A local limit theorem for the Poisson distribution and its application to the Le Cam distance between Poisson and Gaussian experiments and asymptotic properties of Szasz estimators

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Abstract

In this paper, we develop a precise local limit theorem for the Poisson distribution. We then apply the result to prove an upper bound on the Le Cam distance between Poisson and Gaussian experiments. We also use it to derive the asymptotics of the variance for Bernstein c.d.f. and density estimators with Poisson weights on the positive half-line (also called Szasz estimators).

Keywords: Poisson distribution, local limit theorem, asymptotic statistics, Gaussian distribution, Le Cam distance, deficiency, comparison of experiments, Bernstein estimator, Szasz estimator, distribution function estimation, density estimation

2020 MSC: Primary: 62E20 Secondary: 62B15, 62G05, 62G07

1. Introduction

Given $m \in \mathbb{N}$ and $x \geq 0$, the Poisson($mx$) probability mass function is defined by

$$V_{k,m}(x) = \frac{(mx)^k}{k!} e^{-mx}, \quad k \in \mathbb{N}_0.$$  \hfill (1.1)

The purpose of this paper is to establish an asymptotic expansion for (1.1) in terms of the Gaussian density with the same covariance profile, namely:

$$\phi_{mx}(y) := \frac{1}{\sqrt{2\pi mx}} \exp \left( - \frac{y^2}{2mx} \right), \quad y \in \mathbb{R}.$$  \hfill (1.2)

This kind of expansion can be useful in all sorts of estimation problems; we give two examples in Section 3. For a general presentation on local limit theorems, see e.g. Borovkov (2013).

Remark 1.1. Throughout the paper, the notation $u = \mathcal{O}(v)$ means that $\limsup |u/v| < C$, as $m \to \infty$ or $n \to \infty$ depending on the context, where $C \in (0, \infty)$ is a universal constant. Whenever $C$ might depend on a parameter, we add a subscript (for example, $u = \mathcal{O}_x(v)$).

Similarly, $u = o(v)$ means that $\lim |u/v| = 0$ as $m \to \infty$ or $n \to \infty$, and subscripts indicate which parameters the convergence rate can depend on.

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\textsuperscript{1}F. O. is supported by a postdoctoral fellowship from the NSERC (PDF) and the FRQNT (B3X supplement).
2. Main result

General local asymptotic expansions of probabilities related to the sums of lattice random variables are well-known in the literature, see e.g. Theorem 1 in Bikyalis (1969), Theorem 1 in Osipov (1969), Theorem 1 in Lazakovićius (1969), Theorem 22.1 in Bhattacharya & Ranga Rao (1976), etc. However, the error terms in these expansions are often not explicit enough for applications. By using the specificity of the distribution at hand, it is often possible to refine those results and obtain explicit and exact rates of convergence with a fraction of the mathematical machinery.

Our main theorem below is a local limit theorem for the Poisson distribution.

**Theorem 2.1.** For all \( m \in \mathbb{N} \) and \( x \in (0, \infty) \), let

\[
\phi(z) := \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{and} \quad \delta_k := \frac{k - mx}{\sqrt{mx}}. \tag{2.1}
\]

For any \( \eta \in (0, 1) \), we have that as \( n \to \infty \), and uniformly for \( k \in \mathbb{N}_0 \) such that \( \frac{\delta_k}{\sqrt{mx}} \leq \eta \),

\[
V_{k,m}(x) = 1 + \frac{1}{\sqrt{mx}} \left( \frac{1}{6} \delta_k^3 - \frac{1}{2} \delta_k \right) + \frac{1}{mx} \left( \frac{1}{72} \delta_k^6 - \frac{1}{6} \delta_k^4 + \frac{3}{8} \delta_k^2 + \frac{1}{12} \right) + O_{x,\eta}\left( \frac{1 + \delta_k}{m^{3/2}} \right). \tag{2.2}
\]

**Proof.** By taking the logarithm on the left-hand side of (2.1), we have

\[
\log \left( \frac{V_{k,m}(x)}{1/\sqrt{mx} \phi(\delta_k)} \right) = \frac{1}{2} \log(2\pi) + (k + \frac{1}{2}) \log(mx) - \log(k!) - \frac{mx}{2} + \frac{1}{2} \delta_k^2. \tag{2.3}
\]

Stirling’s formula yields

\[
\log k! = \frac{1}{2} \log(2\pi) + (k + \frac{1}{2}) \log k - k + \frac{1}{12k} + O(k^{-3}), \tag{2.4}
\]

(see e.g. (Abramowitz & Stegun, 1964, p.257)). Hence, we get

\[
\log \left( \frac{V_{k,m}(x)}{1/\sqrt{mx} \phi(\delta_k)} \right) = -k \log \left( \frac{k}{mx} \right) - \frac{1}{2} \log \left( \frac{k}{mx} \right) \\
\quad + (k - mx) + \frac{1}{2} \delta_k^2 + \frac{1}{12m} \left( \frac{k}{m} \right)^{-1} + O\left( \frac{1}{m^3} \left( \frac{k}{m} \right)^{-3} \right) \\
= -mx \left( 1 + \frac{\delta_k}{\sqrt{mx}} \right) \log \left( 1 + \frac{\delta_k}{\sqrt{mx}} \right) \\
\quad + (k - mx) + \frac{1}{2} \delta_k^2 - \frac{1}{2} \log \left( 1 + \frac{\delta_k}{\sqrt{mx}} \right) \\
\quad + \frac{1}{12mx} \left( 1 + \frac{\delta_k}{\sqrt{mx}} \right)^{-1} + O\left( \frac{1}{(mx)^{3/2}} \left( 1 + \frac{\delta_k}{\sqrt{mx}} \right)^{-3} \right). \tag{2.5}
\]

