Abstract

Local graph clustering and the closely related seed set expansion problem are primitives on graphs that are central to a wide range of analytic and learning tasks such as local clustering, community detection, nodes ranking and feature inference. Prior work on local graph clustering mostly falls into two categories with numerical and combinatorial roots respectively. In this work, we draw inspiration from both fields and propose a family of convex optimization formulations based on the idea of diffusion with \( p \)-norm network flow for \( p \in (1, \infty) \).

In the context of local clustering, we characterize the optimal solutions for these optimization problems and show their usefulness in finding low conductance cuts around input seed set. In particular, we achieve quadratic approximation of conductance in the case of \( p = 2 \) similar to the Cheeger-type bounds of spectral methods, constant factor approximation when \( p \to \infty \) similar to max-flow based methods, and a smooth transition for general \( p \) values in between. Thus, our optimization formulation can be viewed as bridging the numerical and combinatorial approaches, and we can achieve the best of both worlds in terms of speed and noise robustness. We show that the proposed problem can be solved in strongly local running time for \( p \geq 2 \) and conduct empirical evaluations on both synthetic and real-world graphs to illustrate our approach compares favorably with existing methods.

1 Introduction

Graphs are ubiquitous when it comes to modeling relationships among entities, e.g. social [30] and biology [32] networks, and graphs arising from modern applications are massive yet rich in small-scale local structures [20, 16, 9]. Exploiting such local structures is of central importance in many areas of machine learning and applied mathematics, e.g. community detection in networks [25, 35, 20, 16] and PageRank-based spectral ranking in web ranking [27, 12]. Somewhat more formally, we consider local graph clustering as the task of finding a community-like cluster around a given set of seed nodes, where nodes in the cluster are densely connected to each other while relatively isolated to the rest of the graph. Moreover, an algorithm is called \textit{strongly local} if it runs in time proportional to the size of the output cluster rather than the size of the whole graph.

Strongly local algorithms for local graph clustering are predominantly based on the idea of diffusion, which is the generic process of spreading mass among vertices by sending mass along edges. This connection has been formalized in many previous results [29, 1, 34]. Historically, the most popular diffusion methods are spectral methods (e.g. random walks) based on the connection between graph structures and the algebraic properties of the spectrum of matrices associated with the graph [22, 23, 6, 31]. Spectral diffusion methods are widely applied in practice due to the ease of implementation, efficient running time and good performances in many contexts. However, it is also known in theory and in practice that spectral methods can
spread mass too aggressively and not find the right cluster when structural heterogeneities exist, and thus are not robust on real-world graphs constructed from very noisy data \[14, 33, 20, 16\]. A more recent line of work on diffusion is based on the combinatorial idea of max flow exploiting the canonical duality between flow and cut \[34, 10, 26\]. These methods offer improved theoretical guarantees in terms of locating local cuts, and have been shown to be important for pathological cases where spectral methods do not perform well in practice. However, combinatorial methods are generally accepted to be more difficult to understand and implement in practice due to the more complicated underlying dynamics.

In this paper, we propose and study a family of primal and dual convex optimization problems for local graph clustering. We call the primal problem \(p\)-norm flow diffusion, parameterized by the \(L_p\)-norm used in the objective function, and the problem defines a natural diffusion model on graphs using network flow. We refer the dual problem as the \(q\)-norm local cut problem where \(q\) is the dual norm of \(p\) (i.e. \(1/p + 1/q = 1\)). The optimal solution to the \(q\)-norm local cut problem can be used to find good local clusters with provable guarantees. Through our discussion, we use \(p\)-norm diffusion to refer both the primal and the dual problems since our main inspiration comes from diffusion, even though our results are technically for the \(q\)-norm local cut.

We note that almost all previous diffusion methods are defined with the dynamics of the underlying diffusion procedure, i.e. the step-by-step rules of how to send mass, and the analysis of these methods is based on the behaviors of the algorithm. Although there are some efforts on interpreting the algorithms with optimization objectives \[10, 11, 24\], this line of research remains predominantly bottom-up starting with the algorithmic operations. On the other hand, our work starts with a clear optimization objective, and analyze the properties of the optimal solution independent from what method is used to solve the problem. This top-down approach is distinct in theory, and is also very valuable in practice since the de-coupling of objective and algorithm gives the users the freedom at implementation to choose the most suitable solver based on availability of infrastructure and code-base.

### 1.1 Our Main Contributions

We refer readers to Section 2 for formquent discussion. Our first main result is a novel theoretical analysis for local graph clustering using the optimal solution of \(p\)-norm diffusion. In particular, suppose there exists a cluster \(B\) with conductance \(\phi(B)\), and we are given a set of seed nodes that overlaps reasonably with \(B\). Then the optimal solution of \(p\)-norm diffusion can be used to find a cluster \(A\) with conductance at most \(O(\phi(B)^{1/q})\). For \(p = 2\), this result resembles the Cheeger-type quadratic guarantees that are well-known in spectral-based local graph clustering literature \[29, 1\]. When \(p \to \infty\), our conductance guarantee approaches a constant factor approximation similar to max flow methods, while achieving a smooth transition for general \(p\) values in between. We observe in practice that our optimization formulation can achieve the best of both worlds in terms of speed and robustness to noise for when \(p\) lies in the range of small constants, e.g. \(p = 4\).

On the algorithm side, we show that a randomized coordinate descent method can obtain an \(\epsilon\) accurate solution of \(p\)-norm diffusion for \(p \geq 2\) in strongly local running time. The running time of our algorithm captures the effect that it takes longer for the algorithm to converge for larger \(p\) values. Although the iteration complexity analysis is not entirely new, we show a crucial result on the monotonicity of the dual variables, which establishes the strongly local running time of the algorithm.

Our analysis illustrates a natural trade-off as a function of \(p\) between robustness to noise and the running time for solving the dual of the \(p\)-norm diffusion problem. In particular, for \(p = 2\) the diffusion problem can be solved in time linear in the size of the local cluster, but may have quadratic approximation error \(O(\sqrt{\phi(B)})\). On the other hand, the approximation error guarantee improves when \(p\) increases, but it also takes longer to converge to the optimal solution. We believe the regime of \(p\) being small constants offer the best trade-offs in general.
1.2 Previous Work

The local clustering problem is first proposed and studied by [29]. Their algorithm Nibble is a truncated power method with early termination, and their result were later improved to $\tilde{O}(\sqrt{\phi(B)})$ conductance approximation and $\tilde{O}\left(\frac{\text{vol}(B)}{\phi(B)}\right)$ time using approximate personalized PageRank [1], which is one of the most popular local clustering methods. The EvoCut algorithm [4] is the fastest spectral based local clustering method with running time $\tilde{O}(1)$. Some examples include the fasted spectral based local clustering method with running time $\tilde{O}(1)$, which is one of the most popular local clustering methods. The EvoCut algorithm [4] is the fastest spectral based local clustering method with running time $\tilde{O}(1)$. Some examples include the flow-improve method [3], the local flow-improve method [26] and the capacity releasing diffusion [34]. The former relies on black-box max flow primitives, while the latter two require specialized max flow algorithms with early termination to achieve running time that is nearly linear in the size of the local cluster. These algorithms achieve constant approximation error, i.e., the output cluster has conductance $O(\phi(B))$, as opposed to quadratic approximation error for spectral methods.

2 Preliminaries and Notations

We consider un-directed connected graph $G$ with $V$ being the set of nodes and $E$ the set of edges. For simplicity we focus on un-weighted graphs in our discussion, although our result extends to the weighted case in a straightforward manner. The degree $\deg(v)$ of a node $v \in V$ is the number of edges incident to it, and we denote $\vec{d}$ as the vector of all nodes’ degrees and $D = \text{diag}(\vec{d})$. We refer to $\text{vol}(C) = \sum_{v \in C} \deg(v)$ as the volume of $C \subseteq V$. We use subscripts to indicate what graph we are working with, while we omit the subscripts when the graph is clear from context.

A cut is treated as a subset $S \subset V$, or a partition $(S, \overline{S})$ where $\overline{S} = V \setminus S$. For any subsets $S, T \subset V$, we denote $E(S, T) = \{\{u, v\} \in E \mid u \in S, v \in T\}$ as the set of edges between $S$ and $T$. The size of a cut $S$ in $G$ is $\Phi_G(S) = \frac{\delta(S)}{\min(\text{vol}_{G}(S), \text{vol}_{G}(V \setminus S))}$. Unless otherwise noted, when speaking of the conductance of a cut $S$, we assume $S$ to be the side of smaller volume. The conductance of a graph $G$ is $\Phi_G = \min_{S \subset V} \Phi_G(S)$.

A routing (or flow) is a function $f : E \to \mathbb{R}$. For each un-directed edge $e$ with endpoints $u, v$, we arbitrarily orient the edge as $e = (u, v)$, i.e. from $u$ to $v$. The magnitude of the flow over $e$ specifies the amount of mass routed over $e$, and the sign indicates whether we send flow in the forward or reverse direction of the orientation of edge $e$, i.e. $f(u, v)$ positive if mass is sent from $u$ to $v$ and vice versa. We abuse the notation to also use $f(v, u) = -f(u, v)$ for an edge $e = (u, v)$. We denote $B$ as the signed incidence matrix of the graph of size $|E| \times |V|$ where the row of edge $e = (u, v)$ (again, using the arbitrary orientation) has two non-zeros entries, $-1$ at column $u$ and $1$ at column $v$. Through our discussion we refer to a function over edges (or nodes) and its explicit representation as an $|E|$-dimensional (or $|V|$-dimensional) vector interchangeably.

3 Diffusion as Optimization

Given a graph $G$ with signed incidence matrix $B$, and two functions $\Delta, T : V \to \mathbb{R}_{\geq 0}$, we propose the following pair of convex optimization problems, which are the $p$-norm flow diffusion

$$\begin{align*}
\text{minimize } & \|f\|_p \\
\text{s.t. } & B^T f + \Delta \leq T
\end{align*}$$

(1)
and its dual formulation with $q$ such that $1/q + 1/p = 1$

\[
\begin{align*}
\text{maximize} \quad & (\Delta - T)^T x \\
\text{s.t.} \quad & \|Bx\|_q \leq 1 \\
& x \geq 0
\end{align*}
\]

The solution to the dual problem $x \in \mathbb{R}_{\geq 0}^{\lvert V \rvert}$ gives an embedding of the nodes on the (non-negative) real line. This embedding is what we actually compute in the context of local clustering, and we use the primal problem and its flow solution $f \in \mathbb{R}^{\lvert E \rvert}$ mostly for insights and analysis purposes. We discuss the interpretation of the primal problem as a diffusion next.

**The Primal Problem.** As mentioned earlier, we consider a diffusion on a graph $G = (V, E)$ as the task of spreading mass from a small set of nodes to a larger set of nodes. More formally, the function $\Delta$ will specify the amount of initial mass starting at each node, and the function $T$ will give the sink capacity of each node, i.e. the most amount of mass we allow at a node after the spreading. We denote the density (of mass) at a node $v$ as the ratio of the amount of mass at $v$ over $\deg(v)$, and when we use density without any specific node, we mean the maximum density at any node. Naturally in a diffusion, we start with $\Delta$ having small support and high density, and the goal is to reach a state with bounded density enforced by the sink function. This gives a clean physical interpretation where paint (i.e. mass) spills from the source nodes and spreads over the graph and there is a sink at each node where up to a certain amount of paint can settle\(^1\).

In our work, we will always use the particular sink capacity function where $T(v) = \deg(v)$ for all nodes, i.e. density at most 1. We extend the notation to write $\Delta(S) = \sum_{v \in S} \Delta(v)$ and $T(S) = \sum_{v \in S} T(v)$ for a subset of nodes $S$, and for our particular choice of sink capacity we have $T(S) = \text{vol}(S)$. We also write $\lvert \Delta \rvert = \Delta(V)$ as the total amount of mass we start with, which remains constant throughout the diffusion as flow routing conserves mass.

