Self-dual Maxwell field in 3D gravity with torsion

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Abstract
We study the system of self-dual Maxwell field coupled to 3D gravity with torsion, with Maxwell field modified by a topological mass term. General structure of the field equations reveals a new, dynamical role of the classical central charges, and gives a simple correspondence between self-dual solutions with torsion and their Riemannian counterparts. We construct two exact self-dual solutions, corresponding to the sectors with massless and massive Maxwell field, and calculate their conserved charges.

1 Introduction
The three-dimensional (3D) gravity has been used for nearly three decades as a laboratory for exploring basic features of the realistic gravitational dynamics, with a number of outstanding results [1]. In particular, one should mention here the discovery of the Bañados, Teitelboim and Zanelli (BTZ) black hole [2], the study of which helped us to improve our basic understanding of the classical and quantum black hole dynamics.

In the early 1990s, Mielke and Baekler (MB) [3] introduced a new element into this structure by replacing the traditional Riemannian geometry of general relativity (GR) by the Riemann-Cartan geometry, in which the gravitational dynamics is characterized by both the curvature and the torsion. Such an approach is expected to give us a new insight into the relationship between geometry and the dynamical structure of gravity. Recent developments along these lines led to a number of interesting results related to the Chern-Simons formulation, conformal asymptotic structure, black hole solutions, thermodynamics and supersymmetry [4, 5, 6, 7], which confirm that the topological MB model has a reach dynamical structure. Here, we would like to single out one of these results—that 3D gravity with torsion possesses the BTZ-like black hole solution [4], which is electrically neutral.

In an attempt to extend this result to the electrically charged sector, we analyzed static configurations of the MB model and found an exact solution with azimuthal electric field [8]. In the present paper, we continue this investigation of the charged sector by constructing two rotating solutions, corresponding to the self-dual Maxwell field as a source. Standard dynamics of the Maxwell field is modified by introducing a topological, gauge-invariant mass term [9], which has a “screening” effect and eliminates logarithmic divergences. The results

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obtained here can be compared with similar investigations in GR \cite{10, 11, 12, 13, 14}, with a possibility to recognize dynamical effects of torsion.

The paper is organized as follows. In section 2, we give a brief account of 3D gravity with torsion and examine general structure of the field equations, with Maxwell field modified by a topological mass term ($\mu$). In section 3, we restrict the Maxwell field to be self-dual and examine the related dynamical structure. As a result, we discover a significant dynamical role of the classical central charge of 3D gravity with torsion \cite{5}: two possible values of the central charge are directly related to the two self-duality sectors of the Maxwell field. Moreover, we prove a general theorem which establishes a simple correspondence between self-dual solutions with torsion, and their Riemannian counterparts. In sections 4 and 5, relying on the results of Appendix A, we construct two exact, stationary and spherically symmetric solutions, characterized by $\mu = 0$ and $\mu \neq 0$. They are recognized as natural generalizations of the Kamata-Koikawa \cite{10} and Fernando-Mansouri \cite{13} solutions, respectively, found earlier in Riemannian theory. For both of these solutions, we use the canonical approach to calculate their conserved charges—energy, angular momentum and electric charge. Section 5 is devoted to concluding remarks. In Appendix A, we show that every self-dual solution is determined by a single function of radial coordinate (compare with \cite{14}), while Appendix B contains some technical details.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters ($i, j, k, \ldots; \mu, \nu, \lambda, \ldots$) run over 0,1,2, the first letters of the Greek alphabet ($\alpha, \beta, \gamma, \ldots$) run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor $\varepsilon_{ijk}$ and the related tensor density $\varepsilon_{\mu\nu\rho}$ are both normalized so that $\varepsilon^{012} = 1$; the Hodge-star operation is $\star$.

## 2 The field equations

We begin this section with a brief account of the basic structure of 3D gravity with torsion, then we introduce the action containing the Maxwell field modified by a topological (Chern-Simons) mass term and derive the general form of the field equations.

Theory of gravity with torsion can be naturally described as a Poincaré gauge theory (PGT), with an underlying spacetime structure corresponding to Riemann-Cartan geometry. Basic gravitational variables in PGT are the triad $b^i$ and the Lorentz connection $A_{ij} = -A_{ji}$ (1-forms), and the corresponding field strengths are the torsion $T^i$ and the curvature $R^{ij}$. In 3D, one can introduce the notation $A_{ij} = -\varepsilon_{ijk}^\ell b^\ell$ and $R^{ij} = -\varepsilon_{ijk}^\ell R^\ell$, which yields:

\[ T^i = db^i + \varepsilon^i_{jk} \omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \varepsilon^i_{jk} \omega^j \wedge \omega^k. \quad (2.1) \]

PGT is characterized by a useful identity:

\[ \omega^i \equiv \bar{\omega}^i + K^i, \quad (2.2a) \]

where $\bar{\omega}^i$ is the Levi-Civita (Riemannian) connection, and $K^i$ is the contortion 1-form, defined implicitly by $T^i =: \varepsilon^i_{mn} K^m \wedge b^n$. Using this identity, one can express the curvature $R_i = R_i(\omega)$ in terms of its Riemannian piece $\bar{R}_i = R_i(\bar{\omega})$ and the contortion $K_i$:

\[ 2R_i \equiv 2\bar{R}_i + 2\nabla K_i + \varepsilon_{imn} K^m \wedge K^n. \quad (2.2b) \]
The covariant derivative $\nabla = \nabla(\omega)$ acts on a tangent-frame spinor/tensor in accordance with its spinorial/tensorial structure; when $X$ is a form, $\nabla X := \nabla \wedge X$, and $\tilde{\nabla} = \nabla(\tilde{\omega})$.

