And what if gravity is intrinsically quantic?

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Abstract. Since the early days of search for a quantum theory of gravity the attempts have been mostly concentrated on the quantization of an otherwise classical system. The two most contentious candidate theories of gravity, string theory and quantum loop gravity are based on a quantum field theory - the latter is a quantum field theory of connections on a $SU(2)$ group manifold and former a quantum field theory in two dimensional spaces. Here we argue that there is a very close relation between quantum mechanics and gravity. Without gravity quantum mechanics becomes ambiguous. We consider this observation as the evidence for an intrinsic relation between these fundamental laws of nature. We suggest a quantum role and definition for gravity in the context of a quantum universe, and present a preliminary formulation for gravity in a system with a finite number of particles.

1. Introduction and motivations
Since the beginning of search for quantum theory of gravity all approaches have considered the quantization of a classical field theory. Nonetheless the idea of an intrinsic relation between gravity and quantum mechanics is not new. The first evidence can be claimed to be the black hole entropy and its analogy with thermodynamics. However, it is well known that without considering Hawking radiation [1] and Bekenstein entropy limit [3], this analogy seems to be just a mathematical similarity. In fact the black hole entropy is a purely geometrical property related to the diffeomorphism symmetry of and independent of the special case of the Einstein gravity Lagrangian [4, 2]. The discovery of similarity between Hawking radiation temperature and the equivalent temperature obtained from black hole entropy gave another dimension to this as a possible mediator between classical and quantum gravity theories. More recently it has been proved that in D-brane models [5] - a subclass of compactified string models - at low energies and weak couplings the direct counting of the states of an extremal or near extremal black hole lead to an entropy similar to the classical black holes [6]. These results enforce the relation between the classical black hole entropy and the quantum nature of gravity.

On the way to a quantum theory or a quantum connection for gravity various strategies have been taken. Apart from investigations of candidate models such as string theory, loop quantum gravity, causal set models, etc., in which the metric $g_{\mu\nu}$ or connection $\Gamma^\rho_{\mu\nu}$ are quantum fields, many authors tried to derive a gravity theory from a field theory in a curved space time. For instance T. Padmanabahan [7] has used Rindler metric - a frame with constant acceleration - as background rather than Minkowski metric and a somehow general Lagrangian for gravity. It includes a non-trivial surface term on space-like 3-surfaces. By postulating the relation between entropy, temperature, and surface gravity obtained from Einstein gravity, a dynamical equation
similar to one from Einstein gravity has been obtained. This exercise shows explicitly that there is a close relation between the laws of black hole thermodynamics and Einstein gravity. However, black hole entropy is the Noether charge in the models with diffeomorphism symmetry [4]. Therefore it is not a surprise that in a given background the assumption of entropy relation obtained from Einstein gravity leads to Einstein equation. This is analogue to gauge symmetry of electrodynamics in which the presence of a global conserved charge (surface gravity) and the application of Gauss theorem permits to define a 2-form (connection) on a 3-surface. Then by using the gauge symmetry (diffeomorphism) and its conserved current, one can obtain Maxwell equation and its corresponding Lagrangian.

Another popular approach is the construction of a class of theories generally called emergent gravity models in which gravity is considered to be a collective low energy/classical effect of a fundamental microscopic model in a flat Euclidean or Lorentzian background and very different from classical gravity [8]. In this case, gravity is considered to be a classical effect similar to condensation process in condense matter physics. A curved space time with pseudo-Riemannian metric and diffeomorphism invariance have been found in a subset of these models notably models called analogue gravity [9]. For the time being, in none of these models a tensor field theory with properties similar to what we know from Einstein gravity has been found. But there have been progresses in this direction. For instance, recently in a work by F. Girelli, et al. [10] a Nordström scalar gravity model has been constructed from a N-scalar field theory on an Euclidean background.

In all the models mentioned above the background space is classical and a quantum field theory is defined on this space. The same line of thinking has been considered also in noncommutative field theory models with a quantum spacetime [11]. The main physical motivation for introducing noncommutative geometries is the fact that due to Heisenberg uncertainty, a very precise measurement of spacetime will create very large uncertainty in energy-momentum that can produce a black hole preventing further investigation of smaller distances. To solve this problem coordinates are considered to have a non-zero commutation. New progresses [12, 13] in this class of models show that a classical gravity model similar to Einstein gravity can emerge from these models under some conditions and simplifications. As the effect of a quantum spacetime in these models is measurable only at Planck scales, at present there is no direct way to verify them.

