Influence of Dark Matter on Light Propagation in Solar System

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Abstract

We investigated the influence of dark matter on light propagation in the solar system. We assumed the spherical symmetry of spacetime and derived the approximate solution of the Einstein equation, which consists of the gravitational attractions caused by the central celestial body, i.e. the Sun, and the dark matter surrounding it. We expressed the dark matter density in the solar system in the following simple power-law form, \( \rho(t, r) = \rho(t)(\ell/r)^k \), where \( t \) is the coordinate time; \( r \), the radius from the central body; \( \ell \), the normalizing factor; \( k \), the exponent characterizing \( r \)-dependence of dark matter density; and \( \rho(t) \), the arbitrary function of time \( t \). On the basis of the derived approximate solution, we focused on light propagation and obtained the additional corrections of the gravitational time delay and the relative frequency shift caused by the dark matter. As an application of our results, we considered the secular increase in the astronomical unit reported by Krasinsky and Brumberg (2004) and found that it was difficult to provide an explanation for the observed \( d\text{AU}/dt = 15 \pm 4 \text{[m/century]} \).

Key words: Dark Matter, Gravitation, Light Propagation, Ephemerides, Astronomical Unit

1 Introduction

The existence of dark matter was first indicated by Zwicky (1933) and subsequently by Rubin and Ford (1970); Rubin et al. (1980). According to the recent cosmological observation, i.e. Wilkinson Microwave Anisotropy Probe (WMAP) (Spergel et al., 2003), it is suggested that the majority of mass in our

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Universe is the dark matter, which is approximately 6 times that of ordinary (baryonic) matter. Furthermore, it is considered that dark matter has played an important role in the large-scale structure formation of the Universe. Actual observations such as 2dF\(^1\) and SDSS\(^2\) and results of numerical simulations, which assume the existence of dark matter such as Virgo Consortium\(^3\), are in good agreement with each other. Therefore, dark matter is considered to be the fundamental material of our Universe, even though its details are presently cloaked in mystery.

If dark matter is an essential component in the Universe, it is interesting and worthy to investigate its existence and detectability in our neighborhood area, the solar system. The number of recent astronomical and astrophysical measurements in the solar system have drastically increased, and they have led to (i) deep understanding of planetary dynamics and fundamental gravitational physics, (ii) significant improvement in lunar and planetary ephemerides, and (iii) precise determination of various astronomical constants.

To date, the gravitational influence of dark matter on planetary motion, such as the additional perihelion advance, has been studied by several authors. For instance, the effect of galactic dark matter has been considered by Braginsky et al. (1992), Klioner and Soffel (1993), Nordtvedt (1994), and Nordtvedt and Soffel (1995). On the other hand, the upper limit of the dark matter density in the solar system has been estimated within a range (Anderson et al., 1989; Grøn and Soleng, 1996; Khriplovich and Pitjeva, 2006; Iorio, 2006; Sereno and Jetzer, 2006; Khriplovich, 2007; Frére et al., 2008) such that

\[ \rho_{\text{dm}}^{(\text{max})} < 10^{-16} \sim 10^{-20} \, [\text{g/cm}^3]. \]  

Recently, the upper limit of the planet-bound dark matter was also evaluated (Adler, 2008a,b).

However, its contribution to light propagation has hardly been examined, in spite of the fact that the current accurate observations have been archived by an improvement in the observation of light/signal; round-trip time (radar/laser gauging techniques and spacecraft ranging), amelioration of atomic clocks, radio links of spacecraft, and increasing stability of frequency standard. Moreover, planned space missions, such as GAIA\(^4\), SIM\(^5\), LISA\(^6\), LATOR (Turyshev et al., 2004), and ASTROD/ASTROD-1 (Ni, 2007), require accurate light propaga-

\[^1\] http://www.aao.gov.au/2df
\[^2\] http://www.sdss.org/
\[^3\] http://www.virgo.dur.ac.uk/
\[^4\] http://www.rssd.esa.int/index.php?project=GAIA&page=index
\[^5\] http://planetquest.jpl.nasa.gov/SIMLite/sim_index.cfm
\[^6\] http://lisa.nasa.gov/
tion models. Therefore, it is noteworthy to examine how dark matter in the solar system affects light propagation and whether its traces can be detected if it really exists in our solar system.

On the other hand, because of a drastic improvement in the measurement techniques used in the solar system, some unexplained problematic phenomena occurred, such as pioneer anomaly (Anderson et al., 1998), Earth flyby anomaly (Anderson et al., 2008), secular increase in the astronomical unit (Krasinsky and Brumberg, 2004), and anomalous perihelion precession of Saturn (Iorio, 2009). Currently, the origins of these phenomena are far from clear, nevertheless, dark matter may cause some significant contribution to these phenomena (Nieto, 2008; Anderson and Nieto, 2009; Adler, 2009).

