2-Cluster fixed-point analysis of mean-coupled Stuart–Landau oscillators in the center manifold

Felix P Kemeth, Bernold Fiedler, Sindre W Haugland and Katharina Krischer

1 Department of Chemical and Biomolecular Engineering, Johns Hopkins University, Baltimore, MD 21218, United States of America
2 Institut für Mathematik, Freie Universität Berlin, 14195 Berlin, Germany
3 Physik-Department, Nonequilibrium Chemical Physics, Technische Universität München, 85748 Garching, Germany

* Author to whom any correspondence should be addressed.
E-mail: fkemeth1@jh.edu

Keywords: globally coupled oscillators, center manifold reduction, SN-equivariant systems

Abstract
We reduce the dynamics of an ensemble of mean-coupled Stuart–Landau oscillators close to the synchronized solution. In particular, we map the system onto the center manifold of the Benjamin–Feir instability, the bifurcation destabilizing the synchronized oscillation. Using symmetry arguments, we describe the structure of the dynamics on this center manifold up to cubic order, and derive expressions for its parameters. This allows us to investigate phenomena described by the Stuart–Landau ensemble, such as clustering and cluster singularities, in the lower-dimensional center manifold, providing further insights into the symmetry-broken dynamics of coupled oscillators. We show that cluster singularities in the Stuart–Landau ensemble correspond to vanishing quadratic terms in the center manifold dynamics. In addition, they act as organizing centers for the saddle-node bifurcations creating unbalanced cluster states as well for the transverse bifurcations altering the cluster stability. Furthermore, we show that bistability of different solutions with the same cluster-size distribution can only occur when either cluster contains at least 1/3 of the oscillators, independent of the system parameters.

1. Introduction

Long-range interactions play a crucial role in various dynamical phenomena observed in nature. In a swarm of flashing fireflies, they may act as a synchronizing force, causing the swarm to flash in unison. Analogously, in an audience clapping, the acoustic sound of the clapping can be recognized by each individual, leading to clapping in unison. In these cases, long-range interactions lead to the synchronization of individual units [1].

On the other hand, long-range interactions may also lead to a split up of the individuals into two or more groups, also called dynamical clustering. In electrochemistry, a stirred electrolyte or a common resistance may induce long-range coupling, leading to spatial clustering on the electrode [2–8]. In biology, this may explain the formation of different genotypes in an otherwise homogeneous environment [9, 10].

The individual units which experience this long-range or global coupling may be oscillatory, as in the case of flashing fireflies or in a clapping audience, or, as in the case of sympatric speciation, stationary genotypes. Here, we focus on the former case of oscillatory units with long-range interactions.

Clustering in oscillatory systems with long-range interactions has been subject to theoretical investigation for many years [11–15]. See also reference [16] for a recent review on globally coupled oscillators. In particular when the long-range interactions are weak compared to the intrinsic dynamics of the oscillator, it suffices to describe the phase evolution of each unit, and the analysis greatly simplifies [17–19]. If, however, the influence of the coupling is strong, as in the case considered here, such a reduction is no longer feasible and the amplitude dynamics must be considered. Our work aims to add to the theoretical understanding of clustering in this case of strong coupling.

© 2021 The Author(s). Published by IOP Publishing Ltd
From the viewpoint of symmetry, if the coupling between $N$ identical oscillators is global (i.e. all-to-all), then the governing equations are equivariant under the symmetric group $S_N$. This means that the evolution equations $f$ commute with elements $\sigma$ from the symmetry group,

$$f(\sigma x) = \sigma f(x) \quad \forall \sigma \in S_N. \quad (1)$$

In addition, this implies that the system has a trivial solution which is invariant under $S_N$, that is, in which all oscillators are synchronized. Cluster states composed of two clusters, also called two-cluster states, can then be viewed as states with the reduced symmetry $S_{N_1} \times S_{N_2}$, with $N_1$ and $N_2$ being the number of oscillators in each cluster. Using the equivariant branching lemma, it can then be shown that these two-cluster states bifurcate off the trivial solution [9, 20]. The bifurcation at which the synchronized motion becomes unstable and the two-cluster branches (also called primary branches) emerge is commonly referred to as the Benjamin–Feir instability [21, 22].

The intrinsic dimensionality of each oscillatory unit may range from $d = 2$ for FitzHugh–Nagumo [23] and Van der Pol oscillators [24], via $d = 3$ for the Oregonator [25] to $d = 4$ for the original Hodgkin–Huxley model [26], and even higher for more detailed physical models [27]. A system composed of $N$ of these oscillators thus lives in a $d \cdot N$-dimensional phase space, making its full investigation unfeasible even for small $d$ and $N$. One can, however, circumvent this problem of increasingly large dimensions by restricting the dynamics to the center manifold of certain bifurcations. In particular, it is known that the center space of the Benjamin–Feir instability is $N = 1$ dimensional [20, 28], and thus a reduction to the center manifold at this bifurcation allows for reducing the dimension of the problem to $N - 1$ and thus by a factor of $\approx d$. As we show below, such a reduction lets us reveal invariant sets and bifurcation curves analytically—a difficult task in the original $d \cdot N$-dimensional space.

In this work, we focus on a particular example of a globally coupled system, in which the network is composed of oscillating units called Stuart–Landau oscillators, each represented by a complex variable $x_k \in \mathbb{C}$. As opposed to phase oscillators, each Stuart–Landau oscillator has two degrees of freedom, i.e. an amplitude $W_k$ and a phase. With a linear global coupling, the dynamics are then given by

$$\dot{W}_k = W_k - (1 + i\gamma) |W_k|^2 W_k + (\beta r + i\beta i) (\langle W \rangle - W_k), \quad (2)$$

with the complex coupling constant $\beta r + i\beta i$ and the real parameter $\gamma$, also called the shear [29]. $\langle \cdot \rangle$ indicates the ensemble mean and $\dot{W} = dW/dt$. Bold face $W$ indicates a vector containing the ensemble values $|W_1, W_2, \ldots, W_N|$. For $\beta r + i\beta i = 0$ the ensemble is decoupled, and each Stuart–Landau oscillator oscillates with unit amplitude and angular velocity $-\gamma$. For $\beta r + i\beta i \neq 0$, however, a plethora of different dynamical states can be observed. These states include fully synchronized oscillations, in which all oscillators maintain an amplitude equal to one and have a mutual phase difference of zero [30], cluster states, in which the ensemble splits up into two or more sets of synchrony [14, 31, 32], and a variety of quasi-periodic and chaotic dynamics [12, 33].

Two-cluster states can be born and destroyed at saddle-node bifurcations if the number of oscillators in each cluster is different, that is, when they are unbalanced [13]. Balanced solutions with $N_1 = N_2$ emerge from the synchronized solution at the Benjamin–Feir instability. For $N = 16$ oscillators and $\gamma = 2$, the saddle-node bifurcations for different unbalanced cluster distributions $N_1 \neq N_2$ and the Benjamin–Feir instability are depicted in figure 1, as a function of the coupling parameters $\beta r$ and $\beta i$. Here, all the two-cluster solutions exist locally in parameter space below their respective saddle-node bifurcation curve, that is for smaller $\beta r$ values. Up to the Benjamin–Feir instability they coexist with the stable synchronized solution. Descending from large $\beta r$ values, notice that the most-unbalanced cluster state with $N_1 : N_2 = 1 : 15$ is created first. The more balanced cluster states are born subsequently, depending on their distribution, until eventually the balanced cluster state $N_1 : N_2 = 8 : 8$ is born at the Benjamin–Feir instability. At $\beta r = -(1 - \sqrt{3})/2$, $\beta i = (-\gamma - \sqrt{3})/2$, there exists a codimension-two point where the saddle-node bifurcations of all cluster distributions coincide. This point is called a cluster singularity [32]. Note that the qualitative picture in figure 1 does not change when increasing the total number of oscillators $N$. For large numbers $N \to \infty$ we expect a bow-tie-shaped band of saddle-node bifurcation curves, ranging from the saddle-node bifurcation of the most unbalanced cluster state to the Benjamin–Feir instability. As argued in reference [32], the cluster singularity can thus be viewed as an organizing center. By projecting the dynamics close to the Benjamin–Feir instability onto its center manifold, we aim to obtain further insights into the properties of this organizing center, and to elucidate the clustering behavior near it.

The remainder of this article is organized as follows: in section 2, we pass to a corotating frame and introduce the average amplitude $R$, the deviations from the average amplitude $r_k$, the deviations from the mean phase $\phi_k$. Using this corotating system, we discuss how one can describe the dynamics in the center manifold, see section 3. In section 4, we derive the parameters for the dynamics of $x_k$. Detailed calculations are provided
The Benjamin–Feir instability involving the 8 : 8 cluster (dark blue) and the different saddle-node curves creating the unbalanced cluster solutions, $N_1 \neq N_2$, in the $\beta_r, \beta_i$ plane with $\gamma = 2$ and $N = 16$. Each curve belongs to a particular cluster distribution $N_1 : N_2$, and is obtained with numerical continuation using AUTO-07P [34, 35]. Note the position of the cluster singularity at $\beta_r = -(1 - \sqrt{3})/2 \approx 1.23$, $\beta_i = (-\gamma - \sqrt{3})/2 \approx -1.87$ as indicated.

### Table 1. Abbreviations

| $\langle x \rangle = 1/N \sum_{n=1}^{N} x_n$ | $\tilde{x} = x - \langle x \rangle$ |
| $\langle e^x \rangle = 1/N \sum_{n=1}^{N} e^{x_n}$ | $\tilde{e} = e^x - \langle e^x \rangle$ |

in appendix C, for convenience. Based on the parameters in the center manifold, we study the bifurcations of two-cluster states and the role of the cluster singularity in the center manifold, in section 5. We conclude with a detailed discussion of our results and an outlook on future work. For a detailed mathematical analysis of the dynamics of two-cluster states in the center manifold, see the companion paper [36].

### 2. Variable transformation into corotating frame

Notice that equation (2) is invariant under a rotation in the complex plane $W_k \rightarrow W_k \exp(i\phi)$. This invariance can be eliminated by choosing variables in a corotating frame, thus effectively reducing the dimensions of the system from $2N$ to $2N - 1$.

