Quasi-periodic vs. irreversible dynamics of an optically confined Bose-Einstein condensate

Pierre Villain, Patrik Öhberg and Maciej Lewenstein
Institut für Theoretische Physik, Universität Hannover
Appelstr.2, 30167 Hannover, Germany

We consider the evolution of a dilute Bose-Einstein condensate in an optical trap formed by a doughnut laser mode. By solving a one dimensional Gross-Pitaevskii equation and looking at the variance and the statistical entropy associated with the position of the system we can study the dynamical behavior of the system. It is shown that for small condensates nonlinear revivals of the macroscopic wave function are expected. For sufficiently large and dense condensates irreversible dynamics takes place for which revivals of regular dynamics appear as predicted in [9]. These results are confirmed by a two dimensional simulation in which the scales of energy associated with the two different directions mimic the experimental situation.

03.75.Fi, 42.50.Fx, 32.80.-t

I. INTRODUCTION

The experimental achievement of Bose-Einstein condensation (BEC) in cold alkali atomic samples [1–3] has opened up new directions in atom optics. The development of optical (laser) devices to manipulate coherently atomic beams has been particularly spectacular. In the recent years it has led to the advances in atom cooling, providing thus the first steps towards the realization of coherent matter waves. Atomic beam splitters, or atomic mirrors are now common tools in quantum optics laboratories [4,5]. The Bose-Einstein condensates find their natural place in this research field as a source of intense coherent matter waves. Recently the (coherent) bouncing of a BEC on an atomic mirror has been carefully studied [6]. Pursuing the analogy with pure optical phenomena, a four wave mixing experiment using BECs has been successfully realized [7]. However, contrary to a standard four wave mixing of electromagnetic waves, the creation of a fourth wave does not require any interaction with an external nonlinear medium, but results directly from the intrinsic inter atomic interactions. This fact marks a dramatic difference between the behavior of matter waves and light.

For future applications of those atomic coherent matter waves, it is of great importance to study in detail the role of the interactions in the dynamics of the condensate. Such studies are of crucial interest especially when using atomic waveguides and atomic resonators. The main element of such atom optics tools is a far off resonant hollow laser beam which can be loaded with a Bose-Einstein condensate from an atomic trap [8]. For a sufficiently large detuning, spontaneous emission may be neglected leading to a coherent manipulation of the motion of the atoms. If used alone, this doughnut laser beam may be viewed as a matter wave guide, see Fig. 1. If two other blue detuned lasers are added transversally, an atomic de Broglie cavity can be formed.

FIG. 1. A schematic view of the condensate in the optical trap.

In a previous paper [9] we studied the dynamics of a condensate inside a simple resonator (modelled by a one dimensional square box) and focused particularly on the influence of the inter atomic interactions on the behavior of the system. We showed that starting from a localized wave packet and by increasing the nonlinearities we pass from a quasi linear regime, where phenomena like nonlinear revivals and fractional revivals are expected [10], to a regime where stochastization of the movement takes place and where the nonlinear revivals of the wave packet are replaced by revivals of regular dynamics.

In this paper we are interested in finding the same kind of phenomena but using a more realistic trapping potential. In particular the rigid boundary conditions corresponding to the square box potential have to be modified, since this kind of discontinuities cannot be experimentally realized. Moreover, in order to probe the dynamics of the system, some experimentally accessible tools, more adequate than those used in [9], have to be introduced. It must be stressed that the (nonlinear) revivals investigated in this paper concern a mesoscopic object following a nonlinear evolution equation. So far the revivals of a wave packet have been studied [11,12], and observed only for microscopic and linear systems [13,14]. On the other hand, the irreversible behavior which may appear in the dynamics, concerns the evolution of an isolated system.
and is only due to the nonlinear character of the evolution; it is not caused by any damping originating from some interactions with an external reservoir.

The plan of the paper is the following: in section II, after describing briefly the model which we use, we introduce some experimentally accessible observables, namely the variance and the entropy associated with the position, which will be used to check the dynamical behavior of the system. We will show how to recover the results of Ref. [9] (obtained for a box potential) in the case of a more realistic potential. We discuss how the different regimes of the condensate behavior is reflected in the behavior of the variance and the entropy. In section IV, in order to support the validity of the one dimensional model we present some results obtained in a 2D simulation. Finally, we conclude in Sec. V.

