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Résumé. Soit $K$ un corps local de caractéristique résiduelle $p$ et soit $L/K$ une extension séparable finie totalement ramifiée de degré $n$. Soient $\sigma_1, \ldots, \sigma_n$ les $K$-plongements de $L$ dans une clôture séparable de $K$. Pour tout $1 \leq h \leq n$, soit $e_h(X_1, \ldots, X_n)$ le polynôme symétrique élémentaire en $n$ variables de degré $h$, et pour tout $\alpha \in L$, soit $E_h(\alpha) = E_h(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$. Soit $\mathcal{P}_K$ l'idéal maximal de l'anneau des entiers de $K$ et soit $j = \min\{v_p(h), v_p(n)\}$.

Nous montrons que $E_h(\mathcal{P}^r_L) \subset \mathcal{P}_K^{\left\lceil (i_j + hr)/n \right\rceil}$ pour tout $r \in \mathbb{Z}$, où $i_j$ est l'indice d'inséparabilité d'ordre $j$ de l'extension $L/K$. Dans certains cas, nous montrons également que $E_h(\mathcal{P}^r_L)$ n'est contenu dans aucune puissance supérieure de $\mathcal{P}_K$.

Abstract. Let $K$ be a local field whose residue field has characteristic $p$ and let $L/K$ be a finite separable totally ramified extension of degree $n$. Let $\sigma_1, \ldots, \sigma_n$ denote the $K$-embeddings of $L$ into a separable closure of $K$. For $1 \leq h \leq n$ let $e_h(X_1, \ldots, X_n)$ denote the $h$th elementary symmetric polynomial in $n$ variables, and for $\alpha \in L$ set $E_h(\alpha) = e_h(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$. Let $\mathcal{P}_K$ be the maximal ideal of the ring of integers of $K$ and let $j = \min\{v_p(h), v_p(n)\}$.

We show that for $r \in \mathbb{Z}$ we have $E_h(\mathcal{P}^r_L) \subset \mathcal{P}_K^{\left\lceil (i_j + hr)/n \right\rceil}$, where $i_j$ is the $j$th index of inseparability of $L/K$. In certain cases we also show that $E_h(\mathcal{P}^r_L)$ is not contained in any higher power of $\mathcal{P}_K$.

1. The problem

Let $K$ be a field which is complete with respect to a discrete valuation $v_K$. Let $\mathcal{O}_K$ be the ring of integers of $K$ and let $\mathcal{P}_K$ be the maximal ideal of $\mathcal{O}_K$. Assume that the residue field $\overline{K} = \mathcal{O}_K/\mathcal{P}_K$ of $K$ is a perfect field of characteristic $p$. Let $K^{\text{sep}}$ be a separable closure of $K$, and let $L/K$ be a finite totally ramified subextension of $K^{\text{sep}}/K$ of degree $n = up^r$, with $p \nmid u$.

Let $\sigma_1, \ldots, \sigma_n$ denote the $K$-embeddings of $L$ into $K^{\text{sep}}$. For $1 \leq h \leq n$ let $e_h(X_1, \ldots, X_n)$ denote the $h$th elementary symmetric polynomial in $n$ variables, and define $E_h : L \rightarrow K$ by setting $E_h(\alpha) = e_h(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$. 

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for $\alpha \in L$. We are interested in the relation between $v_L(\alpha)$ and $v_K(E_h(\alpha))$. In particular, for $r \in \mathbb{Z}$ we would like to compute the value of

$$g_h(r) = \min\{v_K(E_h(\alpha)) : \alpha \in \mathcal{P}_L^r\}.$$ 

The following proposition shows that $g_h(r)$ is a well-defined integer:

**Proposition 1.1.** Let $L/K$ be a totally ramified extension of degree $n$. Let $r \in \mathbb{Z}$ and let $h$ satisfy $1 \leq h \leq n$. Then $E_h(\mathcal{P}_L^r) \subset \mathcal{P}_K^{[hr/n]}$ and $E_h(\mathcal{P}_L^r) \neq \{0\}$.

**Proof.** For the first claim we observe that if $\alpha \in \mathcal{P}_L^r$ then $v_L(E_h(\alpha)) \geq hr$, and hence $v_K(E_h(\alpha)) \geq hr/n$. To prove the second claim let $\pi_L$ be a uniformizer for $L$ and let

$$f(X) = X^n - c_1X^{n-1} + \cdots + (-1)^{n-1}c_{n-1}X + (-1)^nc_n$$

be the minimum polynomial for $\pi_L$ over $K$. By Krasner’s lemma [6] there is $D > 1$ with the following property: For every Eisenstein polynomial

$$\tilde{f}(X) = X^n - \tilde{c}_1X^{n-1} + \cdots + (-1)^{n-1}\tilde{c}_{n-1}X + (-1)^n\tilde{c}_n$$

in $\mathcal{O}_K[X]$ such that $\tilde{c}_i \equiv c_i \pmod{\mathcal{P}_K^D}$ for $1 \leq i \leq n$, there is a root $\tilde{\pi}_L$ of $\tilde{f}(X)$ in $K^{sep}$ such that $K(\tilde{\pi}_L) = K(\pi_L) = L$. By choosing $\tilde{c}_h$ to be nonzero we get a uniformizer $\tilde{\pi}_L$ for $L$ such that $E_h(\tilde{\pi}_L) = \tilde{c}_h \neq 0$. Let $\pi_K$ be a uniformizer for $K$. Then for $t$ sufficiently large we have $\pi_K^t\tilde{\pi}_L \in \mathcal{P}_L^r$ and

$$E_h(\pi_K^t\tilde{\pi}_L) = \pi_K^{ht}E_h(\tilde{\pi}_L) = \pi_K^{ht}\tilde{c}_h \neq 0.$$ 

Therefore $E_h(\mathcal{P}_L^r) \neq \{0\}$. □

Since $L/K$ is totally ramified, for $\alpha \in L$ we have

$$v_K(E_n(\alpha)) = v_K(N_{L/K}(\alpha)) = v_L(\alpha).$$

Therefore $g_n(r) = r$ for $r \in \mathbb{Z}$. The map $E_1 = \text{Tr}_{L/K}$ is also well-understood, at least when $L/K$ is a Galois extension of degree $p$ (see [8, V §3, Lem. 4] or [1, III, Prop. 1.4]).

**Proposition 1.2.** Let $L/K$ be a totally ramified extension of degree $n$ and let $\mathcal{P}_L^d$ be the different of $L/K$. Then for every $r \in \mathbb{Z}$ we have $E_1(\mathcal{P}_L^r) = \mathcal{P}_K^{[(d+r)/n]}$. Therefore $g_1(r) = [(d + r)/n]$.

**Proof.** Since $E_1(\mathcal{P}_L^r)$ is a nonzero fractional ideal of $K$ we have $E_1(\mathcal{P}_L^r) = \mathcal{P}_K^s$ for some $s \in \mathbb{Z}$. By Proposition 7 in [8, III §3] we have

$$\mathcal{P}_L^{d+r} \subset \mathcal{O}_L \cdot \mathcal{P}_K^s = \mathcal{P}_L^{ns}$$

$$\mathcal{P}_L^{d+r} \not\subset \mathcal{O}_L \cdot \mathcal{P}_K^{s+1} = \mathcal{P}_L^{n(s+1)}.$$ 

It follows that $ns \leq d + r < n(s + 1)$, and hence that $s = [(d + r)/n]$. □
In this paper we determine a lower bound for \( g_h(r) \) which depends on the indices of inseparability of \( L/K \). When \( h = p^j \) with \( 0 \leq j \leq \nu \) and \( K \) is large enough we show that \( g_h(r) \) is equal to this lower bound. This leads to a formula for \( g_{p^j}(r) \) which can be expressed in terms of a generalization of the different of \( L/K \) (see Remark 5.4).

In Sections 2 and 3 we prove some preliminary results involving symmetric polynomials. The main focus is on expressing monomial symmetric polynomials in terms of elementary symmetric polynomials. In Section 4 we prove our lower bound for \( g_h(r) \). In Section 5 we show that \( g_h(r) \) is equal to this lower bound in some special cases.

The author thanks the referee for suggesting improvements to the proofs of Propositions 1.1 and 3.3.

2. Symmetric polynomials and cycle digraphs

Let \( n \geq 1 \), let \( w \geq 1 \), and let \( \lambda \) be a partition of \( w \). We view \( \lambda \) as a multiset of positive integers such that the sum \( \Sigma(\lambda) \) of the elements of \( \lambda \) is equal to \( w \). The number of parts of \( \lambda \) is called the length of \( \lambda \), and is denoted by \( |\lambda| \). For \( k \geq 1 \) we let \( k \ast \lambda \) be the partition of \( kw \) which is the multiset sum of \( k \) copies of \( \lambda \), and we let \( k \cdot \lambda \) be the partition of \( kw \) obtained by multiplying the parts of \( \lambda \) by \( k \). If \( |\lambda| \leq n \) let \( m_\lambda(X_1, \ldots, X_n) \) be the monomial symmetric polynomial in \( n \) variables associated to \( \lambda \), as defined for instance in Section 7.3 of [9]. For \( 1 \leq h \leq n \) let \( e_h(X_1, \ldots, X_n) \) denote the \( h \)th elementary symmetric polynomial in \( n \) variables.

