Series of the solutions to Yang-Baxter equations: Hecke type matrices and descendant R-, L-operators

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Abstract

We have constructed series of the spectral parameter dependent solutions to the Yang-Baxter equations defined on the tensor product of reducible representations with a symmetry of quantum (super)algebra. These series are produced as descendant solutions from the $sl_q(2)$-invariant Hecke type $R^r(u)$-matrices. The analogues of the matrices of Hecke type with the symmetry of the quantum super-algebra $osp_q(1|2)$ are obtained precisely. For the homogeneous solutions $R^{r^2-1}\times(r^2-1)$ there are constructed Hamiltonian operators of the corresponding one-dimensional quantum integrable models, which describe rather intricate interactions between different kind of spin states. Centralizer operators defined on the products of the composite states are discussed. The inhomogeneous series of the operators $R^{r\mathcal{R}}(u)$, extended Lax operators of Hecke type, also are suggested.

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1 Introduction

The Yang-Baxter equations (YBE), appeared in the early investigations on the problem of the exactly solvability in 2d statistical physics, as well as in (1 + 1)-dimensional scattering theory, and being one of the key relations of the QISM (Quantum Inverse Scattering Method), still remain an actual and attractive subject of the statistical and mathematical physics, with expanding area of applications [1]-[17]. In studying the solutions to the YBE with the symmetry of quantum algebra as basic constructions the universal $R$ matrices are considered, which can be achieved either by the quantum double principle of Drinfeld or by the Jimbo’s compositions involved algebra invariant matrices - projection operators, and different schemes of affinization (or baxterization) are developed for obtaining the spectral parameter dependent solutions [7, 9, 12, 13, 16]. It turns out that the range of the spectral parameter dependent solutions to Yang-Baxter equations with the given quantum algebra symmetry is richer, than that which can be constructed via the universal $R$ matrices of the corresponding quantum algebras. For finding the full class of the symmetric YBE solutions it is sufficient to consider the $R$-matrix in the expansion of the whole basis of invariant operators (projectors), which must be specified for the given set of the representations [12, 13, 14, 28]. In particular, investigating solutions defined on the cyclic and indecomposable representations of the quantum algebra $sl_q(2)$ at roots of unity [30, 31], we find new solutions and yet a rich variety of the solutions, which are characterised by different structures of decompositions into the projectors and as well by additional (spectral) parameters [34]. As a yet another confirmation of the mentioned observation could be served the existence of a series of solutions to the YBE with symmetry of the quantum super-algebra $osp_q(1|2)$ defined on the spin-irreps, which differs from the known solutions [20, 22, 26], and the discussion done in the Section 2 of this work demonstrates the exact derivation of this series. The similar solutions (Hecke type $R$-operators), as it is known, exist for the quantum algebra $sl_q(2)$ [8, 10, 19, 29, 28], and this reflects the circumstance that there is an explicit correspondence between the representations of the quantum algebras $sl_q(2)$ and $osp_q(1|2)$, providing that $q \rightarrow i\sqrt{q}$ [23, 24, 26, 27]. Then in Section 3 a descendant series of the men-
tioned solutions is constructed. The non usual behavior of these $R$-matrices is the reducible character of the vector spaces on which the operators act for general values of deformation parameter $q$. The integrable models corresponding to these $R$-matrices describe interactions between different spins (Section 4), however the Hamiltonian operator derived in accordance to the principles of the Algebraic Bethe Ansatz [4, 5, 15] has not the conventional form of the superposition of "spin-spin" operators. New formal operators can be proposed for describing these nearest neighborhood rather entangled interactions. In Section 5 an approach determining the centralizer operators defined on the tensor products of the reducible states is developed, necessary and sufficient relations for them are deduced. By fusion procedure we can find out from the Hecke solutions the descendant inhomogeneous $R$-matrices also, and it turns out that the matrices $R^R_r$ defined on the tensor product $V^r \otimes U^R$, with $V^r$ being an irrep and $U^R$ being a series of the reducible states, formed by the truncation of the tensor products of the irreps $V^r$, may constitute "Hecke type" matrices by their structure. In Section 6 we sketch the scheme of the obtainment of such $R^R_r(u)$ operators, defined for each $r$-dimensional irrep and corresponding series of composite representations with definite $R_n$ dimensions. These matrices we can refer to as the series of the "extended Lax operators", as for the case of the fundamental representation of $sl_q(2)$, when $r = 2$, they just coincide with the matrix representations of the ordinary Lax operator. In Section 7 the summary and some propositions are presented regarded the "extended" $R$-, $L$- operators, and also there are discussed further developments and possible applications of the integrable structures defined on the composite representations. In the next part of the Introduction (Section 1), as well as in the Appendix some preliminary definitions, descriptions and formulas are presented. Also in the Introduction some questions on the baxterization are analyzed.

Quantum super-algebra $osp_q(1|2)$. This graded quantum algebra is constituted by the generators $e$, $f$ (odd generators) and $h$ (even generator), which satisfy to the following commutation $[,]$ and anti-commutation $\{,\}$ relations
\[
\{e, f\} \equiv ef + fe = [h]_q, \quad [h, e] = e, \quad [h, f] = -f. \tag{1.1}
\]
Here, as usual, \([a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}\). Sometimes different definitions for the anti-commutation relation in (1.1) are used, which are equivalent to this one by simple re-scaling of the generators and the deformation parameter [20, 22, 23, 24]. Co-product is defined by the following relations

\[
\Delta[e] = e \otimes q^{-\frac{1}{2}} + q^{\frac{1}{2}} \otimes e, \quad \Delta[f] = f \otimes q^{-\frac{1}{2}} + q^{\frac{1}{2}} \otimes f, \quad \Delta[h] = h \otimes I + I \otimes h. \tag{1.2}
\]

Here \(\otimes\) denotes the graded tensor product, and \(I\) is a unity operator. The quadratic Casimir operator can be written as

\[
c = \left((q^{\frac{1}{2}} + q^{-\frac{1}{2}})ef - [h - \frac{1}{2}]_q \right)^2. \tag{1.3}
\]

Below we shall use the notation \(q_r = \left((-1)^r + 1\right)\frac{i\pi}{4\log q}\) for the factor arising in the case of even dimensional irreps. The odd-dimensional representations are in the full analogy with the non-deformed algebra situation, meanwhile the even-dimensional representations have no well defined limit at \(q \to 1\) [23]. The description of the irreducible representations is brought in the Appendix. The decomposition of the tensor products of two irreps is presented by the following linear combination

\[
V^{r_1} \otimes V^{r_2} = \sum_{r = |r_1 - r_2| + 1, \Delta r = 2}^{r_1 + r_2 - 1} V^r. \tag{1.4}
\]

Let us denote \(j = 2j_r - q_r\) and \(j_k = 2j_{r_k} - q_{r_k}, k = 1, 2\). The Clebsh-Gordan \(q\)-coefficients (CGC) \(C^{(j_1 j_2 j)}_{i_1 i_2 i}\) are defined by this decomposition, where it is assumed \(\{i_1 + i_2 = i\}\) and also we suppose \(r_1 \leq r_2\),

\[
v_i^r = \sum_{i_1 = -j_1}^{j_1} C^{(j_1 j_2 j)}_{i_1 i_2 i} v_{i_1}^{r_1} \otimes v_{i_2}^{r_2}. \tag{1.5}
\]

Here we have presented the formulae of \(C^{(j_1 j_2 j)}_{i_1 i_2 i}\) in such a way to have integer (half-integer) values of the variables \(j, i\) for odd (even) dimensional representations as in the case of \(sl_q(2)\)-algebra (for details see Appendix A2). It slightly differs from the notations we have used in [26], [27]. The inverse CG coefficients are defined by

\[
v_{i_1}^{r_1} \otimes v_{i_2}^{r_2} = \sum_{r = |r_1 - r_2| + 1}^{r_1 + r_2 - 1} \bar{C}^{(j_1 j_2 j)}_{i_1 i_2 i} v_i^r. \tag{1.6}
\]
**R-matrix and YB equations.** As a quasi-triangular Hopf algebra this algebra is equipped with an intertwiner \( R \)-matrix, which ensures the operation

\[
R \Delta[a] = \Delta'[a] R, \quad \Delta' = \sigma \Delta,
\]

(1.6)

where \( a \) is an arbitrary element of the algebra and \( \sigma \) is the graded permutation operator acting on the elements of the algebra: \( \sigma \cdot (a \otimes c) = (c \otimes a) \). \( R \) satisfies to the triangle relation (YBE)

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

(1.7)

Here the right and left sides of the equation are acting on the space \( V_1 \otimes V_2 \otimes V_3 \). \( R_{ij} \) is defined on the product of the spaces \( V_i \) and \( V_j \), and acts on the remaining state as unity operator. In the so-called ”check”-formalism it is used \( \check{R} = PR \) - matrix, where \( P \) is a graded permutation operator acting on the representation spaces as follows, \( P : V_i \otimes V_j = (-1)^{p_i p_j} V_j \otimes V_i \). The sign \( (-1)^{p_i} \) takes into account the grading of the vector states, the parity \( p \) has the values \( p_i = 0/1 \) for the even/odd states. By means of the ”check” \( R \)-matrices the above formulas look like

\[
\check{R} \Delta[a] = \Delta[a] \check{R}, \quad \check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23}.
\]

(1.8)

And note, that in the matrix formulation the ”check” YB equations coincide with the non-graded case and do not contain signs, in contrast to the matrix representation of the YBE in (1.7). This can be achieved by using the matrix form of the graded tensor products.
- (a ⊗ c)_{ij}^{kr} = a_k^l c_l^r (-1)^{p_k+p_r}. The solution to YBE as universal R-matrix in terms of the operators e, f and h, is considered by Drinfeld in the context of the quantum double principle [9]. For the super-algebra osp_q(1|2) it is expressed by the following formula [22, 23]

\[ R^+ = q^{-\frac{\hbar}{2}} \sum_{n=0}^{\infty} \frac{(-q^{1/2})^{n(n-1)}(q-q^{-1})^n}{[n]_+!} q^{-\hbar n/2} e^n \otimes f^n q^{\hbar n/2}.\] (1.9)

Here \([n]_+ = \frac{(-1)^{n-1}q^{n/2}+q^{-n/2}}{q^{1/2}+q^{-1/2}}\). The transpose of this matrix with the change \(q \to 1/q\) also is an intertwiner matrix and is denoted by \(R^-\).

Spectral parameter dependent \(R^{r_1r_2}(u)\)-matrices. The appearance of the Yang-Baxter equations in the quasi-triangular Hopf algebras reflects the connection with the integrable models [10]. By means of the affinization of the quantum groups [7, 12, 16] it becomes possible to construct spectral parameter dependent \(R(u)\)-matrices satisfying to the equations

\[ R_{12}(u)R_{13}(u+w)R_{23}(w) = R_{23}(w)R_{13}(u+w)R_{12}(u).\] (1.10)

As it is well known, the Yang-Baxter equations with spectral parameter dependence play a crucial role in the theory of the two dimensional integrable models [1-15]. However the "baxterization" (supplementing with the spectral parameters) of the constant solutions in general is not simple task.

