Construction of the Conserved Non-linear $\zeta$ via the Effective Action for Perfect Fluids

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Abstract

We consider the problem of how to construct the curvature perturbation $\zeta$ to non-linear levels, which is expected to evolve time independently on super-horizon scales; in particular we concentrate on the situation where the universe is dominated by a perfect fluid. We have used a low energy/long wavelength effective action to model the fluid sector. Different from previous work, our approach assumes neither the absence of vector and tensor perturbations nor “local homogeneity and isotropy”. As a corollary, we also show that the nonlinearly defined graviton field $\gamma_{ij}$ is conserved outside the horizon in the same manner as $\zeta$ is.

1 Introduction

Despite the significant success of the linear perturbation theory in cosmology over the last few decades, more and more efforts have been devoted in recent years to understanding the cosmological perturbations beyond the linear level — both their origins and evolutions. This is partially motivated by the extraordinary improvement in the accuracy of experiments; for instance, both the CMB data and the large scale structure data have indicated that there might be a detectable non-Gaussian feature in the primordial curvature perturbation. Therefore to fully exploit the data, one has to study various cosmological perturbations at non-linear level.

Among these quantities, enormous attention has been paid to the curvature perturbation $\zeta$. Originally it was defined at linear order as the scalar metric perturbation on the uniform-density hyper-surfaces by Bardeen [1] and was shown to evolve time-independently outside the horizon (cf. for instance [2]). Many studies have been done recently in seek of extending the construction of a conserved $\zeta$ to non-linear level in different contexts. Roughly speaking, there are three classes of methods employed to achieve that, which are summarized as follows:

- The first one is the standard perturbative approach — a perturbative expansion in fields [12]. It is straightforward and the equations governing the evolution of $\zeta$ are valid on all scales. However the price to pay is that the equations usually become cumbersome once one goes beyond the linear order in fields and there is a lack of systematic ways to demonstrate the conservation of $\zeta$ on super-horizon scales to arbitrary orders in the field expansion.
The second class of approaches instead focuses on the spatial gradient expansion — i.e. an expansion in powers of $\sigma \equiv k/aH$, where $k$ denotes the wavenumber, — since the super-horizon evolution of $\zeta$ is usually of most interest. At the leading order, it corresponds to the separate universe approach. [7, 10] investigated the cases with an inflating background. There the authors showed that in single field inflation models, $\zeta$ can be defined as the (only) dynamical scalar metric perturbation in the unitary gauge (in which the matter field is demanded unperturbed) and that such a $\zeta$ was conserved to all orders in fields on super-horizon scales. The proofs were based on a Lagrangian formalism, for the matter contents in these cases were specified by explicit matter actions. On the other hand, the cases with a general FRW background were studied in [3, 4, 15] (known as the $\delta N$ formalism). It was shown that the conservation of the nonlinear $\zeta$ on large scales was assured if the pressure of the matter content was only a function of the energy density or if the matter content could be modeled by a single scalar field. As opposed to the aforementioned field theoretic formalism used in inflation scenarios, these analyses relied on the field equations of motion; in the context of the Einstein gravity the energy conservation alone would be sufficient.

A third approach was proposed in [5, 6] by invoking the purely geometrical description of the curvature perturbation. The covariant quantity they constructed there would satisfy an exact, non-perturbative, valid-to-all-scale conservation equation and would eventually reduce to usual $\zeta$ on large scales. The equivalence between this covariant formalism and the $\delta N$ formalism was established in [14, 13].

In this paper, we will present a systematic way of constructing the nonlinear conserved $\zeta$ in a universe dominated by a perfect fluid. A low energy effective action for an ordinary perfect fluid [8, 9] will be employed to model the matter content. So our method here is parallel to that used in the single field inflation cases [7, 10]. On the other hand, although this scenario was investigated to some extent in previous literature, we believe that there is still some novelties in our approach: i) It does not assume that on a sufficiently large smoothing scale, the universe looks locally homogeneous and isotropic (like a FRW patch), as was a crucial assumption in [11, 4]; in fact, such a feature, though can be derived, does not play a vital role in our construction. And ii) our approach works in the presence of vector and tensor perturbations. Moreover it enables us to construct the vector and tensor counterparts of $\zeta$, — i.e. the nonlinear vector and tensor perturbation that evolve time-independently outside the horizon.

2 Setup

We begin with a brief review of the low energy effective description of an ordinary perfect fluid system in Minkowskian spacetime, which was proposed and developed in [8, 9, 16, 17]. For the purpose of this paper, we focus on a fluid with no conserved charges and neglect

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1 The conservation of the vector field does not contradict the usual intuition that vector modes decay in the absence of anisotropic stress tensor. As we will see later, due to the symmetry requirements of a fluid, any constant (in time) transverse vector field configuration is not physical and hence can be set to vanish, leaving only the decaying configuration.
all the dissipative effects. The readers already familiar with our notations are welcome to skip directly to Section 2.2, in which the effective fluid action is used as the matter action in cosmological context and in which some formulas are derived in preparation for the construction of the conserved (nonlinear) quantities on large scales.

2.1 Effective Theory for Perfect Fluids in Flat Spacetime

Our goal is to construct a Lorentz invariant low energy effective theory for an ordinary perfect fluid system. For an ordinary fluid system with no conserved charges, we can specify as the long wavelength degrees of freedom the comoving coordinates of fluid volume elements, parametrized by $\phi^I$, with $I = 1, 2, 3$. So at any fixed time $t$, the physical position occupied by each volume element is given by $\mathbf{x}(\phi^I, t)$. In this description, known as the Euclidean description, the physical spatial coordinates $x^i$ serve as dynamical fields while $t$ and $\phi^I$ are analogous to world sheet coordinates.

