On the $h$-adic Quantum Vertex Algebras Associated with Hecke Symmetries

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Abstract: We study the quantum vertex algebraic framework for the Yangians of RTT-type and the braided Yangians associated with Hecke symmetries, introduced by Gurevich and Saponov. First, we construct several families of modules for the aforementioned Yangian-like algebras which, in the RTT-type case, lead to a certain $h$-adic quantum vertex algebra $V_c(R)$ via the Etingof–Kazhdan construction, while, in the braided case, they produce ($\phi$-coordinated) $V_c(R)$-modules. Next, we show that the coefficients of suitably defined quantum determinant can be used to obtain central elements of $V_c(R)$, as well as the invariants of such ($\phi$-coordinated) $V_c(R)$-modules. Finally, we investigate a certain algebra which is closely connected with the representation theory of $V_c(R)$.

1. Introduction

The Yangians for classical Lie algebras present an important class of quantum groups which goes back to Drinfeld [2]. Before the introduction of this family of Hopf algebras, the algebra structure of the Yangian for the general linear Lie algebra $\mathfrak{gl}_N$ was already intensively studied in the context of inverse scattering method; see, e.g., the papers by Takhtajan–Faddeev [33], Kulish–Sklyanin [25] and Tarasov [34,35]. It is a canonical deformation of the universal enveloping algebra of the infinite-dimensional Lie algebra $\mathfrak{gl}_N[t]$. Moreover, its presentation can be given in terms of single ternary defining relation on the matrix generators, the so-called RTT-relation, which employs the Yang $R$-matrix. For more information on the Yangians for classical Lie algebras see the book by Molev [28].

In this paper, we consider two classes of generalizations of the Yangian for $\mathfrak{gl}_N$, which were introduced by Gurevich and Saponov [13], the Yangians of RTT-type $\mathcal{Y}_{RTT}(R)$ and the braided Yangians $\mathcal{Y}(R)$. They are defined in terms of Yangian-like RTT-relations, which, instead of the Yang $R$-matrix, employ different $R$-matrices obtained by the so-called Baxterization procedure from the involutive and Hecke symmetries. Such algebras exhibit properties similar to Yangians and have important applications to Gaudin-type
models; see [13,14]. For simplicity, we consider only the (skew-invertible) Hecke symmetry case, although we briefly discuss the major differences which occur for involutions as well.

The goal of this paper is to associate the quantum vertex algebra theory with the aforementioned classes of Yangian-like algebras. Our main motivation is the rich interplay, which appears in the classical theory, between representations of vertex algebras and infinite-dimensional Lie algebras, see, e.g., the books by Frenkel and Ben-Zvi [6], Frenkel, Lepowsky and Meurman [7] and Kac [20]. Hopefully, in analogy with the classical case, construction of such new examples of quantum vertex algebras and their applications might lead to the better understanding of the general theory. For more information on the theory of quantum vertex algebras see, e.g., the papers by Etingof and Kazhdan [5], Li [26,27], De Sole, Gardini and Kac [1] and references therein.

Both classes of the aforementioned generalizations of the Yangian for \( \mathfrak{gl}_N \) from [13] can be associated with the \( R \)-matrix with additive and with multiplicative spectral parameter (i.e., more precisely, which satisfies the additive or the multiplicative version of the quantum Yang–Baxter equation). In the additive (resp. multiplicative) case we denote these algebras by \( \mathcal{Y}_{RTT}(R) \) and \( \mathcal{Y}(R) \) (resp. \( \mathcal{Y}_{RTT}(\overline{R}) \) and \( \mathcal{Y}(\overline{R}) \)). In all such cases, we construct certain \( \mathbb{C}[[h]] \)-modules generated by the quantum analogues of creation operators, such that they are equipped with a certain Yangian-like action which resembles the annihilation operators; see Propositions 3.1, 4.2, 5.2 and 5.5.

In the case of algebra \( \mathcal{Y}_{RTT}(R) \) associated with the additive \( R \)-matrix, this is essentially the Etingof–Kazhdan construction [5], so that we obtain a certain \( h \)-adic quantum vertex algebra \( \mathcal{V}_c(R) \), where \( c \in \mathbb{C} \) and \( R \) is the corresponding skew-invertible Hecke-symmetry; see Theorem 3.4. However, in contrast with [5], the underlying braiding no longer needs to be of type \( 1 + \mathcal{O}(h) \), due to the different form of the \( R \)-matrix. On the other hand, the case of braided Yangian for the additive \( R \)-matrix leads to a family of \( \mathcal{V}_c(R) \)-modules; see Theorem 4.7.

As for the multiplicative \( R \)-matrix, the actions of the corresponding algebras \( \mathcal{Y}_{RTT}(\overline{R}) \) and \( \mathcal{Y}(\overline{R}) \) produce families of \( \phi \)-coordinated modules for \( \mathcal{V}_c(R) \); see Theorems 5.8 and 5.9. This is a certain class of structures in the vertex algebra theory introduced by Li [27] in order to establish the connection with quantum groups; see also the more recent paper [18] for more information on the theory of \( \phi \)-coordinated modules. Furthermore, for the multiplicative algebra \( \mathcal{Y}_{RTT}(\overline{R}) \), we introduce in Sect. 5.4 a certain new algebra \( \mathbb{D}(\overline{R})_c \) via the defining relations resembling the RTT-presentation of the quantum affine algebra in type A. We show that a certain wide family of its modules is naturally equipped with the structure of \( \phi \)-coordinated \( \mathcal{V}_c(R) \)-module.

In Sect. 6, motivated by the constructions from [13] and using the properties of the skew-symmetrizer established therein, we study the quantum determinant, a certain formal power series in \( \mathcal{V}_c(R)[[z]] \). We show that, under certain assumptions on the Hecke symmetry \( R \), its coefficients belong to the center of the \( h \)-adic quantum vertex algebra \( \mathcal{V}_c(R) \). Moreover, their images with respect to the \( \phi \)-coordinated module maps lead to the \( \phi \)-coordinated module invariants. Finally, we apply such construction to the algebra \( \mathbb{D}(\overline{R})_c \), thus obtaining a family of elements of its center.

2. Preliminaries

In this section, we recall some basic properties of skew-invertible Hecke symmetries and their \( R \)-matrices. All concepts are introduced over the commutative ring \( \mathbb{C}[[h]] \) so that
they are compatible with the $h$-adic quantum vertex algebra theory. They are obtained from the usual notions, defined over the field $\mathbb{C}(q)$, by setting $q = e^h$.

2.1. Hecke symmetries. Let $N \geq 2$ be an integer. Denote by $I$ and $P$ the identity and the permutation operator on $\mathbb{C}^N \otimes \mathbb{C}^N$, respectively,

$$I = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} \quad \text{and} \quad P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji},$$

where $e_{ij}$ are matrix units. Let $h$ be a formal parameter and $\mathbb{C}[[h]]$ the commutative ring of formal Taylor series in $h$. Define a Hecke symmetry over $\mathbb{C}[[h]]$ as an element of $\text{End} \; \mathbb{C}^N \otimes \text{End} \; \mathbb{C}^N[[h]]$ which satisfies the braid relation

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}$$

(2.2)

and the condition

$$\left( R - e^h I \right) \left( R + e^{-h} I \right) = 0.$$  

(2.3)

In the braid relation we use the standard tensor notation, where for any $A = \sum A_1 \otimes A_2$ in $\text{End} \; \mathbb{C}^N \otimes \text{End} \; \mathbb{C}^N[[h]]$ and distinct indices $1 \leq r, s \leq m$ we write $A_{rs}$ for the element which acts as $A_1$ (resp. $A_2$) on the $r$-th (resp. $s$-th) tensor copy of $(\mathbb{C}^N)^{\otimes m}$, i.e. we have

$$A_{rs} = \sum \left( 1^{\otimes (r-1)} \otimes A_1 \otimes 1^{\otimes (m-r)} \right) \left( 1^{\otimes (s-1)} \otimes A_2 \otimes 1^{\otimes (m-s)} \right).$$

In particular, in (2.2) we have $m = 3$ and $(r, s) \in \{(1, 2), (2, 3)\}$. Regarding the identity in (2.3), the term $e^{\pm h}$ stands for the formal power series $\sum_{k \geq 0} (\pm h)^k/k! \in \mathbb{C}[[h]]$. Clearly, the identity implies that $R$ is invertible and, furthermore, that its inverse is given by $R^{-1} = R - (e^h - e^{-h})I$. Thus the constant term of $R$ with respect $h$, i.e. $R|_{h=0}$ is nonzero.

The Hecke symmetry $R$ is said to be skew-invertible if there exists an element $\Psi$ in $\text{End} \; \mathbb{C}^N \otimes \text{End} \; \mathbb{C}^N[[h]]$ such that

$$\text{tr}_2 R_{12} \Psi_{23} = P = \text{tr}_2 \Psi_{12} R_{23},$$

(2.4)

where the trace is taken over the second tensor factor. As with $R$, this implies that $\Psi|_{h=0}$ is nonzero. Throughout this paper, we consider only skew-invertible Hecke symmetries. One well-known example of such symmetry is given by (cf. [3,17])

$$R = \sum_{i,j=1}^{N} e^{\delta_{ij}h} e_{ij} \otimes e_{ji} + (e^h - e^{-h}) \sum_{i < j} e_{ii} \otimes e_{jj}. $$

(2.5)

It can be easily checked that it satisfies the skew-invertibility condition (2.4) with

$$\Psi = \sum_{i,j=1}^{N} e^{-\delta_{ij}h} e_{ij} \otimes e_{ji} - (e^h - e^{-h}) \sum_{i < j} e^{2(i-j)h} e_{ii} \otimes e_{jj}. $$

Observe that the evaluation of $R$ at $h = 0$ equals the permutation operator $P$, which is an example of skew-invertible involutive symmetry, i.e. it is an involution which satisfies the braid relation (2.2) and the skew-invertibility condition (2.4) (with $\Psi = P$).
In this paper, we shall often use the ordered product notation, where the subscript of the product symbol determines the order of tensor factors. More precisely, for any elements \( A = \sum_i A_i^{(i)} \otimes A_2^{(i)} \) and \( B = \sum_j B_1^{(j)} \otimes B_2^{(j)} \) in \( \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N \) we write
\[
A_{LR} \cdot B = \sum_{i,j} A_1^{(i)} B_1^{(j)} \otimes B_2^{(j)} A_2^{(i)} \quad \text{and} \quad A_{RL} \cdot B = \sum_{i,j} B_1^{(j)} A_1^{(i)} \otimes A_2^{(i)} B_2^{(j)}.
\]

Such notation directly extends to multiple tensor factors, as well as to formal power series with coefficients in the tensor product algebra. One easily checks that the skew-invertibility condition (2.4) is equivalent to
\[
(PR)_{RL} \cdot (P \Psi) = I = (PR)_{LR} \cdot (P \Psi). \quad (2.6)
\]
Conjugating the above equalities by the permutation operator \( P \) we also find
\[
(RP)_{LR} \cdot (\Psi P) = I = (RP)_{RL} \cdot (\Psi P). \quad (2.7)
\]
Finally, by applying the matrix transposition \( ^t \): \( e_{ij} \mapsto e_{ji} \) to the \( i \)-th tensor factor of the identities in (2.6) and (2.7) with \( i = 1, 2 \) we get
\[
(PR)^t (P \Psi)^t = (P \Psi)^t (PR)^t = I \quad \text{and} \quad (RP)^t (\Psi P)^t = (\Psi P)^t (RP)^t = I. \quad (2.8)
\]
For more information on symmetries and their properties we refer the reader to [12,30].

2.2. Normalizing series. Let us denote by \( \mathbb{C}_*(x_1, \ldots, x_n) \) the localization of the ring of formal Taylor series \( \mathbb{C}[[x_1, \ldots, x_n]] \) at nonzero polynomials \( \mathbb{C}[x_1, \ldots, x_n]^\times \). There exists a unique embedding \( \mathbb{C}_*(x_1, \ldots, x_n) \to \mathbb{C}((x_1)) \cdots ((x_n)). \) By extending this embedding to \( \mathbb{C}_*(x_1, \ldots, x_n)[[h]] \) we obtain the map
\[
\iota_{x_1,\ldots,x_n} : \mathbb{C}_*(x_1, \ldots, x_n)[[h]] \to \mathbb{C}((x_1)) \cdots ((x_n))[[h]]. \quad (2.9)
\]
Let \( M > 0 \) be an integer. There exists a unique series \( \overline{g}(x) \) in \( 1 + x \mathbb{C}[[x, h]] \) satisfying
\[
\overline{g}(x) \overline{g}(xe^{-2h}) \cdots \overline{g}(xe^{-2(M-1)h}) = \frac{1 - xe^{-2(M-1)h}}{1 - x}; \quad (2.10)
\]
see [8] and [23, Sect. 2] for more information. Therefore, we have the identity
\[
\overline{f}(x) \overline{f}(xe^{-2h}) \cdots \overline{f}(xe^{-2(M-1)h}) = \frac{1 - xe^{-2(M-1)h}}{1 - xe^{-2Mh}} \quad \text{for} \quad \overline{f}(x) = \frac{1 - x}{1 - xe^{-2h}} \overline{g}(x). \quad (2.11)
\]
The expressions in (2.10) and (2.11) are understood as elements of \( \mathbb{C}[[x, h]] \subset \mathbb{C}((x))[[h]] \) via the embedding \( \iota_x \), so that, in particular, we have
\[
\iota_x (1 - xe^{-ah})^{-1} = \sum_{k \geq 0} x^k e^{-ah} \quad \text{for} \quad a \in \mathbb{C}. \quad (2.12)
\]

---

\(^1\) Equivalently, \( \cdot \) is the standard multiplication in the algebra \( \text{End} \mathbb{C}^N \otimes (\text{End} \mathbb{C}^N)^{\text{op}} \) and \( \cdot \) is the product in \( (\text{End} \mathbb{C}^N)^{\text{op}} \otimes \text{End} \mathbb{C}^N \), where \( A^{\text{op}} \) denotes the opposite algebra of \( A \).
As with the aforementioned expressions, throughout the paper we usually omit the embedding symbol. In the multiple variable case, we employ the expansion convention where the choice of the embedding is determined by the order of the variables. More specifically, if \( \sigma \) is a permutation in the symmetric group \( S_n \), then \((x_{\sigma_1} + \ldots + x_{\sigma_n})^r\) stands for \(x_{\sigma_1, \ldots, x_{\sigma_n}}\) for \( r = 0 \). For example, note that we have \((x_1 + x_2)^r \neq (x_2 + x_1)^r\) for \( r < 0 \), as, by the convention, the former expression should be expanded in the negative powers of \( x_1 \) and the latter in the negative powers of \( x_2 \).

