ON CHARACTERISTIC CURVES OF DEVELOPABLE SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. We investigate the relationship among characteristic curves on developable surfaces. In case parameter curves coincide with these curves, we show that the base curve of a developable surface could be either a plane curve, a circular helix, a general helix or a slant helix.

1. Introduction

Characteristic curves on surfaces are particular curves such as, geodesic and asymptotic curves, or lines of curvature. A regular ruled surface in Euclidean 3-space $\mathbb{E}^3$ whose the Gaussian curvature vanishes is called a developable surface.

Izumiya and Takeuchi [4] studied special curves, like cylindrical helices and Bertrand curves from the viewpoint of the theory of curves on ruled surfaces. They enlightened that cylindrical helices are related to Gaussian curvature and Bertrand curves are related to mean curvature of ruled surfaces. The same authors [5] defined new special curves that called as slant helices and conical geodesic curves which are generalizations of the notion of helices and studied them on developable surfaces. They also introduced the tangential Darboux developable surface of a space curve which is defined by the Darboux developable surface of the tangent indicatrix of the space curve and researched singularities of it.

Leite [6] determined that the orthogonal systems of cycles (curves of constant geodesic curvature) on the hyperbolic plane $\mathbb{H}^2$, aiming at the classification of maximal surfaces with planar lines of curvature in Lorentz–Minkowski 3-space $\mathbb{L}^3$ and indicated that a line of curvature on the spacelike surface $\mathbb{M}$ is planar if and only if its normal image in the hyperbolic plane $\mathbb{H}^2$ is a planar curve in $\mathbb{L}^3$ as well as a regular curve in $\mathbb{H}^2$ is planar if and only if it has constant geodesic curvature.

Lucas and Ortega-Yagües [7] presented the notion of rectifying curve in the three-dimensional sphere $\mathbb{S}^3(r)$ and denoted that a curve $\gamma$ in $\mathbb{S}^3(r)$ is a rectifying curve if and only if $\gamma$ is a geodesic curve of a conical surface as well as the rectifying developable surface of a unit speed curve $\gamma$ is a conical surface if and only if $\gamma$ is a rectifying curve.

Theisel and Farin [8] showed how to compute the curvature and geodesic curvature of characteristic curves on surfaces, such as contour lines, reflection lines, lines of curvature, asymptotic curves, and isophote curves. The conditions of also being characteristic curves of isophotes are studied in [2,3].

This paper is organized as follows. Section 2 is devoted to some basic concepts with regard to theory of curves and surfaces in $\mathbb{E}^3$ and particular curves on surfaces.

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In section 3, both the coefficients of the first and second fundamental forms of developable surfaces and the Gaussian curvatures of normal and binormal surfaces are obtained. Finally, in section 4, the main theorems for particular curves on developable surfaces are given.

2. Preliminaries

We shortly give some basic concepts concerning theory of curves and surfaces in $\mathbb{E}^3$ that will use in the subsequent section. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a curve with $\|\alpha'(s)\| = 1$, where $s$ is the arc-length parameter of $\alpha$ and $\alpha'(s) = \frac{d\alpha}{ds}(s)$. The function $\kappa : I \longrightarrow \mathbb{R}$, $\kappa(s) = \|\alpha''(s)\|$ is defined as the curvature of $\alpha$. For $\kappa > 0$, the Frenet frame along the curve $\alpha$ and the corresponding derivative formulas (Frenet formulas) are as follows.

$$T(s) = \alpha'(s),$$

$$N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},$$

$$B(s) = T(s) \times N(s),$$

where $T$, $N$, and $B$ are the tangent, the principal normal, and the binormal of $\alpha$, respectively.

$$T'(s) = \kappa(s)N(s),$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)B(s),$$

$$B'(s) = -\tau(s)N(s),$$

where the function $\tau : I \longrightarrow \mathbb{R}$, $\tau(s) = \frac{\langle \alpha'(s) \times \alpha''(s), \alpha'''(s) \rangle}{\kappa^2(s)}$ is the torsion of $\alpha$; "$\langle , \rangle$" is the standard inner product, and "$\times$" is the cross product on $\mathbb{R}^3$.