The representation of the matrix \(R^{r_1r_2}(u)\) acting on the tensor product of two irreps \(V^{r_1} \otimes V^{r_2}\) is more convenient to write in the "check"-formalism, \(\check{R}^{r_1r_2}(u)\). In this case the YB equations are written as

\[ \check{R}_{12}(u)\check{R}_{23}(u+w)\check{R}_{12}(w) = \check{R}_{23}(w)\check{R}_{12}(u+w)\check{R}_{23}(u),\] (1.11)

The commutation relations in (1.8) and the fusion rules (1.3) give a hint, that the \(\check{R}\)-matrix must be a linear superposition of the invariant matrices - projection operators \(\check{P}_r^{r_1r_2}\) [12, 19, 22],

\[ \check{R}^{r_1r_2}(u) = \sum_{r=|r_1-r_2|+1}^{r_1+r_2-1} r_r(u)\check{P}_r^{r_1r_2},\] (1.12)
the operator $\hat{P}^{r}_{r_1 r_2}$ vanishes on the spaces $V^{r'}$, $r' \neq r$ in the decomposition (1.3), and acts on the space $V^r$, mapping (imaging) it to the space $V^{r_2} \otimes V^{r_1}$ [26]. For the homogeneous case the matrices $\hat{P}^{r}_{r_1 r_1}$ are the ordinary projection operators $P^r$ acting on $V^r$ as unity operator, $P^r V^{r'} = \delta_{rr'} V^{r'}$. For the quantum super algebra $osp_q(1|2)$ the $r_r(u)$-functions (which are the polynomials as for the case of $sl_q(2)$ algebra) when both of $r_1$ and $r_2$ are odd or even dimensional irreps, can be found e.g. in [12, 19, 22, 26]. The general case of $r_1 \neq r_2 (\text{mod}) 2$ is analyzed in detail in [26].

All the solutions with higher spin representations ($r_{1,2} > 2$) in the series (1.12), which satisfy to (1.11) can be obtained by another way, by means of the so called "fusion" technique or "descendant" procedure [8, 12] from the fundamental $R^{2,2}(u)$ solution. This is the consequence of two factors. One factor is that there is a point $u_0$ for which $\hat{R}^{r_1 r_2}(u_0)$ is proportional to the projector with the maximal spin $\hat{P}^{(r_1+r_2-1)}$. The second one is that the expression on left (or right) hand side of the YBE (1.10) itself can serve as a solution of YBE at $u = u_0$ (or at any point $u = \bar{u}_0$, for which $\hat{R}(\bar{u}_0)$ is a projector or direct sum of the projectors, and thus has the property $\hat{R}(\bar{u}_0)\hat{R}(\bar{u}_0) = \hat{R}(\bar{u}_0)$ for the homogeneous case; for the general case the permutation must be taken into account in the multiplication of the projectors), in the form of $\hat{R}^{(r_1 \times r_2)}_{r_3}$-matrix acting on the product of the vector spaces $V^{r_1 \times r_2} \otimes V^{r_3}$, where $V^{r_1 \times r_2} = (V^{r_1} \otimes V^{r_2})$.

Note, that as a rule, in this paper we are using the upper indexes for R-matrices for denoting the dimensions of the representation spaces on which the $R$ matrices are acting and the down indexes for denoting the positions of the states.

**Baxterization: some observations.** For the two-dimensional fundamental representations of $sl_q(2)$ (definition of this algebra is brought in A3), the spectral parameter dependent $R(u)$-matrix (1.12) defined on them can just be presented by the following sum, up to multiplication by an arbitrary function (see for example [16, 22])

$$R^{2,2}(u) = q^u R^{+2}_{-2} - q^{-u} R^{-2}_{+2}. \tag{1.13}$$
This is true also for the two dimensional representations of the \(osp_q(1|2)\) super-algebra [26]. From this form it means that the matrices \(R_f^\pm\) on the fundamental representations besides of the ordinary constant YBE

\[
R_{f12}^+ R_{f13}^+ R_{f23}^+ = R_{f23}^+ R_{f13}^+ R_{f23}^+; \tag{1.14}
\]

must satisfy also the other equations, which are

\[
R_{f12}^+ R_{f13}^+ R_{f23}^- = R_{f23}^+ R_{f13}^+ R_{f23}^-; \tag{1.15}
\]

\[
R_{f12}^+ R_{f13}^+ R_{f23}^- - R_{f12}^- R_{f13}^+ R_{f23}^- = -R_{f23}^- R_{f13}^- R_{f23}^- + R_{f23}^+ R_{f13}^- R_{f23}^+. \tag{1.16}
\]

For the representations with the higher spins the expansion of \(R^{r_1 r_2}(u)\) (we suppose \(r_1 \leq r_2\)) to the series in terms of the parameter \(q^u\) can contain more terms, i.e. at \(r_1 > 2\)

\[
R^{r_1 r_2}(u) = q^{u(r_1-1)} R^{(r_1)} + q^{u(r_1-3)} R^{(r_1-1)} + \ldots + q^{-u(r_1-1)} R^{(1)}, \tag{1.16}
\]

\[
R^{(r_1)} \approx R^{+r_1 r_2}, \quad R^{(1)} \approx R^{-r_1 r_2}. \tag{1.17}
\]

The last two matrices \((R^{+r_1 r_2}, R^{-r_1 r_2})\) are the braid limits of the corresponding \(R\)-matrix and satisfy to the equations (1.14). The matrices \(R^{2 r}(u)\) constitute the matrix representations of the Lax operator \(L(u)\) [12, 19, 26], which keeps the form \(L(u) = q^u L^+ - q^{-u} L^-\). The matrices \(R^{r_1 r_2}(u)\) which are obtained by the fusion method from the YBE solutions \(R^{2 2}(u)\) defined on the fundamental representations admit the form (1.16) with non vanishing terms \(q^{u(2p-r_1-1)} R^{(p)}\) for all variables \(p\), meanwhile the Hecke type homogeneous matrices \(R^{r_1 r}(u)\) contain only the terms with braid limit matrices \(R^{+ r}\) [8, 12, 19, 13, 29, 28]. In this article we shall demonstrate the existence of the series \(R^{r_1 r}(u)\) of YBE solutions with minimal number of terms in the decomposition (1.16) also for the quantum super-algebra \(osp_q(1|2)\). Then by the fusion method the descendant solutions of such kind matrices will be considered, which for the case \(r > 2\) will be defined on the reducible representations.

Let us see what relations are imposed by the YBE (1.10) on the expansion matrices in (1.16). Denoting \(R^{r_1 r_j}(u) = \sum_{p=1}^{r_i} q^{u(2p-r_1-1)} R_{ij}^{(p)} (r_i \leq r_j)\), and supposing \(r_1 \leq r_2 \leq r_3\), we
have from the YBE
\[
\sum_{p=1}^{r_1} q^{u(2p-r_1-1)} R_{12}^{(p)} \sum_{k=1}^{r_1} q^{(u+v)(2k-r_1-1)} R_{13}^{(k)} \sum_{t=1}^{r_2} q^{v(2t-r_2-1)} R_{23}^{(t)} = \\
\sum_{t=1}^{r_2} q^{v(2t-r_2-1)} R_{23}^{(t)} \sum_{k=1}^{r_1} q^{(u+v)(2k-r_1-1)} R_{13}^{(k)} \sum_{p=1}^{r_1} q^{u(2p-r_1-1)} R_{12}^{(p)},
\]
(1.18)

the following set of the equations
\[
\sum_{p=1}^{r_1} R_{12}^{(p')} R_{13}^{(p)} R_{23}^{(p'')} = \sum_{p=1}^{r_1} R_{23}^{(p'')} R_{13}^{(p)} R_{12}^{(p')},
\]
(1.19)

\[1 \leq (p' - p) \leq r_1, \quad 1 \leq (p'' - p) \leq r_2, \quad \text{i.e.} \quad p' \in [2, 2r_1], \quad p'' \in [2, r_1 + r_2].\]

These equations can be investigated step by step, starting from the braid limit matrices. Of course the expansion (1.16) is valid for the trigonometric (or rational, at \( q \to 1 \)) solutions.

For more general solutions infinite series must be considered, i.e. the summation must not be limited by \( r_1 \) in (1.16), it must be extended to \( \infty \).

The solutions to the equations (1.19) are not unique. As an example we can consider the R-matrices with the symmetry of the quantum algebra \( sl_q(2) \) for the case \( r_{1,2} = 3 \). There are three spectral-parameter dependent solutions \( R^{33}_{33}(u) \) to the homogeneous YBE. Presenting them in the form \( \tilde{R}^{33}_{33}(u) = q^u \tilde{R}^+ + q^{-u} \tilde{R}^- \), the first solution can be written as
\[
\tilde{R}_{13}^{23}(u) = \frac{1}{\tilde{a}} \left( q^u(q^3 P^5 - q P^3 + q^3 P^1) + \tilde{R}_1^0 + q^{-u}(q^3 P^5 - q^{-1} P^3 + q^{-3} P^1) \right),
\]
\[\tilde{a} = (q^2 - q^{-2})(q - q^{-1}), \quad \tilde{R}_1^0 = -(q + q^{-1})(P^5 + P^1) + (q^3 - q^{-3})P^3, \quad (1.20)\]

which is the case \( n = 3 \) of the universal solution \( \tilde{R}^{nn}_{nn}(u) \) [12]. The next solution, associating with the Berman-Wenzl-Murakami algebra [29], reads as
\[
\tilde{R}_{23}^{33}(u) = \frac{1}{a'} \left( -q^u q^2(q^3 P^5 - q P^3 + q^3 P^1) + \tilde{R}_2^0 + q^{-u} q^2(q^3 P^5 - q^{-1} P^3 + q^{-3} P^1) \right),
\]
\[a' = (q^2 - q^{-2})(q^3 + q^{-3}), \quad \tilde{R}_2^0 = (q^5 - q^{-5})(P^3 + P^1) + (q^{-1} - q)P^5. \quad (1.21)\]

This matrix has the same braid limits \( R^\pm \) as the previous one. We see that having the same braid limit constant matrices \( R^\pm \) it is possible to construct different spectral parameter dependent solutions (1.20.1.21). And the third solution
\[
\tilde{R}_{33}^{33}(u) = \frac{1}{1 + a} \left( q^u[P^5 + P^3 + a P^1] + q^{-u}[a(P^5 + P^3) + P^1] \right),
\]
(1.22)
\[ a = \frac{\sqrt{q^4 + q^{-4} - 1 + 2q^2 + 2q^{-2} + q^2 + 1 + q^{-2}}}{\sqrt{q^4 + q^{-4} - 1 + 2q^2 + 2q^{-2} - (q^2 + 1 + q^{-2})}} , \] is just the solution belonging to the so called Hecke type [16, 12] series. All the solutions are brought in the normalized form \( \tilde{R}^{33}(0) = I \).

