Off-shell Ward identities for N-gluon amplitudes

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Abstract – Off-shell Ward identities in non-abelian gauge theory continue to be a subject of active research, since they are, in general, inhomogeneous and their form depends on the chosen gauge-fixing procedure. For the three-gluon and four-gluon vertices, it is known that a relatively simple form of the Ward identity can be achieved using the pinch technique or, equivalently, the background-field method with quantum Feynman gauge. The latter is also the gauge-fixing underlying the string-inspired formalism, and here we use this formalism to prove the corresponding form of the Ward identity for the one-loop $N$-gluon amplitudes.

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Introduction: Ward-Takahashi and Slavnov-Taylor identities. – Ward identities, also known as Ward-Takahashi identities, are the quantum counterparts to Noether’s theorem in classical physics. They are identities between correlation functions stemming from the global and gauge symmetries of the theory, introduced first by Ward \cite{Ward1950} in 1950 and later generalized by Takahashi \cite{Takahashi1950}.

The original work of Ward and Takahashi was concerned with $U(1)$ gauge symmetry and current conservation in QED. Here the “Ward identity” refers to on-shell matrix elements, and is usually written as

$$ k^\mu \mathcal{M}_\mu = 0, \quad (1) $$

where $\mathcal{M}_\mu$ is the matrix element defined by $\mathcal{M} = \varepsilon_\mu \mathcal{M}_{\mu}$, indicating that the longitudinal components of the photon’s polarizations do not contribute to scattering amplitudes. See, e.g., \cite{Itzykson1980} for detailed discussion.

The “Ward-Takahashi” identity is more involved, since it concerns off-shell quantities. In QED, in its most basic form it can be written as

$$ \Gamma_\mu(p,p) = -\frac{\partial}{\partial p^\mu} \Sigma(p), \quad (2) $$

and allows one to relate the electron wave function renormalization factor $Z_2$ to the vertex renormalization factor $Z_1$.

After the development of QCD, in the Seventies the generalization of these QED identities to the non-Abelian case became an active field of research. With respect to the on-shell $S$-matrix identities (1) one finds no essential differences between QED and QCD, except that in the non-Abelian case the vanishing of the effect of the longitudinal gluon polarizations usually involves intricate cancellations between one-particle irreducible and one-particle reducible diagrams (see, e.g., \cite{Collins1984}).

To the contrary, the generalization of the Ward-Takahashi identities to the non-Abelian case leads to the so-called Slavnov-Taylor identities \cite{Slavnov1973, Taylor1974}, and those still remain a subject of active investigation (see, e.g., \cite{Chetyrkin1997} and references therein). This is because these identities in general not only provide relations between different $N$-point functions, but also mix up the physical gauge bosons with the ghosts, and in a way that depends on the gauge-fixing procedure. Thus they tend to be much more non-trivial than their QED prototype. Moreover, they should hold perturbatively and non-perturbatively, and in the bare theory as well as in the renormalized one. Therefore Slavnov-Taylor identities also put restrictions on the renormalized coupling constants for vertex functions which has been studied in detail by many authors, see, e.g., \cite{Kataev1985, Kuraev1985, Kuraev1990}.

The gauge-fixing dependence constitutes a serious problem for applications of the Schwinger-Dyson equations in QCD. Those equations couple an infinite number of Green’s functions, and attempts at explicit solution normally require a truncation to a finite subset of them.
This truncation should be gauge-invariant, which is the non-Abelian case is not easily achieved. This triggered the development of the “pinch technique” (PT) [12,13], a systematic procedure that allows one to construct, starting from the standard Green’s functions derived from the gauge-fixed QCD Lagrangian, improved “gauge-invariant” vertices that fulfill simple QED-like off-shell Ward identities, not involving the ghosts. For the N-gluon vertices, which are our subject of interest in this letter, this procedure has, to the best of our knowledge, been carried out only for \( N = 3 \) and \( N = 4 \), and only at the one-loop level. The three-point vertex is special in that it involves the color indices only as a global prefactor:

\[
\Gamma_{\mu_1\mu_2\mu_3}^{abc}(k_1, k_2, k_3) = -i f^{abc} \Gamma_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3),
\]

(3)

where the \( f^{abc} \) are the structure constants of the Lie algebra, \([T^a, T^b] = if^{abc}T^c\). As shown by Cornwall and Papavassiliou [14], when constructed using the PT it will obey the identity

\[
k_1^{\mu_1} \Gamma_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = -(k_2^{\mu_2}g_{\mu_3\nu} - k_3^{\mu_3}g_{\mu_2\nu})(1 - \Pi(k_2^2)) + (k_3^{\mu_3}g_{\mu_2\nu} - k_2^{\mu_2}g_{\mu_3\nu})(1 - \Pi(k_3^2)),
\]

