Clebsch-Gordan coefficients and the binomial distribution.

Paul O’Hara

Dept. of Mathematics, Northeastern Illinois University, 5500 North St. Louis Avenue, Chicago, IL 60625-4699, USA. email: pohara@neiu.edu

Abstract

A class of Clebsch-Gordan coefficients are derived from the properties of conditional probability using the binomial distribution. In particular, in the case of \( l = l_1 + l_2 \) it is shown that

\[
\langle l_1/2 - k_1, l_2/2 - k_2 | l/2, k = l_1 + l_2 \rangle^2 = \binom{l_1}{k_1} \binom{l_2}{k_2} \binom{l}{k}
\]

Pacs: 3.65, 2.50.Cw.

1 Introduction

There appear to be two standard ways of calculation C-G coefficients in quantum mechanics. One method is to combine the “ladder” operator approach with orthogonality conditions. The second method is to use some type of closed form \( \cite{2} \) which allows the coefficients to be calculated directly. However, the latter approach is at times considered “tedious” \( \cite{2} \) and does not reveal any new information into the nature of these coefficients.

In this paper we prove a theorem which not only permits a special class of C-G coefficients to be calculated from a simple formula but also directly connects them to both the hypergeometric and binomial distributions of classical probability.

Before formulating and proving the theorem, we first define some notation. Let \( L = (L_1, L_2, L_3) \) denote the angular momentum operator and define

\[
L^\pm = L_x \pm iL_y.
\]

Then

\[
L^2 = L_x^2 + L_y^2 + L_z^2 = L^- L^+ + L_z^2 + L_z,
\]

from which it follows that

\[
L^2 |l, m\rangle = l(l + 1) |l, m\rangle,
\]
and

\[ L^\pm |l, m\rangle = [(l \mp m)(l \pm m + 1)]^{1/2} |l, m \pm 1\rangle, \quad (5) \]

where \(|l, m\rangle\) is an eigenvector of \(L^2\) and \(L_z\). Similarly, the basis vectors \(|l, m\rangle \ldots |l - n, m\rangle\) are eigenvectors of \(L^2\) and \(L_z\) with \(-l \leq m \leq l\). Now consider the operator \(L = L_1 + L_2\), \(L_1\) and \(L_2\) being angular momentum operators as defined above. Denote the basis vectors of \(L^2, L_z\) by \(|LM\rangle\), where \(|l_1 - l_2| \leq L \leq |l_1 + l_2|\) and denote the joint basis vector of \((L_1^2, L_1z)\) and \((L_2^2, L_2z)\) by \(|l_1, l_2, m_1, m_2\rangle\) or \(|m_1, m_2\rangle\), if there is no ambiguity.

2 Clebsch-Gordan Coefficients and Binomial Distribution

With notation in place we now state and prove the following theorems:

**Theorem 1** Let \(L = L_1 + L_2\) be as above and let \(l = l_1 + l_2\). If

\[ |L = l/2, M = l/2\rangle = |m_1 = l_1/2, m_2 = l_2/2\rangle \]

then

\[ \langle m_1 = l/2 - k_1, m_2 = l_2/2 - k_2|L = l/2, M = l/2 - k\rangle^2 = \left( \begin{array}{c} l_1 \k_1 \\ k_2 \end{array} \right) \left( \begin{array}{c} l_2 \\ k \end{array} \right) \]

where \(k = k_1 + k_2\).

**Proof:** First note that if the operator \(L^-\) is applied \(k\) times to \(|L = 1/2, M = 1/2\rangle\) we get from equation (5)

\[ (L^-)^k |L = l/2, M = l/2\rangle = \left[ \left( \begin{array}{c} l_1 \k_1 \\ k_2 \end{array} \right) \left( \begin{array}{c} l_2 \\ k \end{array} \right) \right]^{1/2} (L^-)^{k-1} |l_1 l_2, l_2 - 1\rangle \]

\[ = \left[ (l - 1)(l - 2) \ldots (l - k + 1)k! \right]^{1/2} |l \frac{l}{2}, l_2 - k\rangle \]

\[ = \left( \begin{array}{c} l \k \end{array} \right) \frac{k!}{k!} |l \frac{l}{2}, l_2 - k\rangle \]

