Some Remarks on Gravitational Global Monopoles

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Abstract

Using mainly analytical arguments, we derive the exact relation \( \eta_{\text{max}} = \sqrt{\frac{3}{8\pi}} \) for the maximal vacuum value of the Higgs field for static gravitational global monopoles. For this value, the global monopole bifurcates with the de Sitter solution obtained for vanishing Higgs field. In addition, we analyze the stability properties of the solutions.
Figure 1: Schematic of the space of static solutions. For large $\eta^2$, only de Sitter exists and is stable. As $\eta^2$ decreases, a bifurcation occurs and de Sitter exchanges stability with the static monopoles.

In a recent publication [1], one of us (SL) presented numerical results on static, self-gravitating global monopoles. Considered as topological defects, global monopoles have been discussed in connection with ‘topological inflation’ [2, 3]. More recently perturbations of static global monopoles were studied in the context of ‘critical behaviour’ in spherically symmetric gravitational collapse [4].

In the present paper, we shall derive some of the qualitative and quantitative results of [1] from the field equations – a set of non-linear ODEs – obeyed by these solutions. As described in [1] there is a one-parameter family of static global monopoles parametrized by the Higgs vacuum value $\eta$ (taken in units of $M_{Pl} = 1/\sqrt{G}$) for $0 \leq \eta < \eta_{max} \approx \sqrt{3}/8\pi$.

As long as $\eta < \sqrt{1/8\pi}$ the space-time of these solutions has a deficit solid angle at infinity becoming equal to $4\pi$ for $\eta = \sqrt{1/8\pi}$. Solutions with $\sqrt{1/8\pi} < \eta < \eta_{max}$ were found to have a cosmological horizon, outside of which the solutions oscillate with decreasing amplitude about their asymptotic values. As will be demonstrated, this behaviour can be readily understood from the field equations.

Our main result is a simple derivation of the value $\eta_{max}$ as being exactly equal to $\sqrt{3}/8\pi$. At this value of $\eta$ the numerically determined global monopoles are found to bifurcate with the de Sitter solution obtained for vanishing Higgs field – the non-vanishing vacuum energy yielding a cosmological constant.

A linear stability analysis of both types of solutions shows the stability of the global monopole for all values $0 \leq \eta < \eta_{max}$, whereas the de Sitter solution (considered as a solution of the Einstein-Higgs system) is stable against perturbations with support inside the horizon for $\eta > \sqrt{3}/8\pi$ and unstable for smaller values of $\eta$ with an accumulation of unstable modes for $\eta \to 0$. The stability of the global monopole is in agreement with numerical studies of time-dependent solutions performed in [4].

Following the notation of [1], we put $\phi^a = f(r)\hat{r}^a$ for the Higgs field and

$$ds^2 = -A^2 \mu \, dt^2 + \frac{dr^2}{\mu} + r^2 d\Omega^2$$

for the spherically symmetric line element in Schwarzschild coordinates. The resulting static field equations are (we prefer to use rationalized units putting $\bar{f} = \sqrt{4\pi f}$, $\bar{\eta} = \sqrt{\eta}$).
$\sqrt{4\pi\eta}$, $\lambda = 2\pi$ as compared to $f$ and $\eta$ from [1])

\[ f' = \psi \] (2)

\[ \psi' = \frac{f}{r^2\mu} [2 + \frac{r^2}{2}(f^2 - \bar{\eta}^2)] - \psi \left[ \frac{2}{r} + r\psi^2 + \frac{\mu'}{\mu} \right] \] (3)

\[ \mu' = \frac{1 - \mu}{r} - r\mu\psi^2 - \frac{2f^2}{r} - \frac{r}{4}(f^2 - \bar{\eta}^2)^2 \] (4)

\[ A' = r\psi^2 A. \] (5)

Solutions with a regular origin obey the boundary conditions

\[ \bar{f}(r) = ar + O(r^3), \quad \psi(r) = a + O(r^2), \quad \mu(r) = 1 + O(r^2), \quad A(r) = A_0 + O(r^2) \] (6)

and are uniquely specified by the choice of $a = \psi(0)$ and $A_0$.

Such solutions stay regular as long as $\mu > 0$. Their global behaviour is characterized
by three possibilities:

1. $\mu$ has a zero for finite $r = r_0$ with diverging $\psi$, while $A, \bar{f}$ and $\sqrt{\mu}\psi$ stay finite. This type of singularity is just a coordinate singularity related to the stationarity of $r$ considered as a metrical function. In terms of a ‘geodesic’ coordinate $\tau = \int_0^r \frac{dr}{\sqrt{\mu}}$ the function $r(\tau)$ reaches a maximum $r_0 = r(\tau_0)$ and then runs back to $r = 0$ developing a curvature singularity there. This type of behaviour is generic for solutions with a regular origin.

