On the Bipartiteness Constant and Expansion of Cayley Graphs

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Abstract

Let \( G \) be a finite, undirected, \( d \)-regular graph and \( A(G) \) its normalized adjacency matrix, with eigenvalues \( 1 = \lambda_1(A) \geq \cdots \geq \lambda_n \geq -1 \). It is a classical fact that \( \lambda_n = -1 \) if and only if \( G \) is bipartite. Our main result provides a quantitative separation of \( \lambda_n \) from \(-1\) in the case of Cayley graphs, in terms of their expansion. Denoting \( h_{out} \) by the (outer boundary) vertex expansion of \( G \), we show that if \( G \) is a non-bipartite Cayley graph (constructed using a group and a symmetric generating set of size \( d \)) then \( \lambda_n \geq -1 + \frac{c h_{out}^2}{d^2} \), for \( c \) an absolute constant. We exhibit graphs for which this result is tight up to a factor depending on \( d \). This improves upon a recent result by Biswas and Saha [4] who showed \( \lambda_n \geq -1 + \frac{h_{out}^4}{2^9 d^8} \). We also note that such a result could not be true for general non-bipartite graphs.

1 Introduction

It is well-known that the leading eigenvalue \( \lambda_1 \) of the normalized adjacency matrix of a regular graph is 1. A topic of much interest is the second-largest absolute eigenvalue, \( \max(\lambda_2, -\lambda_n) \). The famous Cheeger inequalities relate the spectral gap \( 1 - \lambda_2 \) to the isoperimetric constant (see e.g., [1, 7]). We will be more concerned with what is in some sense the “other spectral gap”, namely the gap between \( \lambda_n \) and \(-1\), as studied in \([2, 12]\).

In a recent work, Breuillard et al. [6] argued that if a non-bipartite Cayley graph is a combinatorial expander - in the sense that \( h_{out} \) is bounded away from 0 - then it must also be a spectral “expander” in the sense that \( \lambda_n \) is bounded away from \(-1 \) \([6]\), Proposition E.1). Combining this result with the Cheeger inequality, it is seen that \( \max_{i > 1} |\lambda_i| \) is bounded away from 1. Biswas [3, Theorem 1.4), building on that argument, gave a bound of the form

\[
\lambda_n \geq -1 + \frac{c h_{out}^2}{d^2},
\]

where \( c \) is an absolute constant.
$1 + \lambda_n \geq \frac{h_{\text{out}}^2}{d^2}$, and in a very recent work Biswas and Saha (4, Theorem 2.12) refined the bound to $1 + \lambda_n \geq \frac{h_{\text{out}}^2}{d^2}$. In this paper we improve on these results (see Theorem 2.6 below), by proving that for every non-bipartite Cayley graph,

$$1 + \lambda_n \geq \frac{C h_{\text{out}}^2}{d^2},$$

where $C > 0$ is a universal constant.

Trevisan (12), and independently Bauer and Jost (2), introduced $\beta$ (defined below), a combinatorial parameter which measures the fraction of edges contributing to the non-bipartiteness of a graph. $\beta$ is also a modification of a related constant developed much earlier by Desai and Rao (8). An analogue of the Cheeger inequalities relates $\beta$ to $\lambda_n$:

**Theorem 1.1 (Trevisan [12], Equation (8)).** For any regular graph,

$$2\beta \geq (1 + \lambda_n) \geq \frac{1}{2}\beta^2.$$ 

Recall that $\lambda_n = -1$ if and only if the graph is bipartite, and $\beta \geq 0$ which was introduced by [12] to capture non-bipartiteness, is also zero whenever the graph is bipartite.

One can think of $\beta$ as serving the same role regarding $\lambda_n$ as the isoperimetric constant $h$ does for $\lambda_2$. Following this analogy, we will define the outer vertex bipartiteness constant $\beta_{\text{out}}$, just as $h_{\text{out}}$ is the outer (vertex) boundary isoperimetric constant. In Theorem 2.3 we demonstrate simple bounds relating $\beta_{\text{out}}$ to $\beta$, and by extension to $\lambda_n$.

A brief outline of the overall proof strategies is as follows. The proofs of the previous results [3, 6] involve, for the Cayley graph $G = (X, S)$, examining the multigraph $G^2 = (X, S^2)$ with edges consisting of 2-walks in $G$. Then letting $A$ be the $h_{\text{out}}(G^2)$-achieving set, the outer (vertex) boundary $\partial_{\text{out},G^2}(A)$ of $A$ in $G^2$ is bounded above by a function of $\lambda_n(G)$. Following a method introduced by Freiman (10), it is observed that if $\partial_{\text{out},G^2}A$ is sufficiently small (with respect to $h_{\text{out}}(G)$), then there is a bipartition of $G$ which approximates $\{A, A^C\}$, contradicting the assumption that $G$ is not bipartite.

Our innovation to this method is, rather than an $A$ as above, to consider sets $L, R$ which achieve $\beta_{\text{out}}(G)$; i.e., the best almost-bipartition of $G$. In the proof of Theorem 2.4 we will demonstrate an upper bound for $\partial_{\text{out}}L$ (or $\partial_{\text{out}}R$) in terms of $\beta_{\text{out}}$, and then, following the same method of Freiman, we argue that if $\beta_{\text{out}}$ is sufficiently small with respect to $h_{\text{out}}$ then $\{L, R\}$ approximates an actual bipartition of $G$, which gives a contradiction. Our main result in Theorem 2.6 follows by combining this proof with Trevisan’s above-mentioned lower bound on $\lambda_n$.

