A Uniformisation of Weighted Maps on Compact Surfaces

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This article is a comprised version of my doctoral thesis completed under the supervision of Norbert A’Campo at Basel University in 1995. For various reasons, I never published the thesis up to now. After all with a delay of two-and-a-half decades, I decided to catch up on it.

Abstract

In 1970 Andreev [An] proved a theorem concerning the existence of polyhedra with preassigned dihedral angles in the hyperbolic 3-space $\mathbb{H}^3$. Thurston [Tu] reinterpreted this theorem in terms of patterns of disks on the 2-sphere and he observed the existence of disk patterns with preassigned overlap angles not exceeding $\pi/2$ on any compact surface $X$. These disk patterns can be interpreted as convex subsets of $\mathbb{H}^3$ which are invariant under a group action $\pi_1(X) \times \mathbb{H}^3 \rightarrow \mathbb{H}^3$. In this paper we prove the existence and uniqueness of disk patterns on compact surfaces with preassigned angles in $]0, \pi[$, provided that the system of preassigned angles fulfill an additional condition. In terms of the corresponding convex subset of $\mathbb{H}^3$ this condition states that the extreme points lie on the sphere at infinity. We prove the existence and uniqueness of disk patterns by a refinement of a variational method introduced in [Br]. In this process we will characterize a disk pattern as a critical point of a functional. Furthermore, it will turn out that its critical value is the volume of a fundamental domain of the corresponding convex subset of $\mathbb{H}^3$. 

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## Introduction

Let $T$ be a cell decomposition of a compact surface $X$ without boundary, $E(T)$ its set of edges and $\theta : E(T) \rightarrow ]0, \pi[$ a function. Roughly speaking, we study the existence and uniqueness of a Riemannian metric of constant curvature on $X$ and of a collection of disks such that
- the centers of the disks are the vertices of $T$,
- two disks whose centers are joined by an edge $e$ meet in an angle $\theta(e)$,
- if the centers of some disks form the vertices of a cell of $T$, then the bounding circles intersect in a point.

For our purpose it is more convenient to consider such collections of disks from a slightly different viewpoint (Chapter I): Let $H^3$ be the hyperbolic 3-space and $\partial H^3$
its boundary. The metric structure of $\mathbb{H}^3$ endows $\partial \mathbb{H}^3$ with a natural conformal structure such that the isometries of $\mathbb{H}^3$ correspond to the conformal transformations of $\partial \mathbb{H}^3$. We call an open subset $\mathcal{O}$ of $\partial \mathbb{H}^3$ a homogeneous domain if its stabilizer $(\text{Iso} \mathbb{H}^3)_\mathcal{O}$ in the isometry group of $\mathbb{H}^3$ acts transitively on $\mathcal{O}$. Let $\tilde{X}$ be a simply connected homogeneous domain and $\pi_1(X)$ a discrete subgroup of $(\text{Iso} \mathbb{H}^3)_{\tilde{X}}$ such that $\tilde{X}/\pi_1(X)$ is homeomorphic to $X$. Furthermore, let $P$ be a $\pi_1(X)$-invariant subset of $\tilde{X}$ such that $P/\pi_1(X)$ is finite. We denote the convex hull of the set $P$ in $\mathbb{H}^3$ by $\lvert P \rvert$. If $\dim \lvert P \rvert = 3$ we call this convex set an ideal $\pi_1(X)$-polyhedron. Every ideal $\pi_1(X)$-polyhedron has an ‘external representation’ as the intersection of the halfspaces supporting the facets of $\lvert P \rvert$. Since there is a canonical bijection between the set of halfspaces of $\mathbb{H}^3$ and the set of conformal disks in $\partial \mathbb{H}^3$, this external representation yields a $\pi_1(X)$-periodic collection of conformal disks in $\tilde{X}$. Projecting these disks to $\tilde{X}/\pi_1(X)$, we get a finite collection of ‘disks’ in $\tilde{X}/\pi_1(X)$. The combinatorics of this projected disk collection is described by a cell decomposition of $X$ such that each vertex corresponds to a disk and each cell corresponds to a point of the set $P/\pi_1(X)$. The dihedral angles of the $\pi_1(X)$-polyhedron can be considered as weights on the edges of the cell decomposition.

Provided that $\theta$ fulfills a necessary condition, we prove the existence of an ideal $\pi_1(X)$-polyhedron whose combinatorics correspond to $T$ and whose dihedral angles are described by the weight function $\theta$ (Chapters 2 - 4). This ideal $\pi_1(X)$-polyhedron is unique up to isometries of $\mathbb{H}^3$ and yields a simply connected homogeneous domain $\tilde{X}$ in such a way that the conformally flat surface $\tilde{X}/\pi_1(X)$ is uniquely determined by $(T, \theta)$.

In Chapter 5 finally, we express the volume of a fundamental domain of a $\pi_1(X)$-polyhedron in terms of the corresponding disk collection.

1 Preliminaries

1.1 A Model for hyperbolic 3-space $\mathbb{H}^3$.

A sphere $S$ in $\mathbb{H}^3$ endowed with the induced Riemannian metric is a surface of positive constant curvature. With geodesic rays perpendicular to $S$, we can export this metric to the boundary $\partial \mathbb{H}^3$. Two such metrics on $\partial \mathbb{H}^3$ are similar if and only if the corresponding spheres have the same center. Hence, we get a model characterizing the points of $\mathbb{H}^3$ as similarity classes of Riemannian metrics. In this section we will briefly show how to express the attributes and objects of $\mathbb{H}^3$ in this model.

Let $S^2$ be the standard conformal 2-sphere and conf$S^2$ its group of conformal automorphisms, i.e. $S^2$ is conformally equivalent to $\mathbb{P}^1 \mathbb{C}$ and the subgroup of orientation preserving elements of conf$S^2$ is isomorphic to $\text{PSL}(2, \mathbb{C})$. A topological circle in $S^2$ is called a conformal circle if it is the set of fixed points of an orientation reversing involution in conf$S^2$. A connected component of the complement of
a conformal circle is called a (conformal) open disk.

We call a non-empty connected open subset $\mathcal{O}$ of $S^2$ a homogeneous domain if its stabilizer $(\text{conf} S^2)_0$ in $\text{conf} S^2$ acts transitively on $\mathcal{O}$. There are three types of simply connected homogeneous domains. Namely, any such domain $\mathcal{O}$ has the form $\mathcal{O} = S^2 \setminus A$, where $A$ is either empty, a point or a closed conformal disk. Accordingly, $\mathcal{O}$ equipped with the conformal structure induced by $S^2$ is conformally equivalent to $S^2$, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$. The group of conformal automorphisms of $\mathcal{O}$ is just $(\text{conf} S^2)_0$.

For every simply connected homogeneous domain which is a proper subset of $S^2$ there is only one similarity class of complete Riemannian metrics of constant curvature having the conformal structure induced by $S^2$. We will consider these metrics as ‘degenerated’ metrics on $S^2$. This leads to the following definition: An inner product structure $n$ on $S^2$ (i.e. an inner product on every fiber of the tangent bundle) is called a singular (Riemannian) metric if there exists a simply connected homogeneous domain $\text{reg}(n)$ of $S^2$ such that the following two conditions are satisfied:

- The restriction of $n$ to $\text{reg}(n)$ is a complete Riemannian metric of constant curvature (called the curvature of $n$) having the conformal structure induced by $S^2$.
- For every $q \in S^2 \setminus \text{reg}(n)$ the inner product of any two tangent vectors at $q$ is $+\infty$.

We call $\text{reg}(n)$ the regular domain of $n$ and $S^2 \setminus \text{reg}(n)$ the singular domain. In the following we consider similar singular metrics (i.e. similar on their regular domains) as equal and we denote the set of similarity classes of singular metrics by $M_{\text{sing}}$. We identify every singular metric $m \in M_{\text{sing}}$ with a representative of constant curvature $-1$, 0 or $+1$ and we define the metric space $(\text{reg}(m), m)$ as the set $\text{reg}(m)$ equipped with the metric $m$. The similarities of $(\text{reg}(m), m)$ (i.e. isometries if the curvature is non-zero) are the elements of the stabilizer $(\text{conf} S^2)_m$.

Since $\text{conf} S^2$ maps homogeneous domains to homogeneous domains, $\text{conf} S^2$ acts on $M_{\text{sing}}$. Corresponding to the type of the regular domain of a singular metric, we divide $M_{\text{sing}}$ into three disjoint subsets: We define $M_{\text{reg}}$, $\partial M_{\text{reg}}$, $M_{\text{disk}}$ to be the set of all singular metrics whose singular domain is empty (respectively, a point of $S^2$, a disk in $S^2$). Note that these three sets are just the orbits of the group action $\text{conf} S^2 \times M_{\text{sing}} \to M_{\text{sing}}$. We call the elements of $M_{\text{reg}}$ (respectively, $M_{\text{disk}}$) regular metrics (respectively, disk metrics). We already mentioned that a metric $m \in M_{\text{sing}} \setminus M_{\text{reg}}$ is determined by its regular domain. Henceforth we will no more distinguish between a metric $m \in M_{\text{disk}}$ and the open disk $\text{reg}(m)$. In the same sense we identify a metric $m \in \partial M_{\text{reg}}$ with its singular domain. This yields a bijection between $S^2$ and $\partial M_{\text{reg}}$.

Our next task is to equip $M_{\text{sing}}$ with a topology: A sequence $m^i \in M_{\text{sing}}$ converges if there exists a singular metric $m \in M_{\text{sing}}$, a compact subset $K \subset S^2$ with non-empty interior, and a sequence $c_i$ of positive real numbers such that

- $K$ is a subset of $\text{reg}(m)$, $\text{reg}(m^i)$ for all but finitely many $i$, and
- the restriction of $c_i m^i$ to $K$ converges to the restriction of $m$ to $K$.
With the topology given by this definition, $M_{\text{reg}}$ and $M_{\text{disk}}$ are open sets in $M_{\text{sing}}$ bounded by $\partial M_{\text{reg}}$. Furthermore, the compactification of $M_{\text{reg}}$ in $M_{\text{sing}}$ is just $M_{\text{reg}} \cup \partial M_{\text{reg}}$.

Since any two conformal structures on $S^2$ are conformally equivalent, the group $\text{conf} S^2$ acts transitively on $M_{\text{reg}}$. The stabilizer subgroup of any point $m \in M_{\text{reg}}$ is isomorphic to $\mathbb{O}(3)$ and therefore a maximal compact subgroup of $\text{conf} S^2$. Hence, $M_{\text{reg}}$ is a homogeneous space. In fact, we now define a distance function $\text{dist} : M_{\text{reg}} \times M_{\text{reg}} \rightarrow \mathbb{R}_+$ such that $(M_{\text{reg}}, \text{dist})$ is a metric space of constant curvature. Let $n, m \in M_{\text{reg}}$. If $n_x$ denotes the inner product induced by $n$ at the point $x \in S^2$ there is a smooth function $f : S^2 \rightarrow \mathbb{R}_+$ such that $m_x = f(x) \cdot n_x$. We define

$$\text{dist}(m, n) := \frac{1}{2} \max_{x \in S^2} \log \frac{m_x}{n_x}.$$  

With this notation the metric space $(M_{\text{reg}}, \text{dist})$ is isometric to the 3-dimensional hyperbolic space $H^3$ and its isometries are the elements of $\text{conf} S^2$. The above compactification then corresponds to the Busemann compactification of $H^3$.

Note that the topological space $M_{\text{sing}}$ is homeomorphic to $\mathbb{R}^3$. In fact, using the Poincaré model of $H^3$, we first identify $M_{\text{reg}}$ with the open unit ball $B^3$ of $\mathbb{R}^3$. Let $l$ be a line passing through $0 \in \mathbb{R}^3$ and define $z, z'$ to be the two piercing points of $l$ and $\partial B^3$. If $x$ is a point on $l$ traveling from $0$ towards $z$, then the point $x$ corresponds to a regular metric on the boundary $\partial B^3 = S^2$. If $x = z$ this metric explodes at the point $z \in \partial B$, i.e. $z$ corresponds to a metric with singular domain $\{z\}$. Continuing the travel along $l$, we define $x$ to be a singular metric whose singular domain is a closed disk in $\partial B^3$ centered at $z$. This disk increases if $x$ moves away from $z$ and tends to $\partial B^3 \setminus \{z'\}$ if $x$ tends to infinity.

Summarizing all these identifications, we get the following diagram

$$
\begin{array}{c}
M_{\text{reg}} \subset M_{\text{reg}} \cup \partial M_{\text{reg}} \subset M_{\text{sing}} \subset M_{\text{disk}} \\
\parallel \quad \parallel \quad \parallel \quad \parallel \\
H^3 \quad H^3 \cup S^2 \quad \mathbb{R}^3 \\
\end{array}
\{\text{conformal disk}\}
$$

For every disk metric $m$ there is a transformation $\varphi_m \in \text{conf} S^2$ such that $\partial \text{reg}(m) = \{x \in S^2 \mid \varphi_m(x) = x\}$. We define $H(m)$ as the hyperplane $\{x \in M_{\text{reg}} \mid \varphi_m(x) = x\}$. Two disk metrics $m, n$ are said to overlap if the bounding circles of their regular domains cut in two distinct points, i.e. $H(m) \cap H(n) \neq \emptyset$. In this case the closure of $\text{reg}(m) \cap \text{reg}(n)$ is a two-gon in $S^2$. The angles at the two vertices of this two-gon coincide. We call this angle the angle enclosed by $m$ and $n$.

### 1.2 Maps on Compact Surfaces.

Let $X$ be a compact surface, $\tilde{X} \rightarrow X$ its universal covering and $\pi_1(X)$ the group of covering transformations. Henceforth we consider only compact surfaces which are connected and have no boundary. Let $\tilde{T}$ be a $\pi_1(X)$-invariant decomposition of $\tilde{X}$ into simply connected closed subsets (called the cells of $\tilde{T}$) by some
arcs (called the edges of $\tilde{T}$) joining pairs of points (called the vertices of $\tilde{T}$). It is understood that no two edges have a common interior point, and that the intersection of two different cells is empty, a vertex or an edge. Projecting the cells of $\tilde{T}$ to $X$ yields a decomposition $T$ of $X$ into a finite number of closed subsets. We call $T$ a cell decomposition of $X$ or a map on $X$, $\tilde{T}$ the lifted cell decomposition or lifted map, and the projection of a vertex (respectively, edge, cell) of $\tilde{T}$ a vertex (respectively, edge, cell) of $T$. We denote the set of vertices (respectively, edges, cells) of $T$ by $V(T)$ (respectively, $E(T)$, $F(T)$). Let $a, b \in V(T) \cup E(T) \cup F(T)$. We call $a$ and $b$ incident if $a \subset b$ or $b \subset a$ and we define

$$\langle a, b \rangle := \begin{cases} 1 & \text{if } a \text{ is incident to } b \\ 0 & \text{else.} \end{cases}$$

The sets $V(\tilde{T})$, $E(\tilde{T})$, $F(\tilde{T})$ and the incidence relations in $\tilde{T}$ are defined in the same manner.

In the current paper we assume that every edge of a cell decomposition is incident to two different vertices - the results can be extended to the more general case without difficulties.

### 1.3 Disk Configurations.

In this section we will give a precise definition of collections of disks producing $\pi_1(X)$-polyhedra. Let $T$ be a cell decomposition of a compact surface $X$ and $\theta : E(T) \rightarrow [0, \pi]$ a so called weight function. We lift $\theta$ to a function $\tilde{\theta} : E(\tilde{T}) \rightarrow [0, \pi]$ by defining $\tilde{\theta}(e) := \theta \circ \varpi(e)$. A function $\mathcal{A} : V(\tilde{T}) \rightarrow \text{M}_{\text{disk}}$ is called a $(T, \theta)$-configuration if there is a homeomorphism $\Phi$ from $X$ to a simply connected homogeneous domain $\text{reg}(\mathcal{A})$ of $S^2$ (called the regular domain of $\mathcal{A}$) such that the following conditions hold:

A1) The elements of the group $\{ \Phi \circ g \circ \Phi^{-1} \mid g \in \pi_1(X) \}$ are restrictions of elements of conf $S^2$,

A2) If $g \in \pi_1(X)$ and $g_\Phi := \Phi \circ g \circ \Phi^{-1}$, then $\mathcal{A} \circ g = g_\Phi \circ \mathcal{A}$

A3) If two vertices $v, w$ of $\tilde{T}$ are joined by an edge $e$, then the disk metrics $\mathcal{A}(v)$ and $\mathcal{A}(w)$ overlap and enclose an angle $\tilde{\theta}(e)$,

A4) If $v_1, v_2, \ldots, v_n$ are the vertices of a cell $f$ of $\tilde{T}$, then the circles bounding the regular domains of $\mathcal{A}(v_1), \mathcal{A}(v_2), \ldots, \mathcal{A}(v_n)$ intersect in a point $\mathcal{A}(f)$.

