The Fermion Generations Problem In The GUST
In The Free World-Sheet Fermion Formulation

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ABSTRACT

In the framework of the four dimensional heterotic superstring with free fermions we present a revised version of the rank eight Grand Unified String Theories (GUST) which contain the $SU(3)_H$-gauge family symmetry. We also develop some methods for building of corresponding string models. We explicitly construct GUST with gauge symmetry $G = SU(5) \times U(1) \times (SU(3) \times U(1))_H$ and $G = SO(10) \times (SU(3) \times U(1))_H$ or $E(6) \times SU(3)_H$ \textcopyright{} $E(8)$ and consider the full massless spectrum for our string models. We consider for the observable gauge symmetry the diagonal subgroup $G_{\text{symm}}$ of the rank 16 group $G \times G \subset SO(16) \times SO(16)$ or $E(8) \times E(8)$. We discuss the possible fermion matter and Higgs sectors in these theories. We study renormalizable and nonrenormalizable contributions to the superpotential. There has to exist "superweak" light chiral matter ($m_H^f < M_W$) in GUST under consideration. The understanding of quark and lepton mass spectra and family mixing leaves a possibility for the existence of an unusually low mass breaking scale of the $SU(3)_H$ family gauge symmetry (some TeV).
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1 Introduction

There are no experimental indications which would impel one to go beyond the framework of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ Standard Model (SM) with three generations of quarks and leptons. None of the up-to-date experiments contradicts, within the limits of accuracy, the validity of the SM predictions for low energy phenomena. However SM includes a large number of arbitrary parameters and many fundamental questions remain unanswered in its framework. The number of generations, the fermion mass origin, generation mixing, neutrino mass uniqueness, CP-violation problems are among most exciting theoretical puzzles in SM. All these questions stimulate the searching for more fundamental theories that have the SM as an effective low energy limit and predict some new particles and phenomena.

One of the interesting steps towards possible explanation of generation problem and others phenomena is the including a non-abelian horizontal factor like $SU(3)$ into gauge group of the model. The benefit of this approach could be a clarifying of such a question as generation number, family mixing nature, quark-lepton mass hierarchy. It is interesting to note that there are some horizontal extensions that do not add too many extra arbitrary parameters. These models allow to analyse the rare processes experimental data by the universal way and to obtain experimentally testable predictions. The constraints of horizontal model parameters followed from this approach permits the existence of the interesting flavour-changing physics in the TeV region. Also these models gives rise to a rather natural way of the superweak-like CP-violation.

The superstring theories are appear to be the most perspective candidates for the Theory of Everything. That is why we consider this approach to be useful in the investigations of GUTs that include horizontal gauge symmetry. For the heterotic string the groups $E_8 \otimes E_8$ and $spin(32)/Z_2$ are characteristic. Hence it is interesting to consider GUT based on its various rank 8 and 16 subgroup. String theories possess infinite dimensional symmetries that place many specific constraints on the theory spectrum. These symmetries origin from 2 dimensional conformal invariance, modular invariance, and Virasoro and Kac-Moody algebras.

In this paper we analyse several Grand Unification models based on 4 dimensional heterotic string in free fermions approach.

We consider possible ways of breaking the "string" gauge group $E_8 \otimes E_8$ down to low energy supersymmetric model that includes Standard Model group and horizontal factor $SU(3)$ and naturally describes 3 or 3 + 1 generations.

The paper is organized as follows. In Chapter 2 we consider supersymmetric extension of the Standard Model including horizontal $SU(3)$ factor. This model could be the low energy limit of the string GUT considered further in Chapter 5. The details of the model structure are described in Appendix A. For such an effective model we present estimates of horizontal gauge symmetry breaking scale and of contributions of horizontal interactions to various rare process.

When we demand that the quark-lepton generations mixing and the splitting of the horizontal gauge boson mass spectrum have the same nature this model gain minimal
number of extra new parameters like $g_{3H}$, $M_{H_0}$ and Yukawa coupling. This allows us to carry complete analysis of the experimental data of all known rare processes. The horizontal gauge symmetry breaking in these models is seemed to be considerably low ($\sim$ some TeV).

For estimating the experimental data of the rare processes in framework of these models Appendix B gives an justification of validity of our $g_{3H}$ choosing basing on the renormalization group analysis of string GUTs.

In Chapter 3 we discuss different ways of breaking of string rank 16 gauge group $E_8 \otimes E_8$ and its symmetric diagonal subgroup of rank 8. We outline the perspective way including the symmetric subgroup on the intermediate stage that does not involve higher level of Kac-Moody algebra representations. Namely in Chapter 4 we analyse restrictions on various unitary representations in the superstring GUTs from the point of the level of the Kac-Moody algebras (KMA) and conformal weights. It is emphasized that the breaking of gauge group down to symmetric diagonal subgroup effectively increases KMA level to 2 and makes the existence of the representations for Higgs fields needed for GUT symmetry breaking possible.

Chapter 5 presents various string models including Grand Unification group $SO(10)$ or $U(5)$ along with horizontal gauge symmetry $U(3)$ and describing $3 + 1$ generations. For all these models it is essential that gauge group is breaking down to the symmetric diagonal subgroup. The rules of building the basis of spin boundary conditions and GSO projections based on modular invariance are given in Appendix C. We use a computer program for calculating the full massless spectrum of states of the considered models. The full spectra for the presented models are given in corresponding tables.

The further analysis of root and weight lattices and their relations with the GSO projection is given in Chapter 6. Based on the method proposed there we build 4D superstring GUT model with $[E_6 \otimes SU(3)]^2$ and $SU(3)^{\otimes 4}$ gauge groups that seems to be perspective in the way to obtain 3 generation model. Using that technique we build these models with $N = 2$ supersymmetry.

In the Chapter 7 the Model 1 is considered in details. Corresponding gauge symmetry breaking is analysed, superpotential and nonrenormalizable contribution permitted by string dynamics are presented. The construction of electromagnetic charge for diagonal symmetric subgroup allows to avoid states with exotic fractional charges that is a typical problem in string GUTs on the level 1. The form of obtained superpotential implies that 2 generations remain massless comparing with the $m_W$ scale. Using the condition of $SU(3)_H$ anomaly cancellation the theory predicts the existence of the Standard Model singlet "neutrino-like" particles that participate only in horizontal interactions. As following from the form of the superpotential some of them could be light (less than $m_W$) that will be very interesting in sense of experimental searches. At last we calculate for model1 the N=4,5 vertex non-renormalizable contributions to the superpotential.
2 Towards a low energy gauge family symmetry with the small number of parameters

2.1 The family problems in SM, family mixing and quark/lepton mass hierarchy

One has ten parameters in the quark sector of the SM with three generations: six quark masses, three mixing angles and the Kobayashi-Maskawa (KM) CP violation phase \(0 < \delta_{KM} < \pi\). The CKM (Cabibbo-Kobayashi-Maskawa) matrix in Wolfenstein parameterization is determined by the four parameters — Cabibbo angle \(\lambda \approx 0.22\), A, \(\rho\) and \(\eta\):

\[
V_{\text{CKM}} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix} = \begin{pmatrix}
1 - 1/2\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - 1/2\lambda^2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}.
\] (1)

In the complex plane the point \((\rho, \eta)\) is a vertex of the unitarity triangle and describes the CP violation in SM. The unitarity triangle is constructed from the following unitarity condition of \(V_{\text{CKM}}\):

\[
V_{ub}^* + V_{td} \approx A\lambda^3.
\]

Recently, the interest in the CP-violation problem was excited again due to the data on the search for the direct CP-violation effects in neutral K-mesons [1],[2]:

\[
\text{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = (7.4 \pm 6) \times 10^{-4},
\] (2)

\[
\text{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = (23 \pm 7) \times 10^{-4},
\] (3)

The major contribution to the CP-violation parameters \(\varepsilon_K\) and \(\varepsilon'_K\) (\(K^0\)-decays), as well as to the \(B^0_d - \bar{B}^0_d\) mixing parameter \(x_d = \frac{\Delta m_{B_d}}{\Gamma(B_d)}\) is due to the large t-quark mass contribution. The same statement holds also for some amplitudes of K- and B-meson rare decays. The CDF collaboration gives the following region for the top quark mass:

\(m_t = 174 \pm 25\) GeV [3]. The complete fit which is based on the low energy data as well as the latest LEP and SLC data and comparing with the mass indicated by CDF measurements gives \(m_t = 162 \pm 9\) GeV [4].

The main undrawbacks of SM now are going from our non-understanding the generation problem, their mixing and hierarchy of quark and lepton mass spectra. For example, for quark masses \(\mu \approx 1\text{GeV}\) we can get the following approximate relations [4]:

\[
m_{ik} \approx (q^u_H)^{2k}m_0, \ k = 0, 1, 2; \ i_0 = u, i_1 = c, i_2 = t,
m_{ik} \approx (q^d_H)^{2k}m_0, \ k = 0, 1, 2; \ i_0 = d, i_1 = s, i_2 = b,
\] (4)
where \( q_H^d \approx (q_H^d)^2 \), \( q_H^d \approx 4 - 5 \approx 1/\lambda \) and \( \lambda \approx \sin \theta_C \).

Here we used the conventional ratios of the ”running” quark masses \([6]\)

\[
m_d/m_s = 0.051 \pm 0.004, \quad m_u/m_c = 0.0038 \pm 0.0012, \quad m_s/m_b = 0.033 \pm 0.011, \quad m_c(\mu = 1 GeV) = (1.35 \pm 0.05) GeV \quad \text{and} \quad m_t^{phys} \approx 0.6 m_t(\mu = 1 GeV). \quad (5)
\]

This phenomenological formula (6) predicts the following value for the \( t \)-quark mass:

\[
m_t^{phys} \approx 180 - 200 GeV. \quad (6)
\]

In SM these mass matrices and mixing come from the Yukawa sector:

\[
L_Y = QY_u \bar{u} h + QY_d \bar{d} h + LY_e \bar{e} h + H.C.,
\]

where \( Q_i \) and \( L_i \) are three quark and lepton isodoublets, \( \bar{u}_i, \bar{d}_i \) and \( \bar{e}_i \) are three right-handed antiquark and antilepton isosinglets respectively. \( h \) is the ordinary Higgs doublet.

In SM the \( 3 \times 3 \) family Yukawa matrices, \((Y_u)_{ij}\) and \((Y_d)_{ij}\) have no any particular symmetry. Therefore it is necessary to find some additional mechanisms or symmetries beyond the SM which could diminish the number of the independent parameters in Yukawa sector \( L_Y \). These new structures can be used for the determination of the mass hierarchy and family mixing.

To understand the generation mixing origin and fermion mass hierarchy several models beyond the SM suggest special forms for the mass matrix of ”up” and ”down” quarks (Fritzsch ansatz, ”improved” Fritzsch ansatz, ”Democratic” ansatz, etc.\([7]\)). These mass matrices have less than ten independent parameters or they could have some matrix elements equal to zero (”texture zeroes”)\([8]\). This allows us to determine the diagonalizing matrix \( U_L \) and \( D_L \) in terms of quark masses:

\[
Y_u^{diag} = D_L Y_d^R, \quad Y_d^{diag} = U_L Y_d^R. \quad (8)
\]

For simplicity the symmetric form of Yukawa matrices has been taken, therefore: \( D_L = D_R^*, \quad U_L = U_R^* \). These ansatzes or zero ”textures” could be checked experimentally in predictions for the mixing angles of the CKM matrix: \( V_{CKM} = U_L D_L^* \). For example, one can consider the following approximate form at the scale \( M_X \) for the symmetric ”texture” used in paper \([8]\):

\[
Y_u = \begin{pmatrix}
0 & \lambda^6 & 0 \\
\lambda^6 & 0 & \lambda^2 \\
0 & \lambda^2 & 1
\end{pmatrix}, \quad Y_d = \begin{pmatrix}
0 & 2\lambda^4 & 0 \\
2\lambda^4 & 2\lambda^3 & 2\lambda^3 \\
0 & 2\lambda^3 & 1
\end{pmatrix}. \quad (9)
\]

Given these conditions it is possible to evolve down to low energies via the renormalization group equations all quantities including the matrix elements of Yukawa couplings \( Y_{u,d} \), the values of the quark masses (see (4)) and the CKM matrix elements (see (1)).
Also, using these relations we may compute $U_L$ (or $D_L$) in terms of CKM matrix and/or of quark masses.

In GUT extensions of the SM with the family gauge symmetry embedded Yukawa matrices can acquire particular symmetry or an ansatz, depending on the Higgs multiplets to which they couple.

The family gauge symmetry could help us to study in an independent way the origin of the up- ($U$) and down- ($D$) quark mixing matrices and consequently the structure of the CKM matrix $V_{CKM} = UD^+$. Let us see the example of Abelian gauge U(1) horizontal symmetry in papers [9]. In this approach the mechanism of mass and mixing hierarchy follows from nonrenormalizable terms in Yukawa potential [34, 9].

\[
L_{eff}^{yf} = \sum_{ij} S_{mij} \phi d_i \phi u_j + H.C.,
\]

where the spontaneous breaking of the horizontal symmetry by VEV of a scalar field $S$, $S \approx \Lambda_H$, which is a singlet of the SM. The mass scale $M$ is connected with a mass of the new massive particles. The powers $m_{ij}$ ($n_{ij}$) are the horizontal charge difference between $Q_i$ and $\bar{d}_j$ ($Q_i$ and $\bar{u}_j$): $H(Q_i) + H(\bar{d}_j) = m_{ij}$, $H(Q_i) + H(\bar{u}_j) = n_{ij}$ [34, 9]. The hierarchy in mixing angles and quark (charged lepton) masses appears due to a small parameter \( \epsilon = \Lambda_H / M << 1 \) [34, 9].

Due to the local gauge family symmetry with a low energy breaking scale $\Lambda_H \approx 1 TeV$ gives us a chance to define the quantum numbers of quarks and leptons and thus establishes a link between them in families. For the mass fermion ansatz considered above in the extensions of SM there could exist the following types of the $SU(3) \times SU(2)_L$ Higgs multiplets: (1,2), (3,1), (8,1), (3,2), (8,2), (1,1),..., which in turn could exist in the spectra of the String Models.

In the framework of the rank eight Grand Unified String Theories we will consider an extension of SM due to local family gauge symmetry, $G_H = SU(3)_H$, $SU(3)_H \times U(1)_H$ models and its developments and their possible Higgs sector. Thus, for understanding the quark mass spectra and the difference between the origins of the up- (or down) quark and charged lepton mass matrices in GUSTs we have to study the Higgs content of the model, which we must use from the one hand for breaking the GUT, Quark-Lepton -, $G_H = SU(3)_H$, ..., $SU(2)_L \times U(1)$- symmetries and from the other hand — for Yukawa matrix constructions.

2.2 The ”bootstrap” $SU(3_H)$ gauge family models.

In the paper [5] we investigated the samples of different scenarios of $SU(3)_H$ breakings down to the $SU(2)_H \times U(1)_3H$, $U(1)_3H \times U(1)_8H$ and $U(1)_8H$- subgroups, as well as the mechanism of the complete breaking of the base group $SU(3)_H$. We tried to realize the SUSY conserving program (see Appendix A) on the scales where the relevant gauge symmetry is broken. In the framework of these versions of the gauge symmetry breaking, we were searching for the spectra of horizontal gauge bosons and gauginos and calculated
the amplitudes of some typical rare processes. Theoretical estimates for the branching ratios of some rare processes obtained from these calculations have been compared with the experimental data on the corresponding values \[10, 11, 12, 5\]. Further we have got some bounds on the masses of $H_\mu$-bosons and the appropriate $H$-gauginos. Of particular interest was the case of the $SU(3)_H$ group which breaks completely on the scale $M_{H_0}$. We calculated the splitting of eight $H$-boson masses in a model dependent fashion. This splitting, depending on the quark mass spectrum, allows us to reduce considerably the predictive ambiguity of the model -"almost exactly solvable model".

We assume at first that all of the 8 gauge bosons of $SU(3)_H$ group acquire the same mass $M_{H_0}$. Such a breaking is not difficult to get by, say, introducing the Higgs fields transforming in accordance with the triplet representation of the $SU(3)_H$ group. These fields are singlet under the Standard Model symmetries: $(z_i \in (3,1,1,0)$ and $\bar{z}_i \in (\bar{3},1,1,0)$, $\langle \bar{z}_i^\alpha \rangle_0 = \delta_\alpha^i V$, $\langle z_i^\alpha \rangle_0 = \delta_i^\alpha V$, $i, \alpha = 1, 2, 3$, where $V = M_{H_0}$). In this case the fermion mass matrices are proportional to a unit matrix because the fermions have a global $SU(3)_H$ symmetry.

The degeneracy of the masses of 8 gauge horizontal vector bosons is eliminated by including the VEV’s of the Higgs fields violating the electroweak symmetry and determining the hierarchical structure of up- and down-quarks (leptons) mass matrices. Thus, there is a set of the Higgs fields (see corresponding Table 14): $H(8, 2), h(8, 2)$ or $Y(\bar{3}, 2), X(3, 2)$ and $\kappa_{1,2}(1, 2)$ which could determine the mass matrix of up- and down-quarks. On the other hand, in order to calculate the splitting between the masses of horizontal gauge bosons, one has to take into account the VEV’s contributions of the Higgs fields $H, h$ or $X, Y$. For example, the VEV’s of the Higgs fields $H$ or $X$ can give the corresponding contributions:

\[
\begin{align*}
(\Delta M^2_u)_{ab}^{1/2} &= g^2_H \sum_{d=1}^8 f^{d a c} f^{b d' c'} \langle H^c \rangle \langle H^{c'} \rangle^*, \quad (11) \\
(\Delta M^2_u)_{ab}^{3/2} &= g^2_H \sum_{k=1}^3 \lambda^a_k \lambda^b_k \langle X^i \rangle \langle X^j \rangle^*, \quad (12)
\end{align*}
\]

Now we can come to constructing the horizontal gauge boson mass matrix $M^{ab2}$ ($a,b=1,2,...,8$):

\[
(M^2_H)_{ab} = M^2_{H_0} \delta_{ab} + (\Delta M^2_d)_{ab}^{1/2} + (\Delta M^2_u)_{ab}^{1/2}.
\]

Here $(\Delta M^2_d)_{ab}$ and $(\Delta M^2_u)_{ab}$ are the "known" functions of heavy fermions, $(\Delta M^2_d)_{ab} = F^{ab}(m_t, m_b, ...)$, which mainly get the contributions due to the vacuum expectations of the Higgs bosons that were used for construction of the mass matrix ansatzes for d- (u-) quarks.

For example in the case of $N_g = 3 + 1$ families with Fritzsch ansatz for quark mass matrices and using $SU(3)_H \times SU(2)$ Higgs fields, $(8, 2)$, [5], we can write down some rough relations between the masses of horizontal gauge bosons ("bootstrap" solution):
\[
M^2_{H_1} \approx M^2_{H_2} \approx M^2_{H_3} \approx M^2_{H_0} + \frac{g^2_H}{4} \left[ \frac{1}{\lambda^2} \frac{m_t m_t'}{1 - m_t/m_t'} \right] + \ldots,
\]
\[
M^2_{H_4} \approx M^2_{H_5} \approx M^2_{H_6} \approx M^2_{H_7} \approx M^2_{H_0} + \frac{g^2_H}{4} \left[ \frac{1}{\lambda^2} m_t m_t' \right] + \ldots,
\]
\[
M^2_{H_8} \approx M^2_{H_0} + \frac{g^2_H}{3} \left[ \frac{1}{\lambda^2} m_t m_t' \right] + \ldots,
\]
(14)

where \(\lambda\) and \(\tilde{\lambda}\) are Yukawa couplings.

We were interested how does the unitary compensation for the contributions of horizontal forces to rare processes [5] depend on different versions of the \(SU(3)_H\)-symmetry breaking. The investigation of this dependence allows, firstly, to understand how low may the horizontal symmetry breaking scale \(M_H\) be, and, secondly, how is this scale determined by a particular choice of a mass matrix ansatz both for quarks and leptons.

We would like to stress a possible existing of the local family symmetry with a low energy symmetry breaking scale, i.e. the existence of rather light H-bosons: \(m_H \geq (1 - 10) TeV\) [5]. We have analyzed, in the framework of the ”minimal” horizontal supersymmetric gauge model, the possibilities of obtaining a satisfactory hierarchy for quark masses and of connecting it with the splitting of horizontal gauge boson masses. We expect that due to this approach the horizontal model will become more definite since it will allow to study the amplitudes of rare processes and the CP-violation mechanism more thoroughly. In this way we hope to get a deeper insight into the nature of interdependence between the generation mixing mechanism and the local horizontal symmetry breaking scale.

2.3 The flavor-changing rare processes and superweak-like source of CP-violation in GUST with the non-Abelian gauge family symmetry.

The existence of horizontal interactions with low energy breaking scale \((\Lambda_H < 10 TeV)\) might lead to large flavor changing neutral currents (FCNC). This interaction is described by the relevant part of the SUSY \(SU(3)_H\)-Lagrangian and has the form

\[
\mathcal{L}_H = g_H \bar{\psi}_d \Gamma_\mu \left( D^\mu \frac{\lambda^a}{2} D^\nu \right) \psi_d O^{ab} Z^{b}_\mu.
\]
(15)

Here we have \((a,b=1,2,...,8)\). The matrix \(O^{ab}\) determines the relationship between the bare, \(H^b_\mu\), and physical, \(Z^b_\mu\), gauge fields and is calculated for the mass matrix \((M^2_H)_{ab}\) diagonalized; \(\bar{\psi}_d = (d , s , b)\) (similarly for ”up”-quarks and charged leptons); \(g_H\) is the gauge coupling of the \(SU(3)_H\) group.

