Weighted biharmonic equations involving continuous potential under exponential nonlinear growth

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Abstract. We deal with a weighted biharmonic problem in the unit ball of \(\mathbb{R}^4\). The non-linearity is assumed to have critical exponential growth in view of Adam’s type inequalities. The weight \(w(x)\) is of logarithm type and the potential \(V\) is a positive continuous function on \(\overline{B}\). It is proved that there is a nontrivial positive weak solution to this problem by the mountain Pass Theorem. We avoid the loss of compactness by proving a concentration compactness result and by a suitable asymptotic condition.

Keywords: Adams inequality, Moser-Trudinger’s inequality, Nonlinearity of exponential growth, Mountain pass method, Compactness level.

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1 Introduction

In this paper, we study the following weighted fourth order equation

\[
\begin{cases}
L := \Delta (w(x)\Delta u) + V(x)u = f(x, u) & \text{in } B \\
u > 0 & \text{in } B \\
u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B,
\end{cases}
\]

(1.1)

where \(B = B(0, 1)\) in \(\mathbb{R}^4\), \(f(x, t)\) is continuous in \(B \times \mathbb{R}\) and behaves like \(\exp\{\alpha t^{\frac{2}{N}}\}\) as \(t \to +\infty\), for some \(\alpha > 0\).

The weight \(w(x)\) is given by

\[
w(x) = (\log \frac{e}{|x|})^{1-\beta}, \beta \in (0, 1).
\]

(1.2)

The potential \(V : \overline{B} \to \mathbb{R}\) is a positive continuous function satisfying some conditions.

Problems of critical exponential growth in second order elliptic equations without weight in dimension \(N = 2\)

\[-\Delta u = f(x, u) \text{ in } \Omega \subset \mathbb{R}^2.\]


have been studied considerably [2, 19, 22, 26].
In dimension \( N \geq 2 \), the critical exponential growth is given by the well known Trudinger-Moser inequality [28, 33]
\[
\sup_{f_\Omega |\nabla u|^N \leq 1} \int_{\Omega} e^{\alpha |u|^N} dx < +\infty \quad \text{if and only if} \quad \alpha \leq \alpha_N,
\]
where \( \alpha_N = \frac{1}{\omega_{N-1}} \) with \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \).

Later, the Trudinger-Moser inequality was improved to weighted inequalities [9, 10]. The influence of the weight in the Sobolev norm was studied as the compact embedding [21].
When the weight is of logarithmic type, Calanchi and Ruf [11] extend the Trudinger-Moser inequality and proved the following results in the weighted Sobolev space
\[
W^{1,N}_{0,\text{rad}}(B, \rho) = \{ u \in C_0^{\infty}(B) : \int_B |\nabla u|^N \rho(x) dx < \infty \}:
\]

**Theorem 1.1 [10]**

(i) Let \( \beta \in [0,1) \) and let \( \rho \) given by \( \rho(x) = \left( \log \frac{1}{|x|} \right)^{\beta} \), then
\[
\int_B e^{\gamma |u|^N} dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,\text{rad}}(B, \rho), \quad \text{if and only if} \quad \gamma \leq \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta},
\]
and
\[
\sup_{u \in W^{1,N}_{0,\text{rad}}(B, \rho), \int_B |\nabla u|^N \rho(x) dx \leq 1} \int_B e^{\alpha |u|^N} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N,\beta} = N \omega_{N-1}^{1-\beta} \left( 1 - \beta \right)^{\frac{1}{1-\beta}}
\]
where \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \) and \( N' \) is the Hölder conjugate of \( N \).

(ii) Let \( \rho \) given by \( \rho(x) = \left( \log \frac{1}{|x|} \right)^{N-1} \), then
\[
\int_B \exp\{e^{\gamma |u|^N} \} dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,\text{rad}}(B, \rho)
\]
and
\[
\sup_{u \in W^{1,N}_{0,\text{rad}}(B, \rho), \|u\|_\rho \leq 1} \int_B \exp\{\beta e^{\omega_{N-1} |u|^N} \} dx < +\infty \quad \Leftrightarrow \quad \beta \leq N,
\]
where \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \) and \( N' \) is the Hölder conjugate of \( N \).
These results opened the way to study second order weighted elliptic problems in dimension $N \geq 2$. We cite the work of Calanchi et al.\[12\], i.e., the following problem

\[
\begin{aligned}
&-\nabla \cdot (\nu(x)\nabla u) = f(x, u) \quad \text{in} \quad B \\
u > 0 &\quad \text{in} \quad B \\
u = 0 &\quad \text{on} \quad \partial B,
\end{aligned}
\]

with the weight $\nu(x) = \log(\frac{E}{|x|})$ and where the function $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^2}\}$ as $t \to +\infty$, for some $\alpha > 0$. Also, recently, Deng et al.\[15\] and Zhang\[35\] studied the following problem

\[
\begin{aligned}
&-\text{div} (\rho(x)|\nabla u|^{N-2}\nabla u) = f(x, u) \quad \text{in} \quad B \\
u > 0 &\quad \text{in} \quad B \\
u = 0 &\quad \text{on} \quad \partial B,
\end{aligned}
\]

where $N \geq 2$, the function $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{N}{N-1}}}\}$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved that there is a non-trivial solution to this problem using Mountain Pass theorem.

Also, we mention that Baraket et al.\[6\] studied the following non-autonomous weighted elliptic equations

\[
\begin{aligned}
&-\text{div}(\rho(x)|\nabla u|^{N-2}\nabla u) + \xi(x)|u|^{N-2}u = f(x, u) \quad \text{in} \quad B \\
u > 0 &\quad \text{in} \quad B \\
u = 0 &\quad \text{on} \quad \partial B,
\end{aligned}
\]

where $B$ is the unit ball of $\mathbb{R}^N$, $N > 2$, $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{N}{N-1}}}\}$ as $t \to +\infty$, for some $\alpha > 0$. $\xi : B \to \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\rho(x)$ is given by $\rho(x) = (\log \frac{E}{|x|})^{N-1}$.

The biharmonic equation in dimension $N > 4$

\[
\Delta^2 u = f(x, u) \quad \text{in} \quad \Omega \subset \mathbb{R}^N,
\]

where the nonlinearity $f$ has subcritical and critical polynomial growth of power less than $\frac{N+4}{N-4}$, have been extensively studied\[7, 17, 20, 31\].

For bounded domains $\Omega \subset \mathbb{R}^4$, in\[1, 29\] the authors proved the following Adams’ inequality

\[
\sup_{u \in S} \int_{\Omega} (e^{\alpha u^2}) - 1)dx < +\infty \iff \alpha \leq 32\pi^2
\]

where

\[
S = \{u \in W^{2,2}_0(\Omega) \mid \left(\int_{\Omega} |\Delta u|^2 dx\right)^{\frac{1}{2}} \leq 1\}.
\]

This last result opened the way to study fourth-order problems with subcritical or critical nonlinearity involving continuous potential (see\[30, 13\]).

We study the existence of the nontrivial solutions when the nonlinear terms have the critical exponential growth in the sense of Adams’ inequalities\[34\]. Our approach is variational methods such as the Mountain Pass theorem.
Pass Theorem with Palais-Smale condition combining with a concentration compactness result. More precisely, let \( \Omega \subset \mathbb{R}^4 \) be a bounded domain and \( w \in L^1(\Omega) \) be a nonnegative function. We introduce the Sobolev space

\[
W^{2,2}_0(\Omega, w) = \text{cl}\{ u \in C_0^\infty(\Omega) \mid \int_B |\triangle u|^2 w(x) dx < \infty \}.
\]

We will focus on radial functions and consider the subspace

\[
W^{2,2}_{0,rad}(\Omega, w) = \text{cl}\{ u \in C_0^\infty_{rad}(\Omega) \mid \int_B |\triangle u|^2 w(x) dx < \infty \}.
\]

The choice of the weight and the space \( W^{2,2}_{0,rad}(B, w) \) are motivated by the following exponential inequality.

