Generic instability of the dynamics underlying the Belinski–Khalatnikov–Lifshitz scenario

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Received: 12 December 2021 / Accepted: 20 February 2022 / Published online: 11 March 2022
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Abstract A class of exact solutions to the Belinski–Khalatnikov–Lifshitz (BKL) scenario is derived and tested for their stability against small perturbations. These are the only regular solutions in the Painlevé sense. We prove that they are unstable in the vicinity of the cosmological singularity. The regularity of the dynamics is also examined with the dynamical systems method. Our results confirm the BKL conjecture that the dynamics near the singularity becomes generically chaotic.

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1 Introduction

By the BKL scenario, we mean the scenario proposed by Belinski, Khalatnikov, and Lifshitz to describe the evolution of the universe towards the cosmological singularity. This scenario, derived within general relativity, leads to the conclusion that the Einstein equations imply the existence of a generic solution with a gravitational singularity \cite{1,2}. By the generic solution, the authors mean, roughly speaking, that it corresponds to a non-zero measure subset of all initial data and depends on the proper number of arbitrary functions of space.

The derivation of this scenario is based on the general (non-diagonal) Bianchi VIII and IX models of spacetime evolving towards the singularity. Here the dynamics can be simplified by assuming that some stress–energy tensor components may be ignored, some Ricci tensor components have negligible influence, and anisotropy of space may grow without bounds. These assumptions lead to enormous simplification of the mathematical form of the dynamics. It can be well approximated by the following system of equations \cite{3–6}:

\begin{align}
\frac{d^2 \ln a}{dt^2} &= \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \\
\frac{d^2 \ln c}{dt^2} &= a^2 - \frac{c}{b},
\end{align}

subject to the constraint

\begin{align}
\frac{d \ln a}{dt} + \frac{d \ln b}{dt} + \frac{d \ln a}{dt} + \frac{d \ln c}{dt} - \frac{d \ln b}{dt} - \frac{d \ln c}{dt} &= a^2 + \frac{b}{a} + \frac{c}{b},
\end{align}

where \(a = a(t), b = b(t)\) and \(c = c(t)\) are the so-called directional scale factors, while \(t\) is the time parameter in the synchronous reference system. Their evolution defines the dynamics of the characteristic lengths in three directions, while the universe tends to the singularity. Due to time-reversibility of Eqs. (1)–(2), it may also describe expansion of the universe away from the singularity.

These scale factors include contributions from standard matter fields, e.g., the perfect fluid with equation of state \(p = k \varepsilon, 0 \leq k < 1\), where \(p\) and \(\varepsilon\) denote the pressure and energy density of the fluid, respectively. The case \(k = 1\) (describing, e.g., a massless scalar field) is excluded, as it does not lead to oscillatory dynamics inherent in the Bianchi models. It is likely that other gravity sources may lead to the asymptotic form (1)–(2) as well. For instance, it may include...
an electromagnetic field or the Yang–Mills fields, but further examination is required to confirm that expectation (see Sec. 4 of [6] for discussion of this issue).

The system of equations (1)–(2) has never been solved analytically, in spite of its importance in the context of the BKL scenario. In this article, we find an explicit solution to this dynamics, analyse its stability and regularity, and provide its physical interpretation.

The BKL scenario [1,2] proposes a mechanism that leads to the generic singularity via a stochastic process. We confirm the existence of this scenario by showing that the only regular solution, in the Painlevé sense, to the dynamics (1)–(2) is unstable and leads to chaos.

The paper is organised as follows: In Sect. 2, the exact solution is presented and its stability is examined. The regularity analysis of the dynamics is carried out in Sec. 3 within the dynamical systems method. The last section presents the conclusions. The appendices contain one of the possible derivations of our solution (Appendix A) and discuss the issue of the monotonicity of the space volume (Appendix B).

2 Solution

2.1 Special exact solution

The analytical solution of Eqs. (1)–(2) reads

\[ a(t) = \frac{3}{|t - t_0|}, \quad b(t) = \frac{30}{|t - t_0|^3}, \quad c(t) = \frac{120}{|t - t_0|^5}, \]

(3)

where \(|t - t_0| \neq 0\) and \(t_0\) is an arbitrary real number.

