KIRILLOV–RESHETIKHIN MODULES ASSOCIATED TO $G_2$

VYJAYANTHI CHARI AND ADRIANO MOURA

Abstract. We define and study the Kirillov–Reshetikhin modules for algebras of type $G_2$. We compute the graded character of these modules and verify that they are in accordance with the conjectures in [7], [8]. These results give the first complete description of families of Kirillov–Reshetikhin modules whose isotypical components have multiplicity bigger than one.

Introduction

In [2] we defined and studied a family of restricted modules for the current and twisted current algebras associated to a finite–dimensional classical simple Lie algebra $g$ and a diagram automorphism of $g$ of order two. These modules, which we called the restricted Kirillov–Reshetikhin modules, are given by generators and relations and were denoted by $KR^\sigma(m\omega_i)$, where $\sigma$ is the diagram automorphism, $i$ is a node of the Dynkin diagram of the subalgebra $g_0$ of $g$ consisting of the fixed points of $\sigma$, and $m$ is a non–negative integer. Here we understand $\sigma$ to be the identity in the untwisted case. They admit a natural grading which is compatible with the grading on the current algebras. In particular, the graded pieces are finite–dimensional modules for $g_0$. It was proved in [2] that, regarded as $g_0$–modules, there were no non–zero maps between the distinct graded pieces and, moreover, the multiplicity of any irreducible representation in a particular graded piece was at most one. In fact, the graded character was computed in [2] and verified to be in accordance with the conjectures in [7, Appendix A] and [8, Section 6] for the usual Kirillov–Reshetikhin modules for the corresponding quantum affine algebras. When $g_0$ is an exceptional Lie algebra, the conjectures in these papers make it clear that for some nodes of the Dynkin diagram one or both of the aforementioned properties of the graded pieces may fail. The modules $KR^\sigma(m\omega_i)$ are known to be isomorphic to the Demazure modules, further details can be found in [1], [2], [5], [6].

In this paper we define and study the modules for the current algebra associated to $G_2$ and to the twisted current algebra associated to $D_4$ and a diagram automorphism of order three. In both cases the fixed point subalgebra $g_0$ is of type $G_2$. We prove that the conjectures of [7] and [8] are true in these cases. In particular, there are now maps of $g_0$–modules between the distinct non–zero graded pieces for $KR^\sigma(m\omega_i)$ for some $i$ and the multiplicity of an irreducible module in a graded piece can be greater than one. Moreover, our result on the graded character of the module $KR(m\omega_1)$ for $G_2$ is actually an improvement on the conjectural graded–character formula in [7] which has some multiplicity–zero terms.

The overall scheme of the proof is very similar to the one in [2]: we prove that the conjectural character formula is an upper bound for the character and then we prove that it is also a lower

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bound. However, one runs into difficulty almost immediately as the underlying combinatorics is rather more complicated. In order to prove the upper bound we use an elementary but useful result on representations of the 3-dimensional Heisenberg algebra. For the lower bound, as in [2], we first study some “fundamental” Kirillov-Reshetikhin modules and then realize the other modules as a submodule of a tensor product of the fundamental ones. But this time the fundamental modules are too big to be constructed explicitly as in [2]. To solve this we use the notion of fusion product of modules of the current algebra, which was introduced and studied in [3], [4]. The second step, in which involves studying graded quotients of tensor products of the fundamental Kirillov-Reshetikhin modules, is really much more complicated, since one has to prove not only that a particular representation occurs in a given grade, but also one has to determine its multiplicity. Identifying these quotients and proving that the isotypical components occur is non-trivial, since the projection of the natural vectors do not generate the desired \( g_0 \)-submodule. To solve this part we use the explicit description of some highest-weight vectors in tensor products of representations of \( \mathfrak{sl}_2 \) and in tensor products of fundamental representations for \( g_0 \).

The paper is organized as follows. In section 1 we fix the basic notation and collect the results we will need for the proof. In section 2 we define the Kirillov-Reshetikhin modules, state the main theorem, and make the connection with the conjectures in [7] and [8]. We prove the theorem in sections 3 and 4.

1. Preliminaries

1.1. The Lie algebra \( G_2 \) and its representations. Throughout this paper \( g_0 \) will denote the Lie algebra of type \( G_2 \), \( h_0 \) a Cartan subalgebra of \( g_0 \) and \( \alpha_i, i = 1, 2 \), a set of simple roots where we assume that \( \alpha_1 \) is short and \( \alpha_2 \) is long. Let \( R_i^+ \) and \( R_s^+ \) be the set of positive long and positive short roots respectively,

\[
R_i^+ = \{ \alpha_2, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1 \}, \quad R_s^+ = \{ \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \}.
\]

Given \( \alpha \in R^+ \) we denote by \( x_\alpha^\pm \) any non-zero element of \( (g_0)^{\pm} \). The subalgebras \( n_0^{\pm} \) are defined in the obvious way by \( n_0^+ = \oplus_{\alpha \in R^+} C x_\alpha^+ \). Let \( \omega_i, i = 1, 2 \), be the fundamental weights and note that \( \omega_1 = 2\alpha_1 + \alpha_2 \) and \( \omega_2 = 2\alpha_1 + 3\alpha_2 \). Let \( P \) (resp. \( Q \)) be the integer lattice spanned by the fundamental weights (resp. simple roots) and let \( P^+ \) (resp. \( Q^+ \)) be the \( \mathbb{Z}_+ \) span of the fundamental weights (resp. simple roots). Fix elements \( h_{\alpha_i}, i = 1, 2 \), such that \( \omega_j(h_{\alpha_i}) = \delta_{ij} \) for \( i, j = 1, 2 \). Then it is easy to see that \( [x_{\alpha_i}^+, x_{\alpha_i}^-] \) is a non-zero multiple of \( h_{\alpha_i} \).

Given a finite-dimensional \( g_0 \)-module \( V \), we have

\[
V = \oplus_{\lambda \in \mathcal{P}} V_\lambda, \quad V_\lambda = \{ v \in V : hv = \lambda(h)v \forall h \in h \}.
\]

Let \( \text{wt}(V) = \{ \mu \in P : V_\mu \neq 0 \} \) and given \( 0 \neq v \in V_\mu \) set \( \text{wt}(v) = \mu \). Let \( \mathcal{Z}[P] \) be the integral group ring of \( P \) with basis \( e(\mu), \mu \in P \), and set

\[
\text{ch}(V) = \sum_{\mu \in \mathcal{P}} \dim(V_\mu)e(\mu).
\]

For \( \lambda \in P^+ \), let \( V(\lambda) \) be the irreducible \( g_0 \)-module with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). Thus \( V = U(g_0)v_\lambda \), where

\[
n_0^+ v_\lambda = 0, \quad hv = \lambda(h)v, \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1}v = 0.
\]
Note that
\[ \text{ch}(V(\omega_1)) = e(0) + \sum_{\pm \alpha \in R_+^d} e(\alpha), \quad \text{ch}(V(\omega_2)) = 2e(0) + \sum_{\alpha \in R^+} e(\alpha). \]

We shall need the following result which is trivially proved.

**Lemma.** Given \(0 \leq p \leq s \in \mathbb{Z}_+\), there exists \(a, b \in \mathbb{C}^\times\) such that the following holds in \(V(\omega_2)^{\otimes s} \otimes V(\omega_1)\):
\[ n_0^+ \left( (x_{\alpha_1+\alpha_2}^+ v_2^{\otimes s-p}) \otimes v_2^{\otimes p} \otimes v_1 + (ax_{\alpha_1+\alpha_2}^- + bx_{\alpha_1+\alpha_2}^-) (v_2^{\otimes s} \otimes v_1) \right) = 0. \]
\[ \square \]

1.2. The associated current and twisted current algebras. Given a Lie algebra \(\mathfrak{a}\), let \(\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]\) be the polynomial current algebra of \(\mathfrak{a}\) with bracket \([x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}\).

From now on, let \(\mathfrak{g}\) be a Lie algebra of type \(D_4\) and \(\sigma\) the automorphism of \(\mathfrak{g}\) induced by an automorphism of order three of the Dynkin diagram. Let \(\xi\) be a primitive cube root of unity. Then,
\[ \mathfrak{g} = \bigoplus_{j=0}^{2} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} : \sigma(x) = \xi^j x\}. \]

Notice that the notation \(\mathfrak{g}_0\) is unambiguous since it is known that the fixed point subalgebra of \(\sigma\) is isomorphic to \(G_2\). Further, the subspaces \(\mathfrak{g}_r, r = 1, 2\), are clearly representations of \(\mathfrak{g}_0\) and in fact \(\mathfrak{g}_r \cong V(\omega_1)\). For \(\alpha \in R_+^d\), we let \(y_{\alpha}^{\pm}, z_{\alpha}^{\pm}\) be any non-zero elements in \((\mathfrak{g}_1)_{\pm \alpha}\) and \((\mathfrak{g}_2)_{\pm \alpha}\), respectively.

Extend \(\sigma\) to an automorphism \(\sigma_t\) of \(\mathfrak{g}[t]\) by \(x \otimes t^s \to \sigma(x) \otimes \xi^s t^s\). Let \(\mathfrak{g}[t]^{\sigma}\) be the set of fixed points of \(\sigma_t\). Then,
\[ \mathfrak{g}[t]^{\sigma} = \mathfrak{g}_0 \otimes C[t^3] \oplus \mathfrak{g}_1 \otimes t C[t^3] \oplus \mathfrak{g}_2 \otimes t^2 C[t^3]. \]

Set
\[ n^{\pm}[t]^{\sigma} = n_0^{\pm} \otimes C[t^3] \bigoplus_{\alpha \in R_+^d} \left( C y_{\alpha}^{\pm} \otimes t C[t^3] \oplus C z_{\alpha}^{\pm} \otimes t^2 C[t^3] \right). \]

We shall use the fact that as vector spaces
\[ U(\mathfrak{g}_0[t]) \cong U(n_0^{\pm}[t]) U(\mathfrak{h}_0[t]) U(n_0^{\pm}[t]), \quad U(\mathfrak{g}[t]^{\sigma}) \cong U(n^{\pm}[t]^{\sigma}) U(\mathfrak{h}[t]^{\sigma}) U(n^{\pm}[t]^{\sigma}) \]
without further comment.

