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Junction between a plate and a rod of comparable thickness in nonlinear elasticity. Part II

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Abstract

We analyze the asymptotic behavior of a junction problem between a plate and a perpendicular rod made of a nonlinear elastic material. The two parts of this multi-structure have small thicknesses of the same order \(\delta\). We use the decomposition techniques obtained for the large deformations and the displacements in order to derive the limit energy as \(\delta\) tends to 0.

KEY WORDS: nonlinear elasticity, junctions, straight rod, plate.
Mathematics Subject Classification (2000): 74B20, 74K10, 74K30.

1 Introduction

In a former paper [12] we derive the limit energy of the junction problem between a plate and a rod under an assumption that couples their respective thicknesses \(\delta\) and \(\varepsilon\) to the order of the Lamé’s coefficients of the materials in the plate and in the rod. This assumption precludes the case where the thicknesses have the same order and the structure is made of the same material (see equation 1.1 in the introduction of [12]). The aim of the present paper is to analyze this specific case for a total energy of order \(\delta^5\). As in [12], the structure is clamped on a part of the lateral boundary of the plate and it is free on the rest of its boundary.

The main difference here is the behavior in the rod in which, for this level of energy (which is higher than the maximum allowed in [12]), the stretching-compression is of order \(\delta\) while the bending is of order \(\delta^{1/2}\). The most important consequence is that in the limit model for the rod the stretching-compression is actually given by the bending in the rod (through a nonlinear relation) and by the bending in the plate at the junction point

\(^3\)We were just finishing this paper when suddenly two days later my friend the Professor Dominique Blanchard died. We worked seven years together, our collaboration was very successful for both.
(see (6.15) and (6.16)). The bending and torsion models in the rod are the standard linear ones. In the plate the limit model is the Von Kármán system in which the action of the rod is modelized by a punctual force at the junction.

Let us emphasize that in order to obtain sharp estimates on the deformations in the junction area, see Lemma 4.2, we use the decomposition techniques in thin domains (see [12], [24], [8], [7]). In order to scale the applied forces which induce a total energy energy of order $\delta^5$, from Lemma 4.2 and [7], we derive a nonlinear Korn’s inequality for the rod (as far as the plate is concerned this type of inequality is already established in [12]). The nonlinear character of these Korn’s inequalities prompt us to adopt smallness assumptions on some components of the forces. Then, we are in a position to study the asymptotic behavior of the Green-St Venant’s strain tensors in the two parts of the structure. At last this allows us to characterize the limit of the rescaled infimum of the 3d energy as the minimum of a functional over a set of limit admissible displacements which includes the nonlinear relation between the stretching-compression and the bending in the rod.

In Section 2 we introduce a few general notations. Section 3 gives a few recalls on the decomposition technique of the deformations in thin structures. In Section 4, we derive first estimates on the terms of the decomposition of a deformation in the rod and sharp estimates in the junction area. In the same section we also obtain Korn’s inequality in the rod. In Section 5 we introduce the elastic energy and the assumptions on the applied forces in order to obtain a total elastic energy of order $\delta^5$. In Section 6 we analyze the asymptotic behavior of the Green-St-Venant’s strain tensors in the plate and in the rod. In Section 7 we prove the main result of the paper namely the characterization of the limit of the rescaled infimum of the 3d energy.

As general references on the theory of elasticity we refer to [2] and [14]. The reader is referred to [1], [32], [22] for an introduction of rods models and to [17], [16], [13], [19], [31] for plate models. As far as junction problems in multi-structures we refer to [15], [16], [28], [29], [30], [3], [26], [27], [23], [20], [21], [4], [5], [6], [25], [10], [11]. For the decomposition method in thin structures we refer to [22], [23], [24], [25], [7], [8], [9], [11].

2 Notations.

Let us introduce a few notations and definitions concerning the geometry of the plate and the rod. Let $\omega$ be a bounded domain in $\mathbb{R}^2$ with lipschitzian boundary included in the plane $(O; e_1, e_2)$ and such that $O \in \omega$. The plate is the domain

$$\Omega_\delta = \omega \times ]-\delta, \delta[.$$

Let $\gamma_0$ be an open subset of $\partial \omega$ which is made of a finite number of connected components (whose closure are disjoint). The corresponding lateral part of the boundary of $\Omega_\delta$ is

$$\Gamma_{0,\delta} = \gamma_0 \times ]-\delta, \delta[.$$
The rod is defined by
\[ B_\delta = D_\delta \times ] - \delta, L[, \quad D_\delta = D(O, \delta), \quad D = D(O, 1) \]
where \( \delta > 0 \) and where \( D_r = D(O, r) \) is the disc of radius \( r \) and center the origin \( O \). We assume that \( D \subset \omega \). The whole structure is denoted
\[ S_\delta = \Omega_\delta \cup B_\delta \]
while the junction is
\[ C_\delta = \Omega_\delta \cap B_\delta = D_\delta \times ] - \delta, \delta[. \]
We denote \( I_d \) the identity map of \( \mathbb{R}^3 \). The set of admissible deformations of the structure is
\[ \mathcal{D}_\delta = \left\{ v \in H^1(S_\delta; \mathbb{R}^3) \mid v = I_d \text{ on } \Gamma_{0,\delta} \right\}. \]
The Euclidian norm in \( \mathbb{R}^k \) (\( k \geq 1 \)) will be denoted \( | \cdot | \) and the Frobenius norm of a square matrix will be denoted \( ||| \cdot ||| \).

3 Some recalls.

To any vector \( F \in \mathbb{R}^3 \) we associate the antisymmetric matrix \( A_F \) defined by
\[ \forall x \in \mathbb{R}^3, \quad A_F x = F \wedge x. \quad (3.1) \]
From now on, in order to simplify the notations, for any open set \( O \subset \mathbb{R}^3 \) and any field \( u \in H^1(O; \mathbb{R}^3) \), we set
\[ G_\delta(u, O) = ||\nabla u + (\nabla u)^T||_{L^2(O; \mathbb{R}^3 \times \mathbb{R}^3)} \]
and
\[ d(u, O) = ||\text{dist}(\nabla u, SO(3))||_{L^2(O)}. \]

3.1 Recalls on the decompositions of the plate-displacement.

We know (see [23] or [24]) that any displacement \( u \in H^1(\Omega_\delta; \mathbb{R}^3) \) of the plate is decomposed as
\[ u(x) = U(x_1, x_2) + x_3 R(x_1, x_2) \wedge e_3 + \overline{u}(x), \quad x \in \Omega_\delta \quad (3.2) \]
where \( U \) is defined by
\[ U(x_1, x_2) = \frac{1}{2\delta} \int_{-\delta}^{\delta} u(x_1, x_2, x_3) dx_3 \quad \text{for a.e. } x_3 \in \omega \]
and where \( R \) is also defined via an average involving the displacement \( u \) (see [23] or [24]). The fields \( U \) and \( R \) belong to \( H^1(\omega; \mathbb{R}^3) \) and \( \overline{u} \) belongs to \( H^1(\Omega_\delta; \mathbb{R}^3) \). The sum
of the two first terms $U_e(x) = U(x_1, x_2) + x_3 R(x_1, x_2) \land e_3$ is called the elementary
displacement associated to $u$.

The following Theorem is proved in [23] for the displacements in $H^1(\Omega_\delta; \mathbb{R}^3)$ and in [24]
for the displacements in $W^{1,p}(\Omega_\delta; \mathbb{R}^3)$ ($1 < p < +\infty$).

**Theorem 3.1.** Let $u \in H^1(\Omega_\delta; \mathbb{R}^3)$, there exists an elementary displacement $U_e(x) =
U(x_1, x_2) + x_3 R(x_1, x_2) \land e_3$ and a warping $\pi$ satisfying (3.2) such that

$$
\|\pi\|_{L^2(\Omega_\delta; \mathbb{R}^3)} \leq C\delta G_s(u, \Omega_\delta), \quad \|\nabla \pi\|_{L^2(\Omega_\delta; \mathbb{R}^3)} \leq CG_s(u, \Omega_\delta),
$$

$$
\|\partial R / \partial x_\alpha\|_{L^2(\omega; \mathbb{R}^3)} \leq \frac{C}{\delta^{3/2}} G_s(u, \Omega_\delta),
$$

$$
\|\partial U / \partial x_\alpha - R \land e_\alpha\|_{L^2(\omega; \mathbb{R}^3)} \leq \frac{C}{\delta^{1/2}} G_s(u, \Omega_\delta),
$$

$$
\|\nabla u - A_R\|_{L^2(\omega; \mathbb{R}^3)} \leq CG_s(u, \Omega_\delta),
$$

(3.3)

where the constant $C$ does not depend on $\delta$.

The warping $\pi$ satisfies the following relations

$$
\int_{-\delta}^{\delta} \pi(x_1, x_2, x_3) dx_3 = 0, \quad \int_{-\delta}^{\delta} x_3 \pi_\alpha(x_1, x_2, x_3) dx_3 = 0 \quad \text{for a.e. } (x_1, x_2) \in \omega.
$$

(3.4)

If a deformation $v$ belongs to $\mathbb{D}_\delta$ then the displacement $u = v - I_d$ is equal to 0 on $\Gamma_{0,\delta}$. In this case the the fields $U$, $R$ and the warping $\pi$ satisfy

$$
U = R = 0 \quad \text{on } \gamma_0, \quad \pi = 0 \quad \text{on } \Gamma_{0,\delta}.
$$

(3.5)

Then, from (3.3), for any deformation $v \in \mathbb{D}_\delta$ the corresponding displacement $u = v - I_d$ verifies the following estimates (see also [23]):

$$
\|R\|_{H^1(\omega; \mathbb{R}^3)} + \|U\|_{H^1(\omega)} \leq \frac{C}{\delta^{3/2}} G_s(u, \Omega_\delta),
$$

$$
\|R\|_{L^2(\omega; \mathbb{R}^3)} + \|U\|_{H^1(\omega)} \leq \frac{C}{\delta^{1/2}} G_s(u, \Omega_\delta).
$$

(3.6)

The constants depend only on $\omega$. From the above estimates we deduce the following
Korn’s type inequalities for the displacement $u$

$$
\|u_\alpha\|_{L^2(\Omega_\delta)} \leq C_0 G_s(u, \Omega_\delta), \quad \|u_3\|_{L^2(\Omega_\delta)} \leq \frac{C_0}{\delta} G_s(u, \Omega_\delta),
$$

$$
\|u - U\|_{L^2(\Omega_\delta; \mathbb{R}^3)} \leq \frac{C}{\delta} G_s(u, \Omega_\delta), \quad \|\nabla u\|_{L^2(\Omega_\delta; \mathbb{R}^3)} \leq \frac{C}{\delta} G_s(u, \Omega_\delta).
$$

(3.7)

Due to Theorem 3.3 established in [8], the displacement $u = v - I_d$ is also decomposed as

$$
u(x) = U(x_1, x_2) + x_3 (R(x_1, x_2) - I_3) e_3 + \bar{u}(x), \quad x \in \Omega_\delta
$$

(3.8)
where \( R \in H^1(\omega; \mathbb{R}^{3 \times 3}) \), \( \overline{\mathbf{u}} \in H^1(\Omega_\delta; \mathbb{R}^3) \) and we have the following estimates

\[
\begin{align*}
|\overline{\mathbf{u}}|_{L^2(\Omega_\delta; \mathbb{R}^3)} & \leq C\delta \mathbf{d}(v, \Omega_\delta) \\
|\nabla \overline{\mathbf{u}}|_{L^2(\Omega_\delta; \mathbb{R}^9)} & \leq C \mathbf{d}(v, \Omega_\delta)
\end{align*}
\]

\[
\begin{align*}
\left\| \frac{\partial R}{\partial x_\alpha} \right\|_{L^2(\mathbb{R}^3)} & \leq \frac{C}{\delta^{3/2}} \mathbf{d}(v, \Omega_\delta) \\
\left\| \frac{\partial U}{\partial x_\alpha} - (R - I_3)e_\alpha \right\|_{L^2(\mathbb{R}^3)} & \leq \frac{C}{\delta^{1/2}} \mathbf{d}(v, \Omega_\delta) \\
\left\| \nabla v - R \right\|_{L^2(\Omega_\delta; \mathbb{R}^9)} & \leq C \mathbf{d}(v, \Omega_\delta)
\end{align*}
\]  

(3.9)

where the constant \( C \) does not depend on \( \delta \). The following boundary conditions are satisfied

\[
\begin{align*}
U = 0, & \quad R = I_3 \quad \text{on } \gamma_0, \\
\overline{\mathbf{u}} = 0 \quad & \text{on } \Gamma_{0,\delta}.
\end{align*}
\]  

(3.10)

Due to (3.9) and the above boundary conditions we obtain

\[
\left\| R - I_3 \right\|_{H^1(\mathbb{R}^3)} + \left\| U \right\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\delta^{3/2}} \mathbf{d}(v, \Omega_\delta). 
\]  

(3.11)

### 3.2 Recall on the decomposition of the rod-deformation.

Now, we consider a deformation \( v \in H^1(B_\delta; \mathbb{R}^3) \) of the rod \( B_\delta \). This deformation can be decomposed as (see Theorem 2.2.2 of [7])

\[
v(x) = \mathcal{V}(x_3) + Q(x_3)(x_1 e_1 + x_2 e_2) + \overline{\mathbf{u}}(x), \quad x \in B_\delta,
\]  

(3.12)

where \( \mathcal{V}(x_3) = \frac{1}{|D_\delta|} \int_{D_\delta} v(x)dx_1dx_2 \) belongs to \( H^1(-\delta, L; \mathbb{R}^3) \), where \( Q \) belongs to \( H^1(-\delta, L; SO(3)) \) and \( \overline{\mathbf{u}} \) belongs to \( H^1(B_\delta; \mathbb{R}^3) \). Let us give a few comments on the above decomposition. The term \( \mathcal{V} \) gives the deformation of the center line of the rod. The second term \( Q(x_3)(x_1 e_1 + x_2 e_2) \) describes the rotation of the cross section (of the rod) which contains the point \((0, 0, x_3)\). The sum of the terms \( \mathcal{V}(x_3) + Q(x_3)(x_1 e_1 + x_2 e_2) \) is called an elementary deformation of the rod.