This solution may be obtained by a systematic method rather than a smart guess. For instance, one can use (i) extension of the Painlevé test applied to Eqs. (1)–(2), (ii) expansion of these equations about \(t = \infty\), or (iii) a search for their self-similar solution. In Appendix A, we describe the first of these methods.

Equations (1)–(2) have been derived from the general dynamics of the Bianchi VIII and IX models under the condition that near the singularity one has [3,4])

\[ a \gg b \gg c > 0. \]

Therefore, the physically relevant part of the special solution to (1)–(2) should satisfy that condition as well. Our solution (3) satisfies this condition, provided that \(|t - t_0|\) is sufficiently large, which is true near the singularity (corresponding to \(|t| \to \infty\)).

2.2 Canonical structure

It is shown in [14] that Eq. (1) can be derived from the Lagrangian

\[ \mathcal{L} = \dot{x}_1 \dot{x}_2 + \dot{x}_2 \dot{x}_3 + \dot{x}_3 \dot{x}_1 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2), \]

(5)

where the dot over a symbol denotes its time derivative \(d/dt\), and

\[ x_1 = \ln a, \quad x_2 = \ln b, \quad x_3 = \ln c. \]

(6)

The dynamical constraint (2) corresponds to the condition that the “energy”

\[ \mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L} = \dot{x}_1 \dot{x}_2 + \dot{x}_2 \dot{x}_3 + \dot{x}_3 \dot{x}_1 - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2) \]

(7)

vanishes [14].

In order to interpret our solution in terms of the canonical variables, we perform an orthogonalisation of the “kinetic” part of the energy by a linear transformation

\[ x_1 = u_1 - u_2 - u_3, \quad x_2 = u_1 + 2u_3, \quad x_3 = u_1 + u_2 - u_3, \]

(8)

which yields the Lagrangian in the form diagonal in the “velocities” \(\dot{u}_1, \dot{u}_2, \dot{u}_3\)

\[ \mathcal{L} = 3\dot{u}_1^2 - \dot{u}_2^2 - 3\dot{u}_3^2 + \exp(2(u_1 - u_2 - u_3)) + \exp(u_2 - 3u_3) + \exp(u_2 + 3u_3). \]

(9)

It also provides an analogous expression for the “energy” \(\mathcal{H}\), which differs from the Lagrangian \(\mathcal{L}\) by having the opposite signs of the “potential” part, which consists of the terms with exponential functions. Thus, the variables \(u_1, u_2, u_3\) define the principal directions in the velocity space. If the “energy” \(\mathcal{H}\) is expressed in terms of the momenta \(p_i = \partial \mathcal{L}/\partial \dot{u}_i, \ i = 1, 2, 3\), then it becomes the Hamiltonian, which is also diagonal in the momenta.

Since the “energy” is zero, the whole dynamics of the system takes place in the inner part of the cone, which means that

\[ 3\dot{u}_1^2 - \dot{u}_2^2 - 3\dot{u}_3^2 > 0, \]

(10)

as shown in Fig. 1.

Expression (10) is zero on the conical surface. Together with the constraint \(\mathcal{H} = 0\), it means that the exponential terms in (9) turn to zero, which is possible only for all \(u_1\)’s
tending to $-\infty$. The latter may happen only for $t \to \infty$ (for the case $t > t_0$).

The exact solution (3) corresponds to

$$
\begin{align*}
    u_1 &= \frac{1}{3} \ln \frac{10800}{|t - t_0|^9}, \\
    u_2 &= \frac{1}{3} \ln \frac{40}{|t - t_0|^4}, \\
    u_3 &= \frac{1}{6} \ln \frac{5}{2}.
\end{align*}
$$

(11)

Note that the dynamics leaves the third component, $u_3$, unchanged, which means that the evolution is two-dimensional. Geometrically, the exact solution (3) describes the dynamics within a planar coaxial section $\dot{u}_3 = 0$ of the cone (10) along the line $\dot{u}_2 = \frac{2}{3} \dot{u}_1$, which lies inside the cone (see Fig. 1). For $t \to \infty$, the solution (11) tends to the vertex of the cone as $|t - t_0|^{-1}$. The line escapes to infinity, at some finite $t_0$, also as $|t - t_0|^{-1}$; however, this region lies beyond the range of applicability of the BKL scenario.

