Higher dimensional generalization of the Benjamin-Ono equation: 2D case

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Abstract
We consider a higher-dimensional version of the Benjamin-Ono (HBO) equation in the 2D setting: $u_t - \mathcal{R}_1 \Delta u + \frac{1}{2} (u^2)_x = 0$, $(x, y) \in \mathbb{R}^2$, which is $L^2$-critical, and investigate properties of solutions both analytically and numerically. For a generalized equation (fractional 2D gKdV) after deriving the Pohozaev identities, we obtain nonexistence conditions for solitary wave solutions, then prove uniform bounds in the energy space or conditional global existence, and investigate the radiation region, a specific wedge in the negative $x$-direction. We then introduce our numerical approach in a general context, and apply it to obtain the ground state solution in the 2D critical HBO equation, then show that its mass is a threshold for global versus finite time existing solutions, which is typical in the focusing (mass-)critical dispersive equations. We also observe that globally existing solutions tend to disperse completely into the radiation in this nonlocal equation. The blow-up solutions travel in the positive $x$-direction with the rescaled ground state profile while also radiating dispersive oscillations into the radiative wedge. We conclude with examples of different interactions of two solitary wave solutions, including weak and strong interactions.
1 | INTRODUCTION

We study the following higher dimensional version of the Benjamin-Ono (HBO) equation

\[ u_t - R_1 \Delta u + \frac{1}{2} (u^2)_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad (1) \]

where the operator \( R_1 \) denotes the Riesz transform operator with respect to the first variable defined by the singular integral

\[ R_1 f(x, y) = \frac{1}{2\pi} \text{p.v.} \int \frac{(x - z_1) f(z_1, z_2)}{((x - z_1)^2 + (y - z_2)^2)^{3/2}} \, dz_1 \, dz_2, \]

and \( \hat{R_1} f(\xi_1, \xi_2) = \frac{-i \xi_1}{|\xi_1, \xi_2|} \hat{f}(\xi_1, \xi_2) \) with \( |(\xi_1, \xi_2)| = \sqrt{\xi_1^2 + \xi_2^2} \).

One of the first mentioning of this equations was by Shrir in Ref. 1, where he was describing the 2d long-wave perturbations in a boundary-layer type shear flow. These perturbations were weakly nonlinear, the flow did not have any inflection points, and the perturbations would be valid for the boundary layers along an inviscid boundary for free surface flows. That model reduced to an equation for the amplitude \( u \) of the longitudinal velocity of the fluid, which is exactly Equation (1). There are various extensions or reductions of Equation (1) that have been studied since then, for some initial studies, see Refs. 2–5, and for recent investigations, refer to Refs. 6–8 and references therein.

In the one-dimensional case, the multiplier associated to the Riesz transform coincides with that of the Hilbert transform operator. From this standpoint, (1) can be regarded as a two-dimensional extension of the Benjamin-Ono equation (BO)

\[ u_t - H \partial_x^2 u + \frac{1}{2} (u^2)_x = 0, \quad (x, \xi) \in \mathbb{R}, \quad (2) \]

where \( \hat{H} f(\xi) = -i \text{sign}(\xi) \hat{f}(\xi) \). We remark that Equation (2), including other nonlinearities, is of interest in various water wave models such as waves in deep water, e.g., see Refs. 9–14 and reviews.15,16

On the other hand, Equation (1) can also be seen as a particular case of the higher dimensional fractional generalized KdV equation

\[ u_t - \partial_x (-\Delta)^s u + \frac{1}{m} (u^m)_x = 0, \quad (x, \xi) \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad m > 1, \quad (3) \]

where \( d \geq 2, 0 < s < 1, m \) is integer, and \((-\Delta)^s\) denotes the fractional Laplacian of order \( s \) defined by the Fourier multiplier with symbol \( |\xi|^{2s} = (\xi_1^2 + \cdots + \xi_d^2)^s \). This generalization is more evident
by recalling that $R_1(u) = P^{-1}(-\frac{i\xi_1}{|\xi|} \hat{u})(x) = -\partial_x(-\Delta)^{-\frac{1}{2}} u$, which yields

$$R_1(-\Delta u) = P^{-1}\left(\frac{-i\xi_1}{|\xi|} \cdot |\xi|^2 \hat{u}\right)(x) = -\partial_x(-\Delta)^{-\frac{1}{2}} u,$$

and (3) generalizes the HBO equation (1) with a fractional dispersion operator of order $s$ and nonlinearity $m$. Setting $s = \frac{1}{2}$ and $m = 2$ in (3) yields (1), whereas $s = 1$ and $m = 2$ in (3) agrees with the Zakharov–Kuznetsov equation (ZK), which in 3d describes the propagation of ionic–acoustic waves in magnetized plasma and in 2d, for example, it serves as the amplitude equation for long waves on the free surface of a thin film in a specific fluid and viscosity parameters.

The family of Equations (3) is useful to measure the competition between the effects of dispersion and nonlinearity in a $d$-dimensional model.

In general, the power $m - 1 > 0$ in (3) does not need to be an integer number. One can take, for instance, $m - 1 = k/p$, where $k$ and $p$ are relatively prime and $p$ is odd. Consequently, it is possible to set a branch of the map $\omega \mapsto \omega^{1/p}$ real on the real axis. A similar condition has been used before in Ref. 19. Alternatively, the nonlinearity in (3) can be replaced with $\partial_x(|u|^{m-1} u)$, and thus, one can consider the equation

$$u_t - \partial_x(-\Delta)^s u + \frac{1}{m} \partial_x(|u|^{m-1} u) = 0, \quad (x, \ldots) \in \mathbb{R}^d, \; t \in \mathbb{R}, \; m > 1. \quad (4)$$

In what follows, when $m$ is not an integer, we will consider (4).

Real solutions of (1) formally satisfy at least three conservation laws: the $L^2$-norm (or mass) conservation

$$M[u(t)] = \int_{\mathbb{R}^2} |u(x,y,t)|^2 \, dx \, dy = M[u(0)], \quad (5)$$

the energy (or Hamiltonian) conservation

$$E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} (-\Delta)^{\frac{1}{4}} u(x,y,t)^2 \, dx \, dy - \frac{1}{6} \int_{\mathbb{R}^2} (u(x,y,t))^3 \, dx \, dy = E[u(0)], \quad (6)$$

and the $L^1$-type conservation

$$\int_{\mathbb{R}} u(x,y,t) \, dx = \int_{\mathbb{R}} u(x,y,0) \, dx, \quad (7)$$

which can also be stated in a 2$d$ form

$$\int_{\mathbb{R}^2} u(x,y,t) \, dx \, dy = \int_{\mathbb{R}^2} u(x,y,0) \, dx \, dy. \quad (8)$$

We mention that no other conserved quantities are known for (1). In contrast, the BO equation (2) is a completely integrable Hamiltonian system, see Refs. 20, 21.

Equation (1) is invariant under the scaling: If $u$ solves (1), then so does

$$u_\lambda(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^2 t) \quad (9)$$
for any positive $\lambda$. Consequently, the homogeneous Sobolev space $H^r_c$ is invariant under the scaling (9) when $r_c = 0$, in other words, Equation (1) is $L^2$-critical. (For a general case of (3) and (4), see Section 2.)

We next recall some results regarding the well-posedness for the Cauchy problem associated to (1) and (3) in Sobolev spaces. In Ref. 22, it was proved that (1) is locally well-posed in $H^r(\mathbb{R}^2)$ whenever $r > 5/3$. In Ref. 23, the local well-posedness (lwp) theory was extended for regularities $r > 3/2$. Furthermore, the well-posedness results in weighted Sobolev spaces as well as some unique continuation principles for Equation (1) were studied in Ref. 24. We remark that the above lwp results were obtained via compactness methods as one cannot solve the initial value problem associated to (1) by a Picard iterative method implemented on its integral formulation for any initial data in the Sobolev space $H^r(\mathbb{R}^2), r \in \mathbb{R}$ (see Ref. 22, Theorem 4.1 and Corollary A.1). As far as Equation (3), the following well-posedness results hold. When $d \geq 2, \frac{1}{2} \leq s < 1$ and $m = 2$ in (3), it was proved in Ref. 23 that (3) is locally well-posed in $H^r(\mathbb{R}^d)$ provided that $r > \frac{d + 3}{2} - 2s$. This same result was proved before for $d \geq 3$ and $s = \frac{1}{2}$ in Ref. 22. On the other hand, by the standard parabolic regularization argument (see Refs. 9, 25), the Cauchy problem associated to (3) is locally well-posed in $H^r(\mathbb{R}^d)$ whenever $r > \frac{d}{2} + 1$ for any $m > 1$ integer, and $0 < s < 1$ fixed. To the best of our knowledge there are no results concerning the global well-posedness (GWP) for the Cauchy problem associated to (3) or (4) with $d \geq 2$ in the current literature. Regarding the GWP for (3) in $d = 1$, see Refs. 26, 27 and references therein.

The solitary-wave solutions for Equation (1) are of the form

$$u(x, y, t) = Q_c(x - ct, y),$$

where $c > 0$ denotes the speed of propagation or the scaling factor for $Q_c(x, y) = c Q(cx, cy)$, and $Q$ is a real-valued vanishing at infinity solution of

$$Q + (-\Delta)^{\frac{1}{2}} Q - \frac{1}{2} Q^2 = 0.$$  \hspace{1cm} (10)

(For a general case of (3), see Sections 2 and 4.) The existence and spatial decay of solutions for (10) were considered in Ref. 28. The uniqueness of positive solutions can be deduced as a particular case of the results obtained in Ref. 29 (see also Ref. 30 for $Id$) for a class of nonlocal equations

$$\Psi + (-\Delta)^s \Psi - |\Psi|^r \Psi = 0, \quad \text{in } \mathbb{R}^d,$$  \hspace{1cm} (11)

with $d \geq 1, s \in (0, 1)$ and $0 < r < r_* = r_*(d, s)$, where

$$r_* = \begin{cases} \frac{4s}{d-2s} & \text{for } 0 < s < \frac{d}{2}, \\ +\infty & \text{for } s \geq \frac{d}{2}. \end{cases}$$

In the general $2d$ setting we present some further results regarding solitary waves in Section 2. In particular, from Pohozaev identities the nonexistence of solitary wave solutions for the fractional $2d$ generalized KdV equation is obtained.

Furthermore, in the same general fractional setting we review existence of solutions in sufficiently regular space and then obtain uniform bounds or global existence criteria in the
$L^2$-subcritical, critical, and supercritical cases, see Theorem 2, though this result is conditional on the lwp in the energy space $H^s(\mathbb{R}^2)$, $0 < s < 1$, because we only have the lwp in $H^r(\mathbb{R}^2)$, $r > 2$. We note that in the $L^2$-critical case, the threshold for global existence is given by the mass of the ground state, and we investigate this threshold more closely in the later part of the paper via numerical simulations. In particular, we show that all sufficiently localized data above the threshold blow up in finite time, confirming the Conjecture 1, see Section 2.2 (we tried initial data with exponential and polynomial decays, with the polynomial decay as low as $r^{-2}$, which is below the ground state decay). We also studied the global existence and observed that such solutions tend to disperse completely into the radiation in the nonlocal $L^2$-critical 2d HBO equation, see Section 5.1. The radiation region is formed as the wedge around the negative $x$-direction with the opening angle as big as $\tan \theta = 2\sqrt{2}$ in this equation, and we show that in general this radiation region only depends on the dispersion operator in the linear equation, see Section 2.3.

After obtaining some results about a single maximum initial data, we turn to the interaction of the solitary waves, and show that various interactions are possible, which depends on the initial geometrical configuration and the distribution of mass in both solitary waves. In particular, there can be no significant (or only weak) interaction, and we also observe strong interactions, where both solitary waves can merge into one and either blow up in finite time, or disperse (eventually both of them), see Section 5.4.

The paper is organized as follows: In Section 2 we study a generalized fractional KdV in 2d setting and review the conserved quantities, scaling invariance, derive Pohozaev identities, which leads to the results about the nonexistence of solitary wave solutions in various contexts. After that in Section 2.2 we consider $L^2$-subcritical, critical, and supercritical cases in the general fractional setting and obtain uniform bounds in the energy space in Theorem 2, parts (C1), (C2), and (C3), respectively; in particular, noticing that in the $L^2$-critical case of the HBO equation the mass of the ground state solution plays the role of the threshold for the global existence. In Section 2.3 we consider a linear 2d fractional KdV equation and show the radiation region, which is a wedge with a specific angle, depending only on the dispersion operator and not on the nonlinearity or dimension. Next, in Section 3 we describe our numerical approach, including space discretization via the rational basis (eigen)functions, discretization of the fractional Laplacian, in particular, using the Dunford–Taylor formula to change the fractional Laplacian into the full Laplacian in Galerkin formulation, and the extension to the higher dimensional computations. In Section 4 we use Petviashvili’s iteration method to obtain the ground state profile (or its rescaled versions). Finally, in Section 5 we show the numerical results confirming the ground state mass threshold for global existence versus finite time blow-up, and study the behavior of globally existing solutions more carefully, finding that even if a solitary wave-type solution starts traveling to the right, it eventually stops moving and subdues into the radiation via dispersive oscillations in the negative $x$-directions. Blow-up solutions, on the other hand, travel in the positive $x$-direction and blow up with the rescaled ground state profiles. Lastly, in Section 5.4 we examine interaction of two solitary waves in different geometrical settings and of different sizes and show weak and strong interactions.