The antisymmetry of the Lorentz connection $A^i_j$ implies that the geometric structure of PGT corresponds to Riemann-Cartan geometry, in which $b^i$ is an orthonormal coframe, $g := \eta_{ij} b^i \otimes b^j$ is the metric of spacetime, $\omega^i$ is the Cartan connection, and $T^i, R^i$ are the torsion and the Cartan curvature, respectively. In what follows, we will omit the wedge product sign $\wedge$ for simplicity.

General gravitational dynamics in Riemann-Cartan spacetime is determined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, we arrive at the topological MB model for 3D gravity [3]:

$$L_0 = 2a b^i R_i - \frac{A}{3} \varepsilon_{ijk} b^j b^k + \alpha_3 L_{CS}(\omega) + \alpha_4 b^i T_i,$$  

(2.3a)

where $a = 1/16\pi G$ and $L_{CS}(\omega) = \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \omega^j \omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection. The MB model is a natural generalization of GR with a cosmological constant ($\text{GR}_\Lambda$). The complete dynamics of the topologically massive Maxwell field coupled to 3D gravity with torsion is described by the Lagrangian

$$L = L_0 + L_M,$$  

(2.3b)

where $F = dA$.

By varying $L$ with respect to $b^i$ and $\omega^i$, one obtains the gravitational field equations. In the nondegenerate sector with $\Delta := \alpha_3 \alpha_4 - a^2 \neq 0$, they have the form [8]

$$2T_i - p \varepsilon_{ijk} b^j b^k = u \Theta_i,$$  

(2.4a)

$$2R_i - q \varepsilon_{ijk} b^j b^k = -v \Theta_i,$$  

(2.4b)

where $\Theta_i = -\delta L_M/\delta b^i$ is the Maxwell energy-momentum current (2-form), and

$$p := \frac{\alpha_3 A + \alpha_4 a}{\Delta}, \quad u := \frac{\alpha_3}{\Delta},$$

$$q := -\left(\alpha_4\right)^2 + a A \frac{1}{\Delta}, \quad v := \frac{a}{\Delta}.$$

After introducing the energy-momentum tensor,

$$T^i_k := (b^i \Theta_k) = -F^i m F_{km} + \frac{1}{4} \delta^i_k F^2,$$

where $F^2 = F^m n F_{mn}$, we can express $\Theta_i$ as

$$\Theta_i = \varepsilon_{imn} t^m b^n, \quad t^m := -\left(T^m_k - \frac{1}{2} \delta^m_k T\right) b^k,$$

with $T = T^k_k$, whereupon the gravitational field equations take the simple form:

$$T_i = \varepsilon_{imn} K^m b^n, \quad K^m = \frac{1}{2} (p b^m + u t^m),$$  

(2.5a)

$$2R_i = q \varepsilon_{imn} b^m b^n - v \varepsilon_{imn} t^m b^n.$$  

(2.5b)
These equations, together with the modified Maxwell equations
\[ d^* F + \mu F = 0, \quad (2.5c) \]
define the complete dynamics of our system. The Cartan curvature \( R_i \) is calculated using the identity (2.2b),
\[ 2R_i = 2\tilde{R}_i + u\tilde{\nabla}t_i + \varepsilon_{imn} \left( \frac{\nu^2}{4} b^m b^n + \frac{up}{2} t^m b^n + \frac{u^2}{4} t^m t^n \right), \quad (2.6) \]
where the Maxwell field contribution is compactly represented by the 1-form \( t^i \).

3 Dynamical characteristics of self-dual solutions

General structure of the field equations (2.5) depends essentially on the form of the Maxwell field. In this section, we discuss some important dynamical characteristics of exact solutions corresponding to the self-dual Maxwell field as a source.

We are looking for a spherically symmetric and stationary solution. Choosing the local coordinates \( x^\mu = (t, r, \varphi) \), we make the following ansatz for the triad field,
\[ b^0 = N dt, \quad b^1 = B^{-1} dr, \quad b^2 = K(d\varphi + Cdt), \quad (3.1) \]
and for the Maxwell field:
\[ F = Eb^0 b^1 - H b^1 b^2. \quad (3.2) \]
Here, \( N, B, C, K \) and \( E, H \) are six unknown functions of the radial coordinate \( r \).

