MODULAR SYMBOLS FOR \( \mathbb{Q} \)-RANK ONE GROUPS AND
VORONOÏ REDUCTION

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Abstract. Let \( G \) be a reductive algebraic group of \( \mathbb{Q} \)-rank one associated to a
self-adjoint homogeneous cone defined over \( \mathbb{Q} \), and let \( \Gamma \subset G \) be a torsion-free
arithmetic subgroup. Let \( d \) be the cohomological dimension of \( \Gamma \). We present
an algorithm to compute the action of the Hecke operators on \( H^d(\Gamma; \mathbb{Z}) \). This
generalizes the classical modular symbol algorithm, when \( \Gamma \subset SL_2(\mathbb{Z}) \), to a setting
including Bianchi groups and Hilbert modular groups. In addition, we generalize
some results of Voronoï for real positive-definite quadratic forms to self-adjoint
homogeneous cones of arbitrary \( \mathbb{Q} \)-rank.

1. Introduction

1.1. Let \( \mathcal{H}_2 = SL_2(\mathbb{R})/SO(2) \) be the upper-half plane, and let \( \Gamma(N) \subset SL_2(\mathbb{Z}) \) be
the principal congruence subgroup of level \( N > 2 \), acting on \( \mathcal{H}_2 \) from the left by
linear fractional transformations. Then the cohomology group \( H^1(\Gamma(N) \setminus \mathcal{H}_2; \mathbb{C}) \) is
closely related to the space of all weight-two modular forms of level \( N \). The modular
symbols provide a concrete approach to the group \( H^1(\Gamma(N) \setminus \mathcal{H}_2; \mathbb{C}) \) (§2.1) that has
allowed the testing of many conjectures in number theory and has led to explicit
formulas for \( L \)-functions and their derivatives [9][11][18]. Important to applications
is the modular symbol algorithm developed by Manin [18]. An algebra of Hecke
operators acts on \( H^1(\Gamma(N) \setminus \mathcal{H}_2; \mathbb{C}) \), and using the algorithm one may compute their
eigenvalues. Essentially this algorithm is the euclidean algorithm applied to pairs of
integers (§2.2).

Now consider the case where \( \Gamma \) is a torsion-free Bianchi subgroup, that is, \( \Gamma \) is of
finite index in \( SL_2(\mathfrak{o}_K) \), where \( \mathfrak{o}_K \) is the ring of integers in an imaginary quadratic
extension \( K/\mathbb{Q} \). The group \( \Gamma \) acts on hyperbolic three-space \( \mathcal{H}_3 = SL_2(\mathbb{C})/SU(2) \),
and we consider the cohomology group \( H^2(\Gamma \setminus \mathcal{H}_3; \mathbb{C}) \). As before, an algebra of Hecke
operators acts on the cohomology, and one is interested in this Hecke-module for
many reasons. For example, results of Grunewald and Schwermer [13] imply that,
for all but a finite set of \( K \), the rational cohomology of \( SL_2(\mathfrak{o}_K) \) contains cus-
pidal cohomology, which is important in the theory of automorphic forms. Also,
the “Langlands philosophy” predicts a “Shimura-Taniyama-Weil” correspondence between Hecke eigenclasses and certain algebraic varieties defined over $\mathbb{K}$. More precisely, let $\Gamma \subset SL_2(\mathcal{O}_K)$ be a congruence subgroup and let $\xi \in H^2(\Gamma \backslash \mathfrak{H}_3; \mathbb{C})$ be a cuspidal Hecke eigenclass. Then one hopes to associate to $\xi$ an algebraic variety $V_{\mathbb{K}}$—specifically an elliptic curve or an abelian variety of dimension two—so that the zeta function of $V$ is assembled from the eigenvalues of $\xi$ in a precise way. Results of Cremona [4] and Cremona and Whitley [5] when $\mathbb{K}$ has class number one support this. Hence one wishes to compute Hecke eigenvalues for general $\mathbb{K}$. But in general the ring $\mathcal{O}_K$ is not a euclidean domain, and one cannot directly apply the modular symbol algorithm as described in [18] (however, see §1.2).

In this paper we present an analog of the modular symbol algorithm for $H^d(\Gamma; \mathbb{Z})$, where $\Gamma$ is a torsion-free arithmetic group associated to certain self-adjoint homogeneous cones, and $d$ is the cohomological dimension of $\Gamma$. This includes finite index subgroups of $SL_2(R)$, where $R$ is

- $\mathbb{Z}$,
- the ring of integers in a $CM$ field, or
- the ring of integers in a totally real field.

We replace the continued fractions of [18] with a study of the geometry of self-adjoint homogeneous cones. Thus our algorithm does not require that $R$ be a euclidean domain.

Here is the organization of this paper. Section 2 contains a review of the classical modular symbol algorithm from [18] and presents our algorithm in that case. Section 3 contains a review of the reduction theory of self-adjoint homogeneous cones, and in Section 4 we generalize some results of [22] to this setting (Theorems 2 and 3). These results are valid for cones of any $\mathbb{Q}$-rank $\geq 1$. Finally, Section 5 contains a description of our algorithm, in Theorem 4. Throughout the paper we comment on implementation issues related to the algorithm.

1.2. Related work. Let $\Gamma \subset SL_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in an imaginary quadratic extension $K = \mathbb{Q}(\sqrt{-m})$. For $K$ a non-euclidean ring with class number one ($m = 19, 43, 67, 163$), Whitley developed a “pseudo-euclidean algorithm” that allowed implementation of the modular symbol algorithm [23].

Also, I learned upon completion of this work that Jeremy Bygott has independently studied the modular symbol algorithm for the non-PID imaginary quadratic case in his forthcoming Ph.D. thesis [6].

1.3. Acknowledgments. The results in this paper depend heavily on results from the work of Avner Ash. I thank him for graciously and patiently explaining his work to me. I also thank Mark McConnell for a careful reading of an early version of this paper and many helpful comments. I also thank the referee for many helpful suggestions.
Finally, the results in this paper are an extension of some of the results in my Ph.D. thesis [14]. I thank heartily my advisor, Robert MacPherson, for the encouragement and inspiration he has given me.

2. A motivating example

In this section we illustrate our algorithm for \( \Gamma \subset SL_2(\mathbb{Z}) \). As in the introduction let \( S_2 = SL_2(\mathbb{R})/SO_2 \), and let \( S_2^* = S_2 \cup \mathbb{Q} \cup \{ \infty \} \) be the usual partial compactification of \( S_2 \) by adding cusps. We assume that \( S_2^* \) is given the Satake topology, and we extend the action of \( SL_2(\mathbb{R}) \) to the cusps. We denote the quotient \( \Gamma \backslash S_2^* \) by \( X_\Gamma \). We assume that \( \Gamma \) is torsion-free, so that \( X_\Gamma \) is smooth.

