On the Kuznetsov Trace Formula for PGL$_2$(C)

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Abstract. In this note, using a representation theoretic method of Cogdell and Piatetski-Shapiro, we prove the Kuznetsov trace formula for an arbitrary discrete group $\Gamma$ in PGL$_2$(C) that is cofinite but not cocompact. An essential ingredient is a kernel formula, recently proved by the author, on Bessel functions for PGL$_2$(C). This approach avoids the difficult analysis in the existing method due to Bruggeman and Motohashi.

Contents

1. Introduction 1
2. Notations and Statement of Theorem 3
3. Spectral Analysis of $L^2(\Gamma \backslash G)$ 6
4. Kloosterman Sums, Poincaré Series and the Kloosterman-Spectral Formula 9
5. An Explicit Kloosterman-Spectral Formula - the Kuznetsov Trace Formula 14
References 18

1. Introduction

In the paper [Kuz], Kuznetsov discovered his trace formula for $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2 \cong \text{PSL}_2(\mathbb{Z}) \backslash \text{PSL}_2(\mathbb{R})/K$, in which $\mathbb{H}^2$ denotes the hyperbolic upper half-plane and $K = \text{SO}(2)/\{\pm 1\}$. There are two forms of his formula. The approach to the first formula is through a spectral decomposition formula for the inner product of two (spherical) Poincaré series. Then, using an inversion formula for the Bessel transform, Kuznetsov obtained another form of his trace formula (the version in [DI] is more complete in the sense that holomorphic cusp forms occurring in the Petersson trace formula are also involved). On the geometric side, a weighted sum of Kloosterman sums arises from computing the Fourier

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coefficients of Poincaré series. The spectral side involves the Fourier coefficients of holomorphic and Maass cusp forms and Eisenstein series along with the Bessel functions associated with their spectral parameters. The second form plays a primary role in the investigation of Kuznetsov on sums of Kloosterman sums in the direction of the Linnik-Selberg conjecture.

Along the classical lines, the Kuznetsov trace formula has been studied and generalized by many authors (see, for example, [Bru1, Bru2, Pro, DI, BM2]). Their ideas of generalizing the formula to the non-spherical case are essentially the same as Kuznetsov. It should however be noted that the pair of Poincaré series is chosen and spectrally decomposed in the space of a given $K$-type.

In the framework of representation theory, the second form of the Kuznetsov formula for an arbitrary Fuchsian group of the first kind $\Gamma \subset \text{PGL}_2(\mathbb{R})$ was proved straightforwardly by Cogdell and Piatetski-Shapiro [CPS]. Their computations use the Whittaker and Kirillov models of irreducible unitary representations of $\text{PGL}_2(\mathbb{R})$. They observe that the Bessel functions occurring in the Kuznetsov formula should be identified with the Bessel functions for irreducible unitary representations of $\text{PGL}_2(\mathbb{R})$ given by [CPS Theorem 4.1], in which the Bessel function $\beta_\pi$ associated with a $\text{PGL}_2(\mathbb{R})$-representation $\pi$ satisfies the following kernel formula,

$$ W(\left( \begin{array}{cc} a & \pi \cr 1 & \end{array} \right) \left( \begin{array}{cc} \pi & \cr 1 & \end{array} \right) ) = \int_{\mathbb{R} \times \mathbb{R}} \beta_\pi(ab) W(b) \, d^\times b, $$

for all Whittaker functions $W$ in the Whittaker model of $\pi$. Note that their idea of approaching Bessel functions for $\text{GL}_2(\mathbb{R})$ using local functional equations for $\text{GL}_2 \times \text{GL}_1$-Rankin-Selberg zeta integrals over $\mathbb{R}$ (see [CPS §8]) also occurs in [Qi].

In the book of Cogdell and Piatetski-Shapiro [CPS], the Kuznetsov trace formula is derived from computing the Whittaker functions (Fourier coefficients) of a (single) Poincaré series $P_f(g)$ in two different ways, first unfolding $P_f(g)$ to obtain a weighted sum of Kloosterman sums, and secondly spectrally expanding $P_f(g)$ in $L^2(\Gamma \backslash G)$ and then computing the Fourier coefficients of the spectral components in terms of basic representation theory of $\text{PGL}_2(\mathbb{R})$. The Poincaré series in [CPS] arise from a very simple type of functions that are supported on the Bruhat open cell of $\text{PGL}_2(\mathbb{R})$ and split in the Bruhat coordinates, which is surely not of a fixed $K$-type. In other words, [CPS] suggests that, instead of the Iwasawa coordinates, it would be more pleasant to study the Kuznetsov formula using the Bruhat coordinates. For this, [CPS] works with the full spectral theorem rather than a version, used by all the other authors, that is restricted to a given $K$-type.

In the direction of generalization to other groups, Miatello and Wallach [MW] gave the spherical Kuznetsov trace formula for real semisimple groups of real rank one, which include both $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{C})$. It is however much more difficult to extend the formula

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1It is Selberg who introduced Poincaré series and realized the intimate connections between Kloosterman sums and the spectral theory of the Laplacian on $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$ in [Sel].
to the non-spherical setting for $SL_2(\mathbb{C})$. The first breakthrough is the work of Bruggeman and Motohashi [BM3], where the Kuznetsov trace formula for $PSL_2(\mathbb{Z}[i]) \backslash PSL_2(\mathbb{C})$ was found.

Let $\mathbb{H}^3$ denote the three dimensional hyperbolic space and let $K = SU(2)/\{\pm 1\}$. Featuring a combination of the Jacquet and the Goodman-Wallach operators, the analysis carried out in [BM3] is considerably hard. Nevertheless, similar to [Kuz], the approach of [BM3] is also from considering the inner product of two certain sophisticatedly chosen Poincaré series of a given $K$-type. It is however remarked without proof in [BM3], §15 that their Bessel kernel should be interpreted as the Bessel function of an irreducible unitary representation of $PSL_2(\mathbb{C})$.

In this note, we shall prove the Kuznetsov trace formula for an arbitrary discrete group $\Gamma$ in $PGL_2(\mathbb{C})$ that is cofinite but not cocompact. An essential ingredient is a kernel formula for $PGL_2(\mathbb{C})$ (see (4.3)) that is identical to (1.1). This is a consequence of the representation theoretic investigations on a general type of special functions, the (fundamental) Bessel kernels for $GL_n(\mathbb{C})$, arising in the Voronoï summation formula; see [Qi], §18. Our derivation of the Kuznetsov formula will be in parallel with that in [CPS], at least for the Kloosterman-spectral formula (see §4). As such, this note should be viewed as the supplement and generalization of the work of Cogdell and Piatetski-Shapiro over the complex numbers. With this method, we can avoid the very difficult and complicated analysis in [BM3].

Finally, we remark that the only reason for considering $PGL_2(\mathbb{C})$ is to keep notations simple. Without much difficulty, we may extend the Kuznetsov formula to $SL_2(\mathbb{C})$. In another direction, we hope to implement the Cogdell-Piatetski-Shapiro method to $PGL_2$ or $SL_2$ over an arbitrary number field.

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2. Notations and Statement of Theorem

2.1. We shall adopt the notations in [CPS]. Let $G = PGL_2(\mathbb{C}) (= PSL_2(\mathbb{C}))$. Let

$$
N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{C} \right\} \subset G,
$$

$$
A = \left\{ \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^{\times} \right\} \subset G.
$$
Define \( A_+ = \{ r \in A : r \in \mathbb{R}_+ \} \). Let \( B = NA \) denote the Borel subgroup of \( G \). Let \( w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) be the long Weyl element of \( G \). Then we have the Bruhat decomposition \( G = B \cup NuB \). Let 
\[
K = SU(2)/\{ \pm 1 \} = \left\{ k_{v,w} = \begin{pmatrix} w & 0 \\ -\overline{w} & \overline{w} \end{pmatrix} : |w|^2 + |\bar{w}|^2 = 1 \right\}/\{ \pm 1 \}.
\]

We have the Iwasawa decomposition \( G = NA_+ K \).

Let \( q \) denote the Lie algebra of \( G \) and \( \mathfrak{u}(q) \) its universal enveloping algebra.

