OPTIMAL GRADIENT ESTIMATES FOR A CONDUCTIVITY PROBLEM WITH DIMENSIONS MORE THAN TWO

LINJIE MA

Abstract. In high-contrast composite materials, the electric (or stress) field may blow up in the narrow region between inclusions. The gradient of solutions depend on \( \varepsilon \), the distance between the inclusions, where \( \varepsilon \) approaches to 0. We using the maximum principle techniques to give another proof of the Dong-Li-Yang estimates \([15]\) for any convex inclusions of arbitrary shape with \( n \geq 3 \). This result solves the problem raised by \([29]\), which considered the case of disk inclusions with \( n \geq 4 \).

1. Introduction and main results

1.1. Background. Let \( D \) be a bounded open set in \( \mathbb{R}^n, n \geq 3 \), containing two subdomains \( D_1 \) and \( D_2 \), with \( \varepsilon \)-apart, for a small positive constant \( \varepsilon \). For a given appropriate function \( g \), we consider the following conductivity problem with Dirichlet boundary data

\[
\begin{aligned}
&-\nabla (a_k(x) \nabla u_{k,\varepsilon}) = 0, & \text{in } D,
&u_{k,\varepsilon} = g, & \text{on } \partial D,
\end{aligned}
\]

where

\[
a_k(x) = \begin{cases} 
 k \in [0, 1) \cup (1, \infty], & \text{in } D_1 \cup D_2, \\
 1, & \text{in } D_0 := D \setminus D_1 \cup D_2. 
\end{cases}
\]

In the context of electric conduction, the elliptic coefficients \( a_k \) refer to conductivity, and the solution \( u_k \) represent voltage potential. From an engineering point of view, the most important quality is \( \nabla u_k \), representing the electric field. The above model arises from the study of composite material \([7]\), where Babuška etc. analyzed numerically that the high concentration of extreme electric field will occur in the narrow region between the adjacent inclusions or between the inclusions and boundaries. Bonnetier and Vogelius \([13]\) proved that \( \nabla u_k \) is bounded for a fixed \( k \), which is far away from 0 and \( \infty \), and circular inclusions in dimension \( n = 2 \). Later, Li and Vogelius \([24]\) extended the results to the general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions in any dimension. In \([23]\), Li and Nirenberg extended the results in \([24]\) to general second order elliptic systems of divergence form.

When \( k \) equals to \( \infty \) (perfect conductor) or 0 (insulator), the gradient of solutions is much different. It was shown in \([14, 18, 28]\) that the gradient general become unbounded, as \( \varepsilon \to 0 \). Ammari et.al. in \([4, 5]\) considered the perfect and insulate conductivity problem for the disk inclusions in dimension two, and gave the blow up rate \( \varepsilon^{-1/2} \) in both cases. They also showed that that the blow up rate is optimal. For the perfect conductivity problem in high dimensions, Yun extended

Date: February 25, 2022.
the results to any bounded strictly convex smooth domains \([30,32]\). Bao, Li and Yin in \([8,9]\) considered the higher dimensions and give the optimal blow up rate, \(\varepsilon^{-1/2}\) for \(n = 2\), \(\varepsilon^{-1/2}\ln\varepsilon\) for \(n = 3\), \(\varepsilon^{-1}\) for \(n \geq 4\). For further works, see e.g., \([1–3,6,8,10–12,16,17,19–22,27,30,31]\) and their references therein.

When \(k\) goes to 0, \(u_k\) converges to the solution of the following insulated conductive problem:

\[
\begin{cases}
\Delta u = 0, & \text{in } D_0, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_1 \cup \partial D_2, \\
u = g & \text{on } \partial D.
\end{cases}
\] (1.2)

where \(\nu\) is the outward unit normal vector. For the insulated conductivity problem, it was proved in \([9]\) that the optimal blow up rate is \(\varepsilon^{-1/2}\) in \(\mathbb{R}^2\). Yun in \([32]\) considered two circle balls and give the optimal blow rate \(\varepsilon^{\frac{\sqrt{2}-2}{2}}\). Li and Yang in \([25,26]\) improved the upper bound in dimension \(n \geq 3\) to be of order \(\varepsilon^{-1/2+\beta}\) for some \(\beta > 0\). Later, Weinkove in \([29]\) used another method for \(n \geq 4\) and improve the results in \([25]\). Recently, Dong, Li and Li considered the optimal gradient estimates for \(n \geq 3\) in \([15]\) and gave the optimal blow up rate for \(n \geq 3\):

\[
-(n-1) + \sqrt{(n-1)^2 + 4(n-2)}
\]

(1.3)

Until now, for the insulated conductivity problem with any dimensions, the blow-up rate has been determined. But there are still some questions need to be solved. In \([29]\), Weinkove used the maximum principle techniques to deal with the problem, but the result has two shortcomings: it won’t give the blow-up rate for \(n = 3\) and only deal with the case for inclusions are two balls. It is worth to be considering that whether we can use the techniques in \([29]\) to get the optimal blow up rate \((1.3)\) for any convex inclusions of arbitrary shape with any dimensions \(n \geq 3\), which is

\[\text{Figure 1.}\]
also an open problem raised in [29]. In our paper, we give a positive answer for this question.

