Method of constructing braid group representation and entanglement in a $9 \times 9$ Yang-Baxter system

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Abstract. In this paper we present reducible representation of the $n^2$ braid group representation which is constructed on the tensor product of $n$-dimensional spaces. Specifically, it is shown that via a combining method we can construct more $n^2$ dimensional braiding matrix $S$ which satisfy the braid relations. By Yang-Baxter relation approach, we derive a $9 \times 9$ unitary $\hat{R}$-matrix according to a $9 \times 9$ braiding $S$-matrix we have constructed. The entanglement properties of $\hat{R}$-matrix is investigated, and the arbitrary degree of entanglement for two-qutrit entangled states can be generated via $\hat{R}$-matrix acting on the standard basis.

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1. Introduction

Quantum entanglement is the most surprising nonclassical property of composite quantum systems that Schrödinger singled out many decades ago as "the characteristic trait of quantum mechanics". Recently, entanglement has become one of the most fascinating topics in quantum information theory, entanglement is recognized as an essential resource for quantum processing and quantum communications and it play a crucial role in quantum computation. It is believed that the protocols based on the entangled states have an exponential speedup over the classical ones. Besides, in highly correlated states in condensed-matter systems such as superconductors and fractional quantum Hall liquids, the entanglement serves as a unique measure of quantum correlations between degrees of freedom. Leveraging the entanglement and using quantum coherence, certain problems may be solved faster by a quantum computer than a classical one.

Recently, it has been revealed that there are natural and profound connections between quantum computations and braid group theory as well as the Yang-Baxter equation (YBE). During the investigation of the relationships among quantum entanglement, topological entanglement and quantum computation, Kauffman and Lomonaco have explored the important role of unitary braiding operators. It is shown that the braid matrix can be identified as the universal quantum gate. This motivates a novel way to study quantum entanglement based on the theory of braiding operators, as well as YBE. The first step along this direction is initiated by Zhang et al. In , the Bell matrix generating two-qubit entangled states has been recognized to be a unitary braid transformation. Later on, an approach to describe Greenberger-Horne-Zeilinger (GHZ) states or N-qubit entangled states based on the theory of unitary braid representations has been presented in Chen and his co-workers used unitary braiding operators to realize entanglement swapping and generate the GHZ states, as well as the linear cluster states. These literatures introduce the braiding operators and Yang-Baxter equations to the field of quantum information and quantum computation. In a very recent work, it has been found that any pure two-qudit entangled state can be achieved by a universal Yang-Baxter equation.

In our paper we present the method of constructing $n^2$ dimensional matrix solutions of braid group algebra relation. The paper is organized as follows. In sec II, we present the reducible representations of $n^2$ braid group algebra. Specifically, more $n^2$ dimensional braiding matrix S which satisfy the braid relations can be obtained by the combining method, and we get some well known and some new braiding matrix S. In sec III, By Yang-Baxter relation approach, we derive a $9 \times 9$ unitary $\tilde{R}$-matrix according to a $9 \times 9$ S-matrix we have constructed. we investigate the entanglement properties of $\tilde{R}$-matrix. It shows that the arbitrary degree of entanglement for two-qudit entangled states can be generated via the unitary matrix $\tilde{R}$-matrix acting on the standard basis. The summary is made in the last section.

2. Method of constructing braiding S-Matrixs

In a recent paper, a reducible representation of the Temperley-Lieb algebra is constructed on the tensor product of n-dimensional spaces. In fact, Temperley-Lieb algebra is a subalgebra of braid algebra. Motivated by this, we investigated the methods of constructing braid representation to get more useful braid representations.
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We first review the theory of braid groups. Let $B_n$ denotes the braid group on $n$ strands. $B_n$ is generated by elementary braids $\{b_1, b_2, \cdots, b_{n-1}\}$ with the braid relations,

\[
\begin{align*}
& b_i b_{i+1} b_i = b_{i+1} b_i b_i \quad 1 \leq i < n-2 \\
& b_i b_j = b_j b_i \quad |i-j| \geq 2
\end{align*}
\]

(1)

where the notation $b_i \equiv b_{i,i+1}$ is used, $b_{i,i+1}$ represents $1 \otimes 1 \otimes 1_3 \cdots \otimes S_{i,i+1} \otimes \cdots \otimes 1_n$, and $1_j$ is the unit matrix of the $j$-th particle.