Given initial mass $\Delta$ and routing $f$, the vector $\vec{m} = B^T f + \Delta$ gives the amount of mass $m(v)$ at each node $v$ after the routing $f$, i.e. $m(v) = \Delta(v) + \sum_u f(u, v)$ is the sum of initial mass and the net amount of mass routed to $v$. We say $f$ is a feasible routing for a diffusion problem when $m(v) \leq T(v)$ for all nodes, i.e. the mass obeys the sink capacity at each node. We say $v$’s sink is saturated if $m(v) \geq T(v)$ and $\text{ex}(v) = \max (m(v) - T(v), 0)$ the excess at $v$. Note there always exists some feasible routing as long as the total amount of mass $\lvert \Delta \rvert$ is at most $\text{vol}(G)$, i.e. there is enough sink capacity over the entire graph to settle all the mass. This will be the case in the context of local clustering, and will we assume this implicitly through our discussion.

The goal of our diffusion problem is to find a feasible routing with low cost. We consider the $p$-norm of a routing $\|f\|_p = (\sum_e f_e^p)^{1/p}$ as its cost. For example, when $p = 2$ we can view the flow as electrical current, then the cost is the square root of the energy of the electrical flow; when $p = \infty$ the cost corresponds to the most congested edge’s congestion of the routing. For $p < \infty$, the cost will naturally encourage the diffusion to send mass to saturate nearby sinks before expanding further, and thus our model inherently looks for local solutions.

For reader familiar with the network flow literature, in the canonical $p$-norm flow problem, we are given the exact amount of mass required at each sink, i.e. the inequality constraint is replaced by equality, and the high level question is how to route mass efficiently from given source(s) to destination(s). In our diffusion problem, as we have the freedom to choose the destination of mass as long as we obey the sink capacities, the essential question becomes where to route the mass so the spreading can be low cost. Despite their similarity and close connection, we believe the distinct challenge of our $p$-norm diffusion problem poses a novel and meaningful direction to the classic problem of network flow.

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\(^1\)There is also a “paint spilling” interpretation for personalized PageRank where instead of sinks holding paint, the paint dries (and settles) at a fixed rate when it pass through nodes. These are two very different mechanisms on how the mass settles.
The Dual Problem. It is straightforward to check problem (2) is the dual of $p$-norm flow diffusion, and strong duality holds for our problems so they have the same optimal value. For the dual problem, we view a solution $x$ as assigning heights to nodes, and the goal is to separate the nodes with source mass (i.e. seed nodes) from the rest of the graph. This is reflected in the objective where we gain by raising nodes with large source mass higher but loss by raising nodes in general. If we consider the height difference between an edge’s two endpoints as the length of an edge, i.e. $|x(u) - x(v)|$, we constrain the separation of nodes with a budget of how much we can stretch the edges. More accurately, the $q$-norm of the vector of edge lengths (i.e. $\|Bx\|_q$) is at most 1. This naturally encourages stretching edges along a bottleneck (i.e. cut with small number of edges crossing it) around the seed nodes, since we can stretch each edge more when the number of edges is smaller (and thus raise seed nodes higher). The dual problem also inherently looks for local solutions as raising nodes far away from the source mass only hurts the objective.

In contrast to random walk based linear operators such as personalized PageRank and heat kernel, even 2-norm diffusion is non-linear due to the combinatorial constraint $B^T f + \Delta \leq T$. More generally, introducing non-linearity has proved to be very successful in machine learning, most notably in the context of deep neural networks. This may offer some intuition why 2-norm diffusion achieves better results empirically comparing to personalized PageRank despite the connection between 2-norm diffusion and spectral methods.\(^2\)

4 Local Graph Clustering

In this section we discuss the optimal solutions of our diffusion problem (and its dual) in the context of local graph clustering. At a high level, we are given a set of seed nodes $S$ and want to find a cut of low conductance close to these nodes. Following prior work in local clustering, we formulate the goal as a promise problem, where we assume the existence of an unknown good cluster $C \subset V$ with $\text{vol}(C) \leq \text{vol}(V)/2$ and conductance $\phi_G(C) = \phi^*$. We consider a generic fixed $G = (V, E)$ and $p \in (1, \infty)$ through our discussion. We keep our discussion at a high level, and defer missing proofs and technical details to the supplementary material.

4.1 Diffusion Setup

To specify a particular diffusion problem and its dual, we need to provide the source mass $\Delta$, and recall we always set the sink capacity $T(v) = \text{deg}(v)$. Given a set of seed nodes $S$, we pick a scalar $\delta$ and let

$$\Delta(v) = \begin{cases} \delta \cdot \text{deg}(v) & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

(3)

Note this gives the total amount of mass $|\Delta| = \delta \cdot \text{vol}(S)$. We will discuss the choice of $\delta$ shortly, but we start with a simple lemma on the locality of the optimal solutions for the primal and dual problems.

Lemma 1. Let $f^*$ and $x^*$ be optimal solutions of (1) and (2) respectively, $\text{supp}(f^*)$ be the subset of edges with non-zero mass crossing them (i.e. the support of vector $f^*$), and $\text{supp}(x^*)$ be the subset of nodes with strictly positive dual value. We have

1. $|\text{supp}(f^*)| \leq \delta \cdot \text{vol}(S)$, and

2. $\text{vol}_G(\text{supp}(x^*)) \leq \delta \cdot \text{vol}(S)$.

3. $x^*(u) > 0$ only if $(B^T f^* + \Delta)(u) = \text{deg}(u)$

\(^2\)We note that algorithms (e.g. [1]) based on random walks nonetheless introduce non-linearity (and also strong locality) to the underlying linear model in the form of approximation or regularization, whereas our model is intrinsically non-linear and strongly local.
Proof sketch. For $p < \infty$, the optimal routing will never push mass out of a node $u$ unless $u$’s sink is saturated, i.e. $f^*(u,v) > 0$ for $u,v$ only if $(BTf^* + \Delta)(u) = \deg(u)$, since otherwise we can reverse the mass routed out of the un-saturated node to reduce the cost of the routing. The total amount of mass $\delta \cdot \text{vol}(S)$ upper-bounds the total volume of the saturated nodes since it takes $\deg(v)$ amount of mass to saturate the node $v$. This observation proves the first claim. The second claim also follows from the same observation and the additional property of complementary slackness, which is exactly the third claim. □

Now we discuss how to set $\delta$. The intuition is that we want the total amount of source mass starting in $C$ to be a constant factor larger than the volume of $C$, say $\Delta(C) \geq 2\text{vol}(C)$ (any constant reasonably larger than 1 would work). The reason is that in such scenario, at least $\Delta(C) - \text{vol}(C) \geq \text{vol}(C)$ amount of mass has to be routed out of $C$ since the nodes in $C$ have total sink capacity $\text{vol}(C)$. When $C$ is a cut of low conductance, any feasible routing must incur a large cost since $\text{vol}(C)$ amount of mass has to get out of $C$ using a relatively small number of discharging edges. In this case, the optimal dual solution $x^*$ will certify the high cost of any feasible primal solution. Naturally, the appropriate value of $\delta$ to get $\Delta(C) \geq 2\text{vol}(C)$ depends on how well the seed set $S$ overlaps with $C$. Suppose $\text{vol}(S \cap C) \geq \alpha\text{vol}(C)$, then we can set $\delta = 2/\alpha$. Without loss of generality, we assume the right value of $\delta$ is known since otherwise we can employ binary search to find a good value of $\delta$.

More formally, we derive a low conductance cut from $x^*$ using the standard sweep cut procedure. In our case, because $x^*$ has bounded support, the procedure can be implemented in $O(\delta \cdot \text{vol}(S))$ total work.

1. Sort the nodes in decreasing order by their values in $x^*$.
2. For each $i \geq 1$ such that the $i$-th node still has strictly positive dual value, consider the cut containing the first $i$ nodes. Among all these cuts (also called level cuts) output the one with the smallest conductance.

Figure 1: The Sweep Cut Procedure.

4.2 Local Clustering Guarantee

**Theorem 2.** Given a set of seed nodes $S$, suppose there exists a cluster $C$ such that

1. $\text{vol}(S \cap C) \geq \alpha\text{vol}(C)$ for some $\alpha \in (0,1]$,
2. $\text{vol}(S \cap C) \geq \beta\text{vol}(S)$ for some $\beta \in (0,1]$

Then the cut $\tilde{C}$ returned by the sweep cut procedure on the optimal dual solution $x^*$ satisfies

$$\phi(\tilde{C}) \leq O \left( \frac{\phi(C)^{1/q}}{\alpha \beta} \right)$$

where $q \in (1, \infty)$ is the norm used in (2)

Note the sweep cut computation only requires the dual solution $x^*$, while the primal solution $f^*$ and the values of $\alpha, \beta$ are only for analysis. Recall we want to set $\delta = 2/\alpha$ in (3) to formulate the dual problem, but we assume $\delta$ is known via binary search. We also assume the entire graph is larger than the total amount of source mass so the primal is feasible and the dual is bounded. As summarized below,

**Assumption 1.** The source mass function $\Delta$ in our problem formulation as specified in (3) satisfies $\delta = 2/\alpha$, which gives $\Delta(C) \geq 2\text{vol}(C)$ and $|\Delta| = \Delta(S) \leq 2\text{vol}(C)/\beta < \text{vol}(G)$. 

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Not surprisingly, our theorem is more meaningful when the given seed set \( S \) has a good overlap with some low conductance cut \( C \), i.e. when \( \alpha, \beta \) are bounded away from 0. In particular, suppose \( \alpha, \beta \) are both at least \( \frac{1}{\log(\text{vol}(C))} \) for some constant \( t \), then the bound in our theorem becomes \( \tilde{O}(\phi(C)^{1/q}) \), where we follow the tradition of using \( \tilde{O} \) to hide poly-logarithmic factors. In particular, for 2-norm diffusion (i.e. \( p = q = 2 \)) this matches the bound achieved by spectral and random walk based methods in this setting, and for \( p \)-norm diffusion with \( p \) approaches \( \infty \) (i.e. \( q \) tends to 1), this matches the guarantees of previous flow based methods in this setting, e.g. [34, 26]. We prove Theorem 2 in the rest of the section. We will keep the discussion at high-level and defer details to the supplementary material.

We start with the simple lemma stating that the objective value of the optimal dual (and primal) solution must be large.

Lemma 3. Suppose \( k = |E(C, V \setminus C)| \) is the cut-size of \( C \), then
\[
(\Delta - \bar{d})^T x^* = \|f^*\|_p \geq \text{vol}(C)/k^{1/q}
\]

Proof sketch. This directly follows from Assumption 1 that at least 2\( \text{vol}(C) \) amount of source mass is trapped in \( C \) at the beginning, so \( \text{vol}(C) \) amount of excess needs to get out of \( C \) using the \( k \) cut edges, and we focus on the cost of \( f^* \) restricted to these edges alone. Since \( p > 1 \), the cost is the smallest if we distribute the routing load evenly on the \( k \) edges, and it is simple to see this incurs cost \( \text{vol}(C)/k^{(p-1)/p} \). The total cost of \( f^* \) must be at least the cost incurred just by routing the excess out of \( C \). \( \square \)

Recall that we define the length of an edge \( e = (u, v) \) to be \( l(e) = |x^*(u) - x^*(v)| \). The actual dual solution may incur edges with tiny non-zero length which causes difficulties in the analysis. Thus, we define the following perturbed edge length so that any non-zero edge length is at least 1/\( \text{vol}(C)^{1/q} \). Note this is only for analysis purpose and doesn’t require changing \( x^* \).

\[
\text{l}(e) = \begin{cases} 
\max \left( \frac{1}{\text{vol}(C)^{1/q}}, l(e) \right) & \text{if } l(e) > 0 \\
0 & \text{otherwise}
\end{cases}
\] (4)

Note the constraint in the dual problem gives
\[
\sum_{e=(u,v)} |x^*(u) - x^*(v)| \cdot l(e)^{q-1} \\
= \sum_{e=(u,v)} |x^*(u) - x^*(v)|^q = \|Bx^*\|_q \leq 1
\]

The next lemma states that the perturbations on edges lengths are small enough so the above quantity remains small.

