The third law of thermodynamics and black holes

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Working in the framework of generalized statistics, the problem of availability of the third law of thermodynamics in the black hole physics is studied by focusing on Schwarzschild black hole which easily and clearly exposes the violation of this law in the common approach based on Bekenstein entropy. Additionally, it is addressed that some inconsistencies between the predictions of quantum field theory and those of thermodynamics on the black hole temperature may be reconciled by using the thermodynamics laws in order to broaden energy definition. It claims that thermodynamics should be employed as a powerful tool in looking for more comprehensive energy definitions in high energy physics, still mysterious.

I. INTRODUCTION

The third law of thermodynamics states that the entropy of a system should approach a constant value \((C)\) at absolute zero temperature, or equally \(S(T \to 0) \to C\). Bekenstein entropy \((S_B)\) is proportional to the horizon area \((A = 4\pi r_h^2)\), and correspondingly, for a Schwarzschild black hole of mass \(M(\equiv E)\) and Hawking temperature \(T_H\) for which \(r_h = 2M\) and \(T = \frac{\partial M}{\partial S} = \frac{1}{8\pi M} = T_H\), we have
\[
S_B = \frac{A}{4} = 4\pi M^2 = \frac{1}{16\pi T_H^2},
\]
in the units of \(c = \hbar = k_B = G = 1\), where \(k_B\) denotes the Boltzmann constant. It clearly indicates that the third law of thermodynamics is not satisfied or briefly, \(T_H \to 0 || M \to \infty \Rightarrow S_B \to \infty\). Although we considered Schwarzschild black hole, the behavior of \(S_B(T(M \to \infty) \to 0) \to \infty\) is common in other black holes such as Kerr-Newman and Reissner-Nordström metrics \[\text{[4, 5]}\].

\(S_B\) is non-extensive, a trait which reminds the generalized statistics such as Tsallis statistics \[\text{[6, 7]}\] including entropies which are not extensive. Indeed, gravitational systems include long-range interaction (gravity), it is proposed that the Boltzmann entropy (leading to Eq. \((1)\)) should be replaced with generalized entropies which leads to substantial consequences in both gravitational and cosmological setups (see for example Refs. \[\text{[8–16]}\] their references and citations). It also seems that there is a connection between deviation from the Boltzmann statistics and the quantum features of gravity \[\text{[16–20]}\], and consequently, the generalized and loop quantum gravity entropies can be classified as the subclasses of a general entropy \[\text{[22]}\].

Our first aim is to study the possibility of satisfying the third law by employing some new entropies proposed that provide respectable solutions in cosmological and gravitational setups. Hereof and in subsequent sections, we respectively study the problem by considering Tsallis and Cirto entropy, Tsallis entropy and Kaniadakis entropy. Throughout the survey, we also address a thermodynamic energy definition for a black hole of mass \(M\), corresponding to each entropy, which admits Hawking temperature. The black hole remnant and its decay time in mentioned entropy formalisms have been studied in fifth section. The last section is also devoted to summary.

II. TSALLIS AND CIRTO ENTROPY AND THE THIRD LAW OF THERMODYNAMICS

Motivated by the non-extensivity of \(S_B\), and also the long-range nature of gravity, Tsallis and Cirto \[\text{[8]}\] introduce a new entropy for black holes as
\[
S_T = \gamma A^\delta,
\]
where \(\gamma\) and \(\delta\) are two unknown free constants evaluated by other parts of physics or observations. It is also useful to note here that this form of entropy is also supported in the framework of quantum gravity \[\text{[24]}\]. It means that two different approaches and motivations lead to the same result which increases the credit of this new proposal for the black hole entropy. Moreover, the equality of results can be considered as the sign of a deep
connection between non-extensivity and quantum gravity helping us build a relation between their parameters, a result noted in Refs. [17, 18]. Considering $r_h = 2M$ and $T = \frac{\delta M}{\delta S_T} = \frac{1}{2\gamma(16\pi)^{\delta/2}M^{\delta-1}}$, one easily finds that the third law of thermodynamics is met whenever $0 < \delta < \frac{1}{2}$, and in summary, $S_T \to 0 \parallel M \to 0 \parallel T \to 0$.

