LOCAL CALABI AND CURVATURE ESTIMATES FOR THE
CHERN-RICCI FLOW†

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Abstract. Assuming local uniform bounds on the metric for a solution of the Chern-Ricci
flow, we establish local Calabi and curvature estimates using the maximum principle.

1. Introduction

Let $(M, \hat{g})$ be a Hermitian manifold. The Chern-Ricci flow starting at $\hat{g}$ is a smooth flow
of Hermitian metrics $g = g(t)$ given by

$$\frac{\partial}{\partial t} g = -R^C_i^j, \quad g|_{t=0} = \hat{g},$$

where $R^C_i^j := -\partial_i \partial_j \log \det g$ is the Chern-Ricci curvature of $g$. If $\hat{g}$ is Kähler, then the
Chern-Ricci flow coincides with the Kähler-Ricci flow.

The Chern-Ricci flow was introduced by Gill [11] and further investigated by Tosatti and
the second-named author [25, 26]. This flow has many of the same properties as the Kähler-Ricci flow.
For example: on manifolds with vanishing first Bott-Chern class the Chern-Ricci
flow converges to a Chern-Ricci flat metric [11]; on manifolds with negative first Chern
class, the Chern-Ricci flow takes any Hermitian metric to the Kähler-Einstein metric [25];
when $M$ is a compact complex surface and $\hat{g}$ is $\partial \partial^\perp$-closed, the Chern-Ricci flow exists until
either the volume of the manifold goes to zero or the volume of a curve of negative self-
intersection goes to zero [25]; if in addition $M$ is non-minimal with nonnegative Kodaira
dimension, the Chern-Ricci flow shrinks exceptional curves in finite time [26] in the sense of
Gromov-Hausdorff. These results are closely analogous to results for the Kähler-Ricci flow
[3, 10, 23, 19, 20].

In this note, we establish local derivative estimates for solutions of the Chern-Ricci flow
assuming local uniform bounds on the metric, generalizing our previous work [21] on the
Kähler-Ricci flow. Our estimates are local, so we work in a small open subset of $\mathbb{C}^n$. Write $B_r$
for the ball of radius $r$ centered at the origin in $\mathbb{C}^n$, and fix $T < \infty$. We have the
following result (see Section 2 for more details about the notation).

Theorem 1.1. Fix $r$ with $0 < r < 1$. Let $g(t)$ solve the Chern-Ricci flow (1.1) in a
neighborhood of $B_r$ for $t \in [0, T]$. Assume $N > 1$ satisfies

$$1/N \hat{g} \leq g(t) \leq N \hat{g} \quad \text{on } B_r \times [0, T].$$

Then there exist positive constants $C, \alpha, \beta$ depending only on $\hat{g}$ such that

(i) $|\tilde{\nabla} g|_\hat{g}^2 \leq \frac{CN^\alpha}{r^2}$ on $B_{r/2} \times [0, T]$, where $\tilde{\nabla}$ is the Chern connection of $\hat{g}$.

(ii) $|\text{Rm}|_\hat{g}^2 \leq \frac{CN^\beta}{r^4}$ on $B_{r/4} \times [0, T]$, for $\text{Rm}$ the Chern curvature tensor of $g$.

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author was a member of the mathematics department of the University of California, San Diego.
Note that the estimates are independent of the time $T$ and so the results holds also for time intervals $[0, T)$ or $[0, \infty)$. The dependence of the constants on $\hat{g}$ is as follows: up to three derivatives of torsion of $\hat{g}$ and one derivative of the Chern curvature of $\hat{g}$ (see Remarks 3.1 and 3.1). We call the bound (i) a local Calabi estimate [2] (see [29] for a similar estimate in the elliptic case).

As a consequence of Theorem 1.1 we have local derivative estimates for $g$ to all orders:

**Corollary 1.2.** With the assumptions of Theorem 1.1 for any $\varepsilon > 0$ with $0 < \varepsilon < T$, there exist constants $C_m$, $\alpha_m$ and $\gamma_m$ for $m = 1, 2, 3, \ldots$ depending only on $\hat{g}$ and $\varepsilon$ such that

$$|\hat{\nabla}_R^m g|_g^2 \leq \frac{C_m N^{\alpha_m}}{r^{\gamma_m}} \quad \text{on } B_{r/8} \times [\varepsilon, T],$$

where $\hat{\nabla}_R$ is the Levi-Civita covariant derivative associated to $\hat{g}$.

Note that our assumption (1.2) often holds for the Chern-Ricci flow on compact subsets away from a subvariety. For example, this always occurs for the Chern-Ricci flow on a non-minimal complex surface of nonnegative Kodaira dimension [25, 26]. It has already been shown by Gill [11] that local derivative estimates exist using the method of Evans-Krylov [9, 14] adapted to this setting. The purpose of this note is to give a direct maximum principle proof of Gill’s estimates, and in the process identify evolution equations for the Calabi quantity $|\nabla g|^2_g$ and the Chern curvature tensor $R_{ijkl}^\alpha$, which were previously unknown for this flow. In addition, we more precisely determine the form of dependence on the constants $N$ and $r$. We anticipate that this may be useful, for example in generalizations of arguments of [20].

In the case when $\hat{g}$ is Kähler, so that $g(t)$ solves the Kähler-Ricci flow, the above result follows from results of the authors in [21]. The more general case we deal with here leads to many more difficulties, arising from the torsion tensors of $g$ and $\hat{g}$. For these reasons, our conclusions here are slightly weaker: for example, we cannot obtain the small values ($\alpha = 3$ and $\beta = 8$) in the estimates of (i) and (ii) that we achieved in [21].

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### 2. Preliminaries

In this section we introduce the basic notions that we will be using throughout the paper. We largely follow notation given in [25]. Given a Hermitian metric $g$ we write $\nabla$ for the Chern connection associated to $g$, which is characterized as follows. Define Christoffel symbols $\Gamma_{ik}^l = g^{pl} \partial_i g_{kl}$. Let $X = X^l \frac{\partial}{\partial x^l}$ be a vector field and let $a = a_k dz^k$ be a $(1, 0)$ form. Then

$$\nabla_i X^l = \partial_i X^l + \Gamma_{ir}^l X^r, \quad \nabla_i a_j = \partial_i a_j - \Gamma_{ij}^r a_r. \quad (2.1)$$

We can, in a natural way, extend $\nabla$ to act on any tensor. Note that $\nabla$ makes $g$ parallel: i.e. $\nabla g = 0$. Similarly we let $\hat{\nabla}$ denote the Chern connection associated to $\hat{g}$.

