Quantum Groverian geodesic paths with gravitational and thermal analogies

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Abstract We present a unifying variational calculus derivation of Groverian geodesics for both quantum state vectors and quantum probability amplitudes. In the first case, we show that horizontal affinely parametrized geodesic paths on the Hilbert space of normalized vectors emerge from the minimization of the length specified by the Fubini–Study metric on the manifold of Hilbert space rays. In the second case, we demonstrate that geodesic paths for probability amplitudes arise by minimizing the length expressed in terms of the Fisher information. In both derivations, we find that geodesic equations are described by simple harmonic oscillators (SHOs). However, while in the first derivation the frequency of oscillations is proportional to the (constant) energy dispersion $\Delta E$ of the Hamiltonian system; in the second derivation the frequency of oscillations is proportional to the square root $\sqrt{F}$ of the (constant) Fisher information. Interestingly, by setting these two frequencies equal to each other, we recover the well-known Anandan–Aharonov relation linking the squared speed of evolution of an Hamiltonian system with its energy dispersion. Finally, upon transitioning away from the quantum setting, we discuss the universality of the emergence of geodesic motion of SHO type in the presence of conserved quantities by analyzing two specific phenomena of gravitational and thermodynamical origin, respectively.

1 Introduction

The concept of Fisher information plays a key role in both physics [1] and information theory [2]. Methods of Fisher information have been widely employed for both classical and quantum physical systems [3]. The increasing importance of the concept of Fisher information in both statistical physics and quantum computing was recently pointed out in [4]. In statistical physics for instance, the application of Fisher information in the kinetic theory of gases is specified by its decrease along the solutions of the Boltzmann equation for Maxwellian molecules in the two-dimensional case [5]. In quantum physics, for example, the output state in Grover’s quantum search algorithm follows a geodesic path emerging from the Fubini–Study metric on the manifold of Hilbert-space rays [6–8].

In Ref. [4], the authors presented an information geometric characterization of the oscillatory or monotonic behavior of statistically parametrized squared probability amplitudes...
originating from special functional forms of the Fisher information function: constant, exponential decay, and power-law decay. Furthermore, for each case, the authors computed both the computational speed and the availability loss of the corresponding physical processes by exploiting a convenient Riemannian geometrization of useful thermodynamical concepts. In Ref. [4], the authors also commented on the possibility of using the proposed methods of information geometry to help identify a suitable trade-off between speed and thermodynamic efficiency in quantum search algorithms. The authors remarked that a deeper understanding of the connection between the Fisher information and the schedule of the quantum algorithm remained to be discovered in order to provide a rigorous mapping between our information geometric analysis and the Hamiltonian formulation of the quantum search problem. In particular, it remained an open problem to describe in an exact quantitative manner how the speed at which the search Hamiltonian can drive the system toward the target state is related to both the functional forms of the schedule and the Fisher information. Fortunately, the comprehension of the connection between the Fisher information and the schedule of a quantum algorithm has been highly enhanced in Refs. [9–11]. More specifically, a detailed investigation concerning the physical connection between quantum search Hamiltonians and exactly solvable time-dependent two-level quantum systems is presented in Ref. [9]. In the work shown in Ref. [9], the transition probabilities from a source state to a target state in a number of physical scenarios specified by a spin-1/2 particle immersed in an external time-dependent magnetic field were computed in an exact analytical manner. Both the periodic oscillatory and the monotonic temporal behaviors of such transition probabilities were analyzed, and their analogy with characteristic features of Grover-like and fixed-point quantum search algorithms was explored, respectively. Finally, the connection between the schedule of a search algorithm in both adiabatic and non-adiabatic quantum mechanical evolutions, and the control fields in a time-dependent driving Hamiltonian was explored. In Ref. [12], motivated by the lack of any comparative thermodynamical analysis of quantum search algorithms and building on the previous works presented in Refs. [4,9,10], the authors borrowed the idea of Riemannian geometrization of the concepts of efficiency and speed within both quantum and thermodynamical settings in order to provide a theoretical perspective on the trade-off between speed and efficiency in terms of minimum entropy production paths emerging from quantum mechanical evolutions. Specifically, they presented an information geometric analysis of entropic speeds and entropy production rates in geodesic evolution on statistical manifolds of parametrized quantum states arising as outputs of su(2; C) Hamiltonian models mimicking different types of continuous-time quantum search evolutions. Interestingly, building upon the work presented in Ref. [12], the same authors have recently presented an information geometric analysis of off-resonance effects on classes of exactly solvable generalized semiclassical Rabi systems in Ref. [13].

After a serious reconsideration of the underlying geometrical structure of our previously mentioned works in Refs. [4,8–13], we have arrived at the conclusion that it is the effective and serious exploitation of the link between quantum search algorithms and geodesics in the complex projective Hilbert space that serves as the essential ingredient giving rise to an increasing number of intriguing interdisciplinary investigations in the literature connecting concepts from information geometry, quantum computing, and thermodynamics. For that reason, we propose to reconsider in this article the mathematical derivation of such an important mathematical link that provides an ever-increasing number of penetrating physical insights.

In this article, we provide a unifying variational calculus computation of Groverian geodesics for both quantum state vectors and quantum probability amplitudes. In the first case, following the works in Refs. [14–16], we verify that horizontal affinely parametrized geodesic paths on the Hilbert space of normalized vectors can be obtained from the mini-
mization of the length specified by the Fubini–Study metric on the manifold of Hilbert space rays. In the second case, inspired by the works in Refs. [6,17,18], we show that geodesic paths for probability amplitudes emerge by minimizing the length specified by means of the classical Fisher information $F$. For the sake of completeness, we point out that the fully quantum scenario arises when phase factors are also considered. In such a case, quantum geodesics show a behavior different from that of geodesics emerging from the classical Fisher information [19]. For both derivations, we observe that geodesic equations describe simple harmonic oscillators (SHOs) with distinct frequencies. Specifically, in the first derivation the frequency of oscillations is proportional to the (constant) energy dispersion $\Delta E$ of the Hamiltonian system. In the second derivation, by contrast, the frequency of oscillations is proportional to the square root $\sqrt{F}$ of the (constant) Fisher information. As a pleasant result, we find that upon equating these two frequencies, we retrieve the well-known Anandan–Aharonov relation (for instance, see Ref. [20]) linking the squared speed $v_H^2$ of evolution of an Hamiltonian (H) system with its energy dispersion $\Delta E$ together with the neat link between speed of evolution $v_H$ and Fisher information $F$, namely

$$\frac{\Delta E^2}{\hbar^2} = v_H^2 = \frac{F}{4},$$

with $\hbar \equiv h/(2\pi)$ being the reduced Planck constant. Finally, moving away from the quantum mechanical setting, we elaborate on the universality of the emergence of geodesic motion of SHO type in the presence of conserved quantities by discussing two specific phenomena of gravitational and thermodynamical flavor, respectively.

The layout of the remainder of this article is as follows: In Sect. 2, we briefly summarize the essential features of Grover’s quantum search algorithm [21]. In particular, we focus on its discrete output quantum state after $k$-iterations and its continuous version in terms of a parametric quantum state vector. In Sect. 3, we discuss two distinct variational calculus derivations yielding Grover-like (Groverian) geodesics. In the first computation, we find the most general parametrization of an horizontal geodesic path in the space of unit rays, assuming that the energy dispersion $\Delta E$ of the systems is constant. In the second computation, using the fact that the Fisher information $F$ for a Groverian probability path is constant, we derive the most general parametrization for the quantum probability amplitudes specifying the quantum state vector tracing the geodesic path. Both geodesic equations describe a simple harmonic oscillator (SHO). Moreover, upon equating the frequencies of the two oscillators, we recover (as pointed out earlier) the Anandan–Aharonov speed–energy relation. In Sect. 4, upon transitioning away from quantum physics, we move to general relativity and thermodynamics. We focus on two physical scenarios where the geodesic motion becomes of SHO type under specific physical conditions. In the gravitational [22] and thermal [23] settings, the frequencies of the SHOs are expressed in terms of the mass density $\rho$ of an ideal liquid and the adiabatic coefficient $\gamma$ of an ideal gas, respectively. Our discussion and final remarks appear in Sect. 5. Finally, some more technical details can be found in Appendices A, B, and C.

2 Grover’s quantum search algorithm

In this section, we present a brief summary of Grover’s quantum search protocol [21]. The searching problem addressed by Grover’s algorithm may be re-stated as follows: assume we seek to retrieve a specific database entry subject to some previously specified condition,
provided the entry in question belongs to an unsorted database (i.e., oracle) containing \( N = 2^n \) elements with \( N \) denoting the dimensionality of the complex Hilbert space \( \mathcal{H} \) with \( n \)-qubit quantum states. One step is required to determine whether the entry that has been examined is the one satisfying the given condition. We assume further that finding the selected entry is not aided by sorting of the database. In a situation such as that described above, the maximally efficient classical algorithm capable of implementing the search scheme necessarily requires examination of database elements one at a time. Thus, if a classical computer is utilized to carry out the searching protocol, then the oracle must be queried on average \( \frac{N}{2} \) times (i.e., \( \mathcal{O}(N) \) classical steps). By utilizing the same hardware as in the classical case however, but requiring the input and output be in a superposition of states, Grover developed a quantum algorithm capable of implementing this searching problem in approximately \( \frac{\pi}{4} \sqrt{N} \) steps (i.e., \( \mathcal{O}(\sqrt{N}) \) quantum mechanical steps) \([21]\).

When considering the case of \( n \)-qubit quantum states, the construction of Grover’s search algorithm can be described as follows \([24]\). The initialization (i.e., step-0) of Grover’s algorithm commences with an application of the Hadamard transform for the purpose of constructing an initial state with uniform amplitude. This initial state is comprised of an equal superposition of all orthonormal computational basis states \( \{|w\rangle\} \) belonging to the \( N \)-dimensional Hilbert space,

\[
|s\rangle \equiv \frac{1}{\sqrt{N}} \sum_{w=1}^{N} |w\rangle = \sin \left( \frac{\varphi}{2} \right) |\bar{w}\rangle + \cos \left( \frac{\varphi}{2} \right) |w_\perp\rangle .
\]  

The state \( |w_\perp\rangle \) in Eq. (2) is defined as,

\[
|w_\perp\rangle \equiv \sqrt{\frac{1}{N-1}} \sum_{w \neq \bar{w}} |w\rangle
\]

while the angle \( \varphi \) quantifies the overlap between the (source) state \( |s\rangle \) and the (target) state \( |\bar{w}\rangle \) and is given by

\[
\sin \left( \frac{\varphi}{2} \right) \equiv \frac{1}{\sqrt{N}} = \langle \bar{w}|s\rangle .
\]

For example, given an initial input, a single iteration of Grover’s algorithm induces a rotation by angle \( \varphi \) in the two-dimensional space spanned by states \( |w_\perp\rangle \) and \( |\bar{w}\rangle \). After \( k \)-iterations, the algorithm arrives at the state \( |\psi_{\text{Grover}}(k)\rangle \) written as

\[
|\psi_{\text{Grover}}(k)\rangle \equiv G^k |s\rangle = \sin \left[ \left( k + \frac{1}{2} \right) \varphi \right] |\bar{w}\rangle + \cos \left[ \left( k + \frac{1}{2} \right) \varphi \right] |w_\perp\rangle ,
\]

with \( G \) denoting the so-called Grover iterate \([24]\). In the limit where \( N \gg 1 \), the number of iterations \( \tilde{k} \) for which \( |\psi_{\text{Grover}}(\tilde{k})\rangle \) is identical to the target state \( |\bar{w}\rangle \) (i.e., when the algorithm achieves success probability equal to one) is approximated by

\[
\tilde{k} \approx \frac{1}{N} \frac{\pi}{4} \sqrt{N} .
\]

