INVESTIGATION OF A NEW ANALYTIC RUNNING COUPLING IN QCD

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The mathematical properties of the new analytic running coupling (NARC) in QCD are investigated. This running coupling naturally arises under “analytization” of the renormalization group equation. One of the crucial points in our consideration is the relation established between the NARC and its inverse function. The latter is expressed in terms of the so-called Lambert W function. This relation enables one to present explicitly the NARC in the renorminvariant form and to derive the corresponding \( \beta \) function. The asymptotic behavior of this \( \beta \) function is examined. The consistent estimation of the parameter \( \Lambda_{QCD} \) is given.

1. Introduction

The description of hadron interactions at small characteristic momenta remains an actual problem of elementary particle theory. The asymptotic freedom in quantum chromodynamics (QCD) enables one to investigate processes at large momentum transfers by making use of standard perturbation theory. But there is a number of phenomena which description lies beyond such calculations, namely quark confinement, gluon and quark condensates and many others. For these purposes nonperturbative approaches are used.

In the late 50’s the so-called analytic approach to quantum electrodynamics was proposed.\(^1\) Its basic idea is the explicit imposition of the causality condition which implies the requirement of the analyticity in the \( Q^2 \) variable for the relevant physical quantities. Recently this approach has been extended to QCD.\(^3\) This leads to the following essential advantages: absence of unphysical singularities at any loop level, stability in the infrared (IR) region, stability with respect to loop corrections, and extremely weak scheme dependence. The analytic approach has been applied successfully to such QCD problems as the \( \tau \) lepton decays, \( e^+e^- \)-annihilation into hadrons, sum rules.\(^4\)

Recently the analytic approach has been employed to the renormalization group (RG) equation.\(^5\) The analyticity requirement was imposed on the RG equation itself, before deriving its solution. Solving the RG equation, “analytized” in the above-mentioned way (i.e., the proper analytical properties of the RG equation as a whole have been recovered), one arrives at the new analytic running coupling.

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An essential point, that plays a crucial role in description of a number of nonperturbative phenomena in our approach, is the IR enhancement of the new analytic running coupling at $Q^2 = 0$. It should be stressed here that such a behavior of the invariant charge is in agreement with the Schwinger–Dyson equations and, as it was demonstrated recently, provides description of quark confinement without invoking any additional assumptions.

The objective of this paper is to study in details the mathematical properties of the new analytic running coupling in QCD. This enables one to present the NARC in the explicitly renorminvariant form and to construct manifestly the corresponding $\beta$ function. We restrict ourself to the one-loop level calculations, taking into account the following consideration. Due to the higher loop stability all the essential features of the new analytic running coupling become apparent at the one-loop level already. The account of higher–loop corrections does not lead to any qualitative changes. At the same time, at the one-loop level one able to carry out all the calculations in an explicit form, that is one of the self–evident merits of the current consideration.

The layout of the paper is as follows. In Sec. 2 the new analytic running coupling is briefly discussed. In Sec. 3 the function $N(a)$, that plays a key role in our analysis, is introduced and its properties are investigated. In turn, the function $N(a)$ is expressed in terms of the so-called Lambert $W$ function. Proceeding from this, in Sec. 4 the one-loop new analytic running coupling is presented manifestly in the renorminvariant form and its basic properties are revealed. Further, the corresponding $\beta$ function is derived in an explicit form and its properties are examined. The consistent estimation of the parameter $\Lambda_{QCD}$ is given. In the Conclusion (Sec. 5) the obtained results are formulated in a compact way, and further studies in this approach are outlined.

2. New analytic running coupling in QCD

In the analytic approach, extended to QCD by Shirkov and Solovtsov, the basic idea is the explicit imposing of the causality condition, which implies the requirement of the analyticity in the $Q^2$ variable for the relevant physical quantities. Later this idea was applied to the “analytization” of the perturbative series (i.e., the recovering of the proper analyticity properties of this series) when calculating the QCD observables.