Let $\mathbb{M}$ be a regular surface and $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{M}$ be a unit-speed curve. Some particular curves lying on $\mathbb{M}$ are characterized as follows.

a) A curve $\alpha$ lying on $\mathbb{M}$ is a geodesic curve if and only if the acceleration vector $\alpha''$ is normal to $\mathbb{M}$, in other words,

$$U \times \alpha'' = 0.$$

b) A curve $\alpha$ lying on $\mathbb{M}$ is an asymptotic curve if and only if the acceleration vector $\alpha''$ is tangent to $\mathbb{M}$, that is,

$$\langle U, \alpha'' \rangle = 0.$$

c) A curve $\alpha$ lying on $\mathbb{M}$ is a line of curvature if and only if $S(T)$ and $T$ are linearly dependent, where $T$ is the tangent of $\alpha$, $U$ is the unit normal, and $S$ is the shape operator of $\mathbb{M}$.

A ruled surface in $\mathbb{R}^3$ is (locally) the map

$$F_{(\gamma, \delta)}(t, u) = \gamma(t) + u\delta(t),$$

where $\gamma : I \longrightarrow \mathbb{R}^3$, $\delta : I \longrightarrow \mathbb{R}^3 \setminus \{0\}$ are smooth mappings and $I$ is an open interval or a unit circle $S^1$ [5], where $\gamma$ and $\delta$ are called the base and generator (director)
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curves, respectively. For \( \| \delta(t) \| = 1 \), the Gaussian curvature of \( F(\gamma, \delta) \) is [4]

\[
K = -\frac{\det(\gamma'(t), \delta(t), \delta'(t))}{(EG - F^2)^2},
\]

where \( E = E(t, u) = \left\| \gamma'(t) + u\delta'(t) \right\| \), \( F = F(t, u) = \langle \gamma'(t), \delta(t) \rangle \), and \( G = G(t, u) = 1 \) are the coefficients of the first fundamental form of \( F(\gamma, \delta) \).

**Theorem 1** ([1]). A necessary and sufficient condition for the parameter curves of a surface to be lines of curvature in a neighborhood of a non-umbilical point is that \( F = f = 0 \), where \( F \) and \( f \) are the respective the first and second fundamental coefficients.

**Theorem 2** (The Lancret Theorem). Let \( \alpha \) be a unit-speed space curve with \( \kappa(s) \neq 0 \). Then \( \alpha \) is a general helix if and only if \( \left( \frac{T}{\kappa}(s) \right) \) is a constant function.

**Theorem 3** ([5]). A unit-speed curve \( \alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3 \) with \( \kappa(s) \neq 0 \) is a slant helix if and only if

\[
\sigma(s) = \pm \left( \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{T}{\kappa} \right)'(s) \right)
\]

is a constant function.

### 3. Developable, Normal, and Binormal Surfaces

In this section, we introduce developable, normal and binormal surfaces associated to a space curve and after that obtain the coefficients of the first and second fundamental forms of them, respectively. The non-singular ruled surfaces whose the Gaussian curvature vanish are called developable surfaces. Now, we firstly get the unit normal \( U \) and the acceleration vectors of the parameter curves of developable surfaces.

Let

\[
K(s, v) = \alpha(s) + v\delta(s)
\]

be a ruled surface with \( \| \delta(s) \| = 1 \). Then, we have

\[
\begin{align*}
K_s &= T + v\delta', \\
K_v &= \delta, \\
U &= \frac{K_s \times K_v}{\|K_s \times K_v\|} = \frac{1}{\left\| (T + v\delta') \times \delta \right\|} [T + v\delta'] \times \delta, \\
K_{ss} &= \delta', \\
K_{sv} &= \kappa N + v\delta'', \\
K_{vv} &= 0,
\end{align*}
\]

where \( T \) is the tangent, \( N \) is the principal normal, and \( \kappa \) is the curvature of \( \alpha \). From Eq.(2.1), the ruled surface \( K(s, v) \) is developable if and only if

\[
\det(T, \delta, \delta') = \langle T \times \delta, \delta' \rangle = 0.
\]
By differentiating the last equation with respect to $s$, we get

$$\left\langle T' \times \delta, \delta' \right\rangle + \left\langle T \times \delta', \delta' \right\rangle + \left\langle T \times \delta'', \delta' \right\rangle = 0$$

(3.2)

$$\left\langle T \times \delta, \delta'' \right\rangle = -\kappa \left\langle N \times \delta, \delta' \right\rangle.$$

Secondly, we obtain the coefficients of the first and second fundamental forms of special developable surfaces.

