U(1) Problem at Finite Temperature

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Abstract. We model the effects of a large number of zero and near-zero modes in the QCD partition function by using sparse chiral matrix models with an emphasis on the quenched topological susceptibility in the choice of the measure. At finite temperature, the zero modes are not affected by temperature but are allowed to pair into topologically neutral near-zero modes which are gapped at high temperature. In equilibrium, chiral and U(1) symmetry are simultaneously restored for total pairing, evading mean-field arguments. We analyze a number of susceptibilities versus the light quark masses. At the transition point the topological susceptibility vanishes, and the dependence on the vacuum angle $\theta$ drops out. Our results are briefly contrasted with recent lattice simulations.

INTRODUCTION

The current theoretical resolution of the U(1) problem in QCD relies on the assumption that the QCD vacuum supports a finite topological susceptibility [1–3]. This assumption is supported by current lattice simulations [4], anomalous Ward identities [5], chiral effective Lagrangians [6] and canonical quantization [7], although there are questions in covariant quantization [8].

In this paper, we will adopt the current view and proceed to analyze what happens to the U(1) problem at finite temperature. At infinite temperature, both the anomaly and topological effects become negligible, so that the U(1) symmetry is effectively restored. In this limit there is an exact chiral and U(1) degeneracy modulo quark masses. The question then is what happens at finite temperature? Does the U(1) restoration coincide or differ from the conventional chiral restoration [9]?
Recently, a number of lattice simulations [10–13] and model calculations [14] have attempted to answer this and other questions with somewhat opposite conclusions. It is our purpose in this letter to try to address some of these issues in a lattice motivated matrix model, focusing on the interplay between zero and near-zero modes, the effects of light quark masses and the importance of the thermodynamical limit. In many ways our analysis will parallel instanton-like calculations in a solvable context, with interesting lessons for these calculations as well as lattice simulations. Indeed, one of the main thrust of the present letter is to provide a minimal framework for a model analysis of current lattice results.

In section 2, we motivate the use of a class of chiral matrix models by reviewing some recent lattice calculations. In section 3, we discuss the saddle point results following from the present model under some generic assumptions on the interplay between zero and near-zero modes. In section 4, we address certain aspects of the chiral and U(1) transitions, including a number of susceptibilities. In section 5, we comment on a number of recent lattice simulations in light of our results. Our conclusions are in section 6.

FORMULATION

FIGURE 1. Lattice results from Ref. [10] for quenched $\beta = 6.2$ (left) and staggered, $N_f = 2$, $\beta = 5.55$, and $mqa = 0.00625$ (right) lattices for topological sectors 0 (crosses), 1 (open circles) and 2 (full boxes).

Lattice Motivation

Recently Kogut, Lagae and Sinclair [10] have studied the chirality content $r_n \equiv \langle n|\gamma_5|n\rangle$ of the low-lying quark eigenstates $\lambda_n$ for staggered fermions with $D|n\rangle = \lambda_n|n\rangle$. Their results after cooling are displayed in Fig. 1 in the $(r, \lambda)$ plane for $\beta = 6/g^2 = 5.55$ and 6.2, on a $16^3 \times 8$ lattice. $|r| = 1$ in the continuum (about $1/4$ on the lattice) corresponds to an eigenvalue
with topological charge ±1, while \( r = 0 \) corresponds to a non-topological eigenvalue. The high temperature configurations (\( \beta = 5.55 \)) are characterized by a depletion in the zero modes (\( r = 1/4 \)), and an enhancement in the near-zero modes (\( r = 0 \)). Throughout we will refer to the modes with definite chirality as zero modes, and those without as near-zero modes. We note that at high temperature and in the continuum, the near-zero modes are gapped by \( \pm \pi T \).

