ON PAIRS OF FINITELY GENERATED SUBGROUPS
IN FREE GROUPS

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Abstract. We prove that for two arbitrary finitely generated subgroups $A$ and $B$ having infinite index in a free group $F$, there is a subgroup $H \leq B$ with finite index $[B:H]$ such that the subgroup generated by $A$ and $H$ has infinite index in $F$. The main corollary of this theorem says that a free group of free rank $r \geq 2$ admits a faithful highly transitive action, whereas the restriction of this action to any finitely generated subgroup of infinite index in $F$ has no infinite orbits.

1. Introduction

The most-known property of a pair of finitely generated subgroups $A$ and $B$ in a free group $F$ is the theorem of Howson ([10], [12], I.3.13) saying that the intersection $A \cap B$ is also finitely generated. The sharp estimate of the (free) rank of $A \cap B$ in terms of ranks of $A$ and $B$ was recently obtained by Mineyev [14] and Friedman [4]. It confirms the old conjecture of Hanna Neumann [16].

On the contrary, the subgroup $\langle A, B \rangle$ generated by $A \cup B$ has finite rank for the obvious reason, and certainly this subgroup can have finite index in $F$ or can just coincide with $F$ when both $A$ and $B$ are “small”, that is, have infinite index in $F$. However we prove in this paper that $\langle A, H \rangle$ is still “small” for some subgroup $H$ which is virtually equal to $B$. A more precise formulation is given by

**Theorem 1.1.** Let $A$ and $B$ be finitely generated subgroups of infinite index in a free group $F$. Then there is a subgroup $H \leq B$ with finite index in $B$ such that the subgroup $\langle A, H \rangle$ has infinite index in $F$.

In addition, for every finite subset $S \subset F \setminus A$, the subgroup $H$ can be chosen so that $\langle A, H \rangle \cap S = \emptyset$.

A particular case of Theorem 1.1 with a cyclic subgroup $B$ is contained in Theorem 5 of the paper [1]. Theorem 1.1 is proved in Section 3, and in Section 4, we prove some corollaries obtained by the iterated applications of this theorem. The main corollary is the following.

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Corollary 1.2. Every noncyclic finitely generated free group $F$

(1) admits a highly transitive action $\circ$ on an infinite set $S$, such that
(2) the restriction of this action to any finitely generated subgroup of infinite index in $F$ is locally finite.

Furthermore, every action of $F$ with properties (1) and (2) is faithful.

Recall that an action of a group $G$ is faithful if its kernel is trivial. It is called highly transitive if it is $k$-transitive for every integer $k \geq 1$. We call an action locally finite if all the orbits of this action are finite.

Thus we have antipodal properties (1) and (2) of the action $\circ$ of $F$. The assumptions “finitely generated of infinite index” are essential in (2) because an easy observation shows that the restriction of a faithful transitive action of $F$ to a subgroup of finite index or to a non-trivial normal subgroup of $F$ cannot be locally finite (see Remark 1.3). On the other hand, the free group $F$ is saturated with finitely generated subgroups of infinite index. The $(r - 1)$-generated subgroups, where $r$ is the free rank of $F$, constitute a minor part of this enormous family.

2. Preliminaries: Coset Graphs, Cores, and Coverings

Let $F = F(X)$ be a free group with a free basis $X$. If $H$ is a subgroup of $F$, then the vertices of the coset graph $\Gamma = \Gamma(H, X)$ are the right cosets $Hg$ of the subgroup $H$ in $F$, and for every coset $Hg$ and $x \in X^{\pm 1}$, we have an edge $e = e(Hg, x)$ with the initial vertex $e_- = Hg$ and the terminal vertex $e_+ = Hgx$. The edge $e$ is labeled by $\text{Lab}(e) = x$, while the inverse edge $e^{-1}$ is labeled by $x^{-1}$, and $e_+^{-1} = Hgx$, $e_-^{-1} = Hg$.

Recall that if $e_i$ and $e_{i+1}$ are two consecutive edges in a path $p = e_1 \ldots e_n$ of (combinatorial) length $n$, then $(e_{i+1})_-(e_i)_+$ for $i = 1, \ldots, n - 1$. The path $p$ of $\Gamma$ is reduced if $e_{i+1} \neq e_i^{-1}$ for $i = 1, \ldots, n - 1$. This is equivalent to saying that the word $\text{Lab}(p) = \text{Lab}(e_1) \ldots \text{Lab}(e_n)$ is reduced. One defines the initial and the terminal vertices of $p$ by the rules $p_- = (e_1)_-$ and $p_+ = (e_n)_+$. A path $p$ is closed if $p_- = p_+$. By the definition of inverse path, we have $p^{-1} = e_{n-1}^{-1} \ldots e_1^{-1}$. For every vertex $v$, there is a path $q$ of length 0 with $q_- = q_+ = v$.

