A MODULE ISOMORPHISM BETWEEN $H^*_T(G/P) \otimes H^*_T(P/B)$ AND $H^*_T(G/B)$

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Abstract. We give an explicit (new) morphism of modules between $H^*_T(G/P) \otimes H^*_T(P/B)$ and $H^*_T(G/B)$ and prove (the known result) that the two modules are isomorphic. Our map identifies submodules of the cohomology of the flag variety that are isomorphic to each of $H^*_T(G/P)$ and $H^*_T(P/B)$. With this identification, the map is simply the product within the ring $H^*_T(G/B)$. We use this map in two ways. First we describe module bases for $H^*_T(G/P)$ that are different from traditional Schubert classes and from each other. Second we analyze a $W$-representation on $H^*_T(G/B)$ via restriction to subgroups $W_P$. In particular we show that the character of the Springer representation on $H^*_T(G/B)$ is a multiple of the restricted representation of $W_P$ on $H^*_T(P/B)$.

1. Introduction

The equivariant cohomology of a variety is a version of the ordinary cohomology ring that conveys additional information about an underlying group action. There are computational tools that often make it easier to construct equivariant cohomology than ordinary, even though we can recover ordinary cohomology from equivariant (Knutsen and Tao’s work gives one example [9]). These computational tools apply in many important cases, including generalized flag varieties $G/B$ and partial flag varieties $G/P$ with the left-multiplication action of the torus $T \subseteq B \subseteq P$.

In this paper we consider a particular presentation of the equivariant cohomology $H^*_T(G/B)$ due Kostant and Kumar [10]. We construct a module isomorphism between the tensor product $H^*_T(G/P) \otimes H^*_T(P/B)$ and $H^*_T(G/B)$. The map naturally descends to a module isomorphism on the ordinary cohomology as well. The fact that these modules are isomorphic is not new [6] Theorem 2.1]. Our map, however, is.

To construct our map, we identify each factor in the tensor product $H^*_T(G/P) \otimes H^*_T(P/B)$ with a submodule of $H^*_T(G/B)$. The bilinear module isomorphism is multiplication of classes inside the ring $H^*_T(G/B)$. More precisely our main theorem states:

Theorem 1.1. Identify $H^*_T(G/P)$ and $H^*_T(P/B)$ isomorphically with submodules of $H^*_T(G/B)$ as described in Section 2.1. Then the multiplication map

$$p \otimes q \mapsto pq$$

induces a bilinear isomorphism of modules

$$H^*_T(G/P) \otimes H^*_T(P/B) \cong H^*_T(G/B).$$

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In Section 3 we prove Theorem 1.1 in the equivariant setting. The result then descends to ordinary cohomology (see Corollary 3.7). Section 4 presents two applications.

Our explicit module isomorphism gives rise to a large collection of module bases of $H^*_T(G/B)$ indexed by the parabolic subgroups $B \subset P \subset G$. One way to state the core problem of Schubert calculus is: analyze combinatorially and explicitly the cohomology ring of a generalized flag variety $G/B$ in terms of the basis of Schubert classes. Theorem 4.3 proves that the bases we obtain are distinct from the Schubert basis and from each other. Our parabolic bases allow us to optimize the choice of basis to make particular computations in Schubert calculus as simple as possible.

As another application we show how these bases can be used to analyze a well-known action of $W$ on $H^*_T(G/B)$ called the Springer representation. In particular Theorem 4.4 says that the character of the restricted action of $W_P$ on $H^*_T(G/B)$ is the scalar multiple $|W_P| \cdot \chi$ where $\chi$ is the character of the $W_P$–representation on $H^*_T(P/B)$.

Our proofs use what many call GKM theory, which describes equivariant cohomology rings in an algebraic–combinatorial fashion. The GKM presentation comes with an explicit formula for the Schubert classes that is due to Billey [3, Theorem 4] and Anderson-Jantzen–Soergel [1, Remark p. 298]. These tools permit an elegant combinatorial and linear-algebraic proof of Theorem 1.1.

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2. Background

We denote by $G$ a complex reductive linear algebraic group and fix a Borel subgroup $B$. We denote the maximal torus in $B$ by $T$ and the Weyl group associated to $G/B$ by $W$. Let $P$ be any parabolic subgroup containing $B$.

Let $W_P$ denote the subgroup of $W$ associated to $P$. This is also a Weyl group, specifically the Weyl group of $P/B$. For elements $w \in W$ the length $\ell(w)$ refers to the minimal number of simple reflections required to write $w$ as a word in the generators $\{s_i : i = 1, 2, \ldots, n\}$ of $W$. Let $W^P$ denote the subset of minimal-length coset representatives of $W/W_P$. The following fact is so essential to our work that we state it explicitly here; many texts give proofs, including Björner-Brenti [4, Lemma 2.4.3].