By calculation, we get the reducible representations of braiding matrix which is defined by two $n \times n$ matrix $A$ and $B \in GL(n, \mathbb{C})$ which all can also be seen as an $n^2$ dimensional vector $\{A^a_b, B^a_b\} \in \mathbb{C}^n \otimes \mathbb{C}^n$.

The braiding matrix $S$ can be expressed as

\[
S^a_b = A^a_c B^c_b \in Mat(\mathbb{C}^n \otimes \mathbb{C}^n)
\]

(2)

where we explicitly write the indices corresponding to the factors in the tensor product space $\mathcal{H}=\otimes^N \mathbb{C}^n$. Substituting the relation into braid relations eq (1), the limited conditions can be derived. The $S$ in eq (2) is a solution of braid relation if and only if (the detail calculation is given in Appendix A)

\[
AB = BA, \quad \text{namely} \quad [A,B] = 0_{n \times n}
\]

(3)

For example, $n=2$, in order to get significative result we set that every row and array of two $2 \times 2$ convertible matrix-A and matrix-B have only one element which is equal to 1 for convenience.

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(4)

Substituting eq (8) into eq (2) we get

\[
S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

(5)

the first $S$ we get is the standard swap gate[18], in order to obtain more useful braiding S-matrix we do the further combination as follows:

\[
S = \sum_{i=1}^{2} a_i S^{(i)}
\]

(6)

where $S^{(1)}$ and $S^{(2)}$ all have the reducible representations as eq (2), $a_1$ and $a_2$ are the corresponding coefficients.

\[
(S^{(1)})^a_c = (A^{(1)})^a_d (B^{(1)})^b_c \quad \quad (S^{(2)})^a_c = (A^{(2)})^a_d (B^{(2)})^b_c \quad \quad (S^{(1)})^a_c = (A^{(2)})^a_d (B^{(2)})^b_c
\]

(7)

according to eq (1), eq (6) and eq (7) we get when $[A^{(i)}, B^{(i)}] = 0, [A^{(i)}, A^{(j)}] = 0, [B^{(i)}, B^{(j)}] = 0$ ($i,j=1,2$) the constructed S-matrix in eq (5) is a braiding-matrix which satisfy the braid relation eq (1).

According to the limitation, we set

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}
\]

(8)
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The coefficient $a_1$ and $a_2$ don’t have restriction and we set them equal to $\frac{1}{\sqrt{2}}$ for convenience. We let $A^{(1)} = B^{(2)} = A$, $B^{(1)} = B$, $A^{(2)} = C$ and $A^{(1)} = B^{(1)} = A$, $A^{(2)} = B$, $B^{(2)} = C$ respectively, according to this combining method we can get two $4 \times 4$ models as follows:

$$
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & i & 0 \\
1 & 0 & 0 & 1 \\
-i & 0 & 0 & i \\
0 & 1 & -i & 0
\end{pmatrix} \quad \text{and} \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & i \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
-i & 0 & 0 & 1
\end{pmatrix}
$$

one can see by the combination we obtain two $4 \times 4$ braiding model while the first $S$-matrix is a new braiding model which is found to be locally equivalent to the DCNOT gate [25]. This motivates us to find generalized $n^2$ ($n \geq 2$) dimensional braiding-matrix representation by the combining method. We do the similar combination as follows:

$$
S = \sum_{i=1}^{n} a_i S^{(i)}
$$

where $S^{(i)}$ also have the reducible representation

$$
(S^{(i)})_{cd}^{ab} = (A^{(i)})_{a}^{\alpha} (B^{(i)})_{\beta}^{\beta}
$$

substituting eq (10) and eq (11) into eq (1), we find $A^{(i)}$ and $B^{(i)}$ subject to the limited conditions as follows (the detail calculation is given in Appendix A):

$$
[A^{(i)}, B^{(j)}] = 0, [A^{(i)}, A^{(j)}] = 0, [B^{(i)}, B^{(j)}] = 0
\quad (i, j = 1, 2, 3 \ldots n)
$$

coefficients $a_i$ are not restricted, when eq (12) is satisfied, $S$-matrix in eq (10) satisfy the braid relation eq (1). Namely we can obtain more $n^2$ dimensional braiding-matrix representation by this combining method.

