Recent developments in quantum mechanics with magnetic fields

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Abstract

We present a review on the recent developments concerning rigorous mathematical results on Schrödinger operators with magnetic fields. This paper is dedicated to the sixtieth birthday of Barry Simon.

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Running title: Recent developments on magnetic fields

1 Introduction

The mathematical formulation of quantum mechanics, given by Schrödinger, Pauli and Dirac, has posed an enormous challenge: can mathematics, with its own tools and standards, rigorously justify or even predict physical phenomena of the quantum world? Similarly to the development of the differential and integral calculus, strongly motivated by Newton’s classical mechanics, new mathematical tools have been created (most notably by von Neumann, Weyl, Wigner and later by Kato). Functional analysis, representation theory and partial differential equations would have been much poorer mathematical disciplines without quantum mechanics.

Electromagnetic fields play a central role in quantum physics; their rigorous inclusion in the theory is certainly one of the key goals of mathematical physics. Quantum electrodynamics (QED) postulates that electric and magnetic fields are to be described within a unified

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relativistic theory. Although the framework for QED has been clear since the 30’s, the mathematical difficulties even to formulate the theory rigorously have not yet been resolved. In the low energy regime, however, massive quantum particles can be described non-relativistically. Electric and magnetic fields, with a good approximation, can be considered decoupled. Since typical magnetic fields in laboratory are relatively weak, as a first approximation one can completely neglect magnetic fields and concentrate only on quantum point particles interacting via electric potentials.

The rigorous mathematical theory of Schrödinger operators has therefore started with studying the operator $H = \frac{1}{2m}p^2 + V(x)$ on $L^2(\mathbb{R}^d)$ and its multi-particle analogues. Here $x \in \mathbb{R}^d$ is the location of the particle in the $d$-dimensional configuration space, $p = -i\nabla_x$ is the momentum operator and $m$ is the mass, that can be set $m = \frac{1}{2}$ with convenient units. The Laplace operator describes the kinetic energy of the particle and the real-valued function $V(x)$ is the electric potential. Although both the kinetic and potential energy operators are very simple to understand separately, their sum exhibits a rich variety of complex phenomena which differ from their classical counterparts in many aspects. The mathematical theory of this operator is the most developed and most extensive in mathematical physics: the best recent review is by Simon [152].

As a next approximation, classical magnetic fields are included in the theory, but spins are neglected. The kinetic energy operator is modified from $p^2$ to $(p + A)^2$ by the minimal substitution rule: $p \mapsto p - eA$ and we set the charge to be $e = -1$. Here $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic vector potential that generates the magnetic field $B$ according to classical electrodynamics. In $d = 2$ or $d = 3$ dimensions $B = \nabla \times A$ is a scalar or a vector field, respectively. In $d = 1$ dimension the vector potential can be removed by a unitary gauge transformation, $e^{i\varphi}(p + A)^2e^{-i\varphi} = p^2$, $\varphi = \int A$, therefore magnetic phenomena in $\mathbb{R}^1$ are absent (they are present in the case of $S^1$).

We will call the operator $(p + A)^2 + V$ the magnetic Schrödinger operator. In general, even the kinetic energy part contains noncommuting operators, $[(p + A)_k, (p + A)_l] \neq 0$, and the theory of $(p + A)^2$ itself is more complicated than that of $p^2 + V$. The simplest case of constant magnetic field, $B = \text{const}$, is explicitly solvable. The resulting Landau-spectrum consists, in two dimensions, of infinitely degenerate eigenvalues at energies $(2n+1)|B|$, $n = 0, 1, \ldots$. Notice that the magnetic spectrum is characteristically different from that of the free Laplacian. The eigenfunctions are localized on a scale $|B|^{-1/2}$; this corresponds to the cyclotronic radius in classical mechanics (Landau orbits).

