Abstract

Eguchi, Ooguri and Tachikawa have observed that the elliptic genus of type II string theory on K3 surfaces appears to possess a Moonshine for the largest Mathieu group. Subsequent work by several people established a candidate for the elliptic genus twisted by each element of $M_{24}$. In this paper we prove that the resulting sequence of class functions are true characters of $M_{24}$, proving the Eguchi-Ooguri-Tachikawa conjecture. The integrality of multiplicities is proved using a small generalisation of Sturm's Theorem, while positivity involves a modification of a method of Hooley. We also prove the evenness property of the multiplicities, as conjectured by several authors. We also identify the role group cohomology plays in both K3-Mathieu Moonshine and Monstrous Moonshine; in particular this gives a cohomological interpretation for the non-Fricke elements in Norton’s Generalised Monstrous Moonshine conjecture. We investigate the proposal of Gaberdiel-Hohenegger-Volpato that K3-Mathieu Moonshine lifts to the Conway group $Co_1$.

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1 Introduction

The elliptic genus (a.k.a. partition function) of a nonlinear sigma model with K3 target space is a very special function. On general grounds, it is a weak Jacobi form of index one and weight zero for \( \mathbb{Z}^2 \times \text{SL}_2(\mathbb{Z}) \), and is therefore equal to the unique (up to scaling) such function, \( \phi_{0,1}(z, \tau) \). This implies it is topological in the sense that it is independent of where you are on the 20-(complex) dimensional K3 moduli space, or indeed of where you are on the 40-dimensional moduli space of K3 sigma models.

The world-sheet description of these string theories is as an \( N = 4 \) superconformal field theory, and thus the whole state space can be organised as an \( N = 4 \) superconformal representation. In particular, the elliptic genus can be decomposed into sums of elliptic genera of \( N = 4 \) superconformal representations. Eguchi, Ooguri and Tachikawa [24] observed that the multiplicities with which these \( N = 4 \) elliptic genera contribute to the elliptic genus are dimensions of \( M_{24} \)-group representations.

According to Conway, the sporadic simple group \( M_{24} \) is the most remarkable of all finite groups (p.300, [9]), with a great wealth of structure and applications. For instance it is the automorphism group of the Golay code, a stabilizer in the Leech lattice, and all symplectic symmetries of K3 surfaces can be embedded in it (in fact in a subgroup \( M_{23} \) [42, 39]). Its order (size) is 244 823 040, and class number (i.e. number of conjugacy classes) is 26. The character table is given in Table 1; we retain the order of irreps (= irreducible representations) in [8], relate with a bar the complex conjugate irrep (when nonisomorphic), and write \( \rho_0 = 1 \) for the trivial representation. We write there \( \alpha_s^t = (s + ti\sqrt{7})/2 \), \( \beta_s^t = (s + ti\sqrt{15})/2 \), and \( \gamma_s^t = (s + ti\sqrt{23})/2 \), for all choices of signs \( s, t \in \{\pm 1\} \).

K3-Mathieu Moonshine was pushed further — indeed, made well-defined — by the work of Cheng [5], Gaberdiel, Hohenegger & Volpato [28, 29] and Eguchi & Hikami [20] who calculated the analogue \( \phi_g(\tau, z) \) of the McKay-Thompson series here. The resulting functions are weak Jacobi forms for \( \mathbb{Z}^2 \times \Gamma_0(|g|) \) up to certain phases (\(|g| \) denotes the order of \( g \in M_{24} \)). At the conclusion of this paper (see also [31]) we identify the cohomological source of these phases. The K3-Mathieu Moonshine fits beautifully into this general cohomological framework of conformal field theory (CFT) [31].

Once we know the weak Jacobi forms \( \phi_g \), we obtain modular forms \( f_g \) from (2.10) below and can read off the class functions \( H_n \) from (2.11) and (2.13) (a class function is a function constant on conjugacy classes). These \( H_n \) can also be obtained explicitly from (1.4) below. In any case, our most important result is the proof that these class functions \( H_n \) are all true representations of \( M_{24} \). This can be regarded as a proof of the weak form of the K3-Mathieu moonshine conjecture.

**Theorem A.** Each \( H_n \) \( (n \geq 1) \) is a true character of \( M_{24} \) (i.e. a sum of irreducible characters of \( M_{24} \)).
The strong conjecture says that these $\phi_g$ are the twisted elliptic genera of an $N = 4$ vertex operator superalgebra — we have nothing to say about this. One would have liked to prove Theorem A by explicitly constructing these $M_{24}$-representations $H_n$. This remains an important challenge. Instead, we prove Theorem A in two steps.

The first step in showing this is to prove that the $H_n$ are virtual characters, i.e. linear combinations over the integers of irreducible characters. It can be easily shown that the coefficients $H_{\alpha}(g)$ will be integer-valued class functions, but this only implies the class function is a linear combination over $\mathbb{Q}$ of irreducibles, with denominators dividing the order $|G|$ of the group (244 million, in our case!). We prove the $H_n$ are virtual by verifying that for each $n$ the quantities $H_{\alpha}(g)$ satisfy certain congruences; we can verify this simultaneously for all $n$ by studying mod $p^n$ reductions of associated modular forms. (Our result Lemma 3 on mod $n$ reductions of modular forms is a refinement of older results in the literature.)

The other and more difficult step in proving Theorem A, is to show that each $H_n(g)$ is a nonnegative (real) linear combination of irreducible characters. It is easy to reduce this to showing that for any $g \neq 1$, $|H_n(g)|$ is small compared to $H_n(1)$ for all $n > N$ for some $N$ (the coefficients for $n \leq N$ are then checked explicitly). Thanks to [6] we know

$$H_n(g) = O \left( \frac{e^{\pi\sqrt{8n-1}/2|g|}}{\sqrt{8n-1}} \right) \quad (1.1)$$

where $|g|$ denotes the order of $g$, so for all $n > N$, where $N$ is sufficiently large, $H_n(1)$
will dwarf $H_n(g)$ for $g \neq 1$, but we need to make this effective by finding that $N$, and this is the hard part of this positivity argument. It is tempting to guess we could follow Rademacher’s calculation of effective bounds for the partition numbers, which looks like it should be similar, but Rademacher’s series was absolutely convergent whereas those in [6] aren’t, so the argument is much more delicate. But the analogy with partition numbers is still useful: generalised Kloosterman sums arise here, and the corresponding zeta functions are what we need to bound; but Whiteman [51] showed long ago that the classical Kloosterman sum has an elegant expression as a sparse sum, and Hooley [37] explained how to use the theory of binary quadratic forms to bound series associated to similar sums, so we follow their lead. This involves though finding significant generalisations of Whiteman’s and Hooley’s results. As an aside, our inequalities yield an independent proof of the convergence of the Rademacher sum expressions in [6].

Integrality of these multiplicities is crucial. In contrast, the significance of positivity is not as clear to this author — after all, elliptic genus is a signed trace. What happens here should be contrasted with the $N = 4$ $c = 6$ toroidal theories, where the elliptic genus vanishes. Positivity could be a consequence of the minimality of this string theory (see e.g. the discussions in [44], [52]) — we return to this important question in the conclusion. In any case, it is intriguing to note that the elliptic genera of the $M_{24}$-twisted modules (these have been recently obtained in [31]) also appear to be true representations (of the appropriate central extension of centralisers in $M_{24}$).

Our methods of proving weak Moonshine are quite robust. In particular, they can surely be applied to the twisted twining Mathieu Moonshine elliptic genera of [31], the ‘Umbral Moonshine’ of [7] and the $PSL_2(\mathbb{Z}_{11}) N = 2$ Moonshine of [21] (as well of course to Monstrous Moonshine itself).

It is a consequence of our proof that the multiplicities $\text{mult}_{\rho_i}(H_n)$ tend to infinity with $n$, for each $i$, and all $\text{mult}_{\rho_i}(H_n)$ are strictly positive for $n \geq 25$. See Theorem 4 below.

It is common in the Mathieu Moonshine literature to emphasise the presence of mock modular forms. Indeed, the sums $q^{-1/8} \sum_{n=0}^{\infty} H_n(g)q^n$ are mock modular for each $g$, not (usually) modular. We had no use however of mock modularity in this paper; however the true modularity of the derived functions we call $f_g$ plays a crucial role.

In Section 3 we also obtain an evenness result conjectured by several authors:

**Theorem B.** Each head character $H_n$ is a sum of

$$\{2, 2\rho_1, \rho_2 + \rho_3, 2\rho_4, 2\rho_5, 2\rho_6, \rho_7 + \rho_8, 2\rho_9, \\ \rho_{10} + \rho_{11}, 2\rho_{12}, 2\rho_{13}, 2\rho_{14}, 2\rho_{15}, 2\rho_{16}, 2\rho_{17}, 2\rho_{18}, 2\rho_{19}, 2\rho_{20}\}$$

Theorems A and B are both used in [12] to prove a conjecture inUmbral Moonshine [7]. Indeed, their Theorem 1.2 is far stronger than Conjecture 5.11 in [7] (specialised to $M_{24}$), and a little weaker than Conjecture 5.12 in [7]. Incidentally, it is curious that imaginary quadratic fields play a role in Umbral Moonshine (see Section 5.4 of [7]) while the equivalent theory of positive-definite quadratic forms plays...
a crucial role in our Section 4 proof. Perhaps this can supply a deeper explanation for Conjecture 5.12 than would arise from arguments as in [12].

Theorems A and B also imply that the elliptic genus of Enriques surfaces can be decomposed into true characters of $M_{12}$ [22] (their elliptic genus is half that of K3 surfaces, so positivity comes from Theorem A and integrality from Theorem B). The VOA or string theoretic interpretation of this observation seems quite obscure.

[30] made the intriguing proposal that in fact the much larger Conway groups $Co_0$ or $Co_1$ may act. We prove (Gerald Höhn notified us that he also has a proof):

**Theorem C.** All $H_n$, as well as $H_{00}$, are restrictions of virtual representations of $Co_0$.

This was a nontrivial obstruction to their suggestion. However, as we explain below, the virtual $Co_0$-representations are much larger than the underlying $M_{24}$-representations (i.e. in the restriction a large representation with negative multiplicity cancels a large representation with positive multiplicity, leaving the small remainder $H_n$). Indeed, it now seems few people expect Mathieu Moonshine to extend to $Co_0$. Of course Theorem C implies that the same conclusion necessarily holds when $Co_0$ is replaced there with any subgroup between $M_{24}$ and $Co_0$. In the conclusion, we mention the other group that should be considered a candidate for enhanced symmetry in Mathieu Moonshine.

In our paper, and indeed in much work on the subject, no relation involving K3 is used or obtained. It seems possible to us that the connection with K3 may be illusory. This would be very disappointing. We return to this a little more in the concluding section.

Because a fairly wide range of readers are potentially interested in parts of this paper, and may wish to extend it to other Moonshines, we have endeavoured to keep it as accessible and as detailed as possible.

## 2 K3-Mathieu Moonshine: Review

Throughout this paper, write $e(x) = \exp(2\pi ix)$. For readability we will often collapse $a \equiv b \pmod{m}$ to $a \equiv_m b$. We will write $|g|$ for the order $|\langle g \rangle|$ of the element $g$ of a group. An excellent review of superconformal field theories and their moduli spaces and elliptic genera, including $\mathcal{N} = 4$ $c = 6$, is included in [52].

The elliptic genus of a sigma model with Calabi-Yau target is defined to be [54]

$$\phi(\tau, z) = \text{Tr}_{H_{RR}}(q^{L_0-c/24}e(zJ_0)(-1)^F \overline{q}^{\overline{L}_0-c/24}) ,$$  \hspace{1cm} (2.1)

where $q = e(\tau)$. Here, $L_0$ resp. $\overline{L}_0$ are the holomorphic resp. antiholomorphic Virasoro generators defining the grading, $J_0$ is an operator of $\mathcal{N} = 2$ supersymmetry defining the charge, and $F$ is the Fermionic number operator. The central charge $c$ equals $3d$, where $d$ is the (complex) dimension of the Calabi-Yau target. $\phi(\tau, z)$ will be independent of $\overline{q}$, by virtue of the supersymmetry in the Ramond sector, and depends only on the topological type (rather than the complex structure) of the target. It is
a weak Jacobi form of index $m = d/2$ and weight $k = 0$ for $\mathbb{Z}^2 \rtimes \Gamma$. This means that $\phi$ is a holomorphic function of $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the upper half-plane, which is modular with respect to $\tau$ and quasi-periodic with respect to $z$: for all $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$ and $l, l' \in \mathbb{Z}$,

\[
\phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k c(mcz^2/(c\tau + d)) \phi(\tau, z), \tag{2.2}
\]

\[
\phi(\tau, z + l\tau + l') = e(-im (l^2\tau + 2lz)) \phi(\tau, z), \tag{2.3}
\]

and in addition has Fourier expansion

\[
\phi(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} c(n, l)q^n y^l, \tag{2.4}
\]

for $y = e(z)$. Consistency of these conditions requires $c(n, l) = (-1)^k c(n, -l)$. The prefix ‘weak’ denotes that the sum over $n$ is allowed to start at 0, i.e. $\phi$ is merely holomorphic at the cusp $\tau = i\infty$; for historical reasons to be a true Jacobi form requires a sum over $l^2 \leq 4mn$.

The theory of Jacobi forms is developed in [25]. The definition extends trivially to weak Jacobi forms for other $\mathbb{Z}^2 \rtimes \Gamma$, where $\Gamma$ is any subgroup of $\text{SL}_2(\mathbb{Z})$ of finite index. Note that because we have insisted here on the full group $\mathbb{Z}^2$ of translations, $\Gamma$ must be a subgroup of $\text{SL}_2(\mathbb{Z})$ because it must act on that group of translations. This is significant because it means we will lose the genus-0 property which played so important a role in Monstrous Moonshine — recall that many of the Monstrous Moonshine Fuchsian groups aren’t subgroups of $\text{SL}_2(\mathbb{Z})$.

Nevertheless, the genus-0 property is what made Monstrous Moonshine special (and rather mysterious), so we should seek an analogue for it in Mathieu Moonshine. [6] have an intriguing proposal: that the mock modular form $q^{-1/8} \sum_n H_n(g) q^n$ equals a certain regularised Rademacher sum. The relation of such a property (in weight-0) to the genus-0 property was established in [17].

The algebra of weak Jacobi forms for $\mathbb{Z}^2 \rtimes \Gamma$ of even weight and integral index, is the polynomial algebra $\mathfrak{m}_\Gamma[\phi_{0,1}, \phi_{-2,1}]$, where $\mathfrak{m}_\Gamma$ is the ring of holomorphic modular forms for $\Gamma$, and

\[
\phi_{0,1}(\tau, z) = 4 \left( \frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2} \right), \quad \phi_{-2,1}(\tau, z) = -\frac{\theta_4(\tau, z)^2}{\eta(\tau)^6}, \tag{2.5}
\]

for the classical Jacobi theta series $\theta_i$ and Dedekind eta function $\eta$. (For $\Gamma = \text{SL}_2(\mathbb{Z})$ this is Proposition 9.3 in [25], but the proof for arbitrary finite-index $\Gamma$ is the same.) In particular, every weight-0 index-1 weak Jacobi form $F(\tau, z)$ for $\mathbb{Z}^2 \rtimes \Gamma$ can be expressed as $F = h\phi_{0,1} + f\phi_{-2,1}$, where $h \in \mathbb{C}$ and $f$ is a holomorphic weight-2 modular form for $\Gamma$. For nonlinear $\sigma$-models on K3, this implies the elliptic genus $\phi(\tau, z)$ must be proportional to $\phi_{0,1}$ and therefore will be constant throughout the 40-dimensional moduli space of K3 sigma models, something we already knew.
The worldsheet description is of an $N = 2$ vertex operator super algebra (VOSA). The holonomy of K3 surfaces extends the $N = 2$ to $N = 4$. Therefore, the space $H_{RR}$ in fact carries a representation of the $N = 4$ superconformal algebra at $c = 6$, so

$$
\phi_{K3}(\tau, z) = A_{00}ch^s_{1/4,0}(\tau, z) + \sum_{n=0}^{\infty} A_n ch^l_{n+1/4,1/2}(\tau, z), \quad (2.6)
$$

where $ch^s_{h,j}(\tau, z)$ is the elliptic genus of the appropriate level 1 $N = 4$ representation ($s = \text{short} = \text{massless} = \text{BPS}$, $l = \text{long} = \text{massive} = \text{non-BPS}$) and $ch^l_{1/4,1/2}$ is to be interpreted as $ch^s_{0,0} + 2ch^s_{1/4,1/2}$. Eguchi, Ooguri & Tachikawa [21] remarked that these coefficients $A_n$, previously calculated in [23], appear to be dimensions of $M_{24}$-representations (except for $n = 0$). More precisely, $A_{00} = 20$ is the dimension of the virtual representation $H_{00} = \rho_1 - 3\rho_0 = \rho_1 - 3$, and $A_n$ is the dimension of $H_n$, where $H_0$ is the virtual representation $-2$, and the next few are

$$
H_1 = \rho_2 + \rho_5, \quad H_2 = \rho_3 + \rho_7, \quad H_3 = \rho_7 + \rho_7, \quad H_4 = 2\rho_{14}, \quad H_5 = 2\rho_{19}, \quad H_6 = 2\rho_{20} + 2\rho_{16}, \quad H_7 = 2\rho_{20} + 2\rho_{19} + 2\rho_{18} + 2\rho_{17} + 2\rho_{13} + 2\rho_{12}, \quad H_8 = 6\rho_{20} + 2\rho_{19} + 2\rho_{18} + 4\rho_{17} + 2\rho_{16} + 2\rho_{15} + 2\rho_{14} + 2\rho_{11} + 2\rho_{10} + \rho_{10} + \rho_{10} + \rho_{8} + \rho_{8}, \quad H_9 = 10\rho_{20} + 8\rho_{19} + 8\rho_{18} + 4\rho_{17} + 4\rho_{16} + 4\rho_{15} + 2\rho_{14} + 2\rho_{13} + 2\rho_{12} + 2\rho_{10} + 2\rho_{10} + 2\rho_{10} + 2\rho_{9} + 2\rho_{7} + 2\rho_{7} + 2\rho_{6}.
$$

(Incidentally, [21] note that expanding the elliptic genus into $N = 2$ characters (rather than $N = 4$) seems to carry a representation of $\text{PSL}_2(\mathbb{Z}_{11})$, a simple group of order 660.)

