Abstract

We study the assignment of indivisible objects to individuals when transfers are not allowed. Previous literature has mainly focused on efficiency (from an ex-ante and ex-post perspective), and individually fair assignments. Consequently, egalitarian concerns have been overlooked. We are inspired by the assignment of apartments in housing cooperatives where families regard the egalitarianism of the assignments as a first-order requirement. In particular, they want to avoid assignments where some families get their most preferred apartment, while others get options ranked very low in their preferences. Based on Rawls’ idea of fairness, we introduce the notion of Rawlsian assignments. We prove that there always exists a unique Rawlsian assignment, which is ordinally efficient, and satisfies equal treatment of equals. We illustrate our analysis with preference data from housing cooperatives. Our results show that the Rawlsian assignment substantially improves, from an egalitarian perspective, both the probabilistic serial mechanism, and the mechanism currently in use.

JEL classification: C70, D63.

Keywords: random assignment, ordinal efficiency, fairness, Rawls.

1 Introduction

Economics deals with the allocation of resources in the economy. Although in many markets prices serve to coordinate this allocation, there are many allocation problems for which the use of prices is neither desirable nor feasible. This is the case for public housing where a set of apartments are to be allocated to families without the use of transfers. One challenge in such situations is how to ensure that resources are allocated efficiently. Also, the indivisibility

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of objects makes it impossible in general to attain fairness from a ex-post perspective. As a remedy, an ex-ante perspective is generally adopted and random allocations where each agent receives a lottery over the set of objects are considered.

Our primary motivation is the assignment of apartments in housing cooperatives. A housing cooperative consists of a group of families that take part in the construction of a building. Once the building is finished, the apartments have to be distributed among the families. Prices cannot be used, and the assignment is computed based on families’ (ordinal) preferences. The main concern of the cooperatives’ members in Uruguay is the egalitarianism of the final assignment. They want to avoid unequal assignments where, for example, a family gets their most preferred apartment, while other families are assigned to apartments they ranked very low.

There have been many important contributions in the literature regarding the efficiency and individual fairness of the assignments. However, not much attention has been paid to this kind of distributional concerns. In this paper, we introduce a new concept to capture this issue and we study its relation with others properties.

We follow the idea of justice by Rawls, and evaluate an assignment by the well-being of the worst-off individuals. The analysis begins by asking, given a certain assignment, who are the worst-off agents. If we were focusing on deterministic assignments, we could consider the rank of the assigned object at each agent’s preferences. The worst-off agents would be those associated with the highest rank. Then, we might select the assignments that minimize the highest rank. In general, there are multiple such assignments, so one could apply the same criterion recursively as proposed by Sen (2017).

When we enlarge the analysis by considering random assignments, the definition of the worst-off agents is more challenging. If agents’ cardinal utilities were known, one might consider the expected utility of each agent, and then look at the agent with the lowest expected utility. But in our problem, only agents’ ordinal preferences are known. Thus, for each agent we consider, among the objects the agent gets with positive probability, the rank of the least preferred object. Then, we focus on those agents associated with the highest rank, and among them we consider those who get the object with highest probability.

*Rawlsian assignments* are defined as follows. Given an assignment, we first look at the probability with which each agent is assigned to her least preferred. With $n$ agents, this is a vector of dimension $n$, and we re-arrange its entries in a non-increasing order. Second, we consider the probability with which each agent gets her last two objects, and once again we re-arrange its entries in a non-increasing order. We proceed in the same way until the first object of each agent (at which step we get a vector of ones). Finally, we *concatenate* all these vectors, beginning with the one associated with the least preferred objects. Then, to compare two given assignments we construct the vectors associated with each of them as defined previously. We say that an assignment is Rawlsian-dominated by other assignment, if the vector associated with the first assignment is lexicographically greater than the one associated with the second assignment. A Rawlsian assignment is an assignment which is not Rawlsian dominated by any
We first show that, in any problem, there always exists a unique Rawlsian assignment. Roughly, if there were two Rawlsian assignments, it would be possible to decrease the probability with which the worst-off agents receive their least preferred objects by considering the average assignment. We also prove that the Rawlsian assignment is ordinally efficient, and satisfies a minimum requirement in terms of fairness (equal treatment of equals). The drawback is that the Rawlsian mechanism is manipulable by agents (it is not strategyproof).

The next question is how to compute the Rawlsian assignment for a given problem. We introduce an algorithm to compute it in polynomial time by solving at most $n^2 \cdot 2^{n+1}$ linear programs.

Finally, we illustrate the analysis with a set of preferences from different housing cooperatives in Uruguay. We compare our solution with the outcome of the probabilistic serial (PS) by Bogomolnaia and Moulin (2001), and with the assignment that was implemented (called the MTAV mechanism). The results show that the Rawlsian solution substantially improves the assignment of the least favored agents. For example, under the PS assignment and for all but two cooperatives out of 24, there is at least one family who gets the least preferred apartment with positive probability. In the Rawlsian assignment, it is the other way around: only in three cases there is a family who gets the least preferred apartment with positive probability. Moreover, the rest of the families get an apartment ranked between their top 28% and top 88% positions of their preferences (and on average in the top 48%). For the PS, the average maximum rank is 84%. There are winners and losers when we change from PS to the Rawlsian mechanism, but the number of agents who prefer their assignment under the Rawlsian mechanism is always strictly higher. Thus, our solution stands for an alternative to both the PS and the MTAV mechanisms as it considerably improves upon them from an egalitarian perspective.

In the next section, we place our contributions within the related literature. Section 3 presents the model and definitions. In Section 4, we define the concept of Rawlsian assignments. Section 5 contains our main results and discuss the relation of Rawlsian assignments with other concepts analyzed in the literature. In Section 6, we describe an algorithm to find the Rawlsian assignment in any problem, and in Section 7 we use it to illustrate the results with data from different housing cooperatives. Finally, in Section 8 we conclude.

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1Equivalently, we compare two assignments by first considering the agent who receives her least preferred object with highest probability (the first entry of the concatenated vector). If the probability is the same under both assignments, we compare the agent who receives her least preferred object with the second highest probability. If all probabilities associated with the least preferred object of each agent are the same, we conduct the same comparison with the least and second-least preferred object of each agent. If at some point, an entry of a vector is strictly lower than the same entry of the other vector, we say that the first assignment Rawlsian-dominates the second.

2The mechanism which is currently used, called the MTAV, was proposed by Prino, Sánchez, and Cancela (2016). First, it selects among all the deterministic assignments, those that minimize the rank of the objects assigned to the worst-off families. Second, among the selected assignments, it considers those that maximize the sum of the families’ utilities, assuming that the utility of the apartment ranked in position $k$ is $n - k$ for all families. Finally, if multiple assignments are selected, one is randomly chosen. See Appendix 9.4 for a formal description of the mechanism.
2 Related Literature

There have been many important contributions since the introduction of the assignment problem by Hylland and Zeckhauser (1979). In addition to the study of efficiency, fairness, and incentive compatibility, different mechanisms have been proposed. Zhou (1990) and Abdulkadiroğlu and Sönmez (1998) were the first papers to introduce the random serial dictatorship (or random priority) mechanism. This mechanism performs very well in terms of incentives, but its outcome may be stochastically dominated with respect to individual preferences (i.e., it may not be ordinally efficient). Bogomolnaia and Moulin (2001) proposed the PS mechanism which is ordinally efficient, but not incentive compatible. We introduce the Rawlsian mechanism which retains the efficiency of the PS and introduces an egalitarian requirement.

The Rawlsian idea of justice (Rawls, 1971) has been applied to two-sided matching markets, see for example Masarani and Gokturk (1989) and Romero-Medina (2005). These papers study the compatibility of this idea of justice and stability, and use the Rawlsian criterion to select a stable matching that treats both sides of the market equally. The assignment problem that we study differs from this literature as it is a one-sided problem (objects do not have priorities) where the concept of stability does not apply. Klaus and Klijn (2010) adapt the Rawlsian criterion to the roommate problem and show that the concept is compatible with stability. Duddy (2021) introduces the notion of egalitarian assignments, also motivated by fairness concerns. We refine his definition by choosing a particular egalitarian assignment (see Appendix 9.3 for a more detailed discussion).

The construction of our concept resembles the welfarist definition of the PS mechanism by Bogomolnaia (2015, 2018). The author presents an alternative description of PS that relies on the cumulative probabilities of being assigned to the first \( k \) indifference classes for each agent \( i \). More specifically, she shows that the PS mechanism is the unique mechanism that lexicographically maximizes the vector containing the cumulative probabilities for all agent-preference pairs \((i,k)\). The Rawlsian criterion performs a similar procedure but starts from the least preferred objects, and minimizes the probability of the worst-off agent. Moreover, the Rawlsian mechanism we introduce lexicographically minimizes the cumulative probabilities preference by preference, instead of lexicographically minimizing the entire vector of cumulative probabilities.

The notion of Rawlsian assignments is also related to the downward lexicographic extension of stochastic dominance (Cho and Doğan, 2016; Cho, 2018). Different from Cho (2018) who conducts the comparison from an individual perspective, that is, he compares two lotteries from an agent’s perspective, we use a similar criterion but applied to the whole assignment. Also, Cho and Doğan (2016) show that upward lexicographic (ul) efficiency and ordinal efficiency are equivalent. Then, a Rawlsian assignment is ul-efficient, but not the other way around.

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3 See also Basteck (2018) for a discussion of other fairness concepts.

4 The analysis of Bogomolnaia (2015, 2018) allows for indifference classes in agents’ preferences. The same extension applies in our case.
3 Primitives and definitions

Let $I$ be the set of agents and $O$ the set of objects (both of size $n$). Each agent $i \in I$ has preferences over the set of objects, denoted by $\succeq_i$. We assume that all objects are acceptable for the agents (every object is preferred to the non-object option). A preference profile is denoted by $\succeq = (\succeq_i)_i \in I$. Sometimes we will represent agent $i$'s preferences as an $n$-dimensional vector $r_i \in \{1, \ldots, n\}^n$, where $r_i = k$ means that object $o$ is ranked $k$-th by agent $i$.5

We study the assignment of objects to agents, where each agent is to be assigned at most one object, and each object cannot be assigned to more than one agent. An assignment problem is defined by the tuple $(I, O, \succeq)$. We fix the sets of agents and objects, and define a problem by agents' preferences, and denoted as: $(\succeq)$.

