DRINFELD–SOKOLOV REDUCTION FOR QUANTUM GROUPS
AND DEFORMATIONS OF W–ALGEBRAS

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Abstract. We define deformations of W–algebras associated to complex semisimple Lie algebras by means of quantum Drinfeld–Sokolov reduction procedure for affine quantum groups. We also introduce Wakimoto modules for arbitrary affine quantum groups and construct free field resolutions and screening operators for the deformed W–algebras. We compare our results with earlier definitions of q-W–algebras and of the deformed screening operators due to Awata, Kubo, Odake, Shiraiishi, Feigin, E. Frenkel, and E. Frenkel, Reshetikhin. The screening operator and the free field resolution for the deformed W–algebra associated to the simple Lie algebra \( \mathfrak{sl}_2 \) coincide with those for the deformed Virasoro algebra introduced in [60].

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INTRODUCTION

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, \( \mathfrak{n}_+ \subset \mathfrak{g} \) a maximal nilpotent subalgebra, \( \chi_0 : \mathfrak{n}_+ \to \mathbb{C} \) a non-singular character of \( \mathfrak{n}_+ \). In paper [46] Kostant proved that the center \( Z(U(\mathfrak{g})) \) of the universal enveloping algebra \( U(\mathfrak{g}) \) is isomorphic to the algebra

\[
\text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes U(\mathfrak{n}_+) \mathbb{C}_{\chi_0}),
\]

where \( \mathbb{C}_{\chi_0} \) is the one–dimensional representation of the algebra \( U(\mathfrak{n}_+) \) corresponding to the character \( \chi_0 \).

Now let \( \hat{\mathfrak{g}}' = \mathfrak{g}[z, z^{-1}] + \mathbb{C}K \) be the nontwisted affine Lie algebra corresponding to \( \mathfrak{g} \). Let \( k \in \mathbb{C} \) be a complex number and denote by \( U(\hat{\mathfrak{g}}')_k \) the quotient of the universal enveloping algebra \( U(\hat{\mathfrak{g}}') \) by the two–sided ideal generated by \( K - k \), where \( K \) is the central element of \( \hat{\mathfrak{g}}' \). It is well known that for generic \( k \) the algebra \( U(\hat{\mathfrak{g}}')_k \) has trivial center, and hence the center \( Z(U(\mathfrak{g})) \) has no an affine counterpart. Remarkably, the algebra (1) has a nontrivial affine generalization \( W_k(\mathfrak{g}) \) called the W–algebra associated to the complex semisimple Lie algebra \( \mathfrak{g} \).

First examples of W–algebras were introduced in papers [17, 18, 19, 48] by Fateev, Lukyanov and Zamolodchikov. Later Feigin and E. Frenkel gave a more conceptual definition of W–algebras using the quantum Drinfeld–Sokolov reduction procedure (see [23, 24, 25]) and generalized the notion of W–algebras to the case of arbitrary nontwisted affine Lie algebras. In fact, algebra (1) and the W–algebra \( W_k(\mathfrak{g}) \) are particular examples of Hecke algebras introduced in [53, 58]. This allows to develop a unified approach to such algebras and to give an invariant functorial definition of W–algebras (see also [58]).

In the terminology introduced in [58] the algebra (1) is the Hecke algebra associated to the triple \( (U(\mathfrak{g}), U(\mathfrak{n}_+), \chi_0) \), and the W–algebra \( W_k(\mathfrak{g}) \) is the semi–infinite Hecke algebra associated to the triple \( (U(\hat{\mathfrak{g}}')_k, U(\mathfrak{n}_+[z, z^{-1}]), \chi) \), where \( \chi \) is a nontrivial character of the algebra \( U(\mathfrak{n}_+[z, z^{-1}]) \) (see Section 3.1 for the exact definition). In the simplest case \( \mathfrak{g} = \mathfrak{sl}_2 \) the algebra \( W_k(\mathfrak{g}) \) is isomorphic to the restricted completion of the quotient of the universal enveloping algebra of the Virasoro algebra by the two–sided ideal generated by \( C - (1 - 6 \frac{(k+1)^2}{k+2}) \), where \( C \) is the central element of the Virasoro algebra.

In [60] Shiraishi, Kubo, Awata and Odake introduced a deformation \( \text{Vir}_{k,h} \) of the Virasoro algebra. The definition of the deformed Virasoro algebra given in [60] was motivated by the theory of symmetric functions. Later a deformed analog for the algebra \( W_k(\mathfrak{sl}_N) \) was introduced by the same authors in [6, 7] and independently by Feigin and E. Frenkel in [22]. In [24] E. Frenkel and Reshetikhin introduced a deformation of the algebra \( W_k(\mathfrak{g}) \) for an arbitrary complex semisimple Lie algebra \( \mathfrak{g} \). In this paper we call the deformed W–algebras introduced in [60, 6, 7, 24] the q–W–algebras since, in fact, they are not deformations of the W–algebras \( W_k(\mathfrak{g}) \) but
quantizations of Poisson algebras of functions on certain reduced Poisson manifolds (see below).

Up to present the relation of the $q$-W–algebras to affine quantum groups was not clear. In this paper we give a definition of deformed W–algebras associated to complex semisimple Lie algebras by generalizing the quantum Drinfeld–Sokolov reduction procedure to the case of affine quantum groups. The $q$-Virasoro algebra $Vir_{k,h}$ is a subalgebra in the deformed W–algebra $W_{k,h}(sl_2)$ introduced in this paper. The same fact is certainly true for the $q$-W–algebra associated to $sl_N$ and the deformed W–algebra $W_{k,h}(sl_N)$ defined in this paper. However the relation of the general definition of $q$-W–algebras given in [34] to quantum groups is still not clear.

Now we make a few historical remarks on development of the W–algebra theory. First we note that the algebra $W_k(g)$ has a natural “quasiclassical” counterpart, the Poisson algebra $W(g)$. This algebra is the algebra of functions on a reduced space obtained by Hamiltonian reduction in the dual space ($\hat{g}'^*$) to the nontwisted affine Lie algebra $\hat{g}' = g[z,z^{-1}] + CK$, equipped with the standard Kirillov–Kostant Poisson structure, with respect to the restriction of the coadjoint action of the Lie group $\tilde{G}$ of the Lie algebra $\hat{g}'$ to the Lie group $\tilde{N}$ corresponding to the Lie subalgebra $n[z,z^{-1}] \subset \hat{g}'$. This coadjoint action is Hamiltonian and has a moment map $\mu : (\hat{g}')^* \to n[z,z^{-1}]^* = n_-[z,z^{-1}]$, where $n_-$ is the opposite nilpotent subalgebra in $g$, and we have identified the space $n_+[z,z^{-1}]$ with $n_-[z,z^{-1}]$ using the standard scalar product on $\hat{g}'$. The reduced space entering the definition of the algebra $W(g)$ corresponds to the value $f$ of the moment map $\mu$, where $f$ is a regular nilpotent element in $n_- \subset n_-[z,z^{-1}]$ regarded as a Lie subalgebra in $g$. The reduction procedure described above was introduced in [14] and is called now the Drinfeld–Sokolov reduction procedure.

The definition of this reduction procedure given in [14] was motivated by the study of certain hamiltonian integrable systems. In fact, important examples of these systems associated to the algebra $W(sl_N)$ as well as the algebra $W(sl_N)$ itself had been known before due to Adler, Gelfand and Dickey [1, 37] who studied some natural Poisson structures on the space of ordinary differential operators on the lines related to integrable nonlinear equations.

The quantum W–algebra $W_k(sl_N)$, $k \in \mathbb{C}$ was introduced in [48] using straightforward quantum extrapolation of the classical formula for the so–called Miura transform obtained in [14]. The Miura transform is the affine counterpart of the Harish–Chandra homomorphism. Unfortunately due to technical difficulties this approach to the definition of quantum W–algebras can not be applied in the general case.

The general definition of quantum W–algebras was given by Feigin and E. Frenkel in [23, 24, 25] by defining the quantum analog of the classical Drinfeld–Sokolov reduction procedure. Feigin and E. Frenkel used the quantum BRST reduction technique developed in [26].

The deformed analog $W_h(sl_N)$ for the algebra $W(sl_N)$ was obtained in [28] by studying the structure of the center of the affine quantum group $U_h(\hat{sl}_N)_k$ at the critical level $k = -h^\vee$ of the central charge (here $h^\vee$ is the dual Coxeter number of $g$). In papers [34, 52] a Poisson–Lie group analog for the Drinfeld–Sokolov reduction procedure was defined. Using the Drinfeld–Sokolov reduction procedure for Poisson–Lie groups one can define certain deformations of the Poisson algebras
$W(g)$. In case $g = \mathfrak{sl}_N$ the deformed Poisson algebra obtained by this procedure coincides with the Poisson algebra $W_h(\mathfrak{sl}_N)$ defined in [33].

The Poisson algebra $W_h(\mathfrak{sl}_N)$ is the quasiclassical limit of the $q$-$W$–algebra defined in [6, 7, 22] in case of $\mathfrak{sl}_N$. Since quantum groups are certain quantizations of Poisson–Lie groups it was natural to believe that the deformed $W$–algebras defined in [6, 7, 34, 60] may be obtained by quantizing the Poisson–Lie group analog of the Drinfeld–Sokolov reduction procedure. An alternative definition of the Drinfeld–Sokolov reduction for Poisson–Lie groups suitable for quantization was given in [57].

The particular construction of the quantum BRST complex (see [47]) used in [23, 24, 25] for the quantum Drinfeld–Sokolov reduction may be only applied in the Lie algebra case, and the generalization of the notion of the quantum Drinfeld–Sokolov reduction to quantum groups requires a more complicated technique. In papers [53, 58] the author developed the general theory of Hecke algebras, a deep generalization of the classical notion of Hecke–Iwahori algebras and of the algebraic BRST reduction technique for Lie algebras (see [17]).

There is another obstruction for direct generalization of the quantum Drinfeld–Sokolov reduction to the case of quantum groups. The problem is that the natural quantum group counterpart of the algebra $U(n_+[z, z^{-1}])$ has no nontrivial characters. In paper [56], motivated by the quasiclassical picture presented in [52], the author introduced other quantum group counterparts of the algebra $U(n_+[z, z^{-1}])$ having nontrivial characters.

In this paper we use the results of [56, 58] to define the deformed $W$–algebras $W_{k,h}(g)$. The paper is organized as follows.

The definition of the deformed $W$–algebras and the study of the properties of these algebras require an extended algebraic background including the semi–infinite cohomology theory and the theory of Verma and Wakimoto modules over affine Lie algebras and quantum groups. In Section 1 we recall general facts on graded algebras, their representations and semi–infinite cohomology for these algebras including semi–infinite Hecke algebras. Using semi–infinite induction procedure we also give the algebraic definition of Wakimoto modules over graded algebras (see Section 1.7).

The material presented in Section 2 on affine Lie algebras and their representation is essentially standard. We only mention that we use the algebraic definition of Wakimoto modules given in [64]. In Section 2.3 we also study in detail some particular properties of Wakimoto modules that are important for the theory of $W$–algebras.

In Section 3 we recall the Hecke algebra definition of $W$–algebras (see [58]). Using purely algebraic approach we also construct the resolution of the vacuum representation for the $W$–algebra $W_k(g)$ defined in [23, 24, 25] and explicitly calculate the differential in this resolution in case $g = \mathfrak{sl}_2$. In the form presented in Sections 3.2 and 3.3 these results are easy to generalize to the deformed case.

In Section 4 we recall some facts about quantum groups and their representations. We also introduce Wakimoto modules over arbitrary affine quantum groups. We prove that in case of affine quantum group $U_h(\hat{\mathfrak{sl}}_2)$ our definition agrees with the explicit bosonic realization of Wakimoto modules (see [59, 3]) for some special set of highest weights.
In Section 5 we define deformations of W–algebras and study properties of the deformed W–algebras. This section is organized similarly to Section 3 except for Section 5.1 where we recall the definition of Coxeter realizations for affine quantum groups and of the quantum group counterparts of the algebra \( U(n[z, z^{-1}]) \) having nontrivial characters.

In conclusion we note that the analog of algebra (1) for finite–dimensional quantum groups was introduced in [54, 55].

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1. Generalities on graded associative algebras

1.1. A class of graded associative algebras. In this paper we consider a class of \( \mathbb{Z} \)–graded associative algebras with unit over a ring \( k \) with unit. All modules and algebras over \( k \) considered in this paper are supposed to be \( k \)–free. Let \( A \) be such an algebra,

\[
A = \bigoplus_{n \in \mathbb{Z}} A_n.
\]

The category of left (right) \( \mathbb{Z} \)–graded modules over \( A \) with morphisms being homomorphisms of \( A \)–modules preserving gradings is denoted by \( A \)–mod (mod \( A \)). For both of these categories the set of morphisms between two objects is denoted by \( \text{Hom}_A(\cdot, \cdot) \). For \( M, M' \in \text{Ob} A \)–mod (Ob mod \( A \)) we shall also frequently use the space of homomorphisms of all possible degrees with respect to the gradings on \( M \) and \( M' \) introduced by

\[
\text{hom}_A(M, M') = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A(M, M'_{\langle n \rangle}),
\]

where the module \( M'_{\langle n \rangle} \) is obtained from \( M' \) by grading shift as follows:

\[
M'_{\langle n \rangle}_k = M'_{k+n}.
\]

In this paper we shall deal with the full subcategory of \( A \)–mod (mod \( A \)) whose objects are modules \( M \in \text{Ob} A \)–mod (Ob mod \( A \)) such that their gradings are bounded from above, i.e.

\[
M = \bigoplus_{n \leq K(M)} M_n, \quad K(M) \in \mathbb{Z}.
\]

This subcategory is denoted by \((A \text{–mod})_0 ((\text{mod} – A)_0)\). We also denote by \( \text{Vect}_k \) the category of \( \mathbb{Z} \)–graded vector spaces over \( k \).

All tensor products of graded \( A \)–modules and graded vector spaces will be understood in the graded sense.

Lemma 1.1.1. ([58], Lemma 2.1.1) Let \( M \) and \( M' \) be two objects of the category \( \text{Vect}_k \) such that \( M = \bigoplus_{n \leq K} M_n, \quad M' = \bigoplus_{n \geq L} M'_n, \quad L \in \mathbb{Z}, \) and for every \( n \) \( \dim M'_n < \infty \). Then

\[
\text{hom}_k(M', M) = M'^* \otimes M, \quad \text{where } M'^* = \text{hom}_k(M', k).
\]
We shall also suppose that the algebra $A$ satisfies the following conditions:

(i) $A$ contains two graded subalgebras $N^+$ and $B^-$ such that $N^+ \subset \bigoplus_{n \geq 0} A_n$, $B^- \subset \bigoplus_{n \leq 0} A_n$.

(ii) $N^+_0 = k$.

(iii) $\dim N^+_n < \infty$ for any $n > 0$.

In particular, $N^+$ is naturally augmented. We denote the augmentation ideal $\bigoplus_{n>0} N^+_n$ by $N^+$. 

(iv) The multiplication in $A$ defines isomorphisms of graded vector spaces

$$B^- \otimes N^+ \rightarrow A \quad \text{and} \quad N^+ \otimes B^- \rightarrow A.$$ 

(v) The mappings (1.1.3) are continuous in the following sense: for every $m, n \in \mathbb{Z}$ there exist $k_+, k_- \in \mathbb{Z}$ such that\n
$$N^+_m \otimes B^-_n \rightarrow \bigoplus_{k_- \leq k \leq k_+} B^-_{n-k} \otimes N^+_{m+k} \quad \text{and} \quad B^-_n \otimes N^+_m \rightarrow \bigoplus_{k_- \leq k \leq k_+} N^+_{m-k} \otimes B^-_{n+k}.$$ 

We shall also need a certain completion $\widehat{A}$ of the algebra $A$ called the restricted completion. The restricted completion may be defined as follows. For any homogeneous component $A_n$ of the algebra $A$ let $A_{n,k}$ be the subspace given by

$$A_{n,k} = \sum_{i=0}^{\infty} A_{n-k-i} N^+_{k+i}.$$ 

The set of subspaces $\{A_{n,k}, k \geq 0\}$ can be regarded as a family of neighborhoods of 0 in some topology on $A_n$. This gives rise to a topology on $A$. Clearly,

$$A_{n,k} A_{m,l} \subset A_{m+n,l},$$

and hence the multiplication on $A$ is continuous in this topology. Therefore the completion $\widehat{A}$ of $A$ with respect to this topology is an associative algebra called the restricted completion of $A$.

An important property of the algebra $\widehat{A}$ is that for every left(right) $A$–module $M \in (A – \text{mod})_0((\text{mod} – A)_0)$ the action of $A$ on $M$ may be uniquely extended to an action of $\widehat{A}$.

1.2. Semiregular bimodule. In this section we recall the definition of the semiregular bimodule for the algebra $A$. The semiregular bimodule plays an important role in the semi–infinite cohomology theory. This module is also a basic ingredient for the algebraic definition of Wakimoto modules.
The notion of the semiregular bimodule was introduced by Voronov in [62] (see also [61]) in the Lie algebra case and generalized in [2] to the case of graded associative algebras satisfying conditions (i)-(v) of Section 1.1.

First consider the left graded $N^+\cdot N^{+*} = \text{hom}_k(N^+, k)$, where the action of $N^+$ on $N^{+*}$ is defined by

$$(n \cdot f)(n') = f(n'n) \text{ for any } f \in N^{+*}, \; n \in N^+.$$ 

The left $A$–module $S_A = A \otimes_{N^+} N^{+*}$

is called the left semiregular representation of $A$ (see [62], Sect 3.2; [2], Sect. 3.4).

Clearly that $S_A = B^- \otimes_{N^{+*}}$ as a left $B^-$-module. The space $S_A = B^- \otimes_{N^{+*}}$ is non–positively graded, and hence $S_A \in (A^- \text{-}\text{mod})_0$.

Now we obtain another realization for the left semiregular representation. Consider another left $A$-module $S'_A = \text{hom}_{B^-}(A, B^-)$, where $B^-$ acts on $A$ by left multiplication. The left action of $A$ on the space $S'_A$ is given by

$$(a \cdot f)(a') = f(a'a), \; f \in \text{hom}_{B^-}(A, B^-), \; a \in A.$$ 

**Lemma 1.2.1.** ([2], Lemma 3.5.1) Let

$$(1.2.1) \quad A = B^- \otimes N^+$$

be the decomposition provided by the multiplication in $A$. Let $\phi : S_A \to S'_A$ be the map defined by

$$\phi(a \otimes f)(a') = (a'a)_{B^-} f((a'a)_{N^+}),$$

where $f \otimes a \in S_A, \; a' \in A$ and $a'a = (a'a)_{B^-} ((a'a)_{N^+}$ is the decomposition (1.2.1) of the element $a'a$. Then $\phi$ is a homomorphism of left $A$–modules.

We shall suppose that the algebra $A$ satisfies the following additional condition:

(vi) The homomorphism $\phi : S_A \to S'_A$ constructed in the previous lemma is an isomorphism of left $A$–modules.

Finally we have two realizations of the left $A$–module $S_A$:

$$(1.2.2) \quad S_A = A \otimes_{N^+} N^{+*},$$

and

$$(1.2.3) \quad S_A = \text{hom}_{B^-}(A, B^-).$$

Now we define a structure of a right module on $S_A$ commuting with the left semiregular action of $A$. First observe that using realizations (1.2.2) and (1.2.3) of the left semiregular representation one can define natural right actions of the algebras $N^+$ and $B^-$ on the space $S_A$ induced by the natural right action of $N^+$ on $N^{+*}$ induced by multiplication in $N^+$ from the left and the right regular representation of $B^-$, respectively. Clearly, these actions commute with the left action of the algebra $A$ on $S_A$. Therefore we have natural inclusions of algebras

$$N^+ \hookrightarrow \text{hom}_A(S_A, S_A), \; B^- \hookrightarrow \text{hom}_A(S_A, S_A).$$

Denote by $A^2$ the subalgebra in $\text{hom}_A(S_A, S_A)$ generated by $N^+$ and $B^-$. 
Proposition 1.2.2. ([2], Corollary 3.3.3, Lemma 3.5.3 and Corollary 3.5.3) $A^\sharp$ is a $\mathbb{Z}$–graded associative algebra satisfying conditions (i)–(v) of Section 1.1. Moreover, $S_A \in (\text{mod} - A^\sharp)_0$ and
\begin{equation}
S_A = N^{+\ast} \otimes_{N^{+}} A^\sharp = \text{hom}_{B^-}(A^\sharp, B^-)
\end{equation}
as a right $A^\sharp$–module.

Using Proposition 1.2.2 the space $S_A$ is equipped with the structure of an $A - A^\sharp$ bimodule. This bimodule is called the semiregular bimodule associated to the algebra $A$. The right action of the algebra $A^\sharp$ on the space $S_A$ is called the right semiregular action.

1.3. Semiproduct. In this section, following [58], we recall the definition and properties of the functor of semiproduct. This functor is a generalization of the functor of semivariants (see [62], Sect. 3.8) to the case of associative algebras. The semi–infinite Tor functor, that we use in this paper, is a two–sided derived functor of the functor of semiproduct.

Let $M \in A - \text{mod}$ be a left graded $A$–module and $M' \in \text{mod} - A^\sharp$ a right graded $A^\sharp$–module. Consider the subspace $M' \otimes_{N^+} M$ in the tensor product $M' \otimes M$ defined by
$$M' \otimes_{N^+} M = \{m' \otimes m \in M' \otimes M : m'n \otimes m = m' \otimes nm' \text{ for every } n \in N^+\}.$$

The semiproduct $M' \otimes_{B^-} M$ of modules $M \in A - \text{mod}$ and $M' \in \text{mod} - A^\sharp$ is the image of the subspace $M' \otimes_{N^+} M \subset M' \otimes M$ under the canonical projection $M' \otimes M \to M' \otimes_{B^-} M$,
\begin{equation}
M' \otimes_{B^-} M = \text{Im}(M' \otimes_{N^+} M \to M' \otimes_{B^-} M).
\end{equation}

Thus the semiproduct $\otimes_{B^-}^N$ is a mixture of the tensor product $\otimes_{B^-}$ over $B^-$ and of the functor $\otimes_{N^+}$ of “$N^+$–invariants”. However the following lemma shows that properties of the semiproduct are rather closely related to those of the usual tensor product.

Lemma 1.3.1. ([58], Lemma 2.3.1) Let $M \in (A - \text{mod})_0$ be a left graded $A$–module, $M' \in (\text{mod} - A^\sharp)_0$ a right graded $A^\sharp$–module and $S_A$ the semiregular bimodule associated to $A$. Then
$$S_A \otimes_{B^-}^N M = M$$
as a left $A$–module, and
$$M' \otimes_{B^-}^N S_A = M$$
as a right $A^\sharp$–module.

In conclusion we remark that the semiproduct of modules naturally extends to a functor $\otimes_{B^-}^N : (\text{mod} - A^\sharp) \times (A - \text{mod}) \to \text{Vect}_k$. 

1.4. Semi–infinite cohomology. In this section we recall, following [32, 58], the definition of the semi–infinite Tor functor for associative algebras. This functor is a derived functor of the functor of semiproduct with respect to a certain class of adapted objects called semijective complexes.

First we formulate the main theorem of semi–infinite homological algebra for an arbitrary associative algebra \( \mathcal{A} \) containing subalgebra \( \mathcal{N} \). We start by recalling the definition of semijective complexes (see [32, Definition 3.3]).

Let \( \mathcal{O}(\mathcal{A}) \) be a full subcategory in the category of left (or right) \( \mathcal{A} \)–modules. Denote by \( \text{Kom}(\mathcal{O}(\mathcal{A})), K(\mathcal{O}(\mathcal{A})) \) and \( D(\mathcal{O}(\mathcal{A})) \) the category of complexes over \( \mathcal{O}(\mathcal{A}) \), the corresponding homotopy and derived category, respectively. We also denote by \( \text{Kom}(\mathcal{N}), K(\mathcal{N}) \) and \( D(\mathcal{N}) \) the category of complexes over the category of \( \mathcal{N} \)–modules \( \mathcal{N} \) – mod, the corresponding homotopy and derived category, respectively.

A complex \( S^\bullet \in \text{Kom}(\mathcal{O}(\mathcal{A})) \) is called semijective (with respect to the subalgebra \( \mathcal{N} \)) if

1. \( S^\bullet \) is \( K \)-injective as a complex of \( \mathcal{N} \)–modules, i.e., for every acyclic complex \( A^\bullet \in \text{Kom}(\mathcal{N} \text{– mod}), \text{Hom}_{K(\mathcal{N})}(A^\bullet, S^\bullet) = 0; \)

2. \( S^\bullet \) is \( K \)-projective relative to \( \mathcal{N} \), i.e., for every complex \( A^\bullet \in \text{Kom}(\mathcal{O}(\mathcal{A})), \text{such that } A^\bullet \text{ is isomorphic to zero in the category } K(\mathcal{N}), \text{Hom}_{K(\mathcal{O}(\mathcal{A}))}(S^\bullet, A^\bullet) = 0. \)

An \( \mathcal{A} \)–module \( M \in \mathcal{O}(\mathcal{A}) \) is called semijective if the corresponding 0–complex \( \ldots \to 0 \to M \to 0 \to \ldots \) is semijective. We also say that \( M \) is projective relative to \( \mathcal{N} \) if the corresponding 0–complex is \( K \)-projective relative to \( \mathcal{N} \). For the 0–complex \( \ldots \to 0 \to M \to 0 \to \ldots \) condition 1 of the definition of semijective complexes is equivalent to the usual \( \mathcal{N} \)-injectivity of \( M \).

