Calculation of the two–photon decay width of the $f_0(980)$ scalar meson

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(Dated: July 4, 2018)
Abstract

The applicability of the quasi–static approximation for calculating the two–photon annihilation rate of the scalar $f_0(980)$ meson envisaged as a $K\bar{K}$ molecule is critically re–examined. It is shown that the validity of this approximation depends on the detailed interplay between the momentum dependence of the annihilation amplitude and the momentum space transform of the bound state wavefunction of the annihilating pair. The approximation becomes invalid when these two scales of variation are similar. An improved method of calculation based on the inclusion of electromagnetic corrections to the kernel of the Bethe–Salpeter equation for the interacting $K\bar{K}$ pair is outlined to cover this case and applied to re-evaluate the two–photon decay width for $f_0(980)$ in a one boson exchange model for the interkaon interaction. The corrections are significant and result in a much better agreement with experiment.

PACS numbers: 11.10.St, 14.40.Aq, 14.40.Cs, 36.10.-k

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1. Introduction

The structure of the lowest mass scalar meson \( f_0(980) \) has been under debate for some time. As possible candidates a \( q\bar{q} \) state [1], a \( q^2\bar{q}^2 \) state [2], a \( K\bar{K} \) molecule [3], or perhaps some combination of these structures [4] have been suggested. However, the large two–pion decay width predicted for the first option, \( \Gamma(q\bar{q} \rightarrow \pi\pi) \sim 500 \text{ MeV} \) from flux tube–breaking [5] to \( \sim 660 \text{ MeV} \) using current algebra [6], does not favor a pure \( q\bar{q} \) configuration in view of the typical experimental width \( \sim (40 – 100) \text{ MeV} \) for the \( f_0(980) \rightarrow \pi\pi \) decay.

On the other hand Barnes [8], in an earlier paper, advanced cogent arguments in support of the molecular picture, using various experimental data to obtain an estimate of the two–photon decay width of the \( f_0(980) \) that agrees qualitatively with experiment when this scalar meson is viewed as a \( K\bar{K} \) molecule. As in the case of positronium [9], Barnes’ calculation relies on the validity of the quasi-static, or wavefunction at contact approximation, where the annihilation rate is computed in Born approximation from the free annihilation cross–section multiplied by a kaon current that is proportional to the probability of having a bound \( K^+K^- \) pair at the origin. The latter quantity in turn depends on the assumed two–body potential that produces the molecular binding. In [8] this was taken to have a gaussian shape, with a depth and range typical of the interkaon Weinstein–Isgur potential [3].

Repeating the same calculation for the one boson exchange potential developed in [4] one finds that the two–photon width comes out an order of magnitude larger than for the Barnes’ potential, although the two–pion width calculated in the same quasi–static approximation agrees with experiment. We have therefore re–examined the validity of the quasi–static approximation by implementing a more complete calculation using the Bethe–Salpeter equation. It is shown that the applicability of this approximation depends on the detailed interplay between the momentum dependence scales of the annihilation amplitude and the momentum space transform of the bound state wave function \( \psi(r) \) of the annihilating pair. If the momentum space transform of \( \psi(r) \) has a much smaller range of variation than that of the annihilation amplitude, the quasi–static approximation becomes applicable. In the opposite situation where these two scales of variation are similar, the quasi–static approximation can be seriously in error and needs to be replaced by the method of calculation outlined in Section 3 below.

2. The \( K\bar{K} \) bound state

A non–relativistic approximation for the Bethe–Salpeter (BS) equation for weakly bound states [10] was employed in [4] to study the mass and strong decays of the \( f_0(980) \) scalar meson considered as a bound kaon–antikaon pair of isospin zero interacting via vector meson exchange. In momentum space, this equation reads

\[
\left( \frac{p'^2}{M_K} + 2M_K - P_0 \right) \phi(p') = \frac{1}{4M_K^2} \int \frac{d^3p}{(2\pi)^3} \tilde{\Gamma}(p', p) \phi(p)
\]  

(1)

where \( \tilde{\Gamma}(p', p) \) is the two–particle irreducible interaction kernel, or transition amplitude, appearing in the BS equation that describes the non–relativistic scattering of the pair in the center of mass (CM) system with a momentum change \( p \rightarrow p' \) for either kaon of mass \( M_K \).