The interaction of the spin with a magnetic field is proportional to the field strength. In the low energy regime this effect is comparable with the energy shift due to inclusion of $A$ into $p^2$. Since electrons are spin-$\frac{1}{2}$ particles, the spin, in principle, should not be neglected whenever magnetic fields are considered. Nevertheless, magnetic Schrödinger operators constitute an important intermediate step to understand magnetic phenomena.
The state space of a spin-$\frac{1}{2}$ particle is $L^2(\mathbb{R}^d, \mathbb{C}^2)$ (in $d = 2, 3$) and the momentum operator is the Dirac operator, $D_A := \sigma \cdot (p + A)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the Pauli matrices. The kinetic energy is given by the Pauli operator,

$$H_P := D_A^2 = [\sigma \cdot (p + A)]^2 = (p + A)^2 + \sigma \cdot B,$$

and external potential may be added as before. The last identity is a special case of the Lichnerowicz formula known from spinor-geometry.

In most of this review we restrict ourselves to these operators and their multi-particle generalizations. However, we briefly mention that in dimensions higher than 3 or on configuration spaces with a non-flat Riemannian metric, the vector potential is canonically defined as a one-form, $\alpha$, and the magnetic 2-form, $\beta = d\alpha$, is its exterior derivative. In the conceptually most general setup for the spinless case, the Hilbert space of states consists of the $L^2$-sections of a $U(1)$-bundle over an orientable Riemannian manifold, $M$, representing the configuration space, and the momentum operator is the covariant derivative, $\nabla$, on this bundle. In this formulation, the vector potential does not appear directly but the magnetic field is $(i$-times) the curvature 2-form of $\nabla$. Proper description of the spin involves covariant derivatives on sections of an Spin$^c$-bundle with Pauli matrices replaced by Clifford multiplication [46].

In relativistic theories, electron-positron pair-creations cannot be neglected and one studies the full relativistic Dirac operator, $\alpha \cdot (p + A) + \beta m$, where $(\alpha, \beta)$ is the vector of the Dirac matrices and $m$ is the mass. Due to the lack of semiboundedness of the Dirac operator, its definition, even without a magnetic field, is a complex issue that is not yet satisfactorily resolved in the many-body situation (“filling the Dirac sea”). We will not pursue this direction here since the current research focuses more on the non-magnetic aspects of the Dirac operator.

A consistent quantum theory requires to quantize the electromagnetic field as well. Ideally, this should be done within the framework of the Dirac operator (relativistic QED) but this problem is beyond the reach of the current techniques. A more tractable model is the nonrelativistic QED, where quantized electromagnetic field is introduced in the Pauli operator, i.e. pair-creations are neglected.

This overview gives an admittedly biased summary of a few recent key results involving magnetic Hamiltonians. Many people have contributed to these questions and a selection was unavoidable; the author apologizes to everyone whose work have been left out. The choice reflects the author’s taste and the pressure of the editors to keep the page limit.

In Section 2 we present results related to the proper definitions of these operators. In Section 3 we discuss one-particle spectral theory, including Lieb-Thirring type bounds and semiclassical methods. In Section 4 we consider multi-particle problems, including stability of matter, large atoms and scattering. Finally, Section 5 is devoted to random Schrödinger operators with magnetic fields.
Barry Simon was undoubtedly one of the initiators and most important contributors of the endeavor to put Schrödinger operators on a solid mathematical ground. His work was especially pioneering in the theory of magnetic fields. Among many of his achievements in this area, here I would just mention those two that had the biggest impact on my own work. Barry was the first who systematically exploited path integral methods for magnetic fields upon an initial suggestion of Nelson (see e.g. [147]). Secondly, his seminal papers with Avron and Herbst [6] have become the classical reference “handbook” about magnetic fields. This overview is dedicated to his 60-th birthday.

2 Basic qualitative properties

2.1 Definitions

Along the development of the rigorous theory of Schrödinger operators without magnetic fields, it was apparently Kato who first initiated the natural program to extend this theory to the most general magnetic fields. The unique self-adjoint extension of the operator \((p + A)^2 + V\) without any growth condition on \(A\) was shown in 1962 by Ikeda and Kato [88]. This result indicated that magnetic operators should not simply be viewed as second order differential operators with variable coefficients. For most mathematical purposes it is misleading to look at \((p + A)^2\) as \(p^2 + A \cdot p + p \cdot A + A^2\). The \(A\)-field plays a special role in magnetic problems: it balances the derivative of the phase of the wave function. This effect is inherently present in the form \((p + A)^2\).

