Junction problems for thin inclusions in elastic bodies

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Abstract. Equilibrium problems for a 2D elastic bodies with thin Euler-Bernoulli and Timoshenko elastic inclusions are considered. It is assumed that inclusions have a joint point, and a junction problem for these inclusions is analyzed. Existence of solutions is proved, and different equivalent formulations of problems are discussed. In particular, junction conditions at the joint point are found. A delamination of the elastic inclusions is also assumed. In this case, inequality type boundary conditions are imposed at the crack faces to prevent a mutual penetration between crack faces. A convergence to infinity of a rigidity parameter of the elastic inclusions is investigated. Limit problems are analyzed.

1. Setting of the problem
Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with Lipschitz boundary \( \Gamma \). Denote \( \gamma_b = (0,1) \times \{0\}, \gamma_t = (1,2) \times \{0\}, \gamma = \gamma_b \cup \gamma_t \cup \{(1,0)\} \) assuming \( \bar{\gamma} \subset \Omega \), see Figure 1

Denote by \( \nu = (0,1) \) a unit normal vector to \( \gamma \); \( \tau = (0,1) \), \( \Omega_{\gamma} = \Omega \setminus \bar{\gamma} \).

The domain \( \Omega \) represents a region with an elastic material, \( \gamma_b \) and \( \gamma_t \) are thin elastic Euler-Bernoulli and Timoshenko inclusions, respectively, incorporated in the elastic material. This means that a behavior of the inclusions is described by the Euler-Bernoulli and Timoshenko equations.

Let \( B = \{b_{ijkl}\} \), \( i,j,k,l = 1,2 \), be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

\[
 b_{ijkl} = b_{jikl} = b_{klij}, \quad i,j,k,l = 1,2, \quad b_{ijkl} \in L^\infty(\Omega),
\]

\[
 b_{ijkl} \xi_{ij} \xi_{kl} \geq c_0 |\xi|_2^2 \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 = \text{const} > 0.
\]

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in these indices. Let \( f = (f_1, f_2) \in L^2(\Omega)^2 \) be a given function.

We first provide a variational formulation of the equilibrium problem for the elastic body with thin inclusions \( \gamma_b, \gamma_t \). A space is introduced

\[
 W = \{ (u,v,w,\varphi) \mid u \in H^1_0(\Omega)^2, (v,w) \in H^1(\gamma)^2, v \in H^2(\gamma_b), \varphi \in H^1(\gamma_t); \quad v = u_\nu, \quad w = u_\tau \quad \text{on} \quad \gamma; \quad v_x(1-) + \varphi(1+) = 0 \}
\]

with the norm

\[
 \|(u,v,w,\varphi)\|_W^2 = \|u\|_{H^1_0(\Omega)^2}^2 + \|(v,w)\|_{H^1(\gamma)^2}^2 + \|v\|_{H^2(\gamma_b)}^2 + \|\varphi\|_{H^1(\gamma_t)}^2.
\]
Here, \( u = (u_1, u_2) \), \( u_\nu = \nu \), \( u_\tau = \tau \). The standard notations \( H^1_0(\Omega) \), \( H^1(\gamma) \), etc., are used
for Sobolev spaces. We identify functions defined on \( \gamma \) with functions of the variable \( x; \ x = x_1 \),
\( h_x = \frac{dh}{dx}, (x_1, x_2) \in \Omega \). For a simplicity, we write \( \sigma(u) \varepsilon(u) = \sigma_{ij}(u) \varepsilon_{ij}(u) \).

The identity is considered

\[
(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{\varphi}) \in W,
\]

\[
\int_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u}) - \int_{\Omega} f \mathbf{u} + \int_{\gamma_b} \mathbf{v}_{xx} \mathbf{v}_{xx} + \int_{\gamma} w_x \mathbf{w}_x + \int_{\gamma} \left\{ (\mathbf{v}_x + \mathbf{\varphi})(\mathbf{\nu}_x + \mathbf{\varphi}) + \mathbf{\varphi}_x \mathbf{\varphi}_x \right\} = 0 \quad \forall (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{\varphi}) \in W.
\]

**Theorem 1.** The problem (1)-(2) has a unique solution.

We are able to provide a differential formulation of the problem (1)-(2). It is necessary to
find functions \( u = (u_1, u_2), v, w, \varphi, \sigma = \{\sigma_{ij}\}, i, j = 1, 2 \), as follows

\[
- \text{div} \ \sigma = f, \ - \varepsilon - B \varepsilon(u) = 0 \quad \text{in} \ \Omega_\gamma,
\]

\[
v_{xxx} = [\sigma_{ij}] \text{ on } \gamma_b, \ - w_{xx} = [\sigma_x] \text{ on } \gamma, \quad v_x = \varphi \quad \text{on } \gamma_t,
\]

\[
(w_x(1-) = w_x(1+); \ v_x + \varphi = \varphi_x = w_x = 0 \quad \text{on } x = 2,
\]

\[
-v_{xxx}(1-) = (v_x + \varphi)(1+); \ v_{xx}(1-) = -\varphi_x(1+).
\]

