On the 1 + 2 Dimensional Isotropic Landau-Lifshitz Equation

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Abstract

By using the geometric concept of PDEs with prescribed curvature representation, we prove that the 1+2 dimensional isotropic Landau-Lifshitz equation is gauge equivalent to a 1 + 2 dimensional nonlinear Schrödinger-type system. From the nonlinear Schrödinger-type system, we construct blowing up $H^3(\mathbb{R}^2)$-solutions to the Landau-Lifshitz equation, which reveals the blow up phenomenon of the equation.

§1. Introduction

The isotropic Landau-Lifshitz equations, or in other words, the generalized Heisenberg models for a continuous ferromagnetic spin vector $s = (s_1, s_2, s_3) \in S^2 \hookrightarrow \mathbb{R}^3$ (see, for example, [22, 25, 6, 29]),

$$s_t = s \times \Delta_{\mathbb{R}^n}s, \quad x \in \mathbb{R}^n, \quad n = 1, 2, 3, \cdots$$  \hspace{1cm} (1)

are important equations in spin magnetic fields in physics. These equations exhibit a rich variety of dynamical properties of a spin vector in different backgrounds.

Though Eq.(1) have a unified version of expressions for $n \geq 1$, there are great differences between dynamical properties of Eq.(1) with $n = 1$ and those of Eq.(1) with $n \geq 2$. When $n = 1$, Eq.(1) is integrable and it can be solved by the method of inverse scattering techniques ([14, 19]). Furthermore, Eq.(1) with $n = 1$ is gauge equivalent to the nonlinear Schrödinger equation of attractive type: $i\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0$ ([34]) (its dual version was proved in [8, 9]). When $n \geq 2$, Eq.(1) are non-integrable and the understanding of their dynamical properties becomes much difficulty. In 1975, Belavin and Polyakov ([1]) were the first who paid attention to the construction of topological static solutions (or in other words, Belavin-Polyakov instantons) to Eq.(1) with $n = 2$, i.e., the 1+2 dimensional isotropic Landau-Lifshitz equation, by applying the technique of the stereographic projection $S^2 \to \mathbb{C}$. These solutions are used for the description of domain walls, magnetic
bubbles and the metastatic states in ferromagnetic spin fields. The consequences of such a study are referred to [25] and references therein. In 1986, Sulem, Sulem and Bardos proved in [29] the local $H^{m+1}(\mathbb{R}^n)$-existence of solutions to the Cauchy problem of Eq.(1) by the difference method and also the global $W^{m+1,6}(\mathbb{R}^n)$-existence if initial data is small enough. Very recently, in [24] Nahmod, Stefanov and Uhlenbeck proved the local well-posedness of the Cauchy problem of the 1+2 dimensional Landau-Lifshitz equation and its dual equation by applying their equivalent equations so-called the modified Schrödinger map equations.

The isotropic Landau-Lifshitz equations (1) are special cases of so-called Schrödinger maps ([6, 18, 24]) or Schrödinger flows ([15, 13, 31, 8]) in geometry. The Schrödinger map from a Riemannian manifold $(M, g)$ to a Kähler manifold $(N, \omega)$ is defined to be the (infinite dimensional) Hamiltonian system of the energy function $E(u) = \int_M |\nabla u|^2 dv_g$ on the mapping space $C^k(M,N)$ for some $k > 0$. More explicitly, let $J$ be a compatible complex structure on $N$ such that $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric on $N$ and we denote by $\nabla E(u)$ the gradient of the function $E(u)$ with respect to the inner product $\langle v, w \rangle_u = \int_M h(u)(v, w)$, $\forall u \in X, \forall v, w \in T_u X$, on $X$, then the corresponding Hamiltonian vector field $V_{E(u)}$ can be expressed explicitly as $V_{E(u)} = J(u)\nabla E(u)$. Thus the Schrödinger map from $(M, g)$ into $(N, \omega)$ is represented by

$$u_t = J(u)\nabla E(u).$$

It is easy to verify that the gradient $\nabla E(u)$ is exactly the tension field $\tau(u)$ of map $u : M \to N$. In a local coordinates

$$\tau^l(u) = \Delta_M u^l + \Gamma^l_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}, \quad 1 \leq l \leq \dim N,$$

where $\Delta_M$ denotes the Laplacian operator on $(M, g)$ with the given metric $g = (g_{\alpha\beta})$, $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$ and $\Gamma^l_{jk}$ are the Christoffel symbols of the target manifold $(N, h)$. So the Schrödinger map from $M$ into $N$ can also be written as:

$$u_t = J(u)\tau(u).$$

It is a straightforward verification that the Schrödinger map from an Euclidean $n$-space $\mathbb{R}^n$ to the 2-sphere $S^2 \hookrightarrow \mathbb{R}^3$ is exactly the isotropic Landau-Lifshitz equation (1) (for example see [6]).

The main object in the study of Schrödinger maps is, of course, the solvability of the corresponding Cauchy or initial-boundary value problem and its solutions’ behaviors. However, comparing to those of heat flows or wave maps [28], this study is still at the beginning stage. Except the results stated above, the following results dealing with Schrödinger maps should also be mentioned. In 1999, Terng and Uhlenbeck showed the global existence of the Cauchy problem of Schrödinger maps from $\mathbb{R}^1$ to complex compact Grassmannians in [31]. Chang, Shatah and Uhlenbeck proved in [6] the global existence and uniqueness of smooth solution to the Cauchy problem of Schrödinger maps from $\mathbb{R}^1$ to compact Riemannian surfaces and also the $W^{2,4}(\mathbb{R}^2)$-global existence result of the radial Schrödinger maps from $\mathbb{R}^2$ to an arbitrary compact Riemann surface for small initial
data. W.Y. Ding and Wang proved in [13] the existence of local smooth or global weak solutions to the Cauchy problem of Schrödinger maps from a compact Riemannian manifold or Euclidean space $\mathbb{R}^n$ to a compact Kähler manifold. Grillakis and Stefanopoulos displayed conservation laws and localized energy estimates of Schrödinger maps to Riemannian surfaces. However, it is widely believed that a Schrödinger map with dimensions of the starting manifold are greater than 1 may develop singularities in finite time in general. This fundamental problem in the study of Schrödinger maps was proposed by W.Y. Ding as a unsolved question in [12]. In fact, the same question for the higher dimensional Landau-Lifshitz equations has been existed for a long time. Blow up corresponds to the self-trapping and intense focusing phenomena of classical ferromagnetic spin fields.

In this paper, we first display that the 1 + 2 dimensional isotropic Landau-Lifshitz equation:

$$s_t = s \times \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) s$$

is gauge equivalent to the following 1 + 2 dimensional nonlinear Schrödinger-type system

$$\begin{align*}
iq_t - q_{zz} + 2uq - 2(\bar{p}q)_z + 2pq_z + 4|p|^2q &= 0 \\
nr_t + r_{zz} - 2ur - 2(\bar{p}r)_z + 2pr_z - 4|p|^2r &= 0 \\
ir_t &= (qr)_z - uz
\end{align*}$$

with the additional restrictions: $\bar{p}_z + p_z = |q|^2 - |r|^2$, $\bar{r}_z + q_z = 2(p\bar{r} - \bar{p}q)$ by using the geometric concept of gauge equivalence for PDEs with prescribed curvature representation developed in [10, 11]. This nonlinear Schrödinger-type system is very different from the modified Schrödinger map equation obtained by Nahmod, Stefanov and Uhlenbeck in [24]. Then, by characterizing some analytic properties of the above nonlinear Schrödinger-type system, we show the existence of blowing up $H^3(\mathbb{R}^2)$-solutions to the 1 + 2 dimensional Landau-Lifshitz equation if the initial data is chosen to be technically small. This blowing up result does not contradict to the global existence of $W^{m+1,6}(\mathbb{R}^2)$-solutions due to Sulem, Sulem and Bardos in [29]. On the contrary, it reflects some new interesting and mysterious properties of the Landau-Lifshitz equation.

This paper is organized as follows. In the section 2, we shall transfer the 1 + 2 dimensional isotropic Landau-Lifshitz equation to its gauge equivalent nonlinear Schrödinger-type system. In section 3, by applying the nonlinear Schrödinger-type system, we construct blowing up $H^3(\mathbb{R}^2)$-solutions to the 1 + 2 dimensional isotropic Landau-Lifshitz equation and give some remarks.

§2. Gauge Equivalence

In almost known results dealing with the Landau-Lifshitz equations and Schrödinger maps, the first step of the study is to transform the original equation to a nonlinear Schrödinger-type equation. In this section we shall apply the geometric concept of gauge transformations to transfer the 1+2 dimensional Landau-Lifshitz equation into a nonlinear Schrödinger-type system in the category of (nonzero) prescribed curvature formulations. The zero curvature formulation in integrable theory is a main indication of integrability of a soliton equation. Zarkharov and Takhtajan introduced in [34] the geometric concept
of gauge equivalence between two soliton equations which provides a useful tool in the study of integrable equations. In [10, 11] the author and his collaborator found that the geometric concept of gauge equivalence can be generalized to differential equations with prescribed curvature representation and then displayed the gauge equivalent structures of the 1+1 dimensional anisotropic Landau-Lifshitz equation (which give an affirmative answer to a question proposed in [2]) and the modified nonlinear Schrödinger equation. Now we find that it is also applicable to the present 1+2 dimensional isotropic Landau-Lifshitz equation.

Let us explicitly write down the 1+2 dimensional isotropic Landau-Lifshitz equation Eq.(1) as follows:

\[ s_t = s \times \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)s, \]  

where \((x, y)\) is the standard coordinates of the Euclidean plane \(\mathbb{R}^2\). We convert it into the matrix form,

\[ S_t = -\frac{1}{2i}[S, S]\bar{z}, \]  

where \(S = \begin{pmatrix} s_3 & s_1 + is_2 \\ s_1 - is_2 & -s_3 \end{pmatrix}\) with \(S^2 = I\) (\(I\) denotes the unit matrix as usual) and \(z = \frac{x+iy}{2}, \ \bar{z} = \frac{x-iy}{2}\) which are \(\frac{1}{2}\) scaling of the usual complex versions of the real variables \(x\) and \(y\). In order to present Eq.(3) as an equation with prescribed curvature representation, let’s set

\[ A = Vd\bar{z} - i\lambda Sdz + \lambda(2V + SS\bar{z} + 2\alpha S)dt \]  

and

\[ K = \left\{ -V_t + \lambda(2V + SS\bar{z} + 2\alpha S)\bar{z} - [V, \lambda(2V + SS\bar{z} + 2\alpha S)] \right\}d\bar{z} \wedge dt + \lambda\left(\frac{1}{2}[S\bar{z}, S\bar{z}] + 2(\alpha S)z\right)dz \wedge dt, \]  

where \(\lambda\) is a spectral parameter which is independent of \(t, z\) and \(\bar{z}\), \(V = V(\lambda, z, \bar{z}, t)\) is a \(2 \times 2\)-matrix satisfying the equation:

\[ (-i\lambda S)\bar{z} - V_z + [-i\lambda S, V] = 0 \]  

and \(\alpha = \frac{1}{2}\text{tr}(G^{-1}G\bar{z}\sigma_3)\) is a scalar function, where \(G\) is an \(SU(2)\)-matrix solving (10) below. \(d+A\) can be geometrically interpreted as defining a connection on a trivial \(SU(2)\)-principal bundle over \(\mathbb{R}^3\) (the space of the independent variables \(x, y\) and \(t\)). Then it is a straightforward computation that the curvature \(F_A\) of the connection \(d+A\) is

\[ F_A = dA - A \wedge A \]  

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\[ \left\{ -V_t + \lambda (2V + SS_z + 2\alpha S)\bar{z} - \left[ V, \lambda (2V + SS_z + 2\alpha S) \right] \right\} d\bar{z} \wedge dt \]
\[ + \left\{ \lambda \left( iS_t + (SS_z + 2\alpha S)\bar{z} \right) \right\} dz \wedge dt \]

and hence Eq.(3) is equivalent to holding the following prescribed curvature condition:
\[ F_A = dA - A \wedge A = K. \quad (7) \]

We would like to point out that, though we cannot give an explicit solution \( V \) to Eq.(6), it still OK for us to prove our desired conclusion, as we shall see below.