3. A $9 \times 9$ braiding S-matrix, Yang-Baxterization and Entanglement

In section II, we present that we can get arbitrary $n^2$ dimensional braiding-matrix representation by the reducible representation and the combining method. Now we emphasize on one $9 \times 9$ braiding $S$-matrix we have constructed to investigate it’s application on quantum entanglement.

For $n=3$, let three $3 \times 3$ matrix $A$, $B$ and $C$ as follows (we choose $\{|0\rangle, |1\rangle, |2\rangle\}$ as the standard basis),

$$
A = \begin{pmatrix}
0 & 0 & e^{i\varphi_1} \\
1 & 0 & 0 \\
0 & e^{-i\varphi_2} & 0
\end{pmatrix} \quad B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & e^{i\varphi_2} \\
e^{-i\varphi_1} & 0 & 0
\end{pmatrix} \\
C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

here $[A,B]=0$, $[A,C]=0$, $[B,C]=0$ satisfy the eq (12), the parameters $\varphi_1$ and $\varphi_2$ are both real. we let $A^{(1)} = B^{(2)} = A$, $A^{(2)} = B^{(1)} = B$ and $A^{(3)} = B^{(3)} = C$, and we set coefficient $a_i$, $(i=1,2,3)$ all equal to 1 for convenience. By combination, In terms of
we obtain a set solution of $G(x)$ and $\rho$ to eq (15) and according to $S_\rho$ one can choose appropriate $q_1$ here. Addition, the initial condition $\tilde{\mathcal{Y}}$ and $S_\rho$ be constructed by using the approach of Yang-Baxterization. Let the unitary Yang-Baxter matrix take the form,

$$\tilde{\mathcal{Y}}_n(x,y)\tilde{\mathcal{Y}}_n(y) = \tilde{\mathcal{Y}}_n(xy)\tilde{\mathcal{Y}}_n(x).$$ (15)

The spectral parameters $x$ and $y$ which are related with the one-dimensional momentum play an important role in some typical models [16]. The asymptotic behavior of $\tilde{\mathcal{Y}}(x,\varphi_1,\varphi_2)$ is $x$-independent, i.e. $\lim_{x\to\infty} \tilde{\mathcal{Y}}_{i+1}(x,\varphi_1,\varphi_2) \propto b_i$, where $b_i$ are braiding operators, which satisfy the braiding relations eq (11). From a given solution of the braid relation $S$, a $\tilde{\mathcal{Y}}(x)$ can be constructed by using the approach of Yang-Baxterization. Let the unitary Yang-Baxter matrix take the form,

$$\tilde{\mathcal{Y}}(x) = \rho(x)(I + G(x)S).$$ (16)

This is a trigonometric solution of YBE, where $\rho(x)$ is a normalization factor. One can choose appropriate $\rho(x)$ to ensure that $\tilde{\mathcal{Y}}(x)$ is unitary. Substituting eq (16) to eq (15) and according to $S^2 = 3S$, one has $G(x)+G(y)+3G(x)G(y)=G(xy)$. In addition, the initial condition $\tilde{\mathcal{Y}}_i(x) = I_i$ yields $G(x=1)=0$ and $\rho(x) = 1$. The unitary condition (i.e., $\tilde{\mathcal{Y}}_i(x) = \tilde{\mathcal{Y}}_{i-1}(x) = \tilde{\mathcal{Y}}(x^-)$) can be tenable only on condition that $\rho(x)\rho(x^-)(G(x) + G(x^-) + 3G(x)G(x^-))=0$. Take account into these condition, we obtain a set solution of $G(x)$ and $\rho(x)$,

$$\rho(x) = x, \quad G(x) = -\frac{x-x^-}{3x}.$$ (17)

