On the distribution of distances in homogeneous compact metric spaces

Mark Herman and Jonathan Pakianathan

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Abstract

We provide a simple proof that in any homogeneous, compact metric space of diameter $D$, if one finds the average distance $A$ achieved in $X$ with respect to some isometry invariant Borel probability measure, then

$$\frac{D}{2} \leq A \leq D.$$ 

This result applies equally to vertex-transitive graphs and to compact, connected, homogeneous Riemannian manifolds.

We then classify the cases where one of the extremes occurs. In particular any homogeneous compact metric space where $A = \frac{D}{2}$ possesses a strict antipodal property which implies in particular that the distribution of distances in $X$ is symmetric about $\frac{D}{2}$ which is hence both mean and median of the distribution.

In particular, we show that the only closed, connected, positive-dimensional Riemannian manifolds with this strict antipodal property are spheres.

Keywords: homogeneous space, metric space, diameter, Riemannian manifold.

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1 Introduction

Good general references for the mathematics used in this paper are [Rud] for measure theory, [Mun] for general topology, [Hat] for algebraic topology and [Lee] for Riemannian geometry.

This paper was motivated by a question of Alan Kaplan that appears on the webpage (www.math.uiuc.edu/~west/openp): For any finite, connected, vertex-transitive graph, is the average distance between vertices at least as large as half the diameter of the graph?

We answer this question in the affirmative by establishing:
Theorem 1.1. Let $X$ be a finite, connected, vertex transitive graph with $n$ vertices and diameter $D$. Let $A = \frac{1}{n^2} \sum_{x,y \in V} d(x,y)$ be the average distance between vertices in this graph, then

$$\frac{D}{2} \leq A \leq (1 - \frac{1}{n})D$$

If we define the average distance instead via $\bar{A} = \frac{1}{n(n-1)} \sum_{x \neq y} d(x,y)$ we have

$$\frac{D}{2} \frac{n}{n-1} \leq \bar{A} \leq D.$$  

These bounds are sharp: For any complete graph $K_n$, $\bar{A} = D = 1$. On the other hand, for a $d$-dimensional hypercube graph $Q_d$, we have $D = d$, and

$$A = \frac{1}{2^d} \sum_{k=0}^d \binom{n}{k} = \frac{d^{d-1}}{2^{d-1}} = \frac{d}{2} = \frac{D}{2}.$$  

A similar bound for the expected distance squared in a vertex transitive graph in terms of the diameter $D$ was obtained in [NRa09]: $\frac{D^2}{8} \leq E[d^2] \leq D^2$. However, when this argument is applied to the expected distance it results in a weaker result than Theorem 1.1.

Theorem 1.1 follows from a more general theorem on homogeneous compact metric spaces, i.e., compact metric spaces whose isometry group acts transitively on points. (Indeed we prove the theorem for antipodal compact metric spaces - see section 2 for definitions.)

Theorem 1.2. Let $X$ be a homogeneous compact metric space and $m$ a Borel probability measure on $X$ invariant under all isometries. Let

$$A = \int_{X \times X} d(x,y)m(x)m(y)$$

be the average distance in the metric space with respect to the product measure $m \times m$ on $X \times X$, then

$$\frac{D}{2} \leq A \leq \mu D$$

where $1 - \mu$ is the $m \times m$ measure of the diagonal in $X \times X$. $\mu = 1$ if points have $m$-measure zero in $X$ and $\mu = (1 - \frac{1}{n})$ if $|X| = n$ is finite.

Theorem 1.2 applies to any connected, finite, vertex-transitive graph under the uniform probability measure on vertices. It also applies to any compact, connected, homogeneous Riemannian manifold $M$. If $G$ is the isometry group of $M$ then $G$ is a Lie group and there is closed subgroup $K$ and continuous bijection $\theta : G/K \rightarrow M$. The push forward of a suitably normalized Haar measure on $G$ hence provides a $G$-invariant measure $m$ on $M$. Examples include spheres with spherical measure under the round spherical metric or under the Euclidean metric (the isometry group is $O(n)$ for either metric). Any compact Lie group such as torii under the probability Haar measure and under a left invariant Riemannian metric provide other examples for which Theorem 1.2 applies. Other examples include Grassmann spaces of various flavors.
In the case of unit spheres $S^d = \{ x \in \mathbb{R}^{d+1} ||x|| = 1 \}$ under the spherical metric and measure, one has $A = \frac{\pi}{2}, D = \pi$ and so $A = \frac{D}{2}$.

