Robustness of Discrete Flows and Caustics in Cold Dark Matter Cosmology

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Abstract

Although a simple argument implies that the distribution of dark matter in galactic halos is characterized by discrete flows and caustics, their presence is often ignored in discussions of galactic dynamics and of dark matter detection strategies. Discrete flows and caustics can in fact be irrelevant if the number of flows is very large. We estimate the number of dark matter flows as a function of galactocentric distance and consider the various ways in which that number can be increased, in particular by the presence of structure on small scales (dark matter clumps) and the scattering of the flows by inhomogeneities in the matter distribution. We find that, when all complicating factors are taken into account, discrete flows and caustics in galactic halos remain a robust prediction of cold dark matter cosmology with extensive implications for observation and experiment.

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I. INTRODUCTION AND OUTLINE

A central problem in dark matter studies today [1] is the question of how the dark matter is distributed in the halos of galaxies, and in particular in the halos of spiral galaxies such as our own Milky Way. Indeed, knowledge of this distribution is of crucial importance in trying to understand galactic dynamics and in predicting signals for direct and indirect searches for dark matter on Earth. In this paper, we argue on general grounds that the distribution of cold collisionless dark matter (CDM) in galactic halos is characterized by discrete flows and caustics [2–6]. The latter are found to be a robust prediction of the cold dark matter hypothesis without any further assumptions, whether of symmetry, self-similarity or anything else.

In Section II we give the basic argument for the existence of discrete flows and caustics, which is that cold dark matter particles lie at all times on a 3-dim. hypersurface in 6-dim. phase space [2]. We call this 3-dim. hypersurface the “phase space sheet”. The number of discrete flows at a given physical location is the number of times this sheet covers 3-dim. physical space at that location. At the boundaries between regions in physical space with differing numbers of flows, the phase space sheet is tangent to velocity space. As a result, the dark matter density is very large on these surfaces, which are called caustics. The density diverges at the caustics in the limit of zero velocity dispersion.

There can be no real doubt that CDM particles form discrete flows and caustics for the reason just stated. However, it is possible that this is not relevant in practice because the number of flows is so large that they form effectively a 6-dim. phase space continuum. In Sections III to V, we estimate the number of flows and consider the ways in which this number can become very large.

In Section III, we estimate the number of flows when the presence of structure in the primordial dark matter fluid on scales less than galactic scales is neglected and scattering of the dark matter flows by inhomogeneities in the galaxy is neglected. We provide a formula for estimating the number of flows at an arbitrary location in a galactic halo in these idealized circumstances. At our location in the Milky Way halo the number of flows is of order 100.

In Section IV, we consider the effect of small scale structure in the dark matter fluid. We show that small scale structure gives an effective velocity dispersion $\delta v_{\text{eff}}$ to the flows. However $\delta v_{\text{eff}}$ is expected to be far less than what is required to smear out the 100 flows at our location.

In Section V, we consider the effect on the flows of dark matter of gravitational scattering by inhomogeneities in the galactic matter distribution. We show that the known inhomogeneities in the luminous matter of our galaxy are far insufficient to thermalize the flows of particles that have fallen in and out of the Galaxy less than 10 times in the past, implying that there are at least 20 flows at our location in the Milky Way which are not thermalized. We also consider the effect of inhomogeneities in the dark matter distribution. They can diffuse the remaining flows of dark matter only if a fraction of order one of the dark matter is in clumps of mass $10^{10} M_\odot$ or more.

In Section VI, we point to the existence of ripples in the distribution of light around large elliptical galaxies [7,8] as proof that flows do not get diffused by gravitational scattering, or by anything else for that matter. In particular the ripples are inconsistent with the dark matter being in clumps of mass $10^{10} M_\odot$ or more.
In Section VII, we consider the effect of our galactic neighbor M31 on the flows and caustics in the Milky Way. We also discuss the ability of N-body simulations to resolve dark matter flows and caustics.

Finally, in Section VIII, we consider the implications of dark matter flows and caustics for experiment and observation.