After the calculations in ”bootstrap” model with the Higgs fields \(\langle H \rangle = (\lambda^a \varphi^a)/2\), \(\langle h \rangle = (\lambda^a \bar{\varphi}^a)/2\) the expressions for the \((K^b_L - K^b_R), (B^0_{dL} - B^0_{dS}), -(B^0_{sL} - B^0_{sS}), - (D^0_L - \ldots\)
meson mass differences (pure quark processes) at tree level take the following general forms:

\[
\begin{align*}
\left[ \frac{(M_{12})^K_{ij}}{m_K} \right]_H & = \frac{1}{2M_{h0}} \left\{ \left[ \phi^a(D) \frac{\lambda^a}{2} D^+ \right]_{ij}^2 + \left[ \phi^a(D) \frac{\lambda^a}{2} D^+ \right]_{ij}^2 \right\} f_{Kij}^2 R_{Kij}, \\
\left[ \frac{(M_{12})^D_{ij}}{m_D} \right]_H & = \frac{1}{2M_{h0}} \left\{ \left[ \phi^a(U) \frac{\lambda^a}{2} U^+ \right]_{ij}^2 \right\} f_{Dij}^2 R_{Dij}.
\end{align*}
\]

where \((i,j) = (1,2),(1,3),(2,3)\) – the \(K\) or \(D\), \(B_d\) or \(T_u\), \(B_s\) or \(T_c\) - meson systems.

The coefficients in formulas (16) are calculated from (13) using formula (14).

For example, for \(K\)-meson systems we find the following contribution (if \(D_L = D_R = D\))

\[
g_H^2/4(D\lambda^aO^{ab}D^+)^{12} = \frac{1}{M_0^2 + \Delta M_b^2}(D\lambda^aO^{cb}D^+)^{12} = 0.
\]

We consider a case of the complex value for VEVs, \(\phi^a\). Also we used the next formula:

\[
(D\lambda^a/2D^+)_{ij}(D\lambda^a/2D^+)_{kl} = 1/2(\delta_{il}\delta_{jk} - 1/3\delta_{ij}\delta_{kl}).
\]

It is interesting, that if a difference between the gauge boson’s masses is generated by Higgs fields in representation \((3, 2)\) (see (2)), than the contribution in \(\left[ \frac{\Delta m}{m} \right]_H\) equal to zero in considering order (for case \(D_L = D_R\)), since we will use Higgs fields \((8, 2)\) for these evaluations. However, for processes including three equivalent index (like \(\mu \rightarrow 3e\)) Higgs fields \((3, 2)\) give nonzero contribution \(\sim (\varphi D^+)_{ij}(D\varphi)_{ij}\).

Note, that formula (17) is true for the case when \(D_L\) differs from \(D_R\) by diagonal phase multiply too. For us the case \(D_L = -D_R\) which corresponds to axial-vector terms is important. In general if \(D_L \neq D_R\) (or \(U_L \neq U_R\)) than in formulas (13) there is a quadratic term \(g_H^2/M_0^2(DLD^+ix(DRD^+)_{ij}, i \neq j)\).

So, we could analyse the ratios (similar for \(B_d,s\)-meson system):

\[
\left[ \frac{\Delta m_K}{m_K} \right]_H = \frac{g_H^2}{M_{h0}^2} Re[C_K] f_K^2 R_K < 7 \cdot 10^{-15}
\]

(18)
and
\[ \left[ \frac{1}{m_K} \frac{IM_{12}}{M_{H_0}} \right]_H = \frac{1}{2} \frac{g^2_H}{M_{H_0}^2} IM[C_K] f^2_K R_K < 2 \cdot 10^{-17}. \]  

(19)

In these formulas in "bootstrap" models \[5\] the expression for \( C_{K,D} \), namely
\[ C_{K,D} = \frac{g^2_H}{2\lambda_Y} \frac{m^2_{t}}{M_{H_0}^2} \times f \left( \frac{m^up_{i}}{m^down_{j}}; \frac{m^down_{k}}{m^down_{l}} \right), \]
contains known complex functions \( f \)'s and their forms depend on quark fermion mass ansatzes \[5\].

Substituting in formula (16) the expressions for \( \varphi \), \( \tilde{\varphi} \) and the elements \( d_{ij} \) of the \( D \) mixing matrix ("bootstrap" solution), we can obtain the lower limit for the value \( M_{H_0} \) \[5\]:
\[ M_{H_0} < O(1 \text{TeV}). \]  

(21)

Here noteworthy are the following two points: a) The appearance of the phase in the CKM mixing matrix may be due to new dynamics working at short distances (\( r \ll \frac{1}{M_W} \)). Horizontal forces may be the source of this new dynamics \[4\]. Using this approach, we might have the CP violation effects both due to electroweak and horizontal interactions.

(b) The CP is conserved in the electroweak sector (\( \delta_{KM} = 0 \)), and its breaking is provided by the structure of the horizontal interactions. Let us consider the situation when \( \delta_{KM} = 0 \). In the SM, such a case might be realized just accidentally. The vanishing phase of the electroweak sector (\( \delta_{KM} = 0 \)) might arise spontaneously due to some additional symmetry. Again, such a situation might occur within the horizontal extension of the electroweak model.

In particular, this model gives rise to a rather natural mechanism of superweak-like CP-violation due to the \((CP = -1)\) part of the effective Lagrangian of horizontal interactions \( (\epsilon'/\epsilon) \leq 10^{-4} \). That part of \( \mathcal{L}_{\text{eff}} \) includes the product of the \( SU(3)_H \)-currents \( I_{\mu_i} I_{\mu_j} \) (\( i=1,4,6,3,8; j=2,5,7 \) or, vice versa, \( i\leftrightarrow j \)) \[4\]. In the case of a vector-like \( SU(3)_H \)-gauge model the CP violation could be only due to the charge symmetry breaking.

The space-time structure of horizontal interactions depends on the \( SU(3)_H \) quantum numbers of quark and lepton superfields and their C-conjugate superfields. One can obtain vector (axial)-like horizontal interactions as far as the \( G_H \) particle quantum numbers are conjugate (equal) to those of antiparticles. The question arising in these theories is how such horizontal interactions are related with strong and electroweak ones. All these interactions can be unified within one gauge group, which would allow to calculate the value of the coupling constant of horizontal interactions. Thus, an unification of horizontal, strong and electroweak interactions might rest on the GUTs \( \tilde{\mathcal{G}} \equiv G \times SU(3)_H \) (where, for example, \( \tilde{\mathcal{G}} \equiv E(8), G \equiv SU(5), SO(10) \) or \( E_6 \)), which may be further broken down to \( SU(3)_H \times SU(3)_C \times SU(2)_L \times U(1)_Y \). For including "vector"-like horizontal gauge symmetry into GUT we have to introduce "mirror" superfields. Speaking more definitely, if we want to construct GUTs of the \( \tilde{\mathcal{G}} \equiv G \times SU(3)_H \) type, each generation must encompass double \( G \)-matter supermultiplets, mutually conjugate under the \( SU(3)_H \)-group. In this approach the first supermultiplet consists of the superfields \( f \) and \( f^c_m \in 3_H \).
while the second is constructed with the help of the supermultiplets $f^c$ and $f_m \in \bar{3}_H$. In this scheme, proton decays are only possible in the case of mixing between ordinary and "mirror" fermions. In its turn, this mixing must, in particular, be related with the $SU(3)_H$-symmetry breaking.

The GUSTs spectra also predict the existing of the new neutral neutrino-like particles interacting with the matter only by "superweak"-like coupling. It is possible to estimate the masses of these particles, and, as will be shown further, some of them have to be light (superlight) to be observed in modern experiment.

A variant for unusual nonuniversal family gauge interactions of known quarks and leptons could be realized if for each generation we introduce new heavy quarks ($F = U, D$), and leptons ($L, N$) which are singlets (it is possible to consider doublets also) under $SU(2)_L$- and triplets under $SU(3)_H$-groups. (This fermion matter could exist in string spectra. See the all three models with $SU(3)_H \times SU(3)_H$ family gauge symmetry). Let us consider for concreteness a case of charged leptons: $\Psi_l = (e, \mu, \tau)$ and $\Psi_L = (E, M, T)$. Primarily, for simplicity we suggest that the ordinary fermions do not take part in $SU(3)_H$-interactions ("white" color states). Then the interaction is described by the relevant part of the SUSY $SU(3)_H$-Lagrangian and gets the form

$$\mathcal{L}_H = g_H \bar{\Psi}_L \gamma^\mu \frac{\Lambda^{a \text{6x6}}}{2} \Psi_L O_{ab} Z^b_{\mu}, \quad (22)$$

where

$$\Lambda^{a \text{6x6}} = \begin{pmatrix} S(L\lambda^a L^+)S & -S(L\lambda^a L^+)C \\ -C(L\lambda^a L^+)S & C(L\lambda^a L^+)C \end{pmatrix}.$$}

Here we have $\Psi_L = (\Psi_l; \Psi_L)$. The matrix $O_{ab}$ ($a, b = 1, 2, 3...8$) determines the relationship between the bare, $H^b_{\mu}$, and physical, $Z^b_{\mu}$, gauge fields. The diagonal $3 \times 3$ matrices $S = \text{diag} \left( s_e, s_\mu, s_\tau \right)$ and $C = \text{diag} \left( c_e, c_\mu, c_\tau \right)$ define the nonuniversal character for lepton horizontal interactions, as the elements $s_i$ depend on the lepton masses, like $s_i \sim \sqrt{m_i}/M_0$ ($i=e, \mu, \tau$). The same suggestion we might accept for local quark family interactions.

For the family mixing we might suggest the next scheme. The primary $3 \times 3$ mass matrix for the light ordinary fermions is equal to zero: $M^0_{ff} \approx 0$. The $3 \times 3$-mass matrix for heavy fermions is approximately proportional to unit $M^0_{FF} \approx M^0_Y \times 1$, where $M^0_Y \approx 0.5 - 1.0 TeV$ and might be different for $F_{up}$-, $F_{down}$- quarks and for $F_{L}$-leptons. We assume that the splitting between new heavy fermions in each class $F_Y$ ($Y=\text{up}, \text{down}, \text{L}$) is small and, at least in quark sector, might be described by the t-quark mass. Thus we think that at the first approximation it is possible to neglect the heavy fermion mixing. The mixing in the light sector is completely explained by the coupling of the light fermions with the heavy fermions. As a result of this coupling the $3 \times 3$-mass matrix $M^0_{FF}$ could be constructed by "democratic" way which could lead to the well known mass family hierarchy:

$$M^0_{\text{6x6}} = \begin{pmatrix} M^0_{ff} & M^0_{ff} \\ M^0_{FF} & M^0_{FF} \end{pmatrix},$$
where
\[ M_{ff}^0 \approx M_{ff}^{dem} + M_{ff}^{corr}. \] (23)

The diagonalization of the \( M_{ff}^0 \)- mass matrix \( XM_{ff}^0 X^+ \) (X = L-, D-, U- mixing matrices) gives us the eigenvalues, which define the family mass hierarchy- \( n_Y^1 << n_Y^2 << n_Y^3 \) and the following relations between the masses of the known light fermions and a new heavy mass scale:

\[ n_i^Y = \sqrt{m_i M_0^Y}, \quad i = 1_g, 2_g, 3_g; \quad Y = up-, down - fermions. \]

In this "see-saw" mechanism the common mass scale of new heavy fermions might be not very far from the energy \( \sim 1TeV \), and as a consequence of it the mixing angles \( s_i \)- might be not too small.
3 The Heterotic Superstring Theory with Rank 8 and 16 Grand Unified Gauge Groups.

3.1 Conformal symmetry in heterotic superstring.

In the heterotic string theory in left-moving (supersymmetric) sector there are $d-2$ (in the light-cone gauge) real fermions $\psi^\mu$, their bosonic superpartners $X^\mu$, and $3(10-d)$ real fermions $\chi^I$. In the right-moving sector there are $d-2$ bosons $\bar{X}^\mu$ and $2(26-d)$ real fermions.

In heterotic string theories \cite{17,18} ($N=1$ SUSY$_{\text{LEFT}}$ ($N=0$ SUSY)$_{\text{RIGHT}} \oplus \mathcal{M}_{c_L,c_R}$ with $d \leq 10$, the conformal anomalies of the space-time sector are canceled by the conformal anomalies of the internal sector $\mathcal{M}_{c_L,c_R}$, where $c_L = 15 - 3d/2$ and $c_R = 26 - d$ are the conformal anomalies in the left- and right–moving string sectors respectively.

One can consider the operator product expansion between the energy-momentum tensor $T(z)$:

$$T(w)T(z) \sim \frac{c/2}{(w-z)^4} + \frac{2}{(w-z)^2}T(z) + \frac{1}{w-z}\partial_z T(z),$$

(24)

where $c$ is a central charge or conformal anomaly.

If we take the moments of the energy-momentum operator $T(z)$ we will get the conformal generators with the following Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n (n^2 - 1) \delta_{n,-m}. $$

(25)

Using Virasoro algebra we can construct representations of the conformal group where highest weight state is specified by two quantum numbers, conformal weight $h$ and central charge $c$, such that:

$$L_0 |h,c\rangle = h |h,c\rangle$$

$$L_n |h,c\rangle = 0, \quad n = 1, 2, 3, ....$$

(26)

For massless state the conformal weight $h = 1$.

In the left supersymmetric sector world-sheet supersymmetry is non-linearly realized via the supercharge

$$T_F = \psi^\mu \partial X_\mu + f_{IJK} \chi^I \chi^J \chi^K,$$

(27)

where $f_{IJK}$ are the structure constants of a semi-simple Lie group $G$ of dimension $3(10-d)$. The operator product expansions $T(z)$ and $T_F(z)$ give the N=1, (1,0), superconformal algebra:
\[ T(w)T(z) \sim \frac{3\hat{c}/4}{(w-z)^4} + \frac{2}{(w-z)^2} T(z) + \frac{1}{w-z} \partial_z T(z), \]
\[ T(w)T_F(z) \sim \frac{3/2}{(w-z)^2} T_F + \frac{1}{w-z} \partial_z T(z), \]
\[ T_F(w)T_F(z) \sim \frac{\hat{c}/4}{(w-z)^4} + \frac{1/2}{(w-z)^2} T(z), \]
where \( \hat{c} = 2/3c = 6 \). The possible Lie algebras of dimension 18 for \( d = 4 \) are \( SU(2)^6 \), \( SU(3) \times SO(5) \), and \( SU(2) \times SU(4) \).

In papers \cite{40} it has been shown that the \( N=1 \) space-time SUSY vacuum of heterotic string with local \((1,0)\) worldsheet superconformal invariance extends to a global \( N=2 \), \((2,0)\), superconformal invariance:

\[ T_F^\pm(w)T_F^\mp(z) \sim \frac{\hat{c}}{(w-z)^3} + \frac{2\partial J}{(w-z)^2} + \frac{2T + \partial J}{w-z} \]
\[ J(w)T_F^\pm(z) \sim \pm \frac{T_F^\pm}{w-z} \]
\[ J(w)J(z) \sim \frac{\hat{c}/2}{(w-z)^2}. \]

A Sugawara- Sommerfeld construction of the energy- momentum tensor \( T(z) \) algebra in terms of bilinears in the Kac-Moody generators \( J^a_n(z) \) \cite{14,15,16},

\[ T(z) = \sum_n L_n z^{-n-2} = -\frac{1}{2k + Q_\psi} \sum_{n,m} J^a_{n-m} J^a_m : z^{-n-2}, \]

allows to get, commuting two generators of the Virasoro algebra, the following expression for the central Virasoro ”charge”:

\[ c_g = \frac{2k \text{dim}g}{2k + Q_\psi} = \frac{\text{dim}g}{x + \tilde{h}}. \]

In the fermionic formulation of the four-dimensional heterotic string theory in addition to the two transverse bosonic coordinates \( X_\mu, \bar{X}_\mu \) and their left-moving superpartners \( \psi_\mu \), the internal sector \( \mathcal{M}_{c_L,c_R} \) contains 44 right-moving \((c_R = 22)\) and 18 left-moving \((c_L = 9)\) real fermions (each real world- sheet fermion has \( c_f = 1/2 \)).

For a couple of years superstring theories, and particularly the heterotic string theory, have provided an efficient way to construct the Grand Unified Superstring Theories \((\text{GUST})\) of all known interactions, despite the fact that it is still difficult to construct unique and fully realistic low energy models resulting after decoupling of massive string modes. This is because of the fact that only 10-dimensional space-time allows existence of two consistent (invariant under reparametrization, superconformal, modular, Lorentz and
SUSY transformations) theories with the gauge symmetries $E(8) \times E(8)$ or $spin(32)/Z_2$ [17, 18] which after compactification of the six extra space coordinates (into the Calabi-Yau [19, 20] manifolds or into the orbifolds) can be used for constructing GUSTs. Unfortunately, the process of compactification to four dimensions is not unique and the number of possible low energy models is very large. On the other hand, constructing the theory directly in 4-dimensional space-time requires including a considerable number of free bosons or fermions into the internal string sector of the heterotic superstring [21, 22, 23, 24]. This leads to as large internal symmetry group such as e.g. rank 22 group. The way of breaking this primordial symmetry is again not unique and leads to a huge number of possible models, each of them giving different low energy predictions.

Because of the presence of the affine Kac-Moody algebra (KMA) $\hat{g}$ (which is a 2-dimensional manifestation of gauge symmetries of the string itself) on the world sheet, string constructions yield definite predictions concerning representation of the symmetry group that can be used for low energy models building [14, 16]. Therefore the following longstanding questions have a chance to be answered in this kind of unification schemes:

1. How are the chiral matter fermions assigned to the multiplets of the unifying group?

2. How is the GUT gauge symmetry breaking realized?

3. What is the origin of the fermion mass hierarchy?

The first of these problems is, of course, closely connected to the quantization of the electromagnetic charge of matter fields. In addition, string constructions can shed some light on the questions about the number of generation and possible existence of mirror fermions which remain unanswered in conventional GUTs [25].

There are not so many GUSTs describing the observable sector of Standard Models. They are well known: the SM gauge group, the Pati-Salam ($SU(4) \times SU(2) \times SU(2)$) gauge group, the flipped SU(5) gauge group and SO(10) gauge group, which includes flipped SU(5) [24, 20].

There are good physical reasons for including the horizontal $SU(3)_H$ group into the unification scheme. Firstly, this group naturally accommodates three fermion families presently observed (explaining their origin) and, secondly, can provide correct and economical description of the fermion mass spectrum and mixing without invoking high dimensional representation of conventional $SU(5)$, $SO(10)$ or $E(6)$ gauge groups. Construction of a string model (GUST) containing the horizontal gauge symmetry provides additional strong motivation to this idea. Moreover, the fact that in GUSTs high dimensional representations are forbidden by the KMA is a very welcome feature in this context.

### 3.2 The possible ways of E(8)-GUST breaking leading to the $N_G = 3$ or $N_G = 3 + 1$ families

All this leads us naturally to consider possible forms for horizontal symmetry $G_H$, and $G_H$ quantum number assignments for quarks (anti-quarks) and leptons (anti-leptons) which
Figure 1: The possible ways of E(8) gauge symmetry breaking leading to the 3+1 or 3 generations.

\[ E(8) \rightarrow SO(16) \]

\[ E(6) \times SU(3)_H \rightarrow SO(10) \times SU(3)_H \times U(1)_H \]

\[ SU(3)^4 \rightarrow SU(5) \times U(1) \times SU(3)_H \times U(1)_H \]

\[ N_g = 3, \quad N_g = 3 + 1 \]

can be realized within GUST’s framework. To include the horizontal interactions with three known generations in the ordinary GUST it is natural to consider rank eight gauge symmetry. We can consider \( SO(16) \) (or \( E(6) \times SU(3) \)) which is the maximal subgroup of \( E(8) \) and which contains the rank eight subgroup \( SO(10) \times (U(1) \times SU(3))_H \) \cite{27}. We will be, therefore, concerned with the following chains (see Fig. 1):

\[ E(8) \rightarrow SO(16) \rightarrow SO(10) \times (U(1) \times SU(3))_H \rightarrow SU(5) \times U(1)_{Y_5} \times (SU(3) \times U(1))_H \]

or

\[ E(8) \rightarrow E(6) \times SU(3) \rightarrow (SU(3))^4. \]
According to this scheme one can get $SU(3)_H \times U(1)_H$ gauge family symmetry with $N_g = 3 + 1$ (there are also other possibilities as e.g. $E(6) \times SU(3)_H \subset E(8)$ $N_g = 3$ generations can be obtained due to the second way of $E(8)$ gauge symmetry breaking via $E(6) \times SU(3)_H$, see Fig.[1]), where the possible additional fourth massive matter superfield could appear from 78 as a singlet of $SU(3)_H$ and transforms as 16 under the $SO(10)$ group.

