Numerical Models for the Simulation of the Fractional-Order Control Systems

I. Dvorka

November 1994
## Contents

| Section | Page |
|---------|------|
| 1 Abstract | 2 |
| 2 Introduction | 2 |
| 3 Definition of the fractional-order control system | 3 |
| 4 Numerical and analytical computation of the unit step response of a fractional-order system | 3 |
| 5 Approximation of fractional-order system with integer-order system | 6 |
| 6 Design of the regulator | 6 |
| 7 Conclusion | 11 |

References
NUMERICAL MODELS FOR THE SIMULATION OF THE FRACTIONAL-ORDER CONTROL SYSTEMS

Ľubomír Dorčák

Department of Management and Control Engineering,
BERG Faculty, Technical University of Kosice
Bozeny Nemcovej 3, 042 00 Kosice, Slovakia
e-mail: Lubomir.Dorcad@tuke.sk, phone: (+42155) 6025172

1 Abstract

This contribution deals with the creation of numerical models for the simulation of the dynamic characteristics of fractional-order control systems and their comparison with analytical models. We give the results of the comparison of dynamic properties in fractional- and integer-order systems with a controller, designed for an integer-order system as the best approximation to given fractional-order system. Other open questions are pointed out, which should be answered in this area of research.

2 Introduction

The standard control systems used so far were all considered as integer-order systems, regardless of the reality. In their analysis and design, the Laplace transform was used heavily for simplicity. Because of the higher complexity and the absence of adequate mathematical tools, fractional-order dynamical systems were only treated marginally in the theory and practice of control systems, e.g. [1, 2, 3]. Their analysis requires familiarity of work with fractional-order derivatives and integrals [4, 5, 6, 9, 10, 11, 12].

By removing the restrictions to integer-order systems it is possible to obtain systems whose properties are a combination of systems of the closest integer-order, but also intermediate types of systems, which broadens the class of the systems considerably [1]. With different fractional-order systems the notions arise such as weak or strong integrator or differentiator, weak or strong fractional-type pole, or zero, with interesting contribution to the dynamics of the system (stability, phase shift etc.), as some properties are retained, others are eliminated. A fractional-order system combines some characteristics of systems of the order \(N\) and \((N + 1)\). Hence by changing the order as a real and not only integer value we have more possibilities for an adjustment of the roots of the characteristic equation according to special requirements.

In this contribution we will analyze dynamic properties of systems in the time domain with an emphasis on the numerical methods of simulation of fractional-order systems. We will point out problems of inadequate approximation of fractional-order
systems with integer-order systems and the differences in dynamic properties of such systems in closed control systems with controller.

3 Definition of the fractional-order control system

For the definition of the control system we consider a simple unity feedback control system shown in Fig.1. \( G_s(s) \) denotes the transfer function of the control system which is either integer-type \( (G^i_s(s)) \) or more generally fractional-type \( (G^f_s(s)) \) and \( G_r(s) \) is the transfer function of the controller, also either integer-type \( (G^i_r(s)) \) or fractional-type \( (G^f_r(s)) \).

![Figure 1: Simple unity feedback control system](image)

For later purposes consider a fractional-order controlled system, which represents our real system, with the transfer function

\[
G^f_s(s) = \frac{1}{a_2 s^\alpha + a_1 s^\beta + a_0}
\]  

(1)

where \( \alpha \) and \( \beta \) are in general real \( (\alpha > \beta) \). In the simulation, the coefficient values \( a_2 = 0.8, a_1 = 0.5, a_0 = 1, \alpha = 2.2, \beta = 0.9 \) were chosen \([10, 12]\). To the fractional-type transfer function \([10, 12]\) there corresponds, in time domain, the fractional-order differential equation

\[
a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + a_0 y(t) = u(t)
\]  

(2)

with initial conditions \( y^{(\beta)}(0) = 0 \) and \( y(0) = 0 \).

4 Numerical and analytical computation of the unit step response of a fractional-order system

For the numerical calculation of the unit step response of the fractional-order system \([8]\) we employ, for the approximation of the fractional derivatives in equation \([8]\), the relation \([3]\) with "short memory" principle, formulated in \([11]\)

\[
y^{(\alpha)}(t) \approx (t-L)D^\alpha_L y(t) = h^{-\alpha} \sum_{j=0}^{N(t)} b_j y(t-jh),
\]  

(3)

where \( L \) is "memory length", \( h \) is time step,

\[
N(t) = \min \left\{ \left[ \frac{L}{h} \right], \left[ \frac{L}{h} \right] \right\}
\]