### 2.2

Let \( \Gamma \subset G \) be a discrete subgroup that is cofinite but not cocompact. Let \( C \subset \partial \mathbb{H}^2 \) denote the set of cusps of \( \Gamma \). We assume that \( \infty \in C \). For each cusp \( \alpha \in C \), we fix \( g_\alpha \in G \) such that \( g_\alpha \cdot \alpha = \infty \). Let \( P_\alpha \) denote the parabolic subgroup of \( G \) stabilizing \( \alpha \). Let \( P_\alpha = U_\alpha M_\alpha \) be its Langlands decomposition. Then the conjugation by \( g_\alpha \) provides an isomorphism \( P_\alpha \cong B \), which induces isomorphisms \( U_\alpha \cong N \) and \( M_\alpha \cong A \). For each \( \alpha \), define \( \Gamma'_\alpha = \Gamma \cap P_\alpha \), \( \Gamma_\alpha = \Gamma \cap U_\alpha \). Let \( g_\alpha \Gamma_\alpha g_\alpha^{-1} = \{ u : u \in \Lambda_\alpha \} \), with \( \Lambda_\alpha \) a lattice in \( \mathbb{C} \). Let \( |\Lambda_\alpha| \) denote the area of \( \Lambda_\alpha \mathbb{C} \). According to [EGM Theorem 2.1.8 (3)], we have \( \Gamma'_\alpha \cong \hat{\eta}_m \times \Lambda_\alpha \), where \( m_\alpha \in \{ 1, 2, 3, 4, 6 \} \), \( \eta_m \) denotes the group of \( m_\alpha \)-th roots of unity, and \( \eta_m \) acts on \( \Lambda_\alpha \) by multiplication.

Let \( \mathfrak{h}(\mathbb{C}^\times) \) denote the group of multiplicative characters of \( \mathbb{C}^\times \). Every character \( \mu \in \mathfrak{h}(\mathbb{C}^\times) \) is of the form \( \mu_{s,d} = |z|^{2s}|z|^d \), with \( s \in \mathbb{C} \), \( d \in \mathbb{Z} \) and the notation \( [z] = z/|z| \). Hence \( \mathfrak{h}(\mathbb{C}^\times) \) has a structure of a complex manifold, \( \mathfrak{h}(\mathbb{C}^\times) \cong \mathbb{Z} \times \mathbb{C} \). We define \( \Re \mu_{s,d} = \Re (s) \). Each \( \mu \in \mathfrak{h}(\mathbb{C}^\times) \) defines a character of \( A \), by \( \mu(\alpha) = \mu(a) \), and a character of \( B \) through \( B \to N \setminus B \cong A \). Through the isomorphisms give above, \( \mu \) also defines a character of \( M_\alpha \) and \( P_\alpha \). Let \( \mathfrak{h}_m(\mathbb{C}^\times) \) denote the set of characters \( \mu_{s,d} \) such that \( m|d \). Note that characters in \( \mathfrak{h}_m(\mathbb{C}^\times) \) are trivial on \( \Gamma'_\alpha \).

Let \( \mathfrak{X}_\infty(\mathbb{C}) \) denote the group of additive characters on \( \Lambda_\infty \setminus \mathbb{C} \). We define the dual lattice \( \Lambda'_\infty \) of \( \Lambda_\infty \) by
\[
\Lambda'_\infty = \left\{ \omega \in \mathbb{C} : \text{Tr}(\omega \lambda) = \omega \lambda + \bar{\omega} \lambda \in \mathbb{Z} \right\}.
\]
Then every \( \psi \in \mathfrak{X}_\infty(\mathbb{C}) \) is of the form \( \psi_\omega(z) = e(\text{Tr}(\omega z)) \) for some \( \omega \in \Lambda'_\infty \), with \( e(x) = e^{2\pi i x} \). Furthermore, each \( \psi \in \mathfrak{X}_\infty(\mathbb{C}) \) defines a character of \( \Gamma_\infty \mathbb{C} \setminus \mathbb{N} \), by \( \psi(\alpha) = \psi(u) \).

### 2.3

In the notations of [Qi §18.2], the irreducible infinite dimensional unitary representations of \( \text{PGL}_2(\mathbb{C}) \) are
- (principal series) \( \pi_{s,t}^+(it) \) for \( t \in \mathbb{R} \) and \( d \in \mathbb{Z} \), with \( \pi_{s,t}^+(it) \cong \pi_{-s,-t}^+(-it) \),
- (complementary series) \( \pi(t) \) for \( t \in (0, \frac{1}{2}) \).

In order to unify notations, let us put \( \pi_d(it) = \pi_{s,t}^+(it) \) and \( \pi_0(t) = \pi(t) \) so that we may denote by \( \pi_d(s) \) either principal series for \( s = it \) or complementary series for \( s = t \) and \( d = 0 \).

\(^{IV}\) When \( m_\alpha = 4 \), respectively 3 or 6, \( \Lambda_\alpha \) must be the ring of integers in the quadratic number field \( \mathbb{Q}(i) \), respectively \( \mathbb{Q}(\sqrt{-3}) \).
2.4. Let $dz$ be twice the ordinary Lebesgue measure on $\mathbb{C}$, and choose the standard multiplicative Haar measure $d^\times z = dz/|z|^2$ on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

We take the Haar measure $dk$ on $K$ of total mass 1. Writing an element of $G$ in the Iwasawa decomposition as $g = z \gamma k$, with $z \in N$, $\gamma \in A_+$ and $k \in K$, we let $dg = 2r^{-3} dz d\gamma dk$. This Haar measure $dg$ on $G$ induces the hyperbolic measure on the hyperbolic space $\mathbb{H}^3 = \{z + rj : z \in \mathbb{C}, r \in \mathbb{R}_+\}$. Moreover, if we use the Bruhat coordinates $g = \underline{u}_1 \omega \underline{a}$ on the open Bruhat cell $NuB$, with $\underline{u}_1, \underline{a} \in N$ and $\underline{a} \in A$, then the measure $dg = (1/4\pi^2) du_1 du_2 da a$.

2.5. Main Theorem.

**Theorem 2.1.** Let $F \in C_c^\infty (\mathbb{C}^\times)$ be a smooth compactly supported function on $\mathbb{C}^\times$. Let $\omega_1, \omega_2 \in \mathcal{N}_G \setminus \{0\}$ and $\psi_1 = \psi_{\omega_1}, \psi_2 = \psi_{\omega_2}$. Then

$$
\sum_{c \in \mathbb{Z}} K_{1} F(c; \psi_1, \psi_2) = \pi |\Lambda_{\infty}|^2 \sum_{\pi \in \mathcal{P}(\mathbb{G})} \frac{A_{\omega_2} (\varphi_\pi) \overline{A_{\omega_1} (\varphi_\pi)}}{(2|d| + 1)G_{s,d}} \hat{F}(s, d)
$$

$$
= \pi |\Lambda_{\infty}|^2 \sum_{\pi \in \mathcal{P}(\mathbb{G})} \frac{A_{\omega_2} (\varphi_\pi) \overline{A_{\omega_1} (\varphi_\pi)}}{(2|d| + 1)G_{s,d}} \hat{F}(s, d)
$$

$$
= \frac{1}{4} \sum_{\omega} \frac{1}{m_\omega |\Lambda_{\omega}|} \sum_{d \in \mathbb{Z}} \mu_{\omega,d} \left( \frac{\omega_2}{\omega_1} \right) Z_{\omega} (\mu_{\omega,d}; 0, \omega_2) Z_{\omega} (\mu_{\omega,d}; 0, \omega_1) \hat{F}(i t, d) dt,
$$

where $K_{1} (c; \psi_1, \psi_2)$ are the Kloosterman sums at infinity associated with $\Gamma$, $\psi_1$ and $\psi_2$ given in §4.1. $A_{\omega} (\varphi_\pi)$ are the Fourier coefficients of a certain $L^2$-normalized automorphic form in the space of the representation $\pi$ that occurs discretely in $L^2 (\Gamma \backslash G)$ (see §5.2). $Z_{\omega} (\mu_{\omega,d}; 0, \omega)$ are the Kloosterman-Selberg zeta functions arising in the Fourier coefficients of Eisenstein series (see §5.3).

$$
G_{s,d} = \begin{cases} 
1, & \text{if } s = i t, \\
\Gamma (1 + 2 t) / \Gamma (1 - 2 t), & \text{if } s = i t, d = 0,
\end{cases}
$$

and finally $\hat{F}(s, d)$ is the Bessel transform of $F$ given by

$$
\hat{F}(s, d) = \frac{1}{\sin (2\pi s)} \int_{\mathbb{C}^\times} F(z) (J_{s,2d} (4\pi \sqrt{z}) - J_{-s,-2d} (4\pi \sqrt{z})) d^\times z,
$$

with

$$
J_{s,2d}(z) = J_{-2s,d} (z) J_{-2s+2d} (z),
$$

and $J_{s}(z)$ the classical Bessel function of the first kind.

