THE SYNTOMIC REGULATOR FOR K–THEORY OF FIELDS

AMNON BESSER AND ROB DE JEU

Abstract. We define complexes analogous to Goncharov’s complexes for the $K$–theory of discrete valuation rings of characteristic zero. Under suitable assumptions in $K$–theory, there is a map from the cohomology of those complexes to the $K$–theory of the ring. In case the ring is the localization of the ring of integers in a number field, there are no assumptions necessary. We compute the composition of our map to the $K$–theory with the syntomic regulator. The result can be described in terms of a $p$–adic polylogarithm. Finally, we apply our theory in order to compute the regulator to syntomic cohomology on Beilinson’s cyclotomic elements. The result is again given by the $p$–adic polylogarithm. This last result is related to one by Somekawa and generalizes work by Gros.

1. Introduction

Let $K$ be a complete discrete valuation field of characteristic zero, $R$ its valuation ring, and $\kappa$ its residue field. Assume $\kappa$ has positive characteristic $p$ and is algebraic over $\mathbb{F}_p$. If $X/R$ is smooth, separated and of finite type, there is a regulator map from $K$-theory to syntomic cohomology

$$K_n^{(j)}(X) \to H^{2j-n}_{\text{syn}}(X,j),$$

see [Bes00a]. In many interesting cases the target group of the regulator is isomorphic to the rigid cohomology group, in the sense of Berthelot, $H^{2j-n-1}_{\text{rig}}(\text{Spec}(\kappa)/K)$, where $X_\kappa$ is the special fiber of $X$. We will be most interested in the situation where $X = \text{Spec}(R)$, and the $K$–group is $K_{2n-1}^{(n)}(R)$ for $n \geq 2$. The target group for the regulator in this case is $H^0_{\text{rig}}(\text{Spec}(\kappa)/K) \cong K$ (see Definition 4.6 for the precise identification). Because $\kappa$ is algebraic over $\mathbb{F}_p$, $K_n(\kappa)$ is torsion for all $n \geq 1$, so from the long exact localization sequence

$$\cdots \to K_n^{(j-1)}(\kappa) \to K_n^{(j)}(R) \to K_n^{(j)}(K) \to K_n^{(j-1)}(\kappa) \to \cdots$$

we get an isomorphism $K_{2n-1}^{(n)}(R) \cong K_{2n-1}^{(n)}(K)$ for $n \geq 2$. Hence we get a regulator map (for $n \geq 2$)

$$\text{reg} : K_{2n-1}^{(n)}(K) \cong K_{2n-1}^{(n)}(R) \to K.$$ 

In this paper we try to explicitly compute this regulator map. We note that if $F$ is a number field with an embedding $F \to K$, we can combine the natural map $K_{2n-1}^{(n)}(F) \to K_{2n-1}^{(n)}(K)$ with this regulator map to obtain a regulator map on $K_{2n-1}^{(n)}(F)$. Also, for a number field $F$, all $K_n(F)$ are torsion if $n$ is even and positive. For the odd ones, all $K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ are $K_{2n-1}^{(n)}(F)$, so the computation for $K_{2n-1}^{(n)}(K)$ is the most interesting from the point of view of number fields. Our
principal tool of study will be the complexes $\widetilde{M}_n^{\bullet}(K)$ constructed in [dJ95]. Write $K^*_{\mathbb{Q}}$ for $K^* \otimes_{\mathbb{Z}} \mathbb{Q}$. The complex $\widetilde{M}_n^{\bullet}(K)$ for $n \geq 2$ is of the form

$$\widetilde{M}_n \to \widetilde{M}_{n-1} \otimes K^*_{\mathbb{Q}} \to \widetilde{M}_{n-2} \otimes \bigwedge^2 K^*_{\mathbb{Q}} \to \cdots \to \widetilde{M}_2 \otimes \bigwedge^{n-2} K^*_{\mathbb{Q}} \to \bigwedge^n K^*_{\mathbb{Q}}$$

where $\widetilde{M}_k = \widetilde{M}_k(K)$ is a $\mathbb{Q}$–vector space generated by symbols $[x]_k$, with $x$ in $K$, $x \neq 0$ or 1, and the differential is given by

$$d([x]_k \otimes y_1 \wedge \ldots \wedge y_{n-k} ) = [x]_{k-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-k}$$

if $k \geq 3$, and

$$d([x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-k} ) = (1 - x) \wedge x \wedge y_1 \wedge \ldots \wedge y_{n-k}$$.

We give this complex a cohomological grading in degrees 1 through $n$. Under suitable assumptions about weights in $K$–theory (as formulated in the Beilinson–Soulé conjecture, see Definition 3.2), there is a map

$$H^r(\widetilde{M}_n^{\bullet}(K)) \to K^{(n)}_{2n-r}(K) \quad (1.1)$$

In Section 3, we construct analogous complexes $\widetilde{M}_n^{\bullet}(R)$ for the ring $R$, whose cohomology (again under suitable assumptions) maps directly to the $K$–theory of $R$, and in Section 7 we compute the regulator map on its image. In the cases we are interested in $\widetilde{M}_n^{\bullet}(R)$ can be identified with the subcomplex of the complex for $K$ spanned in degree $k+1 = 1, \ldots, n-1$ by all $[u]_{n-k} \otimes v_1 \wedge \cdots \wedge v_k$ with $v_1, \ldots, v_k$ in $R^*$, $u$ in $R^*$ such that $1 - u$ is also in $R^*$, and in degree $n$ by all $v_1 \wedge \cdots \wedge v_n$ with all $v_i$ in $R^*$. (A $u$ in $R^*$ such that $1 - u$ is also in $R^*$ is called a special unit.) But redoing the construction has the advantage that we can work over $R$ all the time, which is required for the computation of the regulator.

The case that the field is a number field deserves special mentioning. First of all, no assumptions about weights are necessary in this case. Furthermore, it is known that if $F$ is a number field, the map $H^1(\widetilde{M}_n^{\bullet}(F)) \to K^{(n)}_{2n-1}(F)$ is an isomorphism for $n = 2$ and $n = 3$, as well as when $F$ is a cyclotomic field for all $n \geq 2$. (There is also substantial numerical evidence that it should be an isomorphism for all $n$ for number fields, which is part of a conjecture by Zagier, as well as a corresponding conjecture for infinite fields by Goncharov.) Therefore one would have a complete description of the syntomic regulator for our discrete valuation ring $R$ if we knew that the image of $H^1(\widetilde{M}_n^{\bullet}(R))$ in $K^{(n)}_{2n-1}(R)$ would be everything. This may not be the case, as perhaps $R$ does not have enough special units. One can try to overcome this difficulty by rewriting elements in the image of $H^1(\widetilde{M}_n^{\bullet}(F))$ as being part of the image of $H^1(\widetilde{M}_n^{\bullet}(R'))$ where $F'/F$ is a finite field extension, $R'$ the corresponding ring in $F'$. We do this in the case of cyclotomic fields, so that we obtain a full description of the syntomic regulator in this case. We also state a Conjecture that the formulas found for the regulator on the complex for $R$ generalize to be the regulator on the complex for $F$.

In order to present our results, we shall need the following functions. Let $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ be a branch of the $p$–adic logarithm. This means we define $\log$ on the elements $z$ with $|1 - z| < 1$ by the usual power series, and we extend this to $\mathbb{C}_p^*$ by choosing $\pi \in \mathbb{C}_p$ with $|\pi| < 1$, declaring $\log(\pi) = 0$, and extending to a homomorphism from $\mathbb{C}_p^*$ to $\mathbb{C}_p$ (see Definition 2.1). Note that the values on the
elements in $\mathbb{C}_p^*$ with $|z| = 1$ is independent of the choice of $\pi$, but log and the functions $\text{Li}_n(z)$ about to be described depend on this choice. For the relation, see Proposition 2.6.

Let $\text{Li}_1(z) = -\log(1 - z)$ for $z \neq 0$ or 1. We follow Coleman to recursively define, using his integration theory, functions $\text{Li}_n(z)$ for $n \geq 2$. The defining relations are $d\text{Li}_n(z) = \text{Li}_{n-1}(z) \log z$ and $\lim_{z \to 0} \text{Li}_n(z) = 0$, and they have a unique solution in the class of functions defined by Coleman. It is shown in [Col82] that those functions are locally analytic in the naive topology on $\mathbb{C}_p$, and are given by a convergent power series $\sum_{k=1}^{\infty} z^k/k^n$ on the open unit disc in $\mathbb{C}_p$. The function $\text{Li}_n(z)$ extends to a locally analytic function on $\mathbb{C}_p \setminus \{1\}$ with $\text{Li}_n(z) = 0$ for $n \geq 1$.

These functions satisfy the functional equation

\[
\text{Li}_n(z) + (-1)^n \text{Li}_n(1/z) = -\frac{1}{n!} \log^n(z),
\]

see Proposition 6.4 of [Col82]. We also introduce the function $L_n$, defined as

\[
L_n(z) = \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \text{Li}_{n-m}(z) \log^m(z).
\]

In order to state the Theorems below easily, we shall need linear combinations of these functions. Namely, we want a suitable combination that satisfies a clean functional equation for $z$ and $1/z$. It follows from (1.2) that $L_k(z) + (-1)^k L_k(1/z) = \frac{(-1)^k}{k!} \log^k(z)$. Therefore the function

\[
L_{\text{mod}, n}(z) = \sum_{m=1}^{n} a_m L_m(z) \log^{n-m}(z)
\]

with $a_n = 1$ satisfies

\[
L_{\text{mod}, n}(z) + (-1)^n L_{\text{mod}, n}(1/z) = 0
\]

if $\sum_{m=1}^{n} a_m (-1)^m/m! = 0$. Below, $L_{\text{mod}, n}$ will mean any of those choices. For $n = 2$, there is a unique such function, namely

\[
L_2(z) + \frac{1}{2} \log(z)L_1(z) = L_2(z) - \frac{1}{2} \log(z)L_1(z),
\]

which is studied in Section VI and beyond in [Col82], where it is called $D(z)$.

It is easily deduced from Coleman’s theory (see Remark 2.3) that $\text{Li}_n$ is Galois equivariant. In particular, if $K \subset \mathbb{C}_p$ is a complete subfield, then $\text{Li}_n$, and as a result also $L_n$ and $L_{\text{mod}, n}$, send $K$ to $K$ provided log was defined such that $\log(\pi) = 0$ with $\pi \in K$.

Remark 1.5. By considering the coefficients of the terms $\text{Li}_m(z) \log^{n-m}(z)$ in the functions above, one sees that the functions $L_m(z) \log^{n-m}(z)$ for $m = 1, \ldots, n$ and $L_{\text{mod}, m}(z) \log^{n-m}(z)$ as above for $m = 1, \ldots, n$ span the same $\mathbb{C}_p$-vector space (or even $\mathbb{Q}$-vector space in case all $a_i$’s are in $\mathbb{Q}$), namely the space spanned by $\text{Li}_m(z) \log^{n-m}(z)$ for $m = 1, \ldots, n$. Therefore one can consider any function of the form $\sum_{j=0}^{n-1} b_j \text{Li}_{n-j}(z) \log^j(z)$ with all $b_j$ in $\mathbb{C}_p$ as a candidate for $L_{\text{mod}, n}$, provided $b_0 = 1$ and $\sum_{j=0}^{n-1} \frac{b_j}{(n-j)!} = 0$. Let $B_i$ for $i = 0, 1, \ldots$ be the Bernoulli numbers,
defined by the identity of formal power series
\[ \sum_{i=0}^{\infty} \frac{B_i}{i!} t^i = \frac{t}{e^t - 1}. \]

Then the functions \( L_{\text{mod},n}(z) \) defined by \( \sum_{j=0}^{n-1} \frac{B_j}{j!} \text{Li}_{-j}(z) \log^j(z) \) satisfy the above requirements as \( B_0 = 1 \), and the other identity holds by definition of the \( B_j \) if \( n \geq 2 \). Note that this formula is different from the classical case, where one uses the real or imaginary part of the functions \( \sum_{j=0}^{n-1} \frac{B_j 2^j}{j!} \text{Li}_{-j}(z) \log^j|z| \), see [Zag91] and [dJ95, Remark 5.2]. Another possible natural candidate for the function \( L_{\text{mod},m}(z) \) is \( L_{\text{mod},m}(z) = L_m(z) + L_{m-1}(z) \log(z)/m \). This function is distinguished by the following fact proved in [Bes01b, Theorem 1.1]: It is the unique combination of type \( L_{\text{mod},m} \) with coefficients independent of \( p \) such that the function \( -mp^{1-m}(1-z) dL_{\text{mod},m}(z)/dz \) has a reduction modulo \( p \), for sufficiently large \( p \), which is the so called \( m-1 \)-polylogarithm function introduced in the \( m = 2 \) case by Kontsevich [Kon] and by Elbaz-Vincent and Gangl [EVG00] in general.

If \( R \) is a ring with 1, let \( R^\circ \) be the set of elements \( u \) in \( R \) such that both \( u \) and \( 1-u \) are units. We shall refer to those elements as special units.

We are now ready to state our main results.

**Theorem 1.6.** Let \( F \) be a field of characteristic zero. Let \( \mathcal{O} \subset F \) be a discrete valuation ring, and let \( \kappa \) be the residue field. Assume that the Beilinson–Soulé conjecture holds for fields of characteristic 0 and for \( \kappa \). For \( n \geq 2 \) let \( \tilde{\mathcal{M}}_n^{\bullet}(\mathcal{O}) \) be the subcomplex of the complex \( \tilde{\mathcal{M}}_n^{\bullet}(F) \) constructed in [dJ95] (see also Section 3) generated by symbols of the form \([x]_k \otimes y_1 \wedge \ldots \wedge y_{n-k}\), where all \( y_i \) are elements in \( \mathcal{O}^* \), and \( x \) is in \( \mathcal{O}^\circ \). Then

1. There is a map \( H^r(\tilde{\mathcal{M}}_n^{\bullet}(\mathcal{O})) \to K_{2n-r}^{(n)}(\mathcal{O}) \) such that the diagram
   \[
   \begin{array}{ccc}
   H^r(\tilde{\mathcal{M}}_n^{\bullet}(\mathcal{O})) & \to & K_{2n-r}^{(n)}(\mathcal{O}) \\
   & \downarrow & \\
   H^r(\tilde{\mathcal{M}}_n^{\bullet}(F)) & \to & K_{2n-r}^{(n)}(F)
   \end{array}
   \]
   commutes, where the lower horizontal map is the map in (1.1).

2. If in addition \( \sigma : F \to K \) is an embedding with \( \sigma(\mathcal{O}) \subset R \), then for \( r = 1 \), the regulator map
   \[ H^1(\tilde{\mathcal{M}}_n^{\bullet}(\mathcal{O})) \to K_{2n-1}^{(n)}(\mathcal{O}) \xrightarrow{\sigma} K_{2n-1}^{(n)}(R) \to K \]
   is given by mapping \([x]_n\) to \( \pm(n-1)! L_{\text{mod},n}(\sigma(x)) \).

Moreover, if \( n = 2 \), those results hold without any assumptions on the Beilinson-Soulé conjecture.

**Remark 1.7.** The indeterminacy of the sign in Theorem 1.6 comes from the fact that the maps \( H^r(\tilde{\mathcal{M}}_n^{\bullet}(\mathcal{O})) \to K_{2n-r}^{(n)}(\mathcal{O}) \) and \( H^r(\tilde{\mathcal{M}}_n^{\bullet}(F)) \to K_{2n-r}^{(n)}(F) \) in Theorem 1.6 are natural only up to a choice of sign (see the introduction to Section 3). This indeterminacy of the sign will occur every time those maps are referred to in the paper.
Theorem 1.10. Let $F$ be a number field. Let $\mathcal{O}$ be a localization of the ring of integers of $F$ at a nonzero prime ideal. Then

1. There is a map $H^r(\hat{\mathcal{M}}_{(n)}^\bullet(\mathcal{O})) \to K_{2n-r}^{(n)}(\mathcal{O})$ such that the diagram

\[
\begin{array}{ccc}
H^r(\hat{\mathcal{M}}_{(n)}^\bullet(\mathcal{O})) & \longrightarrow & K_{2n-r}^{(n)}(\mathcal{O}) \\
\downarrow & & \downarrow \\
H^r(\hat{\mathcal{M}}_{(n)}^\bullet(F)) & \longrightarrow & K_{2n-r}^{(n)}(F)
\end{array}
\]

commutes, where the horizontal map is the map in (1.1).

2. If in addition $\sigma : F \to K$ is an embedding with $\sigma(\mathcal{O}) \subset R$, then for $r = 1$, the regulator map

$\text{reg}_\sigma : H^1(\hat{\mathcal{M}}_{(n)}^\bullet(\mathcal{O})) \to K_{2n-1}^{(n)}(\mathcal{O}) \to K_{2n-1}^{(n)}(R) \to K$

is given by mapping $[x]_n$ to $\pm(n-1)! L_{\text{mod},n}(\sigma(x))$.

Remark 1.11. For any fixed element in $H^1(\hat{\mathcal{M}}_{(n)}^\bullet(F))$, all elements involved are special units for almost all $p$, so that this Theorem applies to any given element in $H^1(\hat{\mathcal{M}}_{(n)}^\bullet(F))$ for almost all $p$.

If we try to apply Theorem 1.10 to cyclotomic elements, i.e., to the elements $[\zeta]_n$ corresponding to a $m$-th root of unity $\zeta$ we notice that it applies directly only if $m$ is prime to $p$ since otherwise $\zeta$ may not be special. However, using relations between symbols (the distribution relation, see Proposition 2.10) we are able to prove the following Theorem.
Theorem 1.12. Under the assumptions of Theorem 1.10, the regulator map
\[ \text{reg}_{\sigma} : H^1(\tilde{M}^\bullet_{(n)}(F)) \to K^{(n)}_{2n-1}(F) \cong K^{(n)}_{2n-1}(O) \xrightarrow{\sigma} K^{(n)}_{2n-1}(R) \to K \]
maps \([\zeta]_n\) to \(\pm(n-1)!L_{\text{mod},n}(\sigma(\zeta))\) if \(\zeta\) is any root of unity in \(F^*\).

Remark 1.13. Because it is known that the elements \([\zeta]_n\) for \(n \geq 2\) where \(\zeta\) runs through the primitive \(m\)-th roots of unity generate \(K^{(n)}_{2n-1}(\mathbb{Q}(\mu_m))\), this gives a complete description of the syntomic regulator for cyclotomic fields. Note also that this particular result extends the results of [Gro94], where the corresponding result was proved only for roots of unity of order \(m\) with \((m,p) = 1\) (see Théorème 2.22), and is equivalent to the results of [Som99]. That paper has a different formulation, with another version of a syntomic regulator and also using a specialized version of the polylogarithm at roots of unity. The relation with Coleman’s polylogarithm was proved by Barsky (unpublished). The result of Gros is that the element \([\zeta]_n\) is mapped under the syntomic regulator to \(\text{Li}^{(p)}_n(\zeta)\), where \(\text{Li}^{(p)}_n\) is defined by
\[ \text{Li}^{(p)}_n(z) = \text{Li}_n(z) - \frac{1}{p^n}\text{Li}_n(z^p). \]

Note that the expansion of \(\text{Li}^{(p)}_n\) at 0 is \(\sum_{k=1}^{\infty} z^k/k^n\). The difference between the results is caused by the different normalizations of the regulators. One has the relation
\[ \text{reg}_{\text{Gros}} = \left(1 - \frac{\text{Frob}}{p^n}\right)\text{reg}, \]
where \(\text{Frob}\) is the Frobenius automorphism (The Gros regulator is only defined for unramified fields.) From Galois equivariance it follows that
\[ \text{Frob}(\text{Li}_n(\zeta)) = \text{Li}_n(\text{Frob}(\zeta)) = \text{Li}_n(\zeta^p). \]
The relation between the two results is therefore clear.

We now state the following Conjecture.

Conjecture 1.14. Let \(K \subset \mathbb{C}_p\) be a discrete valuation subfield (i.e., the valuation is induced from the one on \(\mathbb{C}_p\)). Let \(R\) be the valuation ring of \(K\). Assume the Beilinson–Soulé conjecture holds in characteristic zero if \(n \geq 3\). Then, for all \(n \geq 2\), the regulator map
\[ H^1(\tilde{M}^\bullet_{(n)}(K)) \to K^{(n)}_{2n-1}(K) \cong K^{(n)}_{2n-1}(R) \to K \]
is given by the same formula as before, \([x]_n\) being mapped to \(\pm(n-1)!L_{\text{mod},n}(x)\).

Remark 1.15. Of course, Conjecture 1.14 would imply that the map
\[ \tilde{M}^\bullet_{(n)}(K) \to K \]
given by mapping \([x]_n\) to \(L_{\text{mod},n}(x)\) is well defined, as any relation among the \([x]_n\) would give rise to the zero element in \(H^1(\tilde{M}^\bullet_{(n)}(K))\). Moreover, it implies that the map
\[ H^1(\tilde{M}^\bullet_{(n)}(K)) \to K \]
given by mapping \([x]_n\) to \(L_{\text{mod},n}(x)\) is independent of the choice of the branch of the logarithm. We shall verify this last statement under the assumption that the map \(\tilde{M}^\bullet_{(n)}(K) \to K\) given by mapping \([x]_k\) to \(L_{\text{mod},k}(x)\) is well defined for all \(k \leq n\).
in Proposition 2.8 below, after determining the dependence of \( \text{Li}_n(z) \) on the choice of the logarithm.

**Remark 1.16.** As will be described in Section 3, the complexes \( \tilde{\mathcal{M}}^*_n(K) \) and \( \tilde{\mathcal{M}}^*_n(R) \) are quotient complexes of corresponding complexes \( \mathcal{M}^*_n(K) \) and \( \mathcal{M}^*_n(R) \), obtained by imposing the relations \( [x]_k + (-1)^k[1/x]_k \) for all \( k \geq 2 \). The general assumptions about the Beilinson–Soulé conjecture are necessary in order to prove this quotient map to be a quasi–isomorphism. There is a map \( H^*(\mathcal{M}^*_n(R)) \to K_{2n-1}^n(R) \) assuming only the Beilinson–Soulé conjecture for \( K \) and \( \kappa \). Therefore, assuming only the Beilinson–Soulé conjecture for \( K \) and \( \kappa \), we get a commutative diagram

\[
\begin{array}{ccc}
H^1(\mathcal{M}^*_n(R)) & \longrightarrow & H^1(\tilde{\mathcal{M}}^*_n(R)) \\
\downarrow & & \downarrow \\
K_{2n-1}^n(R) & \xrightarrow{\text{reg}} & K.
\end{array}
\]

As each of the steps in the proofs of Theorems 1.6 and 1.10 is fairly technical, we give a brief outline of the main steps and where in the paper they occur.

Using multi relative \( K \)-theory and localization (both discussed in Section 3), we get a diagram

\[
\begin{array}{ccc}
K_{2n-1}^n(\mathcal{O}) & \xrightarrow{\cong} & K_{2n-1}^n(X_{\mathcal{O}}^{n-1};[\square]^{n-1}) \\
\downarrow & & \downarrow \\
K_{2n-1}^n(R) & \xrightarrow{\cong} & K_{2n-1}^n(X_{\mathcal{O}}^{n-1};[\square]^{n-1}) \\
\end{array}
\]

The rows (except the first term) will be used to construct the complexes \( \tilde{\mathcal{M}}^*_n(\mathcal{O}) \) and \( \tilde{\mathcal{M}}^*_n(R) \) in Section 3. If the Beilinson–Soulé conjecture holds generally enough, there is a map \( H^1(\tilde{\mathcal{M}}^*_n(\mathcal{O})) \to K_{2n-1}^n(X_{\mathcal{O}}^{n-1};[\square]) \cong K_{2n-1}^n(\mathcal{O}) \). On the other hand, there is a syntomic regulator \( K_{2n-1}^n(R) \to K \). Using the embedding \( \mathcal{O} \to R \) as in Theorem 1.6 gives us the map

\[
H^1(\tilde{\mathcal{M}}^*_n(\mathcal{O})) \to K_{2n-1}^n(\mathcal{O}) \to K_{2n-1}^n(R) \to K.
\]

Similar results will be proved if \( K \) is replaced with a number \( F \), but without assumptions on weights in algebraic \( K \)-theory. We will also compare those complexes with the complexes \( \tilde{\mathcal{M}}^*_n(K) \) constructed in [dJ95, Section 3].

After reviewing syntomic regulators in Section 4, we analyze this in Sections 5 through 7 by extending the regulator map over the maps

\[
K_{2n-1}^n(R) \cong K_{2n-1}^n(X_{\mathcal{O}}^{n-1};[\square]^{n-1}) \to K_{2n-1}^n(X_{R,\text{loc}}^{n-1};[\square]^{n-1}).
\]

The computation involves Coleman integration, and we start the paper by reviewing it, and tying up some loose ends, in Section 2.

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**Notation:** Throughout the paper, if \( A \) is an Abelian group, we shall write \( A_Q \) for \( A \otimes \mathbb{Q} \). Throughout the paper, \( R \) will be a complete discrete valuation ring of
characteristic zero, with field of fractions $K$, and residue $\kappa$ of characteristic $p > 0$ and algebraic over $\mathbb{F}_p$. (Please note that in the Appendix A $K$ will have a different meaning, whereas $R$ will not.)

2. Some Preliminary Material

We begin with recalling Coleman’s integration theory in the form and to the extent that it will be needed for this work. References for the theory are [Col82] and [CdS88]. There is also a short summary in [Bes00b].

Our basic data is a “basic wide open” in the sense of Coleman. The data defining such an object consist of a complete curve $C/\mathbb{C}_p$, which is defined over some discretely valued subfield and which has good reduction $\overline{C}$ (the reader may take $\mathbb{P}^1$ for $C$ since this is the only case that will be used in this paper), together with a finite nonempty set of points $S \subset \overline{C}(\mathbb{F}_p)$ where $\mathbb{F}_p$ is the algebraic closure of $\mathbb{F}_p$.

