Online Process Monitoring Using Incremental State-Space Expansion: An Exact Algorithm

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Abstract. The execution of (business) processes generates valuable traces of event data in the information systems employed within companies. Recently, approaches for monitoring the correctness of the execution of running processes have been developed in the area of process mining, i.e., online conformance checking. The advantages of monitoring a process’ conformity during its execution are clear, i.e., deviations are detected as soon as they occur and countermeasures can immediately be initiated to reduce the possible negative effects caused by process deviations. Existing work in online conformance checking only allows for obtaining approximations of non-conformity, e.g., overestimating the actual severity of the deviation. In this paper, we present an exact, parameter-free, online conformance checking algorithm that computes conformance checking results on the fly. Our algorithm exploits the fact that the conformance checking problem can be reduced to a shortest path problem, by incrementally expanding the search space and reusing previously computed intermediate results. Our experiments show that our algorithm outperforms comparable state-of-the-art approximation algorithms.

Keywords: Process mining · Conformance checking · Event streams.

1 Introduction

Modern information systems support the execution of different business processes within companies. Valuable traces of event data, describing the various steps performed during process execution can be easily extracted from such systems. The field of process mining \cite{3} aims to exploit such information captured within event data to better understand the overall execution of the process. For example, in process mining, several techniques have been developed that allow us to (i) automatically discover process models, (ii) compute whether the process, as reflected by the data, conforms to a predefined reference model, and (iii) detect performance deficiencies, e.g., bottleneck detection.

Existing process mining approaches work in an offline setting, i.e., data is captured during the process execution over time and process mining analyses
Fig. 1. Overview of online process monitoring. Activities are performed for different process instances over time. Whenever a new activity is executed, the sequence of already executed activities is checked for conformance w.r.t. a reference process model are performed a posteriori. However, some of these techniques benefit from an online application scenario, i.e., analyzing events at the moment they occur. Reconsider conformance checking, i.e., computing whether a process execution conforms to a reference model. When checking conformance in an online setting, the process owner is able to observe and counteract non-conformity at the moment it occurs (Fig. 1). Therefore, potential negative effects on an organization caused by a process deviation can be mitigated or eliminated. This observation inspired the development of novel conformance checking algorithms working in an online setting. However, such algorithms provide approximations of non-conformity and/or use high-level abstractions of the reference model and the event data, i.e., not allowing us to obtain an exact quantification of non-conformance.

In this paper, we propose a novel and exact solution for online incremental conformance checking. Extending existing work in both offline- and online conformance checking, we present a parameter-free algorithmic framework that computes exact conformance checking results and provides an exact quantification of non-conformance. We evaluate our algorithm on the basis of publicly available real event data. Our experiments show that we are able to obtain conformance checking results under reasonable time and memory consumption. Furthermore, our approach has a comparable computational efficiency w.r.t. the approximate variant while guaranteeing exact results.

The remainder of this paper is structured as follows. First, we present existing work and results in the area of conformance checking. Subsequently, we present preliminary concepts and notations. Afterwards, we present the main algorithm, prove its correctness, evaluate the proposed algorithm and present the results of the experiments conducted.
2 Related Work

In this section, we primarily focus on (online) conformance checking techniques. For a detailed overview of various process mining techniques we refer to [3].

One of the first techniques designed to compute conformance statistics is token-based replay [14], which tries to replay the observed behavior on a reference model. Another early conformance checking technique is footprint-based comparison [3], in which the event data and the process model are translated into the same abstraction and then compared. As described in [3, Chapt. 8], both techniques have several drawbacks. As an alternative, alignments have been introduced [2,4] that map the observed behavioral sequences to a feasible execution sequence as described by the reference process model. Furthermore, alignments indicate whether behavior is missing and/or whether inappropriate behavior is observed. The problem of finding an alignment was shown to be reducible to the shortest path problem [4,5]. For a complete overview, we refer to [9].

The aforementioned techniques are designed for offline use, i.e., they work on static (historical) data. In [15] an approach is presented to monitor ongoing process executions based on the notion of a live stream of process events. The authors propose a framework that computes partially completed alignments for the ongoing process instances each time a new event is observed. The approach results in approximate solutions, i.e., false negatives occur in terms of deviation detection. In [7] the authors propose to pre-calculate a transition system based on the model that supports replay of the ongoing process. Costs are assigned to the transition system’s edges and replaying a deviating process instance leads to larger (non-zero) costs. Finally, [8] propose to compute conformance of a process execution based on all possible behavioral patterns of the activities of a process. However, the use of such patterns leads to a loss of expressiveness in deviation explanation and localization.

3 Background

In this section, we present basic notations and concepts used within this paper.

We let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0$ includes 0. Given an arbitrary set $X$, a multiset $B$ over $X$ allows us to assign a multiplicity to the elements of $X$, i.e., $B: X \rightarrow \mathbb{N}_0$. We write a multiset $B$ as $[x^i, y^k, z^n]$, where $B(x)=i$, $B(y)=k$ etc. In case $B(x)=1$, we omit $x$’s superscript from the multiset notation. If $B(x)=0$, we omit $x$ from the multiset notation. For instance, $[x^5, y]$ contains 5 times $x$ and once $y$.

