Local energy-momentum conservation in scalar-tensor-like gravity with
generic curvature invariants

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Abstract

For a large class of scalar-tensor-like gravity whose action contains nonminimal couplings be-
tween a scalar field $\phi(x^\mu)$ and generic curvature invariants $\{\mathcal{R}\}$ beyond the Ricci scalar $R = R_{\mu\nu}$, we prove the covariant invariance of its field equation and confirm/prove the local energy-momentum con-
servation. These $\phi(x^\mu) - \mathcal{R}$ coupling terms break the symmetry of diffeomorphism invariance under a particle transformation, which implies that the solutions to the field equation should satisfy the con-
sistency condition $\mathcal{R} \equiv 0$ when $\phi(x^\mu)$ is nondynamical and massless. Following this fact and based on
the accelerated expansion of the observable Universe, we propose a primary test to check the viability of the modified gravity to be an effective dark energy, and a simplest example passing the test is the “Weyl/conformal dark energy”.

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1. Introduction

An important problem in relativistic theories of gravity is the divergence-freeness of the field equation and the covariant conservation of the energy-momentum tensor. In general relativity (GR), Einstein’s equation $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T^{(m)}_{\mu\nu}$ has a vanishing covariant divergence due to the contracted Bianchi identities $\nabla_\mu G_{\mu\nu} \equiv 0$, which guarantees the local energy-momentum conservation $\nabla_{\mu} T^{(m)}_{\mu\nu} = 0$ or $\partial_\mu (\sqrt{-g} T^{(m)}_{\mu\nu}) = 0$ for the matter tensor $T^{(m)}_{\mu\nu}$. In modified gravities beyond GR and its Hilbert-Einstein action, the conservation problem becomes more complicated and has attracted a lot of interest.

In Ref.[1], the generalized Bianchi identities were derived for the Palatini formulation of the non-linear $f(R)$ gravity, and its local energy-momentum conservation was further confirmed in Ref.[2] by the equivalence between Palatini $f(R)$ and the $\omega = -3/2$ Brans-Dicke gravity. Ref.[3] investigated a mixture of $f(R)$ and the generalized Brans-Dicke gravity, and proved the covariant conservation from both the metric and the Palatini variational approaches. For Einstein-Cartan gravity which allows for spacetime torsion, both the energy-momentum and the angular momentum conservation were studied in Ref.[4] by decomposing the Bianchi identities in Riemann-Cartan spacetimes. In Refs.[3, 5, 6, 7], the nontrivial divergences $\nabla_\mu T^{(m)}_{\mu\nu}$ were analyzed for the situations where the matter Lagrangian density is multiplied by different types of curvature invariants in the action. Also, interestingly in Ref.[8],

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the possible consequences after dropping the energy-momentum conservation in GR, such as the modified evolution equation for the Hubble parameter, were investigated.

Besides the covariant invariance $\nabla^\mu T^{(m)}_{\mu\nu} = 0$ for the matter tensor $T^{(m)}_{\mu\nu}$ that has been standardly defined in GR and modified gravities (cf. Eq.(20) below), the conservation problem has also been studied for more fundamental definitions of energy-momentum tensors from a wider perspective, i.e. from a first-principle approach making use of Noether’s theorem and the classical field theory. For example, the Noether-induced canonical energy-momentum conservation for the translational invariance of the Lagrangian was studied in Ref.[9] for general spacetimes with torsion and nonmetricity. The conservation equations and the Noether currents for the Poincaré-transformation invariance were studied in Ref.[10] for the $3+1$ and $2+1$ dimensional Einstein gravity and the $1+1$ dimensional string-inspired gravity. Also, Refs.[11] and [12] extensively discussed the diﬀeomorphically invariant metric-torsion gravity whose action contains first- and second-order derives of the torsion tensor, and derived the full set of Klein-Noether differential identities and various types of conserved currents.

In this paper, our interest is the covariant invariance of such modified gravities whose actions involve nonminimal couplings between arbitrary curvature invariants $\{\mathcal{R}\}$ and a background scalar field $\phi(x^\alpha)$. For example, $\phi(x^\alpha)$ is coupled to the Ricci scalar $R = R^\alpha_{\alpha}$ in Brans-Dicke and scalar-tensor gravity in the Jordan frame [13], to the Chern-Pontryagin topological density in the Chern-Simons modification of GR [14], and to the Gauss-Bonnet invariant $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ in the Gauss-Bonnet effective dark energy [15]. In theory, one could consider the nonminimal coupling of $\phi(x^\alpha)$ to an arbitrarily complicated curvature invariant beyond the Ricci scalar. In such situations, however, the covariant invariance of the field equation has not been well understood, so we aim to carefully look into this problem by this work. Note that it might sound more complete to analyze the global conservation $\nabla^\mu (T^{(m)}_{\mu\nu} + t_{\mu\nu}) = 0$ or $\partial^\mu [\sqrt{-g}(T^{(m)}_{\mu\nu} + t_{\mu\nu})] = 0$, where $t_{\mu\nu}$ refers to the energy-momentum pseudotensor for the gravitational field, but to make this paper more clear and readable, we choose to concentrate on the local conservation $\nabla^\mu T^{(m)}_{\mu\nu} = 0$, while the incorporation of $t_{\mu\nu}$ will be discussed separately.

This paper is organized as follows. In Sec. 2, we introduce the generic class of modified gravity with the nonminimal $\phi(x^\alpha)$–couplings to arbitrary Riemannian invariants $\{\mathcal{R}\}$, calculate the divergence for different parts of the total action, prove the covariant invariance of the field equation, and confirm the local energy-momentum conservation. Section 3 investigates the reduced situations that the scalar field is nondynamical and massless, and derives the consistency constraint $\mathcal{R} \equiv 0$ which suppresses the breakdown of diﬀeomorphism invariance. Finally, applications of the theories in Secs. 2 and 3 are considered in Sec. 4. Throughout this paper, we adopt the geometric conventions $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$, $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} - \cdots$ and $R_{\mu\nu} = R^a_{\mu\nu}$ with the metric signature $(-, + + +)$.

