Experimental superposition of time directions

Teodor Strömberg, Peter Schiansky, Marco Túlio Quintino, Michael Antesberger, Lee Rozema, Iris Agresti, Časlav Brukner, and Philip Walther

1University of Vienna, Faculty of Physics & Vienna Doctoral School in Physics, Boltzmanngasse 5, A-1090 Vienna, Austria
2University of Vienna, Faculty of Physics & Research Network Quantum Aspects of Space Time (TURIS), Boltzmanngasse 5, 1090 Vienna, Austria
3Institute for Quantum Optics and Quantum Information, Boltzmanngasse 3, 1090 Vienna, Austria
4Sorbonne Université, CNRS, LIP6, F-75005 Paris, France

In the macroscopic world, time is intrinsically asymmetric, flowing in a specific direction, from past to future. However, the same is not necessarily true for quantum systems, as some quantum processes produce valid quantum evolutions under time reversal. Supposing that such processes can be probed in both time directions, we can also consider quantum processes probed in a coherent superposition of forwards and backwards time directions. This yields a broader class of quantum processes than the ones considered so far in the literature, including those with indefinite causal order. In this work, we demonstrate for the first time an operation belonging to this new class: the quantum time flip. Using a photonic realisation of this operation, we apply it to a game formulated as a discrimination task between two sets of operators. This game not only serves as a witness of an indefinite time direction, but also allows for a computational advantage over strategies using a fixed time direction, and even those with an indefinite causal order.

INTRODUCTION

In recent years, the framework of quantum theory has been generalised to describe agents interacting through quantum processes with indefinite causal orders [1–3]. These processes have been realised experimentally using photonic platforms [4–8], thereby witnessing the implementation of causally non-separable series of events. Remarkably, these are not the most general processes allowed by quantum mechanics. Take, for example, the quantum SWITCH [2]: even though the causal order of the constituent events is indefinite, each operation is accessed only in a single time direction. By considering processes where the time direction of the underlying operations is indefinite, one can go beyond the framework of indefinite causality. Indeed, a quantum superposition of evolutions with opposite thermodynamic arrows of time was first proposed in [9]. By associating the “forwards” temporal direction with a positive change in the entropy generated in a thermodynamic process, and its “time-reversing” counterpart with a negative change, the corresponding superposition of processes exhibits a quantum-mechanically undefined thermodynamic arrow of time.

More generally, processes with an indefinite time direction can be studied by considering operations that exhibit a time symmetry; these operations admit a change of reference frame that yields a valid quantum evolution in which the time coordinate is inverted. Unitary channels are an example of such operations, and in particular they admit the following time-reversal symmetries: for every evolution $U$, both the inverse $U \rightarrow U^\dagger$ and the transpose $U \rightarrow U^T$ are valid time-reversal operations.

Given quantum operations that can in principle be accessed in both time directions, we can consider coherent superpositions of transformations made in the forwards and backwards time-directions. This amounts to a new kind of process, which we will refer to as being inseparable in the arrow of time, an example of which - called the quantum time flip - was recently introduced in Ref. [10]. This process cannot be realised within the quantum circuit model. In this work we nevertheless present a photonic implementation of the quantum time flip by exploiting device dependent symmetries of our experimental apparatus. A quantum state undergoing a time evolution is encoded in the polarization degree of freedom of a single photon, while a control qubit determining the time direction is encoded in its path degree of freedom. We show that polarization operations with waveplates naturally implement different time directions for forwards and backwards propagation directions through the waveplates, given the correct Stokes-parameter convention. This results in a deterministic time-reversal, in contrast to more general approaches which may involve multiple uses of the input operation in combination with probabilistic or non-exact methods [11–20]. We can furthermore realise the quantum time flip deterministically by passing the photon through the waveplates in a superposition of the two propagation directions.

We certify the indefinite time direction by demon-
stratizing an information-theoretic advantage of the quantum time flip in the context of a computational game. In this setting, the quantum time flip not only outperforms strategies that utilise operations with a fixed time direction, but even strategies that exploit operations with an indefinite causal order [4, 21].

QUANTUM CIRCUITS, UNITARY TRANSPOSITION, AND PROCESSES WITH INDEFINITE TIME DIRECTION

The standard quantum circuit formalism provides solid grounds for quantum computing and forms the basis for quantum complexity theory [22, 23]. However, it also imposes limitations on how we apply quantum theory. In a circuit, operations necessarily respect a definite causal order and the strict notion of input and output. The existence of time reversal processes such as unitary transposition is forbidden by the standard circuit formalism when given access to one or even two [10] uses of an unknown unitary. However, for practical and foundational reasons, researchers have been designing and pursuing non-exact and probabilistic schemes aimed towards this goal [11–19]. Remarkably, a very recent work shows that in the qubit case, when four uses of the input operation are available, there exists a quantum circuit to invert arbitrary unitary operations [20].

