Superpositions of the dual family of nonlinear coherent states and their non-classical properties

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Abstract

Nonlinear coherent states (CSs) and their dual families were introduced recently. In this paper we want to obtain their superposition and investigate their non-classical properties such as antibunching effect, quadrature squeezing and amplitude squared squeezing. For this purpose two types of superposition are considered. In the first type we neglect the normalization factors of the two components of the dual pair, superpose them and then we normalize the obtained states, while in the second type we superpose the two normalized components and then again normalize the resultant states. As a physical realization, the formalism will then be applied to a special physical system with known nonlinearity function, i.e., Hydrogen-like spectrum. We continue with the (first type of) superposition of the dual pair of Gazeau-Klauder coherent states (GKCSs) as temporally stable CSs. An application of the proposal will be given by employing the Pöschl-Teller potential system. The numerical results are presented and discussed in detail, showing the effects of this special quantum interference.

keywords: nonlinear coherent states; dual family of nonlinear coherent states; superposition of states.
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1 Introduction

Following the development of quantum theory of radiation field, the notion of standard coherent states (CSs) as the states that are most nearly close to classical description of radiation field, were introduced [1, 2]. These states contain important concepts in quantum optics and have numerous applications in different branches of modern physics
The standard CSs of a radiation field are usually constructed using one of the three following manners: i) they are eigenstates of the standard annihilation operator of the harmonic oscillator, i.e., \( a|\alpha\rangle = |\alpha\rangle \), ii) they can be reproduced by the action of a unitary displacement operator on the vacuum of radiation field, i.e., \( \exp(\alpha a^\dagger - \alpha^* a)|0\rangle = |\alpha\rangle \). iii) they minimize the Heisenberg uncertainty relation \((\Delta x)^2(\Delta p)^2 \geq \frac{1}{16}\), with equal variances as the limit of vacuum in both quadratures. In recent years CSs have been generalized in various ways. One of the most important generalization based on algebraic aspects is the notion of nonlinear CSs [6] or \( f\)-CSs [7]. These states defined as the right eigenstates of an \( f\)-deformed annihilation operator \( A = af(n) \), where \( f \) is an operator-valued function of number operator \( n = a^\dagger a \) characterizes the nonlinearity nature of the system. The nonlinear CSs approach provides a powerful method for classification and unification of a vast classes of generalized CSs [8]. They are also useful in the construction of CSs associated to inverse bosonic and \( f\)-deformed annihilation operators [9]. A question may naturally arise: is there any displacement operator whose the action on the vacuum state can reproduce nonlinear CSs? It is shown that the answer is affirmative. It is possible to define a displacement type operator using the auxiliary operators \( B = \alpha \frac{1}{f(n)} \) and its conjugate \( B^\dagger \). Indeed it is proved that two different displacement type operators may be constructed. While the action of one of them on the vacuum reproduce the original nonlinear CSs, the other one produces a new set of nonlinear CSs [10] known as the dual pair with respect to each other [11].

On the other side in recent years much attention has been paid to the quantum superposition of CSs [12, 13] and nonlinear CSs [14] due to their interference effects [15]. Even and odd CSs [16, 17] (even and odd nonlinear CSs [18]) respectively as symmetric and anti-symmetric superposition of CSs (nonlinear CSs) are along this subject of researches. Generally, interference effects in \( f\)-deformed fields were discussed in [19]. The generation of non-classical states possessing properties such as sub-Poissonian behavior, quadrature squeezing and higher order squeezing is the main motivation of this subject of researches.

As the main goal of the present paper we want to construct the general formalism of superposition of the dual pair corresponding to any nonlinear CS class with known \( f(n) \). For this purpose two types of superposition are considered. In the first type we neglect the normalization factors of the dual family of nonlinear CSs, superpose them and then normalize the obtained states. To distinguish this kind of superposition from the second one we call it as "combination". It will be easily shown that this combination is a kind of "superposition of the normalized states" with different weights, where the weights relate to the normalization factors of the individual states (see equation (11) of the paper). But in the second type we superimpose the two "normalized components" and then again normalize the resultant states. It may be recognized that the second type is more customary in the literature. Altogether, some words seem to be necessary about the introduction of the first type. This combination kind of superposition has been done previously in the literature for different purposes [20, 21, 22, 23, 24]. The most general case
of superposition of two canonical CSs $|\alpha\rangle$, $|\beta\rangle$ is introduced as $|\psi\rangle = c_1|\alpha\rangle + c_2|\beta\rangle$, where $c_1$ and $c_2$ are complex numbers restricted by the normalization constraint. Squeezing of these states is investigated by considering $\langle \psi | (\Delta X_\theta)^2 |\psi\rangle$, where $X_\theta = X_1 \cos \theta + X_2 \sin \theta$, and $X_1 + iX_2 = a$ is the annihilation operator [20]. Searching for maximum simultaneous squeezing and antibunching effects of these states has been done in [21]. Later, the latter states have been again studied by the same authors in [22] for investigating the (maximum) amplitude squared squeezing of the hermitian operator $Z_\theta = Z_1 \cos \theta + Z_2 \sin \theta$, where $Z_{1,2}$ operators are defined by $Z_1 + iZ_2 = [a - \langle \psi | a | \psi \rangle]^2$. A rather different motivation for this kind of superposition may outcome from the teleportation scheme. Teleportation of two-mode CSs studied in [23]. The authors considered a particular form of the two-mode CSs of the form $|\psi(x, y)\rangle = x|\alpha, \alpha\rangle + y|\alpha, -\alpha\rangle$, where $x, y$ are coefficients constrained by the normalization condition. Interestingly, such a particular two-mode CSs can always be obtained by first the teleportation a single-mode CSs of the form $|\psi(x, y)\rangle = x|\alpha \sqrt{2}\rangle + y|\alpha, -\sqrt{2}\rangle$ and then superposing it on a 50 : 50 beam-splitter with the vacuum state $|0\rangle$ (see [24] and references therein). According to our terminology, in all of the latter mentioned cases the ”combinations of states” have been considered and studied. In addition to these, the first type has this mathematical property that can exactly be classified in the ”nonlinear CSs” family with a particular nonlinearity function.