Define the torsion tensor $T$ of $g$ by

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k \quad (2.2)$$

We note that $g$ is Kähler precisely when $T = 0$. We write $T_{ij}^k := \Gamma_{ij}^k - \Gamma_{ji}^k := \Gamma_{ij}^k - \Gamma_{ji}^k$ for the components of the tensor $T$. We lower and raise indices using the metric $g$. For example, $T_{i}^{jk} = g^{mi} g^{pj} g_{kl} T_{mj}^{ik}$.

We define the Chern curvature tensor of $g$ to be the tensor written locally as

$$R_{ijkl} = -\partial_{[j} \Gamma_{ik]}^l \quad (2.3)$$
Then
\[ R_{ij} = -\partial_i \partial_j g_{kl} + g^{pr} \partial_i g_{k\ell} \partial_j g_{\ell r}, \]
where again we have lowered an index using the metric \( g \). Note that \( R_{ij} = R_{ji} \) holds.

The commutation formulas for the Chern connection are given by
\begin{equation}
\begin{aligned}
[\nabla_i, \nabla_j] X^l &= R_{jk}^i X^k, \\
[\nabla_i, \nabla_j] X^k &= -R_{jk}^i X^l
\end{aligned}
\end{equation}
\[ (2.5) \]

Because \( g \) is not assumed to be a Kähler metric the Bianchi identities will not necessarily hold for \( R_{ijkl} \). However their failure to hold can be measured with the torsion tensor \( T \) defined above:
\begin{equation}
\begin{aligned}
R_{ijkl} - R_{klij} &= -\nabla_j T_{ikl}, \\
R_{ij} - R_{ji} &= -\nabla_k T_{ikl}, \\
R_{ijkl} - R_{klij} &= -\nabla_j T_{ikl} - \nabla_k T_{ikl} = -\nabla_j T_{ikl} - \nabla_l T_{ikj}, \\
\nabla_p R_{ijkl} - \nabla_i R_{plkj} &= -T_{pi} R_{ijkl}, \\
\nabla_l R_{ijkl} - \nabla_j R_{ikl} &= -T_{ij} R_{ijkl}.
\end{aligned}
\end{equation}
\[ (2.6) \]

These identities are well-known (see [27] for example). Indeed, it is routine to verify the first line, and the second and third lines follow directly from it. Furthermore the fifth line follows directly from the fourth. For the fourth line we calculate:
\[ \nabla_p R_{ijkl}^\ell = -\nabla_p (\partial_j \Gamma^\ell_{ik}) = -\partial_p \partial_j \Gamma^\ell_{ik} - \Gamma^s_{pm} \partial_j \Gamma^r_{sk} + \Gamma^s_{pm} \partial_j \Gamma^r_{sk} + \Gamma^s_{pm} \partial_j \Gamma^r_{sk} \]
\[ \nabla_p R_{ijkl}^\ell = -\nabla_i R_{pijk}^\ell = -T_{pi} R_{ijkl} + \partial_i \left( \partial_j \Gamma^\ell_{ik} - \partial_p \Gamma^r_{ik} + \Gamma^s_{pm} \Gamma^r_{sk} - \Gamma^s_{pm} \Gamma^r_{sk} \right). \]

Now one checks that the quantity in parentheses vanishes.

We define the Chern-Ricci curvature tensor \( R^C_{ij} \) by
\[ \nabla_p R_{ijkl}^\ell = -\nabla_i \partial_j \partial_k \log \det g. \]
\[ (2.7) \]

Note that \( \sqrt{-1} R^C_{ij} dz^i \wedge d\bar{z}^j \) is a real closed (1,1) form. We will suppose that \( g = g(t) \) satisfies the Chern-Ricci flow:
\[ \frac{\partial}{\partial t} g_{ij} = -R^C_{ij}, \quad g_{ij}|_{t=0} = \bar{g}_{ij}, \]
\[ (2.8) \]

for \( t \in [0, T] \) for some fixed positive time \( T \). We will use \( \bar{\nabla}, \bar{\Gamma}^l_{ik}, \bar{T}_{ik} \), \( \bar{R}_{ijkl} \) etc to denote the corresponding quantities with respect to the metric \( \bar{g} \). Define a real (1,1) form \( \omega = \omega(t) \) by \( \omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j \) and similarly for \( \bar{\omega} \). From \( (2.8) \) we have that
\[ \omega = \bar{\omega} + \eta(t) \]
\[ (2.9) \]

for a closed (1,1) form \( \eta \). Hence
\[ T_{ik} = \bar{T}_{ik}. \]
\[ (2.10) \]

Here we raise and lower indices of \( \bar{T} \) using the metric \( \bar{g} \), in the same manner as for \( g \) above. Note that \( T_{ik} = g_{ij} T_{ik}^r = \partial_i g_{k\ell} - \partial_k g_{i\ell} \) and \( \bar{T}_{ik} = \bar{g}_{ij} \bar{T}_{ik}^r = \partial_i \bar{g}_{k\ell} - \partial_k \bar{g}_{i\ell} \).

It is convenient to introduce the tensor \( \Psi_{ik} = \Gamma^l_{ik} - \bar{\Gamma}^l_{ik} \). We raise and lower indices of \( \Psi \) using the metric \( g \), and write \( \Psi^R_{ik} \) for the components of \( \overline{\Psi} \). We note here that \( \Psi \) can be...
used to switch between the connections $\nabla$ and $\hat{\nabla}$. For example given a tensor of the form $X_{ij}$ we have
\begin{equation}
\nabla_p X_{ij} - \hat{\nabla}_p X_{ij} = -\Psi_{pr}^i X_{rj} + \Psi_{pr}^j X_{ri}.
\end{equation}

Observe that
\begin{equation}
\nabla_j \Psi_{ikl} = -R_{i jk}^l + \hat{R}_{i jk}^l.
\end{equation}

We write $\Delta$ for the “rough Laplacian” of $g$, $\Delta = \nabla^q \nabla_q$, where $\nabla_q = g^{qp} \nabla_p$. Finally note that we will write all norms $| \cdot |$ with respect to the metric $g$.