In the case of the $p$-adic integers, a compact metric Abelian group equipped with its Haar measure $m$ and the standard $p$-adic metric, it is easy to compute that $D = 1$ and $A = \frac{p}{p+1}$. As the prime $p \to \infty$ we have $A \to D$.

We also characterize the homogeneous compact metric spaces which achieve the bounds in Theorem 1.2.

**Theorem 1.3.** Let $X$ be a homogeneous compact metric space and $m$ a Borel probability measure on $X$ invariant under all isometries and let $A$ be the average distance in $X$, and $D$ the diameter of $X$.

If $A = \mu D$, then $X$ is finite and the metric is a scaling of the standard discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

and $m$ is the uniform probability measure.

On the other hand, $A = \frac{D}{2}$ if and only if $X$ is a strictly antipodal space, i.e., every point $x$ has a unique antipode $O_x$ such that $d(x, O_x) = D$ with the additional conditions that there is an isometry of $X$ taking $x$ to $O_x$ and such that for any $y \in X$ one has

$$D = d(x, y) + d(y, O_x).$$

Furthermore, in any strictly antipodal space, the antipodal map $O : X \to X$ taking each $x$ to its unique antipode is an isometry of $X$ which is a central element of order exactly two in the isometry group of $X$. It equips $X$ with a free $\mathbb{Z}/2\mathbb{Z}$-action. In such a space the distribution of distances is symmetric about the mean (which is hence also the median) $\frac{D}{2}$. In other words, if $0 \leq a \leq D$ then

$$Pr(d(x, y) \leq a) = Pr(d(x, y) \geq D - a).$$

where $Pr$ is probability with respect to the $m \times m$ probability measure on $X \times X$.

In the case of unit spheres $S^d$ with spherical measure and metric, the distances are uniformly distributed in the interval $[0, \pi]$. The antipodal map is the standard antipodal map $x \to -x$. In the case of distances in the hypercube graph in dimension $d$, the distances are distributed according to the binomial distribution with parameters $n = d$ and $p = \frac{1}{2}$. The antipodal map is the map that swaps zeros and ones in binary strings. In the case of distances in a cycle graph $C_{2n}$, the average distance is $A = \frac{1}{2n}(1(0) + 2(1) + 2(2) + \cdots + 2(n-1) + 1(n)) = \frac{n}{2} = \frac{D}{2}$. The probability distribution of distances is

$$Pr(d = j) = \begin{cases} \frac{1}{n} & \text{if } 1 \leq j \leq n-1 \\ \frac{1}{2n} & \text{if } j = 0, n \end{cases}$$

Finally, we show in Theorem 3.2 that the only closed, connected, positive-dimensional Riemannian manifolds with this strict antipodal property are spheres.
2 Proof of Theorem 1.2

We first define various concepts of antipodal spaces which will be useful.

**Definition 2.1.** Let $X$ be a compact metric space of diameter $D$. We say that $X$ is **antipodal** if for every $x \in X$ there is at least one antipode $O_x$ such that $d(x, O_x) = D$. We further require that there is an isometry of $X$ taking $x$ to $O_x$.

We say that $X$ is **uniquely antipodal** if it is antipodal and if each $x \in X$ has a unique antipode $O_x$.

Finally we say that $X$ is **strictly antipodal** if it is antipodal and if for any $x \in X$, and antipode $O_x$ of $x$ we have:

\[ D = d(x, y) + d(y, O_x) \]

for all $y \in X$.

Note that any homogeneous compact metric space is antipodal. This is because compactness guarantees a pair of points $u, v$ such that $d(u, v) = D$. Then the transitivity of the action of the isometry group on points, guarantees that every $x \in X$ has at least one $O_x$ such that $d(x, O_x) = D$. Furthermore transitivity also guarantees an isometry taking $x \rightarrow O_x$ and hence ensures the property of being an antipodal space.