Lemma 4. \( \sum_{e=(u,v)} |x^*(u) - x^*(v)| \cdot \text{l}(e)^{q-1} \leq 1 + \frac{2}{\beta} \)

Proof sketch. This follows from Assumption 1 that the total amount of mass is at most \( \frac{\beta}{\text{vol}(C)} \), which also upper-bounds \( \text{vol}(\text{supp}(x^*)) \) by Lemma 1. Thus, the number of edges with positive \( l(e) \) cannot be too large, and our perturbation only increases the lengths for these edges by at most \( \frac{1}{\text{vol}(C)^{1/q}} \). The lemma follows from these two facts. \( \square \)

Consider the sweep cut procedure where we order the nodes by their dual values in \( x^* \), and for any \( h > 0 \) denote the cut \( S_h = \{ u | x^*(u) \geq h \} \) to be the set of nodes with dual value larger than \( h \). We only need to consider \( S_h \) when \( h \) equals to the strictly positive dual value of some node in the support of \( x^* \), and the sweep cut procedure will examine all such cuts. We proceed to argue that among these level cuts, there must exists some \( h \) where \( \phi(S_h) \) satisfies the bound in Theorem 2, and thus prove the main theorem.

We start with rewriting the dual objective and constraint using the level cuts.
Claim 1. With level cuts $S_h$ as defined above, we have

$$\int_{h=0}^{\infty} (\Delta(S_h) - \text{vol}(S_h)) \, dh \geq \text{vol}(C)/k^{1/q}$$

and

$$\int_{h=0}^{\infty} \sum_{e \in E(S_h \setminus V \setminus S_h)} \tilde{l}(e)^{q-1} \, dh \leq 1 + \frac{2}{\beta}$$

Proof Sketch. Both claims follow from changing the order of summation to get

$$(\Delta - \vec{d})^T x^* = \int_{h=0}^{\infty} (\Delta(S_h) - \text{vol}(S_h)) \, dh,$$

and

$$\sum_{e=(u,v)} |x^*(u) - x^*(v)|^{q-1} \cdot \tilde{l}(e)$$

$$= \int_{h=0}^{\infty} \sum_{e \in E(S_h \setminus V \setminus S_h)} \tilde{l}(e)^{q-1} \, dh,$$

and then invoke Lemma 3 and Lemma 4 respectively. To see the change of summation, pick any node $v$ in the first equation. $v$ contributes $(\Delta(v) - \deg(v)) \cdot x^*(v)$ to the left hand side, and the same amount to the right hand side as $v$ is in the level cuts for all $h \in (0, x^*(v)]$. The second equation follows similar reasoning. □

Using Claim 1, we take the ratio to get

$$\frac{\int_{h=0}^{\infty} \sum_{e \in E(S_h \setminus V \setminus S_h)} \tilde{l}(e)^{q-1} \, dh}{\int_{h=0}^{\infty} (\Delta(S_h) - \text{vol}(S_h)) \, dh} \leq \frac{3k^{1/q}}{\beta \text{vol}(C)}$$

which means there must exist some $h$ with $S_h$ non-empty and

$$\frac{\sum_{e \in E(S_h \setminus V \setminus S_h)} \tilde{l}(e)^{q-1}}{\Delta(S_h) - \text{vol}(S_h)} \leq \frac{3k^{1/q}}{\beta \text{vol}(C)}$$

(5)

All it remains is to connect the left hand side in the above inequality to the conductance of $S_h$. For the denominator, since the source mass has density at most $\delta = 2/\alpha$ at any node, we get

$$\Delta(S_h) - \text{vol}(S_h) \leq \frac{2}{\alpha} \text{vol}(S_h)$$

(6)

For the numerator, any edge $e = (u,v)$ crossing a level cut $S_h$ must have dual values $x^*(u), x^*(v)$ on different sides of $h$, thus having non-zero length $l(e)$, which means $l(e)$ is at least $1/\text{vol}(C)^{1/q}$. This gives

$$\sum_{e \in E(S_h \setminus V \setminus S_h)} \tilde{l}(e)^{q-1} \geq \frac{|E(S_h, V \setminus S_h)|}{\text{vol}(C)^{(q-1)/q}}$$

(7)

Put (5), (6) and (7) together we get

$$\phi(S_h) = \frac{|E(S_h, V \setminus S_h)|}{\text{vol}(S_h)} \leq \frac{6k^{1/q}}{\alpha \beta \text{vol}(C)^{1/q}} = \frac{6\phi(C)^{1/q}}{\alpha \beta}$$

which proves our main theorem.
Algorithm 1 Coordinate solver for smoothed dual problem

\begin{algorithmic}
\STATE \textbf{Initialize}: $x_0 = 0$
\FOR{$k = 0, 1, 2, \ldots$}, do
\STATE Set $S_k = \{i \in V \mid \nabla f_{\mu}(x_k) < 0\}$.
\STATE Pick $i_k \in S_k$ uniformly at random.
\STATE Update $x_{k+1} = x_k - \frac{\mu^2-q}{\deg(i_k)}\nabla i_k f_{\mu}(x_k)e_{i_k}$.
\IF{$S_k = \emptyset$} then return $x_k$.
\END{algorithmic}

5 Strongly Local Algorithm

For general constrained convex optimization problem, the state of the art solvers are by interior point techniques. However, these methods start with a fully dense initial point and then iteratively solve a linear system to obtain the update direction. Such paradigm already becomes prohibitive when the input size is moderately large. For example, the iterates of CVX [13] fail to converge when solving the dual problem (2) on a $60 \times 60$ grid graph with initial mass $|\Delta| = 12000$ and $p = 4$.

As noted in Lemma 1, the support of optimal primal and dual solutions are bounded by $|\Delta|$. Hence, we propose a randomized coordinate descent variant that solves the equivalent regularized formulation of (2) (formal argument on equivalence in the supplementary material):

$$
\min_{x \geq 0} f(x) := \frac{1}{q}\|Bx\|_q^q - x^T(\Delta - \tilde{d}),
$$

(8)

Our algorithm is strongly local, that is, its running time depends on $|\Delta|$ rather than the size of the whole graph.

Because the function $f(x)$ in the objective of (8) has non-Lipschitz gradient for any $q < 2$, directly minimizing $f(x)$ results in exponential iteration complexity since step-sizes need to converge to zero. To cope with this, we further smooth $f(x)$ by replacing $\|Bx\|_q^q$ with $\|Bx\|_{\mu,q}^q$ where

$$
\|y\|_{\mu,q}^q := \sum_{i \in \text{supp}(\tilde{y})} (y_i^q + \mu^2)^{q/2} + \sum_{i \notin \text{supp}(\tilde{y})} |y_i|^q,
$$

where $\tilde{y}$ is a fixed vector. Since $1 < q < 2$, the function $h(s) = s^{q/2}$ is concave on $\mathbb{R}_{\geq 0}$, and thus $\|y\|_{\mu,q}^q \leq \|y\|_{\mu,q}^q \leq \|y\|_q + \|\text{supp}(\tilde{y})\|\mu^q$. Let $f_{\mu}(x) := \|Bx\|_{\mu,q}^q - x^T(\Delta - d)$, then

$$
f(x) \leq f_{\mu}(x) \leq f(x) + \mu^q|\text{supp}(B\tilde{x})|/q,
$$

(9)

where $\tilde{x}$ is the fixed vector, which is carefully chosen such that the proposed algorithm has local running time. Furthermore, $\nabla f_{\mu}(x)$ is Lipschitz continuous on $\text{supp}(\tilde{x})$. Details are provided in the supplementary material.

The proposed numerical scheme, which we layout the basic steps in Algorithm 1, solves the smoothed problem

$$
\min_{x \geq 0} f_{\mu}(x).
$$

(10)

In the context of $p$-norm flow diffusion, coordinate method enjoys a natural combinatorial interpretation\(^3\): each coordinate (node) update corresponds to routing mass through incident edges and distributing mass across its neighbours. This combinatorial characteristic of coordinate method distinguishes it from other gradient type methods and ensures the solution path is strongly local. We state the following lemma which guarantees the monotonicity of our algorithm. The claim follows a straightforward computation of the gradient and Lipschitz continuity, and we include the detail in the supplementary material.

\(^3\)From primal-dual optimality condition, for any node $v$, the coordinate gradient $\nabla_v f$ measures the difference between the current mass at node $v$ and its sink capacity $\deg(v)$, i.e., $-\nabla_v f(x) = m(v) - \deg(v)$. Hence $S_k$ can be interpreted as the set of nodes that currently have positive excess (cf. Section 3).
Lemma 5 (Monotonicity). For any iteration \( k \) and node \( i \) in Algorithm 1, \( \nabla_i f_\mu(x_k) \leq 0 \), \( \forall i \in \text{supp}(x_k) \).

The above lemma guarantees that nodes with positive gradient all have value 0 in \( x \) (i.e. not in the support), and cannot be decreased due to the \( x \geq 0 \) constraints. Thus, \( x_k \leq x_{k+1} \) for all \( k \) and \( \text{supp}(x_0) \subseteq \text{supp}(x_1) \subseteq \ldots \subseteq \text{supp}(x^*_\mu) \).

Note that \( f_\mu(x) \) is not strongly convex in general, but locality gives us strong convexity: when restricting \( f_\mu \) to the iterates generated by Algorithm 1, \( f_\mu \) has a strong convexity parameter \( \gamma \) which is the minimum eigenvalue of the sub-matrix of the Laplacian defined by the nodes in \( \text{supp}(x^*_\mu) \) [8], multiplied by a positive weight constant, and satisfies

\[
\gamma \geq \frac{1}{|\Delta|^2} \left( \frac{(q-1)|\Delta|^{2p-2} - \mu^2}{(|\Delta|^{2p-2} + \mu^2)^2 - q^2} \right)^{1/2} \tag{11}
\]

where \( p \) is such that \( 1/p + 1/q = 1 \). The result is pessimistic because we do not make any assumption about the internal conductance of the target cluster. Lower bounding \( \gamma \) under stronger assumptions are beyond the scope of this paper. In practice, we observe much better performance, because most clusters are well connected internally.

Locality, Lipschitz continuity, strong convexity, and error bound (9) give us linear convergence rate to the dual problem (8) and the following running time guarantee.

Theorem 6 (Running time). If we pick \( \mu = \mathcal{O}(\frac{1}{|\Delta|^2}) \), the total running time of Algorithm 1 to obtain an \( \epsilon \) accurate solution of (8) is \( \mathcal{O}(\frac{d_{\text{max}}^2}{\gamma} (|\Delta|)^{2/q - 1} \log \frac{1}{\epsilon}) \), where \( d_{\text{max}} = \max_{i \in \text{supp}(x^*_\mu)} \deg(i) \), and \( \gamma \) is the strong convexity parameter that satisfies (11).

Finally, we remark that Algorithm 1 is easily parallelizable, as all coordinates in \( S_k \) can be updated at the same time, without sacrificing locality.

6 Empirical Results

The goal of our experiments is two-fold. First, we carry out experiments on various LFR synthetic graphs [19] and demonstrate that the theoretical guarantees of \( p \)-norm flow diffusion are well reflected in practice. Second, we show the advantage of \( p \)-norm diffusion for local graph clustering tasks on four real datasets. We compare the performance of \( p \)-norm flow diffusion with \( \ell_1 \)-regularized PageRank [11] and nonlinear diffusion under power transfer [15]. Although our theoretical analysis holds for \( p \in (1, \infty) \) and Algorithm 1 converges linearly for any \( p \geq 2 \), we think in practice the most interesting regime is when \( p \) is a small constant, e.g. \( p \in [2, 8] \), as our theory suggests that the marginal gain in terms of conductance guarantee diminishes as \( p \) grows large. We elaborate this in the synthetic experiment by comparing marginal improvements in the performance by raising \( p \) from 2 to 4, and from 4 to 8.