3. Local Calabi estimate

In this section we prove part (i) of Theorem 1.1. We consider the Calabi-type \[2, 28\] quantity
\begin{equation}
S := |\Psi|^2 = |\nabla g|^2.
\end{equation}

Our goal in this section is to uniformly bound $S$ on the set $B_{r/2}$, which we will do using a maximum principle argument. First we compute its evolution. Calculate
\begin{equation}
\Delta S = g^{\pi p} \nabla_p \nabla_q \left( g^{\pi q} g^{ij} g_{k\pi} \Psi_{ij}^k \nabla_{ab}^c \right)
= g^{\pi p} g^{ij} g_{k\pi} \nabla_p \left( \nabla_q \Psi_{ij}^k \nabla_{ab}^c + \Psi_{ij}^k \nabla_q \Psi_{ab}^c \right)
= |\nabla \Psi|^2 + |\nabla \Psi|^2 +\nabla \Psi_{ij}^k \left( \Delta \Psi_{ij}^k \Psi_{ab}^c \right)
+ \Psi_{ij}^k \left( \Delta \Psi_{ab}^c + g^{\pi p} \hat{R}_{\pi q} \Psi_{rk}^c + g^{\pi p} \hat{R}_{\pi q} \Psi_{ar}^c - g^{\pi p} \hat{R}_{\pi q} \Psi_{ab}^c \right)
= |\nabla \Psi|^2 + |\nabla \Psi|^2 + 2 \text{Re}\left( \left( \Delta \Psi_{ij}^k \right) \Psi_{ij}^k \right)
+ \left( \hat{R}_{\pi q} \Psi_{ij}^k + \hat{R}_{\pi q} \Psi_{ij}^k - \hat{R}_{\pi q} \Psi_{ij}^k \right) \Psi_{ij}^k.
\end{equation}

From [2.12] we have
\begin{equation}
\Delta \Psi_{ij}^k = -\nabla_q \hat{R}_{\pi q} + \nabla_q \hat{R}_{\pi q}.
\end{equation}

For the time derivative of $S$, first compute (cf. [17] in the Kähler case),
\begin{equation}
\frac{\partial}{\partial t} \Psi_{ij}^k = \frac{\partial}{\partial t} \Gamma_{ij}^k = -\nabla_i (R^C)_{jk}^k.
\end{equation}

Then
\begin{equation}
\frac{\partial}{\partial t} S = \frac{\partial}{\partial t} \left( g^{\pi p} g^{ij} g_{k\pi} \Psi_{ij}^k \nabla_{ab}^c \right)
= \left( \frac{\partial}{\partial t} g^{\pi p} \right) \Psi_{ij}^k \Psi_{ab}^c + \left( \frac{\partial}{\partial t} g^{ij} \right) \Psi_{ij}^k \Psi_{ab}^c
+ \left( \frac{\partial}{\partial t} g_{k\pi} \right) \Psi_{ij}^k \Psi_{ab}^c + 2 \text{Re}\left( \left( \frac{\partial}{\partial t} \Psi_{ij}^k \right) \Psi_{ij}^k \right)
= \left( R^C \right)^{\pi \pi} \Psi_{ij}^k \Psi_{ab}^c + \left( R^C \right)^{\pi \pi} \Psi_{ij}^k \Psi_{ab}^c
- \left( R^C \right)^{\pi \pi} \Psi_{ij}^k \Psi_{ab}^c - 2 \text{Re}\left( \left( \nabla_i (R^C)_{jk}^k \right) \Psi_{ij}^k \right).
\end{equation}

Therefore
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta \right) S = -|\nabla \Psi|^2 - |\nabla \Psi|^2 + \left( R^C_{\pi p} - R_{\pi p} \right) \Psi_{ij}^k \Psi_{ab}^c + \left( R^C_{\pi p} - R_{\pi p} \right) \Psi_{ij}^k \Psi_{ab}^c
- \left( R_{\pi p} + \Delta \Psi_{ij}^k \right) \Psi_{ij}^k.
\end{equation}
By (2.6) we can re-write the terms involving a difference in curvature using the torsion tensor $T$. For the term in square brackets we compute, using (3.2) and again (2.6) that

\[
\nabla_i R_{jk}^p + \Delta \Psi_{ij}^k = \nabla_i \left( R_{jp}^{\mu k} + \nabla_j T_{pk}^p + \nabla^p T_{pj}^k \right) - \nabla^\mathcal{J} R_{i\mathcal{J}j}^k + \nabla^\mathcal{J} \tilde{R}_{i\mathcal{J}j}^k
\]

\[
= \left( \nabla_p R_{ij}^k - T_{ip}^r R_{rj}^k + \nabla_i \nabla_j T_{pk}^p + \nabla_i \nabla^p T_{pj}^k \right) - \nabla^\mathcal{J} R_{i\mathcal{J}j}^k + \nabla^\mathcal{J} \tilde{R}_{i\mathcal{J}j}^k
\]

\[
= - T_{ip}^r R_{rj}^k + \nabla_i \nabla_j T_{pk}^p + \nabla_i \nabla^p T_{pj}^k + \nabla^\mathcal{J} \tilde{R}_{i\mathcal{J}j}^k.
\]

Hence $S$ satisfies the following evolution equation

\[
\left( \frac{\partial}{\partial t} - \Delta \right) S = - |\nabla \Psi|^2 - |\nabla \Psi|^2
\]

\[
+ \left( \nabla_i T_{ip}^q + \nabla_q T_{ip}^r \right) \nabla_j \Psi_{ij}^k + \left( \nabla_i T_{ip}^q + \nabla_q T_{ip}^r \right) \nabla_j \Psi_{ij}^k
\]

\[
- \left( \nabla_i T_{ip}^q + \nabla_q T_{ip}^r \right) \nabla_j \Psi_{ij}^k
\]

\[
- 2 \text{Re} \left[ \left( \nabla_i \nabla_j T_{pk}^p + \nabla_i \nabla_q T_{ip}^r \right) - T_{ip}^r R_{rj}^k + g_{i\mathcal{J}j} \nabla^\mathcal{J} \tilde{R}_{i\mathcal{J}j}^k \right] \Psi_{ij}^k \right].
\]

(3.4)

There are similar calculations to (3.4) in the literature which generalize Calabi’s argument \cite{Calabi} in the elliptic Hermitian case \cite{Outs}; in the case of the Kähler-Ricci flow (see also \cite{Chen}); and in other settings \cite{Hsu, Lu, Sun}.