2. General theory

2.1. Scalar-tensor-like gravity

Consider a theory of modified gravity or effective dark energy given by the following action,

$$S = \int d^4x \sqrt{-g} \left( \mathcal{L}_{\text{HE}} + \mathcal{L}_G + \mathcal{L}_{\text{NC}} + \mathcal{L}_\phi \right) + S_m,$$

where $\mathcal{L}_{\text{HE}}$ refers to the customary Hilbert-Einstein Lagrangian density as in GR,

$$\mathcal{L}_{\text{HE}} = R,$$
while $\mathcal{L}_G$ denotes the extended dependence on generic curvature invariants $\mathcal{R}$,

$$\mathcal{L}_G = f (R, \cdots , \mathcal{R}).$$

Here $\mathcal{R} = R (g_{\alpha\beta} , R_{\mu\nu\alpha\beta}, \nabla_\gamma R_{\mu\nu\alpha\beta} , \cdots)$ is an arbitrary invariant function of the metric as well as the Riemann tensor and its derivatives up to any order. For example, $\mathcal{R}$ can come from the fourteen\(^2\) algebraically independent real invariants of the Riemann tensor [16] and their combinations, say $R_{\alpha\beta} R^{\alpha\beta} + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + R_{\mu\nu\beta\gamma} R^{\mu\nu\beta\gamma}$, which will yield fourth–order field equations; or differential Riemannian invariants that will lead to sixth– or even higher–order field equations, like $R^\alpha\nu \nabla_\alpha R + R^\alpha\nu R^{\beta\gamma} \nabla_\alpha \nabla_\beta R^{\mu\nu}$. In the total Lagrangian density, $\mathcal{L}_{\text{NC}}$ represents the nonminimal coupling effects,

$$\mathcal{L}_{\text{NC}} = h(\phi) \cdot \tilde{f} (R, \cdots , \mathcal{R}),$$

where $h(\phi)$ is an arbitrary function of the scalar field $\phi = \phi(x^\alpha)$, and $\tilde{f} (R, \cdots , \mathcal{R})$ has generic dependence on curvature invariants, with the dots “· · ·” in $\tilde{f} (R, \cdots , \mathcal{R})$ and the $f (R, \cdots , \mathcal{R})$ above denoting different choices of $\mathcal{R}$. Moreover, the kinetics of $\phi(x^\alpha)$ is governed by

$$\mathcal{L}_\phi = -\lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi).$$

In the $(-, + , + )$ system of conventions, $\phi(x^\alpha)$ is canonical if $\lambda(\phi) > 0$, noncanonical if $\lambda(\phi) < 0$, and nondynamical if $\lambda(\phi) = 0$.

Finally, as usual, the matter action $S_m$ in Eq.(1) is given by the matter Lagrangian density via

$$S_m = 16 \pi G \int d^4 x \sqrt{-g} \mathcal{L}_m (g_{\mu\nu}, \psi_m, \partial_\mu \psi_m),$$

where the variable $\psi_m$ collectively describes the matter fields, and $\psi_m$ is minimally coupled to the metric tensor $g_{\mu\nu}$. Unlike the usual dependence on $\partial_\mu \psi_m$ in its standard form, $\mathcal{L}_m = \mathcal{L}_m (g_{\mu\nu}, \psi_m, \partial_\mu \psi_m)$ does not contain derivatives of the metric tensor – such as Christoffel symbols or curvature invariants, in light of the minimal gravity-matter coupling and Einstein’s equivalence principle; physically, this means $\mathcal{L}_m$ reduces to the matter Lagrangian density for the flat spacetime in a freely falling local reference frame (i.e. a locally geodesic coordinate system).

To sum up, we are considering the modifications of GR into the total Lagrangian density $\mathcal{L} = R + f (R, \cdots , \mathcal{R}) + h(\phi) \cdot \tilde{f} (R, \cdots , \mathcal{R}) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16 \pi G \mathcal{L}_m$, which has been rescaled so that the numerical coefficient $16 \pi G$ is associated to $\mathcal{L}_m$. It can be regarded as a mixture of the nonlinear higher-order gravity $\mathcal{L} = R + f (R, \cdots , \mathcal{R}) + 16 \pi G \mathcal{L}_m$ in the metric formulation for the curvature invariants, and the generalized scalar-tensor gravity $\mathcal{L} = h(\phi) \cdot \tilde{f} (R, \cdots , \mathcal{R}) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16 \pi G \mathcal{L}_m$ in the Jordan frame for the scalar field. Hereafter we will mainly work with actions, and for simplicity we will sometimes adopt the total Lagrangian density in place of the corresponding action in full integral form.

\(^2\)When one combines the spacetime geometry with matter fields in the framework of GR, the amount of independent algebraic invariants will be extended to sixteen in the presence of electromagnetic or perfect-fluid fields [17].
2.2. Divergence-freeness of gravitational field equation

2.2.1. Pure curvature parts

For the Hilbert-Einstein part of the total action, i.e. \( S_{HE} = \int d^4x \sqrt{-g} \mathcal{L}_{HE} \), its variation with respect to the inverse metric yields the well-known result \( \delta S_{HE} \equiv \int d^4x \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \). By the symbol \( \equiv \) we mean the equality after neglecting all total derivatives in the integrand or equivalently boundary terms of the action when integrating by parts, and the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \) respects the twice-contracted Bianchi identity \( \nabla^\mu G_{\mu\nu} = 0 \).

For the gravitational action \( S_G = \int d^4x \sqrt{-g} \mathcal{L}_G \) for the extended dependence on generic Riemannian invariants, formally we write down the variation as \( \delta S_G \equiv \int d^4x \sqrt{-g} H^{(G)}_{\mu\nu} \delta g^{\mu\nu} \), where \( H^{(G)}_{\mu\nu} \) resembles and generalizes the Einstein tensor by

\[
H^{(G)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta \left[ \sqrt{-g} f(R, \cdots, R) \right]}{\delta g^{\mu\nu}}. \tag{7}
\]

Due to the coordinate invariance of \( S_G \), \( H^{(G)}_{\mu\nu} \) satisfies the generalized contracted Bianchi identities \[18, 19\]

\[
\nabla^\mu \left( \frac{1}{\sqrt{-g}} \frac{\delta \left[ \sqrt{-g} f(R, \cdots, R) \right]}{\delta g^{\mu\nu}} \right) = 0, \tag{8}
\]
or just \( \nabla^\mu H^{(G)}_{\mu\nu} = 0 \) by the definition of \( H^{(G)}_{\mu\nu} \). Similar to the relation \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \), one can further expand \( H^{(G)}_{\mu\nu} \) to rewrite Eq.(8) into

\[
\nabla^\mu \left( f_R R_{\mu\nu} + \sum_{f_R} f_R R_{\mu\nu} - \frac{1}{2} f(R, \cdots, R) g_{\mu\nu} \right) = 0, \tag{9}
\]

where \( f_R \equiv \partial f(R, \cdots, R) / \partial R \), \( f_R \equiv \partial f(R, \cdots, R) / \partial R \), and \( R_{\mu\nu} \equiv (f_R \delta R) / \delta g^{\mu\nu} \) – note that in the calculation of \( R_{\mu\nu} \), \( f_R \) will serve as a nontrivial coefficient if \( f_R \) is constant and should be absorbed into the variation \( \delta R \) when integrated by parts.

2.2.2. Nonminimal \( \phi(x^\alpha) \)-curvature coupling part

For the componential action \( S_{NC} = \int d^4x \sqrt{-g} \mathcal{L}_{NC} \) for the nonminimal coupling effect, formally we have the variation \( \delta S_{NC} \equiv \int d^4x \sqrt{-g} H^{(NC)}_{\mu\nu} \delta g^{\mu\nu} \), where

\[
H^{(NC)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta \left[ \sqrt{-g} \phi f(R, \cdots, R) \right]}{\delta g^{\mu\nu}}. \tag{10}
\]

Unlike \( H^{(G)}_{\mu\nu} \) with Eq.(8), \( H^{(NC)}_{\mu\nu} \) does not respect some straightforward generalized Bianchi identities; this is because \( S_{NC} \) involves the coupling with the background scalar field \( \phi(x^\alpha) \) and is no longer purely tensorial gravity. Thus, we will analyze the divergence of \( H^{(NC)}_{\mu\nu} \) by the diffeomorphism of \( S_{NC} \).

