Statistics of addition spectra of independent quantum systems

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Motivated by recent experiments on large quantum dots, we consider the energy spectrum in a system consisting of \(N\) particles distributed among \(K < N\) independent sub-systems, such that the energy of each sub-system is a quadratic function of the number of particles residing on it. On a large scale, the ground state energy \(E(N)\) of such a system grows quadratically with \(N\), but in general there is no simple relation such as \(E(N) = aN + bN^2\).

The deviation of \(E(N)\) from exact quadratic behavior implies that its second difference (the inverse compressibility) \(\chi_N \equiv E(N+1) - 2E(N) + E(N-1)\) is a fluctuating quantity. Regarding the numbers \(\chi_N\) as values assumed by a certain random variable \(\chi\), we obtain a closed-form expression for its distribution \(F(\chi)\). Its main feature is that the corresponding density \(P(\chi) = \frac{dF(\chi)}{d\chi}\) has a maximum at the point \(\chi = 0\). As \(K \to \infty\) the density is Poissonian, namely, \(P(\chi) \to e^{-\chi}\).
I. MOTIVATION

Statistics of spectra is an efficient tool for elucidating properties of various physical systems. So far, most of the effort is focused on the study of energy levels of a system with a fixed number of particles. In this context, one of the central earlier results is that the spectral statistics of many-body systems such as complex nuclei agree with the predictions of random matrix theory \cite{1,2}. On the other extreme, it was found that level statistics of a single particle in chaotic or disordered system also obeys a Wigner-Dyson statistics \cite{3,4}. Recently, experiments are designed to get information on the statistics of the addition spectra of electrons in quantum dots \cite{5}. The pertinent energy levels \( E(N) \) are the ground state energies of a system consisting of \( N \) electrons residing on a quantum dot, which is coupled capacitively to its environment.

Let us single out two properties of the addition spectra of quantum dots. The first one is that, on a large scale, the energy \( E(N) \) grows quadratically with \( N \), while the second one is a consequence of charge quantization, namely, there is, in general, no simple relation such as \( E(N) = aN + bN^2 \). In this context, an appropriate quantity whose statistics is of interest is then the inverse compressibility,

\[
\chi_N \equiv E(N + 1) - 2E(N) + E(N - 1).
\]

It is the deviation of \( E(N) \) from exact quadratic behavior which makes its second difference \( \chi_N \) non-constant. Indeed, in a recent experiment on large quantum dot \cite{6} it was found that the inverse compressibility vanishes for numerous values of electron number \( N \).

In the present work we study the statistics of the addition spectrum of a simple physical system with the two basic properties mentioned above. One example of such a system is motivated by considering the electrostatic energy of large quantum dots (although it should be mentioned that the model is too simple to describe the actual physics). To be specific, we have in mind a system of \( K \) metallic grains such that the number of electrons on the \( i^{th} \) grain is \( n_i \) (\( i = 0, 1, 2, \ldots, K - 1 \)) and their sum equals \( N \). The electrostatic energy of the pertinent system is a bilinear form in the numbers \( n_i \) with a \( K \times K \) matrix \( w \equiv \frac{1}{2}C^{-1} \). Here \( C \) is a positive-definite symmetric matrix of capacitance and inductance coefficients. If the metallic grains are very far apart, the matrix \( C \) is nearly diagonal. Thus, we concentrate on the special case \( C = \text{diag}[C_i] \), for which the energy of the system is given by

\[
E(N) = \min_{\sum_{i=0}^{K-1} n_i = N} \sum_{i=0}^{K-1} \frac{1}{2C_i} n_i^2 \tag{2}
\]

The minimum in (2) is taken over all possible partitions \( n_i \) of \( N \).

Another example is the energy of a system composed of \( K \) different harmonic oscillators, among which one distributes \( N \) spinless fermions. If there are \( n_i \) fermions on oscillator
\(i\) (whose frequency is \(\omega_i\)), then the energy of this oscillator (up to a constant) is \(E_i = \hbar \omega_i n_i (n_i + 1)\), and hence the ground state energy of the system is

\[
E(N) = \min \sum_{i=0}^{K-1} E_i, \text{ (subject to } \sum_{i=0}^{K-1} n_i = N) \tag{3}
\]

We will concentrate on the first example, which is borrowed from the electrostatics of quantum dots (2), and refer to the constants \(C_i\) as capacitors. Some results pertaining to the second example (the system of oscillators (3)) are also presented.

