HAMILTONIAN PRISMS ON 5-CHORDAL GRAPHS

GAO MOU

ABSTRACT. In this paper, we provide a method to find a Hamiltonian cycle in the prism of a 5-chordal graph, which is \((1 + \epsilon)\)-tough, with some special conditions.

1. Preliminaries

The toughness of graphs is a concept introduced by Chvátal [6], when he was doing research on Hamiltonicity of graphs. A graph \(G\) is called \(\beta\)-tough, if for any \(p \geq 2\), it cannot be split into \(p\) components by deleting less than \(n\beta\) vertices. It is not difficult to prove that every \(k\)-tough graph is \(2k\)-connected. A graph \(G\) is called \(k\)-chordal, if for any circle \(C\) in \(G\) with length \(|C| \geq k\), \(C\) has a chord in \(G\). Usually, we call a 3-chordal graph a chordal graph for convenience. Clearly, a chordal graph is also \(k\)-chordal for any \(k \geq 3\).

Chvátal posed a famous conjecture, which is still open today, saying that there exists a constant \(\beta\) such that every \(\beta\)-tough graph is Hamiltonian. Clearly, being 1-tough is a necessary condition for being Hamiltonian. What is more, there exist 2-tough graphs which are not Hamiltonian [4]. In recent decades, researchers found many special classes of graphs for which Chvátal’s conjecture is true, for example, chordal graphs [9], \(2K_2\)-free graphs [5], and planar graphs [12]. Moreover, researchers are also interested in many kinds of analogies of Hamiltonian cycles, such as \(k\)-walks, Hamiltonian-prisms, and 2-factors. A \(k\)-walk in a graph \(G\) is a closed walk visiting each vertex of \(G\) at least once but at most \(k\) times. Clearly, a Hamiltonian cycle can be considered as a 1-walk. And a \(p\)-walk is trivially a \(q\)-walk, for integers \(p \leq q\). The prism over a graph \(G\) is the Cartesian product \(G \times K_2\) of \(G\) with the complete graph \(K_2\). If \(G \times K_2\) is Hamiltonian, then we say \(G\) is prism-Hamiltonian, and we call \(G \times K_2\) the Hamilton-prism of \(G\). It is not difficult to prove that being prism-Hamiltonian is a property stronger than admitting 2-walk but weaker than being Hamiltonian [10]. For \(k\)-walks in graphs, there is also a well-known open conjecture, which is posed by Jackson and Wormald, saying that every \(\frac{1}{e-1}\)-tough graph admits a \(k\)-walk [8]. A 2-factor in a graph is a spanning subgraph, which is consisted of several disjoint cycles. An edge-dominating cycle \(C\) in a graph \(G\) is a cycle such that the induced subgraph on \(V(G) - V(C)\) contains no edge.

Here we list several well-known results relative chordal graphs and 5-chordals on the topic of Hamiltonicity.

**Theorem 1.** [9] Every 10-tough chordal graph is Hamiltonian.

**Theorem 2.** [1] There exists a \((\frac{4}{3} - \epsilon)\)-tough chordal non-Hamiltonian graphs, for any \(\epsilon > 0\).
Theorem 3. Every $(1 + \epsilon)$-tough chordal planar graph is Hamiltonian, for any $\epsilon > 0$.

Theorem 4. Every $(3/4 + \epsilon)$-tough chordal planar graph admits a 2-walk, for any $\epsilon > 0$.

Theorem 5. Every $3/2$-tough 5-chordal graph contains a 2-factor.

2. Main Results

The main result in this paper is following:

Theorem 6. Let $G$ be a $(1 + \epsilon)$-tough 5-chordal graph, if $G$ contains an edge-dominating cycle, then $G$ is prism-Hamiltonian.

The proof of this theorem is consisted of the following two lemmas, one of which is from [7], while another is new.

Lemma 1. Let $G$ be $(1 + \epsilon)$-tough, for some $\epsilon > 0$.

1. If $G$ contains an edge-dominating cycle $C$ with even number of vertices, then the prism over $G$ is Hamiltonian.

2. If $G$ contains an edge-dominating cycle $C = v_1v_2 \cdots v_{2p+1}v_1$ of odd length, and there are three vertices $v_1$, $v_{2q}$ and $v_{2q+1}$, for some $1 \leq q \leq p$, inducing a triangle in $G$, then the prism over $G$ is Hamiltonian.

Lemma 2. Assuming $C = v_1v_2 \cdots v_pv_1$ is a cycle of odd length in a 5-chordal $G$, then there exist three vertices $v_i, v_{i+2q-1}, v_{i+2q}$ of $C$ (the index is in module $p$), which induce a triangle in $G$.

Proof. If $p = 3$, then let $i = 1$ and $q = 1$, so the result holds trivially. If $p \geq 5$, then by 5-chordality, there must be a chord on $C$. Without loss of generality, we can assume that $v_1$ is one endpoint of a chord $e$. If another endpoint of $e$ is $v_3$, then we have proved the result. Otherwise, $e = v_1v_t$ divides $C$ into two cycles: $C_1 = v_1v_2 \cdots v_tv_1$ and $C_2 = v_1v_1 \cdots v_{t+1}v_pv_1$.

If $t$ is odd, then by mathematical induction, we can find triangle with the property we want in $C_1$.

