Sequential detection of genuine tripartite entanglement in semi-device-independent scenarios

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We investigate the possibility of sequential detection of genuine tripartite entanglement by multiple observers performing non-projective or unsharp measurements. A tripartite initial state of either the GHZ- or W-type is shared between two fixed observers in two wings and a sequence of multiple observers in the third wing who act independently of each other. The choice of measurement setting of each of the multiple observers is independent and uncorrelated with the choices of measurement settings and outcomes of the previous observers. We find the upper limit on the number of observers for whom the detection of entanglement is certified through the violation of a genuine tripartite steering inequality in each sequential step of our protocol. Our study is performed in both the one-sided device independent (1SDI) and two-sided device independent (2SDI) scenarios. We further consider two different cases wherein the multiple observers in the third wing are trusted or untrusted, performing characterized and uncharacterized measurements, respectively. We show that for both the choice of initial states, the range of measurement parameters for which genuine entanglement is detected turns out to be less in the 2SDI scenario, while the upper limit on the number of trusted observers is higher in the 1SDI scenario.

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I. INTRODUCTION

Quantum network, as its name suggests, is composed of multiple parties, and is the future of quantum communication tasks where multipartite quantum correlation serves as the resource. Detection and characterization of such resources, in particular, genuine multipartite entanglement [1] is rather important for information theoretic applications [2–11], as well as from foundational perspectives. A multipartite state is called genuinely entangled [1] if and only if (iff) it is not separable across any partition. However, detecting and characterizing entanglement across each partition become more complex when the number of parties is increased.

Well known methods of detecting entanglement include quantum state tomography [12–17] and entanglement witnesses [18–20]. These methods though require some prior information about the state, and more importantly, the underlying assumption that the preparation and measurements devices used are well characterized and trusted. Alternatively, the approach of entanglement detection based on the violations of Bell inequalities [21,22] may be adopted. Though entanglement is necessary but not sufficient for Bell-violation, such an approach is device-independent (DI), since it involves untrusted (black-box) devices by all parties. DI entanglement detection has several applications [23] such as in quantum key distribution [24,25] and randomness generation [26]. However, detection of Bell-violations require high detector efficiencies and low levels of noise. Hence, in practical scenarios, it may not always be feasible to implement DI entanglement detection [27].

An intermediate method to detect entanglement relies upon quantum steering [28,29]. The idea of steering was first introduced by Schrödinger in the context of the Einstein-Podolsky-Rosen (EPR) argument [30], where he demonstrated that the choice of measurement settings on one side can ‘steer’ the state on the other side [28,29]. Much later, Reid proposed a criterion for experimentally demonstrating the EPR argument using the Heisenberg uncertainty relation [31]. Subsequently, Wiseman et al. [32,33] presented an information theoretic perspective of EPR steering. Several criteria have been since proposed in order to detect bipartite EPR steering where one party performs a trusted measurement and the other party performs an untrusted (black-box) measurements [34–41]. Detection of EPR steering certifies the presence of entanglement in a semi-device-independent (SDI) way with lower detector efficiencies and more tolerance to noise compared to the fully DI approach.

In multipartite scenarios, detection of genuine entanglement based on entanglement witnesses has been proposed [1,42–44]. On the other hand, Bell-type inequalities have also been used in order to certify genuine entanglement in a fully DI way [45–54]. There also exist approaches to certify genuine entanglement in asymmetric scenarios. To certify genuine tripartite entanglement in an asymmetric scenario, multi party entanglement witnesses [18–20] have been used. Such an approach may be semi-device independent (SDI) and is often called as multi party entanglement witnessing.

In this work, we propose a novel method to certify genuine tripartite entanglement in a semi-device-independent (SDI) way. In contrast to existing approaches, our method allows for sequential detection of genuine tripartite entanglement. Our approach is based on the violation of a genuine tripartite steering inequality in each sequential step of our protocol. Our study is performed in both the one-sided device independent (1SDI) and two-sided device independent (2SDI) scenarios. We further consider two different cases wherein the multiple observers in the third wing are trusted or untrusted, performing characterized and uncharacterized measurements, respectively. We show that for both the choice of initial states, the range of measurement parameters for which genuine entanglement is detected turns out to be less in the 2SDI scenario, while the upper limit on the number of trusted observers is higher in the 1SDI scenario.

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networks where some of the parties’ measurement devices are uncharacterized whereas others perform trusted measurements, i.e., in SDI ways [55–59]. This idea of SDI genuine entanglement detection, being an intermediate concept between entanglement witnesses and fully DI entanglement detection, is a lot more noise resistant than standard fully DI genuine entanglement detection and less experimentally demanding than the genuine entanglement witness methods as precise control over measurement devices at the untrusted parties’ sides is not required.

Due to decoherence and other difficulties present in generating and detecting genuine entanglement [60–62], which acts as the resource in various information theoretic tasks, it is natural to ask whether one copy of genuinely entangled state can be sequentially used multiple times when some quantum advantage is gained in each round. In other words, the obstacles in preparing and preserving genuine entanglement in real experimental scenarios, is a strong motivation to explore how one can partially preserve genuine entanglement even after performing a few rounds of local operations. In the present study our goal is to address whether genuine entanglement can be sequentially detected multiple times following SDI approaches.

The question of sharing of quantum correlations by multiple sequential observers was first posed by Silva et al. [63] in the context of bipartite Bell nonlocality [21, 22]. The scenario contains of an entangled pair of two spin-$\frac{1}{2}$ particles, shared between two spatially separated wings. Alice performs projective measurement on one half of the entangled state and multiple Bobs perform weak measurements on the other half sequentially and independently of each other. Considering unbiased frequencies of the inputs of the each Bob, it was conjectured [63] that at most two Bobs can demonstrate quantum violation of Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [22] with a single Alice. This result was subsequently confirmed [64] using the unsharp measurement formalism [65, 66]. Note that it has been claimed recently using numerical optimization techniques that the maximum number of Bobs may go up to three [67]. Note also, that it may be possible to increase further the number of Bobs by choosing a different sharpness parameter for the two inputs of each Bob, as shown recently [68].

The approach enunciated in the work of Silva et al. [63] based on unbiased frequencies of inputs and unbiased measurements settings for each Bob, has been realized experimentally [69, 70], and also been employed in many different contexts [64, 71–85]. The idea of sharing of nonlocal quantum correlations by multiple sequential observers has been applied to EPR steering [71, 72], entanglement detection [73–75], steerability of local quantum coherence [76], violations of various Bell-type inequalities [77–79], preparation contextuality [80, 81], unbounded randomness generation [82], distinguishing quantum theory from classical simulations [83], quantum teleportation [84], and random access codes [85].

The above studies dealing with the issue of bipartite quantum correlations are restricted to two spatially separated particles. Recently, the possibility of sequential detection of genuine tripartite entanglement by multiple observers in the fully DI scenario has been studied by Saha et al. [86]. On the other hand, Maity et al. [87] have addressed sequential detection of genuine tripartite entanglement by multiple observers using genuine tripartite entanglement witnesses [1, 42, 43] for W-type and GHZ-type [88] states in a scenario where each of the parties’ measurement settings are trusted. Additionally, DI detection of genuine tripartite entanglement by multiple observers has also been studied in the same work [87]. However, sequential detection of genuine tripartite entanglement by multiple observers in the SDI framework has hitherto remained an open question.

With the above motivations, the scenario investigated in the present work consists of three spin-$\frac{1}{2}$ particles, spatially separated and shared between three wings. Two observers on the first two wings perform projective measurements on their respective particles. On the other hand, there are multiple observers on the third wing who perform non-projective or unsharp measurements on the third particle sequentially and independently of each other. In order to detect genuine tripartite entanglement in SDI scenarios, we demonstrate genuine tripartite steering. For this purpose, we consider the genuine EPR steering inequalities given by Cavalcanti et al. [57] and investigate how many observers on the third wing can detect genuine tripartite entanglement in both one-sided as well as two-sided device independent ways. We study here two separate cases: 1) all the multiple observers on the third wing are untrusted, 2) all the multiple observers on the third wing are trusted.

