RUIN PROBLEM FOR BROWNIAN MOTION RISK MODEL WITH INTEREST RATE AND TAX PAYMENT

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Abstract: Let \( \{B(t), t \geq 0\} \) be a Brownian motion. Consider the Brownian motion risk model with interest rate collection and tax payment defined by

\[
\tilde{U}_\delta^\gamma(t) = \tilde{X}^\delta(t) - \gamma \sup_{s \in [0,t]} \left( \tilde{X}^\delta(s)e^{\delta(t-s)} - ue^{\delta(t-s)} \right), 
\]

with

\[
\tilde{X}^\delta(t) = u e^{\delta t} + c \int_0^t e^{\delta(t-v)} dv - \sigma \int_0^t e^{\delta(t-v)} dB(v),
\]

where \( c > 0, \gamma \in [0,1) \) and \( \delta \in \mathbb{R} \) are three given constants. When \( \delta = 0 \) and \( \gamma \in (0,1) \) this is the risk model introduced from Albrecher and Hipp in [2] where the ruin probability in the infinite time horizon has been explicitly calculated. In the presence of interest rate \( \delta \neq 0 \), the calculation of ruin probability for this risk process for both finite and infinite time horizon seems impossible. In the paper, based on asymptotic theory we propose an approximation for ruin probability and ruin time when the initial capital \( u \) tends to infinity. Our results are of interest given the fact that this can be used as benchmark model in various calculations.

Key Words: Brownian motion; force of interest; tax payment; ruin probability; ruin time.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

The risk reserve process of an insurance company without interest can be modelled by a stochastic process \( \{X(t), t \geq 0\} \) given as

\[
X(t) = u + ct - \sigma B(t), 
\]

see [15, 20], where \( u \geq 0 \) is the initial reserve, \( c > 0 \) is the rate of premium received by the insurance company, and \( \sigma B(t) \) is frequently referred as the loss rate of the insurance company. If we add the effect of tax into the model, the new claim surplus process is

\[
U_\gamma(t) = X(t) - \gamma \sup_{s \in [0,t]} (X(s) - u), 
\]

Date: June 14, 2018.
where \( \gamma \in [0, 1] \) is referred to the rate of tax. One of the most important characteristics in risk theory is the ruin probability and [2] shows that
\[
P\left\{ \inf_{t \in [0, T]} U_\gamma(t) < 0 \right\} = 1 - \left( 1 - e^{-\frac{\sigma^2}{2}} \right)^{1/\gamma}.
\]

Due to the nature of the financial market, we shall consider a more general surplus process including interest rate, see [20], called a risk reserve process with constant force of interest and tax, i.e., \( \tilde{U}_\gamma^\delta(t), \ t \geq 0 \), in (1). For \( T \in (0, \infty) \), this contribution is concerned with the exact extreme of
\[
\psi_{\gamma, T}^\delta(u) := P\left\{ \inf_{t \in [0, T]} \tilde{U}_\gamma^\delta(t) < 0 \right\},
\]
as \( u \to \infty \). See [20, 9, 16] for more studies on risk models with force of interest. Figure 1 in Appendix depicts the ruin scenario.

When \( \gamma = 0 \) and \( \delta > 0 \), i.e. the risk reserve process with positive constant force of interest but without tax, \( \psi_{0, T}^\delta(u) \) with \( T \in (0, \infty) \) is investigated in [6] and \( \psi_{0, \infty}^\delta(u) \) is derived in [4, 3].

Complementary, we investigate the asymptotic properties of the first passage time (ruin time) of \( \tilde{U}_\gamma^\delta(t) \) on the time interval \( [0, \infty) \), given the ruin has ever happened during \( [0, T] \). For any \( u \geq 0 \), and any \( T \in (0, \infty) \), define the ruin time of the risk process \( \tilde{U}_\gamma^\delta(t) \) by
\[
\tau(u) = \inf\{ t \geq 0 : \tilde{U}_\gamma^\delta(t) < 0 \}.
\]

We are interested in the approximate distribution of \( \tau(u) | \tau(u) \leq T \), as \( u \to \infty \).

Brief organization of the rest of the paper: In Section 2 we first present our main results on the asymptotics of \( \psi_{\gamma, T}^\delta(u) \) as \( u \to \infty \) and then we display the approximation of the ruin time. All the proofs are relegated to Section 3.

## 2. Main Results

Note that
\[
\psi_{\gamma, T}^\delta(u) = P\left\{ \inf_{t \in [0, T]} U_\gamma^\delta(t) < 0 \right\} = P\left\{ \sup_{t \in [0, T]} (u - U_\gamma^\delta(t)) > u \right\},
\]
where
\[
U_\gamma^\delta(t) = \tilde{U}_\gamma^\delta(t)e^{-\delta t}, \ t \geq 0.
\]

Thus in the analysis of our main results, we consider \( P\left\{ \sup_{t \in [0, T]} (u - U_\gamma^\delta(t)) > u \right\} \).

In the following theorem, \( \Psi(u), u \in \mathbb{R} \) denotes the survival function of the standard normal distribution \( N(0, 1) \). Throughout this paper we write \( f(u) = h(u)(1 + o(1)) \) or \( f(u) \sim h(u) \) if \( \lim_{u \to \infty} \frac{f(u)}{h(u)} = 1 \) and \( f(u) = o(h(u)) \) if \( \lim_{u \to \infty} \frac{f(u)}{h(u)} = 0 \).

We prefer to state our new results first, i.e., \( \psi_{\gamma, T}^\delta(u) \) as \( u \to \infty \). Cases \( T = \infty \) and \( T \in (0, \infty) \) are very different and will therefore be dealt with separately. We shall analyse first the case \( T \in (0, \infty) \).
Theorem 2.1. We have for $\gamma \in [0, 1), \delta \in (-\infty, 0) \cup (0, \infty)$ and $T \in (0, \infty)$
\begin{equation}
\psi_{\gamma,T}^\delta(u) \sim \frac{2(1 + e^{-2\delta T})}{1 - \gamma + e^{-2\delta T}} \Phi \left( \frac{u + \frac{\sqrt{2}}{\sigma} (1 - e^{-\delta T})}{a} \right), \quad u \to \infty,
\end{equation}
where $a^2 = \frac{\sigma^2}{T}(1 - e^{-2\delta T})$.