General form of the modified Maxwell field equations (2.5c) reads:
\[ E'B + \gamma E + \mu H = 0, \]
\[ H'B + \alpha H + 2\beta E + \mu E = 0, \quad (3.3) \]
where prime denotes the derivative with respect to the radial coordinate \( r \), and \( \alpha, \beta, \gamma \) are components of the Riemannian connection, defined in (A.1). We assume a generalized self-duality of the Maxwell field:
\[ E = \epsilon H, \quad \epsilon^2 = 1. \quad (3.4) \]
Taking the difference of the two Maxwell equations in conjunction with the self-duality condition leads to
\[ 2\beta \epsilon = \gamma - \alpha. \quad (3.5a) \]
Introducing a new radial coordinate \( \rho = \rho(r) \), defined by
\[ d\rho = \frac{dr}{B}, \]
we find that the first integrals of these equations are given as
\[ EK = -Q_e \exp(-\epsilon \mu \rho), \]
\[ -HN = (Q_m - Q_e C) \exp(-\epsilon \mu \rho), \quad (3.5b) \]
where \( Q_e, Q_m \) are integration constants. Inserting here the self-duality condition yields

\[
\epsilon Q_e N = K(Q_m - Q_e C). \tag{3.5c}
\]

The self-duality condition implies \( F^2 = 0 \) and simplifies the form of the electromagnetic energy-momentum tensor,

\[
T^i_j = \begin{pmatrix}
E^2 & 0 & EH \\
0 & 0 & 0 \\
-EH & 0 & -H^2
\end{pmatrix},
\]

and the form of \( t^i \):

\[
t^i = -T^i_j b^j, \\
t^0 = -E(Eb^0 + Hb^2), \quad t^1 = 0, \quad t^2 = -\epsilon t^0.
\]

This defines the contortion as in (2.5a), and exhausts the content of the first field equation.

Going now to the second field equation (2.5b), we combine the relations

\[
\epsilon_{imn} t^m t^n = 0, \quad \delta t_i = \epsilon \frac{B(NK)'}{NK} \epsilon_{imn} t^m b^n,
\]

with (A.5) and calculate the Cartan curvature (2.6):

\[
2R_i = 2\tilde{R}_i + \frac{p^2}{4} \epsilon_{imn} b^m b^n + \left( \frac{pu}{2} + \epsilon \frac{u}{\ell} \right) \epsilon_{imn} t^m b^n. \tag{3.6}
\]

Then, by substituting this result into (2.5b), we obtain:

\[
2\tilde{R}_i = \Lambda_{eff} \epsilon_{imn} b^m b^n - V \epsilon_{imn} t^m b^n, \tag{3.7}
\]

\[
V := v + \frac{pu}{2} + \epsilon \frac{u}{\ell}.
\]

Note that the factor \( V = V(\epsilon) \) is proportional to the classical central charge \( c(\epsilon) \), characterizing the asymptotic conformal structure of 3D gravity with torsion [5]:

\[
V(\epsilon) = \frac{1}{\Delta} \frac{1}{24\pi\ell} c(\epsilon), \quad c(\pm 1) = 24\pi \left[ a\ell + \alpha_3 \left( \frac{p\ell}{2} \mp 1 \right) \right].
\]

(a) Equation (3.7) reveals a new feature of the central charges \( c(\mp 1) \), showing that they have a direct dynamical influence on the self-duality modes \( \epsilon = \mp 1 \) of the system.

The set of equations (3.7) consists of 9 ordinary differential equations for only 3 unknown functions of \( r \): \( N, B \) and \( K \) (\( C, E \) and \( H \) are determined in terms of \( N \) and \( K \) as in (3.5b)). Is there any consistent solution of this overdetermined set of equations? To answer this question, one can solve the system (3.7) directly, as shown in Appendix A, but we follow here another approach, based on some simple properties of (3.7). Namely, in the limit \( u \to 0 \), or equivalently, \( V \to v = -1/a \), equation (3.7) becomes equivalent to the Einstein equation for the self-dual Maxwell field in Riemannian 3D gravity. This property can be formulated as the following constructive statement:

(b) Starting with any self-dual solution in Riemannian 3D gravity, one can generate the related self-dual solution with torsion by making the replacement \( v \to V \).

In what follows, we shall illustrate the power of this theorem by constructing two different solutions of (3.7), belonging to the sectors with \( \mu = 0 \) and \( \mu \neq 0 \), respectively.
4 A self-dual solution with $\mu = 0$

4.1 Construction

The condition $\mu = 0$ refers to the standard, massless Maxwell field. As shown in Appendix A, the field equations define the form of $K^2$ as in (A.6),

$$K^2 = g_1 + g_2 \exp \left( 2\rho / \ell \right) + \frac{2}{\ell} r_0^2 \rho ,$$

whereupon the general solution follows from equations (A.4) and (3.5b). To realize this construction, we first conveniently fix $g_2$ by demanding $g_2 = r_0^2$. Then, choosing the radial coordinate $r$ by

