NEARLY EQUAL DISTANCES IN THE PLANE, II

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Abstract. Let \( \{p_1, \ldots, p_n\} \subset \mathbb{R}^2 \) be a separated point set, i.e., any two points have a distance at least 1. Let \( k \geq 1 \) be an integer, and \( 1 \leq t_1 < \ldots < t_k \) be real numbers. Let \( \delta > 0 \). Suppose for all \( 1 \leq \ell(1) \leq \ell(2) < \ell(3) \leq k \) that \( |t_{\ell(3)}/(t_{\ell(1)}+t_{\ell(2)})-1| \geq \delta \).

Then for \( n \geq n_{k, \delta} \), the number of pairs \( \{p_i, p_j\} \), for which \( d(p_i, p_j) \in [t_1, t_1+1] \cup \ldots \cup [t_k, t_k+1] \), is at most \( n^2/4 + C_{k, \delta} n \). This is sharp, up to the value of the constant \( C_{k, \delta} > 0 \).

1. Introduction

We call a finite set \( \{p_1, \ldots, p_n\} \subset \mathbb{R}^d \) separated if \( \min_{i \neq j} d(p_i, p_j) \geq 1 \), where \( d(x, y) \) denotes the distance of \( x \) and \( y \). We write \( [x, y] \) for the closed segment with endpoints \( x, y \), and \( \text{diam} (\cdot) \) for the diameter of a set in \( \mathbb{R}^d \). We write \( e_d := (0, \ldots, 0, 1) \in \mathbb{R}^d \). In the paper \( c, C, c_d, \) etc. will denote positive constants, which may have different values at different occurrences.

It was a favourite question of the first author, what is the maximal number of equal distances determined by \( n \) points in \( \mathbb{R}^d \). This question induced much research in combinatorial geometry. For the plane he conjectured that this maximum is \( \Omega(n^{1+c/\log \log n}) \), for some constant \( c > 0 \), which is attained for a \( \sqrt{n} \times \sqrt{n} \) square grid (for \( n \) a square), cf. [E1]. The best known upper bound for this question in the plane is \( O(n^{4/3}) \), cf. [SST]. For \( \mathbb{R}^3 \) the best known lower bound is \( \Omega(n^{4/3} \log \log n) \), cf. [E2], and the best known upper bound is \( O_{\varepsilon}(n^{3/2-1/394+\varepsilon}) \), cf. [Z]. The case of

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\( \mathbb{R}^4 \) is completely solved in [Br] and [vW]. Their results, taken together, say that for \( n \geq 5 \) this maximum is \([n^2/4] + n\) if \( 8|n \) or \( 10|n \), and \([n^2/4] + n - 1\) else. Let \( d \geq 6 \) be even and let \( n \geq n_d \). Then this maximum is \((n^2/2)(1 - 1/|d/2|) + n - O_d(1)\), and for \( 2d|n \) it equals \((n^2/2)(1 - 1/|d/2|) + n\), cf. [E3]. Again let \( d \geq 6 \) be even and let \( n \geq n_d \). Then the exact value of this maximum is determined by [S].

About 1989 the first author observed that one can modify this question so that the solution of the modified question is probably much simpler. Rather than the number of equal distances, he asked for the number of almost equal distances. Here several distances are almost equal if they fall in a unit interval. To avoid trivialities, when all distances are close to 0, he supposed that the system of \( n \) points is separated.

In [EMPS] it has been shown that for each \( d \geq 2 \) there exists a constant \( n_d \), such that the following holds. Let \( n \geq n_d \), let \( t \geq 1 \) and let \( \{p_1, \ldots, p_n\} \subset \mathbb{R}^d \) be a separated set. Then the number of pairs \( \{p_i, p_j\} \), for which \( d(p_i, p_j) \in [t, t + 1] \), is at most \((n^2/2)(1 - 1/d) - O_d(1)\). This bound can be attained for any \( d \). We have equality for the set \( \{s_\mu + \nu e_d \mid 1 \leq \mu \leq d, 1 \leq \nu \leq n_\mu\} \), where \( s_1 \ldots s_d \) is a regular simplex in the \( x_1 \ldots x_{d-1} \)-coordinate hyperplane, with vertices \( s_\mu \), and of edge length \( t \). Moreover, for \( 1 \leq \mu \leq d \) we have \( |n/d| \leq n_\mu \leq |n/d| \) and \( \sum_{\mu=1}^d n_\mu = n \). Further, we suppose that \( t \) is sufficiently large \( (t \geq cn^2 \text{ suffices}) \).