2.1. We begin by paraphrasing aspects of Manin’s work [18]. By Poincaré duality, \( H^1(X_\Gamma; \mathbb{C}) \) may be identified with \( H^1(X_\Gamma; \mathbb{C}) \), and so we may study the space of weight-two modular forms for \( \Gamma \) by studying the latter. Let \( q_1 \) and \( q_2 \) be cusps equivalent modulo \( \Gamma \). Then any smooth path \( \gamma \) from \( q_1 \) to \( q_2 \) descends to a closed path on \( X_\Gamma \) representing a class in \( H^1(X_\Gamma; \mathbb{Z}) \). Furthermore, this class is independent of \( \gamma \), and in fact depends only on the ordered pair \((q_1, q_2)\).

More generally, suppose \( q_1 \) is not necessarily equivalent to \( q_2 \) modulo \( \Gamma \). Then integration of one-forms \( \omega \in H^1(X_\Gamma; \mathbb{R}) \) along \( \gamma \) yields a functional \( \int: H^1(X_\Gamma; \mathbb{R}) \to \mathbb{R} \), and this allows us to associate to the pair \((q_1, q_2)\) a class in \( H^1(X_\Gamma; \mathbb{R}) \). By the theorem of Manin-Drinfeld ([18], p. 61), this class actually lies in \( H^1(X_\Gamma; \mathbb{Q}) \). We define a modular symbol to be the rational homology class constructed from an ordered pair of cusps in this way, and denote this class by \([q_1, q_2]\). This class agrees with the class in the previous paragraph when \( q_1 \) and \( q_2 \) are equivalent modulo \( \Gamma \).

Proposition 1. [18] The modular symbols satisfy the following:

1. \([q_1, q_2] = -[q_2, q_1]\).
2. \([q_1, q_2] = [q_1, q_3] + [q_3, q_2]\).

Furthermore, \( H^1(X_\Gamma; \mathbb{Z}) \) is spanned by modular symbols modulo \( \Gamma \).

2.2. The Hecke operators act on \( H^1(X_\Gamma; \mathbb{C}) \), and hence by duality on \( H^1(X_\Gamma; \mathbb{C}) \). On a modular symbol, an operator acts by

\[
[q_1, q_2] \mapsto \sum_{\alpha \in A} [\alpha q_1, \alpha q_2],
\]

where \( A \) is a finite set of \( 2 \times 2 \) integral matrices that depends on the operator. For example, let \( \Gamma = \Gamma(N) \), and let \( p \) be a prime not dividing \( N \). For the classical operator \( T_p \) we may take

\[
\alpha \in \left\{ \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix}, \ldots, \begin{pmatrix} 1 & p - 1 \\ 0 & p \end{pmatrix} \right\}.
\]
Here $\sigma_p \in SL_2(\mathbb{Z})$ is a fixed matrix satisfying

$$\sigma_p \equiv \left( \begin{array}{cc} p^{-1} & 0 \\ 0 & p \end{array} \right) \mod N$$

([20], Prop. 3.36). Note that $\det \alpha \neq \pm 1$.

A finite basis of $H_1(X_\Gamma; \mathbb{C})$ is provided by the set of unimodular symbols. Write a cusp $q$ in lowest terms as $m/n$, where the cusp $\infty$ is written formally as $1/0$. Then the unimodular symbols are the symbols $[q_1, q_2]$ satisfying

$$\det \left( \begin{array}{cc} m_1 & m_2 \\ n_1 & n_2 \end{array} \right) = \pm 1.$$

The Hecke operators do not preserve unimodularity, and it is necessary for eigenvalue computations to construct an explicit homology between a non-unimodular symbol and a cycle of unimodular symbols. This is done by the modular symbol algorithm. Assume that a non-unimodular symbol has the form $[0, q]$, where $q$ is a positive rational number. Let $[a_1, \ldots, a_k]$ be the simple continued fraction expansion of $q$, i.e.

$$q = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}.$$  

Let $q_i$ be the $i^{th}$ convergent $[a_1, \ldots, a_i]$. Then by applying (2) from Proposition [], we have

$$[0, q] = [0, \infty] + [\infty, q_1] + \cdots + [q_{k-1}, q].$$  

Furthermore, the basic properties of simple continued fractions imply that the modular symbols on the right are unimodular. Figure [] illustrates the result for the modular symbol $[0, 12/5]$.

![Figure 1](image)

**Figure 1.** $12/5 = [2, 2, 2]$ implies $[0, 12/5] = [0, \infty] + [\infty, 2] + [2, 5/2] + [5/2, 12/5]$.

To complete the discussion, we note that $SL_2(\mathbb{Z})$ acts transitively on the cusps. Since the above algorithm is $SL_2(\mathbb{Z})$-equivariant, any modular symbol can be written as a sum of unimodular symbols.
2.3. Now we present our technique for writing a modular symbol as an equivalent sum of unimodular symbols. No use will be made of continued fractions; instead, we look at the relationship between a geodesic representing a modular symbol and a certain tessellation of \( \mathfrak{H}_2 \). In this simple case our algorithm will appear needlessly complicated, but it is formulated in a way that will generalize to other settings. It is also quite practical for machine computations.

To begin, we tile \( \mathfrak{H}_2 \) with the \( SL_2(\mathbb{Z}) \)-translates of the ideal geodesic triangle with vertices 0, 1, and \( \infty \) (see Figure 2). This tessellation descends to a finite triangulation of \( X_\Gamma \). The edges of this tessellation are geodesics inducing the unimodular symbols, and every unimodular symbol arises in this way.

Given any point \( x \in \mathfrak{H}_2 \), let \( R(x) \) be the set of vertices of the triangle (or edge) of the tessellation meeting \( x \).

Let \([0, q]\) be a modular symbol as before, and let \( \gamma \) be the ideal geodesic in \( \mathfrak{H}_2 \) from 0 to \( q \). Because \( \gamma \) is a geodesic between two rational cusps, one can show that \( \gamma \) will only meet a finite number of triangles in the tessellation. Hence we may choose a finite subset \( x_1, \ldots, x_r \in \gamma \) (as in Figure 3) so that

1. \( 0 \in R(x_1) \),
2. \( q \in R(x_r) \), and
3. \( R(x_i) \cap R(x_{i+1}) \neq \emptyset \).

We call such a collection a **sufficiently fine partition** of \( \gamma \). From each \( R(x_i) \cap R(x_{i+1}) \) choose a cusp \( q_i \). Then we claim that we have a homology

\[
[0, q] = [0, q_1] + [q_1, q_2] + \cdots + [q_r, q],
\]

and that each term on the right is a unimodular symbol.

