The dual Jacobian of a generalised tetrahedron, and volumes of prisms

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Abstract

We derive an analytic formula for the dual Jacobian matrix of a generalised hyperbolic tetrahedron. Two cases are considered: a mildly truncated and a prism truncated tetrahedron. The Jacobian for the latter arises as an analytic continuation of the former, that falls in line with a similar behaviour of the corresponding volume formulae.

Also, we obtain a volume formula for a hyperbolic $n$-gonal prism: the proof requires the above mentioned Jacobian, employed in the analysis of the edge lengths behaviour of such a prism, needed later for the Schl"afli formula.

Key words: hyperbolic polyhedron, Gram matrix, volume.

1 Introduction

Let $T$ be a generalised hyperbolic tetrahedron (in the sense of [19, 21]) depicted in Fig. 1. If the truncating planes associated with its ultra-ideal vertices do not intersect, we call such a tetrahedron mildly truncated, otherwise we call it intensely truncated. If only two of them intersect, we call such a tetrahedron prism truncated [12]. Let us note that a prism truncated orthoscheme is, in fact, a Lambert cube [11].

The volumes of the tetrahedron and its truncations are of particular interest, since they are the simplest representatives of hyperbolic polyhedra. Over the last decade an extensive study produced a number of volume formulae suitable for analytic and numerical exploration [3, 5, 11, 12, 20, 21]. A similar study was done for the spherical tetrahedron [14, 17], which can be viewed as a natural counterpart of the hyperbolic one. Many analytic properties of the volume formula for a hyperbolic tetrahedron came into view concerning the Volume Conjecture [10, 18].

However, other geometric characteristics of a generalised hyperbolic tetrahedron $T$ are also important and bring some useful information. In particular,
Jac(\(T\)), the Jacobian of \(T\), which is the Jacobian matrix of the edge length with respect to the dihedral angles, is such. This matrix enjoys many symmetries [15] and can be computed out of the Gram matrix of \(T\) [7].

In the present paper, we consider \(\text{Jac}^*(T)\), the dual Jacobian of a generalised hyperbolic tetrahedron \(T\). By the dual Jacobian of \(T\) we mean the Jacobian matrix of the dihedral angles with respect to the edge length. Such an object behaves nicely when \(T\) undergoes both mild and intense truncation: the dual Jacobian of a prism truncated tetrahedron is an analytic continuation for that of a mildly doubly truncated one. Let us mention, that the respective volume formulae are also connected by an analytic continuation, in an analogous manner [12, 19].

As an application of our technique, we give a volume formula for a hyperbolic \(n\)-gonal prism, c.f. [4].

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2 Preliminaries

Let \(T\) be a mildly truncated hyperbolic tetrahedron with vertices \(v_k, k \in \{1, 2, 3, 4\}\), dihedral angles \(a_{ij}\) and edge lengths \(\ell_{ij}, i, j \in \{1, 2, 3, 4\}, i < j\). Depending on whether the vertex \(v_k\) is proper, ideal or ultra-ideal, let us
set the quantity $\varepsilon_k$ to be $+1$, $0$ or $-1$, respectively. Let us consider a vertex $v_i$ of $T$ and the face $F_{jkl}$ opposite to it, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let us consider the link $L(v_i)$ of the vertex $v_i$. Such a link is either a spherical triangle ($\varepsilon_k = +1$), a Euclidean triangle ($\varepsilon = 0$) or a hyperbolic triangle ($\varepsilon_k = -1$). Let us define the quantity $b_{jk}^i$ as follows:

$$b_{jk}^i := \begin{cases} 
\text{the plane angle of } F_{jkl} \text{ opposite to the side } v_jv_k, & \text{if } \varepsilon_l = +1; \\
0, & \text{if } \varepsilon_l = 0; \\
\text{the length of the common perpendicular to the sides } v_jv_l \\
\text{and } v_kv_l \text{ of } F_{jkl}, & \text{if } \varepsilon_l = -1.
\end{cases}$$

Let us also define the quantity $\mu_{jk}^i$ by means of the formula

$$\mu_{jk}^i := \int_0^{b_{jk}^i} \cos(\sqrt{\varepsilon_l}s)ds.$$

Let $\mu_{jk}^i$ denote the derivative of $\mu_{jk}^i$ with respect to $b_{jk}^i$, which means that

$$\mu_{jk}^i = \cos(\sqrt{\varepsilon_l}b_{jk}^i).$$

Let $\sigma_{kl}$ denote the following quantity associated with an edge $v_kv_l$, $k, l \in \{1, 2, 3, 4\}$, $k < l$,

$$\sigma_{kl} := \frac{1}{2}e^{\ell_{kl}} - \frac{1}{2}\varepsilon_k\varepsilon_l e^{-\ell_{kl}}.$$

Let $\sigma'_{kl}$ denote the derivative of $\sigma_{kl}$ with respect to $\ell_{kl}$, so we have that

$$\sigma'_{kl} = \frac{1}{2}e^{\ell_{kl}} + \frac{1}{2}\varepsilon_k\varepsilon_l e^{-\ell_{kl}}.$$

Let us define the momentum $M_i$ of the vertex $v_i$ opposite to the face $F_{jkl}$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$ by the following equality (c.f. [6, VII.6]):

$$M_i := \mu_{jk}^i \mu_{jl}^i \sigma_{kl}.$$

The quantity above is well defined grace to the following theorem.

**Theorem 1 (The Sine Law for faces)** Let $F_{jkl}$ be the face of $T$ opposite to the vertex $v_i$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $F_{jkl}$ is a hyperbolic triangle and the following equalities hold:

$$\frac{\mu_{jk}^i}{\sigma_{jk}} = \frac{\mu_{jl}^i}{\sigma_{jl}} = \frac{\mu_{kl}^i}{\sigma_{kl}}.$$
Let us also define the momentum $M_{jkl}$ of the face $F_{jkl}$ opposite to the vertex $v_i, \{i, j, k, l\} = \{1, 2, 3, 4\}$ by setting (c.f. [6, VII.6])

$$M_{jkl} := \mu^i_{jkl} \sin a_{ik} \sin a_{il}.$$

The quantity above is well defined, according to the following theorem.

**Theorem 2 (The Sine Law for links)** Let $v_i$ be the vertex of $T$ opposite to the face $F_{jkl}, \{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $L(v_i)$ is either a spherical, a Euclidean or a hyperbolic triangle and the following equalities hold:

$$\frac{\sin a_{ij}}{\mu^i_{kl}} = \frac{\sin a_{ik}}{\mu^i_{jl}} = \frac{\sin a_{il}}{\mu^i_{jk}}.$$

Both Theorem 1 and Theorem 2 are paraphrases of the spherical, Euclidean or hyperbolic sine laws (for a generalised hyperbolic triangle, see [9]). The following theorems are the cosine laws for a generalised hyperbolic triangle adopted to the notation of the present paper.

