THE DIMENSIONS AND EULER CHARACTERISTICS OF M. KONTSEVICH’S GRAPH COMPLEXES

THOMAS WILLWACHER AND MARKO ŽIVKOVIĆ

Abstract. We provide a generating function for the (graded) dimensions of M. Kontsevich’s graph complexes of ordinary graphs. This generating function can be used to compute the Euler characteristic in each loop order. Furthermore, we show that graphs with multiple edges can be omitted from these graph complexes.

1. Introduction

The graph complexes in its various guises are some of the most mysterious and fascinating objects in mathematics. They are combinatorially very simple to define, as linear combinations of graphs of a certain kind, with the operation of edge contraction or dually, vertex splitting as differential. Still, their cohomology is very hard to compute and largely unknown at present.

There are various versions of graph cohomology, each playing a central role in one or more fields of algebra, topology or mathematical physics.

• Ordinary (non-decorated) graph cohomology describes the deformation theory of the $E_n$ operads and plays a central role in many quantization problems [14, 9, 10].
• Ribbon graph complexes describe the cohomology of the moduli spaces of curves [9].
• Lie decorated graph complexes describe the cohomology of the automorphisms of a free group and play a central role in many results in low dimensional topology [7, 9].
• Graph complexes of the above sorts, but with external legs, also have a wide range of applications in topology, knot theory and algebra [11, 13, 2, 1, 4, 6].

In all cases, the differential leaves the genus (i.e., the first Betti number, or loop order) of the graph invariant. Hence the graph complexes split into a direct sum of subcomplexes according to genus, and all the above graph cohomology spaces are (at least) bigraded, by the cohomological degree and the genus.

In none of the above cases is the graph cohomology known. Our current knowledge essentially comes from the following sources:

• In low degrees, one knows the cohomology by computer experiments. [3]
• By degree considerations one can show that the graph cohomology is concentrated in a certain range of allowed bidegrees.
• For several, but not all flavors of graph cohomology there are formulas for the Euler characteristics. [12, 11, 8]
• There are many families of known graph cocycles. Some of these cocycles are known to represent non-trivial cohomology classes, while for others this is a conjecture.
• On some graph complexes there is known to be additional algebraic structure. For example the complex $GC_n$ of ordinary graphs (see below) is naturally a differential graded Lie algebra. Hence the algebraic operations may be used to construct new cohomology classes out of known classes.
• The first author showed in [14] that $H^0(GC_2) \cong sl_1$. This is the only case we know in which a classical graph cohomology is known in one degree where it is non-trivial.

In this paper we focus on the “ordinary” graph complexes introduced by M. Kontsevich. The elements of these complexes are linear combinations of isomorphism classes of undirected graphs, with at least trivalent vertices, and without “odd” symmetries. There are two natural choices of when to count a symmetry of a graph as odd, thus yielding two distinct complexes. First, a symmetry may be counted as odd if its induced permutation on the set of edges is odd. Secondly, a symmetry may be considered odd if its induced permutation on the set of vertices and half-edges is odd. For example, the left hand graph in the following picture has an odd symmetry according to the first convention, but not the second, and vice versa for the right hand graph.

\[ \Theta := \begin{array}{c}
    \quad
  \end{array} \]
We will denote by $V_{even}$, respectively $V_{odd}$ the vector spaces spanned by isomorphism classes of at least trivalent graphs without “odd” symmetries for the first, respectively the second notion of oddness, with $v$ vertices and $e$ edges. We allow multiple edges or tadpoles (short cycles). The first main result of this paper are generating functions for the dimensions of $V_{even}$, respectively $V_{odd}$, i. e., for the numbers of isomorphism classes of graphs as above.

**Theorem 1.** Define

$$p_{odd}(s, t) := \sum_{v, e} \dim(V_{odd}) s^v e^e$$

$$p_{even}(s, t) := \sum_{v, e} \dim(V_{even}) s^v e^e$$

Then

$$p_{odd}(s, t) := \frac{1}{(-s, (st)^2)_{\infty}} \sum_{j_1, j_2, \geq 0} \prod_{a} \frac{((-s)^{j_a} (st)^{2\alpha})_{\infty}}{(1 - (-s)^{j_a} (st)^{2\alpha})_{\infty}} \frac{((2e-1)_{\infty})_{\infty}}{(1 - ((2e-1)_{\infty})_{\infty})_{\infty}}^{l_{2e-1/2}}$$

$$p_{even}(s, t) := \frac{1}{(-s, (st)^2)_{\infty}} \sum_{j_1, j_2, \geq 0} \prod_{a, \beta} \frac{((a, q)_{\infty})_{\infty}}{(1 - (a, q)_{\infty})_{\infty}} \frac{((-t)^{j_a} (st)^{2\alpha})_{\infty}}{((-t)^{j_a} (st)^{2\alpha})_{\infty}} \frac{((2e-1)_{\infty})_{\infty}}{(1 - ((2e-1)_{\infty})_{\infty})_{\infty}}^{l_{2e-1/2}}$$

where

$$(a, q)_{\infty} = \prod_{k \geq 0} (1 - aq^{k/2})$$

is the $q$-Pochhammer symbol.

These formulas may be used to compute the Euler characteristics of the graph complexes, which can be found in Table 1 below for loop orders up to 30. These results in particular allow us to probe the graph cohomology far beyond the region where it is currently accessible to direct computer calculation.

The second main result of this paper is to show that the graph complexes $GC_{2e+1}$ (see below for the definition) can be significantly simplified essentially without altering their cohomology by omitting all isomorphism classes of graphs with multiple edges. Let $GC_{2e+1} \subseteq GC_{2e}$ the subcomplex spanned by graphs without multiple edges.

**Theorem 2.** The inclusion $k \Theta \oplus GC_{2e+1}^{odd} \rightarrow GC_{2e+1}$ is a quasi-isomorphism of complexes.

**Remark.** S. Morita, T. Sakasai and M. Suzuki recently computed the Euler characteristics of the commutative graph complexes up to weight 16, using different methods [?].

1.1. Structure of the paper. In section 2 we recall the definitions of the graph complexes we study in this paper. Section 3 is dedicated to the proof of Theorem 2 while in section 4 we give a proof of Theorem 1.

Acknowledgements. We thank P. Etingof, who suggested the problem of computing the Euler characteristics of the graph complexes to T.W. some years ago. We also thank V. Turchin for valuable discussions. This work was partially supported by the Swiss National Science Foundation, grants PDAMP2_137151 and 200021_150012.

2. Graph complexes

2.1. Graph vector spaces. For $v, e \in \mathbb{N}$ let $gra_{v, e, 0}$ be the set of all graphs with $v$ distinguishable vertices and $e$ distinguishable oriented edges between vertices, i.e., $gra_{v, e, 0}$ is the set of all maps $g: K \rightarrow E \times E$ where $V = \{1, 2, \ldots, v\}$ is the set of vertices and $E = \{1, 2, \ldots, e\}$ is the set of edges. We say that edge $e$ connects vertex $g_1(e)$ to the edge $g_2(e)$. Let $V_{v, e}$ be vector spaces freely generated by graphs in $gra_{v, e, 0}$.

