Microscopic Approach to Shear Viscosities in Superfluid Gases: From BCS to BEC

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(Dated: December 14, 2010)

We compute the shear viscosity, \( \eta \), at general temperatures \( T \), in a BCS-BEC crossover scheme which is demonstrably consistent with conservation laws. The study of \( \eta \) is important because it constrains microscopic theories by revealing the excitation spectra. The onset of a normal state pairing gap and the contribution from pair degrees of freedom imply that \( \eta \) at low \( T \) becomes small, rather than exhibiting the upturn predicted by most others. Using the local density approximation, we find quite reasonable agreement with just-published experiments.

PACS numbers: 03.75.Ss,67.10.Jn, 67.85.De

The ultracold Fermi gases near the unitary limit are thought to be related to quark-gluon plasmas [1]. Much attention has focused on particle-physics-based calculations of the anomalously low shear viscosity, \( \eta \) [2,3]. However, considerable insight on the thermodynamics [4] and various spectroscopic studies [5,6] has also been obtained via a condensed matter perspective. This paper belongs to the second school in which BCS theory is extended to accommodate arbitrarily strong interactions. We apply BCS to Bose Einstein condensation (BEC) crossover theory [8] to compute \( \eta \) demonstrating consistency with central sum rules [7] and conservation laws.

The shear viscosity is a powerful probe for testing microscopic theories, because it reflects the normal fluid component. As a result it is extremely sensitive to the nature of the excitation spectrum. In a low \( T \) normal Fermi liquid phase with scattering lifetime \( \gamma^{-1} \) and effective mass \( m^* \), \( \eta = \frac{\pi}{3} n_F \gamma m^* \). More generally, one can think of \( \eta \) as characterized by the effective number of the normal excitations \( n \to n_{eff} (T) \) as well as their lifetime which we emphasize here is a many body effect. Crucial is an understanding of how \( n_{eff} \) depends on \( T \). BCS-based approaches have addressed the behavior of the viscosity in a notable class of fermionic superfluids–helium-3 [8,9]. Here experiments [10] indicate that the viscosity in a notable class of fermionic superfluids–helium-3 decreases with decreasing \( T \). When quantitatively compared with very recent shear viscosity experiments [1] (we independently infer [15] from radio frequency data) the agreement is reasonable. Using previous thermodynamical calculations [4] of the trap energy \( E \) and entropy density \( s \), a plot of the trap-integrated \( \eta \) decreases with decreasing \( E \), while \( \eta / s \) appears to be roughly constant at low \( E \), but above the universal quantum limit [3].

We compute viscosities using the more well controlled current-current correlation functions [7], \( \chi_{JJ} \) rather than stress-tensor correlation functions, via

\[
\eta = -m^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} \chi_T(\omega, q)
\]

which is importantly constrained by the sum rule [7]

\[
\lim_{q \to 0} \int_{-\infty}^{\infty} d\omega \frac{d\omega}{\pi} \left( -\frac{\text{Im} \chi_T(\omega, q)}{\omega} \right) = \frac{n_m(T)}{m}, \tag{1}
\]

The transverse susceptibility \( \chi_T = (\sum_{\alpha=x} \chi_{JJ}^{\alpha\alpha} - \chi_L) / 2 \) with the longitudinal \( \chi_L = \hat{q} \cdot \chi_{JJ} \cdot \hat{q} \). Here \( n_m(T) \) cor-
responsible for the number of particles in the normal fluid and \( n_n(T) \to n \) above \( T_\ast \), where \( n \) is the total particle number. Because dissipative transport in BCS-BEC theory is complex, among the most persuasive checks on a proper characterization of the “normal fluid” is consistency with sum rules. Equation (1) is more general and fundamental than sum rules derived in Ref. [16].

Our theoretical scheme is based on the BCS-Leggett ground state, extended [5] to non-zero temperature \( T \). There are then two contributions to the square of the pairing gap \( \Delta^2(T) = \Delta_{sc}^2(T) + \Delta_{pg}^2(T) \), corresponding to condensed (sc) and non-condensed (pg) pairs, which are associated with a pseudogap. The fermions have dispersion \( E_p \equiv \sqrt{\xi^2_p + \Delta^2(T)} \), where \( \xi_p = \epsilon_p - \mu \) (In what follows, we omit the subscript \( p \) for convenience).