However, it is often more convenient to use the inverse functions $\phi^I(\mathbf{x}, t)$ as dynamical degrees of freedom (known as the Lagrangian description), since the spacetime symmetry can be straightforwardly implemented — we simply demand $\phi^I$ to transform as scalars under Poincare transformations. Also we are allowed to choose the comoving coordinates in such a manner that when the fluid system is at rest, in equilibrium and in a homogeneous state at some given external pressure, $\phi^I = x^I$ — in the field theoretical language, this is equivalent to specifying the ground state of our theory to be

$$\langle \phi^I \rangle = x^I \quad (1)$$

What are the other symmetries, in addition to the Poincare invariance, required to make the system behave like an ordinary fluid? Notice that our choice of the ground state (1) breaks both spatial translational and rotational invariance. In order that the energy momentum tensor of the system in equilibrium remains homogeneous and isotropic, as is indeed the case for a fluid, we shall impose internal symmetries to compensate the spontaneously broken spacetime symmetries. More precisely, we demand that the theory be invariant under such internal transformations (i.e. all fields evaluated at the same spacetime point) as:

$$\mathcal{T}_i : \phi^I \rightarrow \phi'^I = \phi^I + a^I, \quad a^I \text{constant} \quad (2)$$

$$\mathcal{R}_i : \phi^I \rightarrow \phi'^I = O^I_J \phi^J, \quad O^I_J \in SO(3) \quad (3)$$

It is easy to show that the background configuration (1) is invariant under the diagonal translation and rotation, respectively, which are defined as a linear combination of the spatial and internal transformations: $\mathcal{T}_d \equiv \mathcal{T}_s + \mathcal{T}_i$ and $\mathcal{R}_d \equiv \mathcal{R}_s + \mathcal{R}_i$; it is these residual symmetries that correspond to the homogeneity and isotropy of our background configuration.

Moreover an ordinary fluid is insensitive to incompressional deformations — it costs no energy to displace fluid volume elements if they are not compressed or dilated. Expressed in terms of a symmetry requirement, we demand that the theory be invariant under volume

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For space filling fluid, there is a diffeomorphism between the comoving coordinates $\phi^I$ and physical spatial coordinates $x^i$ so that for any fixed time $t$, the inverse functions $\phi^I(\mathbf{x}, t)$ exist.
preserving diffeomorphisms of the comoving coordinates, defined as

\[ \mathcal{D} : \phi^I \rightarrow \xi^I(\phi), \quad \text{with} \quad \det \frac{\partial \xi^I}{\partial \phi^J} = 1. \] (4)

So now we are ready to construct the effective action for an ordinary fluid from \( \phi^I \)'s, compatible with the symmetry properties mentioned above. It is organized as a derivative expansion. Since the internal translation symmetry (2) mandates each field to be accompanied by at least one derivative, at the leading order in the derivative expansion, the only invariant is

\[ B \equiv \det(B^IJ), \quad \text{with} \quad B^IJ = \partial_\mu \phi^I \partial^\mu \phi^J \] (5)

and hence the effective action takes the form of

\[ S_{\text{Fluid}} = \int d^4x \, F(B) \] (6)

where \( F \) is a generic function, which, as we will see, characterizes the equation of state of the fluid in question.

To illustrate that this effective action indeed describes a perfect fluid, we need to check that the (conserved) energy moment tensor takes the famous form:

\[ T_{\mu \nu} = (\rho + p)u_\mu u_\nu + p \eta_{\mu \nu} \] (7)

Before doing that let’s work out the four-velocity field \( u^\mu(x) \) for the fluid system in our field theoretical language, which by definition is given by

\[ 0 = \frac{d}{d\tau} \phi^I(x) \equiv u^\mu(x)\partial_\mu \phi^I(x) \] (8)

where \( \tau \) parametrizes the streamline and the derivative with respect to \( \tau \) vanishes because the comoving coordinate (or the label) of each fluid volume element is fixed. By solving the above equation for \( u^\mu \), we obtain

\[ u^\mu = -\frac{1}{\sqrt{B}} \epsilon^{\alpha \beta \gamma \delta} \partial_\alpha \phi^1 \partial_\beta \phi^2 \partial_\gamma \phi^3 \] (9)

where \( \epsilon \) is the 4d Levi-Civita symbol, with the convention \( \epsilon_{0123} = -\epsilon^{0123} = 1 \). The normalization and overall sign of \( u^\mu \) are chosen such that \( u^\mu u_\mu = -1 \) and \( u^0 > 0 \).

The energy momentum tensor following from the effective action (6) reads

\[ T_{\mu \nu} = -2F'(B)B^{-1}B_{IJ}\partial_\mu \phi^I \partial_\nu \phi^J + \eta_{\mu \nu} F(B). \] (10)

With the aid of (9), the energy momentum tensor above indeed can be recast into the perfect fluid form (7), if we identify \( \rho = -F'(B) \) and \( p = F(B) - 2F'(B)B \). This also justifies our preceding claim that the generic function \( F \) determines the equation of state for fluids. For instance, for an ultra-relativistic fluid with \( p = \rho/3 \), one has \( F(B) \propto B^{2/3} \).

In the rest of this subsection, we will consider small fluctuations about the homogeneous equilibrium background configuration (1); they are associated with Goldstone excitations.

\[ \phi^I = x^I + \pi^I(x). \] (11)
Not all the $\pi^I$ fields feature propagating wave solutions. This can be seen by expanding the effective Lagrangian (6) to the quadratic order:

$$\mathcal{L}^{(2)} = \frac{1}{2} w_0^{(2)} \left( \pi_L^2 - c_s^2 (\partial \pi_L)^2 \right) + \frac{1}{2} w_0 \pi_T^2$$

where $\pi_L$ and $\pi_T$ are the longitudinal (curl-free) and transverse (divergence-free) components of $\pi^I$:

$$\pi^I = \frac{\partial^I}{\sqrt{-\partial^2}} \pi_L + \pi_T^I$$

and where $w_0$ is the equilibrium enthalpy density $\bar{\rho} + \bar{p}$. The (squared) speed of sound $c_s^2$ is given by

$$c_s^2 = \frac{d\rho}{d\rho} \bigg|_{B=B} = 2F''(B)B + F'(B) \bigg|_{B=B}$$

Indeed, we see that only the longitudinal Goldstone field $\pi_L$ admits the standard propagating mode at a finite speed $c_s$, with a dispersion relation $\omega = c_s k$, while the other Goldstone field $\pi_T$ has a degenerate dispersion relation $\omega = 0$ and thus does not propagate. For this reason, we usually interpret $\pi_L$ as the sound wave d.o.f. and $\pi_T$ the vortex d.o.f.

