Stationary Wave Profiles for Nonlocal Particle Models of Traffic Flow on Rough Roads

Jereme Chien and Wen Shen

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Abstract

We study a nonlocal particle model describing traffic flow on rough roads. In the model, each driver adjusts the speed of the car according to the condition over an interval in the front, leading to a system of nonlocal ODEs which we refer to as the FtLs (follow-the-leaders) model. Assuming that the road condition is discontinuous, we seek stationary wave profiles (see Definition 1.1) for the system of ODEs across this discontinuity. We derive a nonlocal delay differential equation with discontinuous coefficient, satisfied by the profiles, together with conditions on the asymptotic values as $x \to \pm \infty$. Results on existence, uniqueness, and local stability are established for all cases. We show that, depending on the case, there might exist a unique profile, infinitely many profiles, or no profiles at all. The stability result also depends on cases. Various numerical simulations are presented. Finally, we establish convergence of these profiles to those of a local particle model, as well as those of a nonlocal PDE model.

Keywords: traffic flow, follow-the-leaders, particle model, traveling waves, stationary wave profiles, existence and uniqueness, stability, convergence.

1 Introduction and derivation of the model

Mathematical models for traffic flow has always been an active research field. Commonly used models include the microscopic particle models and the macroscopic PDE models. See the classical book [32] for a variety of models. Among the particle models, many adopt the popular Follow-the-Leader (FtL) principle, where the behavior of a car depends on the leader/leaders ahead. Such models usually give rise to a large system of ODEs. In the macroscopic models one treats the density of car distribution as the main unknown, and obtains nonlinear conservation laws where the total mass of cars is conserved. The micro-macro convergence is of fundamental interests, and results on first order models can be found in [12,14,15,17,23–25], and a recent work [18] for space dependent flux.

In recent year, increasing research interests have been focused on nonlocal traffic models and other related nonlocal models. At the microscopic level, these models describe that the behavior of each car depends on the traffic pattern over an interval of...
road of certain length. Such models are commonly referred to as the Follow-the-Leaders (FtLs) models. Formally, the corresponding macroscopic models are typically conservation laws with nonlocal flux functions. When the road condition is uniform, various results are achieved on many aspects of the topic. Well posedness of the nonlocal conservation laws was obtained in [3] with a Lax-Friedrich type numerical approximation, and in [22] using a Godunov type scheme. The micro-macro limit, for suitable road condition, is treated [16]. For nonlocal crowd dynamics and pedestrian flow, see [2, 8–10]. On other models that result in nonlocal conservation laws, we refer to [4, 11, 21]. See also results for several space dimensions [1], and other related works in [13,33].

In this work we propose a nonlocal particle model for traffic flow with rough conditions in one space dimension. We now derive the model. We assume that all cars have the same length \( \ell \in \mathbb{R}^+ \), and let \( z_i(t) \) be the position of the \( i \)th car at time \( t \). We order the indices of the cars such that
\[
z_i(t) \leq z_{i+1}(t) - \ell \quad \forall t \geq 0, \quad \text{for every } i \in \mathbb{Z}. \tag{1.1}
\]
For a car with index \( i \), we define the local discrete density perceived by the driver, depending on its relative position to its leader,
\[
\rho_i(t) = \frac{\ell}{z_{i+1}(t) - z_i(t)}. \tag{1.2}
\]
Note that if \( \rho_i = 1 \), then the two cars with indices \( i \) and \( i + 1 \) are bumper-to-bumper. This physical constraint requires that \( 0 \leq \rho_i(t) \leq 1 \) for all \( i \in \mathbb{Z} \) and \( t \geq 0 \). However, we remark that this constraint might fail for some models, see the discussion in section 6 for a model that leads to crashing when \( \rho_i \) becomes larger than 1.

For a given \( t \geq 0 \), based on the car distribution \( \{z_i(t) : i \in \mathbb{Z}\} \) and its corresponding discrete densities \( \{\rho_i(t) : i \in \mathbb{Z}\} \), we construct a piecewise constant density function as
\[
\rho^L(t, x) = \rho_i(t) \quad \text{for } x \in [z_i(t), z_{i+1}(t)) \quad \text{for } i \in \mathbb{Z}, \quad \forall t \geq 0. \tag{1.3}
\]
We let \( w \) denote a weight function, with support on the interval \( x \in [0, h] \) for some given \( h \in \mathbb{R}^+ \). We assume that \( w \) is bounded and Lipschitz continuous on its support and satisfies the assumptions
\[
w(x) \geq 0 \quad \forall x, \quad \int_0^h w(x) \, dx = 1, \quad w(h) = 0, \quad \text{and} \quad w'(x) < 0 \quad \forall x \in [0, h]. \tag{1.4}
\]
Although \( w \) has bounded support, we define it on the whole real line. Note also that the assumption in (1.4) implies that \( w(x) \) is discontinuous at \( x = 0 \), but continuous at \( x = h \).

We assume that the road condition is varying, and let the speed limit \( V(x) \) represent the condition at location \( x \). More specifically, we consider rough road condition where \( V \) is discontinuous, and set
\[
V(x) = \begin{cases} 
V^- > 0, & \text{if } x < 0, \\
V^+ > 0, & \text{if } x \geq 0.
\end{cases} \tag{1.5}
\]
Let $\phi : [0, 1] \rightarrow [0, 1]$ be a $C^2$ function which satisfies the assumptions
\[
\phi(1) = 0, \quad \phi(0) = 1, \quad \text{and} \quad \phi'(\rho) < 0, \quad \phi''(\rho) \leq 0 \quad \text{for} \quad \rho \in [0, 1]. \tag{1.6}
\]

In this particle model, we assume that the speed of the car with index $i$ depends on an average velocity $v^*$, where the average is taken over an interval of length $h$ in front of the car position $z_i$ with weight $w$. To be specific, we let
\[
\dot{z}_i(t) = v^*(z_i; \rho^\ell(t, \cdot)), \quad v^*(z_i; \rho^\ell(t, \cdot)) \doteq \int_{z_i}^{z_i+h} V(y)\phi(\rho^\ell(t, y))w(y - z_i) dy. \tag{1.7}
\]

The dot notation $\dot{z}_i$ denotes the time derivative of $z_i$. The dependence of $v^*$ on $\rho^\ell$ is nonlocal, through an integration. Given an initial distribution of car positions $\{z_i(0)\}$, the system given by (1.7) indicates that the velocity of each car depends on a group of cars in front of it. We refer to (1.7) as the "follow-the-leaders" (FtLs) model.

This family of countably many ODEs (1.7) can be regarded as a nonlinear dynamical system on an infinite dimensional space. For example, one could set $y_i(t) = z_i(t) - z_i(0)$ and write the system (1.7) as an evolution equation on the Banach space of bounded sequences of real numbers $y = \{y_i\} \in \mathbb{Z}$, with norm $\|y\| = \sup_i |y_i|$. For each $i \in \mathbb{Z}$, the right hand side of (1.7) is Lipschitz continuous, even though $V(\cdot)$ is a discontinuous function. For a given initial datum, the existence and uniqueness of solutions to this system follow from the standard theory of evolution equations in Banach spaces, see for example [26,28].

The study on traveling wave profiles is fundamental for flow models. In this work we focus our attention on the stationary wave profiles for the FtLs model (1.7), defined as follows.

**Definition 1.1.** Let $\{z_i(t)\}$ be a solution of the FtLs model (1.7) with initial condition $\{z_i(0)\}$. We say that $P(\cdot)$ is a stationary wave profile for the FtLs model (1.7) if
\[
P(z_i(t)) = \rho_i(t) = \frac{\ell}{z_{i+1}(t) - z_i(t)} \quad \forall i \in \mathbb{Z}, \ t \geq 0. \tag{1.8}
\]

The equation satisfied by the profile $P$ will be derived in section 2 with suitable conditions on the asymptotic values $(\rho^-, \rho^+)$, see (2.18)-(2.19). We obtain a delay integro-differential equation with discontinuous coefficient. Such equations can be studied by an adapted version of the method of step [19,20].

