LEAST AREA PLANES IN HYPERBOLIC 3-SPACE ARE PROPERLY EMBEDDED

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ABSTRACT. We show that if \( \Sigma \) is an embedded least area (area minimizing) plane in \( \mathbb{H}^3 \) whose asymptotic boundary is a simple closed curve with at least one smooth point, then \( \Sigma \) is properly embedded in \( \mathbb{H}^3 \).

1. INTRODUCTION

The asymptotic Plateau problem in hyperbolic space asks the existence of a least area (area minimizing) plane \( \Sigma \subset \mathbb{H}^3 \) asymptotic to given simple closed curve \( \Gamma \subset S^2_\infty(\mathbb{H}^3) \). This problem is solved by Michael Anderson in his seminal papers [A1], [A2]. He proved the existence of a solution for any given simple closed curve in the sphere at infinity. Later, by using topological techniques, Gabai proved a similar result for \( \mathbb{H}^3 \) with any cocompact metric in [Ga]. Then, the author generalized these results to Gromov hyperbolic 3-spaces with cocompact metric [Co1].

Properly embeddedness of the solution has been questioned by both Anderson and Gabai. Nevertheless, only known results about the properly embeddedness of least area planes in \( \mathbb{H}^3 \) is the existence of some properly embedded least area plane for a given simple closed curve in \( S^2_\infty(\mathbb{H}^3) \) by [So1], [So2], [Co3]. It is still not known if there exists a nonproperly embedded least area plane in \( \mathbb{H}^3 \) whose asymptotic boundary is a simple closed curve in \( S^2_\infty(\mathbb{H}^3) \).

On the other hand, recently Colding and Minicozzi proved a very powerful result about properly embeddedness of complete embedded minimal disks in \( \mathbb{R}^3 \) in [CM]. They proved Calabi-Yau Conjectures for embedded surfaces by relating intrinsic distances and extrinsic distances of the minimal disk. As a corollary, they proved that any complete embedded minimal plane in \( \mathbb{R}^3 \) must be proper.

In this paper, we prove an analogous result in \( \mathbb{H}^3 \). We show that if \( \Sigma \) is an embedded least area plane in \( \mathbb{H}^3 \) whose asymptotic boundary is a simple closed curve with at least one smooth point, then \( \Sigma \) is properly embedded in \( \mathbb{H}^3 \). Instead of relating the intrinsic and extrinsic distances as Colding and Minicozzi did, we use powerful topological arguments. The main result of the paper is as follows:

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Theorem 5.2. Let $\Sigma$ be a complete embedded least area plane in $\mathbb{H}^3$ with $\partial_{\infty} \Sigma = \Gamma$ where $\Gamma$ is a simple closed curve in $S^2_{\infty}(\mathbb{H}^3)$ with at least one smooth ($C^1$) point. Then, $\Sigma$ must be proper.

The organization of the paper is as follows: In the next section we will cover some basic results which will be used in the following sections. In section 3, we will analyze the intersection of a least area plane $\Sigma$ with balls exhausting $\mathbb{H}^3$. Then in section 4, we will prove the key lemma which is the most important step for the main result. In section 5, we will prove the main result. Finally in section 6, we will have some concluding remarks.

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2. Preliminaries

In this section, we will overview the basic results which we will use in the following sections. First, we will give the definitions of least area (area minimizing) planes.

Definition 2.1. A least area disk is a disk which has the smallest area among the disks with the same boundary. A least area plane is a plane such that any compact subdisk in the plane is a least area disk.

Definition 2.2. An immersed surface $S$ in $\mathbb{H}^3$ is proper if the preimage of any compact subset of $\mathbb{H}^3$ is compact in the surface $S$. If an embedded surface $S$ in $\mathbb{H}^3$ is proper, we will call $S$ as properly embedded.

Definition 2.3. Let $A$ be a subset of $S^2_{\infty}(\mathbb{H}^3)$. Then the convex hull of $A$, $CH(A)$, is the smallest closed convex subset of $\mathbb{H}^3$ which is asymptotic to $A$. Equivalently, $CH(A)$ can be defined as the intersection of all supporting closed half-spaces of $\mathbb{H}^3$.

It is a well-known fact in minimal surface theory that if $M$ is a minimal surface in $\mathbb{H}^3$ with $\partial_{\infty} M = \Gamma$, then $M \subset CH(\Gamma)$.

Now, we will quote the basic results on asymptotic Plateau problem.

Theorem 2.1. [A2] Let $\Gamma$ be a simple closed curve in $S^2_{\infty}(\mathbb{H}^3)$. Then there exist a complete least area plane $\Sigma$ in $\mathbb{H}^3$ asymptotic to $\Gamma$ at infinity.