Let \( a \in \mathbb{C} \) be nonzero. As demonstrated in [23, Sect. 2], one can apply the substitution \( x = e^{-2a/u} \) to \( f(x) \) and then the embedding \( \iota_u \), which produces the formal power series

\[
\tilde{f}(u) = \iota_u \tilde{f}(x)\Big|_{x = e^{-2u/a}} \in \mathbb{C}((u))[[h]].
\]

The series \( \tilde{f}(u) \) is invertible in \( \mathbb{C}((u))[[h]] \). Moreover, by (2.11) it satisfies

\[
f(u) f(u + ah) \ldots f(u + a(M - 1)h) = \frac{1 - e^{-2u/a} e^{-2(M-1)h}}{1 - e^{-2u/a} e^{-2Mh}},
\]

where the right hand side is again understood as an element of \( \mathbb{C}((u))[[h]] \) via \( \iota_u \).

The series \( f(u) \) and \( \tilde{f}(x) \) depend on the choice of the integer \( M \). In Sect. 6, we shall require that \( M \) is equal to the rank of a certain Hecke symmetry. However, the results of other sections do not depend on the choice of \( M \), so we suppress it in our notation.

2.3. \( R \)-Matrix with additive spectral parameter. We follow [13, Sect. 4] to associate the \( R \)-matrix with a skew-invertible Hecke symmetry \( R \). As before, \( a \) denotes an arbitrary nonzero complex number. Consider the \( R \)-matrix \( R(u) = R_{12}(u) \) defined by

\[
R(u) = \psi f(u) \left( PR + \frac{e^h - e^{-h}}{e^{2u/a} - 1} P \right) \in \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N((u))[[h]],
\]

where \( \psi \) is the invertible element of \( \mathbb{C}[[h]] \) such that \( R(u) \) possesses the unitarity property,

\[
R_{12}(u) R_{21}(-u) = 1.
\]

The constant \( \psi \) can be found as in the proof of [23, Prop. 2.1]. More specifically, consider the product

\[
\alpha(u) g(u) R_{12}'(u) \alpha(-u) g(-u) R_{21}'(-u),
\]

where

\[
R'(u) = PR + \frac{e^h - e^{-h}}{e^{2u/a} - 1} P, \quad g(u) = \iota_u \tilde{g}(x)\Big|_{x = e^{-2u/a}},
\]

\[
\alpha(u) = \iota_u \left( \frac{1 - x}{1 - xe^{-2h}} \right)\Big|_{x = e^{-2u/a}}.
\]

One easily verifies by a direct calculation which employs (2.3) that

\[
\alpha(u) R_{12}'(u) \alpha(-u) R_{21}'(-u) = e^{2h}.
\]
On the other hand, by the proof of [23, Prop. 2.1] the product \( g(u)g(-u) \) belongs to \( 1 + h\mathbb{C}[[h]] \). Hence we can choose \( \psi \in 1 + h\mathbb{C}[[h]] \) such that
\[
\psi^2 g(u) g(-u) = e^{-2h}.
\] (2.17)

Note that by (2.11) we have \( f(u) = a(u)g(u) \). Therefore, by multiplying the identities (2.16) and (2.17) and using (2.14) we obtain the unitarity property (2.15).

The \( R \)-matrix (2.14) satisfies the Yang–Baxter equation,
\[
R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u). \tag{2.18}
\]

In addition to these properties, we shall often use the identities such as
\[
R_{21}(u)(RP)_{13}(RP)_{23} = (RP)_{23}(RP)_{13} R_{21}(u), \tag{2.19}
\]
\[
R_{31}(u)(RP)_{23}(RP)_{21} = (RP)_{21}(RP)_{23} R_{31}(u). \tag{2.20}
\]

They can be easily verified by using the braid relation (2.2), which is satisfied by \( R \) and \( P \), along with the explicit form (2.14) of the \( R \)-matrix. Finally, note that \( R(u) \) is obtained from [13, Prop. 12] by multiplying the corresponding \( R \)-matrix by the permutation operator \( P \) (from the left) and the normalization term \( \psi f(u) \).²

To simplify the following calculation, we write
\[
f_1(u) = \psi f(u) \text{ and } f_2(u) = \frac{e^h - e^{-h}}{e^{2u/a} - 1}, \text{ so that } R(u) = f_1(u) \left( PR + f_2(u)P \right).
\]

The first family of identities in (2.8) implies that the transposed \( R \)-matrices
\[
R(u)^{t_i} = f_1(u) \left( PR \right)^{t_i} \left( I + f_2(u) \left( P\psi \right)^{t_i} P^{t_i} \right) \text{ with } i = 1, 2
\]
are invertible. Indeed, one easily checks that for
\[
S^{(i)}(u) = f_1(u)^{-1} \sum_{l \geq 0} \left( - f_2(u) \left( P\psi \right)^{t_i} P^{t_i} \right)^l \left( P\psi \right)^{t_i} \in \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N((u))[[h]] \tag{2.21}
\]
we have
\[
R(u)^{t_i} S^{(i)}(u) = S^{(i)}(u) R(u)^{t_i} = 1 \text{ for } i = 1, 2. \tag{2.22}
\]

Notice that the infinite sum in (2.21) is convergent with respect to the \( h \)-adic topology since the series \( f_2(u) \) belongs to \( h\mathbb{C}((u))[[h]] \). Let \( S(u) = S^{(i)}(u)^{t_i} \). The index \( i \) is redundant in this expression as \( S^{(1)}(u)^{t_1} = S^{(2)}(u)^{t_2} \). By applying the transposition on the \( i \)-th tensor factor of (2.22) we rewrite the identities in terms of ordered products as
\[
R(u) \cdot S(u) = R(u) \cdot S(u) = 1. \tag{2.23}
\]

² Regarding the normalization, the term \( f(u) \) in (2.14), as well as \( \overline{T}(u) \) in (2.24) below, is used as it leads to nice properties of quantum determinants, as we demonstrate in Sects. 6.2 and 6.3 below.
2.4. R-Matrix with Multiplicative Spectral Parameter. Consider the R-matrix

\[ \overline{R}(x) = \overline{f}(x) \left( PR + \frac{(e^h - e^{-h})x}{1 - x} P \right) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N [[x, h]]; \]  

(2.24)

see [13, Prop. 12]. Roughly speaking, it is obtained from \( \psi^{-1} R(u) \) by replacing \( e^{2u/a} \) by \( x^{-1} \) and then applying the embedding \( \iota_x \); recall (2.9) and (2.12). More precisely, the connection between these \( R \)-matrices is established as follows. For any \( n > 0 \) there exists \( r \geq 0 \) such that \( (1 - x)^r \overline{R}(x) \) belongs to \( \mathbb{C}[x, h] \) modulo \( h^n \). Indeed, the integer \( r \) can be chosen so that \( (1 - x)^r \) cancels all the poles of \( R(x) \) at \( x = 1 \) modulo \( h^n \). Thus, we have

\[ \left( (1 - x)^r \overline{R}(x) \right) \bigg|_{x=e^{-2u/a}} \equiv (1 - e^{-2u/a})^r \psi^{-1} R(u) \mod h^n. \]  

(2.25)

The superscript mod \( h^n \) on the left hand side indicates that the expression inside the brackets is regarded modulo \( h^n \), so that the substitution \( x = e^{-2u/a} \) is indeed well-defined.

The \( R \)-matrix \( \overline{R}(x) \) satisfies the multiplicative version of the Yang–Baxter equation,

\[ \overline{R}_{12}(x) \overline{R}_{13}(xy) \overline{R}_{23}(y) = \overline{R}_{23}(y) \overline{R}_{13}(xy) \overline{R}_{12}(x) \]  

(2.26)

and the identities (2.19) and (2.20). Moreover, arguing as in the end of Sect. 2.3, one easily checks that there exists \( \overline{S}(x) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N [[x, h]] \) such that

\[ \overline{R}(x) \cdot \overline{S}(x) = \overline{R}(x) \cdot \overline{S}(x) = 1. \]  

(2.27)

As with \( \overline{R}(x) \), for any \( n > 0 \) there exists \( r \geq 0 \) such that

\[ \left( (1 - x)^r \overline{S}(x) \right) \bigg|_{x=e^{-2u/a}} = (1 - e^{-2u/a})^r \psi^{-1} S(u) \mod h^n. \]  

(2.28)

Observe that multiplying (2.14) and (2.24) by suitable factors we obtain the \( R \)-matrices

\[ R'(u) = (1 - e^{-2u/a}) PR + (e^h - e^{-h})e^{-2u/a} P \]  

and

\[ \overline{R}'(x) = (1 - x) PR + (e^h - e^{-h})x P \]

which posses only nonnegative powers of \( u \) and \( x \), respectively. Furthermore, they satisfy

\[ \overline{R}'(x) \bigg|_{x=e^{-2u/a}} = R'(u). \]  

(2.29)

Throughout the paper we shall often write bar on the top of the symbol to indicate that the corresponding object comes with the multiplicative spectral parameter. In addition, the letters \( u, v \) will usually indicate additive and \( x, y \) multiplicative variables.
3. $h$-adic Quantum Vertex Algebra $\mathcal{V}_c(R)$

In this section, we use the structure of the algebra $Y_{RTT}(R)$ to introduce the creation and annihilation operators over a certain $\mathbb{C}[[h]]$-module $\mathcal{V}_c(R)$. By employing these operators we carry out the Etingof–Kazhdan-type construction of the $h$-adic quantum vertex algebra over $\mathcal{V}_c(R)$ associated with the skew-invertible Hecke symmetry $R$.

The definition of the following algebra is motivated by the algebras $Y_{RTT}(R)$ from [13]. We discuss a connection between these two classes of algebras in Remark 3.2 below. Also, this algebra can be regarded as the quantum function algebra associated to the $R$-matrix $R(u)$; cf. [4]. Define $Y_{RTT}^+(R)$ as the topologically free associative algebra over the ring $\mathbb{C}[[h]]$ generated by the elements $t_{ij}^{(-r)}$, where $i, j = 1, \ldots, N$ and $r = 1, 2, \ldots$, subject to the defining relations

$$R(u - v) T^+_1(u) T^+_2(v) = T^+_2(v) T^+_1(u) R(u - v). \quad (3.1)$$

The elements $T^+(u)$ are defined by

$$T^+(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}^+(u), \quad \text{where} \quad t_{ij}^+(u) = \sum_{r \geq 1} t_{ij}^{(-r)} u^{r-1}, \quad (3.2)$$

while the subscripts in (3.1) indicate the factors in the tensor product algebra, so that

$$T^+_1(u) = \sum_{i,j=1}^N e_{ij} \otimes 1 \otimes t_{ij}^+(u) \quad \text{and} \quad T^+_2(v) = \sum_{i,j=1}^N 1 \otimes e_{ij} \otimes t_{ij}^+(v). \quad (3.3)$$

In the above definition, we require that the given algebra is topologically free as a $\mathbb{C}[[h]]$-module; cf. [21, Ch. XVI]. This is easily accomplished by suitably modifying the ideal of its defining relations; see, e.g., [24, Rem. 4.1] for more information. We should say that this modification of the ideal is necessary as it is unknown whether the topological $\mathbb{C}[[h]]$-algebra generated by the elements $t_{ij}^{(-r)}$ with $i, j = 1, \ldots, N$ and $r = 1, 2, \ldots$, subject to (3.1) and (3.2) is torsion free.

Let $V$ be a $\mathbb{C}[[h]]$-module. We shall denote by $V((z))_h$ (resp. $V[z^{-1}]_h$) the $\mathbb{C}[[h]]$-module of all formal power series $a(z) = \sum_{r \geq 1} a_r z^r$ in $V[[z^{\pm 1}]]$ (resp. in $V[[z^{-1}]]$) such that we have $a_r \rightarrow 0$ when $r \rightarrow -\infty$ with respect to the $h$-adic topology. Such notation naturally extends to the multiple variable case, so we write, for example, $V((z_1, \ldots, z_n))_h$. Note that if $V$ is topologically free then $V((z))_h$ (resp. $V[z^{-1}]_h$) is just the $h$-adic completion of $V((z))$ (resp. $V[z^{-1}]$).

The following proposition can be proved by a direct calculation which relies on the properties of the $R$-matrix and the defining relations for the algebra $Y_{RTT}^+(R)$; cf. [5, Lemma 2.1].

