HAUSDORFF OPERATORS ON THE SOBOLEV SPACES $W^{k,1}$

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Abstract. We first study the Hausdorff operators on Sobolev spaces. The sharp conditions are given for the boundedness of Hausdorff operators on the Sobolev spaces $W^{k,1}$. As applications, some bounded and unbounded properties of Hardy operator and adjoint Hardy operator on $W^{k,1}$ are deduced.

1. Introduction and Preliminary

For a suitable function $\Phi$, the corresponding Hausdorff operator $H_\Phi$ is defined by

$$H_\Phi f(x) := \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy,$$  \hspace{1cm} (1.1)

where the above integral makes sense for $f$ belongs to some classes of nice functions. The study of Hausdorff operator, originated from some classical summation methods, has a long history in real and complex analysis setting. The interested reader can refer to [3] and [9] for a survey of some historic background and recent developments in this topic. Particularly, Hausdorff operator is an interesting operator in harmonic analysis. It contains some important operators when $\Phi$ is taken suitably, such as Hardy operator, adjoint Hardy operator (see [2, 4, 5]), and the Cesàro operator [13, 15] in one dimension. The Hardy-Littlewood-Pólya operator and the Riemann-Liouville fractional integral can also be derived from the Hausdorff operator.

In recent years, there is an increasing interest on the the boundedness of Hausdorff operators on function spaces, see for example [6, 8, 10, 11]). However, the boundedness of Hausdorff operator can be characterized in only few cases. One can see [7, 18] for the characterization of the bounded Hausdorff operators on Lebesgue spaces, and see [3, 14] for the characterization of the bounded Hausdorff operators on Hardy spaces $H^1$ and $h^1$. Very recently, we establish the characterization of the bounded Hausdorff operators on modulation and Wiener amalgam spaces (see [19]).

As we know, in the fields of harmonic analysis and PDE, it is quite important to study whether the regularity of functions (or initial datas) can be persisted through certain operators. There are many function spaces that can be used to described the regularity of functions, among which Sobolev space is a very popular one. One can see [1, 12] for the boundedness of maximal operator on Sobolev spaces, and see [17] for the study of evolution equations on Sobolev spaces. Note that none of the previous results is about the boundedness property of Hausdorff operator on Sobolev space. Thus, our main aim is to serve as a first contribution about this topic.

Now, we give the definition of Sobolev space $W^{k,p}$ where $k$ is a nonnegative integer and $1 \leq p < \infty$. Let $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ be a differential monomial, whose total order is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Suppose we are given two locally integrable functions on $\mathbb{R}^n$, $f$ and $g$. Then we say that $\frac{\partial^\alpha f}{\partial x^\alpha} = g$ (in the weak sense), if

$$\int_{\mathbb{R}^n} f(x) \frac{\partial^\alpha \varphi}{\partial x^\alpha} (x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x) \varphi(x) dx,$$  \hspace{1cm} (1.2)

where $\mathcal{D}$ is the space of indefinitely differential functions with compact support. Integration by parts shows us that this is indeed the relation that we would expect if $f$ had continuous partial derivatives up to order $|\alpha|$, and $\frac{\partial^\alpha f}{\partial x^\alpha} = g$ had the usual meaning. It is of course not true that every locally

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integrand function has partial derivatives in this sense. However when the partial derivatives exist they are determined almost everywhere by the defining relation.

For any non-negative integer \( k \), the Sobolev space \( W^{k,p}(\mathbb{R}^n) := W^{k,p} \) is defined as the space of functions \( f \), with \( f \in L^p(\mathbb{R}^n) \) and where all \( \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \) exist and \( \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in L^p(\mathbb{R}^n) \) in the weak sense for \( |\alpha| \leq k \). This corresponding norm for the function space \( W^{k,p} \) is defined by

\[
\|f\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|_{L^p} \quad \left( \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = f \right).
\]

With the norm defined above, \( W^{k,p} \) is a Banach space. One can see [10] for a nice description of the basic definitions and properties about Sobolev spaces.