Several recent works on analysis of traveling waves for related models are available. For example, when $h \to 0^+$ and the weight function $w$ tends to a Dirac delta, and (1.7) reduces to a local particle model. For this local particle model, in the simple case with uniform road condition where $V(x) \equiv 1$, the discrete traveling wave profiles are studied in [31]. When $V$ is piecewise constant as in (1.5), the discrete stationary wave profiles are treated in [29]. For the current nonlocal models (1.7) with $V(x) \equiv 1$, discrete traveling waves for particle models as well as traveling waves for nonlocal conservation laws are analyzed in [27]. Those traveling waves might be stationary or travel with a constant velocity. The existence, uniqueness (up to a horizontal shift), and local stability of the profiles are established. The result for the corresponding nonlocal PDE models with piecewise constant $V$ can be found in [30].
In this paper we study the case where $V(x)$ is piecewise constant with a jump at $x = 0$, and we focus our attention on the stationary wave profiles across the discontinuity in $V$. We study existence, uniqueness, and local stability of the stationary wave profiles. Depending on the speed limits $(V^-, V^+)$ and asymptotic values $(\rho^-, \rho^+)$, the results vary. We show that, depending on the cases, (i) there exists a unique profile; (ii) there exist infinitely many profiles; or (iii) there exist no profiles at all. For the cases where the profiles exist, we show that some are time asymptotic solutions for the FtLs model (1.7), while others are unstable and do not attract any nearby solutions of (1.7).

We also address the topic of various limits for the model (1.7), in connection with traveling waves. Formally, there are several limits of interests.

**Limit 1: micro-macro limit to nonlocal conservation law.** Let $h$ be fixed and let $\ell \to 0$, the solutions $\rho^\ell$ of the particle model (1.7) formally converges to the solution of a nonlocal scalar conservation law with discontinuous flux function

$$\rho_t + [\rho A(t, x; V, \rho)]_x = 0, \quad A(t, x; V, \rho) \equiv \int_{x}^{x+h} V(y)\phi(\rho(t, y))w(y-x) \, dy. \quad (1.9)$$

**Limit 2: nonlocal to local particle model.** Let $\ell$ be fixed and let $h \to 0$ and $w \to \delta_0$ (a dirac delta function), one obtains a local particle model with

$$\dot{z}_i(t) = V(z_i)\phi(\rho_i(t)), \quad \rho_i(t) = \frac{\ell}{z_{i+1}(t) - z_i(t)} \quad \forall i \in \mathbb{Z}, t \geq 0. \quad (1.10)$$

In this model, the behavior of the car at $z_i$ depends only on a single leader in front. This is commonly referred to as the follow-the-leader (FtL) model.

**Limit 3: double-limit to local conservation law.** Furthermore, if we take the double limits $\ell \to 0$ and $h \to 0$, formally the model (1.7) converges to a local scalar conservation law with discontinuous flux

$$\rho_t + [V(x)\rho\phi(\rho)]_x = 0. \quad (1.11)$$

Rigorous theoretical results on the convergence of various limits are fundamental questions. Several recent results are of great interests, for the simpler case where the road condition is uniform with $V(x) \equiv 1$. The micro-macro limit of a nonlocal interaction equation with nonlinear mobility and symmetric kernel is studied in [16]. For the PDE models, the nonlocal to local limit (i.e. Limit 3) seems to be a complicated issue. With downstream model, where the support of $w$ is ahead of the driver, in [6] a counterexample is constructed where the total variation of the density blows up instantly for certain initial data in BV. Furthermore, when the support of $w$ is around the driver, counter examples in [5] indicate that solutions for the model with nonlocal fluxes fail to converge to those for the local models. See also [7] for the effect of numerical viscosity in the study of this limit. Convergence results (positive or negative) are still open for nonlocal follow-the-leaders models with weight defined on $[0, h]$ and $w' < 0$.

For the model (1.7) where the road condition are discontinuous with $V$ in (1.5), and the assumptions on the parameters (1.4) and (1.6), these convergence results (micro-macro and nonlocal-local) are still open. In this paper, as a first attempt in this direction, we establish the convergence for the stationary wave profiles for Limits 1 and 2.
The rest of the paper is organized as follows. In section 2 we derive the equation satisfied by the profiles, and we establish several technical Lemmas. The case $V^- > V^+$ is treated in section 3, and the case $V^- < V^+$ in section 4. Each case has 4 subcases, and various results on profiles are proved. In section 5 we prove the convergence of the profiles for Limit 1 and Limit 2. Final concluding remarks are given in section 6, where we also discuss an alternative (possibly faulty) FtLs model on rough roads.

2 Derivation of the profile equation and technical lemmas

In this section we first derive the equation satisfied by the profile, then we present some technical lemmas and previous results which will be useful in the analysis later.

2.1 Derivation of the profile equation

We now derive the equation satisfied by a stationary wave profile. Let $t \geq 0$ be given.

Note that (1.7) can be rewritten as a system of ODEs for the discrete density functions $\rho_i(\cdot)$, such that

$$\dot{\rho}_i(t) = -\frac{\ell}{(z_{i+1} - z_i)^2} \left[v^*(z_i; \rho^f) - v^*(z_{i+1}; \rho^f)\right].$$

(2.12)

Here the piecewise constant function $\rho^f(t, x) = \rho_i$ satisfies

$$\rho_i(x, t) = P_i,$$

for $x \in [z_i, z_{i+1})$, $\forall i \in \mathbb{Z}$.

Differentiating both sides of (1.8) in $t$ and using (1.7) and (2.12), one gets

$$P'(z_i) = \frac{\dot{\rho}_i}{\dot{z}_i} = \frac{P(z_i)^2}{\ell \cdot v^*(z_i; \rho^f(t, \cdot))} \left[v^*(z_i; \rho^f(t, \cdot)) - v^*(z_{i+1}; \rho^f(t, \cdot))\right].$$

(2.13)

We now introduce some notations. For a given profile $P$, we define an operator

$$L^P(x) \doteq x + \frac{\ell}{P(x)},$$

(2.14)

where $L^P(x)$ is the location of the leader for the car at $x$. We remark that, in the particle model with the local density defined in (1.2), we have $\rho_i > 0$ for all $i$. The only exception is for the leader which could have an empty road in front and therefore zero density, but in our discussion on the stationary profile this never occurs. Thus we have $P(x) > 0$ and (2.14) is well-defined.

Furthermore, when the operator $L^P$ is composed with itself multiple times, we use the notation

$$(L^P)^k(x) \doteq L^P \circ L^P \cdots \circ L^P(x), \quad k \in \mathbb{Z}^+.$$  

(2.15)

Given a profile $P$, we define a piecewise constant function $P^f_{\{x\}}$ as

$$P^f_{\{x\}}(y) = P((L^P)^k(x)) \quad \text{for } y \in [(L^P)^k(x), (L^P)^{k+1}(x)], \quad \forall k \in \mathbb{Z}.$$  

(2.16)
With these notations, we now define

\[ v^*(x; P^\ell_{\{\tilde{z}\}}) = \int_x^{x+h} V(y) \cdot \phi(P^\ell_{\{\tilde{z}\}}(y)) \cdot w(y - x) \, dy. \]  

(2.17)

Note that \( V \) and \( P^\ell_{\{\tilde{z}\}} \) are bounded piecewise constant functions so \( x \mapsto v^* \) is Lipschitz continuous. Since the \( z_i \) in (2.13) is arbitrarily chosen, we can replace it with \( x \). Using the notations introduced above, we can rewrite (2.13) as a delay integro-differential equation

\[ P'(x) = \frac{P^2(x)}{\ell \cdot v^*(x; P^\ell_{\{x\}})} \left[ v^*(x; P^\ell_{\{x\}}) - v^*(L^P(x); P^\ell_{\{x\}}) \right], \]

(2.18)

where \( v^* \) is defined in (2.17). We seek continuously differentiable solutions \( P \) of (2.18) with the following asymptotic values

\[ \lim_{x \to -\infty} P(x) = \rho^-, \quad \lim_{x \to +\infty} P(x) = \rho^+. \]

(2.19)

Suitable conditions on \( \rho^-; \rho^+ \) will be specified later. We refer to (2.18)-(2.19) as the asymptotic value problem.

We remark that the delay in (2.18) is strictly positive, of value larger than \( \ell \). Fix a value \( x_0 \in \mathbb{R} \). If \( P \) is given on the half line \( x \geq x_0 \), the equation (2.18) can be solved backward in \( x \), as an “initial value problem”. The existence and uniqueness of solutions to this initial value problem is a key step for the analysis of the asymptotic value problem for the stationary profiles.