In [EMP] the following has been proved. Let \( k \geq 1 \) be a fixed integer, let \( 1 \leq t_1 < \ldots < t_k \) be arbitrary, and let \( \{p_1, \ldots, p_n\} \subset \mathbb{R}^2 \) be any separated set. Then the number of pairs \( \{p_i, p_j\} \), for which \( d(p_i, p_j) \in \bigcup_{\ell=1}^k [t_\ell, t_\ell + 1] \), is at most \((n^2/2)(1 - 1/(k + 1)) + o_k(n^2)\). This estimate is sharp, up to the summand \( o_k(n^2) \).

We have equality \( (\text{with } -O_k(1) \text{ rather than } o_k(n^2)) \) for the set \( \{(\mu t, \nu) \mid 0 \leq \mu \leq k, 1 \leq \nu \leq \mu\} \). Here \( |n/(k + 1)| \leq n_\mu \leq |n/(k + 1)| \) and \( \sum_{\mu=0}^k n_\mu = n \), and \( t_1 = t, \ldots, t_k = kt \), and \( t \) is sufficiently large \( (t \geq cn^2 \text{ suffices}) \).

Still we note that the above two results remain valid, if we consider an interval of the form \([t, t + c_d n^{1/d}]\), or intervals of the form \([t_\ell, t_\ell + c_{k, \varepsilon} \sqrt{n}]\). Here \( c_d \) and \( c_{k, \varepsilon} \) are suitable constants. In the second case \( o_k(n^2) \) has to be substituted by \( \varepsilon n^2 \), and we have to suppose that \( n \geq n_{k, \varepsilon} \).

A common generalization of these two theorems is the following problem: generalize the result of [EMP] to \( \mathbb{R}^d \). This has been treated for \( k = 2 \) in [EMP2]. Their results have been generalized for an arbitrary \( k \) by [FK1] and [FK2]. The last two papers in a sense have settled the whole problem, by reducing it to simpler problems ([FK1] is an extended abstract of [FK1]). Also here one can take, for \( n \geq n_{d, k, \varepsilon} \), intervals of the form \([t_\ell, t_\ell + c_{d, k, \varepsilon} n^{1/d}]\). Under these hypotheses, [FK1] determined for each \( d \) and \( k \) the maximum number of our point pairs, up to a summand \( cn^2 \). Moreover, for \( d \geq d_k \) and \( n \geq n_{d, k} \) and for intervals \([t_\ell, t_\ell + c_{d, k, \varepsilon} n^{1/d}]\), [FK1] determined the exact maximum of the number of our point pairs. This turned out to be a certain Turán number.

[PRV1] and [PRV2] showed the following. Let \( \{p_1, \ldots, p_n\} \subset \mathbb{R}^d \), and let \( cn^2 \) pairs \( \{p_i, p_j\} \) have distances in some interval \([t, t + 1]\). Then for some \( f_d(c) > 0 \) we have \( \text{diam} \{p_1, \ldots, p_n\} \geq f_d(c)n^{2/(d-1)} \). (Actually both papers showed \( t \geq f_d(c)n^{2/(d-1)} \).) The order of magnitude is sharp. An example for \( n = 2m^d - 1 \),
where \( m \geq 1 \) is an integer, is the set \( \{(i_1, \ldots , i_{d-1}, x) \mid 1 \leq i_1, \ldots , i_{d-1} \leq m \text{ are integers, and } x \in \{0, c_d m^2\}\} \).

Two, or \( k \) nearly equal distances in arbitrary \( n \)-point metric spaces have been investigated in [OP]. However, there nearness of distances was measured by the closeness of their quotients to 1 (like in [EMP2], Theorem 2). Moreover, their question was to estimate from below the number \( k \) of pairs of points \( \{p_i, q_i\} \), with all quotients \( d(p_i(1), q_i(1))/d(p_i(2), q_i(2)) \) in a small neighbourhood of 1.

There arises the question, what can we tell for the planar case, for a “general” system of values \( t_1, \ldots , t_k \). This is answered by our theorem, which surprisingly asserts that the bound for any fixed \( k \) is essentially the same as for \( k = 1 \). Our result also can be interpreted as “having many distances almost equal to some \( d \) distances creates some order in the numbers \( t_1, \ldots , t_k \)”.

2. The theorem and some problems

**Theorem.** Let \( k \geq 2 \) be a fixed integer, and let \( \delta \in (0, 1) \) be fixed. Then there exist constants \( n_{k,\delta}, C_{k,\delta} > 0 \), depending on \( k \) and \( \delta \), with the following property. Suppose that \( n \geq n_{k,\delta} \), and \( 1 < t_1 < \ldots < t_k \) satisfy the following hypothesis:

\[
\left\{ \begin{array}{l}
\text{for any } 1 \leq \ell(1) \leq \ell(2) < \ell(3) \leq k \text{ we have } \\
t_{\ell(3)} \not\in [(1-\delta)(t_{\ell(1)} + t_{\ell(2)}), t_{\ell(1)} + t_{\ell(2)} + 2].
\end{array} \right.
\]

Then for any separated set \( \{p_1, \ldots , p_n\} \subset \mathbb{R}^2 \), the number of pairs \( \{p_i, p_j\} \) for which \( d(p_i, p_j) \in \bigcup_{\ell=1}^{k} [t_\ell, t_\ell+1] \), is at most \( n^2/4 + C_{k,\delta} n \).