The set of all possible multisets of a set $X$ is written as $B(X)$. For a given multiset $B$, we write $x \in_+ B$ if $x$ is contained at least once in $B$.

Given an arbitrary set $X$, a sequence $\sigma$ of length $n$ over $X$ assigns an element to each index in $\{1, ..., n\}$, i.e., $\sigma: \{1, ..., n\} \rightarrow X$. The empty sequence is denoted by $\epsilon$. We let $|\sigma|$ denote the length of $\sigma$. We write a sequence $\sigma$ as $\langle \sigma(1), \sigma(2), ..., \sigma(|\sigma|) \rangle$. Concatenation of sequences $\sigma$ and $\sigma'$ is written as $\sigma \cdot \sigma'$, e.g., $\langle x, y \rangle \cdot \langle z \rangle = \langle x, y, z \rangle$. The set of all possible sequences over base set $X$ is written as $X^*$. For element inclusion, we overload the notation for sequences, i.e., given $\sigma \in X^*$ and $x \in X$, we write $x \in^* \sigma$ if $\exists 1 \leq i \leq |\sigma| (\sigma(i) = x)$, e.g., $b \in \langle a, b \rangle$. 
Table 1. Example event log fragment (fictional)

| Event-id | Case-id | Activity       | Resource | Time-stamp     |
|----------|---------|----------------|----------|----------------|
| 12215    | 13152   | create account | Wil      | 2019-04-08 10:45 |
| 12216    | 13153   | create account | Jan      | 2019-04-08 11:12 |
| 12217    | 13154   | request quote  | Marcello | 2019-04-08 11:14 |
| 12218    | 13155   | request quote  | Marlon   | 2019-04-08 11:40 |
| 12219    | 13152   | submit order   | Wil      | 2019-04-08 11:49 |

Fig. 2. Schematic example of an event stream

In the following paragraph we introduce projection functions that will be used later to formally define alignments. Let $\sigma \in X^*$ and let $X' \subseteq X$. We let $\sigma_{|X'} \in X'^*$ with $\epsilon_{|X'} = \epsilon$ and $(\langle x \rangle \cdot \sigma)_{|X'} = \langle x \rangle \cdot \sigma_{|X'}$, if $x \in X'$ and $\sigma_{|X'}$ otherwise.

Let $t = (x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$, we let $\pi_1(t) = x_1, \pi_2(t) = x_2, ..., \pi_n(t) = x_n$ denote the corresponding projection functions. Correspondingly, given a sequence $\sigma = (x_1^1, ..., x_1^m), ..., (x_n^1, ..., x_n^m)$ of length $m$, we define projection functions $\pi_1^*(\sigma) = (x_1^1, ..., x_1^m), ..., \pi_n^*(\sigma) = (x_n^1, ..., x_n^m)$.

**Event Logs** The data used in process mining are event logs that record the execution of activities throughout the execution of a process. In Tab. 1 we present a fictional fragment of an event log. Each row corresponds to an event describing the execution of an activity in the context of an instance of the process. For example, resource Wil has executed the create account activity at 10:45 a.m. on April 8th, which was executed for a process instance identified by Case-id 13152. In the remainder, for simplicity, we use short-hand activity names, e.g., $a$ for “create account”, $b$ for “submit order”, etc. The events related to Case-id 13152 describe the activity sequence $(a, b)$. Since we focus on the online dimension in this paper, we do not provide a formal definition of event logs. It suffices to know that an event log $L$ represents a multiset of sequences of activities that are referred to as traces. Note that multiple cases may describe the same trace.

**Event streams** In the context of this paper, we are interested in observing the process during its execution. Thus, we assume to use an event stream rather than an event log. Conceptually, an event stream is a (infinite) sequence of events.

In Fig. 2, we schematically depict an event stream. We observe multiple events over time, related to different process instances. For example, the first event, $(13152, a)$, indicates that for a process instance with case-identifier 13152 activity $a$ was performed. Furthermore, we expect to observe new events for the same instance in the future. Hence, our knowledge of a process instance is typically incomplete at a given point in time. Moreover, observe that multiple process instances run concurrently.
Definition 1 (Event; Event Stream). Let $\mathcal{C}$ denote the universe of case identifiers, let $\mathcal{A}$ denote the universe of activities and let $\mathcal{E}$ denote the universe of events. An event $e \in \mathcal{E}$ describes the execution of an activity $a \in \mathcal{A}$ in the context of a process instance identified by some case identifier $c \in \mathcal{C}$. An event stream $S$ is a sequence of events, i.e., $S \in \mathcal{E}^*$. 

