**Bounds on Fractional Repetition Codes using Hypergraphs**

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**Abstract**—In the *Distributed Storage Systems* (DSSs), encoded fraction of information is stored in the distributed fashion on different chunk servers. Recently a new paradigm of *Fractional Repetition* (FR) codes have been introduced, in which, encoded data information is stored on distributed servers, where encoding is done using a *Maximum Distance Separable* (MDS) code and a repetition code. In this work, we have shown that an FR code is equivalent to a hypergraph. Using the correspondence, the properties and the bounds of a hypergraph are directly mapped to the associated FR code. Some of the bounds are not satisfied by any FR codes in the literature with equality. In general, the necessary and sufficient conditions for the existence of an FR code is obtained by using the correspondence. It is also shown that any FR code associated with a linear hypergraph is universally good.

**Index Terms**—Distributed Storage Systems, Fractional Repetition Code, Coding for Distributed Storage.

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**I. INTRODUCTION**

The distributed storage is a well studied area, which deals with storing the data on distributed nodes in such a manner so that it allows data accessibility at anytime and anywhere. Many companies such as Microsoft and Google etc. provide such storage services by using distributed data centers. In the storage systems like Google file system [1], multiple copies of data fragments are stored. Thus, the system becomes reliable and the file can be retrieved from the system. At the same level of redundancy, coding techniques can improve the reliability for the storage system [2].

In [3], Dimakis et al. proposed a coding scheme called *regenerating codes* which reduces the repair bandwidth, where repair bandwidth is total number of communicated packets to repair a failed node. In the distributed storage system (DSS), encoded data packets are distributed among *n* nodes such that any data collector can get the complete file by collecting packets from any *k* (*≤ n*) nodes. Also, a failed node can be repaired by replacing it with a new node having at-least same amount of the data information. The newcomer node is constructed by downloading packets from any *d* (*≤ n*) active nodes in the DSS. The Minimum Bandwidth Regenerating (MBR) codes are optimal with respect to the repair bandwidth but not with respect to the computational complexity. Motivated by this, in [4], Rashmi et al. proposed a simple construction of Minimum Bandwidth Regenerating codes, where a survivor node is only required to read and transfer the exact amount of data to the replacement node without extra computation. The construction is extended to new storage code called *Fractional Repetition* (FR) code [5], [6]. A typical FR code is a two layer code with an outer MDS code and an inner repetition code, where a linear [n, k, d] code is called a Maximum Distance Separable (MDS) code if *k* = *n* − *d* + 1. Initially, the FR code is defined on homogeneous DSS (DSS with symmetric parameters). In such systems, each node has same amount of encoded packets and the same number of copies of each packets [5]. In many cases, heterogeneous systems are preferred, in which the node storage capacity and the packet replication factor need not be uniform. In the literature, FR codes are constructed with asymmetric replication factor [7], [8] and asymmetric node storage capacity [9]–[12]. In [13], an FR codes with non-uniform node storage capacity are constructed using uniform hypergraphs. A bound on the maximum file size is calculated for a Locally Repairable Fractional Repetitions (Locally Repairable FR) code in [14], where a Locally Repairable FR code is an FR code in which the repair degree of a node is less than the node storage capacity. In [15], the minimum distance on an FR code is defined as the minimum number of nodes such that if the nodes fail simultaneously then the system can not recover the complete file. FR codes with asymmetric parameters, are also constructed using sequences [16], graphs [17], [18] and combinatorial designs [19], [20].

For the FR code, the asymmetric node storage capacity helps to reduce the total storage cost and the asymmetry on replication factor helps for availability of favorable packets, where favorable packets are the packets which are downloaded with more frequency. The asymmetric repair degree and repair bandwidth help to reduce the repair cost of a node in the FR code.

In this work, an FR code with asymmetric parameters (node storage capacity, packet replication factor, repair degree and repair bandwidth) are considered. In general, it is shown that any FR code is equivalent to a hypergraph. The equivalence between a hypergraph and an FR code leads to the several interesting properties of the FR code. In general, we provide the existence condition for FR codes using the correspondence. Bounds on the node storage capacity and the packet replication factor, are obtained for various FR codes which are equivalent to some known classes of hypergraphs. We have obtain a bound on size of a sub-hypergraph of a linear hypergraph. A new bound on the replication factor, the number of nodes and the number of packets are obtained such that all the FR codes satisfying the bound are the universally good. Using the correspondence, we have identified some bounds which are not satisfied by any FR codes with equality in the literature.
It is observed that an FR code constructed using a linear hypergraph is universally good code. We have proved that the availability of favorable packets can be increased by permuting packets in FR code with asymmetric parameters, where the availability of all packets in FR code with symmetric parameters are constant. It is also shown that the asymmetric parameters of an FR code helps to reduce the total cost associated with node storage capacity.

The paper is organized as follows. In Section II we collect some preliminaries on FR codes and hypergraphs with their parameters and properties. In Section III, the relation of a hypergraph and an FR code is studied in detail. Using the relation, we obtain various bounds on the parameters of FR codes in Section IV. At the end of the Section, the comparison graph is plotted between the reconstruction degree and file size for the FR codes. Section V concludes the paper with some general remarks.

II. PRELIMINARIES

In this section, background on FR codes and hypergraphs are given.

A. Background on Fractional Repetition Codes

Let a file be divided into B distinct information packets, where information packet symbols are associated with a finite field \( \mathbb{F}_q \). By using \( (\theta, B) \) MDS code on the information packets, the B information packets are encoded into \( \theta \) distinct packets \( P_1, P_2, \ldots, P_\theta \). Because of the property of MDS code, the complete file information can be extract from any B packets. Each packet \( P_j \) (\( 1 \leq j \leq \theta \)) is replicated \( \rho_j \) times in the system. All the packets are distributed on \( n \) distinct nodes \( U_1, U_2, \ldots, U_n \) such that a node \( U_i \) (\( i = 1, 2, \ldots, n \)) contains \( \alpha_i \) packets. For the FR code, the maximum node storage capacity and the maximum replication factor are given by \( \alpha = \max\{\alpha_i : i = 1, 2, \ldots, n\} \) and \( \rho = \max\{\rho_j : j = 1, 2, \ldots, \theta\} \) respectively. The FR code is denoted by \( \mathcal{C}(n, \theta, \alpha, \rho) \).