We demonstrate as Example 2.7 that for the odd cycle $1 + \lambda_n = \Theta(h_{\text{out}}^2)$, our result is tight up to a factor depending only on $d$, and therefore the term $h_{\text{out}}^2$ in our main result cannot be improved. With Example 2.8 we demonstrate that there is no converse to our theorem; that is, there is no lower bound for $h_{\text{out}}$ in terms of $\lambda$ (or $\beta$) and $d$ even in the special case of non-bipartite Cayley graphs.
1.1 Notations

Recall that a $d$-regular graph is one in which each vertex has exactly $d$ neighbors. Throughout this paper we will let $G = (V, E)$ be a $d$-regular, connected, non-bipartite simple graph on $n$ vertices. For a subset $S \subset V$, let $\partial(S)$ denote the edge boundary of $S$, namely the set of edges with precisely one endpoint in $S$. $\partial_{out}(S)$ is the outer vertex boundary, the set of vertices that are not in $S$ but do have a neighbor in $S$. Let $h(G)$ denote the (edge) Cheeger constant of $G$ defined as

$$h(G) = \min_{S \subset V: 0 < |S| \leq |V|/2} \frac{|\partial(S)|}{d|S|}.$$ 

The classical isoperimetric constant expansion is defined using the outer vertex boundary:

$$h_{out}(G) = \min_{S \subset V: 0 < |S| \leq |V|/2} \frac{|\partial_{out}(S)|}{|S|}. $$

Let $A = \left[ \begin{array}{cccc}
1 & d & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array} \right]$ be the normalized adjacency matrix and $\Delta = I - A$ the normalized Laplacian. Write $1 = \lambda_1(A) \geq \cdots \geq \lambda_n =: -1 + \mu$. Observe that $\mu > 0$, since $G$ is assumed to be non-bipartite.

For a set $S \subset V$ we denote by $S^C$ its complement $V - S$. For a pair of disjoint sets $L, R \subset V$, define the bipartiteness ratio of $L$ and $R$ to be

$$b(L, R) = \frac{e(L, R^C) + e(R, L^C)}{d|L \cup R|} = \frac{e(L, L) + e(R, R) + |\partial(L \cup R)|}{d|L \cup R|}. $$

Note: Our convention is that $e(A, B)$ counts the ordered pairs in $E(A, B) := \{(a, b) \in A \times B : a \sim b \}$, so that $e(A, A)$ equals twice the number of edges in the subgraph induced by $A$.

Following Trevisan [12], the bipartiteness constant of $G$ is $\beta(G) = \min_{L, R} b(L, R)$. As mentioned above, for a bipartite graph $G$, $\beta(G) = 0$, since $L$ and $R$ can be chosen to be the bipartition of the graph, giving the numerator in $b(L, R)$ to be zero.

One may also observe that the definition of $\beta$ can be obtained by restricting the minimization in (the variational definition of) $\lambda_n$ (or rather that of $1 + \lambda_n$)

$$1 + \lambda_n = \min_{x \in \mathbb{R}^n: x \neq \bar{0}} \frac{1}{d} \sum_{(i,j) \in E} |x_i + x_j| \sum_i |x_i|$$

to functions taking values in $\{-1, 0, +1\}$.

In this work, we define the Cayley graph $(X, S)$, with $X$ being a finite group and $S$ a generating set of $X$, to have edges $g \sim gs$ for all $g \in X, s \in S$.

2 Results

In this section we investigate the problem of relating the bipartiteness constant $\beta$ to the isoperimetric constant $h$. First, we will give a simple example that shows there cannot be a relationship in the general case of non-bipartite regular graphs. But we find that there is no such obstruction for non-bipartite Cayley graphs, and those will be our main focus.
2.1 General results

There is no universal lower bound for $\beta$ (and, because $\mu \geq \frac{1}{2} \beta^2$, neither is there a bound for $\mu$) in terms of $h$ and $d$. To see this we give an example that cannot follow any such bound.

**Example 2.1.** There is a non-bipartite graph for which $h$ is constant and $\beta = O(\frac{1}{nd})$.

**Proof.** Suppose $G$ is a bipartite expander, so that $h = c \in (0, 1)$, $\beta = 0$. Let $L, R$ be the bipartition of $G$ and let $(l_1, r_1), (l_2, r_2) \in L \times R$ be edges of $G$. Now create the $d$-regular graph $G^*$ by replacing these two edges with $(l_1, l_2)$ and $(r_1, r_2)$. Then $\beta(G^*) = \frac{4}{nd}$, achieved by the sets $L, R$. Because we only changed a constant set of edges, $h(G^*) = h(G) + o_d(1) = c + o_d(1)$; i.e., $\beta$ is the smallest possible value and $h$ is constant.

We will try to apply a similar idea to find an obstruction for Cayley graphs.

**Example 2.2.** There is a non-bipartite Cayley graph for which $h$ is constant and $\beta = O(\frac{1}{d})$.

**Proof.** Let $G$ be a bipartite Cayley graph for group $X$ with self-invertible set $S$, where $|S| = d$, and assume $h(G) = c \in (0, 1)$. Let $g \in X$ be an element in the same side of the bipartition as $e$. Define $G^*$ to be the Cayley graph on $X$ generated by $S \cup \{g, g^{-1}\}$. Then $\beta(G^*) = \frac{2}{d+2}$ (or $\beta(G^*) = \frac{1}{d+1}$ in the case $g = g^{-1}$), which is achieved by $L, R$, while $h(G^*) \leq \frac{cd}{d+2}$.

This example does not give as strong an obstruction, because we might have $\beta(G^*) = \Theta(\frac{1}{d})$, then $\beta(G^*) = \Omega(\frac{h(G^*)}{d})$; however a theorem stating that $\beta = \Omega(\frac{h}{d})$ for general Cayley graphs would certainly be of interest. In the next section we will investigate what bounds are possible for this problem.

2.2 Results for Cayley graphs

For the proofs in this section it will be useful to define a bipartiteness parameter $\beta_{\text{out}}$ which involves a count of vertices that violate bipartiteness. This is in contrast to the definition of $\beta$, which uses counts of edges.