1.2. Our domain. Before stating our main results, we first fix our domain. We use \( x = (x', x_n) \) to denote a point in \( \mathbb{R}^n \), \( x' = (x_1, x_2, \ldots, x_{n-1}) \), \( n \geq 3 \). Let \( D \) be a bounded open set in \( \mathbb{R}^n \) that contains a pair of subdomain \( D_1 \) and \( D_2 \) with \( \varepsilon \) distance.

\[
\begin{aligned}
\Omega_r &:= \left\{(x', x_n) \in \mathbb{R}^n \mid -\varepsilon + h_2(x') < x_n < \varepsilon + h_1(x'), |x'| < r\right\}.
\end{aligned}
\]

For \( 0 < r \leq R_0 \), define

\[
\begin{aligned}
\Gamma_+ &= \left\{x_n = \frac{\varepsilon}{2} + h_1(x'), |x'| < R_0\right\}, \\
\Gamma_- &= \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'), |x'| < R_0\right\},
\end{aligned}
\]

where \( h_1 \) and \( h_2 \) satisfy the following assumptions:

\[
\begin{aligned}
h_1(x') > h_2(x'), \quad &\text{for } |x'| < R_0; \\
\|h_1\|_{C^2(\mathbb{B}_{2R_0})} + \|h_2\|_{C^2(\mathbb{B}_{2R_0})} \leq \mu,
\end{aligned}
\]

for some constant \( \mu \).

For \( 0 < r \leq R_0 \), define

\[
\begin{aligned}
\Delta u = 0, &\quad \text{in } D_0, \\
\frac{\partial u}{\partial n} = 0 &\quad \text{on } \Gamma_+ \cup \Gamma_-, \\
\|u\|_{L^\infty} \leq 1.
\end{aligned}
\]
The idea of this paper comes from [29], which using the maximum principle to deal with a specific quantity in the narrow region between the insulators:

\[
\left[ (|x'|^2 + \sigma)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)} - A(bx_n^2 + |x'|^4 + \sigma)\gamma^{-1/2} \right] |\nabla u|^2. \tag{1.8}
\]

\(|x'|^2 + \sigma)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)}\) is the main term, and \(A(bx_n^2 + |x'|^4 + \sigma)\gamma^{-1/2}\) is a small term which is used to adjust the quantity in boundary and interior. \(\sigma\) and \(\delta\) are small positive constants. We revise (1.8) by

\[
\left\{ (|x'|^2 + b\varepsilon)^{1-\gamma} - (b\varepsilon)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)} - A(|x'|^2 + b\varepsilon)^{-\gamma} x_n^2 \right\} |\nabla u|^2.
\]

Here, \((|x'|^2 + b\varepsilon)^{1-\gamma} - (b\varepsilon)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)}\) is our main term. Compared to the main term in (1.8), they are equivalent. But our main term has two advantages: On the one hand, it keeps the parameters \(A\) do not have to be too large, which relate to the second derivative in the interior of the narrow region. On the other hand, it will cause the negative terms smaller, which is much useful to prevent the second derivative blow up, especially for \(n\) small. The term \(-A(|x'|^2 + b\varepsilon)^{-\gamma} x_n^2\) has the same effect compared to the small term in (1.8). The constant \(b\) is chosen to optimize the estimates.

Next, we give our main results.

**Theorem 1.1.** Let \(D, D_1, D_0\) be defined as above and satisfies (1.4)-(1.6). \(g \in C^{1,\alpha}(\partial D)\). Assume that \(u \in H^1(D) \cap C^1(\overline{D_0})\) to system (1.2), then for \(n \geq 3\), we have

\[
\|\nabla u\|_{L^\infty(\Omega_{R/2})} \leq \frac{C\|g\|_{C^{1,\alpha}(\partial D)}}{(\varepsilon + |x'|^2)^{(1-\gamma)/2}}, \quad 0 < \gamma \leq \gamma^*, \tag{1.9}
\]

where

\[
\gamma^* = \gamma^*(n) := \frac{-n(n - 1) + \sqrt{(n - 1)^2 + 4(n - 2)}}{2} \in (0, 1). \tag{1.10}
\]