The LFR model [19] is a widely used benchmark for evaluating community detection algorithms. It is essentially a stochastic block model with the additional property that nodes’ degrees follow the power law distribution, and there is a parameter \( \mu \) controlling what fraction of a node’s neighbours is outside the node’s block. Our theory (and also experiment on LFR graph in the supplementary material) indicates (not surprisingly) that better overlap of the input seed set and a target cluster will result in output cluster having better conductance and F1 measure. In all subsequent experiments, however, when we compare recovery results with \( \ell_1 \)-regularized PageRank and nonlinear power diffusion, we always start the diffusion process from one seed node, as this is the most common practice for semi-supervised local clustering tasks. We set the parameter for \( p \)-norm diffusion so \( |\Delta| \) is a constant factor of the volume of some target cluster (recall from Assumption 1 this is WLOG). We use the same parameter setting of nonlinear power diffusion as what the authors suggested [15]. For \( \ell_1 \)-regularized PageRank, we allow it to “cheat” in the sense that we use ground truth to choose the parameter giving the best conductance result.
We compare these methods on 5 datasets. First on LFR synthetic graphs with different mixing parameter $\mu$. We pick $\mu$ between 0.1 and 0.4 as this range gives rise to graphs that contain reasonable noisy (but not completely noise) clusters. For each graph, we start from a random seed node and we repeat the experiment 100 times. Figure 2 shows the mean and the variance for the conductance and the F1 measure while varying $\mu$. Notice that $p$-norm diffusion behaves similarly to the tuned $\ell_1$-regularized PageRank in both conductance and F1 measure when $p = 2$, whereas it significantly outperforms other methods when $p = 4$ and $p = 8$. We observe a slight gain in terms of conductance by raising $p = 4$ to $p = 8$, but such improvement is marginal. This is not really surprising, since qualitatively the 4-norm unit ball is already very close to the $\infty$-norm unit ball (i.e. the box).

The other four datasets are real-world graphs: the Facebook college graphs of John Hopkins (FB-Johns55) and Colgate (Colgate88) [30]), the social network Orkut [36]) and the biological network Sfld [5]. For each dataset, we filter the ground truth clusters by setting reasonable thresholds on volume, conductance, and the ratio between its internal connectivity and cut conductance. We include the statistics of the ground truth clusters in the supplementary material. For each dataset, we run algorithms starting at each vertex in each cluster and report the average conductance and F1 measure. We omit nonlinear power diffusion for Orkut, as it does not scale well to reasonably large graphs.

The Facebook datasets contain ground truth clusters ranging from low to medium conductance. For almost all clusters, $p$-norm flow diffusion methods have both the best F1 measure and conductance result. The six clusters in the biological dataset Sfld are very noisy, having median conductance around 0.8. The performance of the algorithms is mixed. Observe that nonlinear power diffusion yields the lowest conductance but also the lowest F1 measure at the same time in half of all cases. On the other hand, $p = 4$ and $\ell_1$-regularized PageRank seem to be the most robust. On Orkut dataset, 4-norm diffusion gives best result on all clusters. We defer more details to the supplementary material.
Table 1: Results for real-world graphs

| dataset       | feature       | measure | \( p = 2 \) | \( p = 4 \) | \( \ell_1 \)-reg. pr | nonlinear |
|---------------|---------------|---------|-------------|-------------|---------------------|-----------|
| FB-Johns55    | year 2006     | f1      | 0.36        | 0.35        | 0.31                | 0.31      |
|               |               | cond    | 0.34        | 0.34        | 0.50                | 0.40      |
|               | year 2007     | f1      | 0.39        | 0.59        | 0.38                | 0.36      |
|               |               | cond    | 0.31        | 0.30        | 0.45                | 0.41      |
|               | year 2008     | f1      | 0.51        | 0.51        | 0.51                | 0.37      |
|               |               | cond    | 0.34        | 0.34        | 0.44                | 0.41      |
|               | year 2009     | f1      | 0.84        | 0.85        | 0.83                | 0.49      |
|               |               | cond    | 0.23        | 0.22        | 0.24                | 0.40      |
| major index   | 217           | f1      | 0.85        | 0.87        | 0.83                | 0.75      |
|               |               | cond    | 0.23        | 0.22        | 0.25                | 0.29      |
| Colgate88     | year 2004     | f1      | 0.50        | 0.51        | 0.43                | 0.25      |
|               |               | cond    | 0.66        | 0.66        | 0.71                | 0.36      |
|               | year 2005     | f1      | 0.35        | 0.45        | 0.41                | 0.37      |
|               |               | cond    | 0.51        | 0.51        | 0.53                | 0.37      |
|               | year 2006     | f1      | 0.35        | 0.45        | 0.45                | 0.39      |
|               |               | cond    | 0.37        | 0.36        | 0.50                | 0.38      |
|               | year 2007     | f1      | 0.49        | 0.49        | 0.51                | 0.45      |
|               |               | cond    | 0.34        | 0.34        | 0.45                | 0.39      |
|               | year 2008     | f1      | 0.76        | 0.80        | 0.71                | 0.55      |
|               |               | cond    | 0.31        | 0.30        | 0.35                | 0.40      |
|               | year 2009     | f1      | 0.96        | 0.97        | 0.96                | 0.82      |
|               |               | cond    | 0.13        | 0.12        | 0.13                | 0.24      |
| Sfold         | urease        | f1      | 0.02        | 0.75        | 0.68                | 0.69      |
|               |               | cond    | 0.95        | 0.45        | 0.41                | 0.46      |
|               | AMP           | f1      | 0.85        | 0.83        | 0.83                | 0.84      |
|               |               | cond    | 0.39        | 0.38        | 0.38                | 0.39      |
|               | phosphotriesterase | f1 | 1.00 | 1.00 | 1.00 | 0.13 |
|               | adenosine     | f1      | 0.41        | 0.44        | 0.45                | 0.34      |
|               |               | cond    | 0.86        | 0.42        | 0.43                | 0.41      |
|               | dihydroorotase3 | f1 | 0.49 | 0.96 | 0.96 | 0.07 |
|               |               | cond    | 0.93        | 0.82        | 0.82                | 0.40      |
|               | dihydroorotase2 | f1 | 0.13 | 0.39 | 0.20 | 0.18 |
|               |               | cond    | 0.88        | 0.78        | 0.85                | 0.46      |
| Orient        | A              | f1      | 0.56        | 0.58        | 0.49                |           |
|               |               | cond    | 0.48        | 0.47        | 0.51                |           |
|               | B              | f1      | 0.71        | 0.73        | 0.66                |           |
|               |               | cond    | 0.35        | 0.33        | 0.37                |           |
|               | C              | f1      | 0.63        | 0.64        | 0.57                |           |
|               |               | cond    | 0.33        | 0.32        | 0.35                |           |
|               | D              | f1      | 0.73        | 0.76        | 0.72                |           |
|               |               | cond    | 0.49        | 0.48        | 0.51                |           |
|               | E              | f1      | 0.61        | 0.62        | 0.56                |           |
|               |               | cond    | 0.52        | 0.51        | 0.54                |           |
|               | F              | f1      | 0.79        | 0.81        | 0.76                |           |
|               |               | cond    | 0.52        | 0.51        | 0.54                |           |
|               | G              | f1      | 0.72        | 0.73        | 0.68                |           |
|               |               | cond    | 0.52        | 0.50        | 0.53                |           |
|               | H              | f1      | 0.68        | 0.70        | 0.67                |           |
|               |               | cond    | 0.52        | 0.51        | 0.54                |           |
|               | I              | f1      | 0.60        | 0.62        | 0.56                |           |
|               |               | cond    | 0.49        | 0.48        | 0.52                |           |
|               | J              | f1      | 0.52        | 0.54        | 0.47                |           |
|               |               | cond    | 0.53        | 0.52        | 0.56                |           |
|               | K              | f1      | 0.54        | 0.56        | 0.51                |           |
|               |               | cond    | 0.57        | 0.56        | 0.60                |           |
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Appendices

A Missing Proofs of Section 4

Proof of Lemma 1. For $p < \infty$, the optimal routing will never push mass out of a node $u$ unless $u$’s sink is saturated, i.e. $f^*(u,v) > 0$ for $u,v$ only if $(B^T f^* + \Delta)(u) = \deg(u)$. To see this, take the optimal primal solution $f^*$, and consider the decomposition of $f^*$ into flow paths, i.e., the path that the diffusion solution used to send each unit of mass from its source node to the sink at which it settled. If any node other than the last node on this path has remaining sink capacity, we can truncate the path at that node, and strictly reduce the total cost of the diffusion solution. As each unit of mass is associated with the sink of one node, the total amount of mass $\delta \cdot \vol(S)$ upper-bounds the total volume of the saturated nodes since it takes $\deg(v)$ amount of mass to saturate the node $v$. This observation proves the first claim.

The dual variable $x^*(u)$ corresponds to the primal constraint $(B^T f^* + \Delta)(u) \leq \deg(u)$, and it is easy to see the third claim of the lemma is just complementary slackness. The second claim follows from the first and the third claim. □

Proof of Lemma 3. By Assumption 1 there is least $\Delta(C) \geq 2\vol(C)$ amount of source mass trapped in $C$ at the beginning. Since the sinks of nodes in $C$ can settle $\vol(C)$ amount of mass, the remaining at least $\vol(C)$ amount of excess needs to get out of $C$ using the $k$ cut edges. We focus on the cost of $f^*$ restricted to these edges alone. Since $p > 1$, the cost is the smallest if we distribute the routing load evenly on the $k$ edges, and it is simple to see this incurs cost

$$\left(k \cdot \left(\frac{\vol(C)}{k}\right)^p\right)^{1/p} = \frac{\vol(C)}{k^{(p-1)/p}}.$$ 

The total cost of $f^*$ must be at least the cost incurred just by routing the excess out of $C$. □

Proof of Lemma 4.

$$\sum_{e=(u,v)} |x^*(u) - x^*(v)| \cdot \tilde{l}(e)^{q-1} \leq \sum_{e=(u,v)} \sum_{e \in l(e) = \tilde{l}(e)} l(e)^q + \sum_{e \in l(e) < \tilde{l}(e)} \frac{1}{\vol(C)} \leq 1 + \frac{2}{\beta}$$

The second to last equality follows from the fact that our perturbation only increases the lengths to $\frac{1}{\vol(C)}^{1/q}$. The last inequality follows from that we only increase the length of an edge when its original edge is positive, which means at at least one of its endpoints has positive dual variable value. From Assumption 1 that $\delta = 2/\alpha$ we know that the total amount of mass is at most $\frac{2}{\alpha} \vol(S)$. Together with the conditions in Theorem 2 we get $\frac{2}{\alpha} \vol(S) \leq \frac{2}{\beta} \vol(C)$. This upper-bounds $\vol(C)(\supp(x^*))$ by Lemma 1. Thus, the number of edges with positive $l(e)$ is also at most $\frac{2}{\beta} \vol(C)$. □

Proof of Claim 1. Both claims follow from changing the order of summation to get

$$(\Delta - \bar{d})^T x^* = \int_{h=1}^{\infty} (\Delta(S_h) - \vol(S_h)) \, dh,$$

and

$$\sum_{e=(u,v)} |x^*(u) - x^*(v)| \cdot \tilde{l}(e) = \int_{h=1}^{\infty} \sum_{e \in (S_h, V \setminus S_h)} \tilde{l}(e)^{q-1} \, dh,$$

and then invoke Lemma 3 and Lemma 4 respectively. To see first claim, pick any node $v$ in the first equation. $v$ contributes $(\Delta(v) - \deg(v)) \cdot x^*(v)$ to the left hand side, and the same amount to the right hand side as $v$ is in the level cuts for all $h \in (0, x^*(v)]$.

For the second claim, pick any edge $e = (u,v)$, the edge will cross the level cuts $E(S_h, V \setminus S_h)$ for all $h \in (x^*(v), x^*(u)]$ (assuming wlog $x^*(u) > x^*(v)$), so the contribution from any edge will be the same to both sides of the equation. □
B Constrained and penalized problems from Section 5

We are interested in solving the $q$-norm cut problem

$$
\begin{align*}
\max & \ (\Delta - \dd)^T x \\
\text{s.t.} & \quad \|Bx\|_q \leq 1 \\
& \quad x \geq 0.
\end{align*}
$$

(B.1)

We turn the constrained formulation into a regularized problem and show their equivalence,

$$
\begin{align*}
\min_{x \geq 0} \ f(x) := (\dd - \Delta)^T x + \frac{1}{q} \|Bx\|_q^q.
\end{align*}
$$

(B.2)

Lemma 7 (Equivalence). The solution sets of problems (B.1) and (B.1) are scaled versions of each other.

