From convergence in distribution to uniform convergence

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Dedicated with thanks to Sergei Grudsky, who has left his imprint on all the three of us, on his sixtieth birthday

Abstract We present conditions that allow us to pass from the convergence of probability measures in distribution to the uniform convergence of the associated quantile functions. Under these conditions, one can in particular pass from the asymptotic distribution of collections of real numbers, such as the eigenvalues of a family of $n$-by-$n$ matrices as $n$ goes to infinity, to their uniform approximation by the values of the quantile function at equidistant points. For Hermitian Toeplitz-like matrices, convergence in distribution is ensured by theorems of the Szegő type. Our results transfer these convergence theorems into uniform convergence statements.

Keywords convergence in distribution, quantile function, Toeplitz matrix, eigenvalue asymptotics

Mathematics Subject Classification Primary 60B10, Secondary 15B05, 15A18, 28A20, 47B35

1 Introduction and main results

It was exactly 100 years ago when Szegő published his seminal paper [13] on Toeplitz determinants. Only five years later, his theorem on the asymptotic distribution of the eigenvalues of Hermitian Toeplitz matrices appeared [14]. Since then spectral properties of Toeplitz matrices, in particular the collective behavior of eigenvalues, have been extensively studied by many authors; see, for example, the books [4, 6]. However, it was only recently that asymptotic formulas for individual eigenvalues inside the spectrum backed in the interest; see [3, 5, 7]. This topic is still in its infancy, because the results so far available cover very particular classes of generating functions only.

In our paper [2] with Sergei Grudsky, which was in fact inspired by the papers [3, 19], we proved a result on the uniform approximation of the singular values of Toeplitz matrices, which are the eigenvalues in the case of positive definite Hermitian matrices. The purpose of the present paper is to simplify some proofs from [2] and to put the approach into a more abstract setting, thus extending the range of possible applications.
A probability measure \( \mu \) is called a Borel probability measure on \( \mathbb{R} \) if its domain contains the Borel \( \sigma \)-algebra over \( \mathbb{R} \). Given a Borel probability measure \( \mu \) on \( \mathbb{R} \), the corresponding cumulative distribution function \( F_\mu : \mathbb{R} \to [0,1] \) and quantile function \( Q_\mu : (0,1) \to \mathbb{R} \) are defined by

\[
F_\mu(v) := \mu(-\infty,v], \quad Q_\mu(p) := \inf\{v \in \mathbb{R} : F_\mu(v) \geq p\}.
\]

The support of \( \mu \) is the set

\[
\text{supp}(\mu) := \{v \in \mathbb{R} : \mu(v-\varepsilon,v+\varepsilon) > 0 \quad \forall \varepsilon > 0\}.
\]

If \( \mu \) has a bounded support, then the function \( Q_\mu \) has finite limits at the points 0 and 1, and we extend \( Q_\mu \) to \([0,1]\) by continuity.

Herewith our first main result.

**Theorem 1.1.** Let \( \Lambda \) be a Borel probability measure on \( \mathbb{R} \) and \( (\mu_n)_{n=1}^{\infty} \) be a sequence of Borel probability measures on \( \mathbb{R} \) that converges to \( \Lambda \) in distribution, i.e.,

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \varphi \, d\mu_n = \int_{\mathbb{R}} \varphi \, d\Lambda
\]

for every \( \varphi \in C_b(\mathbb{R}) \). Moreover, suppose that \( \text{supp}(\Lambda) \) is a bounded and connected set and that \( \text{supp}(\mu_n) \subseteq \text{supp}(\Lambda) \) for every \( n \in \mathbb{N} \). Then the sequence \( (Q_{\mu_n})_{n=1}^{\infty} \) converges uniformly to \( Q_\Lambda \):

\[
\lim_{n \to \infty} \sup_{p \in [0,1]} |Q_{\mu_n}(p) - Q_\Lambda(p)| = 0.
\]

In this theorem, the class \( C_b(\mathbb{R}) \) of bounded continuous functions of \( \mathbb{R} \) to \( \mathbb{C} \) can be substituted by the class \( C_c(\mathbb{R}) \) of continuous functions with compact support, because \( \text{supp}(\Lambda) \) is supposed to be a segment of \( \mathbb{R} \).

Theorem 1.1 makes precise what we mean by passing from convergence in distribution to uniform convergence. We emphasize that this passage is based on two assumptions: first, \( \text{supp}(\Lambda) \) is required to be a segment and secondly, all supports \( \text{supp}(\mu_n) \) must be contained in this segment. These assumptions are not caused by our proof but are essential. In [2] we considered a concrete realization of the setting and showed that the conclusion of Theorem 1.1 is no longer true if one of the two assumptions is violated.

