Waste Not, Want Not: Heisenberg-Limited Metrology With Information Recycling

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Information recycling has been shown to improve the sensitivity of interferometers when the input quantum state has been partially transferred from some donor system. In this paper we demonstrate that when the quantum state of this donor system is from a particular class of Heisenberg-limited states, information recycling yields a Heisenberg-limited phase measurement. Crucially, this result holds irrespective of the fraction of the quantum state transferred to the interferometer input and also for a general class of number-conserving quantum-state-transfer processes, including ones that destroy the first-order phase coherence between the branches of the interferometer. This result could have significant applications in Heisenberg-limited atom interferometry, where the quantum state is transferred from a Heisenberg-limited photon source, and in optical interferometry where the loss can be monitored.

When performing an interferometric measurement with a limited number of particles, $N$, there can be significant benefit to using a nonclassical input state to improve the phase sensitivity beyond the standard quantum noise limit (QNL) (shot-noise limit) of $\Delta \phi \sim 1/\sqrt{N}$ [1,2]. The ultimate limit to sensitivity is the Heisenberg limit $\Delta \phi \sim 1/N$ [3,4]. In particular, a Mach-Zehnder (MZ) interferometer can achieve Heisenberg-limited phase sensitivity if the input state has perfect number correlations between the two interferometer modes [5]. An example is the two-mode squeezed vacuum state [6], which is routinely generated in quantum optics laboratories [2].

There exist metrological devices, however, where Heisenberg-limited input states are difficult to generate, such as inertial sensors based on atom interferometry. In such cases, Heisenberg-limited interferometry might still be possible provided a Heisenberg-limited state from a donor system (e.g., two-mode squeezed optical vacuum) can be mapped to this acceptor system. This possibility was demonstrated theoretically in [7], where quantum state transfer (QST) between squeezed light and atoms was shown to enhance the sensitivity of atom interferometry well below the QNL. Similar results are also possible in other contexts, as proposals exist for achieving QST between donor photons and a range of acceptor systems, including atomic motional states [8], room-temperature and laser cooled atomic vapours [9], Bose-Einstein condensates of dilute atomic vapors [10,14], ions [15], solid state systems [16], and mechanical oscillators [16].

Unfortunately, in practice any QST process is imperfect, and even a small degree of imperfection results in a large degradation of the acceptor system’s phase sensitivity from the Heisenberg limit [17]. If the QST efficiency is not too low, however, the degradation in sensitivity can be somewhat ameliorated by measuring the donor state not mapped to the acceptor system and applying the technique of information recycling [17,17]. Here we show that if the donor source displays perfect number correlations, then the acceptor particles give Heisenberg-limited sensitivity regardless of the QST efficiency when used in a Mach-Zehnder (MZ) interferometer, provided information recycling is applied. This holds regardless of the physical mechanism for QST, provided that the QST process is number conserving.

Number-correlated MZ interferometer. To determine the best phase sensitivity possible for a given interferometry scheme, we appeal to the quantum Fisher information. As discussed in [5,15], the quantum Fisher information $\mathcal{F}$ places an absolute lower bound on the phase sensitivity, $\Delta \phi \geq 1/\sqrt{\mathcal{F}}$, called the quantum Cramér-Rao bound (QCRB), which applies regardless of the choice of measurement and phase estimation procedure; the bound depends only on the input state.

It was shown in [19,20] that when a pure state is used as the input to a lossless MZ interferometer (i.e., beamsplitter-mirror-beamsplitter configuration), the quantum Fisher information for estimating a differential phase shift is given by $\mathcal{F} = 4(\langle \hat{L}_y^2 \rangle - \langle \hat{L}_y \rangle^2)$, where $\hat{L}_k \equiv \frac{1}{2} \hat{b}_1^\dagger \sigma_k \hat{b}_2$ defines pseudo-spin operators, $\hat{b}_j = (\hat{b}_j, \hat{b}_j^\dagger)^T$, $\hat{b}_j$ are the usual bosonic annihilation operators for the two modes, and $\sigma_k$ are the set of Pauli spin matrices.

Consider now a two-mode state that displays perfect number correlations between the two input modes,

$$|\Psi_b^\prime\rangle = \sum_{N=0}^\infty c_N |N, N\rangle.$$  \hspace{1cm} (1)

When used as the input to a MZ interferometer, the quantum Fisher information is given by

$$\mathcal{F}_b = \frac{V(\hat{N}_t) + \hat{N}_t(\hat{N}_t + 2)}{2},$$  \hspace{1cm} (2)

where $\hat{N}_t = \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2$ is the operator for the total number of particles, $\hat{N}_t = \langle \hat{N}_t \rangle$ is its expectation value, and $V(X)$ denotes the variance of $X$. For the twin-Fock state $|\Psi_{TF}\rangle = |N/2, N/2\rangle$, the variance is zero, so...
Two-mode squeezed vacuum \cite{21,23},
\[
|\Psi_{sq}(r)\rangle = \text{sech}|r| \sum_{N=0}^{\infty} (-e^{i\delta} \tanh|r|)^N |N, N\rangle ,
\]
with \( r = |r|e^{i\theta} \), has variance \( V(\hat{N}_t) = N_t(N_t + 2) \) and thus \( \mathcal{F}_b = N_t(N_t + 2)/2 \). For a particular choice of measurement signal, \( \hat{S} \), the phase uncertainty is given by \( \Delta \phi = \sqrt{\frac{2}{V(\hat{N}_t) + N_t(N_t + 2)}} \) for sensing small changes away from the operating point. This signal choice thus achieves the QCRB.

Since the MZ interferometer does not require first-order coherence between the branches, the phase uncertainty \( \Delta \phi \) is achieved by any input (mixed) state of the form \cite{25}
\[
\hat{\rho}_b = \sum_{M,N=0}^{\infty} \rho_{MN}|M,M\rangle\langle N,N| ,
\]
not just by the pure states \( |1\rangle \), for which \( \rho_{MM} = c_M e^{2N} \). We define \( \rho_{NN} = p_N \). When \( \rho_{MN} \) is diagonal, i.e., \( \rho_{MN} = p_N \delta_{MN} \), the number correlations between the input branches are purely classical.