Proof. First note that constant non-zero vectors cannot be solutions to any of the problems (B.1) and (B.2). This is because we pick $\Delta$ such that $\sum_i \Delta(i) \leq \sum_i \deg(i)$ and an all-zero vector is feasible and has better objective function value than any constant non-zero vector. Let $x^*$ denote a non-constant solution of (B.2). Then $x^*$ satisfies the optimality conditions of (B.2)

\begin{itemize}
  \item $- (\Delta - \dd) - y + \frac{1}{q} \nabla \|Bx\|_q^q = 0$
  \item $y(i)x(i) = 0 \ \forall i \in V$
  \item $x, y \geq 0$,
\end{itemize}

for some optimal dual variables $y^*$. The optimality conditions of problem (B.1) are

\begin{itemize}
  \item $- (\Delta - \dd) - y + \lambda \nabla \|Bx\|_q^q = 0$
  \item $y(i)x(i) = 0 \ \forall i \in V$
  \item $\lambda(1 - \|Bx\|_q^q) = 0$
  \item $\|Bx\|_q^q \leq 1$
  \item $x, y \geq 0, \ \lambda \geq 0$,
\end{itemize}

where $y$ are the dual variables for the constraint $x \geq 0$, and $\lambda$ is the dual variable for $\|Bx\|_q^q \leq 1$. Observe that by setting $x := x^*/\|Bx^*\|_q^q$, $\lambda := \frac{1}{q}(\|Bx^*\|_q^q)^{q-1}$ and $y := y^*$, then the triplet $x, \lambda, y$ satisfies the latter optimality conditions.

Let us now prove the reverse. Note that because we pick $\Delta$ such that there exists at least one negative component in $-(\Delta - \dd)$, then $\lambda = 0$ can never be an optimal dual variable. Thus $\lambda > 0$. Let $\hat{x}, \hat{\lambda}, \hat{y}$ be a solution to the latter optimality conditions. Observe that $x := (q\lambda)^{\frac{1}{q-1}} \hat{x}$ and $y := \hat{y}$ satisfy the former optimality conditions. This means that the solution sets of the two problems are scaled versions of each other.

Lemma 7 is important because the output of the Sweep Cut procedure is equal for both solutions. This is because the output of Sweep Cut depends only on the ordering of dual variables and not on their magnitude. Therefore, we can use the solution of any of these problems for the local graph clustering problem.

Following Lemma 7, it is easy to see that if one of (B.1) or (B.2) is unbounded, the other must also be unbounded. Moreover, one can easily verify that the $q$-norm regularized problem (B.2) is the dual of an equivalent $p$-norm flow problem

$$
\begin{align*}
\min & \ \frac{1}{p} \|f\|_p^p \\
\text{s.t.} & \quad B^T f + \Delta \leq \dd
\end{align*}
$$

(B.3)
and that strong duality holds, as any $x > 0$ is a Slater point for (B.2). Throughout the discussions in this section, we assume that $|\Delta| \leq \text{vol}(G)$, so (B.3) is feasible, and hence (B.2) is bounded.

Although the two formulations (B.1) and (B.2) are inherently equivalent, the computational advantage of (B.2) permits the use of many off the shelf first order optimization methods, which is crucial for obtaining strongly local algorithms. In all subsequent discussions in the appendix, our goal is prove the running time guarantee of Algorithm 1 in the main text that finds $\epsilon$ accurate solution to (B.2). For convenience we have copied the algorithmic steps here, in Algorithm B.1.

\begin{algorithm}
\caption{Coordinate solver for (C.2)}
\begin{algorithmic}
\State Initialize: $x_0 = 0$
\For{$k = 0, 1, 2, \ldots$}
\State Set $S_k = \{i \in V \mid \nabla_i f_\mu(x_k) < 0\}$.
\State Pick $i_k \in S_k$ uniformly at random.
\State Update $x_{k+1} = x_k - \frac{\mu^{2-q}}{\deg(i_k)} \nabla_i f_\mu(x_k) e_{i_k}$.
\If{$S_k = \emptyset$} then return $x_k$.
\EndFor
\end{algorithmic}
\end{algorithm}

\section{Smoothed objective function from Section 5}

In this section we smooth the original objective function and we prove some important properties of the smoothed problem that will be used to obtain a local running time result for Algorithm B.1.

The objective of (B.2) has non-Lipschitz gradient for any $q < 2$, therefore we smooth it by perturbing the $q$-norm term around zero. Consider the following globally smoothed problem

\begin{equation}
\min_{x \geq 0} \hat{f}_\mu(x) := \frac{1}{q} \sum_{(i,j) \in E} ((x(i) - x(j))^2 + \mu^2)^{q/2} - x^T(\Delta - \vec{d}), \tag{C.1}
\end{equation}

where $\mu > 0$ is a smoothing parameter. Let $x^*_\mu$ denote the optimal solution of (C.1). We further localize the global smoothing effort to only the edges defined by $\text{supp}(Bx^*_\mu)$, and obtain a locally smoothed problem

\begin{equation}
\min_{x \geq 0} f_\mu(x) := \frac{1}{q} \|Bx\|^q_{\mu,q} - x^T(\Delta - \vec{d}), \tag{C.2}
\end{equation}

where

$$
\|Bx\|^q_{\mu,q} := \sum_{(i,j) \in \text{supp}(Bx^*_\mu)} ((x(i) - x(j))^2 + \mu^2)^{q/2} + \sum_{(i,j) \notin \text{supp}(Bx^*_\mu)} |x(i) - x(j)|^q.
$$

In the following lemma we show that the cardinality of $\text{supp}(Bx^*_\mu)$ is bounded by $|\Delta|$, which means that the optimal solution of the smoothed problem is sparse.

\begin{lemma}[Locality]
The number of edges defined by $\text{supp}(Bx^*_\mu)$ is bounded by the total amount of initial mass, i.e.,

$$
|\text{supp}(Bx^*_\mu)| < |\Delta|.
$$
\end{lemma}

\begin{proof}
By the first order optimality condition of (C.1),

$$
\nabla_i \hat{f}_\mu(x) = \sum_{j \sim i} ((x(i) - x(j))^2 + \mu^2)^{q/2-1}(x^*_\mu(i) - x^*_\mu(j)) - \Delta_i + \deg(i) = 0, \quad \forall i \in \text{supp}(x^*_\mu).
$$
\end{proof}
Hence,

\[ |\text{supp}(Bx^*_\mu)| \leq \text{vol}_G(\text{supp}(x^*_\mu)) \]

\[ = \sum_{i \in \text{supp}(x^*_\mu)} \deg(i) \]

\[ = \sum_{i \in \text{supp}(x^*_\mu)} \Delta(i) - \sum_{i \in \text{supp}(x^*_\mu)} \sum_{j \sim i} ((x^*_\mu(i) - x^*_\mu(j))^2 + \mu^2)^{q/2-1}(x^*_\mu(i) - x^*_\mu(j)) \]

\[ = \sum_{i \in \text{supp}(x^*_\mu)} \Delta(i) - \sum_{i,j \in \text{supp}(x^*_\mu)} ((x^*_\mu(i) - x^*_\mu(j))^2 + \mu^2)^{q/2-1}(x^*_\mu(i) - x^*_\mu(j)) \]

\[ \leq |\Delta|. \]

The following lemma establishes lower and upper bounds for the smoothed objective function with respect to the original objective function.

**Lemma 9 (Smooth approximation).** For any \( x \), we have that

\[ f(x) \leq f_\mu(x) \leq f(x) + \frac{1}{q} \mu^q|\Delta|. \]

**Proof.** Since \( q \in (1, 2] \), the function \( h: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) given by \( h(s) = s^{q/2} \) is concave on \( \mathbb{R}_{\geq 0} \) and \( h(0) = 0 \). So \( h \) is sub-additive. Thus

\[ \|Bx\|_q^q = \sum_{(i,j) \in E} ((x(i) - x(j))^2)^{q/2} \]

\[ \leq \sum_{(i,j) \in \text{supp}(Bx^*_\mu)} ((x(i) - x(j))^2 + \mu^2)^{q/2} + \sum_{(i,j) \notin \text{supp}(Bx^*_\mu)} ((x(i) - x(j))^2)^{q/2} \]

\[ = \|Bx\|_{\mu,q}^q \]

\[ \leq \sum_{(i,j) \in \text{supp}(Bx^*_\mu)} ((x(i) - x(j))^2)^{q/2} + \sum_{(i,j) \in \text{supp}(Bx^*_\mu)} (\mu^2)^{q/2} \]

\[ + \sum_{(i,j) \notin \text{supp}(Bx^*_\mu)} ((x(i) - x(j))^2)^{q/2} \]

\[ = \|Bx\|_q^q + \mu^q|\text{supp}(Bx^*_\mu)|, \]

and the result follows by replacing \( |\text{supp}(Bx^*_\mu)| \) with \(|\Delta| \) (cf. Lemma 8). \( \square \)

The approximation bound in Lemma 9 means that any solution of the locally smoothed problem (C.2) can be easily turned into an \( \epsilon \) accurate solution of the regularized \( q \)-norm cut problem (B.2), by setting out \( \mu \) appropriately. Therefore, we focus on solving (C.2) next.

A key construct in our analysis is the globally smoothed problem (C.1) and its relation to the local version (C.2). In particular, we show that the two problems (C.1) and (C.2) have identical optimum, and Algorithm B.1 solves both at the same time, by maintaining strong locality and touching only the nodes in \( \text{supp}(x^*_\mu) \). This means that in practice we never require prior knowledge of \( \text{supp}(x^*_\mu) \) in order to solve (C.2), as minimizing (C.1) is already sufficient for obtaining the optimal solution of (C.2). Hence, when designing and analyzing Algorithm B.1, we
may use, without loss of generality, \(f_\mu(x)\) and \(\hat{f}_\mu(x)\) interchangeably. In the following discussions we start by analyzing important properties of \(\hat{f}_\mu(x)\) which also extend to \(f_\mu(x)\) on \(\text{supp}(x^*_\mu)\).

In the following Lemmas 10, 11 and 12 we provide some technical results which will be used to prove convergence of Algorithm B.1.

\textbf{Lemma 10} (Lipschitz continuity). \(\nabla \hat{f}_\mu(x)\) is Lipschitz continuous with coordinate Lipschitz constant \(L_i = \deg(i)\mu^{q-2}\).