A5) $\text{reg}(\mathcal{A})$ is the disjoint union of $\bigcup_{v \in V(\tilde{T})} \text{reg}(\mathcal{A}(v))$ and $\bigcup_{f \in F(\tilde{T})} \mathcal{A}(f)$.

**Remarks.** Henceforth we identify $g \in \pi_1(X)$ with $g_\Phi$ and we consider $\pi_1(X)$ as a subgroup of conf $S^2$. The set $\text{M}_{\text{disk}}$ is just the set of conformal disks of $S^2$. Hence, a $(T, \theta)$-configuration is a $\pi_1(X)$-equivariant assignment of open disks. If $f$ is a cell of $\tilde{T}$ then $\mathcal{A}(f)$ is the only point contained in the closure of every disk assigned to a vertex incident to $f$. Therefore, the extension of $\mathcal{A}$ to the set $F(\tilde{T})$ remains
\( \pi_1(X) \)-equivariant.
Since \( X \cong \text{reg}(\mathcal{A})/\pi_1(X) \), the group \( \pi_1(X) \) acts properly discontinuously on \( \text{reg}(\mathcal{A}) \).
We claim that there is always a metric \( m \in \mathcal{M}_{\text{sing}} \) such that \( \text{reg}(m) = \text{reg}(\mathcal{A}) \) and \( \pi_1(X) \) is a subgroup of the isometry group of \( (\text{reg}(\mathcal{A}), m) \). In fact, if \( \text{reg}(\mathcal{A}) = S^2 \), then the group \( \pi_1(X) \) is trivial or generated by an involution according as \( X \) is homeomorphic to the sphere or to the projective plane. Since \( \pi_1(X) \) acts without fixed point on \( S^2 = \partial \mathcal{M}_{\text{reg}} \), Brouwer’s Fixed-Point Theorem states that there is an element \( e \in \mathcal{M}_{\text{reg}} \) fixed under \( \pi_1(X) \). Hence, \( \pi_1(X) \) is a subgroup of the isometry group \( (\text{conf}(S^2), m) \) of \( (S^2, m) \). If \( \text{reg}(\mathcal{A}) \neq S^2 \) there is a metric \( m \in \mathcal{M}_{\text{sing}} \) whose regular domain equals the regular domain of \( \mathcal{A} \). Hence, \( \pi_1(X) \) is a subgroup of \( (\text{conf}(S^2), m) \). The elements of this group are the similarities of the metric space \( (\text{reg}(\mathcal{A}), m) \). Since every similarity which is not an isometry fixes a point in \( \text{reg}(\mathcal{A}) \), the elements of \( \pi_1(X) \) have to be isometries in \( (\text{reg}(m), m) \).

1.4 Polyhedral Weight Functions.

Our next aim is to develop some necessary conditions for a function \( \theta : E(T) \rightarrow [0, \pi[ \) to be the weight function of a \((T, \theta)\)-configuration. We start with some notation. A nonempty ordered family \( \mathcal{F} = (e_1, \ldots, e_n) \) of edges of \( T \) is called a chain of edges if there exists a continuous, locally injective path \( \gamma_{\mathcal{F}} : [0, n] \rightarrow X \) such that for every \( i \in \{1, \ldots, n\} \) its restriction to the interval \([i - 1, i]\) is injective and \( \gamma_{\mathcal{F}}([i - 1, i]) = e_i \). If \( \gamma_{\mathcal{F}} \) is a loop, then we call \( \mathcal{F} \) a loop of edges.

A loop \( \mathcal{F} \) of edges is called contractible if \( \gamma_{\mathcal{F}} \) is contractible. A contractible loop of edges is called reduced if there is no subfamily which is a contractible loop of edges. Observe, that the lifts of \( \gamma_{\mathcal{F}} \) are simple closed curves in \( \tilde{X} \) if \( \mathcal{F} \) is a reduced contractible loop of edges. Furthermore, if \( f \) is a cell of \( \tilde{T} \), then there is a reduced contractible loop of edges \( \mathcal{F} \) such that \( f \) is bounded by a lift of \( \gamma_{\mathcal{F}} \).

Let \( \theta : E(T) \rightarrow [0, \pi[ \) be a weight function and assume that \( \mathcal{A} \) is a \((T, \theta)\)-configuration. Let \( \mathcal{F} = (e_1, \ldots, e_n) \) be a reduced contractible loop of edges, \( \tilde{\gamma}_{\mathcal{F}} \) a lift of \( \gamma_{\mathcal{F}} \) and \( v_1, \ldots, v_n \) the vertices of \( \tilde{T} \) along the simple closed curve \( \tilde{\gamma}_{\mathcal{F}} \). Conditions A3-A5 imply that the circles \( \partial \text{reg}(\mathcal{A}(v_1)), \ldots, \partial \text{reg}(\mathcal{A}(v_n)) \) have a point in common if and only if \( \tilde{\gamma}_{\mathcal{F}} \) forms the boundary of a cell of \( \tilde{T} \). In terms of the weight function \( \theta \) this can be expressed in the following way (Figure 1):

B1) \((\pi - \theta(e_1)) + (\pi - \theta(e_2)) + \cdots + (\pi - \theta(e_n)) \geq 2\pi.
B2) \((\pi - \theta(e_1)) + (\pi - \theta(e_2)) + \cdots + (\pi - \theta(e_n)) = 2\pi \) if and only if \( \tilde{\gamma}_{\mathcal{F}} \) bounds a cell of \( \tilde{T} \).

We call a weight function \( \theta \) of an arbitrary cell decomposition of a compact surface polyhedral if Conditions B1 and B2 are fulfilled for every reduced contractible loop of edges.
Remark. There are cell decompositions on compact surfaces which do not admit a polyhedral weight function. As an example consider the truncated tetrahedron, i.e. a tetrahedron whose vertices are cut off by planes parallel to the opposite cell (Figure 2). The surface of this figure admits a cell decomposition $T$ with four triangles and four hexagons. Let $E(T)$ be the set of edges of $T$, $E$ the subset of those edges incident to a triangle and assume that $\theta : E(T) \rightarrow [0, \pi]$ is a polyhedral weight function. Since every edge is incident to a hexagon, we get the following contradiction:

$$4 \cdot 2\pi = \sum_{e \in E} \pi - \theta(e) < \sum_{e \in E(T)} \pi - \theta(e) < 4 \cdot 2\pi.$$  

Therefore, the existence of a polyhedral weight function is a combinatorial characteristic of a cell decomposition. Nevertheless, on every compact surface there exist numerous cell decompositions admitting a polyhedral weight function. As an example we set up cell decompositions with regular cells, i.e. there is an integer $n$ such that every cell is homeomorphic to an Euclidean polygon with precisely $n$ edges. The function $\theta : E(T) \rightarrow [0, \pi[ : e \mapsto 2\pi/n$ is then polyhedral. For more details about polyhedral weight functions of $S^2$ see [Ho].

![Figure 2: Truncated Tetrahedron](image-url)
1.5 Disk Packings.

Let $T$ be a cell decomposition of a compact surface $X$ such that every cell is homeomorphic to an Euclidean quadrangle. Then the weight function $e \mapsto \theta(e) = \pi/2, \forall e \in E(T)$ is polyhedral. If $A$ is a $(T, \theta)$-configuration and $v, w \in V(\tilde{T})$ are incident to a common cell but not joined by an edge, then the disks $\text{reg}(A(v))$ and $\text{reg}(A(w))$ are tangent. Hence, the existence of a $(T, \theta)$-configuration yields the existence of a disk packing, i.e. a collection of tangent disks. For more details about disk packings see [CV]. Figure 3b shows a disk packing on a torus whose combinatorics is described by a triangulation. Figure 3a shows the associated $(T, \theta)$-configuration.

![Figure 3a: (T, \theta)-configuration](image1)

![Figure 3b: Disk packing](image2)

1.6 The Main Theorem.

In this work we establish that Conditions B1 and B2 are sufficient for the existence of a $(T, \theta)$-configuration. Note that if $A$ is a $(T, \theta)$-configuration and $\Phi$ is an element of $\text{conf} S^2$, then $\Phi \circ A$ is again a $(T, \theta)$-configuration. Hence, the group $\text{conf} S^2$ acts on the set of all $(T, \theta)$-configurations. We will prove the following:

**Theorem 1** Let $T$ be a cell decomposition of a compact surface $X$, $E(T)$ its set of edges and $\theta : E(T) \rightarrow [0, \pi]$ a polyhedral weight function. Then there exists a $(T, \theta)$-configuration which is unique up to $\text{conf} S^2$.

**Remark.** If every cell of $T$ is homeomorphic to a triangle or a quadrangle, and $\theta(e) \leq \pi/2, \forall e \in E(T)$ the above theorem is a special case of the Theorem of Andreev and Thurston ([A1], [A2], [Tu]). If $X$ is homeomorphic to $S^2$ compare with the results of Igor Rivin ([R1], [R2]).

Given a cell decomposition $T$ of a compact surface and a polyhedral weight function $\theta$ we will construct a convex space $F_{\Delta, \Sigma}$ and a functional $L_{\Delta, \Sigma}$ on $F_{\Delta, \Sigma}$ such that the $(T, \theta)$-configurations can be identified with the critical points of
L_{\Delta, \Sigma}. The study of this functional leads to the existence and uniqueness of \((T, \theta)\)-
configurations. In the next section we will show that any disk configuration \(\mathcal{A}\)
can be interpreted as a \(\pi_1(X)\)-polyhedron \(|\mathcal{A}|\). If \(\psi\) is a critical point of \(L_{\Delta, \Sigma}\)
corresponding to the disk configuration \(\mathcal{A}\), then it will turn out that \(L_{\Delta, \Sigma}(\psi)\) is the
volume of a fundamental domain of \(|\mathcal{A}|\).

1.7 The Hyperbolic Hull of a Disk Configuration.

Let \(T\) be a cell decomposition of a compact surface \(X\) and \(\mathcal{A}\) a \((T, \theta)\)-configuration.
We define the \textit{hyperbolic hull} \(|\mathcal{A}|\) of \(\mathcal{A}\) as the smallest subset of \(M_{\text{reg}}\) such that
\(|\mathcal{A}| \cup \{\mathcal{A}(f) \mid f \in F(\overline{T})\}\) is a closed convex subset of \(M_{\text{reg}} \cup \text{reg}(\mathcal{A})\) (with the
topology induced by \(M_{\text{sing}}\)). Hence, \(|\mathcal{A}|\) is a \(\pi_1(X)\)-polyhedron. The elements of
the set \(\{\mathcal{A}(f) \mid f \in F(\overline{T})\}\) are called the \textit{vertices} of \(|\mathcal{A}|\). The set of all \textit{vertices}
of \(|\mathcal{A}|\) is a discrete subset in \(\text{reg}(\mathcal{A})\). If this subset is finite, i.e. \(\text{reg}(\mathcal{A}) = S^2\), then
\(|\mathcal{A}|\) is called a finite ideal polyhedron.

For every \(k \in \mathcal{A}(V(\overline{T}))\) the intersection of \(\mathcal{H}(k)\) with \(|\mathcal{A}|\) is a closed convex
subset of \(M_{\text{reg}}\). We call these sets the \textit{facets} of \(|\mathcal{A}|\). If \(\text{reg}(\mathcal{A}) = S^2\) or \(\text{reg}(\mathcal{A}) = S^2 \setminus \{x\}, x \in S^2\), then the boundary of \(|\mathcal{A}|\) in \(M_{\text{reg}}\) is just the union of all facets
of \(|\mathcal{A}|\). If \(\text{reg}(\mathcal{A})\) is an open disk and \(m \in M_{\text{disk}}\) with \(\text{reg}(m) = \text{reg}(\mathcal{A})\), then the
boundary of \(|\mathcal{A}|\) in \(M_{\text{reg}}\) is the union of all facets and the hyperplane \(\mathcal{H}(m)\).

An \textit{edge} of \(|\mathcal{A}|\) finally is a non-empty intersection of two different facets of \(|\mathcal{A}|\).
The edges of \(|\mathcal{A}|\) can also be characterized in the following way: Let \(v, w\) be two vertices of \(\overline{T}\) joined by an edge \(e\). We define \(\mathcal{A}(e)\) to be the intersection of the
hyperplanes \(\mathcal{H}(\mathcal{A}(v))\) and \(\mathcal{H}(\mathcal{A}(w))\). With this notation the edges of \(|\mathcal{A}|\) are just
the geodesics \(\mathcal{A}(e), e \in E(\overline{T})\).

We get a duality between the vertices (respectively, edges, facets) of \(|\mathcal{A}|\) and the
cells (respectively, edges, vertices) of \(\overline{T}\). In fact, for every vertex (respectively, edge)
\(e\) of \(|\mathcal{A}|\) there is one and only one cell (respectively, edge) \(e \in \overline{T}\) with \(e = \mathcal{A}(e)\) and
for every facet \(f\) of \(|\mathcal{A}|\) there is one and only one vertex \(v \in \overline{T}\) with \(f \in \mathcal{H}(\mathcal{A}(v))\).

The discrete group \(\pi_1(X)\) acting on \(S^2\) induces a discrete group of isometries of
\((M_{\text{reg}}, \text{dist})\) and therefore acts properly discontinuously on \(M_{\text{reg}}\). Hence, \(|\mathcal{A}|/\pi_1(X)\)
is a hyperbolic orbifold. If in addition \(\text{reg}(\mathcal{A}) \neq S^2\), then the group \(\pi_1(X)\) acts
freely on \(M_{\text{reg}}\) and \(|\mathcal{A}|/\pi_1(X)\) is a hyperbolic manifold. In fact, let \(m \in M_{\text{sing}}\)
such that \(\text{reg}(\mathcal{A}) = \text{reg}(m)\) and assume that \(p \in M_{\text{reg}}\) is fixed under an element
\(g \in \pi_1(X)\). If \(m \in M_{\text{disk}}\) (respectively, \(m \in \partial M_{\text{reg}}\)) consider the geodesic line
passing through \(p\) and perpendicular to \(\mathcal{H}(m)\) (respectively, passing through \(p\) and
converging to \(m\)). This geodesic is invariant under \(g\) and has a limit point in
\(\text{reg}(\mathcal{A})\). Hence, this limit point is fixed by \(g\). Since \(\pi_1(X)\) acts freely on \(\text{reg}(\mathcal{A})\),
the element \(g\) has to be the identity.
2 A Characterization of Disk Configurations

Let $T$ be a cell decomposition of a compact surface $X$. The aim of this chapter is to characterize $(T, \theta)$-configurations as critical points of some functionals. By $S(T)$ we denote the set of oriented edges, i.e. $S(T) := \{(e, v) \in E(T) \times V(T) \mid \langle v, e \rangle = 1\}$. We say that the edge $e$ and the vertex $v$ are incident to the oriented edge $(e, v)$, and vice versa. For $s \in S(T)$, $v \in V(T)$ we define the bracket:

$$\langle v, s \rangle := \begin{cases} 1 & \text{if } v \text{ is incident to } s \\ 0 & \text{else.} \end{cases}$$

Furthermore, let $|\cdot| : S(T) \rightarrow E(T)$ be the canonical projection and $- : S(T) \rightarrow S(T)$ the orientation reversing function, i.e. for every $s \in S(T)$ the edge $|s|$ is incident to the oriented edges $s$ and $-s$. If $\tilde{T}$ denotes the lifted cell decomposition, then the set of oriented edges $S(\tilde{T})$ of $\tilde{T}$ and the incidence relations are defined likewise. We extend the covering projection $\pi$ to the set of oriented edges by defining $\pi((e, v)) := (\pi(e), \pi(v))$, for all $(e, v) \in S(\tilde{T}) \subset E(\tilde{T}) \times V(\tilde{T})$.