As known, the $Q^2$-evolution of QCD observables is usually described in the framework of the RG approach. However, the standard RG calculations involve, as a rule, the use of the perturbative expansions. As a result, the RG equation at any given loop level does not meet the analyticity requirement. In order to avoid such situation, the procedure of analytization of the RG equation as a whole has been elaborated. This procedure implies the recovering of the correct analytic properties of the RG equation itself, before deriving its solution. As a result, the solution of the analytized RG equation is expressed in terms of a new analytic
running coupling. At the one-loop level this running coupling reads

$$N_{\alpha_{\text{an}}}^{(1)}(Q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}, \quad z = \frac{Q^2}{\Lambda^2},$$  \hspace{1cm} (1)$$

where $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of the $\beta$ function, $n_f$ is the number of active quarks. At the higher loop levels only the integral representation for $N_{\alpha_{\text{an}}}(Q^2)$ has been obtained. So, at the $k$-loop level we have

$$N_{\alpha_{\text{an}}}^{(k)}(Q^2) = N_{\alpha_{\text{an}}}^{(k)}(Q_0^2) \frac{z_0}{z} \exp \left[ \int_0^\infty R^{(k)}(\sigma) \ln \left( \frac{\sigma + z}{\sigma + z_0} \right) \frac{d\sigma}{\sigma} \right],$$  \hspace{1cm} (2)$$

where $R^{(k)}(\sigma) = (2\pi i)^{-1} \sum_{j=0}^{k-1} \frac{\beta_j}{\beta_0} \left\{ \left[ \tilde{\alpha}^{(k)}(-\sigma - i\epsilon) \right]^{j+1} - \left[ \tilde{\alpha}^{(k)}(-\sigma + i\epsilon) \right]^{j+1} \right\}$ is the spectral density, $\tilde{\alpha}^{(k)}$ is the $k$-loop perturbative running coupling, $\tilde{\alpha} \equiv \alpha \beta_0/(4\pi)$, and $z_0 = Q_0^2/\Lambda^2$ is the normalization point.

It is worth making here a short historical comment. When considering the analytic approach to quantum electrodynamics, it was noted that the method proposed in Refs.\(^1\),\(^2\) is not unique. For example, in paper\(^9\) the ambiguity of that method was shown explicitly. Furthermore, it was stressed that for avoiding the ambiguity of the approach developed in Refs.\(^1\),\(^2\) one should impose an additional condition, in particular, by making use of the equations of motion. Specifically, following this way the boson propagator has been derived\(^10\) in which the singularity was removed.

![Graph of the normalized new analytic running coupling](image-url)  \hspace{1cm} \text{Fig. 1.} The normalized new analytic running coupling $\alpha(z) = N_{\alpha_{\text{an}}}(Q^2)/N_{\alpha_{\text{an}}}(Q_0^2)$ at the one-, two-, and three-loop levels. The normalization point is $Q_0^2 = 10 \Lambda^2$, $z = Q^2/\Lambda^2$. \hspace{1cm} (image-url)
in a multiplicative way (i.e., similar to Eq. (1)). Moreover, another common feature of our consideration and that of paper \textsuperscript{10} is the following. Unlike the method \textsuperscript{2,3} in our approach one is also able to catch the nonperturbative contributions to the spectral density for the running coupling (see also Eq. (23)).

Figure 1 presents the new analytic running coupling calculated at the one-, two- and three-loop levels. These curves show that the analytic running coupling \textsuperscript{1} possesses the higher loop stability. Moreover, it can be demonstrated that the singularity of the NARC at the point $Q^2 = 0$ is of the universal type at any loop level.

3. The Lambert $W$ function

For the representation of the new analytic running coupling \textsuperscript{1} in the renorm-invariant form and for the construction of the relevant $\beta$ function, it proves to be convenient to use the so-called Lambert $W$ function. As long ago as the middle of 18th century this function is being employed in diverse physical problems. In current researches in QCD the interest to this function arose just a few years ago. So, it was revealed\textsuperscript{3,12} that the exact solution to the perturbative RG equation for the invariant charge at the two-loop approximation can be expressed in terms of the Lambert $W$ function. At the three-loop level, when the Padé approximation for the $\beta$ function is used, the solution to this equation can be expressed in terms of the Lambert $W$ function also\textsuperscript{12} But in this case one has to take into account that the application of the Padé approximation to the $\beta$ function drastically changes the behavior of perturbative running coupling at small $Q^2$.