Our main results is as follows.

**Theorem 1.1.** Let \( k \geq 0 \) be an integer and \( \Phi \geq 0 \). Then \( H_\Phi \) is bounded on \( L^1(\mathbb{R}^n) \) if and only if

\[
\int_{\mathbb{R}^n} |y|^n (1 + |y|^{-k}) \Phi(y)dy < \infty.
\]

Furthermore, if (1.3) is satisfied, we have

\[
\frac{\partial^{|\alpha|} H_\Phi f}{\partial x^\alpha}(x) = \int_{\mathbb{R}^n} |y|^{-|\alpha|}\Phi(y) \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \quad \text{for} \quad f \in W^{k,1}
\]

hold for all \( |\alpha| \leq k \).

We will give the proof in Section 2. As applications, in Section 3 we get some bounded and unbounded properties for Hardy operator and adjoint Hardy operator.

Throughout this paper, we will adopt the following notations. \( \mathcal{X} \lesssim \mathcal{Y} \) denotes the statement that \( \mathcal{X} \leq C \mathcal{Y} \), with a positive constant \( C \) that may depend on \( n, p \), but it might be different from line to line. The notation \( \mathcal{X} \sim \mathcal{Y} \) means the statement \( \mathcal{X} \lesssim \mathcal{Y} \lesssim \mathcal{X} \). We use \( \mathcal{X} \lesssim_\lambda \mathcal{Y} \) to denote \( \mathcal{X} \leq C_\lambda \mathcal{Y} \), meaning that the implied constant \( C_\lambda \) depends on the parameter \( \lambda \).

## 2. Proof of main theorem

Firstly, we give some basic properties of Gaussian function.

**Lemma 2.1** (Derivatives of Gaussian function). Let \( g(t) = e^{-t^2} \) for \( t \in \mathbb{R} \), \( g^{(m)} \) be the \( m \)-th derivative of \( g \). Then

1. \( g^{(m)}(t) = P_m(t)g(t) \) where \( P_m(t) \) is a \( m \)-order polynomial with respect to the variable \( t \);
2. the sign of the highest order term of \( P_m \) is equal to \((-1)^m\).

**Proof.** Denote by \( Q(m) \) the conclusion of this lemma. Since \( g'(t) = -2te^{-t^2} \), \( Q(1) \) is correct.

Next, we assume \( Q(l-1) \) holds. Write \( P_{l-1}(t) \) by

\[
P_{l-1}(t) = (-1)^{l-1} A_{l-1} t^{l-1} + A_{l-2} t^{l-2} + \cdots + A_0
\]

where \( A_{l-1} > 0, A_j \in \mathbb{R} \) for \( j = 0, 1, \ldots, l-2 \). Then by derivative formula of multiplication we can calculate \( g^{(l)} \) by

\[
g^{(l)}(t) = \left(g^{(l-1)}(t)\right)' = (P_{l-1}(t)g(t))' = (P_{l-1}(t))' g(t) + P_{l-1}(t)(g(t))'
\]

\[
=|(-1)|^{l-1} A_{l-1}(l-1) t^{l-2} + (l-2) A_{l-2} t^{l-3} + \cdots + A_1 g(t)
\]

\[
+ |(-1)|^{l-1} A_{l-1} t^{l-1} + A_{l-2} t^{l-2} + \cdots + A_0 (-2t) g(t)
\]

\[
=|(-1)|^{l-1} (2A_{l-1} t^l + (2A_{l-2}) t^{l-1} + \cdots + A_1) g(t).
\]

Thus, \( P_l(t) = (-1)^l 2A_{l-1} t^l + (-2A_{l-2}) t^{l-1} + \cdots + A_1 \) is a \( l \)-order polynomial and the leading term is \((-1)^l 2A_{l-1} t^l\). Noticing that \( A_{l-1} \) is positive, we have the sign of the leading coefficient of \( P_l \) is \((-1)^l\).