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a unit-speed curve with $\kappa \neq 0$ and let $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of $\alpha$.

1) The ruled surface

$K(s, v) = \alpha(s) + vN(s)$

is called the principal normal surface of $\alpha$. The partial derivatives of $K(s, v)$ with respect to $s$ and $v$ are as follows.

$K_s = (1 - v\kappa)T + v\tau B,$

$K_v = N,$

$K_{ss} = -v\kappa' T + [\kappa - v(\kappa^2 + \tau^2)]N + v\tau' B,$

$K_{sv} = -\kappa T + \tau B,$

$K_{vv} = 0.$

Thus, the unit normal $U$ and the Gaussian curvature $K$ of $K(s, v)$ are obtained as follows.

$$U = \frac{K_s \times K_v}{\|K_s \times K_v\|} = \frac{1}{\sqrt{(1 - v\kappa)^2 + v^2\tau^2}}[-v\tau T + (1 - v\kappa)B],$$

$$E = \left\langle K_s, K_s \right\rangle = (1 - v\kappa)^2 + v^2\tau^2,$$

$$F = \left\langle K_s, K_v \right\rangle = 0,$$

$$G = \left\langle K_v, K_v \right\rangle = 1,$$

$$e = \left\langle U, K_{ss} \right\rangle = \frac{v\tau' + v^2\tau^2(\frac{K_s'}{\tau})}{\sqrt{(1 - v\kappa)^2 + v^2\tau^2}},$$

$$f = \left\langle U, K_{sv} \right\rangle = \frac{\tau}{\sqrt{(1 - v\kappa)^2 + v^2\tau^2}},$$

$$g = \left\langle U, K_{vv} \right\rangle = 0,$$

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{\tau^2}{\|v\|^2}.$$  (3.3)

2) The ruled surface

$K(s, v) = \alpha(s) + vB(s)$
is called the binormal surface of \( \alpha \). The partial derivatives of \( K(s, v) \) with respect to \( s \) and \( v \) are as follows.

\begin{align*}
K_s &= T - v\tau N, \\
K_v &= B, \\
K_{ss} &= v\kappa T + (\kappa - v\tau')N - v\tau^2 B, \\
K_{sv} &= -\tau N, \\
K_{vv} &= 0.
\end{align*}

Thus, the unit normal \( U \) and the Gaussian curvature \( K \) of \( K(s, v) \) are obtained as follows.

\begin{align*}
U &= \frac{K_s \times K_v}{\|K_s \times K_v\|} = \pm \frac{1}{\sqrt{1 + v^2\tau^2}}[-v\tau T - N], \\
E &= \langle K_s, K_s \rangle = 1 + v^2\tau^2, \\
F &= \langle K_s, K_v \rangle = 0, \\
G &= \langle K_v, K_v \rangle = 1, \\
e &= \langle U, K_{ss} \rangle = \frac{-v^2\kappa\tau^2 - \kappa + v\tau'}{\sqrt{1 + v^2\tau^2}}, \\
f &= \langle U, K_{sv} \rangle = \frac{\tau}{\sqrt{1 + v^2\tau^2}}, \\
g &= \langle U, K_{vv} \rangle = 0, \\
\mathbb{K} &= \frac{eg - f^2}{EG - F^2} = -\frac{\tau^2}{[1 + v^2\tau^2]^2}.
\end{align*}

As we can see above, the normal and binormal surfaces of \( \alpha \) are developable if and only if the base curve \( \alpha \) is a plane curve.

