Lefschetz Numbers and Geometry of Operators in $W^*$-modules

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1 Introduction

The main goal of the present paper is to generalize the results of [18, 19] in the following way: To be able to define $K_0(A) \otimes \mathbb{C}$-valued Lefschetz numbers of the first type of an endomorphism $V$ on a $C^*$-elliptic complex one usually assumes that $V = T_g$ for some representation $T_g$ of a compact group $G$ on the $C^*$-elliptic complex. We try to refuse this restriction in the present paper. The price to pay for this is twofold:

(i) We have to define Lefschetz numbers valued in some larger group as $K_0(A) \otimes \mathbb{C}$.

(ii) We have to deal with $W^*$-algebras instead of general unital $C^*$-algebras.

To obtain these results we have got a number of by-product facts on the theory of Hilbert $W^*$- and $C^*$-modules and on bounded module operators on them which are of independent interest.

The present paper is organized as follows: In §2 we prove the necessary facts on Hilbert $W^*$-modules and their bounded module mappings extending results of W. L. Paschke [14], J.-F. Havet [5] and the first author [3]. In §3 we define Lefschetz numbers of two types and show the main properties of them. In §4 we discuss the $C^*$-case and obstructions to refine the main results of §3.

Our standard references for the theory of Hilbert $C^*$-modules are the papers [14, 15, 2, 4, 3, 11, 11] and the book of E. C. Lance [8]. The topological considerations are based on the publications [12, 13, 17, 18, 19, 11]. We are going to continue the investigations presented therein.
2 Hilbert W*-modules and module mappings

We want to show some more very nice properties of Hilbert W*-modules which often do not appear in the general C*-case. This partial class of Hilbert C*-modules was brought to the attention of the public by W. L. Paschke in his classical paper \[14\], and they are of use in many cases. The facts below can be reproved for the class of monotone complete C*-algebras carrying out much technical work, cf. \[4\], but not for larger classes of C*-algebras, in general. However, since we are going to understand the structure of general Hilbert C*-modules and their C*-duals better it suffices to treat the W*-case, and we can avoid these technicalities. Let us start with a property generalizing the (double) annihilator property of arbitrary subsets of W*-algebras.

**Lemma 1** Let $A$ be a W*-algebra and $\{M, \langle.,.\rangle\}$ be a Hilbert $A$-module. For every subset $S \subseteq M$ the bi-orthogonal set $S^\perp \subseteq M$ is a Hilbert $A$-submodule and a direct summand of $M$, as well as the orthogonal complement $S^\perp$.

**Proof:** The property of $S^\perp \subseteq M$ to be a Hilbert $A$-submodule is obvious by the definition of orthogonal complements. Since the $A$-dual Banach $A$-module $M'$ of $M$ is a self-dual Hilbert $A$-module by \[14\], Th. 3.2] one can consider the Hilbert $A$-submodule $N$ of $M'$ consisting of the direct sum of $S^\perp \rightarrow M'$ and of the Hilbert $A$-module of all $A$-linear bounded mappings from $M$ to $A$ vanishing on $S^\perp$. The second summand is the orthogonal complement of $S^\perp$ with respect to $M'$ by construction and hence, it is a self-dual Hilbert $A$-submodule and direct summand of $N$ by \[3\], Th. 3.2, Th. 2.8]. Consequently, the canonical embedding of $S^\perp \subseteq M \hookrightarrow N$ is a direct summand. Example \[3\] below shows that situations different to that described at Lemma 1 can appear e. g. for Hilbert C*-modules over the C*-algebra $A = C([0,1])$.

**Lemma 2** Let $A$ be a W*-algebra, $\{M, \langle.,.\rangle\}$ be a Hilbert $A$-module and $\phi$ be a bounded module operator on it. Then the kernel $\text{Ker}(\phi)$ of $\phi$ is a direct summand of $M$ and has the property $\text{Ker}(\phi) = \text{Ker}(\phi)^\perp \perp$.

**Proof:** By \[13\], Prop. 3.6] every bounded module operator $\phi$ on $M$ continues to a bounded module operator on its $A$-dual Hilbert $A$-module $M'$. The kernel of the extended operator is a direct summand of $M'$ because of the completeness of its unit ball with respect to the $\tau_2$-convergence induced by the functionals $\{f(\langle.,y\rangle) : f \in A_{*1}, y \in M'\}$ there, (cf. \[3\] Th. 3.2]). Consequently, the kernel of $\phi$ inside $M$ has to coincide with its bi-orthogonal complement in $M$, and by Lemma \[3\] it is a direct summand. •
Example 1 Note, that the kernel of bounded $A$-linear operators on Hilbert $A$-modules over arbitrary C*-algebras $A$ is not a direct summand, in general. For example, consider the C*-algebra $A = C([0,1])$ of all continuous functions on the interval $[0,1]$ as a Hilbert $A$-module over itself equipped with the standard inner product $\langle a, b \rangle_A = ab^*$. Define the mapping $\phi_g$ by the formula $\phi_g(f) = g \cdot f$ for a fixed function $g(x) = \begin{cases} -2x + 1 & : x \leq 1/2 \\ 0 & : x \geq 1/2 \end{cases}$ and for every $f \in A$. Then $\text{Ker}(\phi_g)$ equals to the Hilbert $A$-submodule and (left) ideal $\{ f \in A : f(x) = 0 \text{ for } x \in [0,1/2] \}$, being not a direct summand of $A$, but nevertheless, coinciding with the bi-orthogonal complement of itself with respect to $A$.

