INTRODUCTION

The notion of projective connections traces back to E.Cartan. The existence of a holomorphic projective connection on a compact complex manifold $M$ is a strong condition as soon as $\dim M > 1$. A result of Kobayashi and Ochiai ([KoOc80], Theorem 2.1) says that a compact Kähler Einstein (KE) manifold $M_m$ admits a holomorphic projective connection if and only if $M$ is

1.) $\mathbb{P}_m(\mathbb{C})$
2.) a finite étale quotient of a torus or
3.) a ball quotient, i.e., the universal covering space of $M$ can be identified with the open unit ball $\mathbb{B}_m(\mathbb{C})$ in $\mathbb{C}^m$.

Equivalently, the holomorphic sectional curvature is constant on $M$ or, in purely numerical terms, the Chen–Ogiue inequality is an equality:

$$2(m + 1)c_2(M) = mc_1(M)^2.$$ 

Kobayashi and Ochiai also gave a classification of all compact complex surfaces carrying such a connection ([KoOc80]). For Kähler surfaces their result says that any such surface is KE.

Before the authors’ previous work in [JaRa04] it was not known whether there exist non–KE examples of compact Kähler manifolds carrying such a connection. In [JaRa04] the complete classification of projective threefolds with a holomorphic projective connection was given with one additional example: modular families of fake elliptic curves over a Shimura curve.

In this article the classification is extended to projective manifolds of arbitrary dimension – up to the question of abundance (see explanations below):

0.1. Theorem. Let $M$ be an $m$–dimensional projective manifold with a holomorphic projective connection $\Pi$. Assume that $M$ is not Kähler Einstein, i.e., not one of the examples above. Then $K_M$ is nef. If $K_M$ is even abundant and if $\Pi$ is flat, then, up to a finite étale covering, $M$ is an abelian group over a compact Shimura curve $C$, i.e., there exists a holomorphic submersion

$$f : M \longrightarrow C$$

such that every fiber is smooth abelian and such that $f$ has a smooth section. Moreover, the following equivalent conditions are satisfied:

1.) the Arakelov inequality ([FaX3])

$$2 \deg f_* \Omega^1_{M/C} \leq (m - 1) \deg K_C = 2g_C - 2$$
is an equality.

2.) \( M \simeq Z \times_C Z \times_C \cdots \times_C Z \) where \( Z \to C \) a Kuga fiber space constructed from the rational corestriction \( \text{Cor}_{F/Q}(A) \) of a division quaternion algebra \( A \) defined over a totally real number field \( F \) such that
\[
A \otimes \mathbb{Q} \iso M_2(\mathbb{R}) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}.
\]

Here \( \mathbb{H} \) denotes the Hamiltonian quaternions (see section 3).

Conversely, any abelian scheme as in 1.) or 2.) admits a flat projective connection.

The equivalence of 1.) and 2.) for abelian group schemes over curves is a result of Viehweg and Zuo ([ViZu04]). The construction of the family \( Z \to C \) in 2.) is a generalization of a construction of modular families of abelian varieties due to Mumford ([Mu66]). For the proof of Theorem 0.1 see section 4. Note that Theorem 0.1 gives a result analogous to the KE case: an explicit method of how to construct the examples as well as a description in purely numerical terms. Some explanations:

The canonical divisor \( K_M = \det \Omega^1_M \) is called nef if \( K_M \cdot C \geq 0 \) for any irreducible curve \( C \) in \( M \). It is called abundant if \( |dK_M| \) is spanned for some \( d \gg 0 \) defining a holomorphic map \( f : M \to Y \) onto some (perhaps singular) projective variety \( Y \). The map is called Iitaka fibration of \( M \).

Abundance Conjecture ([KoMo98]). Let \( M \) be a projective manifold. Does \( K_M \) nef imply \( K_M \) abundant?

The conjecture is known to hold true in dimensions \( \leq 3 \) and in many particular cases. In different terms, Theorem 0.1 gives a complete classification of projective manifolds carrying a flat holomorphic projective structure in any dimension in which Abundance Conjecture is true.

The construction of the examples is explained in detail in section 5. Here we only give some details. Let \( F \) be a totally real number field, \( F : \mathbb{Q} = d \). Let \( A/F \) be as in 2.) of Theorem 0.1. The rational corestriction \( \text{Cor}_{F/Q}(A) \) is a central simple \( \mathbb{Q} \)-algebra of dimension \( 4d \) and either
\[
\text{Cor}_{F/Q}(A) \simeq M_{2d}(\mathbb{Q}) \quad \text{or} \quad \text{Cor}_{F/Q}(A) \simeq M_{2d-1}(B),
\]
for some division quaternion algebra \( B/\mathbb{Q} \). We refer to the first case as 'B splits'. From this setting one constructs ([Mu66], [ViZu04]) a modular family \( f : Z \to C \) of abelian varieties over some compact Shimura curve \( C \) with simple general fiber \( Z_\tau \) satisfying

- \( \dim Z_\tau = 2d-1 \) and \( \text{End}_Q(Z_\tau) \simeq \mathbb{Q} \) in the case \( B \) split,
- \( \dim Z_\tau = 2d \) and \( \text{End}_Q(Z_\tau) \simeq B \) in the case \( B \) non-split.

The smallest possible dimension is obtained for \( F = \mathbb{Q}, A = B \) an indefinite division quaternion algebra. Here \( f : Z \to C \) is a (PEL-type) family of abelian surfaces, \( \text{End}_Q(Z_\tau) \simeq B \) for \( \tau \) general. Such abelian surfaces are called fake elliptic curves above, it is the example found in [JaRa04]. Since the abundance conjecture is true in dimension \( \leq 3 \) we have the following corollary to Theorem 0.1.

0.2. Corollary. Let \( M \) be a projective manifold with a flat holomorphic normal projective connection of dimension at most 3 which is not Kähler Einstein. Then \( \dim M = 3 \) and \( M \) is, up to étale coverings, a modular family of fake elliptic curves.

The Corollary includes Kobayashi and Ochiai’s classification in the (flat, projective) surface case and the authors’ previous result on projective threefolds.
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Notations. We consider only complex manifolds $M$; $T_M$ denotes the holomorphic tangent bundle, $\Omega^1_M$ the dual bundle of holomorphic one forms. We do not distinguish between line bundles and divisors on $M$. $K_M = \det \Omega^1_M$ denotes the canonical divisor.

1. Holomorphic normal projective connections

There are various definitions of projective structures and connections. We essentially follow Kobayashi and Ochiai ([KoOc80]).

1.1. Holomorphic normal projective connections (H.n.p.c.) First recall the definition of the Atiyah class ([A57]). Associated to a holomorphic vector bundle $E$ on the complex manifold $M$ one has the first jet sequence

$$0 \longrightarrow \Omega^1_M \otimes E \longrightarrow J_1(E) \longrightarrow E \longrightarrow 0.$$ 

Obstruction to the holomorphic splitting is a class $b(E) \in H^1(M, \text{Hom}(E,E) \otimes \Omega^1_M)$ called Atiyah class of $E$. For properties of $b(E)$ see [A57]. We only mention that if $\Theta^{1,1}$ denotes the $(1,1)$–part of the curvature tensor of some differentiable connection on $E$, then, under the Dolbeault isomorphism, $b(E)$ corresponds to $(\Theta^{1,1}) \in H^{1,1}(M, \text{Hom}(E,E))$. In particular, for $M$ Kähler, $\text{tr}(b(E)) = -2\pi \chi(E) \in H^1(M, \Omega^1_M)$. This is why we normalise and put $a(E) := -\frac{1}{2\pi}b(E)$.

1.1. Definition. $M_m$ carries a holomorphic normal projective connection (h.n.p.c.) if the (normalised) Atiyah class of the holomorphic cotangent bundle has the form

$$a(\Omega^1_M) = \frac{c_1(K_M)}{m+1} \otimes id_{\Omega^1_M} + id_{\Omega^1_M} \otimes \frac{c_1(K_M)}{m+1} \in H^1(M, \Omega^1_M \otimes T_M \otimes \Omega^1_M).$$

(Note $\Omega^1_M \otimes T_M \otimes \Omega^1_M \simeq \text{End}(\Omega^1_M) \otimes \Omega^1_M \simeq \Omega^1_M \otimes \text{End}(\Omega^1_M)$.)