First, we may see we have a homology either by repeatedly applying (2) of Proposition \( \mathbb{l} \), or by continuously deforming \( \gamma \) into geodesics inducing the classes on the right-hand side (see Figure 4).

Finally, each nontrivial term on the right of (1) corresponds to an edge in the tessellation because \( q_i \) and \( q_{i+1} \) are both vertices of a triangle containing \( x_{i+1} \). In
fact, as shown in [14], if $\gamma$ is oriented then one may choose the $q_i$ canonically, and our algorithm in this case is equivalent to the modular symbol algorithm.

3. Self-adjoint homogeneous cones

In this section we present the geometric context of our algorithm. This specifies the arithmetic groups to which our algorithm applies and describes the constructions replacing $\mathfrak{H}_2$ and its tessellation from §2.

The results in §3.1–3.4 and §4.1 are due to A. Ash and originally appeared in [1] and [3]. Our exposition closely follows the former.

3.1. Let $V$ be a real vector space defined over $\mathbb{Q}$, and let $C \subset V$ be an open cone. That is, $C$ contains no straight line, and $C$ is closed under homotheties: if $x \in C$ and $\lambda \in \mathbb{R}^>$, then $\lambda x \in C$. The cone $C$ is called self-adjoint if there exists a scalar product $\langle \cdot, \cdot \rangle$ on $C$ such that

$$C = \{ x \in V \mid \langle y, x \rangle > 0 \text{ for all } y \in \overline{C} - \{0\} \}.$$ 

Let $G$ denote the connected component of the identity of the linear automorphism group of $C$, i.e. $G = \{ g \in GL(V) \mid gC = C \}^0$. The cone $C$ is called homogeneous
if $G$ acts transitively on $C$. If $K$ denotes the isotropy group of a fixed point in $C$, then we may identify $C$ with $G/K$. The fact that $C$ is self-adjoint implies that $G$ is reductive and $C$ modulo homotheties is a Riemannian symmetric space.

We also assume that all these notions are compatible with the $\mathbb{Q}$-structure on $V$. That is, as a subgroup of $GL(V)$, $G$ is defined by rational equations, and the scalar product $\langle \cdot, \cdot \rangle$ is defined over $\mathbb{Q}$. This is stronger than saying that $G$ is defined over $\mathbb{Q}$.

In particular, the group of real points $G(\mathbb{R})$ must be isomorphic to a product of the following groups ([10], p. 97):

1. $GL_n(\mathbb{R})$
2. $GL_n(\mathbb{C})$
3. $GL_n(\mathbb{H})$
4. $O(1, n - 1) \times \mathbb{R}^\times$
5. The noncompact Lie group with Lie algebra $\mathfrak{e}_6(-26) \oplus \mathbb{R}$

In each case $V$ is a set of hermitian symmetric matrices. In other words, $V$ is the set of $n \times n$ matrices over an appropriate $\mathbb{R}$-algebra with involution $\tau$, in which $A \in V$ if and only if $A^t = A^\tau$. The cone $C$ is then the subset of “positive-definite” matrices in an appropriate sense. For details we refer to ([10], Ch. V).

3.2. Let $H$ be a hyperplane in $V$. We say that $H$ is a supporting hyperplane of $C$ if $H$ is rational and $H \cap C = \emptyset$ but $H \cap \bar{C} \neq \emptyset$.

Given a supporting hyperplane $H$ of $C$, let $C' = \text{Int}(H \cap C)$. (Here $\text{Int}(A)$ is the interior of $A$ in its linear span.) Then $C'$ is called a rational boundary component, and is a self-adjoint homogeneous cone of smaller dimension than $C$.

**Definition 1.** The cusps of $C$ are the one-dimensional rational boundary components of $C$. The set of cusps is denoted $\Xi(C)$.

3.3. Let $L \subset V(\mathbb{Q})$ be a lattice, i.e. a discrete subgroup of $V(\mathbb{Q})$ such that $L \otimes \mathbb{Q} = V(\mathbb{Q})$. Let $\Gamma_L$ denote the subgroup of $G(\mathbb{Q})$ preserving $L$. An arithmetic subgroup of $G$ is a discrete subgroup commensurable with $\Gamma_L$ for some $L$. Any torsion-free subgroup $\Gamma \subset \Gamma_L$ of finite index will act properly discontinuously and freely on $C$. Thus the quotient $\Gamma \backslash C$ is an Eilenberg-Mac Lane space for $\Gamma$, and the group cohomology $H^*(\Gamma)$ is $H^*(\Gamma \backslash C)$. In fact, since homotheties commute with the action of $\Gamma$, we may pass to $X := \mathbb{R}^{>0} \backslash C$, and compute $H^*(\Gamma \backslash X)$ instead.

3.4. Let $A \subset V(\mathbb{Q})$ be a finite set of nonzero points. The closed convex hull $\sigma$ of the rays $\{\mathbb{R}^{\geq 0} x \mid x \in A\}$ is called a rational polyhedral cone. The rays through the vertices of the convex hull of $A$ are called the spanning rays of $\sigma$. We denote the set of spanning rays by $R(\sigma)$. The group $G(\mathbb{Q})$ acts naturally on the set of rational polyhedral cones, and we denote the action by a dot: $\sigma \mapsto g \cdot \sigma$.

Now we would like to partition $C$ into convex subsets using a collection of rational polyhedral cones, in a manner compatible with the $\Gamma_L$-action. This requires some care, as $C$ is open and any $\sigma$ as above is closed.
Definition 2. ([3], p. 117) Let $\Gamma \subset \Gamma_L$ be an arithmetic subgroup of $G$. A set of closed polyhedral cones $\{\sigma_\alpha\}$ is called a $\Gamma$-admissible decomposition of $C$ when the following hold:

1. Each $\sigma_\alpha$ is the span of a finite number of rational rays.
2. For each $\alpha$, the cone $\sigma_\alpha \subset \bar{C}$.
3. Every face of a $\sigma_\alpha$ is some $\sigma_\beta$ in the decomposition.
4. $\sigma_\alpha \cap \sigma_\beta$ is a common face of $\sigma_\alpha$ and $\sigma_\beta$.
5. For any $\sigma_\alpha$ and any $\gamma \in \Gamma$, $\gamma \sigma_\alpha$ is some $\sigma_\beta$ in the decomposition.
6. Modulo $\Gamma$, there are only a finite number of $\sigma_\alpha$'s.
7. $C = \bigcup_\alpha (\sigma_\alpha \cap C)$.