In particular, we express the complex variables $W_k$ in log-polar coordinates $W_k = \exp(R_k + i\Phi_k)$. Then equation (2) turns into

\begin{align}
\dot{R} &= 1 - e^{2R} \langle e^{2x} \rangle + \text{Re} \left( (\beta_r + i\beta_i) \left( \langle e^x \rangle e^{-R} - 1 \right) \right) \tag{3a} \\
\dot{r}_k &= -e^{2R} \tilde{e}^{R_k} + \text{Re} \left( (\beta_r + i\beta_i) \left( \langle e^x \rangle \tilde{e}^{-R_k} \right) \right) \tag{3b} \\
\dot{\phi}_k &= -\gamma e^{2R} \tilde{e}^{R_k} + \text{Im} \left( (\beta_r + i\beta_i) \left( \langle e^x \rangle \tilde{e}^{-R_k} \right) \right), \tag{3c}
\end{align}

with $k = 1, \ldots, N - 1$, the abbreviations shown in table 1 and the new coordinates summarized in table 2 (see appendix A for a derivation). Hereby, $\tilde{e}$ symbolizes the deviation from the ensemble mean $\langle \cdot \rangle$, and $R$ and $\Phi$ are the ensemble mean logarithmic amplitude and phase, respectively. The logarithmic amplitude and phase deviation of each oscillator from their averages are $r_k$ and $\phi_k$. Notice that through this construction, the averages of these deviations vanish. Furthermore, bold face of a variable, e.g. $x$, symbolizes the set of the respective ensemble variables $\{x_1, x_2, \ldots, x_N\}$. 

---

### Figure 1

The Benjamin–Feir instability involving the 8 : 8 cluster (dark blue) and the different saddle-node curves creating the unbalanced cluster solutions, $N_1 \neq N_2$, in the $\beta_r, \beta_i$ plane with $\gamma = 2$ and $N = 16$. Each curve belongs to a particular cluster distribution $N_1 : N_2$, and is obtained with numerical continuation using AUTO-07P [34, 35]. Note the position of the cluster singularity at $\beta_r = -(1 - \sqrt{3})/2 \approx 1.23$, $\beta_i = (-\gamma - \sqrt{3})/2 \approx -1.87$ as indicated.
To simplify notation, \( r_k + i\varphi_k \) is abbreviated by the complex variable \( z_k \). The transformation into equations (3a) to (3c) has the advantage that the resulting equations are independent of the mean phase \( \Phi \). A change of \( \Phi \) corresponds to a uniform phase shift of the whole ensemble in the complex plane, which in turn means that periodic orbits in the Stuart–Landau ensemble, equation (2), correspond to stationary solutions \( \Phi \) and \( \varphi_k = 0 \). The stability of this equilibrium can be investigated using the eigenspectrum of the Jacobian evaluated at this point. Due to the \( S_\infty \)-symmetry of the solution and the \( S_\infty \)-equivariance of the governing equations, the Jacobian becomes block-diagonal, and thus has a degenerate eigenvalue spectrum \([15, 21]\), see appendix B:

- There is one singleton eigenvalue \( \lambda_1 = -2 < 0 \), corresponding to an eigendirection affecting all oscillators identically. That is, this direction \( \vec{e}_1 \) shifts the amplitude of the synchronized motion but does not alter its symmetry.
- There is the eigenvalue \( \lambda_+ = -1 - \beta_+ + \sqrt{1 - \beta_+^2 - 2\beta_+ \gamma} : = -1 - \beta_+ + d \) which becomes zero at the Benjamin–Feir instability and is of geometric multiplicity \( N - 1 \). The corresponding directions correspond to two-cluster states, with each direction corresponding to one cluster distribution \( \Phi_1 : \Phi_2 \). Up to conjugacy, we arrange here the units such that the first \( N_1 \) oscillators correspond to the same cluster. All two-clusters with the same distribution but different assignments of the oscillators then belong to the same conjugacy class.
- Finally, there is the eigenvalue \( \lambda_- = -1 - \beta_+ - d \) which is negative close to the synchronized solution, which has a geometric multiplicity of \( N - 1 \) and whose eigendirections also have \( S_{\Phi_1} \times S_{\Phi_2} \)-symmetry.

Here, \( d = \sqrt{1 - \beta_+^2 - 2\beta_+ \gamma} \) abbreviates the root of the discriminant where we assume \( 1 - \beta_+^2 - 2\beta_+ \gamma > 0 \), i.e. real \( \lambda_+ \). Notice that the Benjamin–Feir instability \( \lambda_+ = 0 \), alias \( \beta_+ = d - 1 \), i.e. the dark blue curve in figure 1, is of codimension one.

### 3. Center manifold reduction

In the following, we calculate an expansion to third order of the dynamics in the \((N - 1)\)-dimensional center manifold which corresponds to the Benjamin–Feir instability at \( \lambda_+ = 0 = -1 - \beta_+ + d \). In order to do so, it is useful to introduce the coordinates

\[
x_k = \frac{-r_k + \frac{d+1}{2} \varphi_k}{2d} \quad (4)
\]

\[
y_k = \frac{r_k + \frac{d-1}{2} \varphi_k}{2d} \quad (5)
\]

such that

\[
r_k = (1 - d)x_k + (1 + d)y_k \quad (6)
\]

\[
\varphi_k = \gamma' x_k + \gamma' y_k \quad (7)
\]

Here we use the notations \( \gamma' = 2\gamma + \beta_+ \) and \( d \) as defined above. See appendix B for a derivation. The variables \( x_k \) describe the dynamics in the \((N - 1)\)-dimensional center manifold tangent to \( y_k = 0 \forall k \), while \( y_k \) together with \( R \) describe the dynamics in the stable manifold tangent to \( x_k = 0 \forall k \).

Note that the center-manifold must be \( S_\infty \)-invariant. In addition, the global restrictions \( \langle r \rangle = \langle \varphi \rangle = 0 \) and thus \( \langle x \rangle = \langle y \rangle = 0 \) must hold. Therefore, the general form of the center manifold up to quadratic order must follow

\[
y_k = y_k(x) = a_{x^2} + O(x^3)
\]

\[
R = R(x) = b(x^2) + O(x^3),
\]

### Table 2. Coordinate transformations.

| \( R = \langle R \rangle \) | \( r_k = \tilde{R}_k \) | \( \Rightarrow \langle r \rangle = 0 \) |
| \( \Phi = \langle \Phi \rangle \) | \( \varphi_k = \tilde{\Phi}_k \) | \( \Rightarrow \langle \varphi \rangle = 0 \) |
| \( z_k = r_k + i\varphi_k \) | \( \Rightarrow \langle z \rangle = 0 \) |
with the coefficients $a = a (\beta_1, \gamma)$ and $b = b (\beta_1, \gamma)$. Here, we use the tangency of our coordinates $R$ and $y_k$, that is, $R \bigg|_{x_h = 0} = 0$ and $y_k \bigg|_{x_h = 0} = 0$. Since the Benjamin–Feir instability $\beta_1 = d - 1$ is of codimension one, the three-dimensional parameter space $(\beta_1, \beta_1, \gamma)$ becomes two-dimensional. The parameters in the center manifold thus only depend on $\beta_1$ and $\gamma$. By $S_N$-equivariance, the reduced dynamics $\dot{x}_k$ in the center manifold, up to cubic order, must be of the form

$$\dot{x}_k = \lambda_+ x_k + Ax_k^2 + Bx_k^3 + C(x^2)x_k + O (x^4) ,$$

see also references [10, 20], with the parameters $A = A (\beta_1, \gamma)$ and $B = B (\beta_1, \gamma)$ and $C = C (\beta_1, \gamma)$.

4. Derivation of the parameters $a, b, A, B$ and $C$

In this section, we discuss the approach to calculate the coefficients $a, b, A, B$ and $C$ for the dynamics in the center manifold. See appendix C for complete details.

First, we determine $b$. In particular we observe that

$$\dot{R} = \left( \frac{d}{dx_k} R \right) \dot{x}_k = 2b\langle xx \rangle + O (x^2) = 2b\lambda_+\langle x^2 \rangle + O (x^4)$$

holds. Since $\lambda_+ = 0$ at the bifurcation, $\dot{R}$ up to second order in $x_k$ must vanish. Therefore, expressing $\lambda_k = \gamma_k + i\varphi_k$ and $\gamma_k, \varphi_k$ in terms of $x_k$ in equation (3a), we can compute $b$ by comparing the coefficients of the $\langle x^2 \rangle$: the terms in front of $\langle x^2 \rangle$ must thereby vanish. This allows us to estimate $b = b (\beta_1, \gamma)$ as

$$b = \frac{1 - d}{2} \left( \gamma^2 + d^2 + 4d - 5 \right)$$

(11)

with $\gamma'$ and $d$ as defined above.

