Entropy, subentropy and the elementary symmetric functions

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Abstract

We use complex contour integral techniques to study the entropy $H$ and subentropy $Q$ as functions of the elementary symmetric polynomials, revealing a series of striking properties. In particular for these variables, derivatives of $-Q$ are equal to derivatives of $H$ of one higher order and the first derivatives of $H$ and $Q$ are seen to be completely monotone functions. It then follows that $\exp(-H)$ and $\exp(-Q)$ are Laplace transforms of infinitely divisible probability distributions.

It is a striking fact that the entropy $H$ and subentropy $Q$ (cf. [5, 6]) are symmetric functions of their arguments:

$$H(x_1, \ldots, x_d) = -\sum_{i=1}^{d} x_i \ln x_i$$

$$Q(x_1, \ldots, x_d) = -\sum_{i=1}^{d} \frac{x_i^d}{\prod_{j \neq i} (x_i - x_j)} \ln x_i.$$  (2)

It is thus perhaps natural to study them as functions of the associated elementary symmetric polynomials defined by

$$e_1 = \sum_j x_j, \quad e_2 = \sum_{i<j} x_i x_j, \quad e_3 = \sum_{i<j<k} x_i x_j x_k, \ldots$$

Here we will lift the probability condition $e_1 = \sum x_j = 1$ and view $e_1, e_2, \ldots, e_d$ as independent variables. Without loss of generality we will list the $x_j$’s in non-increasing order $0 < x_1 \leq x_2 \leq \ldots \leq x_d$.

For the case of $H$ Fannes [1] recently obtained the following elegant expression for the derivatives of $H$ with respect to $e_2, e_3, \ldots, e_d$ (cf. eq. (9) of [1]):

$$\frac{\partial H}{\partial e_k} = \int_0^{\infty} dt t^{d-k} \frac{t^d + e_1 t^{d-1} + e_2 t^{d-2} + \ldots + e_d}{(t^d + e_1 t^{d-1} + e_2 t^{d-2} + \ldots + e_d)} \quad k = 2, \ldots, d,$$

and in particular deduced that $\frac{\partial H}{\partial e_k} \geq 0$ for $k \geq 2$ (which had been shown previously by other means in [3]). Fannes established eq. [1] by starting with an inscrutably ingenious
integral identity (eq. (6) in [1]). Here we will give an alternative derivation based on complex contour integration techniques and we will also treat the case of subentropy as a function of the elementary symmetric polynomials. Our formulae will reveal a remarkable relationship between the derivatives of entropy and subentropy viz.

\[-\frac{\partial Q}{\partial e_k} = \frac{\partial^2 H}{\partial e_l \partial e_m}\] for any \(k, l, m\) with \(k = l + m\) and \(l, m \geq 1\).

We will also point out a series of further properties of (higher order) derivatives of \(H\) that follow directly from eq. (3) (and also from our contour integral expressions) and which establish the property of complete monotonicity of \(\frac{\partial H}{\partial e_k}\) for \(k \geq 2\) on \(\{ (e_1, \ldots, e_d) : e_k > 0 \text{ for all } k \}\).

We begin with the fundamental relation between the \(e_k\)'s and \(x_j\)'s viz. that \(x_1, \ldots, x_d\) are the roots of the polynomial equation

\[x^d - e_1 x^{d-1} + e_2 x^{d-2} - \ldots + (-1)^d e_d = 0.\] (4)

This defines each \(x_j\) implicitly as a function of the \(e_k\)'s and implicit differentiation gives

\[\frac{\partial x_j}{\partial e_k} = \frac{(-1)^{k+1} x_j^{d-k}}{\prod_{i \neq j} (x_j - x_i)}\]

so then the chain rule gives (as elaborated in [3] eqs. (10) - (16))

\[\frac{\partial H}{\partial e_k} = (-1)^k \sum_{j=1}^{d} \frac{x_j^{d-k} \ln x_j}{\prod_{i \neq j} (x_j - x_i)}\] for \(k \geq 2\) (5)

and

\[\frac{\partial H}{\partial e_1} = - \sum_{j=1}^{d} \frac{x_j^{d-1} \ln x_j}{\prod_{i \neq j} (x_j - x_i)} - 1\] for \(k = 1\) (6)