The following theorem (see Theorem 2.2.2 of [7]) gives a decomposition (3.12) of a deformation and estimates on the terms of this decomposition.

**Theorem 3.2.** Let \( v \in H^1(B_\delta; \mathbb{R}^3) \), there exists an elementary deformation \( \mathcal{V}(x_3) + Q(x_3)(x_1 e_1 + x_2 e_2) \) and a warping \( \overline{\mathbf{u}} \) satisfying (3.12) and such that

\[
\begin{align*}
\left\| \overline{\mathbf{u}} \right\|_{L^2(B_\delta; \mathbb{R}^3)} & \leq C\delta \mathbf{d}(v, B_\delta), \\
\left\| \nabla \overline{\mathbf{u}} \right\|_{L^2(B_\delta; \mathbb{R}^9)} & \leq C \mathbf{d}(v, B_\delta), \\
\left\| \frac{dQ}{dx_3} \right\|_{L^2(-\delta, L; \mathbb{R}^{3 \times 3})} & \leq \frac{C}{\delta^2} \mathbf{d}(v, B_\delta), \\
\left\| \frac{d\mathcal{V}}{dx_3} - Qe_3 \right\|_{L^2(-\delta, L; \mathbb{R}^3)} & \leq \frac{C}{\delta} \mathbf{d}(v, B_\delta), \\
\left\| \nabla v - Q \right\|_{L^2(B_\delta; \mathbb{R}^{3 \times 3})} & \leq C \mathbf{d}(v, B_\delta),
\end{align*}
\]  

(3.13)

where the constant \( C \) does not depend on \( \delta \) and \( L \).
4 Preliminaries results

Let \( v \) be a deformation in \( \mathbb{D}_\delta \). We set \( u = v - I_d \). We decompose \( u \) as \((3.2)\) and \((3.8)\) in the plate and we decompose the deformation \( v \) as \((3.12)\) in the rod.

4.1 A complement to the mid-surface bending.

Let us set

\[
H^1_{\gamma_0}(\omega) = \{ \varphi \in H^1(\omega) ; \varphi = 0 \text{ on } \gamma_0 \}.
\]

We define the function \( \tilde{U}_3 \) as the solution of the following variational problem:

\[
\begin{align*}
\tilde{U}_3 & \in H^1_{\gamma_0}(\omega), \\
\int_{\omega} \nabla \tilde{U}_3 \nabla \varphi &= \int_{\omega} (R - I_3) e_3 \cdot e_3 \frac{\partial \varphi}{\partial x_3}, \\
\forall \varphi & \in H^1(\omega)
\end{align*}
\]

(4.1)

where \( R \) appears in the decomposition \((3.8)\) of \( u \). Due to \((3.9)-(3.11)\), the function \( \tilde{U}_3 \) belongs to \( H^1_{\gamma_0}(\omega) \cap H^2(D) \) (remind that \( D \) is the disc of radius 1 and center the origin \( O \); and we assumed that \( D \subset \subset \omega \)). The function \( \tilde{U}_3 \) satisfies the estimates:

\[
\begin{align*}
||\tilde{U}_3||_{H^1(\omega)} & \leq C \frac{\delta}{\delta^{3/2}} d(v, \Omega_\delta), \\
||\tilde{U}_3 - \tilde{U}_3||_{H^1(\omega)} & \leq C \frac{\delta}{\delta^{3/2}} d(v, \Omega_\delta), \\
||\tilde{U}_3||_{H^2(D)} & \leq C \frac{\delta}{\delta^{3/2}} d(v, \Omega_\delta), \\
|\tilde{U}_3(0,0)| & \leq C \frac{\delta}{\delta^{3/2}} d(v, \Omega_\delta).
\end{align*}
\]

(4.2)

The constants do not depend on \( \delta \).

4.2 A complement to the rod center-line displacement.

Let \( \mathcal{V} \) given by \((3.12)\), we consider \( \mathcal{W}(x_3) = \mathcal{V}(x_3) - x_3 e_3 = \frac{1}{|D_\delta|} \int_{D_\delta} u(x) dx_1 dx_2 \) the rod center-line displacement. From the above Theorem \(3.2\) the estimate below holds true

\[
\left\| \frac{d\mathcal{W}}{dx_3} - (Q - I_3) e_3 \right\|_{L^2(-\delta,L;\mathbb{R}^3)} \leq C \frac{\delta}{\delta} d(v, B_\delta). \tag{4.3}
\]

As in [7] we split the center line displacement \( \mathcal{W} \) into two parts. The first one \( \mathcal{W}^{(m)} \) stands for the main displacement of the rod which describes the displacement coming from the bending and the second one for the stretching of the rod.

\[
\forall x_3 \in [0, L], \quad \mathcal{W}^{(m)}(x_3) = \mathcal{W}(0) + \int_0^{x_3} (Q(t) - I_3) e_3 dt, \tag{4.4}
\]

\[
\mathcal{W}^{(s)}(x_3) = \mathcal{W}(x_3) - \mathcal{W}^{(m)}(x_3).
\]
In the lemma below we give estimates on $\mathcal{W}^{(s)}$ and $\mathcal{W}^{(m)}$.

**Lemma 4.1.** We have

$$||\mathcal{W}^{(s)}||_{H^1(-\delta,L;\mathbb{R}^3)} \leq \frac{C}{\delta} d(v, B_\delta),$$

(4.5)

and

$$||\mathcal{W}^{(m)}_a - \mathcal{W}_a(0)||_{H^2(-\delta,L)} \leq \frac{C}{\delta^2} d(v, B_\delta) + C||Q(0) - I_3||,$$

(4.6)

$$||\mathcal{W}^{(m)}_3 - \mathcal{W}_3^{(m)}(0)||_{H^1(-\delta,L)} \leq \frac{C}{\delta^4} \left[d(v, B_\delta)\right]^2 + C||Q(0) - I_3||,$$

(4.7)

$$||\frac{d\mathcal{W}^{(m)}_3}{dx_3}||_{L^2(-\delta,\delta)} \leq \frac{C}{\delta^{5/2}} \left[d(v, B_\delta)\right]^2 + C\delta^{1/2}||Q(0) - I_3||.$$

The constants do not depend on $\delta$.

**Proof.** Taking into account the facts that $\mathcal{W}^{(s)}(0) = 0$ and $\frac{d\mathcal{W}^{(s)}(s)}{dx_3} = \frac{d\mathcal{W}(s)}{dx_3} - (Q - I_3)e_3$, the estimate (4.3) leads to (4.5). From the third estimate in (3.13) we obtain

$$||Q - Q(0)||_{L^2(-\delta,L;\mathbb{R}^{3x3})} \leq \frac{C}{\delta^2} d(v, B_\delta),$$

(4.8)

Due to the definition (4.4) of $\mathcal{W}^{(m)}$ and estimate (3.13) we get

$$||\frac{d\mathcal{W}^{(m)}_3}{dx_3}||_{H^1(-\delta,L)} \leq \frac{C}{\delta^2} d(v, B_\delta) + C||Q(0) - I_3||$$

and thus (4.6). A straightforward calculation gives

$$\frac{d\mathcal{W}^{(m)}_3}{dx_3} = (Q - I_3)e_3 \cdot e_3 = -\frac{1}{2}||Q - I_3||^2.$$ 

(4.9)

Besides we have

$$\frac{d}{dx_3}((Q - I_3)e_3) = \frac{dQ}{dx_3}e_3.$$ 

(4.10)

We recall that for $\phi \in H^1(0,L)$ and $\eta \in [0,L]$ we have

$$\int_0^\eta |\phi(t) - \phi(0)||^2 dt \leq \frac{\eta^2}{2} \left||\frac{d\phi}{dt}\right||_{L^2(0,L)}^2, \quad \int_0^\eta |\phi(t) - \phi(0)||^4 dt \leq \frac{\eta^3}{3} \left||\frac{d\phi}{dt}\right||_{L^2(0,L)}^4.$$ 

(4.11)

Then, the estimates (3.13) and (4.11) and the equality (4.10) give

$$||Q - I_3||e_3 - (Q(0) - I_3)e_3||_{L^2(-\delta,L;\mathbb{R}^{3x3})} \leq \frac{C}{\delta^2} d(v, B_\delta),$$

(4.12)

$$||Q - I_3||e_3 - (Q(0) - I_3)e_3||_{L^2(-\delta,\delta;\mathbb{R}^3)} \leq \frac{C}{\delta} d(v, B_\delta).$$
Now, again (3.13) and (4.11) lead to
\[
\|(Q - I_3)e_3 - (Q(0) - I_3)e_3\|_{L^4(-\delta, \delta; \mathbb{R}^3)} \leq \frac{C}{\delta^2} d(v, B_\delta),
\]
\[
\|(Q - I_3)e_3 - (Q(0) - I_3)e_3\|_{L^4(-\delta, \delta; \mathbb{R}^3)} \leq \frac{C}{\delta^{5/4}} d(v, B_\delta).
\] (4.13)

Finally, from (4.9) and the above inequality we obtain (4.7). \(\square\)

### 4.3 First estimates in the junction area.

**Lemma 4.2.** We have the following estimate on \(Q(0) - I_3\):

\[
|\langle Q(0) - I_3 \rangle e_3 \cdot e_3 | \leq \frac{C}{\delta^{3/2}} (G_s(u, \Omega_\delta) + d(v, B_\delta)),
\]
\[
|||Q(0) - I_3||| \leq \frac{C}{\delta^{7/4}} G_s(u, \Omega_\delta) + \frac{C}{\delta^{3/2}} d(v, B_\delta)
\] (4.14)

and those about \(W(0)\)

\[
|W_\alpha(0)| \leq \frac{C}{\delta^{3/4}} G_s(u, \Omega_\delta) + \frac{C}{\delta^{1/2}} d(v, B_\delta)
\] (4.15)

and

\[
|W_3(0) - \tilde{U}_3(0, 0)| \leq \frac{C}{\delta^2} [d(v, B_\delta)]^2 + \frac{C}{\delta^{1/2}} (d(v, B_\delta) + G_s(u, \Omega_\delta)) + \frac{C}{\delta} d(v, \Omega_\delta),
\]
\[
|W_3(0)| \leq \frac{C}{\delta^{3/2}} d(v, \Omega_\delta) + \frac{C}{\delta^2} [d(v, B_\delta)]^2 + \frac{C}{\delta^{1/2}} (d(v, B_\delta) + G_s(u, \Omega_\delta)).
\] (4.16)

The constants are independent of \(\delta\).

**Proof.** Step 1. We prove the estimate on \(Q(0) - I_3\). We consider the last inequalities in Theorems 3.1 and 3.2. They give

\[
\|Q - I_3 - A_R\|_{L^2(\mathbb{R}^3)} \leq C (G_s(u, \Omega_\delta) + d(v, B_\delta)).
\] (4.17)

Now, from the third estimate in (3.13), we get

\[
\|Q - Q(0)\|_{L^2(-\delta, \delta; \mathbb{R}^3)} \leq C \delta \left\| \frac{dQ}{dx} \right\|_{L^2(-\delta, \delta; \mathbb{R}^3)} \leq \frac{C}{\delta} d(v, B_\delta).
\]

Hence

\[
\|Q(0) - I_3 - A_R\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C}{\delta} \left( [G_s(u, \Omega_\delta)]^2 + [d(v, B_\delta)]^2 \right).\] (4.18)

We recall that the matrix \(A_R\) is antisymmetric, then (4.18) leads to the first estimate in (4.14). Due to (3.6) we have

\[
||R||_{L^2(D_\delta; \mathbb{R}^3)} \leq C \delta^3 ||R||_{L^4(D_\delta; \mathbb{R}^3)} \leq C \delta^3 ||R||_{H^1(\omega; \mathbb{R}^3)} \leq \frac{C}{\delta^3} [G_s(u, \Omega_\delta)]^4.
\] (4.19)
Then, using the above estimate and \((4.18)\) we deduce the second estimate in \((4.14)\).

**Step 2. We prove the estimate \((4.15)\) on \(W_\alpha(0)\).**

The two decompositions of \(u = v - I_d\) \((3.2)\) and \((3.12)\) give, for a.e. \(x \in C_\delta\)

\[
\mathcal{U}(x_1, x_2) + x_3 \mathcal{R}(x_1, x_2) \wedge e_3 + \bar{u}(x) \\
= \mathcal{W}(x_3) + (Q(x_3) - I_3)(x_1 e_1 + x_2 e_2) + \bar{v}(x).
\]  

(4.20)

Taking the averages on the cylinder \(C_\delta\) of the terms in this equality \((4.20)\) give

\[
\mathcal{M}_{D_\delta}(\mathcal{U}) = \frac{1}{|D_\delta|} \int_{D_\delta} \mathcal{U}(x_1, x_2) dx_1 dx_2 = \mathcal{M}_{I_\delta}(\mathcal{W}) = \frac{1}{2\delta} \int_{-\delta}^{\delta} \mathcal{W}(x_3) dx_3.
\]  

(4.21)

Besides, proceeding as for \(\mathcal{R}\) in \((4.19)\) and from \((3.6)\) we have

\[
||U_\alpha||_{L^2(D_\delta)} \leq C\delta^{-1/4}G_s(u, \Omega_\delta).
\]

From this estimate we get

\[
|M_{I_\delta}(W_\alpha)| = |M_{D_\delta}(U_\alpha)| \leq C \delta^{-1/4}G_s(u, \Omega_\delta). \quad (4.22)
\]

We set \(y_\alpha(x_3) = W_\alpha(x_3) - x_3(Q(0) - I_3)e_3 \cdot e_\alpha\). The estimates \((4.3)\) and \((4.12)\) lead to

\[
\left\| \frac{dy_\alpha}{dx_3} \right\|_{L^2(-\delta, \delta)} \leq \frac{C}{\delta} d(v, B_\delta)
\]

which in turn implies

\[
\|y_\alpha - y_\alpha(0)\|_{L^2(-\delta, \delta)} \leq C d(v, B_\delta).
\]

Taking the average, it yields

\[
|M_{I_\delta}(W_\alpha) - W_\alpha(0)| \leq C \delta^{1/2} d(v, B_\delta). \quad (4.23)
\]

Finally, from \((4.22)\) and \((4.23)\) we obtain \((4.15)\).