**Theorem 3 (The first Cosine Law for faces)** Let $F_{jkl}$ be the face of $T$ opposite to the vertex $v_i, \{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $F_{jkl}$ is a generalised hyperbolic triangle and the following equality holds:

$$\sigma'_{kl} = \frac{\mu^i_{kl} + \mu^i_{jk} \mu^i_{jl}}{\mu^i_{jk} \mu^i_{jl}}.$$

**Theorem 4 (The second Cosine Law for faces)** Let $F_{jkl}$ be the face of $T$ opposite to the vertex $v_i, \{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $F_{jkl}$ is a generalised hyperbolic triangle and the following equality holds:

$$\mu^i_{jk} = \frac{-\varepsilon \sigma'_{kl} + \sigma'_{jl} \sigma'_{kl}}{\sigma_{jl} \sigma_{kl}}.$$

**Theorem 5 (The Cosine Law for Links)** Let $v_i$ be the vertex of $T$ opposite to the face $F_{jkl}, \{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $L(v_i)$ is either a spherical, a Euclidean or a generalised hyperbolic triangle and the following equality holds:

$$\mu^i_{kl} = \frac{\cos a_{ik} + \cos a_{id} \cos a_{il}}{\sin a_{ik} \sin a_{il}}.$$
3 Auxiliary lemmata

In the present section we shall consider various partial derivatives of certain geometric quantities associated with either the faces or the vertex links of a generalised hyperbolic tetrahedron $T$. These derivatives will be used later on in the computation of the entries of $\text{Jac}^*(T)$.

**Lemma 1** For $\{i, j, k, l\} = \{1, 2, 3, 4\}$ we have

$$\frac{\partial \ell_{kl}}{\partial b_{kl}^i} = -\varepsilon_j \frac{\mu_{kl}^i}{M^i},$$

$$\frac{\partial \ell_{kl}}{\partial b_{jk}^i} = -\sigma_{jl}^i \frac{\mu_{kl}^i}{M^i},$$

$$\frac{\partial \ell_{kl}}{\partial b_{jl}^i} = -\sigma_{jk}^i \frac{\mu_{kl}^i}{M^i}.$$

**Proof.** By taking derivatives on both sides of the first Cosine Law for Faces, we get the following formulae:

$$\sigma_{kl} \frac{\partial \ell_{kl}}{\partial b_{kl}^i} = \frac{\partial \sigma_{kl}^i}{\partial b_{kl}^i} = \frac{1}{\mu_{jk}^i \mu_{kl}^j} = -\varepsilon_j \frac{\mu_{kl}^i}{\mu_{jk}^i \mu_{jl}^j},$$

since

$$\frac{\partial \sigma_{kl}^i}{\partial b_{kl}^i} = \sigma_{kl} \frac{\partial \ell_{kl}}{\partial b_{kl}^i}$$

and

$$\frac{\partial \mu_{kl}^i}{\partial b_{kl}^i} = -\varepsilon_j \mu_{kl}^i$$

by a direct computation. This implies the first identity of the lemma.

Now we compute

$$\sigma_{kl} \frac{\partial \ell_{kl}}{\partial b_{jk}^i} = \frac{\sigma_{kl}^i}{\partial b_{jk}^i} = \frac{\left((\mu_{jl}^i)^2 + \varepsilon_l (\mu_{jk}^i)^2\right) \mu_{jl}^i \mu_{kl}^j + \mu_{jl}^i \mu_{jk}^i \mu_{kl}^l}{(\mu_{jl}^i)^2} =$$

$$= \frac{-\mu_{jl}^i + \mu_{jk}^i \mu_{kl}^l \mu_{jl}^i - \mu_{jl}^i \mu_{jk}^i \mu_{kl}^l}{\mu_{jl}^i \mu_{jk}^i \mu_{kl}^l} = -\frac{\partial \ell_{jk}^i}{\partial b_{jk}^i}.$$

where we use the identity $(\mu_{jk}^i)^2 + \varepsilon_l (\mu_{jk}^i)^2 = 1$ and, as before, the fact that $\frac{\partial \mu_{jk}^i}{\partial b_{jk}^i} = -\varepsilon_l \mu_{jk}^i$. Then the second identity follows. The third one is analogous to the second one under the permutation of the indices $k$ and $l$. □
Lemma 2 For \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) we have

\[
\frac{\partial b_{kl}^j}{a_{ij}} = \varepsilon_i \frac{\sin a_{ij}}{M_{jkl}},
\]

\[
\frac{\partial b_{kl}^j}{a_{ik}} = \varepsilon_i \frac{\sin a_{ij} \mu_{jk}}{M_{jkl}},
\]

\[
\frac{\partial b_{kl}^j}{a_{il}} = \varepsilon_i \frac{\sin a_{ij}}{M_{jkl}} \mu_{jk}.
\]

Proof. By taking derivatives on both sides of the Cosine Law for Links, we get the following formulae:

\[
-\varepsilon_i \mu_{kl} \frac{b_{kl}^j}{a_{ij}} = \frac{\partial \mu_{kl}^j}{\partial a_{ij}} = -\cos a_{il} + \cos a_{ij} \cos a_{ik} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{kl}}.
\]

The first identity of the lemma follows. Then we subsequently compute

\[
-\varepsilon_i \mu_{kl} \frac{\partial b_{kl}^j}{a_{ik}} = \frac{\partial \mu_{kl}^j}{\partial a_{ik}} = \mu_{jk} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}}.
\]

The second identity follows. The third one is analogous under the permutation of the indices \( k \) and \( l \). □

Now we shall prove several identities that relate the principal minors of the Gram matrix \( G(T) \) of \( T \) with its face of vertex momenta.

Lemma 3 For \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), we have that

\[
det G_{ii} = \varepsilon_i M_{jkl}.
\]

Proof. Let us perform the computation for \( G_{11} \) and other cases will follow by analogy. We have that

\[
det \left( \begin{array}{ccc} 1 & -\cos a_{14} & -\cos a_{13} \\ -\cos a_{14} & 1 & -\cos a_{12} \\ -\cos a_{13} & -\cos a_{12} & 1 \end{array} \right) =
\]

\[
= \det \left( \begin{array}{ccc} 1 & -\cos a_{14} & -\cos a_{13} \\ 0 & \sin^2 a_{14} & \mu_{34}^2 \sin a_{13} \sin a_{14} \\ 0 & \mu_{34}^2 \sin a_{14} \sin a_{13} \sin a_{14} & \sin^2 a_{13} \sin a_{14} \end{array} \right) =
\]

\[
= (1 - (\mu_{34}^2)^2) \sin^2 a_{13} \sin^2 a_{14} = \varepsilon_1 (\mu_{34}^2)^2 \sin^2 a_{13} \sin^2 a_{14} = \varepsilon_1 M_{234}.
\]