Equation (6) is obtained by requiring \( (\tilde{k} + \frac{1}{2}) \varphi = \frac{\pi}{2} \) and by recognizing that when \( N \gg 1 \), Eq. (3) implies \( \varphi \approx 2/\sqrt{N} \). We observe that Grover’s search algorithm evolves with discrete \( k \). However, the temporal interval between two consecutive discrete steps \( \Delta t \equiv t_{k+1} - t_k = \varphi/2 \approx N^{-1/2} \) become infinitesimally small when \( N \) assumes sufficiently large values \([7]\). Thus, in this particular limiting scenario, we can approximately identify \( (k + 1/2) \varphi \) with a continuous parameter \( \theta \) in such a manner that no skipping occurs along an hypothetical
geodesic motion. Specifically, the output state (5) can be well approximated by a state vector \(|\psi_{\text{Grover}}(\theta)\rangle\) that depends upon a continuous parameter \(\theta\) in the limit where \(N \gg 1\), namely

\[
|\psi_{\text{Grover}}(\theta)\rangle \equiv \sum_{m=1}^{N} \sqrt{p_m(\theta)} |m\rangle,
\]

where

\[
|m|m'\rangle = \delta_{mm'}, p_1(\theta) \equiv \sin^2 \theta \text{ and, } p_l(\theta) \equiv \frac{\cos^2 \theta}{N-1} \text{ with } l \neq 1.
\] (8)

Indeed, the \(N\)-dimensional vector of probability distributions \(\mathbf{p} \equiv (p_1(\theta), p_2(\theta), \ldots, p_N(\theta))\) with \(p_j(\theta)\) defined in (8) can be interpreted as a path induced by Grover’s search algorithm on some suitable probability manifold. In the second part of the following section, we demonstrate that such a probability path is in fact a geodesic curve for which the Fisher information action functional \(S[p_m(\theta)] \equiv \frac{1}{2} \int \sqrt{\mathcal{F}(\theta)} \, d\theta\) with the Fisher information function \(\mathcal{F}(\theta)\) defined as [1],

\[
\mathcal{F}(\theta) \equiv 4 \sum_{m=1}^{N} \left( \frac{\partial \sqrt{p_m}}{\partial \theta} \right)^2,
\]

is extremized. For further details on Grover’s quantum search algorithm and Fisher’s information, we refer to Ref. [24] and Ref. [1], respectively.

3 Groverian geodesics

In this section, we present two distinct variational calculus derivations yielding Groverian geodesics. In the first derivation, assuming that the energy dispersion \(\Delta E\) of the systems is constant, we obtain the most general parametrization of an horizontal geodesic path in the space of unit rays. In the second derivation, exploiting the fact that the Fisher information \(\mathcal{F}\) for a Groverian probability path is constant \([6,17,18]\), we obtain the most general parametrization for the quantum probability amplitudes specifying the quantum state vector tracing the geodesic path. We emphasize that technical details on geodesics in the space of unit rays with special focus on horizontal affinely parametrized geodesics appear in “Appendix A.”

3.1 Geodesic paths for quantum state vectors

Geodesic paths \(\gamma_{\text{geo}}(s)\) with \(s_1 \leq s \leq s_2\) in the Hilbert space rays are those for which the length functional \(\mathcal{L}[\gamma_{\text{geo}}]\) [14],

\[
\mathcal{L}[\gamma_{\text{geo}}] \equiv \int_{s_1}^{s_2} \sqrt{dS_{\text{FS}}^2} = \int_{s_1}^{s_2} \langle u_{\perp}(s) | u_{\perp}(s) \rangle^{1/2} \, ds,
\]

is stationary. Note that \(dS_{\text{FS}}^2\) is the Fubini–Study metric on the projective Hilbert space \(\mathcal{P}(\mathcal{H}) \simeq \mathbb{C}P^{N-1}\) with \(\mathcal{H} \simeq \mathbb{C}^{N}\) defined as \([25,26],\)

\[
dS_{\text{FS}}^2 \equiv \langle u_{\perp}(s) | u_{\perp}(s) \rangle \, ds^2 = \langle \dot{\psi}_{\perp}(s) | \dot{\psi}_{\perp}(s) \rangle \, ds^2,
\]

where \(|u_{\perp}(s)\rangle \equiv |\dot{\psi}_{\perp}(s)\rangle\) with \(\dot{\psi}_{\perp}(s) \equiv \partial_s \psi_{\perp}\) and \(\partial_s \equiv \partial / \partial s\). Furthermore, \(|d\psi_{\perp}\rangle \equiv |d\psi\rangle - \langle \psi | d\psi \rangle |\psi\rangle\) is the projection of \(|d\psi\rangle\) orthogonal to \(|\psi\rangle\). We remark that \(|d\psi\rangle \equiv \)
\[ |\psi' \rangle - |\psi \rangle \] is the difference between the two neighboring normalized pure states \( |\psi \rangle \) and \( |\psi' \rangle \). Stationarity of the length functional requires that \( \delta \mathcal{L} \left[ \gamma_{\text{geo}} \right] = 0 \) for arbitrary variations \( \delta |\psi \rangle \), subject only to the constraint equation \( \text{Re} \{ \langle \delta |\psi \rangle |\psi \rangle \} = 0 \). By carrying out a variational calculation, it is possible to show that when the uncertainty \( \Delta E (t) \) in the energy,

\[
\Delta E (t) \overset{\text{def}}{=} \left[ \langle \psi (t) |H^2 (t) |\psi (t) \rangle - \langle \psi (t) |H (t) |\psi (t) \rangle \right]^{1/2}, \tag{12}
\]
does not depend on time, the most general equation in horizontal and affinely parametrized form \( |\psi_h (s) \rangle \) is given by a simple harmonic oscillator equation,

\[
\frac{d^2}{ds^2} |\psi_h (s) \rangle + v_H^2 |\psi_h (s) \rangle = 0. \tag{13}
\]

It can be shown that Eq. (13) emerges from a variational calculus computation where one seeks to extremize the length functional \( \mathcal{L} \left[ \gamma_{\text{geo}} \right] \) and, at the same time, exploits both the gauge invariance and the reparametrization invariance of such a functional. Then, \( |\psi_h (s) \rangle \) in Eq. (13) denotes the most general geodesic in horizontal and affinely parametrized form [14]. In particular, while gauge invariance leads to the so-called horizontality condition, reparametrization invariance motivates affine parametrizations. An explicit computation with all technical details yielding Eq. (13) together with its most general solution \( |\psi_h (s) \rangle \) appears in “Appendix A.” The generally \( s \)-dependent quantity \( v_H (s) \) in Eq. (13),

\[
v_H (s) \overset{\text{def}}{=} \left[ \langle \psi_h (s) |\dot{\psi}_h (s) \rangle \right]^{1/2} = \frac{\Delta E (s)}{\hbar}, \tag{14}
\]
denotes the speed of transportation of the parallel transported horizontal state vector \( |\psi_h (s) \rangle \) in the Hilbert space of normalized vectors satisfying the parallel transport rule, \( \langle \psi_h (s) |\psi_h (s) \rangle = 0 \). For the sake of completeness, we point out that the state vector \( |\psi_h (s) \rangle \) is connected to the dynamical state vector \( |\psi (s) \rangle \) satisfying the Schrödinger evolution equation by the relation \( [15,16], \)

\[
|\psi_h (s) \rangle \overset{\text{def}}{=} \exp \left[ \left( \frac{i}{\hbar} \int_0^s \langle \psi (s') |H (s') |\psi (s') \rangle \, ds' \right) \right] |\psi (s) \rangle. \tag{15}
\]

Finally, assuming that \( \langle \psi_h (0) |\psi_h (0) \rangle = 1 \), \( \langle \psi_h (0) |\dot{\psi}_h (0) \rangle = 0 \), and \( \langle \dot{\psi}_h (0) |\dot{\psi}_h (0) \rangle = v_H^2 = \text{constant} \), the general solution to Eq. (13) becomes

\[
|\psi_h (s) \rangle = \cos (v_H s) |\psi_h (0) \rangle + \frac{\sin (v_H s)}{v_H} |\dot{\psi}_h (0) \rangle, \tag{16}
\]

where \( v_H \) in Eq. (16) is defined as,

\[
v_H \overset{\text{def}}{=} \frac{\Delta E}{\hbar} = \text{constant}. \tag{17}
\]

For the sake of completeness, we remark that the information encoded into the three conditions right before Eq. (16) determines completely the curve \( s \mapsto |\psi_h (s) \rangle \). In particular, these conditions specify that the curve is traced by starting on the sphere \( \langle \psi_h (0) |\psi_h (0) \rangle = 1 \) with constant velocity \( v_H \) \( \langle \dot{\psi}_h (0) |\dot{\psi}_h (0) \rangle = v_H^2 = \text{constant} \) in a direction tangent to the sphere \( \langle \dot{\psi}_h (0) |\dot{\psi}_h (0) \rangle = 0 \). From Eq. (16), we conclude that the horizontal geodesic can be geometrically interpreted as a real two-dimensional rotation on the plane spanned by the state vectors \( |\psi_h (0) \rangle \) and \( |\dot{\psi}_h (0) \rangle \). It therefore follows that we can view the transition probability \( P_{|\psi_h (0) \rangle \rightarrow |\psi_h (s) \rangle} (s) \) from \( |\psi_h (0) \rangle \) to \( |\psi_h (s) \rangle \),

\[
P_{|\psi_h (0) \rangle \rightarrow |\psi_h (s) \rangle} (s) \overset{\text{def}}{=} |\langle \psi_h (s) |\psi_h (0) \rangle|^2, \tag{18}
\]
in terms of the distance $s$ along the geodesic joining $|\psi_h(0)\rangle$ and $|\psi_h(s)\rangle$. The transition from the digital (discrete) to the analog (continuous time) evolution of Grover’s quantum search algorithm occurs in the limit of $N \gg 1$ where the interval of skip $\Delta k/k = [(k+1/2)\varphi - \varphi/2]/k = \varphi \approx 2/\sqrt{N}$ between two consecutive steps becomes infinitesimally small. In such a limit, the Riemannian geometric formulation of quantum mechanics suggests that Grover’s dynamics yields the shortest paths in $\mathbb{CP}^{N-1}$. Indeed, upon setting $v_H = 1$, Groverian geodesics $|\psi_{\text{Grover}}(s)\rangle \overset{\text{def}}{=} \cos(s)|w_\perp\rangle + \sin(s)|w\rangle$ can be formally obtained from Eq. (16) by taking $|\psi_h(0)\rangle = |w_\perp\rangle$, $|\psi_h(0)\rangle = |w\rangle$, and $s = (k+1/2)\varphi$.

In the following subsection, we obtain the most general parametrization for the quantum probability amplitudes specifying the quantum state vector tracing a quantum Groverian geodesic path.

