RANDOMIZED COORDINATE SUBGRADIENT METHOD FOR NONSMOOTH OPTIMIZATION*

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Lei Zhao
Institute of Translational Medicine
National Center for Translational Medicine
Shanghai Jiao Tong University
Shanghai, China 200240
l.zhao@sjtu.edu.cn

Ding Chen
Alibaba Group
Hangzhou, China 310000
ppposchen@gmail.com

Daoli Zhu
Antai College of Economics and Management
Sino-US Global Logistics Institute
Shanghai Jiao Tong University
Shanghai, China 200030
dlzhu@sjtu.edu.cn

Xiao Li
School of Data Science
The Chinese University of Hong Kong, Shenzhen
Shenzhen, China 518172
lixiao@cuhk.edu.cn

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ABSTRACT

Nonsmooth optimization finds wide applications in many engineering applications. In this work, we propose the Randomized Coordinate Subgradient method (RCS) for solving nonsmooth convex and nonsmooth nonconvex (nonsmooth weakly convex) optimization problems. RCS randomly selects one block coordinate to update at each iteration, making it more practical than updating all coordinates. We consider the linearly bounded subgradients assumption for the objective function, which is more general than the traditional Lipschitz continuity assumption, to account for practical scenarios. We then conduct thorough convergence analysis for RCS in both convex and nonconvex cases based on this generalized Lipschitz-type assumption. Specifically, we establish the $\tilde{O}(1/\sqrt{k})$ convergence rate in expectation and the $\tilde{o}(1/\sqrt{k})$ almost sure asymptotic convergence rate in terms of suboptimality gap when $f$ is nonsmooth convex. If $f$ further satisfies the global quadratic growth condition, the improved $O(1/k)$ rate is shown in terms of the squared distance to the optimal solution set. For the case when $f$ is nonsmooth weakly convex and its subdifferential satisfies the global metric subregularity property, we derive the $O(1/T^{1/4})$ iteration complexity in expectation, where $T$ is the total number of iterations. We also establish an asymptotic convergence result. To justify the global metric subregularity property utilized in the analysis, we establish this error bound condition for the concrete (real valued) robust phase retrieval problem, which is of independent interest. We provide a convergence lemma and the relationship between the global metric subregularity properties of a weakly convex function and its Moreau envelope, which are also of independent interest. Finally, we conduct several experiments to demonstrate the possible superiority of RCS over the subgradient method.

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1 Introduction

Coordinate-type methods tackle optimization problems by successively solving simpler (even scalar) subproblems along coordinate directions. These methods are widely used in signal processing, machine learning, and other engineering fields due to their simplicity and efficiency, as highlighted in the comprehensive review [29]. Coordinate methods are particularly useful when the problem size (measured as the dimension of the variable) is enormous and the computation of function values or full gradients can exhaust memory, or when the problem data is incrementally received or distributed across a network, requiring computation with whatever data is currently available. For more motivational statements, we refer to [21, 24, 29].

1.1 Literature of coordinate methods and motivations

Existing works in this field mainly focus on extending gradient-based methods (e.g., gradient descent, primal-dual method, etc) to coordinate methods; see, e.g., [1, 15, 19, 21, 24, 29, 35] and the references therein. In contrast, a parallel line of research on implementing coordinate-type extension for subgradient oracle is largely unexplored and the related works in this line are surprisingly limited (to the best of our knowledge). In the pioneering work [22], Nesterov proposed a randomized coordinate extension of the subgradient method for solving piece-wise linear convex optimization problems. The convergence rate result in expectation was reported by utilizing the impractical Polyak’s step sizes rule. In the analysis, randomization plays the key role for linking the analysis to that of the full subgradient method. Later, the work [6] follows the same randomization idea and establishes similar convergence rate result in expectation with a more practical constant step size that is proportional to the target accuracy. Unfortunately, these results only apply to convex and Lipschitz continuous optimization, which cannot cover many signal processing and machine learning problems; see Applications 1–3 listed below. In addition, only convergence result in expectation was reported.

Motivated by the above observations, we introduce the Randomized Coordinate Subgradient Method (RCS) (see Algorithm 1) for solving the more general nonsmooth convex / nonsmooth nonconvex optimization problem

\[ \min_{x \in \mathbb{R}^d} f(x), \]  

where \( f : \mathbb{R}^d \to \mathbb{R} \) is assumed to be proper and lower-semicontinuous. Throughout this work, we will not assume \( f \) is Lipschitz continuous. We decompose \( \mathbb{R}^d \) into \( N \) subspaces \( \mathbb{R}^d = \bigotimes_{i=1}^{N} \mathbb{R}^{d_i} \), with \( d = \sum_{i=1}^{N} d_i \). RCS randomly selects one block coordinate with dimension \( d_i \) rather than all the coordinates with dimension \( d \) to update at each iteration, which is the main difference to the full subgradient method.

Problem (1) covers many nonsmooth signal processing and machine learning problems as special instances. We list three of them below.

**Application 1: Robust M-estimators.** The Robust M-estimators problem is defined as follows [17]:

\[ \min_{x \in \mathbb{R}^d} f(x) := \ell(Ax - b) + \sum_{i=1}^{d} \phi_p(x_i), \]  

where \( A \in \mathbb{R}^{n \times d} \) is a matrix, \( b \in \mathbb{R}^n \) is a vector, \( \ell \) is a loss function that could be \( \ell_1 \)-norm or some nonconvex loss functions such as MCP [30] and SCAD [10], and \( \phi_p \) is a penalty function such as \( \ell_1 \)-norm, MCP, SCAD, etc.

**Application 2: SVM.** Support vector machine (SVM) is a popular supervised learning method, which is widely used for classification. The linear SVM problem can be expressed as [4, 31]

\[ \min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} \max \left\{ 0, 1 - b_i (a_i^\top x) \right\} + \frac{p}{2} \|x\|^2, \]  

where \( p > 0, a_i \in \mathbb{R}^d \) is a vector, and \( b_i \in \{\pm 1\}, i = 1, \ldots, n \).

**Application 3: (Real-valued) Robust phase retrieval problem.** Phase retrieval is a common computational problem with applications in diverse areas such as physics, imaging science, X-ray crystallography, and signal processing [11]. The (real-valued) robust phase retrieval problem amounts to solving [9]

\[ \min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \|(Ax)^{\circ 2} - b^{\circ 2}\|_1, \]  

where \( A = (a_1, \ldots, a_n)^\top \in \mathbb{R}^{n \times d} \) is the data matrix, \( b = (b_1, \ldots, b_n)^\top \in \mathbb{R}^n \) is a vector, and \( \circ 2 \) is the Hadamard power.
Algorithm 1 RCS: Randomized Coordinate Subgradient Method for Solving (1)

Initialization: $x^0$ and $\alpha_0$;
1: for $k = 0, 1, \ldots$ do
2: Choose the coordinate index $i(k)$ from $\{1, \ldots, N\}$ uniformly at random;
3: Compute a coordinate subgradient $r^k_{i(k)} \in \partial_i f(x^k)$, where $\partial_i f(x^k)$ is the $i$-th coordinate subdifferential;
4: Update the step size $\alpha_k$ according to a certain rule;
5: Update $x^k_{i(k)} = x^k_{i(k)} - \alpha_k r^k_{i(k)}$, while keep $x^k_j = x^0_j$ for all other $j \neq i(k)$.
6: end for

It is interesting to observe that if $f$ in problem (2) is Lipschitz continuous but can be nonsmooth weakly convex. $f$ in problem (3) is convex. However, it is not Lipschitz continuous due to the weight decay term $\|x\|^2$. $f$ in problem (4) is simultaneously nonsmooth weakly convex and non-Lipschitz continuous. Apart from these examples, we remark that many other practical problems are not Lipschitz continuous and convex, rendering the algorithms and results in [6, 22] not applicable. In this work, our goal is to apply RCS to these more general nonsmooth optimization problems and establish a series of strong convergence results.

1.2 Linearly bounded subgradients assumption

Let us present a more general condition than the traditional Lipschitz continuity assumption in this subsection. A fundamental result in variational analysis and nonsmooth optimization is the equivalence between Lipschitz continuity and bounded subgradients for a very general class of functions [26, Theorem 9.13]. Thus, in order to generalize the Lipschitz continuity or bounded subgradients assumption (they are equivalent as outlined), we impose the following linearly bounded subgradients assumption.

Assumption 1.1. There exist constants $L_1 \geq 0$, $L_2 > 0$ such that the function $f$ in (1) satisfies

$$\|r\| \leq L_1 \|x\| + L_2, \quad \forall x \in \text{dom}(f), \ r \in \partial f(x).$$

When $L_1 = 0$, Assumption 1.1 reduces to the traditional Lipschitz continuity assumption, i.e., $f$ is $L_1$-Lipschitz continuous [26, Theorem 9.13]. Nonetheless, it is significantly more general with $L_1 > 0$. For instance, problem (3) satisfies our assumption trivially due to the linearly bounded term $L_1 \|x\|$. Moreover, considerably many weakly convex problems satisfy our assumption. By checking the subdifferential, it is easy to verify that our assumption holds for problem (4) (see the proof of Proposition 3.1 for its subdifferential) and the robust low-rank matrix recovery problem [14]. Indeed, any weakly convex problem of the composite form $f(x) = h(c(x))$ with $h$ being convex and Lipschitz continuous and $c$ being quadratic satisfies our assumption. This problem class covers most of the weakly convex examples listed in [7]. To be more specific, the subdifferential of such a $f$ can be represented as $\partial f(x) = \nabla c(x)^\top \partial h(c(x))$, in which the convex subdifferential $\partial h(c(x))$ is bounded and the Jacobian $\nabla c(x)^\top$ is a linear function in $x$, and hence $\partial f(x)$ satisfies Assumption 1.1. In a slightly more general sense, a $\rho$-weakly convex function $f$ can be decomposed as $f(x) = \psi(x) - \frac{\rho}{2} \|x\|^2$ for some convex function $\psi$ and its subdifferential can be represented as $\partial f(x) = \partial \psi(x) - \rho x$, where $\partial \psi$ is the convex subdifferential; see Subsection 2.1 for more details. Thus, our assumption holds whenever the decomposed convex function $\psi$ has bounded subgradients (i.e., Lipschitz continuous).

1.3 Main contributions

In this work, we propose the Randomized Coordinate Subgradient method (RCS) for solving nonsmooth convex and nonsmooth weakly convex optimization problems under the linearly bounded subgradients assumption (see Assumption 1.1). Our contributions are as follows:

(a) When $f$ is nonsmooth convex: • We derive the $\tilde{O}(1/\sqrt{k})$ converge rate in expectation in terms of the suboptimality gap (see Theorem 4.1). • We show $O(1/k)$ convergence rate in expectation with respect to the squared distance between the iterate and the optimal solution set under an additional global quadratic growth condition (see Theorem 4.2). • We establish the almost sure asymptotic convergence result (see Theorem 4.3) and the almost sure $\tilde{O}(1/\sqrt{k})$ asymptotic convergence rate result (see Corollary 4.1) in terms of the suboptimality gap, which are valid for each single run of RCS with probability 1. One important step for deriving these convergence results is to establish boundedness of the iterates in expectation so that the technical difficulty introduced by the more general linearly bounded subgradients assumption can be tackled by using diminishing step sizes. We refer to Section 4 for more details.
(b) When $f$ is nonsmooth weakly convex and its subdifferential satisfies the global metric subregularity property:

\(\bullet\) We provide the iteration complexity of $O(\varepsilon^{-4})$ in expectation for driving the Moreau envelope gradient below $\varepsilon$ (see Theorem 5.1). \(\bullet\) We derive the almost sure asymptotic convergence result (see Theorem 5.2) and the almost sure $O(1/k^{1/4})$ asymptotic convergence rate result (in the sense of liminf; see Corollary 5.1) in terms of the Moreau envelope gradient. Our analysis for nonsmooth weakly convex case deviates from the standard one [7] due to the more general Assumption 1.1. Our key idea is to utilize an error bound condition (global metric subregularity) to derive the (approximate) descent property of RCS (see Lemma 5.1). Towards establishing the desired results, we derive a convergence lemma for mapping (see Lemma 3.1) and the relationship between the global metric subregularity properties of a weakly convex function and its Moreau envelope (see Corollary 3.1), which are of independent interests.