In this paper we shall actually deal with a class of relatively to \( \mathcal{N} \) projective modules described in the next lemma (see [52, Sect. 3.1]).

Lemma 1.4.1. ([58, Lemma 2.4.2]) Every left \( \mathcal{A} \)–module \( M \in \mathcal{O}(\mathcal{A}) \) induced from an \( \mathcal{N} \)–module \( V \in (\mathcal{N} \text{– mod}), M = \mathcal{A} \otimes_{\mathcal{N}} V, \) is projective relative to \( \mathcal{N} \).

The following fundamental property of the semiregular bimodule \( S_A \) together with Lemma 1.3.1 shows that \( S_A \) is an analogue of the regular representation in semi–infinite homological algebra.

Proposition 1.4.2. ([58, Proposition 2.4.3]) Let \( \mathcal{A} \) be an associative \( \mathbb{Z} \)–graded algebra over a ring \( k \) with unit satisfying conditions (i)–(vi) of Sections 1.3, and 1.4. Then the semiregular bimodule \( S_A \) is semijective, with respect to the subalgebra \( \mathcal{N}^+, \) as a left \( \mathcal{A} \)–module and a right \( \mathcal{A}^+ \)–module.

The main difficulty in dealing with semijective complexes is that in general position the complex of semijective modules is not semijective. However in some particular cases described in the next proposition \( K \)–injectivity (\( K \)-projectivity relative to \( \mathcal{N} \) or semijectivity) of the complex follows from the corresponding property of the individual terms of this complex.

Proposition 1.4.3. ([32, Proposition 3.7])
1. Any complex \( S^\bullet \in \text{Kom}(\mathcal{O}(\mathcal{A})) \) of \( \mathcal{N} \)-injective modules bounded from below is \( K \)-injective as a complex of \( \mathcal{N} \)–modules.
2. Any complex \( S^\bullet \in \text{Kom}(\mathcal{O}(\mathcal{A})) \) of projective relative to \( \mathcal{N} \) modules bounded from above is \( K \)-projective relative to \( \mathcal{N} \).
3. Any bounded complex \( S^\bullet \in \text{Kom}(\mathcal{O}(\mathcal{A})) \) of semijective modules is semijective.
The definition of the semijective resolution of the complex is also different from the usual one. In general position the complex of left $A$–modules from the category $\text{Kom}(\mathcal{O}(A))$ is not quasiisomorphic to a semijective complex. However one can establish such an isomorphism in the corresponding derived category. This isomorphism is provided by the main theorem of semi–infinite homological algebra.

In order to formulate this theorem we recall that an epimorphism of $A$–modules is called strong if it is split as an epimorphism of $N$–modules. An $A$–module $M$ is called a strong quotient of a projective relative to $N$ if there exists a strong $A$–module epimorphism $P \to M$.

**Theorem 1.4.4.** ([62], Theorem 3.3) Let $A$ be an arbitrary associative algebra containing subalgebra $N$ and let $\mathcal{O}(A)$ be a full subcategory in the category of left (right) $A$–modules. Denote by $\text{Kom}(\mathcal{S}_J(A))$ the category of semijective complexes, with respect to the subalgebra $N$, associated to the abelian category $\mathcal{O}(A)$, and by $\mathcal{K}(\mathcal{S}_J(A))$ the corresponding homotopy category. Suppose that every $A$–module $M \in \mathcal{O}(A)$ is a submodule of an $N$–injective module $M' \in \mathcal{O}(A)$ and a strong quotient of a relative to $N$ projective $A$–module $P \in \mathcal{O}(A)$. Then the functor of localization by the class of quasi–isomorphisms is an equivalence of categories:

$$\mathcal{K}(\mathcal{S}_J(A)) \cong \mathcal{D}(\mathcal{O}(A)).$$

In particular, we have the following important corollary of Theorem 1.4.4.

**Corollary 1.4.5.** ([62], Theorem 3.2) Suppose that the conditions of Theorem 1.4.4 for the algebra $A$ and the category $\mathcal{O}(A)$ are satisfied. Then for every complex $K \in \text{Kom}(\mathcal{O}(A))$ there exists an isomorphism $S \to K$ in the derived category $\mathcal{D}(\mathcal{O}(A))$, where $S \in \text{Kom}(\mathcal{O}(A))$ is a semijective complex. The complex $S$ is called a semijective resolution of $K$.

Properties of semijective resolutions are summarized in the following proposition that is also a corollary of Theorem 1.4.4.

**Proposition 1.4.6.** ([62], Corollaries 3.1 and 3.2) Suppose that the conditions of Theorem 1.4.4 for the algebra $A$ and the category $\mathcal{O}(A)$ are satisfied and let $\phi : K \to L$ be a morphism in $\mathcal{D}(\mathcal{O}(A))$, and $S_\bullet$, $S'_\bullet$ semijective resolutions of $K_\bullet$ and $L_\bullet$, respectively. Then there exists a morphism of complexes $\phi^\bullet : S_\bullet \to S'_\bullet$ in the category $\text{Kom}(\mathcal{O}(A))$ such that the square

$$\begin{array}{ccc}
S_\bullet & \longrightarrow & K_\bullet \\
\downarrow \phi & & \downarrow \phi \\
S'_\bullet & \longrightarrow & L_\bullet
\end{array}$$

is commutative in $\mathcal{D}(\mathcal{O}(A))$. This morphism is unique up to a homotopy.

In particular, any two semijective resolutions of a complex $K_\bullet$ are homotopically equivalent. This equivalence is unique up to a homotopy.

**Corollary 1.4.7.** ([62], Corollary 3.3) Suppose that the conditions of Theorem 1.4.4 for the algebra $A$ and the category $\mathcal{O}(A)$ are satisfied. Then each acyclic semijective complex from the category $\text{Kom}(\mathcal{S}_J(A))$ is homotopic to zero.

By definition a semijective resolution of a left $A$–module $M \in \mathcal{O}(A)$ is a semijective resolution of the corresponding 0–complex $\ldots \to 0 \to M \to 0 \to \ldots$. Next we formulate properties of semijective resolutions of left $A$–modules.
Proposition 1.4.8. (**[62], Corollaries 3.1 and 3.2**) Suppose that the conditions of Theorem 1.4.4 for the algebra \(A\) and the category \(O(A)\) are satisfied. Then
(a) Any left \(A\)-module \(M \in O(A)\) has a semijective resolution.
(b) Any morphism of \(A\)-modules \(M, M' \in O(A)\) \(\phi : M \to M'\) gives rise to a morphism (in the category \(\text{Kom}(O(A))\)) of their semijective resolutions \(\phi^\bullet : S^\bullet \to S'^\bullet\) that is unique up to a homotopy.
(c) In particular, any two semijective resolutions of a module \(M \in O(A)\) are homotopically equivalent. This equivalence is unique up to a homotopy.

Now we suppose that the algebra \(A\) satisfies conditions (i)–(vi) of Sections 1.1 and 1.2. In order to define the semi–infinite Tor functor for \(A\) we shall apply Theorem 1.4.4 to the algebra \(A\), the subalgebra \(N = N^+\) and the subcategory \(O(A) = (A - \text{mod})_0((\text{mod} - A)_0)\) of the category of left (right) \(A\)-modules.

Proposition 1.4.9. Let \(A\) be a \(\mathbb{Z}\)-graded associative algebra satisfying conditions (i) and (iv) of Section 1.1. Then Theorem 1.4.4 holds for the algebra \(A\), the subalgebra \(N = N^+\) and the subcategory \(O(A) = (A - \text{mod})_0((\text{mod} - A)_0)\) of the category of left (right) \(A\)-modules.

Proof. We verify that the conditions of Theorem 1.4.4 are satisfied. Indeed, every module \(M \in (A - \text{mod})_0\) is a submodule of the \(N^+\)-injective module
\[(1.4.1) \quad M' = \text{hom}_{B^-}(A, M),\]
the embedding is given by
\[1 : M \to \text{hom}_{B^-}(A, M),\]
\[1(m)(a) = am, \quad m \in M, \quad a \in A.\]

\(M'\) is \(N^+\)-injective and belongs to the category \((A - \text{mod})_0\) since \(M' = \text{hom}_k(N^+, M)\) as a left \(N^+\) module.

Every module \(M \in (A - \text{mod})_0\) is also a strong quotient of the relative to \(N^+\) projective module
\[(1.4.2) \quad P = A \otimes_{N^+} M,\]
the projection is given by
\[p : A \otimes_{N^+} M \to M,\]
\[p(a \otimes m) = am, \quad m \in M, \quad a \in A,\]
and the \(N^+\)-splitting of this projection is given by
\[(1.4.3) \quad s : M \to A \otimes_{N^+} M,\]
\[s(m) = 1 \otimes m, \quad m \in M.\]

By Lemma 1.4.1 \(P\) is relative to \(N^+\) projective. \(P\) also belongs to the category \((A - \text{mod})_0\) since \(P = B^- \otimes M\) as a left \(B^-\) module.

We define the semi–infinite Tor functor on modules \(M \in (A - \text{mod})_0\), \(M' \in (\text{mod} - A^\sharp)_0\) as the cohomology space of the complex \(S^\bullet(M') \otimes_{B^-} S^\bullet(M),\)
\[
\text{Tor}_A^{N^+}(M', M) = H^\bullet(S^\bullet(M') \otimes_{B^-} S^\bullet(M)),
\]

\[\text{Tor}_A^{N^+}(M', M) = H^\bullet(S^\bullet(M') \otimes_{B^-} S^\bullet(M)),\]
where \( S^\bullet(M), S^\bullet(M') \) are semijective resolutions of \( M \) and \( M' \). By Propositions 1.4.3 and 1.4.8, the space \( \text{Tor}^\bullet_A(M', M) \in \text{Kom(Vect}_k) \) does not depend on the resolutions \( S^\bullet(M), S^\bullet(M') \).

Using Proposition 1.4.8, \( \text{Tor}^\bullet_A(M', M) \) naturally extends to a functor

\[
\text{Tor}^\bullet_A: (\text{mod} - A^\dagger)_0 \times (A - \text{mod})_0 \to \text{Kom(Vect}_k).
\]

The following important theorem is a semi–infinite analogue of the classical theorem about partial derived functors.

**Theorem 1.4.10.** ([58], Theorem 2.5.1) The following three definitions of the spaces \( \text{Tor}^\bullet_A(M', M) \) in \( \text{Kom(Vect}_k) \) are equivalent:

(a) \( \text{Tor}^\bullet_A(M', M) = H^\bullet(S^\bullet(M') \otimes_{B^+} S^\bullet(M)) \);

(b) \( \text{Tor}^\bullet_A(M', M) = H^\bullet(M' \otimes_{B^+} S^\bullet(M)) \);

(c) \( \text{Tor}^\bullet_A(M', M) = H^\bullet(S^\bullet(M') \otimes_{B^-} M) \),

where \( M \in (A - \text{mod})_0, M' \in (\text{mod} - A^\dagger)_0 \), and \( S^\bullet(M), S^\bullet(M') \) are semijective resolutions of \( M \) and \( M' \), respectively.

**Corollary 1.4.11.** ([58], Corollary 2.5.2) Suppose that one of modules \( M \in (A - \text{mod})_0, M' \in (\text{mod} - A^\dagger)_0 \) is semijective. Then

\[
\text{Tor}^\bullet_A(M', M) = M' \otimes_{B^-} M.
\]

Now we recall the definitions of standard semijective resolutions for calculation of the semi–infinite Tor functor. We start by recalling the definition of the standard (normalized) relative bar resolution (see [58], Appendix C and [2], Sect. 2.2).

Let \( B \subset A \) be an arbitrary subalgebra in \( A \). The standard bar resolution \( \overline{\text{Bar}}(A, B, M) \) of a left \( A \)-module \( M \) with respect to the subalgebra \( B \subset A \) is defined as follows:

\[
\overline{\text{Bar}}^{-n}(A, B, M) = A \otimes_B \ldots \otimes_B A \otimes_B M, \quad n \geq 0,
\]

\[
d(a_0 \otimes \ldots \otimes a_n \otimes v) = \\
\sum_{s=0}^{n-1} (-1)^s a_0 \otimes \ldots \otimes a_s a_{s+1} \otimes \ldots \otimes v + \\
+(-1)^n a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n v,
\]

where \( a_0, \ldots, a_n \in A, v \in M \).

In order to define the standard normalized relative bar resolution one needs the following simple lemma.

**Lemma 1.4.12.** ([2], Lemma 2.2.1) The subspace \( \overline{\text{Bar}}^\bullet(A, B, M) \),

\[
\overline{\text{Bar}}^{-n}(A, B, M) = \\
\{ a_0 \otimes \ldots \otimes a_n \otimes v \in \overline{\text{Bar}}^{-n}(A, B, M) \mid \exists s \in \{1, \ldots, n\} : a_s \in B \}
\]

is a subcomplex in \( \overline{\text{Bar}}^\bullet(A, B, M) \).
The quotient complex $\text{Bar}^*(A, B, M) = \frac{\text{Bar}^-(A, B, M)}{\text{Bar}^+(A, B, M)}$ is called the normalized bar resolution of the $A$–module $M$ with respect to the subalgebra $B$.

Now the standard semijective resolutions of modules are defined as follows. First, to any two complexes $X^*, Y^* \in \text{Kom} (\mod - A^*)_0$ we associate a complex $\text{hom}^n_{A^*}(X^*, Y^*)$,

$$\text{hom}^n_{A^*}(X^*, Y^*) = \bigoplus_{n \in \mathbb{Z}} \text{hom}^n_{A^*}(X^*, Y^*),$$

$$\text{hom}^n_{A^*}(X^*, Y^*) = \prod_{n \in \mathbb{Z}} \text{hom}_{A^*}(X^p, Y^{p+n})$$

with the differential given by

(1.4.5) 

$$df = dy \cdot f - (-1)^n f \circ dx \cdot, \quad f \in \text{hom}^n_{A^*}(X^*, Y^*) .$$

**Proposition 1.4.13.** ([58], Proposition 2.6.3) Let $M \in (\mod - A^*)_0$ be a right $A^*$-module. Then the complex $\text{Bar}^{\pm*}(A^*, N^+, M)$ defined by

$$\text{Bar}^{\pm*}(A^*, N^+, M) = \text{hom}_{A^*}^{\pm*} (\text{Bar}^*(A^*, B^-, A^2), M) \otimes_{A^*} \text{Bar}^{\pm*}(A^*, A^2, N^+, A^1)$$

is a semijective resolution of $M$ with respect to $N^+$.

**Proposition 1.4.14.** ([58], Proposition 2.6.4) Let $M \in (A \mod)_0$ be a left $A$-module. Then the complex $\text{Bar}^{\pm*}(A, N^+, M)$ defined by

$$\text{Bar}^{\pm*}(A, N^+, M) = \text{Bar}^{\pm*}(A, A^2, N^+, S_A) \otimes_{B^0} N^+_M$$

is a semijective resolution of $M$ with respect to $N^+$.

### 1.5. Semi-infinite Hecke algebras

In this section we recall, following [58], the definition and properties of semi–infinite Hecke algebras. These algebras play the key role in this paper. Let $A$ be an associative $\mathbb{Z}$ graded algebra over a ring $k$. Suppose that the restricted completion of the algebra $A$ contains a graded subalgebra $A_0$, and both $A$ and $A_0$ satisfy conditions (i)–(vi) of Sections 1.1 and 1.2. We denote by $N^+, B^-$ and $N^+_0$, $B_0^-$ the graded subalgebras in $A$ and $A_0$, respectively, providing the triangular decompositions of these algebras (see condition (iv) of Section 1.1).

Denote by $S - \text{Ind}^A_{A_0}$ the functor of semi-infinite induction

$$S - \text{Ind}^A_{A_0^+} : (\mod - A_0^+)_0 \to (\mod - A^+)_0$$

defined on objects by

$$S - \text{Ind}^A_{A_0^+} (V) = V \otimes_{B_0^+} S_A, \quad V \in (\mod - A_0^+)_0,$$

the structure of a right $A^*$-module on $V \otimes_{B_0^+} S_A$ being induced by the right semiregular action of $A^*$ on $S_A$. In the Lie algebra case this functor was introduced in [64].

One can introduce the derived functor of the functor of semi-infinite induction defined on objects $V^* \in D(\mod - A_0^+_0)$ by $(S - \text{Ind}^A_{A_0})^D (V^*) = S^* \otimes_{B_0^+} S_A$, where $S^*$ is a semijective resolution of the complex $V^*$ (see [58], Section 3.1 for details).

Now assume that the algebra $A_0^+$ is augmented, i.e. we have a character $\varepsilon : A_0^+ \to k$. Denote the corresponding one–dimensional $A_0^+$–module by $k$.
Let \( \text{hom}^\bullet_{D(\text{mod} - A^t)} \) be the double graded Hom in the derived category \( D(\text{mod} - A^t) \) introduced by

\[
\text{hom}^\bullet_{D(\text{mod} - A^t)}(X^\bullet, Y^\bullet) = \bigoplus_{m, n \in \mathbb{Z}} \text{Hom}_{D(\text{mod} - A^t)}(X^\bullet, Y[n](m)^\bullet),
\]

where the complex \( Y[n](m)^\bullet \) is defined by

\[
Y[n](m)^k = Y_{m+n}^k, \quad d_{Y[n](m)^\bullet} = (-1)^n d_{Y^\bullet}.
\]

We shall also use the space \( \text{hom}^\bullet_{K(\text{mod} - A^t)}(X^\bullet, Y^\bullet) \) defined in a similar way.

**Definition 1.** The \( Z^2 \)-graded algebra

\[
(1.5.1) \quad H^\oplus_{\text{mod} - A^t}(A, A_0, \varepsilon) = \text{hom}^\bullet_{D(\text{mod} - A^t)_0}((S - \text{Ind}_{A_0}^{A^t})^D(k_\varepsilon), (S - \text{Ind}_{A_0}^{A^t})^D(k_\varepsilon))
\]

is called the semi–infinite Hecke algebra of the triple \((A, A_0, \varepsilon).\)

The following simple and important property of semi–infinite Hecke algebras follows immediately from definition (1.5.1).

**Proposition 1.5.1.** (\[58\], Proposition 3.1.1) Assume that

\[
H^\bullet((S - \text{Ind}_{A_0}^{A^t})^D(k_\varepsilon)) = \text{Tor}_{A_0}^{\oplus \bullet}(k_\varepsilon, S_A) = k_\varepsilon \otimes_{B_0} S_A.
\]

Then

\[
H^\oplus_{\text{mod} - A^t}(A, A_0, \varepsilon) = \text{hom}^\bullet_{D(\text{mod} - A^t)_0}(k_\varepsilon \otimes_{B_0} S_A, k_\varepsilon \otimes_{B_0} S_A).
\]

In particular,

\[
H^\oplus_{\text{mod} - A^t}(A, A_0, \varepsilon) = \text{hom}_{A_1}(k_\varepsilon \otimes_{B_0} S_A, k_\varepsilon \otimes_{B_0} S_A).
\]

Another important property of the semi–infinite Hecke algebras is that they act in semi–infinite cohomology spaces. For every left \( A_0 \)-module \( M \), \( M \in (A_0 - \text{mod})_0 \), we introduce the semi–infinite cohomology space \( H^\oplus_{\text{mod} - A^t}(A_0, M) \) of \( M \) by

\[
(1.5.2) \quad H^\oplus_{\text{mod} - A^t}(A_0, M) = \text{Tor}_{A_0}^{\oplus \bullet}(k_\varepsilon, M).
\]

**Proposition 1.5.2.** (\[58\], Proposition 3.1.2) For every left \( A \)-module \( M \in (A - \text{mod})_0 \) the algebra \( H^\oplus_{\text{mod} - A^t}(A_0, \varepsilon) \) naturally acts in the semi–infinite cohomology space \( H^\oplus_{\text{mod} - A^t}(A_0, M) \) of \( M \) regarded as a left \( A_0 \)-module,

\[
H^\oplus_{\text{mod} - A^t}(A_0, \varepsilon) \times H^\oplus_{\text{mod} - A^t}(A_0, M) \to H^\oplus_{\text{mod} - A^t}(A_0, M).
\]

This action respects the bigradings of \( H^\oplus_{\text{mod} - A^t}(A_0, \varepsilon) \) and \( H^\oplus_{\text{mod} - A^t}(A_0, M) \).

**1.6. Modules over graded algebras.** In this section we recall general facts about modules over graded associative algebras (see, for instance, \[58\]). We suppose that the algebra \( A \) satisfies conditions (i), (ii) and (iv) of Section 1.1 and the following two additional conditions

(vii) The subalgebra \( B^- \subset A \) contains two graded subalgebras \( N^- \), and \( H \) such that \( N^- \subset \bigoplus_{n \leq 0} B^-_n, N^-_0 = k, H \subset B_0 \) and the multiplication in \( B^- \) defines isomorphisms of graded vector spaces

\[
N^- \otimes H \to B^- \quad \text{and} \quad H \otimes N^- \to B^-.
\]

(viii) There exists an involutive antiautomorphism \( \omega : A \to A \) such that \( \omega|_H = \text{id}, \omega : N^+ \to N^- \) and \( \omega : N^- \to N^+ \).
For every left \( A \)-module \( M \in A - \text{mod} \) we define the corresponding dual module and the contragradient module denoted by \( M^* \) and \( M^\vee \), respectively. Both \( M^* \) and \( M^\vee \) are \( \mathbb{Z} \)-graded \( A \)-modules, \( M^* \in \text{mod} - A, \ M^\vee \in A - \text{mod} \) and

\[
M_n^* = (M_{-n})^*, \quad M_n^\vee = (M_n)^*.
\]

The action of the algebra \( A \) on these modules is defined as follows

\[
\langle \xi \cdot a, v \rangle = \langle \xi, a \cdot v \rangle \quad \text{for any } v \in M, \xi \in M^*, a \in A,
\]

\[
\langle a \cdot \zeta, v \rangle = \langle \zeta, \omega(a) \cdot v \rangle \quad \text{for any } v \in M, \zeta \in M^\vee, a \in A,
\]

where \( \langle \cdot, \cdot \rangle \) stands for the natural paring between \( M^*(M^\vee) \) and \( M \).

Note that if \( M \in (A - \text{mod})_0 \) then \( M^\vee \in (A - \text{mod})_0 \).

Let \( M \) be a left \( A \)-module, \( \lambda : H \to k \) a character. A nonzero vector \( v \in M \) is called a singular vector of weight \( \lambda \) if

\[
n \cdot v = 0 \quad \text{for any } n \in \overline{N}^+ \quad \text{and}
\]

\[
h \cdot v = \lambda(h) v \quad \text{for any } h \in H.
\]

A nonzero vector \( w \in M \) is called a cosingular vector of weight \( \lambda \) if the dual vector is singular of weight \( \lambda \) in the contragradient module \( M^\vee \). From the definition of the contragradient module and of the antiautomorphism \( \omega \) it follows that this condition is equivalent to the following ones:

\[
w \not\in \overline{N}^- M,
\]

\[
h \cdot w = \lambda(h) w \quad \text{for any } h \in H,
\]

where \( \overline{N}^- = \oplus_{n < 0} N_n^- \) is the natural augmentation ideal in \( N^- \).

For any character \( \lambda : H \to k \) we denote by \( I(\lambda) \) the left ideal in \( A \) generated by elements \( h - \lambda(h) \) and \( n \), where \( h \in H \) and \( n \in \overline{N}^+ \). Both \( A \) and \( I(\lambda) \) are naturally left \( A \) modules. The quotient module \( A/I(\lambda) \) is called the Verma module and denoted by \( M_\lambda \).

Denote by \( v_\lambda \in M_\lambda \) the image of \( 1 \in A \) under the natural projection \( A \to A/I(\lambda) \). The vector \( v_\lambda \) is called the vacuum vector of \( M_\lambda \). Clearly, \( M_\lambda \) is generated by the vacuum vector as an \( A \)-module. Moreover the map

\[
\overline{N}^- \to M_\lambda, \ n \mapsto n \cdot v_\lambda
\]

is an isomorphism of \( \overline{N}^- \)-modules. Therefore the \( \mathbb{Z} \)-grading on \( \overline{N}^- \) induces a natural \( \mathbb{Z} \)-grading on \( M_\lambda \). Note that by definition \( M_\lambda \in (A - \text{mod})_0 \).

The module \( M_\lambda^\vee \) contragradient to the Verma module \( M_\lambda \) has also the following explicit description. Let \( k_\lambda \) be the one–dimensional representation of the algebra \( H \) that corresponds to the character \( \lambda : H \to k \). Since \( N^- \) is an ideal in \( B^- \) this representation naturally extends to a representation of the algebra \( B^- \), the action of the subalgebra \( N^- \) on the extended representation being trivial. We denote this \( B^- \)-module by the same symbol. The contragradient Verma module \( M_\lambda^\vee \) is isomorphic to the coinduced representation \( \text{hom}_{B^-}(A, k_\lambda) \):

\[
M_\lambda^\vee = \text{hom}_{B^-}(A, k_\lambda).
\]

Here the left action of \( A \) on \( \text{hom}_{B^-}(A, k_\lambda) \) is induced by multiplication in \( A \) from the right.
By construction the Verma and the contragradient Verma modules possess the following universal property. Let $V$ be a left $A$–module, $v \in V$ a singular vector in $V$ of weight $\lambda$. Then there exist unique homomorphisms

\begin{align*}
(1.6.1) & \quad M_\lambda \rightarrow V, \\
(1.6.2) & \quad V \rightarrow M_\lambda^\vee.
\end{align*}

Homomorphism (1.6.1) is defined as the unique homomorphism that sends $v_\lambda$ into $v$, and homomorphism (1.6.2) is induced by the unique morphism $V \rightarrow k_\lambda$ of $B^-$modules that sends $v$ into the unit of $k$.