### 3.2 Geodesic paths for quantum probability amplitudes

The natural notion of distance between two neighboring pure physical states, the so-called angle in Hilbert space, is specified by the Fubini–Study metric. Except for a constant factor, this is the only Riemannian metric on the set of rays in Hilbert space which is invariant under all unitary transformations. The probabilistic nature of quantum mechanics in its geometric formulation can be made more transparent by considering the equivalence between the angle in Hilbert space and the statistical distance introduced by Wootters to statistically distinguish between two different rays in the same Hilbert space [27]. Within this statistical geometric framework of quantum mechanics, to a greater distance between two neighboring pure states $|\psi\rangle + |d\psi\rangle$ and $|\psi\rangle$ [26],

$$
|\tilde{\psi}\rangle \overset{\text{def}}{=} |\psi\rangle + |d\psi\rangle = \sum_{m=1}^{N} \sqrt{p_m + dp_m e^{i(\phi_m + d\phi_m)}} |m\rangle, \quad \text{and} \quad |\psi\rangle \overset{\text{def}}{=} \sum_{m=1}^{N} \sqrt{p_m e^{i\phi_m}} |m\rangle,
$$

(19)

there corresponds a higher degree of distinguishability of the two states. Here $\{|m\rangle\}_m \in \{1,\ldots, N\}$ is an orthonormal basis of the $N$-dimensional complex Hilbert space $\mathcal{H}$. By using Eq. (19) and the Fubini–Study metric, $d_{FS}^2$ in Eq. (11) reads as follows [26],

$$
d_{FS}^2 \overset{\text{def}}{=} \langle d\psi_\perp | d\psi_\perp \rangle = 1 - |\langle \tilde{\psi} | \psi \rangle|^2,
$$

(20)

with $|d\psi_\perp\rangle \overset{\text{def}}{=} |d\psi\rangle - |\psi\rangle \langle \psi | d\psi \rangle$. Now consider that both, $|\tilde{\psi}\rangle$ and $|\psi\rangle$, are parametrized by a family $M = \{\theta = (\theta^1, \ldots, \theta^n)\} \subset \mathbb{R}^n$, i.e., $p_m = p_m(\theta)$ and $\phi_m = \phi_m(\theta)$. Assuming that the parametrization is one to one, the representation given in Eq. (19) provides an embedding of the family $M$ into the Hilbert space of rays. Therefore, the Fubini–Study metric (11) can be pulled-back to $M$ to obtain the following expression [19]:

$$
\mathcal{F}_q(\theta) = \frac{1}{4} \left\{ \sum_{m=1}^{N} \frac{(dp_m)^2}{p_m} + 4 \left[ \sum_{m=1}^{N} p_m (d\phi_m)^2 - \left( \sum_{m=1}^{N} p_m d\phi_m \right)^2 \right] \right\},
$$

(21)

which is the quantum version of the Fisher information [28]. Obviously, if $M \subset \mathbb{R}$ the expression above becomes

$$
\mathcal{F}_q(\theta) = \frac{1}{4} \left\{ \sum_{m=1}^{N} \frac{(\dot{p}_m)^2}{p_m} + 4 \left[ \sum_{m=1}^{N} p_m (\dot{\phi}_m)^2 - \left( \sum_{m=1}^{N} p_m \dot{\phi}_m \right)^2 \right] \right\} d\theta^2,
$$

(22)
with \( \dot{p}_m \equiv d p_m / d \theta \) and \( \dot{\phi}_m \equiv d \phi_m / d \theta \). We emphasize that it is always possible to assume the variance of phase changes \( \sigma^2_\phi \),

\[
\sigma^2_\phi \overset{\text{def}}{=} \sum_{m=1}^{N} p_m \dot{\phi}_m^2 - \left( \sum_{m=1}^{N} p_m \dot{\phi}_m \right)^2,
\]

(23)
to be equal to zero upon selecting an appropriate choice of basis \( \{|m\} \) [26]. In what follows, we assume to be reasoning with Eq. (22) under such a working condition (for further details on this particular matter, see Refs. [4,13]). Then, geodesic probability paths \( \gamma_{\text{geo}}(\theta) \) with \( \theta_1 \leq \theta \leq \theta_2 \) in the probability space associated with the complex projective Hilbert space \( \mathbb{C}P^{N-1} \) are determined by minimization of the action \( S\left[ p_m(\theta) \right] \),

\[
S\left[ p_m(\theta) \right] = \int \mathcal{L} \left( \dot{p}_m(\theta), p_m(\theta), \theta \right) d\theta,
\]

(24)
where the Lagrangian-like quantity \( \mathcal{L} \left( \dot{p}_m(\theta), p_m(\theta), \theta \right) \) is given by,

\[
\mathcal{L} \left( \dot{p}_m(\theta), p_m(\theta), \theta \right) \overset{\text{def}}{=} \frac{1}{2} \sqrt{\mathcal{F}(\theta)} = \frac{1}{2} \left[ \sum_{m=1}^{N} \dot{p}_m^2(\theta) / p_m(\theta) \right]^{1/2},
\]

(25)
subject to the normalization constraint on the parametrized probability distribution functions \( p_m(\theta) \),

\[
\sum_{m=1}^{N} p_m(\theta) = 1.
\]

(26)
Note that the Fisher information \( \mathcal{F} \) entering Eq. (26) is the classical one. This should come as no surprise since requiring (23) amounts to neglect the phase factor from our analysis. For the ease of analysis, we consider the change of variable \( p_m(\theta) \rightarrow q_m^2(\theta) \) [27]. Then by making the use of the method of Lagrange multipliers with the new variable \( q_m(\theta) \), we seek to minimize the new action \( S'[q_m(\theta)] \)

\[
S'[q_m(\theta)] = \int \mathcal{L}' \left( \dot{q}_m(\theta), q_m(\theta), \theta \right) d\theta
\]

\[
= \int \left\{ \left[ \sum_{m=1}^{N} q_m^2(\theta) \right]^{1/2} - \lambda_{\text{FS}} \left( \sum_{m=1}^{N} q_m^2(\theta) - 1 \right) \right\} d\theta.
\]

(27)
In Eq. (27), \( \lambda_{\text{FS}} \) is a Lagrange multiplier and \( \mathcal{L}' \left( \dot{q}_m(\theta), q_m(\theta), \theta \right) \) acts as a new Lagrangian-like quantity. The path that serves to minimize the action \( S'[q_m(\theta)] \) satisfies the “actuality constraint”,

\[
\delta S'[q_m(\theta)] / \delta q_m(\theta) = 0,
\]

(28)
leading to the following Euler–Lagrange (EL) equation in \( q_m = q_m(\theta) \), namely

\[
\ddot{q}_m - \frac{\dot{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)} \dot{q}_m + 2 \lambda_{\text{FS}} \mathcal{L}(\theta) q_m = 0,
\]

(29)
where $\mathcal{L} (\theta) \overset{\text{def}}{=} \mathcal{L} (\dot{q}_m (\theta), p_m (\theta), \theta)$ appears in Eq. (25) while $\dot{\mathcal{L}} (\theta) \overset{\text{def}}{=} d\mathcal{L}/d\theta$. Given that $\mathcal{F} (\theta) = 4\mathcal{L}^2 (\theta)$, Eq. (29) can be recast as

$$\ddot{q}_m - \frac{1}{2} \frac{\dot{\mathcal{F}} (\theta)}{\mathcal{F} (\theta)} \dot{q}_m + \lambda_{FS} \mathcal{F}^{1/2} (\theta) q_m = 0. \quad (30)$$

Assuming $\mathcal{F} (\theta) = \mathcal{F}_0$ is constant, Eq. (30) reduces $\ddot{q}_m + \lambda_{FS} \mathcal{F}^{1/2} (\theta) q_m = 0$. The Lagrange multiplier $\lambda_{FS}$ is fixed by satisfying conservation of probability in Grover’s dynamics. Recalling that $\mathcal{F}_0 = 4$ in Grover’s dynamics, this constraint demands that $\lambda_{FS}$ satisfies the condition $\lambda_{FS} \mathcal{F}_0^{1/2} = \mathcal{F}_0/4$. Thus, $\lambda_{FS} = \mathcal{F}_0^{1/2}/4$ and Eq. (30) becomes a simple harmonic oscillator equation,

$$\frac{d^2}{d\theta^2} q_m (\theta) + \frac{\mathcal{F}_0}{4} q_m (\theta) = 0. \quad (31)$$

The most general solution $q_m = q_m (\theta)$ of Eq. (31) is,

$$q_m (\theta) = \cos (v_F \theta) q_m (0) + \frac{\sin (v_F \theta)}{v_F} \dot{q}_m (0), \quad (32)$$

with $v_F$ in Eq. (32) defined as,

$$v_F \overset{\text{def}}{=} \mathcal{F}_0^{1/2}/2. \quad (33)$$

Finally, the Groverian probability path vector $p \overset{\text{def}}{=} q \cdot q$ with $q \overset{\text{def}}{=} (q_1, \ldots, q_N)$ can be obtained by imposing the constraints $q_{\bar{w}} (0) = 0$ and $\dot{q}_{\bar{w}} (0) = 1$. These two conditions yield $q_{\bar{w}} (\theta) = \sin (\theta)$ and $q_j (\theta) = \cos (\theta)$ for any $j \neq \bar{w}$ with $1 \leq \bar{w} \leq N$. It therefore follows that the success and failure probabilities for the quantum search become $p_{\bar{w}} (\theta) = \sin^2 (\theta)$ and $p_{w\perp} (\theta) = \cos^2 (\theta)$, respectively.

For the sake of clarity, we emphasize that Groverian geodesic paths are referred to in this article as the continuous-time analogue of the output quantum state emerging from Grover’s algorithm viewed as the horizontal lift of a geodesic in the complex projective Hilbert space. However, in the geometric formulation of adiabatic quantum computation (AQC, [29]), Groverian geodesic paths denote the time-optimal paths on a parameter manifold for time-dependent control parameters that specify the time-dependent interpolating Hamiltonian

$$H [x (t)] = H [1 - x (t), x (t)] [30],$$

where $0 \leq t \leq T$. In the particular case of quantum search by local adiabatic evolution [31], $H_I = H (0) \overset{\text{def}}{=} I - |s\rangle \langle s|$ is the initial Hamiltonian, $H_P = H (T) \overset{\text{def}}{=} I - |w\rangle \langle w|$ is the problem Hamiltonian, and $I$ denotes the identity operator. In order to adiabatically drive the quantum system from the ground state $|s\rangle$ of $H_I$ (that is, the source state) to a final state $|w\rangle$ that is close to the ground state of $H_P$ (that is, the target state) in the shortest possible time, the optimal trajectory of the control parameter $x (t)$ is given by a Groverian geodesic path [30,32],

$$x (\tau) \overset{\text{def}}{=} \frac{1}{2} - \frac{1}{2\sqrt{N-1}} \tan \left[ (1 - 2\tau) \cos^{-1} \left( \frac{1}{\sqrt{N}} \right) \right]. \quad (35)$$

In Eq. (35), $\tau$ denotes a dimensionless natural parameter $\tau = \tau (t)$ with $\tau (0) = 1$ and $\tau (T) = 1$, while $N$ is the dimensionality of the search space.
4 Harmonic geodesic motion beyond quantum settings

In this section, upon transitioning away from the quantum setting, we explore the motivations behind the emergence of SHO geodesic motions arising in general relativistic and thermodynamical settings. In particular, we shall focus on gravitational and thermal analogues of Eqs. (16) and (32).