Analogously, we can calculate $a$ using equations (3a) and (3b) up to second order in $x_k$ and employing

$$\dot{y}_k = \left( \frac{d}{dx_k} y_k \right) \dot{x}_k = O (x^4) .$$

This means we can use $2\dot{y}_h = \dot{r}_k + (d - 1) / \gamma' \varphi_0$, substitute the $z_k$ with $x_k$ in equations (3a) and (3b) and keep terms up to $O (x^4)$. Comparing the coefficients in front of $x_k^3$ then results in

$$a = \frac{(1 - d) \left( \gamma^2 + (1 - d)^2 \right) \left( 3 (d - 1)^2 + \gamma^2 \right)}{8d^2 \gamma'^2} \tag{12}$$

Finally, we can calculate $A, B$ and $C$ using

$$2\dot{x}_k = -\dot{r}_k + \frac{d + 1}{\gamma'} \varphi_k$$

$$= \lambda_+ x_k + Ax_k^2 + Bx_k^3 + C(x^2)x_k .$$

Taking equations (3b) and (3c) and the coefficients $a$ and $b$ obtained above, we can evaluate this equality up to cubic order, yielding the coefficients

$$A = \frac{(d - 1) \left( \gamma^2 + (1 + d)^2 \right) \left( \gamma^2 - 3(d - 1)^2 \right)}{4\gamma'^2 d} \tag{13}$$

$$B = -\frac{(d - 1)^2 \left( \gamma^2 + (1 - d)^2 \right) \left( \gamma^2 + (d + 1)^2 \right) \left( \gamma^2 - 2\gamma' d + 3 (d^2 - 1) \right) \left( \gamma^2 + 2\gamma' d + 3 (d^2 - 1) \right)}{16\gamma'^4 d^2} \tag{14}$$

$$C = \frac{(d - 1)^2}{16d^3 \gamma'^4} \left( \gamma'^6 - 4\gamma'^6 (2d - 1) - 2\gamma'^4 (8d^6 + d^4 - 56d^3 + 22d^2 + 1) \right.$$

$$- 4\gamma'^2 \left( 2d^2 + 5d^2 - 4d^2 - 13d^4 + 2d^2 + 11d^2 - 3 \right) + 9 (d^2 - 1) ^4 \right) \tag{15}$$
Figure 2. The Benjamin–Feir instability (blue, $\lambda_+ = 0$) and the different saddle-node curves creating the unbalanced cluster solutions in the $A, \lambda_+$ plane. The dashed curves belong to particular cluster distributions $N_1 : N_2$ obtained by projecting the curves from the Stuart–Landau ensemble shown in figure 1 using the expressions for $\lambda_+ (\beta_r, \beta_i, \gamma) = -1 - \beta_r + d$, and $A(\beta_r, \beta_i, \gamma)$, cf equation (13). The solid curves $\lambda_{sn}$ indicate the saddle-node bifurcations of the unbalanced cluster states obtained analytically in the center manifold, see equation (16). Note that close to the cluster singularity, where analytical expansions work best, numerical continuation fails due to the concentration of solutions in phase space.

Together with $\lambda_+$, the expressions for $A$, $B$ and $C$ fully specify the dynamics in the center manifold based on the original parameters $\gamma$, $\beta_r$ and $\beta_i$. By rescaling time and $x_k$ in equation (10), the number of independent parameters can be reduced to two, see reference [36]. For simplicity, we use the unscaled equation as in equation (10) here.

5. Clustering and cluster singularities in the center manifold

As shown in figure 1 for $N = 16$ oscillators, we observe a range of saddle-node bifurcations creating the different two-cluster states. The expressions for $\lambda_+$, $A$, $B$ and $C$ above determine the corresponding parameter values in the center manifold. The respective $\lambda_+$ and $A$ values for the numerical curves shown in figure 1 are depicted in figure 2 as dashed curves. Notice that the Benjamin–Feir curve corresponds to the line $\lambda_+ = 0$. Furthermore, we can derive the saddle-node curves creating unbalanced two-cluster states in the center manifold analytically, see appendix D. In particular,

$$\lambda_{sn} = \frac{A^2(1 - \alpha)^2}{4 (B(1 - \alpha + \alpha^2) + C\alpha)}$$

(16)

for unbalanced cluster solutions, with $\alpha = N_1/N_2$. The respective analytical curves for $N = 16$ are shown as solid curves in figure 2. Notice the close correspondence between the mapped bifurcation curves from the full system and the bifurcation curves determined in the center manifold. For less balanced solutions, the saddle-node curves obtained from the Stuart–Landau ensemble depart more strongly from the saddle-node curves calculated analytically in the center manifold. We expect this to be due to the cubic truncation of the flow in the center manifold, thus limiting its accuracy away from the Benjamin–Feir curve.

Note that to obtain the curves in figure 2, we fix $\gamma = 2$ and vary $\beta_i$, $\beta_r$. We then use the expressions for $A(\beta_r, \beta_i)$, $B(\beta_r, \beta_i)$ and $C(\beta_r, \beta_i)$ to get the parameters in the center manifold. Thus the parameters $A$, $B$ and $C$ lie on a two-dimensional manifold. For the curves shown in figure 2, we furthermore use equation (16), yielding one-dimensional curves. The curves are, however, not exactly parabolas, since $B$ and $C$ vary in addition to $A$, which is not shown in figure 2. For all subsequent figures, we use the values of $C = -1$ and $B = -2/(2\sqrt{3} - 3)$ at the cluster singularity for $\gamma = 2$, which can be obtained analytically. See reference [36] p 36 for a derivation.
Figure 3. The bifurcation curves $\lambda_{+,1} (\mu_1 = 0, \text{dotted orange})$ and $\lambda_{+,2} (\mu_2 = 0, \text{dash-dotted orange})$ for the 4 : 12 cluster state in the $A, \lambda_+$ plane and the parameters $B = -2/(2\sqrt{3} - 3), C = -1$. The saddle-node curve creating the 4 : 12 cluster is shown as a solid orange curve. The Benjamin–Feir line is shown in blue, with the $\lambda_{+,1} = \lambda_{+,2}$ curve for the balanced 8 : 8 cluster state depicted as a dotted blue curve. The 4 : 12 cluster is stable in the two regions between the respective $\lambda_{+,1} = 0$ and $\lambda_{+,2} = 0$ curve. The balanced cluster state is stable above the dotted blue curve.

Furthermore, from figure 2 we observe that $A = 0$, in addition to $\lambda_+ = 0$, at the cluster singularity. This means that this codimension-two point is distinguished by vanishing quadratic dynamics in the center manifold, cf equation (10). In addition, it serves as an organizing center for the saddle-node bifurcations of the unbalanced cluster states: at the saddle-node bifurcation, we have in the center manifold for a cluster state $x^*_1 = -A (1 - \alpha) \frac{(1 - 2 \alpha) B - \alpha C}{2 (B (1 - \alpha + \alpha^2) + C \alpha)}$, with $B < 0$ and $C < 0$ for the range of $\beta_l, \beta_r$ considered here (not shown), see appendix D. This means that for negative $A$ values, the saddle-node curves occur at positive $x_1$, for positive $A$ values at negative $x_1$, and for $A = 0$, at the cluster singularity, all saddle-node bifurcations occur at the synchronized solution $x_0 = 0$. This behavior can indeed be observed in the Stuart–Landau ensemble, see figure 6 of reference [32].

The unbalanced cluster states do, in general, not emerge as stable states from the saddle-node bifurcations. Rather, one of the two branches created at the saddle-node bifurcation is subsequently stabilized through transverse bifurcations involving three-cluster solutions with symmetry $S_{N_1} \times S_{N_2} \times S_{N_3}$, also called secondary branches [10]. For a more detailed discussion on secondary branches, see also references [28, 37].

In order to explain this in more detail, we follow reference [37] section 4. Note that each $N_1:2 N_2$ two-cluster solution is invariant under the action of the group $S_{N_1} \times S_{N_2}$. From this, it follows that one can block-diagonalise the Jacobian at the two-cluster solutions $S_{N_1} \times S_{N_2}$. In doing so, one can calculate the $(N_1 - 1)$-degenerate eigenvalue $\mu_1$ describing the intrinsic stability of cluster $\Xi_1$, that is its stability against transverse perturbations. Note, however, that a cluster of size 1 cannot be broken up. Following reference [10] p 23 and using isotypic decomposition, the eigenvalue $\mu_1$ can be expressed as

$$\mu_1 = \frac{J_{11}}{\Xi_1} - \frac{J_{12}}{\Xi_1^2}.$$ 

Here, $J_{ij}/\Xi_i$ denotes $\partial f_i/\partial x_j$, with the respective $x_i$ and $x_j$ in cluster $\Xi_i$ and $f_i$ being the right-hand side of equation (10). Without loss of generality, we assume in the following that $\Xi_1$ is the cluster with the smaller number of oscillators, that is, $N_1 < N_2$ or $\alpha < 1$. Evaluating the Jacobian, one obtains that the eigenvalue $\mu_1$ changes sign at

$$\lambda_{+,1} = \frac{(1 - 2 \alpha) B - \alpha C}{(\alpha - 2)^2 B^2} A^2.$$ 

(17)
Figure 4. The theoretical bifurcation curves $\lambda_{+,1}$ ($\mu_1 = 0$, dotted) and $\lambda_{+,2}$ ($\mu_2 = 0$, dash-dotted) for the different cluster size distributions in the $\lambda, \lambda_+$ plane and the parameters $B = -2/(2\sqrt{3} - 3), C = -1$. The saddle-node curves creating the unbalanced cluster solutions are represented as solid curves, which correspond to the shaded curves in figure 2 with the same color coding. The Benjamin–Feir line is shown in blue. The unbalanced cluster states are stable above the respective dotted curve and below the dash-dotted curve, except for the 1 : 15 cluster, which is stable already at the saddle-node bifurcation. For the 2 : 14 cluster, the dotted and solid curves do not coincide but lie very close in parameter space.

Analogously, the transverse stability of cluster $\Xi_2$ is described by

$$\mu_2 = |J_{11}|_{\Xi_2} - |J_{12}|_{\Xi_2},$$

which changes sign at

$$\lambda_{+,2} = \frac{(\alpha - 2) B - C}{(4\alpha^2 - 4\alpha + 1) B^2} A^2.$$  \hfill (18)$$

Hereby, $\mu_2$ describes the intrinsic stability of cluster $\Xi_2$. Furthermore notice that for the balanced cluster, $\alpha = 1$ and therefore $\lambda_{+,1} = \lambda_{+,2}$. Since both clusters contain an equal number of units, their respective intrinsic stabilities change simultaneously.

In figure 3, $\lambda_{mn}, \lambda_{+,1}$ and $\lambda_{+,2}$ are shown as solid, dotted and dash-dotted orange curves, respectively, for the 4 : 12 two-cluster state. The Benjamin–Feir instability, where the balanced cluster state is born, is drawn as a solid blue line at $\lambda^+ = 0$, and the transverse bifurcation curve $\lambda_{+,1} = \lambda_{+,2}$, where the balanced cluster state is stabilized, is drawn as a dotted blue curve. See figure 4 for the respective curves for a range of cluster distributions.

Figure 3 can be interpreted as follows: coming from negative $\lambda_+$ values, the unbalanced 4 : 12 cluster state is born at $\lambda_{mn}(4 : 12)$ (solid orange). However, this two-cluster state is unstable for the parameter values considered here: the cluster $\Xi_1$ with 4 units is intrinsically unstable with $\mu_1 > 0$ and $\mu_2 < 0$. At the dotted orange curve, $\mu_1$ changes sign, rendering the 4 : 12 cluster state stable. Subsequently, at the dash-dotted orange curve, $\mu_2$ changes sign, leaving the cluster $\Xi_2$ with 12 units intrinsically unstable and thus the 4 : 12 cluster unstable.