**Proposition 2.1.** Every minimal-length word for each element $v \in W^P$ ends in a simple reflection $s_i \not\in W_P$.

2.1. **Restricting to fixed points.** We use a presentation of torus-equivariant cohomology that is often referred to as GKM theory, after Goresky, Kottwitz, and MacPherson [7], though key ideas are due to many others [2] [8] [5] (see [7, Section 1.7] for a fuller history). For suitable spaces $X$ the inclusion map of fixed points $X^T \hookrightarrow X$ induces an injection on cohomology $H^*_T(X) \hookrightarrow H^*_T(X^T)$. Straightforward algebraic conditions determine the image of the injection explicitly [5, Lemma 2.3], though we do not use them in this manuscript.
Through this map $H^*_T(X) \hookrightarrow H^*_T(X^T)$ we think of equivariant classes $p \in H^*_T(X)$ as collections of polynomials in $\bigoplus_{v \in X^T} \mathbb{C}[t^*] \cong H^*_T(X^T)$. We use functional notation to describe the elements $p \in H^*_T(X)$ meaning that for each $v \in X^T$ we have $p(v) \in \mathbb{C}[t^*]$. GKM theory applies to varieties like $G/B$, $G/P$, and $P/B$ that have only even-dimensional ordinary cohomology [7, Theorem 14.1(1)]. In fact each of $G/B$, $G/P$, and $P/B$ is a CW-complex whose cells are Schubert cells indexed by the elements of $W$, $W^P$, and $W_P$ respectively. The fixed point sets of $G/B$, $G/P$, and $P/B$ are also naturally isomorphic to $W$, $W^P$, and $W_P$.

As modules over $\mathbb{C}[t^*]$ the equivariant cohomology of $G/B$, $G/P$, and $P/B$ each have a basis of (equivariant) Schubert classes that are again indexed by the elements of $W$, $W^P$, and $W_P$ respectively. The restrictions $\sigma_w(u)$ of each Schubert class $\sigma_w$ to each fixed point $u$ are given explicitly by what we call Billey’s formula (see Section 2.2). The formula is the same in all three cases $G/B$, $G/P$, and $P/B$. Thus the map that sends the Schubert class $\sigma_w \in H^*_T(G/P)$ to the corresponding Schubert class $\sigma_w \in H^*_T(G/B)$ is a module isomorphism onto its image, and similarly for $P/B$. We identify the images of $H^*_T(G/P)$ and $H^*_T(P/B)$ in $H^*_T(G/B)$ with the modules $H^*_T(G/P)$ and $H^*_T(P/B)$ themselves, so

$$H^*_T(G/P) \cong \text{span}_{\mathbb{C}[t^*]} \langle \sigma_v : v \in W^P \rangle \subseteq H^*_T(G/B)$$

and

$$H^*_T(P/B) \cong \text{span}_{\mathbb{C}[t^*]} \langle \sigma_w : w \in W_P \rangle \subseteq H^*_T(G/B).$$

For $G/P$ this inclusion is only a homomorphism of modules and not a homomorphism of rings.

The map $H^*_T(G/P) \otimes H^*_T(P/B) \to H^*_T(G/B)$ that we consider is the ordinary product of classes inside $H^*_T(G/B)$.

### 2.2. Billey’s formula.

This section describes an explicit combinatorial formula for evaluating the polynomial $\sigma_v(u)$ in $\mathbb{C}[t^*]$. Anderson, Jantzen, and Soergel originally discovered this formula [1]; Billey independently found it as well [3, Theorem 4]. While proven originally for $G/B$ it also holds for $G/P$ [22, Theorem 7.1] and $P/B$ [11, Corollary 11.3.14]. Fix a reduced word for $u = s_{b_1} s_{b_2} \cdots s_{b_{\ell(u)}}$ and define $r(i, u) = s_{b_1} s_{b_2} \cdots s_{b_{i-1}}(\alpha_{b_i})$. Then

$$\sigma_v(u) = \sum_{\text{reduced words } v = s_{b_1} s_{b_2} \cdots s_{b_{\ell(u)}}} \left( \prod_{i=1}^{\ell(v)} r(j_i, u) \right).$$