Further, speaking on other graphs labeled over some alphabet $X$, we always imply that for every edge $e$ we have a unique inverse $e^{-1}$ with $(e^{-1})^{-1} = e$ and $e^{-1} \neq e$. We see from the definition of the coset graph that it is connected, it has a base point corresponding to the subgroup $H$, and every vertex $v$ of $\Gamma$ is standard, i.e., $v$ has a standard star $\text{star}_\Gamma(v) = \text{star}(v)$, that is the set of edges with the initial vertex $v$ has exactly one $x$-edge (= the edge labeled by $x$) for every $x \in X^{\pm 1}$. The following converse statement is known.

Lemma 2.1. Let $F = F(X)$ be a free group with free basis $X$, and let $\Gamma$ be a connected graph, labeled over $X$, having base point, and with all vertices being standard. Then $\Gamma$ is the coset graph of a subgroup $H \leq F$, and the base point corresponds to the coset $H$.

Proof. Since every vertex $v$ is standard, for every $x \in X$, we have a bijection $f_x$ on the set of all vertices $V$: it maps each $v$ to the terminal vertex of the $x$-edge $e \in \text{star}(v)$. Since $F$ is free, the collection of all bijections $f_x$ extends to a (right) action $\circ$ of the entire $F$ on $V$.

The action $\circ$ is transitive since $\Gamma$ is a connected graph. Let $H$ be the stabilizer of the base point $o \in \Gamma$. Since for $g, g' \in F$ and for any action $\circ$, one has $o \circ g = o \circ g'$
if and only if $Hg = Hg'$, we have obtained the bijection $o \circ g \leftrightarrow Hg$, and the action $f_x$ of a free generator $x$ corresponds to the right multiplication by $x$. $\square$

If $H \leq H_1 \leq F$, then for $g, g' \in F$, the equality $Hg = Hg'$ implies $H_1g = H_1g'$, and therefore the mapping $Hg \mapsto H_1g$ induces a label-preserving surjective graph morphism $\phi_H^{H_1} : \Gamma \to \Gamma_1$ of the corresponding coset graphs, preserving the base point. The restriction of the mapping $\phi_H^{H_1}$ to every star is bijective because all the labeled stars are standard over the alphabet $X^{\pm 1}$. In other words, $\phi_H^{H_1}$ is a covering of labeled graphs. Obviously, the multiplicity of this covering (or its index = the number of preimages of any vertex or edge of $\Gamma_1$ in $\Gamma$) is equal to the number of cosets of $H$ in any coset of $H_1$, i.e., it is equal to the index $[H_1 : H]$ of the subgroup $H$ in $H_1$. We will apply the easy converse statement:

**Lemma 2.2.** Let $\Gamma$ and $\Gamma_1$ be connected graphs with standard stars over the alphabet $X^{\pm 1}$, and let $\phi : \Gamma \to \Gamma_1$ be a covering preserving the base point and having index $j$. Then we have the inclusion $H \leq H_1$ for the subgroups corresponding to these graphs in Lemma 2.1 and $[H_1 : H] = j$.

**Proof.** By Lemma 2.1 one may identify the graphs $\Gamma$ and $\Gamma_1$, respectively, with coset graphs of some subgroups $H$ and $H_1$ of the free group $F = F(X)$, where $o = H$ and $o_1 = H_1$ are the base points. Since every star in $\Gamma$ is standard, for every word $w$ over the alphabet $X^{\pm 1}$ and every vertex $v$ of $\Gamma$, there is a unique path $p$ with $p_- = v$ and $\text{Lab}(p) = w$. It follows from the definition of coset graph that a path $p$ of $\Gamma$ originated at $o$ is closed if and only if $H \cdot \text{Lab}(p) = H$, i.e., iff $\text{Lab}(p)$ represents an element from the subgroup $H$. Since the projection $p_1$ of a closed path $p$ to $\Gamma_1$ is closed too, we have $w = \text{Lab}(p) = \text{Lab}(p_1) \in H_1$ for every $w \in H$. Hence $H \leq H_1$. Finally, the index of the covering is equal to $[H_1 : H]$ as this was remarked before the lemma. $\square$

Thus we see that a reduced word over the alphabet $X^{\pm 1}$ represents an element of $H$ if and only if $w$ is the label of a reduced closed path of the coset graph $\Gamma$ with origin at the base point $o$. So one can consider the smallest subgraph $C$ of $\Gamma$ containing $o$ and containing all the reduced closed paths of $\Gamma$ originated at $o$. This labeled graph $C = \text{core}(H, X) = \text{core}(H)$ with the base point $o$ is called the core of $\Gamma$.

It follows from the definition that no vertex of the connected graph $C$, except for the base point $o$, has degree $\leq 1$ in $C$. (Recall that the degree of a vertex $v$ in a graph $C$ is the number of edges in $\text{star}_C(v)$.) If the degree of $o$ is 1, then there is a unique maximal path $p$ in $C$ such that $p_- = o$ and the degree of every its vertex in $C$, except for the terminal vertex $p_+$, does not exceed 2. Let us call this path the handle of $C$. We will suppose that the handle of the core is of length 0 if $C$ has no vertices of degree 1.