2.2 Technical Lemmas and previous results

We now present some technical lemmas and previous results. We define the function

\[ f(\rho) = \rho\phi(\rho), \quad \rho \in [0, 1]. \]

(2.20)

By the assumptions (1.6), we see that \( f'' < 0 \) in the domain. Furthermore, we let \( \hat{\rho} \) be the unique value \( \hat{\rho} \) such that

\[ f'(\hat{\rho}) = 0. \]

(2.21)

Next Lemma establishes the ordering property for the car distribution.

**Lemma 2.1.** Let \( P \) be a continuously differentiable function that satisfies (2.18), and assume that \( 0 < P(x) < 1 \) for all \( x \in \mathbb{R} \). Then

\[ P'(x) < \frac{1}{\ell} P^2(x), \quad \forall x \in \mathbb{R}. \]

(2.22)

Moreover, for every car at \( x \), there exists a unique follower \( x^b \) such that \( L^P(x^b) = x \).

**Proof.** Since \( 0 < P(x) < 1 \) for all \( x \), we have that \( 0 < \phi(P(x)) < 1 \) and therefore \( 0 < v^*(x; P^\ell_{\{x\}}) \) for all \( x \). Then (2.22) follows immediately from (2.18).

Next, fix an \( x \in \mathbb{R} \) and let \( x^b \) be the follower of \( x \) such that \( L^P(x^b) = x \). Then, it holds

\[ (L^P)'(x) = 1 - \frac{\ell}{P^2(x)} P'(x) > 0, \quad \forall x \in \mathbb{R}. \]

Thus the operator \( L^P \) is monotone increasing, and therefore there exists a unique follower \( x^b \). □
We remark that the monotonicity of \( x \mapsto L^P \) implies the ordering property, such that \( x < y \) if and only if \( L^P(x) < L^P(y) \).

**Definition 2.2.** Let \( P \) be a continuously differentiable function such that
\[
0 < P(x) < 1, \quad P'(x) < \frac{1}{\ell}P^2(x), \quad \forall x \in \mathbb{R}.
\]
We call a sequence of car positions \( \{ z_i \} \) a **distribution generated by** \( P \), if
\[
z_{i+1} = z_i + \frac{\ell}{P(z_i)}, \quad \forall i \in \mathbb{Z}.
\]

Note that each given function \( P \) can generate infinitely many distributions of \( \{ z_i \} \). However, if we fix the position of one car, say \( z_0 \), then the distribution is unique.

Given a profile \( P \) and a distribution \( \{ z_i \} \) generated by \( P \) with \( z_j = x \) for any fixed index \( j \), the piecewise constant function \( P^\ell_{\{x\}} \) defined in (2.16) satisfies
\[
P^\ell_{\{x\}}(y) = P(z_i) \quad \text{for} \quad y \in [z_i, z_i+1) \quad \forall i \in \mathbb{Z}, \quad z_j = x \quad \text{for some} \quad j. \tag{2.23}
\]
In the rest of the paper we denote by \( P^\ell_{\{x\}} \) the piecewise constant function associated with \( P \).

The next Lemma is immediate.

**Lemma 2.3.** Let \( P \) be a stationary profile that satisfies (2.18), and let \( \{ z_i(0) \} \) be a distribution generated by \( P \). Let \( \{ z_i(t) \} \) be the solution of the FtLs model (1.7) with initial condition \( \{ z_i(0) \} \). Then, \( \{ z_i(t) \} \) is a distribution generated by \( P \) for all \( t \geq 0 \).

In other words, let \( \{ \rho_i(t) \} \) be the corresponding discrete density for \( \{ z_i(t) \} \), then \( P(z_i(t)) = \rho_i(t) \) for all \( t \geq 0 \) and for all \( i \in \mathbb{Z} \).

We now define the concept of periodic solutions for the FtLs model.

**Definition 2.4.** Let \( \{ z_i(t) : i \in \mathbb{Z} \} \) be the solution of (1.7) with initial condition \( \{ z_i(0) : i \in \mathbb{Z} \} \). We say that \( \{ z_i(t) : i \in \mathbb{Z} \} \) is **periodic** if there exists a constant \( t_p \in \mathbb{R}^+ \), independent of \( i \) and \( t \), such that
\[
z_i(t + t_p) = z_{i+1}(t), \quad \forall i \in \mathbb{Z}, \forall t \geq 0. \tag{2.24}
\]
We call \( t_p \) the **period**.

Definition 2.4 indicates that, in a periodic solution \( \{ z_i(t) : i \in \mathbb{Z} \} \), after a time period of \( t_p \), each car takes over the position of its leader. In the next Lemma we show that this is closely related to stationary wave profiles.

**Lemma 2.5.** (i) Let \( P \) be a continuously differentiable function and \( 0 < P(x) < 1 \) for all \( x \in \mathbb{R} \), and let \( P^\ell_{\{x\}} \) be the associated piecewise constant function. Then, \( P \) satisfies (2.18) if and only if
\[
\int_x^{L^P(x)} \frac{1}{v^*(z; P^\ell_{\{z\}})} dz = C, \quad \forall x \in \mathbb{R}, \tag{2.25}
\]
for some constant \( C \in \mathbb{R}^+ \).

(ii) Moreover, let \( \{ z_i(t) \} \) be the solution of (1.7) with initial data \( \{ z_i(0) \} \) which is a distribution generated by \( P \). Then, \( \{ z_i(t) \} \) is periodic if and only if \( P \) satisfies (2.18). The period \( t_p \) equals the constant \( C \) in (2.25).
Proof. (i) We assume that (2.25) holds. Differentiating (2.25) in $x$ on both sides, we get

$$ (L^P)'(x) \frac{1}{v^*(L^P(x); P^t_{\{x\}})} - \frac{1}{v^*(x; P^t_{\{x\}})} = 0. $$

(2.26)

Using $(L^P)'(x) = 1 - tP'(x)/P^2(x)$ and $0 < P(x) < 1$, we easily deduce (2.18).

Now assume that (2.18) holds, which implies (2.26), and further implies (2.25). Therefore, (2.25) and (2.18) are equivalent.

(ii) Assume that $P$ satisfies (2.18), and let $\{z_i(t)\}$ be the solution of (1.7) with initial data $\{z_i(0)\}$ generated by $P$. Fix any time $t \geq 0$, and an index $i \in \mathbb{Z}$. Since $v^*(z_i; P^t_{\{z_i\}})$ does not depend on $t$ explicitly, (1.7) is separable. Let $t_{p,i}$ be the time it takes for a car at $z_i$ to reach it leader’s position $z_{i+1} = z_i + \ell/P^t_{\{z_i\}}$, we compute

$$ t_{p,i} = \int_t^{t + t_{p,i}} dt = \int_{z_i}^{z_i + \ell/P^t_{\{z_i\}}} \frac{1}{v^*(z_i; P^t_{\{z_i\}})} dz_i. $$

By (2.25) we conclude that $t_{p,i} = C = t_p$ is constant, therefore $\{z_i(t)\}$ is periodic. The reverse implication is easily proved by reversing the above argument. \hfill \Box

We are interested in the asymptotic value problem of (2.18) with the asymptotic conditions (2.19). Defining $z^b$ as

$$ L^P(z^b) = -h, $$

(2.27)

we observe that on $x \in (-\infty, z^b) \cup (0, \infty)$ the equation (2.18) for $P$ is the same as the one for the stationary profile for the FtLs model with $V(x) \equiv 1$, studied in [27]. We denote by $W$ the profiles studied in [27]. We further recall that in [27] the existence, uniqueness (up to a horizontal shift) and local stability of the stationary profiles $W$ were established.

Thanks to the follow-the-leaders principle, we conclude that on $x \geq 0$, the profiles $P$ (if they exist) must match $W$. Furthermore, similar results are valid on asymptotic behaviors as $x \to \pm \infty$, for $P$ and for $W$. We recall the following result from [27] Lemma 2.2.

**Lemma 2.6.** Let $\rho^-, \rho^+ \in \mathbb{R}^+$ be given and assume $\rho^\pm \in (0,1)$. Assume that $P$ is a solution of (2.18), continuously differentiable on $x > 0$ and $x < z^b$, satisfying the asymptotic conditions (2.19). Then, the following holds.