For $v, e \in \mathbb{N}$ and $i \in \mathbb{N}$ let $gra_{v, e, i}$ be the set of all graphs in $gra_{v, e, 0}$ such that every vertex is adjacent to at least $i$ edges, i.e., for every vertex $v \in N$ there are at least $i$ pairs $(e, f) \in K \times \{1, 2\}$ such that $g_j(e) = v$. We call the number of adjacent edges the valence of a vertex. Let $V_{v, e, i}$ be vector space freely generated by graphs in $gra_{v, e, i}$.

However, we are not interested in distinguishing vertices, or edges, or directions of edges. First, if we reverse the orientation of an edge of a graph $\Gamma$ and obtain the graph $\Gamma'$, we identify either $\Gamma = \Gamma'$ or $\Gamma = -\Gamma'$. To do this we introduce the representations $\nu^+$ of the group $(S_2)^v$ on the space $V_{v, e, i}$ by reversing the orientation of edges. Let
\[ v^-(g) = (-1)^tv^+(g), \] where \( t \) is the number of edges reversed by \( g \in (S_2)^t \), be another representation of the group \( S_2 \) on the space \( V^e_{x,.} \). Let \( V^e_{x,\nu} = \) for \( \nu \in \{+, -\} \) be the space of coinvariants of the representation \( v^\nu \).

Secondly, let \( \mu^e \) be the representations of the symmetric group \( S_e \) on the space \( V^e_{x,.} \) which interchange edges. Let \( \mu^e(g) = \text{sgn}(g)\mu^e(g) \), where \( \text{sgn}(g) \) is the sign of the permutation \( g \in S_e \), be another representations of the group \( S_e \) on the space \( V^e_{x,.} \). It is easy to see that representations \( \mu^\nu \) for \( \nu \in \{+, -\} \) descend to representations on the spaces of coinvariants \( V^e_{x,\nu} \). Let \( V^e_{x,\nu} \) be the space of coinvariants of the representation \( \mu^\nu \) on \( V^e_{x,.} \).

Finally, let \( \rho^e \) be the representations of the symmetric group \( S_v \) on the space \( V^e_{x,.} \) by interchanging vertices. Let \( \rho^e(g) = \text{sgn}(g)\rho^e(g) \) be another representation of the group \( S_v \) on the space \( V^e_{x,.} \). It is easy to see that representations \( \rho^\nu \) for \( \nu \in \{+, -\} \) descend to the spaces of coinvariants \( V^e_{x,\nu} \). We are interested in the space \( V^e_{x,\nu} \) of the space of coinvariants of the representation \( \rho^\nu \) on \( V^e_{x,.} \).

A basis of \( V^e_{x,\nu} \) consists of all graphs with \( v \) indistinguishable vertices and \( e \) unoriented indistinguishable edges (i.e., isomorphism classes of graphs), such that every vertex is adjacent to at least \( i \) edges. A basis of for instance \( V^e_{x,\nu} \) consists of the subset of (isomorphism classes of) graphs that do not have automorphisms which act by an odd permutation on the edges.

Note that if \( s_v = - \), then every graph which contains multiple edge has an odd automorphism, by interchanging the constituent edges of the multiple edge. Hence such graphs are zero in the complexes \( V^e_{x,\nu} \). Similarly, if \( s_v = + \), then reverting the direction of a tadpole, i.e., an edge \( e \) such that \( g_1(e) = g_2(e) \), is an odd automorphism and hence such graphs are zero in the complexes \( V^e_{x,\nu} \). We may be interested in excluding tadpoles or multiple edges even in the cases \( s_v = + \) and \( s_v = - \) by starting with the space of graphs which exclude them instead of \( \text{gra}_e \). Let \( V^e_{x,\nu} \subseteq V^e_{x,\nu} \) and \( V^e_{x,\nu} \subseteq V^e_{x,\nu} \) be the subspaces spanned by graphs without tadpoles, and let \( V^e_{x,\nu} \subseteq V^e_{x,\nu} \) and \( V^e_{x,\nu} \subseteq V^e_{x,\nu} \) be the subspaces spanned by graphs without multiple edges. Thus, from now on we consider \( s_v, s_v \in \{+, -, 0\} \).

### 2.2. Chain complexes.

Let \( d : V^e_{x,\nu} \rightarrow V^e_{x,\nu} \) for \( s_v, s_v \in \{+, -\} \) be the map

\[
d^e = \sum_{x \in V(x)} \frac{1}{2} (\text{splitting of } x) - (\text{adding an edge at } x),
\]

where "splitting of \( x \)" means putting \( \bullet \rightarrow \bullet \) instead of the vertex \( x \) and summing over all possible ways how to connect edges that have been connected to \( x \) to the new \( x + 1 \). Note that the expression is 0 unless \( s_v = s_v \), so we consider only that case. "Adding an edge at \( x \)" means adding \( \bullet \) on the vertex \( x \). Unless \( x \) is an isolated vertex, it will cancel one term of the splitting. One can check that \( d^2 = 0 \) if and only if \( s_v = - s_v \). Therefore, for \( n \in \mathbb{Z} \) we can define the full graph complexes with a differential \( d \):

\[
fGC_n = \prod_{i \geq 1} \prod_{e \geq 0} \left\{ \begin{array}{ll} V^e_{x,\nu,0}[(n - 1)e - n(v - 1)] & \text{for } n \text{ even} \\ V^e_{x,\nu,0}[(n - 1)e - n(v - 1)] & \text{for } n \text{ odd} \end{array} \right. 
\]

The degree shifts are chosen such that there is a natural differential graded Lie algebra structure on \( fGC_n \), see [13].

It can be seen that \( d \) can not produce 1-valent and 2-valent vertices, tadpoles and multiple edges, if there were none of them before. Also, the differential obviously does not alter the number of connected components of a graph. Therefore we may define several smaller subcomplexes. Let \( V^e \) := \( V^e_{x,\nu} \) be the space generated by all connected graphs, and similarly let \( V^e_{x,\nu} \) := \( V^e_{x,\nu} \) be the space generated by all connected graphs, and similarly let \( V^e_{x,\nu} \) := \( V^e_{x,\nu} \) and \( V^e_{x,\nu} \) := \( V^e_{x,\nu} \). Then we define the following subcomplexes of \( fGC_n \):

3
In a connected graph a **separating vertex** is a vertex which if removed makes the graph disconnected. A graph without a separating vertex is called one-vertex irreducible. The differential $d$ can not form a separating vertex, so there are subcomplexes generated by the one-vertex irreducible graphs, which we denote by adding a letter $n$ (non-separable) to the name, e.g. $GC_n$.

All complexes split into a product of subcomplexes according to the number of edges minus number of vertices $b = e - v$, which is preserved by the differential $d$. For them, the number $n$ (up to degree shift) does not really matter, only its parity. If we restrict to graphs with at least trivalent vertices those subcomplexes are finitely dimensional. Therefore, we define subcomplexes $GC_{n,b} \subset GC_n$ etc., generated by graphs with fixed $e - v = b$.