Substituting Eq (14), Eq (17) into Eq (16), the unitary solution of YBE can be obtained as following,

$$\tilde{\mathcal{Y}}_i(x,\varphi_1,\varphi_2) = \begin{pmatrix} b & 0 & 0 & 0 & 0 & aq_1 & 0 & aq_1 & 0 \\
0 & b & 0 & a & 0 & 0 & 0 & 0 & aQ \\
0 & 0 & b & 0 & \frac{a}{q_2} & 0 & a & 0 & 0 \\
0 & a & 0 & b & 0 & 0 & 0 & 0 & aQ \\
0 & 0 & aq_2 & 0 & b & 0 & aq_2 & 0 & 0 \\
\frac{a}{q_1} & 0 & 0 & 0 & 0 & b & 0 & a & 0 \\
\frac{a}{q_1} & 0 & a & 0 & \frac{a}{q_2} & 0 & b & 0 & 0 \\
\frac{a}{q_1} & 0 & 0 & 0 & 0 & a & 0 & b & 0 \\
\frac{a}{q_1} & \frac{a}{q_2} & 0 & 0 & 0 & 0 & b \\
\end{pmatrix}$$ (18)

where $a=x^{-1} - x$, $b=2x + x^{-1}$. The Gell-Mann matrices, a basis for the Lie algebra SU(3) [27], $\lambda_n$ satisfy $[I_\lambda, I_\mu] = if_{\lambda\mu\nu}I_\nu(\lambda, \mu, \nu = 1, \cdots, 8)$, where $I_\mu = \frac{1}{2}\lambda_\mu$. As a
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resent paper having done we denote Iₙ by, Iₙ ± iI₂, Vₙ = V₄ ± iV₅, Uₙ ± iI₇, Y = \frac{2}{\sqrt{3}}I₈. we also generate three sets of realization of SU(3) as:

\[
\begin{align*}
I^{(1)}_± &= I^{±}_1 I^{±}_2, & U^{(1)}_± &= U^{±}_1 V^{±}_2, & V^{(1)}_± &= V^{±}_1 U^{±}_2, \\
I^{(2)}_± &= U^{±}_1 U^{±}_2, & U^{(2)}_± &= V^{±}_1 I^{±}_2, & V^{(2)}_± &= I^{±}_1 V^{±}_2, \\
I^{(3)}_± &= V^{±}_1 V^{±}_2, & U^{(3)}_± &= I^{±}_1 U^{±}_2, & V^{(3)}_± &= U^{±}_1 I^{±}_2, \\
\end{align*}
\]

\[
\begin{align*}
I^{(1)}_± &= (I^n_1 - I^n_2) + \frac{1}{2}(Y_1 Y_2 - Y_1 I^n_2), \\
Y^{(1)}_± &= \frac{1}{3}(Y_1 + Y_2) - \frac{2}{3}I^n_1 I^n_2 - \frac{1}{2}Y_1 Y_2; \\
I^{(2)}_± &= (I^n_1 + I^n_2) + \frac{1}{6}(Y_1 + Y_2) + \frac{2}{9}I^n_1 I^n_2 + \frac{1}{2}Y_1 Y_2; \\
Y^{(2)}_± &= -\frac{1}{3}(Y_1 Y_2 - Y_1 I^n_2), \\
I^{(3)}_± &= (I^n_1 + I^n_2) - \frac{1}{3}(Y_1 + Y_2) - \frac{2}{9}I^n_1 I^n_2 - \frac{1}{2}Y_1 Y_2. \\
Y^{(3)}_± &= \frac{1}{2}(I^n_1 + I^n_2) - \frac{1}{3}(Y_1 Y_2 - Y_1 I^n_2). \\
\end{align*}
\]

We denote I^{(k)}_± = I^{(k)}_1 \pm iI^{(k)}_2, V^{(k)}_± = V^{(k)}_4 \mp iV^{(k)}_5, U^{(k)}_± = U^{(k)}_1 \pm iU^{(k)}_7, Y^{(k)}_± = \frac{2}{\sqrt{3}}I^{(k)}_8 (k = 1, 2, 3). These realizations satisfy the commutation relation [I^{(i)}_λ, I^{(j)}_μ] = iδ^{ij}f_{λμν}I^{(k)}_ν (λ, μ, ν = 1, · · · , 8; i, j = 1, 2, 3).