The external four–momenta of the BS equation for bound states are off–shell but still obey four–momentum conservation. In particular the sum of the initial, equal to final, time
components appears explicitly as the energy eigenvalue parameter $P_0$ on the left hand side of Eq. (1) and also in the kernel $\hat{\Gamma}(p,p') = \hat{\Gamma}(p,p'; P_0)$. This eigenvalue lies on the unphysical sheet of the complex energy plane where the four–momenta of the colliding particles are off their mass shell. It gives the mass and decay half–width $P_0 = M - i \Gamma / 2 = 2M_K + \varepsilon - i \Gamma / 2$ of the bound system in the presence of interactions, where $\varepsilon < 0$ is the binding energy of the $K\bar{K}$ molecule. In obtaining the non–relativistic form Eq. (1) it has to be assumed in addition that these four–momenta become “almost” physical for weakly bound states [10].

The corresponding eigenvalue equation in coordinate space is given by the Fourier transform of Eq. (1). This is just the usual Schrödinger wave equation containing a non–local, generally complex, potential with the bound state wavefunction $\psi(r)$ having the transform $\phi(p)$. In [4], an expression for $\hat{\Gamma}(p',p)$ for $K\bar{K}$ scattering and annihilation has been derived using a standard $SU_V(3) \times SU_A(3)$ invariant interaction Lagrangian [11] to describe vector meson exchange between the kaons in the non–relativistic limit. In coordinate space this procedure leads to a real, local one boson exchange potential, plus a pure imaginary contact term that describes the $K\bar{K}$ annihilation into the $\pi\pi$ or $\pi\eta$ isoscalar and isovector channels respectively, i.e. to an optical potential of the form

\[
\mathcal{V}_{opt} \psi(r) = -\frac{1}{4M_K} \int d^3r' \hat{\Gamma}(r,r') \psi(r') \approx -\left[\mathcal{V}_{OBE}(r) + \frac{i}{2} \lim_{v_0 \to 0} (v_r \sigma_a^{(I)}) \delta^3(r)\right] \psi(r).
\]

Here $\hat{\Gamma}(r,r')$ is the coordinate space representation of $\hat{\Gamma}(p',p)$ and $\sigma_a^{(I)}$ the kaon–antikaon annihilation cross section for isospin $I$.

Explicit expressions for $\mathcal{V}_{OBE}^{(0)}$ and $\sigma_a^{(0)}$ in the isoscalar channel that are relevant for the present discussion can be found in [4]. The resulting optical potential leads to a bound state solution for the $K\bar{K}$ system in an $s$–state that gives $P_0 = (981 - 25i)$ MeV for the mass and half width of $f_0(980)$ in good agreement with experiment [7]. While direct measurements of $K\bar{K}$ scattering and annihilation cross sections have not been reported against which the scattering predictions given by $\mathcal{V}_{opt}$ can be compared, one can do so indirectly by using two–channel unitarity to express the cross section for the inverse process $\pi^+\pi^- \rightarrow K\bar{K}$, which has been measured [12], in terms of the common inelasticity parameter of the $K\bar{K}$ and $\pi\pi$ channels. This parameter may then be calculated from the $S$–matrix given by $\mathcal{V}_{opt}$ for the isoscalar channel, and leads to a reasonable accord with the experimental $\sigma(\pi^+\pi^- \rightarrow K\bar{K})$ cross section [4].

Taken together, these facts suggest that the optical potential of Eq. (2) provides an adequate representation of the low energy dynamics of the $K\bar{K}$ system, at least for $I = 0$.

3. Two–photon decay of $f_0(980)$

We now revisit the two–photon decay width problem for $f_0(980)$. One can first implement the quasi–static approximation by replacing the $v_r \sigma_a^{(0)}$ isoscalar cross section for annihilation into two pions with that for $K^+\bar{K}^-$ annihilation into two photons and then calculating the resulting imaginary part $-i\Gamma(\gamma\gamma)/2$ of the energy shift given by Eq. (2) in perturbation theory. The result is [4, 8]

\[
\Gamma(\gamma\gamma) \approx \Gamma_{qs}(\gamma\gamma) = \frac{2\pi\alpha^2}{M_K^2} \psi_{K+\bar{K}^-}^2(0), \quad \psi_{K+\bar{K}^-}(0) = \frac{1}{\sqrt{2}} \psi(0).
\]

\[
\epsilon < 2500, \quad \epsilon \approx 2500 \quad \text{if} \quad \epsilon \approx 2500.
\]