The qualitatively same behavior can be observed for any cluster distribution $\alpha < 1/2$, except for the most unbalanced state (1 : 15). There, cluster $\Xi_1$ cannot be intrinsically unstable, since it contains only one unit. This means that this cluster solution is born stable in its saddle-node bifurcation, and becomes unstable only at $\lambda_3$ when $\mu_2 = 0$. See also the bottom right plot in figure 4. In particular, $\lambda_{mn} = A^2/4B$ for $\alpha = 0$, see equation (16), coincides with $\lambda_{+,1} = A^2/4B$ for $\alpha = 0$, cf equation (17). Furthermore, it is worth noting that the stable patches in parameter space overlap for different cluster distributions. This means that there is a multistability of different two-cluster states.
clusters, with stability of one of the two solutions born in the saddle-node bifurcation, and in particular renders the smaller that is, at the Benjamin–Feir instability the larger cluster of the most unbalanced solution becomes unstable exactly when the balanced solution is born, see also figure 6 for the respective solution curves.

Notice that these results are in close correspondence with the behavior observed in the full Stuart–Landau ensemble, compare, for example, figure 3 with figures 4(b) and 5(b) in reference [32].

\[ \lambda_n = A \sqrt{\frac{N_1}{N}} \] for the balanced cluster state \( \lambda_{+1} = \lambda_{+2} \) as a function of the cluster size \( N_1/N \), see figure 5. It depicts the \( \lambda_n \) curves of the saddle-node bifurcations creating the two-cluster states (\( \lambda_n \), blue) and of the two transverse bifurcations (equations (17) and (18)) altering the stability of the two-clusters, with \( \lambda_{+1} \) in green and \( \lambda_{+2} \) in orange.

The bifurcation scenario can be better visualized by plotting \( \lambda_{+1} \) and \( \lambda_{+2} \) as a function of the cluster size \( N_1/N \), see figure 5. It depicts the \( \lambda_n \) values of the saddle-node bifurcations creating the two-cluster states (\( \lambda_n \), blue) and of the two transverse bifurcations (equations (17) and (18)) altering the stability of the two-clusters, with \( \lambda_{+1} \) in green and \( \lambda_{+2} \) in orange.

When increasing \( \lambda_n \) coming from negative values, all cluster states with \( N_1/N \neq 1/2 \) are born in the saddle-node bifurcation \( \lambda_n \). Note that in fact two solutions for each \( N_1/N \) are created this way. In figure 5, one can observe that for the most unbalanced state \( N_1/N \rightarrow 0 \), the transverse bifurcation stabilizing the smaller cluster \( \lambda_{+1} \) occurs immediately after the saddle-node bifurcation creating that cluster. This bifurcation alters the stability of one of the two solutions born in the saddle-node bifurcation, and in particular renders the smaller

\[ A = -0.2, \ B = -2/(2\sqrt{3} - 3), \ C = -1 \]
of the two clusters in that solution stable to transverse perturbations. For the parameter regime considered here \((A = -0.2, B = -2/(2\sqrt{3} - 3)\) and \(C = -1\)), this solution is in fact stabilized at this bifurcation, that is for \(\lambda_+ > \lambda_{+,1}\).

For \(N_1/N < 1/3\), the respective two-cluster solution remains stable until \(\lambda_{+,2}\), where the larger cluster becomes unstable, thus rendering the whole solution unstable. This can, for example, be observed for the 4:12 cluster-size distribution, see figure 6 (top). There, the variable of one cluster, \(x_1\), is plotted as a function of the bifurcation parameter \(\lambda_+\). The blue dot on the left marks the saddle-node bifurcation wherein the two 4:12 solutions are created. Initially, both solutions are unstable. At \(\lambda_{+,1}\) (orange dot), one of them is stabilized, and at \(\lambda_{+,2}\) (green dot), it is subsequently destabilized.

For \(N_1/N > 1/3\), the scenario is different. There the solution that got stabilized at \(\lambda_{+,1}\) remains stable for all \(\lambda_+ > \lambda_{+,1}\). The bifurcation \(\lambda_{+,2}\) instead occurs at the second cluster solution created at the saddle-node bifurcation. This is illustrated more clearly in figure 6 (bottom) for the 7:9 cluster solution. One of the two solutions becomes stable at \(\lambda_{+,1}\), marked by an orange dot and as discussed above. Since \(N_1/N = 7/16 > 1/3\), this solution remains stable for all \(\lambda_+ > \lambda_{+,1}\). The second solution (upper curve in the bottom part of figure 6) first passes the synchronized solution at the Benjamin–Feir bifurcation \(\lambda_+ = 0\) and finally becomes stabilized at \(\lambda_{+,2}\) marked by a green dot. \(\lambda_{+,2}\) diverges at the pole \(N_1/N = 1/3\), separating the two scenarios shown in figure 6. There the bifurcation switches from the solution with negative \(x_1\) (which, for \(\lambda_+ \to \infty\), diverges to \(-\infty\)) to the solution with positive \(x_1\) (which, for \(\lambda_+ \to \infty\), diverges to \(+\infty\)).

Notice how for the cluster distribution \(N_1/N = 7/16\) the two two-cluster solutions are bistable for \(\lambda_+ > \lambda_{+,2}\). That is, there exist two stable two-cluster solutions with different \(x_1\) but the same cluster size ratio 7:9 that are both stable. This, in fact, has also been observed in the Stuart–Landau ensemble, see for example figure 6 in reference [32]. Note that the singularity of \(\lambda_{+,2}\) at \(N_1/N = 1/3\) \((\alpha = 1/2)\) is independent of the parameters \(A, B\) and \(C\), see equation (18). This means that bistable solutions created as described above can in general only exist for \(N_1/N > 1/3\).

6. Conclusion and outlook

In this paper, we showed how one can map a system of globally coupled Stuart–Landau oscillators onto the \((N-1)\)-dimensional center manifold at the Benjamin–Feir instability. Thereby, we observed that the
bifurcation curves at which two-cluster solutions are born closely resemble their counterparts in the original oscillatory system. This allowed us to investigate a codimension-two point called cluster singularity, from which all these bifurcation curves emanate. In the center manifold, we saw that this point corresponds to a vanishing coefficient \( A = 0 \) in front of the quadratic term of the equations of motion. Due to the reduced dynamics in this manifold, we were able to obtain stability boundaries for two-cluster states analytically. This allows for the more detailed investigation of the bow-tie-shaped cascade of transverse bifurcations that govern the stability of these two-cluster states, highlighting the role of the cluster singularity as an organizing center. The observed behavior is hereby independent of the oscillatory nature of each Stuart–Landau oscillator, but a result of the \( S_N \)-equivariance of the full system. These findings may thus facilitate our understanding of this codimension-two point, and of clustering in general, even beyond oscillatory ensembles.

Through this reduction to the center manifold, we could calculate the bifurcation curves creating the cluster solutions (\( \lambda_{\text{sn}} \)) and altering their stability (\( \lambda_{+1} \) and \( \lambda_{+2} \)) analytically. This allowed us to investigate when stable two-cluster solutions exist more systematically, and in particular revealed when different solutions with the same cluster-size distribution are bistable (cf figure 6). The relative cluster size \( N_1/N = 1/3 \) seems to be a general lower limit for such a bistable behavior. The bifurcation scenario of how states with different cluster size ratios \( N_1/N \) are created is thereby different from the Eckhaus instability \([38]\) in reaction–diffusion systems. There, solutions of different wavelengths are created through supercritical pitchfork bifurcations at the trivial solution and subsequently stabilized through a sequence of subcritical pitchfork bifurcations involving mixed-mode states. In our case, the different two-cluster states are created in saddle-node bifurcations and stabilized at \( \lambda_{+1} \) at a single equivariant bifurcation point involving three-cluster states. However, the detailed interaction between two- and three-cluster states still remains an open topic for future research.

Note that the cubic truncation of the flow in the center manifold has a gradient structure \([36]\). This means that we can assign an abstract potential to each of the cluster distributions for a particular set of parameters \( \lambda_{+1}, A, B \) and \( C \). Is there a particular cluster distribution with a minimal potential value? What is its role in the dynamics between these cluster distributions? The companion paper \([36]\) addresses some of these dynamical questions.

Here, we fixed the parameter \( \gamma = 2 \) in the full Stuart–Landau system, and varied the coupling parameters \( \beta_x, \beta_z \). This restricts our analysis to a small region in parameter space. It is important to mention that for different parameter regimes, a qualitatively different behavior close to the cluster singularity might be observed \([36]\).

As discussed in section 2, the Stuart–Landau ensemble permits the transformation into a corotating frame. This turns limit-cycle dynamics into fixed-point dynamics and thus greatly facilitates the reduction onto the center manifold. For more general oscillatory ensembles, such as systems composed of van der Pol or Hodgkin–Huxley type units, the transformation to a corotating frame may be more cumbersome or not even possible. If the coupling between such units is of a global nature, we expect, however, that the nesting of bifurcation curves creating different cluster distributions, cf figure 2, can also be observed in these systems.

This directly links to the fact that we focused on oscillatory dynamics in this article. An exciting further question is the possibility of equivalent dynamics, such as clustering and cluster singularities, in systems composed of bistable or excitable units.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

F P K thanks B F for the hospitality and the exciting discussions at the Freie Universität Berlin. B F gratefully acknowledges the deep inspiration by, and hospitality of, his coauthors in München who initiated this work. This work has also been supported by the Deutsche Forschungsgemeinschaft, SFB910, project A4 ‘Spatio-Temporal Patterns: Control, Delays, and Design’, and by KR1189/18 ‘Chimera States and Beyond’.