Note that a $\Gamma$-admissible decomposition descends to a decomposition of $X$ into open cells.

We now describe a technique to construct $\Gamma$-admissible decompositions. The technique originates with Voronoï, and was generalized by Ash to all self-adjoint homogeneous cones. Let $L'$ be $L - \{0\}$.

Definition 3. The Voronoï polyhedron $\Pi$ is the closed convex hull of $L' \cap \Xi(C)$.

Theorem 1. ([3], p. 143) The cones over faces of $\Pi$ form a $\Gamma$-admissible decomposition of $C$.

We call the cones in this $\Gamma$-admissible decomposition of $C$ the Voronoï cones, the decomposition of $X$ associated to $\Pi$ the Voronoï decomposition, and the open cells in $X$ the Voronoï cells.\footnote{Our terminology is nonstandard. In [3], the polyhedron $\Pi$ is referred to as a “kernel comparable to a $\Gamma$-polyhedral cocore” and is a specific example of a more general theory.} Here are two examples of this construction.

Example 1. The original example investigated by Voronoï [22] is the following. Let $V$ be the vector space of symmetric $n \times n$ matrices, and let $C \subset V$ be the cone of positive-definite matrices. Then $G = GL_n(\mathbb{R})^0$, which acts on $V$ by $A \mapsto gAg^t$, and $K$ is $SO_n$. The scalar product is given by $\langle A, B \rangle = \text{Tr}(AB)$.

Let $L$ be the lattice of integral symmetric matrices. Then $\Gamma_L = SL_n(\mathbb{Z})$. The set of cusps $\Xi(C)$ is obtained as follows. Any nonzero integral (column) $n$-vector $v$ determines a rank one quadratic form by $v \mapsto vv^t$. The cusps are the rays generated by all points in $\bar{C}$ of this form. Suppose that $z \in L' \cap \Xi(C)$ is a cusp arising in this manner from the $n$-vector $v$. Then if $y \in C$, the scalar product $\langle z, y \rangle$ is equal to the quadratic form $y$ evaluated on $v$.

For $n = 2$ we have that $X = \mathbb{H}_2$, and the Voronoï decomposition is the tessellation described in §2.3. For $n \geq 4$ there is more than one type of top-dimensional Voronoï cone modulo $\Gamma_L$, and not all the top-dimensional cones are simplicial. Complete information about these decompositions for $n = 3$ and 4 can be found in [17] and [19].
Example 2. Let $K/\mathbb{Q}$ be an imaginary quadratic extension, and let $\mathcal{O}_K$ be the ring of integers of $K$. Let $V$ be the vector space of $2 \times 2$ hermitian symmetric matrices over $\mathbb{C}$, and let $C \subset V$ be the cone of positive-definite matrices. Then $V$ is a four-dimensional vector space over $\mathbb{R}$, and $X$ is three-dimensional hyperbolic space $H^3$. For $L$ we may take the matrices in $V$ with entries in $\mathcal{O}_K$, and then $\Gamma_L = \text{SL}_2(\mathcal{O}_K)$. In the classical picture of $H^3 \subset \mathbb{C} \times \mathbb{R} \geq 0$, the rays generated by the vertices of $\Pi$ become the points $K \cup \{\infty\}$, where $K$ is pictured as a subset of $\mathbb{C} \times \{0\}$ and $\infty$ is pictured infinitely far above $\mathbb{C} \times \{0\}$ along $\mathbb{R} \geq 0$.

The Voronoï decomposition becomes a tessellation of $H^3$ into ideal three-polytopes. In general these polytopes will not be simplices. For example, if $K = \mathbb{Q}(\sqrt{-1})$, then the unique polytope modulo $\Gamma_L$ is an octahedron $[7][12]$. If $\mathcal{O}_K$ is euclidean there is only one type of top-dimensional polytope in the tessellation, but for general $K$ there will be more than one type.

3.5. Here is the connection between the Voronoï decomposition of $X$ and $H^*(\Gamma; \mathbb{Z})$. Let $N$ be the dimension of $X$ and $d$ the cohomological dimension of $\Gamma$. Let $C^k$ be the set of Voronoï cells of codimension $k$. The group $\Gamma$ acts naturally on $C^k$ by its action on rational polyhedral cones. We want to construct a $\Gamma_L$-equivariant “coboundary” map $\delta^k : \mathbb{Z}[C^k] \to \mathbb{Z}[C^{k+1}]$ so that the resulting chain complex modulo $\Gamma$ computes $H^*(\Gamma; \mathbb{Z})$. We will call $(C^*, \delta^*)$ a cocell complex, and will say that the Voronoï decomposition gives $X$ a cocell structure.

According to [1], there is a topological space $W \subset X$ such that the following hold:

1. $W$ admits the structure of a $\Gamma$-equivariant regular cell complex with top-dimensional cells of dimension $d$.
2. $W$ is a deformation retract of $X$, so that the homology of the chain complex associated to $\Gamma \setminus W$ is the homology of $\Gamma \setminus X$, and hence of $\Gamma$.
3. This cell structure is dual to the Voronoï decomposition in the following sense: every $k$-cell of $W$ transversely intersects exactly one Voronoï cell of codimension $k$.

Let $W_k$ denote the set of $k$-cells of $W$. Given $\tau \in W_k$, we denote its dual cell by $\hat{\tau} \in C^k$. Let $\mathbb{Z}[W_k]$ (respectively $\mathbb{Z}[C^k]$) denote the free abelian group on the elements of $W_k$ (resp. $C^k$).

We may choose orientations compatibly between the cell and cocell structures in the following sense: for each pair $(\tau, \hat{\tau})$ we may fix orientations so that in the homeomorphism

$$\text{Int } \tau \times \hat{\tau} \longrightarrow \mathbb{R}^N$$

the product of the orientations is carried to a fixed orientation of $\mathbb{R}^N$. This constructs a map $\mathbb{Z}[C^k] \to \text{Hom}_\mathbb{Z}(\mathbb{Z}[W_k], \mathbb{Z})$. 
Now to construct $\delta^*$, we use the boundary map from $W_*$. Given two cells $\sigma, \tau \in W_*$, write $\sigma < \tau$ if $\sigma$ appears in the closure of $\tau$. Then $\partial_k : \mathbb{Z}[W_k] \to \mathbb{Z}[W_{k-1}]$ has the form

$$\tau \mapsto \sum_{\sigma < \tau} \sigma : \tau \sigma,$$

where the $[\sigma : \tau] = \pm 1$ keeps track of the relative orientation between $\sigma$ and $\tau$. (Saying $W$ is a regular cell complex makes $[\sigma : \tau]$ well defined.) We define $\delta^k : \mathbb{Z}[C^k] \to \mathbb{Z}[C^{k+1}]$ by

$$\hat{\tau} \mapsto \sum_{\tau < \sigma} [\tau : \sigma] \hat{\sigma}$$

**Proposition 2.** With the above coboundary map, $H^k(\Gamma; \mathbb{Z})$ is naturally isomorphic to the $k$th cohomology of the quotient modulo $\Gamma$ of $(\mathbb{Z}[C^*], \delta^*)$.

