Predicting visibility of interference fringes in X-ray grating interferometry

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Abstract: The interference fringe visibility is a common figure of merit in designs of x-ray grating-based interferometers. Presently one has to resort to laborious computer simulations to predict fringe visibility values of interferometers with polychromatic x-ray sources. Expanding the authors’ previous work on Fourier expansion of the intensity fringe pattern, in this work the authors developed a general quantitative theory to predict the intensity fringe pattern in closed-form formulas, which incorporates the effects of partial spatial coherence, spectral average and detector pixel re-binning. These formulas can be used to predict the fringe visibility of a Talbot-Lau interferometer with any geometry configuration and any source spectrum.

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1. Introduction

X-ray grating interferometry is a differential x-ray phase-contrast imaging technique that has many potential applications in medical imaging and material science [1–24]. Figure 1 shows a schematic of an x-ray interferometer, which consists of a two-dimensional checkerboard phase grating, a micro-source-array, and a high-resolution imaging detector. The phase grating serves as a beam splitter and divides the incident beam into different diffraction orders. The interference between diffraction orders generates intensity fringes. The shape and modulation
of the interference fringes depend actually on grating’s phase shift, x-ray spectrum, spatial coherence degree of x-ray illumination, and system geometry such as the phase grating to detector distance. The intensity modulation of a fringe pattern usually is characterized by the visibility $V$ of the fringe:

$$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}.$$ (1)

With spatially coherent illumination, interference fringes of the maximal modulation are formed only at certain discrete grating-to-detector distances, which are called the self-image Talbot distances [1–4, 20]. For example, with monochromatic and plane wave illumination, following values of grating-to-detector distances correspond to the 2D self-image Talbot distances:

$$Z_{T-2D} = \frac{np_1^2}{8\lambda},$$ (2)

where $p_1$ is the phase grating period, $\lambda$ is the x-ray wavelength, $n$ takes odd integer values for a $\pi$-gratings, and otherwise $n = 4q + 2$, where $q$ is an integer [25].

The pattern and modulation of interference fringes depend on the spatial coherence degree of x-ray illumination as well. Spatially coherent illumination, such as generated by using synchrotron radiation or micro-focus tubes, allows high visibility of intensity fringes. However, in medical imaging applications micro-focus tube cannot offer sufficient x-ray flux to shorten exposure times. In order to fulfill the requirements of spatial coherence and adequate radiation output, one may employ a periodic array $G_0$ of mutually incoherent micro-anode-sources embedded in the anode (Fig.1) [26, 27]. Equivalently one may utilize an x-ray tube equipped with an absorbing source grating $G_0$, which serves as an aperture mask to break the focus spot into an array of mutually incoherent micro-sources [4, 8, 26, 28]. With appropriate geometrical arrangement, as is explained below, the diffraction fringes generated by these micro-sources are all superimposed on each other. This kind of system setup is called the Talbot-Lau interferometer [4, 28].

In x-ray grating interferometry fringe visibility $V$, as is defined in Eq. (1), is a common figure of merit in designs of x-ray grating-based interferometers. This is because the formation of high-modulation fringe pattern is a prerequisite for robust grating interferometry. When x-ray sources are spatially coherent and monochromatic, the self-image Talbot distances given by Eq.(2) may provide guidance for designs of grating interferometers. But in many applications such as medical imaging, broad-band polychromatic x-ray from x-ray tubes renders the concept of Talbot distance ill-defined. Firstly, Talbot distances are wavelength-dependent. A grating-detector distance may fall onto a Talbot distance for one wavelength but it just mismatch the Talbot distances for other wavelengths. Secondly, Talbot distances depend on grating phase shift values as well. But phase-shift value of a grating varies with x-ray wavelength. For example, a $\pi$-grating for 15-keV x-ray becomes $\pi/2$-grating for 30-keV x-ray, assuming the absence of absorption edges in the energy interval. Consequently, with a broad band x-ray beam the sharp discrete Talbot distances of the self-image regime would be stretched into several downstream intervals of relatively high fringe modulations. The locations and lengths of these intervals depend on the x-ray spectrum and the grating’s phase shift values [25,26]. On the other hand, the broadening of self-image regime with polychromatic x-ray offers design optimization opportunities. Broadband x-ray makes fringe modulations comparably high for a large range of grating-to-detector distances, which can be selected for system optimization [25]. Presently in design optimization of grating interferometers one has to resort to laborious computer simulations to predict the fringe visibility values for interferometers with polychromatic x-ray sources [26,27,29,30]. Even in these computer simulation studies, the spatially coherence effect with the source grating or micro-source array had not been included because of the complexity in simulating spatial coherence effect. Hence for x-ray interferometry design optimization there
is pressing need for a quantitative theory that is able to predict the fringe visibility attainable with any given Talbot-Lau configuration and x-ray spectrum. In this work we present such a general theory to predict the fringe visibility in closed-form formulas.

