Bose-representation for a strongly coupled nonequilibrium fermionic superfluid in a time-dependent trap

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Using the functional integral formulation of a nonequilibrium quantum many-body theory we develop a regular description of a Fermi system with a strong attractive interaction in the presence of an external time-dependent potential. In the strong coupling limit this fermionic system is equivalent to a nonequilibrium dilute Bose gas of diatomic molecules. We also consider a nonequilibrium strongly coupled Bardeen-Cooper-Schrieffer (BCS) theory and show that it reduces to the full nonlinear time-dependent Gross-Pitaevski (GP) equation, which determines an evolution of the condensate wave function.

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A remarkable progress in creation of long-lived cold diatomic molecules in trapped Fermi gases [1, 2, 3, 4, 5] has been recently crowned with experimental demonstration of the molecular Bose-Einstein condensation (BEC) [6, 7, 8]. One of the main fundamental goals of these studies is to investigate experimentally the problem of a crossover from weakly coupled BCS superfluidity to the molecular BEC. Possibly the first experimental realization of the crossover regime was reported recently in Ref. 9. In the last decade a theoretical description of the crossover problem also attracted a considerable interest mainly in the context of high temperature superconductivity [10, 11, 12, 13, 14, 15]. The results of these works show that if the interaction supports two-particle bound states and the density is low enough, the system behaves as a dilute Bose gas of diatomic molecules. This regime corresponds to a strongly coupled fermionic superfluid and a molecular side of the above crossover. The description of a spatially inhomogeneous condensed Bose gas is commonly based on GP equation for the condensate wave function (see Refs. 16, 17 and references therein). Generalization of the previous theoretical works to spatially inhomogeneous Fermi systems demonstrates that in the strong coupling limit the equilibrium BCS theory reduces the common stationary GP equation [18].

One of important features of the experiments with trapped atomic systems is that measurements are frequently performed under nonequilibrium conditions. The popular time-of-flight technique represents an example of experimental methods of this type. Therefore it is desirable to have a consistent kinetic theory of a spatially inhomogeneous Fermi system with strong pair correlations. Recently it has been shown that linear response dynamics of strongly coupled BCS and bosonic GP systems are also equivalent [19]. However, physically it seems to be quite likely that the bosonic description of a strongly coupled nonequilibrium Fermi system should be valid to any order of nonlinearity provided that external fields are slow functions of time on the scale of the inverse molecular binding energy. Despite physical simplicity of this argument a formal nonequilibrium theory of a superfluid Fermi system in the strong coupling limit is still lacking.

The present paper is aimed to fill this gap. We formulate a regular quantum kinetic description of a fermionic system with a strong attractive interaction in the presence of time-dependent trapping potential \( U(x, t) \). Specifically we consider a two-component system that is defined by the following Hamiltonian:

\[
H = \int d\mathbf{x}_1 \left\{ \sum_{j=1}^{2} \Psi_j^\dagger(x_1) \left[ -\frac{\nabla^2}{2m} - \mu + U(x, t) \right] \Psi_j(x_1) - \int d\mathbf{x}_2 \Psi_1^\dagger(x_1) \Psi_2^\dagger(x_2) V(x_1 - x_2) \Psi_2(x_2) \Psi_1(x_1) \right\}.
\]

where \( \Psi_j^\dagger(\Psi_j) \) is the creation (annihilation) operator of a Fermi particle of type \( j \), \( V(x) \) is the interparticle interaction potential and \( \mu \) is the chemical potential (the Lagrange multiplier, which is defined by the condition for conservation of the average number of particles \( N \)). For simplicity the mass of particles \( m \) is assumed to be independent of \( j \).

First we briefly summarize some formalities of a nonequilibrium many-body theory. The complete description of a system with a time-dependent Hamiltonian \( H(t) \) is given by the many-particle density matrix \( \rho(t) \), which satisfies the common equation of motion \( i\hbar \partial_t \rho(t) = \{\rho(t), H(t)\} \) with initial condition \( \rho(t_0) = \rho_0 \).