II. MODEL AND NEW VARIABLES

We recall here briefly the mathematical model we use in order to describe the evolution of a Bose-Einstein condensate in an atomic cavity. We consider the zero temperature case and use the standard Hartree-Bogoliubov approach [18]. All the atoms are described by the same wave function $\Psi(\vec{r}, t)$, the evolution of which is given by the time dependent Gross-Pitaevskii equation

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\vec{r}, t)\right] \Psi(\vec{r}, t) + Nu_0 |\Psi(\vec{r}, t)|^2 \Psi(\vec{r}, t).$$

We denote by $u_0 = 4\pi\hbar^2 a_{sc}/m$, with $a_{sc}$ being the s-wave scattering length, and by $N$ the total number of atoms. The core of the atomic resonator we have in mind is a Laguerre-Gauss laser mode. The radial confinement due to this optical potential is very strong compared to the longitudinal one. Typically the internal radius of the "cylinder" is of the order of a few micrometers and the length $L$ of the "box" may vary from 75 to 100 micrometers. This leads to a radial confinement of the order of several hundreds Hz, while the longitudinal scale of frequency will be of the order of few Hz. Unless we use a very large number of atoms, the dynamics in the radial direction can be considered as frozen; the system behaves effectively as a 1D one, relevant dynamics occurs then in the longitudinal direction (we will come back to this assumption in Sec. IV). Therefore we use a wave function in a product form $\Psi(\vec{r}, t) = \psi(z, t)\chi_0(x, y, t)$, with $\chi_0$ denoting the ground state in the transverse direction. We simplify now the trapping potential by using a square box potential. The evolution equation for $\psi(z, t)$ is then

$$i\hbar \frac{\partial \psi(z, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(z, t)}{\partial z^2} + \frac{Nu_0}{S} |\psi(z, t)|^2 \psi(z, t),$$

with $\psi$ fulfilling rigid boundary conditions, and $S$ denoting the (effective) transversal section that we assume to be constant over the total length of the box. This is of course an additional approximation, since the doughnut laser beam is always focused in the region where the loading of the condensate takes place. Consequently, rigorously speaking we have $S = S(z)$ with $S(z)$ exhibiting a minimum in the center, and a maximum at the borders. Since the relative variance of $S(z)$ is rather small [19], we may neglect it in the first approximation.

Depending on the value of the nonlinear parameter, the dynamics described by Eq. (2) can attain very different character [18]. The relevant nonlinear parameter is given by the ratio $\frac{\Omega_{\text{int}}}{\omega_1}$, where $\Omega_{\text{int}} = Nu_0/\hbar$ is the characteristic frequency related to the interaction term and $\omega_1 = \frac{\pi^2 \hbar}{2mL^2}$ is the frequency of the fundamental mode. We investigate the dynamical behavior for the values $\frac{\Omega_{\text{int}}}{\omega_1} = 3.08$ and $\frac{\Omega_{\text{int}}}{\omega_1} = 61$ which correspond to a typical number of atoms from several hundreds to few thousands in the first case, and from few thousands to few tens of thousands in the second one. Independently of the chosen initial state, we observe that for $\frac{\Omega_{\text{int}}}{\omega_1} = 3.08$ the dynamics remains regular and quasi-periodic (nonlinear revivals of the wave function occur), whereas for $\frac{\Omega_{\text{int}}}{\omega_1} = 61$ the motion is stochastic. In the latter case the wave function does not exhibit any strong spatial relocalization, but the previously mentioned nonlinear revivals are replaced now by revivals of regular dynamics characterized by a return of most of the energy in the few initially populated modes. These results were obtained in Ref. [18] by checking the appearance of energy transfer between the eigenmodes of the system and by looking at the temporal spectrum of the density.

Experimentally the access to the quantities discussed in Ref. [18] is difficult, if not hardly possible. The simplest quantity to measure experimentally is the local density $N|\psi(z, t)|^2$. In order to check the appearance of revivals of the wave function or revivals of regular dynamics the most natural time dependent quantity to study is the variance $\sigma_z^2$ of the mean position of the condensate cloud, defined as $\sigma_z^2 = \langle z^2 \rangle - \langle z \rangle^2$ where $\langle ... \rangle_\psi = \int dz |\psi(z, t)|^2$. If this function exhibits a return to its initial value we will be able to conclude about the appearance of nonlinear revivals of the wave function, or equivalently to some spatial relocalization to its initial position. Although this quantity is adequate in order to probe the revivals, it is insufficient to conclude without ambiguity that a complete delocalization of the wave function has occurred. Such delocalization may correspond either to a relaxation into another regular state with a density peaked only at the cavity ends, or to irreversible dynamics leading to a more or less uniform density distribution. We need therefore another dynamical quantity which would be more sensitive to various kinds of spatial delocalization and relocalization of the wave packet. The entropy $S_z$ associated with the position of the system fulfills such a requirement [20]. This
Statistical entropy is defined as

$$S_z(t) = -\sum_n P_n(t) \log P_n(t),$$

with $P_n$ being the probability of presence in the interval $[z_n, z_n + dz]$ contained in the interval $[0, L]$,

$$P_n(t) = |\psi(z_n, t)|^2 dz.$$  \(3\)