The plan of the paper is as follows: in Section II we present the basic tools for detecting genuine tripartite entanglement in SDI scenarios through the demonstration of genuine EPR steering. The measurement framework involving multiple sequential observers used in this paper is also described in this Section. In Section III, we present the main analysis of this paper, and discuss the results obtained in context of sequential detection of genuine tripartite entanglement of the GHZ-state as well as the W-state in two kinds of SDI scenarios. Finally, we conclude in Section IV.

II. PRELIMINARIES

In this section we discuss in brief the concept of genuine tripartite steering and the measurements employed in order to probe sequential detection of genuine tripartite entanglement by multiple observers in both one-sided-device-independent (1SDI) and two-sided-device-independent (2SDI) scenarios.
A. Detection of Genuine Tripartite Steering

In order to detect genuine tripartite steering, we adopt the procedure and inequalties as presented by Cavalcanti et al. [57]. Let us consider that a tripartite state $\rho$ is shared among three observers, say, Alice, Bob and Charlie. Among them some of the parties' measurement devices are trusted or characterized whereas the rest are untrusted or uncharacterized. The task is to certify that the shared state, $\rho$ is genuinely entangled. Here two different scenarios may arise: First, the situation when only one party's device is untrusted which is known as 1SDI scenario. Secondly, when only two parties' devices are untrusted, i.e., 2SDI scenario.

In the first scenario (1SDI), without loss of generality, one can consider that Alice’s measurement device is untrusted and her measurement operators are denoted by $A_{a|x}$, where $x$ is the choice of input and $a$ is the outcome. After Alice’s measurement, the unnormalized conditional states (assemblages) on Bob and Charlie’s end is given by,

$$\sigma_{a|x}^{BC} = \text{tr}_A \left[ \left( A_{a|x} \otimes I_B \otimes I_C \right) \rho \right].$$

(1)

For the second case (2SDI), let us consider that the measurement devices of Alice and Bob are untrusted. Alice’s measurement operators are denoted by $A_{a|x}$ and Bob’s measurement operators are denoted by $B_{b|y}$. Here $x$ and $y$ are the choices of inputs by Alice and Bob respectively, and $a$ and $b$ are the outcomes of Alice’s and Bob’s measurements respectively. After their measurements, the unnormalized conditional states (assemblages) prepared on Charlie’s side is given by,

$$\sigma_{a|x}^{C} = \text{tr}_{AB} \left[ \left( A_{a|x} \otimes B_{b|y} \otimes I_C \right) \rho \right].$$

(2)

Now if the initial state $\rho$ is not genuinely entangled, then it is in the following bi-separable form,

$$\rho = \sum_{\lambda} \rho^A_{\lambda} \otimes \rho^{BC}_{\lambda} + \sum_{\mu} \rho^B_{\mu} \otimes \rho^C_{\mu}$$

(3)

Here $\rho^A_{\lambda}$, $\rho^{BC}_{\lambda}$ and $\rho^B_{\mu}$ are probability distributions where A:BC, B:AC and AB:C symbolise the type of bipartition of the state $\rho$. For example, A:BC represents the partition between Alice and (Bob, Charlie). Similarly, AB:C represents the partition between (Alice, Bob) and Charlie. No distinction is made between AB:C and C:AB. When the initial state $\rho$ is not genuinely entangled, the assemblage in the first scenario is of the following form,

$$\sigma_{a|x}^{BC} = \text{tr}_A \left[ \left( A_{a|x} \otimes I_B \otimes I_C \right) \rho \right]$$

$$= \sum_{\lambda} \rho^A_{\lambda} \rho^B_{\lambda} + \sum_{\mu} \rho^B_{\mu} \sigma_{a|x}^{BC} + \sum_{\nu} \rho^{AB,C}_{\nu} \sigma_{a|x}^{BC}.$$  

Here $\rho^A_{\lambda}$ denotes the probability of getting the outcome $a$ when Alice performs the measurement denoted by $x$ on the state $\rho^A_{\lambda}$; $\sigma_{a|x}^{BC}$ denotes the unnormalized conditional state on Charlie’s side when Alice gets the outcome $a$ by performing the measurement denoted by $x$ on the bipartite state $\rho^{BC}_{\lambda}$ shared between Alice and Charlie; and $\sigma_{a|x}^{BC}$ denotes the unnormalized conditional state on Bob’s side when Alice gets the outcome $a$ by performing the measurement denoted by $x$ on the bipartite state $\rho^{AC}_{\lambda}$ shared between Alice and Bob. If any assemblage $\rho^{BC}_{\mu}$ prepared by the state $\rho$ cannot be written in the above form (4), then $\rho$ demonstrates genuine EPR steering in 1SDI scenario [57, 59].

In this scenario, Alice, the untrusted party performs uncharacterized measurements on the first particle. Bob and Charlie, the trusted parties perform characterized measurements on the second and third particle, respectively. We consider a tripartite state $\rho$ (either GHZ state or W state) consisting of three spin-$\frac{1}{2}$ particles initially shared among Alice, Bob and Charlie. Alice performs dichotomic measurement of spin component observable on her particle in the direction $\hat{x}$ or $\hat{y}$, or $\hat{z}$. Similarly, Charlie performs dichotomic measurement of spin component observable on his particle in the direction $\hat{z}_1$, or $\hat{z}_2$. The outcomes of each measurement are $\pm 1$.

Here

$$\dot{x}_i = \sin \theta_i \cos \phi_i \hat{x} + \sin \theta_i \sin \phi_i \hat{y} + \cos \theta_i \hat{z},$$

(5)

$$\dot{y}_j = \sin \theta_j \cos \phi_j \hat{x} + \sin \theta_j \sin \phi_j \hat{y} + \cos \theta_j \hat{z},$$

(6)

and

$$\dot{z}_k = \sin \theta_k \cos \phi_k \hat{x} + \sin \theta_k \sin \phi_k \hat{y} + \cos \theta_k \hat{z},$$

(7)

where $i, j, k \in \{0, 1, 2\}$; $0 \leq \theta_i \leq \pi; 0 \leq \phi_i \leq 2\pi; 0 \leq \theta_j \leq \pi; 0 \leq \phi_j \leq 2\pi; 0 \leq \theta_k \leq \pi; 0 \leq \phi_k \leq 2\pi$. We look for the violation of the state specific inequality (9, 11) to demonstrate genuine tripartite steering in 1SDI scenario. Similarly, when the initial state $\rho$ is not genuinely entangled, then the assemblage in the second scenario has the following form,

$$\sigma_{a|x}^{C} = \text{tr}_{AB} \left[ \left( A_{a|x} \otimes B_{b|y} \otimes I_C \right) \rho \right]$$

$$= \sum_{\lambda} \rho^A_{\lambda} \rho^{BC}_{\lambda} \rho^C_{\lambda}$$

(8)