In this note starting from the rank 16 grand unified gauge group (which is the minimal rank allowed in strings) of the form $G \times G$ [28, 29] and making use of the KMA which select the possible gauge group representations we construct the string models based on the diagonal subgroup $G^{symm} \subset G \times G \subset SO(16) \times SO(16)$ ($\subset E_8 \times E_8$) [28]. We discuss and consider $G^{symm} = SU(5) \times U(1) \times (SU(3) \times U(1))_H \subset SO(16)$ where the factor $(SU(3) \times U(1))_H$ is interpreted as the horizontal gauge family symmetry. We explain how the unifying gauge symmetry can be broken down to the Standard Model group. Furthermore, the horizontal interaction predicted in our model can give an alternative description of the fermion mass matrices without invoking high dimensional Higgs representations. In contrast with other GUST constructions, our model does not contain particles with exotic fractional electric charges [30, 28]. This important virtue of the model is due to the symmetric construction of the electromagnetic charge $Q_{em}$ from $Q^I$ and $Q^{II}$ – the two electric charges of each of the $U(5)$ groups [28]:

$$Q_{em} = Q^{II} \oplus Q^I. \tag{32}$$

We consider the possible forms of the $G_H = SU(3)_H , SU(3)_H \times U(1), G_{HL} \times G_{HR}$...- gauge family symmetries in the framework of Grand Unification Superstring Approach. Also we will study the matter spectrum of these GUST, the possible Higgs sectors. The form of the Higgs sector it is very important for GUST-, $G_H$- and SM - gauge symmetries breaking and for constructing Yukawa couplings.
4 World-Sheet Kac-Moody Algebra And Main Features of Rank Eight GUST

4.1 The representations of Kac-Moody Algebra

Let us begin with a short review of the KMA results [14, 16]. In heterotic string the KMA is constructed by the operator product expansion (OPE) of the fields $J^a$ of the conformal dimension $(0,1)$:

$$J^a(w)J^b(z) \sim \frac{1}{(w-z)^2}k \delta^{ab} + \frac{1}{w-z}i f^{abc} J^c + ....$$  \hspace{1cm} (33)

The structure constants $f^{abc}$ for the group $g$ are normalized so that

$$f^{acd} f^{bcd} = Q_{\psi} \delta^{ab} = \tilde{h} \psi^2 \delta^{ab}$$ \hspace{1cm} (34)

where $Q_{\psi}$ and $\psi$ are the quadratic Casimir and the highest weight of the adjoint representation and $\tilde{h}$ is the dual Coxeter number. The $\frac{\psi}{\psi^2}$ can be expanded as in integer linear combination of the simple roots of $g$:

$$\frac{\psi}{\psi^2} = \sum_{i=1}^{\text{rank} g} m_i \alpha_i.$$ \hspace{1cm} (35)

The dual Coxeter number can be expressed through the integers numbers $m_i$

$$\tilde{h} = 1 + \sum_{i=1}^{\text{rank} g} m_i$$ \hspace{1cm} (36)

and for the simply laced groups (all roots are equal and $\psi^2 = 2$): $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ they are equal $n + 1$, $2n - 2$, 12, 18 and 30, respectively.

The KMA $\hat{g}$ allows to grade the representations $R$ of the gauge group by a level number $x$ (a non negative integer) and by a conformal weight $h(R)$. An irreducible representation of the affine algebra $\hat{g}$ is characterized by the vacuum representation of the algebra $g$ and the value of the central term $k$, which is connected to the level number by the relation $x = 2k/\psi^2$. The value of the level number of the KMA determines the possible highest weight unitary representations which are present in the spectrum in the following way

$$x = \frac{2k}{\psi^2} \geq \sum_{i=1}^{\text{rank} g} n_i m_i,$$ \hspace{1cm} (37)

where the sets of non-negative integers $\{m_i = m_1, ..., m_r\}$ and $\{n_i = n_1, ..., n_r\}$ define the highest root and the highest weight in terms of fundamental weights of a representation $R$ respectively [14, 16]:

$$\mu_0 = \sum_{i=1}^{\text{rank} g} n_i \lambda_i$$ \hspace{1cm} (38)
In fact, the KMA on the level one is realized in the 4-dimensional heterotic superstring theories with free world sheet fermions which allow a complex fermion description \[22, 23, 24\]. One can obtain KMA on a higher level working with real fermions and using some tricks \[31\]. For these models the level of KMA coincides with the Dynkin index of representation \(M\) to which free fermions are assigned,

\[
x = x_M = \frac{Q_M \ dimM}{\psi^2 \ dimg}
\]

\((Q_M\) is a quadratic Casimir eigenvalue of representation \(M\)) and equals one in cases when real fermions form vector representation \(M\) of \(SO(2N)\), or when the world sheet fermions are complex and \(M\) is the fundamental representation of \(U(N)\) \[14, 16\].

Thus, in strings with KMA on the level one realized on the world-sheet, only very restricted set of unitary representations can arise in the spectrum:

1. singlet and totally antisymmetric tensor representations of \(SU(N)\) groups, for which \(m_i = (1, \ldots, 1)\);
2. singlet, vector, and spinor representations of \(SO(2N)\) groups with \(m_i = (1, 2, 2, \ldots, 2, 1, 1)\);
3. singlet, \(27\), and \(\overline{27}\)-plets of \(E(6)\) corresponding to \(m_i = (1, 2, 2, 3, 2, 1)\);
4. singlet of \(E(8)\) with \(m_i = (2, 3, 4, 6, 5, 4, 3, 2)\).

Therefore only these representations can be used to incorporate matter and Higgs fields in GUSTs with KMA on the level 1.

In principle it might be possible to construct explicitly an example of a level 1 KMA-representation of the simply laced algebra \(\hat{g}\) (A-, D-, E - types) from the level one representations of the Cartan subalgebra of \(g\). This construction is achieved using the vertex operator of string, where these operators are assigned to a set of lattice point corresponding to the roots of a simply-laced Lie algebra \(g\).

### 4.2 The features of the level one KMA in matter and Higgs representations in rank 8 and 16 GUST Constructions

For example, to describe chiral matter fermions in GUST with the gauge symmetry group \(SU(5) \times U(1) \subset SO(10)\) the following sum of the level-one complex representations: \(1((-5/2) + 5(+3/2) + 10(-1/2)) = 16\) can be used. On the other side, as real representations of \(SU(5) \times U(1) \subset SO(10)\), from which Higgs fields can arise, one can take for example \(5 + \bar{5}\) representations arising from real representation \(10\) of \(SO(10)\). Also, real Higgs representations like \(10(-1/2) + \bar{10}(+1/2)\) of \(SU(5) \times U(1)\) originating from \(16 + \bar{16}\) of \(SO(10)\), which has been used in ref. \[8\] for further symmetry breaking, are allowed.

Another example is provided by the decomposition of \(SO(16)\) representations under \(SU(8) \times U(1) \subset SO(16)\). Here, only singlet, \(v = 16\), \(s = 128\), and \(s' = 128'\) representations
of $SO(16)$ are allowed by the KMA ($s = 128$ and $s' = 128'$ are the two nonequivalent, real spinor representations with the highest weights $\pi_{7,8} = 1/2(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_7 + \epsilon_8)$, $\epsilon_i \epsilon_j = \delta_{ij}$). From the item 2, we can obtain the following $SU(8) \times U(1)$ representations: singlet, $8 + \bar{8} (= 16)$, $8 + 56 + \bar{56} + \bar{8} (= 128)$, and $\bar{1} + 28 + 70 + + 28 + \bar{1} (= 128')$. The highest weights of $SU(8)$ representations $\pi_1 = \pi(8)$, $\pi_7 = \pi(\bar{8})$ and $\pi_3 = \pi(\bar{56})$, $\pi_5 = \pi(56)$ are:

\[
\begin{align*}
\pi_1 &= 1/8(7\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8), \\
\pi_7 &= 1/8(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 - 7\epsilon_8), \\
\pi_3 &= 1/8(5\epsilon_1 + 5\epsilon_2 + 5\epsilon_3 - 3\epsilon_4 - 3\epsilon_5 - 3\epsilon_6 - 3\epsilon_7 - 3\epsilon_8), \\
\pi_5 &= 1/8(3\epsilon_1 + 3\epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + 3\epsilon_5 - 5\epsilon_6 - 5\epsilon_7 - 5\epsilon_8).
\end{align*}
\]

Similarly, the highest weights of $SU(8)$ representations $\pi_2 = \pi(28)$, $\pi_6 = \pi(\bar{28})$ and $\pi_4 = \pi(\bar{70})$ are:

\[
\begin{align*}
\pi_2 &= 1/4(3\epsilon_1 + 3\epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - 6\epsilon_6 - \epsilon_7 - \epsilon_8), \\
\pi_6 &= 1/4(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 - 3\epsilon_7 - 3\epsilon_8), \\
\pi_4 &= 1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5 - 6\epsilon_6 - \epsilon_7 - \epsilon_8).
\end{align*}
\]

However, as we will demonstrate, in each of the string sectors the generalized Gliozzi–Scherk–Olive projection (the GSO projection in particular guarantees the modular invariance and supersymmetry of the theory and also give some nontrivial restrictions on gauge groups and its representations) necessarily eliminates either $128$ or $128'$. It is therefore important that, in order to incorporate chiral matter in the model, only one spinor representation is sufficient. Moreover, if one wants to solve the chirality problem applying further GSO projections (which break the gauge symmetry) the representation $10$ which otherwise, together with $\bar{10}$, could form real Higgs representation, also disappears from this sector. Therefore, the existence of $10_{1/2} + \bar{10}_{1/2}$, needed for breaking $SU(5) \times U(1)$ is incompatible (by our opinion) with the possible solution of the chirality problem for the family matter fields.

Thus, in the rank eight group $SU(8) \times U(1) \subset SO(16)$ with Higgs representations from the level-one KMA only, one cannot arrange for further symmetry breaking. Moreover, construction of the realistic fermion mass matrices seems to be impossible. In old-fashioned GUTs (see e.g. [25]), not originating from strings, the representations of the level two were commonly used to solve these problems.

The way out from this difficulty is based on the following important observations. Firstly, all higher-dimensional representations of (simple laced) groups like $SU(N)$, $SO(2N)$ or $E(6)$, which belong to the level two representation of the KMA (according to equation [37]), appear in the direct product of the level one representations:

\[
R_G(x = 2) \subset R_G(x = 1) \times R_G'(x = 1).
\]
For example, the level-two representations of $SU(5)$ will appear in the corresponding direct products of

$$15, 24, 40, 45, 50, 75 \subset 5 \times 5, 5 \times 5 \times 10, etc.$$ (43)

In the case of $SO(10)$ the level two representations can be obtained by the suitable direct products:

$$45, 54, 120, 126, 210, 144 \subset 10 \times 10, \bar{16} \times 10, 16 \times 16, \bar{16} \times 16.$$ (44)

The level-two representations of $E(6)$ are the corresponding factors of the decomposition of the direct products:

$$78, 351, 351', 650 \subset 27 \times 27$$ or $$27 \times 27.$$ (45)

The only exception from this rule is the $E(8)$ group, two level-two representations (248 and 3875) of which cannot be constructed as a product of level-one representations [27]. Secondly, the diagonal (symmetric) subgroup $G_{symm}$ of $G \times G$ effectively corresponds to the level-two KMA $g(x = 1) \oplus g(x = 1)$[28, 29] because taking the $G \times G$ representations in the form $(R_G, R'_G)$ of the $G \times G$, where $R_G$ and $R'_G$ belong to the level-one of $G$, one obtains representations of the form $R_G \times R'_G$ when one considers only the diagonal subgroup of $G \times G$. This observation is crucial, because such a construction allows one to obtain level-two representations. (This construction has implicitly been used in [29] (see also [28] where we have constructed some examples of GUST with gauge symmetry realized as a diagonal subgroup of direct product of two rank eight groups $U(8) \times U(8) \subset SO(16) \times SO(16)$.)

In strings, however, not all level-two representations can be obtained in that way because, as we will demonstrate, some of them become massive (with masses of order of the Planck scale). The condition ensuring that states in the string spectrum transforming as a representation $R$ are massless reads:

$$h(R) = \frac{Q_R}{2k + Q_{ADJ}} = \frac{Q_R}{2Q_M} \leq 1,$$ (46)

where $Q_i$ is the quadratic Casimir invariant of the corresponding representations, and M has been already defined before (see eq. [39]). Here the conformal weight is defined by $L_0|0\rangle = h(R)|0\rangle$;

$$L_0 = \frac{1}{2k + Q_\psi} \times \left( \sum_{a=1}^{dim g} (T^a_0 T^a_0 + 2 \sum_{n=1}^{\infty} T^a_n T^a_{-n}) \right),$$ (47)

where $T^a_n|0\rangle = 0$ for $n > 0$. The condition (46), when combined with (47), gives a restriction on the rank of GUT’s group ($r \leq 8$), whose representations can accommodate chiral matter fields. For example, for antisymmetric representations of $SU(n = l + 1)$ we have the following values correspondingly : $h = p(n - p)/(2n)$. More exactly, for $SU(8)$ group: $h(8) = 7/16$, $h(28) = 3/4$, $h(56) = 15/16$, $h(70) = 1$; for $SU(5)$,
correspondingly $h(5) = 2/5$ and $h(10) = 3/5$; for $SU(3)$ group $h(3) = 1/3$ although for adjoint representation of $SU(3)$ - $h(8) = 3/4$; for $SU(2)$ doublet representation we have $h(2) = 1/4$. For vector representation of orthogonal series $D_l$ $h = 1/2$, and, respectively, for spinor - $h(spion) = l/8$.

There are some another important cases. The values of conformal weights for $G = SO(16)$ or $E(6) \times SU(3)$, representations $128, (27, 3)$ ($h(128) = 1, h(27, 3) = 1$) respectively, satisfy both conditions. Obviously, these (important for incorporation of chiral matter) representations will exist in the level-two KMA of the symmetric subgroup of the group $G \times G$.

In general, condition (46) severely constrains massless string states transforming as $(R_G(x = 1), R_G'(x = 1))$ of the direct product $G \times G$. For example, for $SU(8) \times SU(8)$ and for $SU(5) \times SU(5)$ constructed from $SU(8) \times SU(8)$ only representations of the form

$$R_{N,N} = \left( (N, N) + \text{h.c.}, \ (N, \bar{N}) + \text{h.c.} \right);$$

with $h(R_{N,N}) = (N - 1)/N$, where $N = 8$ or $5$ respectively can be massless. For $SO(2N) \times SO(2N)$ massless states are contained only in representations

$$R_{v,v} = (2N, 2N)$$

with $h(R_{v,v}) = 1$. Thus, for the GUSTs based on a diagonal subgroup $G_{symm} \subset G \times G$, $G_{symm}$ - high dimensional representations, which are embedded in $R_G(x = 1) \times R'_G(x = 1)$ are also severely constrained by condition (46).

For spontaneous breaking of $G \times G$ gauge symmetry down to $G_{symm}$ (rank $G_{symm} =$ rank $G$) one can use the direct product of representations $R_G(x = 1) \times R_G(x = 1)$, where $R_G(x = 1)$ is the fundamental representation of $G = SU(N)$ or vector representation of $G = SO(2N)$. Furthermore, $G_{symm} \subset G \times G$ can subsequently be broken down to a smaller dimension gauge group (of the same rank as $G_{symm}$) through the VEVs of the adjoint representations which can appear as a result of $G \times G$ breaking. Alternatively, the real Higgs superfields (48) or (49) can directly break the $G \times G$ gauge symmetry down to a $G_{1_{symm}} \subset G_{symm}$ (rank $G_{1_{symm}} \leq$ rank $G_{symm}$). For example when $G = SU(5) \times U(1)$ or $SO(10) \times U(1)$, $G \times G$ can directly be broken in this way down to $SU(3) \times G_{I_{EW}} \times G_{II_{EW}} \times ....$

The above examples show clearly, that within the framework of GUSTs with the KMA one can get interesting gauge symmetry breaking chains including the realistic ones when $G \times G$ gauge symmetry group is considered. However the lack of the higher dimension representations (which are forbidden by (46)) on the level-two KMA prevents the construction of the realistic fermion mass matrices. That is why we consider an extended grand unified string model of rank eight $SO(16)$ or $E(6) \times SU(3)$ of $E(8)$.

The full chiral $SO(10) \times SU(3) \times U(1)$ matter multiplets can be constructed from $SU(8) \times U(1)$-multiplets

$$(8 + 56 + 8 + \bar{56}) = 128$$

of $SO(16)$. In the 4-dimensional heterotic superstring with free complex world sheet fermions, in the spectrum of the Ramond sector there can appear also representations
which are factors in the decomposition of $128'$, in particular, $SU(5)$-decoupled $(10 + \bar{10})$ from $(28 + 28)$ of $SU(8)$. However their $U(1)_5$ hypercharge does not allow to use them for $SU(5) \times U(1)_5$-symmetry breaking. Thus, in this approach we have only singlet and $(5 + \bar{5})$ Higgs fields which can break the grand unified $SU(5) \times U(1)$ gauge symmetry. Therefore it is necessary (as we already explained) to construct rank eight GUST based on a diagonal subgroup $G^{\text{symm}} \subset G \times G$ primordial symmetry group, where in each rank eight group $G$ the Higgs fields will appear only in singlets and in the fundamental representations as in (see [18]).

A comment concerning $U(1)$ factors can be made here. Since the available $SU(5) \times U(1)$ decoupled have non-zero hypercharges with respect to $U(1)_5$ and $U(1)_H$, these $U(1)$ factors may remain unbroken down to the low energies in the model considered which seems to be very interesting.
5 Modular Invariance in GUST Construction with Non-Abelian Gauge Family Symmetry

5.1 Spin-basis in free world-sheet fermion sector.

The GUST model is completely defined by a set $\Xi$ of spin boundary conditions for all these world-sheet fermions (see Appendix C). In a diagonal basis the vectors of $\Xi$ are determined by the values of phases $\alpha(f) \in (-1, 1]$ fermions $f$ acquire ($f \rightarrow -\exp(i\pi\alpha(f))f$) when parallel transported around the string. To construct the GUST according to the scheme outlined at the end of the previous section we consider three different bases each of them with six elements $B = b_1, b_2, b_3, b_4 \equiv S, b_5, b_6$. (See Tables 1, 4, 7.)

Following [23] (see Appendix C) we construct the canonical basis in such a way that the vector $\bar{1}$, which belongs to $\Xi$, is the first element $b_1$ of the basis. The basis vector $b_4 = S$ is the generator of supersymmetry [24] responsible for the conservation of the space-time SUSY.

In this chapter we have chosen a basis in which all left movers ($\psi_\mu; \chi_i, y_i, \omega_i; i = 1, \ldots 6$) (on which the world sheet supersymmetry is realized nonlinearly) as well as 12 right movers ($\bar{\varphi}_k; k = 1, \ldots 12$) are real whereas $(8 + 8)$ right movers $\bar{\Psi}_A, \bar{\Phi}_M$ are complex. Such a construction corresponds to $SU(2)^6$ group of automorphisms of the left supersymmetric sector of a string. Right- and left-moving real fermions can be used for breaking $G^{\text{comp}}$ symmetry [24]. In order to have a possibility to reduce the rank of the compactified group $G^{\text{comp}}$, we have to select the spin boundary conditions for the maximal possible number, $N_{LR} = 12$, of left-moving, $\chi_{3,4,5,6}, y_{1,2,5,6}, \omega_{1,2,3,4}$, and right-moving, $\bar{\phi}^{1, \ldots 12}$ ($\bar{\phi}^p = \bar{\varphi}_p, p = 1, \ldots 12$) real fermions. The KMA based on 16 complex right moving fermions gives rise to the "observable" gauge group $G^{\text{obs}}$ with:

$$\text{rank}(G^{\text{obs}}) \leq 16.$$  \hspace{1cm} (51)

The study of the Hilbert spaces of the string theories is connected to the problem of finding all possible choices of the GSO coefficients $C$ [\[\alpha \beta\]] (see Appendix C), such that the one-loop partition function

$$Z = \sum_{\alpha, \beta} C_{\alpha \beta} \prod_f Z_{\alpha_f \beta_f}$$  \hspace{1cm} (52)

and its multiloop counterparts are all modular invariant. In this formula $C_{\alpha \beta}$ are GSO coefficients, $\alpha$ and $\beta$ are $(k+l)$–component spin–vectors $\alpha = [\alpha(f_1^+), \ldots, \alpha(f_k^+); \alpha(f_1^-), \ldots, \alpha(f_l^-)]$, the components $\alpha_f, \beta_f$ specify the spin structure of the $f$th fermion and $Z[\ldots] –$ corresponding one-fermion partition functions on torus: $Z[\ldots] = \text{Tr} \exp[2\pi i H_{(\text{sect.})}]$.

The physical states in the Hilbert space of a given sector $\alpha$ are obtained acting on the vacuum $|0\rangle_\alpha$ with the bosonic and fermionic operators with frequencies

$$n(f) = 1/2 + 1/2\alpha(f), \quad n(f^*) = 1/2 - 1/2\alpha(f^*)$$  \hspace{1cm} (53)
and subsequently applying the generalized GSO projections. The physical states satisfy the Virasoro condition:

\[ M_2^2 = -1/2 + 1/8 (\alpha_L \cdot \alpha_L) + N_L = -1 + 1/8 (\alpha_R \cdot \alpha_R) + N_R = M_2^R, \]

where \( \alpha = (\alpha_L, \alpha_R) \) is a sector in the set \( \Xi \), \( N_L = \sum_L(frequencies) \) and \( N_R = \sum_R(freq.) \).

We keep the same sign convention for the fermion number operator \( F \) as in [24]. For complex fermions we have \( F_\alpha(f) = 1, \ F_\alpha(f^*) = -1 \) with the exception of the periodic fermions for which we get \( F_{\alpha=1}(f) = -1/2(1 - \gamma_5f) \), where \( \gamma_5 |\Omega\rangle = |\Omega\rangle \), \( \gamma_5 b_0^+ |\Omega\rangle = -b_0^+ |\Omega\rangle \).

The full Hilbert space of the string theory is constructed as a direct sum of different sectors \( \sum_i m_i b_i \), \((m_i = 0, 1, ..., N_i)\), where the integers \( N_i \) define additive groups \( \mathbb{Z}(b_i) \) of the basis vectors \( b_i \). The generalized GSO projection leaves in sectors \( \alpha \) those states, whose \( b_i \)-fermion number satisfies:

\[ \exp(i\pi b_i F_\alpha) = \delta_\alpha C^* \left[ \begin{array}{c} \alpha \\ b_i \end{array} \right], \]

(55)

where the space-time phase \( \delta_\alpha = \exp(i\pi \alpha(\psi_\mu)) \) is equal \(-1\) for the Ramond sector and +1 for the Neveu-Schwarz sector.

5.2 \( SU(5) \times U(1) \times SU(3) \times U(1) \)- Model 1.