II. THE BASIC ARGUMENT

Consider the phase space distribution of CDM particles at an early time $t_{in}$, long before density perturbations have become non-linear. Because CDM is ‘cold’ all the dark matter particles at the same physical location $\vec{q}$ have the same velocity $\vec{v}(\vec{q})$ up to a small dispersion $\delta v$, hereafter called the “primordial velocity dispersion”. The primordial velocity dispersion of the leading cold dark matter candidates is very small. For axions, it is

$$\delta v_a(t) \sim 3 \cdot 10^{-17} c \left( \frac{10^{-5} \text{eV}}{m_a} \right)^{\frac{3}{2}} \left( \frac{t_0}{t} \right)^{\frac{2}{3}},$$

(2.1)

whereas for weakly interacting massive particles (WIMPs)

$$\delta v_W(t) \sim 10^{-11} c \left( \frac{\text{GeV}}{m_W} \right)^{\frac{1}{2}} \left( \frac{t_0}{t} \right)^{\frac{2}{3}}.$$

(2.2)

Here $m_a$ and $m_W$ are respectively the axion mass and WIMP mass, $t$ is the age of the universe and $t_0$ is its present age. Eqs. (2.1,2.2) assume that $t$ is after the time of equal energy density in matter and radiation. Since the particles at the same physical location have the same velocity, they lie on a 3-dim. hypersurface in 6-dim. phase space [2,6]. Let us call this hypersurface the “phase space sheet”. The thickness of the sheet is $\delta v$. The sheet is continuous because the density of particles is huge on the scale over which the sheet is bent in phase space. Being collisionless, the particles move under the influence of purely gravitational forces. At a later time $t$, all the particles that were initially at position $\vec{q}$ have moved to position $\vec{x}(\vec{q}, t)$, where they have velocity $\vec{v}(\vec{q}, t) = \frac{\partial \vec{x}}{\partial t}(\vec{q}, t)$. The function $\vec{x}(\vec{q}, t)$ therefore determines the position of the phase space sheet at all times.

When the density perturbations become non-linear the sheet begins to fold in phase space, i.e. it begins to cover physical space multiple times whereas it covered physical space only once when density perturbations were small. Mathematically this is expressed by stating that, at late times $t$, for a given physical location $\vec{r}$ there are in general multiple solutions $\vec{q}_j, j = 1, 2, .. n$, to the equation $\vec{r} = \vec{x}(\vec{q}_j, t)$. Each solution corresponds to a distinct flow at $\vec{r}$ with velocity $\vec{v}_j(\vec{r}, t) = \frac{\partial \vec{x}}{\partial t}(\vec{q}_j, t)$ and density

$$d_j(\vec{r}, t) = \frac{d(\vec{q}_j, t_{in})}{|D(\vec{q}_j, t)|},$$

(2.3)

where $d(\vec{q}_j, t_{in})$ is the density at location $\vec{q}_j$ at the early time $t_{in}$, and

$$D(\vec{q}_j, t) = \text{det} \left( \frac{\partial x_k}{\partial q_l} \right).$$

(2.4)
The magnitude of $D$ is the Jacobian of the map $\vec{q} \rightarrow \vec{x}$.

Caustics occur where $D = 0$, i.e. where the map is singular [5]. In particular, there is a caustic wherever the number $n$ of flows changes. The physical density is very large at the caustic because the phase space sheet is tangent to velocity space there. The density diverges at the caustic in the limit of zero velocity dispersion. Since the map $\vec{q} \rightarrow \vec{x}$ is singular where the number of flows changes, caustics lie generically at the boundaries between regions which have different numbers of flows. On one side of a caustic surface are two more flows than on the other.

The existence of discrete flows and caustics was noticed in past investigations. In particular, caustics appear as simple fold catastrophes in the distribution of dark matter on very large scales in the context of the Zel’dovich approximation [10], and in the distribution of dark matter on galactic scales in the context of the self-similar model of Filmore and Goldreich [11], and of Bertschinger [12]. That these investigations involve some approximations may have led to the mistaken belief, sometimes expressed, that discrete flows and caustics are a consequence of the approximations made rather than of just the collisionless dark matter hypothesis, i.e. that the discrete flows and caustics disappear when one goes beyond the Zel’dovich approximation in the treatment of density perturbations, or when one abandons the assumptions of spherical symmetry and self-similarity in modeling galactic halos.

However, as the argument given above shows, the existence of discrete flows and caustics follows from just the hypothesis of cold collisionless dark matter. No other assumptions are needed.