(c) We establish that the (real-valued) robust phase retrieval problem (4) satisfies the global metric subregularity property under a very mild condition (see Proposition 3.1). This finding validates the key assumption we have made for analyzing RCS in the weakly convex case for this specific problem. This established error bound condition is expected to be useful for analyzing other robust phase retrieval algorithms.

Finally, we remark that if we set the number of blocks $N = 1$ in RCS, the above convergence results yield new results for the subgradient method under the linearly bounded subgradients assumption.

### 1.4 Literature of subgradient-type methods for optimizing non-Lipschitz functions

The linearly bounded subgradients assumption dates back to [3], in which Cohen and Zhu showed the global convergence of the subgradient method for nonsmooth convex optimization. This assumption was later imposed in [5] to analyze stochastic subgradient method for stochastic convex optimization. However, both works concern the global convergence property while no explicit rate result is reported.

Apart from the assumption we used, there are several other directions for generalizing the theory of subgradient-type methods to non-Lipschitz convex optimization. In [18], the author proposed the concept of relative continuity of the objective function, which is to impose some relaxed bound on the subgradients by using Bregman distance. Then, the author invoked this Bregman distance in the algorithmic design. The resultant mirror descent-type algorithm has a different structure from the classic subgradient method. Similar idea was later used in [33] for analyzing online convex optimization problem. The works [12, 23] transform the original non-Lipschitz objective function to a radial function, which introduces a new decision variable. Then, their algorithms are based on alternatively updating the original variable (by a subgradient step) and the new variable. In [13], a growth condition rather than Lipschitz continuity of $f$ was utilized to analyze the normalized subgradient method. The author obtained asymptotic convergence rate result in terms of the minimum of the suboptimality gap through a rather quick argument that originates in Shor’s analysis [27, Theorem 2.1]. This convergence result finally applies to any convex function that only needs to be locally Lipschitz continuous.

The common feature of the above mentioned works lies in that they all consider convex optimization problems. By contrast, our work studies both convex and nonconvex cases.

### 1.5 Notations

The notation in this paper is mostly standard. Throughout this work, we use $X^*$ to denote the set of optimal solutions of (1). We assume $X^* \neq \emptyset$ without loss of generality. For any $x^* \in X^*$, we denote $f^* = f(x^*)$. If problem (1) is nonconvex, the set of its critical points is denoted by $X$. Note that the indices $i(k), k = 0, 1, 2, \ldots$ generated in RCS are random variables. Thus, RCS generates a stochastic process $\{x^k\}_{k \geq 0}$. Throughout this work, let $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k \geq 0}, \mathbb{P})$ be a filtered probability space and let us assume that the sequence of iterates $\{x^k\}_{k \geq 0}$ is adapted to the filtration $\{\mathcal{F}_k\}_{k \geq 0}$, i.e., each of the random vectors $x^k : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_k$-measurable, which is automatically true once we set $\mathcal{F}_k = \sigma(i(0), i(1), \ldots, i(k))$. We use $\mathbb{E}_{i(k)}$ and $\mathbb{E}_{\mathcal{F}_k}$ to denote the expectations taken over the random variable $i(k)$ and the filtration $\mathcal{F}_k$, respectively. We $\tilde{O}$ and $\tilde{o}$ to denote $O$ and $o$ with hidden log terms, respectively.

### 2 Preliminaries

#### 2.1 Convexity, weak convexity, and subdifferential

A function $\psi : \mathbb{R}^d \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^d$ and $0 \leq \theta \leq 1$, we have $\psi((1-\theta)x + \theta y) \leq (1-\theta)\psi(x) + \theta \psi(y)$. A vector $v \in \mathbb{R}^d$ is called a subgradient of $\psi$ at point $x$ if the subgradient inequality $\psi(y) \geq \psi(x) + \langle v, y - x \rangle$ holds
for all \( y \in \mathbb{R}^d \). The set of all subgradients of \( \psi \) at \( x \) is denoted by \( \partial \psi(x) \), and is called the (convex) subdifferential of \( \psi \) at \( x \).

A function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) is \( \rho \)-weakly convex if there exists a convex function \( \psi \) such that \( \varphi(x) = \psi(x) - \frac{\rho}{2} \|x\|^2 \). For such a \( \rho \)-weakly convex function \( \varphi \), its subdifferential is given by [28, Proposition 4.6]

\[
\partial \varphi(x) = \partial \psi(x) - \rho x,
\]

where \( \partial \psi \) is the convex subdifferential defined above. Additionally, it is well known that the \( \rho \)-weak convexity of \( \varphi \) is equivalent to [28, Proposition 4.8]

\[
\varphi(y) \geq \varphi(x) + \langle \nu, y - x \rangle - \frac{\rho}{2} \|x - y\|^2
\]

for all \( x, y \in \mathbb{R}^d \) and \( \nu \in \partial \varphi(x) \).

### 2.2 Moreau envelope of weakly convex function

If \( f \) is \( \rho \)-weakly convex, proper, and lower semicontinuous, then the Moreau envelope function \( f_\lambda(x) \) and the proximal mapping \( \text{prox}_{\lambda f}(x) \) are defined as [26]:

\[
f_\lambda(x) := \min_y \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\},
\]

\[
\text{prox}_{\lambda f}(x) := \arg \min_y \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.
\]

As a standard requirement on the regularization parameter \( \lambda \), we will always use \( \lambda < \frac{1}{\rho} \) in the sequel to ensure that \( \nabla f_\lambda \) of a \( \rho \)-weakly convex function \( f \) is well defined.

We collect a series of important and known properties of the Moreau envelope in the following proposition; see, e.g., [34, Propositions 1-4].

**Proposition 2.1.** Suppose that \( f \) is a \( \rho \)-weakly convex function. The following assertions hold:

(a) \( \text{prox}_{\lambda f}(x) \) is well defined, single-valued, and Lipschitz continuous.

(b) \( f_\lambda(x) \leq f(x) - \frac{1 - \lambda \rho}{2\lambda} \|x - \text{prox}_{\lambda f}(x)\|^2 \).

(c) \( \lambda \text{dist}(0, \partial f(\text{prox}_{\lambda f}(x))) \leq \|x - \text{prox}_{\lambda f}(x)\| \leq \frac{\lambda}{1 - \rho} \lambda \text{dist}(0, \partial f(x)) \).

(d) \( \nabla f_\lambda(x) = \frac{1}{\lambda}(x - \text{prox}_{\lambda f}(x)) \) and it is Lipschitz continuous.

(e) \( x = \text{prox}_{\lambda f}(x) \) if and only if \( 0 \in \partial f(x) \).

We have the following corollaries of Proposition 2.1, which show that the weakly convex function and its Moreau envelope mapping have the same set of optimal solutions and stationary points.

**Corollary 2.1.** Suppose that \( f \) is a \( \rho \)-weakly convex function. Let \( X_\lambda^* \) denote the set of minimizers of the problem \( \min_{x \in \mathbb{R}^d} f_\lambda(x) \). Then, the following statements hold:

(a) \( f_\lambda(x) \geq f^* \) for all \( x \in \mathbb{R}^d \).

(b) \( f_\lambda(x_\lambda^*) = f^* \) for all \( x_\lambda^* \in X_\lambda^* \). Consequently, we have \( X_\lambda^* = X^* \).

**Proof.** We first show part (a). By the definition of \( f_\lambda(x) \), we have

\[
f_\lambda(x) = \min_{y \in \mathbb{R}^d} \left\{ f(y) + \frac{\|x - y\|^2}{2\lambda} \right\} = f(\text{prox}_{\lambda f}(x)) + \frac{\|x - \text{prox}_{\lambda f}(x)\|^2}{2\lambda} \geq f(\text{prox}_{\lambda f}(x)) \geq f^*.
\]

We now show part (b). By argument (b) of Proposition 2.1, we have for all \( x^* \in X^* \) and \( x_\lambda^* \in X_\lambda^* \) that

\[
f^* = f(x^*) \geq f_\lambda(x^*) \geq f_\lambda(x_\lambda^*) \geq f^*.
\]

where the first inequality follows from argument (b) of Proposition 2.1 and the last inequality is due to part (a) of this corollary. The above inequality yields that \( f^* = f(x^*) = f_\lambda(x_\lambda^*) \) for all \( x^* \in X^* \) and \( x_\lambda^* \in X_\lambda^* \). Thus, we have \( X^* = X_\lambda^* \).

**Corollary 2.2.** Suppose that \( f \) is a \( \rho \)-weakly convex function. Let \( X_\lambda \) be the set of critical points of the problem \( \min_{x \in \mathbb{R}^d} f_\lambda(x) \). Then, we have \( X = X_\lambda \).

**Proof.** This corollary is a direct consequence of parts (c) and (d) of Proposition 2.1.
2.3 The supermartingale convergence theorem

Next, we introduce a well known convergence theorem in stochastic optimization society.

**Theorem 2.1 (Supermartingale convergence theorem [25]).** Let \( \{ \Lambda^k \}_{k \in \mathbb{N}}, \{ \mu^k \}_{k \in \mathbb{N}}, \{ \nu^k \}_{k \in \mathbb{N}}, \) and \( \{ \eta^k \}_{k \in \mathbb{N}} \) be four positive sequences of real-valued random variables adapted to the filtration \( \{ \xi_k \}_{k \in \mathbb{N}} \). Suppose the following recursion holds true:

\[
\mathbb{E}_{\xi_k} [\Lambda^{k+1}] \leq (1 + \mu^k)\Lambda^k + \nu^k - \eta^k, \quad \forall \ k \in \mathbb{N},
\]

where

\[
\sum_{k \in \mathbb{N}} \mu^k < \infty \quad \text{and} \quad \sum_{k \in \mathbb{N}} \nu^k < \infty \quad \text{almost surely.}
\]

Then, the sequence \( \{ \Lambda^k \}_{k \in \mathbb{N}} \) almost surely converges to a finite random variable\(^2 \bar{\Lambda} \) and \( \sum_{k \in \mathbb{N}} \eta^k < \infty \) almost surely.

2.4 A preliminary recursion for RCS

In the following lemma, we establish a preliminary recursion for RCS, which serves as the starting point of our analysis.

**Lemma 2.1 (basic recursion).** Let \( \{ x^k \}_{k \in \mathbb{N}} \) be the sequence of iterates generated by RCS for solving problem (1). Then, for all \( x \in \mathbb{R}^d \), the following hold:

(a) \( \| x^k - x^{k+1} \| \leq \alpha_k \| r^k \| \).