Remarks 2.2. i) When $\gamma = 0$, the result of Theorem 2.1 reduce to asymptotic ruin probability of the risk model without tax, i.e.
\begin{equation}
\psi_{\gamma,T}^\delta(u) \sim 2\Phi \left( \frac{u + \frac{\sqrt{2}}{\sigma} (1 - e^{-\delta T})}{a} \right),
\end{equation}
which corresponds to the result in [6].

ii) In (3), note that $\gamma$ is just related to the denominator of the constant part. The asymptotic result is increase as $\gamma$. In fact, when $\gamma$ is bigger, it means that a company need to pay more tax before the ruin happens, thus the ruin probability should be increasing. Table 1 is the simulated asymptotic results of $\psi_{\gamma,T}^\delta(u)$ in Theorem 2.1, which also shows the increasing about $\gamma$.

| Table 1. The simulated asymptotic results of $\psi_{\gamma,T}^\delta$ |
|------------------|-----|-----|-----|-----|-----|------------------|
| $u$              | $c$ | $\sigma$ | $\delta$ | $T$ | $\gamma$ | asymptotic results |
| 5                | 0.1 | 1     | 0.05     | 20  | 0.1     | 0.0363            |
| 5                | 0.1 | 1     | 0.05     | 20  | 0.2     | 0.0402            |
| 5                | 0.1 | 1     | -0.05    | 20  | 0.2     | 0.0455            |
| 5                | 0.1 | 1     | 0.07     | 20  | 0.1     | 0.0210            |
| 5                | 0.1 | 1     | 0.07     | 30  | 0.2     | 0.0229            |
| 5                | 0.1 | 1     | -0.07    | 30  | 0.2     | 0.0349            |
| 5                | 0.1 | 1     | 0.1      | 20  | 0.1     | 0.0090            |
| 5                | 0.1 | 1     | 0.1      | 30  | 0.2     | 0.0096            |
| 5                | 0.1 | 1     | -0.1     | 30  | 0.2     | 0.0136            |
| 4                | 0.1 | 1     | 0.1      | 20  | 0.1     | 0.0312            |
| 4                | 0.1 | 1     | 0.1      | 30  | 0.2     | 0.0333            |
| 4                | 0.1 | 1     | -0.1     | 30  | 0.2     | 0.0453            |

Theorem 2.3. We have for $\gamma \in [0, 1), \delta > 0$ and $T = \infty$
\begin{equation}
\psi_{\gamma,\infty}^\delta(u) \sim \frac{1}{1 - \gamma} \bar{P}^{\frac{\sigma^2}{2\pi T}}[0, \infty) \Phi \left( \frac{\sqrt{2}}{\sigma} \sqrt{\delta u^2 + 2cu} \right), \quad u \to \infty,
\end{equation}
where for $-\infty \leq S_1 < S_2 \leq \infty$
\begin{equation}
\bar{P}^{\frac{\sigma^2}{2\pi T}}[S_1, S_2] = \mathbb{E} \left\{ \sup_{t \in [S_1, S_2]} e^{\sqrt{\frac{2}{\pi T}} \bar{B}(t) - \frac{\sigma^2}{\pi T} \bar{B}(t)^2} \right\} \in (0, \infty).
\end{equation}
Remarks 2.4. i) We have that
\[
\frac{1}{1 - \gamma} \tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) \Psi \left( \frac{\sqrt{2}}{\sigma} \sqrt{\delta u^2 + 2cu} \right) \sim \frac{1}{1 - \gamma} \tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) \Psi \left( \frac{\sqrt{2}(du + c)}{\sigma \sqrt{\delta}} \right),
\]
where
\[
\tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) = \mathbb{E} \left\{ \sup_{t \in [0, \infty)} e^{\sqrt{2}B(t) - 2t + \frac{2\delta}{\sigma \sqrt{\delta}} \sqrt{t}} \right\}.
\]

The asymptotic result decreases when \( \delta \) increases. Since \( \tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) \) is a decreasing function of \( \delta > 0 \) and \( \Psi \left( \frac{\sqrt{2}(du + c)}{\sigma \sqrt{\delta}} \right) \) is decreasing when \( \delta \) increases and \( u \geq \frac{c}{\sqrt{2}\delta} \), the asymptotic of \( \psi_{\delta, \infty}(u) \) is also a decreasing function of \( \delta \). The effect of \( \delta \) is not an intuitionistic result from the original risk model.

Furthermore, comparing this result with it in [4, 3], the scenario with tax is just \( \frac{1}{1 - \gamma} \) multiple of that without tax. Table 2 is the simulated asymptotic results of \( \psi_{\delta, \infty}(u) \) in Theorem 2.3.

ii) \( \tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) \) and \( \tilde{P}_{\frac{\sigma^2}{\delta^2}}[0, \infty) \) can be considered as the generalised Piterbarg constants, see e.g., [19, 11, 13, 5] for various properties including the finiteness of related constants.

iii) We here interpret that the analysis of \( \psi_{\delta, \infty}(u) \) for the case \( \delta < 0 \) is meaningless. We have
\[
\sup_{t \in [0, \infty)} (u - U_{\delta,\gamma}(t)) \geq \sup_{t \in [0, \infty)} \left( \sigma \int_0^t e^{-\delta v} dB(v) - c \int_0^t e^{-\delta v} dv \right) = \infty, \quad a.s.,
\]
where \( \text{Var} \left( \sigma \int_0^t e^{-\delta v} dB(v) \right) = \frac{\sigma^2}{2\delta}(1 - e^{-2\delta t}) \).

| \( u \) | \( c \) | \( \sigma \) | \( \delta \) | \( \gamma \) | asymptotic results |
|--------|--------|--------|--------|--------|------------------|
| 5      | 0.1    | 1      | 0.05   | 0.1    | 0.0467           |
| 5      | 0.1    | 1      | 0.05   | 0.2    | 0.0526           |
| 5      | 0.1    | 1      | 0.07   | 0.1    | 0.0256           |
| 5      | 0.1    | 1      | 0.07   | 0.2    | 0.0288           |
| 5      | 0.1    | 1      | 0.1    | 0.1    | 0.0113           |
| 5      | 0.1    | 1      | 0.1    | 0.2    | 0.0128           |
| 4      | 0.1    | 1      | 0.1    | 0.1    | 0.0378           |
| 4      | 0.1    | 1      | 0.1    | 0.2    | 0.0425           |

We present below the approximation of the conditional passage time \( \tau(u) | \tau(u) \leq T \) with \( \tau(u) \) defined in (2).

Theorem 2.5. For \( T \in (0, \infty), \delta \in (-\infty, 0) \cup (0, \infty) \) and \( x > 0 \), we have as \( u \to \infty \)
\[
P \left\{ u^2(T - \tau(u)) > x | (\tau(u) \leq T) \right\} \sim \exp \left( -\frac{\sigma^2 e^{-2\delta T} x}{2\delta^2} \right).
\]
For \( T = \infty, \delta > 0 \) and \( x \in (-\frac{c}{\sigma \sqrt{\delta}}, \infty) \), we have

\[
P \left\{ u^2 \left( e^{-2\delta \tau(u)} - \left( \frac{c}{\partial u + c} \right)^2 \right) \leq x \big| \tau(u) < \infty \right\} \sim \frac{\hat{P}_{\alpha^2\tau} [0, x + \frac{c}{\sigma \sqrt{\delta}}]}{\hat{P}_{\alpha^2\tau} [0, \infty]},
\]
as \( u \to \infty \).