The average value of any operator \( \hat{O} \) can be calculated as follows:

\[
\langle \hat{O}(t) \rangle = \frac{\text{Tr} \rho(t) \hat{O}}{\text{Tr} \rho(t)} = \frac{\text{Tr} \rho_0 U^\dagger(t_0, t_0) \hat{O} U(t_0, t)}{\text{Tr} \rho_0},
\]

where \( U(t_2, t_1) = T \exp \left\{-i \int_{t_1}^{t_2} H(t)dt\right\} \) is the evolution operator and \( T \) means the usual chronological ordering. Calculation of the average can be reformulated in a more
compact way if we introduce the following generating functional

$$Z_J = \text{Tr} \rho_0 T_C \exp \left\{-i \int_{C'} H_J(t) dt \right\}, \quad (3)$$

where $H_J(t) = H(t) + J(t) \hat{O}$ is the Hamiltonian in the presence of a source field $J(t)$. The time integration in Eq. (3) runs over contour $C'$, which is shown in Fig. 1a. Contour $C'$ consists of two branches $C_-$ and $C_+$ and goes from initial time $t_0$ to infinity ($C_-$ branch) and backwards ($C_+$ branch). Operator $T_{C'}$ in Eq. (3) orders all times along the integration contour. It is worth to mention that generating functional $Z_J$, Eq. (3), is nontrivial (differs from $\text{Tr} \rho_0$) only if the source $J(t^-)$ at the upper branch differs from the source $J(t^+)$ at the lower branch.

Using the generating functional of Eq. (3) one can represent the average of Eq. (2) in terms of the following functional derivative

$$\langle \hat{O}(t) \rangle = i \left[ \frac{\delta \ln Z}{\delta J(t)} \right]_{J=0}.$$  

Introducing sources for any other operator we can calculate the corresponding observables as well as any higher order correlation function. The formalism is simplified if the initial condition corresponds to the thermal equilibrium: $\rho_0 = \exp[-\beta H(t_0)]$, where $\beta = 1/T$. In this case the generating functional can be written as a trace of the only chronological exponent

$$Z_J = \text{Tr} T_C \exp \left\{-i \int_C H_J(t) dt \right\}, \quad (4)$$

where all times are ordered along new three-branch contour $C$, which is shown in Fig. 1b (for discussion of the three-branch contour in the context of quantum kinetics see, for example, Refs. 20, 21 and references therein). The main advantage of Eq. (4) is the possibility to represent it as a coherent state functional integral. The derivation of this representation is, in fact, independent of particular time contour. Therefore, it simply reproduces the corresponding derivation for the partition function in the equilibrium statistical mechanics [22]. For our nonequilibrium system we get the result

$$Z_J = \int_{\psi(t_0^- - i \beta) = -\psi(t_0^+)} \prod_{j=1}^2 D \psi_j^* D \psi_j e^{i S[\psi_j^*, \psi_j]}, \quad (5)$$

$$S = \int_C dt \int dx \sum_{j=1}^2 \psi_j^*(x, t) \tilde{\mathcal{L}}(x, t) \psi_j(x, t) + S_{\text{int}}. \quad (6)$$

Here $\psi(x, t)$ is a Grassmann field, $t_0^-$ and $t_0^+ - i \beta$ are the initial and the final points of integration contour $C$ respectively (see Fig. 1b). One particle operator $\tilde{\mathcal{L}}(x, t)$ in Eq. (6) is defined as follows

$$\tilde{\mathcal{L}}(x, t) = i \partial_t - \hat{H}(x, t) = i \partial_t + \frac{\nabla^2}{2m} - U_J(x, t) + \mu. \quad (7)$$

For simplicity we introduced only one source $J(x, t)$ for the density operator and redefined the external potential $U_J = U + J$. The second term, $S_{\text{int}}$, in the action of Eq. (6) is given by the second term in Eq. (1) with an additional contour integration and with all $\Psi$-operators being replaced by the corresponding Grassmann fields $\psi$.