A consistent set of diagrams which satisfy Eq. (1), include [5, 17, 18], both Maki Thompson (MT) and two Aslamazov-Larkin (AL) diagrams. In general, the current-current correlation function is \( \chi_{JJ} = \tilde{P} + \frac{3}{m} C J \), where \( C J \) is known [17, 18] and associated with excitations of the collective modes. All transport expressions in this paper reduce to those of strict BCS theory when the attraction is weak and \( \Delta_{pg} = 0 \).

Consistency in linear response theory is based on the use of Ward identities which connect transport to the fermionic self energy,

\[
\Sigma(p, \omega) = -i\gamma + \frac{\Delta_{pg}^2}{\omega + \xi_p + i\gamma} + \frac{\Delta_{sc}^2}{\omega + \xi_p + i0^+}.
\]

Here we use a broadened BCS self energy which was adopted experimentally in Refs. [19] and theoretically [6] to address radio frequency (RF) based studies in the cold gases. The condensed pairs have the usual BCS self energy contribution, \( \Sigma_{sc} \), while the self energy of the non-condensed pairs \( \Sigma_{pg} \) contains an additional damping term parameterized by \( \gamma \).

To keep the equations simple and transparent we proceed in two stages. We begin in the weak dissipation limit, by which we mean the lifetime of the fermionic excitations is very long, so that to leading order we may set \( \gamma \approx 0^+ \) in Eq. (2). In this weak dissipation limit, using the MT and AL diagrams [5, 17, 18] one can write more simply

\[
\tilde{P}(\omega, q) = \sum_{p} \frac{p^2}{m^2} \left[ \frac{E_p + E_-}{E_p + E_-} (1 - f_+ - f_-) \right. \\
\times \frac{E_p + E_- - \xi_+ \xi_- - \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}}{E_p + E_- - \xi_+ \xi_- - \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}} \\
\left. \times \frac{E_p + E_- + \xi_+ \xi_- + \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}}{E_p + E_- + \xi_+ \xi_- + \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}} \right],
\]

where \( h = 1, E_{\pm} = E_{p+m/2}, f_\pm = f(E_{\pm}) \) and \( \delta \Delta^2 = \Delta_{sc}^2 - \Delta_{pg}^2 \).

With these diagrams, one can prove consistency with the sum rule for the transverse susceptibility (Eq. [1]). For the more general case the proof depends on the fact that Eq. (1) is closely tied to the absence (above \( T_\ast \)) and presence (below \( T_\ast \)) of a Meissner effect. More explicitly, in the weak dissipation limit, we note that the total number of particles can be written as \( n = \sum_p (1 - \frac{1}{2}(1 - 2f_\pm)) \).

The superfluid density at general temperatures is given by \( n_s = m \text{Re} P^{xx}(0, 0) + n = \frac{2 \Delta^2}{\pi^2} \sum_p \frac{E_p^2}{(\omega^2 + \Delta_p^2)} \).

The transverse susceptibility at \( q \to 0 \) contains no collective mode contributions; thus using \( \tilde{P} \) in Eq. (3), the left hand side of Eq. (1) is

\[
\sum_{p} \frac{p^2}{m^2} \left( \frac{2 \Delta_{sc}^2}{\omega^2 + \Delta_{sc}^2} - \frac{1}{2} \right) - \frac{\omega^2 - \Delta_{sc}^2}{\omega^2} \frac{\partial f}{\partial E} = \frac{n - n_s}{m} = \frac{4 \pi}{m} \],

thereby proving the sum rule.