Now, let us employ Hawking temperature ($T_H = \frac{1}{8\pi M}$) instead of $T = \frac{\delta M}{\delta S_T}$ which leads to $S_T \propto T^{-\delta}$. It is obvious that the third law is fulfilled only if $0 < \delta < \frac{1}{2}$ and for this case, we briefly have $M \to \infty \parallel S_T, T \to 0$. In Tsallis statistics, an intrinsic temperature discrepancy between real temperature and the temperature obtained by thermodynamic relation may emerge depending on the expectation values definition (averaging methods) used in obtaining quantities such as energy [21], and only, having in hand the system temperature, one can decide on true temperature discrepancy may be more understandable by bearing in mind the intrinsic temperature discrepancy of Tsallis statistics. Consequently, since $\delta$ is a free parameter estimated from observations [6, 7] or probably other parts of physics [17, 18], we cannot further in choosing one of these temperatures, and thus the corresponding thermodynamics, unless we encapsul detailed observations, data and info on black holes.

Of course, a way to reconcile the above inconsistency between temperatures is to redefine energy. In both cases above, we assumed $E = M$, while if we assume $T = T_H = \frac{\partial E}{\partial S}$, and use Eq. [2], then we reach

$$E_T = \int_0^M \frac{1}{8\pi m} \frac{\partial S_T}{\partial m} dm,$$

finally leading to

$$E_T = 4\gamma\delta(4\pi)^{\delta-1} \frac{1}{\delta-1} M^{\delta-1},$$

as the energy of a Schwarzschild black hole of mass $M$ in Tsallis formalism which recovers $E = M$ by inserting $\delta = 1$ and $\gamma = \frac{1}{4}$ (the Bekenstein limit). It is also obvious that $E_T$ is positive if $\delta > \frac{1}{2}$ or $\delta < 0$. For the $\delta > \frac{1}{2}$ case, the third law is not satisfied and we face a situation like what we obtained in the case of Bekenstein entropy (i.e., $T_H \to 0 \parallel S_B, M, E_T \to \infty$). The third law is met for $\delta < 0$ and in parallel $E_T \to 0$ or briefly, $E_T, S_T, T \to 0 \parallel M \to \infty$.

Finally, we think that since the Hawking temperature is also supported by other parts of physics such as quantum field theory in curved spacetime and so on [23, 24], and as there is not any common agreement on the energy definition in high energy physics [25, 26, 27], it is probably reasonable to rely on this approach compared to the two previously mentioned cases. The latter means that thermodynamics may be used to find a more proper energy definition in high energy physics.

In summary, we obtained 3 cases recaped as

i) $E = M$, $T = \frac{\partial E}{\partial S_T} = \frac{1}{2\gamma(16\pi)^{\delta/2}M^{\delta-1}} \neq T_H$,

ii) $E = M$, $T = T_H = \frac{1}{8\pi M} \neq \frac{\partial E}{\partial S_T}$,

iii) $E_T = 4\gamma\delta(4\pi)^{\delta-1} \frac{1}{\delta-1} M^{\delta-1}$, $T = T_H = \frac{1}{8\pi M} = \frac{\partial E_T}{\partial S_T}$,

where the third law is satisfied for the first case when $0 < \delta < \frac{1}{2}$, and for the remaining cases when $\delta < 0$.