It was shown in [MoMo96] how a holomorphic cocycle solution to (1.2) can be thought of as a $\mathbb{C}$–bilinear holomorphic connection map $\Pi : T_M \times T_M \to T_M$ satisfying certain rules modelled after the Schwarzian derivative. Conversely, the existence of such a connection implies (1.2).

Let $\tilde{M} \to M$ be finite étale. Then $M$ carries a h.n.p.c. if and only if $\tilde{M}$ carries a h.n.p.c.

1.2. Holomorphic projective structures. The manifold $M$ is said to admit a holomorphic projective structure or a flat h.n.p.c. if there exists an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ with holomorphic maps $\varphi_i : U_i \to \mathbb{P}_m(\mathbb{C})$ such that

$$\varphi_i \circ \varphi^{-1}_j : \varphi_j(U_{ij}) \to \varphi_i(U_{ij})$$

is the restriction of some $g_{ij} \in \text{PGl}_m(\mathbb{C})$ whenever $U_{ij} = U_i \cap U_j \neq \emptyset$. The following fact is fundamental: If $M$ admits a holomorphic projective structure, then $M$ admits a h.n.p.c. Whether the converse is true is not known. $M$ admits a holomorphic projective structure if and only if $\Pi = 0$ is a cocycle solution to (1.2).

Let $\tilde{M} \to M$ be étale. Then $\tilde{M}$ and $M$ are locally isomorphic and $\tilde{M}$ carries a holomorphic projective structure if and only if $M$ does.
1.3. Example. Examples of projective or Kähler manifolds admitting a (flat) h.n.p.c. are:

1.) $\mathbb{P}_m(\mathbb{C})$.

2.) Any manifold $M = M_m$ whose universal covering space $\tilde{M}$ can be embedded into $\mathbb{P}_m(\mathbb{C})$ such that $\pi_1(M)$ acts by restricting automorphisms from $PGl_{m+1}(\mathbb{C})$.

3.) Ball quotients, i.e., $M$ with $\tilde{M} = \mathbb{B}_m(\mathbb{C})$, the non compact dual of $\mathbb{P}_m(\mathbb{C})$ in the sense of hermitian symmetric spaces. This is a special case of 2.).

4.) Finite étale quotients of tori.

There are more examples without the assumption $M$ projective/Kähler:

5.) tori, certain Hopf manifolds, twistor spaces over conformally flat Riemannian fourfolds.

1.4. Remark. There is an analogous notion of an $S$–structures, $S$ an arbitrary hermitian symmetric space of the compact type ([Ko2]). The rank one case $S = \mathbb{P}_m(\mathbb{C})$ is somewhat special in this context.

1.3. Development. ([KoOc80], [KoWu83]) Let $M_m$ carry a flat h.n.p.c. and let $\{(U_i, \varphi_i)\}_{i \in I}$ be a projective atlas. Choose one coordinate chart $(U_0, \varphi_0)$. Given a point $p \in M_m$, choose a chain of charts $(U_i, \varphi_i)_{i = 0, \ldots, r}$ such that $U_i \cap U_{i-1} \neq \emptyset$, $i = 1, \ldots, r$ and $p \in U_r$. We have

$$\varphi_{i-1} \circ \varphi_i^{-1} = g_i|_{\varphi_i(U_i \cap U_{i-1})}$$

for projective transformations $g_i \in PGl_m(\mathbb{C})$. Set $\psi(p) := (g_1 \circ g_2 \circ \cdots \circ g_r \circ \varphi_r)(p) \in \mathbb{P}_m(\mathbb{C})$. This gives a multivalued map $\psi : M \to \mathbb{P}_m(\mathbb{C})$ which is single valued in the case $M$ simply connected. It is called a development of $M$ to $\mathbb{P}_m(\mathbb{C})$. By construction, $\psi$ is locally an isomorphism.

If $M$ is not simply connected, then let $\tilde{M}$ be its universal covering space and let $\psi : \tilde{M} \to \mathbb{P}_m(\mathbb{C})$ be a development of $\tilde{M}$. We get a natural map $\rho : \pi_1(M) \to PGl_m(\mathbb{C})$ such that $\psi(\gamma(p)) = \rho(\gamma)(\psi(p))$ for any $\gamma \in \pi_1(M)$ ([KoWu83]).

1.5. Example. Two examples of neighborhoods of an elliptic curve:

1.) Let $U \subset \mathbb{P}_2$ be an open neighborhood of an elliptic curve $E$ such that $U$ is a retract of $E$. Then $U$ inherits a holomorphic projective structure from $\mathbb{P}_2$. Let $\mu : \tilde{U} \to U$ be the universal covering space of $U$. Then $\tilde{U}$ inherits a holomorphic projective atlas from $U$ such that $\mu$ is a development of $\tilde{U}$.

2.) Let $Z \longrightarrow C$ be a modular family of elliptic curves as in example 5.2. Then the embedding

$$\iota : \tilde{Z} = \mathbb{C} \times \mathcal{H}_2 \hookrightarrow \mathbb{P}_2(\mathbb{C}), \ (z, \tau) \mapsto [z : \tau : 1],$$

is a development of $\tilde{Z}$.

1.4. Chern Classes. Let $M_m$ be as above compact Kähler with a h.n.p.c. Similar to projectice space one has ([KoOc80]):

$$c_r(M) = \frac{1}{(m+1)^r} \binom{m+1}{r} c_1^r(M), \ r = 0, \ldots, m \quad \text{in} \ H^r(M, \Omega^r_M).$$

In particular, $2(m+1)c_2(M) = mc_1^2(M)$.

2. Birational Classification (flat case)

In the Kähler Einstein case, $2(m+1)c_2(M) = mc_1^2(M)$ if and only if the holomorphic sectional curvature of $M_m$ is constant. This gives the following result due to Kobayashi and Ochiai ([KoOc80]) from the introduction:
2.1. **Theorem.** Let $M$ be Kähler-Einstein with a h.n.p.c. Then

1.) $M \simeq \mathbb{P}_m(\mathbb{C})$ or
2.) $M$ is an étale quotient of a torus or
3.) $M$ is a ball quotient.

Recall that Kähler-Einstein implies $\pm K_M$ ample or $K_M \equiv 0$. Conversely, by Aubin and Yau’s proof of the Calabi conjecture, $K_M$ ample implies $M$ Kähler-Einstein.

Consider the projective case from now on, i.e., let $M_m$ be a projective manifold with a h.n.p.c., not necessarily flat. If $K_M$ is not nef, then $M$ contains a rational curve by the cone theorem ([KoMo98]) meaning that there exists a non-constant holomorphic map $
u : \mathbb{P}_1(\mathbb{C}) \to M$. Then already $M \simeq \mathbb{P}_m(\mathbb{C})$.

2.2. **Proposition.** Let $M_m$ be a projective manifold with a h.n.p.c. If $M$ contains a rational curve, then $M \simeq \mathbb{P}_m(\mathbb{C})$.

We include here a proof in the case of a flat connection, for the general case see [JaRa04].

**Proof.** Let $\psi : \tilde{M} \to \mathbb{P}_m(\mathbb{C})$ be a development of the universal covering space $\mu : M \to \tilde{M}$. We have an induced map $\tilde{\nu} : \mathbb{P}_1(\mathbb{C}) \to \tilde{M}$ such that $\mu \tilde{\nu} = \nu$. Then $\nu^* T_M = (\psi \circ \tilde{\nu})^* T_{\mathbb{P}_m(\mathbb{C})}$ is ample. This forces $M \simeq \mathbb{P}_m(\mathbb{C})$ by Mori’s proof of Hartshorne’s conjecture ([Mori79], in particular [MiPe97 I, Theorem 4.2.]). \qed

2.3. **Remark.** Proposition 2.2 holds mutatis mutandis for any projective manifold with a flat $\mathcal{S}$–structure, $\mathcal{S}$ some irreducible hermitian symmetric space of the compact type: the existence of a rational curve implies $M$ uniruled and $M \simeq \mathcal{S}$ by [HwMo97].

Thus if $M \not\simeq \mathbb{P}_m(\mathbb{C})$, then $K_M$ is nef and $M$ does not contain any rational curve. This has strong consequences: any rational map

$$M' \to M$$

from a manifold $M'$ must be holomorphic. Indeed, if we first had to blow up $M'$ in order to make it holomorphic, then we could find a rational curve in $M$. A manifold with this property is sometimes called strongly minimal or absolutely minimal ([Mori87], §9).