It is possible, nonetheless, that discrete flows have little relevance in practice. The central reason for the practical irrelevance of discrete flows and caustics would have to be that the number of flows is very large, i.e. that the phase space sheet covers physical space so many times that, for all practical purposes, it forms a 6 dim. continuum. We will address this issue now by estimating the number of flows and considering all conceivable ways in which that number can become large.

III. HOW MANY FLOWS?

First, let us estimate the number of flows at an arbitrary location in a galactic halo as a function of galactocentric distance. This first estimate [2] is made under idealized conditions where we neglect the formation of structure on scales smaller than that of the galaxy as a whole, as well as the diffusion of the flows by gravitational scattering off inhomogeneities in the galaxy. After estimating the number of flows in these idealized conditions we will consider the effect of the complicating factors.

The number of flows at a given location inside a galactic halo is the number of ways dark matter particles can reach that location from the far past. If the gravitational potential is smooth on the scale ($\sim 100 \text{ kpc}$) of the galaxy, the number of ways particles can reach a given location from the far past is limited by, and of order, the number of oscillations through the galactic potential that particles at that location may have had since $t = 0$. Let us explain.

Very far from the galactic center, say at $r \sim 1 \text{ Mpc}$ in the case of our own Milky Way, there is only one flow because particles can reach such a location from the far past in only
one way, namely by falling there on the way to the galactic center. We are for the time being ignoring the presence of our galactic neighbor M31, but will consider later any role it may have. At somewhat smaller distances from the galactic center, there are three flows, corresponding to three ways such a position can be reached from the far past. Qualitatively the three ways are 1) by falling there while on the way to the galactic center for the first time, 2) by falling through the galaxy from the opposite side and then reaching the position under consideration on the way out, and 3) by falling through the galaxy from the opposite side, going all the way out on the same side as the position under consideration, and then reaching that position on the way back in.

Note that we can be absolutely certain that the number of flows changes from one to three at some point when approaching the galaxy because the number of solutions $\vec{q}_j$ of the equation $\vec{r} = \vec{x}(\vec{q}, t)$ increases when $r = |\vec{r}|$ decreases and the number of solutions can only change by two at a time. We are also absolutely certain that there is a caustic at the boundary between the region with one flow and the region closer to the galactic center with three flows. Indeed the map $\vec{q} \to \vec{x}$ is singular on that boundary and hence the density diverges there in the limit of zero velocity dispersion.

Further in there is a region with five flows. For every point in that region there are five ways in which particles can reach that given location from the far past. The first three are the same as 1) - 3) above. The additional two are (qualitatively): 4) by falling to the galactic center from the same side as the location under consideration, falling out on the opposite side, falling back in on the opposite side, and now going past the location while falling out of the galaxy for the second time, and 5) by falling in from the same side as the location under consideration, falling out on the opposite side, falling back in from the opposite side, falling out for the second time, turning around and now going past the location while falling onto the galaxy for the third time. Still closer to the galactic center is a region with seven flows, then a region with nine flows, and so on. At each boundary where the number of flows increases by two on the way in, there is a caustic. These caustics are simple fold catastrophes located on topological spheres surrounding the galaxy. We call those the “outer caustics” of the galaxy. In addition there are “inner caustics”, as discussed in refs. [3–5,13].

Let us consider the region where there are seven flows. The particles there oscillated through the galaxy up to three times. In this discussion, we are calling “one oscillation” the motion by which a particle starts with zero radial velocity at some galactocentric distance $r$, approaches the galactic center and then moves back out to a galactocentric distance of order $r$. The reason there are less than nine flows in this region is that to produce nine flows some particles would have to oscillate four times, and this would take longer than the age of the universe at that distance from the galactic center. Closer to the galactic center there are nine flows because the oscillation time there is less than approximately one fourth the age of the universe. Still closer to the galactic center there are eleven flows because the oscillation time there is less than one fifth the age of the universe. And so on. So we may estimate the number of flows $n(r)$ at galactocentric distance $r$ by the formula

$$n(r) \sim 2 \frac{t_0}{T(r)}$$

where $T(r)$ is the oscillation period through the galaxy with amplitude $r$. Eq. (3.1) is only valid in order of magnitude because it does not take account of the fact that the galactic
potential, and hence the oscillation period, is time dependent.