Moreover, the same inequality holds for intervals \( [t_\ell, t_\ell+\alpha] \) rather than \( [t_\ell, t_\ell+1] \), for any fixed constant \( \alpha \). However, then we have \( n_{k,\delta,\alpha} \) and \( C_{k,\delta,\alpha} \), rather than \( n_{k,\delta} \) and \( C_{k,\delta} \).

The estimate of the theorem is essentially sharp. Even, for any fixed \( \varepsilon > 0 \), for each \( k \) and each sufficiently small \( \delta > 0 \) and each \( n \) we have the following. There exist examples with intervals \( [t_\ell, t_\ell+\varepsilon] \) rather than \( [t_\ell, t_\ell+1] \), for which the number of the above pairs is \( n^2/4 + (k-1)n - O_k(1) \). Moreover, this holds for all sufficiently large \( t_k \), depending on \( k, \varepsilon \) and \( n \) (\( t_k \geq \max\{3^{k-1}, cn^2/\varepsilon\} \) suffices).

**Remark 1.** The simplified statement of the Theorem in the abstract follows from this form of the Theorem. If however, \( t_{\ell(1)} \geq t_1 \) is bounded, then 1 of the proof of the Theorem shows the simplified statement of the Theorem in the abstract.

**Remark 2.** If we allow \( t_{\ell(3)} = t_{\ell(1)} + t_{\ell(2)} \) for \( 1 \leq \ell(1) \leq \ell(2) < \ell(3) \leq k \), then we may have \([n^2/3]\) pairs with distances in \( \bigcup_{\ell=1}^{k} [t_\ell, t_\ell+1] \). An example is \( \{(0,1), \ldots , (0, n_1), (t_{\ell(1)}, 1), \ldots , (t_{\ell(2)}, 2), (t_{\ell(3)}, 1), \ldots , (t_{\ell(3)}, n_3)\} \). Here \( [n/3] \leq n_\mu \leq [n/3] \) and \( \sum_{\mu=1}^{3} n_\mu = n \), and \( t_{\ell(1)} \) is sufficiently large (\( t_{\ell(1)} \geq cn^2 \) suffices). Actually an analogous example exists if \( |t_{\ell(3)} - t_{\ell(1)} - t_{\ell(2)}| \) is at most some fixed number in \((0, 1)\).

**Problem 1.** Can one prove a similar theorem under the following weaker hypothesis: for \( 1 \leq \ell(1) \leq \ell(2) < \ell(3) \leq k \) we have \( t_{\ell(3)} \not\in [t_{\ell(1)} + t_{\ell(2)} - C(t_{\ell(1)} + t_{\ell(2)})^c, t_{\ell(1)} + t_{\ell(2)} + 2] \), for some \( c \in [0, 1) \) and \( C > 0 \)?

**Problem 2.** Analogously like in [EMPS], [EMP1], [EMP2], [FK1], is it possible
to extend our Theorem to intervals of the form \([t, t + f(n)]\), for some \(f(n)\) tending to \(\infty\) for \(n \to \infty\)?

**Problem 3.** For \(k = 2\) our theorem settles “everything”. For \(t_2/t_1 = 2\) we have the essentially sharp bound \(n^2/3 + \varepsilon n^2\) ([EMP1]). However, for \(t_2/t_1\) far from \(2\) we have a bound \(n^2/4 + Cn\). Now let \(k \geq 3\). Suppose that we want to exclude only the systems \((t_1, \ldots, t_k) = \((t, \ldots, kt)\), and those “close” to them. (E.g., those which satisfy \(|t_\ell/t_1 - \ell| \leq \delta\) for all \(2 \leq \ell \leq k\).) Then conjecturally, for \(n\) sufficiently large, \(n \geq n_k\), say, the maximum number of pairs of points with distances in \(\cup_{t_\ell=1}^k [t_\ell, t_\ell + 1]\) is obtained as follows. We let \(\{p_1, \ldots, p_n\} := \{((\mu, \nu) | 1 \leq \mu \leq k, 1 \leq \nu \leq n_\mu\}. Here for \(1 \leq \mu \leq k\) we have \([n/k] \leq n_\mu \leq [n/k]\) and \(\sum_{\mu=1}^k n_\mu = n\). Moreover, we take \(t_1 = 1, t_2 = t, \ldots, t_k = (k-1)t\). The number of pairs in question for this example is \((n^2/2)(1 - 1/k) + 2n - O_k(1)\). Actually this was the original question of the first author. Can one formulate analogous questions, with suitable hypotheses, for which the maximum number of pairs with distances in \(\cup_{t_\ell=1}^k [t_\ell, t_\ell + 1]\) is conjecturally \((n^2/2)(1 - 1/s) + o_k(n^2)\), for \(3 \leq s \leq k - 1\)? (Heuristically: if the number of our distances almost equal to some \(k\) numbers \(t_1, \ldots, t_k\) is \((n^2/2)(1 - 1/s) + o_k(n^2)\), then with increasing \(s\) we have ever more structure in \(\{t_1, \ldots, t_k\}\).