This function reaches its maximum value $S_z^{\text{max}}$ for a uniform probability distribution corresponding to the case of a spatially uniform system. On the other hand any spatial localization of the wave function will induce a strong decrease of $S_z$ compared to the maximum value. As usual it is better not to use $S_z$ but a normalized entropy $\eta_z$ defined as

$$\eta_z(t) = \frac{S_z^{\text{max}} - S_z(t)}{S_z^{\text{max}} - S_z(0)}.$$  \(4\)

Experimentally, both $\sigma_z$ and $S_z$ correspond to measurable quantities. The only restriction is coming from the (inevitably) limited spatial resolution in a given experiment. But this limitation affects only the dynamics associated with high frequency modes and the interesting phenomena (revivals of the wave packet or revivals of regular dynamics) are in fact associated to the low modes of the system. A typical spatial resolution can be of the order $3\mu m$ which, in a box of $100\mu m$ limits the resolution of the modes higher than the 30th.

We simulate the evolution of a wave packet initially centered in the box with no initial velocity and with an initial width of $L/5$. This corresponds to the first case investigated in \[9\] where mainly the modes 1 and 3 of the box potential are populated. We use the usual split operator method to solve Eq.(2) \[21\]. We check the conservation of the norm of the wave function and the conservation of the total energy. The former was conserved better than $10^{-4}\%$, and the latter has shown fluctuations smaller than $1\%$. In the figures 2a and 2b we present respectively the time evolution of the variance compared to its value for a uniform system $\sigma_{\text{uni}}$ and the time evolution of $\eta_z$. These plots correspond to a nonlinear coefficient $\frac{\Omega_{\text{uni}}}{\omega_1} = 3.08$. The time is given in units of the fundamental period of the box potential $T_1 = 2\pi/\omega_1$. We restrict ourself to a time interval of one period roughly since for the experimental system it would correspond to a time of the order of $18s$ which is beyond, or in the best case close to the usual lifetime of the condensate in this situation.

![Graph](image1)

**FIG. 2.** Evolution of (a) $\sigma_z/\sigma_{\text{uni}}$, and (b) $\eta_z$ in the case $\frac{\Omega_{\text{uni}}}{\omega_1} = 3.08$ and for an initially centered wave packet.

The evolution of $\sigma_z$ shows clearly some quasi-periodic revivals at times close to those expected in the linear case with this initial state. The nonlinear character of the evolution appears through the position of this revival. The behavior of $\eta_z$ confirms the regular spatial relocalization of the wave function. This function exhibits clear returns to its maximal value 1. Notice that the fractional revivals corresponding to a localization of the atoms near the boundaries are not at all exhibited by $\sigma_z$, but appear very clearly in the time dependence of $\eta_z$, as expected. They are represented by the small peaks preceding a quasi-revival of the system. The occurrence of quasi periodic nonlinear revivals of a mesoscopic wave packet is thus confirmed. From the experimental point of view the bad point is of course the time scale of these phenomena. The first nonlinear revival is shown at a time close to $0.1T_1$ ($\simeq 1.8s$), which is already rather long. We will see in section III that in a more realistic potential than a box like potential the revival time is significantly reduced.

In the figures 3a and 3b we present the time evolution of the same quantities but for a nonlinear parameter $\frac{\Omega_{\text{uni}}}{\omega_1} = 61$. 

3
The variance does not exhibit any clear return to its initial value. On the contrary it oscillates around the value \( \sigma_{uni} \). The absence of strong spatial relocalizations of the wave function is confirmed by the behavior of \( \eta_z \) which stays far from 1 and oscillates around a low value. Notice that this value is not zero, which means that the system has effectively relaxed but is still following some dynamics. The change from the nonlinear revivals of the wave packet with a zero mean velocity for the two same values of the nonlinear parameter. The width of the wave packet is still \( L/5 \) and we choose a \( \sigma_{laser} = L/12 \) which corresponds in a typical experiment to a focused waist of roughly 16\( \mu m \) and the height of the barrier is chosen to be of the order a few mK. We show the time variations of the same quantities than before in order to compare the results with the case of the perfect box. The variance \( \sigma_z/\sigma_{uni} \) is still expressed in units of the uniform case which is calculated using the box potential. We keep also the same unit of time which will allow us to compare the time scale obtained now for the different phenomena we want to investigate. In Figs. 4a and 4b we show the variations of the variance and the normalized entropy \( \eta_z \) for \( \frac{\omega_0}{\omega_1} = 3.08 \).

