Derivation of the Schrödinger equation based on a fluidic continuum model of vacuum and a sink model of particles

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Abstract

We propose a fluidic continuum model of vacuum and a sink flow model of microscopic particles. The movements of a microscopic particle driven by a stochastic force was studied based on stochastic mechanics. We show that there exists a generalized Schrödinger equation for the microscopic particle.

keywords: Schrödinger equation; stochastic mechanics; Hamilton-Jacobi equation; Langevin equation; sink; ether; Planck constant.

1 Introduction

The Schrödinger equation for a non-relativistic particle moving in a potential \( V(x) \) can be written as

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi, \tag{1}
\]

where \( t \) is time, \( x \) is a point in space, \( \psi(x,t) \) is the wave function, \( m \) is the mass of the particle, \( V(x) \) is the potential, \( \hbar \) is the Planck constant, \( \hbar = h/2\pi \), and \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \) is the Laplace operator.

In a remarkable paper \cite{2}, E. Nelson derived the Schrödinger equation Eq. (1) based on two main hypotheses \cite{2}. The first hypothesis is that the position of a Brownian particle satisfies the Smoluchowski equation. The second hypothesis is that the diffusion coefficient \( \nu \) of the Wiener process can be written as \( \nu = h/2m \), where \( m \) is the mass of the particle, \( h = \hbar/2\pi \), and \( \hbar = h/2\pi \) is the Planck constant. Nelson’s stochastic mechanics \cite{2,4} was further developed \cite{3}.

In previous works of stochastic mechanics \cite{5}, these two hypotheses can not be explained. Is it possible for us to derive Nelson’s two hypotheses based on some assumptions about the nonempty vacuum? The purpose of this paper is to propose a derivation of Nelson’s hypotheses. Further, considering the mass-increasing effects, we show that there exists a nonlinear Schrödinger equation for microscopic particles. As a byproduct, the Planck constant \( h \) is calculated theoretically.

2 A fluidic continuum model of vacuum and a sink flow model of microscopic particles

Many philosophers and scientists, such as Laozi \cite{6}, Thales, Anaximenes, etc., believed that everything in the universe is made of a kind of fundamental substance \cite{7}. Descartes was the first to bring the concept of ether into science by suggesting that it has mechanical properties \cite{7}. Descartes interpreted the celestial motions of celestial bodies based on the hypothesis that the universe is filled by an fluidic vortex ether. After Newton’s law of gravitation was published in 1687 \cite{8}, this action-at-a-distance theory was criticized by the French Cartesians \cite{7}. Newton admitted that his law did not touch on the mechanism of gravitation \cite{9}. He tried to obtain a deriva-
tion of his law based on Descartes’ scientific research program. At last, he proved that Descartes’ vortex ether hypothesis could not explain celestial motions properly [8]. Newton himself even suggested an explanation of gravity based on the action of an ether pervading the space [9][10]. Euler attempted to explain gravity based on some hypotheses of a fluidic ether [7].

Since quantum theory shows that the vacuum is not empty and has physical effects, e.g., the Casimir effect, it is valuable to probe the vacuum by introducing the following assumption [11].

Assumption 1 Suppose the universe is filled by a fluidic substratum.

This fluidic substratum may be named the Ω(0) substratum in order to distinguish it from the Cartesian ether.

Suppose that a velocity field of a fluid is continuous and finite at all points of the space, with the exception of individual isolated points. Then these isolated points are called singularities in this fluid. Suppose there exists a singularity at point \( P_0 = (x_0, y_0, z_0) \). If the velocity field of the singularity at point \( P = (x, y, z) \) is \( \mathbf{u}(x, y, z, t) = (Q/4\pi r^2)\hat{\mathbf{r}} \), where \( r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \), \( \hat{\mathbf{r}} \) denotes the unit vector directed outward along the line from the singularity to the point \( P = (x, y, z) \), then we call this singularity a sink if \( Q < 0 \). \( Q \) is called the strength of the sink.

Further, we introduce the following assumption [11].

Assumption 2 All the microscopic particles were made up of a kind of elementary sinks in the Ω(0) substratum. These elementary sinks were created simultaneously. The initial masses and the strengths of the elementary sinks are the same.

