I. INTRODUCTION

As we approach three decades after the achievement of Bose-Einstein condensation (BEC) [1, 2], experiments can for some time now controllably create BECs possessing internal degrees-of-freedom [3–7]. These multi-component systems, due to the Zeeman splitting of the involved magnetic sublevels are known as spinor condensates and have been discussed in dedicated reviews [8, 9], as well as in books [2, 10, 11]. Among spinors with hyperfine spin $F = 1$ or 2, spin-1 BECs represent arguably the most studied class. The two-body interaction of spin-1 bosons is known to contain two interaction parameters accounting for density (or interparticle) and spin-interactions. By engineering the internal states using optical and magnetic fields, various magnetic ground states and the related to them first and second order phase transitions are now accessible [8]. For instance a $^{23}$Na spinor gas experiences antiferromagnetic (AF) interactions [3, 4] whilst $^{87}$Rb [12] and $^7$Li [13,14] feature weak and strong ferromagnetic (FM) ones.

The ground state (GS) phase diagram has been exhaustively studied and summarized, e.g., in Ref. [8]. However, alterations of the latter due to confinement have been very recently explored within the mean-field [15] and the many-body framework [16]. Additionally, due to the presence of internal degrees-of-freedom a plethora of nonlinear excitations bearing a non-topological and a topological character have been proposed theoretically. A partial list of the latter contains: (i) one-dimensional (1D) magnetic and unmagnetized spinor solitons [15, 21], (see also [22] for a two-component analogue of the magnetic solitons), as well as dark-antidark structures [23] (see also [24, 25] for the two-component case). (ii) The realization [26] and the ensuing phase diagram [27] of spinor dark-dark-bright and dark-bright-bright solitary waves (see also [28] for collisions of such solitons) as well as twisted magnetic solitons [29]. (iii) Spin domains [30, 31], textures [32, 33], monopoles [34, 35], as well as three- [37] and two-dimensional (2D) skyrmions (coreless vortices) [38–40] and quantum knots [41, 42]. To the coreless vortex entities belongs also the FM Mermin-Ho vortex [43]. Moreover, half-quantum vortices [44–46] such as those realized in the pure polar (PO) phase of AF spinors [47] and filled-core vortices [48, 49], along with the very recently detected singular SO(3) line vortex [50] can also be included in this list.

Given the enhanced experimental recent interest in spinor BECs and the different excitations that can form in their distinct magnetic phases, we hereby consider quasi-2D, spin-1 BECs featuring either AF or FM spin-interactions. The two distinct spinor systems are harmonically trapped while the choice of parameters utilized throughout our work is based on current state-of-the-art experiments consisting of sodium and rubidium atoms. Concerning the static properties of the setups under consideration, we tackle spinorial stationary states that bear at least one vortex component being filled by bright solitons. The understanding of the stability properties of such configurations is still far from complete and this knowledge will allow to infer the respective phase diagram of nonlinear excitations as it was recently done in one-dimensional settings [24, 51]. The above information can also be valuable for designing certain topolog-
cal states in the different spinor phases in order to examine their dynamical response. Indeed, recent experiments strongly suggest the dynamical accessibility of different patterns. Furthermore, the stability of the solutions that we obtain is addressed via a generalized Bogoliubov-de Gennes (BdG) linearization analysis.

Our findings indicate that vortex-bright-vortex (VBV) and bright-vortex-bright (BVB) nonlinear excitations exist for either AF or FM spin-dependent interactions. Specifically, VBV solutions appear in the AF and the easy-axis (EA) phase of the spinor system for AF and FM interactions respectively. Moreover, they are seen to deform into highly localized vortices (the GS of the PO phase) occupying the symmetric hyperfine components (the zeroth hyperfine state) following a decrease (increase) of the quadratic Zeeman energy shift. On the contrary, BVB structures appear in the PO phase for either AF or FM interactions while for the latter case, they are further evidenced in the easy-plane (EP) phase. Also these configurations experience deformations into a single-component vortex as the quadratic Zeeman energy shift increases and into the GS of the relevant magnetic phase depending on the AF or FM nature of the Bose gas. The aforementioned findings complement the relevant phase diagram of nonlinear excitations that can arise in 2D harmonically confined spin-1 BECs and which, to the best of our knowledge, has not been previously detailed. It is relevant to mention here that the properties of specific vortex structures such as the elliptic one characterized by broken axisymmetry were recently discussed for the PO phase in Refs. [58, 59] and the so-called nematic spin vortices in the EP of AF condensates were analyzed in Ref. [58]. In the same context, the stability of coreless vortices when the longitudinal magnetization is preserved has been exposed [59].

Regarding the stability of the above-obtained spinorial entities, it is explicated that both VBV and BVB stationary states feature stable intervals of existence in terms of the quadratic Zeeman coefficient. These are, in turn, interrupted by narrow windows where oscillatory instabilities take place [57], due to the excited state nature of the relevant configurations. This holds equally also when higher charge vortices are considered, although in the latter setting such instabilities are present even in a single-component setting (as per the seminal work of Ref. [60]). Monitoring the spatiotemporal evolution of the perturbed VBV or BVB entities entails, among others, their robust irregular (out-of-phase) or regular precessional motion. Propagation of triangular and cross-shaped spinor patterns but also structural deformations in which spiralling of the ensuing waveforms is identified stemming from oscillatory instabilities present in the spinor settings at hand. These findings evince that spinor condensates provide a fruitful platform for probing instability-related spontaneous pattern formation.

Further on the dynamical side, quench-induced spin-mixing processes are unveiled under quadratic Zeeman energy shift variations when thermal effects are taken into account. Our main results here can be delineated as follows. We expose the appearance and efficiency of population transfer mechanisms upon dynamically crossing the phase transitions while also showcasing the impact of temperature present in experiments. Larger values of the quadratic energy shift, i.e., entering deeper into the relevant magnetic phase of the spinor systems addressed herein, lead to a more prominent population transfer from the zeroth to the symmetric hyperfine components. This is a result that is found to be independent of the flavor of the ensuing spinorial configurations but is slightly faster for AF when compared to FM spin-dependent interactions. Additionally, spin-mixing is enhanced for higher temperatures. Finally, the nonequilibrium dynamics of the spinor configurations of individual magnetic phases reveals, among others, the generic activation of the precession of the spinors. Deformations of the nonlinear entities exhibit, besides precession, also characteristic spatially anisotropic elongations.

The workflow of the present effort is as follows. Section II deals with the model mean-field equations of motion, the linearization method utilized herein, and the observables used to capture the magnetic properties of the spinor phases. Section III contains our main findings regarding the existence and stability of AF and FM spin-1 BECs. Their quench dynamics at finite temperatures is discussed in Sec. IV. In Sec. V we provide a summary of our results and a list of interesting perspectives for future investigations.