**Step 3. We prove the estimate on \(W_3(0)\).** Using \((4.2)\) we deduce that

\[
||U_3 - \tilde{U}_3||_{L^2(D_\delta)} \leq C\delta^{1/2}||U_3 - \tilde{U}_3||_{L^1(\omega)} \\
\leq C\delta^{1/2}||U_3 - \tilde{U}_3||_{H^1(\omega)} \leq C d(v, \Omega_\delta). \quad (4.24)
\]

Then we replace \(U_3\) with \(\tilde{U}_3\) and \(W_3\) with \(W_3^{(m)}\) in \((4.21)\). Taking into account \((4.5)\) we obtain

\[
|M_{D_\delta}(\tilde{U}_3) - M_{I_\delta}(W_3^{(m)})| \leq \frac{C}{\delta} d(v, \Omega_\delta) + \frac{C}{\delta^{1/2}} d(v, B_\delta). \quad (4.25)
\]

We carry on by comparing \(M_{D_\delta}(\tilde{U}_3)\) with \(\tilde{U}_3(0, 0)\). Let us set

\[
r_\alpha = \frac{1}{\pi \delta^2} \int_{D_\delta} (R(x_1, x_2) - I_3) e_\alpha \cdot e_3 \, dx_1 dx_2
\]
and consider the function $\Psi(x_1, x_2) = \tilde{U}_3(x_1, x_2) - M_{D_\delta}(\tilde{U}_3) - x_1 r_2 - x_2 r_1$. Due to (4.2) we first obtain

$$\left\| \frac{\partial^2 \Psi}{\partial x_\alpha \partial x_\beta} \right\|_{L^2(D_\delta, \mathbb{R}^3)} \leq \frac{C}{\delta^{3/2}} d(v, \Omega_\delta).$$

(4.26)

Then, applying twice the Poincaré-Wirtinger inequality in the disc $D_\delta$ and using (3.3) and the fourth estimate in (4.2) lead to

$$\left\| \nabla \Psi \right\|_{L^2(D_\delta, \mathbb{R}^6)}^2 \leq \frac{C}{\delta} [d(v, \Omega_\delta)]^2, \quad \left\| \Psi \right\|_{L^2(D_\delta, \mathbb{R}^3)}^2 \leq C \delta [d(v, \Omega_\delta)]^2. \quad (4.27)$$

From the above inequalities (4.26) and (4.27) we deduce that

$$\left\| \Psi \right\|_{L^\infty(D_\delta, \mathbb{R}^3)} \leq \frac{C}{\delta^{1/2}} d(v, \Omega_\delta) \implies \left| \Psi(0, 0) \right| = \left| \tilde{U}_3(0, 0) - M_{D_\delta}(\tilde{U}_3) \right| \leq \frac{C}{\delta^{1/2}} d(v, \Omega_\delta).$$

From this last estimate and (4.25) we obtain

$$\left| \tilde{U}_3(0, 0) - M_{I_3}(\mathcal{W}_3^{(m)}) \right| \leq \frac{C}{\delta} d(v, \Omega_\delta) + \frac{C}{\delta} d(v, B_\delta). \quad (4.28)$$

Then using the second estimate in (4.7) and (4.14) we have

$$\left\| \frac{d\mathcal{W}_3^{(m)}}{dx_3} \right\|_{L^2(-\delta, \delta)} \leq \frac{C}{\delta^{3/2}} [d(v, B_\delta)]^2 + \frac{C}{\delta} \left( G_s(u, \Omega_\delta) + d(v, B_\delta) \right). \quad (4.29)$$

Finally, recalling that $\mathcal{W}_3(0) = \mathcal{W}_3^{(m)}(0)$, the above inequality leads to

$$\left| M_{I_3}(\mathcal{W}_3^{(m)}) - \mathcal{W}_3(0) \right| \leq C \delta^{1/2} \left\| \frac{d\mathcal{W}_3^{(m)}}{dx_3} \right\|_{L^2(-\delta, \delta)} \leq \frac{C}{\delta^2} [d(v, B_\delta)]^2 + \frac{C}{\delta^{1/2}} \left( G_s(u, \Omega_\delta) + d(v, B_\delta) \right)$$

which in turn with (4.28) and (4.2) lead to (4.16).

\[\square\]

4.4 Global estimates of $u$: Korn’s type inequality.

Now, we give the last estimates of the displacement $u = v - I_d$ in the rod $B_\delta$.

**Lemma 4.3.** For any deformation $v$ in $\mathcal{D}_\delta$ we have the following inequalities for the displacement $u = v - I_d$ in the rod $B_\delta$:

$$\left\| u - \mathcal{W} \right\|_{L^2(B_\delta; \mathbb{R}^3)} \leq C \left( d(v, B_\delta) + \delta^{1/4} G_s(u, \Omega_\delta) \right),$$

$$\left\| \mathcal{W}_3 \right\|_{L^2(-\delta, L)} + \left\| \mathcal{W}_3^{(m)} \right\|_{L^2(-\delta, L)} \leq \frac{C}{\delta^{3/4}} \left( d(v, B_\delta) + G_s(u, \Omega_\delta) \right),$$

$$\left\| \mathcal{W}_3 \right\|_{L^2(-\delta, L)} + \left\| \mathcal{W}_3^{(m)} \right\|_{L^2(-\delta, L)} \leq \frac{C}{\delta^3} \left[ d(v, B_\delta) \right]^2 + \frac{C}{\delta^{3/2}} \left[ G_s(u, \Omega_\delta) + d(v, \Omega_\delta) + d(v, B_\delta) \right]. \quad (4.30)$$

The constants do not depend on $\delta$. 

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Proof. From (4.8) and (4.14) we get
\[ \| Q - I_3 \|_{L^2(-\delta,L;\mathbb{R}^{3\times 3})} \leq C \left( \frac{d(v,B_\delta)}{\delta^2} + \frac{G_s(u,\Omega_\delta)}{\delta^{3/4}} \right). \tag{4.31} \]

Then, from (3.13) and the above inequality we deduce that
\[ \| u - W \|_{L^2(B_\delta;\mathbb{R}^3)} \leq C (d(v,B_\delta) + \delta^{1/4} G_s(u,\Omega_\delta)). \tag{4.32} \]

From (4.6) again (4.14) and (4.15) we obtain
\[ \| W^{(m)} \|_{H^1(-\delta,L)} \leq C \left( d(v,B_\delta) + \delta^{3/2} G_s(u,\Omega_\delta) + d(v,B_\delta) \right). \tag{4.33} \]

Then since \( W = W^{(m)} + W^{(s)} \), (4.33) and (4.34) give the second estimate in (4.30).

From (4.7) and (4.14) we deduce that
\[ \left\| \frac{dW^{(m)}}{dx_3} \right\|_{L^2(-\delta,L)} \leq C \left[ d(v,B_\delta) \right]^2 + C \left[ \delta^{3/2} \left( G_s(u,\Omega_\delta) + d(v,B_\delta) \right) \right]. \]

which in turn using (4.16) lead to
\[ \| W^{(m)}_3 \|_{L^2(-\delta,L)} \leq C \left[ d(v,B_\delta) \right]^2 + C \left[ \delta^{3/2} \left( G_s(u,\Omega_\delta) + d(v,B_\delta) \right) \right]. \]

and then due to (4.5) we get the last estimate in (4.30). \( \square \)

Corollary 4.4. For any deformation \( v \) in \( \mathbb{D}_\delta \) we have the following Korn’s type inequality for the displacement \( u = v - I_d \) in the rod \( B_\delta \):
\[ \left\| \nabla u \right\|_{L^2(B_\delta;\mathbb{R}^{3\times 3})} \leq C \frac{d(v,B_\delta)}{\delta} + C \frac{G_s(u,\Omega_\delta)}{\delta^{3/4}}, \]
\[ \left\| u_\alpha \right\|_{L^2(B_\delta)} \leq C \frac{d(v,B_\delta)}{\delta} + C \frac{G_s(u,\Omega_\delta)}{\delta^{3/4}}, \]
\[ \left\| u_3 \right\|_{L^2(B_\delta)} \leq C \left[ \frac{d(v,B_\delta)}{\delta^3} \right]^2 + C \left[ \delta^{1/2} \left( G_s(u,\Omega_\delta) + d(v,B_\delta) \right) \right]. \tag{4.34} \]

The constants do not depend on \( \delta \).

Proof. From (3.13) and (4.31) we obtain
\[ \left\| \nabla u \right\|_{L^2(B_\delta;\mathbb{R}^{3\times 3})} \leq C \frac{d(v,B_\delta)}{\delta} + C \frac{G_s(u,\Omega_\delta)}{\delta^{3/4}}. \tag{4.35} \]

The second and third inequalities are immediate consequences of Lemma 4.3. \( \square \)
5 Elastic structure

5.1 Elastic energy.

In this section we assume that the structure $S_\delta$ is made of an elastic material. The associated local energy $\hat{W} : X_3 \rightarrow \mathbb{R}^+$ is the following St Venant-Kirchhoff’s law\(^2\) (see also [14])

\[
\hat{W}(F) = \begin{cases} 
Q(F^T F - I_3) & \text{if } \det(F) > 0 \\
+\infty & \text{if } \det(F) \leq 0.
\end{cases} \tag{5.1}
\]

where $X_3$ is the space of $3 \times 3$ symmetric matrices and where the quadratic form $Q$ is given by

\[
Q(E) = \frac{\lambda}{8}(tr(E))^2 + \frac{\mu}{4}tr(E^2), \tag{5.2}
\]

and where $(\lambda, \mu)$ are the Lamé’s coefficients of the material. Let us recall (see e.g. [19] or [7]) that for any $3 \times 3$ matrix $F$ such that $\det(F) > 0$ we have

\[
[tr(F^T F - I_3)]^2 = |||F^T F - I_3|||^2 \geq \text{dist } (F, SO(3))^2. \tag{5.3}
\]

5.2 Assumptions on the forces and final estimates.

Now we assume that the structure $S_\delta$ is submitted to applied body forces $f_\delta \in L^2(S_\delta; \mathbb{R}^3)$ and we define the total energy $J_\delta(v)$\(^3\) over $D_\delta$ by

\[
J_\delta(v) = \int_{S_\delta} \hat{W}(\nabla v)(x)dx - \int_{S_\delta} f_\delta(x) \cdot (v(x) - I_d(x))dx. \tag{5.4}
\]

Assumptions on the forces. To introduce the scaling on $f_\delta$, let us consider $f_r, g_1, g_2$ in $L^2(0, L; \mathbb{R}^3)$ and $f_p \in L^2(\omega; \mathbb{R}^3)$. We assume that the force $f_\delta$ is given by

\[
f_\delta(x) = \delta^{5/2}\left[f_{r,1}(x_3)e_1 + f_{r,2}(x_3)e_2 + \frac{1}{\delta^{1/2}}f_{r,3}(x_3)e_3 + \frac{x_1}{\delta^2}g_1(x_3) + \frac{x_2}{\delta^2}g_2(x_3)\right]
\]

$x \in B_\delta, \quad x_3 > \delta,$

\[
f_{\delta,\alpha}(x) = \delta^2 f_{p,\alpha}(x_1, x_2), \quad f_{\delta,3}(x) = \delta^3 f_{p,3}(x_1, x_2), \quad x \in \Omega_\delta.
\]

We denote

\[
F_{r,3}(x_3) = \int_{x_3}^{L} f_{r,3}(s)ds, \quad \text{for a. e. } x_3 \in ]0, L[. \tag{5.6}
\]

\(^2\)With a more general assumption on the nonlinear elasticity law (see for example [19] page 1466) we would obtain the same asymptotic behavior as in our case.

\(^3\)For later convenience, we have added the term $\int_{S_\delta} f_\delta(x) \cdot I_d(x)dx$ to the usual standard energy, indeed this does not affect the minimizing problem for $J_\delta$. 

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Theorem 5.1. There exist two constants $C_0$ and $C_1$, which depend only on $\omega$, $L$ and $\mu$, such that if
\[ \|f_p\|_{L^2(\Omega;\mathbb{R}^3)} \leq C_0 \] (5.7)
and if either
\[ \text{Case 1: for a. e. } x_3 \in [0, L[$, \quad F_{r,3}(x_3) \geq 0, \]
or
\[ \text{Case 2: } \|f_{r,3}\|_{L^2(0,L)} \leq C_1 \]
then for $\delta$ small enough and for any $v \in D_\delta$ satisfying $J_\delta(v) \leq 0$ we have
\[ d(v, B_\delta) + d(v, \Omega_\delta) \leq C\delta^{5/2} \] (5.9)
where the constant does not depend on $\delta$.

