Weighted Inequalities for Bilinear Rough Singular Integrals from $L^2 \times L^2$ to $L^1$

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Abstract  We establish a weighted inequality for bilinear rough singular integrals with bound controlled by the cube of the characteristic constant of the pair of weights.

Keywords  Bilinear operators · Weighted inequalities · Rough singular integrals

Mathematics Subject Classification 42B20

1 Introduction

The optimal norms of linear and multilinear weighted inequalities of kinds of operators attracted a lot of attention in past decades. The $A_2$ conjecture asks if the weighted norm of a (smooth) Calderón–Zygmund operator depends on the weight constant $[\omega]_{A_2}$ linearly. This was solved by Hytönen [12]. Lerner [19] attacked this problem later using sparse operators. That is why this method is often referred as the sparse method/control. A shorter proof of the sparse control for singular integral operators with kernels satisfying the Dini condition was given by Lacey [17], which inspired a lot of other papers, [4, 7, 16, 18], just to name a few.
There is a natural question after the solution to the $A_2$ conjecture. Does the weighted norm of a rough singular integral depend on the weight constant linearly as well? This question is partially answered by Hytönen et al. [15], who proved that the weighted norm of a rough singular integral is bounded by $C[\omega]^2$. Their method depends on a modification of a classical dyadic decomposition, see [8] for instance, and Lacey [17]. In this note, we generalize the result of [15] to the bilinear setting.

A bilinear rough singular integral is defined by the following:

$$T_\Omega(f, g)(x) = \int_{\mathbb{R}^{2n}} \frac{\Omega((y', z'), (y, z))}{|(y, z)|^{2n}} f(x - y) g(x - z) dy dz,$$

(1)

where $(y, z)' = \frac{(y, z)}{|(y, z)|} \in S^{2n-1}$, the unit sphere in $\mathbb{R}^{2n}$, and $\Omega$ is an $L^\infty$ function defined on $S^{2n-1}$ with vanishing integral, namely $\int_{S^{2n-1}} \Omega = 0$. The study of this operator goes back to Coifman and Meyer [3], whose boundedness of all points except for endpoints was proved for by Grafakos et al. [11].

We are interested in the weighted norm inequality for $T_\Omega$, namely

$$\|T_\Omega(f, g)\|_{L^1(\nu)} \leq C[\tilde{\omega}]_{A_{(2, 2)}} \| f \|_{L^2(\omega_1)} \| g \|_{L^2(\omega_2)},$$

(2)

where $(\omega_1, \omega_2)$ is an $A_{(2, 2)}$ weight and $\nu = \omega_1^{1/2} \omega_2^{1/2}$; see (6) below for the definition. We are concerning how the constant $C[\tilde{\omega}]_{A_{(2, 2)}}$ depends on $[\tilde{\omega}]_{A_{(2, 2)}}$.

The weighted norm inequality for bilinear rough singular integrals has been addressed by [1,6]. Cruz-Uribe and Naibo [6] obtained the first weighted inequality of bilinear rough singular integrals via interpolation between measures. Barron [1] obtained a sparse control of $T_\Omega$, which implies that the weighted norm of a bilinear rough singular integral depends on the weight constant of the bilinear weight. However, no explicit expression was provided in both papers for classical multiple weights introduced in [21].

We choose a different way here to give an explicit expression showing how the weighted norm $\|T_\Omega\|_{L^2(\omega_1) \times L^2(\omega_2) \to L^1(\nu)}$ depends on the corresponding weight constant. Our method could be modified to other points $(p_1, p_2, p)$ beyond $(2, 2, 1)$ which we study in this note, but we will not pursue them here, since even for the $(2, 2, 1)$ case, we cannot obtain the best result, which we conjecture as $[\omega_1, \omega_2]_{A_{(2, 2)}}$. The reader will find that our method relies heavily on the idea Hytönen et al. [15] used to handle the linear version.

Our main result is the following theorem.