Now, note that for \( |y| \leq \eta < 1 \),

\[
(1 + y) \log(1 + y) = y + \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^4}{12} + O_\eta(y^5), \\
\log(1 + y) = y - \frac{y^2}{2} + O_\eta(y^3), \\
(1 + y)^{-1} = 1 + O_\eta(y). \tag{2.6}
\]
By applying these estimates in (2.5), we obtain

\[
\log \left( \frac{V_{k,m}(x)}{\sqrt{m}x^{\phi(\delta_k)}} \right) = -m \left\{ \frac{\delta_k}{\sqrt{m}x} \left( \frac{\delta_k}{\sqrt{m}x} \right)^2 \right\} + \frac{k - mx}{2} \delta_k^2 - \frac{1}{2} \left( \frac{\delta_k}{\sqrt{m}x} \right)^2 + O_x,\eta \left( \left( \frac{\delta_k}{\sqrt{m}x} \right)^3 \right) \right. \\
+ \frac{1}{12mx} \left\{ 1 + O_x,\eta \left( \frac{\delta_k}{\sqrt{m}x} \right) \right\} + O_{x,\eta}(m^{-3}).
\]

(2.7)

The terms \( k - mx \) and \( \frac{1}{2} \delta_k^2 \) cancel out with the first two terms inside the braces on the first line. Therefore,

\[
\log \left( \frac{V_{k,m}(x)}{\sqrt{m}x^{\phi(\delta_k)}} \right) = \left\{ \frac{1}{6 \sqrt{m}x} + \frac{1}{12 mx} \right\} + O_x,\eta \left( \frac{\delta_k^5}{m^{3/2}} \right) \\
+ \left\{ -\frac{1}{2} \frac{\delta_k}{\sqrt{m}x} - \frac{1}{4} \delta_k^2 + O_x,\eta \left( \frac{\delta_k^3}{m^{3/2}} \right) \right\} + \frac{1}{12mx} + O_{x,\eta}(1 + \delta_k) \\
= \frac{1}{6 \sqrt{m}x} + \frac{1}{12 \delta_k} - \frac{1}{2} \delta_k + \frac{1}{4} \delta_k^2 + \frac{1}{12} \left( \frac{1 + \delta_k}{m^{3/2}} \right).
\]

To conclude the proof, we take the exponential on both sides of the last equation, and we expand the right-hand side with

\[
e^y = 1 + y + \frac{y^2}{2} + O_\eta(y^3), \quad \text{for } |y| \leq \eta < 1.
\]

(2.8)

This ends the proof. \( \square \)

3. Applications

In this section, we present two applications of Theorem 2.1 related to the Le Cam distance between Poisson and Gaussian experiments (Section 3.1) and asymptotic properties of Bernstein estimators with Poisson weights (Section 3.2).

3.1. The Le Cam distance between Poisson and Gaussian experiments

In Carter (2002), the author finds an upper bound on the Le Cam distance (called \( \Delta \)-distance in Le Cam & Yang (2000)) between multinomial and multivariate Gaussian experiments. Carter achieves his goal by looking at the total variation between the measure of a multinomial vector for which the components were jittered by uniforms and a multivariate Gaussian measure with the same covariance profile. Carter’s bound was later improved in Ouimet (2020b) by developing a precise local limit theorem for the multinomial distribution (analogous to Theorem 2.1) and applying it to remove the inductive part of Carter’s argument. We use Carter’s strategy below by jittering a Poisson random variable to obtain an upper bound on the Le Cam distance between (unidimensional) Poisson and Gaussian experiments. For an excellent and concise review on Le Cam’s theory for the comparison of statistical models, we refer the reader to Mariucci (2016).

The following result is analogous to Lemma 2 in Carter (2002). It bounds the total variation between a Poisson\( (mx) \) random variable and a Normal\( (mx, mx) \) random variable. The Le Cam bound appears in Theorem 3.2 right after.
Lemma 3.1. Let $K \sim \text{Poisson}(mx)$ and $U \sim \text{Uniform}(-\frac{1}{2}, \frac{1}{2})$, where $K$ and $U$ are assumed independent. Define $Y := K + U$ and let $\tilde{P}_x$ be the law of $Y$. In particular, if $\mathbb{P}_x$ is the law of $K$, note that

$$\tilde{P}_x(B) := \int_{\mathbb{N}_0} \int_{(-\frac{1}{2}, \frac{1}{2})} 1_B(k + u) \mathbb{P}_x(du) \mathbb{P}(dk), \quad B \in \mathcal{B}(\mathbb{R}).$$

Let $Q_x$ be the law of the $\text{Normal}(mx, mx)$ distribution. Then, for all $x \in (0, 1)$, we have

$$\|\tilde{P}_x - Q_x\| \leq \frac{C}{\sqrt{mx}}, \quad \text{as } m \to \infty,$$

where $\| \cdot \|$ denotes the total variation norm and $C > 0$ is a universal constant.

Proof. By the comparison of the total variation norm with the Hellinger distance on page 726 of Carter (2002), we already know that, for any Borel set $A \in \mathcal{B}(\mathbb{R})$,

$$\|\tilde{P}_x - Q_x\| \leq \sqrt{2 \mathbb{P}(Y \in A^c) + \mathbb{E} \left[ \log \left( \frac{d\tilde{P}_x}{dQ_x}(Y) \right) 1_{\{Y \in A\}} \right]}.$$ \hspace{1cm} (3.3)