To improve on this, let us assume that the galactic potential is such that the rotation curve is flat with present rotation velocity \( v_{\text{rot}}(t_0) \) and that in the past the rotation velocity increased according to the power law \( v_{\text{rot}}(t) = v_{\text{rot}}(t_0)(\frac{t}{t_0})^p \). It is easy to show that in such a potential, the product \( r v_{\text{rot}} \) is an adiabatic invariant, and that the oscillation period of a particle decreases in time as \( T \propto t^{-2p} \). The improved version of Eq. (3.1) is then found to be

\[
n(r) \simeq \frac{2t_0}{(2p+1)T(r, t_0)} \tag{3.2}
\]

where \( T(r, t_0) \) is the present oscillation period with amplitude \( r \). At our distance, \( r_\odot \simeq 8.5 \) kpc, from the center of the Milky Way the oscillation period is approximately \( T(r_\odot, t_0) \simeq 1.3 \cdot 10^8 \) years. Since \( t_0 \simeq 1.4 \cdot 10^{10} \) years, the number of flows at our location in the Galaxy \( n_\odot \simeq 215/(2p + 1) \). The exponent \( p \) is related to the parameter \( \epsilon \) [11] of the self-similar infall model by: \( p = \frac{\epsilon}{9\epsilon - \frac{1}{3}} \). A favored range for \( \epsilon \) is 0.2 to 0.35 [3], which yields \( n_\odot \) in the range 84 to 134. In conclusion, we expect the number of flows at the Sun’s position in the Milky Way halo to be of order 100. This estimate should be valid within a factor 2 at least.

We estimate of order 100 flows on Earth in the idealized situation where the gravitational potential is smooth on galaxy scales and where the infalling dark matter has no small scale structure of its own. We will consider below the effect of small scale structure in the gravitational potential and in the infalling dark matter, both of which tend to increase the number of flows.

Before we get there, however, let us ask whether any effect can reduce the number of flows below the above estimate. In particular, it may seem that angular momentum keeps some of the infalling particles from reaching us since angular momentum introduces for each particle a distance of closest approach to the galactic center. However, angular momentum does not in fact reduce the number of flows. To see this, note that angular momentum is a vector field tangent to the turnaround sphere of particles which are about to fall in. A vector field on a topological sphere must have at least two (simple) zeros. That means that every infalling shell includes particles which will pass through the galactic center. In fact, one can show - see for example the discussion in ref. [4] - that the continual infall of particles from all directions in and out of a gravitational potential produces at least two flows at every point which is inside both the initial and final turnaround spheres. The only way the particles would fail to reach such a point is for it to be near the top of a potential barrier. However, because gravity is attractive, the gravitational potential does not have any maxima.

**IV. SMALL SCALE STRUCTURE IN THE DARK MATTER FLUID**

In the above description we have assumed that the dark matter falling onto a galaxy is without structure of its own. In CDM cosmology, however, the spectrum of primordial density perturbations has power on all scales. So the dark matter that is falling onto a galaxy has in general clustered previously on smaller scales. What effect does that have? We show in this section that, provided the infalling clumps are not too large, the description of a dark matter halo in terms of a phase space sheet, discrete flows and caustics is still valid.
[4,5]. It is of course understood here that the flows exist now in an average sense (i.e. after averaging over the clumps).

Where a clump has formed the sheet is wound up in phase space on the scale of the clump. The sheet has acquired at that location a number of sublayers which give it a certain thickness. It describes therefore a flow with an effective velocity dispersion $\delta v_{\text{eff}}$. $\delta v_{\text{eff}}$ is equal to the velocity dispersion of the particles in the clump. From the point of view of an observer located at some point in physical space, the clumpiness of a flow passing by means that instead of a single flow with a unique velocity there is a set of subflows, odd in number, all with the same velocity up to the dispersion $\delta v_{\text{eff}}$. $\delta v_{\text{eff}}$ generally varies with location on the phase space sheet since the size of the clumps that make up the sheet generally varies with location. Also the number of sheet sublayers varies with location for the same reason.