By mathematical induction, \( Q(m) \) holds for all \( m \in \mathbb{N} \). \( \square \)

Next, we establish the following two propositions for the proof of Theorem 1.1.
Proposition 2.2. Let \( k \geq 0 \) be an integer and \( \Phi \geq 0 \). If \( H_\Phi : W^{k,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) is bounded, we have
\[
\int_{\mathbb{R}^n} |y|^n \Phi(y) dy < \infty.
\]

Proof. Take \( f \) to be a nonnegative Schwartz function, then \( f \in W^{k,1} \). Recalling \( \Phi \geq 0 \), we use the Fubini theorem to deduce that
\[
\|H_\Phi f\|_{L^1} = \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f \left( \frac{x}{|y|} \right) dy dx
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f \left( \frac{x}{|y|} \right) dxdy
\]
\[
= \|f\|_{L^1} \int_{\mathbb{R}^n} |y|^n \Phi(y) dy.
\]

A combination of the above inequality and the boundedness of \( H_\Phi : W^{k,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) yield that
\[
\|f\|_{L^1} \int_{\mathbb{R}^n} |y|^n \Phi(y) dy = \|H_\Phi f\|_{L^1} \lesssim \|f\|_{W^{k,1}} < \infty.
\]

Since \( \|f\|_{L^1} \neq 0 \), we get the desired conclusion immediately. \( \square \)

Proposition 2.3. Let \( k \geq 0 \) be an integer and \( \Phi \geq 0 \). If the Hausdorff operator \( H_\Phi \) is bounded on \( W^{k,1}(\mathbb{R}^n) \), then
\[
\int_{\mathbb{R}^n} |y|^n (|y|^{-|\alpha|} + 1) \Phi(y) dy < \infty, \tag{2.2}
\]
and
\[
\frac{\partial^n H_\Phi f}{\partial x^\alpha}(x) = \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^n f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \quad \text{for } f \in W^{k,1}, \tag{2.3}
\]
hold for all \( |\alpha| \leq k \).

Proof. Assume \( k \geq 1 \), since the proof for \( k = 0 \) is trivial. By the definition of \( W^{k,1} \), we have \( W^{k,1} \subset L^1 \), then the boundedness of \( H_\Phi \) on \( W^{k,1} \) implies that \( H_\Phi \) is bounded from \( W^{k,1} \) to \( L^1 \). Using Proposition 2.2, we have
\[
\int_{\mathbb{R}^n} |y|^n \Phi(y) dy < \infty. \tag{2.4}
\]

Step 1: \( |\alpha| = 1 \).

For \( f \in W^{k,1} \), since the boundedness of \( H_\Phi \) on \( W^{k,1} \), we have \( H_\Phi f \in W^{k,1} \). By the definition of \( W^{k,1} \), the weak derivative of \( H_\Phi f \) exists and belongs to \( L^1 \). Moreover over, for every \( \varphi \in \mathcal{D} \), we have
\[
\int_{\mathbb{R}^n} \frac{\partial H_\Phi f}{\partial x_j}(x) \varphi(x) dx = - \int_{\mathbb{R}^n} H_\Phi f(x) \frac{\partial \varphi}{\partial x_j}(x) dx = - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y) f \left( \frac{x}{|y|} \right) dy \frac{\partial \varphi}{\partial x_j}(x) dx. \tag{2.5}
\]

Combine (2.4) with the fact that \( \varphi \in \mathcal{D} \) and \( f \in L^1_k \subset L^1 \), we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(y)| f \left( \frac{x}{|y|} \right) \frac{\partial \varphi}{\partial x_j}(x) dx dy \leq \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty} \int_{\mathbb{R}^n} |\Phi(y)| \int_{\mathbb{R}^n} \left| f \left( \frac{x}{|y|} \right) \right| dx dy
\]
\[
= \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty} \int_{\mathbb{R}^n} |y|^n \Phi(y) dy \cdot \|f\|_{L^1} < \infty.
\]