Process Models Process models allow us to describe the (intended) behavior of a process. Several process modeling formalism exist ranging from informal descriptions to mathematical models. In this paper, we focus on sound Workflow nets\[1\]. A Workflow net (WF-net), e.g., Fig. 3 is a subclass of Petri nets\[13\], i.e., a process modeling formalism that allows for compact representation of concurrent behavior. Sound WF-nets, in turn, are a subclass of WF-nets with favorable behavioral properties, guaranteeing the absence of deadlocks, live-locks, and other types of errors. We use WF-nets, since a large number of high-level process modeling formalism used in practice, e.g. BPMN\[10\], are easily translated into WF-nets. W.r.t. soundness, it is, to some degree, reasonable to assume that an experienced business process designer defines sound process models. Furthermore, there exist methods to verify the soundness of a given model.

Petri nets consist of two elements, i.e., a set of places $P$, visualized as circles, and a set of transitions $T$, visualized as rectangular boxes. Places and transitions are connected by means of arcs. However, places are only connected to transitions and transitions are only connected to places. We write a Petri net as $N=(P,T,F,\lambda)$, where $F = (P \times T) \cup (T \times P)$ describes the arcs of the Petri net. Additionally, a labeling function $\lambda: T \rightarrow \mathcal{A} \cup \{\tau\}$ assigns an (possibly invisible, i.e., $\tau$) activity label (in the universe of activities $\mathcal{A}$) to each transition, e.g., $\lambda(t_1) = a$ in Fig. 3. Given an element $x \in P \cup T$, we write $x \bullet = \{y \in P \cup T \mid (x,y) \in F\}$, i.e., all elements $y$ that have an incoming arc from $x$, and, symmetrically, $\bullet x = \{y \in P \cup T \mid (y,x) \in F\}$.

The state of a Petri net, i.e., a marking $M$, is described by means of a multiset of places, i.e., $M \in \mathcal{B}(P)$. Graphically, we represent the marking of a Petri net by drawing black dots, e.g., for the visualized marking $M$ in $N_1$ (Fig. 3) $M(p_1)=1$. Given a Petri net $N$ with a set of places $P$ and a marking $M \in \mathcal{B}(P)$, a marked net is written as $(N,M)$. We denote the initial marking of a Petri net with $M_i$. 

Fig. 3. Example WF-net $N_1$ with visualized initial marking $[p_1]$ and final marking $[p_3]$ describing a simplified ordering process.
The transitions of a Petri net allow for changing the state. Given a marking \( M \in B(P) \), a transition \( t \) is enabled if \( \forall p \in \cdot t (M(p) > 0) \). An enabled transition can fire, yielding marking \( M' \in B(P) \), where \( M'(p) = M(p) + 1 \) if \( p \in t \cdot \) \( t \), \( M'(p) = M(p) - 1 \) if \( p \in \cdot t \setminus t \cdot \), otherwise \( M'(p) = M(p) \). Informally, a transition \( t \) is enabled if there is at least one token in each incoming place. By firing \( t \), one token is removed/consumed from each incoming place and one token is put into each outgoing place. We write \( (N, M) \xrightarrow{t} (N, M') \) to denote that firing transition \( t \) in marking \( M \) yields marking \( M' \). In Fig. 3, we have \( (N_1, [p_1]) \xrightarrow{t_1} (N_1, [p_2]) \). If, given marking \( M \), sequence \( \sigma = (t_1, \ldots, t_n) \in T^* \) leads to \( M' \), we overload the notation and write \( (N, M) \xrightarrow{T^*} (N, M') \). If marking \( M' \) is reachable from marking \( M \), i.e., \( \exists \sigma \in T^* \left( (N, M) \xrightarrow{T^*} (N, M') \right) \), we write \( (N, M) \leadsto (N, M') \). We let \( \mathcal{R}(N, M) = \{ M' \in B(P) \mid (N, M) \leadsto (N, M') \} \) denote all possible states/all reachable markings of \( N \) given an initial marking \( M \).

A WF-net \( N = (P, T, F, p_s, p_o, \lambda) \) is a Petri net with labeling function \( \lambda \). It has a unique source place \( p_s \in P \), the initial marking \( M_s = [p_s] \), and a unique sink place \( p_o \in P \), the final marking \( M_f = [p_o] \). Moreover, every element \( x \in P \cup T \) is on a path from \( p_s \) to \( p_o \). Since we assume the WF-nets used in this paper to be sound, the state space, i.e., \( \mathcal{R}(N, [p_s]) \), is finite.

Alignments To explain traces in an event log, which are assumed to be finished, w.r.t. a reference model, we use alignments [4]. Alignments map traces onto an execution sequence of a reference model. In Fig. 3, we present three possible alignments for the trace \( \langle a, b, c \rangle \) and the WF-net \( N_1 \). The first row of each alignment (ignoring the skip symbol \( \gg \gg \)) corresponds to the trace and the second row corresponds to a sequence of transitions (ignoring the skip symbol \( \gg \gg \)) leading from the initial- to the final marking.