The idea is to choose a set $A'$ that excludes the bulk of the Poisson distribution so that the probability $\mathbb{P}(Y \in A')$ is small by a concentration bound. Precisely, we can take the set

$$A' := \{ y \in (-\frac{1}{2}, \infty) : |y - mx| > (mx)^{2/3} \}.$$

Then, if we notice that $K \stackrel{\text{law}}{=} \sum_{i=1}^{m} K_i$ where $K_i \stackrel{\text{i.i.d.}}{\sim} \text{Poisson} (x)$, a standard large deviation bound (see e.g. Shiryaev, 1996, eq.16) shows that, for $m$ large enough,

$$\mathbb{P}(Y \in A^c) \leq \mathbb{P} \left( \left| \frac{1}{m} \sum_{i=1}^{m} K_i - x \right| > \frac{(mx)^{2/3} - 1}{m} \right) \leq 100 \exp \left( - \frac{(mx)^{2/3}}{100} \right).$$ \hspace{1cm} (3.5)

For the expectation in (3.3), if $y \mapsto V_{y,m}(x)$ denotes the density function associated with $\tilde{P}_x$ (i.e. it is equal to $V_{k,m}(x)$ whenever $k \in \mathbb{N}_0$ is closest to $y$), then

$$\mathbb{E} \left[ \log \left( \frac{d\tilde{P}_x}{dQ_x}(Y) \right) 1_{\{Y \in A\}} \right] = \mathbb{E} \left[ \log \left( \frac{V_{y,m}(x)}{\sqrt{mx}} \phi(\delta_y) \right) 1_{\{Y \in A\}} \right]$$

$$= \mathbb{E} \left[ \log \left( \frac{V_{K,m}(x)}{\sqrt{mx}} \phi(\delta_K) \right) 1_{\{K \in A\}} \right]$$

$$+ \mathbb{E} \left[ \log \left( \frac{1}{\sqrt{mx}} \phi(\delta_y) \right) 1_{\{K \in A\}} \right]$$

$$+ \mathbb{E} \left[ \log \left( \frac{V_{K,m}(x)}{\sqrt{mx}} \phi(\delta_y) \right) (1_{\{Y \in A\}} - 1_{\{K \in A\}}) \right]$$

$$=: (I) + (II) + (III).$$ \hspace{1cm} (3.6)

By Theorem 2.1 (note that $|\delta_y/\sqrt{mx}| \leq \frac{1}{2}$ for all $k \in A \cap \mathbb{N}_0$ if $m$ is assumed large enough),

$$(I) = \frac{1}{\sqrt{mx}} \cdot \mathbb{E} \left[ \left( \frac{1}{6} \mathbb{E}[K - mx]^3 \right) \frac{1}{(mx)^{3/2}} - \frac{1}{2} \cdot \mathbb{E}[K - mx] \right] 1_{\{K \in A\}}$$

$$+ \mathcal{O} \left( \frac{1}{mx} \left( \frac{\mathbb{E}|K - mx|^6}{(mx)^3} + \frac{\mathbb{E}|K - mx|^4}{(mx)^2} + \frac{\mathbb{E}|K - mx|^2}{mx} + 1 \right) \right) + \mathcal{O}(m^{-3/2}).$$ \hspace{1cm} (3.7)
By Lemma A.1, the first $O(\cdot)$ term above is $O((mx)^{-1})$. By Corollary A.2, we can also control the $\asymp m^{-1/2}$ term on the right-hand side of (3.7). We obtain

$$ (I) = O((mx)^{-1/2}(P(K \in A^c))^{1/2}) + O((mx)^{-1}) + O_x(m^{-3/2})$$

$$ = O((mx)^{-1}).$$

(3.8)

For the term (II) in (3.6),

$$ \log \left( \frac{1}{\sqrt{mx}} \phi(\delta_K) \right) = \frac{(Y - m)^2}{2mx} - \frac{(K - m)^2}{2mx}$$

$$ = \frac{(Y - K)^2}{2mx} + \frac{(Y - K)(K - m)}{mx}. $$

(3.9)

With our assumption that $K$ and $Y - K = U \sim \text{Uniform}(-\frac{1}{2}, \frac{1}{2})$ are independent, we get

$$ (II) = \frac{1/12}{2mx} - \frac{E[(Y - K)^2 I_{\{K \in A^c\}}]}{2mx} - \frac{E[(Y - K)(K - m) I_{\{K \in A^c\}}]}{mx}$$

$$ = \frac{1}{24mx} + O\left( \frac{P(K \in A^c)}{mx} \right) + O\left( \sqrt{\frac{E[(K - m)^2]}{mx}} \right). $$

(3.10)

$$ = O((mx)^{-1}).$$

For the term (III) in (3.6), the following rough bound from Theorem 2.1 and (3.9),

$$ \log \left( \frac{V_{K,m}(x)}{\sqrt{mx}} \phi(\delta_Y) \right) = \log \left( \frac{V_{K,m}(x)}{\sqrt{mx}} \phi(\delta_K) \right) + \log \left( \frac{1}{\sqrt{mx}} \phi(\delta_Y) \right)$$

$$ = O\left( \frac{|K - m|^3}{(mx)^2} + \frac{|K - m|}{mx} + \frac{1}{mx} \right),$$

(3.11)

yields, by Cauchy-Schwarz and Lemma A.1,

$$ (III) = O\left( \sqrt{\frac{E[(K - m)^6]}{(mx)^4} + \frac{E[(K - m)^2]}{(mx)^2}} \right) + O\left( \sqrt{\frac{E[I_{\{Y \in A\}} - I_{\{K \in A\}}]}{2}} \right). $$

(3.12)

Putting (3.8), (3.10) and (3.12) in (3.6), together with the exponential bound

$$ E\left[ I_{\{Y \in A\}} - I_{\{K \in A\}} \right]^2 = E\left[ I_{\{Y \in A, K \in A^c\}} + I_{\{Y \in A, K \notin A\}} - I_{\{K \in A\}} \right]^2$$

$$ = E\left[ I_{\{Y \in A, K \in A^c\}} - I_{\{Y \in A, K \notin A\}} \right]^2$$

$$ \leq 2 P(K \in A^c) + 2 P(Y \in A^c)$$

$$ \leq 4 \cdot 100 \exp\left( - \frac{(mx)^{2/3}}{100} \right),$$

(3.13)

yields, as $m \to \infty$,

$$ E\left[ \log \left( \frac{d_{Q_x}}{d_{Q_{x}}} (Y) \right) I_{\{Y \in A\}} \right] = (I) + (II) + (III) = O((mx)^{-1}).$$

(3.14)

Now, putting (3.5) and (3.14) together in (3.3) gives the conclusion. □
By inverting the Markov kernel that jitters the Poisson random variable, we obtain the aforementioned Le Cam distance upper bound between Poisson and Gaussian experiments.