**Theorem 1.1** Let $T_\Omega$ be a bilinear rough singular integral operator with $\Omega \in L^\infty(S^{2n-1})$ and $\int_{S^{2n-1}} \Omega = 0$, then

$$\|T_\Omega(f, g)\|_{L^1(\nu)} \leq C \|\Omega\|_{L^\infty} [\omega_1, \omega_2]_{A_{(2, 2)}}^3 \| f \|_{L^2(\omega_1)} \| g \|_{L^2(\omega_2)}$$

(3)

whenever $(\omega_1, \omega_2) \in A_{(2, 2)}$ and $\nu = \omega_1^{1/2} \omega_2^{1/2}$. 

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2 Reverse Hölder Inequalities

Let us recall some basic definitions. A local integrable nonnegative function $\omega$ is an $A_p$ weight for $1 < p < \infty$ if

$$[\omega]_{A_p} = \sup_Q \int_Q \omega \left( \int_Q \omega^{1-p} \right)^{p-1} < \infty.$$  

The constant $[\omega]_{A_p}$ is referred as the weight constant of $\omega$. For the case $p = \infty$, we take the definition

$$[\omega]_{B_\infty} = \sup_Q \frac{1}{\omega(Q)} \int Q M(\omega \chi_Q) < \infty;$$

see [14] for instance. The class $B_\infty$ coincides with the classical weight class $A_\infty = \cup_{1 < p < \infty} A_p$, since $[\omega]_{B_\infty} \leq [\omega]_{A_p}$ (see [13]) and $\omega \in B_\infty$ satisfies the reverse Hölder inequality (Lemma 2.1). For some technical reason, we introduce also

$$[\omega]_{A_p} = \max([\omega]_{B_\infty}, [\omega^{1-p}]_{B_\infty}) \leq C[\omega]_{A_p} \max\left(1, \frac{1}{\omega} \right).$$

A remarkable property of weights is that they satisfy the reverse Hölder inequality (RHI), which states that there exists a positive $\epsilon$, such that

$$\int_Q \omega^{1+\epsilon} dx \leq C \left( \int_Q \omega dx \right)^{1+\epsilon}.$$

As a simple corollary, we see that $\omega^{1+\epsilon} \in A_p$ when $\omega \in A_p$.

A range of $\epsilon$ is given by the following lemma proved in [14].

Lemma 2.1 [14, Theorem 2.3] Let $\omega \in B_\infty$ and let $Q_0$ be a cube. Then

$$\int_{Q_0} \omega^{1+\epsilon} dx \leq 2 \left( \int_{Q_0} \omega dx \right)^{1+\epsilon}$$

for any $\epsilon > 0$, such that $0 < \epsilon \leq \frac{1}{2^{n+1} [\omega]_{B_\infty}^{-1}}$.

In particular, for $p \geq 2$, (4) holds for $\epsilon \leq C[\omega]_{A_p}^{-1}$.

Lemma 2.2 [15, Corollary 3.16] Let $\omega \in A_p$ with $p \in (1, \infty)$, then there exists a constant $\delta \sim \frac{1}{[\omega]_{A_p}}$, such that

$$[\omega^{1+\delta}]_{A_p} \leq 4[\omega]_{A_p}^{1+\delta}.$$  

This result follows from the reverse Hölder inequality. We refer abusively (5) as a reverse Hölder inequality, as well. A main reason for doing this as we will see below.
is that this is a good substitute for the reverse Hölder inequality of multiple weights, while the generalization of (4) to the multiple weights is unclear.

A multiple weight is defined as follows. Let \( 1 \leq p_1, \ldots, p_m < \infty, \frac{1}{\tilde{p}} = \sum_{j=1}^{m} \frac{1}{p_j}, \)
\( \tilde{P} = (p_1, \ldots, p_m), \) and \( \tilde{\omega} = (\omega_1, \ldots, \omega_m). \) Set \( v = v_{\tilde{\omega}} = \prod_{j=1}^{m} \omega_j^{p_j/p_j}. \) We say \( \tilde{\omega} \)
satisfies \( A_{\tilde{p}} \) condition if
\[
[\tilde{\omega}]^{1/p}_{A_{\tilde{p}}} = \sup_Q \left( \frac{1}{|Q|} \int_Q v \right)^{1/p} \prod_{j=1}^{m} \left( \frac{1}{|Q|} \int_Q \omega_j^{1-1/p_j} \right)^{1/p_j} < \infty. \tag{6}
\]

This coincides with the classical weight when \( m = 1 \) and the supremum is \([\omega]^{1/p}_{A_p}. \)

There is an interesting characterization of multiple weights.