**Donor-enhanced MZ interferometer.** Now suppose we want to map the Heisenberg-limited state \( \hat{\rho}_d \) from this donor system to some two-mode acceptor system. This scenario is depicted in Fig. 1. At \( t = t_0 \), the quantum state of the system is prepared such that the state of the donor system is \( \hat{\rho}_b \), while the two modes of the acceptor system (amplification operators \( \hat{a}_1 \) and \( \hat{a}_2 \)) are unoccupied, giving a total state
\[
\hat{\rho}(t_0) = \sum_{M,N=0}^{\infty} \rho_{MN}|M,0,M,0\rangle\langle N,0,N,0| .
\]
A QST process is implemented such that at \( t = t_1 \), some or all of the particles are transferred from mode 1(2) of our donor system to mode 1(2) of our acceptor system. The acceptor particles are then used as the input to a MZ interferometer.

A perfect QST process performs the map \( |N,0\rangle \rightarrow |0,N\rangle \) in each branch of the interferometer, and consequently the MZ interferometer composed of the two acceptor modes is Heisenberg limited. In practice, however, the QST process is imperfect. Some particles remain in the donor modes at time \( t_1 \), and this results in a loss of correlations when considering only the acceptor modes. As was shown in \cite{5,7}, even a small loss of correlations can severely degrade sensitivity. Fortunately, we can reduce this degradation by monitoring those donor particles still remaining after the QST process and incorporating this information as part of our phase-estimation procedure. This technique of *information recycling* has been shown to enhance the sensitivity within specific atom interferometric schemes \cite{7,17}. The surprising result we show here is that a Heisenberg-limited donor source coupled with information recycling yields Heisenberg-limited interferometry with the acceptor modes irrespective of the QST efficiency or the physical mechanism of the QST process.

To show this, we now consider the state after incomplete QST. Without specifying the physical mechanism of the QST process, we apply the following physically motivated constraints:

1. The QST process occurs in two independent branches; i.e., donor mode \( \hat{b}_1(\hat{b}_2) \) can only exchange particles with acceptor mode \( \hat{a}_1(\hat{a}_2) \), and neither branch is affected by the other.
2. Each branch of the QST process conserves particle number; i.e., \( \hat{b}_j\hat{b}_j + \hat{a}_j\hat{a}_j \) is a conserved quantity for \( j = 1,2 \).
3. The QST process is symmetric with respect to the exchange \( \hat{b}_1 \leftrightarrow \hat{b}_2 \) and \( \hat{a}_1 \leftrightarrow \hat{a}_2 \); i.e., the two inde-
pendent branches of the QST process are identical.

Although a beamsplitter transformation satisfies these requirements, these conditions are also satisfied by a broad class of QST processes, both unitary and nonunitary. For example, they allow very complicated QST processes where the QST Hamiltonian contains higher-order couplings; heuristically, this might result in a QST efficiency that depends on the number of particles in the donor mode. Furthermore, the constraints allow for situations where the QST process is mediated by some other set of modes \(\hat{c}_k\) (e.g., a reservoir), which might be depleted and thus reduce the QST efficiency as more particles are of modes \(\hat{c}_k\). Heuristically, this might result in a QST efficiency that depends on the number of particles in the donor mode. Notice that we only require that number correlations be independent branches of the QST process are identical.

A general QST process that satisfies the conditions 1–3 performs the following map in each branch:

\[
\hat{S} = \sum_{i=1,2} \sum_{n,m,l}^{N_m, N_l} A_{Mm, Nn} |M - m, m\rangle \langle N - n, n|.
\]

(8)

There are no constraints on \(A_{Mm, Nn}\) other than the usual physical constraints of normalization and complete positivity. \(p_{n|M} \equiv A_{Nn, NN}\) is the conditional probability that there are \(n\) particles in an acceptor mode, given \(N\) particles initially in the corresponding donor mode.

Under the QST map \(\hat{S}\), the state \(\hat{\rho}(t_0)\) of Eq. (6) is mapped to the (generally mixed) state

\[
\hat{\rho}(t_1) = \sum_{M, N=0}^{\infty} \sum_{N, N=0}^{M, N} A_{M_m, N_{n1}, A_{M_{m2}, N_{n2}}} \times |M - m1, m1, M - m2, m2\rangle \langle N - n1, n1, N - n2, n2|.
\]

Notice that we only require that number correlations between the branches be maintained; dephasing within or between the branches is perfectly acceptable.

Introducing the pseudo-spin operators for the acceptor modes, \(\hat{J}_k \equiv \frac{1}{2} \hat{a}_a \sigma_k a\), where \(a = (\hat{a}_t, \hat{a}_f)^T\), the unitary operator for the MZ interferometer performs the following transformations:

\[
\hat{J}_z(t_f) = U^\dagger \hat{J}_z(t_1) U = \hat{J}_z(t_1) \cos \phi - \hat{J}_x(t_1) \sin \phi, \quad \hat{L}_z(t_f) = U^\dagger \hat{L}_z(t_1) U = \hat{L}_z(t_1),
\]

since only the acceptor particles take part in the interferometric process. As in [7], we estimate the phase by measuring the number of particles at the four output ports (see Fig. 1) and constructing the signal

\[
S = (\hat{J}_z(t_f) + \hat{L}_z(t_f))^2.
\]

Although only \(\hat{J}_z\) contains phase information, the noise in \(\hat{J}_z\) is anticorrelated with \(\hat{L}_z\), so measuring both quantities allows us to correct for this noise and therefore improve sensitivity.

To evaluate the phase sensitivity, we need the first and second moments of \(\hat{S}\) in the state [24]. Since the QST process and the angular-momentum operators preserve total particle number, there is no interference between sectors with different numbers of particles; the desired moments are averages over \(p_{N} \equiv p_{N, N}\). The anticorrelation of \(\hat{J}_z\) and \(\hat{L}_z\), expressed by \(\hat{J}_z \hat{\rho}(t_1) = -\hat{L}_z \hat{\rho}(t_1)\), allows us to convert \(\hat{L}_z\) in these moments to \(\hat{J}_z\). The anticorrelation implies that \(\hat{\rho}(t_1)\) is invariant under rotations about the \(z\) axis; in particular, a rotation by \(\pi\), which takes \(\hat{J}_z\) to \(-\hat{J}_z\), implies that \(\hat{J}_z\) is the average number of acceptor particles and thus the number of particles that take part in the interferometric process.