The \( osp_q(1|2) \)-invariant \( R^{33} \)-matrices, which are equivalent to the solutions (1.20, 1.21) are discussed in details in [26]. And it is known, that in contrast to the case (1.20), the solution (1.21) has no generalization for higher dimensional irreps [26, 28].

The \( osp_q(1|2) \)-invariant analog of the solution (1.22) will be obtained in the next section.

We see, that this solution has the simplest decomposition with two constant braid limit \( R^\pm \)-matrices and \([u(r-1) \to u]\) (1.16), i.e. it preserves the form of (1.13). Note that the spectral parameter dependent YBE and R-matrices are defined up to the re-normalization of the spectral parameter.

2 A series of Hecke type homogeneous solutions to YBE with \( osp_q(1|2) \)-symmetry

One can try to make the generalization of the fundamental representation (1.13) for the higher dimensional cases in such a way, that to keep the form \( R(u) = R^+(q^u) - R^-(q^{-u}) \). It is known, that the Hecke type R-matrices, i.e. the matrices, which satisfy the Hecke relation \((\tilde{R} - q)(\tilde{R}^{-1} + q) = 0\), after ”baxterization” obtain the mentioned form (e.g. [29] and citations therein). And surely for higher dimensions these \( R^{+/−} \)-matrices do not coincide with the braid limit matrices \( R^{+/−} \) obtained from the universal R-matrix.

Although for the quantum super-algebras \( osp_q(1|2N) \) the role of the Hecke algebra is played by the Birman-Wenzl-Murakami algebra, taking into account the equivalence of the quantum algebras \( sl_q(2N) \) and the quantum super-algebras \( osp_q(1|2N) \) in respect to their representation spaces [24], one can also expect the existence of the series of Hecke type R-matrices with the symmetries of these super-algebras. Now let us concentrate our attention on the case of \( osp_q(1|2) \). For the irreps with the dimensions \( r = 2 \) and \( r = 3 \) all the solutions \( \tilde{R}^{rr} \) one can obtain by direct matrix calculations and verify that there are the counterparts of the solutions (1.13, 1.22). In the case of the general \( r \) let us look for the solution of (1.11)
in a special form

\[ \tilde{R}^{rr}(u) = \sum_{r' = 3}^{2r-1} P_{rr}' + \bar{f}(u)P^1 = I^r \otimes I^r + f(u)P^1, \quad (2.1) \]

as the solutions at \( r = 2, 3 \) admit such expansion (and this is valid in the case of \( sl_q(2) \) invariant Hecke type operators, too). Here \( I^r \) is unity operator in the space \( V^r \) and \( f(u) = \bar{f}(u) - 1, \) as \( \sum_{r'=1, \triangle r'=2} P_{rr}' = I^r \otimes I^r. \)

Note, that dealing with the homogeneous matrices for simplicity we use the notations \( P_{ij} \equiv P_{ik}(r_i = r_k), \) and in some cases just \( P_{ij}, \) without the indexes \( i, k \) denoting the spaces.

It is possible to derive the functions \( f(u) \) by various methods exploiting the algebra relations. Here we shall demonstrate an explicit computation in a rather detailed way, using the Clebsh-Gordan coefficients.

The procedure is standard. The right and left sides of the YBE are acting on the space \( V^r \otimes V^r \otimes V^r. \) Let us take an arbitrary vector state in that space, suppose \( v_k^r \otimes v_p^r \otimes v_t^r. \) The projector \( P_{ij}^1 \) acts as non-vanishing (unity) operator only on this kind of the products - \( v_k^r \otimes v_{-k}^r, \) which have 0-value of the operator \( h. \) Using definitions in (1.4, 1.5) and denoting \( j_0 = (r - 1)/2, \) we can write

\[ (P^1)_{12} \cdot v_k^r \otimes v_p^r \otimes v_t^r = (P^1 \otimes I) \cdot v_k^r \otimes v_p^r \otimes v_t^r = \]

\[ \delta_{k,-p} \bar{C} (j_0 j_0 0 -j_0 0 0) \sum_{i=-j_0}^{j_0} C (j_0 j_0 0 -i -i 0) v_i^r \otimes v_{-i}^r \otimes v_t^r, \quad (2.2) \]

\[ (P^1_{rr})_{23} \cdot v_k^r \otimes v_p^r \otimes v_t^r = (I \otimes P^1_{rr}) \cdot v_k^r \otimes v_p^r \otimes v_t^r = \]

\[ \delta_{p,-l} \bar{C} (j_0 j_0 0 -p -p 0) \sum_{i=-j_0}^{j_0} C (j_0 j_0 0 -i -i 0) v_k^r \otimes v_i^r \otimes v_{-i}^r. \quad (2.3) \]

Taking into account these relations let us write down the non-trivial equations which follow from the action of the left and right hand sides of the YBE with the \( R \)-matrices described by (2.1). For definiteness we take \( p = -k \)

\[ \{ f(u) + f(w) - f(u+w) + f(u)f(w)+ \]

\[ f(u)f(w)f(u+w)C (j_0 j_0 0 i-t 0) \bar{C} (j_0 j_0 0 i-t 0) C (j_0 j_0 0 -t -t 0) \bar{C} (j_0 j_0 0 -t -t 0) \} = 0. \]

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This equation has solution for all $t$-s, if the factor $\chi(j_0) = C^{(j_0 j_0)_0} \bar{C}^{(j_0 j_0)_0} C^{(j_0 j_0)_0} \bar{C}^{(j_0 j_0)_0}$ is not dependent from the value of $t$. For defining that factor, let us write explicitly the CG-coefficients for the given case

$$C^{(j_0 j_0)_0} = \prod_{i' = j_0 + 1}^{i} \frac{(-1)^{p_{i'}^r} q^{-\frac{1}{2} - q_{r0} \beta_{i'}^r}}{\beta_{i'}^r} C^{(j_0 j_0)_0}, \quad (2.5)$$

$$\bar{C}^{(j_0 j_0)_0} = (-1)^{p_{i_1}^r} q^{-\frac{1}{2} - q_{r0} \beta_{i_1}^r} \prod_{i' = j_0 + 1}^{i} \frac{(-1)^{p_{i'}^r} q^{-\frac{1}{2} - q_{r0} \beta_{i'}^r}}{\beta_{i'}^r} \bar{C}^{(j_0 j_0)_0}. \quad (2.6)$$

Hence, using that $\left(\frac{\beta_{i'}^r}{\beta_{i}^r}\right)^2 = 1$ and $(\varepsilon_i^j)^2 = 1$ for all possible $i$, we arrive at

$$C^{(j_0 j_0)_0} \bar{C}^{(j_0 j_0)_0} C^{(j_0 j_0)_0} \bar{C}^{(j_0 j_0)_0} = \prod_{i' = j_0 + 1}^{i} q^{-1 - 2q_{r}} \prod_{i' = j_0 + 1}^{i} q^{-1 - 2q_{r}} C^{(j_0 j_0)_0}^{4} = q^{-(1+2q_{r})2j_0} C^{(j_0 j_0)_0}^{4}. \quad (2.7)$$

If to choose the $j_0$-state having even norm, then the state $\{v_0^1\}$ would have 0 parity, and combining the relations (A.5) and (2.6) we deduce

$$C^{(j_0 j_0)_0}^{2} \sum_{i} (-1)^{p_{i}^r} p_{i}^{r_0} q^{-(1+2q_{r})(i+j_0)} = 1. \quad (2.8)$$

When $r$ is odd, then $(-1)^{q_{r0}} = (-1)^{j_0+i}$, $q_{r} = 0$ and

$$\sum_{i = -j_{0}}^{j_{0}} (-1)^{p_{i}^{r_0}} q^{-(1+2q_{r})(i+j_0)} = \sum_{i = -j_{0}}^{j_{0}} (-q)^{-(i+j_0)} = (-q)^{-j_{0}[2j_{0} + 1]_{q^1/2}}.$$  

When $r$ is even, then $(-1)^{p_{i}^{r_0}} = 1$, and again

$$\sum_{i = -j_{0}}^{j_{0}} (-1)^{p_{i}^{r_0}} q^{-(1+2q_{r})(i+j_0)} = \sum_{i = -j_{0}}^{j_{0}} q^{-(1+\frac{i}{2})_{q^1/2}(i+j_0)} = (-q)^{-j_{0}[2j_{0} + 1]_{q^1/2}}.$$  

So, we have, using the relation $(-1)^{2j_{0}} = 1$ for odd dimensional representations, and $q^{-(1+2q_{r})2j_{0}} = (-q)^{2j_{0}}$ for even dimensional ones,

$$q^{-(1+2q_{r})2j_{0}} C^{(j_0 j_0)_0}^{4} = q^{-(1+2q_{r})2j_{0}} / \left((-q)^{-j_{0}[2j_{0} + 1]_{q^1/2}}\right)^2 = \frac{1}{[2j_{0} + 1]_{q^1/2}}. \quad (2.9)$$

The solutions of the equations (2.4) so have dependence only from the dimension of the representation $V^r$. Equations can easily be solved by passing to the corresponding differential equations, expanding the expressions around a fixed point, e.g. at $w = 0$. We can write
finally the spectral parameter dependent Hecke type solutions as

\[ \tilde{R}_{ij}(u_{ij}) = I^r \otimes I^r + f(u)P^1, \quad f(u) = \frac{2}{-1 + \sqrt{1 - \frac{4}{|r|^2 q^{1/2}} \coth au}}. \]  

(2.10)

Here \( a \) is an arbitrary number. The "braid"-limits \( u \to \pm \infty \) of (2.10) coincide with the corresponding Hecke type constant solutions, satisfying to (1.14).

The investigation of the case with symmetry of quantum algebra \( sl_q(2) \) would differ from this consideration only by the gradings of the states and the spin values of the even-dimensional irreps. For the algebra \( sl_q(2) \) the Hecke type solutions have been discussed in the works [11, 13, 29]. The purpose of this section has been to insist that such kind of series of the solutions exists for the super-algebra \( osp_q(1|2) \), thus proofing that there is a full correspondence between the YBE solutions with the symmetries of \( sl_q(2) \) and \( osp_q(1|2) \).