**Theorem 1.2** [34] Let \( \beta \in (0, 1) \) and let \( w \) given by (1), then

\[
\sup_{u \in W^{2,2}_{0,rad}(B, w)} \int_B e^{\alpha |u|^{\frac{2\beta}{1-\beta}}} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_\beta = 4[8\pi^2(1-\beta)]^\frac{1}{1-\beta} \quad (1.3)
\]

Let \( \gamma := \frac{2}{1-\beta} \). In view of inequality (1.3), we say that \( f \) has critical growth at \(+\infty\) if there exists some \( \alpha_0 > 0 \),

\[
\lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^{\gamma}}} = 0, \quad \forall \alpha \text{ such that } \alpha > \alpha_0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^{\gamma}}} = +\infty, \quad \forall \alpha < \alpha_0. \quad (1.4)
\]

To study the solvability of the problem (1.1), consider the space

\[
E = \{ u \in W^{2,2}_{0,rad}(B, w) \mid \int_B V(x)|u|^2 dx < +\infty \},
\]

endowed with the norm

\[
\|u\| = \left( \int_B w(x)|\triangle u|^2 dx + \int_B V(x)|u|^2 dx \right)^\frac{1}{2},
\]

where

\[
w(x) = \left( \log \frac{e}{|x|} \right)^\beta, \beta \in (0, 1).
\]

We note that this norm is issued from the product scalar

\[
\langle u, v \rangle = \int_B (\triangle u \cdot \triangle v) w(x) + V(x) uv \ dx.
\]

Let us now state our results. In this paper, we always assume that the nonlinearities \( f(x, t) \) satisfies these conditions:
\( H_1 \) \( f : B \times \mathbb{R} \to \mathbb{R} \) is continuous, radial in \( x \), and \( f(x, t) = 0 \), for \( t \leq 0 \).

\( H_2 \) There exist \( t_0 > 0 \), \( M > 0 \) such that \( 0 < F(x, t) = \int_0^t f(x, s) ds \leq M|f(x, t)| \), \( \forall t > t_0, \forall x \in B \).

\( H_3 \) \( 0 < F(x, t) \leq \frac{1}{2} f(x, t) t, \forall t > 0, \forall x \in B \).

\( H_4 \) \( \lim_{t \to 0} \sup \frac{2F(x, t)}{t^2} < \lambda_1 \) uniformly in \( x \), where \( \lambda_1 \) is the first eigenvalue of the operator with Dirichlet boundary condition that is

\[
\lambda_1 = \inf_{u \in E, u \neq 0} \frac{\|u\|^2}{\int_B |u|^2 dx}.
\] (1.5)

This eigenvalue \( \lambda_1 \) exists and the corresponding eigen function \( \phi_1 \) is positive and belongs to \( L^\infty(B) \) [16].

and the potential \( V \) is continuous on \( \overline{B} \) and verifies

\( V_1 \) \( V(x) \geq V_0 > 0 \) in \( B \) for some \( V_0 > 0 \).

We say that \( u \) is a solution to the problem (1.1), if \( u \) is a weak solution in the following sense.

**Definition 1.1** A function \( u \) is called a solution to (1.1) if \( u \in E \) and

\[
\int_B (w(x) \Delta u \Delta \varphi + Vu \varphi) dx = \int_B f(x, u) \varphi dx, \quad \text{for all } \varphi \in E.
\]

It is easy to see that seeking weak solutions of the problem (1.1) is equivalent to find nonzero critical points of the following functional on \( E \):

\[
J(u) = \frac{1}{2} \int_B |\Delta u|^2 w(x) dx + \frac{1}{2} \int_B V(x) u^2 dx - \int_B F(x, u) dx,
\] (1.6)

where \( F(x, u) = \int_0^u f(x, t) dt \).

We prove the following result.

**Theorem 1.3** Assume that \( V \) is continuous and verifies \( V_1 \). Assume that the function \( f \) has a critical growth at \( +\infty \) and satisfies the conditions \( H_1 \), \( H_2 \), \( H_3 \) and \( H_4 \). If in addition \( f \) verifies the asymptotic condition

\( H_5 \) \( \lim_{t \to \infty} \frac{f(x, t) t}{e^{\alpha_0 t^2}} \geq \gamma_0 \) uniformly in \( x \), with \( \gamma_0 > \frac{1024(1 - \beta)}{\alpha_0^{-\beta}} \),

then the problem (1.1) has a nontrivial solution.
In general the study of fourth order partial differential equations is considered an interesting topic. The interest in studying such equations was stimulated by their applications in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see [14, 18, 27]. However many applications are generated by the weighted elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example [4, 23]).

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In section 3, we give some useful lemmas for the compactness analysis. In section 4, we prove that the energy $J$ satisfies the two geometric properties, and the compactness condition but under a given level. Finally, we conclude with the proofs of the main results in section 5.

Through this paper, the constant $C$ may change from line to another and we sometimes index the constants in order to show how they change.

### 2 Weighted Lebesgue and Sobolev Spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W^{m,p}_0(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on $\Omega$ and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner in [16], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable}; \int_{\Omega} w(x)|u|^p\,dx < \infty \}$$

for any real number $1 \leq p < \infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \left( \int_{\Omega} w(x)|u|^p\,dx \right)^{\frac{1}{p}}.$$

For $m \geq 2$, let $w$ be a given family of weight functions $w_\tau$, $|\tau| \leq m$,

$$w = \{w_\tau(x) \ x \in \Omega, \ |\tau| \leq m\}.$$

In [16], the corresponding weighted Sobolev space was defined as

$$W^{m,p}(\Omega, w) = \{ u \in L^p(\Omega) \text{ such that } D^\tau u \in L^p(\Omega, w) \text{ for all } 1 \leq |\tau| \leq m \}$$

endowed with the following norm:

$$\|u\|_{W^{m,p}(\Omega, w)} = \left( \sum_{|\tau| \leq m-1} \int_{\Omega} |D^\tau u|^p\,dx + \sum_{|\tau|=m} \int_{\Omega} |D^\tau u|^p w(x)\,dx \right)^{\frac{1}{p}}.$$
If we suppose also that \( w(x) \in L^1_{\text{loc}}(\Omega) \), then \( C_0^\infty(\Omega) \) is a subset of \( W^{m,p}(\Omega, w) \) and we can introduce the space \( W_0^{m,p}(\Omega, w) \) as the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega, w) \).

\((L^p(\Omega, w), \| \cdot \|_{p,w})\) and \((W^{m,p}(\Omega, w), \| \cdot \|_{W^{m,p}(\Omega, w)})\) are separable, reflexive Banach spaces provided that \( w(x) \frac{x}{|x|} \in L^1_{\text{loc}}(\Omega) \).

For \( w(x) = 1 \), one finds the standard Sobolev spaces \( W^{m,p}(\Omega) \) and \( W_0^{m,p}(\Omega) \).

Then the space \( E = \left\{ u \in W_0^{2,2}(B, w) \mid \int_B V(x)|u|^2 \, dx < +\infty \right\} \) is a Banach and reflexive space provided \((V_1)\) is satisfied. The space \( E \) is endowed with the norm

\[ \|u\| = \left( \int_B w(x)|\Delta u|^2 \, dx + \int_B V(x)|u|^2 \, dx \right)^{\frac{1}{2}} \]

which is equivalent to the following norm (see lemma 1)

\[ \|u\|_{W_0^{2,2}(B, w)} = \left( \int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B |\Delta u|^2 w(x) \, dx \right)^{\frac{1}{2}}. \]

### 3 Preliminary for the compactness analysis

In this section, we will derive several technical lemmas for our use later. First we begin by the radial lemma.

**Lemma 1** Assume that \( V \) is continuous and verifies \((V_1)\).

(i) Let \( u \) be a radially symmetric function in \( C_0^\infty(B) \). Then, we have

\[ |u(x)| \leq \frac{1}{2\sqrt{2\pi}} \frac{\| \log(\frac{|x|}{r}) \|^{1-\beta} - 1}{\sqrt{1-\beta}} \int_B w(x)|\Delta u|^2 \, dx \leq \frac{1}{2\sqrt{2\pi}} \frac{\| \log(\frac{|x|}{r}) \|^{1-\beta} - 1}{\sqrt{1-\beta}} \|u\|^2. \]

(ii) There exists a positive constant \( C \) such that for all \( u \in E \)

\[ \int_B V|u|^2 \, dx \leq C\|u\|^2 \]

and then the norms \( \| \cdot \| \) and \( \|u\|_{W_0^{2,2}(B, w)} = \left( \int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B |\Delta u|^2 w(x) \, dx \right)^{\frac{1}{2}} \) are equivalents.
(iv) The following embedding is continuous
\[ E \hookrightarrow L^q(B) \text{ for all } q \geq 1. \]

(v) \( E \) is compactly embedded in \( L^q(B) \) for all \( q \geq 1 \).