$$B = \frac{r^2 - r_0^2}{\ell r} ,$$

we find

$$\rho = \frac{\ell}{2} \ln \left| \frac{r^2 - r_0^2}{r_0^2}\right| .$$

Finally, using (A.4) and (3.5b), we obtain the general solution for $\mu = 0$:}

$$B = \frac{r^2 - r_0^2}{\ell r} , \quad N = \frac{r^2 - r_0^2}{\ell K} ,$$

$$K^2 = r^2 + r_0^2 \ln \left| \frac{r^2 - r_0^2}{r_0^2} \right| + h_1 ,$$

$$C = \frac{Q_m}{Q_e} - \epsilon \frac{N}{K} , \quad E = \epsilon H = - \frac{Q_e}{K} , \quad (4.1a)$$

where $h_1 = g_1 - r_0^2$. The boundary condition $C \to 0$ for $r \to \infty$ yields $Q_m/Q_e = 1/\ell$, and consequently:

$$C = \frac{\epsilon}{\ell} \left( 1 - \frac{r^2 - r_0^2}{K^2} \right) . \quad (4.1b)$$

To complete the solution, we display also the electromagnetic potential:

$$A = \left( \frac{Q_e}{2} \ln \left| \frac{r^2 - r_0^2}{r_0^2} \right| + h_2 \right) (dt + \epsilon \ell d\varphi) . \quad (4.1c)$$

In the limit $u \to 0$, or $V \to -1/a$, the solution (4.1) coincides with the Kamata-Koikawa self-dual solution, found in Riemannian GR [10] (see also [12]). Thus, the field configuration (4.1) represent a generalization of the Kamata-Koikawa solution to the self-dual solution with torsion, in accordance with the theorem (b), Section 3. Our result also confirms that Kamata and Koikawa indeed found a correct solution, in contrast to the opinion presented in [11]. In the limit $Q_e \to 0$, the solution (4.1) reduces asymptotically to the vacuum state of the BTZ-like black hole with torsion [5].

The torsion and the Cartan curvature of the self-dual solution (4.1) can be calculated with the help of equations (2.5). It follows that the scalar Cartan curvature is $R = -6q$, while $\tilde{R} = -6A_{\text{eff}}$. Although $\tilde{R}$ is constant, the form of $\tilde{R}$ implies that the metric of the solution is not maximally symmetric. The logarithmic function appearing in the solution stems from the dimensionality of spacetime.
4.2 The conserved charges

To gain a deeper insight into the nature of the self-dual solution (4.1), we now turn our attention to its conserved charges.

(A) The family of solutions (4.1) is parametrized by \((Q_e, h_1, h_2)\). Considering the neutral limit \(Q_e \to 0\) in Riemannian \(\Lambda\)\(\text{GR}\), Kamata and Koikawa [10] concluded that one should fix the parameter \(h_1\) to zero, as it leads to \(K^2 \to r^2\) in this limit. Accepting the choice \(h_1 = 0\), Chan [10] used the quasi-local formalism to calculate its energy and angular momentum. Quasi-local charges are seen to have logarithmic divergences for large \(r\), stemming from the logarithmic behavior of the electromagnetic potential in 3D. Our calculations confirmed this result in the canonical formalism, where the logarithmic terms produce divergent surface terms in the improved canonical generator.

(B) An interesting attempt to handle these divergences by a suitable regularization procedure was proposed in [15]. In this procedure, we enclose the system in a circle \(C_{\text{as}}\) having a large, but finite radius \(r_{\text{as}}\). This circle represents a regularized spatial boundary at infinity, and the asymptotic region is defined by \(r \to r_{\text{as}}\). Then, we make a choice of the boundary conditions by fixing the values of dynamical variables at \(C_{\text{as}}\). Finally, at the end of our calculations, we take the limit \(r_{\text{as}} \to \infty\).

To be more specific, let us consider the regularization of \(K\). After introducing \(C_{\text{as}}\), we fix the integration constant \(h_1\) to the value \(\bar{h}_1 := r_0^2 \ln \left| \frac{r^2 - r_0^2}{r_{\text{as}}^2 - r_0^2} \right|\), so that

\[
K^2 = r^2 + r_0^2 \ln \left| \frac{r^2 - r_0^2}{r_{\text{as}}^2 - r_0^2} \right|. \tag{4.2a}
\]

This choice is equivalent to the boundary condition \(K^2 = r_{\text{as}}^2\) at \(C_{\text{as}}\). In other words, the logarithmic term is effectively eliminated from the boundary \(C_{\text{as}}\). In a similar manner, we regularize the Maxwell potential \(A\) by choosing \(h_2 = \bar{h}_2 := (Q_e/2r_0^2)\bar{h}_1\):

\[
A = \frac{Q_e}{2} \ln \left| \frac{r^2 - r_0^2}{r_{\text{as}}^2 - r_0^2} \right| (dt + \epsilon \ell d\varphi). \tag{4.2b}
\]

This is equivalent to the boundary condition \(A = 0\) at \(C_{\text{as}}\).

Going to the canonical formalism, we formulate the asymptotic conditions as follows:

(i) the fields \(b^i, \omega^i\) and \(A\) belong to the family of self-dual configurations (4.1), parametrized by \(Q_e\), with \(h_1 = \bar{h}_1\) and \(h_2 = \bar{h}_2\),

(ii) asymptotic symmetries have well-defined canonical generators.