III. STUDIES OF THE DYNAMICS IN A REALISTIC TRAPPING POTENTIAL

Experimentally the optical box will be closed transversally by using two blue detuned lasers. The trapping potential at the boundaries of the box is no more discontinuous and may be modelled in a good approximation as the sum of two identical Gaussians centered at \( z = 0 \) and \( z = L \). The characteristic parameters of the potential are then its height \( V_{laser} \) and the width of the Gaussian \( \sigma_{laser} \). The requirement on \( V_{laser} \) is that it provides a sufficiently high barrier in order to prevent the atoms from leaving the trap. We take it as a given parameter of the problem. The value of \( \sigma_{laser} \) directly depends on the transversal focusing of the lasers at \( z = 0 \) and \( z = L \). The typical value cannot be smaller than a few \( \mu m \). Clearly a too small \( \sigma_{laser} \) will take us back to the case of the perfect box. It is beneficial to use a large \( \sigma_{laser} \) in order to reduce the time scale on which the nonlinear revivals of the wave packet or of the regular dynamics appear. On the other hand, too large \( \sigma_{laser} \) does not interest us because it will correspond to a situation, where the atoms are still very confined and the cloud will just oscillate as in a slightly modified harmonic trap. In other words, the overlapping of the boundary barriers with the initial state must remain as small as possible in order to remain in a situation similar to a box potential from the dynamical point of view.

The study of the dynamics of the system appears to be simpler if one uses the quantities \( \eta_z \) and \( \sigma_z \), since it is difficult to have access to a large range of different modes of the potential (which would be necessary in order to compute for instance the statistical entropy associated with the energy distribution among the eigenmodes as in Ref. [9]). We present here the results of the simulations for the same case as before, namely an initially centered wave packet with a zero mean velocity for the two same values of the nonlinear parameter. The width of the wave packet is still \( L/5 \) and we choose a \( \sigma_{laser} = L/12 \) which corresponds in a typical experiment to a focused waist of roughly 16\( \mu m \) and the height of the barrier is chosen to be of the order a few mK. We show the time variations of the same quantities than before in order to compare the results with the case of the perfect box. The variance \( \sigma_z/\sigma_{uni} \) is still expressed in units of the uniform case which is calculated using the box potential. We keep also the same unit of time which will allow us to compare the time scale obtained now for the different phenomena we want to investigate. In Figs. 4a and 4b we show the variations of the variance and the normalized entropy \( \eta_z \) for \( \frac{\omega_0}{\omega_1} = 3.08 \).
The regular and quasi-periodic behavior of the two quantities is striking. The wave packet clearly exhibits nonlinear revivals at almost regular times. Although the maximal value of $\sigma_z$ is as expected smaller than for the perfect box, the most important point is that the time scale for the nonlinear revivals has been divided by a factor 2. This makes the first quasi-revival of the condensate to appear before 1s. Although it remains long, it is now sufficiently shorter than the lifetime of the condensate. This time scale can be decreased even more by playing with the value of $\sigma_{\text{laser}}$ as well as with the initial state of the system. In particular a spatially more squeezed initial wave packet will populate higher excited states at the beginning of the evolution leading to a smaller linear revival time, and, consequently, to a smaller nonlinear revival time (with these values of the nonlinearities the two regimes are not too much different and the time scales are comparable). It is important to note that, as before, some fractional revivals corresponding to an accumulation of atoms at the boundaries are still visible in the variations of $\eta_z$.

We turn now to the case of strong nonlinearities. The simulations of the evolution $\sigma_z$ and $\eta_z$ for a nonlinear parameter $\Omega_{\text{int}}/\omega_1 = 61$ are presented in Figs. 5a and 5b keeping the same conventions as before.

The behavior is similar to those of Fig. 3. The variance does not come back to its initial value and $\eta_z$ stays far from 1 which proves that the wave function does not exhibit any spatial relocalization. The variations of $\sigma_z$ still exhibit the revivals of regular dynamics. Although the time scale on which they appear has been reduced it is still of the order of 0.1$T_1$ which makes them difficult to observe experimentally.