To every point $y \in \overline{C}(\mathbb{F}_p)$ corresponds a “residue disc” $U_y$, a subspace of the rigid analytic space associated with $C$, consisting of all points in $C$ whose reduction is $y$. The basic wide open $U = U_\lambda$ associated with the data above is a rigid analytic space obtained from $C$ by “removing discs of radius $\lambda < 1$ from the insides of the residue discs $U_y$ for $y \in S$”. Technically this means that if the point $y$ is locally defined by the equation $\bar{z} = 0$, with $z = z_y$ some local parameter near $y$, then one removes the points in $U_y$ where $|z| \leq \lambda$. This procedure depends on the choice of $z$ but becomes independent of this choice as $\lambda$ approaches 1. We will not fix $\lambda$ but think of it as approaching 1 and will take it as large as needed. From now on we will use $U_y$ to denote the residue disc of $y$ in $U$, which is the intersection of the residue disc with $U$. This is the same as before unless $y \in S$ in which case $U_y$ is an annulus given by the equation $\lambda < |z_y| < 1$. Our final basic datum is a Frobenius endomorphism. This is a rigid analytic map $\phi : U_{\lambda_1} \to U_{\lambda_2}$, for some $\lambda_1$ and $\lambda_2$, whose reduction $\bar{\phi}$ is some power of the Frobenius endomorphism of some model of $\overline{C}$ over a finite field, extended $\mathbb{F}_p$ linearly. A good example of such a morphism is $\phi(z) = z^q$ on $\mathbb{P}^1$ for some power $q$ of $p$.

The goal of Coleman’s theory is to integrate certain differential forms on $U$. This is first done locally, on each residue disc $U_y$. If $y \notin S$ this residue disc is isomorphic to the open disc $\{|z| < 1\}$. A rigid differential form on such a disc has a convergent power series expansion $\sum_{n \geq 0} z^n dz$ and integration is done term by term. When $y \in S$ the form $dz_y/z_y$ is also analytic on $U_y$ and so there is no choice but to introduce a logarithm.

Definition 2.1. Let $\pi \in \mathbb{C}_p^*$ be such that $|\pi| < 1$. The branch of the $p$-adic logarithm determined by $\pi$ is the unique function $\log = \log_\pi : \mathbb{C}_p^* \to \mathbb{C}_p$ which is multiplicative, defined by the usual power series when $|z - 1| < 1$ and satisfies $\log(\pi) = 0$.

We fix once and for all a branch of the logarithm. Then the integral of $dz_y/z_y$ can be taken to be $\log(z_y)$ and that allows integration of an arbitrary form on any $U_y$.

Let $A(U)$ (respectively $\Omega^1(U)$) be the ring of rigid analytic functions (respectively one forms) on $U$ and let $A_{\text{loc}}(U)$ (resp. $\Omega^1_{\text{loc}}(U)$) be the ring of $\mathbb{C}_p$-valued functions (resp. module of one forms) on $U$ which are in $A(U_y)$ (resp. $\Omega^1(U_y)$) for each $y \notin S$ and are in the polynomial ring $A(U_y)[\log z_y]$ (resp. in $A(U_y)[\log z_y]dz_y$) when $y \in S$. It is implicit in this definition that it is independent of the choice of the local
parameter \( z_y \), a fact which follows because for any two choices of \( z_y \) the difference between the \( \log(z_y) \) is in \( A(U_y) \).

Each \( \omega \in \Omega^1_{\text{loc}}(U) \) can be integrated in \( A_{\text{loc}}(U) \) in many ways, because we can choose a different constant of integration for each \( U_y \). Coleman’s theory finds a subclass of forms for which one can assign canonically an integral in \( A_{\text{loc}}(U) \) defined up to a global constant. This is done recursively as follows. First one finds integrals to all forms \( \omega \). At each stage one integrates all forms that can be written as \( \sum f_i \omega_i \) where \( f_i \) are integrals which have been found in previous stages and \( \omega_i \in \Omega^1(U) \). The rules for finding the integrals are:

1. The integral is additive.
2. When \( g \in A(U) \), \( \int dg = g + C \), for some constant \( C \).
3. We have \( \phi^* \int \omega = \int \phi^* \omega + C \).

The fact that these rules suffice to carry out the integration process uniquely and that it is independent of the choice of \( \phi \) is the main result of Coleman (see [Col82] and [CdS88]). One other result about Coleman integration that will be used is the following.

**Proposition 2.2.** Let \( f \in A(U)^* \). Then the Coleman integral of the form \( \int df/f \) is \( \log(f) \).

**Proof.** See [CdS88, Lemma 2.5.1].

The original reason that Coleman integrals were introduced is probably to give a \( p \)-adic analogue of complex iterated integrals. Let \( \omega_1, \omega_2, \ldots, \omega_r \) be forms in \( \Omega^1(U) \) and let \( x \in U \). Then we can define an iterated integral

\[
f_r(x) = \int_x^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_r
\]

by defining \( f_1(z) = \int \omega_1 \) normalized so that \( f_1(x) = 0 \) and then by induction \( f_k(z) = \int f_{k-1} \omega_k \) again normalized so that \( f_k(x) = 0 \).

The definition of \( p \)-adic polylogarithms is a slight modification of the above. Here we take \( \omega_1 = -d\log(1-z) \) and \( \omega_i = d\log z \) for \( i > 1 \). Notice that \( d\log z \) has a simple pole at 0. However, if we normalize \( f_i \) at each step to vanish at 0 this zero will cancel with the pole and we will obtain a form which is also integrable at the residue disc of 0. This gives the definition of the introduction.

**Remark 2.3.** If \( U \) is a wide open defined over a field \( L \), containing at least one \( L \)-rational point \( x \) and suppose we have chosen the branch \( \log_\pi \) such that \( \pi \in L \). Then, if one has forms \( \omega_1, \ldots, \omega_r \) which are all defined over \( L \) again, then an iterated Coleman integral \( f = \int \omega_1 \circ \cdots \circ \omega_r \), where the constants are fixed so that all the intermediate integrals \( \int \omega_1 \circ \cdots \circ \omega_k \) take an \( L \)-rational value at \( x \) are Galois equivariant in the sense that for every automorphism \( \sigma \) of \( \mathbb{C}_p \) over \( L \) we have for every \( z \in U \) that \( f(z^\sigma) = (f(z))^\sigma \). In particular, if \( z \) is defined over \( L \) then \( f(z) \in L \). For \( \Li_n \), since the forms are either \( d\log z \) or \( d\log(1-z) \), which are all defined over \( \mathbb{Q}_p \), this means that if we take a branch \( \log_\pi \), with \( \pi \in \mathbb{Q}_p \), then \( \Li_n \) is Galois equivariant over \( \mathbb{Q}_p \).

We now want to collect some facts about the functions \( \Li_n \) and other things here that we need in the rest of the paper.

We begin with recalling some results from [Col82]. The following is contained in Proposition 6.1 and Corollary 7.1a of loc. cit. (Note that Proposition 6.1 of loc. cit. contains an obvious misprint.)
The functions $\text{Li}_n(z)$ are defined on $\mathbb{C}_p \setminus \{1\}$. If $L$ is a finitely ramified extension of $\mathbb{Q}_p$, then the limit $\lim_{z \to \infty} \text{Li}_n(z)$ exists for $n \geq 2$, and is independent of $L$. Using this limit as the value for $\text{Li}_n$ at $1$, $\text{Li}_n$ extends to a function on $\mathbb{C}_p$, which is continuous on finitely ramified extensions of $\mathbb{Q}_p$.

If $m$ and $n$ are integers at least equal to $2$, then on $\mathbb{C}_p$

$$\text{(2.4)} \quad \text{Li}_n(z^m) = m^{n-1} \sum_{\zeta = 1} \text{Li}_n(\zeta z).$$

Clearly the same formula holds for $n = 1$ provided $1 - z^m \neq 0$.

Let $\log_a$ and $\log_b$ be two different branches of the logarithm. Denote the corresponding different branches of $\text{Li}_n$ by $\text{Li}_{n,a}$ and $\text{Li}_{n,b}$. Let $\beta = \log_a p - \log_b p$, and let $v$ be the valuation such that $v(p) = 1$. Note that

$$\text{(2.5)} \quad \log_a(z) - \log_b(z) = v(z)\beta.$$  

**Proposition 2.6.** We have

$$\text{(2.7)} \quad \text{Li}_{n,a}(z) - \text{Li}_{n,b}(z) =$$

$$- \frac{1}{n!} v(1 - z)\beta (\log_a^{n-1} z + \log_a^{n-2} z \log_b z + \cdots + \log_a z \log_b^{n-2} z + \log_b^{n-1} z).$$

**Proof.** We first remark that by the construction of Coleman integrals the polylogarithm depends on the branch of the log chosen only on residue discs where one of the forms involved in the definition, i.e., $dz/z$ and $dz/(z-1)$, has a pole. This means that $\text{Li}_{n,a}$ and $\text{Li}_{n,b}$ can differ at most on the residue discs of $0$, $1$ and $\infty$, and in fact only on the latter two discs because $\text{Li}_n(z)$ is analytic on $|z| < 1$. We note that apriori it would seem that because the constant of integration is determined by the value at $0$ the function could depend on the branch of the log everywhere, but this is not the case exactly because logs do not appear in $\text{Li}_n$ at the residue disc of $0$. Because $v(1 - z) \neq 0$ only on the residue discs of $1$ and $\infty$ the formula is proved except in the cases $|z| > 1$ and $|z-1| < 1$. Suppose $|z| > 1$. Using (1.2) we obtain

$$\text{Li}_{n,a}(z) - \text{Li}_{n,b}(z) =$$

$$(-1)^n (\text{Li}_{n,b}(1/z) - \text{Li}_{n,a}(1/z)) - \frac{1}{n!} (\log_a^n z - \log_b^n z)$$

$$= -\frac{1}{n!} (\log_a^n z - \log_b^n z)$$

$$= -\frac{1}{n!} (\log_a z - \log_b z)(\log_a^{n-1} z + \log_a^{n-2} z \log_b z + \cdots + \log_b^{n-1}(z))$$

$$= -\frac{1}{n!} v(1 - z)\beta (\log_a^{n-1} z + \log_a^{n-2} z \log_b z + \cdots + \log_b^{n-1}(z))$$

because $v(z) = v(1-z)$ for such $z$. It remains to consider the case $|z-1| < 1$. Note that here $\log(z)$ is independent of the branch so the formula to be proved reads

$$\text{Li}_{n,a}(z) - \text{Li}_{n,b}(z) = -\frac{1}{(n-1)!} v(1 - z)\beta \log^{n-1} z.$$  

We prove this by induction on $n$. For $n = 1$ this follows immediately from (2.5). Assume $n > 1$. According to [Col82, Proposition 7.1], $\text{Li}_{n,a}(z) - \text{Li}_{n-1,a}(z) \log(z)$ extends to an analytic function on $|1 - z| < 1$. (Note that $B(0,1)$ should be replaced with $B(1,1)$ everywhere in the formulation and the proof of loc. cit.). The result
will follow from the induction hypothesis if we show that
\[
\gamma_n(z) := (\text{Li}_{n,a}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-1,a}) - (\text{Li}_{n,b}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-1,b}) = 0.
\]

When we differentiate \(\gamma_n(z)\) we find
\[
d(\text{Li}_{n,a}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-1,a}) - d(\text{Li}_{n,b}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-1,b})
\]
\[
= \left(\left(1 - \frac{1}{n-1}\right)\text{Li}_{n-1,a}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-2,a}(z)\right) d\log(z)
\]
\[
- \left(\left(1 - \frac{1}{n-1}\right)\text{Li}_{n-1,b}(z) - \frac{1}{n-1}\log(z)\text{Li}_{n-2,b}(z)\right) d\log(z)
\]
\[
= \frac{n-2}{n-1} \gamma_{n-1}(z) d\log(z) = 0
\]
by induction. So the \(\gamma_n(z)\) is a constant on \(|z-1| < 1\), call it \(C\), and we must show that \(C = 0\). But \(\gamma_n(z)\) satisfies the distribution relation corresponding to (2.4). For \(|z-1| < 1\) this relation now reads \(C = m^{n-1} \cdot m \cdot C\), which shows \(C = 0\) as required.

**Proposition 2.8.** Assume the Beilinson–Soulé conjecture holds for fields of characteristic zero. Let \(\log_n\) and \(\log_b\) denote two branches of the logarithm, and denote the corresponding functions involving \(\text{Li}\)'s by a subscript \(a\) or \(b\). If the maps
\[
\tilde{M}_k(K) \to \mathbb{C}_p
\]
given by mapping \([x]_k\) to \(L_{\text{mod},ka}(x)\) (resp. \(L_{\text{mod},kb}(x)\)) are well defined for \(k \leq n\), then the induced maps
\[
H^1(\tilde{M}_{(n)}(K)) \to \mathbb{C}_p
\]
given by mapping \([x]_n\) to \(L_{\text{mod},na}(x)\) (resp. \(L_{\text{mod},nb}(x)\)) are the same.

**Remark 2.9.** \(M_k(K)\) will be constructed below in Section 3, but for a heuristic approach to working with it see the beginning of the Introduction. Our computation of the regulator map in Sections to come will show that, for a fixed choice of \(\log\), the map
\[
\tilde{M}_n(R) \to \mathbb{C}_p
\]
given by mapping \([x]_n\) to \(L_{\text{mod},n}(x)\) is well defined, but we have to assume this for \(\tilde{M}_n(K)\). Note also that for the special units, the function \(L_{\text{mod},n}(x)\) does not depend on the branch of the logarithm by Proposition 2.8.

**Proof of Proposition 2.8.** Because \(M_n(K)\) is by definition \([\text{dJ}95, \text{Corollary 3.22}]\) the quotient of \(M_n(K)\) by the subspace spanned by elements of the form \([x]_n + (-1)^n[1/x]_n\), and \(L_{\text{mod},n}\) vanishes on those elements by (1.4), we can just as well prove the corresponding result for \(H^1(\tilde{M}_{(n)}(K))\) as the assumption about the Beilinson–Soulé conjecture implies that the map \(H^1(\tilde{M}_{(n)}(K)) \to H^1(\tilde{M}_{(n)}(K))\) is a quasi–isomorphism. Note that all functions of the form \([x]_n \mapsto \log^{n-k}(x) L_{\text{mod},k}(x)\) are well defined on \(M_n(K)\) by assumption as they factor through the map
\[
M_n(K) \to M_{n-1}(K) \otimes K_Q^\times \to \cdots \to M_k(K) \otimes (K_Q^\times)^{\otimes n-k}.
\]
The function $\text{Li}_n(z)$ can be expressed as a linear combination of the functions $L_{\text{mod},k}(z)\log^{n-k}(z)$ with $1 \leq k \leq n$, by the definition of $L_{\text{mod},n}$, and on elements in $H^1(M^\bullet_{(n)}(K))$ those functions vanish for $k < n$. Therefore the map from $M_n(K)$ to $\mathbb{C}_p$, given by mapping $[x]_n$ to $\text{Li}_n(x)$ is well defined under our assumptions, and it suffices to show that it induces a map on $H^1(M^\bullet_{(n)}(K))$, which is independent of the branch of the logarithm. Proposition 2.6 shows that this is true, as the right hand side of equation 2.7 is zero on elements in $H^1(M^\bullet_{(n)}(K))$, as this map factors through the map $M_n(K) \rightarrow M_{n-1}(K) \otimes K^\times_\mathbb{Q} \rightarrow \cdots \rightarrow (K^\times_\mathbb{Q})^{\otimes n}$ above, which maps $[x]_n$ to $(1-x) \otimes x \otimes \cdots \otimes x$.

We shall also need the distribution relation for elements in $M_n(F)$, as given in [dJ95, Proposition 6.1].

**Proposition 2.10.** If $F$ is a field of characteristic zero that contains the $m$–th roots of unity, then in $M_n(F)$ (and hence $\tilde{M}_n(F)$) we have

$$[x^m]_n = m^{n-1} \sum_{\alpha^m=1} [\alpha x]_n.$$  

3. Some $K$–Theory

In this Section we construct the complexes $\tilde{M}^\bullet_{(n)}(O)$ as quotient complexes of complexes $M^\bullet_{(n)}(O)$ for $n \geq 2$. The main idea is the same as in [dJ95], but the fact that we will be working with a discrete valuation ring rather than a field gives rise to complications. In order to highlight the idea we start with a rather gentle exposition. For the proofs of the statements that are used in the construction, we refer the reader to loc. cit., especially Sections 2.1 through 2.3, and 3. In loc. cit. most of the work was done over $\mathbb{Q}$, but in fact the proofs hold over our base $O$, a discrete valuation ring of characteristic zero, without any change. There is also a very brief introduction to multi relative $K$–theory in Appendix A.

The idea of the whole construction is the following. If $B$ is a regular Noetherian scheme, then the pullback $K_\ast(B) \rightarrow K_\ast(A_B^1)$ is an isomorphism. We shall be using an Adams decomposition with respect to weights, $K_m(X)_\mathbb{Q} = \oplus_i K_i(X)^{(i)}$. The weight behaves naturally with respect to pullback, and under suitable hypotheses for a closed embedding, there is a pushforward Gysin map with a shift in weights corresponding to the codimension (see, e.g., [dJ95, Proposition 2.3]).

Let $X_B = \mathbb{P}^1_B \setminus \{t = 1\}$ with $t$ the standard affine coordinate on $\mathbb{P}^1$. Write $[\square]_B$ for the closed subset $\{t = 0, \infty\}$ in $\mathbb{P}^1_B$. Then the relative exact sequence for the couple $(X_B,[\square]_B)$ gives us

$$\cdots \rightarrow K_{n+1}(X_B) \rightarrow K_{n+1}([\square]_B) \rightarrow K_n(X_B;[\square]_B) \rightarrow K_n(X_B) \rightarrow K_n([\square]_B) \rightarrow \cdots$$

for $n \geq 0$. Because the map pullback $K_{n+1}(B) \rightarrow K_{n+1}(X_B)$ is an isomorphism, combining it with the pullback $K_{n+1}(X_B) \rightarrow K_{n+1}([\square]_B) = K_{n+1}(B)^2$ shows that the map $K_{n+1}(X_B) \rightarrow K_{n+1}([\square]_B)$ corresponds to the diagonal embedding $K_{n+1}(B) \rightarrow K_{n+1}(B)^2$. As this holds for all $n \geq 0$, we get that we have an isomorphism $K_n(X_B;[\square]_B) \cong K_{n+1}(B)$ for $n \geq 0$. Note that we have a choice of sign here in the isomorphism of the cokernel of $K_n(B) \rightarrow K_n(B)^2$ with $K_n(B)$. This results in similar choices of signs in the maps $H^p(M^\bullet_{(n)}(O)) \rightarrow H^p_{2n-p}(O)$ (resp. $H^p(\tilde{M}^\bullet_{(n)}(O)) \rightarrow K^{(n)}_{2n-p}(O)$) later on in this Section.
We can iterate this procedure using multi relative $K$–theory. (The construction of this is recalled in Appendix A.) For the sake of exposition we give the argument here for the next level of relativity. If we let $\square_B^n = \{t_1 = 0, \infty\}; \{t_2 = 0, \infty\} \rightarrow \cdots \rightarrow K_{n+1}(X_B^2; \{t_1 = 0, \infty\}) \rightarrow K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \rightarrow K_n(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \rightarrow \cdots.

Using induction on the degree of relativity one sees that the composition

$$K_{n+1}(X_B; \{t_1 = 0, \infty\}) \rightarrow K_{n+1}(X_B^2; \{t_1 = 0, \infty\}) \rightarrow K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \rightarrow K_n(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \rightarrow \cdots.$$

(with the first the pullback along the projection $(t_1, t_2) \mapsto t_2$) is the diagonal embedding, hence we obtain an isomorphism $K_n(X_B^m; \square_B^n) \cong K_{n+1}(X_B^m; \square_B^n)$ for $n \geq 0$. Therefore we get $K_n(X_B^m; \square_B^n) \cong K_{n+1}(X_B; \square_B^n) \cong K_{n+1}(B)$ for $n \geq 0$. By induction one proves that

$$(3.1) \quad K_n(X_B^m; \square_B^n) \cong K_{n+m}(B)$$

for $n \geq 0$ and $m \geq 1$, with $\square_B^n$ shorthand for $\{t_1 = 0, \infty\}; \ldots; \{t_m = 0, \infty\}$ ($m$–th order relativity). One can also do this with weights, and as the weight are compatible with pullbacks, we get isomorphisms $K_n^{(j)}(X_B^m; \square_B^n) \cong K_{n+m}^{(j)}(B)$ for $n \geq 0$ and $m \geq 1$.

In order to get elements in groups like $K_{n+m}(X_B^m; \square_B^n)$, we use localization sequences. We shall explain the idea for $m = 1$. (For $m \geq 2$ the localization sequences get replaced by a spectral sequence, see below.) If $u$ is an element in our discrete valuation ring $\mathcal{O}$ such that both $u$ and $1 - u$ are units, then we get an exact localization sequence

$$\cdots \rightarrow K_m(\mathcal{O}) \rightarrow K_m(X_{\mathcal{O}, \text{loc}}; \square_1) \rightarrow K_m(X_{\mathcal{O}, \text{loc}}; \square_0) \rightarrow K_{m-1}(\mathcal{O}) \rightarrow \cdots$$

where $X_{\mathcal{O}, \text{loc}} = X_{\mathcal{O}} \setminus \{t = u\}$ and we identified $\{t = u\} \subset X_{\mathcal{O}}$ with $\mathcal{O}$ (or rather $\text{Spec}(\mathcal{O})$). We used here that $u$ and $1 - u$ are units in $\mathcal{O}$ so that $\{t = u\}$ does not meet $\square_2$ or $\{t = 1\}$, and that $\mathcal{O}$ is regular in order to identify $K_m(\mathcal{O})$ with $K_m^{(0)}(\mathcal{O})$. (If we want to leave out $\{t = u\}$ or $\{t = v\}$ simultaneously for two distinct elements $u$ and $v$ in $\mathcal{O}$ such that all of $u$, $v$, $1 - u$ and $1 - v$ are units, which we shall do below, this already becomes far more complicated and one is force to use a spectral sequence.) The image of $K_2(\mathcal{O}) \rightarrow K_2(X_{\mathcal{O}, \text{loc}}; \square_2)$ can be controlled by looking at the weights, which for the bit that we are interested in gives us

$$\cdots \rightarrow K_2^{(1)}(\mathcal{O}) \rightarrow K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_3) \rightarrow K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_0) \rightarrow K_1^{(1)}(\mathcal{O}) \rightarrow \cdots.$$

Because of weights in $K$–theory, one knows that $K_2^{(1)}(\mathcal{O}) = 0$, so we can analyze $K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_2)$ as subgroup of $K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_0)$. In [dJ95, Section 3.2] universal elements $[S]_n$ were constructed, of which we want to use $[S]_2$ here. It gives rise to an element $[u]_2$ in $K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_2)$ with boundary $(1 - u)^{-1}$ in $K_1^{(1)}(\mathcal{O})$. If we use this for various $u$ (suitably modifying the localization sequence above into a spectral sequence) and consider elements coming from the cup product

$$K_1^{(1)}(X_{\mathcal{O}, \text{loc}}; \square_1) \times K_1^{(1)}(\mathcal{O}) \rightarrow K_2^{(2)}(X_{\mathcal{O}, \text{loc}}; \square_2)$$
we can get part of $K_2^2(X;\square^1_0) \cong K_3^2(\mathcal{O})$ by intersecting the kernel of the map corresponding to $K_2^2(X_{\mathcal{O},\text{loc}};\square^1_0) \rightarrow K_1^1(\mathcal{O})$ with the space generated by the symbols $[u]_2$ and the image $K_1^1(X_{\mathcal{O},\text{loc}};\square^1_0) \cup K_1^1(\mathcal{O})$ of the cup product.

Unfortunately, this gets fairly technical, but after this gentle introduction we are now ready to begin. The reader is encouraged to compare this construction with the simpler construction for fields, which is carried out in [dJ95, Section 3].

To ease the notation somewhat, we will drop the subscript (indicating the base scheme) from $\square^n$.

**Definition 3.2.** A scheme $B$ has no low weight $K$–theory if the Beilinson–Soulé conjecture holds for $B$, i.e., $K_{2j}^j(B) = 0$ if $2j \leq m$ and $m > 0$. A ring $A$ is said to have no low weight $K$–theory if Spec($A$) does.

We shall use the following notation. Let $t$ be the standard affine coordinate on $\mathbb{P}^1$. We let $X = \mathbb{P}^1_{\mathbb{F}} \setminus \{ t = 1 \}$. If $B$ is any scheme, we let $X_B = X \times_\mathbb{Z} B$, and $X_B^n = X_B \times_B \cdots \times_B X_B$. If $U$ is a subset of $\Gamma(B,\mathcal{O}^*)$ such that if $b$ is in $U$, then $1 - b$ is also in $\Gamma(B,\mathcal{O}^*)$, we let $X_{B,\text{loc}} = X_B \setminus \{ t = b, b \in U \}$, and $X_{B,\text{loc}}^n = X_{B,\text{loc}} \times_B \cdots \times_B X_{B,\text{loc}}$. The set $U$ will normally be clear from the context. We shall also abuse notation by writing $X_{B,\text{loc}}^n$ even after we took direct limits over finite sets $U$. In the multi–relative $K$–theory below, we shall write $\square^n$ for $\{ t_1 = 0, \infty \}; \ldots ; \{ t_n = 0, \infty \}$.

**Notation 3.3.** For the remainder of the section, $\mathcal{O}$ will be a discrete valuation ring with field of fractions $F$ and residue field $\kappa$.