2.1 Angular Datum of a Disk Configuration.

Let $A$ be a $(T, \theta)$-configuration and $v, w \in V(\tilde{T})$ two vertices incident to an edge $e \in E(\tilde{T})$. If $m$ is an element of $M_{\text{sing}}$ such that the closure of $\text{reg}(A(v)) \cup \text{reg}(A(w))$ is contained in $\text{reg}(m)$, then the conformal disks $\text{reg}(A(v))$ and $\text{reg}(A(w))$ are metric disks in $(\text{reg}(m), m)$, i.e. for $k \in \{A(v), A(w)\}$ there is a point $C_{m}(k) \in \text{reg}(k)$ and a number $g_{m}(k) \in \mathbb{R}_{+}$ such that

$$\text{reg}(k) = \{z \in \text{reg}(A) \mid |z - C_{m}(k)|_{m} < g_{m}(k)\}.$$

We call $C_{m}(k)$ the $m$-center of the disk metric $k$. If $f, g \in F(\tilde{T})$ are the cells incident to $e$, then we define $Q_{m}(e)$ to be the geodesic quadrangle in $(\text{reg}(m), m)$ with vertices $C_{m}(A(v)), A(f), C_{m}(A(w)), A(g)$ and contained in the closure of $\text{reg}(A(v)) \cup \text{reg}(A(w))$ (Figure 1h). The geodesic line through $C_{m}(A(v))$ and $C_{m}(A(w))$ cut $Q_{m}(e)$ in two congruent triangles (Figure 1b).

We denote the congruence class of these triangles by $\Delta_{m}(e)$. If $s$ is the oriented edge of $\tilde{T}$ incident to the vertex $v$ and the edge $e$, then we define $\psi_{m}(s)$ to be half the angle of $Q_{m}(e)$ at the vertex $C_{m}(A(v))$. Thus, $\psi_{m}(s)$ and $\psi_{m}(-s)$ are two angles of the triangle $\Delta_{m}(e)$. The third angle does not depend on the metric $m$. It is just $\pi - \theta(e)$. 


Our next goal is to describe the configuration $\mathcal{A}$ by a set of numbers. Let $m \in M_{\text{sing}}$ such that the regular domain of $\mathcal{A}$ equals the regular domain of $m$ and the elements of $\pi_1(X) \subset \text{conf}\mathbb{S}^2$ are isometries in $(\text{reg}(\mathcal{A}), m)$. The set of quadrangles $\{Q_m(e) \mid e \in E(\tilde{T})\}$ is a decomposition of $\tilde{X}$ (Figure 4a). Since $\mathcal{A}$ is $\pi_1(X)$-equivariant and the elements of $\pi_1(X)$ are isometries in $(\text{reg}(\mathcal{A}), m)$, this decomposition is $\pi_1(X)$-invariant. Hence, if $\{\Delta_m\}$ denotes the set of congruence classes of geodesic triangles relative to the metric $m$, then the functions

$$\Delta_m : E(T) \rightarrow \{\Delta_m\}, \quad \Delta_m(e) := \Delta_m(\omega^{-1}(e))$$

and

$$\psi_m : S(T) \rightarrow \mathbb{R}, \quad \psi_m(s) := \psi_m(\omega^{-1}(s))$$

are well defined. We call $\psi_m : S(T) \rightarrow \mathbb{R}$ the angular $m$-datum of $\mathcal{A}$.

Let $\psi : S(T) \rightarrow \mathbb{R}$ be the angular $m$-datum of a $(T, \theta)$-configuration $\mathcal{A}$, $v$ a vertex of $\tilde{T}$ and $s_1, \ldots, s_n$ the oriented edges incident to $v$. The $m$-center of $\mathcal{A}(v)$ is a vertex of the decomposition $\{Q_m(e) \mid e \in E(\tilde{T})\}$ and $Q_m(|s_1|), \ldots, Q_m(|s_n|)$ are just the cells incident to this vertex. Thus,

$$\sum_{i=1}^{n} 2\psi(s_i) = 2\pi.$$

For a function $\psi : S(T) \rightarrow \mathbb{R}$ we get the following two necessary conditions to be the angular $m$-datum of a $(T, \theta)$-configuration:

C1) For every $s \in S(T)$ there is a non-degenerate geodesic triangle in the metric space $(\text{reg}(m), m)$ with angles $\psi(s)$, $\psi(-s)$ and $\pi - \theta(|s|)$,

C2) $\sum_{s \in S(T)} \langle v, s \rangle \psi(s) = \pi, \quad \forall v \in V(T)$.

In the next section we will analyze Condition C1 in more detail, i.e. we examine under what conditions there exists a non-degenerate geodesic triangle in $(\text{reg}(m), m)$ with prescribed angles $\alpha, \beta, \gamma$. 

---

**Figure 4a**

**Figure 4b**
2.2 Geodesic Triangles.

For a metric \( m \in M_{\text{disk}} \cup \partial M_{\text{reg}} \) any three numbers \( \alpha, \beta, \gamma \in \mathbb{R} \) fulfilling the inequalities

\[
\alpha, \beta, \gamma \in ]0, \pi[,
\pi - \alpha - \beta - \gamma \in \begin{cases} ]0, \pi[ & \text{if } m \in M_{\text{disk}}, \\ [0,0] & \text{if } m \in M_{\text{reg}}, \end{cases}
\]

define a similarity class of geodesic triangles in the metric space \((\operatorname{reg}(m), m)\) and vice versa. In this section we will show that an equivalent statement is true for a metric \( m \in M_{\text{reg}} \cup \partial M_{\text{reg}} \).

**Lemma 1** Let \( m \in M_{\text{sing}} \) and let \( c_m \in \{-1, 0, +1\} \) be the curvature of \( m \).

A) Assume that \( d \) is a non-degenerate geodesic triangle in \((\operatorname{reg}(m), m)\) with angles \( 0 < \alpha, \beta, \gamma < \pi \) and define

\[
\hat{\eta} := \frac{1}{2}(\alpha + \beta + \gamma - \pi), \quad \hat{\alpha} := \alpha - \hat{\eta}, \quad \hat{\beta} := \beta - \hat{\eta}, \quad \hat{\gamma} := \gamma - \hat{\eta}.
\]

Then

\[
\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in ]0, \pi[, \quad \hat{\eta} = \pi - \hat{\alpha} - \hat{\beta} - \hat{\gamma} \in \begin{cases} ]0, \pi[ & \text{if } c_m = 1, \\ [0,0] & \text{if } c_m = 0, \\ (-\pi, 0] & \text{if } c_m = -1. \end{cases}
\]

B) If \( m \in M_{\text{reg}} \cup \partial M_{\text{reg}} \) and \( \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\eta} \in \mathbb{R} \) are four numbers fulfilling (1) and (2), then there is a non-degenerate geodesic triangle in \((\operatorname{reg}(m), m)\) with angles \( \alpha = \hat{\alpha} + \hat{\eta}, \beta = \hat{\beta} + \hat{\eta} \) and \( \gamma = \hat{\gamma} + \hat{\eta} \).

**PROOF:** If \( m \in \partial M_{\text{reg}} \), then the assertions holds since \( \hat{\eta} = 0 \).

If \( m \in M_{\text{disk}} \), then \(-2\pi < 2\hat{\eta} = -\operatorname{area}(d) < 0 \). Since \( \hat{\alpha} = \alpha - \hat{\eta} \) etc. we get \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} > 0 \). From \( \hat{\alpha} + \hat{\beta} = \pi - \gamma \) etc. we then conclude that \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} < \pi \).

Assume that \( m \in M_{\text{reg}} \). For \( \zeta \in ]0, \pi[ \) we define a \( \zeta \)-biangle to be the intersection of two closed conformal disks which are bounded by geodesic lines in \((\operatorname{reg}(m), m)\) and enclose an angle \( \zeta \). The area of a \( \zeta \)-biangle is just \( 2\zeta \).

Let \( d \) be a triangle as in A). Since the angles of \( d \) are smaller than \( \pi \), we have \( 0 < \operatorname{area}(d) = 2\hat{\eta} < 2\pi \). Let \( \text{bi}(\alpha) \supset d \) be an \( \alpha \)-biangle with vertices \( A, A' \) as in Figure 5a. Then \( \alpha, \pi - \beta, \pi - \gamma \) are the angles of the triangle \( d_\alpha := \text{bi}(\alpha) \setminus d \) and \( \operatorname{area}(d_\alpha) = 2\hat{\alpha} \). Thus, \( 0 < 2\hat{\alpha} < 2\alpha = \operatorname{area}(\text{bi}(\alpha)) < 2\pi \). In the same way we conclude that \( 0 < \hat{\beta}, \hat{\gamma} < \pi \).

Let \( \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\eta} \) as in B). Since \( 0 < \hat{\alpha} + \hat{\eta} < \hat{\alpha} + \hat{\beta} + \hat{\gamma} + \hat{\eta} = \pi \) we have \( \alpha \in ]0, \pi[ \)
and in the same way we conclude that $\beta, \gamma \in [0, \pi[$. Let $bi(\alpha)$ be an $\alpha$-biangle with vertices $A, A'$ and $B$ a point in the boundary of $bi(\alpha)$ such that $B \neq A, B \neq A'$. Furthermore let $bi(\beta)$ be a $\beta$-biangle with vertices $B, B'$ as in Figure 5a. The intersection of $bi(\alpha)$ and $bi(\beta)$ is a non-degenerate geodesic triangle $d$ with angles $\alpha, \beta, \gamma'$ and area$(d) = \alpha + \beta + \gamma' - \pi$. Varying the point $B$ in our construction, the area of $d$ runs through the interval $]0, \min\{2\alpha, 2\beta\}[$. Hence, for every $\gamma' \in \mathbb{R}$ such that

$$0 < \alpha + \beta + \gamma' - \pi < \min\{2\alpha, 2\beta\}$$

(3)

there is a non-degenerate triangle with angles $\alpha, \beta, \gamma'$. Since

$$0 < \hat{\eta} = \alpha - \hat{\alpha} = \beta - \hat{\beta} < \min\{\alpha, \beta\}$$

inequality (3) holds if $\gamma' = \gamma$ i.e. there is a non-degenerate triangle with angles $\alpha, \beta, \gamma$.

At the end of this section we state some trigonometric relations which we will need later. We keep the above notation and we denote by $a$ the length of the side opposite to $\alpha$. With this notation the half-side formulas of non-Euclidean trigonometry can be written as

$$\tan^2 \frac{a}{2\sqrt{c_m}} = \frac{\sin \hat{\eta} \sin \hat{\alpha}}{\sin \hat{\gamma} \sin \hat{\beta}}.$$  

(4)

If $c_m = -1$ this yields

$$\tanh^2 \frac{a}{2} = \frac{\sin(\hat{\eta}) \sin \hat{\alpha}}{\sin \hat{\gamma} \sin \hat{\beta}}.$$  

(5)

If $d_i$ is a sequence of non-Euclidean triangles converging to an Euclidean one, i.e. $\lim_{i \to \infty} \hat{\eta}_i = 0, \lim_{i \to \infty} \alpha_i = \alpha$ etc. these half-side formulas reduce to the Euclidean law of sines, namely $a/b = \sin \alpha/\sin \beta$ (where $b$ denotes the length of the edge opposite to $\beta$).
2.3 Coherent Angle Systems.

In this section we define a set $F_{\Delta, \Sigma}(T, \theta)$ containing all functions $\psi : S(T) \to \mathbb{R}$ fulfilling Conditions C1 and C2. Our main tool will be Lemma 1. Consider an arbitrary function $\psi : S(T) \to \mathbb{R}$. In accordance with Lemma 1 we define

$$\hat{\psi} : E(T) \to \mathbb{R}, \quad e \mapsto \frac{1}{2} (\psi(s) + \psi(-s) - \theta(e)), \quad \text{where } s \in S(T) \text{ with } |s| = e,$$

$$\hat{\psi} : S(T) \to \mathbb{R}, \quad s \mapsto \psi(s) - \hat{\eta}(|s|),$$

$$\hat{\gamma} : E(T) \to \mathbb{R}, \quad e \mapsto \pi - \theta(e) - \hat{\eta}(e).$$

Let $s \in S(T)$ and $m \in M_{\text{sing}}$. If there exists a non-degenerate triangle in the metric space $(\text{reg}(m), m)$ with angles $\psi(s)$, $\psi(-s)$, $\pi - \theta(|s|)$, then the numbers $\hat{\psi}(s)$, $\hat{\psi}(-s)$, $\hat{\eta}(|s|)$, $\hat{\gamma}(|s|)$ lie in the intervals prescribed by Lemma 1. Hence, if $\psi$ is the angular datum of a $(T, \theta)$-configuration $A$ and $m \in M_{\text{sing}}$ such that $\text{reg}(A) = \text{reg}(m)$, then Condition C1 together with Lemma 1 imply that

$$\hat{\psi}(s) \in [0, \pi], \quad \forall s \in S(T),$$

$$\hat{\gamma}(e) \in [0, \pi], \quad \forall e \in E(T),$$

$$\hat{\eta}(e) \in \begin{cases} [0, c_m \pi] & \text{if } c_m \neq 0, \\ [0, 0] & \text{if } c_m = 0, \end{cases} \quad \forall e \in E(T),$$

where $c_m$ denotes the curvature of the metric $m$. Note that $c_m$ is determined by the Euler characteristic $\chi(X)$ of $X$. In fact, since $\theta$ is polyhedral, we get

$$\pi \cdot \chi(X) = \pi (\#V(T) - \#E(T) + \#F(T))$$

$$= \sum_{v \in V(T)} \sum_{s \in S(T)} \langle v, s \rangle \cdot \psi(s) - \pi \#E(T) + \frac{1}{2} \sum_{f \in F(T)} \sum_{e \in E(T)} \langle e, f \rangle \cdot (\pi - \theta(e))$$

$$= \sum_{s \in S(T)} \psi(s) - \pi \#E(T) + \sum_{e \in E(T)} (\pi - \theta(e))$$

$$= 2 \cdot \sum_{e \in E(T)} \hat{\eta}(e).$$

Since $2 \hat{\eta}(e) = c_m \cdot \text{area}(\Delta_m(e))$ if $c_m \neq 0$, the curvature of $m$ equals the sign of $\chi(X)$. Together with the equalities

$$\hat{\psi}(s) + \hat{\psi}(-s) = \theta(|s|), \quad \forall s \in S(T),$$

$$\hat{\eta}(e) + \hat{\gamma}(e) = \pi - \theta(e), \quad \forall e \in E(T)$$

this simplifies (8) to

$$\hat{\psi}(s) \in [0, \theta(|s|)], \quad \forall s \in S(T),$$

$$\hat{\eta}(e) \in \begin{cases} [0, \pi - \theta(e)], & \text{if } \chi(X) > 0, \\ [0, 0], & \text{if } \chi(X) = 0, \\ (- \theta(e), 0], & \text{if } \chi(X) < 0, \end{cases} \quad \forall e \in E(T).$$
We define $F_\Delta(T, \theta)$ as the set of all functions $\psi : S(T) \rightarrow \mathbb{R}$ satisfying [9] and $F_{\Delta, \Sigma}(T, \theta)$ as the subset of those elements of $F_\Delta(T, \theta)$ satisfying Condition C2. An element of the set $F_{\Delta, \Sigma}(T, \theta)$ is called a coherent angle system. If there is no danger of confusion, we write $F_\Delta$ and $F_{\Delta, \Sigma}$ instead of $F_\Delta(T, \theta)$ and $F_{\Delta, \Sigma}(T, \theta)$. If $\chi(X) \geq 0$, then Lemma[1] states that $F_{\Delta, \Sigma}$ is the set of all functions $\psi : S(T) \rightarrow \mathbb{R}$ satisfying Conditions C1 and C2. But note that for $\chi(X) < 0$ a function $\psi \in F_{\Delta, \Sigma}$ may not satisfy Condition C1. Since $F_\Delta$ as well as $F_{\Delta, \Sigma}$ are subsets of $\mathbb{R}^{#S(T)}$, defined by linear equations and inequalities, they are both convex.