Proof

(i) see [34]

(ii) For all \( u \in E \),
\[ V_0 \int_B |u|^2 dx \leq \int_B V(x)|u|^2 dx \leq m \int_B |u|^2 dx \]
where \( m = \max_{x \in B} V(x) \). Then, (ii) follows.

(iii) Since \( w(x) \geq 1 \), then following embedding are continuous
\[ E \hookrightarrow W^{2,2}_{0,rad}(B, w) \hookrightarrow W^{1,2}_{0,rad}(B) \hookrightarrow L^q(B) \quad \forall q \geq 2. \]
We also have by the Hölder inequality,
\[ \int_B |u|^{\beta_0} dx \leq \int_B (|u| + |u|^2) dx \leq (\int_B \frac{1}{V} dx)^{\frac{1}{2}} \left( \int_B V |u|^2 dx \right)^{\frac{1}{2}} \leq (\int_B \frac{1}{V} dx)^{\frac{1}{2}} \|u\| + \frac{1}{V_0} \|u\|^2. \]

Thus, we get the continuous embedding \( E \hookrightarrow L^p(B) \) for all \( q \geq 1 \).

(iv) The above embedding is also compact. Indeed, let \( u_k \subset E \) be a sequence such that \( \|u_k\| \leq C \) for all \( k \). Then \( \|u_k\|_{W^{2,2}_{0,rad}(B, w)} \leq C \), for all \( k \). On the other hand, we have the following compact embedding
\[ W^{2,2}_{0,rad}(B, w) \hookrightarrow W^{1,2}_{0,rad}(B) \hookrightarrow L^q(B) \text{ for all } q \text{ such that } 1 \leq q < 4, \]
then up to a subsequence, there exists some \( u \in W^{2,2}_{0,rad}(B, w) \), such that \( u_k \) convergent to \( u \) strongly in \( L^q(B) \) for all \( q \) such that \( 1 \leq q < 4 \). Without loss of generality, we may assume that

\[
\begin{align*}
  u_k &\rightharpoonup u \quad \text{weakly in } E \\
  u_k &\rightarrow u \quad \text{strongly in } L^1(B) \\
  u_k(x) &\rightarrow u(x) \quad \text{almost everywhere in } B.
\end{align*}
\]

For \( q > 1 \), it follows from (3.1) and the continuous embedding \( E \hookrightarrow L^p(B) \) \( (p \geq 1) \) that
\[
\int_B |u_k - u|^q dx = \int_B |u_k - u|^q |u|^q \frac{1}{|u|^q} dx \\
\leq \left( \int_B |u_k - u| dx \right)^{\frac{q}{2}} \left( \int_B |u_k - u|^{2q-1} dx \right)^{\frac{1}{2}} \\
\leq C \left( \int_B |u_k - u| dx \right)^{\frac{1}{2}} \rightarrow 0.
\]
This concludes the lemma.
Second, we give the following useful lemma.

**Lemma 2** [19] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \Omega \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to $u$ in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If
\[
\int_{\Omega} |f(x, u_n)u_n| dx \leq C,
\]
where $C$ is a positive constant, then
\[
f(x, u_n) \to f(x, u) \text{ in } L^1(\Omega).
\]

In the sequel, we prove a concentration compactness result of Lions type.

**Lemma 3** Let $(u_k)_k$ be a sequence in $E$. Suppose that,
\[
\|u_k\| = 1, u_k \rightharpoonup u \text{ weakly in } E, u_k(x) \to u(x) \text{ a.e } x \in B, \text{ and } u \neq 0.
\]
Then
\[
\sup_k \int_B e^{\alpha_\beta|u_k|^\gamma} dx < +\infty, \text{ where } \alpha_\beta = 4(8\pi^2(1 - \beta))^{\frac{1}{1 - \beta}},
\]
for all $1 < p < U(u)$ where $U(u)$ is given by:
\[
U(u) := \begin{cases} 
1 & \text{if } \|u\| < 1 \\
\frac{1}{(1 - \|u\|^2)^\frac{1}{p}} & \text{if } \|u\| = 1 
\end{cases}
\]

**Proof**
Since $\|u\| \leq \liminf_k \|u_k\| = 1$, we will split the evidence into two cases.

**Case 1**: $\|u\| < 1$. We assume by contradiction for some $p_1 < U(u)$, we have
\[
\sup_k \int_B \exp(\alpha_\beta p_1 u_k^\gamma) dx = +\infty.
\]
Set
\[
B_k^L = \{ x \in B : u_k(x) \geq L \}
\]
where $L$ is a constant that we will choose later. Let $v_k = u_k - L$. We have
\[
(1 + a)^q \leq (1 + \varepsilon)a^q + (1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q - 1}}})^{1 - q}, \forall a \geq 0, \forall \varepsilon > 0 \forall q > 1.
\]
So, using (3.2), we get
\[
|u_k|^\gamma = |u_k - L + L|^\gamma
\leq \left(|u_k - L| + |L|\right)^\gamma
\leq (1 + \varepsilon)|u_k - L|^\gamma + \left(1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q - 1}}}ight)^{1 - \gamma}|L|^\gamma
\leq (1 + \varepsilon)v_k^\gamma + C(\varepsilon, \gamma)L^\gamma.
\]
We have

\[ \int_B \exp (\alpha \beta p_1 u_k^2) \, dx = \int_{B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx + \int_{B \setminus B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx \]

\[ \leq \int_{B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx + c \exp (\alpha \beta p_1 \mathcal{L}^\gamma) \]

\[ \leq \int_{B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx + c(L, \gamma, |B|), \]

and then

\[ \sup_k \int_{B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx = \infty. \]

By (3.3) we have

\[ \int_{B^+_k} \exp (\alpha \beta p_1 u_k^2) \, dx \leq \exp (\alpha \beta p_1 C(\varepsilon, \gamma) \mathcal{L}^\gamma) \times \int_{B^+_k} \exp ((1 + \varepsilon) \alpha \beta p_1 u_k^2) \, dx. \]

Since, \( p_1 < U(u) \), there exists \( \varepsilon > 0 \) such that \( \tilde{p}_1 = (1 + \varepsilon)p_1 < U(u) \). Thus

\[ \sup_k \int_{B^+_k} \exp (\tilde{p}_1 \alpha \beta v_k^2) \, dx = \infty \quad (3.4) \]

Now, we define

\[ T^\mathcal{L}(u) = \min \{ \mathcal{L}, u \} \text{ and } T^\mathcal{L}_u(u) = u - T^\mathcal{L}(u) \]

and choose \( \mathcal{L} \) such that

\[ \frac{1 - \|u\|^2}{1 - \|T^\mathcal{L} u\|^2} > \left( \frac{\tilde{p}_1}{U(u)} \right)^\frac{2}{\tilde{p}_1}. \quad (3.5) \]

We claim that

\[ \limsup_k \int_{B^+_k} (\omega(x) |\Delta v_k|^2 + V(x) v_k^2) \, dx < \left( \frac{1}{\tilde{p}_1} \right)^{\frac{2}{\tilde{p}_1}}. \]

If this is not the case, then up to a subsequence, we get

\[ \int_{B^+_k} (\omega(x) |\Delta v_k|^2 + V(x) v_k^2) \, dx = \int_B (\omega(x) |\Delta T^\mathcal{L}_u k|^2 + V(x)(T^\mathcal{L}_u k)^2) \, dx \geq \left( \frac{1}{\tilde{p}_1} \right)^{\frac{2}{\tilde{p}_1}} + o_k(1). \]