The meaning of these conditions is quite clear: (i) means that our calculations refer to the regularized solution, while (ii) ensures that we have a well-defined phase space.

As shown in Appendix B, using the asymptotic form of the regularized solution, one can find the related asymptotic parameters, which describe rigid time translations, axial rotations and \(U(1)\) transformations. After that, the standard canonical procedure yields the following expressions for the energy \(E\), the angular momentum \(M\) and the electric charge \(Q\) of our self-dual solution:

\[
\ell E = \epsilon Q_e \frac{u_1}{12 \ell} c(-\epsilon) = \epsilon M, \quad Q = 2\pi Q_e. \tag{4.3}
\]
Although the regularized self-dual solution has the same leading asymptotic terms as the black hole with torsion (compare Appendix B with [5]), its conserved charges $E$ and $M$ are quite different. The reason for this lies in the fact that $E$ and $M$ are basically determined by the sub-leading asymptotic terms (as noted by Chan [10]). This is clearly seen in the canonical formalism, where the analysis of the relevant surface terms shows that $E$ and $M$ in (4.3) stem entirely from $ut'$, the Maxwell field contribution to the contortion. Thus, the nonvanishing $E$ and $M$ are generated by a combined effect of the Maxwell field ($t'$) and the gravitational Chern-Simons term ($u \sim \alpha_3$). This should be compared with Riemannian GR, where the regularization procedure would yield $E = M = 0$.

The regularized solution and the one with $h_1 = h_2 = 0$, are two different members of the same family (4.1), which have different conserved charges. Another difference is found in their geometric properties. Namely, the form of the Maxwell field in (4.1a) implies that $K$ must be real, and consequently, $K^2 > 0$. By inspecting the regularized $K^2$, one finds that $K^2$ is positive for $r = r_{\text{as}}$, and it has a zero at some $r = r_* < r_{\text{as}}$, such that

$$(r_*^2 - r_0^2) \exp(r_*^2/r_0^2) = r_{\text{as}}^2 - r_0^2 .$$

We see that in the limit $r_{\text{as}} \to \infty$, $r_*$ also goes to infinity. Hence, the form the regularized solution (4.2) is valid only in the asymptotic region. One expects that a reasonable extension of this region can be found by going to more suitable local coordinates.

5 A self-dual solution with $\mu \neq 0$

5.1 Construction

The topological mass $\mu$ is introduced to regularize the asymptotic behavior of the Maxwell field. As shown in Appendix A, the solution for $K^2$ in the case $\mu \neq 0$ (with $\epsilon \mu \ell \neq -1$), takes the form

$$K^2 = g_1 + g_2 \exp(2\rho/\ell) - \frac{r_0^2}{\mu^2 \ell^2 + \epsilon \mu \ell} \exp(-2\epsilon \mu \rho) ,$$

where we used $r_0^2 = -\ell^2 Q_e^2 V/4$, as in the previous section. This form of $K^2$, combined with (A.4) and (3.5b), leads to the general self-dual solution with torsion (for $\mu \neq 0$, $\epsilon \mu \ell \neq -1$):

$$N = \frac{g_3 \exp(2\rho/\ell)}{K} , \quad C = \frac{Q_m}{Q_e} - \epsilon \frac{N}{K} ,$$

$$E = \epsilon H = -\frac{Q_e}{K} \exp(-\epsilon \mu \rho) ,$$

where $g_1$, $g_2$ and $g_3$ are constants of integration.

For a fixed $\epsilon$, the solution is characterized by eight parameters: $g_1$, $g_2$, $g_3$, $\mu$, $\ell$, $r_0$, $Q_e$ and $Q_m$. This number can be significantly reduced by imposing various physical/geometric requirements. To begin with, we normalize the time coordinate by setting $g_3 = \ell$. Next, demanding that $C$ vanishes at spatial infinity, we obtain $Q_m/Q_e = \epsilon/\ell$. Then, we choose $g_2 = \ell^2$ to ensure that in the limit $Q_e \to 0$, $g_1 \to 0$, the solution (5.1) reduces to the black hole vacuum. Finally, we note that solutions with vanishing Maxwell field at spatial infinity are characterized by

$$\epsilon \mu > 0 .$$
As we shall see, this restriction ensures that we have finite conserved charges (compare with [14]). Without losing generality, we impose the requirement $\epsilon\mu = 1/\ell$, which simplifies the calculations [13] (see the comment at the end of this section).

Since the solution (5.1) is defined only up to a choice of the radial coordinate, we impose the requirement $K = r$, which is equivalent to

$$
\exp(2\rho/\ell) = \frac{1}{2\ell^2} \left( r^2 - g_1 + \sqrt{(r^2 - g_1)^2 + 2\ell^2r_0^2} \right).
$$

Expressed in terms of $r$, the solution (5.1) reads:

$$
K = r, \quad B = \frac{1}{\ell r} \sqrt{(r^2 - g_1)^2 + 2\ell^2r_0^2}, \quad N = \frac{1}{2\ell r} (r^2 - g_1 + \ell r B), \quad C = \epsilon \left( \frac{1}{\ell} - \frac{N}{r} \right), \quad E = -\frac{Q_e}{r} \sqrt{\frac{\ell}{N^r}}, \quad A = -Q_e \sqrt{\frac{\ell}{N^r}} (dt + \epsilon \ell d\varphi). \tag{5.3}
$$

This result represents a natural generalization of the self-dual solution found by Fernando and Mansouri [13], in the context of Riemannian theory. An equivalent form of the Fernando-Mansouri solution has been derived independently by Clement [12]. In the limit $Q_e \to 0$, the solution reduces to the extreme black hole with torsion [5] (see the next subsection). The parameter $g_1$ will be interpreted in terms of the conserved charges.