IV. ANALYSIS OF THE TWO DIMENSIONAL PROBLEM.

A. Analysis of the model.

In the previous sections we have studies the simplified 1D dynamics arguing that the radial confinement due to the doughnut laser mode was much stronger than the longitudinal one. In consequence the structure of the condensate in the radial direction will not be modified by the nonlinearities. Although this hypothesis is reasonable, it has to be checked at the level of the physical quantities we are using to describe the behavior of the system. The ratio of the typical energies associated with the radial degrees of freedom and the longitudinal ones is of the order of $L^2/R^2 \simeq 100$ where $R$ denotes the internal mean radius of the Laguerre-Gauss laser mode and $L$ the length in the longitudinal direction. Qualitatively, for small nonlinearities, when the dynamics is already quasi-periodic in the longitudinal direction, there are no serious
worries concerning the influence of the other degrees of freedom. On the other hand, for stronger nonlinearities corresponding to \( \frac{L_2}{\omega_1} \) of the same order of magnitude as \( L^2 / R^2 \), a strong coupling between the different degrees of freedom associated with different directions may appear, and it is not sure that we can get the same behavior as before for the variance and the entropy associated with the longitudinal direction. However, we must keep in mind that this simple energetic estimate does not take into account the population of the modes which are important parameters in nonlinear problems. The derivation of more accurate criteria is then necessary. We will propose some of them in the next paragraph.

We turn now to the discussion of the inclusion of an additional dimension. For this purpose we consider the evolution of a condensate in a two-dimensional box. We denote the directions by \( z \) and \( y \) and by \( L_z \) and \( L_y \) the associated lengths. We take \( L_z = 4L_y \) where the energy scale in the \( y \) direction is 16 times bigger than the one of the \( z \) direction and will mimic our radial confinement in the real system. Of course as we are working in Cartesian coordinates the problem is simplified compared to the cylindrical one but it should give us a good representation of the real system. Moreover the typical ratio of the energies is one order of magnitude smaller than the one of our system but it will still provide the same kind of behavior and allows us to test values of nonlinearities sufficiently small to have a good spatial resolution in our numerical grid.

The starting point is still the time dependent Gross-Pitaevskii equation \((4)\). Assuming a very strong confinement in the \( x \) direction, we can neglect the degrees of freedom associated with the \( x \) direction by assuming a wave function of the form \( \Psi(\vec{r}, t) = \psi(z, y, t)\xi_0(x, t) \) where \( \xi_0 \) is the fundamental state of the \( x \) direction. The evolution equation of \( \psi \) is then given by

\[
\begin{align*}
\frac{i\hbar}{\partial t} \psi(z, y, t) &= -\frac{\hbar^2}{2m}\nabla^2 \psi(z, y, t) \\
&+ \frac{Nu_0}{L_x}|\psi(z, y, t)|^2 \psi(z, y, t),
\end{align*}
\]

where \( L_x \) denotes the length in the \( x \) direction.

As in the one dimensional case we can derive a set of coupled differential equations for the coefficients \( c_{nm} \). They can be derived as the Hamiltonian equations of the motion for the complex degree of freedom \( \{c_{nm}, \frac{i\hbar c_{nm}^*}\} \). It is however more interesting to work in action-angle variables \( \{I_{nm}, \theta_{nm}\} \) defined such that

\[ c_{nm} = \sqrt{\frac{I_{nm}}{\hbar}} e^{i\theta_{nm}}. \]

The Hamiltonian leading to the differential equations of motion for the \( c_{nm} \) can be written in action-angle representation as

\[
H = \sum_{nm} \omega_{nm} I_{nm}
+ \frac{\tilde{g}}{2} \sum_{nm} V_{\tilde{n}\tilde{m}} (I_{n1m1} I_{n2m2} I_{n3m3} I_{n4m4})^{1/2}
\]

\[
e^{-i(\theta_{n1m1} + \theta_{n2m2} - \theta_{n3m3} - \theta_{n4m4})},
\]

where \( \tilde{g} = \frac{Nu_0}{L_x\hbar^2} \) and where

\[
V_{\tilde{n}\tilde{m}} = \int_0^{L_z} dz \phi_{n1}\phi_{n2}\phi_{n3}\phi_{n4} \int_0^{L_y} dy \chi_{m1}\chi_{m2}\chi_{m3}\chi_{m4}.
\]

Here \( \tilde{n} \) and \( \tilde{m} \) correspond to \( (n_1, n_2, n_3, n_4) \) and \( (m_1, m_2, m_3, m_4) \). We have also introduced the frequencies \( \omega_{nm} \) corresponding to the mode labelled \( (n, m) \).