**Theorem 2.2.** Let $X$ be a compact antipodal metric space of diameter $D$. Let $m$ be a Borel probability measure on $X$ invariant under isometries and let

\[ A = \int_{X \times X} d(x, y) m(x) m(y). \]

Then \( \frac{D}{2} \leq A \leq \mu D \) where $\mu = 1 - (m \times m)(\Delta)$ where $\Delta$ is the diagonal in $X \times X$.

**Proof.** First note $A = \int_{X \times X - \Delta} d(x, y) m(x) m(y) \leq D(m \times m)(X \times X - \Delta)$ and so \( A \leq \mu D \).

On the other hand as $X \times X$ is compact and $d : X \times X \rightarrow \mathbb{R}$ continuous, we have $d \in L^1(m \times m)$ and so we may apply Fubini’s Theorem:

\[ A = \int_X \left( \int_X d(x, y) m(x) m(y) \right). \]

Thus to establish the lower bound, it is enough to show that $A_y = \int_X d(x, y) m(x) \geq \frac{D}{2}$ for all $y \in X$.

To do this let $O_y$ be an antipode of $y$, as there is an isometry $g$ which takes $y$ to $O_y$, and as $m$ is isometry invariant, we have $A_y = A_{O_y}$.

Then

\[ A_y = \int_X d(x, y) m(x) \geq \int_X (d(y, O_y) - d(O_y, x)) m(x) = D - A_{O_y}. \]
Thus \( 2A_y \geq D \) for any \( y \in Y \) and we are done.

When \( X \) is a finite set of size \( n \) and \( m \) is uniform measure, the factor \( \mu \) in Theorem 2.2 is \( \mu = (1 - \frac{1}{n}) \). On the other hand if points have \( m \)-measure zero in \( X \) then the diagonal \( \Delta \) has \( m \times m \) measure zero in \( X \times X \) by Fubini’s Theorem and so \( \mu = 1 \).

We have already discussed examples in the intro that show the bounds in Theorem 2.2 are sharp. We now characterize the examples which achieve the extremes of these bounds at least in the case of compact homogeneous metric spaces.

**Proposition 2.3.** Let \( X \) be a compact homogeneous metric space of diameter \( D \) and \( m \) an isometry invariant Borel probability measure on \( X \). Suppose additionally that the average distance \( A \) satisfies \( A = \mu D \) where \( \mu = 1 - (m \times m)(\Delta) \).

Then \( X \) is a finite set equipped with a scalar multiple of the discrete metric and \( m \) is uniform measure.

**Proof.** First let us handle the case \(|X| = n \) is finite. Then as \( X \) is homogeneous and \( m \) is an isometry invariant Borel probability measure, it must assign every point of \( X \) measure \( \frac{1}{n} \) and hence \( \mu = 1 - \frac{1}{n} \). Writing \( X = \{x_1, \ldots, x_n\} \) we have

\[
A = \frac{1}{n^2} \sum_{j,k} d(x_j, x_k) = \mu D
\]

if and only if \( d(x_j, x_k) = D \) when \( j \neq k \) i.e. if the metric \( d \) is \( D \) times the discrete metric on \( X \).

On the other hand, when \( X \) is infinite, points must have \( m \)-measure zero as every point has equal measure by homogeneity and \( m(X) = 1 < \infty \). Fubini’s Theorem then shows \( (m \times m)(\Delta) = 0 \) and so \( \mu = 1 \).

Now if \( A = \mu D = D \) in this case we would have

\[
\int_{X \times X} d(x, y)m(x)m(y) = \int_X A_m(y) = D.
\]

By homogeneity \( A_y \) is independent of \( y \in X \) and so this equation implies \( A_y = \int_X d(x, y)m(x) = D \) for all \( y \in X \).