Corollary 1 Let $A$ be a W*-algebra, $\mathcal{M}$ and $\mathcal{N}$ be two Hilbert $A$-modules and $\phi : \mathcal{M} \rightarrow \mathcal{N}$ be a bounded $A$-linear mapping. Then the kernel $\text{Ker}(\phi)$ of $\phi$ is a direct summand of $\mathcal{M}$ and has the property $\text{Ker}(\phi) = \text{Ker}(\phi)^\perp \perp$.

Proof: Consider the Hilbert $A$-module $\mathcal{K}$ formed as the direct sum $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$ equipped with the $A$-valued inner product $\langle .,. \rangle_{\mathcal{M}} + \langle .,. \rangle_{\mathcal{N}}$. The mapping $\phi$ can be identified with a bounded $A$-linear mapping $\phi'$ on $\mathcal{K}$ acting on the direct summand $\mathcal{M}$ as $\phi$ and on the direct summand $\mathcal{N}$ as the zero operator. Since the kernel of $\phi'$ is a direct summand of $\mathcal{K}$ containing $\mathcal{N}$ by Lemma 2 its orthogonal complement is a direct summand of $\mathcal{M}$. The desired result turns out. ⊗

Now we are in the position to give a description of the inner structure of arbitrary Hilbert W*-modules generalizing an analogous statement for self-dual Hilbert W*-modules by W. L. Paschke ([14, Th. 3.12]).

Proposition 1 Let $A$ be a W*-algebra and $\{ \mathcal{M}, \langle .,. \rangle \}$ be a left Hilbert A-module. Then $\mathcal{M}$ is the closure of a direct orthogonal sum of a family $\{ D_\alpha : \alpha \in I \}$ of norm-closed left ideals $D_\alpha \subseteq A$, where the closure of this direct sum is predetermined by the given on $\mathcal{M}$ A-valued inner product $\langle .,. \rangle$ and the A-valued inner products on the ideals are the standard A-valued inner product on $A$. Moreover, for every bounded $A$-linear mapping $r : \mathcal{M} \rightarrow A$ there is a net $\{ x_\beta : \beta \in J \}$ of elements of $\mathcal{M}$ for which the limit

$$\| . \|_A - \lim_{\beta \in J} \langle y, x_\beta \rangle$$

exists for every $y \in \mathcal{M}$ and equals $r(y)$. 
Proof: Fix an arbitrary bounded \( A \)-linear mapping \( r : \mathcal{M} \to A \). The kernel of \( r \) is a direct summand of \( \mathcal{M} \) by Corollary \([1]\). Consider its orthogonal complement. Since \( r \) can be continued to an bounded \( A \)-linear mapping \( r(\cdot) = \langle \cdot, x_r \rangle \) on the \( A \)-dual (self-dual) Hilbert \( A \)-module \( \mathcal{M}' \) of \( \mathcal{M} \) (\( x_r \in \text{ Ker}(r)^{\perp} \subseteq \mathcal{M}' \)) and since the orthogonal complement of the kernel of \( r \) inside \( \mathcal{M}' \) is a direct summand isomorphic to \( \{ Ap, \langle \cdot, \cdot \rangle \} \) for some projection \( p \in A \) by the structural theorem \([4]\), Th. 3.12] for self-dual Hilbert \( W^* \)-modules the orthogonal complement of the kernel of \( r \) with respect to \( \mathcal{M} \) is isomorphic to the Hilbert \( A \)-module \( \{ I, \langle \cdot, \cdot \rangle \} \) for some norm-closed left ideal \( I \subseteq Ap \) of \( A \), where the left-strict closure of the left ideal \( I \) is the \( W^* \)-closed ideal \( Ap \) of \( A \). Now, \( r \) can be identified with the element \( x_r \in Ap \), and \( x_r \in Ap \) is the left-strict limit of a net \( \{ x_{\beta} : \beta \in J \} \) of elements of \( I \cap \mathcal{M} \), cf. \([10], \S 3.12\).

Finally, by transfinite induction one has to decompose \( \mathcal{M} \) into a sum of pairwise orthogonal direct summands of type \( \text{ Ker}(r)^{\perp} \) for bounded \( A \)-linear functionals \( r \) on \( \mathcal{M} \), where \( \text{ Ker}(r)^{\perp} \) is always isomorphic to a left norm-closed ideal \( I \) of \( A \) with the standard \( A \)-valued inner product on it. •

We go on to investigate the image of bounded module mappings between Hilbert \( W^* \)-modules. In general, many quite non-regular things can happen as the example below shows, but embeddings of self-dual Hilbert \( W^* \)-modules into other Hilbert \( W^* \)-modules can be shown to be mappings onto direct summands in contrast to the situation for general Hilbert \( C^* \)-modules.