We have of course \( \hbar \omega_{nm} = \frac{\pi^2 \hbar^2}{2mL_x^2} + \frac{\pi^2 \hbar^2}{2mL_y^2} = (n^2 + 16m^2)\hbar\omega_{1z} \) where we introduced the fundamental frequency \( \omega_{1z} \) in the \( z \) direction.

Compared to the equations derived in \((4)\) we have just a doubling of the indices due to the inclusion of a second dimension. The derivation of the criteria which indicate the borderline between two different dynamical behaviors is then straightforward. The first criterium indicates for what conditions one may expect that the dynamics of the systems will remain close to the linear one. In order to derive this criterium we have to compare the frequencies \( \omega_{nm} \) of the linear problem with their first order correction given by the nonlinearities \((5)\). In this case we have

\[
I_{nm} \ll \frac{8}{9} \frac{\omega_{1z}}{\Omega_{int}} \hbar (n^2 + 16m^2)
\]

where the natural frequency of the nonlinearities is denoted by \( \Omega_{int} = \frac{\tilde{g}}{\hbar} \). As the \( I_{nm} \)'s are normalized to \( \hbar \) the high modes of the box will satisfy this inequality more easily than the low ones for a given value of the nonlinear parameter \( \frac{\tilde{g}}{\Omega_{int}} \).

The Chirikov criterium \((2)\) on the other hand, gives us an indication about the conditions for which the dynamics may stay regular. This condition follows from the resonance overlap criterion which expresses the condition that separatrices associated with two consecutive resonances are touching each other. We compare the difference between two consecutive bare frequencies with the frequency associated with the first order correction. In our case it leads to

\[
I_{nm} \leq \frac{16}{9} \frac{\omega_{1z}}{\Omega_{int}} \hbar (n + 16m + \frac{17}{2}).
\]

As the previous inequality, this one is also more easily fulfilled for the highly excited modes, which consequently may preserve a regular motion easier than the low ones for an increasing value of the nonlinearities. Clearly for a given nonlinear parameter which is at best of the order of
the typical ratio of the two energy scale (say less than 30), only the modes with very low \( m \) may not satisfy these inequalities. As the initial state we choose is a centered wave packet of widths \( L_z/10 \) in the \( z \) direction and \( L_y/10 \) in the \( y \) direction without any mean velocity, we initially populate the modes \((n, m)\) with \( n, m = 1 \) or 3 essentially. This choice is coherent with what has been done in the one dimensional problem and is one of the most favorable in order to see the appearance of characteristic nonlinear dynamics.

We consider three cases for which the ratio \( \Omega \Omega \frac{\omega_1}{\omega_1} \) takes the values: (a) 4, (b) 10, (c) 101. Experimentally this would correspond to condensates whose number of atoms are: (a) few hundreds of atoms, (b) from few hundreds to 2000, (c) from few thousands to 15000. As for the one dimensional case, we want to compute the variance and the statistical entropy associated with a given direction. In the following paragraph we give the definition of \( \sigma_z \) and \( S_z \). Exchanging \( z \leftrightarrow y \) in the following expression gives the expressions of \( \sigma_y \) and \( S_y \). In the case of an initially centered wave packet, the preservation of the symmetry of the wave function simplifies a little bit the expressions. We obtain

\[
\sigma_z^2(t) = \int_0^{L_z} dz \ z^2 \Pi(z, t)
\]

where \( \Pi(z, t) \) is the density of probability of presence at \( z \) at the time \( t \), \( \Pi(z, t) = \int_0^{L_y} dy \ |\psi(z, y, t)|^2 \). For the entropy we use the probability distribution \( P(z_n, t) = \Pi(z_n, t)dz \), giving \( S_z(t) = -\sum_k P(z_k, t) \log P(z_k, t) \), where the \( dz \) is just the numerical step used to define the grid. As before it is more convenient to work with a normalized entropy \( \eta_z \) whose definition is the same as \( \eta \) and with a variance compared to its value in the uniform case.

**B. Numerical results**

We have numerically solved the two dimensional time dependent Gross-Pitaevskii equation \( \Box \) using an explicit method \( \Box \). We checked the conservation of the norm of the wave function (better than \( 10^{-4} \)) and of the energy (better than 1\%). In the Figs. 6 and 7 we present the time variations of the variance and the normalized entropy associated with the different directions for the case (a). The time is given in unit of \( T_1 = 2\pi/\omega_{1z} \). This is the reason why the dynamics in the \( y \) direction is presented on a time scale ten times shorter than for the \( z \) direction. As in the one dimensional case, it is however still long from an experimental point of view.