Consider an arbitrary infinitesimal coordinate transformation \( x^\mu \mapsto x^\mu + \delta x^\mu \), where \( \delta x^\mu = k^\mu \) is an infinitesimal vector field that vanishes on the boundary, \( k^\mu = 0 \mid_{\partial \Omega} \), so that the spacetime manifold…
Taking its contravariant derivative, we immediately obtain the nontrivial divergence
\[ \lambda \]
the dynamical tensor field
\[ 2.2.3. \text{Purely scalar-field part} \]
formation where the observer or equivalently the coordinate system transforms, both the tensor fields
\[ \delta x \]
metric; moreover, the coordinate shift
\[ \delta \phi \] and the coordinate system parameterizing the spacetime remain una
\[ h \]
is mapped onto itself. \( S_{NC} \) responds to this transformation by
\[ \delta S_{NC} = - \int d^4 x h(\phi) \cdot \partial_\mu \left[ k^\mu \sqrt{-g} \bar{f}(R, \cdots, \mathcal{R}) \right] \]
\[ \equiv \int d^4 x \sqrt{-g} \bar{f}(R, \cdots, \mathcal{R}) \cdot \left( h_\phi \partial_\mu \phi \right) k^\mu, \]
where \( h_\phi := dh(\phi)/d\phi \). For Eq.(11), one should note that \( \phi(x^\mu) \) acts as a fixed background, as it only relies on the coordinates (i.e. spatial location and time) and is independent of the spacetime metric; moreover, the coordinate shift \( x^\mu \mapsto x^\mu + k^\mu \) is a particle/active transformation, under which the dynamical tensor field \( g_{\mu\nu} \) and thus \( \sqrt{-g} \bar{f}(R, \cdots, \mathcal{R}) \) transform, while the background field \( \phi(x^\mu) \) and the coordinate system parameterizing the spacetime remain unaffected [20].

Under the particle transformation \( x^\mu \mapsto x^\mu + k^\mu \), the metric tensor varies by \( g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu} \) with
\[ \delta g_{\mu\nu} = \delta_\alpha^\mu g_{\alpha\nu} = -\nabla_\mu k_\nu - \nabla_\nu k_\mu, \]
and therefore \( g^{\mu\nu} \mapsto g^{\mu\nu} + \delta g^{\mu\nu} \) with \( \delta g^{\mu\nu} = -\delta_\alpha^\mu g^{\alpha\nu} = \nabla^\mu k_\nu + \nabla^\nu k_\mu \).

Recalling the definition of \( H_{\mu\nu}^{(NC)} \) with \( H_{\mu\nu}^{(NC)} \) being symmetric for the index switch \( \mu \leftrightarrow \nu \), one has
\[ \delta S_{NC} = 2 \int d^4 x \sqrt{-g} H_{\mu\nu}^{(NC)} \nabla^\mu k^\nu = -2 \int d^4 x \sqrt{-g} \left( \nabla^\mu H_{\mu\nu}^{(NC)} \right) k^\nu, \]
Comparing Eq.(12) with Eq.(13), we conclude that \( H_{\mu\nu}^{(NC)} \) has a nontrivial divergence for \( \phi(x^\mu) \neq \text{constant}, \)
\[ \nabla^\mu H_{\mu\nu}^{(NC)} = -\frac{1}{2} \bar{f}(R, \cdots, \mathcal{R}) \cdot h_\phi \partial_\nu \phi. \]
In fact, Eq.(14) reflects the breakdown of diffeomorphism invariance in the presence of a fixed background scalar field. As a comparison, it is worthwhile to mention that under an observer/passive transformation where the observer or equivalently the coordinate system transforms, both the tensor fields and the background scalar field will be left unchanged, so the symmetry of observer-transformation invariance continues to hold [20].

2.2.3. Purely scalar-field part

Next, for the purely scalar-field part \( S_\phi = \int d^4 x \sqrt{-g} \mathcal{L}_\phi \) with the variation \( \delta S_\phi = \int d^4 x \sqrt{-g} H_{\mu\nu}^{(\phi)} \delta g^{\mu\nu}, \)
explicit calculations find
\[ H_{\mu\nu}^{(\phi)} = -\lambda(\phi) \cdot \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \left( \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi + V(\phi) \right) g_{\mu\nu}. \]
Taking its contravariant derivative, we immediately obtain the nontrivial divergence
\[ \nabla^\mu H_{\mu\nu}^{(\phi)} = -\frac{1}{2} \left( \lambda_\phi \cdot \nabla_\alpha \phi \nabla^\alpha \phi + 2\lambda(\phi) \cdot \Box \phi - V_\phi \right) \cdot \nabla_\nu \phi, \]
where \( \lambda_\phi := d\lambda(\phi)/d\phi, V_\phi := dV(\phi)/d\phi, \) and \( \Box \) denotes the covariant d’Alembertian with \( \Box \phi = g^{\rho\beta} \partial_\rho \nabla_\beta \phi = \frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^{\rho\beta} \partial_\beta \phi). \) On the other hand, extremizing the entire action Eq.(1) with respect to the scalar field, i.e. \( \delta S/\delta \phi = 0 \), one could obtain the kinematical wave equation
\[ 2\lambda(\phi) \cdot \Box \phi = -\bar{f}(R, \cdots, \mathcal{R}) \cdot h_\phi - \lambda_\phi \cdot \nabla_\alpha \phi \nabla^\alpha \phi + V_\phi. \]
We regard it as “kinematical” because it does not explicitly relate the propagation of \( \phi(x^\mu) \) to \( T^{(m)} = g^{\mu\nu} T^{(m)\mu\nu} \) for the matter distribution, while the “dynamical” wave equation can be obtained after comb-
ing Eq.(17) with the trace of the gravitational field equation. Substitute Eq.(17) into the right hand side of Eq.(16), and it follows that

$$\nabla^\mu H_{\mu\nu}^{(\phi)} = \frac{1}{2} \tilde{f} (R, \cdots, \mathcal{R}) \cdot h_{\phi} \nabla_{\nu} \phi,$$

(18)

which exactly cancels out the divergence of $H_{\mu\nu}^{(NC)}$ in Eq.(14) for the nonminimal-coupling part $\mathcal{S}_{\text{NC}}$.