**Lemma 3.4.** Assume $F$ and $\kappa$ have no low weight $K$–theory. Then for $2j \leq q + m$ and $m > q$, all of $K_m^{jq}(X_F;\square^q)$, $K_m^{jq}(X_{\mathcal{O},\text{loc}};\square^q)$, and $K_m^{jq}(X_{\kappa,\text{loc}};\square^q)$ are zero.

**Proof.** Lemma 3.4 of [dJ95] shows the statement to be true for $F$ or $\kappa$. The result for $\mathcal{O}$ follows immediately from the exact localization sequence

$$
\cdots \rightarrow K_m^{jq-1}(X_{\kappa,\text{loc}};\square^q) \rightarrow K_m^{jq}(X_{\mathcal{O},\text{loc}};\square^q) \rightarrow K_m^{jq}(X_F;\square^q) \rightarrow \cdots
$$

\[\square\]

**Remark 3.5.** $\kappa$ has no low weights $K$–theory if $\kappa$ is algebraic over $\mathbb{F}_p$, because all $K_n(\kappa)$ are torsion for $n \geq 1$. It also holds if $\kappa$ is of transcendence degree 1 over $\mathbb{F}_p$ by a result of Harder, see [Har77, Korollar 2.3.2]. Because $K_n(F)$ is torsion for $n \geq 1$ for a finite field $F$, using localization it is enough to show that $K_n(\mathcal{O})$ is torsion for a Dedekind ring in a function field of transcendence degree 1 over a finite field, which is the result quoted.

**Remark 3.6.** $F$ has no low weights $K$–theory if $F$ is a number field, or more generally a subfield of the algebraic closure of $\mathbb{Q}$. As the residue field is an algebraic extension of $\mathbb{F}_p$ in this case, the conditions of Lemma 3.4 are certainly satisfied, and all constructions in this Section go through without assumptions about the weights on the $K$–groups involved.

Consider the divisors on $X^n_0$ defined by putting $t_i = u_j$ for some $u_j$ in $\mathcal{O}^\circ$. Put $W^0 = X^n_0$, and let $W^1$ be the union of divisors $\{ t_i = u_j \}$ for all $u_j$ in some finite set $U \subset \mathcal{O}^\circ$. Considering the singular locus of $W^1$, it is easy to see that one can extend this to a stratification $W^0 \supset W^1 \supset \cdots \supset W^{n+2} = \emptyset$ on $X^n_0$, with all $W^s \setminus W^{s+1}$ for
s = 0, . . . , n + 1 consisting of a finite union of $X_{\mathcal{O}, \text{loc}}^n$, $X_{F, \text{loc}}^n$, and $X_{\kappa, \text{loc}}^{n-1}$'s, which are regular. Using the localization sequences

$$\cdots \to K_{n,W+1}^{(j+s)}(X_{\mathcal{O}}^n; \mathfrak{m}^n) \to K_{n,W}^{(j+s)}(X_{\mathcal{O}}^n; \mathfrak{m}^n) \to \cdots$$

where $K_{n,W}^{(j+s)}(X_{\mathcal{O}}^n; \mathfrak{m}^n)$ etc. is K-theory with supports. we get an exact couple, which gives rise to a spectral sequence converging to $K_n^{(j)}(X_{\mathcal{O}}^n; \mathfrak{m}^n)$. We have isomorphisms

$$K_{n,W+1}^{(j+s)}(X_{\mathcal{O}}^n \setminus W^{s+1}; \mathfrak{m}^n) \cong K_n^{(j)}(W_s \setminus W^{s+1}; \mathfrak{m}^n)$$

and we can identify the terms in the spectral sequence with terms of this type. Note that the components of $(W_s \setminus W^{s+1}; \mathfrak{m}^n)$ are of the form $(X_{\mathcal{O}, \text{loc}}^n, \mathfrak{m}^{n-s})$, $(X_{F, \text{loc}}^n, \mathfrak{m}^{n-s})$ and $(X_{\kappa, \text{loc}}^{n-1}, \mathfrak{m}^{n-s})$. Taking $j = n+1$ we get a spectral sequence with $E_1^{i,t}$ equal to

$$K_{n-s-t}^{(n-s)}(X_{\mathcal{O}, \text{loc}}^n \setminus W^{s+1}; \mathfrak{m}^{n-s-1}) \prod K_{n-s-t}^{(n-s)}(X_{F, \text{loc}}^n \setminus W^{s+1}; \mathfrak{m}^{n-s-1}) \prod K_{n-s-t}^{(n-s)}(X_{\kappa, \text{loc}}^{n-1}; \mathfrak{m}^{n-s})$$

and converging to $K_{n-s-t}^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1})$. If we write $K_{n,\mathcal{O}}^{(j,m)}$ for $K_n^{(j)}(X_{\mathcal{O}, \text{loc}}^m; \mathfrak{m}^m)$ for typographical reasons, and similarly for $F$ and $\kappa$, this looks as

$$\begin{array}{cccccc}
K_{n,\mathcal{O}}^{(n,1)} & K_{n-1,\mathcal{O}}^{(n-1,1)} & \cdots & K_{n-2,\mathcal{O}}^{(n-1,1)} & K_{n-3,\mathcal{O}}^{(n-2,1)} & K_{n-4,\mathcal{O}}^{(n-3,1)} & \cdots \\
K_{n,F}^{(n,1)} & K_{n-1,F}^{(n-1,1)} & \cdots & K_{n-2,F}^{(n-1,1)} & K_{n-3,F}^{(n-2,1)} & K_{n-4,F}^{(n-3,1)} & \cdots \\
K_{n,\kappa}^{(n,1)} & K_{n-1,\kappa}^{(n-1,1)} & \cdots & K_{n-2,\kappa}^{(n-1,1)} & K_{n-3,\kappa}^{(n-2,1)} & K_{n-4,\kappa}^{(n-3,1)} & \cdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \end{array}$$

Observe that, due to the choice of the stratification, $K_{n,s}^{(n-s)}(X_{\kappa, \text{loc}}^{n-1}; \mathfrak{m}^{n-s})$ occurs only when $s \geq 2$. Also, by Lemma 3.4, if both $F$ and $\kappa$ have no low weights $K$-theory, then in the spectral sequence (3.7) converging to $K_n^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1})$ there are no nonzero terms in the row below the one beginning with $K_n^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1})$ (i.e., the middle row of (3.7), where $K_n^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1})$ is denoted $K_{n,\mathcal{O}}^{(n,n-1)}$).

**Lemma 3.8.** *If $\kappa$ has no low weight $K$-theory, then the map*

$$K_n^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1}) \to K_{n-1}^{(n-1)}(X_{F, \text{loc}}^{n-1}; \mathfrak{m}^{n-1})$$

*is injective.*

**Proof.** Immediate from the localization sequence

$$\cdots \to K_{n}^{(n)}(X_{\kappa, \text{loc}}^{n-1}; \mathfrak{m}^{n-1}) \to K_{n}^{(n)}(X_{\mathcal{O}, \text{loc}}^{n-1}; \mathfrak{m}^{n-1}) \to K_{n}^{(n)}(X_{F, \text{loc}}^{n-1}; \mathfrak{m}^{n-1}) \to \cdots$$

as the first term here is zero by Lemma 3.4. \qed
We now notice that all our localizations are compatible with localizing in a larger set \( U \), and that we can take direct limits of our localizations over finite sets \( U \) if we want. In order not to overburden the notation we shall suppress \( U \) from the notation. Notice that this means also that all components in the spectral sequence (3.7) of codimension at least one with \( \mathcal{O} \) as base become the corresponding components with \( F \) as base.

**Assumption 3.9.** We assume for the remainder of the section that the discrete valuation ring \( \mathcal{O} \) has characteristic zero.

We now define symbols in \( K \)-theory. Let \( G = \text{Spec}(\mathbb{Z}[S, S^{-1}, (1 - S)^{-1}]) \).

In [dJ95, Section 3.2] universal symbols \( [S]_n \in K^{(n)}_n(X_{G, \text{loc}}^{n+1}; \square^{n-1}) \)

were constructed, with boundary

\[
\sum_{i=1}^{n-1} (-1)^i [S]_{n-1} |_{t_i = S}
\]

in \( \prod_i K^{(n-1)}_{n-1}(X_{G, \text{loc}}^{n-2}; \square^{n-2}) \) under the boundary in the spectral sequence corresponding to (3.7) for \( G \). (Although the proofs in loc. cit. were formulated over \( \mathbb{Q} \), the constructions hold for a much larger class of base schemes without any change.) Recall that we denote by \( \mathcal{O}^* \) the units in \( \mathcal{O} \), and by \( \mathcal{O}^\flat \) the set of elements \( u \) in \( \mathcal{O}^* \) such that \( 1 - u \) is also in \( \mathcal{O}^* \).

**Definition 3.11.** For \( u \in \mathcal{O}^\flat \) we define the symbol

\[
[u]_n \in K^{(n)}_n(X_{\mathcal{O}, \text{loc}}^{n+1}; \square^{n-1})
\]

as the pullback of the universal symbol \([S]_n\) under the map \(\text{Spec}(\mathcal{O}) \to G\) induced by mapping \(S\) to \(u\).

It was also shown in loc. cit. that the symbol \([1]_n\) exists for \(n \geq 2\), but we shall tacitly ignore this symbol here, as it can also be defined by the distribution relation Proposition 2.10 if there are other roots of unity in \(F\).

We define inductively the symbolic part of the \( K \)-theory. Let

\[
(1 + I)^* = K^{(1)}_1(X_{\mathcal{O}, \text{loc}}^1; \square^1) = \left\{ \prod_j \left( \frac{t - u_j}{1 - 1} \right)^{n_j} \text{ such that } \prod_j u_j^{n_j} = 1 \right\},
\]

where the \(u_j\) are in \(\mathcal{O}^*\) and the \(n_j\) are in \(\mathbb{Z}\), and let \(\text{Symb}_k(\mathcal{O}) \subseteq K^{(k)}_k(X_{\mathcal{O}, \text{loc}}^{k+1}; \square^{k-1})\)

be defined by setting \(\text{Symb}_1(\mathcal{O}) = \mathcal{O}^*\), and

\(\text{Symb}_{k+1}(\mathcal{O}) = \langle [u]_{k+1}, u \in \mathcal{O}^\flat \rangle + (1 + I)^* \cap \text{Symb}_k(\mathcal{O})\)

for \(k \geq 1\). The notation \(\cap\) means the following. There are \(k\) projections of \(X_{\mathcal{O}, \text{loc}}^{k-1}\) to \(X_{\mathcal{O}, \text{loc}}^{k+1}\), giving rise to \(k\) cup products

\[
(1 + I)^* \times K^{(k)}_k(X_{\mathcal{O}, \text{loc}}^{k-1}; \square^{k-1}) \to K^{(k+1)}_{k+1}(X_{\mathcal{O}, \text{loc}}^k; \square^k).
\]

\((1 + I)^* \cap \text{Symb}_k(\mathcal{O})\) is the \(\mathbb{Q}\)-subspace spanned by the image of all those \(k\) cup products.
Because $d[x]_n = \sum_{i=1}^{n-1} (-1)^i [x]_{n-1}t_i = 2$ where $d$ is the differential in the spectral sequence, we get a complex $Symb_{(n)}(O)$ in the row of (3.7) which starts with $K^{(n)}_{n-1}(O) = K^{(n)}_n(X_{\text{O, loc}}^{n-1}; \square^{n-1})$, given by

$Symb_n(O) \to \coprod Symb_{n-1}(O) \to \coprod Symb_{n-2}(O) \to \cdots \to \coprod Symb_2(O) \to \coprod O$.

If $\kappa$ has no low weight $K$-theory, then by Lemma 3.8 we can view those groups as subgroups of the corresponding spaces for $F$. As the components corresponding to $\kappa$ in (3.7) will never play a role in the boundary for elements in $Symb_{(n)}(O)$, we can view the above complex as a subcomplex of the complex

$K^{(n)}_n(X_{\text{F, loc}}^{n-1}; \square^{n-1}) \to \coprod K^{(n-1)}_{n-1}(X_{\text{F, loc}}^{n-2}; \square^{n-2}) \to \coprod K^{(n-2)}_{n-2}(X_{\text{F, loc}}^{n-3}; \square^{n-3}) \to \cdots$

where all coproducts for codimension $r$ are taken over $r$-tuples $(u_1, \ldots, u_r)$ in $(O^\vee)^r$.

**Lemma 3.13.** Assume $F$ has no low weight $K$-theory. Then the map corresponding to further localization from $K^{(n)}_n(X_{\text{F, loc}}^{n-1}; \square^{n-1})$ for one set of localizing elements to $K^{(n)}_n(X_{\text{F, loc}}^{n-1}; \square^{n-1})$ for a larger one is injective.

**Proof.** For $n = 1$ there is nothing to prove. For $n \geq 2$, we use the exact sequence

$0 \to K^{(n)}_n(X_F^{n-1}; \square^{n-1}) \to K^{(n)}_n(X_{\text{F, loc}}^{n-1}; \square^{n-1}) \to \coprod K^{(n-1)}_{n-1}(X_{\text{F, loc}}^{n-2}; \square^{n-2})$

for two different set of localizing elements. As clearly the right most term injects under localizing more (i.e., make the coproduct larger as well), we are done by induction. \qed

By Lemmas 3.8 and 3.13, if both $\kappa$ and $F$ have no low weight $K$-theory, then we also have an inclusion $Symb_n(O) \subset Symb_n(F)$, so that $Symb_{(n)}(O)$ is a subcomplex of $Symb_{(n)}(F)$. We can also forget about exactly which finite subset $U$ of $O^\vee$ or $F^\vee \setminus \{1\}$ we use, and work in the direct limit for such $U$ from now on.

If both $F$ and $\kappa$ have no low weight $K$-theory, then all this takes place in the lowest non-zero row of the spectral sequence (3.7) above, and if we give $Symb_{(n)}$ a cohomological grading in degrees $1$ through $n$, we get a commutative diagram of maps

\[
\begin{array}{ccc}
H^r(Symb_{(n)}(O)) & \longrightarrow & K^{(n)}_{n-r+1}(X_{\text{O, loc}}^{n-1}; \square^{n-1}) \\
\downarrow & & \downarrow \\
H^r(Symb_{(n)}(F)) & \longrightarrow & K^{(n)}_{n-r+1}(X_{\text{F, loc}}^{n-1}; \square^{n-1})
\end{array}
\]

because the differentials on $Symb_n(O)$ and $Symb_n(F)$ are induced from the spectral sequence (3.7).

**Remark 3.14.** Note that if both $\kappa$ and $F$ have no low weight $K$-theory, then the horizontal maps here are injections by construction for $r = 1$.

The complex $Symb_{(n)}$ can be changed into a tensor complex, using the following subcomplex. Define $J_k$ as $(1 + I)^* \cap Symb_k(O)$. Let $J_{(n)}$ be the subcomplex of $Symb_{(n)}$ given by

$J_n \to dJ_n + \coprod J_{n-1} \to d(\ldots) + \coprod J_{n-2} \to \cdots \to d(\ldots) + \coprod J_2 \to d(\ldots)$.
Proposition 3.15. The subcomplex $\mathcal{J}^\bullet_{(n)}$ is acyclic.

Proof. The same as the proof of Lemma 3.7 of [dJ95], see also Remark 3.10 in loc. cit.

Note that the symmetric group $S_{n-1}$ acts on $\text{Sym}^\bullet_{(n)}$ and $\mathcal{J}^\bullet_{(n)}$. Denote the parts of those complexes on which $S_{n-1}$ acts alternatingly by the superscript alt.

Let the complex $\mathcal{M}^\bullet_{(n)}$ be the quotient complex $\left(\text{Sym}^\bullet_{(n)}\right)^{\text{alt}} / \left(\mathcal{J}^\bullet_{(n)}\right)^{\text{alt}}$. It has the form

$$M_n \to M_{n-1} \otimes \mathcal{O}_Q \to M_{n-2} \otimes \bigwedge^2 \mathcal{O}_Q \to \cdots \to M_2 \otimes \bigwedge^{n-2} \mathcal{O}_Q \to \bigwedge^n \mathcal{O}_Q$$

with $M_k(\mathcal{O}) = \text{Sym}_k(\mathcal{O})^{\text{alt}} / J_{(k)}^{\text{alt}}$, which is generated by the classes of the elements $[u]_k$, where $u \in \mathcal{O}^p$, and similar for $F$. Denote the class of $[u]_k$ simply by $[u]_k$. Then the differential is given by

$$d([x]_k \otimes y_1 \wedge \ldots \wedge y_{n-k}) = [x]_{k-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-k}$$

if $k \geq 3$, and

$$d([x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-k}) = (1 - x) \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-k}.$$

Proposition 3.16. If $\kappa$ and $F$ have no low weight $K$-theory, then the localization map $\mathcal{M}^\bullet_{(0)}(\mathcal{O}) \to \mathcal{M}^\bullet_{(0)}(F)$ is injective.

Proof. From the localization sequence

$$\cdots \to K^{(n-1)}_{2n-1}(\kappa) \to K^{(n)}_{2n-1}(\mathcal{O}) \to K^{(n)}_{2n-1}(F) \to \cdots$$

we get that the map $K^{(n)}_{2n-1}(\mathcal{O}) \to K^{(n)}_{2n-1}(F)$ is injective. Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H^1(\mathcal{M}^\bullet_{(k)}(\mathcal{O})) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\mathcal{M}^\bullet_{(k)}(F)) \\
\end{array}
\begin{array}{ccc}
& & \\
& \longrightarrow & \longrightarrow \\
M_{(k)}(\mathcal{O}) & \longrightarrow & M_{(k-1)}(\mathcal{O}) \otimes \mathcal{O}_Q \\
M_{(k)}(F) & \longrightarrow & M_{(k-1)}(F) \otimes F_\mathcal{Q}.
\end{array}
$$

Because the maps to $K^{(n)}_{2n-1}(\mathcal{O})$ (resp. $K^{(n)}_{2n-1}(F)$) the $H^1$’s are injective by Remark 3.14, we find that the map $H^1(\mathcal{M}^\bullet_{(k)}(\mathcal{O})) \to H^1(\mathcal{M}^\bullet_{(k)}(F))$ is injective. It follows by induction on $k$ that the maps $M_k(\mathcal{O}) \to M_k(F)$ are injective, as this is clear for $k = 2$, where the vertical map on the right is the inclusion $\mathcal{O}_Q^2 \otimes \mathcal{O}_Q \to F_\mathcal{Q} \otimes F_\mathcal{Q}$.

Let $N_k = N_k(\mathcal{O}) = <[u]_k + (-1)^k[u^{-1}]_k>$ where the $u$ runs through $\mathcal{O}^p$. This is the analogue of the groups $N_k(F)$ defined in [dJ95, Proposition 3.20]. We consider the subcomplex $\mathcal{N}^\bullet_{(n)}(\mathcal{O})$ of $\mathcal{M}^\bullet_{(n)}(\mathcal{O})$ given by

$$N_n \rightarrow N_{n-1} \otimes \mathcal{O}_Q^2 \rightarrow \cdots \rightarrow N_2 \otimes \bigwedge^{n-2} \mathcal{O}_Q \rightarrow d \left( N_2 \otimes \bigwedge^{n-2} \mathcal{O}_Q \right).$$

Lemma 3.17. If $\kappa$ has more than two elements, or equivalently, if $\mathcal{O}^p \neq \emptyset$, then $\text{d}N_2(\mathcal{O}) = \text{Sym}^2 \mathcal{O}_Q^2$, and similarly for $F$. 
Proof. Because $d[u_2 + d[u_1]]_2 = u \otimes u$ in $\text{Sym}^2 \mathcal{O}_Q^*$, $dN_2 = <u \otimes u, u \in \mathcal{O}^*>$. $\text{Sym}^2 \mathcal{O}_Q^*$ is spanned by elements of the form $v \otimes v$ where $v$ runs through $\mathcal{O}^*$. If $v$ is a special unit, then it is clear from the formula above that $v \otimes v$ is in $dN_2$. If not, $v$ reduces to 1 in $\mathcal{O}$. Let $w$ be a non-special unit in $\mathcal{O}^*$. then $wv, wv^{-1}$ and $wv^2$ are all non-special, and they give the elements $(wv) \otimes (wv), (wv^{-1}) \otimes (wv^{-1})$ and $(wv^2) \otimes (wv^2)$ in $\text{Sym}^2(\mathcal{O}_Q^*)$, and a linear combination of them gives $v \otimes v$. \[\square\]

Proposition 3.18. If the Beilinson–Soulé conjecture holds for $\kappa$ and for fields of characteristic zero, then the complex $\mathcal{N}_{n}^*(\mathcal{O})$ is acyclic.

Proof. If there are no special units, there is nothing to prove as the complex $\mathcal{N}_{n}^*(\mathcal{O})$ is zero. If $\mathcal{O}^*$ is non-empty, we show that $\mathcal{N}_{n}^*(\mathcal{O})$ is quasi-isomorphic to the complex

$$\text{Sym}^n(\mathcal{O}_Q^*) \rightarrow \text{Sym}^{n-1} \otimes \mathcal{O}_Q^* \rightarrow \text{Sym}^{n-2}(\mathcal{O}_Q^*) \otimes \mathcal{O}_Q \rightarrow \cdots$$

$$\cdots \rightarrow \text{Sym}^2(\mathcal{O}_Q^*) \otimes \bigwedge^{n-2} \mathcal{O}_Q^* \rightarrow d \left( \text{Sym}^2(\mathcal{O}_Q^*) \otimes \bigwedge^{n-2} \mathcal{O}_Q^* \right).$$

It is well known that this last complex is acyclic, with an explicit homotopy operator given in Corollary 3.22 of [dJ95]. It was proved in Proposition 3.20 of [dJ95] that the map $[u]_n + (-1)^n[u^{-1}]_n \mapsto u \otimes \ldots \otimes u$ induces an isomorphism between $N_n(F)$ and the subspace of $\text{Sym}^n(F^2_Q^*)$ generated by the elements $u \otimes \ldots \otimes u$ with $u$ in $F^*$. Considering Proposition 3.16, our complex is a subcomplex of the corresponding complex for $F$. So it will suffice to check that the image of $N_n(\mathcal{O})$ is $\text{Sym}^n(\mathcal{O}_Q^*)$, which is done as in the proof of Lemma 3.17. \[\square\]

Remark 3.19. In case $F$ is a number field, we can prove the statement of Proposition 3.18 without assuming the Beilinson–Soulé conjecture. It is known that $F$ satisfies the Beilinson–Soulé conjecture, and it was shown in [dJ95, Proposition 5.1] that the map $[u]_n + (-1)^n[u^{-1}]_n \mapsto u \otimes \ldots \otimes u$ gives an injection from $N_n(F)$ into $\text{Sym}^n(F^2_Q^*)$. Because $\kappa$ is finite, $K_m(\kappa)$ is torsion for $m \geq 1$, we have inclusions $N_n(\mathcal{O}) \subset N_n(F)$ by Proposition 3.16. The proof that $\mathcal{N}_{n}^*(\mathcal{O})$ is acyclic then proceeds as in the general case.

Let $\tilde{M}_{n}^*(\mathcal{O})$ be the quotient complex $\mathcal{M}_{n}^*(\mathcal{O})/\mathcal{N}_{n}^*(\mathcal{O})$. It has the form

$$\tilde{M}_n(\mathcal{O}) \rightarrow \tilde{M}_{n-1}(\mathcal{O}) \otimes \mathcal{O}_Q^* \rightarrow \tilde{M}_{n-2}(\mathcal{O}_Q^*) \otimes \mathcal{O}_Q^* \rightarrow \cdots \rightarrow \tilde{M}_2(\mathcal{O}) \otimes \bigwedge^2 \mathcal{O}_Q^* \rightarrow \tilde{M}_1(\mathcal{O}) \otimes \bigwedge^1 \mathcal{O}_Q^* \rightarrow \tilde{M}_0(\mathcal{O})$$

with $\tilde{M}_k(\mathcal{O}) = M_k(\mathcal{O})/N_k(\mathcal{O})$, and is clearly still generated by the classes of the elements $[u]_k$, where $u \in \mathcal{O}^*$. We have similarly the complex $\tilde{M}_{n}^*(F)$ defined in [dJ95, Corollary 3.22]. In both cases, the differential is now given by

$$d([x]_k \otimes y_1 \wedge \ldots \wedge y_{n-k}) = [x]_{k-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-k}$$

if $k \geq 3$, and

$$d([x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-k}) = (1-x) \wedge x \wedge y_1 \wedge \ldots \wedge y_{n-k}.$$
We have now proved the assertions (1) in Theorem 1.6 and Theorem 1.10. Namely, either assume the Beilinson–Soule conjecture is true for fields of characteristic zero as well as for \( \kappa \), or that \( F \) is a number field. We then have a map

\[
H^r(\tilde{\mathcal{M}}_{(n)}^\bullet(O)) \to K_{2n-r}^{(n)}(O),
\]

which is obtained as the composition of the maps

\[
H^r(\tilde{\mathcal{M}}_{(n)}^\bullet(O)) \overset{\sim}{\to} H^r(\mathcal{M}_{(n)}^\bullet(O)) \overset{\sim}{\to} H^r(Symb_{(n)}^\bullet(O)) \to K_{2n-r}^{(n)}(O).
\]

From left to right, those maps are justified by Proposition 3.18 or Remark 3.19, Proposition 3.15, and Lemma 3.4 as it implies that we are working in the lowest nonvanishing row of the spectral sequence (3.7).