We introduce the abbreviated notation \( \delta_1(\omega) = \delta(\omega - E_+ - E_-), \delta_2(\omega) = \delta(\omega - E_+ + E_-), \) and \( \theta \) is the polar angle. Then based on \( \chi_T(\omega, q) \) the shear viscosity is

\[
\eta = -m^2 \lim_{q \to 0} \frac{\pi \omega}{2 q^2} \sum_{p} \frac{p^2 \sin^2 \theta}{m^2} \left[ (1 - f_+ - f_-) \right.
\times \frac{E_p E_- - \xi_+ \xi_- - \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}}{E_p E_- - \xi_+ \xi_- - \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}} \\
\left. \times \frac{E_p E_- + \xi_+ \xi_- + \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}}{E_p E_- + \xi_+ \xi_- + \frac{\delta \Delta^2}{\omega^2 - (E_p + E_-)^2}} \right],
\]

If we now take the low \( \omega, q \) limits in Eq. (4), \( \eta \) assumes a form similar to a stress tensor correlation function. We incorporate lifetime effects [14] (which preserve the analytic sum rule consistency) by writing \( \delta(\omega \pm q \cdot \nabla_E) = \lim_{\gamma \to 0} \frac{\frac{1}{\omega \pm q \cdot \nabla_E}}{E^2 + \gamma^2} \). After regularizing

\[
\eta = \int_0^\infty dp \frac{p^6}{15 \pi^2 m^2} \frac{E^2 - \Delta_{pg}^2}{E^2} \frac{\xi^2_\omega}{E^2 - \frac{\partial f}{\partial E}} \frac{1}{\gamma^2 + \gamma^2}
\]

from which one can identify the effective carrier number \( (n_{eff}(T)) \propto \eta \gamma \) discussed in the introduction, and verify that it is dramatically suppressed at low \( T \).

Physically, the two types of terms which appear in Eq. (4) are well known in standard BCS theory. The first \( \delta_1 \) refers to processes which require a minimal frequency of the order of \( 2 \Delta(T) \); they arise from the contribution of fermions which are effectively liberated by the breaking of pairs. The second of these terms, involving \( \delta_2 \), arises from scattering of fermionic quasi-particles and is the only surviving contribution to the viscosities, which are defined in the \( \omega \to 0 \) limit. Note that both contributions involve the difference of the condensed and non-condensed components \( (\delta \Delta^2 = \Delta_{sc}^2 - \Delta_{pg}^2) \) with opposite overall signs. Importantly, the low \( \omega \) quasi-particle scattering processes are reduced by the presence of non-condensed pairs – simply because more pairs lead to fewer fermions. By contrast in the high \( \omega \approx 2 \Delta \) limit the number of contributing fermions will be increased by the presence of non-condensed pairs when they are broken, although this effect is only present in a \( \omega \neq 0 \) response.
Equation (5) is a generally familiar BCS expression except for the effects associated with non-zero $\Delta_{pg}$ which appears as a prefactor $1 - \frac{\Delta_{pg}^2}{E^2}$. This deviation from unity can be traced to the AL diagrams. Note that all terms are reduced by a coherence-effect-prefactor associated with $\frac{\delta^2}{E^2}$. The fact that the non-condensed pairs suppress $\eta$ derives from the same physics discussed surrounding Eq. (4) and associated with the scattering term $\delta_1$. The negative sign for this bosonic term comes physically from the fact that when pairs are present there are fewer fermions to contribute to the viscosity. In a more formal sense, this contribution is required for number conservation and the sum rule involving $\chi_T$.

Importantly, the expression in Eq. (5) can be generalized to the stronger dissipation limit, by introducing generalized Green’s functions defined in Ref. (24), which represent the various $pg$ and $sc$ contributions. $\eta = -\sum n_{\pm}^{2} \left[ (1 - \frac{\Delta_{pg}^2}{E^2}) F_{pg}^+ F_{pg}^- + G^+ G^- + F_{sc}^+ F_{sc}^- \right]_{\Omega_m \rightarrow 0^+}$, where $\pm$ is $P \pm Q/2$, $P = (p, i\omega_n), Q = (q \rightarrow 0, i\Omega_m)$, $\omega_n, \Omega_n$ are (fermionic, bosonic) Matsubara frequencies. We emphasize that the lifetime, $\gamma^{-1}$, is a temperature dependent many body effect here found to be associated with pair-fermion inter-conversion, and partially quantified by the cold atom Radio Frequency (RF) “photomission” experiments [6, 10] for $^{40}$K. In $^{6}$Li, where the viscosity experiments are performed [1, 21, 22], we have previously estimated [15] this parameter by fitting non-momentum-resolved RF experiments. We note that the introduction of this fermionic lifetime into a broadened BCS form for the non-condensed pairs (or pseudogap contribution) of Eq.(2) is widely used in the high $T_c$ literature [3] as well.