It is well-known that, in the Yang-Mills gauge theory, there are gauge transformations
\[ A \rightarrow \hat{A} = dGG^{-1} + GAG^{-1}, \quad \text{for } G \in \mathcal{C}^\infty(\mathbb{R}^3, SU(2)) \text{ such that } F_{\hat{A}} = GF_AG^{-1} \]
under the gauge transformation.

**Theorem 1** There is a gauge function \( G(t, z, \bar{z}) \in SU(2) \) such that any given solution \( S(t, z, \bar{z}) \) to the 1+2 dimensional Landau-Lifshitz equation (3) is transformed to a solution \( (p(t, z, \bar{z}), q(t, z, \bar{z}), r(t, z, \bar{z}), u(t, z, \bar{z})) \) to the following nonlinear Schrödinger-type system:
\[ \begin{cases} iq_t - q_{\bar{z}z} + 2uq - 2(\bar{p}q) + 2pq_z + 4|p|^2q = 0 \\ ir_t + r_{\bar{z}z} - 2ur - 2(\bar{p}r) + 2pr_z - 4|p|^2r = 0 \\ ip_t = (qr) - u_z \end{cases} \quad (8) \]

by the gauge transformation, where \( u \) is a unknown real function and \( p, q, r \) are unknown complex functions satisfying the following additional restrictions
\[ \bar{p}_z + p_{\bar{z}} = |q|^2 - |r|^2, \quad \bar{r}_z + q_{\bar{z}} = 2(p\bar{r} - \bar{p}q). \quad (9) \]

**Proof.** Let \( S = S(t, z, \bar{z}) \) be a solution to Eq.(3). We come to choose an \( SU(2) \) matrix \( G(t, z, \bar{z}) \) such that
\[ \sigma_3 = -G^{-1}SG, \quad G^{-1}G = -\begin{pmatrix} p & q \\ r & -p \end{pmatrix} := -U \quad (10) \]
for some complex functions \( p = p(t, z, \bar{z}) \) and \( q = q(t, z, \bar{z}) \) and \( r = r(t, z, \bar{z}) \), where
\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
Indeed, by a direct computation, we see that the general solutions to \( \sigma_3 = -G^{-1}SG \) are of the forms:
\[ G = \frac{1}{\sqrt{2(1 - s_3)}}(S - \sigma_3)\text{diag}(\gamma, \bar{\gamma}), \quad (11) \]
where \( \gamma \) is a complex function of \( z, \bar{z} \) and \( t \) (or in other words, \( x, y \) and \( t \)) with \( |\gamma| = 1 \).
For a fixed \( SU(2) \)-matrix \( G \) given in (11), we have
\[ G_x = -G \begin{pmatrix} i\bar{s} & \psi \\ -\bar{\psi} & -is \end{pmatrix}, \quad G_y = -G \begin{pmatrix} il & \phi \\ -\bar{\phi} & -il \end{pmatrix} \quad (12) \]
for some real functions \( s, l \) and complex functions \( \phi, \psi \). It is easy to see from (12) that, for the complex variables \( z = (x + iy)/2 \) and \( \bar{z} = (x - iy)/2 \), we have

\[
G_z = -G \left( \begin{array}{c} p \\ -\bar{p} \end{array} \right) \left( \begin{array}{c} \psi - i\phi \\ \psi + i\phi \end{array} \right), \quad G_{\bar{z}} = -G \left( \begin{array}{c} -\bar{p} \\ \bar{p} \end{array} \right) \left( \begin{array}{c} \psi - i\phi \\ \psi + i\phi \end{array} \right),
\]

where \( p = l + is \) is a complex function of \( x, y, t \). Thus for any fixed \( G \) given in (11) we have not only (10) with complex functions \( p, q = \psi - i\phi \) and \( r = -\bar{\psi} + i\bar{\phi} \), but also

\[
G_{\bar{z}} = GP := G \left( \begin{array}{c} \bar{p} \\ \bar{q} \\ -\bar{p} \end{array} \right).
\]

Furthermore, from the integrability condition \( P_z + U_{\bar{z}} + [P, U] = 0 \) of the linear system:

\[
G_z = -GU, \quad G_{\bar{z}} = GP,
\]

we have

\[
\bar{p}_z + p_\bar{z} = |q|^2 - |r|^2, \quad \bar{r}_z + q_\bar{z} = 2(p\bar{r} - \bar{p}q),
\]

which is exactly the restrictions (9). Meanwhile, note that \( \alpha = \bar{p} \) by the definition of the connection \( A \) given in (4).

Now, for the connection \( A \) given in (4) with \( S \) being fixed above, we define a connection 1-form as follows

\[
A^G = -G^{-1}dG + G^{-1}AG.
\]

From \( S = -G\sigma_3G^{-1}, \ G_{\bar{z}} = GP \) and \( \alpha = \bar{p} \), we have

\[
SS_\bar{z} + 2\alpha S
= -G\sigma_3G^{-1}(-G\sigma_3G^{-1} + G\sigma_3PG^{-1}) + 2\bar{p}(-G\sigma_3G^{-1})
= -2GP \text{ (off-diag)} G^{-1} - 2\bar{p}G\sigma_3 G^{-1}
= -2G_{\bar{z}} G^{-1}.
\]

So (14) can be re-expressed as follows

\[
A^G = \left( -G^{-1}G_z + G^{-1}VG \right) d\bar{z} + (i\lambda\sigma_3 + U)dz \\
+ \left( -G^{-1}G_t + \lambda(2G^{-1}VG - 2G^{-1}G_{\bar{z}}) \right) dt.
\]

Since \( A \) satisfies the prescribed curvature condition:

\[
F_A = dA - A \wedge A = K,
\]

where \( K \) is given by (5), from gauge theory we know that \( A^G \) must satisfies

\[
F_{A^G} = dA^G - A^G \wedge A^G = G^{-1}KG.
\]
Comparing respectively the coefficients of $dz \wedge d\bar{z}$, $d\bar{z} \wedge dt$ and $dz \wedge dt$ in the both sides of (17), we have

\[ (-G^{-1}G_{\bar{z}} + G^{-1}VG)_{\bar{z}} - U_{\bar{z}} = [i\lambda \sigma_3 + U, -G^{-1}G_{\bar{z}} + G^{-1}VG] = 0, \quad (18) \]

\[ -\left(-G^{-1}G_{\bar{z}} + G^{-1}VG\right)_{\bar{t}} + \left(-G^{-1}G_t + \lambda(2G^{-1}VG - 2G^{-1}G_{\bar{z}})\right)_{\bar{z}} \]

\[ -\left[ -G^{-1}G_{\bar{z}} + G^{-1}VG, -G^{-1}G_t + \lambda(2G^{-1}VG - 2G^{-1}G_{\bar{z}}) \right] \]

\[ = -G^{-1}V_tG + 2\lambda G^{-1}V_{\bar{z}}G + \lambda G^{-1}\left(SS_{\bar{z}} + 2\alpha S\right)G \]

\[ -G^{-1}\left[V, 2\lambda V + \lambda(SS_{\bar{z}} + 2\alpha S)\right]G, \quad (19) \]

and

\[ -U_t + \left(-G^{-1}G_t + \lambda(2G^{-1}VG - 2G^{-1}G_{\bar{z}})\right)_{\bar{z}} \]

\[ -\left[i\lambda \sigma_3 + U, -G^{-1}G_t + \lambda(2G^{-1}VG - 2G^{-1}G_{\bar{z}})\right] \]

\[ = G^{-1}\lambda\left(\frac{1}{2}[S_z, S_{\bar{z}}] + 2(\alpha S)_{\bar{z}}\right)G. \quad (20) \]

Setting

\[ \tilde{V} = -G^{-1}G_{\bar{z}} + G^{-1}VG, \quad (21) \]

and by a straightforward computation, we obtain

\[ G^{-1}\left((i\lambda S)_{\bar{z}} + V_{z} + [i\lambda S, V]\right)G = -U_{\bar{z}} + \tilde{V}_{\bar{z}} - [i\lambda \sigma_3 + U, \tilde{V}], \]

which implies that (18) is automatically satisfied from (6). By using the definition $\tilde{V} = -G^{-1}G_{\bar{z}} + G^{-1}VG$ again, we see that (19) is equivalent to

\[ -\tilde{V}_t + \left(-G^{-1}G_t + 2\tilde{V}\right)_{\bar{z}} - [\tilde{V}, -G^{-1}G_t + 2\tilde{V}] \]

\[ = G^{-1}\left\{-V_t + \lambda(2V + SS_{\bar{z}} + 2\alpha S)_{\bar{z}} - [V, \lambda(2V + SS_{\bar{z}} + 2\alpha S)]\right\}G. \quad (22) \]

It is also a straightforward verification that (22) is an identity too. Finally, we come to treat (20). First, by using the identities: $S_z = G(U\sigma_3 - \sigma_3 U)G^{-1}$ and $S_{\bar{z}} = -GP\sigma_3 G^{-1} + G\sigma_3 P G^{-1}$ deduced from (10) and (13) respectively, $\alpha = \tilde{p}$ and the first equation of (9), we have

\[ \frac{1}{2}[S_z, S_{\bar{z}}] + 2(\alpha S)_{\bar{z}} \]

\[ = G2\left(U^{(off-diag)}\sigma_3, \sigma_3 P^{(off-diag)}\right) - 2\tilde{p}_z\sigma_3 + 4\tilde{p} U^{(off-diag)}\sigma_3 \]

\[ = G\left(2p_z\sigma_3 + 4\tilde{p} U^{(off-diag)}\sigma_3 \right)G^{-1} = G\left(2p_z\sigma_3 + [\sigma_3, Q]\right)G^{-1}, \quad (23) \]
where \( Q = \begin{pmatrix} 0 & -2q \bar{p} \\ -2r \bar{p} & 0 \end{pmatrix} \). Thus, (20) is equivalent to

\[-U_t + (-G^{-1}G_t + 2\lambda \bar{V})_z - [i\lambda \sigma_3 + U, -G^{-1}G_t + \lambda 2\bar{V}] = 2\lambda p_2 \sigma_3 + \lambda [\sigma_3, Q],\]

or equivalently,

\[-U_t + (-G^{-1}G_t)_z + [U, G^{-1}G_t] \\
+ \lambda \left( 2U_z - 2p_2 \sigma_3 + i[\sigma_3, G^{-1}G_t] - [\sigma_3, Q] \right) = 0 \tag{24}\]

here we have used the identities (18): \( U_z - \bar{V}_z + [i\lambda \sigma_3 + U, \bar{V}] = 0 \). Comparing the coefficients of \( \lambda \) and the constant term in (24), we obtain

\[-U_t + (-G^{-1}G_t)_z + [U, G^{-1}G_t] = 0, \tag{25}\]

\[2U_z - 2p_2 \sigma_3 + i[\sigma_3, G^{-1}G_t] - [\sigma_3, Q] = 0. \tag{26}\]