3 Descendant \( R^{(r^2-1)\times(r^2-1)} \)-matrices

We can try to construct descendant \( R \)-matrices corresponding to the discussed \( r \times r \)-dimensional solutions. As it is mentioned already, in the standard fusion (descendant) procedure developed for the algebras under consideration [8, 12] one leans on the property that there is a point \( u_0 \) at which the usual solution \( \tilde{R}_{r \times r}(u) \) (1.12) defined on the tensor product of the
spin-irreps $V^{r_1} \otimes V^{r_2}$ is proportional to the projector $\hat{P}_{r_1 + r_2 - 1}$ on the space with the maximal spin in the fusion (1.3). Thus from the matrices $R^{r_1 r_2}$ and $R^{r_3 r_4}$ satisfying the YBE one can construct the matrix $R^{(r_1 + r_2 - 1)(r_3 + r_4 - 1)}$ on the product with maximal spins (Fig. 2). And such solution is exactly equivalent to the matrix obtained by Jimbo's constructions. In the present situation (2.10) there is a point $u_0$ at which

$$\hat{R}^{r r}(u_0) = I - P^{1} = \sum_{r' = 3}^{2r-1} P^{r'},$$

which can produce $(r^2 - 1) \times (r^2 - 1)$-dimensional $R$-matrices satisfying the YBE defined on the composite spaces. As these solutions can be interesting in the context of the integrable models which describe interactions between different kind of spins, we think it is worthy to obtain the exact form of such matrices. Of course the construction of the intertwiner matrices on reducible spaces has also a mathematical interest. Especially we shall investigate the series of the homogeneous solutions defined on the spaces $U^{r^2 - 1} \otimes U^{r^2 - 1}$, as well as the series of the inhomogeneous descendant solutions, which can be treated as the "extended" versions of the ordinary Lax operators.

The discussion hereinafter is proper for both of the symmetries of quantum algebra $sl_q(2)$ and super-algebra $osp_q(1|2)$, only for the second case the grading of the states must be taken into account. Particularly the tensor product must be replaced by the graded tensor product, and the YBE, when $R$-matrices are written in non check formulation, would contain additional signs conditioned by the parities (e.g. see [26]). For clarity, in the next sections when concretization will be needed we shall use the terminology of the $sl_q(2)$-algebra (basic definitions are brought in the Appendix), but obviously the extension to the case of the quantum superalgebra $osp_q(1|2)$ is straightforward.

The descendant solution on the product $V^{r_1} \otimes V^{r_2} \otimes V^{r_3} \otimes V^{r_4}$ can be presented by the following product of the $R$-matrices (for this case $r_1 = r_2 = r_3 = r_4 = r$)

$$\hat{R}_{1234}(u) = \hat{R}_{12}(u_0)\hat{R}_{34}(u_0)\left[\hat{R}_{23}(u)\hat{R}_{12}(u-u_0)\hat{R}_{34}(u-u_0)\hat{R}_{23}(u-2u_0)\right]\hat{R}_{12}(u_0)\hat{R}_{34}(u_0). \quad (3.1)$$

Recall that every $\hat{R}_{ij}(u)$-matrix has the following decomposition $\hat{R}_{ij}(u) = I_{ij} + f(u)P^{1}_{ij}$ and the point $u_0$ is fixed from the equation $f(u_0) = -1$.

Further the following notations are used $f_{kn} = f(u_{kn})$, where $u_{kn} \equiv u_k - u_n$. Here to each state, indexed by $i$, $j$ (lines in the graphical representations, Figures 1, 2) of the
\( \hat{R}_{ij}(u_{ij}) \)-matrix there are attached ‘velocities’ \( u_i, u_j \). From the figure 2 it is seen, that the ‘flows’ of the velocities correspond to the ‘flows’ of the indexes for the non-check matrices \( R_{ij} = P \hat{R}_{ij} \). And we obtain the following relations among the spectral parameters \( u_{14} = u, u_{12} = u_{34} = u_0, u_{24} = u_{13} = u - u_0, u_{23} = u - 2u_0 \). The proof, that the matrix (3.1) satisfies the YBE, can be done just by successively using the YBE on the matrices \( \hat{R}_{ij}(u) \). And of course, the operator \( \hat{R}_{1234}(u) \) has the invariance of the corresponding quantum (superalgebra), as it is a product of the centralizer operators defined on the space \( \bigotimes_{i=1}^{4} V_i^r \).

Expanding the product of the operators in the big parenthesis into the sum of projection operators, and taking into account that the terms, which are equivalent to \( (I - P_{12})(I - P_{34}) \) or \( P_{12}P_{34} \), vanish after multiplying by \( (I - P_{12})(I - P_{34}) \), we come to the expression

\[
\hat{R}_{1234}(u) = (I - P_{12})(I - P_{34}) \times \left( I + [f_{23} + f_{14} + f_{24}f_{14}]P_{23} + f_{14}f_{24}P_{23}[f_{24}P_{12} + f_{13}P_{34} + f_{24}f_{13}P_{12}P_{34}P_{23}] \right) \times (I - P_{12})(I - P_{34})
\] (3.2)

In the course of the calculations done in the previous section we have obtained that the following relations hold (we denote here \( \mathcal{X} \equiv \mathcal{X}(j_0) = C (j_0 j_0) C (j_0 j_0) \))

\[
P_{12}P_{12}P_{12} = \mathcal{X}P_{12}, \quad P_{23}P_{12}P_{23} = \mathcal{X}P_{23}, \quad P_{12}P_{23}P_{12} = \mathcal{X}P_{12}.
\] (3.3)

And in the same way, acting on the vector \( v_k \otimes v_p \otimes v_i \otimes v_j \) of the space \( V_1 \otimes V_2 \otimes V_3 \otimes V_4 \), we find out that

\[
P_{23}P_{12}P_{34}P_{23} = \mathcal{X}P_{14}P_{23}.
\] (3.4)

Thus we have

\[
\hat{R}_{1234}^{(r^2-1)(r^2-1)}(u) = (I - P_{12})(I - P_{34}) \times \left( I + [f_{23} + f_{14} + f_{24}f_{14} + \mathcal{X}f_{14}f_{23}(f_{24} + f_{13})]P_{23} + \mathcal{X}f_{14}f_{23}f_{24}f_{13}P_{14}P_{23} \right) \times (I - P_{12})(I - P_{34}).
\] (3.5)

The functional dependence of these matrices for all values of \( r \) coincides with the one presented in (1.20), which corresponds to the descendant matrix of the fundamental irrep.
with \( r = 2 \). The descendant matrix for the case with \( r = 3 \), which corresponds to the fundamental irrep of \( osp_q(1|2) \), is defined on the product of the spaces \((V^3 \oplus V^5) \otimes (V^3 \oplus V^5)\).

And the invariant operators in the expansion

\[
\tilde{R}^8(u) = q^u \tilde{R}^{+8} + \tilde{R}^{0,8} + q^{-u} \tilde{R}^{-8},
\]

(3.6)

consist of the linear superpositions of the following projection operators:

\[
\tilde{R}^{\pm/0,8} = \sum_{n,i,j} r_{i,j}^n P^n_{i,j}, \quad n = 1, 3, 5, 7, 9,
\]

(3.7)

where by \( P^n_{i,j} \) the projectors are denoted, which act on the irreps with dimension \( n \) as follows - \( P^n_{i,j} V^n_i = V^n_j \). Note, that in the case of the tensor product of the reducible states (which is true when \( r > 2 \)) in the decomposition there are few irreps with the same dimensions, and only the irrep with the maximal dimension has multiplicity 1,

\[
(V^3 \oplus V^5) \otimes (V^3 \oplus V^5) = \bigoplus_{i=1}^2 V^1_i \oplus \bigoplus_{i=1}^4 V^3_i \oplus \bigoplus_{i=1}^4 V^5_i \oplus \bigoplus_{i=1}^3 V^7_i \oplus V^9_i.
\]

(3.8)

4 Integrable model on 1D chain \( \bigotimes V^{(r^2-1)} \)

As it is known, the spectral parameter dependent Yang-Baxter equations ensure the integrability of 1 + 1 quantum spin models constructed via the R-matrices being solutions to YBE.

And according to the ideas of the Algebraic Bethe Ansatz the Hamiltonian operators of the corresponding models are defined by means of the first order logarithmic derivatives of the transfer matrices \([4, 11, 15]\), \( \tau(u) = tr_a \prod_i R_{ai}(u) \) at the point, where \( \tilde{R} \)-matrix coincides with unity operator (this ensures the locality). The index \( a \) denotes an auxiliary space, and the index \( i \) - the space situated on the \( i \)-th site of a chain. We are interested here in the chain models connected with the obtained series of the \( R^{(r^2-1)(r^2-1)} \)-matrices, for which the quantum spaces defined on each sites are the states \( (V^r \otimes V^r - V^1) = V^{2r-1} \oplus V^{2r-3} \oplus \cdots V^3 \).

From the operator form of the matrix (3.5) we can check that at \( u = u_0 \) the operator \( \bar{R}_{1234} \) is \((I - P^{1}_{12})(I - P^{1}_{34})\), which equals to unity operator in the considered \((r^2 - 1) \times (r^2 - 1)\)-
dimensional space. This follows from the functional dependence of the expressions in (3.5)

\[ f_{23} + f_{14} + f_{23}f_{14} + \mathcal{X} f_{14}f_{23}(f_{24} + f_{13}) = f(u_{14} + u_{23})(1 - \mathcal{X} f_{23}f_{14}) + \mathcal{X} f_{14}f_{23}(f_{24} + f_{13}) = f(2u - 2u_0)(1 - \mathcal{X} f(u)f(u - 2u_0)) + 2\mathcal{X} f(u)f(u - 2u_0)f(u - u_0), \quad (4.1) \]

and

\[ \mathcal{X} f_{14}f_{23}f_{24}f_{13} = \mathcal{X} f(u)f(u - 2u_0)[f(u - u_0)]^2, \quad (4.2) \]

and also from the observations, that the first order series expansion of the function \( f(u) \) near the point \( u = 0 \) equals to \( f(u) \approx f_0 u \), meanwhile the point \((-u_0)\) is singular, as \( f(-u_0 + u) \approx \frac{1}{u(1/f_0 - f_0/4)} \). It means that at the point \( u_0 \) the following expansion is true

\[ f(u)f(u - 2u_0)[f(u - u_0)]^2 = -\frac{(u - u_0)}{f_0^2(1/f_0 - f_0/4)} + O(u - u_0)^2. \quad (4.3) \]

So we obtain the following expansion near \( u_0 \),

\[ \hat{R}_{1234}(u) \approx (I - P_{12})^1(I - P_{34})^1[I + 2(u - u_0)f_0P_{23} - \mathcal{X} \frac{(u - u_0)}{f_0^2(1/f_0 - f_0/4)}P_{23}P_{14}]^1(I - P_{12})^1(I - P_{34})^1. \quad (4.4) \]

And hence, we can formulate the corresponding quantum 1d Hamiltonian defined on a cyclic chain having on each site \( r^2 - 1 \) dimensional vector space (superposition of the spin states) with the following nearest neighborhood interactions arising from the first order expansion of the \( R \)-matrix (4.4)

\[ H = f_0 \sum_i \left( \hat{P}_{i,i+1} + \mathcal{X} \hat{P}_{i,i+1} \right). \quad (4.5) \]

Here the notation \( \mathcal{X} = \mathcal{X} \frac{2f_0^2}{(4f_0 - 2)} \) is used, and the nearest-neighborhood interactions are described by the operators \( P_{i,i+1}, \hat{P}_{i,i+1} \). Provided that the lattice has double substructure, \( i = \{2s, 2s + 1\}, \) i.e. \( \mathcal{V}_i = V_{2s}^r \otimes V_{2s+1}^r \), the following relations take place

\[ \hat{P}_{i,i+1} = [(I - P_{2s, 2s+1}^1)(I - P_{2s+2, 2s+3}^1)] P_{2s+1, 2s+2}^1 [(I - P_{2s, 2s+1}^1)(I - P_{2s+2, 2s+3}^1)] \]

and

\[ \hat{P}_{i,i+1} = [(I - P_{2s, 2s+1}^1)(I - P_{2s+2, 2s+3}^1)] P_{2s+1, 2s+2}^1 [(I - P_{2s, 2s+1}^1)(I - P_{2s+2, 2s+3}^1)]. \]

The internal part of the Hamiltonian operator describes spin interactions on two sub-chains, which can be schematically presented as \( \sum_s (H_{2s, 2s+1} + H_{2s-1, 2s+2}) \). The existence of the external
projection operators \( [(I - P_{2s,2s+1}^1)(I - P_{2s+2,2s+3}^1)] \) however indicates that there are mixed interactions between four neighboring spins on the sub-chains positions.