(4)

where \( \Pi(k^2) \) is the gluon vacuum polarization. This form of the Ward identity holds unambiguously for the scalar algebra, \([\Pi, T^a] = i\pi T^a\). In the present letter, we will use these properties to show that the identities (4), (5) generalize to the N-gluon case in the simplest possible way, namely as

\[
k_1^{\mu_1} \Gamma_{\mu_1\mu_2\ldots\mu_N}(k_1, k_2, \ldots, k_N) = -ig f_{a_1a_2\ldots a_N} \Gamma_{\mu_1\mu_2\ldots\mu_N}^{a_1a_2\ldots a_N}(k_2, k_3, \ldots, k_N)
\]

\[
-ig f_{a_1a_3\ldots a_N} \Gamma_{\mu_2\mu_3\ldots\mu_N}^{a_2a_3\ldots a_N}(k_3, k_4, \ldots, k_N)
\]

\[
-ig f_{a_1a_N\ldots a_N} \Gamma_{\mu_{N-1}\mu_N}^{a_1a_N\ldots a_N}(k_N, k_1, \ldots, k_{N-1}) + \cdots \]

\[
-ig f_{a_1a_N} \Gamma_{\mu_1\mu_N}^{a_1a_N}(k_1, k_N + 1, \ldots, k_{N-1}).
\]

(6)

String-inspired representation of gluon amplitudes. Around 1990, Bern and Kosower used the field theory limit of string theory to derive new parameter integral representations for the QCD one-loop N-gluon amplitudes [25,26]. In its original form this formalism was restricted to on-shell matrix elements, but it was soon extended to the off-shell case by Strassler using worldline path integral representations of the same amplitudes [23,24,27,30]. More recently, this version of the formalism has been found particularly suitable for the study of non-Abelian form factor decompositions [29,31,32]. Let us briefly summarize how the one-loop off-shell 1PI N-gluon amplitudes are constructed in the string-inspired formalism for a scalar, spinor or gluon loop (for details see [27]). The starting point is the following path-integral representation of the scalar loop contribution to this amplitude:

\[
\Gamma_{\text{scal}}(k_1, \varepsilon_1, a_1; \ldots; k_N, \varepsilon_N, a_N) =
\]

\[
(i g)^{\frac{N}{2}} \int_0^\infty \frac{dT}{T} e^{-m^2T} \int D\mathbf{x} e^{-\int_0^T d\tau \dot{\mathbf{x}}^2} \times V_{\text{scal}}[k_1, \varepsilon_1, a_1; \ldots; k_N, \varepsilon_N, a_N].
\]

(7)

Here \( m \) is the mass and \( T \) the proper time of the scalar in the loop. The integral \( \int D\mathbf{x} \) runs over all closed loops in space-time with periodicity \( T \). At fixed \( T \), each gluon is represented by a vertex operator

\[
V_{\text{scal}}[k_i, \varepsilon_i, a_i] = T^{a_i} \int_0^T d\tau \varepsilon_i \cdot \dot{x}_i e^{ik_i \cdot x_i},
\]

(8)

\(^1\)Equation (6) was already stated in [20] but not proven there.
where \( k_i \) and \( \varepsilon_i \) are the gluon momentum and polarization, \( T^{a_i} \) denotes a generator of the color group, and we have abbreviated \( \dot{\tau}_i \equiv \frac{d}{d\tau} \varepsilon(\tau_1) \). The polarization vectors \( \varepsilon_i \) at this stage are just book-keeping devices, and do not fulfill any on-shell constraints.

This is the full amplitude; each gluon vertex operator is integrated along the loop independently, so that the color generators \( T^{a_i} \) appear under the color trace in all possible orderings. It will be sufficient to consider the contribution corresponding to the standard ordering \( \tau_1 \geq \tau_2 \geq \cdots \geq \tau_N \), to be denoted by \( \Gamma_{\text{scal}}^{a_1 \ldots a_N} \). It can be written as

\[
\Gamma_{\text{scal}}^{a_1 \ldots a_N}(k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N) = (ig)^N \int_0^\infty dT \frac{T}{e^{-m^2 T}} \int D\hat{X} e^{-\int_0^T d\tau \frac{1}{2} i^2} \\
\times V_{\text{scal}}[k_1, \varepsilon_1, \ldots; k_N, \varepsilon_N] \\
\times \theta(\tau_1 - \tau_2) \delta(\tau_2 - \tau_3) \cdots \delta(\tau_{N-1} - \tau_N) \delta \left( \frac{T_N}{T} \right). \tag{9}
\]

Apart from imposing the proper-time ordering, one can use here also the translation invariance in proper time to reduce the number of integrations by setting \( \tau_N = 0 \). The full amplitude (7) is obtained from the ordered one (9) by summing over all \((N-1)!\) inequivalent permutations.