1.3. Graded modules and graded characters. The algebras \(\mathfrak{g}_0[t]\) and \(\mathfrak{g}[t]^{\sigma}\) are obviously \(\mathbb{Z}_+\)-graded algebras. Given a \(\mathbb{Z}_+\)-graded module \(V_t = \bigoplus_{n \in \mathbb{Z}_+} V_t[n]\) for \(\mathfrak{g}_0[t]\) or \(\mathfrak{g}[t]^{\sigma}\), it is easy to see that \(V_t[n]\) is a \(\mathfrak{g}_0\)-module. If \(V_t[n]\) is finite-dimensional for all \(n \in \mathbb{Z}_+\), the graded character of \(V_t\) is defined by
\[ \text{ch}_t(V_t) = \sum_{n \in \mathbb{Z}_+} t^n \text{ch}(V_t[n]) = \sum_{n \in \mathbb{Z}_+} t^n \left( \sum_{\mu \in P^+} m_{\mu,n}(V_t[n]) \text{ch}(V(\mu)) \right), \]
where \( m_{\mu,n}(V_t[n]) \) are given by
\[
V_t[n] \cong_{\mathfrak{g}_0} \bigoplus_{\mu \in P^+} V(\mu)^{\oplus m_{\mu,n}(V_t[n])}.
\]

Set
\[
V_t(n) = V_t/ \bigoplus_{s > n} V_t[s].
\]

We end this section with some results which are used crucially later in the paper.

1.4. A result on representations of \( \mathfrak{sl}_2 \). Let \( x^+, x^-, h \) be the standard basis for the Lie algebra \( \mathfrak{sl}_2 \) and let \( V(s) \) be the \((s + 1)\)-dimensional representation of \( \mathfrak{sl}_2 \) with highest weight vector \( v_s \).

**Lemma.** Given \( 0 \leq p \leq s \in \mathbb{Z}_+ \) and \( j \leq \min(p, s - p) \), there exist \( c_1, \ldots, c_j \in \mathbb{C}^\times \) such that the following holds in \( V(1)^{\otimes s} \).

\[
x^+ \left( \left( (x^-)^j v_1^{\otimes s-p} \right) \otimes v_1^{\otimes p} + \sum_{\ell=1}^{j} c_\ell (x^-)^\ell \left( \left( (x^-)^{j-\ell} v_1^{\otimes s-p} \right) \otimes v_1^{\otimes p} \right) \right) = 0.
\]

**Proof.** Notice first that the \( \mathfrak{sl}_2 \)-module of \( V(1)^{\otimes s} \) generated by \( v_1^{\otimes s-p} \otimes v_1^{\otimes p} \) can be identified with the submodule of \( V(s - p) \otimes V(p) \) generated by \( v_{s-p} \otimes v_p \). Hence it suffice to prove that for \( 0 \leq j \leq \min(p, s - p) \) there exist \( c_1, \ldots, c_j \in \mathbb{C}^\times \) such that

\[
x^+ \left( \left( (x^-)^j v_{s-p} \right) \otimes v_p + \sum_{\ell=1}^{j} c_\ell (x^-)^\ell \left( \left( (x^-)^{j-\ell} v_{s-p} \right) \otimes v_p \right) \right) = 0.
\]

But this is immediate from the Clebsch–Gordon formulas. \( \square \)

1.5. A result on representations of the Heisenberg algebra.

**Lemma.** Consider the three dimensional Heisenberg algebra \( \mathfrak{H} \) spanned by elements \( x, y, z \) where \( z \) is central and \( [x, y] = z \). Suppose that \( V \) is a representation of \( \mathfrak{H} \) and let \( 0 \neq v \in V \) be such that \( x^r v = 0 \). Then for all \( k, s \in \mathbb{Z}_+ \) the element \( y^k z^s v \) is in the span of elements of the form \( x^a y^b z^c v \) with \( a > 0 \) and \( 0 \leq c < r \).

**Proof.** Suppose first that \( r = 1 \), then

\[
y^k z^s v = y^k z^{s-1} xyv = y^k xyz^{s-1} v = xy^{k+1} z^{s-1} v - ky^k z^s v,
\]

i.e.,

\[
(k + 1)y^k z^s v = xy^{k+1} z^{s-1} v.
\]

Assume that we know the result for \( r' < r \). Then, we have

\[
y^k z^s v = y^k z^{s-1} xyv - y^k z^{s-1} yxv = xy^{k+1} z^{s-1} v - ky^k z^s v - y^{k+1} z^{s-1} x v,
\]

i.e.,

\[
(k + 1)y^k z^s v = xy^{k+1} z^{s-1} v - y^{k+1} z^{s-1} x v.
\]
Since \( x^{r-1}xv = 0 \) it follows by induction that \( y^{k+1}z^{s-1}xv \) is in the span of elements of the form \( x^ay^bz^cv \) with \( a > 0 \) and \( c < r - 1 \). But such elements are clearly in the span of elements of the form \( x^ay^bz^cv \) with \( a > 0 \) and \( c < r \). An induction on \( s \) again gives the result.

1.6. Fusion Products. We shall need the following result which was proved in [3], [4]. We state it in the form and in the case of interest to us.

**Proposition.** Let \( V_i, i = 1, 2 \), be finite–dimensional graded \( g_0[t] \) modules generated by elements \( v_i, i = 1, 2 \), satisfying the relations: \( n_0^+v_i = 0 \), \( h \otimes t^s = \delta_{r,0} \lambda_i(h)v_i \) for some \( \lambda_i \in P^+ \), \( i = 1, 2 \), and all \( h \in \mathfrak{h}_0 \). Then, there exists a graded \( g_0[t] \)–module denoted \( V_1 \ast V_2 \) which is generated by an element \( v \) satisfying:

\[
n_0^+v = 0, \quad h \otimes t^s = \delta_{r,0} (\lambda_1 + \lambda_2)(h)v, \quad \forall h \in \mathfrak{h}_0
\]

and

\[
V_1 \ast V_2 \cong_{g_0} V_1 \otimes V_2.
\]

□

2. The Kirillov–Reshetikhin modules

In this section we define and prove some elementary properties of the Kirillov–Reshetikhin modules for the algebras of type \( g_0[t] \) and \( g[t]^\sigma \).

2.1. The KR–modules for \( g_0[t] \).

**Definition.** For \( m \in \mathbb{Z}_+ \), let \( KR(m\omega_i) \) be the \( g_0[t] \)–module generated by an element \( v_i,m \) with relations,

\[
n_0^+[t]v_i,m = 0, \quad (h \otimes t^s)v_i,m = \delta_{s,0} m\omega_i(h)v_i,m,
\]

for all \( h \in \mathfrak{h}_0, s \in \mathbb{Z}_+ \), and

\[
x_{\alpha_j}^-v_i,m = 0, \quad i \neq j, \quad (x_{\alpha_i}^-)^{m+1}v_i,m = 0, \quad (x_{\alpha_i}^- \otimes t)v_i,m = 0.
\]

□

2.2. The KR–modules for \( g[t]^\sigma \).

**Definition.** For \( m \in \mathbb{Z}_+ \), let \( KR^\sigma(m\omega_i) \) be the \( g[t]^\sigma \)–module generated by an element \( v_i,m \) with relations,

\[
n^+[t]^\sigma v_i,m = 0, \quad (h \otimes t^s)v_i,m = \delta_{s,0} m\omega_i(h)v_i,m,
\]

for all \( h \otimes t^s \in \mathfrak{h}[t]^\sigma \), and

\[
x_{\alpha_j}v_i,m = (x_{\alpha_j}^-)^{m+1}v_i,m = (y_{\alpha_j}^- \otimes t)v_i,m = (x_{\alpha_j}^- \otimes t^3)v_i,m = 0,
\]

where \( j \neq i \).

□
2.3. The Main Theorem. Let $e_i \in \mathbb{Z}_+^4$, $1 \leq i \leq 4$ be the standard basis and set $0 = (0, 0, 0, 0)$. Define $\text{wt}, \text{wt}^\sigma : \mathbb{Z}_+^4 \to P^+$ and $\text{gr}, \text{gr}^\sigma : \mathbb{Z}_+^4 \to \mathbb{Z}_+$ by

\[(2.5) \quad \text{wt}(r) = (m - r_1 - 3r_2 - 3r_3)w_1 + (r_2 + r_3 - r_4)w_2, \quad \text{gr}(r) = r_1 + r_2 + 2r_3 + 2r_4,
\]

\[(2.6) \quad \text{wt}^\sigma(r) = (r_1 + r_2 - r_3)w_1 + (m - r_1 - r_2 - r_4)w_2, \quad \text{gr}^\sigma(r) = r_1 + 2r_2 + 2r_3 + 3r_4,
\]

where $r = (r_1, r_2, r_3, r_4)$. Set

\[(2.7) \quad A_1 = \{r \in \mathbb{Z}_+^4 : r_4 \leq r_2, 2r_1 + 3r_2 + 3r_3 \leq m\},
\]

\[(2.8) \quad A_2^\sigma = \{r \in \mathbb{Z}_+^4 : r_3 \leq r_1, r_1 + r_2 + r_3 + r_4 \leq m\}.
\]

The main result of this paper is the following.