Appendix A. Variable transformation

Using log-polar coordinates \( W_k = \exp(\mathbf{R}_k + i\Phi_k) \), equation (2) turns into

\[
\left( \mathbf{R}_k + i\Phi_k \right) e^{R_k+i\Phi_k} = e^{R_k+i\Phi_k} - (1 + i\gamma) e^{2R_k+i\Phi_k} + (\beta_x + i\beta_z) \left( e^{R_k+i\Phi_k} - e^{R_k+i\Phi_k} \right).
\]
Dividing by $W_k$ this becomes
\[
\dot{R}_k + i \dot{\Phi}_k = 1 - (1 + i \gamma) e^{2R_k} + (\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right).
\]

We average over $k$ and separate real and imaginary parts. The mean amplitude $R$ and the mean phase $\Phi$ then satisfy
\[
\dot{R} = 1 - \langle e^{2R} \rangle + Re \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right)
\]
\[
\dot{\Phi} = -\gamma \langle e^{2R} \rangle + Im \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right).
\]

Substituting the variables listed in table 2, one obtains $\langle \exp(2R) \rangle = \langle \exp(2R+2\beta) \rangle = \exp(2R) \langle \exp(2\beta) \rangle$, and $\langle \exp(R+i\Phi) \rangle = \langle \exp(R+R+i\varphi+i\Phi) \rangle = \exp(R+i\Phi) \langle \exp(\varphi) \rangle$. Therefore
\[
\dot{R} = 1 - e^{2R} \langle e^{2\beta} \rangle + Re \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right)
\]
\[
\dot{\Phi} = -\gamma e^{2R} \langle e^{2\beta} \rangle + Im \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right).
\]

For the deviations $r_k = R_k - R$ and $\varphi_k = \Phi_k - \Phi$ one may write
\[
r_k = \dot{R}_k - \dot{R}
\]
\[
= 1 - e^{2R_k} e^{2\beta} + Re \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right) - \dot{R}
\]
\[
= -e^{2R_k} e^{2\beta} + Re \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right)
\]
\[
\varphi_k = \dot{\Phi}_k - \dot{\Phi}
\]
\[
= -\gamma e^{2R_k} e^{2\beta} + Im \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right) - \dot{\Phi}
\]
\[
= -\gamma e^{2R_k} e^{2\beta} + Im \left((\beta_k + i \beta) \left((e^{R+i\Phi}) e^{-R_k-i\Phi_k} - 1\right)\right)
\]

with the notations as defined in table 1. The equations for $\dot{R}_k$ and $\dot{\Phi}_k$ then constitute the corotating system equations (3a) to (3c).

**Appendix B. Linearization**

Linearizing the dynamics of the transformed system, equations (3a) to (3c), at the equilibrium $R = 0$, $r_k = \varphi_k = 0$, and using the fact that $\langle r \rangle = 0$, $\langle z \rangle = 0$, see table 2, one gets
\[
\begin{pmatrix}
\dot{R} \\
\dot{r}_k \\
\dot{\varphi}_k
\end{pmatrix}
= \begin{pmatrix}
-2R \\
-2r_k - Re(kz_k) \\
-2\gamma r_k - Im(kz_k)
\end{pmatrix}
= \begin{pmatrix}
-2R \\
-(2 + \beta_k) r_k - \beta \varphi_k \\
-(2\gamma + \beta_\beta) r_k - \beta \varphi_k
\end{pmatrix}
= \begin{pmatrix}
-2 & 0 & 0 \\
0 & -2 - \beta_k & \beta_k \\
0 & -2\gamma - \beta_\beta & -\beta_\beta
\end{pmatrix}
\cdot
\begin{pmatrix}
R \\
r_k \\
\varphi_k
\end{pmatrix}
= J
\cdot
\begin{pmatrix}
R \\
r_k \\
\varphi_k
\end{pmatrix}.
\]

The Jacobian thus has the eigenvalues
- Eigenvalue $\lambda_1 = -2$ with eigenvector $\bar{v}_1 = (1, 0, 0)$.

and two eigenvalues of geometric multiplicity $N - 1$ given by the eigendecomposition
\[
\text{eig} \begin{pmatrix}
-2 - \beta_k & \beta_k \\
-2\gamma - \beta_\beta & -\beta_\beta
\end{pmatrix},
\]

which gives
- The eigenvalue $\lambda_+ = -1 - \beta_k + \sqrt{1 - \beta_k^2 - 2\beta_k\gamma} = -1 - \beta_k + d$.
- And the eigenvalue $\lambda_- = -1 - \beta_k - \sqrt{1 - \beta_k^2 - 2\beta_k\gamma} = -1 - \beta_k - d$. 


Here, we assume $1 - \beta_1^2 - 2\beta_1\gamma > 0$, that is real $\lambda_{+}$. For an analysis of the case $1 - \beta_1^2 - 2\beta_1\gamma < 0$, see reference [39]. The eigenvectors corresponding to these two eigenvalues can be obtained using

$$
\begin{pmatrix}
-2\gamma - \beta_1 & \beta_i \\
2\gamma - \beta_1 & -\beta_i
\end{pmatrix} - \lambda_{\pm} 1_{(N-1)\times(N-1)} \varphi_k = \vec{0},
$$

For $\lambda_{+}$, one thus obtains

$$
\begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 - d
\end{pmatrix} \varphi_k = \begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 - d
\end{pmatrix} \begin{pmatrix} r_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} (1 - d) r_k + \beta_i \varphi_k \\ (-2\gamma - \beta_1) r_k + (1 - d) \varphi_k \end{pmatrix} = 0.
$$

Choosing $\varphi_k = (1 + d)/\beta_i r_k$, we get

$$
\begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 - d
\end{pmatrix} \begin{pmatrix} r_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} (1 + d) r_k + (1 + d) r_k \\ (2\gamma + \beta) r_k + (1 - d) \varphi_k \end{pmatrix} = 0,
$$

thus solving the equality above. The constraint equation (A.1), together with $\langle r \rangle = \langle \varphi \rangle = 0$, defines an $(N - 1)$-dimensional subspace of $\mathbb{R}^{2N - 1}$.

For $\lambda_{-}$, one thus obtains

$$
\begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 + d
\end{pmatrix} \varphi_k = \begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 + d
\end{pmatrix} \begin{pmatrix} r_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} (1 - d) r_k + \beta_i \varphi_k \\ (-2\gamma - \beta_1) r_k + (1 + d) \varphi_k \end{pmatrix}.
$$

Choosing $\varphi_k = (1 - d)/\beta_i r_k$, solves the conditions above. In particular,

$$
\begin{pmatrix}
-1 - d & \beta_i \\
-2\gamma - \beta_1 & 1 + d
\end{pmatrix} \begin{pmatrix} r_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} (1 - d) r_k + (1 - d) r_k \\ (2\gamma + \beta) r_k + (1 + d) \varphi_k \end{pmatrix} = 0,
$$

The constraint equation (A.2), together with $\langle r \rangle = \langle \varphi \rangle = 0$ define an $(N - 1)$-dimensional subspace of $\mathbb{R}^{2N - 1}$. Now, one can define the eigencoordinates $x_k$ describing the dynamics in the space defined by the constraint equation (A.1), the center space of the bifurcation, and eigencoordinates $y_k$, describing the dynamics in the space defined by the constraint equation (A.2). These two sets of variables, together with $R$, can then be used to describe the full system.

**Appendix C. Parameter derivation**

In this section of the appendix, we derive expressions for the parameters $a$, $b$, $A$, $B$ and $C$ as a function of the parameters $\gamma$, $\beta$, and $\beta_i$ from the Stuart–Landau ensemble. Hereby, we will use the condition that $R$ and the $y_k$ are tangential, that is, $\left. \frac{d}{dx_k} R \right|_{x_k=0} = 0$ and $\left. \frac{d}{dx_k} y_k \right|_{x_k=0} = 0$.

### C.1. $a$ and $b$

In order to calculate $a$ and $b$, it is useful to write out the following expressions

$$
\begin{align*}
z_k &= r_k + i\varphi_k \\
&= (1 - d) x_k + (1 + d) y_k + i(\gamma' x_k + \gamma' y_k) \\
&= (1 - d + i\gamma') x_k + a(1 + d + i\gamma') x_k^2 + O(x_k^3)
\end{align*}
$$

$$
\begin{align*}
z_k' &= (r_k + i\varphi_k)^2 \\
&= ((1 - d) x_k + (1 + d) y_k + i(\gamma' x_k + \gamma' y_k))^2 \\
&= ((1 - d + i\gamma') x_k + (1 + d + i\gamma') y_k)^2 \\
&= (1 - d + i\gamma')^2 x_k^2 + 2a(1 - d + i\gamma')(1 + d + i\gamma') x_k x_k^2 + O(x_k^3)
\end{align*}
$$

$$
\begin{align*}
z_k^3 &= (r_k + i\varphi_k)^3
\end{align*}
$$
With the expression for $\gamma$ and the notation $\gamma' = 2\gamma + \beta$, similarly, we expand the following parts and keep terms up to cubic order:

\[
e^{z_k} = 1 + z_k + \frac{z_k^2}{2} + \frac{z_k^3}{6} + \mathcal{O}(x_k^3)
\]

\[
e^{-z_k} = 1 - z_k + \frac{z_k^2}{2} - \frac{z_k^3}{6} + \mathcal{O}(x_k^3)
\]

\[
\langle e^x \rangle = (1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \mathcal{O}(x_k^3))
\]

\[
= 1 + \frac{1}{2}(z^2) + \frac{1}{6}(z^3) + \mathcal{O}(x_k^3)
\]

\[
\tilde{e}^{-z_k} = e^{-z_k} - \langle e^{-z_k} \rangle
\]

\[
= 1 - z_k + \frac{z_k^2}{2} - \frac{z_k^3}{6} - 1 - \frac{1}{2}(z^2) + \frac{1}{6}(z^3) + \mathcal{O}(x_k^3)
\]

\[
= -z_k + \frac{\gamma}{2z_k} - \frac{\gamma^2}{6z_k^2} + \mathcal{O}(x_k^3)
\]

\[
\langle e^x \rangle \tilde{e}^{-z_k} = \left(1 + \frac{1}{2}(z^2) + \frac{1}{6}(z^3)\right) \left(-z_k + \frac{\gamma}{2z_k} - \frac{\gamma^2}{6z_k^2}\right) + \mathcal{O}(x_k^3)
\]

\[
= -z_k + \frac{\gamma}{2z_k} - \frac{\gamma^2}{6z_k^2} + \mathcal{O}(x_k^3)
\]

