Non-analyticity of the Callan-Symanzik $\beta$-function of O($N$) models.

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In the framework of the 1/$N$ expansion we show that the Callan-Symanzik $\beta$-function associated with the four-point coupling $g$ is non-analytic at its zero, i.e. at the fixed-point value $g^*$ of $g$. This singular behavior can be interpreted by renormalization group arguments, and written in terms of scaling correction exponents.

We obtain accurate determinations of $g^*$ in 3-d and 2-d by exploiting two alternative approaches: the $\epsilon$-expansion in the $\phi^4$ formulation of the O($N$) model, and the high-temperature expansion of the lattice $N$-vector (O($N$) nonlinear $\sigma$) model. These results are compared with the available estimates by other approaches, such as the fixed-dimension perturbative expansion, Monte Carlo simulations, etc...

We also present results for the $n$-point renormalized coupling constants that parameterize the behavior of the effective potential in the high- and low-temperature phases.

The renormalization-group theory of critical phenomena provides a description of statistical models in the neighbourhood of the critical point. For O($N$) models calculations are based on the $\phi^4$-field theory. A strategy, which has been largely employed in the study of the symmetric phase, relies on a perturbative expansion in powers of the zero-momentum four-point coupling $g$ performed at fixed dimension. The theory is renormalized at zero momentum by requiring

\begin{align}
\Gamma^{(2)}(p)_{\alpha\beta} &= \delta_{\alpha\beta} Z_G^{-1} [m^2 + p^2 + O(p^4)] \\
\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} &= Z_G^{-2} m g \delta_{\alpha\beta} \delta_{\gamma\delta}/3
\end{align}

where $\delta_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}$. When $m \to 0$ the coupling $g$ is driven toward an infrared stable zero $g^*$ of the Callan-Symanzik $\beta$-function $\beta(g) \equiv m \partial g/\partial m|_{g_0\Lambda}$. (3)

$g^*$ is also obtained as the critical limit of

$g_\sigma = -\frac{3N}{N + 2\chi^2 \xi^d} \to g^*$

where $\chi$ is the magnetic susceptibility, $\xi$ the second-moment correlation length, and $\chi_4 = \sum_{i,j,k}(\phi(0) \cdot \phi(x_i) \phi(x_j) \cdot \phi(x_k))_c$. We recall that $N$-vector (nonlinear O($N$) $\sigma$) and $\phi^4$ models describe the same critical behavior.

In the framework of the 1/$N$ expansion the analysis of the next-to-leading order shows that the Callan-Symanzik $\beta$-function is non-analytic at its zero, i.e. at the fixed-point value $g^*$ of $g$. The large-$N$ result agrees with the singular behavior

\begin{align}
\beta(g) &= -\omega(g^*-g) + \text{analytic terms} \\
&+ c_1 (g^*-g)^{1+\frac{\Delta}{2}} + \ldots + d_1 (g^*-g)^{\Delta_2} + \ldots 
\end{align}

(\Delta = \omega \nu$ and $\Delta_2$ are scaling correction exponents) that can be derived using renormalization group arguments$^4$.

A precise determination of $g^*$ is crucial in the field-theoretic approach based on the $g$-expansion, where the critical exponents are obtained by evaluating appropriate (resummed) anomalous dimensions at $g^*$. In this approach the resummation of the $g$-expansion is usually performed following the Le Guillou Zinn-Justin (LZ) procedure$^5$, which assumes the analyticity of the $\beta$-function. The presence of confluent singularities may then cause a slow convergence to the correct fixed-point value, leading to an underestimation of the uncertainty. A more general analysis explicitly allowing for the presence of confluent singularities would slightly change the value of $g^*$ for small values of $N$ and consequently the values of the critical exponents$^6$. It is therefore important to exploit other approaches to the study of O($N$) models, which can provide a check of the estimates of $g^*$ from the resummations of the $g$-expansion. We considered two al-
ternative approaches: the ε-expansion in the continuum $\phi^4$ formulation and the high-temperature (HT) expansion of the $N$-vector (lattice $O(N)$ σ) model. We extended the ε-expansion of $g^*$ to $O(\epsilon^4)$. Accurate estimates of $g^*$ in 3-d and 2-d were obtained by a constrained analysis of the ε-series using its known values at lower dimensions. Moreover we reanalyzed the available HT expansion of $g_\sigma$ in the $N$-vector models, by a method able to handle the leading confluent singularity (for a more recent analysis using longer series see Ref. [8]). In Table 1 and 2 we present our 3-d and 2-d results respectively. For comparison we also report some of the available estimates from other approaches. The agreement among the various estimates of $g^*$ is globally good.