- As $x \to +\infty$, $P(x)$ approaches $\rho^+$ with an exponential rate if and only if $\rho^+ > \hat{\rho}$.
- As $x \to -\infty$, $P(x)$ approaches $\rho^-$ with an exponential rate if and only if $\rho^- < \hat{\rho}$.

Lemma 2.6 implies that, if $\rho^-$ and $\rho^+$ are stable asymptotic values at $x \to -\infty$ and $x \to +\infty$ respectively, for monotone profiles, then we must have $\rho^- < \hat{\rho} < \rho^+$. In the sequel we say that $\rho^-$ (and $\rho^+$ respectively) is a **stable asymptote** if $\rho^- < \hat{\rho}$ (and $\rho^+ > \hat{\rho}$ respectively). On the other hand, we say that $\rho^-$ (and $\rho^+$ respectively) is an **unstable asymptote** if $\rho^- > \hat{\rho}$ (and $\rho^+ < \hat{\rho}$ respectively). Furthermore, combined with the periodic behavior in Lemma 2.5 we immediately have the next Lemma, which establishes properties on the flux and density values at $x \to \pm \infty$. 

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Let graphs of \( P \) distributions generated by \( P \) for some value \( \bar{f} \) for some value \( \bar{f} \).

Proof. We prove by contradiction. Assume the opposite, that the graphs of \( P \) cross each other, and let \( \{z_i(0)\} \) be the solution of

\[
\begin{align*}
\text{Lemma 2.7.} \quad & \text{Assume that } P \text{ is a piecewise continuously differentiable function that satisfies (2.18), and let } \rho^-, \rho^+ \text{ be asymptotic values such that (2.19) holds. Then there exists a value } \bar{f} \in \mathbb{R}^+ \text{ such that} \end{align*}
\]

\[
V^+ \cdot f(\rho^+) = V^- \cdot f(\rho^-) = \bar{f}.
\]  

\[\text{(2.28)}\]

Let \( \{z_i(0)\} \) be a distribution generated by this \( P \), and let \( \{z_i(t)\} \) be the solution of (1.7) with initial condition \( \{z_i(0)\} \). Then \( \{z_i(t)\} \) is periodic with period \( t_p = \ell/\bar{f} \), i.e.,

\[
t_p = \int_x^{L_p(x)} \frac{1}{v^*(x; P^f_{\{z\}})} \, dz = \frac{\ell}{\bar{f}}, \quad \forall x \in \mathbb{R}.
\]  

\[\text{(2.29)}\]

For (2.18)-(2.19) we will show later that, for some cases there exist infinitely many stationary profiles. If this happens, the next Lemma shows that these profiles never cross each other.

**Lemma 2.8.** Let \( P_1 \) and \( P_2 \) be two distinct profiles that satisfy (2.18)-(2.19). Then the graphs of \( P_1 \) and \( P_2 \) never cross each other.

**Proof.** We prove by contradiction. Assume the opposite, that the graphs of \( P_1 \) and \( P_2 \) cross each other, and let \( y \) be the rightmost crossing point. Without loss of generality we assume that

\[
P_1(y) = P_2(y), \quad P_1(x) > P_2(x) \quad \forall x > y.
\]  

\[\text{(2.30)}\]

Let \( y^\sharp \simeq L_{P_1}(y) = L_{P_2}(y) \) be the position of the leader for the car at \( y \), in both car distributions generated by \( P_1 \) and \( P_2 \), and let \( t_{p,1} \) and \( t_{p,2} \) be the periods

\[
t_{p,1} = \int_y^{y^\sharp} \frac{1}{v^*(x; P^f_{1,\{z\}})} \, dx, \quad t_{p,2} = \int_y^{y^\sharp} \frac{1}{v^*(x; P^f_{2,\{z\}})} \, dx.
\]

By our assumptions (2.30) we have \( t_{p,1} > t_{p,2} \). But since \( P_1, P_2 \) have the same asymptotic values, by Lemma 2.3 and Lemma 2.7 we have that \( t_{p,1} = t_{p,2} \), a contradiction. \( \Box \)

### 3 Case 1: \( V^- > V^+ \)

In this section we study the case \( V^- > V^+ \), i.e., the speed limit has a downward jump at \( x = 0 \). We prove results related to existence, uniqueness and stability of the stationary profiles. To simplify the notations, we introduce the functions

\[
f^-(\rho) \equiv V^- \rho \phi(\rho) = V^- f(\rho), \quad f^+(\rho) \equiv V^+ \rho \phi(\rho) = V^+ f(\rho), \quad \rho \in [0, 1].
\]  

By Lemma 2.7, the asymptotic values \( (\rho^-, \rho^+) \) must satisfy \( f^-(\rho^-) = f^+(\rho^+) = \bar{f} \), for some value \( \bar{f} \in \mathbb{R}^+ \) in the range of both functions \( f^- \) and \( f^+ \). The horizontal line \( f = \bar{f} \) intersects twice with each graph of \( f^- \) and \( f^+ \), see Figure 1. We have

\[
0 \leq \rho_1 < \rho_2 \leq \hat{\rho} \leq \rho_3 < \rho_4 \leq 1, \quad f^-(\rho_1) = f^-(\rho_4) = f^+(\rho_2) = f^+(\rho_3) = \bar{f}.
\]  

\[\text{(3.2)}\]

There are 4 subcases in total:
We first observe that the cases with $\bar{f} = 0$ are trivial. In this case we have $\rho_1 = \rho_2 = 0$ and $\rho_3 = \rho_4 = 1$, and the following:

| subcase | 1A | 1B | 1C | 1D |
|---------|----|----|----|----|
| Profile | $P(x) \equiv 0$ | unit step function | $P(x) \equiv 1$ | no profile, see section 3.3 |

For the rest we only consider the nontrivial cases with $\bar{f} > 0$, and $0 < \rho^+ < 1$.

3.1 Subcase 1A: $0 < \rho^- < \rho^+ \leq \hat{\rho}$

In this case $\rho^+ \leq \hat{\rho}$ is an unstable asymptote as $x \to +\infty$, so the only possible profile on $x \geq 0$ is the constant function $P(x) \equiv \rho^+$. Given this as the “initial condition”, in the next Theorem we solve the initial value problem and construct a unique stationary wave profile on $x < 0$.

**Theorem 3.1.** Given $V^\pm, \rho^\pm$ as in subcase 1A. There exists a unique monotone profile $P$ which satisfies (2.18)–(2.19). We have $P(x) = \rho^+$ on $x \geq 0$.

**Proof.** Given initial condition $P(x) \equiv \rho^+$ on $x \geq 0$, the equation (2.18) can be solved backward in $x$, as an initial value problem, for $x < 0$. The existence and uniqueness of the solution for this initial value problem can be established using the method of steps [19,20] for delay differential equations.

**Step 1. Method of steps.** First we note that $P'(0) > 0$ since $V^- > V^+$. Now, consider the intervals $I_k = [(k-1)\ell, k\ell]$ with $k = 0, -1, -2, \ldots$. Fix an index $k$. Assume that the solution $P$ is given on $x \geq k\ell$, and we extend the solution on $I_k$. We claim that, if

$$P'(x) \geq 0, \quad \rho^+ \geq P(x) > \rho^-,$$

holds for all $x \geq k\ell$, then (3.3) also holds for $x \in I_k$.

Indeed, fix an $x \in I_k$. Since the right hand side of (2.18) is a given Lipschitz continuous function, the existence and uniqueness of the solution follow from standard theory...
on scalar ODE. It remains to establish the desired properties. Since $V$ is monotone decreasing, $P$ is monotone increasing and $\phi$ is a monotone decreasing function, therefore $x \mapsto V\phi(P)$ is monotone decreasing. We have

$$v^*(L^P(x); P^\ell_{\{\ell\}}) \leq v^*(x; P^\ell_{\{\ell\}}),$$

and thus using (2.18) we conclude $P'(x) \geq 0$ for $x \in I_k$.