**Proof.** We must show that $\delta^2 \equiv 0$ and that $\delta$ is the adjoint of $\partial$ with respect to the pairing between $\mathbb{Z}[W_*]$ and $\mathbb{Z}[C^*]$. The former is purely formal using (2) and the fact that $\hat{\tau}$ is the map $\mathbb{Z}[W_k] \to \mathbb{Z}$ that takes $\tau$ to 1 and all others to 0. The latter is easily verified from the definitions and the choice of orientations. \(\square\)

### 4. Voronoï reduction

In this section we address two questions:

1. How do we construct $\Pi$ in practice?
2. Given a point $x \in C$, can we determine a top-dimensional Voronoï cone containing it? (Such a cone is unique for generic $x$, and for any given $x$ there are at most a finite number of such cones containing it.)

In [22], Voronoï answers these in the setting of Example [4], where $C$ is the cone of real positive-definite symmetric matrices. In this section we prove that Voronoï’s results remain true in our more general context.

#### 4.1. First we describe some geometric properties of $\Pi$ which are proved in [4].

Let $F$ be a facet of $\Pi$, that is, a codimension-one face of $\Pi$. Then there is a unique point $y_F \in C \cap V(\mathbb{Q})$ such that

1. $F = \{ x \in \Pi \mid \langle x, y_F \rangle = 1 \}$, and
2. for all $x \in \Pi - F$, we have $\langle x, y_F \rangle > 1$.

We say that $y_F$ defines a supporting hyperplane of $\Pi$.

Given a facet $F$, let $Z_F$ be the finite set of points $z \in L' \cap \Xi(C)$ such that $\langle z, y_F \rangle = 1$. Then $F$ is the convex hull of $Z_F$, and as we range over all $w \in L' \cap \Xi(C)$ such that $w \not\in Z_F$, the set of numbers $\langle w, y_F \rangle$ is bounded below away from 1. We call $y_F$ the perfect form associated to $F$ and $Z_F$ the set of minimal vectors of $y_F$. In the case of Example [4], the $y_F$ are perfect quadratic forms in the classical sense, with minimal vectors $Z_F$ [22].
Let \( \sigma \subset \bar{C} \) be a rational polyhedral cone. Then \( \sigma \) satisfies the “property of Siegel” with respect to the Voronoï decomposition. Specifically, the intersection \( \sigma \cap \Pi \) is cut out from \( \Pi \) by a finite number of supporting hyperplanes ([1], p. 73). This implies that for any \( \sigma \) and for any \( x \in C \), the orbit \( \Gamma_Lx \) meets \( \sigma \) in a finite set.

Given any \( y \in C(\mathbb{Q}) \), let \( \pi(y) : V \to \mathbb{R} \) denote the linear map \( x \mapsto \langle x, y \rangle \). We also need the following finiteness result.

**Proposition 3.** Let \( y \in C(\mathbb{Q}) \). Then for any \( \mu > 0 \), the set
\[
\{ z \in L' \cap \Xi(C) \mid 0 < \langle z, y \rangle \leq \mu \}
\]
is finite.

**Proof.** Given any \( \lambda \in \mathbb{R} \), let \( H_\lambda \) be the affine hyperplane \( \{ x \in V \mid \langle x, y \rangle = \lambda \} \). Then \( H_0 = \ker(\pi) \) is rational, since \( y \in C(\mathbb{Q}) \subset V(\mathbb{Q}) \). Hence the map \( \pi(y) \) takes \( L \) onto a lattice in \( \mathbb{R} \). Since some multiple of \( y \) lies in \( L' \), this lattice is nontrivial. Thus to prove the claim, it is enough to show that for any \( \lambda > 0 \), the set \( H_\lambda \cap L' \cap \Xi(C) \) is finite.

To see this, consider the set \( \bar{C} \cap H_\lambda \). Note that \( H_\lambda \cap \Pi \) is a closed and bounded subset of \( V \). Since \( \Gamma_L \) meets \( \sigma \) in a finite set, \( H_\lambda \cap L' \cap \Xi(C) \), if nonempty, is finite.

4.2. Now we describe the construction of \( \Pi \). Call two facets of \( \Pi \) neighbors if they meet along a codimension-two face of \( \Pi \). We show how, given a facet \( F \), one may systematically find all neighbors of \( F \).

**Lemma 1.** Let \( F \) and \( G \) be neighboring facets of \( \Pi \) with perfect forms \( y_F \) and \( y_G \). Let \( v \in V(\mathbb{Q}) \) be orthogonal to the affine span of the origin and the polytope \( F \cap G \), and such that \( \langle x, v \rangle \geq 0 \) for all \( x \in F \). Then \( y_G = y_F + \rho v \) for a unique \( \rho \in \mathbb{R}^{>0} \).
Proof. First note that the affine span of $F \cap G$ and the origin is a hyperplane, since $F \cap G$ is a codimension-two face of $\Pi$. Thus $v$ is unique up to a scalar. Let $v' = y_G - y_F$. Then $\langle x, v' \rangle = 0$ for all $x \in F \cap G$, and $v' \neq 0$. Thus $\rho v = v'$ for some nonzero $\rho \in \mathbb{R}$, which shows that $y_G = \rho v + y_F$. We must show $\rho$ is positive.

Let $x \in F - (F \cap G)$. Then $1 < \langle x, y_G \rangle = \langle x, y_F \rangle + \rho \langle x, v \rangle = 1 + \rho \langle x, v \rangle$, which means that $\rho \langle x, v \rangle > 0$. Since $\langle x, v \rangle \geq 0$ for all $x \in F$, the result follows.

Suppose that we are given a facet $F$ with corresponding perfect form $y_F$ and minimal vectors $Z_F$. Choose a maximal proper face $E \subset F$, and let $Z_E \subset Z_F$ be the minimal vectors affinely spanning $E$. Let $G$ be the facet neighboring $F$ along $E$, and write $y_G = y_F + \bar{\rho}v$, where $v$ is a vector satisfying the conditions of Lemma 1. Let $Z_G$ be the set of minimal vectors $y_G$. Define the function $\rho(x)$ by
\[
\rho(x) := 1 - \frac{\langle x, y_F \rangle}{\langle x, v \rangle}
\]
and define $S$ by
\[
S := \{ x \in L' \cap \Xi(C) \mid \langle x, v \rangle < 0 \text{ and } y_F + \rho(x)v \in C \}.
\]
Note that $S$ is nonempty. This follows because whenever $z \in Z_G \setminus Z_E$, a computation shows $\rho(z) = \bar{\rho}$ and $y_F + \rho(z)v = y_G$, implying $z \in S$. This same computation shows conversely that if $z \in L' \cap \Xi(C)$ and $\rho(z) = \bar{\rho}$, then $z$ is a minimal vector of $y_G$ not in $Z_E$.

