Kinetic theory of point vortex systems
from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy

Mitsusada M. Sano
Graduate School of Human and Environmental Studies,
Kyoto University,
Sakyo, Kyoto 606-8501, Japan
(Dated: February 1, 2008)

Kinetic equations are derived from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy for point vortex systems in an infinite plane. As the level of approximation for the Landau equation, the collision term of the kinetic equation derived coincides with that by Chavanis (Phys. Rev. E 64, 026309 (2001)). Furthermore, we derive a kinetic equation corresponding to the Balescu-Lenard equation for plasmas, using the theory of the Fredholm integral equation. For large $N$, this kinetic equation is reduced to the Landau equation above.

PACS numbers: 47.32.C-, 05.20.Dd, 05.20.Jj, 05.20.-y

I. INTRODUCTION

A point vortex system is a model of continuous two-dimensional (2D) inviscid fluid dynamics. In fact, a point vortex is an idealized vortex of real vortex in 2D fluids. However, the point vortex system carries some important properties of continuous 2D inviscid fluid dynamics. It is formulated as a Hamilton dynamical system. Therefore, a standard statistical mechanical theory was developed by Onsager. In his formulation, states of a point vortex system are classified into two categories: one is positive temperature states, and the other is negative temperature states. In the positive temperature states, point vortices are distributed in a scattered way. However, in the negative temperature states, huge vortices are formed in time-evolution. The negative temperature states are phenomena observed in earth's surface. That is a formation of huge vortices, such as hurricanes and typhoons. Onsager's ideas are recently reviewed by Eyink and Sreenivasan. Since Onsager, researchers considered the equilibrium statistical mechanics of point vortex systems. The main purpose of their studies is to construct the equilibrium states of point vortex system. The Poisson-Boltzmann equation is used to obtain the equilibrium states. For some cases, the equilibrium states are analytically obtained.

The next stage of the statistical mechanics of point vortex systems focuses on nonequilibrium properties. Standard nonequilibrium statistical mechanics goes toward kinetic theory. Two authors have studied kinetic theory of point vortex system, i.e., by Marmanis and by Chavanis. Marmanis considered a gas of binary pairs of positive and negative point vortices. Chavanis considered a gas of point vortices with the same circulation. In this paper, we are interested in non-neutral plasma in the experimental situation, as an idealized model, we should consider a point vortex system in a circular domain. From experiments on non-neutral plasmas, many interesting properties of vortex dynamics of the 2D Euler equation are now known: (1) Diocotron instability (i.e., in other words, Kelvin-Helmholtz instability), (2) Violent relaxation, (3) Slow decay, (4) Vortex crystals, and (5) Merger of vortices. As a theoretical aspect, recently the slow decay was numerically analyzed using the point vortex system. Although there is a difference in the boundary condition, these properties introduced here are common in the point vortex systems in an infinite plane.

Back to the point vortex systems in an infinite plane, Chavanis derived several kinetic equations for the point vortex system in an infinite plane, in which the point vortices have the same circulation, and estimated interesting physical quantities, like the diffusion coefficient and the drift term, by using his kinetic equations. In this paper, we develop a kinetic theory through the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. The organization of this paper is as follows: (1) We derive a kinetic equation (i.e., the Landau equation), which is identical to that by Chavanis. (2) Furthermore, we derive a kinetic equation (i.e., the Balescu-Lenard equation), which includes more correlation, i.e., the collective effects. This is a new kinetic equation. (3) We show that for large $N$, this kinetic equation is reduced to the Landau equation.

The organization of this paper is as follows. In §III the equations of motion for a point vortex system in an infinite plane are presented. In §IV the treatment of the BBGKY hierarchy is shown. Two key equations for time-evolution of the distribution function and the correlation function are derived. Using these equations, we derive the Vlasov equation, the Landau equation and the Balescu-Lenard equation for point vortex system in an infinite plane. It is shown that for large $N$, the Balescu-Lenard equation is reduced to the Landau equation. In
we summarize the results of this paper and give
future problems.

II. EQUATIONS OF MOTION

Let us consider a point vortex system, which consists of \( N \) point vortices with the same circulation \( \gamma \) in an
infinite plane. The hamiltonian of this system is given
by
\[
H = -\frac{\gamma^2}{4\pi} \sum_{i \neq j}^{N} \ln |\mathbf{r}_i - \mathbf{r}_j|, \tag{1}
\]
where \( \mathbf{r}_i = (x_i, y_i) \). The equations of motion are written
by using the hamiltonian:
\[
\gamma \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \gamma \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}. \tag{2}
\]
Thus, the velocities of the \( i \)th point vortex in the \( x \)- and
\( y \)-directions are given by
\[
v_i^{(x)} = \frac{\gamma}{2\pi} \sum_{j \neq i}^{N} \frac{(\mathbf{r}_i - \mathbf{r}_j)_y}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{3}
\]
\[
v_i^{(y)} = -\frac{\gamma}{2\pi} \sum_{j \neq i}^{N} \frac{(\mathbf{r}_i - \mathbf{r}_j)_x}{|\mathbf{r}_i - \mathbf{r}_j|^2}. \tag{4}
\]
It is convenient to rewrite the velocity in the following form.
\[
\mathbf{v}_i = \sum_{j \neq i} \mathbf{v}(j \rightarrow i), \tag{5}
\]
where
\[
\mathbf{v}(j \rightarrow i) = -\frac{\gamma}{2\pi} z \times \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} = \frac{\gamma}{2\pi} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^2} J \times \mathbf{r}_{ij}, \tag{6}
\]
\( J \) is the \( 2 \times 2 \) symplectic matrix
\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \tag{7}
\]
\( z \) is the unit vector along the z-axis, and \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \).