The main tool for the study of the question of reducibility of $A$–modules is the so–called contravariant bilinear(Shapovalov) form defined on Verma modules. To introduce this form we need the notion of the Harish–Chandra map $\phi$ that is defined, in the abstract setting, as the projection onto $H$ in the direct vector space decomposition

$$
A = \mathcal{N}^- \otimes H \otimes \mathcal{N}^+ + \mathcal{N}^- \otimes H \otimes \mathcal{N}^+ + H \otimes \mathcal{N}^- \otimes \mathcal{N}^+ + \mathcal{N}^+ \otimes \mathcal{N}^- \otimes \mathcal{N}^+
$$

induced by the triangular decomposition $A = \mathcal{N}^- \otimes H \otimes \mathcal{N}^+$ and by the direct vector space decompositions

$$
\mathcal{N}^- = \mathcal{N}^- \oplus k, \quad \mathcal{N}^+ = \mathcal{N}^+ \oplus k,
$$

where $\mathcal{N}^\pm$ are the natural augmentation ideals in $\mathcal{N}^\pm$, respectively.

We define an $H$–valued form on $A$ as follows:

$$
(a, b) = \phi(\omega(a)b).
$$

This form is symmetric (see [49], Lemma 2.2).

The contravariant symmetric bilinear form $S(\cdot, \cdot)$ on the Verma module $M_\lambda$ is defined by

$$
S(v, w) = \lambda((n_v, n_w)),
$$

where $n_v$, $n_w$ are unique elements of $\mathcal{N}^-$ such that $v = n_v \cdot v_\lambda$, $w = n_w \cdot v_\lambda$.

The study of the question of reducibility for the Verma module $M_\lambda$ is based on the following simple observation: the kernel $\text{Ker}(S)$ of the contravariant form $S(\cdot, \cdot)$ defined by

$$
\text{Ker}(S) = \{v \in M_\lambda : S(v, w) = 0 \text{ for any } w \in M_\lambda\}
$$

coincides with the proper maximal submodule in $M_\lambda$. Therefore the Verma module $M_\lambda$ is irreducible if and only if its contravariant form is nondegenerate.

In conclusion we note that the Shapovalov form gives rise to an $A$–module homomorphism

$$
M_\lambda \rightarrow M_\lambda^\vee.
$$

1.7. *Wakimoto modules.* In this section, following the idea of [64], we give an algebraic definition of Wakimoto modules for associative algebras. The notion of Wakimoto modules is important, in particular, for explicit description of $W$–algebras.

In this section we suppose that the algebra $A$ contains two graded subalgebras $A_0$, $A_1$ such that multiplication in $A$ defines isomorphisms of graded vector spaces

$$
A = A_0 \otimes A_1, \quad A = A_1 \otimes A_0,
$$
A, A₀ and A₁ satisfy conditions (i)–(vi) of Sections 1.1, 1.2 and conditions (vii) and (viii) of Section 1.6. We denote by \( N^\pm, N_0^\pm, H_0 \) and \( N_1^\pm, H_1 \) the graded subalgebras in \( A, A₀, A₁ \), respectively, providing the triangular decompositions of these algebras. We also denote \( B^\pm = N^\pm H, B_0^\pm = N_0^\pm H_0, B_1^\pm = N_1^\pm H_1 \). In addition, we suppose that multiplication in \( A \) provides the following decompositions of graded vector spaces

\[
B = B_0 \otimes B_1, \quad N = N_0 \otimes N_1, \quad N = N_1 \otimes N_0.
\]

Let \( S−\text{Ind}_{A₀}^A : (A₀ − \text{mod})₀ → (A − \text{mod})₀ \) be the functor of semi-infinite induction, defined on objects by

\[
S−\text{Ind}_{A₀}^A(V) = S \otimes_{N_0}^{N_0} B_0 V, \quad V ∈ (A₀ − \text{mod})₀,
\]

the structure of a left \( A \)-module on \( S \otimes_{N_0}^{N_0} B_0 V \) being induced by the left semiregular action of \( A \) on \( S_A \).

**Definition 2.** Let \( \lambda : A₀ → k \) be a character of \( A₀ \). Denote by \( k_\lambda \) the corresponding one–dimensional representation of \( A₀ \). The semi–infinite induced representation of the algebra \( A \),

\[
W_\lambda = S−\text{Ind}_{A₀}^A k_\lambda = S \otimes_{B_0}^{N_0} k_\lambda,
\]

is called a Wakimoto module over \( A \).

Note that by definition the natural grading of \( W_\lambda \) induced by that of \( S_A \) is nonpositive, i.e. \( W_\lambda ∈ (A − \text{mod})₀ \).

The following proposition describes the structure of \( W_\lambda \) as an \( A₁ \)-module.

**Proposition 1.7.1.** Let \( k_\lambda \) be a one–dimensional representation of \( A₀ \). The corresponding Wakimoto module \( W_\lambda \) is isomorphic to the semiregular representation \( S_{A₁} \) as an \( A₁ \)-module,

\[
W_\lambda = S_{A₁}.
\]

**Proof.** The proof of this proposition is similar to that of Lemma 2.3.1 in [58]. We only mention that one should use realizations (1.2.4) and (1.2.5) of the semiregular representation \( S_A \) and refer the reader to [58] for further details. \( \square \)

From Proposition 1.4.2 and the previous proposition we deduce the following fundamental property of Wakimoto modules that explains their role in the semi–infinite cohomology theory.

**Corollary 1.7.2.** The Wakimoto module \( W_\lambda \) is semijective as an \( A₁ \)-module, with respect to the subalgebra \( N_1^+ \).

2. **Affine Lie algebras and their representations**

2.1. **Notation.** In this section we recall, following [40], basic facts about affine Lie algebras.

Let \( h^* \) be an \( l \)-dimensional complex vector space, \( a_{ij}, i,j = 1, \ldots, l \) an indecomposable Cartan matrix of finite type , \( \Delta \subset h^* \) the corresponding root system, \( \Delta^+ \) the set of positive roots relative to the set \( \Pi_0 = \{ α_1, ..., α_l \} \) of simple roots. Denote
by $W$ the Weyl group of the root system $\Delta$, and by $s_1, \ldots, s_l \in W$ the reflections corresponding to the simple roots. Let $d_1, \ldots, d_l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non-degenerate $W$–invariant scalar product $(,)$ on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = b_{ij}$.

Let $\mathfrak{g}$ be the complex simple Lie algebra associated to the Cartan matrix $a_{ij}$. The Lie algebra $\mathfrak{g}$ is generated by elements $H_i$, $X^+_i$, $X^-_i$, $i = 1, \ldots, l$ with the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X^\pm_j] = \pm a_{ij} X^\pm_j,$$

$$(2.1.1)$$

$$[X^+_i, X^-_j] = \delta_{ij} H_i, \quad (\text{ad}_{X^\pm_j})^{1-a_{ij}}(X^\pm_i) = 0 \quad \text{for } i \neq j.$$ 

The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by the elements $H_i$ is called the Cartan subalgebra. The nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. The induced nondegenerate bilinear form on $\mathfrak{h}$ extends to an invariant symmetric bilinear form on $\mathfrak{g}$.

The elements $X^\pm_i$ are called the simple positive(negative) root vectors of $\mathfrak{g}$. The subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}$ generated by the simple positive(negative) root vectors of $\mathfrak{g}$ and by the elements $H_i$ is called the positive(negative)Borel subalgebra. The subalgebra $\mathfrak{n}_\pm = [\mathfrak{b}_+, \mathfrak{b}_\pm]$ generated by the simple positive(negative) root vectors is a maximal nilpotent subalgebra in $\mathfrak{g}$.

Let $\hat{\mathfrak{g}}$ be the non-twisted affine Lie algebra corresponding to $\mathfrak{g}$. Recall that the commutant $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathfrak{g}[z, z^{-1}] + \mathfrak{ck}$ is the central extension of the loop algebra $\mathfrak{g}[z, z^{-1}]$ with the help of the standard two–cocycle $\omega_{st}$,

$$\omega_{st}(x(z), y(z)) = \text{Res}\left(\frac{d}{dz} x(z), y(z)\right)dz,$$

where $\langle \cdot, \cdot \rangle$ is the standard invariant normalized bilinear form of the Lie algebra $\mathfrak{g}$. The algebra $\hat{\mathfrak{g}}$ is the extension of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\mathfrak{b}_+ \subset \mathfrak{g}$ generated by the simple positive(negative) root vectors of $\mathfrak{g}$, and $\mathfrak{b}_0 = [\mathfrak{h}, \mathfrak{b}_\pm]$.

The algebra $\hat{\mathfrak{g}}'$ is generated by elements $H_i$, $X^+_i$, $X^-_i$, $i = 0, \ldots, l$ subject to the relations $(2.1.1)$, where $i$ and $j$ run from 0 to $l$ and $a_{ij}$ is the Cartan matrix of $\hat{\mathfrak{g}}$.

We denote by $\hat{\mathfrak{h}} = \mathfrak{h} + \mathfrak{ck} + \mathfrak{cst}$ the Cartan subalgebra in $\hat{\mathfrak{g}}$. Let $\Delta \subset \hat{\mathfrak{h}}^*$, $\Delta_+$, $\Delta_+^r$ and $\Delta_+^i$ be the root system of $\hat{\mathfrak{g}}$, the set of positive roots, the set of positive and imaginary roots of $\Delta_+$, respectively. For any root $\gamma \in \Delta$ we denote by $\mu_\gamma$ and $\text{ht} \gamma$ the multiplicity and the height of $\gamma$. Recall that $\Delta_+^r = \Delta_+^i \cup \{\alpha + m\delta, \alpha \in \hat{\Delta}, m \in \mathbb{N}\}$ and $\Delta_+^i = \{m\delta, m \in \mathbb{N}\}$, where $\delta$ is the positive imaginary root of minimal possible height. We also define an element $\rho(\alpha_i) = 1$, $i = 0, \ldots, l$.

Let $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ be the set of simple positive roots of the root system $\Delta$. One can order the simple roots $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ and the generators $H_i$, $X^+_i$, $X^-_i$, $i = 0, \ldots, l$ of $\hat{\mathfrak{g}}$ in such a way that $H_i$, $X^+_i$, $X^-_i$, $i = 1, \ldots, l$ generate the Lie subalgebra $\mathfrak{g} \subset \hat{\mathfrak{g}}$. So that if $a_{ij}$, $i, j = 0, \ldots, l$ is the Cartan matrix of $\mathfrak{g}$ then $a_{ij}$, $i, j = 1, \ldots, l$ is the Cartan matrix of $\mathfrak{g}$. We shall suppose that such an ordering is chosen.
We also denote by $\hat{b}_\pm$, $\hat{n}_\pm$ the Borel and the maximal nilpotent subalgebras in $\hat{g}$ defined similarly to the finite–dimensional case.

As in the finite–dimensional case there exist coprime positive integers $d_1, \ldots, d_l$ such that the matrix $b_{ij} = d_i a_{ij}$, $i, j = 0, \ldots, l$ is symmetric. The corresponding symmetric bilinear form $(\cdot, \cdot)$ on $\hat{h}$, such that $(\alpha_i, \alpha_j) = b_{ij}$, $i, j = 0, \ldots, l$ induces an isomorphism of vector spaces $\hat{h} \simeq \hat{h}^*$. The induced nondegenerate bilinear form on $\hat{h}$ extends to an invariant symmetric bilinear form on $\hat{g}$.

In conclusion we note that one can introduce other important nilpotent subalgebras in $\hat{g}$. Namely, consider the Lie subalgebras $\hat{n}_\pm = n_\pm[z, z^{-1}] \subset g[z, z^{-1}]$. Since the standard cocycle $\omega$ vanishes when restricted to these subalgebras, $\hat{n}_\pm$ are also Lie subalgebras in $\hat{g}'$ and in $\hat{g}$.

2.2. Verma and Wakimoto modules over affine Lie algebras. In this section we recall particular details of the construction of Verma and Wakimoto modules over affine Lie algebras. The notion of Verma modules became important in the representation theory of complex semisimple Lie algebras after a classical paper by D. Verma on embeddings of Verma modules. Wakimoto modules over the affine Lie algebra $\mathfrak{sl}_2$ were first introduced in [65] using an explicit bosonic realization of $\mathfrak{sl}_2$. The structure of these modules was studied in detail in [8]. In [27–30] B. Feigin and E. Frenkel developed a geometric approach to bosonization and generalized the notion of Wakimoto modules to the case of arbitrary untwisted affine Lie algebras. In this paper we use the algebraic definition of Wakimoto modules given in [14]. At present it is not proved that these two definitions of Wakimoto modules are equivalent. However in this paper we only use Wakimoto modules of highest weight of finite type (see Section 2.3 for the definition of these modules). In this case algebraically defined Wakimoto modules are isomorphic to contragradient Verma modules (see Proposition 2.3.2 in Section 2.3). The same fact is true for geometrically defined Wakimoto modules (see Lemma 4 in [27]).

One can apply the general scheme of Sections 1.6 and 1.7 to define Verma, contragradient Verma and Wakimoto modules over the affine Lie algebra $\hat{g}$ and over the finite–dimensional Lie algebra $g$. Here we consider the case of the affine Lie algebra $\hat{g}$ in detail.

First observe that the Lie algebra $\hat{g}$ is naturally graded, the grading being defined on generators as follows: $\deg h = 0$ for $h \in \hat{n}$, $\deg (X^+_i) = 1$, $\deg (X^-_i) = -1$. The universal enveloping algebra $U(\hat{g})$ inherits a grading from $\hat{g}$ and satisfies conditions (i)–(viii) of Sections 1.1, 1.2 and 1.3 with $N^\pm = U(\hat{n}_\pm)$, $H = U(\hat{h})$. Here the symbol $U(\cdot)$ stands for the universal enveloping algebra of the corresponding Lie algebra. The involutive antiautomorphism $\omega : U(\hat{g}) \to U(\hat{g})$, called the Cartan antiainvolution, is defined on generators by $\omega |_{\hat{h}} = id$, $\omega (X^+_i) = X^-_i$, $i = 0, \ldots, l$.

Each character $\lambda : \hat{g} \to \mathbb{C}$ gives rise to a character of the algebra $H = U(\hat{h})$. Therefore, following the construction of Section 1.4 one can define the corresponding Verma and contragradient Verma module for the algebra $A = U(\hat{g})$. As in Section 1.4 we denote these modules by $M(\lambda)$ and $M(\lambda)^\vee$, respectively.

Let $V$ be a $\hat{g}$–module. One says that $V$ admits a weight space decomposition if

$$V = \bigoplus_{\eta \in \hat{h}^*} (V)_\eta,$$
where

\[(V)_\eta = \{ v \in V : h \cdot v = \eta(h)v \text{ for any } h \in \hat{h} \}\]

is the subspace of weight \(\eta\) in \(V\).

If all the spaces \(V_\eta\) are finite–dimensional then one can introduce the formal character of \(V\) by

\[
\text{ch}(V) = \sum_{\eta \in \hat{\eta}} \dim((V)_\eta)e^\eta.
\]

Let \(Q = \sum_{i=0}^{l} Z\alpha_i\) be the root lattice of \(\hat{g}\), and \(Q_+ = \sum_{i=0}^{l} Z_+\alpha_i\). By construction the Verma module \(M(\lambda)\) admits the weight space decomposition

\[
M(\lambda) = \bigoplus_{\eta \in Q_+} (M(\lambda))_{\lambda - \eta}.
\]

The weight \(\lambda\) is called the highest weight of \(M(\lambda)\).

The module \(M(\lambda)^\vee\) admits the same decomposition. We also note that by construction the weight subspaces of \(M(\lambda)\) and \(M(\lambda)^\vee\) are finite–dimensional and these modules have the same character. This character is equal to

\[
\text{ch}(M(\lambda)) = \text{ch}(M(\lambda)^\vee) = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\text{mult}\alpha},
\]

where \text{mult}\(\alpha\) is the multiplicity of root \(\alpha\).

Recall that the Verma module \(M(\lambda)\) contains a unique maximal proper submodule \(J_\lambda\). We denote by \(L_\lambda\) the irreducible quotient \(M(\lambda)/J_\lambda\).

The problem of reducibility of Verma and contragradient Verma modules is connected with the study of zeroes of the determinant of the corresponding Shapovalov form. If this determinant has a zero then the Verma module has a singular vector and is reducible. More precisely we have the following

**Proposition 2.2.1.** ([1], Theorem 1 and Proposition 3.1) Let \(M(\lambda)\) be the Verma module over the Lie algebra \(\hat{g}\) of highest weight \(\lambda\). Then up to a nonzero constant factor depending on the basis the determinant of the restriction of the Shapovalov form to the weight subspace \((M(\lambda))_{\lambda - \eta}\) is equal to

\[
\prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left( (\lambda + \rho, \alpha) - n \left(\frac{\alpha, \alpha}{2}\right) \right)^{\dim(M(\lambda))_{\lambda - \eta + n\alpha}}.
\]

The module \(M(\lambda)\) \((M(\lambda)^\vee)\) is reducible if and only if

\[
2(\lambda + \rho, \alpha) = n(\alpha, \alpha)
\]

for some \(\alpha \in \Delta_+, n \in \mathbb{N}\). In this case \(M(\lambda)\) \((M(\lambda)^\vee)\) contains a singular(cosingular) vector of weight \(\lambda - n\alpha\).

Equation (2.2.1) is called the Kac–Kazhdan equation.

Now we turn to the algebraic definition of Wakimoto modules over \(\hat{g}\) (see [22]). First, following the general scheme presented in Section 1.7 we have to choose two graded subalgebras \(A_0, A_1 \subset A = U(\hat{g})\) satisfying certain additional conditions. We take

\[
A_0 = U(\overline{\mathfrak{g}}), A_1 = U(\mathfrak{a}),
\]

where

\[
\mathfrak{a} = \hat{n}_+ + z^{-1}\mathfrak{h}[z^{-1}], \overline{\mathfrak{a}} = \hat{n}_- + \hat{h} + \mathfrak{z}[z].
\]
Since $\hat{g} = a + \bar{a}$ as a linear space the conditions imposed on the algebras $A_0$ and $A_1$ in Section 1.7 are satisfied with $N_0 = U(zn_−[z] + z\hbar[z]), \ B_0 = (n_−[z^{-1}] + \hbar), \ N_1 = U(n_+[z]), \ B_1 = (z^{-1}n_+[z^{-1}] + z^{-1}\hbar[z^{-1}]).$

Let $\lambda : \hbar \to \mathbb{C}$ be a character of $\hbar$. Since $\hat{n}_− + z\hbar[z]$ is an ideal in $\bar{a}$ this character uniquely extends to a representation of the Lie algebra $\bar{a}$, the action of the ideal $\hat{n}_− + z\hbar[z]$ on the extended representation being trivial. We denote this representation by $\mathbb{C}_\lambda$. The corresponding Wakimoto module

$$W(\lambda) = S - \text{Ind}_{A_0}^A \mathbb{C}_\lambda = S_A \otimes_{B_0} \mathbb{C}_\lambda,$$

is called a Wakimoto module over $\hat{g}$.

By construction the Wakimoto module $W(\lambda)$ has the same weight space decomposition and the same character as the Verma module $M(\lambda)$ and the contragradient Verma module $M(\lambda)^\vee$ (see [64], Proposition 2.2). Moreover, the vector $w_\lambda \in W(\lambda)$ of highest possible weight $\lambda$ is singular in $W(\lambda)$. Therefore there exist unique homomorphisms (see formulas (1.6.1) and (1.6.2) in Section 1.6)

$$M(\lambda) \to W(\lambda) \to M(\lambda)^\vee.$$  

The composition of these two maps is given by the Shapovalov form of $M(\lambda)$ (see [20], §2.1 for similar construction in case of the Virasoro algebra). Therefore from Proposition 2.2.1 we deduce that the module $W(\lambda)$ is reducible iff $M(\lambda)$ is reducible. Moreover $W(\lambda)$ has singular and cosingular vectors of the same weights as the singular vectors of $M(\lambda)$.

2.3. Wakimoto modules with highest weights of finite type. Let $\lambda : \hbar \to \mathbb{C}$ be a character, $M(\lambda)$, $M(\lambda)^\vee$ and $W(\lambda)$ the corresponding Verma, contragradient Verma and Wakimoto modules over $\hat{g}$ of highest weight $\lambda$. The number $\lambda(K) = k \in \mathbb{C}$ is called the level of $\lambda$. We say that $\lambda$ is of finite type if the corresponding Kac–Kazhdan equation (2.2.1) has only solutions $n, \alpha$ such that $\alpha \in \Delta_+^\vee$.

Remark 2.3.1. For instance if $\lambda|_h$ is an integral weight, i.e. $\lambda(H_i) \in \mathbb{Z}$ for $i = 1, \ldots, l$ then $\lambda$ is of finite type if $k \in \mathbb{C} \setminus \{-h^\vee + \mathbb{Q}\}$, where $h^\vee$ is the dual Coxeter number of $\hat{g}$ and $\mathbb{Q}$ is the set of rational numbers (In this case the level $k$ is called generic). Indeed, let $\beta = \alpha + m\delta, \alpha \in \hat{\Delta}_+, m \in \mathbb{N}$ be a positive root that does not belong to $\hat{\Delta}_+$. Then the corresponding Kac-Kazhdan equation (2.2.1) takes the form

$$(\lambda + \rho, \alpha) + m(k + h^\vee) = \frac{n}{2}(\alpha, \alpha),$$

where we used the equality $(\rho, \delta) = h^\vee$. If $\lambda|_h$ is an integral weight this equation has nontrivial solutions $\alpha \in \hat{\Delta}$ if and only if $k$ is not generic.

Therefore if $\lambda$ is of finite type then singular and cosingular vectors of $M(\lambda)$, $M(\lambda)^\vee$ and $W(\lambda)$ may only appear in the subspaces of weights $\lambda - na, \ n \in \mathbb{N}, \ \alpha \in \hat{\Delta}_+, \ i.e. \ in \ the \ ‘\text{finite–dimensional}’ \ parts \ of \ M(\lambda), \ M(\lambda)^\vee \ and \ W(\lambda)$ (we recall that they appear in subspaces of the same weights simultaneously). Moreover, we have the following proposition describing the structure of Wakimoto modules of highest weight of finite type.

Proposition 2.3.2. Let $\lambda : \hbar \to \mathbb{C}$ be a character of finite type. Then the canonical map

$$W(\lambda) \to M(\lambda)^\vee$$
is an isomorphism of $\mathfrak{g}$–modules. Let $M(\lambda_0)^\vee$ be the contragradient Verma module over $\mathfrak{g}$ of highest weight $\lambda_0 = \lambda|_\mathfrak{h}$. This module is uniquely extended to a $\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial$–module $(M(\lambda_0)^\vee)_{k,\lambda(\partial)}$ in such a way that $\mathfrak{g}[z]$ trivially acts on $(M(\lambda_0)^\vee)_{k,\lambda(\partial)}$, $K$ and $\partial$ act by multiplication by $k = \lambda(K)$ and by $\lambda(\partial)$, respectively. Then both $M(\lambda)^\vee$ and $W(\lambda)$ are isomorphic to the induced representation $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M(\lambda_0)^\vee)_{k,\lambda(\partial)}$.

First we prove the following Lemma which describes the “finite–dimensional” part of $W(\lambda)$.

**Lemma 2.3.3.** The $\mathfrak{g}$–submodule $W(\lambda_0) = \bigoplus_{\eta \in Q^+} (W(\lambda))_{\lambda - \eta}$, $Q^+ = \sum_{i=1}^l \mathbb{Z}_+ \alpha_i$, of the Wakimoto module $W(\lambda)$ is isomorphic to the contragradient Verma module $M(\lambda_0)^\vee$ over $\mathfrak{g}$, where $\lambda_0 = \lambda|_\mathfrak{h}$.

**Proof.** First observe that by construction the $\mathfrak{g}$–module $W(\lambda_0)$ is isomorphic to $S_{U(\mathfrak{g}) \otimes U(\mathfrak{b}_-)} \mathbb{C}\lambda_0$, where $\mathbb{C}\lambda_0$ is the one–dimensional representation of $\mathfrak{b}_-$ obtained by trivial extension of the character $\lambda_0 : \mathfrak{h} \to \mathbb{C}$. Using realization (1.2.3) of the semiregular representation $S_{U(\mathfrak{g})}$, with $A = U(\mathfrak{g})$ and $B^- = U(\mathfrak{b}_-)$, we conclude that the representation $S_{U(\mathfrak{g}) \otimes U(\mathfrak{b}_-)} \mathbb{C}\lambda_0$ is isomorphic to $\text{hom}_{U(\mathfrak{b}_-)}(U(\mathfrak{g}), \mathbb{C}\lambda_0)$. By definition the last $\mathfrak{g}$–module is the contragradient Verma module $M(\lambda_0)^\vee$.

