K-THEORY OF CREPANT RESOLUTIONS OF COMPLEX ORBIFOLDS WITH SU(2) SINGULARITIES

CHRISTOPHER SEATON

ABSTRACT. We show that if $Q$ is a closed, reduced, complex orbifold of dimension $n$ such that every local group acts as a subgroup of $SU(2) < SU(n)$, then the $K$-theory of the unique crepant resolution of $Q$ is isomorphic to the orbifold $K$-theory of $Q$.

1. INTRODUCTION

Let $Q$ be a reduced, compact, complex orbifold of dimension $n$; i.e. a compact Hausdorff space locally modeled on $\mathbb{C}^n/G$ where $G$ is a finite group which acts effectively on $\mathbb{C}^n$ with a fixed-point set of codimension at least 2 (for details of the definition and further background, see [3]). Then a crepant resolution of $Q$ is given by a pair $(Y, \pi)$ where $Y$ is a smooth complex manifold of dimension $n$ and $\pi : Y \to Q$ is a surjective map which is biholomorphic away from the singular set of $Q$, such that $\pi^*K_Q = K_Y$ where $K_Q$ and $K_Y$ denote the canonical line bundles of $Q$ and $Y$, respectively (see [7] for details). In [11], it is conjectured that if $\pi : Y \to Q$ is a crepant resolution of a Gorenstein orbifold $Q$ (i.e. an orbifold such that all groups act as subgroups of $SU(n)$), then the orbifold $K$-theory of $Q$ is isomorphic to the ordinary $K$-theory of $Y$. For the case of a global quotient of $\mathbb{C}^n$, this has been verified for $n = 2$ in [10] and, for Abelian groups and a specific choice of crepant resolution for $n = 3$ in [5]. Here, we apply the ‘local’ results in the case $n = 2$ to the case of a general orbifold with such singularities.

The $K$-theory of an orbifold can be defined in several different ways. First, it can be defined in the usual way in terms of equivalence classes of orbifold vector bundles (see [1]). As well, it is well-known that a reduced orbifold $Q$ can be expressed as the quotient $P/G$ where $P$ is a smooth manifold and $G$ is a compact Lie group [8]. In the case of a real orbifold, $P$ can be taken to be the orthonormal frame bundle of $Q$ with respect to a Riemannian metric and $G = O(n)$. Similarly, in the complex case, $P$ can be taken to be the unitary frame bundle and $G = U(n)$. Hence, the orbifold $K$-theory of $Q$ is defined as the $G$-equivariant $K$-theory $K_G(P)$. See [11] or [9] for more details.

In Section 2 we describe the structure of the singular set $\Sigma$ of $Q$ in the case in question and state the main result. In Section 3 we interpret this decomposition in terms of ideals of the $C^*$-algebra of $Q$ and prove the result.

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2. The Decomposition of $\Sigma$ and Statement of the Result

Let $Q$ be a closed, reduced, complex orbifold with $\dim \mathbb{C} Q = n$, and fix a hermitian metric on $TQ$ throughout. Then each point $p \in Q$ is contained in a neighborhood modeled by $\mathbb{C}^n / G_p$ where $p$ corresponds to the origin in $\mathbb{C}^n$ and $G_p < U(n)$. Suppose that each of the local groups $G_p$ act as a subgroup of $SU(2) < SU(n)$, and then each point $p$ is locally modeled by $\mathbb{C}^n / G_p \cong \mathbb{C}^{n-2} \times (\mathbb{C}^2 / G_p)$. Suppose further that $Q$ admits a crepant resolution $\pi : Y \to Q$ so that $Y$ is a closed complex $n$-manifold. By Proposition 9.1.4 of [7], $(Y, \pi)$ is a local product resolution, which in this context means the following (see 9.1.2 of [7] for the general definition):