Lemma 2.3 [21, Theorem 3.6] \( \tilde{\omega} \in A_{\tilde{p}} \) if and only if \( \omega_j^{1-p'_j} \in A_{m p'_j} \) and \( v \in A_{mp}. \)

In particular, we have the following corollary.

Corollary 2.4 \( \tilde{\omega} \in A_{(2,2)} \) if and only if \( \omega_j^{-1} \in A_4 \) for \( j = 1, 2 \) and \( v = \omega_1^{1/2} \omega_2^{1/2} \in A_2. \)

By Lemma 2.2, we know that for
\[
\|v\|_{L^2} \lesssim \min \left( [\omega_1^{-1}]_{A_4}, [\omega_2^{-1}]_{A_4}, [v]_{A_2} \right) \sim 1/\max \left( [\omega_1^{-1}]_{A_4}, [\omega_2^{-1}]_{A_4}, [v]_{A_2} \right),
\]
we have \( \omega_1^{-(1+\delta)}, \omega_2^{-(1+\delta)} \in A_4 \) and \( v^{1+\delta} \in A_2. \) Consequently, by Corollary 2.4,
\( \tilde{\omega}^{1+\delta} = (\omega_1^{1+\delta}, \omega_2^{1+\delta}) \in A_{(2,2)}. \) This indicates the possible validity of the reverse Hölder inequality of multiple weights of the following form:
\[
\left[ \int_Q v^{1+r} \left( \int_Q \omega_1^{-(1+r)} \right)^{\frac{1}{2}} \left( \int_Q \omega_2^{-(1+r)} \right)^{\frac{1}{2}} \right]^{\frac{1}{1+r}} \leq C \int_Q v \left( \int_Q \omega_1^{-1} \right)^{\frac{1}{2}} \left( \int_Q \omega_2^{-1} \right)^{\frac{1}{2}}. \tag{7}
\]

In particular, we have \( (\omega_1^{1+r}, \omega_2^{1+r}) \in A_{(2,2)} \) and
\[
[\omega_1^{1+r}, \omega_2^{1+r}]_{A_{(2,2)}} \leq [\omega_1, \omega_2]_{A_{(2,2)}}^{1+r}. \tag{8}
\]
Remark 1 By the proof of [21, Theorem 3.6], we see
\[ [\omega_1^{-1}]_{A_4}, [\omega_2^{-1}]_{A_4}, [v]_{A_2} \leq [\omega_1, \omega_2]^2_{A_{2(2)}}, \] (9)
which implies that
\[ [\omega_1, \omega_2]^2_{A_{2(2)}} \leq \min \left( [\omega_1^{-1}]_{A_4}, [\omega_2^{-1}]_{A_4}, [v]_{A_2} \right). \]

Moreover, by Lemma 2.1, we see that \([\omega^{1+r}]_{A_{2(2)}} \leq C[\omega]^{(1+r)}_{A_{2(2)}}\) holds at least for \(r \sim [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\).

We are concerning the largest possible number \(r\), such that \(\omega^{1+r} \in A_{2(2)}\). The example \(\omega_1 = \omega_2 \in A_2\) suggests that \(r = [\omega_1, \omega_2]^{-1}_{A_{2(2)}}\) might be a reasonable conjecture, which unfortunately turns out to be wrong.

Remark 2 We observe that (9) is sharp in the sense that the smallest \(t\), such that \([\omega_1^{-1}]_{A_4}, [\omega_2^{-1}]_{A_4}, [v]_{A_2} \leq [\omega_1, \omega_2]^t_{A_{2(2)}}\) is 2, although the number \(t\) could be 1 in the special case \(\omega_1 = \omega_2 \in A_2\). Therefore, it is impossible to obtain a larger \(r\) by improving the exponent \(t\).

