GENERALIZED SOLUTIONS OF THE VLASOV-POISSON SYSTEM WITH SINGULAR DATA

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ABSTRACT. We study spherically symmetric solutions of the Vlasov-Poisson system in the context of algebras of generalized functions. This allows to model highly concentrated initial configurations and provides a consistent setting for studying singular limits of the system. The proof of unique solvability in our approach depends on new stability properties of the system with respect to perturbations.

1. INTRODUCTION

In kinetic theory one often considers collisionless ensembles of classical particles which interact only by fields which they create collectively. This situation is commonly referred to as the mean field limit of a many particle system. More precisely such ensembles are described by a phase-space distribution function \( f: \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+^+ \), where \( \int_D f(t, x, v) dx dv \) gives the number of particles which at time \( t \) have their position \( x \) and velocity \( v \) in the region \( D \) of phase-space \( \mathbb{R}^6 \). The Vlasov equation expresses the fact that \( f \) is constant along particle paths—which is a direct consequence of the absence of collisions—and reads

\[
\partial_t f(t, x, v) + \partial_x f(t, x, v) v + \partial_v f(t, x, v) F(t, x) = 0,
\]

where \( F \) is some force, which will emerge via some field equation with its source given by the spatial particle density \( \rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v) dv \). In the case of non-relativistic gravitational or electrostatic fields the corresponding system of partial differential equations is the Vlasov-Poisson system, in the case of relativistic electrodynamics it is the Vlasov-Maxwell system and in the case of general relativistic gravity the Vlasov-Einstein system. Such systems have been studied extensively in the literature; for an overview see [5, 16].

The system which is best understood is the Vlasov-Poisson system where \( \rho \) acts as a source term for the Poisson equation. For this system global-in-time classical solutions for general (compactly supported) initial data have been established in [17, 12, 23]; for a review see [18, 19]. The existence-theory for the other systems mentioned above is not equally well understood. For the relativistic Vlasov-Maxwell system global classical solutions are so far only known in special cases (e.g. [7, 6, 20]), while global weak solutions for general data were obtained in [4]. An investigation of the Vlasov-Einstein system was initiated in [21].
It is also interesting to note that the Vlasov-Poisson system possesses as (formal) singular limit cases the Euler-Poisson system with pressure zero and the classical n-body problem, for both of which in general no global-in-time solutions exist. More precisely, in the first case if one considers a phase-space density function which is concentrated in v-space, i.e., \( f(t,x,v) = \rho(t,x)\delta(v-w(t,x)) \), where \( \delta \) denotes the Dirac \( \delta \)-function and \( w \) is a velocity field, then \( f \) formally solves the Vlasov-Poisson system iff \((f, w)\) solves the pressure-less Euler-Poisson system. Similarly a density function \( f \) concentrated in position and velocity space, i.e.,

\[
\begin{align*}
f(t,x,v) &= \sum_{k=1}^{N} \delta(x-x_k(t))\delta(v-v_k(t)),
\end{align*}
\]

formally solves the Vlasov-Poisson system iff \( x_k, v_k \) solve the n-body problem. One main problem here is—of course—the use of distributions in the context of nonlinear equations. Only few rigorous results relating (approximating sequences of) such concentrated solutions of the Vlasov-Poisson system to the solutions of the respective limit systems have been achieved; see [22] for the first case and [13] for the second. The main interest of course would be to use the far better existence-theory in the case of the Vlasov-Poisson system to learn something about the solutions of the related systems, e.g., in the context of shell crossing singularities in the case of the pressure-less Euler-Poisson system. In this work we propose the use of algebras of generalized functions (in the sense of J.F. Colombeau [2, 3]) to study these singular limits of the Vlasov-Poisson system.

As a first step we prove an existence and uniqueness result for singular solutions, i.e., solutions concentrated either in position-space or in momentum-space (or both), to the spherically symmetric Vlasov-Poisson system in a suitable algebra of generalized functions, where the latter provides us with a consistent framework for treating singular i.e., distributional solutions of nonlinear PDEs. The fundamental strategy of solving PDEs with singular initial data in the setting of algebras of generalized functions is regularization of singularities by convolution with a mollifier depending on a regularization parameter \( \varepsilon \) and first solving the equation for fixed \( \varepsilon \) using existence theory in the smooth setting. Proving existence and uniqueness of generalized solutions then amounts to deriving asymptotic estimates with respect to the regularization parameter. This process may alternatively be seen as uncovering new asymptotic stability results of smooth solutions to the system under perturbations of the initial data, which in our view is of independent interest. For a general discussion of applications of Colombeau theory to PDEs see [15]. Recent investigations into linear PDEs in this framework can be found in [9, 10].

We organize our presentation in the following way. In section 2 we collect some well-known facts on the spherically symmetric Vlasov-Poisson system which will be used later on and recall the basic definitions of generalized function algebras in the sense of J.F. Colombeau. Our main results are stated and proved in section 3. Finally we collect some facts on solutions of the Poisson equation in this setting of generalized functions in an appendix.

Although our notation is mostly standard or self-explaining we explicitly mention the following conventions: For a function \( h = h(t,x,v) \) or \( h = h(t,x) \) we denote for given \( t \) by \( h(t) \) the corresponding function of the remaining variables. By \( \| \cdot \|_p \) we denote the usual \( L^p \)-norm for \( p \in [1,\infty] \). The index \( \varepsilon \) in function spaces refers to compactly supported functions. Constants denoted by \( C \) may change their value from line to line but never depend on \( \varepsilon \).
2. Preliminaries

We start by collecting some preliminaries (for a comprehensive presentation including full proofs see [19]) from the existence-theory of the (spherically symmetric) Vlasov-Poisson system, which from now on we shall abbreviate by (VP)

\[
\begin{align*}
\partial_t f + v \partial_x f - \partial_x u \partial_v f &= 0 \\
\Delta u &= 4 \pi \gamma \rho \\
\rho &= \int_{\mathbb{R}^3} f \, dv
\end{align*}
\]

where \( \gamma = \pm 1 \). We suppose the following initial resp. boundary conditions

\[
\begin{align*}
f(0, x, v) &= \hat{f}(x, v) \geq 0 \in C^\infty_c(\mathbb{R}^6) \\
\lim_{|x| \to \infty} u(t, x) &= 0.
\end{align*}
\]

We shall often combine position and velocity into a single variable \( z = (x, v) \) and denote by \( Z(s) = Z(s, t, z) = (X(s, t, z), V(s, t, z)) \) the solutions of the characteristic system of (1),

\[
\begin{align*}
\dot{X}(s) &= V(s) \\
\dot{V}(s) &= -\partial_x u(s, X(s))
\end{align*}
\]

with initial condition \( Z(t, t, z) = z \). The solution of the Vlasov equation is then given by \( f(t, z) = \hat{f}(Z(0, t, z)) \), hence all \( L^p \)-norms \( 1 \leq p \leq \infty \) of \( f \) are constant in time, as is the \( L^1 \)-norm of \( \rho \), i.e., the mass, which will be denoted by \( M \).