Thus for any \( y \in Y \), the set \( S_y = \{x \in X | d(x, y) < D\} \) has \( m \)-measure zero. As \( d : X \times X \to \mathbb{R} \) is continuous, \( S_y \) is an open neighborhood of \( y \). The collection \( \{S_y | y \in X\} \) is an open cover of \( X \) by open sets of \( m \)-measure zero. As \( X \) is compact, it is covered by a finite number of these and hence has \( m \)-measure zero which contradicts that \( m \) is a probability measure. Thus it is impossible for \( A = \mu D \) when \( X \) is infinite. \( \square \)

**Proposition 2.4.** Let \( X \) be a compact homogeneous metric space of diameter \( D \). Let \( m \) be an isometry invariant Borel probability measure on \( X \). Then the average distance \( A \) satisfies \( A = \frac{D}{2} \) if and only if \( X \) is a strictly antipodal space.
Proof. Again \( A_y = \int_X d(x,y) m(x) \) is independent of \( y \in X \) by homogeneity and invariance of \( m \) under isometries. Thus \( A = A_y \) for all \( y \in X \) so if \( A = \frac{D}{2} \) we have \( A_y = \int_X d(x,y) m(x) = \frac{D}{2} \). As \( X \) is homogeneous, it is antipodal so let \( O_y \) be an antipode of \( y \). Then the inequality:

\[
A_y = \int_X d(x,y) m(x) \geq \int_X (d(y,O_y) - d(O_y, x)) m(x) = D - AO_y
\]

is an equality and so we conclude that the set \( T_y = \{ x \in X | d(x, y) > d(y, O_y) - d(O_y, x) \} \) is a set of \( m \)-measure zero. If this set were nonempty, it would provide an \( m \)-measure zero open neighborhood of some point in \( X \). By homogeneity, every point of \( X \) would have a measure zero open neighborhood. Compactness of \( X \) would then imply a finite open cover of \( X \) by measure zero open sets contradicting that \( m \) is a probability measure on \( X \). Thus we conclude \( A = \frac{D}{2} \) implies that \( d(y, O_y) = D = d(y, x) + d(x, O_y) \) for all \( x, y \in X \) i.e., that \( X \) is a strictly antipodal space. Conversely it is easy to check that if \( X \) is strictly antipodal, then \( A = \frac{D}{2} \). \( \square \)

In the next section, we further study the structure of strictly antipodal spaces and show that the distribution of distances in such spaces is symmetric around the mean distance.

## 3 Strictly antipodal spaces

In this section we study the structure of strictly antipodal spaces. By Proposition 2.3 these are the primary class of compact metric spaces whose average distance (with respect to any isometry invariant Borel probability measure) is equal to half their diameter.

**Theorem 3.1.** Let \( X \) be a strictly antipodal compact metric space. Then:

1. \( X \) is uniquely antipodal.
2. The antipodal map \( O : X \to X \) which takes each element \( x \) to its unique antipode \( O_x \) is an isometry and is a central element of order two in the isometry group of \( X \).
3. \( X \) has a free \( \mathbb{Z}/2\mathbb{Z} \)-action via this antipode. Thus if its Euler characteristic is defined, it is even.
4. If \( m \) is a Borel probability measure on \( X \) invariant under isometries, then the distribution of distances in \( X \) is symmetric about its average \( \frac{D}{2} \) which is hence also the median distance. More precisely for any \( 0 \leq a \leq D \) we have \( \Pr(d(x,y) \leq a) = \Pr(d(x,y) \geq D - a) \) where \( \Pr \) is probability with respect to the measure \( m \times m \) on \( X \times X \).