Thus,
\[
\left( \frac{1}{p_1} \right)^{\frac{1}{\gamma}} + \int_B (\omega(x) |\Delta T^L u_k|^2 + V(x)(T^L u_k)^2) \, dx + o_k(1) \leq \\
\int_B (\omega(x) |\Delta T^L u_k|^2 + V(x)(T^L u_k)^2) \, dx + \int_{B \setminus B_k^L} (\omega(x) |\Delta T^L u_k|^2 + V(x)(T^L u_k)^2) \, dx \\
= \int_{B_k^L} (\omega(x) |\Delta u_k|^2 + V(x)u_k^2) \, dx + \int_{B \setminus B_k^L} (\omega(x) |\Delta u_k|^2 + V(x)u_k^2) \, dx = 1.
\]

For \( \mathcal{L} > 0 \) fixed, \( T^L u_k \) is also bounded in \( E \). Hence, up to a subsequence, \( T^L u_k \rightarrow T^L u \) weakly in \( E \) and \( T^L u_k \rightarrow T^L u \) almost everywhere in \( B \). By the lower semicontinuity of the norm in \( E \) and the last inequality, we have

\[
\tilde{p}_1 \geq \frac{1}{\left( 1 - \liminf_{k \to +\infty} \|T^L u_k\|^2 \right)^{\frac{1}{\gamma}}} \geq \frac{1}{\left( 1 - \|T^L u\|^2 \right)^{\frac{1}{\gamma}}},
\]

combining with (3.5), we obtain

\[
\tilde{p}_1 \geq \frac{1}{\left( 1 - \|T^L u\|^2 \right)^{\frac{1}{\gamma}}} \geq \frac{\tilde{p}_1}{U(u)} \left( \frac{1}{1 - \|T^L u\|^2} \right)^{\frac{1}{\gamma}} = \tilde{p}_1,
\]

which is a contradiction. Therefore

\[
\limsup_k \int_{B_k^L} (\omega(x) |\Delta v_k|^2 + V(x)v_k^2) \, dx < \left( \frac{1}{\tilde{p}_1} \right)^{\frac{1}{\gamma}}.
\]

By the Adam’s inequality (1.3), we deduce that

\[
\sup_k \int_{B_k^L} \exp(\tilde{p}_1 \alpha \beta v_k^2) \, dx < \infty
\]

which is also a contradiction. The proof is finished in this case.

**Case 2:** \( \|u\| = 1 \). We can then proceed as in case 1 and obtain

\[
\sup_k \int_{B_k^L} \exp(\tilde{p}_1 \alpha \beta v_k^2) \, dx = \infty
\]

where \( \tilde{p}_1 = (1 + \varepsilon)p_1 \). Then we have

\[
\limsup_k \int_{B_k^L} (\omega(x) |\Delta v_k|^2 + V(x)v_k^2) \, dx = \limsup_k \int_B (\omega(x) |\Delta T^L u_k|^2 + V(x)(T^L u_k)^2) \, dx \geq \left( \frac{1}{\tilde{p}_1} \right)^{\frac{1}{\gamma}}
\]

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thus,

\[
\|T^\mathcal{L}u\|^2 \leq \liminf_k \int_B (\omega(x) |\Delta T^\mathcal{L}u_k|^2 + V(x)(T^\mathcal{L}u_k)^2) dx
\]

\[
\leq 1 - \limsup_k \int_B (|\Delta T^\mathcal{L}u_k|^2 + V(x)(T^\mathcal{L}u_k)^2) dx
\]

\[
\leq 1 - \left(\frac{1}{\rho_1}\right)^{\frac{2}{p_1}}.
\]

On the other hand, since \(\|u\| = 1\), we can take \(\mathcal{L}\) large enough such that

\[
\|T^\mathcal{L}u\|^2 > 1 - \frac{1}{2} \left(\frac{1}{\rho_1}\right)^{\frac{2}{p_1}}
\]

which is a contradiction, and the proof is complete in this case. This complete the proof and lemma 3 is proved.

4 The variational formulation

Since the reaction term \(f\) is of critical exponential growth, there exist positive constants \(a\) and \(c\) such that

\[
|f(x,t)| \leq Ce^{a|t|^\gamma}, \quad \forall |t| > t_1.
\]  

(4.1)

and so, by using \((H_1)\), the functional \(J\) given by (1.6) is \(C^1\).

4.1 The mountain pass geometry of the energy

In the sequel, we prove that the functional \(J\) has a mountain pass geometry.

**Proposition 4.1** Assume that the hypothesis \((H_1), (H_2), (H_3), (H_4),\) and \((V_1)\) hold. Then

(i) there exist \(\rho, \beta_0 > 0\) such that \(J(u) \geq \beta_0\) for all \(u \in E\) with \(\|u\| = \rho\).

(ii) Let \(\varphi_1\) be a normalized eigenfunction associated to \(\lambda_1\) in \(E\). Then, \(J(t\varphi_1) \rightarrow -\infty\) as \(t \rightarrow +\infty\).

**Proof** It follows from the hypothesis \((H_4)\) that there exists \(t_2 > 0\) and there exists \(\varepsilon \in (0, 1)\) such that

\[
F(x,t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0)|t|^2, \quad \text{for } |t| < t_2.
\]  

(4.2)

Indeed,

\[
\limsup_{t \rightarrow 0} \frac{2F(x,t)}{t^2} < \lambda_1
\]
or
\[ \inf_{\tau > 0} \sup_{0 < t < \tau} \frac{2F(x, t)}{t^2}, \quad 0 < t < \tau \} < \lambda_1. \]

Since this inequality is strict, then there exists \( \varepsilon_0 > 0 \) such that
\[ \inf_{\tau > 0} \sup_{0 < t < \tau} \frac{2F(x, t)}{t^2}, \quad 0 < t < \tau \} < \lambda_1 - \varepsilon_0, \]
hence, there exists \( t_2 > 0 \) such that
\[ \sup_{0 < t < t_2} \frac{2F(x, t)}{t^2}, \quad 0 < t < t_2 \} < \lambda_1 - \varepsilon_0. \]

Hence
\[ \forall |t| < t_2 \quad F(x, t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0) |t|^2. \]

From \((H3)\) and \((4.1)\) and for all \( q > 2 \), there exist a constant \( C > 0 \) such that
\[ F(x, t) \leq C |t|^q e^{\alpha t^\gamma}, \quad \forall |t| > t_1. \quad (4.3) \]

So
\[ F(x, t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0) |t|^2 + C |t|^q e^{\alpha t^\gamma}, \quad \text{for all} \ t \in \mathbb{R}. \quad (4.4) \]

Since
\[ \mathcal{J}(u) = \frac{1}{2} \|u\|^2 - \int_B F(x, u) dx, \]
we get
\[ \mathcal{J}(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{2} \lambda_1 (1 - \varepsilon_0) \|u\|^2 - C \int_B |u|^q e^{\alpha u^\gamma} dx. \]

But \( \lambda_1 \|u\|^2 \leq \|u\|^2 \) and from the Hölder inequality, we obtain
\[ \mathcal{J}(u) \geq \frac{\varepsilon_0}{2} \|u\|^2 - C \left( \int_B e^{2\alpha |u|^\gamma} dx \right)^{\frac{1}{q}} \|u\|_{2q}^q. \quad (4.5) \]

From the Theorem 1.2, if we choose \( u \in E \) such that
\[ 2\alpha \|u\|^\gamma \leq \alpha_3, \quad (4.6) \]
we get
\[ \int_B e^{2\alpha |u|^\gamma} dx = \int_B e^{2\alpha \|u\|^\gamma (\frac{\|u\|}{\alpha_3})^\gamma} dx < +\infty. \]

On the other hand \( \|u\|_{2q} \leq C \|u\| \) (Lemma 1), so
\[ \mathcal{J}(u) \geq \frac{\varepsilon_0}{2} \|u\|^2 - C \|u\|^q, \]
for all $u \in E$ satisfying (4.6). Since $2 < q$, we can choose $\rho = \|u\| > 0$ as the maximum point of the function $g(\sigma) = \frac{\alpha}{2} \sigma^2 - C \sigma^q$ on the interval $[0, (\frac{\alpha \beta}{2 \alpha})^\frac{1}{q}]$ and $\beta_0 = g(\rho) \cdot J(u) \geq \beta_0 > 0$.