We can put a lower bound on \(\langle \hat{N}_1 \hat{N}_2 \rangle\) by noting that a state of the form [24] gives

\[
\langle \hat{N}_1 \hat{N}_2 \rangle = \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle \langle \hat{N}_2 \rangle = \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle^2\).
\]

Here \(\langle \hat{N}_j \rangle = \sum_{n=0}^{N} n_j p_{n_j} \) is the conditional expectation value of the number of particles in acceptor mode \(j\), given \(N\) initial particles in donor mode \(j\). That the conditional probabilities are the same in the two branches ensures that \(\langle \hat{N}_1 \rangle = \langle \hat{N}_2 \rangle\). Convergence implies that

\[
\langle \hat{N}_1 \hat{N}_2 \rangle \geq \left( \sum_{N=0}^{\infty} p_N \langle \hat{N}_1 \rangle \right)^2 = \langle \hat{N}_1 \rangle^2 = 1 + N_a^2,
\]

which gives an upper bound on the phase sensitivity of any QST process applied to the initial state \(|\Psi(t_0)\rangle\),

\[
\Delta\phi \leq \sqrt{\frac{2}{N_a (N_a + 2)}} \simeq \frac{\sqrt{2}}{N_a}.
\]

The important feature of this result is that the Heisenberg limit is recovered, with respect to the number of particles, \(N_a\), taking part in the interferometer, rather than the total number of particles \(N\). Although the absolute sensitivity is less than with perfect QST, this is purely due to loss of particles, rather than to loss of correlations. We stress that this is not the true Heisenberg limit, in the sense that we have used \(N_l \geq N_a\) particles to make the measurement, but only \(N_a\) of them have passed through the interferometer. Without the application of information recycling, however, the sensitivity is significantly worse than \(1/N_a\) [23].
For the specific case when the donor source is a twin-Fock state, $|\Psi_b\rangle = |\Psi_{TF}\rangle$, we get $\langle N_1 N_2 \rangle = \langle N_1 \rangle \langle N_2 \rangle$, which gives a phase sensitivity that saturates the bound (13) and is entirely independent of the QST efficiency or even the form of the number-conserving QST interaction. For other initial states, there might be a weak dependence on the QST process (as seen for the beamsplitting case below); nevertheless the phase sensitivity is guaranteed to be at least as good as that given by the twin-Fock state. To be more quantitative about the performance of states other than $|\Psi_{TF}\rangle$, we need to specify a particular Hamiltonian governing the QST process.

**Beamsplitter QST process.** We now consider the simplest possible QST process, a beamsplitter. The Hamiltonian describing this process, $H \propto \sum_{j=1,2} (\hat{a}_j \hat{b}_j^\dagger + \hat{a}_j^\dagger \hat{b}_j)$, leads to the unitary transformation

$$\hat{a}_j(t_1) = \sqrt{1-Q} \hat{a}_j(t_0) - i\sqrt{Q} \hat{b}_j(t_0), \quad (14a)$$

$$\hat{b}_j(t_1) = \sqrt{1-Q} \hat{b}_j(t_0) - i\sqrt{Q} \hat{a}_j(t_0). \quad (14b)$$

Here $Q$ is the **QST efficiency**, i.e., the fraction of donor particles mapped to the acceptor modes.

The transformation (14) allows us to evaluate Eq. (10) explicitly to determine the precise dependence on the QST efficiency. With the initial state (5), we get $\langle N_1 N_2 \rangle = (Q^2 V(\hat{N}_1) + \langle N_0 \rangle^2)/2$, and the phase sensitivity in the presence of information recycling is

$$\Delta \phi = \sqrt{\frac{2}{Q^2 V(\hat{N}_1) + N_a(N_a + 2)}}. \quad (15)$$

For a twin-Fock input, which has $V(\hat{N}_1) = 0$, the phase sensitivity does not depend on $Q$ and is given by the bound in Eq. (13). When the donor state is two-mode squeezed vacuum, $|\Psi_b\rangle = |\Psi_{sq}\rangle$, we find that $\Delta \phi = 1/\sqrt{N_a + 1 + Q}$, which has only a weak dependence on $Q$. Indeed, it is clear that to leading order in the total number of acceptor particles, $N_a = Q N_i$, the sensitivity (15) has Heisenberg scaling for any donor input state (5), regardless of the QST efficiency $Q$. This gives a clear illustration of the power of information recycling as a tool to enable quantum metrology.

It is instructive to compute the quantum Fisher information $\mathcal{F}_a$ for the donor-acceptor interferometer. With the pure initial state (1) and a beamsplitter QST process, the state remains pure, and the quantum Fisher information is simply $\mathcal{F}_a = 4[(\langle J_x(t_1) \rangle^2 - \langle J_y(t_1) \rangle^2)/2].$ The transformations (14) allow us to compute these expectations with respect to the initial state. Since the acceptor modes are initially vacuum, we obtain

$$\mathcal{F}_a = Q^2 \mathcal{F}_b + (1 - Q) N_a = \frac{Q^2 V(\hat{N}_1) + N_a(N_a + 2)}{2}. \quad (16)$$

Comparing with the sensitivity (15), it is clear that our information-recycled signal achieves the best possible Heisenberg scaling, i.e., by saturating the QCRB.

In contrast to these results, when information recycling is not applied, the beamsplitter QST process acts as a linear loss mechanism and Heisenberg scaling is lost (see Fig. 2). This loss of Heisenberg scaling occurs for relatively small deviations of $Q$ from perfect QST and affects any initial state of the form (5) (see also 26,27).

**Applications.** Donor-enhanced interferometry with information recycling requires the following: (i) a correlated source of donor particles, (ii) partial QST between the donor particles and some acceptor system, and (iii) the ability to detect both donor and acceptor particles. It might be particularly useful in situations where there are abundant donor particles and a limited number of acceptor particles [such as QST from photons (donor) to atoms (acceptor) for the purposes of atom interferometry], since the QST efficiency becomes irrelevant once $N_a$ equals the total number of available acceptor particles. In addition to Heisenberg-limited atom interferometry, another potential application for this scheme is optical interferometry which requires coupling into optical fibers before the interferometer. Here, coupling between the freely propagating modes (donor system) and the fiber modes (acceptor system) represents the QST process. Typically there will be some scattering into other modes, which is a source of inefficient QST. Information recycling could be implemented by detecting the scattered photons.