\([z]\) is the integer part \( z \),

3
\[ b_j = (-1)^j \binom{\alpha}{j} \]  
(4)

where \( \binom{\alpha}{j} \) is binomial coefficient. To calculate \( b_j \) it is convenient to use the following recurrent relation

\[ b_0 = 1, \quad b_j = \left(1 - \frac{1 + \alpha}{j}\right) b_{j-1} \]  
(5)

It follows from the estimates derived in [11] that in our case the normed error of such approximation is

\[ \delta_0 = \frac{\left| y^{(\alpha)}(t) - (t-L)D_t^{\alpha}\gamma(t) \right|}{M} = \frac{1}{\sqrt{L} \Gamma(\alpha)}, \quad M = \max_{[0, \infty]} |\gamma(t)| \]  
(6)

whence we have the following constraint for the choice of "memory length" \( L \):

\[ L \geq \frac{1}{\delta_0^2 \Gamma^2(\alpha)} \]  
(7)

where \( \delta_0 \) is the maximum admissible normalized error and \( \Gamma(\alpha) \) is the Gamma function.

By using the relation (3) we can approximate the differential equation (2) in a different mode. Our approximation [10, 12] of equation (2) in discrete time steps \( t_m \) \((m = 2, 3, \ldots)\) has the following form

\[ a_2h^{-\alpha} \sum_{j=0}^{m} b_j y_{m-j} + a_1h^{-\beta} \sum_{j=0}^{m} c_j y_{m-j} + a_0y_m = u_m \]  
(8)

or

\[ a_2h^{-\alpha}(b_0y_m + \sum_{j=1}^{m} b_j y_{m-j}) + a_1h^{-\beta}(c_0y_m + \sum_{j=1}^{m} c_j y_{m-j}) + a_0y_m = u_m \]  
(9)

From the approximation (8) we can derive [10, 12], the following explicit recurrent relation for the calculation of the values \( y_m \) \((m = 2, 3, \ldots)\)

\[ y_m = \frac{u_m - a_2h^{-\alpha} \sum_{j=1}^{m} b_j y_{m-j} - a_1h^{-\beta} \sum_{j=1}^{m} c_j y_{m-j}}{a_2h^{-\alpha}b_0 + a_1h^{-\beta}c_0 + a_0} \]  
(10)

with \( y_0 = 0, \ y_1 = 0, \ u_0 = 0 \) and \( u_m = 1 \) for \( m = 1, 2, \ldots \).

This algorithm does not require iterational calculations, in contrast to the procedure given in [4].

For the analytical calculation of the unit step response of fractional-order systems [3] we apply the analytical form of the impulse response of such system [5]. By integrating the impulse response of such system we obtain the following analytical form of the unit step response of such a fractional-order system

\[ y(t) = \frac{1}{a_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{a_0}{a_2} \right)^m t^{\alpha(m+1)} E_{\alpha-\beta, \alpha+\beta+1}^\alpha \left( -\frac{a_1}{a_2} t^{\alpha-\beta} \right) \]  
(11)

where \( E_{\lambda,\mu}(z) \) is the Mittag-Leffler function in two parameters,

\[ E_{\lambda,\mu}^{(k)}(z) = \frac{d^k}{dz^k} E_{\lambda,\mu}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!}{j! \Gamma(\lambda j + \lambda k + \mu)} z^j, \quad (k = 0, 1, 2, \ldots) \]  
(12)
For the calculation of the Gamma function we used the algorithm from [7]. The calculation of the unit step response directly according to (11) is numerically difficult. For the system (2), at least 20 terms of the series were needed along with the extended precision of the real variables. To employ (11) further it is necessary to consider asymptotic methods of calculations.

Fig. 2 Comparison of unit step responses (10), (11)

Fig. 2 shows a graphical comparison of numerically and analytically computed unit step responses of the system (2) with time step 0.1 seconds. The calculations show a good agreement of both methods. The differences depend on the size of the time step and on the method of approximation of the value $y_1$.

The value $y_1 = 0$ in (10) was calculated using the initial conditions for differential equation (2) and relation (3) with $m = 1$

$$y_1 = \frac{y_0^{(\beta)} - y_0}{b_0 h^{-\beta}}$$  \hspace{1cm} (13)

Better results (see Fig. 3) we can obtain by approximation the value $y_1$ from initial condition $y_0$ directly with equation (10) for $m = 1$. The advantage of such approximation is more evident for differential equations (17), (18) and (23) with derivatives of the function $u(t)$ (unit step).