We distinguish three main types of moves in an alignment. A synchronous move (light-gray) matches an observed activity to the execution of a transition. The label of the transition must correspond to the activity name, e.g., given the first move of the first alignment depicted in Fig. 4 which is a synchronous move, the transition label corresponds to the activity name, i.e., \( \lambda(t_1) = a \). Log moves (dark-gray) indicate that an activity is not re-playable in the current state of the process model. Model moves (white) indicate that the execution of a transition cannot be mapped onto an observed activity. Model moves can be further differentiated into invisible- and visible model moves. An invisible model move consists of an inherently invisible transition \( (\lambda(t) = \tau) \), e.g., in the third alignment in Fig. 4 the first model move \((\gg, t_2)\) is invisible. Visible model moves suggest that an activity should have taken place according to the model but was
not observed at that time. In general, we aim to avoid using log- and visible model moves in an alignment as much as possible.

Next to alignments, there is the concept of prefix-alignments. The first row of a prefix-alignment (ignoring $\gg$) also corresponds to the trace, but the second row corresponds to a sequence of transitions (ignoring $\gg$) leading from the initial marking to a marking from which the final marking can be still reached.

**Definition 2 ((Prefix-)Alignment).** Let $A$ denote the universe of activities, $\sigma \in A^*$, $N = (P, T, F, p_1, p_o, \lambda)$ a sound WF-net and $\gg \notin A \cup T$.

Sequence $\gamma \in ((A \cup \{\gg\}) \times (T \cup \{\gg\}))^*$ is an alignment iff:

1. $\sigma = \pi_1^*(\gamma) \downarrow_A$
2. $(N, [p_i]) \xrightarrow{\pi_2^*(\gamma) \downarrow T} (N, [p_o])$
3. $(\gg, \gg) \notin \gamma$

Similarly, a sequence $\sigma \in ((A \cup \{\gg\}) \times (T \cup \{\gg\}))^*$ is a prefix-alignment if the second requirement is relaxed to:

$$(N, [p_i]) \xrightarrow{\pi_2^*(\gamma) \downarrow T} (N, M) \rightsquigarrow (N, [p_o])$$

For a trace $\sigma$ and a WF-net $N$ let $\Gamma(\sigma, N)$ denote the set of all possible alignments (prefix-alignments).

Given a WF-net and a trace, multiple (prefix-)alignments exist. Hence, we are interested in finding an alignment that minimizes the number of mismatches between the trace and the model. We do so by assigning costs to moves. We typically use the standard cost function that assigns cost 0 to synchronous moves and invisible model moves, and cost 1 to log- and visible model moves.

**Definition 3 (Optimal (Prefix-)Alignment).** Let $\sigma \in A^*$ be a trace, $N = (P, T, F, p_1, p_o, \lambda)$ a sound WF-net and $\kappa : (A \cup \{\gg\}) \cup (T \cup \{\gg\}) \rightarrow \mathbb{R}_{\geq 0}$ a move cost function. An alignment $\gamma \in \Gamma(\sigma, N)$ of $\sigma$ and $N$ is optimal iff:

$$\exists \gamma' \in \Gamma(\sigma, N) \left( \sum_{x \in \gamma} \kappa(x) < \sum_{x \in \gamma'} \kappa(x) \right)$$

Optimality for prefix-alignments is defined analogously.

To compute a (prefix-)alignment, we search for a shortest path in the state-space of the synchronous product net (SPN) of the given trace and the WF-net. In Fig. 5 we show an example SPN of the WF-net $N_1$ (Fig. 3) and the trace $\langle a, b, c \rangle$. The upper highlighted part of the SPN is called the trace net part and the lower highlighted part is called the process net part. Each transition in the SPN corresponds to a (prefix-)alignment move. Therefore, we can assign costs to each transition. Any path in the state-space (sequence of transitions in the SPN), starting at $[p_0', p_1]$ and ending at $[p_3', p_3]$ corresponds to an alignment of $N_1$ and $\langle a, b, c \rangle$. A shortest path corresponds to an optimal alignment.
For a given trace composed of a trace net and a given WF-net, prefix-alignment is used to compute a marking to a marking \( M \) of the given example, a possible shortest path with cost 1 (assuming the standard corresponding trace net \( N \)) is:

\[
\begin{align*}
(N^S, [p_0, p_1]) & \xrightarrow{(11,1)} (N^S_1, [p_1, p_2]) \xrightarrow{(12,\gg)} (N^S_1, [p_2, p_3]) \xrightarrow{(13,\gg)} (N^S_1, [p_3, p_3])
\end{align*}
\]

To compute a prefix-alignment, we look for a shortest path from the initial marking to a marking \( M \in \mathcal{R}(N^S_1, [p_0, p_1]) \) such that \( M(p_1) = 1 \), i.e., the last place of the trace net part is marked. Next we formally define the SPN that is composed of a trace net and a given WF-net.