**Remark 1.2.** From (1.10), when \(n = 3\), \(\gamma^* = \sqrt{2} - 1\), the blow up rate is \(\frac{\sqrt{2} - 2}{\varepsilon}\), which is consistent with [15] and [32]. In the following table we give the exact and approximate numerical values of blow up rate \(-\frac{1-\gamma^*}{2}\) for \(n = 3, 4, 5, 6, \infty\):

| \(n\) | \(\gamma^*\) | \(-\frac{1-\gamma^*}{2}\) | approx. |
|-------|---------------|----------------|--------|
| 3     | \(\sqrt{2} - 1\) | \(-\frac{2-\sqrt{2}}{2}\) | -0.2929 |
| 4     | \(\frac{\sqrt{7} - 3}{2}\) | \(-\frac{5-\sqrt{7}}{4}\) | -0.2192 |
| 5     | \(\sqrt{7} - 2\) | \(-\frac{3-\sqrt{7}}{2}\) | -0.1771 |
| 6     | \(\frac{\sqrt{11} - 3}{2}\) | \(-\frac{7-\sqrt{11}}{4}\) | -0.1492 |
| \(\infty\) | 1 | 0 | 0 |

Obviously, the blow up rate is monotonically increasing about \(n\), that means the electric field concentration phenomenon will disappear as \(n \to \infty\).

The proof of Theorem 1.1 will be given in section 2.
2. Proof of Theorem 1.1

In the following, without loss of generality, we assume that \( \lambda_1 = \lambda_2 = \frac{1}{2} \). Firstly, we have the following lemma, which is similar to Lemma 2.1 in [29].

**Lemma 2.1.** At any point of \( \Gamma_+ \) and \( \Gamma_- \), we have

\[
\frac{\partial}{\partial \nu}(|\nabla u|^2) = 2|\nabla u|^2. \tag{2.1}
\]

**Proof.** Since \( \lambda_1 = \lambda_2 = \frac{1}{2} \), from (1.5), the outward normal vector to \( \Gamma_+ \) is given by

\[\nu = (-x_1, \cdots, -x_{n-1}, 1).\]

So \( \frac{\partial u}{\partial \nu} = 0 \) on \( \Gamma_+ \) becomes

\[u_n - \sum_{j=1}^{n-1} x_j u_{x_j} = 0. \tag{2.2}\]

Differentiating (2.2) with respective to \( x_i, i = 1, \cdots, n \), we have

\[u_{x_n x_i} - u_{x_i} - \sum_{j=1}^{n-1} x_j u_{x_j x_i} = 0,
\]

that is

\[\sum_{j=1}^{n-1} x_j u_{x_j x_i} = u_{x_n x_i} - u_{x_i}.
\]

Hence,

\[
\frac{\partial}{\partial \nu}(|\nabla u|^2) = -2 \sum_{j=1}^{n-1} \sum_{i=1}^{n} x_j u_{x_i} u_{x_j x_i} + 2 \sum_{i=1}^{n} u_{x_i} u_{x_i x_n}
\]

\[= -2 \sum_{i=1}^{n} u_{x_i} (u_{x_n x_i} - u_{x_i}) + 2 \sum_{i=1}^{n} u_{x_i} u_{x_i x_n}
\]

\[= 2 \sum_{i=1}^{n} u_{x_i} u_{x_i} = 2|\nabla u|^2,
\]

as required. \(\square\)

**Proof of Theorem 1.1.** Since we assume that \( \lambda_1 = \lambda_2 = \frac{1}{2} \), we have

\[-\frac{1}{2}|x'|^2 - \frac{\varepsilon}{2} < x_n < \frac{1}{2}|x'|^2 + \frac{\varepsilon}{2} < 1 \text{ in } \Omega_r.\]

Consider the quantity

\[Q = F|\nabla u|^2, \text{ in } \Omega_r,
\]

where

\[F := (|x'|^2 + b\varepsilon)^{1-\gamma} - (b\varepsilon)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)} - A(|x'|^2 + b\varepsilon)^{-\gamma} x_n^2,
\]

\(A, b \) are uniform constants satisfying

\[2\gamma < A < \left(2 + \frac{1}{10}\right) \gamma, \quad b \gg 1. \tag{2.3}\]
We assume that the quantity \( Q \) achieves a maximum at \( p \) in \( \Omega_r \). If \( p \) is in \( D_0 \cap \partial B_r \), we have
\[
\left( (|x'|^2 + b\varepsilon)^{1-\gamma} - (b\varepsilon)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)} \right) |\nabla u|^2 \leq C||g||_{C^{1,\alpha}(\partial D)}^2 \text{ on } \Omega_r,
\]
since
\[
A(|x'|^2 + b\varepsilon)^{-\gamma}x_n^2 \ll (|x'|^2 + b\varepsilon)^{1-\gamma} - (b\varepsilon)^{1-\gamma} + \varepsilon^{1-\gamma(1-\delta)}.
\] (2.4)
Hence (1.9) holds.

Now we prove that \( Q \) can only achieve its maximum on \( D_0 \cap \partial B_r \).