Remark 2.6. When \( \gamma = 0 \), the result of the scenario \( T = \infty \) corresponds to that in [3].

3. Proofs

Before giving the proofs of our main theorems, we need to introduce some notation which play a pivotal role in the proofs, starting with

\[
\mathcal{P}^a[0, S] = \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{\sqrt{2}B(t) - (1+a)t} \right\} \in (0, \infty),
\]
where \( S, a \) are positive constants and

\[
\mathcal{P}^a[0, \infty) := \lim_{S \to \infty} \mathcal{P}^a[0, S] = 1 + \frac{1}{a}
\]
where is known, see e.g., [19] or [11]. Moreover, Pickands constant defined by

\[
\mathcal{H}[0, S] = \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{\sqrt{2}B(t) - t} \right\} \in (0, \infty).
\]
It is known that \( \lim_{T \to \infty} \frac{1}{T} \mathcal{H}[0, T] = 1 \), see [18, 7, 19, 12, 17, 8, 14] for various properties of \( \mathcal{P}^a[0, S], \mathcal{H}[0, S] \) and its generalizations.

Proof of Theorem 2.1 We define for any \( \gamma \in (0, 1) \) the random process \( Z \) by

\[
Z(s, t) := \sigma \int_0^t e^{-\delta v} dB(v) - c \int_0^t e^{-\delta v} dv - \gamma \left( \sigma \int_0^s e^{-\delta v} dB(v) - c \int_0^s e^{-\delta v} dv \right) \frac{1 + \gamma}{1 + \gamma(e^{-\delta s} - 1)}, \quad s, t \geq 0,
\]
which is crucial for our analysis, then for any \( u \) positive

\[
\psi^\gamma_{\delta, T}(u) = \mathbb{P} \left\{ \sup_{0 \leq s \leq t \leq T} Z(s, t) > u \right\}.
\]
Define next the mean function of \( Z(s, t) \)

\[
m(s, t) := \mathbb{E} \{ Z(s, t) \} = \frac{1}{1 + \gamma(e^{-\delta s} - 1)} \left( -c \int_0^t e^{-\delta v} dv + \gamma c \int_0^s e^{-\delta v} dv \right)
\]
and its variance function

\[
V^2_Z(s, t) = \mathbb{E} \{ Z(s, t) - \mathbb{E} \{ Z(s, t) \} \}^2
\]

\[
= \frac{1}{(1 - \gamma + \gamma e^{-\delta s})^2} \mathbb{E} \left\{ \left( \sigma \int_0^t e^{-\delta v} dB(v) - \gamma \sigma \int_0^s e^{-\delta v} dB(v) \right)^2 \right\}
\]
where the random field \( V \) as \((11)\)

\[ \nu(s,t) = \frac{u - m(s,t)}{V_Z(s,t)}, \quad W(s,t) = Z(s,t) - m(s,t). \]

Setting \( V_W(z) = \text{Var}(W(s,t)) \) and \( W(s,t) = \frac{W(s,t)}{V_W(s,t)} \), it is clear that \( V_W = V_Z \), for any \( u > 0 \)

\[ \Pi(u) \leq P \left\{ \sup_{(s,t) \in B} Z(s,t) > u \right\} \leq \Pi(u) + \Pi_1(u), \]

where

\[ \Pi(u) = P \left\{ \sup_{(s,t) \in B} \frac{W(s,t)}{\nu_u(s,t)} > \nu_u(0,T) \right\}, \]

\[ \Pi_1(u) = P \left\{ \sup_{(s,t) \in B} \frac{W(s,t)}{\nu_u(s,t)} > \nu_u(0,T) \right\}. \]

Since

\[ \nu_u(0,T) \]

\[ = 1 - \frac{V_W(0,T) - V_W(s,t)}{V_W(0,T)} + \frac{[m(s,t) - m(0,T)]V_W(s,t)}{(u - m(s,t))V_W(0,T)}, \]

\[ = 1 - \frac{V_W(0,T) - V_W(s,t)}{V_W(0,T)} + \frac{ce^{-\delta T}[T - t + \gamma e^\delta T s]V_W(s,t) + o((T - t) + s)}{(u - m(s,t))V_W(0,T)}, \]

as \((s,t) \to (0,T)\), we have, in view of (8), for any \( \varepsilon \in (0,1) \), and sufficiently large \( u \)

\[ 1 - \frac{V_W(0,T) - V_W(s,t)}{V_W(0,T)} \leq \frac{\nu_u(0,T)}{\nu_u(s,t)} \leq 1 - (1 - \varepsilon) \frac{V_W(0,T) - V_W(s,t)}{V_W(0,T)}, \]

uniformly in \((s,t) \in B_\theta\). Consequently

\[ \mathbb{P} \left\{ \sup_{(s,t) \in B_\theta} W_0(s,t) > \nu_u(0,T) \right\} \leq \Pi(u) \leq \mathbb{P} \left\{ \sup_{(s,t) \in B_\theta} W_\varepsilon(s,t) > \nu_u(0,T) \right\}, \]