The vanishing of the diagonal part of (25) and the equation (26) lead to

\[G^{-1}G_t = -i \left\{ \left( u + U_z^{\text{off-diag}} \right) \sigma_3 + Q \right\} \tag{27}\]

\[p_t = i(u_z - (qr)_z) \tag{28}\]

for some real function \( u = u(t, x, y) \). Here we have used the second equation of (9) to verify that \(-i(U_z^{\text{off-diag}})\sigma_3 + Q \in su(2)\). Substituting (21) and (27,28) into (16) and (17) respectively, we obtain

\[A^G = \bar{V}d\bar{z} + \left( i\lambda \sigma_3 + U \right)dz + \left( 2\lambda \bar{V} + i \left( u + U_z^{\text{off-diag}} \right) \sigma_3 + iQ \right) dt \ tag{29}\]

and

\[K^G = G^{-1}KG = \left\{ -\bar{V}_t + \left( 2\lambda \bar{V} + iu\sigma_3 + iU_z^{\text{off-diag}} \sigma_3 + iQ \right)_z \right\} \\
- \left( \bar{V}, 2\lambda \bar{V} + iu\sigma_3 + iU_z^{\text{off-diag}} \sigma_3 + iQ \right)_z dz \land dt \\
+ \lambda \left( 2U_z^{\text{diag}} + [\sigma_3, Q] \right) dz \land dt \tag{30}\]

where \( \lambda \) is the same spectral parameter as in (4), \( \bar{V} = \bar{V}(\lambda, z, \bar{z}, t) \) is a \( 2 \times 2 \)-matrix satisfying

\[U_z - \bar{V}_z + [i\lambda \sigma_3 + U, \bar{V}] = 0. \tag{31}\]

So, it is a direct computation, from the prescribed curvature representation: \( F_{AC} = K^G \), that the corresponding equation for unknown functions \( (p, q, r, u) \) is just the nonlinear Schrödinger-type system (8). We would like to point out that one should apply the identity (31) in the computation. This completes the proof of Theorem 1. \( \square \)
Now we come to consider the nonlinear Schrödinger-type system (8) with the restriction (9). As indicated in Theorem 1, it is a PDE with prescribed curvature representation:

\[ F_{\tilde{A}} = d\tilde{A} - \tilde{A} \wedge \tilde{A} = \tilde{K} \]  

(32)

in which

\[ \tilde{A} := \tilde{V} \, d\bar{z} + \left( i \lambda \sigma_3 + U \right) dz + \left\{ 2\lambda \tilde{V} + i(u + U_{\frac{3}{2}}^{\text{off-diag}})\sigma_3 + iQ \right\} dt \]  

(33)

and

\[ \tilde{K} := \left\{ -\tilde{V}_t + \left( 2\lambda \tilde{V} + i(u + U_{\frac{3}{2}}^{\text{off-diag}})\sigma_3 + iQ \right) \right\} d\bar{z} \wedge dt \]

\[ + \lambda \left( 2U_{\frac{3}{2}}^{\text{diag}} + [\sigma_3, Q] \right) dz \wedge dt, \]  

(34)

where all the notations have the same meanings indicated above, i.e., \( U = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \), \( \tilde{V} \) solves the equation (31) and \( Q = \begin{pmatrix} 0 & -2q\bar{p} \\ -2r\bar{p} & 0 \end{pmatrix} \).

Next we shall prove that the above gauge transformation from the 1 + 2 dimensional isotropic Landau-Lifshitz (3) to the nonlinear Schrödinger-type system (8) with the restrictions (9) is in fact reversible.

**Theorem 2** There is a gauge matrix function \( G(t, x, y) \in SU(2) \) such that any solution \( (p, q, r, u) \) to the nonlinear Schrödinger-type system (8) with the restriction (9) can be transformed to a solution \( S \) to the 1 + 2 dimensional isotropic Landau-Lifshitz equation (3) by the gauge transformation of \( G \). Moreover, if we require that the gauge matrix \( G \) satisfies \( G|_{x=y=t=0} = I \). Then any \( C^m \)-solution \((m \geq 2)\) to the nonlinear Schrödinger-type system (8) with the restriction (9) corresponds uniquely to a \( C^{m+1} \)-solution to the Schrödinger map (3) and vice versa.

**Proof:** Let \( (p, q, r, u) \) be a solution to Eq.(8) with the restrictions (9). From the prescribed curvature formulation (32), it is a key observation that Eq.(8) with the restrictions (9) is in fact the integrability condition of the following linear system:

\[ G_z = -GU, \quad G_t = -Gi \left( (u + U_{\frac{3}{2}}^{\text{off-diag}})\sigma_3 + Q \right), \]  

(35)

or equivalently,

\[
\begin{align*}
G_x &= -G \begin{pmatrix} i\text{Im}p & \psi \\ -\bar{\psi} & -i\text{Im}p \end{pmatrix} \\
G_y &= -G \begin{pmatrix} i\text{Re}p & \phi \\ -\bar{\phi} & -i\text{Re}p \end{pmatrix} \\
G_t &= G \begin{pmatrix} -iu & iq \bar{z} + 2i\bar{p}q \\ -ir_\bar{z} + 2i\bar{r}q & iu \end{pmatrix}
\end{align*}
\]  

(36)

9
where $\psi = (q-r)/2$ and $\phi = i(q+r)/2$ (here we have used (9) to verify that the coefficient matrix in righthand side of the third equation of (36) is an $su(2)$-matrix). This implies that (35) or (36) is a compatible linear differential system. Since the coefficient matrices in (36) are $su(2)$-matrices, it indicates that general solutions $G(t, x, y)$ to (36) or equivalently (35) belong to $SU(2)$ group. Now let $G(t, z, \bar{z}) = G(t, x, y) \in SU(2)$ be a fundamental solution to (35) or equivalently (36), and we consider the following gauge transformation,

$$A = (dG)G^{-1} + G\tilde{A}G^{-1},$$

where $\tilde{A}$ is the 1-form connection given in (33) with $(p, q, r, u)$ being given above. We try to show that the 1-form $A$ defined by (37) is exactly the connection of Eq.(3) given in (4) when $S$ and $\alpha$ are suitably determined. In fact, substituting the coefficient $-i\lambda S$ of $dz$ of (4) into (37) and comparing the coefficients of $\lambda$ of $dz$ in the both sides of (37), we obtain

$$G_z = -GU, \quad S = -G\sigma_3 G^{-1} \quad (\text{hence} \quad S^2 = I).$$

The first equation of (38) is automatically satisfied because of the first equation of (35). The second one of (38) is regarded as defining $S$. Now, we have to prove that the coefficients of $d\bar{z}$ and $dt$ of $A$ defined by (37) are respectively the same coefficients of $d\bar{z}$ and $dt$ of the connection given in (4), that is,

$$V = G_z G^{-1} + G\tilde{V} G^{-1},$$

$$\lambda \left( 2V + SS_{\bar{z}} + 2\alpha S \right) = G_i G^{-1} + G \left( 2\lambda \tilde{V} + i \left( (u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + Q \right) \right) G^{-1}.$$  (40)

Eq. (39) can be regarded as defining $V$ if we can show that such a $V$ solves Eq.(6), i.e., for the $S$ being given in (38) we have

$$(-i\lambda S)_{\bar{z}} - V_z + [-i\lambda S, V] = 0.$$  (41)

The proof of (41) is a direct computation. Indeed, by using the expression of $V$ given in (39) and the fact that $G$ fulfills (36) (this equivalent to having (10), (13) and the third equation of (36)), we have

$$(i\lambda S)_{\bar{z}} + V_z + [i\lambda S, V] = G \left( U_{\bar{z}} - \tilde{V}_z + [i\lambda \sigma_3 + U, \tilde{V}] \right) G^{-1}. $$  (42)

Since $\tilde{V}$ satisfies (31), this establishes (41). For proving (40), since $G$ satisfies the second equation of (35), it is easy to see that the proof of (40) is equivalent to

$$2V + SS_{\bar{z}} + 2\alpha S = 2G\tilde{V} G^{-1},$$

which, because of (39), is equivalent to

$$SS_{\bar{z}} + 2\alpha S = -2G_z G^{-1}. $$  (43)
Now we take $\alpha = \bar{\rho}$ which fulfills the requirement of $\alpha$ in the definition of the connection (4). From $G_z = -GU$ $G_{\bar{z}} = GP$ and $\alpha = \bar{\rho}$, we have
\[
-(SS_{\bar{z}} + 2\alpha S)
= -G\sigma_3 G^{-1}(-GP\sigma_3 G^{-1} + G\sigma_3 PG^{-1}) + 2(-\bar{\rho})(-G\sigma_3 G^{-1})
= 2GP(\text{off-diag})G^{-1} + 2\bar{\rho}G\sigma_3 G^{-1}
= 2GP G^{-1} = 2G_{z} G^{-1}.
\]
This proves (43). Thus we have proved that the two connections given by (37) and (4) respectively are actually the same one when $S = -G\sigma_3 G^{-1}$ and $\alpha = \bar{\rho}$. What’s the remainder for us to do is to prove that the curvature formula
\[
F_A = K = G\tilde{K} G^{-1} = GF_A G^{-1} \tag{44}
\]
derived under the gauge transformation is satisfied too, where $K$ is given by (5) and $\tilde{K}$ is given by (34). In fact, on the one hand, we see that
\[
G\tilde{K} G^{-1} = G \left\{ -\tilde{V}_t + \left(2\tilde{\lambda} \tilde{V} + i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + iQ\right)_{\bar{z}} - \left[\tilde{V}, 2\tilde{\lambda} \tilde{V} + i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + iQ\right] d\bar{z} \wedge dt + \lambda(2U_{\bar{z}}^{(\text{diag})} + [\sigma_3, Q]) dz \wedge dt \right\} G^{-1}. \tag{45}
\]
On the other hand, by using (39, 43) and (35), it is a straightforward calculation that the coefficient of $dz \wedge dt$ in $K$ is
\[
-V_t + \lambda(2V + SS_{\bar{z}} + 2\alpha S)_{\bar{z}} = [V, \lambda(2V + SS_{\bar{z}} + 2\alpha S)]
= -V_t + 2\lambda(G\tilde{V} G^{-1})_{\bar{z}} - [V, 2\lambda G\tilde{V} G^{-1}]
= -G_{t\bar{z}} G^{-1} + G_{z\bar{z}} G_{t\bar{z}} G^{-1} - G_{t\bar{z}} G^{-1} - G\tilde{V} G^{-1} G_{t\bar{z}} G^{-1} + 2\lambda G_{z\bar{z}} \tilde{V} G^{-1}
+ 2\lambda(2\lambda G\tilde{V} G^{-1} - 2\lambda G\tilde{V} G^{-1} G_{z\bar{z}} G^{-1} - 2\lambda(G_{z\bar{z}} \tilde{V} G^{-1} - G\tilde{V} G^{-1} G_{z\bar{z}} G^{-1})
= G \left\{ -\tilde{V}_t + \left(2\tilde{\lambda} \tilde{V} + i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + iQ\right)_{\bar{z}} - \left[\tilde{V}, 2\tilde{\lambda} \tilde{V} + i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + iQ\right] \right\} G^{-1}. \tag{46}
\]
(45) and (46) indicate that the two coefficients of $d\bar{z} \wedge dt$ in the both sides of (44) are the same one. Meanwhile, by applying the similar argument in getting (23), we have
\[
\frac{1}{2}\left[S_z, S_{\bar{z}}\right] + 2(\alpha S)_{z} = G(2p_{z} \sigma_3 + \left[\sigma_3, Q\right]) G^{-1}, \tag{47}
\]
where $Q = \begin{pmatrix} 0 & \gamma \bar{p} \\ -2\gamma \bar{p} & 0 \end{pmatrix}$ as before. (47) implies
\[
\lambda \left(\frac{1}{2}\left[S_z, S_{\bar{z}}\right] + 2(\alpha S)_{z}\right) = \lambda G \left(2U_{\bar{z}}^{(\text{diag})} + \left[\sigma_3, Q\right]\right) G^{-1},
\]
which shows that the two coefficients of $dz \wedge dt$ in $K$ and $G\tilde{K}G^{-1}$ are also the same one. Thus we have proved the desired identity (44), which implies the holding of the prescribed curvature representation (7). Hence we obtain that $S$ solves Eq.(3). This indicates that $S$ defined by the second equation of (38) from a solution $(p, q, r, u)$ to (8) with the restrictions (9) satisfies the 1+2 dimensional Landau-Lifshitz equation (3).