Now let us explore whether this Hamiltonian is as an operator describing superposition of the pure spin-spin interactions between the different spins defined at \( \mathbb{V}_i \) and \( \mathbb{V}_{i+1} \). A task is to ascertain the possibility of decomposition of the operators \( \hat{P}_{i+1} \) and \( \hat{P}_{i+1} \) in terms of the algebra invariant polynomials of the spin operators defined on the spaces \( \mathbb{V}_i \) and \( \mathbb{V}_{i+1} \).

The expansion of the Hamiltonian operators in terms of the algebra generators at the spaces \( V_{2s} \) and \( V_{2s+1} \) is obvious just by construction.

The question about the spin structure of \( H_{i+1} \) we can achieve either by the quantum \( 6j \)-symbols or directly by the Clebsh-Gordan coefficients [19, 33]. A brief description how to use the quantum \( 6j \)-symbols for obtaining the decomposition is brought in the Appendix A3. From the analysis of the dimensions in the expansions done therein, it follows, that actually, we must clarify whether the following decompositions - \((I - P_{12}^1)(I - P_{34}^1)P_{23}^1(I - P_{12}^1)(I - P_{34}^1) = \sum_{r,r',r''} a_k^r r' r'' A_{12}^r A_{34}^r P_{1234}^k, (I - P_{12}^1)(I - P_{34}^1)P_{23}^1 P_{14}^1(I - P_{12}^1)(I - P_{34}^1) = \sum_{r} a'^r A_{12}^r A_{34}^r P_{1234}^k, \) are valid, where \( A^r \) denotes some combinations of algebra operators defined on the spaces \( \mathbb{V}_{12} \) and \( \mathbb{V}_{34} \). It is known that the quantum \( 6j \)-symbols are expressed by the sums of the quartic products of the Clebsh-Gordan coefficients. And here we prefer operate immediately with the Clebsh-Gordan coefficients as in the previous sections.

### 4.1 The structure of the Hamiltonian operator

Here we use for the orthogonalized vector states \( v^r_k \) the "ket", "bra" notations, \( |j, k\rangle \), \( r = 2j + 1 \). The unity operator defined on the space \( V^r \) can be expressed as \( I^r = \sum_k \frac{|j, k\rangle \langle j, k|}{|j, k\rangle} \).

The projector operator \( P_{r1,r2}^{r_0} \) acting on the tensor product \( V^{r_1} \otimes V^{r_2} \) and distinguishing the space \( V^{r_0}, r_0 = 2j_0 + 1 \), we can write in this way, by using the formula (1.4)

\[
P_{r1,r2}^{r_0} = \sum_{i=-j_0}^{j_0} \sum_{i_1=-j_1}^{j_1} \sum_{i_1'=-j_1}^{j_1} C(j_1,j_2,j_0,i_1,i,i_1') C(j_1,j_2,j_0,i_1,i,i_1') \frac{|j_1,i,i_1\rangle \langle j_2,i,i_1'|j_1,i_1'\rangle}{|j_0,i_0,j_0,i_0\rangle} =
\[
(4.6)
\]

\[
\sum_{i=-j_0}^{j_0} \sum_{i_1=-j_1}^{j_1} \sum_{i_1'=-j_1}^{j_1} C(j_1,j_2,j_0,i_1,i,i_1') C(j_1,j_2,j_0,i_1,i,i_1') \frac{|j_1,i,i_1\rangle \langle j_2,i,i_1'|j_1,i_1'\rangle}{|j_0,i_0,j_0,i_0\rangle}.
\]
The unity operator \( I_{r_1 \times r_2} = \sum_{r_0=|r_1-r_2|+1}^{(r_1+r_2)-1} P_{r_0}^{r_1 \times r_2} \) defined on the space \( V_{r_1} \otimes V_{r_2} \) can be written as

\[
I_{r_1 \times r_2} = \sum_{i_1=-j_1}^{j_1} \sum_{i_2=-j_2}^{j_2} \frac{|j_1, i_1 \rangle \langle j_2, i_2|}{\langle j_1, i_1|j_1, i_1 \rangle \langle j_2, i_2|j_2, i_2 \rangle}.
\]  

(4.7)

For the orthosymplectic algebra one must take into account the grading of the vectors, and appropriate signs would appear in the above formulas. The vector states \( v_{r_0}^{i_k} \) can be chosen to be normalized, then the formulas would be more compact. The first term of the Hamiltonian operator corresponding to the cell \( V_1 \otimes V_2 \otimes V_3 \otimes V_4 \) can be presented as the following, taking into account that \( j_1 = j_2 = j_3 = j_4 \equiv j \),

\[
(I-P_{12}^1)(I-P_{34}^1)[P_{23}^1(I-P_{12}^1)(I-P_{34}^1) = \sum_{j_1=1}^{2j} \sum_{j_2=1}^{2j} \sum_{j_3=1}^{j} \sum_{j_4=1}^{j} \sum_{p,p',k,k'=j}^j C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) \times

\langle j_1, i_1 \rangle \langle j_2, i_2| \langle j_3, i_3 \rangle \langle j_4, i_4 \rangle.
\]

Correspondingly, the second term of the Hamiltonian operator will be

\[
(I-P_{12}^1)(I-P_{34}^1)P_{23}^1[P_{34}^1(I-P_{12}^1)(I-P_{34}^1) = \sum_{j_1=1}^{2j} \sum_{j_2=1}^{2j} \sum_{j_3=1}^{j} \sum_{j_4=1}^{j} \sum_{p,p',k,k'=j}^j C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) C \left( j_1 j_2 \right) C \left( j_3 j_4 \right) \times

\langle j_1, i_1 \rangle \langle j_2, i_2| \langle j_3, i_3 \rangle \langle j_4, i_4 \rangle.
\]

The obtained operators constitute superpositions of the projectors \( (P_{r_0}^o)_{a,b} : (V_{r_0}^o)_a \rightarrow (V_{r_0}^o)_b \), which are algebra invariant operators mapping the different spaces \( (V_{r_0}^o)_{a,b} \) with the same spin \( j_0 = (r_0-1)/2 \) (\( j_1=34 = j_2=34 \) in (4.8)) arising in the fusion of \( V^r \otimes V^r \otimes V^r \otimes V^r \) one to other. For the expression in Eq. (4.8) \( r_0 = 1, \ldots, 2r - 1 \), and for the case of Eq.(4.9) - \( r_0 = 1 \).

Now let us turn to the question arisen just before this subsection. We can see that the mentioned decomposition in general does not take place. In the case of the \( H \)-operator (4.8) the action on the space \( \mathbb{V}_{12} \otimes \mathbb{V}_{34} \) is performed by the linear superpositions of the
projectors $\sim \sum \prod C() \bar{C}() \left[ |v_{p}^{r_{12}}\rangle\langle v_{p'}^{r_{12}'}| \right] \left[ |v_{i-p}^{r_{12}i}'\rangle\langle v_{i-p'}^{r_{12}i}'| \right]$ (emerging at the intermediate stage of the decompositions in Eq. (4.8), in $\prod C() \bar{C}()$ only the first six Clebsh-Gordan coefficients from Eq. (4.8) are included), which means that the action of the Hamiltonian changes not only the values of the algebra operator $h$ (spin’s projectors), but also the kind of the irreps belonging to the spaces $V_{12} = V_{1} \otimes V_{2}$ and $V_{34} = V_{3} \otimes V_{4}$. In the chain $\otimes_{l} V_{i}$, where $i = \{2k; 2k + 1\}$, the action of the corresponding part of the Hamiltonian operator $H = \sum_{i} H_{i,i+1}$ can be schematically presented by the formula

$$H_{i,i+1} \simeq \sum_{r,r'} a_{r_{i},r'_{i+1}} J_{r_{i},r'_{i+1}} J_{r_{i+1},r'_{i+1}},$$

(4.10)

here we have used this formal notation - $J_{r'} = |v_{r}\rangle\langle v_{r'}|$, where the indexes of the spin projections are omitted. In case of (4.9) $r_{i} = r_{i+1}$, $r'_{i} = r'_{i+1}$, as $r_{1234} = 1$ and $r_{12} = r_{34}$, $r'_{12} = r'_{34}$. And clearly, the operators $J_{r'}$ in general ($r \neq r'$) are not expressed by the algebra generators defined on the states $V_{i}$.

Detailed expressions and the study of such quite large Hamiltonian operators for specific cases we purpose to do in subsequent work.

The actual spaces on which the Hamiltonian operator (4.5) is acting, are the truncated products, and in the next discussions we shall use new notations $U^{R}$ for denoting such $R$-dimensional composed spaces. Particularly, by the action of the projectors the following product of the irreps $V_{i} = V_{r_{2s_{i}}} \otimes V_{r_{2s_{i}+1}}$ turns into $r^{2} - 1$-dimensional state -

$$(I - P_{2s_{i}2s_{i}+1}^{1}) V_{i} (I - P_{2s_{i}2s_{i}+1}^{1}) \equiv V_{r_{2s_{i}}} \otimes V_{r_{2s_{i}+1}} - V_{i}^{1},$$

and will be denoted as $U^{r^{2}-1}$.