We acknowledge useful discussions with W. Bowen,
Phase sensitivity of Mach-Zehnder interferometer with number-correlated input

For a MZ interferometer with number-correlated input
\[ \hat{\rho}_b = \sum_{M,N} \rho_{MN} |M,M\rangle \langle N,N|, \]

an optimal signal is \( \hat{S} = \hat{L}_z^2 \). The uncertainty in the measured phase is
\[ \Delta \phi = \sqrt{V(\hat{S}) \left| \frac{\partial}{\partial \phi}(\hat{S}) \right|}. \]

The interferometer transforms \( \hat{L}_z \) to \( \hat{L}_z \cos \phi - \hat{L}_x \sin \phi \), so the moments of the signal are given by expectation values of even powers of \( \hat{L}_z \cos \phi - \hat{L}_x \sin \phi \) in the state \( \hat{\rho}_b \). Because \( \hat{L}_z \) and \( \hat{L}_x \) preserve the total photon number, only the diagonal terms of the density operator contribute to the relevant moments, which are moments of \( N \) of even powers of \( \hat{L}_x \).

Using these facts to calculate the expectation value and variance of \( \hat{S} \) gives
\[ \langle \hat{S} \rangle = \langle \hat{L}_z^2 \rangle \sin^2 \phi, \]
\[ V(\hat{S}) = \langle \hat{L}_z^2 \hat{L}_x^2 \rangle \sin^2 \phi \cos^2 \phi + V(\hat{L}_x^2) \sin^4 \phi, \]
which leads to a measured-phase variance
\[ (\Delta \phi)^2 = \frac{1}{4} \left( \frac{\langle \hat{L}_z^2 \hat{L}_x^2 \rangle}{\langle \hat{L}_x^2 \rangle^2} + \frac{V(\hat{L}_x^2)}{\langle \hat{L}_x^2 \rangle^2} \tan^2 \phi \right). \]
The optimal operating points are $\phi = 0, \pi$.

The angular-momentum moments are

$$\langle \hat{L}_z^2 \rangle = \frac{1}{2} \sum_N N(N+1) p_N = \frac{\langle \hat{N}_z^2 \rangle + 2\langle \hat{N}_x \rangle}{8},$$

$$\langle \hat{L}_z \hat{L}_z^2 \hat{L}_x \rangle = \frac{1}{2} \sum_N N(N+1) p_N = \langle \hat{L}_z^2 \rangle,$$

$$V(\hat{L}_z) = \sum_N \left( \frac{3}{8} N^4 + \frac{3}{4} N^3 + \frac{1}{8} N^2 - \frac{1}{4} N \right) p_N - (\hat{L}_z^2)^2$$

$$= \frac{3}{128} \langle \hat{N}_z^2 \rangle + \frac{3}{32} \langle \hat{N}_x^3 \rangle + \frac{1}{32} \langle \hat{N}_z^4 \rangle - \frac{1}{8} \langle \hat{N}_x \rangle - (\hat{L}_z^2)^2.$$  

Therefore, at the optimal operating points the phase sensitivity is

$$\Delta \phi = \frac{1}{2\langle \hat{L}_z^2 \rangle^{1/2}} = \sqrt{\frac{2}{\langle \hat{N}_z^2 \rangle + 2\langle \hat{N}_x \rangle}}.$$  

**Phase sensitivity for the QST process in the absence of information recycling**

The donor-acceptor state after the QST process (i.e. at time $t_1$) is

$$\hat{\rho}(t_1) = \sum_{M,N=0}^{\infty} \rho_{MN} \sum_{m_{1,1},m_{2,2}} A_{M,m_1,N,n_1} A_{M,m_2,N,n_2} |M-m_1,m_1,M-m_2,m_2,N-n_1,n_1,N-n_2,n_2\rangle.$$  

Without information recycling, the signal we measure is $\hat{S} = [\hat{J}_z(t_f)]^2$, where

$$\hat{J}_z(t_f) = U_{MZ}^\dagger \hat{J}_z(t_1) U_{MZ} = \hat{J}_z(t_1) \cos \phi - \hat{J}_x(t_1) \sin \phi.$$  

In what follows, we condense the notation by writing $\hat{J}_z \equiv \hat{J}_z(t_1)$ and $\hat{J}_x \equiv \hat{J}_x(t_1)$.

In order to calculate the phase sensitivity $\Delta \phi$ of Eq. (15), we need to evaluate the first and second moments of $\hat{S}$. Since the QST process that leads to the state (24) and the angular-momentum operators preserve total particle number, there is no interference between sectors with different numbers of particles; the desired moments are thus averages over $p_N = \rho_{NN}$. Moreover, the anticorrelation of $\hat{J}_z$ and $\hat{L}_z$, expressed by $\langle \hat{J}_z + \hat{L}_z \rangle \hat{\rho}(t_1) = 0 = -\hat{\rho}(t_1) \langle \hat{J}_z + \hat{L}_z \rangle$, means that $\hat{\rho}(t_1)$ is invariant under rotations about the $z$ axis; in particular, a rotation by $\pi$, which takes $\hat{J}_z$ to $-\hat{J}_z$, implies that all terms with an odd number of $\hat{J}_z$ operators have vanishing expectation value.

