Bipartite noisy hypercubes have large higher-order Cheeger separation

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Abstract

The expansion of a graph is typically associated with its spectral properties - testing whether a graph is an expander is usually done using Cheeger’s inequality. One can also use multiple eigenvalues in a higher-order Cheeger’s inequality to test a deeper set of properties on the graph. However, Cheeger’s inequality, and the higher-order Cheeger’s inequality, can be imprecise tools. Recently, Lee, Gharan, and Trevisan constructed the Noisy Hypercube to prove the sharpness of the gap between spectral expansion and edge expansion in the higher-order Cheeger’s inequality.

We are concerned with the dual problem: using the upper end of the Laplacian spectrum to test a graph’s bipartite nature. This has been shown to have several applications, and recently a dual version of Cheeger’s inequality and a dual version of the higher-order Cheeger’s inequality have been presented. We construct the Bipartite Noisy Hypercube and use it to prove the sharpness of the gap between spectral bipartite expansion and bipartite edge expansion in the dual version of the dual higher-order Cheeger’s inequality.

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1 Introduction

Let $G(V, E)$ be a graph where $V$ is the set of vertices and $E$ is the set of edges of the graph. For vertex subsets $A, B \subseteq V$ define $E(A, B)$ to be the set of edges with one endpoint in $A$ and the other endpoint in $B$; edges contained inside $A \cap B$ are counted twice. For vertex subset $T \subseteq V$, let $\overline{T} = V - T$. The expansion of a vertex set $T$ is $\phi(T) = \frac{|E(T, \overline{T})|}{|E(T, V)|}$. Finding a vertex subset of a graph with small expansion is one version of community detection, which has applications to circuit layout design, social networking analysis, web search, recommendation systems, traffic routing, parallel computing, and biology (see [5]). An expander graph is a graph with no small sets with small expansion; these graphs have applications to coding theory, computational algebra, and derandomization of probabilistic algorithms (see [15] for a short gentle introduction and [7] for a more thorough presentation).

The most prominent method to find sets with small expansion involves investigating the spectral properties of a matrix representation of the graph. If we order the vertices such that $V = \{1, 2, \ldots, n\}$, then the adjacency matrix $A$ has a 1 in position $(i, j)$ if $ij \in E$ and 0 otherwise. The degree of a vertex $i$, denoted $d(i)$, is the sum of the entries in row $i$ of $A$, and the degree matrix $D$ is the diagonal matrix with value $d(i)$ in row $i$. The normalized Laplacian is the matrix $L = I - D^{-1/2} A D^{-1/2}$. The normalized Laplacian has been studied extensively [2] and has many nice properties. Among these properties is that the eigenvectors of $L$ form an orthonormal basis of $\mathbb{R}^n$ and have eigenvalues $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n \leq 2$. Moreover, Cheeger's Inequality states that
\[
\frac{\lambda_2}{2} \leq \min_{\emptyset \not= T \subseteq V} \frac{|E(T, \overline{T})|}{|E(T, V)|} \leq \sqrt{2\lambda_2}.
\]
(1)

Some of the papers referenced above write (1) in relation to the eigenvalues of $A$, but those manuscripts restrict their attention to regular graphs - equivalently when $D$ is a scalar times the identity matrix - and in that special case the results are equivalent. Finding the best possible $T$ is NP-hard [6], but the Cheeger Inequality is algorithmic in that it finds a set $T$ that satisfies the upper bound of (1). Some of the applications listed above require finding multiple distinct sets with small expansion (and forcing the sets to be disjoint is one method to ensure that they are distinct). Two teams recently proved a higher-order Cheeger Inequality for this exact reason.
Theorem 1.1 ([10], [12]). There exists a constant \( c < 1 \) such that for any \( k \) and graph \( G \), there exist disjoint vertex sets \( T_1, \ldots, T_c k \) such that for each \( i \), we have that \( \phi_G(T_i) \leq O(\sqrt{\log(k)\lambda_k}) \).