Example 1. Let a file be split into 5 (= B) information packets and encoded into 6 (= \( \theta \)) distinct packets \( P_1, P_2, P_3, P_4, P_5 \) and \( P_6 \) using a \((6, 5)\) MDS code. The 13 replicates of the \( 6 (= \theta) \) distinct packets are stored in the 5 distinct nodes \( U_1, U_2, U_3, U_4 \) and \( U_5 \) such that \( U_1 = \{P_1, P_2, P_3, P_4\}, U_2 = \{P_5, P_6\}, U_3 = \{P_2, P_5\}, U_4 = \{P_3, P_6\} \) and \( U_5 = \{P_4, P_6\} \). The node packet distribution of the FR code \( \mathcal{C}(n, \theta, \alpha, \rho) \): (6, 5, 3) is illustrated in Table 1. Hence, \( \alpha_1 = |U_1| = 4, \alpha_2 = |U_2| = 3, \alpha_4 = |U_4| = 2 \) (\( i = 3, 4, 5 \)) and replication vector \( \mathbf{\rho} = (\rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \rho_6) = (2 2 2 2 2 3) \). In the FR code, \( \alpha = \max\{\rho_j : j = 1, 2, \ldots, 6\} = 3 \) and \( \alpha = \max\{\alpha_i : i = 1, 2, \ldots, 5\} = 4 \). Hence, the FR code is denoted by \( \mathcal{C}(6, 5, 3) \).

Example 2. In an FR code, a failed node \( U_i \) can be repair by replacing a new node \( U_i' \), where both the nodes \( U_i \) and \( U_i' \) carry the same information. For the new node, the information packets are downloaded from some active nodes called helper nodes. The repair degree of a failed node \( U_i \) is the cardinality of the set of the helper nodes. For a node \( U_i \) in an FR code \( \mathcal{C}(n, \theta, \alpha, \rho) \), more then one such helper node set exists with different repair degree. For the node \( U_i \), the maximum repair degree is \( d_i \) and \( d = \max\{d_i : i = 1, 2, \ldots, n\} \). In the FR code \( \mathcal{C}(5, 6, 4, 3) \) (Example 1), if node \( U_2 \) fails then the new node \( U_2' \) can be created by downloading packets from nodes of set \( \{U_1, U_3, U_4\} \) or set \( \{U_1, U_3, U_5\} \). Hence, \( d_2 = 3 \) and \( d = \max\{d_i : i = 1, 2, 3, 4, 5\} = 4 \).

Table 1

| \( \mathcal{C} \) | Distribution of replicated packets on nodes for the FR code \( \mathcal{C} = (5, 6, 4, 3) \) |
| --- | --- |
| \( U_1 \) | \( P_1, P_2, P_3, P_4 \) |
| \( U_2 \) | \( P_1, P_5, P_6 \) |
| \( U_3 \) | \( P_2, P_5 \) |
| \( U_4 \) | \( P_3, P_6 \) |
| \( U_5 \) | \( P_4, P_6 \) |

In an FR code, an \( \mathcal{C}(n, \theta, \alpha, \rho) \) code satisfies the inequality \( \sum_{i \in I} d_i - \binom{|I|}{2} \geq \binom{\alpha}{2} \) for any \( k = n, d_1 \leq d_2 \leq \ldots \leq d_n \) and \( \sum_{i \in I} d_i \leq \alpha \). An FR code \( \mathcal{C}(n, \theta, \alpha, \rho) \) with \( d_1 \leq d_2 \leq \ldots \leq d_n \) and \( |U_i \cap U_j| \leq 1 \) (\( i, j = 1, 2, \ldots, n \) and \( i \neq n \)) satisfies the inequality \( \sum_{i=1}^{k} d_i - \binom{k}{2} \geq \binom{\alpha}{2} \).

B. Background on Hypergraphs

For a finite set \( V = \{v_1, v_2, \ldots, v_n\} \), the pair \( \mathcal{H} = (V; E) \) is called a hypergraph with hypervertex set \( V \) and hyperedge set \( E \), where \( E = \{E : E \subseteq V\} \) is a family of some subset of \( V \). For a hypergraph \( \mathcal{H} = (V; E) \), \( |E| \) is called the size of the hyperedge \( E \in E \) and \( |E(v)| \) is the degree of the hypervertex \( v \in V \), where \( E(v) = \{E : v \in E\} \).

In an FR code, if a data collector connects any \( k \) nodes and downloads \( B \) distinct packets then the file can be retrieved by using the MDS property. The parameter \( k \) is called the reconstruction degree of the FR code. Note that the reconstruction degree of the FR code \( \mathcal{C}(5, 6, 4, 3) \) (Example 1) is \( k = 3 \).
An example of such hypergraph is shown in Figure 1. In this hypergraph \( \mathcal{H} = (V; E) \), hypervertices are shown by simply bold dots and the hyperedges are represented by the covered area which contains some hypervertices. In the particular example, there are 7 hypervertices and 4 hyperedges such that \( E(v_1) = \{E_1\}, E(v_2) = \phi, E(v_3) = \{E_3\}, E(v_4) = \{E_1, E_2\}, E(v_5) = \{E_1, E_2, E_3\}, E(v_6) = \{E_2\} \) and \( E(v_7) = \{E_4\} \).

In a hypergraph \( \mathcal{H} = (V; E) \), a hypervertex \( v \in V \) is called isolated hypervertex if \( E(v) = \phi \). A hypergraph \( \mathcal{H} = (V; E) \) does not have any isolated hypervertex if \( V = \bigcup E \). In a hypergraph \( \mathcal{H} = (V; E) \), a loop is a hyperedge \( E \in E \) with \( |E| = 1 \). If two distinct hyperedges share same hypervertices then those hyperedges are called parallel hyperedges for the hypergraph. Similarly, if two distinct hypervertices are shared by same hyperedges then those hypervertices are called parallel hypervertices.