where the random field \( \{W_\varepsilon(s,t), s, t \geq 0\} \) is defined as

\[ W_\varepsilon(s,t) := \frac{W(s,t)}{V_W(0,T)} \left( 1 - (1 - \varepsilon) \frac{V_W(0,T) - V_W(s,t)}{V_W(0,T)} \right), \quad \varepsilon \in [0,1). \]
Direct calculations show that the standard deviation function \( \sigma_{W, \varepsilon}(s, t) := \sqrt{\mathbb{E}\{(W_{\varepsilon}(s, t))^2\}} \) attains its unique maximum over \( B_\theta \) at \( (0, T) \) with \( \sigma_{W, \varepsilon}(0, T) = 1 \). Thus, in the light of (7), we have
\[
\sigma_{W, \varepsilon}(s, t) = 1 - (1 - \varepsilon) \left( \frac{\gamma \sigma^2 (1 - \gamma + e^{-2\delta T})}{2a} s + \frac{\sigma^2 e^{-2\delta T}}{2a^2} (T - t) \right) (1 + o(1)),
\]
as \( (s, t) \to (0, T) \). Furthermore, it follows that
\[
1 - \text{Cov}(W_{\varepsilon}(s, t), W_{\varepsilon}(s', t')) = \frac{\sigma^2}{2a^2} (e^{-2\delta T} | t - t' | + \gamma^2 | s - s' |) (1 + o(1))
\]
as \( (s, t), (s', t') \to (0, T) \). In addition, we obtain
\[
\mathbb{E}\{(W_{\varepsilon}(s, t) - W_{\varepsilon}(s', t'))^2\} \leq C(2e^{-2\delta T}|t - t'| + 2\gamma^2|s - s'|)
\]
for \( (s, t), (s', t') \in B_\theta \). Consequently, by Theorem 8.2 of [19]
\[
\mathbb{P}\left\{ \sup_{(s, t) \in B_\theta} W_{\varepsilon}(s, t) > \nu_u(0, T) \right\} \sim \frac{2}{1 - \gamma - e^{-2\delta T}} \Psi \left( \frac{u + \frac{\theta}{\delta}(1 - e^{-\delta T})}{a} \right)
\]
as \( u \to \infty, \varepsilon \to 0 \). Thus we obtain the asymptotic upper bound for \( \Pi(u) \) on the set \( B_\theta \). The asymptotic lower bound can be derived using the same arguments. In order to complete the proof we need to show further that
\[
\Pi_1(u) = o(\Pi(u)) \quad \text{as} \quad u \to \infty.
\]
In the light of (10) for all \( u \) sufficiently large
\[
\sup_{(s, t) \in B \setminus B_\theta} \text{Var}\left( \frac{W_{\varepsilon}(s, t)}{\nu_u(s, t)} \right) \leq (\rho(\theta))^2 < 1,
\]
where \( \rho(\theta) \) is a positive function in \( \theta \) which exists due to the continuity of \( V_W(s, t) \) in \( B \). Therefore, a direct application of Borell-TIS inequality as in [1] implies
\[
\mathbb{P}\left\{ \sup_{(s, t) \in B \setminus B_\theta} \frac{W_{\varepsilon}(s, t)}{\nu_u(s, t)} > \nu_u(0, T) \right\} \leq 2\Psi \left( \frac{\nu_u(0, T) - b}{\rho(\theta)} \right) = o(\Pi(u)), \quad u \to \infty,
\]
where \( b = \sup_{(s, t) \in B \setminus B_\theta} \mathbb{E}\left\{ \frac{W_{\varepsilon}(s, t)}{\nu_u(s, t)} \right\} < \infty \). Consequently, Eq. (15) is established. \( \square \)

**Proof of Theorem 2.3** We have
\[
\psi_{\gamma, \infty}(u) = \mathbb{P}\left\{ \sup_{0 < t \leq s < 1} \frac{Z(s, t)}{G_u(s, t)} > 1 \right\},
\]
where
\[
Z(s, t) = \sigma \int_0^{\frac{1}{2}\ln t} e^{-\delta v} dB(v) - \gamma \sigma \int_0^{\frac{1}{2}\ln s} e^{-\delta v} dB(v),
\]
\[
G_u(s, t) = u + \gamma \left( s^{\frac{1}{2}} - 1 \right) u + c \int_0^{\frac{1}{2}\ln t} e^{-\delta v} dv - c\gamma \int_0^{\frac{1}{2}\ln s} e^{-\delta v} dv
\]
\[
= u - \gamma \left( u + \frac{c}{\delta} \right) (1 - s^{\frac{1}{2}}) + \frac{c}{\delta} \left( 1 - t^{\frac{1}{2}} \right).
\]
The variance function of $Z(s, t)$ is given by

$$V_Z^2(s, t) = \text{Var} \left( \sigma \int_0^t e^{-\delta u} dB(u) - \gamma \sigma \int_0^s e^{-\delta u} dB(u) \right)$$

$$= \frac{\sigma^2}{2\delta} \left( (1 - t) - \gamma(2 - \gamma)(1 - s) \right).$$

Let $M_u(s, t) = \frac{G_u(s, t)}{V_Z(s, t)}$, then for $t_u$ in Lemma 4.2

$$M_u := M_u(1, t_u) = \frac{\sqrt{2}}{\sigma} \sqrt{\delta u^2 + 2cu} = \frac{\sqrt{2\delta}}{\sigma} u(1 + o(1)),$$

as $u \to \infty$.

$$\frac{M_u(s, t)}{M_u} - 1 = \frac{[G_u(s, t)V_Z(1, t_u)]^2 - [G_u(1, t_u)V_Z(s, t)]^2}{V_Z(s, t)G_u(1, t_u)[G_u(s, t)V_Z(1, t_u) + V_Z(s, t)G_u(1, t_u)]}$$

Since

$$[G_u(s, t)V_Z(1, t_u)]^2 - [G_u(1, t_u)V_Z(s, t)]^2 = \{(u + \frac{c}{\delta})[1 - \gamma(1 - \sqrt{s})] - \frac{c}{\delta} \sqrt{t_u} \gamma^2 \}_{2/\delta} (1 - t_u)$$

$$- \{(u + \frac{c}{\delta}) - \frac{c}{\delta} \sqrt{t_u} \gamma^2 \}_{2/\delta} [1 - t - \gamma(2 - \gamma)(1 - s)]$$

$$\sim \frac{\sigma^2}{2\delta} u^2 [\sqrt{t - \sqrt{t_u}}^2 + \gamma(1 - \gamma)(1 - s)],$$

then

$$\frac{M_u(s, t)}{M_u} - 1 \sim \frac{1}{2} [\sqrt{t - \sqrt{t_u}}^2 + \gamma(1 - \gamma)(1 - s)].$$

Now we rewrite

$$\psi_{\gamma, \infty}^\delta(u) = P \left\{ \sup_{0 < t \leq s \leq 1} \frac{Z(s, t)}{G_u(s, t)} M_u > M_u \right\}.$$

The correlation function of $Z(s, t)$ is

$$r(s, s', t, t') \sim 1 - \frac{1}{2} |t - t'| - \frac{1}{2} \gamma^2 |s - s'|$$

for $s, s' \to 1$, $t - t_u, t' - t_u \to 0$.

In addition, we obtain for some $\theta_0, C$,

$$E(Z(s, t) - Z(s', t'))^2 \leq C(|t - t'| + \gamma^2 |s - s'|)$$

for $s, t \in [1 - \theta_0, 1] \times [0, \theta_0]$.

Note that for any small $\theta_1, \theta_2 \in (0, 1)$, set $B = \{ (s, t) : 0 < t \leq s \leq 1 \}$ and $\Delta_\theta = [1 - \theta_1, 1] \times [0, \theta_2]$,

$$\Pi(u) := P \left\{ \sup_{(s, t) \in \Delta_\theta} \frac{Z(s, t)}{G_u(s, t)} M_u > M_u \right\} \leq \psi_{\gamma, \infty}^\delta(u) \leq \Pi(u) + \Pi(u).$$
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\[ \tilde{\Pi}(u) := \mathbb{P} \left\{ \sup_{t \in B \setminus \Delta} \frac{Z(s,t)}{G_u(s,t)} M_u > M_u \right\} \]

In the following, we shall focus on the asymptotics of \( \Pi(u) \) as \( u \to \infty \), and finally we show that

(20) \( \tilde{\Pi}(u) = o(\Pi(u)), \quad u \to \infty. \)

