ON $p$-ADIC INTERPOLATING FUNCTION ASSOCIATED WITH MODIFIED DIRICHLET’S TYPE OF TWISTED $q$-EULER NUMBERS AND POLYNOMIALS WITH WEIGHT $\alpha$

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ABSTRACT. The $q$-calculus theory is a novel theory that is based on finite difference re-scaling. The rapid development of $q$-calculus has led to the discovery of new generalizations of $q$-Euler polynomials involving $q$-integers. The present paper deals with the modified Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$. We apply the method of generating function and $p$-adic $q$-integral representation on $\mathbb{Z}_p$, which are exploited to derive further classes of $q$-Euler numbers and polynomials. To be more precise we summarize our results as follows, we obtain some combinatorial relations between modified Dirichlet’s type of twisted $q$-Euler numbers and polynomials with weight $\alpha$. Furthermore we derive Witt’s type formula and Distribution formula (Multiplication theorem) for modified Dirichlet’s type of twisted $q$-Euler numbers and polynomials with weight $\alpha$. Moreover we will find a link between modified twisted Hurwitz-zeta function and $q$-analogue of modified twisted $q$-Euler polynomials with weight $\alpha$. In section three, by applying Mellin transformation we define $q$-analogue of modified twisted $q$-functions of Dirichlet’s type and also we deduce that it can be written as modified Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$. Moreover we will find a link between modified twisted Hurwitz-zeta function and $q$-analogue of modified twisted $q$-functions of Dirichlet’s type which yields a deeper insight into the effectiveness of this type of generalizations. In addition we consider $q$-analogue of partial zeta function and we derive behavior of the modified $q$-Euler $L$-function at $s = 0$. In final section, we construct $p$-adic twisted Euler $q$-$L$ function with weight $\alpha$ and interpolate Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$ at negative integers. Our new generating function possess a number of interesting properties which we state in this paper.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

$p$-adic numbers and $L$-functions theory plays a vital and important role in mathematics. $p$-adic numbers were invented by the German mathematician Kurt Hensel \[1\], around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. The $p$-adic integral was used in mathematical physics, for instance, the functional equation of the $q$-Zeta function, $q$-Stirling numbers and $q$-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$, together with Iwasawa’s $p$-adic $q$-$L$ functions. A $p$-adic zeta function, or more generally a $p$-adic $L$-function, is a function analogous to the Riemann zeta function, or more general $L$-functions, but whose domain and target are $p$-adic (where $p$ is a prime number).
For example, the domain could be the $p$-adic integers $\mathbb{Z}_p$, a profinite $p$-group, or a $p$-adic family of Galois representations, and the image could be the $p$-adic numbers $\mathbb{Q}_p$ or its algebraic closure. For a prime number $p$ and for a Dirichlet character defined modulo some integer, the $p$-adic $L$-function was constructed by interpolating the values of complex analytic $L$-function at non-positive integers. In this paper our main focus will be on $p$-adic interpolation of modified Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$. Actually interpolation is the process of defining a continuous function that takes on specified values at specified points. During the development of $p$-adic analysis, researches were made to derive a meromorphic function, defined over the $p$-adic number field, which would interpolate the same or at least similar values as the Dirichlet $L$-function at non-positive integers. Finding the interpolation functions of special orthogonal numbers and polynomials started by H. Tsumura [29], and P. T. Young [34], for the Bernoulli and Euler polynomials. After Taekyun Kim and Yilmaz Simsek, studied on $p$-adic interpolation functions of these numbers and polynomials. L. C. Washington [30], constructed one-variable $p$-adic $L$-function which interpolates generalized classical Bernoulli numbers at negative integers. Diamond [35], obtained formulas which express the values of $p$-adic $L$-function at positive integers in terms of the $p$-adic log gamma function. Next Fox in [32], introduced the two-variable $p$-adic $L$-functions and T. Kim [21], constructed the two-variable $p$-adic $q$-$L$-function, which is interpolation function of the generalized $q$-Bernoulli polynomials. P. T. Young [34], gave $p$-adic integral representations for the two-variable $p$-adic $L$-function introduced by Fox. T. Kim and S.-H. Rim [17], introduced twisted $q$-Euler numbers and polynomials associated with basic twisted $q$-$l$-functions by using $p$-adic $q$-invariant integral on $\mathbb{Z}_p$ in the fermionic sense. Also, Jang et al. [33], investigated the $p$-adic analogue twisted $q$-$l$-function, which interpolates generalized twisted $q$-Euler numbers attached to Dirichlet’s character. In this paper, we will construct a $p$-adic interpolation function of modified Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$.