### III. TSALLIS ENTROPY AND THE THIRD LAW

Recently, focusing on the relation between Tsallis and Boltzmann statistics, a new entropy has been derived for black holes as [13]

$$S_q = \frac{1}{1-q}[\exp((1-q)S_B) - 1] = \frac{2\exp((1-q)S_B)}{(1-q)} \sinh\left((1-q)S_B\right),$$

where $q$ is a free unknown parameter and this result is also confirmed by calculating the Tsallis entropy content of black holes in the framework of quantum gravity [13, 19]. Following the recipe of previous section, if we assume $E = M$ then we have

$$T = \frac{\partial E}{\partial S_q} = T_H \exp(4\pi M^2),$$

which recovers $T_H$ whenever $q = 1$ (the Bekenstein limit of [49, 13]). Briefly, $T \to 0$ only if $q < 1$, and in this manner $M, S_q \to \infty$ meaning that the third law is not satisfied. On the other hand, if we assume $T = T_H$ (case ii), then we see that the third law is satisfied only if $q > 1$, or briefly $M \to \infty \Rightarrow S_q(T \to 0) \to 0$. For the case iii, where $T = T_H = \frac{1}{8\pi M} = \frac{\partial E_N}{\partial S_N}$, we reach

$$E_q = \int_0^M \exp((1-q)4\pi m^2) dm,$$

as the energy content of a black hole of mass $M$. Clearly, the third law is again satisfied only if $q > 1$ and moreover, $E = \int_0^M dm = M$ is recovered at the limit of $q \to 1$. The above integration can be performed with the solution given as

$$E_q = \frac{\text{erf}(2\sqrt\pi M \sqrt{q-1})}{4\sqrt{q-1}},$$

in which erf(x) denotes the error function [28].
IV. KANIADAKIS ENTROPY AND THE THIRD LAW

Kaniadakis entropy of a black hole is also reported as
\[ S_{\kappa} = \frac{1}{\kappa} \sinh \left( \kappa S_B \right), \]  
where \( \kappa \) is an unknown parameter evaluated by observations and probably, the other parts of physics [13]. Here, simple calculations lead to
\[ T = \frac{\partial E}{\partial S_{\kappa}} = \frac{T_H \cosh(\kappa S_B)}{\kappa}, \]  
for the \( i \)-th case. The result indicates that, independent of the value of \( \kappa \), the third law is not satisfied \( (S_{\kappa} \rightarrow \infty \parallel M \rightarrow \infty \parallel T \rightarrow 0) \). For the second case \( (T = T_H, E = M) \), we can write
\[ S_{\kappa} = \frac{1}{\kappa} \sinh \left( \frac{\kappa}{16\pi T^2} \right), \]  
which shows the third law is satisfied only if \( \kappa < 0 \). If \( \kappa < 0 \), then the third law is also met for case \( iii \), where \( T = T_H \) and energy content of black hole is obtainable by using
\[ E_{\kappa} = \int_0^M \cosh(\kappa 4\pi m^2)dm, \]  
which recovers the \( E = \int_0^M dm = M \) results at the limit of \( \kappa = 0 \). The solution to the above integral is also given as
\[ E_{\kappa} = \frac{1}{8\sqrt{\kappa}} \left[ \text{erf} \left( 2\sqrt{\kappa\pi}M \right) + \text{erfi} \left( 2\sqrt{\kappa\pi}M \right) \right], \]  
where
\[ \text{erfi}(x) = -i\text{erf}(ix). \]  

In Fig. 1, \( E_{\kappa} \) and \( E_q \) are plotted for different values of \( q \) and \( \kappa \), the \( E = M \) case has also been depicted to have a comparison. It is worthwhile to mention that, as it is obvious from this figure, there is an asymptote for \( E_q \) when \( q > 1 \) as \( E_q(M \gg 1) \rightarrow \frac{1}{4\sqrt{\kappa}-1} \).