In what follows a brief review on the $f$-CSs and their dual pairs required for our further manipulations is presented in section 2. Some explanations about the resolution of the identity and the domain of the dual family of states are presented in section 3. Then in section 4 the combination of the dual pair are constructed and the nonlinearity signature of the combination of states is established through driving the corresponding nonlinearity function. In section 5 the (second kind of) superposition of the dual pair has been introduced. The non-classical criteria needed for our further discussion will be introduced in section 6. Section 7 deals with a particular physical realization of the presented formalism by applying to ”Hydrogen-like spectrum”. In section 8 we deal with the Gazeau-Klauder CSs (GKCSs) as the temporally stable CSs [25]. The combination of these states with their dual family will be introduced and their non-classical properties are studied and the nonlinearity signature of the combination of GKCSs is established. A summary of the results is presented in section 9. Finally, in the appendix some expectation values were required for our numerical calculations can be found.

2 The dual family of nonlinear CSs

The nonlinear CSs method is based on the deformation of standard annihilation and creation operators with an intensity dependent function $f(n)$, according to the relations [6, 7]

\[ A = af(n), \quad A^\dagger = f^\dagger(n)a^\dagger, \quad (1) \]
where \( a, a^\dagger \) and \( n = a^\dagger a \) are bosonic annihilation, creation and number operators, respectively. The commutators between \( A \) and \( A^\dagger \) read as:

\[
[A, A^\dagger] = (n+1)f^\dagger(n+1)f(n+1) - nf^\dagger(n)f(n).
\]

(2)

Now we choose \( f(n) \) to be real, i.e., \( f^\dagger(n) = f(n) \) and nonnegative until we deal with GKCSs in section 5. Nonlinear CSs satisfy the typical eigenvalue equation

\[
A|\alpha, f\rangle = \alpha|\alpha, f\rangle, \quad \alpha \in \mathbb{C}
\]

(3)

The Fock space representation of these states is explicitly expressed as

\[
|\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!} [f(n)]!} |n\rangle,
\]

(4)

where \( N_f \) is some appropriate normalization factor determined from \( \langle \alpha, f | \alpha, f \rangle = 1 \). By convention \([ f(n)]! \doteq f(1)f(2)...f(n) \) and \([ f(0)]! \doteq 1 \). On the other hand by considering (2) we see that \([ A, A^\dagger ] \) is neither \( c \)-number nor linear in the generator \( n \) and therefore based on BCH theorem [26] a displacement operator does not exist as \( \exp(\alpha A^\dagger - \alpha^* A) \) to construct nonlinear CSs. But in [10, 11] the operator \( B \) and it’s conjugate \( B^\dagger \) were defined as follow

\[
B = a \frac{1}{f(n)}, \quad B^\dagger = \frac{1}{f(n)} a^\dagger,
\]

(5)

which satisfy the canonical commutation relations \([ A, B^\dagger ] = \hat{I} = [ B, A^\dagger ] \). Now we can consider two displacement-type operator \( D(\beta) = \exp(\beta B^\dagger - \beta^* A) \) and \( \tilde{D}(\beta) = \exp(\beta A^\dagger - \beta^* B) \) and apply them on the vacuum state of radiation field. The results are respectively the states in (4) and a new class of nonlinear CSs which named the dual pair of (4) with the following expansion [10, 11]

\[
|\tilde{\beta}, f\rangle = \tilde{D}(\beta)|0\rangle = \tilde{N}_f \sum_{n=0}^{\infty} \frac{\beta^n [f(n)]!}{\sqrt{n!}} |n\rangle, \quad \beta \in \mathbb{C},
\]

(6)

where again the normalization factor \( \tilde{N}_f \) can be determined from \( \langle \tilde{\beta}, f | \tilde{\beta}, f \rangle = 1 \). It is clear that \( |\alpha, f\rangle = |\alpha, f^{-1}\rangle \). Therefore, \( |\alpha, f\rangle \) and \( |\tilde{\alpha}, f\rangle \) as a dual pair may be generated from two different types of interaction of atom-field, due to the fact that each of them corresponds to a distinct nonlinearity function which mathematically are inverse of each other. The dual pair which will constitute the two components of our further superposition in the present paper are not orthogonal, with the overlap

\[
\langle \alpha, f | \tilde{\alpha}, f \rangle = N_f \tilde{N}_f \exp(|\alpha|^2).
\]

(7)
3 Resolution of the identity for the dual family of nonlinear CSs

Requiring the over-completeness of any class of CSs, the dual pair of CSs should satisfy the resolution of the identity, in addition to the nonorthogonality condition. But the nonorthogonality of the dual family of states (i.e., $\langle \alpha, f|\alpha', f \rangle \neq 0$ and similarly for the dual pair) is so obvious subject that we pay no attention to it. Moreover, the most important requirement for these states is the necessity to satisfy the resolution of the identity, i.e.,

$$\int_{D} d^{2}\alpha|\alpha, f\rangle W(|\alpha|^{2})\langle \alpha, f \rangle = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}$$ (8)

where $d^{2}\alpha = dx dy$, $W(|\alpha|^{2})$ is a positive weight function may be found after specifying $f(n)$, $\hat{I}$ is the unity operator and $D$ is the domain of the states in the complex plane defined by the disk

$$D = \{ \alpha \in \mathbb{C}, |\alpha| \leq \lim_{n \to \infty} [n f^{2}(n)] \},$$ (9)

centered at the origin in the complex plane. Inserting the explicit form of the states (4) in (8) with $|\alpha|^{2} \equiv x$ it can be easily checked that the resolution of the identity holds if the following moment problem is satisfied:

$$\pi \int_{0}^{R} dx \sigma(x)x^{n} = [n f^{2}(n)]!, \quad n = 0, 1, 2, \ldots ,$$ (10)

where $\sigma(x) = \frac{W(x)}{N(x)}$ and $R$ is the radius of convergence determined by the relation (9). The condition (10) presents a severe restriction on the choice of $f(n)$. In fact, only a relatively small number of $f(n)$ nonlinearity functions are known for which the proper functions $\sigma(x)$ exist.