![Fig. 1. A hypergraph \( \mathcal{H} = (V; E) \) is shown, where \( V = \{v_i : i \in \{7\}\} \) and \( E = \{E_1 = \{v_1, v_4, v_5\}, E_2 = \{v_4, v_5\}, E_3 = \{v_3, v_5, v_6\}, E_4 = \{v_7\}\}. \)

For a hypergraph \( \mathcal{H} = (V; E) \), a subset of \( V \) is called a hypervertex cover if each hyperedge is incident to at least one hypervertex of the subset. For an example, the set \( \{v_4, v_6, v_7\} \) is a hypervertex cover of the hypergraph given in Figure 1. Note that the hypervertex cover of any hypergraph exists but it is not unique for some hypergraph.

A hypergraph \( \mathcal{H}' = (V'; E') \) is called a sub-hypergraph of a hypergraph \( \mathcal{H} = (V; E) \) if \( V' \subseteq V \) and \( E' \subseteq \{E' \neq \phi : E' \subseteq E \cap V' \} \). For an example, a hypergraph \( \mathcal{H}' = (V'; E') \) with \( V' = \{v_1, v_3, v_4, v_5, v_6\} \) and \( E' = \{E_1' = \{v_1, v_4, v_5\}, E_2' = \{v_1\}, E_3' = \{v_3, v_5\}\} \), is a sub-hypergraph of the hypergraph \( \mathcal{H} = (V; E) \) (as shown in Figure 1). For a hypergraph \( \mathcal{H} = (V; E) \), hypergraph \( \mathcal{H}' = (V'; E') \) is called sub-hypergraph induced by the set \( V' \subseteq V \) if \( E' = \{E' \neq \phi : E' \subseteq E \cap V' \} \). A hypergraph is called connected hypergraph if there exists a sub-hypergraph with the same hypervertex set such that the sub-hypergraph is isomorphic to a connected graph. The hypergraph \( \mathcal{H} = (V; E) \) (Figure 1) is not a connected hypergraph since there is an isolated hypervertex.

Consider a hypergraph \( \mathcal{H} = (V; E) \) without isolated hypervertices. For a finite set \( V^* = \{v_1^*, v_2^*, \ldots, v_{|E|}^*\} \), a hypergraph \( \mathcal{H}^* = (V^*; E^*) \) is called the dual of the hypergraph \( \mathcal{H} = (V; E) \) if they exist a bijection \( g : E \rightarrow V^* \) such that \( E_j = \{g(E_i) : v_j \in E_i \} \), where \( j = 1, 2, \ldots, |V| \). For an example, a hypergraph \( \mathcal{H}^* = (V^*; E^*) \) with \( V^* = \{v_1^*, v_2^*, v_3^*, v_4^*, v_5^*\} \) and \( E^* = \{E_1^* = \{v_1^*\}, E_2^* = \{v_2^*\}, E_3^* = \{v_1^*, v_2^*\}, E_4^* = \{v_1^*, v_2^*, v_3^*\}, E_5^* = \{v_3^*\}\} \), is the dual of a hypergraph \( \mathcal{H} = (V; E) \) with \( V = \{v_1, v_2, v_3, v_4, v_5\} \) and \( E = \{E_1 = \{v_1, v_3, v_4\}, E_2 = \{v_3, v_4\}, E_3 = \{v_2, v_4\}, E_4 = \{v_5\}\} \), where \( g(E_j) = v_j^* \) and \( i = 1, 2, \ldots, 5 \).

In a hypergraph, if two hyperedges shares one node maximum then the hypergraph is called linear hypergraph. For such hypergraph, one can find bound on the size of the hypergraph as given in the following Lemma.

**Lemma 3.** For a linear hypergraph \( \mathcal{H} = (V; E) \),

\[
|E| \geq \sum_{v \in V} \lfloor |E(v)| \rfloor - \left( \frac{|V|}{2} \right).
\]

**Proof:**

In a linear hypergraph \( \mathcal{H} = (V; E) \), at most \( \left( \frac{|V|}{2} \right) \) hyperedges are repeated in the hyperedge count \( \sum_{v \in V} |E(v)| \).

So, the total number of hyperedges of \( \mathcal{H} \) is bounded by \( \sum_{v \in V} |E(v)| - \left( \frac{|V|}{2} \right) \).

A sub-hypergraph of a linear hypergraph is also a linear hypergraph. For a positive integer \( k \) (not more than the size of the hypergraph), one can find a bound on the size of sub-hypergraph induced by any \( k \)-element subset. Formally, the bound is discussed in the following Lemma with a proof in the Appendix.

**Lemma 4.** For \( V = \{v_i : i = 1, 2, \ldots, n\} \), consider a linear hypergraph \( \mathcal{H} = (V; E) \) with \( |E(v_i)| \leq |E(v_j)| \) for \( i < j \) and \( i, j = 1, 2, \ldots, n \). For a given positive integer \( k < n \), let \( \mathcal{H}' = (V' \subseteq V; E') \) be an induced sub-hypergraph of the hypergraph \( \mathcal{H} = (V; E) \) with \( E' \neq \phi : E' \subseteq E \cap V' \) and \( |V'| = k \). Then

\[
\min\{|E'| : \mathcal{H}' = (V' \subseteq V; E'), |V'| = k\} \geq \sum_{i=1}^{k} |E(v_i)| - \left( \frac{k}{2} \right).
\]

### III. HYPERGRAPHS AND FRACTIONAL REPETITION CODES

In this Section, a bijection (called Hyper-FR mapping) between a hypergraph and an FR code is established. Using the Hyper-FR mapping, the relation between the parameters of FR codes and hypergraphs are studied at the end of the Section.

In an FR code, the distribution of nodes and packets are similar to the incidence of hypervertices and hyperedges in a hypergraph.

