DIMORPHIC MERSENNE NUMBERS AND THEIR APPLICATIONS

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ABSTRACT. The Mersenne primes are primes which can be written as some prime power of 2 minus 1. These primes were studied from antiquity in that their close connection with perfect numbers and even to present day in that their easiness for primality test. In this paper, we introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers and investigate some of their properties in connection with the degenerate Bernoulli polynomials and the incomplete Bell polynomials.

1. INTRODUCTION

In recent years, there have been active explorations for various degenerate versions of many special numbers and polynomials with diverse tools such as generating functions, combinatorial methods, $p$-adic analysis, umbral calculus, differential equations, probability theory, operator theory, analytic number theory and quantum physics. These were initiated by Carlitz in [3,4] and yielded many interesting results of arithmetical and combinatorial nature (see [7-9,12-14,16-18] and the references therein).

The aim of this paper is to introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers and to find some of their applications in connection with the degenerate Bernoulli polynomials. The novelty of this paper is that it is the first paper which introduces the dimorphic Mersenne numbers and shows some of their applications associated with the degenerate Bernoulli numbers and the incomplete Bell polynomials.

The outline of this paper is as follows. In Section 1, we recall Mersenne numbers, the degenerate exponentials and the degenerate Bernoulli polynomials. We remind the reader of the degenerate Stirling numbers of the second kind, the incomplete Bell polynomials and the complete Bell polynomials. Section 2 is the main result of this paper. We first introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers. Then, by using the generating function of the dimorphic Mersenne numbers in (16), in Theorem 1, we derive a recurrence formula for the degenerate Bernoulli polynomials involving the dimorphic Mersenne numbers. We express a factor of the generating function of the dimorphic Mersenne numbers in terms of the incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials (see (16), (20)). In Theorem 2, we obtain from this expression a representation of the dimorphic Mersenne numbers in terms of those incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials. Finally, in Theorem 3 we get an expression of the degenerate Bernoulli polynomials in terms of the same incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials from Theorem 2. In the rest of this section, we recall the facts that are needed throughout this paper.

The Mersenne numbers are defined by

$$M_n = 2^n - 1, \quad (n \geq 0),$$

(see [2]).
where the sum runs over all nonnegative integers $l$. More explicitly, they are given by

\[ B_{n,k}(x_1,\ldots,x_{n-k+1}) = \sum_{l_1+l_2+\cdots+l_{n-k+1}=k} \frac{n!}{l_1!l_2!\cdots l_{n-k+1}!(n-k+1)!} \left( \frac{x_1}{1!} \right)^{l_1} \left( \frac{x_2}{2!} \right)^{l_2} \cdots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{l_{n-k+1}}, \]

where the sum runs over all nonnegative integers $l_1,\ldots,l_{n-k+1}$ satisfying $l_1+\cdots+l_{n-k+1}=k$ and $l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n$. In [3,4], Carlitz introduced the degenerate Bernoulli polynomials defined by

\[ \beta_{n,x} = \left( \frac{x^t}{t^n} \right) = \sum_{n=0}^{\infty} \beta_{n,x} \frac{t^n}{n!}, \]

where

\[ \lim_{x \to 0} \beta_{n,x} = \beta_{n,0} = x^n, \]

Note that

\[ \lim_{x \to 0} \beta_{n,x} = x^n, \ n \geq 0. \]
The complete Bell polynomials are given by

\[
\exp \left( \sum_{i=1}^{\infty} \frac{x_i t^i}{i!} \right) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} B_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}, \quad (\text{see } [1 - 8, 10 - 13, 15]).
\]

Then we have

\[
B_0(x_1, \ldots, x_n) = 1, \quad B_n(x_1, \ldots, x_n) = \sum_{l_1 + 2l_2 + \cdots + nl_n = n} \frac{n!}{l_1! \cdots l_n!} \left( \frac{x_1}{1!} \right)^{l_1} \cdots \left( \frac{x_n}{n!} \right)^{l_n}, \quad (n \geq 1),
\]

where the sum runs over all nonnegative integers \(l_1, \ldots, l_n\) satisfying \(l_1 + 2l_2 + \cdots + nl_n = n\).