**Remark 2.2.** When $\Gamma$ is a congruence group for an imaginary quadratic field, the automorphic forms $\varphi_\pi$ will usually be chosen to be common eigenfunctions of Hecke operators. Furthermore, according to [10], there will be no residual spectrum if $\Gamma$ is congruence.

Note that our definition of Kloosterman sums is slightly different from the usual one. Taking the simplest example of the full modular group $\Gamma = \text{PSL}_2 (\mathbb{C})$, with $\mathbb{C}$ the ring of
integers of an imaginary quadratic field, then for \( c \in \mathbb{O} \setminus \{0\} \) and \( \omega_1, \omega_2 \in \mathbb{O}' \setminus \{0\} \) we have

\[
K_l(c^2; \psi_1, \psi_2) = \sum_{a_1, a_2 \in \mathbb{O}/\mathbb{O}} e \left( \text{Tr} \left( \frac{\omega_1 a_1 + \omega_2 a_2}{c} \right) \right).
\]

For \( \Gamma = \text{PSL}_2(\mathbb{Z}[i]) \) we obtain the summation formula in \([BM3]\).

3. Spectral Analysis of \( L^2(\Gamma \backslash G) \)

The spectral decomposition of \( L^2(\Gamma \backslash G) \) is a consequence of the general theory of Eisenstein series due to Langlands in \([Lan]\). Here we shall follow the expositions in \([CPS]\).

Let \( L^2(\Gamma \backslash G) \) be the space of all square integrable functions on \( \Gamma \backslash G \) with respect to the measure induced by the Haar measure on \( G \). Let \( \langle \cdot, \cdot \rangle \) be the (Petersson) inner product on \( L^2(\Gamma \backslash G) \). \( G \) acts on this space by right translation.

Let \( L^2_0(\Gamma \backslash G) \) denote the space of \( L^2 \)-cusp forms. We have a decomposition

\[
L^2(\Gamma \backslash G) = L^2_0(\Gamma \backslash G) \oplus L^2_\infty(\Gamma \backslash G),
\]

where \( L^2_\infty(\Gamma \backslash G) \) is the orthogonal complement of \( L^2_0(\Gamma \backslash G) \).

3.1. Spectral Decomposition on \( L^2_0(\Gamma \backslash G) \). It is a well known theorem of Gelfand and Piatetski-Shapiro that \( L^2_0(\Gamma \backslash G) \) decomposes into a discrete countable direct sum of irreducible unitary representations \( (\pi, V_\pi) \) of \( G \), each isomorphism classes occurring with finite multiplicity (see \([GPS, HC, Lan]\)). We let \( \Pi_0(\Gamma) \) be the set of irreducible constituents of \( L^2_0(\Gamma \backslash G) \).

3.2. Eisenstein Series. For each \( \mu \in \mathcal{X}(G^\times) \) and cusp \( a \in C \), let \( V_a(\mu) \) denote the Hilbert space of functions \( f : G \to \mathbb{C} \) such that

- \( f(pg) = \mu(p) \delta_a^\frac{1}{2}(p) f(g) \), \( p \in P_a \),
- \( f \) is square integrable on \( K \).

Here \( \delta_a \) is the modulus character of \( P_a \) acting on \( U_a \) by conjugation. For the matrix \( p = g_a^{-1} u a g_a \) in \( P_a \), we have \( \delta_a(p) = |a|^2 \). \( G \) acts on this space by right translation. We denote this representation by \( \pi_a(\mu) \). There is a non-degenerate \( G \)-invariant Hermitian paring on \( V_a(\mu) \times V_a(\overline{\mu}^{-1}) \) given by

\[
\langle f, f' \rangle_a = \int_{P_a \backslash G} f(g) \overline{f'(g)} dg, \quad f \in V_a(\mu), \ f' \in V_a(\overline{\mu}^{-1}).
\]

Note that if \( \Re \mu = 0 \) then \( \pi_a(\mu) \) is unitary with respect to this inner product.

For \( f \in V_a \), we form the Eisenstein series

\[
E_a(g; f, \mu) = \sum_{y \in \Gamma} f(y g).
\]
Note that \( E_a(g; f, \mu) \equiv 0 \) if \( \mu \notin X_m(\mathbb{C}^\times) \). It converges uniformly and absolutely on compact subsets of \( G \) for \( \Re \mu > \frac{1}{2} \). \( E_a(g; f, \mu) \) is analytic in \( \mu \) and admits a meromorphic continuation to all \( \mu \in \mathbb{X}(\mathbb{C}^\times) \). Furthermore, there exist intertwining operators
\[
M(a; b; \mu) : V_b(\mu) \rightarrow V_b(\mu^{-1})
\]
such that the Eisenstein series satisfy the functional equation
\[
(3.1) \quad E_a(g; f, \mu) = \sum_b E_b(g; M(a, b; \mu) f, \mu^{-1}).
\]

3.3. The Residual Spectrum. The poles of the \( E_a(g; f, \mu) \) in \( \Re \mu \geq 0 \), which are identical with the poles of \( M(a, b; \mu) \), are all simple and their \( s \)-coordinates lie in a finite subset of the segment \( (0, \frac{1}{2}] \). The residues of these Eisenstein series at such a pole form a non-cuspidal irreducible representation occurring discretely in \( L^2(\Gamma \backslash G) \). The point \( \mu = \delta_0^2 \) always yields the trivial representation, which represents the constant functions, whereas all the other components in the residual spectrum are complementary series. Let \( \Pi_{\mu}(\Gamma) \) denote the representations \( (\pi, V_\pi) \) which so occur discretely in \( L^2(\Gamma \backslash G) \) coming from the residues of all Eisenstein series \( E_a(g; f, \mu) \).

3.4. The Continuous Spectrum. It is known that \( L^2_c(\Gamma \backslash G) \) is the closure of the space spanned by incomplete Eisenstein series \( E_a(g; f) \) of the form
\[
E_a(g; f) = \sum_{\Gamma \subseteq \Gamma} f(\gamma g),
\]
with \( f \in C_c^\infty(U_a \backslash G) \). For \( f \in C_c^\infty(U_a \backslash G) \), \( E_a(g; f) \) is compactly supported on \( \Gamma \backslash G \), then we may form the inner product
\[
( E_a(:, f), E_b(:, f') )
\]
with the Eisenstein series for \( f' \in V_b(\mu) \) with \( \Re \mu = 0 \). This linear functional will be presented by some \( \Phi_{\mu}(\Pi_\mu) (E_a(:, f)) \in V_b(\mu) \) in the sense that
\[
( E_a(:, f), E_b(:, f'), \mu ) = ( \Phi_{\mu}(\Pi_\mu) (E_a(:, f)), f' )_b, \quad \text{all } f' \in V_b(\mu).
\]
Therefore \( \Phi_{\mu}(\Pi_\mu) \) gives a \( G \)-intertwining projection \( L^2_c(\Gamma \backslash G) \rightarrow V_b(\mu) \).

3.5. The Spectral Decomposition of \( L^2(\Gamma \backslash G) \). For convenience, let \( \Pi_\mu(\Gamma) = \Pi_0(\Gamma) \cup \Pi_\mu(\Gamma) \) denote the complete discrete spectrum of \( L^2(\Gamma \backslash G) \).

**Theorem 3.1.** The spectral decomposition of \( L^2(\Gamma \backslash G) \) is
\[
L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \Pi_\mu(\Gamma)} V_\pi \oplus \sum_a \frac{1}{2\pi i m_a} \frac{1}{m_a} \int_{\Re \mu = 0} V_b(\mu) d\mu,
\]
where the sections of the continuous spectrum must satisfy the functional equation \( \Re s = 0 \) and the integral over \( d\mu \) is on the vertical lines \( \mathbb{X}_m(\mathbb{C}^\times) \cong m_a \mathbb{Z} \times \{ s \in \mathbb{C} : \Re s = 0 \} \).