The Lambert $W$ function is defined as a many–valued function $W(x)$, that satisfies the equation

$$W(x) \exp [W(x)] = x. \tag{3}$$

Only two real branches of this function, the principal branch $W_0(x)$ and the branch $W_{-1}(x)$ (see Fig. 3), will be used in our consideration. The other branches of the Lambert $W$ function take an imaginary values. One can show that for the branches $W_0$ and $W_{-1}$ the following expansions hold:

$$W_0(\varepsilon) = \varepsilon - \varepsilon^2 + O(\varepsilon^3), \quad \varepsilon \to 0, \tag{4}$$

$$W_{-1}(-\varepsilon) = \ln \varepsilon + O(\ln |\ln \varepsilon|), \quad \varepsilon \to 0_+, \tag{5}$$

$$W_0 \left(-\frac{1}{e} + \varepsilon\right) = -1 + \sqrt{2e\varepsilon} + O(\varepsilon), \quad \varepsilon \to 0_+, \tag{6}$$

$$W_{-1} \left(-\frac{1}{e} + \varepsilon\right) = -1 - \sqrt{2e\varepsilon} + O(\varepsilon), \quad \varepsilon \to 0_+. \tag{7}$$

Details concerning the mathematical properties of the Lambert $W$ function can be found in the review\textsuperscript{11}.
In order to represent the NARC (1) in a renorminvariant form and to derive the β function corresponding to $\alpha_{\text{an}}^{(1)}$ one has to solve the equation (see Sec. 4)

$$\frac{z - 1}{z \ln z} = a, \quad z > 0, \quad a > 0$$

with respect to the variable $z$. Let us multiply this equation through by the factor $a^{-1} \ln z$,

$$\ln z = \frac{b}{z} - b, \quad z > 0, \quad b = -\frac{1}{a} < 0.$$  

(9)

The equation obtained has a trivial solution $z = 1$ that does not satisfy initial Eq. (8) when $a \neq 1$. Obviously, this is a consequence of the multiplication of Eq. (8) by $\ln z$. Therefore, when solving Eq. (8) for the variable $z$ one has to discard its trivial solution $z = 1$. Next, we can represent Eq. (9) in the form

$$\frac{b}{z} \exp \left( \frac{b}{z} \right) = b e^{b}.$$  

(10)

Taking into account the definition (3), solution to Eq. (10) can be expressed in terms of the Lambert $W$ function

$$\frac{b}{z} = W_{k} \left( b e^{b} \right).$$  

(11)

The branch index $k$ of the $W$ function will be specified below.

In the physically relevant range $a > 0$ the argument of the Lambert $W$ function in Eq. (11) takes the values $-1/e \leq b e^{b} < 0$. The only two real branches, the
principle branch $W_0$ and the branch $W_{-1}$ (see Fig. 2) correspond to this interval. Here one has to handle carefully with the interchange between branches $W_0(x)$ and $W_{-1}(x)$ at the point $x = -1/e$ (this corresponds to the value $a = 1$).

So, the nontrivial solution to Eq. (9) for the variable $z$ is

$$\frac{1}{z} = \begin{cases} b^{-1} W_0(b e^b), & b < -1, \\ b^{-1} W_{-1}(b e^b), & -1 \leq b < 0 \end{cases}$$  \hspace{1cm} (12)

(another choice of branches will be considered below). Thus, the solution to Eq. (8) we are interested in can be written in the form:

$$z = \frac{1}{N(a)}$$  \hspace{1cm} (13)

where we have introduced the function $N(a)$ (see Fig. 3)

$$N(a) = \begin{cases} N_0(a), & 0 < a \leq 1, \\ N_{-1}(a), & a > 1, \end{cases} \quad N_k(a) = -a W_k \left[ -\frac{1}{a} \exp \left( -\frac{1}{a} \right) \right].$$  \hspace{1cm} (14)

![Fig. 3. The function $N(a)$ (bold curve), and the functions $N_0(a)$ (solid curve) and $N_{-1}(a)$ (dashed curve) (see Eq. (14)).](image)

As follows from the definition (3) the function $x e^x$ and the Lambert $W$ function are mutually inverse. Here one has to distinguish precisely the branches $W_0$ and $W_{-1}$ (see Fig. 2). From this figure it follows directly that for $x > -1$ the function $x e^x$ “corresponds” to the branch $W_0$, and for $x \leq -1$ the function $x e^x$ “corresponds” to the branch $W_{-1}$. Let us introduce for convenience the notations of the inverse
functions for these branches

\[ x e^x = \begin{cases} (W_0)^{-1}(x), & x > -1, \\ (W_{-1})^{-1}(x), & x \leq -1. \end{cases} \] (15)