Example 2 Let \( A \) be the set of all bounded linear operators \( B(H) \) on a separable Hilbert space \( H \) with basis \( \{ e_i : i \in \mathbb{N} \} \). Denote by \( k \) the operator \( k(e_i) = \lambda_i e_i \) for a sequence \( \{ \lambda_i : i \in \mathbb{N} \} \) of non-zero positive real numbers converging to zero. Then the mapping

\[
\phi_k : A \to A , \quad \phi_k : a \to a \cdot k
\]

is a bounded \( A \)-linear mapping on the left projective Hilbert \( A \)-module \( A \). But the image is not a direct summand of this \( A \)-module and is not even Hilbert because direct summands of \( A \) are of the form \( Ap \) for some projection \( p \) of \( A \), and \( 1_A \cdot k \) should equal \( p \). The image of \( \phi_k \) is a subset of the set of all compact operators on \( H \). Note, that the mapping \( \phi_k \) is not injective.

The following proposition was proved for arbitrary \( C^* \)-algebras \( A \), countably generated Hilbert \( A \)-modules \( \mathcal{M}, \mathcal{N} \) without self-duality restriction and an injective bounded module mapping \( \alpha : \mathcal{M} \to \mathcal{N} \) with norm-dense range by H. Lin \([10]\), Th. 2.2]. We present another variant for a similar situation in the \( W^* \)-case.

Proposition 2 Let \( A \) be a \( W^* \)-algebra, \( \mathcal{M} \) be a self-dual Hilbert \( A \)-module and \( \{ \mathcal{N}, \langle \cdot, \cdot \rangle \} \) be another Hilbert \( A \)-module. Suppose, there exists an injective bounded module mapping
\[ \alpha : \mathcal{M} \to \mathcal{N} \text{ with the range property } \alpha(\mathcal{M})^{\perp} = \mathcal{N}. \text{ Then the operator } \alpha(\alpha^*\alpha)^{-1/2} \text{ is a bounded module isomorphism of } \mathcal{M} \text{ and } \mathcal{N}. \text{ In particular, they are isomorphic as Hilbert } A\text{-modules.} \]

**Proof:** The mapping \( \alpha \) possesses an adjoint bounded module mapping \( \alpha^* : \mathcal{N} \to \mathcal{M} \) because of the self-duality of \( \mathcal{M} \), cf. [12, Prop. 3.4]. Since \( \alpha^*\alpha \) is a positive element of the C*-algebra \( \text{End}_A(\mathcal{M}) \) of all bounded (adjointable) module mappings on the Hilbert \( A\)-module \( \mathcal{M} \) the square root of it, \( (\alpha^*\alpha)^{1/2} \), is well-defined by the series

\[
(\alpha^*\alpha)^{1/2} = \|\cdot\| - \lim_{n \to \infty} \|(\alpha^*\alpha)^{1/2}\|^{1/2} \left( \text{id}_\mathcal{M} - \sum_{k=1}^{n} \lambda_k \left( \text{id}_\mathcal{M} - \frac{(\alpha^*\alpha)}{\|(\alpha^*\alpha)\|} \right)^k \right)
\]

with coefficients \( \{\lambda_k\} \) taken from the Taylor series at zero of the complex-valued function \( f(x) = \sqrt{1-x} \) on the interval \([0,1]\). Moreover, because

\[
\langle (\alpha^*\alpha)^{1/2}(x), (\alpha^*\alpha)^{1/2}(x) \rangle = \langle \alpha(x), \alpha(x) \rangle
\]

and because of the injectivity of \( \alpha \) the mapping \( (\alpha^*\alpha)^{1/2} \) has trivial kernel. At the contrary one can only say that the range of \( (\alpha^*\alpha)^{1/2} \) is \( \tau_1 \)-dense in \( \mathcal{M} \), (cf. [3]). Indeed, for every \( A\)-linear bounded functional \( r(\cdot) = \langle \cdot, y \rangle \) on the self-dual Hilbert \( A\)-module \( \mathcal{M} \) mapping the range of \( (\alpha^*\alpha)^{1/2} \) to zero one has

\[
0 = \langle (\alpha^*\alpha)^{1/2}(x), y \rangle = \langle x, (\alpha^*\alpha)^{1/2}(y) \rangle
\]

for every \( x \in \mathcal{M} \). Hence, \( y = 0 \) since \( (\alpha^*\alpha)^{1/2} \) is injective and \( x \in \mathcal{M} \) was arbitrarily chosen.

Now, consider the mapping \( \alpha(\alpha^*\alpha)^{-1/2} \) where it is defined on \( \mathcal{M} \). Since \( (\alpha^*\alpha)^{1/2} \) has both \( \tau_1 \)-dense range and trivial kernel by the assumptions on \( \alpha \) its inverse unbounded module operator \( (\alpha^*\alpha)^{-1/2} \) is \( \tau_1 \)-densely defined. One obtains

\[
\langle \alpha(\alpha^*\alpha)^{-1/2}(x), \alpha(\alpha^*\alpha)^{-1/2}(y) \rangle = \langle x, y \rangle
\]

for every \( x, y \) from the \( (\tau_1 \text{-dense}) \) area of definition of \( (\alpha^*\alpha)^{-1/2} \). Consequently, the operator \( \alpha(\alpha^*\alpha)^{-1/2} \) continues to a bounded isometric module operator on \( \mathcal{M} \) by \( \tau_1 \)-continuity. The range of it is \( \tau_1 \)-closed (i.e., a self-dual direct summand of \( \mathcal{N} \)) and hence, equals \( \mathcal{N} \) by assumption. Finally, since the range of \( (\alpha^*\alpha)^{-1/2} \) is norm-closed and \( \tau_1 \)-dense in \( \mathcal{M} \) and since \( \mathcal{M} \) is self-dual the mapping \( \alpha \) is a (non-isometric, in general) Hilbert \( A\)-module isomorphism itself. \( \bullet \)