Regarding the numbers \(\chi_N\) of (1) as values assumed by a certain random variable, the distribution of this random variable is the main focus of the present work, which culminates in Theorem 1, where we find a closed-form expression for the distribution.

The problem of elucidating the (addition) spectral statistics of a many-body system, consisting of several independent sub-systems (whose dependence of \(E\) on \(n_i\) is known), looks deceptively simple. As will be evident shortly, this is not the case, and finding the distribution in question is quite a non-trivial task. Note that, even for a single particle system composed of several independent sub-systems (e.g., a system of a particle in several boxes), the derivation of level statistics requires a large degree of mathematical effort [7].

The rest of the paper is therefore devoted to a rigorous derivation of our main results.

## II. FORMALISM

**Definition 1.** Let \((\theta_n)_{n=1}^{\infty}\) be a sequence of real numbers and \(F\) a distribution function. The sequence \((\theta_n)\) is asymptotically \(F\)-distributed if

\[
\frac{|\{1 \leq n \leq M : \theta_n \leq x\}|}{M} \rightarrow F(x)
\]

for every continuity point \(x\) of \(F\) (where \(|S|\) denotes the cardinality of a finite set \(S\)).

An equivalent condition is the following. Denote by \(\delta_t\) the point mass at \(t\), and let \(\mu\) be the probability measure corresponding to the distribution \(F\) (namely, \(\mu(A) = \int 1_A dF(x)\) for any Borel set \(A\)). Then \((\theta_n)\) is asymptotically \(F\)-distributed if

\[
\frac{1}{M} (\delta_{\theta_1} + \delta_{\theta_2} + \ldots + \delta_{\theta_M}) \rightarrow \mu
\]

(the convergence being in the weak*-topology).

The notion of asymptotic distribution has a stronger version whereby, instead of requiring only that initial pieces of the sequence behave in a certain way, we require this to happen for any large finite portion of the sequence. This leads to
Definition 2. In the setup of Definition 1, \((\theta_n)\) is asymptotically well \(F\)-distributed if
\[
\lim_{M-L \to \infty} \frac{|\{L < n \leq M : \theta_n \leq x\}|}{M-L} \rightarrow F(x)
\]
for every continuity point \(x\) of \(F\).

Recall that the density of a set \(A \subseteq \mathbb{N}\) is given by
\[
D(A) = \lim_{M \to \infty} \frac{|A \cap [1, M]|}{M}
\]
if the limits exists. If, moreover, the limit
\[
BD(A) = \lim_{M-L \to \infty} \frac{|A \cap (L, M]|}{M-L}
\]
exists (in which case it is certainly the same as \(D(A)\)), then it is called the Banach density of \(A\) (cf. [9, p.72]).

The following lemma is routine.

**Lemma 1.** Let \((\theta_n)_{n=1}^{\infty}\) be a sequence of real numbers. Suppose \(N = \bigcup_{j=1}^{r} A_j\), where the union is disjoint. Let \((\theta^{(j)}_n)_{n=1}^{\infty}\) be the subsequence of \((\theta_n)\), consisting of those elements \(\theta_n\) with \(n \in A_j\), \(1 \leq j \leq r\).

1. If each \((\theta^{(j)}_n)\) is asymptotically \(F_j\)-distributed for some distribution functions \(F_j\), \(1 \leq j \leq r\), and \(D(A_j) = d_j\), \(1 \leq j \leq r\), then \((\theta_n)\) is asymptotically \(F\)-distributed, where \(F = \sum_{j=1}^{r} d_j F_j\).

2. If each \((\theta^{(j)}_n)\) is asymptotically well \(F_j\)-distributed and \(BD(A_j) = d_j\), then \((\theta_n)\) is asymptotically well \(F\)-distributed.

Obviously, a general sequence on the line does not have to be asymptotically distributed according to some distribution function, but one would expect it of sufficiently “regular” bounded sequences. In our case, one might expect \(\chi_N\) to be distributed according to some distribution function corresponding to a measure centered at about \(1/C\). However, this is not the case. In fact, the measure in question is supported on a finite interval, and is a convex combination of an absolutely continuous measure with decreasing density function on some interval \([0, a]\) and the point mass \(\delta_a\) at the right end \(a\) of that interval.