We set \( \delta_1(u) = \left( \frac{\ln u}{u} \right)^q \), \( \delta_2(u) = 2\sqrt{u} \left( \frac{(\ln u)^q}{u} \right) \) for some \( q > 1 \) and

\[ D_u = [1 - \delta_1(u), 1] \times [0, t_u + \delta_2(u)] = \left[ 0, \left( \frac{(\ln u)^q}{u} \right)^2 \right] \times \left[ 0, \left( \sqrt{t_u} + \frac{(\ln u)^q}{u} \right)^2 \right], \quad \Theta_u = \Delta_\theta \setminus D_u. \]

Clearly, for \( u \) large enough,

\[ \Pi_1(u) := \mathbb{P} \left\{ \sup_{(s,t) \in D_u} \frac{Z(s,t)}{G_u(s,t)} M_u > M_u \right\} \leq \Pi(u) \leq \Pi_1(u) + \Pi_2(u), \]

with

(21) \( \Pi_2(u) = o(\Pi_1(u)), \quad u \to \infty. \)

By (19) we have that for any small \( \varepsilon > 0 \), there exists some small \( \theta_1, \theta_2 > 0 \) such that

\[ \frac{M_u}{M_u(s,t)} \leq \frac{1}{1 + \frac{1 - \varepsilon_0}{2 \gamma}} \frac{1}{\gamma(1 - \gamma)(1 - s)} \]

holds for all \( (s,t) \in \Delta_\theta \). Furthermore, for any \( t \in \Theta_u \)

\[ 1 + \frac{1 - \varepsilon_0}{2} \gamma(1 - \gamma)(1 - s) \geq 1 + \frac{1 - \varepsilon_0}{2} \min(1, \gamma(1 - \gamma)) \left( \frac{(\ln u)^q}{u} \right)^2 \]

implying (set \( \bar{Z}(s,t) = \frac{Z(s,t)}{\sqrt{Z(s,t)}} \))

\[ \Pi_2(u) = \mathbb{P} \left\{ \sup_{(s,t) \in \Theta_u} \frac{Z(s,t)}{\sqrt{Z(s,t)}} \frac{M_u}{M_u(s,t)} > M_u \right\} \]

(21) \[ \leq \Omega_1 M_u^4 \Psi \left( M_u \left( 1 + \frac{1 - \varepsilon_0}{2} \min(1, \gamma(1 - \gamma)) \left( \frac{(\ln u)^q}{u} \right)^2 \right) \right) \]

holds for all \( u \) large, with some constant \( \Omega_1 > 0 \). For any small \( \varepsilon > 0 \), define below

\[ \mathcal{B}^{\pm}_u(\Delta) := \left\{ \sup_{(s,t) \in \Delta} \frac{\xi^{\pm}(s,t)}{1 + \frac{(1 \pm \varepsilon)}{2}(\sqrt{1 - t_u})^2[1 + \frac{(1 + (1 - \gamma)^{1/2} / 2 \varepsilon)(1 - s)]} > M_u \right\}, \Delta \subset \mathbb{R}^2, \]
where \( \{\xi^\pm(s, t), s, t \geq 0\} \) is a zero-mean stationary Gaussian field with continuous sample paths and correlation function
\[
\gamma^\pm_\xi(s, t) = \exp \left( -\frac{1}{2} \left( \frac{\gamma^2}{\epsilon^2} \right) |s| - \frac{1}{2} \left( \frac{1}{\epsilon^2} \right) |t| \right).
\]

Next we analyse \( \Pi_1(u) \), as \( u \rightarrow \infty \). By (19), when \( u \) large enough, for any \( t \in D_u \),
\[
\frac{M_u}{M_u(s, t)} \geq \frac{1}{1 + \left( \frac{1}{2} + \epsilon \right)(\sqrt{t} - \sqrt{t_u})^2 \left( 1 + \left( \frac{1}{2} \gamma(1 - \gamma) + \epsilon \right)(1 - s) \right)},
\]
\[
\frac{M_u}{M_u(s, t)} \leq \frac{1}{1 + \left( \frac{1}{2} - \epsilon \right)(\sqrt{t} - \sqrt{t_u})^2 \left( 1 + \left( \frac{1}{2} \gamma(1 - \gamma) - \epsilon \right)(1 - s) \right)}.
\]

Then
\[
\mathbb{P}\{B^+_u(D_u)\} \leq \Pi_1(u) \leq \mathbb{P}\{B^-_u(D_u)\}, \quad u \rightarrow \infty.
\]

Thus we just need establish the asymptotic behavior of \( \pi^+(u) := \mathbb{P}\{B^+_u(D_u)\} \), then according to the continuous of the results which can be seen from the following calculation, setting \( \epsilon \rightarrow 0 \), we will gain the precision estimates of \( \Pi_1(u) \). Below we mainly show the calculation of \( \pi^+(u) \).

\[
D_u = [1 - \delta_1(u), 1] \times [0, t_u + \delta_2(u)] = \left[ 0, \left( \frac{(\ln u)^q}{u} \right)^2 \right] \times \left[ 0, \left( \sqrt{t_u} + \left( \frac{\ln u}{u} \right)^2 \right)^2 \right].
\]

For any positive constant \( S_1, S_2, \) define
\[
\Delta^1_k = 1 - u^{-2}[(k + 1)S_1, kS_1], \quad k = 0, 1, 2, 3, \ldots
\]
\[
\Delta^2_k = [0, t_u], \quad \Delta^3_k = [(\sqrt{t_u} + \sqrt{kS_2}u^{-1})^2, (\sqrt{t_u} + \sqrt{(k + 1)S_2}u^{-1})^2], \quad k = 0, 1, 2, \ldots
\]

and let further for \( u > 0 \)
\[
h_1(u) = [S_1^{-1}(\ln u)^{2q}] + 1, \quad h_2(u) = \left[ \frac{(\ln u)^q}{u} \right]^2 - t_u
\]
\[
\left. \left\{ \sqrt{t_u} + \sqrt{(k + 1)S_2}u^{-1} \right\}, \left( \sqrt{t_u} + \sqrt{kS_2}u^{-1} \right)^2 \right] + 1,
\]

where \( h_1(u), h_2(u) \rightarrow \infty \), as \( u \rightarrow \infty \). By Bonferroni’s inequality we have
\[
\pi^+_1(u) := \mathbb{P}\{B^+_u(\Delta^1_0 \times (\Delta^2_1 \cup \Delta^3_0))\} \leq \pi^+(u) \leq \pi^+_1(u) + \pi^+_2(u),
\]

where
\[
\pi^+_2(u) = \sum_{k_1=1}^{h_1(u)} \mathbb{P}\{B^+_u(\Delta^1_{k_1} \times (\Delta^2_1 \cup \Delta^3_{k_1}))\} + \sum_{k_2=1}^{h_2(u)} \mathbb{P}\{B^+_u(\Delta^1_{k_2} \times \Delta^3_{k_2})\}
\]
\[
+ \sum_{k_1=1}^{h_1(u)} \sum_{k_2=1}^{h_2(u)} \mathbb{P}\{B^+_u(\Delta^1_{k_1} \times \Delta^3_{k_2})\}
\]
\[
=: I_1 + I_2 + I_3.
\]
By Lemma 4.3, we get as $u \to \infty$