Now the function \( N(a) \) in Eq. (14) can be written in a more compact form

\[
N(a) = \begin{cases} -a W_0 \left[ (W_{-1})^{-1}(-a^{-1}) \right], & 0 < a \leq 1, \\ -a W_{-1} \left[ (W_0)^{-1}(-a^{-1}) \right], & a > 1. \end{cases}
\] (16)

It is worth noting that another choice of the branches of the function \( \tilde{N}(a) \)

\[
\tilde{N}(a) = \begin{cases} N_{-1}(a), & 0 < a \leq 1, \\ N_0(a), & a > 1 \end{cases}
\] (17)

leads to the trivial solution of Eq. (13) (in this case \( \tilde{N}(a) \equiv 1 \)). Therefore, the solution \( z = 1/\tilde{N}(a) \) does not satisfy Eq. (8) when \( a \neq 1 \), and consequently we have to reject it.

Let us briefly consider the basic properties of the function \( N(a) \) introduced in

\[ N(a) = 4 \pi \beta_0 \left( 1 - \frac{1}{z} \ln \frac{z}{\Lambda^2} \right) \] (21)

For the convenience we omit the group factor and proceed with the expression

\[ a(Q^2) \equiv \tilde{N}_{an}^{(1)}(Q^2) = \frac{z - 1}{z \ln z}. \] (22)

Let us consider first of all the asymptotics of this function. In the ultraviolet (UV) limit the standard behavior of the invariant charge is reproduced: \( a(Q^2 \to \infty) \to 1/\ln z, \ z = Q^2/\Lambda^2 \), i.e., the asymptotic freedom of the theory is taken into account. In the IR region there is an enhancement of the running coupling \( a(Q^2 \to 0_+) \to -[\varepsilon \ln \varepsilon]^{-1}, \ \varepsilon = Q^2/\Lambda^2 \). One should note here that such a behavior of the invariant charge is in agreement with the Schwinger–Dyson equations.
and as it was demonstrated recently, provides quark confinement without invoking any additional assumptions. Thus the running coupling (21) incorporates both the asymptotic freedom behavior and the IR enhancement in a single expression. Obviously it is an essential advantage of our approach. It is easy to show that the function (22) has smooth behavior in the vicinity of the point $Q^2 = \Lambda^2$:

$$a(Q^2 \to \Lambda^2) = 1 - \delta/2 + O(\delta^2), \quad \delta = Q^2/\Lambda^2 - 1.$$  

It is worth noting also that for the running coupling (22) the causal representation of the Källén-Lehmann type holds:

$$N_{\text{an}}(1)(Q^2) = \int_0^{\infty} \frac{\gamma(\sigma)}{\sigma + z} d\sigma, \quad \gamma(\sigma) = \left(1 + \frac{1}{\sigma}\right) \frac{1}{\ln^2 \sigma + \pi^2}.$$  

(23)

In general, any expression for the running coupling makes sense only if the relevant definition of the parameter $\Lambda$ is provided. Otherwise, a running coupling may be not a renorm-invariant quantity at all. Thus, it is of a primary importance to present both the running coupling and the corresponding definition for its parameter $\Lambda$ explicitly.

So, let us represent the running coupling $N_{\text{an}}(1)$ in the renorm-invariant form. In the general case for the invariant charge $\tilde{g}^2(Q^2/\mu^2, g)/(4\pi) \equiv \alpha(Q^2)$ the normalization condition $\tilde{g}^2(1, g) = g^2$ must be fulfilled. In our case this normalization condition acquires the form

$$\frac{(\mu^2/\Lambda^2) - 1}{\ln(\mu^2/\Lambda^2) - \mu^2/\Lambda^2} = \frac{\beta_0}{4\pi} \alpha(\mu^2) \equiv \alpha(\mu^2).$$  

(24)

Therefore, we have to resolve this equation with respect to the parameter $\Lambda$. The solution to such equations was considered in details in Sec. 3. With the help of the function $N(a)$ (see Eq. (14)) the solution to Eq. (24) can be presented in the form

$$\Lambda^2 = \mu^2 N\left[\frac{\beta_0}{4\pi} \alpha(\mu^2)\right].$$  

(25)