A solution of an assignment problem is a (random) assignment $x = (x_i)_i \in I$, where each $x_i$ is a probability distribution over $O$. We interpret $x_{io} \in [0, 1]$ as the probability with which agent $i$ is allocated object $o$. An assignment is deterministic if all of its entries are either zero or one.6 We will usually describe an assignment by a matrix where the rows represent the agents and the columns the objects, and each row $i$ has the assignment of agent $i$: $x_i$. Let $X$ be the set of assignments (or equivalently, the set of $n \times n$ bi-stochastic matrices). A mechanism is a function $\phi$ that maps every problem to an assignment: for every $\succeq$, $\phi(\succeq) \in X$. We denote by $\phi_i(\succeq)$ the assignment of agent $i$ by mechanism $\phi$ in problem $(\succeq)$.

We will use the following concept of efficiency for random assignments due to Bogomolnaia and Moulin (2001).

**Definition 1** (Ordinal Efficiency). 1. An assignment $x_i$ for agent $i$ first-order stochastically dominates (fosd) an assignment $x_i'$ if, for each $o \in O$:

$$\sum_{o' : o' \succeq_i o} x_{io'} \geq \sum_{o' : o' \succeq_i o} x_{io'}'.$$

2. An assignment $x'$ is stochastically dominated by another assignment $x$ if for every agent $i$, $x_i$ fosd $x_i'$, and $x \neq x'$.

3. An assignment is ordinally efficient if it is not stochastically dominated by any assignment.

The following is a minimal requirement of fairness usually used in the literature.

**Definition 2** (Equal treatment of equals). Given a problem $(\succeq)$, an assignment $x$ satisfies equal treatment of equals if for all $i, j \in I$ we have:

$$\succeq_i = \succeq_j \implies x_i = x_j.$$

5 These assumptions fit our main motivation. Indeed, families in the case of the cooperatives have to rank all the apartments, and ties in preferences are not allowed.

6 We know by the Birkhoff-von Neumann theorem (Birkhoff, 1946; von Neumann, 1953) that any random assignment can be represented as a lottery over the set of deterministic assignments.
A stronger notion of fairness is sd-envy-freeness, which requires that each agent should prefer her assignment over anyone else’s assignment.

**Definition 3 (Sd-envy-freeness).** Given a problem \((\succeq)\), an assignment \(x\) is sd-envy-free if for all \(i,j \in I\), we have \(x_i \flosd x_j\).

If instead of requiring that any individual assignment fosd the assignment of all other agents, we require that there is no other agent’s assignment that fosd the agent’s assignment, we have a weaker notion, called weak sd-envy-freeness.

Finally, we define strategyproofness.

**Definition 4 (Strategyproofness).** A mechanism \(\phi\) is strategyproof if at any preference profile no agent can benefit by misreporting her preferences: for each \(i \in I\), for each \(\succeq\), and for each \(\succeq'_i\) the following holds:

\[\phi_i(\succeq) \flosd \phi_i(\succeq'_i, \succeq_{-i}).\]

As with sd-envy freeness, if in the last definition we require that there is no manipulation such that its outcome fosd the assignment the individual gets from truth-telling, we get a weaker notion, called weak strategyproofness.

### 4 Rawlsian assignments

In this section, we define our main concept of Rawlsian assignments. Given an assignment \(x\), we denote by \(b^x_i(k)\) the total probability with which agent \(i\) gets objects in positions \(n\) to \(k\) in her preferences. That is:

\[b^x_i(k) = \sum_{o \in O} 1\{r_{io} \geq k\}x_{io}.\]

Note that the previous definition is different from the standard concept of stochastic dominance where the probabilities are added from the most to the least preferred object. In particular, \(b^x_i(n)\) is the probability with which agent \(i\) receives her least preferred object, and, by definition, \(b^x_i(1) = 1\). We denote by \(b^x_i\) the vector with the cumulative probability from the least to the most preferred object: \(b^x_i = (b^x_i(n), b^x_i(n-1), \ldots, b^x_i(1) = 1)\).

Given an assignment \(x\), and the vectors \((b^x_i)_{i \in I}\), we define the vector \(B^x \in [0,1]^n^2\) as follows.

1. The first elements \((B^x_1, \ldots, B^x_n)\) are the elements \((b^x_1(n), \ldots, b^x_n(n))\) sorted in a non-increasing order.
2. Elements \((B^x_{n+1}, \ldots, B^x_{2n})\) are the elements \((b^x_1(n-1), \ldots, b^x_n(n-1))\) sorted in a non-increasing order.
3. In general, elements \((B^x_{(k-1)n+1}, \ldots, B^x_{kn})\) for \(k = 1, \ldots, n\), are the elements \((b^x_1(n - k + 1), \ldots, b^x_n(n - k + 1))\) sorted in a non-increasing order.

Vector \(B^x\) describes the cumulative distribution of probabilities induced by the assignment \(x\), from the least preferred object of each agent, to her most preferred object. It orders the cumulative probabilities, grouped by preference, in a non-increasing order. Then, the first \(n\) entries of the vector are the probabilities with which each agent receives her least preferred object. And, in particular, the first entry corresponds to the agent who receives her least preferred object with (weakly) higher probability. The following example illustrates the definitions.\(^7\)

**Example 1.** Consider the following problem with three agents, \(I = \{1, 2, 3\}\), and three objects \(O = \{a, b, c\}\). Agents’ preferences are:

\[
\begin{array}{ccc}
\preceq_1 & \preceq_2 & \preceq_3 \\
a & a & b \\
b & b & c \\
c & c & a
\end{array}
\]

Consider the assignment:

\[
x = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Then: \(b^x_1 = b^x_2 = (0, \frac{1}{2}, 1), b^x_3 = (0, 1, 1), \) and \(B^x = (0, 0, 0, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)\). \(\square\)

To compare two assignments, \(x\) and \(y\), we begin with those agents who are worst-off. We first compare the agent who receives her least preferred object with highest probability (the first entry of vectors \(B^x\) and \(B^y\)). If the probability is the same under both assignments, we compare the agent who receives her least preferred object with the second highest probability. If all probabilities associated with the least preferred object of each agent are the same, we conduct the same comparison with the least and second-least preferred object of each agent. If at some point, an entry of the vector \(B^x\) is strictly lower than the same entry of vector \(B^y\), we say that \(x\) Rawlsian-dominates \(y\). Formally, we compare the vectors \(B^x\) and \(B^y\) using a lexicographic order. This is the idea of the following definition.

**Definition 5.** Given two assignments \(x\) and \(y\), consider the vectors \(B^x\) and \(B^y\). We say that \(x\) Rawlsian-dominates \(y\) (\(x\ R\)-dominates \(y\)) if there is \(j \in \{1, \ldots, n^2\}\) such that \(B^x_j < B^y_j\), and for all \(i < j\), \(B^x_i = B^y_i\).

The following is the key concept of our analysis.

\(^7\)Note that this is different from the leximin order. \(B^x\) is not ordered in a non-increasing way but we order its elements by considering blocks of \(n\) elements, and within each block, the elements are placed in a non-increasing order.
Definition 6. An assignment $x$ is Rawlsian if it is not Rawlsian-dominated (R-dominated) by any other assignment.

The previous definition captures the idea of justice in the spirit of Rawls in the sense that we first improve the assignment of the agent who receives her least preferred object with the highest probability; conditionally on that, we improve the assignment of the second-worst agent, and so on, and so forth.

Example 2. In the problem of Example 1, consider the assignment:

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

associated with the vector $B^y = (1, 0, 0, 1, 0, 0, 1, 1, 1)$. Clearly, $x$ R-dominates $y$. Moreover, $x$ is a Rawlsian assignment. Note first that if another assignment is such that $x_{3c} < 1$, then we should have $x_{1c} > 0$ or $x_{2c} > 0$. And then, this new assignment will be R-dominated by $x$. So, if an assignment $R$-dominates $x$, it should that the $x_{1b} < \frac{1}{2}$ or $x_{2b} < \frac{1}{2}$. In the first case, then $x_{2b} > \frac{1}{2}$, and the new assignment is $R$-dominated by $x$. In the first case, then $x_{1b} > \frac{1}{2}$, and the new assignment is $R$-dominated by $x$. Finally, there is no assignment that $R$-dominates $x$.

5 Results

It is easy to see that in every problem there always exists a Rawlsian assignment. In principle, there might exist multiple Rawlsian assignments. Suppose this is indeed the case, and $x$ and $y$ are two Rawlsian assignments. But then, one may consider the assignment $\frac{1}{2}x + \frac{1}{2}y$, which R-dominates $x$ and $y$. Thus, as the following proposition shows, in every problem there is a unique Rawlsian assignment.

Proposition 1. Given a market, there exists a unique Rawlsian assignment.

A minimum requirement of fairness is the equal treatment of equals: agents with the same preferences should receive the same assignment. In the next proposition we prove that the Rawlsian assignment satisfies this property.

Proposition 2. A Rawlsian assignment satisfies equal treatment of equals.

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8Given a problem, consider an assignment $x^1$, and its associated vector $B^1$. If $x^1$ is not a Rawlsian assignment, there exists another assignment $x^2$ that $R$-dominates $x^1$. Then, there is an entry $j$ of $B^2$ such that $B^2_j < B^1_j$ and $B^1_l = B^2_l$ for all $l < j$. If $x^2$ is not a Rawlsian assignment, there is another assignment $x^3$ that $R$-dominates $x^2$. Then, there is an entry $j'$ of $B^3$ such that $B^3_{j'} < B^2_{j'}$ and $B^2_l = B^3_l$ for all $l < j'$. If $x^3$ is not a Rawlsian assignment, we continue in the same way. Note that, at some point, we should find a Rawlsian assignment because each entry of the vectors $B^k$ is bounded from below by zero, and the number of entries is finite.