**Theorem 3.2** (Bound on the Le Cam distance). For any given $R > 0$, define the experiments

\[ P := \{ P_k \}_{k \geq R}, \quad P_k \text{ is the measure induced by Poisson}(mx), \]

\[ Q := \{ Q_k \}_{k \geq R}, \quad Q_k \text{ is the measure induced by Normal}(mx,mx). \]

Then, we have the following bound on the Le Cam distance $\Delta(P, Q)$ between $P$ and $Q$,

\[ \Delta(P, Q) := \max\{ \delta(P, Q), \delta(Q, P) \} \leq C \sqrt{mR}, \]  

(3.15)

where $C > 0$ is a universal constant,

\[ \delta(P, Q) := \inf_{T_1} \sup_{x \geq R} \left\| P_x - \int_{N_0} T_1(k, \cdot) P_x(dk) - Q_x \right\|, \]

(3.16)

and the infima are taken, respectively, over all Markov kernels $T_1 : N_0 \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ and $T_2 : \mathbb{R} \times \mathcal{B}(N_0) \to [0, 1]$.

**Proof.** By Lemma 3.1, we get the desired bound on $\delta(P, Q)$ by choosing the Markov kernel $T_1^*$ that adds $U$ to $K$, namely

\[ T_1^*(k, B) := \int (-\frac{1}{2}, \frac{1}{2}) \mathbb{1}_B(k + u) du, \quad k \in N_0, \quad B \in \mathcal{B}(\mathbb{R}). \]  

(3.17)

To get the bound on $\delta(Q, P)$, it suffices to consider a Markov kernel $T_2^*$ that inverts the effect of $T_1^*$, i.e. rounding off $Z \sim \text{Normal}(mx,mx)$ to the nearest integer. Then, as explained in Section 5 of Carter (2002), we get

\[ \delta(Q, P) \leq \left\| P_x - \int_{\mathbb{R}} T_2^*(z, \cdot) Q_x(dz) \right\| \]

\[ \quad = \left\| \int_{\mathbb{R}} T_2^*(z, \cdot) \int_{N_0} T_1^*(k, dz) P_x(dk) - \int_{\mathbb{R}} T_2^*(z, \cdot) Q_x(dz) \right\| \]

\[ \quad \leq \left\| \int_{N_0} T_1^*(k, \cdot) P_x(dk) - Q_x \right\|, \]

(3.18)

and we get the same bound by Lemma 3.1. \qed

If we consider the following Gaussian experiment with constant variance

\[ Q^* := \{ Q_k^* \}_{k \geq R}, \quad Q_k^* \text{ is the measure induced by Normal}(\sqrt{mx}, 1/4), \]

then (Carter, 2002, Section 7) showed that

\[ \Delta(Q, Q^*) \leq \frac{C}{\sqrt{mx}}, \]  

(3.19)

using a variance stabilizing transformation, with proper adjustments to the deficiencies in (3.16).

**Corollary 3.3.** With the same notation as in Theorem 3.2, we have

\[ \Delta(P, Q^*) \leq \frac{C}{\sqrt{mx}}, \]  

(3.20)

for a universal constant $C > 0$.

**Proof.** This is a direct consequence of Theorem 3.2, Equation (3.19) and the triangle inequality for the pseudometric $\Delta(\cdot, \cdot)$. \qed
3.2. Asymptotic properties of Bernstein estimators with Poisson weights

In the literature, various asymptotic properties for Bernstein estimators of density functions and cumulative distribution functions (c.d.f.s) with Poisson weights were studied, namely: Gawronski & Stadtmüller (1980, 1981) studied the bias, variance and mean squared error for the density estimator, and Hanebeck & Klar (2020) studied the bias, variance, mean squared error, mean integrated squared error, asymptotic normality, uniform strong consistency and relative deficiency with respect to the empirical c.d.f. for the c.d.f. estimator. The estimators are defined as follows. Assume that the observations \(X_1, X_2, \ldots, X_n\) are independent, \(F\)-distributed (with density \(f\)) and supported on \([0, \infty)\). Then, for \(n, m \in \mathbb{N}\), let

\[
\hat{F}_{m,n}(x) := \sum_{k=0}^{\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(-\infty, \frac{k}{m}]}(X_i) \right\} V_{k,m}(x), \quad x \geq 0,
\]

be the Bernstein c.d.f. estimator with the Poisson weights

\[
V_{k,m}(x) := \frac{(mx)^k}{k!} e^{-mx}, \quad k \in \mathbb{N}_0,
\]

(\(\hat{F}_{m,n}\) was introduced in Hanebeck & Klar (2020) as the Szasz estimator, since it can be seen as the Szasz-Mirakyan operator applied to the empirical c.d.f.), and let

\[
\hat{f}_{m,n}(x) := \sum_{k=0}^{\infty} \left\{ \frac{m}{n} \sum_{i=1}^{n} \mathbb{1}_{\left(\frac{k}{m}, \frac{k+1}{m}\right]}(X_i) \right\} V_{k,m}(x), \quad x \geq 0,
\]

be the Bernstein density estimator with Poisson weights (which was introduced in Gawronski & Stadtmüller (1980)). Assuming that \(F\) and \(f\) are respectively two-times and one-time continuously differentiable on \((0, \infty)\), the theorem below computes the asymptotics of the variance of both estimators. The variance of \(\hat{f}_{m,n}\) was previously calculated using known asymptotics of modified Bessel functions of the first kind in Gawronski & Stadtmüller (1980), so (3.27) gives an alternative proof. As for the variance of \(\hat{F}_{m,n}\), it was incorrectly stated in the Theorem 5 of (Hanebeck & Klar, 2020, arXiv v.1), so (3.26) below fixes their statement and proof. For an extensive review of the literature on Bernstein estimation, we refer the interested reader to Section 2 in Ouimet (2020a).