\(^\text{V}Here we follow the terminologies and notations of [Iwa] rather than [CPS].\)
To be concrete, for a function \( \varphi \in L^2(\Gamma \backslash G) \), we may write

\[
(3.2) \quad \varphi = \sum_{\pi \in \Pi_0(\Gamma)} F_\pi(\varphi) + \sum_a \frac{1}{4\pi^2 m_a |\Lambda_a|} \int_{\mathbb{R} \mu = 0} F_{\pi_a(\mu)}(\varphi) d\mu.
\]

For \( \pi \in \Pi_0(\Gamma) \), \( F_\pi(\varphi) \) is the unique element in \( V_\pi \) satisfying

\[
(3.3) \quad (F_\pi(\varphi), \varphi') = (\varphi, \varphi'), \quad \text{all } \varphi' \in V_\pi.
\]

Let \( \Phi_{\pi_a(\mu)} \) be the \( G \)-intertwining projection from \( L^2(\Gamma \backslash G) \to V_\pi(\mu) \), which extends onto \( L^2(\Gamma \backslash G) \), then

\[
(3.4) \quad F_{\pi_a(\mu)}(\varphi) = E_\pi(g; \Phi_{\pi_a(\mu)}(\varphi), \mu).
\]

The spectral decomposition theorem in general is due to Langlands ([Lan]). For PGL\(_2(\mathbb{C})\), we may however prove the theorem following [EGM Chapter 6], [GJ] or [Iwa] for PSL\(_2(\mathbb{R})\). A Fourier series expansion on the circle \( B \cap SU(2)/\{\pm 1\} \cong SU(1)/\{\pm 1\} \) is needed at the end.

Let \( S(\Gamma \backslash G) \) be the space of smooth vectors in \( L^2(\Gamma \backslash G) \), which is endowed with a finer topology than that induced by the \( L^2 \)-norm. This topology may be defined by the semi-norms \( \nu_X(\varphi) = \|R(X)\varphi\|_2 \) for all \( X \in \mathfrak{u}(g) \). Recall that \( \mathfrak{g} \) is the Lie algebra of \( G \) and \( \mathfrak{u}(g) \) is the universal enveloping algebra. Let \( V_\pi^Z \) and \( V_\pi(\mu)^Z \) denote the spaces of smooth vectors in \( V_\pi \) and \( V_\pi(\mu) \), respectively. As consequences of the theorem of Dixmier-Malliavin [DM], for \( \varphi \in S(\Gamma \backslash G) \), each of its spectral components is smooth and the spectral decomposition \((3.2)\) converges in \( S(\Gamma \backslash G) \) (see [CPS Proposition 1.3, 1.4]).

### 3.6. The Whittaker-Spectral Decomposition

Let \( \psi \in \mathfrak{x}_\varphi(\mathbb{C}^\times) \setminus \{1\} \) and \( \varphi \in S(\Gamma \backslash G) \). We define the \( \psi \)-Whittaker function associated with \( \varphi \) as

\[
W_{\psi, \varphi}(g) = \int_{\Gamma \backslash N} \varphi(ng)\psi^{-1}(n)dn.
\]

Let \( C(N \backslash G; \psi) \) denote the space of continuous functions \( f \) on \( G \) such that \( f(ng) = \psi(n)f(g) \) for all \( n \in N \). Let \( S(N \backslash G; \psi) \) denote the space of smooth vectors in \( C(N \backslash G; \psi) \). It is clear that if \( \varphi \in S(\Gamma \backslash G) \) then \( W_{\psi, \varphi} \in S(N \backslash G; \psi) \).

Using Sobolev’s lemma ([Ev §5.6.3 Theorem 6]), we may prove the following analogue of [CPS Lemma 1.1]. For all \( \varphi \in S(\Gamma \backslash G) \) we have

\[
|W_{\psi, \varphi}(\underline{a})| \ll (|a|^4 + |a|^{-4}) \sum_{X=\underline{X}} \|R(X)\varphi\|_2,
\]

where, on choosing a basis \( \{X_i\}_{i=1}^6 \) of \( \mathfrak{g} \), \( X^{\underline{a}} = \prod_{i=1}^6 X_i^{a_i} \in \mathfrak{u}(g) \) (the order of \( X_i \) in the product is fixed), \( |a| = \sum_{i=1}^6 a_i \), and the implied constant depends only on \( \Gamma \). Consequently, the linear functional \( \varphi \mapsto W_{\psi, \varphi}(1) \) is continuous on \( S(\Gamma \backslash G) \) with its natural topology.
THEOREM 3.2. Let $\psi \in \mathcal{X}_\infty(\mathbb{C}^\times) \setminus \{1\}$. Suppose $\phi \in S(\Gamma \backslash G)$ has spectral expansion as in (3.2). Then
\begin{equation}
W_{\phi, \psi}(1) = \sum_{\pi \in \Pi_\Gamma} W_{\pi(\phi), \psi}(1) + \sum_{a} \frac{1}{4\pi i m_a |N_a|} \int_{\Re \mu = 0} W_{\pi(\mu)}(\phi)(1) d\mu,
\end{equation}
with $\mu \in \mathcal{M}_m(\mathbb{C}^\times)$.

4. Kloosterman Sums, Poincaré Series and the Kloosterman-Spectral Formula

In the following, we fix two nontrivial characters $\psi_1, \psi_2 \in \mathcal{X}_\infty(\mathbb{C})$. Let $\kappa \in \mathbb{C}^\times$ be such that $\psi_2(\kappa z) = \psi_1(z)$.

4.1. Kloosterman Sums. We first introduce
\[ \Omega(\Gamma) = \{ c \in \mathbb{C}^\times : N a c N \cap \Gamma \neq \emptyset, \ c \in A \}, \]
and for $c \in \Omega(\Gamma)$ define $\Gamma_c = N a c N \cap \Gamma$. $\Gamma_c$ is both right and left invariant under $\Gamma_\infty$. For each $\gamma \in \Gamma_c$, we decompose $\gamma$ according to the Bruhat decomposition, namely
\[ \gamma = n_1(\gamma) a n_2(\gamma), \]
with $n_1(\gamma), n_2(\gamma) \in N$.

For $c \in \Omega(\Gamma)$ and $\psi_1, \psi_2 \in \mathcal{X}_\infty(\mathbb{C}^\times)$, we define the associated Kloosterman sum as
\[ K\ell(c; \psi_1, \psi_2) = \sum_{\gamma \in \Gamma_c \backslash \Gamma} \psi_1(n_1(\gamma)) \psi_2(n_2(\gamma)). \]

4.2. Poincaré Series. Given any $a, b > 0$, we let $S_{a, b}(N \backslash G; \psi_1)$ denote the function space consisting of functions $f \in S(N \backslash G; \psi_1)$ satisfying
\[ |f(\tau k)| \leq a \min \{ r^{1+a}, r^{1-\beta} \} \]
for all $\tau \in N, k \in A_+$ and $k \in K$. For $f \in S_{a, b}(N \backslash G; \psi_1)$, we form the Poincaré series
\[ P_f(g) = \sum_{\Gamma \subseteq \Gamma} f(\gamma g). \]
Using the (spherical) Eisenstein series in [EGM §3.2] as majorant, it is readily verified that the series for $P_f(g)$ is absolutely convergent, uniformly on compact subsets. Moreover, applying similar arguments as in the proof of [EGM Proposition 3.2.3], we may estimate $P_f(g)$ near the cusps of $\Gamma$ and prove that $P_f(g) \in S(\Gamma \backslash G)$\footnote{In [CPS §2.3], the notion of rapid decreasing modulo $N$ is introduced in terms of the norm $\| \cdot \|_N$ on $N \backslash G$. However, their arguments in the proofs of Proposition 2.1 and 2.2 are incorrect. Nevertheless, the arguments here work in the $\text{PGL}_2(\mathbb{R})$ context and corrects the errors in [CPS].}

Later, we shall apply the Whittaker-spectral formula (3.5) in Theorem 3.2 to a specific Poincaré series $P_f$ that will be constructed in a moment. With this in mind, we need the following results that follow from standard unfolding computations.

By decomposing $\Gamma$ according to the Bruhat decomposition,
\[ \Gamma = \Gamma_\infty \cup \bigcup_{c \in \Omega(\Gamma)} \Gamma_c, \]
we find on the left hand side of (3.5) a weighted sum of Kloosterman sums.