**Theorem.** Let $m \in \mathbb{Z}_+$. The modules $KR(mw_1)$ and $KR^\sigma(mw_1)$ are $\mathbb{Z}_+$-graded and

\[(i) \quad \text{ch}_t(KR(mw_1)) = \sum_{r \in A_1} t^{\text{gr}(r)} \text{ch}(V(\text{wt}(r))), \quad \text{ch}_t(KR(mw_2)) = \sum_{r=0}^m t^{m-r} \text{ch}(V(rw_2)),
\]

\[(ii) \quad \text{ch}_t(KR^\sigma(mw_1)) = \sum_{r=0}^m t^{m-r} \text{ch}(V(rw_1)), \quad \text{ch}_t(KR^\sigma(mw_2)) = \sum_{r \in A_2^\sigma} t^{\text{gr}^\sigma(r)} \text{ch}(V(\text{wt}^\sigma(r))).
\]

As a consequence of the proof of the theorem we also have:

**Corollary.** (i) Let $m \in \mathbb{Z}_+$, $m = 3m_0 + m_1$. The module $KR(mw_1)$ is isomorphic to the submodule of $KR(3w_1)^{\otimes m_0} \otimes KR(m_1w_1)$ generated by the element $v_{1,3}^{m_0} \otimes v_{1,m_1}$.

(ii) The module $KR(mw_2)$ (resp. $KR^\sigma(mw_i)$, $i = 1, 2$) is isomorphic to the submodule of $KR(\omega_2)^{\otimes m}$ (resp. $(KR^\sigma(\omega_1))^\otimes m$) generated by the element $v_{2,1}^{m}$ (resp. $(v_{1,1}^\sigma)^\otimes m$).

We prove the theorem in the next two sections.

2.4. The connection with the conjectures in [7, 8]. The following formulas were conjectured in [7] Appendix A] and [8 Section 6].

\[\text{ch}_t(KR(mw_1)) = \sum_{k=0}^{[m/3]} \sum_{j=2k}^{m-k} p_{j,k}(t) \text{ch}(V((m - j - k)w_1 + kw_2)),\]

\[\text{ch}_t(KR(mw_2)) = \sum_{k=0}^{m} t^{m-k} \text{ch}(V(kw_2)),\]

\[\text{ch}_t(KR^\sigma(mw_1)) = \sum_{k=0}^{m} t^{m-k} \text{ch}(V(kw_1)),\]

\[\text{ch}_t(KR^\sigma(mw_2)) = \sum_{\substack{j, k \in \mathbb{Z}_+ \\cap k \geq j \leq m}} p_{j,k}^\sigma(t) \text{ch}(V(jw_1 + kw_2)),\]
where \([s]\) denotes the biggest integer smaller than or equal to \(s\),

\[
p_{j,k}(t) = \left(1 + \left[\frac{j - 2k}{3}\right] + \min(0, \left[\frac{m + k - 2j}{3}\right])\right) \sum_{s=0}^{k} t^{j-k+s},
\]

and

\[
p_{j,k}^\sigma(t) = (1 + \min(k, m - j - k)) \sum_{s=0}^{j} t^{3m - 2j - 3k + s}.
\]

For the modules \(KR(m\omega_2)\) and \(KR^\sigma(m\omega_1)\) it is clear that Theorem 2.3 establishes the conjectures.

In order to establish the conjecture for \(KR(m\omega_1)\), write \(m = 3m_0 + m_1\) with \(m_1 = 0, 1, 2\). Define an equivalence relation on \(\mathbb{Z}^4\) by \(r \sim r'\) iff \(wt(r) = wt(r')\) and \(gr(r) = gr(r')\). It is easy to see that \(r \sim r'\) iff \(r - r' = \ell(3e_1 - e_2 - e_4)\) for some \(\ell \in \mathbb{Z}\).

Let \(\bar{r}\) be the equivalence class of \(r\).

Let \(j, k \in \mathbb{Z}_+\) be such that \(0 \leq k \leq m_0 = [m/3], 2k \leq j \leq m - k\). Assume also that \(p_{j,k}(t) \neq 0\), i.e., \(r_4 + k \leq m_0 + [(m_1 - 2r_1)/3]\), where \(r_4 \in \mathbb{Z}_+\) and \(0 \leq r_1 \leq 2\) are defined by \(r_4 = [(j - 2k)/3]\) and \(j - 2k = r_1 + 3r_4\). For \(0 \leq s \leq k\), set \(r_{j,k,s} = (r_1, k + r_4 - s, s, r_4)\).

It is easy to check that,

(i) \(r_{j,k,s} \in A_1\)
(ii) \(wt(r_{j,k,s}) = (m - j - k)\omega_1 + k\omega_2\), \(gr(r_{j,k,s}) = j - k + s\),
(iii) \(# \bar{r}_{j,k,s} \cap A_1 = 1 + \left[\frac{j - 2k}{3}\right] + \min(0, \left[\frac{m + k - 2j}{3}\right])\).

Here \(#S\) is the cardinality of the set \(S\). In other words, we see that

\[
p_{j,k}(t) = \sum_{k=0}^{m_0} \sum_{j=2k}^{m-k} \sum_{s=0}^{k} (# \bar{r}_{j,k,s} \cap A_1) t^{gr(r_{j,k,s})}.
\]

Thus to show that the conjecture in [7] coincides with Theorem 2.3 in the case of \(KR(m\omega_1)\), it suffices to show that

\[
\{r_{j,k,s} : 0 \leq s \leq k, 2k \leq j \leq m - k\}
\]

is a complete set of representatives for the equivalence classes of \(A_1\), i.e.,

\[
A_1 = \bigcup_{0 \leq s \leq k \leq m_0, 2k \leq j \leq m - k} \bar{r}_{j,k,s} \cap A_1.
\]

But this is now easy to do.

In the case of \(KR^\sigma(m\omega_2)\) we proceed similarly. Namely, we define an equivalence relation on \(\mathbb{Z}^4\) by \(r \sim r'\) iff \(wt^\sigma(r) = wt^\sigma(r')\), \(gr^\sigma(r) = gr^\sigma(r')\)

and we let \(\bar{r}\) be the equivalence class of \(r\).
It is easy to see that

\[ r \sim r' \iff r - r' = \ell (e_1 + e_3 - e_4) \quad \text{for some } \ell \in \mathbb{Z}. \]

Given \( j, k, s \in \mathbb{Z}_+ \) satisfying \( j + k \leq m, \ 0 \leq s \leq j \), set \( r_{j,k,s} = (j - s, s, 0, m - j - k) \). Then,

(i) \( r_{j,k,s} \in A_2^\sigma \)
(ii) \( \text{wt}^\sigma(r_{j,k,s}) = j\omega_1 + k\omega_2 \), \( \text{gr}^\sigma(r_{j,k}) = 3m - 2j - 3k + s \),
(iii) \( \# r_{j,k,s} \cap A_2^\sigma = 1 + \min(k, m - j - k) \).

In other words, we see that

\[ p_{j,k}(t) = \sum_{j+k=0}^{m} \sum_{s=0}^{j} (\# r_{j,k,s} \cap A_2^\sigma) t^{\text{gr}(r_{j,k,s})}. \]

Thus to show that the conjecture in [7] coincides with Theorem 2.3 in the case of \( KR^\sigma(m\omega_2) \), it suffices to show that

\( \{r_{j,k,s} : 0 \leq j + k \leq m, \ 0 \leq s \leq j\} \),

is a complete set of representatives for the equivalence classes of \( A_2^\sigma \), i.e.,

\[ A_2^\sigma = \bigcup_{0 \leq j+k \leq m} \bigcup_{0 \leq s \leq j} r_{j,k,s} \cap A_2^\sigma, \]

which is easily done.

2.5. We conclude this section with some elementary properties of the modules \( KR(m\omega_i) \) and \( KR^\sigma(m\omega_i) \). The proof of the next proposition is standard (see [2]) and we omit the details.

**Proposition.** Let \( m \in \mathbb{Z}_+ \), \( i = 1, 2 \), and assume that \( K_m \) (resp. \( v_m \)) is either \( KR(m\omega_i) \) or \( KR^\sigma(m\omega_i) \) (resp. \( v_{i,m} \) or \( v_{i,m}^\sigma \)).

(i) \( K_0 \cong \mathbb{C} \).
(ii) For all \( \alpha \in \mathfrak{h}_0^+ \), we have

\[ (x_\alpha^- \otimes 1)^{m\omega_i(h_\alpha)+1} v_m = 0. \]

(iii) We have

\[ K_m = \bigoplus_{\mu \in \mathfrak{h}_0^+} (K_m)_\mu \]

and \( (K_m)_\mu \neq 0 \) only if \( \mu \in m\omega_i - Q_0^+ \).

(iv) Regarded as a \( \mathfrak{g}_0 \)-module, \( K_m \) and \( K_m[s], s \in \mathbb{Z}_+ \), are isomorphic to a direct sum of irreducible finite-dimensional representations.

(v) For all \( 0 \leq r \leq m \), there exists a canonical homomorphism \( K_m \to K_r \otimes K_{m-r} \) of graded \( \mathfrak{g}_0[t] \)-modules (resp. \( \mathfrak{g}[t]^\sigma \)-modules) such that \( v_m \mapsto v_r \otimes v_{m-r} \).

**Corollary.**

\[ \text{ch}_t(K_m) = \sum_{r \in \mathbb{Z}_+} t^r \left( \sum_{\mu \in P^+} m_{\mu,r}(K_m) \text{ch}(V(\mu)) \right) \]

for some \( m_{\mu,r}(K_m) \in \mathbb{Z}_+ \).
Lemma. Let $m \in \mathbb{Z}_+$, $i = 1, 2$. Let $\alpha \in R^+_0$ and assume that $\alpha = s_i \alpha_i + s_j \alpha_j$, $i \neq j$.