We may call these elementary sinks as monads. Suppose that a particle with mass \( m \) is composed of \( N \) monads. We have the following relationships [11]:

\[
\frac{dm}{dt} = \frac{\rho_0 q_0}{m_0(t)} m(t),
\]

where \( m(t) \) is the mass of a particle at time \( t \), \( m_0(t) \) is the mass of monad at time \( t \), \( -q_0(q_0 > 0) \) is the strength of a monad, \( \rho_0 \) is the density of the substratum, \( t \geq 0 \).

If Assumption 2 is valid, then, the equation of motion of a particle is [11]

\[
m(t) \frac{d\mathbf{v}_p}{dt} = -\frac{\rho_0 q_0}{m_0(t)} m(t) \mathbf{v}_p + \mathbf{F},
\]

where \( m_0(t) \) is the mass of monad at time \( t \), \( -q_0 \) is the strength of a monad, \( m(t) \) is the mass of the particle at time \( t \), \( \mathbf{v}_p \) is the velocity of the particle, \( \mathbf{F} \) denotes other forces.

From Eq. (4), we see that there exists a universal damping force

\[
\mathbf{F}_d = -\frac{\rho_0 q_0}{m_0} m \mathbf{v}_p
\]

exerted on each particle by the Ω(0) substratum.

3 The Langevin equation model of a Brownian particle moving in an external force field

In this section, we study the stochastic Newtonian mechanics of a particle in a potential by a method similar to Nelson’s stochastic mechanics [2][4].

Suppose that a microscopic particle is moving in an external force field \( \mathbf{F}(x, t) \) in the Ω(0) substratum. In order to describe the motion of the microscopic particle, let us introduce a Cartesian coordinate system \( \{0, x_1, x_2, x_3\} \) which is attached to the static substratum at infinity. Let \( x(t) \) denote the position of the Brownian particle at time \( t \). We assume that the velocity \( \mathbf{v}_p = d\mathbf{x}/dt \) exists. Suppose there are also a damping force \( \mathbf{F}_2 \) and a random force \( \xi(t) \) exerting on the particle. Then, according to Eq. (6), the motion of the particle can be described by the Langevin equation [12]

\[
m \frac{d^2 \mathbf{x}}{dt^2} = -\frac{\rho_0 q_0}{m_0(t)} m(t) \mathbf{v}_p + \mathbf{F}_2 + \mathbf{F}(x, t) + \xi(t),
\]

where \( m \) is the mass of the particle.

We assume that the force \( \mathbf{F}(x, t) \) is a continuous function of \( x \) and \( t \). Suppose that the damping force \( \mathbf{F}_2 \) exerted on the microscopic particle by

\[
\mathbf{F}_2 = -f_2 m \mathbf{v}_p,
\]

where \( f_2 \geq 0 \) is a constant.

Using Eq. (6), Eq. (5) can be written as

\[
m \frac{d^2 \mathbf{x}}{dt^2} = -f \mathbf{v} + \mathbf{F}(x, t) + \xi(t),
\]

where \( f = (\rho \rho_0 q_0/m_0 + f_2) m \).

Next, we need a proper mathematical model of the rapidly fluctuating and highly irregular force \( \xi(t) \) to ensure that Eq. (7) is mathematically explicit. Inspired by the Ornstein-Uhlenbeck theory [13][14] of microscopic motion, it is natural to assume that the
random force $\xi(t)$ exerted on a microscopic particle by the substratum $\Omega(0)$ has similar properties as the random force exerted on a microscopic particle immersed in a classical fluid by the fluid. Thus, we make the following assumptions.

**Assumption 3** We assume that the random force $\xi(t)$ exerted on the particle by the substratum $\Omega(0)$ is a three-dimensional Gaussian white noise and the strength $\eta_i^2$, $\eta_i > 0$, $i = 1, 2, 3$, of the $i$th component of $\xi(t)$ is

$$\eta_i^2 = 2fk_0T_0, \quad i = 1, 2, 3, \quad (8)$$

where $f$ is the damping coefficient of the damping force exerted on the particle by the medium, $k_0$ is a constant similar to the Boltzmann constant $k_B$ which depends on the particles which constitute the substratum $\Omega(0)$, $T_0$ is the equilibrium temperature of the substratum $\Omega(0)$.