This observation of [24] is very reminiscent of the Monstrous Moonshine observation of McKay who noted that the expansion coefficients of the Hauptmodul $J$-function seem to be dimensions of Monster group $\mathbb{M}$ representations $V_n$ (see e.g. [32] for a review and references). Thompson suggested considering what are now called the $\text{McKay-Thompson series}$, defined formally for all $g \in \mathbb{M}$ by

$$
T_g(\tau) = \sum_{n=-1}^{\infty} ch_{V_n}(g) q^n, \quad (2.7)
$$

where $q = e^{2\pi i \tau}$. Note that these $T_g$ are constant on each of the 194 conjugacy classes of $\mathbb{M}$ — they are called $\text{class functions}$. After staring at their first few coefficients, Conway & Norton [9] conjectured that these $T_g$ are modular functions (in fact Hauptmoduls=normalised generators of the field of modular functions) for some genus-0 Fuchsian group $\Gamma_g$ commensurable with (but not in general a subgroup of) $\text{SL}_2(\mathbb{Z})$. To make this conjecture more precise, we should turn the logic around and associate to each conjugacy class $K_g$ in $\mathbb{M}$ a uniquely specified modular function $T_g$. From their Fourier coefficients we obtain a sequence $\chi_n$ of $\text{class functions}$, i.e. linear combinations over $\mathbb{C}$ of irreducible characters of $\mathbb{M}$. Because the coefficients of the $T_g$ are rational (in fact integral), it is immediate that this linear combination can be taken over $\mathbb{Q}$, but it is very hard to show they can be taken over $\mathbb{Z}$ (i.e. that these
class functions $\chi_n$ are virtual characters) and even harder to show that they are combinations over $\mathbb{Z}_{\geq 0}$ (i.e., that the $\chi_n$ are actually characters of $\mathbb{M}$-representations $M_n$). Atkin, Fong & Smith (see [48]) tried to prove with the aid of a computer that these $\chi_n$ are indeed true characters, and managed to reduce the proof to a fairly plausible statement they called Conjecture 2.3 (it is often claimed in the literature — see e.g. [32] — that [48] proves the $\chi_n$ are true characters, but this is false). That Conway-Norton conjecture was finally proved in [2], independently of [48]; thanks to this work, we now know much more: there is a vertex operator algebra (VOA) $V^\natural$ (constructed in [27]) with automorphism group $M$, whose twining characters $\text{Tr}_{V^\natural}gq^{L_0-c/24}$ equal the McKay–Thompson series $T_g$ (the ‘$-1$’ in the exponent is the usual shift by $c/24$). In other words, the $M$-modules $V_n$ are the eigenspaces of $V^\natural$ with respect to the Virasoro operator $L_0$. (The adjective ‘twining’ is short for ‘intertwining’, and is used as an alternative to the over-used word ‘twisted’.)

Thus we are led to test further this Mathieu Moonshine observation of [24], by introducing for each $g \in M_{24}$

$$
\phi_g(\tau, z) = \text{ch}_{H,00}(g) \text{ch}_{1,0,0}(\tau, z) + \sum_{n=0}^{\infty} \text{ch}_{H_n}(g) \text{ch}_{n+1,4,1/2}(\tau, z). \tag{2.8}
$$

These should be the twining elliptic genera

$$
\phi_g(\tau, z) = \text{Tr}_{H,RR}gq^{L_0-c/24}c(z, J_0)(-1)^F q^{L_0-c/24}, \tag{2.9}
$$

although that expression is purely formal as none of these K3 sigma models will have a symmetry consisting of all of $M_{24}$. In fact [42], the group of symplectic automorphisms of any K3 surface will typically be trivial, will never exceed order 960, and will only contain elements of order $\leq 8$. And no automorphism, symplectic or otherwise, of a K3 surface can have order 23. But ignore these subtleties for now. Twining elliptic genera should have good modular properties — we would expect them to be weight-0 index-1 Jacobi forms for $\mathbb{Z}^2 \times \Gamma_0(|g|)$ (recall we write $|g|$ for the order of $g$) though possibly with multiplier. This is enough to guess with effort these $\phi_g$ from the first few coefficients (just as was done in Monstrous Moonshine by Conway & Norton).

Conjectured expressions for $\phi_g$, for all $g \in M_{24}$ (constant on conjugacy classes of course) were obtained in [3, 28, 29, 20]. We have

$$
\phi_g(\tau, z) = \frac{w_g}{12} \phi_{0,1}(\tau, z) + f_g(\tau) \phi_{-2,1}(\tau, z), \tag{2.10}
$$

where $w_g \in \mathbb{Z}$ is the Witten index given in Table 2, and $f_g(\tau)$ is some holomorphic weight-2 modular form (with trivial multiplier) for $\Gamma_0(|g|)h_g$ for $h_g$ in Table 2. $w_g$ equals the character value of $\rho_1 + 1$ at $g$; note that it vanishes iff $K_g$ doesn’t intersect $M_{23}$ — this isn’t deep: $\rho_1 + 1$ is the permutation representation, so $K_g$ intersects $M_{23}$ iff $\rho_1 + 1$ has a fixed point, iff its character value (which equals the number of fixed-points) is nonzero. $h_g$ is the length of the shortest cycle in the cycle shape of $g$. Note that in all cases, $h_g|\gcd(N, 12)$ (where $N$ is the order of the element), and
$h_g = 1$ iff the conjugacy class $K_g$ intersects $M_{23}$. We include for later use the order of the centraliser $C_{M_{24}}(g)$ of $g$ in $M_{24}$, and the index of $\Gamma_0(|g|)$ in $\text{SL}_2(\mathbb{Z})$ (the index of $\Gamma_0(\prod p_i^{a_i})$, where $p_i$ are distinct primes, is $\prod (p_i + 1)p_i^{a_i-1}$), as well as the bound $a \times 10^b$ obtained by Theorem 3 below. In Table 2 and elsewhere, we write $7AB$ for $7A \cup 7B$ etc; we collect together conjugacy classes like this when the corresponding Jacobi forms $\phi_g$ coincide.

**Table 2. Data for $M_{24}$ conjugacy classes**

| $K_g$ | 1A  | 2A  | 2B  | 3A  | 3B  | 4A  | 4B  | 4C  | 5A  | 6A  | 6B  | 7AB | 8A  | 10A | 11A | 12A | 12B | 14A | 15AB | 21AB | 23AB |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $a \times 10^b$ | 5 6 | 5.76 | 2.194667 | 2.3162911 | 2.18631146 | 6.88 | 2.99 | 1.518759 | 2.21065142 | 2.611515 | 5.8231712 | 2.9128718 | 8.213 |

Extracting $f_g$ from (2.10), we obtain

$$f_g(\tau) = w_g \frac{\eta^4 \tau^4}{\eta^4} - w_g \sum_{\eta^4} \left( \frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1 + q^n} \right) - \frac{\eta^3 q^{-1/8}}{24} \sum_{n=0}^{\infty} H_n(g) q^n$$

These modular forms $f_g(\tau)$ are given explicitly (i.e. independently of knowing the $H_n(g)$) for all $g \in M_{24}$ in (30, 20) (see also Table 2 of [6]). Indeed, recall the Eisenstein series $E_2(\tau) = 1 - 24q - 72q^2 - \cdots$, a quasi-modular form for $\text{SL}_2(\mathbb{Z})$. Then for each integer $n > 1$,

$$E_2(n)(\tau) := \frac{1}{n-1}(E_2(n\tau) - E_2(\tau)) = 1 + \frac{24}{n-1} \sum_{k=1}^{n} \sigma_1(k) (q^k - q^{nk})$$

is a holomorphic modular form of weight 2 for $\Gamma_0(n)$ with trivial multiplier ($\sigma_1(k) = \sum_{d|k} d$). Writing $\eta(n) := \eta(n\tau)$ and $\eta = \eta(\tau)$, we have: $f_{1A} = 0$,

$$f_{2A} = \frac{4}{3} E_2^{(2)} = \frac{4}{3} + 32q + 32q^2 + 128q^3 + 32q^4 + 192q^5 + 128q^6 + 256q^7 + 32q^8 + 416q^9 \cdots,$$

$$f_{2B} = \frac{2}{\eta^2} E_2^{(2)} = 2 - 16q + 48q^2 - 64q^3 + 36q^4 - 96q^5 + 192q^6 - 128q^7 + 48q^8 - 208q^9 + \cdots,$$

$$f_{3A} = \frac{3}{2} E_2^{(3)} = \frac{3}{2} + 18q + 54q^2 + 18q^3 + 126q^4 + 108q^5 + 54q^6 + 144q^7 + 270q^8 + 18q^9 + \cdots,$$

$$f_{3B} = \frac{2}{\eta^6} E_2^{(3)} = 2 - 12q + 18q^2 + 24q^3 - 84q^4 + 36q^5 + 72q^6 - 96q^7 + 90q^8 + 24q^9 + \cdots,$$

$$f_{4A} = \frac{2}{\eta^8} E_2^{(4)} = 2 - 16q^2 + 48q^4 - 64q^6 + 48q^8 + 0q^9 + \cdots,$$

$$f_{4B} = 2E_2^{(4)} - \frac{1}{3} E_2^{(2)} = \frac{5}{3} + 8q + 40q^2 + 32q^3 + 40q^4 + 48q^5 + 160q^6 + 64q^7 + 40q^8 + 104q^9 + \cdots,$$

$$f_{4C} = \frac{2}{\eta^2} E_2^{(2)} = 2 - 8q + 32q^3 - 16q^4 - 48q^5 + 64q^7 + 48q^8 - 104q^9 + \cdots,$$
\[ f_{5A} = \frac{5}{3} E_2^{(5)} = \frac{5}{3} + 10q + 30q^2 + 40q^3 + 70q^4 + 10q^5 + 120q^6 + 80q^7 + 150q^8 + 130q^9 + \cdots, \]
\[ f_{6A} = \frac{5}{2} E_2^{(6)} - \frac{1}{2} E_2^{(3)} - \frac{1}{6} E_2^{(2)} = \frac{11}{6} + 2q + 14q^2 + 26q^3 + 38q^4 + 12q^5 + 38q^6 + 16q^7 + 86q^8 + 98q^9 + \cdots, \]
\[ f_{6B} = \frac{2}{\eta(6)^2} \eta(2)^2 \eta(3)^2 \eta(6)^2 = 2 - 4q - 6q^2 + 8q^3 + 12q^4 + 12q^5 + 24q^6 + 32q^7 - 6q^8 + 8q^9 + \cdots, \]
\[ f_{7A} = \frac{7}{4} E_2^{(7)} = \frac{7}{4} + 7q + 21q^2 + 28q^3 + 49q^4 + 42q^5 + 84q^6 + 7q^7 + 105q^8 + 91q^9 + \cdots, \]
\[ f_{8A} = \frac{7}{3} E_2^{(8)} - \frac{1}{2} E_2^{(4)} = \frac{11}{6} + 4q + 12q^2 + 16q^3 + 44q^4 + 24q^5 + 48q^6 + 32q^7 + 44q^8 + 52q^9 + \cdots, \]
\[ f_{10A} = \frac{2}{\eta(10)} \eta(2)^2 \eta(5) = 2 - 6q - 2q^2 + 16q^3 - 2q^4 - 6q^5 - 8q^6 - 8q^7 - 2q^8 + 2q^9 + \cdots, \]
\[ f_{11A} = \frac{11}{6} E_2^{(11)} - \frac{22}{5} \eta^2 \eta(11)^2 = \frac{11}{6} + 22q^2 + 22q^3 + 22q^4 + 22q^5 + 44q^6 + 44q^7 + 66q^8 + 66q^9 + \cdots, \]
\[ f_{12A} = \frac{2}{\eta(12)^2} \eta(2)^3 \eta(3)^2 \eta(12)^2 = 2 - 6q + 2q^2 + 6q^3 - 6q^4 + 12q^5 - 10q^6 - 6q^7 - 6q^9 + \cdots, \]
\[ f_{12B} = \frac{2}{\eta(12)^2} \eta(2)^3 \eta(6) \eta(12)^2 = 2 - 8q + 6q^2 + 8q^3 - 4q^4 - 24q^5 + 16q^7 - 8q^8 + 16q^9 + \cdots, \]
\[ f_{14AB} = \frac{91}{36} E_2^{(14)} - \frac{7}{12} E_2^{(7)} - \frac{1}{36} E_2^{(2)} - \frac{14}{3} \eta(2) \eta(7) \eta(14) = \frac{23}{12} - 3q + 11q^2 + 16q^3 + 11q^4 + 10q^5 + 16q^6 + 25q^7 + 39q^8 + 17q^9 + \cdots, \]
\[ f_{15AB} = \frac{35}{16} E_2^{(15)} - \frac{5}{24} E_2^{(5)} - \frac{1}{16} E_2^{(3)} - \frac{15}{4} \eta(3) \eta(5) \eta(15) = \frac{23}{12} - 2q + 9q^2 + 13q^3 + 16q^4 + 13q^5 + 24q^6 + 14q^7 + 15q^8 + 28q^9 + \cdots, \]
\[ f_{21AB} = \frac{7}{3} \eta^3 \eta(21)^3 - \frac{1}{3} \eta^6 \eta(3)^2 = 2 - 5q - 3q^2 + 10q^3 + 7q^4 - 6q^5 - 12q^6 - 5q^7 + 6q^8 + 10q^9 + \cdots, \]
\[ f_{23AB} = \frac{23}{12} E_2^{(23)} - 69 \eta^2 \eta(23)^2 + 92 \eta(2)^2 \eta(46)^2 + 92 \eta(2) \eta(23) \eta(46) + 23 \frac{\eta^3 \eta(23)^3 \eta(2) \eta(46)}{\eta(2) \eta(46)} = \frac{23}{12} + 23q^3 + 23q^4 + 23q^6 + 46q^8 + 23q^9 + \cdots. \]

Equation (7.16) of [13] tells us
\[ \eta^3 q^{-1/8} \sum_{n=0}^{\infty} H_n(1) q^n = 48 F_2^{(2)}(\tau) - 2 E_2(\tau), \quad (2.13) \]
where \( F_2^{(2)}(\tau) = \sum_{n>m>0,n\not\equiv 2m} (-1)^n m q^{mn/2} \). We read off from (2.12) and (2.13) that
\[ E_2^{(2)} \equiv 1 \pmod{24}, \quad (2.14) \]
\[ E_2^{(3)} \equiv 1 \pmod{12}, \quad (2.15) \]
\[ \eta^3 q^{-1/8} \sum_{n=0}^{\infty} H_n(1) q^n \equiv -2 \pmod{48}. \quad (2.16) \]
These will be used in the Section 3 proofs. (We thank Thomas Creutzig and Gerald Höhn for bringing (2.13) and its consequence (2.16) to our attention.)

Lemma 1. [29] Let \( g \in M_{24} \) have order \(|g|\), with parameter \( h \) given in Table 2. Then the twining character \( \phi_g(\tau, z) \) is a Jacobi form of index 1 and weight 0 under \( \Gamma_0(|g|) \) with multiplier defined by

\[
\phi_g \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = e^{2\pi i \text{od}(|g|h)} e^{2\pi i cz^2/(c\tau + d)} \phi_g(\tau, z). \tag{2.17}
\]

The function \( f_g(\tau) \) is a holomorphic modular form of weight 2 for \( \Gamma_0(|g|) \) with multiplier \( e^{2\pi i \text{od}(|g|h)} \), i.e. \( f_g \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 e^{2\pi i \text{od}(|g|h)} f_g(\tau) \forall \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(|g|) \).

\( \Gamma_0(n) \) as usual consists of all matrices \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \) with \( n|c \). The appearance of both \( \Gamma_0(|g|) \) and the multiplier in Lemma 1 are not at all mysterious, as explained in the concluding section.

Again, reversing the logic, we can use these conjectured expressions for \( \phi_g \) to define class functions \( H_{00}(g) \) and \( H_n(g) \). It was checked by brute force that these class functions are in fact true characters, for all \( n \leq 500 \) (except \( H_0 \) and \( H_{00} \) which are merely virtual). This is our Theorem A.

One would certainly want more than weak Moonshine (Theorem A) — e.g. we should have an \( M_{24} \)-worth of twisted twining elliptic genera with nice modularity. This is now done [31]. Unlike Monstrous Moonshine, where the algebraic content comes from the VOA \( V^2 \), the algebraic (not to mention geometric and physical) meaning of K3-Mathieu Moonshine is still unclear. Although we have learned much in the two years or so since [24], we still don’t really know the right questions to ask.

## 3 Weak K3-Mathieu Moonshine I: Integrality

In this section and the next, we prove Theorems A and B, which were stated in Section 2. Our proof falls into two independent steps: integrality (this section) and positivity (next section).

Let \( G \) be any finite group. Let \( \mathcal{P}_G \) be the set of all pairs \((p, K_g)\) where \( p \) is a prime dividing the order \(|G|\), \( K_g \) is a \( p \)-regular conjugacy class in \( G \) (i.e. the order \(|\langle g \rangle|\) of \( g \) is coprime to \( p \)), and \( p \) divides the order \(|C_G(g)|\) of the centraliser. There are precisely 22 pairs in \( \mathcal{P}_G \) for \( G = M_{24} \), which we list in Table 3. For reasons to become clear shortly, we include the highest power \( p^\pi \) of \( p \) dividing \(|C_G(g)|\), and the \( p' \)-section \( S \) which we define next paragraph.

![Table 3](image)

Data for virtual character proof
For any \( h \in G \) of order \( n = p^km \), where \( \gcd(p, m) = 1 \), find \( a, b \in \mathbb{Z} \) such that \( 1 = ap^k + bm \). Then \( h = h_p^a h_n^b \) where the \( p' \)-part \( h_p' = h_p^{ap^k} \) has order \( m \) and the \( p \)-part \( h_n = h_n^{bm} \) has order \( p^k \). The \( p' \)-section \( S \) for the pair \( (p, K_g) \in \mathcal{P}_G \) is defined to be \( S = \{ h \in G \mid h_p' \in K_g \} \). \( S \) is clearly the union of conjugacy classes; in Table 3 we list those conjugacy classes.

Suppose we have an integer-valued class function \( c : G \to \mathbb{Z} \), i.e. \( c \) is constant on conjugacy classes. We are interested in \( H \) being one of the head characters \( H_n \), and we want to prove it is the character of an \( M_{24} \)-representation. The hard part is to prove that \( A \) is a virtual character, i.e. a linear combination over \( \mathbb{Z} \) of irreps. The starting point is the following characterisation of virtual characters, attributed to Thompson and based on the classical theorem of Brauer. It is a refinement of the following basic idea, which the reader can verify for himself: If \( \chi \) is an integer-valued character and \( g \) is an element of order a prime power \( p^r \), then \( \chi(g) \equiv \chi(e) \pmod{p} \).