\textit{Proof.} The first and second order derivatives of \(\hat{f}_\mu(x)\) are
\[
\nabla \hat{f}_\mu(x) = B^T C B x - \Delta + \tilde{d},
\]
\[
\nabla^2 \hat{f}_\mu(x) = B^T \hat{C} B,
\]
where both \(C\) and \(\hat{C}\) are diagonal matrices of size \(|E| \times |E|\) whose diagonal entries correspond to edges \((i, j) \in E\),
\[
C_{(i,j),(i,j)} = ((x(i) - x(j))^2 + \mu^2)^\frac{q}{2} - 1,
\]
\[
\hat{C}_{(i,j),(i,j)} = ((x(j) - x(j))^2 + \mu^2)^\frac{q}{2} - 2 ((q - 1)(x(i) - x(j))^2 + \mu^2).
\]

In order to obtain coordinate Lipschitz constant \(L_i\) of \(\nabla_i \hat{f}_\mu(x)\), we upper bound \(\hat{C}_{(i,j),(i,j)}\) by
\[
\hat{C}_{(i,j),(i,j)} = ((x(i) - x(j))^2 + \mu^2)^\frac{q}{2} - 2 ((q - 1)(x(i) - x(j))^2 + \mu^2)
\]
\[
\leq ((x(i) - x(j))^2 + \mu^2)^\frac{q}{2} - 2 (q - 1)(x(i) - x(j))^2 + \mu^2
\]
\[
= (x(i) - x(j))^2 + \mu^2
\]

Since \(q \in (1, 2]\), we get that
\[
\frac{(q - 1)(x(i) - x(j))^2 + \mu^2}{(x(i) - x(j))^2 + \mu^2} \leq 1,
\]
and thus
\[
\hat{C}_{(i,j),(i,j)} \leq ((x(i) - x(j))^2 + \mu^2)^\frac{q}{2} - 1 \leq \mu^{q-2}.
\]
This means that
\[
\nabla^2 \hat{f}_\mu(x) = B^T \hat{C} B \preceq \mu^{q-2} B^T B.
\]
Therefore
\[
L_i = \mu^{q-2} e_i^T B^T B e_i = \deg(i)\mu^{q-2}.
\]

\textbf{Lemma 11} (Monotone gradients). For any \(x\), the following hold
\begin{itemize}
  \item \(h_{ii}(t) := \nabla_i \hat{f}_\mu(x + te_i)\) is strictly monotonically increasing in \(t\),
  \item \(h_{ij}(t) := \nabla_i \hat{f}_\mu(x + te_j)\) is strictly monotonically decreasing in \(t\) if \(j \sim i\) and constant if \(j \not\sim i\).
\end{itemize}

\textit{Proof.} Expand \(\nabla_i \hat{f}_\mu(x)\) as
\[
\nabla_i \hat{f}_\mu(x) = \sum_{j \sim i} ((x(i) - x(j))^2 + \mu^2)^{q/2-1} (x(i) - x(j)) - \Delta(i) + \deg(i).
\]
So
\[
h_{ii}(t) = \sum_{j \sim i} ((x(i) + t - x(j))^2 + \mu^2)^{q/2-1} (x(i) + t - x(j)) - \Delta(i) + \deg(i).
\]
Each term in the above sum is a function \(g(y) := (y^2 + \mu^2)^{q/2-1} y\). The derivative of \(g(y)\) is
\[
g'(y) = (y^2 + \mu^2)^{q/2-2} ((q - 1)y^2 + \mu^2) > 0,
\]
therefore, \( g(y) \) is strictly monotonically increasing, and hence so it \( h_{ii}(t) \). Similarly, \( h_{ij}(t) \) is equal to

\[
h_{ij}(t) = \sum_{j \sim i} ((x(i) - t - x(j))^2 + \mu^2)^{q/2-1} (x(i) - t - x(j)) - \Delta(i) + \deg(i).
\]

Using the same reasoning as above we get that function \( h_{ij}(t) \) is strictly monotonically decreasing if \( j \sim i \) and is constant if \( j \neq i \) and \( j \neq i \).

Lemma 12 (Bounded gradients). For any iteration \( k \) and node \( i \) in Algorithm B.1, \( \nabla_i \hat{f}_\mu(x_k) \leq 0, \forall i \in \text{supp}(x_k) \).

Proof. Suppose for some \( k \geq 0 \) we have that \( \nabla_i \hat{f}_\mu(x_k) \leq 0 \) for every \( i \in \text{supp}(x_k) \), and that \( S_k \neq \emptyset \). Say we pick some \( i_k \in S_k \). This means \( \nabla_{i_k} \hat{f}_\mu(x_k) < 0 \). It follows from Lipschitz continuity (cf. Lemma 11) that

\[
|\nabla_{i_k} \hat{f}_\mu(x_{k+1}) - \nabla_{i_k} \hat{f}_\mu(x_k)| = |\nabla_{i_k} \hat{f}_\mu(x_k) - \frac{\mu^2-q}{\deg(i_k)} \nabla_{i_k} \hat{f}_\mu(x_k) e_{i_k} - \nabla_{i_k} \hat{f}_\mu(x_k)|
\leq L_{i_k} \left| \frac{\mu^2-q}{\deg(i_k)} \nabla_{i_k} \hat{f}_\mu(x_k) \right| = |\nabla_{i_k} \hat{f}_\mu(x_k)|.
\]

Since \( \nabla_{i_k} \hat{f}_\mu(x_k) < 0 \), we must have \( \nabla_{i_k} \hat{f}_\mu(x_{k+1}) \leq 0 \). Moreover, since

\[
x_{k+1}(i_k) = x_k(i_k) - \frac{\mu^2-q}{\deg(i_k)} \nabla_{i_k} \hat{f}_\mu(x_k) > x_k(i_k),
\]

and \( x_{k+1}(j) = x_k(j) \) for all \( j \neq i_k \), by Lemma 11 we know that \( \nabla_j \hat{f}_\mu(x_{k+1}) < \nabla_j \hat{f}_\mu(x_k) \leq 0 \) for every \( j \neq i_k \), and \( \nabla_j \hat{f}_\mu(x_{k+1}) = \nabla_j \hat{f}_\mu(x_k) \) for every \( j \neq i_k \). The proof is complete by noting that \( \nabla \hat{f}_\mu(x_0) \leq 0 \) holds trivially.

D Strong convexity from Section 5

In this section we prove that the smoothed objective function is strongly convex and we provide a lower bound on the strong convexity parameter. The results in this section will also be used to obtain a local running time result for Algorithm B.1.

Although not entirely necessary, we provide an equivalent, but more intuitive, combinatorial description of Algorithm B.1 (applied to the globally smoothed problem (C.1)\(^4\), as shown in Figure 3. With this combinatorial interpretation we obtain a trivial upper bound on the gap \( |x_k(i) - x_k(j)| \) for any \( (i, j) \in E \) at any point in the sequence \( \{x_k\}_{k=0}^\infty \) generated by Algorithm B.1. We will use it to demonstrate a desired strong convexity result of \( \hat{f}_\mu(x) \).

The basic setting is as follows. Each node \( v \) maintains a height \( x(v) \). For any fixed \( q \in (1,2] \), the flow of mass from node \( u \) to node \( v \) is determined by the relative heights \( x(u) \) and \( x(v) \) and a smoothing parameter \( \mu \geq 0 \),

\[
\text{flow}_q(u, v; x, \mu) := (x(u) - x(v))^2 + \mu^2 (x(u) - x(v)).
\]

Therefore, given \( x \) and \( \mu \), the mass and the excess at node \( v \) are

\[
m(v; x, \mu) = \Delta(v) + \sum_{u \sim v} \text{flow}_q(u, v; x, \mu),
\]

\[
\text{ex}(v; x, \mu) = \max\{0, m(v; x, \mu) - \deg(v)\}.
\]

One can easily verify that, under these definitions,

\[
-\nabla_i \hat{f}_\mu(x) = m(i; x, \mu) - \deg(i),
\]

and that the steps we layout in Figure 3 are indeed equivalent to the steps in Algorithm B.1.

\(^4\)In Theorem 15 we show that there is no difference between applying Algorithm B.1 to either (C.1) or (C.2).
1. Initially, all nodes have height 0, i.e., \( x(v) = 0 \) for all \( v \).

2. While \( \text{ex}(v; x, \mu) > 0 \) for some node \( v \):
   
   (a) Pick any \( v \) where \( \text{ex}(v; x, \mu) > 0 \) uniformly at random.
   
   (b) Route the flow around node \( v \) by raising it to a new height
   
   \[
   x(v) = x(v) + \mu^{\frac{\text{ex}(v; x, \mu)}{\deg(v)}},
   \]

3. Return \( x \).

Figure 3: A combinatorial description of Algorithm B.1

Let \( \{x_k\}_{k=0}^\infty \) be a sequence generated by Algorithm B.1. Let \( k \geq 0 \) be arbitrary. By Lemma 12, we know that \( \text{ex}(i; x_k, \mu) \geq 0 \) for all nodes \( i \) with nonzero height \( x_k(i) > 0 \). This means that at iteration \( k \) when we route a flow around node \( i_k \), we never remove more mass from \( i_k \) than its current excess \( \text{ex}(i_k; x_k, \mu) \), so \( m(v; x_k, \mu) \geq 0 \) for all node \( v \). On the other hand, since the directions of flow, i.e., \( \text{sign(}_{\text{flow}}_q(u, v; x_k, \mu) \), are completely determined by the ordering of current heights \( x_k \), we know that flows cannot cycle. This is because if there is a directed cycle in the induced sub-graph on \( \text{supp}(Bx_k) \), where edges are oriented according to the directions of flow, then it means that there must exist a set of heights \( \{x_k(i_j)\} \) where \( x_k(i_1) < x_k(i_2) < \ldots < x_k(i_1) \), which is not possible. Now, since all nodes have non-negative mass and there is no cycling of flows, the net flow on any edge \((i, j) \in E \) cannot be larger than the total amount of initial mass \( |\Delta| - 1 \),

\[
|(x_k(i) - x_k(j))^2 + \mu^2 q/2| |x_k(i) - x_k(j)| = |\text{flow}_q(i, j; x_k, \mu)| \leq |\Delta| - 1, \quad \text{for all} \ (i, j) \in E.
\]

(D.1)

Lemma 13 (Strong convexity parameter). We have that

\[
\hat{f} \mu(y) \geq \hat{f} \mu(x) + \nabla \hat{f} \mu(x)^T (y - x) + \frac{\gamma}{2} \|y - x\|^2, \quad \forall \ x, y \in U,
\]

where

\[
U := \{ x \in \mathbb{R}^{|V|} \mid \text{supp}(x) \subseteq \text{supp}(x^*_\mu) \text{ and } |x(i) - x(j)| < |\Delta|^{p-1} \forall(i, j) \in E \},
\]

and the strong convexity parameter \( \gamma \) satisfies

\[
\gamma > \frac{1}{|\Delta|^2} \left( \frac{(q - 1)|\Delta|^{2p-2} + \mu^2}{(|\Delta|^{2p-2} + \mu^2)^{2q/2}} \right).
\]

Proof. Recall the second order derivative of \( \hat{f} \mu(x) \) is

\[
\nabla^2 \hat{f} \mu(x) = B^T \tilde{C}(x)B,
\]

where \( \tilde{C}(x) \) is a diagonal matrix of size \( |E| \times |E| \) whose diagonal entries correspond to edges \((i, j) \in E \),

\[
\tilde{C}(x)_{(i,j),(i,j)} = ((x(i) - x(j))^2 + \mu^2)^{q/2}((q - 1)(x(i) - x(j))^2 + \mu^2).
\]

Let \( S := \text{supp}(x^*_\mu) \). Let \( B_S \) denote the sub-matrix with columns chosen such that they correspond to nodes in \( \text{supp}(x^*_\mu) \). Similarly, let \( \tilde{C}_S(x) \) denote the sub-matrix of \( \tilde{C}(x) \) that corresponds to the edges \( (i, j) \in \text{supp}(Bx^*_\mu) \). In order to lower bound \( \gamma \), it suffices to lower bound the smallest eigenvalue of \( B_S^T \tilde{C}_S(x)B_S \) for all \( x \in U \). Note that

\[
\lambda_{\min}(B_S^T \tilde{C}_S(x)B_S) \geq \min_{(i,j) \in \text{supp}(Bx^*_\mu)} \tilde{C}_S(x)_{(i,j),(i,j)} \lambda_{\min}(B_S^T B_S).
\]

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Each diagonal entry in $\bar{C}_S(x)$ is a function $h(s) = (s^2 + \mu^2)^{q/2-2}((q-1)s^2 + \mu^2)$, and since $h'(s) \leq 0$ for all $s \geq 0$, we know that
\[
\min_{(i,j) \in \text{supp}(Bx_\mu^*)} \bar{C}_S(x)_{(i,j),(i,j)} = \frac{(q-1)|\Delta|^{2p-2} + \mu^2}{(|\Delta|^{2p-2} + \mu^2)^{2-q/2}}.
\]
and the result follows because
\[
\lambda_{\min}(B_S^TB_S) > \frac{1}{|\Delta|^2}.
\]
The last inequality can be easily shown by using the local Cheeger-type result in [8].

Note that the strong convexity result implies that the optimal solution $x_\mu^*$ of the globally smoothed problem (C.1) is unique. Furthermore, $\hat{f}_\mu(x)$ is strongly convex on all possible sequences of iterates generated by Algorithm B.1, as shown in the following corollary.

**Corollary 14.** Let $\{\hat{x}_k\}_{k=0}^\infty$ be a sequence generated by applying Algorithm B.1 to the globally smoothed problem (C.1). If $\mu \leq 1$, then $\hat{f}_\mu(x)$ is strongly convex on $X := \{\hat{x}_k\}_{k=0}^\infty \cup \{x_\mu^*\}$.