**Problem 4.** Can one generalize our theorem to \(\mathbb{R}^d\), in the following form. Suppose that there is no “close to degenerate” simplex with all edge lengths some \(t_\ell\)'s (degenerate meaning that it lies in a hyperplane). Then the number of pairs \(\{p_i, p_j\}\), for which \(d(p_i, p_j) \in \cup_{t_\ell=1}^k [t_\ell, t_\ell + 1]\) is at most \((n^2/2)(1 - 1/d) + C_{d,k} n\). Here of course \(C_{d,k}\) should depend also on the “measure of non-degenerateness” (this is \(\delta\) in our Theorem). A possible measure of degeneracy could be \((d, \alpha)\)-Flatness of point-sets from [FK1].

This estimate, if true, would be essentially sharp. In fact, the example from [EMPS], cited in the introduction, with number of pairs in question being \((n^2/2)(1 - 1/d) - O_d(1)\), works for any \(k \geq 1\). (Moreover, we could choose \(t_1 := 1, t_2 := 3, \ldots, t_{k-1} := 3^{k-2}\), and \(t_k := t \geq \max\{3^{k-1}, cn^2\}\), and any \(\delta \leq 1/3\), provided that with edge lengths these \(t_\ell\)'s there is no “close to degenerate” simplex. Then we would have even \((n^2/2)(1 - 1/d) + 2(k - 1)n - O_{d,k}(1)\) pairs in question.)

Now suppose that there is a degenerate simplex \(s_1 \ldots s_{d+1}\) with vertices \(s_\mu\) and with all edge lengths some \(t_\ell\)'s, all these \(t_\ell\)'s being at least \(cn^2\). Let \(s_1 \ldots s_{d+1}\) lie in the \(x_1 \ldots x_{d-1}\)-coordinate hyperplane. Then there exist examples for which the number of the pairs in question is \((n^2/2)(1 - 1/(d + 1)) - O_{d,k}(1)\). Our example is the \(d\)-dimensional generalization of the example in Remark 2. Suppose that for \(1 \leq \mu \leq d + 1\) we have \([n/(d + 1)] \leq n_\mu \leq [n/(d + 1)]\) and \(\sum_{\mu=1}^{d+1} n_\mu = n\). Then our example is \(\{s_\mu + ve_d | 1 \leq \mu \leq d + 1, 1 \leq \nu \leq n_\mu\}\).

Cf. also Remark 3 in the end of §3.

### 3. The proof of the theorem

**Proof. 1.** First we suppose that

\[
(1) \quad t_1 \leq C, \text{ where } C \text{ is some (large) constant.}
\]

Then, like in [EMP1], we use induction. The base of induction is \(k = 1\), for
which the statement holds by [EMPS], cited in the introduction. For \( k \geq 2 \) the hypothesis of the theorem is satisfied for \( 1 \leq t_2 < \ldots < t_k \). Hence the number of pairs \( \{p_i, p_j\} \), such that \( d(p_i, p_j) \in \cup_{t=2}^{k} [t_\ell, t_\ell + 1] \), is at most \( n^2/4 + C_{k-1, \delta} n \). The number of pairs \( \{p_i, p_j\} \) such that \( d(p_i, p_j) \in [t_1, t_1 + 1] \), and \( p_i \) is fixed, is at most the quotient of the area of a circular ring of radii \( t_1 - 1/2 \) and \( t_1 + 3/2 \), and of \( \pi/4 \). Namely, the open circles of radius \( 1/2 \) centred at these \( p_j \)'s are disjoint, and lie in this circular ring. The above quotient is \( 2n(2t_1 + 1)/(\pi/4) \leq 8(2C + 1) \). Thus the number of all pairs \( \{p_i, p_j\} \), such that \( d(p_i, p_j) \in [t_1, t_1 + 1] \), is at most \( 8(2C + 1)n \).

