Quantum Simulation of the Dissipative Anderson Model

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The interplay of Anderson localisation and decoherence results in intricate dynamics but is notoriously difficult to simulate on classical computers. We develop the framework for a quantum simulation of such an open quantum system making use of time-varying randomised gradients, and show that even an implementation with limited experimental resources results in accurate simulations.

Decoherence due to environmental noise can fundamentally change the character of dynamics in quantum systems, from the decay of otherwise stable states up to the emergence of classical dynamics. Our understanding of decoherence processes is limited by the difficulties that we are facing trying to simulate the dynamics of open quantum systems. The advent of quantum simulators offers the possibility to simulate coherent \cite{1} or dissipative \cite{2} dynamics of physical models that are prohibitively expensive to simulate on existing classical computers.

A particularly striking interference phenomenon is the transition of perfectly delocalised Bloch waves, in periodic potentials, to exponentially localised eigenstates in the presence of weak disorder that breaks a system’s periodicity. This Anderson localisation \cite{3} typically degrades in the presence of decoherence such that an initially localised wave-packet can spread over the entire system. Since generically coherent superpositions of wave-packets with large spatial separation decay faster than coherent superpositions of wave-packets with small spatial separation, such a dephasing Anderson system can feature an intricate interplay of coherent dynamics on short spatial scales but incoherent dynamics on larger scales. In one-dimensional systems, this is expected to result in the growth and subsequent decay of interference peaks that exist neither in the perfectly coherent nor in the dephased system \cite{4}.

For higher-dimensional or even interacting systems, there is extremely limited knowledge about the interplay between Anderson localisation and decoherence. While the analysis of Anderson localisation in two or three dimensions is a formidable computational challenge on its own \cite{5}, the inclusion of decoherence effects is likely to keep exceeding our computational capabilities for the foreseeable future, and calls for a quantum simulation.

With Anderson localisation observed in several highly controllable systems including (classical) acoustic and light waves \cite{6,8}, Bose-Einstein condensates in speckle potentials and optical lattices \cite{9,10}, the experimental prerequisites for a quantum simulation of the dephasing Anderson model seem to be available. We therefore develop the framework for the realisation of such a quantum simulation. In order to remain specific, we will discuss this in reference to atoms trapped in an optical lattice, but most of the concepts derived here are platform agnostic and apply equally well to e.g. networks of superconducting qubits \cite{11} or photonic circuits \cite{12}.

The Anderson Hamiltonian describes the dynamics of a single particle through an approximately periodic potential. As long as the disorder is sufficiently weak, one can define states \(|x\rangle\) that correspond to the particle being localised in the potential minimum at position \(x\). With the disorder resulting in energy shifts of the different potential minima, the Anderson Hamiltonian \cite{3} reads

\[
H = \sum_x \epsilon_x |x⟩⟨x| + \tau \sum_{\langle x,y \rangle} (|y⟩⟨x| + |x⟩⟨y|)
\] (1)

where \(\epsilon_x\) are the random energy shifts, \(\tau\) is the rate of tunnelling processes between different lattice sites, and the summation \(\sum_{\langle x,y \rangle}\) is performed over nearest neighbour sites. The dimensionality of the underlying model enters Eq. \(\text{(1)}\) only in terms of the connectivity, i.e. the number of nearest neighbour sites. The Anderson Hamiltonian arises naturally for bosonic atoms in optical lattices with an additional speckle potential. Typically these systems are described by the Bose-Hubbard Hamiltonian

\[
H_H = \sum_x \epsilon_x a_x^\dagger a_x + \tau \sum_{\langle x,y \rangle} (a_y^\dagger a_x + a_x^\dagger a_y) + H_I,
\]

with annihilation(creation) operators \(a_x^\dagger\) for a particle at site \(x\), satisfying bosonic commutation relations, and an onsite interaction \(H_I\). In the single-particle limit this Hubbard Hamiltonian reduces to the above Anderson Hamiltonian with the states \(|x⟩\) defined as \(|x⟩ = a_x^\dagger |0⟩\).

Dephasing will be taken into account in terms of a Lindbladian \(\mathcal{L}\) of the form

\[
\mathcal{L}(|x⟩⟨y|) = -\gamma f(x,y) |x⟩⟨y|,
\] (2)

where \(\gamma\) defines the time-scale of the dephasing processes and \(f(x,y)\) is a function that captures the difference in dephasing rate for different coherent superpositions. The condition \(f(x,x) = 0\)
reflects that $\mathcal{L}$ includes only loss of phase coherence but no diffusive processes, and a large small value of $f(x,y)$ implies that the phase coherence between the states $|x\rangle$ and $|y\rangle$ decays particularly fast (slow).

A key element of optical lattice experiments is the close-to-perfect elimination of decoherence [13]. It is thus necessary to introduce a mechanism resulting in decoherence, and to do so in a fashion that permits tuning $\gamma$, the strength of decoherence. This can be achieved in terms of an energy gradient (or equivalently an acceleration) [14] as depicted in Fig. 1, that is made to fluctuate. Such a tilt, in one-dimension, is described in terms of the Anderson Hamiltonian

$$H_T = \alpha \sum_x x |x\rangle \langle x|,$$

with the parameter $\alpha$ characterising the strength of the tilt.