The question is: how large can the effective velocity dispersion become before the phase space structure described earlier loses its meaning? The answer is that the phase space sheet should not become so thick that successive layers of the sheet touch one another and thus lose their identity. Where the different layers of the sheet touch, the flows overlap in velocity space and become confused with one another. So let us ask: how large can the effective velocity dispersion be before the approximately 100 flows at the Earth’s location in the Galaxy become confused with one another. That question is simple to answer because the velocities are all in a 3-dim box in velocity space whose size is determined by the escape velocity from the Galaxy at our position. That escape velocity is of order 600 km/s. So the box is, roughly speaking, a cube of size 1200 km/s. There is no reason for the 100 velocities to cluster in a particular region of this cube. If $\delta v_{\text{eff}}$ is equal or less than say 30 km/s, then clearly many of the flows on Earth will be distinct from one another. Now, it is very unlikely that the clumps of infalling dark matter have velocity dispersion as large as 30 km/s. Indeed 30 km/s is of order the velocity dispersion of the large magellanic cloud (LMC). If the infalling dark matter is in clumps as large as the LMC, one would expect the clumps to be luminous, as is the LMC.

In summary, provided the effective velocity dispersion of the dark matter falling onto the galaxy is small enough (30 km/s or less in the case of the Milky Way), the phase space structure of galactic halos is still characterized by discrete flows and caustics. However each flow may be divided into an odd number of subflows with velocity spread $\delta v_{\text{eff}}$. Likewise each caustic may be spread into an odd number of subcaustics, which are identical in shape but displaced relative to one another by distances proportional to $\delta v_{\text{eff}}$.

V. GRAVITATIONAL SCATTERING BY GALACTIC INHOMOGENEITIES

When a dark matter flow passes by a clump of matter, whether dark matter or baryonic matter, the particles in the flow are scattered by the gravitational potential of the clump. Examples of clumps of baryonic matter include stars, globular clusters and molecular clouds. For every flow upstream of the clump, there are three flows downstream [14]. Let us call them daughters of the original flow. Downstream of two clumps there will be nine granddaughter flows, and after $p$ clumps, the number of descendant flows will be $3^p$. Since the number of descendants is an exponentially growing function of the number of clumps, the total
number of flows is huge. This would appear to destroy any notions of phase space sheet, discrete flows and caustics. But it doesn’t. The reason is that not all descendants are equal. Although there are three daughter flows downstream of a clump for every flow upstream, there is very little density in the daughter flows except where their velocity vectors have the same direction as the original flow.

Consider a flow of particles passing through a region populated by a class of objects of mass $M$ and density $n$. Gravitational scattering by the objects causes the velocity of each particle in the flow to have a random walk in velocity space. This results in a diffusion of the flow over a cone of angle $\Delta \theta$. One readily finds that

$$\langle \Delta \theta \rangle^2 = \int dt \int_{b_{\min}}^{b_{\max}} \frac{4G^2M^2}{b^2v^4}nvt2\pi b \, db$$

$$= 1.8 \cdot 10^{-7} \left( \frac{10^{-3}c}{v} \right)^3 \left( \frac{M}{M_{\odot}} \right)^2 \ln \left( \frac{b_{\max}}{b_{\min}} \right) \left( \frac{t}{10^{10}\text{year}} \right) \left( \frac{n}{\text{pc}^{-3}} \right),$$

where $v$ is the velocity of the flow and $t$ is the time over which it encountered the objects in question. In the galactic disk, giant molecular clouds are most likely the main contributors. With $M \sim 10^6 M_{\odot}$, $n \sim 3/\text{kpc}^3$, $b_{\max} \sim \text{kpc}$ and $b_{\min} \sim 20 \text{pc}$, they yield $\Delta \theta \sim 0.05$ for dark matter particles that have spent most of their past in the galactic disk. The contributions from globular clusters ($M \sim 5 \cdot 10^5 M_{\odot}$, $n \sim 0.3/\text{kpc}^3$) and stars ($M \sim M_{\odot}$, $n \sim 0.1/\text{pc}^3$) are less important. Therefore, flows of dark matter particles that have spent most of their past in the central parts of the Galaxy may well be washed out by scattering.