In quantum theory, reversible operations are described by unitary operators. Processes which reverse a composition of such operations may be expressed by a function $f$ satisfying:

$$f(UV) = f(V)f(U), \quad \forall U, V,$$

for all unitary operators $U$ and $V$ (see Fig. 1). Under natural assumptions, it can be proven that, up to a unitary transformation, there are only two time reversal functions $f$, unitary transposition $f(U) = U^T$ and unitary inversion $f(U) = U^{-1}$ [10]. For two-dimensional systems, unitary transposition and unitary inversion are unitarily equivalent via a Pauli $\sigma_Y$ operation. This follows from the identity, $U^{-1} = \sigma_Y U^T \sigma_Y$ which holds for all operators $U \in SU(2)$. Hence, for qubits, universal unitary transposition is possible if and only if unitary inversion is possible. When focusing on a particular physical implementation, the general aspects of the standard quantum circuit formalism may limit our view and lead to an apparent mismatch between theory and practice. A known illustrative example is the universal coherent control of unitary operations, where an arbitrary unitary $U$ is applied to the target system conditional on the state of a control qubit: $U \mapsto 1 \otimes |0\rangle\langle 0|_C + U \otimes |1\rangle\langle 1|_C$. While it is not possible to design a quantum circuit to perform universal control, a simple Mach-Zehnder optical interferometer can be used for this task [24–26]. Indeed, experimental control of black box quantum gates has been demonstrated [27, 28]. Such experimental implementations exploit the knowledge of the position of the physical device per-
forming the gate, circumventing this apparent limitation imposed by the quantum circuit formalism.

Although time reversal processes such as unitary transposition are not possible within the standard circuit formalism when given access to one \cite{15,16} or even two \cite{10} uses of an unknown unitary, in this work we implement general qubit unitary transposition, as well as the quantum time flip process, using a particular optical construction. Similar to the case of universal coherent control, we make use of knowledge about our specific experimental apparatus and realise a gadget implementing an arbitrary unitary \( U \) which may be used in two different directions. In the ‘forwards’ direction, this gadget implements the standard \( U \), while in the ‘backwards’ one it has the effect of the transposed \( U^T \).

Moreover, in addition to “simply” reversing a quantum evolution, we also coherently superpose the forwards and backwards time evolutions, and in so doing perform an optical implementation of a process with an indefinite time direction \cite{10}, i.e. one which cannot be described as a convex mixture of processes where time flows forwards and processes where time flows backwards. The process that we implement optically is the quantum time flip for unitary transposition, a process which acts on unitary operations as:

\[
U \mapsto U \otimes |0⟩⟨0|_C + U^T ⊗ |1⟩⟨1|_C. \tag{2}
\]

We then compose the time flip process of Eq. (2) with its flipped version, \( V \mapsto V^T \otimes |0⟩⟨0|_C + V \otimes |1⟩⟨1|_C \), to obtain a process which acts on a pair of unitary operators as:

\[
(U, V) \mapsto UV^T \otimes |0⟩⟨0|_C + U^T V \otimes |1⟩⟨1|_C. \tag{3}
\]

In addition to having an indefinite time direction, the process described in Eq. (3) cannot be described by general process matrices with indefinite causality such as the quantum switch \cite{2} or the Oreshkov-Costa-Brukner (OCB) process \cite{3}. In the next section, we will explain how to witness this property.

**GAME DESCRIPTION**

We now describe a discrimination task, first introduced in Ref. \cite{16}, where the quantum time flip process will be used as a resource to increase our performance. In this game, a referee provides the player with two black box unitaries, \( U \) and \( V \), belonging to either the set \( M_+ \) or \( M_- \), which are known to respect the property:

\[
M_+ := \{(U, V) : UV^T = +U^T V\} \tag{4}
\]

\[
M_- := \{(U, V) : UV^T = -U^T V\}. \tag{5}
\]

The player is then challenged to determine which of the two sets the gates were picked from, while only being allowed to access each of the black boxes once.

As discussed in the previous section, a player able to perform the quantum time flip may implement the process:

\[
(U, V) \mapsto UV^T \otimes |0⟩⟨0|_C + U^T V \otimes |1⟩⟨1|_C. \tag{6}
\]

Consider as a strategy an initial state of the form \( |ψ⟩_T \otimes +⟩_C \), where \( |±⟩_C = \frac{|0⟩_C ± |1⟩_C}{\sqrt{2}} \), \( |ψ⟩_T \) is an arbitrary state, and the subscripts \( C \) and \( T \) refer to the control and target qubits. Sending this state through the gate in Eq. (6) gives the state:

\[
\frac{1}{2} \left[ UV^T + U^T V \right] |ψ⟩_T + +⟩_C + \frac{1}{2} \left[ UV^T - U^T V \right] |ψ⟩_T + -⟩_C. \tag{7}
\]

Since the states \( |±⟩ \) are orthogonal, a player using this strategy can always correctly determine which set was chosen by the referee. In contrast, players who do not have access to indefinite time strategies may not be able
to ascertain with certainty to which set a given pair of unitaries \((U, V)\) belongs. In order to make this claim concrete, Ref. [10] considers a particular game where the set \(\mathcal{M}_+\) has 13 pairs of unitary operators respecting \(UV^T = +U^TV\), and \(\mathcal{M}_-\) has 8 pairs of unitary operators respecting \(UV^T = -U^TV\); these two sets of unitary operators are presented in the Methods. Here, we consider an average case variation of the aforementioned game, which goes as follows: with uniform probability \(p = \frac{1}{13}\), the referee picks a pair of unitary operators \((U, V)\) from \(\mathcal{M}_+\) or \(\mathcal{M}_-\) and lets the player make a single use of each. We then consider the optimal success probability of players who have access to different kinds of resources. As indicated by Eq. (7), players who have access to the quantum time flip can always win with unity probability. The three other classes of strategies, shown in Fig. 2, only have access to a single time direction, forwards or backwards, and convex combinations of these strategies will be called separable in the arrow of time; a detailed mathematical characterisation of these strategies is presented in the methods. Employing the computer-assisted proof methods of Ref. [26] we obtain upper bounds on the maximal success probabilities for players restricted to particular classes of strategies. The code for this is openly available in our online repository, see Methods for details.

