The study of the duration of transient processes in growing random graphs with a non-linear preferential attachment rule

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Abstract. A study of the duration of transients processes in random graphs with preferred attachment was carried out, during which it was found out that for some random graphs transient processes can last very long. For some known growing graphs the absence of stationary distribution of the vertices attachment degree has been found out for the first time. This makes the calculation methods and the techniques for analyzing transient processes up-to-date, because real networks modelled by this graphs, do not have stationary characteristics either. Real networks having stationary characteristics, can for a long time, even if they are too large, stay in the transient mode which does not let estimate these networks by stationary solutions. While modeling these networks, the methods for analyzing the transient modes are also urgent as they provide for controlling the errors of stationary solutions application. The paper gives the example of the stationary solution error control for a random graph, widely used in modeling of growing networks.

1. Introduction
Man is surrounded by a variety of networks. This includes transport, telecommunications, electrical, information networks, spreading rumours, collaboration, social networks, etc. Researchers pay special attention to social networks, which today have turned from a simple means of interpersonal communication into an important tool of information impact, sometimes more effective than the traditional media – press, radio, and the television. A well-known example of the use of social networks for political protest purposes is the so-called "Arab spring" of 2011, when social networks became one of the most important mechanisms for organizing and promptly regulating the actions of the population protesting against the governments of several Arab States [1].

It should be noted that the direct study of social networks is complicated by the fact that these networks are virtual, some of them include billions of nodes, there is a constant change of networks – appear and disappear as network members and particular connections between them. An effective method of studying social networks is the method of modeling, when the real network is replace by a model that has the basic properties of this network. Generally recognized models of social networks are random graphs of various kinds. The most well-known are random graphs with preferential attachment, proposed in the work of A. Barabashi and R. Albert [2]. Various types of random graphs have been described in the papers of a number of researchers [3-7]. In [8-10], random graphs with a non-linear preferential attachment rule (graphs with NPAR) were proposed and the basis of the theory of these random graphs was built. Within the framework of the theory of random graphs with NPAR, methods [8] have been developed that allow, on the basis of retrospective data, for example, on a social network, to choose such parameters of a graph with NPAR that the random graph, generated on
the basis of these parameters, will have characteristics identical to the characteristics of the simulated social network.

Methods of the parameters selection and the developed software [11, 12] allow to predict the change in time of such important characteristics of the real social network as the distribution of attachment degrees (hereinafter DAD) of its nodes, the distribution of probabilities of the state of the selected network node, and the degree of attachment of the node.

2. Elements of the theory of random graphs with NPAR

A random graph with NPAR is grown from a small arbitrary seed graph. At each moment of time, an increment \( t = t_1, t_2, \ldots \) of the graph is added to the it – that is a new vertex with several incident edges. The free ends of the increment edges are attached to the vertices of the graph, which are chosen randomly with probabilities proportional to the given function (weight) \( f(k) \), where \( k \) is the association strength of the vertex. An arbitrary non-negative function of the degree of attachment of a vertex can be chosen as the vertex preferential function of a random graph with NPAR. The probability \( p_i \) that the edge of the increment chooses for attachment vertex \( i \) of the graph is proportional \( f(k_i) \):

\[
p_i = \frac{f(k_i)}{\sum f(k_j)}, \quad i = 1, \ldots, N.
\]

The number of edges \( x \) in the increment of a random graph with NPAR is a random variable that has a discrete probability distribution \( \{r_k\} \). The probability \( r_k = \mathbb{P}(x = k) \geq 0 \) of occurrence of edges in increment \( k \) is given for \( g \leq k \leq h \), where the value \( g \geq 1 \) determines the minimum number of edges in increment, while \( h \) – the maximum number of edges. In addition, \( r_g + \ldots + r_h = 1 \). The probability distribution \( \{r_k\} \) determines the average number of edges \( m = M(x) = \sum kr_k < \infty \) in increment.

A function of preference of the \( f(k) \) graph vertices and the probability \( \{r_k\} \) distribution of the number of edges per increment specify the algorithm used to generate infinite random graph with NPAR. A random graph with NPAR is a model of a growing network in which the network participants are represented by the vertices of the graph, while the connections between them are the edges of the graph.