Firstly, we assume that \( Q \) achieves its maximum at a point \( p \in \Gamma_+ \), then by (2.1), we have
\[
0 \leq \frac{\partial Q}{\partial \nu} = \left( \frac{\partial F}{\partial \nu} + 2F \right) |\nabla u|^2 \text{ on } \Gamma_+.
\] (2.5)
Since \( x_n = \frac{1}{2} |x'|^2 + \frac{r}{2} \) on \( \Gamma_+ \), we have
\[
\frac{\partial (|x'|^2)}{\partial \nu} = (2x_1, \ldots, 2x_{n-1}, 0)(-x_1, \ldots, -x_{n-1}, 1) = -2|x'|^2 \text{ on } \Gamma_+.
\]
\[
\frac{\partial x_n}{\partial \nu} = (0, \ldots, 0, 1)(-x_1, \ldots, -x_{n-1}, 1) = 1 \text{ on } \Gamma_+.
\]
Then by Lemma 2.1 and above,
\[
\frac{\partial F}{\partial \nu} + 2F
= \left[ -2(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma}|x'|^2 - 2A\gamma(|x'|^2 + b\varepsilon)^{-\gamma-1}|x'|^2 x_n^2 - 2A(|x'|^2 + b\varepsilon)^{-\gamma} x_n^2 
+ 2(|x'|^2 + b\varepsilon)^{-1-\gamma} - 2(b\varepsilon)^{-1-\gamma} + 2\varepsilon^{1-\gamma(1-\delta)} - 2A(|x'|^2 + b\varepsilon)^{-\gamma} x_n^2 \right]
\leq \left( 2\gamma - A \right) (|x'|^2 + b\varepsilon)^{-1-\gamma} + 2\varepsilon^{1-\gamma(1-\delta)} < 0,
\] (2.6)
where for the last line, we used the inequalities (2.3) and neglect the term \( 2\varepsilon^{1-\gamma(1-\delta)} \) on the right hand side of (2.6), this is because
\[
2\varepsilon^{1-\gamma(1-\delta)} \ll (|x'|^2 + b\varepsilon)^{1-\gamma}.
\]
Combining (2.5) and (2.6),
\[
0 \leq \frac{\partial Q}{\partial \nu} < 0, \text{ on } \Gamma_+,
\]
which is a contraction.

Similarly, we can also prove that the maximum cannot attained on \( \Gamma_- \).

Next, we assume that \( Q \) achieves a maximum at a point \( p \). If \( p \in B_r \), we have
\[
0 \geq \Delta Q = \Delta F |\nabla u|^2 + 2\nabla F \cdot \nabla(|\nabla u|^2) + 2F |\nabla \nabla u|^2.
\] (2.7)
In the following, we prove that
\[
\Delta Q > 0.
\] (2.8)

**Step 1. Estimates of** \( 2\nabla F \cdot \nabla(|\nabla u|^2) \).
For $2\partial_{x_n}F \cdot \partial_{x_n}(|\nabla u|^2)$, by Cauchy inequality, immediately we have

$$2\partial_{x_n}F \cdot \partial_{x_n}(|\nabla u|^2) = 4F_{x_n} \sum_{j=1}^{n} u_{x_j} u_{x_j x_n}$$

$$\geq -4\eta |\nabla u|^2 - \frac{4}{\eta} |F_{x_n}|^2 |\nabla \nabla u|^2,$$

(2.9)

where $\eta$ is some small positive constant which may differ from line to line, and which can be shrunk at the expense of shrinking $\varepsilon$ or $r$.

Since at maximum of $Q$, for $i = 1, \ldots, n - 1$, it holds that

$$0 = Q_{x_i} = F_{x_i} |\nabla u|^2 + F \partial_{x_i}(|\nabla u|^2).$$

Thus, we have

$$\partial_{x_i}(|\nabla u|^2) = -\frac{F_{x_i} |\nabla u|^2}{F},$$

which leads

$$2\partial_{x_i}F \cdot \partial_{x_i}(|\nabla u|^2) = -\frac{2}{F} F_{x_i}^2 |\nabla u|^2.$$

Sum above for $i$ from 1 to $n - 1$, one has

$$\sum_{i=1}^{n-1} 2\partial_{x_i}F \cdot \partial_{x_i}(|\nabla u|^2) = -\frac{\sum_{i=1}^{n-1} 2F_{x_i}^2}{F} |\nabla u|^2.$$

(2.10)

Combining (2.10) and (2.9) together, we get

$$F_1 := 2\nabla F \cdot \nabla(|\nabla u|^2) \geq - \left[ \frac{\sum_{i=1}^{n-1} 2F_{x_i}^2}{F} + 4\eta \right] |\nabla u|^2 - \frac{4}{\eta} |F_{x_n}|^2 |\nabla \nabla u|^2.$$

(2.11)