In this study, we will examine the influence of dark matter on light propagation in the solar system. First, we assume the spherical symmetry of spacetime and derive a simple approximate solution of the Einstein equation, which consists of gravitational attractions caused by the central celestial body, i.e., the Sun, and the dark matter surrounding it. Then, we will focus on formulating a light propagation model and estimating the additional effects of gravitational time delay and the relative frequency shift of a signal. As an application of our results, we will consider the secular increase in the astronomical unit (of length), AU, reported by Krasinsky and Brumberg (2004).

This paper is organized as follows. In Section 2, we explain the model of spacetime and some assumptions. In Section 3, we derive the approximate solution of the Einstein equation. In sections 4 and 5, we investigate time delay and relative frequency shift, respectively. In section 6, we focus on the application of our results to the secular increase in the astronomical unit. Finally, in section 7, we provide the summary of our study.

2 Model and Assumptions

Before deriving the approximate solution of the Einstein equation, we explain the model of spacetime and some assumptions. First, we suppose that the spacetime is characterized by the gravitational attractions caused by the central celestial body, i.e., the Sun, and dark matter surrounding it. Then, we express the spherically symmetric form of metric as

\[ ds^2 = -e^\mu c^2 dt^2 + e^\nu dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

where \( e^\mu \) and \( e^\nu \) are functions of time \( t \) and radius \( r \), and \( c \) is the speed of light in vacuum.
As the stress-energy tensor \( T^\alpha_\beta \), we presume the following form

\[
T^0_0 = -\varrho(t, r)c^2, \quad T^1_0 = \sigma(t, r)c, \quad T^i_j = 0.
\]  

\( T^0_0 \) is related to the dark matter density \( \varrho(t, r) \). At this time, since we do not have any knowledge about the actual distribution of dark matter in the solar system, then we suppose the following simple power-law form:

\[
\varrho(t, r) = \rho(t) \left( \frac{\ell}{r} \right)^{k},
\]

where \( \ell \) is the normalizing factor that is chosen as \( \ell \equiv r_E \) and \( r_E \) is the orbital radius of Earth, \( k \) is the exponent characterizing \( r \)-dependence of \( \varrho(t, r) \), and \( \rho(t) \) represents the time variation of the dark matter density\(^7\). The test particle, i.e. photon, is subjected to the gravitational attractions caused by the central body and dark matter, which is confined within a spherical shell of radius \( r \) at time \( t \). \( \rho(t) \) is an arbitrary function of time \( t \), however, for the sake of simplicity, we assume that the time variation of the dark matter density in the solar system is considerably slow;

\[
\rho(t) \approx \rho_0 + \frac{d\rho}{dt} \bigg|_{t_0} (t - t_0).
\]

where subscript 0 denotes the initial epoch of planetary ephemerides. As the one possibility, the time variation of the dark matter density \( \varrho(t, r) \) may be caused by the motion of the solar system in our Galaxy, if the distribution of galactic dark matter is inhomogeneous.

The dark matter density \( \varrho(t, r) \) observed in the solar system may be generally expressed as a sum of solar system-bound (or local) dark matter \( \varrho^{(\text{solar})}(t, r) \) and galactic dark matter \( \varrho^{(\text{galactic})}(t, r) \) as follows:

\[
\varrho(t, r) = \varrho^{(\text{solar})}(t, r) + \varrho^{(\text{galactic})}(t, r).
\]

Here, to simplify the situation, we assume that the spacetime is spherically symmetric and the time variation of the dark matter density \( \varrho(t, r) \) is caused by the inhomogeneity of the galactic dark matter as mentioned above. Therefore it is possible to express\(^8\)

\(^7\) In this paper, Greek indexes run from 0 to 3, and Latin ones do from 1 to 3.
\(^8\) \( \rho(t) \) can be considered to be the dark matter density observed around the Earth’s orbit since \( \ell = r_E \).
\[ \rho^{(\text{solar})}(t, r) = \rho^{(\text{solar})}(r) = \rho_0^{(\text{solar})} \left( \frac{\ell}{r} \right)^k, \quad (7) \]

\[ \rho^{(\text{galactic})}(t, r) = \left[ \rho_0^{(\text{galactic})} + \frac{d\rho^{(\text{galactic})}}{dt} \bigg|_0 (t - t_0) \right] \left( \frac{\ell}{r} \right)^k. \quad (8) \]

In this case, \( \rho_0 \) and \( d\rho/dt|_0 \) in (5) are

\[ \rho_0 = \rho_0^{(\text{solar})} + \rho_0^{(\text{galactic})}, \quad \frac{d\rho}{dt} \bigg|_0 = \frac{d\rho^{(\text{galactic})}}{dt} \bigg|_0. \quad (9) \]