Condition $u_3 = \text{const.}$ of (11) in terms of the original variables $u, b, c$ is equivalent to the requirement that $b$ is proportional to the geometric mean of $a, c$, i.e. $b = k \sqrt{ac}$ with a constant coefficient ($k = \frac{1}{2} \sqrt{10}$). This property is easy to observe in (3). A closer look shows that solution (3) is the only one satisfying the geometric mean condition and the constraint (2).

One can conclude, a posteriori, that (3) is a special solution to the dynamics (1)–(2) corresponding to the choice of the following initial data:

$$
\begin{align*}
    a(0) &= -3 t_0^{-1}, & \dot{a}(0) &= -3 t_0^{-2}, \\
    b(0) &= -30 t_0^{-3}, & \dot{b}(0) &= -90 t_0^{-4}, \\
    c(0) &= -120 t_0^{-5}, & \dot{c}(0) &= -600 t_0^{-6},
\end{align*}
$$

(12)

for instance, in the case $t > t_0$ and $t_0 < 0$.

2.3 Stability analysis

In what follows, we consider a linear approximation to the general solution in terms of a small perturbation of the solution (3).

To check how the small perturbation to the solution (3) develops in time, we substitute the following functions into (1)–(2):

$$
\begin{align*}
    a(t) &= 3(t - t_0)^{-1} + \varepsilon \alpha(t), \\
    b(t) &= 30(t - t_0)^{-3} + \varepsilon \beta(t), \\
    c(t) &= 120(t - t_0)^{-5} + \varepsilon \gamma(t).
\end{align*}
$$

(13)

In the first order in the small parameter $\varepsilon$, we obtain

$$
\begin{align*}
    \ddot{a} + \frac{2\dot{a}}{t - t_0} + \frac{28\alpha}{(t - t_0)^2} - \beta &= 0, \\
    \ddot{b} + \frac{6\dot{b}}{t - t_0} + \frac{20\beta}{(t - t_0)^2} - \frac{280\alpha}{(t - t_0)^4} - \gamma &= 0, \\
    \ddot{c} + \frac{10\dot{c}}{t - t_0} + \frac{24\gamma}{(t - t_0)^4} - \frac{16\beta}{(t - t_0)^4} - \frac{720\alpha}{(t - t_0)^6} &= 0,
\end{align*}
$$

(14)

with the constraint

$$
\begin{align*}
    \dot{\gamma} + \frac{6\dot{\gamma}}{t - t_0} + \frac{6\dot{\beta}}{t - t_0} + \frac{24\gamma}{(t - t_0)^4} + \frac{80\dot{\alpha}}{(t - t_0)^4} + \frac{160\alpha}{(t - t_0)^5} &= 0.
\end{align*}
$$

(14d)

The system of equations (14) is linear and homogeneous. Its general solution may be simply written in terms of the rescaled evolution (time) parameter $\theta = \ln|t - t_0|$ and two
The incommensurability of the frequencies $\omega_1$ and $\omega_2$ results in the ergodic character of the evolution which starts from solution (3) perturbed as in (13). Namely, the phase spaces of $(\alpha, \dot{\alpha})$, $(\beta, \dot{\beta})$ and $(\gamma, \dot{\gamma})$ are densely covered with the trajectories of the perturbed solution for almost all initial conditions (the exceptions are of measure zero). Obviously, the six-dimensional phase space of all three variables and their time derivatives, even though reduced to five dimensions by constraint (2), cannot be densely covered by the two-frequency sinusoidal oscillations.

From the point of view of dynamical systems, this means that evolution of the instability leads to chaotic behaviour. This way, even the only regular solution, unique up to translations in $t$, decays to chaos. This confirms the conjecture of BKL that the chaotic behaviour inevitably accompanies an approach to the cosmological singularity.

