Let $S$ be an irreducible smooth projective surface defined over an algebraically closed field $k$. For a positive integer $d$, let $\text{Hilb}^d(S)$ denote the Hilbert scheme that parametrizes the zero-dimensional subschemes of $S$ of length $d$. Let $E$ be a vector bundle on $S$, and let $\mathcal{H}(E) \to \text{Hilb}^d(S)$ be its Fourier–Mukai transform constructed using the structure sheaf of the universal subscheme of $S \times \text{Hilb}^d(S)$ as the kernel. We prove that two vector bundles $E$ and $F$ on $S$ are isomorphic if the vector bundles $\mathcal{H}(E)$ and $\mathcal{H}(F)$ are isomorphic.

1. Introduction

Let $S$ be an irreducible smooth projective surface defined over an algebraically closed field. For a positive integer $d$, let $\text{Hilb}^d(S)$ denote the Hilbert scheme that parametrizes the zero-dimensional subschemes of $S$ of length $d$. Let

$$Z \subset S \times \text{Hilb}^d(S)$$

be the universal subscheme. Let

$$\beta : S \times \text{Hilb}^d(S) \to S \quad \text{and} \quad \gamma : S \times \text{Hilb}^d(S) \to \text{Hilb}^d(S)$$

be the natural projections. Given a coherent sheaf $E$ on $S$, we have the Fourier–Mukai transform

$$\mathcal{H}(E) = \gamma_*(\mathcal{O}_Z \otimes \beta^*E) \to \text{Hilb}^d(S).$$

If $E$ is locally free, then $\mathcal{H}(E)$ is also locally free because the restriction

$$\gamma|_Z : Z \to \text{Hilb}^d(S)$$

is a finite and flat morphism. Therefore, this Fourier–Mukai transform gives a map from the isomorphism classes of vector bundles on $S$ to the isomorphism classes of vector bundles on $\text{Hilb}^d(S)$.

A natural question to ask is whether this map is injective or surjective. Note that since $\dim \text{Hilb}^d(S) > \dim S$ if $d \geq 2$, this map can’t be surjective when $d \geq 2$. Our aim here is to prove that this map is injective. More precisely, we prove the following:

**Theorem 1.1.** Two vector bundles $E$ and $F$ on $S$ are isomorphic if and only if $\mathcal{H}(E)$ and $\mathcal{H}(F)$ are isomorphic.
Theorem 1.1 was proved earlier under the assumption that $S$ is a K3 or abelian surface; this was done by Addington, Markman–Mehrotra and Meachan (see [Ad], [MM], and [MC]).

2. Vector bundles on curves and its symmetric product

Let $k$ be an algebraically closed field. Let $C$ be an irreducible smooth projective curve defined over $k$ of genus $g_C$, with $g_C \geq 2$. The canonical line bundle of $C$ will be denoted by $K_C$. Fix an integer $d \geq 2$. Let $S_d$ denote the group of permutations of $\{1, \cdots, d\}$. The symmetric product

$$\text{Sym}^d(C) := C^d/S_d$$

is the quotient for natural action of $S_d$ on $C^d$. Let $D \subset C \times \text{Sym}^d(C)$ be the universal divisor which consists of all $(x, \{y_1, \cdots, y_d\})$ such that $x \in \{y_1, \cdots, y_d\}$.

Let

$$p_1 : D \longrightarrow C \quad \text{and} \quad p_2 : D \longrightarrow \text{Sym}^d(C)$$

be the projections defined by

$$(x, \{y_1, \cdots, y_d\}) \mapsto x \quad \text{and} \quad (x, \{y_1, \cdots, y_d\}) \mapsto \{y_1, \cdots, y_d\}$$

respectively.

For any algebraic vector bundle $E$ on $C$, define the direct image

$$S(E) := p_2^*p_1^*E \longrightarrow \text{Sym}^d(C),$$

where $p_1$ and $p_2$ are defined in (2.1). This $S(E)$ is locally free because $p_2$ is a finite and flat morphism.

If $0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$ is the Harder–Narasimhan filtration of $E$, then define

$$\mu_{\text{max}}(E) := \frac{\text{degree}(E_1)}{\text{rank}(E_1)} \quad \text{and} \quad \mu_{\text{min}}(E) := \frac{\text{degree}(E/E_{m-1})}{\text{rank}(E/E_{m-1})}.$$ 

So $\mu_{\text{max}}(E) \geq \mu_{\text{min}}(E)$, and $\mu_{\text{max}}(E) = \mu_{\text{min}}(E)$ if and only if $E$ is semistable.

