GUP in presence of extra dimensions and lifetime of mini black holes

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Based on the general considerations of quantum mechanics and gravity the generalized uncertainty principle (GUP) is determined in higher-dimensional case and on the brane, respectively. The result is used to evaluate the effect of GUP on the dynamics of evaporation and lifetime of mini black holes in the brane-world models.

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I. GUP IN FOUR DIMENSIONS

For a quite long time since the appearance of Heisenberg uncertainty relation in 1923 the gravitational interaction between photon and the particle being observed has not been considered as it is usually assumed to be negligible. But at increasingly large energies this interaction becomes more and more important. The main conceptual point concerning GUP is that there is an additional uncertainty in quantum measurement due to gravitational interaction. We focus on different considerations of this problem presented in [1] and [2]. Both of these considerations rely on classical gravitational theory. Let us give here a brief critical review of GUP that comes from the general arguments of quantum mechanics and gravity. We set $\hbar = c = k_B = 1$ in what follows. The idea based on micro-black hole gedanken experiment goes as follows [1]. As it was derived by Heisenberg the uncertainty in the position of the electron when it interacts with the photon is given by $\Delta x \Delta p \geq \frac{1}{2}$. By taking in the high energy situation $\Delta E \approx \Delta p$ the Heisenberg relation takes the form $\Delta E \Delta x \geq \frac{1}{2}$. So that to measure the position of electron more precisely the energy of photon should be increased. But this procedure is limited because there is the gravitational radius associated with $\Delta E$, $r_g = 2G_4 \Delta E$, for which the $2r_g$ becomes greater than $\Delta x$ for $\Delta E > 0.35 G_4^{1/2}$ and therefore the region where $\Delta E$ is located becomes hidden by the event horizon. ($G_4$ is the four-dimensional Newton’s constant). Based on this discussion the combination of the above inequalities gives

$$\Delta x \geq \begin{cases} 1/2\Delta E & \text{if } \Delta E \leq 0.35G_4^{-1/2} \\ 4G_4\Delta E & \text{if } \Delta E > 0.35G_4^{-1/2} \end{cases} . \quad (1)$$

The Eq. (1) implies a minimal attainable uncertainty in position $\Delta x_{min} = 1.43G_4^{1/2}$. While in ordinary quantum mechanics $\Delta x$ can be made arbitrarily small by letting $\Delta E$ grow correspondingly. Combining the Eq. (1) into a single one in the linear way one gets the expression similar to what was obtained previously in the string theory framework |

$$\Delta x \geq \frac{1}{2\Delta E} + 4G_4 \Delta E . \quad (2)$$

The approach proposed in [2] is to calculate the displacement of electron caused by the gravitational interaction with the photon and add it to the position uncertainty. The photon due to gravitational interaction imparts to electron the acceleration given by $a = G_4 \Delta E/r^2$. Assuming $r_0$ is the size of the interaction region the variation of the velocity of the electron is given by $\Delta v \sim G_4 \Delta E/r_0$ and correspondingly $\Delta x_g \sim G_4 \Delta E$. Therefore the total uncertainty in the position is given by

$$\Delta x \geq \frac{1}{2\Delta E} + G_4 \Delta E . \quad (3)$$

As one sees the mechanisms proposed in [1] and [2] giving rise to the gravitational uncertainty are quite distinct though they provide qualitatively the same gravitational uncertainty term which in general has to be taken with some numerical factor of order unity [2]. At a first glance they complement one another in that for energies $\Delta E \leq 0.4G_4^{1/2}$ the GUP is given by Eq. (2) while for $\Delta E > 0.4G_4^{1/2}$ one can take the second mechanism Eq. (3) as it was proposed in [1]. But to be more precise the collapse of $\Delta E$ puts simply the limitation on the measurement procedure. So in principle it is no longer conceivable for $\Delta E > 0.4G_4^{1/2}$ to proceed the measurement. The minimum position uncertainty then becomes $\Delta x_{min} = 1.63G_4^{1/2}$.