**Lemma 4.1.** Let \( f \in S_{a,b}(N\backslash G; \psi_1) \). Then
\[
W_{\psi_1}(1) = \delta_{\psi_1, \psi_1}[A\mathbb{X}/f(1)] + \sum_{c \in \mathbb{Z}(\Gamma)} Kl(c; \psi_1, \psi_2)K(f, c; \psi_1, \psi_2),
\]
where \( \delta_{\psi_1, \psi_2} \) is the Kronecker symbol for \( \psi_1 \) and \( \psi_2 \) lying in the same orbit under the action of \( \mathfrak{m}\mathcal{X} \), \( |A\mathbb{X}| \) is the area of \( A\mathbb{X} \subseteq \mathbb{C} \), and
\[
K(f, c; \psi_1, \psi_2) = \int_N f(u^{-1}n) \psi_2^{-1}(n)dn
\]
is called the Kloosterman-Jacquet distribution.

For the right hand side of (3.5), we shall need two identities on the inner products with Poincaré series. First,
\[
(P_f, \varphi) = \int_{N\backslash G} f(g)W_{\varphi, \psi_1}(g)dg, \quad \varphi \in S(\Gamma \backslash G).
\]
Second, for \( f \in S_{a,b}(N\backslash G; \psi_1) \), as alluded to above, with the arguments in the proof of [EGM Proposition 3.2.3], one may show that the Poincaré series \( P_f(g) \) has sufficient decay to take its inner product with the Eisenstein series \( E_\alpha(g; f', \mu) \), and
\[
(P_f, E_\alpha(; f', \mu)) = \int_{N\backslash G} f(g)W_{E_\alpha(; f', \mu), \psi_1}(g)dg, \quad f' \in V_\alpha(\mu).
\]

### 4.3. The Specific Choice of Poincaré Series

Let \( \eta \in \mathcal{S}(\mathbb{C}) \) be a Schwartz function on \( \mathbb{C} \) and let \( \nu \in C^\infty_c(\mathbb{C}^\times) \) be a smooth function of compact support on \( \mathbb{C}^\times \). To \( \eta \) and \( \nu \) we shall associate a function \( f_{\eta, \nu} \in S(N\backslash G; \psi_1) \) by defining
\[
f_{\eta, \nu}(g) = \begin{cases} \psi_1(u_1)\eta(u_2)\nu(a), & \text{if } g = u_1wu_2a \in Nw\mathbb{A}, \\ 0, & \text{if } g \in B = NA. \end{cases}
\]
We have \( f_{\eta, \nu}(ng) = \psi_1(n)f_{\eta, \nu}(g) \) for \( n \in N \) and it may be shown that \( f_{\eta, \nu} \in S_{a,1}(N\backslash G; \psi_1) \) for all \( a > 0 \). Note that \( f_{\eta, \nu}(1) = 0 \).

Subsequently, for the specific choice \( \varphi = P_{f_{\eta, \nu}} \), the left and the right hand side of the identity (3.5) will be referred to as the geometric and the spectral side, respectively.

### 4.4. The Geometric Side

The Kloosterman-Jacquet distribution \( K(f, c; \psi_1, \psi_2) \) in Lemma 4.1 is \( f = f_{\eta, \nu} \) can be computed very easily.

Since
\[
f_{\eta, \nu} \left( w \begin{pmatrix} c \\ 1 \\ u \\ 1 \end{pmatrix} \right) = f_{\eta, \nu} \left( w \begin{pmatrix} 1 & cu \\ 1 & 1 \end{pmatrix} \right) = \eta(cu)\nu(c),
\]
by definition, we have
\[
K(f_{\eta, \nu}, c; \psi_1, \psi_2) = \int_{\mathbb{C}} \eta(cu)\nu(c)\psi_2^{-1}(u)du = \frac{1}{|c|^2}\nu(c) \int_{\mathbb{C}} \eta(u)\psi_2^{-1}(\frac{u}{c})du.
\]
Lemma 4.2. We have
\[ K(f_{\pi,c};\psi_1,\psi_2) = \frac{1}{|c|^2} v(c) \hat{\eta} \left( \frac{1}{c} \right), \]
where \( \hat{\eta} \) is the Fourier transform with respect to \( \psi_2 \).

\[ \hat{\eta}(z) = \int_{\mathbb{C}} \eta(u) \psi^{-1}_2(uz) du. \]

4.5. The Spectral Side. In the following, we shall compute the spectral side and show that it can be expressed in terms of Bessel transforms of the weight function.

4.5.1. Bessel functions. Let \( \psi \) be a nontrivial additive character of \( \mathbb{C} \). For an infinite dimensional unitary representation \( \pi \) of \( G \), let \( W(\pi,\psi) \) be the Whittaker model of \( \pi \), that is, the image of a nonzero embedding of \( V^\infty_\pi \) into the space \( \text{Ind}^G_N(\psi) \) of \( C^\infty \) functions \( W: G \to \mathbb{C} \) satisfying \( W(ng) = \psi(n)W(g) \) for all \( n \in N \), with \( G \) acting on \( \text{Ind}^G_N(\psi) \) by right translation. Actually, it is known from Shalika’s multiplicity one theorem that such a nonzero embedding exists and is unique up to constant. Let \( \mathcal{K}(\pi,\psi) \) be the associated Kirillov model of \( \pi \) on \( L^2(\mathbb{C}^\times, d^\times z) \), whose smooth vectors are all of the form \( W \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) \) for \( W \in W(\pi,\psi) \).

It is shown in [Qi] (18.4) that there exists a (complex-valued) real analytic function \( \partial_{\pi,\psi} \) on \( \mathbb{C}^\times \) which acts as an integral kernel of the action of the Weyl element \( w \) on the Kirillov model \( \mathcal{K}(\pi,\psi) \). Namely, for \( W \in W(\pi,\psi) \),

\[ W \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) = \int_{\mathbb{C}^\times} \partial_{\pi,\psi}(ab) W \left( \begin{pmatrix} b \\ 1 \end{pmatrix} \right) d^\times b. \]

A simple fact is that the Bessel function is conjugation invariant, namely, \( \overline{\partial_{\pi,\psi}(z)} = \partial_{\pi,\psi^{-1}}(z) = \partial_{\pi,\psi}(\overline{z}) \) (see [Qi] (18.2)).

4.5.2. The constants \( c_{\psi}(\pi,\Gamma) \) and \( c_{\phi}(\pi_\alpha(\mu),\Gamma) \). Let \( \psi \) be a nontrivial character of \( \mathbb{C} \).

First, let \( (\pi,V_\pi) \) be a discrete component of \( L^2(\Gamma\backslash G) \). The inner product on \( V_\pi \) is the Petersson inner product inherited from \( L^2(\Gamma\backslash G) \). For smooth vector \( \varphi \in V^\infty_\pi \subset S(\Gamma\backslash G) \), the map

\[ \varphi \mapsto W_\varphi(g) = \int_{\Gamma\backslash N} \varphi(ng) \psi^{-1}(n) dn \]
defines a Whittaker model \( W(\pi,\psi) \) of \( (\pi,V^\infty_\pi) \). Let \( \mathcal{K}(\pi,\psi) \) be the associated Kirillov model. It comes equipped with its canonical \( L^2 \) inner product on \( \mathbb{C}^\times \). Hence there is a positive constant, which we shall denote \( c(\pi,\Gamma) = c_{\psi}(\pi,\Gamma) \), such that for all \( \varphi,\varphi' \in V_\pi \)

\[ \int_{\mathbb{C}^\times} W_\varphi \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) \overline{W_{\varphi'}} \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) d^\times a = c(\pi,\Gamma) \int_{\Gamma\backslash G} \varphi(g) \overline{\varphi'(g)} dg. \]

Second, we consider the representation \((\pi_\alpha(\mu),V_\alpha(\mu))\). For \( f \in V_\alpha(\mu) \), the function

\[ W_{E_\alpha(\cdot;f,\mu)}(g) = \int_{\Gamma\backslash N} E_\alpha(ng;f,\mu) \psi^{-1}(n) dn \]
defines a Whittaker model $W(\pi_a(\mu), \psi)$ of $\pi_a(\mu)$. Recall that there is also a canonical inner product on $(\pi_a(\mu), V_a(\mu))$ given by

$$\langle f, f' \rangle_a = \int_{P_1} f(g)f'(g)dg.$$ 

Hence there exists a constant $c(\pi_a(\mu), \Gamma) = c_\psi(\pi_a(\mu), \Gamma)$ such that for all $f, f' \in V_a(\mu)^\infty$ we have

$$\int_{C^\infty} W_{E_+(\cdot, \mu)}(1) W_{E_-(\cdot, \mu)}(1) d^\infty a = c(\pi_a(\mu), \Gamma) \int_{P_1} f(g)f'(g)dg.$$

4.5.3. The constants $c(\pi; \psi_2/\psi_1)$ and $c(\pi_a(\mu); \psi_2/\psi_1)$. In what follows we shall have to compare Whittaker models associated with $\psi_1$ and $\psi_2$.