To illustrate the claimed sharpness, we consider the special case \(\omega_1^{-1} = |x|^a \in A_4, \omega_2 = 1\) and \(v = |x|^{-\frac{a}{2}} \in A_2\). A simple calculation [9, p. 506] shows that \(a \in (-n, 2n)\), \([\omega]_{A_4} \sim \frac{1}{(a+n)(3n-a)^3}\), and \([\omega_1, 1]_{A_{2(2)}} = [\omega]^{1/2}_{A_3} \sim \frac{1}{(a+n)^{1/2}(2n-a)^3}\).

By the example in last remark, we are able to show that \(r \sim [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\) we obtained in Remark 1 is sharp.

Lemma 2.5 If \(r \lesssim [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\), then (8) holds.

If \(r\) is a positive number, such that (8) holds for all \((\omega_1, \omega_2) \in A_{2(2)}\), then \(r \lesssim [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\).

Proof The first statement is just Remark 1, so we will focus on the converse. We prove the converse by showing that for \(r \gtrsim [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\), there exists \((\omega_1, \omega_2) \in A_{2(2)}\), such that (8) fails.

Take \(\omega_1^{-1} = \omega = |x|^a \in A_4, \omega_2 = 1,\) and \(v = \omega^{-1/2} \in A_2,\) or equivalently \(a \in (-n, 2n)\). We know that in this case \([\omega_1, \omega_2]_{A_{2(2)}} = [\omega]^{1/2}_{A_3}\), and (8) becomes
\[ [\omega^{1+r}]_{A_3} \leq C[\omega]^{1+r}_{A_3}. \]

A weaker version is that \(\omega^{1+r} \in A_3\), which is equivalent to that \((1+r)a \in (-n, 2n)\). Consider the case \(a\) is close to \(-n\), then \(r \leq \frac{a+n}{a} \sim [\omega]^{-1}_{A_3} = [\omega_1, \omega_2]^{-2}_{A_{2(2)}}\), since \([\omega]_{A_3} \sim \frac{1}{(n+a)(2n-a)^2}\). \(\square\)

Remark 3 There is a different property which is also called RHI for multiple weights. We refer interested readers to [5].
3 A Quantitative Weighted Inequality

In this section, we prove Theorem 1.1, which relies on an improved Dini estimate.

Let \( K_0 = \frac{\Omega((y,z))}{|y,z|^{2n}} \chi_{1 \leq |y,z| \leq 2} \) be the truncated kernel of the bilinear rough singular integral. Take \( \varphi \in \mathcal{S} \), such that \( \text{supp} \hat{\varphi} \subset B(0,1) \) and \( \hat{\varphi}(y,z) = 1 \) when \( |y,z| \leq 1/2 \), and define \( \hat{\psi} = \hat{\varphi}(-) \cdot \hat{\varphi}(2\cdot) \) Define the kernel \( K_k = 2^{-2nk} K_0(2^{-k}(y,z)) \), \( \varphi_k = 2^{-2kn} \varphi(2^{-k}(y,z)) \), and \( \psi_k = 2^{-2kn} \psi(2^{-k}(y,z)) \). Define

\[
T_0(f, g)(x) = \sum_k K_k * (f \otimes g) * \varphi_k(x, x),
\]

and

\[
T_j(f, g)(x) = \sum_k K_k * (f \otimes g) * \psi_{k-j}(x, x)
\]

for \( j \geq 1 \). We remark that this decomposition is essentially the same as the one used in [10], where \( K_0 \) is a smooth truncation. Both truncations satisfy the same decay condition (in frequency side), so the argument in [10] could be applied here, as well.

We have the following lemma on \( T_j \).