Here we want to consider the relation between classical gravity and quantum mechanics in a more intuitive way and see what physical properties join them and what make them inconsistent with each other.

There are a number of unsolved issues in gravity and in quantum mechanics that have fundamental consequences for understanding these phenomena. Followings are some of them:

(i) Why is gravity a universal force ?
(ii) Why the Plank Constant $\hbar$ is universal ?
(iii) Why there is no fundamental mass/length scale in QM ?

The first question above is another way of asking the origin of the Equivalence Principle. This issue has been studied intensively in the framework of classical and quantum gravity models, and in grand unified theories. In fact in many candidate models of quantum gravity such as string theory Equivalence Principle is a low energy effect that breaks at high energies. Nonetheless, at present the upper limit on the breaking of equivalence strongly constrains some of low energy quantum gravity models such as brane models inspired from D-brane compactification of strings. This rises the issue of what is the most fundamental principle in gravity ? Its dependence only on energy-momentum or its geometrical origin as the metric of a pseudo-Riemannian space, or both. Breaking of Equivalence Principle violates the first condition. Therefore if we consider
this principle as part of the definition of what we call gravity, models violating it cannot be considered as a genuine quantum gravity model.

The second universality means that the amount of uncertainty or randomness in all physical systems is the same regardless of their mass, size, and couplings. Let’s assume that the value of $\hbar$ for a particle of mass $m$ is different by a factor of $\alpha$. The Schrödinger/Klein-Gordon equation becomes:

$$(\alpha^2 \hbar^2 \Box - m^2) |\psi\rangle = 0 \iff (\hbar^2 \Box - m'^2) |\psi\rangle = 0 \quad \text{where} \quad m' = \frac{m}{\alpha}$$

Therefore the non-universality of $\hbar$ can be removed by redefinition of mass. But mass is the gravity charge! In the same way a different coupling $G$ to gravity can be removed by redefinition of mass which in its turn modifies Schrödinger equation. Note that if mass is generated by interaction and is related to the Vacuum Expectation Value (VEV) of a field this scaling can be performed on the VEV. This operation is allowed in quantum mechanics because the energy reference is arbitrary. But considering gravity, scaling modifies the coupling of fields to gravity, except in conformal models where the scaling can be absorbed in the space volume. In fact it has been shown that Einstein and higher order gravity models are related to each other by conformal transformation if these models have a scalar field in their matter sector [14]. This is the case for all gauge models where the mass of particles are due to VEV of a higgs type scalar fields. In this case additional terms from non-conformal matter can be considered as non-minimal interaction terms.

The origin of the close dependence of quantum mechanics on gravity is the third item above: QM lacks any fundamental energy or length scale\(^1\). Therefore a theory as fundamental as QM, its extension QFT, and thereby Standard Model (SM) of particle physics need the presence of gravity for being a complete and applicable theory to Nature. On the other hand, gravity is supposed to be just an interaction and the general tendency is to see it with the same eye as the other interactions (except in emergent gravity models mentioned above). Quantum mechanics does not need any of other known interaction e.g. electromagnetism to be meaningful. Thus we can conclude that gravity is fundamentally different and must have a much closer connection to quantum mechanics.

Although emergent gravity models seem to have shown this connection, gravity at low energies rests classical and its connection to a quantum universe appears only at Planck scale. Nonetheless, with simple examples we can show that even at low energies the non-deterministic nature of quantum mechanics is in conflict with a classical (deterministic) universal force like gravity.

2. Ambiguity of a classical universal force in a quantum universe

Consider an empty Minkovski space. We are only interested in low energy scales, therefore we neglect vacuum fluctuations and all the phenomena related to quantum field theory. We add to this space a single particle of mass $m$. To remove the ambiguity due to black hole formation, we can consider a small but finite size for the particle larger than its Schwarzschild radius. In this case if the resolution of the measurement is much larger than the size of the particle it can be considered as point-like and the argument below is correct up to this approximation.

As there is no other observer/particle in this space, without loss of generality we can consider that the particle is at rest. Because we know the energy and momentum of this particle, according

\(^1\) Evidently any energy or length dependent quantity can be used as a scale. For instance $\Lambda_{\text{QCD}}$ the scale which separates perturbative and non-perturbative QCD can be also considered as a natural energy scale. However, we do not know if this point is unique or cover a range of energies with $\alpha_{\text{QCD}} \approx 1$. Moreover, for the time being the only fundamental interaction with a dimensional coupling constant is gravity, and therefore this constant presents the only natural energy scale in physics.
to the Heisenberg uncertainty rule we lose all information about its location, i.e. at any instant of time the particle can be at any point in the space. This has a number of consequences:

(i) A classical particle in a flat Minkovski space breaks the translation symmetry to a spherical symmetry and changes the flat metric to Schwarzschild.