5 Centralizers and reducible representations.

In fact in (4.10) we deal with a centralizer operator defined on the tensor product of two mixed states $U^{r^{2}-1} \otimes U^{r^{2}-1}$. Here we intend by a straightforward construction to reveal the structure of such operators. Let us write down the conditions which the algebra relations put on the centralizers defined on the product $U \otimes U$, with composite representations spaces $U$, and then the extension to general case $U \otimes U \otimes ... \otimes U$ can be done by similar calculations.
Let $U$ consists of some set of irreducible representations: $U = \bigoplus_{r_k} V^{r_k}$. The thorough formulation of the operators $J^\prime_{j,i}$ could be done by means of the ortho-normalized elementary operators $J_{j,a}^{\prime,j',i'}$, $i' \equiv |j',i'\rangle\langle j,i|$, $\bar{J}_{j,a}^{\bar{j},\bar{i}}$, $\bar{J}_{j,a}^{\bar{j},\bar{i}} = \delta_{\bar{j}} \delta_{\bar{i}} J_{j,a}^{j,i}$. (5.11)

If in $U$ there are more than one copies of the irreps with the given same spin $-j$, one must add an additional index for differentiating them, e.g. $-j_a$. Each linear operator $a$, evaluating in $U$ can be presented as a superposition

$$a = \sum_{j_a,i_a,j'_a} a_{j_a,i_a,j'_a} \mathcal{J}_{j_a,i_a}^{j'_a,i'_a}.$$ (5.12)

The algebra generators on this basis can be presented as (the coefficients $\beta$, $h$, $\gamma$ below denote the usual matrix elements of the corresponding operators, for the quantum superalgebra $osp_q(1|2)$ see the Appendix A.1)

$$e = \sum_{j_a,i_a} \beta_{j_a} J_{j,a}^{j_a,i_a} + 1, \quad f = \sum_{j_a,i_a} \gamma_{j_a} J_{j,a}^{j_a,i_a} - 1, \quad h = \sum_{j,a,i} h_{j,a} J_{j,a}^{j,i}.$$ (5.13)

The matrix elements of the generators just by definition are the same (up to some elementary transformations, admissible by the algebra relations, see e.g. [26]) for the irreps with the same spin $\beta_{j_a} \equiv \beta_i, \ldots$. Every element of the center defined on $U$ consists of these elementary projection operators

$$P_{j,a}^{j,b} = \sum_{i=-j}^{j} \mathcal{J}_{j,a}^{j,b,i}, \quad \forall j, \forall a, b.$$ (5.14)

The quadratic Casimir operator is just the sum $c = \sum_{j,a} c_{j,a} P_{j,a}^{j,a}$. The centralizer operators $c$ defined on the tensor products of $n$-copies of the mixed states $U$, $U \otimes U \otimes \cdots \otimes U$ have more rich structure. The sufficient and necessary conditions for them can be obtained straightly from the equations $[\Delta(\Delta \otimes \cdots (\Delta \otimes I))[g], c] = 0$, presenting the commutation of the operators $c$ with the algebra generators defined on the tensor product of the representations by means of the associative co-product operation. In general one can write any linear operator defined
in $\otimes^n U$ as
\[
a = \sum_{j_{ak}, i_{ak}, j_{bk}, i_{bk}} a_{j_{ak}, i_{ak}, j_{bk}, i_{bk}} J_{j_{ak}, i_{ak}} J_{j_{bk}, i_{bk}} \otimes J_{j_{ak}, i_{ak}} \otimes \cdots \otimes J_{j_{ak}, i_{ak}}, \tag{5.15}
\]
or briefly $a = \sum_{\{j_{a\mu}\}} a_{\{j_{a\mu}\}}^{\{i_{a\mu}\}} \otimes^n J_{j_{a\mu}, i_{a\mu}}$. For the generic centralizer operators $c$ in the simplest case $n = 2$ (which is enough to consider if we are interested in the nearest-neighbourhood interaction Hamiltonians) the thorough calculations give the following relations on the coefficients $c_{j_{a1}, i_{a1}; j_{a2}, i_{a2}}^{i_{a1}' i_{a2}'}$, ensuring the commutation of $c = \sum_{\{j_{a\mu}\}} c_{\{j_{a\mu}\}}^{\{i_{a\mu}\}} \otimes^n J_{j_{a\mu}, i_{a\mu}}, n = 2$, with the algebra generators,
\[
i_{a1} + i_{a2} = i_{b1} + i_{b2}, \tag{5.16}
\]
\[
\beta_{j_1}^{i_1} j_{a1}, i_{a1} + j_{a2}, i_{a2} + 1 + q h_{j_1}^{i_1} j_{a2}, i_{a2} + q h_{j_1}^{i_1} j_{a2}, i_{a2} + \gamma_{j_2}^{i_2} j_{a1}, i_{a1} j_{a2}, i_{a2} + \gamma_{j_2}^{i_2} j_{a1}, i_{a1} j_{a2}, i_{a2} + 1.
\]
Here there is taken into account that $h_{j_1}^{i_1} = i$. For the case of the quantum super-algebra $\text{osp}_q(1|2)$ one must take into account the parities of the states, the graded character of the tensor products and that $h_{j_1}^{i_1} = i + i$ constant for the even dimensional irreps.

To obtain the corresponding relations for the general case with arbitrary $n$ is an obvious task, which brings to the evident extension of the Eqs.\,(5.16), with $n$ summands at the r.h.s. and l.h.s. of the equations.
\[
\sum_{k=1}^n i_{ak} = \sum_{k=1}^n i_{bk}, \tag{5.17}
\]
\[
\sum_{k} q^{p<k} h_{j_p}^{i_p} \beta_{j_k}^{i_k} j_{a1}, i_{a1} \cdots j_{ak}, i_{ak} \cdots j_{bk}, i_{bk} + 1 \cdots j_{bn}, i_{bn} + 1 = \sum_{k} q^{p<k} h_{j_p}^{i_p} \beta_{j_k}^{i_k} j_{a1}, i_{a1} \cdots j_{ak}, i_{ak} \cdots j_{bk}, i_{bk} \cdots j_{bn}, i_{bn} + 1,
\]
\[
\sum_{k} q^{p<k} h_{j_p}^{i_p} \gamma_{j_k}^{i_k} j_{a1}, i_{a1} \cdots j_{ak}, i_{ak} \cdots j_{bk}, i_{bk} + 1 \cdots j_{bn}, i_{bn} - 1 = \sum_{k} q^{p<k} h_{j_p}^{i_p} \gamma_{j_k}^{i_k} j_{a1}, i_{a1} \cdots j_{ak}, i_{ak} \cdots j_{bk}, i_{bk} \cdots j_{bn}, i_{bn} - 1.
\]
Particularly the first equation in Eqs.\,(5.17) ensures the conservation of the spin projection. The next equations put the relations on the coefficients $c_{\{j_{a\mu}\}}^{\{i_{a\mu}\}}$. Note, that for the $n$-th term in the sum of the l.h.s of the second equation in (5.17) one must take $\{i_{bn}' + 1 \rightarrow i_{bn}'\}$, and correspondingly for the similar term of the third equation of (5.17) $\{i_{bn}' - 1 \rightarrow i_{bn}'\}$.  

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6 Extended Lax operators

The Hecke type matrices $R^{rr}(u)$ do not allow generalizations to the inhomogeneous $R^{rr'}(u)$ acting on $V^r \otimes V^{r'}$ with $V^{r'}$ being an irrep, so that $R^{rr} R^{rr'} R^{rr'} = R^{rr'} R^{rr'} R^{rr}$, besides of the case $r = 2$ with the fundamental irrep $V^2$, which gives standard universal Lax operator $L$, obeying the quantum YBE (see for the references in [26], where the corresponding operator is constructed for the case of $osp_q(1|2)$)

$$RLL = LLR.$$  \hspace{1cm} (6.1)

However by descendant procedure we can get algebra invariant matrices $R^R : V^r \otimes U^R$ satisfying to YBE with $U^R$ to be a composite representation. The simplest $U^R$ has been discussed in the previous sections. Applying fusion method further one can find descendant $R$ operators defined on the representations larger than $V^r \otimes V^r - V^1$. And by means of these operators one can construct new quantum integrable models on the 1d chains with the action space $\bigotimes U^R_i$. One can expect that the corresponding local quantum Hamiltonian operators would describe interactions between different spins in a rather entangled way, relying to the discussed example of the series of solvable models with homogeneous $R^{r^2-1,r^2-1}$-matrices.

So, we can construct the series of the ”extended Lax operators” $L = R^{rr}$, satisfying (6.1) with Hecke type $R^{rr}$, and of course the case of $r = 2$ would be the usual Lax operator. In the same way, as in the previous sections the descendant series can be constructed by the products of the $R^{rr}(u)$-operators appropriately fixing the values of the spectral parameters [13]. The matrix deduced from the action of the operators on the space $V^r \otimes V^r \otimes V^r$ will be

$$\tilde{R}^{r^2-1}(u) = [I \otimes (I - P^1)](\tilde{R}^{rr}(u + u_0) \otimes I)(I \otimes \tilde{R}^{rr}(u))[I - P^1] \otimes I],$$  \hspace{1cm} (6.2)

defined eventually on $V^r \otimes (V^r \otimes V^r - V^1)$. Note, that the projection operator $(I - P^1)$ at the right of this expression could be omitted due to YBE, which actually ensures its existence. The extension to the space with $n$-product $V^r \otimes V^r \otimes \ldots \otimes V^r$ we can perform repeatedly using YBE and truncating by the appropriate projectors, achieved by taking step
Figure 3: $R^{r \mathcal{R}_n}$-matrix ($R_{i_1 \cdots i_n}(u)$)

by step $u = u_0$ for the right-edge $R^{rr}(u)$-matrices in the Fig.[3]. In the resulting matrix the
all spectral parameters are established in accordance to the summation rule of the additive
spectral parameters in YBE, and, as it was hinted above, only left-hand side projections are
taken into account. So, the matrix $R^{rx \mathcal{R}}(u)$ defined on the truncation of the tensor product
$V^r_i \otimes (V^r_{i_1} \otimes \cdots V^r_{i_n})$ can be written formally as the following expression:

$$R^{r \mathcal{R}_n}_{i_1 \cdots i_n}(u) = [R^{rr}_{i_121}(u_0)R^{rr}_{i_231}(2u_0)R^{rr}_{i_332}(u_0) \cdots R^{rr}_{i_k2}(u_0)] \times R^{rr}_{i_{k+1}i_{k+2}}(u_0) \cdots R^{rr}_{i_{n-1}i_n}(u),$$

where the matrices are presented in "non-check" form, and the low indexes of the R-matrices
show the spaces on which the operators act in the tensor product. The product of the
operators in the second row (6.4) itself is a solution of the YBE defined on $V^r_i \otimes (V^r_{i_1} \otimes \cdots V^r_{i_n})$
the expression in (6.3) realizes the projection operation (recall that $\tilde{R}^{rr}(u_0) = (I - P^1)$).
Step by step acting on the tensor product of the R-matrices these projectors narrow (restrict)
the action space from $r^{n+1}$-dimensional space to the $r \times \mathcal{R}_n$-dimensional space $V^r \times U^{\mathcal{R}_n}$, where $U^{\mathcal{R}_n}$ is a reducible space with dimension $\mathcal{R}_n$, which can be obtained as for the case of fundamental irrep ($r = 2$), from the recurrence formulas: $\mathcal{R}_0 = 1$, $\mathcal{R}_1 = r$, $\mathcal{R}_2 = r^2 - 1$, ... $\mathcal{R}_n = r \times \mathcal{R}_{n-1} - \mathcal{R}_{n-2}$. Correspondingly we can retrieve the structure of the composite space $U^{\mathcal{R}_n}$: $U^{\mathcal{R}_0} = I$, $U^{\mathcal{R}_1} = V^1$, ..., $U^{\mathcal{R}_n+1} \oplus U^{\mathcal{R}_{n-1}} = V^r \otimes U^{\mathcal{R}_n}$, repeatedly using the fusion rules (1.3). For the case $r = 2$, of course we recover $\mathcal{R}_n = n + 1$ and in this case $U^{\mathcal{R}_n} = V^{n+1}$ is the $(n + 1)$-dimensional irreducible representation - $\frac{n}{2}$-spin irrep. Generally we can refer to the space $U^{\mathcal{R}_n}$ as a truncated product of the irreps $\otimes^n V^r$.