The DAD \( Q_k \) for graphs with non-linear preferential function \( f(k) \) can be accurately described by recurrent relations derived in [8]:

\[
Q_m = \frac{\langle f \rangle}{\langle f \rangle + mf(m)},
\]

\[
Q_k = \frac{mf(k-1)}{\langle f \rangle + mf(k)} Q_{k-1}, \quad k \geq m + 1.
\]

Where \( \langle f \rangle \) is the final average weight of vertices, which in general case is easily calculated by the numerical method described in [8], which is below called «Double parameter selection» (DPS).

In the theory of random graphs with NPAR, methods for calculating transients have been developed [13, 14], which make it possible to predict the change in time of a number of characteristics of random graphs, and hence the real networks simulated by them.

Transition DAD of a graph \( \{q_g(t)\} \) are calculated according to recurrent formulas:

\[
q_g(t+1) = \frac{tq_g(t) + r_g - m \frac{q_g(t)f_g}{f(t)}}{t+1},
\]
The final average weight of the graph vertices after \( t \) steps of its growing.

If a stationary regime exists, then with the growth of \( t \) DAD \( \{q(t)\} \) converges to the final DAD \( \{Q_f\} \).

Transition DAD \( \{u(t)\} \) of the selected vertex ("probabilities of states of the selected vertex") are calculated according to the formulas:

\[
\begin{align*}
q_k(t + 1) &= r_k + \frac{m}{f(t)}[q_{k-1}(t)f_{k-1} - q_k(t)f_k], \quad k > g, \quad (5)
\end{align*}
\]

where \( t \) is the number of steps performed to grow the graph (coincides with the number of \( N \) vertices in the graph),

\[
f(t) = \sum_{k \geq g} q_k(t)f_k \text{ is the average weight of the graph vertices after } t \text{ steps of its growing.}
\]

Thus, particular calculations associated with the transitional processes in random graphs with NPAR in the form of formulas (2) and (3), based on the assumption that the graphs are moving in the stationary state, and assuming finiteness of the final average weight \( a = \langle f \rangle \), can be confirmed or disproved by numerical tests using equations (4-7). On the basis of the equations of the transition process can be developed and analytical criteria of the existence of final average weight \( a = \langle f \rangle \). In addition, these equations make it possible to determine the duration of transients by the criterion of the significance of their deviation from the stationary solution in cases where such a solution exists. Knowledge of the duration of transients allows us to determine for graphs of what size these stationary solutions become suitable.

### 3. Problem statement

The calculations of network transient processes [13, 14] show that in particular cases, the DAD \( \{q(t)\} \) quickly converges to the final distribution of \( \{Q_f\} \), calculated using the formulas (2) and (3) for which estimate \( a \) of the final average weight \( \langle f \rangle \) of the graph vertices is preliminary calculated using DAD.

However, in a general case, it is difficult to determine by numerical calculations, including the DAD method, whether the final mean weight \( \langle f \rangle \) is finite. And if \( \langle f \rangle = \infty \), then the value of \( a \), selected by the finite part of the sequence of recurrent formulas (2), (3) and the resulting finite \( a < \infty \), is determined with infinite error. The double precision control used in the DAD shows that the required accuracy has been achieved (for example, that 6 exact significant decimal digits have been obtained), since the double precision control is also performed by a truncated sequence of formulas (2), (3). The application of such an incorrect value \( a \) as the final mean weight \( \langle f \rangle \) in both formulas (2), (3) for the calculation of the DAD \( \{Q_f\} \) and in other formulas of the theory of random graphs with NPAR leads to erroneous results. Thus, the correct solution of the question of whether the final average weight of the graph vertices under study is finite is of fundamental importance for the theory of random graphs with NPAR and its methods. This defines the main task, solved in this article, and consisting in the development of methods for the solution of the question of the existence of final average weight \( \langle f \rangle \) the count with NPAR with the specified parameters. The second important problem solved in the article for the first time is the study of the duration of transients in cases where the final average weight of the vertices of a growing graph is finite.
4. The analysis of application possibility for stationary solutions of final Barabashi-Albert graphs

