Operator Representations of a $q$-Deformed Heisenberg Algebra

Konrad Schmüdgen
Universität Leipzig,
Fakultät für Mathematik und Informatik und NTZ,
Augustusplatz 10/11, D–04109 Leipzig, Germany

Abstract. A class of well-behaved $\ast$-representations of a $q$-deformed Heisenberg algebra introduced in refs. 10 and 3 is studied and classified.

The idea to develop a $q$-deformed quantum mechanics by using quantum groups has been investigated in several papers$^{2,3,6,11,13}$. Such approaches are usually based on a $q$-deformed phase space algebra which is derived from the noncommutative differential calculus of the $q$-deformed configuration space$^{7,14}$. Following the standard procedure in quantum mechanics one has to represent the $q$-deformed position and momentum operators by essentially self-adjoint operators acting on a Hilbert space. More precisely, one has to find appropriate $\ast$-representations of the phase space $\ast$-algebra by unbounded operators in a Hilbert space. In the case of general Euclidean or Minikowski phase spaces the study and classification of these $\ast$-representations turns out to be technically complicated because of the many relations and also because of the various difficulties concerning unbounded operators.

The aim of this paper is to give a rigorous treatment of well-behaved operator representations for one of the simplest example - the one-dimensional $q$-deformed Heisenberg algebra which was invented in refs. 11 and 3. Representations of this algebra have been investigated in ref. 3. Since this $\ast$-algebra occurs as a subalgebra of other larger $\ast$-algebras, the study of general not necessarily irreducible $\ast$-representations seems to be important as well. We shall develop and analyze an operator-theoretic model for such general representations of the $q$-deformed Heisenberg algebra. This model might be used as a tool kit for the study of representations of larger $\ast$-algebras.

This paper is organized as follows. Section I contains the definition and some simple algebraic properties of the $q$-deformed Heisenberg algebra $A(q)$. In Section II we develop a general operator-theoretic model for certain triples of operators which will lead in Section
V to representations of the ∗-algebra \( A(q) \). In Section III the irreducibility and the unitary equivalence of these operator triples are investigated and a number of examples are treated. In Section IV we give a characterization of these operator triples by a number of natural conditions. In Section V we define the self-adjoint ∗-representations of the ∗-algebra \( A(q) \) obtained by means of these operator triples.

In a forthcoming paper we shall study the spectrum of the operator \( X \). For this analysis the \( q \)-Fourier transform\(^5\) will play a crucial role.

**I. The \( q \)-Heisenberg algebra**

For a positive real number \( q \neq 1 \), let \( A(q) \) denote the complex unital algebra with four generators \( p, x, u, u^{-1} \) subject to the defining relations

\[
up = q pu, \quad ux = q^{-1} xu, \quad uu^{-1} = u^{-1} u = 1, \tag{1}
\]

\[
px - q xp = i(q^{3/2} - q^{-1/2})u, \quad xp - q px = -i (q^{3/2} - q^{-1/2})u^{-1}, \tag{2}
\]

where \( i \) denotes the imaginary unit. An equivalent set of relations is obtained if (2) is replaced by

\[
px = i q^{1/2} u^{-1} - i q^{-1/2} u, \quad xp = i q^{-1/2} u^{-1} - i q^{1/2} u. \tag{2}'
\]

From (1) and (2)' it follows that the set of elements \( \{ p^r u^s x^s u^n; r \in \mathbb{N}_0, s \in \mathbb{N}, n \in \mathbb{Z} \} \) is a vector space basis of \( A(q) \).

The algebra \( A(q) \) becomes a ∗-algebra with involution defined on the generators by

\[
p = p^*, \quad x = x^*, \quad u^* = u^{-1}. \tag{3}
\]

Indeed, it suffices to check that the defining relations (1) and (2)' of \( A(q) \) are invariant under the involution (3) which is easily done.

From (1), (2)' and (3) we conclude that there are ∗-isomorphisms \( \rho_1 \) and \( \rho_2 \) of the ∗-algebras \( A(q) \) and \( A(q^{-1}) \) such that

\[
\rho_1(x) = p, \quad \rho_1(p) = x, \quad \rho_1(u) = u \quad \text{and} \quad \rho_2(x) = x, \quad \rho_2(p) = p, \quad \rho_2(u) = -u^*. \]

Because the ∗-algebras \( A(q) \) and \( A(q^{-1}) \) are isomorphic, we shall assume in what follows that \( 0 < q < 1 \).
II. An operator-theoretic model

II.1. Let $\mu_1$ be a finite positive Borel measure on the interval $[q, 1)$. The measure $\mu_1$ extends uniquely to a Borel measure $\mu$ on the half-axis $\mathbb{R}_+ = (0, +\infty)$ by setting $\mu(q^n M) := q^n \mu_1(M)$ for any Borel subset $M$ of $[q, 1)$. Then $\mu$ has obviously the property that $\mu(q^N) = q \mu(N)$ for an arbitrary Borel subset $N$ of $\mathbb{R}_+$ or equivalently that $d\mu(q t) = q d\mu(t)$ for $t \in \mathbb{R}_+$. We shall work with the Hilbert spaces $H := L^2(\mathbb{R}_+, \mu)$ and $\mathcal{H} := L^2([q, 1), \mu_1)$. First we define three linear operators $U, P$ and $X$ on the Hilbert space $H$:

(i) $(Uf)(t) = q^{1/2} f(qt)$ for $f \in \mathcal{H}$,
(ii) $(Pf)(t) = tf(t)$ for $f \in \mathcal{D}(P) := \{ f \in \mathcal{H} : tf(t) \in \mathcal{H} \}$,
(iii) $(Xf)(t) = i t^{-1} (f(q^{-1} t) - f(qt))$ for $f \in \mathcal{D}(X) := \{ f \in \mathcal{H} : t^{-1} f(t) \in \mathcal{H} \}$.

These operators will play a crucial role throughout this paper. Roughly speaking and ignoring technical subtleties (domains, boundary conditions etc.), we shall show that for all "well-behaved" $^*$-representations of the $q$-deformed Heisenberg algebra $A(q)$ the images of the generators $u, p$ and $x$ act by the same formulas as the operators $U, P$ and $X$, respectively.

Obviously, $P$ is an unbounded self-adjoint operator on $\mathcal{H}$. Using the relation $d\mu(q t) = q d\mu(t)$ one easily verifies that $U$ is a unitary operator and that $X$ is a symmetric operator on $\mathcal{H}$. Let $\mathcal{D}_0$ be the set of functions $f \in \mathcal{H}$ such that $\text{supp } f \subseteq [a, b]$ for some $a > 0$ and $b > 0$. (Note that $a$ and $b$ may depend on $f$.) Clearly, $\mathcal{D}_0$ is dense linear subspace of $\mathcal{H}$ which is invariant under $U, P$ and $X$. It is straightforward to check that the operators $P, X, U$ applied to functions $f \in \mathcal{D}_0$ satisfy the defining relations (1), (2) and (3) of the $^*$-algebra $A(q)$. In turns out that the symmetric operator $X$ is not essentially self-adjoint.