II. EMBEDDING NONLINEAR EXCITATIONS IN THE SPINOR SYSTEM

A. Mean-Field equations

We consider a spin-1 BEC of $N = 10^4$ $^{87}$Rb or $^{23}$Na atoms of mass $M$. A uniform magnetic field $B$ is applied along the transversal $z$-direction, and the system is confined in a quasi-2D harmonic trap. The quasi-2D trap is of the form $V(x, y, z) = M\omega^2(x^2 + y^2)/2 + M\omega_z^2z^2/2$, obeying the condition $\omega_z >> \omega$. Here $\omega_z$ denotes the out-of-plane oscillator frequency, i.e., the one along the $z$-direction, and $\omega$ refers to the frequency in the $x - y$ plane (alias in-plane oscillator frequency). The corresponding three-component wavefunction, $\Psi(r; t) = (\psi_1(r; t), \psi_0(r; t), \psi_{-1}(r; t))$ with $r \equiv \{x, y, z\}$, represents the distinct spin-components, $m_F = \pm 1, 0$, of a spin-1 BEC. However, due to the quasi-2D geometry of the potential considered herein (i.e. $\omega_z >> \omega$) the aforementioned three-dimensional wavefunction can be factorized as follows $\Psi_{m_F}(x, y, z; t) = \psi_{m_F}(x, y, t)\phi_{m_F}(z)$, $\phi_{m_F}(z)$ is the normalized GS wavefunction in the $z$-direction for the $m_F$ component. Additionally, throughout this work we choose as characteristic length and energy scales the in-plane oscillator length $l_{osc} = \sqrt{\hbar/M\omega}$.
and $\hbar \omega$ respectively, and thus densities are measured in units of $l_{\text{osc}}^{-2}$. Accordingly, space and time coordinates are rescaled as $x' = x/l_{\text{osc}}$, $y' = y/l_{\text{osc}}$ and $t' = \omega t$ respectively and the quasi-2D wavefunction as $
abla = (\nabla_{x}'/N)|\psi_{mF}(x,y)$. The latter, with the above choices and rescaling (and dropping the primes for convenience) is described within the mean-field framework by the following dimensionless system of three coupled Gross-Pitaevskii equations (GPE)\(^{(2)}\):\(^{(3)}\):\(^{(4)}\):

\[
i \partial_t \psi_1 = \mathcal{H} \psi_1 + q \psi_1 + c_0 (|\psi_{+1}|^2 + |\psi_0|^2 + |\psi_{-1}|^2) \psi_1 + c_1 (|\psi_{+1}|^2 + |\psi_0|^2 - |\psi_{-1}|^2) \psi_1 + c_1 \psi_{-1}^* \psi_0^2, \tag{1}
\]

\[
i \partial_t \psi_0 = \mathcal{H} \psi_0 + c_0 (|\psi_{+1}|^2 + |\psi_0|^2 + |\psi_{-1}|^2) \psi_0 + c_1 (|\psi_{+1}|^2 + |\psi_0|^2) \psi_0 + 2c_1 \psi_1 \psi_0 \psi_{-1}, \tag{2}
\]

\[
i \partial_t \psi_{-1} = \mathcal{H} \psi_{-1} + q \psi_{-1} + c_0 (|\psi_{+1}|^2 + |\psi_0|^2 + |\psi_{-1}|^2) \psi_{-1} + c_1 (|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_{-1}|^2) \psi_{-1} + c_1 \psi_{-1}^* \psi_0^2. \tag{3}
\]

In the above equations, $\mathcal{H} = -\frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) + V(x,y)$ is the single particle Hamiltonian with $V(x,y) = (x^2 + y^2)/2$ denoting the 2D harmonic potential. The component densities correspond to $n_{mF} = |\psi_{mF}|^2$. Moreover, $c_0$ and $c_1$ are the so-called spin-independent and spin-dependent interaction coefficients given by $c_0 = \frac{2N\sqrt{\pi}a_{\text{osc}}^2|\omega|}{2\hbar}$ and $c_1 = \frac{2N\sqrt{\pi}a_{\text{osc}}^2|\omega|}{8\hbar}$ respectively, in the units adopted herein. $\kappa = \omega_x/\omega$ is the anisotropy parameter, while the scattering lengths $a_0$ and $a_2$ account for collisions between two atoms belonging to the scattering channels with total spin $F = 0$ and $F = 2$ respectively. Additionally, $c_0 > 0$ ($c_0 < 0$) accounts for repulsive (attractive) interatomic interactions, while $c_1 > 0$ and $c_1 < 0$ designate AF and FM spin-interactions, respectively. Furthermore, the quadratic Zeeman energy shift, $q$, can be determined via the relation $q = \mu_B^2 B^2/(4\hbar \omega E_{\text{hfs}})$, where $\mu_B$ denotes the Bohr magneton and $E_{\text{hfs}}$ is the hyperfine splitting. Notably, $q$ can be tuned experimentally either by adjusting the external magnetic field $B$ or the hyperfine splitting $E_{\text{hfs}}$ by utilizing a microwave dressing field.

B. Vortex-bright spinor ansatz and BdG approach

Initially [Sec. III], we focus on obtaining stationary solutions of the spinor system of Eqs. (1)-(3) in the form of vortex-bright (VB) solitons by utilizing a Newton-Krylov iterative scheme\(^{(7)}\). Specifically, in order to introduce a vortex (V) of charge $S$ and a bright (B) soliton in the desired $m_F$ component, the following ansatz is applied to the relevant wavefunctions:

$$
\psi_{m_F}^V(r) = \mathcal{H}_m(x) \mathcal{H}_n(y) e^{-(\kappa x^2 + \kappa y^2)/2m}, \tag{4}
$$

$$
\psi_{m_F}^B(r) = \exp \left[ -(x^2 + y^2)/2 \right]. \tag{5}
$$

In Eq. (4), $\mathcal{H}_m(x) = (-1)^m e^{x^2} \frac{d^{2m}}{dx^{2m}} e^{-x^2}$ and $\mathcal{H}_n(y) = (-1)^n e^{y^2} \frac{d^{2n}}{dy^{2n}} e^{-y^2}$ are the Hermite polynomials. A singly quantized vortex can be obtained by employing as an initial guess the $(m,n) = (1,0)$ polynomial namely the first excited state for the real part of the relevant wavefunction, and the $(m,n) = (0,1)$ for the imaginary part, respectively. In a similar vein, e.g. a doubly quantized vortex ($S = 2$) is realized by a suitable combination of $(m,n)$ i.e. by using $(m,n) = (2,0) - (0,2)$ for the real part, while $(m,n) = 2(1,1)$ for the imaginary one. Subsequently, in sections III and IV the stability properties and the quench-induced dynamics of the previously identified equilibrium states are investigated. Notice that we restrict our investigations to the case where the components contain vortices of the same charge $S$. However, it would be worthwhile to consider in the future also cases in which e.g. the symmetric spin states include oppositely charged vortices in order to unravel the creation of patterns analogous to the monopoles appearing in three-dimensions.

For studying the stability of the VBV and BVB configurations found herein, a spectral BdG analysis suitably generalized for 2D spinorial BECs is performed\(^{(11)}\):\(^{(27)}\):\(^{(54)}\):\(^{(55)}\). In delineating the latter, we note that it consists of perturbing the iteratively identified stationary states, $\Psi_{m_F}^0$, of each phase via the ansatz

$$
\Psi_{m_F}(r,t) = \left[ \psi_{m_F}^0(r) + \epsilon \left( a_{m_F} \epsilon e^{-i\Omega t} + b_{m_F}^* \epsilon e^{i\Omega t} \right) \right] \times e^{-i\mu_{m_F} t}. \tag{6}
$$

Here, $\epsilon$ is a small amplitude perturbation parameter and $\mu_{m_F}$ with $m_F = 0, \pm 1$ is the chemical potential of each spin-component. $\Omega$ and $(a_{m_F}, b_{m_F})^T$ denote, respectively, the eigenfrequencies and eigenfunctions of the resulting eigenvalue problem that one obtains upon substituting Eqs. (3) into the system of Eqs. (1)-(3) and keeping terms of order $O(\epsilon \Omega)\(^{(11)}\):\(^{(54)}\):\(^{(55)}\). The resulting eigenvalue problem is subsequently solved numerically.