Kato has proved his celebrated distributional inequality, \(\Delta|\psi| \geq \text{Re} \left[ \text{sgn}\psi \Delta\psi \right]\), for any \(\psi \in L^1_{\text{loc}}\), \(\Delta\psi \in L^1_{\text{loc}}\) in 1973 [93]. Simon has realized its connection to the semigroup inequality, \(|e^{t\Delta}\psi| \leq e^{t\Delta}|\psi|\) in 1977 [146]. A more general abstract setup was considered in [149], and independently in [80], leading to the magnetic versions of these inequalities.

For regular vector potential, a simple proof of the semigroup diamagnetic inequality,

\[
|e^{-t(p+A)^2}\psi| \leq e^{t\Delta}|\psi|, \tag{2.2}
\]

via the Feynman-Kac formula was given in Simon’s paper [146] quoting an argument of Nelson in a private communication. It was apparently Nelson who pointed out the probabilistic approach to Simon (see the history in Simon’s book [147]), but the real power behind the rigorous path integral method for magnetic fields was realized in a series of work of Simon and collaborators [6] (see also [25]).

More singular vector potentials were considered with analytic methods in [94]. Finally, in his seminal paper [148], Simon has given a simple proof of the diamagnetic inequality (2.2) where the operator \(H = (p + A)^2\) was defined under the most general conditions, namely for
$A \in L^2_{\text{loc}}$ and for $H$ defined as the operator associated with the maximal quadratic form. The
domain of the maximal form contains all $\psi \in L^2$ with $(p + A) \psi \in L^2$ in distributional sense. Using
\textsuperscript{(2.2)} and semigroup smoothing, Simon showed that the $C_0^\infty$ is a form core for $H$. A
non-negative potential $V \in L^1_{\text{loc}}$ can be added to $H$ without any difficulty.

The optimal conditions for $C_0^\infty$ being the operator core for $H$ are $A \in L^4_{\text{loc}}$ and $\text{div} A \in L^2_{\text{loc}}$. This was conjectured by Simon and proved by Leinfelder and Simader. Leinfelder has also showed the unitarity equivalence under any gauge transformation, $A \rightarrow A + \nabla \varphi$, that stays within the above classes. Again, a non-negative potential $V \in L^2_{\text{loc}}$ can be added to $H$ without any difficulty and the Leinfelder-Simader theorem extends to a certain class of negative potentials as well ($V = V_1 + V_2 \leq 0$, $V_1, V_2 \in L^2_{\text{loc}}$, $V_1(x) \geq -(\text{const.})|x|^2$, $V_2$ bounded relative to $-\Delta$ with a bound smaller than one).

The most general conditions on potentials with non-trivial negative part, $V_- \neq 0$, are hard
to use directly. The typical argument uses the KLMN theorem (Theorem X.17 \textsuperscript{139}) that
defines self-adjoint operators by adding a relative form bounded perturbation (with bound less than 1) to a semi-bounded closable quadratic form. The boundedness of $V_-$ relative to $(p + A)^2 + V_+$ is, however, hard to check. With the help of the diamagnetic inequality, the boundedness of $V_-$ relative to $p^2 + V_+$ is sufficient. We recall that $V_-$ being in the Kato class, $V_- \in K$, implies even infinitesimal boundedness. Using Stratonovich stochastic integrals, the Feynman-Kac formula can be extended to $A \in L^2_{\text{loc}}$ vector potentials if $V_-$ is relative bounded by $p^2 + V_+$ \textsuperscript{(84)}. If one prefers to use Ito stochastic integrals, then, additionally, $\text{div} A \in L^2_{\text{loc}}$ is also necessary for the Feynman-Kac formula. The definition of the magnetic operator with Neumann boundary conditions was carefully worked out recently in \textsuperscript{87} and the proof of the
diamagnetic inequality and Kato’s inequality were extended to this case using the method of Simon \textsuperscript{148}. The most general diamagnetic result for the Neumann case is obtained in \textsuperscript{88} that uses no regularity assumptions on the domain and on $V_+$.