The first alternative strategy we consider is one in which the player is restricted to using \(U\) and \(V\) in parallel, and this results in a maximal success probability that is bounded by \(\frac{89}{100} \leq p_{\text{par}} \leq \frac{89}{100}\). Next, we consider players restricted to causally ordered strategies, whose maximal success probability is found to be bounded by \(\frac{90}{100} \leq p_{\text{causal}} \leq \frac{91}{100}\). Finally, players given access to process matrices with indefinite causality (also called indefinite testers [30]), but with definite time direction, have their maximal success probability bounded by \(\frac{91}{100} < p_{\text{lc}} \leq \frac{92}{100}\). This game is hence an example of a channel discrimination task with strict hierarchy between four different classes of strategies. Additionally, while the operations selected by the referee are treated as being fully characterised in the above analysis, there are no assumptions made about the measurements performed by the player, and these can be chosen freely. This is therefore an example of a semi-device-independent certification of an indefinite causal order [31, 32] and indefinite time direction.

EXPERIMENT

Our photonic implementation of the game described in the previous section makes use of the quantum time-flip strategy from Eq. (7) to achieve a success probability exceeding that of any strategy only using the gates in one time direction. To coherently apply the quantum time flip, we employ polarization optics in a partially common-path interferometer, depicted in Fig. 3, with the control and target qubits being encoded in the path.
and polarization degrees of freedom of a single-photon, respectively. Our experiment makes use of two quantum time flips, sequentially applied to the two unitaries $V$ and $U$. The resulting controlled channel is the one of Eq. (7) where the gates $UV^T$ and $UT^TV$ act on the target (polarization) qubit and are implemented using two Simon-Mukunda polarization gadgets consisting of three waveplates each [33], for which the transpose operation is obtained by reversing the propagation direction.

Such polarization gadgets generally do not realise the transpose operation in the backwards propagation direction, but rather a related operation:

$$U_{fw} \rightarrow U_{bw} = PU_{fw}^T P^t, \quad (8)$$

where $P$ is a matrix describing the change of reference frame to the backwards direction, and the subscripts indicate the propagation direction. While it is possible to construct a gadget that implements the transpose by introducing time-reversal symmetry breaking elements (T. Strömberg, R. Peterson, and P. Walther, SU(2) gadgets for counterpropagating polarization optics, Manuscript under preparation), here we instead exploit the fact that the transpose is a basis-dependent operation. More concretely, by adopting the convention $(S_1, S_2, S_3) \leftrightarrow (-Z, -Y, -X)$ for our Stokes parameters [34] we find that $P = 1$, and the polarization gadgets transform as the transpose under counterpropagation (see Methods). Superimposing two propagation directions through a gadget therefore allows us to implement the quantum time flip, with the photon path acting as a control degree of freedom. The specific coherent superposition of time flips in Eq. (3) is achieved through the use of fiber optic circulators.

The optical circuit in Fig. 3 begins with a bulk beam-splitter that initializes the control qubit into the state $|\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle_C + |1\rangle_C)$, after which two fiber circulators guide the photons through the $V$ gadget in two different directions, giving the joint control-target state:

$$\frac{1}{\sqrt{2}}(V_1 |\psi\rangle_T \otimes |0\rangle_C + V_1^T |\psi\rangle_T \otimes |1\rangle_C). \quad (9)$$

Entering the circulators from a different port, the photons are then directed to the $U$ gadget, which they once again propagate through in opposite directions, transforming the joint state to:

$$\frac{1}{\sqrt{2}}(U^T V |\psi\rangle_T \otimes |0\rangle_C + U^T |\psi\rangle_T \otimes |1\rangle_C). \quad (10)$$

At the end of the optical circuit, a fiber beam-splitter applies a Hadamard gate on the control qubit, giving the state:

$$\left[\frac{U^T V + U^T V}{2}\right] |\psi\rangle_T |+\rangle_C + \left[\frac{U^T V - U^T V}{2}\right] |\psi\rangle_T |-\rangle_C. \quad (11)$$

A projective measurement on the control (path) qubit in the computational basis then reveals whether $(U, V)$ belong to $M_+$ or $M_-.$

The partially common-path structure of the interferometer has two distinct advantages: (1) photons in the two different propagation directions of the interferometer hit exactly the same spots on the waveplates and the physical symmetries of the gadget therefore ensures the faithful implementation of the time flip independently of any imperfections in the waveplates, (2) the paths traversed in both directions do not contribute any phase noise to the interferometer, thereby simplifying the phase stabilization. More specifically, only the paths connecting the two beam-splitters with the fiber circulators, as well as the fibers directly between the circulators, add phase noise to the interferometer. These fiber components, as well as the bulk beam-splitter at the interferometer input, are housed in a thermally and acoustically insulated box. The passive stabilization of these elements is sufficient to bring the phase drift down to a value of approximately $10 \text{ mrad min}^{-1}$. The use of a bulk-beam-splitter at the input was chosen in order to balance the losses induced by the fiber circulators through the free-space to fiber coupling, and to give control over the interferometer phase through a piezo-electric actuator, while the fiber beam-splitter at the output ensures perfect spatial mode overlap for high interferometric visibility.