9Randomization is key for this result. If we restrict the analysis to deterministic assignments, there are problems with multiple Rawlsian assignments.
There is no relation between Rawlsian assignments and sd-envy-free assignments. In Example 1, the outcome of PS which is sd-envy-free is not Rawlsian. Also, as we show in the next example, a Rawlsian assignment in general is not sd-envy-free (moreover, the same example shows that it is neither weak sd-envy-free).

**Example 3.** Consider the following problem with three agents, $I = \{1, 2, 3\}$, and three objects $O = \{a, b, c\}$.

\[
\begin{array}{ccc}
\geq_1 & \geq_2 & \geq_3 \\
a & b & b \\
b & a & c \\
c & c & a
\end{array}
\]

The Rawlsian assignment is:

\[
x = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Note that agent 3’s assignment is first-order stochastically dominated by the assignment of agent 2 (considering agent 3’s preferences).

Duddy (2021) introduces a notion of egalitarian assignments. An assignment $y$ is **inegalitarian** if there exists another assignment $x$ and an agent $j$ such that the assignment of all the agents at $x$ fosd the assignment of $j$ at $y$. That is, there is another assignment at which all the agents receive an assignment preferred than the assignment of $j$ at $y$. An egalitarian assignment is an assignment that is not inegalitarian. In Appendix 9.3 we define this concept, and show that the Rawlsian assignment is always egalitarian, but that the converse does not hold.

**5.1 Efficiency**

We now analyze the efficiency of the Rawlsian assignments. In contrast to many economic environments where there is a tension between fairness and efficiency, these concepts are compatible in our framework.\(^{10}\)

**Proposition 3.** A Rawlsian assignment is ordinally efficient.

It is easy to see that the converse of the proposition is not true: there are ordinally efficient assignments that are not Rawlsian. We show this in the following example.

\(^{10}\)This was already pointed out by Bogomolnaia and Moulin (2001) as the mechanism they propose, probabilistic serial, is ordinally efficient and sd-envy-free (which implies equal treatment of equals).
Example 4. Consider the problem of Example 1. The outcome of the probabilistic serial $^{11}$ (which coincides in this case with the random serial dictatorship) is:

$$y^{PS} = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$  

The outcome of PS is ordinally efficient and for this problem is different from the Rawlsian assignment.

There is another concept of efficiency, called rank efficiency, recently proposed by Featherstone (2020). In Appendix 9.2 we give the formal definition and show that there is no relation between Rawlsian and rank efficient assignments.

5.2 Strategy-proofness

We have shown that the Rawlsian mechanism, the function that maps every problem to the Rawlsian assignment, is ordinally efficient and satisfies equal treatment of equals. When there are at least four agents, we know by Bogomolnaia and Moulin (2001), that no mechanism is ordinally efficient, strategyproof, and satisfies equal treatment of equals. Thus, we have the following corollary.

Corollary 1. The Rawlsian mechanism is not strategyproof.

To illustrate the possible manipulations of the Rawlsian mechanism, suppose a problem where one object $o$ is regarded by all agents but $i$ and $j$ as the least preferred object. Moreover, assume that agents $i$ and $j$ rank this object as the second least preferred. Then, the Rawlsian mechanism assigns object $o$ to agents $i$ and $j$. In this situation, it is easy to complete the example such that when agent $i$ or $j$ ranks object $o$ higher in the last position, she gets a preferred assignment. This is the idea of the following example.

Example 5. Consider the following problem with three agents, $I = \{1, 2, 3\}$, and three objects $O = \{a, b, c\}$. $^{12}$

$$\succeq_1 \succeq_2 \succeq_3$$

|   | a | b | c |
|---|---|---|---|
| 1 | a | b |   |
| 2 | b | c | c |
| 3 | c | a |   |

$^{11}$ The probabilistic serial is defined as follows. Given a problem, think of each object as an infinitely divisible object with supply 1. **Step 1**: Each agent “eats away” from her favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted. **Step s**, for $s \in \{2, \ldots, S\}$: Each agent eats away from her remaining favorite object at the same speed. Proceed to the next step when an object is completely exhausted. The procedure terminates when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The random allocation of an agent $i$ is given by the amount of each object she has eaten until the algorithm terminates.

$^{12}$ The idea of this example is due to Paleo (2021).
The Rawlsian assignment is:

\[ x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \]

Suppose agent 2 reports the following preferences: \( \succeq_2' = (b, a, c) \), so the Rawlsian assignment is:

\[ y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Agent 2 prefers her assignment under \( y \) to her assignment under \( x \).

As we mentioned before, a weaker notion of strategyproofness requires that the outcome under an individual manipulation cannot dominate the assignment obtained when the agent reveals her true preferences. It is weaker because it allows for the case where the assignment the agent gets when she manipulates is not comparable with the assignment under truth-telling. The previous example shows that the Rawlsian mechanism is not weakly strategyproof.

5.3 The Rawlsian assignment and social welfare functions

We conclude this section by discussing the relation between the Rawlsian assignment and agents’ cardinal utilities. McLennan (2002) characterizes ordinal efficiency in terms of the existence of cardinal utilities that maximize agents’ total welfare. He shows that an assignment is ordinally efficient if, and only if, there exists a profile of cardinal utilities consistent with agents’ ordinal preferences for which the assignment maximizes the utilitarian welfare (see also Manea (2008)).

One may wonder if an equivalent result holds for the Rawlsian assignment. Is it possible to characterize the Rawlsian assignment by the existence of cardinal utilities that maximize the expected utility of the worst-off agents? It turns out that ordinal efficiency implies the existence of such cardinal representation. That is, if \( x \) is an ordinally efficient assignment, then there exist cardinal utilities (consistent with the ordinal preferences) such that \( x \) verifies

\[ x \in \arg \max_{x' \in \mathcal{X}} \{ \min_{i \in I} U_i(x') \}, \tag{1} \]

where \( U_i(x') = \sum_{o \in O} x'_{io} u_{io} \), and \( u_{io} \) is agent \( i \)'s cardinal utility for object \( o \). The Rawlsian assignment is ordinally efficient, so the result implies that for each instance (\( \succeq \)) there exist cardinal utilities for which the Rawlsian assignment maximizes the expected utility of the worst-off agent among all possible assignments.\(^{13}\)

\(^{13}\)See Appendix 9.6 for a proof.
What can we say about the converse of the last result? Consider an assignment that maximizes the welfare of the worst-off agent for some cardinal representation of the ordinal preferences. Is it a Rawlsian assignment? We show in the next proposition that this is not the case. Moreover, Proposition 4 implies a stronger result: there is no mechanism that treats equals equally and maximizes the utility of the worst-off agent for all cardinal representations of the ordinal preferences. In particular, we include an example where for different cardinal utilities (each consistent with the ordinal preferences), a different assignment maximizes the utility of the worst-off agent. And for some of these cardinal utilities, the resulting probabilistic assignment is not equal to the Rawlsian assignment.

**Proposition 4.** Given an instance \((\succeq)\), there is no mechanism that treats equals equally and that maximizes the utility of the worst-off agent for all cardinal utilities that imply ordinal preferences \(\succeq\).

6 Computing the Rawlsian assignment

We now define an algorithm to compute the Rawlsian assignment in polynomial time. The proposed algorithm is an extension of the algorithm by Airiau, Aziz, Caragiannis, Kruger, Lang, and Peters (2019), which computes a leximin distribution of a divisible object, to our setting with multiple objects. The pseudo-code is shown in Algorithm 1.

Intuitively, the algorithm finds the Rawlsian assignment \(x\) by lexicographically minimizing the corresponding vector \(B^x \in [0,1]^n^2\) as defined in Section 4. To do so, it first finds the lowest probability with which any agent can be assigned to her \(n\)-th choice, i.e., \(\min_{i \in I} b^x_i(n)\), by solving the first linear program. Denote by \(b^*\) the optimal value of this linear program, which we can solve efficiently using solvers like CPLEX or Gurobi.

Now that we know that all agents can be assigned to their \(n\)-th preference with a probability of at most \(b^*\), we find out for which agents this probability is exactly equal to \(b^*\). There must be at least one agent for which this is the case, otherwise the optimal solution of the first linear program would have been strictly lower than \(b^*\). If the optimal solution \(\epsilon^*\) of the second linear program is equal to zero for an agent \(i'\), this means that it is not possible to assign that agent to her \(n\)-th choice with a probability strictly lower than \(b^*\) while ensuring that all other agents are assigned to their \(n\)-th choice with a probability of at most \(b^*\). Accordingly, we add an additional constraint to the remaining linear programs that are solved in Algorithm 1 to impose that agent \(i'\) is assigned to her \(n\)-th choice with a probability equal to \(b^*_i(n) = b^*\). Moreover, we add agent \(i'\) to \(I_n\), which is the set of agents for which the probability of being assigned to their \(n\)-th choice is already fixed by the algorithm.

Next, we solve the upper linear program again to find the lowest probability with which any of the remaining agents in \(I \setminus I_n\) is assigned to their \(n\)-th choice, and we solve the lower linear program for the remaining agents in \(I \setminus I_n\) to determine for whom this is the exact probability
of being assigned to their $n$-th choice in the Rawlsian assignment. We repeatedly solve the first and second linear programs until $I_n = I$.

Finally, we repeat the above procedure for each of the $n$ preferences. In general, $I_k$ is the set of agents for which the algorithm has already found the exact probability with which they are assigned to an object of preference $k, \ldots, n$ by the Rawlsian assignment. It can be checked that Algorithm 1 needs to solve at most $n^2 \cdot \frac{n+1}{2}$ linear programs to find the Rawlsian assignment.

**Algorithm 1** Computing the Rawlsian assignment

$I_k \leftarrow \emptyset$ for $k \in \{1, \ldots, n\}$.

