ON \( J \)-SELF-ADJOINT EXTENSIONS OF THE PHILLIPS
SYMMETRIC OPERATOR

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Dedicated to the blessed memory of I. Gohberg.

Abstract. \( J \)-self-adjoint extensions of the Phillips symmetric operator
\( S \) are studied. The concepts of stable and unstable \( C \)-symmetry are introduced in the extension theory framework. The main results are the following: if \( A \) is a \( J \)-self-adjoint extension of \( S \), then either \( \sigma(A) = \mathbb{R} \) or \( \sigma(A) = \mathbb{C} \); if \( A \) has a real spectrum, then 
\( A \) has a stable \( C \)-symmetry and \( A \) is similar to a self-adjoint operator; there are no \( J \)-self-adjoint extensions of the Phillips operator with unstable \( C \)-symmetry.

1. Introduction

Let \( \mathcal{H} \) be a Hilbert space with inner product \((\cdot,\cdot)\) and with fundamental symmetry \( J \) (i.e., \( J = J^* \) and \( J^2 = I \)). The space \( \mathcal{H} \) endowed with the indefinite inner product (indefinite metric) \([x,y]_J := (Jx,y)\), \( \forall x,y \in \mathcal{H} \) is called a Krein space \((\mathcal{H},[\cdot,\cdot]_J)\).

An operator \( A \) in \( \mathcal{H} \) is called \( J \)-self-adjoint if \( A \) is self-adjoint with respect to the indefinite metric \([\cdot,\cdot]_J\). It is clear that \( A \) is \( J \)-self-adjoint if and only if
\[
A^*J = JA.
\]

During the past ten years a steady interest in the study of \( J \)-self-adjoint operators has been strongly increased by the necessity of mathematically correct and rigorous analysis of pseudo-Hermitian Hamiltonians arising in \( \mathcal{PT} \)-symmetric quantum mechanics (PTQM) see e.g., [10]–[19], [32, 35, 38].

In many cases, pseudo-Hermitian Hamiltonians admit the representation \( A+V \), where a (fixed) self-adjoint operator \( A \) and a non-symmetric potential \( V \) satisfy certain (Krein space) symmetry properties which allow one to formalize the expression \( A+V \) as a family of \( J \)-self-adjoint operators \( A_\varepsilon \) acting in a Krein space \((\mathcal{H},[\cdot,\cdot]_J)\). Here \( \varepsilon \in \mathbb{C}^m \) is a complex parameter characterizing the potential \( V \).

Let \( \Xi \) be the domain of variation of \( \varepsilon \). One of important problems for the collection \( \{A_\varepsilon\} \), which is directly inspired by PTQM, is the description of quantitative and qualitative changes of spectra \( \sigma(A_\varepsilon) \) when \( \varepsilon \) runs \( \Xi \). Nowadays this topic has been analyzed with a wealth of technical tools (see, e.g., [7, 8, 21, 24, 39]).

In particular, if the potential \( V \) is singular, then operators \( A_\varepsilon \) turn out to be \( J \)-self-adjoint extensions of the symmetric operator \( \mathcal{S} = A \upharpoonright \ker V \) which commutes with \( J \) and spectral analysis of \( A_\varepsilon \) can be carried out by the extension theory methods [2, 3, 4, 22]. Here, the ‘main ingredients’ are: a holomorphic operator function characterizing \( \mathcal{S} \) (the characteristic function \( \Theta(\cdot) \) [26, 28, 37] or the Weyl function \( M(\cdot) \) [16, 17, 18]) and the boundary conditions which distinguish \( A_\varepsilon \) among other \( J \)-self-adjoint extensions of \( \mathcal{S} \).

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\(^1\)Under a special choice of involution \( J \).
such a setting, the spectral analysis of $A_\varepsilon$ is reduced to the routine solution of algebraic equations including $\theta(\cdot)$ and boundary conditions.

In the present paper we are going to study a special case where the characteristic function of a symmetric operator $S$ with finite deficiency indices is equal to zero ($\Theta(\mu) \equiv 0$, $\forall \mu \in \mathbb{C} \setminus \mathbb{R}$).

One of general constructions leading to symmetric operators $S$ with the zero characteristic function is the following: let $U$ be a bilateral shift with a wandering subspace $W_0$ in $\mathcal{H}$ (see [20] for the terminology) and let $V$ be its restriction onto $\mathcal{H} \ominus W_0$, i.e., $V = U \downharpoonright (\mathcal{H} \ominus W_0)$. Then the operator

$$S = i(V + I)(V - I)^{-1}, \quad \mathcal{D}(S) = \mathcal{R}(V - I)$$

is simple\(^2\) symmetric and its deficiency indices coincide with $< \dim W_0, \dim W_0 >$.

In other words, $S$ is the restriction of the Cayley transform of $U$

$$A = i(U + I)(U - I)^{-1}, \quad \mathcal{D}(A) = \mathcal{R}(U - I)$$

onto $\mathcal{D}(S) = \mathcal{R}(V - I)$.

The operator $S$ defined by (1.2), (1.3) was used by Phillips [36] (with $\dim W_0 = 1$) as an example of the symmetric operator, which is invariant with respect to a certain set $\mathfrak{U}$ of unitary operators ($\mathfrak{U}$-invariant) but it has no $\mathfrak{U}$-invariant self-adjoint extensions. For this reason, the simple symmetric operator $S$ determined by (1.2) and (1.3) will be referred as the Phillips symmetric operator.

Due to specific properties of the Phillips operator (the characteristic function is zero, there are no real points of regular type of $S$, etc) we obtain an evolution of $\sigma(A_\varepsilon)$ which differs from the matrix models [21, 24, 25] and models based on $J$-self-adjoint (symmetric) perturbations of the Schrödinger or Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of an $J$-self-adjoint extension $A_\varepsilon$ of $S$ coincides with real line: $\sigma(A_\varepsilon) = \mathbb{R}$ or with complex plane: $\sigma(A_\varepsilon) = \mathbb{C}$ (Theorem 3.7).

One of the key points in PTQM is the description of a hidden symmetry $C$ which exists for a given pseudo-Hermitian Hamiltonian $A$ in the sector of exact $\mathcal{PT}$-symmetry [9, 10, 11]. The operator $C$ has some rough analogy with the charge conjugation operator in the quantum field theory [10] and it is determined non-uniquely [13]. The existence of $C$ gives rise to an inner product $(\cdot, \cdot)_C = [C, \cdot]_J$ and the dynamics generated by $A$ is therefore governed by a unitary time evolution.

For $J$-self-adjoint extensions $A_\varepsilon \supset S$, where $S$ is an arbitrary symmetric operator commuting with $J$, we introduce the concepts of stable and unstable $C$-symmetry (Definition 2.11). These concepts are natural in the extension theory framework. Roughly speaking, if $A_\varepsilon$ belongs to the sector $\Sigma^s_J$ of stable $C$-symmetry, then $A_\varepsilon$ preserves the property of $C$-symmetry under small variation of $\varepsilon$.

It follows from the results of [1, 23] that for some types of singular perturbations of the Schrödinger or the Dirac operator, the sector $\Sigma^\text{unst}_J$ of unstable $C$ symmetry is not empty and operators $A_\varepsilon$ with real spectra and Jordan points correspond to the case where $\varepsilon$ belongs to the boundary of $\Sigma^\text{unst}_J$.

In the case of the Phillips symmetric operator $S$, the spectral picture above can be essentially simplified. Precisely, assuming the deficiency indices $< 2, 2 >$ of $S$, we show that $\Sigma^\text{unst}_J = \emptyset$ and any $J$-self-adjoint extension of $S$ with real spectrum is similar to a self-adjoint operator (Theorem 3.9 and Corollary 3.10).

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator $A$, respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of

\(^2\)An operator is called simple if its restriction to any nontrivial reducing subspace is not a self-adjoint operator.
A to the set $\mathcal{D}$. The set of points of regular type of a symmetric operator $S$ is denoted by $\hat{\rho}(S)$ (i.e., $r \in \hat{\rho}(S) \iff \|(S - rI)u\| \geq k\|u\|$, $\forall u \in \mathcal{D}(S)$, $k > 0$).