In order to establish $\rho^+$ as a lower bound for $P$ on $I_k$, we use contradiction. We assume that there exists a $y \in I_k$ such that $P(y) = \rho^-$. Then we have $v^*(x; P^\ell_{\{\ell\}}) > V^{-}\phi(\rho^-)$ for any $x > y$. The time it takes for the car at $y$ to reach its leader is

$$t_p(y) = \int_y^{y + \ell/\rho^-} \frac{1}{v^*(x; P^\ell_{\{\ell\}})} \, dx > \int_y^{y + \ell/\rho^-} \frac{1}{V^-\phi(\rho^-)} \, dx = \frac{\ell}{\rho^- V^-\phi(\rho^-)} = \ell/\bar{f}.$$

But according to Lemma 2.7 we must have $t_p(y) = \ell/\bar{f}$, a contradiction. We conclude that $P(x) \geq \rho^+$ on $I_k$.

**Step 2. Asymptotic condition.** It remains to establish the asymptotic condition $\lim_{x \to -\infty} P(x) = \rho^-$. Since the function $P$ is monotone and bounded, the limit $\bar{\rho}^- = \lim_{x \to -\infty} P(x)$ exists. By Lemma 2.7, $\bar{\rho}^-$ must satisfy $V^-f(\bar{\rho}^-) = \bar{f} = V^-f(\rho^-)$, where both $\bar{\rho}^-$ and $\rho^-$ are less than $\hat{\rho}$. Since $f$ is monotone on $\rho < \hat{\rho}$, we conclude that $\bar{\rho}^- = \rho^-$. This complete the proof.

**Stability issue.** Since $\rho^+ \leq \hat{\rho}$ is an unstable asymptote, the constant function $P(x) = \rho^+$ on $x > 0$ does not attract nearby solutions of the FtLs model, therefore the profile is not the time asymptotic limit for solutions of the FtLs model. This is further supported by the following numerical results.

**Sample profile and numerical simulations for the FtLs model.** For all the numerical simulations in this paper, we use the following parameters and functions

$$\ell = 0.05, \quad h = 0.5, \quad \phi(\rho) = 1 - \rho \quad \rho \in [0, 1], \quad w(s) = \frac{2}{h} - \frac{2}{h^2 s} \quad s \in (0, h]. \tag{3.4}$$

For subcase 1A, we use the parameters

$$V^- = 2, \quad V^+ = 1, \quad \bar{f} = 3/16, \quad dz = 0.0002. \tag{3.5}$$

A typical profile $P$ for subcase 1A is given in Figure 2 (left plot). We observe that the profile is monotone and continuously differentiable.

A numerical simulation is also performed for the FtLs model (1.7), with the following Riemann-like initial data

$$z_i(0) = \begin{cases} i \frac{\ell}{\rho^+}, & i \geq 0, \\ i \frac{\ell}{\rho^-}, & i < 0, \end{cases} \quad \text{such that} \quad \rho_i(0) = \begin{cases} \rho^+, & z_i(0) \geq 0, \\ \rho^-, & z_i(0) < 0. \end{cases} \tag{3.6}$$
Numerical solutions for the FtLs model for $0 \leq t \leq T_f = 4$ is computed. Let $\tau_p = \frac{\ell}{f(\rho^+)}$. The graphs for the mappings $z_i(t) \mapsto \rho_i(t)$ for $t \in [T_f - \tau_p, T_f]$ are plotted in green in Figure 2 (right plot) together with the points $\{z_i(T_f), \rho_i(T_f)\}$ in red. We observe that, as time grows, some oscillation enters the region $x > 0$, where the profile $P(x) = \rho^+$ is constant. Since $\rho^+ < \hat{\rho}$ is not a stable asymptote, the oscillation persists as time grows, and the solution does not approach the stationary profile $P$. This behavior is typical for subcase 1A and is independent on the parameters $(\ell, dz, \bar{f}, V^-, V^+, \text{etc.})$, as long as we are in subcase 1A.

![Figure 2: Left: Typical profile $P$ for case 1A. Right: Solution of the FtLs model with Riemann-like initial condition.](image)

3.2 Subcase 1B: with $0 < \rho^- < \hat{\rho} < \rho^+ < 1$

For this subcase we have $\rho^- < \hat{\rho}$ and $\rho^+ > \hat{\rho}$, where both $\rho^-, \rho^+$ are stable asymptotes. We will show that there are infinitely many monotone stationary wave profiles.

**Theorem 3.2 (Existence of profiles).** Given $V^\pm, \rho^\pm$ as in subcase 1B. There exist infinitely many monotone stationary wave profiles $P$ which satisfies the equation (2.18) and the asymptotic conditions (2.19).

**Proof.** On $x \geq 0$, we have $V(x) \equiv V^+$, therefore the profile $P$ must be either constant $P(x) \equiv \rho^+$ or some horizontal shift of $W$, i.e., the stationary profile established in [27]. The profile $P$ is smooth and monotone, taking values between $\rho_2$ and $\rho^+$, where $\rho_2$ is defined in (3.2). In particular, it holds that $\rho_2 < P(0) \leq \rho^+$.

Once the profile is given on $x \geq 0$, we solve an initial value problem backward in $x$. By the same argument as in the proof of Theorem 3.1, there exists a unique monotone solution for each initial value problem, which satisfies also the asymptotic condition $\lim_{x \to -\infty} P(x) = \rho^-$. Therefore there exist infinitely many stationary wave profiles. \(\Box\)

**Local stability of the profiles.** Recall the definitions of $\rho_1, \rho_2, \rho_3, \rho_4$ in (3.2). Let $P^\sharp$ denote the profile with $P(x) = \rho^+ = \rho_3$ on $x \geq 0$, as in Theorem 3.2. Let $P^\flat$ denote the profile with $P(x) = \rho_2$ on $x \geq 0$, as in Theorem 3.1. We define the region $D$ as:

$$D = \\{ (x, \rho) : P^\flat(x) < \rho \leq P^\sharp(x), \ x \in \mathbb{R} \}. \quad (3.7)$$

The next Theorem shows that $D$ is a basin of attractions of the stationary profiles.
Theorem 3.3 (Local stability of the profiles). Let \( \{z_i(t), \rho_i(t)\} \) be the solution of the FtLs model \((1.7)\), with initial condition \( \{z_i(0), \rho_i(0)\} \) which satisfies
\[
(z_i(0), \rho_i(0)) \in D \quad \forall i \in \mathbb{Z}.
\]
Then, we have
\[
(z_i(t), \rho_i(t)) \in D \quad \forall i \in \mathbb{Z}, t \geq 0.
\]
Furthermore, there exists a stationary wave profile \( \hat{P} \), whose graph lies between \( P^\rho \) and \( P^\ell \), such that
\[
\lim_{t \to \infty} \left[ \hat{P}(z_i(t)) - \rho_i(t) \right] = 0, \quad \forall i \in \mathbb{Z}.
\]