**Corollary 2** Let \( A \) be a \( \text{W}^*\)-algebra, \( \mathcal{M} \) be a self-dual Hilbert \( A\)-module and \( \{\mathcal{N}, \langle \cdot, \cdot \rangle\} \) be another Hilbert \( A\)-module. Every injective module mapping from \( \mathcal{M} \) into \( \mathcal{N} \) is a Hilbert \( A\)-module isomorphism of \( \mathcal{M} \) and of a direct summand of \( \mathcal{N} \).
For our application in §3 we need the following partial result:

**Corollary 3** Let $A$ be a $W^*$-algebra, $\mathcal{M}$ and $\mathcal{N}$ be countably generated Hilbert $A$-modules and $F : \mathcal{M} \to \mathcal{N}$ be a Fredholm operator (see [13]). Then $\ker F$ and $(\operatorname{Im} F)^\perp$ are projective finitely generated $A$-submodules, and $\operatorname{Ind} F = [\ker F] - [(\operatorname{Im} F)^\perp]$ inside $K_0(A)$.

**Proof:** We denote by $\hat{\oplus}$ the direct orthogonal sum of two Hilbert $A$-modules, whereas $\oplus$ denotes the direct topological sum of two Hilbert $A$-submodules of a given Hilbert $A$-module, where orthogonality of the two components is not required. Let $\mathcal{M} = M_0 \hat{\oplus} M_1$, $\mathcal{N} = N_0 \oplus N_1$ be the decompositions from the definition of $A$-Fredholm operator:

$$F = \left( \begin{array}{cc} F_0 & 0 \\ 0 & F_1 \end{array} \right) : \mathcal{M}_0 \hat{\oplus} \mathcal{M}_1 \to N_0 \oplus N_1,$$

$F_0 : \mathcal{M}_0 \cong N_0$, $F_1 : \mathcal{M}_1 \to N_1$, $\mathcal{M}_1$ and $\mathcal{N}_1$ are the projective finitely generated modules. Let $x = x_0 + x_1$, $x_0 \in \mathcal{M}_0$ and $x_1 \in \mathcal{M}_1$, and $F(x) = 0$, so $0 = F_0(x_0) + F_1(x_1) \in N_0 \oplus N_1$. Thus $F_0(x_0) = 0$, $F_1(x_1) = 0$, so $x_0 = 0$ and $x \in \mathcal{M}_1$. Thus $\ker F = \ker F_1 \subset \mathcal{M}_1$. By Lemma 2 $\ker F$ is a projective finitely generated $A$-module and has an orthogonal complement. So, by Corollary 2

$$F = \left( \begin{array}{ccc} F_0 & 0 & 0 \\ 0 & F_1 & 0 \end{array} \right) : \mathcal{M}_0 \hat{\oplus} \mathcal{M}_1 \oplus \ker F \to \left( N_0 \oplus \overline{F(M_1)} \right) \hat{\oplus} (\operatorname{Im} F)^\perp$$

and $\operatorname{Ind} F = [\ker F] - [(\operatorname{Im} F)^\perp]$. •

The following example shows that the situations may be quite different for general Hilbert $C^*$-modules and injective mappings between them:

**Example 3** Consider the $C^*$-algebra $A = C([0,1])$ of all continuous functions on the interval $[0,1]$ as a self-dual Hilbert $A$-module over itself equipped with the standard $A$-valued inner product $\langle a, b \rangle_A = ab^*$. The mapping $\phi : f(x) \to x \cdot f(x)$, $(x \in [0,1])$, is an injective bounded module mapping. Its range has trivial orthogonal complement, but it is not closed in norm and, consequently, not a direct summand of $A$. Nevertheless, the bi-orthogonal complement of the range of $\phi$ with respect to $A$ equals $A$.

**Lemma 3** Let $A$ be a $W^*$-algebra. Let $\mathcal{P}$ and $\mathcal{Q}$ be self-dual Hilbert $A$-submodules of a Hilbert $A$-module $\mathcal{M}$. Then $\mathcal{P} \cap \mathcal{Q}$ is a self-dual Hilbert $A$-module and direct summand of $\mathcal{M}$. Moreover, $\mathcal{P} + \mathcal{Q} \subseteq \mathcal{M}$ is a self-dual Hilbert $A$-submodule.