Model 1 is defined by 6 basis vectors given in Table 1 which generates the \( Z_2 \times Z_4 \times Z_2 \times Z_2 \times Z_8 \times Z_2 \) group under addition.

| Vectors | \( \psi_{1,2} \) | \( \chi_{1,6} \) | \( \psi_{1,6} \) | \( \omega_{1,6} \) | \( \varphi_{1,12} \) | \( \Psi_{1,8} \) | \( \Phi_{1,8} \) |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( b_1 \) | 11 | 111111 | 111111 | 111111 | 1^{12} | 1^{8} | 1^{8} |
| \( b_2 \) | 11 | 111111 | 000000 | 000000 | 0^{12} | 1/2^{8} | 0^{8} |
| \( b_3 \) | 11 | 111100 | 000011 | 000000 | 0^{4}1^{8} | 0^{8} | 1^{8} |
| \( b_4 = S \) | 11 | 110000 | 001100 | 000011 | 0^{12} | 0^{8} | 0^{8} |
| \( b_5 \) | 11 | 001100 | 000000 | 110011 | 1^{12} | 1/4^{5} - 3/4^{3} | -1/4^{5} 3/4^{3} |
| \( b_6 \) | 11 | 110000 | 000011 | 001100 | 1^{2}0^{4}1^{6} | 1^{8} | 0^{8} |

In our approach the basis vector \( b_2 \) is constructed as a complex vector with the \( 1/2 \) spin-boundary conditions for the right-moving fermions \( \Psi_A, A = 1, ..., 8 \). Initially it generates chiral matter fields in the \( 8 + 56 + 56 + 8 \) representations of \( SU(8) \times U(1) \), which subsequently are decomposed under \( SU(5) \times U(1) \times SU(3) \times U(1) \) to which \( SU(8) \times U(1) \) gets broken by applying the \( b_5 \) GSO projection.

Generalized GSO projection coefficients are originally defined up to fifteen signs but some of them are fixed by the supersymmetry conditions. Below, in Table 2, we present...
a set of numbers
\[ \gamma \begin{bmatrix} b_i \\ b_j \end{bmatrix} = \frac{1}{i\pi} \log C \begin{bmatrix} b_i \\ b_j \end{bmatrix}. \]
which we use as a basis for our GSO projections.

Table 2: The choice of the GSO basis \( \gamma[b_i, b_j] \). Model 1. (\( i \) numbers rows and \( j \) – columns)

|   | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) |
|---|---|---|---|---|---|---|
| \( b_1 \) | 0 | 1 | 1 | 1 | 1 | 0 |
| \( b_2 \) | 1 | 1/2 | 0 | 0 | 1/4 | 1 |
| \( b_3 \) | 1 | -1/2 | 0 | 0 | 1/2 | 0 |
| \( b_4 \) | 1 | 1 | 1 | 1 | 1 |
| \( b_5 \) | 0 | 1 | 0 | 0 | -1/2 | 0 |
| \( b_6 \) | 0 | 0 | 0 | 0 | 1 | 1 |

In our case of the \( Z_2^4 \times Z_4 \times Z_8 \) model, we initially have \( 256 \times 2 \) sectors. After applying the GSO-projections we get only \( 49 \times 2 \) sectors containing massless states, which depending on the vacuum energy values, \( E_{vac}^L \) and \( E_{vac}^R \), can be naturally divided into some classes and which determine the GUST representations.

Generally RNS (Ramond – Neveu-Schwarz) sector (built on vectors \( b_1 \) and \( S = b_4 \)) has high symmetry including \( N = 4 \) supergravity and gauge \( SO(44) \) symmetry. Corresponding gauge bosons are constructed as follows:

\[ \psi_{1/2}[0]\_L \otimes \Psi_{1/2}^I \Psi_{1/2}^J, \]
\[ \psi_{1/2}[0]\_L \otimes \Psi_{1/2}^I \Psi_{1/2}^J, \quad I, J = 1, \ldots, 22 \] (56)

While \( U(1)_J \) charges for Cartan subgroups is given by formula \( Y = \frac{q}{2} + F \) (where \( F \) — fermion number, see (55)), it is obvious that states (56) generate root lattice for \( SO(44) \):

\[ \pm \varepsilon_I \pm \varepsilon_J \quad (I \neq J); \quad \pm \varepsilon_I \mp \varepsilon_J \] (57)

The others vectors breaks \( N = 4 \) SUSY to \( N = 1 \) and gauge group \( SO(44) \) to \( SO(2)^3_{1,2,3} \times SO(6) \times [SU(5) \times U(1) \times SU(3)_H \times U(1)_H]^2 \), see Figure [4].

Generally, additional basis vectors can generate extra vector bosons and extend gauge group that remains after applying GSO-projection to RNS-sector. In our case dangerous sectors are: \( 2b_2 + nb_5, \ n = 0, 2, 4, 6; \ 2b_5; 6b_5 \). But our choice of GSO coefficients cancels all the vector states in these sectors. Thus gauge bosons in this model appear only from RNS-sector.

In NS sector the \( b_3 \) GSO projection leaves \( (5,3) + (\bar{5},3) \) Higgs superfields (see Figure 2):

\[ \chi_{1/2}|\Omega\rangle_L \otimes \Psi_{1/2}^a \Psi_{1/2}^{a*} \]; \[ \Psi_{1/2}^a \Psi_{1/2}^{a*}|\Omega\rangle_R \] and exchange \( \Psi \rightarrow \Phi \),

(58)
Table 3: The list of quantum numbers of the states. Model 1.

| N° | $b_{1,2,3,b_4,b_5,b_6}$ | $SO_{hid}$ | $U(5)^I$ | $U(3)^I$ | $U(5)^{II}$ | $U(3)^{II}$ | $Y^I_2$ | $Y^I_3$ | $Y^{II}_2$ | $Y^{II}_3$ |
|----|--------------------------|-------------|-----------|-----------|-------------|-------------|--------|--------|-----------|-----------|
| 1  | RNS                      | 0 2 0 1 2(6) 0 | 5 3 1 1 1 | -1 -1 0 0 | 1 1 5 3 0 | 0 0 -1 -1 | 5 1 5 1 -1 0 -1 0 | 1 3 1 3 0 1 0 1 | 5 1 1 3 -1 0 0 1 | 1 3 5 1 0 1 -1 0 |
| 2  | 0 1 0 0 0 0              | 0 1 3 0 0 0 | 1 3 1 1 1 5/2 -1/2 0 0 | 1 3 1 1 5/2 -1/2 0 0 | 0 3 1 1 1 1/2 3/2 0 0 | 0 3 1 1 5/2 3/2 0 0 | 5 1 1 1 -3/2 3/2 0 0 | 1 3 1 1 1/2 -1/2 0 0 |
| 3  | 0 0 1 3 0                | 0 0 1 1 7 0 | -1 ± 2 3 1 1 | 0 -3/2 0 1/2 | 1 ± 2 1 3 1 1 | 0 3 2 0 3/2 0 1/2 | 1 1 3 1 1/2 -1/2 0 0 | 0 1 3 1 ± 2 0 1/2 0 |
| 4  | 1 1 0 1 1                | 1 1 0 5 1 | ± 1 ± 3 1 1 1 | 0 -3/2 0 1/2 | 3 ± 1 1 1 1 | 0 3 2 0 3/2 0 1/2 | 1 ± 2 1 3 1 1/2 -1/2 0 0 | 0 1 3 1 ± 2 0 1/2 0 |
| 5  | 0 1(3) 1 0 2(6) 1        | 1 3 1 1 1 | ± 1 ± 3 1 1 1 | ±3/4 ±1/4 ±3/4 1/4 | 5(5) 1 1 1 1 | ±3/4 ±3/4 ±3/4 ±3/4 | ±5/4 ±5/4 ±1/4 3/4 1/4 | ±5/4 ±3/4 ±1/4 ±3/4 |
| 6  | 1 2 0 0 3(5) 1           | 1 1 1 1 1 | ± 1 ± 4 1 1 1 | ±3/4 ±3/4 ±3/4 ±3/4 | 1 3 1 1 1 | ±3/4 ±3/4 ±3/4 ±3/4 | ±5/4 ±5/4 ±3/4 ±3/4 |
| 7  | 0 0 0 1 0 2(6) 0         | ± 1 ± 4 1 1 1 | ±3/4 ±3/4 ±3/4 ±3/4 | 1 3 1 1 1 | ±3/4 ±3/4 ±3/4 ±3/4 | ±5/4 ±5/4 ±3/4 ±3/4 |

where $a, b = 1, \ldots, 5, i, j = 1, 2, 3$. Four $(3_H + 1_H)$ generations of chiral matter fields from $(SU(5) \times SU(3))_I$ group forming $SO(10)$–multiplets $(1,3) + (5,3) + (10,3); (1,1) + (5,1) + (10,1)$ are contained in $b_2$ and $3b_2$ sectors. Applying $b_3$ GSO projection to the $3b_2$ sector yields the following massless states:

\[
\begin{align*}
    b_{\psi_{12}}^+ b_{\chi_{34}}^+ b_{\chi_{56}}^+ |\Omega\rangle_L & \otimes \left\{ \Psi_{3/4}^a, \psi_{1/4}^a, \psi_{1/4}^b, \psi_{1/4}^\epsilon, \psi_{1/4}^\eta, \psi_{1/4}^\iota, \psi_{1/4}^\kappa \right\} |\Omega\rangle_R, \\
    b_{\chi_{12}}^+ b_{\chi_{34}}^+ b_{\chi_{56}}^+ |\Omega\rangle_L & \otimes \left\{ \Psi_{3/4}^a, \psi_{1/4}^a, \psi_{1/4}^b, \psi_{1/4}^\epsilon, \psi_{1/4}^\eta, \psi_{1/4}^\iota, \psi_{1/4}^\kappa \right\} |\Omega\rangle_R.
\end{align*}
\]

with the space-time chirality $\gamma_{5\psi_{12}} = -1$ and $\gamma_{5\psi_{12}} = 1$, respectively. In these formulae the Ramond creation operators $b_{\psi_{12}}^+$ and $b_{\chi_{a,b}}^+$ of the zero modes are built of a pair of real fermions (as indicated by double indices): $\chi_{a,b}$, $(\alpha, \beta) = (1,2), (3,4), (5,6)$. Here, as in $(\ref{58})$ indices take values $a, b = 1, \ldots, 5$ and $i, j = 1, 2, 3$ respectively.
We stress that without using the $b_3$ projection we would get matter supermultiplets belonging to real representations only i.e. "mirror" particles would remain in the spectrum. The $b_6$ projection instead, eliminates all chiral matter superfields from $U(8)^{II}$ group. It is interesting, that without $b_6$-vector the Model 1 is fictitious U(1)-anomaly \[32\] fully free.

Since the matter fields form the chiral multiplets of $SO(10)$, it is possible to write down $U(1)_{Y_5}$–hypercharges of massless states. In order to construct the right electromagnetic charges for matter fields we must define the hypercharges operators for the observable $U(8)^I$ group as

$$Y_5 = \int_0^\pi d\sigma \sum_a \Psi^a \Psi^a, \quad Y_3 = \int_0^\pi d\sigma \sum_i \Psi^i \Psi^i$$

and analogously for the $U(8)^{II}$ group.

Then the orthogonal combinations

$$\tilde{Y}_5 = \frac{1}{4}(Y_5 + 5Y_3), \quad \tilde{Y}_3 = \frac{1}{4}(Y_3 - 3Y_5),$$

play the role of the hypercharge operators of $U(1)_{Y_5}$ and $U(1)_{Y_H}$ groups, respectively. In Table 3 we give the hypercharges $\tilde{Y}_5^I, \tilde{Y}_3^I, \tilde{Y}_5^{II}, \tilde{Y}_3^{II}$.

The full list of states in this model is given in Table 3. For fermion states only sectors with positive (left) chirality are written. Superpartners arise from sectors with $S = b_4$-component changed by 1. Chirality under hidden $SO(2)^{1,2,3} \times SO(6)^1$ is defined as $\pm_1$, $\pm_2$, $\pm_3$, $\pm_4$ respectively. Lower signs in item 5 and 6 correspond to sectors with components given in brackets.

In the next section we discuss the problem of rank eight GUST gauge symmetry breaking. The matter is that according to the results of section 4 the Higgs fields $(10_{1/2} + \overline{10}_{-1/2})$ do not appear.

### 5.3 $SU(5) \times U(1) \times SU(3) \times U(1)$ – Model 2.

Consider then another $[U(5) \times U(3)]^2$ model which after breaking gauge symmetry by Higgs mechanism leads to the spectrum similar to Model 1.

This model is defined by basis vectors given in Table 4 with the $Z_2^4 \times Z_6 \times Z_{12}$ group under addition.

GSO coefficients are given in Table 5.

The given model corresponds to the following chain of the gauge symmetry breaking:

$$E_8^2 \longrightarrow SO(16)^2 \longrightarrow U(8)^2 \longrightarrow [U(5) \times U(3)]^2.$$  

Here the breaking of $U(8)^2$–group to $[U(5) \times U(3)]^2$ is determined by basis vector $b_5$, and the breaking of N=2 SUSY $\longrightarrow$ N=1 SUSY is determined by basis vector $b_6$.

It is interesting to note that in the abscence of vector $b_5$ $U(8)^2$ gauge group is restored by sectors $4b_3, 8b_3, 2b_2 + c.c.$ and $4b_2 + c.c.$
Table 4: Basis of the boundary conditions for Model 2.

| Vectors | $\psi_{1,2}$ | $\chi_{1,6}$ | $y_{1,6}$ | $\omega_{1,6}$ | $\bar{\varphi}_{1,12}$ | $\Psi_{1,8}$ | $\Phi_{1,8}$ |
|---------|--------------|--------------|-----------|----------------|-----------------|-----------|-----------|
| $b_1$   | 11           | $1^6$        | $1^6$     | $1^6$          | $1^{12}$        | $1^8$     | $1^8$     |
| $b_2$   | 11           | $1^6$        | $0^6$     | $0^6$          | $0^{12}$        | $15^1$ $1^3$ | $0^8$     |
| $b_3$   | 11           | $1^22^12^2$  | $0^6$     | $0^21^22^2$    | $0^8$ $1^4$    | $12^{1/2}$ $1^6$ $3$ | $-1^{1/2}2^5$ $1^6$ $3$ |
| $b_4 = S$ | 11         | $1^2$ $0^4$  | $0^21^20^2$ | $0^4$ $1^2$ | $0^{12}$        | $0^8$     | $0^8$     |
| $b_5$   | 11           | $1^4$ $0^2$  | $0^4$     | $0^6$          | $1^8$ $0^4$    | $15^0$ $3$ | $0^5$ $1^3$ |
| $b_6$   | 11           | $0^21^20^2$  | $1^2$     | $0^4$ $1^2$    | $12^0$ $2^{16}0^2$ | $1^8$     | $0^8$     |

Table 5: The choice of the GSO basis $\gamma[b_i, b_j]$. Model 2. ($i$ numbers rows and $j$ – columns)

|   | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ |
|---|-------|-------|-------|-------|-------|-------|
| $b_1$ | 0     | 1     | $1/2$ | 0     | 0     | 0     |
| $b_2$ | 0     | $2/3$ | $-1/6$| 1     | 0     | 1     |
| $b_3$ | 0     | $1/3$ | $5/6$ | 1     | 0     | 0     |
| $b_4$ | 0     | 0     | 0     | 0     | 0     | 0     |
| $b_5$ | 0     | 1     | $-1/2$| 1     | 1     | 1     |
| $b_6$ | 0     | 1     | $1/2$ | 1     | 0     | 1     |

The full massless spectrum for the given model is given in Table 6. By analogy with Table 3 only fermion states with positive chirality are written and obviously vector supermultiplets are absent. Hypercharges are determined by formula:

$$Y_n = \sum_{k=1}^{n} (\alpha_k/2 + F_k).$$

The given model possesses the hidden gauge symmetry $SO(6)_1 \times SO(2)_{3,4}^3$. The corresponding chirality is given in column $SO_{hid}$. The sectors are divided by horizontal lines and without including the $b_5$–vector form $SU(8)$–multiplets.

For example, let us consider row No 2. In sectors $b_2$, $5b_2$ in addition to states $(1, 3)$ and $(5, 3)$ the $(10, 1)$–state appears, and in the sector $3b_2$ besides the $(10, 1)$–the states $(1, 1)$ and $(5, 1)$ survive too. All these states form $8 + 56$ representation of the $SU(8)^I$ group.

Analogically we can get the full structure of the theory according to the $U(8)^I \times U(8)^{III}$–group. (For correct restoration of the $SU(8)^{III}$–group we must invert 3 and 3 representations.)

In Model 2 matter fields appear both in $U(8)^I$ and $U(8)^{III}$ groups. This is the main difference between this model and Model 1. However, note that in the Model 2 simialrly
Table 6: The list of quantum numbers of the states. Model 2.

| No | $b_1, b_2, b_3, b_4, b_5, b_6$ | $SO_{hid}$ | $U(5)^I$ | $U(3)^I$ | $U(5)^{II}$ | $U(3)^{II}$ | $Y^I_5$ | $Y^I_3$ | $Y^I_5$ | $Y^I_3$ |
|----|--------------------------------|-----------|----------|----------|-------------|-------------|------|------|------|------|
| 1  | RNS                            | $6_1\ 2_2$| 1 1 1 1 1| 0 0 0 0 0 |             |             |      |      |      |      |
|    | $2_3\ 2_4$                      | 1 1 1 1 1| 0 0 0 0 0 |             |             |      |      |      |      |
|    | 0 0 4 1 0 0                    | 5 1 5 1 1| 1 0 -1 0 0|             |             |      |      |      |      |
|    | 0 0 8 1 0 0                    | 1 3 1 3 3| 0 -1 0 -1 |             |             |      |      |      |      |
| 2  | 0 1 0 0 0                      | 5 3 1 1 1| -3/2 -1/2 0 0 |             |             |      |      |      |      |
|    | 0 3 0 0 0                      | 1 3 1 1 1| 5/2 -1/2 0 0 |             |             |      |      |      |      |
|    | 0 3 6 0 0                      | 10 1 1 1 1| 1/2 3/2 0 0 |             |             |      |      |      |      |
| 3  | 0 1 1 0 0 0                    | 1 1 10 3 3| 0 0 1/2 1/2 |             |             |      |      |      |      |
|    | 0 3 6 0 0 0                    | 1 1 5 1 1| 0 0 -3/2 -3/2 |             |             |      |      |      |      |
|    | 1 1 1 1 1                      | 0 0 5/2 -3/2 |             |             |      |      |      |      |
| 4  | 0 2 3 0 0 0                    | -3 ±4 1 3 1 1| -5/4 -1/4 5/4 3/4 |             |             |      |      |      |      |
| 5  | 0 0 3 0 0 0                    | +3 ±4 1 1 5 1| -5/4 3/4 1/4 3/4 |             |             |      |      |      |      |
| 6  | 0 0 9 0 0 0                    | +3 ±4 1 1 5 1| 5/4 -3/4 -1/4 -3/4 |             |             |      |      |      |      |
| 7  | 0 4 9 0 0 0                    | -3 ±4 1 3 1 1| 5/4 1/4 -5/4 -3/4 |             |             |      |      |      |      |
| 8,9| 0 5 0 1 0 1                    | -1 ±3 1 3 1 1| 0 -1 0 0 |             |             |      |      |      |      |
|    | 0 3 0 1 0 1                    | 5 1 1 1 1| 1 0 0 0 |             |             |      |      |      |      |
|    | +1 +3 5 1 1 1                 | 0 0 0 0 |             |             |      |      |      |      |
|    | +1 -3 5 1 1 1                 | -1 0 0 0 |             |             |      |      |      |      |
|    | -1 +3 1 1 5 1                 | 0 0 1 0 |             |             |      |      |      |      |
|    | -1 -3 1 1 5 1                 | 0 0 -1 0 |             |             |      |      |      |      |
|    | 0 5 8 1 0 1                    | +1 ±3 1 1 1 3| 0 0 0 1 |             |             |      |      |      |      |
| 10 | 0 3 3 0 0 1                    | +1 ±4 1 1 1 1| -5/4 3/4 5/4 3/4 |             |             |      |      |      |      |
| 11 | 1 0 3 0 0 1                    | ±2 ±3 1 1 5 1| -1/4 3/4 -5/4 -3/4 |             |             |      |      |      |      |
|    | 1 2 1 1 0 1                    | ±2 -3 1 1 1 3| -5/4 3/4 -5/4 1/4 |             |             |      |      |      |      |
| 12 | 1 0 9 0 0 1                    | ±2 +3 1 1 5 1| 1/4 -3/4 5/4 3/4 |             |             |      |      |      |      |
|    | 1 4 9 0 0 1                    | ±2 +3 1 3 1 1| 5/4 1/4 5/4 3/4 |             |             |      |      |      |      |
| 13 | 0 0 0 1 1 1                    | ±2 ±3 1 1 1 1| 0 -3/2 0 3/2 |             |             |      |      |      |      |
|    | 0 2 0 1 1 1                    | ±2 -3 1 3 1 1| 0 1/2 0 3/2 |             |             |      |      |      |      |
|    | 0 2 8 1 1 1                    | ±2 -3 1 1 1 3| 0 -3/2 0 -1/2 |             |             |      |      |      |      |
|    | 0 4 8 1 1 1                    | ±2 +3 1 3 1 3| 0 1/2 0 -1/2 |             |             |      |      |      |      |
|    | 1 0 3 1 1 1                    | ±1 +3 1 1 1 1| 5/4 3/4 -5/4 3/4 |             |             |      |      |      |      |
|    | 1 0 9 1 1 1                    | ±1 +3 1 1 1 1| -5/4 3/4 5/4 -3/4 |             |             |      |      |      |      |
|    | 1 3 3 0 1 1                    | ±1 -3 1 1 1 1| -5/4 -3/4 -5/4 3/4 |             |             |      |      |      |      |
|    | 1 3 9 0 1 1                    | ±1 -3 1 1 1 1| 5/4 3/4 5/4 -3/4 |             |             |      |      |      |      |
to the Model 1 all gauge fields appear in RNS-sector only and 10 + \overline{10} representation (which can be the Higgs field for gauge symmetry breaking) is absent.