It is worth pointing out that after spontaneous breaking, the spacetime symmetries get mixed with the internal ones, so the three Goldstone fields $\pi^I$ transform as a vector field under the diagonal $SO(3)$.

And after the decomposition (13), $\pi_L$ can be regarded as a scalar field and $\pi_T$ as a transverse vector field.

### 2.2 Cosmological Models with Perfect Fluids

In cosmology, the matter contents are often modeled as perfect fluids. Fortunately, in our field theoretic language, we can straightforwardly write the action of the cosmological model as

$$S = S_{EH} + S_{\text{Fluid}}$$

where the first term on the r.h.s. is just the usual Einstein - Hilbert action for gravity and the matter action $S_{\text{Fluid}}$ is given by the fluid effective action (6), with the flat metric $\eta$ replaced by a cosmological spacetime one $g$ and the measure by $\sqrt{-g} \, d^4x$. We also assume that the metric fluctuates around a flat FRW background $\hat{g}_{\mu\nu} = \text{diag}\{-1, a(t)^2, a(t)^2, a(t)^2\}$. The number of dynamical degrees of freedom in question is counted as follows. In addition to the three Goldstone fields $\pi^I$ — which, as argued in last subsection, are eventually regrouped into one longitudinal scalar and one transverse vector under the residual $SO(3)$ group — of the fluid sector, the gravity sector introduces one extra $2-$polarized d.o.f. — the spin-$2$ graviton, a traceless transverse tensor field.

Perhaps a more transparent way to see this is by removing the gauge redundancy of the gravity sector. Using the ADM variables, one knows that only the spatial part of the metric is dynamical, which can be cast into the most general form of

$$g_{ij} \equiv h_{ij} = a(t)^2 \exp \left( A \delta_{ij} + \partial_i \partial_j \chi + \partial_i C_j + \partial_j C_i + D_{ij} \right)$$

For this reason in the rest of the paper, we will not distinguish the spatial label “$i, j,$...” from the internal ones “$I, J,$...”.
with \( C_i, D_{ij} \) satisfying \( \partial_i C_i = 0 \) and \( \partial_i D_{ij} = D_{ii} = 0 \). If the four gauge conditions of coordinate transformations are chosen to set \( A = 0, \chi = 0, C_i = 0, D_{ii} \) — which is known as the spatially flat slicing gauge (SFSG), — we are left with \( \pi_L, \pi_T, \) and \( D_{ij} \) as the dynamical degrees of freedom, which characterizes, respectively, the scalar, vector and tensor cosmological perturbation.

For the interest of this paper, it turns out to be most convenient to work in another gauge — called the unitary gauge (UG), — in which all the perturbations are absorbed into the metric, leaving the matter fields unperturbed: \( \phi^I = x^I \). Said differently, the spatial coordinates are chosen to coincide with the comoving ones. Meanwhile the temporal gauge freedom is used to determine the time slices such that the scalar quantity \( B \) (defined in eqn. \( 5 \)) remains unperturbed on each time slice: \( B(t) = a(t)^{-6} \).

The unitary gauge leads to many conceptual and computational simplifications. First of all, with our choice of the spatial coordinates, the worldlines of fluid volume elements coincide with the threads \( x^i = \text{constant} \). Indeed, this can be seen by considering the spatial components of the velocity field, which vanish since \( u^I \propto \epsilon^{Iabc} \partial_a \phi^1 \partial_b \phi^2 \partial_c \phi^3 = 0 \). Second, our choice of time slices coincides with the uniform density slices, for the energy density of the fluid, given by \( \rho = -F(B) \), is only a function of \( B \) and it is a constant on each time slice. Using the ADM variables, we can parametrize the metric as

\[
ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)
\]

and the inverse metric \( g^{\mu\nu} \) as

\[
g^{00} = -\frac{1}{N^2}, \quad g^{0i} = g^{i0} = \frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}
\]

where the spatial metric \( h_{ij} \) takes the form of \( (16) \) and \( h^{ij} \) is the inverse spatial metric: \( h^{ik} h_{kj} = \delta^i_j \). One can show easily that in the UG \( B^{Ij} = g^{IJ} \) and that

\[
B = \det g^{IJ} = \frac{1}{\det h_{IJ}} \left( 1 - \frac{N^I N^J}{N^2} \right),
\]

where the index of \( N^I \) (\( N_I \)) are lowered (raised) by \( h_{IJ} \) (\( h^{IJ} \)). Therefore the time slicing condition can be expressed as

\[
B(t) = a(t)^{-6} \iff 3A + \nabla^2 \chi = \log \left( 1 - \frac{N^I N_I}{N^2} \right).
\]

Now the number of dynamical d.o.f. in the UG can be counted as follows: since \( N \) and \( N^I \) are just auxiliary fields and can be determined in terms of \( h_{ij} \) after algebraically solving the imposed the constraints, \( (20) \) implies that the two scalar functions \( A \) and \( \chi \) are not independent. So henceforth we shall always regard \( \chi \) as being expressed (perturbatively) in terms of other perturbations in the metric and select \( A, C_i \) and \( D_{ij} \) as the dynamical fields, the number of which is in agreement with that in the SFSG.