For the remainder of this section we will write $C$ for a constant of the form $CN^\alpha$ for $C$ and $\alpha$ depending only on $g$. Our goal is to show that $S \leq C/r^2$. The constant $C$ will be used repeatedly and may change from line to line, and we may at times use $C''$ or $C_1$ etc.

We would like to bound the right-hand side of (3.4). First, from (2.10) and (2.11) we have, for example,

\[
\nabla_i \nabla_j T_{pk}^p = g_{i\mathcal{J}j} \left( \nabla_i \nabla_j T_{pk}^p - \Psi_{ij}^k \tilde{T}_{ipj}^r \right).
\]

(3.5)

This and similar calculations show that the second and third lines of (3.4) can be bounded by $C(S^{3/2} + 1)$. Next we address the terms in the last line of the evolution equation for $S$.

- Building on (3.5) we find

\[
\nabla_a \nabla_b T_{ij}^k = g_{ik} \left( \nabla_a (\nabla_b \tilde{T}_{ij}^l - \Psi_{ab}^r \tilde{T}_{ij}^r) \right)
\]

\[
= g_{ik} \left( \nabla_a \nabla_b \tilde{T}_{ij}^l - \Psi_{ab}^r \tilde{T}_{ij}^r - \Psi_{ai}^r \nabla_b \tilde{T}_{ij}^r \right)
\]

\[
- \left( \nabla_a \Psi_{bl}^r \tilde{T}_{ij}^r - \Psi_{bl}^r \nabla_a \tilde{T}_{ij}^r + \Psi_{bl}^r \tilde{T}_{ij}^r \right),
\]

(3.6)

and hence $|\nabla_i \nabla_j T_{pk}^p|$ can be bounded by $C(S + |\nabla \Psi| + 1)$.

- Similarly,

\[
\nabla_a \nabla_b T_{ij}^k = g_{ik} \nabla_a (\nabla_b \tilde{T}_{ij}^l - \Psi_{bl}^r \tilde{T}_{ij}^r)
\]

\[
= g_{ik} \left( \nabla_a \nabla_b \tilde{T}_{ij}^l - \Psi_{ai}^p \nabla_b \tilde{T}_{ipj}^r - \Psi_{ai}^p \nabla_b \tilde{T}_{ipj}^r - (\nabla_a \Psi_{bl}^r \tilde{T}_{ij}^r) \right),
\]

(3.7)

and so $|\nabla_i \nabla_j T_{ip}^q |$ can be bounded by $C(S + |\nabla \Psi| + 1)$.

- Next, using (2.10) and (2.12):

\[
T_{ip}^r R_{rj}^k = g_{ip} g_{\mathcal{J}r} \tilde{R}_{i\mathcal{J}j}^k - \nabla_j \Psi_{ij}^k,
\]

so we can bound $|T_{ip}^r R_{rj}^k|$ by $C(|\nabla \Psi| + 1)$. 

Finally, compute
\[
\nabla_p \tilde{R}_{\pi j^k} = \hat{\nabla}_p \tilde{R}_{\pi j^k} - \Psi_{\pi i}^r \tilde{R}_{\pi j^k} - \Psi_{\pi j}^r \tilde{R}_{\pi j^k} + \Psi_{\pi r}^k \tilde{R}_{\pi j^k}.
\]

So \( |g^{pq} \nabla_p \tilde{R}_{\pi j^k} | \) can be bounded by \( C(S^{1/2} + 1) \).

Putting this all together we arrive at the bound
\[
(3.8) \quad \left( \frac{\partial}{\partial t} - \Delta \right) S \leq C(S^{3/2} + 1) - \frac{1}{2}(|\nabla \Psi|^2 + |\Psi|^2).
\]

We note here the bounds:
\[
(3.9) \quad |\nabla \text{tr}_g g|^2 \leq CS
\]
\[
(3.10) \quad |\nabla S|^2 \leq 2S(|\nabla \Psi|^2 + |\Psi|^2).
\]

The first follows from \( \nabla_p \left( \tilde{g}^i j g_{\gamma \beta} \right) = \hat{\nabla}_p \left( \tilde{g}^i j g_{\gamma \beta} \right) = \tilde{g}^i j \hat{\nabla}_p g_{\gamma \beta} \) and the second follows from \( |\nabla S|^2 = |\nabla \Psi|^2 |\nabla \Psi|^2 | \leq 2|\Psi|^2 (|\nabla \Psi|^2 + |\Psi|^2) \). Furthermore from [25, Proposition 3.1] (see also [6] in the elliptic case), we also have the following evolution equation for \( \text{tr}_g g \):
\[
(3.11) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g g = -\tilde{g}^p q T_{ki} \nabla_k g_{p \gamma} \nabla^k g_{\gamma q} - 2 \text{Re} \left( \tilde{g}^i T_{ki} \tilde{g}^k g_{p \gamma} \right) \]
\[
\quad + \tilde{g}^i \left( \tilde{\nabla}_i T_{ji} - \tilde{R}_{ji} \tilde{\nabla}_k g_{\gamma \beta} \right) g_{k \gamma} - \tilde{g}^i \left( \tilde{\nabla}_i \tilde{\nabla}_k \tilde{g} - \tilde{\nabla}_k \tilde{\nabla}_j \tilde{g} \right) \tilde{R}_{ij} \]
\[
\quad + \tilde{g}^i \tilde{\nabla}_k \tilde{R}_{ik} g_{p \gamma} \tilde{\nabla}_p \tilde{g} - g_{p \gamma}.
\]

(Here \( \tilde{\nabla}^k = \tilde{g}^k \tilde{\nabla}_k \) and we have raised indices on the tensor \( \tilde{R} \) using \( \tilde{g} \)). This generalizes the second order evolution inequality for the Kähler-Ricci flow [3] (cf. [28, 1]). Hence we have the estimate
\[
(3.12) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g g \leq -\frac{S}{C_0} + C(S^{1/2} + 1),
\]
for a uniform positive constant \( C_0 \) (in fact we can take \( C_0 = N \)).

We now would like to show that the evolution inequalities (3.8, 3.12) imply a uniform bound on \( S = |\tilde{\nabla} g|^2 \) on \( B_{r/2} \times [0, T] \). Choose a smooth cutoff function \( \rho \) which is supported in \( B_r \) and is identically 1 on \( B_{r/2} \). We may assume that \( |\nabla \rho|^2, |\Delta \rho| \) are bounded by \( C/r^2 \).