Equation (16) presents the general solution of (14). It depends on three arbitrary real constants \{K_1, K_2, K_3\} (expected to be small to comply with the linear approximation) and any two real constants \{\varphi_1, \varphi_2\} from the interval $[0, 2\pi]$. The manifold $\mathcal{M}$ defined by \{K_1, K_2, K_3, \varphi_1, \varphi_2\} is a submanifold of $\mathbb{R}^5$. The solution defined by (13) and (16) corresponds to the choice of the set of the initial data $\mathcal{N}$ that is a small neighbourhood of the initial data (12). $\mathcal{N}$ is a submanifold of $\mathbb{R}^5$, as (12) defines five independent constants due to the constraint (2). Thus, Eq. (16) presents a generic solution to (14), in the sense mentioned in the Introduction, as the measures of both $\mathcal{M}$ and $\mathcal{N}$ are non-zero (although the exact solution (3) is obviously of zero measure in the space of all possible solutions of (1)–(2)).

In this context, the term “generic” requires some comments. Usually, this term is regarded as a more precise equivalent to “typical”. However, in the vast sea of solutions to Einstein’s equations, there may exist many typical islands. Therefore, the authors of [1,2] formulated their conditions of the dependence on a sufficient number of arbitrary functions on space and a non-zero measure of the set of initial data.

Our “genericness” is much more modest, as the BKL scenario consists of ordinary differential equations. Moreover, the perturbation of each of the scale factors $a$, $b$, and $c$ is proportional to the parameter $\varepsilon$. This makes the non-zero measure a small quantity vanishing as $\varepsilon \to 0$. Hence, our family of solutions is not typical, though it meets the BKL criterion of being generic.

For the development of the singularity, an important quantity is the evolution of the space volume $V = abc$. The perturbed volume corresponding to solutions (3) and (16), up to first-order terms in the perturbations $\{\alpha, \beta, \gamma\}$, reads

$$V = (a + \alpha)(b + \beta)(c + \gamma) \cong abc + ab\gamma + bca + ca\beta$$

\[\text{We have incorporated the small parameter } \varepsilon \text{ into } \alpha, \beta, \text{ and } \gamma.\]
As seen from (17), the volume tends to zero for \( t \to \infty \). However, it apparently oscillates with the same two characteristic frequencies \( \omega_1 \) and \( \omega_2 \) as \( a \), \( b \), and \( c \). The ratio of the perturbation to the zero-order term grows as \( \exp(\theta/2) \), as that in the evolution in each direction (16).

Figure 3 presents the instability of the space volume without showing the accompanying uniform expansion (i.e. for \( K_3 = 0 \)). The apparent oscillations are non-physical because they correspond to bounces of the space, which contradicts the gravitational singularity inherent in the BKL scenario. However, it turns out that we can eliminate the bouncing if we restrict the manifold \( \mathcal{N} \) to some submanifold \( \mathcal{K} \). Namely, these volume oscillations (17) do not affect the actual monotonicity of \( V(\theta) \), provided that constant \( K_3 \) is sufficiently large compared to \( K_1 \) and \( K_2 \). Appendix B presents a proof of this desired feature of our solution (16). The evolution of the system towards the singularity manifests as a monotonic decrease in the volume, whereas the evolution away from the singularity is described by its monotonic increase. Therefore, the linear perturbation of the volume has the same properties as the unperturbed volume \( V = abc \) corresponding to solution (3).

