A THREE PARAMETER INVARIANT OF ORIENTED LINKS

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ABSTRACT. This paper defines a new sequence of finite dimensional algebras as quotients of the group algebras of the braid groups. This sequence depends on three homogeneous parameters and has a one-parameter family of Markov traces, and so gives a three parameter invariant of oriented links.

INTRODUCTION

The Jones polynomial was originally constructed by defining a Markov trace on the Temperley-Lieb algebras, see [1]. Similarly the HOMFLY polynomial can be constructed by defining a one parameter family of Markov traces on the Hecke algebras, see [2], and the Kauffman polynomial can be constructed by defining a Markov trace on the Birman-Wenzl-Murakami algebras, see [3] and [4].

In this paper we define a link invariant by the same construction. First we define a tower of algebras, $A_n$, for $n > 0$. For each $n$, $A_n$ is a finite dimensional quotient of the group algebra of the braid group on $n$ strings. Then we show that there is a one-parameter family of Markov traces on this tower of algebras.

The image of each of the standard braid group generators in $A_n$ satisfies a fixed cubic relation and so has three eigenvalues. These three eigenvalues are essentially arbitrary and are homogeneous coordinates; so each $A_n$ is a two-dimensional family of algebras. Since there is a one-parameter family of Markov traces this means that the link invariant depends on three parameters. However the invariant is a rational function and not a Laurent polynomial.

This link invariant has two specialisations to the Jones polynomial. This link invariant also specialises to the link polynomial associated to the spin 1, or adjoint, representation of $SU(2)$. Since this representation is also the fundamental representation of $SO(3)$ this specialisation is also a specialisation of the Kauffman polynomial.

A TOWER OF ALGEBRAS

In this section we define the sequence of algebras, $A_n$, for $n > 0$. For each $n > 0$, $A_n$ is defined as a quotient of the group algebra of the braid group on $n$ strings, $B_n$.

Each of these algebras is a finite dimensional unital algebra over the field of homogeneous rational functions in the indeterminates $x$, $y$ and $z$ of degree zero which are invariant under the involution $x \leftrightarrow x^{-1}$, $y \leftrightarrow y^{-1}$, $z \leftrightarrow z^{-1}$. 
The algebra $A_1$ is one dimensional. The algebra $A_2$ has dimension three and is defined to be the quotient of the group algebra of $B_2$ by the relation

\[(\sigma_1 - x)(\sigma_1 - y)(\sigma_1 - z) = 0\]

The algebra $A_3$, as an abstract algebra, is the direct sum of five matrix algebras of ranks 1, 1, 2, 2 and 3. The dimension of $A_3$ is therefore 19. The homomorphism $B_3 \to A_3$ is defined by specifying five irreducible representations of $B_3$ of these dimensions.

The two one dimensional representations are $\sigma_i \mapsto x$ and $\sigma_i \mapsto z$, for $i = 1, 2$.

The two two dimensional representations are

\[
\begin{align*}
\sigma_1 \mapsto & \begin{pmatrix} x & a_1 \\ 0 & y \end{pmatrix} \\
\sigma_2 \mapsto & \begin{pmatrix} y & 0 \\ a_1' & x \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\sigma_1 \mapsto & \begin{pmatrix} z & a_2 \\ 0 & y \end{pmatrix} \\
\sigma_2 \mapsto & \begin{pmatrix} y & 0 \\ a_2' & z \end{pmatrix}
\end{align*}
\]

where the constants are chosen to satisfy $a_1a_1' = -xy$ and $a_2a_2' = -yz$, and the representations are, up to equivalence, independent of this choice.

The three dimensional representation is

\[
\begin{align*}
\sigma_1 \mapsto & \begin{pmatrix} x & b_1 & \frac{b_1b_2y}{y^2+zx} \\ 0 & y & b_2 \\ 0 & 0 & z \end{pmatrix} \\
\sigma_2 \mapsto & \begin{pmatrix} z & 0 & 0 \\ b_1 & y & 0 \\ \frac{b_1b_2y}{y^2+zx} & b_2' & z \end{pmatrix}
\end{align*}
\]

where the constants are chosen to satisfy $b_1b_1' = b_2b_2' = -y^2 - xz$ and the representation is, up to equivalence, independent of this choice.