The following is taken from ([Ko93], 1.7.): $M$ is said to have generically large fundamental group if for any $p \in M$ general and for any irreducible complex subvariety $p \in W \subset M$ of positive dimension

$$\text{Im}[\pi_1(W_{\text{norm}}) \to \pi_1(M)]$$

is infinite, where $W_{\text{norm}}$ denotes the normalization of $W$. In our case:

2.4. **Proposition.** Any $M \not\simeq \mathbb{P}_m(\mathbb{C})$ with a flat h.n.p.c. has generically large fundamental group.
Proof. By Proposition 2.2, $M \not\cong \mathbb{P}_m(\mathbb{C})$ implies $K_M$ nef. Assume to the contrary $\text{Im}[\pi_1(W_{\text{norm}}) \to \pi_1(M)]$ is finite for some $W$ as above. Denote the normalization map by $\nu : W_{\text{norm}} \to W$.

Let $C_{\text{norm}}$ be some general curve in $W_{\text{norm}}$, i.e., the intersection of dim $W - 1$ general hyperplane sections. Think of $C_{\text{norm}}$ as the normalization of $C := \nu(C_{\text{norm}}) \subset W$. We have

$$\pi_1(C_{\text{norm}}) \to \pi_1(W_{\text{norm}}) \to \pi_1(M).$$

Then $\text{Im}[\pi_1(C_{\text{norm}}) \to \pi_1(M)]$ is finite. The kernel of $\pi_1(C_{\text{norm}}) \to \pi_1(M)$ induces a finite étale covering $C' \to C_{\text{norm}}$ from a compact Riemann surface $C'$, such that $\mu : C' \to C$ factors over $\tilde{M}$, the universal covering space of $M$. Let $\psi : \tilde{M} \to \mathbb{P}_m(\mathbb{C})$ be a development. Denote the induced map $C' \to \tilde{M} \to \mathbb{P}_m(\mathbb{C})$ by $\psi_1$. Then $\mu^*T_M = \psi_1^*T_{\mathbb{P}_m(\mathbb{C})}$ is ample, contradicting $K_M$ nef. $\square$

Now assume $M$ is abundant, i.e., that some multiple of $K_M$ is in fact spanned, defining the Iitaka fibration

$$(2.5) \quad f : M \to Y$$

onto some normal projective variety $Y$ of dimension $\dim Y = \kappa(M)$. Since $M$ does not contain any rational curve, $f$ is equidimensional ([Ka91], Theorem 2). Let $F$ be a general (connected) fiber. Then $K_F = K_M + \det N_{F/M}$ is torsion in $\text{Pic}(F)$ and there exists a finite étale map $\tilde{F} \to F$ such that $K_{\tilde{F}}$ is trivial. By Beauville’s decomposition theorem ([Bo74], [Be83]) we may assume

$$\tilde{F} \cong A_y \times B_y$$

where $A_y$ is abelian and $B_y$ is simply connected. By Proposition 2.4, $B_y$ must be a point. Thus $M$ admits a fibration whose general fiber is covered by an abelian variety (see again [JaRa04] for the non-flat case).

We say that the Iitaka fibration defines an abelian group scheme structure on $M$, in the case $Y$ smooth, $f$ submersive, every fiber of $f$ is a smooth abelian variety, and $f$ admits a smooth section. The next proposition is mainly a consequence of a result of Kollár:

2.6. Proposition. Let $M$ be a projective manifold with a flat h.n.p.c. and assume $K_M$ is abundant. Then there exists a commuting diagram

$$
\begin{array}{ccc}
M' & \longrightarrow & M \\
\downarrow & & \downarrow \\
N' & \longrightarrow & N
\end{array}
$$

with the following property: (i) The vertical maps are the Iitaka fibrations of $M'$ and $M$, respectively. (ii) The map $M' \to M$ is finite étale. (iii) $M' \to N'$ is an abelian group scheme.

of Proposition 2.6. The general fiber of the Iitaka fibration $f : M \to Y$ is an étale quotient of an abelian variety (see the explanations before Proposition 2.6). By [Ko93], 6.3. Theorem there exists a finite étale covering $M' \to M$ such that $f' : M' \to Y'$ is birational to an abelian group scheme $\varphi : A \to S$. Here $M' \to Y'$ is the Stein factorization of $M' \to M \to Y$. Note that $M'$ is abundant and $M' \to Y'$ is the Iitaka fibration of $M'$.
After replacing $M$ by $M'$ we may assume that $M \rightarrow Y$ is birational to $A \rightarrow S$. We obtain a digram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & M \\
\varphi \downarrow & & \downarrow f \\
S & \xrightarrow{g} & Y
\end{array}
\]

in which $h$ and $g$ are birational. The map $h$ must be holomorphic, as $M$ is strongly minimal. The map $\varphi$ has a smooth section. Then $g$ must be holomorphic, too. There exists a Zariski open subset $U \subset Y$ such that the two fibrations $f : f^{-1}(U) \rightarrow U$ and $\varphi : \varphi^{-1}(g^{-1}(U)) \rightarrow g^{-1}(U)$ are isomorphic. The claim is that then $Y$ is smooth and $M \rightarrow Y$ is a smooth abelian fibration with a smooth section.

First note that by (2.7) every fiber of $f$ is irreducible and reduced, and $h$ is generically 1:1 on every fiber $A_s$ of $\varphi$ for $s \in S$ arbitrary. We have a generically injective map

\[
T_{A_s} \rightarrow T_A|_{A_s} \rightarrow h^*T_M|_{A_s}
\]

and it remains to show that it is a bundle map. Indeed, then every fiber of $f$ is smooth, implying the smoothness of $Y$. The existence of a smooth section of $f$ then follows from the existence of a section of $\varphi$ and the absence of rational curves in $M$.

By construction, $dK_M = f^*H$ for some $H \in \text{Pic}(Y)$. Then $(h^*c_1(M)|_{A_s} = 0$ in $H^1(A_s, h^*\Omega^1_M|_{A_s})$. By (1.2) the Atiyah class of $h^*T_M|_{A_s}$ vanishes. The tangent bundle $T_{A_s}$ is trivial. The next Lemma shows that (2.8) is a bundle map. \hfill \square

2.9. Lemma. Let $A$ be a torus and $E$ a vector bundle on $A$ with vanishing Atiyah class $a(E) \in H^1(A, E^* \otimes E \otimes \Omega^1_A)$. Let $0 \neq s \in H^0(A, E)$ be a holomorphic section. Then $Z(s) = \{p \in A | s(p) = 0\} = \emptyset$.

Proof. Let $\mathbb{P}(E)$ be the hyperplane bundle associated to $E$ and $\pi : \mathbb{P}(E) \rightarrow A$ be the projection map. The vanishing of $a(E)$ implies the holomorphic splitting of the relative tangent bundle sequence

\[
0 \rightarrow T_{\mathbb{P}(E)/A} \rightarrow T_{\mathbb{P}(E)} \rightarrow \pi^*T_A \rightarrow 0
\]

([PPSch87], 3.1.). Then $T_{\mathbb{P}(E)} \simeq T_{\mathbb{P}(E)/A} \oplus \pi^*T_A$ and $A$ acts as a group on $\mathbb{P}(E)$. For $a \in A$ let $t_a$ be the induced translation map on $A$ and let $t'_a$ be the induced automorphism of $\mathbb{P}(E)$. We get a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(E) & \xrightarrow{t'_a} & \mathbb{P}(E) \\
\downarrow & & \downarrow \\
A & \xrightarrow{t_a} & A.
\end{array}
\]

Then $t'_aO_{\mathbb{P}(E)}(1) \simeq O_{\mathbb{P}(E)}(1) \otimes \pi^*L$ for some $L \in \text{Pic}(A)$. The push forward to $A$ shows $t'_aE \simeq E \otimes L$ ([Ha77], III, 9.3.). Then $t'_a \det E \simeq \det E \otimes L^{\otimes r}$ for $r = rkE$. As $\det E \in \text{Pic}^0(A)$ is invariant under translations, $L^{\otimes r} \simeq O_A$. Then $L$ is trivial as $A$ acts continuously.