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\]
with \( \alpha = e^2/4\pi \), after projecting out the charged channel amplitude given by the overlap
\( \zeta = \langle K^+ K^- | K \bar{K} \rangle = 1/\sqrt{2} \) for good isospin from the \( K \bar{K} \) isoscalar ground state wave
function \( \psi(0) \) at the origin. In obtaining this estimate we have ignored a possible \( K^0 \bar{K}^0 \rightarrow 2\gamma \)
contribution that lies beyond the scope of the present calculations that assume point kaons, but this contribution is
known to be negligible at low energies in any event [3]. Using Eq. (3) one finds values for the annihilation width of
\( \sim 0.6 \) to 0.9 keV for the Barnes’ potential depending on how \( \psi(0) \) is estimated, or the much larger value 5.59 keV for the
one–boson exchange potential. The experimental value is listed as 0.31 \( \pm 0.11 \) keV [7].

This spread of calculated values simply reflects the impact of the details of the assumed \( K \bar{K} \) interaction potential on the value of \( \psi(0) \) used in Eq. (3). The more important question is how well Eq. (3) actually meets the quasi–static conditions used in its derivation, i.e. that
of being able to calculate the annihilation while ignoring binding and vice versa, at the typical binding energies [7] of about 10 to 20 MeV encountered for the \( K \bar{K} \) molecule. In order to address this question we include the contribution of electromagnetic interactions
between the charged kaons present in the \( f_0(980) \) ground state [19] as part of the interaction
kernel of the BS equation by replacing \( \hat{\Gamma}(p', p) \) with \( \hat{\Gamma}(p', p) + \hat{\Gamma}_{em}(p', p) \) in Eq. (4). Since
the \( f_0(980) \rightarrow 2\gamma \) decay width is related to \( Im\hat{\Gamma}_{em} \), we require two–particle diagrams with at least two intermediate state photons. For \( K^+ K^- \) scattering there are five diagrams of
this type to order \( \alpha^2 \) that together form a gauge invariant set. They are shown in Fig. 1.

Call \( i\hat{\Gamma}_{em} \) the sum of these five diagrams. Then one can show after some calculation
using the Cutkosky rules [10] that the imaginary part of the electromagnetic contribution
to Eq. (4) may be expressed in a form reminiscent of an optical theorem as

\[
2i Im \int \frac{d^3p}{(2\pi)^3} \hat{\Gamma}_{em}(p', p) \zeta \phi(p) = \frac{i}{32\pi^2} \int \frac{1}{2} d\omega M^{\mu\sigma}(p') \int \frac{d^3p}{(2\pi)^3} M_{\mu\sigma}(p) \zeta \phi(p)
\]

\[
= \frac{i}{32\pi^2} \int \frac{1}{2} d\omega \sum_{\lambda\lambda'} M^{\ast}_{\lambda\lambda'}(p') \int \frac{d^3p}{(2\pi)^3} M_{\lambda\lambda'}(p) \zeta \phi(p) \quad (4)
\]

where \( M_{\mu\sigma}(p) \) is the tensor amplitude for the annihilation of \( K^+ K^- \) into two photons at kaon momentum \( p \); \( \zeta = 1/\sqrt{2} \) is the isospin projection factor as before. The symmetry factor \( 1/2 \) arises naturally to restrict the integration \( d\omega \) over the scattering direction of one of the photons to half the solid angle. The second form involving the sum over the photon polarization vectors \( e^\mu_{\lambda}(k) \), follows upon introducing their completeness relation and setting

\[
M_{\lambda\lambda'}(p) = [e^\mu_{\lambda}(k) M_{\mu\sigma}(p) e^\sigma_{\lambda'}(-k)]
\]

We now obtain the electromagnetic decay width \( f_0 \rightarrow 2\gamma \) by including \( Im\hat{\Gamma}_{em}(p', p) \)
in the right hand side of Eq. (4) for the charged channel and calculating the additional imaginary shift in the total energy \( P_0 \rightarrow P_0 - i\Gamma(\gamma\gamma)/2 \) that this produces in perturbation theory. Then

\[
\Gamma(\gamma\gamma) = \frac{1}{4M_K^2} \int \int \frac{d^3p'}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \zeta \phi^*(p') [2i Im\hat{\Gamma}_{em}(p', p)] \zeta \phi(p)
\]

\[
= \frac{1}{64\pi M_K^2} \sum_{\lambda\lambda'} \left[ \int \frac{d^3p}{(2\pi)^3} \zeta \phi(p) M_{\lambda\lambda'}(p) \right]^2 ; \quad (6)
\]

Note in passing that this form reduces to the quasi–static result in Eq. (3) if only the static part \( p \rightarrow 0 \) of the annihilation amplitude is kept. For then \( M_{\lambda\lambda'} \) factors out of the momentum
integral to reproduce the free annihilation cross section as \( \sigma_{em} = \sum_{\lambda \lambda'} |M_{\lambda \lambda'}|^2 / (64 \pi M_K^2) \), while the remaining integral squared over \( \zeta \phi(p) \) just equals \( \psi_{K^+K^-}^2(0) \).