Comparing the dual pairs in (4) and (6) it is readily found that requiring the resolution of the identity for the dual pair in (6), the following must hold

$$\int_{\tilde{D}} d^{2}\beta|\beta, \tilde{f}\rangle \tilde{W}(|\beta|^{2})\langle \beta, \tilde{f} \rangle = \sum_{n=0}^{\infty} |n\rangle\langle n| = \tilde{I},$$ (11)

where $\tilde{D}$ is the domain of the dual states in the complex plane defined by

$$\tilde{D} = \{ \beta \in \mathbb{C}, |eta| \leq \lim_{n \to \infty} [n f^{2}(n)] \}. (12)$$

Setting $|\beta|^{2} \equiv x$ and simplifying the left hand side of equation (11) we are lead to the following moment integral

$$\pi \int_{0}^{\tilde{R}} dx \tilde{\sigma}(x)x^{n} = [n f^{2}(n)]!, \quad n = 0, 1, 2, \ldots ,$$ (13)
where $\tilde{\sigma}(x) = \frac{\tilde{W}(x)}{N(x)}$ and $\tilde{R}$ is the radius of convergence determined by the relation (12). We can introduce the ”general solutions” of the moment problems were introduced in (12) and (13) which is known as the inverse Mellin transform. Indeed $\sigma(x) = M^{-1}[\rho_{\infty}(s) - 1; x]$ for the CSs defined in the whole of space $\mathcal{R} = \infty$, and $\sigma(x)H(\mathcal{R} - x) = M^{-1}[\rho_{\mathcal{R}}(s) - 1; x]$ for the states restricted to the unit disk $\mathcal{R} < \infty$, where $H(z)$ is the Heaviside function and we have defined $\rho(s) = [(s - 1)f^2(s - 1)]!$ (for a useful treatment on Mellin and inverse Mellin transforms refer to [34] and references therein). Similar expressions may be followed for $\tilde{\sigma}(x)$.

### 4 Constructing the combination of the dual family of nonlinear CSs

Now we introduce the combination (first type of superposition) of nonlinear CSs with associated dual pair. In fact the explicit form of states can be easily obtained by linear combination of the un-normalized forms of (4) and (6) with the same nonlinearity function tends to the normalized state

$$
|\alpha, \beta, f\rangle = N_{s1}(\|\alpha, f\rangle + \|\beta, f\rangle)
$$

$$
= N_{s1} \sum_{n=0}^{\infty} \frac{\alpha^n + \beta^n ([f(n)]!)^2}{\sqrt{n!} [f(n)]!} |n\rangle,
$$

where the normalization factor $N_{s1}$ can be easily determined from the condition $\langle \beta, \beta, f|\alpha, \beta, f\rangle = 1$. Note that after dropping the normalization factor of $|\psi\rangle$ we have showed it by $\|\psi\rangle$, and the subscript ”$s$” in (14) and what follows indicates the superposition of the first kind. Clearly taking $\beta = 0$ or $\alpha = 0$ in (14) leads to (4) or (6), respectively. In this paper we restrict ourselves to the special case $\alpha = \beta$ and therefore the introduced state in (14) is simplified to

$$
|\alpha, \alpha, f\rangle \equiv |\alpha, f_{s1}\rangle = N_{s1} \sum_{n=0}^{\infty} \frac{\alpha^n (1 + ([f(n)]!)^2)}{\sqrt{n!} [f(n)]!} |n\rangle, \quad \alpha \in \mathbb{C},
$$

where $N_{s1}$ may be determined as follows

$$
N_{s1} = \left[ \sum_{n=0}^{\infty} \frac{\alpha^{2n} (1 + ([f(n)]!)^2)^2}{n!([f(n)]!)^2} \right]^{-\frac{1}{2}}.
$$

It is worth to mention that this form of superposition (called by us as combination) of a dual pair of nonlinear CSs can be classified in the following category of superpositions of nonlinear CSs as

$$
|\alpha, f_{s1}\rangle = c_1|\alpha, f\rangle + c_2|\tilde{\alpha}, f\rangle,
$$

where $c_1$ and $c_2$ are constants.
where \( c_1 \) and \( c_2 \) are now real numbers determined as

\[
c_1 = \frac{N_{s_1}}{N_f}, \quad c_2 = \frac{N_{s_1}}{N_f},
\]

(18)
depending on \( f(n) \) and \( \alpha \). Note that in this case \( c_1 \) and \( c_2 \) are constrained by the requirement of the normalizability of the combined states. The superposition proposed in (15) or equivalently (17) can be considered as the generalization of the superpositions of CSs to nonlinear CSs, referred to in the introduction of the present paper [20, 21, 22, 23, 24]. We hope that these latter states also find their physical applications in relevant schemes on the performance of the teleportation protocols.

It is well-known that there exists a simple relation between the expansion coefficients \( C_n \)’s of the nonlinear CSs with the corresponding nonlinearity function \( f(n) \) as follows [7]

\[
f(n) = \frac{C_{n-1}}{\sqrt{n}C_n}. \tag{19}
\]

To verify the nonlinearity nature of \( |\alpha, f_{s_1}\rangle \) in (15), the relation (19) leads us to the following nonlinearity function expressed in terms of original nonlinearity function

\[
f_{s_1}(n) = \frac{1 + (\lfloor f(n-1) \rfloor!)^2}{1 + (\lfloor f(n) \rfloor!)^2} f(n). \tag{20}
\]

Therefore, by substitution \( f(n) \) corresponding to any nonlinear oscillator algebra in (20) we are able to find \( f_{s_1}(n) \) associated to the special superposed state \( |\alpha, f_{s_1}\rangle \) introduced in (15).

So such a superposition of any dual pair of states can in principle be obtainable as the eigenstate of the generalized annihilation operator \( a_{f_{s_1}}(n) = A_{s_1} \)

\[
A_{s_1}|\alpha, f_{s_1}\rangle = \alpha|\alpha, f_{s_1}\rangle, \quad \alpha \in \mathbb{C}, \tag{21}
\]

as follows

\[
|\alpha, f_{s_1}\rangle = N_{s_1} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}[f_{s_1}(n)]!}|n\rangle. \tag{22}
\]

In this way, a vast classes of nonlinear CSs may be obtained using various nonlinearity functions \( f(n) \) in (15). It is also possible to consider the proposals of Roy et al [10] or Ali et al [11] and reproduce the states in (15), (17) or (22) via the action of a displacement-type operator \( D_{s_1}(\alpha) = \exp(\alpha C_{s_1}^\dagger - \alpha^* A_{s_1}) \) on the vacuum by \( |\alpha, f_{s_1}\rangle = D_{s_1}(\alpha)|0\rangle \), where we have defined \( C_{s_1} = a \frac{1}{f_{s_1}(n)} \) which has the property \([A_{s_1}, C_{s_1}^\dagger] = \hat{I}\). It is a remarkable point that while previously superpositions of CSs for instance even and odd (linear [16] or nonlinear [18]) CSs are not essentially remained in the \( f \)-deformed family of CSs, the proposed combined states in the present paper (as in the special superposition of nonlinear CSs recently introduced by us in [14]) are exactly ”\( f \)-deformed CSs”.