**Example 5.** As shown in Figure 2 consider a hypergraph \( \mathcal{H} = (V; E) \) with \( V = \{v_i : i = 1, 2, 3, 4\} \) and \( E = \{E_j : j = 1, 2, 3, 4, 5, 6\} \), where \( E_1 = \{v_1, v_2\}, E_2 = \{v_1, v_3\}, E_3 = \{v_1, v_4\}, E_4 = \{v_2, v_3\}, E_5 = \{v_2, v_4\} \) and \( E_6 = \{v_3, v_4\} \). For the hypergraph \( \mathcal{H}(v_1) = \{E_1, E_2, E_3\} \), \( \mathcal{H}(v_2) = \{E_1, E_4, E_5\} \), \( \mathcal{H}(v_3) = \{E_2, E_4, E_6\} \) and \( \mathcal{H}(v_4) = \{E_3, E_5, E_6\} \). As given in Figure 2 consider an FR code \( \mathcal{C} : (4, 6, 3, 2) \) with 4 nodes and 6 such that \( U_1 = \{P_1, P_2, P_3\} \), \( U_2 = \{P_1, P_4, P_5\} \), \( U_3 = \{P_2, P_3, P_6\} \) and \( U_4 = \{P_3, P_5, P_6\} \). Note that there exist the bijection \( \varphi : E \rightarrow \{P_j : j = 1, 2, \ldots, 6\} \) such that \( \varphi(E_j) = P_j \) for each \( j = 1, 2, \ldots, |E| \). Hence, \( \varphi(E(v_i)) = \{P_j : \varphi(E_j) \in E_i, v_i \in E_j\} \).
For $\theta \in \mathbb{N}$ and $|\mathcal{E}| = \theta$, an FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ is equivalent to a hypergraph $\mathcal{H} = (V; \mathcal{E})$ if a bijection $\varphi : \mathcal{E} \rightarrow \{P_j : j = 1, 2, \ldots, \theta\}$ exists such that $\forall v_i \in V, \mathcal{E}(v_i) = \{E : v_i \in E, E \in \mathcal{E}\} \Leftrightarrow U_i = \varphi(\mathcal{E}(v_i)) = \{P_j = \varphi(E) : v_i \in E, E \in \mathcal{E}\}$. The bijection is called Hyper-FR mapping.

For $\theta \in \mathbb{N}$ and $|\mathcal{E}| = \theta$, let $\mathcal{C} : (n, \theta, \alpha, \rho)$ be an FR code which is equivalent to a hypergraph $\mathcal{H} = (V; \mathcal{E})$. Hence, a bijection $\varphi : \mathcal{E} \rightarrow \{P_j : j = 1, 2, \ldots, \theta\}$ exists such that $\forall v_i \in V, \mathcal{E}(v_i) = \{E : v_i \in E, E \in \mathcal{E}\} \Leftrightarrow U_i = \varphi(\mathcal{E}(v_i)) = \{P_j = \varphi(E) : v_i \in E, E \in \mathcal{E}\}$. Because of the bijection, one can immediately see that the number of nodes and the number of hypervertices are same i.e. $n = |V|$, the number of distinct packets and the number of hyperedges are same i.e $\theta = |\mathcal{E}|$, the node storage capacity of a node is the same as the hypervertex degree of the respective hypervertex i.e. $\alpha_i = |\mathcal{E}(v_i)|$ $(i = 1, 2, \ldots, n)$ and the replication factor of a packet is equal to the cardinality of the respective hyperedge i.e. $\rho_j = |E_j|$ $(j = 1, 2, \ldots, \theta)$. Hence, the maximum node storage capacity $\alpha = \max\{|\mathcal{E}(v_i)| : v \in V\}$ and the maximum replication factor $\rho = \max\{|E| : E \in \mathcal{E}\}$. A reconstruction set of the equivalent FR code is associated with a hypervertex cover of a sub-hypergraph $\mathcal{H}' = (V'; \mathcal{E}')$ induced by $V' = \bigcup_{E \in \mathcal{E}(v_i)} E$, and $|\mathcal{E}'| \geq B$. Consider a hypervertex cover $\mathcal{V}'$ of a sub-hypergraph $\mathcal{H}' = (V'; \mathcal{E}')$ of the hypergraph $\mathcal{H} = (V; \mathcal{E})$, where $V' = \bigcup_{E \in \mathcal{E}(v_i)} E$ and $\mathcal{E}' = \mathcal{E}(v_i)$. In the equivalent FR code $\mathcal{C}'$, for a node failure $U_i$, a surviving set is associated with the set $\mathcal{V}' \setminus \{v_i\}$, where $i = 1, 2, \ldots, |V|$. In the FR code, the repair bandwidth for a node is equal to the hypervertex degree of respective hypervertex and tolerance factor is equal to the hypergraph size. For the given positive integer $k < n$, Let $\mathcal{H}' = (V'; \mathcal{E}')$ be a sub-hypergraph induced by $V' \subset V$. The code dimension $D_\theta(k)$ of the equivalent FR code is $D_\theta(k) = \min\{|\mathcal{E}'| : V' \subset V, |V'| = k\}$.

The necessary and sufficient condition for a hypergraph to represent an FR code is discussed in the following straightforward Lemma.

Lemma 7. For a hypergraph $\mathcal{H} = (V \neq \emptyset; \mathcal{E})$, an FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ exists if and only if $|\mathcal{E}(v)| > 0 \ (\forall \ v \in V)$ and $|E| > 0 \ (\forall \ E \in \mathcal{E})$.
with $\mathcal{E}' = \{ E' \neq \phi : E' = E \cap V', E \in \mathcal{E} \}$ and $|V'| = k$. For the equivalent FR code, the code dimension

$$D_{\mathcal{E}}(k) = \min\{|\mathcal{E}'| : \mathcal{H}' = (V' \subset V; \mathcal{E}'), |V'| = k\}. \quad (3)$$

An FR code $\mathcal{C}(n, \theta, \alpha, \rho)$ is universally good if the code dimension $D_{\mathcal{E}}(k)$ is more than or equal to $\sum_{i=1}^{k} d_{i} - \binom{k}{2}$ for any $k < n$, where $d_{1} \leq d_{2} \leq \ldots \leq d_{n}$. The following Theorem gives the necessary and sufficient condition for a universally good FR code on hypergraph.