## 5.4 $SO(10) \times SU(4)$ – Model 3.

As an illustration we can consider the GUST construction involving $SO(10)$ as GUT gauge group. We consider the set consisting of seven vectors $B = b_1, b_2, b_3, b_4 \equiv S, b_5, b_6, b_7$ given in Table 7.

### Table 7: Basis of the boundary conditions for the Model 3.

| Vectors | $\psi_{1.2}$ | $\chi_{1...6}$ | $\eta_{1...6}$ | $\omega_{1...6}$ | $\varphi_{1...12}$ | $\Psi_{1...8}$ | $\Phi_{1...8}$ |
|---------|-------------|----------------|---------------|----------------|----------------|----------------|----------------|
| $b_1$   | 11          | 111111         | 111111        | 111111         | $1^{12}$        | $1^8$          | $1^8$          |
| $b_2$   | 11          | 111111         | 000000        | 000000         | $0^{12}$        | $1^{51/3}$     | $0^8$          |
| $b_3$   | 11          | 000000         | 111111        | 000000         | $0^{814}$       | $0^{513}$      | $0^{513}$      |
| $b_4 = S$| 11          | 110000         | 001100        | 000011         | $0^{12}$        | $0^8$          | $0^8$          |
| $b_5$   | 11          | 111111         | 000000        | 000000         | $0^{12}$        | $0^8$          | $1^{51/3}$     |
| $b_6$   | 11          | 001100         | 110000        | 000011         | $1^{302^160^2}$ | $1^8$          | $0^8$          |
| $b_7$   | 11          | 001100         | 100000        | 100011         | $2^{10^12^100^2}$ | $0^8$          | $1^8$          |

GSO projections are given in Table 8.

It is interesting to note that in this model the horizontal gauge symmetry $U(3)$ extends to $SU(4)$. Vector bosons which needed for this appear in sectors $2b_2\ (4b_2)$ and $2b_5\ (4b_5)$. For further breaking $SU(4)$ to $SU(3) \times U(1)$ we need an additional basis spin-vector.

Of course for getting a realistic model we must add some basis vectors which give additional GSO–projections.

### Table 8: The choice of the GSO basis $\gamma[b_i, b_j]$. Model 3. ($i$ numbers rows and $j$ – columns)

|         | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ |
|---------|-------|-------|-------|-------|-------|-------|-------|
| $b_1$   | 0     | 1     | 0     | 0     | 1     | 0     | 0     |
| $b_2$   | 0     | 2/3   | 1     | 1     | 1     | 1     | 0     |
| $b_3$   | 0     | 1     | 0     | 1     | 1     | 1     | 0     |
| $b_4$   | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $b_5$   | 0     | 1     | 1     | 1     | 2/3   | 0     | 0     |
| $b_6$   | 0     | 1     | 0     | 1     | 1     | 1     | 0     |
| $b_7$   | 0     | 1     | 1     | 1     | 0     | 1     | 1     |

The spectrum of the Model 3 is the next:
1. There is the $[U(1) \times SO(6)_{\text{Hid.}} \times [SO(10) \times SU(4)]^2$ gauge group, the $U(1)$ group is anomaly free;

2. The matter fields, $(16, 4; 1, 1)$, are from $3b_2$ and $5b_2$ sectors;

3. There are Higgs fields from RNS-sectors - $(\pm 1)_1(6)(1, 1; 1, 1)$, $(10, 1; 10, 1)$ and 2 total singlets;

4. There also are some Higgs fields from $mb_2 + nb_5$ sectors, where $m, n = 2 - 4$:

   - $(1, 6; 1, 6)$;

5. Another additional fields are $(-)_6(10, 1; 1, 1)$, $(+)_6(1, 1; 10, 1)$, $(-)_6(1, 1; 1, 6)$, $(+)_6(1, 6; 1, 1)$

   $(-1/2)_1(1, 4; 1, 4) + (\pm 1/2)_1(1, 4; 1, 4) + 2 \times (+)_6(1, 1; 1, 1) + (\pm 1) (\pm)_6(1, 1; 1, 1)$.

   The condition of generation chirality in this model results in the choice of Higgs fields as vector representations of $SO(10)$ ($16 + \bar{16}$ are absent). According to conclusion (49)

the only Higgs fields $(10, 1; 10, 1)$ of $(SO(10) \times SU(4))^{\times 2}$ appear in the model (from RNS-sector) which can be used for GUT gauge symmetry.
6 More explicit methods of model building. Self dual charge lattice.

In a previous models we had to guess how to obtain certain algebra representation and select boundary conditions vectors and GSO coefficients basing only on basis building rules. Below we will develop some methods that help to build models for more complicated cases such as $E_6 \times SU(3)$ and $SU(3) \times SU(3) \times SU(3) \times SU(3)$.

As it is known, square of a root represented by state in sector $\alpha$ is $\sum_i (\alpha_i/2 + F_i)$.

Consider then a mass condition. It reads (for right mass only)

$$M_R^2 = -1 + \frac{1}{8}(\alpha_R \cdot \alpha_R) + N_R = 0,$$

In general we can write $n_f$ as

$$n(f) = F^2 \frac{1 + F\alpha(f)}{2} = \frac{F^2}{2} + F\frac{\alpha}{2}$$

for any $F = 0, \pm 1$ ($F^3 = F$ for that values).

Now $M_R^2$ formulae reads

$$M_R^2 = -1 + \frac{1}{8} \sum_{i=1}^{22} (\alpha_i^2) + \sum_{i=1}^{22} \left( \frac{F_i^2}{2} + F_i \frac{\alpha_i}{2} \right)$$

Hence

$$2 = \sum_{i=1}^{22} \left( \frac{\alpha}{2} + F \right)^2$$

Clearly it is the square of algebra root and it equals to 2 for any massless state. Obviously for massive states normalization will differ from that.

6.1 Building GSO-projectors for a given algebra

As we follow certain breaking chain of $E_8$ then it is very naturally to take $E_8$ construction as a starting point. Note that root lattice of $E_8$ arises from two sectors: NS sector gives 120 of $SO(16)$ while sector with $1^8$ gives 128 of $SO(16)$. This corresponds to the following choice of simple roots

$$\pi_1 = -e_1 + e_2$$
$$\pi_2 = -e_2 + e_3$$
$$\pi_3 = -e_3 + e_4$$
$$\pi_4 = -e_4 + e_5$$
$$\pi_5 = -e_5 + e_6$$
$$\pi_6 = -e_6 + e_7$$
\[ \pi_7 = -e_7 + e_8 \]
\[ \pi_8 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8) \]

Basing on this choice of roots it is very clear how to build basis of simple roots for any subalgebra of \( E_8 \). One can just find out appropriate vectors \( \pi_i \) of the form as in \( E_8 \) with needed scalar products or build weight diagram and break it in a desirable fashion to find roots corresponding to certain representation in terms of \( E_8 \) roots.

After the basis of simple roots is written down one can build GSO-projectors in a following way.

GSO-projection is defined by operator \( (b_i \cdot F) \) acting on given state. The goal is to find those \( b_i \) that allow only states from algebra lattice to survive. Note that \( F = \gamma_i - \alpha_i/2 \) (\( \gamma_i \) — components of a root in basis of \( e_i \)), so value of GSO-projector for sector \( \alpha \) depends on \( \gamma_i \) only. So, if scalar products of all simple roots that arise from a given sector with vector \( b_i \) is equal mod 2 then they surely will survive GSO-projection. Taking several such vectors \( b_i \) one can eliminate all extra states that do not belong to a given algebra.

Suppose that simple roots of the algebra are in the form
\[ \pi_i = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8) \]
\[ \pi_j = (\pm e_k \pm e_m) \]

In this choice we have to find vectors \( b \) which gives 0 or 1 in a scalar product with all simple roots. Note that \( (b \cdot \pi_i) = (b \cdot \pi_j) \) mod2 for all \( i, j \) so \( c_i = (b \cdot \pi_i) \) either all equal 0 mod 2 or equal 1 mod 2. ( for \( \pi_j = (\pm e_k \pm e_m) \) it should be 0 mod 2 because they are arise from NS sector ) Value 0 or 1 is taken because if root \( \pi \in \) algebra lattice then \( -\pi \) is a root also. With such choice of simple roots and scalar products with \( b \) all states from sector like \( 1^8 \) will have the same projector value. Roots like \( \pm e_i \pm e_j \) rise from NS sector and are sum of roots like \( \pi_i = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8) \) and therefore have scalar products equal to 0 mod 2 as is needed for NS sector.

Now vectors \( b \) are obtained very simple. Consider
\[ c_i = (b \cdot \pi_i) = b_j A_{ji} \]

where \( A_{ji} = (\pi_i)_j \) — matrix of roots component in \( e_j \) basis. Hence \( b = A^{-1} \cdot c \) where either all \( c_j = 0 \) mod 2 or \( c_j = 1 \) mod 2. One has to try some combination of \( c_j \) to obtain appropriate set of \( b \). The next task is to combine those \( b_i \) that satisfy modular invariance rules and do not give extra states to the spectrum.

### 6.2 Breaking given algebra using GSO-projectors

It appears that this method of constructing GSO-projectors allows to break a given algebra down to its subalgebra.

Consider root system of a simple Lie algebra. It is well known that if \( \pi_1, \pi_2 \in \Delta \), where \( \Delta \) is a set of positive roots then \( (\pi_1 - \pi_2(\pi_1 \cdot \pi_2)) \in \Delta \) also. For simply laced
algebras it means that if $\pi_i, \pi \in \Delta$ and $(\pi_i \cdot \pi) = -1$ where $\pi_i$ is a simple root then $\pi + \pi_i$ is a root also. This rule is hold automatically in string construction: if a sector gives some simple roots then all roots of algebra and only them also exist (but part of them may be found in another sector). Because square of every root represented by a state is 2 then if $(\pi_i \cdot \pi) \neq -1$ then $(\pi + \pi_i)^2 \neq 2$. So one must construct GSO-projectors checking only simple roots. On the other hand if one cut out some of simple roots then algebra will be broken. For example if a vector $b$ has non-integer scalar product with simple root $\pi_1$ of $E_6$ then we will obtain algebra $SO(10) \times U(1)$ ( $(b \cdot \pi_1)$ even could be 1 if others products are equals 0 mod 2).

More complicated examples are $E_6 \times SU(3)$ and $SU(3) \times SU(3) \times SU(3) \times SU(3)$. For the former we must forbid the $\pi_2$ root but permit it to form $SU(3)$ algebra. Note that in $E_8$ root system there are two roots with $3\pi_2$. We will use them for $SU(3)$. So the product $(b \cdot \pi_2)$ must be 2/3 while others must be 0 mod 2.

We can also get GSO-projectors for all interesting subgroups of $E_8$ in such a way but so far choosing of constant for scalar products ($c_i$ in a previous subsection ) is rather experimental so it is more convenient to follow certain breaking chain.

Below we will give some results for $E_6 \times SU(3)$, $SU(3) \times SU(3)$, $SU(3) \times SU(3)$ and $SO(10) \times U(1) \times SU(3)$. We will give algebra basis and vectors that give GSO-projection needed for obtaining this algebra.

$E_6 \times SU(3)$. This case follow from $E_8$ using root basis from a previous subsection and choosing

$$c_i = (-2, -\frac{2}{3}, 0, 2, -2, 2, 0)$$

This gives GSO-projector of the form

$$b_1 = (1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

Basis of simple roots arises from sector with $1^8$ in right part and reads

\[
\begin{align*}
\pi_1 &= \frac{1}{2}(+e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \\
\pi_2 &= \frac{1}{2}(+e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8) \\
\pi_3 &= \frac{1}{2}(+e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8) \\
\pi_4 &= \frac{1}{2}(-e_1 + e_2 + e_3 - e_4 - e_5 - e_6 + e_7 - e_8) \\
\pi_5 &= \frac{1}{2}(+e_1 - e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8) \\
\pi_6 &= \frac{1}{2}(-e_1 + e_2 - e_3 + e_4 - e_5 + e_6 - e_7 + e_8) \\
\pi_7 &= \frac{1}{2}(+e_1 - e_2 + e_3 - e_4 - e_5 - e_6 + e_7 + e_8)
\end{align*}
\]
\[ \pi_8 = \frac{1}{2}(-e_1 + e_2 - e_3 - e_4 + e_5 - e_6 + e_7 + e_8) \]  

(62)

\[ SO(10) \times U(1) \times SU(3) \). This case follow from \( E_6 \times SU(3) \). In addition to \( b_1 \) we must find a vector that cut out \( \pi_3 \). Using

\[ c_i = (0, 0, 1, 0, 0, 0, 0) \]

and inverse matrix of \( E_6 \times SU(3) \) basis we get GSO-projector of the form

\[ b_2 = (0, 0, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}) \]

Basis of simple roots is the same as for \( E_6 \times SU(3) \) excluding \( \pi_3 \).

\( SU(3) \times SU(3) \times SU(3) \times SU(3) \). Using \( E_6 \times SU(3) \) basis inverse matrix with

\[ c_i = (1, -1, -1, \frac{1}{3}, 1, \frac{1}{3}, -1, -1) \]

We get GSO-projector of the form

\[ b_2 = (-\frac{1}{3}, \frac{1}{3}, 1, 1, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) \]

Easy to see that such a \( c_i \) cut out \( \pi_4 \) and \( \pi_6 \) roots but due to appropriate combination in \( E_6 \) root system two \( SU(3) \) groups will remain. Basis of simple roots is

\[ \pi_1 = \frac{1}{2}(+e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \]
\[ \pi_2 = \frac{1}{2}(+e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8) \]
\[ \pi_3 = \frac{1}{2}(+e_1 - e_2 + e_3 - e_4 - e_5 - e_6 + e_7 + e_8) \]
\[ \pi_4 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4 + e_5 + e_6 - e_7 - e_8) \]
\[ \pi_5 = \frac{1}{2}(+e_1 - e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8) \]
\[ \pi_6 = \frac{1}{2}(-e_1 + e_2 - e_3 - e_4 + e_5 - e_6 + e_7 + e_8) \]
\[ \pi_7 = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 - e_5 - e_6 + e_7 - e_8) \]
\[ \pi_8 = \frac{1}{2}(+e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 - e_8) \]  

(63)

Using all this methods we could construct a model described in the next section.
6.3 \( E_6 \times SU(3) \) three generations model – Model 4.

This model illustrates a branch of \( E_8 \) breaking \( E_8 \to E_6 \times SU(3) \) and is an interesting result on a way to obtain three generations with gauge horizontal symmetry. Basis of the boundary conditions (see Table 9) is rather simple but there are some subtle points. In [13] the possible left parts of basis vectors were worked out, see it for details. We just use the notation given in [13] (hat on left part means complex fermion, other fermions on the left sector are real, all of the right movers are complex) and an example of commuting set of vectors.

Table 9: Basis of the boundary conditions for the Model 4.

| Vectors | \( \psi_{1,2} \) | \( \chi_{1...9} \) | \( \omega_{1...9} \) | \( \varphi_{1...6} \) | \( \Psi_{1...8} \) | \( \Phi_{1...8} \) |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( b_1 \) | 11 | \( \widehat{\frac{1}{3}}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) |
| \( b_2 \) | 11 | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) |
| \( b_3 \) | 00 | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) | \( \frac{1}{3}, \frac{1}{3} \) |
| \( b_4 \) | 11 | \( \widehat{1}, \widehat{1}, \widehat{1} \) | \( \widehat{1}, \widehat{1}, \widehat{1} \) | \( \widehat{1}, \widehat{1}, \widehat{1} \) | \( \widehat{1}, \widehat{1}, \widehat{1} \) | \( \widehat{1}, \widehat{1}, \widehat{1} \) |

A construction of an \( E_6 \times SU(3) \) group caused us to use rational for left boundary conditions. It seems that it is the only way to obtain such a gauge group with appropriate matter contents.

The model has \( N = 2 \) SUSY. We can also construct model with \( N = 0 \) but according to [13] using vectors that can give rise to \( E_6 \times SU(3) \) (with realistic matter fields) one cannot obtain \( N = 1 \) SUSY.

Table 10: The choice of the GSO basis \( \gamma[b_i, b_j] \). Model 4. (\( i \) numbers rows and \( j \) – columns).

|     | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) |
|-----|---------|---------|---------|---------|
| \( b_1 \) | 1/3 | 1 | 1 | 1 |
| \( b_2 \) | 1 | 1 | 1 | 1 |
| \( b_3 \) | 1 | 1 | 0 | 1 |
| \( b_4 \) | 1/3 | 1 | 1 | 1 |

Let us give a brief review of the model contents. First notice that all superpartners of states in sector \( \alpha \) are found in sector \( \alpha + b_4 \) as in all previous models. Although the same sector may contain, say, matter fields and gauginos simultaneously.

The observable gauge group \((SU(3)_H^I \times E_6^I) \times (SU(3)_H^I \times E_6^I)\) and hidden group \( SU(6) \times U(1) \) are rising up from sectors NS, \( b_3 \) and \( 3b_2 + b_4 \). Matter fields in representations \((3, 27) + (\overline{3}, \overline{27})\) for each \( SU(3)_H \times E_6 \) group are found in sectors \( 3b_2, b_3 + b_4 \) and \( b_4 \). Also there are some interesting states in sectors \( b_2, b_2 + b_3, 2b_2 + b_3 + b_4, 2b_2 + b_4 \) and \( 5b_2, 5b_2 + b_3, 4b_2 + b_3 + b_4, 4b_2 + b_4 \) that form representations \((\overline{3}, 3)\) and \((3, \overline{3})\) of the
$SU(3)^I_H \times SU(3)^{II}_H$ group. This states are singlets under both $E_6$ groups but have nonzero $U(1)_{\text{hidden}}$ charge.

We suppose that the model permits further breaking of $E_6$ down to other grand unification groups, but problem with breaking supersymmetry $N = 2 \rightarrow N = 1$ is a great obstacle on this way.

7 Self-duality of the charge lattice and possible gauge groups.

Here we present some results based on the important feature of the charge lattice that is self-duality.

As was shown in [?] the charge lattice $Q$ is an odd self-dual lorentzian lattice shifted by a constant vector $S$ constituted by 32-components vectors with components

$$Q_i = \frac{\alpha_i}{2} + F_i,$$

where $\alpha_i$ is a boundary condition for $i$th fermion and $F_i$ is the corresponding fermion number in a particular string state. Vector $S$ takes care of the space-time spin-statistic and in the case of heterotic string is $(1, 0, 0, \ldots, 0)$.

As we will see below this feature apply serious restriction on the possible gauge group and matter spectrum of the GUST. In this section we will consider only models that permit bosonization which means that we write all fermions in terms of complex fermions and consequently can construct fermionic charge. Also we will restrict ourself considering only models that have only periodic or antiperiodic boundary conditions for left moving fermions (supersymmetric sector). Other possible forms of the left sector can be treated by the similar way but our case is more convenient in sense of building a $N = 1$ SUSY model.

Before analyzing particular GUST with appropriate gauge group we will consider some common features of a class of the lattices that we define above. Firstly notice that since we take all $\alpha_i$ in the left sector to be 0 or 1 then all of the scalar products of the left parts of the lattice vectors will be $\frac{n}{4}$, $n \in \mathbb{Z}$. So the scalar product of the right parts must has the same form in order to obtain the integer scalar product.

Secondly with accounting shifting vector $S$ the states that represent space-time gauge bosons will have all 0 in the left part ($\alpha_i = 0$, $F_1 = 1$, $F_j = 0$, $j > 1$). So the whole scalar product of such a vector is determined by the scalar product of the right part that must be integer in this case. On the other hand if we construct a right part that give an integer scalar product with every vector then we will have a gauge boson with this right part.

Thirdly let us consider the problem of chirality. Suppose we have a state with right part $Q_r$ giving integer or half-integer scalar product with all other vectors right parts (left part scalar products must be correspondingly integer or half-integer). Consider then a vector $Q'$ equals $Q$ with conjugated (i.e. multiplied by -1) right part. Obviously $Q'$ also
will have integer scalar products with all other vectors hence we finally have a space-time boson (scalar particle). The same way it is easy to prove that weight of chiral fermion must have a scalar product of the right part with some other vector’s right part equals to \( \frac{n}{2} + \frac{k}{4}, \ k \neq 0 \mod 4 \).

Concluding we have a classification of charge vectors in a sense of their space-time type. If we have a set of vectors that correspond to the content of a particular model (the vectors of the set must have integer scalar product with each other) then we can easily distinguish gauge bosons, chiral fermions and scalars by calculating scalar product of the right part with every vector in the set. The following list present classification.

- All of the right part scalar products are integer. Then we have gauge boson with this charges in the spectrum.

- All of the right part scalar products are integer or half-integer. Then we have scalar particle with this quantum numbers in the spectrum.

- Other vectors (that have right part scalar product with some of the vectors in a set in a form \( \frac{n}{2} + \frac{k}{4}, \ k \neq 0 \mod 4 \)) represent chiral fermions.

Finally notice that the scalar products of the right part is the scalar products of the weight vectors of particular gauge group representation. The structure of a representation is well known and for particular gauge group we can determine the representation that give appropriate model spectrum and necessary scalar products.

As a first simple example that illustrates all above discussion of this section we consider a model with \( E_6 \times SU(3) \) gauge group. Notice that both \( E_6 \) and \( SU(3) \) groups have only representations with scalar products of the weight vectors of the form \( \frac{n}{3} \). If we wish to obtain a model which has representation for matter then we have to include representation \((27, 3)\) in the spectrum so that this states are space-time chiral fermions. But to make this representation to be the chiral fermions one has to include another weight vector that give scalar product of the right part equal to \( \frac{k}{4}, \ k \neq 0 \mod 2 \). Since there is no such weight vector among weight vector of \( E_6 \) and \( SU(3) \) representations then it is impossible to build a model with \( E_6 \times SU(3) \) gauge group and chiral fermions in the representations appropriate for the GUST. However if one take spin structure vector with \( \frac{2}{3} \) in the left part then it will be possible to build a model but the model will have \( N = 2 \) space-time supersymmetry which means that there are no chiral fermions.

