From causality to time and back

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Abstract. In this work the problem of the existence of a (semi-)time function on spacetime is investigated together with the problem of recovering the causal structure from the set of time functions allowed by the spacetime. These problems are solved thanks also to a mathematical correspondence with utility theory.

1. Introduction
When reasoning about time in general relativity we can distinguish between two different notions which intuitively represent what in everyday life we may term the metric and the order sides of time. The former notion is given by proper time $\tau$ which allows us to give a rigorous meaning to the ‘time interval’ between two events on an observer’s worldline. The latter notion is given by the time function, $t : M \to \mathbb{R}$, namely a function defined all over the spacetime which is continuous and increases over causal curves, that is, $x < y \Rightarrow t(x) < t(y)$.

In this work we shall deal with the latter aspect of time. In particular we investigate under which causality conditions a time function exists. If a time function exists it needs not to be unique, a fact which will bring us to the question as to whether the causal structure of a spacetime can be recovered from the set of time functions. We shall point out that mathematically these problems have analogs in the field of microeconomics known as Utility Theory. The drawn connection will allow us to obtain interesting new results on the spacetime side. For a more detailed study with full proofs the reader is referred to [1].

2. Order relations
Let us recall some definitions from the theory of relations. Let $X$ be a set and let $\Delta = \{(x, x) : x \in X\}$. A preorder $\leq$ on a set $X$ is a reflexive ($x \leq x$) and transitive relation. A strict partial order is an irreflexive ($x \not< x$) and transitive relation. A partial order is a preorder which is antisymmetric ($x \leq y$ and $y \leq x \Rightarrow x = y$). A total preorder is a preorder in which every two elements $x, y \in X$ are comparable, that is (totality) $x \leq y$ or $y \leq x$. A total order is a partial order which is total, or equivalently, a total preorder which is antisymmetric.

Szpilrajn [2, 3] proved\(^1\)

Theorem 2.1 (Szpilrajn) Every partial order can be extended to a total order. Moreover, every partial order is the intersection of all the total orders that extend it.

\(^1\) Actually he gave the proof for strict partial orders but this version can be proved similarly.
In a sense in this paper we are going to find an analogous result in which the partial order is replaced by the Seifert relation (see below) and the total orders are replaced by preorderings induced by time functions.

On a spacetime (time oriented Lorentzian manifold) \((M, g)\) let us write \(x < y\) if there is a future directed causal curve from \(x\) to \(y\) and \(x \leq y\) if \(x < y\) or \(x = y\). Let us write \(x \ll y\) if there is a future directed timelike curve from \(x\) to \(y\). On a spacetime manifold it is natural to define the following relations

\[
I^+ = \{(x, y) : x \ll y\},
\]

\[
J^+ = \{(x, y) : x \leq y\}.
\]

called respectively the chronological relation and the causal relation of the spacetime. In a chronological spacetime \(I^+\) is a strict partial order. The relation \(J^+\) is a preorder which becomes a partial order in causal spacetimes. With respect to the product topology on \(M \times M\) \(I^+\) is open [4, 5] while \(J^+\) is not necessarily closed (e.g., remove a point from Minkowski spacetime).

Given two Lorentzian metrics on the manifold \(M\) we write \(g < g'\) if at any point \(x \in M\), the future timelike cone of \(g'\) contains the future causal cone of \(g\). A spacetime is stably causal if causality is preserved under sufficiently small perturbations of the metric. More precisely, a spacetime is stably causal if there is \(g' > g\) such that \((M, g', g)\) is causal [6]. Stable causality is stronger than causality because it not only demands the absence of closed causal curves but also states that the light cones can be widened all over the spacetime without introducing closed causal curves.

Perhaps one of the most well behaved relations that can be defined on a spacetime is the Seifert relation [7]

\[
J^*_S = \bigcap_{g' > g} J^+_{g'},
\]

which turns out to be reflexive, transitive and closed [8]. Moreover, it is a partial order if and only if the spacetime is stably causal [8, 9].