Our next aim is to characterize the domain of the adjoint operator $X^*$.

For $f \in \mathcal{H} = L^2([q, 1), \mu_1)$ let $f^e$ and $f^o$ be the functions on $\mathbb{R}_+$ defined by

$$f^e(q^{2n} t) = f^o(q^{2n+1} t) = f(t) \text{ for } n \in \mathbb{N}_0, \ t \in [q, 1) \text{ and } f^e(t) = f^o(t) = 0 \text{ otherwise.} \ (4)$$

Clearly, $f^e$ and $f^o$ are in $\mathcal{H} = L^2(\mathbb{R}_+, \mu)$ and we have $U(f^e) - q^{1/2} f^o \in \mathcal{D}(X)$ and $U f^o - q^{1/2} f^o \in \mathcal{D}(X)$. Let $\mathcal{H}_e$ and $\mathcal{H}_o$ denote the set of functions $f^e$ and $f^o$, respectively, where $f \in \mathcal{H}_e = L^2([q, 1), \mu_1)$. 

3
Lemma 1. The domain $\mathcal{D}(X^*)$ is the direct sum of vector spaces $\mathcal{D}(X), \mathcal{S}_e$ and $\mathcal{S}_o$.

Proof. It is straightforward to check that $\mathcal{D}(X) + \mathcal{S}_e + \mathcal{S}_o \subseteq \mathcal{D}(X^*)$. In order to prove the converse, let $g \in \mathcal{D}(X^*)$. Then, by definition there is an $h \in \mathcal{H}$ such that $\langle Xf, g \rangle = \langle f, h \rangle$ for all $f \in \mathcal{D}(X)$. Inserting the definition of $X$ and using once more the fact that $\frac{d\mu(qt)}{qt} = \frac{d\mu(t)}{t}$ we easily conclude that $h(t) = it^{-1}(g(q^{-1}t) - g(qt))$. For a function $f \in \mathcal{H}$ let $f_n$ denote the function in $L^2([q, 1), \mu_1^t)$ given by $f_n(t) = f(q^n t)$. Then we get

$$\|h\|_{L^2(\mathbb{R}_+, \mu)}^2 = \sum_{n=-\infty}^{\infty} \|h_n\|^2 q^n \geq \sum_{n=0}^{\infty} \|g_{n+1} - g_{n-1}\|^2 q^{2n}.$$  

For $n \in \mathbb{N}$ we set $\alpha_n := \|g_{n+1} - g_{n-1}\|^q$. Since $h \in L^2(\mathbb{R}, \mu)$, the sequence $(\alpha_n)$ is in $l_2$. From the inequality

$$\|g_{2r} - g_{2s}\| \leq \alpha_{2r+1} q^{\frac{2r+1}{2}} + \cdots + \alpha_{2s+1} q^{\frac{2s+1}{2}}$$

we obtain

$$\|g_{2r} - g_{2s}\|^2 \leq \left( \sum_{i=2s+1}^{\infty} |\alpha_i|^2 \right) q^{2s+1}(1 - q^2)^{-1}, r \geq s.$$  

(5)

Since $(\alpha_n) \in l_2$, this implies that the sequence $(g_{2n})_{n \in \mathbb{N}}$ converges in the Hilbert space $L^2([q, 1), \mu_1^t)$. Let us denote its limit by $\xi$. We extend $\xi$ to a function $\xi^e$ on $\mathbb{R}_+$ by setting $\xi^e(q^{2nt}) := \xi(t)$ and $\xi^e(q^{2n+1}t) := 0$ for $n \in \mathbb{N}_0$, $t \in [q, 1)$ and $\xi^e(t) = 0$ for $t \geq 1$. Replacing even indices by odd indices, a similar reasoning yields functions $\zeta \in L^2([q, 1), \mu_1^t)$ and $\zeta^o$ on $\mathbb{R}_+$ such that $\zeta^o(q^{2nt}) = \zeta(t)$ and $\zeta^o(q^{2n+1}t) = 0$ for $n \in \mathbb{N}$, $t \in [q, 1)$ and $\zeta^o(t) = 0$ for $t \geq 1$. By construction, $\xi^e \in \mathcal{S}_e$ and $\zeta^o \in \mathcal{S}_o$. Our proof is complete once we have shown that $f := g - \xi^e - \zeta^o$ belongs to the domain $\mathcal{D}(X)$ of the operator $X$.

Letting $r \to \infty$ in (5), we get

$$\|\xi - g_{2s}\|^2 \leq q^{2s+1}(1 - q^2)^{-1} \sum_{n=0}^{\infty} |\alpha_{2n}|^2.$$  

(6)

From (6) and the corresponding estimation of $\|\zeta - g_{2s+1}\|^2$ we obtain

$$\sum_{n=0}^{\infty} \|f_n(t)\|^2 q^n \leq \sum_{n=0}^{\infty} \frac{\|f_n\|^2}{q^{2n+2}} q^n = \sum_{\tau=0}^{\infty} \frac{\|\xi - g_{2r}\|^2}{q^{2\tau+2}} + \frac{\|\zeta - g_{2r+1}\|^2}{q^{2\tau+3}} = (q - q^3)^{-1} \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty.$$
Since \( f(t) = g(t) \) for \( t \geq 1 \), this inequality implies that the functions \( t^{-1}f(t) \) and \( f(t) \) are in \( L^2(\mathbb{R}^+, \mu) \). Thus, \( f \in \mathcal{D}(X) \).

As shown in the preceding proof, for any function \( g \in \mathcal{D}(X^*) \) the ”even components” \( g_{2n} \) and the ”odd components” \( g_{2n+1} \) both have ”boundary limits” \( \xi \) and \( \zeta \) in \( L^2([q, 1), \mu_1) \).

By Lemma 1, any element \( f \in \mathcal{D}(X^*) \) is of the form \( f = f_X + f^e + f^o \) with uniquely determined functions \( f_X \in \mathcal{D}(X), f^e \in \mathcal{H}_e \) and \( f^o \in \mathcal{H}_o \). By the definition of \( \mathcal{H}_e \) and \( \mathcal{H}_o \), there exist unique functions \( f_e, f_o \in \mathcal{H} = L^2([q, 1), \mu_1) \) such that \( (f_e)^e = f^e \) and \( (f_o)^o = f^o \), where the function \( (f_e)^e \) and \( (f_o)^o \) on \( \mathbb{R} \) are given by (4). This notation will be kept in the sequel.

Let \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) denote the scalar products of the Hilbert spaces \( L^2(\mathbb{R}^+, \mu) \) and \( L^2([q, 1), t^{-1}\mu_1) \), respectively.