Since the Fourier transforms \( \phi(p) = \phi(p) \) are spherically symmetric in momentum space for \( s \)-waves it is convenient to split the integral in Eq. (6) into radial and angular parts,

\[
I_{\lambda \lambda'} = \int \frac{d^3p}{(2\pi)^3} \phi(p) M_{\lambda \lambda'}(p) = \frac{1}{(2\pi)^3} \int_0^\infty dp p^2 \phi(p) \Theta_{\lambda \lambda'}(p)
\]

where

\[
\Theta_{\lambda \lambda'}(p) = \int d\Omega M_{\lambda \lambda'}(p, \Omega)
\]

The transition amplitudes \( M_{\lambda \lambda'}(p) \) in these expressions are required at off–shell values of the kaon four–momenta. We take these as \( [\frac{1}{2} P_0, \pm \vec{p}] \approx [M_K, \pm \vec{p}] \) in the CM system after ignoring the small binding energy of the kaon pair. The outgoing photons are of course still on–shell with four–momenta \( [M_K, \pm \vec{k}] \) where \( |\vec{k}| = M_K \) to the same approximation; each photon carries away half the total mass of the decaying bound state. We evaluate \( M_{\lambda \lambda'}(p) \) at these off–shell four–momentum values from the standard expression for the annihilation amplitude \( M_{\mu \sigma}(p) \) for charged point-like bosons, without form factors, that can be found in [13] for example. Then from Eq. (11)

\[
M_{\lambda \lambda'}(p) = e^2 \left\{ 2(e_\lambda \cdot e_{\lambda'}) + \frac{(2p \cdot e_\lambda)(-2p \cdot e_{\lambda'})}{M_K^2 + (\vec{p} - \vec{k})^2} + (k \rightarrow -k) \right\}
\]

where \( |k| = M_K \), after introducing the transverse gauge \( e_\alpha''(k) = [0, e_\lambda] \) with \( k \cdot e_\lambda = 0 \) and \( e_\lambda \cdot e_{\lambda'} = \delta_{\lambda \lambda'} \) for the photon polarization vectors.

The angular integral in Eq. (8) over all kaon momentum directions is easily performed relative to a set of axes where the direction of the photon momentum \( k \) defines the polar axis and the two remaining orthogonal directions by the two polarization vectors of either photon. Then \( \Theta_{\lambda \lambda'}(p) \) has a common diagonal value in the polarization indices and is given by [20]

\[
\Theta_{\lambda \lambda'}(p) = 8\pi e^2 \theta_{\gamma \gamma}(p) \delta_{\lambda \lambda'}, \quad \theta_{\gamma \gamma}(p) = \left[ 4 + p^4 \frac{\ln \left( \frac{2 + p^2}{2 + p^2} - 2p - \frac{1}{2} p^2 \right)}{8p} \right]
\]

after expressing the three–momentum \( p \) in units of \( M_K \). Thus the final integral \( I_{\lambda \lambda'} \) that is required for the width calculation is also diagonal, with a common value \( I \) that we factor as

\[
I = 8\pi \alpha \psi(0) R, \quad R = \int_0^\infty dp f(p) \theta_{\gamma \gamma}(p),
\]

with \( f(p) \) defined as the normalized Fourier transform

\[
f(p) = \frac{p^2}{2\pi^2 \psi(0)} \phi(p); \quad \int_0^\infty dp f(p) = 1.
\]

As before \( \psi(0) \) is the radial wave function of the isoscalar \( K \bar{K} \) pair at the origin. Thus \( R = 1 \) if \( \theta_{\gamma \gamma}(p) \) is replaced by its static limit value \( \theta_{\gamma \gamma}(0) = 1 \). The final form of Eq. (4) for the two–photon decay width then reads

\[
\Gamma(\gamma \gamma) = \frac{2\pi \alpha^2}{M_K} \psi_{K^+K^-}^2(0) R^2 = \Gamma_{\phi \bar{\phi}}(\gamma \gamma) R^2
\]