To conduct the finiteness study of the final average weight of the original random graphs vertices let us firstly consider Barabashi-Albert (BA) graph, which preferential function is \( f(k) = k \). For Barabashi-Albert graph, the exact final DAD of vertices in a closed form is known.

\[
Q_k = \frac{2m(m+1)}{k(k+1)(k+2)} \quad k \geq m.
\]

In BA graph, \( \langle f \rangle = \langle k \rangle \) follows from the fact that \( f(k) = k \), while equality \( \langle k \rangle = 2m \) follows from the rules for constructing random graphs with preferential attachment, and this equality holds for any function of preference \( f(k) \). Let us now calculate \( a \) of the final average weight for BA graph using DPS method (figure 1) for \( m = 2 \).

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| Average number \( m \) of edges in graph | 2 | 2 | | |
| differential = | | | | |
| Fault | \( 1000'(\langle k \rangle - 2m) \) | \( -2.09E-11 \) | \( 3.9999999990998 \) | \( 3.99999964001 \) | \( 3.999999984004 \) | \( -3.553E-08 \) | 0.0023% |
| Calculated value | \( \langle k \rangle \) | \( \langle k \rangle \) | 3.9999999990998 | 3.99999964001 | 3.999999984004 | -3.553E-08 | 0.0023% |
| Calculated average weight | \( \langle f \rangle \) | \( \langle f \rangle \) | 3.9999999990998 | 3.99999964001 | 3.999999984004 | -3.553E-08 | 0.0023% |
| Target | \( \text{Relative fault in the calculation} \) | \| | | | | |
| Parameter \( a \) | | | | | | |
| Fault | \( 1000000'(\langle f \rangle - a) \) | \| | | | | |
| Probability \( P_k \) appearance \( k \) of edges in graph | Degree \( k \) | Weight \( f(k) \) of vertex with degree \( k \) | Final probability \( Q_k \) (recurrent formula) | Final probability \( Q_k \) (Dorogovtsev-Mendes formula) | Absolute fault |
| \( 1 \) | 1 | 2 | 0.499994275 | 0.5 | 5.75204E-05 |
| \( 2 \) | 3 | 3 | 0.199999642 | 0.2 | 4.68028E-07 |
| \( 3 \) | 4 | 4 | 0.100000034 | 0.1 | 5.3392E-07 |
| \( 4 \) | 5 | 5 | 0.07143636 | 0.07142657 | 6.79206E-07 |
| \( 5 \) | 6 | 6 | 0.036714915 | 0.0367142486 | 6.28713E-07 |
| \( 6 \) | 7 | 7 | 0.022310694 | 0.0223089524 | 5.4048E-07 |
| \( 7 \) | 8 | 8 | 0.011657121 | 0.011669667 | 4.54571E-07 |
| \( 8 \) | 9 | 9 | 0.012121963 | 0.012121212 | 3.81139E-07 |
| \( 9 \) | 10 | 10 | 0.00959123 | 0.009591099 | 3.2053E-07 |

Figure 1. The calculation of final average weight and probabilities in \( Q_k \) table processor

By applying DPS method, value of \( a = \langle f \rangle \) is calculated simultaneously with probability \( Q_k \). Figure 1 shows the calculation of the BA graph on the part of the sequence (2), (3) bounded by the values \( k \leq K = 65530 \). At the same time, the sum of all \( Q_k \) with \( k > 65530 \) [8] is transferred to the degree \( k = 65531 \) by assigning it a zero weight. As a result of the parameter selection, the value \( a = 3.9999084004 \) of the final average weight of the graph vertices is obtained. The \( Q_k \) probabilities in column E are calculated by recurrent formulas (2)-(3). In column F, the calculation is based on the exact formula (6). The relative error \( \delta_a \) of calculating the value of \( a \) was 0.0023%. If as a criterion for the completion of the transition process to accept the condition \( \delta_a < 0.001% \), corresponding to the receipt of five exact significant digits of the result, then from the calculations performed it should be concluded that the considered graph BA (and the network modeled by it) must have dimensions of more than 65 536 layers of vertices (nodes), so that the application for their analysis of the stationary solution was correct.