Then $t'_aE \simeq E$ for any $a \in A$ implying that $A$ acts on $H^0(A, E)$. The action must be trivial. Then $s(p + a) = s(p)$ for any $a \in A$ implying $Z(s) = \emptyset$. \hfill \square
3. Torus fibrations

In this section \( M_m \) denotes a compact Kähler manifold with a h.n.p.c. Assume the existence of a holomorphic submersion
\[
f : M_m \longrightarrow N_n
\]
such that every fiber is a torus. Assume that \( f \) has a smooth holomorphic section.

We have the exact sequence
\[
0 \longrightarrow f^*\Omega^1_N \xrightarrow{df} \Omega^1_M \longrightarrow \Omega^1_{M/N} \longrightarrow 0
\]
of holomorphic forms and
\[
E = E^{1,0} = f_*\Omega^1_{M/N}
\]
is a holomorphic rank \( m - n \) vector bundle on \( N \) such that \( f^*E \simeq \Omega^1_{M/N} \) via the canonical map \( f^*f_*\Omega^1_{M/N} \to \Omega^1_{M/N} \).

3.2. Proposition. In the above situation, \( N \) admits a h.n.p.c., and
\[
a(E(-\frac{K_N}{n+1})) = 0 \quad \text{in} \quad H^1(N, \text{End}(E) \otimes \Omega^1_N),
\]
where \( a \) denotes the normalised Atiyah class.

3.4. Remark. 1.) The formula is in terms of classes, we do not assume the existence of a theta characteristic on \( N \).

2.) In the case \( N \) a compact Riemann surface, \( a(E) \) implies \( E \simeq U \otimes \theta \) for some flat bundle \( U \) coming from a representation of \( \pi_1(N) \) \((\text{[A57]})\) and some theta characteristic \( \theta \).

Proposition 3.2 will be proved below, we will first derive some consequences. The trace of the (normalised) Atiyah class gives the first Chern class, hence
\[
c_1(E) = \frac{m-n}{n+1} c_1(K_N) \quad \text{in} \quad H^1(N, \Omega^1_N).
\]
Let us usual \( K_{M/N} = K_M - f^*K_N \). Then \( K_{M/N} = \det \Omega^1_{M/N} = f^* \det E \). We may rewrite (3.5) as follows:

3.6. Corollary. In the situation of Proposition 3.2 the following identities hold in \( H^1(M, \Omega^1_M) \):
\[
c_1(K_{M/N}) = \frac{m-n}{n+1} c_1(f^*K_N) \quad \text{and} \quad c_1(K_M) = \frac{m+1}{n+1} c_1(f^*K_N)
\]
In particular, \( c_1(K_M) \) and \( c_1(f^*K_N) \) are proportional.

The next corollary covers in particular degenerate cases when \( M_m \) is a product of tori and \( f \) a projection or when \( M_m = N_n = \mathbb{P}_m(\mathbb{C}) \) and \( f \) is isomorphism of projective space:

3.7. Corollary.
1.) If \( c_1(K_M) = 0 \) or \( c_1(K_N) = 0 \), then \( M \) and \( N \) are étale quotients of tori.
2.) If \( K_M \) or \( K_N \) are not nef, then \( M \simeq N \simeq \mathbb{P}_m(\mathbb{C}) \).
Proof. 1.) By Corollary 3.6 if \( c_1(M) = 0 \) or \( c_1(N) = 0 \), then \( c_1(K_M) = c_1(K_N) = 0 \). By (1.6) all Chern classes of \( M \) and \( N \) vanish. Then \( M \) and \( N \) are Kähler-Einstein and covered by tori by Theorem 2.1. 2.) If \( K_M \) or \( K_N \) are not nef, then \( K_M \) and \( K_N \) are not nef by Corollary 3.6 and \( M \simeq \mathbb{P}_m(\mathbb{C}) \) and \( N \simeq \mathbb{P}_n(\mathbb{C}) \) by Proposition 2.2. As \( n > 0 \) by assumption, \( m = n \) and \( f \) is an automorphism of projective space.

Proof of Proposition 3.2. Denote the section of \( f \) by \( s : N \to M \). Consider the pull back to \( N \) by \( s \) of \( \Omega_M \)

\[
(3.8) \quad 0 \longrightarrow \Omega_N^1 \xrightarrow{s^*df} s^*\Omega_M^1 \longrightarrow s^*\Omega_{M/N}^1 \simeq E \longrightarrow 0.
\]

We have the map \( ds : s^*\Omega_M^1 \to \Omega_N^1 \). As \( (ds)(s^*df) = d(f \circ s) = id_{\Omega_M^1} \), we see that \( (3.8) \) splits holomorphically.

The normalised Atiyah class of \( s^*\Omega_M^1 \) is obtained from the one of \( \Omega_M^1 \) by applying \( ds \) to the last \( \Omega_M^1 \) factor in (1.2). What we get is

\[
(3.9) \quad a(s^*\Omega_M^1) = \frac{s^*c_1(K_M)}{m + 1} \otimes ds + id_{s^*\Omega_M^1} \otimes \frac{c_1(s^*K_M)}{m + 1} \in H^1(N, s^*\Omega_M^1 \otimes s^*T_M \otimes \Omega_{M/N}^1),
\]

where we carefully distinguish between \( s^*c_1(K_M) \in H^1(N, s^*\Omega_M^1) \) and the class \( c_1(s^*K_M) = ds(c_1(K_M)) \in H^1(N, \Omega_N^1) \).

The Atiyah class of a direct sum is the direct sum of the Atiyah classes (\( [A57] \)).

As the pull back of (3.4) splits holomorphically, we get the Atiyah classes of \( \Omega_N^1 \) and \( E \) by projecting (3.3) onto the corresponding summands.

We first compute \( a(E) \). The class \( c_1(K_M) \in H^1(M, \Omega_M^1) \) is the pull back of some class in \( H^1(N, \Omega_N^1) \); it therefore vanishes under \( H^1 \) of \( \Omega_M^1 \to \Omega_{M/N}^1 \). This means the first summand in (3.3) vanishes if we project, while the second summand becomes

\[
(3.10) \quad id_E \otimes \frac{c_1(s^*K_M)}{m + 1} \in H^1(N, E \otimes E^* \otimes \Omega_N^1),
\]

and this is \( a(E) \). The trace is

\[
c_1(E) = rk(E) \frac{c_1(s^*K_M)}{m + 1} \in H^1(N, \Omega_N^1)
\]

The determinant of (3.3) gives the following identities of classes in \( H^1(N, \Omega_N^1) \):

\[
(3.11) \quad c_1(K_N) = c_1(s^*K_M) - c_1(E) = \frac{m + 1 - (m - n)}{m + 1} c_1(s^*K_M)
\]

Now (3.3) follows from (3.10).

Next we compute \( a(\Omega_N^1) \). We have to apply \( ds \) to the first factor of the first summand in (3.3). This gives \( c_1(s^*K_M) \). As the splitting maps give the identity we get

\[
\frac{c_1(s^*K_M)}{m + 1} \otimes id_{\Omega_N^1} + id_{\Omega_N^1} \otimes \frac{c_1(s^*K_M)}{m + 1} \in H^1(N, \Omega_N^1 \otimes T_N \otimes \Omega_N^1)
\]

and this is \( a(\Omega_N^1) \). As we just saw in (3.11)

\[
\frac{c_1(K_N)}{n + 1} = \frac{c_1(s^*K_M)}{m + 1}.
\]

Replacing this in the above formula we see that \( N \) admits a h.n.p.c. as in (1.2). The Proposition is proved.
3.12. Proposition. In the situation of Proposition 3.2 assume that $f$ is the Iitaka fibration of $M$. Then $N$ is a ball quotient.

Proof. If $f$ is the Iitaka fibration defined by $|dK_M|$ for some $d \in \mathbb{N}$, then $dK_M = f^*H$ for some ample $H \in \text{Pic}(N)$. By Corollary 3.6 $c_1(f^*K_N) = \frac{n+1}{d(n+1)}c_1(f^*H)$. The restriction to the section shows that $K_N$ is ample. As $N$ admits a h.n.p.c., $N$ is a ball quotient by Theorem 2.1 \qed

3.13. Example. The above ideas also apply to the following situation: let $f: M \rightarrow N$ be a vector bundle over some manifold $N$. Then $\Omega^1_{M/N} \cong f^*M^*$. Assume that the total space $M$ carries a h.n.p.c.. As in Proposition 3.2

$$a(M^*(- \frac{K_N}{n+1})) = 0 \quad \text{in } H^1(N, \text{End}(M^*) \otimes \Omega_N^1).$$

For example let $M \subset \mathbb{P}_m(\mathbb{C})$ be the complement of a linear subspace $\simeq \mathbb{P}_n(\mathbb{C})$ of $\mathbb{P}_m(\mathbb{C})$. $M$ inherits a projective structure from $\mathbb{P}_m(\mathbb{C})$. Projection to $\mathbb{P}_n(\mathbb{C})$ shows

$$(3.14) \quad M \cong \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(1)^{\oplus m-n},$$

i.e., $M$ carries the structure of a vector bundle over $\mathbb{P}_n(\mathbb{C})$. Here $M^*(- \frac{K_N}{n+1}) \cong \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(\mathbb{P}_n(\mathbb{C}))$ and the Atiyah class of this bundle vanishes.