Now we will present a more complicated example, namely the \( SO(10) \times U(3) \) gauge group. We demand that the model spectrum includes space-time chiral fermions in the representation \((16, 3)\) of the \( SO(10) \times U(3) \) gauge group. Note that \( U(1) \) charge of this representation is defined up to the sign by the massless condition.

Thus we have a weight vector that we have to include in the lattice in addition to the root lattice of \( SO(10) \times U(3) \). Then we can find all the weights (actually the components of the weights that have nonvanishing scalar products with root vector and weight of matter representation and that correspond to the \( SO(10) \times U(3) \) representation) that give appropriate scalar product with weight of matter representation.
Actually one can obtain a formula that gives $U(1)$ hypercharge for arbitrary representation of $SO(10) \times U(3)$ so that scalar product of this weight with weight of matter representation has the form $\frac{n}{4}$. But it appears that representation $(1,3)$ has integer scalar product with all other vectors that are allowed by weight of $(16,3)$. It means that the representation $(1,3)$ will be a space-time gauge bosons that extend the initial $SO(10) \times U(3)$ gauge group to the $SO(10) \times SU(4)$. Now we can see why all attempts to build a model with $SO(10) \times U(3)$ gauge group and chiral matter in $(16,3)$ representation failed. All models obviously must have $SO(10) \times SU(4)$ gauge group with corresponding matter representations.
8 GUST Spectrum (Model 1)

8.1 Gauge Symmetry Breaking

Let us consider Model 1 in detail. In Model 1 there exists a possibility to break the GUST group \((U(5) \times U(3))^I \times (U(5) \times U(3))^H\) down to the symmetric group by the ordinary Higgs mechanism \([38]\):

\[ G^I \times G^H \rightarrow G_{symm} \rightarrow \ldots \]  

(64)

To achieve such breaking one can use nonzero vacuum expectation values of the tensor Higgs fields (see Table 3, row No 1), contained in the \(2b_2 + 2(6) + 2b_5( + S)\) sectors which transform under the \((SU(5) \times U(1) \times SU(3) \times U(1))^{symm}\) group in the following way:

\[
\begin{align*}
(5, 1; 5, 1)(-1, 0; -1, 0) & \rightarrow (24, 1)(0, 0) + (1, 1)(0, 0), \\
(1, 3; 1, 3)(0, 1; 0, 1) & \rightarrow (1, 8)(0, 0) + (1, 1)(0, 0),
\end{align*}
\]  

(65)

\[
\begin{align*}
(5, 1; 5, 1)(-1, 0, 0, 1) & \rightarrow (5, 3)(1, 1), \\
(1, 3; 1, 3)(0, 1; -1, 0) & \rightarrow (5, 3)(-1, -1).
\end{align*}
\]  

(66)

The diagonal vacuum expectation values for Higgs fields \([33]\) break the GUST group \((U(5) \times U(3))^I \times (U(5) \times U(3))^H\) down to the ”skew”-symmetric group with the generators \(\Delta_{symm}\) of the form:

\[
\Delta_{symm}(t) = -t^* \times 1 + 1 \times t,
\]  

(67)

The corresponding hypercharge of the symmetric group reads:

\[
\bar{Y} = \tilde{Y}^H - \tilde{Y}^I.
\]  

(68)

Similarly, for the electromagnetic charge we get:

\[
Q_{em} = \bar{Q}^H - Q^I = (T_5^H - T_5^I) + \frac{2}{5}(\tilde{Y}_5^H - \tilde{Y}_5^I) = T_5 + \frac{2}{5}Y_5,
\]  

(69)

where \(T_5 = diag(\frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{2}{3}, -\frac{2}{3})\). Note, that this charge quantization does not lead to exotic states with fractional electromagnetic charges (e.g. \(Q_{em} = \pm 1/2, \pm 1/6\)).

Thus, in breaking scheme \([37]\) it is possible to avoid colour singlet states with fractional electromagnetic charges, to achieve desired GUT breaking and moreover to get the usual value for the weak mixing angle at the unification scale (see \([107]\)).

Adjoint representations which appear on the rhs of \((65)\) can be used for further breaking of the symmetric group. This can lead to the final physical symmetry

\[(SU(3^c) \times SU(2_{EW}) \times U(1_Y) \times U(1)') \times (SU(3)_H \times U(1)_H)\]  

(70)

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with low-energy gauge symmetry of the quark – lepton generations with an additional $U(1)^\prime$-factor.

Note, that when we use the same Higgs fields as in (65), there exists also another interesting way of breaking the $G^I \times G^{II}$ gauge symmetry:

$$G^I \times G^{II} \rightarrow SU(3) \times SU(2)_{EW} \times SU(2)_{EW}^H \times U(1_{\bar{Y}}) \times SU(3)_H^I \times SU(3)_H^{II} \times U(1_{Y_H}) \rightarrow ....$$ (71)

In turn, the Higgs fields $\hat{h}_{(1,N)}$ from the NS sector

$$(\bar{5}, \bar{3})_{(-1,-1)} + (\bar{5}, \bar{3})_{(1,1)}$$ (72)

are obtained from N=2 SUSY vector representation $6\bar{3}$ of $SU(8)^I$ (or $SU(8)^{II}$) by applying the $b_5$ GSO projection (see Fig. 2 and Appendix B,C). These Higgs fields (and fields (66)) can be used for constructing chiral fermion (see Table 3, row No 2) mass matrices.

The $b$ spin boundary conditions (Tab.3) generate chiral matter and Higgs fields with the GUST gauge symmetry $G_{\text{comp}} \times (G^I \times G^{II})_{\text{obs}}$ (where $G_{\text{comp}} = U(1)^{\prime} \times SO(6)$ and $G^{I,II}$ have been already defined). The chiral matter spectrum, which we denote as $\hat{\Psi}_{(1,N)}$, with ($\Gamma = 1 , \bar{5} , \bar{10} ; N = 3 , \bar{1}$), consists of $N_y = 3_H + 1_H$ families. See Table 3, row No 2 for the $((SU(5) \times U(1)) \times (SU(3) \times U(1))_H)^{\text{symm}}$ quantum numbers.

The $SU(3)_H$ anomalies of the matter fields (row No 2) are naturally canceled by the chiral "horizontal" superfields forming two sets: $\hat{\Psi}^H_{(1,N;1,N)}$ and $\hat{\Phi}^H_{(1,N;1,N)}$, $\Gamma = 1 , \ N = 1 , \bar{3}$, (with both $SO(2)$ chiralities, see Table 3, row No 3, 4 respectively).

The horizontal fields (No 3, 4) cancel all $SU(3)^I$ anomalies introduced by the chiral matter spectrum (No 2) of the $(U(5) \times U(3))^I$ group (due to $b_6$ GSO projection the chiral fields of the $(U(5) \times U(3))^{II}$ group disappear from the final string spectrum). Performing the decomposition of fields (No 3) under $(SU(5) \times SU(3))^{\text{symm}}$ we get (among other) three "horizontal" fields $\hat{\Psi}^H$:

$$2 \times (\bar{1}, \bar{3})_{(0,-1)} , \ (1, \bar{1})_{(0,-3)} , \ (1, \bar{6})_{(0,1)} \ ,$$ (73)

coming from $\hat{\Psi}^H_{(\bar{1}, \bar{3}; 1,1)}$, (and $\hat{\Phi}^H_{(\bar{1}, \bar{3}; 1,1)}$, $\hat{\Phi}^H_{(1, \bar{1}; 1,1)}$, and $\hat{\Phi}^H_{(1, \bar{1}; 1,1)}$ respectively which make the low energy spectrum of the resulting model (74) $SU(3)_H^{\text{symm}}$ anomaly free. The other fields $\hat{\Phi}^H$ arising from rows No 4, Table 3 form anomaly-free representations of $(SU(3)_H \times U(1)_H)^{\text{symm}}$:

$$2 \times (1, \bar{1})_{(0,0)} , \ (1, \bar{3})_{(0,2)} + (\bar{1}, \bar{3})_{(0,-2)} , \ (1, \bar{8})_{(0,0)} \ .$$ (74)

The superfields $\hat{\phi}_{(1,N)} + \text{h.c.}$, where ($\Gamma = 1 , \bar{5} ; N = 1 , \bar{3}$), from the Table 3, row No 5 forming representations of $(U(5) \times U(3))^{II}$ have either $Q^I$ or $Q^{II}$ exotic fractional charges. Because of the strong $G_{\text{comp}}$ gauge forces these fields may develop the double scalar condensate $\langle \phi \bar{\phi} \rangle$, which can also serve for $U(5) \times U(5)$ gauge symmetry breaking. For example, the composite condensate $\langle \hat{\phi}^{(1,1,1)}_{(5,1,1,1)} \hat{\phi}^{(1,1,1)}_{(1,1,1,1)} \rangle$ can break the $U(5) \times U(5)$ gauge symmetry down to the symmetric diagonal subgroup with generators of the form

$$\Delta_{\text{symm}}(t) = t \times 1 + 1 \times t ,$$ (75)
so for the electromagnetic charges we would have the form

\[ Q_{\text{em}} = Q^{II} + Q^{I}. \]  

(76)

leading again to no exotic, fractionally charged states in the low-energy string spectrum.

The superfields which transform nontrivially under the compactified group \( G^{\text{comp}} = SO(6) \times SO(2)^{x3} \), (denoted as \( \hat{\sigma} \)), and which are singlets of \( (SU(5) \times SU(3)) \times (SU(5) \times SU(3)) \), arise in three sectors, see Table 3, row No 6. The superfields \( \hat{\sigma} \) form the spinor representations \( \bar{4} + \bar{4} \) of \( SO(6) \) and they are also spinors of one of the \( SO(2) \) groups. With respect to the diagonal \( G^{\text{symmn}} \) group with generators given by \( \mathbf{57} \) or \( \mathbf{83} \), some \( \hat{\sigma} \)-fields are of zero hypercharges and can, therefore, be used for breaking the \( SO(6) \times SO(2)^{x3} \) group.

Note, that for the fields \( \hat{\phi} \) and for the fields \( \hat{\sigma} \) any other electromagnetic charge quantization different from \( \mathbf{69} \) or \( \mathbf{76} \) would lead to "quarks" and "leptons" with the exotic fractional charges, for example, for the \( 5^1 \) and \( 1^1 \) multiplets according to the values of hypercharges (see Table 3, row No 6) the generator \( Q^{II} \) (or \( Q^{I} \)) has the eigenvalues \((\pm 1/6, \pm 1/6, \pm 1/6, \pm 1/2, \mp 1/2) \) or \((\pm 1/2) \), respectively.

Scheme of the breaking of the gauge group to the symmetric subgroup, which is similar to the scheme of Model 1, works for Model 2 too. In this case vector-like multiplets \( (\mathbf{5}, \mathbf{1}; \bar{\mathbf{5}}, \mathbf{1}) \) from RNS-sector and \( (\mathbf{1}, \mathbf{3}; \mathbf{1}, \mathbf{3}) \) from \( 4b_3 \) \( (8b_3) \) play the role of Higgs fields. Then generators of the symmetric subgroup and electromagnetic charges of particles are determined by formulas:

\[ \Delta^{(5)}_{\text{sym}} = t^{(5)} \times 1 \oplus 1 \times t^{(5)} \]
\[ \Delta^{(3)}_{\text{sym}} = (-t^{(3)}) \times 1 \oplus 1 \times t^{(3)} \]

\[ Q_{\text{em}} = t^{(5)}_5 - 2/5 Y^5, \quad \text{where } t^{(5)}_5 = (1/15, 1/15, 1/15, 2/5, -3/5) \]

(77)

After this symmetry breaking matter fields (see Table 3 rows No 2, 3) as usual for flip models take place in representations of the \( U(5) \)-group and form four generations \( (1^1 + 5^1 + 10^1; \bar{3}^1 + 1^1)_{\text{sym}} \). And Higgs fields form adjoint representation of the symmetric group, similar to Model 1, which is necessary for breaking of the gauge group to the Standard group. Besides, due to quantization of the electromagnetic charge according to the formula \( (77) \) states with exotic charges in low-energy spectrum also do not appear in this model. In this model \( U(1) \)-group in hidden sector has anomalies which is broken by Dine-Seiberg-Witten mechanism \( \mathbf{11} \). The corresponding D- term could break supersymmetry at very large scale however it is possible to show that there exist D-flat directions with respect to some non-anomalous gauge symmetries, canceling the anomalous D- term and restoring supersymmetry. As result the corresponding VEVs imply that the final symmetry will be less than origine, which includes the

\[ (SU(5) \times U(1))^{\text{sym}} \times SU(3_H) \times U(1_H)^{\text{sym}} \]

observable gauge symmetry. Let us note that the models 5 and 3 do not contain the anomalous \( U(1) \)-groups.
8.2 On the problem of states with exotic fractional charges.

Almost all experimental data points to the fact that all particles we can observe have only integer electromagnetic charge. Quarks are assumed to have charges $\frac{1}{3}$ and $\frac{2}{3}$ but there are no indications that there are particles with charges other than that.

Unfortunately many string models include states with fractional charges (e.g. $\frac{1}{2}$, $\frac{1}{6}$). We consider electromagnetic charge $q$ of a particular state to be exotic if $q \not\in \frac{1}{3} \mathbb{Z}$. Our models are free from these states due to symmetric construction of the electromagnetic charge and interesting features of GSO projection. This statement holds for all states in the model (not only massless).

1. Remind that we define electromagnetic charge as follows

$$Q = T_5^{II} - T_5^{I} + \frac{2}{5}(\tilde{Y}_5^{II} - \tilde{Y}_5^{I}) = T_5^{II} - T_5^{I} + \frac{1}{10}(Y_5^{II} - Y_5^{I}) + \frac{1}{2}(Y_3^{II} - Y_3^{I}),$$

$$T_5 = \{\frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{2}{3}, \frac{3}{3}\}$$

Rewrite $T_5$ in a following way

$$T_5 = \{-1, -1, -1, -\frac{2}{3}, -\frac{5}{3}\} + \frac{16}{15}Y_5 = t_5 + \frac{16}{15}Y_5.$$

It is easy to see that eigenvalues of $t_5$ on all the states will be proportional to $\frac{1}{3}$, so $t_5$ does not contribute to fractional charge. Now we have

$$Q = Q' + t_5^{II} - t_5^{I}, \text{ where } Q' = \frac{7}{6}(Y_5^{II} - Y_5^{I}) + \frac{1}{2}(Y_3^{II} - Y_3^{I}).$$

Now note that electromagnetic charge is defined by scalar product of vector $Q$ and weight $\Lambda$ of a state, $\Lambda_i = \alpha_i^2 + F_i$. Otherwise GSO projection is defined by scalar product of a weight vector and basis vector. So we can express the charge vector $Q$ via basis vectors. The difference will be only in left part of scalar product and in the hidden part of basis vectors. But as we will see this additional contributions do not make fractional charge.

Denote $\tilde{b}_i$ is the part of basis vector that forms observable group (for the Model 1 it is the last 16 components). Now we can rewrite $Q'$ as

$$Q' = -4\tilde{b}_2 + \tilde{b}_1 - \frac{2}{3}b_5$$

The GSO projection reads

$$(b_i \cdot \Lambda) = \tilde{\delta}_\alpha + c \begin{bmatrix} b_i \\ \alpha \end{bmatrix}$$

where we take a logarithm of a usual expression and make corresponding redefinitions.

Now it is obvious that all of the $Q$ eigenvalues will $\in \frac{1}{3}\mathbb{Z}$. Indeed if we now express scalar product $(Q' \cdot \Lambda)$ via GSO coefficients and remaining parts of basis vectors we will
see that most of them contribute integers (e.g. \(c\frac{[b_1]}{b_i}, 4c\frac{[b_2]}{b_i}, (b_1^L \cdot \Lambda), 4(b_2^L \cdot \Lambda), \text{etc.}\)) while the others contribute \(\frac{1}{3}\) or \(\frac{2}{3}\) (e.g. \(-\frac{2}{3}(b_5^L \cdot \Lambda), -\frac{2}{3}(b_5^{bid} \cdot \Lambda), \text{etc.}\)).

We see that states with fractional charges are forbidden by GSO projections on all mass levels. Using exactly the same method one can prove analogous statement for Model 2.

### 8.3 Superpotential. Vertex operators. Nonrenormalizable terms.

The ability of making a correct description of the fermion masses and mixings will, of course, constitute the decisive criterion for selection of a model of this kind. Therefore, within our approach one has to

1. study the possible nature of the \(G_H\) horizontal gauge symmetry (\(N_g = 3_H\) or \(3_H + 1_H\)),

2. investigate the possible cases for \(G_H\)-quantum numbers for quarks (anti-quarks) and leptons (anti-leptons), i.e. whether one can obtain vector-like or axial-like structure (or even chiral \(G_{HL} \times G_{HR}\) structure) for the horizontal interactions.

3. find the structure of the sector of the matter fields which are needed for the \(SU(3)_H\) anomaly cancelation (chiral neutral "horizontal" or "mirror" fermions),

4. write out all possible renormalizable and relevant non-renormalizable contributions to the superpotential \(W\) and their consequences for fermion mass matrices.

All these questions are currently under investigation. Here we restrict ourselves to some general remarks only.

With the chiral matter and "horizontal" Higgs fields available in Model 1 constructed in this paper, the possible form of the renormalizable (trilinear) part of the superpotential responsible for fermion mass matrices is well restricted by the gauge symmetry:

\[
W_1 = g\sqrt{2}\left[\hat{\Psi}_{(1,3)}\hat{\Psi}_{(5,1)}\hat{h}_{(5,3)} + \hat{\Psi}_{(1,1)}\hat{\Psi}_{(5,3)}\hat{h}_{(5,3)} + \hat{\Psi}_{(10,3)}\hat{\Psi}_{(5,3)}\hat{h}_{(5,3)} + \hat{\Psi}_{(10,3)}\hat{\Psi}_{(10,1)}\hat{h}_{(5,3)}\right]
\tag{78}
\]

Another strong constraint, which we used for construction superpotential, comes from the interesting observation that a modular invariant, \(N=1\) space-time supersymmetric theory also extends to a global \(N=2\) world sheet superconformal symmetry \([40]\), which contains now two distinct fermionic components to the energy- momentum tensor, \(T^+_F\) and \(T^-_F\) \([24]\), and there is also the \(U_{J(1)}\) current \(J\). This conserved \(U(1)\) current of this \(N=2\) superalgebra may play a key role in constructing of realistic phenomenology. So, all vertex operators have the definite \(U(1)\) charge. For \(J(z)\) we have

\[
J(z) = i\partial_z (S_{12} + S_{34} + S_{56}),
\tag{79}
\]
where \( S_{ij} \) are the bosonized components of superspin generator \( T_F(z) \) [24]. Let's consider on an example in Models 1,5 the three point fermionic- fermionic - bosonic function for the case

\[
\Psi^1_{(1,3)} \Psi^2_{(5,1)} h^3_{(5,3)},
\]

where the fields 1 and 2 are from Ramond-sector and 3 is from NS -sector. In canonical picture for the fermionic (bosonic) vertex operator the \( U_J(1) \) charge equal to \(-1/2(+1)\).

In boson sector in \( \Psi^1_{(1,3)}, \Psi^2_{(5,1)} \) the left Ramond fermions are with the numbers 3, 4, 6, 10. The field \( h^3_{(5,3)} \) from NS-sector has in boson sector the exitation by world sheet left fermion No 2. The nonvanishing \( U_J(1) \) of these three vertex operators are : \( \beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 1/2 \) and \( \alpha_3 = 1 \). So for the corresponding vertex operators one can obtain :

\[
V^f_{1(-1/2)} = e^{-c/2} S_\alpha e^{-i/2} H_2 \sum_3 e \sum_4 e^{j/2} K_1 X_1 \tilde{G}_1 e^{i/2} K_1 \tilde{X}_1 \tag{80}
\]

\[
V^f_{2(-1/2)} = e^{-c/2} S_\alpha e^{-i/2} H_2 \sum_3 e \sum_4 e^{j/2} K_2 X_2 \tilde{G}_2 e^{i/2} K_2 \tilde{X}_2 \tag{81}
\]

\[
V^b_{3(-1)} = e^{-c} e^{iH_2} e^{j/2} K_3 X_3 \tilde{G}_3 e^{i/2} K_3 \tilde{X}_3 \tag{82}
\]

We see that the correlator of these vertex operators is not equal to zero (for details see Kalara et al in [24]). Following to the paper [24] one can easily get that the corresponding coefficient including to the superpotential is equal to \( g\sqrt{2} \).

From the above form of the Yukawa couplings follows that two (chiral) generations have to be very light (comparing to \( M_W \) scale). The construction of realistic quarks and leptons mass matrices depends, of course, on the nature of the horizontal interactions. To construct the realistic fermion mass matrices one has to also use Higgs fields (\( \tilde{H}_3, \tilde{H}_4 \) and (Table 3, No 5) and also to take into account all relevant non-renormalizable contributions [24, 25, 24].

Higgs fields (\( \tilde{H}_3, \tilde{H}_4 \) can be used for constructing Yukawa couplings of the horizontal superfields (No 3 and 4). The superpotential, \( W_2 \), consists of the next \( R^2 NS \)-terms:

\[
W_2 = g \sqrt{2} \left[ \hat{\Phi}^H_{(1,1,1)} \hat{\Phi}^H_{(3,1,1)} \hat{\Phi}^H_{(3,1,3)} + \hat{\Phi}^H_{(1,1,1)} \hat{\Phi}^H_{(3,1,1)} \hat{\Phi}^H_{(3,1,3)} + \right. \\
+ \left. \hat{\Psi}^H_{(3,1,3)} \hat{\Psi}^H_{(1,1,1)} \hat{\Psi}^H_{(3,1,1)} + \hat{\sigma}_{(-1/4)} \hat{\sigma}_{(+1/4)} \hat{\Psi}_{(1,1,1)} \right] \tag{83}
\]

From (83) it follows that some of the horizontal fields in (74) (No 3, 4) remain massless at the tree-level. This is a remarkable prediction: fields (74) interact with the ordinary chiral matter fields only through the \( U(1)_H \) and \( SU(3)_H \) gauge boson and therefore are very interesting in the context of the experimental searches for the new gauge bosons.