**Theorem 11.** For $V = \{v_{i} : i = 1, 2, \ldots, n\}$, consider a hypergraph $\mathcal{H} = (V; \mathcal{E})$ with $|\mathcal{E}(v_{i})| \leq |\mathcal{E}(v_{j})|$, for $i < j$ and $i, j = 1, 2, \ldots, n$. For a given positive integer $k < n$, let $\mathcal{H}' = (V' \subset V; \mathcal{E}')$ be a induced sub-hypergraph of the hypergraph $\mathcal{H} = (V; \mathcal{E})$ with $\mathcal{E}' = \{E' \neq \phi : E' = E \cap V', E \in \mathcal{E} \}$ and $|V'| = k$. For each $k = 1, 2, \ldots, n - 1$, the FR code is universally good if and only if

$$\min\{|\mathcal{E}'| : \mathcal{H}' = (V' \subset V; \mathcal{E}'), |V'| = k\} \geq \sum_{i=1}^{k} |\mathcal{E}(v_{i})| - \binom{k}{2} \quad (4)$$

*Proof:* The proof follows the Equation 3 and 9 Theorem 4].

**B. Fractional Repetition Codes using Linear Hypergraphs**

A hypergraph is called linear hypergraph if any two hyperedges share at most one hypervertex. Note that in a hypergraph, any two hyperedges share at most one hypervertex if and only if any two hypervertices are shared by at most one hyperedge. If a linear hypergraph is a conditional hypergraph then the hypergraph is called conditional linear hypergraph.

Any two nodes in an FR code equivalent to a conditional linear hypergraph do not share more than one packets. An FR code on conditional linear hypergraph will satisfy the following Lemmas.

**Lemma 12.** An FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ with $|U_{i} \cap U_{j}| \leq 1$ (for each $i \neq j$ and $i, j = 1, 2, \ldots, n$), satisfies $\sum_{j=1}^{n} \frac{\theta}{2} \leq \binom{n}{2}$.

*Proof:* The Lemma follows from [21] Chapter 1, Theorem 3].

**Lemma 13.** An FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ with $|U_{i} \cap U_{j}| \leq 1$ (for each $i \neq j$ and $i, j = 1, 2, \ldots, n$), satisfies $\sum_{i=1}^{n} \frac{\theta}{2} \leq \binom{n}{2}$.

*Proof:* The Theorem follows from the duality (with respect to hypergraph) on Lemma 12.

The bound on the code dimension of FR code associated with linear hypergraph is discussed in the following Theorems.

**Theorem 14.** An FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ with $|U_{i} \cap U_{j}| \leq 1$ satisfies $D_{\mathcal{E}}(k) \geq \sum_{i=1}^{k} \alpha_{i} - \binom{k}{2}$, where $\alpha_{i} \leq \alpha_{j}$ ($1 \leq i < j \leq n$).

*Proof:* The proof follows from the Lemma 3 and Lemma 4]
the literature is associated with some conditional hypergraph, where the minimum rank and the minimum degree of the hypergraph are non-zero positive integer. We have compared our method with other known constructions of FR codes. The properties of the hypergraphs are mapped to FR codes by using Hyper-FR mapping. Some new bounds on FR codes have been derived. It would be an interesting work in the future to find the FR codes which satisfy those bounds with equality. Analysis of FR codes on generalized hypergraph could be an another interesting problem.

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VI. APPENDIX

The Appendix includes the bounds on FR codes correspondence to various classes of hypergraphs such as uniform hypergraphs, regular hypergraphs and intersecting hypergraphs. The cost comparison analysis is done for various FR codes at the end of the Appendix.

A. Proof of Lemma 4

Proof: For \( V = \{v_i : i = 1, 2, \ldots, n\} \), let \( \mathcal{H} = (V; \mathcal{E}) \) be a linear conditional hypergraph with \( |E(v_i)| \leq |E(v_j)| \) for \( 1 \leq i < j \leq n \). For a given positive integer \( k < n \), let \( V' \) be an arbitrary \( k \)-element subset of the set \( V \). Consider a hypergraph \( \mathcal{H}' = (V'; \mathcal{E}') \), where \( \mathcal{E}' = \{E' \neq \emptyset : E' = E \cap V', E \in \mathcal{E}\} \). By definition of degree of a hypervertex, we have

\[
\sum_{v \in V'} |\mathcal{E}(v)| \geq \sum_{i=1}^{k} |\mathcal{E}(v_i)|
\]

\[
\Rightarrow \sum_{v \in V'} |\mathcal{E}(v)| - \left(\frac{|V'|}{2}\right) \geq \sum_{i=1}^{k} |\mathcal{E}(v_i)| - \left(\frac{k}{2}\right).
\]

A sub-hypergraph of a linear hypergraph is also linear. So, using Lemma 3, one can conclude that

\[
|\mathcal{E}'| \geq \sum_{v \in V'} |\mathcal{E}(v)| - \left(\frac{|V'|}{2}\right) \geq \sum_{i=1}^{k} |\mathcal{E}(v_i)| - \left(\frac{k}{2}\right).
\]

The chosen \( k \)-element set \( V' \) is an arbitrary subset of set \( V \), so

\[
\min\{|\mathcal{E}'| : \mathcal{H}' = (V' \subset V; \mathcal{E}'), |V'| = k\} \geq \sum_{i=1}^{k} |\mathcal{E}(v_i)| - \left(\frac{k}{2}\right).
\]

Hence, the lemma is proved.

B. Algebraic Entropy on Fractional Repetition Code

An algebraic entropy of a hypergraph is discussed in [22]. Using a Hyper-FR mapping on a hypergraph, one can find the algebraic entropy for an FR code in the following Theorem.