Let $\pi \in \Pi_{\pi}(\Gamma)$. Because $\psi_2(z) = \psi_1(\kappa z)$, if $W_{\varphi, \psi_1} \in W(\pi, \psi_1)$, then the function

$$W'_{\varphi, \psi_2}(g) = W_{\varphi, \psi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right) g$$

satisfies $W'_{\varphi, \psi_2}(ng) = \psi_2(n)W'_{\varphi, \psi_2}(g)$. Since this left multiplication by $\left( \begin{array}{c} \kappa \\ 1 \end{array} \right)$ commutes with the action of $G$, the collection of function $W'_{\varphi, \psi_2}(g)$ form another $\psi_2$-Whittaker model.

By Shalika’s multiplicity one theorem, there is a constant $c(\pi; \psi_2/\psi_1)$ such that for all $\varphi \in V_a^\infty$ we have

$$W_{\varphi, \psi_2}(g) = c(\pi; \psi_2/\psi_1) W_{\varphi, \psi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right) g.$$

Similarly, there exists a constant $c(\pi_a(\mu); \psi_2/\psi_1)$ such that for all $f \in V_a(\mu)^\infty$ we have

$$W_{E_+(\cdot, \mu), \psi_2}(g) = c(\pi_a(\mu); \psi_2/\psi_1) W_{E_+(\cdot, \mu), \psi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right) g.$$

4.5.4. Computing the spectral side.

**Lemma 4.3.** Let $f = f_{\eta, \nu}$. Suppose that $\psi_2(z) = \psi_1(\kappa z)$.

For $(\pi, V_a)$ occurring discretely in $L^2(\Gamma \backslash G)$, we have

$$W_{F_{\pi}(\eta, \nu), \psi_2}(1) = c_\Gamma(\pi; \psi_1, \psi_2) \int_{C^\infty} \nu(z) \hat{\eta}(1) \frac{1}{z} d^\infty z,$$

with

$$c_\Gamma(\pi; \psi_1, \psi_2) = \frac{c(\pi; \psi_2/\psi_1) c_{\psi_1}(\pi, \Gamma)}{4\pi^2}.$$ 

For $(\pi_a(\mu), V_a(\mu))$ occurring in the continuous spectrum of $L^2(\Gamma \backslash G)$, we have

$$W_{F_{\pi_a}(\eta, \nu), \psi_2}(1) = c_\Gamma(\pi_a(\mu); \psi_1, \psi_2) \int_{C^\infty} \nu(z) \hat{\eta}(1) \frac{1}{z} d^\infty z,$$

with

$$c_\Gamma(\pi_a(\mu); \psi_1, \psi_2) = \frac{c(\pi_a(\mu); \psi_2/\psi_1) c_{\psi_1}(\pi_a(\mu), \Gamma)}{4\pi^2}.$$
ON THE KUZNETSOV TRACE FORMULA FOR \( PGL_2(\mathbb{C}) \)

**Proof.** First, in view of (4.6), we have

\[
W_{F_2(P_f), \phi_2}(1) = c(\pi, \psi_2/\psi_1) W_{F_2(P_f), \phi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right).
\]

So it is enough to compute \( W_{F_2(P_f), \phi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right) \). Let us now drop \( \psi_1 \) from notations. By (3.3), (4.1) and (4.4), we see that \( W_{F_2(P_f), \phi_1} \left( \begin{array}{c} \kappa \\ 1 \end{array} \right) \) is characterized by

\[
\int_{\mathbb{C}^\times} W_{F_2(P_f)} \left( \begin{array}{c} a \\ 1 \end{array} \right) \overline{W_\phi} \left( \begin{array}{c} a \\ 1 \end{array} \right) d^\times a = c(\pi, \Gamma) \int_{N_0^1 G} f(g) \overline{W_\phi(g)} dg \quad \text{all} \ \varphi \in V_\pi^\times.
\]

Now substitute the definition of \( f(g) = f_{\eta, \nu}(g) \) on the right and integrate over the open Bruhat cell. We get

\[
\int_{N_0^1 G} f(g) \overline{W_\phi(g)} dg = \frac{1}{4\pi^2} \int_{\mathbb{C}^\times} \int_{\mathbb{C}^\times} \nu(z) \eta(u) \overline{W_\phi} \left( w \left( \begin{array}{c} 1 \\ u \\ 1 \end{array} \right) \left( \begin{array}{c} z \\ 1 \end{array} \right) \right) d^\times z du.
\]

We now use the identity (4.3) for the Bessel function.

\[
W_\phi \left( w \left( \begin{array}{c} 1 \\ u \\ 1 \end{array} \right) \left( \begin{array}{c} z \\ 1 \end{array} \right) \right) = W_{\pi(\xi, \eta)}(w) = \int_{\mathbb{C}^\times} \beta_\pi(a) W_{\pi(\xi, \eta)} \left( \begin{array}{c} a \\ 1 \end{array} \right) d^\times a
\]

\[
= \int_{\mathbb{C}^\times} \phi_1(a \nu) \beta_\pi(a) W_\phi \left( a \zeta \right) d^\times a
\]

\[
= \int_{\mathbb{C}^\times} \phi_2 \left( \frac{1}{\kappa \zeta} \right) \beta_\pi \left( \frac{a}{\zeta} \right) W_\phi \left( a \zeta \right) d^\times a.
\]

Therefore,

\[
\int_{N_0^1 G} f(g) \overline{W_\phi(g)} dg = \frac{1}{4\pi^2} \int_{\mathbb{C}^\times} \int_{\mathbb{C}^\times} \nu(z) \eta(u) \psi_2^{-1} \left( \frac{au}{\kappa \zeta} \right) \beta_\pi \left( \frac{a}{\zeta} \right) \overline{W_\phi} \left( a \zeta \right) d^\times a d^\times z du.
\]

Interchanging the order of integrations, this becomes

\[
\int_{\mathbb{C}^\times} \left\{ \frac{1}{4\pi^2} \int_{\mathbb{C}^\times} \nu(z) \hat{\eta} \left( \frac{a}{\kappa \zeta} \right) \beta_\pi \left( \frac{a}{\zeta} \right) d^\times z \right\} \overline{W_\phi} \left( a \zeta \right) d^\times a.
\]

Note that \( \hat{\eta} \) is the Fourier transform of \( \eta \) with respect to \( \psi_2 \). This yields the identity

\[
W_{F_2(P_f)} \left( \begin{array}{c} a \\ 1 \end{array} \right) = c(\pi, \Gamma) \frac{1}{4\pi^2} \int_{\mathbb{C}^\times} \nu(z) \hat{\eta} \left( \frac{a}{\kappa \zeta} \right) \beta_\pi \left( \frac{a}{\zeta} \right) d^\times z,
\]

and then follows the formula (4.8).

Proceeding exactly as above, we may derive (4.9). Q.E.D.
4.6. The Kloosterman-Spectral Formula. In conclusion, as a consequence of Theorem 3.2 applied to the Poincaré series $P_f$ with $f = f_{q,r}$ as in §4.3 we have the following Kloosterman-spectral formula.

**Theorem 4.4.** Let $\eta \in \mathcal{S}(\mathbb{C})$ and $\nu \in C_c^\infty(\mathbb{C}^\times)$ be such that $\psi_2(z) = \psi_1(\kappa z)$. Let $\kappa \in \mathbb{C}^\times$ be such that $\psi_2(z) = \psi_1(\kappa z)$. Then

$$\sum_{c \in \Omega(\Gamma)} \frac{Kn(c; \psi_1, \psi_2)}{|c|^2} A_{q,r} \left( \frac{1}{c} \right)$$

$$= \sum_{\pi \in \Pi(\Gamma)} c_\pi(\eta; \psi_1, \psi_2) \int_{\mathbb{C}^\times} A_{q,r}(\pi; z) \Omega_{\pi,\psi_1}(\kappa z) d^\times z$$

$$+ \sum_{\varrho} \frac{1}{4\pi i m} \frac{1}{\Lambda_{\varrho}} \int_{\mathbb{R} \mu = 0} c_\pi(\varrho; \mu; \psi_1, \psi_2) \left\{ \int_{\mathbb{C}^\times} A_{q,r}(\pi; \varrho; \mu; z) \Omega_{\pi,\psi_1}(\kappa z) d^\times z \right\} d\mu,$$

with $\mu \in \mathcal{X}_m(\mathbb{C}^\times)$.