Thus, the combination of (2.5) and the Fubini theorem yields that
\[
\int_{\mathbb{R}^n} \frac{\partial H_\Phi f}{\partial x_j}(x) \varphi(x) dx = - \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f \left( \frac{x}{|y|} \right) \frac{\partial \varphi}{\partial x_j}(x) dx dy
\]
\[
= \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) \varphi(x) dx dy, \tag{2.6}
\]

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where in the last inequality we use the definition of weak derivative again. By the boundedness of $H_\Phi$, we now have
\[
\int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) \varphi(x) dxdy = \int_{\mathbb{R}^n} \frac{\partial H_\Phi f}{\partial x_j}(x) \varphi(x) dx \lesssim \left\| \frac{\partial H_\Phi f}{\partial x_j} \right\|_{L^1} \left\| \varphi \right\|_{L^\infty} \lesssim \left\| H_\Phi f \right\|_{W^{k,1}} \left\| \varphi \right\|_{L^\infty} \lesssim \left\| f \right\|_{W^{k,1}} \left\| \varphi \right\|_{L^\infty}
\]
for every $f \in W^{k,1}$ and $\varphi \in \mathcal{D}$.

Set $g(t) = e^{-|t|^2}$, $G_1(x) = -\sum_{l=1}^n g(x_l)$, where $x_l$ is the $l$-th component of $x = (x_1, x_2, \cdots, x_n)$.

Note that $G_1$ is a Schwartz function, and belongs to $W^{k,1}$. By a direction calculation, we obtain
\[
\frac{\partial G_1}{\partial x_j}(x) = 2x_j \sum_{l=1}^n g(x_l).
\]

Note that $\frac{\partial G_1}{\partial x_j}$ is positive in $\Omega := (0, \infty)^n \subset \mathbb{R}^n$. Take $\{\varphi_i\}_{i=1}^\infty$ to be sequence of nonnegative $C^\infty_c$ functions supported in $\Omega$, so that $0 \leq \varphi_i \leq 1$ for all $i \in \mathbb{N}$, $\varphi_i$ is increasing as $i \to \infty$, and $\lim_{i \to \infty} \varphi_i = \chi_\Omega$.

For every $x \in \Omega$, $y \in \mathbb{R}^n$, we have $\frac{\partial G_1}{\partial x_j}(\frac{x}{|x|}) \geq 0$, then
\[
\lim_{i \to \infty} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial G_1}{\partial x_j} \left( \frac{x}{|y|} \right) \varphi_i(x) dxdy = \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\Omega} \frac{\partial G_1}{\partial x_j} \left( \frac{x}{|y|} \right) dxdy = \left\| \frac{\partial G_1}{\partial x_j} \right\|_{L^1(\Omega)} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) dy.
\]

The combination of (2.7) and (2.8) yields that
\[
\left\| \frac{\partial G_1}{\partial x_j} \right\|_{L^1(\Omega)} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) dy = \lim_{i \to \infty} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial G_1}{\partial x_j} \left( \frac{x}{|y|} \right) \varphi_i(x) dxdy \lesssim \lim_{i \to \infty} \|G_1\|_{W^{k,1}} \|\varphi_i\|_{L^\infty} \lesssim \|G_1\|_{W^{k,1}},
\]
which implies
\[
\int_{\mathbb{R}^n} |y|^{-1} \Phi(y) dy < \infty.
\]

Using (2.4), for $\varphi \in \mathcal{D}$ and $f \in W^{k,1}$ we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \left| \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) \right| \varphi(x) dxdy \lesssim \left\| \varphi \right\|_{L^\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \left| \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) \right| dxdy = \left\| \varphi \right\|_{L^\infty} \left\| \frac{\partial f}{\partial x_j} \right\|_{L^1} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) dy < \infty.
\]