**Definition 4 (Trace net).** For a given trace \( \sigma \in \mathcal{A}^* \) of length \( n \in \mathbb{N} \), let \( N^\sigma = (P^\sigma, T^\sigma, F^\sigma, [p^\sigma_0], [p^\sigma_n], \lambda^\sigma) \) be the corresponding trace net such that:

- \( P^\sigma = \{ p_i | 0 \leq i \leq n \}, T^\sigma = \{ t_i | 1 \leq i \leq n \} \)
- \( F^\sigma = \{ (p_i, t_{i+1}) | 0 \leq i \leq n, p_i \in P^\sigma, t_{i+1} \in T^\sigma, p_i \in P^\sigma \} \)
- \( p^\sigma_0 = p_1, p^\sigma_n = p_n \)
- \( \lambda(t_i)^\sigma = \sigma(i) \), for all \( 1 \leq i \leq n, t_i \in T \)

**Definition 5 (Synchronous Product Net (SPN)).** For a given trace \( \sigma \), the corresponding trace net \( N^\sigma = (P^\sigma, T^\sigma, F^\sigma, [p^\sigma_0], [p^\sigma_n], \lambda^\sigma) \) and a WF-net \( \bar{N} = (P, T, F, [\bar{p}_0], [\bar{p}_n], \lambda) \) s.t. \( P^\sigma \cap P = \emptyset \) and \( F^\sigma \cap F = \emptyset \), we define the SPN \( N^S = (P^S, T^S, F^S, M^S, M^F, \lambda^S) \) s.t.:

- \( P^S = P^\sigma \cup P \)
- \( T^S = (T^\sigma \times \{ \gg \}) \cup (\{ \gg \} \times T) \cup \{ (t', t) \in T^\sigma \times T | \lambda(t) = \lambda^\sigma(t') \neq t \} \)
- \( F^S = \{ (p, (t', t)) \in P^S \times T^S | (p, t) \in F^\sigma \cup (t, p) \in F \} \cup \{ ((t', t), p) \in T^S \times P^S | (t', p) \in F^\sigma \cup (t, p) \in F \} \)
- \( M^S = [p^\sigma_0, p_1] \) and \( M^F = [p^\sigma_n, p_0] \)
- \( \lambda^S: T^S \to (\mathcal{A} \cup \{ \tau \} \cup \{ \gg \}) \times (\mathcal{A} \cup \{ \tau \} \cup \{ \gg \}) \) (assuming \( \gg \notin \mathcal{A} \cup \{ \tau \} \)) such that:
  - \( \lambda^S(t', \gg) = (\lambda^\sigma(t'), \gg) \) for \( t' \in T^\sigma \)
  - \( \lambda^S(\gg, t) = (\gg, \lambda(t)) \) for \( t \in T \)
  - \( \lambda^S(t', t) = \lambda^S(\lambda^\sigma(t), \lambda(t)) \) for \( t' \in T^\sigma, t \in T \)
4 Incremental Prefix-Alignment Computation

In this section, we present an exact algorithm to incrementally compute prefix-alignments. First, we present an informal description of the $A^*$ algorithm since our proposed algorithm is based on $A^*$. Afterwards, we present an overview of the proposed approach followed by a detailed description of the main algorithm and corresponding correctness proofs. Finally, we present a heuristic function for the problem of prefix-alignment computation.

The $A^*$ algorithm is an informed search algorithm that computes a shortest path. It efficiently traverses a search-space by exploiting for a given state the estimated remaining distance, referred to as the heuristic/$h$-value, to the closest goal state. The main idea of the $A^*$ algorithm is to maintain a set of states of the search-space in its so-called open-set $O$. For each state in the $O$ set a path from the initial state to such a state is known and thus, the distance to travel to that state is known, which is referred to as the $g$ value. As long as a goal state is not reached, a state from the set $O$ with minimal $f$-value, i.e., $f = g + h$, is selected for further analysis. The selected state itself is moved into the closed set $C$, which contains fully investigated states for which a shortest path to those states is already known. Note that the used heuristic must fulfill certain properties, i.e., admissibility and consistency. Furthermore, all successor states of the selected state are added to the open set $O$.

Overview The core idea of the proposed algorithm is to exploit previously calculated results, i.e., explored parts of the state-space of a SPN. For each process instance, we maintain a SPN, which is extended as new events are observed. After extending the SPN, we continue the search for an optimal prefix-alignment by reusing intermediate results from the previous search.

In Fig. 6 we visualize a conceptual overview of our approach. We observe a new event $(c, a)$ on the event stream. We check our SPN cache and if we previously built an SPN for case $c$, we fetch it from the cache. We then extend the SPN by means of adding activity $a$ to the trace net part. Starting from intermediate results of the previous search, i.e., open- & closed set used in the $A^*$ algorithm, we find a new, optimal prefix-alignment for case $c$. 

Fig. 6. Overview of the proposed incremental prefix alignment approach
Extending SPNs

Reconsider WF-net $N_1$ (Fig. 3) and assume that the first activity we observe for a process instance is activity $a$. The corresponding SPN is visualized by means of the solid elements in Fig. 7 on the left-hand side. On the right-hand side, we show the corresponding state space (using solid elements). Any state in the state-space of the SPN in Fig. 7 containing a token in $p'_1$ is a suitable goal state of the $A^\ast$ algorithm for an optimal prefix-alignment. Hence, to compute a prefix-alignment, we execute the regular $A^\ast$ algorithm with the aforementioned stop criterion. Next, for the same process instance, we observe an event describing activity $b$. The SPN for the process instance now describing trace $\langle a, b \rangle$ as well as its corresponding state-space is expanded. The expansion is visualized in Fig. 7 by means of dashed elements. In this case, any state that contains a token in $p'_2$ corresponds to a suitable goal state of the optimal prefix-alignment search.