3) The ruled surface

\( K(s, v) = \alpha(s) + vT(s) \)

is called the tangent developable surface of \( \alpha \). The partial derivatives of \( K(s, v) \) with respect to \( s \) and \( v \) are as follows.

\begin{align*}
K_s &= T + v\kappa N, \\
K_v &= 0, \\
K_{ss} &= -v\kappa^2 T + (\kappa + v\kappa')N + v\kappa T B, \\
K_{sv} &= \kappa N, \\
K_{vv} &= 0.
\end{align*}

Thus, the unit normal \( U \) and the coefficients of the first and second fundamental forms of \( K(s, v) \) are obtained as follows.

\begin{align*}
U &= \frac{K_s \times K_v}{\|K_s \times K_v\|} = \pm B, \\
E &= \langle K_s, K_s \rangle = 1 + v^2\kappa^2, \\
F &= \langle K_s, K_v \rangle = 1, \\
G &= \langle K_v, K_v \rangle = 1,
\end{align*}
\[ e = \langle U, K_{ss} \rangle = \pm v \kappa \tau, \quad (3.5) \]
\[ f = \langle U, K_{vs} \rangle = 0, \]
\[ g = \langle U, K_{vv} \rangle = 0. \]

4) The ruled surface
\[ K(s, v) = B(s) + vT(s) \]
is called the Darboux developable surface of \( \alpha \). The partial derivatives of \( K(s, v) \) with respect to \( s \) and \( v \) are as follows.
\[ K_s = (v \kappa - \tau)N, \]
\[ K_v = T, \]
\[ K_{ss} = -\kappa(v \kappa - \tau)T + (v \kappa - \tau)'N + \tau(v \kappa - \tau)B, \]
\[ K_{vs} = \kappa N, \]
\[ K_{vv} = 0. \]

Thus, the unit normal \( U \) and the coefficients of the first and second fundamental forms of \( K(s, v) \) are obtained as follows.
\[ U = \frac{K_s \times K_v}{\|K_s \times K_v\|} = \pm B, \]
\[ E = \langle K_s, K_s \rangle = (v \kappa - \tau)^2, \]
\[ F = \langle K_s, K_v \rangle = 0, \]
\[ G = \langle K_v, K_v \rangle = 1, \]
\[ e = \langle U, K_{ss} \rangle = \pm \tau(v \kappa - \tau), \quad (3.6) \]
\[ f = \langle U, K_{vs} \rangle = 0, \]
\[ g = \langle U, K_{vv} \rangle = 0. \]

5) The ruled surface
\[ K(s, v) = \alpha(s) + v\tilde{D}(s) \]
is called the rectifying developable surface of \( \alpha \), where
\[ \tilde{D}(s) = (\frac{\tau}{\kappa})(s)T(s) + B(s) \]
is the modified Darboux vector field of \( \alpha \). The partial derivatives of \( K(s, v) \) with respect to \( s \) and \( v \) are as follows.
\[ K_s = (1 + v(\frac{\tau}{\kappa}'))T, \]
\[ K_v = \frac{\tau}{\kappa}T + B, \]
\[ K_{ss} = v(\frac{\tau}{\kappa}'')T + \kappa(1 + v(\frac{\tau}{\kappa}'))N, \]
\[ K_{vs} = (\frac{\tau}{\kappa}')T, \]
\[ K_{vv} = 0. \]
Thus, the unit normal $U$ and the coefficients of the first and second fundamental forms of $K(s,v)$ are obtained as follows.

$$U = \frac{K_s \times K_v}{\|K_s \times K_v\|} = \pm N,$$

$$E = \langle K_s, K_s \rangle = \left(1 + v\left(\tau \kappa \right)^{\prime}\right)^2,$$

$$F = \langle K_s, K_v \rangle = \frac{\tau}{\kappa}(1 + v\left(\tau \kappa \right)^{\prime}),$$

$$G = \langle K_v, K_v \rangle = 1 + \frac{\tau^2}{\kappa^2},$$

$$e = \langle U, K_{ss} \rangle = \pm \kappa(1 + v\left(\tau \kappa \right)^{\prime}),$$

$$f = \langle U, K_{vs} \rangle = 0,$$

$$g = \langle U, K_{vv} \rangle = 0.$$  \hspace{1cm} (3.7)

6) The ruled surface

$$K(s,v) = -\bar{D}(s) + vN(s)$$

is called the tangential Darboux developable surface of $\alpha$, where

$$\bar{D}(s) = \frac{1}{\sqrt{\kappa^2(s) + \tau^2(s)}}(\tau(s)T(s) + \kappa(s)B(s))$$

is the unit Darboux vector field of $\alpha$. The partial derivatives of $K(s,v)$ with respect to $s$ and $v$ are as follows.