**Definition 2.1.** For disjoint sets $L, R$, we define

$$b_{\text{out}}(L, R) = \frac{I(L) + I(R) + |\partial_{\text{out}}(L \cup R)|}{|L \cup R|},$$

where $I(S)$ is the number of vertices in $S$ with a neighbor also in $S$. The outer vertex bipartiteness constant is

$$\beta_{\text{out}}(G) = \min_{L, R} b_{\text{out}}(L, R).$$

As with $h$ and $b_{\text{out}}$, there is a simple relationship between $\beta$ and $\beta_{\text{out}}$.

**Theorem 2.3.** $\beta_{\text{out}}(G) \geq \beta(G) \geq \frac{1}{d} \beta_{\text{out}}(G)$. 
Proof. Take $L, R$ that achieve $\beta_{\text{out}}$. Then
\[
\beta_{\text{out}} = \frac{I(L) + I(R) + |\partial_{\text{out}}(L \cup R)|}{|L \cup R|} \geq \frac{\frac{1}{d}e(L, L) + \frac{1}{d}e(R, R) + \frac{1}{d}d(L \cup R)}{|L \cup R|} = \beta.
\]

Now take $L', R'$ that achieve $\beta$. Then
\[
\beta = \frac{e(L', L') + e(R', R') + |\partial(L' \cup R')|}{d|L' \cup R'|} \geq \frac{I(L') + I(R') + |\partial_{\text{out}}(L' \cup R')|}{d|L' \cup R'|} \geq \frac{1}{d}\beta_{\text{out}}.
\]

Now, we can prove our main results: first, a bound relating $\beta_{\text{out}}$ to $h_{\text{out}}$ for all Cayley graphs. Then we will demonstrate a similar bound relating $\beta$ to $h$.

Theorem 2.4. Let $G$ be a non-bipartite simple Cayley graph corresponding to a group $X$ and a generating set $S$ which satisfies $S^{-1} = S$ and $id_X \notin S$. Let $n = |X|$ and $d = |S|$. Then
\[
h_{\text{out}}(G) \leq 200\beta_{\text{out}}.
\]

Proof. Choose the disjoint sets $L, R$ that achieve $\beta_{\text{out}}$. Observe that from the definition of $\beta_{\text{out}}$, $|\partial_{\text{out}}(L \cup R)| \leq \beta_{\text{out}}|L \cup R|$. Set $Y = \{g \in X : \text{dist}_G(g, L \cup R) \geq 2\}$, so that (if $Y$ is non-empty) $\partial_{\text{out}}Y = \partial_{\text{out}}(L \cup R)$. Let $\varepsilon > 0$ be a constant that we will fix later. We will first consider the case that $|Y| > \varepsilon n$.

If $\varepsilon n < |Y| < \frac{1}{2}n$, then
\[
h_{\text{out}} \leq \frac{|\partial_{\text{out}}Y|}{|Y|} \leq \beta_{\text{out}}|L \cup R| \leq \frac{\beta_{\text{out}}}{\varepsilon n} \leq \frac{1}{\varepsilon} \beta_{\text{out}}.
\]

If $|Y| \geq \frac{1}{2}n$, then $|L \cup R| < \frac{1}{2}n$ as $Y$ is disjoint from $L$ and $R$. And then
\[
h_{\text{out}} \leq \frac{|\partial_{\text{out}}(L \cup R)|}{|L \cup R|} \leq \beta_{\text{out}}.
\]

For the remainder of the proof we will assume $|Y| \leq \varepsilon n$. For any $g \in X$, define the sets $A(g) = (gL \cap L) \cup (gR \cap R)$ and $B(g) = (gR \cap L) \cup (gL \cap R)$. Where it is clear we will suppress the input $g$, using $A := A(g)$ and $B := B(g)$. Observe that $A$ and $B$ are disjoint, since $L$ and $R$ are disjoint. We will next bound $|\partial_{\text{out}}A|$ and $|\partial_{\text{out}}B|$.

Consider the set $(\partial_{\text{out}}A) \cap B$: any vertex in that set must be counted by one of $I(L), I(R), I(gL)$, or $I(gR)$. Any other vertex in $\partial_{\text{out}}A$ must be in $\partial_{\text{out}}(L \cup R)$ or $\partial_{\text{out}}((L \cup R)g)$. By symmetry the same holds for $\partial_{\text{out}}B$. It follows that $|\partial_{\text{out}}A(g)|, |\partial_{\text{out}}B(g)| \leq 2(I(L) + I(R) + |\partial_{\text{out}}(L \cup R)|)$, and therefore
\[
h_{\text{out}}(G) \leq \frac{2(I(L) + I(R) + |\partial_{\text{out}}(L \cup R)|)}{\min |A(g)|, |B(g)|}.
\]

Here we use the facts that $I(gL) = I(L)$ and $I(gR) = I(R)$. It is simple to see that the numerator is $2\beta_{\text{out}}|L \cup R|$, it remains to bound $\min\{|A(g)|, |B(g)|\}$. 

5
For this step, we will use a technique developed in [10], which was used to prove similar results in [3, 4, 6].

Without loss of generality, assume that $|L| \geq |R|$. Notice that $|L \cup R| = n - |Y| - |\partial_{out}(L \cup R)| \geq n - \varepsilon n - \beta_{out} n$, and hence by assumption $|L| \geq \frac{1 - \varepsilon - \beta_{out}}{n} n$.

Suppose that $|L| \geq \frac{1 + \varepsilon}{2} n$. Observe that $|L| - I(L) \leq |L^c|$ and so $I(L) \geq \varepsilon n$. Also observe that as a general bound $h_{out} \leq n + 1 - n \leq 2$. In this case we have the bound $\beta_{out} \geq \frac{I(L)}{n} \geq \varepsilon$, and so $h_{out} \leq 2 \leq \frac{2}{\varepsilon} \beta_{out}$. We will consider the other case, where $\frac{1 - \varepsilon - \beta_{out}}{2} n \leq |L| \leq \frac{1 + \varepsilon}{2} n$.