2 | REMARKS ON THE FRACTIONAL 2D GENERALIZED KDV EQUATION

In this section we take a more general approach and consider Equations (3) and (4) in the 2d setting. We start with recalling some useful invariances and inequalities such as the conserved
quantities and scaling invariance, as well as the Gagliardo–Nirenberg inequality and discuss some results regarding the existence or nonexistence of solutions to the stationary problems of the form (11). As an application, we show uniform estimates in time for solutions of (3) in the energy space \( H^s(\mathbb{R}^2) \). Additionally, we present a formal analysis of the dispersive relation of (3), which allows us to conjecture and study regions in space where dispersive oscillations, or radiation, occur for solutions of (3). In particular, setting \( s = \frac{1}{2} \) and \( m = 2 \) the results of this section are valid for (1).

2.1 Preliminaries on the fractional 2D gKdV and ground state solutions

We focus our discussion on the following generalization of (3):

\[
\begin{align*}
    u_t + \nu_1 \partial_x (\Delta^s) u + \frac{\nu_2}{m} (u^m)_x &= 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad m > 1, \\
    \nu_1 &\neq 0, \quad \nu_2 \in \{1, -1\}, \quad 0 < s < 1.
\end{align*}
\]  

where \( \nu_1 \neq 0, \nu_2 \in \{1, -1\} \), and 0 < s < 1. We use two parameters \( \nu_1 \) and \( \nu_2 \) to indicate subtle differences in the existence of the ground state solutions and other properties, see Remark 4 below.

During their lifespans, solutions of Equation (12) satisfy the mass conservation (5), the \( L^1 \)-type invariance (7), and the energy conservation, which in this case is given as

\[
E_s[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} \left| (-\Delta)^{\frac{s}{2}} u(x, y, t) \right|^2 dx dy + \frac{\nu_2}{\nu_1 m (m+1)} \int_{\mathbb{R}^2} (u(x, y, t))^{m+1} dx dy = E_s[u(0)].
\]  

Equation (12) is invariant under the scaling

\[
u_1 \partial_x (\Delta^s) u + \frac{\nu_2}{m} (u^m)_x = 0.
\]

where \( \nu_1 \neq 0, \nu_2 \in \{1, -1\} \), and 0 < s < 1. We use two parameters \( \nu_1 \) and \( \nu_2 \) to indicate subtle differences in the existence of the ground state solutions and other properties, see Remark 4 below.

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Equation (12) is invariant under the scaling

\[
u_1 \partial_x (\Delta^s) u + \frac{\nu_2}{m} (u^m)_x = 0.
\]
results. To establish the existence of solutions one can use the Weinstein classical approach, which consists of determining the best constant $C_{GN}$ in the Gagliardo–Nirenberg inequality

$$\|f\|_{L^{m+1}(\mathbb{R}^2)}^{m+1} \leq C_{GN} \left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^2(\mathbb{R}^2)} \left\|f\right\|_{L^2(\mathbb{R}^2)}^{(m+1)-\frac{(m-1)}{s}},$$

(16)

where the sharp constant $C_{GN}$ is obtained by minimizing the functional

$$J(f) = \left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^2(\mathbb{R}^2)} \left\|f\right\|_{L_{m+1}(\mathbb{R}^2)}^{m+1},$$

(17)

defined for $f \in H^s(\mathbb{R}^2)$ with $f \neq 0$. Thus, one can use concentration–compactness arguments to show that $C_{GN}^{-1} = \inf_{f \neq 0} J(f)$ is attained. Moreover, recalling that $c > 0, \nu_1 < 0,$ and $\nu_2 = 1$, by computing $J'(\cdot)$, it follows that any minimizer $\varphi \in H^s(\mathbb{R}^2)$ satisfies Equation (15) after a suitable rescaling, and the inequality $J(|\varphi|) \leq J(\varphi)$ implies that the minimizer $\varphi$ can be chosen to be non-negative (for further properties see Ref. 29, Appendix D) and the reference therein), concluding the existence part. The uniqueness of the ground state (any nonnegative minimizer $\varphi$ of $J(\cdot)$ is a ground state) was established up to translation (or being radially symmetric and decreasing around some point) in Ref. 29 (for Id case see Ref. 30). Summarizing we have the following result:

**Theorem 1**

Let $0 < s < 1, c > 0, \nu_1 < 0, \nu_2 = 1,$ and $1 < m < \frac{1+s}{1-s}$. Then Equation (15) admits a unique, up to translation, positive solution $\varphi$ in $H^s(\mathbb{R}^2)$. Moreover, there exists some $(x_0, y_0) \in \mathbb{R}^2$ such that $\varphi(\cdot - x_0, \cdot - y_0)$ is radial, positive, and strictly decreasing in $|(x-x_0, y-y_0)|$. Additionally, the function $\varphi$ belongs to $H^{2s+1}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ and it satisfies

$$\frac{C_1}{1 + |(x, y)|^{2s+2}} \leq \varphi(x, y) \leq \frac{C_2}{1 + |(x, y)|^{2s+2}},$$

(18)

for all $(x, y) \in \mathbb{R}^2$, with some constants $C_2 \geq C_1 > 0$ depending on $m$ and $\varphi$.

**Remark 1.** For the case when $m = \frac{1+s}{1-s}$, see a corresponding result in Refs. 31, 32.

By rescaling and setting $s = \frac{1}{2}$, $m = 2$, Theorem 1 establishes the existence of the unique positive solution $Q$ for Equation (10), which we consider later in Sections 4 and 5.

We next derive the key Pohozaev identities (we use them later in Section 4 for the verification of the ground state computations).

**Lemma 1.** Assume $0 < s < 1, c \neq 0, m > 1$ with $m \neq \frac{1+s}{1-s}$. Let $\varphi$ be a smooth vanishing at infinity solution of (15). Then the following identities hold true:

$$\left\|(-\Delta)^{\frac{s}{2}} \varphi\right\|_{L^2(\mathbb{R}^2)}^2 = -\frac{c(m-1)}{\nu_1(2 - (1-s)(m+1))} \|\varphi\|_{L^2(\mathbb{R}^2)}^2,$$

(19)
\[ \int_{\mathbb{R}^2} \varphi^{m+1} \, dx \, dy = \frac{scm(m + 1)}{\nu_2(2 - (1 - s)(m + 1))} \| \varphi \|^2_{L^2(\mathbb{R}^2)}. \tag{20} \]

As a consequence, when \( \nu_1 = -1 \) and \( \nu_2 = 1 \),

\[ E[\varphi] = r_c \frac{c(m - 1)}{2(2 - (1 - s)(m + 1))} \| \varphi \|^2_{L^2(\mathbb{R}^2)}, \tag{21} \]

By “smooth,” we mean that the functions have sufficient regularity to justify the arguments below in the proof of Lemma 1. By setting \( \nu_1 < 0, \nu_2 = 1, c > 0, \) and \( 1 < m < \frac{1 + s}{1 - s} \), the conclusion of Theorem 1 determines the existence of a solution for Equation (15), which satisfies the required assumptions of regularity and decay to verify Lemma 1.

Remark 2. For the specific case of Equation (15) with \( s = \frac{1}{2}, m = 2, c = 1, \nu_1 = -1, \) and \( \nu_2 = 1 \), i.e., Equation (10), the identities (19), (20), and (21) reduce to

\[ \left\| (-\Delta)^\frac{1}{2} \varphi \right\|^2_{L^2(\mathbb{R}^2)} = 2 \| \varphi \|^2_{L^2(\mathbb{R}^2)}, \tag{22} \]

\[ \int_{\mathbb{R}^2} \varphi^3 \, dx \, dy = 6 \| \varphi \|^2_{L^2(\mathbb{R}^2)}, \tag{23} \]

\[ E[\varphi] = 0, \tag{24} \]

which we use to verify the accuracy of our computation for the ground state (see \( e_1, e_2, e_3 \)) in Section 4 below.

Proof of Lemma 1. Multiplying (12) by \( \varphi \) and integrating on \( \mathbb{R}^2 \), yields

\[ \int \left( -c \varphi^2 + \nu_1 (-\Delta)^s \varphi \varphi + \frac{\nu_2}{m} \varphi^{m+1} \right) \, dx \, dy = 0. \tag{25} \]

On the other hand, we claim

\[ \int (-\Delta)^s \varphi (x \varphi_x) \, dx \, dy = -s \int (-\Delta)^{s-1} \partial_x^2 \varphi \varphi \, dx \, dy - \frac{1}{2} \int (-\Delta)^s \varphi \varphi \, dx \, dy, \tag{26} \]

\[ \int (-\Delta)^s \varphi (y \varphi_y) \, dx \, dy = -s \int (-\Delta)^{s-1} \partial_y^2 \varphi \varphi \, dx \, dy - \frac{1}{2} \int (-\Delta)^s \varphi \varphi \, dx \, dy. \tag{27} \]
We only show (26) as the same reasoning leads to (27). We write the left-hand side of (26) as follows:

\[
\int \nabla \cdot \left( \nabla (\Delta \varphi \varphi_x) \right) \, dx \, dy = \int \nabla \cdot \left( \nabla (\Delta \varphi \varphi_x) \right) \, dx \, dy
\]

where we use the notation \([A; B] = AB - BA\) for given operators \(A, B\), and the last term of the above identity is obtained after integration by parts and using that \((\Delta)^{\frac{s}{2}}\) determines a symmetric operator. The proof of (26) is now a consequence of the identity

\[
\left[ (\Delta)^{\frac{s}{2}}; x \right] \partial_x f = -s(\Delta)^{\frac{s-2}{2}} \varphi f.
\]  

which follows by computing the Fourier transform of the commutator. Note that

\[
\left\| (\Delta)^{\frac{s-2}{2}} \partial_x^2 f \right\|_{L^2(\mathbb{R}^2)} \leq \left\| (\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)}.
\]

This establishes (26) and (27).

Next, multiplying (15) by \(x \varphi_x\) and using (26), we get

\[
\int \left( c \varphi^2 - 2 \nu_1 (\Delta)^{s-1} \partial_x^2 \varphi \varphi - \nu_1 (\Delta)^s \varphi \varphi - \frac{2 \nu_2}{m(m+1)} \varphi^{m+1} \right) \, dx \, dy.
\]  

Likewise, (27) yields

\[
\int \left( c \varphi^2 - 2 \nu_1 (\Delta)^{s-1} \partial_y^2 \varphi \varphi - \nu_1 (\Delta)^s \varphi \varphi - \frac{2 \nu_2}{m(m+1)} \varphi^{m+1} \right) \, dx \, dy.
\]

By adding (29) and (30), and using that \(\partial_x^2 + \partial_y^2 = \Delta\), we deduce

\[
\int \left( c \varphi^2 + \nu_1 (s-1)(\Delta)^s \varphi \varphi - \frac{2 \nu_2}{m(m+1)} \varphi^{m+1} \right) \, dx \, dy.
\]

Consequently, solving (31) and (25) yields the desired expressions in (19) and (20).

**Remark 3.** We notice that setting \(c = 0\) in (15) and inspecting the above proof of Lemma 1, Equation (29) must be equal to (30). This imposes the condition \(m = \frac{1+s}{1-s}\), and hence, for this case, we get the identity

\[
-\nu_1 \left\| (\Delta)^{\frac{s}{2}} \varphi \right\|_{L^2(\mathbb{R}^2)}^2 = \nu_2 \left( \frac{1-s}{1+s} \right) \int_{\mathbb{R}^2} \varphi^{\frac{2}{1-s}} \, dx \, dy.
\]
Note that $m = \frac{1+s}{1-s}$ is the energy-critical case for solutions of (12), i.e., $r_c = s$ in (14). The function $\varphi$ that gives the equality in (32) is exactly a minimizer for the fractional Sobolev inequality (and the sharp constant that can be obtained from (32)), see more on that in Refs. 31, 32.

As a direct consequence of the Pohozaev identities deduced in Lemma 1 and (32), we establish the following nonexistence criteria for solutions of the equation (15).