We now specialize the measures \( \mu_n \) to be discrete measures associated to collections of real numbers. On the other hand, we allow \( d\Lambda \) to be of the form \( Xd\mathcal{P} \) with a measurable function \( X \) on an abstract probability space \( (\Omega,\mathcal{F},\mathcal{P}) \). In this setting, one makes the following definition (see [9], for example). Let \( (d(n))_{n=1}^{\infty} \) be a sequence of positive integer numbers tending to infinity and let

\[
\alpha = \left( \alpha_1^{(n)}, \ldots, \alpha_{d(n)}^{(n)} \right)_{n=1}^{\infty}
\]

be a sequence of collections of real numbers. In addition, let \( (\Omega,\mathcal{F},\mathcal{P}) \) be a probability space and \( X: \Omega \to \mathbb{R} \) be an \( \mathcal{F} \)-measurable function. The sequence \( \alpha \) is said to be asymptotically distributed as \( (X,\mathcal{P}) \) if, for every function \( \varphi \in C_c(\mathbb{R}) \),

\[
\lim_{n \to \infty} \frac{1}{d(n)} \sum_{j=1}^{d(n)} \varphi(\alpha_j^{(n)}) = \int_{\Omega} \varphi \circ X \, d\mathcal{P}.
\]
Given a probability space \((\Omega, \mathcal{F}, P)\) and an \(\mathcal{F}\)-measurable function \(X: \Omega \to \mathbb{R}\), we denote by \(\mathcal{R}(X)\), \(F_X\), and \(Q_X\) the essential range of \(X\), the cumulative distribution function, and the quantile function associated to \(X\):

\[
\mathcal{R}(X) := \{v \in \mathbb{R}: P(X^{-1}(v - \varepsilon, v + \varepsilon)) > 0 \quad \forall \varepsilon > 0\}, \\
F_X(v) := P(X^{-1}(-\infty, v]), \quad Q_X(p) := \inf\{v \in \mathbb{R}: F_X(v) \geq p\}.
\]

In this situation we have our second main result.

**Theorem 1.2.** Let a sequence \(\alpha\) of collections of real numbers be asymptotically distributed as \((X, P)\). Suppose \(\mathcal{R}(X)\) is connected and bounded and suppose also that, for each \(n \in \mathbb{N}\), the numbers \(\alpha_1^{(n)}, \ldots, \alpha_d^{(n)}\) belong to \(\mathcal{R}(X)\) and are ordered in the ascending manner:

\[
\alpha_1^{(n)} \leq \alpha_2^{(n)} \leq \cdots \leq \alpha_d^{(n)}.
\]

Then

\[
\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} \sup_{\frac{d(n)-1}{d(n)} \leq u \leq \frac{d(n)}{d(n)}} |\alpha_j^{(n)} - Q_\Lambda(u)| = 0.
\]

In particular,

\[
\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} |\alpha_j^{(n)} - Q_\Lambda(j/d(n))| = 0.
\]

We finally consider the special case where \(\Omega\) is a finite interval in \(\mathbb{R}\), \(P\) is the normalized Lebesgue measure on \(\Omega\), and \(X\) is Riemann integrable. In that case we prove the following theorem, which reveals that the values of \(Q_\Lambda\) at equidistant points can be replaced by the ordered values of the original function \(X\) at some points of \(\Omega\).

**Theorem 1.3.** Let \(\Omega\) be a bounded interval of \(\mathbb{R}\), \(P\) be the normalized Lebesgue measure on \(\Omega\), \(X: \Omega \to \mathbb{R}\) be a Riemann integrable function with connected essential range, \(\alpha = (\alpha_1^{(n)}, \ldots, \alpha_d^{(n)})_{n=1}^\infty\) be a sequence of collections of real numbers asymptotically distributed as \((X, P)\) such that, for every \(n \in \mathbb{N}\), the numbers \(\alpha_1^{(n)}, \ldots, \alpha_d^{(n)}\) satisfy (6) and belong to \(\mathcal{R}(X)\). Furthermore, for every \(n \in \mathbb{N}\), let \(\xi_1^{(n)}, \ldots, \xi_d^{(n)}\) be any points belonging to the different parts of the canonical \(d(n)\)-partition of the interval \(\Omega\), let \(v_1^{(n)}, \ldots, v_d^{(n)}\) be the values of \(X\) at these points, and let \(\sigma_n\) be a permutation of \(\{1, \ldots, d(n)\}\) such that

\[
v_{\sigma_n(1)}^{(n)} \leq \cdots \leq v_{\sigma_n(d(n))}^{(n)}.
\]

Then

\[
\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} |\alpha_j^{(n)} - v_{\sigma_n(j)}^{(n)}| = 0.
\]

Here is an outline of the paper. After recalling some general continuity properties of the quantile function in Section 2, we prove the main results stated above in Section 3. In Section 4 we embark on some applications of the theorems to the singular values and eigenvalues of Toeplitz-like matrices, and in Section 5 we give examples of applications to problems from beyond the matrix world.
2 Continuity of the quantile function

In this section we record some continuity properties of the quantile function. This section is very close to the Section 2 in [2], but we changed some technical details. Throughout this section we suppose that $\mu$ is a Borel probability measure on $\mathbb{R}$ with bounded support. We use the simplified notation $F$ and $Q$ for the functions $F_\mu$ and $Q_\mu$, correspondingly. Recall that these functions are defined by (1).