As for the explicit formula for this 'extended' Lax operator, from the discussion above it follows, that $R^{r\mathcal{R}_n}(u)$ possesses the form $P(P^{\mathcal{R}_{n+1}} + P^{\mathcal{R}_{n-1}} + F_n(u)P^{\mathcal{R}_{n-1}})$, where $F_n(u) \approx \prod_k^n f(u + (n - k)u_0)$. The proof also can be done by induction, noting, that the operator $R^{r\mathcal{R}_{n+1}}(u)$ differs from $R^{r\mathcal{R}_n}(u)$ by the product of $(n+1)$-operators $R^{r\mathcal{R}_n}(u + nu_0) \prod_k^n R^{r\mathcal{R}_n}((k-1)u_0)$, and the spectral parameter dependent terms in $R^{r\mathcal{R}_{n+1}}(u)$ proportional to $\sim f(u+nu_0)$ and $\sim F_n(u)$ are eliminating due to relations coming from YBE, and the action of the projectors $(I - P^1)$, and the only spectral parameter dependent term which survives is $\sim f(u+nu_0)F_n(u) \Rightarrow F_{n+1}(u)$. So, we can summarize

$$R^{r\mathcal{R}_n}(u) = P(P^{\mathcal{R}_{n+1}} + P^{\mathcal{R}_{n-1}} + f_{r,n,q} \prod_k^n f(u + (n - k)u_0)P^{\mathcal{R}_{n-1}}). \quad (6.5)$$

The coefficient $f_{r,n,q}$ is conditioned by the action of the mentioned projection operators, as well by the permutation of the spaces, and can be formulated by means of the appropriate set of the CG-coefficients for each case. It is notable to remark, that it is easy to find out the product $\prod_k^n f(u + (n - k)u_0)$, using the recurrent relations, deduced from the equation (2.4)

$$f(u + u_0) = \frac{-1}{1 + \chi(j_0)f(u)}. \quad (6.6)$$

This gives for the mentioned product a polynomial of this kind $\frac{(-1)^n f(u)}{1 + a_n + b_n f(u)}$.

Formally these operators keep the form $[P^a + h(u)P^b]$ with two projectors, which is typical for the Hecke type operators and ensure the availability of the Hecke relations on the eigen-vectors’ space of $[P^{\mathcal{R}_{n+1}} + P^{\mathcal{R}_{n-1}}]$ which plays the role of the unity operator. Only
here one must be careful managing with the multiplication or with the inverse operations, as for the inhomogeneous $R$-matrices (6.5) we deal with the transposition (permutation) of the vector states $V^r$ and $U^R_n$.

For the case of the fundamental irrep, $r = 2$, fixing $a = \log q$ in (2.10) (note, that there for the case of $sl_q(2)$ we must take into account that $q \rightarrow iq^{1/2}$) we shall come to

$$\prod_{k=1}^n f(u + (n - k)u_0) = \frac{(-1-q^2)^n(q^{2n}-1)}{q^{2n}-q^{2n}}$$

which leads to the usual form of the Lax operator for the algebra $sl_q(N)$: $L(u) = q^uL^+ - q^{-u}L^-$ [29]. For the general cases with $r > 2$, the corresponding and similar expansion of the ”extended” Lax operator $R^{R^*}$ is followed by taking into account the presented above polynomial formulae coming from the recurrent relation (6.6).

All the same results are valid in the non-deformed case also, as the corresponding limit ($q \rightarrow 1$) is well defined (recall merely, that for the ortho-symplectic algebra at the classical limit only odd dimensional irreps are existing).

The one-chain Hamiltonian operators, corresponding to the obtained inhomogeneous matrices (6.5), constructed by means of the transfer matrices, where the auxiliary space is the $V^r$-irrep, describe non trivial interactions between different spins, having the structure expressed by the projection operators, similar to the one, discussed in the previous section. Strict investigations of such Hamiltonian structures will be carried out in the future.

The set of the composed states, fitting to the truncated tensor products of the spin-irreps, can be built for each case separately. For the simple case $n = 2$, the normalized (but not orthogonal) states of $U''^{2-1}$, induced from the initial sublattice, are determined elementary, using the relation (1.5):

$$|\psi_{i,k}\rangle = \frac{\sum_{\bar{j} j_{i+0}} \bar{C}(\bar{j} \bar{j}_{i+0})|g', i+k\rangle_{\bar{j}_{i+0}}}{1-\bar{C}(\bar{j} \bar{j}_{i+0})^2\delta_{i+k} \bar{o}}.$$ 

7 Summary and conclusions

Summing up, we can state, that here new type of solutions to Yang-Baxter equations defined on the composite states has been investigated, in particular, the solutions obtained by the fusion method from the solutions of Hecke type. The Hecke type homogeneous $R''$-
matrix’s series has been constructed for the quantum super-algebra $osp_q(1|2)$, defined on the tensor product of two $r$-dimensional irreducible representations, quite analogous to the corresponding matrix with $sl_q(2)$ symmetry. This pattern ascertains that the equivalence of the representation spaces of two algebras implies the equivalence of the solutions to YBE, as here the important role have the basis operators for the $R$-matrices, i.e. the algebra invariant operators - projectors. So, the YBE solutions known for the algebra $sl_q(2n)$ must be valid for the $osp_q(1|2n)$ quantum super algebra after the appropriate changes connected with the gradings and quantum deformation parameter $q$, as there is full correspondence between their representations [24].

For the Hecke type YBE solutions the corresponding descendant series $R^{r^2-1 \cdot r^2-1}$ have been constructed, which are defined on the composite (reducible) $r^2 - 1$-dimensional states of the $sl_q(1|2)$ (or $osp_q(1|2)$) algebra. Also descendant inhomogeneous matrices $R^{rR}$ (”extended” Lax operators), compatible with the mentioned invariant series, have been suggested, with definite series of $R_n$-dimensional composite states for each $r$-dimensional irrep. Of course, more general ”extended” $R$-operators $R^{RR'}$ also could be observed, which would be descendant matrices inherited from the obtained ones. All such type of $R$-operators produce Hamiltonian operators corresponding to 1d quantum integrable spin models describing non-elementary mixed interactions between different kinds of spins situating on the sites of the chains.

We can summarize the results schematically by the following diagram of the series of YBE solutions with the $sl_q(2)$ ($osp_q(1|2)$) symmetry, including the obtained descendant matrices defined on the composite states (and inherited from the Hecke type $R^{rr}$ matrices with $r > 2$) together with the universal $R^{rr'}$ matrices defined on the irreps, which are descendant matrices originated from the matrix on the fundamental irreps $R^{22}$

$$R^{22}_{\text{(universal matrix/Hecke type)}} \stackrel{\text{descendants}}{\longrightarrow} R^{2r}_{\text{(universal, ordinary Lax)}} \rightarrow R^{rr'}_{\text{(universal)},}$$

$$R^{33}_{\text{(Hecke type)}} \stackrel{\text{descendants}}{\longrightarrow} R^{3\{8,21,...\}}_{\text{(Hecke type)}} \rightarrow R^{\{8,21,...\}\{8,21,...\}}$$

$$\vdots$$

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This diagram does not exhaust all the variety of the YBE solutions for the given symmetries, particularly the solution $R^{33}$ (1.21) has not been involved here, but it does represent the interrelated series of the solutions and their descendants.

**Further developments and applications.** One can go further, and together with the descendant series of the Hecke type solutions $R^{rr}$ (such as the "extended" $L$-matrices $R^{rR}$ or $R$-matrices $R^{RR'}$), which are YBE solutions just by construction, consider any $R$-matrix, defined on the tensor product of the arbitrarily composed sets of the irreps. This can be achieved using the method of the construction of the centralizer operators on the tensor product of the reducible representations, brought in this article, and then try to solve the Yang-Baxter equations representing the $R$-matrix in the form of a superposition of the centralizers. Quite equivalently, one can construct all the possible projection operators which exist for the tensor product of the given composite representations, and represent the $R$-matrix as an expansion over these projectors.

Note, that at the exceptional values of the deformation parameter of the quantum group (i.e, when $q$ is a root of unity), the specter of the irreducible representations is restricted, higher spin irreps are deformating, and new indecomposable representations are arising [25], and correspondingly, the fusion rules also are deformed, but however in this case also the solutions of YBE defined on the composed states can be found, properly defining the centralisers or the projection operators (see [30, 27] and the references therein). As example, at $q^{4} = 1$ the descendant matrix (3.5) for $r = 3$ would be defined on the product of the indecomposable representations $T^{8} \otimes T^{8}$.

Consideration of the eigenproblem of Hamiltonian operators of the proposed integrable models with the help of the Quantum inverse scattering method will require certain non-trivial extensions of the well developed methods of the Algebraic Bethe Ansatz (nested Bethe Ansatz), which is caused both by the reducible character of representations and by the complex structure of Lax operators.
As it is known, by means of the YBE solutions the braid group representations can be realised, and they can be employed to obtain the link and knot invariants \[19, 21, 32\]. Thus one can use the $R$-matrices defined on the composite spaces for determining link invariants for such extended cases too. And besides of pure mathematical and theoretical interest, the solutions to Yang-Baxter equations on the reducible representations of the quantum algebras also can have practical usage. Particularly, such ”extended” $R$-matrices can be attractive in the context of the recent developments of the mutual interrelations of the quantum entanglement theory (in topological aspects) and the theory of integrable models, or, more precisely, the solutions to YBE, as the essential instruments in the construction of integrable models \[17\]. As the Hamiltonian operators corresponding to the discussed solutions describe integrable systems having rather large number of degrees of freedom and rich structure of the spin variety (with quite tangled interactions) even for the lattices with few sites, so possible applications may be assumed in different areas of 2d quantum statistical physics, string theories and particle physics.