### 2.4 Stereographic Angular Datum.

Let $T$ be a cell decomposition of $S^2$ and assume that $A$ is a $(T, \theta)$-configuration. Every metric $m \in M_{\text{reg}}$ yields a specific cell decomposition $\{Q_m(e) \mid e \in E(T)\}$ of $S^2$ and therefore a specific angular $m$-datum $\psi_m$ of $A$. Hence, if $m$ moves along a curve in $M_{\text{reg}}$, then $\psi_m$ moves along a curve in $F_{\Delta, \Sigma}$.

Let $f$ be a cell of $T$, $v_1, \ldots, v_n$ the vertices incident to $f$ and assume that a metric $m \in M_{\text{reg}}$ converges on a geodesic line towards the Euclidean metric $p := A(f)$. Since for any disk metric $k$ the function $M_{\text{reg}} \cup \partial M_{\text{reg}} \rightarrow S^2$, $m \mapsto C_m(k)$ is continuous, the cell decomposition $\{Q_m(e) \mid e \in E(T)\}$ tends to a cell decomposition $\{Q_p(e) \mid e \in E(T)\}$ of $S^2$. In this process the $m$-centers of the disks $A(v_1), \ldots, A(v_n)$ tend to $A(f)$. Hence, every quadrangle $Q_m(e)$ incident to the $m$-center of such a disk (i.e. $e$ is incident to $f$) tends to the degenerate quadrangle $Q_p(e)$ in $(S^2 \setminus \{p\}, p)$. Figure [5] shows these degenerate quadrangles for a specific $(T, \theta)$-configuration. It is not difficult to verify that the angles of these degenerate quadrangles are determined by the weight function $\theta$. Hence, the union of all non-degenerate quadrangles is a polygon in the Euclidean plane $(\text{reg}(p), p)$ whose exterior angles are determined by $\theta$. We define the $f$-stereographic angular datum $\psi \in \overline{F}_{\Delta, \Sigma}$ by $s \mapsto \lim_{m \rightarrow p} \psi_m(s)$.

If $s = (e, v) \in E(T) \times V(T)$ is an oriented edge and $-s = (e, w)$, then $\psi(s)$ and $\tilde{\eta}(e)$ have the following properties:

| $(v, f)$ | $(w, f)$ | $\psi(s)$ | $\tilde{\eta}(e)$ | Remark |
|--------|--------|--------|--------|--------|
| 1 1 | $\frac{\pi}{2}$ | $\frac{\pi - \theta(e)}{2}$ | $Q_p(e)$ degenerate in $(\text{reg}(p), p)$ |
| 1 0 | 0 | 0 | $Q_p(e)$ degenerate in $(\text{reg}(p), p)$ |
| 0 1 | $\theta(e)$ | 0 | $Q_p(e)$ degenerate in $(\text{reg}(p), p)$ |
| 0 0 | $\in[0, \theta(e)]$ | 0 | $Q_p(e)$ non-degenerate in $(\text{reg}(p), p)$ |

We define $F_{\Delta, \Sigma,f}$ as the set of all $\psi \in \overline{F}_{\Delta, \Sigma}$ fulfilling the relations prescribed by the above tabular. Defining

$V^* := V(T) \setminus \{v_1, \ldots, v_n\}$,

$E^* := \{e \in E(T) \mid \text{no vertex of } e \text{ is incident to } f \}$,

$S^* := \{s \in S(T) \mid |s| \in E^*\}$,
every \( \psi \in F_{\Delta, \Sigma, f} \) can canonically be identified with its restriction to the set \( S^* \), i.e.

\[
F_{\Delta, \Sigma, f} \equiv \left\{ \psi : S^* \rightarrow [0, \pi] \left| \begin{array}{c}
\sum_{s \in S^*} \langle v, s \rangle \psi(s) = \theta(v), \quad \forall v \in V^* \\
\psi(s) + \psi(-s) = \theta(|s|), \quad \forall s \in S^*
\end{array} \right. \right\},
\]

(10)

where

\[
\theta(v) := \pi - \sum_{e \in E(T) \setminus E^*} \langle v, e \rangle \cdot \theta(e), \quad \forall v \in V^*.
\]

Observe that the numbers \( \theta(v), v \in V^* \) are positive. This follows since \( \theta \) is polyhedral.

### 2.5 A Functional on the Set of Coherent Angle Systems.

In this section we construct an smooth functional on \( F_{\Delta, \Sigma} \). Later on, we will see that configurations of disks occur as critical points of these functionals. First, we introduce the following Lobachevsky Function \( L : \mathbb{R} \rightarrow \mathbb{R} \):

\[
L(x) := -\int_0^x \log |2 \sin \vartheta| d\vartheta.
\]

It is quite easily checked that \( L \) is well defined as the integral converges for all values of \( x \). The Lobachevsky Function has the following properties [Mi]:

**Proposition 1**

1. \( L \) is a continuous and odd function.
2. \( L \) is smooth for all \( x \in \mathbb{R} \) except for \( k\pi, k \in \mathbb{Z} \).
3. \( L \) is \( \pi \)-periodic.
4. For all \( z \in \mathbb{R} \) we have \( L(z) = 2L(\frac{x}{2}) + 2L(\frac{\pi}{2} + \frac{x}{2}) \).

If \( \phi \in (0, \pi) \), then we define a new function \( \mathcal{I}_\phi : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\mathcal{I}_\phi(x) := L(x) + L(\phi - x) - 2L\left(\frac{\phi}{2}\right).
\]

(11)

**Proposition 2** For every \( \phi \in (0, \pi) \) the continuous function \( \mathcal{I}_\phi \) is smooth on \( (0, \phi) \). Its restriction to the interval \( [0, \phi] \) is strictly concave and non-positive with maximum value \( \mathcal{I}_\phi(\phi/2) = 0 \) and minimum value \( \mathcal{I}_\phi(0) = \mathcal{I}_\phi(\phi) = 2L\left(\frac{\pi}{2} + \frac{\phi}{2}\right) \).

**Proof:** The continuity and smoothness follows immediately from the above proposition. For \( x \in (0, \phi) \) we have

\[
\mathcal{I}_\phi''(x) = L''(x) + L''(\phi - x) = -\cot x - \cot(\phi - x) = \frac{-\sin \phi}{\sin x \sin(\phi - x)} < 0.
\]
The derivative of $\mathcal{I}_\phi$ vanishes if $x = \phi/2$. Since $\mathcal{I}_\phi$ is concave on $[0, \phi]$ the point $x$ is a maximum and the minima lie on the boundary of the interval. \hfill \Box

We define a function $L_\Delta : \mathcal{F}_\Delta \longrightarrow \mathbb{R}$ by

$$
L_\Delta(\psi) := \frac{1}{2} \sum_{s \in S(T)} \mathcal{I}_{\theta(s)}(\hat{\psi}(s)) - \sum_{e \in E(T)} \mathcal{I}_{\pi \cdot \theta(e)}(\hat{\eta}(e)).
$$

The above propositions together with (8) imply that $L_\Delta$ is continuous on $\mathcal{F}_\Delta$, smooth on $\mathcal{F}_\Delta$ and fulfills the relation

$$
L_\Delta(\psi) = \sum_{s \in S(T)} \mathcal{L}(\hat{\psi}(s)) - \sum_{e \in E(T)} \mathcal{L}(\hat{\gamma}(e)) - \sum_{e \in E(T)} \mathcal{L}(\hat{\eta}(e)) - \sum_{e \in E(T)} \mathcal{L}(\theta(e)).
$$

We denote the restriction of $L_\Delta$ to the set $\mathcal{F}_{\Delta, \Sigma}$ by $L_{\Delta, \Sigma}$. If $X$ is homeomorphic to $S^2$ and $f$ is a cell of $T$, then we denote the restriction of $L_{\Delta, \Sigma}$ to the set $\mathcal{F}_{\Delta, \Sigma, f}$ by $L_{\Delta, \Sigma, f}$. The main result of this chapter is the following:

**Theorem 2** Let $T$ be a cell decomposition of a compact surface $X$ and $\theta$ a polygonal weight function.

1) Let $p \in M_{\text{sing}}$ such that the curvature of $p$ equals the sign of the Euler characteristic of $X$. A point $\psi \in \mathcal{F}_{\Delta, \Sigma}$ is a critical point of the functional $L_{\Delta, \Sigma}$ if and only if there exists a $(T, \theta)$-configuration with angular $p$-datum $\psi$. This $(T, \theta)$-configuration is unique up to $(\text{conf } S^2)_p$.

2) Assume that $X$ is homeomorphic to $S^2$ and let $f$ be a cell of $T$. A point $\psi \in \mathcal{F}_{\Delta, \Sigma, f}$ is a critical point of the functional $L_{\Delta, \Sigma, f}$ if and only if there is a $(T, \theta)$-configuration with $f$-stereographic angular datum $\psi$. This $(T, \theta)$-configuration is unique up to conf $S^2$.

The proof of this theorem will be divided into three steps:

**Step A:** Proof of 1) if $\chi(X) \neq 0$. Let $T_{\psi} \mathcal{F}_{\Delta, \Sigma}$ denote the tangent space of $\mathcal{F}_{\Delta, \Sigma}$ at the point $\psi \in \mathcal{F}_{\Delta, \Sigma}$, i.e.

$$
T_{\psi} \mathcal{F}_{\Delta, \Sigma} = \left\{ U : S(T) \longrightarrow \mathbb{R} \mid \sum_{s \in S(T)} \langle v, s \rangle U(s) = 0, \ \forall v \in V(T) \right\}.
$$

We look for simple tangent vectors. Let $e_1$, $e_2$ be two different edges of $T$ incident to a vertex $v$. We define $U_{e_1, e_2} \in T_{\psi} \mathcal{F}_{\Delta, \Sigma}$ by (Figure 1b):

$$
U_{e_1, e_2}(s) = \begin{cases} 
1 & \text{if } s = (e_1, v), \\
-1 & \text{if } s = (e_2, v), \\
0 & \text{else}.
\end{cases}
$$

The set of all these tangent vectors span $T_{\psi} \mathcal{F}_{\Delta, \Sigma}$.

Assume that $\psi \in \mathcal{F}_{\Delta, \Sigma}$ is a critical point of $L_{\Delta, \Sigma}$ and let $v$, $e_1$, $e_2$ be defined as above. For $i = 1, 2$ we will use the following notation: $\alpha_i := \psi((e_i, v))$; $\beta_i := \psi(-(e_i, v))$; $\gamma_i := \pi - \theta(e_i)$; $\hat{\alpha}_i := \hat{\psi}((e_i, v))$ etc. (Figure 3b).
Let \((D L_{\Delta,S})_{\psi}\) denote the tangent map of \(L_{\Delta,S}\) at the point \(\psi\). Using equation (13) we obtain:

\[
0 = 2 \cdot (D L_{\Delta,S})_{\psi} U_{e_1,e_2} = -\log \left| 2 \sin \hat{\alpha}_1 \right| + \log \left| 2 \sin \hat{\beta}_1 \right| + \log \left| 2 \sin \hat{\eta}_1 \right| - \log \left| 2 \sin \hat{\gamma}_1 \right|
\]

\[
+ \log \left| 2 \sin \hat{\alpha}_2 \right| - \log \left| 2 \sin \hat{\beta}_2 \right| - \log \left| 2 \sin \hat{\eta}_2 \right| + \log \left| 2 \sin \hat{\gamma}_2 \right|.
\]

Hence,

\[
\frac{\sin |\hat{\eta}_1| \sin \hat{\beta}_1}{\sin \hat{\alpha}_1 \sin \hat{\gamma}_1} = \frac{\sin |\hat{\eta}_2| \sin \hat{\beta}_2}{\sin \hat{\alpha}_2 \sin \hat{\gamma}_2}.
\]

Assume for the moment that there exist non-degenerate geodesic triangles \(\Delta_p(e_1), \Delta_p(e_2)\) in \((\text{reg}(p), p)\) with angles \(\alpha_1, \beta_1, \gamma_1\) respectively \(\alpha_2, \beta_2, \gamma_2\). Comparing equation (15) with the formulas (4) and (5) shows that the legs opposite to the angles \(\beta_1, \beta_2\) have the same length. But recall that we have to prove the existence of these triangles if \(p \in M_{\text{disk}}\) (see Lemma 1).

Assume therefore that the curvature of \(p\) is \(-1\). For \(i \in \{1, 2\}\) we will show that \(\pi > \alpha_i > 0\) and \(2 \hat{\eta}_i > -\pi\). If these inequalities are fulfilled, then there exists a triangle in the hyperbolic plane \((\text{reg}(p), p)\) with angles \(\alpha_i, \beta_i, \gamma_i\). Since \(\hat{\alpha}_i \in [0, \pi[\) and \(\hat{\eta}_i \in ] - \pi, 0[\), we have

\[
\pi > \alpha_i = \hat{\alpha}_i + \hat{\eta}_i > -\pi.
\]

Now suppose that \(\alpha_i \leq 0\). Then the following (pairwise equivalent) inequalities hold:

\[
0 = \cos \beta_i - \cos \hat{\beta}_i \geq 2 \sin \alpha_i \sin \gamma_i = \cos(\alpha_i - \gamma_i) - \cos(\alpha_i + \gamma_i),
\]

\[
\cos \beta_i + \cos(\alpha_i + \gamma_i) \geq \cos \beta_i + \cos(\alpha_i - \gamma_i),
\]

\[
\sin |\hat{\eta}_i| \sin \hat{\beta}_i \geq \sin \hat{\alpha}_i \sin \hat{\gamma}_i.
\]

Since \((D L_{\Delta,S})_{\psi} U_{e_1,e_2} = 0\) for any pair of edges \(e_1, e_2\) incident to \(v\), we conclude from (16) and (15) that \(\psi(s) \leq 0\) for all \(s \in S(T)\) with \(\langle v, s \rangle = 1\). Hence, the sum of all \(\psi(s)\) with \(\langle v, s \rangle = 1\) has to be non-positive, which contradicts Condition 19.
C2. Arguing in the same way we see that $\beta_i > 0$. If $\alpha_i > 0$ and $\beta_i > 0$ the above calculation shows that

$$\frac{\sin |\hat{\eta}_i| \sin \hat{\beta}_i}{\sin \hat{\alpha}_i \sin \hat{\gamma}_i} < 1 \quad \text{and} \quad \frac{\sin |\hat{\eta}_i| \sin \hat{\alpha}_i}{\sin \hat{\beta}_i \sin \hat{\gamma}_i} < 1.$$  

Multiplying these inequalities we get

$$\sin |\hat{\eta}_i| < \sin \hat{\gamma}_i = \sin (\gamma_i + |\hat{\eta}_i|).$$

Hence, $\gamma_i + |\hat{\eta}_i| < \pi - |\hat{\eta}_i|$ and $2|\hat{\eta}_i| < \pi - \gamma_i$.

Summarizing, we showed that for every oriented edge $s$ of $T$ there exists a non-degenerate triangle in the metric space $(\operatorname{reg}(p), p)$ with angles $\psi(s), \pi - \theta(|s|)$ $\psi(-s)$. We denote its congruence class by $\Delta_p(|s|)$ and the length of its leg opposite to $\psi(s)$ by $l(s)$. If $s_1$, $s_2$ are two oriented edges of $T$ incident to a vertex $v$, then $l(-s_1) = l(-s_2)$. Gluing pairs of these triangles as in (2.1) we get a $\pi_1(X)$-invariant decomposition $\{Q_p(e) \mid e \in E(T)\}$ of $\operatorname{reg}(p)$ (Figure 4). Let $v$ be a vertex of $T$ and let $e_1, \ldots, e_n$ to be the edges incident to $v$. Then the quadrangles $Q_p(e_1), \ldots, Q_p(e_n)$ have a vertex $v_p$ in common and the legs incident to $v_p$ have the same length $\varrho_p(v)$. We define $A(v)$ to be the disk metric whose regular domain is the metric disk in $(\operatorname{reg}(p), p)$ with center $v_p$ and and radius $\varrho_p(v)$. The map $v \mapsto A(v)$ is a $(T, \theta)$-configuration.