With the expression for $R$, see equation (9), we can furthermore write

\[
e^{3R} = 1 + 2R + \mathcal{O}(x_k^3)
\]

\[
= 1 + 2b(x^2) + \mathcal{O}(x_k^3)
\]

\[
e^{3s} = 1 + 2s + \frac{4}{3}r_k^3 + \mathcal{O}(x_k^3)
\]

\[
\langle e^{2r} \rangle = 1 + 2\langle r^2 \rangle + \frac{4}{3}\langle r^3 \rangle + \mathcal{O}(x_k^3)
\]

\[
\tilde{e}^{2s} = e^{2s} - \langle e^{2s} \rangle
\]

\[
= 2s + 2r_k^2 + \frac{4}{3}r_k^3 + \mathcal{O}(x_k^3)
\]

\[
e^{2R\langle e^{2r} \rangle} = (1 + 2b(x^2)) \left(1 + 2\langle r^2 \rangle + \frac{4}{3}\langle r^3 \rangle\right) + \mathcal{O}(x_k^3)
\]

\[
= 1 + 2\langle r^2 \rangle + 2b(x^2) + \frac{4}{3}\langle r^3 \rangle + \mathcal{O}(x_k^3)
\]

\[
e^{2\tilde{e}2s} = (1 + 2b(x^2)) \left(2r_k + 2r_k^2 + \frac{4}{3}r_k^3\right) + \mathcal{O}(x_k^3)
\]

\[
= 2r_k + 2br_k(x^2) + 2r_k^2 + \frac{4}{3}r_k^3 + \mathcal{O}(x_k^3).
\]
Using these approximations, we can write for the dynamics of $R$ up to second order in $x_k$

$$\dot{R} = 1 - e^{2\beta} \langle e^{2\gamma} \rangle + \Re \left( \langle \beta_r + i\beta_i \rangle \left( \langle e^{2\gamma} \rangle e^{-2\gamma} \right) \right)$$

$$= 1 - \left( 1 + 2\langle x^2 \rangle + 2b\langle x^2 \rangle \right) + \Re \left( \langle \beta_r + i\beta_i \rangle \left( 1 + \langle x^2 \rangle - 1 \right) \right)$$

$$= -2\langle x^2 \rangle - 2b\langle x^2 \rangle + \Re \left( (\beta_r + i\beta_i) \langle x^2 \rangle \right)$$

$$= -2(1 - d^2)\langle x^2 \rangle - 2b\langle x^2 \rangle + \Re \left( \langle (1 - d^2) + i\gamma \rangle \right) \langle x^2 \rangle$$

Now, we use the tangential property of $R$. In particular, we can write

$$\dot{R} = \left( \frac{d}{dx_k} \right) R = 2b\langle x^2 \rangle + O \left( x_k^3 \right) = 2b\lambda_+ \langle x^2 \rangle + O \left( x_k^3 \right).$$

At $\lambda_+ = 0$, $\dot{R}$ up to second order must vanish. This allows us to calculate $b$ by comparing the terms in front of $\langle x^2 \rangle$ in $\dot{R}$, yielding

$$\Rightarrow b = -\frac{\beta_r}{2} \left( \gamma^2 + 6\beta_r + \beta_i^2 \right)$$

$$= -\frac{1}{2} \left( \gamma^2 + d^2 + 4d - 5 \right).$$

We can derive the expression for $a$ in a similar way. Here, we write out the dynamics of $y_k$ up to second order. This yields

$$2d\dot{y}_k = \dot{r}_k + \frac{d - 1}{\gamma'} \ddot{\varphi}_k$$

$$= -\left( 1 + (d - 1) \frac{\gamma'}{\gamma} \right) e^{2\beta} e^{\frac{2}{\gamma'}} \Re \left( \langle 1 - i\frac{d - 1}{\gamma} \rangle (\beta_r + i\beta_i) \left( \langle e^{2\gamma} \rangle e^{-2\gamma} \langle x^2 \rangle \right) \right)$$

$$= -\left( 1 + (d - 1) \frac{\gamma'}{\gamma} \right) \left( 2\dot{r}_k + 2\ddot{r}_k \right) + \Re \left( \langle 1 - i\frac{d - 1}{\gamma} \rangle (\beta_r + i\beta_i) \left( -\dot{r}_k + \frac{1}{2\gamma} \ddot{r}_k \right) \right)$$

The term of the coupling constant and its parameters in front in can be summarized by

$$\left( 1 - i\frac{\beta_r}{\gamma} \right) (\beta_r + i\beta_i) = \beta_r + \frac{\beta_r^2 + \beta_i^2}{\gamma^2} - i \left( \frac{\beta_r^2}{\gamma} - \beta_i \right)$$

$$= \beta_r + \frac{\beta_r^2 + \beta_i^2}{\gamma^2} - i \left( \frac{\beta_r^2}{\gamma} + \frac{\beta_r^2 + \beta_i^2}{\gamma} \right)$$

$$\beta_r \frac{\gamma'}{\gamma} = \beta_r \gamma^2 - \beta_i$$

$$= \frac{\beta_r}{2} + \frac{\beta_r^2 + \beta_i^2}{2\gamma^2}.$$

This simplifies the expression for $\dot{y}_k$ to

$$2d\dot{y}_k = -\left( 2 + \beta_r + \frac{\beta_r^2 + \beta_i^2}{\gamma^2} \right) \left( \dot{r}_k + \ddot{r}_k \right)$$

$$+ \Re \left( \left( \beta_r - \frac{\beta_r^2 + \beta_i^2}{\gamma^2} \right) - i \left( \frac{\beta_r^2}{\gamma} + \frac{\beta_r^2 + \beta_i^2}{\gamma} \right) \right) \left( -\dot{r}_k - i\dot{\varphi}_k + \frac{1}{2\gamma} \ddot{r}_k \right)$$

$$= -\left( 2 + \beta_r + \frac{\beta_r^2 + \beta_i^2}{\gamma^2} \right) \left( \dot{r}_k + \ddot{r}_k \right) + \Re.$$
\begin{align*}
&\times \left( \left( \beta_r - \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} + i \left( \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'} \right) \right) \left( -r_k - i\varphi_k + \frac{1}{2} \left( (1 - d)^2 - \gamma'^2 \right) \tilde{x}_k^2 + i \left( 1 - d \right) \gamma \tilde{x}_k^2 \right) \right) \\
&= - \left( 2 + \beta_r + \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \left( r_k + \tilde{\xi}^2 \right) \\
&+ \left( \beta_r - \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \left( -r_k + \frac{1}{2} \left( (1 - d)^2 - \gamma'^2 \right) \tilde{x}_k^2 \right) - \left( \frac{\beta^2_r + 2\beta_\gamma}{\gamma'} \right) \left( \varphi_k - (1 - d) \gamma \tilde{x}_k^2 \right) \\
&= -2 \left( \beta_r + 1 \right) \left( -\beta_r x_k + (\beta_r + 2) y_k \right) - \left( 2 + \beta_r + \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \beta^2_r \tilde{x}_k^2 \\
&+ \frac{1}{2} \left( \beta_r - \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \left( \beta^2_r - \gamma'^2 \right) \tilde{x}_k^2 - 2 \left( \beta^2_r + \beta \right) \left( x_k + y_k \right) - 2 \left( \beta^2_r + \beta \right) \beta \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 y_k - \left( 2 + \beta_r + \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \beta^2_r \tilde{x}_k^2 \\
&+ \frac{1}{2} \left( \beta_r - \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \left( \beta^2_r - \gamma'^2 \right) \tilde{x}_k^2 - 2 \left( \beta^2_r + \beta \right) \beta \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 y_k - \left( 4 + 3\beta_r + \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \beta^2_r \tilde{x}_k^2 \\
&+ \frac{1}{2} \left( \beta_r - \frac{\beta^2_r + 2\beta^2_\gamma}{\gamma'^2} \right) \left( \beta^2_r - \gamma'^2 \right) \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 y_k - \left( 4 + 5\beta_r + \frac{3\beta^2_r + 6\beta^2_\gamma}{2\gamma'^2} \right) \beta^2_r \tilde{x}_k^2 - \frac{1}{2} \left( \beta_r \gamma'^2 - \beta^2_r - 2\beta^2_\gamma \right) \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 \tilde{x}_k^2 - \left( 3 + 2\beta_r + \frac{3\beta^2_r + 6\beta^2_\gamma}{2\gamma'^2} \right) \beta^2_r \tilde{x}_k^2 - \frac{1}{2} \beta_r \gamma'^2 \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 \tilde{x}_k^2 - \frac{\beta^2_r}{2\gamma'^2} \left( \gamma'^2 + 6\beta^2_\gamma + 4\beta^2_\gamma \gamma'^2 + 3\beta^4 + 6\beta^3_\gamma \right) \tilde{x}_k^2 \\
&= -4(\beta_r + 1)^2 \tilde{x}_k^2 - \frac{\beta^2_r}{2\gamma'^2} \left( \gamma'^2 + \beta^2_\gamma \right) \left( 3\beta_r (\beta_r + 2) + \gamma'^2 \right) \tilde{x}_k^2.
\end{align*}

Similar to \( R \), the \( y_k \) are tangential to the center manifold. This translates into the fact that

\[
\dot{y}_k = \left( \frac{d}{dx_k} y_k \right) \dot{x}_k
\]

vanishes up to second order in \( x_k \). Therefore, comparing the terms in front of the \( \tilde{x}_k^2 \) above yields

\[
a = - \frac{\beta_r \left( \gamma'^2 + \beta^2_\gamma \right) \left( 3\beta_r (\beta_r + 2) + \gamma'^2 \right)}{8(\beta_r + 1)^2 \gamma'^2} = \left( 1 - d \right) \left( \gamma'^2 + (1 - d)^2 \right) \left( 3 (d^2 - 1) + \gamma'^2 \right)\frac{8d^2 \gamma'^2}{8d^2 \gamma'^2}.
\]