**Proof.** Note that the points generated by Algorithm B.1 satisfy $\text{supp}(\hat{x}_k) \subseteq \text{supp}(x_\mu^*)$. So it suffices to show that
\[
|x(i) - x(j)| < |\Delta|^{p-1}, \quad \forall (i,j) \in E, \quad \forall x \in X.
\]
We use (D.1) and the assumption that $\mu \leq 1$ to get that, for all $x \in X$ and for all $(i,j) \in E$,
\[
((x(i) - x(j))^2 + \mu^2)^{q/2-1} = ((x(i) - x(j))^2 + \mu^2)^{q/2-1}((x(i) - x(j))^2 + \mu^2)^{1/2}
\leq ((x(i) - x(j))^2 + \mu^2)^{q/2-1}(x(i) - x(j))^2 + ((x(i) - x(j))^2 + \mu^2)^{q/2-1}\mu
\leq ((x(i) - x(j))^2 + \mu^2)^{q/2-1}(x(i) - x(j))^2 + \mu^{q-1}
\leq |\Delta| - 1 + 1
= |\Delta|.
\]
Thus
\[
|x(i) - x(j)|^2 < (x(i) - x(j))^2 + \mu^2 \leq |\Delta|^{2/(q-1)} = |\Delta|^{2p-2},
\]
as required. \qed

**E Convergence, iteration complexity and running time from Section 5**

**Theorem 15 (Convergence).** Let $x_\mu^*$ denote the optimal solution for the globally smoothed problem (C.1),
\[
x_\mu^* := \arg\min_{x \geq 0} \hat{f}_\mu(x).
\]
The iterates $\{x_k\}_{k=0}^\infty$ generated by Algorithm B.1 converge to a limit point $\bar{x}$, where
\[
x_\mu^* = \bar{x} = \arg\min_{x \geq 0} f_\mu(x).
\]
**Proof.** Recall the optimality conditions of (C.1),
(a) Dual feasibility$^5$: $x \geq 0$;
(b) Primal feasibility: $\nabla \hat{f}_\mu(x) \geq 0$.
(c) Complementary slackness (under primal feasibility): $\nabla_i \hat{f}_\mu(x) \leq 0$ for every $i \in \text{supp}(x)$.

$^5$We call $x \geq 0$ dual feasibility because $\hat{f}_\mu(x)$ smoothes the dual problem of $p$-norm flow diffusion.
Suppose \( \{\hat{x}_k\}_{k=0}^{\infty} \) is a sequence generated by Algorithm B.1 where we have replaced \( \nabla_i f_\mu \) with \( \nabla_i \hat{f}_\mu \) and \( \nabla_i f_\mu \) with \( \nabla_i \hat{f}_\mu \). By the definition of index set \( S_k \) in Algorithm B.1, the iterates \( \hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots \) are monotone, i.e., \( 0 \leq \hat{x}_1 \leq \hat{x}_2 \leq \ldots \), so \( \hat{x}_k \) satisfies item (a) for all \( k \). By Lemma 12, \( \hat{x}_k \) also satisfies item (c), for all \( k \). We may assume \( S_k \neq \emptyset \) for all \( k \), as otherwise Algorithm B.1 terminates with the optimal solution that satisfies item (b). So assume that \( S_k \neq \emptyset \) for all \( k \), we argue that the sequence \( \{\hat{x}_k\}_{k=0}^{\infty} \) converges. Suppose not, then by Lemma 10,

\[
\hat{x}_{k+1} := \hat{x}_k - \frac{\mu^{2-q}}{\deg(i_k)} \nabla_{i_k} \hat{f}_\mu(x_k)e_{i_k} = \hat{x}_k - \frac{1}{L_{i_k}} \nabla_{i_k} \hat{f}_\mu(x_k)e_{i_k},
\]

for all \( k \), and hence by coordinate Lipschitz continuity we get

\[
\hat{f}_\mu(\hat{x}_{k+1}) \leq \hat{f}_\mu(\hat{x}_k) - \frac{1}{2L_{i_k}} \left( \nabla_{i_k} \hat{f}_\mu(\hat{x}_k) \right)^2,
\]

for all \( k \). Because the sequence of iterates \( \{\hat{x}_k\}_{k=0}^{\infty} \) does not converge, the sequence of gradients \( \{\nabla_{i_k} \hat{f}_\mu(x_k)\}_{k=0}^{\infty} \) must be bounded away from zero. This means that the sequence of function values \( \{\hat{f}_\mu(x_k)\}_{k=0}^{\infty} \) does not converge, either. But then this implies that \( \hat{f}_\mu(x) \) is unbounded below, contradicting our assumption that an optimal solution to (C.1) exists. Therefore the sequence \( \{\hat{x}_k\}_{k=0}^{\infty} \) must converge to a limit point \( \hat{x}^* \). Now, because \( \nabla \hat{f}_\mu \) is continuous, and since each \( \hat{x}_k \) satisfies optimality conditions (a) and (c), \( \hat{x}^* \) must also satisfy items (a) and (c). It is easy to see that \( \hat{x}^* \) satisfies item (b), too, because otherwise there must be some \( \hat{x}_k \) where \( \hat{x}_k(i) > \hat{x}^*(i) \) for some coordinate \( i \), which is not possible. Thus we have that \( \hat{x}^* = x^*_\mu \).

Furthermore, for all \( x \) where \( \text{supp}(x) \subseteq \text{supp}(x^*_\mu) \), we have that \( f_\mu(x) = \hat{f}_\mu(x) - \frac{1}{\mu} \mu^q(|E| - |\text{supp}(Bx^*_\mu)|) \), and hence \( \nabla f_\mu(x) = \nabla \hat{f}_\mu(x) \). So \( x^*_\mu \) must also satisfy the optimality conditions of the locally smoothed problem (C.2). This means \( x^*_\mu \) is also the optimal solution of (C.2).

Finally, because \( f_\mu(x) \) and \( \hat{f}_\mu(x) \) only differ by a constant for all \( x \) where \( \text{supp}(x) \subseteq \text{supp}(x^*_\mu) \), and because the algorithm increases the iterates monotonically starting from the all-zeros vector, then we may use \( \nabla_i f_\mu \) and \( \nabla_i \hat{f}_\mu \) interchangeably in defining the steps in Algorithm B.1, and this means that \( \{x_k\}_{k=1}^{\infty} \) converge to \( x^*_\mu \), as required. \qed

**Theorem 16** (Iteration complexity). Let \( x^*_\mu \) and \( f^*_\mu \) denote the optimal solution and optimal value of (C.2), respectively. For any \( k \geq 1 \), let \( x_k \) denote the \( k \)th iterate generated by Algorithm 1. Let \( \gamma \) be the strong convexity parameter as described in Lemma 13, and let \( L_{\mu} = \max_{i \in \text{supp}(x^*_\mu)} L_i \), where \( L_i = \deg(i)\mu^{d-2} \) be the coordinate Lipschitz constants of \( f_\mu(x) \). After \( K = O\left( \frac{L_{\mu}}{\gamma} \log \frac{1}{\epsilon} \right) \) iterations, one has

\[
\mathbb{E}[f_\mu(x_K)] - f^*_{\mu} \leq \epsilon.
\]

Furthermore, let \( f^* \) denote the optimal objective value of (B.2). If we pick \( \mu = O\left( \left( \frac{\epsilon}{|E|} \right)^{1/q} \right) \), then after

\[
K' = O\left( \frac{d_{\max}}{\gamma} \left( \frac{|E|}{\epsilon} \right)^{2/q-1} \log \frac{1}{\epsilon} \right)
\]

iterations, where \( d_{\max} = \max_{i \in \text{supp}(x^*_\mu)} \deg(i) \), one has

\[
\mathbb{E}[f(x_{K'})] - f^* \leq \epsilon.
\]

**Proof.** Using coordinate Lipschitz constants given in Lemma 10 we have that

\[
f_\mu(x_{k+1}) = f_\mu(x_k - \frac{1}{L_i} \nabla_i f_\mu(x_k)e_i) \leq f_\mu(x_k) - \frac{1}{L_i} \nabla_i f_\mu(x_k)^2 + \frac{1}{2L_i} \nabla_i f_\mu(x_k)^2 = f_\mu(x_k) - \frac{1}{2L_i} \nabla_i f_\mu(x_k)^2.
\]
Let $L_{S_k} := \max_{i \in S_k} L_i$, and take conditional expectation
\[
\mathbb{E}[f_\mu(x_{k+1}) \mid x_k] \leq f_\mu(x_k) - \frac{1}{2} \sum_{i \in V} \frac{1}{|S_k| L_i} \nabla_i f_\mu(x_k)^2
\]
\[
= f_\mu(x_k) - \frac{1}{2|S_k|} \sum_{i \in S_k} \frac{1}{L_i} \nabla_i f_\mu(x_k)^2
\]
\[
\leq f_\mu(x_k) - \frac{1}{2|S_k| L_{S_k}} \|\nabla S_k f_\mu(x_k)\|^2_2
\]
\[
\leq f_\mu(x_k) - \frac{1}{2|S_k| L_{S_k}} \|\nabla f_\mu(x_k)\|^2_2.
\]

From strong convexity of $f_\mu(x)$ on $\{x_k\}_{k=0}^\infty \cup \{x^*_\mu\}$ (Lemma 13 and Corollary 14) we have that
\[
\|\nabla f_\mu(x_k)\|_2^2 \leq 2\gamma (f_\mu(x_k) - f^*_\mu).
\]

Therefore, we get
\[
\mathbb{E}[f_\mu(x_{k+1}) \mid x_k] \leq f_\mu(x_k) - \frac{\gamma}{|S_k| L_{S_k}} (f_\mu(x_k) - f^*_\mu),
\]
and so
\[
\mathbb{E}[f_\mu(x_{k+1}) \mid x_k] - f^*_\mu \leq \left(1 - \frac{\gamma}{|S_k| L_{S_k}}\right) (f_\mu(x_k) - f^*_\mu).
\]

Note that $S_k \subseteq \text{supp}(Bx^*_\mu)$, thus $|S_k| \leq |\text{supp}(Bx^*_\mu)| \leq |\Delta|$. Let $L_\mu := \max_{i \in \text{supp}(x^*_\mu)} L_i$, then $L_\mu \geq L_{S_k}$. Using these simple inequalities we get
\[
\mathbb{E}[f_\mu(x_{k+1}) \mid x_k] - f^*_\mu \leq \left(1 - \frac{\gamma}{|\Delta| L_\mu}\right) (f_\mu(x_k) - f^*_\mu).
\]

Take conditional expectations over all $x_{k-1}, x_{k-2}, \ldots, x_1, x_0$ we get
\[
\mathbb{E}[f_\mu(x_{k+1})] - f^*_\mu \leq \left(1 - \frac{\gamma}{|\Delta| L_\mu}\right)^k (f_\mu(x_0) - f^*_\mu).
\]

Therefore, after $K = O\left(\frac{L_\mu}{\gamma} \log \frac{1}{\epsilon}\right)$ iterations, one has
\[
\mathbb{E}[f_\mu(x_K)] - f^*_\mu \leq \frac{\epsilon}{2}.
\]

Using Lemma 9 we get that, after $K$ iterations,
\[
\mathbb{E}[f_\mu(x_K)] - f^*_\mu \leq \mathbb{E}[f_\mu(x_K)] - f^*_\mu + \frac{1}{2} \mu^q |\Delta| \leq \frac{\epsilon}{2} + \frac{1}{2} \mu^q |\Delta|.
\]

Hence setting $\mu = O\left((\frac{\epsilon}{|\Delta|})^{1/q}\right)$ gives the required iteration complexity. \(\square\)

**Corollary 17** (Running time). *If we pick $\mu = O\left((\frac{\epsilon}{|\Delta|})^{1/q}\right)$, the total running time of Algorithm B.1 to obtain an $\epsilon$ accurate solution of (B.2) is $O\left(\frac{d^2_{max}}{\gamma} (|\Delta|)^{2q-1} \log \frac{1}{\epsilon}\right)$.*

**Proof.** This is straightforward by noticing that at each step in Algorithm B.1, we touch only the nodes $j$ such that $j \sim i_k$, for updating gradient vector to $\nabla f_\mu(x_{k+1})$ and for obtaining $S_{k+1}$. \(\square\)

## F Empirical Set-up and Results

### F.1 Diffusion on a dumbbell

The best way to visualize $p$-norm flow diffusions for various $p$ values is to start the diffusion processes on the same graph and the same set of seed nodes, with equal amount of initial mass. To this end, we run $p$-norm diffusions for $p \in \{2, 4, 8\}$ on a synthetic small scale “dumbbell” graph obtained by removing edges from a $7 \times 7$ grid graph. We pick a single seed node which locates on one side of the “bridge”, and set $|\Delta| = 121$. For each $p$, we plot optimal dual variables in Figure 4, where we use color intensities and circle sizes to indicate the relative magnitude of
Figure 4: Diffusion on a dumbbell: color intensities and circle sizes are chosen to reflect relative magnitude of optimal dual variables.

dual values, i.e., brighter colors and larger circles size means higher dual values for a fixed $p$, and no circle means the corresponding node has zero dual value.