Thus, the renorm-invariant expression for the one-loop new analytic running coupling (1) is the following

$$N_{\text{an}}(1)(Q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}, \quad z = \frac{Q^2}{\Lambda^2}, \quad \Lambda^2 = \mu^2 N\left[\frac{\beta_0}{4\pi} \alpha(\mu^2)\right].$$  

(26)

It is worth noting that when $\mu \to \infty$ (see Eq. (18)) the right hand side of Eq. (25) tends to its standard form corresponding to the perturbative running coupling $\alpha_s^{(1)}(Q^2) = 4\pi/(\beta_0 \ln z)$

$$\Lambda_s^2 = \mu^2 \exp\left[\frac{4\pi}{\beta_0} \frac{1}{\alpha(\mu^2)}\right].$$  

(27)

Let us proceed to the construction of the $\beta$ function corresponding to the running coupling (1). By definition

$$\beta(\alpha) = \frac{\partial \alpha(\mu^2)}{\partial \ln \mu^2}.$$  

(28)
In the case under consideration, for the invariant charge $\tilde{N}_{\alpha}^{\alpha}(1)$ the $\beta$ function takes the form

$$\tilde{N}_{\beta}^{\alpha}(a) = \left[ \frac{1}{z(a)} - a \right] \frac{1}{\ln[z(a)]}, \quad (29)$$

where $z(a)$ is the solution of Eq. (8) with respect to the variable $z$ (see Sec. 3). For the convenience we omit here the group factor, and use the expression $\tilde{N}_{\beta}^{\alpha}(a) = \tilde{N}_{\beta}^{\alpha}(a) \beta_0/(4\pi)$. With the help of the function $N(a)$ defined in Eq. (14) the $\beta$ function for the NARC (22) can be written in the form

$$\tilde{N}_{\beta}^{\alpha}(a) = a - N(a) \ln[N(a)]. \quad (30)$$

Figure 4 presents this $\beta$ function and the standard perturbative result $\tilde{\beta}_s(a) = -a^2$ corresponding to the one-loop perturbative running coupling $\tilde{\alpha}_s^{(1)}(Q^2) = 1/\ln z$. Thus, the $\beta$ function corresponding to the one-loop new analytic running coupling $\tilde{\alpha}_s^{(1)}$ reads as

$$N_{\alpha}^{\alpha}(Q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z} \equiv \frac{4\pi}{\beta_0} a(Q^2), \quad N_{\beta}^{\alpha}(a) = \frac{4\pi}{\beta_0} \frac{a - N(a)}{\ln[N(a)]}. \quad (31)$$

The numerator and denominator on the right hand side of Eq. (30) have the opposite signs for all positive $a$. This follows directly from the expansion (33) (see
below) and from Figs. 3 and 4. Thus, the β function (30) possesses an important property, namely \( \tilde{\beta}_\text{an} (a) \leq 0 \) for all positive \( a \). It is this property that provides the asymptotic freedom of the theory. By making use of the series (18)–(20) one can derive the following expansions of the β function (30). First of all,

\[
\tilde{\beta}_\text{an} (a \to 0) = -a^2 + O \left[ \exp \left( -\frac{1}{a} \right) \right],
\]

i.e., the standard perturbative limit is reproduced in the UV region. Secondly,

\[
\tilde{\beta}_\text{an} (1 + \varepsilon) = -\frac{1}{2} - \frac{2}{3} \varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0,
\]

i.e., the function (30) has a smooth behavior in the vicinity of the point \( a = 1 \).

Next, in the IR region

\[
\tilde{\beta}_\text{an} (a \to \infty) = -a \left[ 1 + O \left( \frac{\ln(\ln a)}{\ln^2 a} \right) \right].
\]

Such a behavior of the β function provides the IR enhancement of the new analytic running coupling. This enhancement leads ultimately to the quark–antiquark potential rising at large distances.5,6

Thus, as one could anticipate, the asymptotics of the obtained β function incorporate both the asymptotic freedom and the IR enhancement of the running coupling (1).