So the handle $p$ is attached at $p_+$, to the part $\overline{C}$ of the core having no vertices of degree 1. Removing the handle from $C$ one obtains the core $\overline{C} = \text{core}(H')$ with base point $p_+$ for a conjugate subgroup $H' = \text{Lab}(p)^{-1}H(\text{Lab}(p))$.

We draw an example of $\text{core}(H)$ depicting only the edges with “positive” labels $x$ and $y$ in Figure 1, but not their inverses labeled by $x^{-1}$ and $y^{-1}$.
Note that the notion of core of a graph was introduced by J. Stallings [17], but our definition is slightly different: Stallings did not include the handle in the core and considered only the subgraph $\overline{C}$, but we cannot do this since we may not replace the subgroup by a conjugate one in the next section. Besides, we work only with labeled graphs.

Every reduced word representing an element of $H$ can be read along a unique closed path starting at the base point. This path belongs to the minimal subgraph containing all the closed paths labeled by the reduced words generating $H$. It follows that the core $C$ is a finite graph if the subgroup $H$ is finitely generated.

As $\text{star}_C(v)$ can be smaller than the star $\text{star}_\Gamma(v)$ in a bigger graph. However if all the stars of $\Gamma$ are standard over some labeling alphabet $X$, then every $\text{star}_\mathcal{E}(v)$ contains at most one $x$-edge for every $x \in X^{\pm 1}$. It follows that every reduced path is labeled by a reduced word in such a graph.

Since all the edges of reduced closed paths of the coset graph $\Gamma$ with origin $o$ belong to the core $C$ and $\Gamma$ is a connected graph, one obtains $\Gamma$ by attaching infinite labeled trees $T_1, T_2, \ldots$ (“hairs” in terminology of [17]) to different vertices $v_1, v_2, \ldots$ of $C$ in such a way that exactly one vertex of each tree $T_i$ is identified with a vertex $v_i$ of $C$. One attaches such a tree to $v_i$ only if $v_i$ is a deficit vertex in the core, i.e., $\text{star}_C(v_i)$ does not contain an $x$-edge for some $x \in X^{\pm 1}$.

In each of the attached trees, all stars $\text{star}_{T_i}(v)$ of its vertices $v$ are standard, except for the root vertex $o(T_i)$, that coincides with the vertex $v_i$ of $C$. Still the star $\text{star}_\Gamma(o(T_i))$ must be standard in the whole coset graph $\Gamma$; so the labeling of the edges of $\text{star}_{T_i}(o(T_i))$ must complement the labeling of the star $\text{star}_C(v_i)$ of the deficit vertex $v_i$ in the core. Hence the reduced graph $\text{core}(H)$ completely determines the coset graph $\Gamma$ of $H$ up to isomorphism. Furthermore, Lemma 2.1 and the above reconstruction of $\Gamma$ from the core by attaching of labeled trees at deficit vertices, prove the following.

**Lemma 2.3.** Let $C$ be a connected reduced graph with a base point $o$, labeled over an alphabet $X$. If every vertex of $C$, except for $o$, has degree at least 2 in $C$, then $C$ is equal to $\text{core}(H)$ with base point $o$ for a subgroup $H \leq F(X)$. □

Now let $H$ be a finitely generated subgroup of $F$. Since $\text{core}(H)$ is a finite graph, the coset graph $\Gamma = \Gamma(H)$ is finite if and only if it has no infinite trees attached to the core. In other words, the index of a finitely generated subgroup $H \leq F$ is finite in $F$ if and only if $\text{core}(H)$ has no deficit vertices.

Note that the notion of core of a graph was introduced by J. Stallings [17], but our definition is slightly different: Stallings did not include the handle in the core and considered only the subgraph $\overline{C}$, but we cannot do this since we may not replace the subgroup by a conjugate one in the next section. Besides, we work only with labeled graphs.
Lemma 2.4. If a connected reduced graph $\Delta$ labeled over an alphabet $X$ has a deficit vertex, then every its nonempty subgraph $\mathcal{E}$ has also a deficit vertex.

Moreover if the subgraph $\mathcal{E}$ is finite and $\mathcal{E}$ has a deficit vertex $v$, where $\text{star}_\mathcal{E}(v)$ has no $x$-edge for some $x \in X^{\pm 1}$, then $\mathcal{E}$ has a vertex $v'$ (possibly equal to $v$) such that

(a) $\text{star}_\mathcal{E}(v')$ has no $x^{-1}$-edge in $\mathcal{E}$, and

(b) the vertex $v$ is connected with $v'$ in $\mathcal{E}$ by a path whose label is a power of $x$.

Proof. If $\mathcal{E}$ is a proper subgraph with standard stars $\text{star}_\mathcal{E}(v)$, then there remain no edges connecting $\mathcal{E}$ with the compliment $\Delta \setminus \mathcal{E}$, contrary the connectedness of $\Delta$. This proves the first assertion.