Similarly to the non-magnetic case worked out in the fundamental paper by Simon \textsuperscript{151},
with the help of the (magnetic) Feynman-Kac formula one can prove smoothing and continuity
properties of the semigroup and its kernel. This work has been carried out in \textsuperscript{166} with great
care and with many fine details. To summarize the results, one assumes that the vector
potential $A$ belongs to the so-called (local) magnetic Kato class, i.e. $A^2, \text{div} A \in K_{\text{loc}}$, and the
potential is Kato decomposable ($V_+ \in K_{\text{loc}}, V_- \in K$). Then the $L^p$-semigroup is continuous
in time and if $A$ and $V$ are approximated locally in the Kato-norm, then the approximating
semigroups converge. Moreover, the Feynman-Kac formula defines a continuous representation
of the semigroup kernel.

The definition of the Pauli operator can be directly reduced to that of the magnetic
Schrödinger operator using \textsuperscript{(1.1)} and treating $\sigma \cdot B$ as a (matrix-valued) potential term. However, the supersymmetric structure of the Pauli operator (at least in even dimensions)
allows one to define the Pauli operator directly and for more general magnetic fields. On topologically trivial domains only the magnetic field has physical relevance. The weakest necessary condition on $\sigma \cdot B$, if considered as a potential, is $B \in L^1_{\text{loc}}$. However, not every $L^1$ field can be generated by an $L^2_{\text{loc}}$ vector potential, hence $(p + A)^2$ might not be defined even as a quadratic form. Therefore it is desirable to define the Pauli operator directly, by circumventing the vector potential. This idea has been worked out in $d = 2$ dimensions in [49], where $A$ was replaced by a scalar potential, $h$, satisfying $\Delta h = B$, and the Pauli quadratic form was given by

$$q(\psi, \psi) := 4 \int |\partial_z (e^{-h} \psi_+)|^2 e^{2h} + 4 \int |\partial_z (e^h \psi_-)|^2 e^{-2h}, \quad \psi = \left( \begin{array} {c} \psi_+ \\ \psi_- \end{array} \right).$$

(2.3)

This definition is applicable for any measure-valued magnetic field and it is consistent with the standard one for fields that can be generated by $L^2_{\text{loc}}$ vector potential. However, for singular fields the form core is not $C_0^\infty$ any more. Strangely enough, similar construction does not seem to apply for the magnetic Schrödinger operator and the higher dimensional generalizations are also open.

### 2.2 Compact resolvent, essential spectrum, absolute continuous spectrum

A basic qualitative fact about magnetic fields is that their inclusion into the free spinless Laplacian, very roughly saying, increases the bottom of the local spectrum by $|B(x)|$. This intuitive statement makes sense only if the spectrum of the localized operator can be defined and if $B(x)$ is sufficiently regular. The key mathematical reason is the Lichnerowicz identity (1.1) that shows that $(p + A)^2$ on spinors is a nonnegative operator plus $-\sigma \cdot B$. Viewing this identity restricted to spinors with a spin direction opposite to the field, one obtains a useful lower bound on the magnetic Schrödinger operator. One can also see this by using the fact that the components of $(p + A)^2$ do not commute: the commutator is the magnetic field (up to a factor $i$). This trivial but crucial observation is the core of many results throughout the next sections. We emphasize that this effect holds only for the spinless magnetic Schrödinger operator and not for the Pauli operator.