It is well known that in case the initial value \( \hat{f} \) of (VP) is spherically symmetric the respective solution \( f(t) \) will also have this property. Moreover, the spatial density \( \rho(t) \) will be spherically symmetric (in the usual sense on \( \mathbb{R}^3 \)—we shall denote it hence by \( \rho(t, r) \), where \( r = |x| \)) and the Poisson equation simplifies to

\[
\Delta u(t, r) = \frac{1}{r^2} \left( r^2 u'(t, r) \right)' = 4 \pi \gamma \rho(t, r).
\]

By a slight abuse of notation, in what follows we will use \( u(t, x) \) and \( u(t, r) \) interchangeably. In addition to the usual key-estimate on the solution of the Poisson equation with compactly supported source term \( \rho(t) \), i.e.,

\[
||\partial_x u(t)||_{\infty} \leq C||\rho(t)||^{1/3} ||\rho(t)||^{2/3}_{\infty}
\]

in the present setup we also obtain the estimates

\[
||\partial_x u(t, r)|| \leq \frac{M}{r^2} \quad \text{and} \quad ||\partial_x^{\alpha+i+j} u(t)||_{\infty} \leq C||\partial_x^{\alpha} \rho(t)||_{\infty} \quad (\alpha \in \mathbb{N}_0^3, \ i, j = 1, 2, 3).
\]

Combining equations (7) and (8) one obtains for all \( r > 0 \)

\[
||\partial_x u(t, r)|| \leq C \min \left( \frac{1}{r^2}, P(t)^2 \right).
\]

Combining equations (7) and (8) we also obtain the estimates

\[
||\partial_x u(t, r)|| \leq \frac{M}{r^2} \quad \text{and} \quad ||\partial_x^{\alpha+i+j} u(t)||_{\infty} \leq C||\partial_x^{\alpha} \rho(t)||_{\infty} \quad (\alpha \in \mathbb{N}_0^3, \ i, j = 1, 2, 3).
\]

Combining equations (7) and (8) one obtains for all \( r > 0 \)

\[
||\partial_x u(t, r)|| \leq C \min \left( \frac{1}{r^2}, P(t)^2 \right).
\]
Note that the latter estimate together with the fact that from \( \xi \in C^2([0, t]) \), \( g \in L^1(\mathbb{R}) \) and \( |\dot{\xi}(s)| \leq g(\xi(s)) \) \( \forall 0 \leq s \leq t \) it follows that \( |\dot{\xi}(t) - \dot{\xi}(0)| \leq 2\sqrt{\mathcal{V}} ||g||_{1/2} \) yields boundedness of \( P(t) \) (cf. [19], proof of Thm. 1.4). Hence global existence of solutions follows by the standard continuation criterion.

We now turn to algebras of generalized functions in the sense of J.F. Colombeau [22]. These are differential algebras containing the vector space of distributions \( D' \) as a subspace and \( C^\infty \) as a subalgebra while displaying maximal consistency with respect to classical analysis according to L. Schwartz’ impossibility result [24]. The main ingredient of the construction is regularization of distributions by nets of smooth functions and asymptotic estimates in terms of the regularization parameter \( \varepsilon \in (0, 1] =: I \), which in our case will be \( L^\infty \)-estimates global in \( z \) on compact time intervals. We shall work within the so-called special version of the theory and use [33] as our main reference. Colombeau algebras are defined as quotients of the spaces of moderate modulo negligible nets \( (u_\varepsilon)_\varepsilon \) in some basic space \( \mathcal{E} \). In the present case we use \( \mathcal{E} = C^\infty(\mathbb{R}^+_0 \times \mathbb{R}^n)^I \) and the following estimates for moderateness and negligibility (where \( O \) denotes the Landau symbol).

\[
\mathcal{E}_M^\mathcal{g}(\mathbb{R}^+_0 \times \mathbb{R}^n) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E} : \forall K \subset \mathbb{R}^+_0 \forall \alpha \in \mathbb{N}_0^{n+1} \exists N \in \mathbb{N} : \\
\sup_{(t,z) \in K \times \mathbb{R}^n} |\partial^\alpha u_\varepsilon(t,z)| = O(\varepsilon^{-N}) \text{ (as } \varepsilon \to 0)\}, \quad (11)
\]

\[
\mathcal{N}_g(\mathbb{R}^+_0 \times \mathbb{R}^n) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E} : \forall K \subset \mathbb{R}^+_0 \forall \alpha \in \mathbb{N}_0^{n+1} \forall m \in \mathbb{N} : \\
\sup_{(t,z) \in K \times \mathbb{R}^n} |\partial^\alpha u_\varepsilon(t,z)| = O(\varepsilon^m) \text{ (as } \varepsilon \to 0)\}, \quad (12)
\]

\[
\mathcal{G}_g(\mathbb{R}^+_0 \times \mathbb{R}^n) := \mathcal{E}_M^\mathcal{g}(\mathbb{R}^+_0 \times \mathbb{R}^n)/\mathcal{N}_g(\mathbb{R}^+_0 \times \mathbb{R}^n).
\]

The index \( \mathcal{g} \) in the above definitions signifies the global estimates with respect to \( x, v \) (contrary to the local estimates w.r.t. \( t \)). Generalized functions shall be denoted by \( u = [(u_\varepsilon)_\varepsilon] \), meaning that \( u \) is the equivalence class of the net \( (u_\varepsilon)_\varepsilon \). In the following section we shall prove existence and uniqueness results for the spherically symmetric (VP)-system in this setting. By a solution of a differential equation in a Colombeau algebra \( \mathcal{G}_g \) we mean an element \( [(u_\varepsilon)_\varepsilon] \) of the algebra such that each \( u_\varepsilon \) solves the equations up to an element of the ideal \( \mathcal{N}_g \). Roughly, establishing solvability of a PDE in this setting therefore amounts to obtaining a classical solution \( u_\varepsilon \) for each \( \varepsilon \) and proving moderateness of the resulting net \( (u_\varepsilon)_\varepsilon \). Proving uniqueness amounts to showing that any two nets solving the equation up to an element of the ideal (with initial data differing by an element of the respective ideal) necessarily belong to the same equivalence class in \( \mathcal{G}_g \) (cf., e.g., [15]).

We note the following fact which will be essential in the uniqueness-part of the proof of our main result (and follows by an easy adaptation of the proof of [8, Thm. 1.2.3]): \( u = [(u_\varepsilon)_\varepsilon] \in \mathcal{E}_M^\mathcal{g}(\mathbb{R}^+_0 \times \mathbb{R}^n) \) is negligible iff

\[
\forall K \subset \mathbb{R}^+_0 \forall m \in \mathbb{N} : \sup_{(t,z) \in K \times \mathbb{R}^n} |u_\varepsilon(t,z)| = O(\varepsilon^m).
\]

We shall also need a suitable algebra of generalized functions containing the initial data \( \tilde{f} \). To this end we consider nets \( (u_\varepsilon)_\varepsilon \in C^\infty(\mathbb{R}^n)^I \) satisfying estimates of the form

\[
\forall \alpha \in \mathbb{N}_0^n \exists N \in \mathbb{N} (\text{resp. } \forall m \in \mathbb{N}) : \sup_{x \in \mathbb{R}^n} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ (resp. } O(\varepsilon^m)).
\]
We denote the respective spaces by $E^g_q$, $N_q$ and $G_g$ (cf. [1],[14]). A function $u$ in $G_g(\mathbb{R}^n)$ is called compactly supported if there exists a representative $(u_\varepsilon)_\varepsilon$ of $u$ and a compact set $L$ containing the supports of all $u_\varepsilon$. In this case we call the representative $(u_\varepsilon)_\varepsilon$ compactly supported. Note however, that since $G_g$ is not a sheaf there is no well-defined notion of support for its elements (see Example [A.4] below).

The space $D'_L(\mathbb{R}^n)$ of bounded distributions (distributional derivatives of bounded functions) can be embedded into $G_g(\mathbb{R}^n)$ by the map

$$w \mapsto [(w \ast \varphi_\varepsilon)_\varepsilon]$$

where $\varphi$ is a rapidly decreasing function with unit integral and all higher order moments vanishing, and $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$. This embedding commutes with partial derivatives. Analogously, $C^\infty(\mathbb{R}_0^+, D'_L(\mathbb{R}^n))$ can be embedded into $G_g(\mathbb{R}_0^+ \times \mathbb{R}^n)$ via convolution.

3. Generalized solutions of the Vlasov-Poisson system

In this section we will state and prove our main results, providing existence and uniqueness of generalized solutions of the spherically symmetric Vlasov-Poisson system. We begin with a discussion of the relevant symmetry properties.

We will call a generalized function $g \in G_g(\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3)$ spherically symmetric if it possesses a representative $(g_\varepsilon)_\varepsilon$ that is spherically symmetric in the sense of [10] for all $\varepsilon$. Likewise we call a function $g \in G_g(\mathbb{R}^3)$ spherically symmetric if it possesses a representative $(g_\varepsilon)_\varepsilon$ that for fixed $\varepsilon$ is spherically symmetric in the usual sense.