By permuting the set \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), one gets all other identities of the lemma. □
Lemma 4 For \{i, j, k, l\} = \{1, 2, 3, 4\}, we have that
\[- \det G = \sin^2 a_{jk} \sin^2 a_{jl} \sin^2 a_{kl} (M^i)^2.\]

Proof. Let us subsequently compute
\[
\det G = \det \begin{pmatrix}
1 & -\cos a_{34} & -\cos a_{24} & -\cos a_{23} \\
-\cos a_{34} & 1 & -\cos a_{14} & -\cos a_{13} \\
-\cos a_{24} & -\cos a_{14} & 1 & -\cos a_{12} \\
-\cos a_{23} & -\cos a_{13} & -\cos a_{12} & 1
\end{pmatrix} =
\det \begin{pmatrix}
1 & -\cos a_{34} & -\cos a_{24} & -\cos a_{23} \\
0 & \sin^2 a_{34} & -\mu'^1_{23} \sin \alpha_{24} \sin \alpha_{34} & -\mu'^1_{23} \sin \alpha_{23} \sin \alpha_{34} \\
0 & -\mu'^1_{23} \sin \alpha_{24} \sin \alpha_{34} & \sin^2 a_{24} & -\mu'^1_{23} \sin \alpha_{23} \sin \alpha_{24} \\
0 & -\mu'^1_{23} \sin \alpha_{23} \sin \alpha_{24} & -\mu'^1_{23} \sin \alpha_{23} \sin \alpha_{24} & \sin^2 a_{23}
\end{pmatrix} =
\sin^2 a_{23} \sin^2 a_{24} \sin^2 a_{34} \det \begin{pmatrix}
1 & -\mu'^1_{23} & -\mu'^1_{24} \\
-\mu'^1_{23} & 1 & -\mu'^1_{34} \\
-\mu'^1_{24} & -\mu'^1_{34} & 1
\end{pmatrix} =
\sin^2 a_{23} \sin^2 a_{24} \sin^2 a_{34} \det \begin{pmatrix}
1 & -\mu'^1_{23} & -\mu'^1_{24} \\
0 & \varepsilon_4 (\mu'^1_{23})^2 & -\sigma'^1_{34} \mu'^1_{23} \mu'^1_{24} \\
0 & -\sigma'^1_{34} \mu'^1_{23} \mu'^1_{24} & \varepsilon_3 (\mu'^1_{24})^2
\end{pmatrix} =
\sin^2 a_{23} \sin^2 a_{24} \sin^2 a_{34} (\varepsilon_3 \varepsilon_4 - (\sigma'^1_{34})^2 (\mu'^1_{23} \mu'^1_{24})^2 =
- \sin^2 a_{23} \sin^2 a_{24} \sin^2 a_{34} (M^i)^2.
\]

Here we used the Cosine Law for Links in the second equality and the first Cosine Law for Faces in the fourth equality. Also, we used the fact that for \{i, j, k, l\} = \{1, 2, 3, 4\} one has 1 - \varepsilon_i (\mu'^1_{ij})^2 = (\mu'^1_{ij})^2 (in the third equality) and \sigma'^2_{ij} - (\sigma'^1_{ij})^2 = \varepsilon_i \varepsilon_j (in the sixth equality). All other identities of the lemma follow by permuting the set \{i, j, k, l\} = \{1, 2, 3, 4\}. □

4 Dual Jacobian of a generalised hyperbolic tetrahedron

In this section we shall compute the entries of the dual Jacobian matrix \(\Jac^*(T)\) of a generalised hyperbolic tetrahedron \(T\).

Theorem 6 Let \(T\) be a generalised hyperbolic tetrahedron. Then
\[\Jac^*(T) := \frac{\partial (\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34})}{\partial (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})} = -\eta \mathcal{D} \mathcal{S} \mathcal{D},\]

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where

\[ \eta = \left( \frac{\Pi_{i=1}^{4} \det G_{ii}}{(-\det G)^3} \right)^{1/2}, \quad \mathcal{S} = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{24} & \sigma_{34} \end{pmatrix} \]

and

\[ \mathcal{S} = \begin{pmatrix} \omega_{12} & \varepsilon_{1} \sigma_{14}' & \varepsilon_{1} \sigma_{13}' & \varepsilon_{2} \sigma_{24}' & \varepsilon_{2} \sigma_{23}' & \varepsilon_{3} \sigma_{34}' & 1 \\ \varepsilon_{1} \sigma_{14}' & \omega_{13} & \varepsilon_{1} \sigma_{12}' & \varepsilon_{3} \sigma_{34}' & 1 & \varepsilon_{3} \sigma_{32}' & \varepsilon_{4} \sigma_{44}' \\ \varepsilon_{1} \sigma_{13}' & \varepsilon_{1} \sigma_{12}' & \omega_{14} & 1 & \varepsilon_{4} \sigma_{43}' & \varepsilon_{4} \sigma_{42}' & \varepsilon_{3} \sigma_{34}' \\ \varepsilon_{2} \sigma_{24}' & \varepsilon_{3} \sigma_{34}' & 1 & \omega_{23} & \varepsilon_{2} \sigma_{23}' & \varepsilon_{2} \sigma_{21}' & \varepsilon_{4} \sigma_{44}' \\ 1 & \varepsilon_{4} \sigma_{44}' & \varepsilon_{2} \sigma_{23}' & \varepsilon_{4} \sigma_{42}' & \omega_{24} & \varepsilon_{4} \sigma_{44}' & \varepsilon_{4} \sigma_{44}' \\ 1 & \varepsilon_{3} \sigma_{34}' & \varepsilon_{4} \sigma_{44}' & \varepsilon_{3} \sigma_{34}' & \varepsilon_{4} \sigma_{43}' & \omega_{34} & \varepsilon_{3} \sigma_{34}' \end{pmatrix} \],

where

\[ \omega_{kl} = \frac{\sigma_{ik}' \sigma_{jk}' + \varepsilon_{i} \sigma_{il}' \sigma_{kl}' + \sigma_{ij}' \sigma_{jk} + \varepsilon_{k} \sigma_{ik}' \sigma_{jk} \sigma_{kl}'}{\sigma_{kl}^2} \]

**Proof.** We compute the respective derivatives, that constitute the entries of \( \text{Jac}^*(T) \). For \( \{i,j,k,l\} = \{1,2,3,4\} \), one has

\[
\frac{\partial \ell_{kl}}{\partial a_{ij}} = \frac{\partial \ell_{kl}}{\partial b_{ij}} \frac{\partial b_{ij}}{\partial a_{ij}} = -\varepsilon_{i} \frac{\mu_{ij}'}{M_{ij}'} \cdot \varepsilon_{i} \frac{\sin a_{ij}}{M_{ijkl}} = -\frac{1}{M_{ij}'} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} \sigma_{ij} \sigma_{kl} = -\frac{1}{M_{ij}'} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} \sigma_{ij} \sigma_{kl} \]

\[
-\frac{M_{ijkl} M_{ijkl} M_{ijkl}}{(-\det G)^3} \sigma_{ij} \sigma_{kl} = -\sqrt{\frac{\Pi_{i=1}^{4} \varepsilon_{i} \det G_{ij}}{(-\det G)^3}} \sigma_{ij} \sigma_{kl} = \eta \sigma_{ij} \sigma_{kl}. \]

Here we used the definitions of vertex and face momenta, as well as Lemmata \[ \text{[H]} \].