This work expands our previous work of Fourier expansion of interference fringes in grating interferometry. In that work, assuming spatially coherent illumination from synchrotron radiation or microfocus tubes, we evolved the Wigner distribution to obtain closed-form expressions of Fourier coefficients of intensity fringes for any grating-to-detector distances [25]. The purpose of this work is to derive closed-form formulas of the coherence-weighted and spectrum-averaged Fourier coefficients of fringes in Talbot-Lau interferometry with polychromatic x-ray sources. Moreover, in order to predict the intensity fringe patterns, these coherence-weighted and spectrum-averaged Fourier coefficients will be further weighted by detector pixel rebinning effects. In this way these closed-form formulas of coherence-weighted and spectrum-averaged Fourier coefficients enable us to predict the fringe visibility of the Talbot-Lau interferometers. We organize the presentation as follows. In section 2, starting from the Fourier analysis of fringe intensity, we briefly review the general theoretical formalism that underlines our new theory. Particularly we discuss how to compute the coherence degree to incorporate partial spatial coherence effects for Talbot-Lau interferometers. In section 3, we present the closed-form formula of the coherence-weighted Fourier coefficients of fringe intensity, which incorporated spatial coherence effects and x-ray spectral average. Furthermore, through two examples of an inverse-geometry Talbot-Lau interferometer we show how to predict fringe visibility values by using these closed-form formulas. We validate the predicted visibility values by comparing them to computer simulation results. We conclude the paper in section 4.

Fig. 1. Schematic of an x-ray phase grating interferometer with a microfocus source

2. Materials and methods

Consider a grating interferometer as schematically depicted in Fig. 1. The wave-splitting phase grating is denoted by \( G_1 \), which is a two-dimensional checkerboard phase grating. The phase grating of a bi-directional period \( p_1 \) is a periodic array of phase shift modulation between values
of $\Delta \phi$ and zero. Mathematically we model such a checkerboard phase grating $G_1$ by:

$$G_1(\vec{r}) = \exp(i\Delta \phi \cdot h(\vec{r}/p_1 - |\vec{r}/p_1| - 1/2),$$

$$h(\vec{s}) = \begin{cases} 
1, & \text{if } \vec{s} \in [-1/2, 0]^2 \cup [0, 1/2]^2, \\
0, & \text{otherwise},
\end{cases}$$

(3)

where the so-called floor-function $[x]$ is defined as the largest integer that is less or equal to $x$. In order to provide spatially coherent illumination, the interferometer employs a rectangular micro-source-array $G_0$ of bi-directional period $p_0$. As is reported in literature, such a micro-source-array $G_0$ can be realized, for example, by employing a periodic array $G_0$ of micro-tungsten targets embedded in a synthetic diamond-substrate [27], and the micro-tungsten targets emit x-ray when they are irradiated by accelerated electrons from the cathode. With the dominant bremsstrahlung generated from the tungsten dots as compared to that from the diamond substrate, this tungsten-dots array effectively functions as an array of microfocus sources. Equivalently one may utilize an x-ray tube combined with a 2D absorbing source grating to effectively break the tube’s focus spot into an array of micro-sources, as each aperture of the source grating functions as a micro-source, and the source grating as an array of micro-sources [27]. A high-resolution detector is assumed such that it well resolves the intensity fringe pattern. The detector pixel rebinning effect on the fringe pattern will be discussed in section 3.

The interferometer configuration with such a micro-sources array is also called the Talbot-Lau interferometer [4, 28]. The distance between gratings $G_0$ and $G_1$ is denoted by $R_1$, and that between $G_1$ and the detector is $R_2$.

Our theoretical formulation of grating interferometer starts from the Wigner distribution formalism for x-ray Fresnel diffraction [31, 32]. The Wigner distribution in the phase space (position space and ray-direction space) is defined as Fourier transform of the mutual intensity [33]. The advantages of using Wigner distribution formalism for x-ray phase-contrast imaging are two-fold. First, Wigner distribution in wave diffraction process is especially simple owing to its covariance in phase-space [31, 32]. The Wigner distribution in the phase space of wavefields with respect to the displacement vector $\vec{m}$ [33]. The advantages of using Wigner distribution formalism for x-ray Fresnel diffraction [31, 32]. The Wigner distribution in the phase space of wavefields with respect to the displacement vector [33]. The advantages of using Wigner distribution formalism for x-ray phase-contrast imaging are two-fold. First, Wigner distribution directly specifies the partial coherence of the x-ray wavefield. Second, evolution of Wigner distribution in wave diffraction process is especially simple owing to its covariance in phase-space [31, 32]. As is shown in our previous work [25], for monochromatic x-ray sources of wavelength $\lambda$, Fourier transform of the fringe intensity at a distance $(R_1 + R_2)$ from the phase grating $G_1$ is given by:

$$I_k(\vec{u}/M_g; R_1 + R_2) = I_{\text{in}, \lambda} \mu_{\text{in}} \left( \frac{\lambda R_2 \vec{u}}{M_g} \right) \int G_1(\vec{s}/M_g) \cdot e^{i(\lambda R_2 \vec{u} \cdot \vec{s})} \cdot \exp(2\pi i \cdot \vec{u} \cdot \vec{s}) d\vec{s},$$