There are several methods known today to construct an expander graph [7], two examples are the Cayley graphs of specific groups and iterative constructions from smaller expander graphs. What these methods have in common is that the proof of expansion for each involves bounding \( \lambda_2 \) instead of directly bounding \( \phi(T) \). This approach to handling expansion has become so common that the family of graphs with large \( \lambda_2 \) has its own name: they are called spectral expanders. In order to prove the sharpness of Theorem 1.1, a graph had to be constructed that is an expander and not a spectral expander. To this end, Lee, Gharan, and Trevisan noted that \( E(T,T) = E(V,T) - E(T,T) \), and constructed a graph with no small dense subsets. Specifically, they applied Fourier analysis techniques to a specific graph to show that when \( |T| \) is small, \( E(T,T) \) is also small. Therefore \( \phi(T) \) is large.

There are natural generalizations to the world of weighted graphs. A weighted graph is a graph with an associated map \( w : E \to \mathbb{R}^+ \). In this case, the adjacency matrix has value \( w(i,j) \) in position \((i,j)\) and the term \( |E(A,B)| \) is replaced by \( w(A,B) = \sum_{a \in A} \sum_{b \in B} w(a,b) \).

Example 1.2 (Noisy hypercubes [10]). Let \( k \in \mathbb{Z}^+, 1 \leq c < k/2, \) and \( \epsilon = \frac{1}{\log_2(k/c)} \in (0,1) \). Let \( G_{k,c} \) be the weighted complete graph on \( 2^k \) vertices, where each vertex corresponds to a finite binary sequence of length \( k \) (in other words \( V = \{0,1\}^k \)), and the weight of edge \( xy \) is \( \epsilon|x-y|_1 \). Then we have that \( \lambda_k \leq 2\epsilon \) and for any set \( T \subset V \) with \( |T| \leq \frac{c}{k}|V| \) satisfies \( \phi(T) \geq 0.5 \).

The norm between \( x, y \in \{0,1\}^k \) in the above example is defined to be the number of entries in which \( x \) and \( y \) are different (denoted by \( \|x-y\|_1 \)).

The original example claimed that the conclusions held for the extended range \( 1 \leq c < k \), which is false. The set \( E(T,T) \) is clearly a subset of \( E(V,T) \), and so when \( G \) is regular we have that \( \phi(T) \leq \frac{|T|}{|V|} \). In particular, if \( |T| > |V|/2 \), then \( \phi(T) < 0.5 \). So it must be that \( c \leq k/2 \). The error in their proof is that they use the Bonami-Beckner inequality (also known as the
Hypercontractivity Theorem) with \( \eta = \frac{1 - \epsilon}{1 + \epsilon} \). The Bonami-Beckner inequality assumes that \( \eta \geq 0 \), which is true if and only if \( c \leq k/2 \). We will investigate these issues further in the next section.

Certain inequalities require bounding both \( \lambda_2 \) and \( \lambda_n \). For example, a generalization of the Expander Mixing Lemma states that if \( \lambda' = \min\{\lambda_2, 2 - \lambda_n\} \), then for any subset of vertices \( A, B \) where \( \Delta_A \) is the maximum of the degrees among vertices in \( A \), we have that

\[
\left| w(A, B) - w(A, V) \frac{|B|}{n} \right| \leq \Delta_A (1 - \lambda') \sqrt{|A||B|} \tag{2}
\]

(the Expander Mixing Lemma usually assumes that \( G \) is regular – we are not sure if this generalization is known, but we could not find a reference and so we present a proof in section 4 for completeness). Furthermore, the value of \( \lambda_n \) has gathered some independent interest. The value of \( \lambda_n \) is related to bounds on MAX-CUT [16], and it has applications to biology, web searches, and social networking [18]. Liu examined the values of \( \lambda_i \) for large \( i \) as a dual problem to the traditional problem of expansion [11].