Another well behaved causality relation is that defined by Sorkin and Woolgar [10] and denoted \(K^+\). By definition \(K^+\) is the smallest relation which is closed, transitive and contains \(J^+\). Thus the relation \(K^+\) is defined exactly through its good properties. They also defined the spacetime to be \(K\)-causal if \(K^+\) is antisymmetric. It turns out [11] that stable causality is equivalent to \(K\)-causality and that in this case \(K^+ = J^*_S\). For this reason in most of what follows \(K^+\) can replace \(J^*_S\) and the other way around.

Let us now introduce the concept of time function. A time function is a continuous map \(t : M \to \mathbb{R}\) such that \(x < y \Rightarrow t(x) < t(y)\). The definition of semi-time function is similar with \(x < y\) replaced by \(x \ll y\). In [6, 4] Hawking proved that a stably causal spacetime admits a time function. The proof was based on the following argument. The spacetime \(M\) can be endowed with a complete Riemannian metric to which corresponds a volume form \(\mu\). Moreover, the Riemannian metric can be chosen in such a way that, \(\mu(M) = 1\). Let \(g_{\lambda}, \lambda \in [1, 2]\), be a one parameter family of causal metrics such that \(g_{\lambda} > g, \lambda' > \lambda \Rightarrow g_{\lambda'} > g_{\lambda}\). The function \(t_{\lambda}(x) = \mu(I_{g_{\lambda}}(x))\) satisfies \(x < y \Rightarrow t_{\lambda}(x) < t_{\lambda}(y)\) but is only lower semi-continuous. Nevertheless the average \(t = \int_1^2 t_{\lambda} d\lambda\) can be proved to be continuous [4].

I shall give here a different and more general argument for the existence of a time function under stable causality. The idea is that this existence follows from the closure of the relation whose antisymmetry implies stable causality, that is \(J^*_S\) or \(K^+\). According to this approach the existence of a time function is in a sense independent of the Lorentzian geometric nature of the manifold. Before we come to this argument let me review some developments in utility theory.
3. Utility theory and its use in Lorentzian geometry

Utility theory began with the solution given by Gabriel Cramer and Daniel Bernoulli to a problem conceived by Daniel’s cousin, Nicolas Bernoulli, in 1713. The problem is today known as St. Petersburg’s paradox. Nicolas considered two people A and B playing the following game. Player A pays a fixed fee \( F \) to enter the game, then the players toss repeatedly a coin till they get an head. If they get the first head at the \( n \)-th tossing then A wins \( 2^n \) dollars. What would be a fair price \( F \) for entering the game? Without the fee player A is expected to win

\[
\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \cdots + \frac{1}{2^n} \cdot 2^n + \cdots = +\infty,
\]

thus when offered to play A should accept whatever is the finite fee. This fact goes contrary to experiments in which it seems that people would hardly pay a mere 25$ fee to enter the game. According to D. Bernoulli the reason is that, roughly speaking, “happiness is not proportional to money”, so it is not the expected value that should be averaged but rather a new function, called utility that quantifies how much an option is preferred over another. Thus in St. Petersburg game the average utility would actually converge if \( u \) is bounded as a function of money

\[
\frac{1}{2} \cdot u(2) + \frac{1}{4} \cdot u(4) + \cdots + \frac{1}{2^n} \cdot u(2^n) + \cdots < +\infty
\]

From the discussion around this problem started the utilitarian approach of economics, according to which an individual behaves so as to maximize the expected utility. In mathematical terms the utility is a function \( u : A \to \mathbb{R} \) over a totally preordered space \( A \) which satisfies \( x \leq y \) if \( u(x) \leq u(y) \). The space \( A \) is the space of alternatives, and the preorder expresses the preferences of the individual for those alternatives. The preorder is total because it is believed that the individual can always tell, when comparing two options, which one is preferred.