According to the recent investigation, i.e. Bertone and Merritt (2005), the galactic dark matter density is of the order of \( 10^{-24} \left[ \text{g/cm}^3 \right] \). This is several orders of magnitude smaller than the evaluated density of dark matter in the solar system (see (1)). Therefore in this study, we suppose that

\[ \rho_0 = \rho_0^{(\text{solar})}, \quad \frac{d\rho}{dt} \bigg|_0 = \frac{d\rho^{(\text{galactic})}}{dt} \bigg|_0. \quad (10) \]

\( T_i^0, T_j^0 \) represent the time variation of energy and momentum flux. We auxiliary introduced these components to preserve the time dependency of the obtained solution. Though \( T_j^i \) represents the stress part attributed to the dark matter, currently, the equation of state of dark matter is not known, therefore, we adopt the standard assumption that the dark matter is pressure-less dust particles \( p \simeq 0 \) and that its time variation is also negligible \( dp/dt \simeq 0 \).

Finally, we consider the choice of exponent \( k \) of \( \rho(t, r) \). Because the distribution of dark matter in the solar system is poorly understood, we adopt the following three indexes as examples: \( k = 1 \) (density decreasing with \( r \)), \( k = 0 \) (constant density), and \( k = -1 \) (density increasing with \( r \)). The density \( \rho(t, r) \) should be damped at a certain radius \( r_0 \) from the Sun (especially when \( k = 0 \) and \( k = -1 \)), and it is natural to imagine that \( \rho(t, r) \) reaches asymptotically for the galactic dark matter density, i.e. \( \sim 10^{-24} \left[ \text{g/cm}^3 \right] \). However, we are now interested in the astronomical observations within the quite inner (planetary) area of the solar system (see Fig. 1 for the conceptual diagram). Therefore, in this study, we do not consider the details of dark matter density distribution far away from the Sun.

### 3 Approximate Solution of the Einstein Equation

On the basis of the assumptions in Section 2 we obtain the approximate solution of the Einstein equation.
Fig. 1. Conceptual diagram of $r$-dependence of dark matter density for $k = 1$, $k = 0$, and $k = -1$. We simply assume that the actual observations are carried out within the quite inner (planetary) area of the solar system $\ll r_d$.

\[ G^\alpha_\beta \equiv R^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta R = \frac{8\pi G}{c^4} T^\alpha_\beta, \]  

where $G$ is the Newtonian gravitational constant, $G^\alpha_\beta$ is the Einstein tensor, $R^\alpha_\beta$ is the Ricci tensor, $R$ is the Ricci scalar, and $T^\alpha_\beta$ is the stress-energy tensor in (3). The non-zero components of the Einstein tensor are expressed as follows:

\[ G^0_0 = e^{-\nu} \left( \frac{1}{r^2} - \frac{1}{4} \frac{\partial \nu}{\partial r} \right) - \frac{1}{r^2}, \]  

\[ G^1_1 = e^{-\nu} \left( \frac{1}{r^2} + \frac{1}{r} \frac{\partial \mu}{\partial r} \right) - \frac{1}{r^2}, \]  

\[ G^2_2 = G^3_3 = e^{-\nu} \left[ n \frac{\partial^2 \mu}{\partial r^2} + \frac{1}{2} \left( \frac{\partial \mu}{\partial r} \right)^2 - \frac{1}{2} \frac{\partial \nu}{\partial r} \frac{\partial \mu}{\partial r} + \frac{1}{2} \frac{\partial \mu}{\partial r} \right] - e^{-\mu} \left[ \frac{\partial^2 \nu}{\partial t^2} + \frac{1}{2} \left( \frac{\partial \nu}{\partial t} \right)^2 - \frac{1}{2} \frac{\partial \mu}{\partial t} \frac{\partial \nu}{\partial t} \right], \]  

\[ G^0_1 = -\frac{1}{cr} e^{-\nu} \frac{\partial \mu}{\partial t}, \quad G^1_0 = \frac{1}{cr} e^{-\mu} \frac{\partial \nu}{\partial t}. \]

From the 00 component of the Einstein equation, we have

\[ r(1 - e^{-\nu}) = m(t) + \frac{8\pi G}{c^2} \rho(t)\ell^k - k r^{3-k}, \]  

where $m(t)$ is an arbitrary function of time $t$, however, we choose $m(t)$ such that it reduces to the Schwarzschild radius $m(t) \rightarrow m = 2GM/c^2 = \text{constant}$ when $\varrho(t, r) = 0$. Therefore, we obtain
\[ e^{-\nu} = 1 - \frac{2GM}{c^2r} - \frac{8\pi G \rho(t)\ell^k}{c^2 3 - k} r^{2-k}. \] (17)