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But also \((L^-)^k = (L_x - iL_y)^k = \sum_{k_1=0}^{k} \binom{k}{k_1} L_x^{k_1} L_y^{k_2}\). And applying each term
\(L_x^{k_1} L_y^{k_2}\) to \(|m_1 = l_1/2, m_2 = l_2/2\) gives

\[
L_x^{k_1} L_y^{k_2} |m_1 = l_1/2, m_2 = l_2/2\) = \left( \binom{l_1}{k_1} \right)^{1/2} k_1! \left( \binom{l_2}{k_2} \right)^{1/2} k_2! \left| \frac{l_1}{2} - k_1, \frac{l_2}{2} - k_2 \rightangle = k! \left( \binom{l_1}{k_1} \right)^{1/2} \left( \binom{l_2}{k_2} \right)^{1/2} \left| \frac{l_1}{2} - k_1, \frac{l_2}{2} - k_2 \rightangle
\]

It now follows that

\[
\left( \binom{k}{k_1} \right) L_x^{k_1} L_y^{k_2} |m_1 = l_1/2, m_2 = l_2/2\) = k! \left( \binom{l_1}{k_1} \right)^{1/2} \left( \binom{l_2}{k_2} \right)^{1/2} \left| \frac{l_1}{2} - k_1, \frac{l_2}{2} - k_2 \rightangle
\]

and equating this with equation (8) gives

\[
\left| \frac{l_1}{2}, \frac{l_2}{2} - k \right\rangle = \sum_{k_1=0}^{k} \binom{l_1}{k_1} \left( \binom{l_2}{k_2} \right)^{1/2} \left| \frac{l_1}{2} - k_1, \frac{l_2}{2} - k_2 \right\rangle
\]

(9)

In particular,

\[
\langle l_1/2 - k_1, l_2/2 - k_2|l/2, l/2 - (k_1 + k_2) \rangle^2 = \binom{l_1}{k_1} \binom{l_2}{k_2} \left( \binom{l}{k} \right)^{2}
\]

(10)

which is the required result. Note also that the above formula is nothing more than a hypergeometric distribution.

**Theorem 2** If \(K_1, K_2\) are independent binomial random variables with distributions \(B(l_1, p)\) and \(B(l_2, p)\) respectively and \(M_i = l_i/2 - K_i\), for each \(i\) then

- (i) \(K = K_1 + K_2\) will have \(B(l_1 + l_2, p)\) distribution
- (ii) \(P(M = \frac{l}{2} - k) = P(K = k) = \binom{l}{k} p^k (1 - p)^{l-k}\)
- (iii) \(P(M_1 = \frac{l_1}{2} - k_1, M_2 = \frac{l_2}{2} - k_2| M = \frac{l}{2} - k) = \langle M_1 = l/2 - k_1, M_2 = l/2 - k_2| L = l/2, M = l/2 - k \rangle^2\)

where \(k = k_1 + k_2\).
Proof: (i) It is a well known result in probability theory that the sum of two independent binomial random variables with common parameter $p$ is itself a binomial random variable with parameter $p$. Indeed, since $K_1$ and $K_2$ are binomial r.v.’s with moment generating functions $[pe^t + (1 - p)]^{l_1}$ and $[pe^t + (1 - p)]^{l_2}$ respectively, then the moment generating function of $K = K_1 + K_2$ is $[pe^t + (1 - p)]^{l_1+l_2}$ which means $K$ is a binomial random variable with binomial distribution $B(p,l_1 + l_2)$.

(ii) $P(M = l_i/2 - k) = P(l_i/2 - K = l/2 - k) = P(K = k) = \binom{l}{k} p^k (1 - p)^{l-k}$ by definition of binomial

(iii) Direct calculation now gives:

$$P(M_1 = \frac{l_1}{2} - k_1, M_2 = \frac{l_2}{2} - k_2 | M = \frac{l}{2} - k) = \frac{P(M_1 = \frac{l_1}{2} - k_1, M_2 = \frac{l_2}{2} - k_2)}{P(M = \frac{l}{2} - k)}$$

$$= \binom{l_1}{k_1} p^{k_1} (1 - p)^{l_1 - k_1} \binom{l_2}{k_2} p^{k_2} (1 - p)^{l_2 - k_2}$$

$$= \binom{l_1}{k_1} \binom{l_2}{k_2} \binom{l}{k} p^k (1 - p)^{l-k}$$

$$= \langle M_1 = l_1/2 - k_1, M_2 = l_2/2 - k_2 | L = l/2, M = l/2 - k \rangle^2$$

by Theorem 1. The result follows.