(ii) Let $\phi_1 \in E \cap L^\infty(B)$ be the normalized eigen function associated to the eigen-value defined by (1.5) such that $\| \phi_1 \| = 1$. We define the function

$$\varphi(t) = J(t \phi_1) = \frac{t^2}{2} \| \phi_1 \|^2 - \int_B F(x, t \phi_1) dx.$$ 

Then using $(H_1), (H_2)$ and integrating, we get the existence of a constant $C > 0$ such that

$$F(x, t) \geq C e^{\frac{\alpha \beta}{4} t}, \quad \forall |t| \geq t_0.$$ 

In particular, for $p > 2$, there exists $C$ such that

$$F(x, t) \geq C|t|^p - C \geq C|t|^p - C, \quad \forall t \in \mathbb{R}, \ x \in B.$$ 

Hence,

$$\varphi(t) = J(t \phi_1) \leq \frac{t^2}{2} \| \phi_1 \|^2 - |t|^p \| \phi_1 \|_p - c_5 \to -\infty, \quad \text{as} \ t \to +\infty,$$

and it’s easy to conclude.

### 4.2 The compactness level of the energy

The main difficulty in the variational approach to the critical growth problem is the lack of compactness. Precisely the global Palais-Smale condition does not hold. Eventually, some partial Palais-Smale condition still holds under a given level. In the following proposition, we identify the first compactness level of the energy.

**Proposition 4.2** Let $J$ be the energy associated to the problem (1.1) defined by (1.6), and suppose that the conditions $(V_1), (H_1), (H_2), (H_3)$ and $(H_4)$ are satisfied. If the function $f(x, t)$ satisfies the condition (1.4) for some $\alpha_0 > 0$, then the functional $J$ satisfies the Palais-Smale condition $(PS)_c$ for any

$$c < \frac{1}{2} \left( \frac{\alpha_0 \beta}{\alpha_0} \right)^\frac{1}{p},$$

where $\alpha_0 \beta = 4[8\pi^2(1 - \beta)]^{\frac{1}{p}}$.

**Proof** Consider a $(PS)_c$ sequence in $E$, for some $c \in \mathbb{R}$, that is

$$J(u_n) = \frac{1}{2} \| u_n \|^2 - \int_B F(x, u_n) dx \to c, \ n \to +\infty \quad (4.7)$$

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and
\[ |(J'(u_n), \varphi)| = \left| \int_B w(x) \Delta u_n \Delta \varphi dx + \int_B V u_n \varphi dx - \int_B f(x, u_n) \varphi dx \right| \leq \varepsilon_n \| \varphi \|, \tag{4.8} \]
for all \( \varphi \in E \), where \( \varepsilon_n \to 0 \), when \( n \to +\infty \).

Also, inspired by [12], it follows from \((H_2)\) that for all \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that
\[ F(x, t) \leq \varepsilon t f(x, t), \quad \text{for all } |t| > t_\varepsilon \text{ and uniformly in } x \in B, \tag{4.9} \]
and so, by (4.7), for all \( \varepsilon > 0 \) there exists a constant \( C > 0 \)
\[ \frac{1}{2} \| u_n \|^2 \leq C + \int_B F(x, u_n) dx, \]
hence
\[ \frac{1}{2} \| u_n \|^2 \leq C + \int_{|u_n| \leq t_\varepsilon} F(x, u_n) dx + \varepsilon \int_B f(x, u_n) u_n dx \]
and so, from (4.8), we get
\[ \frac{1}{2} \| u_n \|^2 \leq C_1 + \varepsilon \varepsilon_n \| u_n \| + \varepsilon \| u_n \|^2, \]
for some constant \( C_1 > 0 \). Since
\[ \left( \frac{1}{2} - \varepsilon \right) \| u_n \|^2 \leq C_1 + \varepsilon \varepsilon_n \| u_n \|, \tag{4.10} \]
we deduce that the sequence \((u_n)\) is bounded in \( E \). As consequence, there exists \( u \in E \) such that, up to subsequence, \( u_n \rightharpoonup u \) weakly in \( E \), \( u_n \to u \) strongly in \( L^q(B) \), for all \( q \geq 1 \) and \( u_n(x) \to u(x) \) a.e. in \( B \).

Furthermore, we have, from (4.7) and (4.8), that
\[ 0 < \int_B f(x, u_n) u_n \leq C, \tag{4.11} \]
and
\[ 0 < \int_B F(x, u_n) \leq C. \tag{4.12} \]

Since by Lemma 2.1 in [19], we have
\[ f(x, u_n) \to f(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty, \tag{4.13} \]
then, it follows from \((H_2)\) and the generalized Lebesgue dominated convergence theorem that
\[ F(x, u_n) \to F(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{4.14} \]

So,
\[ \lim_{n \to +\infty} \| u_n \|^2 = 2(c + \int_B F(x, u) dx). \tag{4.15} \]
Using (4.4), we have
\[ \lim_{n \to \infty} \int_B f(x, u_n)u_n \, dx = 2(c + \int_B F(x, u) \, dx). \] (4.16)

Then by \((H_3)\) and (4.8), we get
\[ \lim_{n \to +\infty} 2 \int_B F(x, u_n) \, dx \leq \lim_{n \to +\infty} \int_B f(x, u_n)u_n \, dx = 2(c + \int_B F(x, u) \, dx). \] (4.17)

As a direct consequence from (4.17) and (4.14), we get \(c \geq 0\).

Also, by the definition of the weak convergence, we get \(\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle\). Then, passing to the limit in (4.8) and using (4.13), we obtain that \(u\) is a weak solution of the problem (1.1) that is
\[ \int_B (w(x) \Delta u \Delta \varphi + Vu \varphi) \, dx = \int_B f(x, u) \, \varphi \, dx, \quad \text{for all } \varphi \in E. \]

Taking \(\varphi = u\) as a test function, we get
\[ \int_B |\Delta u|^2 w(x) \, dx + \int_B Vu^2 \, dx = \int_B f(x, u)u \, dx \geq 2 \int_B F(x, u) \, dx. \]

Hence \(J(u) \geq 0\). We also have by the Fatou’s lemma and (4.14)
\[ 0 \leq J(u) \leq \frac{1}{2} \liminf_{n \to \infty} ||u_n||^2 - \int_B F(x, u) \, dx = c. \]

So, we will finish the proof by considering three cases for the level \(c\).

\textbf{Case 1.} \(c = 0\). In this case
\[ 0 \leq J(u) \leq \liminf_{n \to +\infty} J(u_n) = 0. \]
So,
\[ J(u) = 0 \]

and then by (4.14)
\[ \lim_{n \to +\infty} \frac{1}{2} ||u_n||^2 = \int_B F(x, u) \, dx = \frac{1}{2} ||u||^2. \]

It follows that \(u_n \to u\) in \(E\).

\textbf{Case 2.} \(c > 0\) and \(u = 0\). We prove that this case cannot happen.

From (4.7) and (4.8) with \(v = u_n\), we have
\[ \lim_{n \to +\infty} ||u_n||^2 = 2c \quad \text{and} \quad \lim_{n \to +\infty} \int_B f(x, u_n)u_n \, dx = 2c. \]

Again by (4.11) we have
\[ ||u_n||^2 - \int_B f(x, u_n)u_n \, dx \leq C \varepsilon_n. \]
First we claim that there exists \( q > 1 \) such that
\[
\int_B |f(x, u_n)|^q dx \leq C, \tag{4.18}
\]
so
\[
\|u_n\|^2 \leq C\varepsilon_n + \left( \int_B |f(x, u_n)|^q \right)^{\frac{q}{q'}} dx \left( \int_B |u_n|^q' \right)^{\frac{1}{q'}}
\]
where \( q' \) the conjugate of \( q \). Since \( (u_n) \) converge to \( u = 0 \) in \( L^{q'}(B) \),
\[
\lim_{n \to +\infty} \|u_n\|^2 = 0
\]
which in contradiction with \( c > 0 \).