**FIG. 6.** (a) Temporal variations of \( \sigma_z/\sigma_{uniz} \); (b) Temporal variations of \( \sigma_y/\sigma_{uniz} \) for an initial centered state without mean velocity for \( \Omega \Omega \frac{\omega_1}{\omega_1} = 4 \)

**FIG. 7.** Time variations of the normalized entropy associated with the position for an initially centered wave packet. (a) \( \eta_z \), (b) \( \eta_y \) for \( \Omega \Omega \frac{\omega_1}{\omega_1} = 4 \)
The dynamics is regular as we may expect it for this small value of nonlinearities. For times shorter than 0.05\(T_1\), the behavior associated with the \(y\) direction is however closer to a linear one than in the \(z\) direction. This appears through the returns of the normalized entropy to the value 1 as it will be the case for a linear behavior. This means that the \(m\) mode distribution did not change during the evolution on the time scale presented here. For longer times a decrease in the precision of the revivals appears, but it remains rather small. Indeed, it is only visible on the scale of the normalized entropy although the variance did not exhibit a noticeable change. In the \(z\) direction quasi-periodic revivals of the wave packet in this direction are shown. They correspond exactly to the ones seen in Fig. 2. We can then conclude that for the smallest value of the nonlinearities considered in the one dimensional case, the effects associated with the radial degree of freedom will not prevent the appearance of nonlinear revivals of the wave function, and in fact do not play a relevant role.

We turn now to the case of intermediate nonlinearities corresponding to the case (b). In Figs. 8 and 9 we present the variations of the variances and the normalized entropies.

![FIG. 8](image.png)

**FIG. 8.** (a) Temporal variations of \(\sigma_{\text{z}}/\sigma_{\text{uni}z}\); (b) Temporal variations of \(\sigma_{\text{y}}/\sigma_{\text{uni}y}\) for an initial centered state without mean velocity for \(\Omega_{\text{int}}/\omega_{1z} = 10\).

The dynamics associated with the two directions are completely different. In the \(z\) direction the quasi-periodic spatial relocalizations of the wave function have vanished. As in the one dimensional case, they have been replaced by revivals of regular dynamics as can be seen in the inset for the variance \(\sigma_{\text{z}}\). A stochastic dynamics has taken place for this direction. On the contrary the dynamics in the \(y\) direction has remained regular and quasi-periodic, although compared to the previous case the nonlinearities begin to influence it stronger. Consequently when the nonlinear parameter \(\Omega_{\text{int}}/\omega_{1z}\) is comparable to the ratio between the energies in \(z\) and \(y\) direction, revivals of regular dynamics in the direction of the small energy scale are expected. The predictions done in the approximated one dimensional case are then valid for the typical values of the nonlinearities we have considered (we point out that in this case we have investigated the behavior for \(\Omega_{\text{int}}/\omega_{1z} = 61\) where the ratio of the scale of energies is of the order of 100).

If we increase more strongly the nonlinearities the coupling between the modes can no more be neglected for the \(y\) direction. This is shown in the Figs. 10 and 11, where we present the variance and the normalized entropy for the case (c).
The very fast decrease of the two normalized entropies indicates a strong relaxation of the system towards a nearly uniform state. The variance in the $z$ direction reaches almost a steady state. It oscillates around its value corresponding to the uniform case and the revivals of regular dynamics of the previous case are still visible, but vanish very quickly. In the $y$ direction the situation is similar. The previous revivals of the wave packet in this direction have been replaced by revivals of regular dynamics, whose amplitudes decrease quickly with time. The dynamics have developed stochasticity as may be expected from the application of the Chirikov criterium \cite{8} for this value of the nonlinear parameter.