From (7) and (8), we note that

\[
\exp \left( \sum_{i=1}^{\infty} \frac{x_i t^i}{i!} \right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i=1}^{\infty} \frac{x_i t^i}{i!} \right)^k
= 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!}
= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!}.
\]

By (9) and (10), we get

\[
B_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}), \quad (n \geq 1).
\]

It is known that the Bell polynomials are given by

\[
\phi_n(x) = \sum_{k=0}^{n} S_2(n, k) x^k, \quad (n \geq 0), \quad (\text{see } [16, 17]).
\]

From (6), (7) and (11), we note that

\[
B_{n,k}(1, 1, \ldots, 1) = S_2(n, k), \quad B_n(x, x, \ldots, x) = \phi_n(x), \quad (n \geq 0).
\]

2. DImorphic Mersenne Numbers and Their Applications

Now, we consider the dimorphic Mersenne numbers given by

\[
M_{n, \lambda} = (2)_{n, \lambda} - (1)_{n, \lambda}, \quad (\lambda \in \mathbb{R}, \quad n \geq 0).
\]

Note that \(\lim_{\lambda \to 0} M_{n, \lambda} = M_n\). We observe that

\[
e_{\lambda}^2(t) - e_{\lambda}(t) = \sum_{n=0}^{\infty} (2)_{n, \lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} (1)_{n, \lambda} \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \left( (2)_{n, \lambda} - (1)_{n, \lambda} \right) \frac{t^n}{n!}.
\]

Thus, by (12) and (13), we have

\[
e_{\lambda}^2(t) - e_{\lambda}(t) = \sum_{n=0}^{\infty} M_{n, \lambda} \frac{t^n}{n!}.
\]
Thus, by (14) and noting that \(M_{0,\lambda} = 0\), we see that

\[
\frac{1}{t}(e^{\frac{t}{\lambda}}(t) - e_{\lambda}(t)) = \frac{1}{t} \sum_{n=1}^{\infty} M_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!}.
\]

From (15), we have

\[
\sum_{n=0}^{\infty} \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!} = \frac{1}{t}(e^{\frac{t}{\lambda}}(t) - e_{\lambda}(t)) = e_{\lambda}(t) \left( e_{\lambda}(t) - 1 \right)
\]

\[
= e^{\frac{t}{\lambda}}(t) \left( e_{\lambda}(t) - 1 \right) = \frac{e^{\frac{t}{\lambda}}(t)}{t} (e_{\lambda}(t) - 1).
\]

Now, we observe that

\[
e^{\frac{t}{\lambda} + 1}(t) = \frac{t}{e_{\lambda}(t) - 1} e_{\lambda}^{\frac{t}{\lambda}}(t) \frac{e^{\frac{t}{\lambda} + 1}(t)}{t e_{\lambda}^{\frac{t}{\lambda}}(t)} (e_{\lambda}(t) - 1)
\]

\[
= \sum_{l=0}^{\infty} \beta_{\lambda}(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{M_{m+1,\lambda}}{m+1} \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left( n \choose l \right) \beta_{\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1} \frac{t^n}{n!}.
\]

On the other hand, by (2), we also have

\[
e^{\frac{t}{\lambda} + 1}(t) = \sum_{n=0}^{\infty} (x+1)_{n,\lambda} \frac{t^n}{n!}.
\]

From (17) and (13), we note that

\[
(x+1)_{n,\lambda} = \sum_{l=0}^{n} \left( n \choose l \right) \beta_{\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1} \frac{t^n}{n!}
\]

\[
= \beta_{\lambda}(x) + \sum_{l=0}^{n-1} \left( n \choose l \right) \beta_{\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1}.
\]

Therefore, by (19), we obtain the following theorem.

**Theorem 1.** For \(n \geq 0\), we have

\[
\beta_{\lambda}(x) = (x+1)_{n,\lambda} - \sum_{l=0}^{n-1} \left( n \choose l \right) \beta_{\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1}.
\]
From (7), we note that

\begin{equation}
\frac{e_\lambda(t) - 1}{te_\lambda(t)} = \frac{1}{e_\lambda(t) - 1} = \frac{1}{1 + \sum_{n=1}^{\infty} \beta_n (x) \frac{t^n}{n!}} = \sum_{k=0}^{\infty} (-1)^k \left( \sum_{i=1}^{\infty} \beta_{i, \lambda}(x) \frac{t^i}{i!} \right)^k = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{k!}{k^k} \left( \sum_{i=1}^{\infty} \beta_{i, \lambda}(x) \frac{t^i}{i!} \right)^k = 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (-1)^k k! B_{n,k} (\beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{n-k+1, \lambda}(x)) \frac{t^n}{n!} + 1
\end{equation}