It is well known and readily verified that $F$ and $Q$ are monotonically increasing (in the non-strict sense), that $F$ is continuous from the right, that the infimum in the definition of $Q(p)$ belongs to the set $\{v \in \mathbb{R} : p \leq F(v)\}$ and therefore is the minimum of this set, and that $Q$ is continuous from the left. The one-sided limits $F(v^-)$ and $Q(p^+)$ exist for each $v \in \mathbb{R}$ and each $p \in (0, 1)$. It follows from the definition of $Q$ that, for every $v$ in $\mathbb{R},$

$$Q(F(v)) \leq v$$

and

$$Q(F(v)^+) \geq v.$$  

We thoroughly work under the assumption that $\text{supp}(\mu)$ is compact. We denote by $\alpha$ and $\beta$ the minimum and maximum of $\text{supp}(\mu)$:

$$\alpha := \min \text{supp}(\mu), \quad \beta := \max \text{supp}(\mu).$$

We write $F(-\infty)$ and $F(+\infty)$ for the limits of $F(v)$ as $v \to -\infty$ and $v \to +\infty$, respectively. The next proposition deals with $F$ near $\alpha$ and $\beta$ and with $Q$ near 0 and 1.

**Proposition 2.1.** The functions $F$ and $Q$ have the following properties.

(a) $F(v) = 0$ for every $v$ in $[-\infty, \alpha]$.

(b) $F(\alpha) = \mu\{\alpha\}$.

(c) $0 < F(v) < 1$ for every $v$ in $(\alpha, \beta)$.

(d) $F(v) = 1$ for every $v$ in $[\beta, +\infty]$.

(e) $\alpha \leq Q(p) \leq \beta$ for every $p$ in $(0, 1)$.

(f) $Q(0^+) = \alpha$, $Q(1^-) = \beta$.

**Proof.** Properties (a) to (d) follow directly from the definition of $F$, $\alpha$, and $\beta$. Given $p \in (0, 1)$, the inequality $Q(p) \geq \alpha$ results from (a), while the inequality $Q(p) \leq \beta$ is a consequence of (c). This proves (e), and we are left with (f).

We first turn to the limit of $Q(p)$ as $p \to 0^+$. Given $\varepsilon > 0$, put $q = F(\alpha + \varepsilon)$. Then $q > 0$, and for every $p$ in $(0, q]$ we can apply (9) to obtain

$$\alpha \leq Q(p) \leq Q(q) = Q(F(\alpha + \varepsilon)) \leq \alpha + \varepsilon.$$ 

This implies that $Q(0^+) = \alpha$. We now consider the limit of $Q(p)$ as $p \to 1^-$. For $\varepsilon > 0$, we put $q = F(\beta - \varepsilon)$. Then $q < 1$, and for every $p \in (q, 1)$ we infer from (10) that

$$\beta - \varepsilon \leq Q(F(\beta - \varepsilon)^+) \leq Q(p) \leq \beta.$$ 

Thus $Q(1^-) = \beta$. 

\[\square\]
We extend $Q$ by continuity to $[0, 1]$: $Q(0) := Q(0^+) = \alpha$ and $Q(1) := Q(1^-) = \beta$. Note that we do not define $Q(0)$ by putting $p = 0$ into (1), because the corresponding value would be $-\infty$.

Here are some well known or easily verifiable relations between $F$ and $Q$.

**Proposition 2.2.** The following are true.

(a) $Q(F(v)) \leq v$ for every $v \in \mathbb{R}$.

(b) $F(Q(u)) \geq u$ for every $u \in [0, 1]$.

(c) Let $u \in [0, 1]$ and $v \in \mathbb{R}$. Then $Q(u) \leq v$ if and only if $u \leq F(v)$.

(d) If $v_1, v_2 \in \mathbb{R}$ and $F(v_1) < F(v_2)$, then $v_1 < Q(F(v_2)) \leq v_2$.

**Proposition 2.3.** The distribution function of $Q$ is $F$, i.e., for every $v \in \mathbb{R}$,

$$\mu_{\mathbb{R}}\{u \in [0, 1] : Q(u) \leq v\} = F(v),$$

where $\mu_{\mathbb{R}}$ stands for the Lebesgue measure on $\mathbb{R}$.

The next criterion implies in particular that the connectedness of supp($\mu$) is equivalent to the continuity of $Q$. This condition plays a crucial role in this paper. For a proof, see [2].

**Proposition 2.4.** The following conditions are equivalent:

(i) supp($\mu$) is connected, i.e., supp($\mu$) = $[\alpha, \beta]$.

(ii) $F$ is strictly increasing on $[\alpha, \beta]$.

(iii) $Q(F(v)) = v$ for every $v \in [\alpha, \beta]$.

(iv) $Q([0, 1]) = [\alpha, \beta]$.

(v) $Q$ is continuous on $[0, 1]$.