Define the bipartite expansion of disjoint vertex sets \( T, T' \) as

\[
\tilde{\phi}(T, T') = \frac{w(T \cup T', \overline{T} \cup \overline{T'}) + w(T, T) + w(T', T')}{w(T \cup T', V)}.
\]

Trevisan proved an analogue of the Cheeger inequality [16], showing that

\[
\lambda_n/2 \leq \min_{T,T'} \tilde{\phi}(T, T') \leq \sqrt{2 - \lambda_n}.
\]

Liu extended this to an analogue of Theorem 1.1 [11] by showing that there exists disjoint vertex sets \( T_1, T'_1, \ldots, T_k, T'_k \), where for each \( i \) we have that

\[
\tilde{\phi}(T_i, T'_i) \leq O(k^3 \sqrt{2 - \lambda_{n+1-k}}). \]

This was sharpened by the second author.

**Theorem 1.3** ([18]). There exists a constant \( C \) such that for any graph \( G \) and value of \( k \) there exist disjoint sets \( T_1, T'_1, T_2, T'_2, \ldots, T_{k/4}, T'_{k/4} \) such that for each \( i \) we have that

\[
\tilde{\phi}(T_i, T'_i) \leq C \sqrt{\log(k)(2 - \lambda_{n+1-k})}.
\]

Liu also proves that when \( n > 2k \), cycles \( C_n \) satisfy

\[
0.45 \sqrt{2 - \lambda_{n+1-k}} \leq \min_{T_1, T_2, \ldots, T_k} \max_i \tilde{\phi}(T_i, T'_i) \leq 1.7 \sqrt{2 - \lambda_{n+1-k}}.
\]
Note that cycles are not expanders, so for a fixed $k$ we see that $\lim_{n \to \infty} 2 - \lambda_{n-k} = 0$. Thus this family shows that bipartite expansion may be an unbounded multiple of $2 - \lambda_{n-k}$. Two questions remain: (1) whether it is possible to have a bipartite expander that is not a spectral bipartite expander and (2) whether the $\sqrt{\log(k)}$ term is necessary in Theorem 1.3. We will answer both questions in the affirmative with one example.

**Example 1.4** (Bipartite noisy hypercubes). Let $k$ and $c$ be fixed, with $1 \leq c \leq \frac{10k}{22}$, and let $\epsilon = \frac{1}{\log_2(\frac{k}{c})}$. We define $G^{(o)}$ to be a complete bipartite spanning subgraph of noisy hypercube $G_{k,c}$ such that $xy \in E(G^{(o)})$ (and keeps the same weight) if and only if $\|x-y\|_1$ is odd. In $G^{(o)}$ we have that $2 - \lambda_{n-k} \leq 3\epsilon$ and for any set $T, T' \subset V$ with $|T \cup T'| \leq \frac{c}{k}|V|$ we have that $\tilde{\phi}(T, T') \geq 0.5$.

The assumption $1 \leq c \leq \frac{10k}{22}$ can only be satisfied when $k \geq 3$, but a similar statement easily holds in the remaining trivial cases.

## 2 Background

For a general graph, we may order our vertices $V = \{1, 2, \ldots, n\}$ as we did when we defined the adjacency matrix. We can consider a vector $f$ as a map $f : V \to \mathbb{R}$, where $f(i)$ is the value in coordinate $i$ of the vector. In this notation, we can think of the matrices $A$ and $L$ as operators on real-valued functions whose domain is $V$. This notation - of maps and operators - also holds when we think of $V$ in terms of $\{0, 1\}^k$ instead of $\{1, 2, \ldots, n\}$. For example, the adjacency matrix operator is $Af(x) = \sum_{xy \in E} w(x, y)f(y)$. Let $\mathcal{H}_k$ denote the set of functions defined from $\{0, 1\}^k$ into $\mathbb{R}$ with the inner-product of two functions $f, g \in \mathcal{H}_k$ defined by

$$\langle f, g \rangle = \frac{1}{n} \sum_{x \in V} f(x)g(x).$$

We will drop the subscript when $k$ is clear. The $p$-norm of a function $f$ is $\|f\|_p = \left(\frac{1}{n} \sum_{x \in V} |f(x)|^p\right)^{1/p}$, therefore $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

Our proofs will make use of the rich field of study on maps whose domain is $\{0, 1\}^k$. Our notation follows that of [17] (this is the notation in Example
We also found the course notes [13] that O’Donnel grew into a book [14] to be enlightening. We will need a select few theorems from this field, which we present below.