$$I_1(u) = o(\pi_1^+(u)), I_2(u) = o(\pi_1^+(u)), I_3(u) = o(\pi_1^+(u)).$$

By Lemma 4.3, we drive that

$$I_1(u) = \sum_{k_1 = 1}^{h_1(u)} P \{ B^1_0(\Delta^1_{k_1} \times (\Delta^2_{-1} \cup \Delta^2_{2})) \}$$

$$\leq \sum_{k_1 = 1}^{h_1(u)} P \left\{ \sup_{(s,t) \in (\Delta^1_{k_1} \times (\Delta^2_{-1} \cup \Delta^2_{2}))} \frac{\xi^+(s,t)}{(s-t) \sqrt{\frac{\sigma^2}{\delta}}} > M_u \left[ 1 + \left( \frac{1}{2} \gamma (1 - \gamma) + \epsilon \right) u^{-2} k_1 S_1 \right] \right\}$$

$$\leq H(0, (\gamma^2 - 2\epsilon) \frac{\delta}{\sigma^2} S_1) \mathcal{P}^{(1+2\epsilon) \frac{\sigma^2}{\delta} \sqrt{1-2\epsilon}}[0, (1-2\epsilon) \frac{\delta}{\sigma^2} (\frac{c}{\delta} + \sqrt{S_2})^2]$$

$$\times \sum_{k_1 = 1}^{h_1(u)} \frac{1}{M_u \left[ 1 + \left( \frac{1}{2} \gamma (1 - \gamma) + \epsilon \right) u^{-2} k_1 S_1 \right]} \exp \left( - \frac{M_u^2 [1 + \left( \frac{1}{2} \gamma (1 - \gamma) + \epsilon \right) u^{-2} k_1 S_1]^2}{2} \right) (1 + o(1))$$

$$= H(0, (\gamma^2 - 2\epsilon) \frac{\delta}{\sigma^2} S_1) \mathcal{P}^{(1+2\epsilon) \frac{\sigma^2}{\delta} \sqrt{1-2\epsilon}}[0, (1-2\epsilon) \frac{\delta}{\sigma^2} (\frac{c}{\delta} + \sqrt{S_2})^2]$$

$$\times \Psi(M_u) \sum_{k_1 = 1}^{h_1(u)} \exp \left( - \frac{\delta}{\sigma^2} (\gamma (1 - \gamma) + 2\epsilon) k_1 S_1 \right) (1 + o(1))$$

as $u \to \infty$. Similarly

$$I_2(u) \leq P^{(1+2\epsilon) \frac{\sigma^2}{\delta} \sqrt{1-2\epsilon}}[0, (\gamma^2 - 2\epsilon) \frac{\delta}{\sigma^2} S_1] \mathcal{P}^{(1+2\epsilon) \frac{\sigma^2}{\delta} \sqrt{1-2\epsilon}}[0, (1-2\epsilon) \frac{\delta}{\sigma^2} (\frac{c}{\delta} + \sqrt{S_2})^2]$$

$$\times \Psi(M_u) \sum_{k_2 = 1}^{h_2(u)} \exp \left( - \frac{\delta}{\sigma^2} (1 + 2\epsilon) k_2 S_2 \right) (1 + o(1))$$

as $u \to \infty$. Moreover,

$$I_3(u) = \sum_{k_1 = 1}^{h_1(u)} \sum_{k_2 = 1}^{h_2(u)} P \{ B^1_0(\Delta^1_{k_1} \times \Delta^2_{k_2}) \}$$

$$\leq \sum_{k_1 = 1}^{h_1(u)} \sum_{k_2 = 1}^{h_2(u)} P \left\{ \sup_{(s,t) \in (\Delta^1_{k_1} \times \Delta^2_{k_2} \cup \Delta^2_{k_2})} \frac{\xi^+(s,t)}{(s-t) \sqrt{\frac{\sigma^2}{\delta}}} > M_u \left[ 1 + \left( \frac{1}{2} \gamma (1 - \gamma) + \epsilon \right) u^{-2} k_1 S_1 + \left( \frac{1}{2} + \epsilon \right) u^{-2} k_2 S_2 \right] \right\}$$

$$= H(0, (\gamma^2 - 2\epsilon) \frac{\delta}{\sigma^2} S_1) \mathcal{P}^{(1+2\epsilon) \frac{\sigma^2}{\delta} \sqrt{1-2\epsilon}}[0, (1-2\epsilon) \frac{\delta}{\sigma^2} (\frac{c}{\delta} + \sqrt{S_2})^2]$$

$$\times \Psi(M_u) \sum_{k_1 = 1}^{h_1(u)} \sum_{k_2 = 1}^{h_2(u)} \exp \left( - \frac{\delta}{\sigma^2} \left( \gamma (1 - \gamma) + 2\epsilon \right) k_1 S_1 + (1 + 2\epsilon) k_2 S_2 \right) (1 + o(1)),$$
as \( u \to \infty \). By letting \( S_1, S_2 \to \infty \), (24) is proved.

Thus we finish the calculation of \( \pi^+(u) \).

For \( \pi^-(u) \), we just need notice that the equations (22) are replaced by

\[
\pi_1^-(u) := \mathbb{P}\{B_u^-(\Delta_0^2 \times (\Delta_1^2 \cup \Delta_0^2))\} \leq \pi^-(u) \leq \pi_1^-(u) + \pi_2^-(u)
\]

and

\[
\pi_2^-(u) = \sum_{k_1=1}^{h_1(u)} \mathbb{P}\{B_u^-(\Delta_0^2 \times (\Delta_1^2 \cup \Delta_0^2))\} + \sum_{k_2=1}^{h_2(u)} \mathbb{P}\{B_u^-(\Delta_0^2 \times \Delta_2^2)\}
\]

\[
+ \sum_{k_1=1}^{h_1(u)} \sum_{k_2=1}^{h_2(u)} \mathbb{P}\{B_u^-(\Delta_{k_1}^2 \times \Delta_{k_2}^2)\}
\]

\[
= : J_1 + J_2 + J_3.
\]

Then by Lemma 4.3, we get

\[\pi_1^-(u) \sim \mathcal{P}^{\frac{2(1-\gamma)}{\gamma^2+2\epsilon}}[0, (\gamma^2 + 2\epsilon) \frac{\delta}{\sigma^2}S_1] \mathcal{P}^{\frac{2\epsilon}{\gamma^2+2\epsilon} \times 2} \mathcal{P}^{\frac{2(1+2\epsilon)}{\gamma^2+2\epsilon}[0, (1+2\epsilon) \frac{c}{\delta} + \sqrt{S_2}^2)]\Psi(M_u),\]

and similarly

\[J_1(u) = o(\pi_1^-(u)), J_2(u) = o(\pi_1^-(u)), J_3(u) = o(\pi_1^-(u)), \text{ as } u \to \infty, S_i \to \infty, i = 1, 2.\]