Hence we can choose \( C_{k, \delta} := C_{k-1, \delta} + 8(2C + 1) \).

2. Therefore we may suppose that

\[ t_1 \geq C, \] where \( C \) is a suitable (large) constant.

We consider the graph \( G \) with vertices \( p_1, \ldots, p_n \), and edges those pairs \( \{p_i, p_j\} \), for which \( d(p_i, p_j) \in \cup_{t=1}^{k} [t_\ell, t_\ell + 1] \). Suppose, in contradiction to the statement of the theorem, that the number of edges of \( G \) is greater than \( n^2/4 + C_{k, \delta} n \). Then, for any preassigned number \( N \), for \( C_{k, \delta} \) sufficiently large, depending on \( N \), we have the following. Our graph \( G \) contains a subgraph \( K(1, N, N) \), cf. [Bo1], Cor. 4.7, or [Bo2], Th. 1.5.2. Here \( K(a, b, d) \) denotes the complete 3-partite graph with colour classes \( A, B \) and \( D \), of sizes \( a, b \) and \( d \), resp. Now \( A, B, D \subset \{p_1, \ldots, p_n\} \), and \( |A| = 1 \) and \( |B| = |D| = N \). Let \( \{x\} = A \) and \( y \in B \) and \( z \in D \). Then the triangle \( xyz \) has all side-lengths in \( \cup_{t=1}^{k} [t_\ell, t_\ell + 1] \).

We let correspond to each side of the triangle \( xyz \) the smallest \( \ell \) such that the length of this side lies in \( [t_\ell, t_\ell + 1] \). This \( \ell \) will be denoted, for the sides \( xy, yz \) and \( zx \), by \( \ell(xy), \ell(yz) \) and \( \ell(zx) \), resp.

For \( y \in B \) the number \( \ell(xy) \) can assume \( k \) values. Hence, for some \( B_1 \subset B \) with \( |B_1| = \lceil N/k \rceil \), for \( y \in B_1 \) the number \( \ell(xy) \) is independent of \( y \). Similarly, for some \( D_1 \subset D \) with \( |D_1| = \lceil N/k \rceil \), for \( z \in D_1 \) the number \( \ell(zx) \) is independent of \( z \). Observe that in our theorem \( k \) is fixed.

For \( y \in B_1 \) and \( z \in D_1 \) the number \( \ell(yz) \) can assume \( k \) values. By a variant of Ramsey’s theorem, cf. [PA], Cor. 9.15 (stated there for two colours only, but its proof given there works for any fixed number of colours – alternatively, the statement for two colours implies the statement for any fixed number of colours), we have the following. There exist \( B_2 \subset B_1 \) and \( D_2 \subset D_1 \), such that \( |B_2| = |D_2| = M \), where also \( M \) is arbitrarily large provided \( N \) is arbitrarily large, such that the following holds. For all pairs \( \{y, z\} \), where \( y \in B_2 \) and \( z \in D_2 \), the number \( \ell(yz) \) is independent of \( y, z \).

Hence

\[ \{x, y, z\} = \{u, v, w\}, \] where \( \ell(1) := \ell(uv) \leq \ell(2) := \ell(vw) \leq \ell(3) := \ell(wu) \).

There are two cases.
Either (I): $\ell(1) \leq \ell(2) < \ell(3)$,

or (II): $\ell(1) \leq \ell(2) = \ell(3)$.

3. In case (I) we have

$$t_{\ell(3)} \leq d(w, u) \leq d(u, v) + d(v, w) \leq t_{\ell(1)} + t_{\ell(2)} + 2.$$  

This implies by the hypothesis of the theorem that

$$t_{\ell(3)} \leq (1 - \delta) (t_{\ell(1)} + t_{\ell(2)}).$$

Since by (2)

$$C \leq t_1 \leq t_{\ell(1)} \leq t_{\ell(2)} < t_{\ell(3)},$$

by choosing $C$ sufficiently large ($C \geq c/\delta$ suffices), we will have also

$$d(w, u) \leq t_{\ell(3)} + 1 \leq (1 - \delta/2) (t_{\ell(1)} + t_{\ell(2)}) \leq (1 - \delta/2) (d(u, v) + d(v, w)).$$

Moreover, by the definition of $\ell(uv)$ etc., we have

$$\max\{d(u, v), d(v, w)\} < d(w, u).$$

By (10), the triangle $uvw$ cannot degenerate to the doubly counted segment $[w, u]$, therefore it is not degenerate.