Fix $p \in Q$, and then there is a neighborhood $U_p \ni p$ modeled by $\mathbb{C}^n / G_p$. By hypothesis, $U_p \cong V \times W / G_p$ where $V \times \{0\} \cong \mathbb{C}^{n-2}$ is the fixed point set of $G_p$, $W \cong \mathbb{C}^2$ is the orthogonal complement of $V$ in $\mathbb{C}^n$ (for some choice of $G_p$-invariant metric on $\mathbb{C}^n$), and we identify $G_p < SU(n)$ with its restriction $G_p < SU(2)$. Then for a resolution $(Y_p, \pi_p)$ of $W / G_p$, we let $\phi : V \times W / G_p \to \mathbb{C}^n / G_p$. $T$ be the ball of radius $R > 0$ about the origin in $\mathbb{C}^n / G_p$, and $U := (id \times \pi)^{-1}(T) \subset V \times Y_p$. There is a local isomorphism $\psi : (V \times Y_p) \setminus U \to Y$ such that the following diagram commutes

$$
\begin{align*}
(V \times Y_p) \setminus U & \xrightarrow{\psi} Y \\
\downarrow \text{id} \times \pi_p & \downarrow \pi \\
(V \times W / G_p) \setminus T & \xrightarrow{\phi} \mathbb{C}^n / G_p.
\end{align*}
$$

Hence, each of the singular points in a neighborhood of $p$ is resolved by $V \times Y_p$. Moreover, as $(Y, \pi)$ is a crepant resolution of $Q$, $(Y_p, \pi_p)$ is a crepant resolution of $\mathbb{C}^2 / G_p$ ([7], Proposition 9.1.5), and hence is the unique crepant resolution of $\mathbb{C}^2 / G_p$. It is clear that a crepant resolution of $Q$ can be formed by patching together local products of the unique crepant resolutions of $\mathbb{C}^2 / G_p$, but we now see that this is the only crepant resolution of $Q$. Moreover, if $S$ denotes a connected component of the singular set $\Sigma$ of $Q$, then a neighborhood of $S$ can be covered by a finite number of charts as above, so that the isotropy subgroups of any $p, q \in S$ are conjugate in $SU(2)$. Moreover, each such chart $\mathbb{C}^n / G_p \cong V \times W / G_p$ restricts to a complex manifold chart of dimension $n - 2$ for $S$.

We summarize this discussion in the following.

Lemma 2.1. Let $Q$ be a closed, reduced, complex orbifold of complex dimension $n$, and suppose each of the local groups $G_p$ acts on $Q$ as a subgroup of $SU(2)$. Then there is a unique crepant resolution $(Y, \pi)$ of $Q$. The singular set $\Sigma$ of $Q$ is given by

$$
\Sigma = \bigsqcup_{i=1}^{k} S_i
$$

for some $k$ finite, where each $S_i$ is a connected, closed, complex $(n-2)$-manifold and the (conjugacy class of the) isotropy subgroup $G_p < SU(2)$ of $p$ is constant on $S_i$. Moreover, if $N_i$ is a sufficiently small tubular neighborhood of $S_i$ in $Q$, then $N_i \cong S_i \times \mathbb{C}^2 / G_p$ and $\pi^{-1}(N_i) \cong S_i \times Y_i$ where $Y_i$ is the unique crepant resolution of $\mathbb{C}^2 / G_p$. 

We assume that the hypotheses of Theorem 2.2, and let $C$ the resolution.

For any $n$-dimensional orbifold that admits a crepant resolution, the local groups can be chosen to be subgroups of $SU(n)$ (see [7]). Therefore, we have as an immediate corollary:

**Corollary 2.3.** Let $Q$ be a 2-dimensional complex orbifold which admits a crepant resolution $(Y, \pi)$. Then

$$K^*_{orb}(Q) \cong K^*(Y)$$

as additive groups.

### 3. Proof of Theorem 2.2

In order to prove Theorem 2.2, we will show that $K_*(A) \cong K_*(B)$ where $A$ is the $C^*$-algebra of $Q$ and $B$ the $C^*$-algebra of $Y$. So fix an orbifold $Q$ that satisfies the hypotheses of Theorem 2.2 and let $k, S_i, N_i$, etc. be as given in Lemma 2.1. We assume that the $N_i$ are chosen small enough so that $N_i \cap N_j = \emptyset$ for $i \neq j$.

For each $i$, let $N_i^0$ be a smaller tubular neighborhood of $S_i$ so that $S_i \subset N_i^0 \subset N_i$, and let $N_0 := Q \setminus \bigcup_{i=1}^k N_i$. Then $\{N_i\}_{i=0}^k$ is an open cover of $Q$ such that $N_0$ contains no singular points. Note that the restriction $\pi_{|\pi^{-1}(N_0)}$ is a biholomorphism on $N_0$.