Next note that by Cauchy’s integral formula we have, for any holomorphic function \(g\),

\[\sum_{j=1}^{d} \frac{g(x_j)}{\prod_{i \neq j} (x_j - x_i)} = \frac{1}{2\pi i} \int \frac{g(z)}{(z - x_1) \ldots (z - x_d)} \, dz\] (7)

where the contour surrounds all poles at \(z = x_1, \ldots, x_d\) and \(g\) is holomorphic in and on the contour. Then eqs. [3] and [11] immediately give

\[\frac{\partial H}{\partial e_k} = \frac{1}{2\pi i} \int \frac{(-1)^k z^{d-k} \ln z}{(z^d - e_1 z^{d-1} + \ldots + (-1)^d e_d)} \, dz\] for \(k \geq 2\) (8)

and

\[\frac{\partial H}{\partial e_1} = \frac{1}{2\pi i} \int \frac{-z^{d-1} \ln z}{(z^d - e_1 z^{d-1} + \ldots + (-1)^d e_d)} \, dz - 1\] for \(k = 1\). (9)

In all these cases the contour goes around all \(0 < x_1 \leq x_2 \leq \ldots \leq x_d\) on the real \(z\)-axis but not around the branch point \(z = 0\) of \(\ln z\).
Now to regain Fannes’ formula eq. (3) we distort the contour into a keyhole contour that excludes the negative real $z$-axis i.e. it runs above and below the negative real axis at distance $\epsilon$ between $z = -R \pm i\epsilon$ and $z = 0 \pm i\epsilon$, loops around the origin $z = 0$, and is completed by a circle of (large) radius $R$. Then direct calculation using standard contour integration techniques (cf. [4]) with the limits $\epsilon \to 0$ and $R \to \infty$ gives Fannes’ formula for the case of $k \geq 2$.

The case of subentropy is easier since $Q$ itself is already of the form of the LHS of eq. (7) and we immediately get (with the same contour as used above):

$$Q = -\frac{1}{2\pi i} \oint \frac{z^d \ln z}{(z^d - e_1 z^{d-1} + \ldots + (-1)^d e_d)} dz.$$  \hspace{1cm} (10)

By looking at eqs. (8), (9) and (10) we easily see the following relation.

**Proposition 1.**

$$-\frac{\partial Q}{\partial e_k} = \frac{\partial^2 H}{\partial e_l \partial e_m} \text{ for any } k, l, m \text{ with } k = l + m \text{ and } l, m \geq 1. \Box$$

Returning now to eq. (3) it is easy to similarly see that higher derivatives of $H$ with respect to the $e_k$’s satisfy the properties in the following three propositions.

**Proposition 2.** For $m \geq 2$ we have

$$(-1)^{m-1} \frac{\partial^m H}{\partial e_{i_1} \ldots \partial e_{i_m}} \geq 0 \text{ for all } i_1, \ldots, i_m \geq 1,$n

and for $m = 1$ we have

$$\frac{\partial H}{\partial e_k} \geq 0 \text{ for } k \geq 2. \Box$$

**Proposition 3.** The $m^{th}$ derivative

$$\frac{\partial^m H}{\partial e_{i_1} \ldots \partial e_{i_m}}$$

as a function of $(e_1, \ldots e_d)$ depends only on the sum of indices $i_1 + \ldots + i_m$, and the same property holds for $Q(e_1, \ldots e_d)$ too. \hspace{1cm} \Box$

Thus for example $\partial^2 H/\partial e_1 \partial e_5 = \partial^2 H/\partial e_2 \partial e_4 = \partial^2 H/\partial e_3^2$ since $1 + 5 = 2 + 4 = 3 + 3$.

Some of the above formulae appear to become singular if any of the $x_j$’s coincide (e.g. if $x_1 = x_2$). However closer inspection reveals that the limit of coincidence (e.g. $x_1 \to x_2$) is always finite and in the contour integral formulae we just use Cauchy’s integral formula with higher order poles to provide values of derivatives rather than values of the functions themselves. With this in mind we have the following result.