Later, Hardt and Lin showed regularity at infinity for these solutions in [HL]. Then, in [To], Tonegawa generalized this result to any complete constant mean curvature hypersurfaces in $\mathbb{H}^n$. The following theorem is indeed true for any dimension. For simplicity, we only mention the result which we are interested in.

Theorem 2.2. [HL], [To] Let $\Gamma$ be a $C^1$ regular simple closed curve in $S^2_{\infty}(\mathbb{H}^3)$, and $\Sigma$ be a least area plane in $\mathbb{H}^3$ asymptotic to $\Gamma$. Let $\overline{\mathbb{H}}^3$ be the compactification
of $\mathbb{H}^3$ with $S^2_\infty(\mathbb{H}^3)$. Then, there is a neighborhood $X$ of $S^2_\infty(\mathbb{H}^3)$ in $\overline{\mathbb{H}^3}$ such that $(\Sigma \cup \Gamma) \cap X$ is a finite union of $C^1$ submanifolds of $\overline{\mathbb{H}^3}$.

The following is a simple lemma which states that the intersection of a least area plane with a ball is generically a disjoint union of disks.

**Lemma 2.3.** \cite{A2} Let $\Sigma$ be a complete minimal plane with $\partial_{\infty}\Sigma = \Gamma$ where $\Gamma$ is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$. Then for almost all $r > 0$, $\Sigma \cap B_r(0)$ is a disjoint union of disks.

The following lemma will be used later. The proof basically uses Meeks-Yau exchange roundoff trick for least area disks \cite{MY2}.

**Lemma 2.4.** \cite{Co2} Let $\Gamma_1$ and $\Gamma_2$ be two disjoint simple closed curves in $S^2_\infty(\mathbb{H}^3)$. If $\Sigma_1$ and $\Sigma_2$ are least area planes in $\mathbb{H}^3$ with $\partial_{\infty}\Sigma_i = \Gamma_i$, then $\Sigma_1$ and $\Sigma_2$ are disjoint, too.

### 3. Intersection of Least Area Planes with Balls Exhausting $\mathbb{H}^3$

In this section, we will analyze the intersection of a given least area plane $\Sigma$ with balls with fixed center and increasing radius in $\mathbb{H}^3$. By Lemma 2.3, we know that for a generic radius $r > 0$, the intersection $B_r(0) \cap \Sigma$ is a collection of disjoint disks. If $\Sigma$ is not proper, we will show that there is a $r_0$ such that for a generic $r > r_0$, $B_r(0) \cap \Sigma$ contains infinitely many disjoint disks. In this section, we will analyze these disks, and classify them accordingly.

Let $\Sigma$ be a least area plane with $\partial_{\infty}\Sigma = \Gamma$ where $\Gamma$ is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$. Fix a point $0$ on $\Sigma \subset CH(\Gamma)$. Let $B_r(0)$ be a closed ball with radius $r$ (extrinsic) and center $0$ in $\mathbb{H}^3$. By Lemma 2.3, we know that for a generic radius $r > 0$, the intersection $B_r(0) \cap \Sigma$ is a collection of disjoint disks.

**Lemma 3.1.** Let $\Sigma$ be an embedded least area plane in $\mathbb{H}^3$ with $\partial_{\infty}\Sigma = \Gamma$ where $\Gamma$ is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$. If $\Sigma$ is not proper, then there exist $r_0 > 0$ such that for a generic $r > r_0$, $B_r(0) \cap \Sigma$ contains infinitely many disjoint disks.

**Proof:** Let $\varphi : D^2 \rightarrow \mathbb{H}^3$ be the smooth embedding with $\varphi(D^2) = \Sigma$. If $\Sigma$ is not proper, then there exist a compact subset $K$ of $\mathbb{H}^3$ such that $\varphi^{-1}(K) = E$ is not compact in $D^2$. This implies $E$ is not bounded in $D^2$.

Let $r_0 > 0$ be a generic radius with $K \subset B_{r_0}(0)$. If $\varphi^{-1}(B_{r_0}(0)) = E_{r_0}$, then clearly $E \subset E_{r_0}$ in $D^2$. By genericity, the intersection of $B_{r_0}(0) \cap \Sigma$ is a collection of disjoint disks. Assume that there are finitely many disks in the intersection, i.e $B_{r_0}(0) \cap \Sigma = \bigcup_{i=1}^N D_i$ where $D_i$ is a closed disk in $\Sigma$. Let $C_i$ be the boundary of the disk $D_i$, i.e. $C_i = \partial D_i$. Consider $\gamma_i = \varphi^{-1}(C_i)$ in $D^2$. Since $\gamma_i$ is a simple closed curve in $D^2$, it will enclose a bounded disk in $D^2$ which is the preimage of $D_i$. Hence, if the disks in the intersection $B_{r_0}(0) \cap \Sigma$ are finitely many, then the preimage $\varphi^{-1}(B_{r_0}(0)) = E_{r_0}$ must be bounded in $D^2$. This is a contradiction as $E$ is not bounded in $D^2$ and $E \subset E_{r_0}$. 
Since \( r_0 \) is any generic radius with \( K \subset B_{r_0}(0) \), for any generic \( r > r_0 \), \( B_r(0) \cap \Sigma \) contains infinitely many disjoint disks.