Proof. From (3.7) and the assumptions (5.5) on the body forces, we obtain on the one hand for any $v \in D_\delta$ and with $u = v - I_d$
\[ \left| \int_{\Omega_\delta} f_\delta(x) \cdot u(x) \, dx \right| \leq C\delta^{5/2} \|f_p\|_{L^2(\omega;\mathbb{R}^3)} G_s(u, \Omega_\delta). \] (5.10)
As far as the term involving the forces in the rod are concerned we first have
\[ \int_{B_\delta} f_\delta(x) \cdot u(x) \, dx = \pi \delta^{3/2} \int_\delta^L f_{r,\alpha}(x_3)W_\alpha(x_3) \, dx_3 + \pi \delta^4 \int_\delta^L f_{r,3}(x_3)W_3(x_3) \, dx_3 \]
\[ + \int_{B_\delta} f_\delta(x) \cdot (u(x) - W(x_3)) \, dx. \]
Then, using Lemma 4.3 and (5.5) we first get
\[ \left| \int_\delta^L f_{r,\alpha}(x_3)W_\alpha(x_3) \, dx_3 \right| \leq \frac{C}{\delta^2} \sum_{\alpha=1}^2 \|f_{r,\alpha}\|_{L^2(0,L)} \left( d(v, B_\delta) + \delta^{1/4} G_s(u, \Omega_\delta) \right), \]
\[ \left| \int_{B_\delta} f_\delta(x) \cdot (u(x) - W(x_3)) \, dx \right| \leq C\delta^{5/2} \left( \|g_1\|_{L^2(0,L;\mathbb{R}^3)} + \|g_2\|_{L^2(0,L;\mathbb{R}^3)} \right) \]
\[ \left( d(v, B_\delta) + \delta^{1/4} G_s(u, \Omega_\delta) \right) \] (5.11)
Now we estimate $\int_\delta^L f_{r,3}W_3(x_3) \, dx_3$. From (4.5) we first obtain
\[ \left| \int_\delta^L f_{r,3}(x_3)W_3^{(s)}(x_3) \, dx_3 \right| \leq \frac{C}{\delta} \|f_{r,3}\|_{L^2(0,L)} d(v, B_\delta). \] (5.12)
Then we have
\[ \int_\delta^L f_{r,3}(x_3)W_3^{(m)}(x_3) \, dx_3 = F_{r,3}(\delta)W_3^{(m)}(\delta) + \int_\delta^L f_{r,3}(x_3) \frac{dW_3^{(m)}}{dx_3}(x_3) \, dx_3. \] (5.13)
Taking to account (4.29) and (4.16) we get

$$|W_3^{(m)}(\delta)| \leq |W_3(0)| + \delta^{1/2} \left\| \frac{dW_3^{(m)}}{dx_3} \right\|_{L^2(-\delta, \delta)} + \left( \frac{C}{\delta^{3/2}} d(v, \Omega_\delta) + \frac{C}{\delta^{1/2}} (G_s(u, \Omega_\delta) + d(v, B_\delta)) \right).$$  \hfill (5.14)

Observe now that due to the expression (4.9) of \( \frac{dW_3^{(m)}}{dx_3} \), this derivative is nonpositive for a.e. \( x_3 \in ]0, L[ \) (see (4.9)).

- If we are in Case 1 in (7.34), we have

$$\int_\delta^L f_{r,3}(x_3) W_3(x_3) dx_3 \leq C |f_{r,3}|_{L^2(0,L)} \left[ \frac{d(v, \Omega_\delta)}{\delta^{3/2}} + \left( \frac{d(v, B_\delta)}{\delta^2} + \frac{G_s(u, \Omega_\delta) + d(v, B_\delta)}{\delta^{1/2}} \right) \right].$$

Hence, we obtain

$$\int_{B_\delta} f_\delta(x) \cdot u(x) dx \leq C \delta^{5/2} \sum_{\alpha=1}^2 \left( |f_{r,\alpha}|_{L^2(0,L)} + |g_\alpha|_{L^2(0,L; \mathbb{R}^3)} \right) \left( d(v, B_\delta) + \delta^{1/4} G_s(u, \Omega_\delta) \right) + C |f_{r,3}|_{L^2(0,L)} \left[ \delta^{5/2} d(v, \Omega_\delta) + \delta^2 [d(v, B_\delta)]^2 \right]^{\frac{3}{2}} + \delta^{7/2} (G_s(u, \Omega_\delta) + d(v, B_\delta)) \right].$$

We recall that (see [8])

$$G_s(u, \Omega_\delta) \leq C d(v, \Omega_\delta) + \frac{C}{\delta^{5/2}} [d(v, \Omega_\delta)]^2$$  \hfill (5.15)

where the constant does not depend on \( \delta \). Then due to (5.10) and the above inequalities we obtain that

$$\int_{\Omega_\delta} f_\delta(x) \cdot u(x) dx \leq C |f_r|_{L^2(\omega; \mathbb{R}^3)} \delta^{5/2} d(v, \Omega_\delta) + C^* |f_r|_{L^2(\omega; \mathbb{R}^3)} \left[ d(v, \Omega_\delta) \right]^2$$

$$\int_{B_\delta} f_\delta(x) \cdot u(x) dx \leq C(f_r, g_1, g_2) \delta^{5/2} \left( d(v, B_\delta) + d(v, \Omega_\delta) \right)$$

$$+ C(f_r, g_1, g_2) \delta^{1/4} \left[ d(v, \Omega_\delta) \right]^2 + C |f_{r,3}|_{L^2(0,L)} \delta^2 \left[ d(v, B_\delta) \right]^2.$$  \hfill (5.16)

Now, for any \( v \in \mathbb{D}_\delta \) such that \( J_\delta(v) \leq 0 \), assumptions (5.1), (5.2),(5.3) and the above estimates lead to

$$\frac{\mu}{8} \left[ [d(v, B_\delta)]^2 + [d(v, \Omega_\delta)]^2 \right] \leq \int_{S_\delta} \hat{W}((\nabla v)(x) dx \leq \int_{S_\delta} f_\delta(x) \cdot u(x) dx$$

$$\leq C(f_r, g_1, g_2) \delta^{5/2} \left( d(v, B_\delta) + d(v, \Omega_\delta) \right) + C(f_r, g_1, g_2) \delta^{1/4} \left[ d(v, \Omega_\delta) \right]^2$$

$$+ C |f_{r,3}|_{L^2(0,L)} \delta^2 \left[ d(v, B_\delta) \right]^2 + C \delta^{5/2} |f_p|_{\mathbb{R}^3} d(v, \Omega_\delta) + C^* |f_p|_{\mathbb{R}^3} \left[ d(v, \Omega_\delta) \right]^2.$$
which in turn gives
\[
\left( \frac{\mu}{8} - C ||f_{r,3}||_{L^2(0,L)} \delta^2 \right) \left[ d(v, B_\delta)^2 + \left( \frac{\mu}{8} - C^* ||f_p||_{\omega; \mathbb{R}^3} - C(f_r, g_1, g_2) \delta^{1/4} \right) \left[ d(v, \Omega_\delta) \right]^2 \right] \\
\leq C(f_r, g_1, g_2) \delta^{5/2} \left( d(v, B_\delta) + d(v, \Omega_\delta) \right) + C \delta^{5/2} ||f_p||_{\omega; \mathbb{R}^3} d(v, \Omega_\delta).
\]

Indeed the two quantities \( C ||f_{r,3}||_{L^2(0,L)} \delta^2 \) and \( C(f_r, g_1, g_2) \delta^{1/4} \) tend to 0 as \( \delta \) tends to 0, then, under the condition \( C^* ||f_p||_{L^2(\omega; \mathbb{R}^3)} \leq \mu/32 \) and for \( \delta \) small enough we obtain
\[
d(v, B_\delta) + d(v, \Omega_\delta) \leq C \delta^{5/2}.
\]

The constant does not depend on \( \delta \).

- If we are in Case 2 in (7.34), from (4.30) we immediately have
\[
\int_{\delta}^{L} f_{r,3}(x_3) \mathcal{W}_3(x_3) dx_3 \leq C ||f_{r,3}||_{L^2(0,L)} \left[ \frac{[d(v, B_\delta)]^2}{\delta^4} + \frac{G_s(u, \Omega_\delta) + d(v, \Omega_\delta) + d(v, B_\delta)}{\delta^{3/2}} \right].
\]

Then, proceeding as in Case 1 leads to
\[
\int_{B_\delta} f_\delta(x) \cdot u(x) dx \leq C \delta^{5/2} \sum_{\alpha=1}^{2} (\|f_{r,\alpha}\|_{L^2(0,L)} + \|g_\alpha\|_{L^2(0,L; \mathbb{R}^3)}) (d(v, B_\delta) + \delta^{1/4} G_s(u, \Omega_\delta)) \oplus \frac{\int_{B_\delta} f_\delta(x) \cdot u(x) dx}{\delta^{3/2}} + C ||f_{r,3}||_{L^2(0,L)} \left[ [d(v, B_\delta)]^2 + \delta^{5/2} (G_s(u, \Omega_\delta) + d(v, \Omega_\delta) + d(v, B_\delta)) \right].
\]

Then for \( \delta \) small enough, we get
\[
\frac{\mu}{8} ( [d(v, B_\delta)]^2 + [d(v, \Omega_\delta)]^2 ) \leq \int_{S_\delta} \hat{W}(\nabla v)(x) dx \leq \int_{S_\delta} f_\delta(x) \cdot u(x) dx \\
\leq C \delta^{5/2} C(f_r, g) (d(v, B_\delta) + d(v, \Omega_\delta)) + C^* ||f_{r,3}||_{L^2(0,L)} \left[ [d(v, B_\delta)]^2 + [d(v, \Omega_\delta)]^2 \right] \\
+ C \delta^{5/2} ||f_p||_{\omega; \mathbb{R}^3} d(v, \Omega_\delta) + C^* ||f_p||_{\omega; \mathbb{R}^3} [d(v, \Omega_\delta)]^2.
\]

Hence, under the conditions \( C^* ||f_p||_{L^2(\omega; \mathbb{R}^3)} \leq \mu/32 \) and \( C^* ||f_{r,3}||_{L^2(0,L)} \leq \mu/32 \) we deduce that
\[
d(v, B_\delta) + d(v, \Omega_\delta) \leq C \delta^{5/2}.
\]

In the both cases, we finally obtain (5.9) \( \Box \)

As a consequence of Theorem 5.1 and estimates (5.16)-(5.17), we deduce that for \( \delta \) small enough and for any \( v \in \mathbb{D}_\delta \) satisfying \( J_\delta(v) \leq 0 \) we have \( u = v - I_d \)
\[
\int_{S_\delta} f_\delta \cdot u \leq C \delta^5, \quad \int_{S_\delta} \hat{W}(\nabla v)(x) dx \leq C \delta^5.
\]

(5.18)

From (5.18) we also obtain for any \( v \in \mathbb{D}_\delta \) such that \( J_\delta(v) \leq 0 \)
\[
c \delta^5 \leq J_\delta(v).
\]

(5.19)
where \( c \) is a nonpositive constant which does not depend on \( \delta \). We set
\[
m_\delta = \inf_{v \in D_\delta} J_\delta(v).
\]
As a consequence of (5.19) we have
\[
c \leq \frac{m_\delta}{\delta^5} \leq 0.
\] (5.20)
In general, a minimizer of \( J_\delta \) does not exist on \( D_\delta \).

6 Asymptotic behavior of a sequence of deformations of the whole structure \( S_\delta \).

In this subsection and the following one, we consider a sequence of deformations \((v_\delta)\) belonging to \( D_\delta \) and satisfying
\[
d(v_\delta, B_\delta) + d(v_\delta, \Omega_\delta) \leq C\delta^{5/2}
\] (6.1)
where the constant does not depend on \( \delta \). Setting \( u_\delta = v_\delta - I_d \), then, due to (6.1) and (5.15) we obtain that
\[
G_s(u_\delta, \Omega_\delta) \leq C\delta^{5/2}.
\] (6.2)
For any open subset \( \mathcal{O} \subset \mathbb{R}^2 \) and for any field \( \psi \in H^1(\mathcal{O}; \mathbb{R}^3) \), we denote
\[
\gamma_{\alpha\beta}(\psi) = \frac{1}{2} \left( \frac{\partial \psi_\alpha}{\partial x_\beta} + \frac{\partial \psi_\beta}{\partial x_\alpha} \right), \quad (\alpha, \beta) \in \{1, 2\}.
\] (6.3)

6.1 The rescaling operators

Before rescaling the domains, we introduce the reference domain \( \Omega \) for the plate and the one \( B \) for the rod
\[
\Omega = \omega \times]1, 1[ \quad B = D \times]0, L[\times]0, L[.
\]
As usual when dealing with thin structures, we rescale \( \Omega_\delta \) and \( B_\delta \) using -for the plate- the operator
\[
\Pi_\delta(w)(x_1, x_2, X_3) = w(x_1, x_2, \delta X_3) \text{ for any } (x_1, x_2, X_3) \in \Omega
\]
defined for e.g. \( w \in L^2(\Omega_\delta) \) for which \( \Pi_\delta(w) \in L^2(\Omega) \) and using -for the rod- the operator
\[
P_\delta(w)(X_1, X_2, x_3) = w(\delta X_1, \delta X_2, x_3) \text{ for any } (X_1, X_2, x_3) \in B
\]
defined for e.g. \( w \in L^2(B_\delta) \) for which \( P_\delta(w) \in L^2(B) \).
6.2 Asymptotic behavior in the plate.