4.1 A gravitational example

From a classical Newtonian mechanics perspective, a small mass that vibrates under gravity about the center of a sphere composed of an ideal fluid will exhibit a simple harmonic motion. For instance, consider an hydrometer of mass \( m \) that consists of a cylindrical stem of diameter \( d \) and a spherical bulb of volume \( V_0 \). Assume that the hydrometer is immersed in an ideal fluid of constant density \( \rho \) and the volume of displaced liquid is \( V = V_0 + \pi \left( \frac{d}{2} \right)^2 h \) with \( h \) denoting the height of the portion of the stem immersed in the liquid. At equilibrium, we have \( mg = \rho g \left[ V_0 + \pi \left( \frac{d}{2} \right)^2 h \right] \). When the hydrometer is displaced by an additional distance \( \Delta h \), the system is subject to a net force \( -\pi \rho g \left( \frac{d}{2} \right)^2 \Delta h \). Then, the system moves out of equilibrium and begins floating/oscillating according to the dynamical equation, \( d^2 \Delta h/dt^2 + v_{\text{Newton}}^2 \Delta h = 0 \), where \( v_{\text{Newton}} \) denotes the frequency of oscillation given by,

\[
v_{\text{Newton}} \equiv \left( \frac{\pi (d/2)^2}{m \rho g} \right)^{1/2}.
\]

In what follows, motivated by the Zatzikis analysis in Ref. [22], we show that such a point-like particle undergoes the same type of motion even when the curvature of the space is considered, provided one assumes two specific working conditions [22]: (i) low velocities and (ii) motion limited to be confined near the center of the sphere. The equations of motion of the small mass \( m \) are,

\[
\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0,
\]

with \( \Gamma^a_{bc} \) denoting the Christoffel symbols of the second-kind and \( s \) in Eq. (37) being an affine parameter [33]. For technical details on the possibility of using a non-affine parameter for describing the geodesic motion, we refer to “Appendix B.” For the particular physical scenario being analyzed [22], we consider a stationary physical system that exhibits spherical symmetry. In this case, the line element \( ds^2 \) can be written as

\[
\begin{align*}
&= g_{0}(r) \, dt^2 - g_{1}(r) \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\end{align*}
\]

with \( g_{0}(r) \) and \( g_{1}(r) \) being time-independent quantities and \( x^a \equiv (t, r, \theta, \phi) \). The four geodesic relations in Eq. (37) reduce to two equations since we limit our analysis to linear oscillations and, therefore, set \( \theta = \text{constant} \) and \( \phi = \text{constant} \). We now focus on the interior of a sphere composed of an ideal fluid (liquid) with constant density \( \rho \). Then, we assume that \( g_{0}(r) \) and \( g_{1}(r) \) are given by [34],

\[
g_{0}(r) \equiv \frac{1}{4} c^2 \left[ 3 \cos (\xi_a) - \cos (\xi) \right], \quad \text{and} \quad g_{1}(r) \equiv \frac{1}{\cos^2 (\xi)},
\]

respectively, with \( c \) being the speed of light. We remark that the quantities \( \xi \) and \( \xi_a \) in Eq. (39) are defined as

\[
\xi \equiv \sin^{-1} \left( \frac{r}{R} \right), \quad \text{and} \quad \xi_a \equiv \sin^{-1} \left( \frac{a}{R} \right),
\]
respectively, with $R \equiv c [3/(8\pi G\rho)]^{1/2}$ where $\rho$ is the constant density of the ideal (that is, incompressible) fluid, $G$ is Newton’s universal gravitational constant, and $a$ is the value of $r$ on the surface of the sphere (thus, it represents the maximal value of $r$). By substituting Eqs. (39) and (40) into Eq. (37), recalling that we assumed $\theta =$constant and $\phi =$constant and finally, taking the limit of low velocities ($\dot{\xi}^2 \ll 1$) together with considering dynamical changes near the center of the sphere ($\sin(\theta) \approx \theta$ and $\cos(\theta_a) \approx 1$), the classical equations of motion for $\xi(t)$ become the familiar simple harmonic oscillator equation

$$\frac{d^2\xi}{dt^2} + v_{GR}^2\xi = 0. \tag{41}$$

The most general solution of Eq. (41) can be written as,

$$\xi(t) = \cos(v_{GR}t)\xi(0) + \frac{\sin(v_{GR}t)}{v_{GR}}\dot{\xi}(0), \tag{42}$$

where the angular frequency $v_{GR}$ in Eq. (41) is a constant quantity given by,

$$v_{GR} \equiv \left(\frac{4\pi}{3}G\rho\right)^{1/2}. \tag{43}$$

As a final remark, we point out that the constancy of $v_{GR}$ in Eq. (43) is a consequence of the constancy of $\rho$. The latter property arises due to our consideration of an ideal fluid. Finally, the working assumption of “ideality” enabled us to consider a stationary metric in Eq. (38).

4.2 A thermal example

In what follows, we consider a famous experiment in thermodynamics where a simple harmonic oscillatory motion occurs. Specifically, we take into account the Rüchhardt experiment used to measure the adiabatic coefficient $\gamma$ of an ideal gas [23]. Recall that $\gamma \equiv C_P/C_V$ is the ratio of the heat capacity at constant pressure ($C_P$) to the heat capacity at constant volume ($C_V$). The experiment is described by an adiabatic ($PV^\gamma =$ constant) compression of a mole of an ideal gas ($PV = RT$). The quantities $P$, $V$, and $T$ denote pressure, volume, and temperature, respectively. The quantity $R$ is the universal gas constant. We recall that the relation $PV^\gamma =$constant is a consequence of the first principle of thermodynamics ($dU = -PdV + dQ$) applied to one mole of an ideal gas ($dU = C_VdT$) in the absence of any heat exchange with the external environment ($dQ = 0$) [35]. The quantities $U$ and $Q$ denote energy and heat, respectively.

We remark at this juncture that the Rüchhardt experiment can be explained via the following sequence of steps [23]. A spherical piston of mass $m$ is allowed to fall under uniform gravity in a cylindrical tube of volume $V$ and of cross-section $A$ which is open on one of its end. The piston has a uniform cross-section so as to create an air-tight seal. The gas trapped within the cylindrical tube is adiabatically compressed by the weight of the spherical piston. During this compression, the temperature of the gas increases. Furthermore, as the piston falls, the piston experiences bounce due to the creation of a gas cushion. As a consequence, the piston begins exhibiting simple harmonic oscillations. At equilibrium, the pressure $P_0$ of the trapped gas within the tube equals the sum of the atmospheric pressure ($P_{atm}$) and the pressure ($mg/A$) exerted by the piston on the gas. As the piston moves away from equilibrium by an infinitesimal distance $x$, the pressure changes by $dP$. Then, the Newtonian equation of motion projected along the $x$-axis, $F_x = m\ddot{x} = AdP$, for the piston becomes

$$\frac{d^2x}{dt^2} + v_{th}^2x = 0, \tag{44}$$
where \( x \) denotes the position of the piston. The most general solution of Eq. (44) can be written as,

\[
x(t) = \cos(v_{\text{th}} t) x(0) + \frac{\sin(v_{\text{th}} t)}{v_{\text{th}}} \dot{x}(0),
\]

where the angular frequency of oscillations \( v_{\text{th}} \) in Eq. (44) is a constant quantity given by,

\[
v_{\text{th}} \equiv \left( \frac{P_0 A^2}{m V_0 \gamma} \right)^{1/2},
\]

with \( P_0 \) and \( V_0 \) being the equilibrium pressure and volume, respectively. The adiabatic coefficient \( \gamma \) can be experimentally determined by measuring the period of oscillations of the piston, \( T \) defined as \( T \) \(=\) \( 2\pi/v_{\text{th}} \). We emphasize that \( v_{\text{th}} \) in Eq. (46) is constant since \( \gamma \) is assumed to be constant. In general, for all ideal gases, \( \gamma > 1 \) and depends on the temperature \([36]\). Moreover, for monotonic gases (for instance, He and Ne), \( \gamma \) is constant over a wide range of temperatures. For diatomic (for instance, \( \text{H}_2 \) and \( \text{O}_2 \)) and polyatomic gases (for instance, \( \text{CO}_2 \) and \( \text{NH}_3 \)) however, \( \gamma \) varies with \( T \). In particular, a very large change in temperature can produce a nonnegligible change in \( \gamma \). However, for (quasi-static) adiabatic thermodynamic processes characterized by a small temperature change, the change in \( \gamma \) can be neglected and \( \gamma \) can be regarded as a constant quantity. In the (quasi-static) adiabatic compression considered here, although the temperature of the gas increases, we assume that the overall change in temperature is so small that \( \gamma \) can be essentially considered to be a constant quantity over this temperature range.

As a final observation, we point out that the method of measuring \( \gamma \) developed by Rüchhardt requires only the use of classical Newtonian mechanics. This, in turn, can also be presented in a Riemannian geometric fashion once we consider the Hamiltonian dynamical formulation of Newton’s construction [37]. Indeed, consider a conservative Hamiltonian system specified by an Hamiltonian \( H(p, q) \) defined as \( H(p, q) = p^2/(2m) + V(q) \) with \( p \) and \( q \) being the generalized momentum and coordinate, respectively, while the energy \( E \) is a conserved quantity. Then, its dynamics can be recast in terms of geodesic motion on a Riemannian manifold. Such a manifold is specified by the configuration space of the dynamical system being considered and is equipped with a metric structure defined by a (conformally flat) Jacobi metric tensor \( g_{ab} \) defined as [37],

\[
g_{ab}(q) \equiv 2 [E - V(q)] \delta_{ab},
\]

where \( 1 \leq a, b \leq n_{\text{dof}} \) with \( n_{\text{dof}} \) being equal to the number of degrees of freedom of the dynamical system. The geodesic equations,

\[
\frac{d^2 q^a}{ds^2} + \Gamma^a_{bc} \frac{dq^b}{ds} \frac{dq^c}{ds} = 0,
\]

can be obtained by minimizing the action functional (that is, the length) \( S[q] \),

\[
S[q] \equiv \int \left[ g_{ab}(q) dq^a dq^b \right]^{1/2},
\]

with \( q \equiv (q^1, ..., q^{n_{\text{dof}}}) \) being local coordinates on the curved manifold. For the sake of clarity, we emphasize that \( s \) in Eq. (48) is the arc-length parameter and is related to the physical time \( t \) via the relation \( ds = 2T dt \), with \( T \) being the kinetic energy of the physical system. Finally, the Riemannian geometrization of Rüchhardt classical mechanical description of the measurement yielding the adiabatic coefficient of an ideal gas can be obtained in a straightforward manner once we observe that \( n_{\text{dof}} = 1, q = x \), and the potential \( V(q) \) in Eq.
(47) reduces to the harmonic potential \((1/2)m v_{th}^2 x^2\) with \(v_{th}\) defined in Eq. (46). We refer to “Appendix C” for an explicit derivation of Newton’s equation of motion in Eq. (44) starting from Eq. (48). Finally, for further discussions on the role played by conformally flat Jacobi metrics in physics, we refer to Refs. [38,39].