\footnote{We show in Appendix A that this can always be done via a perturbative construction.}
We are ready to solve in the UG the constraint equations \( \delta S / \delta N = 0, \delta S / \delta N^i = 0 \) and the time slicing condition \( \text{[20]} \) up to the linear order in fields and to obtain the UG quadratic action:

\[
S^{(2)}_T = \int dt \int \frac{M_{Pl}^2}{8} a(t)^3 \left( \dot{D}_{ij}^2 - \frac{k^2}{a^2} D_{ij}^2 \right),
\]

\[
S^{(2)}_V = \int dt \int \frac{-M_{Pl}^2 \dot{H} a(t)^2}{1 - 4 \dot{H} a^2 / k^2} \dot{C}_{ij}^2,
\]

\[
S^{(2)}_S = \int dt \int \frac{9M_{Pl}^2 \dot{H} a(t)^2}{4(3H a^2 - k^2)} \left( \dot{A} - \frac{\dot{H}}{H} A \right)^2 - \frac{9M_{Pl}^2 \dot{H}}{4} \left( 1 + \frac{\dot{H}}{3H} \right) A^2.
\]

where the symbol \( \int_k \) is abbreviation for \( \int \frac{d^3k}{(2\pi)^3} \) and where we have used the Friedmann equation

\[
\bar{\rho} = -F(\bar{B}) = 3M_{Pl}^2 H^2, \quad \bar{\rho} + \bar{p} = -2F'(\bar{B}) \bar{B} = -2M_{Pl}^2 \dot{H}.
\]

In the long wavelength limit \( \sigma \equiv k/aH \to 0 \), the actions (21–23) reduce to

\[
S^{(2)} = S^{(2)}_T + S^{(2)}_V + S^{(2)}_S \simeq \int dt \int \frac{M_{Pl}^2 a(t)^3}{8} \left[ \frac{1}{8} \dot{D}_{ij}^2 + \frac{1}{4} \left( kC_{ij} \right)^2 + \frac{3}{4} \dot{A}^2 \right]
\]

A few comments are in order:  

(i) The (squared) speeds of sound for fluids in curved spacetime are \( c_s^2 = dp/d\rho |_{\bar{B}} = -1 - \dot{H}/3H \dot{H} \) and \( c_T^2 = 0 \), respectively, for the longitudinal and transverse Goldstone excitations. For \( c_s \) to be sub-luminal, we need \( -1 - \dot{H}/3H \dot{H} < 2 \); for fluids with constant \( w = p/\rho \), this condition corresponds to the scale factor evolving as \( a(t) \propto t^n \), with \( \frac{1}{3} < n < \frac{2}{3} \). In particular, this implies that our cosmological model \( \text{[15]} \) with a perfect fluid can not be a consistent inflationary model. Indeed to have inflation, we need to relax some symmetry requirements for our system, using less symmetric object (e.g. a solid \( \text{[18]} \)) as the matter content. 

(ii) The IR quadratic action \( \text{[25]} \) implies that we should really treat \( A \sim kC_i \sim D_{ij} \) as of the same order in the spatial gradient expansion. Also despite the deceptive appearance, \( \nabla^2 \chi \) is not necessarily of higher order than \( A \) in the spatial gradient expansion; in fact as we will show soon, they are both of the leading order. Thus the scalar, vector and tensor metric perturbation in \( \text{[16]} \) are all of the same order in the spatial gradient expansion, i.e.

\[
A \sim k^2 \chi \sim kC_i \sim D_{ij} \sim \mathcal{O}(\sigma^0)
\]

### 3 Construct the Conserved Curvature Perturbation to Non-linear Order

As mentioned in the previous section, if the matter content of the universe can be modeled as an ordinary perfect fluid, there is only one dynamical scalar field. In the UG it is parametrized by \( A(x) \), the (dynamical) scalar perturbation in the spatial metric (eqn. \( \text{[16]} \)). In this section, we will show that the evolution of \( A \) will remain time-independent as long as the mode is outside the horizon. The conservation is preserved up to all orders
in the field expansion; in particular, at the linear order \( A \) coincides with (twice) the usual (linear) curvature perturbation. Thus we shall define our linear curvature perturbation \( \zeta \) by
\[
\zeta = A/2.
\]

The proof of the conservation of \( \zeta \) on large scales will proceed in several steps as follows: Firstly we expand the action (15) to the leading order in the spatial gradient, i.e. to the order \( O(\sigma) \), while keeping all orders in fields. We show that the IR action starts with terms involving at least two time derivatives, each of which acts on a different field, and hence the constant (time-independent) configurations of \( A, C_i, D_{ij} \) are permitted as solutions to the classical equation of motion. And secondly we show that these constant configurations are stable under small fluctuations — i.e. they are indeed attractors. As we will see the proof for the scalar, vector and tensor perturbations are essentially identical.

**Step 1:** To show the conservation of \( A, C_i, D_{ij} \), we expand the Lagrangian (15) up to the first order in the temporal derivative and to the zeroth order in the spatial gradients, while keeping all orders in fields. The main challenge of doing this is to express the non-dynamical quantities, such as \( N, N_i \) and \( \chi \), in terms of the dynamical ones, by using the time slicing condition (20) and the constraint equations which are given by
\[
0 = \frac{\delta S}{\delta N} = \frac{M_{Pl}^2}{2} \left[ R^{(3)} + N^{-2} \left( 6H^2 + 2H \text{Tr} M + \frac{1}{4} \text{Tr} \left( d\frac{d}{dt}e^{-M}d\frac{d}{dt}e^{M} \right) + \frac{1}{4} \left( \text{Tr} M \right)^2 \right) \right. \\
+ \left. \hat{\nabla}_iN^j(\ldots) \right] + F(B) + 2F'(B)B \frac{N_iN_i}{N^2 - N_iN_i} \\
0 = \frac{\delta S}{\delta N_i} = \hat{\nabla}_i \left[ -\frac{2H}{N} - \frac{\text{Tr} M}{2N} + \frac{\hat{\nabla}_kN^k}{N} \right] + \hat{\nabla}_j \left[ \frac{1}{2N} \left( e^{-M} \frac{d}{dt}e^{M} \right)_j - \frac{\hat{\nabla}^jN_i + \hat{\nabla}_iN^j}{2N} \right] \\
- 2F'(B)B \frac{N_iN_i}{N^2 - N_iN_i}
\]

where \( R^{(3)} \) is the spatial Ricci scalar constructed from \( h_{ij} \), \( \hat{\nabla} \) the covariant derivative compatible with \( h_{ij} \), and the matrix \( M_{ij} = \log(h_{ij}a^{-2}) \). The “\( \ldots \)” in the first equation stands for terms that are regular in the limit \( \sigma \to 0 \) — the form of which is irrelevant in the analysis. Inspecting the constraint equation (28), one finds that \( N_i \) starts at least at the order \( O(\sigma) \). Thus, as long as the super-horizon modes are concerned, all the terms in (27) with \( N_i \) or \( \hat{\nabla} \) (as well as \( R^{(3)} \)) can be neglected, which leads to a crucial fact that \( \delta N \equiv N - 1 \) in the super-horizon regime starts with terms involving two time derivatives:
\[
\delta N = \left[ 1 + \frac{1}{24H^2} \text{Tr} \left( d\frac{d}{dt}e^{-M}d\frac{d}{dt}e^{M} \right) \right]^{1/2} - 1, \quad \sigma \to 0
\]

Moreover, we see that for each field, there is at most one time derivative acting on it.