(b) If Assumption 1.1 holds, then we have

\[
\mathbb{E}_{i(k)} [\| x - x^{k+1} \|^2] \leq \| x - x^k \|^2 + \frac{\alpha_k^2 (L_1 \| x^k \| + L_2)^2}{N} - \frac{2\alpha_k}{N} \langle r^k, x^k - x \rangle.
\]

**Proof.** For all \( x \in \mathbb{R}^d \), we have

\[
\| x - x^{k+1} \|^2 = \sum_{j \neq i(k)} \| x_j - x_j^k \|^2 + \| x_{i(k)} - x_{i(k)}^k + \alpha_k r_{i(k)}^k \|^2
\]

\[
= \| x - x^k \|^2 + 2 \langle \alpha_k r_{i(k)}^k, x_{i(k)} - x_{i(k)}^k \rangle + (\alpha_k \| r_{i(k)}^k \|)^2.
\]

We first show part (a). By taking \( x = x^k \) in (7), we obtain

\[
\| x^k - x^{k+1} \| = \alpha_k \| r_{i(k)}^k \| \leq \alpha_k \| r^k \|.
\]

We now show part (b). Taking expectation with respect to \( i(k) \) on both sides of (7), together with \( \mathbb{E}_{i(k)} \langle r_{i(k)}^k, (x^k - x) \rangle = \frac{1}{N} \langle r^k, x^k - x \rangle \) and \( \mathbb{E}_{i(k)} \langle r_{i(k)}^k \| r_{i(k)}^k \|^2 \rangle = \frac{2}{N} \| r^k \|^2 \), yields

\[
\mathbb{E}_{i(k)} [\| x - x^{k+1} \|^2] \leq \| x - x^k \|^2 + \frac{\alpha_k^2}{N} \| r^k \|^2 - \frac{2\alpha_k}{N} \langle r^k, x^k - x \rangle.
\]

Part (b) follows from invoking Assumption 1.1 in the above inequality. \( \square \)

3 Analytical Results

In this section, we introduce several analytical results that are important for our subsequent convergence analysis — especially for nonsmooth weakly convex optimization. These results are new to our knowledge and are of independent interest.

3.1 A convergence lemma

The following lemma is a strict generalization of [3, Lemma 4] from scalar case to mapping case. It will be the key tool for establishing the asymptotic convergence results for RCS. Note that we do not restrict the mapping \( \Theta \) to be the gradient of \( f \), though we use it in this way in our analysis.

\(^2\)A random variable \( X \) is finite if \( \mathbb{P} (\{ \omega \in \Omega : X(\omega) = \infty \}) = 0. \)
Lemma 3.1 (convergence lemma). Consider the sequences \( \{y^k\}_{k \in \mathbb{N}} \) in \( \mathbb{R}^d \) and \( \{\mu_k\}_{k \in \mathbb{N}} \) in \( \mathbb{R}_+ \). Let \( \Theta : \mathbb{R}^d \to \mathbb{R}^m \) be \( L_\Theta \)-Lipschitz continuous over \( \{y^k\}_{k \in \mathbb{N}} \). Suppose further, \( \exists p > 0 \), such that:

\[
\begin{align*}
\exists M \in \mathbb{R}_+ \text{ such that } & \quad \|y^k - y^{k+1}\| \leq M \mu_k, \quad \forall k \in \mathbb{N}, \tag{8} \\
\sum_{k \in \mathbb{N}} \mu_k &= \infty, \tag{9} \\
\exists \hat{\Theta} \in \mathbb{R}^m \text{ such that } & \quad \sum_{k \in \mathbb{N}} \mu_k \|\Theta(y^k) - \hat{\Theta}\|^p < \infty. \tag{10}
\end{align*}
\]

Then, we have \( \lim_{k \to \infty} \|\Theta(y^k) - \hat{\Theta}\| = 0. \)

Proof. For arbitrary \( \varepsilon > 0 \), let \( N_\varepsilon := \{k \in \mathbb{N} : \|\Theta(y^k) - \hat{\Theta}\| \leq \varepsilon\} \). It immediately follows from (9) and (10) that \( \lim \inf_{k \to \infty} \|\Theta(y^k) - \hat{\Theta}\| = 0 \), which implies that \( N_\varepsilon \) is an infinite set. Let \( \bar{N}_\varepsilon \) be the complementary set of \( N_\varepsilon \) in \( \mathbb{N} \). Then, we have

\[
\varepsilon^p \sum_{k \in \bar{N}_\varepsilon} \mu_k \leq \sum_{k \in \bar{N}_\varepsilon} \mu_k \|\Theta(y^k) - \hat{\Theta}\|^p \leq \sum_{k \in \mathbb{N}} \mu_k \|\Theta(y^k) - \hat{\Theta}\|^p < \infty,
\]

where the first inequality is due to the mapping \( x \mapsto x^p \) is non-decreasing for all \( p > 0 \), while the last inequality is from (10). Hence, for arbitrary \( \delta > 0 \), there exists an integer \( n(\delta) \) such that

\[
\sum_{\ell \in \bar{n}(\delta), \ell \in \bar{N}_\varepsilon} \mu_\ell \leq \delta.
\]

Next, we take an arbitrary \( \gamma > 0 \) and set \( \varepsilon = \frac{\gamma}{2} \), \( \delta = \frac{\gamma}{2L_\Theta M} \). For all \( k \geq n(\delta) \), if \( k \in N_\varepsilon \), then we have \( \|\Theta(y^k) - \hat{\Theta}\| \leq \varepsilon < \gamma \), otherwise \( k \in \bar{N}_\varepsilon \). Let \( m \) be the smallest element in the set \( \{\ell \in N_\varepsilon : \ell \geq k\} \). Clearly, \( m \) is finite since \( N_\varepsilon \) is an infinite set. Without loss of generality, we can assume \( m > k \). Then, we have

\[
\begin{align*}
\|\Theta(y^k) - \hat{\Theta}\| &\leq \|\Theta(y^k) - \Theta(y^m)\| + \|\Theta(y^m) - \hat{\Theta}\| \\
&\leq L_\Theta \sum_{\ell=k}^{m-1} \|y^{\ell+1} - y^\ell\| + \varepsilon \\
&\leq L_\Theta M \sum_{\ell=k}^{m-1} \mu_\ell + \frac{\gamma}{2} \\
&= L_\Theta M \sum_{k \leq \ell < m, \ell \in \bar{N}_\varepsilon} \mu_\ell + \frac{\gamma}{2} \\
&\leq L_\Theta M \sum_{\ell \geq n(\delta), \ell \in \bar{N}_\varepsilon} \mu_\ell + \frac{\gamma}{2} \\
&\leq \gamma.
\end{align*}
\]

Here, the second inequality follows from the Lipschitz continuity of \( \Theta \) over \( \{y^k\}_{k \in \mathbb{N}} \), the third inequality is from (8), and the equality is due to the definition of \( m \). Since \( \gamma > 0 \) is taken arbitrarily, we have shown \( \|\Theta(y^k) - \hat{\Theta}\| \to 0 \) as \( k \to \infty \).

3.2 Error bounds

In our later analysis for nonsmooth weakly convex case, the more general linearly bounded subgradients assumption (i.e., Assumption 1.1) introduces additional difficulties. We need the notion of global metric subregularity of \( \nabla f_\lambda \) to tackle the difficulty. In this subsection, we first identify sufficient conditions on when this error bound holds true. Then, we establish that the concrete robust phase retrieval problem (4) satisfies this error bound.

The following lemma connects two types of global error bounds for weakly convex functions.

Lemma 3.2 (relation between two global error bounds). For a weakly convex function \( f : \mathbb{R}^d \to \mathbb{R} \), the following statements are equivalent:

(a) \( \partial f \) satisfies the global metric subregularity, i.e., there exists a constant \( \kappa_1 > 0 \) such that

\[
\dist(x, \overline{x}) \leq \kappa_1 \dist(0, \partial f(x)), \quad \forall x \in \mathbb{R}^d.
\]

(b) \( f \) satisfies the global proximal error bound, i.e., there exists a constant \( \kappa_2 > 0 \) such that

\[
\dist(x, \overline{x}) \leq \kappa_2 \|x - \prox_{\lambda f}(x)\|, \quad \forall x \in \mathbb{R}^d.
\]
The main difficulty is to show that (a) implies (b). We construct a convergent sequence such that $f$ does not satisfy the global proximal error bound at the limit point of this sequence and then, prove this lemma by contradiction; see the following detailed proof.

**Proof.** The direction “(a)$\iff$(b)” directly follows from part (c) of **Proposition 2.1**. It remains to show the direction “(a)$\Rightarrow$(b)”. Let us assume on the contrary that $f$ does not satisfy the global proximal error bound. Then, there exist a point $x_0 \in \mathbb{R}^d$ and $\kappa_2 > 0$, $\kappa_2^2 \to \infty$, $x_j \to x_0$ as $j \to \infty$ such that

$$\operatorname{dist}(x_j, \overline{x}) > \kappa_2^2 \| x_j - \text{prox}_{\lambda f}(x_j) \| \quad \text{as} \quad j \to \infty. \quad (11)$$

Let $\overline{\mathbb{x}}$ denote the closure of $\mathbb{x}$. Note that $\operatorname{dist}(x, \overline{x}) = \operatorname{dist}(x, \operatorname{cl} \overline{x})$ [8, Proposition 1D.4]. Then, by denoting $y \in \text{proj}(\text{prox}_{\lambda f}(x_j), \operatorname{cl} \overline{x})$, we have

$$\operatorname{dist}(\text{prox}_{\lambda f}(x_j), \overline{\mathbb{x}}) = \| \text{prox}_{\lambda f}(x_j) - y \| = \| \text{prox}_{\lambda f}(x_j) - x_j + x_j - y \| \geq \| x_j - y \| - \| x_j - \text{prox}_{\lambda f}(x_j) \| \geq \| x_j \|_1 - \| x_j - \text{prox}_{\lambda f}(x_j) \| \geq (\kappa_2^2 - 1) \| x_j - \text{prox}_{\lambda f}(x_j) \| \geq (\kappa_2^2 - 1) \lambda \operatorname{dist}(0, \partial f(\text{prox}_{\lambda f}(x_j)))),$$

where the second inequality is due to $\| x_j - y \| \geq \| x_j - x_j \| = \operatorname{dist}(x_j, \overline{x})$ with $\bar{x}_j \in \text{proj}(x_j, \operatorname{cl} \overline{x})$, the third inequality is due to (11), and the last inequality follows from part (c) of **Proposition 2.1**. Since $(\kappa_2^2 - 1) > \kappa_1$ as $j \to \infty$, we reach a contradiction to the global metric subregularity (a). \hfill \Box

Invoking $\| \nabla f_\lambda(x) \| = \frac{1}{\lambda} \| x - \text{prox}_{\lambda f}(x) \|$ (see **Proposition 2.1** (d)) in **Lemma 3.2** (b) and using the fact that $f$ and $f_\lambda$ has exactly the same set of critical points (see **Corollary 2.2**), we obtain the following result.

**Corollary 3.1.** Suppose that the subdifferential $\partial f$ of the weakly convex function $f$ satisfies the global metric subregularity property with parameter $\kappa_1$. Then, there exists a parameter $\kappa_2 > 0$ such that $\nabla f_\lambda$ satisfies the global metric subregularity property with parameter $\lambda \kappa_2$.

Thus, in order to establish the global metric subregularity property of $\nabla f_\lambda$, it suffices to show the global metric subregularity of the subdifferential $\partial f$. We give a sufficient condition for the latter in the following lemma.

**Lemma 3.3** (sufficient condition for global metric subregularity of $\partial f$, Theorem 3.2 of [32]). Suppose that $\partial f$ is a piecewise linear multifunction and $\lim_{\operatorname{dist}(x, \overline{x}) \to \infty} \operatorname{dist}(0, \partial f(x)) = \infty$. Then, $\partial f$ satisfies the global metric subregularity.

As a concrete example, the sufficient condition in **Lemma 3.3** holds for our running example, the robust phase retrieval problem (4) and hence, $\nabla f_\lambda$ of this problem satisfies the global metric subregularity due to **Corollary 3.1**. We present the result in the following proposition.

**Proposition 3.1** (global metric subregularity of robust phase retrieval problem). $\partial f$ of the robust phase retrieval problem (4) satisfies the global metric subregularity if $A$ has full column rank.

**Proof.** Recalling the sufficient condition for global metric subregularity of $\partial f$ in **Lemma 3.3**: (i) $\partial f$ is a piecewise linear multifunction; (ii) $\lim_{\operatorname{dist}(x, \overline{x}) \to \infty} \operatorname{dist}(0, \partial f(x)) = \infty$.