**C.2. A, B and C**

Finally, the coefficients \( A, B \) and \( C \) for the dynamics in the center manifold, cf equation (10), can be obtained by expanding the dynamics of \( x_k \),

\[
2d\dot{x}_k = - \left( -1 + (d + 1) \frac{\gamma}{\gamma'} \right) e^{2\phi} \tilde{e} \bar{e}^2 + \text{Re} \left( \left( -1 - \frac{d + 1}{\gamma'} \right) k \left( e^\varphi \tilde{e} \bar{e} \right) \right),
\]

(C.2)
in powers of $x_k$: the terms in front of $\tilde{x}_i^2$, $\tilde{x}_i^3$, and $x_k(x^2)$ correspond to the coefficients $A$, $B$ and $C$, respectively. In order to do so, we approximate several terms as follows:

\[
\langle e^x e^{-\tilde{x}} \rangle = -z_k + \frac{1}{2} \tilde{x}_k^2 - \frac{1}{6} \tilde{x}_k^3 - \frac{1}{2} z_k(x^2) + O(x_k^4)
\]

where 

\[
z_k = (1 - d + i\gamma') x_k + a (1 + d + i\gamma') \tilde{x}_k + O(x_k^2)
\]

\[
\tilde{x}_k^2 = (1 - d + i\gamma')^2 \tilde{x}_k^2 + 2a (1 - d + i\gamma') (1 + d + i\gamma') x_k \tilde{x}_k + O(x_k^2)
\]

\[
\tilde{x}_k^3 = (1 - d + i\gamma')^3 \tilde{x}_k^3 + O(x_k^2)
\]

\[
x_k(x^2) = \left(1 - d + i\gamma'\right) x_k + a (1 + d + i\gamma') \tilde{x}_k^2
\]

\[
\cdot \left( (1 - d + i\gamma')^2 \tilde{x}_k^2 + 2a (1 - d + i\gamma') (1 + d + i\gamma') x_k \tilde{x}_k^2 + O(x_k^2) \right)
\]

\[
= \left( (1 - d + i\gamma')^2 \tilde{x}_k^2 + a (1 + d + i\gamma') \tilde{x}_k^2 \right) \langle (1 - d + i\gamma')^2 x^2 \rangle + O(x_k^4)
\]

Using these terms, we can write

\[
\langle e^x e^{-\tilde{x}} \rangle = -z_k + \frac{1}{2} \tilde{x}_k^2 - \frac{1}{6} \tilde{x}_k^3 - \frac{1}{2} z_k(x^2) + O(x_k^4)
\]

\[
= - (1 - d + i\gamma') x_k - a (1 + d + i\gamma') \tilde{x}_k
\]

\[
+ \frac{1}{2} \left( (1 - d + i\gamma')^2 \tilde{x}_k^2 + a (1 - d + i\gamma') (1 + d + i\gamma') \tilde{x}_k^2 \right) \langle (1 - d + i\gamma')^2 x^2 \rangle
\]

\[
- \frac{1}{6} \left( (1 - d + i\gamma')^3 \tilde{x}_k^3 \right)
\]

\[
- \frac{1}{2} \left( (1 - d + i\gamma')^3 x_k \tilde{x}_k^2 \right)
\]

\[
= - (1 - d + i\gamma') x_k
\]

\[
+ \left( \frac{1}{2} (1 - d + i\gamma')^2 - a (1 + d + i\gamma') \right) \tilde{x}_k^2
\]

\[
+ \left( a (1 - d + i\gamma') (1 + d + i\gamma') - \frac{1}{6} (1 - d + i\gamma')^3 \right) \tilde{x}_k^3
\]

\[
+ \left( \frac{1}{2} (1 - d + i\gamma')^3 - a (1 + d + i\gamma') (1 + d + i\gamma') \right) x_k \tilde{x}_k^2
\]

\[
e^{2b e^{-\tilde{x}}} = \left( 1 + 2b(x^2) \right) \left( 2r_k + 2r_k^2 + \frac{4}{3} r_k^3 \right) + O(x_k^4)
\]

\[
= 2r_k + 4b r_k^2 + 2r_k^2 + \frac{4}{3} r_k^3 + O(x_k^4)
\]

\[
r_k = (1 - d) x_k + (1 + d) y_k
\]

\[
= (1 - d) x_k + (1 + d) \tilde{x}_k^2
\]
\[ r_k^2 = (1 - d)x_k^2 + 2a (1 - d) (1 + d) x_k \bar{x}_k + O \left( x_k^2 \right) \]
\[ r_k^2 = (1 - d)x_k^2 + O \left( x_k^2 \right) \]
\[ \bar{r}_k^2 = r_k^2 - \langle \bar{r} \rangle \]
\[ = (1 - d)^2 \bar{x}_k^2 + 2a (1 - d) (1 + d) \left( x_k^2 - x_k \langle \bar{x} \rangle \right) \]
\[ \bar{r}_k^2 = (1 - d)^2 x_k^2 + O \left( x_k^2 \right) \]

\[ e^{2\bar{r} \bar{z}_i} = 2r_k + 4b r_k (x_k^2) + 2 \bar{r}_k^2 + \frac{4}{3} \bar{r}_k^2 + O \left( x_k^2 \right) \]
\[ = 2 (1 - d) x_k + 2a (1 + d) x_k^2 \]
\[ + 4b (1 - d) x_k \langle \bar{x} \rangle \]
\[ + 2(1 - d)^2 \bar{x}_k^2 + 4a (1 - d) (1 + d) \left( x_k^2 - x_k \langle \bar{x} \rangle \right) \]
\[ + \frac{4}{3} (1 - d)^2 x_k^2 + O \left( x_k^2 \right) \]
\[ = 2 (1 - d) x_k \]
\[ + (2a (1 + d) + 2(1 - d)^2) \bar{x}_k^2 \]
\[ + \left( 4a (1 - d) (1 + d) + \frac{4}{3} (1 - d)^3 \right) x_k^2 \]
\[ + (4b (1 - d) - 4a (1 - d) (1 + d)) x_k \langle \bar{x} \rangle. \]

We can now insert the different orders of \( x_k \) from \( e^{2\bar{r} \bar{z}_i} \) and \( \langle e^r \rangle \bar{e}^{\bar{z}_i} \) in equation (C.2) (here, we use sympy [40] to solve for the coefficients), yielding

\[ 2dx_k = \frac{(d - 1) \left( \bar{z}_i^2 + (1 + d)^2 \right) \left( \bar{z}_i^2 - 3(d - 1)^2 \right)}{2\gamma^2} \]
\[ - \frac{(d - 1)^2 \left( \bar{z}_i^2 + (d - 1)^2 \right) \left( \bar{z}_i^2 + (d + 1)^2 \right) \left( \bar{z}_i^2 - 2\gamma' d + 3 \left( d^2 - 1 \right) \right) \left( \bar{z}_i^2 + 2\gamma' d + 3 \left( d^2 - 1 \right) \right)}{8\gamma^4 d^2} \]
\[ + \frac{(1 - d)^2}{8d^2} \left( \bar{z}_i^2 - 4\gamma^2 \left( 2d^6 - 7d^2 + 1 \right) \right) \]
\[ - 2 \left( 8d^6 + d^4 - 56d^3 + 22d^2 + 1 \right) - \frac{4}{\gamma^2} \left( 2d^6 + 5d^4 - 4d^3 - 13d^2 + 2d + 11d^2 - 3 \right) \]
\[ + \frac{9}{\gamma^2} \left( d^2 - 1 \right)^4 \right) x_k \langle \bar{x} \rangle. \]

Reading off the coefficients then gives the parameters

\[ A = \frac{(d - 1) \left( \bar{z}_i^2 + (1 + d)^2 \right) \left( \bar{z}_i^2 - 3(d - 1)^2 \right)}{4\gamma^2 d} \]
\[ B = - \frac{(d - 1)^2 \left( \bar{z}_i^2 + (d - 1)^2 \right) \left( \bar{z}_i^2 + (d + 1)^2 \right) \left( \bar{z}_i^2 - 2\gamma' d + 3 \left( d^2 - 1 \right) \right) \left( \bar{z}_i^2 + 2\gamma' d + 3 \left( d^2 - 1 \right) \right)}{16\gamma^4 d^3} \]
\[ C = \frac{(d - 1)^2}{16d^3 \gamma^4} \left( \bar{z}_i^2 - 4\gamma^2 \left( 2d^6 - 7d^2 + 1 \right) - 2\gamma^2 \left( 8d^6 + d^4 - 56d^3 + 22d^2 + 1 \right) \right) \]
\[ - 4\gamma^2 \left( 2d^6 + 5d^4 - 4d^3 - 13d^2 + 2d + 11d^2 - 3 \right) + 9 \left( d^2 - 1 \right)^4 \right) x_k \langle \bar{x} \rangle. \]
Appendix D. Two-cluster states in the center manifold

For two-cluster states, we can take $N = N_1 + N_2$ and write

$$
\dot{x}_k = \lambda_+ x_k + A \tilde{x}_k + Bx_k^3 + C(x^2)x_k + O(x_k^4)
$$

$$
= \lambda_+ x_k + A \left( x_k^2 - \frac{1}{N} (N_1 x_1^2 + N_2 x_2^2) \right) + B \left( x_k^3 - \frac{1}{N} \left( N_1 x_1^3 + N_2 x_2^3 \right) \right) + \frac{C}{N} \left( N_1 x_1^2 + N_2 x_2^2 \right) x_k
$$

with the constraint $k \in \{1, 2\}$ and $N_1 x_1 + N_2 x_2 = 0$, that is, $x_2 = -(N_1/N_2)x_1$. Note that $\dot{x}_k$ must vanish at the two-cluster equilibria. The two-cluster therefore satisfies

$$
0 = \lambda_+ x_1 + A \left( x_1^2 - \frac{1}{N} \left( N_1 x_1^2 + N_2 x_2^2 \right) \right) + B \left( x_1^3 - \frac{1}{N} \left( N_1 x_1^3 + N_2 x_2^3 \right) \right) + \frac{C}{N} \left( N_1 x_1^2 + N_2 x_2^2 \right) x_1
$$

$$
= \lambda_+ x_1 + A \left( x_1^2 - \frac{N_1}{N_2} x_1^2 \right) + B \left( x_1^3 - \frac{N_1 (N_2 - N_1)}{N_2^2} x_1^3 \right) + \frac{CN_1}{N_2} x_1^3
$$

$$
= \lambda_+ x_1 + A \frac{N_2 - N_1}{N_2} x_1^2 + B \left( \frac{N_2^2 - N_1 (N_2 - N_1)}{N_2^2} x_1^3 \right) + \frac{CN_1}{N_2} x_1^3,
$$

and writing $\alpha = N_1/N_2$,

$$
0 = \lambda_+ x_1 + A \left( 1 - \alpha \right) x_1^2 + \left( B \left( 1 - \alpha + \alpha^2 \right) + C\alpha \right) x_1^3.
$$