Conversely, if $\psi$ is the angular $p$-datum of a $(T, \theta)$-configuration, then $\psi$ is an element of $\mathcal{F}_{\Delta, \Sigma}$. If the edges $e, e' \in E(T)$ are incident to a vertex $v$, then $l(-e, v) = l(-e', v)$. Reading (14) backwards, we conclude that $(DL_{\Delta, \Sigma})_{\psi}U_{e, e'} = 0$. Since these vectors span the tangent space, the point $\psi$ is critical.

Evidently, the angular $p$-datum determines a disk configuration up to $(\operatorname{conf} S^2)_p$.

**Step B: Proof of 1) If $\chi(X) = 0$.** Since

$$\mathcal{F}_{\Delta, \Sigma} = \left\{ \psi : S(T) \longrightarrow [0, \pi] \mid \sum_{s \in S(T)} \langle v, s \rangle \psi(s) = \pi, \ \forall v \in V(T) \right\},$$

we have

$$T_{\psi} \mathcal{F}_{\Delta, \Sigma} = \left\{ U : S(T) \longrightarrow \mathbb{R} \mid \sum_{s \in S(T)} \langle v, s \rangle U(s) = 0, \ \forall v \in V(T) \right\}.$$

If $\mathcal{F} = (e_1, \ldots, e_n)$ is a chain of edges (see (4.4)), then we define $v_1, \ldots, v_n$ to be the vertices $\gamma_{\mathcal{F}}(1), \ldots, \gamma_{\mathcal{F}}(n)$ along the curve $\gamma_{\mathcal{F}}$ and $s_i := (e, v_i) \in S(T)$. If $\mathcal{F}$ is a loop of edges, then we define a tangent vector $U_{\mathcal{F}}$ in the following way (Figure 7):

$$U_{\mathcal{F}}(s) := \begin{cases} 1 & \text{if there exists a } i \in \{1, \ldots, n\} \text{ with } s = s_i, \\ -1 & \text{if there exists a } i \in \{1, \ldots, n\} \text{ with } s = -s_i, \\ 0 & \text{else}. \end{cases}$$
for every chain of edges $F$ with angles class $\Delta p$

Step C: Sketch of the Proof of 2).

Let $\psi$ be a critical point of $L_{\Delta, \Sigma}$. Using (13) the function $L_{\Delta, \Sigma}$ reduces to

$$L_{\Delta, \Sigma}(\psi) = \sum_{s \in S(T)} \mathcal{L}(\psi(s)).$$

For every oriented edge $s$ there is a similarity class of geodesic triangles in $(\text{reg}(p), p)$ with angles $\psi(s), \pi - \theta(|s|), \psi(-s)$. The next step of the proof is to fix a congruence class $\Delta_p(|s|)$. As usual, we denote the length of the leg of $\Delta_p(|s|)$ opposite to $\psi(s)$ by $l(s)$. We start with an arbitrary edge $e_0 \in E(T)$ and we choose $\Delta_p(e_0)$. Then for every chain of edges $F = (e_1, \ldots, e_n)$ such that $e_1 = e_0$ we successively fix $\Delta_p(e_2), \ldots, \Delta_p(e_n)$ by demanding $l(s_2) = l(-s_1), \ldots, l(s_n) = l(-s_{n-1})$ (Figure 7b).

We claim that the congruence classes $\Delta_p(e_i)$ do not depend on $F$. Therefore assume that $F = (e_1, \ldots, e_n)$ is a loop of edges and choose $\Delta_p(e_1), \ldots, \Delta_p(e_n)$ as described above. We have to show that $l(-s_n) = l(s_1)$. Since $\psi$ is critical, we conclude from (17) that

$$0 = (DL_{\Delta, \Sigma})_{\psi} U_F = + \log |2 \sin \psi(-s_1)| - \log |2 \sin \psi(s_1)|$$
$$+ \ldots + \log |2 \sin \psi(-s_n)| - \log |2 \sin \psi(s_n)|.$$

Hence,

$$1 = \frac{\sin \psi(-s_1) \cdot \sin \psi(-s_2) \cdots \sin \psi(-s_n)}{\sin \psi(s_1) \cdot \sin \psi(s_2) \cdots \sin \psi(s_n)}.$$

Applying the law of sines we get

$$1 = \frac{l(-s_1) \cdot l(-s_2) \cdots l(-s_n)}{l(s_1) \cdot l(s_2) \cdots l(s_n)} = \frac{l(-s_n)}{l(s_1)}.$$

Therefore, the function $e \mapsto \Delta_p(e)$ is well defined. Furthermore, if $s, s' \in S(T)$ are incident to a vertex $v$, then $l(-s) = l(-s')$. The remainder of the proof is the same as in Step A.

Step C: Sketch of the Proof of 2). Using characterization (11) of $\mathcal{F}_{\Delta, \Sigma, f}$,
every fiber of the tangent space can be identified with the set
\[
\left\{ U : S^* \to \mathbb{R} \left| \begin{array}{l}
\sum_{s \in S^*} \langle v, s \rangle U(s) = 0, \forall v \in V^* \\
U(s) + U(-s) = 0, \forall s \in S^*
\end{array} \right. \right\}.
\]

Let \( p \in \partial M_{reg} \) and assume that \( \psi \) is a critical point of \( \mathcal{F}_{\Delta, \Sigma, f} \). We will construct a \((T, \theta)\)-configuration \( \mathcal{A} \) such that \( \psi \) is the \( f \)-stereographic angular datum of \( \mathcal{A} \) and \( p = \mathcal{A}(f) \). For every \( s \in S^* \) we fix a congruence class of triangles \( \Delta_p(|s|) \) with angles \( \psi(s), \pi - \theta(|s|), \psi(-s) \) in the same way as in Step B. Gluing pairs of these triangles in the way prescribed by \((T, \theta)\) yields non-degenerate quadrangles \( Q_p(|s|), s \in S^* \) in \((reg(p), p)\). Let \( \mathcal{P} \) be the union of all these quadrangles. The exterior angles of \( \mathcal{P} \) are determined by the weight function \( \theta \). The quadrangles \( Q_p(|s|), s \in S^* \) imply the existence of a disk metric \( \mathcal{A}(v) \) for every \( v \in V^* \). On the other hand, these disk metrics define the points \( \mathcal{A}(f'), f' \in F(T) \setminus \{f\} \). It remains to define the disk metrics \( \mathcal{A}(v_1), \ldots, \mathcal{A}(v_n) \). Let \( w \in \{v_1, \ldots, v_n\} \) and \( f, f_1, \ldots, f_k \) the cells incident to \( w \). The regular domain of \( \mathcal{A}(w) \) has to be bounded by a circle in \( S^2 \) passing through \( \mathcal{A}(f), \mathcal{A}(f_1), \ldots, \mathcal{A}(f_k) \), i.e. a geodesic line \( g \) in \((reg(p), p)\) which passes through the points \( \mathcal{A}(f_1), \ldots, \mathcal{A}(f_k) \). An inspection of the exterior angles of \( \mathcal{P} \) shows the existence of such a geodesic \( g \).

Fixing \( p \in \partial M_{reg} \) the \( f \)-stereographic angular datum determines a disk configuration with \( p = \mathcal{A}(f) \) up to \((\text{conf} S^2)_p \). If we do not fix the point \( p \), the \( f \)-stereographic angular datum determines a \((T, \theta)\)-configuration up to \( \text{conf} S^2 \). \( \square \)

### 3 Existence and Uniqueness of Disk Configurations

In this chapter we finally prove Theorem 1 provided that there exist coherent angle systems. Their existence will be shown in Chapter 3. Our main tool will be Theorem 2. It reduces the proof of Theorem 1 to a hunt for critical points. Henceforth, \( \text{const} \) will denote a number which is constant for fixed \((T, \theta)\). The proof will be divided into several steps:

**Proof of Theorem 1 if \( \chi(X) < 0 \):** Since \( \tilde{\eta}(e) < 0 \) for all edges \( e \) of \( T \), we get
\[
\mathcal{L}_{\pi, \theta(\langle \rangle)}(\tilde{\eta}(e)) = \mathcal{L}_{\theta(\langle \rangle)}(|\tilde{\eta}(e)|) + \text{const},
\]
and
\[
\mathcal{L}_\Delta(\psi) = \frac{1}{2} \sum_{s \in S(T)} \mathcal{I}_{\theta(|s|)}(\tilde{\psi}(s)) + \sum_{e \in E(T)} \mathcal{I}_{\theta(|e|)}(|\tilde{\eta}(e)|) + \text{const}'.
\]

Proposition 2 yields that \( \mathcal{L}_\Delta : \mathcal{F}_\Delta \to \mathbb{R} \) is strictly concave. The set \( \mathcal{F}_{\Delta, \Sigma} \) is a convex subset of \( \mathcal{F}_\Delta \). Therefore, \( \mathcal{L}_{\Delta, \Sigma} \) must be concave, too.
Since $\mathcal{F}_{\Delta, \Sigma}$ is nonempty (see Chapter \ref{Chapter:4}), the continuous and bounded function $L_{\Delta, \Sigma}$ has a global maximum $\psi \in \mathcal{F}_{\Delta, \Sigma}$. If $\psi$ is not a boundary point, then $\psi$ is the only critical point of $L_{\Delta, \Sigma}$. Assume therefore that the global maximum $\psi$ is a boundary point of $\mathcal{F}_{\Delta, \Sigma}$, i.e. there exists an $s \in S(T)$ and an $x \in \{0, \theta(|s|)\}$ such that $\hat{\psi}(s) = x$ or $-\hat{\eta}(|s|) = x$. Since the function $L_{\theta(|s|)}$ is singular at the point $x$, i.e.
\[ \lim_{x \to 0} \frac{\partial}{\partial x} L_{\theta(|s|)}(x) = \infty \quad \text{and} \quad \lim_{x \to \theta(|s|)} \frac{\partial}{\partial x} L_{\theta(|s|)}(x) = -\infty, \]
the function $L_{\Delta, \Sigma}$ decreases if a point tends to the boundary. Hence, $\psi \in \mathcal{F}_{\Delta, \Sigma}$ and Theorem \ref{Theorem:2} states that there exists a $(T, \theta)$-configuration with angular datum $\psi$.

Assume that $\mathcal{A}, \mathcal{A}'$ are two $(T, \theta)$-configurations and let $p, p'$ be two metrics such that $\text{reg}(p) = \text{reg}(\mathcal{A})$, $\text{reg}(p') = \text{reg}(\mathcal{A}')$. Since the curvature of $p, p'$ equals the Euler characteristic of $X$, the metrics $p$ and $p'$ are elements of $\mathcal{M}_{\text{disk}}$. The group $\text{conf} S^2$ acts transitive on $\mathcal{M}_{\text{disk}}$. Hence, there is a $\Phi \in \mathcal{M}_{\text{disk}}$ such that $\text{reg}(\mathcal{A}) = \text{reg}(\Phi(\mathcal{A}'))$. Theorem \ref{Theorem:2} states that the angular $p$-data of $\mathcal{A}$ and $\Phi \circ \mathcal{A}'$ are both critical points of $L_{\Delta, \Sigma}$. Since there is only one critical point these angular data coincide, i.e. there is a $\Phi' \in (\text{conf} S^2)_p$ with $\mathcal{A} = \Phi' \circ \Phi \circ \mathcal{A}'$.

**Proof of Theorem \ref{Theorem:1} if $\chi(X) = 0$:** Since $\hat{\eta}(e) = 0$ for all $e \in E(T)$, the function $L_{\Delta}$ reduces to
\[ L_{\Delta}(\psi) = \frac{1}{2} \sum_{s \in S(T)} I_{\theta(|s|)}(\psi(s)) + \text{const.} \quad (18) \]
Let $\mathcal{F}$ be the set of all functions $\psi : S(T) \to \mathbb{R}$ such that $\psi(s) \in [0, \theta(|s|)]$, $\forall s \in S(T)$. Using \ref{Equation:18} we extend $L_{\Delta}$ to the set $\mathcal{F}$. The function $L_{\Delta} : \mathcal{F} \to \mathbb{R}$ is again concave. Since $L_{\Delta, \Sigma}$ is just the restriction of $L_{\Delta}$ to the convex set $\mathcal{F}_{\Delta, \Sigma}$, the function $L_{\Delta, \Sigma}$ has to be convex, too. Now, we conclude as in case $\chi(X) < 0$.

**Proof of Theorem \ref{Theorem:1} if $\chi(X) > 0$:** In the above cases a $(T, \theta)$-configuration was unique up to similarity if we fixed its regular domain. We only had to prove the existence and uniqueness of a critical point in $\mathcal{F}_{\Delta, \Sigma}$. If the Euler characteristic is positive, i.e. the regular domain is $S^2$, then the proof is more delicate for two reasons. First, we have no tool to check whether two angular data describe the same $(T, \theta)$-configuration up to $\text{conf} S^2$ and second, the critical points of $L_{\Delta, \Sigma}$ are saddle points and no global extremal points. We can handle these difficulties by using stereographic angular data.

Assume first that $\chi(X) = 2$ and let $f$ be a cell of $T$. We use the notation of \ref{Lemma:2.4}.

The functional $L_{\Delta, \Sigma,f}$ reduces to
\[ L_{\Delta, \Sigma,f}(\psi) = \frac{1}{2} \sum_{s \in S^*} I_{\theta(|s|)}(\psi(s)) + \text{const.} \]
Since $\mathcal{F}_{\Delta, \Sigma,f}$ is nonempty (see Chapter \ref{Chapter:4}), the functional $L_{\Delta, \Sigma,f}$ takes a global maximum $\psi \in \mathcal{F}_{\Delta, \Sigma,f}$. As in case $\chi(X) = 0$ we conclude that $L_{\Delta, \Sigma,f}$ is again
concave and that \( \psi \) is the only critical point in \( F_{\Delta, \Sigma, f} \). Hence, Theorem 2 states that there is one and only one \((T, \theta)\)-configuration up to \( \text{conf} S^2 \).

If \( \chi(X) = 1 \), then \( X \) is covered by \( S^2 \). Since \( \tilde{\theta} : E(\tilde{T}) \rightarrow ]0, \pi[ \) is polyhedral, there is a \((\tilde{T}, \tilde{\theta})\)-configuration \( A \) which is unique up to \( \text{conf} S^2 \). Let \( g \) be the non-trivial element of \( \pi_1(X) \). Then \( A \circ g \) is a \((\tilde{T}, \tilde{\theta})\)-configuration, too. Thus, there is a \( \Phi \in \text{conf} S^2 \) such that \( A \circ g = \Phi \circ A \). Since \( g^2 = \text{id} \) the function \( \Phi^2 \) fixes every element of the set \( A(F(\tilde{T})) \), i.e. \( \Phi^2 = \text{id} \). \( \square \)

4 Existence of Coherent Angle Systems

Let \( T \) be a cell decomposition of a compact surface \( X \) and \( \theta : E(T) \rightarrow ]0, \pi[ \) a polyhedral weight function. In this chapter we will show that the set \( F_{\Delta, \Sigma}(T, \theta) \) is non-empty. We use a procedure proposed by Yves Colin de Verdière [CV] which needs the following theorem of graph theory.

Let \( A \) be an antisymmetric relation on the finite set \( P \). We call the elements of \( P \) points and those of \( A \) arrows. If \((p, q)\) is an arrow, then we call \( p \) its initial point and \( q \) its endpoint. For a set \( Z \) of points we denote by \( \rightarrow Z \) (respectively, \( Z \rightarrow \)) the set of those arrows having only their endpoint (respectively, initial point) in \( Z \). A flow \( \varphi \) on \((P, A)\) is defined to be a function \( \varphi : A \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \). We will use the following Compatible Flow Theorem which can be found in [BE]:

**Theorem 3** Let \( A \) be an antisymmetric relation on the finite set \( P \) and \( b, B : A \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) flows on \((P, A)\) such that

\[
b(a) \leq B(a), \quad \forall a \in A \\
\sum_{a \in \rightarrow Z} b(a) \leq \sum_{a \in Z \rightarrow} B(a), \quad \forall Z \subset A.
\]

Then there exists a flow \( \varphi : A \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) such that

\[
b(a) \leq \varphi(a) \leq B(a), \quad \forall a \in A \\
\sum_{a \in \rightarrow Z} \varphi(a) = \sum_{a \in Z \rightarrow} \varphi(a), \quad \forall Z \subset A.
\]

We call \( \varphi \) a Kirchoff flow compatible with \((P, A, b, B)\).