**Model**

A simple way to analyze the interplay between the zero and near-zero modes around the chiral transition point is through a matrix model. A pertinent example for the fermion matrix \( D \), was discussed in [15] (and references therein)

\[
D = \begin{pmatrix}
ime^{i\theta} & A + d & 0 & \Gamma_R^\dagger \\
A^\dagger + d & ime^{-i\theta} & \Gamma_L^\dagger & 0 \\
0 & \Gamma_L & ime^{i\theta} & B \\
\Gamma_R & 0 & B^\dagger & ime^{-i\theta}
\end{pmatrix},
\]

(1)

and the partition function is

\[
Z[m, \theta] = \langle \det D \rangle .
\]

(2)

For each flavor, the entries in the matrices have respectively \( n, n, n_+ \) and \( n_- \) elements corresponding to the number of right handed near-zero modes, left handed near-zero modes, right handed zero modes and left handed zero modes. Here \( d \) denotes a diagonal matrix with equal entries, \( d \), the matrices assigned to the near-zero modes are square matrices, while the ones assigned to the zero modes are rectangular matrices. The fluctuations in the rectangularity of the matrices induce the proper \( \text{U}(1) \) breaking [16]. The hopping between zero and near-zero modes is characterized by the overlap matrices \( \Gamma \).

The averaging in (2) is done with respect to the local fluctuations in the topological charge \( \chi = (n_+ - n_-) \), with a Gaussian width fixed by the quenched topological susceptibility \( \chi_* \),

\[
e^{-\frac{(n_+ - n_-)^2}{2\chi_*^2}} = e^{-\frac{\chi^2}{2\chi_*^2}}
\]

(3)

and a Gaussian measure for the random matrix elements \( A, \Gamma, B \), with width \( \Sigma = 1 \). The latter is physically tied to the quark condensate \( |\langle q^\dagger q \rangle| \sim (200 \text{MeV})^3 \). For simplicity, we take the width of the Gaussians to be the same since the hopping between the low-lying modes may be random enough not to distinguish between zero and near-zero modes. The temperature effects on the near-zero modes are parameterized by the deterministic and off-diagonal entries \( d \). They cause the near-zero modes to be gapped by typically
\( d = \pm \pi T \) at high temperature, setting the range of validity of the current assumptions [15]. The depletion in the number of zero modes caused by an increase in the temperature will be discussed below.

We observe that the columns and rows in the fermion matrix (1) may be rearranged to give instead

\[
\begin{bmatrix}
ime^{i\theta} & 0 & A+d & \Gamma_R^t \\
0 & ime^{i\theta} & \Gamma_L & B \\
A^\dagger +d & \Gamma_L^* & ime^{-i\theta} & 0 \\
\Gamma_R & B^\dagger & 0 & ime^{-i\theta}
\end{bmatrix}
\]  

(4)

where we grouped the right and left handed modes, respectively. We denote the total number of modes (size of the full matrix) by \( 2N = 2n + n_+ + n_- \). We note that in the limit \( mV \ll 1 \) the matrix model (2) yields sum-rules for the quark eigenvalues that are consistent with those discussed in [17], following the general arguments in [18] with the identification \( N = nV \), where \( n \) is an order 1 quantity measuring the density of modes. Physically, the latter sets the scale for the fermionic contribution to the energy density.

**Distribution of Eigenmodes**

The matrices \( D \) in (1) have (in the chiral limit) a continuous distribution of eigenvalues \( \tilde{\rho}(\lambda) \), with a superimposed Dirac delta function at zero virtuality \( \lambda = 0 \), for fixed topological charge \( \chi \),

\[
2N \rho(\lambda) = |\chi|\delta(\lambda) + 2N \tilde{\rho}(\lambda) .
\]  

(5)

To assess the dependence of \( r_n \) on \( \lambda \) one can use the fact that \( \lambda_n + i\varepsilon r_n \) are the real and imaginary parts of the non-hermitian operator \( D + \varepsilon i \mathbf{1}_5 \), in first order perturbation theory. Here \( \mathbf{1}_5 = \text{diag}(1, -1, 1, -1) \) with each identity assigned to its pertinent subspace. Taking \( \varepsilon \) infinitesimally small and rescaling the imaginary part of the eigenvalue we obtain the abovementioned dependence\(^1\).