For the sake of simplicity, let us consider for the moment a very strong optical lattice such that tunnelling is negligible. Since the onsite energy part $\sum_x \epsilon_x |x\rangle \langle x|$ of the Anderson Hamiltonian commutes with $H_T$, we can thus discuss the dynamics induced by the tilt alone. Any matrix element of the system state $\rho(t)$ after propagation for time $t$ then reads

$$\langle x| \rho(t) |y\rangle = \langle x| \rho(0) |y\rangle \ e^{i\alpha (x-y)t}.$$  

This is still perfectly coherent dynamics, but an ensemble average over different tilt strengths $\alpha$ will result in the desired dephasing process. To this end, one can take the tilt $\alpha$ to be a random variable with Gaussian distribution centred around $\alpha = 0$ with width $\sigma$. For the ensemble-averaged state $\varrho$ this yields

$$\langle x| \varrho(t) |y\rangle = \langle x| \varrho(0) |y\rangle \ \exp\left(-\frac{\sigma^2 t^2}{2} (x-y)^2 \right).$$

Thus, attenuation of phase coherence with a decay depending on the distance between $|x\rangle$ and $|y\rangle$ is obtained. Phase coherence, however, does not decay exponentially in time, as expected for a Lindbladian, but there is a Gaussian time-dependence. This issue can be overcome in terms of a time-dependent width $\sigma$ [15], or by adopting a stroboscopic perspective in which the system is probed only at integer multiples of some time constant $T$. If the ensemble average resulting in the attenuation of phase coherence of Eq. (5) is performed independently in each interval of duration $T$, then observation after $n$ periods, i.e. after the duration $t = nT$, will yield an attenuation of phase coherence by a factor $\exp\left(-na^2 T^2/2 (x-y)^2 \right)$. This is consistent with the Lindbladian defined above in Eq. (2) with the choice $f(x,y) = (x-y)^2$ and $\gamma = \sigma^2 T/2$.

In principle, this dephasing mechanism is enough to implement the desired quantum simulation based on a Trotter decomposition in terms of time windows of tilt dynamics, with suppressed tunnelling, alternating with time windows of dynamics induced by the Anderson Hamiltonian with finite tunnelling but no tilt. In practice, however, it is not even necessary to modulate the depth of the lattice, but the phase-averaging effect described above can also be realised very well in the presence of finite tunnelling, simply because the Trotter decomposition asserts that the dynamics induced by a sequence of two alternating Hamiltonians coincides with the dynamics induced by the sum of the two Hamiltonians. It is thus possible to realise the present quantum simulation in a digital fashion, i.e. alternating between Hamiltonians $H$ and $H_T$, and in an analogue fashion with no alternation between tilt and tunnelling.

Both realisations simulate the desired dynamics in the limit of a perfect ensemble average, infinitely fast switching of tilts (i.e. $T \to 0$ with $\gamma = \sigma^2 T/2$ constant) and in the case of the digital realisation, infinitely short Trotter steps. As we will show in the following, however, even simulations with averages over rather small ensembles, reasonably long time-windows $T$ and few Trotter steps give an excellent account of the desired dynamics.

In all the subsequent discussion we will consider a linear chain with 400 sites and onsite energies $\epsilon_x$ drawn from a uniform distribution in the interval $[-\tau/5, \tau/5]$; the system is initialised in the ground state of the Anderson Hamiltonian $H$ (given in Eq. (1)). Fig. 2 shows the comparison of site oc-
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\( I_p \) (digital) & 0.0036 & 0.013 & 0.032 & 0.014 \\
\hline
\( I_p \) (analogue) & 0.0032 & 0.014 & 0.038 & 0.0057 \\
\hline
\( I_s \) (digital) & 0.049 & 0.098 & 0.20 & 0.095 \\
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\( I_s \) (analogue) & 0.042 & 0.075 & 0.15 & 0.055 \\
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\caption{Population infidelities obtained with different Trotter steps.}
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digital case, and it is rather astonishing that the analogue method, that comes with substantially reduced experimental complexity, is nearly as good as the digital method for the vast majority of disorder realisations.

The comparison between the exact Lindbladian dynamics and averaged Hamiltonian dynamics, presented here, is necessarily restricted to a one-dimensional system that allows for sufficiently efficient numerical simulation. Whereas an explicit implementation of the present protocol would provide the proof of principle for the quantum simulation of the interplay between Anderson localisation and decoherence, any realisation with a higher-dimensional or an interacting system would help us to explore physics that becomes prohibitively difficult to simulate by classical means. Natural questions to be explored with such a platform could include signatures of the mobility edge in the presence of dephasing or interaction induced stabilisation of structures that would decay in the non-interacting system.

The decoherence model considered here can readily be generalised to any dependence on spatial separation in terms of the distribution for the average over different tilts. Controlled decoherence can evidently also be realised with many different mechanisms, such as photon scattering. The quantum simulation envisioned here aims at reproducing the behaviour consistent with a given Lindbladian, but one may similarly also consider a system-environment interaction as an underlying model that could be realised with a second species of trapped atoms serving as environment. Controlling the inter-species interaction and/or temperature of the environmental species would then allow tuning of the decoherence time, and could be used to explore the transition from Markovian to non-Markovian dynamics.

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