**Proposition 1.** Equation (12) cannot have a smooth nontrivial vanishing at infinity solitary-wave solution unless either one of the following holds:

(i) $\nu_1 < 0, c > 0, 1 < m < \frac{1+s}{1-s}$
(ii) $\nu_1 > 0, c < 0, 1 < m < \frac{1+s}{1-s}$
(iii) $\nu_1 > 0, c > 0, m > \frac{1+s}{1-s}$
(iv) $\nu_1 < 0, c < 0, m > \frac{1+s}{1-s}$
(v) $m > 1$ is an odd integer, $\nu_2 c > 0$, and either (i) and (ii) hold,
(vi) $m > 1$ is an odd integer, $\nu_2 c < 0$, and either (iii) and (iv) hold, or
(vii) $m = \frac{1+s}{1-s}$ is an odd integer, $c = 0$, and $\nu_1 \nu_2 < 0$.

**Remark 4.** (1) When $m > 1$ is an odd integer, we know that $\varphi^{m+1} = |\varphi|^{m+1} \geq 0$, thus, we can use (20) to obtain restrictions on the sign of $\nu_2$. This is exactly (v)–(vii) in Proposition 1. This explains why we included the parameter $\nu_2$ in (12) (in part to distinguish the $\int \varphi^{m+1} dx dy$ integral from the $L^{m+1}$-norm $\int |\varphi|^{m+1} dx dy$).

(2) For the case of the nonlinearity in (4), the solitary wave solution of the form $u(x, y, t) = \phi(x - ct, y)$ yields the equation

$$-c\phi + \nu_1 (-\Delta)^s \phi + \frac{\nu_2}{m} |\phi|^{m-1} \phi = 0. \quad (33)$$

Then, replacing $\varphi$ by $\phi$, and $\varphi^{m+1}$ by $|\phi|^{m+1}$, the estimate (19) holds for solutions of (33), and in this case (20) becomes

$$\|\phi\|_{L^{m+1}(\mathbb{R}^2)}^{m+1} = \frac{sc(m+1)}{\nu_2(2-(1-s)(m+1))} \|\phi\|_{L^2(\mathbb{R}^2)}^2.$$ 

Furthermore, Proposition 1 is modified as follows:

**Proposition 2.** Equation (33) cannot have a smooth nontrivial vanishing at infinity solution unless either one of the following holds:

- $\nu_1 < 0, \nu_2 = 1, c > 0, 1 < m < \frac{1+s}{1-s}$
- $\nu_1 > 0, \nu_2 = -1, c < 0, 1 < m < \frac{1+s}{1-s}$
- $\nu_1 > 0, \nu_2 = -1, c > 0, m > \frac{1+s}{1-s}$, or
- $\nu_1 < 0, c < 0, \nu_2 = 1, m > \frac{1+s}{1-s}$,
\( c = 0, \nu_1 \nu_2 < 0, m = \frac{1+s}{1-s}. \)

### 2.2 Uniform bounds (conditional global existence)

For our next result, we first find an explicit relation between the sharp constant and the solution of (16). Let \( \varphi > 0 \) be a local minimizer of the functional \( J \) defined in (17). Then \( J'(\varphi) = 0 \) implies

\[
\left( (m+1) - \frac{(m-1)}{s} \right) c_1 \varphi + \frac{(m-1)}{s} (-\Delta)^{\frac{s}{2}} \varphi - c_2 \varphi^m = 0 \quad (34)
\]

for some (specific) positive constants \( c_1 \) and \( c_2 \). By (34), and using similar arguments as in the proof of Proposition 1, we deduce the following identities:

\[
\int \left[ \left( (m+1) - \frac{(m-1)}{s} \right) c_1 \varphi^2 + \frac{(m-1)}{s} \left| (-\Delta)^{\frac{s}{2}} \varphi \right|^2 - c_2 \varphi^{m+1} \right] \, dx \, dy = 0, \quad (35)
\]

\[
\int \left[ \left( (m+1) - \frac{(m-1)}{s} \right) c_1 \varphi^2 + \frac{(m-1)(1-s)}{s} \left| (-\Delta)^{\frac{s}{2}} \varphi \right|^2 - \frac{2c_2}{m+1} \varphi^{m+1} \right] \, dx \, dy = 0. \quad (36)
\]

Combining (35) and (36), it is seen that

\[
\left\| (-\Delta)^{\frac{s}{2}} \varphi \right\|_{L^2(\mathbb{R}^2)}^2 = c_1 \| \varphi \|_{L^2(\mathbb{R}^2)}^2, \quad (37)
\]

\[
\| \varphi \|_{L^{m+1}(\mathbb{R}^2)}^{m+1} = \frac{c_1 (m+1)}{c_2} \| \varphi \|_{L^2(\mathbb{R}^2)}^2. \quad (38)
\]

Setting \( \beta_1^{m-1} = \frac{c_2 s m}{(s(m+1)-(m-1))c_1} \) and \( \beta_2^{2s} = \frac{(m-1)}{(s(m+1)-(m-1))c_1} \), we find that \( Q_{s,m}(x,y) = \beta_1 \varphi(\beta_2 x, \beta_2 y) \) solves

\[
Q_{s,m} + (-\Delta)^{\frac{s}{2}} Q_{s,m} - \frac{1}{m} Q_{s,m}^{m} = 0. \quad (39)
\]

Note that from (18) it follows that \( |Q_{s,m}(x,y)| \sim \frac{1}{1+|(x,y)|^{2+2s}} \) for all \( 1 < m < \frac{1+s}{1-s} \). Because \( C_{GN} = \frac{1}{J(\varphi)} \), (37) and (38) imply

\[
C_{GN} = \frac{ms(m+1)}{(m-1) \beta_2^{2s} (s(m+1)-(m-1)) \left\| Q_{s,m} \right\|_{L^2(\mathbb{R}^2)}^{m-1}}. \quad (40)
\]

Summarizing the previous discussion, we obtain the following result.
Proposition 3. Let $0 < s < 1$, $f \in H^s(\mathbb{R}^2)$, then $f \in L^{m+1}(\mathbb{R}^2)$ for any $1 < m < \frac{1+s}{1-s}$, and there is a constant $C_{GN}$ such that (16) holds true. Moreover, the sharp constant for which this inequality is valid is given by (40) with $Q_{s,m}$ being a ground state solution of (39).

The Gagliardo–Nirenberg inequality (16) is also convenient to obtain uniform bounds for solutions of (12) in each regime of criticality established by (14). Before, we recall that by a standard parabolic regularization argument, for integer powers $m > 1$ and a given $u_0 \in H^r(\mathbb{R}^2), r > 2$, there exist a time $T > 0$ and a unique solution $u \in C([0, T); H^r(\mathbb{R}))$ of the initial value problem associated to (12) such that $u(0) = u_0$. We use this existence result to formulate the following proposition.

Proposition 4. Let $0 < s < 1$, $m > 1$ be an odd integer and $\nu_1 \nu_2 > 0$. Consider $u_0 \in H^r(\mathbb{R}^2), r > 2$. Then the solution $u \in C([0, T); H^r(\mathbb{R}^2))$ of the initial value problem associated to (12) with initial data $u_0$ is uniformly bounded in $H^s(\mathbb{R}^2)$ for any $t \in [0, T)$.

When $m > 1$ is an odd integer number and $\nu_1 \nu_2 > 0$, the proof of the above proposition is a direct consequence of the energy (13) and the $L^2$ conservation law as follows:

$$\frac{1}{2} \left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|^2_{L^2(\mathbb{R}^2)} = E_s[u_0] - \frac{\nu_2}{\nu_1 m(m+1)} \left\| u \right\|_{L^{m+1}(\mathbb{R}^2)}^{m+1} \leq E_s[u_0].$$

Remark 5. If the nonlinearity in (12) is modified by $\partial_x |u|^{m-1} u$ as in (4), then the above proposition holds for any $m > 1$ for the modified equation.

To obtain a similar result to Proposition 4 in the case when $\nu_1 \nu_2 < 0$, we require the existence of solutions to (39). For simplicity, we consider $\nu_1 = -1$ and $\nu_2 = 1$ in (12), that is, the two-dimensional model in (3).

Theorem 2. Let $0 < s < 1$ and $1 < m \leq \frac{1+s}{1-s}$ be an integer1. Consider $u_0 \in H^r(\mathbb{R}^2), r > 2$, and $u \in C([0, T); H^r(\mathbb{R}^2))$ be the solution of the initial value problem associated to (3) with initial data $u_0$.

(C1) Assume $1 < m < 2s + 1$. Then the solution $u(t)$ is uniformly bounded in $H^s(\mathbb{R}^2)$ for any $t \in [0, T)$.

(C2) Assume $m = 2s + 1$ and

$$\|u_0\|_{L^2(\mathbb{R}^2)} < \|Q_{s,m}\|_{L^2(\mathbb{R}^2)},$$

where $Q_{s,m}$ is the ground state solution of (39). Then the solution $u(t)$ is uniformly bounded in $H^s(\mathbb{R}^2)$ for any $t \in [0, T)$.

1 If the nonlinearity in the equation is modified to $\partial_x |u|^{m-1} u$, then any value $m \in (1, \frac{1+s}{1-s})$ can be considered, conditional on the local well-posedness.
(C3) Let \( \theta = \frac{r_c}{s} \equiv \frac{1}{s} - \frac{2}{m-1} \). Assume \( 2s + 1 < m \leq \frac{1 + s}{1 - s} \), or equivalently, \( 0 < \theta \leq 1 \), and \( E[u_0] \geq 0 \). Suppose

\[
E_s[u_0]^\theta M[u_0]^{1-\theta} < E_s[Q_{s,m}]^\theta M[Q_{s,m}]^{1-\theta},
\]

where \( Q_{s,m} > 0 \) is the ground state solution of (39).

If

\[
\|(-\Delta)^\frac{s}{2} u_0\|_{L^2(\mathbb{R}^2)}^\theta \|u_0\|_{L^2(\mathbb{R}^2)}^{1-\theta} < \|(-\Delta)^\frac{s}{2} Q_{s,m}\|_{L^2(\mathbb{R}^2)}^\theta \|Q_{s,m}\|_{L^2(\mathbb{R}^2)}^{1-\theta},
\]

then the solution \( u(t) \) of (3) with the initial condition \( u_0 \) is uniformly bounded in \( H^s(\mathbb{R}^2) \) for any \( t \in [0, T) \). Moreover, for any \( t \in [0, T) \)

\[
\|(-\Delta)^\frac{s}{2} u(t)\|_{L^2(\mathbb{R}^2)}^\theta \|u_0\|_{L^2(\mathbb{R}^2)}^{1-\theta} < \|(-\Delta)^\frac{s}{2} Q_{s,m}\|_{L^2(\mathbb{R}^2)}^\theta \|Q_{s,m}\|_{L^2(\mathbb{R}^2)}^{1-\theta}.
\]

The proof for (C1) and (C2) of Theorem 2 follows Weinstein’s classical approach,\(^3\) for (C3) see Refs. 34–36, we provide the details below after making several comments.

Remark 6. (1) The conclusion of Theorem 2 is still valid for \( m > 1 \), not necessarily an integer, provided that for any \( u_0 \in H^r(\mathbb{R}) \), \( r \geq s \), there exist \( 0 < T \leq \infty \), and a unique solution \( u \in C([0, T); H^r(\mathbb{R}^2)) \) of (3) with the initial condition \( u_0 \).

(2) We observe that (C1), (C2), and (C3) correspond to the \( L^2 \)-subcritical, critical, and supercritical cases, respectively.

(3) When \( m = 2s + 1 \), we expect the following conjecture to hold:

**Conjecture 1.** Let \( u_0 \in H^r(\mathbb{R}^2) \) and \( Q = Q_{s,2s+1} \) be the ground state solution of (39).

I. If \( \|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)} \), then the solution \( u(t) \) exists globally in time.

II. If \( \|u_0\|_{L^2(\mathbb{R}^2)} > \|Q\|_{L^2(\mathbb{R}^2)} \) and \( u_0 \) is sufficiently localized, then the solution \( u(t) \) blows up in finite time. In particular, if \( E[u_0] < 0 \) (hence, \( \|u_0\|_{L^2(\mathbb{R}^2)} > \|Q\|_{L^2(\mathbb{R}^2)} \)) and \( u_0 \) has some localization implies blow-up in finite time.

In (C2) we prove the part I of Conjecture 1 for \( u_0 \in H^r(\mathbb{R}^2) \), \( r > 2 \), and make a partial progress for \( u_0 \in H^r(\mathbb{R}^2) \) (conditional on the lwp in \( H^3 \)). The second part of Conjecture 1, when \( s = \frac{1}{2}, \ m = 2 \), is confirmed numerically in Section 5.3 for initial data with different decays at infinity (we show that there are blow-up solutions with positive and negative energy). Moreover, a stable blow-up regime is self-similar (in the core region) with the rescaled ground state as the blow-up profile.

\(^2\) An ultimate goal would be \( u_0 \in L^2(\mathbb{R}^2) \).
We note that (C3) is a generalization of Ref. 34 for fKdV (see also Ref. 14 results for the gBO equation), and in a more general sense, is a generalization of the dichotomy first obtained for the NLS in Refs. 36 and 37, where the opposite inequality in (42) was also possible to consider.