One can now use the chiral structure of \( D \) (its anti-commutativity with \( \mathbf{1}_5 \)) to show that the square

\[
(D + i\varepsilon \mathbf{1}_5)^2 = D^2 - \varepsilon^2
\]  

(6)

is a hermitian operator. This means that the pair \( \lambda_n + i\varepsilon r_n \) is either purely real or purely imaginary. It follows that all nonzero eigenvalues (\( \lambda_n^2 \gg \varepsilon \)) have vanishing \( r_n \) while the \( |\chi| \) topological ones have zero eigenvalue and \( r_n = \pm 1 \). In the limit \( mV > 1 \) the random matrix model (2) allows for a model dependent assessment of the distribution of the low-lying modes of the Dirac operator in the infinite volume limit using the methods discussed in [20].

\(^1\) It is interesting to note that non-hermitean operators of this form with \( \varepsilon = 1/\sqrt{2N} \) yield a generic distribution of non-hermitean eigenvalues [19].
PARTITION FUNCTION

Bosonization

For equal masses, the partition function (2) can be readily bosonized. The result for fixed size matrices is

\[ Z_{N,n\pm}[m, \theta] = \int dP dP^\dagger e^{-NP P^\dagger} e^{-\frac{\chi^2}{2\chi^* V}} \times \left[ (z+P)(\bar{z}+P^\dagger) + d^2 \right]^n (z+P)^{n_+} (\bar{z}+P^\dagger)^{n_-}. \]

Here \( z = me^{i\theta} \) stands for degenerate flavors. The number of near-zero modes \( n \), and the number of zero modes \( n_{\pm} \), fix the size of the matrices in (7). However, their distribution is partly fixed by the Gaussian distribution (3), while the remaining part is fixed by equilibrium arguments as we now discuss.

Detailed balance

At finite temperature the change in the total number of zero modes can be argued generically. Indeed, with increasing temperature the zero modes may deplete either by pairing into topologically neutral aggregates of near-zero modes [21] or screening [22]. For the simplest neutral aggregate (molecule [21]), this pairing is reminiscent of the Kosterlitz-Thouless one in four-dimensions [23].

The chemistry of small neutral aggregates can be described by the probability of formation \( p_f \) and breaking \( p_b \). For molecular arrangements in equilibrium, detailed balance implies

\[ p_b n = p_f \left( \frac{n_-}{N} \right) n_+. \]

The l.h.s stands for the number of pairs broken. The r.h.s stands for the number of pairs formed which is the formation probability, times the probability \( (n_-/N) \) to find an unpaired negative charge, times the total number of positive charges \( n_+ \). In equilibrium, the number of pairing matches the number of breaking.

We may now calculate the square of \( (n_+ + n_-) \) from the constraint \( 2N = 2n + n_+ + n_- \) and subtract \( \chi \) to obtain

\[ n_+ n_- = (N-n)^2 - \frac{1}{4} \chi^2. \]

Therefore \( n \) satisfies

\[ \frac{1}{2} n^2 - N(1 + \delta) n + \frac{1}{2} N^2 - \frac{1}{8} \chi^2 = 0 \]
with \( \delta = p_b/2p_f \). Since from (3) \( \chi \sim \sqrt{N} \) we obtain

\[
n = N \left( 1 + \frac{\delta - \sqrt{\delta^2 + 2\delta}}{\alpha} \right) \frac{1}{8N} \frac{1}{\sqrt{\delta^2 + 2\delta}} \chi^2 + \mathcal{O}(\chi^4)
\]

and

\[
n_\pm = N - n \pm \frac{\chi}{2}.
\]

We are effectively left with a ‘filling fraction’ \( \alpha \) and a contribution to the topological susceptibility \( \chi \). For \( \alpha \to 1 \) we have \( \delta \to 0 \) \((p_b \ll p_f)\). In this case, practically all the zero modes are paired (the unpaired ones are of order \( 1/\sqrt{N} \)) with \( U(1) \) effectively restored. For \( \alpha \to 0 \) we have \( \delta \to \infty \) \((p_b \gg p_f)\). In this case, all the zero modes are unpaired and \( U(1) \) is broken.