If $\mathcal{P}$ is projective and finitely generated then the intersection $\mathcal{P} \cap \mathcal{Q}$ is projective and finitely generated, too. If both $\mathcal{P}$ and $\mathcal{Q}$ are projective and finitely generated then the sum $\mathcal{P} + \mathcal{Q}$ is also.
Proof. Let $p : \mathcal{M} = \mathcal{P} \oplus \mathcal{P}^\perp \to \mathcal{P}^\perp$ be the canonical orthogonal projection existing by [3, Th. 2.8], (cf. [4] for the projective case). Let $p_Q = p : Q \to \mathcal{P}$. Since $Q$ is a self-dual Hilbert $A$-module $p_Q$ admits an adjoint operator and $\text{Ker}p_Q \subseteq Q$ is a direct summand by Lemma 2. Consequently, it is a self-dual Hilbert $A$-submodule of $Q \subseteq \mathcal{M}$. But $\text{Ker}p_Q = \mathcal{P} \cap Q$. To derive the second assertion one has to apply the fact again that every self-dual Hilbert $A$-submodule is a direct summand, cf. [3].

If $\mathcal{P}$ is projective and finitely generated then every direct summand of it is projective and finitely generated, what shows the additional remarks.

## 3 Lefschetz numbers

Throughout this section $A$ denotes a $W^*$-algebra. This restriction enables us to apply the results of the previous section being valid only in the $W^*$-case, in general.

Let $U$ be a unitary operator in the projective finitely generated Hilbert $A$-module $\mathcal{P}$. Then

$$U = \int_{S^1} e^{i\varphi} dP(\varphi),$$

(1)

where $P(\varphi)$ is the projection valued measure valued in the $W^*$-algebra of all bounded (adjointable) module operators on $\mathcal{P}$, and the integral converges with respect to the norm. So we have a bounded and measurable function

$$L(\mathcal{P}, U) : S^1 \to K_0(A), \varphi \mapsto [dP(\varphi)],$$

(2)

This function is bounded in the sense that there exists a projection which is greater than all values with respect to the partial order in the space of projections. Let us denote the set of such functions by $K_0(A)_S$.

Let us note that the Lefschetz numbers for compact group action considered in [19] can be thought of as evaluated (for unitary representation) in the subspace of finitely valued (simple) functions:

$$\text{Simple}(S^1, K_0(A)) \subset K_0(A)_S.$$

Suppose, $\mathcal{P} = A^n$. In the case of $L(\mathcal{P}, U) \in \text{Simple}(S^1, K_0(A))$ associate with the integral (1)

$$\int_{S^1} e^{i\varphi} dP(\varphi) = \sum_k e^{i\varphi_k} P(\mathcal{E}_k)$$

the following class of the cyclic homology $HC_{2l}(M(n, A))$:

$$\sum_k P(\mathcal{E}_k) \otimes \ldots \otimes P(\mathcal{E}_k) \cdot e^{i\varphi_k}.$$

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Passing to the limit we get the following element

\[ \tilde{T}U = \int_{S^1} e^{i\varphi} d(P \otimes \ldots \otimes P)(\varphi) \in HC_{2l}(M(n, A)). \]

Then we define

\[ T(U) = \text{Tr}_n^*(\tilde{T}U) \in HC_{2l}(A), \]

where \( \text{Tr}_n^* \) is the trace in cyclic homology.

**Lemma 4 (\cite[Lemma 6.1]{19})**

Let \( J : \mathcal{M} = A^m \rightarrow \mathcal{N} = A^n \) be an isomorphism, \( U_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}, U_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N} \) be \( A \)-unitary operators and \( JU_{\mathcal{M}} = U_{\mathcal{N}}J \). Then

\[ T(U_{\mathcal{M}}) = T(U_{\mathcal{N}}). \]

Similar techniques can be developed for a projective finitely generated \( A \)-module \( \mathcal{N} \) instead of \( A^n \). For this purpose we take \( \mathcal{N} = q(A^n) \), where \( q \) denotes the orthogonal projection from \( A^n \) onto its direct orthogonal summand \( \mathcal{N} \). Then we set

\[ U \oplus 1 : A^n \cong \mathcal{N} \oplus (1-q)A^n \rightarrow \mathcal{N} \oplus (1-q)A^n \cong A^n, \]

\[ \tilde{T}U = \int_{S^1} e^{i\varphi} d(qPq \otimes \ldots \otimes qPq)(\varphi). \]

The correctness is an immediate consequence of the Lemma 4.

Let us consider an \( A \)-elliptic complex \((E, d)\) and its unitary endomorphism \( U \). The results of §1 (cf. Prop. 2, Lemma 3, Lemma 3) and the standard Hodge theory argument help us to prove the following lemma.

**Lemma 5** For the \( A \)-Fredholm operator

\[ F = d + d^* : \Gamma(E_{ev}) \rightarrow \Gamma(E_{od}), \]

we have

\[ \text{Ker} \left( F|_{\Gamma(E_{ev})} \right) \overset{\text{def}}{=} H_{ev}(E) = \oplus H_{2i}(E), \]

\[ \text{Ker} \left( F|_{\Gamma(E_{od})} \right) \overset{\text{def}}{=} H_{od}(E) = \oplus H_{2i+1}(E), \]