We denote the Euler characteristics of these finite dimensional complexes as follows:

$$
\begin{align*}
\hat{\chi}_b^{\text{odd}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{odd}}_{v,b+v} \right) = \chi(fGC_{n,b}^{\geq 3}) & \text{for } n \text{ odd} \\
\hat{\chi}_b^{\text{even}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{even}}_{v,b+v} \right) = \chi(fGC_{n,b}^{\leq 3}) & \text{for } n \text{ even} \\
\hat{\chi}_b^{\text{odd+}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{odd+}}_{v,b+v} \right) = \chi(fGC_{n,b}^{\geq 3,\neq}) & \text{for } n \text{ odd} \\
\hat{\chi}_b^{\text{even+}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{even+}}_{v,b+v} \right) = \chi(fGC_{n,b}^{\leq 3,\neq}) & \text{for } n \text{ even} \\
\hat{\chi}_b^{\text{odd}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{odd}}_{v,b+v} \right) = \chi(GC_{n,b}) & \text{for } n \text{ odd} \\
\hat{\chi}_b^{\text{even}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{even}}_{v,b+v} \right) = \chi(GC_{n,b}) & \text{for } n \text{ even} \\
\hat{\chi}_b^{\text{odd+}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{odd+}}_{v,b+v} \right) = \chi(GC_{n,b}^{\neq}) & \text{for } n \text{ odd} \\
\hat{\chi}_b^{\text{even+}} & := \sum_{v \geq 0} (-1)^v \dim \left( V^{\text{even+}}_{v,b+v} \right) = \chi(GC_{n,b}^{\neq}) & \text{for } n \text{ even}
\end{align*}
$$

(2)

In this paper we are interested in computing the above Euler characteristics. The numeric result is contained in Table 1 below.

3. **Graphs with multiple edges may be omitted**

The cohomologies of the various graph complexes introduced above are highly related. Obviously, the graph complexes with disconnected graphs are just symmetric products of the complexes of connected graphs. Furthermore, it has been shown in [5, Proposition 3.4] that adding the trivalence condition changes the cohomology of the graph complexes only by a list of known classes, and the omission of graphs with tadpoles does not change the cohomology further.

Finally Conant and Vogtman [3] showed that the complexes of one-vertex irreducible graphs are quasi-isomorphic to their non-one-vertex irreducible relatives.

---

1The number $b$ is of course minus the Euler characteristic of the graph as a topological space.
Proposition 1 (cf. Appendix F of [14]),

\[ H(GC_n) = H(GC_{n+1}) \quad H(GC_{n+2}^n) = H(GC_{n+2}^n), \]

We can use these results to show Theorem 2 of the introduction.

Proof of Theorem 2. We actually prove that \( H(GC_n) = H(GC_{n+2}^n) \oplus \mathbb{K}[2n-3] \) and use Proposition 1. We have splitting of complexes

\[ GC_n = GC_n^n \oplus \mathbb{K} \Theta, \]

where \( \Theta \) is the “Theta” graph \( \Theta \), of degree \( 3-2n \) and \( GC_n^n \) is the remainder. It is clear that \( H(\mathbb{K} \Theta) \cong \mathbb{K}[2n-3] \), so the claim reduces to showing that \( H(GC_n^n) = H(GC_{n+2}^n) \).

To be precise, let a multiple edge \( e \) be the set of all edges connecting the same pair of vertices. Let \( N(e) \) be the number of edges in multiple edge \( e \), let \( S(e) := |N(e)| \) be its strength and let the total strength \( S(\Gamma) \) be the sum of the strengths of all multiple edges in a graph \( \Gamma \). The differential \( d \) can not increase the total strength, so we have a filtration of \( GC_n^n \) by the total strength. The subcomplexes of fixed \( b = e - v \) are finitely dimensional, so for all of them the spectral sequences converge to their cohomology, and therefore the original spectral sequence converges to \( H(GC_n^n) \).

The differential \( d^n \) on the first page does not decrease the total strength, or we can say

\[ d^n \Gamma = \sum_{x \in V(\Gamma)} d^n x \quad \text{for} \quad d^n x = \frac{1}{2} \text{strength preserving splittings of } x, \]

where ”strength preserving splittings of \( x \)” are the splittings of \( x \) that do not split multiple edge in two parts with odd number of edges.

Let a good vertex be a trivalent vertex \( x \) two of whose edges form a double edge: \( - \rightarrow \). The other end of the double edge is denoted by \( (x) \) and the other end of the single edge is denoted by \( s(x) \). We require \( t(x) \neq s(x) \).

For every good vertex \( x \) we define a map \( h_x \) such that locally:

\[ h_x \begin{cases} \pm 1 \frac{1}{N+2} & \text{if } N \text{ odd} \\ \pm 1 \frac{1}{N+1} & \text{if } N \text{ even} \end{cases} \quad \text{if } x \neq v, \]

where the thick edge with number \( N \geq 0 \) indicates an \( N \)-fold edge, and the sign is chosen such that

\[ h_x \begin{cases} \frac{1}{N+2} & \text{if } N \text{ odd} \\ \frac{1}{N+1} & \text{if } N \text{ even} \end{cases} \quad \text{if } v \text{ is the last vertex, all other vertices keep their number and all edges keep their orientation. Note that } h_x \text{ does not change the total strength. We put } h_x = 0 \text{ if } x \text{ is not good, and} \]

\[ h \Gamma = \sum_{x \in V(\Gamma)} h_x \Gamma. \]

Lemma 1. It holds that

\[ (d^n h + h d^n) \Gamma = 2S(\Gamma) \Gamma. \]

Proof. We compute

\[ (d^n h + h d^n) \Gamma = \sum_{y \in V(\Gamma)} d^n y \sum_{x \in V(\Gamma)} h_x \Gamma + \sum_{x \in V(\Gamma)} h_x \sum_{y \in V(\Gamma)} d^n y \Gamma = \sum_{x \in V(\Gamma)} \sum_{x \neq y} d^n y h_x \Gamma + h_x d^n y \Gamma + 2 \sum_{x \neq y} h_x d^n y \Gamma. \]

We claim that \( d^n y \) can not change the property of being good of a vertex \( x \neq y \). Clearly, \( d^n y \) can not change a good vertex to become not good. On the other hand, it can not affect non-neighbors, can not change the valence of other vertices and can not produce multiple edge. Therefore, if \( d^n y \) makes \( x \) good, \( x \) was already trivalent with two of the edges pointing to the same vertex before acting of \( d^n y \). The only possibility when \( x \) was not good before acting is when \( x \) was a trivalent neighbour of \( y \) all of whose three edges form a triple edge towards \( y \), that is \( \Theta \). But \( y \)

\[ \text{We note that in these references the result is only shown for one version of the graph complex. However, the proofs do not depend on the presence or absence of tadpoles or multiple edges.} \]
can not be a separating vertex and hence can not be connected to anything else than \( x \), and the whole graph is the theta graph \( \Theta_n \). But this graph has been explicitly excluded. Therefore

\[
(5) \quad (d^0 h + h d^0) \Gamma = \sum_{x \in \Gamma \setminus \{s(x)\}} \left( d^0_x h_x \Gamma + h_x d^0_x \Gamma \right) + 2 \sum_{x \in \Gamma} h_x d^0_x \Gamma = \\
= \sum_{x \in \Gamma \setminus \{s(x)\}} \sum_{y \neq x} \left( d^0_x h_y \Gamma + h_y d^0_x \Gamma \right) + \sum_{x \in \Gamma \setminus \{s(x)\}} \sum_{y \in \Gamma \setminus \{s(x)\}} \left( d^0_x h_y \Gamma + h_y d^0_x \Gamma \right) + \sum_{x \in \Gamma} h_x d^0_x \Gamma.
\]