Choose any \( (p, K_g) \in \mathcal{P}_G \) and write the \( p' \)-section \( S \) as a disjoint union \( \bigcup_{i=1}^k K_i \) of conjugacy classes, for some \( k \) depending on \( (p, S) \). Let \( R_p \) be the tensor product \( \widehat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\xi_G] \), where \( \widehat{\mathbb{Z}}_p \) are the \( p \)-adic integers and \( \xi_n = e^{2\pi i/n} \). \( R_p \) is introduced to make formal sense of the orthogonality relations we’re about to introduce; the ring \( \mathbb{Z}[\xi_G] \) arises because all irreducible \( G \)-characters take values there. Define \( \mathcal{M}_{(p, K_g)} \) to be the set of all \( k \)-tuples \( (\ell_1, \ldots, \ell_k) \in R_p^k \) such that \( \sum_{i=1}^k \ell_i \chi(K_i) \in p^r R_p \) for all irreducible characters \( \chi \). If \( c \) is a virtual character, then \( c \) is likewise orthogonal to \( \mathcal{M}_{(p,S)} \). What is important to us is the converse:

**Lemma 2.** Let \( R_p \) and \( \mathcal{M}_{(p, K_g)} \) be as above. Let \( c : G \to \mathbb{Z} \) be an integer-valued class function of \( G \). Then \( c \) is a virtual character of \( G \), i.e. a linear combination over \( \mathbb{Z} \) of irreducibles, if \( \sum_{i=1}^k \ell_i c(K_i) \in p^r R_p \) for all \( (\ell_1, \ldots, \ell_k) \in \mathcal{M}_{(p, K_g)} \) and all \( (p, K_g) \in \mathcal{P}_G \).

Of course we want to apply this to \( c = H_n \), for each \( n \) > 0. The reason this characterisation is helpful is that much is known about reductions of modular forms modulo powers of primes, as we will see.

Suppose \( f(\tau) = \sum_{n=0}^\infty f_n q^n \), \( g(\tau) = \sum_{n=0}^\infty g_n q^n \) are holomorphic modular forms of weight \( k \in \mathbb{Z} \) for some finite-index subgroup \( \Gamma \) and multiplier \( \mu : \Gamma \to \mathbb{C}^\times \), and \( \mu \) has finite order \( M \). If the Fourier coefficients \( f_n, g_n \) are equal for all \( n \leq k \|\text{SL}_2(\mathbb{Z})/\Gamma\|/12 \), then \( f = g \). This is an immediate consequence of the classical valence formula for \( \Gamma \) applied to \((f-g)^M \). Incidentally, because of this and Lemma 1, the \( q \)-expansions provided earlier are more than enough to uniquely specify \( f_g \) (and hence \( \phi_g \)) — in fact the coefficients \( n \leq 5 \) suffice. What is much more surprising is that this also applies to the modular forms mod prime powers:

**Lemma 3.** Suppose \( f(\tau) = \sum_{l \geq 0} f_l q^l \in \mathbb{Q}[[q^{1/N}]] \) and \( g(\tau) = \sum_{l \geq 0} g_l q^l \in \mathbb{Q}[[q^{1/N}]] \) are holomorphic modular forms of rational weight \( k \in \mathbb{Q} \) and with some multiplier \( \mu \) for some subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \). Let \( m := \|\text{SL}_2(\mathbb{Z})/\Gamma\| \) be the index. We require that \( \mu \) has finite order, i.e. that all values \( \mu(\gamma) \) for \( \gamma \in \Gamma \) are \( M \)th roots of 1 for some \( M \).

(a) Suppose the Fourier coefficients \( f_l \) and \( g_l \) are integral for all \( l \). Suppose we have an integer \( n > 0 \) such that \( f_l \equiv g_l \pmod{n} \) for all \( l \leq km/12 \). Then \( f_l \equiv g_l \pmod{p} \).
forms for $\Gamma(Nf)$ until you've reduced coefficients of $f$ to 1. This means that $lF$ is a trivial multiplier, whose Fourier coefficients are integral and $\theta$ means $\theta(f)$ from comparing ($\theta$). Let $\theta$ be the smallest positive integer such that $\theta Nf/p$ is a prime ($\theta$). Then Sturm's theorem applies, and we find that $p$ divides all coefficients $F_i$. This must imply that $p$ divides all coefficients of $f$ (otherwise, choose the smallest $l$ such that $p$ doesn't divide $f_i$, and note that $p$ would fail to divide $F_{Kl}$). This means that $f/p$ will have integer coefficients, so $f/p$ will obey all hypotheses of part (a) with now $n$ replaced with $n/p$. Repeat with another prime dividing $n/p$, until you've reduced $n$ to 1.

To see Lemma 3(b), let $V$ be the space of all holomorphic weight $k$ modular forms for $\Gamma(N')$ with trivial multiplier. $V$ is finite-dimensional with an integral basis $f(i) \in \mathbb{Z}[q^{1/N'}]$. Therefore we can write $f = \sum a_i f(i)$ where $a_i \in \mathbb{Q}$. This means the coefficients of $f$ have denominator bounded by the lcm of all denominators of the $a_i$. Let $L$ be the smallest positive integer such that $Lf \in \mathbb{Z}[q]$. If $p$ is a prime dividing $L$ then $p$ will divide the first $mk/12$ coefficients of $Lf$ by the hypothesis on $f$, hence will divide all coefficients by Lemma 3(a), contradicting the minimality of $L$. This means $L = 1$, and we're done. QED to Lemma 3

Theorem 1. Each $H_n$ $(n \geq 1)$ is a virtual character of $M_{24}$ (i.e. a linear combination over $\mathbb{Z}$ of irreducible characters of $M_{24}$).

Proof. First, we need to show the class functions $H_n$ are integer-valued. We will do this by studying the $f_g$'s. One complication is that the coefficients of $f_g(\tau)$ are not quite integers, as can be seen by the displayed values in Section 2. The problem is the constant term $\frac{1}{4}$ in the middle term of (2.11), and the 12 in the denominator of its first term. In fact only the constant term of $w_g (\theta^3_3 + \theta^4_4)/12 - w_g \theta_3 \theta_4/4$ can fail to be integral. To see this, it suffices to show that $\theta_3 \theta_4 \in 1 + 4q\mathbb{Z}[q]$ and $\theta^4_3 + \theta^4_4 \in 2 + 12q\mathbb{Z}[q]$. That these both hold modulo 4 follows from $\theta_3 = 1 + 2 \sum \theta_3^3 + 2 \sum'$ and $\theta_4 = 1 + 2 \sum -2 \sum'$, using obvious notation. That $\theta^4_3 + \theta^4_4 \equiv 1$ (modulo 3) follows from comparing $(\theta^3_3 + \theta^3_4)^2$ and $E_4$ modulo 3, using Lemma 3 with $\Gamma = \Gamma(2)$.

From Lemma 3(b) and checking the integrality of the first few coefficients of $f_g$ we learn now that all coefficients of $f_g - w_g (\theta^3_3 + \theta^3_4)/12 + w_g \theta^3_3 \theta^4_4/4$ are integral (the theta function contribution kills the fractional part of the constant term). This means that all coefficients of each $f_g$ are integral except possibly for the constant term, and
also (from the first equality in (2.11)) that all coefficients of \( \eta^3q^{-1/8} \sum H_n(g)q^n \) are integral. Since \( \eta^3q^{-1/8} \) is invertible in the ring \( \mathbb{Z}[[q]] \), we find that \( H_n(g) \in \mathbb{Z} \) for all \( n \geq 0 \) and all \( g \in M_{24} \).

This invertibility of \( \eta^3q^{-1/8} \) directly gives us the useful implication:

\[
l f_g \in m\mathbb{Z}[[q]] \Rightarrow l H_n(g) \equiv \frac{\ell q}{24} H_n(1) \pmod{m}
\]  

(3.1)

for all \( n \geq 0 \), for any choice of \( l, m \in \mathbb{Z} \) and all \( g \in M_{24} \).

We need to verify, for each pair \( (p, K_g) \in \mathcal{P}_{M_{24}} \), that \( \sum_{i=1}^{k} \ell_i H_n(K_i) \in p^\ell R_p \) for all \( \ell_1, \ldots, \ell_k \in \mathcal{M}_{(p,K_g)} \).

A typical example is the pair \((3, 2A)\). From Table 3 we see that \( S = K_{2A} \cup K_{6A} \).

Note that \( \chi(2A) \equiv_3 \chi(6A) \), so we need to show \( H_n(2A) \equiv_3 H_n(6A) \) for all \( n \). We claim that \( f_{2A} - 4f_{6A} \in 3\mathbb{Z}[[q]] \). To see this, first note that \( f_{2A} - 4f_{6A} \) is a modular form for \( \Gamma_0(6) \) with integer \( q \)-expansion and trivial multiplier, so by Lemma 3(a) it suffices to check that \( 3 \text{ divides the } q^l\text{-coefficient of } f_{2A} - 4f_{6A} \text{ for all } 0 \leq l \leq 2, \) which is trivial to do from the expansions collected in Section 2. By (3.1) or otherwise, this implies \( H_n(2A) \equiv_3 H_n(6A) \) for all \( n \) and we’re done. The pairs \((2, 5A), (2, 7AB), (3, 2B), (3, 4AC), (5, 1A), (5, 2B), (11, 1A)\) are all handled similarly.

The pair \((3, 5A)\) is also easy. Here \( S = 5A \cup 15A \cup 15B \), and we find \( \mathcal{M}_{(3,5A)} = R_3-\text{span}\{(0, 1, -1), (1, -1, 0)\} \) using obvious notation. \( H_n(15A) = H_n(15B) \) for all \( n \) follows from \( f_{15A} = f_{15B}; H_n(5A) \equiv_5 H_n(15A) \) follows from \( f_{5A} \equiv_4 f_{15A} \), proved in the usual manner using \( \Gamma_0(15). \) The pairs \((5, 3A), (7, 1A), (7, 2A), (7, 3B), (23, 1A)\) are handled similarly.

For \((3, 7AB)\), we need \( H_n(7AB) \equiv_3 H_n(21AB) \) for all \( n \). Using (2.11), as well as the congruences (2.16), (2.15), this is equivalent to verifying that \( 4f_{7AB} - f_{21AB} \equiv_3 2E_2^{(3)} \), which by Lemma 3 for \( \Gamma = \Gamma_0(63) \) requires checking up to \( q^{16} \). For \((2, 3A), S = 3A \cup 6A \cup 12A \) and \( \mathcal{M}_{(2,3A)} = R_2-\text{span}\{(2, -2, 0), (1, 1, 1)\}, \) so we need to verify that \( H_n(3A) \equiv_2 H_n(6A) \text{ and } H_n(3A) + H_n(6A) \equiv_8 2H_n(12A), \) for all \( n \). Using now (2.14), this is equivalent to verifying that \( f_{3A} \equiv_2 f_{6A} \text{ and } f_{3A} + 9f_{6A} - 2f_{12A} \equiv_8 -2E_2^{(2)} \).

The former requires checking up to \( q^2 \) (use \( \Gamma_0(6) \)), while the latter requires checking up to \( q^8 \) (use \( \Gamma_0(24) \)). The pair \((2, 3B)\) is handled similarly.

The pair \((3, 1A)\) has \( S = 1A \cup 3A \cup 3B \) and \( \mathcal{M}_{(3,1A)} = R_3-\text{span}\{(1, 1, 1), (0, 9, 3), (0, 0, 9)\}, \) so we need to establish \( H_n(1A) \equiv_3 H_n(3B) \) and \( H_n(3A) \equiv_{27} 3H_n(3B) - 2H_n(1A) \) for all \( n \). These are equivalent to \( f_{3B} \equiv_3 2E_2^{(3)} \) and \( 4f_{3A} - 12f_{3B} \equiv_{27} 9E_2^{(3)} \).

Finally, \( \mathcal{M}_{(2,1A)} \) is the span of \((1, 1, 1, 1, 1, 1), (-22, -6, 2, 2, -2, 2, 0), (44, 12, 28, 4, 4, 0, 0), (-24, 8, 24, 8, 0, 0, 0), (-208, -16, 16, 0, 0, 0, 0), (448, 64, 0, 0, 0, 0, 0), \) using obvious no-
Theorem 2. Each head character \( H_n \) is a linear combination over \( \mathbb{Z} \) of
\[
\{2, 2\rho_1, \rho_2 + \overline{\rho_2}, \rho_3 + \overline{\rho_3}, 2\rho_4, 2\rho_5, 2\rho_6, \rho_7 + \overline{\rho_7}, \rho_8 + \overline{\rho_8}, 2\rho_9, \\
\rho_{10} + \overline{\rho_{10}}, 2\rho_{11}, 2\rho_{12}, 2\rho_{13}, 2\rho_{14}, 2\rho_{15}, 2\rho_{16}, 2\rho_{17}, 2\rho_{18}, 2\rho_{19}, 2\rho_{20}\}
\]

Proof. Let \( \chi \) be any \( M_{24} \)-class function, and \( \rho \in \widehat{M_{24}} \). Write \( \text{mult}_\rho(\chi) \) for the multiplicity. Consider first \( \rho \) a complex character (i.e. \( \overline{\rho} \not\equiv \rho \)). We know that each \( f_g(\tau) \in \mathbb{Q} + q\mathbb{Z}[\![q]\!] \). The integrality of \( H_n(g) \) for each \( n \) means that \( \text{mult}_\rho(H_n) = \text{mult}_\rho(H_n) \).

Much more difficult are the real \( M_{24} \)-irreps \( \rho \) (i.e. \( \rho \equiv \overline{\rho} \)). Assume \( \rho \not\equiv 1, \rho_1 \) for now. Define
\[
M_\rho(\tau) := \sum_{n=0}^\infty \text{mult}_\rho(H_n)q^n = -\sum_{K_g}|C_{M_{24}}(g)|^{-1}\overline{\rho(g)f_g(\tau)} q^{1/8}\eta(\tau)^{-3},
\]
using (2.11), where the second sum is over all conjugacy classes in \( M_{24} \). We know from Theorem 1 that each \( M_\rho(\tau) \in q\mathbb{Z}[\![q]\!] \) (at least for \( \rho \not\equiv 1, \rho_1 \)). We want to show that in fact \( M_\rho(\tau) \in 2q\mathbb{Z}[\![q]\!] \). Of course we can ignore the constant terms of the \( f_g \) in the following.

First note from the \( M_{24} \) character table that \( \rho(23A) \) and \( \rho(23B) \) will be equal and integral. Also, the centraliser \( C_{M_{24}}(23AB) \) has odd order (namely, 23). Since
which is far more than is necessary. Finally, the classes \(7AB\) and \(14AB\) can all be dropped from (3.2): they all have the same order (namely 14) of centraliser, \(\rho(7A) + \rho(7B) + \rho(14A) + \rho(14B) \in 4\mathbb{Z}\) (since \(\mathfrak{p} \cong \rho\)), and the \(q^n\)-coefficients \((n > 0)\) of \(f_{7AB}\) and \(f_{14AB}\) are integers congruent mod 2 (to see this, apply Lemma 3(a) to \(3f_{14AB} - f_{7AB}\)). Let \(\widetilde{M}_\rho(\tau)\) be \(23 \cdot 21 \cdot 5 \cdot 11\) times the sum in (3.2) restricted to the remaining classes \(K_g \in \{1A, 2AB, 3AB, 4ABC, 5A, 6AB, 8A, 10A, 12AB\}\). Then \(\widetilde{M}_\rho(\tau) \in q\mathbb{Z}[\![q]\!]\) and \(M_\rho \equiv \widetilde{M}_\rho\) and we need to show that \(\widetilde{M}_\rho(\tau) \in 2q\mathbb{Z}[\![q]\!]\), or equivalently \(\eta^3q^{-1/8}\widetilde{M}_\rho \in 2q\mathbb{Z}[\![q]\!]\).

Note that \(\eta^3q^{-1/8}\widetilde{M}_\rho\) is a linear combination (over \(\mathbb{Q}\)) of \(f_g\) with \(\rho(g) \neq 0\) and \(g \in \{1A, 2AB, 3AB, 4ABC, 5A, 6AB, 8A, 10A, 12AB\}\). Let \(N_\rho\) be the least common multiple of \(h_gN_g\) over those \(g\), where \(h_g\) is in Table 2 and \(N_g\) is the order of \(g\). We know from Lemma 1 that \(f_g\) is a weight-2 modular form for \(\Gamma_0(h_gN_g)\) with trivial multiplier, so \(\eta^3q^{-1/8}\widetilde{M}_\rho\) is a weight-2 modular form for \(\Gamma_0(N_\rho)\) with trivial multiplier. We collect \(N_\rho\) in Table 4, together with the quantity \(m_\rho/6\) where \(m_\rho = ||\text{SL}_2(\mathbb{Z})/\Gamma_0(N_\rho)||\). By Lemma 3, we need to show that the first \(m_\rho/6\) coefficients of \(\eta^3q^{-1/8}\widetilde{M}_\rho\) are even. As in (2.11) this is equivalent to showing that the first \(m_\rho/6\) values of \(\text{mult}_\rho(H_n)\) are even. The evenness of these multiplicities \(\text{mult}_\rho(H_n)\) has been verified for all \(n \leq 500\) and all real \(\rho\), by Gaberdiel-Hohenegger-Volpato (private communication; see also [29]), which is far more than is necessary.