(ii) A quantum particle as defined above does not break the translation symmetry. The wave function has the form of a free wave $\psi(x) \propto \exp(ip_\mu x^\mu)$ and the probability of detecting the particle in any finite volume approaches to zero (up to coarse-grain approximation mentioned before). To estimate the modification of the spacetime due to the presence of this particle, we can use the semi-classical Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle = \eta_{\mu\nu} \Lambda , \quad \Lambda \to 0 \quad (1)$$

The solution of (1) is simply a De Sitter metric which is conformally flat and as $\Lambda = m/V \to 0$, where $V$ is the coarse-grained volume, the metric becomes very close to the metric of the initial flat Minkovski space. Note that up to coarse-grain approximation this result is independent of the mass of the particle. Conclusion: In the context of quantum mechanics a single massive particle does not change the spacetime by its gravity!

(iii) If we add a second particle with only gravitational interaction with the first one, we can use Wheeler-DeWitt equation and the above metric as background to determine the wave function of the second particle. As the background metric is at most De-Sitter, otherwise flat, the wave function would be again a free wave but the wave number gradually decreases due to the expansion of the space time, and at coarse-grain limit it approaches its value in a flat spacetime. Moreover, the homogeneity of the spacetime does not change by the presence of a second particle either. This means that they do not interact gravitationally. In the same way we can add any number of particles and up to coarse-grain limit they do not feel each others! This paradoxical result is evidently inconsistent with the behaviour we know from classical gravity.

(iv) At present we do not have any experimental evidence that tell us what happens in such a setup.

The origin of the conflict between classical and quantum predictions is the fact that gravity and its dynamical equation are local, in contrast to quantum mechanics which is intrinsically nonlocal. Note that a noncommutative spacetime and emergent gravity can not solve these ambiguities. The setup of this example satisfies the noncommutative relation simply because the location of the particle is completely uncertain in all directions. This sort of ambiguities should be added to the well known conflict between diffeomorphism invariance that leads to a null Hamiltonian and the lack of the concept of time evolution in quantum gravity [15].

These observations rise the idea that maybe we should think about gravity or whatever shows itself as gravity at low energies in another way, presumably in a quantic manner. Here we present some ideas about what can be the role of gravity with properties we know in a quantum world, and how possibly a corresponding field theory can be constructed.

3. The role of gravity in a quantic Universe

Consider the phase space of a classical system and the Hilbert space of the same system when it is quantized. We can define quantization as the definition of an equivalence class in the phase space:

$$\mathcal{H} \equiv \Phi/\mathcal{S} \quad (2)$$

where $\Phi$ is the phase space and $\mathcal{S}$ is the set of surfaces with area = $\hbar$. For simplicity we neglect the spacetime indexes and consider the phase space like a 2-dimensional surface. If we have a
system consisting of 2 non-interacting particles, their phase space or Hilbert space is the direct product of their respective single particle spaces:\(^2\)

\[
\Phi = \Phi_1 \otimes \Phi_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2
\]  

(3)

This means that particles don’t feel each other. However, the assumption of no interaction between two particles is not realistic because we know that gravity is a general force and whatever the nature of 2 particles, they interact through their gravity and therefore:

\[
\mathcal{H} \neq \mathcal{H}_1 \otimes \mathcal{H}_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{\text{gravity}} \mathcal{H}_1 \otimes \mathcal{H}_2
\]  

(4)

Here we want to suggest a new definition for gravity in a quantum context:

**Gravity between quantum particles presents the minimum deviation of their Hilbert space from direct products of single-particle Hilbert spaces**

This definition is consistent with the universality of gravity and with semi-classical results for one particle - a single particle with known energy and momentum does not feel its own gravity! A direct consequence of this property is that such a particle does not make a black hole!\(^3\) By contrast, if the same particle is localized, then its energy-momentum would be completely uncertain similar to particles inside a black hole, which for a far observer can be at any place inside the horizon\(^4\).