The first term is trivially zero. We claim that the second term is also zero. It is enough to assume that \( x \) is the last vertex \( v \). We consider separately the cases of odd and even numbers of “bridging” vertices between \( s(v) \) and \( t(v) \). First, in the even case:

\[
\begin{align*}
\text{even case:} \quad & v \xrightarrow{d^0} N + \sum_{k=0}^{N/2} \binom{2N}{2k} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N+1} N + 2N + 2 \sum_{k=0}^{N/2-1} \binom{2N+2}{2k} \sum_{\text{conn}} \xrightarrow{d^0} N + \frac{1}{4N+2} N + 2 \sum_{k=0}^{N/2} \binom{2N+2}{2k} \sum_{\text{conn}} \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v \xrightarrow{h} \frac{1}{2N-2k+1} \sum_{\text{conn}} v
\end{align*}
\]

where \( \sum_{\text{conn}} \) is the sum over all possibilities of connecting remaining edges of \( v(t) \) to new vertices. We have omitted the term of ”Adding an edge” at \( t(v) \) in the action of \( d^0_{t(v)} \), but it trivially cancels. For an odd number of “bridging” edges the situation is similar:
We have to check that this term is not cancelled by "Adding an edge", i.e. that it is not at the same time true that

\[ \text{Interchanging } \bullet \rightarrow \Gamma \text{ with } \bullet \leftrightarrow \bullet, \text{ } s(v) \text{ with } n(v) \text{ and vice versa in the above calculations leads to the conclusion that the third term in (5) is } d_i^0 h_i\Gamma + h_i d_i^0\Gamma = 0. \]

The remaining term is:

\[ (d_i^0 h + h d_i^0)\Gamma = \sum_{x \in \Gamma(1)} h_s d_i^0 \Gamma = \sum_{x \in \Gamma(1)} h_{s+1} d_i^0 \Gamma. \]

It suffices to consider terms for which \( d_i^0 \) makes the new vertex \( v + 1 \) good, otherwise the term is zero. Vertex \( v + 1 \) in \( d_i^0 \Gamma \) has a single edge towards \( x \), so it is good if and only if a multiple edge \( e \) with \( N(e) \geq 2 \) has been split into a double edge heading towards \( v + 1 \) and an \( (N(e) - 2) \)-fold edge heading towards the new \( x \), and all other edges heading towards the new \( x \), i.e.

\[ \text{We have to check that this term is not cancelled by } "\text{Adding an edge}”, \text{ i.e. that it is not at the same time true that } N = 2 \text{ and that there are no other edges towards } x. \text{ But in that case, since the other end of the } N\text{-fold edge is not separating, the whole graph would be } \bullet \leftrightarrow \bullet = 0. \text{ Therefore}
\]

\[ \left( d_i^0 h + h d_i^0 \right)\Gamma = \sum_{e \text{ multiple edge at } x} \frac{1}{2} \left( \frac{N(e)}{2} \right) + \text{(something where } v + 1 \text{ is not good)} \]
4.1. The proof of Theorem 1. Here we calculate the dimension of \( V_{x, i}^{v, e} \) for \( v, e \in \mathbb{N}, i \in \{0, 1, 2, 3\}, s_v, s_\mu \in \{+, *, -\} \) and \( s_\rho \in \{+, -, \} \). It holds that

\[
\dim \left( V_{x, i}^{v, e} \right) = \dim \left( V_{x, i}^{v, e} \right)_{S_v} = \dim \left( V_{x, i}^{v, e} \right)_{V_{x, i}^{v, e}} = \frac{1}{|S_v|} \sum_{g \in S_v} \xi_{x, i}^{v, e} (g),
\]

where \( \xi_{x, i}^{v, e} \) is the character of the representation \( \rho^v \) of \( S_v \) on \( V_{x, i}^{v, e} \). Furthermore, it is enough to calculate characters of conjugacy classes of \( S_v \), i.e. on partitions \( j \) of \( v \):

\[
\dim \left( V_{x, i}^{v, e} \right) = \frac{1}{v!} \sum_{\nu, j, \sum, a_j = v} \prod_{j, \sum, a_j = v} g_j \xi_{x, i}^{v, e} (g),
\]

where \( g_j \) is any element of the conjugacy class \( j \) of \( S_v \). Let

\[
P_i^{v, e} = \sum_{s \geq 0} \dim \left( V_{x, i}^{v, e} \right) s^i r^e
\]

be the generating functions. Then

\[
P_i^{v, e} = \sum_{v, e \geq 0} \sum_{\nu, j, \sum, a_j = v} \prod_{j, \sum, a_j = v} g_j \xi_{x, i}^{v, e} (g) s^i r^e = \sum_{\nu, j, \sum, a_j = v} \left( \prod_{j, \sum, a_j = v} \frac{g_j}{a_j!} \right) \xi_{x, i}^{v, e} (g)
\]

where

\[
\xi_{x, i}^{v, e} = \sum_{\nu, j, \sum, a_j = v} \xi_{x, i}^{v, e} (g)
\]

is the total character.

In the following we only consider \((s_v, s_\mu, s_\rho) = (+, -, *) \) or \((- , + , -) \). For the computer calculations in subsection 4.1, we also used another possibilities without writing down the closed formula, whose calculations are similar.

Lemma 2.

\[
\xi_{x, i}^{v, e} = \prod_{a \geq 1} \left( 1 + t_{2a-1} \right)^{(a-1)f_{2a-1}}
\]

\[
\prod_{a \geq 1} \left( 1 + t_{2a} \right)^{f_{2a}}
\]

\[
\prod_{a \geq 1} \left( 1 + t_a \right)^{\left( \frac{a(a-1)}{2} \right)}
\]

\[
\prod_{\beta \geq a} \left( 1 + t_{2\beta - 2a} \right)^{f_{2\beta - 2a}}
\]

\[
\left( 1 + t_{\text{lcm}(\alpha, \beta)} \right)^{f_{\text{lcm}(\alpha, \beta)}},
\]

where \( t_a^* := (-t)^a \), and

\[
\xi_{x, i}^{v, e} = \prod_{a \geq 1} \left( \frac{1}{1 - t_{2a-1}^*} \right)^{(a-1)f_{2a-1}}
\]

\[
\prod_{a \geq 1} \left( \frac{1}{1 + t_a^* \left( 1 - t_{2a}^* \right)^{-1}} \right)^{f_{2a}}
\]

\[
\prod_{a \geq 1} \left( \frac{1}{1 - t_a^*} \right)^{\left( \frac{a(a-1)}{2} \right)}
\]

\[
\prod_{\beta \geq a} \left( \frac{1}{1 - t_{2\beta - 2a}^*} \right)^{f_{2\beta - 2a}}
\]

\[
\left( 1 - t_{\text{lcm}(\alpha, \beta)}^* \right)^{f_{\text{lcm}(\alpha, \beta)}},
\]

where \( t_a^* := t^a \).