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## A  Appendix

### A.1

The action of the generators of the quantum super-algebra $osp_q(1|2)$ on the spin-$j_r$ irrep $V^r = \{v^r_{-r+1}, v^r_{-r+2}, \ldots v^r_{r-1}\}$ can be described by the following general relations (see also \[27\]), $-\frac{r-1}{2} \leq i \leq \frac{r-1}{2}$

\[
\begin{align*}
e \cdot v_i^r &= \beta_i^r v_{i+1}^r, & \beta_{-1}^r = 0, \\
h \cdot v_i^r &= \left(i + ((-1)^r + 1) \frac{i\pi}{4\log q}\right) v_i^r, \\
f \cdot v_i^r &= \gamma_i^r v_{i-1}^r, & \gamma_{-1}^r = 0.
\end{align*}
\](A.1)
The commutation relations of the algebra put the following constraints on the coefficients
\[ \alpha_r^i \equiv \beta^r_{i-1} \gamma^r_i, \quad (q_r = [(-1)^r + 1]^{i/m}) \]
\[ \alpha_r^i = \sum_{i' = i}^{r-1} (-1)^{r-i}[i' + q_r]_q = \frac{(-1)^{r-1+i}[r/2 + q_r]_q + [i + q_r - 1/2]_q}{\sqrt{q} + 1/\sqrt{q}}, \quad (A.2) \]
\[ \alpha_r^r = -\alpha_{r-1}^r. \quad (A.3) \]

The values of \( \beta, \gamma \) one can fix by normalizing the representation vectors.

### A.2

Usually for finding the CG-coefficients the method of the highest weight and the normalized vectors is used (see for instance [19]). Let us here present the coefficients in a general form [27], with non-fixed \( \beta, \gamma \) coefficients in (A.1), for the particular case \( i = j \)
\[ C(j_1 j_2 j \ i_1 i_2 i) = \prod_{i' = -j_1 + 1}^{j_1} \frac{(-1)^{i'_1} q^{j_1 + i_1 + q_r}}{\beta_j^{i_1} - \beta_{j - i'_1}} C(-j_1 j_2 j \ i_1) . \quad (A.4) \]

By convention we can suggest that the state with the highest weight has even grading.

Combining two relations, (1.4) and (1.5), and using the orthogonality of the \( v^r_i \)-vectors, we obtain that \( C \) and \( \bar{C} \)-coefficients are inverse each to other in the following matrix sense
\[ \sum_{i_1 = -j_1}^{j_1} C(j_1 j_2 j \ i_1 i_2 i) \bar{C}(j_1 j_2 j' \ i_1 i_2 i) = \delta^j j'. \quad (A.5) \]

From the other hand, as the vectors \( \{v^r_i \} \) and \( \{v^{r'}_i \} \) are orthogonal when \( r \) and \( r' \) don’t coincide, then it follows that \( \{C(j_1 j_2 j \ i_1 i_2 i) \} \approx \{\bar{C}(j_1 j_2 j \ i_1 i_2 i) \} \). The proportionality coefficients we can find from the relation (A.5)
\[ C(j_1 j_2 j \ i_1 i_2 i) = (-1)^{p_1 p_2} \varepsilon^j \varepsilon_{i_1 i_2} C_(j_1 j_2 j \ i_1 i_2 i), \quad (A.6) \]
where \( \varepsilon^j_i \) is the norm of the state \( v^r_i \). The norm for the graded representations can be defined as in the work [22]. Let \( v^r_j \) is an even state, then we can take \( \varepsilon^j_i = 1 \) for all \( i \). If \( v^r_j \) is an odd state, then the norm in the irrep \( V_r \) is indefinite: \( \varepsilon^j_i = (-1)^{j-i} \). For definiteness we can take
$v_{j_1}^{r_1}$ and $v_{j_2}^{r_2}$ as even states, then the irreps $V^{r_1+r_2-1-k}$ have positive norms, when $k = 0 + 4\mathbb{Z}_+$ and have indefinite norms when $k = 0 + 2\mathbb{Z}_+$.

In the relations (A.1) the $\beta_{r-i}^j$, $\gamma_{r}^j$-coefficients according to the mentioned normalization can be fixed so, that $\beta_{r-i}^j = \gamma_{r}^j(-1)^{j-i}$, and will be equal to $\sqrt{\alpha_{r}^j}$ up to a sign.

### A.3

#### Quantum algebra $sl_q(2)$.

At the end we give also brief definition of the quantum algebra $sl_q(2)$. Algebra generators are $e$, $f$ and $h$, which satisfy to the following commutation relations

$$[e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad e q^{h^2} = q^h e, \quad f q^h = q^{h+2} f.$$  \hspace{1cm} (A.7)

Co-product can be defined as

$$\Delta[e] = e \otimes I + q^h \otimes e, \quad \Delta[f] = f \otimes q^{-h} + I \otimes f, \quad \Delta[q^h] = q^h \otimes q^h.$$  \hspace{1cm} (A.8)

The quadratic Casimir operator is

$$c = ef + \left(\frac{q^{h+1} - q^{-h-1}}{q - q^{-1}}\right)^2.$$  \hspace{1cm} (A.9)

Finite-dimensional irreducible representations $V^r$, $\text{dim}[V^r] = r$, are describing by their Casimir eigenvalues $c_r = [r/2]^2_q$ and by ”spin” values $j = (r - 1)/2$, with the analogy of the non-deformed algebra situation.

#### Quantum 6j-symbols.

The associativity of the tensor product of the quantum group is expressed by definition of the quantum 6j-symbols $\{j_{11j2j12}^{j_{j1j2j12}j_{j3j2j23}}\}_q$ as follows \cite{19, 16}

$$j_1 \begin{array}{c|c|c} j_2 & j_3 & j \\ \hline j_1 & j_3 & j_2 \\ \hline j_2 & j_3 & j_1 \\ \hline j_3 & j_1 & j_2 \\ \hline \end{array} = \sum_{j_{23}} \{j_{j1j2j12}^{j_{j1j2j12}j_{j3j2j23}}\}_q j_1 \begin{array}{c|c|c} j_2 & j_3 & j \\ \hline j_2 & j_3 & j_2 \\ \hline j_3 & j_1 & j_2 \\ \hline \end{array}$$  \hspace{1cm} (A.10)

Here $j$, $j_k$ are the spin values of the corresponding $r$-dimensional irreps, $j = (r - 1)/2$. The first diagram corresponds to the tensor product $\{V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \} \otimes V_{j_3} \to V_{j}$, the second one in the sum corresponds to $V_{j_1} \otimes \{V_{j_2} \otimes V_{j_3} \} \to V_{j}$. Also let us use the notation $\rho(j_1, j_2; j)$
[16] for denoting c-numbers which distinguish the Clebsh-Gordan coefficients corresponding to the projections $V_j \rightarrow V_{j_1} \otimes V_{j_2}$ and $V_j \rightarrow V_{j_2} \otimes V_{j_1}$. Then for revealing the spin structure of the operator (4.5), one can consider as an elementary cell of the chain lattice (with sublattice structure) the product of the vector spaces $V_1 \otimes V_2 \otimes V_3 \otimes V_4$. And taking into account, that both of the terms of the Hamiltonian (4.5, 3.5) contain the projector $P_{23}^1$, we can compare the following relations for the decomposition of the vector products, in order to express the terms containing the projectors $P_{23}$ and $P_{14}$ acting on the product $V_1 \otimes \{V_2 \otimes V_3\} \otimes V_4$, by means of the operators acting on the tensor product grouped as $\{V_1 \otimes V_2\} \otimes \{V_3 \otimes V_4\}

$$\begin{array}{c|c|c|c|c}
  j_1 & j_2 & j_3 & j_4 & j_{1234} \\
\hline
  j_{23} & j_{12} & j_{132} & j_{134} & \sum_{j_{12}, j_{34}} \rho(j_{23}, j_{123}) \times \\
& & & & \rho(j_{12}, j_{123}) \rho(j_{3}, j_{134}) \rho(j_{34}, j_{14}) \end{array} \ldots (A.11)
$$

From the another hand we have

$$\begin{array}{c|c|c|c|c}
  j_1 & j_2 & j_3 & j_4 & j_{1234} \\
\hline
  j_{23} & j_{12} & j_{134} & \sum_{j_{14}} \rho(j_{14}, j_{1234}) \times \\
& & & & \rho(j_{14}, j_{1234}) \rho(j_{1234}, j_{1234}) \rho(j_{14}, j_{1234}) \end{array} \ldots (A.12)
$$

As the projector operator $P_{23}^1$ acting on the space $V_2 \otimes V_3$ maps it into the one dimensional space, then $j_{23} = 0$. And it means $j_{123} = j_1$, $j_{1234} = j_{14}$, so \{\rho(j_{12}, j_{1234})\} \approx \delta_{j_{12}, j_{1234}}. These equations are valid for each variable $j_{1234}$ which satisfies to $|j_1 - j_2| \leq j_{1234} \leq (j_1 + j_2)$.

The second term in the $\tilde{R}_{1234}$-matrix contains the projector $P_{14}^1$, which means that for this case $j_{14} = 0$, and hence $j_{1234} = 0$ and $j_{12} = j_{14}$. The external projectors $(I - P_{12}^1)(I - P_{34}^1)$ entering into the $\tilde{R}_{1234}$-matrix ensure that $j_{12} \neq 0$ and $j_{34} \neq 0$. As all the states have the same dimension, i.e. the same spin $j = (r - 1)/2$, then $j_{12/34} \in [1, ..., 2j]$.

In the same spirit we can write out the transition operations passing the following steps: $V_1 \otimes V_2 \otimes V_3 \otimes V_4 \xrightarrow{P_E} \sum \{V_1 \otimes V_2\}_{V_{12}} \otimes \{V_3 \otimes V_4\}_{V_{34}} \rightarrow \sum V_1 \otimes \{V_2 \otimes V_3\} \otimes V_4 \xrightarrow{P_I} \sum \{V_1 \otimes V_4\}_{V_{14}} \otimes \{V_2 \otimes V_3\}_{V_{23}} \xrightarrow{P_E} \sum \{V_1 \otimes V_2\}_{V_{12}} \otimes \{V_3 \otimes V_4\}_{V_{14}'},$ which reflects the action of the Hamiltonian, and here we denote the external and internal projection operators, entering into $H_{1234}$, as $P_E = (I - P_{12}^1)(I - P_{34}^1)$, $P_I = P_{23}^1$ or $P_I = P_{23}^1 P_{14}^1$. In terms of the $6j$-symbols and the $\rho$-coefficients the action of the Hamiltonian term $H_{1234}$ would be ob-
tained from the following relation \((P^E P^I P^E) \times \sum_{j_{12}, j_{34}, j_{1234}} \frac{j_2}{j_{12}} \frac{j_3}{j_{34}} \frac{j_4}{j_{1234}}\) \(\sum\). where \(F_r(\rho)\) is a rational function of the coefficients \(\rho\), and the product of the quantum 6\(j\)-symbols can be written as \((j_{1234} = j_{14}) \prod \{q = \{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\} q\{j_1 j_2 j_3 j_4\}\) \(q\).

The explicit values of the quantum 6\(j\)-symbols and numbers \(\rho\) are calculated and can be found in the literature, see for example in [16].

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