3 Dynamical systems analysis

In this section we examine the stability of the dynamics by using the dynamical systems method [16, 17]. First, we determine the critical points of the dynamics (A1a)–(A1d), which characterises the local geometry of the solution space. For this purpose, we rewrite the system (A1a)–(A1c) in the form suitable for the analyses:

\[
\dot{x} = x^2/a + b - a^3, \quad (19)
\]

\[
\dot{y} = y^2/b + a^2b - b^2/a + c, \quad (20)
\]

\[
\dot{z} = z^2/c + a^2c - c^2/b, \quad (21)
\]

\[
\dot{a} = x, \quad (22)
\]

\[
\dot{b} = y, \quad (23)
\]

\[
\dot{c} = z, \quad (24)
\]

and the constraint (A1d) reads

\[
cxy + bxz + ayz - a^3bc - b^2c - ac^2 = 0. \quad (25)
\]

The Eqs. (1)–(2), and consequently (19)–(21), make sense if

\[
a > 0, \quad b > 0, \quad c > 0. \quad (26)
\]
Inserting $\dot{x} = 0 = \dot{y} = \dot{z} = \dot{a} = \dot{b} = \dot{c}$ into the left-hand sides of (19)–(24) leads to the set of equations

\begin{align}
0 &= x^2/a + b - a^3, \quad (27) \\
0 &= y^2/b + a^2b - b^2/a + c, \quad (28) \\
0 &= z^2/c + a^2c - c^2/b, \quad (29) \\
0 &= x, \quad (30) \\
0 &= y, \quad (31) \\
0 &= z. \quad (32)
\end{align}

For $x = 0 = y = z$ and $t < \infty$, the solution to (27)–(29) does not exist if we insist on (26) being satisfied. Quite separate treatment is required for the case $t \to \infty$.

Applying the substitution $t = 1/\tau$ to the system (A1a)–(A1d) allows us to examine that dynamics in the limit $\tau \to 0$ instead of $t \to \infty$, in the context of the gravitational singularity. However, that substitution turns (A1a)–(A1d) into a system explicitly dependent on $\tau$. The latter prevents us from making use of the dynamical systems analysis. Nevertheless, another method is available.

First, it directly follows from (27)–(32) that when the universe collapses, which includes $a \to 0^+$, then $b/a \to 0^+$ and $c/b \to 0^+$, whence we have $a > b > c > 0$ in the neighbourhood of the singularity. A more precise estimation can be obtained by introducing $\epsilon_1 = b/a$ and $\epsilon_2 = c/b$. This substitution turns (27)–(29), for $x = 0 = y = z$, into the system

\begin{align}
0 &= a(\epsilon_1 - a^2), \quad (33) \\
0 &= a^2(\epsilon_1 - \epsilon_1^2) + a \epsilon_1 \epsilon_2, \quad (34) \\
0 &= a^3 \epsilon_1 \epsilon_2 - a \epsilon_1 \epsilon_2^2, \quad (35)
\end{align}

which implies that $\epsilon_1 = \epsilon_2 =: \epsilon$ and $a \sim \epsilon^{1/2}$, so that $b \sim \epsilon^{3/2}$ and $c \sim \epsilon^{5/2}$. The latter is consistent with the constraint (25). Our solution (3) is obviously consistent with this estimation.

Finally, the space of the critical points, $S$, turns out to be

$$S = \{(a, b, c, x, y, z) \in \mathbb{R}^6 \mid (x = 0 = y = z) \land (b = a^3)\}. \quad (36)$$

Now, let us examine the type of the criticality of the elements of the space $S$. Following the method of [16, 17], we first determine the Jacobian, $J$, corresponding to the rhs of (19)–(24). It can be found to be

$$J = \begin{pmatrix}
J_{11} & 1 & 0 & 2x/a & 0 & 0 \\
J_{21} & J_{22} & 1 & 0 & 2y/b & 0 \\
2ac & c^2/b^2 & J_{33} & 0 & 2z & c \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (37)$$

where

\begin{align}
J_{11} &= -x^2/a^2 - 3a^2, \quad J_{21} = 2ab + b^2/a^2, \quad J_{22} = -y^2/b^2 + a^2 - 2b/a, \quad J_{33} = -z^2/c^2 + a^2 - 2c/b. \\
\end{align}

The Jacobian evaluated at any point of $S$ and of order $\epsilon$ turns out to be

$$J_S = \begin{pmatrix}
-3a^2 & 1 & 0 & 0 & 0 & 0 \\
0 & -a^2 & 1 & 0 & 0 & 0 \\
0 & 0 & -a^2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (38)$$

Thus, the characteristic polynomial, $P(\lambda)$, associated with $J_S$ reads

$$P(\lambda) = (-3a^2 - \lambda)(-a^2 - \lambda)^2(1 - \lambda)^3. \quad (39)$$

It is clear that all the eigenvalues of $J_S$ are real numbers, as $a \in \mathbb{R}$, so that the space $S$ consists of the hyperbolic critical points. However, in the limit $t \to \infty$, the three eigenvalues vanish. Therefore, in the latter case the points of $S$ are non-hyperbolic.