The proof for \(\rho = 1\) and \(\rho = \rho_1\) is similar, but (3.2) has to be modified. For those \(\rho\) define

\[
F_\rho(\tau) := \sum_{K_g} |C_{M_{22}}(g)|^{-1}/\rho(g)f_g(\tau) + \frac{1}{12}E_2^{(2)}(\tau)
\]

\[
= -q^{-1/8}\eta(\tau)^3 \sum_{n=0}^{\infty} \left( \text{mult}_\rho(H_n) - \frac{1}{24}H_n(1) \right) q^n + \frac{1}{12}E_2^{(2)}(\tau)
\]

\[
\equiv_2 -q^{-1/8}\eta(\tau)^3 \sum_{n=0}^{\infty} \text{mult}_\rho(H_n) q^n, \tag{3.3}
\]

using (2.16), (2.14), (2.11) and the fact that \(w_g = 1 + \rho_1(g)\). Thus if we can show \(F_\rho(\tau)\) has even coefficients, we will be done. The expression (3.3) together with Theorem 1 tells us \(F_\rho\) has integral Fourier coefficients. Define \(\widetilde{F}_\rho\) as above by restricting the sum to classes \(K_g \in \{1A, 2AB, 3AB, 4ABC, 5A, 6AB, 8A, 10A, 12AB\}\) and multiplying by \(23 \cdot 21 \cdot 5 \cdot 11\); as above, it suffices to show the first 288 coefficients are even, equivalently that the multiplicities of both 1 and \(\rho_1\) are even in all head characters.
$H_n$ for $n \leq 288$. This has been done in the aforementioned computer checks. QED to Theorem 2

Table 4. Data for Theorem B proof

| $\rho$ | 1 | $\rho_3$ | $\rho_5$ | $\rho_6$ | $\rho_9$ | $\rho_{11}$ | $\rho_{12}$ | $\rho_{13}$ | $\rho_{14}$ | $\rho_{15}$ | $\rho_{16}$ | $\rho_{17}$ | $\rho_{18}$ | $\rho_{20}$ |
|-------|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $N_\rho$ | 288 | 288 | 48 | 288 | 90 | 48 | 48 | 288 | 144 | 288 | 72 | 6 | 90 | 48 | 48 |

$m_{\rho}/6$

| 288 |

| 288 |

| 288 |

| 48 |

| 288 |

| 90 |

| 48 |

| 48 |

| 288 |

| 144 |

| 288 |

| 72 |

| 6 |

| 90 |

| 48 |

| 48 |

| 288 |

| 48 |

| 48 |

4 Weak Mathieu Moonshine II: Positivity

In this section we prove that for each $n > 0$ and each irreducible $\rho \in \hat{M}_{24}$, the multiplicities $\text{mult}_{H_n}(\rho)$ are nonnegative. The difficult part of this positivity proof is effectively bounding a certain series (what we call $Z_{n,h}(3/4)$ below) which does not converge absolutely. There are (at least) two approaches for this: interpreting this as a Selberg-Kloosterman zeta function as in [19], [33], etc; or interpreting this as a sum over equivalence classes of quadratic forms as in Hooley [37]. The former method was used by [6] to prove convergence for the Rademacher sum expressions for the mock modular forms $q^{-1/8} \sum_{n=0}^{\infty} H_n(g)q^n$; the latter method (or rather its recent reincarnation [4]) was suggested in [19] as a way to prove convergence for $g = 1$. We need much more than convergence: we need an explicit bound, and for this we have found the second method more useful.

Let’s begin with an elementary observation. By the triangle inequality,

$$\text{mult}_{H_k}(\rho) = \sum_g \frac{H_k(g)}{|C_{M_{24}}(g)|} \rho(g) \geq \frac{H_k(1)}{|M_{24}|} \rho(1) - \sum_{g \neq 1} \frac{|H_k(g)|}{|C_{M_{24}}(g)|} |\rho(g)| \quad (4.1)$$

where the sum is over all conjugacy classes of $M_{24}$. Since there always is the trivial bound $\rho(1) \geq |\rho(g)|$ (which itself follows immediately from the triangle inequality), our strategy to show $\text{mult}_{H_k}(\rho) > 0$ is to show $H_k(1)$ is much larger in modulus than the other character values $H_k(g)$, at least for $k$ sufficiently large.

Choose any $g \in M_{24}$. Then from Lemma 1 we read that the multiplier of $f_g$ is $\rho_{n,h}$, where $\rho_{n,h}$ sends $(a,b \ c \ d) \in \Gamma_0(n)$ to $e\left(\frac{ad}{mn}\right)$. The multiplier $\epsilon$ of $\eta$ sends $(a,b \ c \ d) \in \text{SL}_2(\mathbb{Z})$ (with $c > 0$) to

$$\frac{1}{\sqrt{2}} e^{-\omega_{d,c} \epsilon} \left(\frac{a + d}{24c}\right), \quad (4.2)$$

where

$$\omega_{d,c} = \prod_{\mu=1}^{k} \exp(\pi i \left(\frac{h\mu}{k}\right) \left(\frac{\mu}{k}\right))$$

$$\omega_{d,c} = \left\{ \begin{array}{ll}
\frac{d}{c} e^{-\frac{1}{8} c(c-1) + \frac{1}{24}(c - \frac{1}{2})(2d + d' - d^2d')} & \text{if } c \text{ is odd} \\
\frac{d}{c} e^{-\frac{1}{8} (2 - cd - d) + \frac{1}{24}(c - \frac{1}{2})(2d + d' - d^2d')} & \text{if } c \text{ is even}.
\end{array} \right. \quad (4.3)$$
Here, \((x) = x - \lfloor x \rfloor - 1/2\) unless \(x \in \mathbb{Z}\) in which case \((x) = 0\). We write here \(d'\) for any solution to \(dd' \equiv c \mod 1\). \((\frac{d}{c})\) denotes the Jacobi symbol. The first expression for \(\omega_{d,c}\) is the most familiar; the second (due to Rademacher [45]) is more useful for us.

We learn in eq.(6.1) of [6] (see also [19] for the special case \(g = 1\)) that

\[
H_k(g) = \frac{4\pi}{(8k - 1)^{1/4}} \sum_{c=1}^{\lfloor L \rfloor} \frac{1}{|g|} I_{1/2} \left( \frac{\pi}{2c |g|} \sqrt{8k - 1} \right) S(k, |g| c, \epsilon^{-3} \rho_{|g|}; n, h) ,
\]

where \(I_{1/2}(x) = \sqrt{\frac{2x}{\pi}} \sinh(x)\) and \(S\) is (up to a constant) a generalised Kloosterman sum for \(\Gamma_0(|g|)\):

\[
S(k, nc, \epsilon^{-3} \rho_{n,h}) = \sum_{0 < d < nc \atop \gcd(d, nc) = 1} \omega_{d,nc}^{-3} e \left( \frac{-cd}{h} \right) e \left( \frac{k d}{nc} \right) .
\]

Fix integers \(n, h, k > 0\) and define \(L = \frac{n}{\pi} \sqrt{2k - 1}\) (so \(\pi \sqrt{2k - 1}/(2nc) < 1\) if \(c > L\)). We have the elementary bounds:

\[
\left| I_{1/2}(x) - \sqrt{\frac{2x}{\pi}} \right| \leq 1 \sqrt{\frac{2x^2}{\pi}} \quad \text{for } 0 < x < 1 ;
\]

\[
|I_{1/2}(x)| < \frac{e^x}{\sqrt{2\pi x}} \quad \text{for all } 0 < x ;
\]

\[
|S(k, nc, \epsilon^{-3} \rho_{n,h})| \leq 1 \quad \text{for all } c \in \mathbb{Z}_{>0}
\]

(in fact Lemma 6 below will give us the Weil bound \(|S| \leq O(\sqrt{c})\), but (4.8) is adequate). These inequalities immediately imply the crude bounds

\[
\left| \frac{1}{(8k - 1)^{1/4}} \sum_{c=2}^{\lfloor L \rfloor} \frac{1}{c} I_{1/2} \left( \frac{\pi \sqrt{8k - 1}}{2nc} \right) S(k, nc, \epsilon^{-3} \rho_{n,h}) \right|
\leq \frac{\sqrt{n}}{\pi \sqrt{8k - 1}} \sum_{c=2}^{\lfloor L \rfloor} \frac{1}{c} e^{\pi \sqrt{8k - 1}/(2nc)} < \frac{1}{\sqrt{2n}} e^{\pi \sqrt{8k - 1}/(4n)} ,
\]

\[
\left| \frac{1}{(8k - 1)^{1/4}} \sum_{c=\lfloor L \rfloor}^{\infty} \frac{1}{nc} I_{1/2} \left( \frac{\pi \sqrt{8k - 1}}{2nc} \right) S(k, nc, \epsilon^{-3} \rho_{n,h}) \right|
\leq \sum_{c=\lfloor L \rfloor}^{\infty} (nc)^{-3/2} S(k, nc, \epsilon^{-3} \rho_{n,h}) + \frac{\pi^2 (8k - 1)}{20n^{7/2}} \sum_{c=\lfloor L \rfloor}^{\infty} c^{-7/2}
\leq |Z_{n,h}(3/4)| + \frac{\pi \sqrt{8k - 1}}{n^{5/2}} + \frac{7\pi^2 (8k - 1)}{80n^{7/2}} ,
\]

where \(Z_{n,h}\) is the Selberg-Kloosterman zeta function

\[
Z_{n,h}(s) = \sum_{c=1}^{\infty} S(k, nc, \epsilon^{-3} \rho_{n,h}) (nc)^{-2s} .
\]
Thus it suffices to find an effective bound for a Selberg-Kloosterman zeta function at $s = 3/4$. We know from [6] that the defining series (4.11) converges there, but it follows from Lemma 4 below that this convergence is not absolute, and in any event it seems difficult to use the analysis of [6] to obtain an explicit bound (as a function in $k$). The key first step in identifying such an effective bound is to rewrite these generalised Kloosterman sums more sparsely. Our calculation resembles that of [51]; a more elegant approach attributed to Selberg and worked out by Rademacher [46] is available but we were unable to generalise it to our context. Incidentally, Lemma 4 should imply that these $S$’s have some multiplicative properties (hence their $Z$’s have Euler-like products).

Lemma 4. Let $c \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}$, $h | n$. Then

$$S(k, nc, e^{-3} \rho_{n,h}) = \frac{-i \sqrt{nc}}{2} \sum_{0 \leq m < 4nc \atop m^2 \equiv 8cn^2 \pm 8c^2n/h} e\left(\frac{m}{4nc}\right).$$  (4.12)

Proof. We will begin with the proof of $n = h = 1$. Writing $m = 2\ell + 1$, define

$$B_c(k) := \frac{-i \sqrt{c}}{2} \sum_{m = 0 \atop m^2 \equiv 8c^{1 - sk}}^{4c-1} (-1)^{(m-1)/2} e\left(\frac{m}{4c}\right) = \frac{-i \sqrt{c}}{2} e\left(\frac{1}{4c}\right) \sum_{\ell = 0 \atop \ell^2 + \ell \equiv 2c - 2k}^{2c-1} (-1)\ell e\left(\frac{\ell}{2c}\right).$$

As we manifestly have $B_c(k + c) = B_c(k)$ (i.e. $B_c$ is a class function for $\mathbb{Z}_c$), we can formally write it as a combination of irreducible $\mathbb{Z}_c$-characters:

$$B_c(k) = \sum_{d = 0}^{c-1} R_{d,c} e\left(\frac{dk}{c}\right),$$  (4.13)

for coefficients

$$R_{d,c} = \frac{-i}{2 \sqrt{c}} e\left(\frac{1}{4c}\right) \sum_{j = 0}^{c-1} e\left(-\frac{dj}{c}\right) \sum_{\ell = 0 \atop \ell^2 + \ell \equiv 2c - 2k}^{2c-1} (-1)^\ell e\left(\frac{\ell}{2c}\right)$$

$$= \frac{-i}{2 \sqrt{c}} e\left(\frac{1}{4c}\right) \sum_{\ell = 0}^{2c-1} (-1)^\ell e\left(\frac{d(\ell^2 + \ell)}{2c} + \frac{\ell}{2c}\right).$$  (4.14)

It is elementary to show that the sum on the right-side of (4.14) vanishes when $\gcd(c, d) > 1$: write $m = \gcd(c, d)$ and $\ell = s2c/m + r$, so $\sum_{\ell = 0}^{2c-1} = \sum_{r = 0}^{2c/m-1} \sum_{s = 0}^{m-1}$, and notice that $\sum_{s} = 0$ for each $r$ (when $m > 1$).

Thus we can restrict to $\gcd(c, d) = 1$. We want to show $R_{d,c} = \omega_{d,c}^{-3}$. Choose $d' \in \mathbb{Z}$ so that $1 = e\left(\frac{dd' - 1(c+1)}{2c}\right)$; this permits us to rewrite (4.14) as

$$R_{d,c} = \frac{-ie(1/4c)}{2 \sqrt{c}} G(d, d; 2c)$$  (4.15)
where $\gamma := d'c + d' + 1$ and $G(a, b; c)$ is the generalised Gauss sum $G(a, b; c) = \sum_{\ell=0}^{c} e((\ell a^2 + b\ell)/c)$. Note that $\gamma := d'c + d' + 1$ is even iff $c$ is even. When $c$ is even, we can complete squares and obtain

$$R_{d,c} = \frac{-i e(1/4c)}{2\sqrt{c}} e \left( \frac{-d\gamma^2}{8c} \right) G(d, 0; 2c)$$

$$= \left( \frac{2c}{d} \right) e \left( \frac{-(d-1)^2}{16} + \frac{2 - c - d\gamma^2}{8c} \right). \tag{4.16}$$

When $c$ is odd, use $\frac{1}{2c} = \frac{1}{c} + \frac{(c+1)/2}{c}$ to write $G(d, d\gamma; 2c) = G(-d, -d\gamma; 2) G(d(c + 1)/2, d\gamma(c + 1)/2; c)$; the left generalised Gauss sum equals 2, while the right is evaluated by completing squares as usual, and we obtain (for $c$ odd)

$$R_{d,c} = \left( \frac{d(c + 1)/2}{c} \right) e \left( \frac{(c - 1)^2}{16} + \frac{2 - 2c - d(c + 1)(d' + 1)^2}{8c} \right). \tag{4.17}$$

Now, for $m$ odd we have $(\frac{2}{m}) = (-1)^{(m^2 - 1)/8}$ and $(\frac{1}{m}) = (-1)^{(m^2 - 1)/2}$. Consider first $c$ even. Then directly from (4.3) and (4.16) we obtain

$$\frac{\omega_{d,c}^{-3}}{R_{d,c}} = e \left( \frac{-2 - 2c - d - d' - 2cd + 2dd' - c^2d + c^2d' + d^2 + dd'^2 + 2cdd'^2 + c^2dd'^2 - c^2d^2d'}{8c} \right). \tag{4.18}$$

Because $dd' \equiv 2c \pmod{4}$, when $c$ is even we can define an integer $L$ by $dd' = 1 + 2cL$, and we know that $d \equiv 4d'$. Then (4.18) collapses to

$$\frac{\omega_{d,c}^{-3}}{R_{d,c}} = e \left( \frac{dL + d'L + 2L}{4} \right) = 1, \tag{4.19}$$

as desired. The proof for $c$ odd is similar: the Jacobi symbol $(\frac{d(c+1)/2}{c})$ equals $(\frac{2d}{c})$; define $L$ by $dd' = 1 + cL$ and use the congruences $c^3 \equiv 8c$ $c$ and $4c^2 \equiv 8c$ $4c$.

Replacing $c$ everywhere with $nc$ establishes the $h = 1$ case of Lemma 4. Arbitrary $h|n$ is handled through the elementary observation that $S(k, nc, \epsilon^{-3}n\epsilon) = S(k - c^2n/h, nc, \epsilon^{-3}n\epsilon)$. QED to Lemma 4

Hooley’s method is based on the $n = 1$ case of Lemma 5 below; it was more recently modified slightly by [4]. We need a more serious revision. A binary quadratic form $Q(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ is called integral if $\alpha, \beta, \gamma \in \mathbb{Z}$, and positive-definite if $\alpha$ and $\gamma$ are both positive. (See e.g. Chapter 12 of [38] for a rather complete introduction to the basic theory of quadratic forms, as is relevant here.) The discriminant is $\beta^2 - 4\alpha\gamma$. For any $C \in \mathbb{Z}_{>0}$ and $D \in \mathbb{Z}_{<0}$, let $\mathcal{Q}(C, D)$ denote the set of all triples $(Q; r, s)$, where $Q$ is integral and positive-definite with discriminant $D$ and where $r, s$ are coprime integers satisfying $Q(r, s) = C$. Any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ acts on $(Q; r, s) \in \mathcal{Q}(C, D)$ by sending $Q(x, y)$ to $Q(ax + by, cx + dy)$ and $(r, s)$ to $(dr - bs, -cr + as)$. In particular, it is easy to verify that if $(Q; r, s)$ and $(Q'; \tilde{r}, \tilde{s})$ are in
the same $\text{SL}_2(\mathbb{Z})$-orbit then $Q, Q'$ have the same discriminant and $Q(r, s) = Q'(\tilde{r}, \tilde{s})$. Then Hooley observes that there is a bijection between the integers $0 \leq m < 2C'$ satisfying $m^2 \equiv 4C \pmod{1}$, and $\text{SL}_2(\mathbb{Z})$-orbits in $Q(C, D)$.

We need to generalise this in two ways. First, choose any integer $n \geq 1$. Write $Q_n(C, D)$ denote the set of all triples $(Q; r, ns)$, where $Q = [n\alpha, \beta, \gamma]$ is positive-definite and of discriminant $D$, $\alpha, \beta, \gamma, r, s \in \mathbb{Z}$, $\text{gcd}(r, ns) = 1$, and $Q(r, ns) = nC$. For example, $Q(C, D) = Q_n(C, D)$. It is elementary to show that the group $\Gamma_0(n)$ acts on $Q_n(C, D)$.

Secondly, suppose that $h$ divides $\text{gcd}(n, 24)$ and that $\text{gcd}(n/h - 1, h) = 1$. Write $n' = n/h$. Let $Q_{n,h}(C, D)$ denote the set of all triples $(Q; r, ns)$ where $Q = [n\alpha, \beta, \gamma/h]$ is positive-definite and of discriminant $D$, $\alpha, \beta, \gamma, r, s \in \mathbb{Z}$, $\gamma \equiv_h \alpha$, $\text{gcd}(r, n's) = 1$, and $Q(r, ns) = C$. Of course $Q_{n,1}(C, D) = Q_n(C, D)$. Recall the group $\Gamma_0(n|h)$, which can be defined equivalently as either the conjugate of $\Gamma_0(n')$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or as the set of all determinant-1 matrices of the form $\begin{pmatrix} a & b/h \\ n & d \end{pmatrix}$ for $a, b, c, d \in \mathbb{Z}$. Let $\Gamma_0(n; h)$ denote the set of all $\begin{pmatrix} a & b/h \\ nc & d \end{pmatrix} \in \Gamma_0(n|h)$ for which $ac \equiv_h bd$. Then it is easy to verify $\Gamma_0(n; h)$ is a group, and we see shortly that it acts on $Q_{n,h}(C, D)$.