Analogous to the above, we compute for \( \{i,j,k,l\} = \{1,2,3,4\} \),

\[
\frac{\partial \ell_{kl}}{\partial a_{ik}} = \frac{\partial \ell_{kl}}{\partial b_{ik}} \frac{\partial b_{ik}}{\partial a_{ik}} = -\varepsilon_{k} \frac{\mu_{ik}'}{M_{ijkl}} \sigma_{jk} = \]

\[
-\varepsilon_{k} \frac{\sqrt{\varepsilon_{j} \det G_{jj}}}{M_{ij}'} \frac{\sin a_{ik}}{\sqrt{\varepsilon_{k} \det G_{kk}}} \frac{1}{\sigma_{ik} \sigma_{kl}} \sigma_{jk}' = \]

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The proof is completed. □

By Lemmata 1-2, we get

\[-\varepsilon_k \frac{\varepsilon_j \det G_{jj}}{M^i \sin a_{jk} \sin a_{jl}} + \frac{\mu_{ik} \mu_{il} \mu_{ik} \mu_{il}}{M^i} \sigma_{ik} \sigma_{kl} \sigma'_{jk} = \]

\[-\varepsilon_k \frac{\varepsilon_l \det G_{ll}}{M^i \sin a_{kk} \sin a_{kl}} \left( -\varepsilon_k \frac{\varepsilon_k \det G_{kk}}{M^i} \right) \frac{\sigma_{ik} \sigma_{kl} \sigma'_{jk}}{\sqrt{(- \det G)^3}} = \]

\[-\varepsilon_k \left( \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\varepsilon_l \det G_{ll}}{M^i} \right) \frac{\sigma_{ik} \sigma_{kl} \sigma'_{jk}}{\sqrt{(- \det G)^3}} = \]

\[-\varepsilon_k \left( \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\varepsilon_l \det G_{ll}}{M^i} \right) \frac{\sigma_{ik} \sigma_{kl} \sigma'_{jk}}{\sqrt{(- \det G)^3}} = \]

Finally, for \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), we compute the derivative

\[\frac{\partial \ell_{kl}}{\partial a_{kl}} = \frac{\partial \ell_{kl}}{\partial b_{jk}^i} \frac{\partial b_{jk}^i}{a_{kl}} + \frac{\partial \ell_{kl}}{\partial b_{jl}^i} \frac{b_{jl}^i}{a_{kl}}.\]

Since the two terms of the above sum are symmetric under the permutation of \( k \) and \( l \), we may compute only the first one. The second one will be analogous. By Lemmata 1, 2 we get

\[\frac{\partial \ell_{kl}}{\partial a_{kl}} = \frac{\partial \ell_{kl}}{\partial b_{jk}^i} \frac{\partial b_{jk}^i}{a_{kl}} = -\varepsilon_k \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\mu_{ik} \mu_{il} \mu_{ik} \mu_{il}}{M^i} \sigma_{il} \sigma_{kl} \sigma'_{jl} = \]

\[-(\sigma'_{ik} \sigma'_{jl} + \varepsilon_l \sigma'_{il} \sigma'_{jl} \sigma'_{kl}) \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\varepsilon_l \det G_{ll}}{M^i} \frac{\mu_{ik} \mu_{il} \mu_{ik} \mu_{il}}{M^i} \sigma_{il} \sigma_{kl} \sigma'_{jl} = \]

\[-(\sigma'_{ik} \sigma'_{jl} + \varepsilon_l \sigma'_{il} \sigma'_{jl} \sigma'_{kl}) \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\varepsilon_l \det G_{ll}}{M^i} \frac{\mu_{ik} \mu_{il} \mu_{ik} \mu_{il}}{M^i} \sigma_{il} \sigma_{kl} \sigma'_{jl} = \]

\[-(\sigma'_{ik} \sigma'_{jl} + \varepsilon_l \sigma'_{il} \sigma'_{jl} \sigma'_{kl}) \frac{\varepsilon_j \det G_{jj}}{M^i} \frac{\varepsilon_l \det G_{ll}}{M^i} \left( -\varepsilon_k \frac{\varepsilon_k \det G_{kk}}{M^i} \right) \frac{\sigma_{ik} \sigma_{kl} \sigma'_{jl}}{\sqrt{(- \det G)^3}} = -\eta (\sigma'_{ik} \sigma'_{jl} + \varepsilon_l \sigma'_{il} \sigma'_{jl} \sigma'_{kl}).\]

Here we used the second Cosine Law for Faces in the second equality. Thus, we obtain

\[\frac{\partial \ell_{kl}}{\partial a_{kl}} = \frac{\partial \ell_{kl}}{\partial b_{jk}^i} \frac{\partial b_{jk}^i}{a_{kl}} + \frac{\partial \ell_{kl}}{\partial b_{jl}^i} \frac{b_{jl}^i}{a_{kl}} = \]

\[-\eta (\sigma'_{ik} \sigma'_{jl} + \varepsilon_l \sigma'_{il} \sigma'_{jl} \sigma'_{kl}) - \eta (\sigma'_{il} \sigma'_{jl} + \varepsilon_k \sigma'_{ik} \sigma'_{jk} \sigma'_{kl}) = -\eta \omega_{kl} \sigma'_{kl}.\]

The proof is completed. □
5 Dual Jacobian of a doubly truncated hyperbolic tetrahedron

Let us consider the case when \( T \) is a (mildly) doubly truncated tetrahedron depicted in Fig. 2 with dihedral angles \( \theta_i \) and edge lengths \( \ell_i \), \( i \in \{1, 2, 3, 4, 5, 6\} \). We suppose that the vertices cut off by the respective polar planes are \( v_1 \) and \( v_2 \).