(i) In $KR(m \omega_i)$ we have
\[(x^-_{\alpha} \otimes t^r)v_{i,m} = 0 \quad \forall \quad r \geq s_i.\]

(ii) In $KR^\sigma(m \omega_2)$ we have
\[(x^-_{\alpha} \otimes t^{3r})v^\sigma_{2,m} = (y^-_{\alpha} \otimes t^{3r+1})v^\sigma_{2,m} = (z^-_{\alpha} \otimes t^{3r+2})v^\sigma_{2,m} = 0 \quad \forall \quad r \geq s_2.\]

(iii) In $KR^\sigma(m \omega_1)$ we have
\[(x^-_{\alpha} \otimes t^{3s})v^\sigma_{1,m} = (y^-_{\alpha} \otimes t^{3s-2})v^\sigma_{1,m} = (z^-_{\alpha} \otimes t^{3s-1})v^\sigma_{1,m} = 0 \quad \forall \quad r \geq s_1, \ s \geq \min(1, s_1).\]
Here we set $y^-_{\alpha} = z^-_{\alpha} = 0$ if $\alpha$ is a long root. □

3. Upper bounds

3.1. The main result of this section is the following.

Proposition. Let $\mu \in P^+$, $k \in \mathbb{Z}_+$.

(i) We have
\[m_{\mu,k}(KR(m \omega_1)) \leq \# \{ r \in A_1 : (\mu, k) = (\text{wt}(r), \text{gr}(r)) \},\]
\[m_{\mu,k}(KR(m \omega_2)) \leq 1 \quad \text{and} \quad m_{\mu,k}(KR(m \omega_2)) = 0 \quad \text{if} \quad \mu \neq (m-k)\omega_2.\]

(ii) We have
\[m_{\mu,k}(KR^\sigma(m \omega_1)) \leq 1 \quad \text{and} \quad m_{\mu,k}(KR^\sigma(m \omega_1)) = 0 \quad \text{if} \quad \mu \neq (m-k)\omega_1.\]
\[m_{\mu,k}(KR^\sigma(m \omega_2)) \leq \# \{ r \in A_2^\sigma : (\mu, k) = (\text{wt}^\sigma(r), \text{gr}^\sigma(r)) \}.\]

The proposition is proved in the rest of this section.

3.2. The case of $KR(m \omega_2)$ and $KR^\sigma(m \omega_1)$. We fix an ordered basis of $\mathfrak{n}_0^- [t]$ as follows: the basis consists of elements in the set
\[\{ x^-_{\alpha} \otimes t^s : \alpha \in R^+, s \in \mathbb{Z}_+ \},\]
with any total order that satisfies $x^-_{\alpha} \otimes t^s < x^-_{\beta} \otimes t^r$ if $s < r$ for all $\alpha, \beta \in R^+$. An application of the PBW theorem and Lemma 2.6(i) shows that
\[KR(m \omega_2) = \sum_{r \in \mathbb{Z}_+} U(g_0)(x^-_{3\alpha_1+2\alpha_2} \otimes t)^r v_{2,m},\]
and that $\mathfrak{n}_0^+(x^-_{3\alpha_1+2\alpha_2} \otimes t)^r v_{2,m} = 0$. This immediately implies that
\[KR(m \omega_2) = \bigoplus_{r \in \mathbb{Z}_+} t^r V((m-r)\omega_2)^{\otimes m_r},\]
where $0 \leq m_r \leq 1$. 
We fix an ordered basis of \((n^-)[t]^\sigma\) as follows: the basis consists of elements in the set
\[
\{X^-_\alpha \otimes t^s : \alpha \in R^+, s \in \mathbb{Z}_+\},
\]
where \(X^-_\alpha \in \{x^-_\alpha, y^-_\alpha, z^-_\alpha\}\) and \(s\) are such that \(X^-_\alpha \otimes t^s \in g[t]^\sigma\) with any total order that satisfies \(X^-_\alpha \otimes t^s < X^-_\beta \otimes t^r\) if \(s < r\) for all \(\alpha, \beta \in R^+\). Using Lemma 2.6(iii) and the Poincare Birkhoff–Witt basis we see that
\[
KR^\sigma(m\omega_1) = \sum_{r \in \mathbb{Z}_+} U(g_0)(y^-_{2\alpha+2} \otimes t)^{r} v^\sigma_{1,m}, \quad n^+_0(y^-_{2\alpha+2} \otimes t)^{r} v^\sigma_{1,m} = 0,
\]
and the proposition follows as before in this case.

### 3.3. The case of \(KR(m\omega_1)\) and \(KR^\sigma(m\omega_2)\)

We now fix an ordered basis of \(n^-[t]\) as follows: the basis consists of elements in the set
\[
\{x^-_\alpha \otimes t^s : \alpha \in R^+, s \in \mathbb{Z}_+\}.
\]
Fix any total order on this set that satisfies the following:

(i) for all \(\alpha, \beta \in R^+\) and \(r > 0\), we have \(x^-_\alpha < x^-_\beta \otimes t^r\),

(ii) further, we have
\[
x^-_{3\alpha+2\beta} \otimes t^2 < x^-_{3\alpha+2\alpha} \otimes t^2 < x^-_{3\alpha+2\alpha} \otimes t < x^-_{2\alpha+1+2\alpha} \otimes t < x^-_{3\alpha+2\alpha} \otimes t < x^-_\beta \otimes t^s,
\]
for all \((\beta, s)\) with \(\beta \in R^+, s > 0\) and
\[
(\beta, s) \notin \{(3\alpha + 2\alpha, 2), (3\alpha + 2\alpha, 2), (3\alpha + 2\alpha, 1), (3\alpha + 2\alpha, 1), (2\alpha + 2\alpha, 1)\}.
\]
For \(n^-[t]^\sigma\) we adopt a similar notation. Set \(X^-_\beta \in \{x^-_\beta, y^-_\beta, z^-_\beta\}\),

(i) for all \(\alpha, \beta \in R^+\) and \(r > 0\), we have \(x^-_\alpha < X^-_\beta \otimes t^r\),

(ii) further, we have
\[
x^-_{3\alpha+1+2\alpha} \otimes t^3 < z^-_{2\alpha+1+2\alpha} \otimes t^2 < z^-_{2\alpha+1+2\alpha} \otimes t < y^-_{2\alpha+1+2\alpha} \otimes t < y^-_{2\alpha+1+2\alpha} \otimes t < X^-_\beta \otimes t^s,
\]
for all \((\beta, s)\) with \(\beta \in R^+, s > 0\) and
\[
(\beta, s) \notin \{(3\alpha + 2\alpha, 2), (2\alpha + 2\alpha, 2), (2\alpha + 2\alpha, 1), (2\alpha + 2\alpha, 1), (\alpha + 2\alpha, 1)\}.
\]
Given \(r \in \mathbb{Z}_+^1\), let \(y_r \in U(g_0[t])\) and \(y^\sigma_r \in U(g[t]^\sigma)\) be defined by
\[
y_r = (x^-_{3\alpha+1+2\alpha} \otimes t^2)^{r_1}(x^-_{3\alpha+1+2\alpha} \otimes t^2)^{r_2}(x^-_{2\alpha+2} \otimes t)^{r_3},
\]
and
\[
y^\sigma_r = (z^-_{2\alpha+1+2\alpha} \otimes t^2)^{r_1}(z^-_{2\alpha+1+2\alpha} \otimes t^2)^{r_2}(y^-_{2\alpha+2} \otimes t)^{r_3},
\]
respectively. If \(r \notin \mathbb{Z}_+^1\), then we set \(y_r = 0\) (resp. \(y^\sigma_r = 0\)).

Using Lemma 2.6 and the PBW theorem we see that
\[
KR(m\omega_1) = \sum_{s \in \mathbb{Z}_+, r \in \mathbb{Z}_+^1} U(n_0^-) y_r(x^-_{3\alpha+1+2\alpha} \otimes t)^{r} v_{1,m},
\]
\[
KR^\sigma(m\omega_2) = \sum_{s \in \mathbb{Z}_+, r \in \mathbb{Z}_+^1} U(n_0^-) y^\sigma_r(y^-_{2\alpha+2} \otimes t)^{r} v^\sigma_{2,m}.
\]
It is easy to see that the relations \( x_{\alpha_2}^{-1} v_{1,m} = 0 \) and \( x_{\alpha_1}^{-1} v_{2,m} = 0 \) imply the following:

\[
(x_{\alpha_1+2\alpha_2}^{-1} \otimes t)^s (x_{\alpha_1+2\alpha_2}^{-1} \otimes t)^r v_{1,m} \in C \left( (x_{\alpha_2}^{-1})^s (x_{\alpha_1+2\alpha_2}^{-1} \otimes t)^{s+r} v_{1,m} \right),
\]

\[
(y_{\alpha_1+2\alpha_2}^{-1} \otimes t)^s (y_{\alpha_1+2\alpha_2}^{-1} \otimes t)^r v_{2,m} \in C \left( (x_{\alpha_2}^{-1})^s (y_{\alpha_1+2\alpha_2}^{-1} \otimes t)^{s+r} v_{2,m} \right).
\]

Since

\[
[x_{\alpha_2}^{-1}, y_{\alpha_1+2\alpha_2}^{-1} \otimes t]^s v_{1,m} \in \sum_{s \in \mathbb{Z}_+^1} C y_s,
\]

we see that

\[
y_{\alpha_1}^{-1} (x_{\alpha_1+2\alpha_2}^{-1} \otimes t)^s v_{1,m} \in \sum_{s \in \mathbb{Z}_+^1} U(n_0^{-1}) y_s v_{1,m},
\]

\[
y_{\alpha_1}^{-1} (y_{\alpha_1+2\alpha_2}^{-1} \otimes t)^s v_{2,m} \in \sum_{r \in \mathbb{Z}_+^1} U(n_0^{-1}) y_s v_{2,m}.
\]

In other words we have proved that

\[
(3.1) \quad KR(m\omega) = \sum_{r \in \mathbb{Z}_+^1} U(n_0^{-1}) y_{\alpha_1}^{-1} v_{1,m} = \sum_{r \in \mathbb{Z}_+^1} U(g_0) y_{\alpha_1}^{-1} v_{1,m},
\]

\[
(3.2) \quad KR^g(m\omega_2) = \sum_{r \in \mathbb{Z}_+^1} U(n_0^{-1}) y_{\alpha_1}^{-1} v_{2,m} = \sum_{r \in \mathbb{Z}_+^1} U(g_0) y_{\alpha_1}^{-1} v_{2,m}.
\]

3.4.