Following Section 2 we decompose the restriction of \( u_\delta = v_\delta - I_3 \) to the plate. The Theorem 3.1 gives \( U_\delta, R_\delta \) and \( \overline{u}_\delta \), then estimates (3.6) lead to the following convergences for a subsequence still indexed by \( \delta \) (see [23] for the detailed proofs of the below convergences and equalities)

\[
\begin{align*}
\frac{1}{\delta} U_{\delta,3} & \longrightarrow U_3 \quad \text{strongly in} \quad H^1(\omega), \\
\frac{1}{\delta^2} U_{\delta,\alpha} & \rightharpoonup U_\alpha \quad \text{weakly in} \quad H^1(\omega), \\
\frac{1}{\delta} R_\delta & \rightharpoonup R \quad \text{weakly in} \quad H^1(\omega; \mathbb{R}^3), \\
\frac{1}{\delta^3} \Pi_\delta (\overline{u}_\delta) & \rightharpoonup \overline{u} \quad \text{weakly in} \quad L^2(\omega; H^1(-1, 1; \mathbb{R}^3)), \\
\frac{1}{\delta^2} \left( \partial U_{\delta,\alpha} - R_\delta \wedge e_\alpha \right) & \rightharpoonup Z_\alpha \quad \text{weakly in} \quad L^2(\omega; \mathbb{R}^3).
\end{align*}
\]

(6.4)

Denoting by \( A_R \) the field of antisymmetric matrices associated to \( R \) as in Section 2, we also have

\[
\begin{align*}
\frac{1}{\delta^2} \Pi_\delta (u_\delta - U_\delta) & \longrightarrow X_3 R \wedge e_3 \quad \text{strongly in} \quad L^2(\Omega; \mathbb{R}^3), \\
\frac{1}{\delta} \Pi_\delta (\nabla u_\delta) & \longrightarrow A_R \quad \text{strongly in} \quad L^2(\Omega; \mathbb{R}^9).
\end{align*}
\]

(6.5)

The boundary conditions (3.5) give here

\[
U_3 = 0, \quad U_\alpha = 0, \quad R = 0 \quad \text{on} \quad \gamma_0,
\]

(6.6)

while (6.4) show that \( U_3 \in H^2(\omega) \) with

\[
\begin{align*}
\frac{\partial U_3}{\partial x_1} = -R_2, \\
\frac{\partial U_3}{\partial x_2} = R_1.
\end{align*}
\]

(6.7)

In [8] (see Theorem 7.3) the limit of the Green-St Venant’s strain tensor of the sequence \( v_\delta \) is also derived. Let us set

\[
\overline{u}_p = \overline{u} + \frac{X_3}{2} (Z_1 \cdot e_3) e_1 + \frac{X_3}{2} (Z_2 \cdot e_3) e_2
\]

(6.8)

and

\[
Z_{\alpha\beta} = \gamma_{\alpha\beta}(U) + \frac{1}{2} \frac{\partial U_\alpha}{\partial x_\alpha} \frac{\partial U_3}{\partial x_\beta}.
\]

(6.9)

Then we have

\[
\frac{1}{2\delta^2} \Pi_\delta ( (\nabla v_\delta)^T \nabla v_\delta - I_3 ) \rightharpoonup E_p \quad \text{weakly in} \quad L^1(\Omega; \mathbb{R}^9),
\]

(6.10)
where the symmetric matrix $E_p$ is defined by

$$E_p = \begin{pmatrix} -X_3 \frac{\partial^2 U_3}{\partial x_1^2} + Z_{11} & -X_3 \frac{\partial^2 U_3}{\partial x_1 \partial x_2} + Z_{12} & \frac{1}{2} \frac{\partial p_{1,1}}{\partial X_3} \\ * & -X_3 \frac{\partial^2 U_3}{\partial x_2^2} + Z_{22} & \frac{1}{2} \frac{\partial p_{2,2}}{\partial X_3} \\ * & * & \frac{1}{2} \frac{\partial p_{3,3}}{\partial X_3} \end{pmatrix}. \quad (6.10)$$

### 6.3 Asymptotic behavior in the rod.

Now, we decompose the restriction of $v_\delta = u_\delta + I_d$ to the rod (see Section 2). The Theorem 3.2 gives $W_\delta$, $Q_\delta$, $\overline{v}_\delta$ and thanks to (4.4) we define $W_\delta^{(m)}$ and $W_\delta^{(s)}$. Then the estimates in Theorem 3.2 and Lemma 4.1 allow to claim that

$$|||v_\delta|||_{L^2(B_\delta;\mathbb{R}^3)} \leq C\delta^{7/2}, \quad |||\nabla v_\delta|||_{L^2(B_\delta;\mathbb{R}^3)} \leq C\delta^{5/2}, \quad |||W_\delta^{(s)}|||_{H^1(-\delta,L;\mathbb{R}^3)} \leq C\delta^{3/2},$$

$$|||W_\delta^{(m)} - W_{\delta,a}(0)|||_{H^2(-\delta,L)} \leq C\delta^{1/2} + C|||Q_\delta(0) - I_3|||,$$

$$|||\frac{dW_\delta^{(m)}}{dx_3}|||_{L^2(-\delta,L)} \leq C\delta + C|||(Q_\delta(0) - I_3)e_3 \cdot e_3|||,$$

$$|||Q_\delta - Q_\delta(0)|||_{H^1(-\delta,L;\mathbb{R}^3)} \leq C\delta^{1/2}. \quad (6.11)$$

Moreover from Lemma 4.2 we get

$$|||Q_\delta(0) - I_3||| \leq C\delta^{3/4}, \quad |||(Q_\delta(0) - I_3)e_3 \cdot e_3||| \leq C\delta,$$

$$|||W_\delta^{(m)}(0)||| \leq C\delta^{3/4}, \quad |||W_{\delta,3}(0) - \overline{U}_{\delta,3}(0,0)||| \leq C\delta^{3/2}, \quad (6.12)$$

$$|||W_{\delta,3}(0)||| \leq C\delta.$$

Finally we obtain the following estimates of the terms $W_{\delta,a}^{(m)}$, $W_{\delta,3}^{(m)}$ and $Q_\delta - I_3$:

$$|||Q_\delta - I_3|||_{H^1(-\delta,L;\mathbb{R}^3)} \leq C\delta^{1/2}, \quad |||W_{\delta,a}^{(m)}|||_{H^2(-\delta,L)} \leq C\delta^{1/2},$$

$$|||W_{\delta,3}^{(m)}|||_{H^1(-\delta,L)} \leq C\delta. \quad (6.13)$$

Now we are in a position to prove the following lemma:
Lemma 6.1. There exists a subsequence still indexed by $\delta$ such that

$$
\begin{align*}
\frac{1}{\delta^{1/2}} W_{\delta,\alpha}^{(m)} & \rightharpoonup W_{\alpha} \quad \text{weakly in } H^2(0,L), \\
\frac{1}{\delta^{1/2}} W_{\delta,\alpha}, \quad \frac{1}{\delta^{1/2}} W_{\delta,\alpha}^{(m)} & \rightharpoonup W_{\alpha} \quad \text{strongly in } H^1(0,L), \\
\frac{1}{\delta} W_{\delta,3}, \quad \frac{1}{\delta} W_{\delta,3}^{(m)} & \rightharpoonup W_3 \quad \text{weakly in } H^1(0,L), \\
\frac{1}{\delta^{3/2}} W_{\delta}^{(s)} & \rightharpoonup W^{(s)} \quad \text{weakly in } H^1(0,L;\mathbb{R}^3), \\
\frac{1}{\delta^{1/2}} (Q_{\delta} - I_3) & \rightharpoonup A_Q \quad \text{weakly in } H^1(0,L;\mathbb{R}^9), \\
\frac{1}{\delta^{5/2}} P_{\delta}(\bar{\nu}_{\delta}) & \rightharpoonup \bar{\nu} \quad \text{weakly in } L^2(0,L;H^1(D;\mathbb{R}^3)).
\end{align*}
$$

(6.14)

We also have $W_{\alpha} \in H^2(0,L)$ and for a.e. $x_3 \in ]0,L[\text{ we have}$

$$
\begin{align*}
\frac{dW_1}{dx_3}(x_3) &= Q_2(x_3), \\
\frac{dW_2}{dx_3}(x_3) &= -Q_1(x_3), \\
\frac{dW_3}{dx_3}(x_3) + \frac{1}{2}\left[ \frac{dW_1}{dx_3}(x_3)^2 + \frac{dW_2}{dx_3}(x_3)^2 \right] &= 0.
\end{align*}
$$

(6.15)

The junction conditions

$$
W_{\alpha}(0) = 0, \quad Q(0) = 0, \quad W^{(s)}(0) = 0, \quad W_3(0) = U_3(0,0)
$$

(6.16)

hold true. We have

$$
\frac{1}{2\delta^{3/2}} P_{\delta}(\nabla v_{\delta}^T \nabla v_{\delta} - I_3) \rightharpoonup E_r \quad \text{weakly in } L^1(B;\mathbb{R}^{3 \times 3}),
$$

(6.17)

where the symmetric matrix $E_r$ is defined by

$$
E_r = 
\begin{pmatrix}
\gamma_{11}(\bar{\nu}_{r}) & \gamma_{12}(\bar{\nu}_{r}) & -\frac{1}{2}X_2 \frac{dQ_3}{dx_3} + \frac{1}{2} \frac{\partial \bar{\nu}_{r,3}}{\partial X_1} + \frac{1}{2} \frac{dW_1^{(s)}}{dx_3} \\
\gamma_{12}(\bar{\nu}_{r}) & \frac{1}{2}X_1 \frac{dQ_3}{dx_3} + \frac{1}{2} \frac{\partial \bar{\nu}_{r,3}}{\partial X_2} + \frac{1}{2} \frac{dW_2^{(s)}}{dx_3} \\
& & -X_1 \frac{d^2W_1}{dx_3^2} - X_2 \frac{d^2W_2}{dx_3^2} + \frac{dW_3^{(s)}}{dx_3}
\end{pmatrix}
$$

(6.18)

Proof. First, taking into account (6.11), (6.13) and upon extracting a subsequence it follows that the convergences (6.14) hold true. First, due to the definition of $W_{\delta}^{(m)}$ and the weak convergence in $H^1(0,L;\mathbb{R}^9)$ of the sequence $\frac{1}{\delta^{1/2}}(Q_{\delta} - I_3)$ towards the antisymmetric matrix $A_Q$ we deduce that $\frac{dW}{dx_3} = Q \wedge e_3$ which gives the two first equalities
Then, the strong convergence in \( L^\infty(0, L; \mathbb{R}^3) \) of the sequence \( \frac{1}{8}(Q_\delta - I_3)e_3 \) towards \( Q \wedge e_3 \), hence \( \frac{1}{8}|(Q_\delta - I_3)e_3|^2 \) converges towards \( |Q \wedge e_3|^2 \), strongly in \( L^\infty(0, L) \). Finally using equality (4.9) we obtain the last equality in (6.15).

The junction conditions on \( Q \) and \( W, \) are immediate consequences of (6.12) and the convergences (6.14).

In order to obtain the junction condition between the bending in the plate and the stretching in the rod note first that the sequence \( \frac{1}{8}\tilde{U}_3 \) converges strongly in \( H^1(\omega) \) to \( U_3 \) because of (4.2) and the first convergence in (6.4). Besides this sequence is uniformly bounded in \( H^2(D) \), hence it converges strongly to the same limit \( U_3 \) in \( C^0(D) \). Moreover the weak convergence of the sequence \( \frac{1}{8}W_{3, \delta}^{(m)}(0) \) in \( H^1(0, L) \), implies the convergence of \( \frac{1}{8}W_{3, \delta}^{(m)}(0) = \frac{1}{8}W_{3, \delta}(0) \) to \( W_3(0) \). Using the third estimate in (6.12) gives the last condition in (6.16).

Once the convergences (6.14) are established, the limit of the rescaled Green-St Venant strain tensor of the sequence \( \psi_\delta \) is analyzed in [7] (see Subsection 3.3) and it gives (6.18).

### 7 Asymptotic behavior of the sequence \( \frac{m_\delta}{\delta^5} \).

The goal of this section is to establish Theorem 7.2. Let us first introduce a few notations. We set

\[
\mathcal{D}_0 = \left\{ (U, W, Q_3) \in H^1(\omega; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^3) \times H^1(0, L) \mid 
\begin{align*}
U_3 &\in H^2(\omega), \quad W_\alpha \in H^2(0, L), \quad U = 0, \quad \frac{\partial U_3}{\partial x_\alpha} = 0 \text{ on } \gamma_0, \\
& \frac{dW_3}{dx_3} + \frac{1}{2} \left[ \frac{dW_1}{dx_3} \right]^2 + \left[ \frac{dW_2}{dx_3} \right]^2 = 0 \text{ in } [0, L], \\
W_3(0) = U_3(0, 0), \quad W_\alpha(0) = \frac{dW_\alpha}{dx_3}(0) = Q_3(0) = 0
\end{align*}
\right\}
\]

Let us notice that \( \mathcal{D}_0 \) is a closed subset of \( H^1(\omega; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^3) \times H^1(0, L) \).

We introduce below the "limit" elastic energies for the plate and the rod whose expressions are well known for such structures.