Remark 3.4. The error in the asymptotic expression of the variance in Theorem 5 of (Hanebeck & Klar, 2020, arXiv v.1) ultimately stems from the authors’ adaptation of a method from Leblanc (2012a) to estimate the technical sum \(\sum_{0 \leq k < \ell \leq m} (\frac{k}{m} - x) V_{k,m}(x) V_{\ell,m}(x)\). In Lemma 2 (iv) of Leblanc (2012a), a continuity correction from Cressie (1978) was used to claim that

\[
\sum_{0 \leq k < \ell \leq m} (\frac{k}{m} - x) P_{k,m}(x) P_{\ell,m}(x) + \sqrt{\frac{x(1-x)}{2\pi}} = o_x(1),
\]

where \(P_{k,m}(x) := \binom{m}{k} x^k (1-x)^{m-k}\) are Binomial weights. Unfortunately, there is an error somewhere in Leblanc’s proof because a quick simulation with Mathematica shows that the correct estimate is

\[
\sum_{0 \leq k < \ell \leq m} (\frac{k}{m} - x) P_{k,m}(x) P_{\ell,m}(x) + \sqrt{\frac{x(1-x)}{4\pi}} = o_x(1).
\]

The mathematical proof of (a generalization of) this fact was also given in Lemma A.3 of Ouimet (2020a) by applying a local limit theorem for the multinomial distribution developed in the same article. In Appendix B, a list of articles and theses whose statements (and proofs) have been affected by this error is collected and appropriate fixes are suggested.
Here are the asymptotics of the variance for the Bernstein c.d.f. and density estimators with Poisson weights (or Szasz estimators). With (3.27), we recover the expression previously found in the proof of the theorem in Gawronski & Stadtmüller (1980) (although we use a different method), and (3.26) fixes the statement of Theorem 5 in Hanebeck & Klar (2020).

**Theorem 3.5.** As $n \to \infty$, we have

$$\text{Var}(\hat{F}_{m,n}(x)) = n^{-1}F(x)(1 - F(x)) - n^{-1}m^{-1/2} \sqrt{\frac{x}{\pi}} + o_x(n^{-1}m^{-1/2}),$$

(3.26)

$$\text{Var}(\hat{f}_{m,n}(x)) = n^{-1/2}m^{-1/2} \frac{f(x)}{\sqrt{4\pi x}} + o_x(n^{-1}m^{1/2}).$$

(3.27)

In a first step, we need to estimate some technical sums using our local limit theorem.

**Lemma 3.6.** Let

$$\tilde{R}_{1,m}^S := m^{1/2} \sum_{k,\ell=0}^{\infty} \left( \frac{k\ell}{m} - x \right) V_{k,m}(x) V_{\ell,m}(x), \quad x \in (0, \infty).$$

(3.28)

Then, for any given $x \in (0, \infty)$,

$$\sup_{m \in \mathbb{N}} |\tilde{R}_{1,m}^S(x)| \leq 2\sqrt{x},$$

(3.29)

and we have, as $m \to \infty$,

$$m^{1/2} \sum_{k=0}^{\infty} V^2_{k,m}(x) = \frac{1}{\sqrt{4\pi x}} + o_x(1),$$

(3.30)

$$m \sum_{k=0}^{\infty} V^3_{k,m}(x) = \frac{1}{2\sqrt{3\pi x}} + o_x(1),$$

(3.31)

$$\tilde{R}_{1,m}^S(x) = -\sqrt{\frac{x}{\pi}} + o_x(1).$$

(3.32)
Proof. By the Cauchy-Schwarz inequality, we have, for all \( m \in \mathbb{N} \),

\[
|\tilde{R}_{1,m}(x)| \leq 2m^{-1/2} \sum_{k=0}^{\infty} |k - mx| V_{k,m}(x)
\leq 2m^{-1/2} \sum_{k=0}^{\infty} |k - mx|^2 V_{k,m}(x) = 2m^{-1/2} \cdot \sqrt{mx} \leq 2\sqrt{x},
\]

which proves (3.29). In order to prove (3.30), consider the decomposition

\[
m^{1/2} \sum_{k=0}^{\infty} V_{k,m}^2(x) = m^{1/2} \sum_{k \in \mathbb{N}_0 : \left| \frac{k}{m} \right| \leq \frac{1}{2}}^\infty V_{k,m}^2(x) + m^{1/2} \sum_{k \in \mathbb{N}_0 : \left| \frac{k}{m} \right| > \frac{1}{2}}^\infty V_{k,m}^2(x).
\]