Let \( K \) be a large uniform constant, to be specified later, which is at least large enough so that
\[
\frac{K}{2} \leq K - \text{tr}_g g \leq K.
\]

Let \( A \) denote another large positive constant to be specified later. We will use a maximum principle argument with the function (cf. [5])
\[
\frac{f}{K - \text{tr}_g g} + A \text{tr}_g g
\]
to show that \( S \) is bounded on \( B_{r/2} \).

Suppose that the maximum of \( f \) on \( B_r \times [0, T] \) occurs at a point \( (x_0, t_0) \). We assume for the moment that \( t_0 > 0 \) and that \( x_0 \) does not lie in the boundary of \( B_r \). We wish to show that at \( (x_0, t_0) \), \( S \) is bounded from above by a uniform constant \( C \). Hence we may assume without loss of generality that \( S > 1 \) at \( (x_0, t_0) \). In particular, we have
\[
(3.13) \quad \left( \frac{\partial}{\partial t} - \Delta \right) S \leq CS^{3/2} - \frac{1}{2}(|\nabla \Psi|^2 + |\Psi|^2), \quad \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g g \leq -\frac{S}{2C_0} + C.
\]
We compute at \((x_0, t_0)\),
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f = A \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr} g + (-\Delta \rho^2) + \frac{S}{K - \operatorname{tr} g} + \rho^2 \frac{S}{(K - \operatorname{tr} g)^2} \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr} g
\]
\[
+ \rho^2 \frac{1}{K - \operatorname{tr} g} \left( \frac{\partial}{\partial t} - \Delta \right) S - 4 \text{Re} \left[ \frac{\rho S}{(K - \operatorname{tr} g)^2} \nabla \operatorname{tr} g \cdot \nabla \rho \right]
\]
\[
- 4 \text{Re} \left[ \frac{1}{K - \operatorname{tr} g} \nabla \rho \cdot \nabla \rho \right] - 2 \text{Re} \left[ \frac{\rho^2}{(K - \operatorname{tr} g)^2} \nabla \operatorname{tr} g \cdot \nabla S \right]
\]
\[
- \frac{2 \rho^2 S}{(K - \operatorname{tr} g)^2} |\nabla \operatorname{tr} g|^2.
\]

But since a maximum occurs at \((x_0, t_0)\) we have \(\nabla f = 0\) at this point, and hence
\[
2 \rho \nabla \rho \frac{S}{K - \operatorname{tr} g} + \rho^2 \frac{\nabla S}{K - \operatorname{tr} g} + \rho^2 \frac{S \nabla \operatorname{tr} g}{(K - \operatorname{tr} g)^2} + A \nabla \operatorname{tr} g = 0.
\]

Then at \((x_0, t_0)\),
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f = A \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr} g + (-\Delta \rho^2) + \frac{S}{K - \operatorname{tr} g} + \rho^2 \frac{S}{(K - \operatorname{tr} g)^2} \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr} g
\]
\[
+ \rho^2 \frac{1}{K - \operatorname{tr} g} \left( \frac{\partial}{\partial t} - \Delta \right) S - 4 \text{Re} \left[ \frac{\rho S}{(K - \operatorname{tr} g)^2} \nabla \rho \cdot \nabla \rho \right] - 2 \text{Re} \left[ \frac{\rho^2}{(K - \operatorname{tr} g)^2} \nabla \operatorname{tr} g \cdot \nabla S \right]
\]
\[
- \frac{2 \rho^2 S}{(K - \operatorname{tr} g)^2} |\nabla \operatorname{tr} g|^2.
\]

Making use of (3.9, 3.10, 3.13) and Young’s inequality, we obtain at \((x_0, t_0)\),
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) f \leq \left( -\frac{A}{2C_0} S + CA \right) + \left( \frac{CS}{r^2 K} \right) + \left( -\frac{\rho^2}{2K^2 C_0} S^2 + \frac{C\rho^2}{K^2} S \right)
\]
\[
+ \left( -\frac{\rho^2}{2K} |\nabla \Psi|^2 + |\nabla \Psi|^2 \right) + \left( \frac{\rho^2}{4K} S \right) + \left( \frac{C A}{K} S \right)
\]
\[
\leq -\frac{A}{2C_0} S + CA + \frac{C'}{r^2} S + \frac{CA}{K} S.
\]

Now pick \(K \geq 4C_0 C\) so that at \((x_0, t_0)\),
\[
0 \leq -\frac{A}{4C_0} S + CA + \frac{C'}{r^2} S.
\]

Then choose \(A = \frac{8C C_0}{r^2} \) so that at \((x_0, t_0)\),
\[
\frac{C'}{r^2} S \leq CA,
\]
giving a uniform upper bound for \(S\). It follows that \(f\) is bounded from above by \(Cr^{-2}\) for a uniform \(C\). Hence \(S\) on \(B_{r/2}\) is bounded above by \(Cr^{-2}\).

It remains to deal with the cases when \(t_0 = 0\) or \(x_0\) lies on the boundary of \(B_r\). In either case we have \(f(x_0, t_0) \leq A \operatorname{tr} g (x_0, t_0) \leq Cr^{-2}\) and the same bound holds.

**Remark 3.1.** Tracing through the argument, one can see that the constants only depend on uniform bounds for the torsion and curvature of \(\hat{g}\), and one and two derivatives (with respect to \(\hat{\nabla}\) or \(\nabla\)) of torsion and one derivative of curvature.
4. LOCAL CURVATURE BOUND

In this section we prove part (ii) of Theorem 1.1. As in the previous section, we write $C$ for a constant of the form $CN^3$ for some uniform $C, \gamma$. We compute in the ball $B_{r/2}$ on which we already have the bound $S \leq C/r^2$.