**Proof of Proposition 2.3.2.** We have to prove that the canonical maps $W(\lambda) \to M(\lambda)^\vee$ and

\[
U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M(\lambda_0)^\vee)_{k,\lambda(\partial)} \to M(\lambda)^\vee
\]  

(2.3.1)

are $\mathfrak{g}$–module isomorphisms.

First note that these maps are injective. For if one of these maps has a kernel then this kernel is a $\mathfrak{g}$–submodule in $W(\lambda)$ or in $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M(\lambda_0)^\vee)_{k,\lambda(\partial)}$, respectively. Therefore such kernel must contain a singular vector. But as we observed in the beginning of this section $W(\lambda)$ may only contain singular or cosingular vectors in the finite–dimensional part $W(\lambda_0)$. By Lemma 2.3.3 $W(\lambda_0) = M(\lambda_0)^\vee$. But the module $M(\lambda_0)^\vee$ is a contragradient Verma module, and hence, it may have only cosingular vectors as a module over $\mathfrak{g}$ (except for the highest weight vector). This implies that $W(\lambda)$ may also have only cosingular vectors except for the highest weight vector. By construction the same is true for the $\mathfrak{g}$–module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M(\lambda_0)^\vee)_{k,\lambda(\partial)}$. We conclude that the kernels of the maps (2.3.1) are trivial since by definition they do not contain the highest weight vectors.

The maps (2.3.1) are also surjective since they respect the gradings on $W(\lambda)$, $M(\lambda)^\vee$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M(\lambda_0)^\vee)_{k,\lambda(\partial)}$ and these three modules have the same character.

Next, using Proposition 2.3.2 and Remark 2.3.1 we construct resolutions by Wakimoto modules for a class of $\mathfrak{g}$ modules induced from finite–dimensional irreducible representations of the Lie algebra $\mathfrak{g}$ (see [23, 24, 25]). These resolutions are induced from the Bernstein–Gelfand–Gelfand resolutions of the finite–dimensional irreducible representations of $\mathfrak{g}$ (see [9] for the definition of the Bernstein–Gelfand–Gelfand resolution).
Corollary 2.3.4. Let $\lambda$ be a character of $\hat{h}$ of generic level $k$ such that $\lambda_0 = \lambda|_h$ is an integral dominant weight for $g$, i.e., $\lambda_0 \in P^+$, where $P^+ = \{ \lambda \in h^* : \lambda(H) \in \mathbb{Z}_{>0}, i = 1, \ldots, l \}$. Let $L(\lambda_0)$ be the irreducible finite–dimensional representation of $g$ with highest weight $\lambda_0$ and denote by $C^\bullet(\lambda_0)$ the Bernstein–Gelfand–Gelfand resolution of $L(\lambda_0)$ by contragradient Verma modules over $g$,

$$0 \to C^1(\lambda_0) \to \cdots \to C^{\dim n^+}(\lambda_0) \to 0,$$

where $W^{(i)} \subset W$ is the subset of the elements of length $i$ of the Weyl group of $g$ and $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$.

Then the induced complex of $\hat{g}$–modules

$$0 \to D^1(\lambda) \to \cdots \to D^{\dim n^+}(\lambda) \to 0,$$

$$D^i(\lambda) = \bigoplus_{w \in W^{(i)}} U(\hat{g}) \otimes_{U(\hat{g}[z]+cK+c\partial)} (M(w(\lambda_0 + \rho_0) - \rho_0)^\vee)_{k,\lambda(\partial)}$$

is a resolution of the induced representation $U(\hat{g}) \otimes_{U(\hat{g}[z]+cK+c\partial)} (L(\lambda_0))_{k,\lambda(\partial)}$ by Wakimoto modules, i.e.

$$D^i(\lambda) = \bigoplus_{w \in W^{(i)}} W(w(\lambda + \rho_0) - \rho_0),$$

where the Weyl group $W$ is regarded as a subgroup in the affine Weyl group of the Lie algebra $\hat{g}$.

2.4. Bosonization for $\hat{sl}_2$. In this section we recall the original definition of Wakimoto modules in the simplest case of the Lie algebra $\hat{sl}_2$ [65]. We also prove that for highest weights of finite type this definition is equivalent to the invariance algebraic definition given in Section 2.3.

In [65] Wakimoto modules for $\hat{sl}_2$ were realized in Fock spaces for the complex associative Heisenberg algebra $H$ generated by elements $\omega_n$, $\omega_n^+$ and $a_n$, $n \in \mathbb{Z}$ subject to the following relations

$$[\omega_n, \omega_m^+] = \delta_{n+m,0},$$

$$[\omega_n, a_m] = [\omega_n^+, a_m] = [\omega_n, \omega_m] = [\omega_n^+, \omega_m^+] = 0,$$

$$[a_n, a_m] = 2(k+2)n\delta_{n+m,0}.$$

Note that the algebra $H$ is naturally $\mathbb{Z}$–graded, $\deg \omega_n = \deg \omega_n^+ = \deg a_n = n$.

We introduce generating series for the generators of the algebra $H$ by

$$\omega(w) = \sum_{n \in \mathbb{Z}} \omega_n w^{-n},$$

$$\omega^+(w) = \sum_{n \in \mathbb{Z}} \omega_n^+ w^{-n},$$

$$a(w) = \sum_{n \in \mathbb{Z}} a_n w^{-n}.$$

We also denote by $X^\pm = X z^n$, $H_n = H z^n$ the “loop” generators of the Lie algebra $\hat{sl}_2$, where $X^\pm$ and $H$ are the Chevalley generators of $sl_2$, and introduce generating
series for these generators by
\[
X^\pm(w) = \sum_{n \in \mathbb{Z}} X_n^\pm w^{-n},
\]
\[
H(w) = \sum_{n \in \mathbb{Z}} H_n w^{-n}.
\]

**Proposition 2.4.1.** ([65], Theorem 1) Suppose that \( k \neq -2 \) and denote by \( \hat{H} \) the restricted completion of the algebra \( H \). Then the map \( \phi_k : U(\hat{sl}_2) \to \hat{H} \) defined in terms of generating series by
\[
X^+(w) \mapsto \omega^+(w),
\]
\[
H(w) \mapsto 2 : \omega(z) \omega^+(w) : + a(w),
\]
\[
X^-(w) \mapsto - : \omega(w) \omega^+(w) : - k \frac{d}{dw} \omega(w) - \omega(w) a(w),
\]
\[
K \mapsto k,
\]
\[
\partial \mapsto \sum_{n \in \mathbb{Z}} n : \omega_{-n} \omega_n^+ : - \frac{1}{2(k + 2)} \sum_{n=1}^{\infty} a_{-n} a_n
\]
is a homomorphism of algebras. Here \( : \) stands for the normally ordered product of elements of the algebra \( H \), i.e. a permuted product of elements such that for \( n \geq 0 \) elements \( \omega_n, \omega_n^+ \) stand on the right.

Using this proposition one can construct representations of the Lie algebra \( \hat{sl}_2 \) in Fock spaces for the algebra \( H \). Let \( \mathcal{H}(\lambda_0) \) be the Fock space for the algebra \( H \) generated by the vacuum vector \( v_{\lambda_0} \) satisfying the following conditions
\[
\omega_n \cdot v_{\lambda_0} = 0 \text{ for } n > 0,
\]
\[
\omega_n^+ \cdot v_{\lambda_0} = 0 \text{ for } n \geq 0,
\]
\[
a_n \cdot v_{\lambda_0} = 0 \text{ for } n > 0,
\]
\[
a_0 \cdot v_{\lambda_0} = \lambda_0 v_{\lambda_0}.
\]
Denote by \( W(\lambda_0, k) \) the representation of the algebra \( \hat{sl}_2 \) in this space constructed with the help of the homomorphism \( \phi_k \).

**Proposition 2.4.2.** Let \( \lambda : \hat{h} \to \mathbb{C} \) be the character of the Cartan subalgebra of the Lie algebra \( \hat{sl}_2 \) such that \( \lambda(H) = \lambda_0, \lambda(K) = k, k \neq -2 \) and \( \lambda(\partial) = 0 \). Suppose that \( \lambda \) is of finite type. Then the \( \hat{sl}_2 \)-module \( W(\lambda_0, k) \) is isomorphic to the Wakimoto module \( W(\lambda) \). In this case both \( W(\lambda_0, k) \) and \( W(\lambda) \) are isomorphic to the contragradient Verma module \( M(\lambda)^\vee \).

**Proof.** First we recall that according to Proposition 2.3.2 the Wakimoto module \( W(\lambda) \) is isomorphic to the contragradient Verma module \( M(\lambda)^\vee \). Therefore it suffices to show that \( W(\lambda_0, k) \) is isomorphic to \( M(\lambda)^\vee \).

Observe that the vacuum vector \( v_{\lambda_0} \) of the module \( W(\lambda_0, k) \) is the singular vector of highest possible weight \( \lambda \). Therefore we have a canonical map
\[
W(\lambda_0, k) \to M(\lambda)^\vee.
\]
We have to show that this map is an isomorphism.

Note that the composition of the canonical map \( M(\lambda) \to W(\lambda_0, k) \) and of the map \((2.4.4)\) is given by the Shapovalov form of \( M(\lambda) \). Therefore the singular(cosingular) vectors of \( W(\lambda_0, k) \) and \( M(\lambda)^\vee \) may appear in the subspaces of the same weights simultaneously. By the definition of characters of finite type (see Section 2.3) the singular(cosingular) vectors may only appear in the “finite–dimensional” part of \( M(\lambda)^\vee \). Explicit calculation shows that the module \( W(\lambda_0, k) \) may only have cosingular vector \( \omega_{\lambda_0+1}^0 \cdot v_{\lambda_0} \) when \( \lambda_0 \in \mathbb{Z}_+ \).

The rest of the proof of this proposition is parallel to that of Proposition 2.3.2.

One just has to remark that both \( W(\lambda_0, k) \) and \( M(\lambda)^\vee \) have the same characters.

For any character \( \lambda : \hat{\mathfrak{h}} \to \mathbb{C} \) of finite type such that \( \lambda(H) = \lambda_0, \lambda(K) = k, k \neq -2 \) and \( \lambda(\partial) = 0 \) we shall always identify the \( \mathfrak{sl}_2 \)–modules \( W(\lambda_0, k) \) and \( W(\lambda) \).

In conclusion we recall the definition of screening operators which are certain intertwining operators between Wakimoto modules \( W(\lambda_0, k) \) (see [8, 28, 29]). First we introduce an operator \( V : \mathcal{H}(\lambda_0) \to \mathcal{H}(\lambda_0 - 2) \) that sends the vacuum vector \( v_{\lambda_0} \) of \( \mathcal{H}(\lambda_0) \) to the vacuum vector \( v_{\lambda_0-2} \) of \( \mathcal{H}(\lambda_0 - 2) \), intertwines the action of the elements \( \omega_\alpha, \omega_\alpha^2 \) and commutes with \( a_n \) as follows

\[
[a_n, V] = -2V\delta_{n,0}.
\]

**Proposition 2.4.3.** ([28], Theorem 3.4) The operator \( S = \text{Res}_{w=0} J(w), S : W(\lambda_0, k) \to W(\lambda_0 - 2, k), \) where the generating series \( J(w) \) is defined by

\[
J(w) = w^{-1}\omega^+(w) \exp \left( -\sum_{n=1}^{\infty} \frac{a_n}{(k + 2)n} w^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{(k + 2)n} w^{-n} \right) V w^{-\frac{a_0}{k+2}},
\]

is a homomorphism of \( \mathfrak{sl}_2 \) modules.

The operator \( S \) is called a screening operator.

3. \( W \)-algebras

3.1. Definition of \( W \)-algebras. In this section we give a definition of \( W \)-algebras associated to complex semisimple Lie algebras. We follow the invariant Hecke algebra approach developed in [58] and refer the reader to [23]–[26] for the original definition. We keep the notation introduced in Section 2.1.

Let \( \mathfrak{n}_+ \) be the maximal positive nilpotent Lie subalgebra in the complex semisimple Lie algebra \( \mathfrak{g} \), \( \hat{\mathfrak{n}}_+ = \mathfrak{n}_+ [z, z^{-1}] \) the corresponding loop Lie algebra. Since

\[
\hat{\mathfrak{n}}_+ = \sum_{i=1}^{l} \sum_{n \in \mathbb{Z}} \mathbb{C} X_i^+ z^n \oplus [\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+]
\]

as a vector space each character \( \varepsilon : \hat{\mathfrak{n}}_+ \to \mathbb{C} \) is completely determined by the constants \( \varepsilon(X_i^+ z^n), i = 1, \ldots, l, n \in \mathbb{Z} \).

Let \( \chi : \hat{\mathfrak{n}}_+ \to \mathbb{C} \) be the character such that

\[
\chi(X) = \begin{cases} 
1 & \text{if } X = X_i^+ z^{-1}, i = 1, \ldots, l \\
0 & \text{if } X \notin \sum_{i=1}^{l} \mathbb{C} X_i^+ z^{-1}, i = 1, \ldots, l
\end{cases}
\]

We denote by \( \mathbb{C}_\chi \) the left one–dimensional \( U(\hat{\mathfrak{n}}_+) \)–module that corresponds to \( \chi \).
Let $U(\mathfrak{g}^\prime)_k$ be the quotient of the algebra $U(\mathfrak{g}^\prime)$ by the two-sided ideal generated by $K - k$, $k \in \mathbb{C}$. Note that for any $k \in \mathbb{C} U(\hat{n}_+)$ is a subalgebra in $U(\mathfrak{g}^\prime)_k$ because the standard cocycle $\omega_{st}$ vanishes when restricted to the subalgebra $\hat{n}_+ \subset \mathfrak{g}[z, z^{-1}]$.

Next observe that the algebras $U(\mathfrak{g}^\prime)_k$ and $U(\hat{n}_+)$ are naturally $\mathbb{Z}$-graded and satisfy conditions (i)-(vi) of Sections 1.1, 1.2, with the natural triangular decompositions $U(\mathfrak{g}^\prime)_k = U(\hat{b}^\prime_-) \otimes U(\hat{n}_+)$ and $U(\hat{n}_+) = U(z^{-1}n[z^{-1}]) \otimes U(n[z])$, where $\hat{b}^\prime_-$ is the Lie subalgebra in $\mathfrak{g}^\prime$ generated by $\hat{n}_-$ and $\mathfrak{h}$. Here both $\hat{n}_+$ and $\hat{n}_-$ are regarded as Lie subalgebras in $\mathfrak{g}^\prime$. Hence one can define the algebras $U(\mathfrak{g}^\prime)^*_k$, $U(\hat{n}_+)^*$ and the semi-infinite Tor functors for $U(\mathfrak{g}^\prime)_k$ and $U(\hat{n}_+)$. The algebra $U(\mathfrak{g}^\prime)^*_k$ is explicitly described in the following proposition.

**Proposition 3.1.1.** ([2], Proposition 4.6.7) Let $\mathfrak{g}^\prime = \mathfrak{g}^\prime + K[1]C$ be the central extension of $\mathfrak{g}^\prime$ with the help of the cocycle $\omega_{st}(x, y) = 2\rho(P_0([x, y]), x, y \in \mathfrak{g}^\prime$. Here $P_0$ is the projection operator onto $\mathfrak{h} + C K$ in the direct vector space decomposition $\mathfrak{g}^\prime = \hat{n}_+ + (\mathfrak{h} + C K) + \hat{n}_-$. Then the algebra $U(\mathfrak{g}^\prime)^*_k$ is isomorphic to the quotient $U(\mathfrak{g}^\prime)/I$, where $I$ is the two-sided ideal in $U(\mathfrak{g}^\prime)$ generated by $K - k$ and $K_1 - 1$.

Note also that for any $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ with finite-dimensional graded components the algebra $U(\mathfrak{g})^*_k$ may be described as the universal enveloping algebra of the central extension of $\mathfrak{g}$ with the help of the so-called critical two-cocycle of $\mathfrak{g}$ (see [2], Proposition 4.6.7), the value of the central charge being equal to one. From the explicit description of the critical cocycle it follows that the critical cocycle of the Lie algebra $\hat{n}_+$ vanishes. Therefore the algebra $U(\hat{n}_+)^*_k$ is isomorphic to $U(\hat{n}_+)$. We shall always identify the algebra $U(\hat{n}_+)^*_k$ with $U(\hat{n}_+)$. The algebra $U(\mathfrak{g}^\prime)^*_k$ is explicitly described in the following proposition.

**Definition 3.** The $W$-algebra $W_k(\mathfrak{g})$ associated to the complex semisimple Lie algebra $\mathfrak{g}$ is the zeroth graded component of the semi-infinite Hecke algebra of the triple $(U(\mathfrak{g}^\prime)_k, U(\hat{n}_+), C_\chi)$,

$$W_k(\mathfrak{g}) = H_k^{\mathfrak{g}^\prime+0}(U(\mathfrak{g}^\prime)_k, U(\hat{n}_+), C_\chi).$$

This definition is equivalent to the original definition of Hecke algebras given in [23, 24] (see Proposition 3.2.2 in [25]). Moreover, we have

**Proposition 3.1.2.** ([10], Sect. 2; [32], Theorem 14.1.9) The nonzero graded components of the semi-infinite Hecke algebra of the triple $(U(\mathfrak{g}^\prime)_k, U(\hat{n}_+), C_\chi)$ vanish,

$$H_k^{\mathfrak{g}^\prime+n}(U(\mathfrak{g}^\prime)_k, U(\hat{n}_+), C_\chi) = 0 \text{ for } n \neq 0.$$

**Remark 3.1.3.** In the definition of $W$-algebras given in [25] we used the grading in the Lie algebra $\mathfrak{g}$ by the degree of the loop parameter $z$ and the character $\chi : \hat{n}_+ \rightarrow \mathbb{C}$ such that

$$\chi(X) = \begin{cases} 1 & \text{if } X = X^+_i, \ i = 1, \ldots, l \\ 0 & \text{if } X \not\in \sum_{i=1}^{l} \mathbb{C}X^+_i, \ i = 1, \ldots, l \end{cases}$$

However in [35] it is shown that these two definitions of $W$-algebras are equivalent.

Using Proposition 1.5.1 one can explicitly calculate the algebra $W_k(\mathfrak{g})$.

**Proposition 3.1.4.** ([25], Theorem 3.2.5) The algebra $W_k(\mathfrak{g})$ is canonically isomorphic to

$$W_k(\mathfrak{g}) = \text{hom}_{U(\mathfrak{g}^\prime)_k}(C_\chi \otimes_{U(\mathfrak{g}^\prime)} U(n[z]), S_U(\mathfrak{g}^\prime)_k, C_\chi \otimes_{U(\mathfrak{g}^\prime)} S_U(\mathfrak{g}^\prime)_k),$$

(3.1.1) where $\text{hom}_{U(\mathfrak{g}^\prime)_k}(C_\chi \otimes_{U(\mathfrak{g}^\prime)} U(n[z]), S_U(\mathfrak{g}^\prime)_k, C_\chi \otimes_{U(\mathfrak{g}^\prime)} S_U(\mathfrak{g}^\prime)_k)$.
3.2. Resolutions and screening operators for W–algebras. In this section we suppose that the level $k$ is generic. Recall that by Proposition 1.5.2 the algebra $W_k(g)$ acts in the spaces $\text{Tor}^{\hat{U}(\hat{\mathfrak{n}}_+)}(\mathbb{C}_X, M)$, where $M \in (U(\hat{g}^/) - \text{mod})_0$. In particular for every left $U(\hat{g}^/)–$module $M \in (U(\hat{g}^/) - \text{mod})_0$ such that the the two–sided ideal of the algebra $U(\hat{g}^/) \text{ generated by } K - k$ lies in the kernel of the representation $M$ the algebra $W_k(g)$ acts in the space $\text{Tor}^{\hat{U}(\hat{\mathfrak{n}}_+)}(\mathbb{C}_X, M)$.

Let $\lambda_k : \hat{g} \rightarrow \mathbb{C}$ be the character such that $\lambda|_{\mathfrak{h}} = 0$, $\lambda(K) = k$ and $\lambda(\partial) = 0$. Denote by $V_k$ the representation of the Lie algebra $\hat{g}$ with highest weight $\lambda_k$ induced from the trivial representation of the Lie algebra $g$, $V_k = U(\hat{g}) \otimes U(\mathfrak{g}^0 + C\mathfrak{k} + C\partial) (L(0))_{k,0}$. $V_k$ is called the vacuum representation of $\hat{g}$. Since the two–sided ideal of the algebra $U(\hat{g}^/) \text{ generated by } K - k$ lies in the kernel of $V_k$ the algebra $W_k(g)$ acts in the space $\text{Tor}^{\hat{U}(\hat{\mathfrak{n}}_+)}(\mathbb{C}_X, V_k)$.

The space $\text{Tor}^{\hat{U}(\hat{\mathfrak{n}}_+)}(\mathbb{C}_X, V_k)$ may be explicitly described using the resolution of the $\hat{g}$–module $V_k$ by Wakimoto modules constructed in Corollary 2.3.4. Indeed, let $D^\bullet(\lambda_k)$ be this resolution, $D^i(\lambda_k) = \bigoplus_{w \in W(i)} W(w(\lambda_k + \rho_0) - \rho_0)$.

**Proposition 3.2.1.** The complex $D^\bullet(\lambda_k)$ is a semijective resolution of $V_k$ regarded as a $U(\hat{\mathfrak{n}}_+)–$module, with respect to the subalgebra $U(n[z])$.

This proposition follows from part 3 of Proposition 1.4.3 and the following lemma.

**Lemma 3.2.2.** Every Wakimoto module $W(\lambda)$ is semijective as a $U(\hat{\mathfrak{n}}_+)–$module, with respect to the subalgebra $U(n[z])$.

**Proof.** First observe that by Proposition 1.7.1 every Wakimoto module is isomorphic to the left semiregular representation $S_U(U(a))$ as a $U(a)$–module. By Lemma 1.3.1 this space is also isomorphic to $S_{U(\hat{\mathfrak{n}}_+)} \otimes U(n[z]) U(\hat{\mathfrak{n}}_+) as a U(\hat{\mathfrak{n}}_+)–$module.

Similarly to Lemma 2.3.1 in [28] one can show that $S_{U(\hat{\mathfrak{n}}_+)} \otimes U(n[z]) S_U(a) = S_{U(\hat{\mathfrak{n}}_+)} \otimes U(z^{-1}h[z^{-1}])$ as a $U(n[z])–$module. By Lemma 4.4 this module is relatively to $U(n[z])$–projective. Indeed, using realization (1.2.3) of the semiregular bimodule $S_{U(\hat{\mathfrak{n}}_+)}$ one can establish a $U(n[z])–$module isomorphism,

$$S_{U(\hat{\mathfrak{n}}_+)} \otimes U(z^{-1}h[z^{-1}]) = U(\hat{\mathfrak{n}}_+) \otimes U(n[z]) U(\hat{\mathfrak{n}}_+) \otimes U(z^{-1}h[z^{-1}]),$$

and the last $U(\hat{\mathfrak{n}}_+)–$module is induced from a $U(n[z])–$module.

The $U(\hat{\mathfrak{n}}_+)–$module $S_{U(\hat{\mathfrak{n}}_+)} \otimes U(z^{-1}h[z^{-1}])$ is also $U(n[z])–$injective because using Lemma 1.1.1 and formula (2.3) we have an isomorphism of $U(n[z])–$modules,

$$S_{U(\hat{\mathfrak{n}}_+)} \otimes U(z^{-1}h[z^{-1}]) = \text{hom}_C(U(n[z]), U(z^{-1}n[z^{-1}]) \otimes U(z^{-1}h[z^{-1}]),$$

and the last module is obviously $U(n[z])–$injective.

Now, by the definition of the semi–infinite Tor functor, in order to calculate the space $\text{Tor}^{\hat{U}(\hat{\mathfrak{n}}_+)}(\mathbb{C}_X, V_k)$ one should apply the functor $\mathbb{C}_X \otimes_U U(n[z])$ to the resolution $D^\bullet(\lambda_k)$ and compute the cohomology of the obtained complex.

Denote by $C^\bullet(\lambda_k)$ the complex $\mathbb{C}_X \otimes_U U(n[z]) D^\bullet(\lambda_k)$,

$$C^\bullet(\lambda_k) = \mathbb{C}_X \otimes_U U(n[z]) D^\bullet(\lambda_k).$$
Proposition 3.2.3. ([23], Theorem 1) $H^{\neq 0}(C^\bullet(\lambda_k)) = 0$, i.e., for $n \neq 0$

$$\text{Tor} \beta_+^n (C, V_k) = 0,$$

and the complex $C^\bullet(\lambda_k)$ is a resolution of the $W_k(\mathfrak{g})$-module $\text{Tor} \beta_+^n (C, V_k)$.

The $W_k(\mathfrak{g})$-module $\text{Tor} \beta_+^n (C, V_k)$ is called the vacuum representation of $W_k(\mathfrak{g})$, and the operators $S_i : C \otimes U(\mathfrak{n}[z]) W(\lambda_k) \to C \otimes U(\mathfrak{g}[z]) W(-\alpha_i + \lambda_k)$ induced by the differential of the complex $C^\bullet(\lambda_k)$ in degree 0,

$$d : C \otimes U(\mathfrak{n}[z]) W(\lambda_k) \to \bigoplus_{i=1}^l C \otimes U(\mathfrak{n}[z]) W(s_i(\lambda_k + \rho_0) - \rho_0) = \bigoplus_{i=1}^l C \otimes U(\mathfrak{n}[z]) W(-\alpha_i + \lambda_k),$$

are called the screening operators for the algebra $W_k(\mathfrak{g})$.