This idea has been elaborated by several authors to investigate the location of the essential spectrum and resolvent compactness of the magnetic operators. For the Schrödinger operator it has been shown that for sufficiently regular fields the condition that the strength of the magnetic field goes to infinity is equivalent to the compactness of the resolvent (see [72] and references to previous results therein, in particular [6]). The regularity assumptions were weakened in [144] by using functions belonging to the so-called reverse Hölder class. For the
Pauli operator without external potential it is conjectured that its resolvent is never compact. This has been shown for sufficiently “well-behaving” magnetic fields in [75]. Under stronger conditions about the magnetic field at infinity, the essential spectrum of the Pauli operator was also identified in [76]. In a recent work of Last and Simon [100] a different characterization of the essential spectrum was given in terms of the union of the spectra of certain limit operators at infinity.

The localized eigenfunctions of the Landau spectrum in case of a constant magnetic field indicate that in \( d = 2 \) dimensions the magnetic field has a strong localization effect, while in \( d = 3 \) dimensions the free motion parallel with the field guarantees absolutely continuous spectrum. It is somewhat surprising that by a small change of the constant magnetic field, the spectrum can become purely absolutely continuous even in \( d = 2 \). This was first observed and proved by Iwatsuka [92] for magnetic fields that are translation invariant in one direction and tend to two different values at plus and minus infinity in the other direction. The classical analogue of this model is actually a very simple geometric picture. Since the cyclotronic radius depends on the field strength, the closed Landau orbits become spirals whose average velocity are nonzero and perpendicular to the gradient of the field.

Similar phenomenon can be created by an external potential in constant magnetic field or by Dirichlet boundary conditions along an edge of the sample that extends to infinity. Under suitable conditions the states can be classified as edge states and bulk states. The edge states are localized along the boundary and they give rise to pure absolutely continuous spectrum inside the Landau gaps. They carry nonvanishing chiral edge currents. This picture persists even under perturbations with a small (possibly random) potential [58]. The edge states exhibit a level repulsion that is even stronger than that of the Gaussian ensembles expected for the usual Anderson model in the extended states regime [125].

### 2.3 Zero modes and multiplicity

The supersymmetric structure of the Pauli operator is responsible for the spectrally rigid and typically large kernel of \( H_P \) in \( d = 2 \). The Aharonov-Casher theorem [2, 27, 128] states that \( \dim \ker H_P \) is given (essentially) by the total flux, divided by \( 2\pi, \frac{1}{2\pi} \int B \). As a special case of the Index Theorem, for smooth data and on a compact manifold, it basically relies on algebraic identities. Still, in its most general form on \( \mathbb{R}^2 \) it was only recently proved in [49] (for finite total flux) and [141] (for non-negative field) using (2.3). For arbitrary field it is false [49]. In the strong field limit, under some regularity assumptions, the local density of Aharonov-Casher zero modes converge to \( \frac{B(x)}{2\pi} \) [34].

The elements of the kernel the Pauli operator (the so-called zero modes) in \( d = 2 \) are conceptually much easier to understand than in \( d = 3 \). Naive extensions of the two dimensional
constructions to three dimensions fail, and they even seem to indicate that there are no zero modes in $d = 3$. A fundamental observation of Loss and Yau [124] is that the equation

$$D_A \psi = 0, \quad A, B, \psi \in L^2$$

does have a solution on $\mathbb{R}^3$, albeit quite complicated. This seemingly innocent fact implies, among others, that nonrelativistic matter with a magnetic field cannot be stable unless the fine structure constant is sufficiently small [57].

The explicit construction of [124] does not shed much light on the conceptual origin of the zero modes. It turns out that two dimensional Aharonov-Casher zero modes on $S^2$ can be lifted to $\mathbb{R}^3$ using the Hopf map and spinor geometry [46] (see also [7, 32] for other examples). In particular, magnetic fields with arbitrary number of zero modes can be constructed. Although many zero modes are obtained in this conceptual way, still not all explicit zero modes of [124] are covered. On the other side, it is known that magnetic fields with zero modes form a slim set in the space of all magnetic fields [10], [33]. It is an interesting open question to connect the existence, or even the number of the zero modes with the geometry of the magnetic field. Currently we have even not a conjecture for a general characterization of magnetic fields with zero mode.