2. Preliminaries

2.1. Elements of the Krein space theory. Let $(\mathcal{H}, [\cdot, \cdot], J)$ be a Krein space with fundamental symmetry $J$. The corresponding orthoprojectors $P_\pm = \frac{1}{2}(I \pm J)$ determine the fundamental decomposition of $\mathcal{H}$

$$
(2.1) \quad \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_- = P_- \mathcal{H}, \quad \mathcal{H}_+ = P_+ \mathcal{H}.
$$

A subspace $\mathcal{L}$ of $\mathcal{H}$ is called hypermaximal neutral if

$$
\mathcal{L} = \mathcal{L}^{[\cdot, \cdot]} = \{ x \in \mathcal{H} : [x, y]_J = 0, \forall y \in \mathcal{L} \}.
$$

A subspace $\mathcal{L} \subset \mathcal{H}$ is called uniformly positive (uniformly negative) if $[x, x]_J \geq a^2\|x\|^2$ $(-[x, x]_J \geq a^2\|x\|^2)$ $a \in \mathbb{R}$ for all $x \in \mathcal{L}$. The subspaces $\mathcal{H}_\pm$ in (2.1) are examples of uniformly positive and uniformly negative subspaces and they possess the property of maximality in the corresponding classes (i.e., $\mathcal{H}_+$ ($\mathcal{H}_-$) does not belong as a subspace to any uniformly positive (negative) subspace).

Let $\mathcal{L}_+(\neq \mathcal{H}_+)$ be an arbitrary maximal uniformly positive subspace. Then its $J$-orthogonal complement $\mathcal{L}_- = \mathcal{L}_+^{[\cdot, \cdot]}$ is a maximal uniformly negative and the direct $J$-orthogonal sum

$$
(2.2) \quad \mathcal{H} = \mathcal{L}_+^{[\cdot, \cdot]} \mathcal{L}_-
$$
gives another (then (2.1)) decomposition of $\mathcal{H}$ onto its positive $\mathcal{L}_+$ and negative $\mathcal{L}_-$ parts (the brackets $[\cdot, \cdot]$ mean the orthogonality with respect to the indefinite metric).

An arbitrary decomposition of the Krein space $(\mathcal{H}, [\cdot, \cdot], J)$ onto its positive and negative parts (like (2.2)) is called canonical.

The subspaces $\mathcal{L}_\pm$ in (2.2) can be described as

$$
\mathcal{L}_+ = (I + X)\mathcal{H}_+, \quad \mathcal{L}_- = (I + X^*)\mathcal{H}_-,
$$

where $X : \mathcal{H}_+ \to \mathcal{H}_-$ is a contraction and $X^* : \mathcal{H}_- \to \mathcal{H}_+$ is the adjoint of $X$.

The self-adjoint operator $T = XP_+ + X^*P_-$ acting in $\mathcal{H}$ is called an operator of transition from the fundamental decomposition (2.1) to the canonical one (2.2). Obviously, $\mathcal{L}_+ = (I + T)\mathcal{H}_+$ and $\mathcal{L}_- = (I + T)\mathcal{H}_-$.

Operators of transition admit a simple description. Namely, a self-adjoint operator $T$ in $\mathcal{H}$ is an operator of transition if and only if $\|T\| < 1$ and $JT = -JT$.

The set $\{T\}$ of all possible operators of transition is in one-to-one correspondence (via $\mathcal{L}_\pm = (I + T)\mathcal{H}_\pm$) with all possible canonical decompositions (2.2) of the Krein space $(\mathcal{H}, [\cdot, \cdot], J)$.

The projectors $P_{\mathcal{L}_\pm} : \mathcal{H} \to \mathcal{L}_\pm$ onto $\mathcal{L}_\pm$ with respect to the decomposition (2.2) are determined by the formulas

$$
P_{\mathcal{L}_-} = (I - T)^{-1}(P_- - TP_+), \quad P_{\mathcal{L}_+} = (I - T)^{-1}(P_+ - TP_-).
$$

The bounded operator

$$
(2.3) \quad C = P_{\mathcal{L}_+} - P_{\mathcal{L}_-} = J(I - T)(I + T)^{-1}
$$
also describes subspaces $\mathcal{L}_\pm$ in (2.2)

$$
(2.4) \quad \mathcal{L}_+ = \frac{1}{2}(I + C)\mathcal{H}, \quad \mathcal{L}_- = \frac{1}{2}(I - C)\mathcal{H}.
$$

The set of operators $C$ determined (2.3) is completely characterized by the conditions

$$
(2.5) \quad C^2 = I, \quad JC > 0.
$$
2.2. Elements of the Von Neumann extension theory. Let $S$ be a closed symmetric densely defined operator in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficiency indices. Denote by $\mathcal{M}_i = \mathcal{H} \cap \mathcal{R}(S - iI)$ and $\mathcal{M}_{-i} = \mathcal{H} \cap \mathcal{R}(S + iI)$ the defect subspaces of $S$ and consider the Hilbert space $\mathcal{M} = \mathcal{M}_{-i} + \mathcal{M}_i$ with the inner product

$$(f, g)_{\mathcal{M}} = (f, g_i) + (f_{-i}, g_{-i}) \quad f = f_i + f_{-i}, \quad g = g_i + g_{-i} \quad \{f_{\pm i}, g_{\pm i}\} \subset \mathcal{M}_{\pm i}.$$ 

The operator $Z(f_{-i} + f_i) = f_{-i} - f_i$ is a fundamental symmetry in the Hilbert space $\mathcal{M}$ and its restriction onto $\mathcal{M}_{-i}$ and $\mathcal{M}_i$ coincide, respectively, with $I$ and $-I$.

Let $J$ be a fundamental symmetry in $\mathcal{H}$. In what follows we assume that

$$(2.6) \quad SJ = JS.$$ 

Then the subspaces $\mathcal{M}_{\pm i}$ reduce $J$ and the restriction $J \mid \mathcal{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathcal{M}$. Moreover, according to the properties of $Z$ mentioned above, $JZ = ZJ$. Therefore, $JZ$ is a fundamental symmetry in $\mathcal{M}$ and sesquilinear form

$$(f, g)_{\mathcal{M}} = (JZf, g)_{\mathcal{M}} = (Jf_{-i}, g_{-i}) - (Jf_i, g_i)$$

determines an indefinite metric on $\mathcal{M}$.

According to von-Neumann formulas any closed intermediate extension $A$ of $S$ (i.e., $S \subset A \subset S^*$) is uniquely determined by the choice of a subspace $M \subset \mathcal{M}$. Precisely,

$$(2.7) \quad D(A) = D(S) + M \quad \text{and} \quad A = S^* \mid D(A).$$

We use the notation $A_M$ for $J$-self-adjoint extensions of $S$ determined by (2.7).

Let $A_M$ and $\tilde{A}_\mathcal{M}$ be arbitrary extensions of $S$ that are defined by the subspaces $M$ and \( \tilde{M} \), respectively. Taking (2.6) and (2.7) into account we derive

$$(2.8) \quad [A_M \psi, \phi]_J - [\psi, A_M^* \phi]_J = 2i[f, g]_{JZ}$$

for all $\psi = u + f \in D(A_M)$, $f \in M$, $\phi = v + g \in D(A_{\mathcal{M}}^*)$, $g \in \tilde{M}$.

It follows from (1.1) and (2.8) that an extension $A_M$ of $S$ is $J$-self-adjoint if and only if

$$M = M^{[\pm]_{JZ}} = \{ f \in \mathcal{M} : [f, g]_{JZ} = 0, \forall g \in M \},$$

i.e., if $M$ is a hypermaximal neutral subspace of the Krein space $(\mathcal{M}, [\cdot, \cdot]_{JZ})$. Formalizing this observation we get the well-known result.

**Proposition 2.1.** The correspondence $A \mapsto M$ determined by (2.7) is a bijection between $J$-self-adjoint (self-adjoint) extensions $A$ of $S$ and hypermaximal neutral subspaces $M$ of the Krein space $(\mathcal{M}, [\cdot, \cdot]_{JZ})$ (of the Krein space $(\mathcal{M}, [\cdot, \cdot]_{J})$).