This system have several conserved quantities: (1) Energy, i.e., \( H = E \). (2) Angular impulse, i.e., \( I = \gamma \sum_{i=1}^{N} |\mathbf{r}_i|^2 \). (3) Linear impulse, i.e., \( \mathbf{L} = \gamma \sum_{i=1}^{N} \mathbf{r}_i \).

Now we define the vorticity \( \omega(\mathbf{r}; t) \),
\[
\omega(\mathbf{r}; t) = \sum_{i=1}^{N} \gamma \delta(\mathbf{r} - \mathbf{r}_i), \tag{8}
\]
and the stream function \( \psi(\mathbf{r}; t) \),
\[
\psi(\mathbf{r}; t) = -\frac{\gamma}{2\pi} \sum_{i=1}^{N} \ln |\mathbf{r} - \mathbf{r}_i|. \tag{9}
\]
Using the stream function, the velocity of the \( i \)th point
vortex is given by
\[
\mathbf{v}_i = -z \times \nabla \psi(\mathbf{r} = \mathbf{r}_i; t) = \mathbf{J} \cdot \nabla \psi(\mathbf{r} = \mathbf{r}_i; t). \tag{10}
\]
It is easily confirmed that the vorticity satisfies the two-
dimensional Euler equation.
\[
\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = 0. \tag{11}
\]

III. THE BBGKY HIERARCHY

We define the distribution function of \( N \)-point vortex
systems:
\[
\mathbf{F} = \mathbf{F}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N; t). \tag{12}
\]
The Liouville equation for \( N \)-point vortex systems in an
infinite plane is given by
\[
\frac{\partial \mathbf{F}}{\partial t} = \mathcal{L} \mathbf{F}, \tag{13}
\]
where
\[
\mathcal{L} = \sum_{i=1}^{N} \frac{1}{\gamma} \left( \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} + \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} \right). \tag{14}
\]
The Liouvillian \( \mathcal{L} \) is rewritten as follows.
\[
\mathcal{L} = -\sum_{i=1}^{N} \sum_{j \neq i} \left( v_x(j \rightarrow i) \frac{\partial}{\partial y_i} + v_y(j \rightarrow i) \frac{\partial}{\partial x_i} \right)
\]
\[
= -\sum_{i<j} \mathbf{v}(j \rightarrow i) \cdot \nabla_{ij}, \tag{15}
\]
where \( \nabla_i = \frac{\partial}{\partial x_i} \) and \( \nabla_{ij} = \nabla_i - \nabla_j \). Now we have used
the fact that \( \mathbf{v}(j \rightarrow i) = -\mathbf{v}(i \rightarrow j) \). We define \( \mathcal{L}_{ij} \) as
\[
\mathcal{L}_{ij} = -\mathbf{v}(j \rightarrow i) \cdot \nabla_{ij}
\]
\[
= -\mathbf{v}(j \rightarrow i) \cdot \nabla_i - \mathbf{v}(i \rightarrow j) \cdot \nabla_j. \tag{16}
\]
Thus, the Liouvillian becomes
\[
\mathcal{L} = \sum_{i<j} \mathcal{L}_{ij}. \tag{17}
\]
We define the \( s \)-body reduced distribution function:
Reduced distribution function and the (g
Now we use the 2-body and 3-body correlation functions:
Thus, the correlation function describes the deviation evolution equations for the s-body reduced distribution function $f_s(r_1, \ldots, r_s)$:
\[
\partial_t f_0 = 0, \quad (19)
\]
and
\[
\partial_t f_s(r_1, \ldots, r_s) = \sum_{i<j} \mathcal{L}_{ij} f_s(r_1, \ldots, r_s)
+ \sum_{i=1}^s \int dr_{s+1} \mathcal{L}_{i,s+1} f_{s+1}(r_1, \ldots, r_{s+1}). \quad (20)
\]
This is the BBGKY hierarchy for the s-body reduced distribution functions. The time-evolution of the s-body reduced distribution function is determined by the s-body reduced distribution function and the (s + 1)-body reduced distribution function.

For $s = 1$, we have
\[
\partial_t f_1(r_1) = \int dr_2 \mathcal{L}_{12} f_2(r_1, r_2). \quad (21)
\]

Now we use the 2-body and 3-body correlation functions: $g_2(r_1, r_2)$ and $g_3(r_1, r_2, r_3)$. The 2- and 3-body correlation functions are related to the 2- and 3-body reduced distribution functions as follows:
\[
f_2(r_1, r_2) = f_1(r_1) f_1(r_2) + g_2(r_1, r_2). \quad (22)
\]
\[
f_3(r_1, r_2, r_3) = f_1(r_1) f_2(r_2) f_1(r_3) + f_1(r_1) g_2(r_2, r_3)
+ f_1(r_2) g_2(r_1, r_3) + f_1(r_3) g_2(r_1, r_2)
+ g_3(r_1, r_2, r_3). \quad (23)
\]