4. Abelian schemes

Let $f: M_m \rightarrow N_n$ be an abelian scheme, $E = E_{1,0} = f_*\Omega^1_{M/N}$ as in the preceding section. In the case where $N$ is a compact Riemann surface of genus $g > 0$ the Arakelov inequality says

$$2 \deg E \leq (m-1)\deg(K_N) = (m-1)(2g_N-2).$$

From Corollary 3.6 and results of Viehweg and Zuo we obtain:

4.1. Proposition. Let $f: M \rightarrow N$ be an abelian scheme. Assume that $M$ admits a h.n.p.c. and that $N$ is of general type, $0 < \dim N < \dim M$. Then $N$ is a compact Riemann surface and

$$2 \deg E = (m-1)\deg(K_N).$$

Proof. Let $V := R^1f_*\mathcal{O}_C$. By [ViZu07], Theorem 1 and Remark 2 we have the inequality of slopes $\mu_{K_N}(V) = 2\mu_{K_N}(E) \leq \mu_{K_N}(\Omega^1_N)$, where for any torsion free sheaf $F$ of positive rank and ample $H \in \text{Pic}(N)$ one defines as usual $\mu_H(F) = c_1(F).H^{n-1}/\text{rk}F$. By Corollary 3.6

$$2\mu_{K_N}(E) = \frac{2}{n+1}c_1(K_N)^n \leq \mu_{K_N}(\Omega^1_N) = \frac{1}{n}c_1(K_N)^n.$$

Since $c_1(K_N)^n > 0$ we find $n = 1$. Then $N$ is Riemann surface and we have in fact equality $2\mu_{K_N}(E) = \mu_{K_N}(\Omega^1_N)$, i.e., the Arakelov inequality is an equality. \qed

4.2. Remark. Let $f: M \rightarrow N$ be as above, $N$ a compact Riemann surface. Think of $N$ as a quotient of $\mathbb{H}$ and of $\pi_1(N)$ as a subgroup of $PSL_2(\mathbb{R})$.

As an intermediate result Viehweg and Zuo show that there exists a lift of $\pi_1(N)$ to a subgroup $\Gamma \subset SL_2(\mathbb{R})$, s.t. $R^1f_*\mathcal{O}_C$ comes from the tensor product of the canonical representation of $\Gamma$ and some unitary representation of $\Gamma$ ([ViZu04], 1.4. Proposition, §7). The fact that $\Gamma$ comes from a quaternion algebra is then a consequence of a result of Takeuchi ([Ta75]).
The tensor description of $R^1f_*\mathbb{C}$ will play a role in the next section. Note that it implies, up to étale base change, $E \simeq U \otimes \theta$, where $U$ is a unitary flat bundle and $\theta$ is some theta characteristic on $N$ (compare remark 3.4).

**proof of Theorem 0.1.** Let $M$ be a projective manifold with a h.n.p.c. $\Pi$. which is not Kähler Einstein. Then $M \not\simeq \mathbb{P}_m(\mathbb{C})$ and $K_M$ is nef by Proposition 2.2.

Assume $K_M$ abundant and $\Pi$ flat. By Theorem 2.6 we may assume that the Iitaka fibration $f : M \rightarrow N$ is a smooth abelian group scheme (after some finite étale covering). By Proposition 3.12, $N$ is a ball quotient. Since $M$ is neither a torus nor a ball quotient, $0 < \dim N < \dim M$. By Proposition 4.1, $N$ is a compact Riemann surface. The equivalence of 1.) and 2.) in Theorem 0.1 for such abelian group schemes is [ViZu04], Theorem 0.5.

It remains to prove that the examples as in 2.) indeed admit a flat projective connection. This will be done in the next section. □

**4.3. Remark.** Let $f : M_m \rightarrow C$ be an abelian group scheme as in Theorem 0.1, i.e., $2 \deg E = (m-1)\deg(\mathcal{K}_C) = (m-1)(2g_C - 2)$. In the push forward of

$$0 \longrightarrow f^*\mathcal{K}_C \longrightarrow \Omega^1_M \longrightarrow \Omega^1_{M/C} \longrightarrow 0$$

we have $f_*\Omega^1_{M/C} \simeq \mathcal{K}_C \otimes R^1f_*\mathcal{O}_M$, implying $H^0(M, f^*\mathcal{K}_C) \simeq H^0(M, \Omega^1_M)$. Then $\text{Alb}(M) \sim \text{Jac}(C)$. The Iitaka fibration essentially coincides with the fibration induced from the Albanese map. This might be an idea of how to prove the abundance conjecture in the present case.

## 5. Projective Non-Kähler–Einstein Examples

The explicit examples are well known families of abelian varieties ([Sh59], [Mu66], [ViZu04], [vGe08]). They have been studied from many points of view but apparently not as examples of manifolds carrying a flat projective structure.

As in Theorem 0.1 let $A$ be a division quaternion algebra defined over some totally real number field $F$ of degree $[F : \mathbb{Q}] = d$. Assume that $A$ splits at exactly one infinite place, i.e.,

$$A \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$  

The existence of $A$ follows from Hilbert’s reciprocity law. Let $\text{Cor}_{F/\mathbb{Q}}(A)$ be the rational corestriction of $A$. Then

$$\text{Cor}_{F/\mathbb{Q}}(A) = M_{2^d-1}(B),$$

where $B$ is a quaternionen algebra over $\mathbb{Q}$ (possibly split). From this data we construct in section 5.2

1.) a torsion free discrete subgroup $\Gamma$ of $\text{Sl}_2(\mathbb{R})$ acting canonically on $U_{\mathbb{R}} = \mathbb{R}^2$ and on $\mathcal{H}_1$ as a Fuchsian group of first kind,

2.) an orthogonal representation $\rho : \Gamma \rightarrow O(g)$ on $W_{\mathbb{R}} \simeq \mathbb{R}^g$ (where $g = 2^{d-1}$ in the case $B$ split and $g = 2^d$ in the case $B$ non split), s.t.

3.) the symplectic representation $id \otimes \rho$ of $\Gamma$ fixes some complete lattice in $U_{\mathbb{R}} \otimes W_{\mathbb{R}}$.

See also remark 4.2. We will first explain how the above data leads to an abelian group scheme $Z_\Gamma \rightarrow C_\Gamma = \mathcal{H}_1/\Gamma$ with a projective structure. Here $Z_\Gamma$ will be compact if and only if $C_\Gamma$ is which is the case if the above data is indeed derived from a division quaternion algebra.
The following particular example is not one we are interested in since it is not compact. It is however well known and gives the general idea:

5.2. Example. Let \( \Gamma \subset \text{Sl}_2(\mathbb{Z}) \) be some standard torsion free congruence subgroup. Consider the quotient \( Z = \mathbb{C} \times \mathfrak{H} \) by the action

\[
(z, \tau) \mapsto \left( \frac{a \tau + b}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma, \ m, n \in \mathbb{Z}.
\]

Then \( Z \) is a smooth manifold with a map \( Z \to C = \mathfrak{H}/\Gamma \), a modular family of elliptic curve. By example \( \text{(I)\text{(2)}} \), \( Z \) has a projective structure.