One has the same maps when replacing \( O \) with \( F \) everywhere, and the inclusion of \( O \) into \( F \) induces an injection of complexes \( \mathcal{M}_{(n)}^\bullet(O) \) into \( \mathcal{M}_{(n)}^\bullet(F) \) by Proposition 3.16, which is compatible with all those maps, i.e., the diagram

\[
\begin{array}{ccc}
H^r(\tilde{\mathcal{M}}_{(n)}^\bullet(O)) & \overset{\sim}{\to} & H^r(\mathcal{M}_{(n)}^\bullet(O)) \\
\downarrow & & \downarrow \\
H^r(\tilde{\mathcal{M}}_{(n)}^\bullet(F)) & \overset{\sim}{\to} & H^r(\mathcal{M}_{(n)}^\bullet(F))
\end{array}
\]

commutes. This shows that we can identify the complex \( \mathcal{M}_{(n)}^\bullet(O) \) with a subcomplex of \( \mathcal{M}_{(n)}^\bullet(F) \).

**Remark 3.22.** If the Beilinson–Soule conjecture holds for fields of characteristic zero and for \( \kappa \), then one proves in the same way as in Proposition 3.16 that the map \( \mathcal{M}_{(n)}^\bullet(O) \to \mathcal{M}_{(n)}^\bullet(F) \) is injective, so we can identify \( \mathcal{M}_{(n)}^\bullet(O) \) with a subcomplex of \( \mathcal{M}_{(n)}^\bullet(F) \).

**Remark 3.23.** If \( F'O/F \) is an arbitrary field extension, and \( O' \subset F' \) is a discrete valuation ring with \( O \subset O' \cap F \), then there are obvious maps \( \tilde{\mathcal{M}}_{(n)}^\bullet(O) \to \tilde{\mathcal{M}}_{(n)}^\bullet(O') \), and similarly for \( F \) and \( F' \). The map \( K_{2n-r}^{(n)}(F) \to K_{2n-r}^{(n)}(F') \) is always injective, the corresponding maps for the complexes of \( F \) to the ones for \( F' \) is injective, provided either the Beilinson–Soule conjecture is true in general for fields of characteristic zero, or \( F' \) is algebraic over \( Q \). This is proved like in the proof of Proposition 3.16, as the assumptions mean that the necessary maps to \( K_{2n-1}^{(n)}(F) \) etc. exist and are injective. Similarly, if \( F'/F \) is algebraic over \( Q \), we get the corresponding statement for the complexes for \( O \) and \( O' \) with the Beilinson–Soule conjecture holds for \( F' \) and \( \kappa' \) (and hence for \( F \) and \( \kappa \)). In particular, all those maps are injective if the Beilinson–Soule conjecture is true for fields of characteristic zero, or \( F' \) is algebraic over \( Q \). If this is the case, we shall always view all complexes as being subcomplexes of the corresponding complexes of \( F' \), and view all \( K \)-groups as being contained in the corresponding \( K \)-groups of \( F' \).

**Remark 3.24.** We make a few remarks about the above construction without assuming the Beilinson–Soule conjecture. There are two places where it plays a role, namely the vanishing of the spectral sequence (3.7) below the row starting with \( K_{n}^{(n)}(X_{O,\text{loc.}}^n,\mathbb{Q}^n) \), and the acyclicity of the subcomplex \( N_{(n)}^\bullet \).
As for the first case, it is well known that $K_n^{(1)}(O) = 0$ for $n \geq 2$. Therefore the last two columns (which would correspond to degrees $n$ and $n+1$ for our complexes) of the spectral sequence below our main row are always zero. This means that the maps $H^r(\check{\mathcal{M}}_n^\bullet(O)) \to K_{2n-r}^{(n)}(O)$ always exists for $r = n - 1$ and $r = n$. As for the second case, it follows from the proof of [dJ95, Proposition 3.20] that the map $N_2(K) \to \text{Sym}^2(K_2)$ given by mapping $[x]_2$ to $x \otimes x$ is an injection as $K_2^{(1)}(L)$ and $K_3^{(1)}(L)$ are zero for any field $L$. Therefore the complex $\check{\mathcal{N}}_n^\bullet(F)$ is acyclic in degrees $n - 1$ and $n$. Because we have the exact sequence

$$K_3^{(1)}(\kappa) \to K_3^{(2)}(O) \to K_3^{(2)}(F)$$

and the first term is zero, we see as in Proposition 3.16 that $\text{Symb}_2(O) \subseteq \text{Symb}_2(K)$, $M_2(O) \subseteq M_2(K)$, that $\check{\mathcal{N}}_n^\bullet(R)$ is acyclic in degrees $n - 1$ and $n$ and hence $\widetilde{M}_2(O) \subseteq \widetilde{M}_2(K)$. We also get a commutative diagram

$$
\begin{array}{ccc}
H^r(\check{\mathcal{M}}_n^\bullet(O)) & \sim & H^r(\mathcal{M}_n^\bullet(O)) \\
\downarrow & & \downarrow \\
H^r(\check{\mathcal{M}}_n^\bullet(F)) & \sim & H^r(\mathcal{M}_n^\bullet(F)) \\
\end{array}
$$

for $r = n - 1$ and $r = n$ without any assumptions.

4. Syntomic regulators

In this Section we briefly recall some parts of the theory of rigid syntomic regulators, originally due in the affine case to Gros [Gro94], as described in detail in [Bes00a]. Our goal is to describe the theory in the minimal details required to understand constructions to follow and to develop certain computational tools that are needed in later Sections.

Recall that $R$ is a complete discrete valuation ring with quotient field $K$ of characteristic 0 and residue field $\kappa$ of characteristic $p$. We will assume that $\kappa$ is algebraic over the prime field since this is required for some of the versions of syntomic cohomology we will be using. All schemes will be separated and of finite type over their respective bases. We describe as little as we need of the general theory, sending the interested reader to [Bes00a]. All versions of syntomic cohomology are defined as cohomologies of certain huge complexes. These are needed for the definition of the regulators but are useless when it comes to calculations. The cohomology can, however, be realized, using some auxiliary data, as the cohomology of very explicit complexes, and maps on cohomology can similarly be realized explicitly. The theory developed in loc. cit. guarantees that these explicit maps are indeed the correct maps and we avoid explicit mentioning of that in the sequel.

For the purpose of this work, the version best suited for computations is the Gros style modified syntomic cohomology denoted by $\check{H}_{ms}$ in [Bes00a]. This is the weakest version of syntomic cohomology and all other versions, in particular $H_{syn}$ have natural maps to it [Bes00a, Proposition 9.5] which, by definition, are compatible with Chern classes in algebraic $K$-theory. Fortunately, according to [Bes00a, Proposition 8.6.3], when $X/R$ is proper and $2n \neq i, i-1, i-2$, the canonical map $H_{syn}^i(X,n) \to H_{ms}^i(X,n)$ is an isomorphism. Therefore, for the purpose of detecting syntomic regulators in the cohomology of $\text{Spec}(R)$ working with $\check{H}_{ms}$ is just as good
as working with $H_{\text{syn}}$. To further simplify matters, we only give the description of $H_{\text{ms}}$ given certain additional data that may not exist in general but do exist in our situation.

Suppose first that $X$ is a scheme over a field $\kappa$ of characteristic $p$. Following Berthelot we define the rigid complex of $X$ over $K$ as follows: We choose an open immersion $X \xrightarrow{j} \hat{X}$ into a proper $\kappa$-scheme and a closed immersion $\hat{X} \to \mathcal{P}$ into a $p$-adic formal $R$-scheme which is smooth in a neighborhood of $X$. We remark that in general there may be some difficulty doing this but in the cases we will consider it will be totally obvious how to do so.

In the above situation we can, following Berthelot, define the complex

$$\mathbb{R}\Gamma_{\text{rig}}(X/K)_{\mathcal{P}} := \mathbb{R}\Gamma(\hat{X}|_\mathcal{P}, j^!\Omega^\bullet_{\hat{X}/\kappa}) .$$

Here, the notation $|\hat{X}|_\mathcal{P}$ stands for the tube of $\hat{X}$ in $\mathcal{P}$, which roughly means the space of points in the rigid analytic space associated to $\mathcal{P}$ that reduce to a point in $\hat{X}$. The functor $j^!$ of “sections of overconvergent support” goes from the category of abelian sheaves on $|\hat{X}|_\mathcal{P}$ to itself and is defined by

$$j^!(F) = \lim_{U} (j_U)_*(F|_U) ,$$

where the direct limit is over all $U$ which are strict neighborhoods of $|X|_\mathcal{P}$ in $|\hat{X}|_\mathcal{P}$ in the sense of Berthelot and $j_U$ is the inclusion of $U$ in $|\hat{X}|_\mathcal{P}$. We recall that $U$ is a strict neighborhood if $\{U, |\hat{X}|_\mathcal{P} - |X|_\mathcal{P}\}$ is an admissible cover of $|\hat{X}|_\mathcal{P}$ in the sense of rigid analysis. We have indexed the complex for simplicity by $\mathcal{P}$ but we should remember the entire setup leading up to the definition. In any case, Berthelot shows that in the derived category of $K$-vector spaces this complex is independent of all choices, so its cohomology, $H_{\text{rig}}(X/K)$, is entirely well defined. To simplify notation we will drop the $\mathcal{P}$ subscript from the notation. In the applications it will be clear which additional data is being used.

We will often need to let a ($\kappa$-linear) Frobenius act on our complexes. To do that we will consider a morphism $\varphi : \hat{X} \to \hat{X}$ which is a $\kappa$-linear base change from a model of $\hat{X}$ defined over a finite field with $q = p^r$ elements of the $r$-th power of the absolute Frobenius. We insist that $\varphi$ preserves $X$. Such a $\varphi$ is called a Frobenius endomorphism of $X$. We then assume that there is a lift $\phi$ of $\varphi$ to $\mathcal{P}$. We call $q$ the degree of $\varphi$ and $\phi$. It is then clear that $\phi$ acts on the rigid complex.

Next we describe the construction of the syntomic complexes. Here we assume that $X$ is a smooth $R$-scheme and that we have an open immersion $X \to \hat{X}$ into a proper $R$-scheme and a closed immersion $\hat{X} \to \mathcal{P}$ into a $R$-scheme, smooth in a neighborhood of $X$, and that there is a $R$-morphism $\phi : \mathcal{P} \to \mathcal{P}$ inducing on the special fiber a Frobenius endomorphism. In this situation we can clearly embed $X_\kappa$ into $\hat{X}_\kappa$ and this last scheme into the $p$-adic completion $\hat{P}$ of $\mathcal{P}$ to get to the situation we had when we defined the rigid complex and $\phi$ will induce a lift of a Frobenius endomorphism.

The given data induces a filtration on the complex $\mathbb{R}\Gamma_{\text{rig}}(X_\kappa/K)$ defines as follows: let $J$ be the sheaf of ideals defining the generic fiber $\hat{X}_K$ inside $|\hat{X}_\kappa|_{\hat{P}}$ and consider the filtration of $\Omega^\bullet_{\hat{X}_\kappa|_{\hat{P}}}$ given by the complexes

$$F^n\Omega^\bullet_{\hat{X}_\kappa|_{\hat{P}}} := J^n\Omega^0 \to J^{n-1}\Omega^1 \to \cdots ,$$
where it is understood that $J^r = \mathcal{O}$ for non positive $r$. This filtration induces a filtration on the rigid complex by

$$F^n\mathbb{R}\Gamma_{rig}(X_\kappa/K) := \mathbb{R}\Gamma([\mathcal{X}_\kappa, j^1 F^n \Omega^\bullet_{rig} |_{X_\kappa, [\mathcal{X}_\kappa]}]).$$

Berthelot shows that these complexes are again independent of the additional data up to quasi–isomorphism. We can now define the Gros style modified syntomic complex to be the complex

$$\widehat{\mathbb{R}}\Gamma_{ms}(X, n) := \text{Cone}(F^n\mathbb{R}\Gamma_{rig}(X_\kappa/K) \xrightarrow{1 - \phi^*/q^n} \mathbb{R}\Gamma_{rig}(X_\kappa/K))[-1].$$

The map in the cone is a shorthand for the composition of the indicated map $1 - \phi^*/q^n$ with the natural map of $F^n\mathbb{R}\Gamma_{rig}(X_\kappa/K)$ into $\mathbb{R}\Gamma_{rig}(X_\kappa/K)$. To fix notation for Cones we use the following sign convention here. If $f : A^\bullet \to B^\bullet$, then $\text{Cone}(A^\bullet \to B^\bullet)[-1]$ is given in degree $i$ by

$$A^i \oplus B^{i-1}, \quad \text{with differential } d(a, b) = (da(f(a) - db).$$

One can show that $\widehat{\mathbb{R}}\Gamma_{ms}(X, n)$ is independent of the additional data except for the choice of the Frobenius endomorphism $\phi$. Here, in the general case one takes a direct limit over all possible Frobenius endomorphisms as described in [Bes00a, Definition 8.4]. When $X = R$ all the connecting homomorphisms of the limit are quasi–isomorphisms so we may in fact fix a single $\phi$.

In [Bes00a] syntomic regulators from the K-theory of $X$ into the various versions of syntomic cohomology were constructed. For the cohomology theory we are considering these take the form of a map

$$c_{i,j} : K^1_X \to \widehat{\mathbb{H}}^{2i-j}(X, i).$$

In this work we will need to consider similar maps in the relative and multi-relative situations. These were not constructed in loc. cit. but are constructed in Appendix A.

We recall the computation of the regulator on a part of the K-theory of affine $R$-schemes. Suppose $X = \text{Spec}(A)$ is such a scheme. We will give an explicit description of the rigid and syntomic cohomology of $X$. We can choose an embedding of $X$ as an open subset in the projective $P = \mathcal{X}$. Suppose $X_\kappa$ is defined in $\mathcal{X}_\kappa$ by the nonvanishing of the reductions of functions $h_i$. Then for $\lambda < 1$ we define a rigid space $U_\lambda$ by the conditions $|h_i| > \lambda$. The $U_\lambda$ are strict neighborhoods of $|X_\kappa, [\mathcal{X}_\kappa]|$ in $|\mathcal{X}_\kappa, [\mathcal{X}_\kappa]|$. It follows that there exists a map

$$\lim_{\lambda < 1} \Gamma(U_\lambda, \Omega^\bullet) \to \mathbb{R}\Gamma_{rig}(X_\kappa/K).$$

**Proposition 4.2.** This map is a quasi–isomorphism. In addition, this quasi–isomorphism is functorial with respect of maps of pairs $(X, \mathcal{X})$.

**Proof.** The first statement follows from the proof of Proposition 1.10 in [Ber97]. The second statement is a consequence of the construction of the rigid complexes in [Bes00a].

To obtain the modified syntomic complex, suppose we have a map $\phi : \mathcal{X} \to \mathcal{X}$ whose reduction is a Frobenius endomorphism fixing $X_\kappa$. The ideal $J$ considered above is the 0 ideal in this case. We thus get a quasi–isomorphism

$$\text{Cone}\left(\lim_{\lambda} \Gamma(U_\lambda, \Omega^{\geq n}) \xrightarrow{1 - \phi^*/q^n} \lim_{\lambda} \Gamma(U_\lambda, \Omega^\bullet) \right)[-1] \cong \widehat{\mathbb{R}}\Gamma_{ms}(X, n).$$
We formally write $U$ for the system of spaces $\{U_\lambda\}$ and define

\begin{equation} \Omega(U) := \lim_{\lambda \to} \Gamma(U_\lambda, \Omega^i). \end{equation}

It follows that

\begin{equation} \tilde{H}_{\text{ms}}^i(X, n) = \frac{\{(\omega, \varepsilon) : \omega \in F^n\Omega^i(U), \varepsilon \in \Omega^{i-1}(U), d\omega = 0, d\varepsilon = (1 - \phi^* q^n)\omega\}}{\{(d\omega, (1 - \phi^* q^n)\omega - d\varepsilon) : \omega \in F^n\Omega^{i-1}(U), \varepsilon \in \Omega^{i-2}(U)\}}, \end{equation}

with $F^n\Omega^j(U) = 0$ if $n < j$ and $\Omega^j(U)$ otherwise.

As mentioned in the introduction, in many cases syntomic cohomology becomes isomorphic to rigid cohomology. The normalization of this isomorphism is perhaps not the obvious one and since the computation of the regulator depends on the particular normalization, we describe this here at least in a special case (see [Bes00a] for a fuller discussion). Suppose that $X$ has relative dimension $i - 1$ over $R$. Suppose in the description above that $(\omega, \varepsilon) \in \tilde{H}_{\text{ms}}^i(X, n)$. We see that $\omega = 0$ so $d\varepsilon = 0$. Thus $\varepsilon$ defines a class in $H^{i-1}_{\text{rig}}(X_\kappa/K)$ which is easily seen to be well defined up to an element of $(1 - \varphi^*/q^n)F^nH^{i-1}_{\text{rig}}(X_\kappa/K)$. When $n \geq i > 0$, the map $X$ will be the case for us, the map

\begin{equation} (1 - \varphi^*/q^n) : H^{i-1}_{\text{rig}}(X_\kappa/K)/F^nH^{i-1}_{\text{rig}}(X_\kappa/K) \to H^{i-1}_{\text{rig}}(X_\kappa/K)/(1 - \varphi^*/q^n)F^nH^{i-1}_{\text{rig}}(X_\kappa/K) \end{equation}

is an isomorphism by [Bes00a, Proposition 8.6.3].

**Definition 4.6.** When $n \geq i > 0$, we have a canonical isomorphism,

$$\tilde{H}_{\text{ms}}^i(X, n) \cong H^{i-1}_{\text{rig}}(X_\kappa/K)/F^n, \quad (0, \varepsilon) \mapsto (1 - \varphi^*/q^n)^{-1}(\text{class of } \varepsilon).$$

The justification for this normalization requires a longer tour into the general theory of syntomic cohomology than we would like to present. The reader may refer to [Bes00a, Proposition 10.1.3] for example. In any case, note that this choice is functorial. We will make this definition in relative situations as well.

We now describe the regulator in this special case. First of all, consider $f \in A^*$. If $\tilde{f}$ is the reduction of $f$ one finds that $\varphi^* \tilde{f} = \tilde{f}^q$ and therefore $f_0 := f^q/\phi^* f$ is congruent to $1$ mod the maximal ideal of $R$. One can deduce from that that the function $\log(f_0)$ is analytic on some $U_\lambda$.

**Lemma 4.7.** ([Bes00a, Proposition 10.3]) The syntomic regulator sends the class of $f$ in $K_1(X)$ to the cohomology class of $(d\log f, \log(f_0)/q)$ in the representation (4.4) of $\tilde{H}_{\text{ms}}^i(X, 1)$.

The value of the regulator on a cup product $f_1 \cup \cdots \cup f_r$ in $K_n(X)$ is the cup product of the regulators of the $f_j$’s, so it is enough to describe the cup product on syntomic cohomology. This is given, in the notation of (4.4), by any of the
formulas, depending on the parameter $\gamma$,
\[
(\omega, \varepsilon) \cup (\omega_2, \varepsilon_2) = (\omega_1 \wedge \omega_2, \\
\varepsilon_1 \wedge \left( \gamma + (1 - \gamma) \left( \frac{\phi^*}{q} \right)^k \right) \omega_2 \\
+ (-1)^{\deg \omega_1} \left( \left( 1 - \gamma \right) + \gamma \left( \frac{\phi^*}{q} \right)^k \right) \omega_1 \wedge \varepsilon_2).
\]

(4.8)

We need to describe the pullback map in syntomic cohomology in certain special situations. Suppose that $X$ is an affine scheme and $f : Y \to X$ is a closed embedding on an affine subscheme and choose the same auxiliary data for $X$ as before. We may compactify $Y$ by embedding it into its closure $\overline{Y}$ in $\overline{X}$. The difficulty in describing the pullback map from $X$ to $Y$ is that the lift of Frobenius morphism $\phi$ will not preserve $\overline{Y}$ in general. Note however that we may and do assume that $\phi$ preserves $Y_\kappa$. The way to overcome the difficulty is as follows: we use the embedding of $\overline{Y}$ into $\overline{X}$ to compute the syntomic complex of $Y$. This gives us the following model for $\tilde{\Gamma}_\kappa(Y, n)$,
\[
\tilde{\Gamma}_\kappa(Y, n) \cong \text{Cone} \left( \text{R} \Gamma \left( \overline{Y}_\kappa \left[ \frac{1}{X} \right] f^! \Omega^1_{\overline{X}, \kappa, \left[ \frac{1}{X} \right]} \right) \xrightarrow{1-\phi^*/q^n} \text{R} \Gamma \left( \overline{Y}_\kappa \left[ \frac{1}{X} \right] f^! \Omega^1_{\overline{X}, \kappa, \left[ \frac{1}{X} \right]} \right) \right)[-1],
\]
and the pullback map is now simply obtained by restriction to the tube $\left| \overline{Y}_\kappa \left[ \frac{1}{X} \right] \right.$. Here $J$ is the ideal of $\overline{Y}_K$ in $\overline{X}_K$.

Suppose now that $Y$ is of relative dimension $i - 1$ over $R$ and that we are given an element of $H^1_{\text{ms}}(X, n)$ represented by the pair of forms $(\omega, \varepsilon)$ as in (4.4). We would like to study the pullback of this element to $Y$, identified with an element of $H^1_{\text{rig}}(Y_\kappa/K)/F^n$ as in Definition 4.6. Note that this pullback does not factor through $H^1_{\text{rig}}(X_\kappa/K)/F^n$ because $X$ is of higher dimension than $Y$ in general. Recalling the sets $U_\lambda$ we see that for each $\lambda$ the set $U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right]$ is a strict neighborhood of $\left| \overline{Y}_\kappa \left[ \frac{1}{X} \right] \right.$ in $\overline{Y}_\kappa \left[ \frac{1}{X} \right]$. It follows that we may factor the map $\Gamma_{\text{rig}}(X_\kappa/K) \to \Gamma_{\text{rig}}(Y_\kappa/K)$, respectively $F^n \Gamma_{\text{rig}}(X_\kappa/K) \to F^n \Gamma_{\text{rig}}(Y_\kappa/K)$ as
\[
\lim_{\lambda} \Gamma(U_\lambda, \Omega^*) \to \lim_{\lambda} \Gamma(U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right], \Omega^*) \to \text{R} \Gamma \left( \overline{Y}_\kappa \left[ \frac{1}{X} \right] f^! \Omega^*_{\overline{X}, \kappa, \left[ \frac{1}{X} \right]} \right),
\]
respectively with $\Omega^*$ replaced by $F^n \Omega^*$. We may therefore factor the map of syntomic complexes $\tilde{\Gamma}_\kappa(X, n) \to \tilde{\Gamma}_\kappa(Y, n)$ via
\[
\text{Cone} \left( \lim_{\lambda} \Gamma(U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right], f^! \Omega^*) \xrightarrow{1-\phi^*/q^n} \lim_{\lambda} \Gamma(U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right], \Omega^*) \right)[-1].
\]

(4.9)

Lemma 4.10. In the situation described above let $\theta \in \lim_{\lambda} \Gamma(U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right], F^n \Omega^{i-1})$ be such that $d\theta = \omega \mid \overline{Y}_\kappa \left[ \frac{1}{X} \right]$. Then the image of $f^*(\omega, \varepsilon)$ in
\[
H^1_{\text{rig}}(Y_\kappa/K)/(1 - \phi^*/q^n)F^n H^1_{\text{rig}}(Y_\kappa/K)
\]
is the same as the image of $\varepsilon \mid \overline{Y}_\kappa \left[ \frac{1}{X} \right] - (1 - \phi^*/q^n)\theta \in \lim_{\lambda} \Gamma(U_\lambda \cap \overline{Y}_\kappa \left[ \frac{1}{X} \right], \Omega^{i-1})$ in the same group.

Proof. Subtract the boundary $d(\theta, 0) = (d\theta, (1 - \phi^*/q^n)\theta)$ from $(\omega, \eta) \mid \overline{Y}_\kappa \left[ \frac{1}{X} \right]$ in (4.9).
Finally we specialize even further and show how to compute the difference of the pullbacks at two nearby points. We assume that \( i = n \). Consider a situation where we are given an affine \( X \) together with a smooth affine map \( \pi : X \to B \) to another affine scheme \( B \) smooth over \( R \). Suppose that \( \pi \) extends to \( \tilde{\pi} : \tilde{X} \to B \). Let \( z' \in B(\kappa) \) and let \( D \) be the rigid analytic space of all points in \( B \) reducing to \( z' \) (this is the residue disc of \( z' \) in the terminology of Coleman). For any \( z \in D(K) \) let \( f_z \) be the embedding of \( Y_z := \tilde{\pi}^{-1}(z) \) in \( X \). The \( Y_z \) for \( z \in D(K) \) have a common reduction which we denote by \( Y_{\kappa} \), and the \( \tilde{\pi}^{-1}(z) \) have a common reduction \( \overline{\mathcal{Y}}_{\kappa} \), which is a compactification of \( Y_{\kappa} \). Finally, the tube \( \overline{\mathcal{Y}}_{\kappa}[\overline{\mathcal{X}}] \) is simply \( \tilde{\pi}^{-1}(D) \).

**Proposition 4.11.** In the situation described above let \((\omega, \varepsilon)\) represent a class in \( H^1_{\text{rig}}(X, i) \). Let \( z_1, z_2 \in D(K) \), let \( J_j \) be the ideal defining \( Y_{z_j} \), and let \( \theta_j \) in \( \lim \lambda \Gamma(U_\lambda \cap \tilde{\pi}^{-1}(D), F^j_\lambda \Omega^{-1}) \) be such that \( d\theta_j = \omega|_{\tilde{\pi}^{-1}(D)} \). Then, the difference of the images in \( H^1_{\text{rig}}(Y_{\kappa}/K) \) of the pullbacks \( f_{z_j}^*(\omega, \varepsilon) \), \( j = 1, 2 \), is the image of \( \theta_2 - \theta_1 \) in \( \lim \lambda \Gamma(U_\lambda \cap \tilde{\pi}^{-1}(D), \Omega^{-1}) \).