Let \( r \geq 1 \) and let \( \phi(X) = a_rX^r + a_{r+1}X^{r+1} + \cdots \) be a power series with generic coefficients \( a_i \). Let \( 1 \leq h \leq n \) and let \( \mu = \{\mu_1, \ldots, \mu_h\} \) be a partition with \( h \) parts, all of which are \( \geq r \). Then for every sequence \( t_1, \ldots, t_h \) consisting of \( h \) distinct elements of \( \{1, \ldots, n\} \), the coefficient of \( X_{t_1}^{\mu_1}X_{t_2}^{\mu_2} \cdots X_{t_h}^{\mu_h} \) in \( e_h(\phi(X_1), \ldots, \phi(X_n)) \) is equal to \( a_\mu := a_{\mu_1}a_{\mu_2} \cdots a_{\mu_h} \). It follows that

\[
(2.1) \quad e_h(\phi(X_1), \ldots, \phi(X_n)) = \sum_\mu a_\mu m_\mu(X_1, \ldots, X_n),
\]

where the sum ranges over all partitions \( \mu \) with \( h \) parts, all of which are \( \geq r \). By the fundamental theorem of symmetric polynomials there is \( \psi_\mu \in \mathbb{Z}[X_1, \ldots, X_n] \) such that \( m_\mu = \psi_\mu(e_1, \ldots, e_n) \). In this section we use a theorem of Kulikauskas and Remmel [7] to compute some of the coefficients of \( \psi_\mu \).

The formula of Kulikauskas and Remmel can be expressed in terms of tilings of a certain type of digraph. We say that a directed graph \( \Gamma \) is a cycle digraph if it is a disjoint union of finitely many directed cycles of length \( \geq 1 \). We denote the vertex set of \( \Gamma \) by \( V(\Gamma) \), and we define the sign of \( \Gamma \) to
be \( \text{sgn}(\Gamma) = (-1)^{w-c} \), where \( w = |V(\Gamma)| \) and \( c \) is the number of cycles that make up \( \Gamma \).

Let \( \Gamma \) be a cycle digraph with \( w \geq 1 \) vertices and let \( \lambda \) be a partition of \( w \). A \( \lambda \)-tiling of \( \Gamma \) is a set \( S \) of subgraphs of \( \Gamma \) such that:

1. Each \( \gamma \in S \) is a directed path of length \( \geq 0 \).
2. The collection \( \{V(\gamma) : \gamma \in S\} \) forms a partition of the set \( V(\Gamma) \).
3. The multiset \( \{|V(\gamma)| : \gamma \in S\} \) is equal to \( \lambda \).

Let \( \mu \) be another partition of \( w \). A \((\lambda, \mu)\)-tiling of \( \Gamma \) is an ordered pair \((S, T)\), where \( S \) is a \( \lambda \)-tiling of \( \Gamma \) and \( T \) is a \( \mu \)-tiling of \( \Gamma \). Let \( \Gamma' \) be another cycle digraph with \( w \) vertices and let \((S', T')\) be a \((\lambda, \mu)\)-tiling of \( \Gamma' \). An isomorphism from \((\Gamma, S, T)\) to \((\Gamma', S', T')\) is an isomorphism of digraphs \( \theta : \Gamma \to \Gamma' \) which carries \( S \) onto \( S' \) and \( T \) onto \( T' \). Say that the \((\lambda, \mu)\)-tilings \((S, T)\) and \((S', T')\) of \( \Gamma \) are isomorphic if there exists an isomorphism from \((\Gamma, S, T)\) to \((\Gamma', S', T')\). Say that \((S, T)\) is an admissible \((\lambda, \mu)\)-tiling of \( \Gamma \) if \((\Gamma, S, T)\) has no nontrivial automorphisms. Let \( \eta_{\lambda\mu}(\Gamma) \) denote the number of isomorphism classes of admissible \((\lambda, \mu)\)-tilings of \( \Gamma \).

Let \( w \geq 1 \) and let \( \lambda, \mu \) be partitions of \( w \). Set

\[
d_{\lambda\mu} = (-1)^{|\lambda|+|\mu|} \sum_{\Gamma} \text{sgn}(\Gamma)\eta_{\lambda\mu}(\Gamma),
\]

where the sum is over all isomorphism classes of cycle digraphs \( \Gamma \) with \( w \) vertices. Since \( \eta_{\lambda\mu} = \eta_{\lambda\mu} \) we have \( d_{\mu\lambda} = d_{\lambda\mu} \). Kulikauskas and Remmel [7, Thm. 1(ii)] proved the following:

**Theorem 2.1.** Let \( n \geq 1 \), let \( w \geq 1 \), and let \( \mu \) be a partition of \( w \) with at most \( n \) parts. Let \( \psi_{\mu} \) be the unique element of \( \mathbb{Z}[X_1, \ldots, X_n] \) such that \( m_{\mu} = \psi_{\mu}(e_1, \ldots, e_n) \). Then

\[
\psi_{\mu}(X_1, \ldots, X_n) = \sum_{\lambda} d_{\lambda\mu} \cdot X_{\lambda_1}X_{\lambda_2} \ldots X_{\lambda_k},
\]

where the sum is over all partitions \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) of \( w \) such that \( \lambda_i \leq n \) for \( 1 \leq i \leq k \).

The remainder of this section is devoted to computing the values of \( \eta_{\lambda\mu}(\Gamma) \) and \( d_{\lambda\mu} \) in some special cases.

**Proposition 2.2.** Let \( w \geq 1 \), let \( \lambda, \mu \) be partitions of \( w \), and let \( \Gamma \) be a directed cycle of length \( w \). Assume that \( \Gamma \) has a \( \lambda \)-tiling \( S \) which is unique up to isomorphism, and that \( \text{Aut}(\Gamma, S) \) is trivial. Similarly, assume that \( \Gamma \) has a \( \mu \)-tiling \( T \) which is unique up to isomorphism, and that \( \text{Aut}(\Gamma, T) \) is trivial. Then \( \eta_{\lambda\mu}(\Gamma) = w \).

**Proof.** For \( 0 \leq i < w \) let \( S_i \) be the rotation of \( S \) by \( i \) steps. Then the isomorphism classes of \((\lambda, \mu)\)-tilings of \( \Gamma \) are represented by \((S_i, T)\) for \( 0 \leq i < w \). Since \( \text{Aut}(\Gamma, T) \) is trivial, all these tilings are admissible. \( \Box \)
Proposition 2.3. Let \( a, b, c, \ell, m, w \) be positive integers such that \( \ell a = mb + c = w \) and \( b \neq c \). Let \( \lambda \) be the partition of \( w \) consisting of \( \ell \) copies of \( a \), let \( \mu \) be the partition of \( w \) consisting of \( m \) copies of \( b \) and 1 copy of \( c \), and let \( \Gamma \) be a directed cycle of length \( w \). Then \( \eta_{\lambda\mu}(\Gamma) = a \).

Proof. The cycle digraph \( \Gamma \) has a \( \lambda \)-tiling \( S \) which is unique up to isomorphism, and a \( \mu \)-tiling \( T \) which is unique up to isomorphism. For \( 0 \leq i < a \) let \( S_i \) be the rotation of \( S \) by \( i \) steps. Then the isomorphism classes of \((\lambda, \mu)\)-tilings of \( \Gamma \) are represented by \((S_i, T)\) for \( 0 \leq i < a \). Since \( \text{Aut}(\Gamma, T) \) is trivial, all these tilings are admissible. \( \square \)

Proposition 2.4. Let \( b, c, m, w \) be positive integers such that \( mb + c = w \) and \( b \neq c \). Let \( \lambda \) be the partition of \( w \) consisting of 1 copy of \( w \) and let \( \mu \) be the partition of \( w \) consisting of \( m \) copies of \( b \) and 1 copy of \( c \). Then \( d_{\lambda\mu} = (-1)^{w+m+1}w \).

Proof. If the cycle digraph \( \Gamma \) has a \( \lambda \)-tiling then \( \Gamma \) consists of a single cycle of length \( w \). Hence by (2.2) we get \( d_{\lambda\mu} = (-1)^{w+m+1} \eta_{\lambda\mu}(\Gamma) \). It follows from Proposition 2.3 that \( \eta_{\lambda\mu}(\Gamma) = w \). Therefore \( d_{\lambda\mu} = (-1)^{w+m+1}w \). \( \square \)

Proposition 2.5. Let \( a, b, \ell, m, w \) be positive integers such that \( \ell a = mb = w \). Let \( \lambda \) be the partition of \( w \) consisting of \( \ell \) copies of \( a \), let \( \mu \) be the partition of \( w \) consisting of \( m \) copies of \( b \), and let \( \Gamma \) be a directed cycle of length \( w \).

1. The number of isomorphism classes of \((\lambda, \mu)\)-tilings of \( \Gamma \) is \( \gcd(a, b) \).
2. Let \((S, T)\) be a \((\lambda, \mu)\)-tiling of \( \Gamma \). Then the order of \( \text{Aut}(\Gamma, S, T) \) is \( \gcd(\ell, m) \).