Thus, the correlation function describes the deviation from the product of the reduced distribution functions. Inserting eq. (22) into eq. (21), we obtain
\[
\partial_t f_1(r_1) = \int dr_2 \left[ \mathcal{L}_{12} f_2(r_1, r_2) + \mathcal{L}_{12} g_2(r_1, r_2) \right]. \quad (24)
\]

Inserting eq. (22) and eq. (23) into eq. (20), the time-evolution equations of $f_2(r_1, r_2)$ is obtained:
\[
\begin{align*}
\partial_t f_2(r_1, r_2) & = \mathcal{L}_{12} [ f_1(r_1) f_1(r_2) + g_2(r_1, r_2) ]
+ \int dr_3 \left\{ \mathcal{L}_{13} [ f_1(r_1) f_1(r_2) f_1(r_3) \
+ f_1(r_1) g_2(r_2, r_3) + f_1(r_2) g_2(r_1, r_3) \
+ f_1(r_3) g_2(r_1, r_2) + g_3(r_1, r_2, r_3) + (1 \leftrightarrow 2) \right\}. \quad (25)
\end{align*}
\]

Similarly, using eq. (24), inserting eq. (22) into eq. (25), and manipulating the resulting equations, the time-evolution equation of $g_2(r_1, r_2)$ is obtained:
\[
\begin{align*}
\partial_t g_2(r_1, r_2) & = \mathcal{L}_{12} f_1(r_1) f_1(r_2) + \mathcal{L}_{12} g_2(r_1, r_2) \
& + \int dr_3 \left\{ \mathcal{L}_{13} f_1(r_1) g_2(r_2, r_3) + \mathcal{L}_{23} f_1(r_2) g_2(r_1, r_3) \
+ (\mathcal{L}_{13} + \mathcal{L}_{23}) [ f_1(r_3) g_2(r_1, r_2) + g_3(r_1, r_2, r_3) ] \right\}. \quad (26)
\end{align*}
\]

Equation (26) is slightly different from the recent result by Chavanis [17], who also developed a BBGKY hierarchy for the point vortex gas. The reason for these differences is unknown.

Following [17], we shall close the hierarchy of BBGKY equations by considering an expansion in powers of $1/N$ for $N \to +\infty$. In the large $N$-limit, i.e., the hydrodynamic limit, we should preserve the total circulation $\Gamma$.

Thus, the circulation $\gamma$ is $\gamma = \Gamma/N$, where $\Gamma = \text{const}$. The order estimate of each function is $g_2(r_1, r_2) \sim 1/N$, $g_3(r_1, r_2, r_3) \sim 1/N^2$, and $f_2(r_1, r_2) \sim 1/N^3$. In the following treatment, the function $g_3$ is omitted, since we cut the correlation, i.e., truncate a chain of the BBGKY hierarchy.

A. Vlasov equation

A mean field approximation is performed by neglecting the term of the correlation function in eq. (24):
\[
\partial_t f(r_1) = \int dr_2 \mathcal{L}_{12} f(r_1) f(r_2). \quad (27)
\]

Using eq. (16), we obtain
\[
\frac{\partial f(r_1)}{\partial t} + (v_1) \cdot \nabla f(r_1) = 0, \quad (28)
\]
where
\[
(v_1) = \int dr_2 f(r_2)(v_2(2 \to 1)i + v_y(2 \to 1)j). \quad (29)
\]

This is a mean field equation for the point vortex system in an infinite plane. It is analogous to the Vlasov equation in plasma physics and in stellar dynamics. We should note that this equation, eq. (28), is nothing but the 2D Euler equation.
B. Landau equation

The next higher order approximation is started with eq. (24) preserving the correlation function:

$$\partial_t f(r_1) = \int d\mathbf{r}_2 \left( \mathcal{L}_{12} f(r_1) f(r_2) + \mathcal{L}_{12} g_2(r_1, r_2) \right). \quad (30)$$

The right hand side of eq. (30) is up to the order $1/N^2$. For the correlation function $g_2(r_1, r_2)$, we approximate eq. (26) up to the order $1/N$.

$$\begin{align*}
\partial_t g_2(r_1, r_2) &= \mathcal{L}_{12} f(r_1) f(r_2) \\
&+ \int d\mathbf{r}_3 (\mathcal{L}_{13} + \mathcal{L}_{23}) f(r_3) g_2(r_1, r_2) \\
&= \mathcal{L}_{12} f(r_1) f(r_2) + \int d\mathbf{r}_3 (\mathcal{L}_{13} + \mathcal{L}_{23}) f(r_3) g_2(r_1, r_2).
\end{align*} \quad (31)$$

Therefore, the correlation function is advected by $\langle \mathbf{v}_1 \rangle$ and $\langle \mathbf{v}_2 \rangle$. Equation (31) is formally solved as

$$g_2(r_1, r_2; t) = U_{12}(t) g_2(r_1, r_2; 0) + \int_0^t d\tau U_{12}(\tau) \mathcal{L}_{12} f(r_1; t - \tau) f(r_2; t - \tau). \quad (32)$$

where

$$U_{12}(\tau) = \exp \left[ -\int_0^\tau dt' \langle \mathbf{v}_1 \rangle \cdot \nabla_1 - \int_0^\tau dt' \langle \mathbf{v}_2 \rangle \cdot \nabla_2 \right]. \quad (33)$$