Here, \( \varepsilon_{ij}(u) = \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) \), \( \sigma_{ij} = \sigma_{ij} \varepsilon_{ij} \nu_i \varepsilon_{ij} \), \( \sigma_x = \sigma_{ij} \varepsilon_{ij} \gamma_i \), and \( [h] = h^+ - h^- \) is a jump of a function \( h \) on \( \gamma \), where \( h^\pm \) are the traces of \( h \) on the faces \( \gamma^\pm \). The signs \( \pm \) correspond to positive and negative directions of \( \nu \).

The function \( u = (u_1, u_2) \) describes a displacement field of the elastic body; functions \( w, v \) fit to displacements of the inclusions \( \gamma_b, \gamma_t \) along the axis \( x_1 \) and axis \( x_2 \) respectively; the function \( \varphi \) describes a rotation angle of the inclusion \( \gamma_t \). Observe that a part of boundary conditions for functions \( u, v, w, \varphi \) is included in the condition \( (u, v, w, \varphi) \in W \).

Relations (3) are the equilibrium equations for the elastic body and Hooke’s law; (4)-(5) are the Euler-Bernoulli and Timoshenko equilibrium equations for the inclusions \( \gamma_b, \gamma_t \).

According to the condition \( (u, v, w, \varphi) \in W \), the vertical (along the axis \( x_2 \)) and tangential (along the axis \( x_1 \)) displacements of the elastic body coincide with the inclusion displacements.
at $\gamma$. The right-hand sides $[\sigma_v], [\sigma_r]$ in (4), (5) describe forces acting on $\gamma$ from the surrounding elastic media.

**Theorem 2.** Problem formulations (3)-(8) and (1)-(2) are equivalent provided that the solutions are smooth.

By (8) and the first relations of (6), (7), we can write a complete system of junction conditions at the joint point $(1, 0)$:

$$
w(1-) = w(1+), \quad v(1-) = v(1+), \quad v_x(1-) = -\varphi(1+),$$
$$w_x(1-) = w_x(1+), \quad -v_{xx}(1-) = (v_x + \varphi)(1+), \quad v_{xx}(1-) = -\varphi_x(1+).$$

### 2. Delaminated elastic inclusion

Assume that the Euler-Bernoulli part $\gamma_b$ of the inclusion $\gamma$ is delaminated, please refer to Figure 1. This means that a crack is located between $\gamma_b$ and the elastic matrix. To fix a situation, the delamination is assumed to be at the positive side of $\gamma_b$. In this case, displacements $v, w$ of the inclusion $\gamma_b$ should coincide with displacements of the elastic body at $\gamma_b^-$. In our model, inequality type boundary conditions are considered at the crack faces to prevent a mutual penetration between the faces.

Denote $\Omega_b = \Omega \setminus \gamma_b$ and introduce a set of admissible functions

$$K = \{(u, v, w, \varphi) \mid u \in H^1_b(\Omega_b)^2, (v, w) \in H^1(\gamma)^2, v \in H^2(\gamma_b), \varphi \in H^1(\gamma_t), u|_{\gamma_b^-} = (w, v), [u_v] \geq 0 \text{ on } \gamma_b; v_x(1-) + \varphi(1+) = 0\},$$

where $H^1_b(\Omega_b) = \{\phi \in H^1(\Omega_b) \mid \phi = 0 \text{ on } \Gamma\}$. Notice that the inequality $[u_v] \geq 0$ included in the definition of $K$ provides a mutual nonpenetration between the crack faces $\gamma_b^\pm$. An equilibrium problem for the elastic body with the delaminated Euler-Bernoulli inclusion $\gamma_b$ and the Timoshenko inclusion $\gamma_t$ can be formulated as follows. We have to find functions $u = (u_1, u_2), v, w, \varphi, \sigma = \{\sigma_{ij}\}, i, j = 1, 2$, as follows

$$-	ext{div } \sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma,$$

$$v_{xxx} = [\sigma_v] \text{ on } \gamma_b, \quad -w_{xx} = [\sigma_r] \text{ on } \gamma,$$

$$-v_{xx} - \varphi_x = [\sigma_v], \quad -\varphi_{xx} + v_x + \varphi = 0 \quad \text{on } \gamma_t,$$

$$(u, v, w, \varphi) \in K; \quad v_{xxx} = v_x = w_x = 0 \text{ for } x = 0,$$

$$w_x(1-) = w_x(1+); \quad v_x + \varphi = \varphi_x = w_x = 0 \text{ for } x = 2, $$

$$\sigma_v^+ \leq 0, \quad \sigma^+_\varphi = 0, \quad \sigma_{\nu}^+ [u_v] = 0 \text{ on } \gamma_b,$$

$$-v_{xxx}(1-) = (v_x + \varphi)(1+); \quad v_{xx}(1-) = -\varphi_x(1+).$$

As before, a part of boundary conditions is included in the relation $(u, v, w, \varphi) \in K$.