Denote by $\Sigma_J(S)$ the set of $J$-self-adjoint extensions of $S$. In general, these extensions may have complex spectra and, moreover, the existence of $A \in \Sigma_J(S)$ with empty resolvent set (i.e., $\sigma(A) = \mathbb{C}$) is also possible. To guarantee nonempty resolvent set for any $A \in \Sigma_J(S)$ we need to impose additional constraints. In this way we recall that a $J$-self-adjoint operator $A$ is called definitizable if the resolvent set of $A$ is nonempty and there exists a polynomial $p(\cdot) \neq 0$ such that $p(A)$ is a nonnegative operator in the Krein space $(\mathcal{H}, [\cdot, \cdot]_J)$.

**Proposition 2.2.** ([6]). Let $S$ have finite deficiency indices. Then if there exists a definitizable extension $A \in \Sigma_J(S)$, then an arbitrary operator from $\Sigma_J(S)$ has a nonempty resolvent set and is definitizable.
2.3. Boundary value spaces technique. Proposition 2.1 provides a description of \( \Sigma_J(S) \) in terms of the Krein space \((\mathfrak{M}, [\cdot, \cdot]_J)\). Another approach which allows one to avoid the use of \( \mathfrak{M} \) is based on the concept of boundary triplets (or boundary value spaces, see [22] and the references therein).

Definition 2.3. A triplet \( (\mathcal{H}, \Gamma_0, \Gamma_1) \), where \( \mathcal{H} \) is an auxiliary Hilbert space and \( \Gamma_0, \Gamma_1 \) are linear mappings of \( \mathcal{D}(S^*) \) into \( \mathcal{H} \), is called a boundary triplet of \( S^* \) if the abstract Green identity

\[
(S^* \psi, \phi) - (\psi, S^* \phi) = (\Gamma_1 \psi, \Gamma_0 \phi)_\mathcal{H} - (\Gamma_0 \psi, \Gamma_1 \phi)_\mathcal{H}, \quad \psi, \phi \in \mathcal{D}(S^*)
\]

is satisfied and the map \( (\Gamma_0, \Gamma_1) : \mathcal{D}(S^*) \to \mathcal{H} \oplus \mathcal{H} \) is surjective.

Denote

\[
(\mathfrak{M}_\mu = \mathfrak{H} \oplus \mathcal{R}(S - \mu I) = \ker(S^* - \mu I), \quad \mu \in \hat{\rho}(S)).
\]

The Weyl function \( M(\cdot) \) and the characteristic function \( \Theta(\cdot) \) of \( S \) associated with a boundary triplet \( (\mathcal{H}, \Gamma_0, \Gamma_1) \) are defined as follows [18, 27, 37]:

\[
M(\mu) \Gamma_0 f_\mathfrak{F} = \Gamma_1 f_\mathfrak{F}, \quad \forall f_\mathfrak{F} \in \mathfrak{M}_\mathfrak{F}, \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R},
\]

\[
\Theta(\mu)(\Gamma_1 + i\Gamma_0) f_\mathfrak{F} = (\Gamma_1 - i\Gamma_0) f_\mathfrak{F}, \quad \forall \mu \in \mathbb{C}_+.
\]

It is clear that \( \Theta(\mu) = (M(\mu) - iI)/(M(\mu) + iI)^{-1}, \forall \mu \in \mathbb{C}_+ \).

The Weyl function (or, characteristic function) determines a simple symmetric operator \( S \) up to unitary equivalence.

The simplest (canonical) boundary triplet can immediately be constructed as a triplet \((\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1)\), where

\[
(\mathfrak{M}_{-i} = \mathfrak{H} \oplus \mathfrak{R}(S + iI) = \ker(S^* + iI), \quad \mu \in \hat{\rho}(S)).
\]

(2.11) \( \Gamma_0 \psi = f_- + Q f_i, \quad \Gamma_1 \psi = if_+ - if_i, \quad \psi = u + f_- + f_+ \in \mathcal{D}(S^*) \)

and \( Q \) is an arbitrary unitary mapping \( Q : \mathfrak{M}_i \to \mathfrak{M}_{-i} \).

To underline the dependence of \( \Gamma_j \) on the choice of \( Q \) in (2.11), we denote by \((\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1, Q)\) the corresponding boundary triplet.

If \( Q \) commutes with \( J \), then the boundary operators \( \Gamma_j \) defined by (2.11) satisfy the relations

\[
(2.12) \quad \Gamma_0 J = J \Gamma_0, \quad \Gamma_1 J = J \Gamma_1.
\]

By Proposition 2.1, self-adjoint extensions \( A_M \supset S \) commuting with \( J \) are described by hypermaximal neutral subspaces

\[
(2.13) \quad M_G = \{ f_i + G f_i \mid \forall f_i \in \mathfrak{M}_i \}
\]

of the Krein space \((\mathfrak{M}, [\cdot, \cdot]_J)\) which satisfy the additional relation \( JM_G = M_G \). Here \( G : \mathfrak{M}_i \to \mathfrak{M}_{-i} \) are unitary mappings. Obviously, \( J M_G = M_G \iff JG = GJ \). The latter gives rise to the existence of boundary triplets \((\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1, -G)\) defined by (2.12) with the additional properties (2.12). We prove the following simple statement:

Proposition 2.4. Boundary triplets \((\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1, Q)\) satisfying (2.12) exist if and only if the set of self-adjoint extensions of \( S \) commuting with \( J \) is non-empty.

For such type of boundary triplets, Proposition 2.1 can be rewritten as follows:

Proposition 2.5. Let \((\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1, Q)\) be a boundary triplet of \( S^* \) which satisfies (2.12). Then an arbitrary \( A \in \Sigma_J(S) \) with \( i \not\in \sigma(A) \) coincides with the restriction of \( S^* \) onto the domain

\[
(2.14) \quad \mathcal{D}(A) = \{ f \in \mathcal{D}(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f \},
\]

where \( K \) is a \( J \)-unitary operator in \( \mathfrak{M}_{-i} \) (i.e., \( J = K^*JK \)).
The correspondence $A = A_K \leftrightarrow K$ determined by (2.14) is a bijection between the set of all $J$-self-adjoint extensions $A_K$ of $S$ such that $i \notin \sigma(A_K)$ and the set of $J$-unitary operators in $\mathfrak{N}_1$. Furthermore,

$$A_K = A_{(K^*)^{-1}}.$$

**Remark 2.6.** $J$-Self-adjoint extensions $A_M$ with $i \in \sigma(A_M)$ are characterized by nontrivial intersections $M \cap \mathfrak{N}_{-i}$ of the corresponding subspaces $M$ in (2.7). In that case, the description (2.14) of $D(A_M)$ is impossible (since $\ker(\Gamma_1 - i\Gamma_0) = \mathfrak{N}_{-i}$ and $\ker(\Gamma_1 + i\Gamma_0) = \mathfrak{N}_i$ by (2.11)).