Now, we will categorize these infinitely many disks in the intersection of non-proper least area plane \( \Sigma \) and sufficiently large ball \( B_r(0) \) in \( H^3 \). Consider the intersection \( CH(\Gamma) \cap B_r(0) \) for sufficiently large \( r \), which is a convex body in \( H^3 \). Let \( \partial^+ CH(\Gamma) \) and \( \partial^- CH(\Gamma) \) be the two connected components of the boundary of \( CH(\Gamma) \). Let \( A_r \) be the annulus in \( \partial B_r(0) \) such that \( \partial A_r = \delta^+_r \cup \delta^-_r \) where \( \delta^+_r \subset \partial^+ CH(\Gamma) \cap \partial B_r(0) \) and \( int(CH(\Gamma)) \cap \partial B_r(0) \subset A_r \).

Since \( \Sigma \subset CH(\Gamma) \), for any disk \( D \) in the intersection \( B_r(0) \cap \Sigma \), \( \partial D \) must belong to \( A_r \). We call a disk \( D \) in the intersection \( B_r(0) \cap \Sigma \) separating if \( \partial D \) is essential in \( A_r \), and nonseparating otherwise.

Next, we will analyze the separating and nonseparating disks in the intersection of a ball with the nonproper least area plane. This analysis will play essential role in the main result.

4. Key Lemma

In this section, we will prove the key lemma which is the most important step for the main result. Roughly, we will show that if a disk \( D \) is nonseparating with \( \partial D \subset A_r \), then \( D \) stays close to \( A_r \), and it does not come near 0.

**Key Lemma:** Let \( \Sigma \) be an embedded least area plane in \( H^3 \) with \( \partial_\infty \Sigma = \Gamma \) where \( \Gamma \) is a simple closed curve with at least one smooth \( (C^1) \) point in \( S^2_\infty(H^3) \). Let \( D_r \) be a nonseparating disk in \( B_r(0) \cap \Sigma \). Then there is a function \( F \) which is a monotone increasing function with \( F(r) \rightarrow \infty \) as \( r \rightarrow \infty \), such that \( d(0, D_r) > F(r) \) where \( d \) is the distance.

**Proof:** First, we will give an outline of the proof. Then, we will prove the lemma in 2 steps.

**Outline:** The idea to show that the nonseparating disks \( D_r \) cannot be too close to the center is to construct a barrier. By using the smooth point assumption, we will show the existence of a complete least area annuli \( A_s \) "linking" \( \Gamma \) such that \( \partial_\infty A_s = \Gamma^+_s \cup \Gamma^-_s \) in \( S^2_\infty(H^3) \) with \( \Gamma^\pm_s \rightarrow \Gamma \) as \( s \rightarrow \infty \). Then, such an \( A_s \) will separate \( H^3 \) into two parts, where one part contains \( A_r \) which contains \( \partial D_r \), and other part contains 0. Then, since \( A_s \) is least area annulus, and \( D_r \) is least area disk, by exchange roundoff trick [MY2], \( D_r \) cannot intersect \( A_s \). So, \( A_s \) becomes a barrier between \( D_r \) and 0. Since \( A_s \rightarrow \infty \) as \( \partial_\infty A_s = \Gamma^\pm_s \rightarrow \Gamma \), this defines a monotone increasing function \( F(r) \) with \( F(r) \rightarrow \infty \) as \( r \rightarrow \infty \), such that \( d(0, D_r) > F(r) \).
Step 1: (Existence of Least Area Annuli) Let $\Gamma^+$ and $\Gamma^-$ be two simple closed curves in opposite sides of $\Gamma$ in $S^2_\infty(H^3)$ and sufficiently close to $\Gamma$. Then there exists a complete least area annulus $A$ in $H^3$ with $\partial_\infty A = \Gamma^+ \cup \Gamma^-$. 