5.1. Construction of the abelian scheme \( Z \to C \).

5.1.1. Choice of bases. Point 1.) includes a choice of a basis of \( U_\mathbb{R} \cong \mathbb{R}^2 \). Choose any basis of \( W_\mathbb{R} \). Write the elements of \( U_\mathbb{R} \) and \( W_\mathbb{R} \) as horizontal vectors. The action of \( \Gamma \) on \( U_\mathbb{R} \) is by left multiplication. The standard symplectic form on \( U_\mathbb{R} \)

\[
\langle u, u' \rangle = u^t J_2 u', \quad J_2 := \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)
\]

identifies \( \text{Sl}_2(\mathbb{R}) = \text{Sp}_2(\mathbb{R}) \) and \( U_\mathbb{R} \cong U_\mathbb{R}^* \) as \( \Gamma \)-modules. Write the elements of \( U_\mathbb{R}^* \) with dual base as horizontal vectors. The action of \( \Gamma \) is then given by right multiplication \( \gamma(u) = u \gamma^{-1} \), the \( \Gamma \)-isomorphism \( U_\mathbb{R} \cong U_\mathbb{R}^* \) by \( u \mapsto u^t J_2 \).

The choice of a basis for \( U_\mathbb{R} \) and \( W_\mathbb{R} \) gives an isomorphism

\[
W_\mathbb{R} \otimes U_\mathbb{R}^* \cong M_{g \times 2}(\mathbb{R}).
\]

Fix this isomorphism and think of the elements of \( W_\mathbb{R} \otimes U_\mathbb{R}^* \) as real \( g \times 2 \) matrices from now on. Any \( \gamma \in \Gamma \) acts by

\[
\gamma(\alpha) = \rho(\gamma) \alpha \gamma^{-1}, \quad \alpha \in M_{g \times 2}(\mathbb{R}).
\]

Since \( W_\mathbb{R} \otimes U_\mathbb{R}^* \cong W_\mathbb{R} \otimes U_\mathbb{R} \) as \( \Gamma \)-modules, we find a complete lattice \( \Lambda \subset M_{g \times 2}(\mathbb{R}) \) invariant under the action of \( \Gamma \) by 3.). By 3.) we find a \( \rho(\Gamma) \) invariant symmetric and positiv definit \( S \in M_{g}(\mathbb{R}) \) s.t. the symplectic form on \( M_{g \times 2}(\mathbb{R}) \) given by

\[
E(\alpha, \beta) := tr(\alpha^t S \beta J_2)
\]

takes only integral values on \( \Lambda \). For later considerations note that \( M_{g \times 2}(\mathbb{R}) \cong W_\mathbb{R} \otimes U_\mathbb{R}^* \) is a symplectic \( \text{O}(S) \times \text{Sl}_2(\mathbb{R}) \) module via \( (\delta, \gamma)(\alpha) = \delta \alpha \gamma^{-1} \).

5.1.2. Definition of \( Z \). Let \( \Gamma_\Lambda \) be set of matrices

\[
\gamma_\Lambda := \left( \begin{array}{cc} \rho_{2}(\gamma) & \rho_{2}(\lambda) \\ 0_{2 \times 2} & \gamma \end{array} \right) \in \text{Sl}_{g + 2}(\mathbb{R}), \quad \gamma \in \Gamma, \lambda \in \Lambda.
\]

Then \( \Gamma_\Lambda \cong \Lambda \times \Gamma \) is a subgroup of \( \text{Sl}_{g + 2}(\mathbb{R}) \) and there is an exact sequence

\[
0 \to \Lambda \to \Gamma_\Lambda \to \Gamma \to 1,
\]

given by \( \lambda \mapsto id_\Lambda \) and \( \gamma_\Lambda \mapsto \gamma \), respectively. The projective action on \( \mathbb{C}^g \times \mathfrak{H} \), i.e., where \( \gamma_\Lambda \in \Gamma_\Lambda \) acts by

\[
(z, \tau) \mapsto \left( \frac{\rho_{2}(\gamma)(z + \lambda(\gamma))}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right), \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),
\]

is properly discontinously and free, since the action of \( \Gamma \) is. The quotient is a smooth complex manifold

\[
Z = Z_{\Gamma_\Lambda} := \mathbb{C}^g \times \mathfrak{H}/\Gamma_\Lambda.
\]

By example [13] 2.), Z has a projective structure. There is a natural holomorphic proper submersion
\[ f : Z \longrightarrow C_{1} := \mathcal{H}_{1}/\Gamma \]
with a section given by \([\tau] \mapsto ([0, \tau])\). The fiber \(Z_{\tau} = f^{-1}(\tau)\) is isomorphic to \(\mathbb{C}^{g}\) divided by the corresponding stabilizer subgroup of \(\Gamma_{\Lambda}\). Since the action of \(\Gamma\) on \(\mathcal{H}_{1}\) is free,
\[ Z_{\tau} \simeq \mathbb{C}^{g}/\Lambda_{\tau}, \]
where \(\Lambda_{\tau}\) is the image of \(\Lambda\) under
\[ (5.4) \quad M_{g \times 2}(\mathbb{R}) \simeq W_{\mathbb{R}} \times U_{\mathbb{R}}^{*} \longrightarrow \mathbb{C}^{g}, \quad \alpha \mapsto \alpha_{\tau} := \left( \begin{array}{c} \tau \\ 1 \end{array} \right). \]

5.1.3. Projectivity of \(Z_{\tau}\). \([5.4]\) endows \(M_{g \times 2}(\mathbb{R}) \simeq W_{\mathbb{R}} \otimes U_{\mathbb{R}}^{*}\) with the complex structure given by
\[ J_{\tau} = \frac{1}{3\text{mt}} \left( \begin{array}{cc} -\text{Re} \tau & \tau \bar{\gamma} \\ \bar{\gamma} & \text{Re}(\tau) \end{array} \right) \in SL_{2}(\mathbb{R}), \quad \text{i.e., } \ i \cdot \alpha_{\tau} = (\alpha J_{\tau}^{-1})_{\tau}. \]
If \(\tau' = \gamma(\tau)\) for some \(\gamma \in SL_{2}(\mathbb{R})\), then \(J_{\tau'} = \gamma J_{\tau} \gamma^{-1}\). Hence \(Z_{\tau}\) is projective if \((M_{g \times 2}(\mathbb{R}) \simeq W_{\mathbb{R}} \otimes U_{\mathbb{R}}^{*}, \Lambda, J, E)\) satisfies the Riemann conditions:
Recall that \(M_{g \times 2}(\mathbb{R}) \simeq W_{\mathbb{R}} \otimes U_{\mathbb{R}}^{*}\) is a symplectic \(O(S) \times SL_{2}(\mathbb{R})\)-module. The complex structure is given by \((id, J_{\tau}) \in O(S) \times SL_{2}(\mathbb{R})\). Therefore \(E(\alpha J_{\tau}^{-1}, \alpha' J_{\tau}^{-1}) = E(\alpha, \alpha')\) for any \(\tau \in \mathcal{H}_{1}\). Positivity for \(\tau = i\) follows from \(J_{1} = J_{2}\) and \(E(\alpha, \alpha J_{2}^{-1}) = tr(\alpha J_{2}^{-1}) > 0\) for \(\alpha \neq 0\). Any \(\tau \in \mathcal{H}_{1}\) of the form \(\tau = \gamma(i)\) for some \(\gamma \in SL_{2}(\mathbb{R})\). The invariance of \(E\) and \(J_{\tau} = \gamma J_{\tau} \gamma^{-1}\) shows that \(E(\alpha, \alpha' J_{\tau}^{-1})\) is a positive definite symmetric. Therefore, \(Z_{\tau}\) is projective.

5.1.4. Isomorphic \(Z_{\tau}\)’s. If \(\tau' = \gamma(\tau)\) for some \(\gamma \in \Gamma\), then \((Z_{\tau}, E) \simeq (Z_{\tau'}, E)\), where \(\varphi : Z_{\tau} \longrightarrow Z_{\tau'}\), as a map \(\mathbb{C}^{g} \longrightarrow \mathbb{C}^{g}\), is given by
\[ (5.5) \quad \frac{1}{\epsilon \tau + d \rho(\gamma)} \]
Indeed,
\[ \Lambda_{\tau'} = \Lambda \left( \begin{array}{c} \gamma(\tau) \\ 1 \end{array} \right) = \frac{1}{\epsilon \tau + d} (\Lambda \gamma)_{\tau} = \frac{\rho(\gamma)}{\epsilon \tau + d} (\rho(\gamma^{-1}) \Lambda \gamma)_{\tau} = \frac{\rho(\gamma)}{\epsilon \tau + d} \Lambda_{\tau}. \]
On the underlying real vector space \(M_{g \times 2}(\mathbb{R}) \simeq W_{\mathbb{R}} \otimes U_{\mathbb{R}}\), \(\varphi\) is given by \(\alpha \mapsto \rho(\gamma) \alpha \gamma^{-1}\) showing that \(\varphi\) respects the polarization \(E\).