**Lemma 2.** As $x$ ranges over $S$, we have $\rho(x) \geq \bar{\rho}$.

*Proof.* Assume there is some $x' \in S$ with $\rho(x') < \bar{\rho}$. Then
\[
\langle x', y_G \rangle = \langle x', y_F \rangle + \bar{\rho} \langle x', v \rangle
\]
\[
< \langle x', y_F \rangle + \rho(x') \langle x', v \rangle
\]
\[
= 1,
\]
which is a contradiction. \qed

Now we show how to compute $y_G$. Choose any $\ell \in S$ and consider the point $y_\ell := y_F + \rho(\ell)v$. Let $T$ be the set
\[
T := \{ x \in L' \cap \Xi(C) \mid \langle x, y_\ell \rangle \leq 1 \}.
\]

**Lemma 3.** The set $T \cap (Z_G \setminus Z_E)$ is finite and nonempty.

*Proof.* Since $v \in V(\mathbb{Q})$, we have that $y_\ell \in C(\mathbb{Q})$. Thus $T$ is finite by Proposition \[3].
We now show $T \cap (Z_G \setminus Z_E)$ is nonempty. Let $z \in (Z_G \setminus Z_E) \cap S$, a set we have seen is nonempty. Then
\[
\langle z, y_\ell \rangle = \langle z, y_F \rangle + \rho(\ell) \langle z, v \rangle
\leq \langle z, y_F \rangle + \bar{\rho} \langle z, v \rangle \quad \text{[Lemma 3]}
= \langle z, y_G \rangle
= 1.
\]
Thus $z \in T$, and $T \cap (Z_G \setminus Z_E)$ is nonempty.

**Theorem 2.** Given a facet $F$ of the Voronoï polyhedron, all its neighbors may be determined in a finite number of steps.

*Proof.* Let $y_F$ be the perfect form corresponding to $F$, and let $Z_F$ be the set of minimal vectors of $y_F$. Using standard techniques of convex geometry, such as Fourier-Motzkin elimination ([24], p. 37), we may determine all the maximal proper faces of $F$. Let $E$ be such a face, and let $Z_E \subset Z_F$ be the minimal vectors affinely spanning $E$. Using $Z_E$, we may determine $v$ as in Lemma 1. Let $G$ be the facet of $\Pi$ neighboring $F$ along $E$, and let $y_G$ be the corresponding perfect form. We need to compute $y_G$.

First we find an $\ell \in S$ by searching over $L$, and then using $\ell$ we construct the finite set $T$. The latter can done in a finite number of steps because it is equivalent to finding the set of vectors in a lattice on which a positive-definite quadratic form is less than a constant (cf. Example 1 and the final paragraph of §3.1). By Lemma 3 $T \cap (Z_G \setminus Z_E)$ is nonempty, where $Z_G$ is the set of minimal vectors of $y_G$. Now let $Z \subset (T \setminus Z_E)$ be the set on which $\rho(x)$ attains its minimum. By Lemma 3 and the paragraph preceding it, $Z \subset Z_G \setminus Z_E$. Let $H$ be the affine span of $Z \cup Z_E$. Then $y_G$ is the unique point satisfying $\langle x, y_G \rangle = 1$ for all $x \in H$.

Repeating this procedure for each maximal face of $F$, we may determine all the neighbors of $F$.

*Remark 1.* In practice, it may be the case that $S$ only consists of $Z_G \setminus Z_E$, as a computation with $SL_2(\mathbb{Z})$ shows.

Hence one may find facets of $\Pi$ provided one can construct an initial facet. In the setting of Example 1 Voronoï did this by showing that the quadratic form $A_n$, defined by
\[
\sum_{i=1}^{n} x_i^2 + \sum_{1 \leq i < j \leq n} (x_i - x_j)^2,
\]
is perfect for all $n$ ([22], §29). In our more general setting, one cannot write down a perfect form that works for every case, even if one restricts to Bianchi groups (Example 2). However, in practice one may do the following.

\[2\]Voronoï called this form the *principal perfect form.*
First choose a large bounded set $U \subset V$ containing the origin, and let $\Sigma$ be the convex hull of $L' \cap \Xi(C) \cap U$. Then $\Sigma$ is a bounded polytope in $V$. Furthermore, since the facets of $\Pi$ are bounded, if $U$ is sufficiently large then most facets of $\Sigma$ will be facets of $\Pi$.

To check if a facet $F \subset \Sigma$ is a facet of $\Pi$, one computes $y_F$ and checks whether the $z \in L' \cap \Xi(C)$ such that $\langle z, y_F \rangle = 1$ are vertices of $F$.

**Remark 2.** In the Bianchi case, another possibility is to first construct the retract $W$ from §3.5 using techniques in [21], and then use the duality between them to deduce the structure of $\Pi$.

4.3. To answer question (2), Voronoï describes a reduction algorithm. This algorithm is based on the following:

**Proposition 4.** Fix an $x \in \Pi$ and a real number $\mu \geq 1$. Then there are only a finite number of perfect forms $y_F$ satisfying

$$\langle x, y_F \rangle \leq \mu.$$ 

**Proof.** Choose a facet $F$ such that $\langle x, y_F \rangle \leq \mu$. Since $y_F \in C(\mathbb{Q})$, Proposition 3 implies the set $\{ z \in L' \cap \Xi(C) \mid \langle z, y_F \rangle \leq \mu \}$ is finite. Hence the polyhedron

$$\Sigma = \{ y \in \Pi \mid \langle y, y_F \rangle \leq \mu \}$$

is a bounded polytope in $C$. If $\mu \in \pi(y_F)(L)$, where $\pi$ is defined in §4.1, then the hyperplanes bounding $\Sigma$ will be rational with respect to the $\mathbb{Q}$-structure on $V$, and thus $\Sigma$ will have vertices in $V(\mathbb{Q})$. Hence, replacing $\mu$ by a slightly larger number if necessary, we may assume $\Sigma$ has vertices in $V(\mathbb{Q})$.