The second term on the right-hand side of (3.34) is exponentially small in \( m \) by a standard large deviation bound, and the first term on the right-hand side can be approximated by a Gaussian integral because of the local limit theorem we developed in Theorem 2.1. If \( \phi_{\sigma^2} \) denotes the density of the \( \mathcal{N}(0, \sigma^2) \) distribution, then

\[
m^{1/2} \sum_{k=0}^{\infty} V_{k,m}^2(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{x}} \phi^2(z)dz + o_x(1)
= \frac{2^{-1/2}}{\sqrt{2\pi x}} \int_{\mathbb{R}} \phi_\frac{1}{2}(z)dz + o_x(1)
= \frac{1}{\sqrt{4\pi x}} \cdot 1 + o_x(1),
\]

which proves (3.30). By a similar argument,

\[
m^{1/2} \sum_{k=0}^{\infty} V_{k,m}^3(x) = \int_{\mathbb{R}} \frac{1}{x} \phi^3(z)dz + o_x(1)
= \frac{3^{-1/2}}{2\pi x} \int_{\mathbb{R}} \phi_\frac{1}{3}(z)dz + o_x(1)
= \frac{1}{2\sqrt{3\pi x}} \cdot 1 + o_x(1),
\]

which proves (3.31). Finally, to obtain the asymptotics of \( R_{1,m}^S(x) \), consider the decomposition

\[
\tilde{R}_{1,m}(x) = 2m^{1/2} \sum_{0 \leq k < \ell < \infty} (\frac{k}{m} - x)V_{k,m}(x)V_{\ell,m}(x) + m^{1/2} \sum_{k=0}^{\infty} (\frac{k}{m} - x)V_{k,m}^2(x).
\]

The second term on the right-hand side of (3.37) is negligible by the Cauchy-Schwarz inequality and (3.31):

\[
m^{1/2} \sum_{k=0}^{\infty} (\frac{k}{m} - x)V_{k,m}^2(x) \leq m^{-1/2} \sqrt{\sum_{k=0}^{\infty} (k-mx)^2 V_{k,m}(x)} \sqrt{\sum_{k=0}^{\infty} V_{k,m}^3(x)}
= m^{-1/2} \cdot \sqrt{mx} \cdot O_x(\sqrt{m^{-1}})
= O_x(m^{-1/2}).
\]
For the first term on the right-hand side of (3.37), we can use the local limit theorem (Theorem 2.1) and integration by parts. Together with (3.37) and (3.38), we obtain

\[ \tilde{R}_{1,m}^S(x) = 2 \cdot x \int_{-\infty}^{\infty} \tilde{z} \phi(x) \int_{z}^{\infty} \phi(y) dy \, dz + o_x(1) \]

\[ = 2 \cdot x \left[ 0 - \int_{-\infty}^{\infty} \phi_x^2(z) \, dz \right] + o_x(1) \]

\[ = - \frac{2x}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} \tilde{\phi}_x(z) \, dz + o_x(1) \]

\[ = - \sqrt{\frac{x}{\pi}} + o_x(1). \quad (3.39) \]

This ends the proof. \qed

\textit{Proof of Theorem 3.5.} By the independence of the observations \( X_i \), a Taylor expansion for the c.d.f. \( F \), and the asymptotic expression for the bias in Theorem 5 of Hanebeck & Klar (2020), we have

\[ \Var(\hat{F}_{m,n}^S(x)) = \frac{1}{n} \left\{ \sum_{k=0}^{\infty} F\left( \frac{\min\{k, \ell\}}{m} \right) - \left( \sum_{k=0}^{\infty} F(\frac{k}{m}) V_{k,m}(x) \right)^2 \right\} \]

\[ = \frac{1}{n} \left\{ F(x)(1 - F(x)) + O_x(m^{-1}) \right. \]

\[ + f(x) \sum_{k=0}^{\infty} \left( \frac{\min\{k, \ell\}}{m} - x \right) V_{k,m}(x) V_{\ell,m}(x) \]

\[ \left. + O_x \left( \sum_{k=0}^{\infty} \frac{|k|}{m} - x \right) V_{k,m}(x) V_{\ell,m}(x) \right\}. \quad (3.40) \]

Now, by the Cauchy-Schwarz inequality, the fact that the variance of a \( \text{Poisson}(mx) \) random variable is \( mx \), and the estimate for \( \tilde{R}_{1,m}^S(x) \) in Lemma 3.6,

\[ \Var(\hat{F}_{m,n}^S(x)) = \frac{1}{n} \left\{ F(x)(1 - F(x)) + O_x(m^{-1}) \right. \]

\[ + m^{-1/2} f(x) \cdot \tilde{R}_{1,m}^S(x) \]

\[ + O_x \left( m^{-2} \sum_{k=0}^{\infty} |k - mx|^2 V_{k,m}(x) \right) \]

\[ = \frac{1}{n} \left\{ F(x)(1 - F(x)) + O_x(m^{-1}) \right. \]

\[ + m^{-1/2} f(x) \cdot \tilde{R}_{1,m}^S(x) \]

\[ = n^{-1} F(x)(1 - F(x)) - n^{-1} m^{-1/2} \sqrt{\frac{2}{\pi}} + o_x(n^{-1} m^{-1/2}). \quad (3.41) \]

Similarly, by the independence of the observations \( X_i \), a Taylor expansion for the density \( f \), and the asymptotic expression for the bias in Equation (8) of Gawronski & Stadtmüller (1980), we have

\[ \Var(\hat{f}_{m,n}^S(x)) = \frac{m^2}{n} \left\{ \sum_{k=0}^{\infty} \int_{m \frac{k}{m} + 1} f(y) dy \, V_{k,m}^2(x) \right. \]

\[ - \left( \sum_{k=0}^{\infty} \int_{m \frac{k}{m} + 1} f(y) dy \, V_{k,m}(x) \right)^2 \]

\[ = n^{-1} m^{1/2} \left\{ \sum_{k=0}^{\infty} \left( f(x) + O_x(m^{-1}) + |k/m - x| \right) V_{k,m}^2(x) \right. \]

\[ + O_x(m^{-1/2}) \right\}. \quad (3.42) \]
By the Cauchy-Schwarz inequality, the fact that the variance of a Poisson($mx$) random variable is $mx$, and the estimates (3.30) and (3.31) in Lemma 3.6, we have

$$\text{Var}(f_{m,n}^S(x)) = n^{-1}m^{1/2} \left\{ \left( f(x) + O_x(m^{-1}) \right) m^{1/2} \sum_{k=0}^{\infty} V_{k,m}^2(x) \right\}$$

$$+ O_x(m^{-1/2}) + O \left( \sqrt{m^{-2} \sum_{k=0}^{\infty} |k - mx|^2 V_{k,m}(x)} \cdot \sqrt{m \sum_{k=0}^{\infty} V_{k,m}^3(x)} \right)$$

$$= n^{-1}m^{1/2} \frac{f(x)}{\sqrt{4\pi x}} + o_x(n^{-1}m^{1/2}).$$

(3.43)