If $\text{star}_\mathcal{E}(v)$ has no $x^{-1}$ edge, then we take $v' = v$. Otherwise, since the subgraph $\mathcal{E}$ is finite, there is a maximal path $p$ starting at $v$ and having label of the form $x^n$, where $n \geq 0$. We define $v' = p_+$, and by the maximality of $p$, $\text{star}_\mathcal{E}(v')$ has no $x^{-1}$ edge. □

3. MAIN LEMMAS AND PROOF OF THEOREM 3.1

Let $H$ be a subgroup of a free group $F = F(X)$ with a free base $X$, and let $\Gamma$ be the coset graph of $H$ with respect to $X$. For a subset $Y \subset X$, let $F(Y)$ denote the subgroup of $F$ generated by $Y$. A vertex $v$ of a subgraph $E \subset \Gamma$ will be called a $Y$-deficit vertex in $E$ if there is $y \in Y^{\pm 1}$ such that $\text{star}_E(v)$ has no edge labeled by $y$.

We define the $Y$-frame $\text{frame}(H, Y)$ as the maximal connected subgraph of $\text{core}(H)$ containing the base point and having all edge labels in $Y^{\pm 1}$.

Lemma 3.1. If the subgroup $H$ is finitely generated, has infinite index in $F$, and it is not contained in $F(Y)$, then $H$ contains a subgroup $K$ of finite index in $H$ such that $\text{core}(K)$ has a deficit vertex which does not belong to $\text{frame}(K, Y)$.

Proof. Assume first that the graph $\mathcal{C} = \text{core}(H)$ has an edge $e$ such that

1. the removal of $e^{\pm 1}$ from $\text{core}(H)$ makes the remaining part $\mathcal{C}'$ of $\mathcal{C}$ disconnected (let us call such an edge a bridge in $\mathcal{C}$) and

2. $\text{Lab}(e) \in X \setminus Y$.

Then the graph $\mathcal{C}'$ has two connected components $\mathcal{D}$ and $\mathcal{E}$, where $\mathcal{D}$ contains the base point $o$ of $\mathcal{C}$, and so it contains the whole $\text{frame}(H, Y)$ since $\text{Lab}(e) \notin Y^{\pm 1}$.

By Lemma 2.4 the graph $\mathcal{E}$ must have a deficit vertex $v$ which remains to be deficit when one gets the edges $e^{\pm 1}$ back, i.e., $v$ is deficit in $\mathcal{C}$. It follows that in $\text{core}(H)$, the deficit vertex $v$ is separated from $\text{frame}(H, Y)$ by the edges $e^{\pm 1}$, and so one may choose $K = H$.

Thus we may further assume that every bridge of $\mathcal{C}$ is labeled by a letter from $Y^{\pm 1}$.

Now we can find $K$ as a subgroup of $H$ with arbitrarily prescribed index $j \geq 2$.

To define $K$ we will construct a covering $f : \Delta \to \Gamma = \Gamma(H)$ of index $j$.

Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{j-1}$ be copies of the coset graph $\Gamma$ and let $\mathcal{C}_i$ be the core of $\Gamma_i$ for $i = 0, \ldots, j - 1$. Note that $\mathcal{C}_0$ has an edge $e_0$ labeled by a letter $x \in X \setminus Y$ because $H$ is not a subgroup of $F(Y)$. The edge $e_0$ is not a bridge in $\mathcal{C}_0$ by the above assumption. Hence one obtains connected graphs after removal of $e_0$ from $\mathcal{C}_0$ or from $\Gamma_0$.

Let $e_0$ connect some vertices $u_0$ and $v_0$ of the graph $\mathcal{C}_0$. By $e_i, u_i$, and $v_i$ we denote the copies of $e_0, u_0$, and $v_0$, respectively, in $\mathcal{C}_i$ ($i = 1, \ldots, j - 1$).
To define $\Delta$ we modify the disjoint union $\Gamma_0 \sqcup \cdots \sqcup \Gamma_{j-1}$ as follows. Preserving all the vertices and all the edges except for $e_0, \ldots, e_{j-1}$, we redirect each $e_i$. To be exact, we replace it by a new edge $e'_i$ with the same label $x$, but $e'_i$ connects $u_i$ with $v_{i+1}$ (indices are taken modulo $j$). The edge $(e'_i)^{-1}$ is redirected respectively.

The obtained graph $\Delta$ is connected because for every $i$, the removal of the edges $e_i^\pm 1$ does not break the connectedness of $\Gamma_i$. The covering $f : \Delta \rightarrow \Gamma$ for $j = 2$

By definition, the function $f$ maps every vertex and every non-modified edge of each $\Gamma_i$ to its copy in $\Gamma$, and $f((e'_i)^\pm 1)$ is the copy of $e_i^\pm 1$ in $\Gamma$. Clearly, $f$ is a covering of $\Gamma$ of index $j$, and so by Lemmas 2.1 and 2.2 $\Delta$ is the coset graph for a subgroup $K$ having index $j$ in $H$. (In fact $K$ is normal in $H$ with cyclic quotient $H/K$.)