### Saddle point analysis

We insert (11-12) into the partition function (7), and perform a linear shift in \( \chi \),

\[
\chi = \tilde{\chi} - i2N \cdot y
\]

with the requirement that the term linear in \( \tilde{\chi} \) vanishes in (7). The resulting consistency condition (saddle point) reads

\[
\frac{1}{2} \log \frac{z + P}{\tilde{z} + P^\dagger} + 2iay = 0
\]

where

\[
a = \frac{1}{\chi_*} n + \frac{\alpha}{2(1 - \alpha^2)} \log \frac{|z + P|^2 + d^2}{|z + P|^2}.
\]

The parameter \( y \) is just proportional to the average topological charge

\[
\langle n_+ - n_- \rangle = 2Vny,
\]

while \( P \) and \( P^\dagger \) in the above equations are the saddle point solutions following from the ‘action’

\[
(1 - \alpha_*) \log |z + P|^2 + \alpha_* \log (|z + P|^2 + d^2) - iy \log \frac{z + P}{\tilde{z} + P^\dagger} - PP^\dagger + \frac{2n}{\chi_*} y^2
\]

where the effective ‘filling fraction’ is

\[2) \text{ Eq. (11) holds for } 1 - \alpha \gg N^{-1/4}, \text{ hence in this paper the limit } \alpha \to 1 \text{ is understood always after the thermodynamical limit } N \to \infty.\]
\[ \alpha_s = \alpha \left(1 + \frac{y^2}{1 - \alpha^2}\right). \]  

Writing out the saddle point equations for \( P \) and \( P^\dagger \) and subtracting yield

\[ \bar{z}P - zP^\dagger = 2iy. \]  

This suggests the decomposition \( e^{-i\theta}P = Q + iy/m \), with \( Q \) real, being the chiral condensate \(|\langle q^\dagger q \rangle|\) and satisfying the saddle point equation

\[ (1-mQ-Q^2 - \frac{y^2}{m^2}) \left[(m+Q)^2 + \frac{y^2}{m^2} + d^2\right] = \alpha_s d^2. \]  

This equation will be analyzed next.

**RESULTS**

The model is totally specified by (1-3) and (8). The thermodynamical limit will be understood as \( N, V \to \infty \), with \( N/V \) fixed. The parameters are: the width of the Gaussian \( \Sigma = 1 \), the current mass \( m \), the quenched topological susceptibility \( \chi \), the deterministic entries \( d \), the vacuum angle \( \theta \), the filling fraction \( \alpha \) and the mode-density \( n = N/V \). Generically, the effects of temperature cause \( 0 < d = \pi T \) and \( 0 \leq \alpha \leq 1 \).

**Chiral condensate**

The \( \chi \) saddle point equation (14) has a trivial solution, \( y = 0 \) for \( \theta = 0 \). Inserting this back into the equation for the condensate \( Q \) and setting \( m = 0 \) yields

\[ Q^2 = \frac{1}{2} \left(1 - d^2 + \sqrt{(1-d^2)^2 + 4(1-\alpha) d^2}\right). \]  

This result is similar to the one considered by [24], although our physical interpretation is different. Indeed, in our case \( \alpha \) measures the amount of U(1) breaking and follows from the rectangular character of the matrices as opposed to the square matrices used in [24]. It is fixed by detailed balance. We have ignored the trivial solution with \( Q = -m \), by maximizing the effective action (17)

\[ \frac{F}{N} = -Q^2 + \log (m+Q)^2 + \alpha \log \frac{(m+Q)^2 + d^2}{(m+Q)^2}. \]  