(4)

where $I_k(\vec{u}/M_g)$ denotes the fringe intensity’s Fourier transform valued at $\vec{u}/M_g$, $\vec{u}$ is the spatial frequency vector at the phase grating plane and $M_g = (R_1 + R_2)/R_1$ is the geometric magnification. In Eq. (4) the constant factor $I_{\text{in}, \lambda}$ denotes the incident x-ray intensity, and the factor $\mu_{\text{in}}(\lambda R_2 \vec{u}/M_g)$ denotes the reduced complex degree of x-ray spatial coherence at the entrance of $G_1$ [33]. For sake of convenience we simply call $\mu_{\text{in}}(\lambda R_2 \vec{u}/M_g)$ the coherence degree. Using the property of the Fourier transform of convolution product and Poisson summation formula [25], we found that Fourier transform of the intensity fringes are given by:

$$I_k(\vec{u}/M_g; R_1 + R_2) = I_{\text{in}, \lambda} \sum_{\vec{m} \in \mathbb{Z}^2} \mu_{\text{in}} \left( \frac{\lambda R_2 \vec{m}}{M_g p_1} \right) \cdot C(\vec{m}; \lambda) \cdot \delta \left( \vec{u} - \frac{\vec{m}}{p_1} \right),$$

(5)

where $\delta$ is the Dirac delta function. Therefore, this equation shows that the fringe intensity is a superposition of diffraction orders with discrete spatial frequencies $\{\vec{m}/p_1; \vec{m} = (m,n) \in \mathbb{Z}^2\}$.
While the coefficients

\[ f \]

where

\[ \lambda \]

According to the method explained in above section, we found that Fourier expansion of the

\[ | \]

Since the floor-function \( \lfloor x \rfloor \) is defined as the largest integer that is less or equal to \( x \), the factor \((-1)^{\lfloor k/2 \rfloor} \) will swing between 1 and -1 as its exponent changes. From Eqs. (6) and (7), it is clear that the Fourier coefficient \( C(\vec{m}; \lambda) \) depends on the grating phase shift \( \Delta \phi \), system geometry setting \( R_1 \) and \( R_2 \), x-ray wavelength \( \lambda \), and its diffraction order \( m \). Particularly, \( |C(\vec{m}; \lambda)| \) decreases relatively slowly as \( 1/mn \) with increasing order \( m = (m, n) \). The efforts in derivation of Fourier coefficients \( C(\vec{m}; \lambda) \) can be traced back to earlier optics literatures [34–36]. But no general closed-form formula was found until recently [25]. The details of the derivation can be found in the Appendix of [25].

In order to derive the coherence-weighted Fourier coefficients, we need to derive the formula for coherence degree \( \mu_m(\lambda R_2 \vec{m}/M_g p_1) \) as well. Spatial coherence degree depends on the source intensity distribution \( I_{source}(\vec{s}) \) and system configuration geometry. The coherence degree can be found from the mutual intensity evolution of wavefields [33]. Since micro-sources in an array emit x-ray incoherently, the Van Citters-Zernike theorem of the mutual intensity propagation is applicable [33]. Using this theorem, we found that the reduced complex degree of the transverse coherence is given by

\[ \mu_m(\lambda R_2 \vec{m}/M_g) = \frac{\int I_{source}(\vec{s}) \exp(i2\pi \vec{R}_2 \cdot \vec{\bar{s}}/M_g R_1) d\vec{s}}{\int I_{source}(\vec{s}) d\vec{s}}. \]  

Substitute the source intensity distribution into Eq. (8), we will be able to derive \( \mu_m(\lambda R_2 \vec{m}/M_g) \), as is shown below in the next section.

3. Results

According to the method explained in above section, we found that Fourier expansion of the intensity fringe of a Talbot-Lau interferometer with monochromatic source is given by:

\[ I_\lambda(\vec{r}; R_1 + R_2) = \frac{I_{in,\lambda}}{M_g^2} \sum_{\vec{m} \in \mathbb{Z}^2} \mu_m(\lambda R_2 \vec{m}/M_g) \cdot C(\vec{m}; \lambda) \cdot \exp\left(i2\pi \frac{\vec{m} \cdot \vec{r}}{M_g p_1}\right). \]  