$b_i^*(k)$ is fixed once $i \in I$ is added to $I_k$.

for $k \in \{n, \ldots, 1\}$ do

while $I_k \neq I$ do

Find the minimum value of $b^*$ such that there exists an assignment $x \in [0, 1]^{n^2}$ satisfying

$$
\sum_{o \in O} x_{io} = 1 \quad \forall i \in I
$$

$$
\sum_{i \in I} x_{io} = 1 \quad \forall o \in O
$$

$$
\sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io} \leq b^* \quad \forall i \in I \setminus I_k
$$

$$
\sum_{o \in O} \mathbb{1}\{r_{io} \geq t\} x_{io} = b_i^*(t) \quad \forall i \in I_k : t \geq k
$$

if $b^* = 0$ then $I_k \leftarrow I$
else

for $i' \in I \setminus I_k$ do

Find the maximum value of $\epsilon^*$ such that there exists an assignment $x \in [0, 1]^{n^2}$ satisfying

$$
\sum_{o \in O} x_{io} = 1 \quad \forall i \in I
$$

$$
\sum_{i \in I} x_{io} = 1 \quad \forall o \in O
$$

$$
\sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io} \leq b^* - \epsilon^* \quad \forall i \in I \setminus I_k
$$

$$
\sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io} \leq b^* \quad \forall i \in I \setminus I_k
$$

$$
\sum_{o \in O} \mathbb{1}\{r_{io} \geq t\} x_{io} = b_i^*(t) \quad \forall i \in I_k : t \geq k
$$

if $\epsilon^* = 0$ then add $i'$ to $I_k$, and set $b_i^*(k) = b^*$.
end if

end for

end if

end while

end for

return The solution $x \in [0, 1]^{n^2}$ from the last solved LP.
7 Empirical Application

Our main motivation is the assignment of apartments in housing cooperatives. A housing cooperative is formed by a group of families who join to construct a building. Once it is finished, the apartments are to be distributed. Prices cannot be used, so the situation fits the assignment problem previously described.

Before the mechanism currently in use, cooperatives assigned apartments randomly. Preferences were not considered and the assignment was taken randomly from the set of all assignments. The assignment was in general not efficient, so they decided to change it. A group of researchers from the Engineering School of the University of Uruguay, proposed a new mechanism, called the MTAV, which was finally adopted. More information about the current mechanism can be found in Prino, Sánchez, and Cancela (2016) and Paleo (2021).

One of the main concerns of the families that participate in the cooperatives is the distribution of the final assignment. They do not want inequitable distributions in terms of the ranking of the assigned apartment in each family’s preferences. They want to avoid a situation where, for example, a family gets the most preferred assignment, while others families get apartments very low in their preferences. The MTAV mechanism explicitly considers this demand (we define the mechanism in Appendix 9.4).

In this section, we use the data of families’ preferences from 24 cooperatives to illustrate the differences between the Rawlsian, PS and MTAV mechanisms. The sizes of the cooperatives range from 4 to 42 families (with an average size of 17). We consider the cumulative number of different apartments ranked in the first, second, third, and fourth position of the preferences of each family as an indicator for the correlation of the preferences. The result shows that preferences are not highly correlated. In Appendix 9.5.1 we present detailed information for each cooperative.\(^1\)

We should mention that the MTAV mechanism is not strategyproof (Paleo, 2021). Nonetheless, we take preferences submitted by the families as their true preferences. We are not aware about manipulations by the families, and in general given the information held by the families, it is very difficult to optimally manipulate the mechanism.

7.1 Comparison with PS: Maximum rank

The Rawlsian mechanism is ordinally efficient and it is designed to improve the worse off agents. A first way to measure this improvement is what we call the maximum rank. Given an assignment, we consider for each agent the rank of the least preferred object received with positive probability. Then, we take the maximum rank among all agents. In Table 1 we look at the maximum rank of the Rawlsian and PS mechanisms. In contrast to what might be expected

\(^{14}\)It is worth noting that there are only two cooperatives, \(C_4\) and \(C_{10}\), where the Rawlsian and the PS mechanisms coincide. These are the smallest cooperatives (each consisting of 4 families). Additionally to these 8 families, there is only one more family who receives the same assignment under the two mechanisms. Thus, overall, 9 families out of 408 receive the same lottery over apartments.
(based on the correlation of preferences), in all but two cooperatives, the PS assigns at least one family to their least preferred apartment with positive probability. For the Rawlsian mechanism, this happens only in three cooperatives (those with a small number of families). For the rest, the Rawlsian mechanism uses apartments ranked in the first 48% of the positions.

Table 1: Size and maximum Rank of each cooperative for the Rawlsian and PS assignment.

| Coop. | Size | Max. Rawls | Max. Rawls (%) | Max. PS |
|-------|------|------------|----------------|--------|
| C1    | 26   | 13         | 50             | 26     |
| C2    | 18   | 12         | 67             | 18     |
| C3    | 4    | 2          | 50             | 4      |
| C4    | 4    | 3          | 75             | 3      |
| C5    | 28   | 8          | 29             | 28     |
| C6    | 8    | 3          | 38             | 8      |
| C7    | 29   | 8          | 28             | 29     |
| C8    | 12   | 7          | 58             | 12     |
| C9    | 15   | 6          | 40             | 14     |
| C10   | 4    | 4          | 100            | 4      |
| C11   | 11   | 5          | 45             | 11     |
| C12   | 16   | 6          | 38             | 16     |
| C13   | 39   | 14         | 36             | 39     |
| C14   | 42   | 33         | 79             | 42     |
| C15   | 14   | 9          | 64             | 14     |
| C16   | 6    | 6          | 100            | 6      |
| C17   | 9    | 3          | 33             | 9      |
| C18   | 15   | 8          | 53             | 15     |
| C19   | 9    | 9          | 100            | 9      |
| C20   | 20   | 10         | 50             | 20     |
| C21   | 24   | 7          | 29             | 24     |
| C22   | 7    | 2          | 29             | 7      |
| C23   | 40   | 11         | 28             | 40     |
| C24   | 8    | 7          | 88             | 8      |

Notes: Coop. stands for cooperative, each denoted as $C_i$ for $i = 1, \ldots, 24$. Size is the number of families in each cooperative. For Max. Rawls (Max. PS) we compute the rank of the least preferred object assigned with positive probability by the Rawlsian (PS) mechanism for each family, and then we take the maximum among all families. Max. Rawls (%) expresses Max. Rawls as a percentage of the length of families' preferences (or, equivalently, the size of the cooperative).

The previous analysis shows that the Rawlsian assignment assigns less families to their least preferred apartments. Now we look at the intensive margin, that is, the probabilities with which families are assigned to their least preferred objects. It could be that even when the PS assigns families to apartments ranked very low, this represents a very small probability. To investigate this, we define the expected number of families assigned to apartments ranked in position $k \in \{1, \ldots, n\}$ as the sum of the probabilities with each family is assigned to the apartment ranked in position $k$. Tables 5 - 10 in the Appendix present for all cooperatives the expected number of families assigned to each option by the Rawlsian and PS mechanism. Not only is
the maximum rank higher under PS than under the Rawlsian assignment for each cooperative, but also the cumulative probability of the least preferred apartments is substantially higher. For example, for the cooperative $C_7$ the Rawlsian mechanism assigns all families to apartments ranked 14th or better (out of 39 apartments), while PS assigns (in expectation) 8 families to apartments ranked 15th or worse.

The general picture regarding the expected number of agents assigned to each rank is as follows. The Rawlsian mechanism assigns a lower number of families to their least preferred apartments compared to PS. But, at the same time, it also assigns a lower number of families to their top choices, and especially to their first choice. As an illustration we include in Figure 1 the distribution of the expected number of families assigned to each option by each mechanism for four cooperatives.

![Graphs showing expected number of agents assigned to each option by the Rawlsian and PS mechanism for cooperatives $C_5$, $C_7$, $C_{13}$, and $C_{23}$](image)

**Figure 1:** Expected number of agents assigned to each option by the Rawlsian and PS mechanism.
7.2 Comparison with PS: individual preferences over assignments

Both the Rawlsian and the PS mechanisms are ordinally efficient. Therefore, it is never the case that all families prefer the assignment under one assignment over the other (equivalently, the outcome of one mechanism is not fosd by the outcome of the other for all families). In this section we compare, for each cooperative, the number of families that prefer the assignment of one mechanism over the other. Table 2 presents the results. For every cooperative there are more families that prefer the Rawlsian assignment than the assignment under PS. Moreover, the average percentage of families that prefer the Rawlsian (PS) assignment over the PS (Rawlsian) assignment is 35% (9%).

Table 2: Number of families who prefer the assignment under the Rawlsian (PS) over the PS (Rawlsian) mechanism.

| Coop. | Size | Prefer Rawls | Prefer PS |
|-------|------|--------------|-----------|
| C1    | 26   | 8            | 1         |
| C2    | 18   | 6            | 1         |
| C3    | 4    | 3            | 1         |
| C4    | 4    | 0            | 0         |
| C5    | 28   | 10           | 1         |
| C6    | 8    | 4            | 1         |
| C7    | 29   | 12           | 2         |
| C8    | 12   | 6            | 1         |
| C9    | 15   | 4            | 1         |
| C10   | 4    | 0            | 0         |
| C11   | 11   | 5            | 2         |
| C12   | 16   | 8            | 1         |
| C13   | 39   | 9            | 1         |
| C14   | 42   | 7            | 1         |
| C15   | 14   | 3            | 1         |
| C16   | 6    | 3            | 2         |
| C17   | 9    | 4            | 1         |
| C18   | 15   | 5            | 1         |
| C19   | 9    | 5            | 1         |
| C20   | 20   | 2            | 1         |
| C21   | 24   | 9            | 1         |
| C22   | 7    | 4            | 1         |
| C23   | 40   | 6            | 2         |
| C24   | 8    | 2            | 0         |

7.3 Comparison with PS: Sd-envy-freeness.

In this section we turn to the analysis of sd-envy-freeness. The PS assignment is sd-envy-free, so no family envies the assignment of another family. As we observed before the Rawlsian mechanism is not sd-envy-fee, so there may be families with envy.
We show in Table 3, for each cooperative, the number of families (and the percentage over all the families) that envy the assignment of some other family. Also, among those families that have envy, we show the average number of families that are envied. For each cooperative, the percentage of families with envy ranges from 13% to 78%, with an average of 53%. Apart from the two cooperatives where the mechanisms coincide, there is only one cooperative where no family envies the assignment of another family.