**Proof.** As seen in Lemma 4.10, the image in \( H^1_{\text{rig}}(Y_{\kappa}/K)/(1 - \varphi^*/q^n)F^1H^1_{\text{rig}}(Y_{\kappa}/K) \) is the image of \( (1 - \varphi^*/q^n)(\theta_2 - \theta_1) \) and the result thus follows from Definition 4.6 of the image in \( H^1_{\text{rig}}(Y_{\kappa}/K) \).

In the situation as above the following immediate Corollary will also be useful. For any point \( x \in D(K) \) the fiber \( \tilde{\pi}^{-1}(x) \) is a lift of \( \overline{\mathcal{Y}}_{\kappa} \) and therefore the rigid cohomology of \( Y_{\kappa} \) can be computed as the cohomology of \( \lim \lambda \Gamma(U_\lambda \cap \tilde{\pi}^{-1}(x), \Omega^*) \).

**Corollary 4.12.** In \( H^1_{\text{rig}}(Y_{\kappa}/K) \) the difference \( f_{z_1}^*(\omega, \varepsilon) - f_{z_2}^*(\omega, \varepsilon) \) is the image of \( (\theta_2 - \theta_1)|_{U_\lambda \cap \tilde{\pi}^{-1}(x)} \) in \( \lim \lambda \Gamma(U_\lambda \cap \tilde{\pi}^{-1}(x), \Omega^{-1}) \).

All of our considerations are also valid for the cohomology of diagrams of schemes, and in particular for the relative and multi–relative cohomologies that will be considered in Sections to come.

5. The integration down process

A key ingredient in the computation of the regulator will be a functional on rigid cohomology obtained by repeated integration, which we now go on to describe.

Let \( \kappa \) be a field of characteristic \( p \). Set \( X^n = (\mathbb{P}_\kappa^n - \{ t = 1 \})^n \). Let \( B \) be an affine \( \kappa \)-variety. Let \( Y \) be an open affine subset of \( X^n \times B \). Let \( \mathfrak{O}^Y \) be the subset of \( Y \) where at least one of the coordinates is either 0 or \( \infty \). We write cohomology relative to \( \mathfrak{O}^Y \) to mean the multi relative cohomology taken in exactly the same way as was done in Section 3 for \( K \)-theory. We would like to write an explicit complex computing the multi relative rigid cohomology of \( Y \).

We first choose the rigid data \( X^n \to (\mathbb{P}_\kappa^1)^n \to (\mathbb{P}_{\text{Spf}(R)}^1)^n \) and \( B \to \overline{B} \to P_B \) where \( \overline{B} \) and \( P_B \) are projective spaces of some degree over \( \kappa \) and \( \text{Spf}(R) \) respectively. We thus obtain a rigid datum for \( Y \) as well. As in Section 4 we obtain for \( \lambda < 1 \) a certain inverse system of rigid space \( U = \{ U_\lambda^Y \} \). We know that there exists a canonical quasi–isomorphism \( \Omega^*(U^Y) := \lim \lambda \to 1 \Omega^*_\lambda(U^Y) \to \mathbb{R}\Gamma_{\text{rig}}(X^n/K) \). Similarly, the complexes of rigid forms on the subspace of \( U_\lambda^Y \) cut out by equations of the form \( t_i = 0 \) or \( t_i = \infty \) are quasi–isomorphic to the rigid complexes of the various components in \( \mathfrak{O}^Y \).

As discussed (at length) in Appendix A, we can now write a complex quasi–isomorphic to the multi relative \( \mathbb{R}\Gamma_{\text{rig}}(X^n; \mathfrak{O}^Y/K) \) by taking iterated cones on the
We can similarly define rigid spaces $U_{f,Y}^n$ forming an inverse system $U_{f,Y}^n$ and like (4.3) we can formally define complexes of differential forms $\Omega^\bullet$.

We can similarly define rigid spaces $U_{f,Y}^n$ forming an inverse system $U_{f,Y}^n$ and like (4.3) we can formalize define complexes of differential forms $\Omega^\bullet(U_{f,Y}^n)$. Let us call $n - j$ the degree of $f$ (this includes the empty function $\emptyset$ with degree $n$), which is the same as the codimension of $Y_f$. The complex computing our multi relative cohomology can then be written as $\bigoplus_{i+\deg f=k} \Omega^i(U_{f,Y}^n)$ in degree $k$. Let us write an element in the $f$ component of this complex as $(\omega, f)$. Then the differential is given by

$$d(\omega, f) = (d\omega, f) - (-1)^{\deg \omega} \sum g \chi(f, g)(\omega|_{Y_f}, g)$$

where

$$\chi(f, g) = \begin{cases} (-1)^{f(r)+r} & \text{im } f = \text{im } g \cup f(r) \\ 0 & \text{otherwise,} \end{cases}$$

and where here $\cup$ denotes disjoint union.

**Definition 5.2.** The complex above is denoted $\Omega^\bullet(U_{f,Y}; \square^n)$. We let $F^i \Omega^\bullet(U_{f,Y}; \square^n)$ be the subcomplex having $\bigoplus_{i+\deg f=k} \Omega^i(U_{f,Y}^n)$ in degree $k$.

The following Lemma is mostly an exercise in sign fixing.

**Lemma 5.3.** The complexes $F^i \Omega^\bullet(U_{f,Y}; \square^n)$ and $F^i \Gamma_{\text{rig}}(Y; \square^n/K)$ are quasi-isomorphic.

**Proof.** We prove this without the filtrations. The result for the filtered part is then clear. First we note that by Proposition 4.2 this complex is quasi-isomorphic to the corresponding complex with $\Omega^i(U_{f,Y}^n)$ replaced by the degree $i$ part of $\Gamma_{\text{rig}}(Y_f/K)$ which we now denote by $\Gamma^i$ for simplicity. Consider now the double complex introduced in the appendix (compare (A.13) and (A.11)) For $\beta : [1, k] \to [1, n]$ an increasing function define $Y_{\beta} = \{(x_1, \ldots, x_n) \in Y : x_{\beta(i)} \in \{0, \infty\}\}$. Then the complex in degree $q$ is $\bigoplus_{k+|\beta|=q} \Gamma^k(Y_{\beta})$. We can again write elements there as pairs $(\omega, \beta)$ and the differential is defined by

$$d(\omega, \beta) = (d\omega, \beta) + (-1)^q (-1)^{|\beta|} \sum_{\beta'} \chi'(\beta, \beta')(\omega|_{Y_{\beta'}}, \beta')$$

where

$$\chi'(\beta, \beta') = \begin{cases} (-1)^r & \text{im } \beta' = \text{im } \beta \cup \beta'(r) \\ 0 & \text{otherwise,} \end{cases}$$

Now we want to switch to a dual point of view. The relation is as follows: For $f$ a function as in the Lemma we define $\beta(f)$ to be the increasing function enumerating $[1, \ldots, n] - \text{im } f$. Then we have $|\beta(f)| = \deg f$ and $Y_f = Y_{\beta(f)}$. The key thing to check is the following: If $g$ is obtained from $f$ by deleting $f(r)$, then $\beta = \beta(f)$ is obtained from $\beta' = \beta(g)$ by deleting $\beta'(f(r) - r + 1)$. This easily gives the result.
Consider now the case where $B = \text{Spec}(\kappa)$ and $Y = X^n$. Then the relative rigid cohomology $H^n_{\text{rig}}(X^n; \mathcal{O}^n/K)$ is well known to be isomorphic to $K$. We can explicitly describe this isomorphism. The basic idea (compare [Gro94]) is of iterated integration between $0$ and $\infty$. We can take $U^\alpha_\lambda$ to be $U^n_\lambda$ where $U_\lambda$ denotes the space $\mathbb{P}^1_{K^\lambda} - \{|t-1| < \lambda\}$ and $U^n_\lambda$ is the $n$th power of $U_\lambda$. Let $(\omega, f) \in \Omega^n(U^n_\lambda; \mathcal{O}^n)$ and suppose that
\[
\omega = G(t_{f(1)}, \ldots, t_{f(j)}) dt_{f(1)} \wedge \cdots \wedge dt_{f(j)}
\]
(here the ordering is critical). Define
\[
\pi(\omega, f) = \int_0^\infty \cdots \int_0^\infty G(t_{f(1)}, \ldots, t_{f(j)}) dt_{f(1)} \cdots dt_{f(j)}.
\]

Notice that now the order is not critical and we can integrate in whatever order we want. Let $H_{\text{dR}}(U^n_\lambda; \mathcal{O}^n)$ be the homology of $\Omega^*(U^n_\lambda; \mathcal{O}^n)$. We have the following.

**Lemma 5.4.** There is a unique isomorphism $H^n_{\text{dR}}(U^n_\lambda; \mathcal{O}^n) \to K$ normalized by the condition that on the class of a closed form $(\omega, f)$ with $\deg f = 0$ it is given by $\pi(\omega, f)$. This functional is given as follows: consider a form $(\eta, g)$ where $g$ has degree $m$ and of the $m$ coordinates on $U^n_\lambda$ which are fixed $i$ are fixed to be $\infty$. Set
\[
\Pi((\eta, g)) = (-1)^{i+\sum (g(k)+k)} \pi(\eta, g).
\]
Then the functional is given by the restriction of the $K$-linear extension of $\Pi$ to closed forms.

**Proof.** We can find a form $(\omega, f)$, with $\deg(f) = 0$, whose cohomology class is non-trivial. The required isomorphism is determined by its value on such a form and is therefore unique. To show that $\Pi$ provided the required map, we only need to show, in view of the fact that $\Pi(\omega, f) = \pi(\omega, f)$ when $\deg(f) = 0$, that it kills exact forms. The exact forms are spanned by forms
\[
d\left(\left(\begin{array}{c}
F(t_{g(1)}, \ldots, t_{g(j)}) \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot g
\end{array}\right)\right)
\]
\[= (-1)^{k-1} \left(\frac{\partial}{\partial t_{g(k)}} F \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot g\right)
\]
\[+ (-1)^{j+g(k)+k} \left(F|_{\{t_{g(k)}\in (0, \infty)\}} \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot h\right)
\]
with $h$ obtained from $g$ by removing the $k$-th value and $F$ a function on a component of $U^n_\lambda$ with $i$ coordinates forced to $\infty$. Notice that
\[
\pi\left(\frac{\partial}{\partial t_{g(k)}} F \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(j)} \cdot g\right)
\]
\[= \pi(F|_{t_{g(k)}=0} \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot h)
\]
\[\pi(F|_{t_{g(k)}=\infty} \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot h)
\]
because in the computation of $\pi$ we can begin the integration on the $g(k)$ coordinate. Now call the two terms on the right hand side of the last equation $\alpha_0$ and $\alpha_\infty$ respectively, and write the possible sign in the definition of $\Pi$ as $\text{Sign}(g, i)$. We therefore obtain
\[
\Pi\left(d\left(\left(\begin{array}{c}
F(t_{g(1)}, \ldots, t_{g(j)}) \cdot dt_{g(1)} \wedge \cdots \wedge dt_{g(k)} \wedge \cdots \wedge dt_{g(j)} \cdot g
\end{array}\right)\right)\right)
\]
\[= (-1)^{k-1} \text{Sign}(g, i)(\alpha_0 - \alpha_\infty) + (-1)^{j+g(k)+k} \left(\text{Sign}(h, i)\alpha_0 + \text{Sign}(h, i+1)\alpha_\infty\right).
\]
Thus, clearly, to make this cancel, we need to choose $\text{Sign}(g, i) = (-1)^{\text{sign}(g) + i}$ with $\text{sign}(g)$ satisfying the relation

$$\text{sign}(g) + k - 1 \equiv \text{sign}(h) + j + g(k) + k + 1 \pmod{2}$$

(the last 1 is there to make this alternating) when $h$ is obtained from $g$ by deleting $g(k)$. After cancellations this becomes

$$\text{sign}(g) \equiv \text{sign}(h) + g(k) + j \pmod{2}.$$  

It is easily seen that $\text{sign}(g) = \sum(g(k) + k)$ satisfies this condition, which completes the proof. \[\square\]

Like in $K$–theory (see the derivation of (3.1) or [dJ95, page 220]) the isomorphism $H_{\text{rig}}^n(X^n; \square^K / K) \cong K$ can be obtained by a repeated application of boundary maps. At each stage there is a choice of signs. Here we have taken the approach of writing down the isomorphism $\Pi$ directly and we would now like to know how it can be obtained using boundary maps.

We have a short exact sequence

$$0 \to H_{\text{rig}}^{n-1}(X^n; \square^{n-1} / K) \to H_{\text{rig}}^n(\square^n; \square^{n-1} / K) \to H_{\text{rig}}^n(X^n; \square^n / K) \to 0,$$

and an isomorphism $H_{\text{rig}}^n(\square^n; \square^{n-1} / K) \cong H_{\text{rig}}^n(X^n; \square^{n-1} / K)$ under pullback. It follows from this that the following two composed maps are isomorphisms:

$$H_{\text{rig}}^n(X^n; \square^{n-1} / K) \to H_{\text{rig}}^n(\square^n; \square^{n-1} / K) \to H_{\text{rig}}^n(X^n; \square^n / K).$$

Here, the two maps come from the two possible choices of the embeddings of $(X^n; \square^{n-1})$ in $(\square^n; \square^{n-1})$ as either $x_n = 0$ or $x_n = \infty$ and they differ by a minus sign. Iterating this we get an isomorphism

$$(5.5) \quad K = H_{\text{rig}}^0(* / K) \xrightarrow{\sim} H_{\text{rig}}^1(X^1; \square^{n-1} / K) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H_{\text{rig}}^n(X^n; \square^n / K).$$

**Proposition 5.6.** The composed map $K \xrightarrow{(5.5)} H_{\text{rig}}^n(X^n; \square^n / K) \xrightarrow{\Pi} K$ is the identity provided at each stage we choose the embedding as $x_i = 0$.

**Proof.** From the proof of Proposition A.12 it is not difficult to get the following explicit description of the map $\Omega^\bullet(\square^n; \square^{n-1}) \to \Omega^{\bullet + 1}(X^n; \square^n)$ (dual to the map $\square(C_\bullet) \to \square(Y_\bullet)[-1]$ in the notation of the proof of Proposition A.12): it is simply given by $(\omega, f) \mapsto (\omega, f)$ where $f : [1, \ldots, j] \to [1, \ldots, n-1]$ is considered on the right as a function $f : [1, \ldots, j] \to [1, \ldots, n]$. It follows that the map (5.5) with the choice of signs as in the Proposition corresponds to the map sending $\alpha \in K$ to $(\alpha, \emptyset)$ on the component with $x_i = 0$ for all $i$. Applying $\Pi$ to this we get $\alpha$. \[\square\]

Now comes a crucial point. In applications we will want to consider the cohomology not of $(X^n; \square^n)$ but rather of an open subset $Y$ obtained from $X^n$ by removing subsets of the form $\{t_j = u\}$ with $u \in \kappa^*$ (e.g., sets of the form $X^n_\text{loc}$ as in Section 3. That means that it is no longer possible to perform the integrals needed to construct $\Pi$ (and of course the isomorphism that $\Pi$ represents does not exist). It is sometimes possible, however, to replace the integral by a Coleman integral. We want to show that when this is possible it corresponds to an operation which can be made sense out of in general.

**Lemma 5.7.** There is a short exact sequence

$$0 \to H_{\text{rig}}^n(X^n; \square^n / K) \to H_{\text{rig}}^n(Y; Y \cap \square^n / K) \to E \to 0,$$
where Frobenius acts on $H^n_{\text{rig}}(X^n; \mathbb{Q}_p)$ trivially and on $E$ with strictly positive weights.

**Proof.** Write $H^i(\ast)$ for $H^i_{\text{rig}}(\ast/K)$). From the diagram of pairs

$$(X^n - Y; \mathbb{Q}_p - Y) \rightarrow (X^n; \mathbb{Q}_p^n) \leftarrow (Y; \mathbb{Q}_p^n)$$

We get the standard long exact sequence

$$\cdots \rightarrow H^n(X^n; \mathbb{Q}_p^n) \rightarrow H^n(Y, Y \cap \mathbb{Q}_p^n) \rightarrow H^{n+1}(X^n - Y; \mathbb{Q}_p^n) \rightarrow \cdots$$

The action of Frobenius on $H^n_{\text{rig}}(X^n; \mathbb{Q}_p/K)$ is trivial because the isomorphism $H^n_{\text{rig}}(X^n; \mathbb{Q}_p/K) \cong H^n_{\text{rig}}(pt/K)$ is Frobenius equivariant. To prove the Lemma we need to show that the first arrow indicated in the diagram is not 0 while the last term has strictly positive weights. The first assertion follows because it is easy to see that the same integration process described in Lemma 5.4 also vanishes on exact relative forms on the pair $(Y; \mathbb{Q}_p^n)$. It remains to show the statement about the weights. To do that we “peel off” the relativity step by step: We have a long exact sequence

$$\cdots \rightarrow H^n(t_{\in \{0, \infty\} - Y; \mathbb{Q}_p^n - Y}(\{t_n \in \{0, \infty\}; \mathbb{Q}_p^n) \rightarrow H^{n+1}(X^n - Y; \mathbb{Q}_p^n) \rightarrow \cdots$$

and the two terms on the sides fit into similar sequences. The key observation is that the degree of the cohomology is always one more than the dimension of the space. The final “building blocks” are of the form $H^{i+1}_{X^n - Y}(X^n)$. By [Chi98] such a term has weights between $i + 1$ and $2i$ (because $X^n - Y$ is always of codimension 1 by our assumptions) except that the term with $i = 0$ clearly vanishes. Thus all terms have positive weights.

**Corollary 5.8.** Let $M \subset H^n_{\text{rig}}(Y; \mathbb{Q}_p/K)$ be any Frobenius invariant subspace containing $H^n_{\text{rig}}(X^n; \mathbb{Q}_p/K)$. Then there exists a unique $K$-linear functional

$$\widetilde{\Pi}_M : M \rightarrow K$$

which is fixed under Frobenius and coincides with the functional induced by $\Pi$ on $H^n_{\text{rig}}(X^n; \mathbb{Q}_p/K)$.

Of course the conclusion is also true with $M = H^n_{\text{rig}}(Y; \mathbb{Q}_p/K)$ in which case we will denote $\widetilde{\Pi}_M$ simply by $\widetilde{\Pi}$. We will need the uniqueness statement, however, for possibly different subspaces.

The map $\widetilde{\Pi}$ gives a splitting of $V = H^n_{\text{rig}}(Y; \mathbb{Q}_p/K)$ into a direct sum $V = K \oplus E$ as a $\phi$-module, where $E$ has no $\phi$-fixed vectors. We will need a certain result about $\phi$-modules with such a structure.

**Lemma 5.9.** Let $V_i = K \oplus E_i$ for $i = 1, 2, 3$ be three $\phi$-modules such that $\phi$ has no invariant vectors on $E_i$ for each $i$ and on $E_1 \otimes E_2$. Let $\Pi_1 : V_1 \rightarrow K$ be the natural projection. Suppose there is a $\phi$-equivariant pairing $(\cdot, \cdot) : V_1 \otimes V_2 \rightarrow V_3$ which gives the usual multiplication when restricted to $K \otimes K$. Then we have $\Pi_3(x_1, x_2) = \Pi_1(x_1) \cdot \Pi_2(x_2)$.

**Proof.** The conditions of the Lemma imply that the algebraic multiplicity of 1 as an eigenvalue of $\phi$ on $V_1 \otimes V_2$ is 1. It follows that the space of $\phi$-invariant functionals on $V_1 \otimes V_2$ is 1-dimensional. Therefore the statement of the Lemma has to be true.
up to a multiplicative constant. This constant has to be 1 because the statement is true for $x_1 = 1$. □

By assumption all components of $Y$ are affine. We can therefore compute relative rigid cohomology using the complex $Ω^n(U^Y; \mathbf{k}^n)$ of Definition 5.2.

**Definition 5.10.** A relative form in $Ω^n(U^Y; \mathbf{k}^n)$ is called Coleman integrable if for each of its component $(ω, f)$ the expression defining $π(ω, f)$ makes sense when we replace ordinary integration with Coleman integration. If $x$ is such a form we denote by $Π_{Col}(x)$ the expression derived from the $π(ω, f)$ as in Lemma 5.4.

**Lemma 5.11.** Coleman integrable relative forms form a subspace of $Ω^n(U^Y; \mathbf{k}^n)$ which is closed under $ϕ$. Exact forms and forms extending to $U^n$ are Coleman integrable. The functional $Π_{Col}$ is $ϕ$-invariant.

**Proof.** The only thing which possibly requires proof is the fact that if $x$ is a relative form which is Coleman integrable, then so is $ϕ^∗(x)$ and $Π_{Col}(ϕ^∗(x)) = Π_{Col}(x)$. This is an easy explicit computation. We may assume that

$$x = (G(t_{f(1)}, \ldots, t_{f(j)}) \cdot dt_{f(1)} \wedge \cdots \wedge dt_{f(j)}, f).$$

Then

$$ϕ^∗(x) = (G(t^q_{f(1)}, \ldots, t^q_{f(j)}) \cdot d(t^q_{f(1)}) \wedge \cdots \wedge d(t^q_{f(j)}).$$

The assumption that $x$ is Coleman integrable means the following: There is a function $F_1(t_{f(1)}, \ldots, t_{f(j)})$ which is a Coleman function in the first variable and such that

$$\frac{∂}{∂t_{f(j)}} F_1 = G.$$ 

Setting $G_1(t_{f(2)}, \ldots, t_{f(j)}) = F_1(t_{f(1)} = 0, t_{f(1)} = ∞)$ we can find a function $F_2(t_{f(2)}, \ldots, t_{f(j)})$ which is again Coleman in the first variable such that

$$\frac{∂}{∂t_{f(2)}} F_2 = G_1$$

and we continue like this until we reach $G_j$ which is just a number equaling $Π_{Col}(x)$. Now we start with

$$\tilde{G}(t_{f(1)}, \ldots, t_{f(j)}) = G(t^q_{f(1)}, \ldots, t^q_{f(j)})q^{-1}_{f(1)} \cdots q^{-1}_{f(j)}.$$ 

The functoriality of the Coleman integral implies that we may take

$$\tilde{F}_1(t_{f(1)}, \ldots, t_{f(j)}) = F_1(t^q_{f(1)}, \ldots, t^q_{f(j)})q^{-1}_{f(1)} \cdots q^{-1}_{f(j)}.$$ 

Then, as $0^q = 0$ and $∞^q = ∞$ we get

$$\tilde{G}_1(t_{f(2)}, \ldots, t_{f(j)}) = G_1(t^q_{f(2)}, \ldots, t^q_{f(j)})q^{-1}_{f(2)} \cdots q^{-1}_{f(j)}$$

and we can continue this process until we find $Π_{Col}(ϕ^∗(x)) = Π_{Col}(x).$ □

We call a cohomology class in $H^n_{rig}(Y; \mathbf{k}^n/K)$ Coleman integrable if it is represented by a Coleman integrable form. Let $M_{Col}$ denote the space of Coleman integrable cohomology classes. It is an immediate consequence of the above that $Π_{Col}$ induces a functional $M_{Col} \rightarrow K$ which is Frobenius invariant. By the construction it is also clear that $Π_{Col}$ is just $Π$ on forms that extend to $U^n$. By corollary 5.8 this functional must coincide with the restriction to $M_{Col}$ of $Π$. We therefore obtain
\textbf{Proposition 5.12.} For any Coleman integrable form $x$ representing a cohomology class $[x]$ we have $\Pi_{\text{Gal}}(x) = \Pi([x])$.

6. REGULATORS FOR SPECIAL ELEMENTS

Recall that we have universal symbols (3.10)

$$[S]_n \in K^{(n)}_n(X^{n-1}_{G,\text{loc}}; \square^{n-1})$$

where $G = \text{Spec}(\mathbb{Z}[S, S^{-1}, (1 - S)^{-1}])$. Now let $B = \text{Spec}(R[S, S^{-1}, (1 - S)^{-1}])$. Pulling back via the canonical map $B \rightarrow G$ we obtain elements, for which we retain the notation,

$$[S]_n \in K^{(n)}_n(X^{n-1}_{B,\text{loc}}; \square^{n-1}) .$$

In this Section we obtain some information on the regulators

$$\text{reg}([S]_n) \in \tilde{H}_{\text{ms}}^{n}(X^{n-1}_{B,\text{loc}}; \square^{n-1}, n) .$$

We embed $X^{n-1}_{\text{loc}}$ in $P = (P_{B,\text{loc}})^{n-1} \times P_{B,\text{loc}}$ (B is mapped to the last coordinate). Taking the special fiber corresponds to the compactification discussed at the beginning of Section 5. Therefore, we obtain certain rigid subspaces $U_{\lambda}$ of $P_{K}$. We denote the inverse system of these by $U^{n-1}_{B,\text{loc}}$ and we have complexes $\Omega^{*}(U^{n-1}_{B,\text{loc}}; \square^{n-1})$ and $F^{*}\Omega^{*}(U^{n-1}_{B,\text{loc}}; \square^{n-1})$ as in Definition 5.2. A map $\phi$ whose reduction is a Frobenius endomorphism and which is compatible with all boundaries is given by raising to $q$’th power for a sufficiently large $q$. One checks easily that the sign convention for cones (4.1) is such that it commutes with taking the complex computing multi relative cohomology. From Lemma 5.3 we therefore have a canonical quasi-isomorphism:

$$\text{Cone} \left(F^{j}\Omega^{*}(U^{n-1}_{B,\text{loc}}; \square^{n-1}) \xrightarrow{1-\phi^{*}/q^{j}} \Omega^{*}(U^{n-1}_{B,\text{loc}}; \square^{n-1}) \right) \cong \tilde{\Gamma}_{\text{ms}}(X^{n-1}_{B,\text{loc}}; \square^{n-1}, j)$$

From degree considerations it is very easy to see that

$$F^{n}\Omega^{n}(U^{n-1}_{B,\text{loc}}; \square^{n-1}) = (\Omega^{n}(U^{n-1}_{B,\text{loc}}); [1, \ldots , n - 1] \rightarrow [1, \ldots , n - 1]) .$$

We can identify this space with $\Omega^{n}(U^{n-1}_{B,\text{loc}})$. On the other hand, $F^{n}\Omega^{n-1}(U^{n-1}_{B,\text{loc}}; \square^{n-1}) = 0$. Thus we obtain (compare (4.4)) the following expression, with the identification made above.

\begin{equation}
(6.1) \quad \tilde{H}_{\text{ms}}^{n}(X^{n-1}_{B,\text{loc}}; \square^{n-1}, n)
\end{equation}

$$= \left\{ (\omega, \varepsilon) : \omega \in \Omega^{n}(U^{n-1}_{B,\text{loc}}), \varepsilon \in \Omega^{n-1}(U^{n-1}_{B,\text{loc}}; \square^{n-1}), d\omega = 0, \varepsilon = (1 - \phi^{*}/q^{n})\omega \right\} \quad \text{over} \quad \left\{ (0, d\varepsilon), \varepsilon \in \Omega^{n-2}(U^{n-1}_{B,\text{loc}}; \square^{n-1}) \right\} .$$

We take this opportunity to consider two other situations that will be needed later. In these cases we compute the syntomic cohomology of $(X^{n}_{\text{loc}}; \square^{n})$ and so there is no B present. The corresponding rigid space were already considered in previous Sections. We denote by $U^{n}_{\text{loc}}$ the space $U X^{n}_{\text{loc}}$. We then have

\begin{equation}
(6.2) \quad \tilde{H}_{\text{ms}}^{n}(X^{n}_{R,\text{loc}}; \square^{n}, n)
\end{equation}

$$= \left\{ (\omega, \varepsilon) : \omega \in \Omega^{n}(U^{n}_{\text{loc}}), \varepsilon \in \Omega^{n-1}(U^{n}_{\text{loc}}; \square^{n}), d\omega = 0, \varepsilon = (1 - \phi^{*}/q^{n})\omega \right\} \quad \text{over} \quad \left\{ (0, d\varepsilon), \varepsilon \in \Omega^{n-2}(U^{n}_{\text{loc}}; \square^{n}) \right\} .$$
Proposition 6.7. We have a map for functions, and the cupproduct in syntomic cohomology gives

This is simply the $n$th rigid cohomology of $(X_{n,\text{loc}}^n)$ but note the twisted identification that we have by Definition 4.6. Also note that in some of the computations we will be using an altogether different model of this syntomic cohomology group.