Fig. 3 Comparison of unit step responses (10), (11)
5 Approximation of fractional-order system with integer-order system

In integer-type linear systems, higher-order systems usually are approximated—under certain conditions [8]—with second-order systems for simplicity. So far, however, fractional-order systems usually have not been considered. Regardless of the reality, their approximation is performed with close integer-type systems, albeit with unfavorable consequences.

Let us approximate the fractional-type system (2), with the coefficients as given there, with the integer-type system of second order

\[ a_2^i y''(t) + a_1^i y'(t) + a_0^i y(t) = u(t) \]  

(14)

The coefficients \( a_k^i \) of the integer-type system take on the values \( a_2^i = 0.7414, \ a_1^i = 0.2313, \ a_0^i = 1 \) under the condition

\[ \sum_{j=0}^{m} (y_f^j - y_i^j)^2 \rightarrow \min \]  

(15)

Fig.4 The integer approximation (14) of the system (2)

The comparison of the unit step responses of both systems is shown in Fig.4. It should be noted that a generally better approximation was achieved for \( \alpha, \beta \leq 2 \).

6 Design of the regulator

For the system (14) we choose as an approximation to the system (2) the integer-order regulator [10, 12] with the transfer function

\[ G_r^i(s) = K + T_d s \]  

(16)

The regulator will be designed so that a unit step at the input of the closed regulation system in Fig.1 will induce at the output an oscillatory unit step response with stability measure \( St = 2 \) and damping measure \( Tl = 0.4 \). Then the coefficients for (14) take on the values \( K = 20.5, \ T_d = 2.7343 \). The differential equation of the integer-order closed regulation system has the form

\[ a_2^i y''(t) + (a_1^i + T_d)y'(t) + (a_0^i + K)y(t) = Kw(t) + T_d w'(t) \]  

(17)
It follows from an analysis of the roots of its characteristic equation that the requirements for stability measure and damping measure are satisfied. The permanent regulation deviation is 4.6% . For the closed regulation system (17) it is very simple to derive the analytical form of the unit step response. In a manner mentioned above it is also possible to obtain an approximating recurrent relation for the numerical calculations

\[
y_m = \frac{K w_m + T_d h^{-1} \sum_{j=0}^{m} d_j w_{m-j} + a_2 h^{-2}(2y_{m-1} - y_{m-2}) + (a_1 + T_d) h^{-1} y_{m-1}}{a_2 h^{-2} + (a_1 + T_d) h^{-1} + (a_0 + K)}
\]  

(18)

for \( m = 1, 2, ... \), \( y_0 = 0, y_{-1} = 0 \), \( w_0 = 0 \) and \( w_m = 1 \) for \( m = 1, 2, ... \).

The unit step response is plotted in Fig.5. From the results of numerical simulation it also follows that the regulation area within 10 seconds was 0.71 and the permanent regulation deviation is up to 5%.

We now apply the integer-order regulator, designed for the approximated system (model of the real system), to the original fractional-type controlled system (2). The differential equation of the closed fractional-order regulation system has the form

\[
a_2 y^{(\alpha)}(t) + T_d y'(t) + a_1 y^{(\beta)}(t) + (a_0 + K)y(t) = K w(t) + T_d w'(t)
\]

(19)

However, the analytical analysis of this equation is not as simple as with the equation (17). We will calculate the unit step response numerically with the aid of an approximating recurrent relation derived as mentioned above

\[
y_m = \frac{K w_m + T_d h^{-1} \sum_{j=0}^{m} d_j w_{m-j} - a_2 h^{-\alpha} \sum_{j=1}^{m} b_j y_{m-j} - a_1 h^{-\beta} \sum_{j=1}^{m} c_j y_{m-j} + T_d h^{-1} y_{m-1}}{a_2 h^{-\alpha} b_0 + a_1 h^{-\beta} c_0 + T_d h^{-1} + (a_0 + K)}
\]

(20)

for \( m = 1, 2, ... \), \( y_0 = 0 \), \( w_0 = 0 \) and \( w_m = 1 \) for \( m = 1, 2, ... \).

To compare the numerical calculations we will also derive an analytical form of the unit step response of the closed regulating system with a fractional-order controlled system and an integer-type regulator. We will obtain the form by integrating
the impulse characteristic from [5] for the equation of type (19), whence

\[
y(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{a_0 + K}{a_2} \right)^m \sum_{k=0}^{m} \left( \frac{a_1}{a_0 + K} \right)^k m! \frac{a_1}{a_0 + K} \left( \frac{e^{a_0 + a_2}}{a_2} \right) \cdot \frac{\Gamma(m+1)}{a_2} \cdot \frac{a_0 + a_2}{a_2} \nonumber
\]

with \( b_2 = \frac{a_0}{a_2} \) and \( c_2 = \frac{a_0}{a_2} \).