For i-th and (i + 1)-th lattices, R-matrix can be expressed in terms of above operators,

\[
\tilde{R}(x, \varphi_1, \varphi_2) = \frac{1}{3}a[I^{(1)}_1 + I^{(2)}_1 + Q(V^{(1)}_1 + U^{(1)}_1)] \\
+ Q^{-1}(U^{(1)}_1 + V^{(1)}_1) + I^{(2)}_1 + I^{(3)}_1 \\
+ q_2(V^{(2)}_1 + U^{(2)}_1) + q_1^{-1}(V^{(2)}_1 + U^{(2)}_1) \\
+ I^{(3)}_1 + I^{(3)}_2 + q_2(V^{(3)}_1 + U^{(3)}_1) \\
+ q_2^{-1}(V^{(3)}_1 + U^{(3)}_1)] + \frac{1}{8}(I \otimes I). \\
\]

So the whole tensor space \(C^3 \otimes C^3\) is completely decomposed i.e. \(C^3 \otimes C^3 = C^3 \oplus C^3 \oplus C^3\). In addition, each block of \(R\)-matrix can be represented by fundamental representation of SU(3) algebra.

According to the condition \(\tilde{R}^i_1(x) = \tilde{R}^{-1}_1(x)\) one can get \(x^* = -x\), so we can introduce a new parameter with \(x = e^{iθ}\), and \(θ\) may be related with entanglement degree. When one acts \(\tilde{R}(θ, \varphi_1, \varphi_2)\) on the separable state \(|mn\rangle\), he yields the following family of states \(|ψ\rangle_{mn} = \sum_{ij=00}^{22} \tilde{R}^{ij}_{mn} |mn\rangle\) (m,n=0,1,2). For example, if m=0 and n=0,

\[
|ψ\rangle_00 = \frac{1}{3}(b|00\rangle + aq_1^{-1}|12\rangle + aq_1^{-1}|21\rangle) \\
\]

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In Ref. [28], the generalized concurrence (or the degree of entanglement [29]) for two qudits is given by

\[ C = \sqrt{\frac{d}{d-1} (1 - I_1)} \]  

(20)

where \( I_1 = \text{Tr}[\rho_A^2] = \text{Tr}[\rho_B^2] = |\kappa_0|^4 + |\kappa_1|^4 + \cdots + |\kappa_{d-1}|^4 \), \( \rho_A \) and \( \rho_B \) are the reduced density matrices for the sub-systems, and \( \kappa_j \)'s (\( j = 0, 1, \ldots, d - 1 \)) are the Schmidt coefficients. Then we can obtain the generalized concurrence of the state \( |\psi\rangle_{00} \) as

\[ C = \sqrt{\frac{3}{2} (1 - \frac{1}{81} |2x + x^2 - 4| - \frac{2}{81} |x - x^2|)} \]

\[ = 2\sqrt{\frac{2}{3}} \sin \theta \sqrt{2\cos^2 \theta + 1} \]  

(21)

one can find that when \( \theta = \frac{\pi}{3} \), the state \( |\psi\rangle_{00} \) becomes the maximally entangled state of two qutrits as state \( |\psi\rangle_{00} = \frac{1}{\sqrt{3}} (e^{i\pi/6}|00\rangle - iq_{-1}^{-1} |12\rangle - iq_{-1}^{-1} |21\rangle) \).

In general, if one acts the unitary Yang-Baxter matrix \( \tilde{R}(x) \) on the basis \( \{ |00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle \} \), he will obtain the same generalized concurrence as Eq(21). It is easy to check that the generalized concurrence ranges from 0 to 1 when the parameter \( \theta \) runs from 0 to \( \pi \). But for \( \theta \in [0, \pi] \), the generalized concurrence is not a monotonic function of \( \theta \). And when \( x = e^{i\pi/3} \), he will generate nine complete and orthogonal maximally entangled states for two qutrits. The QE doesn’t depend on the parameters \( \varphi_1 \) and \( \varphi_2 \). So one can verify that parameter \( \varphi_1 \) and \( \varphi_2 \) may be absorbed into a local operation.

4. Summary

In this paper, we have presented the reducible representation of braid group algebra, Specifically that by the further combining method we can get more \( n^2 \) dimensional braiding S-matrices and we obtain some well known and new braiding models. According to a 9×9 braiding S-matrix which we have constructed satisfying the braiding relations we derived a unitary \( \tilde{R} \)-matrix via Yang-Baxterization. We show that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary \( \tilde{R} \) matrix acting on the standard basis.