While the coefficients \( C(\vec{m}; \lambda) \) in Eq.(9) are given by Eqs. (6) and (7), the spatial coherence degree \( \mu_m(\lambda R_2 \vec{m}/M_g p_1) \) is determined by the source intensity distribution and system geometry. Consider a micro-source array consisting of \((2N_x + 1) \times (2N_y + 1)\) identical micro-sources. The centers of these micro-sources form a \((2N_x + 1) \times (2N_y + 1)\) lattice of bidirectional
periodicity of \( p_0 \). As for the intensity distribution of an individual micro-source, it could be uniform circular or Gaussian distribution, or other forms. For sake of discussion, we assume that each of the micro-sources is a small disk of diameter \( 2a \) with uniform intensity. We will discuss Gaussian micro-sources as well in section 4. This being so, we model the source intensity distribution \( I_{\text{source}}(\mathbf{s}) \) generated from this micro-source array as

\[
I_{\text{source}}(\mathbf{s}) = I_0 \sum_{k=-N_x}^{N_x} \sum_{l=-N_y}^{N_y} \text{Circ} \left( \frac{\|\mathbf{s} - k\mathbf{p}_0 \mathbf{e}_x - l\mathbf{p}_0 \mathbf{e}_y \|}{a} \right),
\]

where

\[
\text{Circ}(r) = \begin{cases} 
1, & \text{if } |r| \leq 1, \\
0, & \text{otherwise},
\end{cases}
\]

and \( I_0 \) is a constant specifying the intensity of each of the micro-sources. Substituting Eqs. (10) and (11) into Eq. (8), according to Van Citters-Zernike theorem, we found that the coherence degree is given by:

\[
\mu_{\text{in}} \left( \lambda R_2 \mathbf{u} / M_g \right) = \frac{2J_1 (2\pi a \cdot (M_g - 1) |\mathbf{u}| / M_g)}{2\pi a \cdot (M_g - 1) |\mathbf{u}| / M_g} \cdot W \left( \lambda R_2 \mathbf{u} / M_g \right),
\]

where \( J_1(x) \) is the Bessel function of the first kind, and \( J_1(x)/x \) is the so-called jinc function, and \( a \) is the radius of the micro-source. The second factor \( W(\lambda R_2 \mathbf{u} / M_g) \) in above equation denotes the interference factor, which represents the effective contribution from the interference of x-ray wave fields emitted from all the micro-sources. Through a lengthy calculation we found that

\[
W(\lambda R_2 \mathbf{u} / M_g) = \frac{\sin \left( \left( 2N_x + 1 \right) \pi \cdot (M_g - 1) p_0 \frac{u_x}{M_g} \right) \cdot \sin \left( \left( 2N_y + 1 \right) \pi \cdot (M_g - 1) p_0 \frac{u_y}{M_g} \right)}{(2N_x + 1)(2N_y + 1) \sin \left( \pi (M_g - 1) p_0 \frac{u_x}{M_g} \right) \cdot \sin \left( \pi (M_g - 1) p_0 \frac{u_y}{M_g} \right)}.
\]

The derivation details can be found in the appendix. Since the fringe intensity is a superposition of diffraction orders with discrete frequencies \( \{ \mathbf{u} = \mathbf{m} / p_1; \mathbf{m} = (m, n) \in \mathbb{Z}^2 \} \), we found from Eq. (13) that \( W(\lambda R_2 \mathbf{m} / M_g p_1) \) achieves its maximal value, namely \( W(\lambda R_2 \mathbf{m} / M_g p_1) = 1 \), when the following constructive interference condition is satisfied for any phase shift \( \Delta \phi \)

\[
p_0 = \frac{R_1}{R_2} M_g p_1.
\]

The constructive interference condition Eq. (14) has a simple geometric interpretation: the fringe displacement generated by any off-center micro-source should be equal to a multiple of the fringe period.

It should be pointed that some authors wrote the constructive interference condition as \( p_0 = (R_1 / R_2) \cdot (M_g p_1 / 2) \) for the cases of \( \pi \)-gratings \([4, 8, 26, 28] \). This would be correct only if monochromatic x-ray sources were employed. In these cases, according to Eq. (6), there are no odd diffraction orders generated because \( \sin(\pi) = 0 \). Therefore the fringe period is \( M_g p_1 / 2 \). But for polychromatic x-ray cases the grating phase-shift varies with x-ray photon energy, so the fringe period should be \( M_g p_1 \), as is given by Eq. (6) and (9). Hence the constructive interference condition is generally given by Eq. (14). For polychromatic x-ray cases, if the incorrect condition were used, the odd diffraction orders would be greatly suppressed, and the fringe visibility would be reduced.