Using the 00 and 11 components of the Einstein equation, it follows

\[ \frac{\partial \mu}{\partial r} + \frac{\partial \nu}{\partial r} = \frac{8\pi G \rho(t)\ell^k}{c^2} r^{k-1}, \] (18)

where we kept the \( O(c^{-2}) \) order terms only on the right-hand side. We obtain the following equation by integrating (18) with respect to \( r \), combing it with (17), and omitting the \( O(c^{-4}) \) and higher order terms:

\[ e^\mu = f(t) \left[ 1 - \frac{2GM}{c^2r} + \frac{8\pi G \rho(t)\ell^k}{c^2 (2 - k)(3 - k)} r^{2-k} \right]. \] (19)

Although \( f(t) \) is also an arbitrary function of time \( t \), we replace the time coordinate with \( \sqrt{f(t)}dt \rightarrow dt \) and delete \( f(t) \). Finally, we obtain

\[ ds^2 = - \left( 1 - \frac{2GM}{c^2r} + \frac{8\pi G \rho(t)\ell^k}{c^2 (2 - k)(3 - k)} r^{2-k} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{c^2r} - \frac{8\pi G \rho(t)\ell^k}{c^2 3 - k} r^{2-k} \right)^{-1} dr^2 + r^2 d\Omega^2, \] (20)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). When \( k = 0 \) and \( \rho(t) = \rho_0 \) = constant, (20) reduces to the metric derived by Grøn and Soleng (1996); therefore, our solution (20) is considered to be some extension of the solution obtained by Grøn and Soleng. In the case of a solar system experiment, it is sufficient to use the following approximate form:

\[ ds^2 = - \left[ 1 - \frac{2}{c^2} U(t, r) \right] c^2 dt^2 + \left[ 1 + \frac{2}{c^2} V(t, r) \right] dr^2 + r^2 d\Omega^2, \] (21)

where

\[ U(t, r) = \frac{GM}{r} - \frac{4\pi G \rho(t)\ell^k}{(2 - k)(3 - k)} r^{2-k}, \] (22)

\[ V(t, r) = \frac{GM}{r} + \frac{4\pi G \rho(t)\ell^k}{3 - k} r^{2-k}. \] (23)

We mention here that it is easy to incorporate the cosmological constant \( \Lambda \) in (20) or (21). However, to focus on the effect of dark matter, we omit the \(-\Lambda r^2/3\) term.

7
Fig. 2. Light/signal path. We assume that the first approximation of the light path is rectilinear along the $x$-direction (bold dashed line), $b$ is the impact parameter, and $r = \sqrt{x^2 + b^2}$. The actual light path is drawn by the bold solid line.

4 Gravitational Time Delay

4.1 Time Delay in Coordinate Time

In this section, we calculate the time delay attributed to the dark matter. In the static spacetime, we can easily relate an affine parameter $\lambda$ to coordinate time $t$ using the Euler-Lagrange equation of $g_{00}$, i.e. Chapter 8 of Weinberg (1972). However, (20) or (21) is the non-static or time-dependent then it is not easy to calculate the geodesic equation analytically in general. Therefore, we consider an alternative approach.

To begin with, we transform (21) from spherical coordinates to rectangular coordinates. By the usual coordinate transformation, i.e.

$$x = r \sin \theta \sin \phi, \quad y = r \sin \theta \cos \phi, \quad z = r \cos \theta,$$

(24) is rewritten as (e.g., Brumberg (1991))

$$ds^2 = -\left(1 - \frac{2}{c^2} U\right) c^2 dt^2 + \left(\delta_{ij} + \frac{2}{c^2} V \frac{x^i x^j}{r^2}\right) dx^i dx^j,$$

(25)

where $\delta_{ij}$ is the Kronecker’s delta symbol. We suppose that the actual light path is calculated along with the approximate rectilinear path ($x$-direction) such that

$$y = b = \text{constant}, \quad z = 0, \quad r = \sqrt{x^2 + b^2},$$

(26)

where $b$ is an impact factor (see Fig. 2). Hence, (25) becomes

$$ds^2 = -\left(1 - \frac{2}{c^2} U\right) c^2 dt^2 + \left(1 + \frac{2}{c^2} V \frac{x^2}{r^2}\right) dx^2.$$

(27)
The world line of the light ray is null geodesic $ds^2 = 0$, therefore from (27) we have

$$\frac{dt}{dx} = 1 + \frac{1}{c^2} \left[ \frac{GM}{r} \left( 1 + \frac{x^2}{r^2} \right) - \frac{4\pi G \ell^k \rho(t)}{(2 - k)(3 - k)} r^{2-k} + \frac{4\pi G \ell^k \rho(t) x^2}{3 - k} r^k \right]$$

(28)

We express $\rho(t)$ in the form of (5).