Following the standard route we introduce in Eq. (5) Gaussian integral over nonlocal Bose field $\eta(x_1, x_2, t)$ and decouple the four-fermion term $S_{\text{int}}$ in the action. It is important that field $\eta(t)$ should satisfy boundary condition $\eta(t_0^+ - i \beta) = \eta(t_0^-)$, which follows from the boundary condition imposed on Grassmann variables in Eq. (5). After this Hubbard-Stratonovich transformation $S_{\text{int}}$ takes the form

$$S_{\text{int}} = - \int \left\{ V(x_1 - x_2) \eta^*(x_1, x_2, t) \eta(x_2, t) \eta^*(x_1, t) + c.c. + \eta^*(x_1, x_2, t) V(x_1 - x_2) \eta(x_1, x_2, t) \right\} \quad (8)$$

with the integration over all internal variables.

Obviously, if we now take the integral over field $\eta$ in the stationary phase approximation, we end up with a nonequilibrium (and nonlocal due to general form of the interaction potential) version of BCS theory. We shall, however, postpone this until the very end and proceed further at the formally exact level.

Calculating Gaussian integral over Grassmann variables we obtain the following representation of generating functional $Z_J$

$$Z_J = Z_J^{(0)} \int_{\eta(t_0^+ - i \beta) = \eta(t_0^-)} D \eta^* D \eta e^{i S_{\text{eff}}[\eta^*, \eta]} \quad (9)$$

Factor $Z_J^{(0)}$ in Eq. (9) corresponds to the noninteracting Fermi gas, while the effective action $S_{\text{eff}}$ is given by the expression, which we present in the obvious structural form

$$S_{\text{eff}}[\eta^*, \eta] = -i \text{Tr} \ln \left(1 - \hat{G} V \hat{\eta} \right) - \eta^* V \eta$$

$$= i \sum_{l=1}^\infty \frac{1}{2l} \text{Tr} \left( \hat{G} V \hat{\eta} \right)^{2l} - \eta^* V \eta, \quad (10)$$

where we introduced the notations

$$\hat{G}(t, t') = \begin{bmatrix} G(t, t') & 0 \\ 0 & -G(t, t') \end{bmatrix}, \quad \hat{\eta} = \begin{bmatrix} 0 \ & \eta(t) \end{bmatrix}. \quad (11)$$
One particle propagator \( G(t, t') \) is defined as the inverse of one particle operator \( \hat{\mathcal{L}} \), Eq. (7):

\[
[i\partial_t - \hat{\mathcal{H}}(x, t)]G(t, t') = \delta_C(t - t'),
\]

where \( \delta_C(t - t') \) is the delta-function on contour \( C \). The uniqueness of a solution to Eq. (12) is guaranteed by the boundary condition \( G(t_0^-, t') = -G(t_0^-, i\beta, t') \). Introducing one particle contour evolution operator \( U_C(t, t') = T_C \exp \left\{ -i \int_0^t \hat{\mathcal{H}}(\tau) d\tau \right\} \) we represent the solution to Eq. (12) in the following form

\[
G(t, t') = -iU_C(t, t_0)[(1 - \hat{\eta}(t_0))\theta_C(t - t') - \hat{\eta}(t_0)\theta_C(t' - t)]U_C(t_0, t),
\]

where \( \theta_C(t) \) is the contour step-function and operator \( \hat{\eta}(t_0) = [e^{\beta\hat{\mathcal{H}}(t_0)} + 1]^{-1} \) corresponds to the Fermi-function at initial time \( t_0 \).