Under this constructive interference condition, \( \mu_{\text{in}} \left( \lambda R_2 \mathbf{m} / M_g p_1 \right) \) becomes just a jinc function:

\[
\mu_{\text{in}} \left( \lambda R_2 \mathbf{m} / M_g p_1 \right) = \frac{2J_1 \left( 2\pi \sqrt{m^2 + n^2/a} / p_0 \right)}{2\pi \sqrt{m^2 + n^2/a} / p_0},
\]

where \( |\mathbf{m}| = \sqrt{m^2 + n^2} \).
where the diffraction order is labeled by $\mathbf{m} = (m, n)$. It is important to note that $\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)$ does not depend on x-ray wavelength, because each of the micro-sources in the array emits x-ray independently from each other, so the wavelength factors get canceled out according to the VanCitters-Zernike theorem of Eq. (8). The spatial coherence degree derived above is obviously equally applicable to the cases of the absorbing source gratings. To apply Eq. (15) to a source grating, $p_0$ is its bidirectional period, and $2a$ is the diameter of the individual apertures. As is shown in Fig. 2, $|\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)| \leq 1$ and $|\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)|$ generally decreases with increasing diffraction orders and diameters of individual micro-sources. Since the modulation of a given diffraction order is reduced by the factor $|\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)|$, hence the higher the diffraction order is, the larger the loss in fringe modulation incurs.

Fig. 2. Value of $\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)$ with different $\hat{\mathbf{m}}$ with the constructive interference condition Eq. (14), where $p_0$ is the period of the micro-source array, and $a$ is the radius of the individual micro-source.

For polychromatic x-ray, the fringe intensity is a sum of its spectral components given by Eq. (9). Hence with a polychromatic source the fringe intensity is given by

$$I(\mathbf{r}) = \frac{I_{in}}{M_s} \sum_{\mathbf{m} \in \mathbb{Z}^2} V(\hat{\mathbf{m}}) \cdot \exp \left( i 2\pi \frac{\hat{\mathbf{m}} \cdot \mathbf{r}}{M_s p_1} \right),$$

$$V(\hat{\mathbf{m}}) \equiv \int S(\lambda) \cdot \mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right) \cdot C(\hat{\mathbf{m}}; \lambda) d\lambda,$$

(16)

where $S(\lambda)$ is the normalized source x-ray fluence spectrum, and we call $V(\hat{\mathbf{m}})$ as the spectrum-averaged and coherence-weighted Fourier coefficients of intensity fringes. As we explained earlier, $\mu_{in} \left( \lambda R_2 \hat{\mathbf{m}} / M_s p_1 \right)$ is indeed independent of wavelength.

On the other hand, we may rewrite Eq. (16) as:

$$I(\mathbf{r}) = \frac{I_{in}}{M_s} \left\{ 1 + 2 \sum_{(m>0, n\neq 0)} V(\hat{\mathbf{m}}) \cdot \cos \left( 2\pi \frac{mx + ny}{M_s p_1} \right) +
\right.$$

$$\left. + 2 \sum_{(m>0, n=0)} V(\hat{\mathbf{m}}) \cdot \left[ \cos \left( 2\pi \frac{mx}{M_s p_1} \right) + \cos \left( 2\pi \frac{my}{M_s p_1} \right) \right] \right\}. \quad (17)$$

For sake of discussion, we assume that the detector has pixel size $p_d$ and a filling factor of unity without pixel cross talk. Incorporating the “blurring” effect of pixel binning by multiplying
the sinc functions \( \text{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x} \), we can express the fringe intensity with the detector pixel re-binning effect as:

\[
I(\mathbf{r}) = \frac{I_{in}}{M_g^2} \left\{ 1 + 2 \sum_{(m>0, n \neq 0)} V(\mathbf{m}) \cdot \text{sinc} \left( \frac{mp_d}{M_g p_1} \right) \cdot \text{sinc} \left( \frac{np_d}{M_g p_1} \right) \cdot \cos \left( 2\pi \frac{mx + ny}{M_g p_1} \right) + 
\right.
\]

\[
+ 2 \sum_{(m>0, n=0)} V(\mathbf{m}) \cdot \text{sinc} \left( \frac{mp_d}{M_g p_1} \right) \cdot \left[ \cos \left( 2\pi \frac{mx}{M_g p_1} \right) + \cos \left( 2\pi \frac{my}{M_g p_1} \right) \right] \right\}. \quad (18)
\]

Combining Eq. (18), with Eqs. (6) and (15), we derived the closed-form formulas for computing the intensity fringes. These formulas incorporate the effects of partial spatial coherence, spectral average and detector pixel re-binning. These formulas, Eqs. (6), (15) and (18) are the central results of this work.

Since the quantities \( |\mu_{in}(\lambda R_2 \mathbf{m}/M_g p_1)| \), \( |C(\mathbf{m}; \lambda)| \) and \( |\text{sinc}(mp_d/M_g p_1)| \) all decrease with increasing \( |m| \) and \( |n| \), only several low diffraction orders with small \( |m| \) and \( |n| \) are able to make non-negligible contributions to the intensity fringes. This feature facilitates the computation of the fringe visibility, as is demonstrated in the following example.