7
We end this section with mentioning that to the best of our knowledge in superposing the quantum states it is enough for one to have the main requirements of CSs for the individual components of the superposed states [16, 17, 18] (in this case the dual families). But, due to the existence of an explicit form of the nonlinearity function \( f_{s1}(n) \) in (20) for the first kind of superpositions we hope that the resolution of the identity for the superposed states may be established in future works”.

5 Superposition of a dual pair of nonlinear CSs

In this section we introduce the superposition (of the second type) of the two normalized components of the dual pair of any class of nonlinear CSs with known \( f(n) \) and then normalize the resultant states. For this purpose, let us initially demonstrate the form of this superposition as

\[
|\psi\rangle_{s2} = N_{s2} (|\alpha, f\rangle + |\widetilde{\alpha}, f\rangle). \tag{23}
\]

We denoted the second type of superposition by \( |\psi\rangle_{s2} \) to emphasis the fact that the resultant states are not \( f \)-deformed. Setting the relations (4) and (6) in (23) we arrive at the explicit form of normalized superposition in the Fock space representation as

\[
|\psi\rangle_{s2} = N_{s2} \sum_{n=0}^{\infty} \frac{\alpha^n G(n, |\alpha|^2)}{\sqrt{n!} [f(n)]!} |n\rangle, \tag{24}
\]

where for simplicity we have set \( G(n, |\alpha|^2) = N_f + \check{N}_f [f(n)]^2 \). By using \( s_2 \langle \psi | \psi \rangle_{s2} = 1 \) one finds the normalization factor as

\[
N_{s2} = \left[ \sum_{n=0}^{\infty} |\alpha|^{2n} G^2(n, |\alpha|^2) \right]^{-\frac{1}{2}}. \tag{25}
\]

6 non-classical criteria of states

Generally, a state is known as a non-classical state (with no classical analogue) if the Glauber-Sudarshan \( P(\alpha) \) function [2] can not be interpreted as a probability density. However, in practice one can not directly apply this criterion and this definition may hardly be applied to investigate the non-classicality nature of a state [27]. Altogether, this purpose has been mainly achieved customarily by checking the “quadrature squeezing, amplitude squared squeezing, sub-Poissonian statistics (antibunching) and oscillatory number distribution”. A common feature of all the above mentioned criteria is that the corresponding \( P \)-function of a non-classical states is not positive definite. Therefore, each of the mentioned effects (first and second order squeezing or sub-Poissson statistics which will be considered in the present paper) are sufficient condition for a state to exhibit non-classicality. It is worth mentioning that the necessary condition for the non-classicality of a states is yet the subject of recent researches [28].
6.1 Sub-Poissonian statistics

To investigating the sub-Poissonian statistical behavior of the superpositions of states we will deal with second order correlation function \[ g^2(0) = \frac{\langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle}. \] (26)

Depending on the particular form of \( f(n) \) the state may exhibit super-Poissonian, Poissonian or sub-Poissonian, respectively if \( g^2(0) > 1 \) (bunching effect), \( g^2(0) = 1 \) or \( g^2(0) < 1 \) (antibunching effect).

6.2 Quadrature squeezing

In order to study quadrature squeezing we consider the following hermitian operators

\[ x = \frac{a + a^\dagger}{2}, \quad y = \frac{a - a^\dagger}{2i}. \] (27)

Then the uncertainty relation holds \( \langle (\Delta x)^2 \rangle \langle (\Delta y)^2 \rangle \geq \frac{1}{16} \), where \( \langle (\Delta x_i)^2 \rangle = \langle x_i^2 \rangle - \langle x_i \rangle^2 \), \( x_i = x \) or \( y \). A state is squeezed if any of the following conditions holds:

\[ \langle (\Delta x)^2 \rangle < \frac{1}{4} \quad \text{or} \quad \langle (\Delta y)^2 \rangle < \frac{1}{4}. \] (28)

Now by using equations (27) and (28) the squeezing conditions may be transformed to the following inequalities:

\[ I_1 = \langle a^2 \rangle + \langle a^\dagger \rangle^2 - \langle a \rangle^2 - \langle a^\dagger \rangle^2 - 2 \langle a \rangle \langle a^\dagger \rangle + 2 \langle a^\dagger a \rangle < 0, \] (29)

and

\[ I_2 = -\langle a^2 \rangle - \langle a^\dagger \rangle^2 + \langle a \rangle^2 + \langle a^\dagger \rangle^2 - 2 \langle a \rangle \langle a^\dagger \rangle + 2 \langle a^\dagger a \rangle < 0, \] (30)

respectively for \( x \) and \( y \) quadratures.

6.3 Amplitude squared squeezing

In order to investigate the amplitude squared squeezing effect the following two hermitian operators have been introduced \[ X = \frac{a^2 + a^\dagger \rangle^2}{2}, \quad Y = \frac{a^2 - a^\dagger \rangle^2}{2i}. \] (31)

In fact \( X \) and \( Y \) are the operators corresponding to the real and imaginary parts of the square of the complex amplitude of the electromagnetic field. Heisenberg uncertainty relation of these two conjugate operators is given by \( \langle (\Delta X)^2 \rangle \langle (\Delta Y)^2 \rangle \geq \frac{1}{4} |\langle [X,Y] \rangle|^2. \)
It follows that our introduced states in (15) and (24) will exhibit amplitude squared squeezing if

\[ \langle (\Delta X)^2 \rangle < \frac{1}{2} |\langle [X, Y] \rangle| \quad \text{or} \quad \langle (\Delta Y)^2 \rangle < \frac{1}{2} |\langle [X, Y] \rangle|. \]  

(32)

With the help of equations (31) and (32) one can introduce the following squeezing conditions corresponding to \( X \) and \( Y \), respectively

\[
I_3 = \frac{1}{4} \left( \langle a^4 \rangle + \langle a^*a^* \rangle + \langle a^2a^2 \rangle + \langle a^*a^2 \rangle - \langle a^2 \rangle^2 - \langle a^* \rangle^2 
- 2 \langle a^2 \rangle \langle a^* \rangle \right) - \langle a^*a \rangle - \langle a^*a \rangle - \frac{1}{2} < 0, \]

(33)

and

\[
I_4 = \frac{1}{4} \left( -\langle a^4 \rangle - \langle a^*a^* \rangle + \langle a^2a^2 \rangle + \langle a^2a^* \rangle + \langle a^* \rangle^2 - \langle a^* \rangle^2 
- 2 \langle a^2 \rangle \langle a^* \rangle \right) - \langle a^*a \rangle - \langle a^*a \rangle - \frac{1}{2} < 0. \]

(34)

It is evident that each of the expectation values in the formulas in this section must be evaluated with respect to a particular quantum state which is of interest (see appendix).