For the spinless magnetic Schrödinger operator no supersymmetric structure is available to analyze the ground states and even to compute the bottom of the spectrum is complicated, apart from the strong field regime (Section 3.2.1). Since the Perron-Frobenius theorem does not apply to the magnetic Laplacian, the ground state can be degenerate, although for generic field it is simple. Still, the strength of magnetic field restricts the possible multiplicity. Based upon similar observations by Colin de Verdière on graphs, it was conjectured in [23] that on a two dimensional manifold $M$, the total curvature of the line bundle, i.e. the total flux, $\int_M |B|$, gives an upper bound on the multiplicity of the magnetic ground state. This was proved in [12] modulo constants depending on the geometry of the base manifold. The same bound with constants depending only on the genus of $M$ is still an intriguing open question. The proof in [12] relies on an upper bound on the ground state energy in terms of the total flux and this intermediate result necessarily depends on the geometry of $M$. The construction of an appropriate trial state uses the scalar potential $h$ (with $\Delta h = B$) instead of the vector potential in order to control the energy solely by the $L^1$ norm of $B$.

2.4 Magnetic operators on the lattice

The magnetic Schrödinger (and Pauli) operator can also be defined on the lattice. The magnetic field is defined on the plaquets, while the magnetic vector potential, $A_b$, is a function on the bonds. The magnetic translation operator along the bond $b$ amounts to a multiplication...
by a complex phase \( e^{i\phi_b} \) in addition to the usual hopping. The field on each plaquet is the oriented sum of the vector potentials along the boundary. The field and the vector potential are defined only modulo \( 2\pi \).

Although this definition is very natural, the spectral properties of the lattice magnetic Schrödinger operator differ vastly from the continuous version. Even for a constant magnetic field \( B \) on a regular two dimensional square lattice (Harper operator), the spectrum exhibits a complex fractal behavior (“Hofstadter butterfly”) sensitively depending on the diophantine properties of \( \alpha = B/2\pi \). With a simple transformation, this operator can be reduced to the almost Mathieu operator; a simple prototype of a one-dimensional discrete Schrödinger operator with an almost periodic potential:

\[
H_{\alpha,\lambda} = \lambda \cos(2\pi \alpha D_x) + \cos x .
\]  

(2.4)

The continuous Schrödinger operator with a constant magnetic field and periodic potential leads to a similar equation.

The Cantor-like spectrum of \( H_{\alpha,\lambda} \) was first proven in \[13\] for a dense set of parameter values and later in a series of papers Helffer and Sjöstrand performed a detailed quantitative semiclassical analysis \[76\] initiated by Wilkinson \[166\] to identify a large set of parameter values \( \alpha \) with Cantor spectrum if \( \lambda = 1 \). With quite different techniques, Last obtained a similar result and he also computed the Lebesgue measure of the spectrum \[92\] for all \( \lambda \). Finally, the Cantor spectrum has recently been proven by Puig \[137\] (\( \lambda \neq 0 \), \( \alpha \) is Diophantine) and by Avila and Jitomirskaya \[3\] for all conjectured values of the parameters: \( \lambda \neq 0 \), \( \alpha \) irrational (“Ten martini problem”, as it was named and popularized by Barry Simon).

2.5 Dia- and paramagnetism

Diamagnetism plays a crucial role in the analysis of the magnetic Schrödinger operators since it gives an easy apriori comparison of magnetic and non-magnetic operators, like (2.2). However, the apparent strength of the basic diamagnetic inequality is somewhat misleading when it comes to quantitative results.

On one hand, it completely neglects magnetic effects; operators with two different but nonzero magnetic fields are not comparable with this method. In particular, diamagnetism in a strong sense, i.e. monotonicity of the energy in the magnetic field strength, does not hold in general because of the de Haas-van Alphen oscillation effect (see \[77\] for a rigorous proof in the weak field regime with a periodic external potential).