In Figure 1(a) we plot the shear viscosity for a unitary gas as computed via the strong dissipation approach. We have verified that the results are similar if we use the simpler form of Eq.(4) directly. The plot is for $\alpha$ defined as $\eta \equiv \alpha n$ versus temperature for a homogeneous system at unitarity and for a range of different lifetime parameterizations. The inset to Figure 1(a) presents a plot of the RF-deduced lifetime as black squares along with a few alternative functional forms. Each of these corresponds (via color coding) to the plots for $\alpha$ in the main body of the figure. In all cases $\alpha$ drops to zero at low temperatures, although one can see that this is slightly countered by the fact that the fermions are longer lived at low $T$. This figure should make it clear that the behavior shown here is quite independent of any detailed models for the inverse lifetime $\gamma^{-1}$. This largely follows from the intuition already embedded in Eq.(5) showing the dominant effect is due to exponential decrease in the number of condensate excitations associated with $s$-wave superfluidity.

In the inset of Figure 1(b) we plot the effective carrier number defined as $\propto \eta^{-1} \gamma$ as a function of $T$. The curve, normalized to the high temperature value where $\Delta = 0$, shows a clear suppression of the carrier number associated with the non-condensed pairs. The main body of the figure presents a breakdown of the various contributions to the viscosity coming from this bubble term and from the contributions of condensed (proportional to $\Delta_{sc}^2$) and non-condensed (proportional to $\Delta_{pg}^2$) pairs.

We turn now to calculations in a trap based on the local density approximation. The inset to Figure 2(a) presents a plot of experimental data from Ref. [1]. The main body of Figure 2(a) presents a comparison of the viscosity coefficient $\alpha$ between theory (based on the RF-deduced lifetime), as black dots, and experiment [22] (red triangles) as a function of $E$. Figure 2(b) shows the comparison of $\eta/s$ where $s$ is the entropy density. We find that $\eta/s$ appears to be relatively $T$ independent at
of the lower temperatures. Here we have used the calculated trap thermodynamics [1] to rescale the various axes, and our calculations are based on the same trap averaging procedure as in Ref. [22]. One can anticipate that, particularly at the lower $T$, the trap-integrated viscosity will be artificially higher than for the homogeneous case, since $\eta$ will be dominated by unpaired fermions at the trap edge. Overall, it can be seen that our calculations agree favorably with the preliminary experimental data shown in the inset and figure. Interestingly, the observed behavior appears more consistent with previous helium-3 experiments [10] than those in helium-4 [11].

There appear to be no other BCS-BEC based calculations of $\eta$ in the literature which address the entire range of $T$ below $T_c$ and also the consistency check of Eq. (1). Bruun and Smith [13] have studied the above $T_c$ shear viscosity and recognized [13] that the pseudogap reduces $\eta$. However, the diagram set which was used was “not conserving” [13]. Rupak and Schafer [3] argued that $\eta$ is dominated by the Goldstone bosons or phonons and the pseudogap reduces $\eta/s$. Importantly, in BCS-based fermionic superfluids the Goldstone bosons do not couple to a transverse response such as $\eta$.

In summary, in this paper we have effectively ascribed the low viscosity of the strongly interacting Fermi gases to effects associated with the depression in the normal fluid carrier density ($n_{eff}(T)$) arising from non-condensed pairs and the related pseudogap. Our viscosity decreases monotonically as $T \to 0$, as seen in superfluid helium-3 [10] and in very recent Fermi gas data [1], suggesting the absence of a ground state normal fluid. For these fermionic superfluids, lifetime effects are associated with fermion-boson interconversion. Moreover because of this mixed statistics [12], Boltzmann approaches are not amenable to including these lifetime contributions. We emphasize that all approaches to transport must necessarily demonstrate, as we do here, consistency with number conservation.