The results in Table 1 indicate that the systematic error in the LZ resummation due to the non-analytic terms in Eq. [1] should be small. This may be explained by the fact that, for small values of $N$, the exponents in Eq. [3] are close to integer numbers, indeed $\Delta_2/\Delta \simeq 2$, $\Delta_3/\Delta \simeq 3$, and $1 + 1/\Delta \simeq 3$. However, the results for $N = 0, 1$ are slightly lower than the estimates given by the LZ resummation of the $O(g^\gamma)$ series of $\beta(g)$, thus favouring the more general analysis of Ref. [4]. This would lead to a small change in the estimates of the critical exponents. For instance, in the case $N = 0$ (self avoiding walks) the resummation of the $O(g^7)$ series of $\gamma(g)$ evaluated at $\bar{g}^* = 1.413(6)$ gives $\gamma(\bar{g}^*) \simeq 1.160$. A lower value $\bar{g}^* \simeq 1.39$, as indicated by our calculations, would lead to $\gamma(\bar{g}^*) \simeq 1.158$, in substantial agreement with the recent result of Monte Carlo simulations $\gamma = 1.1575(6)$ [12] and with the analysis of the ε-expansion: $\gamma \simeq 1.158$.

The effective potential is widely used in the field-theoretic description of fundamental interactions and phase transitions. In statistical physics it represents the free-energy density $\mathcal{F}$ as a function of the order parameter. $\mathcal{F}$ can be expanded in powers of the renormalized magnetization $\varphi$

$$\Delta \mathcal{F} = \sum_{j=1}^{m_2} m_2^{m_2+1-j} d \frac{1}{(2j)!} g_{2j} \varphi^{2j}$$

(6)

where $\Delta \mathcal{F} = \mathcal{F}(\varphi) - \mathcal{F}(0)$, and $g_{2j}$ are the zero-momentum $2j$-point renormalized coupling.

By definition $g_2 = 1$ and $g_4 = g$. Setting $\varphi = m_{(d-2)/2} z / \sqrt{\pi}$ and $r_{2j} = g_{2j} / g^{2j-1}$ we write

$$\Delta \mathcal{F} = \frac{m_d}{g} \left( \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \sum_{j=3}^{1} \frac{1}{(2j)!} r_{2j} z^{2j} \right)$$

(7)
Table 3
Three-dimensional estimates of $r_6$ and $r_8$. A more complete list of the estimates of $r_{2j}$ is reported in [13].

| $N$ | $\epsilon$-exp. | $g$-exp. | ERG | H.T. | $\epsilon$-exp. | $g$-exp. | ERG | H.T. |
|-----|----------------|----------|-----|-----|----------------|----------|-----|-----|
| 1   | 2.058(11)      | 2.053(8) | 2.064(36) | 1.99(6) | 2.48(28)      | 2.47(25) | 2.47(5) | 2.7(4) |
| 2   | 1.94(11)       | 1.967    | 1.83 | 2.2(6) | 3.5(1.3)      | 1.4       |       |     |
| 3   | 1.84(9)        | 1.880    | 1.74 | 2.1(6) | 2.1(1.0)      | 0.84      |       |     |
| 4   | 1.75(7)        | 1.803    | 1.65 | 1.9(6) | 1.2(1.0)      | 0.33      |       |     |

In order to evaluate the first few $r_{2j}$ we performed a constrained analysis of the $\epsilon$-expansion of $r_{2j}$. In Table 3 we report our 3-d results for $r_6$ and $r_8$. We compare them with some of the available estimates from other approaches, such as $d = 3$ $g$-expansion [14], approximate solution of the exact renormalization group equation (ERG) [15,16], high-temperature expansion [17,18]. In the case of the Ising model also $r_{10}$ has been roughly estimated: $r_{10} = -20(15)$.

In two dimensions an analysis of the HT expansion of the free-energy of the Ising model in the presence of an external field gave $r_6 = 3.678(2)$, $r_8 = 26.0(2)$ and $r_{10} = 275(15)$. These numbers compare well with the estimates $r_6 = 3.69(4)$ and $r_8 = 26.4(1.0)$ obtained from our constrained analysis of the $\epsilon$-expansion. Moreover we obtained $r_6 = 3.54(7)$ and $r_8 = 25.1(2.0)$ for $N = 2$ (2-d XY model), and $r_6 = 3.33(6)$ and $r_8 = 20.3(1.7)$ for $N = 3$.

In the broken phase of the 3-d Ising model the effective potential at the coexistence curve can be expanded as

$$F(\varphi) - F(\varphi_0) = \sum_{j=2} m^{d-j(d-2)/2} \frac{1}{j!} g_j (\varphi - \varphi_0)^j$$

where $g_j$ are the zero-momentum $j$-point renormalized coupling in the broken phase ($g_2 = 1$ by definition). A constrained analysis of the $\epsilon$-expansion [19] gave the estimates $g_3 = 13.06(12)$ and $g_4 = 75(7)$. The parametrization (8) does not apply to the case $N \neq 1$, due to the presence of Goldstone bosons. One indeed finds

$$F(\varphi) - F(\varphi_0) \approx c (\varphi^2 - \varphi_0^2)^{d/(d-2)}$$

In 3-d, where $d/(d-2) = 3$, logarithms appear in the corrections to the leading behavior.

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