Inserting eq. (32) into eq. (30), we obtain

$$\partial_t f(r_1) = \int d\mathbf{r}_2 \mathcal{L}_{12} f(r_1) f(r_2) + \int d\mathbf{r}_2 \mathcal{L}_{12} U_{12}(t) g_2(r_1, r_2; 0)$$

$$+ \int d\mathbf{r}_2 \int_0^t d\tau \mathcal{L}_{12} U_{12}(\tau) \mathcal{L}_{12} f(r_1; t - \tau) f(r_2; t - \tau). \quad (34)$$

In the right hand side of eq. (34), the second term vanishes for large $t$ (i.e., the correlation decays), thus we have

$$\partial_t f(r_1) = \int d\mathbf{r}_2 \mathcal{L}_{12} f(r_1) f(r_2) + \int d\mathbf{r}_2 \int_0^t d\tau \mathcal{L}_{12} U_{12}(\tau) \mathcal{L}_{12} f(r_1; t - \tau) f(r_2; t - \tau). \quad (35)$$

We have to evaluate the following term:

$$\mathcal{K}_{\text{coll}}^{(L)}(f f) = \int d\mathbf{r}_2 \int_0^t d\tau \mathcal{L}_{12} U_{12}(\tau) \mathcal{L}_{12} f(r_1; t - \tau) f(r_2; t - \tau). \quad (36)$$

Using the approximation,

$$f(r_1; t - \tau) f(r_2; t - \tau) \approx U_{12}(-\tau) f(r_1; t) f(r_2; t), \quad (37)$$

we obtain

$$\mathcal{K}_{\text{coll}}^{(L)}(f f) \approx \int d\mathbf{r}_2 \int_0^t d\tau \mathcal{L}_{12} U_{12}(\tau) \mathcal{L}_{12} f(r_1; t) f(r_2; t) = \nabla_1 \cdot \int d\mathbf{r}_2 \int_0^t d\tau f_1(t) \mathbf{v}_1(t - \tau) \nabla_1 f(r_1; t) f(r_2; t). \quad (38)$$

The kinetic equation obtained here is

$$\frac{\partial f(r_1)}{\partial t} + \langle \mathbf{v}_1 \rangle \cdot \nabla_1 f(r_1) = \nabla_1 \cdot \int d\mathbf{r}_2 \int_0^t d\tau f_1(t) \mathbf{v}_1(t - \tau) \cdot \nabla_1 f(r_1; t) f(r_2; t), \quad (39)$$

where $\mathbf{v}_1(t)$ is advected as $\mathbf{v}_1(t - \tau) = U_{12}(\tau) \mathbf{v}_1(t) U_{12}(-\tau)$ and $\mathbf{r}_1(t - \tau) = \mathbf{r}_1(t) - \int_0^\tau dt' \langle \mathbf{v}_1 \rangle(t) (t - t'), t - t'$. This equation is analogous to the Landau equation in plasma physics and in stellar dynamics. This equation coincides with the result of \[15\] \[17\]. As shown in \[13\], this equation conserves the angular impulse and the linear impulse. If we use the Markovianization, i.e., extending the time integral to infinity, we obtain

$$\mathcal{K}_{\text{coll}}^{(L)}(f f) \approx \nabla_1 \cdot \int d\mathbf{r}_2 \int_0^\infty d\tau f_1(t) \mathbf{v}_1(t - \tau) \cdot \nabla_1 f(r_1; t) f(r_2; t). \quad (40)$$

However, it is not known whether the Markovianization is assured or not, since point vortex dynamics sometimes gives long-time tail, i.e., the strong correlation. In par-
ticular, in [31], it is shown that the diffusion process for the point vortex exhibits Lévy flight.

Chavanis estimated the relaxation time $t_{\text{relax}}$ by using the estimate of the diffusion coefficient $D$ as $t_{\text{relax}} \sim N/(\ln N) t_D$, where the dynamical time is $t_D \sim (\omega)^{-1} \sim R^2/\Gamma$, which is the time determined by the mean rotation time, and $R$ is the size of the vortex. His estimate of $t_{\text{relax}}$ would be incorrect. In the kinetic theory, the $N$-dependence of $t_{\text{relax}}$ is determined by the $N$-dependence of the collision term, i.e., $K_{\text{coll}}(k) \sim O(1/N)$. This gives $t_{\text{relax}} \sim N t_D$. Recently Chavanis and Lemou used this estimate[16, 17]. This estimate is consistent with the numerical result by Kawahara and Nakanishi for the system in a circular domain[31].