Assume for contradiction that there is no element $g \in X$ for which $|gL \cap g| \in (\delta |L|, (1 - \delta) |L|)$, where $\delta$ is a constant we will define later. Define the sets $X_1 = \{g : |L \cap g| \geq (1 - \delta) |L|\}$ and $X_2 = \{g : |L \cap g| \leq \delta |L|\}$. By assumption $X_1, X_2$ is a partition of $X$.

We can show that if $\delta < \frac{1}{3}$, then $X_2^2 \subset X_1$. Let $g, h \in X_1$, then

$$|gL \cap g| \geq |L| - |gL - gh| - |L - gL| \geq (1 - 2\delta) |L| > \delta |L|,$$

and so assuming $\delta < \frac{1}{3}$, $gh \in X_1$.

Similarly we can show that if $\delta < 1 - \frac{2}{3(1 - \varepsilon - \beta_{out})}$, then $X_2^2 \subset X_1$. Let $g, h \in X_2$, then

$$|gL \cap g| \geq |L - gL| - |X - (gL \cap gh)| = 3 |L| - n - |L \cap gL| - |gL \cap ghL| \geq (3 - \frac{2}{3(1 - \varepsilon - \beta_{out})}) |L| - 2\delta |L| > \delta |L|,$$

and so assuming $\delta < 1 - \frac{2}{3(1 - \varepsilon - \beta_{out})}$, $gh \in X_1$.

![Figure 1: Illustration of $gL$, $gR$, and $A(g), B(g)$.](image)
Because $X_1^2 \subset X_1$, $X_1$ is a subgroup of $X$. Suppose that $X_2$ is empty, so that $X_1 = X$. Then if $\delta < \frac{1}{3}$, $|L|^2 = \sum g |gL \cap L| > \frac{2}{3}|L|^n$; i.e., $|L| > \frac{2}{3}n$. We are already assuming that $|L| \leq \frac{1 + \varepsilon}{2} n$, so there is a contradiction as long as we eventually choose $\varepsilon < \frac{1}{3}$.

On the contrary, we have that $X_2$ is non-empty. As $X_1$ is a proper subgroup of $X$ and $X_2$ is its complement with $X_2^2 \subset X_1$, it follows that $X_1$ is a subgroup of index 2 with $X_2$ as its unique non-trivial coset. Because of this,

$$
\frac{n}{2}(1 - \delta)|L| \leq \sum_{g \in X_1} |L \cap gL| = |L \cap X_1|^2 + |L \cap X_2|^2
= |L|^2 - 2|L \cap X_1||L \cap X_2|
\leq \frac{n}{2}(1 + \varepsilon)|L| - 2|L \cap X_1||L \cap X_2|.
$$

It follows that

$$
|L \cap X_1||L \cap X_2| \leq (\varepsilon + \delta)\frac{n}{2}|L|.
$$

This means that there is $i \in \{1, 2\}$ so that $|L \cap X_i| \leq \sqrt{(\varepsilon + \delta)\frac{n}{2}|L|}$. Let $X_j$ be the other coset of $X_1$, for which $|L \cap X_j| = |L| - |L \cap X_i| \geq |L| - \sqrt{(\varepsilon + \delta)\frac{n}{2}|L|}$.

Let $s \in S$, consider the set $X_j s \cap X_j$. If $g \in X_j s \cap X_j$, either (1) $g \in X_j - L$, (2) $g \in (X_j - L)s$ or (3) $\{g, gs^{-1}\}$ is an edge in $E(L, L)$.

$$
|X_j s \cap X_j| \leq 2|X_j - L| + I(L) \leq 2 \left(\frac{n}{2} - |X_j \cap L|\right) + \beta_{out} n
\leq 2 \left(\frac{n}{2} - |L| + \sqrt{(\varepsilon + \delta)\frac{n}{4}|L|}\right) + \beta_{out} n
\leq 2 \left(\frac{n}{2}(\varepsilon + \beta_{out}) + \frac{n}{2}\sqrt{\frac{1}{2}(\varepsilon + \delta)(1 + \varepsilon)}\right) + \beta_{out} n
= \left(\varepsilon + 2\beta_{out} + \sqrt{\frac{1}{2}(\varepsilon + \delta)(1 + \varepsilon)}\right)n.
$$

Assuming we choose $\delta < \left(\frac{1}{2} - \varepsilon - 2\beta_{out}\right)^2 \frac{2}{1 + \varepsilon} - \varepsilon$, then $|X_j s \cap X_j| < \frac{n}{2}$. But, if $s \in X_1$, then $|X_j s \cap X_j| = |X_j| = \frac{n}{2}$. Therefore $S \subset X_2$. If $g \in X_1$ and $s \in S$, then $gs \in gX_2 = X_2$.

Likewise if $g \in X_2$ and $s \in S$, then $gs \in gX_2 = X_1$. Every edge of $G$ is incident to one vertex from $X_1$ and one in $X_2$, in other words, $G$ is bipartite. This is our desired contradiction. So instead, let $g$ be an element of $X$ for which $|gL \cap L| \in (\delta |L|, (1 - \delta)|L|)$.

$$
|A(g)| \geq |gL \cap L| \geq \delta |L| \geq \frac{\delta(1 - \varepsilon - \beta_{out})n}{2}.
$$

$$
|B(g)| \geq |gL \cap R| \geq |L - gL| - |(L \cup R)^c| \geq \frac{\delta(1 - \varepsilon - \beta_{out})n}{2} - (\varepsilon + \beta_{out})n.
$$