For \( d < 1 \), the solution with \( Q \neq 0 \) sets in independently of \( \alpha \). For \( \alpha = 1 \), the zero modes pair into near-zero modes, and \( Q^2 = 1-d^2 \) [15,24,25]. This is a
U(1) symmetric phase with broken chiral symmetry. A qualitative assessment of the range of temperature where this can take place follows by reinstating the dimensionful constants, that is \( d = \pi T < \sqrt{\Sigma} \). Hence \( T < 70 \text{ MeV} \), which is outside the range of validity of our model (see above). However, this points to the fact that the near-zero modes are sufficiently gapped at already moderate temperatures, leaving the zero modes as the only contributors to the chirally broken phase. Indeed, at \( d = 1 \) we have \( Q = \sqrt{1 - \alpha} \), which is zero-mode driven. From here on, only the case with \( d > 1 \) will be discussed unless specified otherwise.

For \( \alpha \to 1 \),

\[
Q(m) = \frac{d}{\sqrt{d^2 - 1}} \sqrt{1 - \alpha} + \mathcal{O}(m)
\]  

while for \( \alpha = 1 \),

\[
Q(m) = \frac{m}{d^2 - 1} \left[ 1 - \left( \frac{d^2}{d^2 - 1} \right)^3 m^2 \right]
\]

The pairing mechanism suggests an integer ‘exponent’ \( \delta = 1 \). We recall that for \( d = \alpha = 1 \), \( Q = m^{1/3} \) and \( \delta = 3 \) which is mean-field [15,24,25].

**Isotriplet susceptibilities**

A measure of U(1) breaking in the matrix model can be assessed by investigating the difference in the \( \pi^0 \) and \( a^0 \) isotriplet susceptibilities [10,11]

\[
\omega = \langle q^\dagger i15^3qq^\dagger i15^3q \rangle_c - \langle q^\dagger \tau^3qq^\dagger \tau^3q \rangle_c
\]

and is amenable to the quark eigenvalue distribution \( \rho(\lambda) \) through [12]

\[
\omega = 4m^2 \int_0^\infty d\lambda \frac{\rho(\lambda)}{(m^2 + \lambda^2)^2}.
\]

For \( \alpha \to 1 \), the matrix model yields \( Q \to 0 \) with a gapped spectrum in the chiral limit, hence \( \omega = 0 \). This observation is similar to the one we made in [26] without due care to the U(1) problem as we noted. In general, \( \omega \) can be related to the resolvent

\[
G(z) = \frac{1}{N} \left\langle \text{Tr} \left( \frac{1}{z - D} \right) \right\rangle.
\]

Specifically,

\[3^) \text{ In fact it would suffice that } \rho(\lambda) \text{ vanishes as } \sim \lambda^a m^b \text{ with } a > -1 \text{ and } a + b > 1.\]
\[ \omega = \frac{\text{Im } G(im)}{m} - \text{Re } G'(im) = \frac{Q(m)}{m} - Q'(m) \] (28)

where \(Q(m)\) follows from (20).

For \(\alpha < 1\) we have

\[ \omega = \frac{d}{d^2 - 1} \frac{\sqrt{1 - \alpha}}{m} + \mathcal{O}(m^2) \] (29)

which is to be compared to \(\omega \sim \sqrt{1 - \alpha/m}\) for \(d = 1\). The \(1/N\) corrections to (29) are

\[ 2 \left| \chi \right| \frac{1}{N m^2} + \frac{\chi^2}{N^2} (\ldots) . \] (30)

The first term is the contribution of the zero modes in (5) through (26). Since \(\chi \sim \sqrt{N}\) both contributions in (30) are subleading in comparison to (29) in the thermodynamical limit. These effects may still be present in current lattice assessments of \(\omega\) as we discuss below.

For \(\alpha = 1\), we have

\[ \omega = \frac{d^6}{(d^2 - 1)^3} m^2 + \mathcal{O}(m^4) \] (31)

implying that \(\omega\) flips from \(1/m\) to \(m^2\) at the transition point. We note that \(\omega \sim 1/m^{2/3}\) for \(d = 1, \alpha = 1\) which is the mean-field result [15]. It is noteworthy that only integer ‘exponents’ are produced by the pairing transition, a point in support of some general arguments made in [11].