We choose the vertex $o_\Delta$ such that $f(o_\Delta) = o_\Gamma$ and $o_\Delta$ belongs to $C_0$ as the base point of $\Delta$. Let $C'_i = C_i \backslash \{e_i^\pm 1\}$ ($i = 0, \ldots, j - 1$). Each graph $C'_i$ is connected and contains the handle $p_i$ of $C_i$ since $e_i$ is not a bridge in $C_i$. Hence the subgraph $C_\Delta$ of $\Delta$ formed by the edges $(e'_i)^\pm 1$ ($i = 0, \ldots, j - 1$) and by the graphs $C'_0, C'_1, \ldots, C'_{j-1}$ without the $j - 1$ handles $p_1^\pm 1, \ldots, p_{j-1}^\pm 1$ is also connected. The graph $C_\Delta$ has at most one vertex of degree $\leq 1$ (must be equal to $(p_0)^- = o_\Delta$ if any exists), and $C_\Delta$ contains all reduced closed paths of $\Delta$ originated at $o_\Delta$ because every edge in $\Delta \setminus C_\Delta$ belongs to some attached infinite tree. Hence $C_\Delta$ is equal to core($K$).

The degrees of the vertices from $C'_i$ in core($K$) do not exceed their degrees in $\Gamma_i$. Now recall that $H$ is finitely generated and has infinite index in $F$; therefore core($H$) has a deficit vertex. So every $C_i$ has a deficit vertex which remains to be deficit not only in $C'_i$ but also in $\Delta$ by Lemma 2.4. So each of the parts $C'_0, \ldots, C'_{j-1}$ of $\Delta$ has a deficit vertex in $\Delta$.

On the other hand, the subgraph frame($K, Y$) is contained in $C'_0$ since $C'_0$ is connected with the parts $C'_1$ and $C'_{j-1}$ by $x$-edges $(e'_0)^\pm 1$ and $(e'_{j-1})^\pm 1$ only, where $x \notin Y^\pm 1$. Thus the deficit vertex of $\Delta$ belonging to $C'_1$ is the desired one. □
For the next step, we need the following weaker form of Theorem 1.1.

**Lemma 3.2.** Let $A$ and $B$ be finitely generated subgroups of infinite index in a free group $F$. Then there exist two subgroups $A_1$ and $B_1$ in $F$ such that

(a) $A_1$ is a subgroup of finite index in $A$;
(b) $B_1$ is a subgroup of finite index in $B$;
(c) the subgroup $C = \langle A_1, B_1 \rangle$ is of infinite index in $F$.

**Proof.** Since $A$ is finitely generated, it is a free factor in a subgroup $E$ having finite index in $F$ by the theorem of M. Hall (see [8], [12], I.3.10). Then $B \cap E$ has finite index in $B$, and therefore it suffices to prove the lemma for the subgroup pair $(A, B \cap E)$ in the free group $E$ rather than for the pair $(A, B)$ in $F$. (Now and subsequently we use that a subgroup of a free group is free and a subgroup of finite index in a finitely generated group is finitely generated itself; see [12], II,4.2.)

So one may assume from the very beginning that $A$ is a free factor of $F$. In other words, $F$ has a free basis $X$ such that $A$ is freely generated by a finite subset $Y \subset X$.

If $B \leq A = F(Y)$, then obviously one can put $A_1 = A$ and $B_1 = B$, which proves the lemma. So we will assume that $B$ is not contained in $F(Y)$. By Lemma 3.1, $B$ has a subgroup $B_1$ of finite index such that the subgroup $frame(B_1, Y)$ does not contain at least one deficit vertex from $core(B_1)$.

Now we are going to embed the graph $core(B_1)$ in the core of the coset graph of a bigger finitely generated subgroup $C$. We will add edges to $core(B_1)$ as follows. Assume that $v$ is a $Y$-deficit vertex in $frame(B_1, Y)$. It is also $Y$-deficit in $core(B_1)$ by the definition of $Y$-frame. Then using Lemma 3.1 we can find a $Y$-deficit vertex $v'$ in the frame $frame(B_1, Y)$ such that after the adding of a $y$-edge ($y \in Y^{\pm 1}$) connecting $v$ and $v'$, the extended core is still reduced.

By Lemma 2.3, after each step of this procedure, we have a core for some subgroup of $F$. We will keep doing such extensions until the $Y$-frame has no $Y$-deficit vertices. There appear only finitely many new edges since we do not add new vertices. Therefore the subgroup $C$ given by the extended core is finitely generated, $core(C)$ contains $core(B_1)$, and so $C \supseteq B_1$. Since the subgraph $frame(C, Y)$ has no $Y$-deficit vertices, it is equal to the core of the subgroup $A_1 = F(Y) \cap C = A \cap C$, having finite index in $A$. (In fact, this argument also proves M. Hall’s theorem: $B_1 \cap A$ is a free factor of a subgroup $A_1$ having finite index in $A$.)

We also have $C = \langle A_1, B_1 \rangle$ since we added only generators from $A_1$ to the generators of $B_1$ when we extended the core of $B_1$.