\footnote{\( E, \nu \) are the Young modulus and the Poisson’s ratio of the plate and the rod.}
\[ J_p(U) = \frac{E}{3(1-\nu^2)} \int_\omega \left[(1-\nu) \sum_{\alpha,\beta=1}^2 \left| \frac{\partial^2 U_3}{\partial x_\alpha \partial x_\beta} \right|^2 + \nu (\Delta U_3)^2 \right] \]
\[ + \frac{E}{(1-\nu^2)} \int_\omega \left[(1-\nu) \sum_{\alpha,\beta=1}^2 |Z_{\alpha\beta}|^2 + \nu (Z_{11} + Z_{22})^2 \right], \quad (7.2) \]
\[ J_r(W_1, W_2, Q_3) = \frac{E\pi}{8} \int_0^L \left[ \left| \frac{d^2 W_1}{dx_3^2} \right|^2 + \left| \frac{d^2 W_2}{dx_3^2} \right|^2 \right] + \frac{\mu \pi}{8} \int_0^L \left| \frac{dQ_3}{dx_3} \right|^2 \]

where \( Z_{\alpha\beta} \) is given by
\[ Z_{\alpha\beta} = \gamma_{\alpha\beta}(U) + \frac{1}{2} \frac{\partial U_3}{\partial x_\alpha} \frac{\partial U_3}{\partial x_\beta}. \]

The total energy of the plate-rod structure is given by the functional \( J \) defined over \( D_0 \)
\[ J(U, W, Q_3) = J_p(U) + J_r(W_1, W_2, Q_3) - L(U, W, Q_3) \quad (7.3) \]

with
\[ L(U, W, Q_3) = 2 \int_\omega f_p \cdot U + \pi \int_0^L f_r \cdot \mathcal{W} dx_3 + \frac{\pi}{2} \int_0^L g_{\alpha} \cdot (Q \wedge e_\alpha) dx_3 \quad (7.4) \]
where
\[ Q = - \frac{dW_2}{dx_3} e_1 + \frac{dW_1}{dx_3} e_2 + Q_3 e_3. \]

Below we prove the existence of at least a minimizer of \( J \).

**Lemma 7.1.** There exist two constants \( C_p^*, C_r^* \) such that, if \( (f_{p,1}, f_{p,2}) \) satisfies
\[ ||f_{p,1}||_{L^2(\omega)}^2 + ||f_{p,2}||_{L^2(\omega)}^2 < C_p^* \quad (7.5) \]
and if \( f_{r,3} \) satisfies
\[ ||f_{r,3}||_{L^2(0,L)} < C_r^* \quad (7.6) \]
then the minimization problem
\[ \min_{(U, W, Q_3) \in D_0} J(U, W, Q_3) \quad (7.7) \]
admits at least a solution.

**Proof.** Due to the boundary conditions on \( U_3 \) in \( D_0 \), we immediately have
\[ ||U_3||_{H^2(\omega)}^2 \leq C J_p(U). \quad (7.8) \]
Then we get
\[
\sum_{\alpha,\beta=1}^2 \|\gamma_{\alpha,\beta}(U)\|_{L^2(\omega)}^2 \leq C J_p(U) + C \left\| \nabla U_3 \right\|_{L^4(\omega;\mathbb{R}^2)}^4
\]
\[ \leq C J_p(U) + C [J_p(U)]^2. \tag{7.9} \]

Thanks to the 2D Korn's inequality we obtain
\[
\|U_1\|_{H^1(\omega)} + \|U_2\|_{H^1(\omega)} \leq C J_p(U) + C_p [J_p(U)]^2. \tag{7.10} \]

Again, due to the boundary conditions on \(W_\alpha\) and \(Q_3\) in \(D_0\), we immediately have
\[
\|W_1\|_{H^2(0,L)} + \|W_2\|_{H^2(0,L)} + \|Q_3\|_{H^1(0,L)}^2 \leq C J_r(W_1, W_2, Q_3). \tag{7.11} \]

Then, due to the definition of \(D_0\) and \(\{7.11\}\) we get
\[
\left\| \frac{dW_3}{dx_3} \right\|_{L^2(0,L)}^2 \leq C \left\{ \left\| \frac{dW_1}{dx_3} \right\|_{L^4(0,L)}^4 + \left\| \frac{dW_2}{dx_3} \right\|_{L^4(0,L)}^4 \right\} \leq C [J_r(W_1, W_2, Q_3)]^2. \tag{7.12} \]

From the above inequality and \(\{7.8\}\) we obtain
\[
\|W_3\|_{L^2(0,L)}^2 \leq C |W_3(0)|^2 + C \left\| \frac{dW_3}{dx_3} \right\|_{L^2(0,L)}^2 \leq C J_p(U) + C_r [J_r(W_1, W_2, Q_3)]^2. \tag{7.13} \]

Since \(J(0,0,0) = 0\), let us consider a minimizing sequence \((U^{(N)}, W^{(N)}, Q_3^{(N)}) \in D_0\) satisfying \(J(U^{(N)}, W^{(N)}, Q_3^{(N)}) \leq 0\)
\[
m = \inf_{(U,W,Q_3) \in D_0} J(U, W, Q_3) = \lim_{N \to +\infty} J(U^{(N)}, W^{(N)}, Q_3^{(N)})
\]
where \(m \in [-\infty, 0]\).

With the help of \(\{7.8\}-\{7.13\}\) we get
\[
J_p(U^{(N)}) + J_r(W_1^{(N)} W_2^{(N)}, Q_3^{(N)}) \leq C \|f_{p,3}\| \sqrt{J_p(U^{(N)})} \\
+ 2 (\|f_{p,1}\|_{L^2(\omega)}^2 + \|f_{p,2}\|_{L^2(\omega)}^2)^{1/2} (C \sqrt{J_p(U^{(N)})} + \sqrt{C_p J_p(U^{(N)})}) \\
+ C \sum_{\alpha=1}^2 (\|g_{\alpha}\|_{L^2(0,L)} + \|g_{\alpha}\|_{L^2(0,L;\mathbb{R}^2)}) \sqrt{J_r(W_1^{(N)} W_2^{(N)}, Q_3^{(N)})} \\
+ \pi \|f_{r,3}\|_{L^2(0,L)} (C \sqrt{J_r(W_1^{(N)} W_2^{(N)}, Q_3^{(N)})} + \sqrt{C_r J_r(W_1^{(N)} W_2^{(N)}, Q_3^{(N)})})
\]
\[ \tag{7.14} \]

Choosing \(C_p = \frac{1}{2C_p}\) and \(C_r = \frac{1}{\pi \sqrt{C_r}}\), if the applied forces satisfy \(\{7.5\}\) and \(\{7.6\}\) then the following estimates hold true
\[
\|U_3^{(N)}\|_{H^2(\omega)} + \|U_1^{(N)}\|_{H^1(\omega)} + \|U_2^{(N)}\|_{H^1(\omega)} + \|W_1^{(N)}\|_{H^2(0,L)} \\
+ \|W_2^{(N)}\|_{H^2(0,L)} + \|Q_3^{(N)}\|_{H^1(0,L)} + \|W_3^{(N)}\|_{H^1(0,L)} \leq C \tag{7.15} \]

22
where the constant $C$ does not depend on $N$.

As a consequence, there exists $(U^{(s)}, W^{(s)}, Q^{(s)}_3) \in H^1(\omega; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^3) \times H^1(0, L)$ such that for a subsequence

\[
\begin{align*}
U^{(N)}_3 & \rightharpoonup U^{(s)}_3 \quad \text{weakly in } H^2(\omega), \\
U^{(N)}_\alpha & \rightharpoonup U^{(s)}_\alpha \quad \text{weakly in } H^1(\omega), \\
W^{(N)}_\alpha & \rightharpoonup W^{(s)}_\alpha \quad \text{weakly in } H^2(0, L), \\
Q^{(N)}_3 & \rightharpoonup Q^{(s)}_3 \quad \text{weakly in } H^1(0, L), \\
W^{(N)}_3 & \rightharpoonup W^{(s)}_3 \quad \text{weakly in } H^1(0, L).
\end{align*}
\]

Notice that we also get the following convergences:

\[
\begin{align*}
Z^{(N)}_{\alpha\beta} & \rightharpoonup Z^{(s)}_{\alpha\beta} = \gamma_{\alpha\beta}(U^{(s)}) + \frac{1}{2} \frac{\partial U^{(s)}_3}{\partial x_\alpha} \frac{\partial U^{(s)}_3}{\partial x_\beta} \quad \text{weakly in } L^2(\omega).
\end{align*}
\]

The above convergences show that $(U^{(s)}, W^{(s)}, Q^{(s)}_3) \in \mathbb{D}_0$. Finally, since $J$ is weakly sequentially continuous in

\[
H^1(\omega; \mathbb{R}^2) \times H^2(\omega; \mathbb{R}^3) \times H^2(0, L; \mathbb{R}^2) \times H^1(0, L; \mathbb{R}^2)
\]

with respect to

$$(U_1, U_2, U_3, Z_{11}, Z_{12}, Z_{22}, W_1, W_2, W_3, Q_3)$$

The above weak and strong converges imply that

\[
J(U^{(s)}, W^{(s)}, Q^{(s)}_3) = m = \min_{(U, W, Q_3) \in \mathbb{D}_0} J(U, W, Q_3)
\]

which ends the proof of the lemma.

\[\square\]

**Theorem 7.2.** We have

\[
\lim_{\delta \to 0} \frac{m_\delta}{\delta^5} = \min_{(U, W, Q_3) \in \mathbb{D}_0} J(U, W, Q_3),
\]  

where the functional $J$ is defined by (7.3).

**Proof.** Step 1. In this step we show that

\[
\min_{(U, W, Q_3) \in \mathbb{D}_0} J(U, W, Q_3) \leq \liminf_{\delta \to 0} \frac{m_\delta}{\delta^5}.
\]

Let $(v_\delta)_{\delta > 0}$ be a sequence of deformations belonging to $\mathbb{D}_\delta$ and such that

\[
\lim_{\delta \to 0} \frac{J_\delta(v_\delta)}{\delta^5} = \liminf_{\delta \to 0} \frac{m_\delta}{\delta^5}.
\]
One can always assume that \( J_\delta(v_\delta) \leq 0 \) without loss of generality. From the analysis of the previous section and, in particular from estimate (5.9) the sequence \( v_\delta \) satisfies
\[
d(v_\delta, \Omega_\delta) + d(v_\delta, B_\delta) \leq C\delta^{5/2}. \tag{7.19}
\]
From (5.1)-(5.2) and estimates (5.18), we obtain
\[
\| \nabla v_\delta^T \nabla v_\delta - I_3 \|_{L^2(\Omega_\delta; \mathbb{R}^{3 \times 3})} \leq C\delta^{5/2}, \quad \| \nabla v_\delta^T \nabla v_\delta - I_3 \|_{L^2(B_\delta; \mathbb{R}^{3 \times 3})} \leq C\delta^{5/2}. \tag{7.20}
\]

Firstly, for any fixed \( \delta \), the displacement \( u_\delta = v_\delta - I_d \), restricted to \( \Omega_\delta \), is decomposed as in Theorem 3.1. Due to estimate (7.19), we can apply the results of Subsection 6.2 to the sequence \( (v_\delta) \). As a consequence there exist a subsequence (still indexed by \( \delta \)) and \( U^{(0)}, R^{(0)} \in H^1(\omega; \mathbb{R}^3) \) and \( \mathbf{r}_p^{(0)} \in L^2(\omega; H^1(-1, 1; \mathbb{R}^3)) \), such that the convergences (6.4) and (6.5) hold true. Due to (6.6) and (6.7) the field \( U_3^{(0)} \) belongs to \( H^2(\omega) \), and we have the boundary conditions
\[
U^{(0)} = 0, \quad \nabla U_3^{(0)} = 0, \quad \text{on} \ \gamma_0. \tag{7.21}
\]
Subsection 6.2 and estimates in (7.20) also show that
\[
\frac{1}{2\delta^2} \Pi_\delta \left( \nabla v_\delta^T \nabla v_\delta - I_3 \right) \rightharpoonup E_p^{(0)} \ \text{weakly in} \ L^2(\Omega; \mathbb{R}^9) \tag{7.22}
\]
where \( E_p^{(0)} \) is defined
\[
E_p^{(0)} = \begin{pmatrix}
-X_3 \frac{\partial^2 U_3^{(0)}}{\partial x_1^2} + \mathcal{Z}_{11}^{(0)} & -X_3 \frac{\partial^2 U_3^{(0)}}{\partial x_1 \partial x_2} + \mathcal{Z}_{12}^{(0)} & \frac{1}{2} \frac{\partial \mathbf{r}^{(0)}_p}{\partial X_3} \\
\ast & -X_3 \frac{\partial^2 U_3^{(0)}}{\partial x_2^2} + \mathcal{Z}_{22}^{(0)} & \frac{1}{2} \frac{\partial \mathbf{r}^{(0)}_p}{\partial X_3} \\
\ast & \ast & \frac{\partial \mathbf{r}^{(0)}_p}{\partial X_3}
\end{pmatrix} \tag{7.23}
\]
with
\[
\mathcal{Z}_{\alpha \beta}^{(0)} = \gamma_{\alpha \beta} (U^{(0)}) + \frac{1}{2} \frac{\partial U_3^{(0)}}{\partial x_\alpha} \frac{\partial U_3^{(0)}}{\partial x_\beta}. \tag{7.24}
\]

Secondly, still for \( \delta \) fixed, the displacement \( u_\delta = v_\delta - I_d \), restricted to \( B_\delta \), is decomposed as in Theorem 3.2 and (4.4). Again due to the estimate in (7.19), we can apply the results of Subsection 6.3 to the sequence \( (v_\delta) \). As a consequence there exist a subsequence (still indexed by \( \delta \)) and \( W^{(0)}, Q^{(0)} \in H^1(0, L; \mathbb{R}^3) \) and \( \mathbf{r}_p^{(0)} \in L^2(0, L; H^1(D; \mathbb{R}^3)) \) such that the convergences (6.14) hold true. As a consequence of (6.15) the components \( W_\alpha^{(0)} \) belong to \( H^2(0, L) \) and we have
\[
\frac{dW^{(0)}_1}{dx_3} = Q^{(0)} \wedge e_3 \quad \text{and} \quad \frac{dW^{(0)}_3}{dx_3} (x_3) + \frac{1}{2} \left[ \left| \frac{dW^{(0)}_1}{dx_3} (x_3) \right|^2 + \left| \frac{dW^{(0)}_2}{dx_3} (x_3) \right|^2 \right] = 0.
\]
The junction conditions (6.16) give
\[ Q^{(0)}(0) = 0, \quad W_{\alpha}^{(0)}(0) = 0, \quad W^{(s,0)}(0) = 0, \quad W_{3}^{(0)}(0) = U_{3}^{(0)}(0,0). \] (7.25)

As a first consequence, the triplet \((U^{(0)}, W^{(0)}, Q_{3}^{(0)})\) belongs to \(\mathbb{D}_{0}\).

