verbatim}
operation elect() is
  repeat
    for each $j \in \{1, \ldots, m\}$ do myview$_i[j] \leftarrow R[j]$ end do // read of the anony mem.
    $\text{sum}_i \leftarrow \text{myview}_i[1] + \cdots + \text{myview}_i[m]$ // successful compare&swap() seen
    if $\exists x \in \{1, \ldots, m\}$ : myview$_i[x] < u$ then
      if $\text{won}_i < v$ then
        if compare&swap($R[x], \text{myview}_i[x], \text{myview}_i[x] + 1$) then $\text{won}_i \leftarrow \text{won}_i + 1$ fi
      fi
    end if
    if $\text{sum}_i \geq v \times n$ then // sum$_i \geq v \times n$ implies $\text{won}_i = v$
      if compare&swap($R[x], \text{myview}_i[x], \text{myview}_i[x] + 1$) then return (leader) fi
    fi
  until $\text{sum}_i = \text{u} \times \text{m}$
end repeat
return (not leader).
\end{verbatim}

Fig. 3. Exact $d$-election for $n$ anonymous processes and $m$ RMW anonymous registers

Remark Let $\alpha$ and $\beta$ be two integers such that $m = \alpha \times \text{gcd}(m,n)$ and $n = \beta \times \text{gcd}(m,n)$. The equations $u \times m = v \times n + d$ and $d = \ell \times \text{gcd}(m,n)$ give rise to the equation $u \times \alpha = v \times \beta + \ell$, which can be used to obtain a more efficient version of the algorithm.

3.3 Proof of Algorithm 2

Theorem 5. Let $m$, $n$ and $d$ be such that gcd($m,n$) divides $d$, and assume all the processes invoke elect(). Algorithm 2 (Fig. 3) solves exact $d$-election in a fully anonymous system where communication is through RMW registers.

Proof Let us first observe that, due to the atomicity of compare&swap(), if several processes invoke compare&swap($X, v, v+1$) on the very same register $X$ whose value is $v$, exactly one of of them succeeds in writing $v+1$. It follows that each of the $u \times m$ squares is captured by only one process. Moreover, due to the predicate of line 5, each process eventually captures $u$ squares. Once this occurs, it remains $d$ squares, which are captured by $d$ distinct processes at line 8 (these processes are distinct because, once a process captured such a square, it returns the value leader and stops executing). Moreover, a process can capture one of the $d$ remaining squares only after each process
has captured \( v \) squares at line 6. It follows that exactly \( d \) processes exit the algorithm at line 7 with a successful \texttt{compare\&swap()}, and the \((n - d)\) other processes exit the algorithm at line 11. \( \square \) Theorem

4 \textit{d-Election in the RMW Model Where Participation is Required}

This section considers the fully anonymous RMW model in which all the processes are required to participate. In such a context, it presents a \( d \)-election algorithm where \( \gcd(n, m) \leq d \). It also shows that this condition is necessary for \( d \)-election in such a system model.

4.1 A necessary condition for \( d \)-election

The following theorem, which considers anonymous memory and non-anonymous processes with the symmetry constraint, has been stated and proved in [10].

\textbf{Theorem 6 (See [10]).} There is no symmetric \( d \)-election algorithm in the RMW communication model for \( n \geq 2 \) processes using \( m \) anonymous registers if \( \gcd(m, n) > d \).

Clearly, this impossibility still holds in the RMW model where both the processes and the memory are anonymous, and in the model where communication is through anonymous RW registers.

4.2 A necessary and sufficient condition for \( d \)-election

The following corollary is an immediate consequence of Theorem [5]

\textbf{Corollary 1.} For any pair \( \langle n, m \rangle \), it is always possible to solve exact \( \gcd(n, m) \)-election in a fully anonymous system where communication is through RMW registers.

Let us also observe that any exact \( d \)-election algorithm trivially solves \( d \)-election (but then the bound is then not tight). We also have the following theorem.