Proof of Theorem 2. From the definition of energy (13) and the $L^2$ conservation, together with (16), we get

$$
\left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}^2 = 2E_s[u(t)] + \frac{2}{m(m+1)} \int (u(x, y, t))^{m+1} dxdy
$$

$$
\leq 2E_s[u_0] + \frac{2C_{GN}}{m(m+1)} \left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}^{\frac{m-1}{\ensuremath{\gamma}}} \left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{\frac{\gamma(m+1)-(m-1)}{s}}. \quad (44)
$$

If $1 < m < 2s + 1$, the above inequality shows that $\left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}$ is uniformly bounded for all $t \in [0, T)$. This completes the proof of (C1).

If $m = 2s + 1$, we deduce the uniform bound if

$$
\left( 1 - \frac{2C_{GN}}{m(m+1)} \left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{2s} \right) > 0. \quad (45)
$$

Plugging the sharp constant (40) into the above expression yields (C2).

Next, we consider $2s + 1 < m \leq \frac{1+\gamma}{1-\gamma}$. Multiplying both sides of (44) by $\left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{\frac{2(1-\gamma)}{\gamma}}$, we get

$$
\left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}^2 \left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{1}{\gamma}-1\right)} \leq 2E_s[u_0]M[u_0]^{\frac{1}{\gamma}} + \frac{2C_{GN}}{m(m+1)} \left( \left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}^2 \left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{1}{\gamma}-1\right)} \right)^{\frac{m-1}{2s}}. \quad (46)
$$

Setting

$$
x(t) = \left\| (-\Delta)^{\frac{s}{2}} u(t) \right\|_{L^2(\mathbb{R}^2)}^2 \left\| u_0 \right\|_{L^2(\mathbb{R}^2)}^{2\left(\frac{1}{\gamma}-1\right)} \quad \text{and} \quad \beta = \frac{2C_{GN}}{m(m+1)},
$$

we rewrite (46) as

$$
x(t) - \beta x(t)^{\frac{m-1}{2s}} \leq 2E_s[u_0]M[u_0]^{\frac{1}{\gamma}}.
$$

We note that the function $f(x) := x - \beta x^{\frac{m-1}{2s}}$, $x \geq 0$, has a local maximum at $x_0 = \left(\frac{1-\gamma}{\beta}\right) \frac{1-r_c}{r_c}$, with the maximum value $f(x_0) = r_c \cdot \left(\frac{1-r_c}{\beta}\right) r_c$. We impose the conditions

$$
2E_s[u_0]M[u_0]^{\frac{1}{\gamma}} < f(x_0) \quad \text{and} \quad x(0) < x_0. \quad (47)
$$
By a continuity argument, it must be the case that \( x(t) \leq x_0 \) for any time \( t \), when the solution \( u(t) \) is defined. Thus, once we have the explicit relations for (47), the above discussion yields the proof of (C3). By (40) we deduce the identity

\[
\|Q_{s,m}\|_{L^2(\mathbb{R}^2)}^2 = \frac{2(2 - (1 - s)(m + 1))}{m - 1 - 2s} E_s[Q_{s,m}],
\]
then a computation shows that the first condition in (47) is equivalent to

\[
E_s[u_0]M[u_0]^{\frac{1}{s} - 1} < E_s[Q]M[Q]^{\frac{1}{s} - 1},
\]
or (42). Likewise, \( x(0) < x_0 \) is equivalent to (41) and \( x(t) < x_0 \) is rephrased as (43), finishing the proof.

2.3 Linear equation and radiation

When studying solitary waves in KdV-type equations, one ultimately encounters such regions as the soliton core region, the fast decaying tail (typically, located to the right of the moving soliton in the 1d problems), and the radiation region, where the dispersive oscillations propagate (typically to the left of the soliton). In higher dimensional problems similar behavior was observed and obtained in ZK models (see Refs. 38–41, and 42). As we deal with the 2d HBO equation, we also investigate a region in the plane, where a solitary-wave type solution of (1) propagates dispersive oscillations, or radiates into that region. For that we consider a linear evolution initial-value problem

\[
\begin{cases}
\partial_t u - \partial_x (-\Delta)^s u = 0, & (x,y,t) \in \mathbb{R}^3, \quad 0 < s < 1, \\
u(x,0) = u_0(x).
\end{cases}
\]

We first remark that Strichartz estimates for Equation (50) are known for any dimension \( d \geq 2 \) when \( s = \frac{1}{2} \), and for \( \frac{1}{2} < s < 1 \) when \( d \geq 3 \), see Refs. 22, 23. As far as we know, Strichartz estimates for (50) have not been determined for dispersions \( 0 < s < \frac{1}{2} \). For dimension 1, see Ref. 43. On the other hand, for a sufficiently regular initial condition \( u_0 \), the solution of (50) is

\[
u(x,y,t) = \frac{1}{(2\pi)^2} \int e^{ix\xi_1 + iy\xi_2 + i\omega(\xi_1, \xi_2)\cdot x} u_0(\xi_1, \xi_2) \, d\xi_1 d\xi_2,
\]
where \( \omega(\xi_1, \xi_2) = \xi_1 |(\xi_1, \xi_2)|^{2s} \) is the dispersion relation. Then the group velocity is given by

\[
\nabla \omega(\xi_1, \xi_2) = \left| (\xi_1, \xi_2) \right|^{2s-2} \left( (1 + 2s)\xi_1^2 + \xi_2^2, 2s\xi_1\xi_2 \right).
\]

The angle \( \theta(\xi_1, \xi_2) \), determined by \( \nabla \omega(\xi_1, \xi_2) \) and the positive \( y \)-axis, satisfies the relation

\[
\tan(\theta(\xi_1, \xi_2)) = \frac{(1 + 2s)\xi_1^2 + \xi_2^2}{2s\xi_1\xi_2}.
\]
The angle $\theta = \theta_{\text{min}}$ in (54), the blue area is the radiation region (in the frame moving with the soliton).

Setting a reference frame centered at the center of a moving soliton, we obtain the minimal angle determined by the above relation, which satisfies the identity

$$\tan(\theta_{\text{min}}) = \frac{(1 + 2s)^{\frac{1}{2}}}{s},$$

see Figure 1 for a depiction.

For the case of the HBO equation ($s = \frac{1}{2}$), we have

$$\tan(\theta_{\text{min}}) = 2\sqrt{2},$$

or approximately, $\theta_{\text{min}} \approx 70.52^\circ$. Therefore, for an appropriately set up reference frame (moving with the solitary wave), we use $\theta_{\text{min}}$ to define the wedge of the radiation region with an angle of a maximum value of 19.48° with the x-axis (or by symmetry a total angle of 38.96°), see the blue region in Figure 1. We provide numerical confirmation of the radiation region in Section 5 (for example, see Figures 18 and 19).

Remark 7. In the case of the ZK equation ($s = 1$ and $m = 2$ in (3)), the argument above gives an angle for the radiative region of 60°, which is compatible with the results presented in Ref. 38 (see Remarks 1.1 and 1.2), also in Refs. 39, 42. In particular, considering the surface determined by the ZK equation, the angle above is related to the region, where the mitigating factor introduced by the Strichartz estimate in Ref. 44 cancels out. Furthermore, the argument above can also be extended to (3) in higher dimensions (note that the nonlinear part does not play a role on the size of the angle of this region). In particular, the same representative angle was obtained for the 3d ZK equation in Ref. 40, see also Ref. 41.
We are now ready to study Equation (1) numerically; for that we first describe the numerical approach that we develop for the 2d HBO (1), then we obtain the ground state solutions, and finally, we show the dynamical solutions of the 2d HBO equation.

3 | NUMERICAL APPROACH

In this section, we describe our numerical method that we develop to solve the HBO equation (1) on the whole real space \( \mathbb{R}^2 \). We start with the description of the rational basis (eigen)functions on \( \mathbb{R} \), also referred to as Wiener functions, proposed and analyzed in Refs. 45 and 46. We then apply the spectral Galerkin approximation to the fractional Laplacian \((-\Delta)^{\frac{1}{2}}\) from Ref. 47. After the spatial discretization, the equation is reduced to a system of nonlinear ordinary differential equations (ODEs), which can be solved by a variety of standard numerical integrators.

3.1 | Rational basis functions

One method for solving the dispersive equations on \( \mathbb{R}^d \) is to use the Fourier spectral discretization in space by taking a sufficiently large domain. Then, the fractional Laplacian is discretized in a straightforward manner on the frequency space, i.e., \((-\Delta)^{s} u = |\xi|^{2s} \hat{u} \) (for example, as it is done in Refs. 48 and 49). However, this requires extremely large periodic domains to ensure a sufficiently good approximation of the whole real space \( \mathbb{R}^d \), as the fractional Laplacian \((-\Delta)^{s} \) maps, say, Schwartz functions into the functions with an algebraic decay, i.e., for \( f \in \mathbb{S} \), \( u = (-\Delta)^{s} f \) decays algebraically. Thus, an extremely large number of grid points is needed for a satisfactory resolution. For example, in Ref. 14, we took a very large (periodic) domain of length \( L = 20000\pi \) with \( N = 2^{22} \) nodes to approximate the real line \( \mathbb{R} \) for the simulation of solutions to Benjamin-Ono (also to mBO and gBO) equation. However, when considering the two-dimensional HBO case, the task becomes computationally prohibitive for the same computational configuration and parameters as we used in Ref. 14 with \( N = 2^{44} \) nodes on the \([-20000\pi, 20000\pi]^2\) torus. Moreover, besides the large size of the computational domain and number of grid points, Fourier spectral method transforms the model into an equation on a torus of length \([-L, L]\) (or \([-L, L]^d\) for the HBO cases in \( \mathbb{R}^d \)). Consequently, even with the fast decaying Fourier coefficients property observed in Ref. 14, this spectral discretization is still unable to capture the correct asymptotic behavior such as the decay rate when the solution approaches the computational boundary, because the transformed numerical model solves the solution on the torus, which wraps around the boundary, while the actual solution decays at \( \infty \). Another possible approach, recently developed, is a multidomain method for computing the Hilbert transform in \( 1d \) in Ref. 50, which gives a good accuracy, however, it is yet to be developed for the two-dimensional simulations.

An alternative spatial discretization from Ref. 45 considers the use of the rational basis functions (sometimes also called Wiener functions), on the whole real line, i.e.,

\[
u(x, t) = \sum_{n=-\infty}^{\infty} \hat{u}_n(t) \rho_n(x), \quad \rho_n(x) = \frac{(\alpha + ix)^n}{(\alpha - ix)^{n+1}},\tag{56}
\]

where \( \alpha \) is a mapping parameter indicating that half of the grid points are located in the interval \([-\alpha, \alpha]\). It is shown in Ref. 45 that \( \{\rho_n(x)\}_{n=-\infty}^{\infty} \) form a complete orthogonal basis in \( L^2(-\infty,\infty) \).
with the orthogonality
\[
\int_{-\infty}^{\infty} \rho_m(x)\overline{\rho_n(x)}dx = \begin{cases} \pi/\alpha, & m = n \\ 0, & m \neq n. \end{cases} := \frac{\pi}{\alpha} \delta_{m,n}.
\]
Therefore, we have
\[
\tilde{u}_n(t) = \frac{\alpha}{\pi} \int_{t=-\infty}^{\infty} u(x,t)\rho_n(x)dx.
\]
The derivatives of \(u(x, t)\) can be easily computed by the relation
\[
u_x(x,t) = \sum_{n=-\infty}^{\infty} \frac{i}{2\alpha} [n\tilde{u}_{n-1} + (2n+1)\tilde{u}_n + (n+1)\tilde{u}_{n+1}]\rho_n(x),
\]
and the higher derivatives can then be done iteratively.

In numerical computations, a truncation of \(N\) terms is used, i.e.,
\[
u(x,t) \approx \tilde{u}^T \rho := \sum_{n=-N/2}^{N/2-1} \tilde{u}_n(t)\rho_n(x),
\]
where \(\tilde{u} = (\tilde{u}_{-N/2}, \tilde{u}_{-N/2+1}, \ldots, \tilde{u}_{N/2-1})^T\) is the vector of the truncated coefficients, and the same for \(\rho\). This leads to the sparse matrix forms
\[
u_x \approx [S_1 \tilde{u}]^T \rho, \quad \nu_{xx} \approx [S_2 \tilde{u}]^T \rho,
\]
where \(S_1\) is given in (57) via the coefficients of \(\tilde{u}_n\), and \(S_2 = S_1 \times S_1\) from using the relation (57) iteratively.