7 Physical realization

In this section we will apply the presented formalism to a generalized CSs with particular nonlinearity function. Then the previously mentioned non-classical signs of the superposed states will be investigated graphically. For this purpose we consider the Hydrogen-like spectrum which their shifted energy eigenvalues are given in [31] by \( e_n = 1 - \frac{1}{(n+1)^2} \).

The corresponding nonlinearity function of this system by considering the method proposed in [8] may be written as

\[
f(n) = \frac{\sqrt{n^2 + 2}}{n + 1}, \]

(35)

where the associated CSs have been defined in the unit disk centered at the origin. The resolution of the identity for the nonlinear CSs of Hydrogen-like spectrum and their dual pair has been recently established in [25] and [35], respectively. Using (35) in the mean values of (26) (see Appendix) we have plotted second order correlation function for various states against \( \alpha \) in figure 1. As it is clear, only the dual family of corresponding states according to (6) shows sub-Poissonian behavior, while for the introduced states \(|\alpha, f_{s1}\rangle\) and \(|\psi\rangle_{s2}\) as well as \(|\alpha, f\rangle\), \(g^2(0)\) is larger than unity (they exhibit super-Poissonian behavior). In fact comparing the curves displayed in figure 1 one may conclude that the quantum statistical behavior of \(|\alpha, f_{s1}\rangle\) and \(|\psi\rangle_{s2}\) are qualitatively similar. From figures 2-a and 2-b it is found that quadrature squeezing occurred for \(|\alpha, f\rangle\) and \(|\psi\rangle_{s2}\) in \(x\), and
for $|\alpha,f\rangle$ as well as combined state $|\alpha,f_{s1}\rangle$ in $y$ quadrature. Clearly, to guarantee the Heisenberg uncertainty relation, the regions of squeezing occurrence in $x$ and $y$ components for $|\psi\rangle_{s2}$ state is different from each other. Figures 3-a and 3-b show that amplitude squared squeezing has been occurred for $|\tilde{\alpha},f\rangle$ in $X$, and for $|\alpha,f\rangle$ as well as the combined state $|\alpha,f_{s1}\rangle$ in $Y$ component, while the superposed state $|\psi\rangle_{s2}$ does not show amplitude squared squeezing in $X$ nor in $Y$ component. Adding our numerical results for this special nonlinearity function, one may conclude that the behavior of the combination of states are qualitatively very similar to original state $|\alpha,f\rangle$, far from the dual states.

8 Superposition (of the first kind) of Gazeau-Klauder CSs with their dual pair

Adopting certain physical criteria rather than imposing selected mathematical requirements, Gazeau and Klauder by reparameterizing the generalized CSs $|\alpha\rangle$ in terms of two independent parameters $J$ and $\gamma$, introduced the generalized CSs $|J,\gamma\rangle$, known ordinarily as GKCSs in the physical literature [25]. On the other hand, in [32] the authors imposed a minor modification on these states via generalizing the Bargmann representation for the standard harmonic oscillator [33]. In the present paper we proceed to consider this form of GKCSs for further superposition scheme of the first kind (combination). The analytical representations of GKCS associated to a physical system with the discrete nondegenerate spectrum $e_n$ have been introduced as follows

$$|z,\gamma\rangle^{\text{GK}} = N^{\text{GK}} \sum_{n=0}^{\infty} \frac{z^n e^{-i\epsilon_n \gamma}}{\sqrt{\rho(n)}} |n\rangle, \quad z \in \mathbb{C}, \quad 0 \neq \gamma \in \mathbb{R}, \quad (36)$$

where the normalization constant and the function $\rho(n)$ are given by

$$N^{\text{GK}} = \left( \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)} \right)^{-\frac{1}{2}}, \quad \rho(n) = [e_n]!. \quad (37)$$

Also, the dual family of GKCSs as the temporally stable CSs of the dual of KPS (Klauder-Penson-Sixdeniers) CSs [34], were introduced by one of us in [35]. Using the analytical representation the following form for the dual family has been deduced

$$|\tilde{z},\gamma\rangle^{\text{GK}} = \tilde{N}^{\text{GK}} \sum_{n=0}^{\infty} \frac{z^n e^{-i\epsilon_n \gamma}}{\sqrt{\mu(n)}} |n\rangle, \quad z \in \mathbb{C}, \quad 0 \neq \gamma \in \mathbb{R}, \quad (38)$$

where the normalization constant may be written as

$$\tilde{N}^{\text{GK}} = \left( \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\mu(n)} \right)^{-\frac{1}{2}}, \quad (39)$$
and $\mu(n)$ and $\varepsilon_n$ are respectively given by
\[
\mu(n) = \frac{(n!)^2}{\rho(n)}, \quad \varepsilon_n = \frac{n^2}{e_n}.
\] (40)

In (40) $\varepsilon_n$ denotes the energy spectrum of the Hamiltonian associated to the dual pair states. The dual family which will constitute the two components of our superposition in the following subsection are not orthogonal, with the overlap
\[
\langle z, \gamma | \tilde{z}, \gamma \rangle = \sum_{n=0}^{\infty} |z^n e^{i(e_n - \varepsilon_n)\gamma} n!
\] (41)

A few words seems to be useful about the over-completeness of these states. In this relation it is a clear fact that the states belong to each pair of the dual family are indeed nonorthogonal. Also, the over-completeness of GKCSs requires satisfying the resolution of the identity according to:
\[
\int_{0}^{R} d\mu(z) |z, \gamma \rangle^GK \langle z, \gamma | = \sum_{n=0}^{\infty} |n\rangle \langle n| = \hat{I}.
\] (42)

Inserting the explicit form of the states (36) in (42) with $|x|^2 \equiv x$ simplifies it to the following moment problem:
\[
\pi \int_{0}^{R} dx \sigma(x)x^n = \rho(n), \quad n = 0, 1, 2, \ldots.
\] (43)

where $R$ is the radius of convergence may be determined similar to nonlinear CSs. The condition (43) again presents a severe restriction on the choice of $\rho(n)$ and consequently $e_n$. Following the same path one may immediately obtain the conditions for the over-completeness of the dual pair of GKCSs.