2.2.4. Covariant invariance of field equation and local energy-momentum conservation

To sum up, for the modified gravity or effective dark energy given by Eq.(1), its field equation reads

$$G_{\mu\nu} + H_{\mu\nu}^{(G)} + H_{\mu\nu}^{(NC)} + H_{\mu\nu}^{(\phi)} = 8\pi G T_{\mu\nu}^{(m)},$$

(19)

where, unlike $G_{\mu\nu}$ and $H_{\mu\nu}^{(\phi)}$, the exact forms of $\{H_{\mu\nu}^{(G)}, H_{\mu\nu}^{(NC)}\}$ will not be determined until the concrete expressions of $\{\mathcal{L}_G, \mathcal{L}_{\text{NC}}\}$ or $\{f(R, \cdots, \mathcal{R}), \tilde{f}(R, \cdots, \mathcal{R})\}$ are set up. In Eq.(19), the energy-momentum tensor $T_{\mu\nu}^{(m)}$ is defined as in GR via [21]

$$\delta S_m = \frac{1}{2} \times 16\pi G \int d^4x \sqrt{-g} \ T_{\mu\nu}^{(m)} \delta g_{\mu\nu} \quad \text{with} \quad T_{\mu\nu}^{(m)} := \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g_{\mu\nu}},$$

(20)

with $S_m$ rescaled by $16\pi G$ in Eq.(6). Instead of the variational definition Eq.(20), it had been suggested that $T_{\mu\nu}^{(m)}$ could be derived solely from the equations of motion $\frac{\partial \mathcal{L}_m}{\partial \psi_m} - \nabla_\mu \frac{\partial \mathcal{L}_m}{\partial (\partial_\mu \psi_m)} = 0$ for the $\psi_m$ field in $\mathcal{L}_m (g_{\mu\nu}, \psi_m, \partial_\mu \psi_m)$ [22]; however, further analyses have shown that this method does not hold a general validity, and Eq.(20) remains as the most reliable approach to $T_{\mu\nu}^{(m)}$ [23].

Adding up the (generalized) contracted Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$ and Eq.(8), and the nontrivial divergences Eqs.(14) and (18), eventually we conclude that the left hand side of the field equation (19) is divergence free, the local energy-momentum conservation $\nabla^\mu T_{\mu\nu}^{(m)} = 0$ holds, and the tensorial equations of motion for test particles remain the same as in GR.

In fact, the matter Lagrangian density $\mathcal{L}_m = \mathcal{L}_m (g_{\mu\nu}, \psi_m, \partial_\mu \psi_m)$ is a scalar invariant that respects the diffeomorphism invariance under the particle transformation $x^\mu \rightarrow x^\mu + k^\mu$, and Noether’s conservation law directly yields

$$\nabla^\mu \left( \frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g_{\mu\nu}} \right) = 0,$$

(21)

which can be recast into $-\frac{1}{2} \nabla^\mu T_{\mu\nu}^{(m)} = 0$. That is to say, under minimal geometry-matter coupling with an isolated $\mathcal{L}_m$ in the total Lagrangian density, the matter tensor $T_{\mu\nu}^{(m)}$ in Eq.(20) has been defined in a practical way so that $T_{\mu\nu}^{(m)}$ is automatically symmetric, Noether compatible, and covariant invariant, which naturally guarantees the local conservation $\nabla^\mu T_{\mu\nu}^{(m)} = 0$. In this sense, one can regard the vanishing divergence $\nabla^\mu \left( G_{\mu\nu} + H_{\mu\nu}^{(G)} + H_{\mu\nu}^{(NC)} + H_{\mu\nu}^{(\phi)} \right) = 0$ for Eq.(19) to either imply or confirm the conservation $\nabla^\mu T_{\mu\nu}^{(m)} = 0$.

One should be aware that in the presence of nonminimal gravity-matter couplings, like $\mathcal{R} \cdot \mathcal{L}_m$ or more generally $F(R, \cdots, \mathcal{R}) \cdot \mathcal{L}_m$ in the total Lagrangian density, the divergence $\nabla^\mu T_{\mu\nu}^{(m)}$ becomes
nonzero as well and obeys the relation \( \nabla^\mu T^{(m)}_{\mu\nu} = F(R, \ldots, \mathcal{R})^{-1} \cdot \left( \mathcal{L}_m g_{\mu\nu} - T^{(m)}_{\mu\nu} \right) \cdot \nabla^\mu F(R, \ldots, \mathcal{R}) \) instead [5, 6, 7], which recovers the local conservation \( \nabla^\mu T^{(m)}_{\mu\nu} = 0 \) for \( F(R, \ldots, \mathcal{R}) = \text{constant} \).

Also, at a more fundamental level, the \( T^{(m)}_{\mu\nu} \) in Eq.(20) for GR and modified gravities, though practical with all desired properties, is not defined from the first-principle approach, i.e. directly from symmetry and Noether’s theorem in the classical field theory. In this larger framework, the \( T^{(m)}_{\mu\nu} \) in Eq.(20) is often referred to as the Hilbert energy-momentum tensor: it symmetrizes the canonical energy-momentum tensor of translational invariance by adding a superpotential term, and it is a special case of the Belinfante energy-momentum tensor that minimally couples to gravity [24].

3. Nondynamical massless scalar field

3.1. Nondynamical massive scalar field

Due to the \( \lambda(\phi) \)-dependence in \( \mathcal{S}_\phi \), its Lagrangian density becomes \( \mathcal{L}_\phi = -V(\phi) \) when \( \lambda(\phi) \equiv 0 \); considering that \( V(\phi) \) is usually related to the mass of the scalar field in cosmology and high energy physics, we will call \( \phi(x^\mu) \) nondynamical and massive for the situation \( \lambda(\phi) \equiv 0 \) and \( V(\phi) \neq 0 \). As such, instead of producing a propagation equation \( \delta \phi \), the extremization \( \delta S / \delta \phi = 0 \) leads to the following constraint for the potential \( V(\phi) \):

\[
V_\phi = \int (R, \ldots, \mathcal{R}) \cdot h_\phi.
\] (22)

In the meantime, Eqs.(15), (16), and (17) reduce to become

\[
H^{(\phi)}_{\mu\nu} = \frac{1}{2} V(\phi) g_{\mu\nu} \quad \text{and} \quad \nabla^\mu H^{(\phi)}_{\mu\nu} = \frac{1}{2} V_\phi \nabla_\nu \phi = \frac{1}{2} \int (R, \ldots, \mathcal{R}) \cdot h_\phi \nabla_\nu \phi.
\] (23)

Thus, for a nondynamical yet massive scalar field, \( \nabla^\mu H^{(\phi)}_{\mu\nu} \) can still balance the nontrivial divergence \( \nabla^\mu H^{(NC)}_{\mu\nu} \) of the nonminimal \( \phi(x^\mu) \)-curvature coupling effect, while the potential or the mass of the scalar field is restricted by the condition Eq.(22).