II. GUP IN HIGHER DIMENSIONAL CASE

Let us make a straightforward generalization of the ideas given in the preceding section to a higher dimensional case. The $D = 4 + n$ dimensional Black hole has the form |

$$ds^2 = -h(r) dt^2 + \frac{dr^2}{h(r)} + r^2d\Omega_{2+n}^2 , \quad (4)$$

where
TABLE I: The region where the collapse of $\Delta E$ occurs is given by $(\Delta E_1$, $\Delta E_2)$. "Length" denotes the minimal position uncertainty and "Mass" stands for the minimal black hole mass that follows from the minimal observable length. All quantities given here are written in the Planck units.

| n  | $\Delta E_1$ | $\Delta E_2$ | Length | Mass |
|----|--------------|--------------|--------|------|
| 2  | 0.715307    | 162.088      | 0.699001 | 0.089433 |
| 3  | 0.719459    | 68.1812      | 0.694967 | 0.049661 |
| 4  | 0.705562    | 43.7701      | 0.708655 | 0.020487 |
| 5  | 0.684841    | 33.3681      | 0.700906 | 0.010706 |
| 6  | 0.662058    | 27.758221    | 0.755221 | 0.0051723 |
| 7  | 0.639293    | 24.290000    | 0.782114 | 0.0024972 |

Equation (6) predicts the minimum position uncertainty where

$$\Delta x \geq \frac{1}{2\Delta E} + 2\left(\frac{16\pi G_D \Gamma[(n + 3)/2] \Delta E}{(n + 2) 2\pi^{(n+3)/2}}\right)^{1/(1+n)}.$$  

Thus, in higher dimensional case the two approaches [4, 5] give qualitatively different answers in contrast to the four-dimensional one. One can see that the careful consideration gives the higher-dimensional GUP different from that one obtained previously in [5] where the Eq. (8) was assumed for all values of $\Delta E$.

### III. GUP ON THE BRANE

Let us not to go into details of the extra-dimensional models [5] but merely recall a few common features relevant for our consideration. The common features are a low fundamental Planck scale $m_p \sim$ TeV, the localization of the standard model particles on the brane and propagation of gravity throughout the higher dimensional space. There is a length scale $L$, much greater than TeV$^{-1}$, beneath of which the gravitational interaction has the form Eq. (5) while beyond this scale we have the standard four-dimensional law. In what follows we restrict ourselves to the ADD model.

From the above discussion one gets the following expression for GUP

$$\Delta x \geq \begin{cases} 
\frac{1}{2\Delta E} + \beta G_D \Delta E^{1+n} & \text{if } \Delta E^{-1} \leq L \\
\frac{1}{2\Delta E} + \alpha G_D \Delta E & \text{if } \Delta E^{-1} > L 
\end{cases}.$$  

where $\alpha$ is of order unity [6] and $\beta$ has to be determined by the junction of these inequalities at $\Delta E = L^{-1}$. In this way one gets

$$\beta = \alpha \frac{G_4}{G_D} L^n.$$  

As it was mentioned in the first section, the GUP obtained in the framework of string theory coincides with a model-independent derivation based on the general considerations of quantum mechanics and gravity. However, as it is shown in section two, the latter approach gives the higher-dimensional GUP different from that one obtained in four-dimensional case and GUP on the brane incorporates both of them Eq. (6). For the ADD model the size and number of extra dimensions are related as follows $L \sim 10^{-17+30/n}$ cm. So in this case one finds $\beta \sim 10^{-4} - n$ and $\Delta x_{min} \sim 10^{(4+n)/(2+n)} l_p$ as it follows from Eq. (7). However, the minimal observable distance is determined rather due to collapse of $\Delta E$ than merely by the Eq. (6). It is natural because the collapse of $\Delta E$
where \( \sigma \) and \( n \)

FIG. 1: ADD \( n = 2 \) model: The mass of the black hole versus time in Planck units.

puts simply the bound on the measurement procedure.

In this way for minimal observable lengths and the black hole remnant masses one finds the values represented in Table I.

IV. BRANE-LOCALIZED BLACK HOLE EMISSION

It is widely believed that if the fundamental scale of gravity actually lies in the TeV range a very spectacular prediction can be made about the creation of small black holes at colliders or in high-energy cosmic ray interactions \([11, 12]\). The size of black hole produced in this way is typically assumed to be much smaller than the characteristic length of extra dimensions. Then it seems reasonable to describe such objects by higher dimensional Schwarzschild solution Eq. (4).