Now we can complete the bound on the vertex expansion:

$$
h_{out}(G) \leq \frac{2\beta_{out}n}{\min |A(g)|, |B(g)|} \leq \frac{2\beta_{out}n}{\frac{\delta(1 - \varepsilon - \beta_{out})n}{2} - (\varepsilon + \beta_{out})n} = \frac{2\beta_{out}}{\frac{\delta(1 - \varepsilon - \beta_{out})}{2} - (\varepsilon + \beta_{out})}.
$$
At this point we will make our choices of $\delta$ and $\varepsilon$. We have previously required that
\[
\delta < \frac{1}{3}, \quad \delta < 1 - \frac{2}{3(1 - \varepsilon - \beta_{\text{out}})}, \quad \text{and} \quad \delta < \left(\frac{1}{2} - \varepsilon - 2\beta_{\text{out}}\right)^2 \frac{2}{1 + \varepsilon} - \varepsilon.
\]
If $\beta_{\text{out}} \geq \varepsilon$, then we have a bound $h_{\text{out}} \leq 2 \leq \frac{2}{\varepsilon} \beta_{\text{out}}$. In the other case, we assume that $\beta_{\text{out}} \leq \varepsilon$; it is now enough to require that
\[
\delta < \frac{1}{3}, \quad \delta < 1 - \frac{2}{3 - 6\varepsilon}, \quad \text{and} \quad \delta < \left(\frac{1}{2} - 3\varepsilon\right)^2 \frac{2}{1 + \varepsilon} - \varepsilon.
\]
To satisfy these restrictions we will set $\varepsilon = \frac{1}{100}$ and $\delta = \frac{1}{5}$. To summarize all the cases,
\[
\begin{align*}
  h_{\text{out}} & \leq \\
  & \begin{cases} \\
    \frac{1}{\varepsilon} \beta_{\text{out}} & \text{if } \varepsilon n \leq |Y| \leq \frac{1}{2} n, \\
    \beta_{\text{out}} & \text{if } |Y| \geq \frac{1}{2} n, \\
    \frac{2}{\varepsilon} \beta_{\text{out}} & \text{if } \max|L|,|R| \geq \left(\frac{1 + \varepsilon}{2}\right) n, \\
    \frac{2 \beta_{\text{out}}}{2(1 - 2\varepsilon)^2} & \text{otherwise.}
  \end{cases}
\end{align*}
\]
Substituting our choices of $\varepsilon$ and $\delta$ into each of these bounds completes the proof.

In this theorem, we derive a similar relationship between the edge versions of $h$ and $\beta$. The proof of this result is almost identical to that of Theorem 2.4.

**Theorem 2.5.** Let $G$ be a non-bipartite simple Cayley graph corresponding to a group $X$ and a generating set $S$ which satisfies $S^{-1} = S$ and $id_X \notin S$. Let $n = |X|$ and $d = |S|$. Assume that $\beta(G) < \frac{1}{15d}$. Then
\[
h \leq \frac{100\beta}{1 - 15d\beta}.
\]

**Proof.** Choose the disjoint sets $L, R$ that achieve $\beta$, so that $|\partial(L \cup R)| \leq d\beta|L \cup R|$. Set $Y = X - (L \cup R)$, so that $\partial Y = \partial(L \cup R)$. Let $\varepsilon > 0$ be a constant that we will fix later. We will first consider the case that $|Y| > \varepsilon n$.

If $\varepsilon n < |Y| < \frac{1}{2} n$, then
\[
h \leq \frac{|\partial Y|}{d|Y|} \leq \frac{d\beta|L \cup R|}{d\varepsilon n} < \frac{1}{\varepsilon} \beta.
\]
If $|Y| \geq \frac{1}{2} n$, then $|L \cup R| < \frac{1}{2} n$ as $Y$ is disjoint from $L$ and $R$. And then
\[
h \leq \frac{|\partial(L \cup R)|}{d|L \cup R|} \leq \beta.
\]
For the remainder of the proof we will assume $|Y| \leq \varepsilon n$. For any $g \in X$, define the sets $A(g) = (gL \cap L) \cup (gR \cap R)$ and $B(g) = (gR \cap L) \cup (gL \cap R)$.

Observe that $A$ and $B$ are disjoint, since $L$ and $R$ are disjoint. Also define $Z(g) = Y \cup gY$ to be the complement of $A(g) \cup B(g)$, so that $\{A, B, Z\}$ is a partition of $X$. Note that
\[ \partial A = E(A, B) \cup E(A, Z), \] similarly \[ \partial B = E(B, A) \cup E(B, Z), \] and so we next bound \(|\partial_{out}A|\) and \(|\partial_{out}B|\).

Consider the set \(E(A, B)\): if \((v, w) \in A \times B\), then \(v\) and \(w\) must be common members of one of \(L, R, gL, gR\). If \((v, w) \in E(A, B)\), then \((v, w)\) must be in one of the following four sets: \(E(L, L), E(R, R), E(gL, gL), E(gR, gR)\). And so \(e(A, B) \leq e(L, L) + e(R, R) + e(gL, gL) + e(gR, gR) = 2(e(L, L) + e(R, R))\).

Similarly, any edge in \(E(A, Z)\) or \(E(B, Z)\) must be in \(E(L \cup R, Y)\) or \(E(g(L \cup R), gY)\), and so \(e(A, Z), e(B, Z) \leq |\partial(L \cup R)| + |\partial g(L \cup R)| = 2|\partial(L \cup R)|\).

It follows that \(|\partial A(g)|, |\partial B(g)| \leq 2(e(L, L) + e(R, R) + |\partial(L \cup R)|),\) and therefore
\[
\begin{aligned}
   h(G) &\leq \frac{2(e(L, L) + e(R, R) + |\partial(L \cup R)|)}{d \cdot \min\{|A(g)|, |B(g)|\}}.
\end{aligned}
\]

Here we use the facts that \(e(gL, gL) = e(L, L)\) and \(e(gR, gR) = e(R, R)\). By our choice of \(L\) and \(R\), the numerator is precisely \(2d\beta|L \cup R|\), hence it remains to bound \(\min\{|A(g)|, |B(g)|\}\).

Once again we will use a technique of \([10]\).

Without loss of generality, assume that \(|L| \geq |R|\). Notice that \(|L \cup R| = n - |Y| \geq n - \varepsilon n\), and hence by assumption \(|L| \geq \frac{1-\varepsilon}{2} n\).