If $t$ is even, then we can relabel the vertices of $C_2$ by $v'_1 = v_1, v'_2 = v_t, v'_3 = v_{t+1}$ and so on. Clearly, this relabelling process does not change the odd-even of the index. So, by mathematical induction, there is a triangle we want in $C_2$, and this triangle is also the one we want in $C$.

Combining the two lemmas above, we have finished the proof of Theorem 6.

3. Applications

Now, we are interested in the question that under what condition, we can find an edge-dominating cycle in a 5-chordal graph? Here we list several well-known results on this topic.

Lemma 3. Let $G$ be a 2-connected graph of order $n$. If

$$\delta_3(G) := \min \left\{ \sum_{i \leq 3} d(x_i) \mid x_1, x_2, x_3 \text{ are independent vertices in } G \right\} \geq n + 2,$$

then all longest cycles in $G$ are edge-dominating.
Two edges are called *remote* if they are disjoint and there is no edge joining them. Let $N_G(v)$ stand for the set of vertices in $G$ which are adjacent to $v$. For an edge $e = uv$ in $G$, we define the degree $d(e)$ of $e$ by $d(e) = |N_G(u) \cup N_G(v) - \{u, v\}|$.

**Lemma 4.** Let $G$ be a $k$-connected graph ($k \geq 2$) such that, for every $k + 1$ mutually remote edges $e_0, e_1, \ldots, e_k$ of $G$,

$$
\sum_{i=0}^{k} d(e_i) > \frac{1}{2}k(|V(G)| - k).
$$

Then $G$ contains an edge-dominating cycle.

**Lemma 5.** Let $G$ be a 2-connected graph. If $d(e_1) + d(e_2) > |V(G)| - 4$ for any remote edges $e_1, e_2$, then all longest cycles in $G$ are edge-dominating cycles.

Combining the three lemmas above and Theorem 6, we get the following corollaries.

**Corollary 1.** Let $G$ be a $(1 + \epsilon)$-tough 5-chordal graph of order $n$. If

$$
\delta_3(G) \geq n + 2,
$$

then $G$ is prism-Hamiltonian.

**Corollary 2.** Let $G$ be a $(1 + \epsilon)$-tough 5-chordal graph such that, for every 3 mutual remote edges $e_1, e_2, e_3$ of $G$,

$$
\sum_{i=0}^{3} d(e_i) > \frac{3}{2}(|V(G)| - 3).
$$

Then $G$ is prism-Hamiltonian.

**Corollary 3.** Let $G$ be a $(1 + \epsilon)$-tough 5-chordal graph. If $d(e_1) + d(e_2) > |V(G)| - 4$ for any remote edges $e_1, e_2$, then $G$ is prism-Hamiltonian.

**Remarks 1.** The results in this paper is almost trivial now. So, we are looking for some “non-trivial” condition for 5-chordal graphs to have edge-dominating cycles. Especially, we hope to find a toughness condition for 5-chordal graphs to have such cycles.

**References**

[1] D. Bauer, H. J. Broersma, and H. J. Veldman. Not every 2-tough graph is Hamiltonian. *Discrete Appl. Math.*, 99(1-3):317–321, 2000.

[2] D. Bauer, G. Y. Katona, D. Kratsch, and H. J. Veldman. Chordality and 2-factors in tough graphs. *Discrete Appl. Math.*, 99(1-3):323–329, 2000.

[3] T. Bohme, J. Harant, and M. Tkáč. More than one tough chordal planar graphs are Hamiltonian. *J. Graph Theory*, 32(4):405–410, 1999.

[4] J. A. Bondy. Longest paths and cycles in graphs of high degree. Tech. Report CORR 80-16, 1980.

[5] H. Broersma, V. Patel, and A. Pyatkin. On toughness and Hamiltonicity of $2K_2$-free graphs. *J. Graph Theory*, 75(3):244–255, 2014.

[6] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:215–228, 1973.

[7] M. Gao and D. V. Pasechnik. Edge-dominating cycles, $k$-walks and hamilton prisms in $2K_2$-free graphs. Accepted by Journal of Knot Theory and Its Ramifications, 2015.

[8] B. Jackson and N. C. Wormald. $k$-walks of graphs. *Australas. J. Combin.*, 2:135–146, 1990. Combinatorial mathematics and combinatorial computing, Vol. 2 (Brisbane, 1989).

[9] A. Kabela and T. Kaiser. 10-tough chordal graphs are hamiltonian (extended abstract). *Electronic Notes in Discrete Mathematics*, 49:331–336, 2015.
[10] T. Kaiser, Z. Ryjáček, D. Král, M. Rosenfeld, and H.-J. Voss. Hamilton cycles in prisms. *J. Graph Theory*, 56(4):249–269, 2007.
[11] J. Teska. On 2-walks in chordal planar graphs. *Discrete Math.*, 309(12):4017–4026, 2009.
[12] W. T. Tutte. A theorem on planar graphs, transactions of the american mathematical society, 82. *Transactions of the American Mathematical Society*, pages 275–284, 1956.
[13] H. J. Veldman. Existence of dominating cycles and paths. *Discrete Math.*, 43(2-3):281–296, 1983.
[14] K. Yoshimoto. Edge degrees and dominating cycles. *Discrete Mathematics*, 308(12):25942599, 2008.

School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore
E-mail address: gaom0002@e.ntu.edu.sg