The family of \(Z_{\tau}\)’s over \(\mathcal{H}_{1}\), i.e., the quotient \(\mathbb{C}^{g} \times \mathcal{H}_{1}/\Lambda\), is projective. We obtain \(Z\) by dividing out the action of \(\Gamma_{\Lambda}/\Lambda \simeq \Gamma\). Fiberwise this is nothing but \([5.5]\) proving that the polarizations glue to a section of \(R_{f}^{2}Z\).

5.1.5. Map to the moduli space. This follows from the above considerations, we include it here for convenience of the reader. General reference is [31a04]:

5.1.6. Moduli of type \(\Delta\) polarized abelian varieties. Let \(\mathcal{H}_{g} = \{ \Pi \in M_{g}(\mathbb{C}) | \Pi = \Pi^{t}, 3m \Pi > 0 \}\). Let \(\Delta\) be a type, i.e., an invertible \(g \times g\) diagonal matrix \((\delta_{1}, \ldots, \delta_{g})\), \(\delta_{i} \in \mathbb{N}\) and \(\delta_{1} | \delta_{i+1}\). Fix a basis of \(\mathbb{R}^{2g}\), write the elements as horizontal vectors in the form \((x, y), x, y \in \mathbb{R}^{g}\). Choose the letters \((m, n)\) for integral entries. Let
\[ \Lambda_{\Delta} := \{(m, n \Delta) | m, n \in \mathbb{Z}^{g}\} \subset \mathbb{R}^{2g} \]
Any $\Pi \in \mathcal{H}_g$ defines a complex structure on $\mathbb{R}^{2g}$ via $j_\Pi : (x,y) \mapsto x\Pi + y$. As above we have the abelian variety $\mathbb{C}^g/j_{\Pi}(\Lambda\Delta)$, where we write the elements of $\mathbb{C}^g$ as horizontal vectors. Let

$$\tilde{\Gamma} = \{ \gamma \in Sp_{2g}(\mathbb{Q})|\Lambda\Delta\gamma = \Lambda\Delta \}.$$ 

Then $\tilde{\Gamma}$ acts on $\mathcal{H}_g$ in the usual way. Let $\tilde{\Gamma}_\Delta$ be the group consisiting of all maps

$$\gamma_{\lambda\Delta} = \begin{pmatrix} 1 & m & n\Delta \\ 0 & A & B \\ 0 & C & D \end{pmatrix}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}, \lambda\Delta = (m,n\Delta) \in \Lambda\Delta.$$ 

As above we have $0 \rightarrow \Lambda\Delta \rightarrow \tilde{\Gamma}_\Delta \rightarrow \tilde{\Gamma} \rightarrow 1$ and $\tilde{\Gamma}_\Lambda \Delta \simeq \Lambda\Delta \times \tilde{\Gamma}$. The group $\tilde{\Gamma}_\Lambda\Delta$ acts on $\mathbb{C}^g \times \mathcal{H}_g$ in 'Grassmanian manner', i.e.,

$$(z,\Pi) \mapsto ((z + m\Pi + n\Delta)(C\Pi + D)^{-1}, (A\Pi + B)(C\Pi + D)^{-1}).$$

The quotient need not be a manifold since the action of $\tilde{\Gamma}$ need not be free. After replacing $\tilde{\Gamma}$ by some appropriate congruence subgroup, the quotient yields a smooth abelian fibration $U_g \rightarrow A_g$. A relatively ample line bundle is for example induced by the factor of automorphy $a(\gamma\lambda\Delta, (z,\Pi)) = \exp(2i\pi(m\Pi m^t + 2zm^t + (z + m\Pi + n\Delta)(C\Pi + D)^{-1}C(z + m\Pi + n\Delta)^t)$.

5.1.7. An equivariant holomorphic map $\mathbb{C}^g \times \mathcal{H}_1 \rightarrow \mathbb{C}^g \times \mathcal{H}_g.$ is defined as follows: let $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g \in M_{g \times 2}(\mathbb{R})$ be a symplectic lattice basis. It means that these elements generate $\Lambda$ as a $\mathbb{Z}$-module and that $E$ from (5.3) is in this basis given by

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \in M_{2g}(\mathbb{Z}),$$

where $\Delta$ is a certain type. Fix the basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g = \frac{1}{\sqrt{\Delta}}\lambda_1, \ldots, \mu_g = \frac{1}{\sqrt{\Delta}}\mu_g$ of $M_{g \times 2}(\mathbb{R})$ in which $E$ is given in standard form. The choice defines a map $\kappa : M_{g \times 2}(\mathbb{R}) \rightarrow \mathbb{R}^{2g}$. Write the elements of $\mathbb{R}^{2g}$ again as horizontal vectors. Then $\kappa(\Lambda) = \Lambda\Delta$.

For any $\gamma \in \Gamma$, $\alpha \mapsto \rho(\gamma^{-1})\alpha\gamma$ is a symplectic automorphism of $M_{g \times 2}(\mathbb{R})$. We find a matrix $\sigma(\gamma) \in Sp_{2g}(\mathbb{Q})$ such that $\kappa(\rho(\gamma^{-1})\alpha\gamma) = \kappa(\alpha)\sigma(\gamma)$. The map $\sigma$ is a group morphism $\Gamma \rightarrow \tilde{\Gamma}$. It extends to

$$\begin{array}{ccc} \Gamma_{\Lambda} & \rightarrow & \tilde{\Gamma}_{\Lambda\Delta}, \\
\gamma_{\lambda} & \mapsto & \sigma(\gamma) \kappa(\lambda). \end{array}$$

The map $\mathbb{C}^g \times \mathcal{H}_1 \rightarrow \mathbb{C}^g \times \mathcal{H}_g$ is finally defined as follows: For any $\tau \in \mathcal{H}_1$, using (5.3), let

$$\Pi_1(\tau) = (\lambda_1,\tau, \ldots, \lambda_g,\tau) \in M_g(\mathbb{C}),$$

$$\Pi_2(\tau) = (\mu_1,\tau, \ldots, \mu'), \tau) \in M_g(\mathbb{C}).$$

Then $(\Pi_1(\tau), \Pi_2(\tau)\Delta)$ is the period matrix of the abelian variety $Z_\tau$ with respect to the chosen bases. By (5.3) we have $(\Pi_1(\gamma(\tau)), \Pi_2(\gamma(\tau)) = \frac{d(\gamma)}{d(\tau)}(\Pi_1(\tau), \Pi_2(\tau))\gamma(\tau)'$. The matrix $\Pi_2(\tau)$ is invertible by and the Riemann bilinear relations in this form read

$$\Pi(\tau) := [\Pi_2(\tau)]^{-1}\Pi_1(\tau) \in \mathcal{H}_g.$$ 

Note that $(\Pi(\tau), \Delta)$ is the period matrix of $Z_\tau$ with respect to the symplectic lattice basis and the basis of $\mathbb{C}^g$ induced by $\Pi_2(\tau)$. The map $\mathbb{C}^g \times \mathcal{H}_1 \rightarrow \mathbb{C}^g \times \mathcal{H}_g$, $(z,\tau) \mapsto (\Pi_2(\tau)^{-1}z, \Pi(\tau))$ is the desired holomorphic map, equivariant with respect to (6.6).
5.2. From $A$ to $Z$. It remains to show how a quaternion algebra $A$ as above leads to a collection of data 1) — 3). We first recall some results

5.2.1. On central simple algebras. Let $A$ a central simple algebra of finite dimension over a field $K$. It is called division if it is a skew field. It is called a quaternion algebra if $[A : K] = 4$. A quaternion algebra is either division or split, i.e., $A \simeq M_2(K)$. Let $Br(K)$ be the Brauer group of $K$. The order $e(A)$ of $[A] \in Br(K)$ is finite and is called the exponent of $A$. A theorem of Wedderburn says $A \simeq M_r(D)$, where $D/K$ is a division algebra. The $K$–dimension $[D : K] = s(D)^2$ for some $s(D) = s(A) \in \mathbb{N}$ called (Schur–) index of $A$. One has $e(A)|s(A)$ and if $K$ is a local or global field, then even $e(A) = s(A)$.