The coefficient \( R^2 \) directly measures the deviation of this width from the quasi–static limit given by Eq. (3).
4. Calculations for two different potentials

Let us now recalculate the two-photon decay width for both the one–boson exchange (OBE) potential and Barnes’ gaussian potential. For computational convenience it is useful to replace both potentials by their Bargmann equivalent potentials \([14]\) that have the same scattering length \(a_0\) and effective range \(r_0\) as calculated numerically for the original potential forms. Then \((a_0, r_0)\) translate into the two parameters \((a, b)\) of the equivalent Bargmann potential form given in the Appendix. Analytic expressions for both the bound state wave function as well as its Fourier transform are then available in closed form, see Eqs. (19), (21) and Fig. 3.

(i) One Boson Exchange Potential. For this case the scattering length and effective range were calculated in \([4]\). They are \(a_0 = 5.835M_K^{-1}\) and \(r_0 = 1.187M_K^{-1}\), giving \(a = -0.1936M_K\) and \(b = 1.491M_K\). For this parameter set there is a single bound state at \(-18.60\) MeV. Then Eq. (19) gives \(\psi(0) = 0.260M_K^{3/2}\), that leads to \(\Gamma_{qs}(\gamma\gamma) = 5.59\) keV. Taking \(\phi(p)\) from Eq. (21) to do the integral for \(R\) numerically, one finds \(R = 0.287\) so that

\[
\Gamma(\gamma\gamma) = 0.46^{+0.10}_{-0.13} \text{ keV},
\]

or a reduction in the quasi–static value by more than an order of magnitude. The error bars have been estimated by considering the experimental uncertainties \([7]\) on \(P_0 = [(980 \pm 10) - i(20\) to \(50)]\) MeV for the \(f_0(980)\) mass and \(2\pi\) decay half width that change the values of \(a\) and \(b\) accordingly \([4]\).

(ii) Gaussian Potential. In \([8]\) Barnes estimated \(\psi(0)\) using a gaussian potential form \(V(r) = -V_0\exp(-r^2/2r_g^2)\) with \(V_0 = 440\) MeV fitted to the typical interkaon potential depth of the Weinstein–Isgur model, and a range of \(r_g = 0.57\) fm = \(1.435M_K^{-1}\) in order to reproduce a binding energy of \(\sim -10\) MeV for the \(KK\) pair. The scattering length and effective range for this potential are nearly double those of the OBE potential: one calculates that \(a_0 = 8.404M_K^{-1}\), \(r_0 = 2.397M_K^{-1}\) giving \(a = -0.1437M_K\), \(b = 0.6910M_K\) for the parameters of its Bargmann equivalent. The larger scattering length translates into a weaker binding energy, \(-10.2\) MeV, and thus a smaller wave function at the origin, \(\psi_g(0) = 0.102M_K^{3/2} \lesssim \frac{1}{\sqrt{V_0}} \psi(0)\). This gives the much smaller value 0.86 keV for \(\Gamma_{qs}(\gamma\gamma)\). Doing the \(R\) integral as under (i) the result is \(R = 0.632\) so that now

\[
\Gamma(\gamma\gamma) = 0.34 \text{ keV}
\]

This is only about \(24\) times smaller than \(\Gamma_{qs}(\gamma\gamma)\). The quasi–static limit is thus more reliable for kaons moving in the Barnes’ gaussian potential than in the one boson exchange potential. The reason for this is clear from Fig. 2. This figure compares the normalized Fourier transforms \(f(p)\) of Eq. (12) for both potentials over the momentum range \(0 < p < 3\) GeV of relevance for the present calculation. The gaussian has a smaller depth and longer range than the OBE potential. Thus the \(f(p)\) that determines \(R\) embraces a much smaller range of kaon momenta for the Barnes’ potential than it does for the OBE potential, and so gives an \(R\) less different from unity in the first case. Moreover, by equating the binding energy of the Barnes potential to that of the OBE potential without changing its range one finds that it predicts a \(\Gamma(\gamma\gamma) = 0.44\) keV, very close to the OBE result. Similarly, the OBE potential with its binding set equal to that of the Barnes’ potential leads to \(\Gamma(\gamma\gamma) = 0.38\) keV that in turn lies close to the Barnes’ result. Thus the calculated two-photon width depends mainly on the binding energy of the decaying state. This is a general feature of
the weak binding limit \( \kappa^2 \ll b^2 \). For then it follows from Eqs. (19), (21) and (12) that
\( \Gamma_{qs} \sim \psi^2(0) \sim \kappa b^2 \) depends on both the binding energy and potential range parameters \( \kappa \) and \((2b)^{-1}\), while the factor \( R \) of Eq. (13) varies mainly with the range like \( R \sim 1/b \). This is so because its integrand behaves like \( 1/b^2 \) times a function of \( p \) that suppresses the momentum integration beyond \( p \sim 2b \). Consequently the range parameter cancels out upon forming the product \( \psi^2(0)R^2 \), leading to the behavior \( \Gamma(\gamma\gamma) \sim \kappa \sim \sqrt{-\epsilon} \).