On the other hand, the time slicing condition (20), in this long wavelength limit, reduces to
\[
\text{Tr} M = 3A + \nabla^2 \chi = \mathcal{O}(\sigma^2),
\]
which verifies our claim that \( \nabla^2 \chi \) is of the same order as \( A \) in the spatial gradient expansion.
It then follows immediately that in the super-horizon regime, the Lagrangian (15) becomes
\[
\lim_{k/aH \to 0} \mathcal{L} \simeq 2a(t)^{3}N F(B) \simeq -6a(t)^{3}M_{\text{Pl}}^{2}H^{2} + \sum_{n \geq 2} \mathcal{L}_{n}(\phi_{a})
\] (31)

In the last step we have denoted the dynamical fields — \( A, C_{i}, D_{ij} \) — collectively by \( \phi_{a} \) and the number of fields contained in \( \mathcal{L}_{n} \) by the subscript “\( n \)”. Notice that schematically \( \mathcal{L}_{n} \)'s take the form of
\[
\mathcal{L}_{2} \sim Q_{2}(t)G^{ab}_{2}\dot{\phi}_{a}\dot{\phi}_{b}, \quad \mathcal{L}_{n \geq 2} \sim Q_{n}(t)G^{ab...kl}_{n}(\phi)\dot{\phi}_{a}\dot{\phi}_{b}...\dot{\phi}_{k}\dot{\phi}_{l}
\] (32)

with \( Q_{n}(t) \) being a function of time consisting of \( a(t), H \) etc.. Besides the irrelevant field-independent term, this long-wavelength Lagrangian starts at two time derivative level and hence the e.o.m. following from it reads
\[
f_{1}(\phi, \dot{\phi})\ddot{\phi}_{a} + f_{2}(\phi, \dot{\phi})\dot{\phi}_{a}\dot{\phi}_{b} = 0
\] (33)

Therefore it indeed admits \( A, C_{i}, D_{ij} = \text{constant} \) as solutions to the classical equation of motion. Since this IR Lagrangian contains all orders in fields, the conservation of these fields on large scales must be preserved nonlinearly.

**Step 2**: Now we show the solutions \( A, C_{i}, D_{ij} = \text{constant} \) are actually attractors. We just work on the scalar case, since the analysis for the other two is identical. Writing \( A \) as \( A = A_{0} + \delta A \), owing to the constancy of \( A_{0} \), the quadratic action for the fluctuation \( \delta A \) in the long-wavelength limit takes the same form as that for \( A \), which is given by
\[
\delta S_{s}^{(2)} = \int dt \int_{k} \frac{3M_{\text{Pl}}^{2}}{4} a(t)^{3} \delta \dot{A}^{2}
\]
from which follows the linearized equation of motion for \( \delta A \)
\[
\delta \ddot{A} + 3H \delta \dot{A} = 0
\] (34)

It admits two general solutions — one decaying mode and one constant mode:
\[
\delta A_{1} = \int \frac{dt}{a(t)^{3}}, \quad \delta A_{2} = \text{const.}
\] (35)

This thus confirms that the solution \( A = \text{constant} \) is an attractor.

As we saw, our approach is more powerful in some aspect than those in previous literature, for it enables us to show that the vector perturbation \( C_{i} \) and the tensor perturbation \( D_{ij} \) are also conserved on super-horizon scales in the same manner as their scalar counterpart \( A \). And in next section, we will show that the nonlinear, conserved \( \zeta \equiv A/2 \) and \( D_{ij} \) we constructed here agrees with that in [4, 12], up to a global spatial coordinates redefinition.

### 4 Discussion

We have constructed the curvature perturbation \( \zeta \) (to nonlinear level) via the following steps:

1. choose a coordinate system such that the spatial coordinates comove with the fluid and
that equal time slice coincides with the uniform density slice. And \( ii \) define \( \zeta \) to be (half) the coefficient of the term in \( \log (a^{-2}g_{ij}) \) that is proportional to \( \delta_{ij} \). That is,

\[
\zeta = \frac{1}{4} \nabla^{-2} (\delta_{ij} \nabla^2 - \partial_i \partial_j) \log [a^{-2}g_{ij}] \tag{36}
\]

We showed that the \( \zeta \) such defined evolves time-independently in the super-horizon regime to all orders in fields.

Our result of the conservation of the non-linear curvature perturbation \( \zeta \) on large scales agrees with previous literatures \([4, 12]\). But in our analysis we did not neglect the presence of the vector and tensor perturbations, nor did we use the property of local homogeneity and isotropy. As we showed explicitly, the spatial metric in UG on large scales \( (\sigma \to 0) \) is given by

\[
g_{ij} = a(t)^2 \exp \left[ 6 \zeta \left( \frac{1}{3} \delta_{ij} - \hat{k}_i \hat{k}_j \right) + \text{vector} + \text{tensor} \right]. \tag{37}
\]