**Lemma 3.1** [11, Proposition 5] \( T_\Omega = \sum_{j \in \mathbb{Z}} T_j \). \( T_j \) is a bilinear Calderón–Zygmund operator, such that \( \|T_j\|_{L^2 \times L^2 \to L^1} \leq C \|\Omega\|_{L^\infty} 2^{-j|\delta|} \) for \( \delta = 1/16 \).

Moreover, for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \leq \frac{C}{\varepsilon} \|\Omega\|_{L^\infty} \), such that \( T_j \) has the Calderón–Zygmund constant \( C_\varepsilon 2^{j|\varepsilon|} \) for all \( j \in \mathbb{Z} \).

**Remark 4** The boundedness of \( T_j \) is exactly [11, Proposition 5].

[11, Lemma 11] gives just the existence of \( C_\varepsilon \) without the form here. To obtain the right bound we need, we have to re-examine the proof to show that \( C_\varepsilon \leq \frac{C}{\varepsilon} \) with the help of [10, Appendix B1].

The Dini condition plays a crucial role in the following argument, which we now define; see, for example, [22] and references therein. A bilinear operator is called an \( \omega \)-Calderón–Zygmund operator if its kernel satisfies the size condition \( |K(x, y, z)| \leq C_K \frac{\omega}{(|x-y|+|x-z|)^{2n}} \), and the smoothness condition

\[
|K(x+h, y, z) - K(x, y, z)| + |K(x+y+h, z) - K(x, y, z)| \\
+ |K(x, y+z+h) - K(x, y, z)| \leq \frac{1}{(|x-y|+|x-z|)^{2n}} \omega \left( \frac{|h|}{|x-y|+|x-z|} \right),
\]

whenever \( |h| \leq \frac{1}{2} \max(|x-y|,|x-z|) \). We concern mainly the case when \( \omega \) is increasing satisfying \( \omega(0) = 0 \), and \( \|\omega\|_{Dini} = \int_0^1 \omega(t) \frac{dt}{t} < \infty \). In this case, we say that the kernel (or equivalently the operator) satisfies the Dini condition.

For \( T_j \) in the previous lemma, we see that the Calderón–Zygmund constant is \( C_\varepsilon 2^{j|\varepsilon|} \), which implies that we may take \( \omega(t) = C_\varepsilon 2^{j|\varepsilon|} t^\varepsilon \), and hence, \( \|\omega\|_{Dini} \leq C_\varepsilon 2^{j|\varepsilon|} \).
This estimate based on the classical decomposition is not good enough, and we need a new decomposition introduced by [15].

Let \( N(\ell) = 2^\ell \), and we should define \( \tilde{T}_\ell \) as follows:

\[
\tilde{T}_0(f, g) = T_0(f, g)(x)
\]

and

\[
\tilde{T}_\ell(f, g)(x) = \sum_k K_k * (f \otimes g) * [\varphi_{k-N(\ell)} - \varphi_{k-N(\ell-1)}](x, x)
\]

for \( \ell \geq 1 \).

We look at these operators one step further. We need their equivalent multiplier definitions, which are

\[
\tilde{T}_\ell(f, g)(x) = \sum_{k \in \mathbb{Z}} \hat{m}_{k, \ell}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i \xi \cdot (\xi + \eta)} d\xi d\eta,
\]

where \( m_{k, \ell}(\xi, \eta) = \hat{K}(2^k \xi, 2^k \eta)[\hat{\varphi}(2^{\ell-N(\ell)}(\xi, \eta)) - \hat{\varphi}(2^{\ell-N(\ell-1)}(\xi, \eta))] \) is supported in the annulus:

\[
\{(\xi, \eta) \in \mathbb{R}^{2n} : 2^{2\ell-1} \leq |(\xi, \eta)| \leq 2^{2\ell}\}.
\]