Generally, we have

$$\hat{S} = [\hat{J}_z(t_f)]^2 = \hat{J}_z^2 \cos^2 \phi + \hat{J}_x^2 \sin^2 \phi - \cos \phi \sin \phi (\hat{J}_z \hat{J}_x \hat{J}_x + \hat{J}_z \hat{J}_x),$$

$$\hat{S}^2 = [\hat{J}_z(t_f)]^4 = \hat{J}_z^4 \cos^4 \phi + \hat{J}_x^2 \hat{J}_z^2 \cos^2 \phi \sin^2 \phi - \hat{J}_z^2 (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z) \cos^3 \phi \sin \phi$$

$$+ \hat{J}_z^2 \hat{J}_x^2 \cos^2 \phi \sin^2 \phi + \hat{J}_z^2 \hat{J}_x^2 \sin^4 \phi - \hat{J}_z^2 (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z) \cos \sin^3 \phi$$

$$- (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z) \hat{J}_z^2 \cos^3 \phi \sin \phi - (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z) \hat{J}_x^2 \cos \phi \sin^3 \phi$$

$$+ (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z) \hat{J}_x^2 \cos^2 \phi \sin^2 \phi.$$  

Applying our rules, we get

$$\langle \hat{S} \rangle = \langle \hat{J}_z^2 \rangle \cos^2 \phi + \langle \hat{J}_x^2 \rangle \sin^2 \phi,$$

$$\langle \hat{S}^2 \rangle = \langle \hat{J}_z^4 \rangle \cos^4 \phi + \langle \hat{J}_x^4 \rangle \sin^4 \phi + \langle \hat{J}_z^2 \hat{J}_x^2 \hat{J}_x^2 \rangle \cos^2 \phi \sin^2 \phi,$$

which gives the squared phase sensitivity

$$(\Delta \phi)^2 = \frac{V(\hat{J}_z^2) \cot^2 \phi + V(\hat{J}_x^2) \tan^2 \phi + C(\hat{J}_z^2, \hat{J}_x^2) + \langle (\hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_x) \rangle}{4(\langle \hat{J}_z^2 \rangle - \langle \hat{J}_x^2 \rangle)^2}.$$  


Here \( V(\hat{X}) = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \) is, as throughout, the variance of \( \hat{X} \), and \( C(\hat{X}, \hat{Y}) = \langle \hat{X} \hat{Y} + \hat{Y} \hat{X} \rangle - 2 \langle \hat{X} \rangle \langle \hat{Y} \rangle \) is the symmetrized covariance of \( \hat{X} \) and \( \hat{Y} \).

We can write these expectation values in terms of the number operators for the two acceptor modes, \( \hat{N}_1 \) and \( \hat{N}_2 \). We again make use of the form of the input state and the restrictions on the state transfer, which make any term vanish whose number of creation operators, \( \hat{a}_1^\dagger \) or \( \hat{a}_2^\dagger \), does not match the corresponding number of annihilation operators, \( \hat{a}_1 \) or \( \hat{a}_2 \). The relevant angular-momentum moments are

\[
\langle \hat{J}_z^2 \rangle = \frac{1}{4} \langle (\hat{N}_1 - \hat{N}_2)^2 \rangle, \\
\langle \hat{J}_x^2 \rangle = \langle \hat{j}_y^2 \rangle = \frac{1}{4} \langle 2\hat{N}_1 \hat{N}_2 + \hat{N}_1 + \hat{N}_2 \rangle, \\
\langle \hat{J}_z^4 \rangle = \frac{1}{16} \langle (\hat{N}_1 - \hat{N}_2)^4 \rangle, \\
\langle \hat{J}_x^4 \rangle = \frac{1}{16} \langle (2\hat{N}_1 \hat{N}_2 + \hat{N}_1 + \hat{N}_2)^2 + \hat{N}_1 \hat{N}_2 + \hat{N}_1 \hat{N}_2 \rangle, \\
\langle \hat{J}_z^6 \rangle = \langle \hat{J}_x^6 \rangle = \frac{1}{16} \langle (2\hat{N}_1 \hat{N}_2 + \hat{N}_1 + \hat{N}_2)^2 + (\hat{N}_1^2 - \hat{N}_1)(\hat{N}_2^2 + 3\hat{N}_2 + 2) \rangle \\
+ (\hat{N}_1^2 + 3\hat{N}_1 + 2)(\hat{N}_2^2 - \hat{N}_2) \rangle, \\
\langle \hat{J}_x \hat{J}_z^2 \rangle = \langle \hat{J}_x^2 \hat{J}_z \rangle = \frac{1}{16} \langle (\hat{N}_1 - \hat{N}_2)^2(2\hat{N}_1 \hat{N}_2 + \hat{N}_1 + \hat{N}_2) \rangle, \\
\langle \hat{J}_x \hat{J}_z^2 \rangle = \langle \hat{J}_x \hat{J}_z^2 \rangle - \langle \hat{J}_z^2 \rangle + \langle \hat{J}_x^2 \rangle, \\
\langle \hat{J}_x \hat{J}_z \hat{J}_z \hat{J}_z \rangle = \langle \hat{J}_x \hat{J}_x \hat{J}_z \hat{J}_z \rangle = \frac{1}{2} \langle \hat{J}_z^2 \rangle. 
\]

As in the main text, we introduce conditional expectation values to write the moments of the signal \( \hat{S} \):

\[
\langle \hat{S} \rangle = \sum_{N=0}^\infty p_N \left( \frac{1}{2} \sin^2 \phi \langle \hat{N}_1 \rangle_N + \frac{1}{2} \cos^2 \phi \langle \hat{N}_1^2 \rangle_N + \frac{1}{2} \sin^2 \phi - \cos^2 \phi \langle \hat{N}_1 \rangle_N \right), \\
\langle \hat{S}^2 \rangle = \sum_{N=0}^\infty p_N \left( \left( \frac{1}{2} \cos^2 \phi \sin^2 \phi - \frac{\sin^4 \phi}{4} \right) \langle \hat{N}_1 \rangle_N + \left( -\cos^2 \phi \sin^2 \phi + \frac{3\sin^4 \phi}{8} \right) \langle \hat{N}_1^2 \rangle_N \right) \\
+ \left( \frac{3}{4} \cos^2 \phi \sin^2 \phi + \frac{\sin^4 \phi}{4} \right) \langle \hat{N}_1 \rangle_N \langle \hat{N}_1 \rangle_N \\
+ \left( -\frac{3}{4} \cos^2 \phi \sin^2 \phi + \frac{3\sin^4 \phi}{4} \right) \langle \hat{N}_1 \rangle_N \langle \hat{N}_1 \rangle_N \\
+ \left( \frac{3\cos^4 \phi}{8} - \frac{3}{2} \cos^2 \phi \sin^2 \phi + \frac{3\sin^4 \phi}{8} \right) \langle \hat{N}_1^2 \rangle_N \langle \hat{N}_1^2 \rangle_N \right). 
\]

Here we use the fact that the conditional expectation values are the same in the two branches of the interferometer to convert all the conditional expectation values to mode 1.