This completes the definition of $A_3$. Next we find generators and defining relations.

The image of each of $\sigma_1$ and $\sigma_2$ in $A_3$ satisfies the cubic relation (1). Hence each of these can be written as a linear combination of three orthogonal idempotents which sum to the identity. Define $e_i$ and $g_i$, for $i = 1, 2$, by

\[
e_i = \frac{(\sigma_i - y)(\sigma_i - z)}{(x - y)(x - z)} \quad \text{and} \quad g_i = \frac{(\sigma_i - x)(\sigma_i - y)}{(z - x)(z - y)}
\]

Then, for $i = 1, 2$, $e_i$ and $g_i$ are orthogonal idempotents. Since

\[
\begin{align*}
\sigma_i &= y + (x - y)e_i + (z - y)g_i \\
\sigma_i^{-1} &= y^{-1} + (x^{-1} - y^{-1})e_i + (z^{-1} - y^{-1})g_i
\end{align*}
\]

the algebra $A_3$ is generated by $1, e_1, g_1, e_2, g_2$.

There are 13 words in these generators of length at most two, and 8 words of the form $abc$ where each of $a$ and $c$ is $e_1$ or $f_1$ and $b$ is $e_2$ or $f_2$. This gives a combined set of 21 words; since the dimension of $A_3$ is 19 these must satisfy at least two linear relations. These 21 words satisfy the two linear relations

\[
x(y^2 - z^2)g_1e_2e_1 = z(x^2 - y^2)g_1g_2e_1
\]
Note that it is sufficient to check these two relations in the representation (3) since both sides of both equations are zero in the other four representations.

Now we show that there are no other linear relations among these 21 elements. The words 1, $e_1$, $e_2$, $e_1e_2$ and $e_2e_1$ are linearly independent in the two representations in which $g_i = 0$, and similarly, the words 1, $g_1$, $g_2$, $g_1g_2$ and $g_2g_1$ are linearly independent in the two representations in which $e_i = 0$. Therefore, it is sufficient to check that the remaining 12 words in the representation (3) span the vector space of $3 \times 3$ matrices. This can be verified by a direct calculation.

It follows from this that each of the 8 words of the form $abc$ where each of $a$ and $c$ is $e_2$ or $f_2$ and $b$ is $e_1$ or $f_1$ can be written as a linear combination of the above 21 words. These relations, together with ones already given, are defining relations for the algebra $A_3$. These relations can be determined explicitly, but are not reproduced here as they are complicated and not particularly illuminating. In fact, defining relations are the braid relations and two further relations; one of these relations is given in [5] and the other can be obtained from this by the involution $e_i \leftrightarrow g_i$ and $x \leftrightarrow y$. However the important feature of these relations is that they show that

$$A_3 = A_2 + A_2e_2A_2 + A_2g_2A_2$$

This ends the discussion for $n = 3$. Next we discuss the general case.

For $n > 3$, the algebra $A_n$ is generated by elements 1, $e_1, e_2, \ldots, e_{n-1}$ and $g_1, g_2, \ldots, g_{n-1}$. For $1 \leq i \leq n - 1$, $e_i$ and $g_i$ are orthogonal idempotents; for $|i - j| > 1$, $e_ie_j = e_je_i$, $e_ig_j = g_je_i$ and $g_ig_j = g_jg_i$; and for $1 \leq i \leq n - 2$ there is a monomorphism $A_3 \to A_n$ with $e_1 \mapsto e_i, g_1 \mapsto g_i, e_2 \mapsto e_{i+1}$ and $g_2 \mapsto g_{i+1}$.