**Lemma 2.** For arbitrary functions \( f, g \in \mathcal{D}(X^*) \) we have

\[
\langle X^*f, g \rangle - \langle f, X^*g \rangle = \frac{1}{2i} \{(f_e + f_o, g_e + g_o) - (f_e - f_o, g_e - g_o)\}. \tag{7}
\]

**Proof.** Let \( h \in L^2([q, 1), \mu_1) \). From the definitions of the operator \( X \) and of the functions \( h^e, h^o \in L^2(\mathbb{R}, \mu) \) we easily derive that \( (X^*h^e)(t) = -it^{-1}h(qt) \) for \( t \in [1, q^{-1}) \), \( (X^*h^e)(t) = 0 \) for \( t \in \mathbb{R}^+ \setminus [1, q^{-1}) \), \( (X^*h^o)(t) = -it^{-1}h(t) \) for \( t \in [q, 1) \) and \( (X^*h^o)(t) = 0 \) for \( t \in \mathbb{R}^+ \setminus [q, 1) \). Inserting these expressions and using the symmetry of the operator \( X \) we compute

\[
\begin{align*}
\langle X^*f, g \rangle - \langle f, X^*g \rangle &= \langle X^*f_o, g_e \rangle - \langle f_e, X^*g_o \rangle \\
&= -i \int_q^1 (f_o(t)g_e(t) + f_e(t)g_o(t))t^{-1}d\mu(t) \\
&= -i\{(f_o, g_e) + (f_e, g_o)\} \\
&= \frac{1}{2i}\{(f_e + f_o, g_e + g_o) - (f_e - f_o, g_e - g_o)\}. \tag*{\blacksquare}
\end{align*}
\]

Let us illustrate the preceding by the simplest example.

**Example 1.** Let \( \mu_1 \) be the Delta measure \( \delta_a \), where \( a \) is a fixed number from the intervall \([q, 1)\). Then the measure \( \mu \) is supported on the points \( aq^n, n \in \mathbb{Z} \), and we have \( \mu(\{aq^n\}) = \)
\[ q^n \mu(\{a\}) = q^n. \] Hence the scalar product of the Hilbert space \( H = L^2(\mathbb{R}_+, \mu) \) is given by the Jackson integral
\[
(f, g) = \sum_{n=-\infty}^{+\infty} f(aq^n) \frac{g(aq^n)}{q^n} q^n.
\]
Let \( e_n \in H \) be the function \( e_n(t) = q^{-\frac{n}{2}} \delta_{aq^n} \), where \( \delta_x \) is the usual Kronecker symbol. Then the vectors \( e_n, n \in \mathbb{Z} \), form an orthonormal basis of \( H \) and the actions of the operators \( U, P, X \) on these vectors are given by
\[
U e_n = e_{n-1}, \quad P e_n = aq^n e_n, \quad X e_n = \frac{i}{aq^n} \left( q^{-1/2} e_{n+1} - q^{1/2} e_{n-1} \right).
\]
These equations are in accordance with formulas (5) in ref. 3. If \( f \) is the function in \( L^2(\mathbb{R}^+, \mu_1) \approx \mathbb{C} \) with \( f(a) = 1 \), then by definition \( f^e(aq^{2n}) = f^o(aq^{2n+1}) = 1, f^e(aq^{2n+1}) = f^o(aq^{2n}) = 0 \) for \( n \in \mathbb{N}_0 \) and \( f^e(t) = f^o(t) = 0 \) for \( t \geq 1 \). Then we have \( D(X^*) = D(X) + \mathcal{C} \cdot f^e + \mathcal{C} \cdot f^o \) by Lemma 1 and formula (7) reads as
\[
\langle X^* f, g \rangle - \langle f, X^* g \rangle = \frac{1}{2i} \{ (\alpha_1 + \beta_1)(\alpha_2 + \beta_2) - (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \}
\]
for \( \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{C} \).

II.2 The above considerations carry over almost verbatim to the case where the positive half-axis \( \mathbb{R}_+ \) is replaced by the negative half-axis \( \mathbb{R}_- = (-\infty, 0) \). Any positive finite Borel measure \( \mu_1 \) on the interval \( [q, 1) \) induces a positive Borel measure \( \mu \) on \( \mathbb{R}_- \) by defining \( \mu(-q^n M) := q^n \mu_1(M) \) for a Borel subset \( M \) of \( [q, 1) \). The operators \( U, P, X \) on the Hilbert space \( H_- := L^2(\mathbb{R}_-, \mu) \) are defined by the same formulas as in the preceding subsection and Lemma 1 and its proof remain valid in this case as well. However, there is an essential difference which will be crucial in the sequel: Since in the proof of Lemma 2 the integration is over the interval \((-1, -q]\), the expression on the right hand side of (7) must be multiplied by \(-1\). That is, instead of (7) we now have
\[
\langle X^* f, g \rangle - \langle f, X^* g \rangle = \frac{1}{2i} \{ (f_e + f_o, g_e + g_o) - (f_e - f_o, g_e - g_o) \}
\]
for \( f, g \in D(X^*). \)
II.3 After the preceding preparations we are now able to develop the operator-theoretic model for the description of *-representations of the $q$-Heisenberg algebra $\mathcal{A}(q)$. For this let us fix two families $\{\mu_{1}^{j,+}; j \in I_{+}\}$ and $\{\mu_{1}^{j,-}; j \in I_{-}\}$ of finite positive Borel measures on the intervall $[q,1)$.

As above, we define the Hilbert spaces $\mathcal{H}_{j,\pm} := L^{2}(\mathbb{R}_{\pm}, \mu_{1}^{j,\pm})$, $j \in I_{\pm}$, and the operators $U_{j,\pm}, P_{j,\pm}, X_{j,\pm}$ acting therein. We shall work with the representation Hilbert space $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$, where $\mathcal{H}_{+} := \bigoplus_{j \in I_{+}} \mathcal{H}_{j,+}$ and $\mathcal{H}_{-} := \bigoplus_{j \in I_{-}} \mathcal{H}_{j,-}$. The elements of $\mathcal{H}$ are pairs $f = (f^{+}, f^{-})$, where $f^{+} = (f_{j,+}^{+}; j \in I_{+}) \in \mathcal{H}_{+}$ and $f^{-} = (f_{j,-}^{-}; j \in I_{-}) \in \mathcal{H}_{-}$. Let $U, P, X$ denote the operators on $\mathcal{H}$ which are defined as the direct sums of the operators $U_{j,+}, U_{j,-} ; P_{j,+}, P_{j,-} ; X_{j,+}, X_{j,-}$, respectively. Clearly, $U$ is a unitary operator and $P$ is a self-adjoint operator on $\mathcal{H}$. The operator $X$ is only symmetric, but not self-adjoint. Our next aim is to describe all self-adjoint extensions $\tilde{X}$ of $X$ on $\mathcal{H}$ which have the property that $UU^{-1} = q\tilde{X}$.