Let us consider the design of a grating imaging system with the inverse geometry configuration [26–28, 28], which was briefly mentioned in section 1. In the inverse geometry configuration, the phase grating \( G_1 \) is placed close to the micro-source array such that \( R_2 \geq R_1 \) and the magnification factor \( M \) can be as high as a few tens. With such high-magnification settings, the resulting interference fringes can be directly resolved by the imaging detector without using an additional absorbing grating to mask the detector. Consequently, the interferometers with such inverse geometry will significantly reduce radiation dose to the subjects. In design of an inverse geometry interferometer, matching \( R_2 \) to a Talbot distance requires long source-detector distances of few meters, owing to the high-magnification factor required in such a system [28]. However, with polychromatic sources, a broadband x-ray spectrum allows a large range of \( R_1 \) and \( R_2 \) values available for comparable fringe visibility values. One can then select shorter \( R_1 \) and \( R_2 \) distances to reduce the system size. In the literature researchers employed laborious computer simulations to compute and compare the visibility values attainable with different system geometries [26, 27, 37]. In contrast to resorting to computer simulations, we show here that the closed-form formulas of Eqs. (6), (15) and (18) provide a versatile tool for computing fringe visibilities.

Fig. 3. Utilized x-ray spectrum
As one example, consider a single phase grating interferometer with a source array of micro-tungsten-targets, which operates at an acceleration voltage 35 kV and the generated x-ray is filtered with 50\(\mu\)m rhodium filter. This tube voltage and filtration are one of typical settings used in breast imaging. The resulting broad x-ray spectrum spreading from 7 kV to 35 kV is shown in Fig. 3. The spectrum peaks at 22.5 kV and is shaped by a 50\(\mu\)m rhodium filter with the k-edge at 23.2 kV. The interferometer is assumed to employ a 25 \(\times\) 25 array of micro-targets of 2\(\mu\)m in diameter each. The phase grating is a two-dimensional checkerboard phase grating of \(p_1 = 5\mu\)m and phase shift \(\Delta\phi = \pi/2\) at 20-keV. We assume that the fringe pattern is to be resolved by using a detector with pixel size \(p_d = 25\mu\)m, and a high magnification factor of 15 is used. While there are many geometry settings can be used to implement the high-magnification setting [25, 28], here we set the source array-to-grating distance of \(R_1 = 8.4\) cm and a grating-to-detector distance of \(R_2 = 117.6\) cm to make the source-detector distance compact. It is easy to check that such a selection of \(R_1\) and \(R_2\) grossly mismatches any of the Talbot distances for 20 keV x-ray [25, 28]. With such a choice we just want to demonstrate the capability of Eqs. (6), (15) and (18) to predict the fringe visibility for any geometry configuration and any x-ray spectrum. In order to satisfy the constructive interference condition of Eq. (14), we set the period of the micro-source array to \(p_0 = 5.36\mu\)m. Using Eqs. (6), (15) and (18), we computed the spectrum-averaged and coherence-weighted Fourier coefficients \(V(\hat{m})\) and the associated sinc-functions. In this example we found that only first few diffraction orders with \(|\hat{m}| \leq 2\) contribute to the fringe intensity, while the other orders with \(|\hat{m}| > 2\) can be neglected within accuracy of 0.2\%. Therefore, we found that the fringe intensity of the exemplary grating interferometer is given by the following expression:

\[
I(\vec{r}; R_1 + R_2) = \frac{I_n}{M g^2} \left(1 + 2 \times 0.168 \times \left[ - \cos \left(2\pi \frac{x + y}{M g p_1}\right) + \cos \left(2\pi \frac{x - y}{M g p_1}\right) \right] - 2 \times 0.0333 \times \left[ \cos \left(2\pi \frac{2x}{M g p_1}\right) + \cos \left(2\pi \frac{2y}{M g p_1}\right) \right] \right). \tag{19}
\]

Equation (19) shows the composition of the fringe pattern. Obviously the first part in the curly bracket on the right hand side of Eq. (19) contributes to the constant background intensity. The second part in the curly bracket, which is a cross-sinusoidal term in x and y, represents a checkerboard pattern, which is superimposed on the background with an intensity-modulation weighting of \((2 \times 0.168)\), and the third part represents a mesh pattern, which is superimposed on the pattern with a tiny intensity-modulation weighting of \((2 \times 0.0333)\).

As for the fringe visibility, we need to consider the down-sampling effects of the pixel rebinning as well. Since in this example \(M g p_1 = 75\mu\)m and \(p_d = 25\mu\)m, there are three sampled points in a fringe period along each of the directions. Obviously the cross-sinusoidal term in Eq. (19) determines the locations of the maximum and minimum intensities. It is easy to see that the intensity reaches its maximum \(I_{\text{max}} = 1.5706 \cdot I_n/M g^2\) at \((x - p_d/2, y - p_d/2) = (k, l) \cdot M g p_1\), and \((x - p_d/2, y - p_d/2) = (2/3 + k, 2/3 + l) \cdot M g p_1\), and its minimum value \(I_{\text{min}} = 0.5626 \cdot I_n/M g^2\) at \((x, y) = (k, 2/3 + l) \cdot M g p_1\) and \((x, y) = (2/3 + k, l) \cdot M g p_1\), where \(k, l \in \mathbb{Z}\). This being so, the visibility \(V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})\) of the fringe pattern is equal to 0.4725. Note also that Eq. (19) predicts not only the fringe visibility for the interferometer, but also the intensity values at all pixels. Hence Eq. (19) is able to predict the whole interference fringe pattern of the interferometer.