Imagine that $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$. In this paper, we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

The $p$-adic absolute value $|.|_p$, is normally defined by

$$|x|_p = \frac{1}{p^r},$$

where $x = p^r \frac{x}{p^r}$ with $(p, s) = (p, t) = (s, t) = 1$ and $r \in \mathbb{Q}$.

As well-known definition, Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n (x) \frac{t^n}{n!} = e^{E(x)t},$$

with the usual convention about replacing $E^n (x)$ by $E_n (x)$ (for more information, see [8, 9, 14, 15, 16]).

A $p$-adic Banach space $B$ is a $\mathbb{Q}_p$-vector space with a lattice $B^0$ ($\mathbb{Z}_p$-module) separated and complete for $p$-adic topology, ie.,

$$B^0 \simeq \lim_{\rightarrow} B^0/p^n B^0.$$
For all $x \in B$, there exists $n \in \mathbb{Z}$, such that $x \in p^n B^0$. Define

$$v_B(x) = \sup_{n \in \mathbb{N} \cup \{+\infty\}} \{ n : x \in p^n B^0 \}.$$ 

It satisfies the following properties:

$$v_B(x + y) \geq \min(v_B(x), v_B(y)),$$

$$v_B(\beta x) = v_p(\beta) + v_B(x), \text{ if } \beta \in \mathbb{Q}.$$ 

Then, $\|x\|_B = p^{-v_B(x)}$ defines a norm on $B$, such that $B$ is complete for $\|\cdot\|_B$ and $B^0$ is the unit ball.

A measure on $\mathbb{Z}_p$ with values in a $p$-adic Banach space $B$ is a continuous linear map

$$f \mapsto \int f(x) \mu = \int_{\mathbb{Z}_p} f(x) \mu(x)$$

from $C^0(\mathbb{Z}_p, \mathbb{C}_p)$, (continuous function on $\mathbb{Z}_p$) to $B$. We know that the set of locally constant functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ is dense in $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ so.

Explicitly, for all $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$, the locally constant functions

$$f_n = \sum_{i=0}^{p^n-1} f(i) 1_{i+p^n \mathbb{Z}_p} \rightarrow f \text{ in } C^0.$$ 

Now if $\mu \in D_0(\mathbb{Z}_p, \mathbb{Q}_p)$, set $\mu(i+p^n \mathbb{Z}_p) = \int_{\mathbb{Z}_p} f(i) \mu(i+p^n \mathbb{Z}_p)$. Then $\int_{\mathbb{Z}_p} f \mu$ is given by the following “Riemann sums”

$$\int_{\mathbb{Z}_p} f \mu = \lim_{n \to \infty} \sum_{i=0}^{p^n-1} f(i) \mu(i+p^n \mathbb{Z}_p).$$ 

T. Kim defined $\mu$ as follows:

$$\mu_q(a+p^n \mathbb{Z}_p) = \frac{(-q)^a}{[p^n]_q},$$

so,

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x) q^x, \text{ (for details, see } [13, 15, 16]).$$ 

Where $[x]_q$ is a $q$-extension of $x$ which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}. $$

Note that $\lim_{q \to 1} [x]_q = x$ cf. [2-35].

If we take $f_1(x) = f(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).$$

By expression (1.2), we readily see that,

$$(-1)^{n-1} I_{-q}(f) + q^n I_{-q}(f_n) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),$$

$$\text{for all } n \in \mathbb{N} \cup \{+\infty\}. $$

$$\text{and } B^0 \text{ is the unit ball. }$$
where $f_n(x) = f(x + n)$.

Recently, Rim et al. [26] defined the modified weighted $q$-Euler numbers $E_{n,q}^{(\alpha)}$ and the modified weighted $q$-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ by using $p$-adic $q$-integral on $\mathbb{Z}_p$ in the form

$$E_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} q^{-\xi} [\xi]_q^n \, d\mu_{-q}(\xi), \text{ for } n \in \mathbb{N}^* \text{ and } \alpha \in \mathbb{Z}.$$

Let $C_{p^n} = \{ w \mid w^{p^n} = 1 \}$ be the cyclic group of order $p^n$, and let $T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}$, note that $T_p$ is locally constant space (for details, see [17, 23, 24, 27-30, 33]).