V. CONCLUSIONS

In this paper we have studied the nonlinear dynamics of a condensate trapped in an optical box like potential. The scheme we have in mind is a condensate located in the center of a box-like potential where we monitor the time evolution of the wave packet by looking at the associated entropy and variance in order to detect the main properties of the dynamics. The behavior of these functions allows to confirm the predictions done previously in \cite{9}, namely that for small nonlinearities (corresponding to situations with few thousands of atoms at most) the system will exhibit nonlinear revivals through quasi-periodic spatial relocalizations. On the other hand, for stronger nonlinearities (corresponding typically to a number of atoms of few tens of thousands ) the system will follow an effective relaxation dynamics, for which the wave function revivals are replaced by revivals of regular dynamics. The observed revival times in a perfect box potential are shown to be rather too long to be experimentally measurable. Fortunately using a more realistic trapping potential with a pair of blue detuned lasers as “end caps” for the doughnut mode laser configuration, there is a substantial reduction of the revival times, which can then be shorter than the typical condensate life time. It is important to note that the present study demonstrates the possibility of observation of the revivals of a mesoscopic object, which is interesting from an experimental point of view. It shows, however, also irreversibility in a closed system which is interesting from a more fundamental point of view.

In the numerical treatment we have first assumed all the dynamics was taking place in one dimension. In other words the transversal dynamics of the condensate was frozen. This assumption, based on an effective strong radial confinement, is valid as long as the (nonlinear) coupling is not too strong. In order to justify the one dimensional treatment and to study the break-down of it, we have performed a two dimensional simulation using a perfect box potential. By choosing a trap aspect ratio significantly lesser than one, we have demonstrated that the coupling between the two dimensions is present only
at large values of the parameter $\Omega_{nt}/\omega_{1z}$ describing the nonlinear interaction. Hence, the one dimensional study provides indeed a good approach for the physical parameters used here. The coupling between the two dimensions is mainly governed by the different energy scales of the eigenmodes in $z$ and $y$ direction due to the aspect ratio.

Finally, we would like to point out that very recent experimental progress of the Hannover group allows to expect that the observation of the condensate wave packet and transition to irregularity will be feasible within the times of the order of 1s \[19\].

**ACKNOWLEDGMENTS**

We are grateful to A. Sanpera, K. Sengstock, G. Birkl and W. Ertmer for helpful discussions. This work has been supported by Deutsche Forschungsgesellschaft (SFB407), the TMR network ERBXCTCT96-0002 and the ESF PESC Program BEC2000.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995).
[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).
[4] T. Esslinger, M. Weidemüller, A. Hemmerich, and T. Hänsch, Opt. Lett. 18, 450 (1993).
[5] W. Seifert, R. Kaiser, A. Aspect, and J. Mlynek, Opt. Commun. 111, 566 (1994)
[6] K. Bongs, S. Burger, G. Birkl, K. Sengstock, W. Ertmer, K. Rzążewski, A. Sanpera and M. Lewenstein, Phys. Rev. Lett. 83, 3577 (1999).
[7] L. Deng, E. W. Wen, M. Trippenbach, Y. Band, P. S. Julienne, J. E. Simsarian, K. Helmerson, S. L. Rolston and W. D. Phillips Nature 398, 218 (1999).
[8] S. Burger, K. Bongs, K. Sengstock, and W. Ertmer, Proc. Int. School of Quant. Electr., 27th Course, Erice, Sicily (1999).
[9] P. Villain and M. Lewenstein (in print in Phys. Rev. A).
[10] J. Ruostekoski, B. Kneer and W. P. Schleich, cond-mat/9908095.
[11] I. Sh. Averbukh and N. F. Perel’man, Phys. Lett. A 139, 449 (1989).
[12] J. H. Eberly, N. B. Narozhny and J. J. Sanchez-Mondragon, Phys. Rev. Lett. 44, 1323 (1980).
[13] J. A. Yeazell and C. R. Stroud Jr., Phys. Rev. A43, 5153 (1991).
[14] D. R. Meacher, P. E. Meyler, I. G. Hughes and P. Ewart, J. Phys. B 24, L63 (1991).
[15] M. J. J. Vrakking, D. M. Villeneuve and A. Stolow, Phys. Rev. A 54, R37 (1995).
[16] D. Meekhof, C. Monroe, B. E. King, W. M. Itano and D. J. Wineland, Phys. Rev. Lett. 76, 1796 (1996).
[17] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, J.-M. Raymond and S. Haroche, Phys.Rev. Lett. 76, 1800 (1996).
[18] P. Nozières and D. Pines, The Theory of Quantum Liquids, Volume 2, (Addison-Wesley, Massachusets, 1990).
[19] G. Birkl and K. Sengstock (private communication).
[20] I. Bialynicki-Birula, Phys. Lett. 103A, 253 (1984).
[21] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes (Cambridge University Press, 1992).
[22] B. V. Chirikov, Phys. Rep. 52, 263 (1983).
[23] M. Kira, I. Tittonen and S. Stenholm, Phys. Rev. B 52, 10972 (1995).