On the other hand, we may make a change of coordinates so that $x_2 = \cdots = x_{n-1} = 0$ and $x_1 \geq 0$ and hence

$$\nabla |x'|^2 = (2|x'|, 0, \ldots, 0).$$

Next, we use the fact that at the maximum of $Q$ we have

$$0 = Q_{x_1} = F_{x_1} |\nabla u|^2 + F \partial_{x_1}(|\nabla u|^2),$$

that is

$$F_{x_1} = -\frac{F \partial_{x_1}(|\nabla u|^2)}{|\nabla u|^2}.$$

Then by Cauchy-Schwarz inequality,

$$2\partial_{x_1}F \cdot \partial_{x_1}(|\nabla u|^2) = -2F \left[ \partial_{x_1}(|\nabla u|^2) \right]^2 \geq -8F \left( \sum_{j=1}^{n} u_{x_j} u_{x_j x_1} \right)^2$$

$$\geq -8F \sum_{j=1}^{n} u_{x_j x_1}^2.$$
Using the fact that \( u \) is harmonic, one has

\[
\sum_{j=1}^{n} \frac{u_{x_{j}x_{j}}}{n} = \frac{n-1}{n} u_{x_{j}x_{j}} + \frac{1}{n} \sum_{j=2}^{n} u_{x_{j}x_{j}}\n
\leq \frac{n-1}{n} u_{x_{j}x_{j}} + \frac{1}{n} \left( \sum_{k=1}^{n} u_{x_{k}x_{k}} \right) + \frac{1}{2} \sum_{i,j=1}^{n} u_{x_{j}x_{j}}
\]

\[
\leq \frac{n-1}{n} u_{x_{j}x_{j}} + \frac{n-1}{n} \sum_{k=1}^{n} u_{x_{k}x_{k}} + \frac{n-1}{n} \sum_{i,j=1}^{n} u_{x_{j}x_{j}}
\]

\[
\leq \frac{n-1}{n} \sum_{i,j=1}^{n} u_{x_{j}x_{j}} = \frac{n-1}{n} |\nabla \nabla u|^2,
\]

we have

\[
2 \partial_{x_{j}} F \cdot \partial_{x_{j}} (|\nabla u|^2) \geq - \frac{8F(n-1)}{n} |\nabla \nabla u|^2. \tag{2.12}
\]

From (2.12) and (2.9), one has

\[
\mathcal{F}_2 := 2 \nabla F \cdot \nabla (|\nabla u|^2) = 2 \partial_{x_{j}} F \cdot \partial_{x_{j}} (|\nabla u|^2) + 2 \partial_{x_{n}} F \cdot \partial_{x_{n}} (|\nabla u|^2)
\]

\[
\geq - 4\eta |\nabla u|^2 - \left[ \frac{8F(n-1)}{n} + \frac{4}{\eta} |F_{x_{n}}|^2 \right] |\nabla \nabla u|^2. \tag{2.13}
\]

Combining (2.11) and (2.13), for \( 0 < \xi < 1 \), we can write

\[
2 \nabla F \cdot \nabla (|\nabla u|^2) = \xi \mathcal{F}_1 + (1 - \xi) \mathcal{F}_2
\]

\[
\geq - \left[ \sum_{i=1}^{n-1} \frac{2F_{x_{i}}^2}{F} \xi + 4\eta \right] |\nabla u|^2
\]

\[
- \left[ \frac{8F(n-1)}{n} (1 - \xi) + \frac{4}{\eta} |F_{x_{n}}|^2 \right] |\nabla \nabla u|^2. \tag{2.14}
\]

Substituting (2.14) into (2.7), we have

\[
\Delta Q \geq \left[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_{x_{i}}^2}{F} \xi - 4\eta \right] |\nabla u|^2
\]

\[
+ \left[ 2F - \frac{8F(n-1)}{n} (1 - \xi) - \frac{4}{\eta} |F_{x_{n}}|^2 \right] |\nabla \nabla u|^2. \tag{2.15}
\]