Proof: Let $x$ be a $C^1$ smooth point in $\Gamma$. Then, by the discussion in Section 1 in [Ha], we can find round circles $\gamma^+$ and $\gamma^-$ in the opposite sides of $\Gamma$ so that $\gamma^+$ and $\gamma^-$ are as close as we want to $x \in \Gamma$. Let $P^+$ and $P^-$ be totally geodesic planes asymptotic to round circles $\gamma^+$ and $\gamma^-$, respectively. Since we can make $\gamma^+$ and $\gamma^-$ as close as we want by using the construction of Hass in [Ha], we can find curves $\alpha^+$ and $\alpha^-$ on $P^+$ and $P^-$ so that they cobound an embedded least area annulus $\Delta$ whose area is strictly less than the area of the two totally geodesic disks bounded by $\alpha^\pm$ on $P^\pm$ [MY1].

Now, let $N(\Gamma)$ be a neighborhood of $\Gamma$ which is an annulus in $S^2_\infty(H^3)$ so that $\gamma^+$ and $\gamma^-$ are disjoint from $N(\Gamma)$. Let $N^+(\Gamma)$ and $N^-(\Gamma)$ be the components of $N(\Gamma) - \Gamma$ in $S^2_\infty(H^3)$. Foliate $N^\pm(\Gamma)$ with $C^1$ pairwise disjoint simple closed curves $\{\Gamma^+_s\}$ where $s \in (C_\infty, \infty)$, and $\Gamma^+_s \to \Gamma$ as $s \to \infty$. We claim that for any pair $\Gamma^+_s$ and $\Gamma^-_s$, there is a complete least area annulus $A_s$ in $H^3$ with $\partial_\infty A_s = \Gamma^+_s \cup \Gamma^-_s$.

First, we fix a $s \in (C_\infty, \infty)$. By Theorem 2.1, for any simple closed curve in $S^2_\infty(H^3)$, there exist a complete least area plane in $H^3$. Let $\Sigma^+_s$ and $\Sigma^-_s$ be the least area planes in $H^3$ with asymptotic boundary $\Gamma^+_s$ and $\Gamma^-_s$, respectively. Since $\Gamma^+_s$ and $\gamma^+$ are disjoint in $S^2_\infty(H^3)$, then by Lemma 2.4, the least area planes $\Sigma^+_s$ and the geodesic planes $P^\pm$ are pairwise disjoint. Hence, the annulus $\Delta$ intersects $\Sigma^+_s$ transversely in simple closed curves. Let $\Omega^+_s$ be a sufficiently large disk in $\Sigma^+_s$ with $\Delta \cap \Sigma^+_s \subset \Omega^+_s$. Let $\beta^+_s = \partial \Omega^+_s$ be simple closed curve in $\Sigma^+_s$. Since $\Gamma^+_s$ is $C^1$ regular, by Theorem 2.2, $\Sigma^+_s$ behave nicely near asymptotic boundary. We foliate $\Sigma^+_s - \Omega^+_s$ with pairwise disjoint simple closed curves $\beta^-_s$ where $t \in [0, \infty)$. Similarly, we define $\Omega^-_s$ and $\beta^-_s$ in $\Sigma^-_s$, and foliate $\Sigma^-_s - \Omega^-_s$ with pairwise disjoint simple closed curves $\beta^-_s$ where $t \in [0, \infty)$.

Now, we claim that each pair $\beta^+_s$ and $\beta^-_s$ cobounds a least area annulus in $H^3$ for any $t$. To prove that, we need to show that there is an annulus with boundary $\beta^+_s \cup \beta^-_s$ whose area is less than the sum of the areas of the least area disks bounded by $\beta^+_s$ and $\beta^-_s$, say $\Omega^+_s$ and $\Omega^-_s$. If this is the case, then by [MY2], there is a least area annulus in $H^3$ with boundary $\beta^+_s \cup \beta^-_s$.

By construction, $\gamma^+$ and $\Gamma^+_s$ are all pairwise disjoint. By Lemma 2.4, the geodesic planes $P^\pm$ and the least area planes $\Sigma^\pm$ are pairwise disjoint, too. This implies the boundary of the annulus $\Delta$, $\alpha^+ \cup \alpha^-$, is disjoint from $\Sigma^\pm$ and $\Sigma^\pm$. This implies $\Delta \cap \Sigma^\pm$ is collection of simple closed curves as they are least area. Let $\alpha^+_s$ be a simple closed curve in $\Delta \cap \Sigma^+_s$ and $\alpha^-_s$ be a simple closed curve in $\Delta \cap \Sigma^-_s$. Since $\Sigma^\pm$ are least area planes, the intersection curves $\alpha^+_s$ and $\alpha^-_s$ must be essential curves in $\Delta$. Otherwise, $\alpha^+_s$ would bound two different disks, one in $\Sigma^+_s$ and the other one is in $\Delta$. Since $\Sigma^\pm$ and $\Delta$ are both least area already, this cannot happen.
by Meeks-Yau exchange roundoff trick [MY2]. Similarly, it is true for \( \alpha^- \). Let \( \Delta_s \subset \Delta \) be the annulus with boundary \( \alpha^+ \cup \alpha^- \).