Here $\rho^A_{\lambda}(a|x)$ denotes the probability of getting the outcome $a$ when Alice performs the measurement denoted by $x$ on the state $\rho^A_{\lambda}$; $\sigma^C_{a|x}$ denotes the unnormalized conditional
conditional state on Charlie’s side when Bob gets the outcome $b$ by performing the measurement denoted by $y$ on the bipartite state $\rho_{BC}^\mu$ shared between Bob and Charlie; $p_{\nu}(b|y)$ denotes the probability of getting the outcome $b$ when Bob performs the measurement denoted by $y$ on the state $\rho_{BC}^\nu$; $\sigma_{a|x}^{\mu}$ denotes the unnormalized conditional state on Charlie’s side when Alice gets the outcome $a$ by performing the measurement denoted by $x$ on the bipartite state $\rho_{AB}^\mu$ shared between Alice and Charlie; $p_{\nu}(ab|x,y)$ denotes the joint probability of getting the outcomes $a$ and $b$ when Alice and Bob perform the measurements denoted by $x$ and $y$, respectively, on the bipartite state $\rho_{AB}^\nu$. For the GHZ state in the 1SDI scenario, these inequalities are nothing but genuine EPR steering in 2SDI scenario \cite{57, 59}. The violation of the state specific inequality (10) for $i, j, k \in \{0, 1, 2\}$. The inequality has the following form:

$$W_1 = 1 + 0.4405(\langle Z_B \rangle + \langle Z_C \rangle) - 0.0037(\langle Z_B Z_C \rangle - 0.1570(\langle X_B X_C \rangle + \langle Y_B Y_C \rangle + (A_1 X_B X_C) + (A_3 Y_B Y_C)) + 0.2424((A_1) + (A_3 Z_B Z_C)) + 0.1848((A_3 Z_B) + (A_3 Z_C)) - 0.2533((A_1 X_B) + (A_1 X_C) + (A_2 Y_B) + (A_2 Y_C) + (A_1 X_B Z_C) + (A_1 Z_B X_C) + (A_2 Y_B Z_C) + (A_2 Z_B Y_C)) \geq 0 \quad (11)$$

with the pure W state achieving the violation $-0.759$. For the W state in 2SDI scenario, the inequality has the following form:

$$W_2 = 1 + 0.2517((A_3) + (B_3)) + 0.3520(Z) - 0.1112((A_1 X) + (A_2 Y) + (B_1 X) + (B_2 Y)) + 0.1296((A_3 Z) + (B_3 Z)) - 0.1943((A_1 B_1) + (A_2 B_2)) + 0.2277((A_3 B_3) - 0.1590((A_1 B_1 Z) + (A_2 B_2 Z)) + 0.2228((A_3 B_3 Z) - 0.2298((A_1 B_3 X) + (A_2 B_3 Y) + (A_3 B_1 X) + (A_3 B_2 Y)) \geq 0 \quad (12)$$

with the pure W state achieving the violation $-0.480$.

## B. Sequential measurement context

We now describe further the measurement context adopted throughout the present paper in the two discussed scenarios:

1. **1SDI scenario**: Let us consider three spatially separated particles ($W_1$, $W_2$ and $W_3$) sharing a tripartite system with state $\rho$ (either GHZ state or W state) consisting of three spin-$\frac{1}{2}$ particles. Here neither all the measurement devices of the three particles are trusted. Rather, we consider a hybrid model where the measurement devices at wing $W_1$ are untrusted and that in $W_2$, $W_3$ are trusted.

2. **2SDI scenario**: Here also consider that three spatially separated particles ($W_1$, $W_2$ and $W_3$) share a tripartite system with state $\rho$ (either GHZ state or W state) consisting of three spin-$\frac{1}{2}$ particles. The measurement devices at wing $W_1$ and $W_2$ are untrusted and that in $W_3$ are trusted.

In each of these two scenarios mentioned above, we consider the following two cases:

**Case A- Multiple untrusted parties performing sequential measurements**: In this case, we consider multiple Alices, all of whose measurement devices are untrusted, at wing $W_1$. Multiple Alices (Alice$^1$, Alice$^2$, ..., Alice$^n$) perform measurements on the first particle...
In a sharp projective measurement, one obtains the max-
malism used in this paper (For details, see [63, 64, 71]).

2SDI scenario? In order to address this issue, we shall
detect genuine tripartite entanglement in the 1SDI or the
ask at most how many Alices or how many Charlies can
1, 2, 3, and 4, respectively. In the above contexts, we
multiple Alices or multiple Charlies) on a single wing are
sure, at wing 1, perform projective measurements on
the second and third particle, respectively. Since our aim is to explore how many Alices can
detect genuine entanglement in the 1SDI or 2SDI sce-
narios through the violation of genuine EPR steering in-
equalities (9, 10, 11, 12), multiple Alices cannot perform
projective measurements. If any Alice performs a projec-
tive measurement, then the genuine entanglement of the
state will be completely lost and there will be no resid-
ual genuine entanglement for the next Alice. However,
no such restriction is required for the measurements per-
formed by the last Alice in the sequence. Hence, for n
number of Alices, the first (n − 1) Alices in the sequence
should perform weak measurements.

Case B- Multiple trusted parties performing
sequential measurements: In this case, we consider
multiple Charlies, all of whose measurement devices
are trusted, at wing W3. Multiple Charlies (Charlie1, Charlie2, ..., Charliem) perform measurements on
the third particle sequentially. Here, the measurements
performed by Charlie1, Charlie2, ..., Charliem−1 are weak
and the measurement performed by Charlie m is projec-
tive. On the other hand, a single Alice at wing W1
and a single Bob at wing W2 perform projective
measurements on the first and second particle, respectively.

We further make the following two assumptions. First,
each of the multiple observers (either multiple Alices or
multiple Charlies) on a single wing performs measure-
ment independently of other prior observers. In other
words, Alice m (Charlie m) with m ∈ {1, 2, ..., n} is igno-
rant of the choices of measurement settings and outcomes
of Alice1, Alice2, ..., Alice m−1 (Charlie1, Charlie2, ..., Charlie m−1). Secondly, we restrict ourselves to the
unbiased input scenario which implies that all possible mea-
surement settings of each of the multiple observers (either
multiple Alices or multiple Charlies) on a single wing are
equally probable. Note that the no-signalling condition (the probability of obtaining one party’s outcome
does not depend on the other spatially separated party’s set-
ting) is satisfied between the observer(s) at three different
wings as the three particles are spatially separated. How-
ever, this condition is not satisfied between the multiple
observers (either multiple Alice s or multiple Charlies) on
a single wing. In fact, Alice1 (Charlie1) implicitly signals
to Alice2 (Charlie2) by her (his) choice of measurement
on the state before she (he) passes it on and, similarly,
Alice2 (Charlie2) signals to Alice3 (Charlie3), and so on.

The aforementioned scenarios are depicted in Figures
1, 2, 3, and 4, respectively. In the above contexts, we
ask at most how many Alices or how many Charlies can
detect genuine tripartite entanglement in the 1SDI or the
2SDI scenario? In order to address this issue, we shall
consider the genuine EPR steering inequalities (9, 10, 11,
12).

Next, let us briefly discuss the weak measurement for-
malism used in this paper (For details, see [63, 64, 71]).
In a sharp projective measurement, one obtains the max-
imum amount of information at the cost of maximum dis-
turbance to the state. On the other hand, in our scenario,
Alice m (or, Charlie m) passes on the respective particle
to Alice m+1 (or, Charlie m+1) after performing suitable
measurement. Hence, in this case, Alice m (or, Charlie m)
needs to demonstrate genuine steering by disturbing
the state minimally so that Alice m+1 (or, Charlie m+1)
can again demonstrate genuine steering. This can be achieved by weak measurement [63] which is characterized by two real parameters: the quality factor $F$ and the precision $G$ of the measurement. The quality factor $F$ quantifies the extent to which the initial state of the system (to be measured) remains undisturbed during the measurement process and the precision $G$ quantifies the information gain due to the measurement. In case of projective measurement, $F = 0$, $G = 1$. For dichotomic measurements on a qubit system, the optimal trade-off relation between precision $G$ and quality factor $F$ is given by, $F^2 + G^2 = 1$ [63]. In other words, for dichotomic measurements on a qubit system, satisfying the condition: $F^2 + G^2 = 1$ implies that the disturbance is minimized for any particular information gain.