3.2. Nondynamical massless scalar field

Within the situation \( \lambda(\phi) \equiv 0 \), it becomes even more interesting when the potential vanishes as well in Eqs.(5), (15), (16), and (17); we will call the scalar field nondynamical and massless for \( \lambda(\phi) = 0 = V(\phi) \). With \( \mathcal{L}_\phi = 0 \), the total action simplifies into

\[
S = \int d^4 x \sqrt{-g} \left( R + \mathcal{L}_G + \mathcal{L}_{\text{NC}} + 16\pi G L_m \right).
\] (24)

Since \( H^{(\phi)}_{\mu\nu} = 0 \) and \( \nabla^\mu H^{(\phi)}_{\mu\nu} = 0 \), the divergence \( \nabla^\mu H^{(NC)}_{\mu\nu} \) for the nonminimal coupling part as in Eq.(14) can no longer be neutralized. Instead, with \( \nabla^\mu G_{\mu\nu} = 0 \), the generalized contracted Bianchi identities Eq.(8), and the covariant conservation \( \nabla^\mu T^{(m)}_{\mu\nu} = 0 \) under minimal geometry-mater coupling, the contravariant derivative of the field equation \( G_{\mu\nu} + H^{(G)}_{\mu\nu} + H^{(NC)}_{\mu\nu} = 8\pi G T^{(m)}_{\mu\nu} \) forces \( \nabla^\mu H^{(NC)}_{\mu\nu} \) to

\footnotesize
\[\text{We simply use “massive” and “massless” to distinguish the situation } V(\phi) \neq 0 \text{ from } V(\phi) = 0 \text{ when the scalar field is nondynamical. However, we do not follow this usage to call } \phi(x^\mu) \text{ “dynamical and massless” when } \lambda(\phi) \neq 0, V(\phi) = 0, \text{ as it sounds inappropriate to from the spirit of relativity.}\]
vanish. Together with Eq.(14), this implies that to be a solution to the gravity of Eq.(24), the metric tensor $g_{\mu\nu}$ must satisfy the constraint

$$\hat{f}(R, \cdots, \mathcal{R}) \equiv 0 \quad \text{for} \quad \phi(x) \neq \text{constant}. \quad (25)$$

Since the nonzero divergence $\nabla^\mu H_{\mu
u}^{(\text{NC})} = -\frac{1}{2} \hat{f}(R, \cdots, \mathcal{R}) \cdot h_{\delta\phi\delta} \phi$ measures the failure of diffeomorphism invariance in the componential action $\mathcal{S}_\text{NC}$, the consistency condition Eq.(25) indicates that the symmetry breaking of diffeomorphism invariance is suppressed in gravitational dynamics of Eq.(24).

Here one should note that the variation $\delta S/\delta \phi = 0$ yields the condition $\hat{f}(R, \cdots, \mathcal{R}) \cdot h_{\delta\phi} \phi = 0$, which also leads to $\hat{f}(R, \cdots, \mathcal{R}) \equiv 0$ if the scalar field is nonconstant. In addition, the constraint $\hat{f}(R, \cdots, \mathcal{R}) \equiv 0$ does not mean $H_{\mu\nu}^{(\text{NC})} = 0$ or the removal of $\mathcal{Z}_\text{NC}$ from the action Eq.(24). This can be seen by an analogous situation in GR: all vacuum solutions of Einstein’s equation have to satisfy the condition $R \equiv 0$, but the GR action $\mathcal{S} = \int d^4x \sqrt{-g} \left( R + 16\pi G \mathcal{L}_m \right)$ still holds in its standard form.

After $\mathcal{Z}_G$ and $\mathcal{Z}_\text{NC}$ get specified in Eq.(24), how can we know whether it yields a viable theory or not? In accordance with Eq.(25), we adopt the following basic assessment.

**Primary test:** For the action Eq.(24) to be a viable modified gravity or effective dark energy carrying a nondynamical and massless scalar field, an elementary requirement is that the function $\hat{f}(R, \cdots, \mathcal{R})$ in $\mathcal{Z}_\text{NC}$ vanishes identically for the flat and accelerating Friedmann-Robertson-Walker (FRW) Universe with the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx_i)^2 \quad \text{and} \quad \dot{a}(t) > 0, \quad (26)$$

where $a(t)$ is the cosmic scale factor and the overdot means derivative with respect to the comoving time.

This primary test is inspired by the fact that the observable Universe is homogeneous and isotropic at the largest cosmological scale, and the discovery that the Universe is nearly perfectly flat and currently undergoing accelerated spatial expansion. These features have been extensively examined and received strong support from the surveys on the large scale structures, the expansion history, and the structure-growth rate of the Universe, such as the measurements of the distance modulus of Type Ia supernovae, peaks of the baryon acoustic oscillation, and temperature polarizations of the cosmic microwave background. Clearly, the primary test is updatable and subject to the progress in observational cosmology.

**3.3. Weyl dark energy**

Following the primary test above, one can start to explore possible modifications of GR into the total Lagrangian density $\mathcal{L} = R + f(R, \cdots, \mathcal{R}) + h(\phi) \hat{f}(R, \cdots, \mathcal{R}) + 16\pi G \mathcal{L}_m$ and then check the consistency condition $f(R, \cdots, \mathcal{R}) \equiv 0$ under the flat FRW metric Eq.(26). In the integrand of the Hilbert-Einstein action for GR, the Ricci scalar $R$ is the simplest curvature invariant formed by second-order derivatives of the metric; similarly, we can start with the simplest situation that $\hat{f}(R, \cdots, \mathcal{R})$ is some quadratic Riemannian scalar. One possible example is the square of the conformal Weyl tensor $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{3} \left( g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\beta\gamma} + g_{\beta\gamma} R_{\alpha\delta} - g_{\beta\delta} R_{\alpha\gamma} \right) + \frac{1}{6} \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right) R$, which is the totally traceless part in the Ricci decomposition of the Riemann tensor. In this case, we consider the action

$$S_{C^2} = \int d^4x \sqrt{-g} \left( R + \gamma \phi C^2 + 16\pi G \mathcal{L}_m \right), \quad (27)$$


Employing the cosmological redshift

\[ S \text{ of the coupling strength } \gamma \text{ in } S_{C^2}, \text{ should be carefully constrained by the observational data from astronomical surveys. Following the field equation of } S_{C^2}, \text{ consider a } C^2-CDM \text{ model (i.e. } C^2 \text{ cold dark matter) for the Universe instead of } \Lambda CDM. \text{ Then the first Friedmann equation under the flat FRW metric reads}
\]

\[ H^2 = \frac{8}{3} \pi G \left[ \rho_{M0} \left( \frac{a_0}{a} \right)^3 + \rho_{r0} \left( \frac{a_0}{a} \right)^4 + \rho_{C^2} \right], \tag{32} \]

where the densities of nonrelativistic matter \( \rho_M(t) \) and relativistic matter \( \rho_r(t) \) have been related to their present-day values \( \rho_{M0} \) and \( \rho_{r0} \) via the continuity equation \( \dot{\rho} + 3H\rho(1 + w) = 0 \), with the equation of state parameters being \( w_M = 0 \) and \( w_r = 1/3 \), respectively. Also, \( H := \dot{a}/a \) is the evolutionary Hubble parameter, and \( \rho_{C^2} \) denotes the effective energy density of the Weyl dark energy,

\[ \rho_{C^2} = \gamma \left[ 5 \phi \frac{\dddot{a}}{a^2} + 2 \phi \frac{\dddot{a}}{a^2} \right] - \phi \frac{\dddot{a}}{a^2} - 2 \phi \frac{\dddot{a}}{a^2} + \frac{5 \phi}{a} \frac{\dddot{a}}{a} + 2 \phi \frac{\dddot{a}}{a^2} - 4 \phi \frac{\dddot{a}}{a^2} + \frac{4 \phi}{a} \frac{\dddot{a}}{a} + 6 \phi \frac{\dddot{a}}{a^2} + 8 \phi \frac{\dddot{a}}{a^2} \]. \tag{33} \]