Remark that the problem (9)-(15) admits an equivalent variational formulation. Indeed, a solution $(u, v, w, \varphi)$ satisfies a variational inequality

$$
\int_{\Omega_b} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_b} f(\bar{u} - u) + \int_{\gamma_b} v_{xx}(\bar{v}_{xx} - v_{xx}) +
$$
$$+ \int_{\gamma} \{\varphi_x(\bar{\varphi}_x - \varphi_x) + (v_x + \varphi)(\bar{\varphi}_x + \varphi - v_x - \varphi)\} +
$$
$$+ \int_{\gamma} w_x(\bar{w}_x - w_x) \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K; \quad (u, v, w, \varphi) \in K.$$

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3. Rigidity of Euler-Bernoulli beam goes to infinity

In practice, a solution of the problem (9)-(15) should depend on the rigidity parameters of the elastic inclusions. In the model (9)-(15), these parameters were taken to be equal to 1. In this section we introduce a rigidity parameter $\delta > 0$ in the Euler-Bernoulli equations of the problem (9)-(15) and analyze its passage to infinity. For a fixed parameter $\delta$, we have to solve the following problem: to find $u^\delta, v^\delta, w^\delta, \varphi^\delta, \sigma^\delta = \{\sigma^\delta_{ij}\}, i, j = 1, 2$, such that

$$-\text{div} \sigma^\delta = f, \quad \sigma^\delta - B\varepsilon(u^\delta) = 0 \quad \text{in} \quad \Omega_\gamma,$$

$$\delta u^\delta_{xxxx} = [\sigma^\delta_{ij}] \text{ on } \gamma_b, \quad -\text{div}(a^\delta u^\delta_x) = [\sigma^\delta_{ij}] \text{ on } \gamma,$$

$$-v^\delta_{xx} - \varphi^\delta_x = [\sigma^\delta_{ij}], \quad -\varphi^\delta_{xx} + v^\delta_{xx} + \varphi^\delta = 0 \quad \text{on} \quad \gamma_t,$$

$$(u^\delta, v^\delta, w^\delta, \varphi^\delta) \in K; \quad v^\delta_{xxx} = v^\delta_{x} = 0 \text{ for } x = 0,
\delta w^\delta_x(1-) = w^\delta_x(1+); \quad v^\delta_x + \varphi^\delta = \varphi^\delta_x = w^\delta_x = 0 \text{ for } x = 2,
\sigma^\delta_{ij} < 0, \quad \sigma^\delta_{ij} = 0, \quad \sigma^\delta_{ij} \{u^\delta_{x} \} = 0 \text{ on } \gamma_b,
-\delta v^\delta_{x} (1-) = (v^\delta_x + \varphi^\delta)(1+); \quad \delta v^\delta_{x} (1-) = -\varphi^\delta_x (1+).$$

Here, $a^\delta(x) = 1$ on $\gamma_b$, and $a^\delta(x) = \delta$ on $\gamma_b$.

The problem (17)-(23) admits an equivalent variational formulation. A unique solution of the variational inequality (with $\sigma(u^\delta) = \sigma^\delta$) exists

$$\int_{\Omega_b} \sigma(u^\delta) \varepsilon(\bar{u} - u^\delta) - \int_{\Omega_b} f(\bar{u} - u^\delta) + \delta \int_{\gamma_b} v^\delta_{xx}(\bar{v}_{xx} - v^\delta_{xx}) +
\int_{\gamma_b} \{\varphi^\delta_x(\bar{\varphi}_x - \varphi^\delta_x) + (v^\delta_x + \varphi^\delta)(\bar{v}_x + \varphi_x - \varphi^\delta_x)\} +
\int_{\gamma_b} a^\delta w^\delta_x(\bar{w}_x - w^\delta_x) \geq 0 \quad \forall \ (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K.$$

Our aim is to justify a passage to the limit as $\delta \to \infty$ in the problem (24)-(25).