Let $V$ and $W$ be two unitary linear transformations of the Hilbert space $\mathfrak{H}_{-} := \bigoplus_{j \in I_{-}} L^{2}([q,1), t^{-1}\mu_{1}^{j,-})$ on the Hilbert space $\mathfrak{H}_{+} := \bigoplus_{j \in I_{+}} L^{2}([q,1), t^{-1}\mu_{1}^{j,+})$. We define a linear operator $X_{V,W}$ as being the restriction of the adjoint operator $X^{*}$ to the domain

$$
\mathcal{D}(X_{V,W}) := \{f = f_{X} + f^{e} + f^{o} \in \mathcal{D}(X^{*}) : f_{X} \in \mathcal{D}(X), \quad f^{e}_{e} = V(f^{e}_{e} + f^{o}_{0}) + W(f^{e}_{e} - f^{o}_{0}), f^{o}_{0} = V(f^{+}_{e} + f^{-}_{0}) - W(f^{+}_{e} - f^{-}_{0}). \}
$$


\textbf{Proposition 3.} $X_{V,W}$ is a self-adjoint operator on $\mathcal{H}$ such that $X \subseteq X_{V,W}$ and $UX_{V,W}U^{*} = qX_{V,W}$. In particular, we have $UD(X_{V,W}) = \mathcal{D}(X_{V,W})$. Conversely, for any self-adjoint extension $\tilde{X}$ of $X$ satisfying $UD(\tilde{X}) \subseteq \mathcal{D}(\tilde{X})$ there exist unitary transformations $V, W$ of $\mathfrak{H}_{+}$ onto $\mathfrak{H}_{-}$ such that $\tilde{X} = X_{V,W}$.

\textbf{Proof.} From (7)$_{+}$ and (7)$_{-}$ we obtain

$$
-2i(\langle X^{*}f, g \rangle - \langle f, X^{*}g \rangle)
= (f^{e}_{e} + f^{o}_{0} + g^{e}_{e} + g^{o}_{o}) + (f^{e}_{e} - f^{o}_{0} + g^{e}_{e} - g^{o}_{o}) - (f^{e}_{e} + f^{o}_{0} + g^{e}_{e} + g^{o}_{o}) - (f^{e}_{e} - f^{o}_{0} + g^{e}_{e} - g^{o}_{o}).
$$

for arbitrary elements $f = f_{X} + f^{e} + f^{o}$ and $g = g_{X} + g^{e} + g^{o}$ of $\mathcal{D}(X^{*})$. Here $f^{e}_{e}$ denotes the sequence $(f^{e}_{j,+}; j \in I_{+}) \in \mathfrak{H}_{+}$ with $f^{e}_{j,+} \in L^{2}([q,1), \mu_{1}^{j})$ such that the extension $(f^{e}_{j,+})^{e}$
of \( f^+_{\ell^+} \) to \( \mathbb{R}_+ \) by means of formula (4) is just the \((j,+)-\)component of the vector \( f^* \in \mathcal{H} \). A similar meaning attached to the other symbols \( f^-_{\ell^-}, f^+_{\ell^+}, f^-_{\ell^+}, g^+_{\ell^-}, g^-_{\ell^-}, g^+_{\ell^+}, g^-_{\ell^+} \) occurring in (9).

If \( f, g \in \mathcal{D}(X_{V,W}) \), then we have \( f^+_e + f^-_o = V(f^-_e + f^-_o) \), \( g^+_e + g^-_o = V(g^-_e + g^-_o) \), and \( f^+_e - f^-_o = W(f^-_e - f^-_o) \) and \( g^+_e - g^-_o = W(g^-_e - g^-_o) \) by (8). Since \( X_{V,W} \subseteq X^* \), we therefore obtain that \( \langle X_{V,W} f, g \rangle - \langle f, X_{V,W} g \rangle = 0 \) by (9), that is, the operator \( X_{V,W} \) is symmetric. Now let \( g \in \mathcal{D}((X_{V,W})^*) \). Since \( X \subseteq X_{V,W} \subseteq (X_{V,W})^* \subseteq X^* \), we then have \( \langle X^* f, g \rangle = \langle f, X^* g \rangle \) and hence

\[
(f^+_e + f^-_o, g^+_e + g^-_o) + (f^-_e - f^-_o, g^-_e - g^-_o) = (f^-_e + f^-_o, g^-_e + g^-_o) + (f^+_e - f^-_o, g^+_e - g^-_o) \tag{10}
\]

for all \( f \in \mathcal{D}(X_{V,W}) \) by (9). Inserting (8) into (10), we get

\[
(f^-_e + f^-_o, V^*(g^+_e + g^-_o)) + (f^-_e - f^-_o, g^-_e - g^-_o) = (f^-_e + f^-_o, g^-_e + g^-_o) + (f^-_e - f^-_o, W^*(g^+_e - g^-_o)).
\tag{11}
\]

From the construction it is clear that for arbitrary \( h, \ell \in \mathcal{H}_- \) there exists \( f \in \mathcal{D}(X_{V,W}) \) such that \( f^-_e + f^-_o = h \) and \( f^-_e - f^-_o = \ell \). Therefore, it follows from (11) that \( V^*(g^+_e + g^-_o) = g^+_e + g^-_o \) and \( W^*(g^+_e - g^-_o) = g^+_e - g^-_o \) which in turn implies that \( g \in \mathcal{D}(X_{V,W}) \). Thus we have shown that the operator \( X_{V,W} \) is self-adjoint. From the relations \( U(f_e) - q^{1/2}f_o \in \mathcal{D}(X) \) and \( U(f_o) - q^{1/2}f_e \in \partial(X) \) we see that \( U \mathcal{D}(X_{V,W}) = \mathcal{D}(X_{V,W}) \). Since \( UX^*U^* = qX \) and hence \( UX^*U^* = qX^* \) and \( X_{V,W} \) is the restriction of \( X^* \) to \( \mathcal{D}(X_{V,W}) \), the latter yields \( UX_{V,W}U^* = qX_{V,W} \).

Conversely, suppose that \( \tilde{X} \) is a self-adjoint extension of \( X \) such that \( U \mathcal{D}(\tilde{X}) \subseteq \mathcal{D}(\tilde{X}) \). Since \( \tilde{X} \) is symmetric, we have equation (10) for arbitrary elements \( f, g \in D(\tilde{X}) \). By assumption, \( Uf \in D(\tilde{X}) \) for all \( f \in D(\tilde{X}) \). Replacing \( f \) by \( Uf \) in (10) we get

\[
(f^+_e + f^+_o, g^+_e + g^+_o) + (f^-_e - f^-_o, g^-_e - g^-_o) = (f^-_e + f^-_o, g^-_e + g^-_o) + (f^-_e - f^-_o, g^+_e - g^-_o). \tag{12}
\]