In [23], let $\chi$ be a Dirichlet's character with conductor $d (= odd) \in \mathbb{N}$ and $w \in T_p$. If we take $f(x) = \chi(x) w^{x^2} e^{tx}$, then we have $f(x + d) = \chi(x) w^{x^2 + dx} e^{td}$. From (1.3), we see that

$$\int_X \chi(x) w^{x^2} e^{tx} \, d\mu_{-q}(x) = [2]_q \sum_{d=1}^{\infty} (-1)^{d-1-i} q^d \chi(i) w^{d} e^{ti}.$$

In view of (1.4), it is considered by (1.5)

$$F_{w,\chi}^q(t) = \frac{[2]_q \sum_{d=0}^{\infty} (-1)^{d-1-i} q^d \chi(i) w^{d} e^{ti}}{q^{d} w^d e^{td} + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha,w)} \frac{t^n}{n!} \left| t + \log(qw) \right| < \frac{\pi}{d}.$$

Let us consider the modified twisted $q$-Euler polynomials with weight $\alpha$ as follows:

$$E_{n,q}^{(\alpha,w)}(x) = \int_{\mathbb{Z}_p} q^{-\xi} [x + \xi]_q^n \, d\mu_{-q}(\xi), \text{ for } n \in \mathbb{N}^*.$$

By (1.6), and applying combinatorial techniques we have,

$$\sum_{k=0}^{n} \binom{n}{k} q^{\alpha(n-k)x} E_{n-k,q}^{(\alpha,w)}[x] q^k$$

where $E_{n,q}^{(\alpha,w)}(0) := E_{n,q}^{(\alpha,w)}$ are called modified twisted $q$-Euler numbers with weight $\alpha$.

By (1.6), we get generating function of modified twisted $q$-Euler polynomials as follows:

$$F(t,x,q) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha,w)}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m e^{t[x+m]q^\alpha}.$$
By using a complex contour integral representation and (1.8), we get modified twisted Hurwitz-zeta function as follows:

\[ \tilde{\zeta}_q^{(\alpha,w)}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty F^{(\alpha)}(-t, x, q, w) t^{s-1} dt \]

\[ = [2]q \sum_{m=0}^\infty (-1)^m w^m \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t|x+m|q^\alpha} \right) \]

\[ = [2]q \sum_{m=0}^\infty (-1)^m w^m \frac{\chi(m)}{[m+x]q^\alpha}. \]

By (1.8) and (1.9), we now establish a relation between \( E^{(\alpha,w)}_{n,q}(x) \) and \( \tilde{\zeta}_q^{(\alpha,w)}(s, x) \) as follows:

\[ (1.10) \quad \tilde{\zeta}_q^{(\alpha,w)}(-n, x) = E^{(\alpha,w)}_{n,q}(x). \]

In this paper, we construct the generating function of modified Dirichlet’s type twisted \( q \)-Euler polynomials with weight \( \alpha \) in the \( p \)-adic case. Also, we give Witt’s formula for this type polynomials. Moreover, we obtain a new \( p \)-adic \( q \)-Euler \( L \)-function with weight \( \alpha \) associated with Dirichlet’s character \( \chi \), as follows:

\[ l^{\alpha,(w)}_{p,q}(-n \mid \chi) = \tilde{E}^{\alpha,(w)}_{n,\chi_n} - \frac{1}{[p^{-1}]_{q^{\alpha},p}} \chi_p(p) \tilde{E}^{\alpha,(w)}_{n,\chi_n} \]

where \( n \in \mathbb{N}^* \).

2. Properties of Modified Dirichlet’s Type of Twisted \( q \)-Euler Numbers and Polynomials

In this section, by using fermionic \( p \)-adic \( q \)-integral equations on \( \mathbb{Z}_p \), some interesting identities and relations of the modified Dirichlet’s type of twisted \( q \)-Euler numbers and polynomials with weight \( \alpha \), are given.

Definition 1. Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \). For each \( n \in \mathbb{N}^* \) and \( w \in T_p \), Modified Dirichlet’s type of twisted \( q \)-Euler polynomials with weight \( \alpha \) defined by means of the following generating function:

\[ F^{(\alpha)}(t, x, q, w \mid \chi) = \sum_{n=0}^\infty \tilde{E}^{(\alpha,w)}_{n,q}(x \mid \chi) \frac{t^n}{n!} \]

where

\[ F^{(\alpha)}(t, x, q, w \mid \chi) = [2]q \sum_{m=0}^\infty (-1)^m w^m \chi(m) e^{[x+m]q^\alpha}. \]

From (2.1) and (2.2) we obtain,

\[ \sum_{n=0}^\infty \tilde{E}^{(\alpha,w)}_{n,q}(x \mid \chi) \frac{t^n}{n!} = \sum_{n=0}^\infty \left( [2]q \sum_{m=0}^\infty (-1)^m w^m \chi(m) [x+m]q^\alpha \right) \frac{t^n}{n!} \]