The following definition singles out classes of scales which can be used to measure the ‘maximal degree of divergence’ admissible in the initial data of (VP) to allow for unique solvability in the Colombeau algebra:

**Definition 3.1.** Let $p > 0$.

(i) $B^1_{\Sigma_p}$ we denote the space of all scales $\sigma : I \to I$ satisfying $\sigma(\varepsilon) \to 0$ for $\varepsilon \to 0$ and

$$\sigma(\varepsilon)^{-1} = O(\|\log(\varepsilon)\|^{1/p}) \quad (\varepsilon \to 0).$$

(ii) $B^2_{\Sigma_p}$ we denote the space of all scales $\sigma : I \to I$ satisfying $\sigma(\varepsilon) \to 0$ for $\varepsilon \to 0$ and

$$\forall C > 0 \exp(C\sigma(\varepsilon)^{-p}) = O(\|\log(\varepsilon)\|) \quad (\varepsilon \to 0).$$

Then $\Sigma_p^{(i)} \subseteq \Sigma_q^{(i)}$ for $p \geq q$ and $i = 1, 2$. Note that using any scaling $\sigma$ satisfying $\sigma(\varepsilon) \to 0$ for $\varepsilon \to 0$, a $\delta$-source can be viewed as the element $[(\varphi_{\sigma(\varepsilon)})_\varepsilon]$ of the Colombeau algebra. Since obviously $\varphi_{\sigma(\varepsilon)} \to \delta$ in $D'$ as $\varepsilon \to 0$, any such delta net is associated to the standard image $[(\varphi_\varepsilon)_\varepsilon]$ of the Dirac measure, hence macroscopically indistinguishable from it (cf. [3],[15],[8] for discussions of the concept of association and its effects on nonlinear modelling of singularities). After these preparations we may state our existence result.

**Theorem 3.2.** (Existence of generalized solutions)

Let $f^\varepsilon \in G_g(\mathbb{R}^3)$ with a spherically symmetric, non-negative and compactly supported representative $(f_\varepsilon)_\varepsilon$ satisfying

(i) $||f_\varepsilon||_1 = M$ (the mass) for all $\varepsilon$, and
(ii) there exists some $\sigma \in \Sigma^{(1)}_2$ such that $\|\hat{f}_\varepsilon\|_\infty \leq \frac{C}{\sigma(\varepsilon)}$.

Then there exists a solution $(f, u)$ of (VP) in $G_6(\mathbb{R}^+_0 \times \mathbb{R}^6) \times G_6(\mathbb{R}^+_0 \times \mathbb{R}^3)$ with $f(0, x, v) = \hat{f}(x, v)$ and $u$ vanishing at infinity (in the sense of Definition A.2).

Uniqueness of generalized solutions needs stronger assumptions on the data. We present two results; the first one requires a $\Sigma^{(2)}_2$-scale.

**Theorem 3.3. (Uniqueness of generalized solutions in $G_6^0$)**

Let the assumptions of Theorem 3.2 be satisfied but strengthen (ii) to

(ii') there exists some $\sigma \in \Sigma^{(2)}_2$ such that $\|\hat{f}_\varepsilon\|_\infty \leq \frac{C}{\sigma(\varepsilon)}$.

Then $(f, u)$ given in Theorem 3.2 is the unique solution of (VP) with $f(0, x, v) = \hat{f}(x, v)$, velocity support of $f$ bounded as in Lemma 3.5 (ii) and $u$ vanishing at infinity (in the sense of Definition A.3).

If we adjust the algebra to the symmetry of our problem we can do without a $\Sigma^{(2)}_p$-scale. More precisely we change the basic space $E$ in the definitions of $E_M$, $N_6$ and $G_6$, respectively to $E^\circ := \{(u_\varepsilon)_\varepsilon \in E = C^\infty(\mathbb{R}^+_0 \times \mathbb{R}^n)^I \mid \forall t \in \mathbb{R}^+_0 \forall \varepsilon \in I : u_\varepsilon(t)$ is spherically symmetric $\}$, where in case $n = 6$ spherical symmetry is to be understood in the sense of (6) and in case $n = 3$ in the usual sense. We denote the resulting algebra by $G_6^0$. Likewise in case of the algebra $G_6$ we take nets $(u_\varepsilon)_\varepsilon \in C^\infty(\mathbb{R}^n)^I$ such that $u_\varepsilon$ is spherically symmetric, again in the respective senses for $n = 6$ and $n = 3$. The resulting algebra is denoted by $G_6^0$. Now we may state.

**Theorem 3.4. (Uniqueness of generalized solutions in $G_6^0$)**

Let $\hat{f} \in G_6^0(\mathbb{R}^6)$ with a compactly supported and non-negative representative $(\hat{f}_\varepsilon)_\varepsilon$ satisfying (i) in Theorem 3.3 and

(ii') there exists some $\sigma \in \Sigma^{(1)}_2$ such that $\|\partial^\alpha_\varepsilon \hat{f}_\varepsilon\|_\infty \leq \frac{C}{\sigma(\varepsilon)^{1+|\alpha|}}$ for $|\alpha| \leq 1$.

Then there exists a unique solution $(f, u)$ of (VP) in $G_6^0(\mathbb{R}^+_0 \times \mathbb{R}^6) \times G_6^0(\mathbb{R}^+_0 \times \mathbb{R}^3)$ with $f(0, x, v) = \hat{f}(x, v)$, velocity support of $f$ bounded as in Lemma 3.5 (ii) and $u$ vanishing at infinity.

Note that the assumptions in the above theorems in particular allow to model concentrated data which lead to the singular limits of the Vlasov-Poisson system described in the introduction.

To prepare the proof of Theorem 3.4 first note that for fixed $\varepsilon$ the classical theory provides us with a unique solution $(f_\varepsilon, u_\varepsilon)$ in $C^\infty(\mathbb{R}^+_0, \mathbb{R}^6) \times C^\infty(\mathbb{R}^+_0 \times \mathbb{R}^3)$ with initial data $f_\varepsilon(0, z) = \hat{f}_\varepsilon(z)$ and $u_\varepsilon(t) \to 0$ as $|x| \to \infty$. Moreover, the solution will inherit the symmetry property of the data, that is $f_\varepsilon(t)$, $\rho_\varepsilon(t)$ as well as $u_\varepsilon(t)$ will be spherically symmetric.

To prove the existence of generalized solutions we have to verify the moderateness estimates in (11). We split this task into two Lemmas collecting the necessary estimates.

**Lemma 3.5. (Zero order estimates)** There exists $C > 0$ such that for $\varepsilon$ sufficiently small we have for all $t \in \mathbb{R}^+$

...
(i) \[ \| f_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)}, \quad \| f_\varepsilon(t) \|_1 = \| \rho_\varepsilon(t) \|_1 = M \]

(ii) \[ P_\varepsilon(t) \leq \frac{C}{\sigma(\varepsilon)^{3/2}} \]

(iii) \[ \| u_\varepsilon(t) \|_\infty, \quad \| \partial_x u_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)^{3/2}} \]

(iv) \[ \| \rho_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)^2} \]

Moreover, for any \( T > 0 \) and \( \varepsilon \) sufficiently small

(v) \[ \sup_{t \in [0,T]} \| Z_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)^{3/2}}. \]

**Proof.** (i) The \( L_\infty \)-estimate follows easily since for \( \varepsilon \) fixed by the smooth result (see Section 2) we have \( f_\varepsilon(t, z) = \hat{f}_\varepsilon(Z_\varepsilon(0, s, z)) \). The \( L^1 \)-estimates are immediate from assumption (i) in the theorem.