$$K_s = (v - \sigma(s))N^\prime,$$

$$K_v = N,$$

$$K_{ss} = -\sigma^\prime(s)N^\prime + (v - \sigma(s))N^{\prime\prime},$$

$$K_{vs} = N^\prime,$$

$$K_{vv} = 0,$$

where $N^{\prime\prime} = -\kappa^\prime T - (\kappa^2 + \tau^2)N + \tau^\prime B$. Thus, the unit normal $U$ and the coefficients of the first and second fundamental forms of $K(s,v)$ are obtained as follows.

$$U = \frac{K_s \times K_v}{\|K_s \times K_v\|} = \pm \bar{D}$$

$$E = \langle K_s, K_s \rangle = (v - \sigma(s))^2(\kappa^2 + \tau^2),$$

$$F = \langle K_s, K_v \rangle = 0,$$

$$G = \langle K_v, K_v \rangle = 1,$$

$$e = \langle U, K_{ss} \rangle = \pm(\kappa^2 + \tau^2)\sigma(s)(v - \sigma(s)),$$

$$f = \langle U, K_{vs} \rangle = 0,$$

$$g = \langle U, K_{vv} \rangle = 0.$$  \hspace{1cm} (3.8)

From now on, we will investigate the relationship among characteristic curves of developable surfaces.
4. Main Results

We give main theorems that characterize the parameter curves are also particular curves on $K(s,v)$ such as geodesic curves, asymptotic curves or lines of curvature.

**Theorem 4.** Let $K(s,v) = \alpha(s) + v\delta(s)$ be a developable surface with $\|\delta(s)\| = 1$. Then, the following expressions are satisfied for the parameter curves of $K(s,v)$.

i) $s$–parameter curves are also asymptotic curves if and only if

$$\frac{\det(\delta, \delta', \delta'')}{\det(T, \delta, N)} = \frac{\kappa}{v^2},$$

where $T$ is the tangent, $N$ is the principal normal, and $\kappa$ is the curvature of $\alpha$.

ii) $s$–parameter curves cannot also be geodesic curves.

iii) $v$–parameter curves are straight lines.

iv) The parameter curves are also lines of curvature if and only if the tangent of the base curve $\alpha$ is perpendicular to the director curve $\delta$.

**Proof.** i) If we use Eq.(3.1) and Eq.(3.2), and then take inner product of $U$ and $K_{ss}$, we obtain

$$\langle U, K_{ss} \rangle = \kappa \langle T \times \delta, N \rangle + \kappa v \left( \delta \times \delta' \right) + v \left( T \times \delta, \delta' \right) + v^2 \left( \delta' \times \delta, \delta'' \right) = 0$$

$$\frac{\det(\delta, \delta', \delta'')}{\det(T, \delta, N)} = \frac{\kappa}{v^2}.$$

ii) If we take cross product of $U$ and $K_{ss}$, we get

$$U \times K_{ss} = -[\kappa \langle T \times \delta, N \rangle + \kappa v \left( \delta \times \delta' \right)]T + [\kappa v \langle T, \delta \rangle + v \langle T, \delta' \rangle + v^2 \langle \delta', \delta'' \rangle] \delta$$

Since $T, \delta$, and $\delta'$ are linearly dependent, the coefficients of them cannot be zero at the same time, i.e., $s$–parameter curves cannot also be geodesic curves.

iii) Since $U \times K_{vv} = 0$ and $\langle U, K_{vv} \rangle = 0$, $v$–parameter curves are both geodesic curves and asymptotic curves, namely, $v$–parameter curves are straight lines.

iv) Since $K(s,v)$ is the developable surface, from Eq.(3.1), we obtain

$$f = \langle U, K_{vv} \rangle = \langle T \times \delta, \delta' \rangle + v \left( \delta' \times \delta, \delta' \right) = 0,$$

$$F = \langle K_s, K_v \rangle = \langle T, \delta \rangle + v \langle \delta, \delta' \rangle.$$

By the last equation, the parameter curves are also lines of curvature if and only if $\langle T, \delta \rangle = 0$, that is, the tangent of the base curve $\alpha$ is perpendicular to the director curve $\delta$.