Proposition 2.1. Let $E$ and $F$ be vector bundles on $C$ such that

$$\mu_{\text{max}}(E) - \mu_{\text{min}}(E) < 2(g_C - 1) \quad \text{and} \quad \mu_{\text{max}}(F) - \mu_{\text{min}}(F) < 2(g_C - 1).$$

If the two vector bundles $S(E)$ and $S(F)$ (defined in (2.2)) are isomorphic, then $E$ is isomorphic to $F$.

Proof. Let

$$\varphi : C \longrightarrow \text{Sym}^d(C)$$

be the morphism defined by $z \mapsto d \cdot z = (z, \cdots, z)$. Then $\varphi^*S(E)$ admits a filtration

$$0 = E(d) \subset E(d-1) \subset E(d-2) \subset \cdots \subset E(1) \subset E(0) = \varphi^*S(E)$$
such that
\[
E(d-1) = E \otimes K_C^\otimes(d-1) \quad \text{and} \quad E(i)/E(i+1) = E \otimes K_C^\otimes i
\]
for all \(0 \leq i \leq d-2\) (see [BN, p. 330, (3.7)]); in [BN] it is assumed that \(k = \mathbb{C}\), but the proof works for any algebraically closed field. Let
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E
\]
be the Harder–Narasimhan filtration of \(E\). For any \(j \in \mathbb{Z}\),
\[
\mu_{\max}(E \otimes K_C^\otimes j) = \mu_{\max}(E) + 2j(g_C - 1) \quad \text{and} \quad \mu_{\min}(E \otimes K_C^\otimes j) = \mu_{\min}(E) + 2j(g_C - 1).
\]
Hence the condition in (2.3) implies that
\[
\mu_{\max}(E \otimes K_C^\otimes j) < \mu_{\min}(E \otimes K_C^\otimes (j+1)).
\]
Therefore, from (2.4) and (2.5) we conclude the following:

- The Harder–Narasimhan filtration of \(\varphi^*S(E)\) has \(md\) nonzero terms.
- If
\[
0 = V_0 \subset V_1 \subset \cdots \subset V_{md-1} \subset V_{md} = \varphi^*S(E)
\]
is the Harder–Narasimhan filtration of \(\varphi^*S(E)\), then for any \(0 \leq j \leq d\),
\[
V_{mj} = E(d-j),
\]
where \(E(d-j)\) is the subbundle in (2.4).

More precisely, for any \(0 \leq j \leq d-1\) and \(0 \leq i \leq m\),
\[
V_{jm+i}/V_{jm} = E_i \otimes K_C^\otimes (d-j-1).
\]
In particular, we have
\[
V_m = E(d-1) = E \otimes K_C^\otimes(d-1).
\]

If \(S(E)\) and \(S(F)\) are isomorphic, comparing the Harder–Narasimhan filtrations of \(\varphi^*S(E)\) and \(\varphi^*S(F)\), and using (2.6), we conclude that \(E \otimes K_C^\otimes(d-1)\) is isomorphic to \(F \otimes K_C^\otimes(d-1)\). This implies that \(E\) is isomorphic to \(F\). \(\square\)

In [BN, Theorem 3.2], Proposition 2.1 was proved under that assumption that both \(E\) and \(F\) are semistable.

2.1. An example. We give an example to show that in general, \(S(E) = S(F)\) does not imply that \(E = F\).

Note that \(\text{Sym}^2(\mathbb{P}^1) \simeq \mathbb{P}^2\). If we identify \(\text{Sym}^2(\mathbb{P}^1)\) with \(\mathbb{P}^2\), then the universal degree two divisor
\[
D_2 \subset \mathbb{P}^1 \times \text{Sym}^2(\mathbb{P}^1) \simeq \mathbb{P}^1 \times \mathbb{P}^2
\]
is the zero locus of a section of the line bundle \(p^*(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes q^*(\mathcal{O}_{\mathbb{P}^2}(1))\), where
\[
p : \mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \quad \text{and} \quad q : \mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2
\]
are the natural projections. From this we see that
\[ S(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \]
\[ S(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \]
\[ S(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1). \]

For any two vector bundles \( E \) and \( F \) on \( \mathbb{P}^1 \) we have \( S(E \oplus F) = S(E) \oplus S(F) \). From these observations it follows that
\[ S(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1). \]