We have discussed in this section the emergence of simple harmonic motion in gravitational and thermodynamical settings. In general, the link between gravity and thermodynamics is a rather fascinating topic [40]. For a specific perspective on a cosmological model of dark energy using an adiabatic fluid satisfying the relation \(P V^\gamma = \text{constant}\) with a constant adiabatic coefficient \(\gamma\) and evolving according to classical laws of thermodynamics, we refer to Ref. [41].

5 Concluding remarks

In this article, we presented a unifying variational calculus derivation of Groverian geodesics for both the horizontal lift of quantum state vectors (see Eqs. (16) and (17)) and quantum probability amplitudes (see Eqs. (32) and (33)). In the first case, following the Mukunda–Simon work in Ref. [14], we demonstrated that horizontal affinely parametrized lift of geodesic paths on the manifold of Hilbert space rays arise from the minimization of the length specified by the Fubini–Study metric (see Eq. (11)). In the second case, inspired by the Alvarez-Gomez work in Ref. [6], we explicitly illustrated that geodesic paths for probability amplitudes emerge as a consequence of minimizing the length expressed in terms of the Fisher information (see Eq. (21) with \(d\Sigma_{FS}^2 \equiv (1/4)\hat{\mathcal{F}}(\theta) d\theta^2\)). In both derivations, we note that geodesic equations are described by simple harmonic oscillators (SHOs). While in the first derivation, the frequency of oscillations \(v_H\) is proportional to the (constant) energy dispersion \(\Delta E\) of the Hamiltonian system, in the second derivation the frequency of oscillations \(v_F\) is proportional to the square root \(\sqrt{\hat{\mathcal{F}}}\) of the (constant) Fisher information. Interestingly, by equating the two frequencies in Eqs. (17) and (33), we are able to recover the well-known Anandan–Aharonov relation connecting the squared speed \(v_H^2\) of evolution of an Hamiltonian system with its energy dispersion \(\Delta E\), together with the concise link between speed of evolution \(v_H\) and Fisher information \(\mathcal{F}\),

\[
\frac{\Delta E^2}{R_0^2} = v_H^2 = \mathcal{F} = \frac{\pi}{4},
\]

We point out that the emergence of simple harmonic motion in our geometrical investigations of quantum mechanical phenomena is not entirely unexpected. For instance, the normalization conditions linked to the probabilistic nature of quantum mechanics, namely \(|\psi|\psi\rangle = 1\) and \(q \cdot q = 1\), have played an important role in the derivations of geodesic trajectories for state vectors and probability amplitudes, respectively. Stated otherwise, in both cases the trajectories were constrained to be on a spherical surface. In the framework of classical Newtonian mechanics, simple harmonic motion can be shown to emerge from the study of a (free) point-particle of mass \(m\) subject to no external force but constrained to move an a spherical surface of constant radius \(R_0\). The dynamical trajectory \(x(t)\) of such a particle on a sphere defined by the condition \(x \cdot x = R_0^2\) can be obtained from the Euler–Lagrange equations emerging from the constrained Lagrangian \(\mathcal{L}(\dot{x}, x, t) \equiv (m/2) (\dot{x} \cdot \dot{x}) - \Lambda(t)\left[1 - (x \cdot x)/R_0^2\right]\) where \(x = x(t), \dot{x} \equiv dx/dt\), with \(\Lambda(t)\) being a Lagrange multiplier. We observe that although Lagrange multipliers are generally time-dependent quantities [42], we could have scenarios where one deals with space–time-dependent Lagrange multipliers. This happens, for instance, in time-dependent
Table 1  Schematic representation of the emergence of SHO motion from different physics frameworks, diverse metric structures, distinct variables satisfying the geodesic equation, various frequencies, and finally distinctive conserved quantities. The common theme in each and every physical scenario is the presence of a peculiar conserved quantity

| Type of physics        | Type of metric                  | Geodesic equation       | Frequency of SHO     | Conserved quantity          |
|------------------------|---------------------------------|-------------------------|----------------------|----------------------------|
| Quantum                | Fubini–Study                    | State vector            | $\frac{\Delta E}{\hbar}$ | $\Delta E$, Energy dispersion |
| Quantum information    | Fubini–Study                    | Probability amplitude   | $\mathcal{F}^{1/2}/2$ | $\mathcal{F}$, Fisher information |
| Gravitational          | Curved Lorentzian spacetime     | Radial position of particle | $\left(\frac{4\pi G}{3}\rho\right)^{1/2}$ | $\rho$, Ideal liquid mass density |
| Thermal                | Jacobi                          | Position of piston      | $\left(\frac{R_0 A^2}{m V_0 \gamma}\right)^{1/2}$ | $\gamma$, Ideal gas adiabatic coefficient |
information-constrained optimization problems such as those tackled with MaxCal inference algorithms [43,44]. Having said that, the trajectory \( x(t) \) is described by a great circle, that is, a geodesic on the sphere along which the free particle moves with constant speed (that is, the magnitude of the velocity vector). Interestingly, in such a classical mechanical scenario, the Lagrange multiplier \( \Lambda(t) \) is determined to be equal to the time-independent (that is, constant) kinetic energy of the particle, \( T \overset{\text{def}}{=} (1/2)mv^2 \). We also emphasize that simple harmonic motion does not arise merely from the condition of constrained motion on a sphere. Instead, it can be shown that the constancy of some relevant physical quantity must be satisfied. In particular, in the three examples being compared in this discussion, the constancy of the kinetic energy \( T \) (that is, \( v_{\text{geo-Newton}} \propto \sqrt{T} \)) in the classical scenario replaces the constancy of the energy dispersion \( \Delta E \) (that is, \( v_{\text{geo-QM}} \propto \sqrt{\Delta E^2} \); in the investigation of geodesics in quantum ray spaces and that of the Fisher information \( F \) (that is, \( v_{\text{geo-Grover}} \propto \sqrt{F} \)) in the geometric analysis of Grover’s algorithm. As a concluding remark in this specific set of remarks, we point out that further enlightening discussions concerning geodesic motion on spheres of relevance in quantum mechanics can be found in Ref. [45].

Finally, upon transitioning away from the quantum setting, we discuss the universality of the emergence of geodesic motion of SHO type in the presence of conserved quantities by analyzing two specific phenomena of gravitational (see Eqs. (42) and (43)) and thermodynamical origin (see Eqs. (45) and (46)), respectively. In the gravitational case, the harmonic motion emerges from consideration of the geodesic motion on a curved manifold with Lorentzian spacetime metric (see Eq.(38)). In the thermodynamical case, by contrast, the harmonic motion emerges by considering the geodesic motion on a configuration manifold equipped with a conformally flat Jacobian metric (see Eq.(47)). A global summary of the various features concerning the four physical scenarios investigated in the present article appears in Table 1.

Despite the pedagogical nature of our variational calculus reconsideration of Groverian paths for both state vectors and probability amplitudes, the decision to discuss them in parallel enabled us to clarify the physical connection among energy dispersion \( \Delta E \), speed of evolution \( (v_H) \) of the system, and the Fisher information \( F \). For the sake of transparency, we emphasize that the physical relevance of the pairs \( (\Delta E, v_H) \), \( (F, v_H) \), and \( (\Delta E, F) \) has been previously described in contexts different from ours in Refs. [20], [46–48], and [49], respectively. Our work is, however, unique in the sense that we provide a unifying physical link for \( (\Delta E, v_H, F) \) in the novel context of a geometric characterization of quantum searching with the underlying physical motivation of finding a good geometric measure of thermodynamic efficiency for very fast quantum transfer phenomena [4,12]. Moreover, by highlighting the “universality” of harmonic geodesic motion in the presence of conserved quantities associated with physical contexts other than the quantum one, we believe our work can help characterize realistic deviations from ideal Groverian paths in quantum searching by mimicking departures from the harmonic condition in more realistic physical settings [4,12].

While our current degree of rational belief requires further physical and mathematical justification, these types of theoretical analogies could well serve as the nascent forms that ultimately lead to significant findings such as the famous link between optimization methods and annealing in solids [50]. We remain, as ever, highly motivated to further develop these avenues of investigation in future scientific efforts.

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Appendix A: Geodesics in the space of unit rays

In this Appendix, we present technical details on the derivation of horizontal affinely parametrized geodesics [14]. This derivation is omitted in Sect. 3. We also briefly discuss the notion of non-horizontal, non-affinely parametrized geodesics in the second part of this Appendix.

1. Horizontal affinely parametrized geodesics

Let \( \mathcal{H} \) be an \( N \)-dimensional Hilbert space endowed with the scalar product \( \langle \cdot | \cdot \rangle \). Consider the subset \( \mathcal{H}_0 \subset \mathcal{H} \) of unitary vectors in \( \mathcal{H} \), i.e., \( \psi \in \mathcal{H}_0 \) if \( \langle \psi | \psi \rangle = 1 \). A one-parameter smooth curve \( \tilde{\gamma}_0 \) in \( \mathcal{H}_0 \) consists of a family of vectors \( \psi(s) \):

\[
\tilde{\gamma}_0 \overset{\text{def}}{=} \{ \psi(s) \in \mathcal{H}_0 | s \in [s_1, s_2] \in \mathbb{R} \} .
\]  

(A1)

From the assumption that \( \langle \psi(s) | \psi(s) \rangle = 1 \), it immediately follows that \( \text{Re}\langle \psi(s) | \dot{\psi}(s) \rangle = 0 \), with \( \dot{\psi}(s) = \frac{\partial}{\partial s} \psi(s) \). This result is equivalent to

\[
\langle \psi(s) | \dot{\psi}(s) \rangle = i \text{Im}\langle \psi(s) | \dot{\psi}(s) \rangle .
\]  

(A2)

A gauge transformation of \( \tilde{\gamma}_0 \) is typically determined by a real phase factor \( \alpha(s) \) by taking the curve \( \tilde{\gamma}_0 \) into a new one \( \tilde{\gamma}'_0 \):

\[
\tilde{\gamma}_0 \to \tilde{\gamma}'_0 , \quad \psi'(s) = e^{i\alpha(s)} \psi(s) . \quad s \in [s_1, s_2] .
\]  

(A3)

This transformation applies to Eq. (A2) as follows:

\[
\text{Im}\langle \psi'(s) | \dot{\psi'}(s) \rangle = \text{Im}\langle \psi(s) | \dot{\psi}(s) \rangle + \dot{\alpha}(s) .
\]  

(A4)

This allows us to construct a functional of \( \tilde{\gamma}_0 \) which is gauge invariant:

\[
\text{arg}(\psi'(s_1) | \psi'(s_2)) - \text{Im}\int_{s_1}^{s_2} \langle \psi'(s) | \dot{\psi'}(s) \rangle \, ds = \text{arg}(\psi(s_1) | \psi(s_2)) - \text{Im}\int_{s_1}^{s_2} \langle \psi(s) | \dot{\psi}(s) \rangle \, ds ,
\]  

(A5)

where \( \text{arg}(\psi'(s_1) | \psi'(s_2)) = \int_{s_1}^{s_2} \dot{\alpha}(s) \, ds \). This property can be interpreted in the space of unit rays \( \mathcal{H}\mathbb{P}^N \simeq \mathbb{C}\mathbb{P}^{N-1} \), which is simply the quotient of \( \mathcal{H}_0 \) under the action \( \psi' = e^{i\alpha}\psi \) with \( \alpha \in \mathbb{R} \). If we denote by \( \pi \) the corresponding projection map:

\[
\pi : \mathcal{H}_0 \to \mathcal{H}\mathbb{P}^N , \quad \pi(\tilde{\gamma}) = \pi(e^{i\alpha}\tilde{\gamma}) , \quad \text{for all } \alpha \in \mathbb{R} \text{ and } \psi \in \mathcal{H}_0 ,
\]  

(A6)

we observe that the curve \( \tilde{\gamma}_0 \) projects onto a smooth curve \( \gamma_0 \) in \( \mathcal{H}\mathbb{P}^N \). Moreover, if \( \tilde{\gamma}'_0 \) is obtained from \( \tilde{\gamma}_0 \) by the transformation (A3) we find that \( \pi(\tilde{\gamma}'_0) = \gamma_0 \).