For convenience, we introduce notations as $\sigma_i = \eta_i$, $i = 1, 2, 3$. According to Assumption 3 the correlation function $R_i(t, s)$ of the $i$th component of $\xi(t)$ is

$$R_i(t, s) = E[\xi_i(t)\xi_i(s)] = \sigma_i^2\delta(t - s), \quad (9)$$

where $\delta(t)$ is the Dirac delta function, $\xi_i(t)$ is $i$th component of $\xi(t)$.

For convenience, we introduce the following notations

$$\nu_1 = \frac{\sigma_1^2}{2} = f k_0 T_0. \quad (10)$$

The three-dimensional Gaussian white noise $\xi(t)$ is the generalized derivative of a Wiener process $N(t)$ \[15\][17]. We can write formally \[15\][18]

$$\xi(t) = \frac{dN(t)}{dt}, \quad (11)$$

where $N(t)$ is a three-dimensional Wiener process with a diffusion constant $\nu_1$.

Now, based on Assumption 3 the mathematically rigorous form of Eq. (11) is the following stochastic differential equations \[14]\:

\[
\begin{cases}
dx(t) = v_p(t)dt, \\
mvd_p(t) = -fv_p(t)dt + F(x, t)dt + dN(t), \\
x(0) = x_0, \\
v_p(0) = v_0.
\end{cases}
\]

\[
\begin{align}
\nu_1 &= \lim_{m/f \to \infty} x(t) = y(t), \quad (13)
\end{align}
\]

where $y(t)$ is the solution of the following Smoluchowski equation

\[
\begin{cases}
dy(t) = b(y, t)dt + dw(t), \\
y(0) = x_0,
\end{cases}
\]

where $x_0 = x(0)$, $w(t)$ is a three-dimensional Wiener process with a diffusion coefficient $\nu_0$ determined by

$$\nu_0 = \frac{h_0}{2m}, \quad (15)$$

and $h_0 = 2m/k_0 T_0/(\rho_0 g + f_2 m_0)$.

The proof of Theorem 5 can be found in the appendix. If $h_0 = h$, then, we notice that Eq. (15) coincides with the hypothesis $\nu = h/2m$ in Nelson’s stochastic mechanics \[2\].

### 4 A generalized Hamilton-Jacobi equation

Following Nelson \[2\], we define the mean forward derivative $D_+ y(t)$ and the mean backward derivative $D_- y(t)$. We also have another Smoluchowski equation as \[2\]:

$$dy(t) = b_*(y, t)dt + dw_*(t), \quad (16)$$

where $w_*(t)$ has the same properties as $w(t)$ except that the $dw_*(t)$ are independent of the $y(s)$ with $s \geq t$.

Following Nelson \[2\], we introduce the following definitions of current velocity $v(t)$ and osmotic velocity $u(t)$.

$$v = \frac{1}{2}(b + b_*), \quad u = \frac{1}{2}(b - b_*) \quad (17)$$

We have the following result \[2\]:

$$u = \nu_0 \frac{\nabla \rho}{\rho} = \nu_0 \nabla (\ln \rho). \quad (18)$$

From Eq. (18), we introduce the following definition of osmotic potential $R_1$

$$mu = \nabla R_1, \quad (19)$$
where the osmotic potential $R_1$ is defined by $R_1 = m
u_0 \ln \rho$.

Following Nelson [2], we introduce the definition of the mean second derivative $a(t)$ of the stochastic process $y(t)$ as

$$ a(t) = \frac{1}{2} DD_s y(t) + \frac{1}{2} D_s D y(t). \quad (20) $$

Similar to the deterministic Newtonian mechanics, we can also introduce the following concept of deterministic momentum field $p_d(x, t)$ and stochastic momentum field $p_s(x, t)$ of the Brownian particle:

$$ p_d(x, t) = m(t) v(x, t), \quad p_s(x, t) = m(t) u(x, t). \quad (21) $$

From Eq. (2), we see that the mass $m(t)$ of the particle is increasing linearly. Thus, our results depart from the Nelson’s stochastic mechanics. We have the following result.