The dynamical systems analysis shows that the exact form of the dynamics (1)–(2) includes, near the singularity, the instabilities connected with the space of the non-hyperbolic critical points. The dynamics near these points bifurcates [16, 17].

4 Conclusions

The best prototype for the BKL scenario was derived by Belinski, Khalatnikov, and Ryan [3, 4] in the context of the BKL conjecture. It is defined by Eqs. (1)–(2) and the initial data satisfying condition $a > b > c > 0$ (see [5, 6] for more detail).

In this paper we present an analytical solution to that prototype. It is the only regular solution in the Painlevé sense. The solution was, to our best knowledge, previously unknown.

\footnote{A critical point is called a hyperbolic critical point if all eigenvalues of the Jacobian matrix of the linearised equations at this point have non-zero real parts. Otherwise, it is called a non-hyperbolic critical point [16, 17].}
The evolution presented in two recent papers [18, 19] are just numerical simulations of the related dynamics in the Bianchi IX model case.

Our special monotonic solution (3) is generically unstable against perturbations of the initial data (12). Interpreting the evolution of our system in two interesting time directions, we see that (i) in the evolution towards the singularity, the oscillating correction becomes dominant compared to the monotonically decreasing solution (3), and (ii) in the direction away from the singularity, the special solution (3) and the uniform expansion proportional to $K_3$ play the dominant role. In both cases, $K_3$ has to be sufficiently large in comparison with $K_1$ and $K_2$ to ensure the monotonicity of the volume $V$.

These are perturbations of the solution to the dynamics derived from the asymptotic dynamics of the non-diagonal Bianchi VIII and Bianchi IX models, which underlie the BKL scenario. Our two-region scenario is similar to, but much more general than, the two-stage evolution of the diagonal Bianchi IX model considered by Grishchuk et al. [20]. It is based on the decomposition of the Bianchi IX metric into the Friedmann background (among other results) and the terms representing gravitational waves.

Exact solitonic gravitational perturbations on the Friedmann background were recently considered by Belinski et al. (see [21] and references therein). They showed that the solitonic perturbations decay into gravitational waves during the evolution away from the singularity. An interpretation of the perturbations (16) as possible seeds of gravitational waves will be published elsewhere [22].

Making use of the dynamical systems techniques, we show that the evolution of the system (1)–(2) is regular (can be locally linearised) for any finite value of the evolution parameter $t$. The dynamics approaches the space of non-hyperbolic critical points in the limit $t \to \infty$, which means that locally the dynamics cannot be linearised near those points. In that limit, the directional scale factors go to zero so that the space volume $V$ goes to zero as well. The latter means (see [18,19] for more detail) that the system approaches the gravitational singularity.

It results from Sects. 2 and 3 that as the system approaches the singularity, its dynamics becomes sensitive to the choice of the initial data, which means that the dynamics becomes chaotic. We have shown that there exists a family of regular solutions to the BKL system which are parameterised by the real number $t_0 \in \mathbb{R}$. These solutions decay into chaos due to their instabilities. Thus, chaotic behaviour is an inevitable companion of the approach to the cosmological singularity. The latter is consistent with the BKL scenario.

The existence of generic singular behaviour, predicted by BKL, means that general relativity is not a complete theory of gravitation. It is commonly expected that quantum gravity (still to be constructed) would be free from singularities. The quantisation of the BKL scenario carried out recently [23, 24] supports such expectation. However, this result should be confirmed by quantising that scenario, possibly within a completely different approach, to verify its robustness. As commonly known, quantisation of a gravitational system is an ambiguous procedure.