**RESULTS**

Before demonstrating the quantum time flip in the context of the game, we first verified the ability of a polarization gadget to implement both a unitary and its transpose simultaneously, in the two different propagation directions of the light. To this end, we performed quantum process tomography on the implemented unitaries from the sets $M_+$ and $M_-$, in both propagation directions. We then compared the fidelity $F = \langle (U_{fw}|\Psi\rangle)^T U_{bw}^T |\Psi\rangle \rangle_{|\Psi\rangle}$ between the reconstructed unitaries in the forward direction, $U_{fw}$ and $V_{fw}$, with the transposed reconstructed unitaries in the backwards direction, $U_{bw}^T$ and $V_{bw}^T$ (see Methods). The results of this are shown in Fig. 4, where the infidelity, defined as $1 - F$, is plotted. The average infidelity is less than $10^{-3}$, indicating that the gadgets correctly implement the unitaries and their transpose. Note that the fidelity of the transpose is independent of any errors in the tardance of the waveplates in the gadget itself. Such imperfections would cause the fidelity in the implementation of a desired unitary to drop, but would affect the forward and backward directions symmetrically. The
same is true for undesired offsets in the waveplate angles, however in the measurements shown in Fig. 4 the unitaries in the two directions were measured in separate runs, causing them to indeed be sensitive to waveplate angle errors, in addition to errors in the tomography itself.

Having verified the ability to implement a given unitary and its transpose with a single black box simultaneously, we then realised the game discussed in the previous sections. Using a single photon for every round of the game, a total of one million rounds were played. The data for the different elements of $M_M$ and $M_M^{-1}$ was collected sequentially to reduce the time spent rotating the waveplates. The game itself was played using the collected data by uniformly sampling from the two sets of unitaries (see Methods). The relative winning probabilities for each setting are shown in Fig. 5, where it can be seen that our implementation exceeds the indefinite tester bound of 0.92 for every setting, and by extension any strategy that is separable in the arrow of time. More specifically, the average winning probability is found to be 0.9945, with the best and worst case probabilities being 0.9993 and 0.9860 respectively.

The formulation of the indefinite-time-direction witness as a game with only two outcomes, win or lose, allows for a straightforward statistical interpretation of the results. Since we have an upper bound on the probability of success for an indefinite tester, we can calculate the probability $P$ of such a player having obtained $v$ or more victories in $N$ rounds:

$$P = \sum_{k=v}^{N} \binom{N}{k} p_{\text{i.c.}}^k (1 - p_{\text{i.c.}})^{N-k}.$$

This probability is exactly the $P$-value for the experimentally implemented process not being indefinite in its time direction. Out of the $N = 10^6$ rounds played in the experiment, $v = 994,512$ were won by successfully identifying the correct set, while 5,488 rounds were lost. Using a Chernoff bound tailored for the binomial distribution, we can provide an upper bound on the $P$-value, given by:

$$\sum_{k=v}^{N} \binom{N}{k} p_{\text{i.c.}}^k (1 - p_{\text{i.c.}})^{N-k} \leq \exp \left( -N D \left( \frac{v}{N} \parallel p_{\text{i.c.}} \right) \right),$$

where $\exp$ is the exponential function and:

$$D \left( \frac{v}{N} \parallel p \right) := \frac{v}{N} \ln \left( \frac{v}{Np} \right) + \left( 1 \frac{v}{N} \right) \ln \left( \frac{1 - v/N}{1 - p} \right)$$

is the relative entropy. Direct calculation shows that $D \left( \frac{v}{N} \parallel p_{\text{i.c.}} \right) \approx 0.0627$, hence the $P$-value is upper bounded by $P \leq e^{-10^4}$, which is an extremely small number. We therefore conclude that a strategy with a definite time direction could not have produced as many wins as were observed, and the implemented process was inseparable in its arrow of time.

**DISCUSSION**

In this work we have demonstrated, for the first time, a process that is inseparable in the arrow of time. Using an optical interferometer, we implemented a coherent superposition of arbitrary unitary transformations and their time-reversal. Such a process can only be probabilistically simulated by a quantum circuit with a definite time direction. Even agents equipped with two copies of the gates and able to combine them in an indefinite order cannot realise the process deterministically, unless they are given the ability of pre- and post-selecting quantum systems [2, 35–40]. It is worth noting that our implementation of controlled unitary transposition is not in contradiction with the no-go theorem,
stating that there is no quantum circuit that can transform an unknown quantum unitary gate to its transpose [10, 15, 41]. Analogously to the impossibility of perfect unitary coherent control [10, 25, 28, 42], our implementation adopts a device that implements an arbitrary gate $U$, and while this gate can remain unknown, the physical device itself is neither arbitrary nor unknown. Indeed, it is the particular symmetries of the physical device that necessarily and deterministically generate the transposed gate $U^T$. Hence, in this context, our demonstration highlights the limitations of the quantum circuit model for describing the full range of quantum information processing protocols. Through a channel discrimination game, in which we outperform any strategy with a definite time direction, we furthermore certify that the coherent superposition of time-directions yields a process that is inseparable in its arrow of time.