Let $\Delta_R = \frac{1}{2} g^{pq} (\nabla_p \nabla_q + \nabla_q \nabla_p)$. First we need an evolution equation for the curvature tensor. We begin with

$$\frac{\partial}{\partial t} R_{ijkl}^t = \frac{\partial}{\partial t} \left( - \partial^t \Gamma_{ik}^l \right) = - \partial^t \frac{\partial}{\partial t} \left( \Gamma_{ik}^l \right) = - \partial^t (-\nabla_i (R^C)^t) = \nabla_j \nabla_i R_{kl}^t.$$

and therefore,

$$\frac{\partial}{\partial t} R_{ijkl} = - R_{iq}^p R_{jk}^q + \nabla_j \nabla_i R_{kl}^t.$$

Now, computing in coordinates where $g$ is the identity, we find

$$\Delta_R R_{ijkl}^t = \frac{1}{2} (\nabla p \nabla q + \nabla q \nabla p) R_{ijkl}^t$$

$$= \nabla_p (\nabla_j R_{qkl} - T_{pqj} R_{qkl}^t) + \frac{1}{2} (\nabla_p R_{qkl} - R_{pqj} R_{qkl}^t - R_{pql} R_{qjk}^t)$$

$$= \nabla_p (\nabla_j R_{qkl} - T_{pqj} R_{qkl}^t) + \frac{1}{2} (\nabla_p R_{qkl} - R_{pqj} R_{qkl}^t - R_{pql} R_{qjk}^t)$$

$$= \nabla_j \nabla_i (R_{ijkl}^t) + \frac{1}{2} (\nabla p R_{qkl} - R_{pqj} R_{qkl}^t - R_{pql} R_{qjk}^t)$$

$$= \nabla_j \nabla_i (R_{ijkl}^t) + \frac{1}{2} (\nabla p R_{qkl} - R_{pqj} R_{qkl}^t - R_{pql} R_{qjk}^t)$$

$$= \nabla_j \nabla_i (R_{ijkl}^t) + \frac{1}{2} (\nabla p R_{qkl} - R_{pqj} R_{qkl}^t - R_{pql} R_{qjk}^t)$$

Hence

$$\left( \frac{\partial}{\partial t} - \Delta_R \right) R_{ijkl}^t = - R_{qk}^p R_{ijql}^t + R_{pqj}^t R_{qkl}^t - R_{pql}^t R_{qjk}^t + R_{pql}^t R_{qjk}^t - R_{pql}^t R_{qjk}^t - R_{pql}^t R_{qjk}^t$$

$$= \nabla_p (\hat{T}_{pqj} R_{qkl}) + \nabla_j (\hat{T}_{pqj} R_{qkl}) + \nabla_j \nabla_i (\hat{T}_{pqj} R_{qkl}^t) + \nabla_j \nabla i (\hat{T}_{pqj} R_{qkl}^t).$$

To estimate this, we first compute

$$\nabla_p (\hat{T}_{pqj} R_{qkl}) = (\nabla_p \hat{T}_{pqj} - \Psi_{pqj} \hat{T}_{pqj}) R_{qkl}^t + \hat{T}_{pqj} \nabla_p R_{qkl}^t,$$

and this is bounded by $C(|\text{Rm}|/r + |\nabla \text{Rm}|)$. Using the fact that $R_{ijkl}^t = - \nabla_j \Psi_{ijkl}^t + \hat{R}_{ijkl}^t$ we have

$$|\text{Rm}| \leq |\nabla \Psi| + C,$$

and hence

$$|\nabla_p (\hat{T}_{pqj} R_{qkl})| \leq C \left( |\nabla \text{Rm}| + \frac{|\nabla \Psi|}{r} + \frac{1}{r} \right).$$

Similarly for the term $\nabla_j (\hat{T}_{pqj} R_{qkl}).$
The last two terms of (4.2) involve three derivatives of torsion. We claim that
\[
(4.5) \quad |\nabla \nabla \nabla T|, |\nabla \nabla \nabla T| \leq C \left( |\nabla Rm| + \frac{|\nabla \Psi|}{r} + \frac{1}{r^3} \right).
\]
Indeed, applying \(\nabla_\tau\) to (3.6), we have
\[
\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ij} = g^{ijl} \left( \nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl} - \nabla_\tau \nabla_a \nabla_b \nabla_d \hat{T}_{djl} - \nabla_\tau \nabla_a \nabla_b \nabla_d \hat{T}_{djl} \right)
- \nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl}) - \nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_d \hat{T}_{djl} - \nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_d \hat{T}_{djl})
(4.6)
\]
which is bounded by \(C|\nabla \Psi| + C\sqrt{S} + CS\) and hence by \(C(|\nabla \Psi| + 1/r^2)\). The same bound holds for the other two terms on the second line of (4.6).

For the third line, compute
\[
\nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl})) = (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl} + \nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl} - \nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl}))
+ (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl})
(4.7)
\]
and using the fact that \(\nabla_\tau \nabla a = -R_{ab} \nabla b + \hat{R}_{ab}\), we obtain
\[
\nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl})) = \nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla b)) + \nabla_\tau (\nabla_\tau (\nabla_\tau \nabla b))
\]
\[
+ (\nabla_\tau \nabla a \nabla b + \nabla_\tau \nabla a \nabla b)
(4.8)
\]
It follows that
\[
(4.9) \quad |\nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl}))| \leq C \left( |\nabla Rm| + \frac{|\nabla \Psi|}{r} + \frac{1}{r^3} \right).
\]
Finally,
\[
\nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl})) = \nabla_\tau (\nabla_\tau (\nabla_\tau \nabla a \nabla b)) + \nabla_\tau (\nabla_\tau (\nabla_\tau \nabla b))
\]
giving
\[
(4.10) \quad |\nabla_\tau (\nabla_\tau (\nabla_\tau \nabla_a \nabla_b \nabla_c \hat{T}_{ijl}))| \leq C \left( |\nabla Rm| + \frac{|\nabla \Psi|}{r} + \frac{1}{r^3} \right).
\]
Putting together (4.6), (4.7), (4.8), (4.9), and making use of (4.3), we obtain
\[
|\nabla_\tau (\nabla_\tau \nabla_\tau \nabla_\tau T)| \leq C \left( |\nabla Rm| + \frac{|\nabla \Psi| + |\nabla \Psi|}{r} + \frac{1}{r^3} \right)
\]
and the bound for \(\nabla_\tau (\nabla_\tau \nabla_\tau \nabla_\tau T)\) follows similarly. This completes the proof of the claim (4.5).