Proof. The total character \( \xi_{x, i}^{v, e} \) is the polynomial in \( t \) with the coefficient of \( t^r \) being the character of \( g_j \in S_{x, \sum, a_j} \) in the representation \( \rho^v \) on \( V_{x, i}^{v, e} \). A basis of \( V_{x, i}^{v, e} \) consists of graphs with \( v \) distinguishable vertices and \( e \) unoriented indistinguishable edges. Switching edges changes the sign, thus excluding double edges.
An element \( g_j \in S_{\sum \alpha_j} \) acts on \( V_{\sum \alpha_j, 0}^+ \) by moving one graph to another. To calculate the character, we need to find graphs \( x \) moved to \( kx \) for a scalar \( k \). By the definition of the representation \( k \in \{1, -1\} \). The element \( g_j \) acts in this way on the graph \( x \) which has a symmetry (up to the sign) of \( g_j \) on vertices, i.e., whose vertices are partitioned into the \( j_a \) cycles of \( \alpha \) edges with circular symmetry in every cycle. The cycles are distinguishable, and the beginning vertex in the cycle is marked.

Let us pick a cycle with odd number \( 2\alpha - 1 \) of vertices, and let us number the vertices by \( 1, 2, \ldots, 2\alpha - 1 \). If there is an edge between vertex 1 and \( l + 1 \), because of the symmetry there should be also an edge between vertex 2 and \( l + 1 \), and so on. We obtain \( 2\alpha - 1 \) edges in total. Graphs containing these edges contribute to the total character \( \xi_{j \alpha}^{2\alpha - 1} \) by multiplication with \( t_{2\alpha - 1}^r = (-1)^{2\alpha - 1} = t^{2\alpha - 1} \). Note that cycling an odd number of edges is an even permutation, so it does not change sign. Graphs not containing these edges contribute to the total character by multiplication with 1, so this possibility contribute by 1.

There are \( \alpha \) possibilities of putting that cycle of edges, so the contribution is \((1 + t_{2\alpha - 1})^\alpha \). The contributions of all \( j_a \) cycles is \((1 + t_{2\alpha - 1})^\alpha \), and the contribution of all odd cycles is \( \prod_{a \geq 1} (1 + t_{2\alpha - 1})^\alpha \). This is the first line of the formula.

The second line is the contribution of even cycles. The third line is the contribution of the connections between two cycles of the same size, and the forth is the same for cycles of different sizes. The detailed derivation is easy and will be left to the reader. The similar calculation of the total character \( \xi_{j \alpha}^{2\alpha} \) will also be left to the reader. \( \square \)

We are indeed interested in \( \xi_{j \alpha}^{2\alpha - 1} \) and \( \xi_{j \alpha}^{2\alpha} \). Let us calculate the first and leave the second to the reader. We proceed similarly to the proof of the previous lemma, but after fixing \( g_j \in S_{\sum \alpha_j} \), we should consider only graphs with more than 2 adjacent edges. Because of the symmetry, the valences of vertices in the same cycle are the same, so we can talk about the valence of the cycle. However, at this point we have not forced the vertices to be at least 3-valent.

What we can do is to construct a graph with some special cycles for which we are sure by the construction that they are at most 2-valent. Let \( \xi_{j \alpha}^{1+} \) be the special total character, i.e. ‘total character’ of the partition \( j \) which allows adding 0, 1 or 2-valent cycles together with edges. So, \( \xi_{j \alpha}^{1+} \) is the polynomial in variables \( t \) and \( s_\alpha^\nu \) for \( \alpha \in \mathbb{N} \) where the coefficient next to \( r^\nu \prod_{a} (s_\alpha^\nu)^{\nu_a} \) is the number of the graphs (counted with appropriate signs) with \( j_a \) distinguishable \( \alpha \)-cycles, \( n_a \) indistinguishable special \( n_a \)-cycles and \( e \) edges. All cycles have a marked “first” vertex. If there is a symmetry of the order \( r \) between indistinguishable cycles, we divide the term with \( r \).

The key fact is that special cycles with valence up to 2 can be added to the fixed cycles of partition \( j \) in a controlled way: they are either disconnected from the rest and form free loops or lines (vacuum), connected to one cycle (antennas) or connect two cycles (connections). This is the reason why we can not calculate the dimension of \( V_{\sum \alpha, 0}^{j \alpha} \) for \( i > 3 \). Careful calculation leads to the following formula. 

\[ \]
\[ \prod_{\beta_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\beta_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\beta_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\beta_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ j_a j_b \alpha \beta (c \lambda (\alpha, \beta))^{-1} \left( s_{\alpha_0}^* \right) \left( t_{\alpha_0} \right)^{l+1} \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ \frac{1}{2} j_a (j_a - 1) \xi (\alpha, \beta) (l_{\alpha_0} \alpha_0) \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ \frac{1}{2} j_a \alpha (\alpha - 1) \xi (\alpha, \beta) (l_{\alpha_0} \alpha_0)^2 \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ j_a \alpha (\alpha - 1) \xi (\alpha, \beta) (l_{\alpha_0} \alpha_0) \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ j_a \alpha (\alpha - 1) \xi (\alpha, \beta) \left( l_{\alpha_0} \alpha_0 \right)^2 \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ j_a \alpha (\alpha - 1) \xi (\alpha, \beta) \left( l_{\alpha_0} \alpha_0 \right)^3 \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ \frac{1}{2} j_a \alpha (\alpha - 1) \xi (\alpha, \beta) \left( l_{\alpha_0} \alpha_0 \right)^3 \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ \frac{1}{2} j_a \alpha (\alpha - 1) \xi (\alpha, \beta) \left( l_{\alpha_0} \alpha_0 \right)^4 \right] \]

\[ \prod_{\alpha_0=1}^{\alpha_0 (c\lambda (\alpha, \beta))} \prod_{\alpha_1=1}^{\alpha_1 (c\lambda (\alpha, \beta))} \prod_{\alpha_2=1}^{\alpha_2 (c\lambda (\alpha, \beta))} \prod_{\alpha_3=1}^{\alpha_3 (c\lambda (\alpha, \beta))} \exp \left[ \frac{1}{2} j_a \alpha (\alpha - 1) \xi (\alpha, \beta) \left( l_{\alpha_0} \alpha_0 \right)^5 \right] \]

The diagrams next to the factors depict the shape from which the factor comes. Full nodes \bullet represent general cycles in the graph, and empty nodes \bigcirc represent special cycles added to the graph, which must be at most 2-valent. Small number next to the node \( \alpha \) represent the order (number of vertices) of the cycle. A symbol \( \ast \) represents a...
special even cycle with opposite vertices connected, and \( \emptyset \) represents a special 2-valent cycle with a set of inside edges (not towards the opposite vertex) respecting the symmetry. A connection \( \emptyset \rightarrow \emptyset \) (with the right-hand cycle always being the special one) represents a set of edges where every vertex of the right-hand cycle is by symmetry forced to be connected to two opposite vertices of the left-hand even cycle. A connection \( \emptyset \rightarrow \emptyset \) represents a single set of edges connecting vertices from different cycles respecting the symmetry, different from the one represented by a harpoon.