The Walsh functions defined by

\[ W_S(x) = (-1)^{\sum_{i \in S} x_i} \]

for \( S \subseteq \{k\} \) form an orthonormal basis for \( \mathcal{H}_k \). Thus, any \( f \in \mathcal{H}_k \) can be written as \( f(x) = \sum_{S \subseteq \{k\}} \hat{f}(S) W_S(x) \) for some set of coefficients \( \hat{f}(S) \). We call \( \hat{f}(S) \) the Fourier coefficients of \( f \) where

\[ \hat{f}(S) = \langle W_S, f \rangle = 2^{-k} \sum_{x \in V} f(x)(-1)^{\sum_{i \in S} x_i}. \]

Also recall Parseval’s Identity which states that \( \|f\|_2^2 = \sum_{S \subseteq \{k\}} \hat{f}(S)^2 \).

We define a noise process \( E_\eta \) to be a randomized automorphism on \( \{0, 1\}^k \), where \( \mathbb{P}[E_\eta(x) = y] = \eta^{|x-y|_1}(1-\eta)^{|x-y|_1} \). This is the standard model for independent bit-flip errors in coding theory. The noise operator is defined to be \( N_\eta f(x) = \mathbb{E}[f(E_\eta(x))] \). The noise operator is suggested to “flatten out” the values of \( f \), although the exact strength to which this is true remains open [1]. This process is intimately linked to the Fourier coefficients and the Walsh functions by \( N_\eta f(x) = \sum_{S \subseteq \{k\}} \hat{f}(S) \eta^{|S|} W_S(x) \) (this is also known as the Bonami-Beckner operator). The final statement that we need is the Bonami-Beckner inequality: if \( 1 \leq p \leq q \) and \( 0 \leq \eta \leq \sqrt{(p-1)/(q-1)} \), then \( \|N_\eta f\|_q \leq \|f\|_p \). We will not need the full generality of this statement, just that if \( 0 \leq \eta \leq 1 \), then

\[ \sum_{S \subseteq \{k\}} \hat{f}(S)^2 \eta^{|S|} \leq \|f\|_{1+\eta^2}^2. \]
3 Details of Example 1.4

3.1 Eigenvalues

We begin by re-calculating the degree of a vertex in $G_{k,c}$. Let $x$ be a fixed vertex, so that the degree of $x$ is

$$\sum_{y \in V} \epsilon^{\|x-y\|_1} = \sum_{i=0}^{k} \epsilon^{k} \left| \{y : \|x-y\|_1 = i \} \right| = \sum_{i=0}^{k} \binom{k}{i} \epsilon^{k} = (1 + \epsilon)^k.$$

Using this generating function, we see that the degree $d^o_k(x)$ of $x$ in $G^{(o)}$ is

$$\frac{1}{2} \left( (1 + \epsilon)^k - (1 - \epsilon)^k \right) = \frac{1}{2} \left( \sum_{i=0}^{k} \binom{k}{i} \epsilon^{i} - \sum_{i=0}^{k} \binom{k}{i} (-\epsilon)^i \right)$$

$$= \frac{1}{2} \sum_{i=0}^{k} \binom{k}{i} \epsilon^{i} (1 - (-1)^i)$$

$$= \sum_{0 \leq i \leq k, i \text{ odd}} \binom{k}{i} \epsilon^{i}$$

$$= d^o_k(x).$$

It will be convenient to define a graph $G^{(e)}$ to be the subgraph of $G_{k,c}$ where $E(G^{(e)}) = E(G_{k,c}) - E(G^{(o)})$. Using a symmetrical argument, we see that each vertex in $G^{(e)}$ has degree $d^e_k(x) = \frac{1}{2} \left( (1 + \epsilon)^k + (1 - \epsilon)^k \right)$. We have chosen our $\epsilon$, $c$, and $k$ such that $d^o_k(x) \geq \frac{1}{2} 2 (1 + \epsilon)^k$.