On the other hand, diamagnetism is applicable only for the exponential statistics, \( \text{tr} \ e^{-\beta H} = \sum_j e^{-\beta \lambda_j} \), of the eigenvalues, \( \lambda_j \), in particular for the ground state (\( \beta \to \infty \)). Going beyond these constraints is notoriously difficult and there are only a few results and many open questions.
Loss and Thaller proved \cite{123} that the heat kernel of a two-dimensional Schrödinger operator $H = (p + A)^2$ with an arbitrary magnetic field $B(x)$ can be estimated by

$$\left| e^{-tH}(x,y) \right| \leq \frac{B}{4\pi \sinh Bt} e^{-(x-y)^2/4t}$$

if $B(x) \geq B(\geq 0)$. The right hand side is smaller than the free heat kernel and its exponential behaviour $\approx e^{-Bt}$ correctly reflects a spectral shift at the ground state energy by at least $B$. However, it does not retain the full Gaussian offdiagonal decay of the magnetic heat kernel with a constant field. With the help of this inequality, sharp $L^p-L^q$ bounds were shown in \cite{123}. The proof heavily uses the Gaussian character of the heat kernel of the constant field operator. Several counterexamples \cite{39} show that this result is basically the best one could hope for: there is no strict diamagnetic comparison between two non-homogeneous magnetic fields or even between two homogeneous magnetic fields with a potential. The Gaussian offdiagonal decay cannot be fully recovered. Such type of decay apparently requires real analyticity of the magnetic field and potential in the angular direction \cite{37,132,159}.

In the large field limit, the diamagnetic effect is so strong that the magnetic Schrödinger operator converges (in resolvent sense) to the free Laplacian with Dirichlet boundary conditions on the regime where the magnetic field vanishes \cite{79}. In other words, strong magnetic fields act like Dirichlet walls, confining the electron motion to regimes where the field is zero.

In contrast to the diamagnetism of the (spinless) magnetic Schrödinger operator, the Pauli operator tends to be paramagnetic. This issue was apparently raised first in \cite{82} and E. Lieb proved (Appendix of \cite{6}) that the ground state energy of the Pauli operator with potential cannot increase as a constant magnetic field is turned on. However, paramagnetism fails for non-homogeneous fields \cite{5,71,39}.

For many-fermion systems, one studies the sum of the low lying eigenvalues of the one-body operator. This statistics is more singular, it is beyond the exponential statistics offered by the heat kernel and surprising phenomena occur. The magnetic Schrödinger operator on a square lattice turns out to be paramagnetic at half-filling. It is the maximal flux ($\pi$ on each plaquet) that minimizes the sum of the first $\Lambda/2$ magnetic eigenvalues on a torus of volume $\Lambda$. The result actually holds on any bipartite graph that has a periodicity at least in one direction. After some special cases presented in \cite{104} and proved in \cite{108}, the general result was proven by Lieb \cite{105}. The proof uses reflection positivity and seemingly it cannot be extended to other filling factors or to graphs without periodicity, leaving the general case as an intriguing open question.

Diamagnetism for sums of the Schrödinger eigenvalues fails in the continuum as well. For a compact domain in $\mathbb{R}^d$ and for a constant magnetic field $B$, let $\lambda_j(B)$ be the $j$-th magnetic eigenvalue. The sum of the first $N$ eigenvalues, $\sum_{j=1}^N \lambda_j(B)$, may decrease by turning on a
nonzero magnetic field, but it can never drop below the semiclassical bound \[43\]. The proof heavily relies on the homogeneity of the magnetic field. For this case a stronger diamagnetic inequality was proven:

$$\text{tr}[\chi f((p + A)^2)] \leq \text{tr}[\chi f(p^2)]$$

for an arbitrary nonnegative, convex function $f$ decaying to zero at infinity. Here $\chi$ is the characteristic function of an open set, the operators are defined in the whole $\mathbb{R}^n$. This stronger diamagnetic inequality fails for non-homogeneous fields \[43\] but still the semiclassical bound for the eigenvalue sum is conjectured to hold.

### 2.6 One-body scattering

A short range magnetic field, $|B(x)| \leq C\langle x \rangle^{-1-\varepsilon}$, does not substantially influence the non-magnetic scattering theory, in particular asymptotic completeness holds. Long range potentials can also be included. The most general result is due to Robert \[140\]; previously Loss and Thaller treated the $|B(x)| \leq C\langle x \rangle^{-3/2-\varepsilon}$ case in \[121\] and they also considered the Dirac operator \[122\].