Let us consider quadratic (Gaussian) part, \( S^{(2)}_{\text{eff}} \), of the effective action, Eq. (10),

\[
S^{(2)}_{\text{eff}} = -\int_C dt dt' \eta^\ast(t)\{V\delta_C(t - t') + VK_0(t, t')V\}\eta(t'),
\]

where

\[
K_0(x_1, x_2; t; x_1', x_2', t') = iG(x_1, t; x_1', t')G(x_2, t; x_2', t')
\]

is the bare two-particle propagator. Diagrammatic representation of the second term in Eq. 14 is shown in Fig. 2a. Full two-particle propagator \( K(t, t') \) in the ladder approximation satisfies Bethe-Salpeter equation

\[
K = K_0 - K_0VK_0.
\]

Using the definition of Eq. (15) we transform Eq. (14) as follows

\[
S^{(2)}_{\text{eff}} = \int_C dt dt' \eta^\ast(t)\{K^{-1} - K^{-1}K_0K^{-1}\}\eta(t').
\]

From this point we concentrate on a strongly coupled superfluid. We assume that the lowest eigenvalue \(-\varepsilon_0\) of the two-particle problem

\[
\left[ -\nabla^2 - V(r) \right] \chi_n(r) = -\varepsilon_n\chi_n(r)
\]

corresponds to a bound state with radius \( a_0 (\varepsilon_0 = 1/m\alpha_0^2) \). In the strong coupling limit \( a_0 \) defines the smallest spatial scale of the problem, which means that \( a_0n^{1/3} \ll 1 \) and in addition \( a_0/L \ll 1 \), where \( L \) is the characteristic length-scale of external potential \( U(x, t) \). Similarly, the binding energy \( \varepsilon_0 \) is the largest energy scale, which should be large than both temperature \( T \) and the inverse characteristic time related to variations of \( U(x, t) \). The above assumptions allow for significant simplification of the effective action, Eq. (10).

In the initial state at \( t = t_0 \) the system is in equilibrium and the chemical potential is negative and close to half of the binding energy \( \mu \sim -\varepsilon_0/2 \). In fact, the strong coupling limit can be formally viewed as a limit \( \mu \to -\infty \), while the quantity \( 2\mu + \varepsilon_0 \) remains finite. In this limit \( n(t_0) \to 0 \), and the one particle propagator of Eq. (13) reduces to the operator of retarded evolution along contour \( C: G(t, t') = -iU_C(t, t')\theta_C(t - t') \). Therefore the bare two particle Green’s function \( K_0(t, t') \) takes a form of the two particle retarded propagator

\[
K_0^{1,2}(t, t') = -iT_Ce^{-i\int_0^t [\hat{\mathcal{H}}(\tau) + \hat{\mathcal{H}}(\tau)]d\tau}\theta_C(t - t'),
\]

where indices 1 and 2 label the propagating particles. Finally, the full two particle contour propagator \( K(t, t') \) of Eq. (15) can be found as a solution to the following equation

\[
\left[ i\partial_t + \frac{\nabla^2}{4m} - U_f \left( x + \frac{r}{2}, t \right) - U_f \left( x - \frac{r}{2}, t \right) + \frac{\nabla^2}{m} + V(r) + 2\mu + i0 \right] K = \delta(x - x')\delta(r - r')\delta_C(t - t').
\]

In Eq. (19) we introduced the notations \( x = (x_1 + x_2)/2 \) and \( r = x_1 - x_2 \) for the center-of-mass and the relative coordinates respectively. The infinitesimally small term, \( i0 \), in Eq. (19) means that this equation should be solved with retarded boundary conditions. By definition, the expression in square brackets in Eq. (19) is the inverse operator \( K^{-1} \), which enters Gaussian part, Eq. (16), of the effective action.

To derive a low energy form of the Gaussian action, Eq. (16), we expand fluctuating Bose field \( \eta(x_1, x_2, t) \) in terms of eigen functions \( \chi_n \) of the two-body problem
Eq. (17)

\[ \eta(x_1, x_2, t) = \sum_n \varphi_n(x, t) \chi_n(r). \]  

(20)

Substitution of this expansion into Eq. (16) leads to the result

\[ S^{(2)}_{\text{eff}} = \int [\phi(x, t)]^* \sum_{m, n} \varphi_n^*(\chi_m | K^{-1} - K^{-1} K_0 K^{-1}| \chi_n) \varphi_n. \]

(21)