**Lemma 2.2.** The polynomial $\sigma_v(u)$ has the following properties:

1. The polynomial $\sigma_v(u)$ does not depend on the choice of reduced word for $u$ [3, Theorem 4].
2. The polynomial $\sigma_v(u)$ is homogeneous of degree $\ell(v)$ [3, Corollary 5.2].
3. The polynomial $\sigma_v(u) \neq 0$ if and only if $v \leq u$ [10, Proposition 4.24].
4. For any $u$ we have $\sigma_e(u) = 1$. 
Example 2.3. Let $G/B$ have Weyl group $W = A_2$ and let $u = s_1s_2s_1$ and $v = s_1$. The word $v$ is found as a subword of $s_1s_2s_1$ in the two places $s_1s_2s_1$ and $s_1s_2s_1$.

$$\sigma_v(u) = r(1, s_1s_2s_1) + r(3, s_1s_2s_1) = \alpha_1 + s_1s_2(\alpha_1) = \alpha_1 + \alpha_2$$

3. Main Theorem

This section proves the main theorem of the paper. First in the equivariant setting, and then for ordinary cohomology, we prove that the module map from $H^*(G/P) \otimes H^*(P/B)$ to $H^*(G/B)$ induced by $p \otimes q \mapsto pq$ is a bilinear isomorphism of modules. We show this first in the equivariant case by proving that it takes a module basis for $H^*_T(G/P) \otimes H^*_T(P/B)$ to a module basis for $H^*_T(G/B)$. The non-equivariant case follows from the equivariant case.

**Theorem 3.1.** The set of Schubert class products $\{\sigma_v\sigma_w : v \in W_P, w \in W_P\}$ is a linearly independent set over $\mathbb{C}[t^*]$. In this section we will prove Theorem 3.1 by arranging these products in the matrix

$$A = (\sigma_v(v'w')\sigma_w(v'w'))_{(v,w),(v',w') \in W_P \times W_P}.$$ 

Our notational convention is to index rows by pairs $(v, w) \in W_P \times W_P$ and columns by pairs $(v', w')$ also in $W_P \times W_P$.

We begin by establishing an order on $W_P \times W_P$. The elements of both $W_P$ and $W_P$ are partially ordered by length; fix a total order on $W_P$ (respectively $W_P$) consistent with this partial order and extend this lexicographically to all of $W_P \times W_P$. For instance all rows and columns corresponding to pairs in $(e, W_P)$ come before any pair in $(s_i, W_P)$.

For the remainder of this section we will consider the matrix $A$ to have rows and columns ordered as above. The proof of Theorem 3.1 is given in Section 3.2.

3.1. Key lemmas. We begin with two lemmas. The first will prove that given the above ordering of its rows and columns, the matrix $A = (\sigma_v(v'w')\sigma_w(v'w'))$ is block upper-triangular. The second lemma will construct a matrix $M \cdot vN$ where $M$ is an invertible matrix and $vN$ is known to have linearly independent rows and columns. The main theorem will then show how $A$ and $M \cdot vN$ are related.

**Lemma 3.2.** The matrix $A = (\sigma_v(v'w')\sigma_w(v'w'))_{(v,w),(v',w') \in W_P \times W_P}$ is block upper-triangular.

**Proof.** Choose $v, v' \in W_P$. Consider the entries of $A$ whose rows are indexed by pairs in $(v, W_P)$ and whose columns are indexed by pairs in $(v', W_P)$. By construction this is a square $|W_P| \times |W_P|$ block. Its entries are $(\sigma_v(v'w')\sigma_w(v'w'))$ where $w, w'$ range over all of $W_P$. As established in Proposition 2.1, the last letter in every reduced word for $v' \in W_P$ is a simple reflection $s_i \notin W_P$. Thus every reduced word for $v \in W_P$ inside $v'w'$ is contained in the prefix $v'$. Therefore $\sigma_v(v'w') = \sigma_v(v')$. By Property 3 of Billey’s formula $\sigma_v(v')$ is non-zero if and only if $v \leq v'$ in the Bruhat order. Therefore whenever $\ell(v) \geq \ell(v')$ and $v \neq v'$ the entire block is zero. 