On the dynamical side, in order to study alterations of the stationary states existing in a specific phase when crossing a phase boundary\(^{(2)}\), a quench of the quadratic Zeeman energy shift is applied. The quench is performed from an initial $q \equiv q_i$ to a postquench value $q \equiv q_f$ in a way that assures penetration to a different phase. To seed population transfer in the quench dynamics, the commutator of the total spin operator with the Hamiltonian has to be nonzero and we achieve this by including dissipation into the system. Such dissipation, that can naturally arise in current BEC experiments when a non-negligible thermal gas component is present in the system, in which case dissipation can be added phenomenologically into Eqs. (1)-(3) leading to the so-called dissipative GPE model that is then solved numerically\(^{(25)}\):\(^{(72)}\). For the dynamical evolution of the spinorial system a fourth-order (in time) Runge-Kutta method is used with temporal and spatial discretization $dt = 10^{-4}$ and $dx = dy = 0.05$ respectively, while a (2nd order)
finite difference scheme is utilized for the spatial derivatives.

C. Magnetization and spin polarization

In the following we define the quantities that have been utilized herein in order to characterize the different equilibrium states occurring in the distinct phases of both AF and FM spin-1 BECs. These observables will also be useful for monitoring the quench-induced dynamics of the spinor system and the emergent spin-mixing processes.

The spinor system of Eqs. (1) - (3) preserves the total number of particles, \( N \equiv \sum_{m_F} \int d\mathbf{r} |\psi_{m_F}(\mathbf{r}, t)|^2 \). Due to the aforementioned conservation, the corresponding population of each spin component being defined as

\[
n_{m_F} = \frac{1}{N} \int d\mathbf{r} |\psi_{m_F}|^2, \quad m_F = 0, \pm 1, \tag{7}
\]
satisfies the inequality \( 0 \leq n_{m_F} \leq 1 \). Moreover, throughout this work we prescribe that the (similarly conserved quantity of the) net magnetization along the \( z \)-direction i.e., \( M_z = (1/N) \int d\mathbf{r} \left( |\psi^+|^2 - |\psi^-|^2 \right) \), remains zero. This, in turn, implies that there is no population imbalance between the symmetric \( m_F = \pm 1 \) components. Additionally, in order to extract the phase diagram of nonlinear excitations that appear in 2D spinor BECs and further tackle population transfer phenomena that can possibly emerge during the dynamics we invoke the polarization

\[
P = \frac{1}{N} \int d\mathbf{r} \left( |\psi_0|^2 - |\psi_1|^2 - |\psi_{-1}|^2 \right). \tag{8}
\]

Evidently, \( P \) acquires values lying in the interval \(-1 \leq P \leq 1\). In particular, if \( P = 1 \) (\( P = -1 \)) then only the \( m_F = 0 \) (\( m_F = \pm 1 \)) state (states) is (are) populated yielding a single (two) component, 1C (2C), configuration. In contrast, when \(-1 < P < 1\), all three \( m_F \) components (3C) are populated.

In the investigations that we pursue, the in-plane trapping frequency is set to \( \omega = 2\pi \times 20 \) Hz and the transverse one to \( \omega_z = 2\pi \times 400 \) Hz, thus leading to an anisotropy parameter \( \kappa \approx 20 \) inspired by recent 2D BEC experiments, see, e.g., Ref. [61]. Additionally, for AF interactions, a BEC of \( ^{23}\text{Na} \) atoms is considered having mass \( M = 23 \text{amu} \), \( s \)-wave scattering lengths \( a_0 = 2.52862 \text{nm} \), \( a_2 = 2.77196 \text{nm} \) and therefore, \( c_0 \approx 0.013 \text{N} \) and \( c_1 \approx 0.00039 \text{N} \). For FM interactions, we consider a BEC composed of \( ^{87}\text{Rb} \) atoms with mass \( M = 87 \text{amu} \), \( a_0 = 5.387 \text{nm} \), \( a_2 = 5.313 \text{nm} \) and thus \( c_0 \approx 0.05 \text{N} \) and \( c_1 \approx -0.00023 \text{N} \), respectively. The quadratic Zeeman coefficient, \( q \), is varied within the interval \([-3, 3]\) and unless stated otherwise, the total particle number and the vortex charge are fixed to \( N = 10^4 \) and \( S = 1 \) respectively.

III. STATIC PROPERTIES OF NONLINEAR EXCITATIONS

A. Antiferromagnetic vortex-bright configurations

To tackle the nonlinear excitations of the VB form that arise in the distinct phases of 2D harmonically confined spin-1 BECs, an initial guess provided by Eqs. (4) - (5) is introduced to the time-independent version of the system of Eqs. (1) - (3). Specifically, for AF interactions (\( c_1 > 0 \)), it is well-known that two distinct phases exist depending on the value of the quadratic Zeeman energy shift \( q \). Namely, for \( q < 0 \) the AF phase is realized while for \( q > 0 \) the system resides in the PO phase. In the former phase and at the GS level, only the symmetric \( m_F = \pm 1 \) spin-components are populated.

Thus, a natural choice for accessing the corresponding excited states is to consider an initial guess where vortices (bright solitons) are embedded in the \( m_F = \pm 1 \) hyperfine states and a bright soliton (vortex) occupies the \( m_F = 0 \) component. It turns out that among these two, i.e. VBV and BVB, configurations only VBV excitations exist in the AF phase. Representative density profiles are illustrated as insets in Fig. 1(a2) – (a4). We note in passing that for all vortex entities to be presented throughout we have verified that they are accompanied by the expected \( 2\pi S \) phase winding (with \( S \) denoting the vortex charge). These 3C stationary states exhibit polarizations \(-1 < P < 1\) (see orange line in Fig. 1) and their interval of existence is provided in the last column of Table I. They are further found to deform upon a \( q \) variation into highly localized vortices occupying the symmetric spin-components as \( q \) decreases [Fig. (a5) – (a6)]. These 2C vortices are characterized by \( P = -1 \) and they exist for all values of \( q < -2.5 \) that we have checked, see also second column of Table I. Yet another deformation occurs for the VBV configurations but upon increasing \( q \). In this case, each vortex core gradually becomes wider in order to effectively trap the accompanying wider bright soliton of the \( m_F = 0 \) spin-component. This alteration holds until the 1C GS of the PO phase is reached that is, in turn, characterized by \( P = 1 \) (first column of Table I).