We have defined \(E(N)\) indirectly by means of the following

**Problem 1.** For each non-negative integer \(N\), find non-negative integers \(n_0, n_1, \ldots, n_{K-1}\), satisfying \(n_0 + n_1 + \ldots + n_{K-1} = N\), for which \(\sum_{i=0}^{K-1} \frac{1}{2x_i} \cdot n_i^2\) is minimal.

It turns out that this problem is intimately related to a second optimization problem. Put \(w_i = \frac{1}{2x_i}\), \(0 \leq i \leq K-1\), and let \(\Delta\) denote the set of all positive odd multiples of the numbers \(\frac{1}{2x_i}\):
\[
\Delta = \{w_0, 3w_0, 5w_0, \ldots, w_1, 3w_1, 5w_1, \ldots, w_{K-1}, 3w_{K-1}, 5w_{K-1}, \ldots\}\]
Here we treat $\Delta$ as a multi-set, or a sequence, in the sense that if some elements appear in this representation of $\Delta$ more than once (which occurs if some ratio $w_i/w_j$ is a rational number with odd numerator and denominator), then we consider $\Delta$ as having several copies of these numbers.

**Problem 2.** For each non-negative integer $N$, minimize $\sum_{m=1}^{N} \delta_m$, where $\delta_1, \delta_2, \ldots, \delta_N$ range over all distinct $N$-tuples in $\Delta$.

Note that, if an element appears several times in $\Delta$, it is allowed to appear the same number of times in the sum as well.

Let us demonstrate the equivalence of the two problems. Given the sum $\sum_{i=0}^{K-1} w_i \cdot n_i^2$, we may use the equality $w_i \cdot n_i^2 = w_i + 3w_i + 5w_i + \ldots + (2n_i - 1)w_i$ to see that any feasible value for the objective function of the first problem is a feasible value for the objective function of the second problem as well. On the other hand, solving Problem 2 is trivial. Namely, one minimizes the sum there simply by taking the $N$ least elements of the set $\Delta$.

In particular, for each $i$, the multiples of $w_i$ present in the optimal solution will be all odd multiples $w_i, 3w_i, 5w_i, \ldots$ up to some $(2n_i - 1)w_i$. Thus, the optimal solution of Problem 2 yields the optimal solution of Problem 1 also. We note in passing that this discussion shows also that the minimum (for each of the problems) is obtained at a unique point unless $\Delta$ contains multiple elements. (However, we shall always refer to the optimal solution, even when there may be several.)

A simple consequence of the above is

**Proposition 1.** Let $n = (n_i)_{i=0}^{K-1}$ be the optimal solution of Problem 1 for some value of $N$. Then the optimal solution of Problem 1, with $N + 1$ instead of $N$, is $n' = (n_i')_{i=0}^{K-1}$, where $n_i' = n_j + 1$ for some $0 \leq j \leq K - 1$ and $n_i' = n_i$ for $i \neq j$.

**Remark.** It is convenient to comment here on the effect a certain change in the original problem would make. One may consider the energies $E_i$ to be $w_i n_i(n_i + 1)$ instead of $w_i n_i^2$. This would change $\Delta$ to be the set of all even multiples of the $w_i$'s. Obviously, this would leave intact the equivalence of Problems 1 and 2. One can check that this would have also no effect on Theorems 1 and 2 infra.

To formulate our main result we need a few definitions and notations. Real numbers $\theta_1, \theta_2, \ldots, \theta_r$ are independent over $\mathbb{Q}$ if, considered as vectors in the vector space $\mathbb{R}$ over the field $\mathbb{Q}$, they are linearly independent. Equivalently, this is the case if the equality $m_1\theta_1 + m_2\theta_2 + \ldots + m_r\theta_r = 0$ for integer $m_1, m_2, \ldots, m_r$ implies $m_1 = m_2 = \ldots = m_r = 0$. Considering the actual physical system (a collection of metallic grains), it is reasonable to assume that the capacitors $C_i$ are random, so that generically they are independent over $\mathbb{Q}$. Without loss of generality we may rearrange the $K$ capacitors such that $C_0 = \max_{0 \leq i \leq K-1} C_i$. It is also useful to divide all the capacitors by the largest one, so that the
scaled capacitors $c_i \equiv C_i/C_0$ with $1 = c_0 > c_1, c_2, \ldots, c_{K-1}$ are dimensionless. Finally, set $s = c_0 + c_1 + \ldots + c_{K-1}$.