**Proposition 6** If there exists a function $S_1(x, t)$ such that

$$ p_d = \nabla S_1, \quad (22) $$

then, the deterministic momentum field $p_d(x, t)$ and stochastic momentum field $p_s(x, t)$ of the particle satisfy the following equations

$$ \frac{\partial p_d(t)}{\partial t} = \omega_0 p_d + F - \frac{1}{2m} \nabla (p_d^2) + \frac{1}{2m} \nabla (p_s^2) + \nu_0 \nabla^2 p_s, \quad (23) $$

$$ \frac{\partial p_s(t)}{\partial t} = \omega_0 p_s - \nu_0 \nabla^2 p_d - \frac{1}{m} \nabla (p_d \cdot p_s). \quad (24) $$

where $\omega_0 \triangleq \rho_0 t/m_0(t)$.

**Proof of Proposition 6** According to Eq. (21), we have the following relationships

$$ \frac{\partial p_d(t)}{\partial t} = \frac{dm(t)}{dt} v + m(t) \frac{\partial v}{\partial t}, \quad (25) $$

$$ \frac{\partial p_s(t)}{\partial t} = \frac{dm(t)}{dt} u + m(t) \frac{\partial u}{\partial t}. \quad (26) $$

Following a similar method of Nelson [2], we obtain the following results

$$ \frac{\partial v}{\partial t} = \frac{F}{m} - \nu_0 \nabla v + (u \cdot \nabla) u + \nu_0 \nabla^2 u, \quad (27) $$

$$ \frac{\partial u}{\partial t} = -\nu_0 \nabla (v \cdot \nabla) - \nabla (v \cdot u). \quad (28) $$

Putting Eq. (27) into Eq. (25), we obtain

$$ \frac{\partial p_d(t)}{\partial t} = \frac{dm(t)}{dt} v + F - m(v \cdot \nabla) v + m(u \cdot \nabla) u + m\nu_0 \nabla^2 u, \quad (29) $$

$$ \frac{\partial p_s(t)}{\partial t} = \frac{dm(t)}{dt} u - m\nu_0 \nabla (v \cdot u) - m \nabla (v \cdot u). \quad (30) $$

Using Eq. (2) and some formula in field theory, we arrive at Eq. (23-24). This ends the proof of Proposition 6. □

We may call the functions $S_1(x, t)$ defined in Eq. (22) as the current potential. The current potential $S_1(x, t)$ is not uniquely defined by the deterministic momentum field $p_d(x, t)$.

**Theorem 7** Suppose that there exist two functions $V(x)$ and $S_1$ such that

$$ F(x, t) = -\nabla V(x), \quad (31) $$

$$ p_d = \nabla S_1. \quad (32) $$

Then, the generalized Hamilton’s principal function $S \triangleq S_1 - iR_1, i^2 = -1$, satisfies the following generalized Hamilton-Jacobi equation

$$ \frac{\partial S}{\partial t} = -\omega_0 S + \frac{1}{2m} \nabla^2 (\nabla S)^2 + V(x) - iv_0 \nabla^2 S + a_1(t) + ia_2(t), \quad (33) $$

where $a_1(t)$ and $a_2(t)$ are two unknown real functions of $t$.

The proof of Theorem 7 can be found in the appendix. The generalized Hamilton’s principal function $S$ is not uniquely defined by $p_d$. The reason is that $p_d = \nabla S_1$. Thus, $S_1$ is not uniquely defined by $p_d$.

It is not surprising that the generalized Hamilton-Jacobi equation Eq. (33) is similar to the following Hamilton-Jacobi equation in classical mechanics [19]

$$ \frac{\partial S}{\partial t} = \frac{1}{2m} \nabla^2 (\nabla S)^2 + V(x). \quad (34) $$

Similar to Bohr’s Correspondence Principle in quantum mechanics, we may also introduce the following correspondence principle in stochastic mechanics.