**Corollary 2.5.** Let $\mu$ be a Borel probability measure on $\mathbb{R}$ such that supp($\mu$) is bounded and connected. Then $F$ is strictly increasing and $Q$ is uniformly continuous on $[0, 1]$.

Here is a result concerning Riemann integrable functions. It is one of the basic ingredients to the proof of Theorem 1.3. It was proved in [2] in slightly different notation.

**Proposition 2.6.** Let $\Omega \subset \mathbb{R}$ be a bounded interval, $P$ be the normalized Lebesgue measure on $\Omega$, and $X : \Omega \to \mathbb{R}$ be a Riemann integrable function with connected essential range. For every $n \in \mathbb{N}$, let $s_1^{(n)}, \ldots, s_d^{(n)}$, $v_1^{(n)}, \ldots, v_d^{(n)}$, and $\sigma$, be as in Theorem 1.3. Then

$$\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} \left| Q_X(j/d(n)) - v_{\sigma(j)}^{(n)} \right| = 0.$$  \hspace{1cm} (12)

### 3 Proofs of the main results

The following proposition is a special version of Alexandroff’s criterion for convergence in distribution, which is also known as the portmanteau lemma (see, for example, Sections 8.1 and 8.2 of [1] or Lemma 2.2 and Lemma 21.2 of [23]). Proofs of the equivalence (i)$\iff$(ii) can be found in Sections 8.1 and 8.2 of [1] or Lemma 2.2 of [23]. We remark that this equivalence was established by A. D. Alexandroff in the more abstract context of metric spaces. The equivalence (ii)$\iff$(iii) is elementary: see Lemma 21.2 of [23].
Proposition 3.1. Let $\Lambda$ be a Borel probability measure on $\mathbb{R}$ and $(\mu_n)_{n=1}^{\infty}$ be a sequence of Borel probability measures on $\mathbb{R}$. Then the following conditions are equivalent.

(i) For every $\varphi \in C_b(\mathbb{R})$, (3) holds.

(ii) $\lim_{n \to \infty} F_{\mu_n}(v) = F_{\Lambda}(v)$ for every point $v \in \mathbb{R}$ at which $F_{\Lambda}$ is continuous.

(iii) $\lim_{n \to \infty} Q_{\mu_n}(p) = Q_{\Lambda}(p)$ for every point $p \in (0, 1)$ at which $Q_{\Lambda}$ is continuous.

The next result says that pointwise convergence on a segment, jointly with monotonicity and continuity, imply uniform convergence.

Proposition 3.2. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function and $(f_n)_{n=1}^{\infty}$ be a sequence of functions $[0, 1] \to \mathbb{R}$ such that the function $f_n$ is increasing (in the non-strict sense) for every $n \in \mathbb{N}$ and $f_n(p) \to g(p)$ as $n \to \infty$ for every $p \in [0, 1]$. Then

$$\lim_{n \to \infty} \sup_{p \in [0,1]} |f_n(p) - g(p)| = 0.$$ 

Proof. Let $\varepsilon > 0$. Since $g$ is uniformly continuous on $[0, 1]$, we can select a positive number $\delta$ such that

$$|g(p_1) - g(p_2)| \leq \frac{\varepsilon}{2} \text{ whenever } p_1, p_2 \in [0, 1] \text{ and } |p_1 - p_2| \leq \delta. \quad (13)$$

Choose $m \in \{1, 2, \ldots \}$ such that $1/m \leq \delta$. Using the convergence $f_n(p) \to g(p)$ at the points $p = j/m$, $j = 0, 1, \ldots, m$, we find an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$ and every $j \in \{0, \ldots, m\}$,

$$\left| f_n \left( \frac{j}{m} \right) - g \left( \frac{j}{m} \right) \right| < \frac{\varepsilon}{2}. \quad (14)$$

Now let $n \geq n_0$ and $p \in [0, 1]$. Pick $j \in \{0, \ldots, m-1\}$ such that $j/m \leq p \leq (j+1)/m$. Then the monotonicity of $f_n$ together with the inequalities (14) and (13) implies that

$$f_n(p) \leq f_n \left( \frac{j+1}{m} \right) < g \left( \frac{j+1}{m} \right) + \frac{\varepsilon}{2} < g(p) + \varepsilon.$$ 

In a similar manner,

$$f_n(p) \geq f_n \left( \frac{j}{m} \right) > g \left( \frac{j}{m} \right) - \frac{\varepsilon}{2} > g(p) - \varepsilon,$$

which completes the proof. \qed

Proof of Theorem 1.1. By Corollary 2.5, $Q_{\Lambda}$ is uniformly continuous on $[0, 1]$. Therefore, by Proposition 3.1, $Q_{\mu_n}$ pointwisely converges to $Q_{\Lambda}$ on $(0, 1)$. We are so left with the points 0 and 1.