This ends the proof. \(\square\)

From Theorem 3.5, other asymptotic expressions can be (and were) derived such as the mean squared error in Gawronski & Stadtmüller (1980) and Hanebeck & Klar (2020) (assuming the correction in (3.26)). We can also optimize the bandwidth parameter $m$ with respect these expressions to implement a plug-in selection method, exactly as we would in the setting of traditional multivariate kernel estimators, see e.g. (Scott, 2015, Section 6.5) or (Chacón & Duong, 2018, Section 3.6).

A. Technical lemmas

Below, we prove a general formula for the central moments of the Poisson distribution, and evaluate the second, fourth and sixth central moments explicitly. This lemma is used to estimate the $\asymp m^{-1}$ errors in (3.7) of the proof of Lemma 3.1, and also as a preliminary result for the proof of Corollary A.2 below.

**Lemma A.1 (Central moments of the Poisson distribution).** Let $K \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$. The general formula for the central moments is given by

$$\mathbb{E}[(K - \lambda)^n] = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{j=0}^{\ell} \binom{\ell}{j} \lambda^j \right) (-\lambda)^{n-\ell}, \quad n \in \mathbb{N},$$

(A.1)

where $\binom{\ell}{j}$ denotes a Stirling number of the second kind. In particular,

$$\begin{align*}
\mathbb{E}[(K - \lambda)^2] &= \lambda, \\
\mathbb{E}[(K - \lambda)^4] &= 3\lambda^2 + \lambda, \\
\mathbb{E}[(K - \lambda)^6] &= 15\lambda^3 + 25\lambda^2 + \lambda.
\end{align*}$$

(A.2)

**Proof.** If we denote the falling factorial by $(x)_j := (x)(x-1)\ldots(x-j+1)$, then it is well known that, for any $x \geq 0$ and $\ell \in \mathbb{N}$,

$$x^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} (x)_j,$$

(see e.g. (Graham et al., 1994, p.262)). Therefore, by the binomial formula, we get

$$\mathbb{E}[(K - \lambda)^n] = \sum_{\ell=0}^{n} \mathbb{E}[(K)_\ell^\ell] (-\lambda)^{n-\ell} = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{j=0}^{\ell} \binom{\ell}{j} \mathbb{E}[(K)_j]\right) (-\lambda)^{n-\ell}.$$  

(A.4)
We obtain the main claim (A.1) from (A.4) since

$$\mathbb{E}[(K)^j] = \sum_{k=j}^{\infty} \binom{k}{j} \frac{\lambda^k}{k!} e^{-\lambda} = \lambda^j e^{-\lambda} \sum_{k=j}^{\infty} \frac{\lambda^{k-j}}{(k-j)!} = \lambda^j. \quad (A.5)$$

The particular examples in (A.2) follow from this Mathematica code,

$$f[n_] := \text{Expand[ FullSimplify[ } \text{Sum[Binomial[n, l] } \ast \text{Sum[StirlingS2[1, j] } \ast \lambda^j, \{j, 0, 1\}] \ast \text{(-}\lambda)^{(n - l)}, \{l, 0, n\}]\text{]};$$

where we evaluate $f(2), f(4)$ and $f(6)$. \hfill \square

We can also estimate the moments of Lemma A.1 on various events. The corollary below is used to estimate the $\asymp m^{-1/2}$ errors in (3.7) of the proof of Lemma 3.1.

**Corollary A.2.** Let $K \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$, and let $A \in \mathcal{B}(\mathbb{R})$ be a Borel set. Then,

$$\mathbb{E}[(K - \lambda) 1_{\{K \in A\}}] \leq \lambda^{1/2}(\mathbb{P}(K \in A^c))^{1/2},$$

$$\mathbb{E}[(K - \lambda)^2 1_{\{K \in A\}}] \leq 2(1 + \lambda)(\mathbb{P}(K \in A^c))^{1/2},$$

$$\mathbb{E}[(K - \lambda)^3 1_{\{K \in A\}}] \leq \sqrt{41}(1 + \lambda)^{3/2}(\mathbb{P}(K \in A^c))^{1/2}. \quad (A.6)$$

**Proof.** Note that (A.2) implies

$$\mathbb{E}[(K - \lambda)^2] = \lambda,$$

$$\mathbb{E}[(K - \lambda)^4] \leq 3(1 + \lambda)^2 + (1 + \lambda)^2 = 4(1 + \lambda)^2,$$

$$\mathbb{E}[(K - \lambda)^6] \leq 15(1 + \lambda)^3 + 25(1 + \lambda)^3 + (1 + \lambda)^3 = 41(1 + \lambda)^3. \quad (A.7)$$

We get (A.6) by applying the Cauchy-Schwarz inequality and bounding using (A.7). \hfill \square

**B. Propagation of errors due to Lemma 2 (iv) of Leblanc (2012a)**

The incorrect estimate in Lemma 2 (iv) of Leblanc (2012a), mentioned in Remark 3.4, has propagated in the literature and caused many subsequent errors in the statements of theorems, propositions and/or lemmas for articles and theses dealing with Bernstein c.d.f. estimators. Below are 14 articles and theses (in alphabetical order of the authors’ last name) where at least one erroneous statement can be traced back to the error in Lemma 2 (iv) of Leblanc (2012a):