Incrementally Performing Regular $A^\ast$

Hereafter, we present the main algorithm to compute prefix-alignments, on the basis of previously executed instances of the $A^\ast$ algorithm.

The main idea of our approach is to continue the search on an extended search space. When we receive a new event $(c, a)$, we apply the regular $A^\ast$ algorithm using the cached open- and closed-set for case identifier $c$ on the corresponding extended SPN. Hence, we incrementally solve shortest path problems on finite, fixed state-spaces by using the regular $A^\ast$ algorithm with predefined open- and closed sets from the previous search.

In Alg. 1, we present an algorithmic description of the $A^\ast$ approach. The algorithm uses a SPN as input, the open- and closed-set of the previously executed instance of the $A^\ast$ algorithm, i.e., for the process instance at hand, a cost-so-far function $g$, a predecessor function $p$, a heuristic function $h$, and a cost function $d$ as input. First, we initialize all states that have not been discovered yet (line 2).
Algorithm 1: Revised $A^*$ algorithm

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{input:} $N^S = (P^S, T^S, F^S, M^S_i, M^S_f, \lambda^S)$, $O, C \subseteq \mathbb{R}(N^S, M^S_i)$, $g: \mathbb{R}(N^S, M^S_i) \rightarrow \mathbb{R}_{\geq 0}$, $p: \mathbb{R}(N^S, M^S_i) \rightarrow T^S \times \mathbb{R}(N^S, M^S_i)$, $h: \mathbb{R}(N^S, M^S_i) \rightarrow \mathbb{R}_{\geq 0}$, $d: T^S \rightarrow \mathbb{R}_{\geq 0}$
\STATE \textbf{begin}
\STATE \hspace{1em} let $p_{|\sigma|}$ be the last place of the trace net part of $N^S$;
\STATE \hspace{1em} \textbf{forall} $m \in \mathbb{R}(N^S, M^S_i) \setminus O \cup C$ \textbf{do}
\STATE \hspace{2em} $g(m) \leftarrow \infty$;
\STATE \hspace{2em} $f(m) \leftarrow \infty$;
\STATE \hspace{1em} \textbf{forall} $m \in O$ \textbf{do}
\STATE \hspace{2em} $f(m) = g(m) + h(m)$;
\STATE \hspace{1em} \textbf{while} $O \neq \emptyset$ \textbf{do}
\STATE \hspace{2em} $m \leftarrow \arg \min_{m \in O} f(m)$;
\STATE \hspace{2em} if $p_{|\sigma|} \in m$ then
\STATE \hspace{3em} $\gamma \leftarrow $ alignment that corresponds to the sequence of transitions $(t_1, \ldots, t_n)$
\STATE \hspace{3em} where $t_n = \pi_1(p(m))$, $t_{n-1} = \pi_1(\pi_2(p(m)))$, etc. until there is a marking that
\STATE \hspace{3em} has no predecessor, i.e., $m^S_i$;
\STATE \hspace{2em} return $\gamma$, $O$, $C$, $g$, $p$;
\STATE \hspace{1em} \textbf{forall} $t \in T^S$ \textbf{s.t.} $(N^S, m)[t](N^S, m')$ \textbf{do}
\STATE \hspace{2em} if $m' \notin C$ then
\STATE \hspace{3em} $O \leftarrow O \cup \{m\}$;
\STATE \hspace{3em} if $g(m) + d(t) < g(m')$ then
\STATE \hspace{4em} $g(m') \leftarrow g(m) + d(t)$;
\STATE \hspace{4em} $f(m') \leftarrow g(m') + h(m')$;
\STATE \hspace{4em} $p(m') \leftarrow (t, m)$;
\STATE \hspace{1em} \textbf{end for}
\STATE \hspace{1em} \textbf{end if}
\STATE \hspace{1em} $C \leftarrow C \cup \{m\}$;
\STATE \hspace{1em} $O \leftarrow O \setminus \{m\}$;
\STATE \textbf{end for}
\STATE \textbf{end while}
\STATE \textbf{end}
\end{algorithmic}
\end{algorithm}

Since the SPN is extended and the goal states are different with respect to the previous run of the algorithm for the same process instance, we recalculate the heuristic values and update the $f$-values for all states in the open set (line 6) because we are now looking for a shortest path to a state that has a token in the newly added place in the trace net part of the SPN. Hence, the new goal states were not present in the previous search problem. Thereafter, we pick a state from the open set with smallest $f$-value (line 4). First, we check if the state is a goal state, i.e., whether it contains a token in the last place of the trace net part. If so, we reconstruct the sequence of transitions lead to the state, and thus, yield a prefix-alignment (using predecessor function $p$). Otherwise, we move the current state from the open- to the closed set and examine all its successor states. If a successor state is already in the closed set, we ignore it. Otherwise, we add them to the open set and update the $f$-value and the predecessor stored in $p$ if a cheaper path was found.