**Theorem 5.** Let $K(s,v)$ be the normal surface of $\alpha$. Then, the following expressions are satisfied for the parameter curves of $K(s,v)$.

i) $s$–parameter curves are also asymptotic curves if and only if the exactly one $s_0$–parameter curve is the base curve, i.e., $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a circular helix.

ii) $s$–parameter curves are also geodesic curves if and only if the base curve $\alpha$ is a circular helix.

iii) $v$–parameter curves are straight lines.
iv) The parameter curves are also lines of curvature if and only if the base curve $\alpha$ is a plane curve.

Proof. i) From Eq.(3.3), we have

$$\langle U, K_{ss} \rangle = 0 \iff v\tau' + v^2\tau^2\frac{(K)}{\tau}' = 0$$

$$v\tau' + v^2\tau^2\frac{(K)}{\tau}' = 0$$

$$v = 0 \text{ or } v = 0 \text{ or } \frac{1}{\tau}' = v\frac{(\kappa)}{\tau}'$$

where $c$ is a constant. Then, the $s_0-$ parameter curve is the base curve $K(s,0) = \alpha(s)$ or $\kappa$ and $\tau$ are constants. As a result, the base curve $\alpha$ becomes a circular helix.

ii) $s-$parameter curves are also geodesic curves if and only if

$$U \times K_{ss} = 0 \iff v\tau' - v^2\kappa \tau^2 - \kappa = 0$$

Since $T$, $N$, and $B$ are linearly independent, we have

$$(\kappa - v(\kappa^2 + \tau^2))(1 - v\kappa) = 0,$$

$$v\kappa' - \frac{v^2}{2}(\kappa^2 + \tau^2)' = 0,$$

$$v\tau(\kappa - v(\kappa^2 + \tau^2)) = 0.$$

From the expression of $U$, we see that both $1 - v\kappa$ and $v\tau$ cannot be zero at the same time. Hence, by the first and last equations above, we obtain $\kappa = v(\kappa^2 + \tau^2)$.

If we substitute this in the second equation, we get $\frac{v^2}{2}(\kappa^2 + \tau^2)' = 0$. As $\kappa \neq 0$, $v \neq 0$. Then we have $\kappa^2 + \tau^2 = constant$. Since $v$ and $\kappa^2 + \tau^2$ are constants, from $\kappa = v(\kappa^2 + \tau^2)$, it follows that $\kappa$ is a constant and thus $\tau$ is a constant. In other words, the base curve $\alpha$ is a circular helix.

iii) Since $U \times K_{v} = 0$ and $\langle U, K_{vv} \rangle = 0$, $v-$parameter curves are both geodesic curves and asymptotic curves, namely, $v-$parameter curves are straight lines.

iv) We have $F = 0$. Moreover,

$$f = \frac{\tau}{\sqrt{(1 - v\kappa)^2 + v^2\tau^2}} = 0 \iff \tau = 0.$$

Then, the parameter curves are also lines of curvature if and only if the base curve $\alpha$ is a plane curve.

\begin{theorem}
Let $K(s,v)$ be the binormal surface of $\alpha$. Then, the following expressions are satisfied for the parameter curves of $K(s,v)$.

i) $s-$parameter curves are also asymptotic curves if and only if the curvatures of $\alpha$ hold the differential equation

$$v\tau' - v^2\kappa \tau^2 - \kappa = 0.$$ 

\end{theorem}
ii) $s$–parameter curves are also geodesic curves if and only if the exactly one $s_0$–parameter curve is the base curve, i.e., $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a plane curve.

iii) $v$–parameter curves are straight lines.

iv) The parameter curves are also lines of curvature if and only if the base curve $\alpha$ is a plane curve.

Proof. i) From Eq.(3.4), we have

$$\langle U, K_{ss} \rangle = 0 \iff -v^2 \kappa \tau^2 - \kappa + v \tau' = 0.$$  

ii) $s$–parameter curves are also geodesic curves if and only if

$$U \times K_{ss} = \frac{1}{\sqrt{1 + v^2 \tau^2}}[v \tau^2 T - v^2 \tau^3 N + v^2 \tau \tau' B] = 0.$$  

Since $T$, $N$, and $B$ are linearly independent, we have

$$v \tau^2 = 0,$$

$$v^2 \tau^3 = 0,$$

$$v^2 \tau \tau' = 0.$$  

Then, it follows that $v = 0$ or $\tau = 0$, i.e., the $s_0$–parameter curve is the base curve $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a plane curve.