Now we formulate our main results.

**Theorem 1.** Suppose $C_0, C_1, \ldots, C_{K-1}$ are independent over $\mathbb{Q}$. Then the sequence $(\chi_N)_{N=1}^\infty$ is asymptotically $F$-distributed, where the distribution $F$ is given by either of the following two representations:

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0}^{i-1} \left(1 - \frac{x}{2w_j}\right), & 0 \leq x < 2w_0, \\ 1, & 2w_0 \leq x, \end{cases}$$

(4)

$$= \begin{cases} 0, & x < 0, \\ 1 - \frac{1}{s} \sum_{S \subseteq \{1, \ldots, K-1\}} (|S| + 1) \prod_{i \in S} c_i \prod_{i \not\in S} (1 - c_i) \cdot \left(1 - \frac{x}{2w_0}\right)^{|S|}, & 0 \leq x < 2w_0, \\ 1, & 2w_0 \leq x. \end{cases}$$

(5)

It is not immediately obvious from the formulas, but $F$ has one discontinuity, namely at the point $2w_0$. The reason is that, as the elements of $\Delta$ are all odd multiples of the $w_i$’s, and as $w_0$ is the smallest of the $w_i$’s, it happens occasionally that there is no odd multiple of $w_1, \ldots, w_{K-1}$ between two consecutive multiples of $w_0$. The size of the atom at $2w_0$ is $\frac{1}{s} \cdot \prod_{i=1}^{K-1} (1 - c_i)$. This is easily explained intuitively. In fact, the “density” of odd multiples of $w_i$ is $c_i$ times the same density for multiples of $w_0$. Hence the “probability” that an interval of the form $[(2n-1)w_0, (2n+1)w_0)$ does not contain an odd multiple of $w_i$ is $1 - c_i$. Assuming that the “events” of containing different $w_i$’s are independent, we conclude that the proportion of multiples of $w_0$ in $\Delta$ whose successors are also such is $\prod_{i=1}^{K-1} (1 - c_i)$. Since the proportion of multiples of $w_0$ in $\Delta$ is $\frac{1}{s}$, we arrive at the required expression for the size of the atom.

Now we would like to study the asymptotic of the distances between consecutive elements of $\Delta$ as the number of capacitors grows. Obviously, as this happens, the distances become smaller. More precisely, on the average we have $\frac{1}{2w_j}$ odd multiples of each $w_j$ in each unit interval, and hence we have there $\sum_{j=0}^{K-1} \frac{1}{2w_j} = \frac{s}{2w_0}$ elements of $\Delta$ altogether. Hence the average distance between consecutive elements is $\frac{2w_0}{s}$. To understand the asymptotic behavior of the gaps, it makes sense therefore to normalize them so as to have mean 1. Thus, we multiply the distances by $\frac{s}{2w_0}$, and ask about the asymptotic behavior

**Theorem 2.** Suppose the capacitances $C_0, C_1, \ldots$ are chosen uniformly and independently in $[0, 1]$. For each $K$, let $F_K$ denote the distribution corresponding to the normalized gaps
when taking into account the first $K$ capacitors only. Then, with probability 1, the distributions $F_K$ converge to an $\text{Exp}(1)$ distribution function.

Remark. As will be seen in the proof, we actually use much less to prove Theorem 2 than is required by the conditions of the theorem. Namely, we need the capacitances $C_i$ to be linearly independent over $\mathbb{Q}$, and that they do not form a fast diminishing sequence.

It is worthwhile mentioning that this type of “Poissonian” asymptotic behavior of consecutive gaps is typical. For example, this is the case for uniformly selected numbers in $[0,1]$, and is conjectured to be the case in other interesting cases as well. (See, for example, [11] and [12] and the references there.)