Theorem 21. (Algebraic Entropy) Consider an FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with the node adjacency matrix \( A(\mathcal{C}) = [a_{ij}]_{n \times n} \). The algebraic entropy associated with the FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \), is given by

\[
I(\mathcal{C}) = - \sum_{i=1}^{n} \lambda_i \log_2 \lambda_i,
\]

where \( \lambda_i \) (for \( i = 1, 2, \ldots, n \)) is a discrete probability distribution. In particular, \( \lambda_i \) (\( i = 1, 2, \ldots, n \)) is the eigenvalue of matrix \( L(\mathcal{C}) = D - A(\mathcal{C}) \) are real such that \( \sum_{i=1}^{n} \lambda_i = 1 \) and \( 0 \leq \lambda_i \leq 1 \) (\( \forall i = 1, 2, \ldots, n \)), where

\[
D = \text{diag} \left( \sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{nj} \right).
\]

Proof: The proof follows from the algebraic entropy of the hypergraph [22 Chapter 1].

C. Bounds on Fractional Repetition Codes using Hypergraphs

The alternating condition for the existence of an FR code is given in the following Theorem.

Theorem 22. (Alternate Existence Condition) Consider two vectors \( (\rho_1, \rho_2, \ldots, \rho_0) \) and \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \) on integers such that \( \rho \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_0 \) for \( j = 1, 2, \ldots, \theta \) and \( i = 1, 2, \ldots, n \). An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) exists such that

\[
\begin{align*}
[U_i] & = \alpha_i \quad \text{for } i = 1, 2, \ldots, n \quad \text{and replication factor of packet } P_j \text{ is } \rho_j \text{ for } j = 1, 2, \ldots, \theta, \\
1) & \sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{\theta} \rho_j \quad \text{and} \\
2) & \min\{\alpha_i, m\} \geq \sum_{j=1}^{m} \rho_j \quad \text{for } m < \theta, m \in \mathbb{Z}.
\end{align*}
\]

Proof: The proof follows from the duality (with respect to hypergraph) on the Theorem 8.

Note that if parameters of an FR code do not satisfy one of the constraint in the Theorem 8 or Theorem 22 then the FR code does not exist.

In an FR code, two packets with same replication factor are called parallel packets if the two packets are shared by same nodes. In an FR code, two nodes with same node storage capacity are called parallel nodes if the two nodes contain same packets. For an FR code without parallel packets and parallel nodes, the bound on node storage capacity and replication factor are given in the following two Lemmas.

Lemma 23. An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) without parallel packets, satisfies \( \alpha \leq 2^{n-1} \).

Proof: The proof follows from the Theorem 4 in [21 Chapter 1].

Lemma 24. An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) without parallel nodes, satisfies \( \rho \leq 2^{n-1} \).

Proof: The proof follows from the duality (with respect to the hypergraph) on Lemma 23.

Theorem 25. For an FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with \( |\{U : P_m \in U\} \cap \{U : P_j \in U\}| \leq 1 \),

\[
D(\mathcal{C}) \geq \sum_{i=1}^{k} \alpha_i - \left(\frac{k}{2}\right),
\]

where \( \alpha_1 \leq \alpha_j \) and \( 1 \leq m < j \leq \theta \).

Proof: The proof follows from the Theorem 14 and \( |U_i \cap U_j| \leq 1 \iff |\{U : P_m \in U\} \cap \{U : P_l \in U\}| \leq 1 \) for \( 1 \leq m < l \leq \theta \) and \( 1 \leq i < j \leq n \).

D. Fractional Repetition Codes using the Uniform Hypergraphs

For a positive integer \( r \), an \( r \)-uniform hypergraph is a hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) with \( |E| = r \), for each \( E \in \mathcal{E} \). If a conditional hypergraph is an \( r \)-uniform hypergraph then the hypergraph is called an \( r \)-uniform conditional hypergraph.

An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) associated with an \( r \)-uniform conditional hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) will have the same
replication factor for each packet i.e. \( \rho = \rho_j \) for each \( j = 1, 2, \ldots, \theta \).

For a positive integer \( r \), an \( r \)-uniform hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) is called an \( r \)-complete hypergraph denoted by \( K^r_n \) if \( \mathcal{E} = \{ E : E \subseteq V, |E| = r \} \). The following Lemma gives a bound on total number of packets for an FR code.

**Lemma 26.** For an FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with \( \rho = \rho_j \) (for each \( j = 1, 2, \ldots, \theta \)), the total number of distinct packets is \( \theta \leq \binom{n}{\rho} \).

**Proof:** In an \( r \)-complete hypergraph \( K^r_n \) with no parallel hyperedges and isolated hypervertices, \( |\mathcal{E}| = \binom{n}{r} \). An \( r \)-uniform hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) is a sub-hypergraph of the \( r \)-complete hypergraph \( K^r_n \). Hence, \( |\mathcal{E}| \leq \binom{\binom{n}{r}}{r} \). The number of distinct edges in FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) equivalent to a \( \rho \)-uniform hypergraph \( \mathcal{H} = (V; \mathcal{E}) \), is \( \theta \leq \binom{n}{\rho} \).

Note that the condition given in Lemma [26] is derived in [27] for FR codes with symmetric parameters. If an \( r \)-uniform hypergraph is a connected hypergraph then the hypergraph is called a connected \( r \)-uniform hypergraph.

An FR code equivalent to a uniform hypergraph will satisfy the following bound.

**Lemma 27.** An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with \( |U_i \cap U_j| \leq 1 \) (for each \( i, j = 1, 2, \ldots, n \)) and \( \rho_m = \rho \) (for each \( m = 1, 2, \ldots, \theta \)) satisfies \( \theta \leq \frac{n(n-1)}{\rho(\rho-1)} \).

**Proof:** The proof follows from [21] Chapter 1, Theorem 3.

**Remark 28.** The FR code associated with a \( \rho \)-uniform linear conditional hypergraph is GFR code [2]. For a \( \rho \)-uniform linear conditional hypergraph, the Theorem [2] follows the bound on code dimension for GFR code [2] Theorem 4.

