CONVEXITY OF ČEBYŠEV SETS THROUGH DIFFERENTIABILITY OF DISTANCE FUNCTION

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Abstract. The aim of this paper is to present some equivalent conditions that ensure the convexity of a Čebyshev set. To do so, we use Gateaux differentiability of the distance function.

AMS Subject Classification(2000) : 46B20

Key Words: Distance function, nearest point, Čebyshev set, strictly convex space, smooth space, Gateaux differentiability, subdifferential.

1. Introduction

The approximation theory is one of the important branch of functional analysis that Čebyshev originated it in nineteenth century. But, convexity of Čebyshev sets is one of the basic problems in this theory. In a finite dimensional smooth normed space a Čebyshev set is convex and for infinite dimensional, every weakly closed Čebyshev set in a smooth and uniformly convex Banach space is convex. Every boundedly compact Čebyshev set in a smooth Banach space is convex and in a Banach space, which is uniformly smooth, each approximately compact Čebyshev set is convex, and that in a strongly smooth space or in a Banach space $X$ with strictly convex dual $X^*$, every Čebyshev set with continuous metric projection is convex, [2]. There are still several open problems concerning convexity of Čebyshev sets. One of them asks whether every Čebyshev set in a strictly convex reflexive Banach space is convex? [2]

2. Basic definitions and Preliminaries

In this section, we collect some elementary facts which will help us to establish our main results. For details the reader is referred to [1]. As the first step, let us fix our notation. Through this paper, $(X,||.||)$ denotes a real Banach space and $S(X) = \{x \in X ; ||x|| = 1 \}$.

For a nonempty subset $K$ in $X$, the distance of $x \in X$ from $K$ is defined as $d_K(x) = \inf\{||x - v|| ; v \in K\}$. The $K$ is said to be a Čebyshev set if, each point in $X$ has a unique nearest point in $K$. In other words, for every $x \in X$, there exist a unique $\overline{x} \in K$ such that $||x - \overline{x}|| = d_K(x)$. (This concept was introduced by S. B. Stechkin in honour of the founder of best approximation theory, Čebyshev).
One interesting and fruitful line of research, dating from the early days of Banach space theory, has been to relate analytic properties of a Banach space to various geometrical conditions on the Banach space. The simplest example of such a condition is that of strict convexity. It is often convenient to know whether the triangle inequality is strict for non collinear points in a given Banach space. We say that the norm $\| \cdot \|$ of $X$ is strictly convex (rotund) if,

$$\|x + y\| < \|x\| + \|y\|$$

whenever $x$ and $y$ are not parallel. That is, when they are not multiples of one another.

Related to the notion of strict convexity, is the notion of smoothness.

We say that, the norm $\| \cdot \|$ of $X$ is smooth at $x \in X \setminus \{0\}$ if, there is a unique $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Of course, the Hahn-Banach theorem ensures the existence of at least one such functional $f$.

The spaces $L^p(\mu), 1 < p < \infty$, are strictly convex and smooth, while the spaces $L^1(\mu)$ and $C(K)$ are neither strictly convex nor smooth except in the trivial case when they are one dimensional.

If the dual norm of $X^*$ is smooth, then the norm of $X$ is strictly convex and if the dual norm of $X^*$ is strictly convex, then the norm of $X$ is smooth. Note that, The converse is true only for reflexive spaces. There are examples of strictly convex spaces whose duals fail to be smooth.

Let $f : X \to \mathbb{R}$ be a function and $x, y \in X$. Then $f$ is said to be Gateaux differentiable at $x$ if, there exists a functional $A \in X^*$ such that $A(y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}$. In this case $f$ is called Gateaux differentiable at $x$ with the Gateaux derivative $A$ and $A$ is denoted by $f'(x)$. In this case, the $A(y)$ is denoted usually by $\langle f'(x), y \rangle$. If the limit above exists uniformly for each $y \in S(X)$, then $f$ is Fréchet differentiable at $x$ with Fréchet derivative $A$. Similarly, the norm function $\| \cdot \|$ is Gateaux (Fréchet) differentiable at non-zero $x$ if the function $f(x) = \|x\|$ is Gateaux differentiable.

In the general, Gateaux differentiability not imply Fréchet differentiability. For example the canonical norm of $l^1$ is nowhere Fréchet differentiable and it is Gateaux differentiable at $x = (x_i)_{i \in \mathbb{N}}$ if and only if $x_i \neq 0$ for every $i \in \mathbb{N}$.

The norm of any Hilbert space, is Fréchet differentiable at nonzero points.