From the above expressions, we conclude that $B > 0$ and $N > 0$ over the whole range of $r$, and moreover, the Cartan curvature is constant, $R = -6q$. Hence, (5.3) is a perfectly regular solution.

### 5.2 The conserved charges

As usual, our calculation of the conserved charges begins by fixing the asymptotic conditions of the fields in (5.3):

$$
N \sim \frac{r}{\ell} - \frac{g_1}{\ell r}, \quad B \sim \frac{r}{\ell} - \frac{g_1}{\ell r}, \quad C \sim \frac{\epsilon g_1}{\ell r^2}, \\
E \sim \frac{Q_e}{r^2}, \quad A_\mu \sim \frac{Q_e}{r}. \tag{5.4a}
$$

Note that the electromagnetic potential has the same asymptotic behavior as in 4D. As one can see from the absence of the logarithmic terms, the topological mass $\mu$ acts as an infrared regulator, which modifies the asymptotics of all the fields.

Since $t^i \sim O_3$, one immediately concludes that the Maxwell field contribution to the asymptotic behavior of $(b^i, \omega^i)$ can be ignored. In other words, due to the fast asymptotic decrease of the Maxwell potential, the electromagnetic contribution to the energy and angular momentum vanishes. Moreover, the asymptotics defined by (5.4a) coincides with that of the extreme BTZ black hole with torsion [5], provided we make the identifications

$$
4G\ell m = \frac{g_1}{\ell} = 4G\epsilon J. \tag{5.4b}
$$

9
Using (5.4b) and the expressions for the conserved charges of the BTZ black hole with torsion [5], we find the energy and angular momentum of our self-dual solution (5.3):

$$\ell E = \frac{g_1}{6\ell} c(-\epsilon) = \epsilon M . \quad (5.5a)$$

In the Riemannian limit $u \to 0$, these expressions agree with those obtained in the quasilocal formalism by Fernando and Mansouri [13], and by Dereli and Obukhov [14].

To calculate the electric charge, we note that the canonical generator of the theory with the topological mass $\mu \neq 0$ contains an additional term in the $U(1)$ sector. Namely, the $U(1)$ piece of the canonical generator has the form (compare with Appendix A)

$$G_3 = \dot{\lambda} \pi^0 - \lambda (\partial_\alpha \pi^\alpha - 2\mu \varepsilon^{0\alpha\beta} \partial_\alpha A_\beta) ,$$

where $\pi^\alpha = -b F^{0\alpha} - \mu \varepsilon^{0\alpha\beta} A_\beta$. The variation of $G_3$ reads:

$$\delta G_3[\lambda] = 2\pi \delta \left( 4Q_e \sqrt{\frac{\ell}{rN}} \right) \sim O_1 ,$$

hence $G_3$ is perfectly regular, and we have

$$Q = 0 . \quad (5.5b)$$

Thus, although the radial electric field exists, its asymptotic fall off, given by $E \sim Q_e/r^2$, is too fast to produce a nonvanishing electric charge. Consequently, the constant $Q_e$ can not be interpreted as the electric charge, contrary to the expectation expressed in [13]. Thus, our Fernando-Mansouri-like solution should be called a neutral self-dual solution.

**Comment.** As we have seen, the assumption $\epsilon \mu \ell = 1$ allows us to find a simple solution $\rho = \rho(r)$ of the condition $K = r$. In the general case $\epsilon \mu \ell \neq 1$, $\rho(r)$ is not an elementary functions. However, in the asymptotic region we have

$$\exp(2\rho/\ell) \sim \frac{r^2 - g_1}{\ell^2} ,$$

so that

$$B = \frac{dr}{d\rho} \sim \frac{r}{\ell} - \frac{g_1}{\ell r} .$$

Thus, the asymptotic form of $N$, $B$ and $C$ is the same as in the case $\epsilon \mu \ell = 1$. Moreover, the asymptotic behavior of the Maxwell variables reads:

$$E \sim \frac{1}{r^{1+\epsilon \mu \ell}} , \quad A_\mu \sim \frac{1}{r^{\epsilon \mu \ell}} .$$

As a consequence, the conserved charges have again the values (5.5). In other words, all solutions in the sector $\epsilon \mu > 0$ have the same values of the conserved charges, as in Riemannian theory [14].
6 Concluding remarks

In this paper, we studied dynamical properties and exact solutions of the self-dual Maxwell field modified by a topological mass term, in interaction with 3D gravity with torsion.