In the string-inspired formalism the path integral (9) is done by Gaussian integration, leading to the following “Bern-Kosower master formula”:

\[
\Gamma_{\text{scal}}^{a_1 \ldots a_N}(k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N) = (ig)^N \int_0^\infty dT (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \\
\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-2}} d\tau_{N-1} \\
\times \exp\left\{ \sum_{i,j=1}^N \frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j \\
+ \frac{1}{2} \dot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right\\right\}_{\text{multi-linear}}. \tag{10}
\]

Here \( D \) is the space-time dimension, and the notation \( _{\text{multi-linear}} \) means that, after the expansion of the exponential, only terms linear in every polarization vector should be retained. \( G_{Bij} \) stands for the “bosonic world-line Green’s function”

\[
G_B(\tau_i, \tau_j) = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T} - \frac{T}{6}. \tag{11}
\]

Writing out the exponential in eq. (10) one obtains an integrand

\[
\exp\left\{ \right\\right\}_{\text{multi-linear}} = (-i)^N P_N(\dot{G}_{Bij}, \dot{G}_{Bij}) \\
\times \exp\left\{ \frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j \right\}, \tag{12}
\]

with a certain polynomial \( P_N \).

Starting from this parameter-integral representation of the scalar loop contribution to the \( N \)-gluon amplitude, one can now generate the contributions of the spinor and gluon loop in the following way. There exists a systematic integration-by-parts procedure that eliminates all second derivatives of \( G_B \) [25,26,31,33]. After this, a parameter-integral representation of the spinor loop contribution can (up to the global normalization) be generated from the scalar-loop one by replacing every “\( r \)-cycle” appearing in \( Q_N \), that is, a product of \( G_B \) whose indices form a cycle, according to the “cycle-replacement rule”

\[
G_{B_{i_1}i_2}G_{B_{i_2}i_3} \cdots G_{B_{i_n}i_1} \rightarrow G_{B_{i_1}i_2}G_{B_{i_2}i_3} \cdots G_{B_{i_n}i_1} \\
- G_{F_{i_1}i_2}G_{F_{i_2}i_3} \cdots G_{F_{i_n}i_1}, \tag{13}
\]

where \( G_{F12} \equiv \text{sign}(\tau_1 - \tau_2) \) denotes the “fermionic” world-line Green’s function. A similar “cycle replacement rule” allows one to generate a parameter-integral representation of the gluon-loop contribution.

For our present purpose it will further be important that in the partially integrated integrand each \( \tau \)-cycle \( G_{B_{i_1}i_2}G_{B_{i_2}i_3} \cdots G_{B_{i_n}i_1} \) appears multiplied with a corresponding “Lorentz-cycle” \( \text{tr}(f_{i_1} f_{i_2} \cdots f_{i_n}) \), where

\[
f_{i}^{\mu} \equiv k_{i}^{\mu} - \varepsilon_{i}^{\nu} k_{i}^{\nu}, \tag{14}
\]

is the field-strength tensor of gluon \( i \) [24,28,31].

**Derivation of the \( N \)-gluon Ward identity: scalar loop.** – Let us now turn to the Ward identity in the \( N \)-gluon case. Starting from the path-integral representation (7) of the scalar contribution to the \( N \)-gluon amplitude, and replacing \( \varepsilon_i \) by \( k_i \), the corresponding vertex operator becomes the integral of a total derivative, and collapses to boundary terms:

\[
V_{\text{scal}}[k_i, \varepsilon_i] \equiv k_i, \varepsilon_i \int_0^{\tau_{N-1}} d\tau k_i \cdot \dot{x} (\tau) e^{ik_i x(\tau)} = -iT^{a_i} k_i \delta \frac{\partial}{\partial \tau} e^{ik_i x(\tau)} = -iT^{a_i} [e^{ik_i x(\tau)} - e^{ik_i x(\tau_{N-1})}]. \tag{15}
\]