At a glimpse, one may think that even though the vector and tensor perturbations are neglected, this metric is anisotropic, which is inconsistent with our intuition about cosmological fluids that no anisotropy in the metric will be generated if the fluid is free of anisotropic stress. However, this puzzle results from our non-conventional choice of coordinates: we can perform a further coordinate transformation (from the UG)

\[
t \to t, \quad x^I \to y^I = x^I + \xi^I(\vec{x}) \tag{38}
\]

such that

\[
\phi^I(t, \vec{y}) = y^I + \zeta^I(\vec{y}), \quad \text{and} \quad g_{ij}(t, \vec{y}) = a(t)^2 e^{2\gamma'(\vec{y})} \exp \gamma'_{ij}(\vec{y}), \quad \text{with} \quad \partial_i \gamma'_{ij} = \gamma'_{ii} = 0 \tag{39}
\]

where the time-independence of \( \zeta^I(\vec{x}) \) follows directly from the time-independence of \( A, C_i \), and \( D_{ij} \) and where the function \( \xi^I(\vec{y}) \) is obtained by inverting the 3-diffeomorphism \( y^I(\vec{x}) = x^I + \xi^I(\vec{x}) \). Notice that the coordinate transformation like \( \xi(38) \) alters neither the time slices nor the threading of the spatial coordinate, the latter of which is because

\[
u^I \propto \epsilon^{\alpha \beta \gamma} \partial_\alpha \phi^0 \partial_\beta \phi^0 \partial_\gamma \phi^0 = 0, \quad u^I_I = g_{0I} u_0 = h_{ij} N^j u^0 \sim O(\sigma). \tag{40}
\]

Therefore this implies that in the new coordinate system and in the super-horizon regime, both the spatial metric \( g_{ij} \) and \( T_{ij} \) (c.f. eqn. \( 37 \)) — which is reduced to \( T_{ij} = pg_{ij} \) on large scales — are isotropic, were the tensor perturbation ignored, and homogeneous since in the limit \( \sigma \to 0 \) the spatial dependent of \( \zeta' \) and \( \gamma' \) is negligible.

That is, although convenient for computational purposes, the UG coordinate is not capable of exhibiting the local homogeneity and isotropy in the long wavelength limit. Fortunately there exists a new coordinate system, which is indistinguishable from the UG coordinate by only inspecting the physical quantities such as \( \rho, p, u^i \) etc. and in which the existence of

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\(^5\)In fact having fixed the UG by \( \phi^I = x^I \) and \( B = a(t)^{-6} \), we have depleted all gauge freedoms for choosing a coordinate system. Furthermore, as we argued earlier, we should not think the terms in \( g_{ij} \) with “superficially” more derivatives such as \( \partial_i \partial_j \chi, \partial_i C_j \) are of higher order in the spatial gradient expansion than \( A \delta_{ij} \). As a consequence of which, as long as the UG is chosen, we are not entitled to claim any property about \( g_{ij} \) on large scales.
local homogeneous and isotropic patches on large scales can be verified (rather than being assumed, as were in previous literatures). The $\zeta'$ in (39) is also time-independent on large scales and it agrees perfectly with the definition in [4, 12].

As we showed in last section, on super-horizon scales, the vector field $C_i$ evolves time-independently. However, this does not correspond to a physical vector mode, as can be understood as follows. Transform to the coordinate system specified by (39). Owing to the internal volume-preserving-diffeomorphism invariance (4) of the fluid system, the configuration in eqn (39) is in fact symmetric to the one free of vortex degrees of freedom (divergence-free vector modes), either in the matter fields or in the dynamical metric components. That is, as the leading contribution, the constant (in time) $C_i$ configuration has no physical significance; the well known decaying vector modes in the universe dominated by perfect fluids come from subdominant, time-dependent terms in $g_{ij}$ in the UG.

It is known that a perfect fluid free of vortex degrees of freedom (divergence-free vector modes) has an equivalent effective description involving only one scalar field — i.e. the $P(X) \equiv P((\partial \psi)^2)$ description [8, 19, 20]. Then we can apply the method in Ref. [7, 10] directly to a non-inflating background and define the curvature perturbation $\zeta$ as the (only) dynamical scalar metric perturbation in unitary gauge:

$$\psi(t, \vec{x}) = \psi_0(t), \quad g_{ij} = a^2 e^{2\zeta} \exp\{\gamma_{ij}\}$$

Details of the proof of the conservation of $\zeta$ on super-horizon scales in the $P((\partial \psi)^2)$ context are collected in Appendix [3]. The nonlinear $\zeta$ constructed via this logic is identical to the $\zeta'$ field in (39), hence agrees with our definition up to a global reshuffle of the spatial coordinates.

5 Conclusions

In this paper we consider the problem of how to extend the definition of curvature perturbation $\zeta$ to nonlinear levels; in particular we concentrate on the situation where the universe is dominated by a perfect fluid. We have used a low energy/long wavelength effective action to model the fluid sector and constructed the nonlinear $\zeta$ as follows: i) Fix the gauge such that the comoving coordinates of the fluid coincide with the spatial ones and that the constant time slices are the uniform density slices; and ii) define $\zeta = A/2$ where $A$ is the isotropic scalar part in the spatial metric [16]. We have shown this $\zeta$ (as well as its tensor counterpart) is conserved outside the horizon to all orders in fields.

Although this topic has been investigated to some extent in previous literature, we believe our approach is novel and has its own merits. For instance, in discussing the conservation on super-horizon scales of $\zeta$ (the scalar mode), we do not need to neglect the vector or tensor modes; on the contrary, we illustrate that there are vector (non-physical) and tensor counterparts of $\zeta$ which are conserved in the same manner as $\zeta$ is. Moreover, our proof does not assume/rely on that the universe looks locally like FRW patches (local homogeneity and isotropy) on sufficiently large scales, which was the key input in previous work. Nevertheless, we have shown our definition of $\zeta$ agrees with that in Lyth et al. (2005) and Malik et al. (2004) up to a global reshuffle of the spatial coordinates.
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Appendix

A the Spatially Flat Slicing Gauge (SFSG) to All Orders in Fields

In this section, we show that starting from an arbitrary $g_{ij}(x) = \bar{g}_{ij} + \delta g_{ij}$, we can always achieve the non-linear SFSG, specified by

$$g_{ij}(x) = a(t)^2 \exp{\gamma_{ij}(x)}, \quad \text{with} \quad \gamma_{ii} = \partial_i \gamma_{ij} = 0$$

(42)

via some appropriate gauge transformation $x^\mu \to y^\mu = x^\mu + \xi^\mu(x)$. Here we only concentrate on the most relevant case to our analysis, in which $\bar{g}_{\mu\nu} = \text{diag}\{-1, a(t)^2, a(t)^2, a(t)^2\}$ and $\delta g$ is treated as small fluctuations around the FRW background. We prove this by using a perturbative construction.