Note that cycles \( \emptyset \) have inner valence 1, and cycles \( \emptyset \) have inner valence 2. The existence of a simple connection towards a special cycle \( \alpha \bullet \emptyset \beta \) implies that \( \alpha \mid \beta \) and increases the valence of the special cycle by one. Therefore a simple connection between two special cycles implies that they have the same order. A "harpooned" connection towards a special cycle \( 2\alpha \bullet \emptyset \beta \) implies that \( \alpha \mid \beta \) and increases the valence of the special cycle by two. If the other cycle is also special, its valence is increased by one, and it has a double order.

For the illustration we explain the first factor, the contribution of connections between different cycles, say \( \alpha \)-cycle and a \( \beta \)-cycle, \( \beta < \alpha \). The two cycles can be connected via a chain of \( l \) special cycles. Because all special cycles are connected to two cycles, there can not be internal connections in the cycles and from each vertex exactly 1 edge goes to the next and to the previous cycle. Because of connecting rules, the order of all special cycles in the chain is the same and it is a multiple of the least common multiple \( \text{lcm}(\alpha, \beta) \). So, the contribution of all connections between different cycles \( \beta \) is the product over all \( \beta > \alpha \geq 1 \), all chain lengths \( l \geq 1 \) and all possibilities of orders of special cycles \( c \cdot \text{lcm}(\alpha, \beta) \) for \( c \geq 1 \), of the contribution of such type of connections, i.e. of connections between \( \alpha \) and \( \beta \)-cycle of length \( l \) and order of special cycle \( c \cdot \text{lcm}(\alpha, \beta) \).

Let \( \xi \) be the contribution of exactly 1 such connection. There can be any number of that type of connections for generally different starting and ending \( \alpha \) and \( \beta \)-cycles. If there are \( n \) of them connecting different pairs of cycles, the contribution is \( \frac{1}{n!} \) in order not to count same cases multiple times. Even if some of them connect the same pair of cycles, because of the symmetry factor the contribution remains \( \frac{1}{n!} \). So, the total contribution of that type of connection is \( 1 + \xi + \frac{1}{2!} \xi^2 + \frac{1}{3!} \xi^3 + \cdots = \exp(\xi) \).

To calculate \( \xi \) we first chose an \( \alpha \)-cycle and \( \beta \)-cycle in \( j_\alpha j_\beta \) possible ways. We can connect the first special cycle with the \( \alpha \)-cycle in \( \alpha \) different ways, and the last with the \( \beta \)-cycle in \( \beta \) different ways. Connections between special cycles can be done in \( (c \cdot \text{lcm}(\alpha, \beta))^{l-1} \) different ways. We also add \( l \) special cycles \( (s^1_{l \cdot \text{lcm}(\alpha, \beta)})^l \) and \( l+1 \) cycles of edges \( (s^2_{l \cdot \text{lcm}(\alpha, \beta)})^{l+1} \). Multiplying everything leads to the total character \( \xi \) of \( \xi \) of the contribution of such type of connections, i.e. to the character of graphs with the same cycles, of which one is special (2- or less-valent). We subtracted graphs with two low-valent cycles twice, so we need to add the character of graphs with 2 special cycles. Then we need to subtract the character of graphs with 3 special cycles, add with 4, etc.

So all special total characters \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \) for \( k \leq j \) contribute to the total character \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \), namely the coefficient (a polynomial in \( t \)) next to \( \prod_{\alpha} (s^1_{l \cdot \text{lcm}(\alpha, \beta)})^{j_\alpha-k_\alpha} \) with a sign \( (-1)^{\sum_{\alpha} j_\alpha-k_\alpha} \). But all cycles in \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \) are distinguishable while the special cycles contributing to \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \) are not. We can put \( j_\alpha-k_\alpha \) special indistinguishable cycles between \( k_\alpha \) ordered cycles in \( j_\alpha！/k_\alpha！ \) ways. So the contribution of the coefficient next to \( \prod_{\alpha} (s^1_{l \cdot \text{lcm}(\alpha, \beta)})^{j_\alpha-k_\alpha} \) in \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \) into \( \xi^1_{l \cdot \text{lcm}(\alpha, \beta)} \) is multiplied by \( (-1)^{\sum_{\alpha} j_\alpha-k_\alpha} k_\alpha！/j_\alpha！ \).

Therefore, if we put \( s^1_0 := -s^0/\alpha \) we arrive at the formula (see (5)):

\[
(14) \quad p^{\text{even}} := \sum_{j_1, j_2 \geq 0} \prod_{\alpha} s^{j_\alpha_{l_{j_1} \cdot \text{lcm}(\alpha, \beta)}}_{j_\alpha} \xi_{l \cdot \text{lcm}(\alpha, \beta)}^{j_\alpha} = \sum_{j_1, j_2 \geq 0} \prod_{\alpha} s^{j_\alpha_{l_{j_1} \cdot \text{lcm}(\alpha, \beta)}}_{j_\alpha} \xi_{l \cdot \text{lcm}(\alpha, \beta)}^{j_\alpha}.
\]

We use the \( q \)-Pochhammer symbol

\[
(15) \quad (a, q)_\infty = \prod_{k \geq 0} \left(1 - aq^k\right)
\]

and simplify

\[
(16) \quad p^{\text{even}}(s, t) = \sum_{j_1, j_2 \geq 0} \prod_{\alpha} s^{j_\alpha_{l_{j_1} \cdot \text{lcm}(\alpha, \beta)}}_{j_\alpha} \frac{1}{j_\alpha ! \cdot l_{j_1} !} \left(\frac{(t^{j_\alpha} - s^{j_\alpha})}{(t^{j_\alpha} - s^{j_\alpha})}ight)_{j_\alpha !} \left(\frac{(t^{j_\alpha} - s^{j_\alpha})}{(s^{j_\alpha})}ight)_{j_\alpha !} \prod_{\alpha, \beta} \left(\frac{(t^{j_\alpha} \cdot \text{lcm}(\alpha, \beta))}{(s^{j_\alpha} \cdot \text{lcm}(\alpha, \beta))}ight)_{j_\alpha !} \prod_{\alpha, \beta} \left(\frac{(t^{j_\alpha} \cdot \text{lcm}(\alpha, \beta))}{(s^{j_\alpha} \cdot \text{lcm}(\alpha, \beta))}ight)_{j_\alpha !}.
\]
By an equally tedious and lengthy computation which we leave to the reader one arrives at the formula for the odd case:

\[ p^{\text{odd}}(s, t) := P_{3}^{++} = \frac{1}{(-s, (st)^{2})_{\infty}} \sum_{j, j_{s}, \geq 0} \prod_{\alpha} \left( -s \right)^{j_{s}} \left( -1 \right)^{j_{s}} \left( (-st)^{3}, (st)^{2} \right)_{\infty}^{j_{2}} \left( \left( 2a-1, (st)^{4a-2} \right)_{\infty} \right)^{j_{2a-1}/2} \left( \left( \left( -1 \right)^{s} i, s \right)_{\infty} \right)^{j_{2s}} \prod_{\alpha, \beta} \left( \left( \left( (-1)^{s} i, t \right)_{\infty} \right) \right)^{j_{2s}}. \]

4.2. A variant of Theorem [1]. By a similar computation as that leading to Theorem [1] we may also compute generating functions for the dimensions of the spaces of graphs \( V_{n, e}^{\text{odd}} \) and \( V_{n, e}^{\text{even}} \).