(ii) We conclude from equation (10) and (i)

\[ \| \partial_x u_\varepsilon(t) \|_\infty \leq \frac{C}{\| \rho_\varepsilon(t) \|_1^{1/3}} \| \rho_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)^{3/2}}. \] (13)

We set

\[ g_\varepsilon(t, r) := \min \left\{ \frac{1}{r^2}, \left( \frac{P_\varepsilon(t)}{\sigma(\varepsilon)^{1/3}} \right)^2 \right\} \]

Note that \( g_\varepsilon(s, r) \leq g_\varepsilon(t, r) \) for \( s \leq t \) since \( P_\varepsilon \) is monotonically increasing. Then combining \( \| \partial_x u_\varepsilon(t, r) \| \leq M/r^2 \) with the above estimate we obtain from the characteristic equation

\[ |\hat{X}_\varepsilon^i(s)| = |\partial_x u_\varepsilon(s, \hat{X}_\varepsilon(s))| \leq C g_\varepsilon(t, |X^i_\varepsilon(s)|) \]

for \( s \leq t \) and \( 1 \leq i \leq 3 \). Therefore by the standard argument mentioned below equation (10) we obtain

\[ |\hat{X}_\varepsilon^i(t) - \hat{X}_\varepsilon^i(0)| \leq 2C \sqrt{2} \| g_\varepsilon(t) \|_1^{1/3} \]

and are left with calculating the \( L^1 \)-norm of \( g_\varepsilon(t) \). We have

\[
\int_{-\infty}^{\infty} |g_\varepsilon(t, r)| dr = 2 \int_{0}^{\infty} |g_\varepsilon(t, r)| dr \leq 2C \int_{0}^{\frac{\sigma(\varepsilon)^{1/3}}{P_\varepsilon(t)}} 1 dr + 2C \int_{\frac{\sigma(\varepsilon)^{1/3}}{P_\varepsilon(t)}}^{\infty} \frac{1}{r^2} dr
\]

\[ = 2C \frac{P_\varepsilon(t)}{\sigma(\varepsilon)^{3/2}} + 2C \frac{P_\varepsilon(t)}{\sigma(\varepsilon)^{3/2}} \leq C \frac{P_\varepsilon(t)}{\sigma(\varepsilon)^{3/2}} \]

Thus we obtain

\[ |\hat{X}_\varepsilon^i(t) - \hat{X}_\varepsilon^i(0)| \leq C \frac{P_\varepsilon(t)^{1/2}}{\sigma(\varepsilon)^{3/2}} \] (14)

and hence from the definition of \( P_\varepsilon \)

\[ P_\varepsilon(t) \leq \hat{P} + CP_\varepsilon(t)^{3/2} \sigma(\varepsilon)^{-1/2}, \]

where \( \hat{P} \) bounds the diameter of the support of \( \hat{f}_\varepsilon \). This in turn implies that \( P_\varepsilon(t) \) is bounded independent of \( t \) for \( \varepsilon \) fixed, and that \( P_\varepsilon(t) \leq \frac{C}{\sigma(\varepsilon)^{3/2}} \), which together with (14) gives (ii) and (v).
We insert (ii) into (13) to prove (iii), i.e.,

$$
\| \partial_x u_\varepsilon(t) \|_\infty \leq \frac{C}{\sigma(\varepsilon)^2}.
$$

The estimate on $u_\varepsilon(t)$ now follows easily by integration (taking into account that $|u_\varepsilon(t, x)| = O(1/|x|)$), while for (iv) we note

$$
\| \rho_\varepsilon(t) \|_\infty \leq C|f|^3 \leq \frac{C}{\sigma(\varepsilon)^2}.
$$

\[ \Box \]

**Lemma 3.6.** (Higher order x, v-estimates) For all $\alpha \in \mathbb{N}_0^n$, all $\beta \in \mathbb{N}_0^n$ and all $T > 0$ there exists $C > 0$ such that for $\varepsilon$ sufficiently small and all $t \in [0, T]$ we have:

(i) $\| \partial^2_x f_\varepsilon(t) \|_\infty \leq e^{C \sigma(\varepsilon)^{-2}}$

(ii) $\| \partial^2_x \rho_\varepsilon(t) \|_\infty \leq e^{C \sigma(\varepsilon)^{-2}}$

(iii) $\| \partial^2_x Z_\varepsilon(t) \|_\infty \leq e^{C \sigma(\varepsilon)^{-2}}$

(iv) $\| \partial^2_x + \varepsilon_i \eta_\varepsilon(t) \|_\infty \leq e^{C \sigma(\varepsilon)^{-2}} \forall i, j.$

Note that compared with the zeroth order estimates we have to use an exponential term in $\sigma$ to bound the respective expressions necessitating the use of the scale $\sigma$ in condition 2 in Theorem 3.2. However, this term, i.e., $\exp(\sigma(\varepsilon)^{-2})$ suffices to bound derivatives of any order. In particular, higher order derivatives do not lead to higher order exponential terms which would cause our approach to fail.

**Proof.** We prove the Lemma by induction on $|\alpha|$ and $|\beta|$. In the case $|\alpha| = |\beta| = 0$ we have shown even stronger estimates on $f_\varepsilon(t)$, $Z_\varepsilon(t)$ and $\rho_\varepsilon(t)$ already in Lemma 3.3 The only remaining estimate is the one on $\partial^2_x u_\varepsilon(t)$ which follows from $\| \partial^2_x u_\varepsilon(t) \|_\infty \leq C |\rho_\varepsilon(t)\|_\infty \leq C \sigma(\varepsilon)^{-2}$.

To carry out the inductive step we assume the Lemma holds for $|\alpha|, |\beta| \leq n$. We have to infer the respective estimates for $|\alpha| = |\beta| = n + 1$. We define

$$
\xi_\varepsilon^{(\alpha)}(s) := \partial^2_x X_\varepsilon(s), \quad \eta_\varepsilon^{(\alpha)}(s) := \partial^2_x V_\varepsilon(s).
$$

Using the characteristic system we obtain (for suitably chosen $i$)

$$
\dot{\xi}_\varepsilon^{(\alpha)}(s) = \frac{d}{ds} \partial^2_x X_\varepsilon(s) = \partial^2_x V_\varepsilon(s) = \eta_\varepsilon^{(\alpha)}(s)
$$

$$
\dot{\eta}_\varepsilon^{(\alpha)}(s) = \frac{d}{ds} \partial^2_x V_\varepsilon(s) = -\partial^2_x \left( \partial_x u_\varepsilon(s, X_\varepsilon(s, t, z)) \right)
$$

$$
= -\partial^2_x (\partial_x u_\varepsilon(s, X_\varepsilon(s, t, z))) \partial_x X_\varepsilon(s, t, z)
$$

$$
= -\partial^2_x \eta_\varepsilon^{(\alpha)}(s, X_\varepsilon(s)) \xi_\varepsilon^{(\alpha)}(s)
$$

$$
- \sum_{0 < \gamma \leq \alpha - \varepsilon_i} \left( C^{\alpha - \varepsilon_i} \gamma \right) \partial^2_x \eta_\varepsilon^{(\alpha)}(s, X_\varepsilon(s)) \partial^2_x \eta_\varepsilon^{(\alpha)}(s, X_\varepsilon(s))
$$

The last expression is a sum of products of terms of the form

$$
\partial^2_x u_\varepsilon(s, X_\varepsilon(s)) \text{ with } |\delta| \leq n + 2
$$

for which we have $\| \partial^2_x u_\varepsilon(t) \|_\infty \leq C |\partial^2_x \rho_\varepsilon(t)\|_\infty \leq \exp(C \sigma(\varepsilon)^{-2})$ by (9) and the induction hypothesis since $|\delta'| \leq n$, and

$$(\partial^2_x X_\varepsilon)^{\nu'} (\partial^2_x X_\varepsilon)^{\omega'} \text{ with } \max(|\nu|, |\omega|) \leq n$$
which by induction hypothesis is also bounded by \(\exp(C\sigma(\varepsilon)^{-2})\) on compact time intervals.

So we find using (11) for \(|\alpha| = 0\) and Lemma 3.5 (iv)

\[
|\eta^\alpha_{\varepsilon}(s)| \leq |\partial^2_x u_\varepsilon(s, X_\varepsilon(s))| |\xi^\alpha_{\varepsilon}(s)| + e^{C\sigma(\varepsilon)^{-2}} \leq C\sigma(\varepsilon)^{-2}|\xi^\alpha_{\varepsilon}(s)| + e^{C\sigma(\varepsilon)^{-2}}.
\]

Hence summing up we obtain

\[
|\eta^\alpha_{\varepsilon}(s)| + |\xi^\alpha_{\varepsilon}(s)| \leq e^{C\sigma(\varepsilon)^{-2}} + C\sigma(\varepsilon)^{-2}(|\eta^\alpha_{\varepsilon}(s)| + |\xi^\alpha_{\varepsilon}(s)|),
\]

which by Gronwall’s Lemma gives

\[
|\eta^\alpha_{\varepsilon}(s)|, |\xi^\alpha_{\varepsilon}(s)| \leq e^{C\sigma(\varepsilon)^{-2}},
\]

i.e., \(\partial^\alpha_x Z_\varepsilon(s) \leq \exp(C\sigma(\varepsilon)^{-2})\) on \([0, T]\) for all \(|\alpha| = n + 1\), which is (iii).