Figure 2: Doubly truncated tetrahedron (mild truncation)

If \( T \) is mildly truncated then the formula from Theorem 6 applies. If \( T \) is a prism truncated tetrahedron, as in Fig. 3, with dihedral angles \( \mu, \theta_i \) and edge lengths \( \ell, \ell_i \), \( i \in \{1, 2, 3, 5, 6\} \) then its Gram matrix is given \cite{12, 13} by

\[
G = \begin{pmatrix}
1 & -\cos \theta_1 & -\cos \theta_2 & -\cos \theta_6 \\
-\cos \theta_1 & 1 & -\cos \theta_3 & -\cos \theta_5 \\
-\cos \theta_2 & -\cos \theta_3 & 1 & -\cosh \ell \\
-\cos \theta_6 & -\cos \theta_5 & -\cosh \ell & 1
\end{pmatrix}.
\]

Each link \( L(v_k), k = 1, 2 \), is a hyperbolic quadrilateral with two right same-side angles, which can be seen as a hyperbolic triangle with a single truncated vertex. Each link \( L(v_k), k = 3, 4 \), is a spherical triangle. In the definitions of Section 2 we change each \( b_{ij} \), with \( i \neq j \in \{2, 3, 4\} \), for \( b_{ij}^{\prime} + \frac{i\pi}{2} \) and each \( b_{2j}^{\prime} \), with \( i \neq j \in \{1, 3, 4\} \), for \( b_{2j}^{\prime} + \frac{i\pi}{2} \). Thus, some of the vertex and face momenta become complex numbers. All the trigonometric rules of Section 2 still hold grace to \cite[Section 4.3]{2}. Computing the respective derivatives in a complete analogy to the proof of Theorem 6 we obtain the following statement.
Theorem 7 Let \( T \) be a prism truncated tetrahedron depicted in Fig. 3. Then by means of the analytic continuation \( a_{12} := i \ell, \ell_{12} = i \mu \) we have

\[
\text{Jac}^\ast(T) := \frac{\partial(\mu, \ell, \theta_1, \theta_2, \theta_3, \theta_5, \theta_6)}{\partial(\ell_1, \ell_2, \ell_3, \ell_5, \ell_6)} = \frac{\partial(\ell_{12}, \ell_{34}, \ell_{13}, \ell_{23}, \ell_{24}, \ell_{14})}{\partial(a_{12}, a_{34}, a_{13}, a_{23}, a_{24}, a_{14})}.
\]

6 Volume of a hyperbolic prism

Let \( \alpha_n \) denote the \( n \)-tuple \( \alpha := (\alpha_1, \ldots, \alpha_n) \) with \( 0 < \alpha_k < \pi, k = 1, \ldots, n \). Let \( \beta_n \) and \( \gamma_n \) be analogous \( n \)-tuples. Let \( \Pi(\alpha_n, \beta_n, \gamma_n) \) be the hyperbolic \( n \)-sided prism depicted in Fig. 4 with the respective dihedral angles, as shown in the picture.

Let \( S_k, k = 1, \ldots, n \), be the supporting hyperplane for the \( k \)-th side face of the prism, let \( S_0 \) be that of the top face and \( S_{n+1}, - \) of the bottom face. Let \( S_k^+, k = 0, \ldots, n + 1 \), be the respective half-spaces, such that \( \Pi := \bigcap_{i=0}^{n+1} S_i^+ \). Let \( T := T(\alpha, \alpha', \beta, \beta'; \ell) \) be the prism-truncated tetrahedron depicted in Fig. 5. Here \( \alpha, \alpha' \) and \( \beta, \beta' \) are the respective dihedral angles, \( \ell \) is the length of the respective edge. The volume \( \text{Vol} T \) of the tetrahedron \( T \) is given by \[ \text{in Section 7 we give a simplified formula for the volume of } T \]

\[ 11 \]
Figure 4: The prism $\Pi(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$

**Theorem 8** Let $\Pi := \Pi(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$ be a hyperbolic $n$-sided prism, as in Fig. 4. If $p_0p_{n+1} \subset \Pi$, then the volume of $\Pi$ is given by the formula

$$\text{Vol} \, \Pi = \sum_{k=1}^{n} v(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k; \ell^*)$$

where $\ell^*$ is the unique solution to the equation $\frac{d\Phi}{d\ell}(\ell) = 0$, with

$$\Phi(\ell) := \pi \ell - \sum_{k=1}^{n} v(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k; \ell).$$

Let $P_k$, $k = 1, \ldots, n$, be the plane containing $p_0p_{n+1}$ and orthogonal to $S_k$. First, we consider the case when $p_0p_{n+1}$ lies inside the prism $\Pi$ and the planes $P_k$, $k = 1, \ldots, n$, divide the prism $\Pi$ into $n$ prism truncated tetrahedra, as shown in Fig. 6. Then each $P_k$ meets the $k$-th side face of the prism $\Pi$. Obviously, the planes $S_0$, $S_k$, $S_{k+1}$ and $S_{n+1}$ together with $P_k$ and $P_{k+1}$ are the supporting planes for the faces of a prism truncated tetrahedron $T_k$. Each $P_k$ is orthogonal to $S_k$, $S_0$ and $S_{n+1}$. The remaining dihedral angles are easily identifiable. Let $\mu_k$ denote the dihedral angle along the edge $p_0p_{n+1}$ and let $\ell^*$ be its length. Then we have $T_k = T(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k; \ell^*)$, $k = 1, \ldots, n$. Clearly,

$$\text{Vol} \, \Pi = \sum_{i=1}^{n} \text{Vol} \, T_k = \sum_{k=1}^{n} v(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k; \ell^*).$$
Thus, we have to prove only the following statement.

**Proposition 1** If the common perpendicular $p_0p_{n+1}$ is inside the prism $\Pi$ and each $P_k$ meets the respective side also inside $\Pi$, $k = 1, \ldots, n$, then the equation $\frac{\partial \Phi}{\partial \ell} = 0$ has a unique solution $\ell = \ell^*$, the length of $p_0p_{n+1}$.

**Proof.** Let us consider the collection of prism truncated tetrahedra $T_k := T(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k, \gamma_{k+1}; \ell)$, $k = 1, \ldots, n$. Each pair $\{T_k, T_{k+1}\}$ of them has an isometric face corresponding to the plane $P_k$. Indeed, each such a face is completely determined by the plane angles (two right angles at the side of length $\ell$, the angles $\alpha_k$ and $\alpha_{k+1}$ at the opposite side) and one side length. We obtain the prism $\Pi(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$ by gluing the tetrahedra $T_k$ together along the faces $P_k$, $k = 1, \ldots, n$, in the respective order. Their edges of length $\ell$ match together, and one obtains a prism if the angle sum of the dihedral angles $\mu_k$, $k = 1, \ldots, n$, along them equals $2\pi$. Under the conditions of the proposition, we have that

$$\frac{\partial \Phi}{\partial \ell} = \pi - \sum_{k=1}^{n} \frac{\partial v}{\partial \ell}(\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}, \gamma_k, \gamma_{k+1}; \ell).$$

Since $v$ if the volume function from [12, Theorem 1], then by applying the Schäfli formula [16, Equation 1] one obtains

$$\frac{\partial \Phi}{\partial \ell} = \pi - \frac{1}{2} \sum_{k=1}^{n} \mu_k.$$
Thus, whenever the tetrahedra $T_k$ constitute a prism, we have $\sum_{k=1}^{n} \mu_k = 2\pi$ or, equivalently, $\frac{\partial \mu}{\partial \ell} = 0$. The length $\ell$ in this case is exactly the length of the common perpendicular $p_0p_{n+1}$ to the planes $S_0$ and $S_{n+1}$.