The above optimal trade-off relation between information gain and quality factor is achieved under unsharp measurement formalism [64, 71]. Unsharp measurement [65, 66] is one particular class of positive operator valued measurements (POVM) [65, 66]. A POVM is nothing but a set of positive operators that add to identity, i.e., $E \equiv \{E_i | \sum_i E_i = 1, 0 < E_i \leq 1 \forall i\}$. Here, each of the effect operators $E_i$ determines the probability $\text{Tr}[\rho E_i]$ of obtaining the $i$th outcome (here $\rho$ is the state of the system on which the measurement is performed).

In the unsharp measurement formalism, the effect operators are defined as,

$$E_+^\lambda = \lambda P_+ + (1 - \lambda) \frac{P_-}{2},$$

where $\lambda$ is the sharpness parameter with $0 < \lambda \leq 1$; $P_+$ ($P_-$) denotes the projectors associated with the outcome +1 (-1); $P_+ + P_- = \mathbb{I}$ and $P_-^2 = P_+$, $E_+^\lambda$ are obtained by mixing projectors with white noise. The probability of getting the outcomes +1 and −1, when the above unsharp measurement is performed on the state $\rho$, are given by $\text{Tr}[\rho E_+^\lambda]$ and $\text{Tr}[\rho E_-^\lambda]$ respectively. Using the generalized von Neumann-L"uders transformation rule [65], the states after the measurements, when the outcomes +1 and −1 occurs, are given by,

$$\sqrt{E_+^\lambda \rho E_+^\lambda} \text{ and } \sqrt{E_-^\lambda \rho E_-^\lambda}$$

and

$$\frac{\text{Tr}[E_+^\lambda \rho]}{\text{Tr}[E_-^\lambda \rho]}$$

respectively.

For the von Neumann-L"uders transformation rule in the unsharp measurement formalism, it was shown [64] that the quality factor $F$ and the precision $G$ are given by, $F = \sqrt{F^2 - X^2}$ and $G = \lambda$. Hence, the optimal trade-off relation between information gain and disturbance, $F^2 + G^2 = 1$ for qubits is compatible with the unsharp measurement formalism [64, 71]. In other words, the unsharp measurement formalism along with the von Neumann-L"uders transformation rule provides the largest amount of information for a given amount of disturbance created on the state due to the measurement.

In our study, we will consider that multiple Alices (in Case A) or multiple Charlies (in Case B) in the sequence, except for the last one, perform unsharp measurements.

### III. DETECTION OF GENUINE TRIPARTITE ENTANGLEMENT

Using the formalism discussed in the earlier sections we are now in a position to explore the maximum num-
ber of parties (trusted or untrusted) who can detect genuine entanglement in the previously mentioned two types of SDI scenarios. Before proceeding, let us recapitulate some previous relevant results on sequential detection of genuine entanglement in other scenarios.

It has been shown [86] that at most two Charlies can sequentially demonstrate genuine tripartite nonlocality [45] with a single Alice and a single Bob when the GHZ state is initially shared. On the other hand, at most one Charlie can sequentially demonstrate genuine tripartite nonlocality with a single Alice and a single Bob when the W state is initially shared [86]. Now, in order to demonstrate genuine nonlocality, genuine entanglement is necessary and hence, it can be concluded that two Charlies can sequentially detect genuine tripartite entanglement of the GHZ state in a fully DI scenario with a single Alice and a single Bob. On the other hand, only one Charlie can detect genuine tripartite entanglement of the W state in a fully DI scenario with a single Alice and a single Bob. These results are probed through the quantum violation of the Svetlichny inequality [45]. Recently, the possibility of sequential detection of genuine tripartite entanglement in the fully DI scenario as well as using entanglement witness has been explored by Maity et al. [87]. Considering both linear as well as non-linear correlation inequalities which detect genuine entanglement in the fully DI scenario, it was shown that at most two Charlies can detect genuine entanglement of the GHZ state. However, using genuine entanglement witnesses, where all parties are treated as trusted, it has been shown that at most four Charlies can detect genuine entanglement sequentially with the single Alice and single Bob when the W state is initially shared among them and at most twelve Charlies can detect genuine entanglement sequentially when the GHZ state is initially shared among them.

Here we ask a similar question, but in the SDI scenario, i.e., in the intermediate context between the entanglement witness scenario and the fully DI scenario. Using the inequalities by Cavalcanti et al. [57] as the criteria for SDI entanglement certification, we explore the possibilities of sequential detection of genuine entanglement of GHZ and W states in the 1SDI and 2SDI scenario. We will consider both the cases: multiple Alices performing uncharacterized (untrusted) measurements sequentially, and multiple Charlies performing characterized (trusted) measurements sequentially.

A. Multiple untrusted parties performing sequential measurements

In this subsection, we discuss sequential detection of genuine tripartite entanglement in the 1SDI and 2SDI scenarios considering Case A mentioned earlier, i.e., when multiple Alices perform sequential uncharacterized measurements on the first particle, single Bob performs characterized or uncharacterized (depending on whether the scenario is 1SDI or 2SDI) measurements on the second particle and single Charlie performs characterized measurements on the third particle. Let Bob perform dichotomic projective measurement of the spin component observable in the direction $y_0$, or $y_1$, or $y_2$. Charlie performs dichotomic projective measurement of the spin component observable in the direction $z_0$, or $z_1$, or $z_2$. Alice $m$ (where $m \in \{1, 2, ..., n\}$) performs dichotomic unsharp measurement of the spin component observable in the direction $\hat{x}^m_i$, or $\hat{x}^m_j$, or $\hat{x}^m_k$. The outcomes of each measurement is $\pm 1$.

The projectors associated with Bob’s sharp spin component measurement in the direction $\hat{y}_j$ (with $j \in \{0, 1, 2\}$) are given by $P_{b[y_j]} = \frac{\mathbb{I}_2 + b \hat{y}_j \cdot \mathbf{\sigma}}{2}$ (with $b \in \{-1, 0, 1\}$ being the outcome of Bob’s sharp measurement). Similarly, the projectors associated with Charlie’s sharp component measurement in the direction $\hat{z}_k$ (with $k \in \{0, 1, 2\}$) can be written as $P_{c[z_k]} = \frac{\mathbb{I}_2 + c \hat{z}_k \cdot \mathbf{\sigma}}{2}$ (with $c \in \{-1, 1\}$ being the outcome of Charlie’s sharp measurement). Here $\mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector composed of three Pauli matrices. The directions $\hat{y}_j$ and $\hat{z}_k$ can be expressed as,

$$\hat{y}_j = \sin \theta^y_j \cos \phi^y_j \hat{X} + \sin \theta^y_j \sin \phi^y_j \hat{Y} + \cos \theta^y_j \hat{Z},$$

(14)

and

$$\hat{z}_k = \sin \theta^z_k \cos \phi^z_k \hat{X} + \sin \theta^z_k \sin \phi^z_k \hat{Y} + \cos \theta^z_k \hat{Z},$$

(15)

where $j, k \in \{0, 1, 2\}; 0 \leq \theta^y_j \leq \pi; 0 \leq \phi^y_j \leq 2\pi; 0 \leq \theta^z_k \leq \pi; 0 \leq \phi^z_k \leq 2\pi$. $\hat{X}, \hat{Y}, \hat{Z}$ are three orthogonal unit vectors in Cartesian coordinates.