Let $P$ denote the unitary frame bundle of $Q$, and then $Q = P/U(n)$. Let $A := C^*(Q)$ denote the $C^*$-algebra $C(P) \rtimes_\alpha U(n)$ of $Q$ where $\alpha$ is the action of $U(n)$ on $C(P)$ induced by the usual action on $P$, and let $A^0$ denote the dense subalgebra $L^1(U(n), C(P), \alpha)$ of $C(P) \rtimes_\alpha U(n)$. Let $I_0^0$ denote the ideal in $A^0$ consisting of functions $\phi$ such that $\phi(g)$ vanishes on $P_{|S_i}$ for each $g \in U(n)$ (i.e. $I_0^0 = L^1(U(n), C_0(P \setminus P_{|S_i}), \alpha)$; as usual, $P_{|S_i}$ denotes the restriction of $P$ to $S_i$), and let $I_1$ be the closure of $I_0^0$ in $A$. Similarly, for each $j$ with $1 < j \leq k$, set $I_j^0 := L^1(U(n), C_0 \left( P \setminus \bigcup_{i=1}^j P_{|S_i} \right), \alpha)$ to be the ideal of functions $\phi$ in $A^0$ such that for each $g \in U(n)$, $\phi(g)$ vanishes on the fibers over $S_1, S_2, \cdots, S_j$, and $I_j$ the closure of $I_j^0$ in $A$. Then we have the ideals

$$I_k \subset I_{k-1} \subset \cdots \subset I_1 \subset I_0 := A.$$

Note that, for each $j$ with $1 \leq j < k$, $I_j/I_{j+1} \cong C(P_{|S_{j+1}}) \rtimes_\alpha U(n)$, and $I_k \cong C_0(P_{|N_0}) \rtimes_\alpha U(n)$.
Similarly, let $B := C(Y)$ denote the algebra of continuous functions on $Y$, and let $J_j$ denote the ideal of functions which vanish on $\pi^{-1}(\bigcup_{i=1}^j S_i)$. Then we have

$$J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 := B,$$

with $J_j/J_{j+1} \cong C(\pi^{-1}(S_{j+1}))$ and $J_k \cong C_0(\pi^{-1}(N_0))$.

Recall that $\pi$ restricts to a biholomorphism

$$\pi_{|\pi^{-1}(N_0)} : \pi^{-1}(N_0) \cong N_0.$$

Hence, as the action of $U(n)$ is free on $P_{|N_0}$,

$$K_*(I_k) = K_*(C_0(P_{|N_0}) \rtimes_\alpha U(n))$$

$$\cong K_*(U(n)(P_{|N_0}))$$

naturally, by the Green-Julg Theorem ([2] Theorems 20.2.7 and 11.7.1),

$$\cong K^*(P_{|N_0}/U(n))$$

as the $U(n)$ action is free on $N_0$,

$$= K^*(N_0)$$

$$= K^*(\pi^{-1}(N_0))$$

$$= K_*(J_k).$$

Therefore, there is a natural isomorphism

$$(3.1) \quad K_*(I_k) \cong K_*(J_k).$$

Hence, Theorem 2.2 holds for orbifolds such that $k = 0$; i.e. manifolds. The next lemma gives an inductive step which, along with the previous result, yields the theorem.

**Lemma 3.1.** Suppose

$$K_*(I_j) \cong K_*(J_j)$$

naturally for some $j$ with $1 \leq j \leq k$. Then

$$K_*(I_{j-1}) \cong K_*(J_{j-1}).$$

**Proof.** Note that $I_j$ is an ideal in $I_{j-1}$, with $I_{j-1}/I_j = C(P_{|S_j}) \rtimes_\alpha U(n)$. Similarly, $J_j$ is an ideal in $J_{j-1}$ with $J_{j-1}/J_j = C(\pi^{-1}(S_j))$. We have the standard exact sequences

$$K_0(I_j) \rightarrow K_0(I_{j-1}) \rightarrow K_0(I_{j-1}/I_j)$$

$$\partial \uparrow \downarrow \partial$$

$$K_1(I_{j-1}/I_j) \leftarrow K_1(I_{j-1}) \leftarrow K_1(I_j)$$
and
\[
K_0(J_j) \to K_0(J_{j-1}) \to K_0(J_{j-1}/J_j)
\]
\[
\partial \uparrow \quad \downarrow \partial
\]
\[
K_1(J_{j-1}/J_j) \leftarrow K_1(J_{j-1}) \leftarrow K_1(J_j).
\]
So if we show that \(K_*(I_{j-1}/I_j) \cong K_*(J_{j-1}/J_j)\) naturally, by the Five lemma, we are done.