V. BLACK BODY RADIATION AND BLACK HOLE EVAPORATION

In the framework of common statistical mechanics based on Gibbs entropy, the black hole evaporation is described by Stefan-Boltzmann (SB) formula [29, 30]. As it is apparent from Eq. 11, we have \( S_B, M \rightarrow \infty \) when \( T \rightarrow 0 \) meaning that we face a catastrophe [30]. This result can be summarized in the language of SB law as
\[ \frac{dM}{dt} = -A\sigma T^4, \]  
where the ordinary energy definition \( (E = M) \) is used and \( \sigma(=\frac{\pi^2}{30}) \) in our units [31, 32] denotes the SB constant [30]. Here, minus sign is also due to the fact that Eq. 16 explains the amount of energy that system loses it. It clearly shows that the decay rate \( (\frac{dM}{dt}) \) diverges as \( M \) (\( T \)) approaches zero (infinity) [30]. Another consequence of this law is our ability to find the decay time \( \tau \) as
\[ \tau = -\frac{1}{\sigma} \int_M^0 \frac{dm}{AT^3} = \frac{(8\pi)^3 M^3}{2\sigma^3}, \]  
and therefore \( \tau \sim M^3 \) for a Schwarzschild black hole of temperature \( T_H \). Eq. 17 also indicates that \( \tau \rightarrow \infty \) when \( M \rightarrow \infty \) while \( T, \frac{dM}{dt} \rightarrow 0 \) [30].
The decay time is finite and for those values of \( \delta \) parameter and those of middle panel, the third law of thermodynamics \((S(T \rightarrow 0) \rightarrow 0)\) is violated.

In Fig. 3, we have plotted decay time of a black hole against its mass for the second case, where we observe that the larger the black hole mass the longer it takes for the black hole to completely evaporate. The slope of \( T_{1}^{(C)} \) curve increases for larger values of black hole mass so that the decay time grows unboundedly and diverges for massive and super massive black holes. Finally, Fig. 4 presents the decay time of black hole for the third case. In the upper panel we observe that, for \( \delta < 0 \) for which the third law is respected, a black hole with initial finite mass will completely evaporate at a finite amount of time. The lower panel shows that for \( \delta > 0 \) the decay time is an increasing function of the black hole mass and the heavier the initial black hole the longer it takes to completely evaporate. However, from the viewpoint of the third case, for this value of \( \delta \) parameter the third law is not respected. As yet there is no agreement on the numeric value of \( \alpha \) parameter we have considered the value of this parameter to be unity [33, 37]. We further note that entropy is a dimensionless quantity in each system of units where the Boltzmann constant is unity. Hence, from Eq. (2) we can deduce that \( \gamma \propto C/\ell_{Pl}^{8} \) and as we work in the system of units for which \( \ell_{Pl} = 1 \), \( \gamma \) parameter is a positive constant value which we have considered it to be unity.

### B. Tsallis black hole

Although Eq. (16) is used in some previous works which investigate the black hole thermodynamics in various non-extensive statistics [14, 17], a comprehensive study in the Tsallis framework should employs the Tsallis counterpart of Eq. (16). The latter is a controversial issue [21, 38, 40], as different averaging methods are usable and are employed in this statistics [21]. These different methods have their own benefits and shortcomings, and indeed, their correctness and accessibility situations are still unsolved and need more attentions and observations [21]. Here, motivated by the fact that

\[
\frac{dE}{dT} = -A_{\sigma q}T^{4} \tag{20}
\]

is obtained by different approaches [38], and is also originated from the black body spectrum in Tsallis statistics [41], we focus on Eq. (20) as the alternative of Eq. (16). Here, \( \sigma_{q} \) is called the generalized SB constant calculated as [38, 39].
FIG. 2: Plot of decay time for the first case, versus black hole mass for different values of parameter $\delta$. We have set $\sigma = \pi^2/60$, $\gamma = 1$ and $\alpha = 1$.

\[ \sigma_q = \frac{1}{\pi^2} \int_0^\infty \left[ \frac{x^3}{\exp(x) - 1} - \frac{1 - q}{2} \frac{x^5 \exp(x)}{\exp(x) - 1} \right] dx \Rightarrow \]

\[ \sigma_q \simeq \sigma (1 - 6 \cdot 15(1 - q)) = \sigma (6 \cdot 15q - 5 \cdot 15), \] (21)

if the integration is numerically solved. It is also obvious that $\sigma_q \to \sigma$ at the appropriate limit of $q \to 1$.