Proof. (1) Identify \( V(\Gamma) \) with \( \mathbb{Z}/w\mathbb{Z} \) and consider the translation action of \( b\mathbb{Z}/w\mathbb{Z} \) on \((\mathbb{Z}/w\mathbb{Z})/(a\mathbb{Z}/w\mathbb{Z})\). The isomorphism classes of \((\lambda, \mu)\)-tilings of \( \Gamma \) correspond to the orbits of this action, and these orbits correspond to cosets of \( a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b) \cdot \mathbb{Z} \) in \( \mathbb{Z} \).

(2) The automorphisms of \((\Gamma, S, T)\) are rotations of \( \Gamma \) by \( k \) steps, where \( k \) is a multiple of both \( a \) and \( b \). Hence the number of automorphisms is \( w/\text{lcm}(a, b) \), which is easily seen to be equal to \( \gcd(\ell, m) \). \( \square \)

The following proposition generalizes the second part of [7, Thm. 6].

Proposition 2.6. Let \( a, b, \ell, m, w \) be positive integers such that \( \ell a = mb = w \). Let \( \lambda \) be the partition of \( w \) consisting of \( \ell \) copies of \( a \) and let \( \mu \) be the partition of \( w \) consisting of \( m \) copies of \( b \). Set \( u = \gcd(a, b) \) and \( v = \gcd(\ell, m) \). Then \( d_{\lambda\mu} = (-1)^{w-v+\ell+m(u)} \). In particular, if \( u < v \) then \( d_{\lambda\mu} = 0 \).

Proof. Set \( i = a/u \) and \( j = b/u \). Then \( m = vi \) and \( \ell = vj \). Let \( \Gamma \) be a cycle digraph which has an admissible \((\lambda, \mu)\)-tiling, and let \( \Gamma_0 \) be one of the cycles which make up \( \Gamma \). Then the length of \( \Gamma_0 \) is divisible by \( \text{lcm}(a, b) = uij \).
Suppose $\Gamma_0$ has length $k \cdot uij$. Let $\lambda_0$ be the partition of $kuij$ consisting of $kj$ copies of $a = ui$, and let $\mu_0$ be the partition of $kuij$ consisting of $ki$ copies of $b = uj$. Then by Proposition 2.5(2) every $(\lambda_0, \mu_0)$-tiling of $\Gamma_0$ has automorphism group of order $\gcd(ki, kj) = k$. Since $\Gamma$ has an admissible $(\lambda, \mu)$-tiling we must have $k = 1$. Therefore $\Gamma$ consists of $v$ cycles, each of length $uij$. By Proposition 2.5(1) the number of isomorphism classes of $(\lambda_0, \mu_0)$-tilings of a $uij$-cycle $\Gamma_0$ is $\gcd(a, b) = u$. An admissible $(\lambda, \mu)$-tiling of $\Gamma$ consists of $v$ nonisomorphic $(\lambda_0, \mu_0)$-tilings of $uij$-cycles. Hence the number of isomorphism classes of admissible $(\lambda, \mu)$-tilings of $\Gamma$ is $\eta_{\lambda\mu}(\Gamma) = \binom{u}{v}$. Hence by (2.2) we get $d_{\lambda\mu} = (-1)^{w-v+\ell+m}\binom{u}{v}$. 

3. Some subrings of $\mathbb{Z}[X_1, \ldots, X_n]$

Let $n \geq 1$. In some cases we can get information about the coefficients $d_{\lambda\mu}$ which appear in the formula for $\psi_\mu$ given in Theorem 2.1 by working directly with the ring $\mathbb{Z}[X_1, \ldots, X_n]$. In this section we define a family of subrings of $\mathbb{Z}[X_1, \ldots, X_n]$. We then study the $p$-adic properties of the coefficients $d_{\lambda\mu}$ by showing that for certain partitions $\mu$ the polynomial $\psi_\mu$ is an element of one of these subrings.

For $k \geq 0$ define a subring $R_k$ of $\mathbb{Z}[X_1, \ldots, X_n]$ by

$$R_k = \mathbb{Z}[X_1^{p^k}, \ldots, X_n^{p^k}] + p\mathbb{Z}[X_1^{p^{k-1}}, \ldots, X_n^{p^{k-1}}] + \cdots + p^k\mathbb{Z}[X_1, \ldots, X_n].$$

We can characterize $R_k$ as the set of $F \in \mathbb{Z}[X_1, \ldots, X_n]$ such that for $1 \leq i \leq k$ there exists $F_i \in \mathbb{Z}[X_1, \ldots, X_n]$ such that

$$F(X_1, \ldots, X_n) \equiv F_i(X_1^{p^i}, \ldots, X_n^{p^i}) \pmod{p^{k+1-i}}. \tag{3.1}$$

**Lemma 3.1.** Let $k, \ell \geq 0$ and let $F \in R_k$. Then $p^\ell F \in R_{k+\ell}$ and $F^{p^\ell} \in R_{k+\ell}$. 

**Proof.** The first claim is clear. To prove the second claim with $\ell = 1$ we note that for $1 \leq i \leq k$ it follows from (3.1) that

$$F(X_1, \ldots, X_n)^p \equiv F_i(X_1^{p^i}, \ldots, X_n^{p^i})^p \pmod{p^{k+2-i}}.$$ 

In particular, the case $i = k$ gives

$$F(X_1, \ldots, X_n)^p \equiv F_k(X_1^{p^k}, \ldots, X_n^{p^k})^p \pmod{p^2} \equiv F_k(X_1^{p^{k+1}}, \ldots, X_n^{p^{k+1}}) \pmod{p}.$$ 

It follows that $F^p \in R_{k+1}$. By induction we get $F^{p^\ell} \in R_{k+\ell}$ for $\ell \geq 0$. \qed

**Lemma 3.2.** Let $k, \ell \geq 0$ and let $F \in R_k$. Then for any $\psi_1, \ldots, \psi_n \in R_\ell$ we have $F(\psi_1, \ldots, \psi_n) \in R_{k+\ell}$. 
Proof. Since $F \in R_k$ we have
\[ F(X_1, \ldots, X_n) = \sum_{i=0}^{k} p^{k-i} \phi_i(X_1^{p^i}, \ldots, X_n^{p^i}) \]
for some $\phi_i \in \mathbb{Z}[X_1, \ldots, X_n]$. Since $\psi_j \in R_\ell$, by Lemma 3.1 we get $\psi_j^{p^i} \in R_{i+\ell}$. Since $R_{i+\ell}$ is a subring of $\mathbb{Z}[X_1, \ldots, X_n]$ it follows that $\phi_i(\psi_1^{p^i}, \ldots, \psi_n^{p^i}) \in R_{i+\ell}$. By Lemma 3.1 we get $p^{k-i} \phi_i(\psi_1^{p^i}, \ldots, \psi_n^{p^i}) \in R_{k+\ell}$. We conclude that $F(\psi_1, \ldots, \psi_n) \in R_{k+\ell}$. \qed

Proposition 3.3. Let $w \geq 1$ and let $\lambda$ be a partition of $w$ with at most $n$ parts. For $j \geq 0$ let $\lambda^j = p^j \cdot \lambda$. Then $\psi_{\lambda^j} \in R_j$.

Proof. We use induction on $j$. The case $j = 0$ is trivial. Let $j \geq 0$ and assume that $\psi_{\lambda^j} \in R_j$. Since $\lambda^{j+1} = p \cdot \lambda^j$ we get
\[
m_{\lambda^{j+1}}(X_1, \ldots, X_n) = m_{\lambda^j}(X_1^p, \ldots, X_n^p) = \psi_{\lambda^j}(e_1(X_1^p, \ldots, X_n^p), \ldots, e_n(X_1^p, \ldots, X_n^p)).
\]
For $1 \leq i \leq n$ let $\theta_i \in \mathbb{Z}[X_1, \ldots, X_n]$ be such that
\[
e_i(X_1^p, \ldots, X_n^p) = \theta_i(e_1, \ldots, e_n).
\]
It follows from the above that
\[
\psi_{\lambda^{j+1}}(X_1, \ldots, X_n) = \psi_{\lambda^j}(\theta_1(X_1, \ldots, X_n), \ldots, \theta_n(X_1, \ldots, X_n)).
\]
Since
\[
e_i(X_1, \ldots, X_n)^p \equiv e_i(X_1^p, \ldots, X_n^p) \pmod{p}
\equiv \theta_i(e_1, \ldots, e_n) \pmod{p}
\]
we have $\theta_i(X_1, \ldots, X_n) \equiv X_i^p \pmod{p}$, and hence $\theta_i \in R_1$. Therefore by Lemma 3.2 we get $\psi_{\lambda^{j+1}} \in R_{j+1}$. \qed

Corollary 3.4. Let $t \geq j \geq 0$, let $w' \geq 1$, and set $w = w' p^j$. Let $\lambda'$ be a partition of $w'$ and set $\lambda = p^j \cdot \lambda'$. Let $\mu$ be a partition of $w$ such that there does not exist a partition $\mu'$ with $\mu = p^{j+1} \cdot \mu'$. Then $p^{t-j}$ divides $d_{\lambda \mu}$. This holds in particular if $p^{j+1} \nmid |\mu|$.