5. An Explicit Kloosterman-Spectral Formula - the Kuznetsov Trace Formula

In this section, we shall express the Bessel functions in terms of the classical Bessel functions and relate the constants $c_F(\pi; \psi_1, \psi_2)$ to classical quantities.

5.1. By [QI Proposition 18.5], the Bessel functions $\Omega_{\pi,\psi}$ may be expressed explicitly in terms of the classical Bessel functions. For $\pi \cong \pi_d(s)$, with either $s = t$ purely imaginary or $s = t$ on the segment $(0, \frac{1}{2})$, and $\psi = \psi_d$, with $a \in \mathbb{C}^\times$, we have

$$\Omega_{\pi,\psi}(z) = \frac{2\pi^2}{\sin(2\pi s)} |a|^2 \left( J_{s,2d}(4\pi a \sqrt{z}) - J_{-s,2d}(4\pi a \sqrt{z}) \right),$$

where $J_{s,2d}(\tau) = J_{-2s-d}(-\tau) J_{-2+s+d}(-\tau)$. Here $\sqrt{z}$ is the principal branch of the square root of $z$ and the expression on the right is independent on the argument of $z$ modulo $2\pi$.

5.2. We first consider the discrete component $(\pi, V_\pi) \subset L^2(\Gamma \backslash \mathbb{G})$ with $\pi \cong \pi_d(s)$, $s = it$ or $s = t$. The restriction of $V_\pi$ on $K$ decomposes into the direct sum of the $(2l + 1)$-dimensional representations $\sigma_l$ of $K$ with $l \geq |d|$. Furthermore, each representation $\sigma_l$ decomposes into the direct sum of the one-dimensional weight spaces of weight $q$ with $|q| \leq l$. Accordingly, we say that a vector of $V_\pi$ is of $K$-type $(l, q)$ if it lies in the $q$-weight space of $\sigma_l$. Therefore, $V_\pi$ contains a unique vector $\phi$ of type $(|d|, d)$ normalized such that $\phi$ has Petersson square norm 1. It is known that $\phi(g)$ will have a Fourier expansion at infinity in terms of Jacquet’s Whittaker function $W_{\omega, d}$ (see, for example, [JL]),

$$\phi \left( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix} \right) = \sum_{\omega \in \mathcal{N}_{\omega}} a_\omega(\phi) \psi(\omega) W_{\omega, d}^d(r, k),$$

VII. By convention, the weights $q$ are the eigenvalues for $H = -\frac{1}{2} i \otimes \begin{pmatrix} i \\ -i \end{pmatrix}$ as an element in the complexified Lie algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}(2)$. Note that the definition of $K$-type coincides with that in [BM] if $q$ is changed into $-q$. 
with \( z \in \mathbb{C}, r \in \mathbb{R}_+, k \in K \). Now, to define \( W^{r,d}_\omega \), we first introduce the function

\[
\phi_{r,d}(\begin{pmatrix} 1 & z \\ r & 1 \end{pmatrix} k_{v,w}) = \begin{cases} r^{2s+1} e^{2i \phi}, & \text{if } d \geq 0, \\
 r^{2s+1} e^{-2\phi}, & \text{if } d < 0. \end{cases}
\]

It is readily verified that \( \phi_{r,d} \) is left \( \mu_s e^z \)-variant, that is, \( \phi_{r,d}(uag) = |a|^{2s+1} |a|^d \phi_{r,d}(g) \), with \( u \in N \) and \( a \in A \). For \( \Re s > 0 \), \( W^{r,d}_\omega(r,k) \) is given by

\[
W^{r,d}_\omega(r,k) = \int_N \psi^{-1}(n) \phi_{r,d}(umrk) dn,
\]

in which the integral converges absolutely, and \( W^{r,d}_\omega(r,k) \) is defined for all \( s \) via meromorphic continuation, except for the poles that occur in the case \( \omega = 0 \).

For computing the constant \( c_1(r, \psi_1, \psi_2) \), we shall be concerned only with those nonzero \( \omega \) and the values of \( W^{r,d}_\omega(r,k_0) \), with \( \nu = e^{\frac{i}{2} \phi} \). With these in mind, we have the following lemma.

**Lemma 5.1.** Let \( \phi \in \mathbb{R}/2\pi \mathbb{Z} \) and \( r \in \mathbb{R}_+ \). Set \( \nu = e^{\frac{i}{2} \phi} \) and \( a = re^{i \phi} \). Then, for \( \omega \in \mathcal{N}_\infty \setminus \{0\} \), we have

\[
W^{r,d}_\omega(r,k_0) = \frac{2(-1)^d (2\pi)^{2s+|d|+1}}{\Gamma(2s+|d|+1)} |a|^{2s+|d|} |a|^d K_{|d|-2s}(4\pi |\omega a|).
\]

**Proof.** On expressing \( ukr_{k_0} \) in the Iwasawa coordinates, changing the variables from \( z \) to \( rz \) and using the polar coordinates for \( z \), we arrive at

\[
W^{r,d}_\omega(r,k_0) = 2r^{1-2s} e^{id\phi} \int_0^{\infty} \frac{x^{2|d|+1} I^d_\omega(rx)}{(1+x^2)^{2s+|d|+1}} dx,
\]

with

\[
I^d_\omega(x) = \int_0^{2\pi} e^{2\pi r(x e^{i \omega})^{-2i d \phi}} d\theta.
\]

We first recall the integral representation of Bessel,

\[
2\pi r^d J_m(x) = \int_0^{2\pi} e^{ix \cos \theta + im \phi} d\theta, \quad m \in \mathbb{Z},
\]

then we find that

\[
I^d_\omega(x) = 2\pi (-1)^d |\omega|^{2d} J_{2|d|}(4\pi |\omega| x).
\]

Consequently,

\[
W^{r,d}_\omega(r,k_0) = 4\pi (-1)^d |\omega|^{2d} e^{1-2s} e^{i d \phi} \int_0^{\infty} \frac{x^{2|d|+1} J_{2|d|}(4\pi |\omega| x)}{(1+x^2)^{2s+|d|+1}} dx.
\]

Using the formula [GR 6.565 4]

\[
\int_0^{\infty} \frac{x^{\mu+1} J_\nu(ax)}{(1+x^2)^{\mu+1}} dx = \frac{\alpha^\mu}{2\pi \Gamma(\mu+1)} K_{\nu-\mu}(a), \quad 2\Re \nu + \frac{3}{2} > 2\Re \nu > -1, \quad a > 0,
\]

we obtain

\[
W^{r,d}_\omega(r,k_0) = \frac{4\pi (-1)^d (2\pi |\omega|)^{2s+|d|} |\omega|^{2d|d|+1} e^{i d \phi}}{\Gamma(2s+|d|+1)} K_{|d|-2s}(4\pi |\omega| r).
\]
Therefore (5.1) is proven for $\Re s > 0$ and remains valid by analytic continuation. Q.E.D.