To obtain the round-trip time from (28), we assume that the lapse time $\Delta t$ is expressed by a linear combination of each effect as follows:

$$\Delta t = \Delta t_{pN} + \Delta t_{(\text{const})}^{(\text{dm})} + \Delta t_{(t)}^{(\text{dm})}$$

(29)

where $\Delta t_{pN}$ corresponds to the Shapiro time delay in 1st post-Newtonian approximation, $\Delta t_{(\text{const})}^{(\text{dm})}$ is attributed to the static part of the dark matter density ($\rho_0$ of (5)), and $\Delta t_{(t)}^{(\text{dm})}$ is the contribution of the time-dependent part of dark matter density ($d\rho/dt\mid_0 t$ of (5)). The post-Newtonian parts are easily obtained as follows:

$$\Delta t_{pN} = \frac{x_2 - x_1}{c} + GM \frac{c^3}{c^3} \left[ 2 \ln \frac{x_2 + \sqrt{x_2^2 + b^2}}{x_1 + \sqrt{x_1^2 + b^2}} - \left( \frac{x_2}{\sqrt{x_2^2 + b^2}} - \frac{x_1}{\sqrt{x_1^2 + b^2}} \right) \right].$$

(30)

Next, we calculate the time delay caused by dark matter. The static part $\Delta t_{(\text{const})}^{(\text{dm})}$ is straightforwardly integrated as

$$\Delta t_{(\text{const})}^{(\text{dm})} = \frac{\pi G \rho_0}{c^3} H(x_1, x_2; k)$$

(31)

$$H(x_1, x_2; k) = \begin{cases} 
-2b^2 \ln \frac{x_2 + \sqrt{x_2^2 + b^2}}{x_1 + \sqrt{x_1^2 + b^2}} & (k = 1) \\
\frac{2}{3} \left[ \frac{1}{3} (x_2^3 - x_1^3) - b^2 (x_2 - x_1) \right] & (k = 0) \\
-\frac{1}{12} b^4 \ln \frac{x_2 + \sqrt{x_2^2 + b^2}}{x_1 + \sqrt{x_1^2 + b^2}} \\
-2(x_2 \sqrt{x_2^2 + b^2} - x_1 \sqrt{x_1^2 + b^2}) \\
+ 3b^2 (x_2 \sqrt{x_2^2 + b^2} - x_1 \sqrt{x_1^2 + b^2}) & (k = -1) \end{cases}$$

(32)

Finally, we compute the time-dependent part $\Delta t_{(t)}^{(\text{dm})}$ ($d\rho/dt\mid_0 t$ part). If light is emitted from Earth at $t = T$ and it reaches the reflector (planet/spacecraft) at $t = T + \Delta t_{(t)}^{(\text{dm})}$, then $\Delta t_{(t)}^{(\text{dm})}$ satisfies
\[ c \int_{T}^{T+\Delta t_{dm}^{(t)}} \frac{1}{t} \, dt = c \ln \frac{T + \Delta t_{dm}^{(t)}}{T} = \frac{d\rho}{dt} \bigg|_{0 \to x_1} \int_{x_1}^{x_2} \cdots \, dx, \quad \text{(33)} \]

where the integral \( \int_{x_1}^{x_2} \cdots \, dx \) is the same as that in the case of the static part \( \Delta t_{dm}^{(\text{const})} \). Since \( \frac{d\rho}{dt}|_{0} \ll 1 \), \( \Delta t_{dm}^{(t)} \) can be expressed as

\[ \Delta t_{dm}^{(t)} = \frac{1}{c} \frac{d\rho}{dt}|_{0} T \int_{x_1}^{x_2} \cdots \, dx. \quad \text{(34)} \]

Therefore, the time delay caused by dark matter is expressed as

\[ \Delta t_{dm} \equiv \Delta t_{dm}^{(\text{const})} + \Delta t_{dm}^{(t)} = \frac{\pi G}{c^3} \left( \rho_0 + \frac{d\rho}{dt}|_{0} T \right) H(x_1, x_2; k). \quad \text{(35)} \]

In this case, the time delay caused by dark matter can be characterized by the density at the emission time of signal \( t = T \), that is \( \rho(T) = \rho_0 + \frac{d\rho}{dt}|_{0} T \).

In order to apply (30) and (35) to two-way light propagation, i.e. Earth → receiver (planet/spacecraft) → Earth, we consider the following situation; the light path used in computation is parallel to the \( x \)-axis, Earth, and the receiver (planet/spacecraft) are located \( x = a_E \) and \( x = -a_R \), respectively (see Fig. 2 again). We suppose that (a) during the round-trip of light, Earth and the receiver are almost at rest and that (b) the time variation of the dark matter density is also considerably slow. In other words, the time lapse \( \Delta T \) can be mainly determined using by the dark matter density at the emission time \( t = T \), that is \( \rho(T) \). Hence, the round-trip time in the coordinate time, \( \Delta T \) is expressed as

\[ \Delta T = 2 \frac{a_E + a_R}{c} + \frac{2GM}{c^3} \left[ 2 \ln \left( a_E + \sqrt{a_E^2 + b^2} \right) \left( a_R + \sqrt{a_R^2 + b^2} \right) \right. \\
\left. - \left( \frac{a_E}{\sqrt{a_E^2 + b^2}} + \frac{a_R}{\sqrt{a_R^2 + b^2}} \right) \right] \\
+ \frac{2\pi G}{c^3} \left( \rho_0 + \frac{d\rho}{dt}|_{0} T \right) H(a_E, a_R; k), \quad \text{(36)} \]

where we substitute \( H(a_E, a_R; k) = H(0, a_E; k) + H(0, a_R; k) \). To calculate (36), we referred to an approach in Section 40.4 and Figure 40.3 shown by Misner et al. (1970).