The effect operators associated with Alice $m$’s $(m \in \{1, 2, ..., n\})$ unsharp measurement of spin component observable in the direction $\hat{x}^m_i$ (with $i \in \{0, 1, 2\}$) are given by,

$$E^\lambda_m[a_i^m] = \mathbb{I}_2 + a^m \hat{x}^m_i \cdot \mathbf{\sigma} + (1 - \lambda_m) \frac{\mathbb{I}_2}{2},$$

(16)

with $a^m \in \{-1, 1\}$ being the outcome of Alice $m$’s unsharp measurement and $\lambda_m (0 < \lambda_m \leq 1)$ is the sharpness parameter corresponding to Alice $m$’s unsharp measurement. For a sequence of $n$ Alices, the measurements of Alice $n$ will be sharp, i.e., $\lambda_n = 1$. The direction $\hat{x}^m_i$ is given by,

$$\hat{x}^m_i = \sin \theta^m_i \cos \phi^m_i \hat{X} + \sin \theta^m_i \sin \phi^m_i \hat{Y} + \cos \theta^m_i \hat{Z},$$

(17)

where $i \in \{0, 1, 2\}; 0 \leq \theta^m_i \leq \pi; 0 \leq \phi^m_i \leq 2\pi$.

There are various types of correlations appearing in the inequalities (9, 10, 11, 12). In the following, we present the detailed calculations of these correlations between Alice $m$, Bob and Charlie.

The joint probability distribution of occurrence of the outcomes $a^1, b, c$, when Alice 1 performs unsharp measurement of spin component observable along the direction $\hat{x}^1_i$, and Bob and Charlie perform projective measurements of spin component observables along the directions $\hat{y}_j$ and $\hat{z}_k$ respectively on the shared tripartite
let state $\rho$, is given by,
\[
P(a^1, b, c|\hat{x}^1, \hat{y}_j, \hat{z}_k) = \text{Tr} \left \{ \left ( E^{\lambda_1}_{a^1|\hat{x}^1} \otimes \frac{\| b_j \| + \hat{y}_j \cdot \hat{\sigma}}{2} \otimes \frac{\| c_z \| + \hat{z}_k \cdot \hat{\sigma}}{2} \right ) \rho \right \}.
\] (18)

In this case, the correlation function between Alice$^1$, Bob and Charlie can be written as
\[
\langle x^1_i y_j z_k \rangle = \sum_{a^1=\pm 1, b=\pm 1, c=\pm 1} a^1 b c P(a^1, b, c|\hat{x}^1, \hat{y}_j, \hat{z}_k).
\] (19)

After performing unsharp measurement, Alice$^1$ passes her particle to Alice$^2$. The unnormalized post-measurement reduced state at Alice$^2$'s end, after Alice$^1$ gets the outcome $a^1$ by performing unsharp measurement of spin component observable along the direction $\hat{x}^1$ and Bob and Charlie get the outcomes $b$ and $c$ by performing sharp measurements of spin component observables along the directions $\hat{y}_j$ and $\hat{z}_k$ respectively, is given by,
\[
\rho_{an}^2 = \text{Tr}_{BC} \left \{ \left ( \sqrt{E^{\lambda_1}_{a^1|\hat{x}^1}} \otimes \frac{\| b_j \| + \hat{y}_j \cdot \hat{\sigma}}{2} \otimes \frac{\| c_z \| + \hat{z}_k \cdot \hat{\sigma}}{2} \right ) \rho \right \}.
\] (20)

where,
\[
\sqrt{E^{\lambda_1}_{a^1|\hat{x}^1}} = \sqrt{\frac{1 + \lambda_1}{2}} \left ( \frac{\| b_j \| + \hat{y}_j \cdot \hat{\sigma}}{2} \right ) + \sqrt{\frac{1 - \lambda_1}{2}} \left ( \frac{\| b_j \| - \hat{y}_j \cdot \hat{\sigma}}{2} \right ).
\] (21)

In order to get the reduced state, the partial trace has been taken over the subsystems of Bob and Charlie.

Now Alice$^2$ again performs unsharp measurement (associated with sharpness parameter $\lambda_2$) of spin component observable along the direction $\hat{x}^2$ on the reduced state $\rho_{an}^2$ and gets the outcome $a^2$. The joint probability distribution of occurrence of the outcomes $a^1$, $a^2$, $b$, $c$, when Alice$^1$, Alice$^2$ perform unsharp measurements of spin component observables along the directions $\hat{x}^1$, $\hat{x}^2$ respectively and Bob and Charlie perform projective measurements of spin component observables along the directions $\hat{y}_j$ and $\hat{z}_k$ respectively, is given by,
\[
P(a^1, a^2, b, c|\hat{x}^1, \hat{x}^2, \hat{y}_j, \hat{z}_k) = \text{Tr} \left [ E^{\lambda_2}_{a^2|\hat{x}^2} \cdot \rho_{an}^2 \right ].
\] (22)

From this expression, the joint probability of obtaining the outcomes $a^2$, $b$, $c$ by Alice$^2$, Bob, Charlie, respectively, can be calculated as,
\[
P(a^2, b, c|\hat{x}^2, \hat{y}_j, \hat{z}_k) = \sum_{a^1=\pm 1} P(a^1, a^2, b, c|\hat{x}^1, \hat{x}^2, \hat{y}_j, \hat{z}_k).
\] (23)

Let $\langle x^2_i y_j z_k \rangle_{x^1}$ denote the correlation between Alice$^2$, Bob and Charlie, when Alice$^1$, Alice$^2$ perform unsharp measurements of spin component observables along the directions $\hat{x}^1$, $\hat{x}^2$ respectively and Bob, Charlie perform projective measurements of spin component observables along the directions $\hat{y}_j$ and $\hat{z}_k$ respectively. The expression for $\langle x^2_i y_j z_k \rangle_{x^1}$ can be obtained as,
\[
\langle x^2_i y_j z_k \rangle_{x^1} = \sum_{a^2=\pm 1} P(a^2, b, c|\hat{x}^1, \hat{x}^2, \hat{y}_j, \hat{z}_k).
\] (24)

Since Alice$^2$ is ignorant about the choice of the measurement setting of Alice$^1$, the above correlation has to be averaged over the three possible measurement settings of Alice$^1$ (unsharp measurement of spin component observables in the directions $\{\hat{x}^1_l, \hat{x}^2_l\}$). This average correlation function between Alice$^2$, Bob and Charlie is given by,
\[
\langle x^2_i y_j z_k \rangle_{av} = \sum_{i=0,1,2} \langle x^2_i y_j z_k \rangle_{x^1_i} P(\hat{x}^1_i).
\] (25)

Here $P(\hat{x}^1_i)$ is the probability of Alice$^1$’s unsharp measurement of spin component observable in the direction $\hat{x}^1_i$ ($i \in \{0,1,2\}$). Since, we restrict ourselves to unbiased input scenario, all the three measurement settings for Alice$^1$ are equally probable, i.e., $P(\hat{x}^1_0) = P(\hat{x}^1_1) = P(\hat{x}^1_2) = \frac{1}{3}$.

Using this general expression (25) for the average three-party correlation function, the terms appearing in inequalities (9, 10, 11, 12) can be easily calculated for probing genuine EPR steering between Alice$^2$, Bob and Charlie in ISDI and 2SDI scenarios. Note that there are several two-party correlation functions and one-party expectation values on the left hand sides of inequalities (9, 10, 11, 12). These can be calculated using the above-mentioned approach invoking the no-signalling condition between the observers at three different wings. For example, the average correlation function between Alice$^2$ and Bob can be calculated as follows.