The rest is to prove that $\ell = \ell^*$ is a unique solution. In order to do so, we shall show that $\frac{\partial \mu}{\partial \ell} > 0$, $k = 1, \ldots, n$. By using Theorem 7 we get the following formulae for a prism truncated tetrahedron (as depicted in Fig. 3):

$$\frac{\partial \ell_2}{\partial \ell} = -\eta \sin \mu \sinh \ell_6 \cosh \ell_3, \quad \frac{\partial \ell_3}{\partial \ell} = -\eta \sin \mu \sinh \ell_5 \cosh \ell_2,$$

$$\frac{\partial \ell_5}{\partial \ell} = -\eta \sin \mu \sinh \ell_2 \cosh \ell_5, \quad \frac{\partial \ell_6}{\partial \ell} = -\eta \sin \mu \sinh \ell_3 \cosh \ell_6.$$

Note that the above derivatives are all negative. In our present notation it means that for each prism truncated tetrahedron $T_k$, $k = 1, \ldots, n$, the edges of the top and bottom faces inherited from the prism $\Pi$ diminish their length if we increase solely the parameter $\ell$. Let us denote $T_k := T(\alpha_k, \alpha_k \oplus 1, \beta_k, \beta_k \oplus 1; \gamma_k \oplus 1; \ell)$ and $T'_k := T(\alpha_k, \alpha_k \oplus 1, \beta_k, \beta_k \oplus 1; \gamma_k \oplus 1; \ell')$ with $\ell' > \ell$.

Let $ABCD$ be the top (equiv., bottom) face of $T_k$, as shown in Fig. 7 and $A'B'C'D'$ be the top (equiv., bottom) face of $T'_k$. Since the dihedral angles accept for $\mu_k$ and $\mu'_k$ remain the same, the plane angles of $ABCD$ at $A$, $B$, $C$ and those of $A'B'C'D'$ at $A'$, $B'$ and $C'$ are respectively equal. One sees easily that we can match then $ABCD$ and $A'B'C'D'$ such that $B$ and $B'$
Figure 7: Prisms $T_k$ and $T'_k$ with top faces marked

coincide, the sides $AB$ and $A'B'$, $BC$ and $B'C'$ overlap and the point $D'$ lies inside the quadrilateral $ABCD$. Then the area of $A'B'C'D'$ is less than that of $ABCD$. Equivalently, by the angle defect formula [1, Theorem 1.1.7], $\mu'_k > \mu_k$. Thus, $\frac{\partial \mu_k}{\partial k} > 0$, $k = 1, \ldots, n$, and the proposition follows. □

However, there is a possibility that, although the common perpendicular $p_0p_{n+1}$ is entirely inside the prism $\Pi$, one (or several) of the planes $P_k$ meets the respective $S_k$ partially outside of the face $S_k$.

First we consider the case when a single plane $P_k$ meets $S_k$ outside, as depicted in Fig. □. Like this, we obtain the figure shaded in grey, that consists of two triangular prisms sharing an edge. Thus the planes $P_0$, $P_k$, $P_{k\oplus 1}$, $S_k$, $S_{k\oplus 1}$ and $S_{n+1}$ bound a “butterfly” prism. We put $k = 1$, for clarity. In the general case, $k \geq 2$, one uses induction on the number of planes $P_k$ meeting $S_k$ outside of $\Pi$.

**Proposition 2** If the common perpendicular $p_0p_{n+1}$ is completely inside the prism $\Pi$, the plane $P_1$ meets the plane $S_1$ outside of $\Pi$, and all other $P_k$, $k = 2, \ldots, n$, meet the respective side faces inside $\Pi$, then the volume of the prism equals

$$\text{Vol } \Pi = \sum_{i=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k; \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell^*)$$
Figure 8: The decomposition of Π (top view, on the left) and the “butterfly” prism truncated tetrahedron $T_k$ (on the right)

where $\ell^*$ is the unique solution to the equation $\frac{\partial \Phi}{\partial \ell}(\ell) = 0$, with

$$\Phi(\ell) := \pi \ell - \sum_{i=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_k; \ell).$$

Proof. We observe that the “butterfly” prism $T_1$ overlaps with the subsequent prism truncated tetrahedron $T_2$ exactly on its part $T_1^{(o)}$ outside of Π. The part of $T_1$ inside Π, called $T_1^{(i)}$, contributes to the total volume of the prism. The volume of $T_1^{(o)}$ is excessive in the respective volume formula and should be subtracted. In fact, we prove that

$$v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^*) = V := \text{Vol } T_1^{(i)} - \text{Vol } T_1^{(o)} ,$$

which implies that the excess in volume brought by $T_2$ is eliminated by the term “$-\text{Vol } T_1^{(o)}$”.

In order to do so, let us denote by $\theta$ the dihedral angle along the common edge of the triangular prisms $T_1^{(o)}$ and $T_1^{(i)}$. Let $\ell_{\theta}$ be the length of this edge. Let $\gamma := \gamma_2$ and let $\ell_{\gamma}$ be the length of the vertical edge with dihedral angle $\gamma$. We know that $\frac{\partial V}{\partial \gamma} = -\frac{1}{2} \ell_{\gamma}$, by the structure of the volume formula for a prism truncated tetrahedron. Indeed, the function $V$ does not correspond to
the volume of a real prism truncated tetrahedron any more, however all the
metric relations defining the dihedral angles between the respective planes
are preserved. Thus, after computing the derivative \( \frac{\partial V}{\partial \ell} \) analogous to \([12]\),
we obtain the latter equality. Now we compute the respective derivatives
for the parts of the “butterfly” prism \( T_1 \).

![Figure 9: A “butterfly” prism truncated tetrahedron](image)

Observe that the parameter \( \theta \) depends on \( \gamma \), while we vary \( \gamma \) and keep all
other dihedral angles fixed. Let us denote \( \hat{\gamma} = \pi - \gamma \) for brevity. We have
that

\[
\frac{\partial \text{Vol} T_1^{(o)}}{\partial \hat{\gamma}} = -\frac{\ell_\gamma}{2} - \frac{\ell_\theta}{2} \frac{\partial \theta}{\partial \hat{\gamma}}
\]

and

\[
\frac{\partial \text{Vol} T_1^{(i)}}{\partial \gamma} = -\frac{\ell_\theta}{2} \frac{\partial \theta}{\partial \gamma},
\]

by the Schl"afli formula \([16] \text{ Equation 1}\).