Since $G$ is a solution to the linear first-order differential system (35), it is well-known from linear theory of differential equations that such a $G$ is unique if we propose the initial condition $G|_{z=y=t=0} = I$ on $G$. Under this circumstance, we see that a solution $(p, q, r, u)$ to Eq.(8) with the restrictions (9) corresponds uniquely to a solution $S$ to Eq.(3) by the gauge transformation and vice versa. Furthermore, because of the relation $S_z = G2U^{(\text{off-diag})}\sigma_3 G^{-1}$ deduced from (38), the remainder part of the theorem is obviously true. □

We would like to point out that the unknown functions in the system (8) can be reduced to $(p, q, r)$ with $p$ being a real function. In fact, we may restrict $G$ given by (11) to satisfy $G_x = -G \begin{pmatrix} 0 & \psi \\ -\bar{\psi} & 0 \end{pmatrix}$ for some complex function $\psi$ which leads to $\gamma = \exp \left( -i \int_0^z \frac{(s_1 s_2 - s_2 s_1)}{2s_1 - 2} dx \right)$. Thus $p$ becomes now a real function of $x, y$ and $t$. Under this situation, it is easy to verify that the vanishing of the diagonal part of (25) and the equation (26) lead to

$$ G^{-1}G_t = -i \left\{ \left( (\text{Re}(qr) - \partial_x^{-1}\partial_y \text{Im}(qr)) + U_z^{(\text{off-diag})} \right) \sigma_3 + Q \right\} + i\tau \sigma_3 \quad (48) $$

$$ p_t = 2(\text{Re}(qr))_y + ((\text{Im}(qr))_x - \partial_x^{-1}\partial_y^2 \text{Im}(qr)) - \tau_y \quad (49) $$

for some real function $\tau = \tau(y, t)$ depending only on $t$ and $y$. Moreover, notice that the restriction (10) on the gauge matrix $G$ allows an arbitrariness in $G$ of the form: $G \rightarrow Ge^{i\sigma \beta(t, y)}$ for an arbitrary real function $\beta(t, y)$. If we require $\beta$ (the existence of such a $\beta$ is easy to verify) to satisfy

$$ \frac{\partial \beta}{\partial y} = -\int_0^t \tau_y, \quad \frac{\partial \beta}{\partial t} = -\tau $$

then $G$ can be modified so that for the new $G$ we have

$$ G_y = -G \begin{pmatrix} ip + i \int_0^t \tau_y & \psi e^{-2i\beta} \\ -\bar{\psi} e^{2i\beta} & -ip - i \int_0^t \tau_y \end{pmatrix}, $$

$$ G_t = G \begin{pmatrix} -i(\text{Re}(qr) - \partial_x^{-1}\partial_y \text{Im}(qr)) - i\tau & \sigma e^{-2i\beta} \\ -\sigma e^{2i\beta} & i(\text{Re}(qr) - \partial_x^{-1}\partial_y \text{Im}(qr)) + i\tau \end{pmatrix}, $$

where $\sigma = iq_z + 2ipq$, which implies that for the new $G$ the second term on the right-hand side of (48) is 0 and meanwhile the third term on the right-hand side of (49) is 0 too. Hence the system (8) is reduced to the following nonlinear Schrödinger type system:

$$ \begin{cases} iq_z - q_{z\bar{z}} + 2q((\text{Re}(qr) - \partial_x^{-1}\partial_y \text{Im}(qr))) + 2pq_{\bar{z}} - 2(pq)_z + 4p^2q = 0 \\
ir_z + r_{z\bar{z}} - 2r((\text{Re}(qr) - \partial_x^{-1}\partial_y \text{Im}(qr))) + 2pr_{\bar{z}} - 2(pr)_z - 4p^2r = 0 \\
p_t = 2(\text{Re}(qr))_y + ((\text{Im}(qr))_x - \partial_x^{-1}\partial_y^2 \text{Im}(qr)), \end{cases} \quad (50) $$
where the real function \( p \) and the complex function \( q, r \) must satisfy the restriction (9) too. Though (50) looks much simpler than (8) in unknown variables, both (8) and (50) with the restrictions (9) are essentially equivalent to each other. However, the natural choice of complex version of \( p \) in the system (8) plays an important role in constructing blow-up \( H^3(\mathbb{R}^2) \)-solutions to the Landau-Lifshitz equation (2), as we shall see in the next section.

**Remark 1** The expression (10) or (38) of the two gauge equivalent solutions gives an explicit relationship of the 1+2 isotropic dimensional Landau-Lifshitz equation (2) and the nonlinear Schrödinger-type system (8), which also plays a key role in constructing blow-up solutions to Eq.(2) in the next section. This explicit relation is not obtained by other transformations, such as the Hasimoto transformation.

§3. Blowing up solutions

We follow the basic conventional notations for Sobolev spaces \( H^k(\mathbb{R}^n) \), \( W^{k,\sigma}(\mathbb{R}^n) \) \((\sigma > 1)\) of real or complex-valued functions or spaces \( C^k(\mathbb{R}^n) \) of continuous differential functions up to order \( k \) on \( \mathbb{R}^n \) for \( n \geq 2 \) and norms \( \| \cdot \|_{W^{k,\sigma}(\mathbb{R}^n)} \) or \( \| \cdot \|_{C^k(\mathbb{R}^n)} \) used in [16]. In this section, we shall construct, by use of its gauge equivalent nonlinear Schrödinger-type equation (8) displayed in the previous section, blowing up \( H^3(\mathbb{R}^2) \)-solutions to the Landau-Lifshitz equation (2). Before doing this, let’s characterize some analytic properties of the system (8) or equivalently Eq.(50) with the restrictions (9).

**Claim 1.** System (50) has the following conservation laws:

\[
\begin{align*}
\int_{\mathbb{R}^2} |q(x,y,t)|^2 dx dy &= \int_{\mathbb{R}^2} |q(x,y,0)|^2 dx dy, \\
\int_{\mathbb{R}^2} |r(x,y,t)|^2 dx dy &= \int_{\mathbb{R}^2} |r(x,y,0)|^2 dx dy.
\end{align*}
\]

In fact, we multiply \( 2\bar{q} \) (or \( 2\bar{r} \)) to the first equation (or the second equation) of (50) and take the imaginary part of the result to get

\[
\frac{\partial}{\partial t} |q|^2 = 2\text{Im}((\Delta q)\bar{q}) - 4p|q|^2_y - 4p_y|q|^2, \quad \frac{\partial}{\partial t} |r|^2 = -2\text{Im}((\Delta r)\bar{r}) - 4p|r|^2_y - 4p_y|r|^2.
\]

Thus the above conservation laws follow from integrating the identities over \( \mathbb{R}^2 \).

**Claim 2.** Let us introduce the polar coordinates \((\rho, \theta)\) of \( \mathbb{R}^2 \), that is \( x = \rho \cos \theta, y = \rho \sin \theta \). Thus we have

\[
\begin{align*}
\frac{\partial}{\partial z} &= e^{-i\theta} \left( \frac{\partial}{\partial \rho} - i \frac{1}{\rho} \frac{\partial}{\partial \theta} \right), \\
\frac{\partial}{\partial \bar{z}} &= e^{i\theta} \left( \frac{\partial}{\partial \rho} + i \frac{1}{\rho} \frac{\partial}{\partial \theta} \right).
\end{align*}
\]

We would like to find following ansatz solutions to (8):

\[
p = 0, q = e^{-i\theta} Q(\rho, t), r = -e^{-i\theta} \bar{Q}(\rho, t), u = -|Q|^2 + 2 \int_{\rho}^{\infty} \frac{|Q|^2(\tau, t)}{\tau} d\tau
\]

\begin{equation}
(51)
\end{equation}
for some suitable functions $Q(\rho, t)$. One may verify that (9) and the third equation of (8) are satisfied automatically (since $\frac{\partial^{-1}}{\partial y} (rq) = \frac{\partial^{-1}}{\partial t}(e^{-2i\theta}|Q|^2(\rho, t)) = -|Q|^2(\rho, t) + 2 \int_{\rho}^{\infty} \frac{|Q|^2(\tau, t)}{\tau} d\tau$) and the first and second equations of (8) lead to

$$iQ_t - \left(Q_{\rho\rho} + \frac{1}{\rho} Q_{\rho} - \frac{1}{\rho^2} Q\right) - 2Q \left(|Q|^2 - 2 \int_{\rho}^{\infty} \frac{|Q(\tau, t)|^2}{\tau} d\tau\right) = 0. \tag{52}$$

This equation, or its equivalent form: $iQ_t + Q_{\rho\rho} + \frac{1}{\rho} Q_{\rho} - \frac{1}{\rho^2} Q + 2Q \left(|Q|^2 - 2 \int_{\rho}^{\infty} \frac{|Q(\tau, t)|^2}{\tau} d\tau\right) = 0$, was deduced by many authors from the (generalized) Hasimoto transformation (see [7, 23, 26, 6]).