Therefore the cone generated by the vertices of $\Sigma$ is rational polyhedral, and only meets a finite subset of the orbit $\Gamma_L x$. By taking the adjoint action of $\Gamma_L$ with respect to the scalar product, it follows that only finitely many facets $F'$ that are $\Gamma_L$-equivalent to $F$ satisfy $\langle x, y_{F'} \rangle \leq \mu$. Since there are only a finite number of facets modulo $\Gamma_L$, the result follows.

The following lemma gives a local condition for when a given point of $C$ lies in a cone over a facet of $\Pi$.

**Lemma 4.** Let $F$ be a facet of $\Pi$, and let $\mathcal{G}$ be the set of neighbors of $F$. Let $x \in C$. Suppose that $\langle x, y_F \rangle \leq \langle x, y_G \rangle$ for all $G \in \mathcal{G}$. Then $x$ lies in the cone over $F$.

**Proof.** Choose $\lambda$ so that $x' := \lambda x$ satisfies $\langle x', y_F \rangle = 1$ and $\langle x', y_G \rangle \geq 1$ for $G \in \mathcal{G}$. For $\epsilon > 0$, let $\Sigma_\epsilon$ be the polyhedron defined by

$$\Sigma_\epsilon = \{ x \mid 1 \leq \langle x, y_F \rangle \leq 1 + \epsilon \ \text{and} \ \langle x, y_G \rangle \geq 1 \ \text{for} \ G \in \mathcal{G} \}. $$

If $\epsilon$ is sufficiently small, then $\Sigma_\epsilon \subset \Pi$. Hence $x' \in \Sigma_\epsilon$, and thus $x' \in \Pi$. Since $x'$ also lies in the supporting affine hyperplane $\{ x \mid \langle x, y_F \rangle = 1 \}$, it must lie in $F$.

Now we describe our Voronoï reduction algorithm.
Theorem 3. Let \( x \in \Pi \), and choose a facet \( F \). Let \( \mu = \langle x, y_F \rangle \). The following algorithm determines a cone in the Voronoï decomposition containing \( x \):

1. For each neighbor \( G \) of \( F \), compute \( \langle x, y_G \rangle \).
2. If there exists a neighbor \( G \) with \( \langle x, y_G \rangle < \mu \), replace \( F \) with \( G \), \( \mu \) with \( \langle x, y_G \rangle \), and return to step one.
3. Otherwise, terminate the procedure: \( x \) lies in the cone generated by \( F \).

Proof. By Lemma 4, if the algorithm terminates then we have determined a cone containing \( x \). We now prove that the algorithm terminates. Suppose that a neighbor \( G \) of \( F \) has \( \langle x, y_G \rangle < \mu \). (Note that this quantity must be positive since \( C \) is self-adjoint and \( x, y \in C \).) Then we return to step one, and we have decreased the scalar product. Since by Proposition 4 the set of facets satisfying \( \langle x, y_{F'} \rangle \leq \mu \) is finite, the algorithm must terminate.

Remark 3. The data needed to implement this algorithm is the same as that needed for the structure of \( \Pi \) modulo \( \Gamma_L \), along with some additional information. In particular, one must determine:

1. A finite set \( \mathcal{F} \) of representatives of the facets of \( \Pi \) modulo \( \Gamma_L \).
2. For each \( F \in \mathcal{F} \) and each neighbor \( G \) of \( F \), an element \( \gamma \in \Gamma_L \) such that \( \gamma G \in \mathcal{F} \).

For an example of the implementation of this algorithm for \( SL_2(\mathbb{Z}) \), we refer to [15].

As far as we know, the computational complexity of this algorithm is unknown. However, in our experience it performs very well for \( SL_3(\mathbb{Z}) \) and \( SL_4(\mathbb{Z}) \).

5. The modular symbol algorithm

In this section we define modular symbols and describe our Hecke algorithm. Our definition of a modular symbol is closely related to the definitions appearing in [2] and [4]. As before, let \( N \) be the dimension of \( X \) and let \( d \) be the cohomological dimension of \( \Gamma \). We assume from now on that \( G \) has \( \mathbb{Q} \)-rank one, so that \( N = d + 1 \).

5.1. Let \( \rho(\Pi) \) be the set of rays in \( \check{C} \) generated by the vertices of the Voronoï polyhedron \( \Pi \). Given an ordered pair \( (u, v) \in \rho(\Pi) \times \rho(\Pi) \), we want to construct a class \([u, v] \in H^d(\Gamma; \mathbb{Z})\).

To this end we recall that \( X \) may be extended to a bordification \( \check{X} \) such that the quotient \( \Gamma \backslash \check{X} \), the Borel-Serre compactification, is a compact manifold with corners with interior \( \Gamma \backslash X \). Let \( \pi: \check{X} \to \Gamma \backslash \check{X} \) be the canonical projection. Given \( u, v \in \rho(\Pi) \), we determine a path in \( \Gamma \backslash \check{X} \) as follows. First \( u \) and \( v \) determine a closed cone \( \sigma \subset V \), and we let \( \bar{\sigma} \) be the closure of \( \sigma \cap V \) mod homotheties in \( \check{X} \). Then \( \pi(\bar{\sigma}) \) is a path with endpoints lying in \( \partial(\Gamma \backslash \check{X}) \). Choosing an ordering \( (u, v) \) fixes an orientation of \( \pi(\bar{\sigma}) \) and thus determines a class in \( H_1(\Gamma \backslash \check{X}, \partial(\Gamma \backslash \check{X}); \mathbb{Z}) \). By Lefschetz duality,

\[
H_1(\Gamma \backslash \check{X}, \partial(\Gamma \backslash \check{X}); \mathbb{Z}) = H^d(\Gamma \backslash \check{X}; \mathbb{Z}),
\]

and since \( \Gamma \backslash \check{X} \) is homotopy equivalent to \( \Gamma \backslash X \), we have actually determined a class in \( H^d(\Gamma; \mathbb{Z}) \).
Definition 4. A modular symbol is a class in $H^d(\Gamma; \mathbb{Z})$ constructed as above from an ordered pair $(u, v) \in \rho(\Pi) \times \rho(\Pi)$. The class is denoted $[u, v]$.

Note that this definition is almost the same as that in §2.1 for $\Gamma \subset SL_2(\mathbb{Z})$, because using $(u, v)$ to generate a cone is the same as choosing a specific path as in §2.1. However, as in the earlier definition, the class $[u, v]$ is independent of this path.

We have the following analogue of Proposition 1:

Proposition 5. Let $u, v \in \rho(\Pi)$. The modular symbols satisfy the following:

1. $[u, v] = -[v, u]$.
2. If $w \in \rho(\Pi)$, then $[u, v] = [u, w] + [w, v]$.
3. The modular symbols span $H^d(\Gamma; \mathbb{Z})$.