We will use the eigenvalues of the adjacency matrix to calculate the eigenvalues of the Laplacian of our graph. Because our graph is regular, the eigenvectors of the Laplacian are the same as the eigenvectors of the adjacency matrix. We can use this information to directly calculate the eigenvalues associated to $G^{(o)}$. For this calculation we will need the eigenvalues of the normalized adjacency matrix. The normalized adjacency matrix is defined by $D^{-1/2}AD^{-1/2}$. If $\rho$ is an eigenvalue of the adjacency matrix for a regular graph, $\overline{\rho}$ will denote the associated eigenvalue of the normalized version.

Let $\rho_S = \frac{(1+\epsilon)^k - |S| (1-\epsilon)^k}{2}$.
Lemma 3.1. Let $S \subseteq [k]$. Then, $AW_S = \rho S W_S$.

Using Lemma 3.1 we see that for each $0 \leq i \leq k$, with multiplicity $\binom{k}{i}$, the normalized Laplacian has eigenvalue

$$\lambda_i = 1 - \rho_i = 1 - \frac{(1 + \epsilon)^{k-i}(1 - \epsilon)^i - (1 - \epsilon)^{k-i}(1 + \epsilon)^i}{(1 + \epsilon)^k - (1 - \epsilon)^k}. \quad (4)$$

By choosing the $k + 1$ sets $S$ with $|S| \geq k - 1$, we find our eigenvalues that satisfy $2 - \lambda_{n-k} \leq 3\epsilon$. Symmetrically, when $|S| \leq \frac{\epsilon}{k} n$,

$$\rho_{|S|} = \frac{(1 + \epsilon)^{k-|S|}(1 - \epsilon)^{|S|} - (1 - \epsilon)^{k-|S|}(1 + \epsilon)^{|S|}}{(1 + \epsilon)^k - (1 - \epsilon)^k} \leq \frac{11}{10} \left( \frac{(1 + \epsilon)^{|S|}(1 - \epsilon)^{k-|S|}}{(1 + \epsilon)^k} \right) \leq \frac{11}{10} \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{|S|}.$$

Now we return to give the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $S \subseteq [k]$. Consider the following:

$$AW_S(x) = \sum_{y \in N(x)} w(x, y) W_S(y)$$

$$= \left( \sum_{\|x - y\|_1 \text{ odd}} \sum_{\sum_{i \in S} x_i = \sum_{y_i \in y} (x_i \text{ mod } 2)} \epsilon^{\|x - y\|_1}(-1)^{\sum_{i \in S} y_i} + \sum_{\|x - y\|_1 \text{ even}} \sum_{\sum_{i \in S} x_i \neq \sum_{y_i \in y} (x_i \text{ mod } 2)} \epsilon^{\|x - y\|_1}(-1)^{\sum_{i \in S} y_i} \right)$$

$$= W_S(x) \left( \sum_{\sum_{i \in S} x_i = \sum_{y_i \in y} (x_i \text{ mod } 2)} \sum_{\sum_{i \in S} x_i \neq \sum_{y_i \in y} (x_i \text{ mod } 2)} \epsilon^{\|x - y\|_1} - \sum_{\sum_{i \in S} x_i \neq \sum_{y_i \in y} (x_i \text{ mod } 2)} \sum_{\sum_{i \in S} y_i (x_i \text{ mod } 2)} \epsilon^{\|x - y\|_1} \right).$$

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First we will concentrate on the first summand in the above expression, call it $T_1$. In $T_1$ we are summing over all $y \in N(x)$ such that the following two conditions hold:

1. $\sum_{i \in S} x_i = \sum_{i \in S} y_i (\mod 2)$

2. $\sum_{i \in S} x_i \neq \sum_{i \in S} y_i (\mod 2)$.