Setting \( f = g \) and combining formulas (10) and (12) we obtain

\[
\|f^+_e + f^+_o\| = \|f^-_e + f^-_o\| \quad \text{and} \quad \|f^+_e - f^+_o\| = \|f^-_e - f^-_o\|. \tag{13}
\]

for all \( f \in D(\tilde{X}) \).
For $f \in \mathcal{D}(X^*)$ we abbreviate $B_{\pm}(f) = (f^+_e + f^+_o, f^-_e - f^-_o)$. The vector space $B_{\pm}(\tilde{X}) = \{B_{\pm}(f) : f \in \mathcal{D}(\tilde{X})\}$ is called the "boundary space" of the operator $\tilde{X}$. We shall show that $B_{+}(\tilde{X}) = \mathfrak{H}_+ \oplus \mathfrak{H}_+$ and $B_{-}(\tilde{X}) = \mathfrak{H}_- \oplus \mathfrak{H}_-$. First let us note that the spaces $B_{\pm}(\tilde{X})$ are closed in $\mathfrak{H}_\pm \oplus \mathfrak{H}_\pm$. Otherwise let $\tilde{X}$ denote the restriction of $X^*$ to the domain $\mathcal{D}(\tilde{X}) = \{f \in \mathcal{D}(X^*) : B_{\pm}(f) \in B_{\pm}(\tilde{X})\}$, where the bar means the closure in the Hilbert space $\mathfrak{H}_\pm \oplus \mathfrak{H}_\pm$. The symmetry of an operator $Y$ such that $X \subseteq Y \subseteq X^*$ is equivalent to the validity of equation (10) for all $f, g \in \mathcal{D}(Y)$. Hence $\tilde{X}$ is symmetric, because $\tilde{X}$ is so. Since a self-adjoint operator has no proper symmetric extension, we conclude that $\tilde{X} = \tilde{X}$ which means that $B_{+}(\tilde{X})$ and $B_{-}(\tilde{X})$ are closed. Next let us suppose that $(\xi, \zeta) \perp B_{\pm}(\tilde{X})$ in $\mathfrak{H}_+ \oplus \mathfrak{H}_+$. We then choose a vector $g \in \mathcal{D}(\tilde{X})$ such that $\xi = g^+_e + g^+_o, \zeta = g^-_e - g^-_o$ and $g^+_e = g^-_o = 0$. Then the right-hand side of (9) vanishes for all $f \in \mathcal{D}(\tilde{X})$, so that $\langle \tilde{X}f, g \rangle = \langle X^*f, g \rangle = \langle f, X^*g \rangle$ for all $f \in \mathcal{D}(\tilde{X})$ by (9). Consequently, $g \in \mathcal{D}(\tilde{X}^*)$. Since $\tilde{X}$ is self-adjoint, $g$ must be in $\mathcal{D}(\tilde{X})$. Because $(\xi, \zeta) \perp B_{\pm}(\tilde{X})$, this implies that $\xi = \zeta = 0$. This proves that $B_{+}(\tilde{X}) = \mathfrak{H}_+ \oplus \mathfrak{H}_+$. Similarly $B_{-}(\tilde{X}) = \mathfrak{H}_- \oplus \mathfrak{H}_-$. 

Since $B_{\pm}(\tilde{X}) = \mathfrak{H}_\pm \oplus \mathfrak{H}_\pm$ as just shown, it follows from (13) that there are unitary operators $V$ and $W$ of $\mathfrak{H}_-$ onto $\mathfrak{H}_+$ such that $f^+_e + f^+_o = V(f^-_e + f^-_o)$ and $f^-_e - f^-_o = W(f^-_e - f^-_o)$ for all $f \in \mathcal{D}(\tilde{X})$. That is, $\mathcal{D}(\tilde{X}) \subseteq \mathcal{D}(X_{V,W})$. Since $\tilde{X}$ and $X_{V,W}$ are self-adjoint, we conclude that $\tilde{X} = X_{V,W}$.

III. Irreducibility and unitary equivalence

III.1 The next two propositions decide when a triple of operators $\{P, X_{V,W}, U\}$ defined in the preceding section is irreducible and when two such triples are unitarily equivalent. Here we shall say that the triple $\{P, X_{V,W}, U\}$ on $\mathcal{H}$ is irreducible if any bounded operator $A$ on $\mathcal{H}$ satisfying

$$PA \subseteq AP, \ X_{V,W}A \subseteq AX_{V,W} \text{ and } AU = UA$$

is a scalar multiple of the identity operator on $\mathcal{H}$.

Recall that the operator triple $\{P, X_{V,W}, U\}$ depends on the two families $\{\mu^j_{l,\pm} : j \in I_{\pm}\}$ of measures on the intervall $[\eta, 1)$ and on the two unitary operators $V, W : \mathfrak{H}_- \to \mathfrak{H}_+$. In order to formulate the corresponding conditions it is convenient to work with the Hilbert
spaces $\mathcal{R}_\pm = \bigoplus_{j \in I_\pm} L^2([q, 1), \mu_1^{j, \pm})$ rather than with $\mathcal{H}_\pm = \bigoplus_{j \in I_\pm} L^2([q, 1), t^{-1}\mu_1^{j, \pm})$. Further, let $P_\pm$ denote the self-adjoint operator on $\mathcal{R}_\pm$ which acts componentwise as the multiplication by the variable $t$. Clearly, $V$ and $W$ are bounded linear operators of $\mathcal{R}_-$ to $\mathcal{R}_+$ such that

$$V' := P^{1/2}_+ V P^{-1/2}_- \quad \text{and} \quad W' := P^{1/2}_+ W P^{-1/2}_-$$

are unitary.

**Proposition 4.** The triple $\{P, X_{V,W}, U\}$ as defined above is irreducible if and only if any bounded self-adjoint operators $A_+$ on $\mathcal{R}_+$ and $A_-$ on $\mathcal{R}_-$ satisfying

$$A_+ P_+ = P_+ A_+, \quad A_- P_- = P_- A_-, \quad A_+ V' = V' A_-, \quad A_+ W' = W' A_-$$

or equivalently

$$A_+ P_+ = P_+ A_+, \quad A_- P_- = P_- A_-, \quad A_+ V = V A_+, \quad A_+ W = W A_-$$

are scalar multiples of the identity.

**Proof.** We only show that the above condition implies the irreducibility of the triple. The proof of the converse implication is easier and will be omitted. Suppose that $A$ is a bounded operator on $\mathcal{H}$ satisfying (14). Since the set of such $A$ is invariant under the involution, we can assume that $A$ is self-adjoint. Let $E(\cdot)$ denote the spectral projections of $P$. Since $PA \subseteq AP$, the subspace $\mathcal{R}_+ = E([q, 1))\mathcal{H}$ of $\mathcal{H}$ reduces $A$ and the restriction $A_+$ of $A$ to $\mathcal{R}_+$ commutes with the restriction $P_+$ of $P$ to $\mathcal{R}_+$. Similarly, the restrictions $\tilde{A}_-$ of $A$ and $\tilde{P}_-$ of $P$ to the reducing subspace $E((-1, q])\mathcal{H}$ commute. Changing the variable from $t$ to $-t$, the Hilbert space $E((-1, q])\mathcal{H}$ and the operator $\tilde{P}_-$ become $\mathcal{R}_-$ and $P_-$, respectively, and the operator $\tilde{A}_-$ goes into an operator, say $A_-$, on $\mathcal{R}_-$. Thus, $A_- P_- = P_- A_-$. From the assumptions $AU = UA$ and $X_{V,W} A \subseteq AX_{V,W}$ it follows easily that $(Af_\pm)^\ast = A_\pm f_\pm^\ast$ and $(Af_\pm)^\ast = A_\pm f_\pm$ for $f \in \mathcal{D}(X_{V,W})$. Since $Af_\in \mathcal{D}(X_{V,W})$ has to satisfy the relation (8), we obtain $A_+ V = V A_-$ and $A_- W = W A_-$. Therefore, by the above condition, $A_\pm = \lambda_\pm I$ for some $\lambda_\pm \in \mathbb{C}$. Since $A_+ V = V A_-$ and $AU = UA$, it follows that $\lambda_+ = \lambda_-$ and $A = \lambda_+ \cdot I$ on $\mathcal{H}$.