Just as the study of indefinite causality led to the discovery and realisation of quantum information protocols with practical advantages [43, 44], we envision that future studies of processes with an indefinite time order will expand both the theoretical and experimental toolkit, and open up new avenues for quantum information processing. The experimental methods in this work could, for instance, be used to implement transposed circuits of higher dimension through the Reck scheme [45]. Finally, the investigation of time reversed quantum processes also holds applications in quantum thermodynamics. Indeed, in [16], it was shown that the processes for which the quantum time flip produces another valid process are exactly those which do not increase the entropy in either time direction, and the application of superpositions of two time directions in the context of thermodynamic work was recently studied in [9, 46].

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**DATA AVAILABILITY**

All data used in this work is openly available in Zenodo under: 10.5281/zenodo.7352614

**CODE AVAILABILITY**

The code used to perform the computer assisted proofs is openly available at the following online repository: https://github.com/mtcq/UnitaryTransposition

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METHODS

Arbitrary unitary transposition

The description of linear retarders depends on the convention used for the polarization states, i.e. which Pauli matrices are associated with which Stokes parameters. The most commonly used convention in quantum optics is:

\[(S_1, S_2, S_3) \leftrightarrow (Z, X, Y),\]

(15)
corresponding to the \{H, V\}, \{+, -\} and \{L, R\} polarizations being the eigenstates of Z, X and Y respectively. Under this convention, a linear retarder, such as a waveplate, at an angle \(\theta\) to the vertical axis, is described by the following matrix:

\[
U(\theta) = e^{-i\theta Y} e^{-i r Z} e^{i \theta Y}
\]

(16)

where \(r\) is the retardance of the element. Note that the matrix \(U(\theta)\) is symmetric since:

\[
U(\theta)^T = (e^{i \theta Y})^T (e^{-i r Z})^T (e^{-i \theta Y})^T = e^{-i \theta Y} e^{-i r Z} e^{i \theta Y}.
\]

(17)

Propagating through such an element backwards has the effect of taking \(\theta \rightarrow -\theta\). This transformation can be written as:

\[
Z U(\theta) Z = U(-\theta)
\]

(18)
since

\[
Z e^{-i \theta Y} = e^{i \theta Y}.
\]

(19)

For a general polarization gadget consisting of several linear retarders described by the unitary \(U_{G,fw}\) in the forwards direction we find the unitary for the backwards propagation direction, \(U_{G,bw}\), by transposing the order of the individual linear retarders and changing the sign of their respective angles:

\[
U_{G,fw} = U_1(\theta_1) \ldots U_n(\theta_n) \rightarrow U_{G,bw} = U_n(-\theta_n) \ldots U_1(-\theta_1)
\]

(20)

which can be written:

\[
U_{G,fw} \rightarrow Z U_{G,fw}^T Z
\]

(21)
since:

\[
Z (U_1(\theta_1) \ldots U_n(\theta_n))^T Z = Z U_n(\theta_n) \ldots U_1(\theta_1) Z
\]

(22)

\[
= Z U_n(-\theta_n) \ldots U_1(-\theta_1).\]

The transformation in Eq. (21) is not useful for realising the transpose, since the Z gates around the unitary \(U_{G,fw}^T\) have to be undone to recover the transpose. However, this problem can be overcome by picking a different convention for the polarization basis states, such as \((S_1, S_2, S_3) \leftrightarrow (X, Y, Z)\) which is a cyclic permutation of the aforementioned one (corresponding to a rotation of the basis vectors by \(\pi/3\) around the vector \([1\ 1\ 1]\)), and which is commonly used in polarimetry. In this work, we chose the convention:

\[(S_1, S_2, S_3) \leftrightarrow (-Z, -Y, -X).\]

(23)

The minus signs are necessary to preserve the handedness of the coordinate system when exchanging X and Y. That this convention yields the desired transformation under counterpropagation can be realised by noting that the
Stokes parameters of a unitary always transform as \((S_1, S_2, S_3) \rightarrow (S_1, -S_2, S_3)\), however for completeness we will perform the calculation explicitly. In the convention of Eq. (23) a linear retarder at an angle \(\theta\) is written as:

\[
U(\theta) = e^{i\theta X} e^{i\theta Z} e^{-i\theta X}
\]

and the corresponding unitary in the backwards direction is

\[
U(-\theta) = e^{-i\theta X} e^{i\theta Z} e^{i\theta X} = (e^{i\theta X} e^{i\theta Z} e^{-i\theta X})^T
\]

It then follows that a general waveplate gadget also transforms as the transpose:

\[
U_{G, fw} = U_1(\theta_1) \ldots U_n(\theta_n) \rightarrow U_{G, bw} = U_n(-\theta_n) \ldots U_1(-\theta_1) = (U_1(\theta_1) \ldots U_n(\theta_n))^T = U_{G, fw}^T. \tag{26}
\]