We claim

$$\angle uvw > \max\{\angle vwu, \angle wuv\} \geq \min\{\angle vwu, \angle wuv\} \geq \delta_1,$$

where $\delta_1 = \delta_1(\delta) > 0$. The first inequality follows from (11). For the last inequality we prove $\angle vwu \geq \delta_1$; the proof of $\angle wuv \geq \delta_1$ is the same.

In fact, let the sides of $\angle vwu$ be fixed, and let $d(v, w) \in [0, d(w, u)]$. Then, by a triangle inequality, the maximum of $d(u, v) + d(v, w)$ is attained for $d(v, w) = d(w, u)$. The value of this maximum is $d(w, u) + 2d(w, u) \sin(\angle vwu/2)$. Hence, by (10),

$$d(w, u) \leq (1 - \delta/2) (d(u, v) + d(v, w)) \leq (1 - \delta/2) (d(w, u) + 2d(w, u) \sin(\angle vwu/2)).$$

Comparing the first and last expressions in (13), we get

$$\angle vwu \geq 2 \arcsin(\delta/(4 - 2\delta)) =: \delta_1 \sim \delta/2,$$

which ends the proof of (12). Then (12) implies that

$$\max\{\angle vwu, \angle wuv\} < \angle uvw = \pi - \angle vwu - \angle wuv \leq \pi - 2\delta_1 =: \pi - \delta_2,$$
where $\delta_2 = \delta_2(\delta)$.

The above considerations apply to all of the triangles $xyz$ from (3). Let us choose one of them, $x_0y_0z_0$, say. Then both $y_0$ and $z_0$ can be replaced by any of $M$ points $y_i \in B_2$ and $z_i \in D_2$, say, so that we obtain a triangle $xy_iz_i$ satisfying (3).

Now we will only use the fact that $y_0$ can be replaced by any of $M$ points $y_i \in B_2$, so that we obtain a triangle $xy_i z_0$ satisfying (3). Then by (4) all distances $d(x, y_i)$ belong to one of the intervals $[t_{\ell(i)}, t_{\ell(i)} + 1]$, $[t_{\ell(2)}, t_{\ell(2)} + 1]$ and $[t_{\ell(3)}, t_{\ell(3)} + 1]$. Similarly, all distances $d(y_i, z_0)$ belong to one of these three intervals (possibly different from the above one).

Above we have chosen some $y_i \in B_2$, namely, $y_0$. Then for each $y_i \in B_2$ we have $|d(x, y_i) - d(x, y_0)| \leq 1$, i.e., $d(x, y_i) \in [d(x, y_0) - 1, d(x, y_0) + 1]$. Similarly $d(y_i, z_0) \in [d(y_0, z_0) - 1, d(y_0, z_0) + 1]$. Moreover, by (12) and (15) we have $\angle xy_0z_0 \in [\delta_1, \pi - \delta_2]$.

That is, all of these $M$ points $y_i$ lie in the intersection of two circular rings, of centres $x$ and $z_0$. These circular rings have inner radii $d(x, y_0) - 1 \geq C - 1$ and $d(y_0, z_0) - 1 \geq C - 1$, and outer radii $d(x, y_0) + 1$ and $d(y_0, z_0) + 1$, resp. Since even $C - 1$ is large, this intersection is the union of two “curvilinear almost rhombs”, symmetric to the line $z_0x$. Moreover, $\angle xy_0z_0 \in [\delta_1, \pi - \delta_2]$.

Any of these two “curvilinear almost rhombs” can be included in an intersection of two parallel strips of width 3, which is a rhomb $R$. These strips are orthogonal to the vectors $y_0 - x$ and $y_0 - z_0$, resp. The boundary lines of the first strip intersect the halfline from $x$, passing through $y_0$, at points with distances $d(x, y_0) + 1$ and $d(x, y_0) - 2$ from $x$. Similarly, the boundary lines of the second strip intersect the halfline from $z_0$, passing through $y_0$, at points with distances $d(y_0, z_0) + 1$ and $d(y_0, z_0) - 2$ from $z_0$. This statement is very intuitive; a formal proof can be found in the first arxiv variant of [EMP2], pp. 9-13. Then all $M$ points $y_i \in B_2$ lie in the union of $R$ and its axially symmetric image w.r.t. the line $z_0x$.

Consider the rhomb $R_1$, whose side lines are obtained from those of the rhomb $R$ by translating them outwards (from the centre of $R$) through a distance $1/2$. The height of $R_1$ is 4. By $\angle xy_0z_0 \in [\delta_1, \pi - \delta_2]$ the angles of $R_1$ are bounded away from 0 and $\pi$. Hence its area, which is $4^2$ divided by the sine of one of its angles, i.e., by $\sin \angle xy_0z_0$, is bounded above by some function of $\delta$. Observe that in our Theorem $\delta$ is fixed.