The (real-valued) robust phase retrieval problem can be formulated as

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \| (Ax)_{\circ 2} - b_{\circ 2} \|_1 = \frac{1}{n} \sum_{i=1}^n | a_i^T x |^2 - b_i^2 |,$$

where $A = (a_1, ..., a_n)^T \in \mathbb{R}^{n \times d}$ is nonzero known sampling matrix, $b = (b_1, ..., b_n)^T \in \mathbb{R}^n$ is observed measurements and $\circ 2$ is the Hadamard power. The subgradient of robust phase retrieval problem (4) is

$$\partial f(x) = \frac{2}{n} A^T (Ax \circ \xi) = \frac{2}{n} \sum_{i=1}^n a_i (a_i^T x) \xi_i,$$

Thus, $\partial f$ is a piecewise linear multifunction and $\lim_{\operatorname{dist}(x, \overline{x}) \to \infty} \operatorname{dist}(0, \partial f(x)) = \infty$. Therefore, $\partial f$ satisfies the global metric subregularity. \hfill \Box


with $\xi_i \in \{-1\} \quad (a_i^T x)^2 - b_i^2 < 0$

with $\xi_i \in [-1, 1] \quad (a_i^T x)^2 - b_i^2 = 0$ and $\odot$ is the Hadamard product. It is easy to see the feasible region of the

with $\xi_i \in \{1\} \quad (a_i^T x)^2 - b_i^2 > 0$

subdifferential function is divided into finitely many polyhedrons, defined by $\{x : a_i^T x < -b_i\}$, $\{x : a_i^T x = -b_i\}$, $\{x : b_i < a_i^T x < b_i\}$, $\{x : a_i^T x = b_i\}$, $\{x : a_i^T x > b_i\}$ for all $i = 1, \ldots, n$. On these regions, the subdifferential function is always linear multifunction. Therefore, robust phase retrieval problem satisfies (i) of the sufficient condition of metric subregularity.

Next we are going to show the real valued phase retrieval problem satisfies (ii) of the sufficient condition of metric subregularity. Intuitively, for any given $x \in \mathbb{R}^d$, we can divided the index of datasets $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ into three subsets:

$$
\begin{align*}
I_1 &= \{i \in \{1, \ldots, n\} | (a_i^T x)^2 = b_i^2\}, \\
I_2 &= \{i \in \{1, \ldots, n\} | (a_i^T x)^2 > b_i^2\}, \\
I_3 &= \{i \in \{1, \ldots, n\} | (a_i^T x)^2 < b_i^2\}.
\end{align*}
$$

(12)

We can easily find the fact that $I_1 \cap I_2 = \emptyset$, $I_2 \cap I_3 = \emptyset$, $I_1 \cap I_3 = \emptyset$, and $I_1 \cup I_2 \cup I_3 = \{1, \ldots, n\}$. For given $x \in \mathbb{R}^d$, the subdifferential of robust phase retrieval problem can be rewritten as

$$
\partial f(x) = \frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot [-1, 1]) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right].
$$

(13)

And

$$
\frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right] \in \partial f(x),
$$

(14)

with $\delta_i \in [-1, 1], i = 1, \ldots, n$. It follows that

$$
\frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right]
= \frac{2}{n} \left[ \sum_{i=1}^n a_i a_i^T x + \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) - \sum_{i \in I_1} a_i (a_i^T x) - 2 \sum_{i \in I_3} a_i a_i^T x \right]
= \frac{2}{n} \left[ A^T A x + \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) - \sum_{i \in I_1} a_i (a_i^T x) - 2 \sum_{i \in I_3} a_i a_i^T x \right]
\geq \frac{2}{n} \left[ \|A^T A x\| - 2 \sum_{i \in I_1} |a_i| \cdot |a_i^T x| - 2 \sum_{i \in I_3} \|a_i\| \cdot |a_i^T x| \right],
$$

(15)

with $\delta_i \in [-1, 1]$. Recalling (12), we have $|a_i^T x| = |b_i|$, $\forall i \in I_1$ and $|a_i^T x| < |b_i|$, $\forall i \in I_3$. Then (15) yields that

$$
\frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right]
\geq \frac{2}{n} \left[ \|A^T A x\| - 2 \sum_{i \in I_1} |a_i| \cdot |b_i| - 2 \sum_{i \in I_3} \|a_i\| \cdot |b_i| \right],
$$

(16)

with $\delta_i \in [-1, 1]$. By the full column rank of $A$, we denote the positive definite matrix $Q = A^T A$, and $\sigma_{\min}(Q^T Q)$ be the minimum eigenvalue of matrix $Q^T Q$. Then (16) yields that

$$
\frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot \delta_i) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right]
\geq \frac{2}{n} \left[ \sqrt{\sigma_{\min}(Q^T Q)} \|x\| - 2 \sum_{i \in I_1} |a_i| \cdot |b_i| - 2 \sum_{i \in I_3} \|a_i\| \cdot |b_i| \right],
$$

(17)
with $\delta_i \in [-1, 1]$. By the combination of (13) and (17), we have that
\[
\text{dist}(0, \partial f(x)) \geq \frac{2}{n} \left[ \sqrt{\sigma_{\min}(Q^\top Q)} \|x\| - 2 \sum_{i \in I} \|a_i\| \cdot |b_i| - 2 \sum_{i \in I} \|a_i\| \cdot |b_i| \right].
\] (18)
Again using the full column rank of $A$, we have that the critical point set $\overline{X}$ is bounded (See Fact 5.1). Then by the combination of (18) and the boundness of $\overline{X}$, we obtain that $\text{dist}(0, \partial f(x)) \to \infty$ as $\text{dist}(x, \overline{X}) \to \infty$.

Then we can conclude that under the assumption $A$ is full column rank, $\text{dist}(0, \partial f(x)) \to \infty$ as $\text{dist}(x, \overline{X}) \to \infty$. Thus, the sufficient condition (ii) is satisfied, which leads to the global metric subregularity of $\partial f(x)$ of the robust phase retrieval problem.

\section{Convergence Analysis for Nonsmooth Convex Optimization}

We first present the $\tilde{O}(1/\sqrt{k})$ convergence rate result in expectation in the following theorem.

\textbf{Theorem 4.1} (convergence rate in expectation). Suppose that $f$ in (1) is convex and Assumption 1.1 holds. Moreover, suppose the step sizes $\{\alpha_k\}_{k \in \mathbb{N}}$ satisfy
\[
\sum_{k \in \mathbb{N}} \alpha_k = \infty \quad \text{and} \quad \sum_{k \in \mathbb{N}} \alpha_k^2 < \infty.
\] (19)
Let $\tilde{x}^k = \sum_{j=0}^k \alpha_j x^j / \sum_{j=0}^k \alpha_j$ for all $k \geq 0$. Then, $\forall k \in \mathbb{N}$, we have
\[
\mathbb{E}_{F_{k-1}} \left[ f(\tilde{x}^k) - f^* \right] \leq \frac{N \text{dist}^2(x^0, X^*) + C_1 \sum_{j=0}^k \alpha_j^2}{2} \sum_{j=0}^k \alpha_j,
\] (20)
where $C_1 > 0$ is a constant defined in the proof. Consequently, if the step sizes $\alpha_k = \frac{\Delta}{\sqrt{k+1} \log(k+2)}$ with some constant $\Delta > 0$ for all $k \in \mathbb{N}$, then, we have for all $k \geq 0$
\[
\mathbb{E}_{F_{k-1}} \left[ f(\tilde{x}^k) - f^* \right] \leq \frac{\log(k + 2) \left( \frac{N \text{dist}^2(x^0, X^*)}{\Delta} + \frac{C_1 \Delta}{\log(2)} \right)}{\sqrt{k + 1}}.
\] (21)

Our Theorem 4.1 generalizes the existing results [6, 22] by assuming linearly bounded subgradients assumption (i.e., Assumption 1.1) rather than the traditional Lipschitz continuity assumption. This generalization is vital to cover more general nonsmooth convex problems like SVM (see (3)), the robust regression problem with weight decay, etc; see Section 1 about the generality of Assumption 1.1. Technically speaking, this more general assumption introduces the additional term $\frac{1}{N} L_1^2 \alpha_k^2 \|x^k - x^*\|^2$ in the recursion obtained in Lemma 4.1 (see below), which introduces additional difficulties in order to derive convergence result. Our main idea in the proof of Theorem 4.1 is to show an upper bound on $\{\mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2]\}_{k \in \mathbb{N}}$ (see Lemma 4.2 below), which allows us to transfer this additional term to $\tilde{O}(\alpha_k^2)$ and absorb it into the error term. Then, we can establish a standard recursion (see Lemma 4.3) and apply classic analysis to derive the desired result. Let us also mention that the diminishing step sizes conditions in (19) is important for bounding the term $\{\mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2]\}_{k \in \mathbb{N}}$. Next, we present the detailed proof of Theorem 4.1.

\textbf{Proof}. We need the following three lemmas as preparations.

\textbf{Lemma 4.1} (preliminary recursion). Suppose that $f$ in (1) is convex and Assumption 1.1 holds. Then, for any $x^* \in X^*$, we have
\[
\mathbb{E}_{x}(\|x^{k+1} - x^*\|^2) \leq \left( 1 + \frac{4}{N} L_1^2 \alpha_k^2 \right) \|x^k - x^*\|^2 + \alpha_k^2 \frac{4 L_1^2 \|x^*\|^2}{N} - \frac{2 \alpha_k}{N} (f(x^k) - f^*).
\] (22)

\textbf{Proof}. Taking $x = x^*$ in part (b) of Lemma 2.1 and by the convexity of $f$, we can compute
\[
\mathbb{E}_{\mathbb{E}_{x}(\|x^{k+1} - x^*\|^2) \leq \|x^k - x^*\|^2 + \frac{\alpha_k^2}{N} (L_1 \|x^k\| + L_2)^2 - \frac{2 \alpha_k}{N} (f(x^k) - f^*)
\leq \|x^k - x^*\|^2 + \frac{\alpha_k^2}{N} (2 L_1^2 \|x^k\|^2 + 2 L_2^2) - \frac{2 \alpha_k}{N} (f(x^k) - f^*)
\leq \|x^k - x^*\|^2 + \frac{\alpha_k^2}{N} (4 L_1^2 \|x^* - x^k\|^2 + 4 L_2^2 \|x^*\|^2 + 2 L_1^2 \|x^*\|^2) - \frac{2 \alpha_k}{N} (f(x^k) - f^*),
\]
which yields the desired result.