Notice that $\|x - y\|_1 = \sum_{i \in S} |x_i - y_i| + \sum_{i \in S} |x_i - y_i|$. Thus,

$$T_1 = \left( \sum_{i \in S} \sum_{y \text{ where } i \in S} x_i = \sum_{y \text{ where } i \in S} y_i (\mod 2) \epsilon \sum_{i \in S} |x_i - y_i| \right) \left( \sum_{i \in S} \sum_{y \text{ where } i \in S} x_i \neq \sum_{y \text{ where } i \in S} y_i (\mod 2) \epsilon \sum_{i \in S} |x_i - y_i| \right).$$

In a similar fashion, the second summand, call it $T_2$, can be seen to be

$$T_2 = \left( \sum_{i \in S} \sum_{y \text{ where } i \in S} x_i = \sum_{y \text{ where } i \in S} y_i (\mod 2) \epsilon \sum_{i \in S} |x_i - y_i| \right) \left( \sum_{i \in S} \sum_{y \text{ where } i \in S} x_i \neq \sum_{y \text{ where } i \in S} y_i (\mod 2) \epsilon \sum_{i \in S} |x_i - y_i| \right).$$

Now, notice that $T_1 = d^e_{|S|} \cdot d^o_{|S|}$ and $T_2 = d^e_{|S|} \cdot d^o_{|S|}$. Pulling everything back together we observe

$$AW_S(x) = \left( d^e_{|S|} \cdot d^o_{|S|} - d^e_{|S|} \cdot d^o_{|S|} \right) W_S(x) = \rho_S W_S(x). \quad \square$$

### 3.2 Expansion

In this section we will prove that $\phi(T) \geq \frac{1}{2}$ for any $T \subset V$ with $|T| < \frac{e}{k} |V| = \frac{e}{k} n$. Recall that $\phi(T) = \frac{w(T, T)}{w(T, V)}$ and let $T = T' \cup T''$. Because $\tilde{\phi}(T', T'') = \phi(T' \cup T'') + \frac{w(T, T) + w(T', T'')}{w(T \cup T', V)}$, this will conclude the details of Example [1,4]. We will require the following two lemmas.
Lemma 3.2. Let $T \subseteq [n]$ and define $1_T \in H$ to be the characteristic function of $T$. If $|T| \leq \frac{c}{k} n$, then

$$\frac{1}{d_k} \langle 1_T, A1_T \rangle \leq \frac{11}{10} \sum_{S \subseteq [k]} \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{|S|} \left( \widehat{1_T}(S) \right)^2.$$

Proof. Let $T \subseteq [n]$. Recall that $AW_S = \rho_S W_S$, and so by (4) we have that $\frac{1}{d_k} AW_S = \frac{1}{|S|} \rho_S W_S \leq \frac{11}{10} \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{|S|}$. Using that the Walsh functions form an orthonormal basis, we observe:

$$\frac{1}{d_k} \langle 1_T, A1_T \rangle = \frac{1}{n} \sum_{x \in \{0,1\}^k} 1_T(x) \frac{1}{d_k} A1_T(x)$$

$$= \frac{1}{n} \sum_{x \in \{0,1\}^k} \left( \sum_{S \subseteq [k]} \widehat{1_T}(S) W_S(x) \right) \left( \sum_{S' \subseteq [k]} \widehat{1_T}(S') \frac{1}{d_k} AW_{S'}(x) \right)$$

$$= \frac{1}{n} \sum_{x \in \{0,1\}^k} \left( \sum_{S \subseteq [k]} \widehat{1_T}(S) W_S(x) \right) \left( \sum_{S' \subseteq [k]} \widehat{1_T}(S') \rho_{|S'|} W_{S'}(x) \right)$$

$$= \sum_{S \subseteq [k]} \sum_{S' \subseteq [k]} \widehat{1_T}(S) \rho_{|S'|} \widehat{1_T}(S') \left( \frac{1}{n} \sum_{x \in \{0,1\}^k} W_S(x) W_{S'}(x) \right)$$

$$= \sum_{S \subseteq [k]} \rho_{|S|} \left( \widehat{1_T}(S) \right)^2$$

$$\leq \sum_{S \subseteq [k]} \frac{11}{10} \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{|S|} \left( \widehat{1_T}(S) \right)^2. \quad \square$$