Obviously, \( \tilde{T}_\ell = \sum_{j=N(\ell-1)}^{N(\ell)} T_j \), so we obtain the following two trivial estimates depending on Lemma 3.1:

\[
\| \tilde{T}_\ell(f, g) \|_{L^1} \leq C \| \Omega \|_{L^\infty} 2^{-N(\ell)\delta'} \| f \|_{L^2} \| g \|_{L^2},
\]

since \( \sum_{j=N(\ell-1)}^{N(\ell)} C 2^{-|j|\delta} \leq C 2^{\ell-1} 2^{-2^{\ell-1}\delta} \leq C \delta 2^{-N(\ell)\delta/4} \). The second estimate is that the Calderón–Zygmund constant \( C_\ell \) related to \( \tilde{T}_\ell \) is bounded by the following:

\[
\sum_{j=N(\ell-1)}^{N(\ell)} C_\ell 2^{j\epsilon} \leq C_\ell N(\ell) 2^{N(\ell)\epsilon}.
\]

By taking \( \epsilon = t \ell N(\ell)^{-1} \), we control the last quantity by \( C_\ell 2^{(2+t)\ell} \), which in anyway is greater than \( 2^{2\ell} = N(\ell)^2 \), a bound we shall improve.

It was essentially proved in [15] the following lemma.

Lemma 3.2 [15, Lemma 3.10] The operator \( \tilde{T}_\ell \) is a bilinear \( \omega_\ell \)-Calderón–Zygmund operator with

\[
C_\ell \leq C \| \Omega \|_{L^\infty}
\]

and

\[
\omega_\ell(t) \leq C \| \Omega \|_{L^\infty} \min(1, 2^{N(\ell)} t),
\]

which implies further that \( \| \omega_\ell \|_{Dini} \leq C \| \Omega \|_{L^\infty} (1 + N(\ell)) \).
We see that the function \( \varphi \) in [15] is compactly supported in the spatial side (for variable \( x \)), while our decomposition uses that \( \varphi \) is compactly supported in the frequency side (for variable \( \xi \)). We take \( \varphi \) of this form due to the method taken in [11]. We are still able to prove Lemma 3.2 in our setting, which is given in the Appendix.

**Remark 5** The Dini constant of \( \tilde{T}_\ell \) that we had was \( N(\ell)^2 \), which is now \( N(\ell) \).

In the linear case, Lacey [17] proved sparse controls for singular integrals whose kernels satisfy the Dini condition, which was reproved later by Hytönen et al. [15], and Lerner [20]. Li [22] generalized Lerner’s result to the multilinear setting, which is useful for us.

**Lemma 3.3** [22, Theorem 1.2] Let \( T \) be a bilinear \( \omega \)-Calderón–Zygmund operator. Then, the norm \( \| T(\varphi, \psi) \|_{L^1(\nu)} \) is bounded by the following:

\[
C[\omega_1, \omega_2]_{A(2,2)}(\| T \|_{L^2 \times L^2 \to L^1} + C_{K} + \| \omega \|_{Dini})\| f \|_{L^2(\omega_1)}\| g \|_{L^2(\omega_2)}
\]

for \( (\omega_1, \omega_2) \in A(2,2) \) and \( \nu = \omega_1^{1/2} \omega_2^{1/2} \).

Another important tool is the interpolation between measures. The version that we need is taken from the classical monograph [2].

**Lemma 3.4** [2, Theorem 4.4.1, Theorem 5.5.3] Let \( T \) be a bilinear operator, such that

\[
\| T(\varphi, \psi) \|_{L^p(\mu_1)} \leq M_1\| f \|_{L^p(\omega_1)}\| g \|_{L^p(\nu_1)}
\]

and

\[
\| T(\varphi, \psi) \|_{L^p(\mu_2)} \leq M_2\| f \|_{L^p(\omega_2)}\| g \|_{L^p(\nu_2)}.
\]

Then, \( T \) can be extended to a bilinear operator bounded from \( L^p(\omega) \times L^p(\nu) \) to \( L^p(\mu) \) with norm bounded by \( M_1^{1-\theta} M_2^\theta \), where \( \mu = \mu_1^{1-\theta} \mu_2^\theta \), and both \( \omega \) and \( \nu \) are defined in a similar way.

Now, we prove the claimed quantitative weighted inequality of the bilinear rough singular integral. We should emphasize again that our argument is parallel to the one previously used in Hytönen et al. [15] for the linear case.