In the course of the proof, we shall make use of the notion of uniform distribution modulo 1 and a few basic results relating to it. (The reader is referred to Kuipers and Niederreiter [10] for more information.) A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is uniform distribution modulo 1 if

$$\left|\left\{1 \leq n \leq M : a \leq \{x_n\} < b\right\}\right| \xrightarrow{M \to \infty} b - a, \quad 0 \leq a < b \leq 1,$$

where $\{t\}$ is the fractional part of a real number $t$. In terms of Definition 1, $(x_n)$ is uniformly distributed modulo 1 if and only if the sequence $(\{x_n\})$ of fractional parts is $F$-distributed, where $F$ is the distribution function of the uniform distribution on $[0,1]$: $F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$

The generalization of the notion of an asymptotically $F$-distributed sequence to that of an asymptotically well $F$-distributed sequence clearly carries over to our case. Instead of requiring only that the dispersion of large initial pieces of the sequence becomes more and more even, we require this to happen at arbitrary locations. This version is termed well-distribution. Thus, $(x_n)_{n=1}^{\infty}$ is well-distributed modulo 1 if

$$\left|\left\{L < n \leq M : a \leq \{x_n\} < b\right\}\right| \xrightarrow{M-L \to \infty} b - a, \quad 0 \leq a < b \leq 1.$$

Both notions have multi-dimensional analogue. A sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^s$ is uniformly distributed modulo 1 in $\mathbb{R}^s$ if

$$\left|\left\{1 \leq n \leq N : a \leq \{x_n\} < b\right\}\right| \xrightarrow{N \to \infty} \prod_{i=1}^{s} (b_i - a_i), \quad 0 \leq a < b \leq 1,$$

where inequalities between vectors in $\mathbb{R}^s$ are to be understood component-wise, $0 = (0,0,\ldots,0) \in \mathbb{R}^s$, $a = (a_1,a_2,\ldots,a_s)$, and so forth.
Perhaps the most basic example of a sequence which is uniformly distributed modulo 1 is \((n\alpha)_{n=1}^\infty\), where \(\alpha\) is an arbitrary irrational. In the multi-dimensional case, the sequence \((n\alpha_1, n\alpha_2, \ldots, n\alpha_s)\) is uniformly distributed modulo 1 in \(\mathbb{R}^s\) if and only if the numbers 1, \(\alpha_1, \alpha_2, \ldots, \alpha_s\) are linearly independent over \(\mathbb{Q}\). Moreover, in this case uniform distribution implies well-distribution (cf. \([10, \text{Example 1.6.1, Exercise 1.6.14}]\)).

Given a partition \(N = \bigcup_{j=1}^l A_j\) and positive integers \(r_j, j = 1, \ldots, l\), we define the \((r_j)_{j=1}^l\)-inflation of the given partition as the partition of \(N\) obtained by inflating each element of each of the sets \(A_j\) into \(r_j\) elements. More precisely, we construct sets \(B_j, j = 1, \ldots, l\), as follows. For a positive integer \(i\), let \(f(i) = j\) if \(i \in A_j\). Given any positive integer \(n\), let \(m\) be defined by \(\sum_{i=1}^{m-1} f(i) < m \leq \sum_{i=1}^m f(i)\). Let \(n \in B_j\) if \(m \in A_j\). The following lemma is routine.

**Lemma 2.** In this setup:

1. If \(D(A_j) = d_j, 1 \leq j \leq l\), then \(D(B_j) = \frac{r_j d_j}{\sum_{i=1}^l r_i d_i}\).

2. If \(BD(A_j) = d_j, 1 \leq j \leq l\), then \(BD(B_j) = \frac{r_j d_j}{\sum_{i=1}^l r_i d_i}\).

**Proof of Theorem 1.** Between any two consecutive odd multiples of \(w_0\), there is at most one odd multiple of each \(w_j, 1 \leq j \leq K - 1\). In fact, one easily verifies that, given a positive integer \(m\), there is an odd multiple of \(w_j\) between \((2m - 1)w_0\) and \((2m + 1)w_0\), namely there exists an integer \(n\) with

\[
(2m - 1)w_0 \leq (2n - 1)w_j < (2m + 1)w_0,
\]

if and only if

\[
mc_j \in \left[\frac{1-c_j}{2}, \frac{1+c_j}{2}\right] \pmod{1}.
\]

Moreover, the relative position of \((2n - 1)w_j\) within the interval \([(2m - 1)w_0, (2m + 1)w_0]\) is the same, but in the opposite direction, as that of \(mc_j \pmod{1}\) within the interval \([\frac{1-c_j}{2}, \frac{1+c_j}{2}]\), that is

\[
(2n - 1)w_j = \alpha \cdot (2m - 1)w_0 + (1 - \alpha) \cdot (2m + 1)w_0, \quad (0 < \alpha \leq 1),
\]

if and only if

\[
mc_j \equiv (1 - \alpha) \cdot \frac{1-c_j}{2} + \alpha \cdot \frac{1+c_j}{2} \pmod{1}.
\]