Suppose that \(|L| \geq \frac{1-\varepsilon}{2} n\). Observe that \(e(L, RC^c) \geq d(|L| - |R|) \geq d\varepsilon n\). In this case we have the bound \(\beta \geq \frac{e(L, RC^c)}{nd} \geq \varepsilon\), and so \(h \leq 1 \leq \frac{1}{\varepsilon} \beta\).

We will consider the other case, where \(\frac{1-\varepsilon}{2} n \leq |L| \leq \frac{1+\varepsilon}{2} n\).

Assume for contradiction that there is no element \(g \in X\) for which \(|L \cap gL| \in (\delta|L|, (1-\delta)|L|)\), where \(\delta\) is a constant we will define later. Define the sets \(X_1 = \{ g : |L \cap gL| \geq (1-\delta)|L| \}\) and \(X_2 = \{ g : |L \cap gL| \leq \delta|L| \}\). By assumption \(X_1, X_2\) is a partition of \(X\).

We can show that if \(\delta < \frac{1}{3}\), then \(X_1^2 \subset X_1\). Let \(g, h \in X_1,\) then
\[
|ghL \cap L| \geq |L| - |gL - ghL| - |L - gL| \\
\geq (1 - 2\delta)|L| > \delta|L| ,
\]
and so assuming \(\delta < \frac{1}{3}\), \(gh \in X_1\).

Similarly we can show that if \(\delta < 1 - \frac{2}{3-3\varepsilon}\), then \(X_2^2 \subset X_1\). Let \(g, h \in X_2,\) then
\[
|ghL \cap L| \geq |L - gL| - |X - (gL \cap ghL)| \\
= 3|L| - n - |L \cap gL| - |gL \cap ghL| \\
\geq (3 - \frac{2}{1-\varepsilon})|L| - 2\delta|L| > \delta|L| ,
\]
and so assuming \(\delta < 1 - \frac{2}{3-3\varepsilon}\), \(gh \in X_1\).

Because \(X_1^2 \subset X_1, X_1\) is a subgroup of \(X\). Suppose that \(X_2\) is empty, so that \(X_1 = X\). Then if \(\delta < \frac{1}{3}\), \(|L|^2 = \sum_g |gL \cap L| > \frac{2}{3}|L| n\); i.e., \(|L| > \frac{2}{3} n\). We already assumed that \(|L| \leq \frac{1+\varepsilon}{2} n\), so there is a contradiction as long as we eventually choose \(\varepsilon < \frac{1}{3}\).
On the contrary, we have that $X_2$ is non-empty. As $X_1$ is a proper subgroup of $X$ and $X_2$ is its complement with $X_2^2 \subset X_1$, it follows that $X_1$ is a subgroup of index 2 with $X_2$ as its unique non-trivial coset. As a result,

$$\frac{n}{2}(1-\delta)|L| \leq \sum_{g \in X_1} |L \cap gL|$$

$$= |L \cap X_1|^2 + |L \cap X_2|^2$$

$$= |L|^2 - 2|L \cap X_1||L \cap X_2|$$

$$\leq \frac{n}{2}(1+\varepsilon)|L| - 2|L \cap X_1||L \cap X_2|.$$ 

It follows that

$$|L \cap X_1||L \cap X_2| \leq (\varepsilon + \delta)\frac{n}{2}|L|.$$ 

This means that $\exists i \in \{1, 2\}$ so that $|L \cap X_i| \leq \sqrt{(\varepsilon + \delta)\frac{n}{2}}|L|$. Let $X_j$ be the other coset of $X_1$, for which $|L \cap X_j| = |L| - |L \cap X_i| \geq |L| - \sqrt{(\varepsilon + \delta)\frac{n}{2}}|L|$.

Let $s \in S$, consider the set $X_js \cap X_j$. If $g \in X_js \cap X_j$, then either (1) $g \in X_j - L$, or (2) $g \in (X_j - L)s$ or (3) $g \in I(L)$.

$$|X_js \cap X_j| \leq 2|X_j - L| + I(L)$$

$$\leq 2\left(\frac{n}{2} - |X_j \cap L|\right) + d\beta n$$

$$\leq 2\left(\frac{n}{2} - |L| + \sqrt{(\varepsilon + \delta)\frac{n}{2}}|L|\right) + d\beta n$$

$$\leq 2\left(\frac{\varepsilon n}{2} + \frac{n}{2}\sqrt{\frac{1}{2}(\varepsilon + \delta)(1 + \varepsilon)}\right) + d\beta n$$

$$= \left(\varepsilon + d\beta + \sqrt{\frac{1}{2}(\varepsilon + \delta)(1 + \varepsilon)}\right)n.$$

Assuming that we chose $\delta < \left(\frac{1}{2} - \varepsilon - d\beta\right)^2\frac{2}{1+\varepsilon} - \varepsilon$, then $|X_js \cap X_j| < \frac{n}{2}$. But, if $s \in X_1$, then $|X_js \cap X_j| = |X_j| = \frac{n}{2}$. Therefore $S \subset X_2$. If $g \in X_1$ and $s \in S$, then $gs \in gX_2 = X_2$.