**Remark 29.** In [12] Fact 1, it is shown that an IFR code is equivalent to a \( \rho \)-uniform hypergraph.

### E. Fractional Repetition Codes using Regular Hypergraphs

An \( s \)-regular hypergraph is a hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) such that \( |\mathcal{E}(v_i)| = s \), for each \( v_i \in V \). An conditional hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) is called an \( s \)-regular conditional hypergraph if \( |\mathcal{E}(v_i)| = s \), for each \( v_i \in V \).

An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) associated with an \( s \)-regular conditional hypergraph \( \mathcal{H} = (V; \mathcal{E}) \) will have the same node storage capacity for each node i.e. \( \alpha = \alpha_i \) for each \( i = 1, 2, \ldots, n \). The following three Lemmas give bound on the FR codes.

**Lemma 30.** An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) associated with connected \( \alpha \)-regular conditional hypergraph, satisfies the following conditions.

1. \( \sum_{j=1}^{\theta} \rho_j \) is multiple of \( \alpha \),
2. \( \sum_{j=1}^{\theta} \rho_j \geq \frac{\alpha(\theta-1)}{\alpha-1} \),
3. \( \rho \leq \frac{\theta}{\alpha} \sum_{j=1}^{\theta} \rho_j \).

**Proof:** The proof follows from the duality (with respect to hypergraphs) on Lemma [9]

**Lemma 31.** An FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with \( |U_i \cap U_j| \leq 1 \) (for each \( i, j = 1, 2, \ldots, n \)) and \( \alpha_m = \alpha \) (for each \( m = 1, 2, \ldots, n \)) satisfies \( \theta \leq \frac{\theta(\theta-1)}{\alpha(\alpha-1)} \).

A hypervertex of an \( s \)-regular hypergraph \( \mathcal{H} = (V; \mathcal{E}) \), shares \( s \) distinct hyperedges so \( |V| \leq \binom{|\mathcal{E}|}{s} \). The following Lemma ensures the existence of an FR code with the same number of nodes and packets and same packet replication factor and node storage capacity.

**Lemma 32.** For a positive integer \( r \), there exists an FR code \( \mathcal{C} : (n, n, r, r) \) such that \( 2 \leq r \leq n-1 \) and \( \alpha_i = \rho_j = r \) for each \( i = 1, 2, \ldots, n \).

**Proof:** The proof follows from the fact that there exists an \( r \)-uniform \( r \)-regular hypergraph with \( n \) hypervertices for \( 2 \leq r \leq n-1 \).

The node packet distribution of the FR code \( \mathcal{C} : (5, 5, r, r) \) with \( 5 \) nodes and \( \alpha_i = \rho_j = r \) (for each \( i = 1, 2, 3, 4, 5 \)) are following.

- For \( r = 2 \), there exists an FR code \( \mathcal{C} : (5, 5, 2, 2) \) with \( U_1 = \{P_1, P_2\}, U_2 = \{P_2, P_3\}, U_3 = \{P_3, P_4\}, U_4 = \{P_4, P_5\} \) and \( U_5 = \{P_5, P_1\} \).
- For \( r = 3 \), there exists an FR code \( \mathcal{C} : (5, 5, 3, 3) \) with \( U_1 = \{P_1, P_2, P_3\}, U_2 = \{P_2, P_3, P_4\}, U_3 = \{P_3, P_4, P_5\}, U_4 = \{P_4, P_5, P_1\} \) and \( U_5 = \{P_5, P_1, P_2\} \).
- For \( r = 4 \), there exists an FR code \( \mathcal{C} : (5, 5, 4, 4) \) with \( U_1 = \{P_1, P_2, P_3, P_4\}, U_2 = \{P_2, P_3, P_4, P_5\}, U_3 = \{P_3, P_4, P_5, P_1\}, U_4 = \{P_4, P_5, P_1, P_2\} \) and \( U_5 = \{P_5, P_1, P_2, P_3\} \).

**Remark 33.** Each FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) considered in [5] is associated with a \( \rho \)-uniform \( \alpha \)-regular hypergraph \( \mathcal{H} = (V; \mathcal{E}) \).

Recall that the storage capacity of nodes in an FR code is related to the replication factor of packets in the dual FR code and vice versa. Following two Lemmas are based on the properties of duality.

**Lemma 34.** If \( \mathcal{C} : (n, \theta, \alpha, \rho) \) is an FR code with \( \alpha_i = \alpha \) (for each \( i = 1, 2, \ldots, n \)) then the dual FR code \( \mathcal{C}^* : (\alpha^*, \theta^*, \alpha^*, \rho^*) \) is an FR code with \( \rho_j^* = \alpha \), for each \( j = 1, 2, \ldots, \theta^* \).

**Proof:** The Lemma follows from the fact that the dual of an \( r \)-uniform hypergraph is an \( r \)-regular hypergraph.

**Lemma 35.** If \( \mathcal{C} : (n, \theta, \alpha, \rho) \) is an FR code with \( \rho_j = \rho \) (for each \( j = 1, 2, \ldots, \theta \)) then the dual FR code \( \mathcal{C}^* : (\alpha^*, \theta^*, \alpha^*, \rho^*) \) is an FR code with \( \alpha_j^* = \rho \), for each \( i = 1, 2, \ldots, \theta^* \).

**Proof:** The Lemma follows from the fact that the dual of an \( s \)-regular hypergraph is an \( s \)-uniform hypergraph.

**Remark 36.** If \( \mathcal{C} : (n, \theta, \alpha, \rho) \) is an FR code with \( \rho_j = \rho \) and \( \alpha_i = \alpha \) (for each \( j = 1, 2, \ldots, \theta \) and \( i = 1, 2, \ldots, \theta \)) then the dual FR code \( \mathcal{C}^* : (n^*, \theta^*, \alpha^*, \rho^*) \) is an FR code with \( \alpha_i^* = \alpha^* \) and \( \rho_j^* = \rho^* \), for each \( i = 1, 2, \ldots, n^* \) and \( j = 1, 2, \ldots, \theta^* \).

**Remark 37.** In [18], the Transpose code of an FR code is
the FR code corresponding to dual hypergraph. All the FR codes in [18] correspond to a $p$-uniform $\alpha$-regular hypergraph. So, all the Transpose codes are corresponding to an $\alpha$-uniform $p$-regular hypergraph.