3.3. The Virasoro algebra. In this section we describe, following [23], the $W$-algebra $W_k(\mathfrak{sl}_2)$. Using the algebraic definition of Wakimoto modules we also obtain the results of [23] on the explicit form of the resolution of the vacuum representation for this algebra for generic $k$. We use the bosonic realization of Wakimoto modules over the Lie algebra $\mathfrak{sl}_2$ and the notation introduced in Section 2.4.

Denote by $\text{Vir}$ the Virasoro algebra, i.e., the complex Lie algebra generated by elements $T_n$, $n \in \mathbb{Z}$ and $C$ with the following defining relations

$$[T_n, T_m] = (n - m)T_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n+m,0}, \quad [C, T_n] = 0, \quad n, m \in \mathbb{Z}.$$  

Note that the Virasoro algebra is naturally $\mathbb{Z}$-graded.

Proposition 3.3.1. ([23], Proposition 4) Let $U(\text{Vir})_c$ be the quotient of the universal enveloping algebra $U(\text{Vir})$ by the two sided ideal generated by the element $C - c$, where $C$ is the central element of the Lie algebra $\text{Vir}$ and $c \in \mathbb{C}$. Suppose that $k$ is generic. Then the algebra $W_k(\mathfrak{sl}_2)$ is isomorphic to the restricted completion of the algebra $U(\text{Vir})_c$, where $c = 1 - \frac{6(k+1)^2}{k+2}$,

$$W_k(\mathfrak{sl}_2) = \tilde{U}(\text{Vir})_c, \quad c = 1 - \frac{6(k+1)^2}{k+2}.$$  

Note that the central charge $c = 1 - \frac{6(k+1)^2}{k+2}$ is invariant under the following transformation of the parameter $k$:

$$k + 2 \mapsto \frac{1}{k+2}.$$  

As a consequence we have the following proposition.

Proposition 3.3.2. Let $k, k' \in \mathbb{C}$ be generic. Suppose that $k' + 2 = \frac{1}{k + 2}$. Then the algebras $W_k(\mathfrak{sl}_2)$ and $W_{k'}(\mathfrak{sl}_2)$ are isomorphic.
Let $\tilde{W}$ be the Wakimoto module of highest weight $\lambda$ of finite type such that $\lambda(H) = \lambda_0$, $\lambda(K) = k$, $k \neq -2$ and $\lambda(\partial) = 0$. Denote by $H^0 \subset H$ the subalgebra in $H$ with generators $a_n$, $n \in \mathbb{Z}$ subject to the relations 

$$[a_n, a_m] = 2(k+2)n\delta_{n+m,0}.$$ 

Let $\pi(\lambda_0, k + h^\vee)$ be the $\hat{h}$ (and $H^0$)-submodule in $W(\lambda_0, k)$ generated by the vacuum vector $v_{\lambda_0}$ under the action of the subalgebra $H^0 \subset H$. Then the natural linear space embedding $\pi(\lambda_0, k + h^\vee) \to W(\lambda_0, k)$ gives rise to a linear space isomorphism 

$$\pi(\lambda_0, k + h^\vee) = C_\chi \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k).$$ 

Proof. In order to prove this lemma we note that there is a linear space isomorphism $C_\chi \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k) = C_{\chi_0} \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k)$, where $\chi_0$ is the trivial character of the Lie algebra $\mathfrak{n}_+$. Since the Cartan subalgebra $\mathfrak{h} \subset \hat{\mathfrak{g}}$ normalizes the Lie subalgebras $\mathfrak{n}[z]$ and $z^{-1}\mathfrak{n}[z^{-1}]$ the space $C_{\chi_0} \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k)$ is naturally an $\hat{\mathfrak{h}}$-module, the module structure being induced by the action of the Lie algebra $\hat{\mathfrak{h}}$ on the space $W(\lambda_0, k)$.

Now recall that in the proof of Lemma 2.2.2 we observed that any Wakimoto module is isomorphic to $S_{U(\mathfrak{n}_+)} \otimes_{U(\mathfrak{z}^{-1}\mathfrak{h}[z^{-1}])} W(\lambda_0, k)$ as an $U(\mathfrak{n}_+)$-module. Therefore from Lemma 3.3.1 we deduce that $C_{\chi_0} \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k) = U(\mathfrak{z}^{-1}\mathfrak{h}[z^{-1}])$ as a linear space. Explicit calculation shows that the induced $\hat{\mathfrak{h}}$-module structure on $U(\mathfrak{z}^{-1}\mathfrak{h}[z^{-1}])$ coincides with that of $\pi(\lambda_0, k + h^\vee)$, and hence we have an isomorphism of $\hat{\mathfrak{h}}$-modules,

$$C_{\chi_0} \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k) = \pi(\lambda_0, k + h^\vee).$$ 

From explicit formulas for the bosonic realization of the Wakimoto module $W(\lambda_0, k)$ (see Proposition 2.4.1) it follows that the natural embedding of $\hat{\mathfrak{h}}$-modules $\pi(\lambda_0, k + h^\vee) \to W(\lambda_0, k)$ gives rise to an embedding of $\hat{\mathfrak{h}}$-modules 

$$\pi(\lambda_0, k + h^\vee) \to C_\chi \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k).$$ 

Finally observe that the space $\pi(\lambda_0, k + h^\vee)$ is decomposed into the direct sum of finite-dimensional weight subspaces with respect to the action of the Lie algebra $\mathfrak{h}$. Therefore, in view of (3.3.3), embedding (3.3.4) is an isomorphism of $\mathfrak{h}$-modules.

Remark 3.3.4. Using linear isomorphism (3.3.2) one can equip the space $C_\chi \otimes_{U(\mathfrak{z}^{-1}\mathfrak{n}[z^{-1}])} W(\lambda_0, k)$ with the structure of an $H^0$-module. This $H^0$-module structure is not natural.
Proposition 3.3.5. The only nontrivial component \( S_1 : \pi(0, k + h^\vee) \to \pi(-2, k + h^\vee) \) of the differential of resolution (3.3.3) is given by \( S_1 = \text{Res}_{z=0} J_1(z) \), where the generating series \( J_1(z) \) is defined as follows

\[
J_1(z) = \exp \left( -\sum_{n=1}^{\infty} \frac{a_{-n}}{(k + 2)n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{(k + 2)n} z^{-n} \right) V,
\]

and the operator \( V : \pi(0, k + h^\vee) \to \pi(-2, k + h^\vee) \) sends the vacuum vector \( v_0 \) of \( \pi(0, k + h^\vee) \) to the vacuum vector \( v_{-2} \) of \( \pi(-2, k + h^\vee) \) and commutes with the elements \( a_n \) as follows

\[
[a_n, V] = -2V\delta_{n,0}.
\]

Proof. First observe that the only nontrivial component of the differential in complex (3.3.3) is induced by that arising from the resolution of the vacuum representation \( V_0 \) by Wakimoto modules (see Corollary 2.3.4). The last differential is an intertwining operator between \( \hat{\mathfrak{g}} \)-modules \( W(0, k) \) and \( W(-2, k) \). If such operator exists then either \( W(0, k) \) has a cosingular vector or \( W(-2, k) \) has a singular vector.

In the proof of Proposition 2.4.3 we observed that for \( \lambda \) of finite type and \( k \neq -2 \) the Wakimoto module \( W(\lambda) = W(\lambda_0, k) \) may only have cosingular vector \( \omega_0 = v_{\lambda_0} \) if \( \lambda_0 \in \mathbb{Z}_+ \). Therefore, using Remark 2.3.1 we conclude that the only nontrivial intertwining operator between \( W(0, k) \) and \( W(-2, k) \) corresponds the cosingular vector \( \omega_0 \cdot v_0 \in W(0, k) \). This operator is the projection operator onto the quotient of \( W(0, k) \) by the submodule generated by the highest weight vector, the image of the cosingular vector being the highest weight vector in \( W(-2, k) \). Explicit calculation shows that this operator coincides with the intertwining operator \( S : W(0, k) \to W(-2, k) \) introduced in Proposition 2.4.3.

Next observe that the algebra \( U(\hat{\mathfrak{n}}_+) \) is commutative and hence for any \( \lambda_0 \) the action of this algebra on the space \( W(\lambda_0, k) \) gives rise to an action on the space \( \mathbb{C}_\chi \otimes_{U(z^{-[n][l]} z)} W(\lambda_0, k) = \pi(\lambda_0, k + h^\vee) \). Using the definition of the character \( \chi \), Lemma 1.3.1 and the fact that any Wakimoto module is isomorphic to \( S_{U(\hat{\mathfrak{n}}_+) \otimes U(z^{-1} h[z^{-1}])} \) as an \( U(\hat{\mathfrak{n}}_+) \)-module we conclude that for \( n \neq -1 \) the elements \( X^+_n = \omega^+_n \) act on the space \( \mathbb{C}_\chi \otimes_{U(z^{-[n][l]} z)} W(\lambda_0, k) = \pi(\lambda_0, k + h^\vee) \) in the trivial way and the element \( X^+_{-1} = \omega^+_{-1} \) acts on this space as the identity operator.

Finally note that the action of the elements \( a_n, n \in \mathbb{Z} \) and of the operator \( V \) on the resolution (3.3.5) commute with the action of the algebra \( U(\hat{\mathfrak{n}}_+) \). Therefore the operator \( S : W(0, k) \to W(-2, k) \) gives rise to the operator \( S_1 : \pi(0, k + h^\vee) \to \pi(-2, k + h^\vee) \).

In conclusion we recall that the action of the generators \( T_n \) of the algebra \( W_k(\mathfrak{sl}_2) \) on the spaces \( W(0, k) \) and \( W(-2, k) \) is given in terms of the generating series \( T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n} \) by

\[
T(z) = \frac{1}{4(k + 2)} : a(z)^2 + \frac{1}{2} \left( 1 - \frac{1}{k + 2} \right) z^2 \frac{d}{dz} (z^{-1} a(z)).
\]
4. Affine quantum groups and their representations

4.1. Affine quantum groups. In this section we recall some basic facts about affine quantum groups \[12\]. We follow the notation of \[12\].

Let \(h\) be an indeterminate, \(\mathbb{C}[[h]]\) the ring of formal power series in \(h\). We shall consider \(\mathbb{C}[[h]]–\)modules equipped with the so–called \(h–\)adic topology. For every such module \(V\) this topology is characterized by requiring that \(\{h^nV \mid n \geq 0\}\) is a base of the neighbourhoods of 0 in \(V\), and that translations in \(V\) are continuous. It is easy to see that, for modules equipped with this topology, every \(\mathbb{C}[[h]]–\)module map is automatically continuous.

A topological algebra over \(\mathbb{C}[[h]]\) is a complete \(\mathbb{C}[[h]]–\)module \(A\) equipped with a structure of \(\mathbb{C}[[h]]–\)algebra (see \[12\], Definition 4.3.1). All tensor products (direct sums) of complete \(\mathbb{C}[[h]]–\)modules and of topological algebras over \(\mathbb{C}[[h]]\) will be understood as completed in the \(h–\)adic topology algebraic tensor products (direct sums).

The standard quantum group \(U_h(\hat{\mathfrak{g}})\) associated to an affine Lie algebra \(\hat{\mathfrak{g}}\) is the algebra over \(\mathbb{C}[[h]]\) topologically generated by elements \(H_i, X_i^+, X_i^−, i = 0, \ldots, l\) and \(\partial\) with the following defining relations:

\[
[H_i, H_j] = 0, \ [H_i, X_j^±] = ±a_{ij}X_j^±,
\]

\[
X_i^+X_j^− - X_j^−X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

\[
[\partial, H_i] = 0, \ [\partial, X_i^±] = ±\delta_{i,0}X_i^±,
\]

where \(K_i = e^{d_i h}, e^h = q, q_i = q^{d_i} = e^{d_i h},\)

and the quantum Serre relations:

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1 - a_{ij}}{r} \right]_{q_i} (X_i^±)^{1-a_{ij}-r}X_j^±(X_i^±)^r = 0, \ i \neq j,
\]

where

\[
\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

We shall also use the weight–type generators defined by

\[
Y_i = \sum_{j=1}^{l} d_i(a^{-1})_{ij} H_j,
\]

and the elements \(L_i = e^{h Y_i}\).

The Cartan antiinvolution \(\omega : U_h(\hat{\mathfrak{g}}) \rightarrow U_h(\hat{\mathfrak{g}})\) is defined on generators by

\[
\omega(X_i^±) = X_i^±, \ \omega(H_i) = H_i, \ \omega(\partial) = \partial, \ \omega(h) = -h.
\]

Note that the algebra \(U_h(\hat{\mathfrak{g}})/hU_h(\hat{\mathfrak{g}})\) is naturally isomorphic to \(U(\hat{\mathfrak{g}})\).

Next following \[12, 13, 14\] and \[16\] we recall the construction of the Cartan–Weyl basis for \(U_h(\hat{\mathfrak{g}})\). In order to construct such a basis one should fix an ordering of the root system \(\Delta_+\).

We say that the system \(\Delta_+\) is in normal (or convex) ordering if its roots are totally ordered in the following way: (i) all simple roots follow each other in an
arbitrary order; (ii) each nonsimple root $\alpha + \beta \in \Delta_+$, where $\alpha \neq n\beta$ is situated between $\alpha$ and $\beta$.

Fix some normal ordering in $\Delta_+$ satisfying an additional condition:

$$\alpha_i + n\delta < k\delta < (\delta - \alpha_j) + l\delta$$

for any simple roots $\alpha_i, \alpha_j \in \Pi_0$, $i, j = 1, \ldots, l$, $n, k \geq 0$, $l > 0$. We apply the following inductive procedure for the construction of real root vectors $X_\gamma, \gamma \in \Delta_+^c$ starting from the simple root vectors $X_\alpha = X_i^+, i = 0, \ldots, l$.

Let $\gamma \in \Delta_+^c$ be a real root and $\alpha_1, \ldots, \gamma, \ldots, \beta$ the minimal subset in $\Delta_+$ containing $\gamma$ such that $\alpha < \gamma < \beta$ and $\gamma = \alpha + \beta$. Then we set

$$X_\gamma = [X_\alpha, X_\beta]_q$$

if $X_\alpha$ and $X_\beta$ have already been constructed. Here

$$[X_\alpha, X_\beta]_q = X_\alpha X_\beta - q^{(\alpha, \beta)} X_\beta X_\alpha.$$ 

When we get the imaginary root $\delta$ we stop for a moment and use the following formulas:

$$X^{(i)}_\delta = \varepsilon_1(\alpha_i)[X_\alpha, X_{\delta-\alpha_i}]_q,$$

$$X_{\alpha_i + m\delta} = \varepsilon_m(\alpha_i)(-\alpha_i)_q) - m(ad X^{(i)}_\delta)^m X_{\alpha_i},$$

$$X_\delta - \alpha_i + m\delta = \varepsilon_m(\alpha_i)(\alpha_i)_q) - m(ad X^{(i)}_\delta)^m X_{\delta-\alpha_i},$$

$$X^{(i)}_{(m+1)\delta} = \varepsilon_{m+1}(\alpha_i)[X_{\alpha_i + m\delta}, X_{\delta-\alpha_i}]_q,$$

for $m > 0$, $i = 1, \ldots, l$, where $(ad x)y = [x, y]$ is the usual commutator, $\varepsilon_m(\alpha_i) = (-1)^m\theta(\alpha_i)$, and the function $\theta : \Pi_0 = \{\alpha_1, \ldots, \alpha_l\} \to \{0, 1\}$ is chosen in such a way that for any pair $i, j$, $i \neq j$ such that $(\alpha_i, \alpha_j) \neq 0$ we have $\theta(\alpha_i) \neq \theta(\alpha_j)$.

Then we use the inductive procedure again to obtain other real root vectors $X_{\gamma + n\delta}, X_{\delta - \gamma + n\delta}$, $\gamma \in \Delta$. We come to the end by defining imaginary root vectors $X^{(i)}_{n\delta}$ via intermediate vectors $X^{(i)}_{n\delta}$ by means of the following (Schur) relations:

$$q_i - q_i^{-1}E^{(i)}(z) = \log \left( 1 + (q_i - q_i^{-1})E^{(i)}(z) \right)$$

where $E^{(i)}(z)$ and $E^{(i)}(z)$ are generating functions for $X^{(i)}_{n\delta}$ and for $X^{(i)}_{n\delta}$;

$$E^{(i)}(z) = \sum_{n \geq 1} X^{(i)}_{n\delta} z^{-n},$$

$$E^{(i)}(z) = \sum_{n \geq 1} X^{(i)}_{n\delta} z^{-n}.$$ 

The root vectors for negative roots are obtained by the Cartan antiinvolution $\omega$:

$$X_{-\gamma} = \omega(X_\gamma)$$

for $\gamma \in \Delta_+$.

For $\gamma = \sum_{i=0}^l l_i \alpha_i$ we also put

$$\gamma^\lor = \sum_{i=0}^l l_i d_i H_i.$$ 

Let $U_h(\widehat{\mathfrak{g}}_+), U_h(\widehat{\mathfrak{g}}_-)$ and $U_h(\widehat{\mathfrak{g}})$ be the $\mathbb{C}[\hbar]$-subalgebras of $U_h(\widehat{\mathfrak{g}})$ topologically generated by the $X^+_i, X^-_i, i = 0, \ldots, l$ and by the $H_i$, $i = 0, \ldots, l$ and $\partial$, respectively.
Now using the root vectors $X_\gamma$ we can construct a topological basis of $U_h(\mathfrak{g})$.

**Proposition 4.1.1.** ([14], Proposition 3.3) The elements

$$(X_{\beta_1}^{(j_1)})^{r_1} \cdots (X_{\beta_p}^{(j_p)})^{r_p},$$

where $r_i > 0$, $j_i = 1, \ldots$, $\text{mult } \beta_i, \beta_i \in \Delta_+$ are positive roots such that

$$\beta_1 < \ldots < \beta_p$$

in the sense of the normal ordering, form a topological basis of $U_h(\mathfrak{n}_+)$. The elements

$$(X_{-\gamma_1}^{(s_1)})^{s_1} \cdots (X_{-\gamma_p}^{(s_p)})^{s_p}$$

where $s_i > 0$, $j_i = 1, \ldots$, $\text{mult } \gamma_i, \gamma_i$ are positive roots such that

$$\gamma_1 < \ldots < \gamma_p$$

in the sense of the normal ordering, form a topological basis of $U_h(\mathfrak{n}_-)$. The elements

$$\partial^t H_0^i \cdots H_l^i,$$

where $t_i, t \geq 0$, form a topological basis of $U_h(\mathfrak{h})$.

Multiplication defines an isomorphism of $\mathbb{C}[[\hbar]]$ modules:

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \rightarrow U_h(\mathfrak{g}).$$

We also denote by $U_h(\mathfrak{n}_\pm)$ the subalgebra in $U_h(\mathfrak{g})$ topologically generated by $X_i^\pm, i = 0, \ldots l$ and by the $H_i, i = 0, \ldots l$ and $\partial$. Clearly, multiplication defines an isomorphism of $\mathbb{C}[[\hbar]]$ modules:

$$U_h(\mathfrak{n}_\pm) \otimes U_h(\mathfrak{h}) \rightarrow U_h(\mathfrak{n}_\pm).$$

Next we introduce analogues of the subalgebras $U(\mathfrak{n}_\pm) \subset U(\mathfrak{g})$ and of the subalgebras $U(\mathfrak{a}), U(\mathfrak{f}) \subset U(\mathfrak{g})$ for the algebra $U_h(\mathfrak{g})$.

First we define other root vectors $\hat{X}_\gamma$ and $\hat{X}_\gamma$ by the following formulas (see [12]):

$$\hat{X}_\gamma = X_\gamma, \quad \hat{X}_\gamma = -\exp(-h\gamma^\vee)X_\gamma, \quad \forall \gamma \in \Delta_+,$$

and

$$\hat{X}_\gamma = X_\gamma, \quad \hat{X}_\gamma = -X_\gamma \exp(h\gamma^\vee), \quad \forall \gamma \in \Delta_+.$$

Denote by $U_h(\mathfrak{n}_+)$ and $U_h(\mathfrak{n}_-)$ the subalgebra in $U_h(\mathfrak{g})$ topologically generated by the elements $\hat{X}_{n\delta + \alpha_i}, n \in \mathbb{Z}, i = 1, \ldots, l$ and $\hat{X}_{n\delta - \alpha_i}, n \in \mathbb{Z}, i = 1, \ldots, l$, respectively. We also denote by $U_h(\mathfrak{a})$ and $U_h(\mathfrak{f})$ the subalgebra in $U_h(\mathfrak{g})$ topologically generated by the elements $\hat{X}_{n\delta + \alpha_i}, n \in \mathbb{Z}, \hat{X}_{r \delta}^{(i)}, r < 0, i = 1, \ldots, l$ and by $\hat{X}_{n\delta - \alpha_i}, n \in \mathbb{Z}, \hat{X}_{r \delta}^{(i)}, r > 0, i = 1, \ldots, l, H_i, i = 0, \ldots, l, \partial$, respectively. To construct topological bases for $U_h(\mathfrak{a})$ and $U_h(\mathfrak{f})$ we fix the following ordering in the root system $\Delta$:

$$\gamma_1, \gamma_2, \ldots, \gamma_N, -\gamma_1, -\gamma_2, \ldots, -\gamma_N,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_N$ is the normal ordering in $\Delta_+$ used in the construction of the root vectors $X_\alpha$.

The following proposition follows immediately from the results of [12] on commutation relations between the elements $\hat{X}_\gamma$ and $\hat{X}_\gamma$. 
Proposition 4.1.2. The elements

\((\hat{X}_{\beta_1})^{r_1} \ldots (\hat{X}_{\beta_p})^{r_p}\),

where

\(r_i > 0, \beta_i \in \{\alpha + n\delta, \alpha \in \Delta_+^\circ, n \in \mathbb{Z}\}\)

and

\(\beta_1 < \ldots < \beta_p\)

in the sense of the normal ordering, form a topological basis of \(U_h(\tilde{n}_+)\).

The elements

\((\hat{X}_{\gamma_1})^{s_1} \ldots (\hat{X}_{\gamma_p})^{s_p}\),

where

\(s_i > 0, \gamma_i \in \{-\alpha + n\delta, \alpha \in \Delta_+^\circ, n \in \mathbb{Z}\}\)

and

\(\gamma_1 < \ldots < \gamma_p\)

in the sense of the normal ordering, form a topological basis of \(U_h(\tilde{n}_-)\).

The products

\((\hat{X}_{\gamma_1}^{(j_1)})^{s_1} \ldots (\hat{X}_{\gamma_q}^{(j_q)})^{s_q}\partial^m H_0^{m_0} \ldots H_l^{m_l}\),

where \(s_i > 0, j_i = 1, \ldots, \text{mult } \gamma_i, \gamma_i \in \{-\alpha + n\delta, \alpha \in \Delta_+^\circ, n \in \mathbb{Z}\} \cup \{r\delta, r < 0\}\) and

\(\beta_1 < \ldots < \beta_p\)

in the sense of the normal ordering, form a topological basis of \(U_h(a)\).

Multiplication defines an isomorphism of \(\mathbb{C}[[h]]\) modules:

\(U_h(a) \otimes U_h(\tilde{a}) \rightarrow U_h(\tilde{g})\).