**Proposition 4.** Consider the $m^{th}$ derivative $\frac{\partial^m H}{\partial e_{i_1} \ldots \partial e_{i_m}}$ for $H(e_1, \ldots e_d)$ with $d$ variables. Introduce the entropy function $\tilde{H}(\tilde{x}_1, \ldots, \tilde{x}_{dm})$ with $dm$ variables and corresponding elementary symmetric polynomials $\tilde{e}_1, \ldots, \tilde{e}_{dm}$. Then for any $(e_1, \ldots, e_d)$ arising from roots $x_1, \ldots x_d$ we have

$$(-1)^{m-1} \frac{\partial^m H}{\partial e_{i_1} \ldots \partial e_{i_m}}(e_1, \ldots, e_d) = \frac{\partial \tilde{H}}{\partial \tilde{e}_K}(\tilde{e}_1, \ldots, \tilde{e}_{md})$$  \hspace{1cm} (11)
where $K = i_1 + \ldots + i_m$ and the RHS is evaluated at the point $(\tilde{e}_1, \ldots, \tilde{e}_{md})$ being the elementary symmetric polynomial values for the md $\tilde{x}_j$'s

$$(\tilde{x}_1, \ldots, \tilde{x}_{dm}) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_d, \ldots, x_d)$$

having each $x_i$ repeated $m$ times. □

**Proof** By factoring $(z^d - e_1 z^{d-1} + \ldots + (-1)^d e_d)$ as $(z - x_1) \ldots (z - x_d)$ we see that

$$(z^d - e_1 z^{d-1} + \ldots + (-1)^d e_d)^m = z^{md} - \tilde{e}_1 z^{md-1} + \ldots + (-1)^{md} \tilde{e}_{md}$$

where the $\tilde{e}_k$'s are the elementary symmetric functions of $md$ variables evaluated at the repeated values of the $x_j$'s. Then eq. (11) follows by differentiating eqs. (8) and (9) $m - 1$ times. □

To conclude, we make a connection with the concept of complete monotonicity and the classical theorem of Bernstein. In [2] it was shown that these concepts apply to a special kind of entropy; here we show how they relate to $H$ and $Q$.

A function $f(t_1, \ldots, t_m)$ is said to be completely monotone if

$$(-1)^j \frac{\partial}{\partial t_{i_1}} \ldots \frac{\partial}{\partial t_{i_j}} f \geq 0,$$

for $t_{i_q} \in [0, \infty)$ and $j = 0, 1, 2, \ldots$. From Proposition 2 it follows that each first derivative $\partial H/\partial e_k$, $2 \leq k \leq d$, is completely monotone in the variables $e_1, e_2, \ldots, e_d$, and, using Proposition 1, the same holds for the derivatives $\partial Q/\partial e_k$.

Bernstein’s theorem [7], in a multivariate form, says that any completely monotone function is the Laplace transform of a positive density, $f(t_1, \ldots, t_m) = \mathcal{L}[\mu(s_1, \ldots, s_m)](t_1, \ldots, t_m)$, or more explicitly

$$f(t_1, \ldots, t_m) = \int_0^\infty e^{-(t_1 s_1 + \ldots + t_m s_m)} \mu(s_1, \ldots, s_m) ds_1 \ldots ds_m.$$
Thus $H$ itself is not completely monotone, since both $H$ and its derivatives are positive: there is no change of sign between the function and its first derivative, as eq. (12) requires. However, if the first derivatives of a function $f$ are completely monotone, then so is $e^{-f}$ [7]. This is easy to check by repeated differentiation of $e^{-f}$. Extending this to many variables, we see that $e^{-H(e_2,...,e_d)}$ is the Laplace transform of a completely positive function $\mu(s_2,\ldots,s_d)$, and since $H(0,\ldots,0) = 0$, $\mu$ is a probability density. Actually, we can say more than this, since $e^{-H} = (e^{-H/m})^m = \mathcal{L}[\nu^m]$, where $e^{-H/m} = \mathcal{L}[\nu]$. This means that, for any integer $m$, $\mu$ is the $m$-fold convolution of a measure $\nu$. This property is called \textit{infinitely divisibility} [7], and is possessed by many fundamental statistical distributions, like the Gaussian.

Thus we know that $e^{-H}$ is the Laplace transform of an infinitely divisible function, and, since all the above remarks apply to $Q$, the same is true of $e^{-Q}$. It would be very desirable to be able to identify these fundamental-seeming underlying distributions. Unfortunately, we have so far been unable to derive them, even for $d = 2$, and we offer it as an intriguing unsolved problem.

\section*{References}

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