2.4. **Description of $\Sigma_J(S)$. The case of deficiency indices $< 2, 2 >$.** We are going to analyze $\Sigma_J(S)$ in more detail for the case where $S$ has deficiency indices $< 2, 2 >$. To avoid the study of self-adjoint extensions we assume $J \neq I$. Then, the following subspaces of the Hilbert space $\mathfrak{M}$:

$$\mathfrak{M}_{11} = (I + Z)(I + J)\mathfrak{M}, \quad \mathfrak{M}_{1-} = (I - Z)(I - J)\mathfrak{M}, \quad \mathfrak{M}_{-1} = (I + Z)(I - J)\mathfrak{M}, \quad \mathfrak{M}_{-} = (I - Z)(I + J)\mathfrak{M},$$

are nontrivial and mutually orthogonal. Therefore, $\dim \mathfrak{M}_{11} = 1$ (since $\dim \mathfrak{M} = 4$) and there exists an orthonormal basis $\{e_{\pm}\}$ of the Hilbert space $\mathfrak{M}$ such that

$$\mathfrak{M}_{11} = e_{1+}, \quad \mathfrak{M}_{1-} = e_{1-}, \quad \mathfrak{M}_{-1} = e_{-1}, \quad \mathfrak{M}_{-} = e_{-}.$$}

In that case

$$Je_{1+} = e_{1+}, \quad Je_{1-} = e_{1-}, \quad Je_{-1} = -e_{1-}, \quad Je_{-} = -e_{-};$$

$$Ze_{1+} = e_{1+}, \quad Ze_{1-} = -e_{1-}, \quad Ze_{-1} = e_{-1}, \quad Ze_{-} = -e_{-}.$$}

Let us consider the boundary triplet $(\mathfrak{M}_{-i}, \Gamma_0, \Gamma_1, Q)$ defined by (2.11), where a unitary mapping $Q : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ acts as follows:

$$Qe_{-1} = e_{1+}, \quad Qe_{-} = e_{-}.$$}

The operator $Q$ commutes with $J$ due to (2.16) and hence, relations (2.12) hold.

Denote by $K = \|k_{ij}\|$ the matrix representation of a $J$-unitary operator $K$ in $\mathfrak{M}_{-i}$ with respect to the basis $\{e_{1+}, e_{1-}\}$. By (2.16), the restriction of $J$ onto $\mathfrak{M}_{-i}$ can be identified (with respect to the basis $\{e_{1+}, e_{1-}\}$) with the matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This means that $\sigma_3 = K^* \sigma_3 K$ (since $K$ is $J$-unitary). The simple analysis of the latter relation leads to the following description of $K$:

$$K = K(\zeta, \phi, \omega, \xi) = e^{-i\xi} \begin{pmatrix} -\cosh(\zeta)e^{-i\phi} & (\sinh(\zeta)e^{-i\omega}) \\ (\cosh(\zeta)e^{i\omega}) & -\sinh(\zeta)e^{i\phi} \end{pmatrix},$$

where $\zeta \in \mathbb{R}$ and $\xi, \phi, \omega \in [0, 2\pi)$. Using Proposition 2.5, we obtain the following

**Proposition 2.7.** The formula

$$S^* \mid D(A_M), \quad D(A_M) = \{ f \in D(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f \},$$

where $K$ is an arbitrary $J$-unitary operator in $\mathfrak{M}_{-1}$ and the boundary triplet $(\mathfrak{M}_{-1}, \Gamma_0, \Gamma_1, Q)$ is defined by (2.11) and (2.17) establishes the one to one correspondence between $J$-self-adjoint extensions $A_M \in \Sigma_J(S)$ with $i \notin \sigma(A_M)$ and matrices $K(\zeta, \phi, \omega, \xi)$ defined by (2.18).

**Remark.** It follows from Proposition 2.1 and relations (2.16) that operators $A_M \in \Sigma_J(S)$ with $i \in \sigma(A_M)$ are described by the two-parameter set of hypermaximal neutral subspaces

$$M(k_1, k_2) = e_{1+} + e^{ik_1}e_{1-}; \quad e_{-1} + e^{ik_2}e_{-}, \quad k_1, k_2 \in \mathbb{R}.$$
of the Krein space \((\mathfrak{M}, [\cdot, \cdot]_J)\). By virtue of (2.11) and (2.17), the subspaces \(M(k_1, k_2)\) can be (formally) described by (2.14) if we put

\[ \zeta = \infty, \quad \xi = 0, \quad \omega = \frac{k_2 + k_1}{2}, \quad \phi = \frac{k_2 - k_1}{2} \]

in (2.18) and consider \(\cosh \infty = \sinh \infty = \infty\) as a number.

To emphasize the relationship \(A_M \equiv \mathcal{K}\) established in Proposition 2.7, we will use the notation \(A_K\) instead of \(A_M\).

**Corollary 2.8.** The adjoint operator \(A_K^*\) of \(A_K \in \Sigma_J(S)\) is defined by \(\mathcal{K}(\zeta, \phi, \xi, \omega)\)

\[ \text{i.e.,} \]

\[ A_K^*(\zeta, \phi, \xi, \omega) = A_K(\zeta, \phi, \xi, \omega). \]

The set of self-adjoint extensions of \(S\) commuting with \(J\) is described by unitary matrices \(\mathcal{K}(0, \phi, \omega, \xi)\).

**Proof.** The relation (2.20) follows from (2.15) and (2.18).

If a self-adjoint extension \(A \supset S\) commutes with \(J\), then \(A\) is also \(J\)-self-adjoint and \(A = A_K(\zeta, \phi, \omega, \xi)\) by Proposition 2.7. Using (2.20) and taking into account (2.18), we get \(\zeta = 0\) that completes the proof of Corollary 2.8. \(\square\)

2.5. **The property of \(C\)-symmetry.** By analogy with [10] the definition of \(C\)-symmetry in the Krein spaces setting can be formalized as follows.

**Definition 2.9.** An operator \(A\) acting in a Krein space \((\mathfrak{H}, [\cdot, \cdot]_J)\) has the property of \(C\)-symmetry if there exists a bounded linear operator \(C\) in \(\mathfrak{H}\) such that:

(i) \(C^2 = I\);
(ii) \(JC > 0\);
(iii) \(AC = CA\).

By virtue of (2.3) and (2.5) the property of \(C\)-symmetry of \(A\) means that \(A\) can be decomposed

\[ A = A_+ [\cdot]_J A_-, \quad A_+ = A \upharpoonright \mathfrak{L}_+, \quad A_- = A \upharpoonright \mathfrak{L}_- \]

with respect to the canonical decomposition (2.2) (with subspaces \(\mathfrak{L}_\pm\) determined by (2.4)).

If a \(J\)-self-adjoint operator \(A\) possesses the property of \(C\)-symmetry, then its counterparts \(A_\pm\) in (2.21) turn out to be self-adjoint operators in the Hilbert spaces \(\mathfrak{L}_\pm\) with the inner products \([\cdot, \cdot]_J\) and \([-\cdot, \cdot]_J\), respectively. This simple observation leads to the following statement.

**Proposition 2.10.** ([1]) A \(J\)-self-adjoint operator \(A\) has the property of \(C\)-symmetry if and only if \(A\) is similar to a self-adjoint operator in \(\mathfrak{H}\). If a \(J\)-self-adjoint operator \(A\) has the property of \(C\)-symmetry then its spectrum is real and the adjoint operator \(C^*\) provides the property of \(C\)-symmetry for \(A^*\).

**Definition 2.11.** Let \(A \in \Sigma_J(S)\) have the property of \(C\)-symmetry realized by an operator \(C\). We will say that \(A\) belongs to the sector \(\Sigma^R_J\) of stable \(C\)-symmetry if the operator \(C\) commutes with \(S\). Otherwise \((AC = CA\ but SC \neq CS)\), the operator \(A\) belongs to the sector \(\Sigma^\ast_J\) of unstable \(C\)-symmetry.

The next statement immediately follows from Theorem 3.1 in [1].

**Proposition 2.12.** Let \(A_M \in \Sigma_J(S)\) be defined by (2.7). Then \(A_M \in \Sigma^R_J\) if and only if \(CM = M\), where \(C\) realizes the property of \(C\)-symmetry for \(S\).

3. **The Phillips symmetric operator**

We are going to specify general results of previous section to the case of Phillips symmetric operator \(S\) defined by (1.2) and (1.3).
3.1. Preliminaries. The general definition (1.2), (1.3) of \( S \) looks rather abstract and, in many cases, it is useful to work with a model realization of \( S \) in \( \mathcal{H} = l_2(\mathbb{Z}, N) \) (\( N \) is an auxiliary finite-dimensional Hilbert space). In that case
\[
U(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots),
\]
\[
V(\ldots, x_{-2}, x_{-1}, 0, x_1, x_2, \ldots) = (\ldots, x_{-3}, x_{-2}, x_{-1}, 0, x_1, x_2, \ldots),
\]
where \( x_j \in N \) and elements at the zero position are underlined.