**Step 2:** Estimates of \[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_{x_{i}}^2}{F} \xi - 4\eta \] |\nabla u|^2.
By simple computation,
\[
\Delta F = 2(n-1)(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma} - 4\gamma(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-1}|x'|^2 \\
+ 2A(n-1)\gamma(|x'|^2 + b\varepsilon)^{-\gamma-1}x_n^2 - 4A\gamma(1+\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-2}x_n^2|x'|^2 \\
- 2A(|x'|^2 + b\varepsilon)^{-\gamma} \\
\geq [2(n-1)(1-\gamma) - 2A] (|x'|^2 + b\varepsilon)^{-\gamma} - 4\gamma(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-1}|x'|^2 \\
- 4A\gamma(1+\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-2}x_n^2|x'|^2 \\
\geq 2(n-1)(1-\gamma) - (4+1/5)\gamma - \eta \eta (|x'|^2 + b\varepsilon)^{-\gamma} \\
- 4\gamma(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-1}|x'|^2, \\
\tag{2.16}
\]
where for the last line, we use (2.3) and the fact that
\[
4A\gamma(1+\gamma)(|x'|^2 + b\varepsilon)^{-\gamma-2}x_n^2|x'|^2 \leq C(|x'|^2 + b\varepsilon)^{1-\gamma} \leq \eta (|x'|^2 + b\varepsilon)^{-\gamma}.
\]
For \(F_{x_i}^2\), one has
\[
F_{x_i}^2 = \frac{2(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma}x_i + 2A\gamma(|x'|^2 + b\varepsilon)^{-\gamma-1}x_n^2x_i}{2} \\
\leq 4(1+\eta)(1-\gamma)^2(|x'|^2 + b\varepsilon)^{-2\gamma}x_i, \\
\tag{2.17}
\]
where we use that
\[
2A\gamma(|x'|^2 + b\varepsilon)^{-\gamma-1}x_n^2x_i \leq 2\eta(1-\gamma)(|x'|^2 + b\varepsilon)^{-\gamma}|x_i|.
\]
Hence, from (2.17) and (2.4), we can write
\[
-\frac{\sum_{i=1}^{n-1} 2F_{x_i}^2}{F} \geq - \frac{8(1+\eta)(1-\gamma)^2(|x'|^2 + b\varepsilon)^{-2\gamma}|x'|^2}{(|x'|^2 + b\varepsilon)^{1-\gamma} + (b\varepsilon)^{1-\gamma}} \\
\geq - \frac{8(1+\eta)(1-\gamma)^2(|x'|^2 + b\varepsilon)^{-2\gamma}|x'|^2}{(|x'|^2)^{1-\gamma} + \eta} \\
= -8(1+\eta)(1-\gamma)^2 \left(\frac{|x'|^2}{|x'|^2 + b\varepsilon}\right)^\gamma (|x'|^2 + b\varepsilon)^{-\gamma}, \\
\tag{2.18}
\]
Combining (2.16) and (2.18), one has
\[
\Delta F - \sum_{i=1}^{n-1} \frac{2F_{x_i}^2}{F} \xi - 4\eta \\
\geq 2(n-1)(1-\gamma) - (4+1/5)\gamma - \eta \eta (|x'|^2 + b\varepsilon)^{-\gamma} \\
- 4\gamma(1-\gamma) \left|\frac{|x'|^2}{|x'|^2 + b\varepsilon}\right| (|x'|^2 + b\varepsilon)^{-\gamma} - 8\xi_0(1-\gamma)^2 \left(\frac{|x'|^2}{|x'|^2 + b\varepsilon}\right)^\gamma (|x'|^2 + b\varepsilon)^{-\gamma}, \\
\tag{2.19}
\]
where \(\xi_0 = \xi(1+\eta)\).
Set
\[
\phi := \phi(x',b) = \frac{|x'|^2}{|x'|^2 + b\varepsilon} \in (0,1),
\]
then \((2.19)\) can be written by
\[
\Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F^2} \xi - 4\eta \\
\geq \left[ 2(n - 1)(1 - \gamma) - 4\gamma - 4\phi\gamma(1 - \gamma) - 8\phi^\gamma \xi_0(1 - \gamma)^2 - \frac{\gamma}{5} - \eta \right] (|x'|^2 + b\varepsilon)^{-\gamma},
\]
(2.20)

Next, we prove that
\[
\left[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F^2} \xi - 4\eta \right] |\nabla u|^2 > 0.
\]
(2.21)

Define
\[
\rho := - \left[ \gamma^2 + (n-1)\gamma - (n-2) \right] \geq 0.
\]
(2.22)
then \((2.20)\) can be written by
\[
\Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F^2} \xi - 4\eta \\
\geq \left[ \left( 4\phi - 8\xi_0\phi^\gamma \right) \gamma^2 + \left( 16\xi_0\phi^\gamma - 2n - 2 - 4\phi \right) \gamma + \left( 2(n - 1) - 8\xi_0\phi^\gamma \right) \right. \\
- \frac{\gamma}{5} - \eta \right] (|x'|^2 + b\varepsilon)^{-\gamma} \\
\geq \left\{ - \left( 4\phi - 8\xi_0\phi^\gamma \right) \rho + \left( 8(n+1)\xi_0\phi^\gamma - 2n - 2 - 4n\phi \right) \gamma \\
+ \left[ 2(n - 1) + 4(n-2)\phi - 8(n-1)\xi_0\phi^\gamma \right] - \frac{\gamma}{5} - \eta \right\} (|x'|^2 + b\varepsilon)^{-\gamma}
\]

