Foldy-Wouthuysen transformation for a non-Hermitian Hamiltonian

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Abstract.
The free Dirac Lagrangian is extended with a non-Hermitian mass term. It is shown that the model has real energies and a conserved current in a given region of parameter space, and the Hamiltonian is mapped on a Hermitian Hamiltonian, using a (non-unitary) Foldy-Wouthuysen transformation.

1. Introduction
We analyse here a model introduced in [1], in the framework of PT-symmetric theories (see [2] for an introduction). We use the the Foldy-Wouthuysen (FW) transformation, which explicitly shows the positive and negative energy states solutions of the equation of motion [3]. Although the Hamiltonian is not Hermitian, it has real energies and a conserved current in a given region of the parameter space. We derive the FW mapping on a Hermitian Hamiltonian, which has the same eigenvalues as the original Hamiltonian. This construction shows the potential relevance of the model, when the energies are real. These results were derived in [4], where more details can be found, together with the study of another non-Hermitian extension of the Dirac Lagrangian.

The non-Hermitian Lagrangian density we focus on is [1]

$$\mathcal{L} = \bar{\psi} \left( i \partial_t - m - \mu \gamma^5 \right) \psi ,$$

and is symmetric under simultaneous time-reversal and parity transformations.

2. Equation of motion
The derivation of the equation of motion is not trivial, since for a non-Hermitian Lagrangian the variation of the action with respect to $\psi$ or $\bar{\psi}$ are not equivalent. We therefore need to derive the equation of motion from the fundamental fermion components, which may be written as $\psi_a = \phi_a + i \chi_a$, where $\phi_a$ and $\chi_a$ are real. The equations of motion can then be correctly obtained by demanding that the variational derivatives of $S$ with respect to $\phi_a$ and $\chi_a$ independently vanish. Using the Dirac representation of gamma matrices, it is shown in [4] that

$$\frac{\delta S}{\delta \phi_a} = 2 \left[ i \gamma^0 \left( \gamma^0 \partial_t + \gamma^1 \partial_1 + \gamma^3 \partial_3 \right) - \mu \gamma^0 \gamma^5 \right]_{ac} \phi_c + 2 \left[ - \gamma^0 \gamma^2 \partial_2 - im \gamma^0 \right]_{ac} \chi_c,$$

$$\frac{\delta S}{\delta \chi_a} = -2 \left[ i \gamma^0 \gamma^2 \partial_2 - m \gamma^0 \right]_{ac} \phi_c + 2 \left[ i \gamma^0 \left( \gamma^0 \partial_t + \gamma^1 \partial_1 + \gamma^3 \partial_3 \right) - \mu \gamma^0 \gamma^5 \right]_{ac} \chi_c.$$
such that
\[
\frac{1}{2} \delta S + i \frac{\delta S}{\delta \phi_a} + i \frac{\delta S}{\delta \chi_a} = (\gamma^0 (i\phi - m - \mu \gamma^5)(\phi + i\chi))_a .
\]

But the previous equation is just equivalent to
\[
\gamma^0 \frac{\delta S}{\delta \bar{\psi}^*} = (i\partial_0 - m - \mu \gamma^5)\psi ,
\]
which agrees with the variation \(\delta S/\delta \bar{\psi}\) performed with \(\bar{\psi}\) and \(\psi\) considered as independent fields. Hence, we may derive the equation of motion by following the formal conventional procedure.

3. Conserved current

Plugging a plane wave in the equation of motion (2) leads to the dispersion relation
\[
\omega^2 = m^2 + p^2 - \mu^2 ,
\]
which exhibits real energies \(\omega\) for all momenta \(\vec{p}\), as long as \(\mu^2 \leq m^2\). It is shown in [4] that the model (1) has a conserved current, which is
\[
j^\nu = \bar{\psi} \gamma^\nu \left(1 + \frac{\mu}{m} \gamma^5\right) \psi ,
\]
and the probability density can be written in terms of the Right- and left-handed components of \(\psi\) as
\[
\rho = \bar{\psi} \left(1 + \frac{\mu}{m} \gamma^5\right) \psi = \left(1 + \frac{\mu}{m}\right) |\psi_R|^2 + \left(1 - \frac{\mu}{m}\right) |\psi_L|^2 ,
\]
(3)
Thus, \(\mu = \pm m\) are interesting special cases in which the contribution to the density is entirely from right- or left-handed degrees of freedom.