Now, we make a surgery to get an annulus with boundary \( \beta^+_s \cup \beta^-_s \). Let \( D_{\alpha^+_s} \) be the least area disk in \( \Sigma^+_s \) with boundary \( \alpha^+_s \). Similarly, let \( D_{\alpha^-_s} \) be the least area disk in \( \Sigma^-_s \) with boundary \( \alpha^-_s \). Since \( \Delta \) is least area annulus, the area of \( \Delta_s \) is strictly less than the sum of the areas of \( D_{\alpha^+_s} \) and \( D_{\alpha^-_s} \). Otherwise, \( X = (\Delta - \Delta_s) \cup D_{\alpha^+_s} \cup D_{\alpha^-_s} \) would be two disks with boundary \( \alpha^+ \cup \alpha^- \). Moreover, \( X \) has singular circles \( \alpha^+_s \cup \alpha^-_s \). By rounding off \( X \) along these circles, we get a smaller area disks, say \( X' \). Now, if we put a very thin tube between the disks with a very small area, we get an annulus \( X'' \) whose area is less than \( \Delta \) with the same boundary. This is a contradiction. Hence, the area of \( \Delta_s \) is strictly less than the sum of the areas of \( D_{\alpha^+_s} \) and \( D_{\alpha^-_s} \).

Consider the disks \( \Omega^+_{s_0} \subset \Sigma^+_s \) with \( \partial \Omega^+_{s_0} = \beta^+_{s_0} \) and \( \Omega^-_{s_0} \subset \Sigma^-_s \) with \( \partial \Omega^-_{s_0} = \beta^-_{s_0} \). By construction, \( D_{\alpha^+_s} \) are subdisks of \( \Omega^+_{s_0} \). Let \( Y_{s_0} = (\Omega^+_{s_0} - D_{\alpha^+_s}) \cup (\Omega^-_{s_0} - D_{\alpha^-_s}) \cup \Delta_s \) be the annulus with boundary \( \beta^+_s \cup \beta^-_s \). Since the area of \( \Delta_s \) is strictly less than the sum of the areas of \( D_{\alpha^+_s} \) and \( D_{\alpha^-_s} \), the area of \( Y_{s_0} \) is less than the sum of the areas of the disks \( \Omega^+_{s_0} \) and \( \Omega^-_{s_0} \). Then by [MY2], there is a least area annulus \( A_{s_0} \) with \( \partial A_{s_0} = \beta^+_s \cup \beta^-_s \). Similarly, for any \( t \in [0, \infty) \), there is a least area annulus \( A_{st} \) such that \( \partial A_{st} = \beta^+_s \cup \beta^-_s \). Let \( \{A_{ti}\} \) be a sequence of least area annuli where \( i \in \mathbb{N} \). Notice that the boundary of each annuli in the sequence is \( \beta^+_s \cup \beta^-_s \), which are simple closed curves in \( \Sigma^+_s \) and \( \Sigma^-_s \). Since \( \partial \Sigma^+_s = \Sigma^+_s \), \( \beta^+_s \rightarrow \Gamma^+_s \) and \( \beta^-_s \rightarrow \Gamma^-_s \) as \( i \rightarrow \infty \). Then, by using the techniques in [A2], we can get a subsequence of \( \{A_{ti}\} \) converging to a complete least area annulus \( A_s \) with \( \partial \Sigma^+_s = \Gamma^+_s \cup \Gamma^-_s \). Hence, Step 1. follows.

**Step 2:** (Nonseparating Disks Stays Away from the Center) There is a function \( F \) which is a monotone increasing function with \( F(r) \rightarrow \infty \) as \( r \rightarrow \infty \), such that if \( D_r \) is a nonseparating disk in \( B_r(0) \cap \Sigma \), then \( d(0, D_r) > F(r) \) where \( d \) is the distance.