The gauge invariance of (A5) implies that it is actually a functional of \( \gamma_0 \). Another important property of the functional (A5) is that it is reparametrization invariant. Combining these two properties, we can write

\[
\varphi[\gamma_0] = \text{arg}(\psi(s_1) | \psi(s_2)) - \text{Im}\int_{s_1}^{s_2} \langle \psi(s') | \dot{\psi}(s') \rangle \, ds' ,
\]  

(A7)
which is called the geometric phase associated with the smooth curve $\gamma_0$. Now, given $\gamma_0 \in \mathcal{H}\mathbb{P}^N$, the horizontal lift $\tilde{\gamma}_0$ of $\gamma_0$ is specified by requiring that

$$\text{Im}(\langle \psi(s') | \dot{\psi}(s') \rangle) = 0 \quad \Leftrightarrow \quad \langle \psi(s') | \dot{\psi}(s') \rangle = 0.$$  \hspace{1cm} (A8)

We can see from Eq. (A7) that this requirement implies that for a given $\gamma_0$ and a given initial $\psi(s_1)$ projecting onto the initial point of $\gamma_0$ there is a unique horizontal lift of $\gamma_0$ starting from $\psi(s_1)$. In particular, this allows us to follow geodesic paths in $\mathcal{H}\mathbb{P}^N$ by studying the behavior of their horizontal lift in the Hilbert space of unit vectors.

We recall that geodesic paths $\gamma_{\text{geo}}(s)$ in $\mathcal{H}\mathbb{P}^N$, with $s_1 \leq s \leq s_2$, are those for which the action functional $\mathcal{L}[\gamma_{\text{geo}}]$ (that is, the length),

$$\mathcal{L}[\gamma_{\text{geo}}] \equiv \int_{s_1}^{s_2} \sqrt{\langle \dot{\gamma}(s) | \dot{\gamma}(s) \rangle} = \int_{s_1}^{s_2} \langle u_\perp(s) | u_\perp(s) \rangle^{1/2} \, ds,$$  \hspace{1cm} (A9)

is stationary. Recalling that $|u_\perp| \equiv |u| - \langle \psi | u \rangle$ with $|u| \equiv |\dot{\psi}|$ and $\langle \psi | \psi \rangle = 1$, $\mathcal{L}[\gamma_{\text{geo}}]$ in Eq. (A9) can be rewritten as,

$$\mathcal{L}[\gamma_{\text{geo}}] = \int_{s_1}^{s_2} \left( |\langle u | \psi \rangle| - \langle \psi | u \rangle \langle \psi | u \rangle \right)^{1/2} \, ds.$$  \hspace{1cm} (A10)

Recalling further that $z + z^* = 2 \text{Re} (z)$ for any $z \in \mathbb{C}$, the variation $\delta \mathcal{L}[\gamma_{\text{geo}}]$ of $\mathcal{L}[\gamma_{\text{geo}}]$ in Eq. (A10) becomes,

$$\delta \mathcal{L}[\gamma_{\text{geo}}] = \int_{s_1}^{s_2} \frac{1}{||u_\perp||} \text{Re} \left( |\langle u | \psi \rangle| - \langle \psi | u \rangle \langle \psi | u \rangle \right) \, ds.$$  \hspace{1cm} (A11)

At this juncture, we emphasize that the quantity $- \text{Re} [\langle \delta \psi | u \rangle \langle u | \psi \rangle]$ in Eq. (A11) equals $\text{Re} [\langle \delta \psi | u_\perp \rangle \langle u | \psi \rangle]$. This equality is a consequence of the fact that the inner products $\langle \delta \psi | \psi \rangle$ and $\langle u | \psi \rangle$ are pure imaginary numbers. These relations, in turn, are ultimately a consequence of the normalization condition $\langle \psi | \psi \rangle = 1$. More explicitly, observe that

$$\text{Re} [\langle \delta \psi | u_\perp \rangle \langle \psi | u \rangle + \langle \delta \psi | u \rangle \langle u | \psi \rangle] = \text{Re} [\langle \delta \psi | u_\perp \rangle \langle u | \psi \rangle^\ast + \langle \delta \psi | u \rangle \langle u | \psi \rangle] = \text{Re} [-\langle \delta \psi | u_\perp \rangle \langle u | \psi \rangle + \langle \delta \psi | u \rangle \langle u | \psi \rangle] = \text{Re} \{\langle u | \psi \rangle [-\langle \delta \psi | u_\perp \rangle + \langle \delta \psi | u \rangle]\} = \text{Re} [-\langle u | \psi \rangle \langle \delta \psi | \psi \rangle \langle u | \psi \rangle] = \text{Re} [-|\langle u | \psi \rangle|^2 \langle \delta \psi | \psi \rangle].$$  \hspace{1cm} (A12)

Thus, since $|\langle u | \psi \rangle|^2$ is real and $\langle \delta \psi | \psi \rangle$ is purely imaginary, we obtain from Eq. (A12) the result that $\text{Re} [\langle \delta \psi | u_\perp \rangle \langle u | \psi \rangle] = - \text{Re} [\langle \delta \psi | u \rangle \langle u | \psi \rangle]$. Therefore, by exploiting this last relation, $\delta \mathcal{L}[\gamma_{\text{geo}}]$ in Eq. (A11) becomes

$$\delta \mathcal{L}[\gamma_{\text{geo}}] = \int_{s_1}^{s_2} \frac{1}{||u_\perp||} \text{Re} \left[ |\langle u | \psi \rangle| + \langle \delta \psi | u \rangle \langle u | \psi \rangle \right] \, ds,$$  \hspace{1cm} (A13)

that is,

$$\delta \mathcal{L}[\gamma_{\text{geo}}] = \int_{s_1}^{s_2} \text{Re} \left[ \left( \frac{d}{ds} \langle \delta \psi | \frac{|u_\perp|}{\sqrt{||u_\perp||}} \rangle \right) \right] \, ds + \int_{s_1}^{s_2} \text{Re} \left[ \langle \delta \psi | \frac{|u_\perp|}{\sqrt{||u_\perp||}} \rangle \langle \psi | u \rangle \right] \, ds.$$  \hspace{1cm} (A14)

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Observe that by eliminating the boundary terms, the first term on the RHS of Eq. (A14) can be written as
\[
\int_{s_1}^{s_2} \text{Re} \left[ \left( \frac{d}{ds} (\delta \psi) \right|_{||u_\perp||} \right] ds = \int_{s_1}^{s_2} \text{Re} \left\{ \frac{d}{ds} \left[ \langle \delta \psi \right|_{||u_\perp||} \right] \right\} ds
\]
\[
- \int_{s_1}^{s_2} \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \frac{d}{ds} \right] \right\} ds
\]
\[
= \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \right] \int_{s_1}^{s_2} ds
\]
\[
- \int_{s_1}^{s_2} \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \frac{d}{ds} \right] \right\} ds
\]
\[
= - \int_{s_1}^{s_2} \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \frac{d}{ds} \right] \right\} ds,
\]
that is,
\[
\int_{s_1}^{s_2} \text{Re} \left[ \left( \frac{d}{ds} (\delta \psi) \right|_{||u_\perp||} \right] ds = - \int_{s_1}^{s_2} \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \frac{d}{ds} \right] \right\} ds.
\]
By combining Eqs. (A14) and (A16), the variation \(\delta \mathcal{L}[\gamma_{geo}]\) becomes,
\[
\delta \mathcal{L}[\gamma_{geo}] = - \int_{s_1}^{s_2} \text{Re} \left[ \left( \delta \psi \right|_{||u_\perp||} \frac{d}{ds} \right] \right\} ds.
\]
Upon imposing the requirement that \(\delta \mathcal{L}[\gamma_{geo}] = 0\) for any variation \((\delta \psi)\) with \((\delta \psi|\psi)\) being purely imaginary, Eq. (A17) reduces to
\[
\frac{d}{ds} \left| u_\perp \right|_{||u_\perp||} - \langle \psi|u \rangle \left| u_\perp \right|_{||u_\perp||} = f(s)|\psi|,
\]
where \(f(s)\) is an arbitrary real-valued function. Recalling that \(\langle u|\psi\rangle\) is purely imaginary, we can set \(\langle u|\psi\rangle \equiv i A_\psi(u)\) with \(A_\psi(u) \in \mathbb{R}\). In this manner, the geodesic equation in Eq. (A18) becomes
\[
\left[ \frac{d}{ds} - i A_\psi(u) \right] \left| u_\perp \right|_{||u_\perp||} = f(s)|\psi|.
\]
At this point, we remark that geodesic paths can be formally obtained by integrating the geodesic relation in Eq. (A19). Since the action functional \(\mathcal{L}[\gamma_{geo}]\) is invariant under reparametrizations and gauge transformations however, it follows that geodesic paths are reparametrization and gauge-covariant quantities. Specifically, we observe that by exploiting the gauge freedom, \(\gamma_{geo}(s)\) is a geodesic path traced by a vector state \(|\psi(s)\rangle\) so that the horizontal lift \(\tilde{\gamma}_{geo}(s)\) of \(\gamma_{geo}(s)\) is traced by the state vector \(|\psi_h(s)\rangle\) with \(\langle \psi_h(s)|\psi_h(s)\rangle = 0\).

Therefore, by taking advantage of the gauge freedom, we set \(A_{\psi_h}(u) \equiv -i \langle \psi_h|u \rangle = 0\) with the consequence being that \(|u_\perp| = |u|\). Thus, Eq. (A19) becomes,
\[
\frac{d}{ds} \left| u_\perp \right|_{||u_\perp||} = f(s)|\psi_h|.
\]
At this stage, by exploiting the parametrization covariance, we can choose a convenient affine parametrization of the curve which is unique (modulo linear inhomogeneous changes in \(s, s \to \tilde{s} = as + b\) with \(a, b \in \mathbb{R}\setminus\{0\}\) such that \(|u|\) is constant along the curve. Therefore, upon setting \(|\psi_h(s)|\psi_h(s)\rangle = \text{constant}, \langle \psi_h(s)|\psi_h(s)\rangle = 1,\) and \(|\psi_h(s)|\psi_h(s)\rangle = 0\) for
any \( s \), we find that an horizontal affinely parametrized geodesic in the space of unit rays satisfies the equation

\[
\frac{d^2}{ds^2} |\psi_h (s)\rangle + \{ \dot{\psi}_h (s) | \dot{\psi}_h (s) \rangle |\psi_h (s)\rangle = 0.
\] (A21)

The differential relation in Eq. (A21) describes a simple harmonic oscillator. Assuming \( \langle \psi_h (0) | \psi_h (0) \rangle = 1 \), \( \langle \psi_h (0) | \dot{\psi}_h (0) \rangle = 0 \), and \( \langle \psi_h (0) | \dot{\psi}_h (0) \rangle = v_H^2 \), the general solution \( |\psi_h (s)\rangle \) of Eq. (A21) becomes,

\[
|\psi_h (s)\rangle = \cos (v_H s) |\psi_h (0)\rangle + \frac{\sin (v_H s)}{v_H} |\dot{\psi}_h (0)\rangle.
\] (A22)

The emergence of the horizontal, affinely parametrized geodesic path in Eq. (A22) concludes our formal derivation.