**Lemma 4.2** (boundedness of \( \{ \mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2] \}_{k \in \mathbb{N}} \)). Suppose that \( f \) in (1) is convex and Assumption 1.1 holds. Suppose further that the step sizes \( \{ \alpha_k \}_{k \in \mathbb{N}} \) satisfy (19). Then, for any \( x^* \in X^* \), we have

\[
\mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2] \leq B_1 \quad \forall k \in \mathbb{N},
\]

with \( B_1 = \left[\|x^0 - x^*\|^2 + \frac{(4L_1^2\|x^*\|^2 + 2L_2^2)}{N} \right] e^{1 + \frac{\alpha_k}{N} L_1^2 \pi} > 0. \)

**Proof.** Upon taking expectation with respect to \( F_{k-1} \) on both sides of Lemma 4.1 and by the fact that \( f(x^k) - f^* \geq 0 \), we obtain

\[
\mathbb{E}_{F_k}[\|x^{k+1} - x^*\|^2] \leq \left(1 + \frac{4}{N}L_1^2 \alpha_k^2 \right) \mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2] + \alpha_k^2 \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N} - \frac{2\alpha_k}{N} \mathbb{E}_{F_{k-1}}[f(x^k) - f^*].
\]

Let \( \beta^k = \mathbb{E}_{F_{k-1}}[\|x^k - x^*\|^2] \prod_{j=0}^{k-1} \left(1 + \frac{4}{N}L_1^2 \alpha_j^2 \right)^{-1} \) with \( \beta^0 = \|x^0 - x^*\|^2 \). Then, multiplying \( \prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j^2 \right)^{-1} \) on both sides of the above inequality yields

\[
\beta^{k+1} \leq \beta^k + \alpha_k^2 \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N} \prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j \right)^{-1}
\]

\[
\leq \beta^k + \alpha_k^2 \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N}.
\]

Unrolling this inequality for \( 0, 1, \ldots, k \) yields

\[
\beta^{k+1} \leq \beta^0 + \left( \sum_{j=0}^{k} \alpha_j^2 \right) \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N}
\]

\[
\leq \beta^0 + \sum_{j=0}^{k} \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N}.
\]

which further implies

\[
\mathbb{E}_{F_k}[\|x^{k+1} - x^*\|^2] \leq \left[\|x^0 - x^*\|^2 + \frac{4L_1^2\|x^*\|^2 + 2L_2^2}{N} \right] \prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j \right).
\]

(23)

Next, we upper bound the term \( \prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j \right) \) as

\[
\prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j \right) = e^{\log \prod_{j=0}^{k} \left(1 + \frac{4}{N}L_1^2 \alpha_j \right)} = e^{\sum_{j=0}^{k} \log \left(1 + \frac{4}{N}L_1^2 \alpha_j \right)} \leq e^{1 + \frac{4}{N}L_1^2 \sum_{j=0}^{k} \alpha_j^2 \leq e^{1 + \frac{4}{N}L_1^2 \pi}. \quad \text{(by (19))}
\]

Invoking this bound in (23) gives

\[
\mathbb{E}_{F_k}[\|x^{k+1} - x^*\|^2] \leq B_1 \quad \forall k \geq 0,
\]

where \( B_1 = \left[\|x^0 - x^*\|^2 + \frac{(4L_1^2\|x^*\|^2 + 2L_2^2)}{N} \right] e^{1 + \frac{4}{N}L_1^2 \pi} > 0. \)

Based on the previous two lemmas, we can derive the following recursion on \( \text{dist}(x, X^*) \).

**Lemma 4.3** (key recursion for convex case). Suppose that \( f \) in (1) is convex and Assumption 1.1 holds. Suppose further that the step sizes \( \{ \alpha_k \}_{k \in \mathbb{N}} \) satisfy (19). Then, for any \( x^* \in X^* \), we have

\[
\mathbb{E}_{F_k}[\text{dist}^2(x^{k+1}, X^*)] \leq \mathbb{E}_{F_{k-1}}[\text{dist}^2(x^k, X^*)] + \alpha_k^2 C_1 \frac{C_1}{N} - \frac{2\alpha_k}{N} \mathbb{E}_{F_{k-1}}[f(x^k) - f^*],
\]

(24)

where \( C_1 := 28L_1^2B_1 + 16L_1^2\|x^*\|^2 + 2L_2^2 \).
Proof. Let \( \text{cl} X^* \) denote the closure of \( X^* \). Let \( x_k^* = \text{proj}(x^k, \text{cl} X^*) \). Taking expectation with respect to \( F_{k-1} \) on both sides of Lemma 4.1 with \( x^* = x_k^* \) provides

\[
E_{F_k} \left[ \text{dist}^2(x^{k+1}, X^*) \right] - E_{F_{k-1}} \left[ \text{dist}^2(x^k, X^*) \right] \\
\leq - \frac{2\alpha_k}{N} E_{F_{k-1}} \left[ f(x^k) - f^* \right] + \left( \frac{4}{N} L_1^2 E_{F_{k-1}} \left[ \text{dist}^2(x^k, X^*) \right] + E_{F_{k-1}} \left[ \frac{4L_1^2\|x_k^*\|^2 + 2L_2^2}{N} \right] \right) \alpha_k^2,
\]

(25)

where we also used the fact \( \text{dist}(x^k, X^*) = \|x^k - x_k^*\| \) [8, Proposition 1D.4] and \( \text{dist}(x^{k+1}, X^*) \leq \|x^{k+1} - x_k^*\| \). Note that Lemma 4.2 ensures \( \text{dist}(x^k, cl X) \leq 3 \|x_k^*\| \). Hence, \( \text{dist}(x^k, cl X) \leq 3 \|x_k^*\| \leq 3N\alpha_k \). Since \( \|x^k\|^2 \leq 2\|x^*\|^2 + 2\|x^k\|^2 \), we have \( E_{F_{k-1}} \left[ \|x^k\|^2 \right] \leq 2B_1 + 2\|x^k\|^2 \). In addition, since \( E_{F_{k-1}} \left[ \text{dist}^2(x^k, X^*) \right] \leq E_{F_{k-1}} \left[ \|x^k - x^*\|^2 \right] \leq B_1 \) and \( \|x_k^*\|^2 \leq 2 \text{dist}^2(x^k, X^*) + 2\|x^k\|^2 \), we can obtain \( E_{F_{k-1}} \left[ \|x_k^*\|^2 \right] \leq 6B_1 + 4\|x^k\|^2 \). Invoking these bounds in (25) yields the desired result:

\[
E_{F_k} \left[ \text{dist}^2(x^{k+1}, X^*) \right] \leq E_{F_{k-1}} \left[ \text{dist}^2(x^k, X^*) \right] + C_1 \alpha_k^2 - \frac{2\alpha_k}{N} E_{F_{k-1}} \left[ f(x^k) - f^* \right],
\]

with \( C_1 = 28L_1^2 B_1 + 16L_1^2 \|x^k\|^2 + 2L_2^2 \).

With these three lemmas, we can provide the proof of Theorem 4.1. Unrolling the recursion in Lemma 4.3 and by the definition of \( \hat{x}^k \) and the convexity of \( f \), we have

\[
E_{F_{k-1}} \left[ f(\hat{x}^k) - f^* \right] \leq E_{F_{k-1}} \left[ \sum_{j=0}^{k} \alpha_j (f(x^j) - f^*) \right] \\
= N \text{dist}^2(x^0, X^*) + C_1 \sum_{j=0}^{k} \alpha_j - \frac{2\alpha_k}{N} E_{F_{k-1}} \left[ f(x^k) - f^* \right],
\]

(20)

which shows (20). To derive (21), let \( \alpha_k = \frac{\Delta}{\sqrt{k+1 \log(k+2)}} \) for all \( k \in \mathbb{N} \). The first condition in (19) is automatically satisfied. By the integral comparison test, we have

\[
\sum_{j=0}^{k} \alpha_j^2 \leq \alpha_0^2 + \int_{0}^{k} \frac{\Delta^2}{(t+1) \log(t+2)} dt \leq \frac{2\Delta^2}{(\log 2)^2}.
\]

Thus, the second condition in (19) is also satisfied. Plugging the above upper bound and the fact that \( \sum_{j=0}^{k} \alpha_j \geq \frac{\Delta}{\sqrt{k+1 \log(k+2)}} \) into (20) yields the result.

The proof is complete.

We will establish more convergence results for RCS in the remaining parts of this section, and all the following results are not reported in the existing works [6, 22].

Next, we consider the situation where \( f \) further satisfies the global quadratic growth condition. Then, we can derive a \( O(1/k) \) convergence rate in expectation using diminishing step sizes.

**Theorem 4.2** (convergence rate in expectation under additional global quadratic growth). Suppose that \( f \) in (1) is convex and Assumption 1.1 holds. Suppose further that \( f \) satisfies the global quadratic growth condition with parameter \( \kappa_3 > 0 \), i.e., there exists a constant \( \kappa_3 > 0 \) such that \( \text{dist}^2(x, X^*) \leq \kappa_3 (f(x) - f^*) \), \forall x \in \mathbb{R}^d \). Consider the step sizes \( \alpha_k = N\kappa_3/(k+1) \) for all \( k \geq 0 \), which satisfies the conditions in (19). Then, we have

\[
E_{F_k} [\text{dist}^2(x^{k+1}, X^*)] \leq \frac{NC_1 \kappa_3^2}{k+1} \quad \forall k \in \mathbb{N}.
\]

**Proof.** By Lemma 4.3, we have

\[
E_{F_k} [\text{dist}^2(x^{k+1}, X^*)] - E_{F_{k-1}} [\text{dist}^2(x^k, X^*)] \leq C_1 \frac{\alpha_k^2}{N} - \frac{2\alpha_k}{N} E_{F_{k-1}} [f(x^k) - f^*].
\]

Since \( f \) satisfy the global quadratic growth condition with \( \kappa_3 \), we obtain

\[
E_{F_k} [\text{dist}^2(x^{k+1}, X^*)] - E_{F_{k-1}} [\text{dist}^2(x^k, X^*)] \leq C_1 \frac{\alpha_k^2}{N} - \frac{2\alpha_k}{N\kappa_3} E_{F_{k-1}} [\text{dist}^2(x^k, X^*)].
\]

(26)

Let \( \alpha_k = \frac{N\kappa_3}{k+1} \). (26) reduces to

\[
E_{F_k} [\text{dist}^2(x^{k+1}, X^*)] - E_{F_{k-1}} [\text{dist}^2(x^k, X^*)] \leq \frac{NC_1 \kappa_3^2}{(k+1)^2} - \frac{2}{k+1} E_{F_{k-1}} [\text{dist}^2(x^k, X^*)].
\]
Applying Theorem 2.1 to (27) also provides
\[
\lim_{n \to \infty} \text{dist}^2(x^{k+1}, X^*) \leq (k + 1)^2 \mathbb{E}_{\mathcal{F}_k} [\text{dist}^2(x^k, X^*]) - [(k + 1)^2 - 2k - 2] \mathbb{E}_{\mathcal{F}_{k-1}} [\text{dist}^2(x^k, X^*)] \leq NC_1 \kappa_3^2.
\]

Note the fact that \((k + 1)^2 - 2k - 2 = k^2 - 1 \leq k^2\). Then, we further have
\[
\lim_{n \to \infty} \text{dist}^2(x^{k+1}, X^*) \leq k^2 \mathbb{E}_{\mathcal{F}_{k-1}} [\text{dist}^2(x^k, X^*)] \leq NC_1 \kappa_3^2.
\]

Therefore,
\[
(k + 1)^2 \mathbb{E}_{\mathcal{F}_k} [\text{dist}^2(x^{k+1}, X^*)] \leq \sum_{j=0}^{k} \left[ (j + 1)^2 \mathbb{E}_{\mathcal{F}_j} [\text{dist}^2(x^{j+1}, X^*)] - j^2 \mathbb{E}_{\mathcal{F}_{j-1}} [\text{dist}^2(x^j, X^*)] \right]
\]
\[
\leq NC_1 \kappa_3^2 (k + 1),
\]
which yields
\[
\mathbb{E}_{\mathcal{F}_k} [\text{dist}^2(x^{k+1}, X^*)] \leq \frac{NC_1 \kappa_3^2}{k + 1}.
\]

The additional assumption in Theorem 4.2, i.e., \(f\) satisfies the global quadratic growth condition, is not stringent for a set of \(\ell_2\)-regularized convex problems such as the robust regression problem with weight decay and the SVM problem. Since these problems are strongly convex (due to the \(\ell_2\)-regularizer), the global quadratic growth condition is automatically satisfied. More generally, the work [20] discusses sufficient conditions for global quadratic growth, which is valid for a much broader class of functions than strongly convex ones.

Both Theorem 4.1 and 4.2 present the convergence rate in the sense of expectation, which holds for the average of infinitely many runs of the algorithm. In the following, we will establish the almost sure convergence results, which hold with probability 1 for each single run of RCS.