**Proof:** In the construction in Step 1, we show that for each pair \( \Gamma^+_s \) and \( \Gamma^-_s \), there is a least area annulus \( A_s \) with \( \partial \Sigma^+_s = \Gamma^+_s \cup \Gamma^-_s \). Since \( \Gamma^+_s \) is \( C^1 \), by [HL] and [16], \( A_s \cup \Gamma^+_s \cup \Gamma^-_s \) is a \( C^1 \) submanifold of the compactification of hyperbolic 3-space \( \overline{\mathbb{H}^3} \). Hence, \( A_s \cup \Gamma^+_s \cup \Gamma^-_s \) separates \( \overline{\mathbb{H}^3} \) into two parts, say \( K^+_s \) and \( K^-_s \), where \( \Gamma \subset K^+_s \). Recall that \( A_r \) is the annulus in \( \partial B_r(0) \) such that \( \text{int}(CH(\Gamma)) \cap \partial B_r(0) \subset A_r \). Define a monotone increasing function \( f : (C', \infty) \rightarrow (C, \infty) \) such that \( A_r \subset K^+_f(r) \) for \( C' \) sufficiently large. Since \( \partial \Sigma^+_s = \Gamma^+_s \cup \Gamma^-_s \rightarrow \Gamma \) as \( s \rightarrow \infty \), the annuli \( \{A_{ti}\} \) escapes to infinity as \( s \rightarrow \infty \). Hence, we can also put the condition \( f(r) \rightarrow \infty \) as \( r \rightarrow \infty \) on \( f \). Now, define a function \( F : (C', \infty) \rightarrow (0, \infty) \) such
that \( F(r) = d(0, A_{f(r)}) \) where \( d \) is the distance in \( \mathbb{H}^3 \). Clearly, \( F \) is a monotone increasing function and \( F(r) \to \infty \) as \( r \to \infty \).

Now, we claim that if \( D_r \) is a nonseparating disk in \( B_r(0) \cap \Sigma \), then \( d(0, D_r) > F(r) \). To prove this claim, all we need to show is \( D_r \subset K^+_{f(r)} \) for any \( r \in (C, \infty) \). In other words, if \( A_r \) is in the positive side of \( A_{f(r)} \), i.e. \( A_r \subset K^+_{f(r)} \), then \( D_r \) stays in the same side of \( A_{f(r)} \), i.e. \( D_r \subset K^+_{f(r)} \) (See Figure 1.). By assumption \( \partial D_r \subset A_r \), and so the boundary of \( D_r \) is in the positive side of \( A_{f(r)} \). Assume that \( D_r \) intersects \( A_{f(r)} \). Since they are both least area, and the \( \partial D_r \cap A_{f(r)} = \emptyset \), the intersection is a collection of simple closed curves. Let \( \eta \) be such a curve. \( \eta \) cannot be essential in \( A_{f(r)} \), since it bounds a disk in \( D_r \), and so \( D_r \) will be a separating disk, which contradicts to the assumption. If \( \eta \) is not essential in \( A_{f(r)} \), then this means \( \eta \) bounds a disk in \( A_{f(r)} \), too. However, since \( A_{f(r)} \) and \( D_r \) are both least area, this is a contradiction by Meeks-Yau exchange roundoff trick [MY2]. Hence, if \( D_r \) is a nonseparating disk in \( B_r(0) \cap \Sigma \), then \( d(0, D_r) > F(r) \).

**Remark 4.1.** This lemma is the key point of the main result. Intuitively, this lemma prevents a least area plane to come into the compact part unnecessarily, where this is very crucial for a plane to be nonproper.
5. MAIN RESULT

In this section, we complete the proof of the main theorem. First, we need a lemma which basically says that if we have a nonproper least area plane, then we can find arbitrarily large ball such that the intersection with the least area plane contains infinitely many separating disks.

**Lemma 5.1.** Let $\Sigma$ be an embedded least area plane in $H^3$ with $\partial_{\infty}\Sigma = \Gamma$ where $\Gamma$ is a simple closed curve with at least one smooth $(C^1)$ point in $S^2_{\infty}(H^3)$. If $\Sigma$ is not proper, then for any $R$, there exist $R' > R$ such that the intersection $B_{R'}(0) \cap \Sigma$ contains infinitely many separating disks.

**Outline:** Assume on the contrary that there exist $R_0 > 0$ such that there is no $R > R_0$ such that $B_R(0) \cap \Sigma$ contains infinitely many separating disjoint disks. Fix a generic $R_1 > R_0$. Let $B_{R_1}(0) \cap \Sigma$ contains infinitely many nonseparating disjoint disks $\{D_i\}$. Firstly, show that $\{D_i\}$ has an infinite subcollection $\{D_{i_j}\}$ such that $\text{Area}(D_{i_j}) > \delta'$ where $\delta' > 0$. Then, fix a generic $R_2 > R_1$ with $F(R_2) > R_1$. Then there is a collection of disjoint disks $\{E_l\}$ in $B_{R_2}(0) \cap \Sigma$ such that for any $i_j$ there is an $l$ with $D_{i_j} \subset E_l$. By using the area bound, show that $\{E_l\}$ is an infinite collection of disjoint disks, and by the assumption, for all but finitely many, they are nonseparating. Let $E_{i_1}$ be a such nonseparating disk. Since $D_{i_k} \subset E_{i_1}$, $d(0, E_{i_1}) < R_1$. Since $E_{i_1}$ is nonseparating, and by Key Lemma, $d(0, E_{i_1}) > F(R_2) > R_1$. This is a contradiction.