As a final remark, we point out that we refer to the third subsection of this Appendix for a brief physical note on the important concepts of dynamical and geometric phases (see Eq. (A7)) in quantum mechanical evolutions.

2. Non-horizontal non-affinely parametrized geodesics

We recall that any two arbitrary non-orthogonal vectors \( |\psi_A\rangle \) and \( |\psi_B\rangle \) in the unit ray space can be connected by a geodesic arc that is generally non-horizontal and non-affinely parametrized [14],

\[
|\psi (s)\rangle = e^{i\beta \frac{s}{2}} \left[ \sin (\theta - s) |\psi_A\rangle + e^{-i\beta} \sin (s) |\psi_B\rangle \right],
\] (A23)

where \( 0 \leq s \leq \theta \) with \( 0 < \theta < \pi/2 \) and \( \langle \psi_A | \psi_B \rangle = |\langle \psi_A | \psi_B \rangle| e^{i\beta} \). Note that if \( \langle \psi_A | \psi_B \rangle = \delta_{AB} \), it is sufficient to take \( \beta = 0 \) and \( \theta = \pi/2 \) in order to obtain a geodesic arc \( |\psi (s)\rangle \) connecting the two orthogonal states \( |\psi_A\rangle \) and \( |\psi_B\rangle \). The geodesic in Eq. (A23) is generally neither horizontal nor affinely parametrized. The horizontality condition \( \langle \psi (s) | \psi (s) \rangle = 0 \) is achieved via gauge freedom, while the affine parametrization condition \( \langle \psi (s) | \psi (s) \rangle = \) constant is obtained via reparametrization freedom.

For example, let \( |\psi (t)\rangle \) represent the state vector of a quantum system that evolves according to the Schrödinger equation \( i\hbar \partial_t |\psi (t)\rangle = H |\psi (t)\rangle \). If one exploits only the gauge freedom, then the horizontal vector \( |\psi (t)\rangle \) satisfies the equation

\[
\frac{d^2}{dt^2} |\psi (t)\rangle + [v_H (t)]^2 |\psi (t)\rangle = 0,
\] (A24)

where \( v_H (t) \equiv \Delta E (t) /\hbar \) with \( \Delta E (t) \) being the uncertainty in the energy of the system and \( t \) is the ordinary physical time parameter. To verify that \( v_H (t) = \Delta E (t) /\hbar \), recall that \( |u_\perp\rangle \equiv |u\rangle - \langle \psi | u \rangle |\psi\rangle \) and \( |u\rangle \equiv |\psi\rangle = -i/\hbar H |\psi\rangle \). Then, we have

\[
[v_H (t)]^2 = |u_\perp|u_\perp\rangle = \langle u | u \rangle - |\langle u | \psi \rangle|^2 = \langle \psi | \psi \rangle - |\langle \psi | \psi \rangle|^2 = \frac{1}{\hbar^2} \left[ |\psi| H^2 |\psi\rangle - \langle \psi | H |\psi\rangle^2 \right] = \frac{[\Delta E (t)]^2}{\hbar^2},
\] (A25)
that is, \( v_H (t) = \Delta E (t) / \hbar \). In general, when a non-affine parameter is used to describe the geodesic curve, the analytical integration of Eq. (A24) can be highly nontrivial.

3. Dynamical and geometric phases in quantum evolutions

From a physics standpoint, the concept of geometric phase (also known as Berry’s phase, [51]) emerges when considering the adiabatic evolution of a quantum mechanical system whose Hamiltonian \( H \) returns to its original value and the state vector evolves as an eigenstate of the Hamiltonian. As the Hamiltonian returns to its original value after a time \( t \), the system will return to its original state, apart from a phase factor \( e^{i \varphi_{\text{tot}} (t)} \). Specifically, we have

\[
| \psi (t) \rangle = e^{i \varphi_{\text{tot}} (t)} | \psi (0) \rangle .
\] (A26)

The phase factor \( e^{i \varphi_{\text{tot}} (t)} \) in Eq. (A26) can be described in terms of a circuit-dependent component \( e^{i \varphi_{\text{geo}} (t)} \) and an usual dynamical component \( e^{i \varphi_{\text{dyn}} (t)} = e^{-i H t / \hbar} \) which specifies the evolution of any stationary state. In terms of phases, we have

\[
\varphi_{\text{geo}} (t) = \varphi_{\text{tot}} (t) - \varphi_{\text{dyn}} (t) .
\] (A27)

The total phase is defined as,

\[
\varphi_{\text{tot}} (t) \equiv \text{arg} [\langle \psi (0) | \psi (t) \rangle] ,
\] (A28)

where \( \text{arg} (z) \equiv \tan^{-1} [\text{Im} (z) / \text{Re} (z)] \) and \( z \in \mathbb{C} \). The dynamical phase is defined for any evolution, cyclic or not, and can be expressed in terms of the time integral of the expectation value of the Hamiltonian \( H \),

\[
\varphi_{\text{dyn}} (t) \equiv -\frac{1}{\hbar} \int_0^t \langle \psi (t') | H (t') | \psi (t') \rangle \, dt'.
\] (A29)

The quantity \( \varphi_{\text{dyn}} (t) \) in Eq. (A29) encodes information about the duration of the evolution of the physical system. Substituting Eqs. (A29) and (A28) into Eq. (A27), the geometric phase becomes

\[
\varphi_{\text{geo}} (t) = \text{arg} [\langle \psi (0) | \psi (t) \rangle] + \frac{1}{\hbar} \int_0^t \langle \psi (t') | H (t') | \psi (t') \rangle \, dt'.
\] (A30)

The geometric phase \( \varphi_{\text{geo}} (t) \) in Eq. (A30) is the so-called Berry phase. It offers relevant information about the geometry of the path of the quantum evolution viewed in the projective Hilbert space of rays. Observe that using Schrödinger’s evolution equation, we have

\[
\text{Im} \int_0^t \langle \psi (t') | \dot{\psi} (t') \rangle \, dt' = \text{Im} \left( -\frac{i}{\hbar} \int_0^t \langle \psi (t') | H (t') | \psi (t') \rangle \, dt' \right)
= \frac{1}{\hbar} \int_0^t \langle \psi (t') | H (t') | \psi (t') \rangle \, dt'
= \varphi_{\text{dyn}} (t) .
\] (A31)

Therefore, identifying the corresponding quantities together with employing Eqs. (A31) and (A30), we can recover Eq. (A7). For further details on phase changes in cyclic and non-cyclic quantum evolutions, we refer to Ref. [52] and Ref. [16], respectively. Finally, for a description of Berry’s phase in terms of natural geometric structures or in terms of the fiber bundle language, we refer to Ref. [53] and Ref. [54], respectively.
Appendix B: Affinely v.s. non-affinely parametrized geodesic paths

In this Appendix, we present some technical details on the possibility of using a non-affine parameter for describing the geodesic motion on a curved spacetime manifold. This technicality was mentioned in Section IV when discussing our gravitational problem.

In the framework of general relativity on curved manifolds [33,55], an affinely parametrized geodesic $\gamma(\tau)$ is a curve whose tangent vector $\dot{\gamma}$ is everywhere nonzero and is parallelly propagated. Therefore, the covariant derivative of the vector field $\dot{\gamma}$ along $\gamma$ is zero, that is $D_\tau \dot{\gamma} = 0$. As a consequence, $\|\dot{\gamma}\| = \text{constant}$ since

$$\partial_\tau \|\dot{\gamma}\|^2 \equiv \langle D_\tau \dot{\gamma}, \dot{\gamma} \rangle + \langle \dot{\gamma}, D_\tau \dot{\gamma} \rangle = 0.$$  \hspace{1cm} (B1)

In particular, the differential equation for an affine geodesic $\gamma(\tau)$, with $\tau$ being the affine parameter along the curve is given by,

$$\frac{d^2 \gamma^a}{d\tau^2} + \Gamma^a_{bc} \frac{d\gamma^b}{d\tau} \frac{d\gamma^c}{d\tau} = 0.$$  \hspace{1cm} (B2)

with $\Gamma^a_{bc}$ being the usual Christoffel connection coefficients of the second kind. The standard form of Eq. (B2) is preserved if and only if we replace $\tau$ with $\tilde{\tau} = \tilde{\tau}(\tau) \equiv A\tau + B$ with $A$, $B$ being constants. Stated otherwise, an affine parameter is defined up to a change of scale ($A \neq 0$) and origin ($B \neq 0$). More generally, a geodesic curve $\gamma(s)$ is a curve whose tangent vector $\dot{\gamma}$ is everywhere nonzero and only needs to be proportional to a parallelly propagated vector. In this case, the non-affinely parametrized geodesic equation becomes,

$$\frac{d^2 \gamma^a}{ds^2} + \Gamma^a_{bc} \frac{d\gamma^b}{ds} \frac{d\gamma^c}{ds} = g(s) \frac{d\gamma^a}{ds}.$$  \hspace{1cm} (B3)

The quantity $g(s)$ in Eq. (B3) is defined as $g(s) \equiv \frac{d}{ds} \ln \left[ \lambda^{-1}(s) \right]$ with $\lambda(s) > 0$ being a differentiable function such that [33],

$$\left[ \Gamma(0, s; \gamma) \right]_a^{\cdot b} \cdot \left[ \frac{d\gamma^a}{ds}(0) \right] = \lambda(s) \left[ \frac{d\gamma^b}{ds}(s) \right],$$  \hspace{1cm} (B4)

with $\Gamma(0, s; \gamma)$ denoting the so-called connector map. As a final remark, we point out that it can be shown that Eq. (B3) reduces to Eq. (B2) by performing a suitable change of variables,

$$s \to \tau : s = \sigma(\tau), \text{ with } \frac{d\sigma}{d\tau} \equiv \lambda(\sigma(\tau)).$$  \hspace{1cm} (B5)

We leave this simple verification as an exercise for the interested reader.

Appendix C: Geometrization of Newtonian Mechanics

In this Appendix, we present a straightforward derivation of Newton’s equation of motion in Eq. (44) starting from Eq. (48). This technicality was mentioned in Section IV when discussing our thermodynamical problem.