The joint probability distribution of occurrence of the outcomes $a^2$, $b$, when Alice$^1$, Alice$^2$ perform unsharp measurements of spin component observables along the directions $\hat{x}^1$, $\hat{x}^2$ respectively and Bob performs projective measurement of spin component observable along the direction $\hat{y}_j$, is given by,
\[
P(a^2, b|\hat{x}^1, \hat{x}^2, \hat{y}_j) = \sum_{c=\pm 1} P(a^2, b, c|\hat{x}^1, \hat{x}^2, \hat{y}_j, \hat{z}_k).
\] (26)

Here, we have used the no-signalling condition between Charlie’s wing and the other two wings. In the above case, the average two-party correlation function $\langle x^2_i y_j \rangle_{av}$
between Alice and Bob is given by,

\[
\langle x_i^2 y_j \rangle_{\text{ave}} = \sum_{i=0,1,2} \left[ \sum_{a=-1}^{+1} \sum_{b=-1}^{+1} a^2 b P(a^2, b|x_i\rangle, \langle x_j|) \right] P(x_j^2).
\]  

(27)

Following the above-mentioned approach, each term appearing on the left hand sides of the inequalities (9, 10, 11, 12) in the context of Alice's, Bob and Charlie can be calculated. We next obtain the maximum number of Alices, who can detect genuine tripartite entanglement in 1SDI scenario when the GHZ state as well as the W-state is initially shared between multiple Alices, single Bob and single Charlie.

1. **When GHZ state is shared**

Let us consider the GHZ-state \(|\psi_{\text{GHZ}}\rangle\) given by \(\rho_{\text{GHZ}} = |\psi_{\text{GHZ}}\rangle \langle \psi_{\text{GHZ}}|\) shared among the three spatially separated wings, where

\[
|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),
\]

(28)

where \(|0\rangle\) and \(|1\rangle\) denotes two mutually orthonormal states in \(\mathbb{C}^2\). Here multiple untrusted parties (Alices) perform sequential weak measurements. We would like to explore the maximum number of Alices who can sequentially detect genuine entanglement of the initially shared GHZ state in both 1SDI as well as 2SDI scenarios. For 1SDI scenario, we shall consider the inequality (9). Any violation of this inequality (9) certifies the presence of genuine entanglement in 1SDI way. For the 2SDI scenario, we shall consider the inequality (10) as any violation of this inequality certifies the presence of genuine entanglement in 2SDI way.

At first, we focus on the 1SDI scenario. Here the measurements performed by each of the multiple Alices are uncharacterized. On the other hand, the measurements by the single Bob and the single Charlie are characterized. Now, we want to find out whether Alice and Alice can sequentially detect genuine tripartite entanglement in 1SDI scenario. In other words, we will find out whether Alice and Alice can sequentially detect genuine tripartite entanglement in 1SDI scenario. In this case, the measurements of the final Alice in the sequence, i.e., Alice will be sharp (\(\lambda_2 = 1\)), and the measurements of Alice will be unsharp. We observe that, for example, when Alice gets \(G_1 = -0.1\), then Alice gets \(G_1 = -0.55\). This happens for the following choices of measurement settings by Alice and Alice:

- \(\theta_0^1, \phi_0^1, \theta_1^1, \phi_1^1, \theta_2^1, \phi_2^1, \theta_0^2, \phi_0^2, \theta_1^2, \phi_1^2, \theta_2^2, \phi_2^2\) \(\equiv (\pi, 0, \pi, \pi, 0, \pi, 0, \pi, 0, \pi, 0, \pi)\), with \(\lambda_1 = 0.627\).

Note that, here the choices of measurement settings by Bob and Charlie are not mentioned as they are already specified in the inequality (9). This is due to the fact that Bob and Charlie perform characterized measurements in this case. Hence, we can conclude that Alice and Alice can sequentially certify genuine tripartite entanglement in 1SDI scenario when the GHZ state is initially shared.

Next, we address whether Alice, Alice and Alice can sequentially violate the inequality (9) with single Bob and single Charlie. In this case, the measurements of the final Alice, i.e., Alice are sharp (\(\lambda_3 = 1\)), and the measurements of Alice and Alice are unsharp. We observe that, when Alice gets \(G_1 = -0.1\) and Alice gets \(G_1 = -0.1\), then Alice gets \(G_1 = -0.183\). This happens for the following choices of measurement settings by Alice and Alice and Alice:

- \(\theta_0^1, \phi_0^1, \theta_1^1, \phi_1^1, \theta_2^1, \phi_2^1, \theta_0^2, \phi_0^2, \theta_1^2, \phi_1^2, \theta_2^2, \phi_2^2\) \(\equiv (\pi, 0, \pi, \pi, 0, \pi, 0, \pi, 0, \pi, 0, \pi)\), with \(\lambda_1 = 0.627\) and \(\lambda_2 = 0.736\). Hence, Alice, Alice and Alice can sequentially certify genuine tripartite entanglement in 1SDI scenario when the GHZ state is initially shared.

Now we want to investigate whether Alice, Alice and Alice can sequentially violate the inequality (9) with single Bob and single Charlie. Here the measurements of Alice are sharp (\(\lambda_4 = 1\)), and the measurements of all other Alices are unsharp. In this case, we observe that for any choices of measurement settings by multiple Alices, Alice, Alice, Alice and Alice cannot sequentially violate the inequality (9). These results are summarized in Table I. The permissible range of each \(\lambda_m\) depends on the values \(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}\). In the table, we have presented the permissible range of each \(\lambda_m\) for the minimum permissible value of each \(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}\). The permissible range of \(\lambda_m\) will be smaller than this if we take other value \(\lambda_i > \lambda_m\) \(\forall i < m\), and the maximum number of Alices may get reduced. It is to be noted here that Alice may obtain quantum mechanical violation of the inequality (9) if the sharpness parameter of any previous Alice is too small not to get a violation. In fact, any three Alices (at most) can sequentially certify genuine tripartite entanglement in 1SDI scenario by violating the inequality (9) with single Bob and single Charlie.

Next, we similarly find out the maximum number of Alices who can sequentially certify genuine tripartite entanglement in the 2SDI scenario by violating the inequality (10) with single Bob and single Charlie. Here, in addition to multiple Alices, the measurements performed by Bob are also uncharacterized. In this case, we find that the maximum number of Alices is three. This result is also summarized in Table I. Thus, for the GHZ state we get the maximum numbers of Alices to be three both in the 1SDI scenario and the 2SDI scenario. However, the allowed range of the sharpness parameter is larger in the 1SDI scenario.

2. **When W state is shared**

Here, let us consider that the three-qubit W state \(\rho_W = |\psi_W\rangle \langle \psi_W|\) is initially shared between multiple Al-
Let us now explore the maximum number of Alices who can sequentially detect genuine entanglement of the initial GHZ-state, the chances of sequential detection of genuine entanglement, is same in both the 1SDI and 2SDI scenarios. Here, the GHZ state is initially shared between multiple Alices, single Bob and single Charlie.

Let us now explore the maximum number of Alices who can sequentially detect genuine entanglement of the initially shared W state in both 1SDI as well as 2SDI scenarios. Here we will use the genuine EPR steering inequalities (11) and (12) for 1SDI and 2SDI genuine entanglement detection, respectively.

Following the approach mentioned in Section III A 1, we observe that at most two Alices can sequentially certify genuine tripartite entanglement in 1SDI scenario by violating the inequality (11) with single Bob and single Charlie. Also, at most two Alices can sequentially certify genuine tripartite entanglement in 2SDI scenario by violating the inequality (12) with single Bob and single Charlie. These results are summarized in Table II. The permissible range of each $\lambda_m$ depends on the values $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$. In the table, we have presented the permissible range of each $\lambda_m$ for the minimum permissible value of each $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$. The permissible range of $\lambda_m$ will be smaller than this if we take other value $\lambda_i > \lambda_{\text{min}}^i \forall i < m$, and the maximum number of Alices may get reduced.

From the aforementioned results, it is evident that the maximum number of Alices, who can sequentially detect genuine entanglement, is same in both the 1SDI and 2SDI scenarios for the W-state too. Again, as in the case of the GHZ-state, the chances of sequential detection of genuine entanglement is higher in 1SDI scenario, since the allowed range of the sharpness parameter is larger.