3. Vector bundles on surfaces and Hilbert scheme

Let \( S \) be an irreducible smooth projective surface defined over \( k \). For any \( d \geq 1 \), let \( \text{Hilb}^d(S) \) denote the Hilbert scheme parametrizing the 0–dimensional subschemes of \( S \) of length \( d \) (see [Fo]). Let \( \mathcal{Z} \subset S \times \text{Hilb}^d(S) \) be the universal subscheme which consists of all \((x, z) \in S \times \text{Hilb}^d(S)\) such that \( x \in z \). Let
\[ q_1 : \mathcal{Z} \longrightarrow S \quad \text{and} \quad q_2 : \mathcal{Z} \longrightarrow \text{Hilb}^d(S) \]
be the projections defined by \((x, z) \mapsto x \) and \((x, z) \mapsto z\) respectively.

For any algebraic vector bundle \( E \) on \( S \), define the direct image
\[ \mathcal{H}(E) := q_2 \ast (q_1^* E) \longrightarrow \text{Hilb}^d(S), \]
where \( q_1 \) and \( q_2 \) are the projections in (3.1). Since \( q_2 \) is a finite and flat morphism, the direct image \( \mathcal{H}(E) \) is locally free. We note that \( \mathcal{H}(E) \) is the Fourier–Mukai transform of \( E \) with respect to the kernel sheaf \( \mathcal{O}_{\mathcal{Z}} \) on \( S \times \text{Hilb}^d(S) \).

**Theorem 3.1.** Let \( E \) and \( F \) be vector bundles on \( S \) such that \( \mathcal{H}(E) \) (defined in (3.2)) is isomorphic to \( \mathcal{H}(F) \). Then the two vector bundles \( E \) and \( F \) are isomorphic.

**Proof.** If \( \iota : C \hookrightarrow S \) is an embedded irreducible smooth closed curve, then \( \iota \) induces a morphism
\[ \text{Sym}^d(C) \hookrightarrow \text{Hilb}^d(S). \]

Fix a very ample line bundle \( \mathcal{L} \) on \( S \). Let
\[ 0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E \]
be the Harder–Narasimhan filtration of \( E \) with respect to \( \mathcal{L} \). Let \( Y \subset S \) be the subset over which some \( E_i \) fails to be a subbundle of \( E \). This \( Y \) is a finite subset because any torsionfree sheaf on \( S \) is locally free outside a finite subset. Also note that \( Y \) is the subset over which the filtration in (3.4) fails to be filtration of subbundles of \( E \).

For \( n \geq 1 \), let
\[ \iota : C \longrightarrow S, \quad C \in |\mathcal{L}^\otimes n| \]
be an irreducible smooth closed curve lying in the complete linear system \( |L^{\otimes n}| \) such that \( \iota(C) \cap Y = \emptyset \). Since \( \mathcal{L} \) is very ample, such curves exist.

For each \( 1 \leq i \leq m \), there is an integer \( \ell_i \) such that \( \iota^*(E_i/E_{i-1}) \) is semistable for a general member of \( C \in |L^{\otimes n}| \) if \( n \geq \ell_i \) [MR, p. 221, Theorem 6.1]. Take

\[
\ell' = \max\{\ell_1, \ldots, \ell_m\}.
\]

If \( n \geq \ell' \), then for a general \( C \in |L^{\otimes n}| \), the pulled back filtration

\[
0 = \iota^*E_0 \subset \iota^*E_1 \subset \cdots \subset \iota^*E_{m-1} \subset \iota^*E_m = \iota^*E
\]

coincides with the Harder–Narasimhan filtration of \( \iota^*E \). Indeed, this follows immediately from the following two facts:

1. \( \iota^*(E_i/E_{i-1}) \) is semistable for a general member of \( C \in |L^{\otimes n}| \) if \( n \geq \ell_i \), and
2. \( \mu(\iota^*(E_i/E_{i-1})) > \mu(\iota^*(E_{i+1}/E_i)) \) because \( \mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i) \).