Now by a change of variable
\[
x = \alpha \tan \frac{\theta}{2}, \quad -\pi \leq \theta \leq \pi,
\]
and a spatial discretization \(x_j = \alpha \tan \frac{\theta_j}{2}, \theta_j = jh, h = 2\pi/N, j = -N/2, \ldots, N/2\), we have
\[
(\alpha - i x_j) u(x_j) \approx \sum_{n=-N/2}^{N/2-1} \tilde{u}_n [(\alpha - i x_j)\overline{\rho_n(x_j)}] = \sum_{n=-N/2}^{N/2-1} \tilde{u}_n e^{in\theta_j}.
\]
The Fast Fourier transform (FFT) can be applied to obtain the coefficients \(\tilde{u}_n\). This approximation can be easily extended to higher dimensions with a tensor product.
3.2 Discretization of the fractional Laplacian on $\mathbb{R}^2$

In this section, we describe the spectral Galerkin approximation of the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^2$. This method was introduced by Shen et al. in Ref. 47 via the Mapped Chebyshev functions (the Chebyshev polynomials mapped on the whole real line). In this work, instead of the Mapped Chebyshev functions, we use the rational basis functions, which could also be used to approximate the fractional Laplacian, and can be easily extended to other dimensions.

We note that $S_1 = iS$, where $S$ is a real symmetric matrix from (57). Therefore, $S$ is diagonalizable with all real eigenvalues written as

$$S = E \Lambda E^T,$$

where $E = (e_{j,k})_{j=-N/2}^{N/2-1} \cdots e_{N/2-1,k}$ is the matrix formed by the orthonormal eigenvectors $\vec{e}_k$ of $S$, and $\Lambda$ is the real diagonal matrix. Then, we have

$$S_1 = iE \Lambda E^T, \quad S_2 = -E(\Lambda^2)E^T. \quad (60)$$

For a matrix $M$, we denote $M(j,k) = m_{j,k}$ to be the $j$th row and $k$th column element. Denote $\lambda_k = \Lambda^2(k, k)$ to be the $k$th eigenvalue of the diagonal matrix $\Lambda^2$. It is easy to see that $\lambda_k \geq 0$.

Consider the set of new basis functions $\{\hat{\rho}_k(x)\}$, which is obtained as the diagonal transformation of the old basis $\{\rho_k\}$, i.e.,

$$\hat{\rho}_k(x) := \sum_{j=-N/2}^{N/2-1} e_{j,k}\rho_j(x), \quad \vec{e}_k = (e_{-N/2,k}, \ldots, e_{N/2-1,k})^T. \quad (61)$$

Then, from the direct adaption of the proof in Ref. [47, Lemma 2.1], we have

$$\langle \hat{\rho}_k, \hat{\rho}_j \rangle_{L^2} = \frac{\pi}{\alpha} \delta_{k,j}, \quad \text{and} \quad \langle \hat{\rho}_k', \hat{\rho}_j' \rangle_{L^2} = \frac{\pi}{\alpha} \lambda_k \delta_{k,j}, \quad (62)$$

which is called the “biorthogonal property” for the basis $\{\hat{\rho}_k\}$.

Let $\hat{u} = (\hat{u}_{-N/2}, \ldots \hat{u}_{N/2-1})^T$ be the coefficients vector with respect to the basis $\{\hat{\rho}_k\}$. One can see the relation $\hat{u} = E^T \vec{u}$ in $1d$. In $2d$, let $\hat{U} = (\hat{u})_{j,k=-N/2}^{N/2-1}$ be the matrix of the coefficients with respect to the basis functions $\{\rho_{j,k}\}$. Then, we have the coefficients matrix with respect to the basis functions $\{\hat{\rho}_{j,k}\}$ obtained by

$$\hat{U} = E^T \hat{U} E.$$

Now, let $(-\Delta)^s u = f$. We show that $\hat{f}_{j,k} = (\lambda_j + \lambda_k)^s \hat{u}_{j,k}$ in $\mathbb{R}^2$. This is achieved by following the argument in Ref. 47.

Indeed, in Ref. 51, the authors use the Dunford–Taylor formula to change the fractional Laplacian $(-\Delta)^s$ into the full Laplacian $(-\Delta)$ in the Galerkin formulation,

$$\left( (-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v \right)_{L^2(\Omega)} = C_s \int_0^\infty t^{1-2s} \int_\Omega (-\Delta)(I - t^2\Delta)^{-1} u(x) v(x) dx dt, \quad (63)$$
where \( C_s = \frac{2\sin(\pi s)}{\pi} \), \( \mathbb{I} \) is the identity operator and \( \Omega \) can be either the bounded domain in \( \mathbb{R}^d \) or \( \Omega = \mathbb{R}^d \). Denote \( w(x) = (\mathbb{I} - t^2 \Delta)^{-1} u(x) \). Then,

\[
-t^2 \Delta w + w = u, \quad x \in \Omega,
\]

and thus,

\[
(-\Delta)(\mathbb{I} - t^2 \Delta)^{-1} u(x) = -\Delta w = t^{-2}(u - w).
\]

Solving Equation (64) and then evaluating the integral (63) with respect to \( t \) becomes a crucial step in accurately evaluating the fractional Laplacian \((-\Delta)^s\). Subsequent works in Ref. 52 proposed the method of evaluating the integral (63) by the sinc functions. Later, the authors in Ref. 47 observed the following integral identity:

\[
\int_0^\infty \frac{t(1-2s)\lambda^{1-s}}{1 + t^2 \lambda} dt = \frac{\pi}{2\sin(\pi s)} = \frac{1}{C_s},
\]

and thus, the integral system (63)–(64) can be evaluated exactly in the frequency space if we can write the inside integral \( \int_\Omega t^{-2}(u - w)v dx \) in the diagonal form. As a consequence, the \((-\Delta)^s u\) is evaluated efficiently.

For example, let \( \Omega \) be the periodic bounded domain \([0, 2\pi]\), and we use the Fourier basis to approximate \( u(x) \), i.e., \( u(x) \approx u_N(x) = \sum_{-N/2}^{N/2-1} \hat{u}_k e^{ixk} \). By setting \( v_k = e^{-ixk} \) for each \( k = -N/2, \ldots, N/2 - 1 \), we have \( \hat{w}_k = \frac{\hat{u}_k}{1 + t^2 k^2} \) from (64), and consequently, \( \int_\Omega t^{-2}(u_N - w_N)v_k dx = \frac{k^2}{1 + t^2 k^2} \hat{u}_k \) from the inner part of the integral in (63) with the orthogonal property of the test function \( v_k \).

Therefore, the equation for (63) for each \( k \) yields

\[
\left( (-\Delta)^s u_N, (-\Delta)^s v_k \right)_{L^2(\Omega)} = C_s \int_0^\infty \frac{t(1-2s)\lambda^{1-s}}{1 + t^2 \lambda} \hat{u}_k dt = k^2 \lambda^s \hat{u}_k C_s \int_0^\infty \frac{t(1-2s)\lambda^{1-s}}{1 + t^2 \lambda} dt = |k^2|^{1-s} \hat{u}_k.
\]

In other words,

\[
(-\Delta)^s u(x) \approx (-\Delta)^s u_N(x) = \sum_{-N/2}^{N/2-1} (|k^2|^{1-s} \hat{u}_k)e^{ixk},
\]

which matches the form \( F((-\Delta)^s u) = |\xi|^{2s} \hat{u} \) in the usual sense.

When \( \Omega = \mathbb{R}^d \), the Fourier basis is no longer preferable, as the domain truncation may lead to large errors. Now, let \( u(x) \approx u_N(x) = \sum_{-N/2}^{N/2-1} \hat{u}_k \hat{v}_k(x) \) be the finite interpolation function of \( u(x) \) with respect to the basis function \( \{\hat{v}_k\} \) in \( \Omega = \mathbb{R} \). Following a similar argument, by equipping the inner product of \( \hat{v}_k \) for each \( k = -N/2, \ldots, N/2 - 1 \) yields \( \hat{w}_k = \frac{\lambda_k}{1 + t^2 \lambda_k} \hat{u}_k \) from (62) and (64).

Then, by setting the test function \( v_k = \hat{v}_k \) in (63), and using the orthogonal property from (62), we obtain

\[
\left( (-\Delta)^s u_N, (-\Delta)^s v_k \right)_{L^2(\mathbb{R})} = C_s \int_0^\infty \frac{t(1-2s)\lambda_k^{1-s}}{1 + t^2 \lambda_k} \hat{u}_k dt = \lambda_k^s \hat{u}_k C_s \int_0^\infty \frac{t(1-2s)\lambda_k^{1-s}}{1 + t^2 \lambda_k} dt = \lambda_k^s \hat{u}_k.
\]
Thus, we obtain

$$(-\partial_{xx})^s u_N = \sum_{k=-N/2}^{N/2-1} |\lambda_k|^s \hat{u}_k \hat{\rho}_k(x).$$

Therefore, the stiff matrix for $(-\partial_{xx})^s u_N$ is $\Sigma = \text{diag}(|\lambda_j|^s)$ for $j = -N/2, \ldots, N/2 - 1$. (A stiff matrix typically means that explicit time integration schemes become inefficient; here, the coefficients from the approximating the derivatives of the function $u$ become large, which typically grow as $(2N)^{2s}$ in our numerical simulations, and thus, result in the inefficiency of the explicit time integration.)

The higher dimensional generalization can be found in Ref. 47, we omit the full details for brevity and give a summary. For $\Omega = \mathbb{R}^2$ we have

$$(-\Delta)^s u_N(x, y) = \sum_{j,k=-N/2}^{N/2-1} (\lambda_j + \lambda_k)^s \hat{u}_{j,k} \hat{\rho}_j(x) \hat{\rho}_k(y).$$

Let $\Sigma_{j,k} = (\lambda_j + \lambda_k)^s$ be the stiff matrix for the fractional Laplacian operator $(-\Delta)^s$ in $\mathbb{R}^2$. Then, the matrix $\Sigma \otimes \hat{U}$ constitutes the matrix of coefficients for $(-\Delta)^s u_N(x, y)$, where $\otimes$ denotes the pointwise product between the matrices, i.e., $A \otimes B = (a_{jk} b_{jk})$. We also denote $\otimes^m$ as the pointwise power of a matrix, e.g., for $m = 2$, $U \otimes^2 = U \otimes U$.

**Remark 8.** In fact, the Galerkin formulation of (63) is not limited to the spectral Galerkin method. It could also be used for finite element method or discontinuous Galerkin method. The crucial part is to solve the BVP problem (64) accurately on a given domain $\Omega$. In this paper, when $\Omega = \mathbb{R}^2$, the spectral Galerkin method with biorthogonal rational basis functions $\{\hat{\rho}_{j,k}\}$ is used.

The spatial discretization procedure is set as follows: We first set $N$ nodes and the mapping parameter $\alpha$, and obtain the matrix $S_1$ from (57). Then, the orthonormal transformation matrix $E$ and the diagonal matrix $\Lambda$ are computed from (60). Let $U(t) = u(x_j, y_k, t)$ be the approximation of the solution $u(x, y, t)$ to Equation (1). The coefficient matrix $\hat{U}_{j,k} = \hat{u}_{j,k}$ can be obtained by (59) with FFT. Then, we use the relation $\hat{U} = E^T \hat{U} E$ to obtain the biorthogonal coefficients $\hat{u}_{j,k}$, and similarly, going backward reversing steps lead us obtaining the value of $U$ from $\hat{U}$. Note that from (60), the first-order stiff matrix, $\hat{S}_1 = E^T S_1 E = E^T (i E \Lambda E^T) E = i \Lambda$ with respect to the basis $\{\hat{\rho}_{j,k}\}$, is also diagonal.

Finally, the semidiscretization of the HBO equation (1) on the frequency space $\{\hat{\rho}_{j,k}\}$ yields

$$\hat{U}_t - \hat{S}_1 (\Sigma \otimes \hat{U}) + \frac{1}{m} \hat{S}_1 (\hat{U} \otimes^m) = 0.$$  

(67)

The matrix $\hat{U}$ and $\Sigma$ can be reordered into an $N^2 \times 1$ long vectors

$$\tilde{U} = (\hat{u}_{-N/2,-N/2}, \hat{u}_{-N/2+1,-N/2}, \ldots, \hat{u}_{-N/2-1,N/2-1})^T;$$

$$\tilde{\Sigma} = (\Sigma_{-N/2,-N/2}, \Sigma_{-N/2+1,-N/2}, \ldots, \Sigma_{-N/2-1,N/2-1})^T.$$
The first-order stiff matrix $\mathbf{\hat{S}}_1$ for $\partial_{xx}$ can be changed as $\mathbf{\hat{S}}_1^x = \text{kron} (\mathbf{I}, \mathbf{\hat{S}}_1)$, where kron is the Kronecker product and $\mathbf{I}$ is the $N \times N$ identity matrix. Then, Equation (67) becomes

$$\mathbf{\hat{U}}_t - \mathbf{\hat{S}}_1^x (\text{diag} (\mathbf{\hat{S}}) \mathbf{\hat{U}}) + \frac{1}{m} \mathbf{\hat{S}}_1^x (\mathbf{\hat{U}} \circ m) = 0. \quad (68)$$

The system (68) only involves the diagonal matrices, and thus, can be solved with the computational cost $\mathcal{O} (N^2)$. However, we recall that $\mathbf{\hat{U}} = \mathbf{E}^T \mathbf{U} \mathbf{E}$, which is the multiplication between full matrices. Therefore, the transformation between the frequency space and the physical space needs $\mathcal{O} (N^3)$ operations, which is the total computational cost of our algorithm.