From here we obtain

\[
||\partial^\alpha_x f_{\varepsilon}(t)||_\infty \leq Ce^{C\sigma(\varepsilon)^{-2}} \text{ for all } |\alpha| = n + 1
\]

since \(\partial^\alpha_x \tilde{f}_{\varepsilon}(t, Z_\varepsilon(0, t, z)))\) is a sum of products of certain \(\partial^\alpha_x \tilde{f}_{\varepsilon}(Z_\varepsilon(0, t, z))\) with products of powers of derivatives of \(Z_\varepsilon(0, t, z, \cdot)\), and we can use (iii) and the moderateness of \(\tilde{f}_{\varepsilon}\). Thereby we have also shown (i).

Item (ii) is now obvious using Lemma 3.5 (ii). Finally, to prove (iv) we combine (ii) with (11).

\(\square\)

**Proof Theorem 3.3**

By the estimates of Lemma 3.6 and Lemma 3.1 above we obtain the necessary bounds on \(\sup_{(t, z) \in K \times \mathbb{R}^6} |\partial^\alpha_x f_{\varepsilon}(t, z)|\) and \(\sup_{(t, z) \in K \times \mathbb{R}^3} |\partial^\alpha_x u_\varepsilon(t, z)|\), where \(K\) is a compact subset of \([0, \infty)\).

To obtain the estimates on \(\partial_t f_{\varepsilon}\) we plug the estimates established so far into the Vlasov equation (using the bounded velocity support of \(f_{\varepsilon}(t)\)). From here the estimate on \(\partial_t \rho_{\varepsilon}\) and hence on \(\partial_t u_{\varepsilon}\) follows. Now differentiating the Vlasov equation we obtain the estimates on terms of the form \(\partial_t \partial^\alpha_x f_{\varepsilon}\). Higher order \(\partial_t\) and mixed \((t, z)\)-estimates of \(f_{\varepsilon}\) are obtained by successively differentiating Vlasov’s equation and in turn imply the respective estimates on \(u_{\varepsilon}\).

Moreover \(u(t) = [(u_{\varepsilon}(t))_\varepsilon]\) is vanishing at infinity in the sense of Definition A.5 since the support of \(\Delta u_{\varepsilon}(t)\) is bounded by \(C + t P_{\varepsilon}(t)\) and \(P_{\varepsilon}(t) \leq C\sigma(\varepsilon)^{-1/3}\) by Lemma 3.5 (ii).

This proves existence of solutions in \(\mathcal{G}_\varepsilon(\mathbb{R}^+ \times \mathbb{R}^6) \times \mathcal{G}_\varepsilon(\mathbb{R}^+ \times \mathbb{R}^3)\) with \(f(0, x, v) = \tilde{f}(x, v)\) and \(u(t)\) vanishing at infinity.

\(\square\)

**Proof Theorem 3.2**

We have to prove uniqueness of the solution obtained above. So assume \((f = [(f_{\varepsilon})_\varepsilon], u = [(u_{\varepsilon})_\varepsilon])\) is a solution as constructed above and let \((\bar{f} = [(f_{\varepsilon})_\varepsilon], \bar{u} = [(ar{u}_{\varepsilon})_\varepsilon])\) be another solution of the (VP) system with the same initial data (i.e., \(\bar{f}(0) = \tilde{f}\), \(\bar{u}\) vanishing at infinity in the sense of Definition A.5 (with distinguished representative \(\bar{u}_{\varepsilon}\), and \(\varepsilon\)-wise bounded (by \(P_{\varepsilon}(t)\), satisfying (ii) of Lemma 3.5) velocity support of \(\tilde{f}_{\varepsilon}(t)\)). Proving uniqueness in our setting amounts to establishing
that the differences \( f_\varepsilon - \tilde{f}_\varepsilon \) and \( u_\varepsilon - \tilde{u}_\varepsilon \) lie in the respective ideals. We have
\[
\partial_t \tilde{f}_\varepsilon + v \partial_x \tilde{f}_\varepsilon - \partial_x \tilde{u}_\varepsilon \partial_t \tilde{f}_\varepsilon = n_\varepsilon
\]
\[
\Delta \tilde{u}_\varepsilon = 4\pi \gamma \int_{\mathbb{R}^3} \tilde{f}_\varepsilon \, dv + n_\varepsilon
\]
(16)
\[
\tilde{f}_\varepsilon(0) = \tilde{f}_\varepsilon^0 + n_\varepsilon =: \tilde{f}_\varepsilon^0,
\]
where \((n_\varepsilon)_\varepsilon\) denotes a "generic" (analogous to the "generic" constant \(C\)) element of the ideal which may denote different negligible quantities in each equation. Denoting by \(\tilde{Z}_\varepsilon\) the characteristics of the above inhomogeneous Vlasov equation we obtain (cf. e.g., [11], appendix A):
\[
\tilde{f}_\varepsilon(t, z) = \tilde{f}_\varepsilon^0(\tilde{Z}_\varepsilon(0, t, z)) + \int_0^t n_\varepsilon(s, \tilde{Z}_\varepsilon(s, t, z)) \, ds
\]
\[
= \tilde{f}_\varepsilon^0(\tilde{Z}_\varepsilon(0, t, z)) + n_\varepsilon(t, z) = \tilde{f}_\varepsilon^0(\tilde{Z}_\varepsilon(0, t, z)) + n_\varepsilon(t, z).
\]
(17)
Consequently we may estimate the difference in the distribution functions using Lemma 3.6 (i)
\[
|f_\varepsilon(t, z) - \tilde{f}_\varepsilon(t, z)| \leq ||\partial_2 \tilde{f}_\varepsilon||_{\infty} |Z_\varepsilon(0, t, z) - \tilde{Z}_\varepsilon(0, t, z)| + |n_\varepsilon(t, z)|
\]
\[
\leq e^{C\sigma(c)^{-2}} |Z_\varepsilon(0, t, z) - \tilde{Z}_\varepsilon(0, t, z)| + |n_\varepsilon(t, z)|.
\]
(18)
For the characteristics we obtain (for \(0 \leq s \leq t\))
\[
|X_\varepsilon(s) - \tilde{X}_\varepsilon(s)| \leq \int_0^t |V_\varepsilon(s') - \tilde{V}_\varepsilon(s')| \, ds'
\]
\[
|V_\varepsilon(s) - \tilde{V}_\varepsilon(s)| \leq \int_0^t |\partial_x u_\varepsilon(s', X_\varepsilon(s')) - \partial_x \tilde{u}_\varepsilon(s', \tilde{X}_\varepsilon(s'))| \, ds'
\]
\[
\leq \int_0^t (|\partial_x u_\varepsilon(s', X_\varepsilon(s')) - \partial_x u_\varepsilon(s', \tilde{X}_\varepsilon(s'))|
\]
\[
+|\partial_x \tilde{u}_\varepsilon(s', \tilde{X}_\varepsilon(s')) - \partial_x \tilde{u}_\varepsilon(s', \tilde{X}_\varepsilon(s'))|) \, ds'
\]
\[
\leq \sup_{s \leq s' \leq t} ||\partial_2^2 u_\varepsilon(s')||_{\infty} \int_0^t |X_\varepsilon(s') - \tilde{X}_\varepsilon(s')| \, ds'
\]
\[
+ \int_0^t |\partial_x u_\varepsilon(s', \tilde{X}_\varepsilon(s')) - \partial_x \tilde{u}_\varepsilon(s', \tilde{X}_\varepsilon(s'))| \, ds.
\]
(19)
Now we turn to the perturbed Poisson equation for \(\tilde{u}_\varepsilon\), i.e.,
\[
\Delta \tilde{u}_\varepsilon(t, x) = 4\pi \gamma \rho_\varepsilon(t, x) + n_\varepsilon(t, x).
\]
(20)
Since \(\tilde{u}\) is strongly vanishing at infinity the right hand side in the above equation has its support in \(B_{\varepsilon^{-N}}(0)\) for some \(N \geq 0\). Furthermore by assumption \(\tilde{f}_\varepsilon(t)\) has its \(v\)-support contained in some \(B_{\tilde{P}_\varepsilon(t)}(0)\) and hence its \(x\)-support bounded by \(\tilde{R} + \int_0^t \tilde{P}_\varepsilon(s) \, ds\), where \(\tilde{R}\) bounds the \(x\)-support of \(\tilde{f}_\varepsilon(0)\). This implies that the
support of $\tilde{f}_\varepsilon(t)$ is bounded by some $B_{Q_\varepsilon(t)}$ with $\tilde{Q}_\varepsilon(t) \leq C(t)\tilde{P}_\varepsilon(t)$ where $C(t)$ depends linearly on time.