Claim 3. If $q(t, s)$ is an arbitrary solution to the following nonlinear Schrödinger equation:

$$iq_t - q_{ss} - 2q|q|^2 = 0 \tag{53}$$

which is an integrable system, one can verify straightforwardly that $(p = 0, q(t, \cos \delta z + \sin \delta \bar{z}), r(t, \cos \delta z + \sin \delta \bar{z}) = -\frac{1+i}{1+i}, u = -|q|^2)$ is a solution to (8) with the restriction (9), where $\delta$ is a free parameter. It is well-known that system (53) has (global) 1 + 1 dimensional $N$-soliton solutions (see [14]) and hence so does (2) correspondingly. For example, the following 1+2 dimensional travelling 1-soliton to Eq.(2),

$$s_1(x, y, t) = \frac{2\text{sh}(\cos \delta x + \sin \delta y)}{\text{ch}^2(\cos \delta x + \sin \delta y)} \cos t$$

$$s_2(x, y, t) = \frac{2\text{sh}(\cos \delta x + \sin \delta y)}{\text{ch}^2(\cos \delta x + \sin \delta y)} \sin t$$

$$s_3(x, y, t) = 1 - \frac{2}{\text{ch}^2(\cos \delta x + \sin \delta y)}$$

is obtained from the 1-soliton solution $q = e^{-it}\text{sech}$ to (53) by the gauge transformation.

The fact that the Landau-Lifshitz equations are related to nonlinear Schrödinger-type equations has been known for a long time. Many authors applied properties of nonlinear Schrödinger-type equations to study the Landau-Lifshitz equations. For example, it was the use of its equivalent nonlinear Schrödinger-type equation obtaining by the technique of the stereographic projection $S^2 \to C$, Sulem, Sulem and Bardos proved in [29] the global $W^{m+1,6}(\mathbb{R}^n)$-existence of the Cauchy problem of the Landau-Lifshitz equation (1) (with $n \geq 2$) for small initial data. So it is very reasonable that we may use the nonlinear Schrödinger-type system (8) to reveal the blow-up phenomenon of the Landau-Lifshitz equation (2), though this nonlinear Schrödinger-type system looks very complicated. Let briefly review some blow-up results and searching methods of nonlinear Schrödinger equations since they will enlighten on our approach. There has been much interest and groundbreaking work within the decades in the study of nonlinear Schrödinger equations with general nonlinearities (see, for example, [4, 5, 21, 30]). Blowing-up solutions to the Cauchy problem of nonlinear Schrödinger equations of the forms

$$\begin{cases}
  iq_t = \Delta q + |q|^\sigma q, & \text{in } \mathbb{R}^n \\
  q(x, 0) = q_0(x),
\end{cases} \tag{54}$$
where $\sigma > 1 + n/4$ is a positive constant, were displayed in [17, 20]. In the proofs of
the mentioned blow-up results for the nonlinear Schrödinger equation (54), the
conservation law $E(q) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla q|^2 dv - \frac{1}{\sigma + 1} \int_{\mathbb{R}^n} |q|^\sigma dv = E(q_0)$ plays a key role. Comparing
the nonlinear Schrödinger equation (54) with the nonlinear Schrödinger-type equation (8) or
(52), we find that there are additional integral terms in our present case. The extra term(s)
prevent us from getting the analogous conservation law as that of Eq.(54). Thus we will
face new difficulties in characterizing the blowing-up property of the present nonlinear
Schrödinger-type system if we go along the way depending on conservation laws. On the
other hand, the Eq.(54) of the critical case $\sigma = 4/n + 1$ admits the following conformal
invariance ([33]),

$$q(x, t) \rightarrow (Cq)(x, t) = \frac{e^{-ib|x|^2}}{(a + bt)^{n/2}} q(X, T)$$

where $X = \frac{x}{a + bt}, T = \frac{c + dt}{a + bt}$ and $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R})$, i.e., $a, b, c, d$ are real numbers
and $ad - bc = 1$. That is to say $Cq$ is a solution to (54) too. Weinstein constructed in
[33] (one can also refer to [5]) by using this conformal invariance (which is absent for
$\sigma \neq 4/n + 1$) blowing-up solutions from localized finite energy solitary waves. And he also
proved the sharpness of a condition for the global existence of solutions in [32]. However,
we find surprisingly that the method used by Weinstein can be modified to our present
nonlinear Schrödinger-type equations (8) and applied to construct blowing-up solutions
to the Landau-Lifshitz equation (2).

**Lemma 1** Assume that $q = q(\rho, t)$, where $(\rho, \theta)$ is the polar coordinates of $\mathbb{R}^2$, solves the
following nonlinear Schrödinger-type equation:

$$iq_t - q_{\rho\rho} - \frac{1}{\rho} q_\rho + \frac{1}{\rho^2} q - q \left( 2|q|^2 - 4 \int_\rho^\infty \frac{|q|^2(\tau, t)}{\tau} d\tau \right) - \frac{ib}{d - bt}(q + \rho q_\rho) = 0. \quad (55)$$

Then the following $(\tilde{\rho}, \tilde{q}, \tilde{r}, \tilde{u})$ giving by the conformal transformation:

$$\tilde{\rho}(x, y, t) = -\frac{ib\tilde{z}}{2(a + bt)},$$

$$\tilde{q}(x, y, t) = e^{-ib(\rho^2 + \rho^2)} q(R, T)e^{-i\theta},$$

$$\tilde{r}(x, y, t) = -\frac{e^{ib(\rho^2 + \rho^2)}}{(a + bt)} \tilde{q}(R, T)e^{-i\theta},$$

$$\tilde{u}(x, y, t) = -\frac{1}{(a + bt)^2} \left( |q(R, T)|^2 - 2 \int_R^\infty \frac{|q(\tau, T)|^2}{\tau} d\tau \right) + \frac{b^2z\tilde{z}}{2(a + bt)^2},$$

where $R = \frac{\rho}{a + bt}, T = \frac{c + dt}{a + bt}$ and $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R})$ (i.e., $ad - bc = 1$), is a solution to
the system (8) with the restrictions (9).
Proof. The proof is just a lengthy verification step by step. For example, we have
\[
\begin{align*}
\frac{d}{dt}q - (q_{xx} + q_{yy}) + 2\dot{u}q - 2(\dot{p}q)_z + 2\ddot{p}q_z + 4|\dot{p}|^2q = \\
e^{-\frac{i(u^2 + q^2)}{(a + bt)^3}} \left( iq_T - q_{RR} - \frac{1}{R} q_R + \frac{1}{R^2} q - q(2|q|^2 - 2 \int_R^\infty \frac{|q|^2}{\tau} d\tau) \right) \\
- \frac{ib}{d-bT} q - \frac{ib}{d-bT} Rq_R e^{-i\theta} = 0;
\end{align*}
\]
and so the other equations. □

Remark 2 Combining the gauge equivalent structure of the 1+2 dimensional Landau-Lifshitz equation (2) displayed in Theorem 1 and 2 with the conformal invariance displayed in Lemma 1, we may obtain a corresponding conformal invariant property to the 1+2 dimensional isotropic Landau-Lifshitz equation (2). We call it the partial conformal invariant of Eq.(2) since it is only proved for some special case. However, it is very interesting to see if the 1+2 dimensional Landau-Lifshitz equation admits a similar conformal invariance in general.

Now we concentrate ourselves on characterizing some useful analytic characters of the Cauchy problem of the following equation:
\[
\begin{align*}
\begin{cases}
\frac{d}{dt}q + q_{\rho\rho} + \frac{1}{\rho} q_{\rho} - \frac{1}{\rho^2} q + q \left( 2|q|^2 - 4 \int_\rho^\infty \frac{|q|^2}{\tau} d\tau \right) - \frac{ib}{d-b\rho} \left( q + \rho q_{\rho} \right) = 0 \quad (56) \\
q(\rho, 0) = q_0(\rho).
\end{cases}
\end{align*}
\]
or equivalently
\[
\begin{align*}
\begin{cases}
\frac{d}{dt}Q + \Delta Q + Q \left( 2|Q|^2 - 4 \int_\rho^\infty \frac{|Q|^2}{\tau} d\tau \right) - \frac{ib}{d-b\rho} \left( Q + \rho Q_{\rho} \right) = 0 \quad (57) \\
Q(\rho, 0) = Q_0(\rho).
\end{cases}
\end{align*}
\]
where \( Q(\rho, \theta, t) = q(\rho, t)e^{-i\theta} \) and \( Q_0(\rho, \theta) = q_0(\rho)e^{-i\theta} \). It is easy to see that Eq.(55) is complex conjugate to Eq.(56) and vice versa. Moreover, if the initial data \( q_0(\rho) \) satisfies \( q_0(\rho)e^{-i\theta} \in H^2(\mathbb{R}^2) \), the local existence of \( H^2(\mathbb{R}^2) \)-solutions of the form \( Q = q(\rho, t)e^{-i\theta} \) to the Cauchy problem (57) (or equivalently the local existence of solutions \( q(\rho, t) \) to the Cauchy problem (56) such that \( q(\rho)e^{-i\theta} \in H^2(\mathbb{R}^2) \)) can be deduced directly from
the known theory of nonlinear Schrödinger-type equations or indirectly from the local $H^3(\mathbb{R}^2)$-existence and uniqueness result of the Landau-Lifshitz equation proved by Sulem, Sulem and Bardos in [29], Theorem 1 and Lemma 1 (in this way, we require additionally $\rho^2 q_0(\rho)e^{-i\theta} \in L^2(\mathbb{R}^2)$, $\rho(q_0(\rho))_\rho e^{-i\theta} \in L^2(\mathbb{R}^2)$ such that $S_z \bigg|_{t=0}, S_\theta \bigg|_{t=0} \in H^2(\mathbb{R}^2)$ in the case of $c = 0$ in Lemma 1).