Then we apply the Fubini theorem to the right side of (2.6), obtain that
\[
- \int_{\mathbb{R}^n} H_\Phi f(x) \frac{\partial \varphi}{\partial x_j}(x) dx = \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) \varphi(x) dxdy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) dy \varphi(x) dx.
\]

By the definition of weak derivative, we get the desired conclusion
\[
\frac{\partial H_\Phi f}{\partial x_j}(x) = \int_{\mathbb{R}^n} |y|^{-1} \Phi(y) \frac{\partial f}{\partial x_j} \left( \frac{x}{|y|} \right) dy.
\]

Obviously, the above inequality is valid for every $j = 1, 2, \cdots, n$.

**Step 2:** $2 \leq |\alpha| \leq k$ if $k \geq 2$. 

Assume by induction that for $|\beta| \leq m - 1$, we have

$$
\int_{\mathbb{R}^n} |y|^n (|y|^{-|\beta|} + 1) \Phi(y)dy < \infty.
$$

(2.10)

and

$$
\frac{\partial^\beta H_\Phi f}{\partial x^\beta}(x) = \int_{\mathbb{R}^n} |y|^{-|\beta|} \Phi(y) \frac{\partial^\beta f}{\partial x^\beta} \left( \frac{x}{|y|} \right) dy
$$

for $f \in W^{k,1}$.

(2.11)

For a fixed multi-index $\alpha$ with total order $m$, there exist a multi-index $\beta$ such that $|\beta| = m - 1$ and $\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^\beta f}{\partial x^\beta}$ for some $j \in \{1, 2, \cdots, n\}$. Using the definition of weak derivative and (2.11), for $\varphi \in \mathcal{D}$ and $f \in W^{k,1}$, we have

$$
\int_{\mathbb{R}^n} \frac{\partial^\alpha H_\Phi f}{\partial x^\alpha}(x) \varphi(x) dx = (-1)^m \int_{\mathbb{R}^n} H_\Phi f \frac{\partial^\beta \varphi}{\partial x^\beta_j}(x) dx
$$

(2.12)

$$
= - \int_{\mathbb{R}^n} \frac{\partial^\beta H_\Phi f}{\partial x^\beta}(x) \frac{\partial \varphi}{\partial x_j}(x) dx
$$

$$
= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\beta|} \Phi(y) \frac{\partial^\beta f}{\partial x^\beta} \left( \frac{x}{|y|} \right) dy \frac{\partial \varphi}{\partial x_j}(x) dx.
$$

Recall (2.10), $\varphi \in \mathcal{D}$ and $f \in W^{k,1}$, then the function $|y|^{-|\beta|} \Phi(y) \frac{\partial^\beta f}{\partial x^\beta} \left( \frac{x}{|y|} \right) \frac{\partial \varphi}{\partial x_j}(x)$ is absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$. So, we use the Fubini theorem to deduce that

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\partial^\beta H_\Phi f}{\partial x^\beta}(x) \varphi(x) dx
$$

$$
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\beta|} \Phi(y) \frac{\partial^\beta f}{\partial x^\beta} \left( \frac{x}{|y|} \right) dy \frac{\partial \varphi}{\partial x_j}(x) dx
$$

(2.13)

$$
= \int_{\mathbb{R}^n} |y|^{-|\beta|} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\beta f}{\partial x^\beta} \left( \frac{x}{|y|} \right) \varphi(x) dx
$$

$$
= \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \varphi(x) dx,
$$

where in the last equality we use the definition of weak derivative and the fact that $f \in W^{k,1}$.