Therefore, we state the following theorem:

Theorem 1. Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \). For each \( n \in \mathbb{N}^* \) and \( w \in T_p \), we have

\[ \tilde{E}^{(\alpha,w)}_{n,q}(x \mid \chi) = [2]q \sum_{m=0}^\infty (-1)^m w^m \chi(m) [x+m]q^\alpha. \]

By using (2.3), we can write

\[ E_{n,q}^{(\alpha,w)}(x | \chi) = [2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} (-1)^{l+m} w^l \chi(l) [x+m]_q^n \]

\[ = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^{l} w^l \chi(l) \sum_{m=0}^{\infty} (-1)^m (w^d)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k(x+l+md)} \]

\[ = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^{l} w^l \chi(l) \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k(x+l)} \frac{-1}{q^\alpha k w^d + 1}. \]

So, we obtain the following corollary:

**Corollary 1.** Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \). For each \( n \in \mathbb{N}^* \) and \( w \in T_p \), we have

\[ E_{n,q}^{(\alpha,w)}(x | \chi) = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^{l} w^l \chi(l) \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k(x+l)} \frac{-1}{q^\alpha k w^d + 1}. \]

By applying \( f(\xi) = q^{-\xi} \chi(\xi) w^\xi [x+\xi]_{q^\alpha}^n \) into (1.1),

\[ \int_{\mathbb{Z}_p} q^{-\xi} \chi(\xi) w^\xi [x+\xi]_{q^\alpha}^n \, d\mu_{-q}(\xi) \]

\[ = \frac{1}{(1-q^\alpha)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k} \int_{\mathbb{Z}_p} \chi(\xi) w^\xi q^{\alpha k - \xi} \, d\mu_{-q}(\xi), \]

where from (1.3), we easily see that

\[ \int_{\mathbb{Z}_p} \chi(\xi) w^\xi q^{\alpha k - \xi} \, d\mu_{-q}(\xi) = \frac{[2]_q}{q^\alpha k w^d + 1}, \]

By using (2.4) and (2.5) we obtain

\[ \int_{\mathbb{Z}_p} q^{-\xi} \chi(\xi) w^\xi [x+\xi]_{q^\alpha}^n \, d\mu_{-q}(\xi) \]

\[ = \frac{1}{(1-q^\alpha)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k} \frac{[2]_q}{q^\alpha k w^d + 1}. \]

(2.6)

Last from equivalent, we obtain Witt’s type formula of modified Dirichlet’s type of twisted q-Euler polynomials with weight \( \alpha \) as follows:

**Theorem 2.** Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \). For each \( n \in \mathbb{N}^* \) and \( w \in T_p \) we obtain

\[ E_{n,q}^{(\alpha,w)}(x | \chi) = \int_{\mathbb{Z}_p} q^{-\xi} \chi(\xi) w^\xi [x+\xi]_{q^\alpha}^n \, d\mu_{-q}(\xi). \]
By \([2,3]\), we obtain *functional equation* as follows:

\[
F^{(\alpha)}(t, x, q, w \mid \chi) = e^{t[x]_{q^\alpha}} F^{(\alpha)}(q^\alpha t, q, w \mid \chi).
\]

By using the definition of the generating function \(F^{(\alpha)}(t, x, q, w \mid \chi)\) as follows:

\[
\sum_{n=0}^{\infty} F^{(\alpha,w)}_{n,q}(x \mid \chi) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} [x]_{q^\alpha} \frac{t^n}{n!}\right) \left(\sum_{\alpha=0}^{\infty} q^{\alpha x} E^{(\alpha,w)}_{n,q}(\chi) \frac{t^n}{n!}\right),
\]

by the Cauchy product in the above equation, we have

\[
\sum_{n=0}^{\infty} E^{(\alpha,w)}_{n,q}(x \mid \chi) \frac{t^n}{n!} = \sum_{\alpha=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} q^{\alpha xl} E^{(\alpha,w)}_{l,q}(\chi) [x]_{q^\alpha}^{n-l}\right) \frac{t^n}{n!},
\]

Therefore, by comparing the coefficients of \(\frac{t^n}{n!}\) on the both sides of the above equation, we can state following theorem:

**Theorem 3.** Let \(\chi\) be a Dirichlet’s character with conductor \(d(=\text{odd}) \in \mathbb{N}\). For each \(n \in \mathbb{N}^*\) and \(w \in T_p\) we have

\[
E^{(\alpha,w)}_{n,q}(x \mid \chi) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha xl} E^{(\alpha,w)}_{l,q}(\chi) [x]_{q^\alpha}^{n-l}.
\]