Introduce a set of infinitesimal rigid displacements

$$R(\gamma_b) = \{\rho = (\rho_1, \rho_2) \mid \rho(x_1, x_2) = d(-x_2, x_1) + (d_1, d_2), \quad (x_1, x_2) \in \gamma_b\}, \quad d, d_1, d_2 \in \mathbb{R},$$

and a set of admissible functions

$$K_r = \{(u, v, w, \varphi) \mid u \in H^1(\Omega_b), (v, w, \varphi) \in H^1(\gamma_b), \quad [u_{\gamma_b} ] \geq 0 \text{ on } \gamma_b; u|_{\gamma_b} = (w, v), \quad u|_{\gamma_b} = (\rho_1, \rho_2) \in R(\gamma_b), \rho_2 x (1) + \varphi(1) = 0\}.$$

It can be proved that $\delta \to +\infty$,

$$u^\delta \to u \text{ weakly in } H^1(\Omega_b), \quad \varphi^\delta \to \varphi \text{ weakly in } \text{ in } H^1(\gamma_b);$$

$$v^\delta \to v \text{ weakly in } H^1(\gamma), \text{ weakly in } H^2(\gamma_b), \quad v_{xx} = 0 \text{ in } \gamma_b,$$

$$w^\delta \to w \text{ weakly in } H^1(\gamma), \quad w_x = 0 \text{ in } \gamma_b.$$
Moreover,

\[
(u, v, w, \varphi) \in K_r, \quad \int_{\Omega_b} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_b} f(\bar{u} - u) + \\
+ \int_{\gamma_b} \left\{ \varphi_x (\bar{\varphi}_x - \varphi_x) + (v_x + \varphi) (\bar{v}_x + \bar{\varphi} - v_x - \varphi) + \\
+ w_x (\bar{w}_x - w_x) \right\} \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K_r.
\]  

(29)

The problem (29) admits an equivalent differential formulation: find displacements \( u = (u_1, u_2) \), \( v, w \), a rotation angle \( \varphi \), a stress tensor \( \sigma = \{\sigma_{ij}\} \), \( i, j = 1, 2 \), and \( \rho^0 \in R(\gamma_b) \) as follows

\[-\text{div} \ \sigma = f, \ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_r, \quad \text{(30)}\]

\[-v_{xx} - \varphi_x = \left[\sigma_{\nu}\right], -\varphi_{xx} + v_x + \varphi = 0, -w_{xx} = [\sigma_{\tau}] \quad \text{on} \quad \gamma_t, \quad \text{(31)}\]

\[u^- = \rho^0, \ \sigma_{\nu}^+ \leq 0, \ \sigma_{\tau}^+ = 0, \ \sigma_{\nu}^+[u_{\nu}] = 0 \quad \text{on} \quad \gamma_b, \quad \text{(32)}\]

\[(u, v, w, \varphi) \in K_r; \ v_x + \varphi = \varphi_x = w_x = 0 \quad \text{for} \quad x = 2, \quad \text{(33)}\]

\[\int_{\gamma_b} [\sigma_{\nu}] \rho + (w_x \rho_1)(1) - (\varphi_x \rho_2)(1) = 0 \quad \forall \rho = (\rho_1, \rho_2) \in R(\gamma_b). \quad \text{(34)}\]

Thus, the following statement can be proved.

**Theorem 3.** The solutions of the problem (24)-(25) converge to the solution of (29) in the sense (26)-(28) as \( \delta \to \infty \).

The model (30)-(34), or (29), describes an equilibrium state for the elastic body with the rigid inclusion \( \gamma_b \) and elastic Timoshenko inclusion \( \gamma_t \). The identity (34) provides equilibrium conditions for the rigid inclusion \( \gamma_b \), i.e. a principal vector of forces and a principal vector of moments acting on \( \gamma_b \) are equal to zero. Indeed, denoting \( (\sigma\nu)^\pm \) by \( (\sigma^1, \sigma^2)^\pm \) on \( \gamma_b^\pm \), the condition (34) can be rewritten in the following form:

\[\int_{\gamma_b} [\sigma^1] = -w_x(1), \quad \int_{\gamma_b} [\sigma^2] = -(v_x + \varphi)(1), \quad \text{(35)}\]

\[\int_{\gamma_b} ([\sigma^2] x_1 - [\sigma^1] x_2) = \varphi_x(1). \quad \text{(36)}\]

We can also write junction conditions included in the definition of \( K_r \):

\[w(1) = \rho_0^b(1), \quad v(1) = \rho_0^b(1), \quad \varphi(1) + \rho_2^b(1) = 0 \quad \text{(37)}\]

and consider (35)-(37) as a complete system of junction conditions at the joint point \( (1, 0) \). Consequently, nonlocal condition (34) can be seen as a part of junction conditions at the joint point \( (1, 0) \).

**References**

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