Proof. Only (2) and (3) require proof. To prove (2), let $\sigma$ be the cone generated by $u$ and $v$, and choose a ray $x \subset \sigma$ distinct from $u$ and $v$. Let $\phi: [0, 1] \to \bar{C}$ be a continuous family of rays such that $\phi(0) = x$ and $\phi(1) = w$. Let $\sigma_1(t)$ (respectively $\sigma_2(t)$) be the cone generated by $u$ and $\phi(t)$ (resp. $\phi(t)$ and $v$). Then $\phi$ provides a continuous deformation of $\sigma$ into $\sigma_1(1) \cup \sigma_2(1)$ that induces the homology $[u, v] = [u, w] + [w, v]$.

Now to prove (3), note that the results of §3.5 imply that any class in $H^d(\Gamma; \mathbb{Z})$ may be written as a cocycle for $\delta$ using the Voronoï cones of codimension-$d$ modulo $\Gamma$. But the Voronoï cones of codimension $d$ are cones generated by pairs of vertices of $\Pi$, and so any such cycle is in the span of the modular symbols.

Remark 5. Although the Voronoï-reduced modular symbols provide a finite spanning set for $H^d(\Gamma; \mathbb{Z})$, they are not a basis of $H^d(\Gamma; \mathbb{Z})$. In fact, they are not even a basis of $\mathbb{Z}[C^\ast]$ modulo $\Gamma$, because they are not necessarily supported on codimension-$d$ Voronoï cones. However, in practice this does not affect their usefulness (cf. Remark 7).
5.3. Let \( u, v \in \rho(\Pi) \), and let \( \sigma \) be the rational polyhedral cone generated by \( u \) and \( v \). We are now ready to describe and prove our algorithm.

Given \( x \in C \), let \( \sigma(x) \) be the unique Voronoï cone containing \( x \), and by abuse of notation let \( R(x) = R(\sigma(x)) \).

Let \( x_1, \ldots, x_r \) be points in \( \sigma \cap C \) such that the rays \( \mathbb{R}^{>0}x_1, \ldots, \mathbb{R}^{>0}x_r \) are distinct. These points subdivide \( \sigma \) into a collection of cones, namely those generated by the pairs \( (u, x_1), \ldots, (x_r, v) \).

**Definition 6.** The decomposition of \( \sigma \) by the \( x_i \) as above is called a **sufficiently fine partition** if

1. \( u \in R(x_1) \),
2. \( v \in R(x_r) \), and
3. for \( i = 1, \ldots, r - 1 \), we have \( R(x_i) \cap R(x_{i+1}) \neq \emptyset \).

**Lemma 5.** Sufficiently fine partitions of \( \gamma \) exist.

**Proof.** Since \( \sigma \) is rational polyhedral, the Siegel property implies that \( \sigma \cap \Pi \) is cut out by finitely many supporting hyperplanes. Hence \( \sigma \cap C \) meets only finitely many top-dimensional cones, and the intersection of these cones with \( \sigma \) subdivides the latter into finitely many 2-cones \( \{\sigma_i\} \). We may take \( x_i \) to be any nonzero point in the interior of \( \sigma_i \). Conditions (1) and (2) of Definition 6 are trivially satisfied. Also, \( R(x_i) \cap R(x_{i+1}) \neq \emptyset \) because \( \sigma_i \subset V_i \) and \( \sigma_{i+1} \subset V_{i+1} \), where \( V_i \) and \( V_{i+1} \) are Voronoï cones that have a face in common.

**Remark 6.** For computational purposes, one may construct sufficiently fine partitions of \( \gamma \) as follows. Let \( u, v \in \rho(\Pi) \) and let \( \sigma \) be the cone generated by \( u \) and \( v \). Choose points \( \bar{u} \in u \) and \( \bar{v} \in v \). Let \( \bar{x} \) be the midpoint of the segment between \( \bar{u} \) and \( \bar{v} \), and let \( x \) be the cone generated by \( \bar{x} \). Now apply Theorem 3 to check whether \( u \in R(\bar{x}) \) and \( v \in R(\bar{x}) \). If these conditions are not satisfied, bisect the segments between \( \bar{u}, \bar{x} \) and \( \bar{x}, \bar{v} \), and check conditions (1), (2), and (3). Eventually, by the Siegel property, after a finite number of iterations one will have constructed a sufficiently fine partition of \( \sigma \).

Now we present our algorithm.

**Theorem 4.** Given a modular symbol \([u, v]\), the following constructs a chain of Voronoï-reduced modular symbols homologous to \([u, v]\):

1. Choose a set of points \( \{x_i\} \) inducing a sufficiently fine partition of the cone generated by \( u \) and \( v \).
2. For \( i = 1, \ldots, r - 1 \), choose a ray \( q_i \in R(x_i) \cap R(x_{i+1}) \).

Then \([u, v] = [u, q_1] + [q_1, q_2] + \cdots + [q_r, v]\).
Proof. First note that each modular symbol on the right hand side is Voronoï-reduced, since \( q_i \) and \( q_i + 1 \) are both rays from \( R(x, \gamma) \). We must show there is a homology between the right side and the left. Notice that
\[
[u, v] = [u, q_1] + [q_1, v]
\]
between Proposition 3. Repeatedly applying this proposition, we see that
\[
[q_i, v] = [q_i, q_{i+1}] + [q_{i+1}, v],
\]
which completes the proof. \( \square \)

Remark 7. For computational purposes, to determine the action of a Hecke operator we must write any modular symbol in terms of a basis of \( H^d(\Gamma; \mathbb{Z}) \), and by Remark 3 the technique in Theorem 4 is not sufficient to do this. However, in practice we may precompute explicit homologies between Voronoï-reduced modular symbols and modular symbols supported on Voronoï cones, as follows.

Let \( \mathcal{F} \) be a set of representatives of the facets of \( \Pi \) modulo \( \Gamma_L \). For each \( F \in \mathcal{F} \), let \( u, v \) be any two vertices of \( F \). Then \([u, v]\) is a Voronoï-reduced modular symbol. To write \([u, v]\) in terms of the basis of codimension-\( d \) Voronoï cones, choose any sequence of vertices \( u = u_0, u_1, \ldots, u_k = v \) of \( F \) such that \( u_i \) and \( u_{i+1} \) are joined by an edge of \( F \). Then
\[
[u, v] = [u, u_1] + \cdots + [u_{k-1}, v]
\]
is the desired homology. Now repeat for all \( u, v \) and all \( F \in \mathcal{F} \).

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MODULAR SYMBOLS FOR $\mathbb{Q}$-RANK ONE GROUPS AND VORONOÏ REDUCTION

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