**Proof of Theorem 1.1** By Lemmas 3.2 and 3.3, we know that

\[
\| \tilde{T}_\ell(\varphi, \psi) \|_{L^1(\nu)} \leq C\| \Omega \|_\infty[\omega_1, \omega_2]_{A(2,2)} N(\ell)\| f \|_{L^2(\omega_1)}\| g \|_{L^2(\omega_2)}
\]

whenever \( (\omega_1, \omega_2) \in A(2,2) \). Moreover, for a fixed \( (\omega_1, \omega_2) \in A(2,2) \), by Lemma 2.5, we have \( (\omega_1^{1+r}, \omega_2^{1+r}) \in A(2,2) \) for \( r \sim [\omega_1, \omega_2]_{A(2,2)}^2 \), and hence

\[
\| \tilde{T}_\ell(\varphi, \psi) \|_{L^1(\nu^{1+r})} \leq C\| \Omega \|_{L^\infty}[\omega_1^{1+r}, \omega_2^{1+r}]_{A(2,2)} N(\ell)\| f \|_{L^2(\omega_1^{1+r})}\| g \|_{L^2(\omega_2^{1+r})}.
\]
Recall also that by (10), we have \( \| \tilde{T}_\ell \|_{L^2 \to L^1} \leq C \| \Omega \|_\infty 2^{-N(\ell)\delta'} \) for a fixed positive \( \delta' \) independent of \( \| \Omega \|_\infty \). Interpolating between this and (13), using Lemma 3.4 and (8), we obtain that

\[
\| \tilde{T}_\ell (f, g) \|_{L^1(\nu)} \leq C \| \Omega \|_{L^\infty}^{-N(\ell)} \| f \|_{L^2(\omega_1)} \| g \|_{L^2(\omega_2)},
\]

which is (3) and we finish the proof of Theorem 1.1.

\[\square\]

Remark 6 If \( r \sim [\omega_1, \omega_2]^{-1}_{A(2,2)} \), as we conjectured right after Remark 1, we obtain the weighted bound \([\omega_1, \omega_2]^{-2}_{A(2,2)}\), similar to the result obtained in the linear case [15]. However, \( r \) can only be \([\omega_1, \omega_2]^{-2}_{A(2,2)}\). This indicates that \([\omega_1, \omega_2]^{-3}_{A(2,2)}\) may be the limit of our method.

Since there exists \( r \sim [\omega]^{-1}_{A_2} \), such that \([\omega]^{1+r}_{A_2} \leq 4[\omega]^{1+r}_{A_2} \) by Lemma 2.2, we may improve the dependence on the weight constant when our weight is a single weight other than a multiple weight. More precisely, we have the following proposition, whose proof is essentially the same as the proof of Theorem 1.1 with the only difference that \( r \sim [\omega]^{-1}_{A_2} \) this time.

Proposition 3.5 Let \( T_\Omega \) be a bilinear rough singular integral operator with \( \Omega \in L^\infty(\mathbb{S}^{2n-1}) \) and \( \int_{\mathbb{S}^{2n-1}} \Omega = 0 \), then

\[
\| T_\Omega (f, g) \|_{L^1(\omega)} \leq C \| \Omega \|_{L^\infty} [\omega]^{3}_{A_2} \| f \|_{L^2(\omega)} \| g \|_{L^2(\omega)}
\]

whenever \( \omega \in A_2 \).

4 Appendix: Proof of Lemma 3.2

In this section, we sketch the proof of Lemma 3.2.

We refer readers to [15, Lemma 3.10] for a detailed proof. Here, we just present a few tiny differences worth explanation.

A careful examination of the proof of [15, Lemma 3.10] shows that once we establish (3.11) and (3.12) of [15], then the remaining argument follows smoothly.