The Higgs fields (see table 3, 13) could give the following \( (NS)^3 \) contributions to the renormalizable superpotential:

\[
W_3 = \sqrt{2} g \left( \hat{\Phi}_{(5,1,1,3)} \hat{\Phi}_{(5,1,5,1)} \hat{\Phi}_{(1,1,5,3)} + \hat{\Phi}_{(5,1,1,3)} \hat{\Phi}_{(1,3,1,3)} \hat{\Phi}_{(5,3,1,1)} \right)
\]

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So, \( W_1 + W_2 + W_3 \) is the most general renormalizable superpotential which includes all non-zero three-point (F-type) expectation values of the vertex operators for corresponding 2-dimensional conformal model.

Now we consider the non-renormalizable four-point vertex operator of the form \((R)^4\), which is the only nonvanishing type of \(N=4\) point operators (see the definitions in paper Kalara et al [24]). We take that \( \frac{1}{\sqrt{2}}(\chi_n \pm i\chi_{n+1}) = e^{\pm iH_k} \), where \( k = 1 + (n + 1)/2 \); \( H_k \) and \( H_{k'} \) similarly for \( y_n \) and \( \omega_n \). Following to [24], the multipliers for \( \chi \) have been written explicitly and for \( y(\omega) \) we have also: \( \sum_{k'}^\pm = e^{\pm \frac{i}{2}H_{k'}}. \)

We need to calculate the correlator: \( \langle V_{1(-1/2)}^b V_{2(-1/2)}^f V_{3(-1)}^b V_{4(0)}^h \rangle \). The 4-th vertex have to be written in noncanonical form (in picture 0). So for R-case we need change vertex \( V_b^b \) from picture –1 to picture 0. According to [24] the formula of changing pictures from \( q \) to \( q + 1 \) is:

\[
V_{q+1}(z) = \lim_{w \to z} e^c(w) T_F(w) V_q(z)
\]

The contribution is going only from \( T_{F^{-1}} \) and for complex case one can obtain:

\[
T_{F^{-1}} = \frac{i}{2\sqrt{2}} \sum_k e^{-iH_k} \left[ (1-i)e^{iH_k'} e^{iH_{k'}} + (1+i)e^{iH_k'} e^{-iH_{k'}} + (1+i)e^{-iH_k'} e^{iH_{k'}} + (1-i)e^{-iH_k'} e^{-iH_{k'}} \right]
\]

Making use (85):

\[
V_{R(-1)}^b = e^{-c} e^{i/2} H_k e^{i/2} H_l \sum_{k'}^{\pm} \sum_{m'}^{\pm} e^{i/2} K X \bar{G} e^{i/2} K X
\]

one can get:

\[
V_{R(0)}^b = \frac{i}{2\sqrt{2}} \left\{ e^{-i/2} H_k e^{i/2} H_l \left( \sum_{k'}^{\pm} \left[ (1 \pm i) e^{iH_{k'}} + (1 \mp i) e^{-iH_{k'}} \right] \right) \sum_{m'}^{\pm} \right\} \times e^{i/2} K X \bar{G} e^{i/2} K X
\]

Another vertex correlator constructions of \( R^4 \) one can see in [24]. Now we can lead to the result of our study of the \( R^4 \) term in model 1:

\[
(1, 3, 1, 1)_{Mat.} \sim e^{-i/2} H_2 \sum_3^{\pm} \sum_4^{\pm} \quad (f. - 1/2)
\]
\[
(+1, -3)(1, 3, 1, 1)_{No4} \sim e^{-i/2} H_6 \sum_5^{\pm} \sum_7^{\pm} \quad (f. - 1/2)
\]
\[
(-1, -4)_{No6} \sim e^{i/2} H_2 e^{i/2} H_10 \sum_4^{\pm} \sum_5^{\pm} \quad (b. - 1)
\]
\[
(+3, +4)_{No6} \sim e^{-i/2} H_6 e^{i/2} H_10 \sum_3^{\pm} \sum_7^{\pm} \left[ (1 - i) e^{iH_9} + (1 + i) e^{-iH_9} \right] + e^{i/2} H_6 e^{-i/2} H_10 \sum_3^{\pm} \sum_7^{\pm} \left[ (1 + i) e^{iH_4} + (1 - i) e^{-iH_4} \right] \quad (b.0)
\]

All others terms of \( R^4 \) contain the correlators like, which are equal to zero: \( \langle e^{-i/2} H_k e^{-i/2} H_k \rangle \equiv 0 \). One can see that in model 1 there is only one nonvanishing term:
\[ W_4 = \hat{\Psi}_{(1,3,1,1)} \hat{\Phi}^H_{(1,3,1,1)} \hat{\sigma}_1 (-,-)\hat{\sigma}_3 (+,+). \]  

(88)

Calculable coefficient from \( W_4 \) have been omitted.

For consideration of the five-point non-renormalizable function we may take into account only the following candidates of the \((NS) \times (R)^4\) vertex configurations:

\[
W_5 = \hat{\Phi}_{(1,1,1,3)} \hat{\Psi}_{(10,3,1,1)} \hat{\Psi}_{(10,1,1,1)} \times \left[ \hat{\Psi}^H_{(1,1,1,3)} \hat{\Psi}^H_{(1,3,1,1)} + \hat{\Phi}^H_{(1,3,1,1)} \hat{\Phi}^H_{(1,1,1,1)} \right] 
\]  

(89)

The left \( N=2 \) superconformal invariance demands that these terms are equal to zero.

Finally, we remark that the Higgs sector of our GUST allows conservation of the \( G_H \) gauge family symmetry down to the low energies \(( \sim \mathcal{O}(1\,\text{TeV}) \) \). Thus in this energy region we can expect new interesting physics (new gauge bosons, new chiral matter fermions, superweak-like CP–violation in \( K^- B^- D^- \)-meson decays with \( \delta_{KM} < 10^{-4} \)).

Table 11: Basis of the boundary conditions for all world-sheet fermions. Model 5.

| Vectors | \( \psi_{1,2} \) | \( \chi_{1..6} \) | \( y_{1..6} \) | \( \omega_{1..6} \) | \( \tilde{\varphi}_{1..12} \) | \( \Psi_{1..8} \) | \( \Phi_{1..8} \) |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( b_1 \) | 11 | 111111 | 111111 | 111111 | 1^{12} | \( \tilde{1}^8 \) | \( \tilde{1}^8 \) |
| \( b_2 = S \) | 11 | 110000 | 001100 | 000011 | 0^{12} | \( \tilde{0}^8 \) | \( \tilde{0}^8 \) |
| \( b_3 \) | 00 | 001111 | 000011 | 001100 | 1^80^4 | \( \tilde{1}^8 \) | \( \tilde{0}^8 \) |
| \( b_4 \) | 11 | 111111 | 000000 | 000000 | 0^{12} | \( \tilde{1}/2^8 \) | \( \tilde{0}^8 \) |
| \( b_5 \) | 11 | 001100 | 000000 | 110011 | \( 1^60^21^20^2 \) | \( 1/4^53/4^3 \) | \( -1/4^53/4^3 \) |
| \( b_6 \) | 00 | 001100 | 100000 | 101100 | \( 1^40^41^0101 \) | \( \tilde{1}^8 \) | \( \tilde{1}^8 \) |

Table 12: The choice of the GSO basis \( \gamma[b_i, b_j] \). Model 5. (\( i \) numbers rows and \( j \) – columns)

| \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) |
|---------|---------|---------|---------|---------|---------|
| \( b_1 \) | 0 | 1 | 0 | 1 | 0 | 1 |
| \( b_2 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( b_3 \) | 0 | 1 | 1 | 1 | -1/2 | 1 |
| \( b_4 \) | 1 | 0 | 0 | 1/2 | 1/4 | 0 |
| \( b_5 \) | 0 | 0 | 0 | 1 | 0 | 1 |
| \( b_6 \) | 1 | 1 | 1 | 1/2 | 1/2 | 0 |
Figure 2: Supersymmetry breaking.

\[ V_{N=2} = (1, \frac{1}{2}) + (\frac{1}{2}, 0) \quad SU(8) \]

\[ \Downarrow \]

\[ V_{N=1} \rightarrow V_{N=1} + S_{N=1} \quad SU(5) \times SU(3) \times U(1) \]

| \( E_{\text{vac}} \)                  | \( J = 1 \) | \( J = 1/2 \) | \( J = 1/2 \) | \( J = 0 \) |
|---------------------------------------|-------------|--------------|--------------|-------------|
| \( [-1/2; -1] \) NS sector           |             |              | (63)         | (63)        |
| \( E_{\text{vac}} = [0; -1] \) SUSY sector | (63)×2      | (63)×2       |              |             |
|                                      |             |              |              |             |
|                                      |            |              |              |             |

\[ \Downarrow b_5 \text{ GSO projection} \]

| \( E_{\text{vac}} \)                  | \( J = 1 \) | \( J = 1/2 \) | \( J = 1/2 \) | \( J = 0 \) |
|---------------------------------------|-------------|--------------|--------------|-------------|
| \( [-1/2; -1] \) NS sector           | (24,1)+(1,1)+(1,8) | (5,\bar{3})+(\bar{5},3) |              |              |
| \( E_{\text{vac}} = [0; -1] \) SUSY sector |             |              | ((24,1)+(1,1)+(1,8))×2 |              |
|                                      |             |              | ((5,\bar{3})+(\bar{5},3))×2 |              |

\[ \text{Gauge multiplets} \quad \text{Higgs multiplets} \]
| \(N^0\) | \(b_1, b_2, b_3, b_4, b_5, b_6\) | \(SO(4) \times U(1)^7_{1,2}\) | \(U(5)^I\) | \(U(3)^I\) | \(U(5)^II\) | \(U(3)^II\) | \(Y_1^L\) | \(Y_3^L\) | \(Y_5^L\) | \(Y_7^L\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | RNS | \(+1_1 + 1_2\) | 5 | 3 | 1 | 1 | -1 | -1 | 0 | 0 |
| \(\hat{\phi}\) | 0 0 0 2 2(6) 0 | \(+1_1 - 1_2\) | 1 | 1 | 5 | 3 | 0 | 0 | -1 | -1 |
| \(\psi^I\) | 0 0 0 1 0 0 | | 1 | 3 | 1 | 1 | 5/2 | -1/2 | 0 | 0 |
| \(\psi^{II}\) | 0 0 0 3 0 0 | | 5 | 3 | 1 | 1 | -3/2 | -1/2 | 0 | 0 |
| 3 | 0 0 0 3 2 0 | | 1 | 1 | 1 | 3 | 0 | 0 | -5/2 | 1/2 |
| \(\psi^{III}\) | 0 0 0 1 6 0 | | 1 | 1 | 5 | 3 | 0 | 0 | 3/2 | 1/2 |
| 4 | 000231 (111071) | \(\pm 1/2_1\) | 1 | 3 | 1 | 1 | 0 | 1/2 | 0 | -1/2 |
| \(\psi^{IV}\) | 000271 (111031) | \(\pm 1/2_1\) | 1 | 3 | 1 | 1 | 0 | 1/2 | 0 | 3/2 |
| 000031 (111271) | \(\pm 1/2_1\) | 1 | 3 | 1 | 3 | 0 | 1/2 | 0 | -1/2 |
| 000071 (111231) | \(\pm 1/2_1\) | 1 | 1 | 10 | 1 | 0 | 0 | -1/2 | 1/2 |
| 5 | \(\pm SO(4), \mp 1/2_1\) | \(\pm SO(4), \pm 1/2_1\) | 1 | 1 | 1 | 1 | \(\pm 5/4\) | \(\pm 3/4\) | \(\pm 5/4\) | \(\pm 3/4\) |
| \(\delta\) | \(\pm SO(4), \pm 1/2_1\) | | 1 | 1 | 1 | 1 | \(\pm 5/4\) | \(\pm 3/4\) | \(\pm 5/4\) | \(\pm 3/4\) |

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9 Appendix A. The N=1 SUSY character of the $SU(3)_H$ gauge family symmetry

We will consider the supersymmetric version of the Standard Model extended by the family (horizontal) gauge symmetry. The supersymmetric Lagrangian of strong, electroweak and horizontal interactions, based on the $SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(3)_H$... (and the Abelian gauge factor also can be taken into consideration), has the standard general form \[ \mathcal{L} = \frac{1}{2} \sum_i m_i^2|\phi_i|^2 + \frac{1}{2} m_2^2 |h|^2 + \frac{1}{2} m_H^2 |H|^2 + \frac{1}{2} \mu_1^2 |\eta|^2 + \frac{1}{2} \mu_2^2 |\xi|^2 + \frac{1}{2} M^2 T r |\Phi|^2 + \frac{1}{2} \sum_k M_k \lambda_k^\alpha \lambda_k^\beta + h.c. + \text{trilinear terms} \] where $H_1 = H$, $H_2 = h$ and $i$ runs over all the scalar matter fields $\tilde{Q}$, $\tilde{u}^c$, $\tilde{d}^c$, $\tilde{L}$, $\tilde{e}^c$, $\tilde{\nu}^c$ and $k$ - runs over all the gauge groups: $SU(3)_H$, $SU(3)_C$, $SU(2)_L$, $U(1)_Y$. At the energies close to the Planck scale all the masses, as well as the gauge coupling, are correspondingly equal (this is true if the analytic kinetic function satisfies $f_{\alpha \beta} \sim \delta_{\alpha \beta}$) \[ [30] \], but at low energies they have different values depending on the corresponding renormgroup equation (RGE). The squares of some masses may be negative, which permits the spontaneous gauge symmetry breaking.

In the "flat" limit, i.e. neglecting gravity, one is left with standard supersymmetric lagrangian and soft SUSY breaking terms, which on the scales $\mu << M_{Pl}$ have the form:

Considering the SUSY version of the $SU(3)_H$-model, it is natural to ask: why do we need to supersymmetrize the model? Basing on our present-day knowledge of the nature of supersymmetry \[ [30], [31] \], the answer will be:

(a) First, it is necessary to preserve the hierarchy of the scales: $M_{EW} < M_{SUSY} < M_H < \cdots < M_{GUT}$. Breaking the horizontal gauge symmetry, one has to preserve SUSY on that scale. Another sample of hierarchy to be considered is: $M_{EW} < M_{SUSY} \sim M_H$. In this case, the scale $M_H$ should be rather low ($M_H \leq$ a few TeV).
Table 14: The Higgs Superfields with their $SU(3)_H$, $SU(3)_C$, $SU(2)_L$, $U(1)_Y$ (and possible $U(1)_H$- factor) Quantum Numbers

|   | H   | C   | L   | Y   | $Y_H$ |
|---|-----|-----|-----|-----|-------|
| $\Phi$ | $8$ | $1$ | $1$ | $0$ | $0$   |
| H     | $8$ | $1$ | $2$ | $-1/2$ | $-y_{H1}$ |
| h     | $8$ | $1$ | $2$ | $1/2$ | $y_{H1}$ |
| $\xi$ | $3$ | $1$ | $1$ | $0$ | $0$   |
| $\eta$ | $3$ | $1$ | $1$ | $0$ | $0$   |
| Y     | $3$ | $1$ | $2$ | $1/2$ | $-y_{H2}$ |
| X     | $3$ | $1$ | $2$ | $-1/2$ | $y_{H2}$ |
| $\kappa_1$ | $1$ | $1$ | $1$ $(2)$ | $0$ $(1/2)$ | $-y_{H3}$ |
| $\kappa_2$ | $1$ | $1$ | $1$ $(2)$ | $0$ $(-1/2)$ | $y_{H3}$ |

(b) To use the SUSY $U(1)_R$ degrees of freedom for constructing the superpotential and forbidding undesired Yukawa couplings.

(c) Super-Higgs mechanism - it is possible to describe Higgs bosons by means of massive gauge superfields [37].

Since the expected scale of the horizontal symmetry breaking is sufficiently large: $M_H >> M_{EW}$, $M_H >> M_{SUSY}$ (where $M_{EW}$ is the scale of the electroweak symmetry breaking, and $M_{SUSY}$ is the value of the splitting into ordinary particles and their superpartners), it is reasonable to search for the SUSY-preserving stationary vacuum solutions.

Let us construct the gauge invariant superpotential $P$. With the fields given in Table 14, the most general superpotential will have the form

$$P = \lambda_0 \left[ \frac{1}{3} Tr(\Phi^3) + \frac{1}{2} M_H Tr(\Phi^2) \right] + \lambda_1 \left[ \eta \Phi \xi + M' \eta \xi \right] + \lambda_2 Tr(\hat{h} \Phi \hat{H}) + \text{(Yukawa couplings)} + \text{(Majorana terms $\nu^c$)}$$

where Yukawa Couplings could be constructed, for example, using the Higgs fields, H and h, transforming under $SU(3)_H \times SU(2)_L$, like (8,2):

$$P_Y = \lambda_3 Q \hat{H} d^c + \lambda_4 L \hat{H} e^c + \lambda_5 Q h u^c.$$  

(92)

Also, one can consider another types of superpotential $P_Y$, using the Higgs fields from Table 14.

Note, the fields $\Phi$, H, h - can be obtained on the level 2 of Kac-Moody algebra $g$ or effectively on the level 1 of algebras $g^I, g^{II}$ after "integration" over heavy fields, when $G^I \times G^{II} \rightarrow G^{symm}$ (see section 5). Higgs fields $X$ and $Y$ are very important in models with forth SU(3)_H-singlet generation. In the construction of the stationary solutions, only the following contributions of the scalar potential are taken into account:

$$V = \sum_i |F_i|^2 + \sum_a |D^a|^2 = V_F + V_D \geq 0$$

(93)
where \[ V_F = \sum \left| \frac{\partial P_F}{\partial F_i} \right|^2 = \left| \frac{\partial P_F}{\partial F_{\Phi a}} \right|^2 + \left| \frac{\partial P_F}{\partial F_{\xi i}} \right|^2 + \left| \frac{\partial P_F}{\partial F_{\eta j}} \right|^2 \] (94)

The case \( \langle V \rangle = 0 \) of supersymmetric vacuum can be realized within different gauge scenarios [3]. By switching on the SUGRA, the vanishing scalar potential is no more required to conserve the supersymmetry with the necessity. The different gauge breaking scenarios do not result in obligatory vacuum degeneracy, as in the case of the global SUSY version. Let us write down each of the terms of formula (94):

\[
P_F(\Phi, \xi, \eta) = \lambda_0 \left[ \frac{i}{4 \times 3} f^{abc} \Phi^a \Phi^b \Phi^c + \frac{1}{4 \times 3} d^{abc} \Phi^a \Phi^b \Phi^c + \frac{1}{4} M_I \Phi^c \Phi^c \right]_F + \\
+ \lambda_1 \left[ \eta_i (T^c)^i_j \xi^j \Phi^c + M' \eta \xi i \right]_F + \\
+ \lambda_2 \left[ \frac{i}{4} f^{abc} h_i^a \Phi^b H_j^c \xi^i j + \frac{1}{4} h_i^a \Phi^b H_j^c \xi^i j \right]_F + h.c. \tag{95}
\]

The contribution of D-terms into the scalar potential will be:

\[
V_D = g_H^2 \left| \eta^i T^a \eta - \xi^i T^a \xi + i/2 f^{abc} \Phi^b \Phi^c + i/2 f^{abc} h_i^a h_j^b \right|^2 + \left| g' \right|^2 \left( \frac{1}{2} h^+ h - \frac{1}{2} H^+ H \right)^2 \tag{96}
\]

The SUSY-preserving condition for scalar potential (93) is determined by the flat \( F_i \) and \( D^a \) directions: \( \langle F_i \rangle_0 = \langle D^a \rangle_0 = 0 \). It is possible to remove the degeneracy of the supersymmetric vacuum solutions taking into account the interaction with supergravity, which was endeavored in SUSY GUT’s, e.g. in the \( SU(5) \) one \[ SU(5) \rightarrow SU(5), SU(4) \times U(1), SU(3) \times SU(2) \times U(1) \).

The horizontal symmetry spontaneous breaking to the intermediate subgroups in the first three cases of [3] can be realized, using the scalar components of the chiral complex superfields \( \Phi \), which are singlet under the standard gauge group. The \( \Phi \)-superfield transforms as the adjoint representation of \( SU(3)_H \). The intermediate scale \( M_I \) can be sufficiently large: \( M_I > 10^5 - 10^6 \text{GeV} \). The complete breaking of the remnant symmetry group \( V_H \) on the scale \( M_H \) will occur due to the nonvanishing VEV’s of the scalars from the chiral superfields \( \eta(3_H) \) and \( \xi(3_H) \). The \( V_{min} \), again, corresponds to the flat directions: \( \langle F_{\eta, \xi} \rangle_0 = 0 \). The version (iv) corresponds to the minimum of the scalar potential in the case when \( \langle \Phi \rangle_0 = 0 \).

As for the electroweak breaking, it is due to the VEV’s of the fields \( h \) and \( H \), providing masses for quarks and leptons. Note that VEV’s of the fields \( h \) and \( H \) must be of the order of \( M_W \) as they determine the quark and lepton mass matrices. On the other hand, the masses of physical Higgs fields \( h \) and \( H \), which mix generations, must be some orders higher than \( M_W \), so that not to contradict the experimental restrictions on FCNC. As a careful search for the Higgs potential shows, this is the picture that can be attained.
Appendix B. The Superstring theory scale of unification and the estimates on the horizontal coupling constant and the Weinberg angle.