Using similar operator-theoretic arguments it is not difficult to prove
Proposition 5. Two triples \( \{P, X_{V,W}, U\} \) and \( \{\tilde{P}, X_{\tilde{V},\tilde{W}}, \tilde{U}\} \) are unitarily equivalent if and only if there unitary operators \( A_+ \) of \( \mathcal{K}_+ \) to \( \tilde{\mathcal{K}}_+ \) and \( A_- \) of \( \mathcal{K}_- \) to \( \tilde{\mathcal{K}}_- \) such that
\[
A_+ P_+ = \tilde{P}_+ A_+, A_- P_- = \tilde{P}_- A_-, A_+ V = \tilde{V} A_- \quad \text{and} \quad A_+ W = \tilde{W} A_-, \tag{18}
\]
where the tilde refers to the corresponding operators and spaces for the triple \( \{\tilde{P}, X_{\tilde{V},\tilde{W}}, \tilde{U}\} \).

III.2. We shall illustrate the preceding by describing a few examples of irreducible representations. We begin with the simplest possible case.

Example 2. Suppose that the Hilbert spaces \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) are one-dimensional. Then the families of measure \( \{\mu_i^{\pm}; i \in I\} \) and \( \{\mu_j^{\pm}; j \in I\} \) consist only of single Dirac measures \( \delta_a \) and \( \delta_b \), respectively, where \( a, b \in [q, 1) \). Then the triples \( \{P, X_{V,W}, U\} \) are parametrized by complex numbers \( V = V' = e^{i\varphi} \) and \( W = W' = e^{i\psi}, \varphi, \psi \in \mathbb{R} \). The self-adjoint extension \( X_{V,W} \) is then characterized by the boundary condition (8), that is,
\[
f^+_c + f^+_o = e^{i\varphi} (f^-_c + f^-_o), \quad f^+_c - f^+_o = e^{i\psi} (f^-_c - f^-_o).
\]
Each such triple is irreducible because the condition in Proposition 4 is trivially fulfilled. Two triples with different pairs of numbers \( (V, W) \) are not unitary equivalent. The case where \( e^{i\varphi} = e^{i\psi} = 1 \) and \( a = b \) has been treated in detail in ref. 3.

Example 3. Let \( P_+ \) be a self-adjoint operator and \( Z \) a unitary operator on a Hilbert space \( \mathcal{H}_+ \) such that the commutant \( \{P_+, Z\}' \) is equal to \( \mathbb{C} \cdot I \). Such operators exist on any separable Hilbert space\(^{12} \). Upon scaling we can assume that the spectrum of \( P_+ \) is contained in \( [q, 1) \). By the spectral representation theorem\(^{1,ch.X,5} \), we can represent \( P_+ \) up to unitary equivalence as the multiplication operator by the independent variable \( t \) on some direct sum Hilbert space \( \mathcal{H}_+ = \bigoplus_{j \in I} L^2([q, 1); \mu_1^{j+}) \). Let \( \{\mu_1^{j-}; j \in I\} \) be an arbitrary family of measures on \( [q, 1) \) such that \( \dim \mathcal{H}_+ = \dim \mathcal{H}_- \), where \( \mathcal{H}_- := \bigoplus_{j \in I} L^2([q, 1); \mu_1^{j-}) \). Let \( W' \) be a unitary operator from \( \mathcal{H}_- \) to \( \mathcal{H}_+ \). We set \( V' := ZW' \) and define \( V \) and \( W \) by (15). Then the triple \( \{P, X_{V,W}, U\} \) is irreducible.

Indeed, if \( A_+ \) and \( A_- \) be bounded self-adjoint operators satisfying (17), then we have \( A_+ Z = A_+ V'W' = V'A_-W' = V'W'A_+ =ZA_+ \) and \( A_+P_+ = P_+A_+ \), so that
$A_+ = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$ and hence $A_- = V'^*A_+V' = \lambda \cdot I$. By Proposition 4, the triple is irreducible.

**Example 4.** For this example we assume that there exist numbers $a, b \in [q, 1)$ such that $\mu_1^{j^+} = \delta_a$ and $\mu_1^{k^-} = \delta_b$ for all $j \in I_+$ and $k \in I_-$. We shall show that in this case an irreducible triple $\{P, X_{V,W}, U\}$ can be only obtained if both index sets $I_+$ and $I_-$ are singletons or equivalently if $\dim \mathfrak{H}_+ = \dim \mathfrak{H}_- = 1$. Indeed, otherwise we take a self-adjoint operator $A_+$ on $\mathfrak{H}_+$ such that $A_+V'W'^* = V'W'^*A_+$ and $A_+ \not\in \mathbb{C} \cdot I$ and set $A_- := V'^*A_+V'$. Then the conditions (16) are fulfilled, hence the triple is not irreducible.

**Example 5.** If the spectra of the operators $P_+$ on $\mathfrak{H}_+$ and $P_-$ on $\mathfrak{H}_-$ are singletons, then we have seen in Example 4 that irreducible triples exist only in the trivial case where $I_+$ and $I_-$ are singletons. To be more precise, we shall consider the following situation: The index sets $I_\pm$ are disjoint union of two countable infinite sets $I_\pm^1$ and $I_\pm^2$ and there are numbers $a_1, a_2, b_1, b_2 \in [q, 1)$, $a_1 \neq a_2$, such that $\mu_1^{j^+} = \delta_{a_j}$ for $j \in I_+^1, \mu_1^{j^+} = \delta_{a_j}$ for $j \in I_+^2, \mu_1^{j^-} = \delta_{b_j}$ for $j \in I_-^1$ and $\mu_1^{j^-} = \delta_{b_j}$ for $j \in I_-^2$. By identifying $I_\pm^j$ with the natural numbers the Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$ become the direct sum $l_2(\mathbb{N}) \oplus l_2(\mathbb{N})$ of two $l_2$-spaces. We choose a bounded operator $T$ on $l_2(\mathbb{N})$ such that $\{T, T^*\}' = \mathbb{C} \cdot I$ and $I \leq 3T^*T \leq 2 \cdot I$ and $I \leq 3T^*T \leq 2 \cdot I$. It is well-known (see ref. 8, Anhang, §4) that the operator matrix

$$Z = \begin{pmatrix} T & \sqrt{T - TT^*} \\ -\sqrt{T - TT^*} & T^* \end{pmatrix}$$

defines a unitary operator $Z$ on $\mathfrak{H}_+ = \mathfrak{H}_- = l_2(\mathbb{N}) \oplus l_2(\mathbb{N})$. Let $W'$ be an arbitrary unitary operator on $\mathfrak{H}_+ = \mathfrak{H}_-$ and set $V' := ZW'$. Then the triple $(P, X_{V,W}, U)$ is irreducible.