Proof. (1). By Lemma 2.8, all stationary profiles in Theorem 3.2 never cross each other. Then, any point \((x, \rho)\) \( \in D \) lies on a unique stationary profile. In other words, the region \( D \) can be parametrized by the profiles \( P \). This motivates the definition of a function
\[
\Psi(x, \rho) \doteq P(0), \quad \text{where } (x, \rho) \in D \text{ and } P \text{ is the profile with } P(x) = \rho.
\]
Let \( \{z_i(t), \rho_i(t)\} \) be the solution of the FtLs model, and define
\[
\Psi_i(t) \doteq \Psi(z_i(t), \rho_i(t)), \quad \forall i \in \mathbb{Z}, \forall t \geq 0.
\]
(2). Fix a time \( t \geq 0 \) and assume that \( \{z_i(t), \rho_i(t)\} \) \( \in D \) for all \( i \in \mathbb{Z} \). Let \( J \) be the index such that
\[
\Psi_J(t) \geq \Psi_i(t) \quad \forall i \in \mathbb{Z} \quad \text{and} \quad \Psi_J(t) > \Psi_i(t) \quad \forall i > J.
\]
Assume that \( J \) is finite, and let \( \hat{P} \) be the profile that satisfies \( \hat{P}(z_J(t)) = \rho_J(t) \). By (3.13) we have
\[
\hat{P}(z_i) > \rho_i \quad \forall i > J.
\]
We claim that
\[
\Psi_J(t) < 0, \quad \text{i.e.,} \quad \hat{P}'(z_J) > \frac{\hat{\rho}_J}{\hat{z}_J}.
\]
Indeed, let \( \{\hat{z}_i\} \) be the unique car distribution generated by \( \hat{P} \) with \( \hat{z}_J = z_J \), and let \( \hat{P}^\ell_{\{z_J\}} \) be the piecewise constant function generated by \( \hat{P} \) according to (2.23). We also let \( \rho^\ell(t, \cdot) \) be the piecewise function defined in (1.3). By (2.18), (1.7) and (2.12) we have
\[
\hat{P}'(z_J) = \frac{\rho^2_J}{\ell \cdot v^*(z_J; \hat{P}^\ell_{\{z_J\}})} \left[ v^*(z_J; \hat{P}^\ell_{\{z_J\}}) - v^*(L\hat{P}(z_J); \hat{P}^\ell_{\{z_J\}}) \right],
\]
\[
\frac{\hat{\rho}_J}{\hat{z}_J} = \frac{\rho^2_J}{\ell \cdot v^*(z_J; \rho^\ell)} \left[ v^*(z_J; \rho^\ell) - v^*(z_{J+1}; \rho^\ell) \right].
\]
We compute
\[
I_1 \doteq \left[ v^*(z_J; \hat{P}^\ell_{\{z_J\}}) - v^*(z_{J+1}; \hat{P}^\ell_{\{z_J\}}) \right] - \left[ v^*(z_J; \rho^\ell) - v^*(z_{J+1}; \rho^\ell) \right]
\]
\[
= \left[ v^*(z_J; \hat{P}^\ell_{\{z_J\}}) - v^*(z_J; \rho^\ell) \right] - \left[ v^*(z_{J+1}; \hat{P}^\ell_{\{z_J\}}) - v^*(z_{J+1}; \rho^\ell) \right]
\]
\[
= \int_{z_J}^{z_{J+1}} V(y) \left[ \phi(\hat{P}^\ell_{\{z_J\}}(y)) - \phi(\rho^\ell(t, y)) \right] w(y - z_J) \, dy
\]
\[
- \int_{z_{J+1}}^{z_{J+1}+h} V(y) \left[ \phi(\hat{P}^\ell_{\{z_J\}}(y)) - \phi(\rho^\ell(t, y)) \right] w(y - z_{J+1}) \, dy.
\]
Since $\hat{P}(z_J) = \rho_J$, we have $z_{J+1} = \hat{z}_{J+1}$. By \[3.14\] we also have
$$\hat{P}_{\{z_J\}}(x) = \rho^\ell(t, x) \quad \forall x \in [z_J, z_{J+1}), \quad \hat{P}_{\{z_J\}}(x) > \rho^\ell(t, x) \quad \forall x > z_{J+1}.$$ We arrive at
$$I_1 = \int_{z_{J+1}}^{z_{J+1}+h} V(y) \left[ \phi(\hat{P}_{\{z_J\}}(y)) - \phi(\rho^\ell(t, y)) \right] \cdot [w(y - z_J) - w(y - z_{J+1})] \, dy.$$ Since $\phi' < 0$, we have
$$\phi(\hat{P}_{\{z_J\}}(y)) - \phi(\rho^\ell(t, y)) < 0 \quad \forall y > z_{J+1}.$$ Recall also that $w' < 0$ on its support $[0, h]$, then we have
$$w(y - z_J) - w(y - z_{J+1}) < 0 \quad \forall y \in [z_{J+1}, z_{J+1} + h).$$ Therefore we conclude that $I_1 > 0$, which implies \[3.15\].

(3). A completely symmetric result holds for the minimum. Let $\hat{J}$ be the index such that
$$\Psi_j(t) \leq \Psi_i(t) \quad \forall i \in \mathbb{Z} \quad \text{and} \quad \Psi_j(t) < \Psi_i(t) \quad \forall i > \hat{J}. \quad (3.16)$$ Assume $\hat{J}$ is finite, and let $\hat{P}$ be the profile that satisfies $\hat{P}(z_j(t)) = \rho_j(t)$. Then
$$\dot{\Psi}_j(t) > 0, \quad \text{i.e.,} \quad \dot{\hat{P}}(z_j) < \frac{\dot{\rho}_j}{\hat{z}_j}. \quad (3.17)$$

(4). We note that, if $\{z_i(0), \rho_i(0)\} \in D$ for all $i \in \mathbb{Z}$, then \[3.15\] and \[3.17\] imply that $\{z_i(t), \rho_i(t)\} \in D$ for all $i \in \mathbb{Z}$ and $t \geq 0$. Furthermore, if $D$ is not empty, then at least one of $J, \hat{J}$ is finite, we have
$$\lim_{t \to \infty} \left[ \max_{i \in \mathbb{Z}} \{\Psi_i(t)\} - \min_{i \in \mathbb{Z}} \{\Psi_i(t)\} \right] = 0,$$ which further implies \[3.10\], completing the proof. 

**Sample profiles and numerical simulations for the FtLs model.** We use the same parameters and functions in \[3.4\]-\[3.5\]. Sample profiles for subcase 1B are given in Figure 3 (left plot). We observe that on $x > 0$, the profiles match the horizontally shifted versions of $W$. The profiles are continuous and monotone.

Numerical simulations for the FtLs model \[1.7\] are performed with the following Riemann-like initial data
$$z_i(0) = \begin{cases} 
\frac{i \ell}{\rho^+} + c_0, & i \geq 0, \\
\frac{i \ell}{\rho^-} + c_0, & i < 0,
\end{cases} \quad \text{such that} \quad \rho_i(0) = \begin{cases} 
\rho^+, & z_i(0) \geq c_0, \\
\rho^-, & z_i(0) < c_0.
\end{cases} \quad (3.18)$$ Here $c_0 \in \mathbb{R}$ is a small number, which we can vary to simulate various cases. Numerical solutions for the FtLs model at $T_f = 4$ is computed, for
$$c_0 = \gamma \cdot \ell / \rho^-, \quad \gamma = \{-0.1, 0.1, 0.5, 1\}.$$ Similar to subcase 1A, we plot the mappings $z_i(t) \mapsto \rho_i(t)$ for $t \in [T_f - \tau_p, T_f]$ in green in Figure 2 (right plot), together with the points $\{z_i(T_f), \rho_i(T_f)\}$ in red. We observe that, after a short time, the solutions $\{z_i(t), \rho_i(t)\}$ approach some stationary wave profile $P$, confirming the stability result in Theorem 3.3.
3.3 Subcase 1C \( \hat{\rho} \leq \rho^+ < \rho^- < 1 \) and subcase 1D \( 0 < \rho^+ < \hat{\rho} < \rho^- < 1 \)

For subcase 1C, \( \rho^- > \hat{\rho} \) is not a stable asymptote for \( x \to -\infty \). For subcase 1D, \( \rho^+ < \hat{\rho}, \rho^- > \hat{\rho} \) are not a stable asymptotes for \( x \to -\infty \). Therefore there are no stationary profiles for either subcases.

**Numerical simulations.** We perform numerical simulation for subcases 1C and 1D, for the FtLs model (1.7) with Riemann-like initial condition (3.6). The results are presented in Figure 4. For each subcase, we plot the mappings \( z_i(t) \mapsto \rho_i(t) \) for \( t \in [T_f - \tau_p, T_f] \) in green, as well the points \( \{z_i(T_f), \rho_i(T_f)\} \) in red. For subcase 1C, we observe that oscillations enter into the region \( x < 0 \) as time grows, and they persist in time. For subcase 1D, we observe that oscillations enter both regions \( x < 0 \) and \( x > 0 \) as \( t \) grow.

4 Case 2. \( V^- < V^+ \)

In this section we establish results on stationary wave profiles for the case \( V^- < V^+ \). The structure and some details of the analysis are similar to case 1, therefore we skip the repetitive details and emphasize the differences. Fix a value \( \bar{f} \), we set (see Figure 5)

\[
0 \leq \rho_1 < \rho_2 \leq \hat{\rho} \leq \rho_3 < \rho_4 \leq 1, \quad f^+(\rho_1) = f^+(\rho_4) = f^-(\rho_2) = f^-(\rho_3) = \bar{f}. \quad (4.1)
\]

We consider the nontrivial case \( \bar{f} > 0 \), in 4 subcases:
Figure 5: Graphs of the functions $f^-, f^+$, and locations of $\rho_1, \rho_2, \rho_3, \rho_4$ and $\hat{\rho}$.

| subcase | 2A | 2B | 2C | 2D |
|---------|----|----|----|----|
| $(\rho^-, \rho^+)$ | $(\rho_2, \rho_1)$ | $(\rho_2, \rho_4)$ | $(\rho_3, \rho_4)$ | $(\rho_3, \rho_1)$ |

4.1 Subcase 2A: $0 < \rho^+ < \rho^- \leq \hat{\rho}$

Since $\rho^+ < \hat{\rho}$, $\rho^+$ is an unstable asymptote as $x \to +\infty$. Similar to subcase 1A, the only possible solution on $x \geq 0$ is the constant function $P(x) \equiv \rho^+$. Following similar arguments as in the proof for Theorem 3.1, there exists a unique stationary wave profile.