$\square$
We defined elements of matrix defined by \( w, u \) where \( v \) isomorphism linearly independent. It is not immediately obvious that the matrices in this lemma are in form \( M \). Then the rows of the matrix \( N \) of the matrix product \( M \) is invertible and so preserves linear independence of the matrix rows. Thus the parabolic subgroup \( w \) has form:

\[
\begin{pmatrix}
  1 & \ast & \ast & \ast \\
  \ast & 0 & \ast & \ast \\
  0 & \ast & 0 & \ast \\
  0 & 0 & \ast & \ast \\
\end{pmatrix}
\]

Example 3.3. Consider the parabolic subgroup \( W_P = \langle s_2 \rangle \) in the type \( A_2 \) Weyl group \( \langle s_1, s_2 \rangle \). The minimal coset representatives are \( W^P = \{ e, s_1, s_1 s_2 \} \). Let \( \sigma_{W_P} \) denote the collection \( \{ \sigma_w : w \in W_P \} \). Then the blocks of the matrix \( A \) are

\[
\begin{pmatrix}
  e_{W_P} & s_1 W_P & s_2 s_1 W_P \\
  \ast & \ast & \ast \\
  0 & \ast & \ast \\
  0 & 0 & \ast \\
\end{pmatrix}
\]

Example 3.4. This example treats pairs \( v, v' \in W^P \) with the same length. Let \( W_P \) be the parabolic subgroup \( \langle s_3 \rangle \subset \langle s_1, s_2, s_3 \rangle \). The elements of \( W^P \) with length two are \( s_1 s_2, s_2 s_1, \) and \( s_3 s_2 \). The restriction of the matrix \( A \) to the diagonal block where \( v \) and \( v' \) both have length two has form:

\[
\begin{pmatrix}
  s_1 s_2 W_P & s_2 s_1 W_P & s_3 s_2 W_P \\
  \ast & 0 & 0 \\
  0 & \ast & 0 \\
  0 & 0 & \ast \\
\end{pmatrix}
\]

In the next lemma we show that the rows of the diagonal blocks of the matrix \( A \) are linearly independent. It is not immediately obvious that the matrices in this lemma are in fact the diagonal blocks; that result is part of the main theorem.

Lemma 3.5 (Linear independence of diagonal blocks). Fix \( v \in W^P \). Assume that the elements of \( W_P \) are ordered consistently with the partial order on length. Let \( M \) be the matrix defined by

\[
M_{wu} = \begin{cases} 
\sigma_{w^{-1}u}(v) & \text{if } u \text{ is a suffix of } w \\
0 & \text{otherwise}
\end{cases}
\]

where \( w, u \in W_P \). Define the matrix \( N \) by \( N = (\sigma_u(v'))_{u, v' \in W_P} \). Consider the algebra isomorphism \( v : \mathbb{C}[t^*] \to \mathbb{C}[t^*] \) induced from the action \( t_\alpha \mapsto t_{v(\alpha)} \). Denote the image of \( N \) under this action of \( v \) by \( vN \).

Then the rows of the matrix \( M \cdot vN \) are linearly independent over \( \mathbb{C}[t^*] \).

Note that \( v \) does not permute the rows or columns of \( N \).

Proof. If \( \ell(u) > \ell(w) \) then by construction \( M_{wu} = 0 \). If \( \ell(u) = \ell(w) \) then \( M_{wu} = 0 \) unless \( w = u \). Therefore \( M \) is an upper-triangular matrix. The entries on the diagonal have the form \( M_{ww} = \sigma_e(v) = 1 \). Since 1 is a unit in \( \mathbb{C}[t^*] \) the matrix \( M \) is invertible.

We defined \( N = (\sigma_u(v'))_{u, v' \in W_P} \) to be the matrix of Schubert classes in \( H^*_T(P/B) \). The rows of \( N \) are the Schubert class basis for \( H^*_T(P/B) \) so the rows and columns of the matrix \( N \) are linearly independent. The function \( v \) acts on \( N \) by sending each \( t_\alpha \) to \( t_{v(\alpha)} \). This operation is invertible and so preserves linear independence of the matrix rows. Thus the new matrix \( vN \) also has linearly independent rows.

Since \( M \) is invertible over \( \mathbb{C}[t^*] \) and \( vN \) has linearly independent rows over \( \mathbb{C}[t^*] \) the rows of the matrix product \( M \cdot vN \) are also linearly independent over \( \mathbb{C}[t^*] \).

\( \square \)
3.2. **Proof of Theorem 3.1.** We now show that each of the diagonal blocks of \( A \) identified in Lemma 3.2 is a scalar multiple of the matrix \( M \cdot vN \) defined by Lemma 3.5. This proves that the rows of matrix \( A \) are linearly independent and thus the collection of Schubert class products \( \{ \sigma_v \sigma_w : v \in W_P, w \in W_P \} \) is linearly independent over \( \mathbb{C}[t^*] \).

**Proof.** Consider the matrix \( A = (\sigma_v(v'w')\sigma_w(v'w'))_{(v,w),(v',w') \in W_P \times W_P} \) with rows and columns ordered lexicographically subordinate to the length partial order on \( W_P \) and \( W_P \) described above.