With this theorem we are in a position to prove the existence of coherent angle systems.

**Lemma 2** If \( \chi(X) \leq 0 \), then the convex set \( F_{\Delta, \Sigma}(T, \theta) \) is non-empty.
\textbf{Proof:} If $\theta : E(T) \to [0, \pi]$ is a polyhedral weight function we have
\[
\pi \cdot \#V(T) = \pi \cdot \#E(T) - \pi \cdot \#F(T) + \pi \chi(X) \\
= \pi \cdot \#E(T) - \sum_{e \in E(T)} (\pi - \theta(e)) + \pi \chi(X) \\
= \sum_{e \in E(T)} \theta(e) + \pi \chi(X). \tag{19}
\]

Consider the finite set $P := V(T) \cup E(T) \cup \{\omega\}$, where $\omega$ is a virtual point, together with the relation
\[ A = S(T) \cup \{(\omega, e) \mid e \in E(T)\} \cup \{(v, \omega) \mid v \in V(T)\}. \]

For $\varepsilon > 0$, $\tau \leq 0$ we define flows $b_\varepsilon, B_\tau : A \to \mathbb{R} \cup \{-\infty, \infty\}$ by (Figure 8a):
\[
\begin{align*}
b_\varepsilon(s) &= \varepsilon, & B_\tau(s) &= \infty, & \forall s \in S(T) \\
b_\varepsilon(\omega,e) &= -\infty, & B_\tau(\omega,e) &= \theta(e) + \tau, & \forall e \in E(T) \\
b_\varepsilon(v,\omega) &= B_\tau(v,\omega) = \pi, & \forall v \in V(T).
\end{align*}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8a}
\caption{Figure 8a}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8b}
\caption{Figure 8b}
\end{figure}

If the sign of $\tau$ equals the sign of $\chi(X)$, then we call $(P, A, b_\varepsilon, B_\tau)$ a flow diagram of $(T, \theta)$. Assume that $\varphi$ is a Kirchoff flow compatible with a flow diagram $(P, A, b_\varepsilon, B_\tau)$. If $\chi(X) < 0$, we have $\varphi(\omega,e) < \theta(e)$, for all $e \in E(T)$. On the other hand, if $\chi(X) = 0$, then Equation (19) together with
\[
\sum_{a \in \rightarrow \{w\}} \varphi(a) = \sum_{a \in \rightarrow \{w\}} \varphi(a)
\]
imply that $\varphi(\omega,e) = \theta(e)$ for every $e \in E(T)$. Hence, for every oriented edge $s$ of $T$ and every $p \in M_{\text{sing}}$ whose curvature equals the sign of $\chi(X)$, there is a geodesic triangle in the metric space $(\text{reg}(p), p)$ with angles $\varphi(s), \varphi(\omega,s), \pi - \theta(|s|)$. Thus, the restriction of $\varphi$ to $S(T)$ is an element of $\mathcal{F}_{\Delta, \Sigma}(T, \theta)$. 

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If we can show that for every subset $Z$ of $A$ there is an $\varepsilon > 0$, $\tau \leq 0$ such that $(P, A, b_\varepsilon, B_\tau)$ is a flow diagram of $(T, \theta)$ fulfilling
\[
\sum_{a \in Z} B_\tau(a) \geq \sum_{a \in \rightarrow Z} b_\varepsilon(a),
\]
then the above theorem states that there is a flow $\varphi$ compatible with a flow diagram $(P, A, b_\varepsilon', B_\tau')$. This implies that $F_{\Delta, \Sigma}(T, \theta) \neq \emptyset$.

Let $Z \subset P, Z_E := Z \cap E(T), Z_V := Z \cap V(T)$. By $[Z_E \to V(T) \setminus Z]$ we denote the set of those elements of $A$ having its initial point in $Z_E$ and its end point in $V(T) \setminus Z$. We distinguish four cases:

**Case 1:** $\omega \not\in Z, Z_E \neq \emptyset$: There is a $y \in \rightarrow Z$ with $b_\varepsilon(y) = -\infty$ and inequality (20) holds for all $\varepsilon, \tau \in \mathbb{R}$.

**Case 2:** $\omega \not\in Z, Z_E = \emptyset$: If $Z_V = \emptyset$ there is nothing to show. Otherwise, we have
\[
\sum_{y \in Z} B_\tau(y) \geq \pi, \quad \sum_{y \in \rightarrow Z} b_\varepsilon(y) \leq \varepsilon \cdot \#V(T)
\]
and (20) holds for some $\varepsilon > 0$.

**Case 3:** $\omega \in Z, [Z_E \to V(T) \setminus Z] \neq \emptyset$: There is a $y \in Z$ with $B_\tau(y) = \infty$.

**Case 4:** $\omega \in Z, [Z_E \to V(T) \setminus Z] = \emptyset$: (i.e. if $e \in Z_E$, then both vertices incident to $e$ are in $Z_V$). we have
\[
\sum_{y \in Z} B_\tau(y) = \sum_{e \in \rightarrow E(T) \setminus Z} \theta(e) + \tau \cdot \#(E(T) \setminus Z)
\]
\[
= \sum_{e \in \rightarrow E(T)} \theta(e) - \sum_{e \in Z_E} \theta(e) + \tau \cdot \#(E(T) \setminus Z)
\]
\[
= \pi \cdot (#V(T) - \chi(X)) - \sum_{e \in Z_E} \theta(e) + \tau \cdot \#(E(T) \setminus Z),
\]
where the last equality follows from (19). On the other hand,
\[
\sum_{y \in \rightarrow Z} b_\varepsilon(y) = \varepsilon \cdot \#[E(T) \setminus Z \to Z_V] + \pi \cdot \#(V(T) \setminus Z).
\]

Hence, we have to show that
\[
\pi \cdot \#Z_V - \pi \cdot \chi(X) > \sum_{e \in Z_E} \theta(e).
\]
Without loss of generality we may assume that for every $v \in Z_V$ there are at least two different edges in $Z_E$ incident to $v$. In fact, because $\theta(e) < \pi, \forall e \in E(T)$, inequality (21) holds if we can prove it under this assumption.

Let $|Z_E| \subset X$ be the union of all edges in $Z_E$ and $X_1, \ldots, X_n$ the connected components of $X \setminus |Z_E|$. Starting with the 1-skeleton $|Z_E|$ we reconstruct $X$ in...
the following way: for every \( i \in \{1, \ldots, n\} \) we attach a closed surface \( \bar{X}_i \) (whose interior is homeomorphic to \( X_i \)) along its boundary \( \partial \bar{X}_i \). Thus,

\[
\chi(X) = \chi(|Z_E|) + \chi(\bar{X}_1) + \cdots + \chi(\bar{X}_n).
\]

We may assume that there is an integer \( k \in \{0, \ldots, n\} \) such that \( \bar{X}_{k+1}, \ldots, \bar{X}_n \) are the only surfaces homeomorphic to a closed disk, i.e. for every \( i \in \{k + 1, \ldots, n\} \) we attach \( \bar{X}_i \) along a reduced contractible loop of edges. Since closed disks are the only surfaces with boundary, connected interior and positive Euler-characteristic, we conclude that

\[
\chi(X) \leq \#Z_V - \#Z_E + (n - k).
\]

If the edges \( e_1, \ldots, e_m \in Z_E \) form a reduced contractible loop of edges in \( X \), then

\[
\sum_{i=1}^{m} (\pi - \theta(e_i)) \geq 2\pi.
\]

Hence,

\[
2 \sum_{e \in Z_E} (\pi - \theta(e)) \geq 2\pi(n - k). \tag{22}
\]

In (22) we have equality if and only if \( \bar{X}_1, \ldots, \bar{X}_n \) are the cells of \( T \), i.e. \( Z_E = E(T) \). In this case \( \rightarrow Z = \rightarrow E = \emptyset \) and we have nothing to show. If \( Z_E \neq E(T) \) we get

\[
\sum_{e \in Z_E} \theta(e) < \pi \cdot (\#Z_E - (n - k)) \leq \pi (\#Z_V - \chi(X)).
\]

\[\square\]

**Lemma 3** If \( \chi(X) > 0 \), then the convex set \( F_{\Delta, \Sigma}(T, \theta) \) is non-empty. Furthermore, if \( X \) is homeomorphic to \( S^2 \) and \( f \) is a cell of \( T \), then \( F_{\Delta, \Sigma, f}(T, \theta) \neq \emptyset \).

**PROOF:** First, assume that \( \chi(X) = 2 \), let \( f \) be a cell of \( T \) and \( e_1, \ldots, e_n \) the edges incident to \( f \). Using characterization (10) of the set \( F_{\Delta, \Sigma, f}(T, \theta) \) we are going to show that \( F_{\Delta, \Sigma, f} \neq \emptyset \).

Consider the finite set \( P = V^* \cup E^* \cup \{\omega\} \), where \( \omega \) is a virtual point, together with the relation

\[
A = S^* \cup \{(\omega, e) \mid e \in E^*\} \cup \{(v, \omega) \mid v \in V^*\}.
\]

We define bounding flows \( b_\epsilon, B_\tau : A \to \mathbb{R} \cup \{-\infty, \infty\} \) as in the proof of Lemma 2. The only modifications are \( \tau = 0 \) and \( b_\epsilon(v, \omega) = B_0(v, \omega) = \theta(v), \forall v \in V^* \) (Figure 8b).
Let \( \varphi \) be a flow compatible with \((P, A, b_\varepsilon, B_0)\). Using (19) we conclude that

\[
\sum_{e \in E^*} \theta(e) = \sum_{e \in E(T)} \theta(e) - \sum_{e \in E(T) \setminus E^*} \theta(e) = \pi(\# V^* + n - \chi(X)) - \sum_{e \in E(T) \setminus E^*} \theta(e)
\]

\[
= \pi \cdot \# V^* - \chi(X) \cdot \pi + \sum_{i=1}^{n} (\pi - \theta(e_i)) - \sum_{e \in E(T) \setminus E^*} \theta(e) = \sum_{v \in V^*} \theta(v).
\]

Hence, the law of Kirchoff at the point \( \omega \) implies that \( \varphi(\omega, e) = \theta(e) \) for all \( e \in E^* \), i.e. the restriction of \( \varphi \) to \( S^* \) is an element of \( \mathcal{F}_{\Delta, \Sigma, J} \). Thus, we have to show that for every subset \( Q \) of \( P \) there is an \( \varepsilon > 0 \) satisfying

\[
\sum_{a \in Q^*} B_0(a) - \sum_{a \in \rightarrow Q} b_\varepsilon(a) \geq 0. \tag{23}
\]

We use the same notation and we distinguish the same cases as in the proof of Lemma 2. Only case 4 is a little bit more delicate. We get

\[
\sum_{y \in \rightarrow Q} B_0(y) = \sum_{e \in E^* \setminus Q_{E^*}} \theta(e) = \sum_{e \in E^*} \theta(e) - \sum_{e \in Q_{E^*}} \theta(e) - \sum_{v \in V^*} \theta(v) - \sum_{e \in E^*} \theta(e),
\]

\[
\sum_{y \in \rightarrow Q} b_\varepsilon(y) = \varepsilon \cdot [E^* \setminus Q_{E^*} \rightarrow Q_{V^*}] + \sum_{v \in V^* \setminus Q_{V^*}} \theta(v).
\]

Hence, we have to show that

\[
\sum_{v \in Q_{V^*}} \theta(v) - \sum_{e \in Q_{E^*}} \theta(e) > 0. \tag{24}
\]

Let \( \partial Q_{E^*} \) be the set of all \( e \in E(T) \setminus E^* \) incident to a \( v \in Q_{V^*} \). Then

\[
\sum_{v \in Q_{V^*}} \theta(v) - \sum_{e \in Q_{E^*}} \theta(e) = \pi \cdot \# Q_{V^*} - \sum_{e \in \partial Q_{E^*}} \theta(e) - \sum_{e \in Q_{E^*}} \theta(e)
\]

\[
= \pi \cdot \# Q_{V^*} - \sum_{e \in \partial Q_{E^*}} \theta(e) - \sum_{e \in Q_{E^*}} \theta(e) + \sum_{i=1}^{n} (\pi - \theta(e_i)) - \pi \chi(X)
\]

and (24) reduces to

\[
\pi \cdot (\# Q_{V^*} + \#\{v_1, \ldots, v_n\}) - \pi \cdot \chi(X) > \sum_{e \in Q_{E^*}} \theta(e) + \sum_{e \in \partial Q_{E^*}} \theta(e) + \sum_{i=1}^{n} \theta(e_i). \tag{25}
\]

Setting \( Z_V := Q_{V^*} \cup \{v_1, \ldots, v_n\} \) and \( Z_E := Q_{E^*} \cup \partial Q_{E^*} \cup \{e_1, \ldots, e_n\} \) this inequality is just inequality (21) found in the proof of Lemma 2. Since we did not use the
restriction $\chi(X) \leq 0$ in the proof of (21), inequality (25) holds.

For every $f \in F(T)$ let $\psi_f$ be an element of $F_{\Delta,\Sigma}.f$. We get an element $\psi \in F_{\Delta,\Sigma}$ by

$$\psi := \frac{1}{\#F(T)} \sum_{f \in F(T)} \psi_f.$$ 

Now assume that $\chi(X) = 1$. Then $\tilde{T}$ is a cell decomposition of $S^2$ and $\tilde{\theta}$ is a polyhedral weight function. Hence, there exists an element $\psi \in F_{\Delta,\Sigma}(\tilde{T}, \tilde{\theta})$. Let $g$ be the non-trivial covering transformation. If $(e,v) \in S(\tilde{T})$ we define $g(e,v) = (g(e),g(v))$. We get an element $\overline{\psi} \in F_{\Delta,\Sigma}(T,\theta)$ by defining $\overline{\psi}(s) := \frac{1}{2}(\psi \circ g + \psi)(\omega^{-1}(s))$ for all $s \in S(T)$.

□

5 Volume of Ideal Polyhedra

The aim of this chapter is to prove the following.

**Theorem 4** Let $T$ be a cell decomposition of a compact surface $X$. If $\psi$ is the angular datum of a $(T,\theta)$-configuration $A$, then

$$\text{vol} \left( \frac{|A|}{\pi_1(X)} \right) = L_{\Delta}(\psi).$$

**Remark.** Let $X$ be homeomorphic to $S^2$ and $f$ a cell of $T$. Since $L_{\Delta}$ is continuous on $F_{\Delta}$, Theorem 4 still holds if $\psi$ is the $f$-stereographic angular datum of $A$.

5.1 Dihedral Angles of Convex Polyhedra.

In this section we will relate the dihedral angles of a polyhedron in $M_{\text{reg}}$ with angles of geodesic polygons. For this purpose we need some preparations:

Let $\Lambda$ be a closed convex subset of $M_{\text{reg}}$. We define the dimension of $\Lambda$ as the dimension of the smallest totally geodesic subset of $M_{\text{reg}}$ containing $\Lambda$. A hyperplane $\mathcal{H}$ of $M_{\text{reg}}$ is called a supporting hyperplane if $\Lambda$ is contained in a closed half-space bounded by $\mathcal{H}$ and $\mathcal{H} \cap \Lambda \neq \emptyset$. If $\mathcal{H}$ is a supporting hyperplane such that $\Lambda$ is not contained in $\mathcal{H}$, then we call $\mathcal{H} \cap \Lambda$ a face of $\Lambda$. If $\dim \Lambda \geq 1$, a face of dimension $\dim \Lambda - 1$ is called a facet of $\Lambda$.