For the proof of the claim, since \( f \) has subcritical or critical growth, for every \( \varepsilon > 0 \) and \( q > 1 \) there exists \( t_\varepsilon > 0 \) and \( C > 0 \) such that for all \( |t| \geq t_\varepsilon \), we have
\[
|f(x, t)|^q \leq Ce^{\alpha_0 (\varepsilon + 1)t^\gamma}. \tag{4.19}
\]
Consequently,
\[
\int_B |f(x, u_n)|^q dx = \int_{\{|u_n| \leq t_\varepsilon\}} |f(x, u_n)|^q dx + \int_{\{|u_n| > t_\varepsilon\}} |f(x, u_n)|^q dx \\
\leq 2\pi^2 \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_0 (\varepsilon + 1)|u_n|^\gamma} dx.
\]

Since \( 2c < \left( \frac{\alpha_0}{\alpha_0} \right)^2 \), there exists \( \eta \in (0, \frac{1}{2}) \) such that \( 2c = (1 - 2\eta)(\frac{\alpha_0}{\alpha_0})^2 \).

On the other hand, \( \|u_n\|^{\gamma} \to (2c)^{\gamma} \), so there exists \( n_\eta > 0 \) such that for all \( n \geq n_\eta \), we get
\[
\|u_n\|^\gamma \leq (1 - \eta) \frac{\alpha_0}{\alpha_0}
\]
Therefore,
\[
\alpha_0 (1 + \varepsilon) \left( \frac{|u_n|}{\|u_n\|} \right)^\gamma \|u_n\|^\gamma \leq (1 + \varepsilon)(1 - \eta)\alpha_0.
\]
We choose \( \varepsilon > 0 \) small enough to get
\[
\alpha_0 (1 + \varepsilon) \|u_n\|^\gamma \leq \alpha_0.
\]
Therefore, the second integral is uniformly bounded in view of (1.3) and the claim is proved.

**Case 3.** \( c > 0 \) and \( u \neq 0 \). In this case, we claim that \( J(u) = c \) and therefore, we get
\[
\lim_{n \to +\infty} \|u_n\|^2 = 2(c + \int_B F(x, u) dx) = 2(J(u) + \int_B F(x, u) dx) = \|u\|^2.
\]
Do not forgot that
\[ J(u) \leq \frac{1}{2} \lim_{n \to +\infty} \|u_n\|^2 - \int_B F(x, u)dx = c. \]

Suppose that \( J(u) < c \). Then,
\[ \|u\|^\gamma < (2(c + \int_B F(x, u)dx))^\frac{\gamma}{2}. \] (4.20)

Set
\[ v_n = \frac{u_n}{\|u_n\|} \]
and
\[ v = \frac{u}{(2(c + \int_B F(x, u)dx))^\frac{\gamma}{2}}. \]

We have \( \|v_n\| = 1 \), \( v_n \rightharpoonup v \) in \( E \), \( v \neq 0 \) and \( \|v\| < 1 \). So, by Lemma 3, we get
\[ \sup_n \int_B e^{p\alpha \|v_n\|^\gamma} dx < \infty, \]
for \( 1 < p < U(v) = (1 - \|v\|^2)^\frac{\gamma}{2} \).

As in the case (2), we are going to estimate \( \int_B |f(x, u_n)|^q dx \).

For \( \varepsilon > 0 \), one has
\[
\int_B |f(x, u_n)|^q dx = \int_{\{|u_n| \leq \varepsilon\}} |f(x, u_n)|^q dx + \int_{\{|u_n| > \varepsilon\}} |f(x, u_n)|^q dx \\
\leq 2\pi^2 \max_{B \times [-\varepsilon, \varepsilon]} |f(x, t)|^q + C \int_B e^{c_0(1+\varepsilon)|u_n|^\gamma} dx \\
\leq C_\varepsilon + C \int_B e^{c_0(1+\varepsilon)|u_n|^\gamma|v_2|^\gamma} dx \leq C,
\]
provided that \( \alpha_0(1+\varepsilon)|u_n|^\gamma \leq p \) \( \alpha_\beta \) and \( 1 < p < U(v) = (1 - \|v\|^2)^\frac{\gamma}{2} \).

Since
\[
(1 - \|v\|^2)^\frac{\gamma}{2} = \left( \frac{2(c + \int_B F(x, u)dx)}{2(c + \int_B F(x, u)dx - \|u\|^2)} \right)^\frac{\gamma}{2} = \left( \frac{c + \int_B F(x, u)dx}{c - J(u)} \right)^\frac{\gamma}{2},
\]
and
\[
\lim_{n \to +\infty} \|u_n\|^\gamma = (2(c + \int_B F(x, u)dx))^\frac{\gamma}{2},
\]
then,
\[ \alpha_0(1+\varepsilon)|u_n|^\gamma \leq \alpha_0(1+2\varepsilon)(2(c + \int_B F(x, u)dx))^\frac{\gamma}{2}. \]

But \( J(u) \geq 0 \) and \( c < \frac{1}{2} \left( \frac{\alpha_\beta}{\alpha_0} \right)^\frac{\gamma}{2} \), then if we choose \( \varepsilon > 0 \) small enough such that
\[
\frac{\alpha_0}{\alpha_\beta}(1+2\varepsilon) < \left( \frac{1}{2(c - J(u))} \right)^\frac{\gamma}{2},
\]
we have...
we get,

\[(1 + 2\varepsilon)\left((c - J(u))^2\right) < \frac{\alpha\beta}{2\pi\alpha_0} \]

So, the sequence \((f(x, u_n))\) is bounded in \(L^q, q > 1\).

Since \(\langle J'(u_n), (u_n - u) \rangle = o(1)\), we have from the boundedness of \(\{f(x, u_n)\}\) in \(L^q(B)\) for \(q > 1\), we can prove that \(u_n \to u\) strongly in \(E\). Indeed, we have

\[\|u_n - u\|^2 = (u_n, u_n - u) - (u, u_n - u) = (u_n, u_n - u) + o_n(1) \to 0 \text{ as } n \to +\infty.\]

From (4.5) and using the Hölder inequality, we get

\[\left|\langle u_n, u_n - u \rangle\right| \leq \varepsilon_n \|u_n - u\| + \int_B f(x, u_n)(u_n - u)dx \leq C\varepsilon_n + \left(\int_B |f(x, u_n)|qdx\right)^{\frac{1}{q}}\left(\int_B |u_n - u|^q \right)^{\frac{1}{q}} \to 0 \text{ as } n \to +\infty.\]

Hence,

\[\lim_{n \to +\infty} \|u_n\|^2 = 2(c + \int_B F(x, u)dx) = \|u\|^2\]

and this contradicts (4.20). So, \(J(u) = c\) and consequently, \(u_n \to u\).

5 Proof of the main results

In the sequel, we will estimate the minimax level of the energy \(J\). We will prove that the mountain pass level \(c\) satisfies

\[c < \frac{1}{2}\left(\frac{\alpha\beta}{\alpha_0}\right)^{\frac{2}{q}}.\]

For this purpose, we will prove that there exists \(z \in E\) such that

\[\max_{t \geq 0} J(tz) < \frac{1}{2}\left(\frac{\alpha\beta}{\alpha_0}\right)^{\frac{2}{q}}.\]  \hspace{1cm} (5.1)

5.1 Adams functions

Now, we will construct particular functions, namely the Adams functions. We consider the sequence defined for all \(n \geq 3\) by

\[\text{(5.2)}\]
Let \( \|\Delta \| = w \) so, we compute \( \zeta_n \) where

\[
\zeta_n = \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) + \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) + \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right)
\]

if \( 0 \leq |x| \leq \frac{1}{\sqrt{n}} \)

\[
\zeta_n = \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) - \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) + \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right)
\]

if \( \frac{1}{\sqrt{n}} \leq |x| \leq \frac{1}{2} \)

\[
\zeta_n = \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) - \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right) + \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right)
\]

if \( \frac{1}{2} \leq |x| \leq 1 \)

where \( \zeta_n \in C_0^\infty(B) \) is such that

\[
\zeta_n|_{x=x} = \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right)
\]

\[
\frac{\partial \zeta_n}{\partial x}|_{x=x} = \frac{1}{2} \left( \frac{\log(e^\sqrt{n})}{\alpha \beta} \right)
\]

\[
\zeta_n|_{aB} = \frac{\partial \zeta_n}{\partial x}|_{aB} = 0 \text{ and } \zeta_n, \nabla \zeta_n, \Delta \zeta_n \text{ are all } o \left( \frac{1}{\log(e^\sqrt{n})} \right).
\]

Let \( v_n(x) = \frac{w_n}{\|w_n\|} \). We have, \( v_n \in E, \|v_n\|^2 = 1 \).