Suppose $f : X \to \mathbb{R}$ is a function and $x \in X$. The functional $x^* \in X^*$ is called a subdifferential of $f$ at $x$ if $\langle x^*, y - x \rangle \leq f(y) - f(x)$, for all $y \in X$. The set of all subdifferentials of $f$ at $x$ is denoted by $\partial f(x)$ and we say that $f$ is subdifferentiable
at $x$ if $\partial f(x) \neq \emptyset$.

The following theorems present relationships between various notions of differentiability for norm and the properties of the related space.

**Theorem 1.** [4] The norm $\|\cdot\|$ is Gateaux differentiable at $x \in X \setminus \{0\}$ if and only if $X$ is smooth at $x$.

**Theorem 2.** [4] If the dual norm of $X^*$ is Fréchet differentiable, then $X$ is reflexive.

**Theorem 3.** [4] Let $f : X \to \mathbb{R}$ be a convex function continuous at $x \in X$ and $\partial f(x)$ is a singleton. Then $f$ is Gateaux differentiable at $x$.

In the last theorem, notice that continuity of $f$ in $x$ is an essential condition. For example, if $f(x) = 1 + \sin \frac{1}{x}$ for all $x \neq 0$ and $f(0) = 0$, then $f$ is not continuous at $x = 0$. Also $\partial f(0) = \{0\}$, while $f$ is not Gateaux differentiable at $x = 0$.

For a real-valued function $\phi$ on $X$ and $x \in X$, set

$$F_\phi(x) = \sup_{\|y\|=1} \sup_{z \in X} \limsup_{t \to 0^+} \frac{\phi(x + tz + ty) - \phi(x + tz)}{t}.$$

**Lemma 1.** [3] Let $\phi$ is a real-valued function on $X$, $x \in X$ and $y_0 \in S(X)$ such that the Gateaux derivative of $\phi$ in $x$ exists and $\langle \phi'(x), y_0 \rangle = F_\phi(x)$. If the norm of $X$ is Gateaux differentiable at $y_0$ with Gateaux derivative $f_{y_0}$, then $\phi$ is Gateaux differentiable at $x$ and for each $y \in X$ we have $\langle \phi'(x), y \rangle = F_\phi(x)f_{y_0}(y)$.

Now the Lemma 1, give us the following corollary, since distance functions are Lipschitz:

**Corollary 1.** Let $K \subseteq X$ is closed and non-empty, $x \in X \setminus K$ and $\overline{x}$ is a nearest point for $x$ in $K$, the distance function $d_K$ is Gateaux differentiable at $x$ and in the direction of $(x - \overline{x})$ with Gateaux derivative $f_{x-\overline{x}}$. Then $d_K$ is Gateaux differentiable at $x$ and $\langle d_K'(x), y \rangle = f_{x-\overline{x}}(y)$ for all $y \in X$.

For nonempty closed subset $K$ of $X$ and $x, y \in X$, set

$$d_K^-(x; y) = \liminf_{t \to 0^+} \frac{d_K(x + ty) - d_K(x)}{t}$$

and

$$d_K^+(x; y) = \limsup_{t \to 0^+} \frac{d_K(x + ty) - d_K(x)}{t}.$$

**Corollary 2.** Suppose $K \subseteq X$ is closed and nonempty, $x \in X \setminus K$, $\overline{x}$ is a nearest point for $x$ in $K$. If the norm of $X$ is Gateaux differentiable at $(x - \overline{x})$ and $\partial^+_K(x; x - $
Theorem 5. Suppose the dual space of set, $x$, is convex, then $d_K(x)$ is Gateaux differentiable at $x$.

Proof. Due to the norm of $X$ is Gateaux differentiable at $x - \pi$, it is sufficient to prove the existence of the limit

$$\lim_{t \to 0} \frac{d_K(x + t(x - \pi)) - d_K(x)}{t}.$$ 

Since $d'_K(x; \pi - x) = -d_K(x)$ and $\lim_{t \to 0} \frac{d_K(x + t(x - \pi)) - d_K(x)}{t} = d_K(x)$, it is sufficient to prove that

$$\lim_{t \to 0^+} \frac{d_K(x + t(x - \pi)) - d_K(x)}{t} = d_K(x).$$

For each $t > 0$ we have $d_K(x + t(x - \pi)) - d_K(x) \leq td_K(x)$. Hence, $d'_K(x; x - \pi) \leq d_K(x)$. If $d'_K(x; x - \pi) = d_K(x)$, then $d_K(x) = d_K(x; x - \pi) \leq d_K(x; x - \pi) \leq d_K(x)$. It follows that $< d'_K(x), x - \pi >$ exists and is equal to $d_K(x)$.