Thus,

\[
\lim_{\epsilon \to 0} \pi^+(u) = \lim_{\epsilon \to 0} \pi^-(u) = \Pi_1(u).
\]

Finally, by (19) and Lemma 4.2, we can choose some small \( \theta_1, \theta_2 > 0 \) so that for any \( u \) sufficiently large

\[
\sup_{(s, t) \in B \setminus \Delta_0} \text{Var}\left\{\frac{Z(s, t)}{G_u(s, t)}M_u\right\} \leq \sup_{(s, t) \in B \setminus \Delta_0} \left(\frac{M_u}{M_u(s, t)}\right)^2 \leq (\rho(\theta_1, \theta_2))^2 < 1
\]

where \( \rho(\theta_1, \theta_2) \) is a positive function in \( \theta_1 \) and \( \theta_2 \) which exists due to the continuity of \( M_u(s, t) \) in \( B \). Additionally, by the almost surely continuity of random field, we have, for some constant \( b > 0 \)

\[\mathbb{P}\left(\sup_{t \in B \setminus \Delta_0} \frac{Z(s, t)}{G_u(s, t)}M_u > b\right) \leq \frac{1}{2}.
\]

Therefore, a direct application of the Borell inequality (e.g., Theorem D.1 of [19]) implies

\[\overline{\Pi}(u) = \mathbb{P}\left(\sup_{t \in B \setminus \Delta_0} \frac{Z(s, t)}{G_u(s, t)}M_u > M_u\right) \leq 2\Psi\left(\frac{M_u - b}{\rho(\theta_1, \theta_2)}\right) = o(\Pi(u)) \text{ as } u \to \infty.
\]

Consequently, Eq. (20) is established, and thus the proof is complete. \(\Box\)

**Proof of Theorem 2.5** For \( T \in (0, \infty) \), first note that

\[
P\left\{u^2(T - \tau(u)) > x | (\tau(u) \leq T)\right\} = \frac{P\left\{\sup_{0 \leq s \leq t \leq T} Z(s, t) > u\right\}}{P\left\{\sup_{0 \leq s \leq t \leq T} Z(s, t) > u\right\}}
\]
for any $x, u > 0$, where $T_x(u) = T - xu^{-2}$ and $Z(s, t)$ is the same as in (5). Next for any $x, u > 0$, we follow the similar argumentation as in the proof Theorem 2.1

$$
P \left\{ \sup_{0 \leq s \leq t \leq T_x(u)} Z(s, t) > u \right\} \sim \mathcal{P}^{\frac{1}{2} \sqrt{\delta u^2 + 2cu}} \lim_{S \to \infty} \mathcal{P}^1 \left[ \frac{\sigma^2 e^{-2ST}}{2a^2}, \frac{\sigma^2 e^{-2ST}}{2a^2} \right] \Psi \left( \frac{\sqrt{2}}{\sigma} \sqrt{\delta u^2 + 2cu} \right),$$

as $u \to \infty$. By Remark 2.3 of [4], we have

$$
\lim_{S \to \infty} \mathcal{P}^1 \left[ \frac{\sigma^2 e^{-2ST}}{2a^2}, \frac{\sigma^2 e^{-2ST}}{2a^2} \right] = \lim_{S \to \infty} \mathbb{E} \left\{ \sup_{t \in \left[ \frac{S}{2a^2}, \frac{S}{2a^2} \right]} e^{\mathcal{T}B(t)-2t} \right\}
= \lim_{S \to \infty} \mathbb{E} \left\{ \sup_{\left[ 0, \frac{2a^2}{S} \right]} e^{\mathcal{T}B(t)-2t} \right\}
= e^{-\frac{\sigma^2 T}{2a^2}} \lim_{S \to \infty} \mathbb{E} \left\{ \sup_{\left[ 0, \frac{2a^2}{S} \right]} e^{\mathcal{T}B(t)-2t} \right\}
= 2e^{-\frac{\sigma^2 T}{2a^2}}.
$$

Therefore, we conclude by an application of Theorem 2.1 and equation(26) that

$$
P \left\{ u^2(T - \tau_1(u)) > x | (\tau_1(u) \leq T) \right\} \to \exp \left( -\frac{\sigma^2 e^{-2ST}x}{2a^2} \right),
$$
as $u \to \infty$, for any $x > 0$.

For the case $T = \infty$, we have

$$
P \left\{ u^2 \left( e^{-2\delta t} - \left( \frac{c}{\delta u + c} \right)^2 \right) \leq x | \tau_u < \infty \right\} = \frac{\mathbb{P} \left\{ \sup_{0 < t \leq s \leq 1} Z(s, t) > 1 \right\}}{\mathbb{P} \left\{ \sup_{0 < t \leq s \leq 1, \frac{Z(s, t)}{G_u(s, t)} > 1 \right\}},
$$

where $Z(s, t)$ and $G_u(s, t)$ are the same as in (16) and (17). We follow the similar argumentation as in the proof Theorem 2.3

$$
P \left\{ \sup_{0 < t \leq s \leq 1, \frac{Z(s, t)}{G_u(s, t)} > 1 \right\} \sim \frac{1}{1 - \gamma} \mathcal{P}^{\frac{1}{2} \sqrt{\frac{\delta^2}{\sigma^2}} \left[ 0, x + \frac{c}{\sigma \sqrt{\delta}} \right]} \Psi \left( \frac{\sqrt{2}}{\sigma} \sqrt{\delta u^2 + 2cu} \right),$$
as $u \to \infty$. Thus we get the results by an application of Theorem 2.3.

\[ \square \]

4. APPENDIX

Here we give several Lemmas which are used in the proofs.
**Lemma 4.1.** The variance function $V_Z^2(s, t)$ in (6) attains its unique global maximum over set $B := \{(s, t): 0 \leq s \leq t < T\}$ at $(s_0, t_0)$, with $s_0 = 0$ and $t_0 = T$. Further $a^2 := V_Z^2(0, T) = \frac{\sigma^2}{2\delta}(1 - e^{-2\delta T})$.