As a consequence $n_\varepsilon(x)$ in equation (20) above has its support also contained in some $B_{\varepsilon-N}(0)$. Therefore we may define $\tilde{u}_\varepsilon(t, x) := \int n_\varepsilon(t, y)/|x-y| \, dy$, which is clearly in the ideal and finally we have found a representative $\tilde{u}_\varepsilon := \tilde{u}_\varepsilon - \tilde{u}_\varepsilon$ of $[(\tilde{u}_\varepsilon)_\varepsilon]$ that satisfies the non-perturbed Poisson equation with source $\tilde{\rho}_\varepsilon$, i.e., $\Delta \tilde{u}_\varepsilon = 4\pi\gamma \tilde{\rho}_\varepsilon$.

This in turn implies $\Delta(u_\varepsilon - \tilde{u}_\varepsilon) = 4\pi\gamma(\rho_\varepsilon - \tilde{\rho}_\varepsilon)$ and using (7) we write

$$
||\partial_x(u_\varepsilon - \tilde{u}_\varepsilon)(t)||_\infty \\
\leq C \int_{\mathbb{R}^3} (f_\varepsilon(t, u) - \tilde{f}_\varepsilon(t, u)) \, du ||_{1/3} \int_{\mathbb{R}^3} (f_\varepsilon(t, u) - \tilde{f}_\varepsilon(t, u)) \, du ||_{2/3}.
$$

On estimating the $L^1$-norm above we use $\tilde{Q}_\varepsilon(t) := \max(Q_\varepsilon(t), \tilde{Q}_\varepsilon(t))$, where $Q_\varepsilon(t)$ denotes the respective bound on the support of $f_\varepsilon(t)$ and write

$$
||\int_{\mathbb{R}^3} (f_\varepsilon(t, u) - \tilde{f}_\varepsilon(t, u)) \, du ||_{1/3} \leq \int_{B_{Q_\varepsilon(t)}} |f_\varepsilon(t, z) - \tilde{f}_\varepsilon(t, z)| \, dz \\
\leq C Q_\varepsilon(t)^6 ||f_\varepsilon(t) - \tilde{f}_\varepsilon(t)||_\infty \\
\leq C ||f_\varepsilon(t) - \tilde{f}_\varepsilon(t)||_\infty,
$$

for $t \in [0, T]$, where we have used Lemma 3.5 (ii) in the last step. So we find

$$
||\partial_x(u_\varepsilon - \tilde{u}_\varepsilon)(t)||_\infty \leq \frac{C}{\sigma(\varepsilon)^{4/3}} ||f_\varepsilon(t) - \tilde{f}_\varepsilon(t)||_\infty.
$$

Now applying Gronwall’s lemma to (19) we obtain for all $q$

$$
|Z_\varepsilon(s) - \tilde{Z}_\varepsilon(s)| \leq C \varepsilon^{\frac{4}{3}q} \sup_{s \leq t} ||\partial^2_x u_\varepsilon(s')||_\infty \int_s^t ||\partial_x u_\varepsilon(s') - \partial_x \tilde{u}_\varepsilon(s')||_\infty ds' \\
\leq C \varepsilon^{\frac{4}{3}q} \int_s^t ||\partial_x u_\varepsilon(s') - \partial_x \tilde{u}_\varepsilon(s')||_\infty ds' + \int_s^t ||n_\varepsilon(s')||_\infty ds' \\
\leq C \varepsilon^{\frac{4}{3}q} \left( \frac{1}{\sigma(\varepsilon)^{4/3}} \int_s^t ||f_\varepsilon(s') - \tilde{f}_\varepsilon(s')||_\infty ds' + \varepsilon^q \right),
$$

where we have again used Lemma 3.5 and (21) above. Finally we combine (18) with (22) and use Gronwall’s lemma for the second time to obtain for all $q$

$$
\sup_{t \in [0, T]} ||f_\varepsilon(t) - \tilde{f}_\varepsilon(t)||_\infty \leq \exp (\sigma^{-4/3} e^{C\sigma(\varepsilon)^{-2}}) \varepsilon^q,
$$

and due to our assumptions on the scale we see that the difference of the distribution functions is in the ideal. From here the respective estimates on the difference of the spatial densities and on $||u_\varepsilon(t) - \tilde{u}_\varepsilon(t)||_\infty$ follow easily. □

**Proof of Theorem 3.4.**

As for existence just observe that (iii”) implies (ii) and that by classical theory the solution inherits the respective symmetry properties of the data.

To prove uniqueness we assume that $(\tilde{f} = [(f_\varepsilon)_\varepsilon], \tilde{u} = [(u_\varepsilon)_\varepsilon]) \in G_\varepsilon^\gamma(\mathbb{R}_0^+ \times \mathbb{R}^d \times \mathbb{R}^d) \times G_\varepsilon^\gamma(\mathbb{R}_0^+ \times \mathbb{R}^3)$ is a solution as constructed above and $(\tilde{f} = [(\tilde{f}_\varepsilon)_\varepsilon], \tilde{u} = [(\tilde{u}_\varepsilon)_\varepsilon])$
is another such solution with the same initial data, \( \tilde{u} \) vanishing at infinity (with distinguished representative \( (\bar{u}, \bar{v}) \)) and the velocity support of \( \bar{f}(t) \) bounded by \( \bar{P}_e(t) \) satisfying (ii) of Lemma 3.3. We follow the proof of Theorem 3.3 up to estimate (19) but now using spherical symmetry we provide a stronger estimate on

\[
\int_{s}^{t} |\partial_x u_x(s', \bar{X}_x(s')) - \partial_x \bar{u}_x(s', \bar{X}_x(s'))| \, ds' \\
\leq 4\pi \int_{s}^{t} \left| \frac{\bar{X}_x(s')}{\bar{r}_x(s')} \right| \int_{0}^{\bar{r}_x(s')} \bar{p}_x(s', l) - \tilde{p}_x(s', l) \, dl \, ds', \tag{23}
\]

where \( \bar{r}_x \) denotes the modulus of \( \bar{X}_x \). Note that this formula does not hold unless we use the algebra \( \mathcal{O} \) since in general \( \tilde{p}_x(t) \) will not be spherically symmetric due to the non-symmetric perturbations in (19). Estimating the difference of the spatial densities we find using \( \tilde{P}_e(t) = \max(\tilde{P}_e(t), P_e(t)) \) as well as (17)

\[
|\rho_x(t, r) - \tilde{\rho}_x(t, r)| \leq \int_{B_{\tilde{P}_e(t)}} |\tilde{f}(t, r, v) - \bar{f}(t, r, v)| \, dv \\
= \int_{B_{\tilde{P}_e(t)}} |\tilde{f}(Z_x(0, t, z)) - \bar{f}(\bar{Z}_x(0, t, z))| \, dv + |n_x(t, r)| \\
\leq \frac{C}{\sigma(\varepsilon)} ||\partial_x \tilde{f}_x||_{\infty} ||Z_x(0, t, .) - \bar{Z}_x(0, t, .)||_{\infty} + |n_x(t, r)|.
\]