The reason 24 arises here (and elsewhere) is because, for any divisor $d$ of 24, any integer $\ell$ coprime to $d$ satisfies $\ell^2 \equiv_d 1$. The defining condition $ac \equiv_h bd$ of $\Gamma_0(n; h)$ is equivalent to requiring that there is some $\ell$ (depending on $a, b, c, d$) coprime to $h$ with $a \equiv_h \ell d$ and $b \equiv_h \ell$. To see this equivalence, run through each prime power $p^r$ exactly dividing $h$; the determinant condition $ad - n'bc = 1$ tells us that either $a$ and $d$ are both coprime to $h$ (in which case require $\ell \equiv (ad \pmod{p^r})$, or $b, c$ are both coprime to $h$ (in which case take $\ell \equiv bc \pmod{p^r}$). Then $a \equiv_{p^r} \ell d$ resp. $b \equiv_{p^r} \ell c$, and the other congruence comes from $ac \equiv_{p^r} bd$. Running through all $p$, we obtain an $\ell$ defined mod $h$ which has all the desired properties.

**Lemma 5(a)** Let $D \in \mathbb{Z}_{<0}$, $C \in \mathbb{Z}_{>0}$. There is a one-to-one correspondence between the set of integers $m$, $0 \leq m < 2nC$, satisfying $m^2 \equiv D \pmod{4nC}$, and $\Gamma_0(n)$-orbits in $Q_n(C, D)$.

(b) There is a one-to-one correspondence between the set of integers $m$, $0 \leq m < 2nC$, satisfying $m^2 \equiv D + 4C^2n' \pmod{4nC}$, and $\Gamma_0(n; h)$-orbits in $Q_{n,h}(C, D)$.

**Proof.** Let’s begin with the simpler part (a). First note that there is an obvious bijection between each $m \in \mathbb{Z}$ with $m^2 \equiv 4nC \pmod{D}$, and each quadratic form $Q = [nC, m, \gamma]$ (necessarily positive-definite) of discriminant $D$, where $\gamma = (m^2 - D)/(4nC)$.

Given any $(Q; r, ns) \in Q_n(C, D)$, $Q = [n\alpha, \beta, \gamma]$, choose any $\tilde{r}, \tilde{s}$ such that $\begin{pmatrix} r & \tilde{r} \\ ns & \tilde{s} \end{pmatrix} \in \Gamma_0(n)$ (this is possible since $\text{gcd}(r, ns) = 1$), and define the root $m := 2mnr\tilde{r} + \beta(r\tilde{s} + ns\tilde{r}) + 2\gamma ns\tilde{s}$. Then $\begin{pmatrix} r & \tilde{r} \\ ns & \tilde{s} \end{pmatrix} \in \Gamma_0(n)$ sends $(Q; r, ns)$ to $([nC, m, \gamma']; 1, 0)$ where $\gamma' = (m^2 - D)/(4nC)$. The other choices of $\tilde{r}, \tilde{s}$ are $\tilde{r} + Lr, \tilde{s} + Lns$ for any $L \in \mathbb{Z}$, which correspond to roots $m + L2nC$, and so the root taken mod $2nC$ is a well-defined function $m(Q; r, ns) \in \mathbb{Z}_{2nC}$ on $Q_n(C, D)$. The
desired bijection in part (a) is this root map (mod \(2nC\)).

Now turn to part (b). Here, there is an elementary bijection between each \(m \in \mathbb{Z}\) with \(m^2 \equiv_{4nC} D + 4n'C^2\), and each positive-definite quadratic form \(Q = [nC, m, \gamma/h]\) of discriminant \(D\), where \(\gamma \equiv_h C\). Choose any \(([na, \beta, \gamma/h]; r, ns) \in Q_{n,h}(C, D)\).

Then \(\left(\frac{a/b/f}{nc/d}\right) \in \Gamma_0(n|h)\) will send \((r, ns)\) to \((ra + nsb, nrc + nsd)\) (the desired form), and \([na, \beta, \gamma/h]\) to

\[n(\alpha a^2 + \beta ac + n'\gamma c^2), 2n'aab + \beta (ad + n'bc) + 2n'c'd, (n'ab^2 + \beta bd + \gamma d^2)/h\]. \(4.20\)

Then \(4.20\) will lie in \(Q_{n,h}(C, D)\) if

\[a^2 \equiv_h d^2, \quad ac \equiv_h bd, \quad n'c^2 \equiv_h n'b^2.\] \(4.21\)

As explained above, the condition \(ac \equiv_h bd\) is equivalent to the existence of an \(\ell\) coprime to \(h\) satisfying \(a \equiv_h \ell d\) and \(b \equiv_h \ell c\), and such an \(\ell\) forces the other two congruences to be satisfied. In other words, the matrices in \(\Gamma_0(n|h)\) satisfying \(4.21\) form the group \(\Gamma_0(n; h)\).

Since \(r\) and \(n's\) are coprime, we can find integers \(\tilde{r}, \tilde{s}\) such that \(r\tilde{s} - n's\tilde{r} = 1\). We claim there is some \(L \in \mathbb{Z}\), unique modulo \(h\), such that \(\left(\frac{r (Lr+\tilde{r})/h}{ns \ Ln's+r+s/h}\right) \in \Gamma_0(n; h)\).

To see this, choose any prime power \(p^e\) exactly dividing \(h\). If \(r\) is coprime to \(p\) choose \(L \equiv_{p^e} 1 - ns'\tilde{s}\) and \(L \equiv_{p^e} r'(-\tilde{r} + \ell s)\) where \(r' r \equiv_{p^e} 1\); otherwise, \(n's\) will be coprime to \(p\), so choose \(s'\) by \(s'n's \equiv_{p^e} 1, \ell p^e\tilde{r} - n'\ell, \) and \(L \equiv_{p^e} s'(-s + \ell r)\).

We are using here that \(1 - n'\) is coprime to \(h\) and hence \(p\). Then \(\ell\) and \(L\) are defined mod \(h\) by running through all \(p\). To see uniqueness of \(L\) modulo \(h\), note that 

\[
\left(\frac{r (Lr+\tilde{r})/h}{ns \ Ln's+r+s/h}\right) = \left(\frac{r (Lr+\tilde{r})/h}{ns \ Ln's+r+s/h}\right)^{-1} = \left(\frac{1 (L-L')/h}{1}ight) \in \Gamma_0(n; h)\
\]

forces \((L - L')/h \in \mathbb{Z}\) by definition of \(\Gamma_0(n; h)\).

So we may assume \(\left(\frac{r \tilde{r}/h}{ns \ \tilde{s}/h}\right) \in \Gamma_0(n; h)\). Define the root \(m := 2n'\alpha r\tilde{r} + \beta (r\tilde{s} + n's\tilde{r}) + 2n'c'd\tilde{s}\) as before. Then \(\left(\frac{r \tilde{r}/h}{ns \ \tilde{s}/h}\right)\) sends \((Q; r, ns)\) to \(([nC, m, \gamma']; 1, 0) \in Q_{n,d}(C, D)\) where \(\gamma' = (m^2 - D)/(4nC)\). From the uniqueness of the previous paragraph, we have that \(\left(\frac{r \ (Lr+\tilde{r})/h}{ns \ Ln's+r+s/h}\right) \in \Gamma_0(n; h)\) iff \(L \in h\mathbb{Z}\), which corresponds to root \(m + L2nC\), and so again the root taken mod \(2nC\) is a well-defined function \(m(Q; r, ns) \in \mathbb{Z}_{2nC}\) on \(Q_{n,d}(C, D)\). **QED to Lemma 5**

We are interested in \(D = 1 - 8k, n = 2n_g, h = h_g\) and \(C = c\). Restrict attention here to \(k \geq 1\) (not a problem since we are only interested in large \(k\)), so \(D < 0\). In fact, to sharpen slightly some of our bounds, we’ll choose \(k \geq 5\). Continue to write \(n' = n/h\).

Note that if both \((Q; r, ns), (Q; r, ns)\) (same \(Q\)) lie in \(Q_{n,k}(C, D)\), then there is an automorphism \(g \in \Gamma_0(n; k)\) of \(Q\) sending \((r, ns)\) to \((\tilde{r}, n\tilde{s})\). For \(D < 0\) (the case we are interested in), each such automorphism \(g\) must have finite order (since completing squares in \(\alpha = a\alpha^2 + \beta ac + n'\gamma c^2\) and \(\gamma = n'\alpha b^2 + \beta bd + \gamma d^2\) bounds the matrix entries \(a, b, c, d\) of \(g\)). Now, \(\Gamma_0(n|h)\) (which contains \(\Gamma_0(n; h)\)) is conjugate to
\(\Gamma_0(n')\), so all of its elements of finite order have orders 2, 4, 6. In fact for us, the stabiliser \(\text{stab}([n\alpha, \beta, \gamma/h]; \Gamma_0(n/h))\) of \([n\alpha, \beta, \gamma/h]\) in \(\Gamma_0(n/h)\) will always be \(\pm 1\). To see this, first note \(\text{stab}([n\alpha, \beta, \gamma/h]; \Gamma_0(n/h))\) is a subgroup of \(\text{stab}([n\alpha, \beta, \gamma/h]; \Gamma_0(n/h))\), which is isomorphic to \(\text{stab}([n'\alpha, \beta, \gamma]; \Gamma_0(n/h))\), which is in turn a subgroup of \(\text{stab}([n'\alpha, \beta, \gamma]; \text{SL}_2(\mathbb{Z}))\). For a primitive form \([n'\alpha, \beta, \gamma]\) (i.e. when \(n'\alpha, \beta, \gamma\) don’t have a common factor), nontrivial stabilisers occur only for discriminant \(D = -3, -4\). But our discriminant satisfies \(D \equiv 1 \pmod{8}\), so even if \([n'\alpha, \beta, \gamma]\) is imprimitive, it can never have a nontrivial stabiliser in \(\text{SL}_2(\mathbb{Z}).\)

We also need a bound on \(\Gamma_0(n/h)\)-equivalence class representatives \([n\alpha, \beta, \gamma/h]\). For this purpose, observe that \(\Gamma_0(n/h)\) contains \(\Gamma_0(nh)\), which has finite index \(nh\prod_p (1 + 1/p)\) in \(\text{SL}_2(\mathbb{Z})\). The cosets for \(\Gamma_0(N)\setminus\text{SL}_2(\mathbb{Z})\) are in bijection with pairs \((c, d)\) \in \(\mathbb{Z}_0^2\) where \(c|N\) and \(1 \leq d \leq N/c\) satisfies \(\gcd(d, c, N/c) = 1\). To any such pair \((c, d)\), a representative of that coset is \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\) where \(d'\in\mathbb{Z}_0\) \(d\) and \(a, b\) are any integers satisfying \(ad' - bc = 1\). Now, \(c = N\) corresponds to the identity coset, so for it take \(c = 0\) instead; then in all cases we have \(c \leq N/2\). We can always choose \(|a| \leq c/2\) by adjusting \(b\) appropriately (at least when \(c \neq 0\)), hence \(|a| \leq N/4\) (true even for \(c = 0\) unless \(N < 4\)). Now, over \(\text{SL}_2(\mathbb{Z})\), any positive-definite discriminant \(D\) quadratic form (with coefficients in \(h^{-1}\mathbb{Z}\)) is equivalent to some \([\alpha', \beta', \gamma']\) (namely Gauss’ reduced form) with \(|\beta'| \leq \alpha' \leq \gamma'\), where \(\alpha' \leq \sqrt{|D|}/3\) and \(\gamma' \leq 11h|D|/39\). Combining with (4.24), and noting that we can always force \(|\beta| \leq n\alpha\) by applying multiples of \((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})\in\Gamma_0(nh)\), we see that any such quadratic form is equivalent over \(\Gamma_0(nh)\), hence over \(\Gamma_0(n/h)\), to some \([n\alpha, \beta, \gamma/h]\) satisfying the (crude but adequate) bounds

\[
0 < \alpha \leq \frac{n^2h^2}{16n} \left(3\sqrt{|D|/3} + 44h|D|/39\right) < \frac{1}{12}nh^3|D|, \tag{4.22}
\]

\[
|\beta| < n^2h^3|D|/12, \tag{4.23}
\]

\[
0 < \gamma \leq (|D| + \beta^2)/(4n'^2) \leq \left\{ \begin{array}{ll}
|D|/7 & \text{if } n = 2 \\
2n'^4|D|/44 & \text{otherwise.} \end{array} \right. \tag{4.24}
\]

(The given bound on \(\alpha\), hence \(\beta\), is also true for the identity coset \(c = 0\).) Implicit in these derivations are 2\(h \leq n\), \(n\) even, and \(|D| \geq 39\).

The point is that the series \(Z_n,h(3/4)\) can be rewritten as a sum over the finitely many \(\Gamma_0(n/h)-\)orbit representatives \([n\alpha, \beta, \gamma/h]\), and over integers \(r, s\). Over-counting by a factor of 2, we can take this to be a free sum over \(r, s\).

**Theorem 3.** Choose any \(k, n \in \mathbb{Z}_{>0}, k \geq 5,\) and any \(h\) dividing \(\gcd(n, 24)\) such that \(\gcd(n/h - 1, 2h) = 1\). Then for \(n \neq 2\),

\[
|Z_{n/2,h}(3/4)| \leq \left( \prod_{p|nh} \frac{p + 1}{p} \right) \left(1 + 2.13|D|^{1/8} \log |D| \times \left( (6.124n^{-35/6}h^{47/6} - 3.09n^{23/4}h^{31/4} + 64.32n^{29/6}h^{7} - 23n^{10/4}h^{7})|D| \right. \right.
\]

\[
\left. + (.146n^{47/6}h^{56/6} - .114n^{31/4}h^{33/4} + 2.51n^{35/6}h^{10} - .74n^{23/4}h^{10})|D|^{3/2} \right). \]
where we write \( D = 1 - 8k \). For \( n = 2 \), this bound on the right-side should be replaced with \((3872|D| + 213|D|^{3/2})(1 + 2.13|D|^{1/8} \log |D|)\).

**Proof.** Note the \( n \) in the statement of Theorem 3 agrees with that of Lemma 5, but is twice that of Lemma 4. We need to bound the limit as \( X \to \infty \) of

\[
Z_{n/2,h}(s; X) = \sum_{c=1}^{X} S(k, nc, e^{-3\rho_{n/2,h}}) (nc/2)^{-2s}.
\]

(4.25)

Thanks to Lemmas 4 and 5, we can write its value at \( s = 3/4 \) as

\[
-\frac{1}{2} \sum_{[n,\beta,\gamma/h]} \sum' \frac{(-1)^{(m(\alpha,\beta,\gamma;r,s)-1)/2}}{nc(\alpha,\beta,\gamma;r,s)} e\left(\frac{m(\alpha,\beta,\gamma;r,s)}{2n c(\alpha,\beta,\gamma;r,s)}\right),
\]

(4.26)

where the first sum is over representatives of the \( \Gamma_0(n;h) \)-orbits of positive-definite quadratic forms of discriminant \( D = 1 - 8k \), and the second sum is over all integers \( r, s \) satisfying \( \gcd(r, n's) = 1 \) and the given inequality (the prime on the sum denotes that coprime condition). The quantities \( m, c \) are defined by

\[
m(\alpha, \beta, \gamma; r, s) = 2n'\alpha r\tilde{r} + \beta (r\tilde{s} + n's\tilde{r}) + 2\gamma n's\tilde{s}, \quad c(\alpha, \beta, \gamma; r, s) = \alpha r^2 + \beta rs + n'\gamma s^2,
\]

where \( \tilde{r}, \tilde{s} \) are any integers satisfying \( \left(\frac{r}{ns}, \frac{\tilde{r}}{\tilde{s}}\right) \in \Gamma_0(n; h) \) — which pair is chosen won’t affect the value of that summand. The factor of 2 in (4.26) comes from the aforementioned redundancy that different \((r, ns)\) can lie in the same orbit.

Write this inner sum \( \sum_{r,s} \) (for each choice of \( \alpha, \beta, \gamma \)) as \( \sum_+ + \sum_- \), depending on whether or not \(|r| > n|s|\) \((|r| = n|s|) \) would contradict \( \gcd(r, s) = 1 \). We will bound \( Z_{n/2,h}(3/4; X) \) by considering separately the contributions of \( \sum_+ \), \( \sum_- \).