(2.8)

So, by using *umbral calculus* convention in equality (2.8), we get

\[
E^{(\alpha,w)}_{n,q}(x \mid \chi) = \left(\sum_{l=0}^{n} \binom{n}{l} q^{\alpha xl} E^{(\alpha,w)}_{l,q}(\chi) [x]_{q^\alpha}^{n-l}\right)^n,
\]

(2.9)

where \(E^{(\alpha,w)}_{l,q}(\chi)\) is replaced by \(E^{(\alpha,w)}_{n,q}(\chi)\).

From (1.3), we arrive at the following theorem:

**Theorem 4.** Let \(\chi\) be a Dirichlet’s character with conductor \(d(=\text{odd}) \in \mathbb{N}, w \in T_p\) and \(m \in \mathbb{N}^*\) we get

\[
\frac{u^n}{E^{(\alpha,w)}_{m,q}}(x \mid \chi) + (-1)^{n-1} E^{(\alpha,w)}_{m,q}(x \mid \chi) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} \chi(l) w^l [l]_{q^\alpha}^m.
\]

(2.10)

So, from (1.3), and some combinatorial techniques we can write

\[
\int_{\mathbb{Z}_p} q^{-\xi} \chi(\xi) w^{\xi} [x + \xi]_{q^\alpha}, d\mu_{\chi}(\xi) = [d]_{q}^{n-d} \sum_{a=0}^{d-1} (-1)^a \chi(a) w^a \int_{\mathbb{Z}_p} q^{-d \xi} w^d \left[\frac{x + a}{d} + \xi\right]_{q^\alpha}^n d\mu_{(-q)^d}(\xi)
\]

(2.10)

Therefore, by (2.10), we obtain the following theorem:

**Theorem 5.** Let \(\chi\) be a Dirichlet’s character with conductor \(d(=\text{odd}) \in \mathbb{N}, w \in T_p\) and \(n \in \mathbb{N}^*\) we have

\[
E^{(\alpha,w)}_{n,q}(x \mid \chi) = [d]_{q^\alpha}^{n} \sum_{a=0}^{d-1} (-1)^a w^a \chi(a) E^{(\alpha,w^d)}_{n,q^d} \left(\frac{x + a}{d}\right).
\]
3. Modified Dirichlet’s type of twisted $q$-Euler $L$-function with weight $\alpha$

In this section, our goal is to consider interpolation function of the generating functions of modified Dirichlet’s type of twisted $q$-Euler polynomials with weight $\alpha$. For $s \in \mathbb{C}$, $w \in T_p$ and $\chi$ be a Dirichlet’s character with conductor $d (= odd) \in \mathbb{N}$, by applying the Mellin transformation in equation (2.2), we obtain

$$\tilde{L}_q^{(\alpha,w)}(x, s \mid \chi) = \frac{1}{\Gamma(s)} \int \limits_{t=0}^{\infty} t^{s-1} F^{(\alpha)}(-t, x, q, w \mid \chi) \, dt$$

so, from above equality, we have

$$\tilde{L}_q^{(\alpha,w)}(x, s \mid \chi) = [2]^q \sum_{m=0}^{\infty} (-1)^m w^m \chi(m) \left( \frac{1}{\Gamma(s)} \int \limits_{t=0}^{\infty} t^{s-1} e^{-t[m+x]q} \, dt \right),$$

Consequently, we are in position to define modified Dirichlet’s type of twisted $q$-Euler $L$-function as follows:

**Definition 2.** Let $\chi$ be a Dirichlet’s character with conductor $d (= odd) \in \mathbb{N}$ and $w \in T_p$ we have

$$\tilde{L}_q^{(\alpha,w)}(x, s \mid \chi) = [2]^q \sum_{m=0}^{\infty} (-1)^m \chi(m) w^m \left[ m + x \right]_q^s,$$

for all $s \in \mathbb{C}$. We note that $\tilde{L}_q^{(\alpha,w)}(x, s \mid \chi)$ is analytic function in the whole complex $s$-plane.