The results are summarized in Fig. 2 and Table I respectively. The conclusion is thus that there is little to distinguish between these two potential models for describing the \( K\bar{K} \) molecule as far the two–photon decay is concerned, once the momentum distribution of relative motion of the interacting pair is taken into account. Both lead to very similar two–photon decay widths that are in fact in semi–quantitative agreement with experiment.

5. Comparison with \( \pi\pi \) decay of \( f_0(980) \)

The calculated value of \( \Gamma(\pi\pi) \approx 50 \text{ MeV} \) for the \( f_0 \rightarrow 2\pi \) decay width already quoted in Section 2 for the OBE potential model using the quasi–static approximation falls well within the rather wide range of experimental values for \( \Gamma(\pi\pi) \) of 40 to 100 MeV reported in the literature [7]. On the other hand the gaussian model scales down this result by \( \psi^2_g(0)/\psi^2(0) \sim 1/6 \) to \( \sim 8.3 \text{ MeV} \) that falls way below experiment. It is therefore important to check whether non–static effects can also lead to significant changes in either of these estimates.

We assume as before [4] that the transition amplitude for the two-pion decay of the \( f_0(980) \) proceeds via \( K^* \) vector meson \( t \)–channel exchange [21]. Then the photons are replaced by pions and the intermediate kaons by \( K^* \)’s in the first two diagrams of Fig. 1. Note that both the direct and crossed diagrams contribute since the the pions are identical bosons in the isospin basis. The imaginary part of this combination leads to the analog of Eq. (4) for pions with the tensor scattering amplitude \( M_{\mu\sigma} \) replaced by a scalar amplitude \( M_{\pi\pi} \). Then Eq. (10) for pions reads \( \Theta_{\pi\pi}(p) = 8\pi g^2_{\pi\piKK}\theta_{\pi\pi}(p) \) where \( g^2_{\pi\piKK} \) is an effective coupling constant that determines the free \( K\bar{K} \rightarrow \pi\pi \) annihilation amplitude. Its value is not needed for the present discussion. Omitting the calculational details, one finds

\[
\theta_{\pi\pi}(p) = \left( \frac{M^2_{K^*}}{4 + p^2_\pi} \right) \left[ \frac{4 + M^2_{K^*} + 2p^2_\pi + 2p^2}{4pp_\pi} \ln \frac{M^2_{K^*} + (p + p_\pi)^2}{M^2_{K^*} + (p - p_\pi)^2} - 1 \right] \quad (16)
\]

in units \( M_K \), that simplifies to

\[
\theta_{\pi\pi}(p) \approx \left[ \frac{5 + p^2_\pi}{2p} \ln \frac{5 + p^2 + 2p}{5 + p^2 - 2p} - 1 \right] \quad (17)
\]

if we use the estimates \( M_{K^*} \approx 2M_K \) and \( p_\pi \approx M_K \) for the \( K^* \) mass and pion threshold momentum in order to compare more directly with \( \Theta_{\gamma\gamma}(p) \). The function \( \theta_{\pi\pi}(p) \) is shown in Fig. 2. Replacing \( \Theta_{\gamma\gamma}(p) \) by either form for \( \theta_{\pi\pi}(p) \) in Eq. (11), one obtains a modification factor of \( R_{\pi\pi} \lesssim 1.1 \) for both the OBE and gaussian potentials, an insignificant change. One can understand this result without any calculation. It comes about because the momentum scale over which \( \theta_{\pi\pi}(p) \) varies is fixed by the mass of the exchanged boson, in this case the \( K^* \), that is nearly a factor two more massive than the exchanged kaon that fixes the momentum scale for \( \Theta_{\gamma\gamma}(p) \). This means that the value of the integral determining \( R_{\pi\pi} \) becomes almost the same as the normalization integral for \( f(p) \) since \( \theta_{\pi\pi}(p) \) hardly varies.
at all over the momentum range of $f(p)$. In contrast with the $2\gamma$ decay problem, the quasi–static approximation introduces an insignificant error into the calculation of the $2\pi$ decay width. The entries for $\Gamma(\pi\pi)$ in Table I are thus essentially unchanged from their quasi–static values.