**Correctness** In this section, we prove the correctness of the approach. We show that states in the closed set do not get new successors upon extending the SPN.

Note that a trivial heuristic function (that does not need any recalculation) is using a value 0 for each state. This results in a search algorithm equivalent to Dijkstra’s Algorithm. The algorithm as described, however, works with any feasible heuristic.
Furthermore, we show that newly added states never connect to “older” states. Finally, we show that the open set always contains a state which is part of an optimal prefix-alignment of the extended trace.

**Lemma 1 (State-space growth is limited to frontier).** Let $\sigma^{i-1} = \langle a_1, \ldots, a_{i-1} \rangle, \sigma^i = \sigma^{i-1} \cdot \langle a_i \rangle$, and $\sigma^{i+1} = \sigma^i \cdot \langle a_{i+1} \rangle$. For a given WF-net $N$ let $N^i_S = (P^i_S, T^i_S, M^i_S, M^f_i, \lambda^i_S)$ be the SPN of $N$ and $\sigma^{i-1}$, $N^i_S$ and $N^i_{S+1}$ analogously. For $M \in B(P^i_S)$, $\forall t \in T^i_S : (N^i_S, M) \vdash t \in T^i_S$

**Proof (By construction of the SPN).** Observe that $P^i_S \subset P^i_{S+1}$ and $T^i_{S-1} \subset T^i_{S+1}$. Let $p_{[\sigma^i]} \in P^i_{S+1}$ be the $i$-th place of the trace net part (note that $p_{[\sigma^i]} \notin P^i_S$) and let $t_{i+1} \in T^i_{S+1} \setminus T^i_S$. By construction of the SPN, we know that $p_{[\sigma^i]} \in \bullet t_{i+1}$ and $\forall j \in \{1, \ldots, i\}$ : $p_{[\sigma^i]} \notin \bullet t_{i+1}$.

Observe that, when searching for an alignment for $\sigma^i$, Alg. 1 returns whenever place $p_{[\sigma^i]}$ is marked. Moreover, the corresponding marking remains in $O$. Hence, each state in $C$ is “older”, i.e., already part of $P^i_{S-1}$. Thus, Lemma 1 proves that states in the closed set $C$ do not get new successors upon extending the SPN.

**Lemma 2 (New states do not connect to old states).** Let $\sigma^i = \langle a_1, \ldots, a_i \rangle$ and $\sigma^{i+1} = \sigma^i \cdot \langle a_{i+1} \rangle$. For a given WF-net $N$, let $N^i_S = (P^i_S, T^i_S, M^i_S, M^f_i, \lambda^i_S)$ (analogously $N^i_{S+1}$) be the SPN of $N$ and $\sigma^i$. For $M \in B(P^i_S)$, $\forall t \in T^i_S : (N^i_S, M) \vdash t \in T^i_S$

**Proof (By construction of the SPN).** Let $t_{i+1} \in T^i_{S+1} \setminus T^i_S$. Let $p_{[\sigma^i]} \in P^i_{S+1}$ be the $(i+1)$-th place of the trace net part. We know that $p_{[\sigma^i]} \in t_{i+1} \bullet$ and $p_{[\sigma^i]} \notin t_{j+1} \bullet \forall j \in \{1, \ldots, i\}$. For all other $t \in T^i_S$ we know that $\exists M \in B(P^i_{S+1}) \setminus B(P^i_S)$ such that $(N^i_S, M) \vdash t$.

From Lemma 1 and 2 we know that states in the closed set are not affected by extending the SPN. Hence, it is feasible to continue the search from the open set and to not reconsider states which are in the closed set.

**Lemma 3 (Exists a state in the O-set that is on the shortest path).** Let $\sigma^i = \langle a_1, \ldots, a_i \rangle$, $\sigma^{i+1} = \sigma^i \cdot \langle a_{i+1} \rangle$, $N^i_S$, $N^i_{S+1}$ the corresponding SPN for a WF-net $N$, $O^i$ and $C^i$ be the open- and closed set after the prefix-alignment computation for $\sigma^i$. Let $\overline{\sigma}_{i+1}$ be an optimal prefix-alignment for $\sigma^{i+1}$.

$$\exists j \in \{1, \ldots, |\overline{\sigma}_{i+1}|\}, \overline{\sigma}^i_{i+1} = (\overline{\sigma}_{i+1}(1), \ldots, \overline{\sigma}_{i+1}(j)) \ s.t.\ (N^i_S, M^i_{i+1}) \overset{\pi^i(\overline{\sigma}^i_{i+1})}{\longrightarrow} (N^i_{S+1}, M_O) \text{ and } M_O \in O^i$$