On the other hand, by (16), we get

\begin{equation}
1 + \sum_{n=1}^{\infty} \frac{M_{n+1, \lambda}}{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{M_{n+1, \lambda}}{n+1} \frac{t^n}{n!} = \frac{e^{t+1}(t)}{te_\lambda(t)} (e_\lambda(t) - 1)
\end{equation}

\begin{equation}
= \frac{e^{t+1}(t)}{te_\lambda(t)} \left( 1 + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k k! B_{j,k} (\beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{j-k+1, \lambda}(x)) \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} (x+1)_{n, \lambda} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{n}{j} \right) (x+1)_{n-j, \lambda} (-1)^k B_{j,k} \left( \beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{j-k+1, \lambda}(x) \right) \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} \left( (x+1)_{n, \lambda} + \sum_{j=1}^{n} \sum_{k=1}^{j} \left( \frac{n}{j} \right) (x+1)_{n-j, \lambda} (-1)^k B_{j,k} \left( \beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{j-k+1, \lambda}(x) \right) \right) \frac{t^n}{n!}.
\end{equation}

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

**Theorem 2.** For \( n \in \mathbb{N} \) and any \( x \), we have

\[ M_{n+1, \lambda} = (x+1)_{n, \lambda} + \sum_{j=1}^{n} \sum_{k=1}^{j} \left( \frac{n}{j} \right) (x+1)_{n-j, \lambda} (-1)^k B_{j,k} \left( \beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{j-k+1, \lambda}(x) \right). \]

We note that

\begin{equation}
B_{n,1} (\beta_{1, \lambda}(x), \beta_{2, \lambda}(x), \ldots, \beta_{n, \lambda}(x)) = \sum_{l_1 + \cdots + l_n = 1}^{1} n! \prod_{i=1}^{n} \frac{1}{l_i !} \left( \frac{\beta_{i, \lambda}(x)}{1!} \right)^{l_i} \left( \frac{\beta_{n, \lambda}(x)}{n!} \right)^{l_n} = n! \frac{\beta_{n, \lambda}(x)}{n!} = \beta_{n, \lambda}(x).
\end{equation}
From Theorem 2 and (22), we further observe that

\begin{equation}
\frac{M_{n+1,\lambda}}{n+1} = (x+1)_{n,\lambda} + \sum_{j=1}^{n} \sum_{k=1}^{j} \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^{k} k! B_{j,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{j-k+1,\lambda}(x))
\end{equation}

= \frac{M_{n+1,\lambda}}{n+1} = (x+1)_{n,\lambda} + \sum_{k=1}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))

= \frac{M_{n+1,\lambda}}{n+1} + \sum_{j=1}^{n-1} \sum_{k=1}^{j} \binom{n-1}{j} \sum_{k=1}^{j} (x+1)_{n-j,\lambda} (-1)^{k} k! B_{j,k} (\beta_{1,\lambda}(x), \ldots, \beta_{j-k+1,\lambda}(x))

= \frac{M_{n+1,\lambda}}{n+1} - B_{n,1}(\beta_{1,\lambda}(x), \ldots, \beta_{n,\lambda}(x)) + \sum_{k=2}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))

= \frac{M_{n+1,\lambda}}{n+1} - B_{n,1}(\beta_{1,\lambda}(x), \ldots, \beta_{n,\lambda}(x)) + \sum_{k=2}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))

= \frac{M_{n+1,\lambda}}{n+1} - B_{n,1}(\beta_{1,\lambda}(x), \ldots, \beta_{n,\lambda}(x)) + \sum_{k=2}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))

Thus, by \((22)\), we get

\begin{equation}
\beta_{n,\lambda}(x) = (x+1)_{n,\lambda} - \frac{M_{n+1,\lambda}}{n+1} + \sum_{k=2}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))
\end{equation}

Therefore, by \((24)\), we obtain the following theorem.