Note that \(I_{j-1}/I_j\) is the \(C^*\)-algebra of the quotient orbifold \(P_{S_j}/U(n)\), which is given by the smooth manifold \(S_j\) with the trivial action of \(G_j\) (here, \(G_j\) denotes a choice from the conjugacy class of isotropy groups \(G_p\) for \(p \in S_j\)). Hence, \(I_{j-1}/I_j \cong C(S_j) \otimes C^*(G_j)\). Similarly, we have
\[
J_{j-1}/J_j = C(\pi^{-1}(S_j))
\]
\[
= C(S_j \times Y_j)
\]
\[
= C(S_j) \otimes C(Y_j),
\]
where \(Y_j\) is the preimage of the origin in the unique crepant resolution of \(\mathbb{C}^2/G_j\).

However, \(K_0(C^*(G_j)) = R(G)\) (\cite{2} Proposition 11.1.1 and Corollary 11.1.2) which is naturally isomorphic to \(K^0(Y_j)\) by \cite{10} (Section 4.3; see also \cite{5}), and \(K^0(Y_j) \cong K_0(C(Y_j))\), so that \(K_0(C^*(G_j))\) and \(K_0(C(Y_j))\) are isomorphic. With this, by the Künneth Theorem for tensor products (\cite{2} Theorem 23.1.3),
\[
0 \to K_0(C(S_j)) \otimes K_0(C^*(G_j)) \to K_0(C(S_j) \otimes C^*(G_j)) \to
\]
\[
\downarrow \quad \downarrow
\]
\[
0 \to K_0(C(S_j)) \otimes K_0(C(Y_j)) \to K_0(C(S_j) \otimes C(Y_j)) \to
\]
\[
\to Tor(K_0(C(S_j)), K_0(C^*(G_j))) \to 0
\]
\[
\downarrow
\]
\[
\to Tor(K_0(C(S_j)), K_0(C(Y_j))) \to 0
\]
and the Five lemma, we have a natural isomorphism
\[
K_0(C(S_j) \otimes C^*(G_j)) \cong K_0(C(S_j) \otimes C(Y_j)).
\]
So
\[
K_0(I_{j-1}/I_j) \cong K_0(J_{j-1}/J_j).
\]

For the \(K_1\) groups, we note that by \cite{4}, Corollary 11.1.2, \(K_1(C^*(G_j)) = 0\). As well, \(K_1(C(Y_j)) \cong K^1(Y_j)\), and it is known that \(Y_j\) is diffeomorphic to a finite collection of 2-spheres which intersect at most transversally at one point (see \cite{7}). Therefore, \(K^1(Y_j) = 0\). Here, the hypothesis that all groups act as subgroups of \(SU(2)\) is crucial. For subgroups of \(SU(3)\), the topology of the resolution is not understood sufficiently to compute the \(K_1\) groups.
With this, we again apply the Künneth theorem and Five lemma

\[
0 \to K_1(C(S_j)) \otimes K_0(C^*(G_j)) \oplus K_0(C(S_j)) \otimes K_1(C^*(G_j)) \to K_1(C(S_j) \otimes C^*(G_j))
\]

\[
0 \to K_1(C(S_j)) \otimes K_0(C(Y_j)) \oplus K_0(C(S_j)) \otimes K_1(C(Y_j)) \to K_1(C(S_j) \otimes C(Y_j))
\]

\[
\cdots \to \text{Tor}(K_1(C(S_j)), K_0(C^*(G_j))) \oplus \text{Tor}(K_0(C(S_j)), K_1(C^*(G_j))) \to 0
\]

\[
\cdots \to \text{Tor}(K_1(C(S_j)), K_0(C(Y_j))) \oplus \text{Tor}(K_0(C(S_j)), K_1(C(Y_j))) \to 0
\]

Therefore, we have a natural isomorphism

\[
K_1(C(S_j) \otimes C^*(G_j)) \cong K_1(C(S_j) \otimes C(Y_j)),
\]

and

\[
K_1(I_{j-1}/I_j) \cong K_1(J_{j-1}/J_j).
\]

\[\square\]

Now, as \(K_*(I_k) \cong K_*(J_k)\), repeated application of Lemma 3.1 yields that \(K_*(A) \cong K_*(B)\), and hence we have proven Theorem 2.2.

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Department of Mathematics and Computer Science, Rhodes College, 2000 N. Parkway, Memphis, TN 38112

E-mail address: seatonc@rhodes.edu