First let us invert the function $y^\mu(x)$ and denote it as

$$x^\mu(y) = y^\mu + v^\mu(y), \quad \text{with} \quad v^\mu(y) = \sum_{n=1}^{\infty} v^\mu_{(n)}(y)$$

(43)

where $v^\mu_{(n)}$ is assumed to be of order $(\delta g)^n$. Under this gauge transformation, the spatial metric transforms as

$$g_{ij}(x) \to \tilde{g}_{ij}(y) = \frac{\partial(y^\alpha + v^\alpha)}{\partial y^i} \frac{\partial(y^\beta + v^\beta)}{\partial y^j} g_{\alpha\beta}(y + v)$$

(44)

Consider $M_{ij} = \log(a^{-2}g_{ij})$. At the first order in $\delta g$, it is straightforward to work out $\tilde{M}_{ij}^{(1)}(y)$, which is given by

$$\tilde{M}_{ij}^{(1)} = a^{-2}\delta g_{ij} + \partial_i v_{(1)}^j + \partial_j v_{(1)}^i + 2H v_{(1)}^0 \delta_{ij}$$

(45)

Notice that any tensor function $\delta g_{ij}(x)$ ($i, j = 1, 2, 3$) can always be put in the form of

$$\delta g_{ij} = a(t)^2 (A \delta_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i + D_{ij})$$

(46)

with $C_i$ and $D_{ij}$ satisfying $\partial_i C_i = D_{ii} = \partial_i D_{ij} = 0$. Thus by choosing

$$v_{(1)}^0 = -A/2H, \quad v_{(1)}^S = -B/2, \quad v_{T(1)}^i = -C_i$$

(49)

6To see this, we define $A$ and $B$ to be the solutions of

$$3A + \nabla^2 B = a^{-2}\delta g_{ii} \quad \text{and} \quad \nabla^2 A + \nabla^2 \nabla^2 B = a^{-2} \partial_i \partial_j \delta g_{ij},$$

(47)

then define $C_i$ as the solution of

$$\partial_i (A + \nabla^2 B) + \nabla^2 C_j = a^{-2} \partial_i \delta g_{ij}$$

(48)

and then use eqn. [46] to determine $D_{ij}$. 

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where $v^S$ and $v^\perp$ are the longitudinal (curl-free) and transverse (divergence-free) components of $v$: $v^{(n)}_i = \partial_i v^S + v^\perp (n)$, we find that $\tilde{M}^{(1)}_{ij} = \mathcal{D}_{ij}$ — i.e. only the transverse traceless part survives.

At the second order the expression for $\tilde{M}^{(2)}_{ij}$ is quite cumbersome, but that doesn’t bother us much. Since it takes the form of

$$\tilde{M}^{(2)}_{ij} = S_{ij} + \partial_i v^\perp_{(2)} + \partial_j v^\perp_{(2)} + 2H v^\perp_0 \delta_{ij}, \quad (50)$$

by the same argument as above, we show that all the fields in $S_{ij}$ can be set to zero except the TT one. And this process can be repeated to all orders in $\delta g$.

Now let $\gamma_{ij} = \sum_{n=1}^{\infty} \tilde{M}^{(n)}_{ij}$, which apparently is also transverse and traceless. The spatial metric in new coordinate system indeed takes the form of eqn. (42).

### B Nonlinear Conserved $\zeta$ in $P(X)$ Fluids

A perfect fluid free of vortex degrees of freedom has a dual low energy effective description involving only one scalar field $\psi$. In the cosmological context, the action is given by

$$S = S_{EH} + \int \sqrt{-g} P(X), \quad X \equiv -\partial_\mu \psi \partial^\mu g^\mu\nu. \quad (51)$$

The energy momentum tensor can be obtained:

$$T_{\mu\nu} = 2P'(X)\partial_\mu \psi \partial_\nu \psi + P(X)g_{\mu\nu}, \quad (52)$$

Noticing that $u_\mu = \partial_\mu \psi X^{-\frac{1}{2}}$, $T_{\mu\nu}$ can be recast into the standard perfect fluid form (7) if we identify

$$\rho = 2XP' - P, \quad p = P. \quad (53)$$

Assuming that the background configuration of the scalar field is only time dependent $\langle \psi(x) \rangle = \psi_0(t)$ and that of the metric takes the usual FRW form, the Einstein equations and the scalar field equation of motion for unperturbed configurations read

$$3M_{Pl}^2 H^2 = -P + 2P' \dot{\psi}_0^2 \quad (54)$$

$$M_{Pl}^2 \dot{H} = -P' \ddot{\psi}_0^2 \quad (55)$$

$$\left(P' + 2P'' \dot{\psi}_0^2\right) \dddot{\psi}_0 + 3HP' \dot{\psi}_0 = 0 \quad (56)$$

Now let us fix the gauge. For the sake of defining a nonlinear $\zeta$, the unitary gauge is of most convenience:

$$\psi(x) = \psi_0(t), \quad g_{ij} = a(t)^2 e^{2\zeta} \exp \gamma_{ij}, \quad (57)$$

We shall show the $\zeta$ such defined is conserved outside the horizon. We adopt similar logic as before: first show that $\zeta = \text{constant}$ is a solution in the long wavelength limit, and then show that this solution is actually an attractor.