**Lemma 2** If $q(\rho, t) \ (0 \leq t < T_0$ for some $0 < T_0 \leq \infty$) is the unique solution to the Cauchy problem (56) which satisfies $Q(\rho, \theta, t) = q(\rho, t)e^{-i\theta} \in H^2(\mathbb{R}^2)$ (i.e., $Q(\rho, \theta, t) = q(\rho, t)e^{-i\theta} \in H^2(\mathbb{R}^2)$ solves the Cauchy problem (57)), then

1. For any $t$ with $0 \leq t < T_0$,
   \[
   \|q\|_{L^2(\mathbb{R}^2)} = \|q_0\|_{L^2(\mathbb{R}^2)} \quad \text{(i.e.} \quad \|Q\|_{L^2(\mathbb{R}^2)} = \|Q_0\|_{L^2(\mathbb{R}^2)}). \tag{58}
   \]

2. There exists a positive constant $C$ such that, if $b < 0$ and $d > 0$,
   \[
   \|\nabla^2 Q\|_{L^2(\mathbb{R}^2)} \geq \frac{1}{C t + \|\nabla^2 q_0\|_{L^2(\mathbb{R}^2)}^2}, \quad 0 \leq t < T_0. \tag{59}
   \]

**Proof.** We multiply Eq.(56) by $2\bar{q}$ and take the imaginary part of the result to get

\[
\frac{d}{dt} |q|^2 + 2 \nabla (\text{Im} \bar{q} \nabla q) - \frac{b}{d - bt} \left(2|q|^2 + \rho |q|^2 \right) = 0.
\]

Integrating the above equation on $\mathbb{R}^2$, we get

\[
\frac{d}{dt} \int_{\mathbb{R}^2} |q|^2 dxdy = 0.
\]

Thus we have (i) by solving this trivial ODE. In order to prove ii), we make the transformation: $Q(\rho, \theta, t) \to \bar{Q}(\rho, \theta, t) = e^{-E t} \tilde{Q}(\rho, \theta, t)$, where $E$ is a positive constant which will be determined latter, to obtain equivalently the following equation for $\bar{Q}$:

\[
i \bar{Q}_t + \Delta_{\mathbb{R}^2} \bar{Q} + \bar{Q} e^{2Et} \left(2|\bar{Q}|^2 - 1 \int_{\rho} \frac{|\bar{Q}(\tau, t)|^2}{\tau} d\tau \right) - \frac{ib}{d - bt} \left(\bar{Q} + \rho \bar{Q}_\rho \right) + iE \bar{Q} = 0 \tag{60}
\]

Taking the derivative with respective to $z$ to the both sides of (60), we see

\[
i(\bar{Q}_z)_t + \Delta_{\mathbb{R}^2} (\bar{Q}_z) + 4e^{2Et} \bar{Q}_z |\bar{Q}|^2 + 2e^{2Et} \bar{Q}^2 (\bar{Q})_z + 4e^{2Et} \bar{Q} \int_{\rho} \frac{|\bar{Q}(\tau, t)|^2}{\tau} d\tau
\]

\[
+ 4e^{2Et} \bar{Q} \frac{|\bar{Q}|^2 e^{-i\theta}}{\rho} - \frac{ib}{d - bt} \left(2\bar{Q}_z + \rho (\bar{Q}_z)_{\rho} \right) + iE \bar{Q}_z = 0. \tag{61}
\]

Similarly we have

\[
i(\bar{Q}_z)_t + \Delta_{\mathbb{R}^2} (\bar{Q}_z) + 4e^{2Et} \bar{Q}_z |\bar{Q}|^2 + 2e^{2Et} \bar{Q}^2 (\bar{Q})_z + 4e^{2Et} \bar{Q} \int_{\rho} \frac{|\bar{Q}(\tau, t)|^2}{\tau} d\tau
\]

\[
+ 4e^{2Et} \bar{Q} \frac{|\bar{Q}|^2 e^{i\theta}}{\rho} - \frac{ib}{d - bt} \left(2\bar{Q}_z + \rho (\bar{Q}_z)_{\rho} \right) + iE \bar{Q}_z = 0. \tag{62}
\]
We continue to take the derivative with respect to $z$ to the both sides of (61) and to have

\[
i(\tilde{Q}_{zz}) + \Delta \tilde{Q}_{zz} + 4e^{2Et}\tilde{Q}_{zz}\tilde{Q} + 4e^{2Et}\tilde{Q}_{zz}^3 + 8e^{2Et}\tilde{Q}_{zz}\tilde{Q} = 0.
\]

We multiply Eq.(63) by $2\tilde{Q}_{zz}$, take the imaginary part of the resulting expression and integrate it on $\mathbb{R}^2$ to have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 + E \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 - 4 \frac{b}{d - bt} \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 = -8e^{2Et} \int_{\mathbb{R}^2} \text{Im}(\tilde{Q}_{zz}^2),
\]

\[
-16e^{2Et} \int_{\mathbb{R}^2} \text{Im}(\tilde{Q}_{zz}^2) - 4e^{2Et} \int_{\mathbb{R}^2} \text{Im}(\tilde{Q}_{zz}^3) = 0.
\]

Since we have the following estimates (here $C$ stands for different constants):

\[
\left\{ \int_{\mathbb{R}^2} \text{Im}\left(\frac{\tilde{Q}_{zz}^2 e^{-2i\theta}}{\tilde{Q}_{zz}^2} \right) \right\} \leq \left( \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 \right)^{1/2} \left( \int_{\mathbb{R}^2} \frac{\tilde{Q}_{zz}^6}{\rho^4} \right)^{1/2}
\]

\[
\leq \|\nabla^2 \tilde{Q}\|_{L^2} \|\tilde{Q}\|_{L^\infty} \left( \int_{\mathbb{R}^2} \frac{\tilde{Q}_{zz}^4}{\rho^4} \right)^{1/2} \text{ (Hardy inequality)}
\]

\[
\leq C^2 \|\nabla^2 \tilde{Q}\|_{L^2}^2 \|\tilde{Q}\|_{L^\infty}^2, \quad (65)
\]

where we have used the Gagliardo-Nirenberg inequality: $\|\nabla f\|_{L^4} \leq C\|f\|_{L^\infty}^{1/2}\|\nabla^2 f\|_{L^2}^{1/2}$ for some constant $C$ in the last inequality,

\[
\left\{ \int_{\mathbb{R}^2} \text{Im}\left(\frac{\tilde{Q}_{zz}^2 e^{-2i\theta}}{\tilde{Q}_{zz}^2} \right) \right\} \leq \left( \int_{\mathbb{R}^2} \tilde{Q}_{zz}^2 \right)^{1/2} \left( \int_{\mathbb{R}^2} \frac{\tilde{Q}_{zz}^8}{\rho^4} \right)^{1/4} \left( \int_{\mathbb{R}^2} \tilde{Q}_{zz}^4 \right)^{1/4}
\]

\[
\leq \|\nabla^2 \tilde{Q}\|_{L^2} \|\tilde{Q}\|_{L^\infty} \left( \int_{\mathbb{R}^2} \frac{\tilde{Q}_{zz}^4}{\rho^4} \right)^{1/4} \left( \int_{\mathbb{R}^2} \tilde{Q}_{zz}^4 \right)^{1/4} \text{ (Hardy)}
\]

\[
\leq C^2 \|\nabla^2 \tilde{Q}\|_{L^2}^2 \|\tilde{Q}\|_{L^\infty}^2 \quad (66)
\]
and
\[ \|Q\|_{L^\infty}^2 \leq C(\|Q\|_{L^2}^2 + \|\nabla^2 Q\|_{L^2}^2), \quad Q \in H^2(\mathbb{R}^2). \tag{67} \]

We substitute (65,66,67) and other easily obtained estimates into (64) to have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 + \left( E - 4 \frac{b}{d - bt} \right) \int_{\mathbb{R}^2} |\tilde{Q}_{zz}|^2 \\
\geq -C\|Q\|_{L^2}^2 \|\nabla^2 \tilde{Q}\|_{L^2}^2 \geq -Ce^{2Et} (e^{-2Et} \|Q_0\|_{L^2}^2 + \|\nabla^2 \tilde{Q}\|_{L^2}^2) \|\nabla^2 \tilde{Q}\|_{L^2}^2
\]
for some constant \(C\). Here we have used the fact: \(\|Q\|_{L^2} = e^{-Et} \|Q_0\|_{L^2}\) from part i) of this lemma. In a completely similar way, after either taking derivative with respect to \(\bar{z}\) to the both sides of (61) or with respect to \(z\) to both sides of (62), and did as above, we still have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla^2 \tilde{Q}|^2 + \left( E + C\|Q_0\|_{L^2}^2 - 4 \frac{b}{d - bt} \right) \int_{\mathbb{R}^2} |\nabla^2 \tilde{Q}|^2 \\
\geq -Ce^{2Et} (e^{-2Et} \|Q_0\|_{L^2}^2 + \|\nabla^2 \tilde{Q}\|_{L^2}^2) \|\nabla^2 \tilde{Q}\|_{L^2}^2
\]
for some positive constant \(C\). From the above two inequalities, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla^2 \tilde{Q}|^2 + \left( E + C\|Q_0\|_{L^2}^2 - 4 \frac{b}{d - bt} \right) \int_{\mathbb{R}^2} |\nabla^2 \tilde{Q}|^2 \geq -Ce^{2Et} \|\nabla^2 \tilde{Q}\|_{L^2}^4
\tag{68}
\]
for some positive constant \(C\). Here we have used the fact that the norm \(\int_{\mathbb{R}^2} (|Q_{zz}|^2 + |Q_{zz}|^2) dx dy\) is equivalent to the norm \(\int_{\mathbb{R}^2} |\nabla^2 Q|^2 dx dy\). That is to say, there is an absolute positive constant \(C_0\) such that \(\int_{\mathbb{R}^2} |\nabla^2 Q|^2 dx dy \leq C_0 \int_{\mathbb{R}^2} (|Q_{zz}|^2 + |Q_{zz}|^2) dx dy \leq \frac{1}{C_0} \int_{\mathbb{R}^2} |\nabla^2 Q|^2 dx dy\). One may see, when we set \(E = C\|Q_0\|_{L^2}^2 - 4 \frac{b}{d} > 0\), that the differential inequality (68) implies
\[
\frac{d}{dt} \|\nabla^2 \tilde{Q}\|_{L^2}^4 + 2E \|\nabla^2 \tilde{Q}\|_{L^2}^2 \geq -Ce^{2Et} \|\nabla^2 \tilde{Q}\|_{L^2}^4,
\]
which is equivalent to
\[
\frac{d}{dt} \frac{1}{\|\nabla^2 \tilde{Q}\|_{L^2}^4} - 2E \frac{1}{\|\nabla^2 \tilde{Q}\|_{L^2}^2} \leq Ce^{2Et}
\]
or
\[
\frac{d}{dt} \left( \frac{1}{e^{2Et}\|\nabla^2 \tilde{Q}\|_{L^2}^2} \right) \leq C.
\]
Therefore
\[
\frac{1}{e^{2Et}\|\nabla^2 \tilde{Q}\|_{L^2}^2} - \frac{1}{\|\nabla^2 Q_0\|_{L^2}^2} \leq Ct, \quad t \geq 0.
\]
This shows (59) by substituting \(\|\nabla^2 Q\|_{L^2}^2 = e^{2Et}\|\nabla^2 \tilde{Q}\|_{L^2}^2\) and \(\|\nabla^2 Q_0\|_{L^2} = \|\nabla^2 \tilde{Q}_0\|_{L^2}\). \(\square\)

We are in the position to prove our main result of this paper.
**Theorem 3** There are $H^3(\mathbb{R}^2)$-solutions to the 1+2 dimensional Landau-Lifshitz equation (2), which blow up in finite time.

**Proof.** We only need to show the existence of some solutions to the 1+2 dimensional isotropic Landau-Lifshitz equation (2) such that their $H^3(\mathbb{R}^2)$-norms blow up in finite time. For this purpose, we first take an initial (complex) radial function $q_0(\rho)$ such that $Q_0(\rho,\theta) = q_0(\rho)e^{-i\theta} \in H^2(\mathbb{R}^2)$ and $||Q_0||_{L^2(\mathbb{R}^2)}$ is chosen to be so small that will be specified below. Then, for any given $b, d$ with $b < 0$ and $d > 0$, we solve the Cauchy problem (56) to get its unique smooth solution $q(\rho, t)$ with $q(\rho, t)e^{-i\theta} \in H^2(\mathbb{R}^2)$. Setting $Q(\rho, \theta, t) = \tilde{q}(\rho, t)e^{-i\theta}$, where $\tilde{q}(\rho, t) = q(\rho, t)$, we see that $Q(\rho, \theta, t)$ a smooth $H^2(\mathbb{R}^2)$-solution to (55). There are only two possibilities to $Q$, say, a) there is a finite $T_0 > 0$ such that $\lim_{t \to T_0^-} ||\nabla^2 Q||_{L^2} = +\infty$ (short time $H^2$-existence) and b) $||\nabla^2 Q||_{L^2} < \infty$ for any $0 < t < \infty$ (long time $H^2$-existence). In the followings we shall show separately that, in either the case of the short or the long time existence, there is a solution $S = S(z, \bar{z}, t)$ to the Landau-Lifshitz equation (3) such that its $H^3(\mathbb{R}^2)$-norm blows up in finite time.