iii) Since $U \times K_{vv} = 0$ and $(U, K_{vv}) = 0$, $v$–parameter curves are both geodesic curves and asymptotic curves, namely, $v$–parameter curves are straight lines.

iv) We have $F = 0$. Moreover,

$$f = \frac{\tau}{\sqrt{1 + v^2 \tau^2}} = 0 \iff \tau = 0.$$  

Then, the parameter curves are also lines of curvature if and only if the base curve $\alpha$ is a plane curve. \hfill $\square$

**Theorem 7.** Let $K(s,v)$ be the tangent developable surface of $\alpha$. Then, the following expressions are satisfied for the parameter curves of $K(s,v)$.

i) $s$–parameter curves are also asymptotic curves if and only if the exactly one $s_0$–parameter curve is the base curve, i.e., $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a plane curve.

ii) $s$–parameter curves are also geodesic curves if and only if the exactly one $s_0$–parameter curve is the base curve, i.e., $K(s,0) = \alpha(s)$.

iii) $v$–parameter curves are straight lines.

iv) The parameter curves cannot also be lines of curvature.

Proof. i) From Eq.(3.5), we have

$$\langle U, K_{ss} \rangle = 0 \iff \pm v \kappa \tau = 0.$$  

Since $\kappa \neq 0$, $v = 0$ or $\tau = 0$, i.e., the $s_0$–parameter curve is the base curve $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a plane curve.

ii) $s$–parameter curves are also geodesic curves if and only if

$$U \times K_{ss} = \mp(v^2 \kappa^2 + \kappa \tau')T + v^2 \kappa^2 N = 0.$$  

Since $T$ and $N$ are linearly independent, we have

$$\kappa + v \kappa' = 0,$$

$$v \kappa^2 = 0.$$  

Then, it follows that $v = 0$ or $\tau = 0$, i.e., the $s_0$–parameter curve is the base curve $K(s,0) = \alpha(s)$ or the base curve $\alpha$ is a plane curve. Therefore, the parameter curves cannot also be lines of curvature.
Since \( \kappa \neq 0 \), it follows that \( v = 0 \), i.e., the \( s_0 \)-parameter curve is the base curve \( K(s, 0) = \alpha(s) \).

iii) Since \( U \times K_{vv} = 0 \) and \( \langle U, K_{vv} \rangle = 0 \), \( v \)-parameter curves are both geodesic curves and asymptotic curves, namely, \( v \)-parameter curves are straight lines.

iv) We have \( F = 0 \) and \( f = 1 \). Therefore, the parameter curves cannot also be lines of curvature. \( \square \)

**Theorem 8.** Let \( K(s, v) \) be the Darboux developable surface of \( \alpha \). Then, the following expressions are satisfied for the parameter curves of \( K(s, v) \).

i) \( s \)-parameter curves are also asymptotic curves if and only if the base curve \( \alpha \) is a plane curve or a general helix.

ii) \( s \)-parameter curves are also geodesic curves if and only if the base curve \( \alpha \) is a general helix.

iii) \( v \)-parameter curves are straight lines.

iv) The parameter curves are also lines of curvature.

**Proof.**

i) From Eq.(3.6), we have

\[
\langle U, K_{ss} \rangle = 0 \iff \pm \tau(v\kappa - \tau) = 0.
\]

Then, \( \tau = 0 \) or \( \frac{\tau}{\kappa} = v \) = constant, i.e., the base curve \( \alpha \) is a plane curve or a general helix.

ii) \( s \)-parameter curves are also geodesic curves if and only if

\[
U \times K_{ss} = \mp(v\kappa - \tau)T \mp \kappa(v\kappa - \tau)N = 0.
\]

Since \( T \) and \( N \) are linearly independent, we have

\[
(v\kappa - \tau)' = 0, \quad \kappa(v\kappa - \tau) = 0.
\]

Then, it follows that \( \frac{\tau}{\kappa} = v \) = constant, i.e., the base curve \( \alpha \) is a general helix.

iii) Since \( U \times K_{vv} = 0 \) and \( \langle U, K_{vv} \rangle = 0 \), \( v \)-parameter curves are both geodesic curves and asymptotic curves, namely, \( v \)-parameter curves are straight lines.

iv) We have \( F = f = 0 \). Consequently, the parameter curves are also lines of curvature. \( \square \)