**Theorem 3.** Define

\[ p^{\text{odd}}(s, t) := \sum_{v, e} \dim \left( V_{n, e}^{\text{odd}} \right) s^{v} t^{e} \quad \text{and} \quad p^{\text{even}}(s, t) := \sum_{v, e} \dim \left( V_{n, e}^{\text{even}} \right) s^{v} t^{e}. \]

Then

\[ p^{\text{odd}}(s, t) = \frac{1}{(-s, (st)^{2})_{\infty}} \sum_{j, j_{s}, \geq 0} \prod_{\alpha} \left( -s \right)^{j_{s}} \left( -1 \right)^{j_{s}} \left( (-st)^{3}, (st)^{2} \right)_{\infty}^{j_{2}} \left( \left( 2a-1, (st)^{4a-2} \right)_{\infty} \right)^{j_{2a-1}/2} \left( \left( \left( -1 \right)^{s} i, s \right)_{\infty} \right)^{j_{2s}} \prod_{\alpha, \beta} \left( \left( \left( (-1)^{s} i, t \right)_{\infty} \right) \right)^{j_{2s}} \cdot \]

\[ p^{\text{even}}(s, t) = \frac{1}{(-s, (st)^{2})_{\infty}} \sum_{j, j_{s}, \geq 0} \prod_{\alpha} \left( -s \right)^{j_{s}} \left( -1 \right)^{j_{s}} \left( (-st)^{3}, (st)^{2} \right)_{\infty}^{j_{2}} \left( \left( 2a-1, (st)^{4a-2} \right)_{\infty} \right)^{j_{2a-1}/2} \left( \left( \left( -1 \right)^{s} i, s \right)_{\infty} \right)^{j_{2s}} \prod_{\alpha, \beta} \left( \left( \left( (-1)^{s} i, t \right)_{\infty} \right) \right)^{j_{2s}}. \]

4.3. The connected part. Let us denote \( n_{b}^{e} := \dim \left( V_{n, b}^{\text{odd}} \right) \) and \( n_{b}^{e} := \dim \left( V_{n, b}^{\text{odd}} \right) \). Basis elements of \( V_{n, e}^{\text{odd}} \) are possibly disconnected graphs. Let one of them consist of \( f_{\alpha}^{e} \) connected graphs in \( V_{n, \alpha}^{\text{odd}} \), for \( \alpha = 1, \ldots, v \). It is \( \sum_{\alpha, \beta} a_{\alpha}^{e} = b \) and \( \sum_{\alpha, \beta} b_{\alpha}^{e} = v \). So we are choosing \( f_{\alpha}^{e} \) elements out of \( n_{b}^{e} \) basis elements of \( V_{n, \alpha}^{\text{odd}} \). This can be done in \( \binom{n_{b}^{e}}{f_{\alpha}^{e}} \) ways if the number of vertices \( \beta \) is odd, and in \( (-1)^{f_{\alpha}^{e}} \binom{n_{b}^{e}}{f_{\alpha}^{e}} \) ways if \( \beta \) is even, respecting the symmetry. All together, we have the formula

\[ n_{b}^{e} = \sum_{\sum_{\alpha, \beta} a_{\alpha}^{e} = b} \prod_{\sum_{\alpha, \beta} b_{\alpha}^{e} = v} \left( -1 \right)^{f_{\alpha}^{e}} \binom{n_{b}^{e}}{f_{\alpha}^{e}}. \]
One then computes

\[ \chi_b^{\text{odd}} = \sum_{v \geq 0} (-1)^v n_b^v = \]

\[ = \sum_{v \geq 0} (-1)^v \sum_{(f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \prod_{a, \beta} \left( -1 \right)^{\alpha, \beta} \left( \sum_{\alpha \neq \beta = 0} \left( -1 \right)^{\alpha, \beta} \frac{a + \beta}{f_a} \right) = \]

\[ = \sum_{(f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \prod_{a, \beta} \left( -1 \right)^{\alpha, \beta} \left( \sum_{\alpha \neq \beta = 0} \left( -1 \right)^{\alpha, \beta} \frac{a + \beta}{f_a} \right) = \]

\[ = \sum_{(i, a, b \geq 1, 2, \ldots) (f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \prod_{\alpha, \beta} \left( -1 \right)^{\alpha, \beta} \left( \sum_{\alpha \neq \beta = 0} \left( -1 \right)^{\alpha, \beta} \frac{a + \beta}{f_a} \right) = \]

\[ = \sum_{i \in \mathbb{N}} \prod_{\alpha, \beta} \left( -1 \right)^{\alpha, \beta} \prod_{\alpha \neq \beta = 0} \left( -1 \right)^{\alpha, \beta} \left( \frac{a + \beta}{f_a} \right) \]

(19)

**Lemma 3.** For every \( i \in \mathbb{N} \) it holds that

\[ \sum_{j \geq 0} \prod_{\beta} \left( -1 \right)^{\beta} \frac{\chi_a^{\text{odd}}}{i} = \left( \sum_{j \geq 0} \prod_{\beta} \left( -1 \right)^{\beta} \frac{\chi_a^{\text{odd}}}{i} \right) \]

Proof.

\[ \sum_{i} \chi^i \prod_{(f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \frac{\chi_a^{\text{odd}}}{i} = \sum_{i} \chi^i \prod_{(f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \frac{\chi_a^{\text{odd}}}{i} = \sum_{i} \chi^i \prod_{(f \geq 0, a, b \geq 1, 2, \ldots) a \neq b} \frac{\chi_a^{\text{odd}}}{i} = \]

\[ = \prod_{\beta} \sum_{j} \left( -1 \right)^{\beta} \left( \frac{\chi_a^{\text{odd}}}{i} \right) \]

\[ = \prod_{\beta} \sum_{j} \left( 1 + x \right)^{-1} \left( -1 \right)^{\beta} \left( \frac{\chi_a^{\text{odd}}}{i} \right) \]

\[ = \prod_{\beta} \left( 1 + x \right)^{-1} \left( -1 \right)^{\beta} \left( \frac{\chi_a^{\text{odd}}}{i} \right) \]

\[ = \prod_{\beta} \left( 1 + x \right)^{-1} \left( -1 \right)^{\beta} \left( \frac{\chi_a^{\text{odd}}}{i} \right) \]

\[ = \prod_{\beta} \left( 1 + x \right)^{-1} \left( -1 \right)^{\beta} \left( \frac{\chi_a^{\text{odd}}}{i} \right) \]

(20)

So the conclusion is that

\[ \chi_b^{\text{odd}} = \sum_{i \in \mathbb{N}} \prod_{a} (-1)^i \left( \frac{\chi_a^{\text{odd}}}{i} \right). \]

A similar argument leads to the same formula for the even case:

\[ \chi_b^{\text{even}} = \sum_{i \in \mathbb{N}} \prod_{a} (-1)^i \left( \frac{\chi_a^{\text{even}}}{i} \right). \]

The same formulas hold for \( \chi_b^{\text{odd}} \) and \( \chi_b^{\text{even}} \). They can be used recursively to calculate the Euler characteristics of the complexes of connected graphs from that of the complexes of all graphs.