### 3.3 Time integration

The given system (67), or (68), can be integrated by various time integrators. When applying the standard explicit time integrators, such as the Runge-Kutta (RK4) method, the time step $\Delta t$ has to be chosen to satisfy the stability condition, which is $\Delta t < \max_j |\lambda_j|^{-s-1/2}$, or $\Delta t \lesssim (2N)^{-2s-1}$ from our numerical simulation.

To allow a larger time step size, the modified fourth-order exponential time differencing (mETDRK4), which evaluates the value of the singular coefficients $\phi(z) = e^{z-1/z}$ when $z$ is close to 0 in the standard ETDRK4 in Ref. 53 by the Cauchy integral from Ref. 54, can be applied. This method allows us to take the time step $\Delta t \sim \max_j (|\lambda_j|)^{-1/2}$ due to the first-order derivative on the nonlinear term $(\frac{1}{m} u^m)_x$. Moreover, because (68) is a diagonal system, the Cauchy integral $\phi(z) = e^{z-1/z}$ is only needed at the point, where $z$ is close to 0, instead of the whole matrix system. We also refer to Ref. 55 for an efficient way to compute the function $\phi(z)$ when $z$ is a matrix for interested readers. Other implicit Runge-Kutta methods can be used for the choice of even larger time steps, for example, the fourth-order Runge-Kutta method with Gauss-Legendre collocation points (IRK4), which has been shown competitively efficient to the mETDRK4 method in simulating KdV equations (see e.g., Refs. 49, 56, and 57). The resulting nonlinear system can be solved by the fixed point iteration, similar to Ref. 49 or Ref. 58. In our simulations, we used both the mETDRK4 and IRK4 methods, and the results match.

### 4 COMPUTATION OF THE GROUND STATE SOLUTION

In this section, we show our numerical results for computing the ground state solution $Q$ that solves $-Q + (\Delta)^s Q + \frac{1}{m} Q^m = 0$, $Q > 0$, $Q \in H^{2s+1}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$, or the rescaled profiles of the solitary waves $Q_c$ in (15), namely,

$$c Q_c + (\Delta)^s Q_c - \frac{1}{m} (Q_c)^m = 0, \quad (69)$$

where $c > 0$ is a constant that generates a family of the rescaled ground state solutions

$$Q_c(x, y) = c^{m-1} Q\left(\frac{1}{c^{2s}} x, c^{\frac{1}{2s}} y\right), \quad (70)$$

producing traveling solitary waves $u(x, y, t) = Q_c(x - c t, y)$. 
We apply the Petviashvili’s iteration to obtain the profiles $Q_c$. We give a brief review of this method, which has been well-studied in the literature and for details we refer the interested reader to Refs. 59–63, and 64.

We denote by $Q_h$ our numerical solution for $Q_c$ in (69) for a given $c > 0$. Next, we define the operator $\mathbb{M} = (-\Delta)^{s} + c \mathbb{I}$, where $\mathbb{I}$ is the identity operator. The idea of the Petviashvili’s iteration is to find a constant to prevent the fixed point iteration

$$Q_{h}^{l+1} = \frac{1}{m} \mathbb{M}^{-1} \left( Q_{h}^{l} \right)^{m}$$

from going to 0 or $\infty$, where $Q_{h}^{l}$ is the numerical solution at the $l$th iteration.

To do so, we consider taking the inner product of (69) with $Q_c$, which yields

$$SL(Q_c) := \int_{\mathbb{R}^2} Q_c^2 dx dy = \int_{\mathbb{R}^2} \left( \frac{1}{m} \mathbb{M}^{-1} Q_c \right) Q_c dx dy =: SR(Q_c).$$

At each iteration, we search for the constant $\gamma_l$ such that

$$SL(\gamma_l Q_h^l) = SR(\gamma_l Q_h^l).$$

Inserting (73) into (72) yields

$$\gamma_l = \left( \frac{m \langle Q_h^l, Q_h^l \rangle}{\langle Q_h^l, \mathbb{M}^{-1} (Q_h^l)^{m} \rangle} \right)^{\frac{1}{m-1}}.$$ 

Thus, we obtain the following fixed point (Petviashvili’s) iteration:

$$Q_{h}^{l+1} = \frac{1}{m} \mathbb{M}^{-1} \left( \gamma_l Q_{h}^l \right)^{m}. $$

In numerical computations, the function $Q$ and the operator $\mathbb{M}$ are discretized by the bioorthogonal rational basis functions described in the previous section. For a comparison, we also discretize the function $Q$ and the operator $\mathbb{M}$ by the standard Fourier spectral method, which can be found in literature, e.g., Refs. 59 or 63. We set the stopping criteria to be $\|Q_{h}^{l+1} - Q_{h}^{l}\|_{L, \infty(\mathbb{R}^2)} < Tol$ with $Tol = 10^{-8}$ in our computations.

### 4.1 Ground state in the 2D (critical) HBO

Until now the discussion in this paper has been for a general equation of type (3) or (4), at this point we completely turn to the 2d HBO equation (1), that is, we only consider $m = 2$ and $s = \frac{1}{2}$ in (69).

Figure 2 shows our numerical solution for $Q_c$ from (69) with $c = 1$, $m = 2$, $s = \frac{1}{2}$ (we set $\alpha = 10$ and $N = 512$). It shows a well-localized, radially symmetric, positive function (in agreement with Theorem 1). To double check the decaying property of the numerical solution $Q_h$, we track, for example, the quantity $x^3 Q(x, 0)$ (i.e., the decay in one of the cross-sections, by $y = 0$) and plot
FIGURE 2 Computation of $Q$ with biorthogonal rational basis functions. Top left: solution $Q_c$ of (69), $c = 1$. Top middle: spatial dependence of $x^3 Q(x, 0)$. Top right: $\|Q^{l+1}_n - Q^l_n\|_{L^\infty(\mathbb{R}^2)}$ on the log scale for each iteration. Bottom left: the coefficients $|\hat{Q}_j|$ for the expansion of $Q$ with respect to the biorthogonal rational basis function $\hat{\rho}_j$. Bottom right: the coefficients $|\tilde{Q}_j|$ for the expansion of $Q$ with respect to the (regular) rational basis function $\rho_j$.

the resulting curve in the top middle graph of Figure 2. Observe the convergence to the horizontal asymptote as $x$ grows large. We obtain similar results in other cross-sections. This confirms that the solution decays as $1/|x|^3$, stated in (18) of Theorem 1. The top right subplot of Figure 2 tracks the difference of each Petviashvili’s iteration on the log scale. One can see that the difference $\|Q^{l+1}_n - Q^l_n\|_{L^\infty(\mathbb{R}^2)}$ decays exponentially, which agrees with theoretical results on the Petviashvili’s iteration method in Refs. 59, 60, and 61.

The bottom two subplots in Figure 2 track the coefficients for the expansion of $Q$ as $Q = \sum \hat{Q}_j \hat{\rho}_j$ (bottom left) and $Q = \sum Q_j \rho_j$ (bottom right), where $\{\hat{\rho}_j\}$ and $\{\rho_j\}$ are the sets of biorthogonal rational basis functions and regular basis functions, respectively, introduced in the previous section. One can see that the coefficients of $|\hat{Q}_j|$ or $|\tilde{Q}_j|$ decay as the index increases (i.e., the number of nodes $N$ grows larger). Comparing these two subplots, one may notice that the coefficients $|\hat{Q}_j|$ decay monotonously in a straightforward manner. Nevertheless, these two bases are equivalently efficient in accuracy, because the biorthogonal rational basis functions $\{\hat{\rho}_j\}$ can be thought of as the linear combination (or, for example, rotation) of the regular rational basis functions $\{\rho_j\}$ from (61). Because the coefficients with respect to the rational basis function $\{\rho_j\}$ are slightly “nicer looking” in terms of the visualization, we will only show the results with respect to $\{\rho_j\}$ when tracking the coefficients in our simulation.

For a comparison, we also show the numerical profiles of the ground state $Q_c$ obtained from the Fourier spectral method with $N = 2048$ on a square $[-50\pi, 50\pi]^2$ in Figure 3. The top left subplot shows the solution profile. The top right subplot shows the Fourier coefficients of $Q$, which also decrease to the level of $10^{-9}$. The bottom left subplot tracks the quantity $x^3 Q(x, 0)$. Compared with the result from Figure 2, one can see the Fourier spectral method fails to capture its asymptotic behavior. The bottom right shows the difference $\|Q^{l+1}_n - Q^l_n\|_{L^\infty}$ between each Petviashvili’s iteration (75).
Computation of $Q$ via the Fourier spectral method. Top left: solution $Q_\alpha$ of (69), $\alpha = 1$. Top right: the coefficients of $Q$. Bottom left: spatial dependence of $x^3 Q(x,0)$. Bottom right: $\|Q_\alpha^{l+1} - Q_\alpha^l\|_{L^\infty(\mathbb{R}^2)}$ on the log scale for each iteration.

Figure 3

To check further the accuracy and consistency of our computations of $Q$, we define the error quantities $e_1$, $e_2$, and $e_3$ from the Pohozaev identities (22), (23) and a multiple of energy (in the $L^2$-critical case $E[Q] = 0$) (24) as

$$e_1 = \left\|(-\Delta)^{\frac{1}{2}} Q_h\right\|_{L^2(\mathbb{R}^2)}^2 - 2\|Q_h\|_{L^2(\mathbb{R}^2)},$$

$$e_2 = \|Q_h\|_{L^3(\mathbb{R}^2)}^3 - 6\|Q_h\|_{L^2(\mathbb{R}^2)}^2,$$

$$e_3 = 3\left\|(-\Delta)^{\frac{1}{2}} Q_h\right\|_{L^2(\mathbb{R}^2)}^2 - 3\|Q_h\|_{L^3(\mathbb{R}^2)}^3.$$ 

We mention that the solution on the torus $[-L, L]^2$ also satisfies the Pohozaev identities. Thus, we omit listing our numerical simulations obtained from the Fourier spectral method for the accuracy comparison.

Table 1 shows the numerical values for $e_1$, $e_2$, and $e_3$ depending on the mapping parameter $\alpha$ and the number of nodes $N$. We can see that the error decreases as we increase the value of $\alpha$, or, in other words, if we increase the length of the computational domain. On the other hand, increasing the number of nodes $N$ will not decrease the error (compare the second column with the last column in Table 1 for $\alpha = 20$ and $N = 512$ vs. 1024).

For later purposes, we compute the $L^2$-norm of $Q$. Table 2 shows how this value depends on the mapping parameter $\alpha$ and the number of nodes $N$. 
Table 1

| N   | 256 | 512 | 1024 | 2048 | 1024 |
|-----|-----|-----|------|------|------|
| α   |     |     |      |      |      |
| e₁  | 0.81473 | 0.21302 | 0.058093 | 0.019039 | 0.21121 |
| e₂  | 1.6295 | 0.42605 | 0.11619 | 0.038078 | 0.42684 |
| e₃  | 0.81473 | 0.21302 | 0.058093 | 0.019039 | 0.20679 |

Table 2

| N   | 256 | 256 | 512 | 512 | 512 |
|-----|-----|-----|-----|-----|-----|
| α   | 10  | 20  | 10  | 20  | 20  |
| ||Q||²_{L²} | 42.6381 | 39.1681 | 42.6406 | 42.7366 | 39.3294 |

5 | Numerical Solutions of the HBO Equation

In this section we discuss our numerical findings for the dynamical HBO equation (1). We first discuss solutions that exist for all times, then we explore the possibility of finite time blow-up, and finish with investigating the interactions between two solitary waves. We recall that Equation (1) is $L²$-critical, and the Conjecture 1 states that the ground state $Q$ would be a possible threshold for the globally versus finite time existing solutions (to be more precise, the $L²$ norm of $Q$). To investigate that we consider various multiples and translations of $Q$ as well as other types of data with different decay rates, and confirm the conjecture. Furthermore, our analysis shows that the blow-up solutions are self-similar (in its core region) with the profiles of the rescaled ground state solutions. As far as the globally existing solutions we observe that eventually they all disperse into the radiation. Even those solutions, which initially start traveling to the right (in the $x$-direction) and try to approach a rescaled ground state profile, due to the outgoing dispersive oscillatory radiation (in the opposite direction or region): The location of the peak of the solution travels to the right (in the positive $x$-direction), but stops (possibly for some time), and then travels to the left, completely shedding via dispersive oscillations into the radiation. Furthermore, we observe the angle of the radiation wedge as it was discussed in Section 2.3. In the interaction of two solitary waves we show different scenarios of interaction, including a strong interaction, where two traveling waves combine into one that will either radiate away or blow up in finite time, depending on the total combined mass and initial geometrical configuration.