**Proposition 3.1.** For any $c \in \mathbb{C}$ there exists a unique formal power series $T^-(u)$ in $\text{End} \mathbb{C}^N \otimes \text{Hom}(Y^+_{RTT}(R), Y^+_{RTT}(R)((u))_h)$ satisfying $T^-(u)1 = 1 \otimes 1$ such that for all integers $n \geq 1$ we have

$$T^{-}_0(u) T^+_1(v_1) \ldots T^{-}_n(v_n) = R_{01}(u - v_1 + hc/2)^{-1} \ldots R_{0n}(u - v_n + hc/2)^{-1} \times T^+_1(v_1) \ldots T^+_n(v_n) R_{0n}(u - v_n - hc/2) \ldots R_{01}(u - v_1 - hc/2). \quad (3.4)$$
Moreover, $T^-(u)$ is invertible in $\text{End} \, \mathbb{C}^N \otimes \text{Hom}(\mathcal{Y}_{RTT}^+(R), \mathcal{Y}_{RTT}^+(R)((u))_h)$ and it satisfies the following identities for operators on $\mathcal{Y}_{RTT}^+(R)$:

$$R(u - v) T_1^- (u) T_2^- (v) = T_2^- (v) T_1^- (u) R(u - v),$$

(3.5)

$$R(u - v + hc/2) T_1^- (u) T_2^+ (v) = T_2^+ (v) T_1^- (u) R(u - v - hc/2).$$

(3.6)

In order to emphasize that the $\mathbb{C}[[h]]$-module $\mathcal{Y}_{RTT}^+(R)$ is regarded with respect to the action of $T^-(u)$, which depends on the choice of $c \in \mathbb{C}$, we denote it by $\mathcal{Y}_{RTT}^+(R)_c$.

**Remark 3.2.** The algebra $\mathcal{Y}_{RTT}(R)$, which generalizes the Yangian for $\mathfrak{gl}_N$, was introduced in [13] via defining relations of the form (3.5),

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v),$$

where the corresponding generator series $T(u)$ consists of nonpositive powers of the variable $u$. Regarding the operator series $T^-(u)$, although satisfying the relation of the same form, it possesses nonnegative powers of $u$ as well, which appears to be common for the annihilation operators in Etingof–Kazhdan’s construction; cf. [5, Lemma 2.1].

Let $u = (u_1, \ldots, u_n)$ be a family of variables and $z$ a single variable. We shall often use the following notation for operators on $(\text{End} \, \mathbb{C}^N)^{\otimes n} \otimes \mathcal{Y}_{RTT}^+(R)$:

$$T_{[n]}^\pm (u) = T_1^\pm(u_1) \cdots T_n^\pm(u_n) \quad \text{and} \quad T_{[n]}^\pm (z + u) = T_1^\pm (z + u_1) \cdots T_n^\pm (z + u_n),$$

(3.7)

where $z + u = (z + u_1, \ldots, z + u_n)$. In addition, to simplify the notation, for any $b \in \mathbb{C}$ we write $z + u + bh$ for the variables $(z + u_1 + bh, \ldots, z + u_n + bh)$. Suppose $v = (v_1, \ldots, v_m)$ is another family of variables. We introduce the following $R$-matrix products with entries in $(\text{End} \, \mathbb{C}^N)^{\otimes n} \otimes (\text{End} \, \mathbb{C}^N)^{\otimes m}$:

$$R_{nm}(z + u - v + bh) = \prod_{1 \leq i \leq n+1 \leq j \leq n+m} R_{ij}(z + u_i - v_j - n + bh),$$

(3.8)

where the arrows indicate the order of factors. If the variable $z$ is omitted in the $R$-matrices on the right side, we denote the resulting expression by $R_{nm}(u - v + bh)$. Finally, we shall use the analogous notation for the products of the matrices $S(u) = S^{(i)}(u)^b$. Using this notation, one can generalize (3.1), (3.5) and (3.6) as follows; cf. [5, Eq. (2.9)].

**Proposition 3.3.** For any $n, m \geq 1$ and the variables $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_m)$ the following identities hold for operators on $\mathcal{Y}_{RTT}^+(R)_c$:

$$R_{nm}^{12}(u - v) T_{[n]}^\pm (u) T_{[m]}^\pm (v) = T_{[m]}^\pm (v) T_{[n]}^\pm (u) R_{nm}^{12}(u - v),$$

$$R_{nm}^{12}(u - v + hc/2) T_{[n]}^{13} (u) T_{[m]}^{23} (v) = T_{[m]}^{23} (v) T_{[n]}^{13} (u) R_{nm}^{12}(u - v - hc/2).$$

From now on, the tensor products of $\mathbb{C}[[h]]$-modules are understood as $h$-adically completed. Essentially, the next theorem is the Etingof–Kazhdan construction [5, Thm. 2.3] for the $R$-matrix (2.14). As with the original version, it can be proved by directly verifying the $h$-adic quantum vertex algebra axioms using the properties of the $R$-matrix and the RTT-relations among the operators $T^\pm(u)$. We omit these calculations since they go in parallel with the case of suitably normalized Yang $R$-matrix, whose details can be found in the proofs of [9, Thm. 2.3.8] and [19, Thm. 4.1]. Regarding the terminology, by the term $h$-adic quantum vertex algebra we refer to the generalization of the notion of quantum VOA [5, Sect. 1.4.1], as given in [26, Def. 2.20].
Theorem 3.4. For any $c \in \mathbb{C}$ there exists a unique structure of $h$-adic quantum vertex algebra on $Y^+_{RTT}(R)_c$ such that the vacuum vector is the unit $1 \in Y^+_{RTT}(R)_c$, the vertex operator map $Y(\cdot, z)$ is given by

$$Y(T^+[n](u) 1, z) = T^+[n](z + u) T^-[n](z + u + hc/2)^{-1}$$

(3.9)

and the braiding $S(z)$ satisfies the identity

$$S^{34}(z) \left( R^{12}_{nm}(z + u - v)^{-1} T^{24}_{[n]}(v) R^{12}_{nm}(z + u - v - hc) T^{+13}_{[n]}(u)(1 \otimes 1) \right)$$

$$= T^{+13}_{[n]}(u) R^{12}_{nm}(z + u - v + hc)^{-1} T^{24}_{[n]}(v) R^{12}_{nm}(z + u - v)(1 \otimes 1).$$

(3.10)

We shall denote the $h$-adic quantum vertex algebra established in Theorem 3.4 by $\mathcal{V}_c(R)$. Regarding the formula for the braiding, note that the superscripts 1, 2, 3, 4 in (3.10) indicate the tensor factors as follows:

$$\frac{1}{(\text{End } \mathbb{C}^N)^\otimes n} \otimes \frac{2}{(\text{End } \mathbb{C}^N)^\otimes m} \otimes \frac{3}{\mathcal{V}_c(R) \otimes \mathcal{V}_c(R)}.\quad (3.11)$$

Remark 3.5. The results of this section, including Theorem 3.4, also hold if $R$ is a skew-invertible involutive symmetry. In this case, one can employ, e.g., the $R$-matrix

$$R(u) = \left( 1 - \frac{ah}{u} \right)^{-1} \left( PR + \frac{ah}{u} P \right) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N[[h/u]]\quad (3.12)$$

from [13, Sect. 4] instead of (2.14). Thus the corresponding operator series $T^-(u)$, defined as in Proposition 3.1, possesses only nonpositive powers of $u$, so that it produces the action of the algebra $Y_{RTT}(R)$ associated with $R$ [13] (defined over the ring $\mathbb{C}[[h]]$); cf. Remark 3.2. In particular, if $R = P$, by suitably rescaling the generators one obtains from $T^-(u)$ the action of the Yangian for $\mathfrak{gl}_N$ on the corresponding dual Yangian, which comes from the structure of the level $c$ double Yangian for $\mathfrak{gl}_N$; see [16,22].

4. $\mathcal{V}_c(R)$-Modules

In this section, we use the structure of the braided Yangian associated to the $R$-matrix (2.14) to construct a family of $\mathcal{V}_c(R)$-modules.

4.1. Braided Yangian-type action (additive case). The definition of the following algebra is motivated by the notion of braided Yangian from [13]. A connection between these two classes of algebras is discussed in Remark 4.3 below. Let $Y^+(R)$ be the topologically free associative algebra over the ring $\mathbb{C}[[h]]$ generated by the elements $l^{(-r)}_{ij}$, where $i, j = 1, \ldots, N$ and $r = 1, 2, \ldots$, subject to the defining relations

$$R_{12}(u - v) L^+_1(u) (RP)_{12} L^+_2(v) = L^+_2(v) (RP)_{21} L^+_1(u) R_{21}(u - v).$$

(4.1)

The elements $L^+(u)$ are defined by

$$L^+(u) = \sum_{i,j=1}^N e_{ij} \otimes l^+_{ij}(u), \quad \text{where} \quad l^+_{ij}(u) = \delta_{ij} - \sum_{r \geq 1} l^{(-r)}_{ij} u^{r-1},$$

(4.2)
while their subscripts indicate the factors in the tensor product algebra as in (3.3).

Let \( u = (u_1, \ldots, u_n) \) be a family of variables and \( z \) a single variable. We introduce the following notation for the formal power series with coefficients in \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes \mathcal{Y}^+(R)\):

\[
L^+_i(z + u) = \prod_{1 \leq i \leq n} L^+_i(z + u_i)(RP)_{ii+1} \cdots (RP)_{in}.
\]

(4.3)

If \( z \) is omitted from the right hand side, we denote the resulting expression by \( L^+_i(u) \).

**Lemma 4.1.** The coefficients of all matrix entries of \( L^+_i(u) = L^+_i(u_1, \ldots, u_n) \) with \( n > 1 \) along with the unit 1 span an \( h \)-adically dense \( \mathbb{C}[[h]] \)-submodule of \( \mathcal{Y}^+(R) \).

**Proof.** The lemma follows from the identity

\[
L^+_i(u) = \prod_{1 \leq i \leq n} L^+_i(u_i)(RP)_{ii+1} \cdots (RP)_{in}.
\]

(4.4)

Indeed, it is clear that the coefficients of all matrix entries of \( L^+_1(u_1) \cdots L^+_n(u_n) \) with \( n > 1 \) along with 1 span an \( h \)-adically dense \( \mathbb{C}[[h]] \)-submodule of \( \mathcal{Y}^+(R) \). However, due to (2.7), we can remove all terms \( RP \) from the right hand side of (4.4) by multiplying the equality by suitable copies of \( \Psi P \). Thus, we can express \( L^+_1(u_1) \cdots L^+_n(u_n) \) in terms of \( L^+_i(u_1, \ldots, u_n) \), which implies the lemma.

We have the following braided Yangian analogue of Proposition 3.1:

**Proposition 4.2.** For any \( c \in \mathbb{C} \) there exists a unique formal power series \( L^-(u) \) in \( \text{End } \mathbb{C}^N \otimes \text{Hom}(\mathcal{Y}^+(R), \mathcal{Y}^+(R)((u))_h) \) satisfying \( L^-(u)1 = I \otimes 1 \) such that for all integers \( n > 1 \) and the variables \( v = (v_1, \ldots, v_n) \) we have

\[
L^-0(u)(RP)_{01} \cdots (RP)_{0n} L^+_{i_1} \cdots L^+_{i_n}(v) = R_{01}(u - v_1 + hc/2)^{-1} \cdots R_{0n}(u - v_n + hc/2)^{-1} \times L^+_{i_1} \cdots L^+_{i_n}(RP)_{10} \cdots (RP)_{n0} R_{n0}(u - v_n - hc/2) \cdots R_{10}(u - v_1 - hc/2).
\]

(4.5)

Moreover, \( L^-(u) \) is invertible in \( \text{End } \mathbb{C}^N \otimes \text{Hom}(\mathcal{Y}^+(R), \mathcal{Y}^+(R)((u))_h) \) and it satisfies the following identities for operators on \( \mathcal{Y}^+(R) \):

\[
R_{12}(u-v)L^-_1(u)(RP)_{12}L^-_2(v) = L^-_2(v)(RP)_{21}L^-_1(u)R_{21}(u-v),
\]

(4.6)

\[
R_{12}(u-v+hc/2)L^-_1(u)(RP)_{12}L^+_2(v) = L^+_2(v)(RP)_{21}L^-_1(u)R_{21}(u-v-hc/2).
\]

(4.7)

**Proof.** First, note that the superscripts in (4.5) indicate tensor factors as follows:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
(\text{End } \mathbb{C}^N) & (\text{End } \mathbb{C}^N)^{\otimes n} & \otimes \mathcal{Y}^+(R).
\end{array}
\]

The fact that \( L^-(u) \) is well-defined by (4.5) can be proved by a direct calculation which shows that it preserves the ideal of defining relations (4.1) for the algebra \( \mathcal{Y}^+(R) \). The calculation relies on the Yang–Baxter equation (2.18) and the properties (2.19), (2.20)
of the $R$-matrix. Next, by Lemma 4.1 the expression (4.5) uniquely determines $L^{-}(u)$. Regarding the invertibility, one easily checks that the inverse of $L^{-}(u)$ is given by

$$L^{-0}(u)^{-1}L_{[n]}^{+12}(v) = \left( S_{n0}(-u + v_n - hc/2) \ldots S_{10}(-u + v_1 - hc/2) \right) \cdot \left( (RP)_{01} \ldots (RP)_{0n} \right)_{RL} \times L_{[n]}^{+12}(v) R_{10}(u - v_1 - hc/2)^{-1} \ldots R_{n0}(u - v_n - hc/2)^{-1} \left( (RP)_{n0}^{-1} \ldots (RP)_{10}^{-1} \right).$$

Hence both $L^{-}(u)$ and its inverse belong to $\text{End} \mathbb{C}^{N} \otimes \text{Hom}(\mathbb{Y}^{+}(R), \mathbb{Y}^{+}(R)((u)))_h$ due to the form of $R(u)$ and $S(u)$. Finally, the identities (4.6) and (4.7) are immediate consequences of (4.5), which can be verified using the properties (2.18), (2.19) and (2.20) of the $R$-matrix.