Before doing these, we first choose some real $a$ and $c$ such that the $2 \times 2$ matrix

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in SL(2, R), \ i.e. \ ad - bc = 1.
$$

The precise choice of $a$ and $c$ will be determined later. From Lemma 1, we see that

$$
\tilde{p}(x, y, t) = -\frac{ib\bar{z}}{2(a + bt)}
$$

$$
\tilde{q}(x, y, t) = \frac{e^{-\frac{b(z^2 + y^2)}{4(a + bt)}}}{a + bt}\tilde{q}(R, T)e^{-i\theta}
$$

$$
\tilde{r}(x, y, t) = -\frac{e^{\frac{bt(z^2 + y^2)}{4(a + bt)}}}{(a + bt)}\tilde{q}(R, T)e^{-i\theta}
$$

$$
\tilde{u}(x, y, t) = -\frac{1}{(a + bt)^2}\left(|\tilde{q}(R, T)|^2 - 2 \int_R^\infty \frac{|\tilde{q}(\tau, T)|^2}{\tau}d\tau \right) + \frac{b^2z\bar{z}}{2(a + bt)^2},
$$

is a smooth solution to Eq.(8) with the restriction (9), where $R = \frac{\sqrt{a + bt}}{a + bt}$ and $T = \frac{\sqrt{a + bt}}{a + bt}$. By Theorem 1 and Theorem 2, there is a smooth solution $S(z, \bar{z}, t)$ to the Eq.(3) which is gauge equivalent to the solution $(\tilde{p}, \tilde{q}, \tilde{r}, \tilde{u})$ of (8) with the restrictions (9). From the formula: $S_z = 20G^\sigma G^{-1}$, $S_{\bar{z}z} = -2GP^\sigma G^{-1}$ deduced by the relation (38), where $U = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{r} & -\tilde{p} \end{pmatrix}$, $P = \begin{pmatrix} \tilde{p} & \tilde{r} \\ -\tilde{q} & \tilde{p} \end{pmatrix}$ and the fact that $G \in SU(2)$ is smooth (which implies that the absolute of every entry of $G$ is not greater than 1), we have, as an entry, $|\tilde{q}| \leq ((s_1)_z| + |(s_2)_z| + |(s_3)_z|)$. Thus we obtain

$$
||S_z||_{L^2(\mathbb{R}^2)}^2 \geq ||Q||_{L^2(\mathbb{R}^2)}^2 = ||Q_0||_{L^2(\mathbb{R}^2)}^2.
$$

In a similar way, we have the formulae:

$$
S_{zz} = 2G \begin{pmatrix} -2\bar{q}\bar{r} & 2\bar{p}\bar{q} - \bar{q}_{\bar{z}} & 2\bar{p}\bar{r} + \bar{r}_{\bar{z}} \\ 2p\bar{r} + \bar{r}_{z} & 2p\bar{q} - \bar{q}_{z} & -2p\bar{q}\bar{r} \end{pmatrix} G^{-1}, \quad S_{\bar{z}\bar{z}} = 2G \begin{pmatrix} -2\bar{q}\bar{r} & 2\bar{p}\bar{q} - \bar{q}_{\bar{z}} & 2\bar{p}\bar{r} + \bar{r}_{\bar{z}} \\ 2p\bar{r} + \bar{r}_{z} & 2p\bar{q} - \bar{q}_{z} & -2p\bar{q}\bar{r} \end{pmatrix} G^{-1}.
$$
\[\begin{align*}
S_{zzz} &= 2G \left( -\frac{3(\overline{q\theta})_z}{B} \ A \ G^{-1}, \quad S_{zzz} = 2G \left( -\frac{3(\overline{q\theta})_z}{E} \ D \ G^{-1}, \quad S_{zzz} = 2G \left( \begin{array}{cc}
F & H \\
J & -F
\end{array} \right) \ G^{-1}
\right) \right.
\end{align*}\]
where
\begin{align*}
A &= \left( 2\tilde{p}\tilde{q} - \tilde{q}_z \right)_z - 2\tilde{p}(2\tilde{p}\tilde{q} - \tilde{q}_z) - 4\tilde{q}^2 \tilde{r} \\
B &= \left( 2\tilde{p}\tilde{r} + \tilde{r}_z \right)_z + 2\tilde{p}(2\tilde{p}\tilde{r} + \tilde{r}_z) + 4\tilde{q}^2 \tilde{q}^2 \\
D &= \left( 2\tilde{p}\tilde{r} + \tilde{r}_z \right)_z + 2\tilde{p}(2\tilde{p}\tilde{r} + \tilde{r}_z) + 4\tilde{r}^2 \tilde{q} \\
E &= \left( 2\tilde{p}\tilde{q} - \tilde{q}_z \right)_z - 2\tilde{p}(2\tilde{p}\tilde{q} - \tilde{q}_z) - 4\tilde{q}^2 \tilde{r} \\
F &= -2(\tilde{q}\tilde{r})_z + \tilde{r}(2\tilde{p}\tilde{r} + \tilde{r}_z)_z - \tilde{q}(2\tilde{p}\tilde{q} - \tilde{q}_z) \\
H &= \left( 2\tilde{p}\tilde{q} - \tilde{q}_z \right)_z + 2\tilde{p}(2\tilde{p}\tilde{q} - \tilde{q}_z) + 4\tilde{r}^2 \tilde{q} \\
J &= \left( 2\tilde{p}\tilde{r} + \tilde{r}_z \right)_z - 2\tilde{p}(2\tilde{p}\tilde{r} + \tilde{r}_z) - 4\tilde{q}^2 \tilde{r}
\end{align*}\]
Thus we see from the above matrices equations that
\[\begin{align*}
\|S_{zzz}\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}\|_{L^2(\mathbb{R}^2)} \\
\geq \|A\|_{L^2(\mathbb{R}^2)} + \|D\|_{L^2(\mathbb{R}^2)} + \|H\|_{L^2(\mathbb{R}^2)} \\
\geq \left( \frac{1}{a + bt} \right)^2 \left( C\|\nabla^2 Q(T)\|_{L^2(\mathbb{R}^2)} - 48\|Q(T)\|_{L^2(\mathbb{R}^2)} \right)^2
\end{align*}\]
for some positive constant \(C\). On the other hand, by using the Sobolev inequality \(\|Q\|_{L^6(\mathbb{R}^2)} \leq C\|Q\|_{L^2(\mathbb{R}^2)}^{1/3}\|\nabla Q\|_{L^2(\mathbb{R}^2)}^{2/3}\) and the Gagliardo-Nirenberg inequality \(\|\nabla Q\|_{L^2(\mathbb{R}^2)} \leq C\|Q\|_{L^2(\mathbb{R}^2)}^{1/2}\|\nabla^2 Q\|_{L^2(\mathbb{R}^2)}^{1/2}\), where \(C\) stands for different positive constants, we have \(\|Q\|_{L^6(\mathbb{R}^2)} \leq C_0\|Q\|_{L^2(\mathbb{R}^2)}\|\nabla Q\|_{L^2(\mathbb{R}^2)} = C_0\|Q\|_{L^2(\mathbb{R}^2)}^{2/3}\|\nabla^2 Q\|_{L^2(\mathbb{R}^2)}\) for some constant \(C_0\). Here we have used the conservation law i) in Lemma 2. Substituting this inequality into (69), we may obtain
\[\begin{align*}
\|S_{zzz}(T)\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}(T)\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}(T)\|_{L^2(\mathbb{R}^2)} \\
\geq \left( \frac{1}{a + bt} \right)^2 \left( C - 48C_0\|Q_0\|_{L^2(\mathbb{R}^2)}^{2/3}\|\nabla^2 Q(T)\|_{L^2(\mathbb{R}^2)} \right)^2
\end{align*}\]
Therefore we may choose the initial data \(Q_0\) in advance such that \(\|Q_0\|_{L^2(\mathbb{R}^2)}^2\) is small enough that \(C - 48C_0\|Q_0\|_{L^2(\mathbb{R}^2)}^2 > 0\). Then we have
\[\begin{align*}
\|S_{zzz}(T)\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}(T)\|_{L^2(\mathbb{R}^2)} + \|S_{zzzz}(T)\|_{L^2(\mathbb{R}^2)}
\end{align*}\]
\[ \geq C \left( \frac{1}{a + bt} \right)^2 \| \nabla^2 Q(T) \|_{L^2(\mathbb{R}^2)} \]  

(70)

for some constant \( C \) depending on \( \| Q_0 \|_{L^2} \).

Now let’s discuss the two different situations of the short or long time existence mentioned before.

a) short time \( H^2 \)-existence case. That is, there is a finite time \( T_0 > 0 \) such that \( \lim_{t \to T_0^-} \| \nabla^2 Q \|_{L^2} = +\infty \). Since \( t = \frac{at_0 - c}{d - bt_0} \), we may choose the entries \( a, b, c, d \) with \( ab < 0 \) in advance such that \( t_0 = \frac{at_0 - c}{d - bt_0} > 0 \) and \( t_0 = \frac{at_0 - c}{d - bt_0} < -\frac{a}{b} \) (for example \( a = 1, b = -1, d = 1 \) is small and \( c = 0 \)) since \( -\frac{a}{b} - \frac{at_0 - c}{d - bt_0} = \frac{d - bc}{bd - bt_0} > 0 \). Thus we see that there is an one to one corresponding between \( t \in [0, t_0] \) and \( T \in [0, T_0) \) and

\[ \lim_{t \to t_0^-} \| \nabla^2 Q(T) \|_{L^2} = \lim_{T \to T_0^-} \| \nabla^2 Q(T) \|_{L^2} = +\infty. \]

Therefore, by noting that \( \| S \|_{H^1(\mathbb{R}^2)}^2 \geq C_0 (\| S_{zzz} \|_{L^2(\mathbb{R}^2)} + \| S_{zz} \|_{L^2(\mathbb{R}^2)} + \| S_{zzzz} \|_{L^2(\mathbb{R}^2)})^2 + \| S \|_{H^2(\mathbb{R}^2)}^2 \) for some positive constant \( C_0 \) and the estimate (70), there is a finite time \( t_0 < -a/b \) such that

\[ \lim_{t \to t_0^-} \| S \|_{H^3(\mathbb{R}^2)} \geq C_0 C \left( \frac{1}{a + bt_0} \right)^2 \lim_{T \to T_0^-} \| \nabla^2 Q(T) \|_{L^2} = +\infty. \]

This shows that \( \| S \|_{H^3(\mathbb{R}^2)} \) blows up in this case of short time \( H^2 \)-existence of \( Q \).