Subsection 6.3 and the second estimate (7.20) also show that
\[ \frac{1}{2\delta^{3/2}} p_{\delta}((\nabla v_{\delta})^{T} \nabla v_{\delta} - I_{3}) \rightarrow E_{r}^{(0)} \quad \text{weakly in} \quad L^{2}(B; \mathbb{R}^{3 \times 3}), \] (7.26)
where the symmetric matrix \(E_{r}^{(0)}\) is defined by
\[
E_{r}^{(0)} = \begin{pmatrix}
\gamma_{11}(\overline{v}_{r}^{(0)}) & \gamma_{12}(\overline{v}_{r}^{(0)}) & -X_{1}^2 \frac{d^2 W_{1}^{(0)}}{dx_3^2} - X_{2}^2 \frac{d^2 W_{2}^{(0)}}{dx_3^2} + \frac{d W_{3}^{(s,0)}}{dx_3} \\
\gamma_{12}(\overline{v}_{r}^{(0)}) & \gamma_{22}(\overline{v}_{r}^{(0)}) & \frac{1}{2} X_{1} \frac{d Q_{3}^{(0)}}{dx_3} + \frac{1}{2} \frac{\partial \overline{v}_{r}^{(0)}}{\partial x_1} + \frac{1}{2} \frac{\partial W_{2}^{(s,0)}}{dx_3} \\
-X_{1} X_{2} \frac{d W_{1}^{(0)}}{dx_3} & \frac{1}{2} X_{1} \frac{d Q_{3}^{(0)}}{dx_3} + \frac{1}{2} \frac{\partial \overline{v}_{r}^{(0)}}{\partial x_1} + \frac{1}{2} \frac{\partial W_{2}^{(s,0)}}{dx_3} & X_{2}^2 \frac{d^2 W_{2}^{(0)}}{dx_3^2} + \frac{d W_{3}^{(s,0)}}{dx_3}
\end{pmatrix}. \] (7.27)

In order to bound from below the quantity \(\liminf_{\delta \to 0} \frac{J_{\delta}(v_{\delta})}{\delta^{3}}\), using the assumptions on the forces (5.5) and the convergences (6.4) and (6.14) we first have
\[ \lim_{\delta \to 0} \frac{1}{\delta^{5}} \int_{S_{\delta}} f_{\delta} \cdot (v_{\delta} - I_{4}) = \mathcal{L}(U^{(0)}, W^{(0)}, Q_{3}^{(0)}) \] (7.28)
where \(\mathcal{L}(U, W, Q_{3})\) is given by (7.24) for any triplet in \(\mathbb{D}_{0}\).

As far as the elastic energy is concerned, we write
\[
\frac{1}{\delta^{5}} \int_{S_{\delta}} \overline{W}_{\delta}((\nabla v_{\delta})^{T} \nabla v_{\delta}) = \frac{1}{\delta^{5}} \int_{\Omega_{\delta}} \overline{W}_{\delta}((\nabla v_{\delta})^{T} \nabla v_{\delta}) + \frac{1}{\delta^{5}} \int_{B_{\delta} \setminus C_{\delta}} \overline{W}_{\delta}((\nabla v_{\delta})^{T} \nabla v_{\delta})
\]
\[ = \int_{\Omega} Q(\Pi_{\delta} \left[ \frac{1}{\delta^2} ((\nabla v_{\delta})^{T} \nabla v_{\delta} - I_{3}) \right]) + \int_{B} Q(\chi_{B \setminus D \times |0,\delta|} \Pi_{\delta} \left[ \frac{1}{\delta^{3/2}} ((\nabla v_{\delta})^{T} \nabla v_{\delta} - I_{3}) \right])
\]

From the weak convergences of the Green-St Venant’s tensors in (7.22), (7.26) and equality (7.28), we obtain
\[ \liminf_{\delta \to 0} \frac{J_{\delta}(v_{\delta})}{\delta^{3}} \geq \int_{\Omega} Q(E_{p}^{(0)}) + \int_{B} Q(E_{r}^{(0)}) - \mathcal{L}(U^{(0)}, W^{(0)}, Q_{3}^{(0)}) \] (7.29)
where \(E_{p}^{(0)}\) and \(E_{r}^{(0)}\) are given by (7.23) and (7.50).

The next step in the derivation of the limit energy consists in minimizing \(\int_{-1}^{1} Q(E_{p}^{(0)}) dX_{3}\) (resp. \(\int_{D} Q(E_{r}^{(0)}) dX_{1} dX_{2}\)) with respect to \(\overline{w}_{p}^{(0)}\) (resp. \(\overline{v}_{r}^{(0)}\)).
First the expressions of $Q$ and of $E_p^{(0)}$ under a few calculations show that

$$
\int_{-1}^{1} Q(E_p^{(0)}) dX_3 \geq \frac{E}{3(1 - \nu^2)} \left[ (1 - \nu) 2 \sum_{\alpha, \beta = 1}^{2} \left| \frac{\partial^2 U_3^{(0)}}{\partial x_\alpha \partial x_\beta} \right|^2 + \nu \left( \Delta U_3^{(0)} \right)^2 \right] + \frac{E}{(1 - \nu^2)} \left[ (1 - \nu) 2 \sum_{\alpha, \beta = 1}^{2} \left| \mathcal{Z}_{\alpha, \beta}^{(0)} \right|^2 + \nu (\mathcal{Z}_{11}^{(0)} + \mathcal{Z}_{22}^{(0)})^2 \right]
$$

(7.30)

the expression in the right hand side of (7.30) is obtained through replacing $\overline{u}_p^{(0)}$ by

$$
\overline{u}_p^{(0)}(\cdot, \cdot, X_3) = \frac{\nu}{1 - \nu} \left[ \left( \frac{X^2}{2} - \frac{1}{6} \right) \Delta U_3^{(0)} - X_3 (\mathcal{Z}_{11}^{(0)} + \mathcal{Z}_{22}^{(0)}) \right] e_3.
$$

(7.31)

Following [7] (see equation (4.56) and (4.57)), choosing $\mathcal{W}_3^{(s,0)} = 0$ and $\overline{u}_r^{(0)}$ such that

$$
\begin{align*}
\overline{u}_{r,1}^{(0)} &= -\nu \left[ \frac{X^2}{2} - X_2 \frac{d\mathcal{W}_1^{(0)}}{dx_3} \right], \\
\overline{u}_{r,2}^{(0)} &= -\nu \left[ \frac{X^2}{2} - X_2 \frac{d\mathcal{W}_2^{(0)}}{dx_3} \right], \\
\overline{u}_{r,3}^{(0)} &= -X_1 \frac{d\mathcal{W}_1^{(s,0)}}{dx_3} - X_2 \frac{d\mathcal{W}_2^{(s,0)}}{dx_3},
\end{align*}
$$

(7.32)

permit to obtain

$$
\int_{D} Q(E_r^{(0)}) dX_1 dX_2 \geq \frac{E \pi}{8} \left[ \left| \frac{d^2 \mathcal{W}_1^{(0)}}{dx_3^2} \right|^2 + \left| \frac{d^2 \mathcal{W}_2^{(0)}}{dx_3^2} \right|^2 \right] + \frac{\mu \pi}{8} \left| \frac{d\mathcal{W}_3^{(0)}}{dx_3} \right|^2.
$$

(7.33)

In view of (7.29), (7.30) and (7.33), the proof of (7.17) is achieved.

**Step 2.** In this step we show that

$$
\limsup_{\delta \to 0} \frac{m_3}{\delta^3} \leq \min_{(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in \mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3).
$$

Let $(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}) \in \mathbb{D}_0$ such that

$$
\min_{(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in \mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) = \mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}).
$$

We consider a sequence $(\mathcal{U}^{(n)}, \mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)})_{n \geq 2}$ of elements belonging to $\mathbb{D}_0$ such that

- $\mathcal{U}_n^{(n)} \in W^{2,\infty}(\omega) \cap H^{1}_{\gamma_0}(\omega)$ and
- $\nabla \mathcal{U}_n^{(n)} = 0$ in $D_{1/n}$, $(\alpha, \beta) \in \{1, 2\}^2$,
- $\mathcal{U}_n^{(n)} \rightharpoonup \mathcal{U}_0^{(1)}$ strongly in $H^{1}(\omega),
$$

(7.34)
\[ \mathcal{U}_3^{(n)} \in W^{3,\infty}(\omega) \cap H^{2,0}_\partial(\omega) \text{ and} \]
\[ \frac{\partial^2 \mathcal{U}_3^{(n)}}{\partial x_\alpha \partial x_\beta} = 0 \text{ in } D_{1/n}, \quad (\alpha, \beta) \in \{1, 2\}^2, \quad (7.35) \]
\[ \mathcal{U}_3^{(n)} \to \mathcal{U}_3^{(1)} \text{ strongly in } H^2(\omega), \]
\[ \mathcal{W}_\alpha^{(n)} \in W^{3,\infty}(-1/n, L) \text{ with } \mathcal{W}_\alpha^{(n)} = 0 \text{ in } [-1/n, 1/n] \text{ and} \]
\[ \mathcal{W}_\alpha^{(n)} \to \mathcal{W}_\alpha^{(1)} \text{ strongly in } H^2(0, L), \quad (7.36) \]
\[ \mathcal{Q}_3^{(n)} \in W^{1,\infty}(-1/n, L) \text{ with } \mathcal{Q}_3^{(n)} = 0 \text{ in } [-1/n, 1/n] \text{ and} \]
\[ \mathcal{Q}_3^{(n)} \to \mathcal{Q}_3^{(1)} \text{ strongly in } H^1(0, L), \quad (7.37) \]

We define \( \mathcal{W}_3^{(n)} \in W^{2,\infty}(-1/n, L) \) by
\[ \mathcal{W}_3^{(n)} = \mathcal{U}_3^{(n)}(0, 0) \quad \text{and} \quad \frac{d\mathcal{W}_3^{(n)}}{dx_3} + \frac{1}{2} \left[ \frac{d\mathcal{W}_1^{(n)}}{dx_3} \right]^2 + \left| \frac{d\mathcal{W}_2^{(n)}}{dx_3} \right|^2 = 0 \text{ in } ] - 1/n, L[. \]

Obviously we have
\[ \mathcal{W}_3^{(n)} \to \mathcal{W}_3^{(1)} \text{ strongly in } H^1(0, L). \quad (7.38) \]

In order to define an admissible deformation of the whole structure, we introduce both fields \( \overline{u}_p^{(1)} \in L^2(\omega; H^1(-1, 1; \mathbb{R}^3)) \) obtained through replacing \( \mathcal{U}^{(0)} \) by \( \mathcal{U}^{(1)} \) in (7.24)- (7.31) and \( \overline{v}_{r,1}^{(1)} \in L^2(0, L; H^1(D)) \) obtained through replacing \( \mathcal{W}^{(0)} \) and \( \mathcal{Q}_3^{(0)} \) by \( \mathcal{U}^{(1)} \) and \( \mathcal{Q}_3^{(0)} \) in (7.32) and taking \( \overline{w}_{r,3}^{(1)} = 0. \)

Then, we consider two sequences of warpings \( \overline{u}_p^{(n)}, \overline{v}_r^{(n)} \) such that
\[ \overline{u}_p^{(n)} \in W^{1,\infty}(\Omega; \mathbb{R}^3) \text{ with } \overline{u}_p^{(n)} = 0 \text{ on } \partial \omega \times ] - 1, 1[ \text{ and} \]
\[ \overline{u}_p^{(n)} \to \overline{u}_p^{(1)} \text{ strongly in } L^2(\omega; H^1(-1, 1; \mathbb{R}^3)), \]
\[ \overline{v}_r^{(n)} \in W^{1,\infty}(-1/n, L[ \times D; \mathbb{R}^3) \text{ with } \overline{v}_r^{(n)} = 0 \text{ in the cylinder } D \times ] - 1/n, 1/n[ \text{ and} \]
\[ \overline{v}_r^{(n)} \to \overline{v}_r^{(1)} \text{ strongly in } L^2(0, L; H^1(D; \mathbb{R}^3)). \]