From (4.1) and the claim we just proved, since the first two lines of (4.2) are of the order \(|\nabla Rm|^2\), we have the bound
\[
(4.10) \quad \left| \left( \frac{\partial}{\partial t} - \Delta_R \right) Rm \right| \leq C \left( |\nabla Rm|^2 + |\nabla Rm| + \frac{|\nabla \Psi| + |\nabla \Psi|}{r} + \frac{1}{r^3} \right).
\]
Now
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\text{Rm}|^2 = g^{jk} g^{ld} (R^C)^{\gamma\xi} R_{\gamma l\xi j} R_{abcd} \]
\[
+ g^{jk} g^{ld} (R^C)^{\gamma j} R_{\gamma k l} R_{abcd} \]
\[
+ g^{jk} g^{ld} (R^C)^{\xi k} R_{\xi j l} R_{abcd} \]
\[
+ g^{jk} g^{ld} (R^C)^{\xi j} R_{\xi k l} R_{abcd} \]
\[
+ 2\text{Re} \left[ g^{jk} g^{ld} g^{\gamma\xi} \left( \frac{\partial}{\partial t} - \Delta R_{\gamma l\xi j} \right) R_{abcd} \right] \]
\[
- 2|\nabla \text{Rm}|^2.
\]
(4.11)

This together with (4.10) and (4.3) implies
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\text{Rm}|^2 \leq C \left( |\text{Rm}|^2 + |\text{Rm}|^3 + |\nabla \text{Rm}| \cdot |\text{Rm}| + \frac{(|\nabla \Xi + |\nabla \Psi|)|\text{Rm}|}{r} + \frac{|\text{Rm}|}{r^3} \right)
\]
\[
\leq C \left( |\text{Rm}|^3 + \frac{1}{r} + \frac{|\nabla \Xi|^2 + |\nabla \Psi|^2}{r} + \frac{|\text{Rm}|}{r^3} \right) - |\nabla \text{Rm}|^2.
\]
(4.12)

To show $|\text{Rm}|^2$ is locally uniformly bounded we will use an argument similar to the previous section. Let $\rho$ now denote a cutoff function which is identically 1 on $B_{r/4}$ and supported in $B_{r/2}$. From the previous section we know that $S$ is bounded by $C/r^2$ on $B_{r/2}$. As before we can assume $|\nabla \rho|^2$ and $|\Delta \rho|$ are bounded by $C/r^2$. Let $K = C_1/r^2$ where $C_1$ is a constant to be determined later, and is at least large enough so that $K \leq K - S \leq K$. Let $A$ denote a constant to be specified later. We will apply the maximum principle argument to the quantity
\[
f = \rho^2 \frac{|\text{Rm}|^2}{K - S} + AS.
\]

As in the previous section, we calculate at a point $(x_0, t_0)$ where a maximum of $f$ is achieved, and we first assume that $t_0 > 0$ and that $x_0$ does not occur at the boundary of $B_{r/2}$. We use the fact that $|\nabla f| = 0$ at this point, giving us
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f = A(\frac{\partial}{\partial t} - \Delta)S + (-\Delta(\rho^2)) \frac{|\text{Rm}|^2}{K - S} + \rho^2 \frac{|\text{Rm}|^2}{(K - S)^2} \left( \frac{\partial}{\partial t} - \Delta \right) S
\]
\[
+ \rho^2 \left( \frac{1}{K - S} \left( \frac{\partial}{\partial t} - \Delta \right) |\text{Rm}|^2 - 4\text{Re} \left( \frac{1}{K - S} \rho \nabla \rho \cdot \nabla |\text{Rm}|^2 \right) + \frac{2A|\nabla S|^2}{K - S} \right).
\]

Our goal is to show that at $(x_0, t_0)$, we have $|\text{Rm}|^2 \leq C/r^4$. Hence without loss of generality, we may assume that $1/r + |\text{Rm}|/r^3 \leq C|\text{Rm}|^3$ and hence (4.12) becomes
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\text{Rm}|^2 \leq C \left( |\text{Rm}|^3 + \frac{Q}{r} \right) - |\nabla \text{Rm}|^2,
\]
where for convenience we are writing $Q = |\nabla \Xi|^2 + |\nabla \Psi|^2$. For later purposes, recall from (4.3) that $|\text{Rm}|^2 \leq Q + C$ and from (3.10) that $|\nabla S|^2 \leq 2SQ$.

Also note that $|\nabla |\text{Rm}|^2| \leq 2|\text{Rm}||\nabla \text{Rm}|$. By (3.8) we find that on $B_{r/2}$ we have
\[
(\frac{\partial}{\partial t} - \Delta)S \leq \frac{C}{r^3} - \frac{1}{2}Q.
\]
Using these, we find at \((x_0, t_0)\),
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq \left( \frac{CA}{r^3} - \frac{AQ}{2} \right) + \left( \frac{C|Rm|^2}{K r^2} \right) + \left( \frac{C \rho^2 |Rm|^2}{K 2r^3} - \frac{\rho^2 |Rm|^2 Q}{2K^2} \right) \\
+ \left( \frac{C \rho^2 |Rm|^3}{K} + \frac{C \rho^2 Q}{Kr} - \frac{\rho^2}{K} |\nabla Rm|^2 \right) + \left( \frac{\rho^2 |\nabla Rm|^2}{2K} + \frac{C |Rm|^2}{Kr^2} \right) + \left( \frac{8ASQ}{K} \right).
\]
First choose \(C_1\) in the definition of \(K\) to be sufficiently large so that
\[
\frac{8ASQ}{K} \leq \frac{AQ}{4},
\]
where we use the fact that \(S \leq C/r^2\). Next observe that
\[
\frac{C \rho^2 |Rm|^3}{K} \leq \frac{\rho^2 |Rm|^2 Q}{2K^2} + C' \rho^2 |Rm|^2,
\]
and hence
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq \frac{CA}{r^3} - \frac{AQ}{4} + C''Q + C.
\]
Now we may choose \(A\) sufficiently large so that \(A \geq 8C''\) and we obtain at \((x_0, t_0)\),
\[
Q \leq \frac{C}{r^3},
\]
which implies that \(|Rm|^2 \leq C/r^3\) at this point. It follows that at \((x_0, t_0)\), \(f\) is bounded from above by \(C/r^3\). The same bound holds if \(x_0\) lies in the boundary of \(B_{r/2}\) or if \(t_0 = 0\). Hence on \(B_{r/4}\) we obtain
\[
|Rm|^2 \leq \frac{C}{r^3},
\]
as required. This completes the proof of Theorem 1.1.

Remark 4.1. In addition to the dependence discussed in Remark 3.1, the constants also depend on three derivatives of the torsion of \(\hat{g}\), with respect to \(\hat{\nabla}\) or \(\hat{\nabla}\).