The self-adjoint operator \( A \) takes the form
\[
Af = i(\ldots, x_{-3} + x_{-2}, x_{-2} + x_{-1}, x_{-1} + x_0, x_0 + x_1, x_1 + x_2, \ldots),
\]
\[
f \in D(A) \iff f = (\ldots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, x_{-1} - x_0, x_0 - x_1, x_1 - x_2, \ldots),
\]
where \( \sum_{i \in \mathbb{Z}} ||x_i||_N^2 < \infty \) and the symmetric operator \( S \) is the restriction of \( A \) onto the set
\[
u \in D(S) \iff \nu = (\ldots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, x_{-1} - x_0, x_0 - x_1, x_1 - x_2, \ldots),
\]
which consists of all \( \nu \in D(A) \) such that \( x_0 = 0 \).

Recalling (2.9) and using (3.2), (3.3), it is easily to see that (see, e.g., [29])
\[
\mathcal{H}_\mu = \{ f_\mu(x) = (\ldots, 0, 0, F, 0, 0, \ldots) : \forall x \in N \},
\]
\[
\mathcal{H}_i = \{ f_i(x) = (\ldots, 0, 0, 0, x, 0, 0, \ldots) : \forall x \in N \}, \quad \mu \in \mathbb{C}_+,
\]
\[
\mathcal{H}_- = \{ f_-(x) = (\ldots, 0, 0, 0, x, 0, 0, \ldots) : \forall x \in N \},
\]
\[
\mathcal{H}_\sigma = \{ f_\sigma(x) = (\ldots, 0, 0, 0, x, r_\sigma x, r_\sigma^2 x, \ldots) : \forall x \in N \},
\]
where \( r_\mu = \frac{\mu^+}{\mu^+} \). Direct calculation with the use of (3.3) and (3.4) gives
\[
f_\mu((1 - r_\mu)x) = u + f_i(x), \quad f_\sigma((1 - r_\mu)x) = v + f_-(x), \quad \forall x \in N,
\]
where \( u, v \in D(S) \). Therefore,
\[
\mathcal{H}_\mu \subset D(S)^+ \mathcal{H}_i, \quad \mathcal{H}_\sigma \subset D(S)^+ \mathcal{H}_-, \quad \forall \mu \in \mathbb{C}_+.
\]

**Lemma 3.1.** Let \( (\mathcal{H}_-, \Gamma_0, \Gamma_1, Q) \) be a boundary triplet of the Phillips symmetric operator \( S \) (defined by (1.2) and (1.3)). Then the corresponding characteristic function \( \Theta(\cdot) \) of \( S \) is equal to zero.

**Proof.** It is sufficient to verify this statement for the case where \( S \) is defined by (3.2) and (3.3). According to (3.6), an arbitrary \( f_\tau \in \mathcal{H}_\sigma \) has the form \( f_\tau = u + f_-, \) where \( u \in D(S) \) and \( f_- \in \mathcal{H}_- \). But then \( (\Gamma_1 + i\Gamma_0)f_\tau = 2if_- \) and \( (\Gamma_1 - i\Gamma_0)f_\tau = 0 \) due to (2.11). Therefore, \( \Theta(\mu) \equiv 0 \) (\( \forall \mu \in \mathbb{C}_+ \)) by (2.10). Lemma 3.1 is proved. \( \square \)

**Lemma 3.2.** Let \( S \) be defined by (3.2) and (3.3) and let \( J \) be a fundamental symmetry in \( l_2(\mathbb{Z}, N) \). Then \( J \) commutes with \( S \) if and only if
\[
J(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, J^-x_{-2}, J^-x_{-1}, J^-x_0, J^+x_1, J^+x_2, \ldots),
\]
where \( J_\pm \) are fundamental symmetries in \( N \).

**Proof.** Let \( J \) commute with \( S \). It follows from (2.9) that defect subspaces \( \mathcal{H}_\mu \) are invariant with respect to \( J \). Taking (3.4) into account we conclude that the restrictions \( J^- := J \upharpoonright \mathcal{H}_i \) and \( J^+ := J \upharpoonright \mathcal{H}_- \) determine two fundamental symmetries \( J^- \) and \( J^+ \) in \( N \). Further, the equality \( JS = SJ \) is equivalent to the relation \( JV = VJ \), where \( V \) is defined by (3.1). Combining this relation with the first and third relations in (3.4) and taking the definition of \( J_\pm \) into account we establish (3.7).

Conversely, if a fundamental symmetry \( J \) is defined by (3.7), then relations (3.2) and (3.3) imply that \( JS = SJ \). Lemma 3.2 is proved. \( \square \)
Lemma 3.3. Let $S$ be defined by (3.2) and (3.3), let $J$ be a fundamental symmetry in $l_2(\mathbb{Z}, N)$ commuting with $S$, and let $C$ be a bounded operator in $l_2(\mathbb{Z}, N)$ such that $C^2 = I$ and $JC > 0$. Then $C$ commutes with $S$ if and only if
\begin{equation}
C(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, C_{-x_{-2}}, C_{-x_{-1}}, C_{x_0}, C_{x_1}, C_{x_2}, \ldots),
\end{equation}
where $C_{\pm}$ are bounded operators in $N$ such that $C_{\pm}^2 = I_N$ and $J_{\pm} C_{\pm} > 0$ where $J_{\pm}$ are taken from the formula (3.7).

Proof. By Lemma 3.2, the operator $J$ is defined by (3.7), where $J_{\pm}$ are fundamental symmetries in $N$.

Assume that $C$ commutes with $S$. Then, using (2.6) one gets $SF = FS$, where $F = JC$ is a bounded self-adjoint operator. Hence,
$$SC^* = SFJ = FJS = C^*S.$$ The obtained relation $C^*S = SC^*$ and $C^2 = I$ imply that the defect subspaces $\mathcal{R}_{\mu}$ of $S$ are invariant with respect $C$. It follows from (3.4) that the restrictions $C_- := C \upharpoonright \mathcal{R}_{\mu}$ and $C_+ := C \upharpoonright \mathcal{R}_{-\mu}$ determine bounded operators $C_{\pm}$ in $N$ such that $C_{\pm}^2 = I_N$ and $J_{\pm} C_{\pm} > 0$. Reasoning by analogy with the proof of Lemma 3.2, we complete the proof. \qed

3.2. Description of $J$-self-adjoint extensions. Using (3.4) we can identify the Hilbert space $\mathcal{M} = \mathcal{R}_{-i} \oplus \mathcal{R}_i$ with
$$N \oplus N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in N \right\}.$$
In that case
\begin{equation}
Z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad J \upharpoonright \mathcal{M} = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix}.
\end{equation}

Proposition 3.4. Let $S$ be defined by (3.2) and (3.3). Then the set $\Sigma_J(S)$ of $J$-self-adjoint extensions of $S$ is non-empty if and only if
\begin{equation}
\dim[(I - J_+)N] = \dim[(I - J_-)N].
\end{equation}

Proof. By Proposition 2.1, $J$-self-adjoint extensions of $S$ exist if and only if the Krein space $(\mathcal{M}, [\cdot, \cdot]_2)$ has hypermaximal neutral subspaces. This is possible only in the case where $\dim[(I + JZ)\mathcal{M}] = \dim[(I - JZ)\mathcal{M}]$ or, that is equivalent (see (3.9)),
$$\dim[(I + J_+)N] + \dim[(I - J_-)N] = \dim[(I - J_+)N] + \dim[(I + J_-)N].$$
This identity is equivalent to (3.10) (since $\dim[(I + J_+)N] + \dim[(I - J_-)N] = \dim N$ and $\dim N < \infty$). Proposition 3.10 is proved. \qed

Corollary 3.5. Let $S$ be defined by (3.2) and (3.3) and let $J$ be a fundamental symmetry commuting with $S$ in $l_2(\mathbb{Z}, N)$. Then self-adjoint extensions of $S$ commuting with $J$ exist if and only if the identity (3.10) holds.

Proof. If $A_M$ is a self-adjoint extension of $S$ commuting with $J$, then $A_M \in \Sigma_J(S)$ and relation (3.10) holds due to Proposition 3.4.