C. Balescu-Lenard equation

In this subsection, we derive a kinetic equation for point vortex systems in an infinite plane, which is analogous to the Balescu-Lenard equation in plasma physics. The starting point is the time-evolution equations of the one-body reduced distribution function $f(r_1)$ and the two-body correlation function $g_2(r_1, r_2)$.

$$
\partial_t f(r_1) = \int dr_2 \left( L_{12} f(r_1) f(r_2) + L_{12} g_2(r_1, r_2) \right),
$$

and

$$
\partial_t g_2(r_1, r_2) = L_{12} f(r_1) f(r_2) + \int dr_3 \left( L_{13} f(r_1) g_2(r_2, r_3) + L_{23} f(r_2) g_2(r_1, r_3) \right) + (L_{13} + L_{23}) [f(r_3) g_2(r_1, r_2)].
$$

The two-body correlation function is formally solved as

$$
g_2(r_1, r_2; t) = \int_0^t dt U_{12}(\tau) \left( L_{12} U_{12}(\tau) f(r_1) f(r_2) + L_{12} U_{12}(\tau) g_2(r_1, r_2) \right)
+ \int dr_3 \left( L_{13} U_{12}(\tau) f(r_1) g_2(r_2, r_3) + L_{23} U_{12}(\tau) f(r_2) g_2(r_1, r_3) \right)
$$

The kinetic equation is formally obtained as

$$
\frac{\partial f_1}{\partial t} + \langle v_1 \rangle \cdot \nabla f_1 = \int dr_2 L_{12} g_2(r_1, r_2; t). \quad (44)
$$

If, as done for plasma systems in [29], we set

$$
g_2(r_1, r_2; t) = g_2(r_1 - r_2; t) = \int dk \exp[i k \cdot (r_1 - r_2)] \tilde{g}_2(k; t), \quad (45)
$$

the right hand side of eq. (44), i.e., the collision term, vanishes.

$$
\int dr_2 L_{12} g_2(r_1, r_2; t)
= \frac{1}{\gamma} \int dr_2 \left( \nabla_1 V(r_1 - r_2) \right)^\top \cdot J \cdot \nabla_1 g_2(r_1, r_2)
= \frac{1}{\gamma} \int dr_2 \int dk \exp[i k \cdot (r_1 - r_2)] \tilde{V}(k) \left( i k \right)^\top \cdot J \nabla_1
\times \int dk' \exp[i k' \cdot (r_1 - r_2)] \tilde{g}_2(k'; t)
= \frac{(2\pi)^2}{\gamma} \int dk \tilde{V}(k) (i k)^\top \cdot J \cdot k \tilde{g}_2(-k; t)
= 0,
$$

(46)

since $k^\top \cdot J \cdot k = 0$. Here “$A^\top$” means the transpose of the vector $A$. In eq. (45), the homogeneity is assumed.

The above result shows that inhomogeneity is important for point vortex systems. To not make the collision term vanish, we change the definition of the Fourier transform of $g_2(r_1, r_2; t)$. Therefore, we set

$$
g_2(r_1, r_2; t) = \int dk \int dk_2 \exp[i k_1 \cdot r_1 + i k_2 \cdot r_2] \tilde{g}_2(k_1, k_2; t) \quad (47)
$$

$L_{12}$ can be rewritten in the form

$$
L_{12} = \frac{1}{\gamma} (\nabla_1 V(r_1 - r_2))^\top \cdot J \cdot \nabla_1, \quad (48)
$$

where

$$
V(r_1 - r_2) = -\frac{\gamma^2}{2\pi} \ln |r_1 - r_2|. \quad (49)
$$

Now we consider the Fourier transform of the function $V(r)$, where $r = r_1 - r_2$:

$$
V(r) = \int dk \tilde{V}(k) e^{ik \cdot r},
\tilde{V}(k) = \frac{1}{(2\pi)^2} \int dr V(r) e^{-ik \cdot r}. \quad (50)
$$

The Fourier transform of $V(r)$ is evaluated as follows:

$$
\tilde{V}(k)
$$
In fact, for large fying its absolute value. Therefore, we suppose that the

Now we evaluate each term in the right hand side of eq.(43).

$$- = \frac{-\gamma^2}{(2\pi)^3} \int \ln |r| e^{-ikr}$$

$$- = - \frac{\gamma^2}{(2\pi)^3} \int_0^\infty r \, dr \int_0^{2\pi} d\theta \, ln r e^{-ikr \cos(\theta)}$$

$$- = - \frac{\gamma^2}{(2\pi)^2} \int_0^\infty \ln r J_0(kr)$$

$$- = - \frac{\gamma^2}{(2\pi)^2} \left\{ \left[ \frac{r}{k} \ln r J_1(kr) \right]_{-1} - \frac{1}{k} \int_0^\infty \ln J_1(kr) \right\} 51$$

We have to evaluate the following limit:

$$\lim_{r \to \infty} \frac{r}{k} \ln r J_1(kr) \quad 52$$

In fact, for large r, this function oscillates with amplifying its absolute value. Therefore, we suppose that the limiting value of this function is zero. Alternatively, we insert a convergence factor:

$$\lim_{\epsilon \to 0} \int_0^\infty r \ln r J_0(kr) e^{-\epsilon r}. \quad 53$$

As a result, we have

$$\tilde{V}(k) = \frac{\gamma^2}{(2\pi)^2} \frac{1}{k^2}. \quad 54$$

The dependence of \(\tilde{V}(k) \sim 1/k^2\) is a typical behavior of Coulomb systems.