In order to validate our prediction, we performed computer simulations. We simulated x-ray Fresnel diffraction through a single phase grating interferometer with a source array of micro-tungsten-targets. We set the simulated system configuration identical to that described earlier in above example. Assuming a point-like source , we first simulated Fresnel diffraction to propagate the wavefront of the phase grating \(G_1\) to the detector plane with extremely small sampling pitch \(p_s = p_d/(2^8 M g)\). From the resulting wavefront we calculated the intensity image
To simulate the fringe intensity pattern generated by the micro-source array, we computed the convolution of intensity image $I_1$ with the source intensity $I_{\text{source}}(\vec{s})$ of Eq. (10). Finally we apply the ray-binning to the convoluted intensity to obtain the final binned and down-sampled intensity image that measured by the detector. Figure 4(a) shows the resulting intensity fringe pattern from the simulation, and Fig. 4(b) is the fringe pattern obtained from Eq. (19). The two agree very well to each other. The maximum and minimum values in the simulated intensity fringe pattern are 1.5746 and 0.5673, respectively. Thereby the visibility of the fringe pattern is equal to 0.4703, which is in good agreement with the analytical prediction presented earlier. This good agreement validates our general theory presented by Eqs. (6), (15) and (18).

As another example, consider a design modification to the Talbot-Lau interferometer described in above example. Suppose the micro-target’s diameter is enlarged from 2 $\mu$m to 2.5 $\mu$m, while the design of Talbot-Lau interferometer is otherwise identical. Using Eq. (18), we found the intensity fringe pattern is changed to:

$$I(\vec{r}; R_1 + R_2) = \frac{I_{\text{in}}}{M_2} \left\{ 1 + 2 \times 0.133 \times \left[ - \cos \left( \frac{2\pi x + y}{M_2 p_1} \right) + \cos \left( \frac{2\pi x - y}{M_2 p_1} \right) \right] - 
- 2 \times 0.0182 \times \left[ \cos \left( \frac{2\pi 2x}{M_2 p_1} \right) \cos \left( \frac{2\pi 2y}{M_2 p_1} \right) \right] \right\}.$$  \hspace{1cm} (20)

Following the same analysis method as that applied to the fringe pattern of Eq. (19), we found that the fringe visibility with this design modification is significantly reduced to 0.385. The computer simulation results in a visibility value of 0.386, which is again in good agreement with our theoretical prediction.

4. Discussion and conclusions

X-ray grating interferometry has many potential applications in medical imaging and material science. The intensity modulation of a fringe pattern is characterized by the visibility of the fringe. Since formation of high-modulation fringe pattern is critical to robust grating
interferometry, the fringe visibility is a common figure of merit in design of x-ray grating-based interferometer. So far there is no quantitative theory yet that is able to predict fringe visibility, which actually depends on x-ray spectrum, spatial coherence degree of x-ray illumination, system geometry, and detector pixel re-binning. In literature intensive and burdensome computer simulations were used to predict the fringe visibility values of interferometers. In this work, we developed a general theory to predict the fringe visibility values of phase grating interferometers. Our theory, as is instilled in Eqs. (6), (15) and (18), provides closed-form formulas to compute the fringe visibility of a phase grating interferometer.

In previous discussion we assume that each of the micro-sources is a small disk of uniform intensity. But our theory applies to the case with Gaussian micro-source array, in which each of the micro-sources has a Gaussian intensity distribution. Consider a micro-source array consisting of $(2N_x + 1) \times (2N_y + 1)$ identical Gaussian micro-sources. The source intensity distribution $I_{\text{source}}(\mathbf{s})$ from a Gaussian micro-source array can be modeled as:

\[
I_{\text{source}}(\mathbf{s}) = \frac{Q}{2\pi \sigma^2} \sum_{k=-N_x}^{N_x} \sum_{l=-N_y}^{N_y} \exp \left[ -\frac{(\mathbf{s} - kp_0 \mathbf{e}_x - lp_0 \mathbf{e}_y)^2}{2\sigma^2} \right],
\]

where $\sigma$ is the standard deviation of the Gaussian distribution, and $Q$ is a constant equal to the area integral of $I_{\text{source}}(\mathbf{s})$. Substituting Eq. (21) into Eq. (8), we found that, when the constructive interference condition of Eq. (14) is satisfied, the coherence degree for such a Gaussian micro-source array is given by:

\[
\mu_{\text{in}}(\frac{\lambda R_2 \mathbf{\bar{m}}}{M_s p_1}) = \exp \left( -\frac{2\pi^2 (m^2 + n^2) \sigma^2}{p_0^2} \right).
\]

Substituting Eq. (22) into Eq. (18), one can compute the fringe visibility of an interferometer with a Gaussian micro-source array.