where \( H_m(E) \) is the orthogonal complement to \( \text{Im} d \subset \text{Ker} d \subset \Gamma(E_m) \) and \( H_m(E) \) are projective \( U \)-invariant Hilbert \( A \)-modules.
Proof. For \( u_{2i} \in \Gamma(E_{2i}) \) while
\[
(d + d^*)(u_0 + u_2 + u_4 + \ldots) = 0
\]
we have
\[
du_0 + d^*u_2 = 0, \quad du_2 + d^*u_4 = 0, \ldots
\]
Together with the equality
\[
(du, d^*v) = (d^2u, v) = 0
\]
one obtains
\[
du_0 = 0, \quad du_2 = 0, \ldots ; d^*u_2 = 0, \quad d^*u_4 = 0, \ldots
\]
what implies \( u_{2i} \in \text{Ker} (d + d^*) \). On the other hand for \( v_2 \in \text{Im} \ d \), \( v_2 = dv_1 \) we have
\[
(v_2, u_2) = (dv_1, u_2) = (v_1, d^*u_2) = 0.
\]
Thus \( u_{2i} \in H_2i(E) \). Conversely, let \( u = u_0 + u_2 + \ldots, \ u_{2i} \in H_2i(E), \ i.e. du_{2i} = 0, \ (i = 0, 1, 2, \ldots) \), and for any \( v_{2i-1} \in E_{2i-1} \) we have
\[
(dv_{2i-1}, u_{2i}) = 0, \quad (v_{2i-1}, d^*u_{2i}) = 0,
\]
so \( d^*u_{2i} = 0 \). Thus \( u \in \text{Ker} \ (d + d^*) \). The invariance and projectivity follow from the proved identification and Corollary 3.

Definition 1 We define the Lefschetz number \( L_1 \) as
\[
L_1(\mathcal{E}, U) = \sum_i (-1)^i T(U|H_i(\mathcal{E})) \in K_0(A)_S.
\]

Definition 2 We define the Lefschetz number \( L_{2i} \) as
\[
L_{2i}(\mathcal{E}, U) = \sum_i (-1)^i T(U|H_i(\mathcal{E})) \in HC_{2i}(A).
\]

After all the following theorem is evident:

Theorem 1 Let the Chern character \( \text{Ch} \) be defined as in [1, 6, 7]. Then
\[
L_{2i}(\mathcal{E}, U) = \int_{S^1} (\text{Ch}^0_{2i})(L_1(\mathcal{E}, U))(\varphi) \, d\varphi.
\]

Remark 1 In situations, when the endomorphism \( V \) of the elliptic C*-complex represents as an element of a represented there amenable group \( G \) acting on the C*-complex then the \( A \)-valued inner products can be chosen \( G \)-invariant, what gives us the unitarity of \( V \) (see [1]). However, there is another obstruction demanding new approaches which will be shown at Example 4 below.
4 Obstructions in the C*-case and related topics

The aim of this chapter is to show some obstructions arising in the general Hilbert C*-module theory for more general C*-algebras than W*-algebras which cause the made restriction of the investigations in section three. The results underline the outstanding properties of Hilbert W*-modules. To handle the general C*-case we often need a basic construction introduced by W. L. Paschke and H. Lin. It gives a link between the W*-case and the general C*-case.

Remark 2 (cf. [4, Def. 1.3], [4, §4])

Let \( \{M, \langle \cdot, \cdot \rangle \} \) be a left pre-Hilbert \( A \)-module over a fixed C*-algebra \( A \). The algebraic tensor product \( A^{**} \otimes M \) becomes a left \( A^{**} \)-module defining the action of \( A^{**} \) on its elementary tensors by the formula \( ab \otimes h = a(b \otimes h) \) for \( a, b \in A^{**}, h \in M \). Now, setting

\[
\sum_{i} a_i \otimes h_i, \sum_{j} b_j \otimes g_j = \sum_{i,j} a_i \langle h_i, g_j \rangle b_j
\]

on finite sums of elementary tensors one obtains a degenerate \( A^{**} \)-valued inner pre-product. Factorizing \( A^{**} \otimes M \) by \( N = \{ z \in A^{**} \otimes M : \langle z, z \rangle = 0 \} \) one obtains a pre-Hilbert \( A^{**} \)-module denoted by \( M^\# \) in the sequel. It contains \( M \) as a \( A \)-submodule. If \( M \) is Hilbert then \( M^\# \) is Hilbert, and vice versa. The transfer of the self-duality is more difficult. If \( M \) is self-dual then \( M^\# \) is self-dual, too. But,

**Problem.** Suppose, the underlying C*-algebra \( A \) is unital. Whether the property of \( M^\# \) to be self-dual implies that \( M \) was already self-dual?

Other standard properties like e.g. C*-reflexivity can not be transferred. But every bounded \( A \)-linear operator \( T \) on \( M \) has a unique extension to a bounded \( A^{**} \)-linear operator on \( M^\# \) preserving the operator norm, (cf. [4, Def. 1.3]).

**Proposition 3** Let \( A \) be a C*-algebra, \( M \) and \( N \) be two Hilbert \( A \)-modules and \( \phi : M \rightarrow N \) be a bounded \( A \)-linear mapping. Then the kernel \( \text{Ker}(\phi) \) of \( \phi \) coincides with its bi-orthogonal complement inside \( M \). In general, it is not a direct summand.