Really, the estimates on the $M_{H_0}$-scale depend on the value of the family gauge coupling. These estimates can be made in GUST using the string scale

$$M_{SU} \approx 0.73 g_{\text{string}} \times 10^{18} \text{GeV}$$

and the renormalization group equations (RGE) for the gauge couplings, $\alpha_{em}$, $\alpha_3$, $\alpha_2$, to the low energies:

$$\alpha_{em}(M_Z) \approx 1/128,$$

$$\alpha_3(M_Z) \approx 0.11,$$

$$\sin^2 \theta_W(M_Z) \approx 0.233.$$ (98)

The string unification scale could be contrasted with the $SU(3c) \times SU(2) \times U(1)$ naive unification scale, $M_{GUT} \approx 10^{16} \text{GeV}$, obtained by running the SM particles and their SUSY-partners to high energies. The simplest solution to this problem is the introduction of the new heavy particles with SM quantum numbers, which can exist in string spectra.

However there are some other ways to explain the difference between scales of string ($M_{SU}$) and ordinary ($M_{GUT}$) unifications. If one uses the breaking scheme $G^I \times G^H \rightarrow G^\text{sym}$ (where $G^{I,H} = U(5) \times U(3)_H \subset E_8$) described above, then unification scale $M_{GUT} \sim 10^{16} \text{GeV}$ is the scale of breaking the $G \times G$ group, and string unification do supply the equality of coupling constant $G \times G$ on the string scale $M_{SU} \sim 10^{18} \text{GeV}$. Otherwise, we can have an addition scale of the symmetry breaking $M_{\text{sym}} > M_{GUT}$. In any case on the scale of breaking $G \times G \rightarrow G^\text{sym}$ the gauge coupling constants satisfy the equation

$$g_{\text{sym}}^2 = 1/4 (g_1^2 + g_H^2).$$

(99)

Thus in this scheme knowing of scales $M_{SU}$ and $M_{U}$ gives us a principal possibility to trace the evolution of coupling constant of the original group $G \times G$ to the low energy and estimate the value of horizontal gauge constant $g_{3H}$.

The coincidence of $\sin^2 \theta_W$ with experiment will show how realistic this model is.

Let us consider some relations which determine the value of $\sin^2 \theta_W$ for different unification groups and for different ways of the breaking.

Firstly let us consider the case of $SO(10) \times U(3) \rightarrow SU(5) \times U(1) \times [SU(3) \times U(1)]_H$ breaking, which can be illustrated by Model 2. In this case matter fields are generated by world-sheet fermions with periodic boundary conditions. Consequently all representations of matter fields can be considered as the result of destruction of $16$ and $\overline{16}$ representations of the $SO(10)$ group.
If we write the general expansion for a world-sheet fermion in the form

$$f(\sigma, \tau) = \sum_{n=0}^{\infty} \left[ b_{n+1/2}^+ \exp[-i(n + \frac{1-\alpha}{2})(\sigma + \tau)] + d_{n+1/2} \exp[i(n + \frac{1+\alpha}{2})(\sigma + \tau)] \right]$$

(100)

where the quantization conditions are given by the anti-commutation relations

$$\{b_a^+, b_b\} = \{d_a^+, d_b\} = \delta_{ab}$$

then the representation $\overline{16}$ of $SO(10)$ in terms of the creation ($b_0^{i+}$) and annihilation ($b_0^i$) operators will have the form

$$\overline{16} = 1 + \overline{10} + 5 = (1 + b_0^{i+} b_0^{j+} + b_0^{i+} b_0^{j+} b_0^{k+} b_0^{l+} |0 \rangle ,$$

(101)

where $i, j, k, l = 1, \ldots, 5$.

The Clifford algebra is realized via the $\gamma$-matrix for $SO(10)$ group $\gamma_k = (b_k + b_k^\dagger)$ and $\gamma_{5+k} = -i(b_k - b_k^\dagger)$. Generators of the $U(5)$ subgroup can be written in terms $b_0^i$ as $T[U(5)] = 1/2 \ [b_i, b_j^+]$. Then the operator of the $U(1)_5$ hypercharge is

$$Y_5 = 1/2 \sum_i [b_i, b_i^+] = 5/2 - \sum b_i^+ b_i .$$

(102)

But this generator is not normalized, since $Y_5(1, \overline{10}, \overline{5}) = 5/2, 1/2, -3/2$ corresponding, and $\text{Tr} \overline{16} Y_5^2 = 20$.

In our scheme the electromagnetic charge is

$$Q_{EM} = T_5 - 2/5 Y_5 ,$$

(103)

where $T_5 = \text{diag}(1/15, 1/15, 1/15, 2/5, -3/5)$. For representation $\overline{5}$ of the $SU(5)$ this means that

$$Q_{EM}(\overline{5}) = \text{diag}(0^3, 1/2, -1/2) + \text{diag}(2/30^3, -3/30^2)] - 2/5 \cdot (-3/2) =$$

$$= \text{diag}(0^3, 1/2, -1/2) + 1/2 \text{diag}(2/15^3, -3/15^2) + 6/5 =$$

$$= t_3 + 1/2 [\tilde{t}_0 - 4/5 Y_5] = t_3 + 1/2 \text{diag}(4/3^2, 1^2) = t_3 + 1/2 y ,$$

(104)

where $y$ is the electroweak hypercharge.

Now let us write down the principal equation for coupling constants

$$g_5 t_0 A_{\mu} + (kg_5) Y A'_{\mu} = g_{1y} B_{\mu} + g_{1'y'} B'_{\mu} .$$

(105)

In this equation $(kg_5)$ is $U(1)_5$ coupling constant on the scale where $U(5)$ is breaking down (on the $SO(10) \rightarrow U(5)$ scale $k = 1$); operators $t_0 \sim \tilde{t}_0 , Y \sim Y_5$ and have equal norm; $A$ and $B$ are gauge fields.

Diagonal generators can be written in terms of creation-annihilation operators as

$$\text{diag}(A_i) = \sum_{i=1}^5 A_i (1 - b_i^+ b_i) = - \sum A_i b_i^+ b_i .$$

(106)
Consequently $\text{Tr}_{16} \tilde{t}_0^2 = 8/15$. If we shall normalize generators as $\text{Tr}_{16} t_0^2 = \text{Tr}_{16} Y^2 = 8$, then $Y(5) = \sqrt{2/5} Y_5(2) = -3/\sqrt{10}$ and $t_0 = 1/\sqrt{15} \text{ diag}(2^3, -3^2)$. Now after rewriting the equation (102) separately for three up components and two down components, and substitution $B_\mu = c A_\mu + s A'_\mu$, $B'_\mu = -s A_\mu + c A'_\mu$ where $c^2 + s^2 = 1$, we find from equation (103)

$$\sin^2 \theta_W = \frac{g_1^4}{g_1^4 + g_5^4} = \frac{15k^2}{16k^2 + 24} \Bigg|_{k^2 = 1} = \frac{3}{8} \quad (107)$$

Now let us consider the breaking $E_6 \rightarrow U(5) \times U(1)$, which corresponds to models like Model 4. The expansion of matter representation $27$ of the $E_6$ group under the group $SU(5) \times U(1)_5$ is

$$27 = (5_{3/2} + 10_{-1/2} + 1_{-5/2}) + (\bar{5}_{1} + \bar{5}_{-1}) + 1_{0} =$$

$$= [(b_0^+ + b_0^0 b_0^k + b_0^+ b_0^0 b_0^k b_0^+ b_0^0 b_0^k)] + \cdots$$

(108)

The generalization of the formula (102) for the case when representation contains states from different sectors with different boundary conditions is

$$Y_5 = \sum_i \left( \frac{\alpha_i}{2} + \sum_\infty [d^+_E(f_i) d_E(f_i) - b^+_E(f_i^*) b_E(f_i^*)] \right) \quad (109)$$

and analogically for formula (106)

$$\text{diag}(A_i) = \sum_i [A_i \cdot \sum_\infty [d^+_E(f_i) d_E(f_i) - b^+_E(f_i^*) b_E(f_i^*)]], \quad (\sum A_i = 0).$$

Now we have $\text{Tr}_{22} Y_5^2 = 30$ and $\text{Tr}_{22} \tilde{t}_0^2 = 4/5$. By comparison with preceding case both norms are 1.5 times greater, hence formula (107) is true for this case too. But now the $B'_\mu$ is some linear combination of gauge fields.

Further, let us consider the Model 1. This case corresponds to breaking $SO(16) \rightarrow U(8) \rightarrow U(5) \times U(3)$. The matter fields arise from sectors with $\alpha = \pm 1/2$ and correspond to chips of the $SU(8)$ representations

$$
\begin{align*}
8 & \rightarrow \left[(1, 3) \right] + (5, 1) \\
56 & \rightarrow \left[(1, 1) + (10, 3) \right] + (10, 1) + (5, 3) \\
56 & \rightarrow \left[(10, 1) + (5, 3) \right] + (1, 1) + (10, 3) \\
8 & \rightarrow \left[(5, 1) \right] + (1, 3)
\end{align*}
\sim 128_{SO(16)},
\$$

where only the fields in square brackets survive after the GSO projection.

For this model it is necessary to change $Y_5 \rightarrow \tilde{Y}_5 = -1/4 (Y_5 + 5 Y_3)$ in formula (104). Now we can calculate the norms of $t_0$ and $\tilde{Y}_5$ operators for this model.

$$\text{Tr}_{128} \tilde{Y}_5^2 = 160 = 20 \times 8$$

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\[ \text{Tr}_{128}t_0^2 = \frac{64}{15} = \frac{8}{15} \times 8 \]

Hence we find again the formula (107), but \( A'_\mu \) is linear combination of gauge fields, which corresponds to \( Y_5 \) hypercharge and \( kg_5 \) is his coupling constant.

The analysis of RG–equations allows to state that horizontal coupling constant \( g_{3H} \) does not exceed electro-weak one \( g_2 \).

For example, if below the \( M_{GUT} \) scale in non-horizontal sector we have effectively the standard model with four generations and two Higgs doublets (like Model 1, 2), then the evolution of gauge coupling constants is described by equations

\[ \alpha^{-1}(\mu) = \alpha_5^{-1}(M_{GUT}) + 8\pi b_3 \ln(\mu/M_{GUT}) \]

\[ \alpha^{-1}(\mu) \sin^2 \theta_W = \alpha_5^{-1}(M_{GUT}) + 8\pi b_2 \ln(\mu/M_{GUT}) \]

\[ \frac{15}{k^2 + 24} \alpha^{-1}(\mu) \cos^2 \theta_W = (k^2\alpha_5(M_{GUT}))^{-1} + 8\pi b_1 \ln(\mu/M_{GUT}), \]

where

\[ b_3 = \frac{1}{16\pi^2}, \quad b_2 = -\frac{3}{16\pi^2}, \quad b_1 = -\frac{21}{40\pi^2}. \]

From these equations and from (98) we can find

\[ M_{GUT} = 1.3 \cdot 10^{16} \text{GeV}, \quad k^2 = 0.9, \quad \alpha_5^{-1} = 14. \]

Now we can get the relation between \( g_{\text{str.}} \equiv g \) and \( M_{\text{sym.}} \) from RG equations for gauge running constants \( g_{5\text{sym.}} \equiv g_5, g_I^5 \) and \( g_{II}^5 \) on \( M_{GUT} - M_{SU} \) scale. For example, for Model 1

\[ b_{5\text{sym.}}^5 = -\frac{3}{4\pi^2}, \quad b_I^5 = -\frac{5}{16\pi^2}, \quad b_{II}^5 = \frac{3}{16\pi^2}, \]

and from RGE we find the following relation

\[ \frac{g_5^2}{4\pi^2 + 6g_5^2 \ln(M_{GUT}/M_{\text{sym}})} = g_2^2 \frac{8\pi^2 - g_2^2 \ln(M_{\text{sym}}/M_{SU})}{[8\pi^2 - 5g_2^2 \ln(M_{\text{sym}}/M_{SU}) + 3g_2^2 \ln(M_{\text{sym}}/M_{SU})]}. \]

(114)

According to this equation we obtain that if \( M_{SU} \approx 10^{18} \) Gev and the scale of breaking down to symmetric subgroup changes in region \( M_{\text{sym.}} \approx 1.5 \cdot 10^{16} \) GeV — \( 10^{18} \) GeV then \( g_{\text{str.}} \sim O(1) \). Note that these values agree with formula (7).

Using RG equations for the running constant \( g_{3H} \) and the value of the string coupling constant \( g_{\text{str.}} \) we can estimate a value of the horizontal coupling constant at low energies. For Model 1 we have

\[ b_{5\text{sym.}}^{3H} = -\frac{5}{2\pi^2}, \quad b_I^{3H} = -\frac{21}{16\pi^2}, \quad b_{II}^{3H} = -\frac{13}{16\pi^2}, \]

and taking into account the relation (99), we find from RGE for \( g_{3H} \) that

\[ g_{3H}^2 \left( O(1\text{TeV}) \right) \approx 0.05, \]

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and this value depends very slightly on the scale $M_{sym}$. However, note that for all our estimation the presence of the breaking $G \times G$ group to diagonal subgroup $G_{sym}$ played the crucial role.

The above calculations show that for evaluation of intensity of a processes with a gauge horizontal bosons at low energies we can use inequality

$$\alpha_{3H}(\mu) \leq \alpha_{2}(\mu).$$
Appendix C. Rules for constructing consistent string models out of free world-sheet fermions.

The partition function of the theory is a sum over terms corresponding to world-sheets of different genus \( g \). For consistency of the theory we must require that partition function to be invariant under modular transformation, which is reparametrizations not continuously connected to the identity. For this we must sum over the different possible boundary conditions for the world-sheet fermions with appropriate weights \([39]\).

If the fermions are parallel transported around a nontrivial loop of a genus-\( g \) world-sheet \( M_g \), they must transform into themselves:

\[
\chi^I \longrightarrow L_g(\alpha)^I_J \chi^J
\]  

and similar for the right-moving fermions. The only constraints on \( L_g(\alpha) \) and \( R_g(\alpha) \) are that it be orthogonal matrix representation of \( \pi_1(M_g) \) to leave the energy-momentum current invariant and supercharge \((27)\) invariant up to a sign. It means that

\[
\psi^\mu \longrightarrow -\delta_\alpha \psi^\mu , \quad \delta_\alpha = \pm 1 , \\
L^I_{gI} L^J_{gJ} L^K_{gK} f_{IJK} = -\delta_\alpha f_{I'J'K'}
\]

and consequently \(-\delta_\alpha L_g(\alpha) \) is an automorphism of the Lie algebra of \( G \).

Farther, the following restrictions on \( L_g(\alpha) \) and \( R_g(\alpha) \) are imposed:

(a) \( L_g(\alpha) \) and \( R_g(\alpha) \) are abelian matrix representations of \( \pi_1(M_g) \). Thus all of the \( L_g(\alpha) \) and all of the \( R_g(\alpha) \) can be simultaneously diagonalized in some basis.

(b) There is commutativity between the boundary conditions on surfaces of different genus.

When all of the \( L(\alpha) \) and \( R(\alpha) \) have been simultaneously diagonalized the transformations like (115) can be written as

\[
f \longrightarrow -\exp(i\pi\alpha_f) f .
\]

Here and in eqs. (116), (117) the minus signs are conventional.

The boundary conditions (115), (116) are specified in this basis by a vector of phases

\[
\alpha = [\alpha(f_1^L), \cdots, \alpha(f_k^L), \alpha(f_1^R), \cdots, \alpha(f_l^R)] .
\]

For complex fermions and \( d = 4, k = 10 \) and \( l = 22 \). The phases in this formula are reduced mod(2) and are chosen to be in the interval \((-1, +1]\).

Modular transformations mix spin-structures amongst one another within a surface of a given genus. Thus, requiring the modular invariance of the partition function imposes constraints on the coefficients \( C \left[ \begin{array}{c|c} \alpha_1 & \cdots & \alpha_g \\ \hline \beta_1 & \cdots & \beta_g \end{array} \right] \) (weights in the partition function sum, for example see eq.(52) which in turn impose constraints on what spin-structures are allowed.
in a consistent theory. In accordance to the assumptions (a) and (b) these coefficients factorize:

\[
\mathcal{C} \left[ \begin{array}{cccc}
\alpha_1 & \cdots & \alpha_g \\
\beta_1 & \cdots & \beta_g
\end{array} \right] = \mathcal{C} \left[ \begin{array}{c}
\alpha_1 \\
\beta_1
\end{array} \right] \mathcal{C} \left[ \begin{array}{c}
\alpha_2 \\
\beta_2
\end{array} \right] \cdots \mathcal{C} \left[ \begin{array}{c}
\alpha_g \\
\beta_g
\end{array} \right]
\]

(120)

The requirement of modular invariance of the partition function thus gives rise to constraints on the one-loop coefficients \( \mathcal{C} \) and hence on the possible spin structures \((\alpha, \beta)\) on the torus.

For rational phases \(\alpha(f)\) (we consider only this case) the possible boundary conditions \(\alpha\) comprise a finite additive group \(\Xi = \sum_{i=1}^{k} \oplus \mathbb{Z}_{N_i}\) which is generated by a basis \((b_1, \cdots, b_k)\), where \(N_i\) is the smallest integer for which \(N_i b_i = 0 \mod(2)\). A multiplication of two vectors from \(\Xi\) is defined by

\[
\alpha \cdot \beta = (\alpha_L^i \beta_L^i - \alpha_R^i \beta_R^i)_{\text{complex}} + 1/2 (\alpha_L^k \beta_L^k - \alpha_R^k \beta_R^k)_{\text{real}}.
\]

(121)

The basis satisfies the following conditions derived in [23]:

(A1) The basis \((b_1, \cdots, b_k)\) is chosen to be canonical:

\[
\sum m_i b_i = 0 \iff m_i = 0 \mod(N_i) \quad \forall i.
\]

Then an arbitrary vector \(\alpha\) from \(\Xi\) is a linear combination \(\alpha = \sum a_i b_i\).

(A2) The vector \(b_1\) satisfies \(1/2 N_1 b_1 = 1\). This is clearly satisfied by \(b_1 = 1\).

(A3) \(N_{ij} b_i \cdot b_j \equiv 0 \mod(4)\) where \(N_{ij}\) is the least common multiple of \(N_i\) and \(N_j\).

(A4) \(N_i b_i^2 \equiv 0 \mod(4)\); however, if \(N_i\) is even we must have \(N_i b_i^2 \equiv 0 \mod(8)\).

(A5) The number of real fermions that are simultaneously periodic under any three boundary conditions \(b_i, b_j, b_k\) is even, where \(i, j, k\) are not necessarily distinct. This implies that the number of periodic real fermions in any \(b_i\) be even.

(A6) The boundary condition matrix corresponding to each \(b_i\) is an automorphism of the Lie algebra that defines the supercharge [27]. All such automorphisms must commute with one another, since they must simultaneously diagonalizable.

For each group of boundary conditions \(\Xi\) there are a number of consistent choices for coefficients \(\mathcal{C}[\cdots]\), which determine from requirement of invariant under modular transformation. The number of such theories corresponds to the number of different choices of \(\mathcal{C} \left[ \begin{array}{c} b_i \\ b_j \end{array} \right] \). This set must satisfy equations:

(B1) \(\mathcal{C} \left[ \begin{array}{c} b_i \\ b_j \end{array} \right] = \delta_{b_i} e^{i 2 \pi m/N_j} = \delta_{b_j} e^{i \pi (b_i \cdot b_j)/2} e^{i 2 \pi m/N_i}\).

(B2) \(\mathcal{C} \left[ \begin{array}{c} b_i \\ b_j \end{array} \right] = \pm e^{i \pi b_i^2/4}\).

The values of \(\mathcal{C} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \) for arbitrary \(\alpha, \beta \in \Xi\) can be obtained by means of the following
rules:

(B3) \[ C \left[ \begin{array}{c} \alpha \\ \alpha \end{array} \right] = e^{i\pi(\alpha \cdot \alpha + 1)/4} C \left[ \begin{array}{c} \alpha \\ b \end{array} \right]^{N_1/2}. \]

(B4) \[ C \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] = e^{i\pi(\alpha \cdot \beta)/2} C \left[ \begin{array}{c} \beta \\ \alpha \end{array} \right]^*. \]

(B5) \[ C \left[ \begin{array}{c} \alpha \\ \beta + \gamma \end{array} \right] = \delta_{\alpha} C \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] C \left[ \begin{array}{c} \alpha \\ \gamma \end{array} \right]. \]

The relative normalization of all the $C[\cdots]$ is fixed in these expressions conventionally to be $C \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \equiv 1$.

For each $\alpha \in \Xi$ there is a corresponding Hilbert space of string states $H_\alpha$ that potentially contribute to the one-loop partition function. If we wrtte $\alpha = (\alpha_L | \alpha_R)$, then the states in $H_\alpha$ are those that satisfy the Virasoro condition:

\[ M_L^2 = -c_L + 1/8 \alpha_L \cdot \alpha_L + \sum_{L\text{-mov.}} (\text{frequencies}) = -c_R + 1/8 \alpha_R \cdot \alpha_R + \sum_{R\text{-mov.}} (\text{freq.}) = M_R^2. \]

Here $c_L = 1/2$ and $c_R = 1$ in the heterotic case. In $H_\alpha$ sector the fermion $f$ ($f^*$) has oscillator frequencies

\[ \frac{1 + \alpha(f)}{2} + \text{integer}. \]

The only states $|s\rangle$ in $H_\alpha$ that contribute to the partition function are those that satisfy the generalized GSO conditions

\[ \left\{ e^{i\pi(b_i \cdot F_\alpha)} - \delta_{\alpha} C \left[ \begin{array}{c} \alpha \\ b_i \end{array} \right]^* \right\} |s\rangle = 0 \]

for all $b_i$. Where $F_\alpha(f)$ is a fermion number operator. If $\alpha$ contains periodic fermions then $|0\rangle_\alpha$ is degenerate, transforming as a representation of an $SO(2n)$ Clifford algebra.
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