We choose
\[
\xi_0 = 1 - \frac{n}{4(n-1)} + \eta,
\]
(2.23)
we have
\[
\Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F^2} \xi - 4\eta \\
\geq \left\{ \left[ 2 \left( 4 - \frac{n}{n-1} \right) \phi^\gamma - 4\phi \right] \rho + \left[ 2(n + 1)\phi^\gamma \left( 4 - \frac{n}{n-1} \right) - 2n - 2 - 4n\phi \right] \gamma \\
+ \left[ 2(n - 1) + 4(n-2)\phi + (8 - 6n)\phi^\gamma \right] - \frac{\gamma}{5} - \eta \right\} (|x'|^2 + b\varepsilon)^{-\gamma} \\
\geq \left\{ \left[ \left( 6n + 4 - \frac{4}{n-1} \right) \phi^\gamma - 4n\phi - 2n - 2 \right] \gamma \\
+ \left[ 2(n - 1) + 4(n-2)\phi + (8 - 6n)\phi^\gamma \right] - \frac{\gamma}{5} - \eta \right\} (|x'|^2 + b\varepsilon)^{-\gamma}
:= M(\phi, \gamma)(|x'|^2 + b\varepsilon)^{-\gamma}.
\]
where
\[ M(\phi, \gamma) := \left[ (6n + 4 - \frac{4}{n - 1}) \phi^7 - 4n\phi - 2n - 2 \right] \gamma + \left[ 2(n - 1) + 4(n - 2)\phi + (8 - 6n)\phi^7 \right] - \frac{\gamma}{5} - \eta. \]

Case 1: \( n = 3. \) At this time, one has
\[ M(\phi, \gamma) = 4 \left( 1 + \phi + 5\phi^7 \gamma - 2\gamma - \frac{5}{2} \phi^7 - 3\phi \gamma \right) - \frac{\gamma}{5} - \eta. \]
Since \( b \gg 1, \) we can choose \( b \) large enough so that \( \phi \) is much small, which ensures \( M > 0. \)
For example, \( \phi = 1/50, \) and \( 0 < \gamma \leq \gamma^* = \sqrt{2} - 1, \) we have
\[ M(\phi, \gamma^*) \approx 0.3 - \eta > 0, \]
since \( \eta \) is so small so that it can be absorbed. Hence, (2.21) holds.

Case 2: \( n \geq 4. \) From (2.22), we know that
\[ \gamma \leq \frac{n - 2}{n + 1} - \frac{\gamma^2}{n + 1}. \]
With above and the fact that \( \phi^\gamma > \phi \) is much small, we have
\[ M(\phi, \gamma) \geq \left[ (2n + 4 - \frac{4}{n - 1}) \phi - 2n - 2 \right] \gamma + \left[ 2(n - 1) + 4(n - 2)\phi - (6n - 8)\phi^7 \right] - \frac{\gamma}{5} - \eta \geq 2 \left\{ \left[ (n + 2 - \frac{2}{n - 1}) \phi - (n + 1) \right] \left( \frac{n - 2}{n + 1} - \frac{\gamma^2}{n + 1} \right) + 2(n - 1) + 4(n - 2)\phi - (6n - 8)\phi^7 \right\} - \frac{\gamma}{5} - \eta \]
\[ = 2 \left\{ 1 + \gamma^2 + (3n - 5)\phi - (3n - 4)\phi^7 \right\} - \left( \frac{3}{n + 1} + \frac{2}{n - 1} - \frac{6}{(n + 1)(n - 1)} \right) \phi - \left( n + 2 - \frac{2}{n - 1} \right) \frac{\phi^2}{n + 1} \frac{\gamma^2}{n + 1} \right\} - \frac{\gamma}{5} - \eta > 0. \]
Thus, (2.21) holds.

Step 3: Estimates of \[ 2F - \frac{8F(n-1)}{n}(1 - \xi) - \frac{4}{\eta}\|F_{x_n}\|^2 \|\nabla u\|^2. \]
For \( \|F_{x_n}\|^2, \) one has
\[ \|F_{x_n}\|^2 = \left[ 2Ax_n(|x'|^2 + b\varepsilon)^{-\gamma} \right]^2 \leq C(|x'|^2 + b\varepsilon)^{2(1-\gamma)} \leq \eta^3 (|x'|^2 + b\varepsilon)^{1-\gamma}. \]
So
\[
\frac{4}{\eta} |F_{x_n}|^2 \leq C\eta^2(|x'|^2 + b\varepsilon)^{1-\gamma}.
\]
From (2.23), we have
\[
1 - \xi = \frac{n}{4(n - 1)(1 + \eta)},
\]
which leads
\[
2F - \frac{8F(n - 1)}{n} (1 - \xi) - \frac{4}{\eta} |F_{x_n}|^2 \geq 2\eta F - 2\eta^2 (|x'|^2 + b\varepsilon)^{1-\gamma} > 0,
\]
that is
\[
\left[ 2F - \frac{8F(n - 1)}{n} (1 - \xi) - \frac{4}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2 > 0. \tag{2.24}
\]
Combining (2.21), (2.24) and (2.15), we have (2.8), which is contradict with (2.7).
Hence, we have ruled out the possibility that \( Q \) obtains its maximum point at the boundary \( \partial \Omega \setminus (\Gamma_+ \cup \Gamma_-) \) Thus, increasing \( r \) if necessary, we have
\[
Q \leq C\|g\|_{\dot{C}^{1.1}(\partial D)},
\]
(1.9) follows.

\[\square\]

References

[1] H. Ammari, E. Bonnetier, F. Triki, M. Vogelius, Elliptic estimates in composite media with smooth inclusions: an integral equation approach. Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 2, 453-495.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity, Arch.Ration.Mech.Anal. 208 (2013), 275-304.