Using Lemma 2.11 for Gaussian function $g$ we have $g^{(l)}(t) = P_l(t)g(t)$, where $P_l(t)$ is a $l$-order polynomial of $t$. Denote $a_m$ the biggest one of all the real roots of $P_l(t)$ if exists, else $a_m = 0$. Set $G_m(x) = (-1)^m \prod_{l=1}^n g(x_l + a_m)$. By Lemma 2.11 we deduce that

$$
\frac{\partial^\alpha G_m}{\partial x^\alpha}(x) = (-1)^m \prod_{l=1}^n P_{|\alpha_l|} (x_l + a_m) g(x_l + a_m),
$$

where the sign of the leading term of $P_{|\alpha_l|}$ is $(-1)^{|\alpha_l|}$. By the choice of $a_m$, we have

$$
\frac{\partial^\alpha G_m}{\partial x^\alpha}(x) \geq 0
$$

for $x \in \Omega = (0, \infty)^n$. Taking $\{\varphi_i\}_{i=1}^\infty$ as in the step 1, we have

$$
\lim_{i \to \infty} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\alpha G_m}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \varphi_i(x) dx dy
$$

(2.14)

$$
= \frac{\partial^\alpha G_m}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) dy.
$$
The combination of (2.12), (2.13) and (2.14) yields that
\[
\left\| \frac{\partial G_m}{\partial x_j} \right\|_{L^1(\Omega)} \int_{\mathbb{R}^n} |y|^{-|\alpha|-\beta} \Phi(y)dy = \lim_{i \to \infty} \int_{\mathbb{R}^n} |y|^{-|\beta|-1} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\alpha G_m}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \varphi_i(x)dxdy
\]
\[
= \lim_{i \to \infty} \left( - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\beta|} \Phi(y) \frac{\partial^\beta G_m}{\partial x^\beta} \left( \frac{x}{|y|} \right) dy \frac{\partial \varphi_i}{\partial x_j}(x)dx \right)
\]
\[
= \lim_{i \to \infty} \int_{\mathbb{R}^n} \frac{\partial^\alpha H \Phi G_m}{\partial x^\alpha}(x) \varphi_i(x)dx
\]
\[
\leq \lim_{i \to \infty} \left\| \frac{\partial^\alpha H \Phi G_m}{\partial x^\alpha} \right\|_{L^1} \left\| \varphi_i \right\|_{L^\infty} \lesssim \|H \Phi G_m\|_{W^{k,1}} \lesssim \|G_m\|_{W^{k,1}}.
\]

We obtain that
\[
\int_{\mathbb{R}^n} |y|^{n-|\alpha|} \Phi(y)dy < \infty.
\] (2.15)

Moreover, thanks to (2.15), we can apply the Fubini theorem to the right side of (2.13), then obtain that
\[
\int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \varphi(x)dxdy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \varphi(x)dx.
\]
Combining with (2.12), we deduce that
\[
(-1)^m \int_{\mathbb{R}^n} H \Phi \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \varphi(x)dx.
\]
Note that \( \varphi \in \mathcal{D} \) can be chosen arbitrary, we get the desired conclusion
\[
\frac{\partial^\alpha H \Phi f}{\partial x^\alpha}(x) = \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy.
\]
By induction, the conclusion is valid for all \(|\alpha| \leq k\). \(\square\)

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** The necessity is a direct conclusion of Proposition 2.3, we only need to prove the sufficiency. If (1.8) holds, for \( f \in W^{k,1} \), we first obtain that
\[
\left\| H \Phi f \right\|_{L^1} = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \Phi(y)f \left( \frac{x}{|y|} \right) dy \right| dx \leq \int_{\mathbb{R}^n} |y|^n \Phi(y)dy \left\| f \right\|_{L^1} \leq \int_{\mathbb{R}^n} |y|^n \Phi(y)dy \left\| f \right\|_{W^{k,1}}.
\]
Moreover, for any \( f \in W^{k,1} \) and \( \varphi \in \mathcal{D} \), thanks to (1.3), for all \( \alpha \) with \(|\alpha| \leq k\) we deduce that
\[
\int_{\mathbb{R}^n} H \Phi f \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y)f \left( \frac{x}{|y|} \right) \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)dxdy
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f \left( \frac{x}{|y|} \right) \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)dxdy
\]
\[
= (-1)^{|\alpha|} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \int_{\mathbb{R}^n} \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) \varphi(x)dxdy
\]
\[
= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \varphi(x)dx,
\] (2.16)