The first property of these algebras is that for $n > 1$,

$$A_{n+1} = A_n + A_ne_nA_n + A_ng_nA_n$$

This follows by a standard inductive argument on $n$. The result (7) is the basis for the induction and is used in the inductive step. In particular this shows that, for $n > 1$, $A_n$ is finite dimensional; more precisely, it follows that

$$\dim(A_n) \leq \prod_{k=1}^{n} (2^k - 1)$$

**Markov traces**

In this section we find the general Markov trace on the tower of algebras $A_n$. Let $\delta$ and $Z$ be new indeterminates. A Markov trace consists of a trace map $\tau_n$ on $A_n$, for each $n > 0$, such that, for all $a \in A_n$,

$$\tau_{n+1}(1) = \delta^{n+1}$$

$$\tau_{n+1}(a) = \delta\tau_n(a)$$

$$\tau_{n+1}(ae_{n+1}^{\pm 1}) = Z^{\pm 1}\tau_n(a)$$

First we find $\tau_3$. The matrix trace of each irreducible representation of $A_3$ is a trace on $A_3$, and any trace map can be written uniquely as a linear combination of these five traces. Using (7) and the relations $(\sigma_1\sigma_2)(\sigma_1\sigma_2)^{-1} = 1 = e$, and
\((σ_1σ_2)g_1(σ_1σ_2)^{-1} = g_2\), a trace is determined by its values on the elements 1, \(e_1\), \(g_1\), \(e_1g_2\), \(e_1e_2\) and \(g_1g_2\). Since there are six of these elements, these six values are not arbitrary but must satisfy one linear relation.

The values of the trace, \(τ_3\), on these six elements are uniquely determined by (10) and are

\[
\begin{align*}
τ_3(1) &= δ^3 \\
τ_3(e_1) &= δ \frac{x(Z - (y + z)δ + yzZ^{-1})}{(x - y)(x - z)} \\
τ_3(g_1) &= δ \frac{z(Z - (x + y)δ + xyZ^{-1})}{(z - y)(z - x)} \\
τ_3(e_1g_2) &= \left( \frac{x(Z - (y + z)δ + yzZ^{-1})}{(x - y)(x - z)} \right) \left( \frac{z(Z - (x + y)δ + xyZ^{-1})}{(z - y)(z - x)} \right) \\
τ_3(e_1e_2) &= \left( \frac{x(Z - (y + z)δ + yzZ^{-1})}{(x - y)(x - z)} \right)^2 \\
τ_3(g_1g_2) &= \left( \frac{z(Z - (x + y)δ + xyZ^{-1})}{(z - y)(z - x)} \right)^2
\end{align*}
\]

(11)

Since there is one linear relation that these six values must satisfy for \(τ_3\) to be a trace map we regard \(Z\) as an additional parameter and the linear relation as an equation for \(δ\). Any linear relation will give a cubic equation for \(δ\), but a direct calculation gives that the equation for \(δ\) is the following quadratic equation:

\[
\begin{align*}
(x^3y + x^2yz - x^2z^2 - xy^2z + xyz^2 + yz^3)Z^2δ^2 \\
-(Z^2 + xz)(2x^2y - x^2z - xy^2 + xyz - xz^2 - y^2z + 2yz^2)Zδ \\
+(x - y)(y - z)(x^2z^2 + xzZ^2 + Z^4) = 0
\end{align*}
\]

(12)

Note that each of the coefficients is homogeneous of degree six and is also invariant under \(x \leftrightarrow z\), \(y \leftrightarrow y\).

Define \(δ\) by (12). It follows from (8) that there is at most one sequence of linear functionals, \(τ_n\), that satisfy

\[
\begin{align*}
τ_{n+1}(a) &= δτ_n(a) \\
τ_{n+1}(ae_nb) &= \frac{x(Z - (y + z)δ + yzZ^{-1})}{(x - y)(x - z)}τ_n(ab) \\
τ_{n+1}(ag_nb) &= \frac{z(Z - (x + y)δ + xyZ^{-1})}{(z - y)(z - x)}τ_n(ab)
\end{align*}
\]

(13)

In order to have a link invariant it remains to show that each of these linear functionals is well-defined and that each of these linear functionals is a trace map.