Recall that in a local coordinate system the equation of an affinely parametrized geodesic is given by

$$\frac{d^2 q^i}{ds^2} + \Gamma^i_{jk} \frac{dq^j}{ds} \frac{dq^k}{ds} = 0.$$  \hspace{1cm} (C1)
The Christoffel symbols $\Gamma_{ij}^k$ in Eq. (C1) are defined as,

$$\Gamma_{ij}^k \equiv \frac{1}{2} g^{im} \left( \partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk} \right), \quad (C2)$$

where $g_{ij} \equiv 2 \left[ E - V(q) \right] \delta_{ij}$. Using Eq. (C2) together with the expression of the Jacobi metric, Eq. (C1) becomes

$$0 = \frac{d^2q_i}{ds^2} + \frac{d}{ds} \frac{\Gamma_{ij}^k dq_j dq_k}{ds}$$

$$= \frac{d^2q_i}{ds^2} + \frac{1}{2} g^{im} \left( \partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk} \right) \frac{dq_j dq_k}{ds}$$

$$= \frac{d^2q_i}{ds^2} + \frac{1}{2} g^{im} \partial_j g_{km} \frac{dq_j dq_k}{ds} + \frac{1}{2} g^{im} g_{jk} \frac{dq_j dq_k}{ds} - \frac{1}{2} g^{im} \partial_m g_{jk} \frac{dq_j dq_k}{ds}$$

$$= \frac{d^2q_i}{ds^2} + \frac{1}{2} \frac{1}{[E - V(q)]} \delta_{ij} \frac{dq_j dq_k}{ds} + \frac{1}{2} \frac{1}{[E - V(q)]} \partial j \left[ E - V(q) \right] \frac{dq_j dq_k}{ds} + \frac{1}{2} \frac{1}{[E - V(q)]} \partial q_i \left( \frac{dq_j}{ds} \right)^2$$

$$= \frac{d^2q_i}{ds^2} + \frac{1}{2} \frac{1}{[E - V(q)]} \partial j \left[ E - V(q) \right] \frac{dq_j dq_k}{ds} - \frac{1}{2} \frac{1}{[E - V(q)]} g^{ij} \frac{dq_i dq_m}{ds} \partial q_j$$

$$= \frac{d^2q_i}{ds^2} + \frac{1}{2} \frac{1}{[E - V(q)]} \left[ 2 \frac{\partial \left[ E - V(q) \right]}{\partial q_j} \frac{dq_j dq_k}{ds} - g^{ij} \frac{\partial \left[ E - V(q) \right]}{\partial q_j} \frac{dq_k dq_m}{ds} \right],$$

that is,

$$\frac{d^2q_i}{ds^2} + \frac{1}{2} \frac{1}{[E - V(q)]} \left[ 2 \frac{\partial \left[ E - V(q) \right]}{\partial q_j} \frac{dq_j dq_k}{ds} - g^{ij} \frac{\partial \left[ E - V(q) \right]}{\partial q_j} \frac{dq_k dq_m}{ds} \right] = 0. \quad (C3)$$

Next, recalling that $ds^2 = g_{ij} dq^i dq^j = 4 \left[ E - V(q) \right]^2 dt^2$, Eq. (C4) becomes

$$\frac{d^2q_i}{dt^2} + \frac{\partial V(q)}{\partial q_i} = 0, \quad (C5)$$

that is,

$$\frac{d^2q_i}{dt^2} = - \frac{\partial V(q)}{\partial q_i}. \quad (C6)$$

Finally, observing that $n_{1d} = 1, q = x$, and the potential $V(q)$ in Eq. (47) reduces to the harmonic potential $(1/2) m v_d^2 x^2$ with $v_d$ defined in Eq. (46), we get Eq. (44). For further details on the Riemannian geometrization of Newtonian mechanics, we refer to Refs. [37,56].

References

1. B.R. Frieden, Physics from Fisher Information (Cambridge University Press, New York, 1998)
2. T.M. Cover, J.A. Thomas, *Elements of Information Theory* (Wiley, New York, 2006)
3. D. Felice, C. Cafaro, S. Mancini, Information geometric methods for complexity. Chaos **28**, 032101 (2018)
4. C. Cafaro, P.M. Alsing, Decrease of Fisher information and the information geometry of evolution equations for quantum mechanical probability amplitudes. Phys. Rev. E **97**, 042110 (2018)
5. C. Villani, Decrease of the Fisher information for the Landau equation with Maxwellian molecules. Math. Mod. Meth. Appl. Sci. **10**, 153 (2000)
6. J. J. Alvarez, C. Gomez, *A comment on Fisher information and quantum algorithms*, arXiv:quant-ph/9910115 (2000)
7. A. Miyake, M. Wadati, Geometric strategy for the optimal quantum search. Phys. Rev. A **64**, 042317 (2001)
8. C. Cafaro, Geometric algebra and information geometry for quantum computational software. Physica A **470**, 154 (2017)
9. C. Cafaro, P.M. Alsing, Continuous-time quantum search and time-dependent two-level quantum systems. Int. J. Quantum Inf. **17**, 1950025 (2019)
10. C. Cafaro, P.M. Alsing, Theoretical analysis of a nearly optimal analog quantum search. Physica Scripta **94**, 085103 (2019)
11. S. Gassner, C. Cafaro, S. Capozziello, Transition probabilities in generalized quantum search Hamiltonian evolutions. Int. J. Geom. Methods Mod. Phys. **17**, 2050006 (2020)
12. C. Cafaro, P.M. Alsing, Information geometry aspects of minimum entropy production paths from quantum mechanical evolutions. Phys. Rev. E **101**, 022110 (2020)
13. C. Cafaro, S. Gassner, P.M. Alsing, Information geometric perspective on off-resonance effects in driven two-level quantum systems. Quantum Rep. **2**, 166 (2020)
14. N. Mukunda, R. Simon, Quantum kinematic approach to the geometric phase. I. General formalism. Ann. Phys. **228**, 205 (1993)
15. A.K. Pati, On phases and length of curves in a cyclic quantum evolution. Pramana-J. Phys. **42**, 455 (1994)
16. A.K. Pati, Geometric aspects of noncyclic quantum evolutions. Phys. Rev. A **52**, 2576 (1995)
17. C. Cafaro, S. Mancini, An information geometric viewpoint of algorithms in quantum computing. AIP Conf. Proc. **1443**, 374 (2012)
18. C. Cafaro, S. Mancini, On Grover’s search algorithm from a quantum information geometry viewpoint. Physica A **391**, 1610 (2012)
19. F.M. Ciaglia, F. Di Cosmo, D. Felice, S. Mancini, G. Marmo, J.M. Pérez-Pardo, Aspects of geodesical motion with Fisher-Rao metric: classical and quantum. Open Syst. Inf. Dyn. **25**, 1850005 (2018)
20. J. Anandan, Y. Aharonov, Geometry of quantum evolution. Phys. Rev. Lett. **65**, 1697 (1990)
21. L.K. Grover, Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. **79**, 325 (1997)
22. H. Zatzikis, Model of a linear harmonic oscillator in the general theory of relativity. Phys. Rev. **114**, 1645 (1959)
23. E. Rüchhardt, Eine einfache methode zur bestimmung von $C_P / C_V$. Physikalische Zeitschrift **30**, 58 (1929)
24. M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)
25. J.P. Provost, G. Vallee, Riemannian structure on manifolds of quantum states. Commun. Math. Phys. **76**, 289 (1980)
26. S.L. Braunstein, C.M. Caves, Statistical distance and the geometry of quantum states. Phys. Rev. Lett. **72**, 3439 (1994)
27. W.K. Wootters, Statistical distance and Hilbert space. Phys. Rev. D **23**, 357 (1981)
28. P. Facchi, R. Kulkarni, V.I. Man’ko, G. Marmo, E.C.G. Sudarshan, F. Ventriglia, Classical and quantum Fisher information in the geometrical formulation of quantum mechanics. Phys. Lett. A **374**, 4801 (2010)
29. E. Farhi, J. Goldstone, S. Gutmann, M. Sipser, Quantum computation by adiabatic evolution, arXiv:quant-ph/0001106 (2000)
30. A.T. Rezakhani, W.-J. Kuo, A. Hamma, D.A. Lidar, P. Zanardi, Quantum adiabatic brachistochrone. Phys. Rev. Lett. **103**, 080502 (2009)
31. J. Roland, N.J. Cerf, Quantum search by local adiabatic evolution. Phys. Rev. A **65**, 042308 (2002)
32. A.T. Rezakhani, D.F. Abasto, D.A. Lidar, P. Zanardi, Intrinsic geometry of quantum adiabatic evolution and quantum phase transitions. Phys. Rev. A **82**, 012321 (2010)
33. F. De Felice, C.J.S. Clarke, *Relativity on Curved Manifolds* (Cambridge University Press, Cambridge, 1990)
34. M. von Laue, *Die Relativitätstheorie*, vol. 2 (F. Vieweg und Sohn, Braunschweig, 1921)
35. E. Fermi, *Termodinamica*, Bollati Boringhieri (1994)
36. M.W. Zemansky, R.H. Dittman, *Heat and Thermodynamics* (The McGraw-Hill Companies, Inc, New York, 1997)
37. L. Casetti, M. Pettini, E.G.D. Cohen, Geometric approach to Hamiltonian dynamics and statistical mechanics. Phys. Rep. 337, 237 (2000)
38. A. Caticha, C. Cafaro, From information geometry to Newtonian dynamics. AIP Conf. Proc. 954, 165 (2007)
39. C. Cafaro, Works on an information geometrodynamical approach to chaos. Chaos, Solitons and Fractals 41, 886 (2009)
40. T. Padmanabhan, *Gravitation: Foundations and Frontiers* (Cambridge University Press, Cambridge, 2010)
41. P.K.S. Dunsby, O. Luongo, L. Reverberi, Dark energy and dark matter from an additional adiabatic fluid. Phys. Rev. D 94, 083525 (2016)
42. H. Goldstein, *Meccanica Classica* (Zanichelli S. p. A., 1971)
43. E.T. Jaynes, Macroscopic prediction, in *Complex Systems- Operational Approaches in Neurobiology, Physics, and Computers*, ed. by H. Haken (Springer, Berlin, 1985), p. 254
44. C. Cafaro, S.A. Ali, Maximum caliber inference and the stochastic Ising model. Phys. Rev. E 94, 052145 (2016)
45. I. Bengtsson, K. Zyczkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, 2006)
46. D.C. Brody, L.P. Hughston, Geometry of quantum statistical inference. Phys. Rev. Lett. 77, 2851 (1996)
47. L. Pezzé, A. Smerzi, Entanglement, nonlinear dynamics, and the Heisenberg limit. Phys. Rev. Lett. 102, 100401 (2009)
48. M.M. Taddei, B.M. Escher, L. Davidovich, R.L. de Matos Filho, Quantum speed limit for physical processes. Phys. Rev. Lett. 110, 050402 (2013)
49. S. Boixo, S.T. Flammia, C.M. Caves, J.M. Geremia, Generalized limits for single-parameter quantum estimation. Phys. Rev. Lett. 98, 090401 (2007)
50. S. Kirkpatrick, C.D. Gelatt Jr., M.P. Vecchi, Optimization by simulated annealing. Science 220, 671 (1983)
51. M.V. Berry, Quantal phase factors accompanying adiabatic changes. Proc. R. Soc. Lond. Ser. A392, 45 (1984)
52. Y. Aharonov, J. Anandan, Phase change during a cyclic quantum evolution. Phys. Rev. Lett. 58, 1593 (1987)
53. D.N. Page, Geometrical description of Berry’s phase. Phys. Rev. A 36, 3479 (1987)
54. A. Bohm, L.J. Boya, B. Kendrick, Derivation of the geometrical phase. Phys. Rev. A 43, 1206 (1991)
55. J. Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, 1991)
56. M. Pettini, *Geometry and Topology in Hamiltonian Dynamics and Statistical Mechanics* (Springer, New York Inc, 2007)