By substituting $s = -n$ into (3.1) we easily get

$$\tilde{L}_q^{(\alpha,w)}(x, -n \mid \chi) = E_{n,q}^{(\alpha,w)}(x \mid \chi),$$

which led to stating following theorem:

**Theorem 6.** Let $\chi$ be a Dirichlet’s character with conductor $d (= odd) \in \mathbb{N}$, $w \in T_p$ and $n \in \mathbb{N}^*$, we define

$$\tilde{L}_q^{(\alpha,w)}(x, -n \mid \chi) = E_{n,q}^{(\alpha,w)}(x \mid \chi).$$

$$\tilde{L}_q^{(\alpha,w)}(1, s \mid \chi) = \tilde{L}_q^{(\alpha,w)}(s \mid \chi)$$ which is the modified Dirichlet’s type of twisted $q$-Euler $L$-function with weight $\alpha$. Now, by applying combinatorial techniques we
can write,

\[
\tilde{L}_{q}^{(\alpha,w)}(s \mid \chi) = [2]_{q} \sum_{m=1}^{\infty} \frac{(-1)^{m} \chi(m) w^{m}}{[m]^{q^\alpha}}
\]

\[
= [2]_{q} \sum_{m=1}^{\infty} \frac{d-1}{m} (-1)^{a+dm} \chi(a + dm) w^{a+dm}
\]

\[
= \frac{[2]_{q}}{[2]_{q^d}} [d]_{q^s}^{-s} \sum_{a=0}^{d-1} (-1)^{a} \chi(a) w^{a} \sum_{m=1}^{\infty} \frac{(-1)^{m} (w^{d})^{m}}{([\frac{a}{d} + m])_{q^{d\alpha}}}
\]

(3.3)

\[
= \frac{[2]_{q}}{[2]_{q^d}} [d]_{q^s}^{-s} \sum_{a=0}^{d-1} (-1)^{a} \chi(a) w^{a} \tilde{\zeta}_{q^d}^{(\alpha,w^{d})}(s, \frac{a}{d}).
\]

So, by previous calculation we can state following theorem:

**Theorem 7.** Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \) and \( w \in T_{p} \) we have

\[
L_{q}^{(\alpha,w)}(s \mid \chi) = [2]_{q} \sum_{m=1}^{\infty} \frac{(-1)^{m} \chi(m) w^{m}}{[m]^{q^\alpha}}
\]

\[
= [2]_{q} \sum_{m=1}^{\infty} \frac{d-1}{m} (-1)^{a+dm} \chi(a + dm) w^{a+dm}
\]

\[
= \frac{[2]_{q}}{[2]_{q^d}} [d]_{q^s}^{-s} \sum_{a=0}^{d-1} (-1)^{a} \chi(a) w^{a} \sum_{m=1}^{\infty} \frac{(-1)^{m} (w^{d})^{m}}{([\frac{a}{d} + m])_{q^{d\alpha}}}
\]

(3.4)

We now consider the partial-zeta function \( \tilde{H}_{q}^{(\alpha)}(s, a, w \mid F) \) as follows:

\[
\tilde{H}_{q}^{(\alpha)}(s, a, w \mid F) = [2]_{q} \sum_{m \equiv \alpha \pmod{F}} \frac{(-1)^{m} w^{m}}{[m]^{q^\alpha}}.
\]

If \( F \equiv 1 \pmod{2} \), then we have

\[
\tilde{H}_{q}^{(\alpha)}(s, a, w \mid F) = [2]_{q} \sum_{m \equiv \alpha \pmod{F}} \frac{(-1)^{m} w^{m}}{[m]^{q^\alpha}}
\]

\[
= [2]_{q} \sum_{m \equiv \alpha \pmod{F}} \frac{(-1)^{mF+a} w^{mF+a}}{[mF+a]^{q^\alpha}}
\]

\[
= \frac{[2]_{q}}{[2]_{q^F}} \frac{(-1)^{a} w^{a}}{[F]^{q^s}_{q^F}} \sum_{m>0} \frac{(-1)^{m} (w^{F})^{m}}{[m + \frac{a}{F}]_{q^{F\alpha}}}
\]

(3.5)

By expressions (3.2) and (3.6) we get the following theorem:

**Theorem 8.** Let \( F \equiv 1 \pmod{2} \), \( w \in T_{p} \), \( q, s \in \mathbb{C}, |q| < 1 \) and \( n \in \mathbb{N}^* \) we have

\[
\tilde{H}_{q}^{(\alpha)}(-n, a, w \mid F) = \frac{[2]_{q}}{[2]_{q^F}} \frac{(-1)^{a} w^{a}}{[F]^{q^s}_{q^F}} \tilde{E}_{n,q^{F}}^{(\alpha,w^{F})}(\frac{a}{F}).
\]

(3.7)