Conversely, since $\dim N < \infty$, relation (3.10) is equivalent to the identity
$$\dim[(I + J_+)N] = \dim[(I + J_-)N].$$
This implies the existence of unitary mappings $G : \mathcal{R}_i \to \mathcal{R}_{-i}$ such that $GJ = GJ_- = J_+ G = JG$. In that case the hypermaximal neutral subspace $M_G$ of the Krein space $(\mathcal{M}, [\cdot, \cdot]_2)$ (defined by (2.13)) satisfies the relation $JM_G = M_G$ and the corresponding self-adjoint extension $A_M$ commutes with $J$. Corollary 3.5 is proved. \qed
Proposition 3.6. Let $S$ be the Phillips symmetric operator (defined by (1.2) and (1.3)) and let $J$ be a fundamental symmetry commuting with $S$ in $\mathcal{F}$. Then boundary triplets $(\mathcal{N}_-,\Gamma_0,\Gamma_1,Q)$ of $S^*$ defined by (2.11) and satisfying (2.12) exist if and only if the set $\Sigma_J(S)$ is non-empty.

Proof. It is sufficient to establish for the Phillips symmetric operator $S$ realized by the formulas (3.2) and (3.3). In that case, by Proposition 3.4 and Corollary 3.5, $\Sigma_J(S) \neq \emptyset \iff$ there exist self-adjoint extensions of $S$ commuting with $J$. Using now Proposition 2.4 we complete the proof. 

\[ \square \]

Theorem 3.7. Let $S$ be the Phillips symmetric operator, let $J$ be a fundamental symmetry commuting with $S$ in $\mathcal{F}$, and let $A_M \in \Sigma_J(S)$. Then the spectrum of $A_M$ either coincides with $\mathbb{R}$ ($\sigma(A_M) = \mathbb{R}$) or covers the whole complex plane ($\sigma(A_M) = \mathbb{C}$) and its non-real part consists of eigenvalues of $A_M$.

Proof. Since an arbitrary $A_M \in \Sigma_J(S)$ is a finite rank perturbation of the self-adjoint operator $A$ (see (1.3)), the non-real spectrum of $A_M$ may include complex eigenvalues.

Without loss of generality we can suppose that $S$ is determined by the formulas (3.2) and (3.3). Assume that $\mu_0 \in \mathbb{C}_+$ is an eigenvalue of $A_M$. Then there exists an element $f_{\bar{\mu}_0} \in \mathfrak{N}_{\bar{\mu}_0} \cap D(A_M)$ and, according to (3.5),
\[ f_{\bar{\mu}_0} = f_{\bar{\mu}_0}(x) = v_0 + f_{-i}\left(\frac{x}{1 - r_{\mu_0}}\right), \quad v_0 \in D(S) \]
for a certain choice of $x \in N$. Using the second relation in (3.5) for an arbitrary $\mu \in \mathbb{C}_+$, we obtain
\[ f_{\pi}(\frac{1 - r_\mu}{1 - \rho_\mu}x) = v + f_{-i}\left(\frac{x}{1 - r_\mu}\right), \quad v \in D(S). \]
Comparing last two relations we arrive at the conclusion that the element
\[ f_{\pi}(\frac{1 - r_\mu}{1 - \rho_\mu}x) = v - v_0 + f_{\bar{\mu}_0}, \quad \mu \in \mathbb{C}_+, \]
belongs to $\mathfrak{N}_{\pi} \cap D(A_M)$. Hence $\mathbb{C}_+ \subseteq \sigma_p(A_M)$. The relation $\mathbb{C}_- \subseteq \sigma_p(A_M)$ is established by the same manner. Thus, $\sigma(A_M) = \mathbb{C}$ and $\mathbb{C} \setminus \mathbb{R}$ contains eigenvalues of $A_M$.

Assume that the spectrum of $A_M$ is real. Since the Phillips symmetric operator has no real points of regular type (see, e.g., [26]), the spectrum of $A_M$ coincides with $\mathbb{R}$. Theorem 3.7 is proved.

\[ \square \]

Corollary 3.8. Let $S$ be the Phillips symmetric operator and let $J$ be a fundamental symmetry commuting with $S$ in $\mathcal{F}$. Then the set $\Sigma_J(S)$ does not contain definitizable operators.

Proof. By Proposition (3.6), if $\Sigma_J(S)$ is a non-empty set, then there exists a boundary triplet $(\mathfrak{N}_{-i},\Gamma_0,\Gamma_1,Q)$ of $S^*$ which satisfies (2.12). It follows from Proposition 2.5 and Theorem 3.7 that operators $A_M \in \Sigma_J(S)$ with real spectrum are described by the formula (2.14) in terms of the boundary triplet $(\mathfrak{N}_{-i},\Gamma_0,\Gamma_1,Q)$. The rest of operators $A_M \in \Sigma_J(S)$ (which can not be described by (2.14)) have empty resolvent set (due to Remark 2.6 and Theorem 3.7). By Proposition 2.2 this means that $\Sigma_J(S)$ does not contain definitizable operators. Corollary 3.8 is proved.

\[ \square \]

3.3. $J$-self-adjoint extensions with $C$-symmetry.

Theorem 3.9. Let $S$ be the Phillips operator with deficiency indices $< 2, 2 >$. Then an arbitrary $J$-self-adjoint extension $A_M \in \Sigma_J(S)$ has the property of stable $C$-symmetry ($A_M \in \Sigma^C_J$) if and only if the spectrum of $A_M$ is real.
Proof. If $A_M$ has $C$-symmetry, then its spectrum is real (see Proposition 2.10).

Conversely, we assume that $A_M \in \Sigma_J(S)$ has a real spectrum. In that case, by Proposition 2.7, $A_M(= A_K)$ is defined by (2.19), where $K = K(\zeta, \phi, \omega, \xi)$ has the form (2.18).

Without loss of generality we can assume that $S$ is determined by the formulas (3.2) and (3.3) in the space $l_2(\mathbb{Z}, N)$. Then, by virtue of Proposition 2.12 and Lemma 3.3, the operator $A_M(= A_K)$ has a stable $C$-symmetry if and only if $CM = M$ for at least one of operators $C$ determined by (3.8).

It follows from (2.19) that $M = \{ f = f_{-i} + f_i \mid K(\Gamma_1 + i\Gamma_0) f = (\Gamma_1 - i\Gamma_0) f \}$. Employing (2.11) we rewrite the latter relation as follows:

$$M = \{ f_{-i} - Q^{-1} K f_{-i} \mid \forall f_{-i} \in \mathfrak{M}_i \}.$$ 

The obtained description of $M$ and Lemma 3.3 imply that

$$(3.11) \quad CM = M \iff KC_+ = \hat{C}_+ K \quad (\hat{C}_+ := QC - Q^{-1}),$$

where $C_+$ and $\hat{C}_+$ act in $N = \mathfrak{M}_i$ and satisfy the relations $C^2 = I$, $JC > 0$ ($C \in \{ C_+, \hat{C}_+ \}$).