After plugging this back into eq. (7) for the natural ordering \( \tau_1 \geq \tau_2 \geq \cdots \geq \tau_{N-1} \geq \tau_N \geq \tau_{N+1} \cdots \geq \tau_{N+N} \) we have (in the following we omit the global energy-momentum conservation factor)

see eq. (16) on top of the next page

Let us now focus on the term from the lower boundary \( \tau_1 = \tau_{N+1} \). If we apply the same replacement to the ordering that differs from the standard one only by an exchange of \( \tau_1 \) and \( \tau_{N+1} \), \( \tau_1 \geq \tau_2 \geq \cdots \geq \tau_{N-1} \geq \tau_{N+1} \geq \tau_N \geq \cdots \geq \tau_{N+N} \), the same term will be generated from the upper boundary of the \( \tau_1 \) integral, only with the opposite sign and an interchange of the color matrices \( T^{a_i} \) and \( T^{a_{i+1}} \). Thus
\[
\Gamma_{\text{scal}}^{a_1 \ldots a_N} [k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] \times k_i \rightarrow \Gamma^{a_1 \ldots a_N} \left( \text{tr}(T^{a_1} \cdots T^{a_{i-1}} T^{a_i} T^{a_{i+1}} \cdots T^{a_N}) \right) \int_{-\infty}^{\infty} \frac{dT}{T} e^{-m^2 T} \\
\times \int \mathcal{D}x e^{-i \int_0^T d\tau e^i x(\tau)} \left\{ \int_0^{T_1} d\tau_1 \varepsilon_1 \cdot x(\tau_1) e^{i k_1 \cdot x(\tau_1)} \cdots \int_0^{T_{i-2}} d\tau_{i-2} \varepsilon_{i-1} \cdot x(\tau_{i-1}) e^{i k_{i-1} \cdot x(\tau_{i-1})} \right. \\
\times \int_0^{T_{i-1}} d\tau_{i-1} \varepsilon_{i-1} \cdot x(\tau_{i-1}) e^{i k_{i-1} \cdot x(\tau_{i-1})} \cdots \int_0^{T_{N-1}} d\tau_N \varepsilon_N \cdot x(\tau_N) e^{i k_N \cdot x(\tau_N)} \left. \right\}.
\]

(16)

in the Abelian case all the boundary terms would cancel in pairs, but in the non-Abelian theory instead each pair produces a color commutator. Inserting the i-th vertex operator in all N possible ways, but keeping the order of the other vertex operators fixed, it is then easy to arrive at (6) (where we have now set \( i = 1 \)).

**Spinor and gluon loop.** The same identity (6) could be derived analogously for the spinor and gluon loop cases at the path-integral level using appropriate supersymmetric generalizations of (7)–(9) (those representations have been summarized, e.g., in [32]).

However, we find it more convenient to show the independence of the Ward identity of spin by the following argument. When substituting any \( \varepsilon_i \) by \( k_i \) in the partially integrated integrand there are two types of terms, those where the index \( i \) belongs to a cycle and those where it does not. For the first type of terms the polarization vector \( \varepsilon_i \) is contained in the field strength tensor \( f_{ij} \), so that they get annihilated by the substitution, and this is independent of the application of the loop replacement rules. The second type of terms are the ones that produce the right-hand side of the Ward identity, however since in those all the cycle factors are unaffected by the substitution \( \varepsilon_i \rightarrow k_i \) they appear as identical factors under the parameter integral on both sides, so that again the form of the identity, once established for the scalar loop, does not get altered by the application of the loop replacement rules.

**Conclusions.** To summarize, we have demonstrated that the one-loop QCD 1PI N-gluon amplitudes off-shell obey the Ward identity (6), as stated in our previous work [29]. This identity holds unambiguously for the scalar and spinor loop cases, but for the spin one case if and only if the gauge fixing is done using the BFM with quantum Feynman gauge (or equivalently using the pinch technique). To the best of our knowledge, this Ward identity has been previously treated in the literature only up to the four-point case [17]. However, an analogous identity has been derived in string theory in a way similar to ours [34].

As a final comment, let us remark that the fact that the BFM for the gluon loop leads to the same simple, ghost-free Ward identities as for the scalar and spinor loop only if the gluon in the loop is taken in Feynman gauge also implies that a generalization of the existing worldline path integral representations of the non-Abelian effective action [23,27] to other covariant gauges must by necessity run into some algebraic complications.

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