By expressions (3.3) and (3.7), we obtain the following corollary:
Corollary 2. Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{N} \), \( w \in T_p \) and \( F \equiv 1 \pmod{2} \) we have

\[
L_q^{(\alpha, w)}(s \mid \chi) = \frac{F-1}{2} \sum_{a=0}^{\chi(a)} H_q^{(\alpha)}(s, a, w \mid F).
\]

By (1.7) and (3.7), we modify the \( q \)-analogue of the partial zeta function with weight \( \alpha \) as follows:

\[
H_q^{(\alpha)}(s, a, w \mid F) = \frac{[2]_q}{[2]_q F} (-1)^a w^a [a]_{q^s}^{-s} \sum_{l=0}^{\infty} \left( \frac{s}{l} \right) q^{\alpha a l} \left( \frac{[l] q^a}{[a]_{q^s}} \right)^l E_{l,q^s}^{(\alpha, w \mid F)}.
\]

Let \( f (= \text{odd}) \) and \( a \) be the positive integer with \( 0 \leq a < F \). Then, (3.8) reduces to

\[
L_q^{(\alpha, w)}(s \mid \chi) = \frac{[2]_q}{[2]_q F} (-1)^a w^a [a]_{q^s}^{-s} \sum_{l=0}^{\infty} \left( \frac{s}{l} \right) q^{\alpha a l} \left( \frac{[l] q^a}{[a]_{q^s}} \right)^l E_{l,q^s}^{(\alpha, w \mid F)}.
\]

By expression (3.10), we see that \( L_q^{(\alpha, w)}(s \mid \chi) \) is an analytic function \( s \in \mathbb{C} \), with except \( s = 0 \). Furthermore, for each \( n \in \mathbb{Z} \), with \( n \geq 0 \), we get

\[
L_q^{(\alpha, w)}(-n \mid \chi) = E_{n,q}^{(\alpha, w)}(\chi).
\]

By using (3.9), (3.10) and (3.11) we derive behavior of the modified Dirichlet’s type of twisted \( q \)-Euler \( L \)-function with weight \( \alpha \) at \( s = 0 \) as follows:

Theorem 9. The following likeable identity

\[
L_q^{(\alpha, w)}(0 \mid \chi) = \frac{1 + q}{1 + w^F} \sum_{a=0}^{\chi(a)} (-1)^a \chi(a) w^a,
\]

is true.

4. Modified \( p \)-Adic Twisted Interpolation \( q \)-l-Function with weight \( \alpha \)

In this section, we construct modified \( p \)-adic twisted \( q \)-Euler \( l \)-function, which interpolate modified Dirichlet’s type of twisted \( q \)-Euler polynomials at negative integers. Firstly, Washington constructed \( p \)-adic \( l \)-function which interpolates generalized classical Bernoulli numbers.

Here, we use some the following notations, which will be useful in reminder of paper.

Let \( \omega \) denote the \textit{Kummer} character by the conductor \( f_{\omega} = p \). For an arbitrary character \( \chi \), we set \( \chi_n = \chi \omega^{-n}, n \in \mathbb{Z} \), in the sense of product of characters.

Let

\[
\langle a \rangle = \omega^{-1}(a) a = \frac{a}{\omega(a)},
\]

\[
\langle a \rangle_q = \frac{[a]_q}{\omega(a)}.
\]

Thus, we note that \( \langle a \rangle \equiv 1 \pmod{p \mathbb{Z}_p} \). Let

\[
A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n, a_{n,j} \in \mathbb{C}_p, j = 0, 1, 2, ...
\]
be a sequence of power series, each convergent on a fixed subset

\[ T = \left\{ s \in \mathbb{C}_p \mid |s|_p < p^{-\frac{2\pi}{p-2}} \right\}, \]

of \( \mathbb{C}_p \) such that

1. \( a_{n,j} \to a_{n,0} \) as \( j \to \infty \) for any \( n \);
2. for each \( s \in T \) and \( \epsilon > 0 \), there exists an \( n_0 = n_0(s, \epsilon) \) such that

\[ \left| \sum_{n \geq n_0} a_{n,j} s^n \right|_p < \epsilon \quad \text{for all} \quad j. \]

So,

\[ \lim_{j \to \infty} A_j(s) = A_0(s), \quad \text{for all} \quad s \in T. \]

This was constructed by Washington [30] to indicate that each function \( \omega^{-s}(a) a^s \) and

\[ \sum_{l=0}^{\infty} \binom{s}{l} \left( \frac{F}{a} \right)^l B_l, \]

where \( F \) is multiple of \( p \) and \( f \) and \( B_l \) is the \( l \)-th Bernoulli numbers, is analytic on \( T \) (for more information, see [30]).