One can also introduce another realization of the algebra \(U_h(\tilde{g})\), called the new Drinfeld realization, in which the elements \(\hat{X}_\gamma, \hat{X}_{-\gamma} \in U_h(\tilde{g}), \gamma \in \Delta_+^\circ\) and \(X_i^{(r)} \in U_h(\tilde{g})\) play the role of generators (see [12]). Namely we have

Proposition 4.1.3. ([12, Theorem 7.1]) The algebra \(U_h(\tilde{g})\) is isomorphic to the associative algebra topologically generated by elements \(X_i^{\pm}, r \in \mathbb{Z}, H_{i,r}, r \in \mathbb{Z}, i = 1, \ldots, l, K, \partial\) with relations given in terms of the generating series

\[X_i^{\pm}(u) = \sum_{r \in \mathbb{Z}} X_i^{\pm r} u^{-r},\]

\[\Phi_i^{\pm}(u) = \sum_{r=0}^{\infty} \Phi_i^{\pm r} u^{\pm r} = K_i^{\pm 1} \exp \left(\pm(q_i - q_i^{-1}) \sum_{s=1}^{\infty} H_{i,\pm s} u^{\mp s}\right),\]

\[K_i = \exp(d_i h H_{i,0}).\]
by
\[ [\partial, X^+_{i,r}] = r X^+_{i,r}, \]
\[ [\partial, H_{i,r}] = r H_{i,r}, \]
\[ [H_{i,0}, H_{j,r}] = 0, \quad r \in \mathbb{Z}, \]
\[ [H_{i,0}, X^\pm_j(u)] = \pm a_{ij} X^\pm_j(u), \]
\[ \Phi^+_i(u) \Phi^-_j(v) = \Phi^+_j(v) \Phi^+_i(u), \]
\[ K \text{ is central }, \]
\[ \Phi^+_i(u) \Phi^-_j(v) = \frac{g_{ij}(\frac{vq^K}{u})}{g_{ij}(\frac{v^K}{u})} \Phi^-_j(v) \Phi^+_i(u), \]
\[ \Phi^-_i(u) X^\pm_j(v) \Phi^-_i(u)^{-1} = g_{ij}(\frac{vq^K}{u}) \pm 1 X^\pm_j(v), \]
\[ \Phi^+_i(u) X^\pm_j(v) \Phi^+_i(u)^{-1} = g_{ij}(\frac{vq^K}{u}) \mp 1 X^\pm_j(v), \]
\[ (u - v q^{b_{ij}}) X^\pm_i(u) X^\pm_j(v) = (q^{b_{ij}} - u) X^\pm_i(v) X^\pm_j(u), \]
\[ X^+_i(u) X^-_j(v) - X^-_j(v) X^+_i(u) = \frac{h_{ij}}{q_i - q_{i'}} \left( \delta(\frac{uq^K}{v}) \Phi^+_i(vq^K) - \delta(\frac{uq^K}{v}) \Phi^-_i(vq^K) \right), \]
\[ \sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ 1 - a_{ij} \right] k \times \]
\[ X^+_i(z_{\pi(1)}) \cdots X^+_i(z_{\pi(k)}) X^+_i(w) X^+_i(z_{\pi(k+1)}) \cdots X^+_i(z_{\pi(1-a_{ij})}) = 0, \quad i \neq j, \]
where \( g_{ij}(z) = \frac{1 - q^{b_{ij}} z}{1 - q^{b_{ij}} z} q^{-b_{ij}} \in \mathbb{C}_q[[z]][[z]] \) and \( S_n \) is the symmetric group of \( n \) elements. The isomorphism is explicitly given by
\[ K = \delta^i, \quad H_{i,0} = H_i, \quad \partial = \partial \]
\[ H_{i,n} = X^{(i)}_{n0} \exp(\frac{h}{2} n \delta^i), \]
\[ X^+_i = \hat{X}_{n0 + \alpha_i}, \]
\[ X^-_i = \hat{X}_{n0 - \alpha_i}. \]
Sometimes it is convenient to use the weight–type generators \( Y_{i,r} \),
\[ Y_{i,r} = \sum_{k=1}^l (a_{i}^r)^{-1} H_{k,r}, \quad a_{ij}^r = \frac{1}{r} [\exp(\frac{h}{r} n \delta^i)], \quad i, j = 1, \ldots, l. \]
The generators \( X^\pm_i, H_{i,r} \) correspond to the elements \( X^\pm_i z^r, H_i z^r \) of the affine Lie algebra \( \hat{\mathfrak{g}} \) in the loop realization (here \( X^\pm_i, H_i \) are the Chevalley generators of \( \mathfrak{g} \)).
4.2. Verma and Wakimoto modules over affine quantum groups. In the h-adic case the definition of \( \mathbb{Z} \)-graded modules is slightly different from the standard one. A complete topological module \( V \) over \( \mathbb{C}[[h]] \) is called \( \mathbb{Z} \)-graded if it is isomorphic to the h-adic completion of the direct sum \( \oplus_{n \in \mathbb{Z}} V_n \), where \( V_n \subset V \) is the subspace of elements of degree \( n \). A topological algebra \( A \) over \( \mathbb{C}[[h]] \) is called \( \mathbb{Z} \)-graded if, as a \( \mathbb{C}[[h]] \)-module, it is isomorphic to the h-adic completion of the direct sum \( \oplus_{n \in \mathbb{Z}} A_n \), where \( A_n \subset A \) is the subspace of elements of degree \( n \), and multiplication in \( A \) defines maps \( A_n \otimes A_m \to A_{n+m} \). A complete topological module \( V \) over \( \mathbb{Z} \)-graded topological algebra \( A \) is called \( \mathbb{Z} \)-graded if it is \( \mathbb{Z} \)-graded as a topological module over \( \mathbb{C}[[h]] \) and the action of \( A \) on \( V \) defines maps \( A_n \times V_m \to V_{n+m} \). A morphism \( \phi : V \to W \) of \( \mathbb{Z} \)-graded topological modules \( V \) and \( W \) over \( \mathbb{Z} \)-graded topological algebra \( A \) is a \( \mathbb{C}[[h]] \)-linear map commuting with the action of \( A \) on \( V \) and \( W \) such that \( \phi(V_n) \subset W_n \) for any \( n \in \mathbb{Z} \).

The category of left (right) \( \mathbb{Z} \)-graded topological modules over a \( \mathbb{Z} \)-graded topological algebra \( A \) with morphisms being morphisms of graded topological \( A \)-modules is denoted by \( A - \text{mod} \) \( (\text{mod } A) \). For both of these categories the set of morphisms between two objects is denoted by \( \text{Hom}_{A}(\cdot, \cdot) \). For \( M, M' \in \text{Ob } A - \text{mod} \) \( (\text{mod } A) \) we shall also use the space of homomorphisms \( \text{Hom}_{A}(M, M') \) of all possible degrees with respect to the gradings on \( M \) and \( M' \) defined as the h-adic completion of the space \( \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{A}(M, M'(n)) \) (see Section 1.4 for the definition of this space).

All the results presented in Section 1 for graded associative algebras and modules hold for complete topological graded algebras and modules if morphisms of modules are understood in the h-adic sense. In particular, Verma and Wakimoto modules over the algebra \( U_h(\mathfrak{g}) \) are defined similarly to the Lie algebra case (see Section 2.2). One should apply the general scheme of Sections 1.3 and 1.7 taking into account that the algebra \( U_h(\mathfrak{g}) \) is naturally \( \mathbb{Z} \)-graded, \( \deg(\mathfrak{h}_i) = \deg(\partial) = 0 \) for \( i = 0, \ldots, l \), \( \deg(X_+^i) = 1 \), \( \deg(X_-^i) = -1 \), and satisfies conditions (i)–(viii) of Sections 1.4, 1.2, 1.6 and 1.7 with \( N^{\pm} = U_h(\mathfrak{h}_{\pm}) \), \( H = U_h(\mathfrak{h}) \), \( A_0 = U_h(\mathfrak{a}) \), \( A_1 = U_h(\mathfrak{a}^\ast) \).

In this paper we only need Verma, contragradient Verma and Wakimoto modules over \( U_h(\mathfrak{g}) \) corresponding to a very special set of characters \( \lambda : U_h(\mathfrak{h}) \to \mathbb{C}[[h]] \). To introduce this set we note that the algebra \( U_h(\mathfrak{h}) \) is isomorphic to \( U(\mathfrak{h})[[h]] \). Let \( \lambda : \mathfrak{h} \to \mathbb{C} \) be a character. This character naturally extends to a character \( \lambda : U_h(\mathfrak{h}) \to \mathbb{C}[[h]] \). We denote by \( M_h(\lambda) \), \( M_h(\lambda)\lor \) and \( W_h(\lambda) \) the Verma, the contragradient Verma and the Wakimoto module corresponding to this character. Observe that the \( U(\mathfrak{g}) \)-modules \( M_h(\lambda)/hM_h(\lambda) \), \( M_h(\lambda)\lor/hM_h(\lambda)\lor \) and \( W_h(\lambda)/hW_h(\lambda) \) are naturally isomorphic to \( M(\lambda) \), \( M(\lambda)\lor \) and \( W(\lambda) \), respectively.

Let \( V \) be a \( U_h(\mathfrak{g}) \)-module. One says that \( V \) admits a weight space decomposition if, as an \( U(\mathfrak{h}) \)-module, \( V \) is isomorphic to the h-adic completion of the \( U(\mathfrak{h}) \) module

\[
\bigoplus_{\eta \in \mathfrak{h}^\ast} (V)_{\eta},
\]

where

\[
(V)_{\eta} = \{ v \in V : h \cdot v = \eta(h)v \text{ for any } h \in U(\mathfrak{h}) \}
\]

is the subspace of weight \( \eta \) in \( V \). Here \( U(\mathfrak{h}) \) is regarded as a subalgebra in \( U_h(\mathfrak{h}) \).
If all the spaces $V_i$ are finite-dimensional over $\mathbb{C}[[h]]$ then one can introduce the formal character of $V$ by

$$\text{ch}(V) = \sum_{\eta \in \mathfrak{h}^*} \dim((V)_\eta)e^\eta.$$

From the definitions of the modules $M_h(\lambda)$, $M_h(\lambda)^\vee$ and $W_h(\lambda)$ and Propositions 4.1.1, 4.1.2, it follows that they have the same weight space decompositions and the same characters as in the nondeformed case.

Clearly, any $U_h(\mathfrak{g})$ module $V$ is always reducible. It contains proper submodule $hV$. Therefore it makes sense to change the definition of reducibility for $U_h(\mathfrak{g})$–modules. We shall say that an $U_h(\mathfrak{g})$–module $V$ is reducible if and only if the corresponding $U(\mathfrak{g})$–module $M(\lambda)$ is reducible, i.e. iff

$$2(\lambda + \rho, \alpha) = n(\alpha, \alpha)$$

for some $\alpha \in \Delta_+$, $n \in \mathbb{N}$. In this case $M_h(\lambda)$ $(M_h(\lambda)^\vee)$ contains a singular (cosingular) vector of weight $\lambda - n\alpha$.

We also note that the composition of the canonical maps

$$M_h(\lambda) \to W_h(\lambda) \to M_h(\lambda)^\vee$$

gives the Shapovalov form of $M_h(\lambda)$. Therefore we obtain the following corollary of the previous proposition.

**Corollary 4.2.2.** The module $W_h(\lambda)$ is reducible if and only if $M_h(\lambda)$ is reducible. Moreover $W_h(\lambda)$ has singular and cosingular vectors of the same weights as the singular vectors of $M(\lambda)$.

Using Proposition 4.2.1 and Corollary 4.2.2 we conclude that all the statements about Wakimoto modules over $U(\mathfrak{g})$ with highest weight of finite type may be carried over to the deformed case. Here we only formulate analogues of Proposition 2.3.2 and Corollary 2.3.4 for $U_h(\mathfrak{g})$. The proofs of these statements are quite similar to those in the undeformed case.

Denote by $U_h(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)$ the subalgebra in $U_h(\mathfrak{g})$ topologically generated by $X_i^\pm, i = 1, \ldots, l, X_0^\pm$ and by the $H_i, i = 0, \ldots, l$ and $\partial$. Let $U_h(\mathfrak{g})$ be the subalgebra in $U_h(\mathfrak{g})$ topologically generated by $X_i^\pm, i = 1, \ldots, l$ and by the $H_i, i = 1, \ldots, l$.

**Proposition 4.2.3.** Let $\lambda : \mathfrak{h} \to \mathbb{C}$ be a character of finite type. Then the canonical map

$$W_h(\lambda) \to M_h(\lambda)^\vee$$

is an isomorphism of $U_h(\mathfrak{g})$–modules. Let $M_h(\lambda_0)^\vee$ be the contragradient Verma module over $U_h(\mathfrak{g})$ of highest weight $\lambda_0 = \lambda|_h$. This module is uniquely extended to a $U_h(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)$–module $(M_h(\lambda_0)^\vee)_{k, \lambda(\partial)}$ in such a way that $K$ and $\partial$ act by multiplication by $k = \lambda(K)$ and by $\lambda(\partial)$, respectively. Then both $M_h(\lambda)^\vee$ and $W_h(\lambda)$ are isomorphic to the induced representation $U_h(\mathfrak{g}) \otimes_{U_h(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M_h(\lambda_0)^\vee)_{k, \lambda(\partial)}$.

$$M_h(\lambda)^\vee = W_h(\lambda) = U_h(\mathfrak{g}) \otimes_{U_h(\mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}\partial)} (M_h(\lambda_0)^\vee)_{k, \lambda(\partial)}.$$
To formulate the quantum group analog of Corollary 4.2.4 we first recall that, according to Theorem 3.3 in [50], for the finite–dimensional irreducible representation \( L_h(\lambda_0) \) of the algebra \( U_h(\mathfrak{g}) \) with integral dominant highest weight \( \lambda_0 \) one can define the Bernstein–Gelfand–Gelfand resolution by contragradient Verma modules \( M_h(w(\lambda_0 + \rho_0) - \rho_0)\), over \( U_h(\mathfrak{g}) \),

\[
0 \to C^1_h(\lambda_0) \to \cdots \to C^{\dim \mathfrak{g}}_{h}(\lambda_0) \to 0,
\]

\[
C^i_h(\lambda_0) = \bigoplus_{w \in W^{(i)}} M_h(w(\lambda_0 + \rho_0) - \rho_0)^\vee,
\]

where \( W^{(i)} \subset W \) is the subset of the elements of length \( i \) of the Weyl group of \( \mathfrak{g} \) and \( \rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta^c} \alpha \).

**Corollary 4.2.4.** Let \( \lambda \) be a character of \( \hat{\mathfrak{h}} \) of generic level \( k \) such that \( \lambda_0 = \lambda|_{\mathfrak{h}} \) is an integral dominant weight for \( \mathfrak{g} \), i.e., \( \lambda_0 \in P^+ \), where \( P^+ = \{ \lambda \in \mathfrak{h}^* : \lambda(H_i) \in \mathbb{Z}_{\geq 0}, i = 1, \ldots, l \} \). Let \( L_h(\lambda_0) \) be the irreducible finite–dimensional representation of \( U_h(\mathfrak{g}) \) with highest weight \( \lambda_0 \) and denote by \( C^i_h(\lambda_0) \) the Bernstein–Gelfand–Gelfand resolution of \( L_h(\lambda_0) \) by contragradient Verma modules. Then the induced complex of \( U_h(\hat{\mathfrak{g}}) \)–modules

\[
0 \to D^1_h(\lambda) \to \cdots \to D^{\dim \mathfrak{g}}_h(\lambda) \to 0,
\]

\[
D^i_h(\lambda) = \bigoplus_{w \in W^{(i)}} U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}[z]+C_{K+C_0}) \left( M_h(w(\lambda_0 + \rho_0) - \rho_0)^\vee \right)_{k,\lambda(\partial)}
\]

is a resolution of the induced representation \( U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}[z]+C_{K+C_0}) \left( L_h(\lambda_0) \right)_{k,\lambda(\partial)} \) by Wakimoto modules, i.e.,

\[
D^i_h(\lambda) = \bigoplus_{w \in W^{(i)}} W_h(w(\lambda + \rho_0) - \rho_0),
\]

where the Weyl group \( W \) is regarded as a subgroup in the affine Weyl group of the Lie algebra \( \hat{\mathfrak{g}} \).

**4.3. Bosonization for \( U_h(\hat{\mathfrak{gl}}) \).** In this section we recall, following [50, 3], the bosonic realization of the Wakimoto module in case of the algebra \( U_h(\hat{\mathfrak{sl}}_2) \).

Let \( k \) be a complex number. Let \( \mathbf{H}_h \) be the topological algebra over \( \mathbb{C}[[h]] \) topologically generated by elements \( a_n, b_n, c_n, Q_a, Q_b, Q_c, n \in \mathbb{Z} \) satisfying the following commutation relations:

\[
[a_m, a_n] = \delta_{n+m,0} \frac{[2n]_q}{n} \frac{[(k+2)n]_q}{n}, \quad [a_0, Q_a] = 2(k+2),
\]

\[
(4.3.1) \quad [b_m, b_n] = -\delta_{n+m,0} \frac{[2n]_q}{n} \frac{[2n]_q}{n}, \quad [b_0, Q_b] = -4,
\]

\[
[c_n, c_m] = \delta_{n+m,0} \frac{[2n]_q}{n} \frac{[2n]_q}{n}, \quad [c_0, Q_c] = 4,
\]

and all the other commutators of the elements \( a_n, b_n, c_n, Q_a, Q_b, Q_c, n \in \mathbb{Z} \) vanish.

For the elements \( a_n \) we introduce formal generating series \( a(L; M, N|z; \alpha) \) carrying parameters \( L, M, N \in \mathbb{N}, \alpha \in \mathbb{C} \),

\[
a(L; M, N|z; \alpha) = -\sum_{n \neq 0} \left[ \frac{L^n}{M^n q^n} \frac{a_n}{[M^n]_q [N^n]_q} z^{-n} q^n \right]^{\alpha} + \frac{La_0}{MN} \log z + \frac{LQ_a}{MN}.
\]
We define generating series $b(L; M, N|z; \alpha)$, $c(L; M, N|z; \alpha)$ in the same way. In case $L = M$ we also write $a(N|z; \alpha) = a(L; L, N|z; \alpha) = -\sum_{n \neq 0} \frac{a_n}{[Nn]_q} z^{-n} q^{|n|}\alpha + \frac{a_0}{N} \log z + \frac{Q_\alpha}{N}$.

and similarly for $b(N|z; \alpha)$, $c(N|z; \alpha)$.

We define a $q$-difference operator with a parameter $n \in \mathbb{N}$ by

$$n \partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n} z)}{(q - q^{-1}) z}.$$

For $p \in \mathbb{C}[[h]]$, $s \in \mathbb{C}$ we define the Jackson integral by

$$\int_0^{s\infty} f(t)dp = s(1 - p) \sum_{m=-\infty}^{\infty} f(sp^m)p^m.$$

We also denote by $\cdots$ the normal ordered product of elements of $H'_h$ in which the elements $a_n, b_n, c_n, n \geq 0$ stand on the right.

**Proposition 4.3.1.** ([59], Proposition 3) Let $H_h$ be the algebra over $\mathbb{C}[[h]]$ topologically generated by elements $a_n, b_n, c_n, V_Q = \exp(\frac{Q_\alpha}{2}), V_Q^{-1} = \exp(-\frac{Q_\alpha}{2})$, $n \in \mathbb{Z}$, where $a_n, b_n, c_n, Q$ satisfy commutation relations (4.3.1). Denote by $\mathcal{H}(\lambda_0)_h$ the representation space for the algebra $H_h$ topologically generated by the vacuum vector $v_{\lambda_0}$ satisfying the following conditions

$$a_n \cdot v_{\lambda_0} = 0 \text{ for } n > 0, \quad b_n \cdot v_{\lambda_0} = 0 \text{ for } n \geq 0, \quad c_n \cdot v_{\lambda_0} = 0 \text{ for } n \geq 0, \quad a_0 \cdot v_{\lambda_0} = \lambda_0 v_{\lambda_0}.$$

Let $k \neq -2$. Then the Fourier coefficients of the generating series

$$H = a_0 + b_0, \quad K = k,$$

$$\partial = \sum_{n=1}^{\infty} \left( \frac{-n^2}{2n} a_{-n} a_n + \frac{n^2}{2n} b_{-n} b_n - \frac{n^2}{2n} c_{-n} c_n \right),$$

$$\Phi^+(z) =: \exp \left\{ (q - q^{-1}) \sum_{n>0} (q^n a_n + q^{-n-1} b_n) z^{-n} + h(a_0 + b_0) \right\} :,$$
\[ \Phi^-(z) =: \exp \left\{ -(q - q^{-1}) \sum_{n < 0} (q^{3n}a_n + q^{2(k+2)} n b_n) z^{-n} - h(a_0 + b_0) \right\}; \]

\[ X^+(z) = -z : [ i \partial_z \exp \{ -c(2|q^{-k-2}z; 0) \} ] \times \exp \{ -b(2|q^{-k-2}z; 1) \}; \]

\[ X^-(z) = z : [ k+2 \partial_z \exp \{ a(k+2|q^{-2}z; - \frac{q^2}{2}) + b(2|q^{-k-2}z; -1) \] 

\[ + c(k+1; 2, k+2|q^{-k-2}z; 0) \} ] \]

\[ \times \exp \{ -a(k+2|q^{-2}z; \frac{k+2}{2}) + c(1; 2, k+2|q^{-k-2}z; 0) \} \] are well-defined operators in the space \( \mathcal{H}(\lambda_0)_h \) and satisfy the defining relations of the algebra \( U_h(\mathfrak{sl}_2) \) in the new Drinfeld realization (see Proposition 4.1.3).

We denote by \( W_h(\lambda_0, k) \) the representation of the algebra \( U_h(\mathfrak{sl}_2) \) in the space \( \mathcal{H}(\lambda_0)_h \) constructed in the previous proposition.

**Proposition 4.3.2.** Let \( \lambda : \widehat{\mathfrak{h}} \to \mathbb{C} \) be a character of the Cartan subalgebra \( \widehat{\mathfrak{h}} \) of the Lie algebra \( \mathfrak{sl}_2 \) such that \( \lambda(H) = \lambda_0, \lambda(K) = k, k \neq -2 \) and \( \lambda(\partial) = 0 \). Denote the natural extension of \( \lambda \) to a character \( \lambda : U_h(\mathfrak{h}) \to \mathbb{C}[[h]] \) by the same letter. Suppose that \( \lambda \) is of finite type. Then the \( U_h(\mathfrak{sl}_2) \)-module \( W_h(\lambda_0, k) \) is isomorphic to the Wakimoto module \( W_h(\lambda) \). In this case both \( W_h(\lambda_0, k) \) and \( W_h(\lambda) \) are isomorphic to the contragradient Verma module \( M_h(\lambda) \).

**Proof.** The proof of this proposition is similar to that of Proposition 2.4.2 in the nondeformed case. We only note here that the module \( W_h(\lambda_0, k) \) may only have cosingular vector \( V_0^{\lambda_0+1} \cdot v_{\lambda_0} \) when \( \lambda_0 \in \mathbb{Z}_+ \).

In conclusion we recall the definition of screening operators which are certain intertwining operators between Wakimoto modules \( W_h(\lambda_0, k) \). First we introduce an operator \( V_h : \mathcal{H}(\lambda_0)_h \to \mathcal{H}(\lambda_0 - 2)_h \) that sends the vacuum vector \( v_{\lambda_0} \) of \( \mathcal{H}(\lambda_0)_h \) to the vacuum vector \( v_{\lambda_0-2} \) of \( \mathcal{H}(\lambda_0 - 2)_h \), intertwines action of the elements \( b_n, c_n, V_Q, V_Q^{-1}, n \in \mathbb{Z} \) and commutes with \( a_n \) as follows

\[ [a_n, V_h] = -2V_h \delta_{n, 0}. \]

**Proposition 4.3.3.** ([13], Proposition 4) The operator \( S_h = \int_0^\infty J_h(w) dw, p = q^{2(k+2)}, S_h : W_h(\lambda_0, k) \to W(\lambda_0 - 2, k)_h, \) where the generating series \( J_h(w) \) is defined by

\[ J_h(w) = - : [ i \partial_w \exp \{ -c(2|q^{-k-2}w; 0) \} ] \exp \{ -b(2|q^{-k-2}w; -1) \} ; \]

\[ \times \exp \left( -\sum_{n=1}^{\infty} \frac{a_n}{(k + 2)n} q^{-(k+2)-3n} w^{n} \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{(k + 2)n} q^{-(k+2)+n} w^{-n} \right) \]

\[ \times V_h z^{-\frac{\lambda_0}{2}}, \]

is a homomorphism of \( U_h(\mathfrak{sl}_2) \) modules.

The operator \( S_h \) is called a screening operator.
5. Deformations of W–algebras

5.1. Coxeter realizations of affine quantum groups. The generalization of Definition 3 of W–algebras to the case of quantum groups is not so direct. The main problem is that the natural deformation $U_h(\tilde{\mathfrak{g}}) \subset U(\mathfrak{g})$ of the subalgebra $U(\tilde{\mathfrak{g}}) \subset U(\mathfrak{g})$ introduced in Section 4.1 has no nontrivial characters (see [56] for details). In order to overcome this difficulty one needs to introduce other quantum counterparts naturally appear in the so–called Coxeter realizations of the quantum group $U_h(\tilde{\mathfrak{g}})$ introduced in [56]. Below we recall the definition of these realizations following [56].

Let $U_h(\tilde{\mathfrak{g}})$ be the subalgebra in $U_h(\tilde{\mathfrak{g}})$ topologically generated by elements $H_i, X_i^+, X_i^−, i = 0, \ldots, l$. Fix $k ∈ \mathbb{C}$ and denote by $U_h(\tilde{\mathfrak{g}}) \kappa$ the quotient of the algebra $U_h(\tilde{\mathfrak{g}})$ by the two–sided ideal generated by $K − k$. Let $U_h(\tilde{\mathfrak{g}}) = \kappa$ be the restricted completion of the algebra $U_h(\tilde{\mathfrak{g}})$.

Let $A_h$ be the free associative topological algebra over $\mathbb{C}[\hbar]$ topologically generated by the Fourier coefficients of generating series

\[ e_i(u) = \sum_{r ∈ \mathbb{Z}} e_i,r u^{−r}, \]
\[ f_i(u) = \sum_{r ∈ \mathbb{Z}} f_i,r u^{−r}, \]
\[ K_i^+(u) = \sum_{r=0}^{∞} K_i^+ u^{−r}, \]
\[ K_i^−(u)^{−1} = \sum_{r=0}^{∞} K_{i,r}^{−1} u^{−r} \]

and by elements $H_i, i = 1, \ldots, l$. Introduce a $\mathbb{Z}$–grading on the algebra $A_h$ by $\deg(e_i,n) = \text{ht}(nδ + α_i)$ for $i = 1, \ldots, l$, $n ≥ 0$, $\deg(e_i,n) = −\text{ht}(−nδ − α_i)$ for $i = 1, \ldots, l$, $n < 0$, $\deg(f_i,n) = \text{ht}(nδ − α_i)$ for $i = 1, \ldots, l$, $n > 0$, $\deg(f_i,n) = −\text{ht}(−nδ + α_i)$ for $i = 1, \ldots, l$, $n ≤ 0$, $\deg(K_i^+)$ = $\pm\text{ht}(nδ)$ for $i = 1, \ldots, l$, $n ≥ 0$, $\deg(H_i) = 0$ for $i = 1, \ldots, l$. We denote by $A_h$ the restricted completion of $A_h$.