**Theorem 9.** Let \( K(s, v) \) be the rectifying developable surface of \( \alpha \). Then, the following expressions are satisfied for the parameter curves of \( K(s, v) \).

i) \( s \)-parameter curves are also asymptotic curves if and only if \( \frac{\tau}{\kappa} \) is a linear function.

ii) \( s \)-parameter curves are also geodesic curves if and only if the exactly one \( s_0 \)-parameter curve is the base curve, i.e., \( K(s, 0) = \alpha(s) \) or \( \frac{\tau}{\kappa} \) is a linear function.

iii) \( v \)-parameter curves are straight lines.

iv) The parameter curves are also lines of curvature if and only if the base curve \( \alpha \) is a plane curve or \( \frac{\tau}{\kappa} \) is a linear function.

**Proof.**

i) From Eq.(3.7), we have

\[
\langle U, K_{ss} \rangle = 0 \iff \pm \kappa(1 + v(\frac{\tau}{\kappa})') = 0.
\]
Since $\kappa \neq 0$, we get $\frac{\tau}{\kappa} = -\frac{1}{v}s + c$, where $c$ is a constant.

ii) $s$–parameter curves are also geodesic curves if and only if

$$U \times K_{ss} = \pm v\left(\frac{\tau}{\kappa}\right)^{\prime\prime} B = 0.$$ 

Then, it follows that $v = 0$ or $(\frac{\tau}{\kappa})^{\prime\prime} = 0$, i.e., the $s_0$–parameter curve is the base curve $K(s, 0) = \alpha(s)$ or $\frac{\tau}{\kappa} = -as + c$, where $a$ and $c$ are constants.

iii) Since $U \times K_{vv} = 0$ and $\langle U, K_{vv} \rangle = 0$, $v$–parameter curves are both geodesic curves and asymptotic curves, namely, $v$–parameter curves are straight lines.

iv) We have $f = 0$. Consequently, the parameter curves are also lines of curvature.

Theorem 10. Let $K(s, v)$ be the tangential Darboux developable surface of $\alpha$. Then, the following expressions are satisfied for the parameter curves of $K(s, v)$.

i) $s$–parameter curves are also asymptotic curves if and only if the base curve $\alpha$ is a general helix or a slant helix.

ii) $s$–parameter curves are also geodesic curves if and only if the base curve $\alpha$ is a plane curve or a slant helix.

iii) $v$–parameter curves are straight lines.

Proof. i) From Eq.(3.8), we have

$$\langle U, K_{ss} \rangle = 0 \iff \pm(\kappa^2 + \tau^2)\sigma(s)(v - \sigma(s)) = 0.$$ 

In the last equation, $\kappa$ and $\tau$ cannot be zero at the same time. Hence, it concludes that $\sigma(s) = 0$ or $\sigma(s) = v = \text{constant}$ in other words the base curve $\alpha$ is a general helix or a slant helix.

ii) $s$–parameter curves are also geodesic curves if and only if

$$U \times K_{ss} = \mp\kappa(v - \sigma(s))\sqrt{\kappa^2 + \tau^2}T\pm(\kappa^2 + \tau^2)(\frac{\sigma(s) - v}{\sqrt{\kappa^2 + \tau^2}})^{\prime} N \mp \tau(v - \sigma(s))\sqrt{\kappa^2 + \tau^2} B = 0.$$ 

Since $T$, $N$, and $B$ are linearly independent, we obtain

$$\kappa(v - \sigma(s))\sqrt{\kappa^2 + \tau^2} = 0,$$

$$(\kappa^2 + \tau^2)(\frac{\sigma(s) - v}{\sqrt{\kappa^2 + \tau^2}})^{\prime} = 0,$$

$$\tau(v - \sigma(s))\sqrt{\kappa^2 + \tau^2} = 0.$$ 

From this, we get $\tau = 0$ or $\sigma(s) = v = \text{constant}$, i.e., the base curve $\alpha$ is a plane curve or a slant helix.

iii) Since $U \times K_{vv} = 0$ and $\langle U, K_{vv} \rangle = 0$, $v$–parameter curves are both geodesic curves and asymptotic curves, namely, $v$–parameter curves are straight lines.

iv) We have $F = f = 0$. Consequently, the parameter curves are also lines of curvature. \qed
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