### Table I: The permissible ranges of the sharpness parameters $\lambda_m$ (where $0 < \lambda_m \leq 1$) of Alice$^m$ (untrusted party) in order to certify genuine entanglement in 1SDI and 2SDI scenarios by demonstrating quantum violations of the inequalities (9) and (10) respectively. Here, the GHZ state is initially shared between multiple Alices, single Bob and single Charlie.

| Alice$^m$ | 1SDI scenario | 2SDI scenario |
|-----------|---------------|---------------|
|           | Permissible $\lambda_m$ range | Permissible $\lambda_m$ range |
| Alice$^1$ | $1 \geq \lambda_1 > \lambda_{\text{min}}^1 = 0.577$ | $1 \geq \lambda_1 > \lambda_{\text{min}}^1 = 0.584$ |
| Alice$^2$ | $1 \geq \lambda_2 > \lambda_{\text{min}}^2 = 0.658$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 2$ | $1 \geq \lambda_2 > \lambda_{\text{min}}^2 = 0.668$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 2$ |
| Alice$^3$ | $1 \geq \lambda_3 > \lambda_{\text{min}}^3 = 0.787$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 3$ | $1 \geq \lambda_3 > \lambda_{\text{min}}^3 = 0.805$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 3$ |
| Alice$^4$ | No valid range for $\lambda_4$ \ for any $\lambda_i$ with $i < 4$ | No valid range for $\lambda_4$ \ for any $\lambda_i$ with $i < 4$ |

### Table II: The permissible ranges of the sharpness parameters $\lambda_m$ (where $0 < \lambda_m \leq 1$) of Alice$^m$ (untrusted party) in order to certify genuine entanglement in 1SDI and 2SDI scenarios by demonstrating quantum violations of the inequalities (11) and (12) respectively. Here, the W state is initially shared between multiple Alices, single Bob and single Charlie.

| Alice$^m$ | 1SDI scenario | 2SDI scenario |
|-----------|---------------|---------------|
|           | Permissible $\lambda_m$ range | Permissible $\lambda_m$ range |
| Alice$^1$ | $1 \geq \lambda_1 > \lambda_{\text{min}}^1 = 0.588$ | $1 \geq \lambda_1 > \lambda_{\text{min}}^1 = 0.678$ |
| Alice$^2$ | $1 \geq \lambda_2 > \lambda_{\text{min}}^2 = 0.674$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 2$ | $1 \geq \lambda_2 > \lambda_{\text{min}}^2 = 0.823$ \ when $\lambda_1 = \lambda_{\text{min}}^1 \forall i < 2$ |
| Alice$^3$ | No valid range for $\lambda_3$ \ for any $\lambda_i$ with $i < 3$ | No valid range for $\lambda_3$ \ for any $\lambda_i$ with $i < 3$ |

Next, we discuss sequential detection of genuine tripartite entanglement in both 1SDI and 2SDI scenarios considering Case B mentioned earlier (II.B), i.e., when single Alice performs uncharacterized measurements on the first particle, single Bob performs characterized or uncharacterized (depending on whether the scenario is 1SDI or 2SDI) measurements on the second particle, and multiple Charlies perform sequential characterized (trusted) measurements on the third particle. Here, a tripartite state $\rho$ (either GHZ state or W state) consisting of three spin-$\frac{1}{2}$ particles is initially shared among Alice, Bob and multiple Charlies. Alice performs dichotomic sharp measurement of spin component observable on her part in the direction $\hat{x}_0$, or $\hat{x}_1$, or $\hat{x}_2$. Bob performs dichotomic sharp measurement of spin component observable on his part in the direction $\hat{y}_0$, or $\hat{y}_1$, or $\hat{y}_2$. Charlie$^m$ (where $m \in \{1, 2, \ldots, n\}$) performs dichotomic unsharp measurement (associated with sharpness parameter $\lambda_m$) of spin component observable in the direction $\hat{z}_m$, or $\hat{z}_m^0$, or $\hat{z}_m^1$. The outcomes of each measurement are $\pm 1$. Here

$$\hat{x}_i = \sin \theta_i^x \cos \phi_i^x \hat{X} + \sin \phi_i^x \sin \sigma_i^x \hat{Y} + \cos \theta_i^x \hat{Z},$$

$$\hat{y}_j = \sin \theta_j^y \cos \phi_j^y \hat{X} + \sin \phi_j^y \sin \phi_j^y \hat{Y} + \cos \theta_j^y \hat{Z},$$

and

$$\hat{z}_k^m = \sin \theta_k^m \cos \phi_k^m \hat{X} + \sin \phi_k^m \sin \phi_k^m \hat{Y} + \cos \theta_k^m \hat{Z},$$

where $i, j, k \in \{0, 1, 2\}$; $0 \leq \theta_i^x \leq \pi; 0 \leq \phi_i^x \leq 2\pi; 0 \leq \theta_j^y \leq \pi; 0 \leq \phi_j^y \leq 2\pi; 0 \leq \theta_k^m \leq \pi; 0 \leq \phi_k^m \leq 2\pi$.

As in earlier cases, genuine tripartite entanglement certification in 1SDI and 2SDI scenarios is probed through the quantum violations of genuine EPR steering inequalities (9, 10, 11, 12). Here, the correlation functions and expectation values can be calculated using the technique.
described in Section III A, with only the role of Alice and Charlie being interchanged. Below, we determine the maximum number of Charlies who can sequentially detect genuine entanglement in 1SDI and 2SDI scenarios when the GHZ state or the W state is initially shared between single Alice, single Bob and multiple Charlies.

1. When GHZ state is shared

Let the GHZ state be initially shared between Alice, Bob and multiple Charlies. At first, we focus on the 1SDI scenario, i.e., Alice performs uncharacterized measurements, Bob and multiple Charlies perform characterized measurements. Now, we want to find out whether Charlie\(^1\) and Charlie\(^2\) can sequentially certify genuine tripartite entanglement in 1SDI scenario by violating the inequality (9) with single Bob and single Alice. In this case, the measurements of Charlie\(^2\) will be sharp (\(\lambda_2 = 1\)). We observe that when Charlie\(^1\) gets \(G_1 = -0.1\), then Charlie\(^2\) gets \(G_1 = 0.706\). This happens for the following choices of measurement settings by Alice: \((\theta_0, \phi_0, \theta_1, \phi_1, \theta_2, \phi_2) \equiv (\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0)\), with \(\lambda_1 = 0.507\). Here, the choices of measurement settings by Bob, Charlie\(^3\) and Charlie\(^2\) are not mentioned. Bob and each of the Charlies perform the particular characterized measurements as specified in the inequality (9). Therefore, Charlie\(^1\) and Charlie\(^2\) can sequentially certify genuine tripartite entanglement in 1SDI scenario when the GHZ state is initially shared.

Next, we enquire whether Charlie\(^1\), Charlie\(^2\) and Charlie\(^3\) can sequentially violate the inequality (9) with single Alice and single Bob. In this case, the measurements of Charlie\(^3\) are sharp (\(\lambda_3 = 1\)). When Charlie\(^1\) gets \(G_1 = -0.1\) and Charlie\(^2\) gets \(G_1 = -0.1\), then Charlie\(^3\) gets \(G_1 = -0.55\). This happens for the following choices of measurement settings by Alice: \((\theta_0, \phi_0, \theta_1, \phi_1, \theta_2, \phi_2) \equiv (\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0)\), with \(\lambda_1 = 0.507\) and \(\lambda_2 = 0.558\). Hence, Charlie\(^1\), Charlie\(^2\) and Charlie\(^3\) can sequentially certify genuine tripartite entanglement in 1SDI scenario when the GHZ state is initially shared. Proceeding further in this way, we find that at most six Charlies can sequentially certify genuine tripartite entanglement in the 1SDI scenario by violating the inequality (9) with single Bob and single Alice.