Finally, $C$ has infinite index in $F$ since our extensions do not touch the deficit vertex of $core(B_1)$ lying outside of $frame(B_1, Y)$. Thus, the properties (a)–(c) are obtained. \qed

We will use the following easy generalization of Lemma 3.2.

**Lemma 3.3.** Let $L_1, L_2, \ldots, L_m$ be finitely generated subgroups of infinite index in a free group $F$ ($1 \leq m < \infty$). Then there exist subgroups $N_1, N_2, \ldots, N_m$ of finite indices in $L_1, L_2, \ldots, L_m$, respectively, such that the subgroup $\langle N_1, N_2, \ldots, N_m \rangle$ has infinite index in $F$.

**Proof.** The statement is obvious if $m = 1$. If $m \geq 2$, then it follows from Lemma 3.2 that one can obtain a subgroup $\langle N_1, N_2 \rangle = C$ of infinite index in $F$ with $N_1$ and $N_2$ having finite indices in $L_1$ and $L_2$, respectively. Therefore the subgroups $N_1, N_2$, and $C$ are finitely generated. In turn, there are a subgroup $H$ of finite
index in $C$ and a subgroup $N_3$ of finite index in $L_3$ such that $\langle H, N_3 \rangle$ is of infinite index in $F$. But $H$ contains some subgroups $N_1'$ and $N_2'$ of finite indices in $N_1$ and $N_2$, respectively, namely $N_1' = N_1 \cap H$ and $N_2' = N_2 \cap H$, and so, in $L_1$ and in $L_2$ as well. Therefore the subgroup $\langle N_1', N_2', N_3 \rangle$ also has infinite index in $F$. Arguing in this way we complete the proof by induction on $m$. \hfill \Box

**Lemma 3.4.** Let $A$ and $B$ be finitely generated subgroups of infinite index in a free group $F$. Then there is a subgroup $D$ of $F$ such that

(a) $D$ is finitely generated;

(b) $D$ has infinite index in $F$;

(c) $D$ contains a subgroup $H$ of $B$ having finite index in $B$;

(d) the normalizer of $D$ in $F$ contains $A$.

**Proof.** Let $A_1, B_1$, and $C$ be the subgroups of $F$ given by Lemma 3.2. We choose some (finite) left transversal $\{a_1, \ldots, a_m\}$ of $A_1$ in $A$ and define $L_i = a_iCa_i^{-1}$ $(i = 1, \ldots, m)$. By Lemma 3.3 there are subgroups $N_1 \leq L_1, \ldots, N_m \leq L_m$ with finite indices $[L_i : N_i]$ such that the subgroup $D = \langle N_1, \ldots, N_m \rangle$ has infinite index in $F$.

Write $N_i = a_iQ_i a_i^{-1}$, where $[C : Q_i] < \infty$. One can replace each $Q_i$ by a single $Q = \bigcap_{i=1}^m Q_i$, and $Q$ also has finite index in $C$. Moreover, decreasing $Q$, we may assume that it is normal in $C$. Then it follows that $D$ is invariant under conjugations by all $a \in A$ since for any $a_i$ there are some $j$ and $a' \in A_1$ such that

$$aa_i = a_j a', \text{ and therefore}$$

$$aN_i a^{-1} = aa_iQ_i a_i^{-1} a^{-1} = a_j a'Q a'^{-1} a_j^{-1} = a_j Q a_j^{-1} = N_j.$$ 

Here $a'Q a'^{-1} = Q$ because $Q$ is normal in $C$ and $a' \in A_1 \leq C$ by Lemma 3.2 (c).

Thus properties (a), (b), and (d) of $D$ are obtained. To complete the proof of the lemma, we may assume that $a_1 = 1$, and so $D \geq N_1 = Q$, but $Q$ is a subgroup of finite index in $C$, and therefore $Q$ must contain a subgroup $H$ of finite index in $B$ by Lemma 3.2 (b). This proves property (c). \hfill \Box

**Proof of Theorem 1.1** One may assume that $B \neq \{1\}$. Let $D$ be the subgroup provided by Lemma 3.4. By Lemma 3.4 (d), $D$ is normal in $\langle A, D \rangle = AD$. It is finitely generated by Lemma 3.3 (a). But a non-trivial finitely generated normal subgroup of a free group must have finite index (see [12], I.3.12), that is $|AD/D| < \infty$. Hence $\langle A, D \rangle$ has infinite index in $F$ by Lemma 3.4 (b). So has $\langle A, H \rangle$, where $H$ is a subgroup of $B \cap D$ provided by Lemma 3.4 (c). The first claim of the theorem is proved.

Now we recall that given a finitely generated subgroup $A$ of a free group $F$ and a finite subset $S \subset F \setminus A$, there is a subgroup $M$ of finite index in $F$ such that $M \geq A$ and $M \cap S = \emptyset$ (see [12], I.3.10). Therefore we have $\langle M \cap \langle A, H \rangle \rangle \cap S = \emptyset$ and consequently,

$$\langle A, M \cap H \rangle \cap S = \langle M \cap A, M \cap H \rangle \cap S \leq (M \cap \langle A, H \rangle) \cap S = \emptyset.$$

So to prove the second claim of the theorem we just replace $H$ by the subgroup $H' = H \cap M$ having finite indices in $H$ and in $B$. \hfill \Box
4. Applications to the actions of free groups

**Corollary 4.1.** (a) A finitely generated free group $F$ has a subgroup $R$ of infinite index, such that for every finitely generated subgroup $L \leq F$ of infinite index in $F$, the index $[L : L \cap R]$ is finite.