The estimation of the number of vertices with degree of attachment \( k \) in the graph BA is found in [16]. A layer is defined as a manyfold of vertices of a graph that have equal degrees of attachment.
The average number of DAD layers $K_{\text{aver}}$ in a BA graph with $N$ vertices when $n=2$ is described by the dependence:

$$K_{\text{aver}} = 3.05 \cdot N^{0.5}. \quad (9)$$

It follows that for the appearance of a vertex layer with the degree of connectivity $k = N = 65531$, the total number of vertices of a random graph must reach a value $N = 65531^2 / 3.052^2 \approx 4.62 \cdot 10^8$ and that for a graph BA with such dimensions, the use of stationary solutions will not be correct yet.

Under the correct application of stationary solutions to graphs (networks) of finite size, we understand such application, in which the researcher can evaluate only the stochastic components of the deviations of the grown finite graph averaged characteristics (finite network under study) from the calculated ones, assuming the deterministic components of the deviations are negligible compared to the stochastic ones. Having five exact digits for the approximation of transient characteristics by stationary characteristics is sufficient for such application and, at the same time, not too redundant, since the experience of growing large random graphs gives many examples when the averaged characteristics converge to some constants with an accuracy of four significant digits.

However, suppose that we are satisfied with the relative error $\delta_a = 0.0023\% \text{ obtained when calculating the value of } a \text{ by } K = 65531 \text{ layers. In this case, the assumption that the sizes of the graph that make up the vertices ensure the correctness of the application of stationary solutions to it is very optimistic. Indeed, the value of } a = 3.9999084004 \text{ is calculated with an acceptable error } \delta_a = 0.0023\% \text{ on the stationary probabilities of a finite graph with } K = 65531 \text{ layers, but in a really grown graph with so many layers their probabilities will be transient, so approximately in the third part of the layers (with higher degrees) will be still far from the stationary probabilities. Quite a large part of the layers with powers } K-1, K-2, \ldots \text{ will be empty at all [16], since the vertex with the leading degree } k = K- \text{ gets a significant preference when binding new vertices of the graph to it. The same abundance of empty layers with nodes preceding the highest degree achieved characterizes real growing networks, for example, the network of Autonomous Internet systems [9].}

The slow convergence of the value $a$ of the BA graph calculated by the DPS method to the final average weight $\langle f \rangle$ is due to the asymptotically-power tail of the distribution (8). In this distribution $Q_k$ converges to $Ck^{-3}$, with increasing $k$, where $C$ is a constant (in this case, $C = 2m \ (m + 1)$). In this distribution, the moments from the second and higher are infinite. The first moment (mathematical expectation) and the moments of a fractional order less than two are also finite. At the same time, as it is well known, in most of the widespread information networks, including in the mentioned above network of autonomous systems, the DAD of the vertices have even heavier tail areas, in the asymptotic of which $Q_k \sim k^{-\alpha}$ the indicator $\alpha$ approaches from above to the values close to 2. In such networks, transitional processes are much slower than in the BA column. This makes the problem of an extensive study of transients in growing networks using formulas (4)-(6) even more relevant.

5. Is the final average weight in graphs with NPAR final: preliminary numerical analysis duration of transient processes

In general, for graphs with NPAR, the exact solutions of the close form are not known. That is why a method is needed to answer the question if the final average weight of the vertices of the graph has a finite value. Therefore, let us make a preliminary numerical calculation of this problem with the help of a specially written program which for graphs with NPAR provides, according to the formulas (2)-(3), to calculate the probabilities $Q_k$, average values $\langle f \rangle$ and $\langle k \rangle$ for $k \leq K$, where $K$ can reach several millions.

The program implements the DPS method.