Now we evaluate each term in the right hand side of eq.(43).

\[ (A) = \int_0^t d\tau \, U_{12}(\tau) \mathcal{L}_{12} U_{12}(-\tau) f_1 f_2 \]

\[ = - \int_0^t d\tau \, v_1(t - \tau) \cdot \nabla_{12} f_1 f_2 \]

\[ = \frac{1}{\gamma} \int_0^t d\tau \int dk \exp \left[ ik \cdot (r_1 - r_2) - ik \cdot \int^\tau dt' \langle v_1 \rangle + ik \cdot \int^\tau dt' \langle v_2 \rangle \right] \tilde{V}(k)(ik) \cdot J \cdot \nabla_{12} f_1 f_2. \quad 55 \]

\[ (B) = \int_0^t d\tau \, U_{12}(\tau) \mathcal{L}_{12} U_{12}(-\tau) g_2(r_1, r_2; t) \]

\[ = - \int_0^t d\tau \, v_1(t) \cdot \nabla_{12} g_2(r_1, r_2; t) \]

\[ = \frac{1}{\gamma} \int_0^t d\tau \int dk \exp \left[ ik \cdot (r_1 - r_2) - ik \cdot \int^\tau dt' \langle v_1 \rangle + ik \cdot \int^\tau dt' \langle v_2 \rangle \right] \tilde{V}(k)(ik) \cdot J \cdot \nabla_{12} g_2(r_1, r_2; t). \quad 56 \]

\[ (C) = \int_0^t d\tau \, U_{12}(\tau) \int dr_3 \mathcal{L}_{13} U_{123}(-\tau) f_1 g_2(r_2, r_3; t) \]

\[ = \frac{1}{(2\pi)^2 \gamma} \int dR \int dR' \int_0^t d\tau \int dk \int dk' \times \exp \left[ ik \cdot (r_1 - R') + ik' \cdot (r_2 - R) - ik \cdot \int^\tau (\langle v_1 \rangle - \langle V' \rangle) dt' \right] \times \tilde{V}(k)(ik) \cdot J \cdot \nabla_{12} f_1 g_2(R, R'; t). \quad 57 \]

\[ (D) = \int_0^t d\tau \, U_{12}(\tau) \int dr_3 \mathcal{L}_{23} U_{123}(-\tau) f_2 g_2(r_1, r_3; t) \]

\[ = \frac{1}{(2\pi)^2 \gamma} \int dR \int dR' \int_0^t d\tau \int dk \int dk' \times \exp \left[ ik' \cdot (r_1 - R) + ik \cdot (r_2 - R') - ik \cdot \int^\tau (\langle v_2 \rangle - \langle V' \rangle) dt' \right] \times \tilde{V}(k)(ik) \cdot J \cdot \nabla_{2} g_2(R, R'; t). \quad 58 \]
The order estimate of these terms is as follows.

\[
(A) \sim \frac{1}{N}, \quad (B) \sim \frac{1}{N^2}, \quad (C) \sim \frac{1}{N}, \quad (D) \sim \frac{1}{N^3}
\]

(59)

Therefore, we can neglect the term of \((B)\).

Then we obtain the following integral equation:

\[
g_2(r_1, r_2; t) = q(r_1, r_2; t) + \int dR \int dR' K([r_1, r_2], [R, R'])g_2(R, R'; t),
\]

where

\[
q(r_1, r_2; t) = \frac{1}{\gamma} \int_0^t \int dk \exp \left[ ik \cdot (r_1 - r_2) - \frac{\gamma}{2} k^2 t \right] dt \langle v_1 \rangle + ik \cdot \int_t^\infty dt' \langle v_2 \rangle
\]

\[
\times \tilde{V}(k)(ik)^T \cdot J \cdot \nabla_{12} f_1 f_2,
\]

and

\[
K([r_1, r_2], [R, R']) = \frac{1}{(2\pi)^2 \gamma} \int_0^t \int dk \int dk' \exp \left[ ik \cdot (r_1 - R') + ik' \cdot (r_2 - R) - \frac{\gamma}{2} k^2 t \right] dt' \langle v_1 \rangle - \langle V' \rangle dt'
\]

\[
\times \tilde{V}(k)(ik)^T \cdot J \cdot \nabla_{12} f_1 + \frac{1}{(2\pi)^2 \gamma} \int_0^t \int dk \int dk' \exp \left[ ik' \cdot (r_1 - R) + ik \cdot (r_2 - R') - \frac{\gamma}{2} k^2 t \right] dt' \langle v_2 \rangle - \langle V' \rangle dt'
\]

\[
\times \tilde{V}(k)(ik)^T \cdot J \cdot \nabla_{22} f_2.
\]

(61)

The function \(K([r_1, r_2], [R, R'])\) is called the integral kernel of the integral equation. This integral equation takes the form of the Fredholm integral equation of the second kind. Thus, how to solve it is known[33]. Now for brevity, we set \(x = (r_1, r_2)\) and \(y = (R, R')\). The integral equation, which should be solved, is

\[
g_2(x; t) = q(x; t) + \lambda \int dy \ K(x, y)g_2(y; t).
\]

(62)

(63)

If the required conditions are satisfied, this integral equation is solved as

\[
g_2(x; t) = q(x; t) + \int dy \ \Xi(x, y; \lambda_0)q(y; t).
\]

(64)

The function \(\Xi(x, y; \lambda)\) is called the resolvent. \(\lambda_0\) is chosen to make the series convergent. For large \(N\), the kernel \(K(x, y)\) is \(\sim 1/N\). Thus, for large \(N\), we can take as \(\lambda_0 = 1\). Therefore, if the kernel is bounded, for large \(N\), we get a convergent series. The resolvent is given by

\[
\Xi(x, y; \lambda) = \frac{D(x, y; \lambda)}{D(\lambda)},
\]