**Theorem 4.1** (Existence of a unique profile). Given $V^\pm, \rho^\pm$ as in subcase 2A. There exists a unique stationary wave profile $P$ which satisfies (2.18)-(2.19). The profile is constant on $x \geq 0$, and monotone decreasing on $x < 0$.

A sample profile can be found in Figure 6 (left plot), using the same parameters and functions in (3.4)-(3.5), except for $V^- = 1, V^+ = 2$. Similar to subcase 1A, since $\rho^+ < \hat{\rho}$ is an unstable asymptote as $x \to \infty$, the profile $P$ does not attract nearby solutions of the FtLs model (1.7). This is confirmed by the numerical evidence in Figure 6 (right plot) for the solutions of (1.7).

Figure 6: Left: Typical profile $P$ for case 2A. Right: Solution of the FtLs model with Riemann-like initial condition.

4.2 Subcase 2B: with $0 < \rho^- < \hat{\rho} < \rho^+ < 1$

This is the counterpart for subcase 1B. Here both $\rho^-$ and $\rho^+$ are stable asymptotes at $x \to -\infty$ and $x \to \infty$ respectively. Similar to subcase 1B, there are infinitely many
stationary wave profiles. However, due to the upward jump in $V$ at $x = 0$, the profiles are no longer monotone, resulting in more involving analysis.

**Theorem 4.2** (Existence of profiles). Given $V^\pm, \rho^\pm$ as in subcase 2B. There exist infinitely many stationary wave profiles $P$ which satisfies (2.18)-(2.19). These profiles are monotone on $x > 0$, but might not be monotone on $x < 0$.

**Proof.**

1. On $x \geq 0$, $P$ must match some horizontal shift of $W$. Recall the definition (4.1). For any profile we have $P(x) > \rho_1$ on $x > 0$.

Let $P^b$ be the unique profile in Theorem 4.1 for subcase 2A, with $P^b(x) = \rho_1$ on $x \geq 0$. The profile is monotone decreasing and satisfies the asymptotic condition $\lim_{x \to -\infty} P^b(x) = \rho^-$. Furthermore, since $W(x) > \rho_1$ on $x \geq 0$, by the ordering property in Lemma 2.8 we conclude that all profiles $P$ will lie above $P^b$ also on $x < 0$. Thus, $P^b$ serves as a lower envelope for all profiles.

2. Consider the interval $I = [z^b, 0]$ where $L^P(z^b) = -h$. Using horizontally shifted versions of $W$ as initial condition for $P$ on $x \geq 0$, we get various solutions of $P$ on $I$. By continuity there exist infinitely many profiles $P$ such that $P(x) < \rho_b$ for all $x \in I$, and $P$ is monotone decreasing on $I$. By a similar proof as for Theorem 4.1 one concludes the existence of infinitely many monotone stationary wave profiles on $x < 0$.

3. It remains to show that, if a profile $P$ lies between $\rho^-$ and $\rho_3$ on $x < z^b$, then we must have

$$\lim_{x \to -\infty} P(x) = \rho^-.$$ (4.2)

Indeed, we first observe that, if $P$ approaches a limit as $x \to -\infty$, then it must be either $\rho^-$ or $\rho_3$ according to the periodic property. If in addition $P$ is monotone, then since $\rho_3$ is an unstable asymptote, then $P$ must be monotone increasing on $x < 0$, and we conclude (4.2).

4. We are left to consider the case that $P$ is oscillatory on $x < z^b$ between $\rho^-$ and $\rho_3$. By the periodic property we derive that

$$\frac{P(x)}{\ell} \int_x^{x+\ell/P(x)} \frac{1}{\nu^*(z; P^L(x))} dz = \frac{P(x)}{f} = \frac{f(P(x))}{f(\rho^-)} \cdot \frac{1}{V^- \phi(P(x))}.$$ (4.3)

Since $f$ is strictly concave, for any $\rho \in [\rho^-, \rho_3]$ we have the estimate

$$\frac{f(\rho)}{f(\rho^-)} = 1 + \frac{f(\rho) - f(\rho^-)}{f} \geq 1 + C_f(\rho)$$ (4.4)

where $C_f$ is a positive function, defined as

$$C_f(\rho) \triangleq \frac{1}{f} \cdot \min \left\{ (\rho - \rho^-), (\rho_3 - \rho) \right\} \cdot \min \left\{ \frac{f(\rho) - f(\rho^-)}{\rho - \rho^-}, \frac{f(\rho) - f(\rho_3)}{\rho - \rho_3} \right\}.$$

By (4.3)-(4.4) we now have

$$\frac{1}{V^- \phi(P(x))} \left( 1 + C_f(P(x)) \right) \leq \operatorname{average}_{y \in [x, L^P(x)]} \left\{ \frac{1}{\nu^*(y; P^L(y))} \right\}.$$
Since φ is a monotone function, this implies that,

\[ \forall x < z^*, \ \exists y \in (x, L^P(x) + h) \ \text{s.t.} \ \ P(x)(1 + C_f(P(x))) < P(y), \ \ (4.5) \]

In particular, (4.5) holds also for the case where \( x \) is a local maximum. Let \( \{x_k\} \) be a sequence of local maximum such that \( x_{k+1} < x_k \) and \( P(y) < P(x_k), \forall y < x_k \). By (4.5) we have that

\[ P(x_{k+1})(1 + C_f(P(x_{k+1}))) \leq P(x_k) \ \ \forall k, \ \ (4.6) \]

where \( C_f(\rho) = 0 \) only when \( \rho = \rho^- \). Thus, these max values (if they exist) are strictly monotone. If \( \{x_k\} \) is a finite sequence, we conclude (4.2). Otherwise, if \( \{x_k\} \) is an infinite sequence, then (4.6) implies that \( \lim_{k \to \infty} P(x_k) = \rho^- \), concluding (4.2).

**Local Stability.** The stability result and the proof for these profiles are the same as Theorem 3.3 for subcase 1B, and we omit the details. Sample profiles and numerical simulations for the FtLs model are presented in Figure 7, similar to subcase 1B.

**Subcases 2C and 2D.** Similar to the subcases 1C and 1D, there are no stationary profiles for subcases 2C and 2D. In Figure 8 we present similar numerical simulations for these subcases for the FtLs model (1.7).

![Figure 7: Left: Typical profiles \( P \) for subcase 2B. Right: Solution of the FtLs model with Riemann-like initial condition for subcase 2B.](image1)

![Figure 8: Solution of the FtLs model with Riemann-like initial condition for subcase 2C (left plot) and subcase 2D (right plot).](image2)

5 Convergence of stationary wave profiles to various limits

As mentioned in the introduction, when the road condition \( V \) is discontinuous, rigorous analysis of convergences for (1.7) to various limits as \( \ell \to 0 \) and/or \( h \to 0 \) are interesting
open problems. As a first step in this direction, we study two of these limits in the setting of stationary wave profiles.