2. $\mu$ has a zero for finite $r = r_h$ with finite $\psi, \bar{f}$ and $A$. Such solutions possess a cosmological horizon at $r = r_h$ enforcing the boundary conditions

\[ \mu'_{h} = \frac{1}{r_h} \left[ 1 - 2\bar{f}_h^2 - \frac{r_h^2}{4}(\bar{f}_h^2 - \bar{\eta}^2)^2 \right] \] (7)

\[ \psi_h = \frac{\bar{f}_h}{r_h^2\mu'_{h}} \left[ 2 + \frac{r_h^2}{2}(\bar{f}_h^2 - \bar{\eta}^2) \right] \] (8)

with $\bar{f}_h = \bar{f}(r_h)$ etc..

3. $\mu > 0$ for all $r > 0$, i.e. solutions staying finite for all $r$. The asymptotic behaviour for $r \to \infty$ is given by

\[ \mu(r) = 1 - 2\bar{\eta}^2 + \frac{d}{r} + O\left(\frac{1}{r^2}\right) \] (9)

requiring $\bar{\eta} < \frac{1}{\sqrt{2}}$

\[ \bar{f}(r) = \bar{\eta} - \frac{2}{\bar{\eta}^2} + O\left(\frac{1}{r^2}\right) + c e^{\frac{\omega(r - \frac{d}{\pi - 2\pi} \ln r)}{r}} (1 + O\left(\frac{1}{r}\right)) \] (10)

with $\omega = \frac{\bar{\eta}}{\sqrt{1 - 2\bar{\eta}^2}}$ and some constants $c$ and $d$ depending on the solution.

The asymptotic behaviour of $\mu$ entails a ‘conical’ singularity at $r = \infty$ corresponding to deficit solid angle $\Delta = 8\pi\bar{\eta}^2$. For $\bar{\eta}^2 = \frac{1}{2}$ this deficit angle becomes $4\pi$ and
the character of the solution for \( r \to \infty \) changes to

\[
\mu(r) = \frac{d}{r}(1 + O\left(\frac{1}{r}\right)) \tag{11}
\]

\[
\bar{f}(r) = \frac{1}{\sqrt{2}} - \frac{2\sqrt{2}}{r^2} + O\left(\frac{1}{r^3}\right) + ce^{-\omega r^{3/2}}(1 + O\left(\frac{1}{r}\right)) \tag{12}
\]

with \( \omega = \frac{1}{3}\sqrt{\frac{2}{d}} \) and again constants \( c \) and \( d \) depending on the solution.

For \( \bar{\eta} > 1/\sqrt{2} \) the asymptotic behavior for \( r \to \infty \) becomes oscillatory due to the sign change of \( \omega^2 \). However, the factor \( 1/r \) in front of the exponential leads to a damping of the amplitude and \( \bar{f}(r) \to \bar{\eta} \) for \( r \to \infty \) as is clearly visible from the numerical results of [1].

Both the solutions of type 2) and 3) require fine-tuning of the parameter \( a \) characterizing solutions regular at \( r = 0 \). The numerical results of [1] indicate that for \( 0 < \bar{\eta} < \bar{\eta}_{\text{max}} \) exactly one such solution with non-vanishing Higgs field exists. The corresponding graph of \( a(\bar{\eta}) \) is shown in Fig.2. In the following we shall try to explain the essential features of \( a(\bar{\eta}) \).

For small values of \( \bar{\eta} \) the self-gravitating monopole can be understood as a perturbation of the flat monopole, which is obtained for \( \bar{\eta} \to 0 \) after a rescaling \( \bar{f} = \bar{\eta} \hat{f} \) and \( r = \hat{r}/\bar{\eta} \) solving the equation

\[
(\hat{r}^2 \hat{f}')' = \hat{f}' \left[ 2 + \frac{\hat{r}^2}{2} (\hat{f}^2 - 1) \right]. \tag{13}
\]

Solving this equation numerically with the boundary condition \( \hat{f}(\hat{r}) = \hat{a}\hat{r} + O(\hat{r}^3) \) one finds a globally regular solution for \( \hat{a} \approx 0, 3578 \). Expressed in terms of \( \bar{f} \) we get

\[
\bar{f}(r) = \bar{\eta}^2 \hat{a} r + O(r^3) = ar + O(r^3) \tag{14}
\]

and thus \( a = \bar{\eta}^2 \hat{a} \). This behavior is clearly visible in Fig.4.