**Lemma.** Let \( k \in \mathbb{Z}_+ \).

(i) We have,

\[
(3.3) \quad (x_{\alpha_1}^{-1})^k y_{\alpha_1}^{-1} v_{1,m} \in \sum_{j=0}^{k} C y_{\alpha_1}^{-1} (k-j) e_1 + (j-k) e_2 + j e_3 v_{1,m}, \quad (x_{\alpha_2}^{-1})^k y_{\alpha_1}^{-1} v \in \sum_{j=0}^{k} C y_{\alpha_1}^{-1} (2k-j) e_1 + (k-j) e_3 + j e_4 v_{2,m}.
\]

The elements \( \{ y_{\alpha_1}^{-1} v_{1,m} : r \in \mathbb{Z}_+^1 \} \) span a finite-dimensional representation of \( n_0^+ \) and hence \( KR(m\omega_1) \) is a finite-dimensional \( g_0[t] \)-module.

(ii) We have,

\[
(3.4) \quad (x_{\alpha_1}^{-1})^k y_{\alpha_1}^{-1} v_{2,m} \in \sum_{j=0}^{k} C y_{\alpha_1}^{-1} (2k-j) e_1 + (k-j) e_3 + j e_4 v_{2,m}.
\]

The elements \( \{ y_{\alpha_1}^{-1} v_{2,m} : r \in \mathbb{Z}_+^1 \} \) span a finite-dimensional representation of \( n_0^+ \) and hence \( KR^g(m\omega_2) \) is a finite-dimensional \( g[t] \)-module.

**Proof.** We prove (ii), the proof of (i) is identical. The observation that

\[
[x_{\alpha_1}^{-1}, z_{2\alpha_1+\alpha_2}^{-1}] = 0, \quad [x_{\alpha_1}^{-1}, z_{2\alpha_1+\alpha_2} \otimes t^2] \in C z_{2\alpha_1+\alpha_2} \otimes t^2,
\]

proves the first inclusion in (3.3). To prove the second, we begin by observing that

\[
[x_{\alpha_1}^{-1}, z_{2\alpha_1+\alpha_2} \otimes t^2] = 0
\]

and

\[
[x_{\alpha_1}^{-1}, z_{\alpha_1+\alpha_2} \otimes t^3] \in C x_{3\alpha_1+\alpha_2} \otimes t^3,
\]

\[
[x_{\alpha_1}^{-1}, z_{\alpha_1+\alpha_2} \otimes t^2] \in C z_{\alpha_1} \otimes t^2,
\]

\[
[x_{\alpha_1}^{-1}, y_{\alpha_1+\alpha_2} \otimes t] \in C y_{\alpha_1} \otimes t.
\]
Lemma 2.6 and the commutation relations in $g[t]^\sigma$ now prove that for any $s \in \mathbb{Z}_1^+$,
\[(x_{3a_1+\alpha_2} t^3) y_s^\sigma v_{2,m} = 0, \quad (z_{\alpha_1} t^2) y_s^\sigma v_{2,m} = 0.\]

Since $[x_2^+, (y_{\alpha_1+\alpha_2} t)^r]$ is in the span of the elements $(y_{\alpha_1+\alpha_2} t)^{r-1} (y_{\alpha_1} t)$, $(z_{\alpha_1+\alpha_2} t^2) (y_{\alpha_1+\alpha_2} t)^{r-2}$ and $(x_{\alpha}^{-} t^3) (y_{\alpha_1+\alpha_2} t)^{r-3}$, we find that $(3.4)$ follows for $k = 1$ from a further application of Lemma 2.6.

In particular, we have shown that the subspace spanned by the elements \{ $v_{r}^\sigma v_{2,m} : r \in \mathbb{Z}_1^+$ \} is a representation of $n_0^\sigma$. To see that the subspace is finite-dimensional, note that for each $\mu \in P$, the set \{ $r \in \mathbb{Z}_1^+ : wt(y_r) = \mu$ \} is finite. Hence if the subspace was infinite-dimensional, there would exist an infinite family of distinct elements $y_r$, $j \geq 1$ with $wt(y_r), y_k$ if $j \neq k$. Since $KR(m_\omega)$ is a direct sum of finite-dimensional irreducible $g_0$-modules, it follows that there must exist an infinite family of distinct elements $\mu_j \in P^+$ such that $KR^\sigma(m_\omega)_{\mu_j} \neq 0$. But this is impossible since there are only finitely many elements in the $m_\omega - Q^+$. The fact that $KR^\sigma(m_\omega)$ is finite-dimensional is immediate from (3.1).

3.5. Let $\pi_0 : KR(m_\omega) \to U(g_0) v_{1,m}$ be the canonical projection of $g_0$-modules so that we have $KR(m_\omega) = U(g_0) v_{1,m} \oplus ker(\pi_0)$. If $\pi_0$ is injective, the proposition is proved. Otherwise there exists $r_1 \in \mathbb{Z}_1^+$ such that the element $y_{r_1} v_{1,m}$ has a non-zero projection onto $ker(\pi_0)$. Moreover, $r_1$ can be chosen so that: $y_s v_{1,m} \in U(g_0) v_{1,m}$ if $s \in \mathbb{Z}_1^+$ is such that either $wt(s) - wt(r_1) \in Q^+ \setminus \{0\}$ or $wt(s) = wt(r_1)$ with $s < r_1$, where $<$ is the lexicographic ordering on $\mathbb{Z}_1^+$ given by
\[(r_1, r_2, r_3, r_4) < (s_1, s_2, s_3, s_4) \iff r_k < s_k, \quad \text{where} \quad k = \min\{1 \leq p \leq 4 : r_p \neq s_p\}.\]

Let $v_1$ be the projection of $y_{r_1} v_{1,m}$ onto $ker(\pi_0)$. Using Lemma 3.4 we see that
\[n_0^+ y_{r_1} v_{1,m} \in U(g_0) v_{1,m},\]
and hence $n_0^+ v_{1,m} = 0$.

Repeating this argument, we see that we can choose $r_0, \ldots, r_k \in \mathbb{Z}_1^+$ and elements $v_j \in KR(m_\omega)_{wt(r_j)}, 0 \leq j \leq k$ such that:

(i) $wt(r_0) = m_\omega \geq wt(r_1) \geq \cdots \geq wt(r_k)$,
(ii) $n_0^+ v_j = 0, 0 \leq j < k, v_0 = v_{1,m}$

such that the following holds:

(a) as $g_0$-modules $KR(m_\omega) = \bigoplus_{j=0}^k U(g_0) v_j \cong \bigoplus_{j=0}^k V(wt(r_j))$,
(b) the projection of $y_{r_j}$ onto $U(g_0) v_j$ is $v_j$. Moreover if $s \in \mathbb{Z}_1^+$ is such that either $wt(s) - wt(r_j) \in Q^+$ or $wt(s) = wt(r_j)$, then $y_s v_{1,m} \in \bigoplus_{p=0}^{j-1} U(g_0) v_p$.

Proposition 3.4 is proved for $KR(m_\omega)$ if we show that $r_j \in A_1$ for all $0 \leq j \leq k$. We first prove that if $r_j = (r_1, r_2, r_3, r_4)$, then $r_4 \leq r_2$. For this, note that
\[x_{\alpha_1}^+(x_{3\alpha_1+\alpha_2} t^2) (x_{2\alpha_1+\alpha_2} t) r_1 v_{1,m} = 0, \quad (x_{\alpha_2}^{-}) r_2 (x_{3\alpha_1+\alpha_2} t^2) (x_{2\alpha_1+\alpha_2} t) r_1 v_{1,m} = 0.\]

The subalgebra of $g_0[t]$ spanned by $(x_{2\alpha_1+\alpha_2} t^2)$, $(x_{3\alpha_1+\alpha_2} t^2)$ and $x_{\alpha_2}^{-}$ is isomorphic to the three dimensional Heisenberg algebra. Lemma 1.6 now implies that if $r_4 > r_2$, then $y_{r_j} v_{1,m}$ is in the span of elements of the form $(x_{\alpha_2}^{-})^a y_s v_{1,m}$ with $a > 0$ and $wt(s) > wt(r_j)$. But
such elements have zero projection on \( U(g_0)v_j \) and hence \( y_{r_j} \) has zero projection onto \( U(g_0)v_j \) which contradicts (b).