Next we define a partition of \(N\) as follows. Write the elements of \(\Delta\) in ascending order: \(\Delta = \{\delta_1 < \delta_2 < \delta_3 < \ldots\}\). Given \(n \in N\), let \(S \subseteq \{1, 2, \ldots, K - 1\}\) denote the set of all
those $j$’s such that the unique interval of the form $[(2m-1)w_0, (2m+1)w_0)$ containing $\delta_n$ contains an odd multiple of $w_j$. The set of all integers $n$ giving rise in this way to any set $S$ is denoted by $B_S$. Consider the partition $\mathcal{N} = \bigcup_{S \subseteq \{1, \ldots, K-1\}} B_S$. To prove the theorem using Lemma 1, we have to find the Banach densities of the sets $B_S$ and the asymptotic distribution of the corresponding subsequences $(\chi_n)_{n \in B_S}$ of $\chi_n$.

The partition of $\mathcal{N}$ into sets of the form $B_S$ is obtained as an inflation of a somewhat more straightforward partition. In fact, let $S$ be any subset of $\{1, 2, \ldots, K-1\}$. Denote by $A_S$ the set of those positive integers $n$ for which the interval $[(2n-1)w_0, (2n+1)w_0)$ contains odd multiples of $w_j$ for $j \in S$ and does not contain such multiples of the other $w_j$’s. Then $\mathcal{N} = \bigcup_{S \subseteq \{1, \ldots, K-1\}} A_S$ is a partition, and its $(\mid S \mid + 1)_{S \subseteq \{1, \ldots, K-1\}}$-inflation yields the partition $\mathcal{N} = \bigcup_{S \subseteq \{1, \ldots, K-1\}} B_S$.

In view of the equivalence of (3) and (5), $A_S$ is the set of those $n$’s for which $nc_j \in \left(\frac{1-c_j}{2}, \frac{1+c_j}{2}\right)$ for $j \in S$ and $nc_j \notin \left(\frac{1-c_j}{2}, \frac{1+c_j}{2}\right)$ for $j \notin S$. By the conditions of the theorem, the numbers $1, c_1, \ldots, c_{K-1}$ are linearly independent over $\mathbb{Q}$, and hence the sequence $c = (nc_1, nc_2, \ldots, nc_{K-1})^\infty_{n=1}$ is well-distributed modulo 1 in $\mathbb{R}^{K-1}$. This means that

$$D(A_S) = BD(A_S) = \prod_{i \in S} c_i \prod_{i \notin S} (1 - c_i). \quad (10)$$

Denote the right hand side of (10) by $p_S$. In view of the above and Lemma 2, this implies

$$D(B_S) = BD(B_S) = \frac{(\mid S \mid + 1)p_S}{\sum_{T \subseteq \{1, 2, \ldots, K-1\}} (\mid T \mid + 1)p_T}. \quad (11)$$

The denominator on the right hand side can be given a simpler form. In fact, let $X_i, \ i = 1, 2, \ldots, K-1$, be independent random variables with $X_i \sim B(1, c_i)$, and $X = \sum_{i=1}^{K-1} X_i$. Then:

$$\sum_{T \subseteq \{1, 2, \ldots, K-1\}} (\mid T \mid + 1)p_T = E(X + 1) = 1 + c_1 + \ldots + c_{K-1} = s. \quad (12)$$

Hence:

$$BD(B_S) = \frac{(\mid S \mid + 1)p_S}{s}. \quad (13)$$

Let $S$ be an arbitrary fixed subset of $\{1, 2, \ldots, K-1\}$, say $S = \{1, 2, \ldots, l\}$, where $0 \leq l \leq K-1$. If $n \in A_S$, then there exist odd integers $a_{1n}, a_{2n}, \ldots, a_{ln}$ such that $a_{jn}w_j \in [(2n-1)w_0, (2n+1)w_0)$. Put:

$$v_n = (a_{1n}w_1, a_{2n}w_2, \ldots, a_{ln}w_l) - (2n-1)w_0 \cdot (1, 1, \ldots, 1) \in [0, 2w_0)^l, \quad n \in A_S.$$ 