[3] H. Ammari, G. Dassios, H. Kang and M. Lim, Estimates for the electric field in the presence of adjacent perfectly conducting spheres, Quart. Appl.Math. 65 (2007), no. 2, 339-355.

[4] H. Ammari, H. Kang and M. Lim, Gradient estimates for solutions to the conductivity problem, Math. Ann. 332 (2005), no. 2, 277-286.

[5] H. Ammari, H. Kang, H. Lee, J. Lee and M.Lim, Optimal estimates for the electric field in two dimensions. J. Math. Pures Appl. (9) 88 (2007), no. 4, 307-324.

[6] H. Ammari, H. Kang, H. Lee, M. Lim and H.Zribi, Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions, J.Differential Equations. 247 (2009), no. 11, 2897-2912.

[7] I. Babuška, B. Andersson, P. Smith and K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale, Comput. Methods Appl. Mech. Engrg. 172 (1999), no 1-4,27-77.

[8] E.S. Bao, Y.Y. Li and B.Yin, Gradient estimates for the perfect conductivity problem, Arch. Ration. Mech. Anal., 193 (2009), no. 1, 195-226.

[9] E.S. Bao, Y.Y. Li and B.Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions, Comm. Partial Differ. Equa., 35 (2010), no.11, 1982-2006.

[10] J.G. Bao, H.G. Li and Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients, Arch. Rational Mech.Anal. 215 (2015), 307-351.

[11] J.G. Bao, H.G. Li and Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two, Adv. Math. 305 (2017), 298–338.

[12] E. Bonnetier and F. Triki, On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D. Arch. Ration. Mech. Anal. 209 (2013), no. 2, 541-567.
E. Bonnetier and M. Vogelius, An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section, SIAM J. Math. Anal. 31 (2000), no. 3, 651-677.

B. Budiansky and G.F. Carrier, High shear stresses in stiff fiber composites. J. App. Mech. 51 (1984), 733-735.

H.J. Dong, Y.Y. Li, Z.L. Yang. Optimal gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. arXiv:2110.11313v1.

H. Kang, M. Lim and K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities, J. Math. Pures Appl. (9) 99 (2013), no. 2, 234-249.

H. Kang, M. Lim and K. Yun, Characterization of the electric field concentration between two adjacent spherical perfect conductors. SIAM J. Appl. Math. 74 (2014), no. 1, 125-146.

J.B. Keller, Stresses in narrow regions, Trans. ASME J. Appl. Mech., 60 (1993), 1054-1056.

H.G. Li, Asymptotics for the electric field concentration in the perfect conductivity problem. SIAM J. Math. Anal. 52(2020), no.4, 3350-3375.

H.G. Li, Y.Y. Li, and Z. Yang, Asymptotics of the gradient of solutions to the perfect conductivity problem, Multiscale Model. Simul. 17 (2019), no. 3, 899-925.

H.G. Li, Y.Y. Li, E.S. Bao and B. Yin, Derivative estimates of solutions of elliptic systems in narrow regions, Quart. Appl. Math. 71(2014) , no.3, 589-596.

H.G. Li, L.J. Xu, Optimal estimates for the perfect conductivity problem with inclusions close to the boundary. SIAM J. Math. Anal. 49 (2017), no. 4, 3125-3142.

Y.Y. Li and L. Nirenberg, Estimates for elliptic systems from composite material. Dedicated to the memory of Jürgen K. Moser, Comm. Pure Appl. Math. 56 (2003), no. 7, 892-925.

Y.Y. Li and M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Ration. Mech. Anal. 153 (2000), no. 2, 91-151.

Y.Y. Li and Z.L. Yang, Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two (2020). arXiv:2012.11405v6.

Y.Y. Li and Z.L. Yang, Gradient estimates of solutions to the conductivity problem with flatter insulators. Anal. Theory Appl. 37 (2021), no. 1, 114-128.

M. Lim and K. Yun, Blow-up of electric fields between closely spaced spherical perfect conductors, Comm. Partial Differ. Equa.,34 (2009), no. 10-12, 1287-1315.

X. Markenscoff, Stress amplification in vanishingly small geometries, Computational Mechanics 19 (1996), no. 1, 77-83.

B. Weinkove, The insulated conductivity problem, effective gradient estimates and the maximum principle. Math. Ann. https://doi.org/10.1007/s00208-021-03214-3.

K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape, SIAM J. Appl. Math., 67 (2007), no. 3, 714-730.

K. Yun, Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections, J. Math. Anal. Appl. 350 (2009), no. 1, 306-312.

K. Yun, An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions. J. Differential Equations 261 (2016), no. 1, 148-188.

(L.J. Ma) Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China.

Email address: mlj0314@126.com