Fig. 6 shows a graphical comparison of analytically (21) and numerically (20) computed unit step responses of the system (19) with time step 0.1 seconds. We can see the better results of the numerical calculations by approximation the value \( y_1 \) directly with equation (21) (a) than by approximation with equation (3) (b).

Fig. 6 Comparison of unit step responses (20a), (20b) and (21)

A comparison of the unit step responses of the closed regulation system with integer-order system (17), with the fractional-order system (19) is shown in Fig. 7.

Fig. 7 Comparison of unit step responses (17), (19)

It can be seen that the dynamic properties of the closed regulation system with a fractional-order system and an integer-order regulator designed for an integer-order system which is an approximation of the fractional-order system are considerably
worse than with the integer-order system and the integer-order regulator. The regulation area during 10 seconds is greater by 50%, the system stabilizes later and has larger surplus oscillations. The system (19) is much more sensitive to changes in parameters. For example [10, 12], at the change of \( T_d \) to value 1 the system (19) is just behind the border of stability (Fig. 8), and with another decrease of \( T_d \) it is already unstable, whereas the system (17) is stable.

![Comparison of unit step responses (17), (19) with changed \( T_d \)](image)

Hence disregarding the fractional order of the original system, its approximation by an integer-type system of second order and an application of a regulator designed for the approximating system to the original fractional-order system is not adequate for our case \( \alpha > 2 \).

For the fractional-order system we now use the fractional-order regulator [10, 12] with the transfer function

\[
G_f^f(s) = K + T_d s^\delta
\]

The differential equation of the closed regulation system with a fractional-order system and a fractional-order regulator has the form

\[
a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + T_d y^{(\delta)}(t) + (a_0 + K) y(t) = K w(t) + T_d w^{(\delta)}(t)
\]

In this case the fractional derivative \( \delta \) of the unit step \( w^\delta(t) \) is no longer equal to zero for \( t > 0 \). The numerical computation of the unit step response will be done through an approximating recurrent relation derived as mentioned above

\[
y_m = \frac{K w_m + T_d h^{-\delta} \sum_{j=0}^{m} d_j w_{m-j} - a_2 h^{-\alpha} \sum_{j=1}^{m} b_j y_{m-j}}{a_2 h^{-\alpha} b_0 + a_1 h^{-\beta} c_0 + T_d h^{-\delta} d_0 + (a_0 + K)} - a_1 h^{-\beta} \sum_{j=1}^{m} c_j y_{m-j} - T_d h^{-\delta} \sum_{j=1}^{m} d_j y_{m-j}
\]

for \( m = 1, 2, ... \), \( y_0 = 0 \), \( w_0 = 0 \) and \( w_m = 1 \), \( m = 1, 2, ... \)

The analytical form of the unit step response of a closed regulation system with fractional-order controlled system and fractional-order regulator can be obtained [10, 12] by dividing equation (23) into two parts, according to the right-hand side. By taking the derivative of the order \( \delta \) of the impulse characteristic for the second part, by subsequent integer-type integration of the impulse characteristic of both
parts, and their adding together, we obtain the following resulting analytical form of the unit step response for equation of the type (23)

\[
y(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{a_0 + K}{a_2} \right)^m \sum_{k=0}^{m} \binom{m}{k} \left( \frac{T_d}{a_0 + K} \right)^k \]

\[
\left( \frac{\mu^{\alpha(m+1) - \delta k}}{b_2} \right) \frac{\mu^{\alpha(m) - \delta (k+1)}}{c_2} \left( -\frac{a_1}{a_2} t^{\alpha - \beta} \right) \]

with \( b_2 = \frac{a_2}{K} \) and \( c_2 = \frac{a_2}{T_d} \).

Fig. 9 shows a graphical comparison of numerically and analytically computed unit step responses of the system (23).

Fig.9 Comparison of unit step responses (24), (25) of the system (23)

In Fig. 10, a comparison of the unit step responses of the closed regulation system with a fractional-order regulated system and the integer-order regulator (19) and the fractional-order regulator (23), with only two, experimentally found, parameters \( T_d = 3.7343 \) and \( \delta = 1.15 \), is given.

Fig.10 Comparison of unit step responses of the systems (19) and (23)

It follows from the simulations (Fig. 10) that the dynamic properties of the
system \cite{19} with an integer-order regulator (Fig.7) improved with the use of a fractional-order regulator \cite{23}. Therefore in the following it is necessary to deal with the methods of synthesizing the structure and parameters of fractional-order regulators.