Let

Our main result in this Section gives the following partial data about the regulator of $[S]_n$.

Proposition 6.5. The regulator of $[S]_n$ in $\tilde{H}^n_{\text{ms}}(X_{B,\text{loc}}^{n-1}, \boxtimes^{n-1}, n)$ is given, in the representation (6.1), by $(\omega_n, \varepsilon_n)$, with some $\varepsilon_n \in \Omega^{n-1}(U_{B,\text{loc}}^{n-1}, \boxtimes^{n-1})$.

Forgetting the relativity gives a map $K^{(n)}_n(X_{B,\text{loc}}^{n-1}, \boxtimes^{n-1}) \rightarrow K^{(n)}_n(X_{B,\text{loc}}^{n-1})$. Let us denote the image of $[S]_n$ by $(S)_n$. The corresponding map in syntomic cohomology, $\tilde{H}^n_{\text{ms}}(X_{B,\text{loc}}^{n-1}, \boxtimes^{n-1}, n) \rightarrow \tilde{H}^n_{\text{ms}}(X_{B,\text{loc}}^{n-1}, n)$ simply takes the pair $(\omega, \varepsilon)$ of (6.1) to $(\omega, \varepsilon')$ in the representation (4.4), where $\varepsilon'$ is the component of $\varepsilon$ corresponding to the index function $[1, \ldots, n-1] \rightarrow [1, \ldots, n-1]$. Thus, our Proposition follows immediately from the following Proposition.

Proposition 6.6. The regulator of $(S)_n$ in $\tilde{H}^n_{\text{ms}}(X_{B,\text{loc}}^{n-1}, n)$ is given, in the representation (4.4), by $(\omega_n, \varepsilon'_n)$, with some $\varepsilon'_n \in \Omega^{n-1}(U_{B,\text{loc}}^{n-1})$.

This last Proposition, again follows easily, using the formulas for the regulator map for functions, and the cupproduct in syntomic cohomology given by Lemma 4.7 and (4.8) respectively, from the following purely K-theoretic result.

Proposition 6.7. We have

Proof. Forgetting the relativity is compatible with the construction of the spectral sequence used in (3.7), so in the map

the element $(S)_n$ will be mapped under the differential in the spectral sequence to $\sum_{i=1}^{n-1}(-1)^i(S)_{n-1}|t_i=S$. From this we can determine $(S)_n$ very easily by induction using Lemma 6.8 below.

Lemma 6.8. For $m > n > 0$, the map

is injective.
Proof. Induction on \( n \). For \( n = 1 \), this is clear from the localization sequence
\[
\cdots \to K_m^{(m)}(X_B) \to K_m^{(m)}(X_{B,\text{loc}}) \to K_{m-1}^{(m-1)}(B) \to \cdots
\]
as \( K_m^{(m)}(X_B) \cong K_m^{(m)}(B) = 0 \) if \( m > 1 \). For the induction step, consider the commutative diagram
\[
\begin{array}{ccc}
K_m^{(m)}(X_{B,\text{loc}}^{n-1} \times_B X_B) & \xrightarrow{\phi_1} & \prod_{i=1,\ldots,n-2} K_{m-1}^{(m-1)}(X_{B,\text{loc}}^{n-2} \times_B X_B) |_{t_i = S} \\
\downarrow \psi_1 & & \downarrow \psi_3 \\
K_m^{(m)}(X_{B,\text{loc}}^{n-1}) & \xrightarrow{\phi_2} & \prod_{i=1,\ldots,n-1} K_{m-1}^{(m-1)}(X_{B,\text{loc}}^{n-2} \times_B X_{B,\text{loc}}) |_{t_i = S}
\end{array}
\]
Here the first vertical column is part of an exact localization sequence. \( \psi_3 \) is injective because \( K_{m-1}^{(m-1)}(X_{B,\text{loc}}^{n-2} \times_B X_B) \) is isomorphic to \( K_{m-1}^{(m-1)}(X_{B,\text{loc}}^{n-2}) \) under pullback from the base, and we can restrict the image in \( K_{m-1}^{(m-1)}(X_{B,\text{loc}}^{n-2} \times_B X_{B,\text{loc}}) \) to \( t_{n-1} = 0 \) in order to find the element back. \( \phi_1 \) is injective because again using pullback from the base this reduces to the case \( n - 1 \), where it is true by induction. In particular, if \( \phi_2(\alpha) = 0 \) for some \( \alpha \), \( \psi_2(\alpha) = 0 \), and \( \alpha = \psi_1(\beta) \) for some \( \beta \). Then \( \psi_3(\phi_1(\beta)) = \phi_2(\alpha) = 0 \), which implies \( \beta = 0 \) as both \( \phi_1 \) and \( \psi_3 \) are injective. Therefore \( \alpha = 0 \).

By Lemma 6.8, \((S)_n \) is determined by its image under the boundary. As \((S)_1 = (1 - S) \) and \((S)_2 \) has boundary \(-[S]_1 = -(S)_1 = (1 - S)^{-1} \), one obtains immediately by induction the required formula,
\[
(S)_n = (1 - S) \cup \frac{t_1 - S}{t_1 - 1} \cup \ldots \cup \frac{t_{n-1} - S}{t_{n-1} - 1}.
\]
(We use normalizations so that the \( K \)-theory acts on the right in localization sequences.)

To end this Section, we give the following Lemma.

Lemma 6.9. Let \( F(t) \) be an element of \( (1 + I)^*(R) = K_1^{(1)}(X_{R,\text{loc}}; \mathbb{Q}^1) \). Then its regulator in \( \tilde{H}_{\text{ms}}^1(X_{R,\text{loc}}; \mathbb{Q}^1, 1) \) is given, in the representation \((6.2)\), by
\[
(\text{dlog } F(t), \text{log}(F_0(t))/q)
\]
\[(6.10)\]

Proof. Note that for \( n = 1 \), \((6.2)\) reduces to
\[
\tilde{H}_{\text{ms}}^1(X_{R,\text{loc}}; \mathbb{Q}^1, 1) = \{ (\omega, \varepsilon) : \omega \in \Omega^1(U_{\text{loc}}^1), \varepsilon \in \Omega^0(U_{\text{loc}}^1), \text{d} \omega = 0, \text{d} \varepsilon = (1 - \phi^*/q)\omega \}.
\]
In this way of writing it looks exactly the same as \( H_{\text{ms}}^1(X_{R,\text{loc}}, 1) \). As remarked after \((6.2)\) this is slightly misleading since the differentials are different and take relativity into account. Here this means that the map \( H_{\text{ms}}^1(X_{R,\text{loc}}; \mathbb{Q}, 1) \to H_{\text{ms}}^1(X_{R,\text{loc}}, 1) \) given simply by \( (\omega, \varepsilon) \mapsto (\omega, \varepsilon) \) embeds \( H_{\text{ms}}^1(X_{R,\text{loc}}; \mathbb{Q}, 1) \) as the subspace of pairs.
$(\omega, \varepsilon)$ where $\varepsilon$ vanishes at 0 and $\infty$. Thus, our Lemma is an immediate consequence of Lemma 4.7.

7. END OF THE PROOF

We denote the composed map

$$K_n^1(X_{R, \text{loc}}, \square^{n-1}) \xrightarrow{\text{reg}} \tilde{H}_{\text{ms}}^n(X_{R, \text{loc}}, \square^{n-1}, \kappa) \xrightarrow{\sim} H_{\text{rig}}^{n-1}(X_{\kappa, \text{loc}}, \square^{n-1}/K) \xrightarrow{\tilde{\Pi}} K$$

by $\mathcal{R}$. Here, the isomorphism is normalized according to Definition 4.6 and the map $\tilde{\Pi}$ is defined immediately following Corollary 5.8.

**Proposition 7.1.** We have the following commutative diagram

$$
\begin{array}{ccc}
K_n^1(X_{R, \text{loc}}, \square^{n-1}) & \xrightarrow{\mathcal{R}} & K_n^1(X_{R, \text{loc}}, \square^{n-1}) \\
\downarrow \cong & & \downarrow \cong \\
K & & K
\end{array}
$$

**Proof.** The vertical maps factor through the regulator maps. By the functoriality of the normalization of Definition 4.6, the commutativity of the diagram will follow if we show the commutativity of the diagram

$$
\begin{array}{ccc}
H_{\text{rig}}^{n-1}(X_{\kappa, \text{loc}}, \square^{n-1}/K) & \xrightarrow{\tilde{\Pi}} & H_{\text{rig}}^{0}(\text{Spec}(\kappa)/K) \\
\downarrow \cong & & \downarrow \cong \\
K & & K
\end{array}
$$

But as explained in Proposition 5.6 the composed map $H_{\text{rig}}^{n-1}(X_{\kappa, \text{loc}}, \square^{n-1}/K) \to K$ is simply the map $\Pi$ and therefore the commutativity follows from Corollary 5.8. \qed

**Proposition 7.2.** The composition

$$\text{Symb}_n(R) \subset K_n^1(X_{R, \text{loc}}, \square^{n-1}) \xrightarrow{\mathcal{R}} K$$

factors through the quotient $\text{Symb}_n(R)/(1 + I)^* \cup \text{Symb}_{n-1}(R) = M_n(R)$.

**Proof.** In fact, we can show that $\mathcal{R}$ vanishes on $(1 + I)^* \cup K_{n-1}^1(X_{R, \text{loc}}, \square^{n-2})$. This will follow by symmetry for all possible products involved in $\cup$ if we show that the composition

$$
\tilde{H}_{\text{ms}}^1(X_{R, \text{loc}}, \square^1, 1) \times \tilde{H}_{\text{ms}}^{n-1}(X_{R, \text{loc}}, \square^{n-2}, n-1) \xrightarrow{\cup} \tilde{H}_{\text{ms}}^n(X_{R, \text{loc}}, \square^{n-1}, n) \xrightarrow{\sim} H_{\text{rig}}^{n-1}(X_{\kappa, \text{loc}}, \square^{n-1}/K) \xrightarrow{\tilde{\Pi}} K
$$

vanishes on pairs where the first coordinate is used for $(1 + I)^*$. Let $F(t)$ be in $(1 + I)^*$. By (6.10), its regulator is given, in the representation (6.2) by $\langle d\log F(t), \gamma \rangle$, where the first coordinate belongs to $\Omega^1(U_{\text{loc}})$ and the precise form of the second coordinate does not matter as we will see in a second. On the other hand, elements of $H_{\text{ms}}^{n-1}(X_{R, \text{loc}}, \square^{n-2}, n-1)$ are, by (6.3), always of the form $(0, \delta)$, with $\delta$ in $\Omega^{n-2}(U_{\text{loc}}^*, \square^{n-2})$. Choosing $\gamma = 0$ in (4.8), we see that $\langle d\log F(t), \gamma \rangle \cup (0, \delta)$, will be of the form $(0, d\log F(t) \wedge \delta)$ where $\wedge$ here means the product in complexes of
Lemma 7.6. We have

\begin{align}
\frac{\Pi([\text{dlog } F(t)])}{\Pi([\text{dlog } F(t)] \cup [\delta])} = \Pi([\text{dlog } F(t)] \cup [\delta])
\end{align}

vanishes, where the cup product on the right is a cup product in multi relative rigid cohomology. By Lemma 5.9 it equals \(\Pi([\text{dlog } F(t)])) \cdot \Pi([\delta])\) and the result follows since

\[\Pi([\text{dlog } F(t)]) = \Pi_{\text{col}}(\text{dlog } F(t)) = \log(F(\infty)) - \log(F(0)) = 0.\]

We continue to denote the induced map by \(\mathcal{R}\),

\begin{align}
\mathcal{R} : M_a(R) \to K.
\end{align}

Recall that in Definition 3.11 we defined for any \(z \in R^a\) a symbol \([z]_n\) in \(K_n^{(n)}(X_{R, \text{loc}}; \mathfrak{m}^{n-1})\) by pullback of the universal symbol \([S]_n\) in \(K_n^{(n)}(X_{\mathbb{G}, \text{loc}}; \mathfrak{m}^{n-1})\) along the map sending \(S\) to \(z\), where \(G = \text{Spec}(\mathbb{Z}[S, S^{-1}, (1 - S)^{-1}])\). We now compute \(\mathcal{R}([z]_n)\). We begin this by exploring some auxiliary functions.

**Definition 7.4.** We define a sequence of functions \(f_k(z, S)\) inductively as follows:

\begin{align}
f_0(z, S) = \frac{S}{1 - S}, \quad f_{k+1}(z, S) = \int_z^S f_k(z, t) \text{dlog } t.
\end{align}

At some later point we will want to interpret the integral as a Coleman integral but at this point it is enough to define it when \(z\) and \(S\) belong to the same residue disc, in which case it is well defined without making any Frobenius equivariance assumptions. It is immediately noticed that \(f_k(z, S)\) vanishes to order \(k\) at \(z = S\).

**Lemma 7.6.** We have

\[f_n(z, S) = \text{Li}_n(S) - \sum_{k=0}^{n-1} \frac{1}{k!}(\log S - \log z)^k \text{Li}_{n-k}(z).
\]

*Proof.* The proof is by induction on \(n\). For \(n = 1\) it is immediately verified that

\[f_1(z, S) = \log(1 - z) - \log(1 - S) = \text{Li}_1(S) - \text{Li}_1(z).
\]

Suppose that the statement is true for \(n\). Then for \(n + 1\) we get

\[f_{n+1}(z, S) = \int_z^S \left( \text{Li}_n(t) - \sum_{k=0}^{n-1} \frac{1}{k!}(\log t - \log z)^k \text{Li}_{n-k}(z) \right) \frac{dt}{t}.
\]

\[= \left( \text{Li}_{n+1}(t) - \sum_{k=0}^{n-1} \frac{1}{(k + 1)!}(\log t - \log z)^{k+1} \text{Li}_{n-k}(z) \right)|_z^S
\]

\[= \text{Li}_{n+1}(S) - \text{Li}_{n+1}(z) - \sum_{k=0}^{n-1} \frac{1}{(k + 1)!}(\log S - \log z)^{k+1} \text{Li}_{n-k}(z)
\]

\[= \text{Li}_{n+1}(S) - \sum_{k=0}^{n} \frac{1}{k!}(\log S - \log z)^k \text{Li}_{n+1-k}(z).
\]
Proposition 7.7. Let $z_1, z_2 \in \mathbb{R}^n$ belong to the same residue disc. Then we have
\[
\mathcal{R}([z_1]) - \mathcal{R}([z_2]) = (-1)^n(n-1)!\left(L_n(z_1) - L_n(z_2)\right).
\]

Proof. For any $z \in \mathbb{R}^n$ we may factor the map $\text{Spec}(\mathbb{R}) \to G$ defined by sending $S$ to $z$ via the map $\text{Spec}(\mathbb{R}) \to B = G \otimes \mathbb{R}$ defined in the same way. By functoriality of the regulator map it follows that $\text{reg}([z])$ equals $i_z^*\text{reg}([S])$, where $i_z : (X^1_{B, \text{loc}}, \square^n) \to (X^1_{B, \text{loc}}, \square^n)$ is the embedding in $(X^1_{B, \text{loc}}, \square^n)$ of the fiber at $z$. Thus we are in position to apply Corollary 4.12, but in the relative case, which, as mention after its statement, also applies. To carry out the computation we also shift the index from $n$ to $n+1$ as the computation seems to come out a bit cleaner this way.

We begin with $\bar{H}^{n+1}_\text{ma}(X^n_{B, \text{loc}}, \square^n, n+1)$ in the representation (6.1) (with $n$ shifted to $n+1$). In there we have the regulator of $[S]_{n+1}$, given, according to Proposition 6.5, by the pair $(\omega, \epsilon)$, where\[
\omega = \omega_{n+1} = \text{dlog}(1-S) \land \text{dlog}(t_1 - S) \land \ldots \land \text{dlog}(t_n - S) \land \frac{t_1}{t_1 - 1} \land \ldots \land \frac{t_n}{t_n - 1},
\]
is the form defined in (6.4), while $\epsilon$ is unknown. Note that $\omega$ should really be thought of as the relative form $\bar{\omega} = (\omega, [1, \ldots, n] \to [1, \ldots, n]) \in \Omega^{n+1}(U^n_{B, \text{loc}}, \square^n)$.

We have the projection $\pi : (X^n_{B, \text{loc}}, \square^n) \to B$, which we can compactify to $\bar{\pi} : (\mathbb{P}^n_B)^n \to B$, where the power is taken over $B$. By assumption, $z_1$ and $z_2$ belong to the same residue disc, which we call $D$.

The recipe for computing
\[
\text{reg}([z_1]) - \text{reg}([z_2]) = i_{z_1}^*\text{reg}([S]) - i_{z_2}^*\text{reg}([S]),
\]
according to Corollary 4.12, calls for computing, for $z = z_1$ and $z = z_2$, a form $\theta_z$ such that
\[
\theta_z \in F^{n+1}_n \Omega^n(U^n_{B, \text{loc}} \cap \pi^{-1}(D); \square^n) \quad \text{and} \quad d\theta_z = \bar{\omega}|_{\pi^{-1}(D)},
\]
where $J$ is the ideal defining $\pi^{-1}(z)$. Such a form is given in the following Lemma.

Lemma 7.9. Let
\[
\theta_z = -\sum_{k=0}^{n} (-1)^k k! \sum_{\deg h = k} \left( (-1)^\sum (h(i)+i) f_{k+1}(z, S) \bigwedge_{i=1}^{n-k} \text{dlog} \frac{t_{h(i)} - S}{t_{h(i)} - 1} \right).
\]

Then $\theta_z$ satisfies (7.8). Here, $f_k$ is the function introduced in Definition 7.4 and the form indicated only for the component in which all the constant coordinates are 0, otherwise the form is 0.

Proof. Recall that the condition for being in $F^{n+1}_n$ is that the sum of the degree of the form and its order of vanishing at $S = z$ is $n+1$. And this is clear for $\theta_z$. Now we prove that the differential is correct. We will show that $d(-\theta_z) = -\bar{\omega}$. Consider first the differential of a single term in $-\theta_z$.
\[
(-1)^k \sum (h(i)+i) k! (f_{k+1}(z, S) \bigwedge_{i=1}^{n-k} \text{dlog} \frac{t_{h(i)} - S}{t_{h(i)} - 1} \right),
\]
with $h$ of degree $k$. Using (5.1) the differential is

$$(-1)^{k+\sum (h(i) + i)} k! \left( f_k(z, S) \right) \text{dlog}(S) \wedge \prod_{i=1}^{n-k} \text{dlog} \left( \frac{t_{h(i)} - S}{t_{h(i)} - 1}, h \right)$$

minus (except when $n = k$) a sum of terms obtained by restricting one of the coordinates $t_{h(j)}$ to be 0 (when we restrict any coordinate to $\infty$ we get 0 and there is no need to keep track of that). In the wedge product at $j$ we get $\text{dlog}(S)$ and moving it to the front gives a sign of $(-1)^{j-1}$. This form is then associated with a function $g$ for which $\chi(h, g) = (-1)^{h(j) + j}$. In addition there is an overall sign of $(-1)^{n-k}$ on the entire sum. Thus, the sign on the component with the function $g$ obtained from $h$ by deleting $h(j)$ has a sign of $(-1)$ to the power

$$n - k + k + \sum (h(i) + i) + h(j) - 1$$

$$\equiv n + 1 + \sum g(i) + \sum i \equiv k + 1 + \sum (g(i) + i) \pmod{2}.$$ 

Thus we find

$$\text{d}(\chi(h, g)\neq 0) (-1)^{k+\sum (h(i) + i)} k! \left( f_k(z, S) \right) \text{dlog}(S) \wedge \prod_{i=1}^{n-k} \text{dlog} \left( \frac{t_{h(i)} - S}{t_{h(i)} - 1}, h \right)$$

$$= (-1)^{k+\sum (h(i) + i)} k! \left( f_k(z, S) \right) \text{dlog}(S) \wedge \prod_{i=1}^{n-k} \text{dlog} \left( \frac{t_{h(i)} - S}{t_{h(i)} - 1}, h \right)$$

$$- \sum_{\chi(h,g)\neq 0} (-1)^{k+1+\sum (g(i) + i)} k! \left( f_{k+1}(z, S) \right) \text{dlog}(S) \wedge \prod_{i=1}^{n-k-1} \text{dlog} \left( \frac{t_{g(i)} - S}{t_{g(i)} - 1}, g \right).$$

Now we consider the coefficient in $\text{d}(\theta_z)$ in the $g$ component when $\deg g = m$. If $m > 0$ then it gets contributions from both lines in the right hand side of the last equation. The contributions from the second line correspond to $k = m - 1$. There are exactly $m$ different $h$'s that would give $g$ and the contributions are identical. Thus it is visibly seen that the contributions from the second line cancel the ones from the first line. The only term that survives is the one with $m = 0$. Here there is only a contribution from the first line. So we find

$$\text{d}(\theta_z) = \left( f_0(z, S) \right) \text{dlog}(S) \wedge \prod_{i=1}^{n} \text{dlog} \left( \frac{t_i - S}{t_i - 1}, \text{id} \right)$$

$$= - \left( \text{dlog}(1 - S) \wedge \prod_{i=1}^{n} \text{dlog} \left( \frac{t_i - S}{t_i - 1}, \text{id} \right) \right)$$

$$= -\tilde{\omega}.$$

Now $i_{z_2}^* \text{reg}(\lceil S \rceil_n) - i_{z_1}^* \text{reg}(\lceil S \rceil_n) = \theta_{z_2} - \theta_{z_1}$ restricted to the fiber above any $S$ in $D(K)$. On this difference we need to apply $\Pi_{\text{Col}}$. We will in fact compute $\Pi_{\text{Col}}(\theta_z)$ for any $z$. We integrate with respect to the $t$'s keeping $S$ fixed. We see that all the terms we need to successively integrate are products of $\text{dlog}$'s, which the integration process converts into logs. The extra sign coming from the formula for $\Pi$ is $(-1)^{\sum (h(i) + i)}$ because we are always in the component where all the fixed
coordinates are 0. So using Lemma 7.6 and the definition of \( L_{n+1}(S) \) as in (1.3), we find \( \Pi_{\text{Col}}(\theta_\zeta) \) equals

\[
- \sum_{k=0}^{n} (-1)^{k} k! \cdot \left( \begin{array}{c} n \\ k \end{array} \right) f_{k+1}(z, S) \log^{n-k}(S)
\]

\[
= -n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{(n-k)!} f_{k+1}(z, S) \log^{n-k}(S)
\]

\[
= -n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{(n-k)!} \text{Li}_{k+1}(S) \log^{n-k}(S)
\]

\[
+ n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{(n-k)!} \left[ \sum_{l=0}^{k} \frac{1}{l!} (\log(S) - \log(z))^l \text{Li}_{k+1-l}(z) \right] \log^{n-k}(S)
\]

\[
= (-1)^{n-1} n! L_{n+1}(S)
\]

\[
+ n! \sum_{k=0}^{n} (-1)^{k} \frac{1}{(n-k)!} \left[ \sum_{r=0}^{k} \frac{1}{(k-r)!} (\log(S) - \log(z))^{k-r} \text{Li}_{r+1}(z) \right] \log^{n-k}(S)
\]

\[
= (-1)^{n-1} n! L_{n+1}(S)
\]

\[
+ n! \sum_{r=0}^{n} \frac{1}{(n-r)!} (\log(S) - \log(z))^{n-r} \text{Li}_{r+1}(z)
\]

\[
= (-1)^{n} n! (L_{n+1}(S) - L_{n+1}(S))
\]

Thus we find

\[
\mathcal{R}([z_1]_{n+1}) - \mathcal{R}([z_2]_{n+1}) = (-1)^n n! ((L_{n+1}(z_2) - L_{n+1}(S)) - (L_{n+1}(z_1) - L_{n+1}(S)))
\]

\[
= (-1)^{n+1} n! (L_n(z_1) - L_{n+1}(z_2)).
\]

\[\square\]

**Proposition 7.10.** For \( z \) in \( R^b \) we have

\[\mathcal{R}([z]_n) = (-1)^n (n-1)! L_n(z).\]

**Proof.** Let \( E_n(z) \) be the difference of the two sides of the equation. By Proposition 7.7 \( E_n(z) \) is constant on each residue disc. The function \( E_n(z) \) satisfies the distribution relation

\[\frac{1}{m} \sum_{\zeta^m = 1} E_n(\zeta z) = \frac{E_n(z^m)}{m^n}\]

for each positive integer \( m \). The left hand side of (7.11) satisfies the relation because by [dJ95, Proposition 6.1] we have the relation

\[\frac{1}{m} \sum_{\zeta^m = 1} [\zeta z]_n = \frac{[z^m]_n}{m^n}\]
Suppose that a certain "motivic" functional equation is satisfied, which then translates into a functional equation for the complex polylog and uses the multivaluedness to show Proposition 7.14.