**Limit 1: Micro-macro limit.** Fix \( h > 0 \). Formally, as \( \ell \to 0 \), the particle model (1.7) converges to the nonlocal PDE with discontinuous coefficient (1.9). Let \( Q \) be a stationary profile for (1.9) around \( x = 0 \). Then, \( Q \) satisfies the following integral equation with discontinuous coefficient

\[
Q(x) \cdot \int_x^{x+h} V(y)\psi(Q(y))w(y-x) \, dy = \bar{f} \quad \forall x \in \mathbb{R}. \tag{5.1}
\]

With a slight abuse of notation, we denote the averaging operator as

\[
\mathcal{A}(x; V, Q) = \int_x^{x+h} V(y)\psi(Q(y))w(y-x) \, dy. \tag{5.2}
\]

Taking the limit \( x \to \pm \infty \) we get

\[
\lim_{x \to \pm \infty} \mathcal{A}(x; V, Q) = V^{-}f^{-}(\rho^{-}) = V^{+}f^{+}(\rho^{+}) = \bar{f}, \quad \text{where } \rho^{\pm} = \lim_{x \to \pm \infty} Q(x).
\]

The stationary profiles \( Q \) were studied in a recent work [30], where results on existence, uniqueness, and stability are established, in a similar setting as for the FtLs model (1.7). For a given set of asymptotic value \( \rho^{\pm} \), for subcases 1A and 2A, there exists a unique stationary wave profile \( Q \), which is not asymptotically stable. For subcases 1B and 2B, there exist infinitely many stationary wave profiles, which are local attractors for the solution of the Cauchy problems of the conservation laws. For all other subcases there are no stationary wave profiles. Sample profiles \( Q \) for subcases 1B and 2B are shown in Figure 9 taken from [30], for comparison. Furthermore, when \( V(x) \equiv 1 \), the micro-macro limit of the traveling wave profiles is proved in [27].

We consider subcases 1A, 1B, 2A, and 2B, where stationary wave profiles exist.

**Theorem 5.1.** Fix \( h > 0 \) and \( \bar{f} \). Let \( V^{\pm}, \rho^{\pm} \) be given such that we are in one of the subcases 1A, 1B, 2A, and 2B. Let \( P^{\ell} \) be a stationary wave profile for the FtLs models such that \( P^{\ell}(0) = \rho_0 \). Then, as \( \ell \to 0^+ \), the sequence \( \{P^{\ell}\} \) converges to a limit profile \( Q \). The profile \( Q \) is a stationary profile for the conservation law (1.9), with \( Q(0) = \rho_0 \).
Proof. On $x \geq 0$, since $V(x) \equiv V^+$ is constant, the micro-macro convergence follows from the result in [27]. On $x \leq 0$, since the set of functions $\{\mathcal{P}^\ell\}$ is equicontinuous, by the Arzelà-Ascoli Theorem, as $\ell \to 0$ the sequence $\{\mathcal{P}^\ell\}$ converges to a limit function $\hat{P}$ uniformly on bounded sets. Furthermore, since the asymptotic condition $\lim_{x \to -\infty} P_\ell(x) = \rho^-$ is satisfied for all $\ell$, we conclude that the convergence is uniform for all $x \in \mathbb{R}$.

It remains to show that $\hat{P}(x) \equiv Q(x)$ for $x \leq 0$, i.e., $\hat{P}$ satisfies the equation (5.1). Indeed, let $P^\ell_{\{x\}}$ be the piecewise constant function generated by $\mathcal{P}^\ell$ according to (2.16), using the periodicity (2.29) we get

$$\frac{\mathcal{P}^\ell(x)}{f} = \left(\frac{\ell}{\mathcal{P}^\ell(x)}\right)^{-1} \int_x^{x+\ell/\mathcal{P}^\ell(x)} \frac{1}{v^*(z; P^\ell_{\{z\}})} dz = \text{average}_{z \in [x, x+\ell/\mathcal{P}^\ell(x)]} \left\{ \frac{1}{v^*(z; P^\ell_{\{z\}})} \right\}.$$

Taking the limit $\ell \to 0$, the above equation gives

$$\frac{\hat{P}(x)}{f} = \lim_{\ell \to 0} \left[ \text{average}_{z \in [x, x+\ell/\mathcal{P}^\ell(x)]} \left\{ \frac{1}{v^*(z; P^\ell_{\{z\}})} \right\} \right] = \frac{1}{A(x; V, \hat{P})} \quad \forall x < 0,$$

which is exactly the equation (5.1), completing the proof.

Limit 2: Nonlocal to local. Fix $\ell > 0$ and let $h \to 0$, we formally obtain the local particle model (1.10). The stationary profile $U$ satisfies a discontinuous delay differential equation (DDDE)

$$U'(x) = \frac{U(x)^2}{\ell V(x) \phi(U(x))} \cdot \left[ V(x) \phi(U(x)) - V(x^\sharp) \phi(U(x^\sharp)) \right], \quad x^\sharp = x + \frac{\ell}{U(x)}. \quad (5.3)$$

The DDDE is studied in detail in [29], where similar results on existence, uniqueness and stability are established. Typical profiles for case 1B and 2B are shown in Figure 10, taken from [29]. We have a similar convergence result.

Theorem 5.2. Fix $\ell > 0$, $\bar{f} > 0$ and let $h > 0$ be given. Let $V^\pm, \rho^\pm$ satisfy the conditions for subcases 1A, 1B, 2A, or 2B. Denote by $\mathcal{P}^h$ the set of profiles for various $h$ such that $\mathcal{P}^h(0) = \tilde{\rho}$ for all $h$. Then, as $h \to 0$, the sequence $\{\mathcal{P}^h\}$ converges to the profile $U$ which satisfies (5.3), the asymptotic conditions $\lim_{x \to \pm\infty} U(x) = \rho^\pm$, and $U(0) = \tilde{\rho}$.  

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Proof. The proof is very similar to the proof of Theorem 5.1. Since the sequence \( \{P^h\} \) is equicontinuous, by the Arzelà-Ascoli Theorem it converges to a limit function \( \tilde{P} \) uniformly on bounded sets, as \( h \to 0 \). To show that \( \tilde{P} \equiv U \), we observe that, as \( h \to 0 \), \( w \to \delta_0 \) (a Dirac delta at the origin), and we have

\[
\lim_{h \to 0} A(x; V, P^h) = V(x) \phi(\tilde{P}(x)), \quad \lim_{h \to 0} A(LP^h(x); V, P^h) = V(L\tilde{P}(x)) \phi(\tilde{P}(L\tilde{P}(x))),
\]

for every \( x \in \mathbb{R} \). Recall that \( L\tilde{P}(x) = x + \frac{\ell(\tilde{P})}{F(x)} \). Thus, by (2.18) we conclude that \( \tilde{P} \) satisfies the equation (5.3), as well as the asymptotic conditions and the condition at \( x = 0 \), completing the proof.

6 Concluding remarks

In this paper we study stationary wave profiles for a nonlocal particle model of traffic flow on rough roads where the speed limit function \( V \) is discontinuous at \( x = 0 \). We establish results on the existence, unique, and stability for these profiles for all cases.

As comparison, we present another nonlocal particle model where

\[
\dot{z}_i(t) = V(z_i) \cdot \phi(\rho^*(t, z_i)), \quad \text{where} \quad \rho^*(t, z_i) = \int_{z_i}^{z_i+h} \rho^t(t, y)w(y - z_i) \, dy, \quad (6.1)
\]

and \( \rho^t \) is the piecewise constant function defined in (1.3). Here the weighted average is taken over the discrete density function.

When the road condition is uniform with \( V(x) \equiv 1 \), travelling wave profiles are studied in [27]. In the case of rough road condition, similar analysis on stationary wave profiles can be carried out and similar results can be proved, with small changes in the detail. In Figure 11 we present sample profiles for subcases 1A, 1B, 2A and 2B.

Unfortunately, model (6.1) has a fatal flaw. In certain situations the model leads to traffic accidents, where the discrete density \( \rho_i(t) \) becomes bigger than 1, even with initial density less than 1. See Figure 12 for two numerical simulations that demonstrate this scenario, where we use \( (V^-, V^+) = (2, 1) \) and the initial conditions

left plot: \( \rho_i(0) = \begin{cases} 0.9 & \text{if } z_i(0) < 0, \\ 0.75 & \text{if } z_i(0) > 0, \end{cases} \) \quad right plot: \( \rho_i(0) = \begin{cases} 0.9 & \text{if } z_i(0) < 0, \\ 0.25 & \text{if } z_i(0) > 0. \end{cases} \)

We observe that, at the origin \( \rho_i(t) \) has a peak larger than 1, for some \( t > 0 \).

Numerical simulations for generating the plots used in this paper are carried out in Scilab. The source codes can be found at:

www.personal.psu.edu/wxs27/SIM/Traffic-CS-2019

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subcases 1A and 1B

subcases 2A and 2B

Figure 11: Typical profiles for the alternative model, for various subcases.

Figure 12: Two simulations at $t = 1$, showing that, when $V^- > V^+$, car density becomes bigger than 1 as time grows, indicating cars crashing.

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