Proof. Since $d_{\lambda \mu}$ does not depend on $n$ we may assume without loss of generality that $n \geq w$. It follows from this assumption that $|\lambda| \leq n$, so by Proposition 3.3 we have $\psi_\lambda \in R_t$. Since $w \leq n$ the parts of $\mu = \{\mu_1, \ldots, \mu_h\}$ satisfy $\mu_i \leq n$ for $1 \leq i \leq h$. Therefore the formula for $\psi_\lambda$ given by Theorem 2.1 includes the term $d_{\mu \lambda} X_{\mu_1} X_{\mu_2} \ldots X_{\mu_h}$. The assumption on $\mu$ implies that $X_{\mu_1} X_{\mu_2} \ldots X_{\mu_h}$ is not a $p^{j+1}$ power. Since $\psi_\lambda \in R_t$ this implies that $p^{t-j}$ divides $d_{\mu \lambda}$. Since $d_{\lambda \mu} = d_{\mu \lambda}$ we get $p^{t-j} \mid d_{\lambda \mu}$.
\qed
**Proposition 3.5.** Let \( w' \geq 1, j \geq 1, \) and \( t \geq 0. \) Let \( \lambda', \mu' \) be partitions of \( w' \) such that the parts of \( \lambda' \) are all divisible by \( p^j. \) Set \( w = w'p^j, \) so that \( \lambda = p^j \cdot \lambda' \) and \( \mu = p^j \cdot \mu' \) are partitions of \( w. \) Then \( d_{\lambda \mu} \equiv d_{\lambda' \mu'} \pmod{p^{j+1}}. \)

**Proof.** As in the proof of Corollary 3.4 we may assume without loss of generality that \( n \geq w'. \) Then \( |\lambda'| = |\lambda| \leq n. \) It follows from Proposition 3.3 that \( m_{\lambda'} = \psi_{\lambda'}(e_1, \ldots, e_n) \) for some \( \psi_{\lambda'} \in R_t. \) Using induction on \( k \) we see that for \( 1 \leq i \leq n \) and \( k \geq 0 \) we have

\[
e_i(X_1^{p^j}, \ldots, X_n^{p^j})^{p^k} \equiv e_i(X_1, \ldots, X_n)^{p^{j+k}} \pmod{p^{j+1}}.
\]

Since \( \psi_{\lambda'} \in R_t \) it follows that

\[
m_{\lambda}(X_1, \ldots, X_n) = m_{\lambda'}(X_1^{p^j}, \ldots, X_n^{p^j})
\]

\[
= \psi_{\lambda'}(e_1(X_1^{p^j}, \ldots, X_n^{p^j}), \ldots, e_n(X_1^{p^j}, \ldots, X_n^{p^j}))
\]

\[
\equiv \psi_{\lambda'}(e_1(X_1, \ldots, X_n)^{p^j}, \ldots, e_n(X_1, \ldots, X_n)^{p^j}) \pmod{p^{j+1}}.
\]

We also have \( m_{\lambda} = \psi_{\lambda}(e_1, \ldots, e_n). \) Therefore there is a symmetric polynomial \( \tau \in \mathbb{Z}[X_1, \ldots, X_n] \) such that

\[
\psi_{\lambda}(e_1, \ldots, e_n) = \psi_{\lambda'}(e_1^{p^j}, \ldots, e_n^{p^j}) + p^{j+1} \tau(X_1, \ldots, X_n).
\]

It follows from the fundamental theorem of symmetric polynomials that \( \tau \in \mathbb{Z}[e_1, \ldots, e_n]. \) Hence we have

\[
\psi_{\lambda}(X_1, \ldots, X_n) \equiv \psi_{\lambda'}(X_1^{p^j}, \ldots, X_n^{p^j}) \pmod{p^{j+1}}.
\]

Since \( w' \leq n \) the parts of \( \mu' \) and \( \mu \) are all \( \leq n. \) Therefore the formula for \( \psi_{\lambda'} \) given by Theorem 2.1 includes the term \( d_{\mu' \lambda}X_{\mu'_{1}}X_{\mu'_{2}} \ldots X_{\mu'_{h}} \), and the formula for \( \psi_{\lambda} \) includes the term

\[d_{\mu \lambda}X_{\mu_{1}}X_{\mu_{2}} \ldots X_{\mu_{p^{j} h}} = d_{\mu \lambda}X_{\mu'_{1}}X_{\mu'_{2}} \ldots X_{\mu'_{h}}.\]

It follows that \( d_{\mu \lambda} \equiv d_{\mu' \lambda'} \pmod{p^{j+1}}. \) Therefore we have \( d_{\lambda \mu} \equiv d_{\lambda' \mu'} \pmod{p^{j+1}}. \)

\[\square\]

4. **Containment**

Let \( L/K \) be a totally ramified extension of degree \( n = up^r, \) with \( p \nmid u. \) Let \( \sigma_1, \ldots, \sigma_n \) be the \( K \)-embeddings of \( L \) into \( K^{sep}. \) Let \( 1 \leq h \leq n \) and recall that \( E_h : L \to K \) is defined by \( E_h(\alpha) = e_h(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)) \) for \( \alpha \in L. \) In this section we define a function \( \gamma_h : \mathbb{Z} \to \mathbb{Z} \) such that for \( r \in \mathbb{Z} \) we have \( E_h(\mathcal{P}_L^r) \subset \mathcal{P}_L^{\gamma_h(r)}. \) The function \( \gamma_h \) will be defined in terms of the indices of inseparability of the extension \( L/K. \) In the next section we show that \( \mathcal{O}_K \cdot E_h(\mathcal{P}_L^r) = \mathcal{P}_L^{\gamma_h(r)} \) holds in certain cases.
Lemma 4.1. Let $\pi_L$ be a uniformizer for $L$ and let 
\[ f(X) = X^n - c_1 X^{n-1} + \cdots + (-1)^{n-1} c_{n-1} X + (-1)^n c_n \]
be the minimum polynomial of $\pi_L$ over $K$. Then $c_h = E_h(\pi_L)$. For $k \in \mathbb{Z}$ define $\overline{v}_p(k) = \min\{v_p(k), \nu\}$. For $0 \leq j \leq \nu$ set 
\[ i_j = \min\{n v_K(c_h) - h : 1 \leq h \leq n, \overline{v}_p(h) \leq j\} \]
\[ = \min\{v_L(c_h \pi_L^{n-h}) : 1 \leq h \leq n, \overline{v}_p(h) \leq j\} - n. \]
Then $i_j$ is either a nonnegative integer or $\infty$. If $\text{char}(K) = p$ then $i_j^{\pi_L}$ must be finite, since $L/K$ is separable. If $i_j^{\pi_L}$ is finite write $i_j^{\pi_L} = a_j n - b_j$ with $1 \leq b_j \leq n$. Then $v_K(c_{b_j}) = a_j, v_K(c_{h}) \geq a_j$ for all $h$ with $1 \leq h < b_j$ and $\overline{v}_p(h) \leq j$, and $v_K(c_{b_j}) \geq a_j + 1$ for all $h$ with $b_j < h \leq n$ and $\overline{v}_p(h) \leq j$.

Let $e_L = v_L(p)$ denote the absolute ramification index of $L$. We define the $j$th index of inseparability of $L/K$ to be 
\[ i_j = \min\{i_j^{\pi_L} + (j' - j) e_L : j \leq j' \leq \nu\}. \]

By Proposition 3.12 and Theorem 7.1 of [4], $i_j$ does not depend on the choice of $\pi_L$. Furthermore, our definition of $i_j$ agrees with Definition 7.3 in [4] (see also [5, Rem. 2.5]; for the characteristic-$p$ case see [2, p. 232–233] and [3, §2]).

The following facts are easy consequences of the definitions:

1. $0 = i_\nu < i_{\nu-1} \leq \cdots \leq i_1 \leq i_0 < \infty$.
2. If $\text{char}(K) = p$ then $e_L = \infty$, and hence $i_j = i_j^{\pi_L}$.
3. Let $m = \overline{v}_p(i_j)$. If $m \leq j$ then $i_j = i_m = i_m^{\pi_L} = i_m^{\pi_L}$. If $m > j$ then $\overline{v}_p(h) = \infty$ and hence $i_j^{\pi_L} = \infty$.

Lemma 4.1. Let $1 \leq h \leq n$ and set $j = \overline{v}_p(h)$. Then $v_L(c_h) \geq i_j^{\pi_L} + h$, with equality if and only if either $i_j^{\pi_L} = \infty$ or $i_j^{\pi_L} < \infty$ and $h = b_j$.

Proof. If $i_j^{\pi_L} = \infty$ then we certainly have $v_L(c_h) = \infty$. Suppose $i_j^{\pi_L} < \infty$.