where we use the Fubini theorem in the first and last inequalities. Since \( \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \in L^1 \), then (2.16) yields that
\[
\frac{\partial^\alpha H \Phi f}{\partial x^\alpha}(x) = \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \in L^1
\] (2.17)
in the weak sense. Note that (2.17) is valid for all multi-index $\alpha$ with total order less than $k$, we conclude that

$$
\|H_\Phi f\|_{W^{k,1}} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha H \Phi f}{\partial x^\alpha} \right\|_{L^1} = \sum_{|\alpha| \leq k} \left( \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \frac{\partial^\alpha f}{\partial x^\alpha} \left( \frac{x}{|y|} \right) dy \right)_{L^1} \leq \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |y|^{-|\alpha|} \Phi(y) \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{L^1} \leq \int_{\mathbb{R}^n} (|y|^n + |y|^{n-k}) \Phi(y) dy \cdot \|f\|_{W^{k,1}}.
$$

Now, we have completed the proof of sufficiency.

3. Applications

As we mentioned before, Hausdorff operators can be regarded as the generalization of some classical operators, such as Hardy operator $H$ and its adjoint operator $H^*$. Thus, by choosing special $\Phi$, we can obtain the bounded and unbounded properties for special operator. In one dimension, we take $\Psi(t) = \frac{\chi_{(1,\infty)}(t)}{t^2}$ and $\Psi^*(t) = \frac{\chi_{(0,1)}(t)}{t}$, we have

$$
H_\Phi f(x) = H f(x) = \frac{1}{x} \int_0^x f(t) dt
$$
and

$$
H_{\Psi^*} f(x) = H^* f(x) = \int_x^\infty \frac{f(t)}{t} dt
$$
respectively.

**Proposition 3.1.** Hardy operator $H$ is not bounded on $W^{k,1}(\mathbb{R})$ for all $k \geq 0$. The adjoint operator of Hardy operator $H^*$ is bounded on on $L^1(\mathbb{R})$, but not bounded on $W^{k,1}(\mathbb{R})$ with $k \in \mathbb{Z}^+$.

**Proof.** For the boundedness of Hardy operator on Sobolev spaces, by Theorem 1.1 we only need to check whether (1.3) holds for $\Psi$. A direct computation yields that

$$
\int_{\mathbb{R}} \Psi(t) \cdot |t|(1 + |t|^{-k}) dt = \int_{\mathbb{R}} \frac{\chi_{(1,\infty)}(t)}{t^2} \cdot |t|(1 + |t|^{-k}) dt \geq \int_{1}^{+\infty} \frac{1}{t} dt = +\infty. \quad (3.1)
$$
Using Theorem 1.1 we know that Hardy operator is not bounded on $W^{k,1}(\mathbb{R})$.

To verify the boundedness of the adjoint operator of Hardy operator, we need to check (1.3) for $\Psi^*$. More precisely, we have

$$
\int_{\mathbb{R}} \Psi^*(t) \cdot |t| dt = \int_{\mathbb{R}} \frac{\chi_{(0,1)}(t)}{t} \cdot |t| dt = 1 \quad (3.2)
$$
and when $k$ is a positive integer

$$
\int_{\mathbb{R}} \Psi^*(t) \cdot |t|(1 + |t|^{-k}) dt = \int_{\mathbb{R}} \frac{\chi_{(0,1)}(t)}{t} \cdot |t|(1 + |t|^{-k}) dt \geq \int_0^1 t^{-k} dt = +\infty. \quad (3.3)
$$
The desired conclusion follows by using Theorem 1.1.

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