However, as was emphasized above, there are in the halo flows of particles which are falling in and out of the galaxy for the first time as well as flows of particles which have fallen in and out of the galaxy only a small number of times in the past. Since the particles involved have spent very little of their past in the inner parts of the galaxy, such flows are not washed out by scattering off molecular clouds or other inhomogeneities in the luminous matter. To be specific, consider particles which have fallen through the inner parts of the Galaxy ten times or less in the past. Such particles spent at most $5 \cdot 10^8$ years in the disk, and hence $\Delta \theta < 10^{-2}$ for the corresponding 20 flows. Hence there are at least 20 dark matter flows on Earth which have not been thermalized by inhomogeneities in the distribution of luminous matter.

We must still consider the effect of scattering by inhomogeneities in the dark matter. For a flow of particles falling in and out of the galaxy the dominant contribution to the integral $\int n \, dt$ in Eq. (5.1) is from the inner regions of the halo because $n$ decreases with $r$, at large $r$, as $r^{-2}$ or faster. Let us assume that a fraction $f$ of the dark matter is in clumps of mass $M$. Then, for a flow of velocity $v = 400 \text{ km/s}$, passing through the inner parts (say $r < 20 \text{ kpc}$) of the Galaxy once, the effect of scattering by dark matter clumps is

$$\langle \Delta \theta \rangle^2 \sim 10^{-11} f \left( \frac{M}{M_{\odot}} \right),$$

where we used $M n = f \frac{1}{3} 10^{-24} \text{ gr/cm}^3$ and $\ln \left( \frac{b_{\max}}{b_{\min}} \right) = 4$. Unless a large fraction of the dark matter is in clumps of mass $10^{10} M_{\odot}$ or larger, the flows of particles which have fallen in and out of the galaxy only a small number of times in the past are not thermalized. We do not know of any evidence that the dark matter is in clumps of $10^{10} M_{\odot}$ or larger. To the contrary, the next subsection presents evidence against this possibility.
VI. RIPPLES AROUND GIANT ELLIPTICAL GALAXIES

Malin and Carter [7] observed that many bright elliptical galaxies are surrounded by ripples in the distribution of light. These ripples have been interpreted [8,12,15] as caustics of luminous matter from a small galaxy that was “eaten” by the giant elliptical. Computer simulations of the infall of the small galaxy in the fixed gravitational potential of the giant elliptical show that the small galaxy gets tidally disrupted, and that its stars end up on a thin ribbon in phase space. This ribbon is similar to the phase space sheet described earlier except that it is more limited in spatial extent. The ripples in the distribution of light around giant ellipticals are the outer caustics caused by the folding of the ribbon of stars in phase space. As far as we know, there is no other successful explanation of the ripples.

The appearance of the ripples provides an existence proof of discrete flows and caustics. Although made of ordinary matter, stars are collisionless when they fall in and out of a galactic gravitational potential well. That caustics occur in the distribution of stars answers the two concerns discussed above about the existence of discrete flows and caustics of dark matter. Indeed we may conclude from the ripples’ existence that 1) the velocity dispersion of a small galaxy is insufficient to erase caustics, and 2) flows do not get diffused by scattering of inhomogeneities. In particular, the dark matter is not in clumps of mass $10^{10} M_{\odot}$ or larger because otherwise those clumps would diffuse the flows of stars that cause ripples around giant elliptical galaxies.

The ripples also answer other concerns that one may have with regard to discrete flows and caustics. One concern sometimes expressed is that the flows are unstable. We do not know of a reason why the flows would be unstable. But at any rate, the occurrence of the ripples proves that the flows are not unstable.