By the equivalence of (3) and (5), the sequence $(v_n)_{n \in A_S}$ is well-distributed modulo $2w_0$ in $\mathbb{R}^l$. Now each $v_n$ gives rise to $l + 1$ terms of $(\chi_n)_{n \in B_S}$, as follows. Let $v_n^{(1)} \leq v_n^{(2)} \leq \ldots \leq v_n^{(l)}$ be
all coordinates of \( v_n \) in ascending order. Set:

\[
u_n = (v_n^{(1)}, v_n^{(2)} - v_n^{(1)}, \ldots, v_n^{(l)} - v_n^{(l-1)}, 2w_0 - v_n^{(l)}), \quad n \in A_S.
\]

The sequence \((\chi_n)_{n \in B_S}\) consists of all coordinates of all vectors \(u_n\). Now we use the fact that if \(X_1, X_2, \ldots, X_r\) are independent random variables, distributed \(U(0, h)\), and \(X^{(1)}, X^{(2)}, \ldots, X^{(r)}\) are the corresponding order statistics, then each of the random variables \(X^{(1)}, X^{(2)} - X^{(1)}, \ldots, X^{(r)} - X^{(r-1)}, h - X^{(r)}\) has the distribution function defined by \(G(x) = 1 - (x/h)^r\) for \(0 \leq x \leq h\) (which follows as a special case from [8, p.42, ex.23]). Consequently, for each \(1 \leq j \leq l + 1\), the sequence given by the \(j\)th coordinate of all vectors \(u_n, n \in A_S\), is asymptotically well \(G_1\)-distributed, where \(G_1(x) = 1 - (x/2w_0)^l\) for \(0 \leq x \leq 2w_0\). Hence the sequence \((\chi_n)_{n \in B_S}\) is asymptotically well \(G_1\)-distributed. Combined with (13), it proves (5).

We shall indicate only briefly the proof of (4), which is quite simpler. This time, we split \((\chi_n)\) into a union of subsequences \((\chi_n^{(i)}), 0 \leq i \leq K - 1\), by putting \(\chi_n\) in the sequence \(\chi_n^{(i)}\) if \(\delta_n\) is a multiple of \(w_i\). Clearly, the proportion of terms of \((\chi_n)\) belonging to \((\chi_n^{(i)})\) is \(c_i/s\). Next, consider the minimal odd multiples of all \(w_j\)'s which are larger than \(\delta_n\). The minimum of these \(K\) numbers is \(\delta_{n+1}\). For each \(j \neq i\), the distance from \(\delta_n\) to the minimal odd multiple of \(w_j\) following \(\delta_n\) is “distributed” \(U(0, 2w_j)\). (For \(i = 0\) it is also possible that the next term will be again a multiple of \(w_0\).) The linear independence of the \(C_i\)'s over \(Q\) implies that these \(K - 1\) distances are (statistically) independent, so that their minimum is distributed according to the function \(G_2(x) = 1 - \prod_{j=0 \atop j \neq i}^{K-1} \left(1 - \frac{x}{2w_j}\right)\) on the interval \([0, 2w_0)\). These considerations can be formalized to prove (4). This completes the proof.

**Remark.** It is possible to shorten the proof by proving directly the equality of the right hand sides of (4) and (5). In fact, it is easy to integrate both forms with respect to \(x\); the equality of the resulting expressions follows easily from the binomial theorem. We have chosen the long way, as it is more instructive.

**Proof of Theorem 2.** The distribution \(F_K\) is obtained from that in Theorem 1 by stretching by the constant factor \(\frac{x}{2w_0}\). Hence:

\[
F_K(x) = \begin{cases} 
0, & x < 0, \\
1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0 \atop j \neq i}^{K-1} \left(1 - \frac{c_j x}{s}\right), & 0 \leq x < s, \\
1, & s \leq x. 
\end{cases} \tag{14}
\]