Before we discuss our numerical findings, we track the decay of the coefficients corresponding to the rational basis functions $\{\rho_j\}$ in our first example $u_0 = 0.9Q$ in Figure 4. The left subplot shows the coefficients at $t = 0$. One can see the decay to the $10^{-6}$ level with $N = 256$ nodes. At the time $t = 10$ (when we terminate our simulation), the coefficients decay to the level of $10^{-4}$. This increase of the coefficients most likely results from the radiation of the solution as time evolves. In other words, the solution radiates to the left with oscillations (see, e.g., Figures 5 and 8), and these oscillations cause worth decay in the coefficients. In fact, this phenomenon was also noticed by using other unbounded orthogonal basis functions such as the Laguerre functions, Hermite functions, and Mapped Chebyshev functions for approximating the function $f(x) = \frac{\sin x}{1+x^2}$, for example, in Ref. 65, Chapter 7. The Fourier spectral method may show a better decaying coefficients property, because the model is being studied on a compact periodic domain. However, the solution will wrap around and the radiation part will show up in the right-hand side as time evolves, which is
supposed to radiate to $-\infty$. In summary, we choose to use the rational basis functions, because it forms the complete orthogonal system in Ref. 45 with the coefficients $|\tilde{u}_N|$ decay to 0 as $N \to \infty$ for oscillatory decaying functions (e.g., $f(x) = \frac{\sin x}{1+x^2}$) from Ref. 46, which indicates the consistency in the spatial discretization.

5.1 Globally existing solutions

We start with considering the initial data of the form

$$u_0(x, y) = A Q(x, y), \quad (76)$$

where $Q(x, y)$ is the solution of (69) with $c = 1$ (or rather its numerical approximation $Q_h$ obtained in the previous section) and the constant $A > 0$. In this part we consider data such that $\|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)}$, thus, we take $A < 1$. For completeness, we mention that $E[u_0] = A^2(1 - A)\|Q\|_{L^2(\mathbb{R}^2)}^2$ by Pohozaev identities.
We first set $A = 0.9$ and track the time evolution of $u(t)$ up to $t = 10$ (the end of the computational time in this simulation), the snapshots of this solution at times $t = 0, 1, 3, 5, 7, 10$ are given in Figure 5. Starting from a radially symmetric initial condition at $t = 0$, the main peak travels along the $x$-axis in its positive direction while decreasing in its $L^\infty$ norm (note that the height is decreasing in time in Figure 5). The dispersive oscillations start developing right away, which we refer to as the radiation; the oscillations are outgoing in the negative $x$-direction. We note that the solution around the solitary wave core preserves its radial symmetry. For that we plot the cross-sections at the initial time and at the ending time of our simulations ($t = 10$).

The cross-sections by the $y = 0$ and $x = 0$ planes of the initial profile are given on the left plot of Figure 6, both coincide, because the initial profile is radially symmetric. The middle plot of Figure 6 shows both cross-sections by $y_c = 0$ (solid blue line) and by $x_c = 2.5$ (dashed red line) at the final time of this simulation $t = 10$. By $(x_c, y_c)$ we denote the coordinate of the peak of the solution (at a given time), i.e.,

$$\|u\|_{L^\infty(\mathbb{R}^2)} = |u(x_c, y_c)|.$$  

(77)

Note that in the middle plot the profile from the $x_c = 2.5$ cross-section (dash red curve) is intentionally shifted to the right to show the symmetry of the profile at $t = 10$ (otherwise, the peak in this slice (dash red curve) would be at $y = 0$). On the same graph we also plot the rescaled and shifted profile of $Q$, that is, $Q_c(x - x_c, 0)$ (dotted yellow line) to show that the solution has a good match (in the area excluding the radiation region to the left). The parameter $c$ is the scaling parameter as defined in (69) and (70) (for the case $s = \frac{1}{2}, m = 2$) and we compute it as follows in our simulations:

$$c = \frac{\|u(t_{max})\|_{L^\infty}}{\|Q\|_{L^\infty}},$$  

(78)

where $t_{max}$ is the maximal time in our computations.

From the right graph in Figure 6 it seems that the solution approaches a rescaled solitary wave that is traveling to the right of the $x$-axis with the decreasing height and decreasing speed shedding some radiation in the negative $x$-direction. It is plausible to suppose that this asymptotic behavior continues (as we showed in (C2) of Theorem 2 that solutions (at least sufficiently smooth) with the mass under the threshold are uniformly $H^5$ bounded globally in time), however, this is not the case. For this specific initial condition $u_0 = 0.9 \, Q$ it is challenging to track reliably the evolution beyond $t_{max} = 10$, therefore, we consider slightly smaller initial amplitude $A$ in (76). We are able
FIGURE 7  Left: time evolution of the kinetic energy for \( u_0 = 0.9Q \). Middle: time dependence of \( \|u(t)\|_{L^\infty} \) for \( u_0 = 0.9Q \). Right: time evolution of the peak location \( x_c \) for \( u_0 = 0.85Q \): observe that the peak stops traveling to the right (in the positive \( x \)-direction) and then moves in the opposite direction (the solution eventually radiates). A similar behavior is expected for \( u_0 = 0.9Q \)

to track the time evolution of \( u_0 = 0.85Q \) (as well as smaller \( A \)) and on the right graph of Figure 7 we show the trajectory of \( x_c \). We note that the peak stops traveling to the right when the location \( x_c \) stops around \( t = 10 \) and, after a short pause (the profile at that time has good matching with the solitary wave \( Q_c \) as in the middle graph of Figure 6), starts moving to the left (though sometimes moving forward and again backward, this is due to dispersive oscillations that can create double peaks, e.g. see top right of Figure 12), and then gets dispersed into the radiation.

We next consider initial data that decays slower than \( Q \) (recall that \( Q \) decays as \( 1/|x|^3 \))

\[
 u_0(x, y) = \frac{A}{1 + x^2 + y^2}, \quad A > 0. 
\] (79)

Noting that \( \|u_0\|_{L^2(\mathbb{R}^2)}^2 = A^2 \pi \), we obtain the threshold value for \( A \), i.e., when \( \|u_0\|_{L^2(\mathbb{R}^2)} = \|Q\|_{L^2(\mathbb{R}^2)} \), or equivalently, \( A_{th} = \|Q\|_{L^2(\mathbb{R}^2)}/\sqrt{\pi} \approx 3.7 \), where we used the value for the norm of \( Q \) from Table 2.

We now take \( A = 3 \), so that \( \|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)} \) (we also compute \( E[u_0] \approx 2.14 \)), and track its time evolution. Figure 8 shows the snapshots of \( u(t) \) at times \( t = 0, 1, 5, 10, 15, 20, 25, 30, 35, 40 \). The height is decreasing while the location of the peak is not moving significantly for some time, there is some shift in the positive \( x \)-direction around \( t = 20 \) (see tracking of \( x_c \) in the middle subplot of Figure 9), the radiation develops immediately in the negative \( x \)-direction, and the peak location after \( t = 30 \) starts moving to the left, or in the negative \( x \)-direction. On the left of Figure 10 the plot shows that the \( L^\infty \) norm of the solution decreases in time.

To understand better what happens with this solution, we check the cross-sections at various times and plot them in Figure 10. On the left graph the radial symmetry of the initial data is obvious; in the middle at \( t = 20 \) the peak has moved to \( x_c = 0.1 \) and one can see some resemblance of both cross-sections to the \( Q_c \) profile; on the right the peak location has moved to left to \( x_c = -0.35 \) (at the final computational time \( t = 40 \)) and the fitting to the \( Q_c \) is much less than in the middle graph for both cross-sections. This indicates that when the peak is traveling to the right, it is trying to approach the rescaled ground state profile, while when the peak of the solution is traveling to the left, it goes into the dispersive oscillatory behavior and, of course, no profile matching is expected.

We also investigate smaller amplitude data

\[
 u_0(x, y) = \frac{A}{1 + (x - a)^2 + y^2}, \quad \] (80)
FIGURE 8  Snapshots of the solution $u(t)$ with $u_0 = \frac{3}{1 + x^2 + y^2}$

FIGURE 9  Evolution of $u_0 = \frac{3}{1 + x^2 + y^2}$: time dependence of $\|u(t)\|_{L^\infty}$ (left), trajectory of $x_c$ in time (middle), errors of conserved quantities (right)

FIGURE 10  Cross-sections of the solution $u(t)$ with $u_0 = \frac{3}{1 + x^2 + y^2}$ at different times. Left: $t = 0$, cross-sections by $y = 0$ and $x = 0$ planes. Middle: $t = 20$, cross-sections by $y = y_c = 0$ (solid blue), and $x = x_c = 0.1$ (dashed red), compared with $Q_c$ (dotted yellow). Right: $t = 40$, cross-sections by $y = y_c = 0$ (solid blue) and $x = x_c = -0.35$ (dashed red), compared with $Q_c$ (dotted yellow). Note much tighter fit to $Q$ in the middle graph when the peak was traveling to the right
**FIGURE 11** Snapshots of the solution $u(t)$ with $u_0 = \frac{1}{1+(x-5)^2+y^2}$

**FIGURE 12** Top left: initial profile of $u_0 = \frac{1}{1+(x-5)^2+y^2}$ given via cross-sections by planes $y = 0$ and $x = 5$. Top right: cross-sections of the solution $u(t)$ at $t = 30$ by $y = y_c = 0$ (solid blue) and $x = x_c = -4$ (dashed red); here, we shifted the second cross-section to $x_c$ to check symmetry and compare with $Q_c$ (dotted yellow). Bottom: time dependence of $\|u(t)\|_{L^\infty}$, $\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^2)}^2$, and $x_c$ trajectory

with $A = 1$ and a shift $a = 5$ (to make sure the main bump is located in the more dense grid point part $[-\alpha, \alpha]^2$ during the simulation). The snapshots of the time evolution of this $u_0$ are provided in Figure 11.

Unlike the previous example (with a larger amplitude $A = 3$), the peak of the solution moves in the left $x$-direction right away (see the graph of the $x_c$ trajectory in the bottom right of Figure 12), meaning that both the peak of the solution moves to the left and the oscillatory radiation develops immediately and disperses to the left.
Furthermore, this is a good example to note the shape of the radiation region: the dispersive oscillations extend into a wedge region (for example, it can be clearly seen in the top right plot of Figure 11). We investigate this more carefully in Section 5.2 below.

Another type of data we consider is the one which has a faster decay than the ground state, that is, an exponential decay,

$$u_0(x, y) = A e^{-(x^2+y^2)}.$$  \hspace{1cm} (81)

We consider $A < A_{th} \approx 5$, because $\|u_0\|^2_{L^2(\mathbb{R}^2)} = \frac{\pi}{2} A^2$.

We show the snapshots of the solution $u(t)$ with $A = 4.5$ (here, $E[u_0] \approx 4.23$) in Figure 13. One can easily notice that the solution starts moving to the right shedding the radiative oscillations to the left of the $x$-axis. The height is decreasing in time, this can be seen in the snapshots and also on the left plot of Figure 14.

In the middle plot of Figure 15 we track the location of the peak $x_c$. One can note that up to about time $t = 12$ the solution moves to the right, though the peak’s location $x_c$ stops for some time around $x_c = 1.5$ up to $t = 22$, and then starts moving in the negative direction. We continue our simulations until $t = 40$ (note that the error of the energy is stable, but the error in the mass conservation is starting to increase after $t = 30$, therefore, for accuracy we stop our simulations at $t = 40$). For comparison we plot the cross-sections at times $t = 0, 20, 40$ in Figure 15, observing some tightness and closeness to $Q_c$ (around the peak location) up to about time $t = 20$ and then getting further away from the ground state profile and becoming asymmetric, especially in the cross-section by $y = 0$ (even around the peak location).
FIGURE 14  Evolution of \( u_0 = 4.5 e^{-(x^2+y^2)} \): time dependence of \( \|u(t)\|_{L^\infty} \) (left), trajectory of \( x_c \) in time (middle), errors in computation of conserved quantities (right)

FIGURE 15  The solution profiles for \( u_0 = 4.5 e^{-(x^2+y^2)} \) for \( t = 0, 20, 40 \). The solution scatters in a radially symmetric manner and \( \|u\|_{L^\infty(\mathbb{R}^2)} \) (or the maximum height) is decreasing in time

We next check the nonradial initial data of the form

\[
    u_0(x, y) = \frac{A}{1 + (x^2 + (0.5y)^2)^2}.
\] (82)

We note that \( \|u_0\|_{L^2(\mathbb{R}^2)} = \frac{\pi}{\sqrt{2}} A \), therefore, to check our conjecture, we consider \( A < A_{th} \approx 2.9 \).