The map occurring: To obtain a functional equation for polylogs [Woj91] a similar phenomenon happens: To obtain a functional equation for $Wojtkowiak$ on functional equations for polylogarithms $[Woj91]$ a similar phenomenon occurs: To obtain a functional equation for $p$-adic polylogs one starts with a functional equation for the complex polylog and uses the multivaluedness to show that a certain "motivic" functional equation is satisfied, which then translates into a $p$-adic functional equation.

Proposition 7.14. The map

$$H^1(\mathcal{M}_{(\alpha)}^*(\mathcal{O})) \xrightarrow{\Phi} K_{2n-1}^{(n)}(\mathcal{O}) \xrightarrow{\sigma} K_{2n-1}^{(n)}(R) \xrightarrow{\text{reg}} K,$$

where the map $\Phi$ is part of (3.20), is induced by the map sending the symbol $[x]_n$ to $(-1)^n(n-1)!L_n(\sigma(x))$.

Proof. Suppose $\sum a_i[x_i]_n \in H^1(\mathcal{M}_{(\alpha)}^*(\mathcal{O}))$. Let

$$\alpha = \Phi(\sum a_i[x_i]_n) \in K_{2n-1}^{(n)}(\mathcal{O}) \approx K_n^{(n)}(X_{\mathcal{O}_{\text{loc}}}^{n-1}; \square^{n-1}).$$

By definition, the image of $\alpha$ in $K_n^{(n)}(X_{\mathcal{O}_{\text{loc}}}^{n-1}; \square^{n-1})$ equals $\sum a_i[x_i]_n \text{ modulo } (1 + I)^* \cup \text{Symb}_{n-1}(\mathcal{O})$. By functoriality, the image of $\sigma(\alpha)$ in $K_n^{(n)}(X_{R_{\text{loc}}}^{n-1}; \square^{n-1})$ equals $\sum a_i[\sigma(x_i)]_n \text{ modulo } (1 + I)^* \cup \text{Symb}_{n-1}(R)$. By Proposition 7.1 and Proposition 7.2 we therefore have

$$\text{reg}(\sigma(\alpha)) = \mathcal{R}(\sum a_i[\sigma(x_i)]_n) = \sum a_i \mathcal{R}([\sigma(x_i)]_n) = \sum a_i(-1)^n(n-1)!L_n(\sigma(x_i)) \text{ by Proposition 7.10}.$$
Proof of Theorems 1.6 and 1.10. Part (1) of each Theorem was already proved in Section 3. To prove part (2), note that any of the functions $L_{\text{mod},n}(z)$ differs from $L_n(z)$ by a linear combinations of the functions $z \mapsto \log^k(z)L_{n-k}(z)$ for $k \geq 1$. Any function in this combination, when composed with $\sigma$, factors through the composed differential

$$M_n(\mathcal{O}) \to M_{n-1}(F) \otimes F_Q^* \to \cdots \to M_{n-k}(F) \otimes (F_Q^*)^{\otimes k}$$

which maps $[x]_n$ to $[x]_{n-k} \otimes x \otimes \ldots \otimes x$. Therefore the functions $L_n(\sigma(x))$ and $L_{\text{mod},n}(\sigma(x))$ (and in fact, also $\text{Li}_n(\sigma(x))$), coincide on $H^1(M(n)(\mathcal{O}))$. But any function $L_{\text{mod},n}(z)$ satisfies $L_{\text{mod},n}(z) + (-1)^nL_{\text{mod},n}(1/z)$. Therefore the map $[x]_n \mapsto (-1)^n(n-1)!L_{\text{mod},n}(\sigma(x))$ factors through the quotient map $M_n(\mathcal{O}) \to \widetilde{M}_n(\mathcal{O})$. But the composition

$$H^1(M(n)(\mathcal{O})) \to H^1(\widetilde{M}(n)(\mathcal{O})) \to K$$

is still given by $[x]_n \mapsto (-1)^n(n-1)!L_n(\sigma(x))$. Thus, the Theorem follows from Proposition 7.14.

Proof of Theorem 1.12. Let $F$ be a number field. Note that roots of unity will not be special units in general, so we have to work in the complex for $F$ rather than for $\mathcal{O}$. Namely, let $\zeta$ in $F$ be a root of unity of order $m > 1$. If $(m,p) = 1$ $\zeta$ is a special unit, and we have the result already. If $m = ps$ with $s > 1$ and $(p,l) = 1$, write $\zeta = \zeta_1\zeta_2$ with $\zeta_1$ of order $p^s$ and $\zeta_2$ of order $l$. As the reduction of $\zeta$ is the same as the reduction of $\zeta_2$, we see that $\zeta$ is special unless $m = p^s$. If $m = p^s$, let $r > 1$ be an integer congruent to 1 modulo $p^s$. Then $\zeta' = \zeta$, and from the distribution relations Proposition 2.10 and (2.4) we find that $[\zeta]_n = r^{n-1}\sum_{\alpha=1}^{r\alpha-1}[\zeta\alpha]_n$ in $M_n(F')$ with $F' = F(\mu_r)$. According to [Bes00a, Lemma 8.8] the modified syntomic regulator commutes with finite base change. This means in the case we are considering that if $R'$ is a finite extension of $R$ there is a commutative diagram

$$K^{(n)}_{2n-1}(R) \longrightarrow K$$

$$\downarrow \quad \downarrow$$

$$K^{(n)}_{2n-1}(R') \longrightarrow K'$$

where the map $K \to K'$ is the natural inclusion. We can therefore do our computations for the regulator just as well in $K' = K(\mu_r)$. As all $\zeta\alpha$ in the sum are special units except when $\alpha = 1$, we can solve for $\text{reg}([\zeta]_n)$ as $r^{n-1} \neq 1$. As $\text{reg}([x]_n) = L_{\text{mod},n}(x)$ if $x$ is a special unit, and $L_{\text{mod},n}$ satisfies the corresponding distribution relation by [Col82, Proposition 6.1], our result follows for $m > 1$. For $\zeta = 1$ one uses the distribution relation similarly.

Remark 7.15. Although not needed for the purposes of this paper, we would like to sketch a somewhat less explicit method of doing the computations of this Section. This was in fact our original method. The idea is quite easy to explain: Out of our regulator computations, we obtained the fact that for certain constants $\alpha_k$ the function

$$P(z,S) = \sum_{k=0}^{n} \alpha_k f_k(z,S) \log^{n-k} S$$
is a sum of a function of $z$ and a function of $S$ (first line of final computation). In other words, its mixed derivatives vanish. We get this relation initially only for $z$ and $S$ which belong to the same residue disc, even though we know that using Coleman integration the functions $f_n$ themselves extend to all $z$ and $S$. What we did was to write explicitly $f_n(z, S)$ in terms of logarithms and polylogarithms in $z$ and $S$ separately and then show that $P(z, S)$ can indeed be written as a sum of a function of $S$ and a function of $z$, the latter being our sought after regulator function. Note that the separation of variables now holds for any $z$ and $S$.

The alternative approach is to deduce the “global” separation of variables from the same result for $z$ and $S$ in the same residue disc using Coleman theory in 2-variables. The theory developed in [Bes01a] defines a notion of a Coleman function in several variables. One then shows that iterated integrals of the kind used to define $f_n(z, S)$ make it a Coleman function in both variables (in particular for fixed $S$ it is a Coleman function in $z$, but note that the notion of being Coleman in two variables is stronger than the notion of begin Coleman in each variable separately). Coleman functions form a ring, which shows that $P(z, S)$ is also a Coleman function, and so are its mixed derivatives. The theory then shows that the fact that the mixed derivatives vanish on some residue disc imply that they vanish identically which in turn implies a global separation of variables.

Knowing a global separation of variables is very convenient, for if $P(z, S) = P(z) + g(S)$, and all we need to know is $P(z)$ up to a constant, then this is supplied by $P(z, S_0)$ for any $S_0$ we take. In our particular situation, if one substitutes $S_0 = -1$, whose log is 0, in the first line of the final computation we find that up to a constant $P(z) = -(-1)^n n! \cdot f_{n+1}(z, -1)$. So one is then left with computing this function.

### Appendix A. Chern classes in relative cohomology

In this appendix we give the necessary constructions for the main paper as far as relative $K$–theory and Chern classes in syntomic cohomology are concerned. Most of this material is rather standard and has to be modified in a rather minor way in order to fit the current context, so we are sketchy in places. One thing that we work out in glorious detail is the description of a complex that computes the multi relative syntomic cohomology encountered in Section 5.

In [Bes00a] the first named author described a theory of rigid syntomic regulators. This is not sufficient in all applications, for example those described in the present work, since one often needs to extend the regulators (= Chern classes) to the relative situation. Our goal in this appendix is to explain how the construction extends to more general “spaces”, yielding in particular Chern classes from relative $K$-theory to relative syntomic cohomology. The construction is completely formal, relying primarily on [GS99] with some input from [dJ95].

We consider the category $S$ of Noetherian finite dimensional schemes over $R$ and the topos $T$ of sheaves on it with respect to the Zariski topology. Following [GS99] we call a simplicial object of $T$ a space. If $X \in S$, $Y \mapsto \text{hom}(Y, X)$ gives us an element of $T$, which we still denote by $X$. With $X_+$ we will denote the space consisting of the disjoint union of the constant sheaf $X$ and the constant simplicial basepoint $*$.
Let \( S' \subset S \) be the subcategory of schemes which are in addition smooth and separated over \( R \), again equipped with the Zariski topology. Following [dJ95, Section 2.2] we make the following definition.

**Definition A.1.** A space \( X_\bullet \) is called a smooth separated pointed simplicial scheme if it is represented in each degree by a scheme in \( S' \) (interpreted as sheaf) together with a disjoint base point and if it is furthermore degenerate above a finite simplicial degree. In the sequel we shall refer to these spaces as pointed simplicial schemes for short, unless stated otherwise.

Clearly, if \( X \in S' \), then \( X_+ \) is a pointed simplicial scheme. According to [dJ95, Lemma 2.1] a pointed simplicial scheme is \( K \)-coherent in the sense of [GS99, Section 3.1, Definition 1] (this notion is also defined in [dJ95, top of page 201]).

When \( X_\bullet \) and \( Y_\bullet \) are spaces, \([X_\bullet, Y_\bullet]\) denotes the set of morphisms between \( X_\bullet \) and \( Y_\bullet \) in the homotopy category of spaces constructed in [GS99]. The point is that there is an element \( K \) in \( T \) such that if \( X \) is a smooth finite type scheme over \( R \), then \([S^m \wedge X_+, K] \cong K_m(X)\) (loc. cit. Proposition 5 of Section 3.2). Gillet–Soulé therefore make the following Definition.

**Definition A.2.** If \( X_\bullet \) is a space, we define
\[
K_m(X_\bullet) = H^{-m}(X_\bullet, K) = [S^m \wedge X_\bullet, K]
\]
for \( m \geq 0 \).

In the body of the paper we have systematically used the \( K \)-theoretic notation for this, but in this Appendix we shall also use the notation \( H^{-m}(X_\bullet, K) \).

In [GS99], it is then shown that if \( X_\bullet \) is \( K \)-coherent, then \( H^{-m}(X_\bullet) \) has a special \( \lambda \)-ring structure. In particular, when \( X_\bullet \) is \( K \)-coherent of cohomological dimension at most \( d \), then according to loc. cit. Proposition 8 of Section 4.4, \( K_m(X_\bullet) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{i=0}^{m+d} K_m^{(i)}(X_\bullet) \), with \( K_m^{(i)}(X_\bullet) = H^{-m}(X_\bullet, K)^{(i)} \) the \( \mathbb{Q} \) sub vector space of \( K_m(X_\bullet) \otimes = H^{-m}(X_\bullet, K) \otimes \mathbb{Q} \) of elements \( x \) such that \( \psi^k(x) = k^i x \) for all \( k \geq 2 \), and \( \alpha = 2 \) if \( m \geq 2 \), \( \alpha = 1 \) if \( m = 1 \) and \( \alpha = 0 \) if \( m = 0 \). This will certainly apply to a pointed simplicial space \( X_\bullet \) which is degenerate above simplicial degree \( N \) and where the maximum relative dimension of scheme components over \( R \) is \( M \), with \( d = M + N + 1 \), cf. loc. cit. Lemma 1.2.2.2 or 3.2.4.

For two spaces \( X_\bullet \) and \( Y_\bullet \) we get a map \( K_m(X_\bullet) \times K_n(Y_\bullet) \rightarrow K_{m+n}(X_\bullet \wedge Y_\bullet) \) from the composition of
\[
S^{m+n} \wedge X_\bullet \wedge Y_\bullet \rightarrow S^m \wedge X_\bullet \wedge S^n \wedge Y_\bullet \rightarrow K \wedge K \rightarrow K
\]
because \( S^m \wedge S^n = S^{m+n} \) and \( K \) comes equipped with a map \( K \wedge K \rightarrow K \). Under this product, we get \( K_m^{(i)}(X_\bullet) \times K_n^{(j)}(Y_\bullet) \rightarrow K_{m+n}^{(i+j)}(X_\bullet \wedge Y_\bullet) \) (loc. cit. top of page 48).

If \( X_\bullet \) and \( Y_\bullet \) are pointed spaces with a base pointed preserving map \( Y_\bullet \rightarrow X_\bullet \), then the reduced mapping cone \( C_\bullet = C(Y_\bullet, X_\bullet) \), whose definition will be recalled later, see (A.9), is also a pointed space, and one gets an exact sequence
\[
\cdots \rightarrow K_{m+1}(X_\bullet) \rightarrow K_{m+1}(Y_\bullet) \rightarrow K_m(C_\bullet) \rightarrow K_m(X_\bullet) \rightarrow \cdots .
\]

The most important applications of these are if \( Y_\bullet \) is a pointed closed simplicial subscheme of \( X_\bullet \) (i.e., the map corresponds to a closed embedding of schemes on all the scheme components of \( X_\bullet \)), in which case one gets the \( K \)-theory of \( X_\bullet \) relative to \( Y_\bullet \): \( K_m(C_\bullet) = K_m(X_\bullet; Y_\bullet) \). Iterating this one gets multi relative \( K \)-groups as
in the body of the paper. E.g., if $Y_{1*}$ and $Y_{2*}$ are closed simplicial subschemes of $X_*$, and $Y_{12*} = Y_{1*} \cap Y_{2*}$, with $C_{1*} = C(Y_{1*}, X_*)$, $C_{2*} = C(Y_{12*}, Y_{2*})$ and $C_{3*} = C(C_{2*}, C_{1*})$ we get an exact sequence
\[ \cdots \rightarrow K_{m+1}(X_*, Y_{1*}) \rightarrow K_{m+1}(Y_{1*}, Y_{2*}) \rightarrow K_m(X; Y_{1*}, Y_{2*}) \rightarrow K_m(X_*, Y_{1*}) \rightarrow \cdots \]
where we write $K_m(X_*, Y_{1*}, Y_{2*})$ for $K_m(C_{3*})$.

The other application is $K$–theory with support, in which case $Y_*$ is an open pointed subscheme of $X_*$. Let $Z_*$ be the closed pointed simplicial scheme complement of $Y_*$ in $X_*$ (i.e., the open subset in every scheme component, together with the base point $*$), and assume all scheme components of $Z_*$ are regular. If also conditions (TC1) and (TC2) of [dJ95, page 202] hold for the embeddings in $Z_* \rightarrow X_*$, then $K_m(Z_*) \cong K_m(C(Y_* \rightarrow X_*))$. The sequence then becomes a localization sequence
\[ \cdots \rightarrow K_{m+1}(X_*) \rightarrow K_{m+1}(Y_*) \rightarrow K_m(Z_*) \rightarrow K_m(X_*) \rightarrow \cdots \]

Under very restrictive hypothesis where $Z_*$ is of codimension $d$ in all scheme components (see [dJ95, Proposition 2.3]) one can prove a Gysin exact sequence
\[ \cdots \rightarrow K_{m+1}^{(i+d)}(X_*) \rightarrow K_{m+1}^{(i+d)}(Y_*) \rightarrow K_m^{(i)}(Z_*) \rightarrow K_m^{(i+d)}(X_*) \rightarrow \cdots \]

In order to be able to define regulators with values in the cohomology of a complex of Abelian groups, we briefly review how this is put into the context of spaces.

When $A_*$ is a homological chain complex of Abelian objects in $T$, $X_*$ is a space, and $n \geq 0$, we write $H^{-n}(X_*, A_*) := [S^n \wedge X_*, K(A_*)]$, where $K$ is the Dold–Puppe functor, see [Qui73, II 4.11].

In [Bes00a] the different versions of syntomic cohomology are constructed as cohomologies of bounded below complexes of presheaves $\Gamma^*_?(i)$ on $S'$, where $? = \circ$ could stand for any of the versions of syntomic cohomology considered. By [Bes00a, Proposition 6.2] these presheaves are pseudo-flasque in the sense that there is a Mayer–Vietoris exact sequence involving $U, V, X = U \cup V$ for two open subsets $U$ and $V$ of $X$. We associate to these presheaves spaces as follows: Consider first the sheaf $\Gamma^*_?(i)'$ on $S'$ associated to the presheaf $\Gamma^*_?(i)$. Next consider the map of sites $r : S \rightarrow S'$ corresponding to the inclusion $S' \subset S$ and set $\Gamma^*_?(i) = r^* \Gamma^*_?(i)'$, a complex of sheaves on $S$. As those complexes are complexes of sheaves with cohomological numbering, we put, for any space $X_*$
\[ H^{2i-n}(X_*, \Gamma^*_?(i)) = [S^n \wedge X_*, K(2i, \Gamma^*_?(i))] \]
where, for any cohomological complex $A^*$ in nonnegative degree, $K(2i, A^*)$ the Dold–Puppe construction applied to the homological complex $A^0 \rightarrow A^1 \rightarrow \cdots \rightarrow A^{2i-1} \rightarrow \ker(A^{2i} \rightarrow A^{2i+1})$ in degrees $2i$ through 0. (We shall see shortly that, as the presheaves are pseudo-flasque, using the associated sheaves does not change much.)

The remainder of this Appendix mainly consists of getting an explicit complex that computes $H^{-m}(X_*, \Gamma^*_?(i))$, first for pointed simplicial schemes and then more specifically for those pointed simplicial schemes underlying the multi-relative $K$–theory. Together with the construction of Chern classes in Proposition A.15 below, this provides the reference for the regulator and the complexes used in the explicit calculations in the body of the paper.
So let $X_\bullet$ be a (smooth separated) pointed simplicial scheme. If we write $DT$ for the derived category of Abelian chain complexes in $T$, 

$$H^{2i-m}(X_\bullet, \Gamma_\bullet^*(i)) = [S^m \wedge X_\bullet, K(2i, \Gamma_\bullet^*(i))]$$

is also isomorphic to $[N_\circ(S^m \wedge X_\bullet), \Gamma_\bullet^*(i)]_{DT}$, cf. [dJ95, (24) on page 213], where $N_\circ(\cdot)$ denotes the reduced chain complex associated to the pointed simplicial objects involved. As in loc. cit., the Alexander–Whitney map, in degree $n$ is also isomorphic to $[U_s, \Gamma_\circ(U_s)]_{DT}$. If $\Gamma_\circ(U_s) \to I^\bullet$ is an injective resolution, this equals $[N_\circ(X_\bullet)[-n], I^\bullet]$ (maps up to chain homotopy), because the complex $I^\bullet$ is bounded below ([Har66, p.67]).

Using the Yoneda lemma, we can compute this as in [dJ95, pages 214—216] as the $2i - n$–th cohomology of the complex $C^\bullet(X_\bullet, I^\bullet)$ given by

$$C^q(X_\bullet, I^\bullet) = \bigoplus_{t+s=q} \hom(X_s, I^t) = \bigoplus_{s+t=q} \Gamma(X_s, I^t)$$

with $d^{(s,t)} = (-1)^q(d_s^{N(X_\bullet)})^* + d_I^*$, where of course we ignore the degenerate part of the scheme component of $X_\bullet$ as well as the basepoint $*$ as we are working with the complex $N_\circ(X_\bullet)$.

We would like to be able to replace $I^\bullet$ with $\Gamma_\circ(U_s)$ in this complex. By using a filtration in the simplicial index and the associated spectral sequence, it follows immediately from the following two Lemmas that we can do this without changing the cohomology of the complex.

**Lemma A.5.** Let $P^\bullet$ be a complex of pseudo flasque presheaves of Abelian groups on $S'$ and let $I^\bullet$ be an injective resolution of the associated complex of sheaves. If $X \in S'$, then the natural map on global sections

$$\Gamma(X, P^\bullet) \to \Gamma(X, I^\bullet)$$

is a quasi isomorphism of complexes of Abelian groups.

**Proof.** The proof follows the proofs of Theorem 4 and Theorem 1' of [BG73] extremely closely, but it is easier as it is in the context of complexes of Abelian groups rather than simplicial sets. Namely, for every open set $U$ of $X$, let $F^\bullet(U) = \text{Cone}(P^\bullet(U) \to I^\bullet(U))$ Then it follows immediately from the five lemma that the cohomology of $F(U)$ satisfies a Mayer–Vietoris exact sequence associated to two open subsets $U$ and $V$, hence is pseudo flasque. If we let $T^q(U)$ be the $q$–th cohomology group of the complex $F^\bullet(U)$, then the proof of Theorem 1' applies verbatim if we replace $*$ with 0 everywhere, and take into account that our indexing is cohomological rather than homological.

**Lemma A.6.** Let $F^\bullet$ be a complex of sheaves on $S'$ and let $X \in S'$. Then the canonical map $H^q_S(X, F^\bullet) \to H^q_S(X, rF^\bullet)$ is an isomorphism.

**Proof.** This is essentially part of [Mil80, Proposition III.3.1], where it is stated for a single sheaf and for the big site of schemes over $X$, but the proof is the same.
Because the complexes $\Gamma^i_s(i)$ are pseudo flasque, by Lemma A.5 and Lemma A.6 we could replace the complex (A.4) with the complex $C^\bullet(X_\bullet, \Gamma^i_s(i))$ given by
\begin{equation}
C^q(X_\bullet, \Gamma^i_s(i)) = \bigoplus_{s+t=q} \Gamma(X_s, \Gamma^i_t(i))
\end{equation}
with $d^{(s,t)}(\Gamma^i_s(i)) = (-1)^q d^s \cdot \Gamma^i_t(i)$.

**Remark A.8.** Note that we are not using the fact that $X_\bullet$ is degenerate above a certain degree. Therefore, the description of cohomology as the cohomology of (A.7) is valid for such spaces as well, provided the components belong to $S'$. In particular, it is valid for the classifying spaces $BGL_n$ over $R$.

In the paper, we have to use this complex on an iterated simplicial reduced mapping cone $C_\bullet$, which we recall is defined as follows.

For $f : X_\bullet \to Y_\bullet$ a map of pointed simplicial schemes, define the reduced mapping cone of $f$ by
\begin{equation}
C(X_\bullet, Y_\bullet) = Y_\bullet \coprod X_\bullet \times I/ \sim,
\end{equation}
where $I$ is the simplicial version of the unit interval, given in degree $s$ by all sequences $\{0, \ldots, 0, 1, \ldots, 1\}$ of length $s + 1$, and pointed by $\{1, \ldots, 1\}$, and $\sim$ are the usual identifications to obtain the reduced mapping cone.

Let $X$ be a scheme, and let $Y_1, \ldots, Y_n$, be subschemes. Denote by $X_+$ the pointed simplicial scheme consisting of $X$ in every degree, together with a disjoint basepoint $\star$. Consider the iterated mapping cone $C(X, \{Y_1, \ldots, Y_s\})$ inductively defined by
\begin{align*}
C(X, \{Y_1\}) &= C(Y_1^+, X) \\
C(X, \{Y_1, \ldots, Y_{s+1}\}) &= C(C(X, \{Y_{s+1}, \{Y_1, \ldots, Y_{s,s+1}\}\}), C(X, \{Y_1, \ldots, Y_s\}))
\end{align*}
where $Y_{i,j} = Y_i \cap Y_j$. Using induction, one sees easily that the space $C_\bullet$ one finds for $X, Y_1, \ldots, Y_n$ is as follows.
\begin{equation}
C_m = \star \coprod \prod_{\alpha_1, \ldots, \alpha_m} Y_{\alpha_1, \ldots, \alpha_m}
\end{equation}
with $\alpha_i \in \{\{0, \ldots, 0\}, \{0, \ldots, 0, 1\}, \ldots, \{0, 1, \ldots, 1\}\}, \{Y_{\alpha_1, \ldots, \alpha_m} = \bigcap_{\alpha_i \neq \{\ldots, 0\}} Y_i \}
\text{and } \cap_0 Y_i = X$. The boundary and degeneracy maps are the natural maps coming from the inclusions and the identity, which we get by deleting or doubling the $i$-th place in the zeroes and ones, with the convention that we identify $Y_{\alpha_1, \ldots, \alpha_m}$ with $\star$ if at least one of the $\alpha$’s consists of only 1’s. Clearly, $C_\bullet$ is a pointed simplicial scheme, smooth if $X$, all $Y_j$ and all of their possible intersections are smooth. Due to our definition of the mapping cone, the reduced chain complex $N_*(C_\bullet)$ no longer looks like an iterated mapping cone of reduced chain complexes as there are too many nondegenerate copies of intersections for $n \geq 2$, and neither does the complex in (A.7).