**Theorem 4.3** (almost sure asymptotic convergence). Suppose that \(f\) in (1) is convex and Assumption 1.1 holds. Consider the step sizes conditions (19). We have \(\lim_{k \to \infty} f(x^k) = f^*\) almost surely, and \(\lim_{k \to \infty} x^k = x^* \in X^*\) almost surely.

In order to deal with the more general Assumption 1.1, we conduct all the arguments in the intersection of the events \(\Omega_1 := \{ w : \{ x^k(w) \}_{k \in \mathbb{N}} \text{ is bounded} \}\) and the events \(\Omega_2 := \{ w : \sum_{k \in \mathbb{N}} \alpha_k (f(x^k(w)) - f^*) < \infty \}\). Applying the supermartingale convergence theorem (see Theorem 2.1) to the recursion in Lemma 4.1 yields \(\mathbb{P}(\Omega_1 \cap \Omega_2) = 1\). Eventually, the desired result in Theorem 4.3 can be derived by applying our convergence lemma (i.e., Lemma 3.1) to any outcome \(w \in \Omega_1 \cap \Omega_2\) since all the requirements in Lemma 3.1 are satisfied for such an \(w\). We provide the detailed proof in the following.

**Proof.** We restate the recursion in Lemma 4.1 below:
\[
\mathbb{E}([x^{k+1} - x^*]^2) \leq \left(1 + \frac{4}{N} L_1^2 \alpha_k^2\right) [x^k - x^*]^2 + \alpha_k^2 \frac{4 L_1^2}{N} [x^*]^2 + \frac{2 L_2^2}{N} - 2 \alpha_k \left(f(x^k) - f^*\right). \tag{27}
\]

Note that \([x^k - x^*]^2 \geq 0\), \(\frac{2 \alpha_k}{N} (f(x^k) - f^*) \geq 0\), and \(\sum_{k \in \mathbb{N}} \alpha_k^2 < \infty\). Therefore, upon applying Theorem 2.1 to (27), we obtain that \(\lim_{k \to \infty} [x^k - x^*]^2\) almost surely exists. It immediately follows that the sequence \(\{x^k\}_{k \in \mathbb{N}}\) is almost surely bounded, i.e., we have \(\Omega_1 \subseteq \Omega\) such that
\[
\mathbb{P}\left(\{ w \in \Omega_1 : \{ x^k(w) \}_{k \in \mathbb{N}} \text{ is bounded} \}\right) = 1. \tag{28}
\]

Applying Theorem 2.1 to (27) also provides \(\sum_{k \in \mathbb{N}} \alpha_k (f(x^k) - f^*) < \infty\) almost surely. Thus, we have \(\Omega_2 \subseteq \Omega\) such that
\[
\mathbb{P}\left(\{ w \in \Omega_2 : \sum_{k \in \mathbb{N}} \alpha_k (f(x^k(w)) - f^*) < \infty \}\right) = 1. \tag{29}
\]

Let us define \(\Omega_3 = \Omega_1 \cap \Omega_2\). By (28), statement (a) of Lemma 2.1 and Assumption 1.1, we have \([x^k(w) - x^k+1(w)]^2 \leq C_3(w) \alpha_k\) and \(f\) is Lipschitz continuous over \(\{x^k(w)\}_{k \in \mathbb{N}}\) for all \(w \in \Omega_3\). Then, we can apply Lemma 3.1 to obtain
\[
\lim_{k \to \infty} f(x^k(w)) = f^*, \quad \forall w \in \Omega_3.
\]
From (28) and (29), it is clear that $\mathbb{P}(\Omega_3) = 1$. Finally, we obtain that $f(x^k) \to f^*$ almost surely.

Next, we show that $x^k \to x^* \in \mathbb{X}^*$ almost surely. Let us consider an arbitrary outcome $w \in \Omega_3$. Since $\{x^k(w)\}$ is bounded and $f(x^k(w)) \to f^*$ as shown above, we can extract a convergent subsequence $\{x^{k_i}(w)\}$ such that $x^{k_i}(w) \to x^* \in \mathbb{X}^*$. This, together with the established fact that $\lim_{k \to \infty} \|x^k - x^*\|^2$ almost surely exists, yields the desired result.

Theorem 4.3 asserts that RCS eventually finds an optimal solution of problem (1) for each single run almost surely. It does not provide a rate result. Actually, we can further establish the almost sure asymptotic rate result.

**Corollary 4.1** (almost sure asymptotic convergence rate). Under the setting of Theorem 4.3. Consider the step sizes $\alpha_k = \frac{\Delta}{\sqrt{k+1} \log(k+2)}$ with $\Delta > 0$ and let $\tilde{x}^k = \frac{1}{k+1} \sum_{j=0}^{k} x^j$ for all $k \geq 0$, then we have

$$f(\tilde{x}^k) - f^* \leq o\left(\frac{\log(k+2)}{\sqrt{k+1}}\right) \text{ as } k \to \infty \text{ almost surely.}$$

**Proof.** Form the proof of Theorem 4.3, we have $\sum_{k \in \mathbb{N}} \alpha_k (f(x^k) - f^*) < \infty$ almost surely. Applying Kronecker’s lemma with the fact that $\{1/\alpha_k\}$ being increasing, we have

$$\lim_{k \to \infty} \alpha_k \sum_{j=0}^{k} (f(x^j) - f^*) = 0.$$ 

Then, the desired result follows by recognizing

$$(k+1)\alpha_k (f(\tilde{x}^k) - f^*) \leq \alpha_k \sum_{j=0}^{k} (f(x^j) - f^*),$$

where we have used convexity of $f$ and definition of $\tilde{x}^k$. $\square$

## 5 Convergence Analysis for Nonsmooth Weakly Convex Optimization

In this section, we turn to establish convergence results for RCS when $f$ in (1) is weakly convex. The iteration complexity in expectation for RCS is established in the following theorem.

**Theorem 5.1** (iteration complexity in expectation). Suppose that $f$ in (1) is $\rho$-weakly convex and Assumption 1.1 holds. Let $f_{\lambda}$ be the Moreau envelope of $f$ with $\lambda < 1/\rho$. Suppose further $\partial f$ satisfies the global metric subregularity property with parameter $\kappa_1 > 0$ (see Lemma 3.2 (i) for definition) and the set of critical points $\mathbb{X}$ is bounded. If the step size $\alpha_k = \frac{\Delta}{\sqrt{1 + k \log(k+2)}}$ with $\Delta > 0$ being a constant and $T$ being the total number of iterations satisfies $\alpha_k \leq \frac{1 - \lambda \rho}{8L(T+1)\kappa_1^{2}}$, then we have

$$\min_{0 \leq t \leq T} \mathbb{E}_{\mathcal{F}_{t-1}} \left[ \|\nabla f_{\lambda}(x^k(t))\|^2 \right] \leq \frac{4N(f_{\lambda}(x^0) - f^*)}{\Delta (1 - \lambda \rho) \sqrt{T + 1}} + 4C_2 \Delta,$$

where $\kappa_2, C_2 > 0$ are constants defined in the proof.

Let us interpret this iteration complexity result in a convention way. In order to achieve $\min_{0 \leq t \leq T} \mathbb{E}_{\mathcal{F}_{t-1}} \left[ \|\nabla f_{\lambda}(x^k(t))\|^2 \right] \leq \epsilon$, RCS needs at most $O(\epsilon^{-4})$ number of iterations in expectation, which is in the same order as other subgradient-type methods for nonsmooth weakly convex minimization. This is reasonable since RCS with $N = 1$ recovers the subgradient method.

Technically speaking, the linearly bounded subgradients assumption (i.e., Assumption 1.1) introduces (considerably) extra technical difficulties. The key idea in our proof is to use the global metric subregularity property of $\nabla f_{\lambda}$ to establish the approximate descent property of RCS (see Lemma 5.1 below). Such an error bound condition is ensured by Corollary 3.1 as $\partial f$ is assumed to be globally metric subregular. Note that Proposition 3.1 shows that $\partial f$ of the concrete robust phase retrieval problem (4) satisfies the global metric subregularity property. Next, we provide the detailed proof of Theorem 5.1.

**Proof.** In order to prove Theorem 5.1, we first derive the recursion for weakly convex case as preparation.
**Lemma 5.1** (key recursion for weakly convex case). *Under the setting of Theorem 5.1. Denote \( \max_{x \in \Xi} \| x \|^2 \leq B_2 \) with some \( B_2 > 0 \). Then, we have

\[
\mathbb{E}_{i(k)} \left[ f(x) - f^* \right] \leq f_\lambda(x^k) - f^* + \frac{\alpha_k C_2}{N} - \frac{(1 - \lambda \rho)\alpha_k}{4N} \| \nabla f_\lambda(x^k) \|^2,
\]

where \( C_2 = \frac{2L_1^2B_2 + L_2^2}{\lambda} \) is a positive constant.

**Proof.** By the \( \rho \)-weak convexity of \( f \) and the statement (b) of Lemma 2.1 (set \( x = \hat{x}^k = \text{prox}_{\lambda \rho}(x^k) \)), we have

\[
\mathbb{E}_{i(k)} \left[ \| x^k - x^{k+1} \|^2 \right] \leq \| x^k - x^{k+1} \|^2 + \frac{\alpha_k^2 (L_1 \| x \|^k + L_2)^2}{2N} - \frac{2\alpha_k}{N} \left[ f(x^k) - f(\hat{x}^k) - \frac{\rho}{2} \| \hat{x}^k - x^k \|^2 \right].
\]

The definitions of Moreau envelope (5) and proximal mapping (6) imply

\[
\mathbb{E}_{i(k)} \left[ f_\lambda(x^k+1) \right] \leq f(\hat{x}^k) + \frac{1}{2\lambda} \mathbb{E}_{i(k)} \left[ \| x^k - x^{k+1} \|^2 \right].
\]

Combining (30) and (31) yields

\[
\mathbb{E}_{i(k)} \left[ f_\lambda(x^k+1) \right] \leq f_\lambda(x^k) + \frac{\alpha_k^2 (L_1 \| x \|^k + L_2)^2}{2N} - \frac{\alpha_k}{2N} \left[ f(x^k) - f(\hat{x}^k) - \frac{\rho}{2} \| \hat{x}^k - x^k \|^2 \right] - \frac{\alpha_k}{2N} \left[ f(x^k) - f(\hat{x}^k) - \frac{\rho}{2} \| \hat{x}^k - x^k \|^2 \right].
\]

This, together with parts (b) and (d) of Proposition 2.1, gives

\[
\mathbb{E}_{i(k)}[f_\lambda(x^k+1)] \leq f_\lambda(x^k) + \frac{\alpha_k^2 (L_1 \| x \|^k + L_2)^2}{2N} - \frac{(1 - \lambda \rho)\alpha_k}{2N} \| \nabla f_\lambda(x^k) \|^2. \tag{32}
\]

Let \( \hat{x}^k \in \text{proj}(x^k, \text{cl} \ X) \), where \( \text{cl} \ X \) denotes the closure of \( X \) as used in the proof of Lemma 3.2. It then follows from (32) that

\[
\begin{align*}
\mathbb{E}_{i(k)}[f_\lambda(x^k+1) - f^*] & \leq (f_\lambda(x^k) - f^*) + \frac{\alpha_k^2 (L_1 \| x \|^k + L_2)^2}{2N} - \frac{(1 - \lambda \rho)\alpha_k}{2N} \| \nabla f_\lambda(x^k) \|^2 \\
& \leq (f_\lambda(x^k) - f^*) + \frac{\alpha_k^2 (2L_1^2 \| x \|^2 + 2L_2^2)}{2N} - \frac{(1 - \lambda \rho)\alpha_k}{2N} \| \nabla f_\lambda(x^k) \|^2, \tag{33}
\end{align*}
\]

where \( C_2 := \frac{2L_1^2B_2 + L_2^2}{\lambda} \). Recall the assumption that \( \partial f \) satisfies the global metric subregularity with \( \kappa_1 > 0 \). It follows from Corollary 3.1 that \( \nabla f_\lambda(x) \) also satisfies the global metric subregularity with some parameter \( \kappa_2 \lambda > 0 \). Note that \( \Xi = \Xi_\lambda \) (see Corollary 2.2). Hence, we have

\[
\text{dist}(x^k, \Xi) \leq \kappa_2 \lambda \| \nabla f_\lambda(x^k) \|. \tag{34}
\]

Then, invoking (34) with \( \alpha_k \leq \frac{1 - \lambda \rho}{8L_1^2 \kappa_2 \lambda} \) in (33) provides the desired result. \( \square \)

With this lemma, we can provide the proof of Theorem 5.1. Taking total expectation to the recursion in Lemma 5.1, we obtain

\[
\frac{1 - \lambda \rho}{4N} \mathbb{E}_{\mathcal{F}_{k-1}} \left[ \| \nabla f_\lambda(x^k) \|^2 \right] \leq \mathbb{E}_{\mathcal{F}_k} \left[ (f_\lambda(x^k) - f^*) - (f_\lambda(x^{k+1}) - f^*) \right] + \frac{C_2}{N} \alpha_k^2.
\]

Summing the above inequality over 0, 1, \ldots, \( T \) and invoking step sizes \( \alpha_k = \frac{\Delta}{\sqrt{T+1}} \) give the desired complexity result. \( \square \)
We can easily find the fact that $\delta A$ where $\sigma \xi$

**Fact 5.1** (bounded $X$ of robust phase retrieval). For the robust phase retrieval problem (4), if the data matrix $A$ has full column rank, then the set of critical points $X$ is bounded.