**Remark 4.3.** The notion of braided Yangian was introduced in [13] via defining relations

$$P_{12} R_{12}(u - v) L_{1}(u) R_{12} L_{1}(v) = L_{1}(v) R_{12} L_{1}(u) P_{12} R_{12}(u - v),$$

where the corresponding generator series $L(u)$ consists of nonpositive powers of the variable $u$ and its constant term equals the identity. This relation is equivalent to

$$R_{12}(u - v) L_{1}(u) (RP)_{12} L_{2}(v) = L_{2}(v) (RP)_{21} L_{1}(u) R_{21}(u - v),$$

i.e. it takes the form of (4.6). However, the operators $L^{-}(u)$ considered in (4.6) contain both positive and negative powers of $u$.

The action of $L^{-}(u)$ given by (4.5) depends on the choice of $c \in \mathbb{C}$. Hence, to indicate that the $\mathbb{C}[[h]]$-module $\mathbb{Y}^{+}(R)_c$ is regarded with respect to this action, we denote it from now on by $\mathbb{Y}^{+}(R)_c$. Generalizing the notation, we write $L_{[n]}^{-}(z + u)$ (resp. $L_{[n]}^{+}(u)$) for the expression obtained from (4.3) (resp. (4.4)) by replacing all $L_{i}^{+}(z + u_i)$ (resp. $L_{i}^{-}(u_i)$) on the right hand side by the operator series $L_{i}^{-}(z + u_i)$ (resp. $L_{i}^{+}(u_i)$). Also, the original series in (4.3) and (4.4) are from now on regarded as operators on $\mathbb{Y}^{+}(R)_c$ with respect to the algebra multiplication.

The relations established in Proposition 4.2 can be generalized as follows. First, denote by $R_{\tilde{n}\tilde{m}}(u - v + bh)$ the expression obtained from $R_{nm}(u - v + bh)$, where $b \in \mathbb{C}$, by conjugating all $R$-matrices by the permutation operator $P$, i.e.

$$R_{\tilde{n}\tilde{m}}(u - v + bh) = \prod_{1 \leq i \leq n} \prod_{n+1 \leq j \leq n+m} R_{ji}(u_i - v_{j-n} + bh). \quad (4.8)$$

Also, we write $R_{\tilde{n}\tilde{m}}(z + u - v + bh)$ for the analogous product of the $R$-matrices $R_{ji}(z + u_i - v_{j-n} + bh)$; recall (3.8). Furthermore, we use the analogous notation for the products of matrices $S(u)$. Thus, for example, by combining (2.15) and (2.23) one easily proves

$$R_{nm}(u - v)^{-1}_{LR} \cdot S_{\tilde{n}\tilde{m}}(-u + v) = R_{nm}(u - v)^{-1}_{RL} \cdot S_{\tilde{n}\tilde{m}}(-u + v) = 1. \quad (4.9)$$

Next, we introduce the following notation

$$(RP)_{nm} = \prod_{1 \leq i \leq n} \prod_{n+1 \leq j \leq n+m} (RP)_{ij} \quad \text{and} \quad (RP)_{\tilde{n}\tilde{m}} = \prod_{1 \leq i \leq n} \prod_{n+1 \leq j \leq n+m} (RP)_{ji}.$$
Proposition 4.4. For any \( n, m \geq 1 \) and the variables \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m) \) the following identities hold for operators on \( Y^+(R)_c \):

\[
R_{nm}^{12}(u - v)L_{[n]}^{±13}(u)(RP)_{nm}^{12}L_{[m]}^{±32}(v) = L_{[n]}^{±23}(v)(RP)_{nm}^{12}L_{[m]}^{±13}(u)R_{nm}^{12}(u - v),
\]

\[
R_{nm}^{12}(u - v + hc/2)L_{[n]}^{−13}(u)(RP)_{nm}^{12}L_{[m]}^{±32}(v) = L_{[n]}^{±23}(v)(RP)_{nm}^{12}L_{[m]}^{−13}(u)R_{nm}^{12}(u - v - hc/2).
\]

Note that by using (4.9) we can rewrite the second identity in Proposition 4.4 as

\[
L_{[n]}^{−13}(u)^{-1}L_{[m]}^{±23}(v) = S_{nm}^{12}(-u + v - hc/2)
\]

\[
(RP)_{nm}^{12}R_{nm}^{12}(u - v - hc/2)^{-1}L_{[n]}^{−13}(u)^{-1}((RP)_{nm}^{12})^{-1}).
\] (4.10)

Lemma 4.5. For any \( c \in \mathbb{C}, n \geq 1 \) and the variables \( u = (u_1, \ldots, u_n) \) we have

\[
L_{[n]}^{−1}(u)^{-1} \in (\text{End } \mathbb{C}^N)^{⊗ n} \otimes \text{Hom}(Y^+(R)_c, Y^+(R)_c((u_1, \ldots, u_n))_h).
\]

Proof. By Lemma 4.1, the coefficients of matrix entries of all \( L_{[m]}^{±1}(v) \) along with \( 1 \) span an \( h \)-adically dense \( \mathbb{C}[[h]] \)-module of \( Y^+(R)_c \). Thus it is sufficient to check that their images under \( L_{[n]}^{−1}(u)^{-1} \) belong to \( (\text{End } \mathbb{C}^N)^{⊗ n} \otimes Y^+(R)_c((u_1, \ldots, u_n))_h \). This follows immediately by applying (4.10) to \( 1 \in Y^+(R)_c \) since \( L_{[n]}^{−1}(u)^{-1} \) is constant with respect to the variables \( u_1, \ldots, u_n \). Indeed, due to the form of \( R(u) \) and \( S(u) \), as given by (2.14) and (2.21), and the expansion convention from Sect. 2.2, the remaining terms on the right hand side, \( S_{nm}^{12}(-u + v - hc/2) \) and \( R_{nm}^{12}(u - v - hc/2)^{-1} \) belong to

\[
(\text{End } \mathbb{C}^N)^{⊗ n} \otimes (\text{End } \mathbb{C}^N)^{⊗ m}((u_1, \ldots, u_n))_h[[v_1, \ldots, v_m]].
\]

4.2. The \( \mathcal{V}_c(R) \)-Module Structure Over \( Y^+(R)_c \). In the next theorem we construct an example of \( \mathcal{V}_c(R) \)-module over the \( \mathbb{C}[[h]] \)-module of \( Y^+(R)_c \). For readers’ convenience, we first recall the definition of a module for an \( h \)-adic quantum vertex algebra [26, Def. 2.23].

Definition 4.6. Let \( (\mathcal{V}, Y, 1, S) \) be an \( h \)-adic quantum vertex algebra. A \( \mathcal{V} \)-module is a pair \( (\mathcal{M}, Y_{\mathcal{M}}) \) such that \( \mathcal{M} \) is a topologically free \( \mathbb{C}[[h]] \)-module and

\[
Y_{\mathcal{M}}(z) = Y_{\mathcal{M}}(\cdot, z) : \mathcal{V} \otimes \mathcal{M} \to \mathcal{M}((z))_h
\]

\[
u \otimes w \mapsto Y_{\mathcal{M}}(z)(u \otimes w) = Y_{\mathcal{M}}(u, z)w = \sum_{r \in \mathbb{Z}} u_r w z^{-r-1}
\]

is a \( \mathbb{C}[[h]] \)-module map which satisfies \( Y_{\mathcal{M}}(1, z)w = w \) for all \( w \in \mathcal{M} \) and the weak associativity: for any \( u, v \in \mathcal{V}, w \in \mathcal{M} \) and integer \( k \in \mathbb{Z}_{\geq 0} \) there exists \( r \in \mathbb{Z}_{\geq 0} \) such that

\[
(z_0 + z_2)^r Y_{\mathcal{M}}(u, z_0 + z_2) Y_{\mathcal{M}}(v, z_2)w = (z_0 + z_2)^r Y_{\mathcal{M}}(Y(u, z_0)v, z_2)w \quad (4.11)
\]

belongs to \( h^k \mathcal{M}[[z_0^{±1}, z_2^{±1}]] \), i.e. such that the above expressions coincide modulo \( h^k \).
To emphasize that \(Y^\ast(R)\) is now regarded as a \(\mathcal{V}_c(R)\)-module, we denote it by \(\mathcal{M}(R)\).

**Theorem 4.7.** For any \(c \in \mathbb{C}\) there exists a unique structure of module for the \(h\)-adic quantum vertex algebra \(\mathcal{V}_c(R)\) on \(\mathcal{M}(R)\) so that \(Y_{\mathcal{M}(R)}(\cdot, z) = 1_{\mathcal{M}(R)}\) and such that for all integers \(n \geq 1\) and the variables \(u = (u_1, \ldots, u_n)\) we have

\[
Y_{\mathcal{M}(R)}(T^+_n(u), z) = L^+_n(z + u) L^-_n(z + u + hc/2)^{-1}. \tag{4.12}
\]

**Proof.** Let us show that the \(\mathbb{C}[[h]]\)-module map is well-defined by (4.12), i.e. that it annihilates the ideal of defining relations (3.1). Given the definition of the algebra \(Y_{RTT}^+(R)\), it is sufficient to check that for any indices \(1 \leq i < n\) and the families of variables

\[
u = (u_1, \ldots, u_n)\quad \text{and} \quad u^{(i)} = (u_1, \ldots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \ldots, u_n)
\]

the image of the expression

\[
R_{i+1}(u_i - u_{i+1}) T^+_n(u) 1 - P_{i+1} T^+_n(u^{(i)}) P_{i+1} R_{i+1}(u_i - u_{i+1})
\]

under \(Y_{\mathcal{M}(R)}(\cdot, z)\) is trivial. In other words, due to (4.12), we have to prove that

\[
R'_{i+1} L^+_n(z + u) L^-_n(z + u + hc/2)^{-1} - P_{i+1} L^+_n(z + u^{(i)}) L^-_n(z + u^{(i)} + hc/2)^{-1} P_{i+1} R'_{i+1}
\]

is zero for \(R'_{i+1} = R_{i+1}(u_i - u_{i+1})\). However, this immediately follows from the identities

\[
R'_{i+1} L^+_n(z + u + bh) = P_{i+1} L^+_n(z + u^{(i)} + bh) P_{i+1} R_{i+1}(u_i - u_{i+1}) \quad \text{with} \quad b \in \mathbb{C},
\]

which can be verified by using (2.20) and the relations (4.1) and (4.6).

Next, as the coefficients of matrix entries of all \(T^+_n(u) 1\) with \(n \geq 1\) along with \(1\) span an \(h\)-adically dense \(\mathbb{C}[[h]]\)-submodule of \(\mathcal{V}_c(R)\), the \(\mathbb{C}[[h]]\)-module map \(Y_{\mathcal{M}(R)}(\cdot, z)\) is uniquely determined by (4.12). Moreover, by Lemma 4.5, its image belongs to \(\mathcal{M}(R)_c((z))_h\). Therefore, it remains to prove that it satisfies the weak associativity, as given in Definition 4.6.

We start the proof of the weak associativity property by considering the image of

\[
T^{+13}_{[n]}(u) T^{+24}_{[m]}(v)(1 \otimes 1) \in (\text{End} \ \mathbb{C}^N)^{\otimes (n+m)} \otimes \mathcal{V}_c(R)^{\otimes 2}[[u_1, \ldots, u_n, v_1, \ldots, v_m]] \tag{4.13}
\]

under the operators from (4.11). Throughout the calculation, we use superscripts 1, 2, 3, 4 to indicate the tensor factors as in (3.11). We shall need the following simple consequence of the second relation in Proposition 3.3:

\[
T^{-13}_{[n]}(u) T^{+23}_{[m]}(v) 1 = S^{12}_{nm} (-u + v - hc/2) \cdot \left( T^{+23}_{[m]}(v) 1 R^{12}_{nm} (u - v - hc/2)^{-1} \right). \tag{4.14}
\]

Due to (4.12), the application of \(Y_{\mathcal{M}(R)}(\cdot, z_0 + z_2)(1 \otimes Y_{\mathcal{M}(R)}(\cdot, z_2))\), which corresponds to the first term in (4.11), to the expression in (4.13) produces

\[
L^{+13}_{[n]}(z_0 + z_2 + u) L^{-13}_{[n]}(z_0 + z_2 + u + hc/2)^{-1} L^{+23}_{[m]}(z_2 + v) L^{-23}_{[m]}(z_2 + v + hc/2)^{-1}. \tag{4.15}
\]
On the other hand, by applying \( \hat{Y}_{\mathcal{M}(R)}(\cdot, z_2)(Y(\cdot, z_0) \otimes 1) \), which corresponds to the second term in (4.11), to (4.13) and then using (3.9) and (4.14) we get
\[
S_{nm}^{12}(-z_0 - u + v - hc) \cdot (L^{+123}_{[n+m]}(z_2 + z_0 + u, z_2 + v) \\
\times L^{-123}_{[n+m]}(z_2 + z_0 + u + hc/2, z_2 + v + hc/2)^{-1} R_{nm}^{12}(z_0 + u - v)^{-1}) \quad (4.16)
\]
where \((z_2 + z_0 + u + bh, z_2 + v + bh)\) with \(b \in \mathbb{C}\) denotes the \((n + m)\)-tuple of variables
\[
(z_2 + z_0 + u_1 + bh, \ldots, z_2 + z_0 + u_n + bh, z_2 + v_1 + bh, \ldots, z_2 + v_m + bh).
\]