F. Analysis

In this subsection, the comparison among FR codes on homogeneous DSSs and heterogeneous DSSs is done by numerical analysis. Consider an FR code with $n = 5$ and $\theta = 10$. For an FR code $C : (n = 5, \theta = 10, \alpha, \rho)$, a data collector connects $k \leq n = 5$ nodes and receives $D_C(k) \leq \theta = 10$ distinct packets. Two plots between the maximum file size (or code dimension) $D_C(k)$ and the reconstruction degree $k$ are given in Figure 3. The * are the points, for the FR code $C_{\text{homo}} : (n = 5, \theta = 10, \alpha = 4, \rho = 2)$ with symmetric parameters [5]. The + are the points, for the codes $C_{\text{hete}} : (n = 5, \theta = 10, \alpha_i \leq 6, \rho_i \leq \rho, i = 1, 2, 3, 4, 5)$ and $\rho_j \leq \rho \leq 4 (j = 1, 2, \ldots, 10)$. The + points are obtained for the FR codes $C_{\text{hete}}$ equivalent to a random hypergraph with $5$ nodes and $10$ distinct packets. In the Figure 3 the code dimension of $C_{\text{homo}}$ and $C_{\text{hete}}$ are same for $k = 0, 4, 5$. For $k = 1, 2, 3$, the code dimension $D_{C_{\text{hete}}}(k) = 1 + D_{C_{\text{homo}}}(k)$. The graph shows that the code dimension of the FR code with symmetric parameters are not more than the code dimension of the FR code with asymmetric parameters.

In an FR code, some encoded packets may be favorable for a data collector because of their functionality and characteristics. For example, if symmetric $(\theta, B)$ MDS code is used in the FR code then the first $B$ packets will be favorable to any user because those packets needs no computation since they are the exact copy of the information packets. One can increase the availability of those packets in the FR code by increasing the replication factor of only those packets. Note that the replication factor of each packet in an FR code with the symmetric parameters are same, so the availability of every packets in the FR code remains the same.

For $n$ nodes and $\theta$ packets, consider an FR code $C_{\text{hete}}$ with asymmetric parameters. In the FR code $C_{\text{hete}}$, a packet $P_j$ is available in $\rho_j$ distinct nodes so, the availability fraction of the packet $P_j$ is $\rho_j/n$. Choosing higher values of the replication factor $\rho_j$, one can increase the availability fraction of the packet $P_j$ in the system. Note that the availability fraction of each packet is fixed with $\rho_j/n$ for an FR code $C_{\text{homo}} : (n, \theta, \ldots, \rho)$ with symmetric parameters.

In an FR code, some nodes may be favorable for a data collector because of their popularity or storage cost. For example, a data collector would like to connect to a node which contains more number of favorable packets and having less storage cost. By permuting the packets, one can find an FR code with the same parameters such that the typical node contains more favorable packets.

For an FR code with asymmetric parameters, the system allows to store more packets in a node. Hence, one can find an FR code with the low storage cost sum by replacing packets between two nodes, where the storage costs of the two packets are distinct.

Now, consider an FR code with $n$ nodes and $\theta$ packets. Let the $c_i \in \mathbb{R}$ be the cost to store one packet in the node $U_i$. For some $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, if $2c_i \leq c_j$ then the cost to store two packets in a node $U_i$ is less then or equal to the cost to store one packet in node $U_j$. Hence, for a given FR code, the following two propositions give the condition of the existence of FR codes with less storage cost sum.

**Proposition 38.** For the given $n, \theta \in \mathbb{N}$, let $C_{\text{homo}} : (n, \theta, \alpha, \rho)$ be an FR code with the symmetric parameters. Let the $c_i \in \mathbb{R}$ $(c_1 \leq c_2 \leq \ldots \leq c_n)$ be the cost to store one packet in the node $U_i$. For some $i, j \in \mathbb{N}$, if $2c_i \leq c_j$ then an FR code $C_{\text{hete}} : (n, \theta, \alpha', \rho')$ with asymmetric parameters exists such that the total cost sum associated with $C_{\text{hete}}$ is less then the total cost sum associated to $C_{\text{homo}}$, where the total sum of node storage capacity of $C_{\text{homo}}$ and $C_{\text{hete}}$ are same.

**Proposition 39.** For given $n, \theta \in \mathbb{N}$, let $C : (n, \theta, \alpha, \rho)$ be an FR code with asymmetric parameters. Let the $c_i \in \mathbb{R}$ $(c_1 \leq c_2 \leq \ldots \leq c_n)$ be the cost to store one packet in the node $U_i$. For some $i, j \in \mathbb{N}$, if $c_i + c_i\alpha_i \leq c_j\alpha_j$ then an FR code $C' : (n, \theta, \alpha', \rho')$ exists with asymmetric parameters such that the total cost sum associated with $C'$ is more then the total cost sum associated to $C$, where the total sum of node storage capacity of $C$ and $C'$ are the same.
### TABLE II
**FR CODES AND RESPECTIVE HYPERGRAPHS**

| FR code | Corresponding conditional hypergraph $\mathcal{H} = (V; \mathcal{E})$ |
|---------|-------------------------------------------------------------------|
| $\mathcal{C} : (n, \theta, \alpha, \rho)$ | $\rho$-uniform and $\alpha$-regular hypergraph |
| FFR code [23] | Linear hypergraph |
| FFR code [23] | $\rho$-uniform hypergraph |
| FR code [12] | $\rho$-uniform and $\alpha$-regular hypergraph |
| FR code [12] | Hypergraph |
| FR code [12] | $\rho$-uniform hypergraph |
| FR code [12] | Hypergraph |
| FR code [12] | $\rho$-uniform and $\alpha$-regular hypergraph |
| FR code [19], [26], [28], [29], [30] | Intersecting, $\rho$-uniform and $\alpha$-regular hypergraph |
| Locally repairable FR code [14], [20] | $\rho$-uniform, $\alpha$-regular and non-linear hypergraph |