Sometimes it is more convenient to present the β function in terms of the invariant charge \( g, \alpha \equiv g^2/(4\pi) \). In this case the asymptotic expansions (32) and (34) can be rewritten in the following way: \( \tilde{\beta}_\text{an} (g) \simeq -\beta_0 g^3/(16\pi^2) \) when \( g \to 0 \) (this is nothing but the well-known standard perturbative result); and \( \tilde{\beta}_\text{an} (g) \simeq -g \) when \( g \to \infty \). It is worth noting here that a similar (i.e., linear in \( g \)) asymptotic behavior \( \beta (g) \simeq -2g \) when \( g \to \infty \) holds, when the so-called Mandelstam approximation is used (see paper 16 and references therein). However the latter leads to the running coupling with more singular behavior in the IR region, namely \( \alpha(Q^2) \sim Q^{-4} \), when \( Q^2 \to 0^+ \).

As it was mentioned in the previous section, the new analytic running coupling (1) incorporates both the asymptotic freedom and the IR enhancement in a single expression. Obviously this is an essential advantage of our approach. It was shown explicitly that invariant charge of the form (1) leads to the rising quark–antiquark \((q\bar{q})\) potential without invoking any additional assumptions. The comparison of this \( q\bar{q} \) potential with the so-called Cornell phenomenological potential gives the following estimation for the \( \Lambda_{\text{QCD}} \) parameter: \( \Lambda = (0.60 \pm 0.1) \) GeV.

The fit has been performed in the physically meaningful region \( 0.1 \text{ fm} \leq r \leq 1.0 \text{ fm} \) (in this interval three active quarks should be taken into account, \( n_t = 3 \)). The analogous comparison of the \( q\bar{q} \) potential derived in Ref. 5 with the lattice data 18 has been carried out recently. This gives a close value, \( \Lambda = (0.57 \pm 0.1) \) GeV. Furthermore, the estimation of the parameter \( \Lambda_{\text{QCD}} \) has also been performed recently when
deducing the value of the gluon condensate by making use of the NARC (1). This gives the value \( \Lambda = (0.65 \pm 0.05) \) GeV. Therefore, we have the acceptable estimation of the parameter \( \Lambda_{\text{QCD}} \) in the framework of our approach: \( \Lambda = (0.60 \pm 0.09) \) GeV.

All this testifies that the new analytic running coupling (1) substantially involves the nonperturbative behavior of quantum chromodynamics.

5. Conclusion

The mathematical properties of the new analytic running coupling in QCD (1) are investigated. This running coupling naturally arises under analytization of the renormalization group equation (1). One of the crucial points in our consideration is the explicit relation established between the NARC (Eq. (1)) and its inverse function \( N(a) \) (Eq. (14)). The latter can be expressed in terms of the so-called Lambert W function. The properties of the function \( N(a) \) are examined in details. As known, when using the RG approach it is important to present the relevant quantities in a renorminvariant form. The function \( N(a) \) enables one to perform this explicitly for the NARC (1). Furthermore, the \( \beta \) function corresponding to the running coupling \( \frac{\delta}{\delta a} \) is expressed in terms of the function \( N(a) \). The explicit form of the function \( \frac{\delta}{\delta a} \) in Eq. (11) provides an essential advantage when considering the gluon condensate. The asymptotics of the \( \beta \) function in Eq. (31) are examined also. As one could anticipate, it incorporates both the asymptotic freedom and the IR enhancement of the invariant charge (1) in a single expression. It was shown that there is a consistent estimation of the parameter \( \Lambda_{\text{QCD}} \) in the framework of our approach: \( \Lambda = (0.60 \pm 0.09) \) GeV. Thus, the new analytic running coupling (1) substantially incorporates the nonperturbative behavior of the quantum chromodynamics.

In further studies it would be interesting to investigate numerically higher–loop corrections to the NARC (1) and to compare the results with the explicit expressions obtained in the present paper.

Acknowledgments

The author would like to thank Professor D. V. Shirkov and Dr. I. L. Solovtsov for interest in this work. The detailed discussions with Dr. L. von Smekal are greatly appreciated. The author is grateful to Professor G. S. Bali for supplying the relevant data on the lattice calculations. The useful references provided by Professor R. M. Corless and stimulating comments by Professor B. A. Arbuzov are thankfully acknowledged.

The partial support of RFBR (Grant No. 99-01-00091) is appreciated.

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