Let

$$C_+ = ||c_{ij}||$$

be the matrix representation of $C_+$ with respect to the basis $\{ e_{++}, e_{+-} \}$. Then the relations $C_+^2 = I$, $J C_+ > 0$ take the form

$$(3.12) \quad C_+^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} C_+ > 0,$$

where $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix representation of $J \mid \mathfrak{M}_i$ with respect to $\{ e_{++}, e_{+-} \}$ (since (2.16)). A simple analysis of (3.12) leads to the following description of $C_+$:

$$(3.13) \quad C_+ = C_{\tilde{\chi}, \tilde{\omega}} \begin{pmatrix} \cosh \tilde{\chi} & (\sinh \tilde{\chi}) e^{-i\tilde{\omega}} \\ - (\sinh \tilde{\chi}) e^{i\tilde{\omega}} & - \cosh \tilde{\chi} \end{pmatrix}, \quad \tilde{\chi}, \tilde{\omega} \in \mathbb{R}.$$

Reasoning by analogy for the matrix representation $\hat{C}_+$ of $\hat{C}_+$ we get

$$(3.14) \quad \hat{C}_+ = C_{\tilde{\chi}, \tilde{\omega}} \begin{pmatrix} \cosh \tilde{\chi} & (\sinh \tilde{\chi}) e^{-i\tilde{\omega}} \\ - (\sinh \tilde{\chi}) e^{i\tilde{\omega}} & - \cosh \tilde{\chi} \end{pmatrix}, \quad \tilde{\chi}, \tilde{\omega} \in \mathbb{R}.$$

Passing to the matrix representation in (3.11) we conclude that $A_M(= A_K)$ has a stable $C$-symmetry if and only if

$$(3.15) \quad K(\zeta, \phi, \omega, \xi) C_{\tilde{\chi}, \tilde{\omega}} = C_{\tilde{\chi}, \tilde{\omega}} K(\zeta, \phi, \omega, \xi),$$

where $K(\zeta, \phi, \omega, \xi)$ is defined by (2.18). A routine analysis of (3.15) with the use of (3.13) and (3.14) shows that (3.15) is equivalent to the system of relations

$$(3.16) \quad \begin{cases} \cosh \tilde{\chi} - \cosh \tilde{\chi} + \tanh \zeta [e^{i(\tilde{\omega} - \omega - \phi)} \sinh \tilde{\chi} - e^{i(\omega - \tilde{\omega} - \phi)} \sinh \tilde{\chi}] = 0, \\ \tanh \zeta [\cosh \tilde{\chi} + \cosh \tilde{\chi}] + e^{i(\phi + \omega - \tilde{\omega})} \sinh \tilde{\chi} + e^{-i(\phi - \omega + \tilde{\omega})} \sinh \tilde{\chi} = 0. \end{cases}$$

Let us set $\tilde{\chi} = \tilde{\chi} = \chi$. Then the first relation in (3.16) is satisfied when

$$(3.17) \quad \omega = \frac{\tilde{\omega} + \tilde{\omega}}{2}$$

and the second one goes over

$$\tanh \zeta + \tanh \chi \cos \left( \phi + \frac{\tilde{\omega} - \tilde{\omega}}{2} \right) = 0.$$
The latter equation can be solved with respect to $\chi$ if and only if

\begin{equation}
|\tanh \zeta| < \left| \cos \left( \phi + \frac{\omega - \bar{\omega}}{2} \right) \right|.
\end{equation}

Since $\bar{\omega}, \omega \in \mathbb{R}$ are independent variables, conditions (3.17) and (3.18) with fixed $\omega$ and $\phi$ can easily be satisfied by a suitable choice of $\omega$ and $\bar{\omega}$. This means that the system (3.16) has a solution $\tilde{x}, \bar{x}, \hat{x}, \tilde{\omega}$ for any fixed $\zeta, \phi, \omega, \xi$. Therefore, $A_{K(\zeta, \phi, \omega, \xi)}$ has a stable $C$-symmetry for any choice of $\zeta, \phi, \omega,$ and $\xi$. Theorem 3.9 is proved. \hfill $\square$

**Corollary 3.10.** Let $A$ be $J$-self-adjoint extension of the Phillips symmetric operator $S$ with deficiency indices $<2,2>$. Then $A$ is similar to a self-adjoint operator if and only if the spectrum of $A$ is real.

**Proof.** It follows from Proposition 2.10 and Theorem 3.9 \hfill $\square$

3.4. Various realizations of the Phillips operator. It follows from (1.2) and (1.3) that the Phillips symmetric operator $S$ can be obtained as the restriction of a self-adjoint operator $A$ with Lebesgue spectrum onto the domain

\begin{equation}
D(S) = \{ f \in D(A) \mid ((A - iI)f, w) = 0, \forall w \in W_0 \},
\end{equation}

where $W_0$ is a wandering subspace of the bilateral shift $U$. In particular, this means that the Phillips symmetric operator $S$ naturally arises in the study of the formal expression $i \frac{d}{dx} + <\delta, \cdot > \delta(x)$ and it coincides with the operator

$$S = i \frac{d}{dx}, \quad D(S) = \{ u(\cdot) \in W^2_2(\mathbb{R}) \mid u(0) = 0 \}$$

acting in $L^2_2(\mathbb{R})$. This example illustrates one of possible general approaches to the construction of the Phillips symmetric operator. Indeed, let $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ and let $S = \mathfrak{N}_1 \oplus \mathfrak{N}_2$, where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are simple maximal symmetric operators in the Hilbert spaces $\mathfrak{N}_1$ and $\mathfrak{N}_2$ with deficiency indices $< m, 0 >$ and $< 0, m >$ $(m \in \mathbb{N}$), respectively. In that case $S$ is a simple symmetric operator in $\mathfrak{N}$ with deficiency indices $< m, m >$ and its characteristic function $\Theta(\cdot)$ associated with an arbitrary boundary triplet $(\mathfrak{N}_-, \Gamma_0, \Gamma_1, Q)$ (see (2.11)) is equal to zero. By Lemma 3.1 this means that $S$ is a Phillips symmetric operator.

Using (1.3) and (3.19) it is easy to calculate the defect subspaces $\mathfrak{N}_{k,1}$ of $S$

$$\mathfrak{N}_{k,1} = W_0 \quad \text{and} \quad \mathfrak{N}_{-k,1} = UW_0.$$

According to (1.2) and (1.3), a bilateral shift $U$ and its wandering subspace $W_0$ are the main ingredients for the determination of the Phillips symmetric operator $S$. To illustrate this point, we have presented below two mathematicical constructions where $U$ appears naturally and $W_0$ admits a simple description.

3.4.1. Multiresolution approximation of $L^2_2(\mathbb{R})$. We recall [33, 34] that a multiresolution approximation (MRA) of $L^2_2(\mathbb{R})$ is a sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2_2(\mathbb{R})$ such that: (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$; (iii) $\cup_{j \in \mathbb{Z}} V_j$ is dense in $L^2_2(\mathbb{R})$; (iv) $f(\cdot) \in V_j \Leftrightarrow f(2^{-j} \cdot) \in V_0$; (v) there exists a function $\varphi(\cdot) \in V_0$ such that the sequence $\{\varphi(-k), k \in \mathbb{Z}\}$ is a Riesz basis of $V_0$.

Let

$$Uf(x) = \frac{1}{\sqrt{2}}f \left( \frac{x}{2} \right), \quad \forall f \in L^2_2(\mathbb{R})$$

be the dilation operator in $\mathfrak{N} = L^2_2(\mathbb{R})$ and let $\{V_j\}_{j \in \mathbb{Z}}$ be a fixed multiresolution approximation of $L^2_2(\mathbb{R})$. Then $U$ is a bilateral shift in $L^2_2(\mathbb{R})$ with a wandering subspace $W_0 = V_1 \oplus V_0$ (due to properties (i)–(iv)).

According to the general results of MRA-based wavelet theory [33, 34] the subspace $W_0$ is a wavelet subspace and relations (i)–(v) imply the existence of a function (wavelet)
ψ(·) ∈ W₀ such that the sequence {ψ(· − k), k ∈ ℤ} forms an orthonormal basis in W₀. This means that (3.19) can be rewritten as follows:

\[ \mathcal{D}(S) = \{ f ∈ \mathcal{D}(A) \mid ((A - iI)f, \psi(· − k)) = 0, \forall k ∈ ℤ \}, \]

where the wavelet ψ(·) is directly constructed by the (scaling) function φ(·) from condition (v).