(b) Any subgroup $R$ with property (a) contains no non-trivial normal subgroups of $F$.

**Proof.** (a) Let $L_1, L_2, \ldots$ be an enumeration of all finitely generated subgroups having infinite index in $F$. Let us put $R_1 = L_1$. For $i > 1$, by induction, we define $R_i = \langle R_{i-1}, H_i \rangle$, where $H_i$ is a subgroup of finite index in $L_i$ such that $R_i$ is a finitely generated subgroup of infinite index in $F$. Such a choice of $H_i$ is possible by Theorem 4.1 since by the inductive hypothesis, $R_{i-1}$ is a finitely generated subgroup of infinite index in $F$.

Now we define $R = \bigcup_{i=1}^{\infty} R_i$. We see that $R$ is a union of the members of the non-decreasing series of the subgroups $R_i$ having infinite index in $F$. Hence the subgroup $R$ has infinite index in $F$ itself because $F$ is finitely generated, and so is every subgroup of finite index in $F$. Since for every $i$, the subgroup $R_i$ contains a subgroup $H_i$ of finite index in $L_i$, statement (a) is proved.

**Remark 4.2.** The subgroup $R$ must be finitely generated because otherwise, by M. Hall’s theorem, there would exist a subgroup $L$ such that $\langle L \rangle = R \ast L$ is of finite index in $F$, and such finitely generated $L$ would violate the property for $R$.

(b) To prove the second statement we need one more lemma.

**Lemma 4.3.** For every non-trivial normal subgroup $N$ of a free group $F = F(x_1, \ldots, x_r)$ of finite rank $r \geq 2$, there exists an $r$-generated subgroup $H$ of infinite index in $F$ such that $F = HN$.

**Proof.** Using conjugation by the generators, we easily obtain a reduced word $w \in N$ of the form $x_1v_{x_1^{-1}}$. Define $H = \langle x_1w, x_2, \ldots, x_r \rangle$. Clearly, there are no cancellations in the products of powers of the distinct generators of $H$. Hence these generators form a free basis of $H$ and, for the same reason, we have $H \cap \langle x_1 \rangle = 1$, which implies that $[F : H] = \infty$. Since modulo $N$, the basis of $H$ is equal to $(x_1, \ldots, x_r)$, the equality $F = HN$ follows.

To finish the proof of statement (b) we may assume that the free rank $r$ of $F$ is at least 2. Assume that the subgroup $R$ contains a non-trivial subgroup $N$ normal in $F$. Let $H$ be the finitely generated subgroup of infinite index in $F$ provided by Lemma 4.3. Since $R$ contains $N$ and, by (a), it contains a subgroup of finite index from $H$, we conclude that $R$ has finite index in $F$ because $F = HN$. The obtained contradiction with property (a) completes the proof of (b).

**Remark 4.4.** Obviously, the index $[L : L \cap R]$ is infinite for every subgroup $L$ of finite index. Also one cannot omit the assumption that $L$ is finitely generated in part (a) of Corollary 4.1 in particular, it follows from (b) that the index $[L : L \cap R]$ is infinite for every non-trivial normal subgroup $L$ of $F$. Indeed, otherwise there is a characteristic subgroup $N$ of $L$ such that $N \leq L \cap R \leq R$ and the factor group $L/N$ has finite exponent dividing $[L : L \cap R]$. Therefore $N$ is a non-trivial normal subgroup of $F$ contrary to Lemma 4.1 (b).

The following reformulation of Corollary 4.1 in terms of the actions of $F$ answers the question raised in Open Problem 3 of [1].
Corollary 4.5. (a) Every non-trivial finitely generated free group $F$ admits a transitive action on an infinite set $S$ such that the restriction of this action to any finitely generated subgroup $L$ of infinite index in $F$ is locally finite.

(b) Every action $\circ$ of $F$ satisfying the conditions from (a) is faithful.

Proof. (a) Let $F$ act by right translations on the set $S$ of right cosets of the subgroup $R$ provided by Corollary 4.4. Then $S$ is infinite and the action on $F$ is transitive. An $L$-orbit $(Rg)L$ of a point $Rg$ with respect to the action of a finitely generated subgroup $L$ with infinite index $[F : L]$ has the same size as the orbit $RL'$ of the action of the conjugate subgroup $L' = gLg^{-1}$, which is also finitely generated and has infinite index in $F$. But the stabilizer of the point $R$ under the action of $L'$ is $L' \cap R$. So the size of the orbit $RL'$ is equal to the index $[L' : L' \cap R]$, which is finite by Corollary 4.1. The local finiteness is proved.