So we also define $\Box(C_\bullet)$ to be the homological chain complex given in degree $k$ by $\bigoplus_{|\beta|=k} \mathbb{Z}[Y_\beta]$, with $\beta$ a subset of $\{1, \ldots, n\}$, $Y_\beta = \cap_{\beta} Y_i$, and $Y_\emptyset = X$. The boundary is given on generators of $\Box(C_\bullet)$ by
\begin{equation}
d(Y_\beta) = (-1)^{k-1} \sum_{j=1}^k (-1)^j Y_{\beta \setminus \{\beta_j\}}
\end{equation}
if \( \beta = \{ \beta_1, \ldots, \beta_k \} \) with \( \beta_1 < \beta_2 < \cdots < \beta_k \). (Just as in the complexes \( N_*(\cdot) \), the maps here are the ones induced from the maps in the pointed spaces, which means the correspond to the maps of sheaves that the scheme component represent in our topos \( T \).)

**Proposition A.12.** \( N_*(C_\bullet) \) and \( \Box(C_\bullet) \) are quasi isomorphic.

*Proof.* Define a map

\[
\Psi : \Box(C_\bullet) \longrightarrow N_*(C_\bullet)
\]

via

\[
Y_\beta \mapsto \sum_{\sigma \in S_k} (-1)^\sigma Y_{(\beta, \sigma)}
\]

in degree \( k \), where \( (-1)^\sigma \) is the sign of \( \sigma \), and \( (\beta, \sigma) = \alpha_1, \ldots, \alpha_n \) is an index defined as follows. We make \( \alpha_j = \{ 0, \ldots, 0 \} \) unless \( j \) is an element of \( \beta \). The remaining \( k \) \( \alpha_j \) are indexed by \( \beta \). We consider the \( k \) standard \( k + 1 \) tuples \( \{ 0, \ldots, 0, 1 \}, \ldots, \{ 0, 1, \ldots, 1 \} \), and put \( \alpha_{|k\beta(1)|} \) equal to the \( j \)-th \( k + 1 \)-tuple in this list.

We have to check that \( \Psi \) defines a map of complexes. This is clear if \( k = 0 \). For \( k \geq 1 \), \( \Psi_{k-1} \circ d \) is given by mapping \( Y_\beta \) (with \( |\beta| = k \)) to

\[
\Psi_{k-1}((-1)^{k-1} \sum_{j=1}^{k} (-1)^{j} Y_{\beta\setminus\{\beta_j\}}) = \sum_{j=1}^{k} (-1)^{k+j-1} \sum_{\tau \in S_{k-1}} (-1)^\tau Y_{(\beta\setminus\{\beta_j\}, \tau)}.
\]

On the other hand, \( \Psi_k \) maps \( Y_\beta \) to \( \sum_{\sigma \in S_k} (-1)^\sigma Y_{(\beta, \sigma)} \), which \( d \) maps to

\[
\sum_{\sigma \in S_k} (-1)^\sigma \sum_{j=0}^{k} (-1)^{j} Y_{d_{j}(\beta, \sigma)}
\]

where \( d_j \) is the \( j \)-th simplicial face. Now notice that the \( j = 0 \) term here is zero, as it introduces \( \{ 1, \ldots, 1 \} \) among the indices so this corresponds to \( \ast \) in \( C_\bullet \), which maps to zero in \( N_*(C_\bullet) \). Also, for \( j = 1, \ldots, k - 1 \) the \( j \)-th and \( j + 1 \)-th indices become the same after applying \( d_j \), so that \( \sigma \) and \( \sigma \circ (j + 1) \) give the same contribution, which cancels due to \( (-1)^\sigma \). Therefore only one term survives, corresponding to \( j = k \), i.e., \( d_k \), which eliminates the last element of all the indices. So we are left with

\[
(-1)^k \sum_{\sigma \in S_k} (-1)^\sigma Y_{d_k(\beta, \sigma)} = (-1)^k \sum_{j=1}^{k} \sum_{\tau \in S_{k-1}} (-1)^{j-1} (-1)^\tau Y_{(\beta\setminus\{\beta_j\}, \tau)}
\]

because if \( \sigma(1) = j \), the \( j \)-th index in \( d_k \beta \) consists only of zeroes as well, and we can write the corresponding term as a contribution in the sum on the right with \( k = j \) and \( \tau = (k \ldots j) \circ \sigma \circ (1 \ldots k) \) due to the renumbering involved.

In order to check that \( \Psi \) defines a quasi isomorphism, we proceed by induction on the degree of relativity \( n \). We investigate how \( \Psi \) behaves with respect to taking cones.

So let \( Y_\bullet \rightarrow X_\bullet \) correspond to taking the last \( (n-\text{th}) \) relativity into account, with \( C_\bullet \) the corresponding reduced mapping cone. Let us first note that \( \Box(C_\bullet) \) is the cone of the map \( \Box(Y_\bullet) \rightarrow \Box(X_\bullet) \). Namely, any component \( C_{\beta} \) of \( \Box(C_\bullet) \) comes from \( \Box(X_\bullet) \) if and only if \( \beta \) does not contain \( n \), and that the components containing \( n \) correspond to \( Y_{\beta \setminus \{ n \}} \)'s, i.e., to \( \Box(Y_\bullet)[-1] \). So \( \Box(C_\bullet) \) is the mapping cone of \( \Box(Y_\bullet) \rightarrow \Box(X_\bullet) \) provided the differential is the one on the cone. As \( \Box(X_\bullet) \)
is a subcomplex, we only need to check what the differential does on $C_{\beta}$ with $n$ in $\beta$. Let $k = |\beta|$. Applying $d$ we get
\[
(-1)^{k-1} \sum_{j=1}^{k} (-1)^j C_{\beta \setminus \{\beta_j\}} = -C_{\beta \setminus \{n\}} - (-1)^{k-2} \sum_{j=1}^{k-1} (-1)^j C_{\beta \setminus \{\beta_j\}},
\]
which is exactly what we want.

We shall verify that we have a map of triangles
\[
\cdots \rightarrow \square(Y_\bullet) \rightarrow \square(X_\bullet) \rightarrow \square(C_\bullet) \rightarrow \square(Y_\bullet)[-1] \rightarrow \cdots
\]
with the maps as follows. The map $\square(Y_\bullet) \rightarrow \square(X_\bullet)$ comes directly from $Y_\bullet \rightarrow X_\bullet$, and similarly for $N_\ast(Y_\bullet) \rightarrow N_\ast(X_\bullet)$. The map $\square(X_\bullet) \rightarrow \square(C_\bullet)$ just views $\beta$, a subset of $\{1, \ldots, n-1\}$, as a subset of $\{1, \ldots, n\}$. The map $N_\ast(X_\bullet) \rightarrow N_\ast(C_\bullet)$ is the natural map from the map $X_\bullet \rightarrow C_\bullet$. The map $\square(C_\bullet) \rightarrow \square(Y_\bullet)[-1]$ maps $Y_\beta$ to 0 if $n$ is not in $\beta$, and to $Y_{\beta \setminus \{n\}}$ if $n$ is in $\beta$. The map $N_\ast(C_\bullet) \rightarrow N_\ast(Y_\bullet)[-1]$ is the composition of the natural map $N_\ast(C_\bullet) \rightarrow N_\ast(S^1 \wedge Y_\bullet)$ corresponding to contracting $X_\bullet$ to $*$, and the Alexander–Whitney map $N_\ast(S^1 \wedge Y_\bullet) \rightarrow N_\ast(Y_\bullet)[-1]$, a quasi isomorphism (see (A.3)) as $N_\ast(S^1)$ is isomorphic to $\mathbb{Z}[-1]$ via the projection to the $\{0,1\}$-component in degree 1.

We shall check below that those maps give a map of triangles. It is well known (and using Mayer–Vietoris for an open cover $U \cup V$ for nonreduced mapping cones of the realization of simplicial sets easily seen) that the bottom triangle gives rise to a long exact sequence in homology. As the top triangle also gives a long exact sequence, we know by induction that $\Psi : \square(C_\bullet) \rightarrow N_\ast(C_\bullet)$ is a quasi isomorphism, as $\Psi$ is clearly an isomorphism if $n = 0$.

In the diagram, the first square commutes because of the naturality of $\Psi$. For the second square, we note that applying $\Psi_{C_\bullet}$ on the image of $\square(X_\bullet)$ inside $\square(C_\bullet)$ is the same as applying $\Psi_{X_\bullet}$ and tagging on an index $\{0, \ldots, 0\}$ to the indices already used in $N_\ast(X_\bullet)$. (The extra $\{0, \ldots, 0\}$ corresponds to $n \in \{1, \ldots, n\}$.) This is exactly the result as going around the second square counterclockwise, as $X_\bullet$ is the simplicial subscheme of $C_\bullet$ given by the components of $C_\bullet$ that acquire a copy of $\{0, \ldots, 0\}$ from the simplicial interval involved in constructing the mapping cone.

For the third square, the map $\square(C_\bullet) \rightarrow \square(Y_\bullet)[-1]$ corresponds to mapping to zero any $Y_\beta$ with $\beta \subseteq \{1, \ldots, n-1\} \subseteq \{1, \ldots, n\}$, and to $Y_{\beta \setminus \{n\}}$ if $\beta \nsubseteq \{1, \ldots, n-1\}$. If $n \not\in \beta$, $\Psi(Y_\beta)$ will always have the last index $\alpha_n$ in $Y_{\alpha_1, \ldots, \alpha_n}$ equal to $\{0, \ldots, 0\}$, which already goes to zero in $N_\ast(S^1 \wedge Y_\bullet)$. If we have a term $Y_\beta$ with $n \in \beta$, let $k = |\beta|$. Going clockwise, we get $\Psi_{k-1}[-1]|_{Y_{\beta \setminus \{n\}}} = \sum_{\sigma \in S_{k-1}} (-1)^{\sigma} Y_{(\beta \setminus \{n\}, \sigma)}$. Going in the other direction, we get
\[
Y_\beta \mapsto \sum_{\sigma \in S_k} (-1)^{\sigma} Y_{(\beta, \sigma)}
\]
in $N_\ast(C_\bullet)$. The Alexander–Whitney map (A.3) maps this to
\[
\sum_{j=0}^{k} \sum_{\sigma \in S_k} (-1)^{\sigma} d_{j+1} \circ d_{j+2} \circ \cdots \circ d_k (\langle \beta, \sigma \rangle_n) \otimes d_0 Y_{(\beta, \sigma) \setminus \{n, \sigma\}_n}.
\]
After the projection to \{0, 1\} in \(N_*(S^1)\), we only get a nonzero contribution if \((\beta, \sigma)_n = \{0, 1, \ldots, 1\}\) and \(j = 1\). Considering the definition of \((\beta, \sigma)\), this means that \(\sigma(n) = n\), so that \(\sigma \in S_{k-1} \subset S_k\). So we find

\[
\sum_{\sigma \in S_{k-1}} (-1)^\sigma \{0, 1\} \otimes d_0 Y_{(\gamma, \sigma)},
\]

with \(\gamma = (\beta, \sigma) \setminus \{(\beta, \sigma)_n\}\). Considering that \(d_0\) removes the first coordinate in all the first \(n-1\) tuples, \(d_0 Y_{(\gamma, \sigma)} = Y_{(\beta \setminus \{n\}, \sigma)}\). Projecting to the \{0, 1\}–component in \(N_*(S^1) = \mathbb{Z}[1]\), we therefore find

\[
\sum_{\sigma \in S_{k-1}} (-1)^\sigma Y_{(\beta \setminus \{n\}, \sigma)}
\]

as before.

\[\square\]

We now return to our original problem of computing the cohomology groups using explicit complexes. For \(C_* = C(X, \{Y_1, \ldots, Y_n\})\) as above, if all scheme components are smooth of finite type over the base ring \(R\), we can replace \(N_*(C_*)\) by the quasi isomorphic complex \(\square(C_*)\) by Proposition A.12 from the very beginning, so instead of (A.4) or (A.7), we can also use the complex \(C^\square_* (C_*, \Gamma^\square(i))\) with

\[
(A.13) \quad C^\square_* (C_*, \Gamma^\square) = \bigoplus_{t+s=q} \oplus_{|\beta|=\alpha} \Gamma(X_\beta, \Gamma^\square(i))
\]

and \(d^{(s,t)} = (-1)^\sigma (d^\square)\cdot^{(C_*)}_{*} + d_t \Gamma^\square(i).

\textbf{Remark A.14.} In order to get products in \(K\)–theory taking the relative into account, we define maps

\[
C(X, \{Y_1, \ldots, Y_{s+t}\}) \rightarrow C(X, \{Y_1, \ldots, Y_s\}) \wedge C(X, \{Y_{s+1}, \ldots, Y_{s+t}\})
\]

by the diagonal embedding \(Y_{a_1, \ldots, a_{s+t}} \rightarrow Y_{a_1, \ldots, a_s} \times Y_{a_{s+1}, \ldots, a_{s+t}}\) and identifying anything of the form \(\cdots \times * \times \cdots\) with \(*\) in the right hand side. Taking reduced chain complexes, and using the Alexander–Whitney map gives us a map

\[
N_* (C(X, \{Y_{s+1}, \ldots, Y_{s+t}\})) \rightarrow N_* (C(X, \{Y_1, \ldots, Y_s\}) \otimes N_* (C(X, \{Y_{s+1}, \ldots, Y_{s+t}\}))
\]

which we want to compare with a similar map using the \(\square(\cdot)\) complexes. Namely, let \(Y_\beta\) be a component in \(\square(C(X, \{Y_1, \ldots, Y_{s+t}\}))\), and let \(\beta_1 = \{1, \ldots, s\} \cap \beta, \beta_2 = \{s+1, \ldots, s+t\} \cap \beta\). Then we define the map

\[
\square(C(X, \{Y_1, \ldots, Y_{s+t}\})) \rightarrow \square(C(X, \{Y_1, \ldots, Y_s\}) \otimes \square(C(X, \{Y_{s+1}, \ldots, Y_{s+t}\}))
\]

via the map \(Y_\beta \mapsto (-1)^{|\beta_1| \cdot |\beta_2|} Y_{\beta_1} \otimes Y_{\beta_2}\). (This is a map of formal sums of sheaves represented by schemes, and the maps are induced from the scheme embeddings \(Y_\beta \rightarrow Y_{\beta_1}\) and \(Y_\beta \rightarrow Y_{\beta_2}\).) We claim that with this definition we have a commutative diagram (where we suppress \(X\) and \(Y\) from the notation for typographical reasons)

\[
\begin{array}{ccc}
\square(C(1, \ldots, s+t)) & \longrightarrow & \square(C(1, \ldots, s)) \otimes \square(C(s+1, \ldots, s+t)) \\
\downarrow \Psi & & \downarrow \Psi \otimes \Psi \\
N_*(C(1, \ldots, s+t)) & \longrightarrow & N_*(C(1, \ldots, s)) \otimes N_*(C(s+1, \ldots, s+t))
\end{array}
\]
Namely, write $k = |\beta|$, $k_1 = |\beta_1|$ and $k_2 = |\beta_2|$. Starting in the top left corner of the diagram, going the bottom left corner, and then to the bottom right corner (using also the Alexander–Whitney map), we get on $Y_\beta$:

$$Y_\beta \mapsto \sum_{\sigma \in S_k} (-1)^\sigma Y_{(\beta, \sigma)}$$

$$\mapsto \sum_{\sigma \in S_k} (-1)^\sigma Y_{(\beta, \sigma)} \otimes Y_{(\beta, \sigma)_{s+1}, \ldots, (\beta, \sigma)_{s+t}}$$

$$\mapsto \sum_{j=0}^k \sum_{\sigma \in S_k} (-1)^\sigma d_{j+1} d_{j+2} \ldots d_k Y_{(\beta, \sigma)_{s+1}, \ldots, (\beta, \sigma)_{s+t}} \otimes d_0^k Y_{(\beta, \sigma)_{s+1}, \ldots, (\beta, \sigma)_{s+t}}.$$

Now observe that the nonzero indices involved are $k$ in total, of length $k+1$, i.e.,

$$\{0, 0, \ldots, 0, 0, 0, \ldots, 0, 1\}$$

$$\vdots$$

$$\{0, 0, \ldots, 0, 1, 1, \ldots, 1\}$$

$$\{0, 0, \ldots, 1, 1, 1, \ldots, 1\}$$

$$\vdots$$

$$\{0, 1, 1, 1, 1, \ldots, 1\}.$$

$d_{j+1} d_{j+2} \ldots d_k$ deletes the last $k-j$ columns, $d_0^k$ deletes the first $j$ columns of all tuples involved. For fixed $j$, if any of the last $j$ tuples ends up among the last $t$ tuples of $(\beta, \sigma)$, then one of the tuples becomes $\{1, \ldots, 1\}$ under $d_0^k$ and the corresponding component is mapped to zero in $N_s$. So for a nonzero contribution, the last $j$ tuples must end up among $(\beta, \sigma)_1, \ldots, (\beta, \sigma)_s$. If any more tuples end up in $(\beta, \sigma)_1, \ldots, (\beta, \sigma)_s$, then after applying $d_{j+1} d_{j+2} \ldots d_k$, we get at least two tuples consisting entirely of zeroes. But as these contributions are summed alternatingly over $S_k$, and the original indices can be swapped by a transposition which yields a minus sign, those contributions cancel. As $k_1$ tuples must end up among $(\beta, \sigma)_1, \ldots, (\beta, \sigma)_s$, this shows that for a nonzero contribution we must have $j = k_1$, the nonzero tuples among $(\beta, \sigma)_1, \ldots, (\beta, \sigma)_s$ are the last $k_1$ rows above, and the nonzero tuples among $(\beta, \sigma)_{s+1}, \ldots, (\beta, \sigma)_{s+t}$ are the first $k - k_1 = k_2$ rows above. The sum then simplifies to

$$= (-1)^{k_1 k_2} \left( \sum_{\tau_1 \in S_{k_1}} (-1)^{\tau_1} Y_{(\beta_1, \tau_1)} \right) \otimes \left( \sum_{\tau_2 \in S_{k_2}} (-1)^{\tau_2} Y_{(\beta_2, \tau_2)} \right)$$

because the permutation $\sigma$ must map $\{1, \ldots, k - k_1\}$ to $\{k_1 + 1, \ldots, k\}$ as well as $\{k - k_1 + 1, \ldots, k\}$ to $\{1, \ldots, k_1\}$, so we must have

$$(1 \ldots k)^{-k_1} \sigma = \tau_2 (1 k_2 + 1) \ldots (k_1 k) \tau_1 (1 k_2 + 1) \ldots (k_1 k)$$
for some $\tau_1$ in $S_k_1$ and $\tau_2$ in $S_k_2$. As this equals $(-1)^{|\beta_1||\beta_2|}\Psi(Y_{\beta_1}) \otimes \Psi(Y_{\beta_2})$ the diagram commutes as required.

Now suppose that $A$ and $B$ are homological chain complexes, with a cup product $A \otimes B \to C$. There is a map $\phi : K(A) \wedge K(B) \to K(A \otimes B)$ (with $K$ the Dold–Puppe construction as before), which gives rise to a map

$$[S^n \wedge X_\bullet, K(A)] \times [S^m \wedge Y_\bullet, K(B)] \to [S^{m+n} \wedge X_\bullet \wedge Y_\bullet, K(A \otimes B)]$$

$$\to [S^{m+n} \wedge X_\bullet \wedge Y_\bullet \to K(C)].$$

It is shown [dJ95, page 215] that under our identifications $[S^n \wedge X_\bullet, K(A)]$ with $[N_s(X_\bullet)[-n], A]_{DR}$ etc., this corresponds to the composition

$$N_s(X_\bullet \wedge Y_\bullet)[-n - m] \to N_s(X_\bullet)[-n] \otimes N_s(Y_\bullet)[-m] \to A \otimes B \to C$$

with the first map the Alexander–Whitney map and the last map the given product. In the cases we are interested in this becomes a cup product of sections in (pre)sheaves, and it follows from these formulae that the product on components corresponds to cup products $\Gamma(X_s, A) \times \Gamma(Y_t, B) \to \Gamma(X_s \times Y_t, C)$ up to signs. In particular, for the explicit map at the very beginning of this Remark, the diagram tells us that the product in the complex (A.13) is up to a sign given by the map

$$\Gamma(Y_{\beta_1}, A) \times \Gamma(Y_{\beta_2}, B) \to \Gamma(Y_{\beta_1 \beta_2}, C),$$

which is the composition of

$$\Gamma(Y_{\beta_1}, A) \times \Gamma(Y_{\beta_2}, B) \to \Gamma(Y_{\beta_1} \times Y_{\beta_2}, C) \to \Gamma(Y_{\beta_1 \beta_2}, C),$$

the last map being the pullback corresponding to the “diagonal” $Y_{\beta_1 \beta_2} \to Y_{\beta_1} \times Y_{\beta_2}$.

The previous constructions are applied in the body of the paper with the schemes

$$X = (\mathbb{P}_B^n \setminus \{t = 1\})^n$$

and $Y_i = \{t_i = 0, \infty\}$, with $t$ the standard affine coordinate on $\mathbb{P}^1$, and $t_i$ the $i$–th coordinate in the $n$–fold product, or localizations of those schemes.

After this rather explicit exercise, we now turn our attention to the theory of Chern classes. The theory of syntomic Chern classes of [Bes00a] can be extended from schemes to arbitrary spaces as follows. In loc. cit. before Theorem 7.5 universal Chern classes

$$c_n \in H^{2i}(BGL, \Gamma^\bullet(i))$$

were constructed. Again this was explicitly done in some of the theories but it can easily be done in all the others. Further, this was done with the cohomology defined as the cohomology of the complex (A.7), but since the components of $BGL_n$ belong to $S'$ it follows from Remark A.8 that this is the same as the definition we have been using here. Now a standard procedure [GS99, 6.1] produces, for each $\alpha \in H^{-m}(X_\bullet, K)$ a Chern class

$$c_i(\alpha) \in H^{2i-m}(X_\bullet, \Gamma^\bullet(i)).$$

More precisely, if $K = \mathbb{Z} \times \mathbb{Z}_\infty BGL$ is the sheaf used to define algebraic $K$-theory of spaces as $K_n(X_\bullet) = [S^n \wedge X_\bullet, K]$, then each $c_i$ defines a map $K \to K(2i, \Gamma^\bullet(i))$. If $\alpha$ is an element in $K_m(X_\bullet)$, then by composition we get the element $c_i(\alpha)$ in $[S^{m+n} \wedge X_\bullet, K(2i, \Gamma^\bullet(i))] = H^{2i+m}(X_\bullet, \Gamma^\bullet(i))$.

For a $K$–coherent space $X_\bullet$, both

$$H^\wedge(X_\bullet, \Gamma^\bullet(i)) := H^0(X_\bullet, \mathbb{Z}) \times (\{1\} \times (\oplus_{i>1} H(X_\bullet, \Gamma^\bullet(i))))$$
and $\oplus_{m \geq 0} H^{-m}(X_\bullet, K)$ have $\lambda$-ring with involution structures described in loc. cit. 6.1. and there is a total Chern class

$$c : \oplus_{m \geq 0} H^{-m}(X_\bullet, K) \to H^\sim(X_\bullet, \Gamma^\dagger) .$$

**Proposition A.15.** When $X_\bullet$ is $K$-coherent the total Chern class in a morphism of $\lambda$-rings with involutions.

**Proof.** Everything is reduced to the properties of the universal Chern classes (see for example the proof of [GS99, Theorem 5] for the $\lambda$-structure). These properties are deduced in the following way. There is a map of complexes of sheaves (in the derived category) $\Gamma^\dagger(n) \to \Gamma^\dagger_{dR}$, where the latter complex is the complex of differential forms on the generic fiber. We get an induced map of cohomology theories which is compatible with cup products and therefore also with $\lambda$-operations. This map gives an injection

$$\oplus_i H^2_i(BGL_N, \Gamma^\dagger(i)) \to \oplus_i H^{2i}_{dR}(BGL_N/K)$$

on the part of the cohomology of $BGL_N$ containing the Chern classes for any $N$. The syntomic universal Chern classes are defined to map to the corresponding de Rham Chern classes. Since both sides of (A.16) are closed under products, all required properties of syntomic universal Chern classes follow from the corresponding results for the universal de Rham classes.

As all the cohomology groups are $\mathbb{Q}$–vector spaces, one gets a Chern character from this in the usual way (cf. [Sch88, §4] or [Gil81, Definition 2.34]), which gives a ring homomorphism

$$\text{reg} : K_*(X_\bullet) = H^\sim^*(X_\bullet, K) \to H^*(X_\bullet, \Gamma^\dagger(*)) .$$

with the property that $\text{reg}(K_m^n(X_\bullet, K)) \subseteq H^{2j-m}(X_\bullet, \Gamma^\dagger(j))$, cf. [Sch88, Corollary on page 28].

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DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER-SHEVA 84105, ISRAEL

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, SCIENCE LABORATORIES, SOUTH ROAD, DURHAM DH1 3LE, UNITED KINGDOM