**Proof:** Let us assume, \( \text{Ker}(\phi) \neq \text{Ker}(\phi)^{\perp\perp} \) with respect to the \( A \)-valued inner product of \( M \). Form the direct sum \( L = M \oplus N \). The mapping \( \phi \) extends to a bounded \( A \)-linear mapping \( \psi \) on \( L \) setting

\[
\psi(x) = \begin{cases} 
\phi(x) & : x \in M \\
0 & : x \in N
\end{cases}
\]

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Extend ψ further to a bounded $A^{**}$-linear operator on the correspondent Hilbert $A^{**}$-module $L^\#$. By Lemma 2 the sets Ker(φ)# and (Ker(φ)⊥⊥)# both are contained in the kernel Ker(ψ) of ψ, which is a direct summand of $L^#$ and fulfils Ker(ψ) = Ker(ψ)⊥⊥. This contradicts the assumption.

The second assertion follows from Example 1.

**Corollary 4** Let $A$ be a $C^*$-algebra and $\{M, ⟨.,.⟩\}$ be a Hilbert $A$-module. The kernel Ker(r) of every bounded module mapping $r : M \rightarrow A$ coincides with its bi-orthogonal complement inside $M$, but it is not a direct summand, in general.

**Corollary 5** Let $A$ be a $C^*$-algebra and $\{M, ⟨.,.⟩\}$ be a Hilbert $A$-module. Suppose, there exists a bounded module mapping $r : M \rightarrow A$ with the property $\text{Ker}(r) = \{0\}$. Then $r$ is the zero mapping.

**Lemma 6** Let $A$ be a $C^*$-algebra and $\{M, ⟨.,.⟩\}$ be a (left) Hilbert $A$-module. For every bounded module mapping $r : M \rightarrow A$ the subset $\text{Ker}(r)^{\perp} \subseteq M$ is a direct summand of $M$ isomorphic to a (left) norm-closed ideal of $A$ as a (left) Hilbert $A$-module.

Proof: By Corollary 4 the set Ker(r)^{\perp} \subseteq M can be assumed to be non-zero, in general. Again, form the Hilbert $A^{**}$-module $M^#$ and extend $r$ to a bounded $A^{**}$-linear mapping $r'$ on it. The kernel of $r'$ is a direct summand of $M^#$ isomorphic to a (left) norm-closed ideal of $A^{**}$ as a Hilbert $A^{**}$-module by Corollary 4 and Proposition 4. Consequently, Ker(r) ⊆ Ker(r')∩M ⊆ M^# is isomorphic to a (left) norm-closed ideal $D$ of $A$ as a (left) Hilbert $A$-module.

We want to get a structure theorem on the interrelation of Hilbert $C^*$-modules and their $C^*$-dual Banach $C^*$-modules. To obtain the full picture define a new topology on (left) Hilbert $C^*$-modules in analogy to the (right) strict topology on $C^*$-algebras $A$:

**Definition 3** Let $A$ be a $C^*$-algebra and $\{M, ⟨.,.⟩\}$ be a (left) Hilbert $A$-module. A norm-bounded net $\{x_\alpha : \alpha \in I\}$ of elements of $M$ is fundamental with respect to the right $*$-strict topology if and only if the net $\{⟨y, x_\alpha⟩ : \alpha \in I\}$ is a Cauchy net with respect to the norm topology on $A$ for every $y \in M$. The net $\{x_\alpha : \alpha \in I\}$ converges to an element $x \in M$ with respect to the right $*$-strict topology if and only if

$$\lim_{\alpha \in I} \|⟨y, x - x_\alpha⟩\|_A = 0$$

for every $y \in M$. 

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Theorem 2 Let $A$ be a $C^*$-algebra and $\{\mathcal{M}, \langle \cdot, \cdot \rangle_A\}$ be a (left) Hilbert $A$-module. The following conditions are equivalent:

(i) $\mathcal{M}$ is self-dual.

(ii) The unit ball of $\mathcal{M}$ is complete with respect to the right $\ast$-strict topology.

Moreover, the linear hull of the completed with respect to the right $\ast$-strict topology unit ball of $\mathcal{M}$ coincides with the $A$-dual Banach $A$-module $\mathcal{M}'$ of $\mathcal{M}$.

Proof: First, let us show the equivalence $(i) \Leftrightarrow (ii)$. Suppose the unit ball of $\mathcal{M}$ is complete with respect to the right $\ast$-strict topology. Consider an arbitrary non-trivial bounded module mapping $r : \mathcal{M} \to A$ of norm one. Restrict the attention to the non-zero Hilbert $A$-submodule $\text{Ker}(r)^\perp \subseteq \mathcal{M}$ being isomorphic as a Hilbert $A$-module to a norm-closed (left) ideal $D$ of $A$ equipped with the standard $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ by Lemma 6. By [3, Th. 3.2] there exist nets $\{x_\alpha : \alpha \in I\} \subset \text{Ker}(r)^\perp$ bounded in norm by one such that $\tau_2 - \lim_{\alpha \in I} x_\alpha = r$ inside the self-dual Hilbert $A^{**}$-module $((\text{Ker}(r)^\perp)^\#)'$. But, the values $r(y), y \in \text{Ker}(r)^\perp$, all belong to $A$ and, in particular, to the set of all right multipliers of the $C^*$-subalgebra and two-sided ideal $B = \langle \text{Ker}(r)^\perp, \text{Ker}(r)^\perp \rangle$ of $A$.