The above identities together with the fact that \( \frac{\partial}{\partial \gamma} = -\frac{\partial}{\partial \hat{\gamma}} \) imply that

\[
\frac{\partial}{\partial \gamma_2} v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^*) = \frac{\partial V}{\partial \gamma_2}.
\]

By analogy, we can prove that

\[
\frac{\partial}{\partial \xi} v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^*) = \frac{\partial V}{\partial \xi},
\]

17
for any $\xi \in \{\alpha_1, \alpha_n, \beta_1, \beta_n, \mu_1\}$. The volume formula for a prism truncated tetrahedron implies that by setting $\alpha_1 = \alpha_n = \pi/2$ and $\beta_1 = \beta_n = \pi/2$ we get $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma; \ell^*) = 0$. In the case of a “butterfly” prism $T_1$, under the same assignment of dihedral angles, we have that the bases of the two triangular prisms become orthogonal to their lateral sides. Thus $T_1^{(i)}$ and $T_1^{(o)}$ degenerate into Euclidean prisms, which means that their volumes tend to zero. Thus, we obtain the identity $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma_2; \ell^*) = V$.

The proof of the monotonicity for the function $\frac{\partial \Phi}{\partial \ell}(\ell)$ is analogous to that in Proposition 1. However, since the part $T_1^{(o)}$ contributes to the function $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma_2; \ell)$ with the negative sign, we have to replace the edge lengths $\ell_3$ and $\ell_6$ with $-\ell_3$ and $-\ell_6$, respectively, as shown in Fig. 9. Then we recompute the respective derivatives of the lengths of the horizontal edges according to Theorem 7. We obtain that although the lengths $\ell_2$ and $\ell_5$ diminish, as before, the lengths $\ell_3$ and $\ell_6$ increase. This implies that the upper (resp., lower) triangular base of $T_1^{(o)}$ can be placed entirely inside the upper (resp. lower) triangular base of $T_1^{(o)}$. By the area comparison argument, we have that $\mu_1' > \mu_1$. The inequality $\frac{\partial \mu_1'}{\partial \ell} > 0$ follows. □

Remark. In the general case, when the common perpendicular $p_0p_{n+1}$ does not lie entirely inside the prism $\Pi$, we expect that an analogue to Theorem 8 holds with an exception that the equation $\frac{\partial \Phi}{\partial \ell}(\ell) = 0$ may have several solutions. However, one of these solutions is geometric and yields the volume of $\Pi$.

7 Modified volume formula

We modify the volume formula for a prism truncated tetrahedron from [12], in order to reduce it to a simpler form. Indeed, the formula in [12, Theorem 1] uses analytic continuation and accounts for possible branching with respect to any variable $a_j = e^\ell$, with some $j \in \{1, 2, \ldots, 6\}$, and $a_k = e^{i \theta_k}$, for any $k \in \{1, 2, \ldots, 6\} \setminus \{j\}$. Usually, we put $j = 4$ for simplicity. However, the formula allows for intense truncation at any edge, since it is invariant under a permutation of the variables $a_l, l = 1, 6$.

In our case, given a prism $\Pi$ and its decomposition into prism truncated tetrahedra $\Pi_i, i = 1, \ldots, n$, we know that only the common perpendicular $p_0p_{n+1}$ is produced by an intense truncation. Thus, we can always put $j = 4$ and, moreover, the variable $a_4$ will be the only one that might cause branching. In this case, we suggest a simplified version of the formula from [12, Theorem 1]. This formula also has less numeric discrepancies and performs faster, if
used for an actual computation.

Let \( a_k = e^{i\theta_k} \), \( k \in \{1, 2, 3, 5, 6\} \), \( a_4 = e^f \). Let \( \mathcal{U} = \mathcal{U}(a_1, a_2, a_3, a_4, a_5, a_6, z) \) denote the function

\[
\mathcal{U} = \text{Li}_2(z) + \text{Li}_2(a_1 a_2 a_4 a_5 z) + \text{Li}_2(a_1 a_3 a_4 a_6 z) + \text{Li}_2(a_2 a_3 a_5 a_6 z) - \text{Li}_2(-a_1 a_2 a_3 z) - \text{Li}_2(-a_1 a_5 a_6 z) - \text{Li}_2(-a_2 a_4 a_6 z) - \text{Li}_2(-a_3 a_4 a_5 z),
\]

where \( \text{Li}_2(\cdot) \) is the dilogarithm function.

Let \( z_- \) and \( z_+ \) be two solutions to the equation \( e^{\frac{a_q z}{a_r}} = 1 \) in the variable \( z \). According to [12, 20], these are

\[
z_- = -\frac{q_1 - \sqrt{\frac{q_1^2}{2} - 4q_0 q_2}}{2q_2} \quad \text{and} \quad z_+ = -\frac{q_1 + \sqrt{\frac{q_1^2}{2} - 4q_0 q_2}}{2q_2},
\]

where

\[
q_0 = 1 + a_1 a_2 a_3 + a_1 a_5 a_6 + a_2 a_4 a_6 + a_3 a_4 a_5 + a_1 a_2 a_4 a_5 + a_1 a_3 a_4 a_6 + a_2 a_3 a_5 a_6,
\]

\[
q_1 = -a_1 a_2 a_3 a_4 a_5 a_6 \left( \left( a_1 - \frac{1}{a_1} \right) \left( a_4 - \frac{1}{a_4} \right) + \left( a_2 - \frac{1}{a_2} \right) \left( a_5 - \frac{1}{a_5} \right) + \left( a_3 - \frac{1}{a_3} \right) \left( a_6 - \frac{1}{a_6} \right) \right),
\]

\[
q_2 = a_1 a_2 a_3 a_4 a_5 a_6 (a_1 a_4 + a_2 a_5 + a_3 a_6 + a_1 a_2 a_4 + a_1 a_3 a_5 + a_2 a_3 a_4 + a_4 a_5 a_6 + a_1 a_2 a_3 a_4 a_5 a_6).
\]

Given a function \( f(x, y, \ldots, z) \), let \( f(x, y, \ldots, z) \mid_{z=\pm} \) denote the difference \( f(x, y, \ldots, z_-) - f(x, y, \ldots, z_+) \). Now we define the following function \( \mathcal{Y} = \mathcal{Y}(a_1, a_2, a_3, a_4, a_5, a_6, z) \) by means of the equality

\[
\mathcal{Y} = \frac{i}{4} \left( \mathcal{U}(a_1, a_2, a_3, a_4, a_5, a_6, z) - z \frac{\partial \mathcal{U}}{\partial z} \log z \right) \mid_{z=\pm}.
\]

In order to avoid excessive branching in numerical computations, we put

\[
q_i' = \frac{q_i}{\prod_{k=1}^6 a_k} \quad \text{and} \quad z_\pm = -\frac{q_1' - \sqrt{\frac{q_1'^2}{2} - 4q_0' q_2'}}{2q_2'}.
\]