6. Discussion and conclusions

We have shown that calculating the annihilation width of a decaying bound system in the quasi–static approximation can be seriously in error when the momentum ranges of variation of the annihilation amplitude and the momentum transform of the bound state wave function are similar, and have given a revised formulation by including electromagnetic corrections in the kernel of the Bethe–Salpeter equation to cover this case. The resulting changes in the calculated two–photon annihilation widths for $f_0(980) \to \gamma\gamma$ in the molecular picture are considerable for one boson exchange model, less so for the Barnes’ potential model. In fact the predictions for these two competing potentials are brought into semi–quantitative agreement with each other and experiment as shown in Table 1. In contrast the calculated two–pion annihilation width for $f_0(980) \to \pi\pi$ is essentially unaffected by non–static corrections, but has completely different values for the two models that favor the one boson exchange model.

Since the two–photon decay width in the OBE model has been decreased by nearly an order of magnitude by the non–static corrections to already agree semi–quantitatively with experiment, the speculation in Ref. [4] of important admixtures of, for example, $q^2\bar{q}^2$ states in the pure $K\bar{K}$ ground state in order to reproduce the two–photon width based on the quasi–static approximation falls away: The one boson exchange model reproduces the mass, as well as reasonable two–pion and two–photon decay widths for $f_0(980)$ without any further assumptions once non–static effects are included in the electromagnetic sector, thereby giving additional support to the suggestion that this meson is predominantly a $K\bar{K}$ molecule.

7. Acknowledgments

This research was supported by the Ernest Oppenheimer Memorial Trust in terms of a Harry Oppenheimer Fellowship Award, which is gratefully acknowledged. I would also like to thank Professor Helmut Hofmann and other members of the Theory Division at the Physik–Department of the Technische Universität München at Garching for their kind hospitality, as well as Dr Veljko Dmitrašinović of the Vinča Institute of Nuclear Sciences, Belgrade, for essential correspondence.

8. Appendix: Equivalent Bargmann Potentials

For calculational simplicity we replace both the expression for $V_{OBE}$ that appears in Eq. (2) as well as Barnes’ gaussian potential by their Bargmann equivalents that have the same respective scattering lengths and effective ranges. These are given by the two–parameter potential [14]

$$V_{Barg}(r) = \frac{1}{M_K} \frac{8b^2}{b^2 - a^2} \left[ \frac{e^{br}}{b-a} + \frac{e^{-br}}{b+a} \right]^2$$

(18)
with \( a = -[1 - \sqrt{1 - 2r_0/a_0}/r_0 \) and \( b = [1 + \sqrt{1 - 2r_0/a_0}]/r_0 \) in order to reproduce the same scattering length \( a_0 \) and effective range \( r_0 \) of the original potential.

If \( a = -\kappa < 0 \) the Bargmann potential in Eq. (18) has a single bound \( s \)–state at energy \( \epsilon = -\kappa^2/M_K \). The normalized bound state eigenfunction belonging to this eigenvalue is given by

\[
\psi(r) = \psi(0) \frac{u(r)}{r}, \quad \psi(0) = \left[ \frac{2\kappa(b^2 - a^2)}{4\pi} \right]^{1/2}
\]

\[
u(r) = \frac{\tanh(br)}{b - \kappa \tanh(br)} e^{-\kappa r} \rightarrow r, \quad r \rightarrow 0
\]

We note that for weak binding in a short range potential, i.e. \( a_0 >> r_0 \), the Bargmann potential parameters below Eq. (18) reduce to \( a \approx -1/a_0 \) and \( b \approx 2/r_0 \) approximately.

In this limit one retrieves the standard universal forms \[16\] \( \epsilon \approx -1/M_K a_0^2 \) and \( r\psi(r) \approx (2\pi a_0)^{-1/2} \exp(-r/a_0) \) for the binding energy and radial wavefunction at large distances \( r >> r_0 \).