**Topological susceptibility**

The topological susceptibility in the matrix model is simply given by

\[ \chi_{\text{top}} = -\frac{\partial^2}{\partial \theta^2} \log Z = -2 \frac{\partial y}{\partial \theta} . \] (32)

Expanding the consistency equation to linear order in \(y\), we obtain

\[ y = \frac{-\theta}{2a + \frac{1}{m(m+Q)}} \] (33)

with \(a\) defined in Eq. (15), which gives

\[ \frac{1}{\chi_{\text{top}}} = \frac{1}{\chi^*} + \sum_{i=1}^{N_f} \frac{1}{2m_i(m_i+Q_i)} + \frac{\alpha}{2(1-\alpha^2)} \log \frac{(m+Q)^2 + d^2}{(m+Q)^2} \] (34)
where we have reinstated the flavor dependence. The first contribution is the quenched susceptibility, the second contribution is the screening caused by the near-zero modes and the unpaired zero modes, and the third contribution stems from the paired zero modes. Note that $\chi_{\text{top}}$ vanishes not only for massless quarks but also for maximal pairing with $\alpha = 1$, as the asymmetry of $D$’s become minimal. This happens as $Q \to 0$, in qualitative agreement with recent lattice simulations [4].

**Pseudoscalar susceptibilities**

The connected and disconnected pseudoscalar susceptibilities associated with $q\dagger 1_5 q$ may be assessed in a similar way. These susceptibilities were recently addressed on the lattice [10]. In our case, the disconnected part $\chi_5^{\text{dis}}$ reads

$$
\chi_5^{\text{dis}} = \frac{1}{N} \left\langle \text{Tr} 1_5 \frac{1}{im - D} 1_5 \frac{1}{im - D} \right\rangle \tag{35}
$$

and is readily amenable to (34) through

$$
\chi_5^{\text{dis}} = \frac{1}{V} \left\langle \frac{(n_+ - n_-)^2}{m^2} \right\rangle = \frac{\chi_{\text{top}} m^2}{m^2}. \tag{36}
$$

In the broken phase $\chi_{\text{top}}$ is dominated by the the second term in (34) for small $m$, hence $\chi_5^{\text{dis}} \sim 1/m$. As $Q \to 0$, the limits $m \to 0$ (chiral) and $\alpha \to 1$ (pairing) do not commute. For fixed mass and $\alpha \to 1$, $\chi_5^{\text{dis}} \sim (\alpha - 1) \log m/m^2 \sim 0$, while for $\alpha = 1$ and $m \to 0$, $\chi_5^{\text{dis}} \sim 1/(1 + Q/m) \sim 1$. In both cases, $\chi_5^{\text{dis}}$ is finite. Note that for $d = 1$, $\chi_5^{\text{dis}} \sim m^2/3$.

The connected part $\chi_5^{\text{conn}}$ follows from the identity [16]

$$
\chi_{\text{top}} = \frac{2m}{N_f^2} Q - \frac{m^2}{N_f^2} \left( \chi_5^{\text{disc}} - \chi_5^{\text{conn}} \right). \tag{37}
$$

This is the random matrix version of the QCD anomalous Ward identity [5]. Hence

$$
\chi_5^{\text{conn}} = (N_f^2 + 1) \chi_5^{\text{disc}} - 2 \frac{Q}{m} \tag{38}
$$

for $m > 0$. Again, the connected part of the susceptibility is plagued with similar ambiguities in the chiral and pairing limits. For $\alpha = 1$ and $m \to 0$, $\chi_5^{\text{conn}}$ is finite.
Scalar susceptibilities

The connected and disconnected isosinglet susceptibility associated with \( q^\dagger q \) may be estimated in our case as well, following the lattice conventions [10,11],

\[
\chi^{\text{conn}}_S = \frac{1}{N} \langle \text{Tr} \frac{1}{im - D} \frac{1}{im - D} \rangle
\]

and

\[
\chi^{\text{disc}}_S = \frac{1}{N} \langle \text{Tr} \frac{1}{im - D} \text{Tr} \frac{1}{im - D} \rangle - \frac{1}{N^2} \langle \text{Tr} \frac{1}{im - D} \rangle^2.
\]