5. Higher order estimates

In this last section, we prove Corollary 1.2 by establishing the estimates for \(|\nabla^m_R^x g|^2_{\hat{g}}\) for \(m = 2, 3, \ldots\). For this part, we essentially follow the method of Gill [11] (cf. [4, 7, 8, 16] in the Kähler case), but since the setting here is slightly more general, we briefly outline the argument. In this section, we say that a quantity is uniformly bounded if it can be bounded by \(CN^{\alpha - \gamma}\) for uniform \(C, \alpha, \gamma\).

We work on the ball \(B_{r/4}\), and assume the bounds established in Theorem 1.1. As in [25], define reference tensors \((\hat{g}_t)_{\hat{\imath} \hat{j}} = \hat{g}_{\hat{\imath} \hat{j}} - t \hat{R}_{\hat{\imath} \hat{j}}^x\), where \(\hat{R}_{\hat{\imath} \hat{j}}^x\) is the Chern-Ricci curvature of \(\hat{g}\). For each fixed \(x \in M\), let \(\varphi = \varphi(x, t)\) solve
\[
\frac{\partial \varphi}{\partial t} = \log \left( \frac{\det g(t)}{\det \hat{g}} \right), \quad \varphi|_{t=0} = 0.
\]
Then \(g_{\hat{\imath} \hat{j}} = (\hat{g}_t)_{\hat{\imath} \hat{j}} + \partial_t \varphi \cdot \partial_{\lambda} \varphi\) is the solution of the Chern-Ricci flow starting at \(\hat{g}\).

Consider the first order differential operator \(D = \frac{\partial}{\partial x^\gamma}\), where \(x^\gamma\) is a real coordinate. Applying \(D\) to the equation for \(\varphi\), we have
\[
\frac{\partial}{\partial t} (D \varphi) = \hat{g}^n D g_{\hat{\imath} \hat{j}} - \hat{g}^n D (\hat{g})_{\hat{\imath} \hat{j}} = \hat{g}^n \partial_t \varphi (D \varphi) + \hat{g}^n D (\hat{g}_t)_{\hat{\imath} \hat{j}} - \hat{g}^n D \hat{g}_{\hat{\imath} \hat{j}}.
\]
Hence, working in real coordinates, the function $u = D(\varphi)$ satisfies a linear parabolic PDE of the form
\begin{equation}
\partial_t u = a^{\alpha\beta} \partial_{x^\alpha} \partial_{x^\beta} u + f,
\end{equation}
where $A = (a^{\alpha\beta})$ is a real $2n \times 2n$ positive definite symmetric matrix whose largest and smallest eigenvalues $\Lambda$ and $\lambda$ satisfy
\begin{equation}
C^{-1} \leq \lambda \leq \Lambda \leq C,
\end{equation}
for a uniform positive constant $C$.

Moreover, the entries of $A$ are uniformly bounded in the $C^{\delta/2, \delta}$ parabolic norm for $0 < \delta < 1$. Indeed our Calabi-type estimate from part (i) of Theorem 1.1, for a uniform positive constant $C$

implies that the Riemannian metric $g_R$ associated to $g$ is bounded in the $C^1$ norm in the space direction. On the other hand, from the curvature bound of Theorem 1.1, we know that $\partial_t g$ is bounded in the $C^{0, \delta}$ parabolic norm for all $\delta < 1$. This implies that $a^{\alpha\beta}$ is uniformly bounded in the $C^{\delta/2, \delta}$ parabolic norm for any $0 < \delta < 1$.

Next, note that $u = \frac{\partial \varphi}{\partial x^\gamma}$ in (5.1) is bounded in the $C^0$ norm since $g(t)$ is uniformly bounded and hence $|\nabla \varphi|_{C^0}$ is uniformly bounded. Moreover, $f$ in (5.1) is uniformly bounded in the $C^{\delta, \delta}$ norm.

We can then apply Theorem 8.11.1 in [15] to (5.1) to see that $u$ is bounded in the parabolic $C^{1+\delta/2, 2+\delta}$ norm on a slightly smaller parabolic domain: $[\varepsilon', T] \times B_{r'}$ for any $\varepsilon'$ and $r'$ with $0 < \varepsilon' < \varepsilon$ and $r'/8 < r' < r/4$. Tracing through the argument in [15], one can check that the estimates we obtain indeed are of the desired form.

Now apply $D$ to the equality $g_{\gamma\gamma}(t) = (\hat{g}_t)_{\gamma\gamma} + \partial_\gamma \partial_{\gamma'} \varphi$. We get
\begin{equation}
Dg_{\gamma\gamma} = D(\hat{g}_t)_{\gamma\gamma} + \partial_\gamma \partial_{\gamma'} u,
\end{equation}
where we recall that $D = \partial / \partial x^\gamma$ for some $\gamma$. Since we have bounds for $u$ in $C^{1+\delta/2, 2+\delta}$ this implies that $\partial_\gamma \partial_{\gamma'} u$ is bounded in $C^{\delta/2, \delta}$. Since $D(\hat{g}_t)_{ij}$ is uniformly bounded in all norms we get that $Dg_{ij}$ is uniformly bounded in $C^{\delta/2, \delta}$ for all $i, j$. Since $D = \partial / \partial x^\gamma$ and $\gamma$ was an arbitrary index, it follows that $\partial_\gamma a^{\alpha\beta}$ is uniformly bounded in $C^{\delta/2, \delta}$ for all $\alpha, \beta, \gamma$. We have a similar estimate for $\partial_\gamma f$. Now apply Theorem 8.12.1 in [15] (with $k = 1$) to see that, for any $\alpha$, $\partial_\alpha u$ is uniformly bounded in $C^{1+\delta/2, 2+\delta}$ on a slightly smaller parabolic domain. This means that $D^\alpha \varphi$ is uniformly bounded in $C^{1+\delta/2, 2+\delta}$ for any multi-index $\alpha \in \mathbb{R}^{2n}$ with $|\alpha| \leq 2$.

We can then iterate this procedure and obtain the required $C^k$ bounds for $g(t)$ for all $k$. This completes the proof of the corollary.

**Remark 5.1.** In [21], we showed how to obtain higher derivative estimates for curvature using simple maximum principle arguments (following [13, 18]). However, in the case of the Chern-Ricci flow, there are difficulties in using this approach because of torsion terms that need to be controlled. An alternative method to proving the estimates in this section may be to generalize the work of Gill on the Kähler-Ricci flow [12]. This could give an
“elementary” maximum principle proof, but the technical difficulties in carrying this out seem to be substantial.

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