We now compare the expressions in (4.15) and (4.16). First, we use (4.10) to swap the order of the two middle factors in (4.15), thus getting
\[
S_{nm}^{12}(-z_0 - u + v - hc) \cdot (L^{+13}_{[n]}(z_0 + z_2 + u)(RP)_{nm}^{12} L^{+23}_{[m]}(z_2 + v) \\
\times R_{nm}^{12}(z_0 + u - v)^{-1} L^{-13}_{[n]}(z_0 + z_2 + u + hc/2)^{-1} ((RP)_{nm}^{12})^{-1} L^{-23}_{[m]}(z_2 + v + hc/2)^{-1}).
\]

Next, using the first family of relations in Proposition 4.4, we reorder the last four factors:
\[
S_{nm}^{12}(-z_0 - u + v - hc) \cdot (L^{+13}_{[n]}(z_0 + z_2 + u)(RP)_{nm}^{12} L^{+23}_{[m]}(z_2 + v) \\
\times L^{-23}_{[m]}(z_2 + v + hc/2)^{-1} ((RP)_{nm}^{12})^{-1} L^{-13}_{[n]}(z_0 + z_2 + u + hc/2)^{-1} R_{nm}^{12}(z_0 + u - v)^{-1}.
\]

Finally, it is clear that this equals to
\[
S_{nm}^{12}(-z_0 - u + v - hc) \cdot (L^{+123}_{[n+m]}(z_0 + z_2 + u, z_2 + v) \\
\times L^{-123}_{[n+m]}(z_0 + z_2 + u + hc/2, z_2 + v + hc/2)^{-1} R_{nm}^{12}(z_0 + u - v)^{-1}). \quad (4.17)
\]

Note that the expressions in (4.16) and (4.17) are not equal as the term \(L^{-123}_{[n+m]}(...)^{-1}\) in the former should be expanded in negative powers of \(z_2\) and in the latter in negative powers of \(z_0\). Let \(w \in \mathcal{M}(R)/c\) and let \(a_1, \ldots, a_n, b_1, \ldots, b_m, k\) be arbitrary positive integers. We shall now consider the coefficients of the monomials
\[
\prod_{i=1}^{n} u_i^{a_i} \cdots \prod_{k=m} u_k^{b_k} \quad \text{for} \quad 0 \leq b_j \leq a_j, \quad 0 \leq k^\prime < k. \quad (4.18)
\]

By Lemma 4.5, there exists \(r \geq 1\) such that the coefficients of monomials (4.18) in
\[
(z_0 + z_2)^r L^{-123}_{[n+m]}(z_0 + z_2 + u + hc/2, z_2 + v + hc/2)^{-1} w \quad \text{and}
\]
\[
(z_0 + z_2)^r L^{-123}_{[n+m]}(z_2 + z_0 + u + hc/2, z_2 + v + hc/2)^{-1} w
\]
coincide. Therefore, suppose we apply (4.16) and (4.17) to the vector \(w\) and then multiply them by \((z_0 + z_2)^r\). Then the coefficients of the monomials (4.18) in the resulting expressions coincide, so we conclude that the weak associativity holds. \(\square\)

**Remark 4.8.** As with Sect. 3, the results of this section can be also recovered when \(R\) is a skew-invertible involutive symmetry by using the \(R\)-matrix (3.12); recall Remark 3.5. In this case, the corresponding series \(L^{-}(u)\) from Proposition 4.2 possesses only non-positive powers of \(u\) and its constant term is the identity. Thus it produces the action of the braided Yangian associated with \(R\) [13]; recall Remark 4.3. Naturally, we assume here that the braided Yangian is regarded over the ring \(\mathbb{C}[[h]]\).
5. \(\phi\)-Coordinated \(\mathcal{V}_c(R)\)-Modules

In this section, we introduce certain modules for the algebras \(Y(R)\) and \(Y_{RTT}(R)\) associated with the \(R\)-matrix (2.24) defined over \(\mathbb{C}[[h]]\). Using the modules we construct two families of \(\phi\)-coordinated \(\mathcal{V}_c(R)\)-modules for \(\phi(z_2, z_0) = z_2 e^{-2z_0/a}\). In the end, we introduce a certain algebra and establish a partial connection between its restricted modules and \(\phi\)-coordinated \(\mathcal{V}_c(R)\)-modules.

5.1. Action of the algebra \(Y(R)\) (multiplicative case). The following construction goes in parallel with Sect. 4. We start with the topologically free associative algebra \(Y\) over the ring \(\mathbb{C}[[h]]\) which is generated by the elements \(\bar{t}^{(-r)}_{ij}\), where \(i, j = 1, \ldots, N\) and \(r = 1, 2, \ldots\). These are subject to the defining relations

\[
\bar{R}_{12}(x/y) \bar{L}^+(x) (RP)_{12} \bar{L}^+_2(y) = \bar{L}^+_2(y) (RP)_{21} \bar{L}^+_1(x) \bar{R}_{21}(x/y). \tag{5.1}
\]

The elements \(\bar{L}^+(x)\) are defined by

\[
\bar{L}^+(x) = \sum_{i,j=1}^N e_{ij} \otimes \bar{t}^+_{ij}(x), \quad \text{where} \quad \bar{t}^+_{ij}(x) = \delta_{ij} - \sum_{r \geq 1} \bar{t}^{(-r)}_{ij} x^{r-1}.
\]

Let \(x = (x_1, \ldots, x_n)\) be a family of variables. We introduce the following notation for the formal power series with coefficients in \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes Y(R)\):

\[
\bar{L}^+_{[n]}(x) = \prod_{1 \leq i \leq n} \bar{L}^+_i(x_i) (RP)_{i;i+1} \ldots (RP)_{i;n}.
\]

The next lemma can be proved analogously to Lemma 4.1.

**Lemma 5.1.** The coefficients of all matrix entries of \(\bar{L}^+_{[n]}(x) = \bar{L}^+_{[n]}(x_1, \ldots, x_n)\) with \(n \geq 1\) along with the unit 1 span an \(h\)-adically dense \(\mathbb{C}[[h]]\)-module of \(Y(R)\).

We omit the proof of the following proposition as it goes in parallel with the proof of Proposition 4.2. In particular, it relies on Lemma 5.1 and \(R\)-matrix properties from Sect. 2.4. The superscripts in the formula (5.3) below are again used to indicate tensor factors analogously to the statement of Proposition 4.2.

**Proposition 5.2.** For any \(c \in \mathbb{C}\) there exists a unique formal power series \(\bar{L}^-(x)\) in \(\text{End } \mathbb{C}^N \otimes \text{Hom}(Y(R), Y(R)[x^{-1}]_h)\) satisfying \(\bar{L}^-(x)1 = I \otimes 1\) such that for all integers \(n \geq 1\) and a family of variables \(y = (y_1, \ldots, y_n)\) we have

\[
\bar{L}^{-0}(x) (RP)_{01} \ldots (RP)_{0n} \bar{L}^{+12}_{[n]}(y) = \bar{R}_{10}(y_1 e^{hc/a}/x) \ldots \bar{R}_{n0}(y_n e^{hc/a}/x) \\
\times \bar{L}^{+12}_{[n]}(y) (RP)_{10} \ldots (RP)_{n0} \bar{R}_{0n}(y_n e^{-hc/a}/x)^{-1} \ldots \bar{R}_{01}(y_1 e^{-hc/a}/x)^{-1}. \tag{5.3}
\]

The series \(\bar{L}^-(x)\) is invertible in \(\text{End } \mathbb{C}^N \otimes \text{Hom}(Y(R), Y(R)[x^{-1}]_h)\) and it satisfies the following identities for operators on \(Y(R)\):

\[
\bar{R}_{12}(x/y) \bar{L}^-(x) (RP)_{12} \bar{L}^+_2(y) = \bar{L}^+_2(y) (RP)_{21} \bar{L}^+_1(x) \bar{R}_{21}(x/y), \tag{5.4}
\]

\[
\bar{L}^+_1(x) (RP)_{12} \bar{L}^+_2(y) \bar{R}_{12}(y e^{-hc/a}/x) = \bar{R}_{21}(y e^{hc/a}/x) \bar{L}^+_2(y) (RP)_{21} \bar{L}^+_1(x). \tag{5.5}
\]
As before, we write $Y^+(\overline{R})_c$ to indicate that the $\mathbb{C}[[h]]$-module $Y^+(\overline{R})$ is regarded with respect to the action (5.3) of the series $\overline{L}(x)$, which depends on the choice of $c \in \mathbb{C}$. Also, generalizing the notation (5.2), we denote by $\overline{L}_{[n]}(x)$ the expression obtained from (5.2) by replacing all $L^+_i(x_i)$ on the right hand side by $L^-_i(x_i)$. The original series in (5.2) are from now on regarded as operators on $Y^+(\overline{R})_c$ with respect to the algebra multiplication.

**Remark 5.3.** Note that (5.4) is equivalent to the defining relations of the braided Yangian $Y(\overline{R})$ [13] associated with the $R$-matrix $\overline{R}(x)$. As the constant term of $\overline{R}(x)$ with respect to $x$ equals $PR$, one easily checks that the constant term of the series $\overline{L}^-(x)$, as given by (5.3), is the identity. Suppose that $Y(\overline{R})$ is defined over $\mathbb{C}[[h]]$ and that its matrix of generators is denoted by $\overline{L}(x)$; cf. [13, Sect. 6]. Then for any $c \in \mathbb{C}$ the assignment $\overline{L}(x) \mapsto \overline{L}^-(x)$ defines the structure of $Y(\overline{R})$-module over $Y^+(\overline{R})_c$.

The commutation relations for operators $\overline{L}^\pm_i(x)$ can be generalized as follows. First, for the families of variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ and $b \in \mathbb{C}$ we write

$$\overline{R}_{nm}(x e^{bh}/y) = \prod_{1 \leq i \leq n+1} \prod_{1 \leq j \leq n+m} \overline{R}_{ij}(x_i e^{bh}/y_{j-n}).$$

As in (4.8), we shall write bar on the top of the subscripts $n$ and $m$ to indicate that the $R$-matrices are conjugated by the permutation operator $P$, so that we have

$$\overline{R}_{nm}(x e^{bh}/y) = \prod_{1 \leq i \leq n+1} \prod_{1 \leq j \leq n+m} \overline{R}_{ji}(x_i e^{bh}/y_{j-n}). \quad (5.6)$$

We write “opp” in the superscript if the $R$-matrix factors appear in the opposite order,

$$\overline{R}_{nm}^{opp}(x e^{bh}/y) = \prod_{1 \leq i \leq n+1} \prod_{1 \leq j \leq n+m} \overline{R}_{ij}(x_i e^{bh}/y_{j-n}).$$

In addition, we apply analogously the above notation to the products of matrices $\overline{S}(x)$; recall (2.27). The next proposition can be verified by a direct calculation which relies on the relations (5.1), (5.4) and (5.5) and the $R$-matrix properties from Sect. 2.4.

**Proposition 5.4.** For any $n, m \geq 1$ and the variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ the following identities hold for operators on $Y^+(\overline{R})_c$:

$$\overline{R}^{12}_{nm}(x/y) \overline{L}^{\pm 13}_{[n]}(x) (RP)^{12}_{nm} \overline{L}^{\pm 23}_{[n]}(y) = \overline{L}^{\pm 23}_{[m]}(y) (RP)^{12}_{nm} \overline{L}^{\pm 13}_{[n]}(x) \overline{R}^{12}_{nm}(x/y),$$

$$\overline{L}^{-13}_{[n]}(x) (RP)^{12}_{nm} \overline{L}^{23}_{[n]}(y) \overline{R}^{opp 12}_{nm} (ye^{-hc/\mathbb{A}}/x) = \overline{R}^{opp 12}_{nm} (ye^{hc/\mathbb{A}}/x) \overline{L}^{23}_{[m]}(y) (RP)^{12}_{nm} \overline{L}^{-13}_{[n]}(x).$$

5.2. **Action of the algebra $Y_{RTT}(\overline{R})$ (multiplicative case).** We proceed in parallel with Sect. 5.1 to construct the action of the algebra $Y_{RTT}(\overline{R})$. Let $Y^+_{RTT}(\overline{R})$ be the topologically free associative algebra over the ring $\mathbb{C}[[h]]$ generated by the elements $x_{ij}^{-r}$, where $i, j = 1, \ldots, N$ and $r = 1, 2, \ldots$. These are subject to the defining relations

$$\overline{R}(x/y) \overline{T}_1^+(x) \overline{T}_2^+(y) = \overline{T}_2^+(y) \overline{T}_1^+(x) \overline{R}(x/y). \quad (5.7)$$
The elements $\mathcal{T}^+ (x)$ are defined by
\[
\mathcal{T}^+ (x) = \sum_{i,j=1}^N e_{ij} \otimes \mathcal{T}^+_{ij} (x), \quad \text{where} \quad \mathcal{T}^+_{ij} (x) = \sum_{r \geq 1} t_{ij}^{(-r)} x^{r-1}. \quad (5.8)
\]

Consider the formal power series $\mathcal{T}^+_{[n]} (x) := \mathcal{T}^+ (x_1) \ldots \mathcal{T}^+ (x_n)$ whose coefficients belong to $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes \mathcal{Y}^+_{\text{RTT}} (\overline{R})$. Clearly, the coefficients of all matrix entries of $\mathcal{T}^+_{[n]} (x)$ with $n \geq 1$ along with the unit 1 span an $h$-adically dense $\mathbb{C}[[h]]$-submodule of $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$.