Partition the matrix \( A \) into blocks according to the pairs \( v, v' \in W_P \). Lemma 3.2 proved that \( A \) is block-upper-triangular with this partition. Now consider the blocks along the diagonal, namely the blocks of the form

\[
(\sigma_v(v'w')\sigma_w(v'w'))_{w,w' \in W_P} = \sigma_v(v) \cdot (\sigma_w(v'w'))_{w,w' \in W_P}
\]

for each \( v \in W_P \). Proposition 3 of Billey’s formula guarantees that \( \sigma_v(v) \) is non-zero so it suffices to consider the matrix \( (\sigma_w(v'w'))_{w,w' \in W_P} \). We will show that

\[
(\sigma_w(v'w'))_{w,w' \in W_P} = M \cdot vN
\]

where \( M \) and \( vN \) are the matrices of Lemma 3.5. Multiplying matrices gives

\[
M \cdot vN = \left\{ \begin{array}{c} \left( \begin{array}{c} M_{wu} \\ \\ \end{array} \right) \\ w \text{ ranges over } W_P \end{array} \right\} \cdot \left\{ \begin{array}{c} \left( \begin{array}{c} v(\sigma_u(w')) \\ \end{array} \right) \\ u \text{ ranges over } W_P \end{array} \right\}
\]

We now show that for any \( w, w' \in W_P \) the polynomial \( \sigma_w(v'w') \) can be decomposed as the sum \( \sum_{u \in W_P} M_{wu} \cdot v(\sigma_u(w')) \). Consider Billey’s formula for \( \sigma_w(v'w') \) and group terms according to which part of \( w \) is a subword of \( v \) and which part is a subword of \( w' \). More precisely:

\[
\sigma_w(v'w') = \sum_{u \text{ a suffix of } w} \sigma_{wu}(v) \cdot \sigma_{u}(w')
\]

By construction of \( M \) this is \( \sum_{u \in W_P} M_{wu} \cdot v(\sigma_u(w')) \). Therefore the matrix \( (\sigma_w(v'w'))_{w,w' \in W_P} \) is equal to \( M \cdot vN \) as desired.

By Lemmas 3.2 and 3.5 the rows of the matrix \( A \) are linearly independent over \( \mathbb{C}[t^*] \).
Thus the Schubert class products \( \{ \sigma_v \sigma_w : v \in W^P, w \in W_P \} \) are linearly independent over \( \mathbb{C}[t^*] \).

Using Theorem 3.1, we can prove Theorem 1.1 in the equivariant setting.

**Theorem 3.6.** The map \( p \otimes q \mapsto pq \) induces a bilinear isomorphism of modules

\[
H^*_T(G/P) \otimes H^*_T(P/B) \cong H^*_T(G/B).
\]

**Proof.** For any pair \((v, w) \in W^P \times W_P\) the polynomial degree of the homogeneous class \( \sigma_v \sigma_w \) is \( \ell(v) + \ell(w) \) just like that of \( \sigma_{vw} \). The map \( W^P \times W_P \to W \) given by \((v, w) \mapsto vw\) is a bijection \([4]\) and induces a bijection \( \sigma_v \sigma_w \mapsto \sigma_{vw} \) which preserves polynomial degree. Thus the set \( \{ \sigma_v \sigma_w : v \in W^P, w \in W_P \} \) contains the correct number of elements of each polynomial degree to be a basis of \( H^*_T(G/B) \).

By Theorem 3.1, the set \( \{ \sigma_v \sigma_w : v \in W^P, w \in W_P \} \) is also linearly independent over \( H^*_T(pt) \). Thus it is a basis for \( H^*_T(G/B) \).

The equivariant isomorphism induces a similar isomorphism in ordinary cohomology, essentially by Koszul duality. We prove the result here.

**Corollary 3.7.** The map \( p \otimes q \mapsto pq \) induces a bilinear isomorphism of modules

\[
H^*(G/P) \otimes H^*(P/B) \cong H^*(G/B).
\]

**Proof.** Let \( M \subseteq \mathbb{C}[t^*] \) be the augmentation ideal, namely \( M = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \). Recall that the ordinary cohomology is the quotient \( H^*(X) \cong \frac{H^*_T(X)}{MH^*_T(X)} \) when \( X \) satisfies certain conditions, for instance, if \( X \) has no odd-dimensional ordinary cohomology \([7]\). Consider the two projections

\[
H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B) \to \frac{H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B)}{M(H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B))}
\]

and

\[
H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B) \to \frac{H^*_T(G/P)}{MH^*_T(G/P)} \otimes_{\mathbb{C}[t^*]} \frac{H^*_T(P/B)}{MH^*_T(P/B)}.
\]