The arguments for \( \sum_+ \) and \( \sum_- \) are completely analogous, and so we will consider in detail only \( \sum_- \). Since then \( r > 0 \), we can write

\[
m(\alpha, \beta, \gamma; r, s) = \frac{2n'\tilde{r}(\alpha r^2 + \beta rs + n'\gamma s^2) + 2n'\gamma s + \beta r}{2nr (\alpha r^2 + \beta rs + n'\gamma s^2)} = \frac{us'}{R} + \frac{2n'\gamma s + \beta r}{2nr (\alpha r^2 + \beta rs + n'\gamma s^2)},
\]

(4.27)

where \( \delta = \gcd(s, h) \), \( h' = h/\delta \), \( R = rh' \), \( s' \in \mathbb{Z} \) is any inverse of \( s/\delta \) mod \( R \), and \( u \in \mathbb{Z} \) is defined mod \( R \) by \( u \equiv_r -\delta^{-1}(n')^{-1}, u \equiv_{h'} 1 - s'n' \), and \( 3|\gcd(r, h') \)

\( u \equiv_9 -(n')^{-1} \) (note that \( \gcd(r, h) \neq 1 \) implies it equals 3, in which case \( s \) is coprime to 3 and \( n' \equiv_3 -1 \)). Writing

\[
\alpha r^2 + \beta rs + n'\gamma s^2 = n'\gamma (s + \frac{\beta r}{2n'\gamma})^2 - \frac{Dr^2}{4n'\gamma} = \alpha (r + \frac{\beta s}{2\alpha})^2 - \frac{Ds^2}{4\alpha},
\]

(4.28)

the first inequality gives

\[
|r| \leq \sqrt{\frac{X4n'\gamma}{|D|}} = C\sqrt{X}, \quad S_- \leq s \leq S_+,
\]

(4.29)
where $C = \sqrt{4n'\gamma/|D|}$ and

\begin{align*}
S_+ &= \min \left\{ \frac{|r|}{n}, -\frac{\beta r}{2n'\gamma} + \frac{1}{2n'\gamma} \sqrt{4n'\gamma X + Dr^2} \right\}, \quad (4.30) \\
S_- &= \max \left\{ -\frac{|r|}{n}, -\frac{\beta r}{2n'\gamma} - \frac{1}{2n'\gamma} \sqrt{4n'\gamma X + Dr^2} \right\}. \quad (4.31)
\end{align*}

We compute

\[ (-1)^m = (-1)^{n'(\alpha r\tilde{\tau} + \beta s\tilde{s} + \gamma s\tilde{\tau}) + (\beta - 1)/2} = (-1)^{(\beta - 1)/2} \quad (4.32) \]

using the evenness of $n'$ and the discriminant relation $r\tilde{\tau} - n's\tilde{\tau} = 1$ ($\beta$ is odd because of the discriminant condition $\beta^2 - 4n'\alpha\gamma = 1 - 8k$). Now put

\[ \varphi(r, s) = \frac{(-1)^{(\beta - 1)/2}}{\alpha r^2 + \beta rs + \gamma n's^2} e \left( \frac{2n'\gamma s + \beta r}{2nr(\alpha r^2 + \beta rs + \gamma n's^2)} \right); \quad (4.33) \]

we need to bound

\[ \sum > \sum_{|r| \leq CVX} \sum_{\delta h} \sum' e \left( \frac{us'}{hr/\delta} \right) \varphi(r, s) = \sum_{|r| \leq CVX} \sum_{\delta h} \sum' e \left( \frac{us'}{R} \right) \varphi(r, \delta S), \quad (4.34) \]

where we use (4.27), we write $R = hr/\delta$ as before and $S' = s'$ is any inverse of $S$ mod $R$. Rewrite the sum over $S$ in (4.34) (for fixed $\alpha, \beta, \gamma, r, \delta$) using partial summation (the discrete analogue of integration by parts):

\[ \sum' e \left( \frac{us'}{R} \right) \varphi(r, \delta S) = \sum\left[ S'/\delta \right] g(\sigma) \left( \varphi(r, \delta \sigma) - \varphi(r, \delta \sigma + \delta) \right) + g([S'/\delta]) \varphi(r, \delta \left[ S'/\delta \right]), \quad (4.35) \]

where

\[ g(\sigma) = \sum' e(uS'/R) \quad (4.36) \]

denotes the incomplete Kloosterman sums ($R$ is implicit). We have for ($n > 2$)

\[ |\varphi(r, s) - \varphi(r, s + \delta)| \leq \frac{16n'^2\gamma^2\delta}{D^2r^4} \left( \frac{|r|}{2} \sqrt{|D|} + n'\gamma\delta \right) + \frac{2\pi(4n'\gamma)^2}{2nr} \left( \frac{2}{|D|^2} \frac{2}{2r|\sqrt{|D|}}, \right. \]

\[ \leq .0042 \frac{n'^2h^2}{|r|^3} \delta \sqrt{|D|} + .000188 \frac{n'^2h^2}{r^4} \delta^2 |D| + .0083 \frac{n'^2h^2}{r^4} \sqrt{|D|}, \quad (4.37) \]

\[ |\varphi(r, \delta [S'/\delta])| \leq 4n'/\gamma/(|D|r^2) \leq .091n'^3h^3r^{-2}, \quad (4.38) \]

where we use repeatedly (4.28), as well as Taylor’s remainder $|e(\theta) - 1| \leq 2\pi\theta_0$ for $|\theta| \leq \theta_0$, and the elementary bound $x/(x^2 + a^2) \leq 1/(2|a|2)$. Thus all that remains is
to bound \( g(\sigma) \). The order of its growth with \(|R|\) is stated (without proof) in Lemma 3 of [37]; an effective bound is:

**Lemma 6.** For any integers \( k, u, h_1, h_2 \) with \( h_1 \leq h_2 \), we have

\[
\left| \sum'_{h_1 \leq h \leq h_2} e\left(\frac{uh'}{k}\right) \right| < \left(\frac{k + h_2 - h_1}{k} + 2 + 2 \ln(k)\right) \sqrt{k \sqrt{\gcd(u, k)}} d(k),
\]

where \( h' \) denotes the inverse of \( h \) (mod \( k \)) and the prime over the summation means to restrict to \( \gcd(h, k) = 1 \).

*Proof.* Assume for now that \( h_2 - h_1 < k \). For any integer \( h, h_1 \leq h \leq h_1 + k - 1 \), the map sending \( h \) to

\[
\left(\frac{h - h_2 - 1/2}{k}\right) - \left(\frac{h - h_1 + 1/2}{k}\right) + \frac{h_2 - h_1 + 1}{k} = \left| \frac{h - h_1 + 1/2}{k} \right| - \left| \frac{h - h_2 - 1/2}{k} \right|
\]

equals 1 for \( h_1 \leq h \leq h_2 \) and 0 otherwise. Now, we claim that for \( 0 < x < 1 \),

\[
\left| (x) + \sum_{j=1}^k \frac{\sin(2\pi jx)}{\pi j} \right| < \frac{1}{2k \min(x, 1 - x)}. \tag{4.40}
\]

To see this, it suffices to consider \( 0 < x \leq 1/2 \). Write \( K(x) = 1 + 2 \sum_{j=1}^k \cos(2\pi jx) = \sin(\pi (2k + 1)x) / \sin(\pi x) \); then

\[
((x)) + \sum_{j=1}^k \frac{\sin(2\pi jx)}{\pi j} = \int_0^x K(y)dy - 1/2 = - \int_x^{1/2} K(y)dy \tag{4.41}
\]

\[
= \frac{-1}{\pi (2k + 1)} \frac{\cos(\pi (2k + 1)x)}{\sin(\pi x)} + \frac{1}{2k + 1} \int_x^{1/2} \frac{\cos(\pi (2k + 1)y)}{\sin^2(\pi y)} dy. \tag{4.42}
\]

Since \( \sin(\pi t) \geq 2t \) for \( 0 \leq t \leq 1/2 \), we get

\[
\left| ((x)) + \sum_{j=1}^k \frac{\sin(2\pi jx)}{\pi j} \right| \leq \frac{1}{2\pi (2k + 1)x} + \frac{1}{4 (2k + 1)x} \tag{4.43}
\]

which implies the weaker (4.40).

Write \( S(v, u; k) \) for the (complete) Kloosterman sum \( \sum_{1 \leq h \leq k} e((vh + uh')/k) \) where the prime denotes restricting the sum to \( \gcd(h, k) = 1 \). Then e.g. Lemma 2 of [36] gives an effective bound for it:

\[
|S(v, u; k)| \leq \sqrt{k \sqrt{\gcd(u, k)}} d(k). \tag{4.44}
\]
Finally, it is elementary that \( \sum_{j=1}^{k} 1/j \leq 1 + \ln(k) \). Therefore, putting all this together, we obtain

\[
\left| \sum_{h=h_1}^{h_2} e \left( \frac{h'}{k} \right) \right| \leq \frac{h_2 - h_1 + 1}{k} |S(0, u; k)| + \sum_{j=1}^{k} \frac{e(-k + 1/2)j/k - e(-j/(2k))}{2\pi j} S(j, u; k) + 2 \sum_{j=1}^{k/2} \frac{1}{2k(j - 1/2)/k} \leq \frac{h_2 - h_1 + 1}{k} \sqrt{k} \sqrt{\text{gcd}(u, k)} d(k) + \frac{1}{\pi} \sqrt{k} \sqrt{\text{gcd}(u, k)} d(k)(1 + \ln(k)) + (1 + \ln(k)). \tag{4.45}
\]

That the inequality \( (4.39) \) also holds when \( h_2 - h_1 \geq k \) now follows from \( (4.44) \).

**QED to Lemma 6**

Effective bounds for the divisor function \( d(n) = \sigma_0(n) \) are \( d(n) \leq C_\epsilon n^\epsilon \), for any \( \epsilon > 0 \), where

\[
C_\epsilon = \prod_{p < 2^{1/\epsilon}} \frac{1}{\epsilon \ln(p) e^{1-\epsilon \ln(p)}}, \tag{4.46}
\]

where the product runs over all primes \( p < 2^{1/\epsilon} \). To see this, recall \( d(n) = \prod_p (a_p + 1) \) when \( n = \prod_p p^{\alpha_p} \) is the prime decomposition, so \( d(n)/n^\epsilon = \prod_p \frac{a_p + 1}{p^{\alpha_p} \ln(p)} \). The primes appearing in \( (4.46) \) are precisely those for which \( \frac{a_p + 1}{p^{\alpha_p} \ln(p)} \) can be > 1; the power \( a_p \) is then chosen to maximise this factor. Any \( \epsilon < 1/2 \) works for us, e.g. \( C_{1/4} \approx 9.11... \) (in fact a slightly more refined analysis shows \( C_{1/4} \) can be taken to be 8.55). Also, from \( \ln y \leq y - 1 \) we obtain \( \ln(k) < 12 k^{1/12} - 12 \). Therefore we obtain as a bound on \( (4.35) \):

\[
\sum_{-\delta \leq S \leq \delta} e \left( \frac{uS^2}{R} \right) \varphi(r, \delta S) \leq (103 \left| \frac{r h^5}{\delta} \right|^{5/6} - 74.8 \left| \frac{r h^{3/4}}{\delta} \right|^{3/4}) (0.0084 \frac{n^5 h_6 r^2}{\delta^2} \sqrt{|D|} + 0.000377 \frac{n^8 h_6^9}{|r|^3} \delta |D|) + 0.0164 \frac{n^4 h_6^2}{\delta^2 |r|^3} \sqrt{|D|} + \frac{0.91 n^3 h^3}{r^2} + (1.065 \frac{n^5 h_6^{41/6}}{\delta^3 |r|^{7/6}} - 0.733 \frac{n^5 h_6^{27/4}}{\delta^{3/4} |r|^{5/4}}) \sqrt{|D|} + (0.0389 \frac{n^8 h_6^{59/6} \delta^{1/6}}{|r|^{13/6}} - 0.028 \frac{n^8 h_6^{39/4} \delta^{1/4}}{|r|^{9/4}}) |D|. \tag{4.47}
\]

We can now bound the limit of \( (4.33) \) as \( X \to \infty \), using the inequalities \( n^{1-\mu}/(\mu - 1) < \sum_{r=n}^{X} r^{-\mu} < n^{-\mu} + n^{1-\mu}/(\mu - 1) \) for \( \mu > 1 \) which follow by comparing series with integral:

\[
\lim_{X \to \infty} \sum_{r>n} \left| \frac{19.136 n^{29/6} h_6^{41/6} - 9.67 n^{19/4} h_6^{27/4}}{\sqrt{|D|} + (0.455 n^{41/6} h_6^{59/6} - 0.359 n^{27/4} h_6^{39/4}) |D|}. \tag{4.48}
\]

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When \( n = 2 \) (so \( h = \delta = 1 \)), we get instead the weaker \( \lim_{X \to \infty} |\sum_\phi| < 2136 \sqrt{|D|} + 61|D| \).

Bounding \( \sum_\phi \) is completely analogous; (4.27) becomes

\[
\frac{m}{2nc} \equiv 1 \quad \frac{ur'}{S} = \frac{\alpha r + \beta s}{2ns(\alpha r^2 + \beta rs + \gamma n's^2)},
\]

where \( \delta = \gcd(r, h) \in \{1, 3\}, S = ns/\delta, r' \) is the inverse of \( r/\delta \text{ mod } S \), and \( u \) is defined mod \( S \) by \( u \equiv_s \delta^{-1}(1 + n's^2 - n'^2s^4) \). The other equations and bounds are obtained by interchanging \( r \leftrightarrow s \) and \( \alpha \leftrightarrow n'\gamma \). In (4.34) and elsewhere it suffices to take \( \delta \in \{1, 3\} \). Then (4.37), (4.38) become

\[
|\varphi(r, s) - \varphi(r + \delta, s)| \leq 0.56\frac{n^2h^6\delta}{|s|^3}\sqrt{|D|} + 0.093\frac{n^3h^9\delta^2}{s^4}|D| + 0.34\frac{nh^6}{s^4}\sqrt{|D|},
\]

\[
|\varphi(\delta|S_+/\delta|, s)| \leq nh^3s^{-2}/3.
\]

Therefore (4.35) becomes

\[
\left| \sum_{S_+/\delta \leq R \leq S_+|/\delta} e\left(\frac{uR'}{S}\right) \varphi(\delta R, s) \right| < (1.916\frac{n^{29/6}\delta^{1/6}}{|s|^{13/6}} - 1.26\frac{n^{19/4}\delta^{1/4}}{|s|^{9/4}})h^9|D|
\]

\[
+ (12.98\frac{n^{23/6}h^6}{\delta^{6/6}|s|^{7/6}} + 70.1\frac{n^{17/6}h^6}{\delta^{11/6}|s|^{13/6}} - 8.56\frac{n^{15/4}h^6}{\delta^{3/4}|s|^{5/4}} - 46\frac{n^{11/4}h^6}{\delta^{7/4}|s|^{9/4}})\sqrt{|D|},
\]

and we obtain

\[
\lim_{X \to \infty} \left| \sum_\phi \right| \leq (7.832n^{29/6}h^9 - 2.33n^{19/4}h^9)|D| + (201n^{23/6}h^6 - 72n^{15/4}h^6)\sqrt{|D|}.
\]

All that remains is the sum over the representatives \([n\alpha, \beta, \gamma/h]\) of \( \Gamma_0(n; h) \)-equivalence classes of quadratic forms of discriminant \( D \). Let \( h'_{n,h}(D) \) denote the number of those (not necessarily primitive) equivalence classes. Note that our sum is now independent of \( \alpha, \beta, \gamma \), so a bound for \(|Z_{n/2,h}(3/4)|\) will be the sum of our bounds for \( \sum_\phi \) and \( \sum_\psi \), multiplied by \( h'_{n,h}(D)/2 \) (the 2 here compensates for the overcounting, as mentioned earlier). Now, since all stabilisers here are \( \pm I \), \( h'_{n,h}(D) \) will equal the number of \( \Gamma_0(n') \)-equivalence classes of forms \([n'\alpha, \beta, \gamma]\), times the index of \( \Gamma_0(n; h) \) in \( \Gamma_0(n|h) \), and so we obtain the (crude) bound

\[
h'_{n,h}(D) \leq h'(D) \frac{\text{SL}_2(\mathbb{Z})}{\Gamma(n, h)}
\]

where \( h'(D) \) is the total number of \( \text{SL}_2(\mathbb{Z}) \)-equivalence classes of integral forms (not necessarily primitive) of discriminant \( D \), and where \( \Gamma(n, h) \) consists of all matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( n \) divides \( c \) and \( h \) divides \( b \) (this clearly forms a group). In deriving (4.52), we are conjugating everything by \( \begin{pmatrix} h_0^t \\ 0 \\ 1 \end{pmatrix} \); \( \Gamma(n, h) \) arises as the conjugate of the
in these class functions $\Gamma_0(nh)$ of $\Gamma_0(n; h)$. Now, the index of $\Gamma(n, h)$ in $\text{SL}_2(\mathbb{Z})$ equals that of its conjugate $\Gamma_0(nh)$, since indices of Fuchsian groups can be expressed as a ratio of areas of fundamental domains and conjugating by $\text{SL}_2(\mathbb{R})$ preserves those areas. Let $h(D)$ denote the class number of $D$, i.e. the number of $\text{SL}_2(\mathbb{Z})$-equivalence classes of *primitive* integral quadratic forms of discriminant $D$. Then we have the bound

$$h(D) < \sqrt{|D|} \pi (2 + \log |D|)$$

(see Theorems 12.14.3 and 12.10.1 in [38]). Hence

$$\frac{h'(D)}{2} = \frac{1}{2} \sum_{m^2 | D} h(D/m^2) < \frac{\sqrt{|D|}}{2\pi} \sum_{m^2 | D} \frac{2 - \log m}{m} + \frac{\sqrt{|D|} \log |D|}{4\pi} \sum_{m^2 | D} 1$$

$$< 0.32\sqrt{|D|} + 0.681|D|^{5/8} \log |D|,$$

using our earlier bound for the divisor function $d(n)$. Putting this all together, we obtain

$$h_{n,h}'(D)/2 < (0.32\sqrt{|D|} + 0.681|D|^{5/8} \log |D|)nh \prod_{p/(nh)} \frac{p + 1}{p},$$

and hence the bound given in the statement of our theorem. **Q.E.D. to Theorem 3**

We can approximate the $2.13|D|^{1/8} \log |D|$ with the upper bound $120(|D|^{1/7} - |D|^{1/8})$ by the usual reasoning. Putting this all together, we obtain the following (very crude) bounds for the character values for each $g \in M_{24}$ and $k \geq 150$:

$$H_k(1) > \frac{4}{K} e^{\pi K/2} - \frac{4\pi}{\sqrt{2}} e^{\pi K/4} - 2.5 \times 10^4 K^{23/7},$$

$$|H_k(g)| < \frac{4}{K \sqrt{n_g}} e^{\pi K/(2n_g)} + \sqrt{8\pi} e^{\pi K/(4n_g)} + a_g \times 10^b K^{23/7},$$

where $n_g = |g|$ is the order of $g$, $K = \sqrt{8k - 1}$, and the values $a_g, b_g$ are computed from Theorem 3 and are collected in Table 2. We find that when $k \geq 390$, the ratio $(H_k(1)/|M_{24}|)/(|H_k(g)|/|C_{M_{24}}(g)|)$ exceeds $1.6 \times 10^5$ for all elements $g \neq 1$ except for $g$ in conjugacy class $12B$, where the ratio is $> 1.3$. In particular we find that

$$\frac{H_k(1)}{|M_{24}|} > \sum_{g \neq 1} \frac{|H_k(g)|}{|C_{M_{24}}(g)|},$$

for all $k \geq 390$, where the sum is over all conjugacy classes in $M_{24}$. As explained in [41], this is sufficient to deduce positivity of the multiplicities of all $M_{24}$-irreps in these class functions $H_k$. Again, positivity (or rather nonnegativity) has been obtained experimentally for $n$ up to 500, far more than we need.

It is somewhat disappointing that we need to check by hand (or rather, by computer!) positivity for so many $k$, when empirically [1.38] is satisfied for $k > 30$. The
cause of this are the bounds of Theorem 3, which apparently are far looser than they
could be. But the value gained in tightening these bounds is far less than the effort
to be spent, since it is so easy to calculate these character multiplicities. The reader
interested in reducing 390 should focus on improving the bound for the case 12B (i.e.
fix \( n = 24, h = 12 \) from the start, and restrict to \( k > 100 \) say; the index of \( \Gamma_0(nh) \) in
\( \Gamma_0(n; h) \) here is 12, so the Theorem 3 bound can be improved immediately by that
factor 12).

The bounds (4.56), (4.57) imply:

**Theorem 4.** Choose any \( M_{24} \)-irrep \( \rho \). Then the multiplicity \( \text{mult}_\rho(H_n) \) of \( \rho \) in \( H_n \)
tends to \( \infty \) as \( n \to \infty \). Moreover, \( \text{mult}_\rho(H_n) > 0 \) for all \( n \geq 25 \).