Lemma 3.3. Let $T \subseteq [n]$. Then,

$$w(T, \overline{T}) = w(T, V) - n \langle 1_T, A1_T \rangle.$$

Proof. Let $T \subseteq [n]$. Notice that what we really are proving is that $n \langle 1_T, A1_T \rangle = w(T, T)$. Now consider
\[
n(1_T, A1_T) = \sum_{x \in \{0,1\}^k} 1_T(x) (A1_T(x)) \\
= \sum_{x \in \{0,1\}^k} 1_T(x) \left( \sum_{y \in N(x)} w(x, y)1_T(y) \right) \\
= \sum_{x \in T} \sum_{y \in T} w(x, y) \\
= w(T, T). \qed
\]

**Theorem 3.4.** The expansion \( \phi(T) \geq \frac{1}{2} \) for any \( T \subset V \) with \(|T| < \frac{\epsilon}{k} |V| = \frac{\epsilon}{k} n \).

**Proof.** Let \( T \subseteq [n] \) such that \(|T| \leq \frac{\epsilon}{k} n \). Then,

\[
\phi(T) = \frac{w(T, \overline{T})}{w(T, V)} = 1 - \frac{n}{|T|d_k^2} (1_T, A1_T).
\]

By (3) with \( \eta = \sqrt{\frac{1-\epsilon}{1+\epsilon}} \), we have that

\[
\frac{n}{|T|d_k^2} (1_T, A1_T) \leq \frac{n}{|T|^{10}} \sum_{S \subseteq [k]} \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{|S|} \left( \overline{1_T}(S) \right)^2 \\
\leq \frac{11n}{10|T|} \|1_T\|^2_{1+\eta^2} \\
= \frac{11n}{10|T|} \left( \frac{|T|}{n} \right)^{2/(1+\eta^2)} \\
= \frac{11}{10} \left( \frac{|T|}{n} \right)^{\frac{1}{1+\eta^2}}.
\]
By choice of $\epsilon$, we have that

\[
\phi(T) \geq 1 - \frac{11}{10} \left( \frac{|T|}{n} \right)^\epsilon
\geq 1 - \frac{11}{10} \left( \frac{c}{k} \right)^\epsilon
\geq 1 - \frac{1}{2}
\]

4 Expander Mixing Lemma Revisited

We are now going to provide the proof to (2), which is that if $\lambda' = \min\{\lambda_2, 2 - \lambda_n\}$ then for any subsets of vertices $A, B$ where $\Delta_A$ is the maximum of the degrees among vertices in $A$, we have that

\[
\left| w(A, B) - w(A, V) \frac{|B|}{n} \right| \leq \Delta_A (1 - \lambda') \sqrt{|A||B|}.
\]

The standard proof of (2) uses the eigenvalues of the adjacency matrix when the graph is regular (and hence is equivalent to using the eigenvalues of the normalized Laplacian). During the study of quasi-random graphs, Chung, Graham, and Wilson [3] analyzed a similar result for non-regular graphs while still using the eigenvalues of the adjacency matrix. Fact 7 of [3] states that

\[
\left| w(A, A) - w(A, V) \frac{|A|}{n} \right| \leq o(n^2),
\]

under a set of assumptions that are analogous to requiring that $(1 - \lambda') \leq o(1)$. Chung and Graham [4] would go on to consider the same problem using the eigenvalues of the normalized Laplacian; their version of the Expander Mixing Lemma is the following

\[
\left| w(A, B) - w(A, V) w(V, B) \frac{w(V, V)}{w(V, V)} \right| \leq (1 - \lambda') \sqrt{w(A, V) w(V, B)}.
\]

This is stronger than (2) when $\frac{w(A, V)}{|A|\Delta_A}$ is small and weaker than (2) when $\frac{w(V, B)}{|B|\Delta_A}$ is large. When we consider $d$-regular graphs, both results reduce to the Expander Mixing Lemma. The main attractiveness of (2) is that it seems to be a half-step towards $(p, q)$-bijumbled graphs, which satisfy

\[
|w(A, B) - p|A||B|| \leq q \sqrt{|A||B|}.
\]
and have several applications [8, 9].