**Assumption 8** If the diffusion constant $\nu_0$ and the parameter $\omega_0$ are small enough, i.e., $\nu_0 \to 0$ and $\omega_0 \to 0$, then, the generalized Hamilton-Jacobi equation Eq. (33) in stochastic mechanics becomes identical to the Hamilton-Jacobi equation Eq. (34) in classical mechanics

**Theorem 9** Suppose that Eq. (31,32) are valid. Then, the generalized Hamilton’s principal function $S$ satisfies the following generalized Hamilton-Jacobi equation

$$ \frac{\partial S}{\partial t} = -\omega_0 S + \frac{1}{2m} \nabla^2 (\nabla S)^2 + V(x) - iv_0 \nabla^2 S. \quad (35) $$
Proof of Theorem 9 Let $\omega_0 = \nu_0 = 0$. Then, from Eq. (18), we have $u = 0$. Thus, from Eq. (21), we have $p_s = 0$. Then, from Eq. (19), $R_1$ is a constant. Thus, Eq. (33) can be written as

$$-\frac{\partial S_1}{\partial t} = \frac{1}{2m} (\nabla S_1)^2 + V(x) + a_1(t) + ia_2(t). \quad (36)$$

According to Assumption 8, Eq. (36) should be identical to the Hamilton-Jacobi equation Eq. (34). Thus, we obtain $a_1(t) = 0$ and $a_2(t) = 0$. This ends the proof of Theorem 9. □

5 Generalized Schrödinger equation in stochastic Newtonian mechanics

We introduce the following definition

$$\psi(x, t) = e^{i \frac{S(x, t)}{2m\nu_0}}. \quad (37)$$

We may call the function $\psi(x, t)$ as wave function following Hamiltonian mechanics [20]. The generalized Hamilton’s principal function $S$ is not uniquely defined by $p$. Therefore, the wave function $\psi(x, t)$ defined by Eq. (37) is not uniquely defined by $p$.

Theorem 10 The wave function $\psi(x, t)$ defined by Eq. (37) satisfies the following Schrödinger like equation

$$i\frac{\partial \psi}{\partial t} = \omega_0 \psi \ln \psi - \nu_0 \nabla^2 \psi + \frac{1}{2m\nu_0} V \psi. \quad (38)$$

Eq. (38) is equivalent to the generalized Hamilton-Jacobi equation Eq. (34).

Proof of Theorem 10 From the definition Eq. (37), we have

$$S(x, t) = \frac{2m\nu_0}{i} \ln \psi(x, t). \quad (39)$$

Putting Eq. (39) into Eq. (35), we obtain a Schrödinger like equation Eq. (38). Conversely, putting Eq. (37) into Eq. (38), we obtain the generalized Hamilton-Jacobi equation Eq. (35). This ends the proof of Theorem 10. □

Putting Eq. (15) into Eq. (10), we obtain the following proposition.

Corollary 11 Suppose that Eq. (31) and (32) are valid. Then, the wave function $\psi(x, t)$ defined by Eq. (37) satisfies the following nonlinear Schrödinger equation

$$i\hbar_0 \frac{\partial \psi}{\partial t} = -\frac{\hbar_0^2}{2m} \nabla^2 \psi + V \psi + \hbar_0 \omega_0 \psi \ln \psi. \quad (40)$$

The mass-increasing effect indicated in Eq. (2) is so small that it may be difficult for us to detect in the time scale of human beings. If we introduce the following assumption: $\omega_0 \to 0$, then, the nonlinear Schrödinger equation Eq. (40) reduces to the Schrödinger equation Eq. (1).

6 Discussion

The wave function $\psi(x, t)$ is not uniquely defined by $p$. Wallstrom [21] points out: “the Madelung equation are not equivalent to the Schrödinger equation unless a quantization imposed. This condition is that the wave function be single valued.” This quantization condition was confirmed by plenty of experiments in quantum mechanics [1, 19]. Thus, a successful stochastic interpretation of nonrelativistic quantum phenomena should derive this quantization condition.

7 Conclusion

In Nelson’s stochastic mechanics, the Schrödinger equation Eq. (1) is derived based on two main hypotheses. In previous works of stochastic mechanics, these two hypotheses can not be explained. Based on a fluidic continuum model of the vacuum and a sink flow model of microscopic particles, we show that Nelson’s two hypotheses can be derived. Similar to Bohr’s Correspondence Principle, we also introduce a correspondence principle. The generalized Hamilton’s principal function satisfies a generalized Hamilton-Jacobi equation. Further, considering the mass-increasing effects, we show that there exists a nonlinear Schrödinger equation for microscopic particles. As a byproduct, the Planck constant $\hbar$ is calculated theoretically.