- **Belalia (2016a)**
  - $V_x(y)$ should be equal to $\sqrt{x(1-x)/(4\pi)}F_x'(y)$ in Proposition 2.2, Theorem 2.2 and Theorem 2.3;
  - $V(x, y)$ should be equal to $[F_x(x, y)(x(1 - x)/\pi)^{1/2} + F_y(x, y)(x(1 - x)/\pi)^{1/2}]$ in Theorem 4.1, Corollary 4.1, Corollary 4.2 and Theorem 4.2;
  - $V_Z(z)$ should be equal to $[F_Z(z)(z(1 - z)/\pi)^{1/2}]$ in Theorem 4.1;
  - $V(x_1, \ldots, x_d)$ should be equal to $\sum_{j=1}^{d} \left\{ F_{x_j}(x_1, \ldots, x_d)(x_j(1-x_j)/\pi)^{1/2} \right\}$ in Remark 4.2;
  - $\psi_2(x)$ should be equal to $\sqrt{x(1-x)/(4\pi)}$ in Lemma 4.1 (ii);

- **Belalia (2016b)**
  - $V(x, y)$ should be equal to $[F_x(x, y)(x(1 - x)/\pi)^{1/2} + F_y(x, y)(x(1 - x)/\pi)^{1/2}]$ in Theorem 1, Corollary 1, Corollary 2 and Theorem 2;
- $v_{Z}(z)$ should be equal to $[F_{Z}(z)(1-z)/\pi]^{1/2}$ in Theorem 1;
- $V(x_{1},\ldots,x_{d})$ should be equal to $\sum_{j=1}^{d} \left\{ F_{x_{j}}(x_{1},\ldots,x_{d})(1-x_{j})/\pi \right\}^{1/2}$ in Remark 2;
- $v_{2}(x)$ should be equal to $\sqrt{x(1-x)/(4\pi)}$ in Lemma 1 (ii);

- **Belalia et al. (2017)**
  - $V_{x}(y)$ should be equal to $\sqrt{x(1-x)/(4\pi)}F_{x}'(y)$ in Proposition 2, Theorem 2 and Theorem 3;

- **Dib et al. (2020)**
  - $V(x,y)$ should be equal to $\left[H_{x}(x,y)(x(1-x)/\pi)^{1/2} + H_{y}(x,y)(x(1-x)/\pi)^{1/2}\right]$ in Proposition 3;

- **Dutta (2016)**
  - $V(x_{0})$ should be equal to $f(x_{0})[x_{0}(1-x_{0})/\pi]^{1/2}$ on page 242 and everywhere inside $m_{\text{opt}}$;

- **Erdoğan et al. (2019)**
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Theorem 2;
  - The r.h.s of (16) should be equal to $W^{2}(m)^{-1/2}\left\{ -[x(1-x)/(4\pi)]^{1/2} + O(1)\right\}$;

- **Hanebeck (2020)**
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Theorem 5.3, Theorem 5.5, Corollary 5.2, Theorem 5.6, Corollary 5.3 and Theorem 5.7;
  - $V_{S}(x)$ should be equal to $f(x)[x/\pi]^{1/2}$ in Theorem 6.4, Theorem 6.6, Corollary 6.2, Theorem 6.7, Corollary 6.3 and Theorem 6.8;
  - $R_{i,m}^{S}(x)$ should be equal to $m^{-1/2}\left\{-x/(4\pi) + o_{x}(1)\right\}$ in Lemma 6.3 (e);

- **(Hanebeck & Klar, 2020, arXiv v.1)**
  - $V_{S}(x)$ should be equal to $f(x)[x/\pi]^{1/2}$ in Theorem 5, Theorem 7, Corollary 2, Theorem 8, Corollary 3 and Theorem 9;
  - $R_{i,m}^{S}(x)$ should be equal to $m^{-1/2}\left\{-x/(4\pi) + o_{x}(1)\right\}$ in Lemma 3 (e);

- **Jmaei (2018)**
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Proposition 1.3.2 and in the expression of the MSE and MISE on page 52;
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Proposition 2.3.1, Proposition 2.3.2, Corollary 2.3.1, (2.3.7), (2.3.8) and Remark 2.3.1;

- **Jmaei et al. (2017)**
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Proposition 3.1, Proposition 3.2, Corollary 3.1, (10), (11) and Remark 3.1;

- **Leblanc (2012b)**
  - $\Psi(x)$ should be equal to $\sqrt{x(1-x)/\pi}$ in (18) and Lemma 8

- **Lyu (2020)**
  - $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Theorem 3.3;
  - $V(x,y)$ should be equal to $\left\{ F_{x}(x,y)(x(1-x)/\pi)^{1/2} + F_{y}(x,y)(x(1-x)/\pi)^{1/2}\right\}$ in Theorem 3.4;
\begin{itemize}
  \item $V(z)$ should be equal to $\left\{ \frac{F(z)(z(1-z)/\pi)^{1/2}}{1} \right\}$ in Theorem 3.4;
  \item $\psi_2(x)$ should be equal to $\sqrt{x(1-x)/(4\pi)}$ in Lemma 3 (iv);
  \item Tchouake Tchuiguep (2013)
    \begin{itemize}
      \item $V(x)$ should be equal to $f(x)[x(1-x)/\pi]^{1/2}$ in Théorème 3.4 and Corollaire 3.6;
      \item $\gamma_2(x)$ should be equal to $\sqrt{x(1-x)/(4\pi)}$ in Lemma 3.5 (ii);
    \end{itemize}
  \item Wang et al. (2019)
    \begin{itemize}
      \item $\psi_2(x)$ should be equal to $\left\{ t(1-t)/(4\pi) \right\}^{1/2}$ in Lemma 2 (ii);
      \item The last term of $V(t)$ in Lemma 3 (iii) should be equal to $\left\{ t(1-t)/(4\pi) \right\}^{1/2}$.
    \end{itemize}
\end{itemize}

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