Therefore, there exists a special net $\{x_\alpha : \alpha \in I\} \subset \text{Ker}(r)^\perp$ such that

$$\|b\mathcal{M} - \lim_{\alpha \in I} b(\langle y, x_\alpha \rangle - r(y)) = 0$$

for every $y \in A$, every $b \in B$, cf. [6, §3.12]. Since the set $\{by : b \in B, y \in \text{Ker}(r)^\perp\}$ is norm-dense in $\text{Ker}(r)^\perp$ one implication is shown. The opposit one follows from the formula

$$r(y) = \|y\|_A - \lim_{\alpha \in I} \langle y, x_\alpha \rangle, \ y \in \mathcal{M},$$

defining a bounded module mapping $r : \mathcal{M} \to A$ for every norm-bounded fundamental with respect to the right $\ast$-strict topology net $\{x_\alpha : \alpha \in I\} \in \mathcal{M}$. By the way one has proved the conclusion that the $A$-dual Banach $A$-module $\mathcal{M}'$ of every Hilbert $A$-module $\mathcal{M}$ arises as the linear hull of the completed with respect to the right $\ast$-strict topology unit ball of $\mathcal{M}$.

Corollary 6 Let $A$ be a $C^*$-algebra and $D$ be a norm-closed (left) ideal of $A$. Then $\{D, \langle \cdot, \cdot \rangle_A\}$ is self-dual if and only if there is a projection $p \in A$ such that $D \equiv Ap$ and $p \in D$.

Proof: If $D$ is self-dual then the identical embedding of $D$ into $A$ is a bounded $A$-linear mapping. It must be represented by an element $p \in D$ with the property $dp^* = d$ for every $d \in D$. That is, $pp^* = p \in D$ is positive and idempotent. The functional property of the mapping $p$ gives the structure of $D$ as $D \equiv Ap$.

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Theorem 3 Let $A$ be a $C^*$-algebra and $\{M, \langle ., . \rangle \}$ be a (left) Hilbert $A$-module. The following conditions are equivalent:

1. $M$ is $A$-reflexive.

2. Every norm bounded net $\{x_\alpha : \alpha \in I\}$ of elements of $M$ for which all the nets $\{r(x_\alpha) : \alpha \in I\}$, $(r \in M')$, converge with respect to $\|\|_A$ has its limit $x$ inside $M$.

Moreover, the linear hull of the completed with respect to this topology unit ball of $M$ coincides with the $A$-bidual Banach $A$-module $M''$ of $M$.

Proof: Suppose $M$ is not self-dual because otherwise one simply refers to Theorem 2. Obviously, the linear hull of the completion of the unit ball of $M$ with respect to this topology is a Banach $A$-module $N$. Continue the $A$-valued inner product from $M$ to $N$ by the rule

$$\langle x, y \rangle = \lim_{\alpha \in I} \langle x_\alpha, y \rangle$$

for every element $\langle ., y \rangle \in M'$, where $y \in M$. Since the net converges with respect to the right $^*$-strict topology on $M$, too, the limit $x$ can be interpreted as an $A$-linear bounded functional on $M$. This lets to the definition of the value $\langle x, x \rangle$ in the same manner. Consequently, $N$ is a Hilbert $A$-module containing $M$ as a Hilbert $A$-submodule and possessing the same $A$-dual Banach $A$-module $M' \equiv N'$. (Cf. [15] for similar constructions.) Moreover, the unit ball of $N$ is complete with respect to the new topology. Since the $A$-valued inner product on $M$ can be continued to an $A$-valued inner product on $M'' \equiv N''$ by [13, Th. 2.4] every element of $M''$ can be described in this way, and $N$ is $A$-reflexive.

Example 4 Consider the $C^*$-algebra $A = C([0,1])$ of all continuous functions on the unit interval as a Hilbert $A$-module over itself. Let $U$ be defined as

$$U(f)(t) = e^{it}f(t) , \ t \in [0, 1] ,$$

a unitary operator. Take this unitary operator as the generator of a unitary representation of the amenable abelian group $\mathbf{Z}$. All complex irreducible representations of $\mathbf{Z}$ are one-dimensional. If we would like to apply A. S. Mishchenko’s theorem in this case then we would have to have a finite spectrum for the generator $U$ of the representation what is not the case. Beside this, the only projections inside $A$ and, therefore, the only self-adjoint idempotent module operators on $A$ are $1_A$ and $0_A$, and there exists no spectral decomposition of elements and no non-trivial direct $A$-module summand inside $A$. 

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Remark 3 As it is known in all sufficient cases the morphism $S$ gives an isomorphism of $HC_2(A)$ and $HC_0(A)$ and we can work only with the second group. In this situation we can define the Lefschetz number $L_0 \in HC_0(A)$ as in [18] for general C*-algebras $A$. But for $K$-groups valued numbers even in the case of an action of an e.g. amenable group $G$ (see Example [3]) we need some kind of infinitness and convergence, so we have to pass to $K_0(A)_S$. The natural expression of this infinitness of eigenvalues is the spectral decomposition, so we have to work with W*-algebras, at least for $L_1$. The crucial moment is that in this situation there is no theorem like [12].

Surely this argument is quite unexplicite and we have a chance for refinement e.g. for the monotone complete C*-algebras. But, the techniques for the monotone complete case are rather complicated and the results do only differ slightly from that of the W*-case, cf. [4].

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