Theorem 4. [4] If the dual space of $X$ is strictly convex, then each closed nonempty subset $K$ in $X$ satisfying $\limsup_{\|y\| \to 0} \frac{d_K(x + y) - d_K(x)}{\|y\|} = 1$ for all $(x \in X \setminus K)$ is convex.

3. Main Result

We start this section with our main result.

Theorem 5. Suppose the dual space of $X$ is strictly convex, $K \subseteq X$ is a Čebyšev set, $x \in X \setminus K$ and $\partial d_K(x)$ is singleton. The following are equivalent:

(i) $K$ is convex.

(ii) $d_K$ is convex.

(iii) $d_K$ is Gateaux differentiable at $x$.

(iv) There is $z \in S(X)$ such that $\lim_{t \to 0^+} \frac{d_K(x + tz) - d_K(x)}{t} = 1$.

(v) $\limsup_{\|y\| \to 0} \frac{d_K(x + y) - d_K(x)}{\|y\|} = 1$.

Proof. (i ⇒ ii) Since $K$ is closed convex set, $d_K$ is convex [2].

(ii ⇒ iii) Since $d_K$ is convex and continuous at $x$ and $\partial d_K(x)$ is singleton, $d_K$ is Gateaux differentiable at $x$ and $\{d'_K(x)\} = \partial d_K(x)$.

(iii ⇒ iv) First note that by the definition of Čebyšev sets there is a unique element $\pi \in K$ such that $\|x - \pi\| = d_K(x)$. It follows from Gateaux differentiability of $d_K$ that, $\liminf_{t \to 0^+} \frac{d_K(x + ty) - d_K(x)}{t}$ exists for every $y \in X$. For each $t > 0$ we have, $d_K(x + t(x - \pi)) - d_K(x) \leq td_K(x)$. Hence for $y = x - \pi$, we set:

$$\liminf_{t \to 0^+} \frac{d_K(x + t(x - \pi)) - d_K(x)}{t} = d_K(x).$$
Since \( x \in X \setminus K \), \( d_K(x) > 0 \) and if \( t' = \frac{t}{d_K(x)} \) as \( t \to 0^+ \), Then by the above:

\[
\liminf_{t' \to 0^+} \frac{d_K(x + t'(x - \overline{x})) - d_K(x)}{t'} = d_K(x),
\]

If now \( z = \frac{x - \overline{x}}{\|x - \overline{x}\|} \), then \( \|z\| = 1 \) and we have \( \liminf_{t \to 0^+} \frac{d_K(x + tz) - d_K(x)}{t} = 1 \). On the other hand, \( d_K \) is a Lipschitz function and so \( \limsup_{t \to 0^+} \frac{d_K(x + tz) - d_K(x)}{t} \leq 1 \).

(iv \Rightarrow v) Since \( d_K \) is a Lipschitz function \( \limsup_{\|y\| \to 0} \frac{d_K(x + y) - d_K(x)}{\|y\|} \leq 1 \). On the other hand for each \( v \in S(X) \),

\[
\lim_{t \to 0^+} \frac{d_K(x + tv) - d_K(x)}{t} \leq \limsup_{\|y\| \to 0} \frac{d_K(x + y) - d_K(x)}{\|y\|},
\]

in particular for \( v = z \) in iv, we have \( 1 \leq \limsup_{\|y\| \to 0} \frac{d_K(x + y) - d_K(x)}{\|y\|} \).

(v \Rightarrow i) This follows from theorem 4.

**Remark 1.** Suppose that the norm of \( X \) and the dual norm of \( X^* \) are Fréchet differentiable, \( K \subseteq X \) is Čebyšev and \( x \in X \setminus K \). Then \( X \) is reflexive, since the dual norm of \( X^* \) is Fréchet differentiable. Moreover \( X \) is smooth, since the norm of \( X \) is Fréchet differentiable. Thus \( X^* \) is strictly convex. If now \( d_K \) is Gateaux differentiable at \( x \), then \( K \) is convex.

**Remark 2.** Suppose that \( K \subseteq X \) is Čebyšev, \( x \in X \setminus K \) and \( X^* \) is strictly convex. By the definition of Čebyšev sets, there is unique \( \overline{x} \in K \) such that \( \|x - \overline{x}\| = d_K(x) \), if now \( d_K'(x; x - \overline{x}) = d_K(x) \), then by corollary 2 and Remark 1, \( K \) is convex.

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