Let us estimate the order of the time delay \( \Delta T_{dm} \). Because \( \frac{d\rho}{dt}|_{0} T \) is now
Fig. 3. $a_R$ dependence of additional time delay $\Delta T_{dm}$. As $\rho_0$, we adopt $\rho_0 \sim 10^{-16} [g/cm^3]$ and fixed $a_E = 1.0$ [AU] (= $1.5 \times 10^{11}$ [m]) and $b = 0.001$ [AU] (= $1.5 \times 10^8$ [m]).

anticipated to be considerably smaller than the dominant part $\rho_0$, we neglect the $d\rho/dt|_0 T$ term here and evaluate

$$\Delta T_{dm} \simeq \frac{2\pi G \rho_0}{c^3} \mathcal{H}(a_E, a_R; k).$$  \hspace{1cm} (37)

Fig. 3 illustrates the $a_R$ dependence of the time delay $\Delta T_{dm}$. We adopted $\rho_0 \sim 10^{-16} [g/cm^3]$, which is the largest upper limit obtained from the dynamical perturbation on planetary motion. We fixed $a_E = 1.0$ [AU] (= $1.5 \times 10^{11}$ [m]) (orbital radius of the Earth) and impact parameter $b = 0.001$ [AU] (= $1.5 \times 10^8$ [m]). If the dark matter is accumulated in the neighborhood of the Sun ($k = 1$), $\Delta T_{dm} \sim 10^{-25}$ [s] in a given range of $a_R$. When $k = 0$ and $k = -1$, $\Delta T_{dm}$ is of the order of $10^{-20}$ [s] in the inner planetary region, while in the outer planetary region, it is of the order of $10^{-19} < \Delta T_{dm} < 10^{-17}$ [s] ($k = 0$) and $10^{-16} < \Delta T_{dm} < 10^{-14}$ [s] ($k = -1$). However, the current observational limit in the solar system is $\sim 10^{-8}$[s] or a few 100 [m] for planetary radar and $10^{-11}$[s] or a few [m] for spacecraft ranging; the internal error of the atomic clocks on Earth is $\sim 10^{-9}$ [s]. Then, at this time, it is difficult to extract the trace of dark matter from the ranging data.

4.2 Time Delay in Proper Time

The round-trip time of the light ray (36) is expressed in the coordinate time. However, the actual measurement is performed by the atomic clocks on the surface of Earth, which shows proper time $\tau$. Therefore, we must transform (36) into proper time. Presently, it is sufficient to use the equation of proper
time for the quasi-Newtonian approximation such that

\[
\frac{d\tau}{dt} = 1 - \frac{1}{c^2} \left( U + \frac{1}{2} v^2 \right).
\] (38)

Evaluating \( \frac{d\tau}{dt} \) around the orbit of Earth and keeping the \( O(c^{-3}) \) terms only, the round-trip time \( \Delta \tau \) measured in proper time is given by

\[
\Delta \tau = \left. \frac{d\tau}{dt} \right|_E \Delta T = 2 \frac{a_E + a_R}{c} + \frac{2GM}{c^3} \left[ 2\ln \left( \frac{a_E + \sqrt{a_E^2 + b^2}}{b} \right) + \frac{2\pi G}{c^3} \left( \rho_0 + \frac{d\rho}{dt} \bigg|_0 T \right) \mathcal{H}(a_E, a_R; k) - 2 \frac{a_E + a_R}{c^3} \left[ \frac{1}{2} v_E^2 + \frac{GM}{a_E} - \frac{4\pi G}{(2 - k)(3 - k)} a_E^{2-k} \left( \rho_0 + \frac{d\phi}{dt} \bigg|_0 T \right) \right] \right] \quad (39)
\]

where \( v_E \) is the orbital velocity of Earth.