Next suppose that there exists \( 0 \leq j \leq k \) such that \( r_j = (r_1, r_2, r_3, r_4) \) and \( 2r_1 + 3r_2 + 3r_3 > m \). Setting, \( s = r_{j_0} + r_1e_2 - r_1e_1 \), we see from Lemma 3.4 that

\[
(r_{\alpha_1}^+)^t y_s v_{1,m} = y_{r_{j_0}} v_{1,m} + \sum_{p=1}^{r_1} y_{r_{j_0}} - p(3e_1 - e_2 - e_4) v_{1,m}.
\]

Now, \( r_{j_0} - p(3e_1 - e_2 - e_4) < r_{j_0} \) if \( p \geq 1 \) it follows that the projection of \( y_{r_{j_0}} - p(3e_1 - e_2 - e_4) v_{1,m} \) for \( p \geq 1 \) onto \( U(g_0)v_j \) is zero. Since \( y_{r_{j_0}} v_{1,m} \) has a non-zero projection onto \( U(g_0)v_{j_0} \), we see using (3.5) that \( y_s v_{1,m} \) also has a non-zero projection onto \( U(g_0)v_{j_0} \). Now,

\[
\text{wt}(s) = \text{wt}(r_{j_0}) - r_1\alpha_1 = (m - 3r_1 - 3r_2 - 3r_3)\omega_1 + (r_1 + r_2 + r_3 - s_4)\omega_2.
\]

Since \( 2r_1 + 3r_2 + 3r_3 > m \) it follows that \( \text{wt}(s) \) is not dominant integral and so we must have that

\[
\text{wt}(s) + (3r_3 + 3r_2 + 3r_1 - m)\alpha_1 \in \text{wt}(U(g_0)v_{j_0}) \subset \text{wt}(r_{j_0}) - Q^+,
\]

i.e.,

\[
\text{wt}(r_{j_0}) + (2r_1 + 3r_2 + 3r_3 - m)\alpha_1 \in \text{wt}(r_{j_0}) - Q^+
\]

which is impossible. Proposition 3.1 is proved for \( KR(m\omega_1) \). The result is deduced for \( KR^\sigma(m\omega_2) \) in exactly the same way. One works with the Heisenberg algebra spanned by \( x_{\alpha_1}^- \), \( z_{\alpha_1 + \alpha_2}^2 \otimes t^2 \) and \( z_{2\alpha_1 + \alpha_2}^2 \otimes t^2 \) and we omit the details.

4. Lower Bounds

4.1. The main result of this section is the following Proposition which together with Proposition 3.1 proves Theorem 2.3.

**Proposition.**

(i) We have

\[
m_{\mu,k}(KR(m\omega_1)) \geq \# \{ r \in A_1 : (\mu, k) = (\text{wt}(r), \text{gr}(r)) \},
\]

\[
m_{(m-k)\omega_2,k}(KR(m\omega_2)) \geq 1.
\]

(ii) We have

\[
m_{(m-k)\omega_1,k}(KR^\sigma(m\omega_1)) \geq 1.
\]

\[
m_{\mu,k}(KR^\sigma(m\omega_2)) \geq \# \{ r \in A_2^\sigma : (\mu, k) = (\text{wt}^\sigma(r), \text{gr}^\sigma(r)) \}.
\]
4.2. The modules $KR(m\omega_2)$. Note that the $\mathfrak{g}_0$ module $V(\omega_2)$ is isomorphic to the adjoint representation of $\mathfrak{g}_0$. Let $\langle , \rangle$ be the Killing form on $\mathfrak{g}_0$. If $m = 1$, then it is straightforward to check that the formulas

$$(x \otimes t^r)(y, a) = (\delta_{r,0}[x, y], \delta_{r,1} < x, y >),$$

define a graded $\mathfrak{g}_0[t]$–module structure on $K = V(\omega_2) \oplus \mathbb{C}$ with $K[0] = V(\omega_2)$, $K[1] = \mathbb{C}$. It is trivial to check that $K$ is a $\mathfrak{g}_0[t]$ module quotient of $KR(\omega_2)$ which proves the proposition for $m = 1$. Moreover, the assignment $x_{3\alpha_1 + 2\alpha_2}^+ \mapsto v_{2,1}$ extends to a $\mathfrak{g}_0[t]$–module isomorphism $K \cong KR(\omega_2)$ and hence,

$$(x_{3\alpha_1 + 2\alpha_2}^+ \otimes t)v_{2,m} \neq 0, \quad n_0^+(x_{3\alpha_1 + 2\alpha_2}^- \otimes t)v_{2,m} = 0.$$

For $m > 1$, consider the module

$$K_m = K(0)^{\otimes m-k} \otimes K^\otimes k.$$ 

Let $\bar{v}_{2,m}$ be the image of $v_{2,m}$ in $K(0)$ and set

$$\bar{v}_m = v_{2,m}^{\otimes m-k} \otimes v_{2,m}^\otimes k.$$

Using the explicit description of the module $K$, it is now easy to see that the module $K_m = \mathcal{U}(\mathfrak{g}_0[t])\bar{v}_m$ is a quotient of $KR(m\omega_2)$. Moreover,

$$(x_{3\alpha_1 + 2\alpha_2}^- \otimes t)^k\bar{v}_m \neq 0, \quad n_0^+((x_{3\alpha_1 + 2\alpha_2}^- \otimes t)^k\bar{v}_m) = 0,$$

which proves that $m_{(m-k)\omega_2,k}(K_m) \neq 0$. Since $KR(m\omega_2)$ is a semisimple $\mathfrak{g}_0$–module it follows that $m_{(m-k)\omega_2,k}(KR(m\omega_2)) \neq 0$, thus proving the proposition.

4.3. The modules $KR(m\omega_1), \ 1 \leq m \leq 3$. Using Proposition 3.1 we see that $m_{\mu,k}(KR(\omega_1)) = 0$ if $k \neq 0$. Since the formula

$$(x \otimes t^r)v = \delta_{r,0}xv, \ \forall \ x \in \mathfrak{g}_0, \ v \in V(\omega_1),$$

defined a $\mathfrak{g}_0[t]$–module action on $V(\omega_1)$ which makes it a quotient of $KR(\omega_1)$, we are done.

For $m = 2$, we see from Proposition 3.1 that

$$m_{\mu,k}(KR(2\omega_1)) = 0, \ (\mu, k) \notin \{(2\omega_1, 0) \ (\omega_1, 1)\}.$$ 

Consider the fusion product $K = KR(\omega_1) \ast KR(\omega_1)$. Using Proposition 1.6 we see that $\bar{v} = (x_{\alpha_1}^- \otimes t)(v_{1,m} \ast v_{1,m})$ is a non–zero element of $K$ and moreover,

$$n_0^+[t]\bar{v} = 0, \quad (h_0 \otimes t\mathcal{C}[t])\bar{v} = 0, \quad h\bar{v} = \omega_2(h)v, \quad (x_{\alpha_2}^- \otimes t)\bar{v} = 0,$$

for all $h \in \mathfrak{h}_0$. In other words, $\tilde{K} = \mathcal{U}(\mathfrak{g}_0[t])\bar{v}$ is a graded $\mathfrak{g}_0[t]$–module quotient of $KR(\omega_2)$, and it follows from Section 4.2 that either

$$\text{ch}_t(\tilde{K}) = \text{ch}(V(\omega_2)) \text{ or } \text{ch}_t(\tilde{K}) = \text{ch}(V(\omega_2)) + t\text{ch}(\mathcal{C}).$$

Since

$$K \cong_{\mathfrak{g}_0} V(2\omega_1) \oplus V(\omega_2) \oplus V(\omega_1) \oplus \mathbb{C},$$

it follows that either

$$K/\tilde{K} \cong_{\mathfrak{g}_0} V(2\omega_1) \oplus V(\omega_1) \text{ or } K/\tilde{K} \cong_{\mathfrak{g}_0} V(2\omega_1) \oplus V(\omega_1) \oplus \mathbb{C}.$$
An application of Proposition 1.6 again shows that $K/K$ is a quotient of $KR(2\omega_1)$. Equation (4.1) implies that

$$K/K \cong_{g_0} V(2\omega_1) \oplus V(\omega_1), \quad KR(2\omega_1) \cong K/K,$$

and proves Proposition 1.1 in this case.

For $m = 3$, we see from Proposition 3.1 that

$$m_{\mu, r}(KR(3\omega_1)) = 0, \quad (\mu, r) \notin \{(3\omega_1, 0), (2\omega_1, 1), (\omega_2, 2), (0, 3)\}.$$

Consider the fusion product $K = KR(\omega_2) \ast KR(\omega_2)$. Set $\tilde{K} = U(g_0[t])(x_{\omega_2} \otimes t)(v_{2,1} \ast v_{2,1})$. Using Proposition 1.6 one checks easily that and $\tilde{K}$ is a quotient of $KR(3\omega_1)$ and that $K/K$ is a quotient of $KR(2\omega_2)$. Since

$$\tilde{K} \oplus K/K \cong_{g_0} K \cong_{g_0} V(2\omega_2) \oplus V(3\omega_1) \oplus V(2\omega_1) \oplus 3V(\omega_2) \oplus 2C,$$

the proposition follows for $m = 3$ from equation (4.2) together with the fact that $K/K \subset_{g_0} V(2\omega_2) \cong_{g_0} V(\omega_2) \oplus V(\omega_2) \oplus C$.

4.4. We shall need the following result.

**Lemma.** Let $r \in \mathbb{Z}_+^1$.

(i) In $KR(\omega_1)$ we have $y_r v_{1,1} = 0$ for all $r \in \mathbb{Z}_+^1, r \neq 0$.