Both susceptibilities follow from (20). Specifically,

\[
\chi^{\text{conn}}_S = Q'(m) = \frac{1}{d^2 - 1}
\]

\[
\chi^{\text{disc}}_S = Q^2(m) = \frac{m^2}{(d^2 - 1)^2}.
\]

for \( \alpha = 1 \). This is to be compared with the mean-field result for \( d = 1 \), \( \chi^{\text{conn}}_S = 1/m^{2/3} \) and \( \chi^{\text{disc}}_S = m^{2/3} \). The factorized result for the disconnected isosinglet susceptibility follows from the absence of correlations in the number \( (n_+ + n_-) \).

\[ \theta \] angle dependence

In the symmetric phase and for small \( m \), the \( \theta \) dependence of the free energy \( \ln Z/V \) is simple. Indeed, since \( y \sim m \), for \( \alpha < 1 \) we may neglect the last term in the consistency equation (14) and obtain

\[
\sum_i \arctan \frac{y/m_i}{Q_i} = -\theta.
\]

The saddle point equation (20) in the chiral limit can be solved. Defining \( Q_i + iy/m_i \equiv |Q_{is}|e^{i\phi_i} \), the result for each flavor is

\[
\frac{y^2}{m_i^2 \sin^2 \phi_i} = Q_{is}^2.
\]

where \( Q_{is} \) follows from \( Q \) in (21) through the substitution \( \alpha \to \alpha_s \) for \( m = 0 \). Hence

\[
\sum_i \phi_i = \theta,
\]

\[
m_1 \sin \phi_1 = \ldots = m_{N_f} \sin \phi_{N_f}.
\]
These equations are analogous to the zero-temperature equations originally derived in QCD [1–3] and more recently in a matrix model [27]. Therefore the dependence of the free energy on $\theta$ in the broken phase is the same as the vacuum one. The temperature dependence is only implicit through $Q_\ast$.

As $Q_\ast$ approaches zero at the critical point and in the chiral limit, the dependence on $\theta$ changes. For small $m$, we may no longer neglect the last term in the consistency equation (14) as it diverges. Geometrically the line that intersects the curves of the $\arctan$’s becomes nearly vertical so that $y$ is for all purposes 0 regardless of the value of the $\theta$ angle. This extends the result $\chi_{\text{top}} = 0$ obtained earlier at $\theta = 0$ to $\theta \neq 0$.

The fact that the free energy no longer depends on $\theta$ at the critical point and beyond, may be traced to the occurrence of a non-analytic term $|\chi|$ in the partition function. Indeed from (10) and for $\alpha = 1$

$$n = N - \frac{1}{2} |\chi|.$$  \hspace{1cm} (47)

Inserting this into the partition function (7), we obtain

$$e^{i\theta \chi} e^{-b|\chi|} e^{-\frac{\chi^2}{2\pi^2}}$$  \hspace{1cm} (48)

with $b$ a positive factor stemming from (7). Performing the integral/sum over $\chi$ gives a vanishing contribution to the free energy $\log Z/V$. Specifically, the sum over $\chi$ is for a range of parameters well approximated by

$$2\text{Re} \frac{1}{1 - e^{i\theta-b}}$$  \hspace{1cm} (49)

which gives zero contribution to the free energy. This is a direct consequence of the total quenching of the topological fluctuations in the paired configurations of zero modes. The simultaneous restoration of chiral and U(1) symmetry at finite temperature yields a symmetric phase that preserves strong CP.

**COMPARISON TO LATTICE**

In a first lattice study by Bernard et al. [11], chiral symmetry restoration was found to precede the U(1) restoration. Their analysis relied on gauge configurations at fixed lattice spacing $a \sim 1/6T_c \sim 0.25$ fm [11] for $N_t = 6$. Since finite volume effects were not investigated, it may be that the small U(1) breaking effects detected in these simulations through a lattice measurement of $\omega$ for staggered fermions are of the type (30). However, simple estimates based on their numbers appear to be on the larger side of their reported results [20].