Table 3: Number of families with envy

| Coop. | Size | Envy Ag. | Envy Ag. (%) | Avg. Envied Ag. |
|-------|------|----------|--------------|-----------------|
| C₁    | 26   | 15       | 58           | 3               |
| C₂    | 18   | 14       | 78           | 4               |
| C₃    | 4    | 0        | 0            | 0               |
| C₄    | 4    | 0        | 0            | 0               |
| C₅    | 28   | 14       | 50           | 2               |
| C₆    | 8    | 4        | 50           | 1               |
| C₇    | 29   | 17       | 59           | 3               |
| C₈    | 12   | 8        | 67           | 2               |
| C₉    | 15   | 7        | 47           | 2               |
| C₁₀   | 4    | 0        | 0            | 0               |
| C₁₁   | 11   | 4        | 36           | 1               |
| C₁₂   | 16   | 6        | 38           | 2               |
| C₁₃   | 39   | 28       | 72           | 4               |
| C₁₄   | 42   | 30       | 71           | 6               |
| C₁₅   | 14   | 8        | 57           | 2               |
| C₁₆   | 6    | 3        | 50           | 1               |
| C₁₇   | 9    | 3        | 33           | 1               |
| C₁₈   | 15   | 7        | 47           | 2               |
| C₁₉   | 9    | 8        | 89           | 3               |
| C₂₀   | 20   | 13       | 65           | 3               |
| C₂₁   | 24   | 7        | 29           | 1               |
| C₂₂   | 7    | 3        | 43           | 1               |
| C₂₃   | 40   | 24       | 60           | 2               |
| C₂₄   | 8    | 1        | 13           | 1               |

7.4 Comparison with MTAV

The MTAV is a deterministic mechanism, so it is hard to compare it with the two other mechanisms. With this caveat in mind, Tables 5 - 10 in the Appendix include the expected number of agents assigned to each rank by each of the three mechanisms. By construction, the maximum rank of the Rawlsian mechanism and MTAV always coincide. However, it is interesting to note that there are cases where the Rawlsian mechanism assigns less families to the least preferred apartment (among those that are received with positive probability). For example, consider $C₁$: under the Rawlsian mechanism only one family receives the maximum rank (which is 13,
out of 25 apartments), while under MTAV two families receive their 13th choice. The same is true for cooperatives \(C_2, C_5, C_9, C_{13}, C_{15}, C_{17}, C_{21}, \) and \(C_{23}\). Overall, it is worth noting that the outcome of the MTAV mechanism is situated in between the two other mechanisms. Indeed, the Rawlsian mechanism outperforms the MTAV under the Rawlsian criterion, but not in terms of the expected number of agents assigned to their top choice. The PS mechanism outperforms the MTAV in terms of the expected number of agents assigned to their top choice, but not under the Rawlsian criterion.

8 Concluding Remarks

We study the assignment of indivisible objects to agents when prices cannot be used. We are inspired by the case of housing cooperatives where families are concerned about the distribution of the final assignment. More specifically, assignments where some families get their first option while others get options very low in their preferences should be avoided. In this context, we introduce a new concept, Rawlsian assignments, under which individuals assignments are improved beginning from those who are worst-off. We prove that there always exists a unique Rawlsian assignment which is ordinally efficient and satisfies equal treatment of equals. We then compare our proposed mechanism with the PS and the mechanism currently in use in this market. The results show that the Rawlsian mechanisms considerably outperforms the other two mechanisms from an egalitarian perspective. However, as Proposition 4 illustrates, it is not the unique mechanism maximizing the utility of the worst-off agent. Hence, studying alternative mechanisms that are egalitarian in nature is an interesting direction for future research.

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9 Appendix

9.1 Proofs

9.1.1 Proof of Proposition 1.

Proof. Suppose there are two Rawlsian assignments, \( x \) and \( y \). Both are associated with the same vector \( B \). Given an assignment \( z \), consider a matrix \( P^z \) where agents are represented in the rows, and in column \( k \) we include the probability with which each receives the object ranked in position \( k \).\(^{15}\) Starting from the last column, consider the first column where the matrices \( P^x \) and \( P^y \) differ (there is such a column as \( x \) and \( y \) are different assignments). Let’s say that it is column \( n - c \).

Note that the probabilities of column \( c \) in each matrix are the same, but distributed differently. Until column \( c \), the two matrices are the same, so the same agents get the same probabilities for the corresponding objects. This implies that, all agents receive the same probability for objects ranked in positions \( n - (c + 1), \ldots, n \) under \( x \) and \( y \). Therefore, they also receive the same probabilities under assignment \( \frac{1}{2}(x + y) \).

Consider the highest element of column \( c \) of each of the assignments \( x \), \( y \), and \( \frac{1}{2}(x + y) \). If the highest element of the assignments \( x \) and \( y \) corresponds to the same agent, then the highest element of the three assignments coincides. If it corresponds to the different agent, then the highest element of the assignments \( x \) and \( y \) is higher than the highest element of the assignment \( \frac{1}{2}(x + y) \). As the assignments \( x \) and \( y \) differ in at least one object, the assignment \( \frac{1}{2}(x + y) \) \( R \)-dominates \( x \) and \( y \). But this contradicts the fact that \( x \) and \( y \) are Rawlsian assignments. \( \square \)

9.1.2 Proof of Proposition 2.

Proof. Consider a problem, and two agents, say \( i \) and \( j \) with the same preferences. Suppose that their assignments under the Rawlsian assignment differ: \( x^R_i \neq x^R_j \). Consider the assignment \( y \) where all the agents receive the same lottery than in \( x^R \), and for agents \( i \) and \( j \), each receives \( \frac{1}{2}(x^R_i + x^R_j) \). Consider the vectors \( B^{x^R} \) and \( B^{\frac{1}{2}(x^R_i + x^R_j)} \), and compare them element by element. They have the same entries until the least preferred object of \( i \) and \( j \) where \( x^R_i \neq x^R_j \), and this entry is lower under \( B^{\frac{1}{2}(x^R_i + x^R_j)} \) than under \( B^{x^R} \). Then, assignment \( y \) \( R \)-dominates \( x^R \), which is a contradiction. \( \square \)

9.1.3 Proof of Proposition 3

Proof. We will use the equivalent definition of ordinal efficiency where we accumulate the probabilities from bottom to top: \( x \) \( fosd \) \( x' \) if the probability of being assigned to her least preferred object is larger under \( x' \) than under \( x \), the probability of being assigned to her two least preferred objects is larger under \( x' \) than under \( x \), and so on so forth.

\(^{15}\) Matrix \( P^z \) is created by reordering each row based on agents’ preferences. Each row of \( x \) and \( P^z \) has the same elements, but ordered differently.
Consider a Rawlsian assignment $x$, and suppose it is not ordinally efficient. Let $y$ be another assignment that stochastically dominates $x$. Consider the first $n$ entries of the vectors $B^x$ and $B^y$: the probability with which each agent is assigned to her least preferred object, under $x$ and $y$, respectively.

Let denote by $(p_{nn}, p_{(n-1)n}, \ldots, p_{1n})$ and $(q_{nn}, q_{(n-1)n}, \ldots, q_{1n})$, the first $n$ elements of vectors $B^x$ and $B^y$, respectively. Note that the agent corresponding to each probability $p_{in}$ may not be same agent corresponding to $q_{in}$. Also, $p_{in} > p_{jn}$ and $q_{in} > q_{jn}$ for $i > j$ (the elements are ordered from the largest to the smallest).

We know that $p_{nn} \leq q_{nn}$ (because $x$ is Rawlsian). Suppose $p_{nn} < q_{nn}$. So these elements cannot correspond to the same agent, otherwise the agent is worst off under $y$ than under $x$, and $y$ does not fosd $x$. So, it should be that the agent corresponding to $q_{nn}$ has a probability $p_{jn} \leq p_{nn}$ in $B^x$. But then, $p_{jn} \leq p_{nn} < q_{nn}$. Then, the agent receives a larger probability of being assigned to her least preferred object under $y$ than under $x$, which contradicts the fact that $y$ fosd $x$. Then, $p_{nn} = q_{nn}$. Moreover, the agent associated with probability $p_{nn}$ is the same agent who is associated with $q_{nn}$. If not, then, the agent associated with probability $q_{nn}$ is associated with a probability $p_{jn}$, but then $p_{jn} < p_{nn} = q_{nn}$, and then the agent is assigned to her least preferred option with higher probability under $y$ than under $x$, which is a contradiction.

We know that $p_{(n-1)n} \leq q_{(n-1)n}$ (because $x$ is Rawlsian). Suppose $p_{n(n-1)} < q_{n(n-1)}$. So these elements cannot correspond to the same agent, otherwise the agent is worst off under $y$. By the same argument as before, it cannot correspond neither to an agent with the probability $p_{jn}$, $j < (n-1)$. So, it should be that it corresponds to the agent associated with $p_{nn}$. Consider now the agent who corresponds to $q_{nn}$, and let’s find her probability in vector $B^x$. It should be an element $p_{i(n)}$ with $i > n$. But then, $q_{nn} = p_{nn} > p_{i(n)}$, then, agent $i$ receives her least preferred object with higher probability in $y$ that in $x$, a contradiction. Then, $p_{(n-1)n} = q_{(n-1)n}$.

By a similar argument we prove that $p_{in} = q_{in}$ for $i = 1, \ldots, n$.