Inserting this into (23) we obtain using Lemma 3.5

\[
\int_{s}^{t} |\partial_x u_x(s', \bar{X}_x(s')) - \partial_x \bar{u}_x(s', \bar{X}_x(s'))| \, ds \\
\leq \frac{C}{\sigma(\varepsilon)} ||\partial_x \tilde{f}_x||_{\infty} \sup_{s \leq s' \leq t} ||\bar{X}_x(s')||_{\infty} \int_{s}^{t} (||Z_x(0, s', .) - \bar{Z}_x(0, s', .)||_{\infty} + |n_x(s', r)|) \, ds' \\
\leq C\sigma(\varepsilon)^{-\frac{q}{4p}} \int_{s}^{t} ||Z_x(0, s', .) - \bar{Z}_x(0, s', .)||_{\infty} \, ds' + C\varepsilon^q
\]

for \( t \in [0, T] \) and all \( q \). Now combining this with (19) we obtain

\[
||Z_x(0, s, .) - \bar{Z}_x(0, s, .)||_{\infty} \leq C\sigma(\varepsilon)^{-\frac{q}{4p}} \int_{s}^{t} ||Z_x(0, s', .) - \bar{Z}_x(0, s', .)||_{\infty} \, ds' + C\varepsilon^q,
\]

which by Gronwall’s lemma gives

\[
\sup_{t \in [0, T]} ||Z_x(0, t, .) - \bar{Z}_x(0, t, .)||_{\infty} \leq \varepsilon^q e^{C\sigma(\varepsilon)^{-\frac{q}{4p}}}.\]
Hence by our assumption on the scale the difference of the characteristics is negligible. By (17) this immediately implies that \([f_\varepsilon] = [\hat{f}_\varepsilon]\) and so the same holds true for the spatial density as well as for the potential. \qed

### Appendix A. Uniqueness for Generalized Solutions of the Poisson Equation

In this appendix we collect some facts on the Poisson equation within the framework of nonlinear generalized functions. We focus on the question of uniqueness, presenting a solution concept providing the existence of unique generalized solutions subject to a boundary condition generalizing the classical condition \(u \to 0\) (\(|x| \to \infty\)). Throughout this appendix we assume that \(n \geq 3\) and write the Poisson equation as \(\Delta u = \rho\). Also, we denote the fundamental solution of the Laplace equation by \(C_n/|x|^{n-2}\).

In addition to the algebra \(\mathcal{G}_d(\mathbb{R}^n)\) used in our main results we also treat the case of the standard (special) Colombeau algebra \(\mathcal{G}^s(\Omega)\) (with \(\Omega \subseteq \mathbb{R}^n\)) which is defined using estimates on compact subsets of \(\Omega\), i.e.,

\[
\forall \alpha \in \mathbb{N}_0^n \forall K \subset \subset \Omega \exists N \in \mathbb{N} \text{ (resp. } \forall m \in \mathbb{N} \text{)} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ (resp. } O(\varepsilon^m)).
\]

We begin with some preliminaries. Let \(u \in \mathcal{G}^s(\Omega)\). Then \(u\) has compact support (that is: \(\exists K \subset \subset \Omega : u|_{\Omega \setminus K} = 0\)) if and only if there exists a representative \((u_\varepsilon)_\varepsilon\) of \(u\) and \(L \subset \subset \Omega\) such that \(\text{supp}(u_\varepsilon) \subseteq L\) for all \(\varepsilon \to 0\). In this case we say that \((u_\varepsilon)_\varepsilon\) itself has compact support.

Indeed for any compactly supported \(u\) we may choose a cut off function \(\chi \in \mathcal{D}(\Omega)\) such that \(\chi \equiv 1\) on a neighborhood of the support of \(u\). Then for any representative \((u_\varepsilon)_\varepsilon\) of \(u\) we construct a new representative \((\chi u_\varepsilon)_\varepsilon\) which vanishes outside the support of \(\chi\).

However, in general \(L\) will properly contain the support of \(u\) in its interior. Indeed let \(u = \iota(\delta)\) (with \(\iota\) denoting the embedding of distributions into the algebra of generalized functions) then there clearly exist representatives that vanish outside any compact neighborhood of the origin. On the other hand there is no representative which vanishes outside the support of \(u\).

Next we note that any generalized function \(u \in \mathcal{G}^s(\mathbb{R}^n)\) has a representative \((u_\varepsilon)_\varepsilon\) which vanishes at infinity, i.e., \(u_\varepsilon(x) \to 0\) (\(|x| \to \infty\)) \(\forall \varepsilon\). Indeed take any representative of \(u\) and multiply it with an \(\varepsilon\)-dependent cut-off function \(\chi_\varepsilon\) which is equal to unity inside a ball of radius \(1/(2\varepsilon)\) and vanishes outside a ball of radius \(1/\varepsilon\). Moreover we have the following warning example of non-uniqueness of generalized solutions to the Laplace equation.

**Example A.1.** We consider \(\Delta u = 0\) in \(\mathcal{G}^s(\mathbb{R}^n)\). Clearly \(u = 0\) is a solution. On the other hand we construct a solution \(\hat{u}\) as follows: Set \(\hat{u}_\varepsilon = \chi_\varepsilon u\) with \(\chi_\varepsilon\) as above. Then \(\hat{u}_\varepsilon\) vanishes at infinity, \(|(\hat{u}_\varepsilon)| = 1\) and \(\Delta \hat{u}_\varepsilon = 0\).

However, there does not exist a representative \((\hat{u}_\varepsilon)_\varepsilon\) of \(\hat{u}\) such that \(\text{supp}(\Delta \hat{u}_\varepsilon)\) is contained in some ball of radius \(R\) for all \(\varepsilon\). Indeed suppose to the contrary that \(\Delta \hat{u}_\varepsilon = n_\varepsilon \in \mathcal{N}^s(\mathbb{R}^n)\) with \(\text{supp}(n_\varepsilon) \in B_R(0)\) for all \(\varepsilon\). Then by classical uniqueness we have that \(\hat{u}_\varepsilon(x) = C_n \int \frac{n_\varepsilon(y)}{|x-y|^{n-2}} dy\) and hence \((\hat{u}_\varepsilon)_\varepsilon\) is in the ideal which is not possible.
This observation motivates the following definition securing uniqueness of solutions to the Poisson equation.

**Definition A.2.** Let \( \rho \in \mathcal{G}^s(\mathbb{R}^n) \) be compactly supported. We call \( u \in \mathcal{G}^s(\mathbb{R}^n) \) a solution of the Poisson equation vanishing at infinity if \( \Delta u = \rho \) and if there exists a representative \( (u_\varepsilon)_\varepsilon \) of \( u \) that satisfies

(i) \( \forall \varepsilon > 0 : \lim_{x \to \infty} u_\varepsilon(x) = 0 \), and

(ii) \( (\Delta u_\varepsilon)_\varepsilon \) is compactly supported.

We may now state the following result.

**Theorem A.3.** Let \( \rho \in \mathcal{G}^s(\mathbb{R}^n) \) be compactly supported. Then there exists one and only one solution of the Poisson equation

\[
\Delta u = \rho
\]

vanishing at infinity.

Note that the assumptions on \( u \) in Definition A.2 are not redundant. Indeed the compact support of \( \rho \) guarantees the existence of a representative \( (u_\varepsilon)_\varepsilon \) satisfying property (ii) and there also exists a representative \( (\tilde{u}_\varepsilon)_\varepsilon \) of \( u \) which vanishes at infinity. However, in general \( u_\varepsilon \neq \tilde{u}_\varepsilon \) and uniqueness may fail as is explicitly demonstrated by the example above.

**Proof.** Existence: By the above we may choose a compactly supported representative \( (\rho_\varepsilon^s)_\varepsilon \) of \( \rho \) and define

\[
u_\varepsilon(x) := C_n \int \frac{\rho_\varepsilon^s(y)}{|x - y|^{n-2}} \, dy \]

By the classical theory \( u_\varepsilon \) satisfies both requirements stated in the theorem.