We consider the situation at the point 0. This is the place where the assumption that $\text{supp}(\mu_n) \subseteq \text{supp}(\Lambda)$ makes its debut. It implies that

$$Q_{\mu_n}(0) = \inf \text{supp}(\mu_n) \geq \inf \text{supp}(\Lambda) = Q_{\Lambda}(0).$$
Then, given $\varepsilon > 0$, there is a $\delta > 0$ such that $Q_\Lambda(\delta) < Q_\Lambda(0) + \varepsilon/2$ and an $n_0 \in \mathbb{N}$ such that $|Q_{\mu_n}(\delta) - Q_\Lambda(\delta)| < \varepsilon/2$ for every $n \geq n_0$. Consequently, for every $n \geq n_0$,

$$Q_\Lambda(0) \leq Q_{\mu_n}(0) \leq Q_{\mu_n}(\delta) < Q_\Lambda(\delta) + \varepsilon < Q_\Lambda(0) + \varepsilon,$$

whence $Q_{\mu_n}(0) \to Q_\Lambda(0)$. The convergence at the point 1 can be proved in a similar manner. Thus, $Q_{\mu_n}$ pointwisely converges to $Q_\Lambda$ on $[0, 1]$. Proposition 3.2 now implies that the convergence is uniform. \qed

**Proof of Theorem 1.2.** We simply translate Theorem 1.2 into the language of Theorem 1.1.

First step. For each $n \in \mathbb{N}$, we denote by $\mu_n$ the normalized counting measure associated to the tuple $(\alpha_1^{(n)}, \ldots, \alpha_{d(n)}^{(n)})$, i.e., for every Borel subset $B$ of $\mathbb{R}$, we put

$$\mu_n(B) = \frac{\# \{ j \in \{1, \ldots, d(n)\} : \alpha_j^{(n)} \in B \}}{d(n)}.$$ 

In other words, $\mu_n$ is nothing but the arithmetic mean of the Dirac measures concentrated at the points $\alpha_1^{(n)}, \ldots, \alpha_{d(n)}^{(n)}$:

$$\mu_n = \frac{1}{d(n)} \sum_{j=1}^{d(n)} \delta_{\alpha_j^{(n)}}.$$ 

Since the tuple $(\alpha_1^{(n)}, \ldots, \alpha_{d(n)}^{(n)})$ is ordered,

$$F_{\mu_n}(v) = \frac{\max \{ j \in \{1, \ldots, d(n)\} : \alpha_j^{(n)} \leq v \}}{d(n)}$$

and

$$Q_{\mu_n}(j/d(n)) = \alpha_j^{(n)}. \quad (15)$$ 

Second step. Denote by $\Lambda$ the pushforward measure on $\mathbb{R}$ associated to $P$ and $X$, i.e., for every Borel subset $B$ of $\mathbb{R}$, put

$$\Lambda(B) = P(X^{-1}(B)).$$

Then $F_X = F_\Lambda$, $Q_X = Q_\Lambda$, and

$$\mathcal{R}(X) = \{ v \in \mathbb{R} : P(X^{-1}((v - \varepsilon, v + \varepsilon))) > 0 \ \forall \varepsilon > 0 \} = \{ v \in \mathbb{R} : \Lambda(v - \varepsilon, v + \varepsilon) > 0 \ \forall \varepsilon > 0 \} = \text{supp}(\Lambda).$$

Third step. Since $\mathcal{R}(X)$ is bounded and the points $\alpha_j^{(n)}$ belong to $\mathcal{R}(X)$, the limit relation (5) holds not only for every $\varphi \in C_c(\mathbb{R})$, but for every $\varphi \in C_0(\mathbb{R})$, i.e., $\mu_n$
converges to $\Lambda$ in distribution. By Theorem 1.1, $Q_{\mu_n}$ uniformly converges to $Q_X$. Using (15) we conclude that

$$\max_{1 \leq j \leq d(n)} |\alpha_j^{(n)} - Q_\Lambda(j/d(n))| = \max_{1 \leq j \leq d(n)} |Q_{\mu_n}(j/d(n)) - Q_\Lambda(j/d(n))| \leq \sup_{p \in [0,1]} |Q_{\mu_n}(p) - Q_\Lambda(p)|,$$

which completes the proof of (8). Finally, from the uniform continuity of $Q_\Lambda$ we obtain

$$\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} \sup_{\frac{j-1}{d(n)} \leq u < \frac{j}{d(n)}} |Q_\Lambda(j/d(n)) - Q_\Lambda(u)| = 0,$$

which jointly with (8) yields (7).

Proof of Theorem 1.3. The assertion of this theorem is immediate from Theorem 1.2 and Proposition 2.6.

4 Applications to Toeplitz-like matrices

Given a matrix $A \in \mathbb{C}^{d \times d}$, we denote by $s_1(A), \ldots, s_d(A)$ its singular values written in the ascending order, $s_1(A) \leq \cdots \leq s_d(A)$, and for a Hermitian matrix $A \in \mathbb{C}^{d \times d}$, we let $\lambda_1(A), \ldots, \lambda_d(A)$ stand for its eigenvalues written in the ascending order, taking multiplicities into account, $\lambda_1(A) \leq \cdots \leq \lambda_d(A)$. In accordance with the definition of asymptotic distribution given in Section 1, we adopt the following terminology. Let $(A_n)_{n=1}^\infty$ be a sequence of square complex matrices, denote the order of $A_n$ by $d(n)$, and suppose that $d(n) \to \infty$ as $n \to \infty$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function. If the sequence of tuples $(s_1(A_n), \ldots, s_{d(n)}(A_n))_{n=1}^\infty$ is asymptotically distributed as $(X, P)$, then we say that the singular values of the sequence $(A_n)_{n=1}^\infty$ are asymptotically distributed as $(X, P)$. A similar terminology is used for the eigenvalues.