4. Foldy-Wouthuysen transformation

The Hamiltonian density \(H\) corresponding to the model (1) can be found from the Schrodinger form of the equation of motion
\[
i\partial_0 \psi = \mathcal{H} \psi , \quad \text{where} \quad \mathcal{H} = \gamma^0 \left[\vec{p} \cdot \vec{\gamma} + m + \mu \gamma^5\right] ,
\]
such that
\[
H = \psi^\dagger \mathcal{H} \psi = \bar{\psi} \left[\vec{p} \cdot \vec{\gamma} + m + \mu \gamma^5\right] \psi .
\]
The FW transformation consists in defining the mapping
\[
U \mathcal{H} U^{-1} = \omega \gamma^0 , \quad \chi = U \psi ,
\]
which, for \(\mu^2 \leq m^2\), leads to the Hermitian FW Hamiltonian
\[
H_{FW} = \chi^\dagger \omega \gamma^0 \chi = \omega \bar{\chi} \chi .
\]
Unlike the Dirac case, the operator \(U\) here is not unitary, and thus \(H_{FW} \neq H\). Indeed,
\[
H = \psi^\dagger \mathcal{H} \psi = \psi^\dagger U^{-1} \mathcal{H} U \psi = \psi^\dagger U^{-1} \omega \gamma^0 \chi \neq (U \psi)^\dagger \omega \gamma^0 \chi = H_{FW} .
\]
By analogy with the Dirac case, we seek $U$ in the form

$$U \equiv \exp \left( \theta \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{-(\vec{p} \cdot \vec{\gamma} + \mu \gamma^5)^2}} \right) = \cos \theta + \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{\sqrt{p^2 - \mu^2}} \sin \theta , \quad (4)$$

and we obtain

$$U \mathcal{H} U^{-1} = \gamma^0 \left[ \cos(2\theta) - \frac{m}{\sqrt{p^2 - \mu^2}} \sin(2\theta) \right] + \gamma^0 \left[ m \cos(2\theta) + \sqrt{p^2 - \mu^2} \sin(2\theta) \right].$$

Requiring the term proportional to $(\vec{p} \cdot \vec{\gamma} + \mu \gamma^5)$ to vanish implies

$$\tan(2\theta) = \frac{\sqrt{p^2 - \mu^2}}{m} , \quad (5)$$

and it is then easy to check that $U \mathcal{H} U^{-1} = \omega \gamma^0$.

In the limit $\mu \to 0$, the FW angle (5) gives the known result in the Dirac case:

$$\tan(2\theta_{\text{Dirac}}) = p/m.$$

We note that, if we start with $\theta = i\theta'$, we find

$$\tan(2\theta') = \frac{\sqrt{\mu^2 - p^2}}{m} ,$$

such that the regimes $p^2 \leq \mu^2$ and $\mu^2 \leq p^2$ can be swapped by the change $\theta \leftrightarrow i\theta'$. Finally, the limit $p^2 \to \mu^2$ of the mapping (4) is not singular, since

$$\lim_{p^2 \to \mu^2} U = 1 + \frac{\vec{p} \cdot \vec{\gamma} + \mu \gamma^5}{2m}.$$

5. Conclusion

We showed that, although the model (1) is not Hermitian, it has the expected properties to potentially describe a physical system, as long as one considers $\mu^2 \leq m^2$. Given the asymmetric role (3) of the left- and right-handed components of the fermion, this non-Hermitian model might be relevant to neutrino propagation, opening a new opportunity to describe chirality. Also, gauging the model (1) might lead to a new description of chiral anomaly, which is currently under study.

References

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