If $b_j < h \leq n$ then $v_L(c_h) = n v_K(c_h) \geq n(a_j + 1)$, and hence 
\[ v_L(c_h) \geq na_j + n > na_j - b_j + h = i_j^{\pi_L} + h. \]

If $1 \leq h < b_j$ then 
\[ v_L(c_h) \geq na_j > na_j - b_j + h = i_j^{\pi_L} + h. \]

Finally, we observe that $v_L(c_{b_j}) = na_j = i_j^{\pi_L} + b_j$. \hfill $\Box$

For a partition $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ whose parts satisfy $\lambda_i \leq n$ for $1 \leq i \leq k$ define $c_\lambda = c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_k}$.

Proposition 4.2. Let $w \geq 1$ and let $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ be a partition of $w$ whose parts satisfy $\lambda_i \leq n$. Choose $q$ to minimize $\overline{v}_p(\lambda_q)$ and set $t = \overline{v}_p(\lambda_q)$. Then $v_L(c_\lambda) \geq i_t^{\pi_L} + w$. If $v_L(c_\lambda) = i_t^{\pi_L} + w$ and $i_t^{\pi_L} < \infty$ then $\lambda_q = b_t$ and $\lambda_i = b_v = n$ for all $i \neq q$. \hfill $\Box$
Proof. If \( i_t^{\pi_L} = \infty \) then \( v_L(c_{\lambda_i}) = \infty \), and hence \( v_L(c_\lambda) = \infty \). Suppose \( i_t^{\pi_L} < \infty \). By Lemma 4.1 we have \( v_L(c_{\lambda_q}) \geq i_t^{\pi_L} + \lambda_q \), and \( v_L(c_{\lambda_i}) \geq \lambda_i \) for \( i \neq q \). Hence \( v_L(c_\lambda) \geq i_t^{\pi_L} + w \), with equality if and only if \( v_L(c_{\lambda_q}) = i_t^{\pi_L} + \lambda_q \) and \( v_L(c_{\lambda_i}) = \lambda_i \) for \( i \neq q \). It follows from Lemma 4.1 that these conditions hold if and only if \( \lambda_q = b_t \) and \( \lambda_i = b_v \) for all \( i \neq q \). \( \square \)

Proposition 4.3. Let \( w \geq 1 \), let \( \mu \) be a partition of \( w \) with \( h \leq n \) parts, and set \( j = \pi_p(h) \). Let \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) be a partition of \( w \) whose parts satisfy \( \lambda_i \leq n \), choose \( q \) to minimize \( \pi_p(\lambda_q) \), and set \( t = \pi_p(\lambda_q) \). Then

1. Suppose \( v_L(d_{\lambda \mu} c_\lambda) \geq i_j + w \).

2. Suppose \( v_L(d_{\lambda \mu} c_\lambda) = i_j + w \). Then \( i_t^{\pi_L} \) is finite, \( \lambda_q = b_t \), and \( \lambda_i = n \) for all \( i \neq q \).

Proof. (1) Suppose \( t \geq j \). Then by Corollary 3.4 we have \( \pi_p(d_{\lambda \mu}) \geq t - j \). Hence by Proposition 4.2 we get

\[
 v_L(d_{\lambda \mu} c_\lambda) \geq (t - j)e_L + i_t^{\pi_L} + w \geq i_j + w.
\]

Suppose \( t < j \). Using Proposition 4.2 we get

\[
 v_L(d_{\lambda \mu} c_\lambda) \geq v_L(c_\lambda) \geq i_t^{\pi_L} + w \geq i_t + w \geq i_j + w.
\]

(2) If \( v_L(d_{\lambda \mu} c_\lambda) = i_j + w \) then all the inequalities above are equalities. In either case it follows that \( i_t^{\pi_L} \) is finite and \( v_L(c_\lambda) = i_t^{\pi_L} + w \). Therefore by Proposition 4.2 we get \( \lambda_q = b_t \) and \( \lambda_i = n \) for all \( i \neq q \). \( \square \)

We now apply the results of Section 2 to our field extension \( L/K \). For a partition \( \mu \) with at most \( n \) parts we define \( M_\mu : L \to K \) by setting \( M_\mu(\alpha) = m_\mu(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)) \) for \( \alpha \in L \).

Proposition 4.4. Let \( r \geq 1 \) and let \( \alpha \in \mathcal{P}_L^r \). Choose a power series

\[
 \phi(X) = a_r X^r + a_{r+1} X^{r+1} + \cdots
\]

with coefficients in \( \mathcal{O}_K \) such that \( \alpha = \phi(\pi_L) \). Then for \( 1 \leq h \leq n \) we have

\[
 E_h(\alpha) = \sum_\mu a_{\mu_1} a_{\mu_2} \cdots a_{\mu_h} M_\mu(\pi_L),
\]

where the sum ranges over all partitions \( \mu = \{\mu_1, \ldots, \mu_h\} \) with \( h \) parts such that \( \mu_i \geq r \) for \( 1 \leq i \leq h \).

Proof. This follows from (2.1) by setting \( X_i = \sigma_i(\pi_L) \) and taking \( a_j \in \mathcal{O}_K \). \( \square \)

Proposition 4.5. Let \( n \geq 1 \), let \( w \geq 1 \), and let \( \mu \) be a partition of \( w \) with at most \( n \) parts. Then

\[
 M_\mu(\pi_L) = \sum_\lambda d_{\lambda \mu} c_\lambda,
\]

where the sum is over all partitions \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) of \( w \) such that \( \lambda_i \leq n \) for \( 1 \leq i \leq k \).
Proof. This follows from Theorem 2.1 by setting $X_i = E_i(\pi_L) = c_i$. \qed

Let $1 \leq h \leq n$ and recall that we defined $g_h : \mathbb{Z} \to \mathbb{Z}$ by setting $g_h(r) = s$, where $s$ is the largest integer such that $E_h(\mathcal{P}_L^r) \subset \mathcal{P}_K^s$.

**Theorem 4.6.** Let $L/K$ be a totally ramified extension of degree $n = up^t$, with $p \nmid u$. Let $r \in \mathbb{Z}$, let $1 \leq h \leq n$, and set $j = \pi_p(h)$. Then

$$E_h(\mathcal{P}_L^r) \subset \mathcal{P}_K^{\lceil(ij + hr)/n\rceil}$$

$$g_h(r) \geq \left\lceil \frac{i_j + hr}{n} \right\rceil.$$

**Proof.** Let $\pi_K$ be a uniformizer for $K$. Then for $t \in \mathbb{Z}$ we have

$$E_h(\mathcal{P}_L^{nt + r}) = E_h(\pi_K^t \cdot \mathcal{P}_L^r) = \pi_K^{ht} \cdot E_h(\mathcal{P}_L^r)$$

$$\left\lceil \frac{i_j + h(nt + r)}{n} \right\rceil = ht + \left\lceil \frac{i_j + hr}{n} \right\rceil.$$

Therefore it suffices to prove the theorem in the cases with $1 \leq r \leq n$. By Proposition 4.4 each element of $E_h(\mathcal{P}_L^r)$ is an $\mathcal{O}_K$-linear combination of terms of the form $M_{\mu}(\pi_L)$, where $\mu$ is a partition with $h$ parts, all $\geq r$. Fix one such partition $\mu$ and set $w = \Sigma(\mu)$; then $w \geq hr$. Using Proposition 4.5 we can express $M_{\mu}(\pi_L)$ as a sum of terms $d_{\lambda\mu}c_{\lambda}$, where $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a partition of $w$ into parts which are $\leq n$. By Proposition 4.3(1) we get $v_L(d_{\lambda\mu}c_{\lambda}) \geq i_j + w \geq i_j + hr$. Since $d_{\lambda\mu}c_{\lambda} \in K$ it follows that $v_K(d_{\lambda\mu}c_{\lambda}) \geq \lceil(ij + hr)/n\rceil$. Therefore we have $v_K(M_{\mu}(\pi_L)) \geq \lceil(ij + hr)/n\rceil$, and hence $E_h(\mathcal{P}_L^r) \subset \mathcal{P}_K^{\lceil(ij + hr)/n\rceil}$. \qed

5. Equality

In this section we show that in some special cases we have $\mathcal{O}_K \cdot E_h(\mathcal{P}_L^r) = \mathcal{P}_K^{\lceil(ij + hr)/n\rceil}$, where $j = \pi_p(h)$. This is equivalent to showing that $g_h(r) = \lceil(ij + hr)/n\rceil$ holds in these cases. In particular, we prove that if the residue field $\mathcal{K}$ of $K$ is large enough then $g_{p^j}(r) = \lceil(ij + rp^j)/n\rceil$ for $0 \leq j \leq \nu$. To prove that $g_h(r) = \lceil(ij + hr)/n\rceil$ holds for all $r \in \mathbb{Z}$, by Theorem 4.6 it suffices to show the following: Let $r$ satisfy

$$\left\lceil \frac{i_j + hr}{n} \right\rceil < \left\lceil \frac{i_j + h(r + 1)}{n} \right\rceil.$$

Then there is $\alpha \in \mathcal{P}_L^r$ such that $v_K(E_h(\alpha)) = \lceil(ij + hr)/n\rceil$. By (4.1) and (4.2) it’s enough to prove this for $r$ such that $1 \leq r \leq n$.