Now we specialize to the case where the QST process is a beamsplitter, with \( Q \) denoting the QST efficiency. In this case the conditional probability distributions in the two branches are binomial distributions, and we can write the number moments in terms of moments \( \langle \hat{N}_k \rangle \) of the total particle number in the initial state. As in the main text,
we use $N_t \equiv \langle \hat{N}_t \rangle$ to denote the average total particle number. The relevant moments take the following forms:

\begin{align}
\langle \hat{J}_z^2 \rangle &= \frac{Q}{4}(1-Q)N_t, \quad (34a) \\
\langle \hat{J}_y^2 \rangle &= \frac{Q}{4} \left( \frac{Q}{2}(\hat{N}_t^2) + N_t \right), \quad (34b) \\
\langle \hat{J}_x^4 \rangle &= \frac{Q}{16} \left( 3Q(1-Q) \langle \hat{N}_t^2 \rangle + (6Q^2 - 6Q + 1) N_t \right), \quad (34c) \\
\langle \hat{J}_x^6 \rangle &= \frac{Q}{16} \left( \frac{3Q^3(\hat{N}_t^2)}{8} + \frac{3Q^2}{2} (2 - Q) (\hat{N}_t^3) + \frac{Q}{2} (3Q^2 - 12Q + 10) \langle \hat{N}_t^2 \rangle + (1 - 3Q) N_t \right), \quad (34d) \\
\langle \hat{J}_z^2 \hat{J}_x^2 \rangle &= \langle \hat{J}_z \hat{J}_x \rangle = \frac{Q}{16} \left( \frac{Q^2}{2} \langle \hat{N}_t^3 \rangle + Q(1-Q) \langle \hat{N}_t^2 \rangle + (1 - 2Q) N_t \right). \quad (34e)
\end{align}

This implies that

\begin{align}
\langle \hat{S} \rangle &= \frac{Q}{4} \left( \frac{Q}{2} \sin^2 \phi \langle \hat{N}_t^2 \rangle + [(1-Q) \cos^2 \phi + \sin^2 \phi] N_t \right), \quad (35) \\
\langle \hat{S}^2 \rangle &= \frac{3}{128} Q^4 \sin^4 \phi \langle \hat{N}_t^4 \rangle - \frac{3}{64} Q^3 \sin^2 \phi (Q \cos 2\phi - 4 + 3Q) \langle \hat{N}_t^3 \rangle \\
&\quad + \frac{1}{256} Q^2 \left( 39Q^2 - 96Q + 64 - 4(4 - 3Q^2) \cos 2\phi - 3Q^2 \cos 4\phi \right) \langle \hat{N}_t^2 \rangle \\
&\quad - \frac{1}{16} Q \left( 6Q \cos^2 \phi (Q \cos^2 \phi - 2) + 2 \cos 2\phi + 5 \right) - 1) N_t. \quad (36)
\end{align}

To find the optimal operating point of the interferometer, we return to Eq. \[30\]. Unlike the information-recycling signal, $\Delta \phi$ does not generally attain a minimum at $\phi = 0$. Here the minimum occurs when $J(\phi) \equiv V(\hat{J}_z^2) \cot^2 \phi + V(\hat{J}_x^2) \tan^2 \phi$ is a minimum, which occurs at

\begin{align}
\phi &= \tan^{-1} \left[ (V(\hat{J}_z^2)/V(\hat{J}_x^2))^{1/4} \right], \quad (37)
\end{align}

and this gives $\min \phi \mathcal{J} = 2 \sqrt{V(\hat{J}_z^2)V(\hat{J}_x^2)}$. Consequently, the minimum phase sensitivity is

\begin{align}
(\Delta \phi_{\min})^2 &= \frac{2 \sqrt{V(\hat{J}_z^2)V(\hat{J}_x^2)} + C(\hat{J}_z^2, \hat{J}_x^2) + (\langle \hat{J}_z \hat{J}_x + \hat{J}_x \hat{J}_z \rangle)^2}{4(\langle \hat{J}_z^2 \rangle - \langle \hat{J}_x^2 \rangle)^2}. \quad (38)
\end{align}

If the QST process is a beamsplitter with QST fraction $Q$, we can use Eqs. \[34\], \[31f\], and \[31g\] to put Eq. \[38\] in the form

\begin{align}
(\Delta \phi_{\min})^2 &= \frac{Q^2}{\mathcal{F}_b} + \frac{1}{4Q^3 \mathcal{F}_b} \left\{ \sqrt{\frac{1}{2} (1-Q) \left( N_t + Q(1-Q) [6(\mathcal{F}_b - 2N_t) - N_t^2] \right)} A(Q, \hat{N}_t) \\
&\quad + (1-Q) \left[ Q^2 (3\langle \hat{N}_t^3 \rangle - 2\mathcal{F}_b N_t) + 4Q(4-Q)(2+Q)(1-Q)\mathcal{F}_b \\
&\quad - 2N_t [5 + (1-Q) (QN_t - 6(1-Q))] \right] \right\}, \quad (39)
\end{align}

where

\begin{align}
A(Q, \hat{N}_t) &\equiv 8[Q \mathcal{F}_b + (1-Q)N_t][10 - 3Q(4-Q) - Q(\mathcal{F}_b + (1-Q)N_t)] \\
&\quad - 24[3 - Q(3-Q)]N_t + 3Q^2[Q\langle \hat{N}_t^3 \rangle + 4(2-Q)\langle \hat{N}_t^3 \rangle]. \quad (40)
\end{align}

and $\mathcal{F}_b = [V(\hat{N}_t) + N_t(N_t + 2)]/2$ is the quantum Fisher information information for perfect QST. Alternatively, we
can write the minimum phase sensitivity as
\[
(\Delta \phi_{\text{min}})^2 = \frac{Q^4}{F_a - (1 - Q)N_a} + \frac{1}{4(F_a - (1 - Q)N_a)^2} \left\{ \sqrt{\frac{1}{2} (1 - Q) \left( (1 - Q) (6F_a - N_a^2) - (5 - 6Q^2) N_a \right)} \tilde{A}(Q, N_t) + (1 - Q) \left[ 3Q^3 \langle \tilde{N}_t^3 \rangle + 4(4 - Q(2 + Q)(1 - Q)) F_a - 2N_a(F_a + 7 - 2Q^4) \right] \right\},
\]
where \( N_a = QN_t, \ F_a = Q^2F_b - (1 - Q)N_a, \) and
\[
\tilde{A}(Q, N_t) \equiv QA(Q, N_t) = 8F_a(10 - 3Q(4 - Q) - F_a) - 24(3 - Q(3 - Q))N_a + 3Q^3(Q(\tilde{N}_t^4) + 4(2 - Q)(\tilde{N}_t^3)).
\]

Importantly, when \( Q = 1, \) the term in the curly braces on the right-hand side of Eq. (39) or (41) vanishes, and we have \( \Delta \phi_{\text{min}} = 1/\sqrt{F_b}, \) so the phase sensitivity saturates the QCRB, as expected.