5 Relative Frequency Shift

We use (36) to derive the relative frequency shift of signal \( y^9 \), which is defined as

\[
y = \frac{\delta \nu}{\nu} \equiv - \frac{d\Delta T}{dt}.
\] (40)

When the light ray passes near the limb of the Sun such as in the Cassini experiment [Bertotti et al. 2003], the conditions \( a_E, a_R \gg b, da_E/dt, \text{and} da_R/dt \ll db/dt \) hold, where \( b = \sqrt{b_0^2 + (vt)^2} \). Then, the relative frequency shift caused by the Sun, \( y_{pN} \), and dark matter, \( y_{dm} \), are expressed as

\[
y = y_{pN} + y_{dm} \quad (41)
y_{pN} = \frac{8GM}{c^3 b} \frac{db}{dt} \quad (42)
\]

Here, \( y \) is not the \( y \)-coordinate, but the relative frequency shift according to Bertotti et al. (2003).
Fig. 4. Additional relative frequency shift caused by dark matter, and $y_{dm}$ is plotted as a function of $a_R$. We set $b_0 = 2R_{\text{Sun}}$, $R_{\text{Sun}} \simeq 6.9 \times 10^8$ [m], $v \simeq 30$ [km/s], and $t = 1$ [day].

$$y_{dm} = \frac{\pi G}{c^3} \left( \rho_0 + \frac{d\rho}{dt} \right) \mathcal{K}(a_E, a_R; k)$$

$$\mathcal{K}(a_E, a_R; k) = \begin{cases} 
8b_\ell \left( \ln \frac{4a_Ea_R}{b^2} - 1 \right) \frac{db}{dt} & (k = 1) \\
\frac{8}{3} b(a_E + a_R) \frac{db}{dt} & (k = 0) \\
\frac{2}{7} b \left[ b^2 \ln \frac{4a_Ea_R}{b^2} - (a_E^2 + a_R^2) \right] \frac{db}{dt} & (k = -1).
\end{cases}$$

Fig. 4 shows the relative frequency shift caused by dark matter, $y_{dm}$ as a function of $a_R$. In this plot, we substitute $b_0 = 2R_{\text{Sun}}$, $R_{\text{Sun}} \simeq 6.9 \times 10^8$ [m], $v \simeq 30$ [km/s], and $t = 1$ [day] ($t = 0$ gives the closest point). The order of magnitude of $y_{dm}$ is $\sim 10^{-25}$; however, currently the stability of frequency standard is of the order of $10^{-15}$ or even higher. Therefore, the expected frequency shift caused by dark matter is approximately 10 orders of magnitude smaller than the present observational limit of frequency.

6 Application to Secular Increase in Astronomical Unit

In this section, we apply the previous results to the secular increase in the astronomical unit (of length) reported by Krasinsky and Brumberg (2004). The astronomical unit (AU) is one of the important scales in astronomy, and it is the basis of the cosmological distance ladder. AU is also a fundamental astronomical constant, which gives the relation between two length units; 1 [AU] in the astronomical system of units and 1 [m] in SI ones. Presently, AU is determined by using the planetary radar and spacecraft ranging data (round-trip time of light ray), and the latest best-fit value is obtained as (Pitjeva,
1 [AU] = 1.495978706960 \times 10^{11} \pm 0.1 \text{ [m].} \quad (45)

We use the calculated planetary ephemerides (solution of equation of motion) to compute the theoretical value of the round-trip time $t_{\text{theo}}$ using the following formula:

$$t_{\text{theo}} = \frac{d_{\text{theo}}}{c} \text{AU \ [s],} \quad (46)$$

where $d_{\text{theo}}$ [AU] is the interplanetary distance. $t_{\text{theo}}$ [s] is compared with the observed round-trip time $t_{\text{obs}}$ [s], and AU is optimized by the least square method.

However, when Krasinsky and Brumberg replaced $t_{\text{theo}}$ with

$$t_{\text{theo}} = \frac{d_{\text{theo}}}{c} \left[ \text{AU} + \frac{d\text{AU}}{dt}(t - t_0) \right] \quad (47)$$

and fitted it to the observational data, they found that $d\text{AU}/dt$ had a non-zero and positive secular value, $15 \pm 4 \text{ [m/century]},$ where $t_0$ is the initial epoch. The evaluated value $d\text{AU}/dt = 15 \pm 4 \text{ [m/century]}$ is approximately 100 times that of the current determination error of AU (see (45)). At present, the time-dependent part $(d\text{AU}/dt)(t - t_0)$ cannot be related to any theoretical predictions, hence, several attempts have been made to explain this secular increase in AU on the basis of various factors such as the effects of cosmological expansion (Krasinsky and Brumberg, 2004; Mashhoon et al., 2007; Arakida, 2009), mass loss of the Sun (Krasinsky and Brumberg, 2004; Noerdlinger, 2008), and time variation of the gravitational constant $G$ (Krasinsky and Brumberg, 2004). However, unfortunately, thus far, none of these factors seem to be responsible for the secular increase in AU.