Let $n \in M_{\text{reg}}$. If $m \in M_{\text{reg}}$ (respectively, $m \in M_{\text{disk}}$), then we define $[n,m]$ to be the shortest geodesic segment joining $n$ and $m$ (respectively, joining $n$ with a point of the hyperplane $\mathcal{H}(m)$). If $m \in \partial M_{\text{reg}}$, then we define $[n,m]$ as the geodesic half-line starting at $n$ and converging to $m$. We call $[n,m]$ the geodesic join of $m$ and $n$. A non-empty convex subset $M$ of $M_{\text{reg}}$ is said to pass through $m \in M_{\text{sing}}$ if for every $n \in M$ the geodesic join $[n,m]$ is a subset of $M$. For
arbitrary \( m, n \in M_{\text{sing}} \) we define the geodesic join \([m, n]\) to be the intersection of all geodesic segments, half-lines and lines passing through \( m \) and \( n \).

Let \( M \) be a subset of \( M_{\text{reg}} \) and \( m \in M_{\text{sing}} \). We project \( M \) to the boundary \( \partial M_{\text{reg}} \) in the following way: we define \( \perp m \partial M \) to be the set of all points \( n \) in \( \text{reg}(m) \) such that the intersection of \([m, n]\) and \( M \setminus \{m\} \) is non-empty. For a viewer ‘sited’ at \( m \), the set \( \perp m \partial M \) is just that part of \( \text{reg}(m) \) which is hidden by \( M \). If \( M \) is a hyperplane passing through \( m \), then there is a unique reflection \( \Phi \in \text{conf} \ S^2 \) fixing \( M \) pointwise. Hence, \( \Phi \in (\text{conf} \ S^2)_{m} \). Since the elements of the group \( (\text{conf} \ S^2)_{m} \) are the similarities of the metric space \((\text{reg}(m), m)\), the reflection \( \Phi \) is an isometry in \((\text{reg}(m), m)\). This isometric reflection fixes \( \perp m \partial M = \partial M \cap \text{reg}(m) \) pointwise. Thus, the set \( \perp m \partial M \) is a geodesic line in \((\text{reg}(m), m)\). Since for any pair of points \( x, y \in \text{reg}(m) \) there is a hyperplane passing through \( x, y \) and \( m \), every geodesic line in \((\text{reg}(m), m)\) arises in this way.

Let \( \Lambda \) be a 3-dimensional closed convex subset of \( M_{\text{reg}} \), \( m \) a point of \( \in M_{\text{sing}} \) and \( f_1, \ldots, f_n \) the facets of \( \Lambda \) passing through \( m \). Furthermore assume that \( \perp m \Lambda \) is a polygon in \( \text{reg}(m) \) bounded by \( \perp m f_1, \ldots, \perp m f_n \). Since each facet \( f_1, \ldots, f_n \) is contained in a hyperplane passing through \( m \), the segments \( \perp m f_1, \ldots, \perp m f_n \) are geodesic segments in \((\text{reg}(m), m)\). Combining this with the fact that every hyperplane intersects \( S^2 \) perpendicularly, we conclude that the angles of the geodesic polygon \( \perp m \Lambda \) in \((\text{reg}(m), m)\) coincide with the dihedral angles of the polyhedron \( \Lambda \) at the ‘vertex’ \( m \).

5.2 Decomposition of \(|\mathcal{A}|\) into a Set of Signed Simplices.

Let \( M_i \subset M_{\text{reg}}, i \in I \) be a family of subsets and \( \varepsilon_i \in \{-1, 0, 1\}, i \in I \) a family of flags. We define the union of the signed sets \( \varepsilon_i M_i \) as

\[
\bigcup_{\{i \in I|\varepsilon_i = +1\}} M_i \setminus \bigcup_{\{i \in I|\varepsilon_i = -1\}} M_i.
\]

If \( m \) is a point of \( M_{\text{sing}} \) and \( \lambda \subset M_{\text{reg}} \) a closed convex set, then we define the cone with base \( \lambda \) and apex \( m \) as the smallest convex subset of \( M_{\text{reg}} \) containing \( \lambda \) and passing through \( m \). We denote this closed convex subset of \( M_{\text{reg}} \) by \( \text{Cone}_m(\lambda) \). Thus, \( \text{Cone}_m(\lambda) \) is just the union of all geodesic joins \([m, x], x \in \lambda \).

We are going to decompose convex sets into signed cones. Let \( \lambda \) be a facet of a closed convex set \( \Lambda \) and \( m \in M_{\text{sing}} \). We define an index \( \langle\langle m, \lambda, \Lambda\rangle\rangle \) indicating the sign of \( \text{Cone}_m(\lambda) \) by

\[
\langle\langle m, \lambda, \Lambda\rangle\rangle := \begin{cases} 
+1 & \text{if } \text{Cone}_m(\lambda) \cap \Lambda \neq \lambda, \\
0 & \text{if } \dim \text{Cone}_m(\lambda) = \dim \lambda, \\
-1 & \text{else.}
\end{cases}
\]

If \( m \in M_{\text{reg}} \) see Figure 9 where \( \dim \Lambda = 2 \), \( H \) denotes the hyperplane carrying \( \Lambda \).
and the dotted line is the geodesic carrying $\lambda$.

\[ \langle \langle m, \lambda, \Lambda \rangle \rangle = +1 \]
\[ \langle \langle m, \lambda, \Lambda \rangle \rangle = 0 \]
\[ \langle \langle m, \lambda, \Lambda \rangle \rangle = -1 \]

**Figure 9**

Let $T$ be a cell decomposition of a compact surface $X$ and $\mathcal{A}$ a $(T, \theta)$-configuration. For every $v \in V(\tilde{T})$ the intersection of the hyperplane $H(\mathcal{A}(v))$ with $|\mathcal{A}|$ is a facet of the set $|\mathcal{A}|$. We denote this facet by $\mathcal{f}(v)$. Let $m \in M_{\text{sing}}$ such that $\text{reg}(m)$ = $\text{reg}(\mathcal{A})$ and $m$ is $\pi_1(X)$-invariant. We decompose $|\mathcal{A}|$ in a set of signed cones with apex $m$ and bases $\mathcal{f}(v), v \in V(\tilde{T})$:

\[
|\mathcal{A}| = \bigcup_{w \in V(\tilde{T})} \text{Cone}_m (\mathcal{f}(w)) \setminus \bigcup_{\langle \langle m, \mathcal{f}(w), |\mathcal{A}| \rangle \rangle = 1} \left( \text{Cone}_m (\mathcal{f}(w)) \setminus \mathcal{f}(w) \right) \bigcup_{\langle \langle m, \mathcal{f}(w), |\mathcal{A}| \rangle \rangle = -1} \left( \text{Cone}_m (\mathcal{f}(w)) \setminus \mathcal{f}(w) \right). \tag{26}
\]

If $m \in M_{\text{reg}}$, then Figure 10 illustrates this decomposition. It shows the intersection of $|\mathcal{A}|$ with a hyperplane containing $m$.

\[
\langle \langle m, \mathcal{f}(v_i), |\mathcal{A}| \rangle \rangle = -1 \]
\[
\langle \langle m, \mathcal{f}(v_i), |\mathcal{A}| \rangle \rangle = 1, \ i = 1, 2, 4
\]

**Figure 10**

If $m \in \partial M_{\text{reg}} \cup M_{\text{disk}}$, then $\text{Cone}_m (\mathcal{f}(w)) \subset |\mathcal{A}|$ for any $w \in V(\tilde{T})$. Hence,
\( \langle m, f(w), |A| \rangle = 1 \), \( \forall w \in V(\tilde{T}) \) and

\[ |A| = \bigcup_{w \in V(\tilde{T})} \text{Cone}_m (f(w)). \]

Our final aim is to decompose \( |A| \) in a set of signed simplices. For that purpose we first decompose the facets \( f(w), w \in V(\tilde{T}) \). Let \( v \) be an arbitrary but fixed vertex of \( \tilde{T} \) and

\[ E(v) := \{ e \in E(\tilde{T}) \mid e \text{ incident to } v \}. \]

The facets of the 2-dimensional closed convex set \( f(v) \) are just the geodesic lines \( A(e), e \in E(v) \) (see 1.7). In the same way as we decomposed \( |A| \) in a set of cones with apex \( m \), we now decompose the facet \( f(v) \). First we project the metric \( m \) to the hyperplane \( H(A(v)) \). Namely, if \( g \) is the geodesic line passing through \( m \) and \( A(v) \), then we define \( m(v) \) as the piercing point of \( g \) with the hyperplane \( H(A(v)) \).

\[
\langle m, f(v), |A| \rangle = 1 = \bigcup_{\langle m, f(v), |A| \rangle = 1} \text{Cone}_m (Cone_{m(v)} (A(|s|))) \]

We now combine (26) and (27). Let \( s \in S(\tilde{T}) \) be incident to \( v \). We define

\[ \Omega_m(s) := \text{Cone}_m (Cone_{m(v)} (A(|s|))) \]

and an index \( \varepsilon_m(s) \in \{-1, 0, 1\} \) by

\[ \varepsilon_m(s) := \langle m, f(v), |A| \rangle \cdot \langle \langle m, f(v), |A| \rangle, f(v) \rangle. \]

Thus, if the symbol ‘\( \approx \)’ means equality up to a set of measure zero, we get

\[
|A| \approx \bigcup_{t \in S(\tilde{T})} \Omega_m(t) \ \bigcup_{\varepsilon_m(t) = -1} \Omega_m(t).
\]

5.3 Proof of Theorem 4

Consider decomposition (28) and let \( t \in S(\tilde{T}) \). Since \( m \) is \( \pi_1(X) \)-invariant and \( A \) is \( \pi_1(X) \)-equivariant, we have \( \varepsilon_m(t) = \varepsilon_m(g(t)) \) and \( g(\Omega_m(t)) = \Omega_m(g(t)) \), \( \forall g \in \pi_1(X) \). Thus, for \( s \in S(T) \) the numbers \( \varepsilon_m(s) := \varepsilon_m(w^{-1}(s)), \text{vol} \Omega_m(s) := \text{vol} \Omega_m(w^{-1}(s)) \) are well defined and

\[ \text{vol} \left( |A|/\pi_1(X) \right) = \sum_{s \in S(T)} \varepsilon_m(s) \cdot \text{vol} \Omega_m(s). \]

Theorem 4 follows now from Lemma 4.
Lemma 4 Let $\psi_m$ be the angular $m$-datum of a $(T, \theta)$-configuration $A$. For every $s \in S(T)$ the following volume formula holds:

$$\varepsilon_m(s) \cdot \text{vol} \Omega_m(s) + \varepsilon_m(-s) \cdot \text{vol} \Omega_m(-s) = I_{\theta_{|s|}}(\hat{\psi}_m(s)) - I_{\theta_{-|s|}}(\tilde{\eta}_m(|s|)).$$

**Proof:** In the remainder of this proof let $s$ be an arbitrary but fixed oriented edge of $T$ and $\tilde{s}$ an element of $S(\tilde{T})$ such that $w(\tilde{s}) = s$. Furthermore let $f, g$ be the cells of $\tilde{T}$ incident to $|\tilde{s}|$, let $v, w$ be the vertices of $\tilde{T}$ incident to $|\tilde{s}|$ and define $k := A(v), l := A(w)$.

Assume first, that $\langle \langle m, f(v), |A| \rangle \rangle \cdot \langle \langle m, f(w), |A| \rangle \rangle \neq 0$, i.e. $m \not\in \mathcal{H}(k) \cup \mathcal{H}(l)$. The geodesic line $g$ passing through $m$ and $k$ is invariant under the group $G := (\text{conf } S^2)_m \cap (\text{conf } S^2)_k$. The elements of $G$ are the isometries of $(\text{reg}(m), m)$ fixing $\text{reg}(k)$. Since the $m$-center $C_m(k)$ of $k$ is the only point in $\text{reg}(k)$ invariant under $G$, the geodesic line $g$ has to pass through this point. An analogous consideration shows that the geodesic line passing through $m$ and $l$ exists also through the $m$-center $C_m(l)$ of the disk metric $l$. If $m \in \partial \mathcal{M}_{\text{reg}}$ we illustrate this in the half-space model $\mathcal{M}_{\text{reg}} = \mathbb{C} \times \mathbb{R}^+$ with boundary $\partial \mathcal{M}_{\text{reg}} = \mathbb{C} \cup \{\infty\}$. In this model the geodesic lines passing through $\infty$ are the Euclidean half-lines $\{z\} \times \mathbb{R}^+$, $z \in \mathbb{C}$ and the geodesic lines not passing through $\infty$ are Euclidean semi-circles centered at a point $z \in \mathbb{C} \times \{0\}$. If $m = \infty$, then Figure 11a shows the intersecting hyperplanes $\mathcal{H}(k)$ and $\mathcal{H}(l)$.

![Figure 11a](image1)

**Figure 11a**

The hyperplane $\mathcal{H}$ passing through $m, k$ and $l$ divides $\Omega_m(\tilde{s})$ in two congruent simplices. Let $O_m(\tilde{s})$ be the one containing $A(f)$. Since $\mathcal{H}$ is perpendicular to $\mathcal{H}(k)$ and $\mathcal{H}(l)$, the geodesic line $A(|\tilde{s}|) = \mathcal{H}(k) \cap \mathcal{H}(l)$ intersects $\mathcal{H}$ perpendicularly. Hence, all but at most three dihedral angles of $O_m(\tilde{s})$ are right. Such simplices are called orthoschemes. Figure 11b shows a schematic view of $O_m(\tilde{s})$.

In addition, the sum of the dihedral angles at the vertex $A(f)$ is $\pi$. In fact, $\int_0^{A(f)} O_m(\tilde{s})$ is a geodesic triangle $d$ in the Euclidean plane $(S^2 \setminus A(f), A(f))$ and the angles of $d$ coincide with the dihedral angles of $O_m(\tilde{s})$ at the vertex $A(f)$ (see Figure 5.1). If $\tau_m(s)$ (respectively, $\delta_m(s)$) denotes the dihedral angles at the edge

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carried by the geodesic line passing through \( m \) and \( k \) (respectively, \( m \) and \( A(f) \)), then the following formula holds (see [Kli]):

\[
\text{vol} O_m(s) = \mathcal{V}(\tau_m(s), \delta_m(s)), \quad \text{where} \\
\mathcal{V}(x, y) := \frac{1}{4} \mathcal{L} \left( x + \frac{\pi}{2} - y \right) + \frac{1}{4} \mathcal{L} \left( -x + \frac{\pi}{2} - y \right) + \frac{1}{2} \mathcal{L}(y). \tag{29}
\]

We first determine the angle \( \tau_m(s) \). In [5.1] we showed that the dihedral angles of \( O_m(s) \) at the ‘vertex’ \( m \) coincide with the angles of the geodesic triangle \( \mathcal{H} \mathcal{O}_m(s) \) in \( \text{reg}(m, m) \). Every leg of this triangle is contained in the boundary of a hyperplane carrying a facet of \( O_m(s) \). Hence, the angle \( \tau_m(s) \) coincides with an angle enclosed by the geodesic line in \( \text{reg}(m, m) \), passing through \( C_m(k) \), \( C_m(l) \) and the geodesic line passing through \( C_m(k) \), \( A(f) \), i.e. \( \tau_m(s) = \psi_m(s) \) or \( \tau_m(s) = \pi - \psi_m(s) \). Since \( 2 \cdot \tau_m(s) \) is a dihedral angle of the convex set \( \Omega_m(s) \), the angle \( \tau_m(s) \) cannot be bigger than \( \pi/2 \). Thus,

\[
\tau_m(s) := \begin{cases} 
\psi_m(s) & \text{if } \psi_m(s) \leq \frac{\pi}{2}, \\
\pi - \psi_m(s) & \text{if } \psi_m(s) \geq \frac{\pi}{2}.
\end{cases} \tag{30}
\]

Our next step will be to express the index \( \langle \langle m(v), A([\bar{s}]), f(v) \rangle \rangle \) in terms of the triangle \( \Delta_m(|s|) \). The geodesic line \( A([\bar{s}]) \) divides \( \mathcal{H}(k) \) in two half-planes (see Figure [11]). We have \( \langle \langle m(v), A([\bar{s}]), f(v) \rangle \rangle = 1 \) (respectively, \( -1 \)) if and only if \( m(v) \) is contained in the open half-plane carrying \( f(v) \backslash A([\bar{s}]) \) (respectively, the half-plane containing no point of \( f(v) \)). In order to relate \( \langle \langle m(v), A([\bar{s}]), f(v) \rangle \rangle \) with \( \psi_m(s) \) we project \( m(v), A([\bar{s}]) \) and \( \mathcal{H}(k) \) to \( \partial \text{reg} \). We have

\[
\mathcal{H}^m \backslash m(v) \in \mathcal{H}^m \mathcal{H}(k) \supset \mathcal{H}^m f(v) \supset \mathcal{H}^m A([\bar{s}]).
\]

The set \( \mathcal{H}^m \mathcal{H}(k) \) is again a conformal disk contained in \( \text{reg}(m) \) and bounded by \( \partial \text{reg}(k) \). We denote the disk metric with regular domain \( \mathcal{H}^m \mathcal{H}(k) \) by \( k^* \). Since \( \mathcal{H}(k) = \mathcal{H}(k^*) \), the geodesic line carrying \([m, m(v)] = [m, k]\) passes through \( C_m(k) \) and \( C_m(k^*) = \mathcal{H}^m m(v) \).