For three obtained cases, we have

\[ \tau_{q}^{q} = \frac{(8\pi)^3}{2\sigma_q} \int_{M}^{0} m^2 \exp((1 - q)4\pi m^2) dm, \]

\[ \tau_{ii}^{q} = \frac{(8\pi)^3}{2\sigma_q} \frac{M^3}{3}, \]

\[ \tau_{iii}^{q} = \tau_{i}^{q}. \] (22)

Since only the second and third cases satisfy the third
FIG. 5: Plot of decay time for Tsallis black hole against its initial mass. We have set $\sigma = \pi^2/60$.

C. Kaniadakis black hole

Black body spectrum in Kaniadakis statistics has recently been studied [31, 42], and it has been shown that

$$ \frac{dE}{dt} = -A \sigma \kappa T^4, $$

(23)

where $\sigma = \frac{J_{\kappa}(0)}{4\pi^2}$ in which $J_{\kappa}(0) = \int_0^\infty \frac{x^3}{\exp_\kappa(x)-1} dx$ while $\exp_\kappa(x) = [\sqrt{1+\kappa^2 x^2} + \kappa x]^\frac{1}{\kappa}$, and we have $\sigma_{\kappa \to 0} = \sigma$ [31]. Finally, we reach

$$ \tau_{\kappa i} = \frac{(8\pi)^3}{2\sigma \kappa} \int_0^M m^2 \cosh (4\kappa \pi m^2) dm, $$

$$ \tau_{\kappa ii} = \frac{(8\pi)^3}{2\sigma \kappa} M^3, $$

$$ \tau_{\kappa iii} = \tau_{\kappa i}, $$

(24)

for three cases we discussed above. In Fig. (6) we have plotted $\tau_{\kappa ii}$ and $\tau_{\kappa iii}$ (family of black curves) for some values of $\kappa < 0$ parameter where we observe that the decay time grows as $\kappa$ tends to larger values in negative direction. Such a behavior is parallel to the satisfaction of the third law. For the second case we observe that the decay time is finite for a finite mass black hole.

VI. SUMMARY

Based on the third law of thermodynamics, it is impossible for a system to touch zero-entropy state (at least a state with minimum and finite value of entropy) as its temperature tends to zero by only experiencing a finite number of thermodynamical processes. As each process spends its own time interval to be completed, we may even swap finite number of thermodynamical processes with finite time in the above statement. The story becomes more complicated in the ordinary black hole physics, generated by Bekenstein entropy, where it is obtained that entropy diverges, while its temperature approaches zero. In this situation, while black hole evaporates at finite time [17], and loses all of its mass, its temperature diverges at its final evolution steps meaning that we face a catastrophe [30]. Here, we only focused on Schwarzschild black hole as a primary solution which clearly exposes the mentioned inconsistency with third law.

Motivated by the long range nature of gravity, some recent works that propose a deep connection between quantum gravity and generalized statistics [16–20], and also the successes of these types of statistics in justifying some cosmological and gravitational phenomena [8–16], we studied the status of third law for a Schwarzschild black hole in the framework of some generalized statistics and we found out that this law may theoretically be settled. Moreover, we obtained that the thermodynamic analysis along with the laws may help us find new energy definitions, thus establishing a consistency between the results of thermodynamics and the predictions of quantum field theory about the black body radiation. The latter means that thermodynamics may eventually shed light on the physics and pave the way to find a comprehensive energy definition [26, 27].

It is finally useful to mention that, a few months after submitting our preprint to arXiv, we found two related papers [43, 44]. While one of them investigates the availability of the generalized second law of thermodynamics...
in a universe whose apparent horizon meets the generalized entropies [44], another one studies the BH temperature and energy by employing the Tsallis and Cirto and Rényi entropies [43]. The approach of Ref. [43], and additionally, its findings about the Tsallis and Cirto entropy are similar to what we did and obtained in Sec. (II), considered as a confirmation for our concern and the strategy adopted here.

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