We can apply the same argument as before to the vectors $(p_{n(n-1)}, p_{(n-1)(n-1)}, \ldots, p_{1(n-1)})$, and $(q_{n(n-1)}, q_{(n-1)(n-1)}, \ldots, q_{1(n-1)})$; \ldots $(p_{n1}, p_{(n-1)1}, \ldots, p_{11})$, and $(q_{n1}, q_{(n-1)1}, \ldots, q_{11})$ to prove that $B^x = B^y$. Moreover, by the same argument, $x = y$, which is a contradiction.

\[\square\]

9.2 Relation with rank efficiency

Given an assignment $x$, we define $M^x(k) = \sum_{i \in I} b^x_i(k)$ for $k = 1, \ldots, n$. So, for example, $M^x(n)$ is the sum of the probabilities with which each agent is assigned to her least preferred option. Equivalently, $M^x(k)$ is the expected number of agents who receive an object ranked in position $k$ or lower in assignment $x$.

An assignment, $x$, is said to rank-dominates another assignment, $y$, if the rank distribution of $y$ first-order stochastically dominates that of $x$, that is, if: \[16\]

\[16\]This is equivalent to the original definition of Featherstone (2020), where the sum of the probabilities are
\[ M_y(k) \geq M_x(k). \]

for all ranks, \( k \), with the inequality strict for at least one \( k \). If an assignment is not rank-dominated by any other assignment, it is rank efficient.

The following example shows a problem where the Rawlsian assignment is not rank-efficient.

**Remark 1.** A rank efficient assignment may not be Rawlsian. Consider the following problem.\(^{17}\)

\[
\begin{array}{cccccc}
\geq_1 & \geq_2 & \geq_3 & \geq_4 & \geq_5 \\
\hline
a & b & c & b & b \\
\hline
d & c & b & a & a \\
\hline
\end{array}
\]

The following assignment is rank efficient:

\[
y = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

but it is not Rawlsian. Indeed, its associated vector is

\[
B^y = (0, 0, 0, 0; 1, 0, 0, 0; 1, 1, 0, 0; 1, 1, 0, 0; 1, 1, 1, 1).\]

Consider the following assignment \( x \) (boxed in agents’ preferences):

\[
x = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

It is easy to see that \( x \) \( R \)-dominates \( y \).

**Remark 2.** A Rawlsian assignment may not be rank-efficient.

**Proof.** Consider the following problem:

\(^{17}\)This example is due to Paleo (2021).
\[
\begin{array}{ccc}
\geq_1 & \geq_2 & \geq_3 \\
a & a & b \\
b & b & a \\
c & c & c \\
\end{array}
\]

The following is the Rawlsian assignment of the problem:
\[
x = \begin{pmatrix}
\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\
0 & \frac{2}{3} & \frac{1}{3}
\end{pmatrix}.
\]

Consider the following assignment:
\[
y = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

Compute \(M_x(k)\) and \(M_y(k)\):

\[
\begin{array}{c|c|c}
 & M_j(3) & M_j(2) & M_j(1) \\
x & 1 & 3 & 3 \\
y & 1 & 1 & 3
\end{array}
\]

Assignment \(y\) Rank-dominates assignment \(x\).

9.3 Relation with egalitarian assignments

In this section, we define the concept of egalitarian assignments (Duddy, 2021), and show that the Rawlsian assignment is always egalitarian, but that the converse does not hold. Let \(p^x_i(\text{Top } k)\) be the probability that \(i\) receives an object from their top \(k\) objects, under assignment \(x\).

**Definition 7.** An assignment \(x\) is inegalitarian if there exists another assignment \(x'\) and an agent \(j\) such that:

\[p^x_i(\text{Top } k) \geq p^{x'}_j(\text{Top } k) \iff 1 - b^x_j(n - k) \geq 1 - b^{x'}_j(n - k) \iff b^x_j(n - k) \geq b^{x'}_j(n - k)\]

for all \(i\) and \(k = 1, \ldots, n\) (with the inequality being strict at least for one \(k\), for every \(i\)).

**Definition 8.** An assignment is egalitarian if it is not inegalitarian.

**Remark 3.** An egalitarian assignment may not be Rawlsian. Consider the following problem.
The following assignment is egalitarian:

\[
y = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

If not, there should be another assignment \( y' \) such that agent 4 is assigned to object \( d \) with probability less than 1 (all agents in \( y \) are assigned to an object in 3rd position or higher). But in that case, one of the other agents, let's say \( i \neq 4 \), should be assigned to object \( d \) with positive probability. Then,

\[
p_{ij}^{y'}(\text{Top 3}) < 1 = p_{ij}^y(\text{Top 3}) \forall l,
\]

which is a contradiction.

The Rawlsian assignment of this problem is:

\[
x = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proposition 5.** A Rawlsian assignment is egalitarian.

**Proof.** Consider a Rawlsian assignment \( x \), and suppose it is not egalitarian. Then, there exists another assignment \( x' \) and an agent \( j \) such that: \( p_{ij}^{x'}(\text{Top } k) \geq p_{ij}^x(\text{Top } k) \) for all \( i \) and \( k \) (with the inequality being strict at least for one \( k \), for every \( i \)).

Let’s consider the agent who receives the larger positive probability of her least preferred object in assignment \( x \). This is the first positive element in vector \( B^x \). Suppose this is agent \( l \), and her least preferred object that she receives with positive probability is in position \( \tilde{k} \).

Note that \( l \) cannot be agent \( j \) because in that case the first non-zero element in \( B^{x'} \) is strictly lower than the element in the same position in \( B^x \), but this contradicts that \( x \) is Rawlsian. In particular, we should have that \( p_{ij}^x(\text{Top } \tilde{k}) > p_{ij}^x(\text{Top } \tilde{k}) \) (if not, the same argument applies: every agent in \( x' \) should be better than agent \( j \) in \( x \), and then all that agents who receive the largest probability for the object in position \( \tilde{k} \) will be better. This implies that all the first
non-zero elements in \( B^x \) will shift part of the probabilities to more preferred objects, which
contradicts that it is a Rawlsian assignment.)

Then, for every agent \( i \) we have that: \( p^x_i (\text{Top} \tilde{k}) \geq p^x_j (\text{Top} \tilde{k}) > p^x_l (\text{Top} \tilde{k}) \). This implies
that the element in the same position where \( B^x \) has its first positive entry, is strictly lower in \( x' \), which is a contradiction.

\[ \Box \]

9.4 The MTAV mechanism.

In this section we define the MTAV mechanism following Paleo (2021). Let \( \mathcal{M} \) be the set of
deterministic assignment. Each assignment in \( \mathcal{M} \) is represented by a \( n \times n \) matrix \( M \) where
\( m_{ij} = 1 \) if, and only if, agent \( i \) receives object \( j \). Given two matrices \( M \) and \( M' \) in \( \mathcal{M} \), we denote
by \( M \odot M' \) the matrix where each element is the product of the corresponding elements of \( M \)
and \( M' \): \( (M \odot M')_{ij} = (m_{ij})(m'_{ij}) \). Also, denote as \( \max(M) \) and \( \sum(M) \) the maximum and
the sum of the elements of \( M \), respectively.

Given a problem \( (\succeq) \), define the matrix \( P \) with agents’ preferences, where \( P_{ij} \) is the rank of
object \( j \) in \( i \)'s preferences (equivalently, \( P_{ij} = r_{ij} \)). The MTAV is defined as follows.

1. For each \( M \in \mathcal{M} \) computes \( P \odot M \).
2. Compute \( \max(P \odot M) \).
3. Select the assignments that minimize \( \max(P \odot M) \).
4. Among the assignments selected in the previous step, select those that minimizes \( \sum(P \odot M) \).
5. If more than one assignment are selected in the last step, take one assignment at random.

9.5 Empirical Analysis: additional information.

9.5.1 Descriptive Statistics

We first illustrate some basic characteristics of the cooperatives we analyze. Table 4 shows
the number of families in each cooperative (which equals the number of apartments), and the
number of different objects ranked in the top \( k \) positions \( (k = 1, 2, 3, 4) \) in absolute terms, and
as percentage of the total number of objects.
Table 4: Size is the number of families in each cooperative. The next four columns, 1st, 2nd, 3rd, and 4th, have the number of different apartments ranked in the top 1, 2, 3 and 4 positions, respectively. The last four columns express the previous four columns as a percentage of the total number of objects.

|   | Size | 1st | 2nd | 3rd | 4th | 1st | 2nd | 3rd | 4th |
|---|------|-----|-----|-----|-----|-----|-----|-----|-----|
| C₁ | 26   | 16  | 20  | 21  | 22  | 61  | 76  | 80  | 84  |
| C₂ | 18   | 9   | 11  | 13  | 14  | 50  | 61  | 72  | 77  |
| C₃ | 4    | 3   | 4   | 4   | 4   | 75  | 100 | 100 | 100 |
| C₄ | 4    | 2   | 2   | 4   | 4   | 50  | 50  | 100 | 100 |
| C₅ | 28   | 17  | 23  | 24  | 25  | 60  | 82  | 85  | 89  |
| C₆ | 8    | 5   | 8   | 8   | 8   | 62  | 100 | 100 | 100 |
| C₇ | 29   | 16  | 22  | 24  | 25  | 55  | 75  | 82  | 86  |
| C₈ | 12   | 6   | 9   | 11  | 11  | 50  | 75  | 91  | 91  |
| C₉ | 15   | 9   | 11  | 13  | 15  | 60  | 73  | 86  | 100 |
| C₁₀| 4    | 3   | 3   | 3   | 4   | 75  | 75  | 75  | 100 |
| C₁₁| 11   | 6   | 7   | 10  | 10  | 54  | 63  | 90  | 90  |
| C₁₂| 16   | 11  | 14  | 15  | 15  | 68  | 87  | 93  | 93  |
| C₁₃| 39   | 13  | 23  | 28  | 31  | 33  | 58  | 71  | 79  |
| C₁₄| 42   | 19  | 24  | 27  | 28  | 45  | 57  | 64  | 66  |
| C₁₅| 14   | 5   | 8   | 9   | 10  | 35  | 57  | 64  | 71  |
| C₁₆| 6    | 3   | 4   | 5   | 5   | 50  | 66  | 83  | 83  |
| C₁₇| 9    | 6   | 8   | 9   | 9   | 66  | 88  | 100 | 100 |
| C₁₈| 15   | 10  | 10  | 12  | 12  | 66  | 66  | 80  | 80  |
| C₁₉| 9    | 4   | 5   | 5   | 5   | 44  | 55  | 55  | 55  |
| C₂₀| 20   | 8   | 11  | 15  | 17  | 40  | 55  | 75  | 85  |
| C₂₁| 24   | 16  | 21  | 23  | 23  | 66  | 87  | 95  | 95  |
| C₂₂| 7    | 4   | 7   | 7   | 7   | 57  | 100 | 100 | 100 |
| C₂₃| 40   | 17  | 26  | 30  | 34  | 42  | 65  | 75  | 85  |
| C₂₄| 8    | 3   | 5   | 5   | 5   | 37  | 62  | 62  | 62  |
### 9.5.2 Expected number of agents assigned to each rank