\textbf{Theorem 7.} For any pair \( \langle n, m \rangle \), it is possible to solve \( d \)-election in a fully anonymous system where communication is through RMW registers if and only if \( \gcd(n, m) \leq d \).

\textbf{Proof} If direction. For any pair \( \langle n, m \rangle \) such that \( \gcd(n, m) \leq d \), it is possible to solve \( d \)-election by running an exact \( \gcd(n, m) \)-election algorithm, which exists due to Corollary [1].

As the fully anonymous model is a weaker model than the symmetric model, the “Only if” direction follows from Theorem [6]. \( \square \) Theorem
5 Impossibility in the RW Communication Model

Theorem 8. There is neither $d$-election nor exact $d$-election algorithms in the process anonymous RW non-anonymous communication model.

Proof Assuming such an algorithm exists, let us order the participating processes in some fixed order, e.g., $p_1, ..., p_x$ ($x = n$ in the case where full participation is required). Let us consider in such a setting a lock-step execution in which $p_1$ executes its first (read or write operation on a shared register) operation, then $p_2$ executes its first operation, etc., until $p_x$ that executes its first (read or write) operation on the shared non-anonymous memory. As all processes have the same code, they all execute the same operation on the same register and are consequently in the same local state after having executed their first operation. The same occurs after they have their (same) second operation, etc. It follows that, whatever the number of steps executed in a lock-step manner by the processes, they all are in the same local state. So, it is impossible to break their anonymity (that would allow us to elect some of them).

Let us consider an anonymous memory in which the memory adversary associates the same address mapping to all the processes (i.e., $\forall i, j \in \{1, \cdots, n\}$ and $x \in \{1, \cdots, m\}$ we have $f_i(x) = f_j(x)$, see Section 1.2). In this case, the model boils down to the process anonymous and non-anonymous memory. The next corollary is then an immediate consequence of the previous theorem.

Corollary 2. There is neither $d$-election nor exact $d$-election algorithms in the fully anonymous RW communication model.

6 Conclusion

This article has investigated the $d$-election problem in fully anonymous shared memory systems. Namely, systems where not only the processes are anonymous but the shared memory also is anonymous in the sense that there is no global agreement on the names of the shared registers (any register can have different names for distinct processes). Assuming RMW atomic registers, it has shown that both the $d$-election problem (at least one and at most $d$ processes are elected) and the exact $d$-election problem (exactly $d$ processes are elected) can be solved in such an adversarial context if and only if the three model parameters $n$ (number of processes), $m$ (size of the anonymous memory), and $d$ (number of leaders) satisfy some properties. These necessary and sufficient conditions are:

- $m \in M(n, d)$ for solving $d$-election when participation is not required,
- $\gcd(m, n)$ divides $d$ for solving exact $d$-election when participation is required, and
- $\gcd(m, n) \leq d$ for solving $d$-election when participation is required.

It has also been shown that,

- neither $d$-election nor exact $d$-election can be solved in a fully anonymous system where communication is through atomic RW registers.
This work complements previously known research on the symmetry-breaking problem (election) in the context of fully anonymous RW/RMW systems. A very challenging problem remains to be solved: are there other non-trivial functions that can be solved in the fully anonymous RW/RMW setting.

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A The case where \( m = 1 \)

When the anonymous memory is made up of a single register \( R \), we have \( \text{gcd}(1, n) = 1 \leq 1 \). In this case there is a very simple \( d \)-election algorithm described below, where the single anonymous register is initialized to 0.

\[ \text{Algorithm 3: code of a process } p_i \text{ when } m = 1 \] 
(Participation required or not required)

operation elect() is
1 repeat forever
2 \( \text{myview}_i \leftarrow R \) // atomic read the anonymous memory
3 if \( \text{myview}_i \geq d \) then return (\text{not leader}) fi
4 if compare&swap(\( R, \text{myview}_i, \text{myview}_i + 1 \)) then return (leader) fi
5 end repeat.

Fig. 4. \( d \)-election for \( n \) anonym. processes when the anonym. memory is a single RMW register

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