b) long time \( H^2 \)-existence case. That is, \( \| \nabla^2 Q(T) \|_{L^2} \) exists for any \( T > 0 \). From (70), we see that

\[ \| S \|_{H^3(\mathbb{R}^2)}^2 \geq C_0 (\| S_{zzz}(T) \|_{L^2(\mathbb{R}^2)} + \| S_{zz}(T) \|_{L^2(\mathbb{R}^2)} + \| S_{zzzz}(T) \|_{L^2(\mathbb{R}^2)})^2 \]

\[ \geq C \left( \frac{1}{a + bt} \right)^4 \| \nabla^2 Q(T) \|_{L^2(\mathbb{R}^2)}^2 \]  

(71)

for some constant \( C \) depending on \( \| Q_0 \|_{L^2(\mathbb{R}^2)} \) when \( \| Q_0 \|_{L^2(\mathbb{R}^2)} \) is chosen suitably small. By lemma 2 ii), we have (if \( b < 0 \) and \( d > 0 \))

\[ \| \nabla^2 Q(T) \|_{L^2(\mathbb{R}^2)}^2 \geq \frac{1}{C_1 T + \frac{1}{\| \nabla^2 Q_0 \|_{L^2(\mathbb{R}^2)}^2}}, \quad T > 0. \]

for some positive constant \( C_1 \) and hence

\[ \| S \|_{H^3(\mathbb{R}^2)}^2 \geq C \left( \frac{1}{a + bt} \right)^3 \frac{1}{C_1 T + \frac{1}{\| \nabla^2 Q_0 \|_{L^2(\mathbb{R}^2)}^2}}. \]  

(72)

Here are have used the identity: \( a + bt = \frac{1}{d - bt} \). Because \( T \) is an increasing function of \( t \), when set the entries \( a > 0, b < 0, c = 0 \) and \( d > 0 \) in advance, we see that there is one
to one corresponding between \( t \in [0, -b/a] \) and \( T \in [0, +\infty) \) and \( T \to +\infty \) as \( t \to -b/a \).

Thus, under this circumstance,

\[
\lim_{t \to -b/a} \left( \frac{1}{a + bt} \right)^3 \frac{(d - bT)}{(C_1 T + \frac{1}{||\nabla^2 Q_0||^2_{L^2(R^2)}})} = +\infty. \quad (73)
\]

Therefore, from (72) and (73), there is a finite time \( t_0 \leq -b/a \) such that

\[
\lim_{t \to t_0^-} ||S||_{H^3(R^2)} = +\infty.
\]

This also shows that \( ||S||_{H^3(R^2)} \) blows up in finite time in this case. The proof of Theorem 3 is completed. \( \square \)

We would like to point out that, not like the nonlinear Schrödinger equation with critical cases ([33]), for blowing-up \( H^3(R^2) \)-solutions to the 1+2 dimensional Landau-Lifshitz equation (2) constructed in the proof of Theorem 3 b) one gets no their (up to the third derivatives) point-wise blow-up information at the origin as \( t \to -a/b \). So our blowing up result does not contradict to the global existence of \( W^{m+1,6}(R^2) \)-solutions to the Landau-Lifshitz equations with small initial data due to Sulem, Sulem and Bardos in [29]. Furthermore, we may see from Theorem 3 that it is impossible to establish the global existence of \( H^m(R^2) \)-solutions \((m \geq 3)\) to the Cauchy problem of the 1+2 dimensional Landau-Lifshitz equation for small initial data in general. That is to say, the result of the global existence of \( W^{m,6}(R^2) \)-solutions in [29] to the Cauchy problem of the Landau-Lifshitz equation (2) for small initial data cannot be generalized to the case of energy estimates in general. This indicates that the higher dimensional Landau-Lifshitz equations may admit some unusual dynamical properties.

In a similar way, the following conformal transformation of a solution \( Q(\rho, t) \) to Eq.(52):

\[
Q(\rho, t) \to \tilde{Q}(\rho, t) = \frac{e^{-\frac{b|\rho|^2}{4(a+bt)}}}{a+bt}Q(R, T)
\]

is invariant, i.e., \( \tilde{Q}(\rho, t) \) is a solution to Eq.(52) too. The proof of this conclusion is a direct computation and we omitted it here. It is well known that nonlinear Schrödinger equations (54) with \( \sigma = 4/n + 1 \) have localized finite energy solutions ([27, 3, 33]) which are called solitary waves. Those are solutions of the form \( q(x; E_0)e^{iE_0t} \) with \( E_0 > 0 \), where \( q(x; E_0) \) solves the semi-linear elliptic equation \( \Delta q - E_0q + |q|^\sigma q = 0, q \in H^1(R^n) \). For our present Eq.(52), the solutions of the form

\[
\psi(\rho, t) = q(\rho, E)e^{iEt},
\]

where \( E \) is a real constant and \( q(\rho, E) \) solves

\[
q_{\rho\rho} + \frac{1}{\rho}q_{\rho} - \frac{1}{\rho^2}q - Eq + 2|q|^2q - 4q \int_0^\infty \frac{|q|^2(\tau)}{\tau} d\tau = 0, \quad q \in W^{1,\sigma}(R^2) \quad (75)
\]
for some $\sigma \geq 2$, are also called solitary waves. We are interested in $W^{1,\sigma}(\mathbb{R}^2)$-solitary wave solutions for general $p \geq 2$ because of the $W^{1,\sigma}(\mathbb{R}^2)$-global existence of the Cauchy problem of the Landau-Lifshitz equations (1) with small initial data obtained in [29] for $\sigma = 6$ and in [6] for $\sigma = 4$ and $n = 2$. However, whether (75) has a nontrivial solution is unknown.

We finally remark that the system (8) provides a new mathematical point of view in investigating the 1+2 dimensional isotropic Landau-Lifshitz equation (2). The existence of blowing-up $H^3(\mathbb{R}^2)$-solutions to the Landau-Lifshitz equation (2) gives also an affirmative answer to the problem proposed by Ding in [12] for Schrödinger maps. However whether there are $H^m(\mathbb{R}^n)$-solutions to the Landau-Lifshitz equation for general $n \geq 3$ which blow up in finite time is still unknown. We believe that some ideas displayed in this paper will be helpful in understanding this problem.

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References

[1] A.A. Belavin and A.M. Polyakov, Metastable states of two dimensional isotropic ferromagnets, JETP Lett. 22 (1975) 503-506.

[2] L.A. Bordag and A.B. Yanovski, Polynomial Lax pairs for the Chiral $O(3)$-field equation and the Landau-Lifshitz equation, J. Phys. A: Math. Gen. 28 (1995) 4007-4013.

[3] H. Berestycki and P.L. Lions, Nonlinear scalar field equations I,II, Arch. Rat. Mech. Anal. 82 (1983) 313-376.

[4] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993) 107-156.

[5] J. Bourgain, Global solutions of nonlinear Schrödinger equations, AMS Colloq. Publ. 46, AMS Providence R.I. 1999.

[6] N. Chang, J. Shatah and K. Unlenbeck, Schrödinger maps, Comm. Pure Appl. Math. 53 (2000) 590-602.

[7] M. Daniel, K. Porsezian and M. Lashmanan, On the integrability of the inhomogeneous spherically symmetric Heisenberg ferromagnet in arbitrary dimensions, J. Math. Phys. 35 no.12 (1994) 6498-6510.
[8] Q. Ding, A note on the NLS and the Schrödinger flow of maps, Phys. Lett. A 248 (1998) 49-56; The NLS equation and its $SL(2, \mathbb{R})$ structure, J. Phys. A: Math. Gen. 323 (2000) L325-329.

[9] Q. Ding, The gauge equivalence of the NLS and the Schrödinger flow of maps in $2 + 1$ dimensions, J. Phys. A: Math. Gen. 32 (1999) 5087-5096.

[10] Q. Ding and Z. Zhu, On the gauge equivalent structure of the modified nonlinear Schrödinger equation, Phys. Lett. A 295 (2002) 192-197.

[11] Q. Ding and Z. Zhu, On the gauge equivalent structure of the Landau-Lifshitz equation and its applications, J. Phys. Soc. of Japan 72 no.1 (2003) 49-53.

[12] W.Y. Ding, On the Schrödinger flows, Proceedings of the ICM, Beijing Vol.II (2002) 283-291.

[13] W.Y. Ding and Y.D. Wang, Schrödinger flows into Kähler manifolds, Science in China A 44 (2001) 1446-1464.

[14] L.D. Faddeev and L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag Berlin Heideberg 1987.

[15] C.A. Faure, D.J. Moore and C. Piron, Deterministic evolutions and Schrödinger flows, Helv. Phys. Acta 68 (1995) 150-157.

[16] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, 1977.

[17] R.T. Glassey, On the blowing-up of solutions to that Cauchy problem for the non-linear Schrödinger equations, J. Math. Phys. 18 (1977) 1794-1797.

[18] M. Grillakis and V. Stefanopoulos, Lagrangian formulation, energy estimate, and the Schrödinger maps problem, Comm. PDE 27 (2002) 1845-1877.

[19] C.H. Gu, Soliton Theory and Its Applications, Zhejiang Science and Technology Publishing House, 1990.

[20] O. Kavian, A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations, Trans. of AMS 299 (1987) 193-203.

[21] C.E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math. 134 (1998) 489-545.

[22] L.D. Landau and E.M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowj. 8 (1935) 153; reproduced in Collected Papers of L.D. Landau, Pergaman Press, New York (1965) 101-114.
[23] M. Lakshmanan and M. Daniel, On the evolution of higher dimensional Heisenberg continuum spins systems, Physica A 107 (1981) 533-552.

[24] A. Nahmod, A. Stefanov and K. Uhlenbeck, On Schröinger maps, Comm. Pure Appl. Math. 56 no.1 (2003) 114-151; Erratum: On Schrödinger maps, Comm. Pure Appl. Math. 57 no.6 (2004) 833-839.

[25] N. Papanicolaou and T.N. Tomaras, Dynamics of magnetic vortices, Nuclear Phys. B 360 (1991) 425-462.

[26] T. Ruijgrok and J. Jukiewicz, On a new formulation of the continuum Heisenberg spin system in a space of arbitrary dimensions, Physica A 103 (1980) 573-585.

[27] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) 149-162.

[28] M. Struwe, Geometric evolution problems, in Nonlinear partial differential equation on differential geometry ed. by R. Hardt and M. Wolf, IAS/Park City Math. Ser. Vol.2 (1996) 257-333.

[29] P.L. Sulem, C. Sulem and C. Bardos, On the continuous limit for a system of classical spins, Comm. Math. Phys. 107 (1986) 431-454.

[30] C. Sulem and P.L. Sulem, The nonlinear Schrödinger equations, Applied Math. Sci. 139, Springer-Verlag, New York 1999.

[31] C.L. Terng and K. Uhlenbeck, Schrödinger flows on Grassmannian, (1999) math.DG /9901086.

[32] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimation, Comm. Math. Phys. 87 (1983) 567-576.

[33] M.I. Weinstein, On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations, Comm. PDE 11 no.5 (1986) 545-565.

[34] V.E. Zarkharov and L.A. Takhtajan, Equivalence of a nonlinear Schrödinger equation and a Heisenberg ferromagnetic equation, Theor. Math. Phys. 8 (1979) 17-23.