We also require the Fourier transform \( \phi(p) \) of \( \psi(r) \) defined by

\[
\psi(r) = \int \frac{d^3p}{(2\pi)^3} \phi(p) e^{i\mathbf{p}\cdot\mathbf{r}}
\]

This only depends on the magnitude of the momentum \( p = |\mathbf{p}| \) since \( \psi(r) \) in Eq. (19) is spherically symmetric. One finds

\[
\phi(p) = \int d^3r \psi(r) e^{-i\mathbf{p}\cdot\mathbf{r}} = \frac{4\pi}{p} \psi(0) Im \int_0^\infty dr \ u(r) e^{ipr}
\]

\[
= \frac{4\pi}{p} \psi(0) \frac{1}{4b^2} Im \left\{ \frac{1}{\rho(1+\rho)} 2F_1[1,2,2+\rho;\frac{1}{2} + \frac{\kappa}{2b}] \right\}, \quad \rho = \frac{\kappa}{2b} - i \frac{p}{2b}
\]

\[
\sim \frac{1}{p^6}, \quad p \rightarrow \infty
\]

in terms of the hypergeometric function \( 2F_1(\alpha, \beta, \gamma; z) \) \[17\]. This result follows immediately upon using the substitution \( x = 1 - \exp(-2br) \) to transform the radial integral into an integral representation of the hypergeometric function.

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[19] We ignore the instantaneous Coulomb attraction between the charged partners due to one-photon exchange, since this is insignificant in comparison with the strong interaction in the $K\bar{K}$ ground state. The roles of the strong and Coulomb interactions are reversed in the $K^+K^-$ (kaonium) atom, where the Coulomb binding is dominant, the strong interaction endowing the pure Coulomb state with a small additional energy shift and decay width \([4]\).

[20] For comparison, using on–shell kaon four–momenta replaces $\theta_{\gamma\gamma}(p)$ by $[p(2 + p^2)]^{-1}\ln((2 + p^2 + 2p)/(2 + p^2 - 2p))$ that only starts to differ from Eq. \([10]\) at $O(p^4)$.

[21] Additional contributions to the $K\bar{K} \rightarrow \pi\pi$ width can arise from $s$–channel scalar meson exchange. Using the relevant coupling constants derived in \([15]\) from the linear $\sigma$ model for the exchange of the $m_\sigma \approx 600$ MeV $f_0(600)$, or $\sigma$ meson, one estimates a contribution of $\lesssim 11$ MeV to the total decay width from this source. This small value is mainly due to suppression of the $K\bar{K}$ to nonstrange $\sigma$ coupling.
TABLE I: Summary of the calculated values for the one–boson exchange and gaussian potentials. The numbers quoted below have been obtained from their Bargmann equivalent potentials as described in the text. The experimental values from various sources have also been listed.

| Source                                      | ψ(0) GeV$^{3/2}$ | $\Gamma_{q\bar{q}}$(γγ) keV | $R_{\gamma\gamma}$ | $\Gamma(\gamma\gamma)$ keV | $\Gamma(\pi\pi)$ MeV |
|---------------------------------------------|------------------|-----------------------------|-------------------|-----------------------------|---------------------|
| One boson exchange potential [4]            | 0.091            | 5.59                        | 0.287             | 0.46$^{+0.10}_{-0.13}$      | ~ 50                |
| Gaussian potential [8]                       | 0.036            | 0.86                        | 0.632             | 0.34                        | ~ 8.3               |
| Particle Data Group [7]                     | -                | -                           | -                 | 0.31$^{+0.08}_{-0.11}$      | 40 to 100           |
| Fermilab E 791 Collaboration [18]           | -                | -                           | -                 | -                           | 44 ± 4              |
FIG. 1: The five diagrams that contribute to order $\alpha^2$ to the electromagnetic part of interaction kernel in the Bethe–Salpeter equation for $K^+K^-$ scattering. Solid lines: charged kaons, broken lines: photons.
FIG. 2: Variation of the angular integrals $\theta(p)$ given by Eqs. (10) and (16) with the kaon CM momentum for $2\gamma$ versus $2\pi$ decay of the $f_0(980)$ (left hand scale). Also shown are the corresponding normalized Fourier transforms $f(p) = p^2 \phi(p)/(2\pi^2 \psi(0))$ of Eq. (12) for the one boson exchange (OBE) and Barnes’ gaussian potentials respectively (right hand scale).
FIG. 3: Comparison of the numerically generated normalized radial bound state wave functions for the one–boson exchange and Barnes’ gaussian potentials (solid curves) respectively, with their analytic counterparts given by Eq. (19) for their equivalent Bargmann potentials (broken curves). The numerical and analytic wave functions essentially coincide in the case of the Barnes’ potential.