Having examined the existence of VBV excitations along with their relevant structural deformations we next explore the stability properties of such configurations. As stated earlier, to perform the BdG analysis the ansatz of Eq. (6) is used for this specific stationary solution. The relevant BdG spectra, obtained upon solving the eigenvalue problem associated to the VBV solutions, are depicted in Fig. 2(a1) – (b1). Note that there exist in the real part, \( \text{Re}(\Omega) \), of the spectrum and at \( \Omega = 0 \) three different pairs of modes. The latter are related to the symmetries preserved by the spinor system. Namely the total particle number conservation, the zero net magnetization considered and the rotational symmetry. Besides the aforementioned modes, two additional negative energy ones appear among the remaining modes of the
discrete spectra that are denoted by light blue circles. The two distinct trajectories, obtained with respect to $q$, of these so-called anomalous modes (AMs) can be discerned in Fig. 2(a1). Each of these modes is known to correspond to the precession of each of the two vortices within the parabolic trap. Additionally, these AMs are quantified through their negative energy or negative Krein signature [54], which for the 2D spinor system reads
\[ K = \omega \int dx \, dy \sum_{m_F = 0, \pm 1} |a_{m_F}|^2 - |b_{m_F}|^2. \] (9)

It should be marked here that the existence of these modes is an immediate byproduct of the fact that the stationary states found herein are excited states of the spinor system. Moreover, as long as these eigenfrequencies maintain their real nature, then their negative Krein signature further indicates that while a stationary solution is dynamically stable, it is simultaneously unstable thermodynamically [11]. The latter, in turn, implies that given a channel of energy dissipation, as in the case of the dissipative spinor system that will be discussed below, these eigendirections will be activated leading to an instability of the ensuing configuration. Notice that upon increasing $q$ so as to reach the phase transition point ($q = 0$), in the vicinity of the latter, the aforementioned negative energy modes decrease in frequency, with both crossing the zero frequency axis around $q \approx -0.2$. At the same time also a decreasing in frequency positive energy mode crosses $\Omega = 0$ and leads to the appearance of the finite imaginary part, $\text{Im}(\Omega) \neq 0$, shown in Fig. 2(b1). The destabilization of the deformed VBV configuration is followed by a change in the Krein signature of the two (previously) negative energy modes from negative (light blue circles) to positive (black circles).

In addition to the above stability analysis results, there exist narrow intervals of $q$ where oscillatory instabilities [22] take place for the VBV solution. In general, this type of instability stems from collision events involving pairs of positive and negative Krein signature modes resulting in eigenfrequency quartets and also possessing a finite imaginary component $\text{Im}(\Omega) \neq 0$ [22, 27]. We must emphasize here, that this is yet another key feature related to the theory of AM: namely, their role in the manifestation of instabilities even in the absence of finite temperatures. Three such collision events can be readily seen in the BdG spectrum of Fig. 2(a1) appearing e.g. at $q = -1.5$, $q = -0.6$ and $q = -0.4$. The first two are associated with the higher-lying anomalous mode whose absence for these values of $q$ is transparent while the last one entails the collision and disappearance of both negative energy modes.

Two case examples are considered below for $q = -0.6$, demonstrating the activation of e.g. the lower-lying anomalous mode ($AM_1$) along with exploring the oscillatory instability present for this value of $q$. Particularly, Fig. 2(c1)−(f1) illustrate density profiles along the $x$ and $y$ axis of a perturbed VBV stationary state. Namely, a perturbation consisting of adding to the VBV solution the eigenvector associated either with $AM_1$ or with the

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FIG. 1. Phase diagram of nonlinear excitations for AF spinor 2D BECs. Polarization, $P$, with respect to the quadratic Zeeman coefficient $q$ for VBV (orange solid line) and BVB (dashed purple line) equilibrium states existing in AF ($c_1 > 0$) spin-1 BECs. Insets (a1)-(a5) and (b1)-(b5) illustrate representative density profiles, $|\Psi_{m_F}(x,y)|^2$, of a VBV [BVB] configuration (3C structure in the middle row) and its corresponding deformations towards a 1C (top row) and a 2C (bottom row) stationary state. The components that are not depicted possess zero population. For both types of solutions singly quantized vortices are considered for the relevant in each case $m_F$ component (see legends).
eigenfrequency quartet identified for \(q = -0.6\). Notice that the AM\(_1\) causes an overall shift to the left and a slight asymmetry of the VBV structure with respect to \(y = 0\) [purple solid lines in Fig. 2(c1)] but it has no significant effect along the \(x\)-direction. Along the latter, the perturbed, \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\) (solid blue lines), and the unperturbed, \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\) (dashed black lines), density profiles fall on top of each other. However, this is not the case when considering the quartet scenario [Fig. 2(d1), (f1)]. Besides a minuscule overall shift of the VBV structure in both directions the predominant effect of this mode is the asymmetric distribution of the density of the vortex in the \(m_F = -1\) component found to be more pronounced along the \(y\)-direction. Importantly, the \(m_F = +1\) component (not shown) has the same effect with that of \(m_F = -1\) when the VBV is perturbed via AM\(_1\) but it is complementary to it for the AM\(_2\) mode. Finally, it is worth commenting here, that dynamical evolution of the excited, with AM\(_1\), VBV entity leads to its precessional motion where the entire VBV rotates around the trap. Whilst, exciting the configuration with AM\(_2\) results in a rotating cross-shaped pattern in which the vortex components perform an anti-phase oscillation among each other and the \(m_F = 0\) bright component remains unaltered. This anti-phase vibration leads, in turn, to an overall breathing of the BEC background.

For AF interactions but for \(q > 0\), namely within the PO phase, the preferable configuration consists of a solely occupied \(m_F = 0\) spin-component. Since this component, according to the GS of the system [3], is expected to become the majority one, in our search for nonlinear excitations arising in this phase we choose to imprint a vortex on it. Consequently, bright solitons are plugged in the remaining \(m_F = \pm 1\) spin-components. With such an initial guess, indeed, VBV stationary solutions are captured for \(0.3 \leq q < 1.95\) (see also the relevant third column of Table I). Characteristic density contours of such a VBV structure are presented as insets in Fig. 2(b2) – (b4). Notice that similarly to the VBV configurations, the VBV stationary states are characterized by \(-1 < P < 1\) (see

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**FIG. 2. BdG spectra of 3C (a1)-(b1) VBV and (a2)-(b2) VBV stationary states upon a \(q\) variation for \(c_1 > 0\). In both cases the anomalous modes (AMs) are depicted by light blue circles while the background ones by black circles. Notice the two AM present for VBV structures when compared to the single pair occurring for BVB configurations. (c1)-(f1) [c2)-(f2)] Density profiles along both spatial directions of a perturbed VBV (BVB) solution for \(q = -0.6\) [\(q = 0.4\)]. For comparison, the density of both the perturbed, \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\) and \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\), and the stationary configurations, \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\) and \(|\tilde{\Psi}_{m_F}(x, y_0 = 0)|^2\), are provided (see legends). The AF BEC consists of \(N = 10^4\) sodium atoms confined in a quasi-2D harmonic trap.**

| \(c_1> 0\) | 1C | 2C | 3C |
|---|---|---|---|
| VBV | \(q \geq -0.25\) | \(q < -2.5\) | \(-2.5 \leq q < -0.2\) |
| \(q \geq 1.95\) | \(q < 0.3\) | \(0.3 \leq q < 1.95\) |
| VBV | \(q \geq -0.14\) | \(q < -2.74\) | \(-2.74 \leq q < -0.14\) |
| \(q \geq 2.07\) | \(q < 0.15\) | \(0.15 \leq q < 2.07\) |
formations with respect to gradual as captured by the slope of the polarization as \(q\) increases, leading to a single highly localized vortex occurring \((q = 0)\) separating the PO and the AF phase, an \(\text{AF} \) state, reminiscent of the GS of the AF phase, occurs \([\text{see Fig. 1(b)}]\) and the relevant first column of Table I. On the contrary, as \(q\) decreases towards the first order transition boundary \((q = 0)\) separating the \(\text{PO}\) and the \(\text{AF}\) phase, an abrupt deformation of the \(\text{BVB}\) configuration to the \(\text{2C}\) state, reminiscent of the GS of the \(\text{AF}\) phase, occurs \([\text{see Fig. 1(b)}]\) and the second column of Table I around \(q = 0.3\). Notice, that both the \(\text{VBV}\) and the \(\text{BVB}\) configurations feature smooth deformations towards the \(\text{2C}\) and the \(\text{1C}\) vortex state respectively. In the opposite \(q\) direction a sharp transition takes place when the relevant phase boundary is approached to \(\text{1C}\) and \(\text{2C}\) zero vortex states respectively. This behavior of the polarization is in direct contrast to the corresponding sharp transition occurring on the GS level, i.e. in the absence of nonlinear excitations (results not shown here for brevity) \([5,6]\).