Likewise if $g \in X_2$ and $s \in S$, then $gs \in gX_2 = X_1$. Every edge of $G$ is incident to one vertex from $X_1$ and one in $X_2$, in other words, $G$ is bipartite. This is our desired contradiction. So instead, let $g$ be an element of $X$ for which $|gL \cap L| \leq (\delta|L|, (1-\delta)|L|)$.

$$|A(g)| \geq |gL \cap L| \geq \delta|L| \geq \frac{\delta(1-\varepsilon)n}{2}.$$ 

$$|B(g)| \geq |gL \cap R| \geq |L - gL| - |Y| \geq \frac{\delta(1-\varepsilon)n}{2} - \varepsilon n.$$ 

Now we can complete the bound of the Cheeger constant;

$$h(G) \leq \frac{2d\beta n}{d \cdot \min |A(g)|, |B(g)|} \leq \frac{2d\beta n}{\frac{\delta(1-\varepsilon)n}{2} - \varepsilon n} = \frac{2\beta}{\frac{\delta(1-\varepsilon)n}{2} - \varepsilon n}.$$ 

At this point we will make our choices of $\delta$ and $\varepsilon$. We have previously required that

$$\delta < \frac{1}{3}, \; \delta < 1 - \frac{2}{3-3\varepsilon}, \; \text{and} \; \delta < \left(\frac{1}{2} - \varepsilon - 2d\beta\right)^2\frac{2}{1+\varepsilon} - \varepsilon.$$ 

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To satisfy these bounds, take \( \varepsilon = \frac{1}{100} \) and \( \delta = \frac{32}{100} - 4d\beta \).

\[
h(G) \leq \frac{2\beta}{\left( \frac{32}{100} - 4d\beta \right) - \frac{99}{200} - \frac{1}{100}} < \frac{2\beta}{\frac{14}{100} - 2d\beta} < \frac{15\beta}{1 - 15d\beta}.
\]

After using our values of \( \varepsilon \) and \( \delta \), we can summarize all the cases:

\[
h \leq \begin{cases} 
100\beta & \text{if } \frac{1}{100} n \leq |Y| \leq \frac{1}{2} n, \\
\beta & \text{if } |Y| \geq \frac{1}{2} n, \\
100\beta & \text{if } \max |L|, |R| \geq (\frac{101}{200}) n, \\
\frac{15\beta}{1 - 15d\beta} & \text{otherwise}.
\end{cases}
\]

As a general bound, we may conclude with

\[
h \leq \frac{100\beta}{1 - 15d\beta},
\]

completing the proof.

Now by combining our results with Trevisan’s bound, we obtain the following theorem that improves on the main result of [3].

**Theorem 2.6.** Let \( G \) be a \( d \)-regular Cayley graph.

1. There is a universal constant \( C_1 \) so that
   \[
   \mu \geq \frac{C_1 h_{\text{out}}^2}{d^2}.
   \]

2. There is a universal constant \( C_2 \) so that the following holds: if \( \beta \leq \frac{1}{30d} \) then
   \[
   \mu \geq C_2 h^2.
   \]
   Otherwise,
   \[
   \mu \geq \frac{C_2}{d^2}.
   \]

**Proof.** To see Item 1, recall from Theorem [1.1] that \( \mu \geq \frac{1}{2} \beta^2 \). From Theorem [2.3] recall that \( \beta \geq \frac{1}{d} \beta_{\text{out}} \). And our result in Theorem [2.4] we have \( \beta_{\text{out}} \geq \frac{1}{200} h_{\text{out}} \). Combining these inequalities gives the desired result, with \( C_1 = \frac{1}{80,000} \).

To see Item 2, we again use \( \mu \geq \frac{1}{2} \beta^2 \). The result of Theorem [2.5] is that \( h \leq \frac{100\beta}{1 - 15d\beta} \). If \( \beta \leq \frac{1}{30d} \), then \( h \leq 200\beta \). On the other hand, if \( \beta > \frac{1}{30d} \), then we see directly that \( \mu \geq \frac{1}{2\left(30d^2\right)} \). These two cases give the desired result, with \( C_2 = \frac{1}{80,000} \).
2.3 Examples

As an illustrative example of our proof method in Theorems 2.4 and 2.5, we will examine an odd cycle.

Example 2.7. Let $X = \mathbb{Z}_{2k+1}$ where $k$ is a positive integer, and consider the Cayley graph $G = (X, \{\pm 1\})$. Using the methods of our theorem, we can use the $\beta_{out}(G)$-achieving almost-bipartition to obtain a vertex cut which approximates $h_{out}(G)$. Similarly we can use the $\beta(G)$-achieving almost-bipartition to obtain a cut which approximates $h(G)$.

Proof. First we solve the vertex-expansion problem. $\beta_{out} = \frac{1}{2k}$, achieved by taking $L$ to be the odd integers $\{1, 3, \ldots, 2k - 1\}$, $R = \{2, 4, \ldots, 2k\}$ and leaving $2k + 1$ uncolored.

Take $g = k$, so that $gL = \{k + 1, k + 3, \ldots, k - 2\}$ and $gR = \{k + 2, k + 4, \ldots, k - 1\}$. $Z(k) = \{0, k\}$. If $k$ is even, then $L \cap gL = \{k + 1, k + 3, \ldots, 2k - 1\}$ and $R \cap gR = \{k + 2, k + 4, \ldots, 2k\}$, thus $A(k) = \{k + 1, \ldots, 2k\}$ and $B(k) = \{1, \ldots, k - 1\}$. On the other hand if $k$ is odd, it can be seen that $A(k) = \{1, \ldots, k - 1\}$ and $B(k) = \{k + 1, \ldots, 2k\}$. In either case, $\partial_{out}A(k) = \partial_{out}B(k) = Z(k)$ and we have the bound

$$h_{out} \leq \frac{|Z(k)|}{\min |A(k)|, |B(k)|} = \frac{2}{k - 1}.$$

Observe that in this case we have approximately achieved the actual value $h_{out} = \frac{2}{k}$ by choosing an optimal value of $k$. It is well known that $\mu = \Theta(\frac{1}{k^2})$, and clearly $d = \Theta(1)$, so our bound $\mu \geq \frac{h_{out}^2}{\mu^2}$ is tight up to a constant factor.

Now, we use similar methods to work with the edge expansion. $\beta = \frac{1}{2k+1}$, achieved by taking $L$ to be the odd integers $\{1, 3, \ldots, 2k + 1\}$, $R = \{2, 4, \ldots, 2k\}$.