Similarly, in the 2SDI scenario (where single Alice and single Bob perform uncharacterized measurements and multiple Charlies perform characterized measurements), we find that at most three Charlies can sequentially certify genuine tripartite entanglement by violating the inequality (10) with single Bob and single Alice. These results are summarized in Table III. The permissible range of each \(\lambda_m\) depends on the values \(\lambda_1, \lambda_2, \ldots, \lambda_m-1\). In the table, we have presented the permissible range of each \(\lambda_m\) for the minimum permissible value of each \(\lambda_1, \lambda_2, \ldots, \lambda_m-1\). The permissible range of \(\lambda_m\) will be smaller than this if we take other value \(\lambda_i > \lambda_{min}^i\) \(\forall i < m\), and the maximum number of Charlies may get reduced. In these cases, we get the maximum numbers of Charlies to be six and three in the 1SDI scenario and 2SDI scenarios, respectively.

2. When W state is shared

Finally, we consider the W state to be initially shared between single Alice (untrusted party), single Bob (trusted or untrusted party depending on whether the scenario is 1SDI or 2SDI) and multiple Charlies (all trusted parties).

Following the approach mentioned in Section III B 1, we observe that at most four Charlies can sequentially certify genuine tripartite entanglement in the 1SDI scenario by violating the inequality (11) with single Bob and single Alice. On the other hand, at most three Charlies can sequentially certify genuine tripartite entanglement in the 2SDI scenario by violating the inequality (12) with single Alice and single Bob. These results are summarized in Table IV. As in their earlier cases, the permissible range of each \(\lambda_m\) depends on the values \(\lambda_1, \lambda_2, \ldots, \lambda_m-1\). In the table, we have presented the permissible range of each \(\lambda_m\) for the minimum permissible value of each \(\lambda_1, \lambda_2, \ldots, \lambda_m-1\). The permissible range of \(\lambda_m\) will be smaller than this if we take other value \(\lambda_i > \lambda_{min}^i\) \(\forall i < m\), and the maximum number of Charlies may get reduce d. We observe that the maximum number of Charlies who can sequentially detect genuine entanglement decreases as we move from the 1SDI scenario to the 2SDI scenario.
**TABLE IV:** The permissible ranges of the sharpness parameters $\lambda_m$ (where $0 < \lambda_m \leq 1$) of Charlie $^m$ (trusted party) in order to certify genuine entanglement in 1SDI and 2SDI scenarios by demonstrating quantum violations of the inequalities (11) and (12) respectively.

Here, the W state is initially shared between single Alice, single Bob and multiple Charlies.

| Charlie$^m$ | Permissible $\lambda_m$ range | Permissible $\lambda_m$ range |
|-------------|-----------------------------|-----------------------------|
| Charlie$^1$ | $1 \geq \lambda_1 > \lambda_{1\text{min}} = 0.522$ | $1 \geq \lambda_1 > \lambda_{1\text{min}} = 0.634$ |
|          | when $\lambda_1 = \lambda_{1\text{min}} \forall i < 2$ | when $\lambda_1 = \lambda_{1\text{min}} \forall i < 2$ |
| Charlie$^2$ | $1 \geq \lambda_2 > \lambda_{2\text{min}} = 0.578$ | $1 \geq \lambda_2 > \lambda_{2\text{min}} = 0.747$ |
|          | when $\lambda_2 = \lambda_{2\text{min}} \forall i < 2$ | when $\lambda_2 = \lambda_{2\text{min}} \forall i < 2$ |
| Charlie$^3$ | $1 \geq \lambda_3 > \lambda_{3\text{min}} = 0.659$ | $1 \geq \lambda_3 > \lambda_{3\text{min}} = 0.962$ |
|          | when $\lambda_3 = \lambda_{3\text{min}} \forall i < 3$ | when $\lambda_3 = \lambda_{3\text{min}} \forall i < 3$ |
| Charlie$^4$ | $1 \geq \lambda_4 > \lambda_{4\text{min}} = 0.882$ | No valid range for $\lambda_4$ |
|          | when $\lambda_4 = \lambda_{4\text{min}} \forall i < 4$ | for any $\lambda_4$ with $i < 4$ |
| Charlie$^5$ | No valid range for $\lambda_5$ | No valid range for $\lambda_5$ |
|          | for any $\lambda_i$ with $i < 5$ | for any $\lambda_i$ with $i < 5$ |

IV. CONCLUSIONS

Multipartite quantum correlations are used as important resources in various quantum network scenarios and other information theoretic applications [2, 3, 5, 6, 8, 9, 11]. However, due to the difficulties present in experimentally producing multipartite quantum correlations, exploring the possibilities of using single multipartite quantum correlation several times is not only interesting for foundational studies, but may also be useful for information theoretic applications.

In the present study, we address the question: whether multiple observers can detect genuine tripartite entanglement sequentially in SDI scenarios? We consider three spatially separated spin-$\frac{1}{2}$ particles in the GHZ or the W state, and shared between three wings. Out of these three wings, two observers on two wings perform sharp projective measurements on their respective particles whereas multiple observers at the third wing perform unsharp measurements on the third particle sequentially and independently.

We consider two separate cases, wherein the multiple observers perform either characterized (trusted) or uncharacterized (untrusted) measurements. Using the genuine EPR steering inequalities [57], we determine the maximum number of (trusted or untrusted) parties who can sequentially detect genuine tripartite entanglement in the 1SDI and 2SDI scenarios. We show that for both the choice of initial states, the range of measurement parameters for which genuine entanglement is detected turns out to be less in the 2SDI scenario when there are untrusted parties on the third wing, while the upper limit on the number of trusted observers on the third wing is lower in the 2SDI scenario. Thus, the chance of sequential detection of genuine entanglement decreases as we move from the 1SDI scenario to the 2SDI scenario.

To summarize, our analysis shows that increasing the number of trusted sides increases the chances of sequential detection of genuine entanglement in the semi-device independent framework. Such a result seems to nicely complement earlier studies [86, 87] from which it follows that the chance of sequential detection of genuine entanglement is less in the fully DI scenario (where all parties are untrusted) compared to that based on the genuine entanglement witness approach (where all parties are trusted). In the fully DI framework [86] at most two Charlies can share genuine nonlocality with a single Alice and a single Bob when they share the GHZ state, while only one Charlie can do so when the shared state is the W state. On the other hand, when all parties are trusted the number of Charlies goes up to 12 for the GHZ state and 4 for the W state, as shown through the witness based approach [87]. The SDI approach adopted in the present study occupies an intermediate status between the above two approaches, as borne out by the fact that the maximum number of Charlies turns out to be 6 and 3 in the 1SDI and 2SDI scenarios, respectively for the GHZ state, and 4 and 3 in the 1SDI and 2SDI scenarios, respectively for the W state.

Before concluding, it may be noted that our results, being valid in the intermediate scenario between the genuine entanglement witness based approach and the fully DI approach, should be more amenable for experimental verification since it is more tolerable to environmental noise than the fully DI scenario, and has less difficulties in realizing than genuine entanglement witnesses where all parties are trusted. As sequential sharing of two qubit nonlocality has already been experimentally demonstrated for two parties [69, 70, 79], our analysis can be experimentally verified in the near future. Finally, it may be worth exploring certain recent ideas regarding numerical optimization of the allowed range of parameters [67], and employing more general unsharp measurement formalism [68], in the context of a variety of nonlocal correlations with the aim of further increasing the number of parties who can share such correlations using a single copy of the prepared quantum state.

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