Recall that a dual variable is nonzero only if the corresponding node is saturated (i.e., the mass it holds equals its degree). Observe that 2-norm diffusion leaks a lot of mass to the other side of dumbbell, whereas 4-norm and 8-norm diffusions saturate entire left-hand side, without leaking much mass to the right. The reason that this happens is because $p = 4$ and $p = 8$ put significantly larger penalty on the flow that passes through the “bridge”, making it difficult to send mass over to the other side.

Note that, if we were to perform the sweep cut procedure described in Section 4, then 2-norm diffusion would fail to recover the “right” cluster, while both 4-norm and 8-norm diffusions return the entire left-hand side of dumbbell, which is the best possible result in terms of conductance guarantee. Although this example is overly simplistic, it does demonstrate that $p$-norm flow diffusion with higher $p$ values are more sensitive to bottlenecks when routing mass around a seed node, and, on the other hand, it is possible to overcome such bottleneck even when $p$ is a reasonably small constant, say $p = 4$ and $p = 8$. Indeed, our subsequent experiments show that there is already a significant improvement in the context of local graph clustering, when raising $p = 2$ to $p = 4$.

F.2 Missing plots from LFR synthetic experiments

The following experiment on an LFR synthetic graph with $\mu = 0.3$ demonstrates that, not surprisingly, the more overlap between an initial set of seed nodes and the target cluster, the better result we can get from $p$-norm flow diffusion. Note that the parameter $\mu = 0.3$ means that 30% of all edges of a node from some target cluster links to the outside of that cluster. Because of such noise level, the goal is not to recover the ground truth exactly, but to obtain a cluster that overlaps well with the target (e.g., has a good F1 score). We have chosen $\mu = 0.3$ because it represents reasonably noisy clusters that are not completely noise. For target clusters that have lower conductance and are connected better internally, the results of $p$-norm diffusion are already very good even when we start from a singe node (cf. Figure 2).

We randomly pick a set of seed nodes $S$ from some target cluster $C$ in the graph and vary the percentage overlap of $S$ in $C$, i.e., $|S|/|C|$. Note that this results in different values of $\alpha$ in Theorem 2. Then for each $p \in \{2, 4, 8\}$, we run $p$-norm flow diffusion and use the sweep cut procedure to find a smallest conductance cluster. Figure 5 shows the mean and the variance for the conductance and the F1 measure while varying the ratio $|S|/|C|$. As expected, as the percentage overlap $|S|/|C|$ increases, which also means $\alpha$ decreases, we recover clusters with lower conductances and higher F1 scores. Observe that we have a large gap in both conductance and F1 measure when $p = 2$ is increased to $p = 4$, but while there is some gain in raising from $p = 4$
to $p = 8$, but it is very marginal.

Figure 5: The conductance and the F1 measure on varying overlaps on an LFR synthetic graph. The black dashed line show ground truth conductance. The bands show the variation over 100 trials.

F.3 Datasets

To compare the performances of different methods on graph clustering tasks, we use four real datasets consisting of both social and biological networks. The graphs that we consider are all unweighted and undirected. Table 2 shows some basic characteristics of these graphs.

| dataset     | number of nodes | number of edges | description                                           |
|-------------|-----------------|-----------------|-------------------------------------------------------|
| FB-Johns55  | 5157            | 186572          | Facebook social network for Johns Hopkins University  |
| Colgate88   | 3482            | 155043          | Facebook social network for Colgate University        |
| Sfld        | 232             | 15570           | Pairwise similarities of blasted sequences of proteins |
| Orkut       | 3072441         | 117185083       | Large-scale on-line social network                    |

The two Facebook graphs are chosen from the Facebook100 dataset based on their assortativity values in the first column of Table A.2 in [30], where the data were first introduced and analyzed. Each of these comes with some features, e.g., gender, dorm, major index and year. We consider a set of nodes with the same feature as a ground truth cluster, and filter the “ground truth” clusters by setting a 0.6 threshold on maximum conductance. We also omit clusters whose volume is larger than one third of the volume of the entire graph, since discovering clusters whose sizes are close to the entire graph is not the purpose of local clustering.

The biology dataset Sfld contains pairwise similarities of blasted sequences of 232 proteins belonging to the amidohydrolase superfamily [5]. A gold standard is provided describing families within the given superfamily. According to the gold standard the amidohydrolase superfamily contains 29 families. We consider each family as a ground truth cluster, and filter them by setting a 0.9 threshold on the maximum conductance.

The last dataset Orkut is a free on-line social network where users form friendship each other. It can be downloaded from [21]. This network comes with 5000 ground truth communities, which we filter by setting minimum community size 50, maximum cut conductance 0.5, and minimum ratio between internal connectivity (i.e., the smallest nonzero eigenvalue of the normalized Laplacian of the subgraph defined by the cluster) and cut conductance to 0.85. This resulted in 11 reasonably noisy clusters, having conductances between 0.4 and 0.5.

We include the statistics of all ground truth clusters that we used in the experiments in Table 3.
Table 3: Filtered “ground truth” clusters for real-world graphs

| dataset    | feature       | volume | nodes | conductance |
|------------|---------------|--------|-------|-------------|
| FB-Johns55 | year 2006     | 81893  | 845   | 0.54        |
|            | year 2007     | 89021  | 842   | 0.49        |
|            | year 2008     | 82934  | 926   | 0.39        |
|            | year 2009     | 33059  | 910   | 0.21        |
|            | major index 217 | 10697  | 201   | 0.26        |
| Colgate88  | year 2004     | 14888  | 230   | 0.54        |
|            | year 2005     | 50643  | 501   | 0.50        |
|            | year 2006     | 62065  | 557   | 0.48        |
|            | year 2007     | 68382  | 589   | 0.41        |
|            | year 2008     | 62430  | 641   | 0.29        |
|            | year 2009     | 35379  | 641   | 0.11        |
| Sfd        | urease        | 31646  | 100   | 0.42        |
|            | AMP           | 3186   | 28    | 0.53        |
|            | phosphotriesterase | 381  | 7    | 0.78        |
|            | adenosine     | 1062   | 10    | 0.83        |
|            | dihydrooorotase3 | 494  | 7    | 0.83        |
|            | dihydrooorotase2 | 3119 | 13   | 0.90        |
| orkut      | A             | 49767  | 383   | 0.42        |
|            | B             | 31912  | 202   | 0.45        |
|            | C             | 16022  | 141   | 0.45        |
|            | D             | 11698  | 113   | 0.46        |
|            | E             | 26248  | 194   | 0.47        |
|            | F             | 4617   | 64    | 0.47        |
|            | G             | 13786  | 128   | 0.47        |
|            | H             | 14109  | 107   | 0.48        |
|            | I             | 18652  | 195   | 0.49        |
|            | J             | 41612  | 318   | 0.50        |
|            | K             | 20204  | 223   | 0.50        |

F.4 Methods and parameter setting

We compare the performance of $p$-norm flow diffusion with $\ell_1$-regularized PageRank [11] and nonlinear diffusion under power transfer [15]. Given a starting node $v_s$, teleportation probability $\alpha$, and tolerance parameter $\rho$, the $\ell_1$-regularized PageRank is an optimization problem whose optimal solution is an approximate personalized PageRank (APPR) vector [1]. This $\ell_1$-regularized variational formulation allow us to apply coordinate method and obtain an APPR vector in linear and strongly local running time. The nonlinear diffusion model applies a nonlinear transformation to node values after each matrix-vector products between the Laplacian matrix and the current node values. Since it is demonstrated in [15] that a nonlinear transfer defined by the power function $u \mapsto u^{0.5}$ has the best overall performance when compared to other nonlinear functions like tanh or heat kernel, we only compare $p$-norm flow diffusion with nonlinear diffusion governed by this power function. We choose $p \in \{2, 4\}$ for $p$-norm diffusion and demonstrate the advantage of our method even when $p$ is a small constant.

Our goal here is to compare the behaviour of different algorithms under a unified setting, and not to fine tune any particular model. Therefore, we always start $p$-norm diffusion from a single seed node, and set $|\Delta| = t \cdot \text{vol}(C)$ for some constant factor $t$ and some target cluster $C$ (recall from Assumption 1 this is WLOG). In particular, on the Facebook datasets we set $t = 3$ because the target clusters already have a large volume, and on Orkut dataset we set
$t = 5$. On the other hand, because the clusters in Sfld are very noisy, we vary $t \in \{1, 2, \ldots, 10\}$ and pick the cluster with the lowest conductance. In all cases, we make sure the choice of $t$ is such that $|\Delta|$ is less than the volume of the whole graph. For the nonlinear diffusion model with power transfer, we use the same parameter setting as what the authors suggested in [15]. The $\ell_1$-regularized PageRank is the only linear model here for comparison, so we allow it to use ground truth information to choose the parameter $\alpha$ giving the best conductance result. We tune $\alpha$ by picking $\alpha = \{\lambda/8, \lambda/4, \lambda/2, \lambda, 2\lambda\}$, where $\lambda$ is the smallest nonzero eigenvalue of the normalized Laplacian for the subgraph that corresponds to the target cluster. Since the support size of APPR vector is bounded by $1/\rho$ [11], and the support size of dual variables for $p$-norm diffusion is bounded by $|\Delta|$, for comparison purposes, we set the tolerance parameter $\rho$ for $\ell_1$-regularized PageRank so that $\rho = 1/|\Delta|$.

F.5 Additional discussion

The clustering results are already listed in Table 1. We comment on some observations from the results of the Sfld dataset. The clusters in this dataset are very noisy, having median conductance 0.8. Hence it is highly probable that a ground truth cluster is contained in some larger clusters (but not one of the 29 true families) that have much lower conductances. This might partially explain why nonlinear power diffusion finds low conductance cuts with poor F1 scores. Note that the nonlinear diffusion model is the only global method here for comparison. On the contrary, $p$-norm diffusions and $\ell_1$-regularized PageRank take advantage of locality and find local clusters that align well with the ground truth.

Interestingly, on cluster “urease” in the Sfld dataset, there is a striking difference between the performances of 2-norm and 4-norm diffusions. This is because the volume of that cluster is more than half the volume of the entire graph, and in order to make the optimization problem of $p$-norm flow feasible, we set $|\Delta|$ only marginally larger than the volume of the cluster, as opposed to other cases where $|\Delta|$ is 3 or 5 times larger. Such difference between 2-norm and 4-norm diffusions demonstrates that higher $p$-norm flows are likely more robust to a change in the amount of initial mass, in the sense that 4-norm still delivers the best F1 score for the cluster “urease”, among all methods, even though it started with a relatively very small amount of initial mass$^6$.

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$^6$The $\ell_1$-regularized PageRank is able to produce a moderately good result for “urease” because, in this case in particular, we have also fine tuned its tolerance parameter $\rho$ by allowing it to be small (i.e., we tried various $\rho$ where $1/\rho$ is much larger than the volume of the whole graph). Had we imposed strict locality on $\ell_1$-regularized PageRank by maintaining $1/\rho < \text{vol}(G)$, the result of this spectral method would be as terrible as 2-norm diffusion.