For \( n \) fixed, let us consider the sequence of deformations of the plate \( \Omega_\delta. \) We set
\[ v_{\delta,1}^{(n)}(x) = x_1 + \delta^2 \left[ \mathcal{U}_1^{(n)}(x_1, x_2) - x_3 \frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1} (x_1, x_2) + \delta \overline{u}_p^{(n)}(x_1, x_2, x_3) \right], \]
\[ v_{\delta,2}^{(n)}(x) = x_2 + \delta^2 \left[ \mathcal{U}_2^{(n)}(x_1, x_2) - x_3 \frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2} (x_1, x_2) + \delta \overline{u}_p^{(n)}(x_1, x_2, x_3) \right], \quad (7.39) \]
\[ v_{\delta,3}^{(n)}(x) = x_3 + \delta \left[ \mathcal{U}_3^{(n)}(x_1, x_2) + \delta^2 \overline{v}_r^{(n)}(x_1, x_2, x_3) \right]. \]
If $\delta$ is small enough (in order to have $\delta \leq 1/n$) the expression of $v_{\delta}^{(n)}$ in the cylinder $C_{\delta}$ is given by

$$
\begin{align*}
v_{\delta,1}^{(n)}(x) &= x_1 + \delta^2 \left[ \frac{U_1^{(n)}(0,0)}{\delta} - \frac{x_3}{\delta} \frac{\partial U_3^{(n)}}{\partial x_1}(0,0) \right], \\
v_{\delta,2}^{(n)}(x) &= x_2 + \delta^2 \left[ \frac{U_2^{(n)}(0,0)}{\delta} - \frac{x_3}{\delta} \frac{\partial U_3^{(n)}}{\partial x_2}(0,0) \right], \\
v_{\delta,3}^{(n)}(x) &= x_3 + \delta \left[ U_3^{(n)}(0,0) + x_1 \frac{\partial U_3^{(n)}}{\partial x_1}(0,0) + x_2 \frac{\partial U_3^{(n)}}{\partial x_2}(0,0) \right].
\end{align*}
$$

(7.40)

We denote

$$
Q^{(n)} = - \frac{dW_2^{(n)}}{dx_3} e_1 + \frac{dW_3^{(n)}}{dx_3} e_2 + Q_3^{(n)} e_3, \quad R^{(n)} = \frac{\partial U_3^{(n)}}{\partial x_2}(0,0)e_1 - \frac{\partial U_3^{(n)}}{\partial x_1}(0,0)e_2.
$$

The field $Q^{(n)}$ belongs to $W^{1,\infty}(-1/n, L; \mathbb{R}^3)$. Let $R^{(n)}_\delta$ be the matrix field defined by

$$
R^{(n)}_\delta(0) = I_3, \quad \frac{dR^{(n)}_\delta}{dx_3} = A_{F^{(n)}_\delta} R^{(n)}_\delta \quad \text{in} \quad [-1/n, L]
$$

(7.41)

where $F^{(n)}_\delta = \delta^{1/2} \frac{dQ^{(n)}}{dx_3} + \delta R^{(n)}$ (see (3.1)) and let $W^{(n)}_\delta$ be defined in $[-1/n, L]$ by

$$
W^{(n)}_\delta(x_3) = \int_0^{x_3} \left( R^{(n)}_\delta(t) - I_3 \right) e_3 dt + \delta^2 U_1^{(n)}(0,0)e_1 + \delta^2 U_2^{(n)}(0,0)e_2 + \delta U_3^{(n)}(0,0)e_3.
$$

(7.42)

We have $R^{(n)}_\delta \in W^{1,\infty}(-1/n, L; SO(3))$, $W^{(n)}_\delta \in W^{2,\infty}(-1/n, L; \mathbb{R}^3)$ and the following strong convergences (as $\delta$ tends towards $0$)

$$
\begin{align*}
R^{(n)}_\delta &\longrightarrow I_3 \quad \text{strongly in} \quad W^{1,\infty}(-1/n, L; SO(3)), \\
\frac{1}{\delta^{1/2}} \frac{dR^{(n)}_\delta}{dx_3} &\longrightarrow A_{F^{(n)}_\delta} \quad \text{strongly in} \quad L^{\infty}(-1/n, L; \mathbb{R}^9), \\
\frac{1}{\delta^{1/2}} (R^{(n)}_\delta - I_3) &\longrightarrow A_{Q^{(n)}} \quad \text{strongly in} \quad W^{1,\infty}(-1/n, L; \mathbb{R}^9), \\
\frac{1}{\delta} (R^{(n)}_\delta - I_3) e_3 \cdot e_3 &\longrightarrow -\frac{1}{2} ||A_{Q^{(n)}} e_3||_2^2 \quad \text{strongly in} \quad L^{\infty}(-1/n, L), \\
\frac{1}{\delta^{1/2}} W^{(n)}_\delta &\longrightarrow W^{(n)}_\delta \quad \text{strongly in} \quad W^{1,\infty}(-1/n, L), \\
\frac{1}{\delta} W^{(n)}_\delta &\longrightarrow W^{(n)}_\delta \quad \text{strongly in} \quad W^{1,\infty}(-1/n, L).
\end{align*}
$$

(7.43)

By definition, $Q^{(n)}$ is equal to $0$ in $[-1/n, 1/n]$, hence we have

$$
\forall x_3 \in [-1/n, 1/n], \quad R^{(n)}_\delta(x_3) = \exp \left( \delta A_{R^{(n)}} x_3 \right).
$$
Now, we consider the fields \( \overline{R}_\delta^{(n)} \in W^{1,\infty}(-\delta, L; SO(3)) \) and \( \overline{W}_\delta^{(n)} \in W^{2,\infty}(-\delta, L; \mathbb{R}^3) \) defined by

\[
\overline{R}_\delta^{(n)}(x_3) = \exp\left( \delta A_{\overline{R}_\delta^{(n)}} x_3 \right) \text{ in } [-\delta, L], \\
\overline{W}_\delta^{(n)}(x_3) = \int_0^{x_3} \left( \overline{R}_\delta^{(n)}(t) - I_3 \right) e_3 dt + \mathcal{W}_\delta^{(n)}(0) \text{ in } [-\delta, L].
\] (7.44)

We introduce a last displacement \( \tilde{v}_\delta^{(n)} \) belonging to \( W^{1,\infty}(B_\delta; \mathbb{R}^3) \) (for \( \delta \leq 1/n \))

\[
\tilde{v}_\delta^{(n)}(x) = \begin{pmatrix}
\delta^2 (U_1^{(n)}(0, 0) - \frac{x_3}{\delta} \frac{\partial U_3^{(n)}}{\partial x_1}(0, 0)) \\
\delta^2 (U_2^{(n)}(0, 0) - \frac{x_3}{\delta} \frac{\partial U_3^{(n)}}{\partial x_2}(0, 0)) \\
\delta (U_3^{(n)}(0, 0) + x_1 \frac{\partial U_3^{(n)}}{\partial x_1}(0, 0) + x_2 \frac{\partial U_3^{(n)}}{\partial x_2}(0, 0)) \\
- \overline{W}_\delta^{(n)}(x_3) - (\overline{R}_\delta^{(n)}(x_3) - I_3)(x_1 e_1 + x_2 e_2).
\end{pmatrix}
\]

We have

\[ ||\nabla \tilde{v}_\delta^{(n)}||_{L^\infty(B_\delta; \mathbb{R}^9)} \leq C^{(n)} \delta^2. \]

Now, we are in a position to define the deformation \( v_\delta^{(n)} \) in the rod \( B_\delta \). We set

\[
v_\delta^{(n)}(x) = x + \mathcal{W}_\delta^{(n)}(x_3) + (\overline{R}_\delta^{(n)}(x_3) - I_3)(x_1 e_1 + x_2 e_2) + \delta^{5/2} \overline{\mathcal{W}}^{(n)}_r(x_3) \left( \frac{x_1}{\delta}, \frac{x_2}{\delta}, x_3 \right) + \tilde{v}_\delta^{(n)}(x).
\] (7.45)

In the cylinder \( C_\delta \), the above expression of \( v_\delta^{(n)} \) matches the one given by (7.40) if \( \delta \) is small enough (\( \delta \leq 1/n \)).

By construction the deformation \( v_\delta^{(n)} \) belongs to \( \mathcal{D}_\delta \) and satisfies

\[ ||\nabla v_\delta^{(n)} - I_3||_{L^\infty(S_\delta; \mathbb{R}^9)} \leq C(n) \delta^{1/2}. \]

Hence, for a.e. \( x \in S_\delta \) we have \( \det \left( \nabla v_\delta^{(n)}(x) \right) > 0 \). Then we have

\[ m_\delta \leq J_\delta(v_\delta^{(n)}). \] (7.46)

The expression (7.33) of the displacement \( v_\delta^{(n)} - I_d \) in the plate is similar to the decomposition (3.2) given in Section 2. Hence the results of Subsection 6.2 and the regularity of the terms \( \mathcal{U}^{(n)} \) and \( \overline{\mathcal{U}}^{(n)} \) lead to

\[
\frac{1}{2\delta^2} \Pi_\delta\left( (\nabla v_\delta^{(n)})^T \nabla v_\delta^{(n)} - I_3 \right) \to E^{(n)}_\rho \text{ strongly in } L^\infty(\Omega; \mathbb{R}^9),
\] (7.47)

29
where the symmetric matrix $E_p^{(n)}$ is defined by

$$E_p^{(n)} = \begin{pmatrix}
-X_3 \frac{\partial^2 U_3^{(n)}}{\partial x_1^2} + Z_1^{(n)} & -X_3 \frac{\partial^2 U_3^{(n)}}{\partial x_1 \partial x_2} + Z_{12}^{(n)} & \frac{1}{2} \frac{\partial \mu_3^{(n)}}{\partial X_3} \\
* & -X_3 \frac{\partial^2 U_3^{(n)}}{\partial x_2^2} + Z_{22}^{(n)} & \frac{1}{2} \frac{\partial \mu_3^{(n)}}{\partial X_3} \\
* & * & *
\end{pmatrix}$$

Remark that here $\mu_p^{(n)} = \overline{\mu}^{(n)}$ (see (6.8) and where the $Z^{(n)}$s are obtained through replacing $U$ by $U^{(n)}$ in (6.9).

Now, in the rod $B_\delta$ we have the following strong convergence in $L^\infty(B; \mathbb{R}^9)$ (as $\delta$ tends towards 0):

$$\frac{1}{\delta^{3/2}} P_{\delta} \left( \nabla \nu_\delta^{(n)} - R_\delta^{(n)} \right) \to \frac{dR^{(n)}}{dx_3} (X_1 e_1 + X_2 e_2) + \left( \begin{array}{ccc}
\gamma_{11}(\overline{v}_r^{(n)}) & \gamma_{12}(\overline{v}_r^{(n)}) & \frac{1}{2} \frac{\partial \overline{\nu}_{r,3}^{(n)}}{\partial X_1} \\
* & \gamma_{22}(\overline{v}_r^{(n)}) & \frac{1}{2} \frac{\partial \overline{\nu}_{r,3}^{(n)}}{\partial X_2} \\
* & * & 0
\end{array} \right).$$

Then we obtain

$$\frac{1}{2\delta^{3/2}} P_{\delta} \left( (\nabla \nu_\delta^{(n)})^T \nabla \nu_\delta^{(n)} - I_3 \right) \to E_r^{(n)} \quad \text{strongly in } L^\infty(0; \mathbb{R}^9),$$

where the symmetric matrix $E_r^{(n)}$ is defined by

$$E_r^{(n)} = \begin{pmatrix}
\gamma_{11}(\overline{v}_r^{(n)}) & \gamma_{12}(\overline{v}_r^{(n)}) & -\frac{1}{2} X_2 \frac{dQ_3^{(n)}}{dx_3} + \frac{1}{2} \frac{\partial \overline{\nu}_{r,3}^{(n)}}{\partial X_1} \\
* & \gamma_{22}(\overline{v}_r^{(n)}) & \frac{1}{2} X_1 \frac{dQ_3^{(n)}}{dx_3} + \frac{1}{2} \frac{\partial \overline{\nu}_{r,3}^{(n)}}{\partial X_2} \\
* & * & -X_1 \frac{d^2 \mathcal{W}_1^{(n)}}{dx_3^2} - X_2 \frac{d^2 \mathcal{W}_2^{(n)}}{dx_3^2}
\end{pmatrix}.$$
then from the expression (7.40) of $v_\delta^{(n)}$ in $C_\delta$ and the strong convergences (7.47)-(7.49) we get

$$\lim_{\delta \to 0} \frac{1}{\delta^5} \int_{S_\delta} \tilde{W}_\delta(\nabla v_\delta^{(n)})(x)dx = \int_{\Omega} Q(E_p^{(n)}) + \int_{B} Q(E_r^{(n)}).$$

From the expressions of $v_\delta^{(n)}$ in the plate and in the rod, from the convergences (7.43) and taking to account the expressions of the applied forces (5.5) we get

$$\lim_{\delta \to 0} \frac{1}{\delta^5} \int_{S_\delta} f_\delta \cdot (v_\delta^{(n)} - I_d) = \mathcal{L}(U^{(n)}, W^{(n)}, Q^{(n)}).$$

Then, from the above limits and (7.46) we finally get

$$\limsup_{\delta \to 0} \frac{m_\delta}{\delta^5} \leq \lim_{\delta \to 0} \frac{J_\delta(v_\delta^{(n)})}{\delta^5} = \int_{\Omega} Q(E_p^{(n)}) + \int_{B} Q(E_r^{(n)}) - \mathcal{L}(U^{(n)}, W^{(n)}, Q^{(n)}). \quad (7.51)$$

Now, $n$ goes to infinity, due to the definitions of the warpings $u_p^{(1)}$ and $v_r^{(1)}$ and the strong convergences (7.34)-(7.35)-(7.36)-(7.37)-(7.38) that give

$$\limsup_{\delta \to 0} \frac{m_\delta}{\delta^5} \leq \int_{\Omega} Q(E_p^{(1)}) + \int_{B} Q(E_r^{(1)}) - \mathcal{L}(U^{(1)}, W^{(1)}, Q^{(1)}) = \mathcal{J}(U^{(1)}, W^{(1)}, Q^{(1)}).$$

This conclude the proof of the theorem. \qed

**Remark 7.3.** Let us point out that Theorem 7.2 shows that for any minimizing sequence $(v_\delta)_{\delta > 0}$ as in Step 1, the convergence of the rescaled Green-St Venant’s strain tensor in (7.22) is a strong convergence in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ and the convergence (7.26) is a strong convergence in $L^2(B; \mathbb{R}^{3 \times 3}).$

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