Multilevel Toeplitz matrices

Let $a$ be a function in $L^\infty$ on $\mathbb{T}^k$, where $\mathbb{T}$ is the complex unit circle. The Fourier coefficients of $a$ are defined by

$$a_{j_1, \ldots, j_k} = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} a(e^{i\theta_1}, \ldots, e^{i\theta_k})e^{-i(j_1\theta_1 + \cdots + j_k\theta_k)}d\theta_1 \cdots d\theta_k.$$

Suppose that for each $n \in \mathbb{N}$ we are given a $k$-tuple $(m_1^{(n)}, \ldots, m_k^{(n)}) \in \mathbb{N}^k$. We denote by $W_n(a)$ the linear operator acting on

$$Y_n := \ell^2(\{1, \ldots, m_1^{(n)}\} \times \cdots \times \{1, \ldots, m_k^{(n)}\})$$
by the rule
\[(W_n(a)x)_{i_1,\ldots,i_k} = \sum_{j_1=1}^{m_1(n)} \cdots \sum_{j_k=1}^{m_k(n)} a_{i_1-j_1,\ldots,i_k-j_k} x_{j_1,\ldots,j_k},\]
and we let \(T_n(a)\) stand for the matrix representation of \(W_n(a)\) in the standard basis of \(Y_n\). The matrix \(T_n(a)\) is called a \(k\)-level Toeplitz matrix. Note that in this case \(d(n) = m_1(n) \cdots m_k(n)\). Tyrtyshnikov [21] showed that if \(\min(m_1(n),\ldots,m_k(n)) \to \infty\) as \(n \to \infty\), then the singular values of \(T_n(a)\) are asymptotically distributed as \(X := |a|\) on \(\mathbb{T}^k\) with normalized invariant measure. In [2], we showed that if the essential range \(\mathcal{R}(|a|)\) is just the segment \([0, \|a\|_\infty]\), then
\[
\lim_{n \to \infty} \max_{1 \leq j \leq d(n)} |s_j(T_n(a)) - Q_{|a|}(j/d(n))| = 0.
\]
Now this result can simply be deduced from Tyrtyshnikov’s in conjunction with Theorem 1.2.

**Sums of products of Toeplitz matrices**

A 1-level Toeplitz matrix is a usual Toeplitz matrix, that is, a matrix of the form \((a_{i-j})_{i,j=1}^n\). If the entries \(a_k \ (k \in \mathbb{Z})\) are the Fourier coefficients of a function \(a \in L^\infty(\mathbb{T})\), then \((a_{i-j})_{i,j=1}^n\) is denoted by \(T_n(a)\) and \(a\) is referred to as the symbol of the matrices \(T_n(a) \ (n \in \mathbb{N})\).

Denote by \(\mu_\mathbb{T}\) the normalized invariant measure on the unit circle \(\mathbb{T}\). For every pair \((p,q)\) with \(p \in \{1,\ldots,M\}, \ q \in \{1,\ldots,N_p\}\), take functions \(a^{(p,q)} \in L^\infty(\mathbb{T})\) and define \(B_n \in \mathbb{C}^{n \times n}\) by
\[
B_n = \sum_{p=1}^M \prod_{q=1}^{N_p} T_n(a^{(p,q)}).
\]
Then it is known from [6, 15, 17, 20, 22] that the singular values of \(B_n\) are asymptotically distributed as \((X, \mu_\mathbb{T})\), where
\[
X := \left| \sum_{p=1}^M \prod_{q=1}^{N_p} a^{(p,q)} \right|.
\]
If \(\mathcal{R}(X)\) is a segment \([0, \beta]\) and \(\|B_n\| \leq \beta\) for every \(n \in \mathbb{N}\), then Theorem 1.2 assures that
\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} |s_j^{(n)}(B_n) - X(j/n)| = 0. \tag{16}
\]
As an example, consider the products \(B_n = T_n(a^{(1)})T_n(a^{(2)})\) of Toeplitz matrices with the symbols \(a^{(1)}\) and \(a^{(2)}\) shown in Figure 1. In that case \(X = a^{(1)}a^{(2)}\), the essential range of \(X\) is \([0,1]\), and the norms of \(B_n\) are bounded by 1. Therefore (16) holds. Note that \(\mathcal{R}(a^{(1)})\) and \(\mathcal{R}(a^{(2)})\) have gaps, which implies that the singular values of
$T(a^{(q)})$ cannot be approximated uniformly by the values of $a^{(q)}$, $q = 1, 2$. Denoting the maximum on the left-hand side of (16) by $\varepsilon^{(n)}$ we get the following table:

| $n$  | 32   | 64   | 128  | 256  | 512  | 1024 |
|------|------|------|------|------|------|------|
| $\varepsilon^{(n)}$ | $5.7 \cdot 10^{-2}$ | $3.1 \cdot 10^{-2}$ | $1.6 \cdot 10^{-2}$ | $8.6 \cdot 10^{-6}$ | $4.4 \cdot 10^{-3}$ | $2.3 \cdot 10^{-3}$ |