If \( a \otimes b \in H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B) \) and \( m \in M \) then \( m(a \otimes b) = (ma) \otimes b = a \otimes (mb) \) so the kernels of the two projections agree. It follows that we have an isomorphism

\[
\frac{H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B)}{M(H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B))} \cong \frac{H^*_T(G/P)}{MH^*_T(G/P)} \otimes_{\mathbb{C}[t^*]} \frac{H^*_T(P/B)}{MH^*_T(P/B)}.
\]

The map \( \phi : H^*_T(G/P) \otimes_{\mathbb{C}[t^*]} H^*_T(P/B) \to H^*_T(G/B) \) is an isomorphism of \( \mathbb{C}[t^*] \)-modules so it commutes with taking the quotient by the augmentation \( MH_T(G/P) \). Combining these results gives

\[
\frac{H^*_T(G/P)}{MH^*_T(G/P)} \otimes_{\mathbb{C}[t^*]} \frac{H^*_T(P/B)}{MH^*_T(P/B)} \cong \frac{H^*_T(G/P)}{MH^*_T(G/P)} \otimes \frac{H^*_T(P/B)}{MH^*_T(P/B)} \]

or in other words \( H^*(G/P) \otimes H^*(P/B) \cong H^*(G/B) \).
4. Applications

The parabolic basis $B_P = \{ \sigma_v \sigma_w : v \in W^P, w \in W_P \}$ is generally not the Schubert basis. In fact we will show that, with the exception of $P = G$ and $P = B$, each of the bases $B_P$ is distinct not only from the Schubert basis but from any other parabolic basis as well. As with different bases of symmetric functions, this is a useful computational tool. As another application we compute the character of a particular Springer representation.

4.1. The parabolic basis $B_P$. We begin with an example illustrating that the basis $B_P$ is not the Schubert basis.

Example 4.1. We again use the $A_2$ example $W_P = \langle s_2 \rangle$ and $W^P = \{ e, s_1, s_2 s_1 \}$. Four of the classes in $B_P$ are also Schubert classes:

\[ \sigma_e \sigma_e = \sigma_e \]
\[ \sigma_e \sigma_{s_2} = \sigma_{s_2} \]
\[ \sigma_{s_1} \sigma_e = \sigma_{s_1} \]
\[ \sigma_{s_2s_1} \sigma_e = \sigma_{s_2s_1} \]

The remaining two classes are not Schubert classes.

\[ \sigma_{s_1} \sigma_{s_2} \]
\[ (\alpha_1 + \alpha_2)^2 \]
\[ \sigma_{s_2} \sigma_{s_1} \]
\[ \sigma_{s_2s_1} \sigma_{s_2} \]
\[ \alpha_1(\alpha_1 + \alpha_2) \]
\[ \alpha_2(\alpha_1 + \alpha_2) \]

The class $\sigma_{s_1} \sigma_{s_2}$ is equal to $\sigma_{s_1} \sigma_{s_2} = \sigma_{s_2} \sigma_{s_1}$ and the class $\sigma_{s_2s_1} \sigma_{s_2}$ is equal to $\sigma_{s_2s_1} \sigma_{s_1}$.

A Schubert class $\sigma_w$ could appear in one of these parabolic bases even if the word $w$ is contained in neither the parabolic subgroup nor the set of minimal coset representatives. If the class $\sigma_w$ does appear in the basis, we know exactly which Schubert classes are multiplied together to obtain it.
Lemma 4.2. Fix a parabolic $P$ and suppose that $v \in W^P, w \in W_P$. If the product class $\sigma_v\sigma_w$ is equal to a single Schubert class $\sigma_u$ then $vw$ is the parabolic decomposition of $u$.

Proof. If $\sigma_v\sigma_w = \sigma_u$ then $\ell(v) + \ell(w) = \ell(u)$ since both sides must have the same polynomial degree. For any $u' \neq u$ at least one of $\sigma_v(u')$ or $\sigma_w(u')$ must be zero since the product $\sigma_v(u')\sigma_w(u')$ is zero. By construction $\ell(v) + \ell(w) = \ell(vw)$ and by Property 3 of Billey’s formula $\sigma_v(vw)$ and $\sigma_w(vw)$ are both nonzero. Thus $\sigma_u(vw)$ is nonzero, implying that $vw \geq u$. But $vw$ has the same length as $u$ so the two words must be equal. \qed

Both $B_G$ and $B_B$ are the classical Schubert basis. Since $W_G = W_B = W$ every class in $B_G$ has the form $\sigma_e\sigma_w = \sigma_w$ and every class in $B_B$ has the form $\sigma_v\sigma_e = \sigma_v$. However all other parabolic bases are distinct.