This equation has the solutions $x_1 = 0, x_2 = 0$ and

$$
x_1^\pm = \frac{1}{2 \left( B \left( 1 - \alpha + \alpha^2 \right) + C\alpha \right)} \left( -A \left( 1 - \alpha \right) \pm \sqrt{A^2 (1 - \alpha)^2 - 4 \lambda_+ (B \left( 1 - \alpha + \alpha^2 \right) + C\alpha) \right) \right)
$$

$$
x_2^\pm = -(N_1/N_2)x_1^\pm.
$$

The saddle-node curves creating the two-cluster solutions are thus parameterized by the vanishing discriminant

$$
0 = A^2 (1 - \alpha)^2 - 4 \lambda_+ \left( B \left( 1 - \alpha + \alpha^2 \right) + C\alpha \right)
$$

$$
\Rightarrow \lambda_+ = \lambda_m = \frac{A^2 (1 - \alpha)^2}{4 \left( B \left( 1 - \alpha + \alpha^2 \right) + C\alpha \right)}
$$

for unbalanced cluster solutions, that is, $\alpha \neq 1$ or $N_1 \neq N_2$. Thus, at the saddle-node bifurcation

$$
x_1^+ = x_1^- = \frac{A \left( 1 - \alpha \right)}{2 \left( B \left( 1 - \alpha + \alpha^2 \right) + C\alpha \right)}.
$$

ORCID iDs

Felix P Kemeth https://orcid.org/0000-0001-8535-8113
Sindre W Haugland https://orcid.org/0000-0002-1642-7952

References

[1] Strogatz S H 2000 From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators *Physica D* 143 1–20
[2] García-Morales V and Krischer K 2008 Normal-form approach to spatiotemporal pattern formation in globally coupled electrochemical systems *Phys. Rev. E* 78 057201
[3] Kiss I Z, Zhai Y and Hudson J L 2006 Characteristics of cluster formation in a population of globally coupled electrochemical oscillators: an experiment-based phase model approach *Prog. Theor. Phys. Suppl.* 161 99–106
[4] Wang W, Kiss I Z and Hudson J L 2000 Experiments on arrays of globally coupled chaotic electrochemical oscillators: synchronization and clustering *Chaos* 10 248–56
[5] Varela H, Beta C, Antoine B and Krischer K 2005 A hierarchy of global coupling induced cluster patterns during the oscillatory H2-electrooxidation reaction on a Pt ring-electrode *Phys. Chem. Chem. Phys.* 7 2429
[6] Plenge F, Varela H and Krischer K 2003 Pattern formation in stiff oscillatory media with nonlocal coupling: a numerical study of the hydrogen oxidation reaction on Pt electrodes in the presence of poisons *Phys. Rev. E* 72 066211
[7] Schönleber K, Zensen C, Heinrich A and Krischer K 2014 Pattern formation during the oscillatory photoelectrooxidation of n-type silicon: turbulence, clusters and chimeras *New J. Phys.* 16 063024
[8] Kim M, Bertram M, Pollmann M, von Oertzen A, Mikhailov A S, Rotermund H U and Ertl G 2001 Controlling chemical turbulence by global delayed feedback: pattern formation in catalytic CO oxidation on Pt(110) Science 292 1357–60

[9] Elmihiri T 2001 Symmetry and emergence in polymorphism and sympatric speciation PhD Thesis Warwick University of Warwick

[10] Stewart I, Elmihiri T and Cohen J 2003 Symmetry-breaking as an origin of species Bifurcation, Symmetry and Patterns (Basel: Birkhäuser)

[11] Okada K 1993 Variety and generality of clustering in globally coupled oscillators Physica D 63 424–36

[12] Nakagawa N and Kuramoto Y 1994 From collective oscillations to collective chaos in a globally coupled oscillator system Physica D 75 74–80

[13] Banaji M 2002 Clustering in globally coupled oscillators Dyn. Syst. 17 263–85

[14] Daido H and Nakanishi K 2007 Aging and clustering in globally coupled oscillators Phys. Rev. E 75 036206

[15] Ku W L, Girvan M and Ott E 2015 Dynamical transitions in large systems of mean field-coupled Landau–Stuart oscillators: extensive chaos and cluster states Chaos 25 123122

[16] Pikovsky A and Rosenblum M 2015 Dynamics of globally coupled oscillators: progress and perspectives Chaos 25 097616

[17] Kuramoto Y 1984 Chemical Oscillations, Waves and Turbulence (Springer Series in Synergetics) vol 19 (Berlin: Springer)

[18] Watanabe S and Strogatz S H 1994 Constants of motion for superconducting Josephson arrays Physica D 74 197–253

[19] Golubitsky M and Stewart I 2002 Linear Stability (Basel: Birkhäuser) pp 33–57

[20] Hakim V and Rappel W-J 1992 Dynamics of the globally coupled complex Ginzburg–Landau equation Phys. Rev. A 46 R7347–50

[21] Benjamin T B and Feir J E 1967 The disintegration of wave trains on deep water part 1. Theory J. Fluid Mech. 27 417–30

[22] FitzHugh R 1955 Mathematical models of threshold phenomena in the nerve membrane Bull. Math. Biophys. 17 257–78

[23] van der Pol B 1926 On relaxation-oscillations London, Edinburgh Dublin Phil. Mag. J. Sci. 2 978–92

[24] Field R J and Noyes R M 1974 Oscillations in chemical systems. IV. Limit cycle behavior in a model of a real chemical reaction J. Chem. Phys. 60 1877–84

[25] Hodgkin A L and Huxley A F 1952 A quantitative description of membrane current and its application to conduction and excitation in nerve J. Physiol. 117 500–44

[26] Andreozzi E, Carannante I, D’Addio G, Cesarelli M and Balbi P 2019 Phenomenological models of Nav1.5. A side by side, procedural, hands-on comparison between Hodgkin–Huxley and kinetic formalisms Sci. Rep. 9 17493

[27] Dias A P S and Rodrigues A 2006 Secondary bifurcations in systems with all-to-all coupling, Part II Dyn. Syst. 21 439–63

[28] Aronson D G, Ermentrout G B and Kopell N 1990 Amplitude response of coupled oscillators Physica D 41 403–49

[29] Pikovsky A, Rosenblum M and Kurths J 2001 Synchronization (Cambridge: Cambridge University Press) pp 222–35

[30] Röhm A, Lüdge K and Schneider I 2018 Extensive chaos and cluster states Physica D 391 257–78

[31] Röhm A, Lüdge K and Schneider I 2018 Bifurcation, Symmetry and Patterns (Basel: Birkhäuser) pp 222–35

[32] Kemeth F, Haugland S W and Krischer K 2019 Cluster singularity: the unfolding of clustering behavior in globally coupled oscillators Chaos 29 023107

[33] Nakagawa N and Kuramoto Y 1993 Collective chaos in a population of globally coupled oscillators

[34] Doedel E J and Wang X J 2007 AUTO-07P: continuation and bifurcation software for ordinary differential equations Technical Report Pasadena CA Center for Research on Parallel Computing California Institute of Technology 91125

[35] Field R J and Noyes R M 1974 Oscillations in chemical systems. IV. Limit cycle behavior in a model of a real chemical reaction J. Chem. Phys. 60 1877–84

[36] Hodgkin A L and Huxley A F 1952 A quantitative description of membrane current and its application to conduction and excitation in nerve J. Physiol. 117 500–44

[37] Andreozzi E, Carannante I, D’Addio G, Cesarelli M and Balbi P 2019 Phenomenological models of Nav1.5. A side by side, procedural, hands-on comparison between Hodgkin–Huxley and kinetic formalisms Sci. Rep. 9 17493

[38] Dias A P S and Rodrigues A 2006 Secondary bifurcations in systems with all-to-all coupling, Part II Dyn. Syst. 21 439–63

[39] Aronson D G, Ermentrout G B and Kopell N 1990 Amplitude response of coupled oscillators Physica D 41 403–49

[40] Pikovsky A, Rosenblum M and Kurths J 2001 Synchronization (Cambridge: Cambridge University Press) pp 222–35

[41] Röhm A, Lüdge K and Schneider I 2018 Extensive chaos and cluster states Physica D 391 257–78

[42] Röhm A, Lüdge K and Schneider I 2018 Bifurcation, Symmetry and Patterns (Basel: Birkhäuser) pp 222–35

[43] Kemeth F, Haugland S W and Krischer K 2019 Cluster singularity: the unfolding of clustering behavior in globally coupled oscillators Chaos 29 023107

[44] Nakagawa N and Kuramoto Y 1993 Collective chaos in a population of globally coupled oscillators

[45] Doedel E J and Wang X J 2007 AUTO-07P: continuation and bifurcation software for ordinary differential equations Technical Report Pasadena CA Center for Research on Parallel Computing California Institute of Technology 91125

[46] Fiedler B, Haugland S W, Kemeth F and Krischer K 2020 Global two-cluster dynamics under large symmetric groups arXiv:2008.06944

[47] Dias A P S and Stewart I 2003 Secondary bifurcations in systems with all-to-all coupling Proc. R. Soc. A 459 19691986

[48] Tuckerman L S and Barkley D 1990 Bifurcation analysis of the Eckhaus instability Physica D 46 57–86

[49] Kemeth F P 2019 Symmetry breaking in networks of globally coupled oscillators: from clustering to chimera states PhD Thesis Garching Technische Universität München

[50] Meurer A et al 2017 SymPy: symbolic computing in python Peerf Comput. Sci. 3 e103