Employing the cosmological redshift \( z := a_0/a - 1 \) as well as the replacements \( \dddot{a}/a = \dot{H} + H^2 \) and \( \dddot{a}/a = \dot{H} + 3HH + H^3 \), Eq.(32) can be parameterized into

\[ H(z; H_0, p) = H_0 \sqrt{\Omega_{M0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_{C^2}}, \tag{34} \]
where $H_0$ represents the Hubble constant $H(z = 0)$, $\Omega_{M0} = 8\pi G \rho_{M0}/(3H_0^2)$, $\Omega_{\sigma0} = 8\pi G \rho_{\sigma0}/(3H_0^2)$, and

$$
\Omega_{C2} = \frac{32\pi G}{H_0^2} \gamma \left\{ \phi H \left[ 5 \left( H + H^2 \right) - 2 \left( H + H^2 \right)^2 - H^2 - 2H^4 \right] + \phi H \left( 5 - 4H - 4H^3 \right) \left( H + 3HH + H^3 \right) + 8\phi H^6 + \phi \left( H + H^2 \right) \left[ \left( 2H + 2H^2 + 4H^2 \right) \left( H + H^2 \right) - 4H^2 - 6H^4 \right] \right\}.
$$

Typically, we can use the Markov-Chain Monte-Carlo engine CosmoMC [25] to explore the parameter space $p = (\Omega_{M0}, \Omega_{\sigma0}, \gamma)$ for the Weyl dark energy $S_{C2}$, and find out how well it matches the various sets of observational data. This goes beyond the scope of this paper and will be analyzed separately.

4. Applications

4.1. Chern-Simons gravity

The four-dimensional Chern-Simons modification of GR was proposed by the action [14] (note that to not confuse with the traditional gauge gravity carrying a three-dimensional Chern-Simons term [26])

$$
S_{CS} = \int d^4x \sqrt{-g} \left( R + \gamma \phi \frac{^*RR}{\sqrt{-g}} + 16\pi G \mathcal{L}_m \right).
$$

(36)

The scalar field $\phi = \phi(x^\mu)$ is nonminimally coupled to the Chern-Pontryagin density $^*RR := ^*R_{a\beta\gamma\delta}R^{a\beta\gamma\delta} = \frac{1}{2} \epsilon_{a\beta\gamma\delta} R^{a\beta\gamma\delta}$, where $^*R_{a\beta\gamma\delta} := \frac{1}{2} \epsilon_{a\beta\gamma\delta} R^{a\beta\gamma\delta}$ is the left dual of the Riemann tensor, and $\epsilon_{a\beta\gamma\delta}$ represents the totally antisymmetric Levi-Civita pseudotensor with $\epsilon_{0123} = \sqrt{-g}$ and $\epsilon^{0123} = 1/\sqrt{-g}$. The field equation reads $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \gamma H^\mu_{\nu}(\phi) = 8\pi G T_{\mu\nu}^{(m)}$, where $H^\mu_{\nu}(\phi) \equiv \frac{1}{\sqrt{-g}} \left( \frac{\delta^*RR}{\delta g^\mu_{\nu}} \right)$ collects the contributions from the $\phi(x^\mu)$-coupled Chern-Pontryagin density,

$$
\sqrt{-g} H^\mu_{\nu}(\phi) = 2 \partial^\nu \phi \left( \epsilon_{\mu\alpha\beta\gamma} \nabla^\alpha R^{\beta\gamma} + \epsilon_{\nu\alpha\beta\gamma} \nabla^\alpha R^{\beta\gamma} \right) + 2 \partial_{\mu} \partial_{\nu} \phi \left( ^*R^{\alpha}_{\mu \nu} + ^*R^{\alpha}_{\nu \mu} \right).
$$

(37)

According to the general theory in Secs. 2.2 and 3.2, the Chern-Simons gravity Eq.(36) involves a nondynamical and massless scalar field. Identifying $\int (R, \cdots, R)$ as $^*RR/\sqrt{-g}$ and with $h(\phi) = \gamma \phi$ in Eqs.(14) and (25), we obtain the divergence

$$
\nabla^\mu H^\mu_{\nu}(\phi) = - \frac{\gamma ^*RR}{2 \sqrt{-g}} \partial_\nu \phi,
$$

(38)

as well as the constraint $^*RR \equiv 0$ for nontrivial $\phi(x^\mu)$. It can be easily verified that $^*RR$ vanishes for the flat and accelerating FRW Universe, and thus passes the primary test in Sec. 3.2. Also the condition $^*RR \equiv 0$ only applies to the action Eq.(36), and is invalid for the massive Chern-Simons gravity $\mathcal{L} = R + \gamma \phi \frac{^*RR}{\sqrt{-g}} - V(\phi) + 16\pi G \mathcal{L}_m$ or the dynamical case $\mathcal{L} = R + \gamma \phi \frac{^*RR}{\sqrt{-g}} - \lambda(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + 16\pi G \mathcal{L}_m$.

4.2. Reduced Gauss-Bonnet dark energy

The Gauss-Bonnet dark energy was introduced by the action $S_{GB}^{(1)} = \int d^4x \sqrt{-g} \left( R + h(\phi)G - \lambda \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m \right)$ [15], where $\lambda \in \{\pm 1, 0\}$, and the scalar field is nonminimally coupled to the Gauss-Bonnet invariant $G := \left( \frac{1}{2} \epsilon_{a\beta\gamma\delta} R^{a\beta\gamma\delta} \right) \cdot \left( \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^{\alpha\nu\mu\beta} \right) \equiv R^2 - 4R_{a\beta}R^{a\beta} + R_{a\mu\nu\beta}R^{a\mu\nu\beta}$. If $\phi(x^\mu)$
is nondynamical with \( \lambda = 0 \), the action \( S^{(1)}_{GB} \) reduces to become \( S^{(2)}_{GB} = \int d^4 x \sqrt{-g} \left( R + h(\phi)\mathcal{G} - V(\phi) + 16\pi G \mathcal{L}_m \right) \), and according to Eq.(22) with \( f(R, \cdots, \mathcal{R}) \) identified as the Gauss-Bonnet invariant, \( V(\phi) \) has to satisfy the constraint \( V_\phi = \mathcal{G} \partial_\phi \). Moreover, the nonminimally coupled \( h(\phi)\mathcal{G} \) part in \( S^{(1)}_{GB} \) and \( S^{(2)}_{GB} \) contributes to the field equation by

\[
H^{(GB)}_{\mu\nu} = 2R\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu \right)h + 4R_{\mu}^{\ a} \nabla_a \nabla_\nu h + 4R_{\nu}^{\ a} \nabla_a \nabla_\mu h
- 4R_{\mu\nu} \Box h - 4g_{\mu\nu} \cdot R^{\alpha\beta\gamma\delta} \nabla_\alpha \nabla_\beta h + 4R_{\alpha\mu\nu\beta} \nabla^\gamma \nabla_\gamma h ,
\]