We now give the RTT-type counterpart of Proposition 5.2. Roughly speaking, it can be regarded as the multiplicative version of Proposition 3.1. It can be easily proved using the properties of the $R$-matrix and the defining relations for the algebra $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$.

**Proposition 5.5.** For any $c \in \mathbb{C}$ there exists a unique formal power series $\mathcal{T}^- (x)$ in $\text{End } \mathbb{C}^N \otimes \text{Hom} (\mathcal{Y}^+_{\text{RTT}} (\overline{R}), \mathcal{Y}^+_{\text{RTT}} (\overline{R})[x^{-1}]_h)$ satisfying $\mathcal{T}^- (x) 1 = I \otimes 1$ such that for all integers $n \geq 1$ and the variables $y = (y_1, \ldots, y_n)$ we have
\[
\mathcal{T}^+_{[n]} (y) = \mathcal{R}_{10}(y_1 e^{hc/a}/x) \ldots \mathcal{R}_{n0}(y_n e^{hc/a}/x) \\
\times \mathcal{R}^{[1]}(y_1 e^{hc/a}/x)^{-1} \ldots \mathcal{R}^{[n]}(y_n e^{hc/a}/x)^{-1}. \quad (5.9)
\]
The series $\mathcal{T}^- (x)$ is invertible in $\text{End } \mathbb{C}^N \otimes \text{Hom} (\mathcal{Y}^+_{\text{RTT}} (\overline{R}), \mathcal{Y}^+_{\text{RTT}} (\overline{R})[x^{-1}]_h)$ and it satisfies the following identities for operators on $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$:
\[
[\mathcal{R}^{[1]}(x/y), \mathcal{T}^-_{[2]}(y)] = [\mathcal{T}^-_{[1]}(x), \mathcal{R}^{[2]}(y)] = [\mathcal{T}^-_{[1]}(x), \mathcal{R}^{[2]}(y)] = [\mathcal{R}^{[1]}(x/y), \mathcal{T}^-_{[2]}(y)] = \mathcal{T}^-_{[2]}(y). \quad (5.10)
\]
\[
\mathcal{T}^-_{[1]}(x) \mathcal{T}^-_{[2]}(y) = \mathcal{R}^{[1]}(x/y) \mathcal{T}^-_{[1]}(x) \mathcal{R}^{[2]}(y), \quad \mathcal{T}^-_{[2]}(y) \mathcal{R}^{[1]}(x/y) = \mathcal{T}^-_{[1]}(x) \mathcal{R}^{[2]}(y). \quad (5.11)
\]

The commutation relations from the proposition can be easily generalized to the multiple operator case, as was done for the braided Yangian setting in Proposition 5.4.

Again, we use the subscript $c$ and write $\mathcal{Y}^+_{\text{RTT}} (\overline{R})_c$ to indicate that the $\mathbb{C}[[h]]$-module $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$ is regarded with respect to the action (5.9) of $\mathcal{T}^- (x)$, which depends on the choice of $c \in \mathbb{C}$. Also, we write $\mathcal{T}^-_{[n]}(x) = \mathcal{T}^-_{[1]}(x_1) \ldots \mathcal{T}^-_{[n]}(x_n)$. The series $\mathcal{T}^-_{[n]}(x)$ are from now on regarded as operators on $\mathcal{Y}^+_{\text{RTT}} (\overline{R})_c$ with respect to the algebra multiplication.

**Remark 5.6.** The relations (5.10) are equivalent to the defining relations of the algebra $\mathcal{Y}^+_{\text{RTT}} (\overline{R})_c$ [13] associated to (2.24). Suppose that $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$ is defined over $\mathbb{C}[[h]]$ and that its matrix of generators is denoted by $\mathcal{T}^- (x)$; cf. [13, Sect. 6]. Then for any $c \in \mathbb{C}$ the assignment $\mathcal{T}^- (x) \mapsto \mathcal{T}^- (x)$ defines the structure of $\mathcal{Y}^+_{\text{RTT}} (\overline{R})$-module over $\mathcal{Y}^+_{\text{RTT}} (\overline{R})_c$.

### 5.3. Constructing $\phi$-coordinated $\mathcal{V}_c (\mathcal{R})$-modules

The notion of $\phi$-coordinated module, where $\phi = \phi (z_2, z_0)$ is an associate of the one-dimensional additive formal group, was introduced by Li [27]. In this paper we consider its straightforward generalization over the ring $\mathbb{C}[[h]]$; cf. [24, Def. 5.1]. Regarding the associate, we set
\[
\phi (z_2, z_0) = z_2 e^{-2z_0/a}. \quad (5.12)
\]

To present the precise definition, we need some notation. Let $V$ be a topologically free $\mathbb{C}[[h]]$-module. Suppose we have some elements $A, C \in \text{Hom} (V, V[[z_1^{\pm 1}, z_2^{\pm 1}]]$ and
Let \((V, Y, 1, S)\) be an \(h\)-adic quantum vertex algebra. A \(\phi\)-coordinated \(V\)-module is a pair \((\mathcal{M}^\phi, Y_{\mathcal{M}^\phi})\) such that \(\mathcal{M}^\phi\) is a topologically free \(\mathbb{C}[[[h]]]\)-module and

\[
Y_{\mathcal{M}^\phi}(z) = Y_{\mathcal{M}^\phi}((\cdot, z)) : \mathcal{V} \otimes \mathcal{M}^\phi \to \mathcal{M}^\phi((z))_h
\]

\[
u \otimes w \mapsto Y_{\mathcal{M}^\phi}(\nu \otimes w) = Y_{\mathcal{M}^\phi}(\nu, z)w = \sum_{r \in \mathbb{Z}} u_r w z^{-r-1}
\]

is a \(\mathbb{C}[[[h]]]\)-module map which satisfies \(Y_{\mathcal{M}^\phi}(1, z)w = w\) for all \(w \in \mathcal{M}^\phi\) and the weak associativity: for any \(u, v \in \mathcal{V}, k \in \mathbb{Z}_{\geq 0}\) there exists \(r \in \mathbb{Z}_{\geq 0}\) such that

\[
(z_1 - z_2)^r Y_{\mathcal{M}^\phi}(u, z_1)Y_{\mathcal{M}^\phi}(v, z_2) \in \text{Hom} (\mathcal{M}^\phi, \mathcal{M}^\phi ((z_1, z_2))) \mod h^k, \tag{5.13}
\]

\[
\left((z_1 - z_2)^r Y_{\mathcal{M}^\phi}(u, z_1)Y_{\mathcal{M}^\phi}(v, z_2)\right)_{z_1 = \phi(z_2, z_0)} \mod h^k
\]

\[
- (\phi(z_2, z_0) - z_2)^r (\phi Y_{\mathcal{M}^\phi}(Y(u, z_0)v, z_2) \in h^k \text{Hom} (\mathcal{M}^\phi, \mathcal{M}^\phi([[z_0^{\pm 1}, z_2^{\pm 1}]]) \tag{5.14}
\]

In the next theorem, we construct an example of \(\phi\)-coordinated \(V_c(R)\)-module using the structure of braided Yangian module from Sect. 5.1. To emphasize that \(Y^*(R)\) is now regarded as a \(\phi\)-coordinated \(V_c(R)\)-module, we denote it by \(\mathcal{M}^\phi(R)_c\).

**Theorem 5.8.** For any \(c \in \mathbb{C}\) there exists a unique structure of \(\phi\)-coordinated \(V_c(R)\)-module on \(\mathcal{M}^\phi(R)_c = Y^*(R)_c\) such that for all integers \(n \geq 1\) and the families of variables \(u = (u_1, \ldots, u_n)\) and \(x = (x_1, \ldots, x_n)\) we have

\[
Y_{\mathcal{M}^\phi(R)_c}(T^+_n[u]) = \bar{L}^+_n(x)_{x_i = \phi(z, u_i)} \bar{L}^-_n(x)_{x_i = \phi(z, u_i + hc/2)}^{-1} \tag{5.15}
\]

**Proof.** We omit some technical details, as they go in parallel with the proof of Theorem 4.7, and only discuss the differences which occur in this case. First, regarding the expression in (5.15), the substitutions \(x_i = \phi(z, u_i)\) and \(x_i = \phi(z, u_i + hc/2)\) are assumed to be carried out simultaneously for all \(i = 1, \ldots, n\). Note that the second family of relations in Proposition 5.4 implies the identity

\[
\bar{L}^-_n(x)^{-1} \bar{L}^+_n(y)^{-1} \bar{L}^{+23}_{nm}(y) y^{hc/a} / x = \bar{L}^{+12}_{nm}(y) y^{hc/a} / x \bar{L}^-_n(x)^{-1} \bar{L}^{+12}_{nm}(y) y^{hc/a} / x \bar{L}^-_n(x)^{-1} \bar{L}^{+12}_{nm}(y) y^{hc/a} / x \tag{5.16}
\]

By applying (5.16) to 1 and then arguing as in the proof of Lemma 4.5 we find

\[
\bar{L}^-_n(x)^{-1} \in (\text{End} \mathbb{C}[[[h]]]) \otimes \text{Hom} (\mathcal{M}^\phi(R), \mathcal{M}^\phi(c), \mathcal{M}^\phi(R)(x_1^{-1}, \ldots, x_n^{-1})_h),
\]

so that the substitutions in (5.15) are well-defined.

The fact that the \(\mathbb{C}[[[h]]]\)-module map \(Y_{\mathcal{M}^\phi(R)_c}(\cdot, z)\) is well-defined by (5.15) is verified as in the proof of Theorem 4.7. However, the corresponding argument also employs (2.29) and the following observation: if we replace the \(R\)-matrix \(R(u - v)\) (resp. \(R(x/y)\)) in (3.1) (resp. (5.1) and (5.4)) by \(R'(u - v)\) (resp. \(R'(x/y)\)), we obtain an equivalent equality.
Regarding (5.13), note that this requirement is necessary in order for the substitution $z_1 = \phi(z_2, z_0)$ in the first term of (5.14) to be well-defined. It is verified in a similar way to the equality between (4.15) and (4.17). Nonetheless, we give some details in order to take care of the differences. First, by applying $Y_{\mathcal{M}_s}(\cdot, z_1)(1 \otimes Y_{\mathcal{M}_s}(\cdot, z_2))$ to (4.13) we obtain

$$\bar{L}_{[n]}^{+13}(x)\big|_{x_i=\phi(z_1,u_i)} T_{[n]}^{-13}(x)^{-1}\big|_{y_j=\phi(z_1,u_i+h/2)c} \bar{L}_{[m]}^{+23}(y)^{-1}\big|_{y_j=\phi(z_2,v_j+hc/2)},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. Next, we employ (5.16) to swap the second and the third term above, thus getting

$$\bar{S}_{\bar{n}\bar{m}}^{12} = \bar{S}_{\bar{n}\bar{m}}(ye^{hc/a}/x)\big|_{x_i=\phi(z_1,u_i+hc/2)} \text{ and } \bar{R}_{nm}^{opp} = \bar{R}_{nm}^{opp}(ye^{-hc/a}/x)\big|_{x_i=\phi(z_1,u_i+hc/2)}.$$

The property (5.13) now follows by examining the coefficients of the monomials (4.18) in (5.17). Indeed, by the form of $\bar{S}_{\bar{n}\bar{m}}^{12}$ and $\bar{R}_{nm}^{opp}$ there exists an integer $r_1 \geq 1$ such that the coefficients of the given monomials in the product $(z_1 - z_2)^r \bar{S}_{\bar{n}\bar{m}}^{12} \bar{R}_{nm}^{opp}$ possess only finitely many negative powers of the variables $z_1$ and $z_2$. Next, by the first family of relations in Proposition 5.4 there exists an integer $r_2 \geq 1$ such that the coefficients of the given monomials in the product

$$(z_1 - z_2)^r \bar{L}_{[n]}^{-13}(x)^{-1}\big|_{x_i=\phi(z_1,u_i+hc/2)} ((R P)_{nm}^{12})^{-1} \bar{L}_{[m]}^{-23}(y)^{-1}\big|_{y_j=\phi(z_2,v_j+hc/2)}$$

possess only finitely many negative powers of $z_1$ and $z_2$. As the remaining terms in (5.17) contain only nonnegative powers of $z_1$ and $z_2$, we conclude that the property (5.13) holds for the monomials in (4.18) for any integer $r \geq r_1 + r_2$.