Once again we let $\pi_L$ be a uniformizer for $L$ whose minimum polynomial over $K$ is

$$f(X) = X^n - c_1X^{n-1} + \cdots + (-1)^{n-1}c_{n-1}X + (-1)^nc_n.
Theorem 5.1. Let $L/K$ be a totally ramified extension of degree $n = up^\nu$, with $p \nmid u$. Let $j$ be an integer such that $0 \leq j \leq \nu$ and $\varpi_p(i_j) \geq j$. Then for all $r \in \mathbb{Z}$ we have

$$
\mathcal{O}_K \cdot E_{p^j}(\mathcal{P}_L^r) = \mathcal{P}_K^{(i_j+rp^j)/n}.
$$

$$
g_{p^j}(r) = \left\lfloor \frac{i_j + rp^j}{n} \right\rfloor.
$$

Proof. Set $m = \varpi_p(i_j)$. Then $i_j = (m - j)e_L + i_m^{\nu}$. In particular, if char$(K) = p$ then $m = j$ and $i_j = i_m = i_m^{\nu}$. We can write $i_m^{\nu} = an - b$ with $1 \leq b \leq n$ and $\varpi_p(b) = m$. Therefore it suffices to prove that $v_K(E_{p^j}(\pi_L^r)) = (m - j)e_K + a + r_1 n$. Let $r_1 \in \mathbb{Z}$ and set $r = b' + r_1 up^{\nu-j}$. Then

$$
i_j + rp^j = (m - j)e_L + an + r_1 n.
$$

Therefore we have

$$
\left\lfloor \frac{i_j + rp^j}{n} \right\rfloor = (m - j)e_K + a + r_1
$$

and

$$
\left\lfloor \frac{i_j + (r + 1)p^j}{n} \right\rfloor = (m - j)e_K + a + r_1 + 1,
$$

with $e_K = v_K(p) = e_L/n$. It follows that the only values of $r$ in the range $1 \leq r \leq n$ satisfying (5.1) are of the form $r = b' + r_1 up^{\nu-j}$ with $0 \leq r_1 < p^j$. Therefore it suffices to prove that $v_K(E_{p^j}(\pi_L^r)) = (m - j)e_K + a + r_1$ holds for these values of $r$.

Let $\mathbf{\mu}$ be the partition of $rp^j$ consisting of $p^j$ copies of $r$. Then $E_{p^j}(\pi_L^r) = M_{\mathbf{\mu}}(\pi_L)$, so it follows from Proposition 4.5 that

$$
E_{p^j}(\pi_L^r) = \sum_{\lambda} d_{\lambda \mathbf{\mu}} c_{\lambda},
$$

where the sum is over all partitions $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ of $rp^j$ such that $\lambda_i \leq n$ for $1 \leq i \leq k$. It follows from Proposition 4.3(1) that $v_L(d_{\lambda \mathbf{\mu}} c_{\lambda}) \geq i_j + rp^j$.

Suppose $v_L(d_{\lambda \mathbf{\mu}} c_{\lambda}) = i_j + rp^j$. Then by Proposition 4.3(2) we see that $\lambda$ has at most one element which is not equal to $n$. Since $\Sigma(\lambda) = rp^j = b + r_1 n$, and the elements of $\lambda$ are $\leq n$, it follows that $\lambda = \mathbf{\kappa}$, where $\mathbf{\kappa}$ is the partition of $rp^j$ which consists of 1 copy of $b$ and $r_1$ copies of $n$. Since $E_{p^j}(\pi_L^r) \in K$ and $d_{\mathbf{\kappa} \mathbf{\mu}} c_{\mathbf{\kappa}} \in K$ it follows from (5.3) and (5.2) that

$$
E_{p^j}(\pi_L^r) \equiv d_{\mathbf{\kappa} \mathbf{\mu}} c_{\mathbf{\kappa}} \pmod{\mathcal{P}_K^{(m-j)e_K+a+r_1+1}}.
$$

Let $\mathbf{\kappa}'$ be the partition of $r$ consisting of 1 copy of $b'$ and $r_1$ copies of $up^{\nu-j}$, and let $\mathbf{\mu}'$ be the partition of $r$ consisting of 1 copy of $p^j$. Then $\mathbf{\kappa}' = p^j \mathbf{\mu}'$ and $\mathbf{\mu} = p^j \ast \mathbf{\mu}'$. Since $v_p(b') = m - j$ it follows from Proposition 3.5 that $d_{\mathbf{\kappa} \mathbf{\mu}} \equiv d_{\mathbf{\kappa}' \mathbf{\mu}'} \pmod{p^{m-j+1}}$. Suppose $m < \nu$. Then $b < n$, so $b' \neq up^{\nu-j}$. Hence by Proposition 2.4 we get $d_{\mathbf{\kappa} \mathbf{\mu}'} = (-1)^{r_1+1} r_1$. Since $r = b' + r_1 up^{\nu-j}$
and \( v_p(b') = m - j \) this implies \( v_p(d_{k'\mu'}) = v_p(r) = m - j \). Suppose \( m = \nu \). Then \( b = n \) and \( b' = p^{-j}b = up^{\nu-j} \), so \( k' \) consists of \( r_1+1 \) copies of \( up^{\nu-j} \). Since \( \gcd(up^{\nu-j}, r) = up^{\nu-j} \) and \( \gcd(r_1+1, 1) = 1 \), by Proposition 2.6 we get \( d_{k'\mu'} = (-1)^{r+r_1+1}up^{\nu-j} \). Hence \( v_p(d_{k'\mu'}) = \nu - j = m - j \) holds in this case as well. Since \( d_{k\mu} \equiv d_{k'\mu'} \pmod{p^{m-j+1}} \) it follows that \( v_p(d_{k\mu}) = m - j \). Therefore

\[
v_K(d_{k\mu}c_{k'}) = v_K(d_{k\mu}) + v_K(c_b c_n^{r_1}) = (m - j)e_K + a + r_1.
\]

Using (5.4) we conclude that

\[
v_K(E_p\ell(\pi_L^{r_1})) = (m - j)e_K + a + r_1.
\]

\[\square\]

**Theorem 5.2.** Let \( L/K \) be a totally ramified extension of degree \( n = up^\nu \), with \( p \nmid u \). Let \( j \) be an integer such that \( 0 \leq j \leq \nu \) and \( v_p(i_j) < j \). Set \( m = v_p(i_j) \) and assume that \( |\overline{K}| > p^m \). Then for all \( r \in \mathbb{Z} \) we have

\[
\mathcal{O}_K \cdot E_p\ell^i(\mathcal{P}_L^r) = \mathcal{P}_K^{(i_j + rp^j)/n}
\]

\[
g_p\ell(r) = \left\lfloor \frac{i_j + rp^j}{n} \right\rfloor.
\]

**Proof.** Since \( m < j \) we have \( i_m = i_j = i_{j L} \). Therefore \( i_j = an - b \) for some \( a, b \) such that \( 1 \leq b < n \) and \( v_p(b) = m \). Hence \( b = b'p^j + b''p^m \) for some \( b', b'' \) such that \( 0 < b'' < p^{j-m} \) and \( p \nmid b' \). Let \( r_1 \in \mathbb{Z} \) and set \( r = b' + r_1up^{\nu-j} \). Then

\[
i_j + rp^j = an + r_1n - b''p^m,
\]

so we have

\[
\left\lfloor \frac{i_j + rp^j}{n} \right\rfloor = a + r_1 + \left\lfloor \frac{-b''p^m}{n} \right\rfloor = a + r_1
\]

\[
\left\lfloor \frac{i_j + (r_1 + 1)p^j}{n} \right\rfloor = a + r_1 + \left\lfloor \frac{p^j - b''p^m}{n} \right\rfloor = a + r_1 + 1.
\]

It follows that the only values of \( r \) in the range \( 1 \leq r \leq n \) satisfying (5.1) are of the form \( r = b' + r_1up^{\nu-j} \) with \( 0 \leq r_1 < p^j \). It suffices to prove that for every such \( r \) there is \( \beta \in \mathcal{O}_K \) such that \( v_K(E_p\ell(\pi_L^{r_1} + \beta\pi_L^{r_1+b''})) = a + r_1 \).

Let \( \eta(X) = E_p\ell(\pi_L^r + X\pi_L^{r_1+b''}) \). We need to show that there is \( \beta \in \mathcal{O}_K \) such that \( v_K(\eta(\beta)) = a + r_1 \). It follows from Proposition 4.4 that \( \eta(X) \) is a polynomial in \( X \) of degree at most \( p^j \), with coefficients in \( \mathcal{O}_K \). For \( 0 \leq \ell \leq p^j \) let \( \mu^\ell \) be the partition of \( rp^j + \ell b'' \) consisting of \( p^j - \ell \) copies of \( r \) and \( \ell \) copies of \( r + b'' \). By Proposition 4.4 the coefficient of \( X^\ell \) in \( \eta(X) \) is equal to \( M_{\mu^\ell}(\pi_L) \). By Proposition 4.5 we have

\[
(5.6) \quad M_{\mu^\ell}(\pi_L) = \sum_\lambda d_{\lambda\mu^\ell}c_{\lambda},
\]
where the sum is over all partitions \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) of \( rp^j + \ell b'' \) such that \( \lambda_i \leq n \) for \( 1 \leq i \leq k \). Using Proposition 4.3(1) and equation (5.5) we get

\[
v_L(d_{\lambda \mu'}c_\lambda) \geq i_j + rp^j + \ell b'' = (a + r_1)n + (\ell - p^m)b'' > (a + r_1 - 1)n.
\]

(5.7)

Since \( d_{\lambda \mu'}c_\lambda \in K \) it follows that \( d_{\lambda \mu'}c_\lambda \in \mathcal{P}_K^{a+r_1} \). Therefore by (5.6) we have \( M_{\mu'}(\pi_L) \in \mathcal{P}_K^{a+r_1} \).