8.1 Mathematical structure of the combination of the dual pair of GKCSs

Now by using the un-normalized form of equations (36) and (38), we construct superposition of a dual pair of GKCSs as $|z, \gamma \rangle_{s1}^{GK} = N_{s1}^{GK} (|z, \gamma \rangle^GK + |\tilde{z}, \gamma \rangle^GK)$. It is straightforward to obtain its explicit representation in Fock space as
\[
|z, \gamma \rangle_{s1}^{GK} = N_{s1}^{GK} \sum_{n=0}^{\infty} \frac{z^n K(n, \gamma)}{\sqrt{n! \rho(n)}} |n\rangle,
\] (44)

where we have introduced $K(n, \gamma) = n! e^{-i\gamma e_n} + [e_n]! e^{-i\gamma e_n}$. The normalization constant $N_{s1}^{GK}$ can be obtained as follows
\[
N_{s1}^{GK} = \left( \sum_{n=0}^{\infty} \frac{|z^n K(n, \gamma)|^2}{[e_n]!(n!)^2} \right)^{-\frac{1}{2}}.
\] (45)
The nonlinearity of the combined states in (44) may be established with the result

\begin{equation}
GK_s^{1}(n, \gamma) = \frac{K(n-1, \gamma)}{K(n, \gamma)},
\end{equation}

and by convention \(GK_s^{1}(0, \gamma)\)\! ≡ 1. Note that on the contrary to the previous combination of the dual pair of nonlinear CSs which perfectly remained in the \(f\)-deformed states, \(|z, \gamma\rangle_{s1}^{GK}\) in (44) are not of GKCSs type, i.e., they are not temporally stable.

As one may recognize the superposition of the second kind of GKCSs need more calculations, but yet with the same procedures of section 5, so we ignore studying it at present.

8.2 Physical realization

Now we use the results of previous subsection to study the non-classical properties of Pöschl-Teller potentials [36]. It must be mentioned that this physical system is widely used in quantum mechanics specially in atomic and molecular physics. The energy spectrum of such system is given by \(\epsilon_n = n(n + \lambda + \kappa)\), where \(\lambda, \kappa > 1\) characterize the potential. Following the path of [8] the corresponding nonlinearity function is obtained as

\begin{equation}
f(n) = \sqrt{n + \lambda + \kappa}.
\end{equation}

The resolution of the identity for the GKCS and their dual pair associated to Pöschl-Teller potential has been recently established in [36] and [35], respectively. Now by substitution (47) in expectation values needed for the non-classical behavior (see Appendix), we can investigate the non-classical properties of a dual pair of GKCSs ((36) and (38)) and the combination of states introduced in (44). In figure 4 we have plotted the curve of second order correlation function in (26) against \(z\) for fixed values \(\gamma = 0.5\) and \(\lambda = 4 = \kappa\). As an interesting feature of our numerical results it is clear that the interference effects in the combination of the dual pair of GKCSs vanish the sub-Poissonian behavior of the individual GKCSs and its dual pair in all ranges of \(z\), i.e., the state in (44) shows super-Poissonian statistics. Figures 5 and 6 show quadrature squeezing and amplitude squared squeezing for the three sets of states, original GKCSs \(|z, \gamma\rangle^{GK}\), their dual pair \(|\tilde{z}, \gamma\rangle^{GK}\) and the combination of states \(|z, \gamma\rangle_{s1}^{GK}\), respectively. In all cases we have chose \(z \in \mathbb{R}\). From figures 5-a, 5-c and 5-e for squeezing in \(x\) quadrature one can see that the original GKCSs, shows oscillatory behavior, no squeezing for the dual pair as well as the combined state. On the other side from figures 5-b, 5-d and 5-f the oscillatory behavior of squeezing in \(y\) quadrature is seen for the original GKCSs, squeezing in all ranges of \(\gamma\) and \(z\) for the dual pair as well as the combined state. Interestingly, similar qualitatively behavior in \(X\) may be observed from figures 6-a, 6-c and 6-e, and in \(Y\) from figures 6-b, 6-d and 6-f when amplitude squared squeezing is evaluated.
9 Summary and Conclusion

In summary we introduced the superposition of the dual pair of $f$-deformed states in two distinct approaches. The presented formalisms have been applied to a familiar physical system, i.e., Hydrogen-like spectrum. We continued the work with the dual pair of GKCSs, the states which were extracted from a more physical insights. For the latter states we have used the Pöschl-Teller potentials as a physical realization. Adding our results some considerable points are remarkable. Firstly, the combination (superposition of the first type) of any dual pair of nonlinear CSs and GKCSs maintained them in the family of nonlinear CSs, although with a modified $f_{s_1}(n)$ and $f_{GKs_1}(n, \gamma)$ rather than original $f(n)$ and $f_{GK}(n, \gamma)$, respectively. This observation is unlike the usual types of superpositions more frequently have been introduced in the literature, also has been done by us in section 4 of the present paper (for instance even and odd, for standard and nonlinear CSs). Finding the explicit forms of $f_{s_1}(n)$ and $f_{GKs_1}(n, \gamma)$ has the potentiality of the reconstruction the combination of states via the two well-known approaches: i) the algebraic method and ii) group theoretical procedures. Secondly, it is noticeable that by considering the formalism introduced by one us in [8], we can give Hamiltonians describe the dynamics of the systems corresponding to the combined states $|\alpha, f_{s_1}\rangle$ in (20) and $|z, \gamma\rangle_{s_1}^{GK}$ in (46). Indeed, using (20) the $f$-deformed ladder operators formalism lead one to the Hamiltonian $\hat{H}_{s_1} = A_{s_1}^{\dagger} A_{s_1} = n f_{s_1}^2 (n)$, and using (16) to $\hat{H}_{s_1}^{GK} = A_{s_1}^{GK\dagger} A_{s_1}^{GK} = n |f_{s_1}^{GK}(n, \gamma)|^2$.