The corestriction $Cor_{F/Q}(A)$ is a $4^d$ dimensional central simple $Q$–algebra. The corestriction induces a map of Brauer groups

$$Br(F) \longrightarrow Br(Q).$$

Since $e(A) = s(A) = 2$ we have $e(Cor_{F/Q}(A)) = 1$ or $2$. Hence

$$\text{Cor}_{F/Q}(A) \simeq M_{2^d}(Q) \quad \text{or} \quad \text{Cor}_{F/Q}(A) \simeq M_{2^{d-1}}(B),$$

where $B$ is a division quaternion algebra over $Q$. We refer to the first case as “$B$ splits” because here $\text{Cor}_{F/Q}(A) \simeq M_{2^d}(Q)$ for $B = M_2(Q)$. Consider the following $Q$–vector spaces

$$V_Q = Q^{2^d}, \text{ in the case } B \text{ split, } V_Q = B^{2^{d-1}}, \text{ in the case } B \text{ non–split.}$$

The $Q$–dimension is $2^d$ and $2^{d+1}$, respectively. The elements of $V_Q$ are considered as vertical vectors and $V_Q$ as a $\text{Cor}_{F/Q}(A)$–module.

From $A \otimes Q \simeq M_2(\mathbb{R}) \oplus \mathbb{H}^{\otimes d-1}$ using $\mathbb{H} \otimes \mathbb{R} \mathbb{H} \simeq M_4(\mathbb{R})$ we obtain

$$\text{Cor}_{F/Q}(A) \otimes Q \simeq M_2(\mathbb{R}) \otimes Q \mathbb{H}^{\otimes d-1} \simeq \begin{cases} M_{2^d}(\mathbb{R}), & \text{d odd} \\ M_{2^{d-1}}(\mathbb{H}), & \text{d even.} \end{cases}$$

Then $B$ is indefinite (i.e., $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$) if and only if $d$ is odd, $B$ is definite (i.e., $B \otimes \mathbb{R} \simeq \mathbb{H}$) if and only if $d$ is even and

$$V_R := V_Q \otimes Q \mathbb{R} \simeq \begin{cases} \mathbb{R}^{2^d}, & \text{d odd and } B \text{ split} \\ M_{2^d \times 2}(\mathbb{R}), & \text{d odd and } B \text{ non split} \\ \mathbb{H}^{2^{d-1}} \simeq \mathbb{C}^{2^d}, & \text{d even} \end{cases}$$

with the obvious action of $\text{Cor}_{F/Q}(A) \otimes Q \mathbb{R}$ from (5.8). Note that $V_R$ is an irreducible $(\text{Cor}_{F/Q}(A) \otimes Q \mathbb{R})^\times$ module in the first and third case, while it is a direct sum of two isomorphic modules in the second.

The corestriction comes with a map

$$Nm : A^\times \longrightarrow \text{Cor}_{F/Q}(A)^\times.$$

Let $x \mapsto x'$ be the canonical involution of $A$ and

$$G = \{x \in A|xx' = 1\}.$$

Then $G$ is an algebraic group over $Q$. Via $Nm$, $G(Q)$ acts on $V_Q$. Let $\Lambda \subset V_Q$ be some complete lattice, $\Gamma \subset G(Q)$ be some torsion free arithmetic subgroup fixing $\Lambda$ via $Nm$. 

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Consider the action over \(\mathbb{R}\): The elements of norm 1 in \(M_2(\mathbb{R})\) and \(\mathbb{H}\) form the groups \(SL_2(\mathbb{R})\) and \(SU(2)\), respectively. Then \(\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \oplus \mathbb{H}^{\otimes d-1}\) shows
\[
G(\mathbb{R}) = SL_2(\mathbb{R}) \times SU(2) \times \cdots \times SU(2)
\]
for \(d\). The isomorphism \(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R})\) induces \(SU(2) \times SU(2) \to SO(4)\). Then \(N\) factors over
\[
\begin{cases}
SL_2(\mathbb{R}) \times SO(2^{d-1}), & B \text{ indefit } / d \text{ odd} \\
SL_2(\mathbb{R}) \times SU(2^{d-1}), & B \text{ definit } / d \text{ even}.
\end{cases}
\]
Projection onto the first factor embeds \(\Gamma\) into \(SL_2(\mathbb{R})\). Projection onto the second factor induces an orthogonal representation \(\rho'\) of \(\Gamma\) (in the case \(d\) even take the real part of the unitary form). In the case \(B\) split or definit put \(\rho = \rho'\), in the case \(B\) non–split indefinite put \(\rho = \rho' \oplus \rho'\) and , the direct sum of. Then \(Nm = id \otimes \rho\) over the reals fixing, by construction, a complete lattice \(\Lambda\). This gives 1)–3).

5.11. Remark. 1.) The group \(G\) can be identified with the special Mumford Tate group of the constructed family of abelian varieties. The ring \(\End_{\mathbb{Q}}(Z_{\tau})\) for \(\tau\) general is isomorphic to the group of endomorphisms of \(V\) commuting with the action of \(G\).

In the case \(B\) non split, \(V_{\mathbb{Q}}\) becomes a \(B\)–module via \(\beta v := v\beta'\) and this action clearly commutes. This gives an embedding \(B \hookrightarrow End_{\mathbb{Q}}(Z_{\tau})\). Together with results from \([\text{VIZ04}]\), where \(B\) is not explicitely mentioned) we find \(\End_{\mathbb{Q}}(Z_{\tau}) \simeq B\) in the case \(B\) non–split and \(\End_{\mathbb{Q}}(Z_{\tau}) \simeq \mathbb{Q}\) in the case \(B\) split.

2.) Assume 1.)–3.) as above are given corresponding to the family \(Z \to C\). In 3.) we may replace \(\rho\) by a direct sum of, say, \(r\) copies of \(\rho\). This again gives 1.)–3.).

Let us illustrate the construction with one more explicit example.

5.12. Example.(False elliptic Curves) \((\text{SI59})\) This is the case \(d = 1\), \(F = \mathbb{Q}\), \(A = B\) indefit. Denote quaternion conjugation in \(B\) by \(\alpha \mapsto \alpha'\). Choose some pure quaternion \(y\) (i.e., \(y = -y'\)) such that \(b := y^2 < 0\). There exists \(x \in B\) such that \(xy = -yx\) and \(a := x^2 > 0\). Then \(B\) is generated by \(x\) and \(y\) as an algebra over \(\mathbb{Q}\) and
\[
x \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}
\]
gives an embedding \(B \hookrightarrow M_{2 \times 2}(\mathbb{R})\). Identify \(B\) with its image in \(M_{2 \times 2}(\mathbb{R})\), s.t. reduced norm and trace are then given by usual matrix determinant and trace, respectively.

Let \(\Lambda\) be some complete lattice in \(V_{\mathbb{Q}} := B\). For the construction we may take a maximal order for \(\Lambda\). Let \(\Gamma\) be a torsion free subgroup of the unit subgroup \(\Lambda_1^* = \{\alpha \in \Lambda | \alpha \alpha' = 1\}\). The matrix \(S := J_2y\) is symmetric and positive definite. Let \(\rho : \Gamma \to O(\mathbb{R}^2, S)\) be the trivial representation. Then all of the above points are satisfied.

Indeed, \(\Gamma\) acts on \(V_{\mathbb{R}} = M_2(\mathbb{R})\) via \(\alpha \mapsto \rho(\gamma)\alpha\gamma^{-1} = \alpha\gamma^{-1}\), fixing \(\Lambda\). The form \(E(\alpha, \beta) = tr(\alpha^t S \beta J_2)\) from \((5.3)\) is \(\Gamma\)–stable. The extension of quaternion conjugation from \(B\) to \(V_{\mathbb{R}} = M_2(\mathbb{R})\) is given by \(\alpha \mapsto J_2^{-1} \alpha^t J_2\). Then (recall \(tr(\alpha \beta) = tr(\beta \alpha)\))
\[
E(\alpha, \beta) = tr(\alpha^t S \beta J_2) = tr(-J_2 \beta^t S \alpha) = tr(-J_2 y^t \alpha J_2 \beta^t) =
\]

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\[ \text{tr}(y_\alpha J_2^{-1}\beta^t J_2) = \text{tr}(y_\alpha \beta). \]

The last description shows that \( E \) only takes rational values on \( B \). Then some multiple of \( E \) only takes integral values on \( \Lambda \) (\( E \) is Shimura’s form from \([Sh59]\)).

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