Table 5: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | Rawls | PS | MTAV | Rawls | PS | MTAV | Rawls | PS | MTAV | Rawls | PS | MTAV |
|----------|-------|----|------|-------|----|------|-------|----|------|-------|----|------|
| 1        | 7.17  | 12.54 | 10   | 2.4   | 6.42 | 8     | 2     | 2.44 | 2     | 2     | 2   | 2    |
| 2        | 3.19  | 2.08  | 5    | 2     | 0.71 | 2     | 2     | 0.89 | 2     | 0     | 0   | 0    |
| 3        | 2.97  | 1.03  | 1    | 0.6   | 0.51 | 0     | 0     | 0.22 | 0     | 2     | 2   | 2    |
| 4        | 3.33  | 0.86  | 2    | 0     | 0.7  | 0     | 0     | 0.44 | 0     | 0     | 0   | 0    |
| 5        | 0.33  | 0.49  | 0    | 3     | 1.07 | 1     | 0     | 0    | 0     | 0     | 0   | 0    |
| 6        | 1     | 0.73  | 1    | 3     | 1.13 | 2     | 0     | 0    | 0     | 0     | 0   | 0    |
| 7        | 2     | 0.96  | 1    | 4     | 1.36 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 8        | 2     | 0.93  | 3    | 0     | 0.79 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 9        | 1     | 0.17  | 0    | 2     | 0.76 | 1     | 0     | 0    | 0     | 0     | 0   | 0    |
| 10       | 0     | 0.23  | 0    | 0     | 0.67 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 11       | 1     | 0.14  | 0    | 0     | 0.53 | 2     | 0     | 0    | 0     | 0     | 0   | 0    |
| 12       | 1     | 0.3   | 1    | 1     | 0.38 | 2     | 0     | 0    | 0     | 0     | 0   | 0    |
| 13       | 1     | 1.04  | 2    | 0     | 0.3  | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 14       | 0     | 0.42  | 0    | 0     | 0.64 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 15       | 0     | 0.6   | 0    | 0     | 0.34 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| 16+      | 0     | 3.49  | 0    | 0     | 1.68 | 0     | 0     | 0    | 0     | 0     | 0   | 0    |
| Total    | 26    | 26    | 26   | 18    | 18   | 18    | 4     | 4    | 4     | 4     | 4   | 4    |
Table 6: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | Rawls C₅ | PS C₆ | MTAV C₇ | Rawls C₈ |
|----------|----------|-------|---------|----------|
| 1        | 7.75     | 13.82 | 11.5    | 3.5      |
| 2        | 5.83     | 4.54  | 4       | 4.17     |
| 3        | 3.42     | 0.61  | 1.8     | 3.33     |
| 4        | 2        | 1.06  | 2.5     | 4.12     |
| 5        | 2.5      | 1.01  | 3.5     | 1.02     |
| 6        | 3        | 0.48  | 1.5     | 0.36     |
| 7        | 2.5      | 0.55  | 4       | 1.25     |
| 8        | 1        | 0.45  | 2       | 0.48     |
| 9        | 0        | 0.53  | 0       | 0.18     |
| 10       | 0        | 0.29  | 0       | 0.12     |
| 11       | 0        | 0.7   | 0       | 0.2      |
| 12       | 0        | 0.51  | 0       | 0.75     |
| 13       | 0        | 0.59  | 0       | 0        |
| 14       | 0        | 0.19  | 0       | 0        |
| 15       | 0        | 0.6   | 0       | 0        |
| 16+      | 0        | 2.07  | 0       | 0        |
| Total    | 28       | 28    | 28      | 12       |

9.6 Rawlsian assignments and cardinal utilities

In this section we first prove in Proposition 6 that if \( x \) is an ordinally efficient assignment, there exists a cardinal representation of the agents’ ordinal preferences such that \( x \) maximizes the expected utility of the worst-off agent, among all possible assignments. Next, we prove in Proposition 4 that the reverse is not true, and that an instance \( (\succeq) \) may admit a continuum of different assignments that maximize the utility of the worst-off agent for some cardinal utilities, even when all agents have the same cardinal utilities.

**Proposition 6.** Consider a problem \( (\succeq) \) and an ordinally efficient assignment \( x \) for this problem. Then, there exists a cardinal representation of the preferences \( U = (U_i)_{i \in I} \) such that:

\[
x \in \arg \max_{x' \in X} \{ \min_{i \in I} U_i(x') \}.
\]

**Proof.** Consider an ordinally efficient assignment \( x \). By definition, \( x \) is not stochastically dominated by any other assignment. Equivalently, for any cardinal representation of the agents’ preferences, there is no assignment such that the expected utility of every agent is greater than or equal to the expected utility in this assignment (strictly greater for one agent). Fix another cardinal representation of the agents’ preferences such that the expected utility of each agent under \( x \) is the same. Then, the minimum cardinal utility in \( x \) is the utility of any agent. Suppose there is another assignment \( y \) such that the minimum expected utility among all agents is strictly greater than the minimum expected utility under \( x \). Therefore, every agent’s expected utility under \( y \) is greater than her expected utility under \( x \) (because it is greater than or equal
Table 7: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | C_9  | C_10 | C_11 | C_12 |
|----------|------|------|------|------|
|          | Rawls | PS  | MTAV | Rawls | PS  | MTAV | Rawls | PS  | MTAV | Rawls | PS  | MTAV |
| 1        | 4.88  | 7.23 | 8    | 2.42  | 2.42 | 3    | 4     | 5.63 | 6    | 7.81  | 9.05 | 8    |
| 2        | 2.38  | 1.37 | 1    | 0.33  | 0.33 | 0    | 3     | 0.71 | 1    | 3.56  | 2.69 | 4    |
| 3        | 1.75  | 0.81 | 1    | 0.25  | 0.25 | 0    | 3     | 2.23 | 2    | 1.63  | 0.71 | 1    |
| 4        | 4     | 1.45 | 2    | 1     | 1    | 1    | 0     | 0.57 | 1    | 0.75  | 0.55 | 1    |
| 5        | 1     | 1.31 | 1    | 0     | 0    | 0    | 1     | 0.46 | 1    | 1.25  | 0.54 | 1    |
| 6        | 1     | 0.87 | 2    | 0     | 0    | 0    | 0     | 0.72 | 1    | 1     | 0.35 | 1    |
| 7        | 0     | 0.26 | 0    | 0     | 0    | 0    | 0     | 0.13 | 0    | 0     | 0.38 | 0    |
| 8        | 0     | 0.06 | 0    | 0     | 0    | 0    | 0     | 0.16 | 0    | 0     | 0.32 | 0    |
| 9        | 0     | 0.34 | 0    | 0     | 0    | 0    | 0     | 0.26 | 0    | 0     | 0.45 | 0    |
| 10       | 0     | 0.78 | 0    | 0     | 0    | 0    | 0     | 0.03 | 0    | 0     | 0.25 | 0    |
| 11       | 0     | 0.27 | 0    | 0     | 0    | 0    | 0     | 0.11 | 0    | 0     | 0.12 | 0    |
| 12       | 0     | 0.06 | 0    | 0     | 0    | 0    | 0     | 0     | 0    | 0     | 0.08 | 0    |
| 13       | 0     | 0.13 | 0    | 0     | 0    | 0    | 0     | 0     | 0    | 0     | 0.16 | 0    |
| 14       | 0     | 0.07 | 0    | 0     | 0    | 0    | 0     | 0     | 0    | 0     | 0.2  | 0    |
| 15       | 0     | 0    | 0    | 0     | 0    | 0    | 0     | 0     | 0    | 0     | 0.06 | 0    |
| 16+      | 0     | 0    | 0    | 0     | 0    | 0    | 0     | 0     | 0    | 0     | 0.09 | 0    |
| Total    | 15    | 15   | 15   | 4     | 4    | 4    | 11    | 11   | 11   | 16    | 16   | 16   |