Using the developed program, we obtained estimates of the final average weight $a = \langle f \rangle$ for random graphs with the number of edges in the increment $m = 2$ and with different preferential functions that are often used in random graph studies. The results of the calculations are presented in table 1.
6. Analytical solution of the finiteness of the final mean weight $\langle f \rangle$

As an analytical method for solving the problem of finiteness of the final average weight $a = \langle f \rangle$, it is proposed to use asymptotic analysis of the peaks of the DAD, based on an approximate analytical description of the DAD obtained by the mean field method (MFM) [2].

Using MFM, we derive, for example, an approximate analytical description of the RCA vertices of the graph grown using the weight function $f(k) = k^s$ (see table 1). Given that the number of $N$ vertices in the graph is always equal to the time $t$ of the next increment, we perform the following steps prescribed by MFM.

### Table 1. The average weight $\langle f \rangle$ of random graphs vertices calculation

| Maximum degree $K$ of attachment used in calculations | Preferential function $f(k) = \ln(k)$ | $f(k) = k, \gamma = 0.9$ | $f(k) = k$ | $f(k) = k + s, s = -1$ | $f(k) = k + s, s = 1$ | $f(k) = k^s, \gamma = 1.1$ | $f(k) = \ln(k)$ |
|-----------------------------------------------------|----------------------------------|---------------------|----------------|---------------------|----------------|---------------------|----------------|
| 1----------------------------------------------------|----------------------------------|---------------------|----------------|---------------------|----------------|---------------------|----------------|
| 10 000                                              | 1.174529113                      | 3.38171357         | 3.999398796    | 2.985883846       | 4.999973611    | 4.890514235      | 8.047009769     |
| 65 530                                              | 1.174529113                      | 3.381713786        | 3.9999908039   | 2.994660388       | 4.999998386    | 4.943445221      | 8.936705781     |
| 100 000                                             | 1.174529113                      | 3.381713786        | 3.9999999991   | 2.995696675       | 4.999999115    | 4.955827097      | 9.138613919     |
| 1 000 000                                           | 1.174529113                      | 3.381713786        | 3.99999418     | 2.998658344       | 4.999999999    | 5.034715146      | 10.23807939     |
| 5 000 000                                           | 1.174529113                      | 3.381713786        | 3.999998617    | 2.999403204       | 4.999999999    | 5.110903341      | 11.01268326     |
| 10 000 000                                          | 1.174529113                      | 3.381713786        | 3.999999079    | 2.999578229       | 5.0000000001   | 5.151546188      | 11.34727074     |

The table allows us to note that for random graphs with the preferential functions from columns 2 and 3, when the number of layers $K = 10000$ a stationary average weight of vertices is calculated with accuracy to 10 and 7 numbers correspondently. For these functions $f(k)$, the degree of vertices in the graph grow slower than in the graph BA, so the final DAD $\{Q_k\}$ has a lighter tail than in the graph BA. For a graph of BA (column 4) final average weight $\langle f \rangle = \langle k \rangle = 4$ determined 6 significant digits only when the number of layers. For a graph with a linear preferential function $f(k) = k + s$ (columns 5-6 of the table), the final mean weights are known: $\langle f \rangle = 3$ for the case, when $s = -1$ and $\langle f \rangle = 5$ for $s = 1$. The average degree of attachment $\langle k \rangle$, according to the rule of construction of random graphs, is equal to 4. As can be seen from the table, for graphs with the preferential function $f(k) = k + s$ the final average weight is achievable, but the transient process in random graphs with this function lasts, for a very long time like in Barabashi-Albert graphs.

For a graph with a linear preferential function $f(k) = k + s$ (columns 5 and 6 of the table) final average weight known: for the case when $s = -1$ have $\langle f \rangle = \langle k - 1 \rangle = 2m - 1 = 3$, if $s = 1$, respectively, $\langle f \rangle = 2m + 1 = 5$. In the first case (column 5) we have a heavier than in the graph BA, tail DAD $\{Q_k\}$, and the calculation of the average weight is completed more slowly, in the second case (column 6) tail DAD easier than in the graph BA, and the calculation is completed faster.

For graphs with preferential functions from columns 7-8, it cannot be confidently stated that in the limit, with the number of layers of the graph tending to infinity, the first digit in the estimation of the stationary average weight will remain the same.