7 Conclusion

In the contribution we gave relations for simulation of dynamic properties of fractional-order systems, via numerical as well as analytical methods. Also presented were the results of some simulations. The calculations showed a good agreement of the new numerical method and analytical method of simulation of the fractional-order control systems.

It was shown that the fractional-order system \cite{2} can be well approximated by the second-order system \cite{14} for $\alpha, \beta \leq 2$. The integer-order regulator designed for the integer-order system \cite{14} can in this case be applied with good results also to the fractional-order system. A worse approximation is achieved with $\alpha$ or $\beta > 2$. An application of the integer-order regulator, designed for the integer-order system as an approximation to such a fractional-order system is inadequate and with a change of system or regulator parameters can lead to system instability.

In the future it is desirable to deal with the asymptotic methods in the analytical method of computation of the dynamic characteristics of fractional-order systems. Further, it is necessary to deal with the methods of identification of the fractional order and of the parameters of such systems. The most important task will be the elaboration of methods of synthesis of the structures and parameters of regulators for such types of systems.

References

[1] M.Axtell and M.E.Bise : Fractional Calculus Applications in Control Systems, in: Proc. of the IEEE 1990 Nat. Aerospace and Electronics Conf., New York, 1990, pp. 563 - 566.

[2] G. D. Kalojanov, Z. M. Dimitrova : Teoretiko-experimentalnoe opredelenie oblastej primenimosti sistemy "PI (I) reguliator-objekt s necelocisleennoj astaticnostiu", (Theoretico-experimental determination of the domain of applicability of the system " PI (I) regulator - fractional-type astatic systems"), in: Izvestia vyssich ucebnych zavedenij, Elektromechanika, No 2, 1992, pp. 65-72.

[3] A. Outstaloup : From Fractality to non integer Derivation through Recursivity, a Property Common to these two Concepts: A Fundamental Idea from a new Process Control Strategy, in: Proc. of 12th IMACS World Congress, Paris, July 18-22, 1988, vol. 3, pp. 203-208.

[4] I. Podlubny, J. Misanek : Využitie derivacie necelociselneho radu pre modelovanie pohybu velkej tenkej dosky vo viskoznjej tekutine, (The Use of Derivatives of Non Integer Order for Modelling the Motion of a Large Thin Plate in Viscous Fluid), 9. celoštátna konferencia "Riadenie procesov", Vysoké Tatry , 1.-3. júna 1993, pp. 274-278.
[5] I. Podlubny: *The Laplace Transform Method for Linear Differential Equations of the Fractional Order*, UEF - 02 - 94, The Academy of Sciences Institute of Experimental Physics, 1994, Kosice.

[6] K. B. Oldham and J. Spanier: *The Fractional Calculus*, Academic Press, New York, 1974.

[7] Jerome Spanier, Keith B. Oldham: *An Atlas of Functions*. SPRINGER-VERLAG Berlin, 1987.

[8] R.C. Dorf: *Modern Control Systems*, Addison-Wesley Publishing Company, New York, 1990.

[9] I. Podlubny, L. Dorcak, J. Misanek: *Application of Fractional Order Derivatives in Calculation of the Change of Intensity in Heat Load of Blast Stove Walls*, Transactions of the TU of Kosice, Riecansky Publishing Co., Cambridge, 1994, (to appear).

[10] L. Dorcak, J. Prokop, I. Kostial: *Investigation of the Properties of Fractional-Order Dynamical Systems*, in: Proceedings of the 11th International Conference on Process Control and Simulation, ASRTP’94, September 19-20, 1994, Kosice - Zlata Idka, Slovak republic, pp. 58-66.

[11] Podlubny I.: *Numerical methods of the fractional calculus*. Trans. of the TU Kosice, 1994, No. 1, Riecansky Science Publ. Co., Cambridge (to appear)

[12] L. Dorcak I. Kostial: *Simulation of Fractional-Type Control Systems*, in: 5th International DAAAM Symposium, University of Maribor - Slovenia, 27-29th October 1994, pp. 99-100.
Názov: Numerical Models for the Simulation of the Fractional-Order Control Systems
Autor: Ing. Šubomír Dorčák, CSc.
Zodp. redaktor: RNDr. P. Samuely, CSc.
Vydavateľ: Ústav experimentálnej fyziky SAV, Košice
Redakcia: ÚEF SAV, Watsonova 3, 04001 Košice, Slovenská republika
Počet strán: 12
Náklad: 120
Rok vydania: 1994
Tlač: OLYMPIA s.r.o., Mánesova 23, 040 01 Košice