(65)

where

\[
D(\lambda) = 1 - \lambda \int ds K(s, s) + \frac{\lambda^2}{2!} \int ds_1 ds_2 K(s_1, s_2) + \cdots + \frac{(-\lambda)^p}{p!} \int \cdots \int ds_1 \cdots ds_p K(x, s_1, \ldots, y, s_p) + \cdots
\]

and

\[
D(x, y; \lambda) = K(x, y) - \lambda \int ds K(x, s) + \frac{\lambda^2}{2!} \int ds_1 ds_2 K(x, s_1, y, s_2) + \cdots + \frac{(-\lambda)^p}{p!} \int \cdots \int ds_1 \cdots ds_p K(x, s_1, \ldots, y, s_p) + \cdots
\]

(66)

(67)
Thus, a derived kinetic equation is

\[ \frac{\partial f_1}{\partial t} + \langle \mathbf{v}_1 \rangle \cdot \nabla f_1 = \int d\mathbf{r}_2 L_{12} g_2(\mathbf{r}_1, \mathbf{r}_2; t), \]  

(69)

where

\[ g_2(\mathbf{r}_1, \mathbf{r}_2; t) = q(\mathbf{r}_1, \mathbf{r}_2; t) + \int d\mathbf{R} \int d\mathbf{R}' \Xi(\{\mathbf{r}_1, \mathbf{r}_2\}; \{\mathbf{R}, \mathbf{R}'\}; \lambda_0) q(\mathbf{R}, \mathbf{R}'; t). \]

(70)

The formal solution is obtained by the expansion.

\[ g_2 = \sum_{n=0}^{\infty} (\lambda T)^n q. \]

(74)

The Fredholm theory for integral equation basically uses the boundedness of the operator and the expansion, i.e., eq. (74). This expansion is assured by the boundedness of the operator \( T \). The expansions of eqs. (66) and (67) are also due to the boundedness of the operator \( T \). As the second point, unlike usual Fredholm integral equations, the integral domain in eq. (63) is infinite. If the integral domain is finite and the kernel is bounded, the boundedness of operators is easily shown. For our case, since the integral domain is infinite, we should treat the operator \( T \) carefully. These two points, i.e., the boundedness of the operator \( T \) and the infinite integral domain, should be checked and be treated in a rigorous way. However, in this paper, we do not pursue a rigorous discussion. These problems are reserved for mathematical physicists.

IV. CONCLUDING REMARK

We have derived a kinetic equation for point vortex systems in an infinite plane. The kinetic equations derived are analogues of the Landau equation and the Balescu-Lenard equation. Equation (69) coincides with the result of Chavanis [13, 17]. Equation (69) possesses several interesting properties, which were shown in [13]. The Balescu-Lenard equation (69) is a new kinetic equation. Furthermore, we have shown that for large \( N \), the Balescu-Lenard equation, i.e., eq. (69) is reduced to the Landau equation, i.e., eq. (69). Therefore, we can conclude that for point vortex systems in an infinite plane, without symmetrical restriction (such as axisymmetric and unidirectional flows), the most generalized kinetic equation is eq. (69).

The following point would be the interesting point of the derived kinetic equation, i.e., eq. (69): The interac-
tion among point vortices is long range, i.e., logarithmic. In addition, the derived kinetic equation is analogous to the Landau equation for 3D plasmas. However, the collision term for point vortex systems may not diverge. In [17], for the axisymmetric case, the collision term does not diverge. This is a symptom of the non-divergence of the collision term for the Landau equation. The reason of this is that the difference between the integration of the collision term, i.e., eq. (38), and that of the Landau collision term for 3D plasmas. The former has the integration with respect to $r_2$ (i.e., position), while the latter has the integration with respect to $r_2$ and $v_2$ (i.e., velocity).

For the point vortex systems, the expression of the energy spectrum was derived for the system in an infinite plane[34] and for the system in a circular domain[35]. The energy spectrum is closely related to the diffusion coefficient[36]. We will be able to compare the theory of [36] with the kinetic theory in this paper.

Another interesting point is the following: our kinetic theory is not directly connected to Onsager’s temperature. As shown in [37] for two-sign point vortex systems, Onsager’s temperature affects nonequilibrium properties, i.e., decaying process. To find this connection leads to understand nonequilibrium properties, i.e., classification of nonequilibrium processes.

The most interesting problem, which we would like to attack with eq. (39), is a decaying property of vortex crystals in non-neutral plasmas. Unfortunately, eq. (39) is for the system in an infinite plane, not for the system in a circular domain. Thus, it is not for an experimental situation. But, eq. (39) surely captures the nature of phenomena for the system in a circular domain in some extent. Many experimental results show that the vortex crystals are quasi-stationary states. To analyze quasi-stationary states, recent advances for long-range interaction systems would be some hints for us, such as a study of the Hamiltonian mean field (HMF) model. The HMF exhibits a slow decay, in which the state is stuck in a quasi-stationary state, as well as in point vortex systems. For the HMF model, the slow decay is analyzed by the Vlasov equation. The quasi-stationary state is very near to a stable solution of the Vlasov equation. The estimate of some quantities, i.e., the algebraic decay and the tail of the velocity distribution function etc., is carried out[38], and is tested by a numerical simulation[39]. Their analysis would be useful for our problems. However, to this end, we should know the behavior of the 2D Euler equation, instead that of the Vlasov equation for usual kinetic theory of particle systems.