Denote by $S_l$ the symmetric group of $l$ elements. Fix an element $π ∈ S_l$ and denote by $F_{ij}(z), i, j = 1, \ldots, l$ the Taylor series in formal variable $z$ given by

\[ F_{ij}(z) = \frac{q_i^{n_{ij}} − z q_j^{n_{ij}}}{1 − zq_{ij}}, a_{ij} ≠ 0, \]
\[ F_{ij}(z) = q_i^{n_{ij}}, a_{ij} = 0, \]

where the coefficients $n_{ij} ∈ \mathbb{C}$ satisfy the equations

\[ d_i n_{ii} − d_j n_{ij} = ε_{ij} b_{ij}, \]

and the matrix $ε^{π}_{ij}, i, j = 1, \ldots, l$ is given by

\[ ε^{π}_{ij} = \begin{cases} -1 & \pi^{−1}(i) < \pi^{−1}(j) \\ 0 & i = j \\ 1 & \pi^{−1}(i) > \pi^{−1}(j) \end{cases} \]

If we associate to the element $π ∈ S_l$ a Coxeter element $s_π$ of the Weyl group $W$ by the formula $s_π = s_{π(1)} \ldots s_{π(l)}$ then Lemma 3 in [56] shows that the coefficients

...
\( \varepsilon_{ij}^\pi b_{ij} \) are the matrix elements of the Cayley transform \( \frac{1 + s_\pi}{1 - s_\pi} \) of the operator \( s_\pi : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \) in the basis of simple roots,

\[
\frac{1 + s_\pi}{1 - s_\pi} \alpha_i, \alpha_j = \varepsilon_{ij}^\pi b_{ij}.
\]

(5.1.6)

We shall also use the following formal power series:

\[
M_{ij}(z) = g_{ij}(zq^{-k})^{-1}F_{ji}(zq^k)F_{ji}(zq^{-k})^{-1},
\]

\[
G_{ij}(z) = M_{ij}(zq^{-k})M_{ij}(zq^k)^{-1},
\]

\[
F_{ij}^-(z) = F_{ij}(zq^{2k}).
\]

Let \( \hat{U}_{h,K}(\mathfrak{g}') \) be the quotient algebra of the algebra \( \hat{A}_h \) by the two-sided ideal topologically generated by the Fourier coefficients of the following generating series:

\[
K^\pm_i(u)K^\pm_j(v) - K^\pm_j(v)K^\pm_i(u), \quad K^\pm_i(u)K^\pm_i(u)^{-1} - 1, \quad K^\pm_i(u)^{-1}K^\pm_i(u) - 1,
\]

\[
H_iH_j - H_jH_i,
\]

\[
H_iK^\pm_j(v) - K^\pm_j(v)H_i
\]

\[
K^\pm_i(u) - \exp(\pm hd_iH_i),
\]

\[
K^\pm_i(u)K^\mp_j(v) - G_{ij}(\mathfrak{g}')K^\mp_j(v)K^\pm_i(u),
\]

\[
[H_i, e_{ij}(u)] - a_{ij}e_{ij}(u),
\]

\[
[H_i, f_{ij}(u)] - a_{ij}f_{ij}(u)
\]

\[
K^\pm_i(u)e_{ij}(v) - M_{ji}(\mathfrak{g}')e_{ij}(v)K^\pm_i(u),
\]

\[
K^\pm_i(u)f_{ij}(v) - M_{ji}(\mathfrak{g}')^{-1}f_{ij}(v)K^\pm_i(u),
\]

\[
K^\pm_i(u)e_{ij}(v) - M_{ji}(\mathfrak{g}')^{-1}e_{ij}(v)K^\pm_i(u),
\]

\[
K^\pm_i(u)f_{ij}(v) - M_{ji}(\mathfrak{g}')^{-1}f_{ij}(v)K^\pm_i(u),
\]

\[
(u - vq^{-b_{ij}})[e_{ij}(u), e_{ij}(v)],
\]

\[
(u - vq^{b_{ij}})[f_{ij}(u), f_{ij}(v)] - (q^{b_{ij}} - 1)f_{ij}(u)f_{ij}(v),
\]

\[
\sum_{\pi \in S_{1-n_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] \prod_{p=1}^q F_{ij}(z_{\pi(p)}) \prod_{r=1}^k F_{ij}(qz_{\pi(r)}) \times \prod_{s=k+1} \frac{F_{ij}(z_{\pi(s)})\cdots e_{ij}(z_{\pi(k+1)})\cdots e_{ij}(z_{\pi(1)\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)})..., i \neq j,
\]

\[
1 \prod_{s=k+1} F_{ij}(\frac{z_{\pi(s)}}{w})e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(k)})e_{ij}(z_{\pi(k+1)})\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)})\cdots e_{ij}(z_{\pi(1)}>
\[
\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ 1 - \frac{a_{ij}}{k} \right] \prod_{p < q} F_{ij}^{-}(\frac{\pi(p)}{\pi(e)}) \prod_{r=1}^{k} F_{ji}^{-}(\frac{w}{\pi(r)}) \times \\
\prod_{s=k+1}^{1} F_{ij}^{-}(\frac{\pi(s)}{\pi(e)}) \tilde{f}_{i}(\pi(1)) \cdots \tilde{f}_{i}(\pi(k)) \tilde{f}_{i}(w) \tilde{f}_{i}(\pi(k+1)) \cdots \tilde{f}_{i}(\pi(1-a_{ij})), \ i \neq j,
\]

Note that this ideal only depends on skew-symmetric combination of the coefficients \( n_{ij} \), and hence by (5.1.4) the algebra \( \tilde{U}_{h,k}(\mathfrak{g}') \) only depends on the Coxeter element \( s_{x} \in W \).

We show that the algebra \( \tilde{U}_{h,k}(\mathfrak{g}') \) is a realization of \( \tilde{U}_{h}(\mathfrak{g}) \).

**Proposition 5.1.1.** ([19], Proposition 8) For every solution of equation (5.1.4) and every solution \( n_{ij}^* \in \mathbb{C}[h] \), \( r \neq 0 \) of the system

\[
(n_{ij}^* - n_{ji}^*) q^{-kr} - n_{im}^* n_{ji}^* r (B')^{-1} q^{-kr} - q^{-kr} = \frac{1}{r} (q^{r_{h_{ij}}} - q^{-r_{h_{ji}}}), \ r \in \mathbb{N},
\]

where \( B'_{ij} = q^{r_{h_{ij}}} - q^{-r_{h_{ji}}} \), there exists an isomorphism of algebras \( \tilde{\psi}(n) : \tilde{U}_{h,k}(\mathfrak{g}') \rightarrow \tilde{U}_{h}(\mathfrak{g}) \) given by:

\[
\tilde{\psi}(n)(e_{i}(u)) = q^{-n_{ij}} \Phi_{i}^{0}(n) \Phi_{i}^{-}(u)^{n} X_{i}(u) \Phi_{i}^{+}(u)^{n},
\]

\[
\tilde{\psi}(n)(f_{i}(u)) = \Phi_{i}^{0}(n)^{-1} \Phi_{i}^{-}(u q^{k})^{n} X_{i}(u) \Phi_{i}^{+}(u q^{-k})^{n},
\]

\[
\tilde{\psi}(n)(K_{i}^{\pm}(u)) = K_{i}^{\pm} \exp(\sum_{s=1}^{\infty} (q_{i} - q_{i}^{-1}) H_{i,\pm,1} q^{-s} u^{\pm s} - Y_{1,\pm,1} n_{ij}^{\pm} q^{-s} u^{\pm s}),
\]

\[
\tilde{\psi}(n)(H_{i}) = H_{i},
\]

where \( \Phi_{i}^{0}(n), \Phi_{i}^{-}(u)^{n}, \Phi_{i}^{+}(u)^{n} \) are defined by

\[
\Phi_{i}^{+}(u)^{n} = \exp \left( \sum_{r=1}^{\infty} Y_{1,\pm,1} n_{ij}^{\pm} u^{\pm r} \right)
\]

\[
\Phi_{i}^{0}(n) = \prod_{j=1}^{l} L_{j}^{n_{ij}}.
\]

The algebra \( \tilde{U}_{h,k}(\mathfrak{g}') \) is called a Coxeter realization of \( \tilde{U}_{h}(\mathfrak{g}) \).

Let \( \tilde{U}_{h,k}^{\ast}(\tilde{\mathfrak{n}}_{+}) \subset \tilde{U}_{h,k}(\mathfrak{g}') \) be the restricted completion in \( \tilde{U}_{h,k}(\mathfrak{g}') \) of the subalgebra of \( \tilde{U}_{h,k}(\mathfrak{g}') \) topologically generated by the Fourier coefficients of the series \( e_{i}(u) \), \( i = 1, \ldots, l \). The defining relations in the subalgebra \( \tilde{U}_{h,k}^{\ast}(\tilde{\mathfrak{n}}_{+}) \) are

\[
(u - v q^{r_{h_{ij}}}) [e_{i}(u), e_{j}(v)] = 0,
\]

\[
\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ 1 - \frac{a_{ij}}{k} \right] \prod_{p < q} F_{ii}^{-}(\frac{\pi(p)}{\pi(e)}) \prod_{r=1}^{k} F_{ji}^{-}(\frac{w}{\pi(r)}) \times \\
\prod_{s=k+1}^{1} F_{ij}^{-}(\frac{\pi(s)}{\pi(e)}) \tilde{e}_{i}(\pi(1)) \cdots \tilde{e}_{i}(\pi(k)) \tilde{e}_{i}(w) \tilde{e}_{i}(\pi(k+1)) \cdots \tilde{e}_{i}(\pi(1-a_{ij})) = 0, \ i \neq j.
\]
Clearly, the algebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+})/U_{h}^{*}(\tilde{n}_{+}) \) is isomorphic to the restricted completion of the algebra \( U(\tilde{n}_{+}) \).

The subalgebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) was introduced in [4] in order to define quantum group counterparts of nontrivial characters of the subalgebra \( U(\tilde{n}_{+}) \subset U(\tilde{g}) \). Some commutative elements in the algebra \( \tilde{U}_{h}(\mathfrak{s}l_{N}) \) similar to the Fourier coefficients of the generating series \( \tilde{\psi}_{(n)}(\varphi(u)) \) were also constructed in [4].

**Proposition 5.1.2.** The map \( \chi_{h}^{*} : U_{h}^{*}(\tilde{n}_{+}) \to \mathbb{C} \) defined by \( \chi_{h}^{*}(\varphi_{i}(u)) = \varphi_{i}(u), i = 1, \ldots, l \), where \( \varphi_{i}(u) \in \mathbb{C}[[h]]((u)) \) are arbitrary formal power series, is a character of the algebra \( U_{h}^{*}(\tilde{n}_{+}) \).

### 5.2. Definition of deformed W–algebras.

In the previous section we defined the quantum group counterparts \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) of the algebra \( \tilde{U}(\tilde{n}_{+}) \) having nontrivial characters. Let \( \chi_{h} : \tilde{U}_{h}^{*}(\tilde{n}_{+}) \to \mathbb{C} \) be the character of the algebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) such that \( \chi_{h}(\varphi_{i}(u)) = u_{i} \), \( i = 1, \ldots, l \) (see Proposition 5.1.2). We denote by \( \mathbb{C} \chi_{h} \) the corresponding one–dimensional representation of the algebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \). It would be natural to define the deformed W–algebra corresponding to the affine quantum group \( U_{h}(\tilde{g}') \) as the semi–infinite Hecke algebra of the triple \( (U_{h}(\tilde{g}'))_{h}, \tilde{U}_{h}^{*}(\tilde{n}_{+}), \chi_{h}) \). However the algebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \subset \tilde{U}_{h}^{*}(\tilde{g}') \) is the restricted completion of the subalgebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \subset \tilde{U}_{h}^{*}(\tilde{g}') \) topologically generated by the Fourier coefficients of the series \( \chi_{h}(u), i = 1, \ldots, l \). Therefore there is no any nontrivial triangular decomposition in the algebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \). This implies that this algebra does not satisfy conditions (i)–(vi) of Sections 1.1, 1.2, and hence the semi–infinite Tor functor for \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) and the semi–infinite Hecke algebra of the triple \( (U_{h}(\tilde{g}'))_{h}, \tilde{U}_{h}^{*}(\tilde{n}_{+}), \chi_{h}) \) do not exist.

In order to overcome this difficulty we shall consider the subalgebra \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \subset U_{h}(\tilde{g}')_{h} \). The defining relations (5.1.10) of the algebra \( U_{h}^{*}(\tilde{n}_{+}) \) only contain commutators. Therefore \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) is the universal enveloping algebra of the Lie algebra \( \tilde{\mathfrak{n}}_{+}^{\pm} \) topologically generated by the Fourier coefficients of the series \( e_{i}(u), i = 1, \ldots, l \) subject to defining relations (5.1.10). The Lie algebra \( \tilde{\mathfrak{n}}_{+}^{\pm} \) is \( \mathbb{Z} \)-graded, \( \tilde{\mathfrak{n}}_{h}^{\pm} = \oplus_{n \in \mathbb{Z}}(\tilde{\mathfrak{n}}_{h}^{\pm})_{n} \). Denote the subalgebras \( \oplus_{n > 0}(\tilde{\mathfrak{n}}_{h}^{+})_{n} \) and \( \oplus_{n < 0}(\tilde{\mathfrak{n}}_{h}^{-})_{n} \) by \( (\tilde{\mathfrak{n}}_{h}^{+})^{\pm} \) and \( (\tilde{\mathfrak{n}}_{h}^{-})^{\pm} \), respectively. Then multiplication in \( \tilde{U}_{h}^{*}(\tilde{n}_{+}) \) defines an isomorphism of vector spaces, \( U_{h}^{*}(\tilde{n}_{+}) = (U_{h}^{*}(\tilde{n}_{+}))^{+} \otimes (U_{h}^{*}(\tilde{n}_{+}))^{-} \), where \( (U_{h}^{*}(\tilde{n}_{+}))^{+} = U((\tilde{\mathfrak{n}}_{h}^{+})^{+}) \) and \( (U_{h}^{*}(\tilde{n}_{+}))^{-} = U((\tilde{\mathfrak{n}}_{h}^{-})^{-}) \).

Note that the quotient algebra \( U_{h}^{*}(\tilde{n}_{+})/U_{h}^{*}(\tilde{n}_{+}) \) is not isomorphic to \( U(\tilde{n}_{+}) \) since the Serre relations (5.1.11) are not satisfied in \( U_{h}^{*}(\tilde{n}_{+}) \).

The algebra \( U_{h}(\tilde{g}')_{h} \) inherits a \( \mathbb{Z} \)-grading from \( U_{h}(\tilde{g}') \) and satisfies conditions (i)–(vi) of Sections 1.1, 1.2, with the natural triangular decomposition \( U_{h}(\tilde{g}')_{h} = U_{h}^{*}(\tilde{g}') \otimes U_{h}(\tilde{n}_{+}), \) where \( U_{h}^{*}(\tilde{g}') \) is the image of the subalgebra of \( U_{h}(\tilde{g}') \) generated by \( X_{i}^{-} \) and \( H_{i}, \) \( i = 0, \ldots, l \) in the quotient \( U_{h}(\tilde{g}')_{h} \). Hence one can define the algebra \( \tilde{U}_{h}(\tilde{g}')_{h} \) and the semi–infinite Tor functor for \( U_{h}(\tilde{g}')_{h} \).

Now we shall define deformed W–algebras. Fix a solution \( n_{ij} \) of equations (5.1.4) and a solution \( n_{ij}', r \neq 0 \) of the system (5.1.7), such that \( n_{ij}' = 0 \pmod{h} \). Such solutions exist. For instance, one can put \( n_{ij}' = 0 \) for \( r < 0 \). Then equations (5.1.7) yield \( n_{ij}' = -\frac{1}{2}q^{\frac{1}{2}} (q^{(b_{ij})} - q^{-(c_{ij}b_{ij})}) \) for \( r > 0 \).
Consider the functor
\[ C_{\chi_h} \otimes (U_{h}^* g(\tilde{a}+,))^{+}_h : (U_h(\tilde{g}'))_h - \text{mod}_0 \to \text{Vect}_k, \] (5.2.1)
\[ M \mapsto C_{\chi_h} \otimes (U_{h}^* g(\tilde{a}+,))^{+}_h M, \]
where the operation \((U_{h}^* g(\tilde{a}+,))^{+}_h \) is defined with the help of formula (1.3.1).

**Definition 4.** The algebra
\[ W_{k,h}^*(g) = \text{hom}_{U_h(\tilde{g}')} (C_{\chi_h} \otimes (U_{h}^* g(\tilde{a}+,))^{+}_h S_{U_h(\tilde{g}')} k, C_{\chi_h} \otimes (U_{h}^* g(\tilde{a}+,))^{+}_h S_{U_h(\tilde{g}')} k). \]
is called the deformed W-algebra associated to the complex semisimple Lie algebra \( g \).

Since Serre relations (5.1.11) are satisfied in the representations \( S_{U_h(\tilde{g}')} k \) and \( C_{\chi_h} \), regarded as a left (right) module over the subalgebra \( \tilde{U}_{h}^* g(\tilde{a}+) \subset U_h(\tilde{g}') \), the quotient algebra \( W_{k,h}^*(g) / hW_{k,h}^*(g) \) is isomorphic to \( W_k g \).

Now we introduce the semi–infinite cohomology spaces for \( U_h(\tilde{g}'))_k – \text{modules from the category } (U_h(\tilde{g}'))_h - \text{mod}_0 \) with respect to the subalgebra \( U_h^*(\tilde{n}+) \subset U_h(\tilde{g}'). \) These semi–infinite cohomology spaces have natural structures of \( W_{k,h}^*(g) – \text{modules}. \)

We shall define the semi–infinite cohomology for \( U_h(\tilde{g}'))_k – \text{modules from the category } (U_h(\tilde{g}'))_h - \text{mod}_0 \) with respect to the subalgebra \( U_h^*(\tilde{n}+) \) as a derived functor of the functor (5.2.1). In order to define this derived functor we shall introduce a suitable class of resolutions for objects from the category \( (U_h(\tilde{g}'))_h - \text{mod}_0 \).

**Proposition 5.2.1.** Let \( U_h(\tilde{n}+)^+ \subset U_h(\tilde{g}'))_k \) be the subalgebra topologically generated by the elements \( X_\gamma = \tilde{X}_\gamma, \gamma \in \{ \alpha \in \Delta, \alpha \in \tilde{\Delta}_+, n \geq 0 \} \). Then every \( U_h(\tilde{g}'))_k – \text{module } M \) from the category \( (U_h(\tilde{g}'))_h - \text{mod}_0 \) has a semi–infinite resolution \( S^\bullet (M) \in \text{Kom}(U_h(\tilde{g}'))_k - \text{mod}_0 ) \) with respect to the subalgebra \( U_h(\tilde{n}+)^+ \). This resolution is unique up to homotopy equivalence.

**Proof.** We shall apply Theorem [4.4] and Proposition [4.8] to the algebra \( U_h(\tilde{g}'))_k \), the subalgebra \( U_h(\tilde{n}+)^+ \subset U_h(\tilde{g}'))_k \), and the category \( (U_h(\tilde{g}'))_k - \text{mod}_0 \) of left \( U_h(\tilde{g}'))_k – \text{modules}. Note that the conditions of Theorem [4.4] are satisfied for these data.

Indeed, let \( M \in (U_h(\tilde{g}'))_k - \text{mod}_0 \) be a left \( U_h(\tilde{g}'))_k – \text{module. Then } M \) is a submodule of the left \( U_h(\tilde{g}'))_k – \text{module } M' \in (U_h(\tilde{g}'))_k - \text{mod}_0 \) defined by formula (4.1) with \( B^+ = U_h(\tilde{B}_-) \). By Proposition [4.1] \( M' \) is isomorphic to \( \text{hom}_{U_h(\tilde{g}')} (U_h(\tilde{n}+)^+, M) \) as a left \( U_h(\tilde{n}+) – \text{module. Now observe that } \) by Proposition [4.1] we also have an isomorphism of right \( U_h(\tilde{n}+)^+ – \text{modules, } U_h(\tilde{n}+) = U_h(\tilde{n}+)^+ \otimes U_h(\tilde{n}+)^+, \) where \( U_h(\tilde{n}+) \) is the subalgebra in \( U_h(\tilde{n}+) \) topologically generated by the elements \( X_\gamma, \gamma \in \{ -\alpha + n\delta, \alpha \in \tilde{\Delta}_+, n > 0 \} \). This implies that \( M' \) is also injective as a \( U_h(\tilde{n}+)^+ – \text{module. \)}

Now let \( P \) be the \( U_h(\tilde{g}'))_k – \text{module defined by formula (4.2) with } \) \( N^+ = U_h(\tilde{n}+) \). Then \( M \) is a strong quotient of \( P \) with respect to \( U_h(\tilde{n}+) \). Since \( U_h(\tilde{n}+)^+ \) is a subalgebra in \( U_h(\tilde{n}+) \) the \( U_h(\tilde{n}+) – \text{splitting } s : M \to P \) defined by formula (4.3) is also a \( U_h(\tilde{n}+)^+ – \text{splitting. Therefore } M \) is a strong quotient of \( P \) with respect to \( U_h(\tilde{n}+)^+ \).
Now Proposition 5.2.1 immediately follows from Proposition 1.4.8.

Now let \( M \in (U_h(\hat{\mathfrak{g}})\xi - \text{mod})_0 \) be a left \( U_h(\hat{\mathfrak{g}})\xi \)-module. We define the semi-infinite cohomology space \( H^\Xi + \bullet(U^*_h(\hat{n}_+), M) \) of \( M \) with respect to the subalgebra \( U^*_h(\hat{n}_+) \) as the cohomology of the complex \( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow S^\bullet(M) \),

\[
(5.2.3) \quad H^\Xi + \bullet(U^*_h(\hat{n}_+), M) = H^\bullet(\mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow S^\bullet(M)),
\]

where \( S^\bullet(M) \) is a semijective resolution of \( M \) with respect to the subalgebra \( U_h(\hat{n}_+) \). By Proposition 5.2.1 this definition does not depend on the resolution \( S^\bullet(M) \).

Definition 5.2.3 of the semi-infinite cohomology spaces is motivated by the following theorem.

**Theorem 5.2.2.** Let \( M \in (U_h(\hat{\mathfrak{g}})\xi - \text{mod})_0 \) be a left \( U_h(\hat{\mathfrak{g}})\xi \)-module. Then the algebra \( W_{k,h}(\mathfrak{g}) \) naturally acts in the space \( H^\Xi + \bullet(U^*_h(\hat{n}_+), M) \),

\[
W_{k,h}(\mathfrak{g}) \times H^\Xi + \bullet(U^*_h(\hat{n}_+), M) \rightarrow H^\Xi + \bullet(U^*_h(\hat{n}_+), M).
\]

This action respects the gradings of \( W^*_h(\mathfrak{g}) \) and \( H^\Xi + \bullet(U^*_h(\hat{n}_+), M) \).

To prove this theorem we shall realize the algebra \( W^*_h(\mathfrak{g}) \) as zeroth cohomology of a certain differential graded algebra which naturally acts on a standard complex for calculation of the semi-infinite cohomology space \( H^\Xi + \bullet(U^*_h(\hat{n}_+), M) \).

**Proposition 5.2.3.** The algebra \( W_{k,h}(\mathfrak{g}) \) is isomorphic to the zeroth cohomology of the differential graded algebra

\[
Y^* = \text{end}^*_U(U_h(\hat{\mathfrak{g}})\xi)^{\bullet} \big( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \big).
\]

The nonzeroradient grades of the cohomology of this differential graded algebra vanish,

\[
H^\neq 0(Y^*) = 0.
\]

**Proof.** First observe that the cohomology of the differential algebra \( Y^* \) is isomorphic to the algebra

\[
\text{end}^*_K((\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0)^{\bullet} \big( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \big),
\]

see [38], III.6.14.

Next, the complex

\[
\mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \in \text{Kom}((\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0)
\]

is semijective with respect to the subalgebra \( U_h(\hat{n}_+) \) by the definition of the complex \( \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \).