Thirdly, the superpositions in the present paper do not always highlight the non-classical effects, there is special cases in which it weaken the non-classical properties. This is not surprising if one recalls that the number states are with highest non-classicality and canonical CSs (which are particular superposition of the number states) are with lowest non-classicality (and highest classicality). To this end we are willing to mention that the presented formalism can be easily extended to any physical system, either with known nonlinearity function or with known discrete non-degenerate spectrum (center of mass motion of trapped ion, harmonious states, anharmonic oscillator and generally any nonlinear oscillator algebra).

10 Appendix

Considering the introduced states in (15), the expectation values needed for the calculation and computation of the non-classicality criteria have been expressed in section 5 corresponding to the first kind of superposition (combination) of states can be easily obtained as

$$\langle a \rangle_{s_1} = |N_{s_1}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+1} \alpha^n (1 + ([f(n+1)]!)^2)) (1 + ([f(n)]!)^2))}{n! \ [f(n)]! \ [f(n+1)]!}.$$ (48)
\[ \langle a^2 \rangle_{s_1} = |N_{s_1}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+2} \alpha^{*n} (1 + ([f(n + 2)])^2) (1 + (f(n)]^2)^2)}{n! [f(n)]! [f(n + 2)]!} \]  

\[ \langle a^4 \rangle_{s_1} = |N_{s_1}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+4} \alpha^{*n} (1 + ([f(n + 4)])^2) (1 + (f(n)]^2)^2)}{n! [f(n)]! [f(n + 4)]!} \]  

\[ \langle a^\dagger a \rangle_{s_1} = |N_{s_1}|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+1)} (1 + ([f(n + 1)])^2)^2}{n! ([f(n + 1)]^2)^2}, \]  

\[ \langle a^\dagger a^2 \rangle_{s_1} = |N_{s_1}|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+2)} (1 + ([f(n + 2)])^2)^2}{n! ([f(n + 2)]^2)^2}, \]  

Similarly, the expectation values needed for the calculation and computation of the non-classicality criteria corresponding to the second kind of superposition of states introduced in [24] are as follows

\[ \langle a \rangle_{s_2} = |N_{s_2}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+1} \alpha^{*n} G(n + 1, |\alpha|^2)G(n, |\alpha|^2)}{n! [f(n)]! [f(n + 1)]!}, \]  

\[ \langle a^2 \rangle_{s_2} = |N_{s_2}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+2} \alpha^{*n} G(n + 2, |\alpha|^2)G(n, |\alpha|^2)}{n! [f(n)]! [f(n + 2)]!}, \]  

\[ \langle a^4 \rangle_{s_2} = |N_{s_2}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{n+4} \alpha^{*n} G(n + 4, |\alpha|^2)G(n, |\alpha|^2)}{n! [f(n)]! [f(n + 4)]!}, \]  

\[ \langle a^\dagger a \rangle_{s_2} = |N_{s_2}|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+1)} G^2(n + 1, |\alpha|^2)}{n! ([f(n + 1)]^2)^2}, \]  

\[ \langle a^\dagger a^2 \rangle_{s_2} = |N_{s_2}|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+2)} G^2(n + 2, |\alpha|^2)}{n! ([f(n + 2)]^2)^2}, \]  

To investigate the non-classical properties of GKCSs introduced in [44], we need the following mean values

\[ \langle a \rangle^{GK}_{s_1} = |N_{s_1}^{GK}|^2 \sum_{n=0}^{\infty} \frac{z^{n+1} z^{*n} K(n + 1, \gamma)K^*(n, \gamma)}{(n!)^2 \sqrt{[\epsilon_n]! [\epsilon_{n+1}]!} (n + 1)}, \]  

\[ \langle a^2 \rangle^{GK}_{s_1} = |N_{s_1}^{GK}|^2 \sum_{n=0}^{\infty} \frac{z^{n+2} z^{*n} K(n + 2, \gamma)K^*(n, \gamma)}{(n!)^2 \sqrt{[\epsilon_n]! [\epsilon_{n+2}]!} (n + 1)(n + 2)}, \]  

\[ \langle a^4 \rangle^{GK}_{s_1} = |N_{s_1}^{GK}|^2 \sum_{n=0}^{\infty} \frac{z^{n+4} z^{*n} K(n + 4, \gamma)K^*(n, \gamma)}{(n!)^2 \sqrt{[\epsilon_n]! [\epsilon_{n+4}]!} (n + 1) \cdots (n + 4)}, \]
\begin{align}
\langle a^\dagger a \rangle^G_{s_1} &= |N_{s_1}^G|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}|K(n, \gamma)|^2}{(n!)^2 [\epsilon_n]!} n, \\
\langle a^2 a^\dagger a^\dagger \rangle^G_{s_1} &= |N_{s_1}^G|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}|K(n, \gamma)|^2}{(n!)^2 [\epsilon_n]!} n(n-1).
\end{align}

It is clear that the mean values of \( \langle a^\dagger \rangle_{s_i}, \langle a^2 \rangle_{s_i}, \langle a^4 \rangle_{s_i} \) and \( \langle a^2 a^\dagger \rangle_{s_i} \) can be obtained by taking the conjugates of \( \langle a \rangle_{s_i}, \langle a^2 \rangle_{s_i}, \langle a^4 \rangle_{s_i} \) and \( \langle a^2 a^\dagger \rangle_{s_i} \), respectively. Note that \( i \) in the subscripts \( s_i \) may be 1 or 2 respectively for the first and second type of superpositions.

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References

[1] R.J. Glauber Phys. Rev. 131 (1963) 2766.

[2] E.C.J. Sudarshan Phys. Rev. Lett. 10 (1963) 277.

[3] J.R. Klauder, B.-S. Skagerstam "Coherent States, Applications in Physics and Mathematical Physics" Singapore: World Scientific (1985).

[4] S. Twareque Ali, J-P. Antoine, J-P. Gazeau, "Coherent States, Wavelets and Their Generalizations" Berlin: Springer 2000.

[5] W.M. Zhang, D.H. Feng, R. Gilmore Rev. Mod. Phys. 62 (1990) 867.

[6] R.L. de Matos Filho, W. Vogel Phys. Rev. A 54 (1996) 4560.

[7] V.I. Man'ko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Physica Scripta 55 (1997) 528.

[8] R. Roknizadeh, M.K. Tavassoly, J. Phys. A: Math. Gen. 37 (2004) 8111.