Let \( W \) be a vector bundle \( S \). Define

\[
d_W := c_1(\mathcal{L}) \cdot c_1(W) \in \mathbb{Z}.
\]

As before, let

\[
\iota : C \to S, \quad C \in |L^{\otimes n}|
\]

be an irreducible smooth closed curve. We have

\[
(3.5) \quad \deg(\iota^*W) = n \cdot d_W.
\]

In other words, \( \deg(\iota^*W) \) depends linearly on \( n \). From the adjunction formula,

\[
2(\text{genus}(C) - 1) = c_1(L^{\otimes n}) \cdot c_1(L^{\otimes n} \otimes K_S),
\]

where \( K_S \) is the canonical line bundle of \( S \) (see [Ha, p. 361, Proposition 1.5]). Hence we have

\[
(3.6) \quad \text{genus}(C) = \frac{n^2(c_1(L) \cdot c_1(L)) + nd_{K_S} + 2}{2}
\]

(see (3.5)). In other words, \( \text{genus}(C) \) is a quadratic function of \( n \).

Comparing (3.5) and (3.6) we conclude that there is an integer \( \ell \geq \ell' \) such that for \( n \geq \ell \), we have

\[
\mu(\iota^*E_1) - \mu(\iota^*(E/E_{m-1})) < 2(\text{genus}(C) - 1),
\]

where \( C \in |L^{\otimes n}| \) is an irreducible smooth closed curve. Note that this implies that \( \text{genus}(C) \geq 2 \).

Consider the embedding in (3.3). The restriction of \( \mathcal{H}(E) \) (respectively, \( \mathcal{H}(F) \)) to \( \text{Sym}^d(C) \) coincides with \( S(\iota^*E) \) (respectively, \( S(\iota^*F) \)) constructed in (2.2). So \( S(\iota^*E) \) and \( S(\iota^*F) \) are isomorphic because \( \mathcal{H}(E) \) and \( \mathcal{H}(F) \) are isomorphic. Since \( S(\iota^*E) \) and \( S(\iota^*F) \) are isomorphic, from Proposition 2.1 it follows that \( \iota^*E \) and \( \iota^*F \) are isomorphic for a general \( C \in |L^{\otimes n}| \) with \( n \geq \ell \).
The line bundle $L$ being ample, there is an integer $\ell''$ such that for every $n \geq \ell''$, we have
\begin{equation}
H^1(S, E \otimes F^* \otimes K_S \otimes L^{\otimes n}) = 0.\tag{3.7}
\end{equation}
Take $n \geq \ell''$, and let
\[ \iota : C \hookrightarrow S \]
be any irreducible smooth closed curve lying in $|L^{\otimes n}|$. Consider the short exact sequence of sheaves
\begin{equation}
0 \to F \otimes E^* \otimes \mathcal{O}_S(-C) \to F \otimes E^* \to (F \otimes E^*)|_C \to 0.\tag{3.8}
\end{equation}
Since $H^1(S, F \otimes E^* \otimes \mathcal{O}_S(-C)) = H^1(S, E \otimes F^* \otimes L^{\otimes n} \otimes K_S)^*$ (Serre duality), from (3.7) it follows that
\[ H^1(S, F \otimes E^* \otimes \mathcal{O}_S(-C)) = 0. \]
Therefore, from the long exact sequence of cohomology groups associated to (3.8) we conclude that the restriction homomorphism
\begin{equation}
\rho : H^0(S, F \otimes E^*) \to H^0(C, (F \otimes E^*)|_C)\tag{3.9}
\end{equation}
is surjective.

Take $n \geq \max\{\ell, \ell''\}$, and let $C \in |L^{\otimes n}|$ be a general member. We know that $\iota^*E$ and $\iota^*F$ are isomorphic. Fix an isomorphism
\[ I : \iota^*E \to \iota^*F. \]
So $I \in H^0(C, \iota^*(F \otimes E^*))$. Since $\rho$ in (3.9) is surjective, there is a homomorphism
\[ \tilde{I} \in H^0(S, F \otimes E^*) \]
such that $\rho(\tilde{I}) = I$. Let $r$ be the rank of $E$ (and also $F$). Consider the homomorphism of line bundles
\[- \bigwedge^r \tilde{I} : \bigwedge^r E \to \bigwedge^r F
\]
induced by $I$. Let
\[ D(\tilde{I}) := \text{Div}(\bigwedge^r \tilde{I}) \]
be the effective divisor for $\bigwedge^r \tilde{I}$. We know that $D(\tilde{I})$ does not intersect $C$ because the restriction $\rho(\tilde{I}) = I$ is an isomorphism. But $C$ is an ample effective divisor, so $C$ intersects any closed curve in $S$. Therefore, $D(\tilde{I})$ must be the zero divisor. Consequently, the homomorphism $\bigwedge^r \tilde{I}$ is an isomorphism. This implies that $\tilde{I}$ is an isomorphism. So the two vector bundles $E$ and $F$ are isomorphic. □

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