Let us calculate the low energy form of matrix elements in Eq. (21). The leading contribution to the low energy action is given by the term \( \sim \varphi_0^* \varphi_0 \). Using the explicit form of \( K^{-1} \) (operator in the brackets in Eq. (19)) and Eq. (22) we obtain the following expression for the matrix element \( \langle \chi_0 | K^{-1} | \chi_0 \rangle \)

\[ \langle \chi_0 | K^{-1} | \chi_0 \rangle \approx i \partial_t + \frac{\nabla^2}{4m} - 2U_J(x, t) + 2\mu + \varepsilon_0. \]

(22)

In the derivation of Eq. (22) we have used the fact that function \( \chi_0(r) \) is localized on the scale \( a_0 \), which by basic assumption satisfies the condition \( a_0/L \ll 1 \). To the same level of accuracy all off-diagonal matrix elements \( \langle \chi_n | K^{-1} | \chi_0 \rangle \) vanish due to orthogonality of functions \( \chi_0 \) with different quantum numbers. The contributions of the type \( \langle \chi_n | K^{-1} K_0 K^{-1} | \chi_0 \rangle \) describe corrections to Eq. (22) (for \( n = 0 \)) and the coupling of the lowest and excited two-particle states (for \( n \neq 0 \)). In the low energy corners these terms are also small since they contain a factor \( K_0 \sim 1/\mu \sim 1/\varepsilon_0 \) which is the smallest parameter of the theory. Matrix elements \( \langle \chi_n | K^{-1} - K^{-1} K_0 K^{-1} | \chi_m \rangle \) with \( n, m \neq 0 \) define contributions of excited states to the effective action. They are at least of the order of the excitation energy \( \varepsilon_0 \). Since the excited states are decoupled from the lowest bound state, they do not change low energy physics. Actually they do contribute only to renormalization of the boson-boson interaction (see below). It is worth noting that in the strong coupling limit the ideal gas contribution \( Z^{(0)}_J \) to the generating functional of Eq. (9) is also irrelevant as it corresponds to the ideal Fermi gas with a large negative chemical potential.

Therefore the low energy contribution to the Gaussian part of the effective action takes a clear physical form

\[ S^{(2)}_{\text{eff}} = \int [\phi(x, t)]^* \left\{ i \partial_t + \frac{\nabla^2}{2M} - 2U_J(x, t) + \lambda \right\} \phi, \]

(23)

which corresponds to the Keldysh action of an ideal nonequilibrium gas of Bose particles with the mass \( M = 2m \) and the chemical potential \( \lambda = 2\mu + \varepsilon_0 \) in the presence of the effective external potential \( U_J(x, t) = 2U(x, t) \). Equation (23) is, in fact, one of the main formal results of the present paper.

Terms with \( l > 1 \) in the series in Eq. (10) describe \( l \)-boson interactions. Diagrammatic representation for the two-boson term is shown in Fig. 2b. Wiggled lines in this figure correspond to fluctuating Bose fields \( \varphi(x, t) \), circles are boson-fermions “vertices” \( \Lambda_0 \), which depend only on relative coordinate \( r = x_1 - x_2 \) (see Eqs. (8) and (20))

\[ \Lambda_0(r) = V(r) \chi_0(r) \]

whereas solid lines stand for the one particle fermionic Green’s functions \( G \). Physically interaction of Fig. 2b corresponds to a scattering via virtual decay of two composite bosons with subsequent exchange by the constituent fermions. The main contribution to internal integrals in the diagram of Fig. 2b comes from a high energy region. In fact, the excitation energy \( \varepsilon_0 \) sets an effective lower cut off for these integrals. Therefore all effects of inhomogeneity and deviations from the equilibrium, which enter the diagram via the one particle Green’s functions \( G \), are irrelevant for the calculation of this term. The above arguments are also applicable to all terms with higher \( l \). To the leading order in the strong coupling limit all coefficients in terms with \( l > l \) simply coincide with those obtained in the equilibrium theory. This should be contrasted to the Gaussian part \( l = 1 \) where the high energy contribution to the diagram of Fig. 2a is canceled out by the term \( \eta^* V \eta \) in the effective action of Eq. (10).