**Proof.** The robust phase retrieval problem (4) can be rewritten as

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n}\|(Ax)^{\circ 2} - b\|_1 = \frac{1}{n} \sum_{i=1}^n \|(a_i^T x)^2 - b_i^2\|_1,$$

where $A = (a_1, ..., a_n)^T \in \mathbb{R}^{n \times d}$, $b = (b_1, ..., b_n)^T \in \mathbb{R}^n$ and $\circ 2$ is the Hadamard power. And the subgradient of robust phase retrieval problem is

$$\partial f(x) = \frac{2}{n} A^T (Ax \circ \xi) = \frac{2}{n} \sum_{i=1}^n a_i (a_i^T x \cdot \xi_i) \quad \text{with} \quad \xi = (\xi_1, ..., \xi_n)^T,$$

$$\xi_i \in \begin{cases} \{-1\} & (a_i^T x)^2 - b_i^2 < 0 \\ [-1, 1] & (a_i^T x)^2 - b_i^2 = 0, \text{ and } \circ \text{ is the Hadamard product.} \\ \{1\} & (a_i^T x)^2 - b_i^2 > 0 \end{cases}$$

Intuitively, for any given $x \in \mathbb{R}^d$, we can divided the index of datasets $\{(a_1, b_1), ..., (a_n, b_n)\}$ into three subsets:

$$I_1 = \{i \in \{1, ..., n\} | (a_i^T x)^2 = b_i^2\},$$
$$I_2 = \{i \in \{1, ..., n\} | (a_i^T x)^2 > b_i^2\},$$
$$I_3 = \{i \in \{1, ..., n\} | (a_i^T x)^2 < b_i^2\}.$$

(36)

We can easily find the fact that $I_1 \cap I_2 = \emptyset, I_2 \cap I_3 = \emptyset, I_1 \cap I_3 = \emptyset$, and $I_1 \cup I_2 \cup I_3 = \{1, ..., n\}$. For given $x \in \mathbb{R}^d$, the subdifferential of robust phase retrieval problem can be rewritten as

$$\partial f(x) = \frac{2}{n} \left[ \sum_{i \in I_1} a_i (a_i^T x \cdot [-1, 1]) + \sum_{i \in I_2} a_i a_i^T x - \sum_{i \in I_3} a_i a_i^T x \right].$$

(37)

If $x = \bar{x} \in X$, (37) yields that

$$\sum_{i \in I_2} a_i a_i^T \bar{x} = \sum_{i \in I_3} a_i a_i^T \bar{x} - \sum_{i \in I_1} a_i (a_i^T \bar{x} \cdot \delta_i),$$

(38)

with $\delta_i \in [-1, 1], i = 1, ..., n$. By the full column rank of $A$, we denote the positive definite matrix $Q = A^T A$, and $\sigma_{\min}(Q^T Q)$ be the minimum eigenvalue of matrix $Q^T Q$. Then we can derive that

$$\sqrt{\sigma_{\min}(Q^T Q)} \|\bar{x}\| \leq \|A^T \bar{x}\|$$

$$= \left\| \sum_{i=1}^n a_i a_i^T \bar{x} \right\|$$

$$= \left\| \sum_{i \in I_1} a_i a_i^T \bar{x} + \sum_{i \in I_2} a_i a_i^T \bar{x} + \sum_{i \in I_3} a_i a_i^T \bar{x} \right\|$$

$$= \left\| 2 \sum_{i \in I_2} a_i a_i^T \bar{x} + \sum_{i \in I_3} a_i a_i^T \bar{x} - \sum_{i \in I_1} a_i (a_i^T \bar{x} \cdot \delta_i) \right\| \quad \text{(by (38))}$$

$$\leq 2 \sum_{i \in I_3} \|a_i\| \cdot \|a_i^T \bar{x}\| + 2 \sum_{i \in I_1} \|a_i\| \cdot |a_i^T \bar{x}|.$$

(39)

By (36), we have $|a_i^T \bar{x}| = |b_i|, \forall i \in I_1$ and $|a_i^T \bar{x}| < |b_i|, \forall i \in I_3$. Then (39) yields that

$$\|\bar{x}\| \leq \frac{2 \sum_{i \in I_3} \|a_i\| \cdot |b_i| + 2 \sum_{i \in I_1} \|a_i\| \cdot |b_i|}{\sqrt{\sigma_{\min}(Q^T Q)}}.$$

Therefore, under the assumption that $A$ is full column rank, the critical points set of the robust phase retrieval problem (4) is bounded, i.e., $\max_{\bar{x} \in X} \|\bar{x}\| \leq B_2 = \frac{2 \sum_{i \in I_3} \|a_i\| \cdot |b_i| + 2 \sum_{i \in I_1} \|a_i\| \cdot |b_i|}{\sqrt{\sigma_{\min}(Q^T Q)}}$. \qed
In the following theorem, we give the almost sure convergence result that holds true for each single run of the algorithm.

**Theorem 5.2** (almost sure asymptotic convergence). Suppose that $f$ in (1) is $\rho$-weakly convex and Assumption 1.1 holds. Let $f_\lambda$ be the Moreau envelope of $f$ with $\lambda < 1/\rho$. Suppose further $\partial f$ satisfies the global metric subregularity property with parameter $\kappa_1 > 0$ (see Lemma 3.2 (i) for definition) and the set of critical points $\mathcal{X}$ is bounded. If the step sizes $\{\alpha_k\}_{k \in \mathbb{N}}$ satisfy (19), then $\lim_{k \to \infty} \|\nabla f_\lambda(x^k)\| = 0$ almost surely and hence, every accumulation point of $\{x^k\}_{k \in \mathbb{N}}$ is a critical point of problem (1) almost surely.

**Proof.** We repeat the recursion in Lemma 5.1 below for convenience (Note that $\alpha_k \to 0$ due to (19). The requirement on $\alpha_k$ in Lemma 5.1 must be satisfied for all large enough $k$):

$$
E_i(k)[f_\lambda(x^{k+1}) - f^*] \leq f_\lambda(x^k) - f^* + \alpha_k^2 C_2 \frac{(1 - \lambda \rho)\alpha_k}{4N} \|\nabla f_\lambda(x^k)\|^2,
$$

(40)

Noted that $f_\lambda(x) - f^* \geq 0$, $(\frac{1 - \lambda \rho)\alpha_k}{4N} \|\nabla f_\lambda(x^k)\|^2 \geq 0$, and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$. Applying Theorem 2.1 to (40) yields $\lim_{k \to \infty} \{f_\lambda(x^k) - f^*\}$ almost surely exists and is finite. By the definition of $f_\lambda$, we have $\{f(\text{prox}(x^k)) + \frac{1}{2\lambda}\|x^k - \text{prox}(x^k)\|^2 - f^*\}_{k \in \mathbb{N}}$ is almost surely bounded, which further implies that $\{\|\nabla f_\lambda(x^k)\|\}_{k \in \mathbb{N}}$ is almost surely bounded due to $f(\text{prox}(x^k)) \geq f^*$ and $\|x^k - \text{prox}(x^k)\| = \lambda \|\nabla f_\lambda(x^k)\|$. Note that $\partial f$ satisfies the global metric subregularity. We have $\text{dist}(x^k, \mathcal{X}) \leq \kappa_3 \alpha_k \|\nabla f_\lambda(x^k)\|$ for some $\kappa_3 > 0$ as derived in (34). Thus, $\{x^k\}_{k \in \mathbb{N}}$ is almost surely bounded since $\mathcal{X}$ is bounded by our assumption. Next, combining Assumption 1.1, the almost surely boundedness of $\{x^k\}$, and statement (a) of Lemma 2.1, we have the events $\Omega_4$ such that

$$
\mathbb{P}\{\{w \in \Omega_4 : \|x^k(w) - x^k+1(w)\| \leq C_4(w)\alpha_k, C_4(w) > 0\}\} = 1.
$$

Applying Theorem 2.1 to (40) also provides $\Omega_5$ such that

$$
\mathbb{P}\{\{w \in \Omega_5 : \sum_{k=0}^{\infty} \alpha_k \|\nabla f_\lambda(x^k(w))\|^2 < \infty\}\} = 1.
$$

(41)

Noted that $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\nabla f_\lambda$ is Lipschitz continuous. By Lemma 3.1, we have

$$
\lim_{k \to \infty} \|\nabla f_\lambda(x^k(w))\| = 0, \quad \forall w \in \Omega_4 \cap \Omega_5.
$$

It is easy to see that $\mathbb{P}(\Omega_4 \cap \Omega_5) = 1$. Hence, we have

$$
\lim_{k \to \infty} \|\nabla f_\lambda(x^k)\| = 0 \quad \text{almost surely.}
$$

Invoking Corollary 2.2, we conclude that every accumulation point of $\{x^k\}_{k \in \mathbb{N}}$ is a critical point of (1) almost surely.

Finally, we can derive the almost sure asymptotic convergence rate in the sense of liminf in the following corollary.

**Corollary 5.1** (almost sure convergence rate). Under the setting of Theorem 5.2. Let $\alpha_k = \frac{\Delta}{\sqrt{k+1} \log(k+2)}$ with some $\Delta > 0$. We have $\liminf_{k \to \infty} (k+1)^{\frac{\Delta}{2}} \|\nabla f_\lambda(x^k)\| = 0$ almost surely.