Actually the totality property was later challenged by economists [12] who dropped it in order to accommodate some evidence of indecisiveness in the individual behavior. With a preorder the definition of utility had to be changed as follows

\[
x \leq y \Rightarrow u(x) \leq u(y), \quad x < y \Rightarrow u(x) < u(y)
\]

which reduces to the above definition for total preorders.

The alternative or prospects space \( A \) can be endowed with a topology which in this way introduces a notion of ‘closeness’ between alternatives. As a consequence in mathematical economics it became important to establish whether a given prospect space admitted a continuous utility function. In particular it was important to investigate the conditions to be imposed on the preorder so as to obtain the existence of a continuous utility function.

The most important theorems in this direction were Peleg’s [13] and Levin’s [14] theorems. We comment on Levin’s theorem because it can be shown that, at least for our purposes, Peleg’s theorem gives results which are weaker than Levin’s. Levin’s theorem in short states that the closure of the order relation is a sufficient requirement. More precisely we have the following

**Theorem 3.1 (Levin)** Let \( X \) be a second countable locally compact Hausdorff space, and \( R \) a closed preorder on \( X \), then there exists a continuous utility function. Moreover, denoting with \( \mathcal{U} \) the set of continuous utilities we have that the preorder \( R \) can be recovered from the continuous utility functions, namely there is a multi-utility representation

\[
(x, y) \in R \Leftrightarrow \forall u \in \mathcal{U}, \ u(x) \leq u(y).
\]

(1)

Actually, the last statement was not included in Levin’s article although its proof is tacitly given in that paper. This utility representation has been pointed out only recently in a preprint by Evren and Ok [15].

In [1] I have investigated the concept of utility for the relations \( I^+ \) and \( K^+ \) proving the following
Theorem 3.2

(i) In a chronological spacetime the continuous $I^+$-utilities are the semi-time functions.

(ii) In a stably causal spacetime the continuous $K^+$-utilities are the time functions.

Using point (i) I obtained what is possibly the only result available on the existence of semi-time functions.

**Theorem 3.3** A chronological spacetime in which $\bar{J}^+$ is transitive admits a semi-time function.

In order to apply this result it can be useful to recall that in reflective $(x \in \bar{I}^-(y) \iff y \in \bar{I}^+(x))$ spacetimes $\bar{J}^+$ is transitive [11] and that spacetimes admitting a complete timelike vector field are reflective [16].

I also proved, without using smoothing techniques, that the existence of a time function implies $K$-causality [11]. As $K^+$ is closed, joining this result with Levin’s theorem we get

**Theorem 3.4** A spacetime is $K$-causal if and only if it admits a time function (as a consequence time functions are always $K^+$-utilities). In this case, denoting with $\mathcal{A}$ the set of time functions we have that the partial order $K^+$ can be recovered from the time functions, that is

$$(x, y) \in K^+ \iff \forall t \in \mathcal{A}, t(x) \leq t(y). \quad (2)$$

I also showed that in this theorem time function can be replaced with temporal function, where a temporal function is a time function which is smooth with a timelike gradient. This theorem tells us under which conditions it is possible to recover the causality or the chronology relation from the set of time (or temporal) functions. In a causally simple spacetime it is possible to recover the causal relation because in these spacetimes $K^+ = J^+$, as $J^+$ is closed. In a class of spacetime which I called causally easy [11] which are characterized by strong causality and the transitivity of $\bar{J}^+$, and which stay between stable causality and causal continuity, it is possible to recover the chronology relation because $K^+ = \bar{J}^+$ and hence $I^+ = \text{Int} \bar{J}^+ = \text{Int} K^+$. Nevertheless, this study suggests that neither $I^+$ not $J^+$ should be regarded as fundamental and that instead at a more fundamental level the spacetime should be regarded as a topological ordered space where the order is provided by the $K^+$ relation.

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