We also see from this figure that the function \( a(\bar{\eta}) \) after reaching a maximum turns back to \( a = 0 \) with diverging derivative. The latter may be understood as a consequence of the invariance \( \bar{f} \to -\bar{f} \) of the field equations in combination with the smooth behaviour of \( a(\bar{\eta}) \) at \( \bar{\eta}_{\text{max}} \). Next we will use this “empirical” observation to derive the exact value for \( \bar{\eta}_{\text{max}} \).

Suppose we take a globally regular solution \( \bar{f}(a(\bar{\eta}), \bar{\eta}, r) \) for some fixed value of \( \bar{\eta} \) and vary \( a(\bar{\eta}) \to a(\bar{\eta}) + \delta a \). Then generically we will get a solution of type 1). However, if \( \partial a(\bar{\eta})/\partial \bar{\eta} = \infty \) an infinitesimal change of \( a \) will carry us to another regular solution for the same value of \( \bar{\eta} \). Thus \( \partial f/\partial a \) will be bounded for \( r \to \infty \) resp. \( r \to r_h \) for points where \( \partial a/\partial \bar{\eta} = \infty \) in contrast to points where this derivative is finite. As seen from Fig.2 the solution for \( \bar{\eta} = \bar{\eta}_{\text{max}} \) corresponds to \( a(\bar{\eta}) = 0 \) and thus \( \bar{f}(\bar{\eta}_{\text{max}}, r) \equiv 0 \). This is nothing but the de Sitter solution given by integrating Eq.(4)

\[
\mu_{\text{dS}}(r) = 1 - \left(\frac{r}{r_h}\right)^2 \quad \text{with} \quad r_h = \frac{2\sqrt{3}}{\bar{\eta}^2}. \tag{15}
\]

Figure 3 shows the approach of the monopole solutions to the de Sitter solution for \( \bar{\eta} \to \bar{\eta}_{\text{max}} \). According to what was said above, the de Sitter solution must have the
Figure 2: \( a = \bar{f}'(0) \) for the static monopole as a function of \( \bar{\eta}^2 \). Observe that \( a \to 0 \) as \( \bar{\eta}^2 \to 3/2 \) signifying the approach to de Sitter space. Note that the critical point in \( a \) does not occur at \( \bar{\eta}^2 = 1 \), which was found to be the critical point in the core radius in [1].
Figure 3: Demonstration of approach to de Sitter solution. The field $\mu'(r)$ is shown as $\bar{\eta}^2$ approaches $3/2$. The solid line denotes the de Sitter solution for $\bar{\eta}^2 = 3/2$ while the other profiles show $\mu'$ for the static monopoles. The solutions approach the de Sitter solution for an increasing domain of $r$ until ultimately driven to oscillate around the regular solution $\mu' = 0$. 
bounded "zero mode" $\partial \bar{f}/\partial a$ for $\bar{\eta} = \bar{\eta}_{\text{max}}$. Linearizing the field Eqs. (2-3) on the de Sitter background we obtain (putting $\varphi = r \delta \bar{f}$)

$$(\mu_{\text{as}} \varphi')' \equiv \left((1 - \frac{r^2}{r_h^2})\varphi'ight)' = \left(\frac{2}{r^2} - \frac{\bar{\eta}^4}{6} - \frac{\bar{\eta}^2}{2}\right)\varphi. \quad (16)$$

Introducing $x = r/r_h$ we get

$$\frac{d}{dx}\left((1 - x^2)\frac{d}{dx}\varphi\right) = \left(\frac{2}{x^2} - 2 - \frac{6}{\bar{\eta}^2}\right)\varphi. \quad (17)$$

The solution with the correct boundary condition $\varphi(x) = x^2 + O(x^4)$ for $x \to 0$ valid for $\varphi = \partial \bar{f}/\partial a$ is given by $\varphi \equiv x^2$ obtained for $\bar{\eta}^2 = 3/2$.