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Appendix A.

The two limited conditions in Sec.2 will be calculated in detail as follows,

If we substitute \( S_{cd}^{ab} = A_{d}^{a}B_{c}^{b} \) into the braid relation in Eq(1) (i.e. \( S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \))

\[ [S_{12}S_{23}S_{12}]_{def}^{abc} = [S_{12}]_{ijk}^{abc}[S_{23}]_{\alpha\beta\gamma}^{ijk}[S_{12}]_{de\bar{f}}^{\alpha\beta\gamma} \]

\[ = S_{ij}^{ab} S_{\beta\gamma}^{\alpha\beta\gamma} S_{\alpha\beta}^{\alpha\beta\gamma} \]

\[ = A_{d}^{a}B_{c}^{b} A_{1}^{e}B_{2}^{f}A_{e}^{b}B_{d}^{d} \]

\[ = (BA)^{a}_c (A^2)^{e}_f (B^2)^{b}_d \]  

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\[ [S_{23} S_{12}]_{edf}^{abc} = [S_{23}]_{ijk} [S_{12}]_{jkl} [S_{23}]_{def}^{ \alpha \beta \gamma} \]

\[ = S_{23}^{bc} S_{12}^{aj} S_{23}^{\alpha jk} \]

\[ = A_{k}^{\alpha} B_{r}^{\beta} A_{r}^{\beta} B_{k}^{\gamma} \]

\[ = (AB)^{j}_{c} (A^{2})^{j}_{f} (B^{2})^{j}_{d} \quad (A.2) \]

According to Eq(A.1) and Eq(A.2), one can see if \( AB = BA \), namely \( [A, B] = 0 \), the braid relation \( S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23} \) holds.

Substitute \( S = \sum_{i=1}^{n} a_{i} S^{(i)} \) into \([S_{12} S_{23}]_{edf}^{abc} \) and \([S_{23} S_{12}]_{edf}^{abc} \) respectively, one has

\[ [S_{12} S_{23}]_{edf}^{abc} = \sum_{ghl} a_{g} a_{h} a_{l} [S^{(g)}_{12}]_{gh} [S^{(h)}_{23}]_{hl} [S^{(l)}_{12}]_{de} \]

\[ = \sum_{ghl} a_{g} a_{h} a_{l} (S^{(g)})_{ij} (S^{(h)})_{jk} (S^{(l)})_{ik} \]

\[ = \sum_{ghl} a_{g} a_{h} a_{l} (B^{(g)} A^{(h)})_{ij} (A^{(h)} B^{(l)})_{jk} \quad (A.3) \]

\[ [S_{23} S_{12}]_{edf}^{abc} = \sum_{\lambda \mu \nu} a_{\lambda} a_{\mu} a_{\nu} [S^{(\lambda)}_{23}]_{\lambda} [S^{(\mu)}_{12}]_{\mu} [S^{(\nu)}_{23}]_{\nu} \]

\[ = \sum_{\lambda \mu \nu} a_{\lambda} a_{\mu} a_{\nu} (S^{(\lambda)})_{bc} (S^{(\mu)})_{ij} (S^{(\nu)})_{lk} \]

\[ = \sum_{\lambda \mu \nu} a_{\lambda} a_{\mu} a_{\nu} (A^{(\lambda)} B^{(\nu)})_{ik} (A^{(\mu)} A^{(\nu)})_{jk} \quad (A.4) \]

Here \((g, h, l = 1, 2, 3 \cdots n)\) and \((\lambda, \mu, \nu = 1, 2, 3 \cdots n)\) respectively. So we can let \( g = \nu, h = \mu \) and \( l = \lambda \), then according to Eq(A.3) and Eq(A.4), we limit \( A^{\lambda} B^{\nu} = B^{\nu} A^{\lambda} \), \( A^{\nu} A^{\mu} = A^{\mu} A^{\nu} \), and \( B^{\mu} B^{\lambda} = B^{\lambda} B^{\mu} \) \((\lambda, \mu, \nu = 1, 2, 3 \cdots n)\). Under this limited condition, the Eq(A.3) is equal to Eq(A.4), namely when the limited condition Eq(12) is satisfied the braid relation \( S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23} \) holds.

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