Indeed, in this section we proved positivity of the multiplicities for \( n \geq 390 \); the
values \( 25 \leq n < 390 \) can be checked explicitly. An immediate corollary of Theorem
4 is the validity of Conjecture 5.11 in Umbral Moonshine [7]. However it should be
remarked that the proof of this in [12] actually established a much stronger statement.
Oddness of McKay-Thompson coefficients is far less trivial than strict positivity of
certain multiplicities, since as we see almost every multiplicity will be strictly positive.

## 5 Is the Conway group the stringy symmetry?

In beautiful work, [30] followed Kondo’s treatment [39] of Mukai’s classification [42]
of symplectic automorphisms of K3 surfaces, to obtain the symmetries of K3 sigma
models. (Similar considerations are considered in [50].) It turns out all these stringy
symmetries are subgroups of the Conway groups \( \text{Co}_0 \) or \( \text{Co}_1 \) (but not \( M_{24} \)), where
they will necessarily generate the full group \( \text{Co}_0 \) (resp. \( \text{Co}_1 \)). This begs the (perhaps
naive) question: should the automorphism group of Mathieu Moonshine actually be
a Conway group?

The Conway group \( \text{Co}_0 \) is the automorphism group of the Leech lattice; the simple
group \( \text{Co}_1 \) is its quotient by its centre \( \pm 1 \). Hence the \( \text{Co}_1 \)-irreps are a subset of those
of \( \text{Co}_0 \), consisting of those \( \text{Co}_0 \)-irreps whose kernel contains \( -1 \). We call a \( \text{Co}_0 \)-irrep
a spinor if it is not trivial on \( -1 \). \( M_{24} \) is a subgroup of both \( \text{Co}_0 \) and \( \text{Co}_1 \). In
this section we give the restrictions of \( \text{Co}_0 \) (hence \( \text{Co}_1 \)) irreps to \( M_{24} \), for irreps with
dimension up to 1 million (there are 32 of these). The notation comes from the
Atlas of Finite Groups [8]; in a couple places it differs slightly from that of [24] (e.g.
the dimension-1035 irreps are in a different order: the conjugate ones we write as
1035’ and 1035’’). The first step was identifying the conjugacy classes; once these are
known, the character multiplicities follow directly from the character table of \( M_{24} \)
together with the orthogonality relations. The arguments are straightforward and
we’ll avoid the details, giving only the results.

| Table 5. Matching of conjugacy classes |

| \( M_{24} \) | 1A | 2A | 2B | 3A | 3B | 4A | 4B | 4C | 5A | 6A | 6B | 7A | 7B | 8A | 10A | 11A | 12A | 12B | 12C | 12M | 14A | 14B | 15A | 15B | 21A | 21B | 23A | 23B |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \text{Co}_0 \) | 1A | 2A | 2C | 3B | 3D | 4D | 4F | 5B | 6E | 6L | 7B | 7B | 8E | 10B | 11A | 12J | 12M | 14A | 14B | 15A | 15B | 21C | 21C | 23A | 23B |

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Table 6. The 32 smallest Conway irreps

| $\tilde{\sigma}_1$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ | $\sigma_7$ | $\sigma_8$ | $\sigma_9$ | $\sigma_{10}$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{14}$ | $\sigma_{15}$ | $\sigma_{16}$ | $\sigma_{17}$ | $\sigma_{18}$ | $\sigma_{19}$ | $\sigma_{20}$ | $\sigma_{21}$ | $\sigma_{22}$ | $\sigma_{23}$ | $\sigma_{24}$ | $\sigma_{25}$ | $\sigma_{26}$ | $\sigma_{27}$ | $\sigma_{28}$ | $\sigma_{29}$ | $\sigma_{30}$ | $\sigma_{31}$ | $\sigma_{32}$ |
|-------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 24                | 276       | 299       | 1771      | 2024      | 2576       | 4570       | 8855       | 17250      | 27300      | 37674       | 40480       | 44275       | 80730       | 94875       | 95680       | 170016      | 290000      | 313950      | 315744      | 345345      | 351624      | 376740      | 388080      | 483000      | 644644      | 673750      | 789360      | 822250      | 871884      |

We find that any virtual character $\rho = m_0 + m_1\rho_1 + \cdots + m_{20}\rho_{20}$ of $M_{24}$, i.e. any linear combination with integer coefficients $m_i$ of $M_{24}$ irreps, is the restriction of a virtual $Co_0$-representation involving the first 32 irreps of $Co_0$, iff the multiplicities obey $m_2 = m_7, m_3 = m_3, m_4 = m_8, m_{10} = m_{10}$, as well as the relation $m_9 + m_{14} \equiv 2 m_{15}$. Thus it is hard for an $M_{24}$-representation to be a restriction of a virtual $Co_0$-representation (0% will be). Although the restriction map is not surjective, Theorem B tells us the Mathieu-Moonshine characters $H_n$ lie in the image of that restriction. This is the content of Theorem C given earlier. In particular we have:

$$\rho_1 = \tilde{\sigma}_1 - 1,$$

$$\rho_2 = \tilde{\sigma}_2 = 3\tilde{\sigma}_{14} + 39\sigma_{13} + 28\tilde{\sigma}_{10} + 12\sigma_{11} + 23\tilde{\sigma}_{8} + 8\sigma_{10} + 25\sigma_{9} + 2\sigma_{5} + 19\tilde{\sigma}_{4} + 24\tilde{\sigma}_3 + 18\sigma_{3} + 26\tilde{\sigma}_{1} - 28\sigma_{18} - 11\tilde{\sigma}_{12} - 42\tilde{\sigma}_{9} - 44\tilde{\sigma}_{6} - 9\sigma_{9} - 100\tilde{\sigma}_{5} - 18\sigma_{7} - 78\sigma_{4} - 144\sigma_{2} - 218\sigma_{1} - 155,$$

$$\rho_3 = \tilde{\rho}_3 = 14\sigma_{13} + 11\tilde{\sigma}_{10} + 15\sigma_{12} + 4\sigma_{11} + 6\tilde{\sigma}_{8} + 15\tilde{\sigma}_{10} + 8\sigma_{9} + 105\sigma_{4} + 14\tilde{\sigma}_{3} + 64\tilde{\sigma}_3 + 12\sigma_{1},$$

$$\rho_4 = \sigma_2 + 1 - 2\tilde{\sigma}_1,$$

$$\rho_5 = \sigma_1 + 1 - \tilde{\sigma}_1,$$

$$\rho_6 = 8\sigma_{14} + 14\sigma_{13} + 11\tilde{\sigma}_{10} + 7\sigma_{12} + 4\sigma_{11} + 6\tilde{\sigma}_{8} + 7\sigma_{10} + 8\sigma_{9} + 89\tilde{\sigma}_{4} + 14\tilde{\sigma}_3 + 32\sigma_{3} + 121\sigma_{1},$$

$$\rho_7 = 36\sigma_{14} + 41\sigma_{13} + 33\tilde{\sigma}_{10} + 6\sigma_{12} + 12\sigma_{11} + 25\tilde{\sigma}_{8} + 6\sigma_{10} + 25\sigma_{9} + 25\tilde{\sigma}_4 + 50\tilde{\sigma}_3 + 55\sigma_{3} + 35\tilde{\sigma}_1,$$

$$\rho_8 = 343\sigma_{1} - 28\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_9 = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{10} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{11} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{12} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{13} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{14} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51,$$

$$\rho_{15} = 30\sigma_{18} - 12\tilde{\sigma}_{12} - 15\tilde{\sigma}_9 - 15\tilde{\sigma}_6 - 12\sigma_{8} - 28\tilde{\sigma}_5 - 12\sigma_{7} - 21\tilde{\sigma}_{6} - 81\sigma_{2} - 73\sigma_{1} - 51.$$
\[2 \rho_{15} = 30 \sigma_{14} + 39 \sigma_{13} + 28 \sigma_{10} + 11 \sigma_{12} + 12 \sigma_{11} + 22 \sigma_{8} + 11 \sigma_{10} + 25 \sigma_{9} + 244 \sigma_{4} + 40 \sigma_{3} + 67 \sigma_{3}
\[+ 326 \rho_1 - 28 \sigma_{18} - 12 \sigma_{12} - 40 \sigma_{9} - 44 \sigma_{6} - 26 \sigma_{8} - 90 \sigma_{5} - 33 \sigma_{7} - 76 \sigma_{6} - 211 \sigma_{2} - 229 \sigma_{1} - 158,
\]
\[\rho_{16} = \sigma_4 + 4 \sigma_1 - 2 \sigma_2 - 2 \sigma_1 - 2,
\]
\[\rho_{17} = 5 \sigma_{18} + 2 \sigma_{12} + 7 \sigma_9 + 7 \sigma_6 + 12 \sigma_8 + 5 \bar{\sigma}_5 + 11 \sigma_7 + 8 \sigma_6 + \sigma_5 + 54 \sigma_2 + 27 \sigma_1 + 15
\]
\[-7 \sigma_{13} - 5 \bar{\sigma}_10 - 7 \sigma_{12} - 2 \sigma_{11} - 4 \bar{\sigma}_8 - 7 \sigma_{10} - 4 \sigma_9 - 57 \bar{\sigma}_4 - 10 \bar{\sigma}_3 - 34 \sigma_3 - 64 \bar{\sigma}_1,
\]
\[\rho_{18} = \sigma_{14} + 6 \sigma_{13} + 4 \bar{\sigma}_10 + 6 \sigma_{12} + 2 \sigma_{11} + 2 \bar{\sigma}_8 + 6 \sigma_{10} + 4 \sigma_9 + 46 \bar{\sigma}_4 + 7 \bar{\sigma}_3 + 24 \sigma_3 + 58 \bar{\sigma}_1
\]
\[-4 \sigma_{18} - 2 \sigma_{12} - 6 \bar{\sigma}_9 - 7 \bar{\sigma}_6 - 7 \sigma_8 - 8 \sigma_5 - 8 \sigma_7 - 8 \sigma_6 - 44 \sigma_2 - 30 \sigma_1 - 19,
\]
\[\rho_{19} = 8 \sigma_{14} + 10 \sigma_{13} + 7 \bar{\sigma}_10 + 2 \sigma_{12} + 3 \sigma_{11} + 6 \sigma_8 + 2 \sigma_{10} + 6 \sigma_9 + 68 \bar{\sigma}_4 + 14 \bar{\sigma}_3 + 18 \sigma_3 + 95 \bar{\sigma}_1
\]
\[-7 \sigma_{18} - 3 \sigma_{12} - 10 \sigma_9 - 11 \sigma_6 - 8 \sigma_8 - 23 \sigma_5 - 9 \sigma_7 - 21 \sigma_6 - 64 \sigma_2 - 67 \sigma_1 - 48,
\]
\[\rho_{20} = 41 \sigma_{13} + 29 \sigma_{10} + 41 \sigma_{12} + 12 \sigma_{11} + 24 \bar{\sigma}_8 + 41 \sigma_{10} + 26 \sigma_9 + 318 \bar{\sigma}_4 + 45 \bar{\sigma}_3 + 191 \sigma_3 + 346 \bar{\sigma}_1
\]
\[-29 \sigma_{18} - 12 \sigma_{12} - 42 \bar{\sigma}_9 - 46 \bar{\sigma}_6 - 59 \sigma_8 - 29 \bar{\sigma}_5 - 64 \sigma_7 - 49 \sigma_6 - 287 \sigma_2 - 149 \sigma_1 - 78.
\]

We considered more \(\text{Co}_0\)-irreps than \(M_{24}\) ones; the additional relations between restrictions are:

\[\bar{\sigma}_2 + \bar{\sigma}_1 = \sigma_3 + \sigma_1 + 1,
\]
\[\bar{\sigma}_5 + \bar{\sigma}_4 + \sigma_3 + 1 = \sigma_7 + \sigma_4 + \sigma_2,
\]
\[\bar{\sigma}_{11} + \bar{\sigma}_3 = \bar{\sigma}_8 + \bar{\sigma}_5 + \sigma_7 + \sigma_4 + 2 \bar{\sigma}_2 + \sigma_2 + \sigma_1 + \bar{\sigma}_1,
\]
\[\sigma_{14} + \sigma_8 = \bar{\sigma}_8 + \sigma_7 + \sigma_2 + 4 \sigma_4 + 2 \sigma_2 + \sigma_2 + \sigma_1,
\]
\[\bar{\sigma}_8 + \sigma_7 + \sigma_2 = \sigma_{12} + \sigma_{10} + \sigma_6 + \sigma_3 + \bar{\sigma}_1,
\]
\[\sigma_{19} + \sigma_8 = \sigma_{18} + 2 \sigma_7 + 2 \sigma_4 + \sigma_2 + 2 \sigma_1,
\]
\[\bar{\sigma}_6 + 4 \bar{\sigma}_4 + 3 \bar{\sigma}_3 + 24535 \bar{\sigma}_2 + 24541 \bar{\sigma}_1 = 2 \sigma_8 + 2 \sigma_5 + 24533 \sigma_3 + 6 \sigma_2 + 24537 \sigma_1 + 24537,
\]
\[85 \sigma_{14} + 88 \sigma_{13} + 62 \bar{\sigma}_{10} + 4 \sigma_{12} + 26 \sigma_{11} + 50 \bar{\sigma}_8 + 4 \sigma_{10} + 56 \sigma_9 + 516 \bar{\sigma}_4 + 97 \bar{\sigma}_3 + 72 \sigma_3 + 754 \bar{\sigma}_1
\]
\[= 62 \sigma_{18} + 26 \bar{\sigma}_{12} + 90 \bar{\sigma}_9 + 99 \bar{\sigma}_6 + 41 \sigma_8 + 234 \bar{\sigma}_5 + 53 \sigma_7 + 190 \sigma_6 + 453 \sigma_2 + 582 \sigma_1 + 428.
\]

Of course we’d prefer true representations to virtual ones, but perhaps we should not be too surprised that virtual representations arise here, because the definition of elliptic genus involves signs, and after all even \(H_{00}\) and \(H_0\) were virtual (but see Section 6).

However, the dimensions of the virtual \(\text{Co}_0\)-representations needed will be extremely large. By the total dimension of a virtual representation \(\rho_+ \otimes \rho_-\) we mean the quantity \(\dim(\rho_+) + \dim(\rho_-)\). For any \(M_{24}\) representation, if there is any \(\text{Co}_0\)-virtual representation which restricts to it, there will be infinitely many (just add to it any of the above relations). For each basic combination of \(M_{24}\)-irreps, we have selected above the \(\text{Co}_0\)-virtual representation of smallest total dimension we could find, which restricts to it. (This ambiguity would be eliminated by identifying if possible the twining elliptic genera \(\phi_g\) for all \(g \in \text{Co}_0\).)

For example, of all \(\text{Co}_0\)-irreps of dimension less than 1 million, only 2 of them contain \(\rho_2\) or \(\rho_2^\sigma\) as a summand: namely, a \(\text{Co}_1\)-irrep of dim 313950, and a spinor irrep of dim 789360. The smallest virtual \(\text{Co}_0\)-representation we could find which
restricts to $\rho_2 + \overline{\rho_2}$ (a 90-dimensional representation of $M_{24}$) has total dimension over 100 billion.

In fact more is true. Suppose for contradiction the natural identification of the Witten index $w_g$ with $\tilde{\sigma}_1$ (the only $\text{Co}_0$-rep which restricts to $1 + \rho_1$). Now, compute (among other things) the twining characters for the automorphism of the Gepner model (1)$^6$; they found two symmetries of order 9 (the bottom two rows of their Table 1), both of whose Witten indices equal 3, and whose twining elliptic genera are nonetheless different. However, $\text{Co}_0$ has only one conjugacy class (namely class 9C) of order 9 with $\tilde{\sigma}_1 = 3$, and this would imply those twining genera should be equal. (We thank Roberto Volpato for sharing this observation.) This contradiction means that $w_g$ cannot equal $\tilde{\sigma}_1(g)$, and so the Witten genus too will be virtual, having total dimension much greater than 24. These large total dimensions make the proposed extension to the Conway groups seem unlikely.

6 Speculations

Because the phenomena underlying Mathieu Moonshine are still obscure, we conclude with assorted questions and speculations. The most obvious challenge suggested by this paper is to explicitly construct these representations $H_n$. The evidence that there is some vertex algebra-like object underlying the Mathieu Moonshine observations now seems overwhelming; perhaps the most satisfying way to construct the $H_n$ would be to construct this vertex (super)algebra. This would provide the algebraic underpinning of Mathieu Moonshine, and it should hint at its still-mysterious geometric and physical meanings.

We regard this vertex superalgebra construction as the most important challenge of Mathieu Moonshine. At $c = 24$ we have an $N = 0$ VOA (namely the Moonshine Module $V^\natural$) with lots of nice properties including an action of $M$. At $c = 12$ we have an $N = 1$ VOSA (namely Duncan’s algebra $\text{VOSA}$), with lots of nice properties including an action of the Conway group. Could there be at $c = 6$ an $N = 2$ or $N = 4$ VOSA (namely the algebra underlying our Mathieu Moonshine) with lots of nice properties including an action of $M_{24}$? After all, $M_{24}$, $\text{Co}_1$, $M$ is the Holy Trinity of sporadic finite simple groups (e.g. Griess constructed the Monster by starting with $M_{24}$, lifting to Conway and then moving on to the Monster). (I thank Gerald Höhn and Chongying Dong for informal discussions on this point).

Another possible construction, which may also bring K3 into Mathieu Moonshine, is through the chiral de Rham complex $\text{C}^{\infty}$ (a sheaf of vertex superalgebras) associated to a K3 surface. The trace over its global section (which is itself a vertex superalgebra) recovers elliptic genus $\text{E}$. The orbifold theory, including the construction of twisted modules, has also been studied $\text{O}$. One of the most intriguing aspects of Mathieu Moonshine is the positivity of the irrep multiplicities. This discussion is already anticipated in Chapter 7.2 of Wendland’s thesis $\text{W}^\natural$. There we find that the partition function in the Ramond-
Ramond sector of an $N = 4$ $c = 6$ superconformal field theory is

$$Z_{RR} = |ch^{s}_{0,0}|^2 + h|ch^{s}_{1/4,0}|^2 + \epsilon (ch^{s}_{0,0}ch^{s*}_{1/4,0} + c.c.) + Fch^{l}_{1/4,1/2}ch^{s*}_{0,0} + F'ch^{s}_{0,0}ch^{l*}_{1/4,1/2} + Gch^{l}_{1/4,1/2}ch^{s*}_{1/4,0} + G'ch^{s}_{1/4,0}ch^{l*}_{1/4,1/2} + H|ch^{l}_{1/4,1/2}|^2,$$

(6.1)

using the same conventions for $N = 4$ superconformal characters as (2.6), where $h, \epsilon$ are nonnegative integers and $F, F', G, G', H$ are functions of $\tau, z$ with only nonnegative integer coefficients in $q, q^{-1}$. The elliptic genus is

$$\phi = (\epsilon - 2)ch^{s}_{0,0} + (h - 2\epsilon)ch^{s*}_{1/4,0} + (G - 2F')ch^{l}_{1/4,1/2}.$$

(6.2)

For the K3 component of moduli space, $\epsilon = 0$ and $h = h^{1,1} = 20$. In particular note that the function $F$ counts holomorphic fields which exist in that theory but which are not contained in the $N = 4$ vacuum representation. We can think of holomorphic fields, very roughly, as additional symmetries of the theory; since we would expect that generically the chiral algebra of the K3 sigma model is simply $N = 4$ superconformal, we should have generically $F = 0$. This would mean the non-BPS part of the elliptic genus is (generically) this function $G$ which has only non-negative coefficients. From this the conjectured non-negativity would follow. This argument also applies to the character-valued elliptic genus, so all multiplicities should be nonnegative. A similar argument can be found in Ooguri [44]. (We thank Katrin Wendland for discussions on this point.)