The snapshots of the time evolution at times \( t = 0, 1, 5, 10, 15, 20 \) for the initial condition (82) with \( A = 2 \) are shown in Figure 16. One can note that the solution decreases in the height, and

FIGURE 16  Snapshots of the solution \( u(t) \) with \( u_0 = \frac{A}{1 + (x^2 + (0.5y)^2)^2} \), \( A = 2 \)
5.2 Angle of the radiation wedge

We investigate the radiative region of solutions, in particular, the angle of the wedge that was obtained in Section 2.3. For that we consider the following initial data:

\[ u_0(x, y) = e^{-(x^2+y^2)}, \quad u_0(x, y) = \frac{1}{1+(x^2+y^2)^2}, \quad u_0(x, y) = \frac{1}{1+(x^2+y^2)}. \quad (83) \]
FIGURE 19  $t = 5$: radiation region snapshots for different data (top); contour plots with an angle estimation of the radiation wedge (black dash lines) (bottom). Left: $u_0 = e^{-(x^2+y^2)}$. Middle: $u_0 = \frac{1}{1+(x^2+y^2)^2}$. Right: $u_0 = \frac{1}{1+x^2+y^2}$.

FIGURE 20  Snapshots of the solution $u(t)$ with $u_0 = 1.1Q(x+1, y)$

Figures 18 and 19 show the solutions profiles for each of the above initial condition at the times $t = 3$ and $t = 5$. The top row in both figures shows a snapshot of the solution at either $t = 3$ or $t = 5$ of the data (83) in the left, middle, and right columns, respectively. The bottom row offers the contour views and shows an estimate for the angle of radiation wedge with the black-dash lines.

For the simplicity of interpretation, we simply measure the tangent of the angle. One can see that for $u_0 = e^{-(x^2+y^2)}$, the dispersive oscillations are restricted to the angle $\theta$ with a crude estimate of $\theta \approx \arctan\left(\frac{13}{40}\right) \approx 18.00^\circ$. For $u_0 = \frac{1}{1+(x^2+y^2)^2}$, the angle is $\theta \approx \arctan\left(\frac{11}{40}\right) \approx 15.38^\circ$. For $u_0 = \frac{1}{1+x^2+y^2}$, which has the slowest decay among the considered initial data here, the angle is $\theta \approx \arctan\left(\frac{10}{40}\right) \approx 14.04^\circ$. Comparing these observations with the wedge in Figure 1 and in (55) in
Section 2.3, we observe that our angle approximations lie within the angle $19.42^\circ$, or $\tan \theta = 2\sqrt{2}$, thus, confirming the result of Section 2.3. (Note that the wedge is traveling with the solitary wave in time, and therefore, is shifting with the solution to the right, or in the positive $x$-direction.)

5.3 Blow-up solutions

In this part we study the second part of Conjecture 1, a possibility to develop a finite time blow-up in the case when $\|u_0\|_{L^2(\mathbb{R}^2)} > \|Q\|_{L^2(\mathbb{R}^2)}$. (We remark that numerically it is impossible to study exactly the threshold case $\|u_0\|_{L^2(\mathbb{R}^2)} = \|Q\|_{L^2(\mathbb{R}^2)}$ and we have already demonstrated that various data with $\|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)}$ generate global in time solutions.)

We start with the initial data of the perturbed ground state (76), noting that for any $A > 1$, the energy of such data is negative, $E[AQ] < 0$. Fixing $A = 1.1$, i.e., $u_0(x, y) = 1.1Q(x, y)$, we compute the time evolution $u(t)$ and plot the details in Figures 20 and 21. For the purposes of staying within the (symmetrical) computational domain, we shift this initial condition in the negative $x$-direction. Figure 20 shows snapshots of the time evolution for $u_0(x, y) = 1.1Q(x + 1, y)$ at $t = 0, 0.5, 1, 1.5, 2.5, 3$. Observe that the solution becomes tighter around its peak and the height is slowly increasing in time (see also Figure 21). Furthermore, the peak is traveling to the right, in the positive $x$-direction. Until the time when the solution travels beyond our “dense” part of the computational domain ($[-\alpha, \alpha]$), we observe that both the $L^\infty$ norm and the kinetic energy keeps
FIGURE 22  Snapshots of the solution $u(t)$ with $u_0 = \frac{4.5}{1 + ((x+2.5)^2 + (0.5y)^2)^2}$.

FIGURE 23  The cross-sections of the solution $u(t)$ with $u_0 = \frac{4.5}{1 + ((x+2.5)^2 + (0.5y)^2)^2}$ at $t = 0$ and $t = 2.07$ (top); growth of the norms in time (bottom) increasing in time (blue solid line in the bottom row graphs of Figure 21). This gives an indication of possible blow-up, however, because the initial data is very close to the threshold, our current numerical simulations do not provide sufficient information in this case.
Therefore, we modify slightly the amplitude in the initial condition and consider $u_0(x, y) = 1.2Q(x + 1, y)$, for which we track the norms $\|u(t)\|_{L^\infty(\mathbb{R}^2)}$ as well as $\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^2)}$ for a comparison. In the bottom graphs of Figure 21, one can see that the time evolution in this case blows up almost immediately (around the time $t = 1.1$). Because the quantities $\|u(t)\|_{L^\infty(\mathbb{R}^2)}$ and $\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^2)}$ have a similar behavior in both cases of initial condition ($A = 1.1$ and $1.2$), we can draw the conclusion that the solution with $u_0 = AQ$ blows up for $A > 1$.

We take a step further in studying the blow-up behavior of solutions in this equation and look at the blow-up profiles. For example, an excellent matching of the cross-sections of the solution generated by $u_0 = 1.1Q$ at time $t = 3$ can be observed on the right top plot of Figure 21. In the case of $A = 1.2$, we obtain a similar matching. This indicates that a stable critical blow-up in Equation (1) follows a self-similar dynamics with the ground state profile.

We next test the nonradial data of the form

$$u_0(x, y) = \frac{A}{1 + ((x + a)^2 + (0.5y)^2)^2},$$

which has the norm $\|u_0\|_{L^2(\mathbb{R}^2)} = \frac{1}{\sqrt{2}} A \pi$. The threshold value for $A$ is $A_{th} \approx 2.9$.

We study the data (84) with $A > A_{th}$ (and the shift $a = 2.5$ for convenience of graphing) and observe the blow-up behavior. For example, for $A = 4.5$ the snapshots of the time evolution at times $t = 0, 0.2, 0.5, 1.5, 2, 2.07$ are shown in Figure 22. In the following Figure 23 we provide the cross-sections of the solution at the beginning ($t = 0$) and at the last computational time before the blow-up ($t = 2.07$), as well as the growth of the norms in time. One can notice that starting with a nonradially symmetric initial data (top left graph shows the asymmetry in the cross-sections), the solution evolves into a radially symmetric solitary wave with a rescaled and shifted ground state profile, indicating the behavior of a radially symmetric self-similar blow-up dynamics (in the core region).

We also check the nonradial data with slower decay

$$u_0(x, y) = \frac{A}{1 + (x + a)^2 + (0.5y)^2},$$

(85)
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FIGURE 25 Top: cross-sections before and after the interaction for $u_0 = 0.9Q(x + 5, y) + Q_{0.25}(x + 1, y)$. As the higher bump obtains sufficient mass after the interaction, it blows up with the rescaled ground state profile $Q_c$. Bottom: time dependence of the kinetic energy and the $L^\infty$ norm

which has the norm $\|u_0\|_{L^2(\mathbb{R}^2)} = \sqrt{2\pi A}$, and hence, the threshold value $A_{th} \approx 2.6$. Taking $A > A_{th}$, we observe a similar blow-up behavior as shown in Figures 22 and 23.

Finally, we mention the gaussian type of data (81), which has an exponential decay. Taking $A > A_{th} \approx 5$, we note that for example, $A = 5.5$ in $u_0 = A e^{-(x^2 + y^2)}$ produces a blow-up in finite time; in this case, $E[u_0] = 0.87 > 0$. Considering $A = 6$ in the same $u_0$ also produces a blow-up solution; in this example, $E[u_0] = -2.11 < 0$. Thus, it is possible to have solutions with the positive and negative energy that blow-up in finite time (see remarks after Conjecture 1).

We conclude that the part 2 of Conjecture 1 holds for all data that we considered. Furthermore, a stable blow-up shows a self-similar dynamics with the ground state profile.

5.4 | Multibump solutions

We next investigate the interaction of two solitary waves, for that we take two rescaled solitary waves $Q_{c_1}$ and $Q_{c_2}$ as defined in (70), and track their evolution and interaction.

We first consider the two solitary waves that are separated along the $x$-axis, i.e.,

$$u_0(x, y) = a_1 Q_{c_1}(x + x_1, y) + a_2 Q_{c_2}(x + x_2, y).$$  (86)
In Figure 24, we show the snapshots of the time evolution for the initial condition
\[ u_0 = 0.9 Q(x + 5, y) + Q_{0.25}(x + 1, y). \]

While the mass for each bump is smaller than our predicted threshold \( \|Q\|_{L^2(\mathbb{R}^2)} \), the total mass is greater than \( \|Q\|_{L^2(\mathbb{R}^2)} \); in the given example \( \|u_0\|_{L^2(\mathbb{R}^2)}^2 \approx 83.06 \).

In Figure 24, we can see the higher solitary wave (which is further to the left on the \( x \)-axis) travels faster than the lower one, and they interact, merging together, between the time \( 2 < t < 4 \). After \( t > 4 \), they split. In the process of interaction, the initial higher bump obtains sufficient amount of mass, and continue traveling in the positive \( x \)-direction, it blows up in finite time, see the norm growth in the bottom row in Figure 25, while the smaller bump loses the mass and completely radiates to the left.

If the two solitary waves are not sufficiently large, the interaction will still be in similar manner: the higher, and thus, faster one will catch up with the slower one, merge into it and then both will split from each other, after the interaction, with the faster one obtaining some additional mass from the slower one; however, (eventually) both will disperse into the radiation, see Figures 26 and 27 for
\[ u_0 = 0.7 Q_{0.5}(x + 4, y) + 0.5 Q_{0.25}(x, y). \]

In this case the total mass \( \|u_0\|_{L^2(\mathbb{R}^2)}^2 \approx 41.56 \).

We next modify the initial data and separate the two solitary waves in the \( y \)-coordinate. For example, consider
\[ u_0 = 0.9 Q(x, y - 5) + 0.9 Q(x, y + 5), \quad (87) \]

so there is about 10 units of separation in \( y \). One can see the two bumps moving parallel along the \( x \)-direction without much of an interaction, and eventually, radiate away, see Figure 28 as if they would just exist on their own. The energy in this case is \( E[u_0] \approx 2.69 > 0 \).
The higher bump obtains some extra mass after the interaction, however, it is not sufficient to develop a blow-up, and thus, both bumps eventually radiate. While decreasing in its height, the solution maintains radial symmetry and is close to the rescaled profile $Q_c$.

In our final example, we consider the same two solitary waves as before, but now they are separated in the $y$-coordinate not as much, so the two bumps are sufficiently close to each other. The initial condition is

$$u_0 = 0.9 Q(x + 5, y - 1) + 0.9 Q(x + 5, y + 1),$$

there is only two units of separation in $y$, for a depiction see Figure 29.

The interaction happens as the two solitary waves merge into one. A joint lump will have sufficient mass and will generate a finite time blow-up solution. One can see that as merging together occurs, the radiation wedge is being generated, which clearly continues after the two bumps merged into one. Because the merged solution has large enough mass, it will blow up in finite time (see the height of snapshots in the bottom row of Figure 29).

We show the cross-sections at the final (computational) time $t = 9.49$ as well as the matching with the rescaled $Q_c$ in the left graph of Figure 30. The dependence on time of the kinetic energy and the $L^\infty$ norm is shown in the middle and right graphs of the same figure. One can notice that the height during the merging drops significantly (around time $t = 2$), but due to the sufficient mass, the solution picks up the growth of its height and its kinetic energy. We note that the energy in the last two examples (87) and (88) is the same and positive, i.e., $E[u] \approx 2.69 > 0$. 

![Cross-sections before and after the interaction](image1)

![Cross-sections before and after the interaction](image2)
**FIGURE 28**  Snapshots of time evolution for $u_0 = 0.9Q(x, y - 5) + 0.9Q(x, y + 5)$

**FIGURE 29**  Snapshots of strong interaction for $u_0 = 0.9Q(x + 5, y - 1) + 0.9Q(x + 5, y + 1)$

**FIGURE 30**  Details on the interaction for $u_0 = 0.9(Q(x + 5, y - 1) + Q(x + 5, y + 1))$. Left: cross-sections after the interaction at the final computational time and matching with the rescaled $Q_c$. Middle and right: time dependence of the kinetic energy and the $L^\infty$ norm
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