Finally, the weak associativity property (5.14) is proved by the same argument as in the proof of Theorem 4.7. However, in this case, its proof also requires the use of explicit connection between the additive and multiplicative $R$-matrices, (2.14) and (2.24), as given in (2.25) and (2.28). Note that the former $R$-matrix governs the $h$-adic quantum vertex algebra structure and the latter the $\phi$-coordinated module structure.\footnote{At this point, it is worth it to recall that, in contrast, the $h$-adic quantum vertex algebra structure and the corresponding module structure from Theorem 4.7 are both given in terms of the same $R$-matrix (2.14) with additive spectral parameter.} In fact, exactly because of the form of the substitution which appears on the left hand side of the identities (2.25) and (2.28), the particular choice of associate (5.12) enables us to establish the $\phi$-coordinated $\mathcal{V}_c(R)$-module structure on $\mathcal{M}_s(R)_c$. \qed

In the following theorem, we construct an example of $\phi$-coordinated $\mathcal{V}_c(R)$-module using the module for the algebra $Y_{\mathcal{M}_s}(\bar{R})$ established in Sect. 5.2. We omit its proof as it goes in parallel with the proofs of Theorems 4.7 and 5.8. To emphasize that $Y_{\mathcal{M}_s}(\bar{R})_c$ is now regarded as a $\phi$-coordinated $\mathcal{V}_c(R)$-module, we denote it by $\mathcal{M}_s^{\phi}(\bar{R})_c$.\footnote{See the previous footnote.}
Theorem 5.9. For any $c \in \mathbb{C}$ there exists a unique structure of $\phi$-coordinated module for the $h$-adic quantum vertex algebra $V_c(R)$ on $M_{RTT}^\phi(R)_c = Y_{RTT}^+(\overline{R})_c$ such that for all integers $n \geq 1$ and the variables $u = (u_1, \ldots, u_n)$, $x = (x_1, \ldots, x_n)$ we have

$$Y_{M_{RTT}^\phi(R)_c}(T^+_n(u)1, z) = T^+_n(x)\big|_{x_i = \phi(z, u_i)} \overline{T}^+_n(x)^{-1}\big|_{x_i = \phi(z, u_i + hc/2)}. \quad (5.18)$$

5.4. On the algebra $D(\overline{R})_c$. In this subsection, we define a certain algebra via relations which closely resemble those from the RTT-presentation of the quantum affine algebra in type $A$; see, e.g., [8, 31]. For any $c \in \mathbb{C}$ let $D(\overline{R})_c$ be the topologically free algebra over $\mathbb{C}[[h]]$ generated by the elements $t_{ij}^{(r)}$ with $i, j = 1, \ldots, N$ and $r \in \mathbb{Z}$. Its defining relations are (5.7), (5.10) and (5.11), where $\overline{T}^+(x)$ is defined as in (5.8) and $\overline{T}^-(x)$ is given by

$$\overline{T}^-(x) = \sum_{i,j=1}^N e_{ij} \otimes \overline{t}_{ij}(x) \quad \text{for} \quad \overline{t}_{ij}(x) = \sum_{r \geq 0} \overline{t}_{ij}^{(r)} x^{-r}. \quad (5.19)$$

The $D(\overline{R})_c$-module $W$ is said to be restricted if it is topologically free as a $\mathbb{C}[[h]]$-module and the action of $\overline{T}^-(x)$ on $W$ is invertible and such that $\overline{T}^-(x)_{\pm 1}$ belongs to $\text{End} \mathbb{C}^N \otimes \text{Hom}(W, W[x^{-1}]_h)$. Clearly, Proposition 5.5 implies

Corollary 5.10. There exists a unique structure of restricted $D(\overline{R})_c$-module on $Y_{RTT}^+(\overline{R})_c$ such that the action of $\overline{T}^-\phi(x)$ is given by (5.9) and $\overline{T}^+(x)$ acts by multiplication.

Using (5.10) one easily checks that on any restricted $D(\overline{R})_c$-module $W$ we have

$$\overline{T}_{n}^-(x_1, \ldots, x_n)_{\pm 1} \in \text{End} \mathbb{C}^N \otimes \text{Hom}(W, W[x_1^{-1}, \ldots, x_n^{-1}]_h) \quad \text{for all} \ n \geq 1.$$ 

Recall that by $\phi = \phi(z_2, z_0)$ we denote the associate given by (5.12). We have

Corollary 5.11. Let $W$ be a restricted $D(\overline{R})_c$-module. There exists a unique structure of $\phi$-coordinated $\mathcal{V}_c(R)$-module on $W$ such that

$$Y_{\mathcal{V}_c(R)}(T^+_n(u)1, z) = \overline{T}^+_n(x)\big|_{x_i = \phi(z, u_i)} \overline{T}^+_n(x)^{-1}\big|_{x_i = \phi(z, u_i + hc/2)}. \quad (5.20)$$

Furthermore, if $W_1 \subset W$ is a restricted $D(\overline{R})_\phi$-submodule, then it is also a $\phi$-coordinated $\mathcal{V}_c(R)$-submodule with respect to the suitable (co)restriction of $Y_{\mathcal{V}_c(R)}(\cdot, z)$.

Proof. The first assertion can be proved by the same arguments as Theorem 5.9. More specifically, using the defining relations for the algebra $D(\overline{R})_c$ and the fact that $W$ is restricted one can directly verify the requirements from Definition 5.7. As for the second assertion, it is sufficient to observe that, due to (5.20), for any $\mathbb{C}[[h]]$-submodule $W_1 \subset W$ the inclusions $\overline{T}^\pm(x)W_1 \subset \text{End} \mathbb{C}^N \otimes W_1[[x^\pm 1]]$ imply that $Y_{\mathcal{V}_c(R)}(v, z)w$ belongs to $W_1[[z^\pm 1]]$ for all $v \in \mathcal{V}_c(R)$ and $w \in W_1$. \hfill $\Box$

By Corollary 5.10, $Y_{RTT}^+(\overline{R})_c = M_{RTT}^\phi(R)_c$ is a restricted $D(\overline{R})_c$-module, so Corollary 5.11 implies that it possesses a structure of $\phi$-coordinated $\mathcal{V}_c(R)$-module. However, in this particular case, the corresponding $\phi$-coordinated module structure coincides with the one which is already established by Theorem 5.9.
Remark 5.12. The braided Yangian counterpart of Corollary 5.11 can be obtained as follows. Using the family of relations (5.1), (5.4) and (5.5) one can again introduce an algebra over \( \mathbb{C}[[h]] \). Then, by arguing as in the proof of Theorem 5.8, one can show that all (suitably defined) restricted modules for the aforementioned algebra are naturally equipped with the structure of \( \phi \)-coordinated \( V_c(\mathcal{R}) \)-module. It might be an interesting problem to investigate whether such algebras, along with braided Yangians, can be realized as subalgebras of \( D(\mathcal{R})_c \), or of the corresponding algebras \( Y_{RTT}(\mathcal{R}) \), as is the case with certain classes of reflection algebras; cf. [29,32].

Remark 5.13. If \( \mathcal{R} \) is a skew-invertible involutive symmetry, one can analogously use relations (3.1), (3.5) and (3.6), given in terms of the \( \mathcal{R} \)-matrix (3.12), to define a certain algebra which resembles the double Yangian in type \( A \); cf. [16,22]. The analogue of Corollary 5.11 for such algebra states that its (suitably defined) restricted modules are equipped with the structure of \( \phi \)-coordinated \( V_c(\mathcal{R}) \)-module. Thus, in particular, the \( \phi \)-coordinated modules are no longer required in such setting.

6. Quantum Determinants

In this section, motivated by the explicit expressions for the determinants for the algebras \( Y_{RTT}(\mathcal{R}) \) and \( Y(\mathcal{R}) \) from [13], we obtain certain families of central elements for the \( h \)-adic quantum vertex algebra \( V_c(\mathcal{R}) \) and invariants for its (\( \phi \)-coordinated) modules from Sects. 4 and 5.

6.1. Preliminaries on skew-symmetrizers. Here we recall some properties of the skew-symmetrizer associated with the skew-invertible Hecke symmetry \( \mathcal{R} \) which we need later on. Our exposition closely follows [13, Sect. 6], which we adapt to our setting by taking \( q = e^h \). To simplify the notation, for any positive integer \( k \) we shall write

\[
[k]_h = \frac{e^{kh} - e^{-kh}}{e^h - e^{-h}}.
\]

Note that \([k]_h\) belongs to \( k + h \mathbb{C}[[h]] \), so that we have \([k]_h^{-1} \in \frac{1}{k} + h \mathbb{C}[[h]]\). Define

\[
\widehat{\mathcal{R}}(x) = R + \frac{(e^h - e^{-h})x}{1 - x} I \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N[[x, h]].
\] (6.1)

Clearly, \( \widehat{\mathcal{R}}(x) \) is obtained from the \( \mathcal{R} \)-matrix (2.24) by removing its normalization term and multiplying it by the permutation operator \( P \) from the left. For any positive integer \( m \) the skew-symmetrizer \( \mathcal{P}^{(m)} \) associated to \( \mathcal{R} \) can be defined as an element of \( (\text{End } \mathbb{C}^N)^{\otimes m}[[h]] \) such that

\[
\mathcal{P}^{(1)} = I \quad \text{and} \quad \mathcal{P}^{(k+1)} = (-1)^k [k + 1]^{-1}_h \widehat{\mathcal{R}}_{12}(e^{2h}) \widehat{\mathcal{R}}_{23}(e^{4h}) \ldots \widehat{\mathcal{R}}_{k-k+1}(e^{2(k-1)h}) P^{(k)}_{1 \ldots k}.
\] (6.2)

On the right hand side, we use the usual notation convention, where the subscripts of elements indicate the tensor copies on which they are applied. For example, \( P^{(k)}_{1 \ldots k} \) means that \( \mathcal{P}^{(k)} \) acts on the first \( k \) tensor factors of \( (\text{End } \mathbb{C}^N)^{\otimes (k+1)}[[h]] \).
From now on, we assume that the rank of the skew-invertible Hecke symmetry \( R \) is \((M|0)\); cf. [12, Def. 2]. Recall that the value of \( M \) uniquely determines the normalization series from Sect. 2.2. By [13, Lemma 19] we have

\[
P^{(M)}_{2...M+1} \hat{R}_{1\to M}(x) = (-1)^{M-1} e^{h} [M]_{h} \frac{1 - x e^{-2(M-1)h}}{1 - x e^{-2Mh}} P^{(M)}_{2...M+1} P^{(M)}_{1...M}, \tag{6.3}
\]

where \( \hat{R}_{1\to M}(x) \) stands for the product

\[
\hat{R}_{1\to M}(x) = \hat{R}_{12}(x) \hat{R}_{23}(x e^{-2h}) \ldots \hat{R}_{M+1}(x e^{-2(M-1)h}).
\]

Finally, due to [15, Lemma 2.1], the skew-symmetrizer possesses the following properties:

\[
P^{(M)}_{1...M} R_{M+1} \ldots R_{23} \hat{R}_{12} = (-1)^{M-1} e^{h} [M]_{h} P^{(M)}_{1...M} P^{(M)}_{2...M+1},
\]

\[
P^{(M)}_{2...M+1} P^{(M)}_{1...M} P^{(M)}_{2...M+1} = [M]_{h}^2 P^{(M)}_{2...M+1}.
\]

As \( R \) is a skew-invertible Hecke symmetry of rank \((M|0)\), the image of the skew-symmetrizer \( P^{(M)} \) is spanned by a single nonzero vector. Let us denote one such vector by \( v_{R} \). It can be proved that there exists a linear map \( N_{R} \) such that for all \( w \in \mathbb{C}^{N} \) we have \( R_{12} R_{23} \ldots R_{M+1}(w \otimes w) = N_{R}(w) \otimes v_{R} \); see [10, 13] for more details and the explicit expression for \( N_{R} \).

**Remark 6.1.** The main results in the next two sections are proved under the assumption that \( N_{R} \) is a scalar matrix. This requirement goes back to [10], where it is proved that quantum determinants in certain quantum matrix algebras are central if and only if \( N_{R} \) is scalar. Aside from that paper, the reader may find different examples of families of Hecke symmetries and involutive symmetries which satisfy this condition, as well as the examples of those which do not, in [11, 13]. In particular, the matrix \( N_{R} \) is scalar for the symmetry \( R \) given by (2.5).

### 6.2. Center of \( \mathcal{V}_{c}(R) \)

Recall that by \( R \) we denote a skew-invertible Hecke-symmetry of rank \((M|0)\) and by \( \Psi \) its skew-inverse (2.4). Let \( C = tr_{2} \Psi \), where the subscript 2 indicates that the trace is taken over the second tensor factor. Thus, \( C = (c_{ij}) \) is the square matrix of order \( N \). The \( R \)-trace \( tr^{R} A \) of an arbitrary \( N \times N \) matrix \( A = (a_{ij}) \) is defined by

\[
tr^{R} A = tr (A \cdot C) = \sum_{i,j=1}^{N} a_{ij} c_{ji}.
\]

It is generalized to multiple tensor factors by setting \( tr^{R}_{1,...,k} = tr^{R}_{1} \ldots tr^{R}_{k} \), where \( tr^{R}_{i} \) denotes the \( R \)-trace taken over the \( i \)-th tensor factor. Motivated by the form of the determinant [13, Rem. 22], we consider the power series

\[
qdet^{+}(u) = tr^{R}_{1,...,M} P^{(M)}_{1} T_{1}^{+}(u) T_{2}^{+}(u + ah) \ldots T_{M}^{+}(u + a(M-1)h) \in \text{End} \mathcal{V}_{c}(R)[[u]].
\]

Note that the given expression is well-defined because \( \mathcal{V}_{c}(R) \) is \( h \)-adically complete.

Before we state our next result, we recall that the *center* \( \mathfrak{z}(V) \) of an \( h \)-adic quantum vertex algebra \( V \) (see [1, 19] for more information) is defined by

\[
\mathfrak{z}(V) = \{ w \in V : Y(v, z) w \in V[[z]] \text{ for all } v \in V \}.
\]
Theorem 6.2. If $N_R$ is a scalar matrix, then all coefficients of the series $\text{qdet } T^+(u) 1$ belong to the center $\mathfrak{z}(\mathcal{V}_c(R))$ for any $c \in \mathbb{C}$.