**Proof.** Part(1): Let \( x \in X \) then if \( O_x \) is an antipode of \( x \), strict antipodality says \( D = d(x,y) + d(y, O_x) \) for all \( y \in X \). If \( y \) were another antipode of \( x \) then \( d(x,y) = D \) also and so \( d(y, O_x) = 0 \) so \( y = O_x \). Thus each element has a unique antipode and \( X \) is a uniquely antipodal space.
Part(2): Let \( x, y \in X \). From strict antipodality \( D = d(x, y) + d(y, O_x) \) and 
\( D = d(O_x, O_y) + d(y, O_x) \). This implies \( d(x, y) = d(O_x, O_y) \) and so the antipodal 
map \( O : X \to X \) taking each \( x \) to its unique antipode \( O_x \) is an isometry. 
\( O \circ O = Id \) so it represents an element of order two in \( Iso(X) \), the group of 
isometries of \( X \). If \( g \) is any isometry of \( X \) then \( d(gx, gO_x) = d(x, O_x) = D \)
which implies \( gO_x = O_{gx} \) by uniqueness of antipodes. Thus \( g \circ O = O \circ g \) and 
\( O \) commutes with any isometry of \( X \) in \( Iso(X) \), i.e., it lies in the center 
of \( Iso(X) \).

Part(3): This part follows immediately from (2) and basic facts about Euler 
characteristics from algebraic topology.

Part(4): As \( O : X \to X \) is an isometry it is \( m \)-measure preserving. Thus 
\[
Pr(d(x, y) \leq a) = \int_X \left( \int_{|d(y, x)| \leq a} m(y) \right) m(x) = \int_X \left( \int_{|d(y, O_x)| \leq a} m(y) \right) m(x)
\]
as \( O \) exchanges the sets \( \{ y | d(y, x) \leq a \} \) and \( \{ y | d(y, O_x) \leq a \} \). Finally strict 
antipodality lets us write the condition \( d(y, O_x) \leq a \) as \( d(x, y) \geq D - a \). This 
completes the proof.

We now show that the only closed, connected, positive dimensional Riemann-
ian manifolds which are strictly antipodal are spheres.

**Theorem 3.2.** Let \( X \) be a closed, connected, Riemannian manifold which is
strictly antipodal. Then if \( d = \dim(X) \geq 1 \) then \( X \) is homeomorphic to \( S^d \).

**Proof.** Fix a point \( p \in X \). As \( X \) is closed and connected, it is geodesically 
complete and so the exponential map \( e : T_p(X) \to X \) is onto, continuous and 
smooth away from the cut locus of \( p \).

If \( D \) is the diameter of \( X \), then the restriction of \( e \) to the closed ball \( B \) of
radius \( D \) in \( T_p(X) \) has \( e : B \to X \) still onto.

If \( y \) is a point of \( X \) other than \( p \) or \( O_p \), the antipode of \( p \), then \( 0 < d(p, y) < D \)
and as \( D = d(p, y) + d(y, O_p) \) one sees that the concatenation of any
minimal geodesic from \( p \) to \( y \) with any minimal geodesic from \( y \) to \( O_p \) has length 
\( D = d(p, O_p) \) and hence is a path of minimal length from \( p \) to \( O_p \). However it
is a standard fact that paths of minimum length between points in a geodesically 
complete Riemannian manifold are unbroken geodesics. Thus any minimal 
geodesic from \( p \) to \( y \) (of unit speed) must have matching velocity vector with 
any minimal geodesic from \( y \) to \( O_p \) at the point \( y \).

From this it follows that there is a unique minimal geodesic from \( p \) to \( y \) and
from \( y \) to \( O_p \) as if there were more than one of either, one could find a path
of length \( D \) from \( p \) to \( O_p \) consisting of a strictly broken geodesic path which
contradicts \( d(p, O_p) = D \).

This then implies that \( e \) is injective when restricted to the interior of \( B \) and
so in particular, \( d(p, e(v)) = |v| \) for all \( v \in Int(B) \). By continuity, this implies
\( d(p, e(v)) = D \) for all points \( v \) on the boundary of \( B \). However \( O_p \) is the unique
point of distance \( D \) from \( p \) and so we conclude that \( e \) takes all of the boundary
of \( B \) to the single point \( O_p \).
This implies that $\epsilon$ induces a bijective continuous map $\overline{\epsilon} : B/\sim \to X$ where $B/\sim$ is the closed ball with its boundary identified to a point. It is well-known that $B/\sim$ is homeomorphic to $S^d$. Finally as $S^d$ is compact, and $X$ is Hausdorff, $\overline{\epsilon}$ is a homeomorphism. Furthermore it is smooth outside a single point corresponding to the identification point of $S^d$.

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