It is noteworthy that the observed $d\text{AU}/dt$ does not imply the expansion of planetary orbit and/or an increase in the orbital period of a planet. As a matter of fact, the determination error of the latest planetary ephemerides is considerably smaller than the reported $d\text{AU}/dt$ (see in Table 4. of Pitjeva (2005)). Hence, $d\text{AU}/dt$ may be caused by some effects on light propagation, and not by the dynamical perturbation on planetary motion.

Moreover, AU denotes not only the conversion constant of the length unit but also the value that characterizes the $GM$ of the Sun in SI units such that

$$GM_{\text{Sun}} = k^2\text{AU}^3/d^2 \text{ [m}^3/\text{s}^2], \quad (48)$$
where \( k = 0.01720209895 \) is the Gaussian gravitational constant, and \( d \) is a day such that \( d = 86400 \, [s] \). Therefore, the observed \( d\text{AU}/dt \) may be related to an increase in the dark matter density such that \( GM(t) = G(M_{\text{Sun}} + M_{\text{dm}}(t)) \), where \( M_{\text{dm}}(t) \) is the total mass of dark matter within a planetary orbit at time \( t \).

Then, let us evaluate the extent of time variation of dark matter density \( d\rho/dt|_0 \) in (36) that is needed to explain the observed \( d\text{AU}/dt \). We have

\[
\frac{d_{\text{theo}}}{c} \frac{d\text{AU}}{dt} T \sim \frac{2\pi G}{c^3} \frac{d\rho}{dt} \bigg|_0 \, TR^3, (49)
\]

where we set \( \mathcal{H}(a_E, a_R; k) \sim R^3 \), and \( R \) is the orbital radius of a planet. In the case of Earth–Mars ranging, we let \( R \sim 1.52 \, [\text{AU}] \) (orbital radius of Mars). To obtain the reported \( d\text{AU}/dt \), \( d\rho/dt|_0 \) must be of the order of \( 10^{-9} \, [\text{g}/(\text{cm}^3\text{s})] \) and \( d\rho/dt|_0 T \sim 1 \, [\text{g}/\text{cm}^3] \) for \( T \sim 100 \, [\text{y}] \). However, this value corresponds to the density of water; therefore, this possibility of achieving such value is unrealistic and should be made an exception.

7 Summary

We investigated the influence of dark matter on light propagation in the solar system. We used the simplified model to derive the approximate solution of the Einstein equation, which consists of the gravitational attractions caused by the central celestial body, i.e. the Sun, and dark matter surrounding it. We found that the derived metric (21) can be considered to be an extension of the previous work by Grøn and Soleng (1996). We assumed that the simple time variation of dark matter density, and focused our discussion on light propagation then computed the additional corrections of gravitational time delay and relative frequency shift. However, the expected effects were considerably smaller than the current observational limits, even when we considered the largest upper limit evaluated from the planetary perturbation caused by dark matter, \( \rho_0 \sim 10^{-16} \, [\text{g}/\text{cm}^3] \).

We applied the obtained results to the secular increase in the astronomical unit reported by Krasinsky and Brumberg (2004) and considered the possibility of explaining the observed \( d\text{AU}/dt = 15 \pm 4 \, [\text{m}/\text{century}] \) on the basis of the time variation of the dark matter density. We found that to induce the obtained \( d\text{AU}/dt \), the change in the dark matter density \( d\rho/dt|_0 \) in (5) must be of the order of \( 10^{-9} \, [\text{g}/(\text{cm}^3\text{s})] \) and that \( d\rho/dt|_0 T \sim 1 \, [\text{g}/\text{cm}^3] \) for the interval \( T \sim 100 \, [\text{y}] \). However, it is completely unrealistic to achieve these values, and the existence of dark matter and its time variation cannot explain \( d\text{AU}/dt \).
As mentioned in the previous section, some attempts were made to show the secular increase in AU. However, the origin of $d\text{AU}/dt$ is presently far from clear. As the one possibility, it is believed that the most plausible reason for the origin of $d\text{AU}/dt$ is the lack of calibrations of internal delays of radio signals within spacecrafts. Nevertheless, as other unexplained anomalies discovered in the solar system, $d\text{AU}/dt$ may be attributed to the fundamental property of gravity, therefore, this issue should be explored in terms of all possibilities.

Though it is currently impossible to detect the evidence of dark matter from light propagation, some planned space missions, especially ASTROD, are aimed to achieve a clock stability of $10^{-17}$ over a travel time of 1000 [s] (Ni, 2007). Improvement in both the laser ranging technique and the clock stability may enable us to observe the trace of dark matter, if it really exists in the solar system. For this purpose, it is very important subjects to develop a rigorous light propagation model.

In particular, since it is not easy to analytically calculate the time-dependent null geodesic equation, in this study, we integrated (28) assuming the simple linear combinations of each effect. However, from the theoretical point of view and some astronomical and astrophysical applications such as formulation of the cosmological gravitational lensing in the expanding background, it is noteworthy to develop a method to analytically compute the time-dependent geodesic equation.

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