Indeed, let $A_+$ and $A_-$ be self-adjoint bounded operators on $\mathfrak{H}_+ = \mathfrak{H}_-$ satisfying (17). Since $a_1 \neq a_2$, the relation $A_+P_+ = P_+A_+$ implies that $A_+$ is given by a diagonal operator matrix

$$A_+ = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$
B commutes with $T$ and $T^*$ and so with $\sqrt{I-T^*T}$ which in turn gives $\sqrt{I-T^*T}B = \sqrt{I-T^*T}C$. Because $\sqrt{I-T^*T}$ is invertible, we get $B = C$. Since $B \in \{T,T^*\}'$, we obtain $B = C = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$. Thus, $A_+ = \lambda \cdot I$ and $A_- = V^*A_+V = \lambda \cdot I$, so that the triple is irreducible by Proposition 4.

IV. A characterization of the operator triples

Let $\{P,X_{V,W},U\}$ be an operator triple as in section II and let $\mathcal{D}_1$ be the set of all vectors $\mathcal{f} = f_X + f_e + f_o \in \mathcal{D}(X_{V,W})$ with $f_X \in \mathcal{D}_o$, where $\mathcal{D}_o$ is as defined in Section II. Then $\mathcal{D}_1$ is a dense linear subspace of the Hilbert space $\mathcal{H}$ such that $\mathcal{D}_1$ is invariant under the operators $P, X_{V,W}, U$ and the restrictions of $P$ and $X_{V,W}$ to $\mathcal{D}_1$ are essentially self-adjoint. Further, the three operators $P, X_{V,W}, U$ applied to vectors $\mathcal{f} \in \mathcal{D}_1$ satisfy the relations (1) and (2). From the construction it is clear that the range $E([q,1])\mathcal{H}(\cong \mathfrak{R}_+)$ of the spectral projection $E([q,1])$ of the operator $P$ is contained in $\mathcal{D}_1$. Our next proposition says that the operator triples $\{P,X_{V,W},U\}$ can be characterized by some of the properties just mentioned.

**Proposition 6.** Let $\{P',X',U'\}$ be a triple of two self-adjoint operators $P'$ and $X'$ and a unitary operator $U'$ on a Hilbert space $\mathcal{H}$. Let $E(\cdot)$ denote the spectral measure of $P'$. Suppose that there exists a linear subspace $\mathcal{D}_1 \subseteq \mathcal{D}(P'X') \cap \mathcal{D}(X'P')$ of $\mathcal{H}$ such that:

(i) $E([q,1])\mathcal{H} \subseteq \mathcal{D}_1$ and $E((-1,-q])\mathcal{H} \subseteq \mathcal{D}_1$.

(ii) The operators $P',X',U'$ satisfy the relations (1) and (2) for vectors in $\mathcal{D}_1$.

(iii) The restrictions $P'|\mathcal{D}_1$ and $X'|\mathcal{D}_1$ of $P'$ and $X'$ to $\mathcal{D}_1$ are essentially self-adjoint.

Then $\{P',X',U'\}$ is unitarily equivalent to an operator triple $\{P,X_{V,W},U\}$ defined in Section II.

**Sketch of proof.** The restriction $P'_1$ of $P'$ to the invariant subspace $\mathcal{H}_1 := E([q,1])\mathcal{H}$ is obviously a bounded self-adjoint operator on the Hilbert space $\mathcal{H}_1$ with spectrum contained in the interval $[q,1]$. By the spectral representation theorem, there is a family $\{\mu_j^{\pm}; j \in I_+\}$ of finite positive Borel measures on $[q,1]$ and a unitary isomorphism of $\mathcal{H}_1$ on $\mathfrak{R}_+ := \bigoplus_j L^2([q,1],\mu_j^{\pm})$ such that $P'_1$ is unitarily equivalent to the operator $P_1$ on $\mathfrak{R}_+$ which acts componentwise as the multiplication by the variable $t$. Since 1 is not an eigenvalue of $P'_1$.
by construction, we have $\mu_{1,j}^+(\{1\}) = 0$ for all $j \in I_+$. For simplicity let us identify $\mathcal{H}_1$ with $\mathcal{K}_+$ and $P_1'$ with $P_1$.

Next we show that $\ker P' = \{0\}$. Let $f \in \ker P'$. Since $P'\mathcal{D}_1$ is essentially self-adjoint by (iii), there exists a sequence $\{f_n\}$ of vectors $f_n \in \mathcal{D}_1$ such that $f_n \to f$ and $P'f_n \to P'f = 0$ in $\mathcal{H}$. Since $X'P'f_n = i(q^{1/2}U'^* + q^{1/2}U')f_n$ by (ii) and the operators $U'$ and $U'^*$ are bounded, we obtain $(q^{-1/2}U'^* + q^{1/2}U')f = 0$ in the limit. This in turn yields that $q\|f\| = \|f\|$ and so $f = 0$.

By (ii), we have $U'P'f = qP'U'f$ for all $f \in \mathcal{D}_1$. Since $P'\mathcal{D}_1$ is essentially self-adjoint, this remains valid for $f \in \mathcal{D}(P')$, so that $P' \subseteq qU'^*P'U'$. Since $P'$ is self-adjoint, we conclude that $P' = qU'^*P'U'$. Hence we have $U^nE(\mathfrak{N}) = E(q^{-n}\mathfrak{N})$ for any Borel subset $\mathfrak{N}$ of $\mathbb{R}$ and arbitrary $n \in \mathbb{Z}$. Let $\mu^{j,+}$ be the extension of the measure $\mu_{1,j}^{+,+}$ to $\mathbb{R}_+$ as in II.1. From the preceding considerations it follows that $E(\mathbb{R}_+)\mathcal{H} = \oplus jL^2(\mathbb{R}_+, \mu^{j,+}) \equiv \mathcal{H}_+$ and that $U'$ acts in each component by formula (i) in subsection II.1. Proceeding in a similar manner, we obtain a family $\{\mu_{1,j}^{-,+}; j \in I_-\}$ of measures on $[q, 1]$ such that $\mu_{1,j}^{-,+}(\{1\}) = 0$ for $j \in I_-, E(\mathbb{R}_-)\mathcal{H} = \oplus jL^2(\mathbb{R}_-, \mu^{j,-}) \equiv \mathcal{H}_-$ in the notation of Section II and $U'$ acts componentwise as given by formula (i) in II.1. Since $E(\{0\})\mathcal{H} = \ker P' = \{0\}$ as proved in the preceding paragraph, we conclude that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

From the construction it is clear that $P'$ and $U'$ are the operators $P$ and $U$, respectively, as in Section II. Let us finally turn to the operator $X'$. Recall that we have $X'P'f = i(q^{-1/2}U'^* + q^{1/2}U')f$ for $f \in \mathcal{D}_1$. By arguing as the paragraph before last, this relation remains valid for all $f \in \mathcal{D}(P')$. If $f$ denotes a component of the vector $f$, then the preceding equation yields that $g := tf \in \mathcal{H}, t^{-1}g = f \in \mathcal{H}$ and $(X'g)(t) = i(q^{-1}f(q^{-1}t) - qf(qt)) = it^{-1}(g(q^{-1}t) - g(qt)) = (Xg)(t)$. Hence $X'f = Xf$ for all $f \in \mathcal{D}(P')$. Since $X'\mathcal{D}_1$ is essentially self-adjoint, the relation $U'X'f = q^{-1}X'U'f$ for $f \in \mathcal{D}_1$ by (ii) extends to vectors $f \in \mathcal{D}(X')$, so that $U'X'U'^* = q^{-1}X'$. Thus, $X'$ is a self-adjoint extension of the operator $X$ such that $U\mathcal{D}(X') = \mathcal{D}(X')$. By Proposition 3, $X'$ is of the form $X_{V,W}$.