**Proof.** $\overline{\sigma}^i_{i+1}$ corresponds to a sequence of markings, i.e., $S = (M^i_{i+1}, \ldots, M', M'', \ldots, M''')$. Let $X^{i+1} = B(P^i_S) \setminus C^i \cup O^i$. It holds that $X^{i+1} \cap O^i = X^{i+1} \cap C^i = O^i \cap C^i = \emptyset$. Note that $M''' \in X^{i+1}$ because $M''' \notin B(P^i_S)$. Assume $\forall M \in S : M \notin O^i$ $\Rightarrow \forall M \in S : M \in C^i \cup X^{i+1}$. Observe that $M_i = M^i_{i+1} \in C^i$ since initially $M_i = M^i_{i+1}$ is the initial marking and in the very first iteration $M^i_{i+1}$ is selected for expansion because it is not a goal state, Algorithm 1. We know that for any state pair $M', M''$ it cannot be the case that $M' \in C^i$, $M'' \in X^{i+1}$. Since we know that at least $M^i_{i+1} \in C^i$ and $M''' \in X^{i+1}$ there $\exists M', M'' \in S$ such that $M' \in C^i$, $M'' \in O^i$. 


Hence, when incrementally computing prefix-alignments, continuing the search from the previous open- and closed set, leads to optimal prefix-alignments.

**Heuristic for Prefix-Alignment Computation** Since the $A^*$ algorithm uses a heuristic function to efficiently traverse the search space, we present in this section a relaxed heuristic for prefix-alignment computation based on a existing heuristic [4] used for alignment computation. Both heuristics can be formulated as an Integer Linear Program (ILP).

Let $N^S = (P^S, T^S, F^S, M^S_i, M^S_f, \lambda^S)$ be a SPN of a WF-net $N = (P, T, F, [p_i], [p_o], \lambda)$ and a trace $\sigma$ with corresponding trace net $N^\sigma = (P^\sigma, T^\sigma, F^\sigma, [p_i^\sigma], [p_o^\sigma], \lambda^\sigma)$. Let $c : T^S \rightarrow \mathbb{R}_{\geq 0}$ be a cost function. We define a revised heuristic function for prefix-alignment computation as an ILP:

- **Variables**: $X = \{x_t | t \in T^S\}$ and $\forall x_t \in X : x_t \in \mathbb{N}_0$
- **Objective function**: $\min \sum_{t \in T^S} x_t \cdot c(t)$
- **Constraints**:
  - Trace net part: $M^S_f(p) = \sum_{t \in p} x_t - \sum_{t \in p^\circ} x_t \forall p \in P^S : p \in P^\sigma$
  - Process model part: $0 \leq \sum_{t \in p} x_t - \sum_{t \in p^\circ} x_t \forall p \in P^S : p \in P$

The revised heuristic presented is a relaxed version of the existing heuristic used for alignment computation. Hence, any solution to the state equation based heuristic is also a solution to the revised heuristic. Since this is a minimization problem and the existing heuristic is proven to be consistent and admissible [4,9], it directly follows that the revised heuristic is admissible and consistent.

**5 Evaluation**

We evaluated the algorithm on real event data. Hereinafter, we present the experimental setup and subsequently discuss the results.

**Experimental Setup** The algorithm introduced in [12] serves as a baseline algorithm. We refer to this algorithm as Online Conformance Checking (OCC). Upon receiving an event, OCC partially reverts previously computed prefix-alignments (using a given maximal window size). Therefore, the algorithm does not guarantee optimality, i.e., it does not search for a global optimum. However, OCC can also be used without partially reverting, i.e., using window size $\infty$. Optimality is then guaranteed but the algorithm essentially computes a prefix-alignment from scratch each time. For our evaluation, the newly proposed algorithm, incremental $A^*$ (IAS), as well as OCC have been implemented in the process mining library PM4Py [6]. Both algorithms use the revised heuristic.

We used a real-life dataset [12] containing event data from a medical training process and a reference process model. To simulate an event stream, we iterated over the traces in the event log and emit each preformed activity as an event.
Results In Fig. 8, we present the results of our experiments. OCC-W\(x\) represents OCC with window size \(x\). We observe that the calculation time of the heuristic computation contributes significantly to the total time. Furthermore, OCC-W\(x\) variants need less time than the variants guaranteeing optimality. We observe that IAS visits a comparable number of states to the OCC-W5 and OCC-W10 variants. The average number of visited states, i.e., states added to the closed set, is an indicator of search efficiency. We even observe that up to a prefix length of 50, IAS performs noticeably better than OCC-W5 and OCC-W10. We also measured the avg. number of traversed arcs and en-queued states. Since both attributes show almost identical relative behavior to the avg. number of visited states, we do not present the corresponding plots. Since IAS and OCC are both optimal, they have zero cost deviation. Furthermore, we see that the larger the window size, the smaller the error. However, this was to be expected.

6 Conclusion

We proposed a novel algorithm to efficiently monitor ongoing processes in an online setting by computing a prefix-alignment once a new event occurs. We have shown that the calculation of a prefix-alignment can be continued from the previous open- and closed set on an extended search space. Moreover, we have shown that the state equation based heuristic can be adapted for prefix-alignment computation. The results show that the proposed algorithm is comparable with existing approaches and additionally ensures optimality.

\(^4\) Note that both heuristics can also be defined as a Linear Program (LP) which leads to faster calculation but less accurate results.
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