Figure 1: The top row shows the symbols $a^{(1)}$, $a^{(2)}$, and $a^{(1)}a^{(2)}$, respectively. The bottom row shows the singular values of $B_n$ for $n = 8, 32, 128$ as points with coordinates $(\frac{j}{n}, s_j^{(n)})$, and the quantile function of $a^{(1)}a^{(2)}$.

**Other Toeplitz-like matrices**

The asymptotic distribution of singular and eigenvalues are known for many other classes of matrices: $g$-Toeplitz matrices [8, 12], locally Toeplitz matrices [15, 18, 24], and also in the more general situation when some “complicated” matrices can be approximated by “simple” matrices with known distribution of singular values or eigenvalues (see [9] or [20, Theorem 2.1]).

In all these cases, the uniform convergence of the eigenvalues holds under the assumption that the matrices $A_n$ are Hermitian, that the essential range of the function $X$ is bounded and connected, and that the eigenvalues of $A_n$ are contained in $R(X)$. For the singular values of (not necessarily Hermitian) matrices $A_n$, it is sufficient to require that $R(X)$ is a segment of the form $[0, \beta]$ and that $\|A_n\| \leq \beta$ for every $n$.

We remark that $Q_X(j/d(n))$ is a very rough approximation to individual singular values (or eigenvalues) because the magnitude of the error is usually comparable with the distance between consecutive singular values (or eigenvalues). However, the quantile approach yields more precise approximations once the sites $j/d(n)$ are substituted by more cleverly chosen points; see [3]. We will not embark on this subtle issue here. Theorem 1.3 should nevertheless be of use for numerical methods since it has the potential to provide us with an initial approximation for iterative algorithms.
Nets instead of sequences

Theorems 1.1, 1.2, 1.3 are also true with sequences replaced by nets. We decided to restrict ourselves to sequences just for the sake of simplicity. But here is a result by I. B. Simonenko [16] where nets are the appropriate language.

Let \( a : T^k \to \mathbb{R} \) be a continuous function. For a finite subset \( M \) of \( \mathbb{Z}^k \), denote by \( W_M(a) \) the linear operator defined by

\[
(W_M(a)x)_i = \sum_{j \in M} a_{i-j}x_j, \quad i \in M
\]

on \( \ell^2(M) \) and let \( T_M(a) \) be the matrix representation of \( W_M(a) \) in the standard basis of \( \ell^2(M) \). Now suppose \((M_\nu)_{\nu \in \mathbb{N}}\) is any net of finite subsets of \( \mathbb{Z}^k \) such that

\[
\lim_{\nu \in \mathbb{N}} \min_{Y \subset M_\nu} \max_{\substack{Z \subset \mathbb{Z}^k \setminus M_\nu \ Z \supset Y}} \left( \frac{\#(M_\nu \setminus Y)}{\#M_\nu}, \frac{1}{\text{dist}(Y, \mathbb{Z}^k \setminus M_\nu)} \right) = 0.
\]

Simonenko showed that then the eigenvalues \( \lambda_j(T_{M_\nu}(a)) \) of the Hermitian matrices \( T_{M_\nu}(a) \) all belong to \( \mathbb{R}(a) = [\min(a), \max(a)] \) and are asymptotically distributed as \( X : \mathbb{R} \to T^k \) with the measure \( \mu_T \times \cdots \times \mu_T \). From Theorem 1.2 (for nets) we therefore conclude that

\[
\lim_{\nu \in \mathbb{N}} \max_{1 \leq j \leq \#M_\nu} |\lambda_j(T_{M_\nu}(a)) - Q_a(j/\#M_\nu)| = 0.
\]

5 Examples from beyond the matrix world

Example 5.1. The purpose of this example is to turn inside out the famous arcsine law for random walks discovered by P. Lévy in 1939 (see [11, Chapter 10]). What results after that procedure is the quantile version of the arcsine law, which might be called the sine law for random walks.