BVB excitations turn out to be linearly stable configurations for all values of \(q \in (0.1, 1.95)\), with a relevant example shown in the BdG spectrum of Fig. 2(b) e.g. for \(q = 0.5\). Due to the single vortex contained in this configuration, only a single pair of negative energy modes is present in this spectrum. According to our discussion above, when activated, i.e., upon adding the associated to it eigenvector to the BVB solution, this mode leads to the precessional motion of the BVB structure. It is only for significantly deformed BVB configurations, namely for states where the bright soliton dominates the configuration corresponding to \(q \leq 0.4\), that oscillatory instabilities \((\text{like the one depicted in Fig. 2} a_2)\) appear. In order to appreciate the effect of the emergent eigenfrequency quartet on the BVB solution, we have added to the latter the corresponding quartet eigenvector. A close inspection of the associated density profiles along both spatial directions, namely \(|\Psi_{m_F}(x, y_0 = 0)|^2\) and \(|\Psi_{m_F}(x_0 = 0, y)|^2\), illustrated in Fig. 2(c)\( - (f_2)\) reveals that such an addition leads to a right-shifted and asymmetric with respect to the origin, \((x, y) = 0\), BVB structure, an outcome that is more pronounced along the \(x\)-direction. With the asymmetry being evident for the \(m_F = 0\) vortex component and the shift becoming visible in the corresponding \(m_F = -1\) bright soliton component. In both cases for comparison we show also the relevant unperturbed BVB solution \([\text{see dashed black lines in Fig. 2(c)} - (f_2)]\) in each case. Finally, the anomalous mode ceases to exist for \(q < 0.3\) signaling the transition to the GS of the \(\text{AF}\) phase.

**B. Ferromagnetic nonlinear excitations**

Turning to FM spin-interactions \((c_1 < 0)\) three phases can be realized as \(q\) is varied, supporting GS with an occupancy ranging from \(\text{1C}\) to \(\text{3C}\) \([8]\). In particular, the so-called \(\text{1C}\) fully magnetized along the \(+z (-z)\)-direction
I of a perturbed VBV \([\text{BVB}]\) solution for present for BVB configurations. (c) the background modes. Contrary to AF interactions, FM VBV entities feature three anomalous mode pairs but a single pair is present for BVB configurations. (c) the corresponding polarization curve is found to be feature larger densities as depicted in the insets of Fig. 2C. These states possess zero net magnetization \(m\) and \(|m| = 0\). Eventually, the 3C BVB structure deforms into an easy-axis (EA) phase exists for \(q < 0\). The 3C easy-plane (EP) phase occurs for \(0 < q < q_T\) and the 1C PO phase is characterized by \(q > q_T\) [8, 13, 27]. In the latter two inequalities \(q_T = 2c_1 n\) (which equals 0.05, for our chosen parameters) designates the threshold between the involved phases with \(n\) being the peak density at the trap center. In this FM spinor setting, VBV stationary states are identified for \(-2.74 \leq q < -0.14\) (third row of Table I). These states possess zero net magnetization and \(-1 < P < 1\) as shown in Fig. 4. They also have density profiles, \(|\Psi_m(x, y)|^2\), similar to their AF siblings [Fig. 3(a2) – (a1)]. Strikingly, FM VBV waves are more persistent configurations when compared to their AF counterparts. They are seen to penetrate deeper into the EA phase before deforming into a 2C vortex structure for smaller \(q\) values (third row of Table I). They further transform slower to the PO GS following an increment of \(q\) towards the phase transition boundary \((q = 0)\). As such the corresponding polarization curve is found to be right-shifted thus being closer to the origin when compared to the relevant AF one.

For \(0 < q < q_T\), i.e. within the EP phase, the existence of BVB stationary states is also unveiled and presented in Fig. 3. It is noteworthy that FM BVB structures also feature larger \(q\) intervals of existence in comparison to their AF analogues (fourth row of Table I). These structures penetrate the PO regime with the underlying 3C densities as depicted in the insets of Fig. 3(b2) – (b1). Recall that at the GS level the PO phase exists for \(q \geq q_T\). Eventually, the 3C BVB structure deforms into the 1C vortex configuration illustrated, e.g., for \(q = 2.5\) in Fig. 3(b1). The existence of these states is (parametrically) prolonged also following a decrease of \(q\) until a 2C Thomas-Fermi state is reached within the EA phase [Fig. 3(b3)-(b6)]. This has as a result, a left-shifted polarization curve that is closer to the origin when compared to the relevant AF one.

Investigating the stability of both configurations we find that, as their AF counterparts, VBV and BVB stationary states experience stable intervals of existence. This result can be verified by inspecting the BdG spectra shown in Fig. 2(a1) for the VBV solution and in Fig. 3(a2) and (b2) for the BVB one. Notice that in both cases and for the parametric intervals shown, all eigenfrequencies maintain their real nature, i.e., \(\text{Im}(\Omega) = 0\). However, these structures further feature narrow \(q\) intervals where oscillatory instabilities occur. One such example is presented regarding the VBV entity for \(q = -1.2\) in the BdG spectrum of Fig. 2(b1). Similarly to the AF cases discussed above, also here the emergence of an eigenfrequency quartet is observed, that owes its presence to the collision of the higher-lying negative energy mode, \(AM_5\), with a positive energy one. Importantly though, and also in sharp contrast to the AF VBV solutions, three instead of two AMs appear in the spectrum of this configuration. As stated earlier, since two vortices participate in this configuration two anomalous mode pairs are to be expected for this stationary state. Thus, we initially investigate further the presence of the lowest-lying mode, namely \(AM_1\). This mode appears remarkably close to the zero eigenfrequency axis and remains in near the latter as \(q\) is varied till its destabilization slightly below the threshold separating the EA and the EP i.e. at \(q = -0.2\). It is found to be responsible, via its eigenvector, for a change in the width of the VBV configuration while it further causes a slight tilting of the VBV entity in both
destabilization takes place at $AM_1$ collides with a positive Krein background mode. In this latter case as it is shown in Fig. 4(d$_1$), (f$_1$), the VBV configuration even though tilted, shifted and lifted along $x = 0$, it further experiences a more flattened one directional density redistribution around the vortex core. This leads in turn to a spiraling of the 2D VBV entity, an outcome caused by the oscillatory instability.