Once produced, the black hole would decay very rapidly to a spectrum of particles by Hawking radiation. For mass shedding formula in \( D \) dimensional spacetime, under assumption that the black hole emits mainly the massless particles we have the following expression \([11, 12]\):

\[
\frac{dM}{dt} \approx -7.4931 r_D^2 T^4 - g_{\text{bulk}} \sigma_D A_D T^D, \tag{11}
\]

where

\[
r_D = \left( \frac{\mu}{2} \right)^{\frac{1}{D-3}} \left( \frac{D-1}{D-3} \right)^{\frac{1}{2}} A_D = \Omega_D^{-2} r_D^{p_D-2},
\]

and \( \sigma_D \) denotes the \( D \) dimensional Stefan-Boltzmann constant

\[
\sigma_D = \frac{\Omega_D^{-2}}{4(2\pi)^{D-1}} \Gamma(D) \zeta(D).
\]

Since only gravity is allowed to propagate in the bulk, \( g_{\text{bulk}} \) simply counts the number of polarization states of the graviton, namely

\[
g_{\text{bulk}} = \frac{D(D-3)}{2}.
\]

In the framework of heuristic approach \([6, 8]\) to the Hawking radiation the black hole is envisioned as a cube with size two times the gravitational radius, inner space of which is inaccessible for outer observable, and the characteristic energy of the emitted particles is estimated with the use of GUP. Then the lower value of energy obtained in this way is identified with the radiation temperature

\[
T = \frac{D-3}{\pi} \Delta E.
\]

The main observation achieved in the framework of this approach is that at the Planck scale black hole ceases to radiate, even though its temperature reaches a maximum. It cannot radiate further and becomes an inert remnant of about Planck mass.

The problem of brane-localized mini black hole evaporation was studied in the framework of stringy induced GUP in \([9]\). Here we focus on the GUP derived in the preceding section. From the expression of GUP derived in section three one sees that its effect on the Hawking temperature becomes negligibly small for the ADD model. The Figure 1 shows the mass decay of the black hole with initial mass \( M = 10m_p \) for ADD model with \( n = 2 \). The corresponding lifetime is estimated as \( \sim 156t_p \). Without GUP one obtains practically the same result.

V. CONCLUSION

Following to the papers \([1, 2]\) we have defined the GUP on the brane which is further used for evaluating of evaporation of brane-localized mini black holes in the framework of heuristic approach given in \([7]\). As it is shown for ADD model the GUP effect on the black hole evaporation is strongly suppressed because of small \( \beta \) factor. We have also evaluated the masses of the black hole remnants the existence of which increase the lower cutoff for the black hole production reducing therefore the rate of corresponding events.

In regard with the GUP approach to the black hole evaporation a few remarks are in order. In general, GUP assumes two values of \( \Delta E \) for a given \( \Delta x \). In four-dimensional case the situation may be somewhat simplified by omitting the branch \( \Delta E > 0.4G_4^{-1/2} \) in GUP as it corresponds to the collapse of \( \Delta E \) (see section one). In this regard the situation in presence of extra dimensions is more complicated because the region of collapse of \( \Delta E \), given by \( (\Delta E_1, \Delta E_2) \), is bounded. Only the lower value of \( \Delta E \) is applicable to the black hole radiation for it gives the correct asymptotic dependence of the Hawking temperature on the black hole mass. The question naturally
arises is why the nature selects the lower one in the case of black hole emission whereas in the framework of GUP both of the solutions have the same right of existence. So we face the challenge of understanding what is the status of that part of the GUP corresponding to the higher values of \(\Delta E\). Does the nature uses the higher value of \(\Delta E\) instead of the lower one in some cases? If so, what is the intrinsic principle that selects which solution of GUP should be used for a given process? Also it is necessary to know what are the properties of remnants left behind the black hole evaporation in the GUP approach. Another important point is the quantum gravity effects since GUP takes into account the gravitational interactions at the Planck scale [13]. Moreover, the black hole remnant may be rather due to quantum gravitational effects (the nature of which is well established in this case [13]) than the GUP [14]. (Certainly, a big obstacle so far is the lack of a direct experimental hint that there is a need for a quantum theory of gravity).

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