(ii) In $KR(2\omega_1)$ we have

$$y_r v_{1,2} = 0 \iff r \notin \{0, e_1, e_2\},$$

and

$$n_0^+ y_{e_1} v_{1,2} = 0, \quad y_{e_2} v_{1,2} \in C x_{\alpha_1^-} y_{e_1} v_{1,2}.$$ 

(iii) In $KR(3\omega_1)$ we have

$$y_r v_{1,3} = 0 \iff r \notin \{0, e_1, e_2, e_3, e_4, 2e_1, 2e_2, e_1 + e_2, e_2 + e_3, 3e_1\}.$$ 

The elements $y_{e_2} v_{1,3}, x_{\alpha_2}^- y_{e_1} v_{1,3}$ are linearly independent and there exists $a \in C^\times$, such that

$$n_0^+ y_{e_1} v_{1,3} = n_0^+ y_{e_3} v_{1,3} = n_0^+ y_{3e_1} v_{1,3} = n_0^+ (y_{e_2} - a x_{\alpha_1^-} y_{e_1}) v_{1,3} = 0.$$

Finally,

$$y_{e_2 + e_3} v_{1,3} \in C(y_{3e_1} v_{1,3}), \quad y_{e_1} v_{1,3} \in C(x_{\alpha_2}^- y_{e_3} v_{1,3}), \quad y_{2e_1} v_{1,3} \in C(x_{\alpha_1}^- a x_{\alpha_2} y_{e_3} v_{1,3}).$$

**Proof.** Part (i) is obvious. For (ii), it is clear from the fact $ch_t(KR(2\omega_1)) = ch(V(2\omega_1)) + tch(V(\omega_1))$ that $y_r v_{1,m} = 0$ if $r \notin \{0, e_1, e_2\}$. For the converse, suppose that $y_{e_1} v_{1,2} = 0$. Since $\text{wt}(e_2) < \omega_1$, this means that if $r \in \mathbb{Z}_+^1$ is such that $\text{wt}(r) = \omega_1$, then $y_r v_{3, m} \in V(2\omega_1)$ and hence proves that $m_{\mu, 1}(KR(2\omega_1)) = 0$ which is a contradiction. A simple calculation proves that $x_{\alpha_1}^+ y_{e_2} v_{1,2} \in C^\times (y_{e_1} v_{1,2})$ and hence it follows that $y_{e_2} v_{1,2} \neq 0$. The second equality in part (ii) is trivially established. The proof of (iii) is a similar detailed analysis based on the graded character of $ch_t(KR(3\omega_1))$. 

\[\square\]
4.5. The modules $KR(m\omega_1)$, $m > 3$. Set $K = KR(3\omega_1)$, $v = v_{1,3}$ and by abuse of notation we also denote by $v$ the image of $v$ in $K(j)$ for $0 \leq j \leq 3$. Let

$$K(1) = K/U(g_0[t])v = K(1)/U(g_0)v.$$ 

For any $\varepsilon \in \{0,1\}$ and $p \in \mathbb{Z}_+^4$ with $\sum_{i=1}^4 p_i \leq m_0$, where $m = 3m_0 + m_1$ with $0 \leq m_1 \leq 2$, set

$$K_{m_1,m_0}^{\varepsilon}(p) = K^{\otimes p_3} \otimes K(2)^{\otimes p_3} \otimes K(1)^{\otimes p_2} \otimes K(0)^{\otimes m_0 - \sum_{i=1}^4 p_i} \otimes K(m_1\omega_1)(\varepsilon).$$

Given an equivalence class $r$ such that $r \cap A_1 \neq \emptyset$ we assume that $r = r_{j,k,s}$ and let $r_1, r_4$ be defined as in section 2.3. Then set

$$r_0 = r + r_3(3e_1 - e_2 - e_4) = (r_1 + 3r_4)e_1 + r_2e_2 + r_3e_3,$$

where $r_2 = k - s$ and $r_3 = s$. For $0 \leq n \leq \# r \cap A_1$, set

$$r_n = r_{n-1} - (3e_1 - e_2 - e_4)$$

and define $p_n(r) \in \mathbb{Z}_+^4$, $\varepsilon(r) \in \{0,1\}$ by:

(i) if $m_1 = 2$, then

$$p_n(r) = (\delta_2, r_1, r_2 + n, r_3 + n, r_4 - n), \quad \varepsilon(r) = 1 - \delta_{0,r_1},$$

(ii) if $m_1 = 1, r_1 = 2$ and $r_2 + r_3 + r_4 = m_0 - 1$ (in particular $\# r \cap A_1 = 1$), set

$$p_0(r) = (0, r_2, r_3 + 1, r_4), \quad \varepsilon(r) = 0,$$

(iii) and in all other cases,

$$p_n(r) = (r_1, r_2 + n, r_3 + n, r_4 - n), \quad \varepsilon(r) = 0.$$

It is now tedious but not hard to see that the modules $K_{m_1,m_0}^{\varepsilon}(p_n(r))$ are defined for all $0 \leq n \leq \# r \cap A_1$. Finally, let $v_{p_n(r)}$ be the image of the tensor product of the elements $v^{\otimes m_0} \otimes v_{1,m_1}$ in $K_{m_1,m_0}^{\varepsilon}(p_n(r))$.

**Proposition.** Let $r \in A_1$ be as above and consider $K_{m_1,m_0}^{\varepsilon}(p_n(r))$ for $0 \leq n \leq \# r \cap A_1$. Write $p_n(r) = (p_1, p_2, p_3, p_4)$ and $p_0 = m_0 - \sum_{i=1}^4 p_i$. If $m_1 = 1, r_1 = 2$ and $r_2 + r_3 + r_4 = m_0 - 1$ we have

$$y_{r_0}v_{p_0(r)} = y_{3r_4}e_1v^{\otimes r_4} \otimes x_{\alpha_1+\alpha_2}y_{(r_3+1)e_3v^{\otimes r_1+1}} \otimes y_{r_2e_2}v^{\otimes r_2} \otimes v_{1,1},$$

and there exists $a, b \in \mathbb{C}$ such that

$$n_0^+(y_{r_0} + (ax_{\alpha_2}x_{\alpha_1} + bx_{\alpha_1+\alpha_2})y_{r_0-2e_1+e_3})v_{p_0(r)} = 0.$$

In all other cases we have

$$y_{r_1}v_{p_n(r)} = 0 \quad \text{if} \quad \ell < n,$$

(4.5)

$$y_{r_0}v_{p_n(r)} = y_{3p_4e_1}v^{\otimes p_4} \otimes (x_{\alpha_2})^n y_{p_3e_3}v^{\otimes p_3} \otimes y_{p_2e_2}v^{\otimes p_2} \otimes y_{p_1e_1}v^{\otimes p_1} \otimes v^{\otimes p_0} \otimes y_{\varepsilon(r)e_1}v_{1,m_1},$$

and there exists $c_1, \ldots, c_n \in \mathbb{C}^*$ such that

$$n_0^+(y_{r_0} + \sum_{\ell=1}^n c_\ell (x_{\alpha_2})^\ell y_{r_0 + \ell(e_3 - e_1)})v_{p_n(r)} = 0.$$
Proof. A straightforward computation using Lemma 1.3 prove equations 1.3), 1.5), and 1.6). To prove (4.7), let $r_{n,\ell} = r_n + \ell(e_3 - e_4)$ for $0 \leq \ell \leq n$. Lemma 1.4 now gives,

$$y_{r_{n,\ell}} v_{n}(f) = y_{3p_4e_1} v_{n}^{\otimes p_4} \otimes (x_{\alpha_2}^{-})^{n-\ell} y_{p_3e_3} v_{n}^{\otimes p_3} \otimes y_{p_2e_2} v_{n}^{\otimes p_2} \otimes y_{p_1e_1} v_{n+1,3}^{\otimes p_1} \otimes v^{\otimes p_0} \otimes y_{e}(f)e_1 v_{1,m_1},$$

and also that

$$x_{\alpha_2}^{+} y_{r_{n,\ell}} v_{n}(f) = x_{\alpha_2}^{+} y_{3p_4e_1} v_{n}^{\otimes p_4} = x_{\alpha_2}^{+} y_{p_1e_1} v_{n}^{\otimes p_1} = x_{\alpha_2}^{+} v^{\otimes p_0} = x_{\alpha_2}^{+} y_{e}(f)e_1 v_{1,m_1} = 0.$$

Since

$$\text{wt}(y_{3p_4e_1} v_{n}^{\otimes p_4})(h_{\alpha_2}) = \text{wt}(y_{p_1e_1} v_{n}^{\otimes p_1})(h_{\alpha_2}) = \text{wt}(v^{\otimes p_0})(h_{\alpha_2}) = \text{wt}(y_{e}(f)e_1 v_{1,m_1})(h_{\alpha_2}) = 0,$$

it follows now that to prove (4.7), it suffices to find $c_1, \cdots, c_n \in C^\times$ such that

$$x_{\alpha_2}^{+} \left( (x_{\alpha_2}^{-})^n y_{p_3e_3} v_{n}^{\otimes p_3} \otimes y_{p_2e_2} v_{n}^{\otimes p_2} + \sum_{\ell=1}^n c_\ell (x_{\alpha_2}^{-})^{\ell} (x_{\alpha_2}^{-})^{n-\ell} y_{p_3e_3} v_{n}^{\otimes p_3} \otimes y_{p_2e_2} v_{n}^{\otimes p_2} \right) = 0$$

in $K(2)^{\otimes p_3} \otimes K(1)^{\otimes p_2}$. Since $\text{wt}(y_{e_3} v) = \text{wt}(y_{e_2} v) = \omega_2$, $x_{\alpha_2}^{+} y_{e_3} v = 0$, and in $K(1)$ we have $y_{e_2} v \neq 0$ and $x_{\alpha_2}^{+} y_{e_2} v = 0$, the result now follows from Lemma 1.4.

To prove (4.8) we first observe that Lemma 1.3 also gives

$$y_{r_0-(2e_1+e_3)} v_{n}(f) = y_{3p_4e_1} v_{n}^{1,3} \otimes y_{(r_3+1)e_3} v_{1,3}^{\otimes r_3+1} \otimes y_{r_2e_2} v_{1,3}^{\otimes r_2} \otimes v_{1,1}^{\otimes r_1}.$$

The rest of the proof is now similar to the previous case using Lemma 1.1. 

4.6. Let $V$ be any $g_0[\ell]$–module quotient of $KR(m\omega_1)$ and $v$ be the image of $v_{1,m}$. Given $r \in Z^4_+$, let $V^{\geq r}$ be the $g_0$–submodule of $V$ generated by the elements $\{y_s v : \text{wt}(s) > \text{wt}(r)\}$ and let $V^{>r}$ be the $g_0$–submodule generated by $V^{>r}$ and the elements $\{y_s v : s \in \bar{r}\}$. For $\mu \in P^+$, and any finite–dimensional $g_0$–module $W$, let $m_\mu(W)$ be the multiplicity of the isotypical component in $W$ corresponding to $\mu$.