Assume that \( \chi \) be a Dirichlet’s character with conductor \( f \in \mathbb{N} \) with \( f \equiv 1 \pmod{2} \). Thus, we consider the modified Dirichlet’s type of twisted \( p \)-adic \( q \)-Euler \( l \)-function with weight \( \alpha \), \( l^{(\alpha,w)}(\chi) \), which interpolate the modified Dirichlet’s type of twisted \( q \)-Euler polynomials with weight \( \alpha \) at negative integers.

For \( f \in \mathbb{N} \) with \( f \equiv 1 \pmod{2} \), let us assume that \( F \) is positive integral multiple of \( p \) and \( f \equiv f \chi \). We are now ready to give definition of \( l^{(\alpha,w)}(\chi) \) as follows:

\[ l^{(\alpha,w)}(\chi) = \sum_{a=0}^{F-1} \chi(a) (-1)^a \sum_{l=0}^{\infty} \binom{s}{l} \left( \frac{F}{a} \right)^l E^{(\alpha,w,F)}_{l,F} \]

By (4.1), we note that \( l^{(\alpha,w)}(\chi) \) is analytic for \( s \in T \).

For \( n \in \mathbb{N} \), we have

\[ E^{(\alpha,w)}_{n,\chi_n} = \left[ F \right]_{q^n}^{F-1} \sum_{a=0}^{F-1} (-1)^a \chi_n(a) E^{(\alpha,w)}_{n,q} \left( \frac{a}{F} \right). \]

If \( \chi_n(p) \neq 0 \), then \( (p, f \chi_n) = 1 \), and thus the ratio \( \frac{F}{p} \) is a multiple of \( f \chi_n \).

Let

\[ \lambda = \left\{ \frac{a}{p} \mid a \equiv 0 \pmod{p} \quad \text{for} \quad a_i \in \mathbb{Z} \quad \text{with} \quad 0 \leq a_i < F \right\}. \]

Thus, we have

\[ \left[ F \right]_{q^n}^{F-1} \sum_{a=0 \atop p \mid a} (-1)^a \chi_n(a) E^{(\alpha,w)}_{n,q} \left( \frac{a}{F} \right) \]

\[ = \frac{1}{\left[ p^{-1} \right]_{q^n}^{p-1} \chi_n(p) \sum_{\eta \in \lambda} (-1)^\eta \chi_n(\eta) E^{(\alpha,w)}_{n,q} \left( \frac{\eta}{p} \right). \]
By (4.3), we can define the second modified twisted generalized Euler numbers attached to $\chi$ as follows:

$$(4.4) \quad \tilde{E}^{\ast}_{n,\chi_n}(\alpha,w) = \left[\frac{F}{p}\right]^n \sum_{\eta \in \lambda} (-1)^{n,\chi} E_{n,q}(\eta) \left(\frac{\eta}{F/p}\right).$$

By (4.2), (4.3) and (4.4), we readily get that

$$\tilde{E}^{\ast}_{n,\chi_n}(\alpha,w) = \left[\frac{F}{p}\right]^n \sum_{\eta \in \lambda} (-1)^{\eta,\chi} E_{n,q}(\eta) \left(\frac{\eta}{F/p}\right).$$

By (4.1) and (1.8), we readily see that

$$l_{p,q}^{(\alpha,w)}(-n | \chi) = \left[\frac{F}{p}\right]^n \sum_{\eta \in \lambda} (-1)^{\eta,\chi} E_{n,q}(\eta) \left(\frac{\eta}{F/p}\right).$$

Consequently, we state the following Theorem:

**Theorem 10.** Let $n \in \mathbb{N}$, the following equalities

$$l_{p,q}^{(\alpha,w)}(s | \chi) = \sum_{a=1}^{F} \chi(a) (-1)^{a} w^{a} \left( \frac{\eta^{a}}{p} \right) \sum_{l=1}^{\infty} \left( \frac{\eta^{a}}{l} \right) q^{al} \left[ \frac{F^{l} q^{a}}{p^{l}} \right] E_{l,q}^{(\alpha,w)}(\eta),$$

and

$$l_{p,q}^{(\alpha,w)}(-n | \chi) = \tilde{E}^{(\alpha,w)}_{n,\chi_n}(\alpha,w) - \left[\frac{F}{p}\right]^{n} \sum_{\eta \in \lambda} (-1)^{\eta,\chi} E_{n,q}(\eta) \left(\frac{\eta}{F/p}\right),$$

are true.

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