**Proof of Lemma 4.1** It is obvious that $t_0 = T$, then
\[
\frac{\partial V_Z^2(s, T)}{\partial s} = \frac{\gamma \sigma^2 e^{-\delta s}}{(1 - \gamma + \gamma e^{-\delta s})^3} \left( (1 - e^{-2\delta T}) - \gamma(2 - \gamma)(1 - e^{-\delta s}) \right) - \frac{\gamma(2 - \gamma)\sigma^2 e^{-2\delta s}}{(1 - \gamma + \gamma e^{-\delta s})^2} \\
= \frac{\gamma \sigma^2 e^{-\delta s}}{(1 - \gamma + \gamma e^{-\delta s})^3} \left( (1 - e^{-2\delta T}) - \gamma(2 - \gamma) - (1 - \gamma)(2 - \gamma)e^{-\delta s} \right),
\]
we have when $\delta < 0$, $V_Z^2(s, T)$ attains the maximum only at $s = 0$ and when $\delta > 0$, $V_Z^2(s, T)$ attains the maximum only at $s = 0$ and $s = T$, since
\[
V_Z^2(0, T) = \frac{\sigma^2}{2\delta}(1 - e^{-2\delta T}), \quad V_Z^2(T, T) = \frac{\sigma^2(1 - \gamma)^2}{2\delta(1 - \gamma + \gamma e^{-\delta T})(1 - e^{-2\delta T})},
\]
and $\frac{(1 - \gamma)^2}{(1 - \gamma + \gamma e^{-\delta T})} < 1$, hence the claim follows. \(\square\)

**Lemma 4.2.** Let $F_u(s, t) = \frac{V_Z(s, t)}{G_u(s, t)}$ with $G_u(s, t)$ in (17) and $V_Z(s, t)$ in (18). Then for $u$ sufficiently large, the function $F_u(s, t), 0 < t \leq s \leq 1$ attains its maximum at the unique point $(s, t) = (1, t_u)$, where
\[
t_u = \left( \frac{c}{\delta u + c} \right)^2.
\]

**Proof of Lemma 4.2** Note that
\[
\frac{\partial V_Z^2(s, t)}{\partial t} = 2V_Z(s, t) \frac{\partial V_Z(s, t)}{\partial t} = \frac{\sigma^2}{2\delta}, \quad \frac{\partial V_Z^2(s, t)}{\partial s} = 2V_Z(s, t) \frac{\partial V_Z(s, t)}{\partial s} = \frac{\sigma^2}{2\delta}(2 - \gamma).
\]
Then we have
\[
\frac{\partial F_u(s, t)}{\partial t} = \frac{\partial V_Z(s, t)}{\partial t} \cdot \frac{1}{G_u(s, t)} - \frac{V_Z(s, t)}{G_u^2(s, t)} \left( -c \frac{t^{-\frac{1}{2}}}{2\delta} \right) \\
= \frac{1}{2G_u^2(s, t)V_Z(s, t)} \left( \frac{\partial V_Z^2(s, t)}{\partial t} \cdot G_u(s, t) + V_Z^2(s, t) \frac{ct^{-\frac{1}{2}}}{\delta} \right) \\
= \frac{\sigma^2 t^{-1/2}}{4G_u^2(s, t)V_Z(s, t)} \left\{ [(1 - \gamma)^2 + (2 - \gamma)\gamma s] \frac{c}{\delta} - (u + \frac{c}{\delta})(1 - \gamma + \gamma s^{\frac{1}{2}}) \right\}.
\]
So $t_u \to 0$, as $u \to \infty$.
\[
\frac{\partial F_u(s, t)}{\partial s} = \frac{\partial V_Z(s, t)}{\partial s} \cdot \frac{1}{G_u(s, t)} - \frac{V_Z(s, t)}{G_u^2(s, t)} \left( \frac{1}{2} \gamma u s^{-\frac{1}{2}} + \frac{c^2}{2\delta} s^{-\frac{3}{2}} \right) \\
= \frac{1}{2G_u^2(s, t)V_Z(s, t)} \left[ \frac{\partial V_Z^2(s, t)}{\partial s} \cdot G_u(s, t) - V_Z^2(s, t)(\gamma u s^{-\frac{1}{2}} + \frac{c^2}{\delta} s^{-\frac{3}{2}}) \right] \\
= \frac{\gamma \sigma^2 s^{-\frac{1}{2}}}{4G_u^2(s, t)V_Z(s, t)} \left( (2 - \gamma)(1 - \gamma)(u + \frac{c}{\delta}) - \frac{c}{\delta} t^{\frac{1}{2}}s^{\frac{1}{2}} - [(1 - \gamma)^2 - t](u + \frac{c}{\delta}) \right).
\]
so for $u$ large enough, $F_u(\cdot, t_u)$ only can reach its maximum only at $s = 1$ or $s = t_u$.

$$F_u(1, t) = \frac{\sqrt{\frac{\sigma^2}{2\gamma}(1-t)}}{u + \frac{\gamma}{2}(1-t^2)}, \quad F_u(t, t) = \frac{\sqrt{\frac{\sigma^2}{2\gamma}(1-t)}}{u + \frac{\gamma}{2}u t^2 + \frac{\gamma}{2}(1-t^2)} < F_u(1, t).$$

So $F_u(s, t)$ attains the maximum point only at $s = 1$. Let $\frac{\partial F_u(1, t)}{\partial t} = 0$, we get $t_u = \left(\frac{c}{\delta u + c}\right)^2$. □

Lemma 4.3. $\{\xi(s, t), s, t \geq 0\}$ is a zero-mean stationary Gaussian field with continuous sample paths and correlation function $\gamma_\xi(s, t) = \exp(-\gamma_1|s| + \gamma_2|t|)$ for some positive constants $\gamma_i$, $i = 1, 2$. Further, $\lim_{u \to \infty} t_u u^2 = c > 0$ and $\lim_{u \to \infty} \frac{f(u)}{u} = d > 0$. Then for some positive constants $S_i, i = 1, 2, b_i, i = 1, 2$,

$$\mathbb{P}\left\{\sup_{(s, t) \in [1-S_1 u^{-2}, 1] \times [0, (\sqrt{S_2 u^{-1} + \sqrt{t_u})^2]} \frac{\xi(s, t)}{1+b_1(1-s)+b_2(\sqrt{t}-\sqrt{t_u})^2} > f(u)\right\} \sim \mathcal{P}_{\xi}^{b_1, d_1, 2} \mathcal{D}^{b_2, c_2} \mathcal{C}^{0, a_2d^2(\sqrt{c} + \sqrt{S_2})^2} \Psi(f(u)).$$

where

$$\mathcal{P}_{\xi}^{b_1, d_1, 2} \mathcal{D}^{b_2, c_2} \mathcal{C}^{0, a_2d^2(\sqrt{c} + \sqrt{S_2})^2} := \mathbb{E}\left\{\sup_{t \in [0, a_2d^2(\sqrt{\tau+\sqrt{S_2})^2]} e^{\sqrt{2B(t)-t-\frac{b_2}{2}(\sqrt{\tau}-\sqrt{\tau a_2c})^2}}\right\}.$$

The proof of this lemma follows along the same lines of [10][Theorem 2.1].

Figure 1. Ruin times

Figure 1 shows the ruin time $\tau(u)$ of a surplus process $U_\gamma^\delta(t)$.

Acknowledgement: Thanks to Enkelejd Hashorva for his suggestions. Thanks to Swiss National Science Foundation Grant no. 200021-166274.
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