3 Application

We now apply the above theorem to calculate the C-G coefficients for a pair of spin-1 particles. However, we will also find that it reveals interesting information about the probability weightings associated with the $|1⟩$, $|0⟩$, $|−1⟩$ states of an individual particle composing the pair. First note that direct calculation using ”ladder” operators gives:

$$|2,2⟩ = |1,1⟩$$

$$|2,1⟩ = \frac{1}{\sqrt{2}} |1,0⟩ + \frac{1}{\sqrt{2}} |0,1⟩$$

$$|2,0⟩ = \frac{2}{3} |0,0⟩ + \frac{1}{\sqrt{6}} |1,−1⟩ + \frac{1}{\sqrt{6}} |−1,1⟩$$
\[ |2, -1\rangle = \frac{1}{\sqrt{2}} |-1, 0\rangle + \frac{1}{\sqrt{2}} |0, -1\rangle \]  
\[ |2, -2\rangle = |-1, -1\rangle. \]  

We now calculate some of the same coefficients using the above theorems. Note that the conditions of Theorem 1 are met, in the sense that \[ |2, 2\rangle = |1, 1\rangle. \] Hence, the formula can be directly applied. Indeed, for \[ |2, 0\rangle = |4/2, 4/2 - 2\rangle, \] the formula gives

\[ \langle m_1 = 2/2 - 1, m_2 = 2/2 - 1| L = 4/2, M = 4/2 - 2\rangle^2 = \frac{\binom{2}{1} \binom{2}{1}}{\binom{4}{2}} = \frac{2}{3}, \]

\[ \langle m_1 = 2/2 - 0, m_2 = 2/2 - 2| L = 4/2, M = 4/2 - 2\rangle^2 = \frac{\binom{2}{0} \binom{2}{2}}{\binom{4}{2}} = \frac{1}{6}, \]

\[ \langle m_1 = 2/2 - 2, m_2 = 2/2 - 0| L = 4/2, M = 4/2 - 2\rangle^2 = \frac{\binom{2}{2} \binom{2}{0}}{\binom{4}{2}} = \frac{1}{6} \]

which clearly correspond to the correct C-G coefficients.

The same result can also be achieved by applying Theorem 2. However, in this case, the use of conditional probability theory also reveals unexpected information about the distribution of the spin spectrum of the spin 1 particle\(^1\). In turns out, the C-G coefficients for two spin 1 particles with \( l = l_1 + l_2 \) can only be derived from conditional probability theory, provided the spectral distribution of an individual spin 1 particle has a probability distribution of the form \( p^2, 2pq, q^2 \), which in the case of \( p = q \) becomes \( 1/4, 1/2, 1/4 \), in contrast to the current belief of \( 1/3, 1/3, 1/3 \). Specifically, let \( M_i \) where \( i = 1, 2 \) be a random variable associated with the spin of two independent particles such that

\[ P(M_i = 1) = P(M_i = -1) = \frac{1}{4}, \quad P(M_i = 0) = \frac{1}{2} \]  

Also, let \( M = M_1 + M_2 \) be the sum of their spins and note that \( M \) is a random variable with values \( 2, 1, 0, -1, -2 \). Then the conditional distribution\(^2\) for the state

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\(^1\)Recall that for two events \( A \) and \( B \) defined on a finite sample space \( S \), the conditional probability of \( A \) given \( B \) is denoted by \( P(A|B) \) and \( P(A|B) = P(A \cap B) / P(B) \) provided \( P(B) \neq 0 \).
$|2, 0\rangle$ associated with the two independent particles gives

$$P\ (M_1 = 0, M_2 = 0| M = 0) = \frac{2}{3},$$

(22)

$$P\ (M_1 = 1, M_2 = -1| M = 0) = P(M_1 = -1, M_2 = 1| M = 0) = \frac{1}{6}$$

(23)

which coincides with the C-G calculation. On the other hand, if

$$P(M_i = 1) = P(M_i = 0) = P(M_i = -1) = \frac{1}{3}$$

(24)

then direct calculation gives

$$P(M_1 = 0, M_2 = 0| M = 0) = P(M_1 = 1, M_2 = -1| M = 0)$$

$$= P(M_1 = -1, M_2 = 1| M = 0) = \frac{1}{3}$$

which are the C-G coefficients associated with the singlet state:

$$|0, 0\rangle = \frac{1}{\sqrt{3}} |1, -1\rangle + \frac{1}{\sqrt{3}} |-1, 1\rangle - \frac{1}{\sqrt{3}} |0, 0\rangle.$$ 

(25)

References

[1] Bickel and Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, 455-456(1977), Holden-Day.

[2] Brink and Satchler, *Angular Momentum*,30-35(1968), Clarendon Press

[3] Paul O’Hara, *The spin-statistics theorem – did Pauli get it right*, arXiv:quant-ph/0109137.