The presence of gravity becomes visible only if we have at least 2 particles in the system. A nontrivial projection of \(\mathcal{H}_1 \otimes \mathcal{H}_2\) to itself defines the new Hilbert space \(\mathcal{H}\). In fact when an interaction is present such a projection happens also in the phase space for both classical and quantum systems. What is more important in this model and different from others is that gravity mixes the equivalent classes:

\[
\mathcal{H} = \Phi_S, \quad S \neq S_1 \otimes S_2
\]  

(5)

This means that uncertainty surfaces for the two particles are not anymore separate but define an inseparable subspace of the total phase space \(\Phi\). This can solve the puzzle of universal \(\hbar\). Assuming that this projection depends on \(G_N\) or equivalently \(M_P\), this provides the missing mass/length scale for quantum mechanics. The extension of this construction to multi-particle systems is trivial and we do not present details here.

The nontrivial mixing of equivalence spaces presenting the Heisenberg uncertainty relation means that the uncertainties of coordinate and momentum of two particles are not any more independent. This is somehow similar to noncommutative models. However, in the latter case the spacetime is considered as an independent entity with noncommutative coordinates. Here the nontrivial commutation relation is not an intrinsic property but induced. It is possible that for large particle number the two models have some sort of relation. We leave the investigation of this eventuality to future works.

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\(^2\) Note that for simplifying the notification, we interchange the meaning of Hilbert space as the vector space to which all the state of a physical system belong and the orbit of the Lagrangian operator of the system for a given initial condition. The meaning should be clear from the context.

\(^3\) At first sight there is an ambiguity in this argument because in both classical and quantum mechanics a particle can be composite. We assume that equation (4) is applied to fundamental particles. In addition, in the framework of this model the concept of a single particle can be even meaningless. See the end of this section for more details.

\(^4\) Note that the Schwarzschild radius here is determined by the energy-momentum uncertainty and not by the invariant mass of the particle.
The nontrivial projection of the combined Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be considered as a basis transformation with the condition that in the new coordinates the transformed particles can be considered as free and separable. This basis would play the role of a local inertia frame in the classical general relativity and should constrain the projection. However, at this level of the development of the model we can not prove that it always exists. Therefore we consider that its existence is granted.

We expect that at classical limit the dynamic equation of this model approaches Wheeler-DeWitt equation with minimal interaction to gravity:

\[
(h^2 \Box + \frac{R}{6} - m^2)\langle \psi \rangle = 0 \tag{6}
\]

The general structure of the operator applied to the wave function is:

\[
\hat{P}^\mu \hat{P}_\mu + f_\mu(x)\hat{P}^\mu + h(x) \quad , \quad \hat{P}_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{7}
\]

We consider the following transformation from one-particle coordinates to free coordinates $X$:

\[
\hat{X}_i = \hat{x}_i \quad , \quad \hat{P}_i = \hat{p}_i + \sum_{j \neq i} f(\hat{x}_j, \hat{p}_j) \tag{8}
\]

where $f(\hat{x}_j, \hat{p}_j)$ is an arbitrary function. It is easy to show that commutations become:

\[
\begin{align*}
[\hat{x}_i, \hat{x}_j] &= [\hat{X}_i, \hat{X}_j] = 0 \quad i, j = 1, 2, \ldots \tag{9} \\
[\hat{p}_i, \hat{p}_j] &= [\hat{P}_i, \hat{P}_j] = 0 \tag{10} \\
[\hat{x}_i, \hat{p}_i] &= -i\hbar \tag{11} \\
[\hat{X}_i, \hat{P}_i] &= -i\hbar \tag{12} \\
[\hat{x}_i, \hat{P}_j] &= [\hat{X}_i, \hat{P}_j] = 0 \quad i \neq j \tag{13} \\
[\hat{x}_i, \hat{P}_j] &= [\hat{x}_i, f(\hat{x}_i, \hat{p}_i)] \tag{14}
\end{align*}
\]

where $f(\hat{x}_i, \hat{p}_i)$ is an arbitrary function. Note that in (9) to (13) indexes in indicate particles and spacetime indexes are neglected. Assuming that the function $f(\hat{x}_i, \hat{p}_i)$ is analytical, we can expand it as a polynomial:

\[
f(\hat{x}_i, \hat{p}_i) = \sum_{m,n} C_{mn} : \hat{x}_m^i \hat{p}_n^i : \Lambda^{n-m-1} \tag{15}
\]