It will be convenient to work with renormalized Fourier coefficients

$$A_\omega(\varphi_2) = \frac{(2\pi)^2|\omega|^2|\varphi_2|^d a_\omega(\varphi_2)}{\Gamma(2s + |d| + 1)}.$$  

Then if $\psi_i(z) = \psi_{\omega_i}(z)$, $i = 1, 2$, we see that

$$W_{\psi_1, \psi_2} \left( \begin{array}{c} a \\ 1 \end{array} \right) = 2\left| \frac{\Lambda_{\omega_2}}{\Lambda_{\omega_1}} \right| \left( \frac{2\pi}{\Gamma(2s + |d| + 1)} \right) A_\omega(\varphi_2) |a||\omega_1||\omega_2|^d K_{|d| - 2\omega_1}^2(4\pi|\omega_1|).$$

Let us first compute $c(\pi; \psi_2/\psi_1)$, which is defined by

$$W_{\psi_1, \psi_2} \left( \begin{array}{c} \omega_1 \\ 1 \end{array} \right) = c(\pi; \psi_2/\psi_1) W_{\psi_1, \psi_1} \left( \begin{array}{c} \omega_2 \\ 1 \end{array} \right).$$

It follows that

$$c(\pi; \psi_2/\psi_1) = \frac{A_{\omega_2}(\varphi_1) |\omega_1|}{A_{\omega_1}(\varphi_1) |\omega_2|}.$$

On the other hand, as $\varphi_1$ has Petersson square norm 1, $c_{\psi_1}(\pi, \Gamma)$ is defined by

$$c_{\psi_1}(\pi, \Gamma) = \int_{C^*} W_{\psi_1, \psi_1} \left( \begin{array}{c} a \\ 1 \end{array} \right) \overline{W_{\psi_1, \psi_1}} \left( \begin{array}{c} a \\ 1 \end{array} \right) d^x a.$$

We first let $s = it$. Integrating in the polar coordinates, we obtain

$$c_{\psi_1}(\pi, \Gamma) = \frac{8|\Lambda_{\omega_2}|^2}{|\Gamma(2s + |d| + 1)|^2} \left| A_{\omega_1}(\varphi_1) \right|^2 |\omega_1|^{2|d|} \int_{0}^{\infty} \int_{0}^{\infty} r^{2|d| + 1} K_{|d| - 2\omega_1}^2(4\pi|\omega_1| r) K_{|d| + 2\omega_2}^2(4\pi|\omega_1| r) dr.$$ 

As a special case of the Weber-Schafheitlin formula for $K$-Bessel functions (see [GR 6.576 4]), we have

$$\int_{0}^{\infty} x^{\rho - 1} K_\nu(ax) K_\mu(ax) dx$$

$$= 2\pi^{\rho - 1} \left( \frac{1}{2} (\rho + \nu + \mu) + (\frac{1}{2} (\rho + \nu - \mu) \right) \Gamma \left( \frac{1}{2} (\rho - \nu + \mu) \right) \Gamma \left( \frac{1}{2} (\rho - \nu - \mu) \right),$$

$$\Re \rho > |\Re \nu| + |\Re \mu|, a > 0.$$ 

This yields

$$c_{\psi_1}(\pi, \Gamma) = \frac{2\pi|\Lambda_{\omega_2}|^2 |A_{\omega_1}(\varphi_1)|^2}{(2|d| + 1)|\omega_1|^2}.$$ 

Therefore, we get

$$c_{\Gamma}(\pi; \psi_1, \psi_2) = \frac{|\Lambda_{\omega_2}|^2 A_{\omega_1}(\varphi_2) A_{\omega_1}(\varphi_2)}{2\pi(2|d| + 1)|\omega_1 \omega_2|}.$$ 

Similarly, when $s = t$ and $d = 0$, with $t \in (0, \frac{1}{2})$, we have

$$c_{\psi_1}(\pi, \Gamma) = \frac{2\pi|\Lambda_{\omega_2}|^2 |A_{\omega_1}(\varphi_1)|^2}{G(t)|\omega_1|^2}.$$
with
\[ G_{t,0} = \frac{\Gamma(1 + 2t)}{\Gamma(1 - 2t)}. \]

Hence
\[ c_T(\pi; \psi_1, \psi_2) = \frac{A_{\omega_1}(\phi_{\tau})A_{\omega_2}(\phi_{\tau})}{2\pi G_{t,0}|\omega_1\omega_2|}. \]

Combining these, if we simply put \( G_{a,d} = 1 \), then
\[ c_T(\pi; \psi_1, \psi_2) = \frac{|\Lambda_x|^2 A_{\omega_1}(\phi_{\tau})A_{\omega_2}(\phi_{\tau})}{2\pi(2|d| + 1)G_{a,d}|\omega_1\omega_2|}. \]

5.3. For the continuous spectrum, we fix a cusp \( a \) and let \( \mu(z) = \mu_{s,d}(z) = [z]^d|z|^{2s} \), with \( s = it \) and \( ma|d \). Choose \( f \in V_{s}(\mu) \) to be
\[ f(g) = \phi_{s,d}(g a g). \]

We have \( \langle f, f \rangle_a = 1/(2|d| + 1) \). With \( f \) is associated the Eisenstein series \( E_a(g; f, \mu) \). To describe the Fourier coefficients of this Eisenstein series we need some notations.

For each cusp \( a \) let \( \Omega_a(\Gamma) = \{ c \in \mathbb{C}^\times : g, \Gamma \cap Na \neq \emptyset, c \in A \} \), and for each \( c \in \Omega_a(\Gamma) \) define \( \Gamma_c = g, \Gamma \cap Na \). Then \( \Gamma_c \) is left invariant under \( g, \Gamma \cap Na^{-1} \) and right invariant under \( \Gamma_\infty \). For each \( \gamma \in \Gamma_c \) we write
\[ \gamma = n_1(\gamma)w_n\gamma_2(\gamma), \]
with \( n_1(\gamma), n_2(\gamma) \in N \).

For \( c \in \Omega_a(\Gamma) \) we define the Ramanujan sum \( K_{a}^c(\gamma; 1, \psi) \) by
\[ K_{a}^c(\gamma; 1, \psi) = \sum_{\gamma \in \gamma, \Gamma \cap \Gamma_c} \psi(n_2(\gamma)), \]
and for \( \omega \in \Lambda_\gamma \) we define the Kloosterman-Selberg zeta series
\[ Z_{a}^\omega(\mu; 0, \omega) = \sum_{c \in \Omega_a(\Gamma)} K_{a}^c(\gamma; 1, \psi_\omega) \frac{\mu(c)\delta_{\sigma}(c)}{2}, \]
if \( \Re \mu \) is sufficiently large. Recall that \( \delta_{\sigma}(c) = |c|^2 \) is the modulus character. These series \( Z_{a}^\omega(\mu; 0, \omega) \) have meromorphic continuation to the whole complex plane and have no poles on the line \( \Re \mu = 0 \).

We may now describe the Fourier expansion of the Eisenstein series \( E_a(g; f, \mu) \). For
\[ g = \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix} k, \]
we have
\[ E_a(g; f, \mu) = \delta_{\sigma}(c) m_x \phi_{s,d}(g) + \frac{1}{|\Lambda_\infty|} Z_{a}^\omega(\mu; 0, 0) W_{a}^{r,d}(r, k) \]
\[ + \frac{1}{|\Lambda_\infty|} \sum_{\omega \in \Lambda_\gamma \setminus \{0\}} \psi_\omega(z) Z_{a}^\omega(\mu; 0, \omega) W_{a}^{r,d}(r, k), \]
where the first term exists only if \( a = \infty \).
Let \( \psi_1 = \psi_{\omega_1} \) and \( \psi_2 = \psi_{\omega_2} \), with \( \omega_1, \omega_2 \in \mathcal{N}_{\infty} \setminus \{0\} \). From Lemma 5.1 for \( i = 1, 2 \), we have

\[
W_{\mu_0, \nu_0, \psi}(a) = \frac{2(-1)^d (2\pi)^{2s+|d|+1}}{\Gamma(2s+|d|+1)} \left[ \frac{\omega_2}{\omega_1} \right]^{2s-1} \frac{\omega_2^d}{\omega_1^d} K_{|d|-2s}(4\pi|\omega|, a).
\]

Then we get

\[
c(\pi_{\sigma}(\mu); \psi_1, \psi_2) = \frac{Z^p_1(\mu; 0, \omega_1, \omega_2)}{Z^p_1(\mu; 0, \omega_1)} \left[ \frac{\omega_2}{\omega_1} \right]^{2s-1} \frac{\omega_2^d}{\omega_1^d}
\]

and

\[
c_{\psi_i}(\pi_{\sigma}(\mu), \Gamma) = \frac{2\pi |Z^p_1(\mu; 0, \omega_1)|^2}{|\omega_1|^2}.
\]

Therefore,

\[
c^T(\pi_{\sigma}(\mu); \psi_1, \psi_2) = \frac{Z^p_1(\mu; 0, \omega_2) Z^p_1(\mu; 0, \omega_1)}{2\pi |\omega_1\omega_2|} \left( \frac{\omega_2}{\omega_1} \right).
\]

5.4. In conclusion, with the computations above, the Kuznetsov trace formula in Theorem 2.1 then follows from Theorem 4.4, with the choice of weight function \( F(z) = A_{\eta, \nu} (z/\omega_1 \omega_2) \left| z/\omega_1 \omega_2 \right| \). It should be remarked that \( F(z) \) of this form actually exhaust all functions in \( C^\infty_{c}(C^\infty) \) if one let \( \eta \) and \( \nu \) vary.

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