**Theorem 3.** For \(n \in \mathbb{N}\), we have

\begin{equation}
\beta_{n,\lambda}(x) = (x+1)_{n,\lambda} - \frac{M_{n+1,\lambda}}{n+1} + \sum_{k=2}^{n} (-1)^{k} k! B_{n,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{n-k+1,\lambda}(x))
\end{equation}

\begin{equation}
+ \sum_{j=1}^{n-1} \sum_{k=1}^{j} \binom{n-1}{j} \sum_{k=1}^{j} (x+1)_{n-j,\lambda} (-1)^{k} k! B_{j,k} (\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \ldots, \beta_{j-k+1,\lambda}(x)).
\end{equation}

3. Conclusion

Study of various degenerate versions of some special numbers and polynomials, which began with the papers [3,4] by Carlitz, regained recent interests of some mathematicians and led to unexpected introduction of degenerate gamma functions and degenerate umbral calculus (see [9,14]).

In this paper, the dimorphic Mersenne numbers were introduced as a degenerate version of the Mersenne numbers and some of their properties was investigated in connection with the degenerate Bernoulli polynomials and the incomplete Bell polynomials.

It is one of our future research projects to continue to study various degenerate versions of some special numbers and polynomials and to find their applications to physics, science and engineering as well as to mathematics.
REFERENCES

[1] Araci, S. A new class of Bernoulli polynomials attached to polyexponential functions and related identities. Adv. Stud. Contemp. Math. (Kyungshang) 31 (2021), No. 2, 195–204.
[2] Burton, D. M. Elementary number theory. Second edition. W. C. Brown Publishers, Dubuque, IA, 1989. xviii+450 pp. ISBN: 0-697-05919-7
[3] Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15 (1979), 51–88.
[4] Carlitz, L. A degenerate Staudt-Clausen theorem. Arch. Math. (Basel) 7 (1956), 28–33.
[5] Comtet, L. Advanced combinatorics. The art of finite and infinite expansions. Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974. xi+343 pp. ISBN: 90-277-0441-4-05-02
[6] Goubui, M. New recurrence relations for Bernoulli polynomials and numbers (preprint), 2022.
[7] Kim, B. M.; Kim, Y.; Park, J.-W. On the reciprocal degenerate Lah-Bell polynomials and numbers. Adv. Stud. Contemp. Math. (Kyungshang) 32 (2022), no. 1, 63–70.
[8] Kim, D. S.; Kim, T. A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. 27 (2020), no. 2, 227–235.
[9] Kim, D. S.; Kim, T. Degenerate Sheffer sequences and λ-Sheffer sequences. J. Math. Anal. Appl. 493 (2021), no. 1, 124521, 21 pp.
[10] Kim, D. S.; Kim, T.; Lee, S.-H.; Park, J.-W. Some new formulas of complete and incomplete degenerate Bell polynomials. Adv. Difference Equ. 2021, Paper No. 326, 10 pp.
[11] Kim, H. K. The incomplete and complete r-central Lah-Bell polynomials. Proc. Jangjeon Math. Soc. 25 (2022), no. 1, 95–111.
[12] Kim, T. Degenerate complete Bell polynomials and numbers. Proc. Jangjeon Math. Soc. 20 (2017), no. 4, 533–543.
[13] Kim, T.; Kim, D. S. On some degenerate differential and degenerate difference operators. Russ. J. Math. Phys. 29 (2022), no. 1, 37–46.
[14] Kim, T.; Kim, D. S. Degenerate Laplace transform and degenerate gamma function. Russ. J. Math. Phys. 24 (2017), no. 2, 241–248.
[15] Kim, T.; Kim, D. S.; Jang, L.-C.; Lee, H.; Kim, H.-Y. Complete and incomplete Bell polynomials associated with Lah-Bell numbers and polynomials. Adv. Difference Equ. 2021, Paper No. 101, 12 pp.
[16] Kim, T.; Kim, D. S.; Kwon, J.; Lee, H. Some identities involving degenerate r-Stirling numbers. Proc. Jangjeon Math. Soc. 25 (2022), no. 2, 245–252.
[17] Kwon, J.; Kim, T.; Kim, D. S.; Kim, H. Y. Some identities for degenerate complete and incomplete r-Bell polynomials. J. Inequal. Appl. 2020, Paper No. 23, 9 pp.
[18] Ma, M.; Lim, D. Some identities on the fully degenerate Bell polynomials of the second kind. Adv. Stud. Contemp. Math.(Kyungshang) 30 (2020), no. 1, 145–154.
[19] Simsek, Y. Construction of generalized Leibnitz type numbers and their properties. Adv. Stud. Contemp. Math. (Kyungshang) 31 (2021), no. 3, 311–323.

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