**Acknowledgements** We would like to thank Vladimir Belinski for the valuable discussions.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: The article concerns entirely theoretical research.]

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**Funded by** SCOAP$^3$.

**Appendix A: Derivation of the solution (3)**

To derive the solution, we are going to use singularity analysis. Similar work was performed for the vacuum Bianchi IX matter (mixmaster universe) in [11]; one of their results was regaining Taub’s special solutions [12].

The analysed Eqs. (1) in their polynomial form read

\[
\begin{align*}
    a \ddot{a} - a^2 - ab + a^4 &= 0, \\
    a \ddot{b} - a \dot{b}^2 - a^3 b^2 + b^3 - ab c &= 0, \\
    b \ddot{c} - b \dot{c}^2 - a^2 b c^2 + c^3 &= 0,
\end{align*}
\]

while the constraint (2) takes the polynomial form

\[
    c \dot{a} b + \dot{b} \dot{c} + a \dot{b} \dot{c} - a^3 b - b^2 c - ac^2 = 0.
\]

The constant coefficients of the system (A1a)–(A1d) are obviously free of singular points. Hence, the positions of the possible singularities of the solution are determined by the initial conditions (“movable singularities” [7,8]). To find a solution with a singularity, which is a pole and thus does not introduce branching (i.e. has the Painlevé property), we proceed in a similar way to that used in the classical Painlevé test [9,10]. Since our system is overdetermined, we need a more involved analysis.
First, we look for a solution in the form of the Laurent series about the assumed pole $t_0$ so that we have

$$a(t) = (t - t_0)^p \sum_{n=0}^{\infty} a_n(t - t_0)^n,$$

(A2a)

$$b(t) = (t - t_0)^q \sum_{n=0}^{\infty} b_n(t - t_0)^n,$$

(A2b)

$$c(t) = (t - t_0)^s \sum_{n=0}^{\infty} c_n(t - t_0)^n.$$  

(A2c)

Substituting series (A2) into the system (A1a)–(A1d), we obtain, in the zero order, conditions of balance of the dominant terms. These conditions may be satisfied in all Eqs. (A1a)–(A1d) provided that the exponents are $p = -1, q = -3, s = -5$ (the other family of solutions with $p = -1, q = 2, s = 0$, does not comply with the physics of the system, as some of the terms are complex for real time).

A search for the initial exponents (only) in the Laurent expansion was performed in [13]. However, to the best of our knowledge, Painlevé analysis for the BKL equations (A1a)–(A1d) or other models including matter has never been done.

The coefficients at the dominant terms obtained from the balance equations prove to be

$$a_0 = 3, \quad b_0 = 30, \quad c_0 = 120.$$  

(A3)

In the Painlevé test, the higher-order coefficients are obtained from algebraic linear recurrence relations, which yield the coefficients $a_n, b_n, c_n$ as functions of the lower-order coefficients and $t_0$. If the general solution was of the type (A2), then there should be five arbitrary constants, as the system consists of three second-order ordinary differential equations (ODE), (A1a)–(A1c), with one constraint (A1d). Bearing in mind that one of the arbitrary constants is $t_0$, we see that the recurrence equations should provide four more constants.

The arbitrary constants occur at the recurrence relation determining those terms of the series (A2) if two conditions are satisfied together: (i) the rank of the coefficient matrix is lower than the number of unknown coefficients and (ii) the system is compatible, i.e. extending the coefficient matrix with the right-hand sides of the algebraic equations does not increase its rank.

In the Painlevé test, the indices of these terms are referred to as “resonances” [9] or simply “indices” [7] (the latter by analogy with the Fuchsian theory of linear differential equations with singularities). For the first three Eqs. (A1a)–(A1c), there is one positive “resonant index” (a compromise between these two names) $r_1 = 2$, such that the determinant of the algebraic recurrence system of three equations vanishes at $n = r_1 = 2$. However, if we also require satisfaction of the constraint (A1d), then the rank of the coefficient matrix of the linear system is always 3, i.e. there are $3 \times 3$ submatrices of the coefficient matrix in (A1a)–(A1d) whose determinants are non-zero. The requirement that the $3 \times 3$ determinants vanish has more solutions: a resonant index $r_2 = -1$, which corresponds to the arbitrariness of $t_0$, and four complex values:

$$r_{3,4} = \frac{1}{2} \left( 1 \pm i \sqrt{95 - 24 \sqrt{6}} \right)$$  

(A4a)

$$r_{5,6} = \frac{1}{2} \left( 1 \pm i \sqrt{95 + 24 \sqrt{6}} \right).$$  

(A4b)