(1) General structure of the field equations implies the following dynamical properties:
– dynamical evolution is directly influenced by the classical central charges,
– there is a simple correspondence between self-dual solutions with torsion and their Riemannian counterparts, and
– any self-dual solution with torsion is completely determined by a single function $K^2$.

(2) We constructed two exact, stationary and self-dual solutions with torsion, characterized by $\mu = 0$ and $\mu \neq 0$. They represent respective generalizations of the Kamata-Koikawa and Fernando-Mansouri solutions, found earlier in Riemannian GR$_\Lambda$. For each of these solutions, we calculated its conserved charges. The expressions for energy and angular momentum are proportional to the classical central charges.

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A Solving the gravitational field equations

In this Appendix, we demonstrate that the set of the gravitational equations (3.7) and the self-dual Maxwell equations (3.3) reduces to a simple differential equation for $K^2$. Integrating this equation, we find that its general solution depends on $\mu$.

The Riemannian connection $\tilde{\omega}^i$ is determined by the condition of vanishing torsion, $db^i + \varepsilon_{ijk} \tilde{\omega}^j b^k = 0$. Starting from the expression (3.1) for $b^i$, we find:

$$\tilde{\omega}^0 = -\beta b^0 - \gamma b^2, \quad \tilde{\omega}^1 = -\beta b^1,$$
$$\tilde{\omega}^2 = -\alpha b^0 + \beta b^2,$$

where $\alpha, \beta, \gamma$ are defined in terms of the triad components as follows:

$$\alpha := \frac{BN'}{N}, \quad \beta := \frac{BKC'}{2N}, \quad \gamma := \frac{BK'}{K}. \quad (A.1)$$

As a consequence, the Riemannian curvature $\tilde{R}^i$ takes the following form:

$$\tilde{R}_0 = (\beta' B + 2\beta \gamma) b^0 b^1 - (\gamma' B + \gamma^2 + \beta^2) b^1 b^2,$$
$$\tilde{R}_1 = -(\alpha \gamma + \beta^2) b^2 b^0,$$
$$\tilde{R}_2 = -(\alpha' B + \alpha^2 - 3\beta^2) b^0 b^1 - (\beta' B + 2\beta \gamma) b^1 b^2. \quad (A.2)$$
Using the explicit form of $\tilde{R}_i$, we find that the essential content of the second gravitational field equation (3.7) is given by the following four equations:

\[
\begin{align*}
\beta' B + 2\beta & = \frac{1}{2} VEH, \\
\gamma' B + \gamma^2 + \beta^2 & = -\Lambda_{\text{eff}} + \frac{1}{2} VH^2, \\
\alpha' B + \alpha^2 - 3\beta^2 & = -\Lambda_{\text{eff}} - \frac{1}{2} VH^2, \\
\alpha \gamma + \beta^2 & = -\Lambda_{\text{eff}}. 
\end{align*}
\]  

\begin{equation}
(A.3)
\end{equation}

In the Riemannian limit $u \to 0$, or equivalently $V \to -1/a = -2$ (in units $8\pi G = 1$), these equations reduce to equations (7)-(10) of Ref. [14]. Substituting the self-duality condition $2\epsilon\beta = \gamma - \alpha$ into the last equation, we find that $\gamma$ and $\alpha$ can be expressed in terms of $\beta$ as

\[
\gamma = \frac{1}{\ell} + \epsilon\beta, \quad \alpha = \frac{1}{\ell} - \epsilon\beta, 
\]

\begin{equation}
(A.4a)
\end{equation}

where we used $\Lambda_{\text{eff}} := -1/\ell^2$. This implies, in particular,

\[
\frac{B(NK)'}{2N} = \frac{1}{2}(\alpha + \gamma) = \frac{1}{\ell}. 
\]

\begin{equation}
(A.4b)
\end{equation}

Now, one can see that only one of the first three equations is independent (we take it to be the first one). Using the radial coordinate $\rho$ introduced in Section 3 by $d\rho = dr/B$, and the relations (3.5b) and (3.4), the first equation in (A.3) reduces to the form

\[
\beta' + 2\beta \frac{K'}{K} = \frac{\epsilon Q_e^2 V}{2K^2} \exp(-2\epsilon \mu \rho), 
\]

where prime denotes now the derivative with respect to $\rho$. Substituting here the expression for $\beta$ obtained from (A.4), $\beta = \epsilon(K'/K - 1/\ell)$, we find

\[
(K^2)'' - \frac{2}{\ell}(K^2)' = Q_e^2 V \exp(-2\epsilon \mu \rho). 
\]

\begin{equation}
(A.5)
\end{equation}

This equation can be easily solved for $K^2$. Indeed,

\[
K^2 = K_h^2 + K_p^2, 
\]

\begin{equation}
(A.6a)
\end{equation}

where $K_h^2$ is the general homogeneous solution, and $K_p$ a particular solution of (A.5):

\[
K_h^2 = g_1 + g_2 \exp(2\rho/\ell), \\
\mu = 0 : \quad K_p^2 = -\frac{\ell V Q_e^2}{2} \rho, \\
\mu \neq 0 : \quad K_p^2 = \frac{V Q_e^2\ell^2/4}{\mu^2\ell^2 + \epsilon \mu \ell} \exp(-2\epsilon \mu \rho). 
\]

\begin{equation}
(A.6b)
\end{equation}

For $\mu \neq 0$, we also have to assume $\epsilon \mu \ell \neq -1$.