All $M$ disjoint open circles of radius $1/2$, of centres $y_i \in B_2$, are contained in the union of $R_1$ and its axially symmetric image w.r.t. the line $z_0x$. Hence by area considerations $M$ must be bounded. However, $M$ can be arbitrarily large, a contradiction.

4. In case (II) (cf. (6)) we choose one of the triangles, $x_0y_0z_0$, say. By (4) and (6) we have $\{x, y_0, z_0\} = \{u_0, v_0, w_0\}$, where

\begin{equation}
(16) \quad \ell(u_0, v_0) = \ell(1) \leq \ell(v_0, w_0) = \ell(2) = \ell(w_0, u_0) = \ell(3).
\end{equation}

Then one of $u_0$ and $v_0$ is different from $x$, hence belongs to $B_2$ or to $D_2$. Let, e.g., $u_0 \in B_2$. Then $u_0$ can be replaced by any of $M$ points $y_i \in B_2$, so that we obtain a triangle $xy_iz_0$ satisfying (3).

Now we claim that $\angle w_0u_0v_0$ lies in a small neighbourhood of the angular interval $[\pi/3, \pi/2]$. In fact, let us write $a := d(w_0, u_0)$, $b := d(u_0, v_0)$ and $c := d(v_0, w_0)$.
Then we have for the angle $\gamma$ of the triangle $u_0v_0w_0$, opposite to its side $c$, that

$$
\cos \gamma \leq \frac{(t_\ell(2) + 1)^2 + (t_\ell(1) + 1)^2 - t_\ell(2)^2)}{2t_\ell(2)t_\ell(1)} = \frac{t_\ell(1)/(2t_\ell(2)) + 1/t_\ell(1) + 1/t_\ell(2) + 1/(t_\ell(1)t_\ell(2))}{2} \leq 1/2 + 2/C + 1/C^2,$$

where $C$ can be anyhow large. Therefore $\gamma$ is at least a value close to $\pi/3$.

On the other hand,

$$
\cos \gamma \geq \frac{[t_\ell(2)^2 + t_\ell(1)^2 - (t_\ell(2) + 1)^2]}{(2ab)}.
$$

If the numerator of (18) is nonnegative, then $\gamma \leq \pi/2$. If this numerator is negative, then the right hand side of (18) can be estimated further from below by

$$
\frac{[t_\ell(1)^2 - 2t_\ell(2) - 1]/(2t_\ell(1)t_\ell(2)) = t_\ell(1)/(2t_\ell(2))}
-1/t_\ell(1) - 1/(2t_\ell(1)t_\ell(2)) \geq 0 - 1/C - 1/(2C^2),
$$

and then $\gamma$ is at most a value close to $\pi/2$.

Summing up: $\gamma$ is bounded away both from 0 and from $\pi$.

Then using $|B_2| = M$, which can be arbitrarily large, and the area considerations from the end of 3, we get a contradiction.

5. Now we prove the assertion with $\alpha$, where we may suppose $\alpha > 1$. In 1 the base of induction is the statement from [EMPS], where we can have intervals of the form $[t, t+c\sqrt{n}]$. So we may take intervals of the form $[t, t+\alpha]$, for $n \geq n_k,\delta,\alpha$. After this point, let us diminish $\{p_1, \ldots, p_n\}$ in ratio $1/\alpha$. Then rather than $[t_\ell, t_\ell + \alpha]$ we will have intervals $[t_\ell/\alpha, t_\ell/\alpha + 1]$, and rather than $t_1 \geq 1$ we will have $t_1/\alpha \geq 1/\alpha$. Furthermore, all the steps of the above proof apply for this diminished point system, except that the minimal distance is at least $1/\alpha$. Then we will have disjoint open circles of radius $1/(2\alpha)$, where $\alpha$ is a constant. The area considerations work for these disjoint open circles as well, in 1, in the end of 3 and in the end of 4.

6. The asserted essential sharpness of the estimate is shown by the example which is the planar case of the example from [EMPS], cited in the introduction. Here we choose $t_1, \ldots, t_k$ as in the second paragraph of Problem 4. The statement about essential sharpness with the intervals $[t_\ell, t_\ell + \varepsilon]$ follows from this by Pythagoras’ theorem. ■

Remark 3. Concerning Problem 4, unfortunately the proof of our Theorem does not carry over even to $d = 3$ and $k = 2$. E.g., we can have tetrahedra with two different edge lengths $t_1 < t_2$, two opposite edges of length $t_1$, the remaining ones of length $t_2$, where $t_2/t_1$ is large. For these for each vertex the absolute value of the determinant formed by the unit vectors pointing from that vertex to all the other vertices is small. (However, all other tetrahedra with these edge lengths can probably be treated as in the proof of our Theorem.) However, using $(d, \alpha)$-Flatness of point-sets from [FK1], this example is excluded. So this gives a chance to apply $(d, \alpha)$-Flatness of point-sets as a measure of non-degeneracy in $\mathbb{R}^d$.