It is commonly expected that the moduli space of $c = 6$ $N = 4$ superconformal field theories consists of two components: a toroidal one with vanishing elliptic genus, and the K3 sigma models with elliptic genus $2\phi_{0,1}$. But if this is true, then there can be no $c = 6$ $N = 4$ superconformal field theory underlying Mathieu moonshine: the work of [30] shows that the corresponding automorphism groups are too small.

But $N = 4$ (more precisely, $N = (4, 4)$) theories possess more supersymmetry than we need. We could deform an $N = (4, 4)$ theory to e.g. an $N = (0, 4)$ heterotic (see e.g. [55] for an introduction to similar theories). These theories are geometrical, corresponding to bundles over K3, and are also related to chiral de Rham. They possess the desired elliptic genus $2\phi_{0,1}$. The moduli space of these theories is 90-dimensional, far larger than that of $N = (4, 4)$, so there is a much greater chance for some larger symmetry groups. However, the breaking of the left-moving $N = 4$ supersymmetry seems to destroy the justification for decomposing the elliptic genus into $N = 4$ superconformal characters — a step crucial to Mathieu moonshine. It is tempting to partially break the supersymmetry, say to $N = (4, 1)$, but such theories seem to possess the full $N = (4, 4)$ supersymmetry. (I thank Ilarion Melnikov for discussions involving this point.)

The difficulty in interpreting Mathieu Moonshine in terms of K3 sigma models (e.g. the interesting work of Taormina-Wendland [50] is still far from realising $M_{24}$) leads one to consider the unhappy possibility that the connection with K3 is perhaps accidental. The relevant (twisted) elliptic genera are so heavily constrained that there are bound to be empty coincidences. For example, any of the 71 or so $c = 24$
holomorphic rational VOA — e.g. one associated to the Leech lattice — will have a VOA character (a.k.a. partition function) equal to $J(\tau) + c$ for some constant $c \in \mathbb{Z}_{\geq 0}$, again because it is so severely constrained. The coefficients of $J$ (with or without $c$) will have an interpretation as dimensions of Monster representations as we know, but conjecturally only one of those VOAs actually carries a nontrivial action of the Monster $\mathbb{M}$. It takes a (slightly) deeper analysis to rule out Monster actions on these other VOAs. Could this relation of K3 sigma models to our Mathieu Moonshine be likewise illusory? After all, it is clear that the Jacobi forms of Umbral Moonshine cannot have a direct interpretation as elliptic genera, when the group is not $\mathbb{M}$.

Similarly, perhaps we shouldn’t regard $\mathbb{M}$ as sacrosanct. The evenness property of Theorem B hints perhaps that the symmetry is somewhat larger. The analysis of [30] and our Theorem C hints that the ‘ultimate’ symmetry lies somewhere between $\mathbb{M}$ and Conway. The split extension $\mathbb{Z}_2^2 \rtimes \mathbb{M}$, a maximal subgroup of $\text{Co}_0$, is the only such group and seems worth a look. (The evenness property was used in [12] to prove one of the Umbral Moonshine conjectures, and together with Theorem A proves that the elliptic genus of Enriques surfaces decomposes into a sum of $M_{12}$ characters.)

The appearance of $\Gamma_0([g])$ in Lemma 1 is what one would expect from the CFT orbifold story. Let us quickly review that basic theory, as developed by [14], [13], [1], and others. Let $\mathcal{V}$ be a (bosonic) rational VOA with automorphisms containing some finite group $G$. By $\mathcal{V}^G$ we mean the vertex operator subalgebra consisting of all fixed-points of $G$ in $\mathcal{V}$. Conjecturally, $\mathcal{V}^G$ is also rational; in the mathematical literature it is called the orbifold of $\mathcal{V}$ by $G$ (the orbifold construction means something a little different in the physics literature).

We are interested here in the simplest case, where $\mathcal{V}$ only has a single irreducible module (namely itself). The twisted modules $M(g)$ of $\mathcal{V}$ are parametrised by (a subset of the) conjugacy class representatives $g$, and (a subgroup of) the centraliser of $g$ in $G$ acts on them, so we can define twisted twining characters $Z_{g,h}(\tau) = \text{Tr}_{M_g}(h^{q_{a \tau - c/24}})$. The special case $Z_{1,h}(\tau)$ are sometimes called McKay-Thompson series in the mathematics literature. Then for any $(a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})$, $Z_{g,h}(a\tau + b \ c \tau + d)$ will equal $Z_{g^a h^c; g^a h^c}(\tau)$ up to some phase. For general $g, h$, $Z_{g,h}(\tau)$ will thus be a modular function with multiplier for the group of all matrices $(a \ b \ c \ d) \in \Gamma(\text{lcm}([g], |h|))$. However, $(a \ b \ c \ d) \in \Gamma_0(|h|)$ will send the McKay-Thompson series $Z_{1,h}$ to another, $Z_{1,h^d}$ (up to a phase). The integer $d$ will be coprime to $|h|$, and sends a character value of $h$ to a Galois associate. When the McKay-Thompson series have integer coefficients (e.g. in Monstrous or Mathieu Moonshine), these Galois associates have the same value, and $Z_{1,h^d} = Z_{1,h}$. In other words, since we have that integrality, our McKay-Thompson series $Z_{1,h}$ will be modular functions (with multiplier) for $\Gamma_0(|h|)$.

To complete the story, we need to consider equivariant cohomology. Let $B_G$ be a classifying space, then the group cohomology $H^*_G(1; N)$ (often denoted $H^*(G; N)$) with values in a $G$-module $N$ is defined by $H^*(B_G; N)$. Just as $H^2_G(1; U(1))$ controls the projective representatives of $G$, and $\alpha \in H^2_G(G; \mathbb{Z})$ ($G$ acting on itself by conjugation) parametrises the possible orbifold fusion rings (all given by twisted
equivariant $K$-theory $\alpha K^0_G(G))$, the representation theory of $\mathcal{V}^G$ is controlled by a $3$-cocycle $\alpha \in H^3_G(1; U(1))$. In particular, when $\alpha$ is trivial, the modules $M$ of $\mathcal{V}^G$ conjecturally are in natural one-to-one correspondence with pairs $(g, \psi)$, where $g$ is a conjugacy class representative in $G$ and $\psi$ is an irrep of the centraliser $C_G(g)$. Their characters $\chi_M(\tau) = \text{Tr}_M q^{L_0-c/24}$ form a vector-valued modular function for $\text{SL}_2(\mathbb{Z})$ with multiplier explicitly defined in terms of $G$ [14]. The twisted modules $M(g)$ of $\mathcal{V}^G$ are also (conjecturally) parametrised by all conjugacy class representatives $g$, and the full centraliser of $g$ in $G$ acts on them. Then $Z_{g^h}(\tau) = Z_{g,h}(\tau)$ and $Z_{g,h}(a\tau + b/c\tau + d) = Z_{g^h, g^h \cdot g}(\tau)$. When the cocycle is not trivial — the generic case — a subset of these pairs $(g, \psi)$ and $(g, h)$ will not be viable. Again, the characters of $\mathcal{V}^G$ should yield a vector-valued modular form, but the multiplier is more complicated (though known [11]). Similarly, the relations $Z_{g^h,k}(\tau) = Z_{g,h}(\tau)$ and $Z_{g,h}(a\tau + b/c\tau + d) = Z_{g^h, g^h \cdot g}(\tau)$ have to be adjusted by phases determined by $\alpha$.

We have order-12 phases in Mathieu Moonshine (e.g. the $e^{2\pi i d/(|g|h)}$ in (2.17)), so we would expect (supposing the orbifold theory extends to our $N = 4$ setting) the relevant 3-cocycle $\alpha$ to have order 12. Indeed, [18] computed that $H^3_{M_{24}}(1; U(1)) \cong \mathbb{Z}_{12}$, and [31] explicitly verified that a generating cocycle yields the phases appearing in Mathieu Moonshine. Moreover, $H^3_{M_{23}}(1; U(1)) \cong 1$ [41], and so the phases for elements lying in $M_{23}$ should be trivial, and this indeed is what is observed. It is also observed that many pairs $(g, h)$ are not viable, as expected since $\alpha$ is nontrivial. If some group $G$ larger than $M_{24}$ (e.g. $\text{Co}_0$ or $2^{12}.M_{24}$) actually acts here, then for this reason we’d require $H^3_G(1; U(1))$ to contain an order-24 element. (The cohomology of $\text{Co}_0$ has not been computed yet). It is observed in [31] that some twisted twining elliptic genera vanish even when there is no cohomological obstruction — this would be surprising for a bosonic theory but elliptic genera (being a signed trace) often vanishes in nontrivial theories so this is not mysterious here.

In summary, the perfect formal fit of twisted twining elliptic genera here with the general theory of VOA orbifolds, lends strong support to the belief that there is a vertex operator superalgebra underlying Mathieu Moonshine.

In the case of Monstrous Moonshine, the phases are 24th roots of 1, so we would expect the orbifold $(\mathcal{V}^\natural)^M$ to be governed by an order 24 cocycle. We would also expect this $\alpha$ to obstruct certain pairs $Z_{g,h}$. Indeed, empirical observations encapsulated in Norton’s Generalised Monstrous Moonshine [43] make exactly this point: whether a pair is viable or not is captured by Norton’s Fricke dichotomy. This should be equivalent to the cohomology condition. This relation between cohomology and Norton’s mysterious Fricke condition seems to be new. It would be very interesting to make that relation more explicit.

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References

[1] P. Bantay: Orbifolds, Hopf algebras and the Moonshine. Lett. Math. Phys. 22 (1991), 187–194.
[2] R. E. Borcherds: Monstrous moonshine and monstrous Lie superalgebras. Invent. Math. 109 (1992), 405–444.
[3] A. Borisov, A. Libgober: Elliptic genera of toric varieties and applications to mirror symmetry. Invent. Math. 140 (2000), 453–485.
[4] K. Bringmann, K. Ono: The f(q) mock theta function conjecture and partition ranks. Invent. Math. 165 (2006), 243–266.
[5] M.C.N. Cheng: K3 Surfaces, N=4 dyons, and the Mathieu group M24. Commun. Number Theory Phys. 4 (2010), 623 [arXiv:1005.5415].
[6] M.C.N. Cheng, J. F. R. Duncan: On Rademacher sums, the largest Mathieu group, and the holographic modularity of Moonshine. [arXiv:1110.3859].
[7] M. C. N. Cheng, J. F. R. Duncan, J. A. Harvey: Umbral moonshine. [arXiv:1204.2779].
[8] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson: The Atlas of Finite Groups. (Oxford University Press, Oxford 2003).
[9] J.H. Conway, S. Norton: Monstrous moonshine. Bull. Lond. Math. Soc. 11 (1979), 308–339.
[10] J.H. Conway, N. J. A. Sloane: Sphere Packings, Lattices and Groups. (3rd edition) (Springer, New York 1999).
[11] A. Coste, T. Gannon, P. Ruelle: Finite group modular data. Nucl. Phys. B581 (2000), 679–717.
[12] T. Creutzig, G. Höhn, T. Miezaki: The McKay-Thompson series of Mathieu Moonshine modulo two. [arXiv:1211.3703v2].
[13] A. Dabholkar, S. Murthy, D. Zagier: Quantum black holes, wall crossing, and mock modular forms. [arXiv:1208.4924v1].
[14] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde: The operator algebra of orbifold models. Commun. Math. Phys. 123 (1989), 485–526.
[15] R. Dijkgraaf, E. Witten: Topological gauge theories and group cohomology. Commun. Math. Phys. 129 (1990), 393–429.
[16] J. F. Duncan: Super-Moonshine for Conway’s largest sporadic group. Duke Math. J. 139 (2007), 255–315.
[17] J. F. R. Duncan, I. B. Frenkel: Rademacher sums, moonshine and gravity. Commun. Number Theor. Phys. 5 (2011) no. 4, 1–128; [arXiv:0907.3529].
[18] M. Dutour Sikirić, G. Ellis: Wythoff polytopes and low-dimensional homology of Mathieu groups. J. Algebra 322 (2009), 4149–4150.
[19] T. Eguchi, K. Hikami: Superconformal algebras and mock theta functions 2. Rademacher expansion for K3 surface. Commun. Number Theory Phys. 3 (2009), 531–554; [arXiv:0904.0911v2].
[20] T. Eguchi, K. Hikami: Note on twisted elliptic genus of K3 surface. Phys. Lett. B694 446–455 (2011) [arXiv:1008.4924].
[21] T. Eguchi, K. Hikami: N = 2 moonshine. Phys. Lett. B717 (2012), 266–273; [arXiv:1209.0610].
[22] T. Eguchi, K. Hikami: Enrique moonshine. [arXiv:1301.5033].
[23] T. Eguchi, H. Ooguri, A. Taormina, S.-K. Yang: Superconformal algebras and string compactification on manifolds with SU(N) holonomy. Nucl. Phys. B315 (1989), 193–221.
[24] T. Eguchi, H. Ooguri, Y. Tachikawa: Notes on the K3 surface and the Mathieu group M24. Exper. Math. 20 91–96 (2011); [arXiv:1004.0950].
[25] M. Eichler, D. Zagier: The Theory of Jacobi Forms. (Birkhäuser, Boston, 1985).
[26] E. Frenkel, M. Szczesny: Chiral de Rham complex and orbifolds. J. Alg. Geom. 16 (2007), 599–624.
[27] I. Frenkel, J. Lepowsky, A. Meurman: Vertex Operator Algebras and the Monster (Academic Press, San Diego 1988).
[28] M. R. Gaberdiel, S. Hohenegger, R. Volpato: Mathieu twining characters for K3. JHEP 09 (2010) 058; [arXiv:1006.0221].
[29] M. R. Gaberdiel, S. Hohenegger, R. Volpato: Mathieu Moonshine in the elliptic genus of K3. JHEP 10 (2010) 062; [arXiv:1007.3778v3].
[30] M. R. Gaberdiel, S. Hohenegger, R. Volpato: Symmetries of K3 sigma models. [arXiv:1106.4315v1].
[31] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch, R. Volpato: Generalised Mathieu Moonshine. [arXiv:1211.7074].
[32] T. Gannon: Moonshine beyond the Monster: The Bridge connecting Algebra, Modular Forms and Physics. (Cambridge University Press, Cambridge 2006).
[33] D. Goldfeld, P. Sarnak: Sums of Kloosterman sums. Invent. Math. 71 (1983), 243–250.
[34] S. Govindarajan: Unravelling Mathieu moonshine. [arXiv:1106.5715v1].
[35] R. L. Griess, Jr.: The friendly giant. Invent. Math. 69 (1982), 1–102.
[36] C. Hooley: An asymptotic formulae in the theory of numbers. Proc. London Math. Soc. 7 (1957), 396–413.
[37] C. Hooley: On the number of divisors of quadratic polynomials. Acta Math. 110 (1963), 97–114.
[38] Hua Loo Keng: Introduction to Number Theory. (Springer, Berlin 1982).
[39] S. Kondo: Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces. Duke Math. J. 92 (1998), 593–603, appendix by S. Mukai.
[40] F. Malikov, V. Schechtman, A. Vaintrob: Chiral de Rham complex. Commun. Math. Phys. 204 (1999), 439–473.
[41] R.J. Milgram: The cohomology of the Mathieu group M23. J. Group Theory 3 (2000), 7–26.
[42] S. Mukai: Finite groups of automorphisms of K3 surfaces and the Mathieu group. Invent. Math. 94 (1988), 183–221.
[43] S. P. Norton: Generalized moonshine. The Arcata Conference on Representations of Finite Groups (Arcata, 1986) Proc. Sympos. Pure Math. 47 (American Mathematical Society, Providence 1987) 208–209.
[44] H. Ooguri: Superconformal symmetry and geometry of Ricci flat Kahler manifolds. Int. J. Mod. Phys. A4 (1989), 4304–4324.
[45] H. Rademacher: Bestimmung einer gewissen Einheitswurzel in der Theorie der Modulfunktionen. J. London Math. Soc. 7 (1932), 14–19.
[46] H. Rademacher: On the Selberg formula for $A_k(n)$. J. Indian Math. Soc. 21 (1957), 41–55.
[47] A. Selberg: On the estimation of Fourier coefficients of modular forms. in: Proc. Sympos. Pure Math., Vol. VIII (American Mathematical Society, Providence 1965), pp. 1–15.
[48] S.D. Smith: On the head characters of the monster simple group. Finite Groups – Coming of Age (Montreal 1982) (American Mathematical Society, Providence 1985) pp. 303–313.
[49] J. Sturm: On the congruence of modular forms. Number theory (New York, 1984–1985) Lecture Notes in Math. 1240 (Springer, Berlin, 1987), 275–280.
[50] A. Taormina, K. Wendland: The overarching finite symmetry group of Kummer surfaces in the Mathieu group $M_{24}$, arXiv:1107.3834.
[51] A. L. Whiteman: A sum connected with the series for the partition function. Pacific J. Math. 6 (1956), 159–176.
[52] K. Wendland: Moduli spaces of unitary conformal field theories. Ph.D. Thesis, Universit¨ at Bonn (2000).
[53] R. A. Wilson: On the 3-local subgroups of Conway’s group $Co_1$. J. Alg. 113 (1988), 261–262.
[54] E. Witten: Elliptic genera and quantum field theory. Commun. Math. Phys. 109 (1987), 525–536.
[55] E. Witten: Two-dimensional models with (0,2) supersymmetry: perturbative aspects. Adv. Theor. Math. Phys. 11 (2007), 1–63.