Suppose \( v_K(d_{\lambda \mu'}c_\lambda) = a + r_1 \). Then \( v_L(d_{\lambda \mu'}c_\lambda) = (a + r_1)n \), so by (5.7) we get \( \ell \leq p^m \). Hence for \( p^m < \ell \leq p^j \) we have \( M_{\mu'}(\pi_L) \in \mathcal{P}_K^{a+r_1+1} \). Let \( w = b + r_1n = rp^j + b''p^m \) and let \( \mu = \mu_{p^m} \) be the partition of \( w \) consisting of \( p^m \) copies of \( r + b'' \) and \( p^j - p^m \) copies of \( r \). Then the coefficient of \( X^{p^m} \) in \( \eta(X) = M_{\mu}(\pi_L) \). Let \( \kappa \) be the partition of \( w \) consisting of 1 copy of \( b \) and \( r_1 \) copies of \( n \). Suppose \( \lambda \) is a partition of \( w \) with parts \( \leq n \) such that \( v_K(d_{\lambda \mu}c_\lambda) = a + r_1 \). Since \( (a + r_1)n = i_j + w \) it follows from Proposition 4.3(2) that \( \lambda \) has at most one element which is not equal to \( n \). Since \( \Sigma(\lambda) = b + r_1n \), and the elements of \( \lambda \) are \( \leq n \), it follows that \( \lambda = \kappa \). Hence by (5.6) we have

\[
M_{\mu}(\pi_L) \equiv d_{\kappa \mu}c_\kappa \pmod{\mathcal{P}_K^{a+r_1+1}}.
\]

(5.8)

Set \( w' = b'p^{j-m} + b'' + r_1up^{\nu-m} = rp^j + b'' \). Let \( \kappa' \) be the partition of \( w' \) consisting of 1 copy of \( b'p^{j-m} + b'' \) and \( r_1 \) copies of \( up^{\nu-m} \), and let \( \mu' \) be the partition of \( w' \) consisting of 1 copy of \( r + b'' \) and \( p^{j-m} - 1 \) copies of \( r \). Then \( \kappa = p^m \cdot \kappa' \) and \( \mu = p^m \cdot \mu' \), so by Proposition 3.5 we have \( d_{\kappa \mu} \equiv d_{\kappa' \mu'} \pmod{p} \).

Let \( \Gamma \) be a cycle digraph which has an admissible \((\kappa', \mu')\)-tiling. Suppose \( \Gamma \) has more than one component. Since \( \Gamma \) has a \( \kappa' \)-tiling, \( \Gamma \) has at least one component \( \Gamma_0 \) such that \( |V(\Gamma_0)| = k \cdot up^{j-m} \) for some \( k \) such that \( 1 \leq k \leq r_1 \). Let \( \kappa'_0 \) be the submultiset of \( \kappa' \) consisting of \( k \) copies of \( up^{\nu-m} \). Then \( \kappa'_0 \) is the unique submultiset of \( \kappa' \) such that \( \Gamma_0 \) has a \( \kappa'_0 \)-tiling. Furthermore there is a submultiset \( \mu'_0 \) of \( \mu' \) such that \( \Gamma_0 \) has a \( \mu'_0 \)-tiling. We will see below that \( \mu'_0 \) is uniquely determined.

Suppose \( r \) does not divide \( kup^{\nu-m} \). Then there is \( \ell \geq 0 \) such that \( \mu'_0 \) consists of \( \ell \) copies of \( r + b'' \) together with \( \ell \) copies of \( r \). By Proposition 2.3 we have \( \eta_{\kappa'_0 \mu'_0}(\Gamma_0) = up^{\nu-m} \). Let \( \Gamma_1 \) be the complement of \( \Gamma_0 \) in \( \Gamma \), let \( \kappa'_1 = \kappa' \setminus \kappa'_0 \), and let \( \mu'_1 = \mu' \setminus \mu'_0 \). Since \( \Gamma_1 \) has no cycle of length \( |V(\Gamma_0)| = b'' + (\ell + 1)r \) we have \( \eta_{\kappa' \mu'}(\Gamma) = \eta_{\kappa'_1 \mu'(\Gamma_0)} \eta_{\kappa'_0 \mu'_0}(\Gamma_1) \). Hence \( \eta_{\kappa' \mu'}(\Gamma) \) is divisible by \( p \) in this case.

On the other hand, suppose \( r \) divides \( kup^{\nu-m} \). If \( r \) also divides \( r + b'' \) then \( p \nmid r \), so \( r \mid ku \). It follows that \( r_1up^{\nu-j} + b' = r < ku \leq r_1u \), a contradiction. Hence there is \( \ell \geq 1 \) such that \( \mu'_0 \) consists of \( \ell \) copies of \( r \). Let \( (S, T) \) be an
admissible \((\kappa', \mu')\)-tiling of \(\Gamma\) and let \((S_0, T_0)\) be the restriction of \((S, T)\) to \(\Gamma_0\). Then \((S_0, T_0)\) is a \((\kappa'_0, \mu'_0)\)-tiling of \(\Gamma_0\). By Proposition 2.5(2) the automorphism group of \((\Gamma_0, S_0, T_0)\) has order \(\gcd(k, \ell)\). Since \(\text{Aut}(\Gamma_0, S_0, T_0)\) is isomorphic to a subgroup of \(\text{Aut}(\Gamma, S, T)\), it follows that \(\gcd(k, \ell)\) divides \(|\text{Aut}(\Gamma, S, T)|\). Therefore the assumption that \((S, T)\) is admissible implies that \(\gcd(k, \ell) = 1\). Since \(k \cdot \upsilon^{r-m} = \ell \cdot r\) we get \(k \mid r \) and \(\ell \mid \upsilon^{r-m}\). It follows that there is \(q \in \mathbb{Z}\) with \(r = kq\) and \(\upsilon^{r-m} = \ell q\). By Proposition 2.5(1) the number of isomorphism classes of \((\kappa'_0, \mu'_0)\)-tilings of \(\Gamma_0\) is 
\[
\eta_{\kappa'_0, \mu'_0}(\Gamma_0) = \gcd(\upsilon^{r-m}, r) = \gcd(\ell q, kq) = q.
\]
If \(p \mid q\) then as above we deduce that \(\eta_{\kappa', \mu'}(\Gamma)\) is divisible by \(p\). On the other hand, if \(p \nmid q\) then \(q \mid u\); in particular, \(q \leq u\). Since \(k \leq r_1\) this gives the contradiction \(r = kq \leq r_1 u\). By combining the two cases we find that if \(\Gamma\) has more than one component then \(\eta_{\kappa', \mu'}(\Gamma)\) is divisible by \(p\).

Finally, suppose that \(\Gamma\) consists of a single cycle of length \(w'\). Then by Proposition 2.2 we have \(\eta_{\kappa', \mu'}(\Gamma) = w'\). Hence by (2.2) we get 
\[
d_{\kappa, \mu} \equiv d_{\kappa', \mu'} \equiv \pm \eta_{\kappa', \mu'}(\Gamma) \equiv \pm w' \pmod{p}.
\]
Since \(w' \equiv b'' \pmod{p}\) it follows that \(p \nmid d_{\kappa, \mu}\). Hence by (5.8) we get 
\[
v_K(M_{\mu}(\pi_L)) = v_K(c_{\kappa}) = a + r_1.
\]
Let \(\pi_K\) be a uniformizer for \(K\) and set \(\phi(X) = \pi_K^{a-r_1} \eta(X)\). Then \(\phi(X) \in \mathcal{O}_L[X]\). Let \(\overline{\phi}(X)\) be the image of \(\phi(X)\) in \(\overline{K} [X]\). We have shown that \(\overline{\phi}(X)\) has degree \(p^m\). Since \(|\overline{K}| > p^m\) there is \(\overline{\beta} \in \overline{K}\) such that \(\overline{\phi}(\overline{\beta}) \neq 0\). Let \(\beta \in \mathcal{O}_K\) be a lifting of \(\overline{\beta}\). Then \(\phi(\beta) \in \mathcal{O}_K^\times\). It follows that 
\[
v_K(E_{p^j}(\pi_L^r + \beta \pi_L^{r+b''})) = v_K(\eta(\beta)) = a + r_1.
\]
Hence if \(r = b' + r_1 \upsilon^{r-j}\) with \(0 \leq r_1 < p^j\) then 
\[
\mathcal{O}_K \cdot E_{p^j}(\mathcal{P}_L^r) = \mathcal{P}_K^{a+r_1} = \mathcal{P}_K^{[i_j+r p^j]/n}.
\]
We conclude that this formula holds for all \(r \in \mathbb{Z}\). \(\square\)

**Remark 5.3.** Theorems 5.1 and 5.2 together imply that if \(\overline{K}\) is sufficiently large then \(g_{p^j}(r) = \lceil (i_j + r p^j)/n \rceil\) for \(0 \leq j \leq \nu\). This holds for instance if \(|\overline{K}| \geq p^\nu\).