With these on-going debates and the possible impact for other physics sub-disciplines, measurements and theories of the shear viscosity take on particular importance. As shown here they constrain microscopic theories through the detailed information they provide about the nature of the superfluid excitation spectrum. Interestingly, calculations of the $\omega \to 0$ conductivity lead, by analogy, to “bad metal” behavior [24] in the cuprates, which may be a counterpart to perfect fluidity in the cold gases.

This work is supported by NSF-MRSEC Grant 0820054. We thank Ben Fregoso and also Le Luo and John Thomas for helpful conversations. C.C.C. acknowledges the support of the U.S. Department of Energy through the LANL/LDRD Program.

[1] C. Cao, E. Elliot, J. Joseph, H. Wu, J. Petricka, T. Schafer, and J. E. Thomas, Science, 9 December 2010 (10.1126/science.1195219).
[2] P. K. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. 94, 111601 (2005).
[3] G. Rupak and T. Schafer, Phys. Rev. A 76, 053607 (2007).
[4] J. Kinast, A. Turlapov, J. E. Thomas, Q. J. Chen, J. Stajic, and K. Levin, Science 307, 1296 (2005).
[5] Q. J. Chen, J. Stajic, S. Tan, and K. Levin, Phys. Rep. 412, 1 (2005).
[6] Q. Chen, Y. He, C.-C. Chien, and K. Levin, Rep. Prog. Phys. 72, 122501 (2009).
[7] L. P. Kadanoff and P. C. Martin, Annals of Physics 24, 419 (1963).
[8] O. Valls, and A. Houghton, Phys. Lett. 50A, 211, (1997); M. A. Shahzamanian, J. Phys. C 21, 553 (1998).
[9] M. Dorffe, H. Brand, and R. Graham, J. Phys. C 13, 3337 (1980).
[10] M. Nakagawa, A. Matsuhara, O. Ishikawa, T. Hata, and T. Kodama, Phys. Rev. B 54, R6849 (1996); C.N. Archie, T.A. Alvesado, J.D. Reppy, and R.C. Richardson, J. Low Temp. Phys. 42, 295 (1981).
[11] A. D. B. Woods and A. C. Hollis Hallet, Can. J. Phys. 41, 596 (1963).
[12] R. A. Brogliab, A. Molinari, and T. Regge, Ann. Phys. 109, 349 (1977).
[13] G.M. Bruun and H. Smith, Phys. Rev. A 72, 043605 (2005); G.M. Bruun and H. Smith, Phys. Rev. A 75, 043612 (2007).
[14] L. Kadanoff and P. Martin, Phys. Rev. 124, 670 (1961).
[15] C. C. Chien, H. Guo, Y. He, and K. Levin, Phys. Rev. A 81, 023622 (2010).
[16] E. Taylor and M. Randeria, Phys. Rev. A 81, 053610 (2010).
[17] I. Kosztin, Q. J. Chen, Y.-J. Kao, and K. Levin, Phys. Rev. B 61, 11662 (2000).
[18] H. Guo, C.-C. Chien, and K. Levin, Phys. Rev. Lett. 105, 120401 (2010).
[19] J. T. Stewart, J. P. Gaebler, and D. S. Jin, Nature 454, 744 (2008).
[20] D. Wulin, Y. He, C.-C. Chien, D. K. Morr, and K. Levin, Phys. Rev. B 80, 134504 (2009).
[21] B. Clancy, L. Luo, and J. E. Thomas, Phys. Rev. Lett. 99, 140401 (2007).
[22] A. Turlapov, J. Kinast, B. Clancy, L. Luo, J. Joseph, and J. E. Thomas, J. Low Temp. Phys. 150, 567 (2008).
[23] M. Shahzamanian and H. Yavary, Physica B 321, 385 (2002).
[24] V. J. Emery and S. A. Kivelson, Nature 374, 434 (1995).
[25] Because it is more difficult to precisely enforce the counterpart longitudinal sum rule, which also involve Goldstone boson effects, in this paper we do not discuss the bulk viscosity.