The preferential functions in these columns provide a faster growth of vertex degrees in the growing graph than in the graph BA, and with the growth of the vertex degree its weight relative to the weight of the same vertex in the graph BA increases to an infinite multiplicity: (column 7) and (column 8). This observation allows us to conjecture that one possible simple indication that the final average weight of vertices of a growing graph with a given preferential function $f(k)$ is infinite is the infinite limit of the ratio $f(k)/k$. However, this and other similar hypotheses have not been proved yet. Therefore, a reliable analytical method is needed to solve the finiteness of the final mean weight for any given weight functions $f(k)$.

...
1. Let us write for the time variation of the average degree \( k_i \) of a vertex \( i \) derived from (1) the differential equation \( \frac{dk_i}{dt} = \frac{mk_i^\gamma}{at} \) and find its solution

\[
\int \frac{dk_i}{mk_i^\gamma} = \int \frac{dt}{at} + C \quad \text{or} \quad \frac{1}{m} \frac{k_i^{1-\gamma}}{1-\gamma} = \frac{1}{a} \ln(t) + C , \ i.e. \ k_i = \left[ \frac{m(1-\gamma)}{a} \ln(t) + C_1 \right]^{\frac{1}{1-\gamma}}. \tag{10}
\]

Let us find constant \( C_1 \) from the initial condition \( k(i) = m \):

\[
\left[ \frac{m(1-\gamma)}{a} \ln(i) + C_1 \right]^{\frac{1}{1-\gamma}} = m , \ i.e. \ C_1 = m^{1-\gamma} - \frac{m(1-\gamma)}{a} \ln(i). \]

Substituting \( C_1 \) in (10), we finally get

\[
k_i = \left[ \frac{m(1-\gamma)}{a} \ln\left( \frac{t}{i} \right) + m^{1-\gamma} \right]^{\frac{1}{1-\gamma}}. \tag{11}
\]

2. Using (11), we find number \( l \) of such a vertex which degree is \( k \):

\[
\left[ \frac{m(1-\gamma)}{a} \ln\left( \frac{t}{l} \right) + m^{1-\gamma} \right]^{\frac{1}{1-\gamma}} = k , \ i.e. \ l = e^{-a \left( \frac{k^{1-\gamma} - m^{1-\gamma}}{1-\gamma} \right)} = \left( \frac{k^{1-\gamma} - m^{1-\gamma}}{1-\gamma} \right). \tag{12}
\]

3. From here we obtain the expression of the distribution function (Df) degree \( k \) of a vertex randomly chosen in a graph:

\[
\hat{F}(k) = \frac{l - l}{t} = e^{-a \left( \frac{k^{1-\gamma} - m^{1-\gamma}}{1-\gamma} \right)} , \ \ \ \ \ k \geq m. \tag{13}
\]

By \( \hat{F}(k) \) we denote the obtained approximate estimate of D.f. \( F(k) \).

4. DAD \( Q_k \) vertices of are found as a derivative \( F'(k) \) :

\[
\hat{Q}_k = \hat{F}'(k) = \frac{a}{m} k^{-\gamma} e^{-a \left( \frac{k^{1-\gamma} - m^{1-\gamma}}{1-\gamma} \right)} , \ \ \ \ \ k \geq m. \tag{14}
\]

By \( \hat{Q}_k \) we denote the obtained approximate estimate of the probability \( Q_k \).

Applying in cases of doubt, the form of the DAD in the form of D.f. (13) or of a distribution (14), we can determine the mathematical expectation (M.E.) \( \langle k \rangle \) and \( a = \langle f \rangle \) by calculating the relevant integrals. Since the expressions (13) and (14) are approximate, no exact values \( \langle k \rangle \) and \( \langle f \rangle \) will be found. But, since the rate of reduction of \( Q_k \) with growth of \( k \) (tail weight) is determined by the expressions (13) and (14) asymptotically exactly, the question of finiteness of the calculated M.E. will be solved by precisely calculating integrals. If we obtain the final estimate \( \hat{f} \) for M.E. \( \langle f \rangle \), we calculate M.E. \( \langle f \rangle \) with the required accuracy now using recurrent formulas (2), (3) by the DPS method.