**Remark 5.4.** Let \(L/K\) be a totally ramified separable extension of degree \(n = \upsilon^\nu\). The different \(\mathcal{P}_L^{d_0}\) of \(L/K\) is defined by letting \(d_0\) be the largest integer such that \(E_1(\mathcal{P}_L^{-d_0}) \subset \mathcal{O}_K\). For \(1 \leq j \leq \nu\) one can define higher order analogs \(\mathcal{P}_L^{d_j}\) of the different by letting \(d_j\) be the largest integer such that \(E_{p^j}(\mathcal{P}_L^{-d_j}) \subset \mathcal{O}_K\). An argument similar to the proof of Proposition 1.2 shows that 
\[
\mathcal{O}_K \cdot E_{p^j}(\mathcal{P}_L^r) = \mathcal{P}_K^{[p^j(d_j+r)/n]}.
\]
This generalizes Proposition 1.2, which is equivalent to the case \( j = 0 \) of this formula. By Proposition 3.18 of [4], the valuation of the different of \( L/K \) is \( d_0 = i_0 + n - 1 \). Using Theorems 5.1 and 5.2 we find that, if \( K \) is sufficiently large, \( d_j \) is the largest integer such that \( [(i_j - d_j p^j)/n] \geq 0 \). Hence \( d_j = [(i_j + n - 1)/p^j] \) for \( 0 \leq j \leq \nu \).

**Example 5.5.** Let \( K = \mathbb{F}_2((t)) \) and let \( L \) be an extension of \( K \) generated by a root \( \pi_L \) of the Eisenstein polynomial \( f(X) = X^8 + tX^3 + tX^2 + t \). Then the indices of inseparability of \( L/K \) are \( i_0 = 3, i_1 = i_2 = 2 \), and \( i_3 = 0 \). Since \( [(i_2 + 2^2 \cdot 1)/2^3] = 1 \), the formula in Theorem 5.2 would imply \( \mathcal{O}_K \cdot E_4(P_L^1) = P_L^1 \). We claim that \( E_4(P_L) \subset P_K^2 \).

Let \( \alpha \in P_L \) and write \( \alpha = a_1 \pi_L + a_2 \pi_L^2 + \cdots \), with \( a_i \in \mathbb{F}_2 \). It follows from Propositions 4.4 and 4.5 that \( E_4(\alpha) \) is a sum of terms of the form \( a_\mu d_\lambda c_\lambda \), where \( \lambda \) is a partition whose parts are \( \leq 8 \) and \( \mu \) is a partition with \( 4 \) parts such that \( \Sigma(\lambda) = \Sigma(\mu) \). We are interested only in those terms with \( K \)-valuation \( 1 \). We have \( v_K(c_\lambda) \geq 2 \) unless \( \lambda \) is one of \( \{5\}, \{6\}, \) or \( \{8\} \). If \( \lambda = \{8\} \) then \( 2 \mid d_\lambda \mu \) for any \( \mu \) by Corollary 3.4. If \( \lambda = \{6\} \) and \( \mu = \{1, 1, 1, 3\} \) then \( d_\lambda \mu = 6 \) by Proposition 2.4. If \( \lambda = \{6\} \) and \( \mu = \{1, 1, 2, 2\} \) then a computation based on (2.2) shows that \( d_\lambda \mu = 9 \). If \( \lambda = \{5\} \) and \( \mu = \{1, 1, 1, 2\} \) then \( d_\lambda \mu = -5 \) by Proposition 2.4. Combining these facts we get

\[
E_4(\alpha) \equiv a_1^3 a_2 t + a_1^2 a_2^2 t \pmod{P_K^2}.
\]

Since \( a_1, a_2 \in \mathbb{F}_2 \) we have \( a_1^3 a_2 + a_1^2 a_2^2 = 0 \). Therefore \( E_4(\alpha) \in P_K^2 \). Since this holds for every \( \alpha \in P_L \), we get \( E_4(P_L) \subset P_K^2 \). This shows that Theorem 5.2 does not hold without the assumption about the size of \( K \).

The following result shows that \( g_h(r) = [(i_j + hr)/n] \) does not hold in general, even if we assume that the residue field of \( K \) is large. It also suggests that there may not be a simple criterion for determining when \( g_h(r) = [(i_j + hr)/n] \) does hold.

**Proposition 5.6.** Let \( L/K \) be a totally ramified extension of degree \( n \), with \( p \nmid n \). Let \( r \in \mathbb{Z} \) and \( 1 \leq h \leq n \) be such that \( n \mid hr \). Set \( s = hr/n, u = \text{gcd}(r, n), \) and \( v = \text{gcd}(h, s) \). Then \( g_h(r) = [(i_0 + hr)/n] = s \) if and only if \( p \) does not divide the binomial coefficient \( \binom{v}{u} \). In particular, if \( u < v \) then \( g_h(r) > s \).

**Proof.** Since \( L/K \) is tamely ramified we have \( \nu = 0, i_0 = 0, \) and

\[
\left\lfloor \frac{i_0 + hr}{n} \right\rfloor = \left\lfloor \frac{hr}{n} \right\rfloor = s.
\]

It follows from Theorem 4.6 that \( g_h(r) \geq s \). If \( r' = nt + r \) then \( s' = hr'/n = ht + s, u' = \text{gcd}(r', n) = u, \) and \( v' = \text{gcd}(h, s') = v \). Hence by (4.1) it suffices to prove the proposition in the cases with \( 1 \leq r \leq n \).
Suppose $p$ does not divide $\binom{v}{u}$. To prove $g_h(r) = s$ it suffices to show that $v_K(E_h(\pi_L^r)) = s$. Let $\mu$ be the partition of $hr$ consisting of $h$ copies of $r$. Then $E_h(\pi_L^r) = M_\mu(\pi_L)$, so it follows from Proposition 4.5 that

$$E_h(\pi_L^r) = \sum_{\lambda} d_{\lambda\mu} c_\lambda,$$

(5.9)

where the sum is over all partitions $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ of $hr$ such that $\lambda_i \leq n$ for $1 \leq i \leq k$. Let $\kappa$ be the partition of $hr$ consisting of $s$ copies of $n$ and let $\lambda$ be a partition of $hr$ whose parts are $\leq n$. Then by Proposition 4.3(1) we have $v_L(d_{\kappa\mu} c_\lambda) \geq hr = sn$. Furthermore, if $v_L(d_{\kappa\mu} c_\lambda) = hr$ then by Proposition 4.3(2) we have $\lambda = \kappa$. Hence by (5.9) we get

$$E_h(\pi_L^r) \equiv d_{\kappa\mu} c_\kappa \quad (\text{mod } \mathcal{P}_K^{s+1}).$$

By Proposition 2.6 we have $d_{\kappa\mu} = \pm \binom{u}{v}$. Since $p \nmid \binom{u}{v}$ and $v_K(c_\kappa) = s$ it follows that $v_K(E_h(\pi_L^r)) = s$. Therefore $g_h(r) = s$.

Suppose $p$ divides $\binom{v}{u}$. By Proposition 4.4, each element of $E_h(\mathcal{P}^r_L)$ is an $\mathcal{O}_K$-linear combination of terms of the form $M_\nu(\pi_L)$ where $\nu$ is a partition with $h$ parts, all $\geq r$. Fix one such partition $\nu$ and set $w = \Sigma(\nu)$; then $w \geq hr = sn$. By Proposition 4.5 we can express $M_\nu(\pi_L)$ as a sum of terms of the form $d_{\lambda\nu} c_\lambda$, where $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a partition of $w$ into parts which are $\leq n$. By Proposition 4.3(1) we have $v_L(d_{\lambda\nu} c_\lambda) \geq w \geq sn$. Suppose $v_L(d_{\lambda\nu} c_\lambda) = sn$. Then $w = sn$, and by Proposition 4.3(2) we see that $\lambda$ consists of $k$ copies of $n$. It follows that $kn = w = sn$, and hence that $k = s$. Therefore $\lambda = \kappa$. Since $\Sigma(\nu) = w = sn = hr$ we get $\nu = \mu$. Since $d_{\lambda\mu} = \pm \binom{u}{v}$ and $p$ divides $\binom{u}{v}$ we have $v_L(d_{\lambda\mu} c_\lambda) > v_L(c_\kappa) = sn$, a contradiction. Hence $v_L(d_{\lambda\mu} c_\lambda) > sn$ holds in all cases. Since $d_{\lambda\mu} c_\lambda \in K$ we get $v_K(d_{\lambda\mu} c_\lambda) \geq s + 1$. It follows that $E_h(\mathcal{P}^r_L) \subset \mathcal{P}_K^{s+1}$, and hence that $g_h(r) \geq s + 1$.

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