Proposition. Let $s \in Z^4_+$. We have,

$$m_\mu(V^{>s}) \neq 0 \implies \mu > \text{wt}(s),$$

$$m_\mu(V^{\geq s}) \neq 0 \implies \mu \geq \text{wt}(s).$$

In particular,

$$m_\mu(V^{\geq s}/V^{>s}) \neq 0 \implies \mu = \text{wt}(s).$$

Proof. Suppose that $m_\mu(V^{>s}) \neq 0$ for some $\mu \in P^+$. Let $p_\mu : V^{>s} \rightarrow V(\mu)^{\otimes m_\mu(V^{>s})}$ be the projection of $g_0$–modules onto the corresponding isotypical component. Since $p_\mu \neq 0$ it follows that there must exist $r \in A_1$ with $\text{wt}(r) > \text{wt}(s)$ such that $p_\mu(y_r v) \neq 0$. This implies that $\mu - \text{wt}(r) \in Q^+$, i.e., $\mu \geq \text{wt}(r)$. The first implication of the Lemma follows. The second is proved similarly. If $m_\mu(V^{\geq s}/V^{>s}) \neq 0$, then $p_\mu(y_r v) \neq 0$ for some $r \in \bar{s}$ and hence $\mu \geq \text{wt}(s)$. If $\mu > \text{wt}(s)$ then there must exist $r'$ with $\text{wt}(r') = \mu$ such that $y_{r'} v$ has non–zero projection onto $V(\mu)$ (see section 3.5). But this is impossible since $y_{r'} v \in V^s$. 


4.7. Completion of the proof of Proposition 4.4 for $KR(m\omega_1)$. Given $r \in A_1$, let $r_n$ and $v_{p_n}(r)$ be defined as in section 4.3.

**Proposition.** Let $r \in A_1$ and $u = \sum_{n=0}^{\# A_1} c_n y_{r_n}$ for some $c_n \in \mathbb{C}$. Then $uv_{1,m} \in KR(m\omega_1)^{>r}$ only if $c_n = 0$ for all $0 \leq n \leq \# A_1$. In particular, we have

$$KR(m\omega_1)^{>r}/KR(m\omega_1)^{>r} \cong V(\text{wt}(r))^{\otimes \ell}$$

for some $\ell \geq \# A_1$.

**Proof.** Suppose that $c_n \neq 0$ for some $0 \leq n \leq \# A_1$ and assume that $n$ is maximal with this property. It is not difficult to see that $V = U(g_0[t])v_{p_n}(r)$ is a quotient of $KR(m\omega_1)$. If $uv_{1,m} \in KR(m\omega_1)^{>r}$, then we also have $wv_{p_n}(r) \in V^{>r}$. On the other hand, it follows from Proposition 4.4 that $uv_{p_n}(r) = c_n y_{r_n}v_{p_n}(r) \neq 0$. But then equations (4.4) and (4.7) contradict Lemma 4.6 since $\text{wt}(uv_{p_n}(r)) = \text{wt}(r)$. Hence, we must have that $c_n = 0$ for all $n$. $\square$

4.8. The modules $KR^\sigma(m\omega_1)$. In what follows we shall write an element of $g$ as a triple $(x_0, x_1, x_2)$ with $x_j \in g_j$, $j = 0, 1, 2$. Let $<,>$ be the Killing form of $g$. It is not hard to check that the following formulas define an action of $g[t]^\sigma$ on $K = g_2 \oplus \mathbb{C}$:

$$y_0 \otimes t^{3r}(x_2, a) = \delta_{r,0}([y_0, x_2], 0),$$

$$y_1 \otimes t^{3r+1}(x_2, a) = \delta_{r,0}(0, <x_2, y_1>) + 0, <x_2, y_1> > 0,$$

$$y_2 \otimes t^{3r+2}(x_2, a) = 0,$$

where $y_j \in g_j$ for $j = 0, 1, 2$. Moreover, since $<,>$ is non degenerate on $g_1 \times g_2$, it is not hard to see that the assignment $x_2^\alpha \mapsto v_{2,1}^\alpha$ extends to an isomorphism of $g[t]^\sigma$-modules $K \cong KR^\sigma(\omega_1)$. For $m > 1$ we proceed exactly as in section 4.2.

4.9. The modules $KR^\sigma(m\omega_2)$. Proceeding as in the previous section we see that the following formulas define an action of $g[t]^\sigma$ on $g \oplus \mathbb{C}$:

$$y_0 \otimes t^{3r}(x_0, x_1, x_2, a) = \delta_{r,0}([y_0, x_0], [y_0, x_1], [y_0, x_2], 0) + (0, 0, 0, \delta_{r,1} <x_0, y_0>)$$

$$y_1 \otimes t^{3r+1}(x_0, x_1, x_2, a) = \delta_{r,0}(0, [y_1, x_0], [y_1, x_1], <x_2, y_1>) + 0, <x_2, y_1> > 0,$$

$$y_2 \otimes t^{3r+2}(x_0, x_1, x_2, a) = \delta_{r,0}(0, 0, [y_2, x_0], <x_1, y_2>) + 0, <x_1, y_2> > 0,$$

where $x_j, y_j \in g_j$ for $j = 0, 1, 2$. Moreover, it is straightforward to check that this module is a quotient of $KR^\sigma(\omega_2)$ and hence proves Proposition 4.4 when $m = 1$.

For $m > 1$ the proof follows the same pattern as that for $KR(m\omega_1)$, $m > 3$, and we just list the relevant modifications and omit the details. Let $r \in \mathbb{Z}_1^4$. Similarly to Lemma 4.4 we see that in $KR^\sigma(\omega_2)$ we have

$$y_r^\sigma v_{2,1}^\sigma = 0 \iff r \notin \{0, e_1, e_2, e_3, e_4, e_1 + e_2\},$$

$$n_0^+ y_{e_1}^\sigma v_{2,1}^\sigma = n_0^+ y_{e_2}^\sigma v_{2,1}^\sigma = n_0^+ y_{e_3}^\sigma v_{2,1}^\sigma = 0, \quad \text{and} \quad y_{e_3}^\sigma v_{2,1}^\sigma \in \mathbb{C}x_{\omega_1}y_{e_2}^\sigma v_{2,1}^\sigma.$$
\( p = (p_1, p_2, p_3) \in \mathbb{Z}_+^3 \) satisfying \( p_1 + p_2 + p_3 \leq m \) define
\[
K_m(p) = K^{\otimes p_3} \otimes K(2)^{\otimes p_2} \otimes K(1)^{\otimes p_1} \otimes K(0)^{\otimes m-p_1-p_2-p_3},
\]
and let \( v_p \) be the image of \( v^{\otimes m} \) in \( K_m(p) \).

Now fix \( \bar{r} \) such that \( \bar{r} \cap A^2_\sigma \neq \emptyset \) and assume that \( r = r_{j,k,s} = (r_1, r_2, 0, r_4) \), where \( r_{j,k,s} \) is defined as in section 2.4. Then, for \( 0 \leq n \leq \# \bar{r} \cap A^2_\sigma \), set \( r_n = r + n(e_1 + e_3 - e_4) \) and \( p_n(\bar{r}) = (r_1 + n, r_2 + n, r_4 - n) \). We have the following analog of Proposition 4.5.

**Proposition.** Let \( p \in A^2_\sigma \) be as above and consider \( K_m(p_n(\bar{r})) \) for \( 0 \leq n \leq \# \bar{r} \cap A^2_\sigma \). Then
\[
y^\sigma_{r_n}v_{p_m(\bar{r})} = 0 \quad \text{if} \quad \ell < n,
\]
\[
y^\sigma_{r_n}v_{p_m(\bar{r})} = y^\sigma_{p_n(\bar{r})} y^\sigma_{r_n} v^\sigma_{p_n(\bar{r})} = y^\sigma_{p_n(\bar{r})} v^\sigma_{p_n(\bar{r})} - y^\sigma_{p_n(\bar{r})} v^\sigma_{p_n(\bar{r})}.
\]
and there exist \( c_1, \ldots, c_n \in C^\times \) such that
\[
n^+_n \left( y^\sigma_{p_n} + \sum_{\ell=1}^n c_\ell \left( y^\sigma_{p_n} - y^\sigma_{p_n} \right) \right) v_{p_m(\bar{r})} = 0.
\]

The proof of Proposition 4.1 is then completed as before by using the obvious modification of Proposition 4.7.

**References**

1. V. Chari, S. Loktev, Weyl, Fusion and Demazure modules for the current algebra of \( sl_{r+1} \), math.QA/0502165.
2. V. Chari, A. Moura, The restricted Kirillov-Reshetikhin modules for the current and twisted current algebras, to appear in Comm. Math. Phys., math.RT/0507584.
3. B. Feigin, S. Loktev, On Generalized Kostka Polynomials and the Quantum Verlinde Rule, Differential topology, infinite–dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, Vol. 194 (1999), p. 61–79, math.QA/9812093.
4. B. Feigin, A.N. Kirillov, S. Loktev, Combinatorics and Geometry of Higher Level Weyl Modules, math.QA/0503315.
5. G. Fourier, P. Littelmann, Tensor product structure of affine Demazure modules and limit constructions, math.RT/0412432.
6. G. Fourier, P. Littelmann, Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions, math.RT/0509276.
7. G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on the Fermionic Formula, Contemp. Math. 248 (1999), 243–291.
8. G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, Paths, Crystals and Fermionic Formulae, Prog.Math.Phys. 23 (2002), 205–272.

Department of Mathematics, University of California, Riverside, CA 92521.

E-mail address: chari@math.ucr.edu

UNICAMP - IMECC, Campinas - SP - Brazil, 13083-970.

E-mail address: aamoura@ime.unicamp.br