The reason it is not clear that these linear functionals are well-defined is that the decomposition in (8) is not a direct sum decomposition due to the relations (6). In order to show these linear functionals are well-defined it is sufficient to show that, for \(a, b \in A_{n-1}\),

\[
\begin{align*}
x(y^2 - z^2)τ_{n+1}(ag_{n-1}e_ne_{n-1}b) &= z(x^2 - y^2)τ_{n+1}(ag_{n-1}g_{n}eb) \\
τ_{n+1}(ag_{n-1}e_ne_{n-1}b) &= τ_{n+1}(ag_{n-1}g_{n}eb)
\end{align*}
\]

(14)
These equations are obviously satisfied since both sides of both equations are zero by (13).

The proof that each of these linear functionals is a trace follows by the same argument as in [2]. The proof is by induction on \( n \) and uses the case \( n = 3 \) as the basis of the induction and in the inductive step.

To prove that \( \tau_{n+1} \) is a trace it is sufficient to prove that for all \( a \in A_{n+1} \), \( \tau_{n+1}(ae_n) = \tau_{n+1}(e_na) \) and \( \tau_{n+1}(ag_n) = \tau_{n+1}(g_na) \). Using (8), there are three cases to consider: \( a \in A_n \), \( a \in A_ne_nA_n \) and \( A_ng_nA_n \). In each of these cases use (8) again to write each \( A_n \) as the sum of \( A_{n-1} \), \( A_{n-1}e_{n-1}A_{n-1} \) and \( A_{n-1}g_{n-1}A_{n-1} \). This gives a large number of special cases to consider, but each of these cases gives the same equation as for the same special case with \( n = 3 \). Since \( \tau_3 \) is a trace it follows that each of these equations is satisfied and so \( \tau_{n+1} \) is a trace.

If \( \beta \) is a braid on \( n \) strings with writhe \( w(\beta) \) and with oriented link closure \( \hat{\beta} \), then define \( L(\hat{\beta}) \) by

\[
L(\hat{\beta}) = Z^{-w(\beta)}\tau_n(\beta)
\]

Then by Markov’s theorem, \( L(\hat{\beta}) \) is an invariant of the isotopy class of the oriented link \( \hat{\beta} \).

The invariant \( L(\hat{\beta}) \) is not a Laurent polynomial but is a rational function. Since \( \delta \) satisfies a quadratic relation, \( L(\hat{\beta}) \) can be written uniquely as \( L_0(\hat{\beta}) + \delta L_1(\hat{\beta}) \). Each of the functions \( L_0(\hat{\beta}) \) and \( L_1(\hat{\beta}) \) is a homogeneous rational function in \( x, y, z \) and \( Z \) of degree 0, and is a Laurent polynomial in \( Z \) with coefficients which are homogeneous rational functions in \( x, y \) and \( z \).

There are two epimorphisms from the tower of algebras \( A_n \) to the tower of Temperley-Lieb algebras one given by \( e_i = 0 \) and the other by \( g_i = 0 \). Hence there are two different specialisations of \( L \) to a Jones polynomial.

This link invariant also specialises to the link polynomial associated to the spin 1, or adjoint, representation of \( SU(2) \). The composition factors of the tensor product of three copies of the spin 1 representation are the representations with spin 0, 1, 2 and 3 with multiplicities 1, 3, 2 and 1 respectively. This gives representations of the braid group on three strings of these dimensions and it is sufficient to check that they are irreducible and isomorphic to ones defined in (2) and (3).

Finally, \( L \) specialises to the link invariant defined in [5]. In this specialisation, \( x, y \) and \( z \) satisfy \( y^2 = \omega xz \) where \( \omega \) is a primitive cube root of 1, and this link invariant has the curious feature that \( D = 0 \).

These properties show the similarities between \( L \) and other known link invariants. There are other two variables link polynomials known with some of these properties, for example, the HOMFLY polynomial, the Kauffman polynomial and the two-cabling of the HOMFLY polynomial. The link invariant \( L \) does not specialise to any of these three polynomials. An elementary, but complicated, way to show this is to show that the relation (12) is not satisfied, in each case.

References

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