We will prove (2) using the eigenvalues of $D^{-1}A$. Note that Theorems 1.1 and 1.3 were not proven using the normalized Laplacian, but a conjugate of it: the random walk Laplacian, or $L_{RW} = D^{-1/2}LD^{1/2} = I - D^{-1}A$. Conjugates have the same eigenvalues, but the eigenvectors of the random walk Laplacian have repeatedly proven to be more useful. The name of $L_{RW}$ comes from the fact that if $v$ is a row vector, then multiplying by $(L_{RW}/2)^k$ with $v$ on the left calculates the state after a $k$-step random lazy walk (although our applications make use of the right-side eigenvalues). Thus while we work with the more convenient matrix $D^{-1}A$, our statements are equivalent for the more common matrix $L$. In fact, we begin by translating known facts on $L$ to $D^{-1}A$:

• $D^{-1}A$ has $n$ orthonormal eigenvectors,
• whose eigenvalues range from $-1 \leq \lambda_1^* \leq \ldots, \lambda_2^* \leq \lambda_1^* = 1$, and
• $(D^{-1}A) \mathbb{1} = \mathbb{1}$.

We will require the following lemma.

**Lemma 4.1.** Fix vertex set $B$, and let $d_B(v) = w(\{v\}, B)$. Let $\lambda_i^*$ be the eigenvalues of $D^{-1}A$ as above, and let $\lambda = \max\{-\lambda_n^*, \lambda_2^*\}$.

We have that

$$\sum_{v \in V} \left( \frac{d_B(v)}{d(v)} - \frac{|B|}{n} \right)^2 \leq \lambda^2 |B| \left( 1 - \frac{|B|}{n} \right).$$

**Proof.** Define $f$ to be the vector such that $f(v) = \left( 1 - \frac{|B|}{n} \right)$ when $v \in B$ and $f(w) = -\frac{|B|}{n}$ when $w \in \overline{B}$. By construction, $f$ is orthogonal to $\mathbb{1}$, and so $\|D^{-1}Af\|^2 \leq \lambda^2 \|f\|^2$. It is a direct calculation that $\|f\|^2 = \frac{|B|}{n} \left( 1 - \frac{|B|}{n} \right) n$. The statement then follows from

$$\|D^{-1}Af\|^2 = \sum_{w \in V} \left( \frac{\left( 1 - \frac{|B|}{n} \right) d_B(w) + \left( -\frac{|B|}{n} \right) d_{\overline{B}}(w)}{d(w)} \right)^2,$$
and $d_B(w) = d(w) - d_B(w)$. □

**Proof of (2).**

Apply the Cauchy-Schwartz Inequality to see that

$$|w(A, B) - w(A, V)| \leq \sum_{u \in A} \left| d_B(u) - d(u) \right| |B|/n \leq \Delta_A \sum_{u \in V} \left| d_B(u) - d(u) \right| |B|/n \leq \Delta_A \left( \sum_{u \in V} \left| d_B(u) - d(u) \right|^2 \right)^{1/2} |A|^{1/2} \leq \Delta_A \lambda \sqrt{|A||B|}. □$$

We used a coarse estimate when we pulled out the $\Delta_A$ term above; if we let $\gamma = \frac{w(A,V)}{|A|\Delta_A} \leq 1$, then a simple argument gives us the better (but more complicated) bound

$$|w(A, B) - w(A, V)| \leq (8\gamma)^{1/3} \Delta_A (1 - \lambda') \sqrt{|A||B|}.$$

Frequently the special case when $A = B$ is of particular interest.

**Corollary 4.2.** Let $G$ be a graph, where $\lambda_i$ are the eigenvalues of the normalized Laplacian. Let $\lambda' = \min\{\lambda_2, 2 - \lambda_n\}$. For any vertex subset $A$, where the maximum degree among the vertices of $A$ is $\Delta_A$, we have

$$|w(A, A) - w(A, V)| \leq (1 - \lambda') \Delta_A |A|.$$

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