Sets of unitary operators used in the game

In this section, we explicitly list the two sets of unitary operators used in the discrimination task considered in this work. These sets of operators were first presented at Ref. [10].

\[
\mathcal{M}_+ := \left\{ (I), (I, X), (I, Z), (X, I), (X, X), (X, Z), (Z, I), (Z, X), (Z, Z), \left( \frac{X-Y}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}} \right), \left( \frac{X+Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}} \right), \left( \frac{Z-Y}{\sqrt{2}}, \frac{Z+Y}{\sqrt{2}} \right), \left( \frac{Z+Y}{\sqrt{2}}, \frac{Z-Y}{\sqrt{2}} \right) \right\}
\]

\[
\mathcal{M}_- := \left\{ (Y, I), (Y, X), (Y, Z), (I, Y), (X, Y), (Z, Y), \left( \frac{I+iY}{\sqrt{2}}, \frac{I-iY}{\sqrt{2}} \right), \left( \frac{I-iY}{\sqrt{2}}, \frac{I+iY}{\sqrt{2}} \right) \right\}. \tag{27}
\]

Obtaining upper bounds for different classes of strategies

We now detail how to obtain an upper bound on the winning probability of the game described in the main manuscript. Let \(N\) be the total number of pairs of unitary operators contained in the set \(\mathcal{M}_+\) and \(\mathcal{M}_-\). Following a uniform distribution, i.e., with probably \(1/N\), the referee picks a pair of unitary operators \((U_i, V_i)\). The player should then employ a quantum strategy to guess whether \((U_i, V_i)\) belongs to \(\mathcal{M}_+\) or \(\mathcal{M}_-\). Let \(p(\pm |(U_i, V_i))\) be the probability that the player guesses \((U_i, V_i) \in \mathcal{M}_\pm\). The probability of such player to win the game is then given by

\[
p = \frac{1}{N} \sum_{(U_i, V_i) \in \mathcal{M}_+} p(+(U_i, V_i)) + \sum_{(U_i, V_i) \in \mathcal{M}_-} p(-|(U_i, V_i)). \tag{28}
\]

For the qubit scenario considered here, we can analyse the case where unitary gates act backwards by simply considering the case where all involved unitary operators are transposed. This is true because, as discussed earlier, there are only two anti-homomorphisms from \(SU(d)\) to \(SU(d)\), and for any \(U \in SU(2)\), we have that \(U^{-1} = \sigma_y U^T \sigma_y\). More explicitly, the winning probability for players using the unitary gates backwards is given by

\[
p = \frac{1}{N} \sum_{(U_i, V_i) \in \mathcal{M}_+} p(+|(U_i^T, V_i^T)) + \sum_{(U_i^T, V_i^T) \in \mathcal{M}_-} p(-|(U_i^T, V_i^T)). \tag{29}
\]
Also, as we show more explicitly later, since the success probability is linear function of the strategies, convex combinations of forward and backwards strategies cannot increase the maximal success probability. Hence it is enough to analyse the forward and backwards case.

When the player is restricted to parallel strategies, the most general approach consists of preparing a quantum state \( \rho \), sending part of this state to the operators \( U_i \) and \( V_i \), and then performing a quantum measurement with outcomes labelled as + or −, that is,

\[
p_{\text{par}}(\pm |(U_i, V_i)) = \text{tr} \left[ M_{\pm} \left( U_i \otimes V_i \otimes 1 \right) \rho \left( U_i^\dagger \otimes V_i^\dagger \otimes 1 \right) \right],
\]

where \( M_+, M_- \geq 0 \) are the POVM operators associated to the outcomes + and −, see Fig. 2 for a pictorial illustration.

Parallel strategies may be analysed in the (parallel) tester formalism \([29,47]\), also known as process POVM \([48]\). Let us label the linear spaces corresponding to the input and output spaces as \( \mathcal{H}_I \) and \( \mathcal{H}_O \) respectively. We can then write \( U_i \otimes V_i : \mathcal{H}_I \to \mathcal{H}_O \) with \( \mathcal{H}_I \cong \mathcal{H}_O \cong C_2 \otimes C_2 \). In the tester formalism, operations are viewed as states and Eq. (30) may be written as the generalized Born’s rule. More formally, we have that:

\[
p_{\text{par}}(\pm |(U_i, V_i)) = \text{tr} \left[ T_\pm |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \right],
\]

where \( T_+, T_- \in L(\mathcal{H}_I \otimes \mathcal{H}_O) \) are test elements and \( |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \) is the Choi vector of \( U_i \otimes V_i \) defined as:

\[
|U_i \otimes V_i \rangle := \sum_l |l \rangle \otimes \left( U_i \otimes V_i |l \rangle \right),
\]

where \( \{|l\rangle\} \) is the computational basis for \( \mathcal{H}_I \). The operators \( T_+ \) and \( T_- \) are parallel testers when \( T_+, T_- \geq 0 \) and their sum respects:

\[
T_+ + T_- = \sigma_I \otimes \mathbb{1}_O,
\]

where \( \sigma \in L(\mathcal{H}_I) \) is a quantum state. As shown in Refs. \([29,47,48]\), all parallel strategies as in Eq. (30) can be represented by testers such as those in Eq. (31), and vice versa. Hence, when optimizing over all possible strategies, instead of considering all possible states \( \rho \) and measurements \( M_+ \) as in Eq. (30), we may optimize over all valid testers \( T_\pm \) as in Eq. (31).