(39)

where, compared with the original field equation in Ref.[15], we have removed the algebraic terms in \( H^{(GB)}_{\mu\nu} \) by the Bach-Lanczos identity \( \frac{1}{2}G_{\mu\nu} \equiv 2RR_{\mu\nu} - 4R_{\mu}^{\ a}R_{\alpha\nu} - 4R_{\alpha\mu\nu\beta}R^\alpha_{\beta} + 2R_{\mu\alpha\beta\gamma}R^\alpha_{\beta\gamma} \) [27]. The divergence of \( H^{(GB)}_{\mu\nu} \), in accordance with Eq.(14), reads

\[
\nabla^\mu H^{(GB)}_{\mu\nu} = -\frac{1}{2} \mathcal{G} : \partial_\nu \phi .
\]

(40)

However, it would be problematic if one further reduces \( S^{(2)}_{GB} \) into

\[
S^{(3)}_{GB} = \int d^4 x \sqrt{-g} \left( R + h(\phi)\mathcal{G} + 16\pi G \mathcal{L}_m \right) ,
\]

(41)

where \( \phi(x^a) \) is both nondynamical and massless. The metric tensor has to satisfy \( \mathcal{G} \equiv 0 \) to be a solution to the field equation \( R_{\mu\nu} - \frac{1}{2}Gg_{\mu\nu} + H^{(GB)}_{\mu\nu} = 8\pi GT^{(m)}_{\mu\nu} \) for the reduced Gauss-Bonnet dark energy \( S^{(3)}_{GB} \). For the flat FRW Universe with the metric Eq.(26), the Gauss-Bonnet invariant is

\[
\mathcal{G} = 24 \frac{\dot{a}^2 \ddot{a}}{a^3} ,
\]

(42)

and thus \( \mathcal{G} \) vanishes only if the Universe were of static state (\( \dot{a} = 0 \)) or constant acceleration (\( \ddot{a} = 0 \)). Hence, the constraint \( \mathcal{G} \equiv 0 \) for \( S^{(3)}_{GB} \) is inconsistent with the cosmic acceleration, which indicates that unlike \( S^{(1)}_{GB} \) and \( S^{(2)}_{GB} \), \( S^{(3)}_{GB} \) is oversimplified and can not be a viable candidate of effective dark energy.

### 4.3. Generalized scalar-tensor theory

Since \( S_{HE} \) and \( S_G \) in Eq.(1) respect the diffeomorphism invariance and the (generalized) contracted Bianchi identities, in this subsection we will ignore them and focus on the following scalar-tensor-type gravity in the Jordan frame:

\[
S_{ST} = \int d^4 x \sqrt{-g} \left( f(R, \phi) + \mathcal{L}_{NC} + \mathcal{L}_\phi + 16\pi G \mathcal{L}_m \right)
= \int d^4 x \sqrt{-g} \left( f(R, \phi) + h(\phi) \cdot \mathcal{F}(R, \cdots, \mathcal{R}) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m \right) ,
\]

(43)

where \( f(R, \phi) \) is a hybrid function of the Ricci scalar and the scalar field. \( f(R, \phi) \) contributes to the field equation by

\[
H^{f(R, \phi)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \delta \left( \sqrt{-g} f(R, \phi) \right) = -\frac{1}{2} f(R, \phi) \cdot g_{\mu\nu} + f_R R_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_R ,
\]

(44)
where \( f_R = f_R(R, \phi) = \partial f(R, \phi) / \partial R \). With the Bianchi identity \( \nabla^\mu \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \right) = 0 \) and the third-order-derivative commutator \((\nabla_{\nu} \Box - \Box \nabla_{\nu}) \), explicit calculations yield

\[
\nabla^\mu H_{\mu \nu}^{(R, \phi)} = - \frac{1}{2} f_\phi \cdot \nabla_{\nu} \phi ,
\]

(45)

where \( f_\phi = f_\phi(R, \phi) = \partial f(R, \phi) / \partial \phi \). On the other hand, the kinematical wave equation \( \delta S_{ST} / \delta \phi = 0 \) reads \( 2 \lambda(\phi) \cdot \Box \phi = - f_\phi - \tilde{f} \cdot h_\phi - \lambda_\phi \cdot \nabla_\alpha \phi \Box \phi + V_\phi \), which recasts the divergence \( \nabla^\mu H_{\mu \nu}^{(\phi)} = - \frac{1}{2} \left( \lambda_\phi \cdot \nabla_\alpha \phi \Box \phi + 2 \lambda(\phi) \cdot \Box \phi \cdot \nabla_{\nu} \phi \right) \) as in Eq. (16) into

\[
\nabla^\mu H_{\mu \nu}^{(\phi)} = \frac{1}{2} \left( f_\phi + \tilde{f} (R, \cdot \cdot, R) \cdot h_\phi \right) \cdot \nabla_{\nu} \phi .
\]

(46)

Hence, with Eqs. (14), (45) and (46), we immediately learn that the field equation \( H_{\mu \nu}^{(R, \phi)} + H_{\mu \nu}^{(NC)} + H_{\mu \nu}^{(\phi)} = 8 \pi G T_{\mu \nu}^{(m)} \) for the scalar-tensor-type gravity \( S_{ST} \) is divergence free. By the total Lagrangian density for the sake of simplicity, the concretization of Eq. (43) includes, for example, standard Brans-Dicke gravity \( L = m \left( \sqrt{\phi} \right) \) [13], generalized Brans-Dicke gravity \( L = m \left( \phi \right) \) [27], Lovelock-scalar-tensor gravity \( L = f_1(\phi) R + f_2(\phi) \Box R + f_3(\phi) \Box^3 \phi \) [27], minimal dilatonic gravity \( L = \phi R - 2 \Lambda U(\phi) \) [28], Gauss-Bonnet dilatonic gravity \( L = R - \nabla_\alpha \phi \Box^2 \phi - e^{-\gamma \phi} G \) or \( L = e^{-\gamma \phi} (R - \nabla_\alpha \phi \Box^2 \phi + G) \) motivated by the low-energy heterotic string theory [29], and the standard scalar-tensor gravity \( L = F(\phi) R - Z(\phi) \cdot \nabla_\alpha \phi \Box^2 \phi - V(\phi) + 16 \pi G L_{m} \) [30]; all these examples satisfy the local energy-momentum conservation \( \nabla^\mu T_{\mu \nu}^{(m)} = 0 \) and have divergence-free field equations.