Since \( F_b = aN_a^2 + O(N_a^3) \) for some positive constant \( a, \) we can use the convexity relations \( \langle \tilde{N}_t^4 \rangle \geq \langle \tilde{N}_t \rangle^4 \) and \( \langle \tilde{N}_t^3 \rangle \geq \langle \tilde{N}_t \rangle^3 \) to determine that, to leading order in \( 1/N_a, \)
\[
(\Delta \phi_{\text{min}})^2 \gtrsim \frac{Q^4}{\alpha N_a^2} + (1 - Q) \left( \frac{\sqrt{2(6a - 1)(6a^2 - 3) + O(1/N_a)} + 2(3 - 2a) + O(1/N_a^2)}{8\alpha N_a} \right).
\]

Therefore, for any deviations from perfect QST on the order of \( (1 - Q) \gtrsim 1/N_a, \) the Heisenberg scaling is lost, and we return to the standard QNL scaling \( \Delta \phi_{\text{min}} \sim \sqrt{(1 - Q)/N_a}. \) Since the initial number of donor particles is typically and desired to be large, in practice Heisenberg scaling is lost for very small departures from perfect QST.

We demonstrate this point more concretely in Fig. 2 of the main text, where we plot the phase sensitivity for the specific cases where the donor modes are initially in a twin-Fock state or a two-mode squeezed vacuum state. For a twin-Fock state, \( \langle \tilde{N}_t \rangle = 0, \langle \tilde{N}_t^3 \rangle = N_t^3, \) and \( \langle \tilde{N}_t^4 \rangle = N_t^4, \) and the sensitivity at the optimal operating point is
\[
(\Delta \phi_{\text{TF}})^2 = \frac{1}{N_a(N_a + 2Q)^2} \left( 2(N_a + 2Q) + \sqrt{\frac{2}{3} (1 - Q)(1 + 2(1 - Q)(N_a - 3Q))} \tilde{A}_{\text{TF}}(Q, N_a) + 2(1 - Q) \left[ 1 + (N_a - 3Q)(N_a + 2) \right] \right),
\]
\[
\tilde{A}_{\text{TF}}(Q, N_a) = (N_a - 2)(N_a + 2)(N_a + 4) + 12(1 - Q) \left[ 2 + N_a(N_a + 3 - Q) \right].
\]

For two-mode squeezed vacuum, \( \langle \tilde{N}_t \rangle = N_t(N_t + 2), \langle \tilde{N}_t^3 \rangle = 2N_t(2 + 3V(\tilde{N}_t)), \) and \( \langle \tilde{N}_t^4 \rangle = 8N_t(N_t + 1)(1 + 3V(\tilde{N}_t)), \) so the sensitivity at the optimal operating point becomes
\[
(\Delta \phi_{\text{sq}})^2 = \frac{1}{2N_a(N_a + 2Q)^2} \left( 2(N_a + 2Q) + \sqrt{1 - Q \left[ 1 + 5(1 - Q)N_a \right]} \tilde{A}_{\text{sq}}(Q, N_a) + (1 - Q) \left[ 1 + N_a \left[ 5(1 - Q) + 8(N_a + 2Q) \right] \right] \right),
\]
\[
\tilde{A}_{\text{sq}}(Q, N_a) = (1 - Q) \left[ 1 - 10Q(1 - Q) + (N_a + 2Q)(9 - 5Q(2 - Q) + 8N_a(N_a + 2)) \right].
\]

**Phase sensitivity for the QST process when using information recycling**

If we use the technique of information recycling, the signal we are interested in is \( \hat{S} = (\hat{J}_z + \hat{L}_z)^2. \) Just as without recycling, there is no interference between sectors with different numbers of particles; the desired moments are thus averages over \( p_N = \rho_{NN}. \) Moreover, the anticorrelation of \( \hat{J}_z \) and \( \hat{L}_z, \) expressed by \( \langle \hat{J}_z + \hat{L}_z \rangle \hat{p}(t) = 0 = \)
\(-\hat{\rho}(t_1)(\hat{J}_y + \hat{L}_z),\) allows us to convert \(\hat{L}_z\) in these moments to \(\hat{J}_x\). The anticorrelation, as before, also implies that \(\hat{\rho}(t_1)\) is invariant under rotations about the \(z\) axis; in particular, a rotation by \(\pi\), which takes \(\hat{J}_x\) to \(-\hat{J}_x\), implies that all terms with an odd number of \(\hat{J}_x\) operators have vanishing expectation value. In converting \(\hat{L}_z\) to \(\hat{J}_x\), we introduce \(\hat{J}_y\) into our expressions, so this last rule becomes that the only nonvanishing moments are those for which the total power of \(\hat{J}_x\) and \(\hat{J}_y\) is even.