Once we have $K^2$, we can go back to (A.4) and (3.5b) and calculate $N, C$ and $E, H$ (when the radial coordinate is $\rho$, we have $B = 1$). Consequently:
(c) The general solution of the system of field equations (A.3) and (3.3) for $\epsilon \mu \ell \neq -1$, is determined by a single function $K^2$.

This is a remarkable dynamical feature of the $\mu$-modified self-dual Maxwell field in 3D gravity with torsion. It shows, in particular, a complete resemblance with the corresponding Riemannian dynamics [14], in accordance with the theorem (b) in Section 3.

B Calculation of the conserved charges (4.3)

Our approach to the conserved charges is based on the canonical formalism, as outlined in Section 4. We start by giving the asymptotics for the regularized field $b^i$,

$$b^i_\mu \sim \begin{pmatrix} \frac{r}{\ell} - \frac{r_0^2}{\ell r} & 0 & 0 \\ 0 & \ell + \frac{\ell_0^2}{r^3} & 0 \\ \frac{\epsilon r_0^2}{\ell r} & 0 & r \end{pmatrix},$$

for $\omega^i$,

$$\tilde{\omega}^i_\mu \sim \begin{pmatrix} -\frac{\epsilon r_0^2}{\ell^2 r} & 0 & -\frac{r}{\ell} \\ -\frac{r}{\ell^2} + \frac{r_0^2}{\ell^2 r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t^i_\mu \sim \begin{pmatrix} -\frac{Q_e^2}{\ell r} & 0 & -\frac{\epsilon Q_e^2}{r} \\ 0 & 0 & 0 \\ \epsilon Q_e^2 & 0 & \frac{Q_e^2}{r} \end{pmatrix},$$

$$\omega^i_\mu \sim \tilde{\omega}^i_\mu + \frac{1}{2}(p b^i_\mu + u t^i_\mu),$$

and for $A$:

$$A_\mu \sim 0.$$

Note that the logarithmic terms are hidden in higher-order terms. In order to ensure consistency, we restrict the original gauge parameters to the form compatible with the adopted asymptotic conditions:

$$\xi^\mu = (\ell T_0, 0, S_0), \quad \theta^i = (0, 0, 0), \quad \lambda = \lambda_0,$$

where $T_0, S_0$ and $\lambda_0$ are constant parameters associated to the rigid time translations, axial rotations and $U(1)$ transformations, respectively.

The calculation of the conserved charges is closely related to the form of the canonical generator $G$. In the asymptotic region, where the transformation parameters are constant,
$G$ has the following effective form:

$$G = -G_1 - G_3,$$

$$G_1 := \xi^\rho \left[ b^i \rho \mathcal{H}_i + \omega^i \rho \mathcal{K}_i + \left( \partial_\rho b^i_0 \right) \pi^0_i + \left( \partial_\rho \omega^i_0 \right) \Pi^i_0 + \left( \partial_\rho A_0 \right) \pi^0 \right],$$

$$G_3 := -\lambda \partial_\alpha \pi^\alpha,$$

were $\mathcal{H}_i$ and $\mathcal{K}_i$ are components of the total Hamiltonian (see Appendix C in Ref. [8]).

The asymptotic generator $G$ acts on basic dynamical variables via the Poisson brackets, and consequently, it must be differentiable. If this is not the case, the form of $G$ can be improved by adding a suitable surface term. To improve $G$, we calculate its variation:

$$\delta G_1 = \xi^\rho \left\{ -2\varepsilon^{0\alpha\beta} \partial_\alpha \left[ b^i_\alpha \delta \alpha b_{i\beta} + \alpha_4 b_{i\beta} \right] + \omega^i \rho \delta \delta (ab_{i\beta} + \alpha_3 \omega_{i\beta}) \right\} + R,$$

$$\delta G_3 = -\lambda \partial_\beta \delta \pi^\beta,$$

where $R$ denotes the contribution of regular (differentiable) terms. Using the adopted asymptotic conditions, performing the integration over the boundary $S_{as}$ and taking the limit $r_{as} \to \infty$, we obtain:

$$\delta G_1[\xi] = \xi^0 \frac{2\pi \varepsilon}{\ell} \delta \left[ \alpha_3 Q^2 V(-\varepsilon) \right],$$

$$\delta G_1[\xi] = \xi^2 2\pi \delta \left[ \alpha_3 Q^2 V(-\varepsilon) \right],$$

$$\delta G_3[\lambda] = 2\pi \lambda \delta Q^\varepsilon.$$

Thus, the improved canonical generator takes the form

$$\hat{G} = G + \xi^0 E + \xi^2 M + \lambda Q,$$

where $E$, $M$ and $Q$ are the canonical charges of the self-dual solution, displayed in (4.3).

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