VII. FURTHER REMARKS

Our large neighbor galaxy M31 is presently at a distance of approximately 730 kpc from us and has a line of sight velocity component in the Milky Way rest frame of order 120 km/s in our direction [15]. What role does M31 play? In the future, 5 Gyr from now say, the halos of M31 and the Milky Way will be passing through each other. The gravitational potential will be more sharply time dependent then than it has been in the past. There will however continue to be discrete flows and caustics. Indeed, as was already emphasized, the existence of discrete flows and caustics follows from just the assumption of cold dark matter. The distribution of caustics in both galaxies will be very much altered by the encounter, but the caustics will not disappear. Once M31 and the Milky Way have joined into one big entity, their joint gravitational potential well will cause a new inflow of surrounding dark matter, and a fresh pattern of discrete flows and caustics will be established. With regard to the flows that exist in the Solar neighborhood today, M31’s influence is small because the particles that form those flows fell onto the Galaxy a long time ago, at $t < t_0/2$, when M31 was much further from us than it is now. M31 does of course have an influence on the first one or three Milky Way flows in its neighborhood, but this is a relatively small fraction of the total phase space structure of the Milky Way halo.
A large part of the accepted lore on the structure of dark matter halos is based on the results of N-body simulations. Are discrete flows and caustics seen in the simulations? Actually, the existence and relevance of discrete flows and caustics are disputed on the basis of N-body simulations in refs. [16,17]. Still, for the reasons stated in the previous sections, discrete flows and caustics must be present in the simulations when the latter have adequate resolution. Ref. [17] argues that caustics have negligible relevance. Yet, the velocity distribution of dark matter particles in the simulation described in that paper shows peaks (Figure 6 of [17]). Such peaks mean that there are discrete flows in the simulated halo. And, if there are discrete flows, caustics are present as well.

It should be emphasized that the resolution of present simulations is far inadequate to describe the phase space structure of the Milky Way halo down to our position in it. Indeed, we found above that the minimum number of flows at our location is of order 100. Since phase space is 6 dimensional, the minimum number of particles required to describe the phase space structure of the Milky Way halo, down to our position, is $100^6 = 10^{12}$. Present simulations have only of order $10^7$ to $10^8$ particles per galactic halo. Also, because the particles in the simulations are few they have to be proportionately heavy. Each has a mass of order $10^6 M_\odot$. As a result, the simulations are afflicted by two-body relaxation [18]. The simulated particles make hard two-body collisions with one another, whereas two-body collisions between axions or WIMPs are negligible.

Finally we note that discrete flows are evident in the N-body simulations of Stiff and Widrow [19] who use a special technique to increase the resolution in the relevant regions of phase space. Caustics are seen in the simulations of refs. [20] and, more recently, ref. [21].

**VIII. IMPLICATIONS FOR OBSERVATION AND EXPERIMENT**

In this section, we briefly list the implications of discrete flows and caustics for observation and experiment and refer to the growing literature on this topic.

According to the arguments of the previous sections, the velocity spectrum of dark matter particles on Earth has at least twenty peaks due to flows of particles that have fallen in and out of the Galaxy ten times or less in the past. The discrete flows have distinct signatures in direct dark matter detectors on Earth. Each flow produces a peak in the spectrum of microwave photons from axion to photon conversion in cavity detectors of dark matter axions [22], and a plateau in the recoil energy spectrum of nuclei struck by WIMPs in WIMP detectors [23]. As a result of the orbital motion of the Earth around the Sun, each of these spectral features has a distinct annual modulation that depends on the velocity vector of the flow. If any of the direct dark matter detectors [24–27] finds a signal, it can in principle provide a wealth of information about the structure of the Milky Way halo by observing the spectral features caused by discrete flows.

If the Sun is close to a caustic, the flows that form that caustic have very large densities on Earth and hence produce prominent signals in the direct dark matter detectors. Although they have only received bare mention in this paper, there are inner caustics [3–5,13] in galactic halos in addition to the outer caustics which we did discuss at some length. Because the inner caustics are located relatively close to the galactic center, it is not unlikely that the Sun is near an inner caustic of the Milky Way halo.
Although made of particles which move with typical halo velocities \( (v \sim 10^{-3}c) \), the caustics themselves move much more slowly. The positions of caustics in galactic halos change only on cosmological time scales. As a result, the caustics can accrete baryonic matter. The positions of the caustics in the halo may be revealed in this way [28]. Caustics can also produce bumps in galactic rotation curves. There is evidence that the galactocentric radii at which bumps occur in the rotation curves of external galaxies [29] and in the rotation curve of the Milky Way [4,28] are distributed in the same way as the radii of inner caustic rings in the self-similar halo model [3].

WIMP annihilation is enhanced by the presence of caustics [30] because the annihilation rate per unit time and unit volume is proportional to the WIMP density squared. If photons from WIMP annihilation are observed the positions of caustics may be revealed as sharp lines and hot spots on the sky.

Finally, dark matter caustics produce gravitational lensing of distant sources [31]. Because caustics have distinct density profiles, they have distinct gravitational lensing signatures as well.

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