Thus we have found a simple derivation of $\bar{\eta}_{\text{max}} = \sqrt{\frac{3}{2}}$. Since the r.h.s. of Eq.(17) becomes more negative for $\bar{\eta}^2 < 3/2$ the solution $\varphi(x)$ fulfilling the correct boundary condition at $x = 0$ develops a zero before reaching the horizon at $x = 1$. According to the "Jacobi criterion" [5] this implies an instability of the de Sitter background for $\bar{\eta} < \sqrt{\frac{3}{2}}$. Linearizing the time-dependent field equations around the static solution we let

$$\bar{f}(r, t) = \bar{f}_0(r) + e^{i\omega t} \delta \bar{f}, \quad (18)$$

where $\bar{f}_0$ represents the static Higgs field. Unstable modes of the static solution correspond to modes with $\omega^2 < 0$. The Jacobi criterion relates the existence of such negative modes to the existence of zeros of the solution for $\omega^2 = 0$ (zero mode) with the correct boundary condition at $r = 0$.

For $\bar{\eta} > \sqrt{\frac{3}{2}}$ the solution $\varphi(x)$ remains positive for all $x$ indicating the stability of the solution under spherically symmetric perturbations with support inside the horizon. On the other hand for $\bar{\eta} \to 0$ the de Sitter solution develops more and more unstable modes manifested by the existence of bounded zero-modes with more and more zeros. The latter are readily obtained with the polynomial ansatz

$$\varphi_K(x) = \sum_{k=1}^{K} c_k x^{2k}. \quad (19)$$

Inserting this ansatz into the Eq.(17) yields the recursion relation

$$c_{k+1} = \frac{2k(2k + 1)}{(2k + 2)(2k + 1) - 2} c_k \quad (20)$$

and the condition (derived from $c_{K+1} = 0$)

$$\bar{\eta}_{K}^2 = \frac{3}{K(2K + 1) - 1} \quad (21)$$

yielding $\bar{\eta}_1^2 = 3/2$, $\bar{\eta}_2^2 = 1/3$, $\bar{\eta}_3^2 = 3/20$ etc..

Eq.(17) can be transformed to the hypergeometric differential equation as follows. Putting $\xi = \frac{1 - x^2}{1 - x^2}$ and $\varphi(x) = \xi(1 + \xi)^m y(\xi)$ with $2m(2m - 1) = 3 + 6/\bar{\eta}^2$ we obtain

$$\xi(\xi + 1)y'' + \left((2m + 3)\xi + \frac{5}{2}\right)y' + (m + 1)^2 y = 0 \quad (22)$$
Figure 4: Perturbations to the de Sitter solution. For $\bar{\eta}^2 > 1.5$, the perturbation is everywhere positive indicating linear stability. For $\bar{\eta}^2 < 1.5$, zero-crossings exist indicating instability of the de Sitter solution. Additional zeros appear as $\bar{\eta}^2$ passes through the values $\bar{\eta}^2_K$ found from Eq. (21), the first few of which are shown here.
Figure 5: Perturbations to the static global monopole. For all \( \eta^2 \), the perturbation is everywhere positive indicating linear stability.

fulfilled by the hypergeometric function \( F(m+1, m+1, 5/2, \xi) \), the polynomial solutions \( \varphi_K(x) \) corresponding to the Jacobi polynomials \( y_K(\xi) \).

The stability of the flat global monopole (under spherically symmetric perturbations) implies its stability for small values of \( \eta \). According to standard arguments for one-parameter families of solutions \( \eta \), we expect no change of stability up to the bifurcation point \( \eta = \eta_{\text{max}} \) with the de Sitter solution. In order to analyze this hypothesis we have performed a detailed numerical stability analysis for the global monopole using again the Jacobi criterion. For solutions with a horizon our stability analysis is restricted to the region inside the horizon. This is justified by the fact that no perturbations can penetrate through the horizon from the region where \( \mu < 0 \) (similar to the case of stability analysis performed for black holes).

As is well-known \( [\text{ref.}] \) the metric perturbations can be integrated out and one obtains a Schrödinger type perturbation equation for \( \varphi \). Introducing the coordinate \( \sigma \) through \( \sigma = \int_0^\sigma \frac{dr}{\sqrt{\mu}} \) one gets

\[
- \frac{d^2 \varphi}{d\sigma^2} + V \varphi = 0
\]  

(23)
with
\[ V = A^2 \mu \left[ \frac{2}{r^2} + \frac{1}{2} (3f^2 - \eta^2) + \frac{2}{r^2 A} (A \mu r^3 \psi^2)' + \frac{(A \mu)'}{r A} \right] \] (24)

and the boundary condition \( \varphi(r) = \sigma^2 + O(\sigma^4) \) for \( \sigma \to 0 \). Our numerical analysis shows (see Fig.5) that the solutions \( \varphi \) have no zero for all \( 0 < \eta < \sqrt{3}/2 \) confirming our hypothesis about the stability of the global monopole.

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