Using the ADM variables, the effective action (51) becomes

$$S = \int \mathcal{N} \sqrt{h} \left( \frac{M_{Pl}^2}{2} R^{(3)} + \frac{M_{Pl}^2}{2N^2} (E_i^j E_j^i - E^2) + P(X) \right) \quad (58)$$
We can then obtain the constraint equations by varying this action with respect to $\mathcal{N}$ and $N^i$:

\[
R^{(3)} - \frac{1}{\mathcal{N}^2} \left\{ -6(H + \dot{\zeta})^2 + 4(H + \dot{\zeta})\nabla_i N^i - \frac{1}{4}\text{Tr}\left(\frac{d}{dt}e^{-\gamma} \frac{d}{dt}e^\gamma\right) + \frac{1}{2}\nabla_i N_j \nabla^j N_j \right\} + \frac{1}{2}\nabla_j N^i \nabla^j N_i - \nabla_j N^i \nabla_j N^j - \frac{1}{2}\left(e^{-\gamma} \frac{d}{dt}e^\gamma\right)_j \left(\nabla^j N_i + \nabla_i N^j\right)
\]

\[
\frac{2}{M_{Pl}^2} P \left(\frac{\dot{\psi}_0^2}{\mathcal{N}^2}\right) - \frac{4}{M_{Pl}^2 \mathcal{N}^2} P' \left(\frac{\ddot{\psi}_0}{\mathcal{N}^2}\right) = 0
\]

\[
\nabla_i \left( -\frac{2}{\mathcal{N}}(H + \dot{\zeta}) + \frac{1}{\mathcal{N}}\nabla_j N^j \right) + \nabla_j \left[ \frac{1}{2\mathcal{N}} \left( e^{-\gamma} \frac{d}{dt}e^\gamma \right)_j - \frac{1}{2\mathcal{N}} \nabla_i N^j - \frac{1}{2\mathcal{N}} \nabla^j N_i \right] = 0
\]

From equation (60), we can see that $\nabla_i N^j \sim \dot{\zeta}$ (or $\dot{\gamma}$), i.e. $\nabla_i N^j$ is of the zeroth order in the spatial gradient expansion ($\sigma = k/aH$). Thus solving $\delta\mathcal{N} \equiv \mathcal{N} - 1$ via equation (59) up to order $O(\sigma^0)$, we have

\[
\delta\mathcal{N} = -M_{Pl}^2 \left( P - P' \dot{\psi}_0^2 + 2P'' \dot{\psi}_0^4 \right)^{-1} \left( 3H\dot{\zeta} - H\nabla_i N^i \right) + O(\dot{\zeta}^2, \dot{\gamma}^2, \dot{\zeta}\dot{\gamma}).
\]

Plugging this into the action (58) and expanding it to $O(\sigma^0)$ while keeping all orders in fields, we have

\[
S = -2M_{Pl}^2 \int \sqrt{h} \left\{ 3H^2 + \dot{H} + \left( 3H\dot{\zeta} - H\nabla_i N^i \right) + O(\dot{\zeta}^2, \dot{\gamma}^2, \dot{\zeta}\dot{\gamma}) \right\}
\]

\[
= -2M_{Pl}^2 \int \frac{d}{dt} \left( a^3 e^{3\zeta} H \right) + O(\dot{\zeta}^2, \dot{\gamma}^2, \dot{\zeta}\dot{\gamma})
\]

where in the first equality we have used the background Einstein equations (54), (55) and the background scalar field equation (56) and in the second equality we have neglected the contribution from $\nabla_i N^i$ since it is a boundary term. Therefore we conclude that the effective action in the long wavelength limit starts with terms involving two times derivatives (acting on different fields) and, by the same argument as before, that $\dot{\zeta} = 0, \dot{\gamma} = 0$ are solutions to the classical e.o.m.’s.

Now we show that the solutions $\zeta = \text{constant}$ and $\gamma = \text{constant}$ are attractors. As before we consider the fluctuations around these classical solutions and work out the quadratic action for $\delta\zeta$ and $\delta\gamma$. Owing to the constancy of the classical solutions, the quadratic action for fluctuations take the same form as that for $\zeta$ and $\gamma$ themselves, which can be obtained by solving the constraint equations (59), (60) up to the linear order in fields, plugging into (58) the linearized solutions $\delta\mathcal{N}_1$, $N^i_1$ and expanding the effective action to the quadratic order. Thus we have

\[
\delta S \simeq M_{Pl}^2 \int a^3 \left[ \frac{\varepsilon}{c_s^2} \delta\zeta^2 - \frac{\varepsilon}{a^2} (\partial_i \delta\zeta)^2 \right] + \frac{1}{8} \left[ \delta\gamma_{ij} - \frac{1}{a^2} (\partial_i \delta\gamma_{jk})^2 \right]
\]

where $\varepsilon \equiv -\dot{H}/H^2$ and $c_s$ is the speed of sound defined as

\[
c_s^2 \equiv \frac{dp}{d\rho}_{X = \dot{\psi}_0^2} = \frac{P'((\dot{\psi}_0^2))}{P''((\dot{\psi}_0^2)) + 2\dot{\psi}_0^2 P''((\dot{\psi}_0^2))}.
\]
Thus in the long wavelength limit $\sigma \to 0$, the linearized equation for $\delta \zeta$ ($\delta \gamma$) possesses two general solutions, one being time independent and the other decaying, which confirms our claim that $\zeta = \text{constant}$ and $\gamma = \text{constant}$ are attractors.

Before ending this section, let us remark that the nonlinear $\zeta$ constructed in $P(X)$ context agrees with that in $F(B)$ context. Notice that in unitary gauge, $X = \dot{\psi}_0(t)^2(1 + \delta N)^{-2}$. Since on super-horizon scales $\delta N \to 0$ due to the constancy of $\zeta$ and $\gamma$, the constant time slices in the unitary gauge coincide with the uniform density slices (for $\rho$ is a function of $X$ only). Similarly, since $N^i$ vanishes on large scales for the same reason, we have $u^i \propto g^{0i} \propto N^i \to 0$, i.e. the threading of the spatial coordinates is chosen such that the threads $x^i = \text{constant}$ coincide with the integral curves of the 4-velocity $u^\mu$ (the comoving worldlines). Therefore $\zeta$ defined in $P(X)$ context is identical to $\zeta'$ in (39).

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