[9] M.K. Tavassoly, J. Phys. A: Math. and Theor. 41 (2008) 285305.

[10] B. Roy, P. Roy, J. Opt. B, Quantum and Semiclass. Opt. 2 (2000) 65.

[11] S. Twareque Ali, R. Roknizadeh and M.K. Tavassoly J. Phys. A: Math. Gen. 37 (2004) 4407.

[12] B. Yurke, D. Stoler, Phys. Rev. Lett. 57 (1986) 13.

[13] R. Zeng, M.A. Ahmad, S. Liu, Opt. Commun. 271 (2007) 162.

[14] O. Abbasi, M.K. Tavassoly, Opt. Commun. 282 (2009) 3737.

[15] P. Domokos, J. Janszky, P. Adam, T. Larsen, Quantum Opt. 6 (1994) 187.

[16] V.V. Dodonov, I.A. Malkin, V.I. Man’ko, Physica 72 (1974) 597.

[17] W. Schleish, M. Pernigo, Fam Le Kien, Phys. Rev. A 44 (1991) 2172.

[18] S. Mancini, Phys. Lett. A 233 (1997) 291.

[19] S. Mancini, V.I. Man’ko and P. Tombesi, Physica Scripta 58 (1998) 486.

[20] H. Prakash, P. Kumar, Physica A 319 (2003) 305.

[21] H. Prakash, P. Kumar, Physica A 341 (2004) 201.

[22] H. Prakash, P. Kumar, Eur. Phys. J. D 46 (2008) 359.
[23] X. Wang, Phys. Rev. A 64 (2001) 022302.

[24] N.B. An, Phys. Lett. A 373 (2009) 1701.

[25] J-P. Gazeau, J.R. Klauder, J. Phys. A: Math. Gen. 32 (1999) 123.

[26] W.H. Louisell, ”Quantum Statistical Properties of Radiation” New York: Wiley (1973).

[27] E. Shchukin, Th. Richter, W. Vogel, J. Opt. B: Quantum Semiclass. Opt. 6 (2004) S597.

[28] Th. Richterand, W. Vogel, Phys. Rev. Lett. 89 ( 2002) 283601-1;
ibid, J. Opt. B: Quantum Semiclass. Opt. 5 (2003) S371.

[29] R.J. Glauber, Phys. Rev. 130 (1963) 2529.

[30] M. Hillery, Opt. Commun. 62 (1987) 135.

[31] J.R. Klauder, J. Phys. A: Math. Gen. 29 ( 1996) 293.

[32] A.H. El Kinani, M. Daoud, J. Math. Phys. 35 (2001) 2279; ibid, Int. J. Mod. Phys. B. (2003) 15 2465; ibid, Int. J. Mod. Phys. B. 16 ( 2002) 3915.

[33] V. Bargmann Commun. Pure. Appl. Math. 14 (1961) 187.

[34] J.R. Klauder, K.A. Penson, J-M. Sixdeniers, Phys. Rev. A. 64 (2001) 013817.

[35] R. Roknizadeh, M.K. Tavassoly, J. Math. Phys. 46 (2005) 042110.

[36] J-P. Antoine, J-P. Gazeau, P. Monceau, J.R. Klauder, K.A. Penson, J. Math. Phys. 42 (2001) 2349.
FIG. 1 The curves of second order correlation function $g^2(0)$ corresponding to the states $|\alpha, f\rangle$, $|\tilde{\alpha}, f\rangle$, $|\alpha, f_{s1}\rangle$ and $|\psi\rangle_{s2}$ against amplitude $\alpha \in \mathbb{R}$ for Hydrogen-like spectrum.

FIGs. 2 The plots of the inequality $I_1$ indicates squeezing in $x$ quadrature according to equation (29) displayed in 2-a, and the inequality $I_2$ indicates squeezing in $y$ quadrature according to equation (30) displayed in 2-b corresponding to the states $|\alpha, f\rangle$, $|\tilde{\alpha}, f\rangle$, $|\alpha, f_{s1}\rangle$ and $|\psi\rangle_{s2}$ against amplitude $\alpha \in \mathbb{R}$ for Hydrogen-like spectrum.

FIGs. 3 The plots of the inequality $I_3$ indicates amplitude squared squeezing in $X$ according to equation (33) displayed in 3-a, and the inequality $I_4$ indicates amplitude squared squeezing in $Y$ according to equation (34) displayed in 3-b corresponding to the states $|\alpha, f\rangle$, $|\tilde{\alpha}, f\rangle$, $|\alpha, f_{s1}\rangle$ and $|\psi\rangle_{s2}$ against $\alpha \in \mathbb{R}$ for Hydrogen-like spectrum.

FIG. 4 The curves of second order correlation function $g^2(0)$ corresponding to the states $|z, \gamma\rangle_{GK}^T$, $|\tilde{z}, \gamma\rangle_{GK}^T$ and $|z, \gamma\rangle_{s1}^G$ against $z \in \mathbb{R}$ for Pöschl-Teller potential at fixed $\gamma = 0.5$ and $\lambda = 4 = \kappa$.

FIGs. 5 The three-dimensional plots of the inequality $I_1$ for squeezing in $x$ quadrature according to equation (29) against $z$ and $\gamma$ using the states $|z, \gamma\rangle_{GK}^T$, $|\tilde{z}, \gamma\rangle_{GK}^T$ and $|z, \gamma\rangle_{s1}^G$, displayed respectively in figures 5-a, 5-c and 5-e, and the inequality $I_2$ for squeezing in $y$ quadrature according to equation (30) against $z$ and $\gamma$ for the above three sets of states in figures 5-b, 5-d and 5-f, respectively. In all cases the Pöschl-Teller potential is considered for $\lambda = 4 = \kappa$.

FIGs. 6 The three-dimensional plots of the inequality $I_3$ for amplitude squared squeezing in $X$ according to equation (33) against $z$ and $\gamma$ using the states $|z, \gamma\rangle_{GK}^T$, $|\tilde{z}, \gamma\rangle_{GK}^T$ and $|z, \gamma\rangle_{s1}^G$, displayed respectively in figures 6-a, 6-c and 6-e, and the inequality $I_4$ for amplitude squared squeezing in $Y$ according to equation (34) against $z$ and $\gamma$ for the above three sets of states in figures 6-b, 6-d and 6-f, respectively. In all cases the Pöschl-Teller potential is considered with $\lambda = 4 = \kappa$. 