**Proof:** By Lemma 3.1, there exist $r_0 > 0$ such that for a generic $r > r_0$, $B_r(0) \cap \Sigma$ contains infinitely many disjoint disks. Assume that there exist $R_0 > r_0$ such that there is no $R > R_0$ such that $B_R(0) \cap \Sigma$ contains infinitely many separating disjoint disks.

Let $R_1 > R_0$ be a generic radius in the sense of Lemma 3.1. i.e. $B_{R_1}(0) \cap \Sigma$ contains infinitely many nonseparating disjoint disks $\{D_i\}$. Let $\gamma_i = \partial D_i$ be the pairwise disjoint simple closed curves in the annulus $A_{R_1} \subset \partial B_{R_1}(0)$. Since $\{D_i\}$ are nonseparating, $\{\gamma_i\}$ are not essential in $A_{R_1}$. Let $\Omega_i$ be the disk in $A_{R_1}$ with $\partial \Omega_i = \gamma_i$. We claim that $\{\gamma_i\}$ has an infinite subsequence $\{\gamma_{i_j}\}$ with $\Omega_{i_j} \supset \Omega_{i_k}$ for any $i_j < i_k$.

Assume on the contrary that there is no such subsequence. Since $\{\gamma_i\}$ is a collection of pairwise disjoint curves in $A_{R_1}$, $\{\gamma_i\}$ must have an infinite subsequence $\{\gamma_{i_j}\}$ with $\Omega_{i_j} \cap \Omega_{i_k} = \emptyset$ for any $i_j \neq i_k$. We can also assume that any curve in the sequence is an outermost curve, i.e. $\Omega_{i_j} \not\subset \Omega_k$ for $k \neq i_j$. Since the area of $A_{R_1}$ is finite, $\sum_{j=1}^{\infty} \text{Area}(\Omega_{i_j}) < \infty$. This implies as $i_j \to \infty$, $\text{Area}(\Omega_{i_j}) \to 0$. Since $\{D_{i_j}\}$ are least area, $\text{Area}(D_{i_j}) \to 0$ as well. Let $\epsilon > 0$ be a sufficiently small number with $R_1 - \epsilon$ is also a generic in the sense of Lemma 3.1. As $R_1 - \epsilon > R_0$, by assumption, $B_{R_1-\epsilon}(0) \cap \Sigma$ must contain infinitely many nonseparating disjoint disks $\{E_l\}$. However, as $i_j \to \infty$, $\text{Area}(\Omega_{i_j}) \to 0$ and $\{\gamma_{i_j}\}$ is outermost curve,
there exist \( \Gamma \) where \( \Sigma \) contains infinitely many disjoint disks. Let \( \Omega \) be the length of \( \gamma \).

Now, let \( R_2 > R_1 \) be also generic in the sense of Lemma 3.1 and \( F(R_2) > R_1 \). Since \( B_{R_1}(0) \subset B_{R_2}(0) \), the \( B_{R_1}(0) \cap \Sigma \subset B_{R_2}(0) \cap \Sigma \). Hence, there is a collection of disjoint disks \( \{ \gamma_i \} \) in \( B_{R_2}(0) \cap \Sigma \) such that for any \( i \), there is an \( l \) with \( D_i \subset E_l \). We claim that the collection \( \{ E_l \} \) contains infinitely many disjoint disks.

If the collection \( \{ E_l \} \) has only finitely many disjoint disks, then there is an \( l_0 \) such that \( E_{l_0} \) contains infinitely many disks in \( \{ D_i \} \). By the proof of Lemma 3.1, for any \( l \), the disk \( E_l \) must have finite area. Since, for any \( i \), \( Area(D_{i_l}) > \delta \) and the area of \( E_{l_0} \) is finite, this is a contradiction. Therefore, the collection \( \{ E_l \} \) contains infinitely many disjoint disks.

Since \( R_2 > R_1 \), all but finitely many disks in the collection \( \{ E_l \} \) must be nonseparating. Let \( E_{l_1} \) be a nonseparating disk in the collection. Let \( D_{i_k} \subset E_{l_1} \). Since \( D_{i_k} \subset B_{R_1}(0) \cap \Sigma \), \( d(0, D_{i_k}) < R_1 \). Hence, \( d(0, E_{l_1}) < R_1 \). However, \( E_{l_1} \) is nonseparating, and by Key Lemma, \( d(0, E_{l_1}) > F(R_2) \). Since \( F(R_2) > R_1 \), this is a contradiction. Hence, the proof follows.