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8 Appendix

8.1 Proof of Theorem 5

Using the following definitions

$$\beta = \frac{f}{m}, \quad K(x, t) = \frac{F(x, t)}{m}, \quad B(t) = \frac{N(t)}{m}, \quad (41)$$
we see that \( B(t) \) is a three-dimensional Wiener process with a diffusion coefficient \[14\]

\[
\nu_2 = \frac{\nu_1}{m^2} = \frac{f k_0 T_0}{m^2} = \frac{\beta k_0 T_0}{m}. \tag{42}
\]

Using Eq. (41), Eq. (12) become

\[
\begin{align*}
\{ dx(t) &= v_p(t)dt, \\
\{ dv_p(t) &= -\beta v_p(t)dt + K(x, t)dt + dB(t).
\end{align*} \tag{43}
\]

We define

\[
b(x, t) = \frac{K(x, t)}{\beta}, \quad w(t) = \frac{B(t)}{\beta}. \tag{44}
\]

Then, we see that \( w(t) \) is a three-dimensional Wiener process with a diffusion coefficient \[14\]

\[
\nu_0 = \frac{\nu_2}{\beta^2} = \frac{k_0 T_0}{m \beta} = \frac{k_0 T_0}{f}. \tag{45}
\]

Using Eq. (44), Eq. (13) can be written as

\[
\begin{align*}
\{ dx(t) &= v_p(t)dt, \\
\{ dv_p(t) &= -\beta v_p(t)dt + \beta b(x, t)dt + \beta dw(t).
\end{align*} \tag{46}
\]

Let \( x(t) \) be the solution of Eq. (46) with \( x(0) = x_0, v_p(0) = v_0 \). According to Assumption \[4\] the functions \( b(x, t) : R^3 \times R_+ \rightarrow R^3 \) also satisfies a global Lipschitz condition. Applying Nelson’s Theorem 10.1 in \[3\], for a time scale of an observer very large compared to the relaxation time \( 1/\beta \), \( x(t) \) converges to the solution \( y(t) \) of the Smoluchowski equation

\[
dy(t) = b(y, t)dt + dw(t) \tag{47}
\]

with \( y(0) = x_0 \).

From Eq. (7) and Eq. (45), we have

\[
f = \left( \frac{\rho_0 q_0}{m_0} + f_2 \right) m \tag{48}
\]

\[
\nu_0 = \frac{k_0 T_0}{f} = \frac{m_0 k_0 T_0}{(\rho_0 q_0 + f_2 m_0) m}. \tag{49}
\]

We introduce the following notations

\[
h_0 = \frac{2 m_0 k_0 T_0}{(\rho_0 q_0 + f_2 m_0)} = \frac{h_0}{2\pi}. \tag{50}
\]

Using Eq. (50), Eq. (49) can be written as

\[
\nu_0 = \frac{h_0}{2m}, \quad \text{or,} \quad \nu_0 = \frac{h_0}{4\pi m}. \tag{51}
\]

This ends the proof of Theorem \[5\] \[5\]

\section*{8.2 Proof of Theorem \[7\]}

Multiplying Eq. (23) with \(-1\) and adding Eq. (24) multiplied by \(i\), we obtain

\[
\begin{align*}
\frac{\partial (p_d - i p_s)}{\partial t} &= -\nu_0 (p_d - i p_s) - F \\
+ \frac{1}{2m} \nabla [(p_d - i p_s)^2] - iv_0 \nabla^2 (p_d - i p_s).
\end{align*} \tag{52}
\]

We introduce the following definition

\[
p \triangleq p_d - i p_s. \tag{53}
\]

Putting Eq. (53) into Eq. (22), we have

\[
\frac{\partial p}{\partial t} = -\nu_0 p - F + \frac{1}{2m} \nabla (p^2) - iv_0 \nabla^2 p. \tag{54}
\]

We introduce the following definition

\[
S \triangleq S_1 - i R_1. \tag{55}
\]

Putting Eq. (19) and Eq. (32) into Eq. (53) and using Eq. (54), we have

\[
p = \nabla S. \tag{56}
\]