Take $g = k$, so that $gL = \{k + 1, k + 3, \ldots, k - 2, k\}$ and $gR = \{k + 2, k + 4, \ldots, k - 1\}$. $Z(k) = \{\}$ If $k$ is even, then $L \cap gL = \{k+1, k_3, \ldots, 2k+1\}$ and $R \cap gR = \{k+2, k+4, \ldots, 2k\}$, thus $A(k) = \{k + 1, \ldots, 2k + 1\}$ and $B(k) = \{1, \ldots, k\}$. On the other hand if $k$ is odd,

![Figure 2: Illustration of $A$ and $B$ in the odd cycle $C_9$.](image-url)
it can be seen that \( A(k) = \{1, \ldots, k\} \) and \( B(k) = \{k + 1, \ldots, 2k + 1\} \). In either case, \( \partial A(k) = \partial B(k) = \{k, k + 1, 2k + 1\} \) and we have the bound

\[
h \leq \frac{2}{2 \min |A(k)|, |B(k)|} = \frac{1}{k - 1}.\]

Observe that in this case we have approximately achieved the actual value \( h = \frac{1}{k} \) by choosing an optimal value of \( k \). It is well known that \( \mu = \Theta(\frac{1}{k^2}) \), so our bound \( \mu \gtrsim h^2 \) is tight up to a constant factor.

**Remark.** Notice that in this example, we did not need to use Freiman’s method, as it is simple to explicitly find a value \( g \) for which \( A(g) \) and \( B(g) \) are both \( \Theta(n) \).

We will now give a simple example that shows our bound \( \beta \gtrsim h \) is not tight for general Cayley graphs, and indeed that there cannot be a reverse inequality of the form \( h \gtrsim f(\beta, d) \) for any non-trivial function \( f \).

**Example 2.8.** Let \( X = Z_3 \times Z_{2k+1} \) where \( k \) is a positive integer, let \( S = \{(\pm 1, 0), (0, \pm 1)\} \), and let \( G = (X, S) \) be the Cayley graph. Then \( h_{\text{out}}(G) \ll \beta_{\text{out}}(G) \).

**Proof.** Consider the set \( A = \{[k] \times Z_3\} \), with \( |\partial_{\text{out}}(A)| = 6 \) and \( |A| = 3k \). Using \( A \) as a candidate we see that \( h_{\text{out}} \leq \frac{2}{k} \).

Let \( L, R \) be a candidate bipartition of \( X \). For any 3-cycle \( C \) in \( G \), if \( L \) or \( R \) intersects \( C \) then at least one vertex of \( C \) must be in \( \partial_{\text{out}}(L \cap R) \) or be counted by \( I(L) \) or \( I(R) \).

That means that

\[
b_{\text{out}}(L, R) = \frac{I(L) + I(R) + |\partial_{\text{out}}(L \cup R)|}{|L \cup R|} \\
\geq \frac{\sum_C I_C(L) + I_C(R) + |\partial_{\text{out}, C}(L \cup R)|}{|L \cup R|} \\
\geq \frac{\frac{1}{3} \sum_C |(L \cup R) \cap C|}{|L \cup R|} = \frac{1}{3}.
\]

So \( \beta_{\text{out}} \geq \frac{1}{3} \gg \frac{2}{k} \geq h_{\text{out}} \).

At a high level we are looking for bounds on \( h \) in terms of \( \beta \) and \( d \). As \( \beta \) and \( d \) are both \( \Theta(1) \), this example tells us that there can be no lower bound on \( h \) that applies to all Cayley graphs. Observe that a similar analysis gives a similar result for \( h \) and \( \beta \) on the same graph.

### 3 Open Questions

- Recall the Cheeger inequalities \( 2h \geq \lambda \geq \frac{1}{2}h^2 \), where \( \lambda := 1 - \lambda_2 \). A problem of general interest is to categorize the graphs for which \( \lambda \approx h \) and those for which
\( \lambda \approx h^2 \). Similarly, we can ask if there are some non-trivial classes of non-bipartite graphs for which \( \mu \approx \beta^2 \) (or alternately \( \mu \approx \beta \)). In particular, there has recently been investigation into various definitions of the discrete curvature. For example, Klartag et al. \cite{11} (see also \cite{9}) demonstrated that if a graph has non-negative curvature in the sense of the curvature-dimension inequality, then \( 16dh^2 \geq \lambda \); that is, \( \lambda \approx h^2 \). A class of graphs for which this curvature bound holds is Cayley graphs of abelian groups. Is there a definition of discrete curvature that permits the characterization of a class of graphs for which \( \mu \approx \beta^2 \)?

- In our result \( \mu \geq \frac{C_{\text{out}}}{d^2} \) our focus was on obtaining the correct dependence of \( \mu \) on \( h_{\text{out}} \) and we did not explore the tightness in terms of degree \( d \). In the proof we first relate \( \mu \) to \( \beta \) and then use the simple bound \( \beta \geq \frac{1}{d} \beta_{\text{out}} \) from Theorem 2.3. Bobkov, Houdré, and the third author \cite{5} introduced a functional constant \( \lambda_{\infty} \) and used the proof methods of Cheeger inequalities to demonstrate an analogous relationship between \( \lambda_{\infty} \) and \( h_{\text{out}} \).

  Is it possible to do the same for \( \beta_{\text{out}} \); that is, can we define a functional constant \( \mu_{\infty} \) (say) and prove directly a relationship between \( \mu_{\infty} \) and \( \beta_{\text{out}} \). This would be in contrast to our current proof which relates \( \mu \) and \( \beta_{\text{out}} \), using \( \beta \) as an intermediary.

- Biswas and Saha \cite{4} proved that for any non-bipartite Cayley sum graph (that is, a graph defined by the relation \((g, h) \in E \text{ iff } gh \in S \) for some generating set \( S \)), \( \mu \geq \frac{C_{\text{out}}}{d^8} \) for a universal constant \( C \). To obtain this result they modified the proof method of Biswas’s similar result for Cayley graphs \cite{3}. The modification is necessary because the original result makes use of the vertex-transitivity of a Cayley graph; a Cayley sum graph need not be transitive. Is it possible to extend our Theorem 2.6 to the setting of Cayley sum graphs in a similar way?

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