(b) The stabilizer $R$ of any point $v$ must have infinite index in $F$ since $F$ transitively acts on an infinite set. Since the restriction of the action $\circ$ to every finitely generated subgroup $H$ of infinite index is locally finite, the $H$-orbit $v \circ H$ is finite. Since the size of this orbit is equal to $[H : H \cap R]$, we conclude by Corollary 4.1 that $R$ does not contain non-trivial normal subgroups of $F$. Hence the kernel of the action $\circ$ is trivial, that is this action is faithful.

The approach used in the proof of Corollaries 4.4 and 4.5 can be employed to unite different extremal properties of group actions. We combine two opposite properties of actions, namely, locally finiteness and multiple transitivity. First faithful highly transitive actions of non-cyclic free groups of finite ranks on infinite sets were obtained in [13] (see other proofs in [3], [1]). Similar results were proved for free products of groups in [6], [9], and [7] (see also [15]). Quite recently this theorem was extended to surface groups [11], to $Out(F_n)$ [5], and finally, to all non-elementary hyperbolic groups with trivial finite radical [2].

Applying Theorem 1.1 and a theorem from [1], one can sharpen Corollary 4.5 and obtain Corollary 1.2.

Proof of Corollary 1.2. As in the proof of Corollary 4.5, the group $F$ acts by right multiplications on the set of the right cosets of a subgroup $R$. However the construction of $R$ is modified in comparison with Corollary 4.4 as follows. We will enumerate both finitely generated subgroups of infinite index in $F$ and all $2k$-tuples $(g_1, \ldots, g_k, g'_1, \ldots, g'_k)$ of elements from $F$ for all $k \geq 1$. In the inductive definition of the increasing series $R_1 \leq R_2 \leq \ldots$, we introduce $R_i$ using the rule from Corollary 4.1 if $i$ is odd. Namely, if $i = 2j - 1$ for $j > 1$, then now $R_{2j-1} = \langle R_{2j-2}, H_j \rangle$, where $H_j$ is a subgroup of finite index in $L_j$ such that $R_{2j-1}$ has infinite index in $F$. For even $i$-s, we will apply the following Theorem 6 from [1]:

Let $H$ be a finitely generated subgroup of infinite index in a free group $F$ of rank $r > 1$. Let $(Hg_1, \ldots, Hg_k)$ and $(Hg'_1, \ldots, Hg'_k)$ $(k \geq 1)$ be two $k$-tuples of pairwise different cosets. Then there is a finitely generated subgroup $H'$ of infinite index in $F$ and an element $b \in F$, such that $H \leq H'$ and $H'g_jb = H'g'_jb$ for every $j = 1, \ldots, k$.

Now if $i$ is even, we take the first tuple $(g_1, \ldots, g_k, g'_1, \ldots, g'_k)$ (if any exists) in our enumeration such that (1) it was not considered at the previous steps, (2) the cosets $R_{i-1}g_1, \ldots, R_{i-1}g_k$ are pairwise different, and (3) the cosets $R_{i-1}g'_1, \ldots, R_{i-1}g'_k$ are pairwise different too. Then we apply the cited theorem from [1] to the subgroup $H = R_{i-1}$ and to the elements $g_1, \ldots, g_k, g'_1, \ldots, g'_k$ and obtain $R_i = H'$. If there are no tuples with properties (1)–(3) then we set $R_i = R_{i-1}$.
Again the subgroup $R = \bigcup_{i=1}^{\infty} R_i$ has infinite index and the properties from Corollary 4.5 hold. In addition, the action is now $k$-transitive for any $k \geq 1$ because if we have two $k$-tuples of pairwise distinct cosets $(Rg_1, \ldots, Rg_k)$ and $(Rg'_1, \ldots, Rg'_k)$, then the corresponding cosets remain pairwise distinct modulo every subgroup $R_i$ for $1 \leq R$, and so by the above construction, we should have $R_i g_i b = R_i g'_i$ for some integer $i, b \in F$, and $j = 1, \ldots, k$. It follows that $R g_j b = R g'_j$, as required for the $k$-transitivity. 

**Remark 4.6.** One can derive an effective bound for the numbers of edges and vertices in the coset graph of the subgroup $H$ in terms of the coset graphs of the subgroups $A$ and $B$ given in the formulation of Theorem 1.1 and for given $A$ and $B$, the set of generators of $H$ can be found algorithmically. Taking into account the decidability of the membership problem for finitely generated subgroups of free groups, one can construct a Turing machine enumerating the generators of the subgroup $R$ in Theorem 1.1 and its corollaries. Furthermore, the last sentence of Theorem 1.1 is a clue to the machine enumeration of the complement $F \setminus R$. So the subgroup $R$ can be recursive in the statements of this paper.

However the upper bounds for the parameters of $H$ based on our proof of Theorem 1.1 are too rough and unsatisfactory. The problem of finding of realistic estimates is open.

Another open problem was formulated in [1]: Can the action with property (a) from Corollary 4.5 have maximal growth? (See details in [1].)

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