Table 8: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | C_13 | C_14 | C_15 | C_16 |
|----------|------|------|------|------|
|          | Rawls | PS  | MTAV | Rawls | PS  | MTAV | Rawls | PS  | MTAV | Rawls | PS  | MTAV |
| 1        | 4.5   | 9.29 | 9    | 3.11  | 12   | 7    | 1.78  | 3.65 | 4    | 2.17  | 2.44 | 3    |
| 2        | 7     | 7.23 | 5    | 5.35  | 3.39 | 9    | 1.97  | 2.23 | 4    | 1.33  | 1.09 | 1    |
| 3        | 4     | 2.48 | 5    | 1.51  | 1.76 | 3    | 0.83  | 0.68 | 0    | 1.33  | 0.39 | 1    |
| 4        | 4     | 1.96 | 5    | 4.18  | 1.51 | 1    | 2.75  | 1.2  | 0    | 1.17  | 0.59 | 0    |
| 5        | 3.5   | 2.25 | 4    | 4.25  | 1.12 | 4    | 2.67  | 1.62 | 2    | 0.74  | 0.69 | 0    |
| 6        | 3     | 1.46 | 2    | 1.58  | 1.23 | 0    | 3     | 1.06 | 1    | 1     | 1    | 1    |
| 7        | 3     | 1.16 | 1    | 2.97  | 1.69 | 2    | 0     | 0.09 | 1    | 0     | 0    | 0    |
| 8        | 0.5   | 1.24 | 0    | 1.83  | 1.32 | 5    | 0     | 0.54 | 0    | 0     | 0    | 0    |
| 9        | 4.5   | 0.67 | 2    | 1.29  | 0.49 | 1    | 1     | 0.64 | 2    | 0     | 0    | 0    |
| 10       | 2     | 0.73 | 1    | 1     | 0.3  | 1    | 0     | 0.31 | 0    | 0     | 0    | 0    |
| 11       | 0     | 0.63 | 1    | 1.5   | 0.67 | 0    | 0     | 0.45 | 0    | 0     | 0    | 0    |
| 12       | 1     | 0.47 | 1    | 1.75  | 1.21 | 1    | 0     | 0.72 | 0    | 0     | 0    | 0    |
| 13       | 1     | 0.86 | 1    | 3.17  | 1.38 | 1    | 0     | 0.33 | 0    | 0     | 0    | 0    |
| 14       | 1     | 0.6  | 2    | 2     | 1.21 | 0    | 0     | 0.48 | 0    | 0     | 0    | 0    |
| 15       | 0     | 0.23 | 0    | 0.5   | 0.75 | 1    | 0     | 0    | 0    | 0     | 0    | 0    |
| 16+      | 0     | 7.73 | 0    | 6     | 11.97| 6    | 0     | 0    | 0    | 0     | 0    | 0    |
| Total    | 39    | 39   | 39   | 42    | 42   | 42   | 14    | 14   | 14   | 6     | 6    | 6    |
Table 9: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | C17 | PS | MTAV | C18 | PS | MTAV | C19 | PS | MTAV | C20 | PS | MTAV |
|----------|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|------|
| 1        | 3.5 | 4.62| 5    | 5.33| 8.39| 9    | 2.8 | 3.14| 4    | 2.21| 5.34| 5    |
| 2        | 4.5 | 2.13| 2    | 1.67| 0.19| 0    | 1   | 0.83| 1    | 3.71| 1.4  | 3    |
| 3        | 1   | 0.93| 2    | 2.5 | 0.51| 0    | 0   | 0.37| 0    | 1.25| 1.92| 1    |
| 4        | 0   | 0.92| 0    | 0   | 0.06| 0    | 1.03| 0.23| 0    | 3.25| 1.91| 3    |
| 5        | 0   | 0.2 | 0    | 0.75| 0.55| 1    | 0.92| 0.76| 0    | 4.33| 1.93| 2    |
| 6        | 0   | 0.04| 0    | 0.25| 0.48| 1    | 1.25| 1    | 2    | 1.25| 1.02| 2    |
| 7        | 0   | 0.07| 0    | 3.5 | 2   | 4    | 0.89| 0.78| 0    | 0    | 0.31| 0    |
| 8        | 0   | 0.07| 0    | 1   | 0.91| 1    | 0.11| 0.89| 1    | 1.1  | 0.8  | 0    |
| 9        | 0   | 0.02| 0    | 0   | 0.18| 0    | 1   | 1    | 1    | 2    | 0.97| 3    |
| 10       | 0   | 0    | 0    | 0.38| 0   | 0    | 0   | 0    | 0    | 1.5  | 0.5  | 1    |
| 11       | 0   | 0    | 0    | 0.53| 0   | 0    | 0   | 0    | 0    | 0.41| 0    | 0    |
| 12       | 0   | 0    | 0    | 0.09| 0   | 0    | 0   | 0    | 0    | 0.63| 0    | 0    |
| 13       | 0   | 0    | 0    | 0.34| 0   | 0    | 0   | 0    | 0    | 0.53| 0    | 0    |
| 14       | 0   | 0    | 0    | 0.29| 0   | 0    | 0   | 0    | 0    | 0.26| 0    | 0    |
| 15       | 0   | 0    | 0    | 0.1 | 0   | 0    | 0   | 0    | 0    | 0.67| 0    | 0    |
| 16       | 0   | 0    | 0    | 0   | 0   | 0    | 0   | 0    | 0    | 1.39| 0    | 0    |
| Total    | 9   | 9    | 9    | 15  | 15  | 15   | 9   | 9    | 9    | 20  | 20  | 20   |

Table 10: Expected number of agents assigned to each rank for the Rawlsian, PS, and MTAV assignments.

| Position | C21 | PS | MTAV | C22 | PS | MTAV | C23 | PS | MTAV | C24 | PS | MTAV |
|----------|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|------|
| 1        | 8.79| 12.44| 12  | 4   | 3.75| 4    | 5.5 | 12.16| 10  | 2.38| 2.83| 3    |
| 2        | 1.92| 0.57 | 2   | 3   | 1.63| 3    | 5.81| 5.27 | 5    | 2.17| 0.57| 2    |
| 3        | 3.13| 1.36 | 3   | 0   | 1.29| 0    | 4.64| 3.01 | 4    | 0.25| 0.57| 0    |
| 4        | 1.33| 0.86 | 0   | 0   | 0.17| 0    | 8.47| 2.71 | 8    | 0    | 0.32| 0    |
| 5        | 6.83| 3.21 | 4   | 0   | 0.04| 0    | 4.06| 2.76 | 2    | 0.2  | 0.71| 0    |
| 6        | 1   | 0.7  | 1   | 0   | 0.04| 0    | 3.89| 2.02 | 5    | 2    | 1.29| 2    |
| 7        | 1   | 0.64 | 2   | 0   | 0.08| 0    | 2.56| 1.31 | 1    | 1    | 1.18| 1    |
| 8        | 0   | 0.45 | 0   | 0   | 0   | 0    | 2.58| 0.93 | 2    | 0    | 0.54| 0    |
| 9        | 0   | 0.27 | 0   | 0   | 0   | 0    | 0.5 | 0.9  | 0    | 0    | 0.0  | 0    |
| 10       | 0   | 0.37 | 0   | 0   | 0   | 0    | 0.5 | 0.47 | 1    | 0    | 0    | 0    |
| 11       | 0   | 0.81 | 0   | 0   | 0   | 0    | 1.5 | 0.7  | 2    | 0    | 0    | 0    |
| 12       | 0   | 0.15 | 0   | 0   | 0   | 0    | 0   | 0.39 | 0    | 0    | 0    | 0    |
| 13       | 0   | 0.36 | 0   | 0   | 0   | 0    | 0   | 0.35 | 0    | 0    | 0    | 0    |
| 14       | 0   | 0.16 | 0   | 0   | 0   | 0    | 0   | 0.7  | 0    | 0    | 0    | 0    |
| 15       | 0   | 0.32 | 0   | 0   | 0   | 0    | 0   | 0.3  | 0    | 0    | 0    | 0    |
| 16+      | 0   | 1.33 | 0   | 0   | 0   | 0    | 0   | 6.01 | 0    | 0    | 0    | 0    |
| Total    | 24  | 24   | 24  | 7   | 7   | 7    | 40  | 40   | 40   | 8    | 8    | 8    |
to the minimum expected utility). But this implies that \( x \) is not ordinally efficient, which is a contradiction.

\[ \]

\textbf{Proof of Proposition 4.}

\textit{Proof.} Consider the following example:

\[
\begin{array}{ccc}
\succeq_1 & \succeq_2 & \succeq_3 \\
a & a & b \\
b & b & a \\
c & c & c
\end{array}
\]

And consider the following assignment with \( \alpha \in [0, \frac{1}{2}] \):

\[
x^\alpha = \begin{pmatrix}
0.5 & 0.5 - \alpha & \alpha \\
0.5 & 0.5 - \alpha & \alpha \\
0 & 2\alpha & 1 - 2\alpha
\end{pmatrix}.
\]

Note that \( x^\alpha \) is indeed a feasible assignment that treats equals equally. Moreover, note that \( x^\alpha_{1a} = x^\alpha_{2a} = 0.5 \) and \( x^\alpha_{3a} = 0 \) must hold, because otherwise agent 3 could decrease the probability of being assigned to object \( a \) while agents 1 and 2 could increase their probability of being assigned to object \( a \). This exchange would increase everyone’s utilities, and any alternative assignment than \( x^\alpha \) is therefore not an assignment that maximizes the utility of the worst-off agent.

Consider the case where agents’ cardinal utilities for being assigned to their first, second, and third object are the same. Then, the utilities that the agents experience in \( x^\alpha \) are:

\[
\begin{align*}
U_1(\alpha) &= U_2(\alpha) = 0.5 \cdot u(1) + (0.5 - \alpha) \cdot u(2) + \alpha \cdot u(3) \\
U_3(\alpha) &= 2\alpha \cdot u(1) + (1 - 2\alpha) \cdot u(3)
\end{align*}
\]

Because both functions are linear in \( \alpha \), we maximize the utility of the worst-off agent by determining \( \alpha^* \) for which \( U_1(\alpha^*) = U_2(\alpha^*) = U_3(\alpha^*) \). It can be checked that we satisfy this condition for

\[
\alpha^* = \frac{0.5 \cdot u(1) + 0.5 \cdot u(2) - u(3)}{2 \cdot u(1) + u(2) - 3 \cdot u(3)}.
\]

Because \( u(1) \geq u(2) \geq u(3) \), it must hold that \( \frac{1}{3} \leq \alpha^* \leq \frac{1}{2} \). Hence, for each assignment \( x^\alpha \) with \( \alpha \in [\frac{1}{4}, \frac{1}{2}] \), there exist cardinal utility functions for which \( x^\alpha \) maximizes the utility of the worst-off agent. Note that \( x^{\frac{1}{3}} \) is the Rawlsian assignment.

\[ \]