As an example for the BVB solution, we choose the one of a significantly deformed, i.e., close to threshold, BVB excitation [Fig. 4(c$_2$)-(e$_2$)]. Notice that the bright soliton hosted in the $m_F = \pm 1$ spin-components dominates the configuration for $q = 0.2$. Perturbing this state with the eigenvector associated with the single —in this case— $AM_1$ pair, leads to a BVB entity significantly tilted along the two spatial directions with the bright soliton featuring also a slight shift along the $x$ axis. Snapshots during the spatiotemporal evolution of this perturbed entity are provided in Fig. 5(a$_1$) – (c$_5$). As expected, the precessional motion of the entire VBV structure is observed from the initial stages of the dynamics, with the bright soliton $m_F = \pm 1$ components remaining trapped in the course of the evolution around the vortex core, see Fig. 5(a$_1$) – (c$_5$). For comparison here, in the bottom panels of Fig. 5(d$_1$) – (f$_1$), a perturbed VBV excitation via the eigenvector of $AM_3$ is presented for $q = -1.0$. Two key findings are worth commenting here. The one concerns the fact that even though the amplitude of the perturbation for both structures is the same, the precession of the VBV excitation is not as pronounced as the one observed for the deformed BVB solution. However, and even more importantly irregular precession is featured by the VBV structure with the two vortices being out-of-phase throughout their motion. This is an outcome that has a drastic effect also on the bright soliton which, contrary to the BVB state, now remains unaffected.

C. Impact of higher-charge vorticity

Next, we aim to generalize our findings by considering different system sizes and vortex charges. In particular, in the former case we systematically vary the total number of particles within the range $N \in [1 \times 10^3, 2 \times 10^4]$ while in the latter situation vortices of $S = 2, 3$ are explored. Experimentally higher-charge vortices can be realized using the topological phase-imprinting technique $[$23$. Remarkably enough, by monitoring the polarization under $(q, N)$ variations reveals that the polarization of the FM spinor system remains insensitive under such parametric changes independently of the stationary configuration (not shown for brevity). Sizable deviations are only present when higher charge vortices are
FIG. 6. Polarization, $P$, in terms of the quadratic Zeeman coefficient $q$ for (a) a VBV and (b) a BVB configuration upon also varying the vortex charge $S$ (see legend). An increasing $S$ prolongs the region of existence of the 3C state with respect to $q$. $(c_1)-(c_3)$ $(d_1)-(d_3)$ Density contours of a stationary VBV state of charge $S = 2\ [S = 3]$ for $q = -1.0\ [q = -2.2]$, i.e. within the EA phase. The FM spin-1 BEC mixture contains $N = 10^4$ $^{87}$Rb atoms.

Vortices have charge $n$ in scalar $[60, 73–76]$ and two-component BECs $[77]$, the vortices are prone to decay into singly quantized vortex pairs $[68]$. Thus, higher charge vortices can effectively ping by nonlinear excitations such as the vortices studied here $[65]$. Notice that in all three cases the bright soliton of the $m_F = 0$ spin-component is not altered in contrast to the vortices of the $m_F = \pm 1$ spin-components (the $m_F = +1$ component is complementary to $m_F = -1$ and as such it is not shown here). Evidently, perturbing the VBV solution with the eigenvector related to $AM_4$ or $AM_5$ leads to an asymmetric and off-centered, predominantly along-$y$, VBV structure. This asymmetry, however, is found to be slightly more pronounced along the $x$-direction when perturbing the solution with $AM_5$ when compared to $AM_4$. Here, dynamical activation of $AM_4$ unveils the formation of anti-phase triangular patterns in the vortex $m_F = \pm 1$ components which along with an intact bright soliton $m_F = 0$ component precess around the trap. Contrary to the above dynamics, perturbing the VBV entity with $AM_5$ leads to the formation and robust propagation of a deformed structure. Here, the two vortices perform an irregular out-of-phase precession leaving in this way the bright soliton in the $m_F = 0$ component intact, but instead of forming triangles, they feature dipolarly elongated density distributions being inverted between the $m_F = +1$ and $m_F = -1$ components. However, addition of the eigenvector associated with $AM_7$ entails a completely different deformation. Here, the vortex develops around its core a density peak along the $x$-direction [not clearly visible in the scales shown in Fig. 6(d1)] and is seen to be lifted from $y = 0$ along the $y$-direction [Fig. 6(g1)]. Additionally, also the width of the vortices changes in both directions, growing wider in the $x$- and narrower in the $y$-direction, implying in turn a breathing core situation. Indeed, dynamical activation of $AM_7$ leads to the irregular (out-of-phase) precession of a spatially anisotropic oppositely elongated with respect to each other $m_F = \pm 1$ components within this breathing core VBV structure. The remaining eigenvectors associated with $AM_2$, $AM_3$ and $AM_6$ result respectively in an up-lifted and slightly shifted (off-centered) with respect to the origin $(x = y = 0)$ VBV entity, to a shifted and slightly asymmetric (tilted) VBV configuration and to a lifted, slightly shifted from $y = 0$ and tilted with respect to $x = 0$ VBV structure. Finally, we

This structure as can be seen for instance in Fig. 7(a1) for $q = -1.0$.

Among these modes the lowest-lying one, $AM_1$, residing close to the zero frequency axis, as in the FM $S = 1$ scenario, is found to be responsible for a slight tilting of the VBV excitation but predominantly causes a change in the width of the participating vortices. Importantly, though, also for higher charges, the number of negative Krein modes is greater than the one anticipated for an $S = 2$ VBV solution. Indeed, it is known $[77]$ that since the two vortices are doubly quantized in this case one can assign two anomalous mode pairs to each of the two participating vortices. These yield in turn four anomalous mode pairs for such a state rather than the seven identified herein. Thus in what follows, a visualization of the effect that three out of the seven modes have on the stationary VBV state is offered in Fig. 6(b1) $-(g_1)$. Notice that

complementary to the VBV configuration in contrast to the three found for a BVB solution. (b) Remarkably seven AMs present in the BdG spectrum of this configuration. The distinct spin-components are dominant effect that the eigenvectors related to AMi with i = 1, 2, . . . , 7, decrease in frequency but only around q = −0.05 cross the zero frequency axis signaling the termination of this nonlinear excitation.

On the other hand, S = 2 BVB solutions destabilize via two eigenfrequency zero crossings of the two principal AMs present in the BdG spectrum of this configuration. Namely, AM3 which is the higher-lying negative energy mode and AM2 being the lowest-lying one. These destabilizations take place at q = 0, i.e., at the threshold (q = 0) separating the EP and the EA phases, and q = 0.15. However, among the two only the second destabilization produces a sizable imaginary component being of the order of Im(Ω) ∼ 10−2. Also an oscillatory instability is identified for the S = 2 BVB entity appearing around q = 0.7. This is an instability that owes its existence to the collision of AM3 with a positive Krein mode giving rise to an eigenfrequency quartet similar to those identified for the S = 1 structures. There exists also a third anomalous mode for this BVB configuration. Namely AM1, that stems from a change in sign of a background mode from positive to negative. This mode appears in the BdG spectrum for q = 0.6 and remains present as q is further lowered towards the phase transition point.