Proof. (1) Suppose that the following identity for operators on $\mathcal{V}_c(R)$ holds:

$$T^-(u) \text{qdet } T^+(v) = \text{qdet } T^+(v) T^-(u). \quad (6.4)$$

It implies that for any integer $n \geq 1$ and the variables $u = (u_1, \ldots, u_n)$ and $z$ we have

$$T_{[n]}^{-}(z + u + hc/2)^{-1} \text{qdet } T^+(v) = \text{qdet } T^+(v) T_{[n]}^{-}(z + u + hc/2)^{-1}. \quad (6.3)$$

By applying this on the vacuum vector $1$ and using $T^-(u)^{-1} 1 = I \otimes 1$ we get

$$T_{[n]}^{-}(z + u + hc/2)^{-1} \text{qdet } T^+(v) 1 = \text{qdet } T^+(v) 1. \quad (6.7)$$

Next, using the explicit expression (3.9) for the vertex operator map we find

$$Y(T^+_{[n]}(u), z) \text{qdet } T^+(v) 1 = T^+_{[n]}(z + u) \text{qdet } T^+(v) 1. \quad (6.5)$$

The right hand side contains only nonnegative powers of the variable $z$. Moreover, as the coefficients of all matrix entries of $T^+_{[n]}(u_1, \ldots, u_n)$ with $n \geq 1$ along with $1$ span an $h$-adically dense $\mathbb{C}[[h]]$-submodule of $\mathcal{V}_c(R)$, we conclude from (6.5) that $Y(w, z) \text{qdet } T^+(v) 1$ belongs to $\mathcal{V}_c(R[[z]])$ for all $w \in \mathcal{V}_c(R)$, so that all coefficients of $\text{qdet } T^+(v) 1$ are elements of the center $\mathfrak{z}(\mathcal{V}_c(R))$. Therefore, in order to prove the theorem, it is sufficient to check that the equality (6.4) holds. This can be done by arguing as in the proof of [13, Prop. 21]; see also [13, Rem. 22]. However, we present the underlying calculations below in order to take care of differences which occur in our setting, such as the dependency on the properties of the normalization series from Sect. 2.2 (when $c$ is nonzero).

(2) Conjugating relation (3.6) by the permutation operator $P$ and using (2.15) we get

$$f(-u + v - hc/2) \hat{\mathcal{R}}_{12}(e^{-\frac{2}{\alpha}(-u+v-hc/2)}) T^+_1(v) T^-_2(u) = f(-u + v + hc/2) T^-_1(u) T^+_2(v) \hat{\mathcal{R}}_{12}(e^{-\frac{2}{\alpha}(-u+v+hc/2)}). \quad (6.6)$$

From this, one easily derives the more general identity

$$F^- \hat{\mathcal{R}}_{1 \rightarrow M}(e^{-\frac{2}{\alpha}(-u+v-hc/2)}) T^+_1(v) T^+_2(v + ah) \cdots T^+_M(v + a(M - 1)h) T^-_{M+1}(u) = T^-_1(u) T^+_2(v) T^+_3(v + ah) \cdots T^+_M(v + a(M - 1)h) \hat{\mathcal{R}}_{1 \rightarrow M}(e^{-\frac{2}{\alpha}(-u+v+hc/2)}) F^+, \quad (6.8)$$

where, due to the property (2.13) of the series $f(u)$, the expressions $F^\pm$ are given by

$$F^\pm = \prod_{i=1}^{M} f(-u + v + a(i - 1)h \pm hc/2) = \frac{1 - e^{-\frac{2}{\alpha}(-u+v\pm hc/2)}}{1 - e^{-\frac{2}{\alpha}(-u+v\pm hc/2)}} e^{-2(M-1)h}. \quad (6.7)$$

By combining (6.3) and (6.7) we find

$$T_{2 \ldots M+1}^{(M)} F^\pm \hat{\mathcal{R}}_{1 \rightarrow M}(e^{-\frac{2}{\alpha}(-u+v\pm hc/2)}) = (-1)^{M-1} e^{h \cdot [M]_h} T_{2 \ldots M+1}^{(M)} \mathcal{P}_{1 \ldots M}. \quad (6.8)$$
Therefore, multiplying (6.6) by $P_{2...M+1}^{(M)}$ from the left gives us

\[
(-1)^{M-1} e^h [M]_h P_{2...M+1}^{(M)} P_{1...M}^{(M)} T_1^+(v) \ldots T_M^+(v + a(M - 1)h) T_{M+1}^{-}(u) = T_1^{-}(u) P_{2...M+1}^{(M)} T_2^+(v) \ldots T_{M+1}^{+}(v + a(M - 1)h) \hat{R}_{1\rightarrow M}(e^{-\frac{2}{h}(-u + v + h^2/2)}) F^+.
\]

Next, by using the identities (3.1) and (6.2) we obtain

\[
P_{1...M}^{(M)} T_1^+(v) T_2^+(v + a h) \ldots T_M^+(v + a(M - 1)h) = T_1^+(v + a(M - 1)h) \ldots T_M^+(v + a h) T_{M+1}^{+}(v) P_{1...M}^{(M)}.
\]

Applying this to the tensor factors $2, \ldots, M + 1$ on the right hand side of (6.9), then using (6.8) and, finally, canceling common factors on both sides, (6.9) takes the form

\[
P_{2...M+1}^{(M)} P_{1...M}^{(M)} T_1^+(v) T_2^+(v + a h) \ldots T_M^+(v + a(M - 1)h) T_{M+1}^{-}(u) = T_1^{-}(u) T_2^+(v + a(M - 1)h) \ldots T_M^+(v + a h) T_{M+1}^{+}(v) P_{2...M+1}^{(M)} P_{1...M}^{(M)}.
\]

As in the corresponding part of the proof of [13, Prop. 21], this turns to the equality

\[
P_{2...M+1}^{(M)} P_{1...M}^{(M)} qdet T^+(v) T_{M+1}^{-}(u) = T_1^{-}(u) qdet T^+(v) P_{2...M+1}^{(M)} P_{1...M}^{(M)}
\]

which can be transformed as in [13, Rem. 22] into

\[
N_R T^{-}(u) qdet T^+(v) = qdet T^+(v) T^{-}(u) N_R.
\]

Hence, if the matrix $N_R$ is scalar, this implies (6.4), as required. \qed

\section{6.3. Invariants of (\phi-coordinated) $V_c(R)$-modules} In this subsection, we consider the image of the constant term of the quantum determinant under the (\phi-coordinated) $V_c(R)$-module maps (4.12), (5.15) and (5.18). We shall write $(RP)_{[I]} = I \in \text{End} \mathbb{C}^N$ and

\[
(RP)_{[M]} = \prod_{1 \leq i < M} (RP)_{i;i+1} \ldots (RP)_{i;M} \in (\text{End} \mathbb{C}^N)^\otimes M[[h]] \text{ for } M > 1,
\]

where, as before, $R$ is a skew-invertible Hecke-symmetry of rank $(M|0)$ and $I$ the identity matrix. Notice that for $u = (u_1, \ldots, u_M)$ and $x = (x_1, \ldots, x_M)$ we have

\[
L_{[M]}^{-1}(u) = L_{[M]}^{-1}(x) = (RP)_{[M]}^{-1} \otimes 1 \text{ and } T_{[M]}^{-1}(x) = I \otimes M \otimes 1, \quad (6.10)
\]

where $I \in \text{End} \mathbb{C}^N$ denotes the identity matrix of order $N$. We now employ the aforementioned (\phi-coordinated) $V_c(R)$-module maps to introduce the following power series

\[
qdet L^+(z) = Y_{M(R)}(\text{qdet } T^+(0)1, z) 1 \in \mathcal{M}(R)_c[[z]],
\]

\[
qdet T^+(z) = Y_{M^\phi(R)}(\text{qdet } T^+(0)1, z) 1 \in \mathcal{M}^\phi(R)_c[[z]],
\]

\[
qdet T^+(z) = Y_{\mathcal{M}_{RTT}(R)}(\text{qdet } T^+(0)1, z) 1 \in \mathcal{M}_{RTT}(R)_c[[z]].
\]

As suggested by our notation, we shall refer to them as quantum determinants.
By using the expressions for the \((\phi\text{-coordinated})\ \mathcal{V}_c(R)\)-module maps (4.12), (5.15) and (5.18) and the identities in (6.10) one easily derives explicit formulae

\[
q\det L^+(z) = \text{tr}_{1, \ldots, M}^{R} P^{(M)} L^+_{[M]}(z, z + a h, \ldots, z + a (M - 1) h) (RP)^{-1}_{[M]},
\]

(6.11)

\[
q\det \overline{L}^+(z) = \text{tr}_{1, \ldots, M}^{R} P^{(M)} \overline{L}^+_{[M]}(z, z e^{-2 h}, \ldots, z e^{-2(M - 1) h}) (RP)^{-1}_{[M]},
\]

(6.12)

\[
q\det \overline{T}^+(z) = \text{tr}_{1, \ldots, M}^{R} P^{(M)} \overline{T}^+_{[M]}(z, z e^{-2h}, \ldots, z e^{-2(M - 1) h}).
\]

(6.13)

Therefore, the determinants (6.11), (6.12) and (6.13) can be naturally regarded as operators on the \(\mathbb{C}[[h]]\)-modules \(\mathcal{M}(R)_c\), \(\mathcal{M}^\phi(R)_c\) and \(\mathcal{M}^\phi_{RTT}(R)_c\) with respect to the multiplication in the corresponding algebra \(Y^+(R)\), \(Y^+(R)\) and \(Y^+_{RTT}(R)\).

**Lemma 6.3.** Let \(c \in \mathbb{C}\). The next identities hold for operators on \(\mathcal{M}(R)_c\) and \(\mathcal{M}^\phi(R)_c\):

\[
q\det L^+(z_1) L^-(z_2) = L^-(z_2) q\det L^+(z_1) \quad \text{and} \quad q\det \overline{L}^+(z_1) \overline{L}^-(z_2) = \overline{L}^-(z_2) q\det \overline{L}^+(z_1).
\]

If \(N_R\) is a scalar matrix, we have the following identity for operators on \(\mathcal{M}^\phi_{RTT}(R)_c\):

\[
q\det \overline{T}^+(z_1) \overline{T}^-(z_2) = \overline{T}^-(z_2) q\det \overline{T}^+(z_1).
\]

**Proof.** The lemma is verified by the arguments which closely follow the proofs of [13, Prop. 21, Rem. 22] and Theorem 6.2 and rely on the properties of the normalizing series and the skew-symmetrizer from Sects. 2.2 and 6.1. In order to employ these arguments, it is useful to observe that, despite seemingly different expressions, \(q\det \overline{L}^+(z)\) is of the same form as the series \(e_M(z)\) which belongs to the center of the braided Yangian [13, Prop. 21]. Indeed, this is easily proved by moving all permutation operators \(P\) which appear in the above expression for \(q\det \overline{L}^+(z)\) to the right and then using the braid relation (2.2), which is satisfied by \(R\). The same remark applies to the other two determinants. \(\square\)

We are now ready to present the vertex algebraic interpretation of quantum determinants. Suppose \(W\) is a \((\phi\text{-coordinated})\) module for the \(h\)-adic quantum vertex algebra \(V\) with respect to the map \(Y_W(\cdot, z)\). Define the \textit{submodule of invariants} \(z(W)\) of \(W\) by

\[
z(W) = \{ w \in W : Y_W(v, z)w \in W[[z]] \text{ for all } v \in V \}.
\]

The following theorem shows that the quantum determinants produce invariants of \((\phi\text{-coordinated})\) \(\mathcal{V}_c(R)\)-modules. Due to Lemma 6.3, it can be verified by arguing as in the first part of the proof of Theorem 6.2.

**Theorem 6.4.** Let \(c \in \mathbb{C}\). All coefficients of \(q\det L^+(z)1\) and \(q\det \overline{L}^+(z)1\) belong to the corresponding submodule of invariants \(z(\mathcal{M}(R)_c)\) and \(z(\mathcal{M}^\phi(R)_c)\), respectively. Moreover, if \(N_R\) is a scalar matrix, then all coefficients of \(q\det \overline{T}^+(z)1\) belong to \(z(\mathcal{M}^\phi_{RTT}(R)_c)\).

Due to (6.13) the series \(q\det \overline{T}^+(x)\) can be also regarded as a power series with coefficients in \(D(R)_c\), i.e. as an element of \(D(R)_c[[x]]\). Thus, we have the following corollary:

**Corollary 6.5.** If \(N_R\) is a scalar matrix, then all coefficients of \(q\det \overline{T}^+(x)\) belong to the center of the algebra \(D(R)_c\) for any \(c \in \mathbb{C}\).
Proof. Clearly, it suffices to show that \( \text{qdet} \overline{T^+}(x) \) commutes with the generator matrices \( \overline{T^-}(y) \) and \( \overline{T^+}(y) \) of \( \mathbf{D}(\overline{R})_c \). Regarding the former matrix, the commutation relations between \( \overline{T^+}(x) \) and \( \overline{T^-}(y) \), when regarded as operators on \( \mathcal{M}_R^{\phi}(R)_c \) and when regarded as series with coefficients in \( \mathbf{D}(\overline{R})_c \), coincide, so this follows from Lemma 6.3. As for the latter matrix, the family of defining relation (5.7) can be written in terms of (6.1) as

\[
\hat{R}_{12}(x/y) \overline{T^+}_1(x) \overline{T^+}_2(y) = \overline{T^+}_1(y) \overline{T^+}_2(x) \hat{R}_{12}(x/y). \tag{6.14}
\]

Indeed, the above equality is found by canceling the normalization series \( \overline{f}(x/y) \) in (5.7) and then multiplying the resulting equality by the permutation operator from the left. As the form of (6.14) coincides with the form of the defining relation for the algebra \( \mathcal{Y}_{RTT}(\overline{R}) \), one shows that \( \text{qdet} \overline{T^+}(x) \) and \( \overline{T^+}(y) \) commute as in [13, Rem. 22]. \( \square \)

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