We compute \( \Delta w_n(x) \), we get

\[
\Delta w_n(x) = \begin{cases} 
-\frac{(1-\beta)(4-2\beta)|x|^{-2\beta}}{(\alpha \beta)^{\frac{3}{4}} \log(e^\sqrt{n})^\frac{2\alpha - 1}{2}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt{n}} \\
-\frac{(1-\beta)(\log(\frac{r}{\sqrt{n}}))^{-\beta}}{(\alpha \beta)^{\frac{3}{4}} \log(e^\sqrt{n})^\frac{2\alpha - 1}{2}} & \text{if } \frac{1}{\sqrt{n}} \leq |x| \leq \frac{1}{2} \\
\Delta \zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1 
\end{cases}
\]

So,

\[
\|\Delta w_n\|^2_{L^2} = 2\pi^2 \int_0^{\frac{1}{\sqrt{n}}} \frac{1}{r^3} \Delta w_n(x)^2 \left( \log \frac{r}{\sqrt{n}} \right)^2 dr + 2\pi^2 \int_{\frac{1}{\sqrt{n}}}^{\frac{1}{2}} \frac{1}{r^3} \Delta w_n(x)^2 \left( \log \frac{r}{\sqrt{n}} \right)^2 dr + 2\pi^2 \int_{\frac{1}{2}}^{1} \frac{1}{r^3} \Delta w_n(x)^2 \left( \log \frac{r}{\sqrt{n}} \right)^2 dr
\]
we have,

\[ I_1 = 2\pi^2 \frac{(1 - \beta)^2(4 - 4\beta)^2}{(\frac{\alpha\beta}{4\pi})^{\frac{2}{3}} (\log(\sqrt{n}))^{\frac{2}{3}}} \int_0^{\frac{1}{\sqrt{n}}} r^{3 - 4\beta} (\log \frac{e}{r})^\beta dr \]
\[ = 2\pi^2 \frac{(1 - \beta)^2(4 - 2\beta)^2}{(\frac{\alpha\beta}{4\pi})^{\frac{2}{3}} (\log(\sqrt{n}))^{\frac{2}{3}}} \left[ \int_0^{\frac{1}{\sqrt{n}}} r^{4 - 4\beta} (\log \frac{e}{r})^\beta \frac{n}{\pi} \right] \]
\[ + 2\pi^2 \frac{8(1 - \beta)^2(4 - 2\beta)^2}{(\frac{\alpha\beta}{4\pi})^{\frac{2}{3}} (\log(\sqrt{n}))^{\frac{2}{3}}} \left[ \int_0^{\frac{1}{\sqrt{n}}} r^{4 - 4\beta} (\log \frac{e}{r})^{\beta-1} dr \right] \]
\[ = o \left( \frac{1}{\log e \sqrt{n}} \right). \]

Also,

\[ I_2 = 2\pi^2 \frac{(1 - \beta)^2}{(\frac{\alpha\beta}{4\pi})^{\frac{2}{3}} (\log(\sqrt{n}))^{\frac{2}{3}}} \int_0^{\frac{1}{\sqrt{n}}} \frac{1}{r} (\log \frac{e}{r})^{-\beta} (2 + \beta (\log \frac{e}{r})^{-1})^2 dr \]
\[ = -2\pi^2 \frac{(1 - \beta)^2}{(\frac{\alpha\beta}{4\pi})^{\frac{2}{3}} (\log(\sqrt{n}))^{\frac{2}{3}}} \left[ \int_0^{\frac{1}{\sqrt{n}}} \frac{\beta^2}{-1 - \beta} (\log \frac{e}{r})^{-\beta-1} + 4 (\log \frac{e}{r})^{-\beta} + \frac{4}{1 - \beta} (\log \frac{e}{r})^{1-\beta} \right] \]
\[ = 1 + o \left( \frac{1}{\log e \sqrt{n}} \right). \]

and \( I_3 = o \left( \frac{1}{\log e \sqrt{n}} \right). \) Then \( \|\Delta w_n\|_{2,w}^2 = 1 + o \left( \frac{1}{\log e \sqrt{n}} \right). \) Also,

for \( 0 \leq |x| \leq \frac{1}{\sqrt{n}}, \) \( u_n^\gamma(x) \geq \left( \frac{\log(\sqrt{n})}{\alpha\beta} \right) + o(1). \)

### 5.2 Key Lemmas

**Lemma 4** Assume \( V(x) \) is continuous and \( (V_1) \) is satisfied. Then there holds \( \lim_{n \to +\infty} \| w_n \|^2 = 1 \)

**Proof** We have

\[
\| w_n \|^2 = \int_B |\Delta w_n|^2 w(x) dx + \int_B V w_n^2 dx
\]
\[ = 1 + o \left( \frac{1}{\log e \sqrt{n}} \right) + \int_{0 \leq |x| \leq \frac{1}{\sqrt{n}}} V w_n^2 dx + \int_{\frac{n}{\pi} \leq |x| \leq \frac{4}{2 \pi}} V w_n^2 dx + \int_{|x| \geq \frac{4}{2 \pi}} V w_n^2 dx.
\]

For \( |x| \leq \frac{1}{\sqrt{n}}, \) \( w_n^2 \leq \left( \frac{\log(\sqrt{n})}{\alpha\beta} \right)^{\frac{\delta}{\beta}} + \frac{1}{2^\delta (\log(\sqrt{n}))^\frac{2\delta}{2\beta}} \). Then,

\[
\int_{0 \leq |x| \leq \frac{1}{\sqrt{n}}} V w_n^2 dx \leq 2\pi^2 m \left( \frac{\log(\sqrt{n})}{\alpha\beta} \right)^{\frac{\delta}{\beta}} + \frac{1}{2^\delta (\log(\sqrt{n}))^\frac{2\delta}{2\beta}} \int_0^{\frac{1}{\sqrt{n}}} r^3 dr = o_n(1)
\]

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Also,
\[
\int_{\frac{1}{\sqrt{n}}}^{1} V w_n^2 dx \leq 2\pi^2 m \frac{1}{(\frac{\alpha \beta}{\alpha_0} \log(e^{\sqrt{n}}))^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt{n}}}^{1} r^3 \left( \log(\frac{e}{r}) \right)^2 dr
\]
So,
\[
\int_{\frac{1}{\sqrt{n}}}^{1} V w_n^2 dx \leq 2\pi^2 m \frac{1}{(\frac{\alpha \beta}{\alpha_0} \log(e^{\sqrt{n}}))^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt{n}}}^{1} e^2 r dr = o_n(1).
\]

Finally,
\[
\int_{\frac{1}{\sqrt{n}}}^{1} V w_n^2 dx \leq m \int_{|x| \geq \frac{1}{\sqrt{n}}} \zeta_n^2 dx = o_n(1).
\]
Hence, \( \|w_n\|^2 \leq 1 + o_n(1) \) and consequently \( \|w_n\|^7 \leq 1 + o_n(1) \).

In the same way, using the fact that for all \( 0 \leq |x| \leq \frac{1}{\sqrt{n}} \), \( w_n^2 \geq \left( \frac{\log(e^{\sqrt{n}})}{\alpha \beta} \right)^{\frac{2}{\gamma}} \), we get
\[
\int_{0 \leq |x| \leq \frac{1}{\sqrt{n}}} V w_n^2 dx \geq V_0 2\pi^2 \left( \frac{\log(e^{\sqrt{n}})}{\alpha \beta} \right)^{\frac{2}{\gamma}} \int_{0}^{\frac{1}{\sqrt{n}}} r^3 dr = o_n(1)
\]
and
\[
\int_{\frac{1}{\sqrt{n}}}^{1} V w_n^2 dx \geq 2\pi^2 V_0 \frac{1}{(\frac{\alpha \beta}{\alpha_0} \log(e^{\sqrt{n}}))^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt{n}}}^{1} r^3 \left( \log(\frac{e}{r}) \right)^2 dr = o_n(1)
\]
Consequently, \( 1 + o_n(1) \leq \|w_n\|^2 \leq 1 + o_n(1) \).