Because we get these complex “indices” instead of the actual positive integer resonant indices, the recurrence relations do not generate new arbitrary constants. Hence, a possible solution with the pole may only be a special one. The general solution does not have the form of a Laurent series (the equations are non-Painlevé, i.e. contain branch points or essential singularities which introduce branching [8]). The first three equations are compatible with the fourth one. Hence, the system of the recurrence relations has exactly one solution at each order.

It is easy to recognise solution (3) in the series (A2) limited to the zero-order terms. Indeed, Eq. (3) is a solution of the system (1) and satisfies the constraint (2) both for $t > t_0$ and $t < t_0$. At the same time, it consists of the zero-order coefficients (A3) divided by the appropriate powers of $t - t_0$. This result is compatible with the recurrence relations: they yield $a_n = 0, b_n = 0$, and $c_n = 0$, for all $n > 0$ (again, provided that we require satisfaction of the constraint (2)).

A similar expansion can be performed in the neighbourhood of $t = \infty$. It results in the conclusion that there are no solutions which tend to infinity while $t \to \infty$. The dominant terms for $a, b, c$ prove to be $\pm 3r^{-1}, \pm 30r^{-3},$ and $\pm 120r^{-5}$, respectively, corresponding to those of the special solution (3). These are the only possible coefficients if $t = \infty$ is a regular point or a pole. There are no indices which are resonant for the whole four-equation system, i.e. the rank of the $4 \times 3$ matrix of the coefficients is never less than 3. The equations following from the requirement that the determinants of all $3 \times 3$ submatrices of the coefficients vanish yield solutions which cannot be the indices of the expansion, namely $r = -1$ and the same irrational complex $r$ as in (A4). This way, the large-$t$ expansion provides the same special solution (3) as the expansion about a hypothetic movable pole $t_0$.

An expansion in an arbitrary function $\Phi(t)$ instead of $t - t_0$ shows that the special solution (3) is the only solution of (1) which has the Painlevé property and satisfies constraint (2). Another family of solutions of (1) having the Painlevé property exists, which are proportional to powers of $\cos \phi(t)$, instead of $(t - t_0)^{-1}$, but they are incompatible with constraint (2). The situation is similar to that obtained in [11], although the model is different.
The Painlevé property is a usual companion to integrability and regular behaviour, which includes lack of bifurcations (or multifurcations) at unknown moments, but the connection is not 1:1. However, this somewhat vague statement may be made stronger in our case, because our system is autonomous and its physically relevant solutions depend on $t$ through $t - t_0$ with $t \in \mathbb{R}$. Hence, $t_0 \in \mathbb{R}$ may also be poles for this class of solutions. This means that a physical non-Painlevé solution has to encounter the branching singularity in its evolution towards $t_0$ or stem from the singularity in its evolution from $t_0$.

Appendix B: Monotonicity of volume

This appendix contains a discussion of the monotonicity of the volume. We first discuss how the volume is affected by the linear evolution of a small perturbation $\alpha$, $\beta$, $\gamma$ of (respectively) $a$, $b$, $c$ given by solution (3). Then, we add a short discussion of the general case. In the first part, we bear in mind that the linear approximation is valid in a limited interval of time, as the system is linearly unstable.

The volume $V = abc$ given by (17) should be a decreasing function of time $t$ or $\theta = \ln(t - t_0)$, $t > t_0$ (which also means increasing in the backward evolution). In the linear approximation, the $\theta$ derivative of $V$ may be written as

$$V'(\theta) = -2160e^{-170/2}\left[150K_3e^{-3\theta/2} + 45e^{-\theta/2}
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