V. *-Representations of the $q$-Heisenberg algebra

V.1 We have considered so far only operator triples and operator relations rather than
representations of the algebra $\mathcal{A}(q)$. But any operator triple $\{P, X_{V,W}, U\}$ gives rise to a self-adjoint representation of the $*$-algebra as follows. Indeed, let $\mathcal{D}_1$ be the domain defined at the beginning of section IV. For vectors in $\mathcal{D}_1$ the operators $P, X_{V,W}, U$ satisfy the defining relations (1) and (2) of the algebra $\mathcal{A}(q)$. Hence there exists a unique $*$-representation $\pi_1$ of the $*$-algebra $\mathcal{A}(q)$ on the domain $\mathcal{D}_1$ such that

$$\pi_1(p) = P_{|\mathcal{D}_1}, \pi_1(x) = X_{V,W}_{|\mathcal{D}_1}, \pi_1(u) = U_{|\mathcal{D}_1}.$$  

(For the notions on unbounded $*$-representations used in what follows we refer to the monograph$^9$. Recall that the symbol $T_{|\mathcal{D}_1}$ means the restriction of $T$ to $\mathcal{D}_1$.)

The $*$-representation $\pi_1$ is not yet self-adjoint (see ref. 9, Definition 8.1.10), because, roughly speaking, $\mathcal{D}_1$ is not the largest possible domain. However, since the operators $\pi_1(p)$ and $\pi_1(x)$ are essentially self-adjoint, it follows at once from Proposition 8.1.12 (v) in ref. 9 that the adjoint representation $\pi := (\pi_1)^*$ is self-adjoint. It is not difficult to verify that the domain $\mathcal{D}$ of the $*$-representation $\pi$ is just the intersection of domains of all possible products of the operators $P, X_{V,W}, U$ (see ref. 9, Proposition 8.1.17). From these facts it follows that the operator triple $\{P, X_{V,W}, U\}$ is irreducible if and only if the $*$-representation $\pi$ is so and that two triples are unitarily equivalent if and only if the corresponding $*$-representations are so. That is, Propositions 4 and 5 provide also the conditions for the irreducibility and the unitary equivalence of these $*$-representations of the $*$-algebra $\mathcal{A}(q)$.

V.2 Finally, we briefly discuss how operator representations of the $q$-deformed Heisenberg algebra $\mathcal{A}(q)$ can be constructed by means of the Schrödinger representation $P := -i\frac{d}{dt}$ and $Q := t$ of the ”ordinary” momentum and position operators.

Let us write $q = e^{-\alpha}$ with $\alpha \in \mathbb{R}$. We define three operators $U, P, X$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$:

$$U = e^{iQ}, P = e^{\alpha P}, X = i(q^{-1/2}e^{-iQ} - q^{1/2}e^{iQ})e^{-\alpha P}. \quad (18)$$

The vector space $\mathcal{D} := \text{Lin}\{e^{\gamma t - t^2}; \gamma \in \mathbb{C}\}$ is a dense linear subspace of $\mathcal{H}$. Since the operator $e^{\beta P}, \beta \in \mathbb{R}$, acts as $(e^{\beta P}f)(t) = f(t - \beta t)$ on functions $f \in \mathbb{C}$ (see, for instance,
ref. 10 for a rigorous proof), the operators $U, P, X$ satisfy the relations (1) and $(2)'$ and the restrictions of these operators to the invariant dense domain $\mathcal{D}$ define a $*$-representation of the $*$-algebra $\mathcal{A}(q)$. This operator representation (18) appears already somewhat hidden in ref. 2. Indeed, if we change the variable $t$ to $e^t$, then the operator triple \( \{U \oplus U, (-P) \oplus P, (-X) \oplus X\} \) on the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H}$ is easily seen to be unitarily equivalent to the triple in formula (2.2) in ref. 2.

The operator representation (18) is irreducible on $\mathcal{H}$. Obviously, $U$ is unitary and $P$ is self-adjoint. However, an essential disadvantage of the representation (18) is that the operator $X$ is only symmetric, but not essentially self-adjoint. The latter can be shown by the argument used in the proof of Proposition A.2 in ref. 10. The reason for this failure is the fact the holomorphic function $h(z) = q^{-1/2}e^{iz} - q^{1/2}e^{-iz}$ admits the zero $z_0 = i\frac{\alpha}{2}$ in the strip \( \{z \in \mathbb{C} : 0 < \text{Im} z < \alpha\} \).

Reference

[1] Dunford, N. and Schwartz, J.: Linear Operators, Part II. Interscience Publishers, New York, 1963

[2] Fichtmüller, M., Lorek, A. and Wess, J.: $q$-Deformed phase space and its lattice structure. Preprint MPI-PhI/95-109, Munich, 1995

[3] Hebecker, A., Schreckenberg, S., Schwenk, J., Weich, W. and Wess, J.: Representations of a $q$-deformed Heisenberg algebra. Z. Phys. C. 64 (1994), 335 - 359

[4] Koelink, H.T. and Swartouw, R.F.: On the zeros of the Hahn-Exton$q$-Bessel function and associated $q$-Lommel polynomials. J. Math. Anal. and Appl. 186 (1994), 690–710

[5] Koornwinder, T.H. and Swartouw, R.F.: On $q$-analogues of the Fourier and Hankel transforms. Trans. Amer. Math. Soc. 333 (1992), 445 - 461

[6] Lorek, A., Weich, W. and Wess, J.: Non-commutative Euclidean and Minkowski structures. Preprint, MPI-PhI/96-124, Munich, 1996

[7] Pusz, W. and Woronowicz, S.L.: Twisted second quantization. Reports Math. Phys. 27 (1989), 231 - 257
[8] Riesz, F. and Sz.-Nagy, B.: Vorlesungen über Funktionalanalysis. DVW, Berlin, 1956

[9] Schmüdgen, K.: Unbounded operator algebras and representation theory. Birkhäuser, Basel, 1990

[10] Schmüdgen, K.: Integrable operator representations of $\mathbb{R}^2_q, X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$. Commun. Math. Phys. 159 (1994), 217-237

[11] Schwenk, J. and Wess, J.: A $q$-deformed quantum mechanical toy model. Phys. Letters B 291 (1992), 273 - 277

[12] Topping, D.M.: Lectures on von Neumann algebras. Van Nostrand, New York, 1971

[13] Weich, W.: The Hilbert space representations for $SO_q(3)$-symmetric quantum mechanics. Preprint, Munich, 1994, hep-th /9404029

[14] Wess, J. and Zumino, B.: Covariant differential calculus on the quantum hyperplane. Nucl. Phys. B. Proc. Suppl. 18 B (1991), 302-312