The geodesic line \( \mathcal{H}^m A([\bar{s}]) \) in \( \text{reg}(m, m) \) divides the disk \( \mathcal{H}^m k^* \) in two open half-disks. Let \( d \) be the one containing no point of \( \mathcal{H}^m f(v) \) (Figure [12]). Then

\[
\langle \langle m(v), A([\bar{s}]), f(v) \rangle \rangle = \begin{cases} 
-1 & \text{if } C_m(k^*) \in d, \\
0 & \text{if } C_m(k^*) \in \mathcal{H}^m A([\bar{s}]), \\
1 & \text{else}.
\end{cases}
\]

In [5.1] we showed that any geodesic line in \( \text{reg}(m, m) \) containing the point \( C_m(k) \) is of the form \( \mathcal{H}^m M \), where \( M \) is a hyperplane in \( \text{M}_{\text{reg}} \) passing through \( m \) and \( k \). Since these hyperplanes passes also through \( C_m(k^*) \), every geodesic through \( C_m(k) \) contains the point \( C_m(k^*) \). In particular we conclude that \( C_m(k) = C_m(k^*) \) if \( m \in \partial \text{M}_{\text{reg}} \cup \text{M}_{\text{disk}} \). It is not difficult to verify that \( C_m(k) = C_m(k^*) \) if and only if \( \langle \langle m, f(v), |A| \rangle \rangle = 1 \).

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For pairwise distinct points $A, B, C \in \text{reg}(m)$ let $[A, B]_m$ denote the geodesic segment in $(\text{reg}(m), m)$ joining $A$ and $B$ and $\angle_m(A, B, C)$ the angle of the triangle $A, B, C$ at the vertex $B$. We have $\angle_m(A, C_m(k), C) = \angle_m(A, C_m(k^*), C)$ (see Figure 5a if $B := C_m(k) \neq C_m(k^*) =: B'$). Hence, $2\psi_m(s)$ is just the angle obtained by turning $[C_m(k^*), A_f]_m$ into $[C_m(k^*), A_g]_m$ without passing through a vertex of $\perp_m f(v)$ (Figure 12) and

$$\langle \langle m(v), A([\tilde s]), f(v) \rangle \rangle = \begin{cases} 1 & \text{if } \psi_m(s) < \frac{\pi}{2}, \\ 0 & \text{if } \psi_m(s) = \frac{\pi}{2}, \\ -1 & \text{if } \psi_m(s) > \frac{\pi}{2}. \end{cases}$$

(31)

Finally we have to determine the angle $\delta_m(s)$. Let $\Delta$ be the triangle in the congruence class $\Delta_m([s])$ with vertices $C_m(k), C_m(l), A_f$, and $\Delta^*$ the triangle contained in $\text{reg}(k^*) \cup \text{reg}(l^*)$ with vertices $C_m(k^*), C_m(l^*), A_f$. To simplify the notation we denote the angles $\psi(s), \psi(-s), \pi - \theta(|s|)$ of $\Delta$ by $\alpha, \beta, \gamma$ and the angles of $\Delta^*$ at the vertices $C_m(k^*), C_m(l^*), A_f$ by $\alpha^*, \beta^*, \gamma^*$. If $m \in M_{\text{reg}}$, then Figure 13 illustrates the relations between the angles $\alpha, \beta, \gamma$ and $\alpha^*, \beta^*, \gamma^*$. It shows the triangles $\Delta$ and $\Delta^*$ in the Poincaré model $M_{\text{reg}} = \{ x \in \mathbb{R}^3 \mid |x| < 1 \}$ with $m = 0 \in \mathbb{R}^3$, i.e. $(\text{reg}(m), m)$ is the standard metric sphere in $\mathbb{R}^3$. 

Figure 12

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If \( m \in \partial M_{\text{reg}} \cup M_{\text{disk}} \), then \( \Delta^* = \Delta \). Since \( \perp_m \Omega_m(s) \) is the convex hull of the set \( \{ A(f), A(g), C_m(k^*) \} \) in \( \text{reg}(m, m) \), the angle \( \delta_m(s) \) is bounded by the altitude of \( \Delta^* \) and the leg passing through \( C_m(k^*), A(f) \) (Figure 14).

A short computation using the trigonometric relations in the triangle \( \Delta^* \) yields the following formula:

\[
\delta_m(s) = \begin{cases} 
\omega_{\gamma^*}(\alpha^*, \beta^*) & \text{if } \alpha^* \leq \frac{\pi}{2}, \\
-\omega_{\gamma^*}(\alpha^*, \beta^*) & \text{if } \alpha^* \geq \frac{\pi}{2},
\end{cases}
\]

(32)

where

\[
\omega_z(x, y) := \arctan \left( \frac{\cos x \sin z}{\cos y + \cos x \cos z} \right) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
\]

From the definition of \( \Delta, \Delta^* \) and the fact that

\[
\omega_{\pi-\gamma}(\pi - \alpha, \beta) = \omega_{\pi-\gamma}(\alpha, \pi - \beta) = -\omega_{\gamma}(\alpha, \beta)
\]

we immediately deduce the following relations (Figure 13):
In particular we conclude that
\[
\omega_\gamma(\alpha^*, \beta^*) = \langle m, f(v), |A| \rangle \cdot \langle m, f(w), |A| \rangle \cdot \omega_\gamma(\alpha, \beta)
\]
and with (31) we get
\[
\alpha^* < \frac{\pi}{2} \iff \langle m, f(w), |A| \rangle \cdot \langle m(v), A(\tilde{s}), f(\epsilon) \rangle = +1
\]
\[
\alpha^* > \frac{\pi}{2} \iff \langle m, f(w), |A| \rangle \cdot \langle m(v), A(\tilde{s}), f(\epsilon) \rangle = -1.
\]

Combining this with (32) yields
\[
\delta_m(s) = \epsilon_m(s) \cdot \omega_\gamma(\alpha, \beta).
\]
Since $\mathcal{V}(x, \pm y) = \mathcal{V}(\pi - x, \pm y) = \pm \mathcal{V}(x, y)$, we finally get
\[
\mathcal{V}(\alpha, \omega_\gamma(\alpha, \beta)) = \mathcal{V}(\tau_m(s), \omega_\gamma(\alpha, \beta)) = \epsilon_m(s) \mathcal{V}(\tau_m(s), \delta_m(s)) = \epsilon_m(s) \operatorname{vol} O_m(\tilde{s})
\]
\[
= \frac{\epsilon_m(s)}{2} \operatorname{vol} \Omega_m(s).
\]

The technical Lemma 5 completes the proof.

If $\langle m, f(v), |A| \rangle \cdot \langle m, f(w), |A| \rangle$ is zero, then $m \in \mathcal{H}(k) \cup \mathcal{H}(l) \subset M_{\text{reg}}$. We proved that for every $m' \in M_{\text{reg}}$ with $m' \not\in \mathcal{H}(k) \cup \mathcal{H}(l)$ the volume formula
\[
\epsilon_{m'}(s) \cdot \operatorname{vol} \Omega_{m'}(s) + \epsilon_{m'}(-s) \cdot \operatorname{vol} \Omega_{m'}(-s) = \mathcal{I}_{\tau_{(|s|)}}(\hat{\psi}_{m'}(s)) - \mathcal{I}_{\tau_{-\theta(|s|)}}(\hat{\tilde{\eta}}_{m'}(|s|))
\]
holds. Since both sides of this formula are continuous in $M_{\text{reg}}$, it still holds if $m = m'$. □

**Lemma 5** For $\gamma \in \mathbb{R}$ we define $\omega_\gamma, \mathcal{V} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by
\[
\omega_\gamma(x, y) := \arctan \left( \frac{\cos x \sin \gamma}{\cos y + \cos x \cos \gamma} \right)
\]
\[
\mathcal{V}(x, y) := \frac{1}{4} \mathcal{L} \left( x + \frac{\pi}{2} - y \right) + \frac{1}{4} \mathcal{L} \left( -x + \frac{\pi}{2} - y \right) + \frac{1}{2} \mathcal{L}(y).
\]
If $\alpha, \beta \in \mathbb{R}$, then
\[
2 \mathcal{V}(\alpha, \omega_\gamma(\alpha, \beta)) + 2 \mathcal{V}(\beta, \omega_\gamma(\beta, \alpha)) = \mathcal{I}_{\tau_{\gamma}} \left( \frac{\alpha - \beta - \gamma + \pi}{2} \right) - \mathcal{I}_\gamma \left( \frac{\alpha + \beta + \gamma - \pi}{2} \right).
\]
PROOF: It is understood that the function arctan is defined on \( \mathbb{R} \cup \{+\infty, -\infty\} \), i.e. we define \( \arctan(+\infty) = \pi/2 \) and \( \arctan(-\infty) = -\pi/2 \). Let

\[
\hat{\eta} := (\alpha + \beta + \gamma - \pi)/2, \quad \hat{\alpha} := \alpha - \hat{\eta}, \quad \hat{\beta} := \beta - \hat{\eta}, \quad \hat{\gamma} := \gamma - \hat{\eta}.
\]

Proposition 2 in 2.5 yields

\[
I_{\pi - \gamma}(\hat{\alpha}) - I_{\gamma}(\hat{\eta}) = L(\hat{\alpha}) - L(\hat{\eta}) = L(\hat{\gamma}) + L(\gamma).
\]

Hence, we have to show that

\[
2 \mathcal{V}(\alpha, \omega_\gamma(\alpha, \beta)) + 2 \mathcal{V}(\beta, \omega_\gamma(\beta, \alpha)) = L(\hat{\alpha}) + L(\hat{\beta}) - L(\hat{\eta}) - L(\hat{\gamma}) + L(\gamma). \tag{33}
\]

Note that both sides of (33) are continuous in \( \alpha, \beta, \gamma \) and that

\[
\omega_\gamma(\alpha, \beta) + \omega_\gamma(\beta, \alpha) = \gamma \mod \pi. \tag{34}
\]

We will prove (33) if \( \omega_\gamma(\alpha, \beta) \equiv 0 \mod \pi \) and we will show that the partial derivatives of both sides of (33) coincide almost everywhere.

We have \( \omega_\gamma(\alpha, \beta) \equiv 0 \mod \pi \) if and only if \( \alpha \equiv \pi/2 \mod \pi \) or \( \gamma \equiv 0 \mod \pi \). If \( \gamma \equiv 0 \mod \pi \) both sides of (33) are zero. If \( \alpha \equiv \pi/2 \mod \pi \) we get \( \mathcal{V}(\alpha, 0) = 0 \) and Proposition 2 yields

\[
L(\hat{\alpha}) - L(\hat{\eta}) = L\left(\frac{\pi}{2} \pm \frac{\pi}{4} - \frac{\beta - \gamma}{2}\right) + L\left(\frac{\pi}{2} - \frac{\beta - \gamma}{2}\right) = \frac{1}{2} L(-\beta + \frac{\pi}{2} - \gamma)
\]

\[
L(\hat{\beta}) - L(\hat{\gamma}) = L\left(\frac{\beta}{2} - \frac{\gamma}{2} \pm \frac{\pi}{4}\right) + L\left(\frac{\beta}{2} + \frac{\gamma}{2} \pm \frac{\pi}{4}\right) = \frac{1}{2} L(\beta + \frac{\pi}{2} - \gamma).
\]

Thus, the right side of (33) is just \( 2\mathcal{V}(\beta, \gamma) \).

It is not difficult to verify that both sides of (33) are differentiable if

\[
\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\eta} \not\equiv 0 \mod \pi \quad \text{and} \quad \omega_\gamma(\alpha, \beta), \omega_\gamma(\beta, \alpha) \not\equiv 0 \mod \pi/2.
\]

In this case we will calculate the derivatives in direction \( \alpha, \beta \) and \( \gamma \). We first state the following two identities which can be verified with the addition formulas in trigonometry:

\[
\left| \frac{\sin(\alpha + \frac{\pi}{2} - \omega_\gamma(\alpha, \beta))}{\sin(-\alpha + \frac{\pi}{2} - \omega_\gamma(\alpha, \beta))} \right| = \left| \frac{\sin(\hat{\alpha} \sin \hat{\gamma})}{\sin \hat{\eta} \sin \hat{\beta}} \right|,
\]

\[
\left| \frac{\sin(\alpha + \frac{\pi}{2} - \omega_\gamma(\alpha, \beta)) \sin(-\alpha + \frac{\pi}{2} - \omega_\gamma(\alpha, \beta))}{\sin^2 \omega_\gamma(\alpha, \beta)} \right| = \left| \frac{4 \sin(\hat{\alpha} \sin \hat{\gamma} \sin \hat{\eta} \sin \hat{\beta})}{\sin^2 \gamma} \right|.
\]

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Hence, we get
\[
4 \frac{\partial}{\partial \alpha} V(\alpha, \omega, \gamma(\alpha, \beta)) = -\log \left| \frac{\sin \hat{\alpha} \sin \hat{\gamma}}{\sin \hat{\eta} \sin \hat{\beta}} \right| + \frac{\partial \omega(\alpha, \beta)}{\partial \alpha} \log \left| \frac{4 \sin \hat{\alpha} \sin \hat{\gamma} \sin \hat{\eta} \sin \hat{\beta}}{\sin^2 \gamma} \right|
\]
\[
4 \frac{\partial}{\partial \alpha} V(\beta, \omega, \gamma(\beta, \alpha)) = \frac{\partial \omega(\beta, \alpha)}{\partial \alpha} \log \left| \frac{4 \sin \hat{\alpha} \sin \hat{\gamma} \sin \hat{\eta} \sin \hat{\beta}}{\sin^2 \gamma} \right|
\]
\[
4 \frac{\partial}{\partial \gamma} V(\alpha, \omega, \gamma(\alpha, \beta)) = \frac{\partial \omega(\alpha, \beta)}{\partial \gamma} \log \left| \frac{4 \sin \hat{\alpha} \sin \hat{\gamma} \sin \hat{\eta} \sin \hat{\beta}}{\sin^2 \gamma} \right|
\]
Using (34), the derivatives of the left side of (33) in direction \(\alpha, \beta\) and \(\gamma\) reduces to
\[
\frac{1}{2} \log \left| \frac{\sin \hat{\alpha} \sin \hat{\gamma}}{\sin \hat{\eta} \sin \hat{\beta}} \right|, \quad \frac{1}{2} \log \left| \frac{\sin \hat{\beta} \sin \hat{\gamma}}{\sin \hat{\eta} \sin \hat{\alpha}} \right|, \quad \frac{1}{2} \log \left| \frac{4 \sin \hat{\alpha} \sin \hat{\gamma} \sin \hat{\eta} \sin \hat{\beta}}{\sin^2 \gamma} \right|
\]
Therefore, they coincide with the derivatives of the right side of (33). \(\square\)

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