One advantage of using the tester formalism, is that the maximal probability of winning the discrimination game can be written in terms of a semidefinite program (SDP) via the following optimisation problem:

\[
\max \frac{1}{N} \left[ \sum_{(U_i, V_i) \in M_+} \text{tr} \left( T_+ |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \right) \right] + \sum_{(U_i, V_i) \in M_-} \text{tr} \left( T_- |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \right)
\]

\[
\text{s.t.: } T_+, T_- \geq 0
\]

\[
T_+ + T_- = \sigma_I \otimes \mathbb{1}_O
\]

\[
\text{tr}(\sigma) = 1.
\]

Following the steps of Ref. \([29]\), the dual problem is given by:

\[
\min \frac{1}{N} \sum_{(U_i, V_i) \in M_+} |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \leq C
\]

\[
\sum_{(U_i, V_i) \in M_-} |U_i \otimes V_i \rangle \langle U_i \otimes V_i| \leq C
\]

\[
\text{tr}_O(C) = \text{tr}_{IO}(C) \frac{\mathbb{1}_I}{d_I},
\]

\[
1
\]

\footnote{Here \( L(\mathcal{H}_I \otimes \mathcal{H}_O) \) denotes the set of linear operators from \( \mathcal{H}_I \otimes \mathcal{H}_O \) (linear endomorphisms).}
where \( d_i \) is the dimension of \( \mathcal{H}_i \) (for our particular problem, \( d_i = 4 \)). By the definition of dual problem, if we find a linear operator \( C \) satisfying the feasibility constraints of inequality \((39)\), inequality \((40)\), and Eq. \((41)\), the quantity \( \text{tr}(C)/d_i \) is an upper bound on the maximal success probability. In order to obtain a computer-assisted-proof upper bound with fraction of integers, we use standard and efficient floating-point arithmetic algorithms to solve the SDP, obtain an operator \( C \) which satisfies the constraints of the dual problem and truncate it in such a way that the feasibility constraints are still satisfied. We refer to our online repository (see Code Availability) for an implementation of this procedure and to Ref. \([29]\) for a detailed explanation on how to perform the truncation step.

When the player is restricted to causal strategies (also referred to as sequential strategies), the most general approach consists of preparing a quantum state \( \rho \), sending part of this state to the operators \( U_i \) (or to \( V_i \)), applying a quantum channel \( \mathcal{E} \), then performing the operation \( V_i \) (or \( U_i \)), and finally performing a quantum measurement with outcomes labelled as \( + \) or \( - \), that is:

\[
p_{\text{seq}}(\pm|(U_i, V_i)) = \text{tr}\left[ M_{\pm} (V_i \otimes 1) \mathcal{E} \left( U_i \otimes 1 \rho U_i^* \otimes 1 \right) (V_i^* \otimes 1) \right].
\]

Using the concept of sequential testers \([29, 47]\), we can also write the problem of finding the optimal causal strategy as an SDP. Since there is a notion of causal order, we label the input and output space of the first operation as \( \mathcal{H}_1 \) and \( \mathcal{H}_{O1} \) respectively. Analogously, we use \( \mathcal{H}_2 \) and \( \mathcal{H}_{O2} \) for the second operations. If the player uses the operation \( U_i \) first and \( V_j \) second, we have that \( U_i : \mathcal{H}_1 \to \mathcal{H}_{O1} \) and \( V_j : \mathcal{H}_2 \to \mathcal{H}_{O2} \). Following Ref. \([29]\), the primal and dual problem for causal strategies are respectively given by

\[
\begin{align*}
\max & \frac{1}{N} \left[ \sum_{(U_i, V_j) \in \mathcal{M}_+} \text{tr}\left( T_+ |U_i \otimes V_j\rangle\langle U_i \otimes V_j|\right) + \sum_{(U_i, V_j) \in \mathcal{M}_-} \text{tr}\left( T_- |U_i \otimes V_j\rangle\langle U_i \otimes V_j|\right) \right] \\
\text{s.t.:} & \quad T_+ + T_- \geq 0 \\
& \quad T_+ + T_- = W_{\mathcal{I}1, \mathcal{I}2} \otimes 1_{\mathcal{O}2} \\
& \quad \text{tr}_{\mathcal{I}1, \mathcal{I}2}(W_{\mathcal{I}1, \mathcal{I}2}) = \sigma \otimes 1_{\mathcal{O}1} \\
& \quad \text{tr}(\sigma) = 1.
\end{align*}
\]

and

\[
\begin{align*}
\min \ & \frac{1}{N} \left[ \sum_{(U_i, V_j) \in \mathcal{M}_+} |U_i \otimes V_j\rangle\langle U_i \otimes V_j| \leq C \right] \\
\text{s.t.:} \quad & \quad \frac{1}{N} \sum_{(U_i, V_j) \in \mathcal{M}_+} |U_i \otimes V_j\rangle\langle U_i \otimes V_j| \leq C \\
& \quad \text{tr}_{\mathcal{O}2}(C) = \frac{\text{tr}_{\mathcal{O}2}(C) \otimes 1_{d_1}}{d_2} \\
& \quad \text{tr}_{\mathcal{O}1, \mathcal{O}2}(C) = \text{tr}_{\mathcal{O}1, \mathcal{O}2}(C) \frac{1_{d_1}}{d_2}.
\end{align*}
\]