What is left after this cancellation gives exactly the action, Eq. (23), of nonequilibrium Bose gas in the presence of the external time dependent inhomogeneity.

For colliding composite particles of low energy (smaller than \( \varepsilon_0 \)) the diagram Fig. 2b is independent of four-momenta. The corresponding constant defines the strength \( g_2^{(0)} \) of an effective short range two-body interaction. Using Green’s functions of the homogeneous system as the solid lines in Fig. 2b and performing frequency integration we arrive at the result (see, for example, similar calculations in Refs. 11, 14)

\[ g_2^{(0)} = 2 \sum_p \left( \begin{array}{c} \Lambda_0^2(p) \\ \eta^* (p) + \varepsilon_0 \end{array} \right) = 2 \sum_p \left( \begin{array}{c} \frac{p^2}{m} + \varepsilon_0 \end{array} \right) \chi_0^4(p). \]

(25)

In the second equality in Eq. (25) we make an explicit use of Eq. (24), which relates the boson-fermion vertex \( \Lambda_0 \) to the bound state wave function \( \chi_0 \). Similarly one can calculate all higher order bare coupling constants

\[ g_l^{(0)} = (-1)^l \frac{(2l - 1)!}{(l - 1)!^2} \sum_p \left( \begin{array}{c} \frac{p^2}{m} + \varepsilon_0 \end{array} \right) \chi_0^{2l}(p). \]

(26)

It is, however, well established that in the strong coupling limit the pairwise interaction always dominates for any dimension \( d \) of space [11, 15]. Consequently the unrenormalized action for low energy fluctuating Bose fields of fermionic pairs take the form

\[ S_{\text{eff}} = \int [\phi(x, t)]^* \left\{ i \partial_t + \frac{\nabla^2}{2M} - 2U_J(x, t) + \lambda \right\} \phi - \frac{1}{2} g_2^{(0)} (\varphi^\ast \varphi)^2. \]

(27)
Let us consider 3d system with a short range interaction potential \( V(r) \). If the characteristic range of interaction \( R \) is much smaller than zero energy scattering length \( a_F \), all relevant quantities can be expressed in terms of \( a_F \). The bound state energy takes a form \( \varepsilon_0 = 1/ma_F^2 \), which means that \( a_0 = a_F \). Substituting an explicit form of the normalized bound state wave function

\[
\chi_0(p) = \frac{\sqrt{8\pi}}{a_F} \frac{1}{p^2 + a_F^2}
\]

into Eq. (25) we arrive at the well known result

\[
g_2^{(0)} = \frac{4\pi a_F}{m} = \frac{4\pi a_B^{(0)}}{M},
\]

where \( a_B^{(0)} = 2a_F \) is the bare (unrenormalized) bosonic scattering length.

Nonequilibrium BCS theory corresponds to the stationary phase approximation for generating functional \( Z_J \) of Eq. (9). In the strong coupling limit it is sufficient to take the stationary point of the integral

\[
Z_J = \int D\varphi^* D\varphi e^{iS_{\text{eff}}[\varphi^*, \varphi]}
\]

with \( S_{\text{eff}}[\varphi^*, \varphi] \) defined by Eq. (27). Stationary point value \( \Phi(x,t) \) of Bose field \( \varphi(x,t) \) satisfies the time-dependent GP equation

\[
i\partial_t \Phi = -\frac{\nabla^2}{2M} + 2U_J(x,t) + \frac{4\pi a_B^{(0)}}{M} |\Phi|^2 - \lambda \Phi.
\]

with initial condition \( \Phi(x,t_0) = \Phi_0(x) \), where \( \Phi_0(x) \) is the solution to the stationary GP equation

\[
\left[ -\frac{\nabla^2}{2M} + 2U_J(x,t) + \frac{4\pi a_B^{(0)}}{M} |\Phi_0|^2 - \lambda \right] \Phi_0 = 0.
\]