Now, we can prove the main result of the paper.

**Theorem 5.2.** Let \( \Sigma \) be a complete embedded least area plane in \( H^3 \) with \( \partial_\infty \Sigma = \Gamma \) where \( \Gamma \) is a simple closed curve in \( S^2_\infty(H^3) \) with at least one smooth \( (C^1) \) point. Then, \( \Sigma \) must be proper.

**Proof:** Assume that \( \Sigma \) is not proper. Then by Lemma 5.1, for any \( R > 0 \), there exist \( R' > R \) such that the intersection \( B_{R'}(0) \cap \Sigma \) contains infinitely many separating disks. Let \( \beta \) be a path from \( \partial^+CH(\Gamma) \) to \( \partial^-CH(\Gamma) \) through \( 0 \). Let \( l \) be the length of \( \beta \). Let \( R_1 > 0 \) be so that \( F(R_1) > l \), and \( B_{R_1}(0) \cap \Sigma \) contains infinitely many pairwise disjoint separating disks \( \{ D_i \} \). Let \( x_i \in D_i \) for any \( i \). Let \( \{ \gamma_{i,j} \} \) be the family of paths in \( \Sigma \) between \( x_i \) and \( x_j \). Let \( R_2 = \inf_{R > R_1} \{ R \mid \exists i, j > 0, \exists \gamma_{i,j} \subset \Sigma, \gamma_{i,j} \subset B_R(0) \} \). In other words, \( B_{R_2}(0) \) is the smallest closed ball among the balls \( B_R(0) \) where at least two of the disjoint disks in \( B_{R_1}(0) \cap \Sigma \) can be connected in \( B_R(0) \cap \Sigma \). Say, we can connect \( D_i \) and \( D_j \) in \( B_{R_2}(0) \) via path \( \gamma_{i,j} \). Let \( E \) be the component of \( B_{R_2}(0) \cap \Sigma \) containing \( D_i \) and \( D_j \), i.e. \( D_i \cup D_j \subset E \subset B_{R_2}(0) \cap \Sigma \). Because of the assumption on \( R_2 \), \( \gamma_{i,j} \cap \partial B_{R_2}(0) \neq \emptyset \). Hence, \( \partial E \) in \( \partial B_{R_2}(0) \) is a nonsimple closed curve (with degenerate point \( \gamma_{i,j} \cap \partial B_{R_2}(0) \)). Then, by changing the center a little bit from \( 0 \) to \( 0' \), if necessary, we can find a
sufficiently small $\epsilon$ such that $R_2' = R_2 + \epsilon$ is a generic radius in the sense of Lemma 3.1, and the component $E'$ in $B_{R_2'}(0)$ containing $D_i$ and $D_j$ is a nonseparating disk. Since $D_i$ is a separating disk in $B_{R_1}(0)$, $d(0, D_i) < l$. This implies $d(0, E') < l$. However, since $E'$ is a nonseparating disk in $B_{R_2'}(0)$, $d(0, E') > F(R_2')$. Since $F(R_2') > F(R_1) > l$, this is a contradiction. The proof follows.

\[\square\]

6. Concluding Remarks

As it is mentioned in the introduction, even though many experts of the field has questioned the problem, there is a very few results about the properly embeddedness of least area planes in $\mathbb{H}^3$. For example, it is still not known if there exists a nonproperly embedded least area plane in $\mathbb{H}^3$ whose asymptotic boundary is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$. On the other hand, there is a construction by Freedman and He communicated to Gabai of a nonproper least area plane in $\mathbb{H}^3$. It is not clear to the author as to how a plane constructed in this manner can have limit set a simple closed curve.

Recently, Colding and Minicozzi solved an analogous question in $\mathbb{R}^3$ in [CM]. They proved Calabi-Yau Conjectures for embedded surfaces by relating intrinsic distances and extrinsic distances of the minimal disk. As a corollary, they proved that any complete embedded minimal plane in $\mathbb{R}^3$ must be proper. Our approach is very different from them as they use purely analytic methods, while our techniques are purely topological. When starting this problem, our aim is to prove the following conjecture.

**Conjecture:** Let $\Sigma$ be a complete embedded least area plane in $\mathbb{H}^3$ with $\partial_\infty \Sigma = \Gamma$ where $\Gamma$ is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$. Then, $\Sigma$ must be proper.

In this paper, we proved this statement with the existence of a smooth point condition on $\Gamma$. We needed this condition for the Key Lemma to show existence of a least area annulus linking $\partial_\infty \Sigma$. If one can bypass this without the smooth point condition, then the theorem can be proved in full generality. On the other hand, since the smooth point condition also means the finite thickness of $\text{CH}(\Gamma)$ in one direction, this might be an essential point for the result.

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