Another sequential strategy would be to use \( V_i \) before \( U_i \). For this case, the semidefinite program is then exactly the same as the one before, but we exchange the roles of \( V_i \) and \( U_i \). Our methods show that, when \( U_i \) precedes \( V_i \), the success probability is bounded by \( \frac{90\%}{91\%} \leq p_{UV} \leq \frac{91\%}{100\%} \), and when \( V_i \) precedes \( U_i \), the success probability is bounded by \( \frac{90\%}{91\%} \leq p_{UV} \leq \frac{91\%}{100\%} \). Since the two bounds coincide, we have \( \frac{90\%}{91\%} \leq p_{\text{causal}} \leq \frac{91\%}{100\%} \).

When the player is restricted to general quantum strategies without a definite causal order, the strategies are described by means of an indefinite tester \([30]\), which are positive semidefinite operators that add up to a process matrix \([3]\), that is \( T_+ + T_- = W \), where \( W \) is a bipartite process matrix. Following Ref. \([29]\), and defining the
trace-and-replace maps as $iX := \text{tr}_i(X) \otimes 1_i$, the primal and the dual problem are respectively given by

$$
\max \frac{1}{N} \left[ \sum_{(U_i, V_i) \in M_+} \text{tr} \left( T_+ |U_i \otimes V_i\rangle\langle U_i \otimes V_i| \right) + \sum_{(U_i, V_i) \in M_-} \text{tr} \left( T_- |U_i \otimes V_i\rangle\langle U_i \otimes V_i| \right) \right] \tag{52}
$$

s.t.: $T_+, T_- \geq 0$

$$
T_+ + T_- = W \tag{53}
$$

$$
I_2 \Omega_2 W = \Omega_1 I_2 \Omega_2 W \tag{54}
$$

$$
\Omega_1 W = \Omega_2 I_1 \Omega_1 W \tag{55}
$$

$$
W = \Omega_1 W + \Omega_2 W - \Omega_1 \Omega_2 W \tag{56}
$$

$$
\text{tr}(W) = \text{tr}(I_1 \Omega_2). \tag{57}
$$

and

$$
\min \text{tr}(C)/d_I \tag{59}
$$

s.t.: $\frac{1}{N} \sum_{(U_i, V_i) \in M_+} |U_i \otimes V_i\rangle\langle U_i \otimes V_i| \leq C \tag{60}$

$$
\frac{1}{N} \sum_{(U_i, V_i) \in M_-} |U_i \otimes V_i\rangle\langle U_i \otimes V_i| \leq C \tag{61}
$$

$$
\Omega_1 C = I_2 \Omega_1 C \tag{62}
$$

$$
\Omega_2 C = I_2 \Omega_2 C. \tag{63}
$$

**Data analysis**

As described in the main text, the game was played by having the referee pick pairs of unitaries from the sets $M_{\pm}$ in a uniformly random way in every round. The player’s outcome was determined by the first unused photon detection event in the event list corresponding to that choice of unitary by the referee. More concretely, let $O_{j \pm}^{k}$ be the $k$-th element in the time ordered list of detection events $O_{j \pm}$ for the implemented pair of unitaries $M_{j \pm}$. Then the outcome of the $n$-th round of the game, in which the referee picked the pair of unitaries $M_{j \pm}$ for the $k$-th time, is $O_{j \pm}^{k}$.

In order to filter out background events resulting from various back-reflections in the experimental setup, as well as detector dark counts, two-fold coincidence events between the signal and idler photons were used to time filter the detection events.

The superconducting nanowire detectors used in the experiment have a slight polarization dependence in their detection efficiency, and due to the different pairs of unitaries generating different target qubit states the event rates for different implemented unitaries varied. This difference in efficiency was not necessary to account for, because the number of events for each pair of unitaries was truncated, in reverse chronological order, to match the setting with the fewest events. To find the numbers of rounds won and lost, the data was sampled from once, drawing $10^6$ different samples from unique, chronologically ordered (for each setting) detection events. The exact number of won and lost rounds in this sampling were 994, 512 won and 5, 488 lost.

A detection efficiency imbalance is also present in the two output ports of the interferometer, corresponding to the two different measurement outcomes of the control qubit. This efficiency difference could quite easily be characterised and corrected for, however such actions are equivalent to classical post-processing and is captured by the indefinite tester. Imbalanced detection efficiency could therefore not lead to a violation of the bound, and is not necessary to correct for since the data already violates the bound.

The measurement of the fidelity between the unitary implemented in one direction and the transpose of the unitary in the other direction was performed with coherent light. To estimate the fidelity, the two unitaries were
first fitted to the data using a maximum likelihood estimation and then the fidelity was calculated by evaluated the following average:

\[ \mathcal{F} = \left\langle \left( (U_{fw}|\Psi\rangle \right)^\dagger U_{bw}^T |\Psi\rangle \right\rangle_{|\Psi\rangle}, \]

(64)

taken over 1000 Haar-random states |Ψ⟩. This was done in every step of a Monte-Carlo simulation to estimate the measurement uncertainties induced by the waveplate errors.