References
NEARLY EQUAL DISTANCES IN THE PLANE, II

[Bo1] B. Bollobás, Extremal graph theory, London Math. Soc. Monographs 11, Academic Press, Inc. [Hartcourt Brace Jovanovic, Publishers], London-New York, 1978. MR 80a:05120.

[Bo2] B. Bollobás, Extremal graph theory, In: Handbook of Combinatorics (Eds. R. Graham, M. Grötschel, L. Lovász), Elsevier, Amsterdam, 1995, Ch. 23, 1231-1292. MR 97b:05073.

[Br] P. Brass, On the maximum number of unit distances among n points in dimension four, In: Intuitive Geometry (Proc. 5th Conf. Budapest, 1995), Bolyai Soc. Math. Stud. 6 (Eds. I. Bárány, K. Böröczky), J. Bolyai Math. Soc., Budapest (1997), 277-290. MR 98j:52030.

[E1] P. Erdős, On sets of distances of n points, Amer. Math. Monthly 53 (1946), 248-250. MR 7,471c.

[E2] P. Erdős, On sets of distances of n points in Euclidean space, Magyar Tud. Akad. Mat. Kutató Int. Közl. (Publ. Math. Inst. Hungar. Acad. Sci.) 5 (1960), 165-169. MR 25#4420.

[E3] P. Erdős, On some applications of graph theory to geometry, Canadian J. Math. 19 (1967), 968-971. MR 35#2520.

[EMP1] P. Erdős, E. Makai, Jr., J. Pach, Nearly equal distances in the plane, Combin. Probab. Comput., 2 (1993) (4), 401–408. MR 95i:52018. Reprinted in: Combinatorics, Geometry and Probability, Papers of Conf. in Honour of Erdős’ 80th birthday, Trinity College, Cambridge, 1993 (Eds. B. Bollobás, A. Thomason), Cambridge Univ. Press, Cambridge, 1997, 283-290. MR 1476450.

[EMP2] P. Erdős, E. Makai, Jr., J. Pach, Two nearly equal distances in $\mathbb{R}^d$, arXiv:1901.01055.

[EMPS] P. Erdős, E. Makai, Jr., J. Pach, J. Spencer, Gaps in difference sets and the graph of nearly equal distances, In: Applied Geometry and Discrete Math., The V. Klee Festschrift, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 4, Amer. Math. Soc., Providence, RI (1991), 265–273. MR 92i:52021.

[EP] P. Erdős, J. Pach, Variations on the theme of repeated distances, Combinatorica 10 (1990) (3), 261-269. MR 92b:52037.

[FK1] N. Frankl, A. Kupavskii, Nearly k-distance sets, arXiv:1906.02574; to appear in Discr. Comput. Geom.

[FK2] N. Frankl, A. Kupavskii, Nearly k-distance sets, Acta Math. Univ. Comenian. (N.S.) 88 (2019) (3), 689-693. MR 4012869.

[OP] A. Ophir, R. Pinchasi, Nearly equal distances in metric spaces, Discrete Appl. Math. 174 (2014), 122-127. MR 3215463.

[PA] J. Pach, P. K. Agarwal, Combinatorial Geometry, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, Whiley and Sons, New York etc., 1995. MR 96j:52001.

[PRV1] J. Pach, R. Radić, J. Vondrák, Nearly equal distances and Szemerédi’s regularity lemma, Compt. Geom., 34 (2006) (1), 11-19. MR 2006i:52022.

[PRV2] J. Pach, R. Radić, J. Vondrák, On the diameter of separated point sets with many nearly equal distances, European J. Combin., 27 (2006) (8), 1321-1332. MR 2007f:51026.

[S] K. J. Swanepoel, Unit distances and diameters in Euclidean spaces, Discr. Comput. Geom. 41 (2009), 1-27. MR 2010f:52031.

[SST] J. Spencer, E. Szemerédi, W. Trotter, Jr., Unit distances in the Euclidean plane, In: Graph Theory and Combinatorics (Cambridge, 1983), (Ed. B. Bollobás), Acad. Press, London (1984), 293-303. MR 86m:52015.

[vW] P. van Wamelen, The minimum number of unit distances among n points in dimension four, Beiträge Algebra Geom., 40 (1999) (2), 475-477. MR 2000i:52031.

[Z] J. Zahl, Breaking the 3/2 barrier for unit distances in three dimensions, Int. Math. Res. Not. IMRN, 2019 (20), 6235-6284. MR 4031237.