It follows from the definition of \( q_i' \), \( i = 1, 2, 3 \), above that the quantity \( q_1'^2 - 4q_0' q_2' \) is a real number, c.f. [20, Section 1.1, Lemma]. This fact prevents computational discrepancies and simplifies any further numerical analysis of the volume formula.
Proposition 3 The volume of a prism truncated tetrahedron $T$ is given by

$$\text{Vol } T = \Re \left( -\mathcal{V} + a_4 \frac{\partial \mathcal{V}}{\partial a_4} \log a_4 \right).$$

Proof. Let us denote

$$f(T) = \Re \left( -\mathcal{V} + a_4 \frac{\partial \mathcal{V}}{\partial a_4} \log a_4 \right),$$

and compute the derivative

$$\frac{\partial}{\partial \ell} \left( f(T) + \frac{\mu \ell}{2} \right) = a_4 \frac{\partial}{\partial a_4} \left( f(T) + \frac{\mu \log a_4}{2} \right) =$$

$$= a_4 \frac{\partial}{\partial a_4} \left( \Re \left( -\mathcal{V} + \left( a_4 \frac{\partial \mathcal{V}}{\partial a_4} + \frac{\mu}{2} \right) \log a_4 \right) \right).$$

From the proof of [12, Theorem 1], we know that the function $a_4 \frac{\partial \mathcal{V}}{\partial a_4} + \frac{\mu}{2}$ has an a.e. vanishing derivative. Hence,

$$\frac{\partial}{\partial \ell} \left( f(T) + \frac{\mu \ell}{2} \right) = a_4 \frac{\partial}{\partial a_4} \left( \Re \left( -\mathcal{V} + \left( a_4 \frac{\partial \mathcal{V}}{\partial a_4} + \frac{\mu}{2} \right) \log a_4 \right) \right) =$$

$$= \Re \left( -a_4 \frac{\partial \mathcal{V}}{\partial a_4} + a_4 \frac{\partial \mathcal{V}}{\partial a_4} + \frac{\mu}{2} \right) = \frac{\mu}{2}.$$

This implies that $\frac{\partial f(T)}{\partial \mu} = -\frac{\ell}{2}$. By analogy to the proof of [12, Theorem 1], we can show that $\frac{\partial f(T)}{\partial \ell} = -\frac{\ell}{2}$, and that if $T$ degenerates into a right Euclidean prism, then $f(T) \to 0$. Thus, $\text{Vol } T = f(T)$ and the proposition follows. □

Also, we have the following way to determine the dihedral angle $\mu$ along the length $\ell$ edge coming from the intense truncation.

Proposition 4 The angle $\mu$ is given by

$$\mu \equiv \frac{i a_4}{2} \frac{\partial \mathcal{V} (a_1, \ldots, a_6, z)}{\partial a_4} \bigg|_{z=-z_-}^{z=z_+} \mod \pi.$$

Proof. From the proof of [12, Theorem 1], we already have that a.e.

$$a_4 \frac{\partial \mathcal{V}}{\partial a_4} + \frac{\mu}{2} = \frac{\pi}{2} k,$$
for some $k \in \mathbb{Z}$. Hence $\mu$ equals $2\alpha_4 \frac{\partial \mathcal{V}}{\partial a_4}$ modulo $\pi$, where $0 < \mu < \pi$.

Moreover, we have that

$$
\left. \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z \pm(a_1, \ldots, a_6))}{\partial a_4} \right|_{z = z_{\pm}} + \left. \frac{\partial}{\partial a_4} \left( z_{\pm} \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial z} \right|_{z = z_{\pm}} \log z_{\pm} \right) =
$$

$$
\left. \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial a_4} \right|_{z = z_{\pm}} + \left. \frac{\partial}{\partial a_4} \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial z} \right|_{z = z_{\pm}}
$$

(3)

$$
\left. \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial a_4} \right|_{z = z_{\pm}},
$$

(4)

since, for some $m \in \mathbb{Z}$,

$$
z_{\pm} \left. \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial z} \right|_{z = z_{\pm}} = 2\pi i m,
$$

by definition of the quantities $z_-$ and $z_+$.

Therefore, we obtain

$$
\mu \equiv 2\alpha_4 \frac{\partial \mathcal{V}}{\partial a_4} \mod \pi \equiv \frac{i \alpha_4}{2} \left. \frac{\partial \mathcal{V}(a_1, \ldots, a_6, z)}{\partial a_4} \right|_{z = z_{\pm}} \mod \pi,
$$

where $0 < \mu < \pi$. □

8 Numerical examples

Finally, we produce some numerical examples concerning an $n$-gonal ($n \geq 5$) prism $\Pi_n$ with the following distribution of dihedral angles: the angles along the vertical edges are $\frac{2\pi}{5}$, the angles adjacent to the bottom face are $\frac{\pi}{3}$, and those adjacent to the top face are $\frac{\pi}{2}$. Indeed, such a prism $\Pi_n$ exists due to [8, Theorem 1.1]. Then we apply Theorem 8 for the cases $n = 5, 6, 7$, and perform all necessary numeric computations with Wolfram Mathematica.

Each of the above prisms $\Pi_n$ is subdivided into $n$ isometric copies of a prism truncated tetrahedron $T_n$. The graph of $\text{Vol} T_n$, with $n = 5$, as a function of $\ell$, is shown in Fig. 10 on the left. The graph of $\frac{\partial \Phi}{\partial \ell}(\ell)$ for the same prism truncated tetrahedron $T_n$ is depicted in Fig. 11 on the right. We observe that the function $\frac{\partial \Phi}{\partial \ell}(\ell)$ is indeed monotone and has a single zero $\ell \approx 0.50672$...
The volume of $T_n$ with $\theta_1 = \frac{2\pi}{5}$, $\theta_2 = \theta_3 = \frac{\pi}{2}$, $\theta_5 = \theta_6 = \frac{\pi}{3}$ and $\ell \approx 0.50672...$ equals $\sim 0.52639...$ by Proposition 3. Thus, we can see that $\text{Vol} \Pi_n = n \cdot \text{Vol} T_n$ (here we set $n = 5$). As well, we have that $\mu = \frac{2\pi}{5} \approx 1.25664...$ by Proposition 4.

Table 1: Left: parameters $(\ell, \mu)$ of $T_n$, right: volume of $\Pi_n$

| $n$ | $(\ell, \mu)$ | $\text{Vol} \Pi_n$ |
|-----|---------------|---------------------|
| 5   | $(0.50672, 2\pi/5)$ | 2.63200             |
| 6   | $(0.38360, \pi/3)$   | 3.43626             |
| 7   | $(0.312595, 2\pi/7)$ | 4.19077             |

Figure 10: Left: $\text{Vol} T_n$, right: $\frac{\partial \Phi}{\partial \ell}$, both as functions of $\ell$

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