Note that some of the values appearing on the right hand side depend on \(K\) implicitly. Namely, since \(w_0\) is assumed in Theorem 1 to be the least \(w_i\), each time a \(C_i\) is selected
which is larger than all the heretofore selected $C_j$'s, we have to rearrange the $C_j$'s, thus changing $w_0$ and the $c_j$'s. We have to show that

$$F_K(x) \xrightarrow{K \to \infty} 1 - e^{-x}, \quad x \geq 0. \quad (15)$$

Indeed, fix $x \geq 0$. Since

$$s = c_0 + c_1 + \ldots + c_{K-1} = \frac{C_0 + C_1 + \ldots + C_{K-1}}{C_0} \geq C_0 + C_1 + \ldots + C_{K-1} \quad (16)$$

and the $C_i$'s are independent and uniformly distributed in $[0, 1]$, we have

$$s \xrightarrow{K \to \infty} \infty. \quad (17)$$

Hence, with probability 1, for sufficiently large $K$ we have

$$F_K(x) = 1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \xrightarrow{K \to \infty} e^{-x}, \quad x \geq 0. \quad (18)$$

Thus, to prove (15) we need to show that

$$\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \xrightarrow{\text{as}} e^{-x}, \quad x \geq 0. \quad (19)$$

Now, on the one hand, using the inequality

$$1 - t \leq e^{-t}, \quad t \in \mathbb{R},$$

we have

$$\prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \leq e^{-x \sum_{j \neq i}^{K-1} \frac{c_j}{s}} \leq e^{-x + x/s}, \quad i = 0, 1, \ldots, K-1,$$

and therefore

$$\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \leq \frac{1}{s} \sum_{i=0}^{K-1} c_i e^{-x + x/s} = e^{-x + x/s} \xrightarrow{\text{as}} e^{-x}. \quad (20)$$

On the other hand, as $t \to 0$ we have

$$e^{-(t+t^2)} = 1 - (t + t^2) + \frac{(t + t^2)^2}{2} + O(t^3) = 1 - t - \frac{t^2}{2} + O(t^3),$$

so that for all $t$ in some sufficiently small neighborhood of 0

$$e^{-(t+t^2)} \leq 1 - t.$$
\[
K - 1 \prod_{j=0 \atop j \neq i}^{K-1} \left(1 - \frac{c_j x}{s}\right) \geq e^{-x \sum_{j=0 \atop j \neq i}^{K-1} c_j x^2 - x^2 \sum_{j=0 \atop j \neq i}^{K-1} c_j^2} \geq e^{-x - Kx^2/s^2}. \tag{21}
\]

Obviously, with probability 1, \( s \) grows linearly with \( K \), namely for all sufficiently large \( K \) we have \( s \geq aK \) for a suitably chosen \( a > 0 \). (In fact, any \( a < \frac{1}{2} \) will do.) By (21):

\[
\prod_{j=0 \atop j \neq i}^{K-1} \left(1 - \frac{c_j x}{s}\right) \geq e^{-x - Kx^2/s^2} \xrightarrow{K \to \infty} e^{-x}. \tag{22}
\]

From (20) and (22) it follows that

\[
\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0 \atop j \neq i}^{K-1} \left(1 - \frac{c_j x}{s}\right) \xrightarrow{K \to \infty} e^{-x}, \tag{23}
\]

which completes the proof.

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[1] E. P. Wigner, Ann. Math. 53, 36, (1951); 62, 548 (1955); 65, 203 (1957); 67, 325 (1958).

[2] F. J. Dyson, Jour. Math. Phys. 3, 140 (1962); 3, 157 (1962); 3, 166 (1962).

[3] O. Bohigas, M.J. Giannoni and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).

[4] B. L. Altshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91, 220 (1986). [Sov. Phys. JETP 64, 127 (1986)].

[5] U. Sivan, R. Berkovits, Y. Aloni, O. Prus, A. Auerbach and G. Ben Yoseph, Phys. Rev. Lett. 77, 1123 (1996); F. Simmel, T. Heinzel, and D. A. Wharam, Europhys. Lett. 38, 123 (1997); S. R. Patel, S. M. Cronenwel, P. R. Stewart, A. G. Huiberg, C. M. Marcus, C. I. Durooz, J. S. Harris, K. C. Kampman and A. C. Gossard, Phys. Rev. Lett.m 80, 4522 (1998).

[6] N.B. Zhitenev, R.C. Ashoori, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 79(1997), 2308.

[7] M.V. Berry and M. Tabor, Proc. Roy. Soc. London 356, 375 (1977).

[8] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed., Wiley, New York, 1971.

[9] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, Princeton, New Jersey, 1981.

[10] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[11] P. Kurlberg and Z. Rudnick, The distribution of spacings between quadratic residues, preprint.

[12] Z. Rudnick and A. Zaharescu, The distribution of spacings between small powers of a primitive root, preprint.