4.4. **Numerical data.** The formulas from Subsection 4.1 can be used to calculate the dimensions of the spaces \( V_{\chi, e, a}^{\text{even}} \) and \( V_{\chi, e, a}^{\text{odd}} \) using the computer. As an example, in Table 2 we list the dimensions of \( V_{\chi, e}^{\text{even}} \) and \( V_{\chi, e}^{\text{odd}} \) for \( \nu \) up to 24 and \( e \) up to 36, modulo the product of prime numbers 3999971 · 39999949 · 399999929 · 39999923 · 39999917 · 39999901 · 3999893 · 3999971. Our results can also be used to calculate the Euler characteristics of the graph complexes \( \chi_b^{\text{even}}, \chi_b^{\text{odd}}, \chi_b^{\text{even}} \), and \( \chi_b^{\text{odd}} \). The formulas from Subsection 4.3 lead us to the Euler characteristics of the connected parts. We have done these calculations for the even and odd case, with and without tadpoles and multiple edges respectively, for the whole complex and for the connected part, with \( b \) up to 30, modulo
15808115832821291933 = 991 · 997 · 3999949 · 3999971. The results are listed in the following table. Note that the omission of tadpoles or multiple edges does not alter the Euler characteristic. This of course follows for all $b$ from Theorem 2 and [14, Proposition 3.4] and is expected, but we nevertheless provide the computed data below as a consistency check.

**Remark.** Note in particular that the Euler characteristics of the even and odd graph complexes are astonishingly similar, up to a conventional sign factor.

| $b$ | Even | Odd |
|-----|------|-----|
|     | $\chi^{even}$ | $\bar{\chi}^{even}$ | $\chi^{odd}$ | $\bar{\chi}^{odd}$ |
| 1   | 0     | 0    | 1    | 1    |
| 2   | 1     | 1    | 2    | 2    |
| 3   | 0     | 0    | 3    | 3    |
| 4   | 2     | 2    | 6    | 6    |
| 5   | -1    | -1   | 8    | 8    |
| 6   | 3     | 3    | 14   | 14   |
| 7   | -1    | -1   | 20   | 20   |
| 8   | 4     | 4    | 32   | 32   |
| 9   | -4    | -4   | 44   | 44   |
| 10  | 6     | 6    | 68   | 68   |
| 11  | -5    | -5   | 93   | 93   |
| 12  | 8     | 8    | 139  | 139  |
| 13  | -10   | -10  | 191  | 191  |
| 14  | 12    | 12   | 274  | 274  |
| 15  | -18   | -18  | 372  | 372  |
| 16  | 12    | 12   | 529  | 529  |
| 17  | -25   | -25  | 713  | 713  |
| 18  | 28    | 28   | 980  | 980  |
| 19  | -25   | -25  | 1300 | 1300 |
| 20  | 62    | 62   | 1759 | 1759 |
| 21  | -22   | -22  | 2318 | 2318 |
| 22  | 56    | 56   | 3119 | 3119 |
| 23  | -74   | -74  | 4107 | 4107 |
| 24  | -396  | -396 | 5914 | 5914 |
| 25  | -3068 | -3068| 10508| 10508|
| 26  | -794  | -794 | 13606| 13606|
| 27  | 35619 | 35619| 38725| 38725|
| 28  | 9349  | 9349 | 10583| 10583|
| 29  | -634587 | -634587 | -667610 | -667610 |
| 30  | -39755 | -39755 | 28305 | 28305 |

Table 1: The table of the Euler characteristics of the various graph complexes as defined in [2].
Table 2: Dimensions of the spaces of connected graphs, $\tilde{V}_{\text{even}}, \tilde{v}_{\text{even}}$, and $\tilde{V}_{\text{odd}}, \tilde{v}_{\text{odd}}$, as computed by Theorem 3 and (a version of) the formulas of section 4.3.
REFERENCES

[1] G. Arone and V. Tourtchine. Graph-complexes computing the rational homotopy of high dimensional analogues of spaces of long knots. 2011. arXiv:1108.1001.
[2] G. Arone and V. Tourtchine. On the rational homology of high dimensional analogues of spaces of long knots. ArXiv e-prints, 2011. arXiv:1105.1576.
[3] Dror Bar Natan and Brendan McKay. Graph Cohomology - An Overview and Some Computations. unpublished, available on http://www.math.toronto.edu/ drorbn/papers/GCOC/GCOC.ps.
[4] A. Berglund and I. Madsen. Homological stability of diffeomorphism groups. 2012. arXiv:1203.4161.
[5] James Conant, Ferenc Gerlits, and Karen Vogtmann. Cut vertices in commutative graphs. Q J Math, 56(3):321–336, September 2005.
[6] James Conant, Martin Kassabov, and Karen Vogtmann. Hairy graphs and the unstable homology of Mod(g, s), Out(F_n) and Aut(F_n). J. Topol., 6(1):119–153, 2013.
[7] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. Invent. Math., 84(1):91–119, 1986.
[8] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. Invent. Math., 85(3):457–485, 1986.
[9] M. Kontsevich. Formal non-commutative symplectic geometry. The Gelfand Mathematical Seminars, 1990-1992, Ed. L.Corwin, I.Gelfand, J.Lepowsky, pages 173–187, 1993.
[10] Maxim Kontsevich. Formality Conjecture. Deformation Theory and Symplectic Geometry, pages 139–156, 1997. D. Sternheimer et al. (eds.).
[11] Pascal Lambrechts and Victor Turchin. Homotopy graph-complex for configuration and knot spaces. Trans. Amer. Math. Soc., 361(1):207–222, 2009.
[12] John Smillie and Karen Vogtmann. A generating function for the Euler characteristic of Out(F_n). In Proceedings of the Northwestern conference on cohomology of groups (Evanston, IL., 1985), volume 44, pages 329–348, 1987.
[13] Victor Turchin. Hodge-type decomposition in the homology of long knots. I. Topol., 3(3):487–534, 2010.
[14] Thomas Willwacher. M. Kontsevich’s graph complex and the Grothendieck-Teichmüller Lie algebra, 2010. arxiv:1009.1654.

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
E-mail address: thomas.willwacher@math.uzh.ch

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
E-mail address: marko.zivkovic@math.uzh.ch