**Acknowledgments**

The author thanks Professor H. Tomita for continuous encouragements and Professor Y. Kiwamoto for introducing him vortex dynamics. The author is grateful to one of anonymous referees for numerous advices to the first manuscript and for letting the author know the reference [17]. Some important changes have been made between the first version and the second one using the result of [17].

[1] A. J. Chorin: *Vorticity and Turbulence* (Springer-Verlag, Berlin, 1994).
[2] P. K. Newton: *The N-Vortex Problem: Analytical Techniques* (Springer, New York, 2000).
[3] H. von Helmholtz: *Phil. Mag.* 4(33), 485 (1858).
[4] G. R. Kirchhoff: *Vorlesungen ¨uber Mathematische Physik* I, (Teubner, Leipzig, 1876).
[5] L. Onsager: *Nuovo Cimento Suppl.* 6, 279 (1949).
[6] G. L. Eyink and K. R. Sreenivasan: *Rev. Mod. Phys.* 78, 87 (2006).
[7] S. Kida: *J. Phys. Soc. Jpn.* 39, 1395 (1975).
[8] G. Joyce and D. Montgomery: *J. Plasma Phys.* 10, 107 (1973).
[9] Y. B. Pointin and T. S. Lundgren: *Phys. Fluids.* 19, 1459 (1976).
[10] T. S. Lundgren and Y. B. Pointin: *J. Stat. Phys.* 17, 323 (1977).
[11] C. E. Seyler, Jr.: *Phys. Fluids* 19, 1336 (1976).
[12] P. A. Smith and T. M. O’Neil: *Phys. Fluids* B2(12), 2961 (1990).
[13] A. C. Ting, H. H. Chen and Y. C. Lee: *Physica D* 26, 37 (1987).
[14] H. Marmanis: *Proc. Roy. Soc. Lond.* A454, 587 (1998).
[15] P. H. Chavanis: *Phys. Rev. E* 64, 026309 (2001).
[16] P. H. Chavanis and M. Lemou: *cond-mat/0703023* (2007).
[17] P. H. Chavanis: *arXiv:0704.3953* (2007).
[18] C. F. Driscoll and K. S. Fine: *Phys. Fluids B* 2, 1359 (1995).
[19] K. S. Fine, A. C. Cass, W. G. Flynn, and C. F. Driscoll: *Phys. Rev. Lett.* 75, 3277 (1995).
[20] D. Z. Jin and D. H. E. Dubin: *Phys. Rev. Lett.* 80, 4434 (1998).
[21] D. Schecter, D. H. E. Dubin, K. S. Fine, and C. F. Driscoll: *Phys. Fluids* 11, 905 (1999).
[22] Y. Kiwamoto, K. Ito, A. Sanpei, and A. Mohri: *Phys. Rev. Lett.* 85, 3173 (2000).
[23] A. Sanpei, Y. Kiwamoto, K. Ito, and Y. Soga: *Phys. Rev. E* 68, 016404 (2003).
[24] Y. Soga, Y. Kiwamoto, A. Sanpei, and J. Aoki: *Phys. Plasmas* 10, 3922 (2003).
[25] N. N. Bogoliubov: *J. Phys. USSR* 10, 257 (1946).
[26] M. Born and H. S. Green: *Proc. Roy. Soc. London* A188, 10 (1946).
[27] J. G. Kirkwood: *J. Chem. Phys.* 14, 180 (1946).
[28] J. Yvon: *La théorie statistique des fluides et l’équation d’état*, Actualités scientifiques et industrielles, No. 203, (Hermann, Paris, 1935).
[29] R. Balescu: *Statistical Dynamics: Matter out of equilibrium* (Imperial College Press, London, 1997).
[30] R. Kawahara and H. Nakanishi: J. Phys. Soc. Jpn 75, 054001 (2006).
[31] R. Kawahara and H. Nakanishi: cond-mat/0611694.
[32] M. M. Sano, Y. Yatsuyanagi, Y. Yoshida and H. Tomita: J. Phys. Soc. Jpn 76, 064001 (2007).
[33] R. Courant and D. Hilbert: Methoden der Mathematischen Physik, (Verlag von Julius Springer, Berlin, 1931), 2nd ed.
[34] E. A. Novikov: JETP 41, 937 (1976).
[35] T. Yoshida and M. M. Sano: J. Phys. Soc. Jpn 74, 587 (2005).
[36] J. B. Taylor and B. McNamara: Phys. Fluids 14, 1492 (1971).
[37] Y. Yatsuyanagi, Y. Kiwamoto, H. Tomita, M. M. Sano, T. Yoshida and T. Ebisuzaki: Phys. Rev. Lett. 94, 054502 (2005).
[38] F. Bouchet and T. Dauxois: Phys. Rev.E 72, 045103(R) (2005).
[39] Y. Y. Yamaguchi, F. Bouchet and T. Dauxois: J. Stat. Mech. P01020 (2007).