4.4. Hybrid metric-Palatini \( f(R) \) gravity

So far we have been using the metric formulation for the curvature invariants; however, the local conservation can be proved for the Palatini or hybrid metric-Palatini \( f(R) \) gravity without referring to the Palatini formulation of the (generalized) Bianchi identities. Consider the following hybrid metric-Palatini \( f(R) \) action

\[
S_{Hf}^{(1)} = \int d^4 x \sqrt{-g} \left( R + f(\hat{R}) + 16 \pi G L_{m} \right) ,
\]

(47)

where \( R \) is the usual Ricci scalar for the metric \( g_{\mu \nu} \), while \( \hat{R} = \hat{R}(g, \hat{\Gamma}) = g^{\mu \nu} \hat{R}_{\mu \nu}(\hat{\Gamma}) \) denotes the Palatini Ricci scalar, with the Palatini Ricci tensor given by \( \hat{R}_{\mu \nu}(\hat{\Gamma}) = \hat{\Gamma}_{\mu \nu}^{\alpha} - \partial_\mu \hat{\Gamma}_\nu^{\alpha} + \hat{\Gamma}_\alpha^{\nu} \hat{\Gamma}_\mu^{\alpha} - \hat{\Gamma}_\mu^{\alpha} \hat{\Gamma}_\nu^{\alpha} \). Variation of \( S_{Hf} \) with respect to the independent connection \( \hat{\Gamma}_{\mu \nu}^{\alpha} \) yields \( \hat{\nabla}_{\rho} \left( \sqrt{-g} f_{\hat{R}} g^{\rho \mu} \right) = 0 \), where \( \hat{\nabla} \) is the covariant derivative of the connection and \( f_{\hat{R}} := df(\hat{R}) / d\hat{R} \). Thus, \( \hat{\nabla} \) is compatible with the auxiliary metric \( \hat{g}_{\mu \nu} = \hat{g}_{\mu \nu} \), as \( \sqrt{-g} \hat{g}_{\mu \nu} = \sqrt{-g} f_{\hat{R}} g^{\mu \nu} \). Relating \( \hat{g}_{\mu \nu} \) to \( g_{\mu \nu} \) by the conformal transformation \( g_{\mu \nu} \leftrightarrow \hat{g}_{\mu \nu} \), and accordingly rewriting \( \hat{R}_{\mu \nu} \) and \( \hat{R} \) in the metric formulation, one could find that \( S_{Hf}^{(1)} \) is equivalent to [31]

\[
S_{Hf}^{(2)} = \int d^4 x \sqrt{-g} \left( R + \phi R + \frac{3}{2 \phi} \nabla_\alpha \phi \Box \phi - V(\phi) + 16 \pi G L_{m} \right) ,
\]

(48)
where $\phi(x') = f_k(\tilde{R})$ and $V(\phi) = f_k\tilde{R} - f(\tilde{R})$. $S^{(2)}_{H_f}$ is just the mixture of GR and the $\omega_{BD} = -3/2$ Brans-Dicke gravity. Recall that Eq.(43) has employed the generic function $f(R, \phi)$ for $S_{ST}$, which includes the hybrid situations like $f(R, \phi) = R + \phi R$. Hence, following Sec. 4.3, it is clear that the hybrid scalar-tensor gravity $S^{(2)}_{H_f}$ and thus the hybrid metric-Palatini $f(\tilde{R})$ gravity $S^{(1)}_{H_f}$ have divergence-free field equations and respect the local energy-momentum conservation.

5. Conclusions

In this paper, we have investigated the covariant invariance of the field equation for a large class of hybrid modified gravity $\mathcal{L} = R + f(R, \cdots, \mathcal{R}) + h(\phi) \cdot \tilde{f}(R, \cdots, \mathcal{R}) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m$. For the four components $\mathcal{L}_{HE} = R$, $\mathcal{L}_{G} = f(R, \cdots, \mathcal{R})$, $\mathcal{L}_{NC} = h(\phi) \cdot \tilde{f}(R, \cdots, \mathcal{R})$, and $\mathcal{L}_{a} = -\lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi)$, we have calculated their contributions $\{G_{\mu\nu}, H^{(G)}_{\mu\nu}, H^{(NC)}_{\mu\nu}, H^{(f)}_{\mu\nu}\}$ to the gravitational field equation along with the respective divergences, which proves the divergence-freeness of the field equation (19) and confirms/proves the local energy-momentum conservation under minimal gravity-matter coupling.

$H_{\mu\nu}^{(NC)}$ and $H^{(f)}_{\mu\nu}$ fail to obey the generalized contracted Bianchi identities due to the presence of the background scalar field $\phi(x')$, but fortunately, the two nontrivial divergences $\nabla^\mu H_{\mu\nu}^{(NC)}$ and $\nabla^\mu H^{(f)}_{\mu\nu}$ exactly cancel out each other. When $\phi(x')$ is nondynamical and massless, i.e. $\lambda(\phi) = 0 = V(\phi)$, the divergence $\nabla^\mu H_{\mu\nu}^{(NC)} = -\frac{1}{2} \tilde{f}(R, \cdots, \mathcal{R}) \cdot h_\phi \delta \phi \cdot \nabla_\nu \phi$ is forced to vanish, which implies the constraint $\tilde{f}(R, \cdots, \mathcal{R}) \equiv 0$ for nonconstant $\phi(x')$. We have suggested a primary viability test for the gravity $\mathcal{L} = R + f(R, \cdots, \mathcal{R}) + h(\phi) \cdot \tilde{f}(R, \cdots, \mathcal{R}) + 16\pi G \mathcal{L}_m$ by requiring that $\tilde{f}(R, \cdots, \mathcal{R})$ vanishes identically for the flat and accelerating FRW Universe, and a simplest example is the Weyl dark energy $\mathcal{L} = R + \gamma \phi \mathcal{C}^2 + 16\pi G \mathcal{L}_m$.

With the general theory developed in Secs. 2.2 and 3.2, we have considered the applications to the Chern-Simons gravity, Gauss-Bonnet dark energy, and various (generalized) scalar-tensor gravities. In fact, the theory $\mathcal{L}_{ST} = f(R, \phi) + h(\phi) \cdot \tilde{f}(R, \cdots, \mathcal{R}) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m$ in Sec. 4.3 can be further extended into $\mathcal{L}_{EST} = f(R, \cdots, \mathcal{R}, \phi) - \lambda(\phi) \cdot \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m$, for which we conjecture that the covariant conservation still holds, with

$$H^{(f)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \delta \left[ \sqrt{-g} f(R, \cdots, \mathcal{R}, \phi) \right]$$

and

$$\nabla^\mu H^{(f)}_{\mu\nu} = -\frac{1}{2} f_{\phi}(R, \cdots, \mathcal{R}, \phi) \cdot \nabla_\nu \phi .$$

However, this divergence relation has not yet been proved in this paper, and we hope it could be solved in future.

In prospective studies, we will take into account the existent candidates of the energy-momentum pseudotensor $t_{\mu\nu}$ for the gravitation field (cf. Ref.[32] for a review), and discuss the global conservation $\nabla^\mu \left( T^{(m)}_{\mu\nu} + t_{\mu\nu} \right) = 0$. Also, we will make use of more fundamental definitions of the energy-momentum tensor, and look deeper into the conservation problem in modified gravities from the perspective of Noether’s theorem and the classical field theory.

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