What we want to estimate is \( | \sum_k K_k \ast \varphi_{k-N(\ell)} | \). We see that

\[
K_k \ast \varphi_{k-N(\ell)} (\bar{x}) = 2^{-(k-N(\ell))2n} \int_{|\bar{y}| \sim 2^k} \frac{\Omega((\bar{y}))}{|\bar{y}|^{2n}} \varphi \left( \frac{\bar{x} - \bar{y}}{2^{k-N(\ell)}} \right) d\bar{y},
\]

where \( \bar{x}, \bar{y} \in \mathbb{R}^{2n} \).
Fix $\vec{x}$ and assume $|\vec{x}| = 2^l$. If $l \leq k + 10$, then

$$
\left| 2^{-(k-N(\ell))2n} \int_{|\vec{y}| \sim 2^k} \frac{\Omega((\vec{y})')}{|\vec{y}|^{2n}} \varphi \left( \frac{\vec{x} - \vec{y}}{2^{k-N(\ell)}} \right) d\vec{y} \right|
\leq C \|\Omega\|_{L^\infty} 2^{-(k-N(\ell))2n} 2^{-2kn} \int_{|\vec{z}| \sim 2^{N(\ell)}} \left( 1 + |2^{N(\ell)-k} \vec{x} - \vec{z}| \right)^{-M} 2^{2n(k-N(\ell))} d\vec{z}
\leq C \|\Omega\|_{L^\infty} 2^{-2kn}.
$$

If $l \geq k + 10$, then $\varphi \left( \frac{\vec{x} - \vec{y}}{2^{k-N(\ell)}} \right)$ is bounded by $C 2^{(2n+1)(k-N(\ell)-l)}$, which implies that

$$
\left| 2^{-(k-N(\ell))2n} \int_{|\vec{y}| \sim 2^k} \frac{\Omega((\vec{y})')}{|\vec{y}|^{2n}} \varphi \left( \frac{\vec{x} - \vec{y}}{2^{k-N(\ell)}} \right) d\vec{y} \right|
\leq C \|\Omega\|_{L^\infty} 2^{-(k-N(\ell))2n} 2^{-2kn} 2^{(2n+1)(k-N(\ell)-l)} 2^{2kn}
= C \|\Omega\|_{L^\infty} 2^{k-N(\ell)} 2^{-(2n+1)l}.
$$

Summing over $k$, we obtain

$$
\left| \sum_k K_k \ast \varphi_{k-N(\ell)} \right|
\leq \sum_{k \geq l-10} C \|\Omega\|_{L^\infty} 2^{-2kn} + \sum_{k \leq l-10} C \|\Omega\|_{L^\infty} 2^{k-N(\ell)} 2^{-(2n+1)l}
\leq C \|\Omega\|_{L^\infty} \left[ 2^{-2ln} + 2^{-(2n+1)l} 2^{-N(\ell)2^l} \right]
\leq C \|\Omega\|_{L^\infty} |\vec{x}|^{-2n}.
$$

Similarly, we can prove that

$$
\var{\nabla} \left( \sum_k K_k \ast \varphi_{k-N(\ell)}(\vec{x}) \right) \leq C \|\Omega\|_{L^\infty} \frac{2^{N(\ell)}}{|\vec{x}|^{2n+1}}.
$$

Notice that the kernel $\tilde{K}_\ell(x, y, z)$ of $\tilde{T}_\ell$ is

$$
\sum_k K_k \ast [\varphi_{k-N(\ell)} - \varphi_{k-N(\ell)-1}][(x, x) - (y, z)],
$$

so a routine argument implies (11) and (12). This completes the proof of Lemma 3.2.

**Acknowledgements** Peng Chen was supported by NNSF of China (No. 11501583), Guangdong Natural Science Foundation (No. 2016A030313351), and the Fundamental Research Funds for the Central Universities (No. 161gpy45). Danqing He was supported by NNSF of China (No. 11701583), Guangdong Natural Science Foundation (No. 2017A030310054), and the Fundamental Research Funds for the Central Universities (No. 17lgpy11). Liang Song was supported by NNSF of China (No. 11471338 and 11622113) and Guangdong Natural Science Funds for Distinguished Young Scholar (No. 2016A030306040).
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