For every \( n \in \mathbb{N} \) and every \( w = (w_1, \ldots, w_n) \in \{-1,1\}^n \), put

\[
G_n(w) = \frac{\#\{k \in \{1, \ldots, n\} : w_1 + \cdots + w_k > 0\}}{n}.
\]

Let \( d(n) = 2^n \) and \( \alpha_1^{(n)}, \ldots, \alpha_d^{(n)} \) be the numbers \( G_n(w) \), \( w \in \{-1,1\}^n \), written in the increasing order. For example, if \( n = 3 \), then we have \( 2^3 = 8 \) elements \( w \) with the following values of \( G_3(w) \):

\[
G_3(\{-1,-1,-1\}) = 0, \quad G_3(\{-1,-1,1\}) = 0, \quad G_3(\{-1,1,-1\}) = 0, \\
G_3(\{-1,1,1\}) = \frac{1}{3}, \quad G_3(\{1,-1,-1\}) = \frac{1}{3}, \quad G_3(\{1,-1,1\}) = \frac{2}{3}, \\
G_3(\{1,1,-1\}) = 1, \quad G_3(\{1,1,1\}) = 1.
\]

The corresponding collection \( \alpha^{(3)} = (\alpha_1^{(3)}, \ldots, \alpha_8^{(3)}) \) is

\[
\begin{pmatrix}
0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, 1, 1
\end{pmatrix}.
\]
Lévy’s arcsine law says that $\alpha^{(n)} = (\alpha^{(n)}_1, \ldots, \alpha^{(n)}_{d(n)})$ is asymptotically distributed as $(X, P)$, where $X$ is defined on $[0, 1]$ by

$$X(v) := \frac{2}{\pi} \arcsin \sqrt{v},$$

and $P$ is the Lebesgue measure on $[0, 1]$. Consequently,

$$Q_X(p) = \sin^2 \frac{\pi p}{2},$$

and by Theorem 1.2,

$$\lim_{n \to \infty} \max_{1 \leq j \leq 2^n} \left| \alpha^{(n)}_j - \sin^2 \frac{\pi j}{2n+1} \right| = 0. \quad (17)$$

Figure 2 shows the values $\alpha^{(n)}_j$ for $n = 3$ and $n = 30$. We see that the convergence is very slow. Denoting the maximum on the left-hand side of (17) by $\varepsilon^{(n)}$, we get the following table:

| $n$ | 5   | 10  | 15  | 20  | 25  | 30  |
|-----|-----|-----|-----|-----|-----|-----|
| $\varepsilon^{(n)}$ | 0.300 | 0.209 | 0.164 | 0.144 | 0.126 | 0.116 |

Figure 2: The left picture shows the points $(j/2^n, \alpha^{(n)}_j)$ for $n = 3$ and the plot of $Q_X(v)$ from Example 5.1. The right picture corresponds to $n = 30$; we there glued together $2^{30}$ points to 31 line segments.

The next result follows from Theorem 1.1 and indicates another class of applications of that theorem, namely, application to asymptotically distributed sequences of numbers.

**Proposition 5.2.** Let $\Lambda$ be a Borel probability measure on $\mathbb{R}$ with bounded connected support and let $(\beta_j)_{j=1}^\infty$ be a bounded sequence of real numbers which are asymptotically distributed as $\Lambda$ in the sense that, for every $v \in \mathbb{R},$

$$\lim_{n \to \infty} \frac{\# \{ j \in \{1, \ldots, n\}: \beta_j \leq v \}}{n} = F_\Lambda(v).$$
Let \((\alpha_1^{(n)}, \ldots, \alpha_n^{(n)})\) denote the collection \((\beta_1, \ldots, \beta_n)\) written in the ascending order. Then
\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} |\alpha_j^{(n)} - Q(\lambda(j/n))| = 0.
\]
In particular, for sequences which are uniformly distributed on \([0, 1]\) (see [10]), one has \(Q(p) = p\) for every \(p \in [0, 1]\), and hence Proposition 5.2 implies that
\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} |\alpha_j^{(n)} - j/n| = 0. \tag{18}
\]

**Example 5.3.** Consider the sequence \(\beta_j = j\sqrt{2} - \lfloor j\sqrt{2} \rfloor\). By Weyl’s equidistribution theorem, it is uniformly distributed on \([0, 1]\). Figure 3 shows the points \((j/64, \alpha_j^{(64)})\) for \(j = 1, \ldots, 64\). Denoting the maximum on the left-hand side of (18) by \(\varepsilon^{(n)}\) we get the following table:

| \(n\) | \(32\) | \(64\) | \(128\) | \(256\) | \(512\) | \(1024\) |
|------|------|------|------|------|------|------|
| \(\varepsilon^{(n)}\) | \(4.6 \cdot 10^{-2}\) | \(2.6 \cdot 10^{-2}\) | \(1.1 \cdot 10^{-2}\) | \(5.6 \cdot 10^{-2}\) | \(3.3 \cdot 10^{-3}\) | \(2.4 \cdot 10^{-3}\) |

If fact, the behavior of \(\varepsilon^{(n)}\) is rather irregular, but we obtained \(\varepsilon^{(n)} \leq \frac{0.7 \ln(n)}{n}\) for \(n = 2, 3, \ldots, 10000\).

![Figure 3: The blue points are the points whose abscissas are \(j/64\) and whose ordinates are the ordered numbers \(j\sqrt{2} - \lfloor j\sqrt{2} \rfloor\) with \(j = 1, \ldots, 64\). The gray line is the graph of the identity function on \([0, 1]\).](image)

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