As we already noted, the pairing mechanism supports integer ‘exponents’ for $\omega$, a point sought in [11].
In a second lattice study by Kogut et al. [10], the low-lying quark eigenvalues of the staggered Dirac operator where investigated. Their analysis shows that the disconnected isosinglet susceptibility $\chi_5^{\text{dis}}$, decreases but remains finite in the high temperature phase. The finite result was shown to follow from the eigenmodes with finite chirality (topological). The conclusion was that the U(1) symmetry was not restored in the symmetric phase, although again finite volume effects were not investigated. In the present matrix model, we have observed that $\chi_5^{\text{dis}}$ remains finite in the chiral and U(1) symmetric phase for $d > 1$, when the thermodynamical limit is carried. Also, we have noted an ambiguity in the limits $m, \alpha \to 0, 1$, suggesting that the cooling procedure may be subtle while carrying the chiral limit. Indeed, lattice cooling affects the “filling fraction” $\alpha$.

In a third lattice study by Chandrasekharan et al. [12], the chiral condensate and $\omega$ were calculated using also staggered fermions for fixed $\beta = 5.3$ and $N_t = 4$. Although their results were found to be consistent with those of Bernard et. al. [11], they concluded that the anomalous effects were small, hinting at the possible restoration of U(1) in the symmetric phase. Although their conclusions are closer to ours in spirit, they differ in content since their small value of $\omega$ was obtained from a linear extrapolation in the current quark mass, as opposed to a quadratic extrapolation suggested by our results. Also, we have observed that the $\theta$-dependence drops in the symmetric phase in distinction to a general assumption they made.

In a fourth lattice study by Vranas et al. [13], lattice simulations with domain wall fermions were carried at $N_t = 4$. It was found that the high temperature phase preserves chiral symmetry with a small amount of U(1) breaking, although with a somehow heavier pion mass. The method preserves flavor symmetry and incorporates the effect of the anomaly at every stage of the simulation. It is indeed encouraging that the results of these simulations are the closest to ours.

**CONCLUSIONS**

We have used a simple matrix model to analyze the interplay between zero and near-zero modes at finite temperature. While the model finds its motivation in the lattice results described above, it was originally argued from an NJL model with U(1) breaking [15]. At finite temperature, the pairing mechanism at work in the zero mode sector is reminiscent of the one originally suggested in the context of instantons [21]. The present model is by no means exhaustive as additional effects, e.g. Debye screening, have been omitted. Their consideration goes beyond the scope of this work.

This notwithstanding, our results indicate that chiral and U(1) symmetry are simultaneously restored for maximum pairing of zero modes. Although the chiral condensate receives contribution from all low-lying modes, its depletion
to zero requires that the zero modes are paired and the near-zero modes are gapped. A simple estimate shows that the near zero modes are substantially gapped at moderately low temperatures, suggesting their early decoupling. This rules out the possibility of a U(1) restoration prior to a chiral restoration, and suggests that both symmetry restorations occur simultaneously.

The transition by pairing the topological charges is followed by a number of observations regarding the topological, scalar and pseudoscalar susceptibilities for small current quark masses. In particular, integer 'exponents' were observed in contrast to the fractional exponents expected from general universality arguments. These susceptibilities have been extensively studied on the lattice. Our comparison with the most recent lattice simulation using domain wall fermions is very encouraging, although some improvements regarding the extrapolation to zero quark mass and finite volume effects are still warranted in the staggered simulations. In many ways, our results should benefit the more complex instanton calculations when they become available.

Finally, we have shown that in the symmetric phase the topological susceptibility vanishes in the thermodynamical limit. As a result, the partition function develops a non-analyticity in the net topological charge that causes the symmetric phase to be CP even whatever the vacuum angle. While admittedly this is a result of the present matrix model, it should be interesting to see whether it carries to QCD in the infinite volume limit.

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