The mean and second moment of the signal are

\[
\langle \tilde{S} \rangle = \langle \hat{J}^2_x \rangle (\cos \phi - 1)^2 + \langle \hat{J}^2_y \rangle \sin^2 \phi, \tag{46}
\]

\[
\langle \tilde{S}^2 \rangle = \langle \hat{J}^4_x \rangle (\cos \phi - 1)^4 + \langle \hat{J}^4_y \rangle \sin^4 \phi + \langle \hat{J}^2_x \rangle \langle \hat{J}^2_x \rangle + \langle \hat{J}^2_y \rangle + 4 \langle \hat{J}_x \hat{J}_y \rangle \langle \hat{J}^2_x \rangle \sin^2 \phi (\cos \phi - 1)^2 + i(\langle \hat{J}_x \hat{J}_x \hat{J}_y \rangle - \langle \hat{J}_y \hat{J}_x \hat{J}_x \rangle) (2 \sin^2 \phi \cos \phi (\cos \phi - 1) + \langle \hat{J}^2_y \rangle \cos^2 \phi \sin^2 \phi. \tag{47}
\]

Using Eq. (46) and Eq. (47) in \((\Delta \phi)^2 = V(\tilde{S})/(\partial_\phi \langle \tilde{S} \rangle)^2\) and taking the limit as \(\phi \to 0\) gives

\[
(\Delta \phi_{\text{min}})^2 = \frac{\langle \hat{J}^2_y \rangle}{4 \langle \hat{J}^2_x \rangle^2}. \tag{48}
\]

Noting that \(\langle \hat{J}^2_y \rangle = \langle \hat{J}^2_x \rangle\), we recover \(\Delta \phi_{\text{min}} = \frac{1}{2 \langle \hat{J}^2_x \rangle^{1/2}}\).

Written in terms of conditional expectation values, Eqs. (46) and (47) become

\[
\langle \tilde{S} \rangle = \sum_{N=0}^{\infty} p_N \left( \frac{1}{2} \sin^2 \phi \langle \hat{N}_1 \rangle_N + \frac{1}{2} (\cos \phi - 1)^2 \langle \hat{N}_1^2 \rangle_N + \frac{1}{2} (\sin^2 \phi - (\cos \phi - 1)^2) \langle \hat{N}_1 \rangle_N \langle \hat{N}_1 \rangle_N \right), \tag{49}
\]

\[
\langle \tilde{S}^2 \rangle = \sum_{N=0}^{\infty} p_N \left( \frac{1}{2} \cos^2 \phi \sin^2 \phi - \frac{\sin^4 \phi}{4} \right) \langle \hat{N}_1 \rangle_N + \left( (1 - \cos \phi) \cos \phi \sin^2 \phi + \frac{3 \sin^4 \phi}{8} \right) \langle \hat{N}_1^2 \rangle_N \]

\[
+ \left( \frac{3}{4} \cos \phi - 1 \right)^2 \langle \hat{N}_1^3 \rangle_N + \frac{1}{8} (\cos \phi - 1)^4 \langle \hat{N}_1^4 \rangle_N \]

\[
+ \left( - \sin^2 \phi \cos \phi + \frac{6}{4} \cos^2 \phi \sin^2 \phi - \frac{\sin^4 \phi}{4} \right) \langle \hat{N}_1 \rangle_N \langle \hat{N}_1 \rangle_N \]

\[
+ \left( \frac{-3}{4} \cos \phi - 1 \right)^2 \sin^2 \phi + \left( \frac{3 \sin^4 \phi}{4} \right) \langle \hat{N}_1^2 \rangle_N \langle \hat{N}_1 \rangle_N \]

\[
+ \left( - \frac{1}{2} \cos \phi - 1 \right)^4 + \frac{3}{2} (\cos \phi - 1)^2 \sin^2 \phi \right) \langle \hat{N}_1^3 \rangle_N \langle \hat{N}_1 \rangle_N \]

\[
+ \left( \frac{3 (\cos \phi - 1)^4}{8} - \frac{3}{2} (\cos \phi - 1)^2 \sin^2 \phi + \frac{3 \sin^4 \phi}{8} \right) \langle \hat{N}_1^2 \rangle_N \langle \hat{N}_1^2 \rangle_N \). \tag{50}
\]

Specifying the QST process as a beamsplitter, we again use the identities in the previous section to get

\[
\langle \tilde{S} \rangle = \frac{1}{4} \sin^2 \phi \left( N_t Q + \frac{\langle \hat{N}_1^2 \rangle Q^2}{2} \right) - \frac{1}{4} N_t (Q - 1) Q (\cos \phi - 1)^2, \tag{51}
\]

\[
\langle \tilde{S}^2 \rangle = \sin^4 \phi \left( \frac{1}{16} N_t (Q - 3 Q^2) + \frac{1}{32} \langle \hat{N}_1^2 \rangle (3 Q^2 - 12 Q + 10) Q^2 - \frac{3}{32} \langle \hat{N}_1^3 \rangle (Q - 2) Q^3 + \frac{3 \langle \hat{N}_1^4 \rangle Q^4}{128} \right) \]

\[
+ \frac{6 \sin^2 \phi (\cos \phi - 1)^2}{16} N_t (2 Q^2 - 3 Q + 1) Q + \frac{1}{16} \langle \hat{N}_1 \rangle (Q - 1)^2 Q^2 - \frac{1}{32} \langle \hat{N}_1^3 \rangle (Q - 1)^2 Q^3 \]

\[
+ (\cos \phi - 1)^4 \left( \frac{3}{16} \langle \hat{N}_1^2 \rangle (Q - 1)^2 Q^2 - \frac{1}{16} N_t (Q - 1) Q (6 Q^2 - 6 Q + 1) \right) \]

\[
+ \frac{1}{4} \sin^2 \phi \cos^2 \phi \left( N_t Q + \frac{\langle \hat{N}_1^2 \rangle Q^2}{2} \right) + \frac{1}{2} N_t (Q - 1) Q \sin^2 \phi \cos \phi (\cos \phi - 1). \tag{52}
\]

As mentioned in the main text, the optimal operating point is \(\phi = 0\). To see this we, can expand \(\langle \tilde{S} \rangle\), \(\langle \tilde{S}^2 \rangle\) and \(\partial_\phi \langle \tilde{S} \rangle^2\) around \(\phi = 0\). As only even powers of \(\phi\) remain, we have an extremal point here; plotting the sensitivity confirms that \(\phi = 0\) is indeed a minimum, given by
\[ (\Delta \phi_{\text{min}})^2 = \frac{1}{4\langle J_2^2 \rangle} = \frac{2}{Q^2 \langle \tilde{N}_t^2 \rangle + 2QN_t} = \frac{2}{Q^2 V(N_t) + N_a (N_a + 2)} = \frac{1}{\mathcal{F}_a}. \]