### 5.3 Min-Max level estimate

We are going to the desired estimate.

**Lemma 5** For the sequence \((v_n)\) identified by (5.2), there exists \( n \geq 1 \) such that
\[
\max_{t \geq 0} J(t v_n) < \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}.
\]

**Proof** By contradiction, suppose that for all \( n \geq 1 \),
\[
\max_{t \geq 0} J(t v_n) \geq \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}.
\]
Therefore, for any \( n \geq 1 \), there exists \( t_n > 0 \) such that
\[
\max_{t \geq 0} J(t v_n) = J(t_n v_n) \geq \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}
\]
and so,
\[ \frac{1}{2} t_n^2 - \int_B F(x, t_n v_n) \, dx \geq \frac{1}{2} \left( \frac{\alpha_\beta}{\alpha_0} \right)^2. \]

Then, by using \((H1)\)
\[ t_n^2 \geq \left( \frac{\alpha_\beta}{\alpha_0} \right)^2. \] (5.4)

On the other hand,
\[ \frac{d}{dt} J(t v_n) \bigg|_{t=t_n} = t_n - \int_B f(x, t_n v_n) v_n \, dx = 0, \]
that is
\[ t_n^2 = \int_B f(x, t_n v_n) t_n v_n \, dx. \] (5.5)

Now, we claim that the sequence \((t_n)\) is bounded in \((0, +\infty)\).
Indeed, it follows from \((H5)\) that for all \(\varepsilon > 0\), there exists \(t_\varepsilon > 0\) such that
\[ f(x, t) \geq (\gamma_0 - \varepsilon) e^{\alpha_0 t^\gamma} \quad \forall |t| \geq t_\varepsilon, \quad \text{uniformly in } x \in B. \] (5.6)

Using (5.4) and (5.5), we get
\[ t_n^2 = \int_B f(x, t_n v_n) t_n v_n \, dx \geq \int_{0 \leq |x| \leq \frac{1}{\sqrt{n}}} e^{\alpha_0 t_n^\gamma} \, dx. \]

Since
\[ \frac{t_n}{\|w_n\|} \left( \frac{\log e^{\sqrt{n}}}{n} \right)^{\frac{1}{n}} \to \infty \quad \text{as } n \to +\infty, \]
then it follows from (5.6) that for all \(\varepsilon > 0\), there exists \(n_0\) such that for all \(n \geq n_0\)
\[ t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0 \leq |x| \leq \frac{1}{\sqrt{n}}} e^{\alpha_0 t_n^\gamma} \, dx. \]

\[ t_n^2 \geq 2\pi^2 (\gamma_0 - \varepsilon) \int_0^{\frac{1}{\sqrt{n}}} r^3 e^{\alpha_0 t_n^\gamma \left( \frac{\log(e^{\sqrt{n}})}{\alpha_\beta} \right)} + o(1) \, dr \] (5.7)

Hence,
\[ 1 \geq 2\pi^2 (\gamma_0 - \varepsilon) e^{\alpha_0 t_n^\gamma \left( \frac{\log(e^{\sqrt{n}})}{\alpha_\beta} \right) + o(1)} - 3 \log n - 2 \log t_n. \]

Therefore \((t_n)\) is bounded. Also, we have from the formula (5.5) that
\[ \lim_{n \to +\infty} t_n^2 \geq \left( \frac{\alpha_\beta}{\alpha_0} \right)^2. \]

Now, suppose that
\[ \lim_{n \to +\infty} t_n^2 > \left( \frac{\alpha_\beta}{\alpha_0} \right)^2. \]
then for $n$ large enough, there exists some $\delta > 0$ such that $t^\gamma_n \geq (\frac{\alpha_0}{\alpha})^{\frac{2}{\gamma}} + \delta$. Consequently the right hand side of (5.7) tends to infinity and this contradicts the boundedness of $(t_n)$. Since $(t_n)$ is bounded, we get

$$\lim_{n \to +\infty} t^2_n = (\frac{\alpha_0}{\alpha})^{\frac{2}{\gamma}}.$$  \hspace{1cm} (5.8)

Let

$$A_n = \{ x \in B | t_n v_n \geq t \} \quad \text{and} \quad C_n = B \setminus A_n,$$

$$t^2_n = \int_B f(x, t_n v_n) t_n v_n dx = \int_{A_n} f(x, t_n v_n) t_n v_n dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx \geq (\gamma_0 - \varepsilon) \int_{A_n} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx$$

$$= (\gamma_0 - \varepsilon) \int_{A_n} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx - (\gamma_0 - \varepsilon) \int_{C_n} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx.$$

Since $v_n \to 0$ a.e in $B$, $\chi_{C_n} \to 1$ a.e in $B$, therefore using the dominated convergence theorem, we get

$$\int_{C_n} f(x, t_n v_n) t_n v_n dx \to 0 \quad \text{and} \quad \int_{C_n} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx \to \frac{\pi^2}{2}.$$

Then,

$$\lim_{n \to +\infty} t^2_n = (\frac{\alpha_0}{\alpha})^{\frac{2}{\gamma}} \geq (\gamma_0 - \varepsilon) \lim_{n \to +\infty} \int_{B} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx - (\gamma_0 - \varepsilon) \frac{\pi^2}{2}.$$

On the other hand,

$$\int_{B} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx \geq \int_{\frac{1}{\sqrt{n}}} \int_{\frac{1}{\sqrt{n}}} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx + \int_{C_n} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx.$$

Then, using (5.4)

$$\lim_{n \to +\infty} t^2_n \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int_{B} e^{\alpha_0 t^\gamma_n} v_n^\gamma dx \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \frac{2\pi^2}{r^\frac{1}{p}} \int_{\frac{1}{\sqrt{n}}} \int_{\frac{1}{\sqrt{n}}} e^{\log(\frac{t + \varepsilon}{\sqrt{n}})} v_n^\gamma dx.$$
Therefore, we get
\[
\lim_{n \to +\infty} t_n^2 \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int_B e^{\alpha_0 t_n^2 v_n^2} dx
\]
\[
\geq \lim_{n \to +\infty} 2\pi^2 (\gamma_0 - \varepsilon) \frac{\|w_n\|^\gamma \log(e \sqrt{n})}{4} e^4 \int \frac{\|w_n\|^\gamma \log(e \sqrt{n})}{4 \log(2e)} \frac{e^{\|w_n\|^\gamma \log(e \sqrt{n})}}{4 \log(2e)} (s^2 - 4\varepsilon) ds
\]
\[
\geq \lim_{n \to +\infty} 2\pi^2 (\gamma_0 - \varepsilon) \frac{\|w_n\|^\gamma \log(e \sqrt{n})}{4} e^4 \int \frac{\|w_n\|^\gamma \log(e \sqrt{n})}{4 \log(2e)} \frac{e^{\|w_n\|^\gamma \log(e \sqrt{n})}}{4 \log(2e)} 4s ds
\]
\[
= \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \frac{\pi^2}{2} e^4 (e^{-4\log(e \sqrt{n})} + e^{-4\log(2e)})
\]
\[
= (\gamma_0 - \varepsilon) \frac{\pi^2}{32}
\]
It follows that
\[
\left(\frac{\alpha_0}{\alpha_0^*}\right)^\frac{4}{\beta} \geq (\gamma_0 - \varepsilon) \frac{\pi^2}{32}
\]
for all \(\varepsilon > 0\). So,
\[
\gamma_0 \leq \frac{1024(1 - \beta)}{\alpha_0^{1 - \beta}},
\]
which is in contradiction with the condition \((H_5)\).

Now by Proposition 4.2, the functional \(J\) satisfies the \((PS)\) condition at a level \(c < \frac{1}{2} \left(\frac{\alpha_0}{\alpha_0^*}\right)^\frac{4}{\beta}\). Also, by Proposition 4.1, we deduce that the functional \(J\) has a nonzero critical point \(u\) in \(W\). From maximum principle, the solution \(u\) of the problem (1.1) is positive. The Theorem 1.3 is proved.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Availability of data**

Data openly available in a public repository that issues data sets with DOIs. We also mention that the documentation to support this study are available from Umm Al-Qura University.

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