3.4.2. Abstract wave equation. Let us consider an operator-differential equation

\[ u_{tt} = -Lu, \]

where L is a positive self-adjoint operator in a Hilbert space H. By \( H_L \) we denote the completion of domain of definition \( \mathcal{D}(L) \) with respect to the norm \( \|u\|_{H_L}^2 := (Lu, u)_H \) and consider the Hilbert space \( \mathcal{H} = H_L ⊕ H \) (the energy space). It is convenient to write elements of \( \mathcal{H} \) as column matrices \( \begin{pmatrix} u \\ v \end{pmatrix} \), where \( u ∈ H_L \) and \( v ∈ H \). Put \( u_t = v \) and rewrite (3.20) as

\[ \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = iQ \begin{pmatrix} u \\ v \end{pmatrix}, \quad Q = i \begin{pmatrix} 0 & -I \\ L & 0 \end{pmatrix}. \]

The operator Q with the domain \( \mathcal{D}(Q) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid \{u, v\} ⊂ \mathcal{D}(L) \right\} \) is essentially self-adjoint in \( \mathcal{H} \). Its closure \( \mathcal{A} \) is a generator of the group of unitary (in \( \mathcal{H} \)) operators \( \mathcal{W}_\mathcal{A}(t) = e^{it\mathcal{A}} \), which determines solutions of the Cauchy problem for the abstract wave equation (3.20).

The equation (3.20) is said to be free (unperturbed) wave equation if there exists a simple maximal symmetric operator \( R \) in \( H \) such that

\[ B^2 ⊂ L \quad \text{and} \quad (Lu, u) = \|B^* u\|_H^2, \quad ∀u ∈ \mathcal{D}(L). \]

Assume that \( \{u_n\} \) belong \( \mathcal{D}(B^2) \) and form a Cauchy sequence in \( H_L \). Then \( \{Bu_n\} \) is the Cauchy sequence in \( H \) (due to the second relation in (3.21)) and hence \( \lim_{n \to \infty} Bu_n = γ ∈ H \). In that case we will say that the sequence \( \{u_n\} \) converges to the element \( x_γ \) in the space \( H_L \). Obviously the Hilbert space \( H_L \) can be identified with the set of elements \( \{x_γ \mid ∀γ ∈ H\} \) and \( (x_γ, x_ζ)_{H_L} = (γ, ζ)_H \).

In what follows, without loss of generality we assume that \( B \) has zero defect number in the lower half-plane. Then \( B \) admits the representation

\[ B = T^{-1} \frac{d}{ds} T, \quad \mathcal{D}(B) = T^{-1} 0_{W_2^1} (\mathbb{R}_+, \mathcal{N}), \]

where \( T \) is an isometric mapping from \( H \) onto \( L_2(\mathbb{R}_+, \mathcal{N}) \), \( \mathcal{N} \) is an auxiliary Hilbert space of dimension equal to the nonzero defect number of \( B \) and \( \mathbb{R}_+ = (0, \infty) \).

Using (3.22), we can define \( B \) in various functional spaces getting, as a result, different specific realizations of the free abstract wave equation. In particular, the classical free wave equation \( u_{tt}(x, t) = Δu(x, t) \) in \( \mathbb{R}^n \) (n is odd) can be obtained from (3.22) if we choose \( \mathcal{N} \) as the Hilbert space \( L_2(S^{n−1}) \) of functions square-integrable on the unit sphere \( S^{n−1} \) in \( \mathbb{R}^n \) and consider the isometric operator \( T : L_2(\mathbb{R}^n) → L_2(\mathbb{R}_+, \mathcal{N}) \) defined on the rapidly decreasing smooth functions \( u(x) ∈ S(\mathbb{R}^n) \) by the formula

\[ (Tu)(s, w) = (\partial_s^m R u)(s, w), \quad m = \frac{(n − 1)}{2}, \quad s ≥ 0, \quad w ∈ S^{n−1}; \]

where \( R \) is the Radon transformation. In that case the Laplace operator \( L = −Δ \) in \( L_2(\mathbb{R}^n) \) satisfies condition (3.21) and equation (3.20) takes the form \( u_{tt}(x, t) = Δu(x, t) \) (see [31] for detail).
Assume that (3.20) is the free wave equation for some choice of $B$. Then the corresponding generator $A$ is the Cayley transform of a bilateral shift $U$ and a wandering subspace $W_0$ can be chosen as follows [30]:

\begin{equation}
W_0 = \left\{ \left( \begin{array}{c} xh \\ -ih \end{array} \right) \mid \forall h \in \ker(B^* + iI) \right\}.
\end{equation}

Substituting (3.23) into (3.19) and taking into account that the domain $D(A)$ can be described explicitly, we find $S$. For instance, let $L$ be the Friedrichs extension of $B^2$. Then $L = B^*B$ and this operator satisfies (3.21). In this case:

\[ A \left( \begin{array}{c} x\gamma \\ p \end{array} \right) = -i \left( \begin{array}{c} xBp \\ -B^*\gamma \end{array} \right), \quad D(A) = \left\{ \left( \begin{array}{c} x\gamma \\ p \end{array} \right) \mid \forall \gamma \in D(B^*), \forall p \in D(B) \right\} \]

and $S$ is the restriction of $A$ onto the set of elements

\[ \left\{ \left( \begin{array}{c} x\gamma \\ p \end{array} \right) \mid \gamma \in D(B^*), p \in D(B) \right\} \]

such that $((B^* + iI)(p - i\gamma), h) = 0$ for all $h \in \ker(B^* + iI)$.

**Corollary 3.11.** Assume that the nonzero defect number of $B$ is 2 and $J$ is a fundamental symmetry in $\mathfrak{N}$ such that $SJ = JS$. Then if $A \in \Sigma_J(S)$ has a real spectrum, then $W_A(t) = e^{iAt}$ is a $C_0$-semigroup.

**Proof.** Immediately follows from Corollary 3.10. \qed

4. Conclusions

In this paper we have studied the collection $\Sigma_J(S)$ of $J$-self-adjoint extensions of the Phillips symmetric operator $S$. Our attention to $\Sigma_J(S)$ was inspired by a steady interest in the spectral analysis of new classes of $J$-self-adjoint operators $A_\varepsilon$ with the aim to illustrate quantitative and qualitative changes of spectra $\sigma(A_\varepsilon)$ when parameters $\varepsilon$ run the domain of variation $\Xi$. Due to specific inherent properties of the Phillips operator $S$ (the zero characteristic function, the absence of real points of regular type, etc) we obtained a spectral picture which differs from the matrix models [21, 24, 25] and models based on $J$-self-adjoint (symmetric) perturbations of the Schrödinger or the Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of $A \in \Sigma_J(S)$ coincides with real line: $\sigma(A) = \mathbb{R}$ or with complex plane: $\sigma(A) = \mathbb{C}$ (Theorem 3.7).

For operators $A_\varepsilon \in \Sigma_J(S)$ (where $S$ is an arbitrary symmetric operator commuting with $J$) we have introduced the concepts of stable and unstable $C$-symmetry (Definition 2.11). These concepts are natural for sets of $J$-self-adjoint operators appearing in the extension theory framework. Roughly speaking, if $A_\varepsilon$ belongs to the sector $\Sigma_J^{st}$ of stable $C$-symmetry, then $A_\varepsilon$ preserves the property of $C$-symmetry under small variation of $\varepsilon$.

For singular perturbations of the Schrödinger or the Dirac operator, the corresponding symmetric operator $S$ has real points of regular type. In that case, the sector $\Sigma_J^{unst}$ of unstable $C$ symmetry is not empty and operators $A_\varepsilon \in \Sigma_J(S)$ with real spectra and Jordan points arise in the case where $\varepsilon$ lies on the boundary of $\Sigma_J^{st}$ [1, 23]. This picture is essentially simplified for the Phillips symmetric operator $S$ since $S$ has no real points of regular type. We have shown that the sector $\Sigma_J^{unst}$ of unstable $C$-symmetry is the empty set and there are no $J$-self-adjoint extensions of $A_{\text{sym}}$ with real spectra and Jordan points (this fact follows from Theorem 3.9 and Corollary 3.10). These results have been obtained under the assumption that $S$ has deficiency indices $< 2, 2 >$. We believe that they remain true for the general case $< n, n >$. However, the corresponding proof requires more cumbersome analysis and the case $< n, n >$ will be considered in a forthcoming paper.
An open problem is finding an adequate physical phenomenon for which $J$-self-adjoint extensions of $S$ can be served as model Hamiltonians. In this way we have just discussed certain representations of $S$ related to abstract wave equation and multiresolution approximation.

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