Now observe that by Proposition 1.4.9, Theorem 1.4.4 holds for the algebra \( U_h(\hat{\mathfrak{g}})\xi \), the subalgebra \( U_h(\hat{n}_+) \) and the category \( (\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0 \). Since the complex \( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \in \text{Kom}((\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0) \) is semijective Theorem 1.4.4 implies an algebraic isomorphism,

\[
\text{end}^*_K((\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0)^{\bullet} \big( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \big) = \text{end}^*_D((\text{mod} - U_h(\hat{\mathfrak{g}})\xi)_0)^{\bullet} \big( \mathbb{C}_h \otimes (U^*_h(\hat{n}_+))^{\bullet} \rightarrow \text{Bar}^\Xi + \bullet(U_h(\hat{\mathfrak{g}})\xi, U_h(\hat{n}_+), S_{U_h(\hat{\mathfrak{g}})\xi}) \big).
\]
Similarly to Lemma A5.1 in [13] one can establish an isomorphism of complexes of right \( U_h(\widehat{g}) \) \( C^- \)–modules,
\[
\mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^- \to \operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^- = \operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^-.
\]

By Proposition 1.4.12 the last complex is a semisimple resolution of the right \( U_h(\widehat{g}) \) \( C^- \)–module \( \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^- \). In particular,
\[
H^\bullet(\operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^-) = \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^-.
\]

This implies that
\[
(5.2.6) \quad H^\bullet(Y^\bullet) = \operatorname{end}_{D((\operatorname{mod} - U_h(\widehat{g}))_0)}(\mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^-) = \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+))^-.
\]

To prove the second part of Proposition 5.2.3 we shall use the following lemma.

**Lemma 5.2.4.** Let \( X^\bullet_n \) be a complex of complete \( \mathbb{C}[[h]] \) \( C^- \)–modules. Denote by \( X^\bullet \) the quotient complex \( X^\bullet_n / h X^\bullet_n \). Suppose that \( H^n(X^\bullet) = 0 \) for some \( n \in \mathbb{Z} \). Then \( H^n(X^\bullet) = 0 \).

**Proof.** Let \( x_h \in X^\bullet_n \) be a cocycle, i.e. \( d_h x_h = 0 \), where \( d_h \) is the differential in \( X^\bullet_n \). We have to prove that \( x_h = d_h y_h, \ y_h \in X^\bullet_{n-1} \).

Denote by \( d \) the differential in the complex \( X^\bullet \) and by \( x \in X^a \) the element \( x_h \pmod{h} \). Since \( d_h x_h = 0 \) we have \( d x = 0 \), and hence \( x = d y_1, \ y_1 \in X^a \) because \( H^0(X^\bullet) = 0 \). Since the operator \( d_h \) coincides with \( d \) \( \pmod{h} \) we also obtain that \( x_h - d_h y_1 = h x_1, \ x_1 \in X^h \) and \( d_h x_1 = 0 \). Now we can apply the same procedure to \( x_1 \). If we continue this process we shall finally obtain an infinite sequence of elements \( y_i \in X^h \) such that \( x_h - \sum_{i=1}^{\infty} d_h y_i = 0 \pmod{h^{m+1}} \). Since the space \( X^h \) is a complete \( \mathbb{C}[[h]] \) \( C^- \)–module the series \( d_h (\sum_{i=1}^{\infty} y_i) \) converges to \( x_h \). This completes the proof.

Now observe that isomorphisms 1.2.0 and Proposition 1.5.1 imply that \( H^\bullet(Y^\bullet) = H^\bullet(\operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+), C_{X_h}) \pmod{h} \). Therefore the algebra \( H^\bullet(\operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+), C_{X_h}) \pmod{h} \) may be calculated as the cohomology of the differential graded algebra \( Y^\bullet / h Y^\bullet \). Now by Proposition 3.1.2 \( H^\bullet(Y^\bullet) = 0 \). In order to prove that \( H^\bullet(Y^\bullet) = 0 \) it remains to apply Lemma 5.2.4 to the complex \( Y^\bullet \).

Next we define a standard complex for calculation of the semi–infinite cohomology space \( H^\bullet(\operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+), M) \).

**Lemma 5.2.5.** Let \( M \in (U_h(\widehat{g})_k - \operatorname{mod})_0 \) be a left \( U_h(\widehat{g})_k \) \( \operatorname{mod} \)–module. Then the semi–infinite cohomology space \( H^\bullet(\operatorname{Bar} \mathbb{C}_{X_h} \otimes (U^+_{\nu_h}(\widehat{n}_+), M) \) may be calculated as the
Lemma 5.3.2. similar to Lemma 3.2.2 in the nondeformed case.

The space

\( H \) restriction of this action to the zeroth cohomology of

\( \mod) \)

this section we construct the resolution of the vacuum representation of the algebra

\( (\mod) \)

the cohomology space \( H \) is a semijective resolution of \( M \) with respect to the subalgebra \( U_h(\widehat{\mathfrak{g}}^\vee)^+ \). The proof of this fact is similar to that of Proposition 2.6.4 in [18].

Proof of Theorem 5.2.2. Theorem 5.2.2 follows from the fact that the differential graded algebra (5.2.5) naturally acts on the complex (5.2.7). By Lemma 5.2.3 this action induces an action of the cohomology of the differential graded algebra \( Y^\bullet \) on the cohomology space \( H \) of the algebra \( U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \). In particular, by Proposition 5.2.3 the restriction of this action to the zeroth cohomology of \( Y^\bullet \) induces action (5.2.4).

5.3. Resolutions and screening operators for deformed \( W \)-algebras. In this section we construct the resolution of the vacuum representation of the algebra \( W_{k+1}^\ast \) similar to the resolution of the vacuum representation of the algebra \( W_k \) defined in Section 3.2. We suppose that the level \( k \) is generic. Recall that by Theorem 5.2.2 the algebra \( W_{k+1} \) acts in the spaces \( H \) of \( k+1 \) the complex \( (U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+, M) \), where \( M \in (U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \mod)_{k-1} \). In particular, for every left \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \)-module \( M \in (U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \mod)_{k-1} \) such that the two-sided ideal of the algebra \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) generated by \( K \) lies in the kernel of the representation \( M \) the algebra \( W_{k+1}^\ast \) acts in the space \( H \) of the algebra \( W_{k+1}^\ast \).

Let \( \lambda_b: \mathfrak{h} \to \mathbb{C} \) be the character such that \( \lambda|_b = 0, \lambda(K) = k \) and \( \lambda(\theta) = 0 \). Denote by \( V_{k+1} \) the representation of the algebra \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) with highest weight \( \lambda_b \) induced from the trivial representation of the algebra \( U_{k+1}(\widehat{\mathfrak{g}}) \), \( V_k = U_{k+1}(\widehat{\mathfrak{g}}) \otimes U_{k+1}(\widehat{\mathfrak{g}}) \sigma \) \((\mathfrak{h}, \mathfrak{g})_{k-1} \). \( V_{k+1} \) is called the vacuum representation of \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \). The \( W_{k+1}^\ast \)-module \( H \) of \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) is called the vacuum representation of the algebra \( W_{k+1}^\ast \).

The space \( H \) of \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) may be explicitly described using the resolution of the algebra \( W_{k+1}^\ast \) modules constructed in Corollary 4.2.4.

Indeed, let \( D_k^\bullet(\lambda_b) \) be this resolution, \( D_k^\bullet(\lambda_b) = \oplus_{w \in W} \) \( W_{k+1}(w(\lambda_b + \rho_0) - \rho_0) \).

Proposition 5.3.1. The complex \( D_k^\bullet(\lambda_b) \) is a \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \)-semjective resolution of \( V_{k+1} \) with respect to the subalgebra \( U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \).

This proposition follows from part 3 of Proposition 4.4.3 and the following lemma similar to Lemma 3.2.2 in the nondeformed case.

Lemma 5.3.2. Every \( \mathfrak{g} \)-module \( W_{k+1}(\lambda_b) \) is semjective as a module over \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) with respect to the subalgebra \( U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \).

Now in order to calculate the space \( H \) of \( U_{k+1}(\widehat{\mathfrak{g}}^\vee) \) one should apply the functor \( C_{\lambda_b} \otimes U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \) to the resolution \( D_k^\bullet(\lambda_b) \) and compute the cohomology of the obtained complex.

Denote by \( C_k^\bullet(\lambda_b) \) the complex \( C_{\lambda_b} \otimes U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ \) of \( D_k^\bullet(\lambda_b) \),

\[
C_k^\bullet(\lambda_b) = C_{\lambda_b} \otimes U_{k+1}(\widehat{\mathfrak{g}}^\vee)^+ D_k^\bullet(\lambda_b). 
\]
In order to prove that $H^{\#0}(C^*_h(\lambda_k)) = 0$ we shall apply Lemma 5.2.4. Observe that the complex $C^*_h(\lambda_k)/hC^*_h(\lambda_k)$ is isomorphic to the resolution $C^*_h(\lambda_k)$ constructed in Section 3.2.

The following theorem follows immediately from Proposition 3.2.3 and Lemma 5.2.4 applied to the complex $C^*_h(\lambda_k)$.

**Theorem 5.3.3.** $H^{\#0}(C^*_h(\lambda_k)) = 0$, i.e., for $n \neq 0$

$$H^\mathbb{Z} + n(U^*_h(\tilde{n}_+), V_{k,h}) = 0,$$

and the complex $C^*_h(\lambda_k)$ is a resolution of the $W^{s\pi}_k(\mathfrak{g})$–module $H^\mathbb{Z} + 0(U^*_h(\tilde{n}_+), V_{k,h})$

The operators $S^h_i : C_{\chi,h} \otimes (U^*_h(\tilde{n}_+))^+ \to C_{\chi,h} \otimes (U^*_h(\tilde{n}_+))^+$ defined with the help of the isomorphism $\tilde{\psi}(n) : \tilde{U}^*_h(\tilde{\mathfrak{g}}_{\tilde{\mathfrak{sl}}_2}) \to \tilde{U}^*_h(\tilde{\mathfrak{gl}}_2)_k$ (see Proposition 4.3.1) are called deformed screening operators.

5.4. **The deformed Virasoro algebra.** In this section we explicitly calculate the deformed screening operator for the deformed W–algebra $W^\xi_{k,h}(\mathfrak{sl}_2)$ \footnote{Note that there is a unique Coxeter element in the Weyl group of the Lie algebra $\mathfrak{sl}_2$.}. We suppose that the level $k$ is generic and use the bosonic realization for Wakimoto modules over the algebra $U_h(\mathfrak{sl}_2)$ and the notation introduced in Section 4.3. The proofs of the statements presented in this section are quite parallel to the proofs of similar results for the Virasoro algebra (see Section 3.3), and we do not repeat these proofs here.

In order to calculate the deformed screening operator for the algebra $W^\xi_{k,h}(\mathfrak{sl}_2)$ we shall need explicit formulas for the bosonic realization of Wakimoto modules $W^\xi_h(\lambda)$ (see Proposition 4.3.1) in terms of the generators of the Coxeter realization of the algebra $\tilde{U}^*_h(\tilde{\mathfrak{sl}}_2)'_k$.

Since for any $\lambda_0 \in \mathbb{C}$ the two–sided ideal of the algebra $\tilde{U}^*_h(\tilde{\mathfrak{sl}}_2)'_k$ generated by $K - k$ lies in the kernel of the representation $W^\xi_h(\lambda_0, k)$ the algebra $\tilde{U}^*_h(\tilde{\mathfrak{sl}}_2)'_k$ indeed acts on the spaces $W^\xi_h(\lambda_0, k)$. Explicit calculation shows that the action of the Fourier coefficients of the generating series $K^\pm(z)$, $e(z)$ and $f(z)$ defined with the help of the isomorphism $\tilde{\psi}(n) : \tilde{U}^*_h(\tilde{\mathfrak{sl}}_2)'_k \to \tilde{U}^*_h(\tilde{\mathfrak{sl}}_2)_k$ (see Proposition 5.1.1) on the space $\mathcal{H}(\lambda_0)_h$ introduced in Proposition 4.3.1 is given by

$$K^\pm(z) := \exp\{(q - q^{-1})\sum_{n > 0} (\tilde{a}_n + q^{\pm(k+1)n}) \frac{[n]_q}{[2n]_q [kn]_q} \} \exp\{\frac{[n]_q}{[2n]_q [kn]_q} ((2k + 1)n)_q - [n]_q \tilde{b}_n\} z^{\pm n}$$

$$\pm h(\tilde{a}_0 + \tilde{b}_0) \}.$$
\( e(z) = -z : [ t \partial_z \exp \{- c(2 [q^{-k-2}z]; 0) \} \exp \{- \bar{b}(2 [q^{-k-2}z]; 0) \} : \]
\[ f(z) = z : [ k+2 \partial_z \exp \{ \bar{a}(k+2 |z; 0) + \bar{b}(2 |q^{-k-2}z; 0)
+ c(k+1; 2, k+2 [q^{-k-2}z]; 0) \} ] \times \exp \{- \bar{a}(k+2 |z; k+2) + c(1; 2, k+2 [q^{-k-2}z]; 0)
- \bar{b}(2 |q^{-k-2}z; k+2) - \bar{b}(2 |q^{-k-2}z; k) \} : ; \]
where
\[ \bar{a}_r = q^{2r} (q^{-\frac{k}{2} |r|} - \frac{|r|}{[2r]_q} n_r \frac{[k |r|]_q}{[kr]_q} (B_{-r} a_r + A_{-r} \frac{[(k+2) r]_q}{[2r]_q} b_r), r \neq 0, \]
\[ \bar{b}_r = A_r a_r + B_r b_r, r \neq 0, \]
\[ A_r = r n_r q^{2r-|r|-(k+2)r}, B_r = q^{|r|} + r n_r q^{-k-2 |r|}, r \neq 0, \]
\[ \bar{a}_0 = a_0, \bar{b}_0 = b_0. \]
The elements \( \bar{a}_r \) and \( \bar{b}_r \) satisfy the following commutation relations
\[ [\bar{a}_r, \bar{a}_s] = \delta_{r+s,0} \frac{[(k+2) r]_q [r]_q}{r [kr]_q} ([2k+1) r]_q - [r]_q), r, s \neq 0, \]
\[ [\bar{b}_r, \bar{b}_s] = -\delta_{r+s,0} \frac{[2r]_q [2r]_q}{r}, r, s \neq 0. \]
Moreover, the elements \( \bar{a}_n, \bar{b}_n, c_n, V_Q = \exp(\frac{Q_k+Q_{k'}}{2}), V_Q^{-1} = \exp(-\frac{Q_k+Q_{k'}}{2}), n \in \mathbb{Z} \) may be regarded as a new system of generators of the algebra \( \mathbf{H}_h \) and the representation space \( \mathcal{H}(\lambda_0)_h \) may be defined as the representation space for the algebra \( \mathbf{H}_h \) topologically generated by the vacuum vector \( \psi_{\lambda_0} \) satisfying the following conditions
\[ \bar{a}_n \cdot \psi_{\lambda_0} = 0 \text{ for } n > 0, \]
\[ \bar{b}_n \cdot \psi_{\lambda_0} = 0 \text{ for } n \geq 0, \]
\[ c_n \cdot \psi_{\lambda_0} = 0 \text{ for } n \geq 0, \]
\[ \bar{a}_0 \cdot \psi_{\lambda_0} = \lambda_0 \psi_{\lambda_0}. \]
Therefore the action of the algebra \( \tilde{U}_{h,k}^* \) in the representation space \( W_h(\lambda_0, k) \) for the algebra \( U_h(\tilde{\mathfrak{g}}_2)_k \) does not explicitly depend on the isomorphism \( \tilde{\psi}_{(\lambda_0)} : \tilde{U}_{h,k} \rightarrow \tilde{U}_{h,k} \).
Now let \( C_{\lambda_k} \) be the resolution of the vacuum representation of the algebra \( \tilde{U}_{h,k}^* \) introduced in the previous section. Using Remark 2.3.7 and Proposition 4.3.3, this resolution may be rewritten as
\[
(5.4.1) \quad 0 \rightarrow \mathbb{C}_{\chi_h} \otimes \langle U_{h,k}^* \rangle^+ \cdot W_h(0, k) \rightarrow \mathbb{C}_{\chi_h} \otimes \langle U_{h,k}^* \rangle^+ \cdot W_h(-2, k) \rightarrow 0.
\]
We shall explicitly calculate the spaces \( C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(0, k) \) and \( C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(-2, k) \) and the induced operator
\[
S^h_1 : C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(0, k) \to C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(-2, k).
\]

**Lemma 5.4.1.** Let \( W_h(\lambda_0, k) \) be the Wakimoto module of highest weight \( \lambda \) of finite type such that \( \lambda(K) = k, k \neq -2 \) and \( \lambda(\theta) = 0 \). Denote by \( H^0_h \subset H_h \) the subalgebra in \( H_h \) with generators \( \tilde{a}_n, n \in \mathbb{Z} \) subject to the relations
\[
[\tilde{a}_r, \tilde{a}_s] = \delta_{r+s,0} \frac{[(k+2)r]_q [r]_q}{r [kr]_q} ((2k+1)r_q - [r]_q).
\]

Let \( \pi_h(\lambda_0, k + h^\vee) \) be the \( U_h(\hat{h}) \) (and \( H^0_h \))–submodule in \( W_h(\lambda_0, k) \) generated by the vacuum vector \( v_{\lambda_0} \) under the action of the subalgebra \( H^0_h \subset H_h \). Then the natural linear embedding \( \pi_h(\lambda_0, k + h^\vee) \to W_h(\lambda_0, k) \) gives rise to a linear space isomorphism
\[
(5.4.2) \quad \pi_h(\lambda_0, k + h^\vee) = C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(\lambda_0, k).
\]

**Remark 5.4.2.** Using linear isomorphism \( [5.4.3] \) one can equip the space \( C_{\chi_h} \otimes (U^+_\chi(\hat{a}_+))^- W_h(\lambda_0, k) \) with the structure of an \( H^0_h \)–module. This \( H^0_h \)–module structure is not natural.

Using the last lemma the individual terms of the resolution \( [5.4.1] \) are equipped with the \( H^0_h \)–module structure, and the resolution takes the form
\[
(5.4.3) \quad 0 \to \pi_h(0, k + h^\vee) \to \pi_h(-2, k + h^\vee) \to 0.
\]

**Proposition 5.4.3.** The only nontrivial component \( S^h_1 : \pi_h(0, k+h^\vee) \to \pi_h(-2, k+h^\vee) \) of the differential of resolution \( [5.4.3] \) is given by \( S^h_1 = \int_{0}^{\infty} J^h_t(w) d\tau, \tau = q^{2(k+2)}, \) where the generating series \( J^h_t(z) \) is defined as follows
\[
J^h_t(w) = - \exp \left( - \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{(k+2)n_q} w^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{(k+2)n_q} w^{-n} \right) V_h,
\]
and the operator \( V_h : \pi_h(0, k + h^\vee) \to \pi_h(-2, k + h^\vee) \) sends the vacuum vector \( v_0 \) of \( \pi_h(0, k + h^\vee) \) to the vacuum vector \( v_{-2} \) of \( \pi_h(-2, k + h^\vee) \) and commutes with the elements \( \tilde{a}_n \) as follows
\[
(5.4.4) \quad [\tilde{a}_n, V_h] = - 2V_h \delta_{n,0}.
\]

**Proof.** The proof of this proposition is similar to that of Proposition \( 1.3.3 \). We only mention that in case of \( \hat{a}_2 \) the Lie algebra \( \tilde{\mathfrak{n}}^+_h \) is isomorphic to \( \mathfrak{n}^+_h \). In particular, the Lie algebra \( \tilde{\mathfrak{n}}^+_h \) is commutative. We also note that the intertwining operator \( S_h : W_h(\lambda_0, k) \to W_h(\lambda_0 - 2, k) \) introduced in Proposition \( 1.3.3 \) may be defined using elements \( e_n, \tilde{a}_n, n \in \mathbb{Z} \) and the operator \( V_h : \mathcal{H}(\lambda_0)_h \to \mathcal{H}(\lambda_0 - 2)_h \) that sends the vacuum vector \( v_{\lambda_0} \) of \( \mathcal{H}(\lambda_0)_h \) to the vacuum vector \( v_{\lambda_0 - 2} \) of \( \mathcal{H}(\lambda_0 - 2)_h \), intertwines the action of the elements \( \tilde{b}_n, \tilde{c}_n, V_Q, V_Q^{-1}, n \in \mathbb{Z} \) and commutes with \( \tilde{a}_n \) according
Proposition 5.4.4. (the following proposition. the algebra $H$ is a homomorphism of algebras. 52 A. SEVOSTYANOV to (5.4.4). Explicit calculation shows that $S_h = \int_0^s J_h^s(w)dt$, $p = q^{2(k+2)}$, where the generating series $J_h^s(w)$ is given by

$$J_h^s(z) = -u^{-1}e(u) \exp\left(-\sum_{n=1}^{\infty} \frac{a_{-n}}{([k+2]n)_q}w^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{([k+2]n)_q}w^{-n}\right) V_h w^{-\frac{a_0}{\sqrt{q}}}.$$

The deformed screening operator $S_h^h$ coincides with the screening operator for the q–Virasoro algebra introduced in [60, 6, 7]. This algebra may be defined as follows.

Let $T_h$ be the free associative topological algebra over $\mathbb{C}[[h]]$ topologically generated by elements $\{T_n| n \in \mathbb{Z}\}$. The algebra $T_h$ is naturally $\mathbb{Z}$-graded. We denote by $\hat{T}_h$ the restricted completion of $T_h$. The q–Virasoro algebra $Vir_{h,k}$ is the quotient of the algebra $\hat{T}_h$ by the two-sided ideal generated by the elements

$$[T_n, T_m] + \sum_{l=1}^{\infty} f_l (T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) + (q - q^{-1})^2 \frac{[k+2]q}{[k+1]q} \sum_{n=0}^{\infty} \frac{[2(k+2)n]_q}{[k+1]_q} \delta_{m+n,0},$$

where $k \in \mathbb{C}$ and the coefficients $f_l$ are defined with the help of the generating series $f(z)$,

$$f(z) = \sum_{l=0}^{\infty} f_l z^l = \exp\left\{- (q - q^{-1})^2 \sum_{n=1}^{\infty} \frac{1}{n q^{(k+1)n}} \left[ \frac{[k+2]q}{[k+1]q} \delta\left(\frac{2(k+1)w}{z}\right) - \delta\left(\frac{2(k+1)w}{z}\right) \right]\right\}.$$

Introducing the generating series $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$ the defining relations of the q–Virasoro algebra may be written as follows

$$f(w/z)T(z)T(w) - T(w)T(z)f(z/w) =$$

$$-(q - q^{-1})\frac{[k+2]q}{[k+1]q} \left[ \delta\left(\frac{2(k+1)w}{z}\right) - \delta\left(\frac{2(k+1)w}{z}\right) \right],$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$.

One can define an action of the algebra $Vir_{h,k}$ in the spaces $\pi_h(\lambda_0, k + h)$ using the following proposition.

**Proposition 5.4.4.** ([60], Section 4) Let $\hat{H}^1_h$ the restricted restricted completion of the algebra $H^1_h$. The map $Vir_{h,k} \rightarrow \hat{H}^1_h$ defined by

$$T(z) \mapsto \Lambda(zq^{k+1}) : + : \Lambda(zq^{-(k+1)})^{-1} ;,$$

where

$$\Lambda(z) = q^{-1}q^{-1}e^{\left\{ \sum_{n=1}^{\infty} \frac{q^{-n}}{([k+1]n)_q} z^n a_{-n} \right\}} \times e^{\left\{ (q - q^{-1})^2 \sum_{n=1}^{\infty} \frac{q^n}{([k+1]n)_q} z^{-n} a_n \right\} q^{-\frac{a_0}{\sqrt{q}}}},$$

is a homomorphism of algebras.

The following proposition shows that the algebra $Vir_{h,k}$ acts in the vacuum representation space $\text{Tor}_{\hat{U}_k^0}^{\mathbb{C}^{\chi_h}, V_{k,h}}$ for the algebra $W_{k,h}^s(\mathfrak{sl}_2)$.
Proposition 5.4.5. ( [60], Section 5) The action of the algebra $\mathcal{V}ir_{h,k}$ on the spaces $\pi_{h}(0, k + h^\vee)$ and $\pi_{h}(-2, k + h^\vee)$ commutes with the operator $S_{h}^{1} : \pi_{h}(0, k + h^\vee) \rightarrow \pi_{h}(-2, k + h^\vee)$. Therefore the algebra $\mathcal{V}ir_{h,k}$ acts in the vacuum representation space $\text{Tor}_{U_{k}^{\ast}(\mathfrak{g}_{2})}(C_{\lambda}, V_{k,h})$ for the algebra $W_{k,h}^{\ast}$. 

Remark 5.4.6. In fact using results of [36, 52] and [57] on the Drinfeld–Sokolov reduction for Poisson–Lie groups, the relation between Hecke algebras and classical Poisson reduction (see [53]), and geometric arguments similar to those presented in [54], Ch.4 one can show that $\mathcal{V}ir_{h,k}$ is a subalgebra in $W_{k,h}^{\ast}$. 

In conclusion we recall (see [7], Section 3 and [51], Section 3.1) that defining relations (5.4.5) are invariant under transformations $\theta$ and $\omega$ of the parameter $k$ and of the formal deformation parameter $h$ defined by

\begin{align}
\theta(k) &= \frac{1}{k + 2} - 2, \quad \theta(h) = -h(k + 2); \\
\omega(k) &= k, \quad \omega(h) = -h.
\end{align}

As a consequence we have the following proposition.

Proposition 5.4.7. Let $k, k' \in \mathbb{C}$ be complex numbers. Suppose that $k$ and $k'$ and formal deformation parameters $h$ and $h'$ are related by one of transformations (5.4.7). Then the algebras $\mathcal{V}ir_{k,h}$ and $\mathcal{V}ir_{k',h'}$ are isomorphic.

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