where $C_{mn}$ is a dimensionless C-number constant and $\Lambda$ is an energy scale, presumably a scale comparable to Planck energy. The symbol :: indicates that operators $\hat{x}_i$ and $\hat{p}_i$ are ordered such that all position operators precede momentum operators. We assume that $n - m - 1$ the power of $\Lambda$ is always positive such that this term becomes important only at energies close to Planck energy. In this case the dominant term in (15) is the term with $m = -1$ and $n = 1$. If we neglect terms with higher power of $\Lambda$ and include the constant $C_{-1,1}$ in $\Lambda$:

\[
\begin{align*}
\hat{P}_i &= \hat{p}_i + \sum_{j \neq i} : \frac{\hat{p}_j}{\Lambda \hat{x}_j} : \\
[\hat{x}_i, \hat{P}_j] &= [\hat{X}_i, \hat{P}_j] = -i\hbar \Lambda \hat{x}_i \tag{17}
\end{align*}
\]

Finally, we apply (16) to the Schrödinger/Klein-Gordon equation (1):

\[
\begin{align*}
\left\{ m_i^2 + 2 \sum_{j \neq i} : \frac{\hat{p}_j \hat{p}_i}{\Lambda \hat{x}_j} : + \sum_{j,k \neq i} : \frac{\hat{p}_j \hat{p}_k}{\Lambda^2 \hat{x}_j \hat{x}_k} : + \sum_{i} m_i^2 \right\} |\psi\rangle = 0 \tag{18}
\end{align*}
\]
The operator part of equation (18) has the same structure as (7) except that it contains discrete operators functioning on different particles. We leave the extension of this formalism to a continuum for another work.

The choice of commutation relations and the definition of the function \( f \) are not unique. For instance, similar to noncommutative models one can assume that \( \hat{x}_i \) and \( \hat{x}_j \) for \( i \neq j \) do not commute. Assuming \( \Lambda \) as the dimensional scale for this commutation, it should be proportional to \( \Lambda^{-2} \). Thus, it is sub-dominant with respect to (17) which is of order \( \Lambda^{-1} \). In addition, one can assume that \( f(\hat{x}_i, \hat{p}_i) \) depends also on \( \hat{p}_j \). This does not change (14) but will add terms proportional to \( \Lambda^{-1} \) to (12). In this case \( f(\hat{x}_i, \hat{p}_i, \hat{p}_j) \) is:

\[
f(\hat{x}_i, \hat{p}_i, \hat{p}_j) = \sum_{m,n,k} C_{mn} : \hat{x}_m^{i} \hat{p}_n^{i} \hat{p}_k^{j} : \Lambda^{n+k-m-1} \tag{19}
\]

Assuming that \( n \) and \( k \) can take half-integer values, the term with \( m = -1 \) and \( n = k = 1/2 \) has the same order in \( \Lambda \) as the term used to obtain equation (16). The half-power of momentums can be related to a supersymmetric transformation and permits a natural extension of this model to supergravity and introduction of spinors in the model. We leave detailed investigation of this case to another work.

In the formulation above we neglected spacetime indexes. Apart from simplifying the notation, there is a deeper reason for this negligence. First of all it proves that this formalism can be applied to any background spacetime. More importantly, the similarity of the role of species/particles index to spacetime index indicates that there is an interchangeable role between what is called a particle and a point in the spacetime. Therefore we can claim that in this model there is a natural unification or embedding of the spacetime with a group manifold determining the symmetries and variety of particles. This aspect is similar to string theories. The hope is that in the extension of this model to continuum, geometrical properties and symmetries determine the signature of spacetime metric, the emergence of a time as well as the dimension of space.

4. outline

We suggested a modified structure of Hilbert space of multi-particle quantum system to present the effect of gravity. We showed that the redefinition of coordinate and momentum operators leads to a dynamical equation similar to the Wheeler-DeWitt equation for quantum mechanics in curved spacetime. A natural extension of the model to supergravity seems possible. Before being able to claim this model as a candidate model for quantum gravity we need to investigate a number of issues such as extension to infinite number of particles in another word a field theory, the emergence of a time coordinate, and possible relation between this model and other candidate quantum gravity models.

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Figure 1. The Universe, presented in this figure by the map of Cosmic Microwave Background (CMB) anisotropy observed by the WMAP satellite, can be the large scale manifestation of an intrinsically indeterministic universe based on the uncertainty rules of Quantum Mechanics, symbolically presented by Schrödinger Cat à la Da Vinci to remind Tuscany, the home country of Da Vinci and the venue of the DICE2008 conference.

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