**Example 1.** For the case of the weight function \( f(k) = k^\gamma \) at \( \gamma > 1 \) we write, using (13), the expression for M.E. \( \langle k \rangle \) :
This integral diverges to infinity for any finite $a$. In the same time in fact $\langle k \rangle$ of course: on construction of the graph $\langle k \rangle = 2m$. Hence, in integral (15) $a \to \infty$. Therefore, if we in table 1 (column 7) the calculation would be stopped at some arbitrarily high $K$ and the resulting finite $a$ will be taken as an approximate solution, we would make an infinite error.

So, when using the function $f(k) = k^{1.1}$ (table 1) $a = \infty$. And, as follows from the formulas (2), (3) for $m = 2$, in the final DAD we get $Q_2 = 1$, the rest $Q_a$ are all equal to zero. For $\gamma = 1.5$, the final DAD is exactly the same, and this becomes obvious when simulating (growing) a graph with a size of only 100,000 vertices.

**Example 2.** When the weighting function $f(k) = k \cdot \ln(k)$ (table 1), applying MFM we obtain D.f. of the degrees of the vertices $F(k) = 1 - \left(\frac{\ln k}{\ln m}\right)^{\frac{a}{m}}$ and therefore $\langle k \rangle = m + \int_0^\infty \left(\frac{\ln k}{\ln m}\right)^{\frac{a}{m}} dk$. This integral diverges to infinity for any finite $a$, hence the parameter $a \to \infty$. The use of the DPS method, which gives only finite values of $a$, will lead here to an infinite error.

**Example 3.** When the weight function $f(k) = k^{0.9} \ln k$ of the MFM can not be applied, since the solution of the original differential equation is not expressed in elementary functions. But the finiteness of the mean weight follows from the fact that the asymptotic rate of increase of the weight function here is less than that of the function $f(k) = k$: for any $c > 0$ we have $\lim_{k \to \infty} \frac{ck^{0.9} \ln k}{k} = 0$. The calculation $\langle f \rangle$ using the method of DPS at $K = 10000$, 40 000, 1 million, 4 million, 8 million gives the approximation $\langle f \rangle = 5.970803$, 6.178251, 6.520930, 6.620860, 6.661795 and, consequently, the approximation $\langle k \rangle = 3.744160$, 3.770692, 3.832100, 3.860479, 3.874862. The slow convergence is explained by the fact that the asymptotic rate of weight growth $f(k) = k^{0.9} \ln k$ becomes less than that of the function $f(k) = k$ at very high $k$: the ratio $\frac{k^{0.9} \ln k}{k^{0.1}} = \frac{\ln k}{k^{0.1}}$ becomes less than one only when $k > 3.43 \cdot 10^{15}$. The number $K$ of rows in the calculation table, in which the weight $\langle f \rangle$ is fixed with 6-7 significant digits, will be several orders of magnitude higher, and the number of vertices in the grown graph becomes higher by several orders of magnitude (not to mention the fact that in most of the layers of such a graph the probability of the degrees of It can be said, therefore, that the DAD of the vertices of a growing graph or network with such a weight function at any practically attainable size will be in a state of transition.

Although we do not yet know how to calculate the final average weight of the vertices of this graph, we know that this weight is finite, and therefore we can set ourselves the task of its calculation and, accordingly, the calculation of the final DAD vertices of this graph.

7. **Conclusion**

The article shows that the duration of transients in the growing of graphs (networks) can be very long, and the final average weight of the vertices can be infinite, which will lead to infinite errors in the calculation of the stationary characteristics of such graphs. The method of analytical solution of the question of weight extremity is developed $\langle f \rangle$. Recommendations for determining the lower estimate of the size of the grown graph with a finite $\langle f \rangle$, in which it is possible to apply stationary solutions. The task of exact transient study in growing graphs, equations (4)-(7) transition of processes to perform such a study.
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