Theorem 4.3. With the exception of $P = G$ and $Q = B$, distinct parabolics $P$ and $Q$ have distinct bases $B_P$ and $B_Q$.

Proof. Assume that $P \neq Q$ and that neither is $B$. There is at least one simple reflection in $W_Q$. Without loss of generality assume that $W_P$ is not contained in $W_Q$. Then there is at least one simple reflection in $W_P \setminus W_Q$. Consider all paths in the Dynkin diagram between simple roots corresponding to reflections in $W_Q$ and simple roots corresponding to reflections in $W_P \setminus W_Q$. Choose a minimal-length such path and denote the endpoints by $\alpha_i$ and $\alpha_j$ where $\alpha_i$ corresponds to $s_i \in W_P \setminus W_Q$ and $\alpha_j$ corresponds to $s_j \in W_Q$. Let $v_R$ be the word $s_j\cdots s_i$ corresponding to that path. The path is minimal so the word $v$ contains no reflections in $W_P$ or $W_Q$.

Since $v_R \in W$ is a word whose letters $s_{i_1} s_{i_2} \ldots s_{i_k}$ are the simple reflections in order corresponding to the vertices of a path in the Dynkin diagram, the letters $s_{i_j}$ and $s_{i_{j+1}}$ do not commute. Each reflection $s_{i_j}$ occurs at most once so no braid moves can be performed on $v_R$. Thus $v_R$ has no other factorization into simple reflections in $W$.

Consider the factorization of $v_R$ in each of $W^P W_P$ and $W^Q W_Q$. Since $v_R$ has a unique minimal word this factorization simply splits $v_R$ into a prefix and a suffix, with the prefix ending in the rightmost occurrence of $s_{i_j} \not\in W_P$ respectively $W_Q$. In particular the reflection $s_{i_j}$ is not in $W_Q$ so $v_R \in W^Q$. If $v \neq e$ then similarly $s_jv \in W^P$ and $s_i \in W_P$.

We will show that either the Schubert class corresponding to $v_R$ is in $B_Q$ or the Schubert class corresponding to $s_j s_i$ is in $B_P$. The two bases $B_P$ and $B_Q$ are equal only if $\sigma_{v_R}$ appears in $B_P$ or $\sigma_{s_i s_j}$ appears in $B_Q$ respectively. Lemma 4.2 showed that the only way that $\sigma_{s_i s_j}$ could equal $\sigma_{v_R}$ is if $ab = v_R$ and similarly for $\sigma_{s_j s_i}$. We will then evaluate at particular Weyl group elements to prove that the classes cannot be equal.

The cases we consider are:

Case 1. If $s_j \not\in W_P$ then $\sigma_{v_R} \in B_Q$ and $\sigma_{s_j v s_i} \in B_P$.

Case 2. If $s_j \in W_P$ then

a) if $v_R = s_j v s_i$ for $v \neq e$ then $\sigma_{v_R}$ is in $B_Q$ and $\sigma_{s_j v s_i} \in B_P$.

b) if $v_R = s_j s_i$ then $\sigma_{s_j s_i}$ is in $B_P$ and $\sigma_{s_j s_i} \in B_Q$.

Case 1. To see that $\sigma_{v_R} \neq \sigma_{s_j v s_i}$ we compare their values at $s_j v s_i s_j v$. We could prove that $s_j v s_i s_j v$ is reduced by an argument involving relations like the one with which we proved $v_R$ has a unique reduced word; alternatively we could observe that $s_j v s_i s_j v$ is
reduced because it is in Björner-Brenti’s normal form [4 Proposition 3.4.2] (with roots ordered in the same order as the path from $\alpha_i$ to $\alpha_j$). On the one hand

$$\sigma_{v_R}(s_jv_s_i s_j v) = \sigma_{s_j v}(s_j v s_i) = \sigma_{v_R}(v_R).$$

On the other hand

$$\sigma_{s_j v}(s_jv_s_i s_j v) \cdot \sigma_{s_i}(s_jv_s_i s_j v) = (\sigma_{s_j v}(s_j v) + s_j v s_i \sigma_{s_j v}(s_j v) + \text{other non-negative terms}) \cdot s_j v(\alpha_i)$$

which is $\sigma_{v_R}(v_R)$ + something positive. This proves the claim in this case.