Function \( \Phi(x,t) \) plays a role of the condensate wave function. The relation of \( \Phi(x,t) \) to the nonlocal BCS order parameter follow Eqs. (8) and (20)

\[
\Delta(r, x,t) = V(r) \chi_0(r) \Phi(x,t).
\]

In the case of short range interatomic interaction BCS theory becomes local with the local order parameter of the form

\[
\Delta(x,t) = \int \Delta(r,x,t) dr = \frac{1}{m} \sqrt{\frac{8\pi}{a_F}} \Phi(x,t).
\]

Equations (30), (31) prove the equivalence of strongly coupled nonequilibrium BCS theory and time-dependent GP theory. As one could expect on the physical grounds, the equivalence goes far beyond the linear response regime.

Stationary phase approximation on the level of generating functional Eq. (9) (BCS theory) neglects high energy (\( \sim \varepsilon_0 \)) fluctuations of Bose fields \( \varphi(x,t) \). To construct a regular description of low energy physics one has first integrate the high energy degrees of freedom out. After the integration we obtain the final effective kinetic theory. This theory is formally defined by the generating functional of Eq. (29) and the effective action of Eq. (27) with \( g_2^{(0)} \) being replaced by the physical interaction constant \( g_2 \). As usual, the calculation of observables with such an effective field theory requires a proper regularization procedure (see, for example, [23]). The regularization should take into account the fact that coupling constant \( g_2 \) in the action is already equal to its physical zero energy value. Numerical value of the physical (renormalized) interaction \( g_2 \) is obtained as a zero energy limit of the scattering matrix \( T \), which satisfies the following equation

\[
T = T^{(0)} + T^{(0)} \mathcal{G} \mathcal{T},
\]

where \( \mathcal{G} = (K^{-1} - K^{-1}K_0K^{-1})^{-1} \) is the full bosonic propagator (see Eq. (16)), and \( T^{(0)} \) is the full momentum dependent bare scattering matrix of Fig. 2b. Since the relevant energy scale in Eq. (32) is defined by the binding energy \( \varepsilon_0 \), slow external potential \( U_J(x,t) \), which enters Eq. (32) via \( T^{(0)} \) and \( \mathcal{G} \), is again irrelevant. In the homogeneous 3d Fermi system with a short range attractive a numerical solution to Eq. (32) has been found by Pieri and Strinati [14]. The result reported in Ref. 14 (\( g_2 = 4\pi a_B/M \) with \( a_B \approx 0.75a_F \)) is somewhat different from the result of an explicit treatment of the four particle problem \((a_B \approx 0.6a_F) \) [24]. This discrepancy, though not strong, still requires a clarification, since Eq. (32) should be formally equivalent to the four particle scattering problem of Ref. 24.

In contrast to the 3d case where \( a_B \sim a_F \) and \( g_2 \sim g_2^{(0)} \), in strongly anisotropic traps the renormalization effects should be more pronounced. For example, in a quasi-2d trap the physical coupling constant, which enters an effective 2d GP equation, should take the form [25, 26]

\[
g_2 = \frac{4\pi}{M|\ln |a_B|}.
\]

In this case \( a_B \) can be quite different from \( a_F \) since the bosonic scattering length \( a_B \) is proportional to the bound state radius \( a_0 \), which strongly depends on the transverse size of a quasi-2d trap.

In conclusion we developed a regular quantum kinetic theory of a dilute Fermi gas with strong pair correlations, which is trapped by a time-dependent external potential. The description of this strongly coupled fermionic superfluid reduces the theory of a dilute nonequilibrium Bose gas of diatomic molecules. In particular, we proved the equivalence of nonequilibrium BCS theory in the strong coupling limit and the full nonlinear time-dependent GP equation. Despite these results may seem to be quite obvious physically, we believe that our general approach can be useful in studying more complicated situations related to new nonequilibrium phenomena in the problem of BCS-BEC crossover.
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