The above-discussed modes are illustrated in Fig. 7(a2) while their activation leads to deformations of the stationary S = 2 BVB state shown in Fig. 7(b2) – (g2). Notice that the bright soliton mF = ±1 component remains almost unaffected when perturbed by either of the eigenvectors associated with AM1, AM2 and AM3, but the mF = 0 vortex component is drastically altered. The predominant effect that the eigenvectors related to AM1 and AM2 feature on this vortex state is its tilting. Namely, the left and right asymmetric redistribution of the density around the vortex core in the x−y−plane that is found to be more pronounced along the x-direction. However, the eigenvector related to AM3 is responsible for a deformation similar to the one found for the VBV structure when
perturbed with the eigenvector associated with $AM_7$ [see here Fig. 4 (d2), (g2)] involving an opposite change of the wavefunction width between the $x$- and $y$-directions, i.e., expansion in one and contraction in the other]. A case example showcasing the dynamical evolution of a perturbed $S = 2$ configuration is provided in Fig. 8 (a)–(d) for $q = 0.6$. Notice the structural deformation of the ensuing BVB structure caused by the addition of the eigenvector related to $AM_7$. Evidently, already at $t = 0$ a triangular pattern [52, 51, 52], breaking the radial symmetry of the trap along the azimuthal direction, is seen in Fig. 8 (a)–(d) whose precessional motion is then followed for times up to $t = 1.0$ [Fig. 8 (d1) – (d3)]. This strongly suggests that this deformation is caused by an azimuthal mode with triangular symmetry (i.e., an $e^{i\theta}$ perturbation mode). It is also worthwhile to mention that similar findings are also present for AF spinor BECs (not shown).

Moreover, we emphasize at this point that the robustness of stable VBV and BVB stationary states has been dynamically confirmed by monitoring their spatiotemporal evolution for times up to $t = 2.0$. Finally, in order to emulate the presence of a finite thermal fraction being usually present in cold atom experiments we introduced the following ansatz $\Psi_{\text{pert}}^{m_F} = \Psi^0_{\text{BEC}}(x,y) \left[1 + \epsilon \delta(x,y)\right]$. In this expression, $\epsilon$ accounts for the thermal fraction and $\delta(x,y)$ denotes a normally distributed perturbation with zero mean and variance unity [51]. Generically, this ansatz allows for the activation of the respective anomalous mode in the course of the evolution. Additionally, it should be noted that the AMs are converted to unstable eigendirections in the presence of a thermal fraction, correspondingly dominating the BEC dynamics, similarly to what is known, e.g., for two-component condensates in the work of [50].

This way, the destabilization mechanisms found above would be evident in a corresponding experimental realization.

### IV. QUENCH DYNAMICS ACROSS MAGNETIC PHASES

Having explicated the static properties of VBV and BVB nonlinear excitations, in the following we aim at addressing alterations of the ensuing waveforms being subjected to quenches of the $q$ parameter in order to cross the distinct magnetic phase boundaries (see also Fig. 1 and Fig. 3). To monitor the quench-induced dynamical evolution of the spinor gases at hand in an experimentally relevant fashion [7], we expose them to finite temperatures. In the mean-field framework in order to qualitatively account for thermal effects we utilize the following coupled system of three dissipative GPEs [27, 72]

\[
(i - \gamma) \partial_t \psi_0 = \hat{H} \psi_0 + c_0 (|\psi_{+1}|^2 + |\psi_0|^2 + |\psi_{-1}|^2) \psi_0 + c_1 (|\psi_{+1}|^2 + |\psi_0|^2) \psi_0 + 2c_2 \psi_1 \psi_0^* \psi_{-1},
\]

\[
(i - \gamma) \partial_t \psi_{\pm 1} = \hat{H} \psi_{\pm 1} + c_0 (|\psi_{\pm 1}|^2 + |\psi_0|^2 + |\psi_{\mp 1}|^2) \psi_{\pm 1} + c_1 (|\psi_{\pm 1}|^2 + |\psi_0|^2 - |\psi_{\mp 1}|^2) \psi_{\mp 1} + q \psi_{+1} + c_1 \psi_1^* \psi_{-1},
\]

Eqs. (10)–(11) $\hat{H} = \mathcal{H} - \mu m_F$ and $\gamma \ll 1$ is a dimensionless dissipative parameter that is connected to the spinors’ temperatures [72]. Typically, $\gamma \in [2 \times 10^{-4}, 2 \times 10^{-3}]$ refers to temperatures $T \in [10, 100]$ nK as has been discussed, e.g., in Ref. [72].

Representative examples among the extensive investigations performed herein, are presented in Fig. 9 and

![Instantaneous density profiles of a BVB solution illustrating the $m_F = -1$ (top), $m_F = 0$ (middle) and $m_F = +1$ (bottom) components, upon considering a quench from the PO to the AF phase with $q_1 = 0.3$ and $q_1 = -1.0$ (see legends). The structural deformation of the BVB entity corresponding to a precession and simultaneous spatial elongation of all components is monitored for times up to $t = 1.1$s. The vortical pattern at $m_F = 0$ acquires a dipolar spatial form. Spin-mixing from $m_F = 0$ to $m_F = \pm 1$ is also initiated from the early stages of the dynamics. The AF ($c_1 > 0$) BEC consists of $N = 10^4$ $^{23}$Na atoms and the damping parameter is $\gamma = 0.0023$.](image-url)
FIG. 10. Density profiles of the distinct spin-components (see legends) during the dynamical evolution of a quenched FM BVB excitation. The pre-quench value is $q_i = 0.15$ while the post-quench one is $q_f = 4.5$, namely deep in the PO phase. The precessional motion of the BVB configuration dominates in the course of the evolution. A simultaneous population transfer from the $m_F = \pm 1$ states to the $m_F = 0$ component also takes place. The FM ($c_1 < 0$) BEC consists of $N = 10^8$ $^{87}$Rb atoms being confined in a quasi-2D harmonic trap, while $\gamma = 0.0023$.

FIG. 11. Spin-mixing processes of an AF spin-1 BEC subjected to quenches of the quadratic Zeeman coefficient in the presence of finite temperature. Temporal evolution of the populations, $n_{m_F}(t)$, of the different spin-components considering quenches from the AF to the PO phase and vice versa for an AF spinor gas. The initial state configuration, having a pre-quench value $q = q_i$, refers to the underlying in each phase 3C stationary state (see legends). $\chi_1$, $\chi_2$ Transitioning from a 3C→2C state and from a $\chi_2$, $\chi_3$ 3C→1C for $\gamma = 0.01$ and for different post-quench quadratic Zeeman coefficients $q = q_f$ (see legends). $\chi_2$] Same as $\chi_2$] but for fixed $q = q_f = 4.5$ $\chi_2$ $q = q_f = 5.0$ and for $\gamma = 0.0023$, 0.01. The AF condensate ($c_1 > 0$) consists of $N = 10^8$ $^{23}$Na atoms.

FIG. 12. Spin-mixing of a FM spin-1 condensate for finite temperatures following quenches of the $q$ parameter. Dynamics of the involved populations, $n_{m_F}(t)$ upon considering quenches from the EP to the EA and the PO phases of a FM spinor gas. The initial state configuration, characterized by a pre-quench value $q = q_i$, is the relevant in each phase 3C stationary state (see legends). $(\chi_1),(\chi_2)$ Transitioning from a 3C→2C state and from a $(\chi_2),\chi_3$ 3C→1C for $\gamma = 0.01$ and for different post-quench quadratic Zeeman coefficients $q = q_f$ (see legends). $\chi_1$] $(\chi_2]$ Same as $\chi_2$] but for fixed $q = q_f = 3.0$ $\chi_2$ $q = q_f = 4.5$ and upon varying $\gamma$ (see legends). The initial state preparation refers to a 3C stationary solution at $q_i = 0.15$ for BVB and at $q = -0.3$ for the VBV. The FM condensate ($c_1 < 0$) contains $N = 10^8$ $^{87}$Rb atoms.