**Case 2a.** In this case we evaluate the classes at $s_i s_j v s_i$ which is reduced by the previous argument. In $B_Q$ we have $\sigma_{v_R}(s_i s_j v s_i) = s_i(\sigma_{v_R}(v_R))$ while in $B_P$ we have

$$\sigma_{s_j v}(s_i s_j v s_i) \cdot \sigma_{s_i}(s_j v s_i) = s_i(\sigma_{s_j v}(v_R)) \cdot (\alpha_i + s_i s_j v(\alpha_i)).$$

Again this equals $s_i(\sigma_{v_R}(v_R)) +$ something positive. The claim holds in this case, too.

**Case 2b.** We look at the classes corresponding to $s_i s_j$. The word $s_i s_j$ is contained in $W_P$ and decomposes into $s_i \in W_Q$ and $s_j \in W_Q$. Evaluating at the reduced word $s_j s_i s_j$ gives

$$\sigma_{s_i}(s_j s_i s_j) \cdot \sigma_{s_j}(s_j s_i s_j) = s_j(\alpha_i) \cdot (\alpha_j + s_j s_i(\alpha_j))$$

but 

$$\sigma_{s_i s_j}(s_j s_i s_j) = s_j(\alpha_i) \cdot s_j s_i(\alpha_j).$$

These are unequal which proves the theorem. \(\square\)

4.2. **Representations of $W_P$.** In this section we use the parabolic basis of $H^*_T(G/B)$ to describe explicitly the character of an action of $W_P$ on $H^*_T(G/B)$. This group action is in fact the restriction of a well-known Weyl group action on $H^*_T(G/B)$ called Springer’s representation; Kostant and Kumar first studied the presentation we use here [10], see Section 4.17 and Proposition 4.24.g.

Kostant and Kumar showed that the Weyl group acts as a collection of algebra homomorphisms on the equivariant cohomology $H^*_T(G/B)$ according to the rule that if $w \in W$ and $p \in H^*_T(G/B)$ then the class $w \cdot p$ is given, in terms of its localizations, by:

$$(w \cdot p)(v) = p(vw^{-1}) \quad \text{for each } v \in W.$$  

We restrict this action to an arbitrary parabolic subgroup $W_P$ of $W$.

**Theorem 4.4.** Fix a subset $P$ of simple roots in $\Delta$. Let $m_P = |W_P|$. Let $\chi_P$ denote the character of the restriction to $W_P$ of Kostant-Kumar’s action of $W$ on $H^*_T(G/B)$. Then

$$\chi_P = m_P \chi$$

where $\chi$ is the character of Kostant-Kumar’s action of $W$ on $H^*_T(P/B)$.

**Proof.** Consider the basis $\{\sigma_w \sigma_v : w \in W^P, v \in W_P\}$ from the previous section. Choose any simple reflection $s_i \in W_P$ and consider the image $s_i \cdot (\sigma_{w'} \sigma_{v'})$ of $s_i$ acting on the basis element $\sigma_{w'} \sigma_{v'}$. Kostant-Kumar’s action is a map of algebras so

$$s_i \cdot (\sigma_{w'} \sigma_{v'}) = (s_i \cdot \sigma_{w'})(s_i \cdot \sigma_{v'}).$$

No reduced word for $w \in W^P$ ends in $s_i$. For each $u \in W$ we conclude by Billey’s formula that

$$\sigma_{w'}(us_i) = \sigma_{w'}(u).$$
A MODULE ISOMORPHISM BETWEEN $H^*_T(G/P) \otimes H^*_T(P/B)$ AND $H^*_T(G/B)$

Hence $s_i \cdot \sigma_{w'} = \sigma_{w'}$ and so for all $s_i, v' \in W_P$ and all $w' \in W^P$ we have

$$s_i \cdot (\sigma_{w'} \sigma_{v'}) = \sigma_{w'} (s_i \cdot \sigma_{v'}).$$

It follows that for each $v'' \in W_P$ we have $v'' \cdot (\sigma_{w'} \sigma_{v'}) = \sigma_{w'} v'' \cdot \sigma_{v'}$. In particular $\chi_P = |W_P| \chi$ since the coefficient of $\sigma_{w'} \sigma_{v'}$ in the expansion $v'' \cdot (\sigma_{w'} \sigma_{v'})$ in terms of the basis $\{\sigma_{w} \sigma_{v}\}$ is the same as the coefficient of $\sigma_{v'}$ in the expansion of $v'' \cdot \sigma_{v'}$ in terms of the basis $\{\sigma_{v}\}$.

□

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