Putting Eq. (56) into Eq. (54), we obtain

\[
\frac{\partial (\nabla S)}{\partial t} = -\nu_0 \nabla S - F + \frac{1}{2m} \nabla [(\nabla S)^2] - iv_0 \nabla^2 (\nabla S). \tag{57}
\]

Noticing \( F = -\nabla V(x) \), Eq. (57) becomes

\[
\frac{\partial (\nabla S)}{\partial t} = -\nu_0 \nabla S + \nabla V + \frac{1}{2m} \nabla [(\nabla S)^2] - iv_0 \nabla^2 (\nabla S). \tag{58}
\]

Eq. (58) can be written as

\[
\nabla \left[ \frac{\partial S}{\partial t} - \nu_0 S + V(x) + \frac{1}{2m} (\nabla S)^2 - iv_0 \nabla^2 S \right] = 0. \tag{59}
\]

Integration of Eq. (59) gives

\[
- \frac{\partial S}{\partial t} = -\nu_0 S + V(x) + \frac{1}{2m} (\nabla S)^2 - iv_0 \nabla^2 S + a_1(t) + ia_2(t), \tag{60}
\]

where \( a_1(t) \) and \( a_2(t) \) are two unknown real functions of \( t \). This ends the proof of Theorem \[7\]. \[5\]

\begin{thebibliography}{9}

[1] L. D. Landau and Lifshitz. \textit{Non-relativistic Theory Quantum Mechanics}, translated from the Russian by J.B. Sykes and J.S. Bell. Pergamon, London, 1958.

[2] E. Nelson. Derivation of the schrodinger equation from newtonian mechanics. \textit{Physical Review}, 150:1079, 1966.
\end{thebibliography}
[3] E. Nelson. *Dynamical Theories of Brownian Motion*. Princeton University Press, Princeton, 1972.

[4] E. Nelson. *Quantum Fluctuations*. Princeton University Press, Princeton, 1985.

[5] E. Nelson. Review of stochastic mechanics. *Journal of Physics: Conference Series*, 361:012011, 2012.

[6] Laozi. *Tao Te King, in seven languages*. Farkas Lorinc Imre Pub., transl. Stephen Mitchell and et al., 1995.

[7] E. Whittaker. *A History of the Theories of Aether and Electricity, Revised and enlarged edition, vol. 1*. Thomas Nelson and Sons Ltd., London, 1951.

[8] Isaac Newton. *Mathematical Principles of Natural Philosophy and His System of the World*. Univ. of Calif. Press, Berkeley, 1962.

[9] I. B. Cohen. *The Newtonian Revolution*. Cambridge University Press, 1980.

[10] Isaac Newton. *Opticks*. Bell London, 1931.

[11] Xiao-Song Wang. Derivation of the newton’s law of gravitation based on a fluid mechanical singularity model of particles. *Progress in Physics*, 4:25–30, 2008.

[12] S. Chandrasekhar. Stochastic problems in physics and astronomy. *Reviews of Modern Physics*, 15:1–89, 1943.

[13] G. E. Uhlenbeck and L. S. Ornstein. On the theory of brownian motion. *Physical Review*, 36:823–841, 1930.

[14] Olav Kallenberg. *Foundations of Modern Probability*. Springer-Verlag, 1997.

[15] I. M. Gel’fand and N. J. Vilenkin. *Generalized Functions, vol. 4, translated from Russian*. Academic Press, New York, 1961.

[16] T. T. Soong. *Random Differential Equations in Science and Engineering*. Academic Press, New York, 1973.

[17] Ludwig Arnold. *Stochastic Differential Equations: Theory and Applications*. John Wiley Sons, New York, 1974.

[18] C. W. Gardiner. *Handbook of Stochastic Methods, 3rd ed*. Springer-Verlag, Berlin, 2004.

[19] J.-Y. Zeng. *Quantum Mechanics, vol. I, 4th edition*. Science Press, Beijing, 2007.

[20] H. Goldstein. *Classical Mechanics*. Addison Wesley, 2002.

[21] T. C. Wallstrom. Inequivalence between the schrödinger equation and the madelung hydrodynamic equations. *Phys. Rev. A*, 49:1613–1617, 1994.