Uniqueness: Let \( u, \tilde{u} \) be two solutions as above and choose representatives \( (u_\varepsilon)_\varepsilon \) and \( (\tilde{u}_\varepsilon)_\varepsilon \) satisfying (i) and (ii) in Definition A.2. From the second property we conclude that \( \Delta (u_\varepsilon - \tilde{u}_\varepsilon) = n_\varepsilon \) is compactly supported. By the first property we have \( u_\varepsilon - \tilde{u}_\varepsilon \to 0 \) \((|x| \to \infty)\). Hence by the classical theory \( (u_\varepsilon - \tilde{u}_\varepsilon)(x) = C_n \int \frac{n_\varepsilon(y)}{|x - y|^{n}} \, dy \) which obviously is in the ideal. \( \square \)

We now turn to the “global” algebra \( \mathcal{G}_g \). The basic difference between \( \mathcal{G}_g \) and \( \mathcal{G} \) is that due to the global estimates defining it, \( \mathcal{G}_g \) is not a sheaf.

**Example A.4.** Define \( u^{(m)}_\varepsilon \in C^\infty((-m, m)) \) to be 1 for \( \varepsilon > 1/m \) and \( \exp(-1/\varepsilon) \) for \( \varepsilon \leq 1/m \). Choose a partition of unity \( (\chi_m)_{m \in \mathbb{N}} \) subordinate to \( ((-m, m))_{m \in \mathbb{N}} \) and set

\[
u_\varepsilon(x) := \sum_{m=1}^{\infty} \chi_m(x) u^{(m)}_\varepsilon(x) .
\]

Then \( (u_\varepsilon)_\varepsilon \in \mathcal{E}^s_M(\mathbb{R}) \setminus \mathcal{N}_{g}(\mathbb{R}) \), so \( u = [(u_\varepsilon)_\varepsilon] \) provides an example of a nonzero element of \( \mathcal{G}_g(\mathbb{R}) \) whose restriction to each \((-m, m)\) is zero.

Clearly in this setting Example A.1 does not work since here \((\tilde{u}_\varepsilon)_\varepsilon\) is not a representative of the function 1. This opens the possibility of relaxing condition (ii) in Definition A.2 which is necessary in the context of the (VP)-system since \( \Delta u_\varepsilon(t) \) as constructed in the proof of Theorem 3.2 will not be compactly supported. On the other hand we have proved \( P_n(t) \leq C \sigma(\varepsilon)^{-1/3} \) in Lemma 3.5 (ii). This motivates the following definition which will provide us with the solution concept used in our main results.
Definition A.5. Let $\rho \in \mathcal{G}_g(\mathbb{R}^n)$ be compactly supported. We call $u \in \mathcal{G}_g(\mathbb{R}^n)$ a solution of the Poisson equation vanishing at infinity if $\Delta u = \rho$ and if there exists a representative $(u_\varepsilon)_\varepsilon$ of $u$ that satisfies

(i) $\forall \varepsilon > 0 : \lim_{x \to \infty} u_\varepsilon(x) = 0$, and
(ii) $\text{supp}(\Delta u_\varepsilon) \subseteq B_{\varepsilon^{-N}}(0)$ for some $N \geq 0$.

Note that again conditions (i) and (ii) are not redundant. Indeed take $u_\varepsilon$ with $u_\varepsilon = 1$ on $B_{\varepsilon^{1/2}}(0)$ and vanishing outside a ball of twice that radius. Then (i) clearly holds but $\Delta u_\varepsilon \neq 0$ near $|x| = e^{1/\varepsilon}$. The desired result in this framework is

Theorem A.6. Let $\rho \in \mathcal{G}_g(\mathbb{R}^n)$ be compactly supported. Then there exists one and only one solution of the Poisson equation

$$\Delta u = \rho$$

vanishing at infinity.

Proof. Existence is proved as in Theorem A.3. To prove uniqueness suppose we have two solutions $u, \tilde{u} \in \mathcal{G}_g(\mathbb{R}^n)$ vanishing at infinity. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be representatives according to Definition A.5. By condition (ii) we have $\Delta (u_\varepsilon - \tilde{u}_\varepsilon) = n_\varepsilon$ with $(n_\varepsilon)_\varepsilon$ in the ideal and $\text{supp}(n_\varepsilon) \subseteq B_{\varepsilon^{-N}}(0)$ for some $N$. So

$$|u_\varepsilon - \tilde{u}_\varepsilon|(x) \leq C_n \int_{B_{\varepsilon^{-N}}(0)} \frac{|n_\varepsilon(y)|}{|x - y|^{n-2}} dy \leq C \varepsilon^m \int_0^{\varepsilon^{-2N}} r dr \quad \forall m,$$

hence is in the ideal. \qed

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REFERENCES

[1] Biagioni, H., Oberguggenberger, M., Generalized solutions to Burgers’ equation. J. Diff. Eqs. 97 263 - 287, (1992).
[2] Colombeau, J.F., New Generalized Functions and Multiplication of Distributions. (North Holland, Amsterdam, 1984).
[3] Colombeau, J.F., Elementary Introduction to New Generalized Functions. (North Holland, Amsterdam 1985).
[4] DiPerna, R. J., Lions, P.-L., Global weak solutions of Vlasov-Maxwell systems. Comm. Pure Appl. Math. 42, no. 6, 729–757 (1989).
[5] Glassey, R., The Cauchy Problem in Kinetic Theory. (SIAM, Philadelphia, PA, 1996).
[6] Glassey, R., Schaeffer, J., Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data. Commun. Math. Phys. 119, 353–384 (1988).
[7] Glassey, R., Strauss, W. A., Absence of shocks in an initially dilute collisionless plasma. Commun. Math. Phys. 113, 191–208 (1987).
[8] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Geometric Theory of Generalized Functions. Mathematics and its Applications 537 (Kluwer Academic Publishers, Dordrecht, 2001.)
[9] Hörmander, G., Oberguggenberger, M., Elliptic regularity and solvability for partial differential equations with Colombeau coefficients, Electron. J. Differential Equations, Vol. 2004, (14), 1–30 (2004).
[10] Hörmander, G., Oberguggenberger, M., Pilipović, S., Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients, Trans. Amer. Math. Soc., 358, 3363–3383 (2006).
[11] Kunzinger, M., Rein, G., Steinbauer, R., On local solutions of the relativistic Vlasov-Klein-Gordon system. *Electron. J. Differential Equations*, Vol. 2005, No. 01, 1-17, 2005.

[12] Lions, P.-L., Perthame, B., Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.* 105, 415–430 (1991).

[13] Neunzert, H., An introduction to the nonlinear Boltzmann-Vlasov equation, in C. Cercignani (Ed.) *Kinetic theories and the Boltzmann equation*, Lecture Notes in Math. 1048, 60–110, (Springer, Berlin, 1984).

[14] Oberguggenberger, M., Case study of a nonlinear, nonconservative, non-strictly hyperbolic system. *Nonlinear Anal.* 19, 53 - 79, (1992).

[15] Oberguggenberger, M., Multiplication of distributions and applications to partial differential equations, *Pitman Research Notes in Mathematics* 259, Longman 1992.

[16] Perthame, B., Mathematical tools for kinetic equations, *Bull. Amer. Math. Soc.* 41(2), 205 - 244, (2004).

[17] Pfaffelmoser, K., Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Diff. Eqns.* 95, 281–303 (1992).

[18] Rein, G., Selfgravitating systems in Newtonian theory—the Vlasov-Poisson system , Mathematics of gravitation, Part 1 (Warsaw, 1996), 179–194, Banach Center Publ., 41, Part I, Polish Acad. Sci. Warsaw, 1997.

[19] Rein, G., Collisionless kinetic equations from astrophysics—The Vlasov-Poisson system. *Handbook of Differential Equations*, Evolutionary Equations. Vol. 3. Eds. C.M. Dafermos and E. Feireisl, Elsevier (2007)

[20] Rein, G., Generic global solutions of the relativistic Vlasov-Maxwell system of plasma physics. *Commun. Math. Phys.* 135, 41–78 (1990).

[21] Rein, G., Rendall, A. D., Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. *Commun. Math. Phys.* 150, 561-583 (1992).

[22] Dietz C., Sandor, V., The hydrodynamical limit of the Vlasov-Poisson System, Transport Theory Statist. Phys. 28(5), 499–520, (1999).

[23] Schaeffer, J., Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Commun. Part. Diff. Eqns.* 16, 1313–1335 (1991).

[24] Schwartz, L., Sur l’impossibilité de la multiplication des distributions. *C. R. Acad. Sci. Paris* 239, 847-848 (1954).