Our theory is a general quantitative theory that is versatile in other usage as well. For example, in the cases with spatially coherent x-ray illumination generated by synchrotron radiation or a point-like focal spot, the theory is applicable by simply letting $\mu_{\text{in}}(\lambda R_2 \mathbf{\bar{m}}/M_s p_1) = 1$ in Eq. (16). Moreover, in the cases with a source of single spot, the fringe intensity pattern equation Eq. (18) remains applicable, provided $p_0$ in Eq. (15) or Eq. (22) is replaced by $(R_1/R_2) \cdot M_s p_1$.

The formulas of Eqs. (6), (15) and (18) derived in this work can be applied to one-dimensional grating interferometers as well, provided some dimensional reduction changes are imposed. The diffraction order label $\mathbf{\bar{m}} = (m, n)$ in these formulas will be reduced to the one-integer label $m$ in one-dimensional (1D) cases. Hence, when apply Eq. (15) and (18) to 1D interferometers, one should let $n = 0$ in these formulas. More importantly, for the 1D cases the two-dimensional Fourier coefficients $C(\mathbf{\bar{m}}; \lambda)$ should be replaced by the one-dimensional $C(m; \lambda)$ given as follows [25]:

\[
C(m; \lambda) = \begin{cases} 
1, & \text{if } m = 0, \\
-(1 - \cos \Delta \phi) \cdot (-1)^{\lfloor k \lambda R_2/M_s p_1 \rfloor} \frac{\sin(\pi \lambda R_2/M_s p_1)}{k \pi}, & \text{if } m = 2k, k \neq 0, \\
i \cdot \sin \Delta \phi \cdot \frac{\sin((4m \lambda R_2/M_s p_1)^2 + (k+1/2)^2)}{\pi (k+1/2)}, & \text{if } m = 2k + 1, 
\end{cases}
\]

where $i = \sqrt{-1}$. With these dimensional reduction changes, the intensity fringe pattern for an
one-dimensional Talbot-Lau interferometer is given by

\[
I(x) = \frac{I_{in}}{M^2} \left\{ 1 + 2 \sum_{k>0} \left[ V(2k) \cdot \frac{2k p_d}{M g p_1} \cdot \cos \left( 2\pi \frac{2k x}{M g p_1} \right) \right. \\
+ \left. V(2k-1) \cdot \frac{2(k-1) p_d}{M g p_1} \cdot \sin \left( 2\pi \frac{2(k-1) x}{M g p_1} \right) \right] \right\}, \tag{24}
\]

where

\[
V(m) \equiv \int S(\lambda) \cdot \mu_{in}(\lambda R_2 m/M g p_1) \cdot D(m;\lambda) \, d\lambda,
\]

and

\[
D(m;\lambda) = C(m;\lambda), \text{ if } m \text{ is even and } D(m;\lambda) = i \cdot C(m;\lambda), \text{ if } m \text{ is odd.}
\]

In conclusion, we developed a general quantitative theory to predict the fringe visibility values of Talbot-Lau interferometers with polychromatic x-ray sources. The derived closed-form expressions for the interference fringes provide a versatile tool for design optimization of grating-based x-ray interferometers. Since polychromatic anode sources with limited partial spatial coherence prevail in medical imaging, the presented theory will be especially useful for future medical applications of x-ray Talbot-Lau interferometry.

Appendix: Derivation of the interference factor in Eq. (13)

The reduced complex degree of spatial coherence \( \mu_{in}(\lambda R_2 \hat{u}/M g) \) is determined by the source intensity distribution and system geometry. In the derivation one encounters a key integral as follows:

\[
\int \text{Circ} \left( \frac{2|\hat{u}|}{a} \right) \exp \left( i 2\pi \frac{R_2 \hat{u}}{M g R_1} \right) \, d\hat{u} = \frac{M g a R_1}{R_2 |\hat{u}|} J_1 \left( 2\pi a \left( M g \right)^{-1} R_2 |\hat{u}| \right), \tag{A1}
\]

where \( J_1(x) \) is the Bessel function of the first kind. Hence, substituting Eqs. (10) and (11) into Eq. (8), one obtains:

\[
\mu_{in} \left( \lambda R_2 \hat{u} / M g \right) = \frac{2 J_1 \left( 2\pi a \cdot (M g - 1) |\hat{u}| / M g \right)}{2\pi a \cdot (M g - 1) |\hat{u}| / M g} \times \\
\times \sum_{n=-N_x}^{N_x} \sum_{n=-N_y}^{N_y} \exp \left[ i 2\pi (M g - 1) p_0 (mu_x + nu_y) / M g \right] / (2N_x + 1)(2N_y + 1). \tag{A2}
\]

Working out the summations in above equation, one is led to Eq. (13).

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