**Proof.** Note that the step sizes $\alpha_k = \frac{\Delta}{\sqrt{k+1} \log(k+2)}$. The two conditions in (19) are automatically satisfied since

$$
\sum_{j=0}^{k} \alpha_j^2 \leq \alpha_0^2 + \int_0^k \frac{\Delta^2}{(t+1) \log^2(t+2)} dt \leq \frac{2\Delta^2}{(\log 2)^2},
$$

and

$$
\sum_{j=0}^{k} \alpha_j \geq \sum_{j=0}^{k} \frac{\Delta}{\sqrt{k+1} \log(k+2)} = \frac{\Delta \sqrt{k+1}}{\log(k+2)}.
$$

We now prove this corollary by contradiction. Let us assume that the statement does not hold. Then, we have

$$
\mathbb{P}\left\{\left\{w \in \Omega : \liminf_{k \to \infty} \sqrt{k+1}\|\nabla f_\lambda(x^k(w))\|^2 \geq \delta \right\}\right\} > 0,
$$

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for some $\delta > 0$. Then, there exists a sufficiently large $\bar{k}$ such that

$$
P\left( \left\{ w \in \Omega : \sqrt{\bar{k} + 1} \| \nabla f_\lambda(x^k(w)) \| \geq \delta \right\} \right) > 0
$$

for all $k \geq \bar{k}$. Hence, we have

$$
P\left( \left\{ w \in \Omega : \sum_{k=\bar{k}}^{\infty} \frac{1}{\sqrt{\bar{k} + 1} \log(k+2)} \| \nabla f_\lambda(x^k(w)) \| \geq \sum_{k=\bar{k}}^{\infty} \frac{\delta}{(k+1) \log(k+2)} \right\} \right) > 0. \tag{42}
$$

By the integral comparison test, we have $\sum_{k=\bar{k}}^{\infty} \frac{1}{(k+1) \log(k+2)} \geq \int_{\bar{k}+2}^{\infty} \frac{1}{t \log(t)} dt = \infty$. Note that $\alpha_k = \frac{\Delta}{\sqrt{\bar{k} + 1} \log(k+2)}$.

Therefore, (42) is a contradiction to (41).

### 6 Numerical Simulations

In this section, we test RCS on Applications 1-3 listed in Section 1. From these experiments, our observations can be summarized as follows: RCS uses much less workspace memory than the subgradient method (SubGrad) per iteration. Moreover, RCS is observed to converge faster than SubGrad in the first few epochs. Here, one epoch means passing through all the $d$ variables in $x$. For instance, one iteration of SubGrad means one epoch, while one epoch of RCS consists of $N$ iteration if we set $N = d$ in Algorithm 1. These observations justify the main spirit of coordinate-type methods: RCS could be very efficient when the dimension of the problem is too high to use SubGrad (out of memory for even one iteration) and when the solution accuracy just needs to be modest (then just run RCS for a few epochs). Note that these two situations are common in signal processing and machine learning areas.

The experiments on the robust M-estimators and linear SVM problems are conducted by using MATLAB R2020a on a personal computer with Intel Core i5-6200U CPU (2.4GHz) and 8 GB RAM. The experiments on the robust phase retrieval problem are conducted by using Python on a computer cluster with 2× Intel Xeon Cascade Lake 6248 (2.5GHz, 20 cores) CPU and 12× Samsung 16GB DDR4 ECC REG RAM.

#### 6.1 Robust M-estimators problem

We compare RCS with SubGrad on the robust M-estimators problem (2) with $\ell_1$-loss and $\ell_1$-penalty, i.e., Application 1 with $\ell(\cdot) = \frac{1}{n} \| \cdot \|_1$ and $\phi_p(\cdot) = p \| \cdot \|_1$:

$$
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \| Ax - b \|_1 + p \| x \|_1,
$$

where $A \in \mathbb{R}^{n \times d}$ is a matrix, $b \in \mathbb{R}^n$ is a vector, and $p > 0$ is the regularization parameter.

In order to implement RCS, we introduce the block-wise partition of the columns of the data matrix $A = (A_1, A_2, \cdots, A_N)$, where $A_i \in \mathbb{R}^{n \times d_i}$ is the $i$-th block. The key factor is to calculate the coordinate subgradient $r_{i(k)}^k$ used in RCS for solving this problem, which is given by:

$$
\begin{cases}
    r_{i(k)}^k = \frac{1}{n} A_{i(k)}^\top \text{sign}(s^k) + p \text{sign}(x_{i(k)}^k), \\
    s^{k+1} = s^k + A_{i(k)}(x^{k+1}_{i(k)} - x^k_{i(k)}),
\end{cases}
$$

where $s^0 = Ax^0 - b$. $s^k$ is an iteratively updated intermediate quantity that is used to compute $r_{i(k)}^k$.

We generate synthetic data for simulation. The elements of $A \in \mathbb{R}^{n \times d}$ are generated in an i.i.d. manner according to Gaussian $\mathcal{N}(0, 1)$ distribution. To construct a sparse true solution $x^*$, we first randomly select $(p_{\text{fail}} \cdot n)$ locations. Then, we fill each of the selected locations with an i.i.d. Gaussian $\mathcal{N}(0, 1000)$ entry, while the remaining locations are set to 0. Here, $p_{\text{fail}}$ is the ratio of outliers. The measurement vector $b$ is obtained by $b = Ax^* + \delta$.

In Figure 1, we show the evolution of $f(x^k) - f^*$ and $\text{dist}(x^k - x^*)$ versus epoch counts with different number of blocks $N$. We can observe that RCS with larger $N$ converges faster than RCS with smaller $N$ and SubGrad. In addition, larger $N$ requires less workspace memory than small $N$ and SubGrad per iteration. Moreover, we find that RCS can effectively recover the sparse coefficients.
We compare RCS (set $N=d$) with SubGrad on the linear SVM problem (3) (i.e., Application 2). We repeat the problem in the following for convenience:

$$
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} \phi_{p_1}(a_i^\top x - b_i) + \frac{p_2}{2} \|x\|_1.
$$

where $a_i \in \mathbb{R}^d$ is a vector, $b_i \in \mathbb{R}$ is a scalar, $p_2 > 0$ is the regularization parameter, and $\phi_{p_1}(z) = \begin{cases} 
|z| - \frac{z}{2p_1} & |z| \leq p_1 \\
\frac{1}{2p_1} |z| & |z| > p_1 
\end{cases}$ is the MCP-loss.

Let $A = (a_1, \ldots, a_n)^\top \in \mathbb{R}^{n \times d}$ be a matrix and $b = (b_1, \ldots, b_n)^\top \in \mathbb{R}^n$ be a vector. In order to implement RCS, we introduce the block-wise partition of the columns of the data matrix $A = (A_1, A_2, \cdots, A_N)$, where $A_i \in \mathbb{R}^{n \times d_i}$ is the $i$-th block. For this problem, the coordinate subgradient used in RCS can be calculated as:

$$
\begin{align*}
\rho_{i(k)}^k &= \frac{1}{n} A_{i(k)}^\top u^k + p_2 \text{sign}(x_{i(k)}^k), \\
\delta_{i(k)}^{k+1} &= s^k + A_i(x_{i(k)}^{k+1} - x_{i(k)}^k), \\
\end{align*}
$$

where $u^k \in \partial\Phi(s^k) = (\partial\phi_{p_1}(s_{1}^k), \ldots, \partial\phi_{p_1}(s_{n}^k))^\top$ with $\partial\phi_{p_1}(z) = \begin{cases} 
\text{sign}(z) - \frac{z}{p_1} & |z| \leq p_1 \\
0 & |z| > p_1 
\end{cases}$ and $s^0 = Ax^0 - b$.

We conduct our simulation based on the synthetic data generated in Subsection 6.1. In Figure 2 and Figure 3, we show the evolution of $f(x^k) - f^*$ and dist$(x^k - x^*)$ versus epoch counts with different number of blocks $N$. Similar to the conclusions draw from Figure 1, we can observe that RCS with larger $N$ converges faster than RCS with smaller $N$ and SubGrad. In addition, larger $N$ requires less workspace memory than small $N$ and SubGrad per iteration. Finally, RCS can effectively recover the sparse coefficients.

### 6.2 Linear SVM problem

We compare RCS (set $N=d$) with SubGrad on the linear SVM problem (3) (i.e., Application 2). We repeat the problem in the following for convenience:

$$
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} \max \left\{ 0, 1 - b_i(x^\top a_i) \right\} + \frac{p}{2} \|x\|^2.
$$

Figure 3: Experiments on Robust M-estimators problem with MCP-loss and $\ell_1$-penalty. Top Left: $f(x^k) - f^*$ versus epoch counts for different choices of $N$ (i.e., the number of blocks in RCS); Top Right: $\text{dist}(x^k, x^*)$ versus epoch counts for different choices of $N$; Bottom Left: Coefficients recovery by RCS with $N = d$; Bottom Right: Comparison on workspace memory consumption per iteration. Here, $n = 1000$, $d = 2000$, $s = 40$, and $p_{\text{fail}} = 0.25$.

Table 1: RCS versus SubGrad on linear SVM.

| Data set       | RCS $f(x^k)$ memory (MB) | SubGrad $f(x^k)$ memory (MB) |
|---------------|--------------------------|-------------------------------|
| colon-cancer  | 0.0379                   | 0.0399                        |
| (62/2000)     |                          | 1.9698                        |
| duke          | 0.0078                   | 0.0719                        |
| (44/7129)     |                          | 5.0593                        |
| leu           | 0.0062                   | 0.1325                        |
| (38/7129)     |                          | 4.4065                        |
| gisette-scale.| 0.1270                   | 0.3166                        |
| (t1000/5000)  |                          | 76.5076                       |
| a1a           | 0.5762                   | 0.5762                        |
| (1605/119)    |                          | 2.9557                        |
| a2a           | 0.5896                   | 0.5896                        |
| (2265/119)    |                          | 4.1692                        |
| a3a           | 0.5718                   | 0.5718                        |
| (3185/122)    |                          | 6.0067                        |
| a4a           | 0.5821                   | 0.5821                        |
| (4781/122)    |                          | 9.0143                        |

where $a_i \in \mathbb{R}^d$ is a vector, $b_i \in \mathbb{R}$ is a scalar, $p > 0$ is a positive number.

Let $A = (a_1, \ldots, a_n)^T \in \mathbb{R}^{n \times d}$ be the data matrix and $b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$ be a vector. We define $\mathbb{R}^{n \times d} \ni \tilde{A} = (b_1 \mathbb{1}_d^T) \circ A$, where $\mathbb{1}_d$ is a $d$-dimensional vector of all 1s and $\circ$ is the Hadamard product. To implement RCS, we introduce the block-wise partition of the columns of the matrix $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_N)$, where $\tilde{A}_i \in \mathbb{R}^{n \times d_i}$ is the $i$-th block. Then, it is key to compute the coordinate subgradient $r_{i(k)}^k$, which can be calculated as:

$$
\begin{align*}
  r_{i(k)}^k &= -\frac{1}{n} \tilde{A}_{i(k)}^T \max\{0, \text{sign}(s^k)\} + px_{i(k)}^k, \\
  s^{k+1} &= s^k + \tilde{A}_{i(k)}(x_{i(k)}^k - x_{i(k)}^{k+1}),
\end{align*}
$$

where $s^0 = 1_n - \tilde{A}x^0$.

We use the real datasets from LIBSVM [2] to test the algorithms. The termination criterion for all algorithms is 200 epochs. We display the results in Table 1, from which we can observe that RCS outperforms SubGrad in terms of both the returned objective value and workspace memory consumption (per iteration).

### 6.3 Robust phase retrieval problem

We compare RCS (set $N = d$) with SubGrad on the robust phase retrieval problem (4) (i.e., Application 3). We display the problem below for convenience.

$$
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \|(Ax)^{\odot 2} - b^\odot 2\|_1,
$$

where $\odot$ denotes the Hadamard product.
We considered a general linearly bounded subgradients assumption and conducted thorough convergence analysis. We also derived a convergence lemma, the relationship between the global metric subregularity properties of a weakly convex function and its Moreau envelope, and the global metric subregularity of the (real valued) robust phase retrieval problem, which are of independent interests. Moreover, we conducted several experiments to show the superiority of RCS over the subgradient method. Our work reveals a provable and efficient subgradient-type method for a general class of large-scale nonsmooth optimization problems.

For nonsmooth weakly convex optimization, we need the global metric subregularity property in order to cope with the general linearly bounded subgradients assumption. It would be interesting to see if one can reduce this condition to a local version, which will enlarge the coverage of our theory. It is also worth considering more variants of RCS like its asynchronous and parallel versions. We leave them as our future directions.

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Figure 5: A supplemental to Figure 4 by showing more epochs.

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