รูปที่ 11 ความขยับระหว่าง спинของ BEC แบบ AF ที่ถูกสัมผัสกับการ kuphen ของค่า 2 ใน dopaque ซึ่งมีอุณหภูมิที่ไม่เป็นตัวอย่าง. ความขยับของค่า $q$ ในแต่ละระยะเวลามีค่า $q_f = -1.0$. การเปลี่ยนแปลงที่ดูเหมือนว่าการส่งผ่านของ популяชัน переходไปใน $m_F = 0$ ที่ $m_F = \pm 1$ ถูกเปลี่ยนแปลง [ก็ตามรูปที่ 11 $\chi_1,b_1$] จากช่วงเริ่มต้นของการ kuphen ของค่า dopaque ของค่า dopaque ช่วงเริ่มต้นของการ kuphen ของค่า dopaque นิยามทำนอง precedes the precession of an initially stationary spinorial BVB structure. This motion is accompanied by a prominent elongation along with the instantaneous rotation of all three spin constituents. Moreover, the vortex experiences a structural deformation reminiscent of a doughnut-like pattern: an outcome that is further captured by the two mode motion of the relevant temporal evolution of the populations of the individual components illustrated in FIG. 11 $\chi_1,b_1$. This two mode motion is characterized by rapid oscillations of the populations and a long-time transfer (not shown in the presented timescales) where exchange of the populations between the $m_F = 0$ and $m_F = \pm 1$ takes place. No-
tice that the bright soliton $m_F = \pm 1$ components remain trapped around the vortex core, following its composite motion throughout the evolution. Turning to FM interactions and upon considering a quench from $q_i = 0.15$ (EP phase) to $q_f = 4.5$ (PO phase) it is observed that the precessional motion constitutes the dominant dynamical mode, entailing an arguably faster spin-mixing process when compared to the aforementioned AF scenario.

In order to shed light onto the underlying spin-mixing processes triggered by the quench, a close inspection of the population of the individual components is performed. Specifically, Figs. 11(a1) – (c2) and Figs. 12(a1) – (c2) capture the essence of our findings for a wide selection of pre- and post-quench quadratic Zeeman energies and for distinct $\gamma$ values. AF ($c_1 > 0$) and FM ($c_1 < 0$) condensates are treated on equal footing. For both spinor settings, transitions across the distinct magnetic phases are initiated from the relevant in each phase 3C VBV and BVB stationary states towards the corresponding 2C or 1C configuration. Irrespectively of the spinorial BEC system, spin-mixing processes are activated from the initial stages of the quench-induced dynamics. We find that population transfer occurs faster for larger post-quench values of $q_f$ accessing this way states that are deeper in the relevant magnetic phase [Figs. 11(a1), (b2) and Figs. 12(a1), (b2)]. Additionally, spin-mixing is accelerated for a larger dissipation parameter $\gamma$ being in turn related to higher temperatures, see for instance Figs. 11(c1), (c2) and Figs. 12(c1), (c2). We also remark that slightly enhanced intercomponent population transfer arises for AF when compared to FM interactions as can be inferred by comparing Fig. 11(a2) and Fig. 12(a2). Finally, it is important to note here, that similar to the aforementioned findings occur during the nonequilibrium dynamics of higher charge excitations. However, in this case, the spin-mixing processes discussed above, are found to be relatively accelerated.

V. CONCLUSIONS AND FUTURE PERSPECTIVES

In the present work the existence, stability as well as the quench-induced dynamics of VB-type nonlinear excitations arising in 2D harmonically trapped spin-1 antiferromagnetic and ferromagnetic BECs have been explored. Our investigation has been focusing on variations of the quadratic Zeeman energy shift so as to access and subsequently cross the distinct magnetic phases of such settings. A systematic Bogoliubov de-Gennes linearization analysis has been utilized for the extraction of the stability properties of the considered nonlinear excitations.

In particular, the existence of VBV and BVB stationary states has been exemplified, with the former being present in the antiferromagnetic and the easy-plane phases for antiferromagnetic and ferromagnetic spin-interactions respectively. On the contrary, BVB solutions appear in the polar phase of either antiferromagnetic or ferromagnetic spinors. In this latter scenario, stable BVB structures are also found within the easy-plane phase too. In both settings deformations of the ensuing waveforms as the associated transition boundary is approached are explicated complementing this way the phase diagram of these type of nonlinear excitations in the $(c_1, q)$—plane. Indeed, we find for both types of structures, where they may degenerate into two-component ones or into single component ones, as the relevant $q$ parameter is varied.

It turns out that independently of their flavor and also of their charge, the aforementioned entities exhibit stable intervals of existence that can be interrupted by narrow windows where oscillatory instabilities take place. Indeed, we have elaborated on the number of anomalous mode eigendirections that the structures bear and thus the number of potential instabilities, as well as illustrated when these instabilities may materialize as a result of collision of these anomalous modes with positive energy ones. We have also illustrated the dynamical outcome of excitation of different ones among these anomalous modes. The robustness or unstable dynamics of the above-described entities are confirmed via the dynamical evolution of the spinor system demonstrating for instance the precessional motion of VBV and BVB spinors and their structural deformation towards —among others— triangular-shaped patterns.

We have further investigated the quench-induced dynamical evolution of the aforementioned three-component spinors at finite temperatures so as to appreciate the system’s dynamical response. Here, it is found that spin-mixing processes occur faster for larger postquench quadratic Zeeman energy shifts and an increasing dissipation parameter. Also, population transfer is slightly enhanced when considering antiferromagnetic instead of ferromagnetic spin-dependent interactions. Monitoring the nonequilibrium dynamics reveals, among others, the activation of the precessional motion along with a spatial elongation of the spinorial nonlinear excitations, irrespectively of their specific nature and spin-interactions. The above processes are accelerated when higher charge vortices are contained in the spinorial configuration. The latter also bear a significantly larger number of anomalous modes and, thus, potentially unstable eigendirections.

There exist several extensions of the present work worth pursuing in future endeavors. A straightforward generalization would be to study the quench dynamics in a $^7\text{Li}$ spin-1 BEC where the strong ferromagnetic spininteraction would certainly enhance the spin-mixing processes which might be possibly associated with a richer pattern formation. Additionally, exploring the interaction effects of vortex lattices as well as their stability and dynamics in spinor setups is of direct relevance, due to the potential of inclusion of external rotation [2, 11]. Indeed, it is already of significant recent interest to explore the interaction of two multi-component vortical patterns, as has been explored recently in two-component settings,
e.g., in \[ \text{81, 82} \] (see also references therein). Moreover, in the current setup the inclusion of three-body recombination processes as a dissipative mechanism in selective spin-channels constitutes a situation that accounts for possible experimental imperfections \[ \text{83}. \] Yet another fruitful perspective is to consider domain-walls formed by two out of the three spin-components with the remaining one being a nonlinear excitation of different flavor, e.g. a vortex \[ \text{84}. \] This setting will enable one to devise particular spin-mixing channels and consequently study dynamical pattern formation.

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