The borderline case, when $B(x)$ decays as $|x|^{-1}$ at large distances is substantially more involved. In this case, there is no decay on the vector potential. In the simplest $d = 2$ dimensional, axially symmetric situation the lack of decay leads to a dense point spectrum in the low energy region while the spectrum is absolutely continuous above an energy threshold \[129\]. To study the scattering, for simplicity one considers $d = 2$ and assumes that the field is homogeneous of degree $-1$, i.e. in polar coordinates $(r, \theta)$ it is given by $B = b(\theta)/r$. The analogous problem for non-magnetic scattering is the case of a homogeneous of degree zero electric potential, $V(x) = U(x/|x|)$, where the generic classical trajectories are asymptotically straight and they select the directions of local extrema of $U$. The classical trajectory in the borderline magnetic case turns out to be a logarithmic spiral if the magnetic field has a definite sign. If the total flux is zero, $\int_0^{2\pi} b(\theta) d\theta = 0$, then the trajectories are approximately straight lines (in the direction of the zeroes of $b(\theta)$); if the total flux is non-vanishing but $b$ has no definite sign, then both types of behavior may occur. The corresponding quantum scattering follows these trajectories. Part of this picture has already been proven in the recent work \[26\], the rest is work under preparation.

Scattering in constant magnetic field was first studied in \[6\] where the asymptotic completeness of the one-particle scattering for short range and Coulomb potential was shown (a different proof given in \[150\]). The general long range potential was treated independently by Laba \[97, 98\] and Iwashita \[91\].
2.7 Miscellaneous

In this section we mention two results whose non-magnetic counterparts are classics but their standard proofs are quite rigid and their extensions to magnetic fields were considerably more involved.

The Rayleigh-Faber-Krahn isoperimetric inequality on the lowest Dirichlet eigenvalue of a domain of given area predates quantum mechanics. Its magnetic analogue asserts \[ \lambda(\Omega, B) \geq \lambda(D, B), \quad \text{Area}(\Omega) = \text{Area}(D), \quad D = \text{disk}. \] (2.5)

The constant magnetic field plays the role of the homogeneity of the membrane in Rayleigh’s original formulation of the problem.

In the non-magnetic case, the minimal eigenvalue in any dimensions is attained for the ball. The minimizing domain for constant magnetic fields in dimensions \( d \geq 3 \), however, is unknown. Isoperimetric results for the magnetic Neumann Laplacians are also unknown. Note that (2.5) does not hold for the Neumann case since the ground state has a tendency to favor nonspherically symmetric geometry (Section 3.2.1), but the disk geometry should be extremal for other spectral variational problems in this case as well.

The standard proof of the original Faber-Krahn inequality uses rearrangement methods that are applicable for positive functions. The magnetic ground state of a general domain is genuinely complex and its amplitude, its phase, and the vector potential must be rearranged separately.

A Schrödinger operator with a periodic external potential has purely absolutely continuous (AC) spectrum by a classical theorem of Thomas [161]. The periodicity of the magnetic field itself does not guarantee AC spectrum (e.g. \( B = \text{const} \neq 0 \) in \( d = 2 \)), but a periodic vector potential does. Note that this latter implies not only the periodicity of the magnetic field but also that the flux is zero in the unit cells.

The absolute continuity of the magnetic Schrödinger spectrum with a small periodic vector potential was first proven in [70]. The proof was reduced by perturbation to the original analyticity argument of Thomas and it could not be extended beyond the perturbative regime. In [15] a representation similar to (2.3) was used to transform a periodic vector potential into a periodic external potential and a modification of Thomas’ argument applied. A periodic metric can also be included [131]. This approach, however, works only in \( d = 2 \) dimensions.

The general case was obtained by Sobolev [158] who proved that the spectrum is purely AC for the magnetic Schrödinger operator with a sufficiently smooth periodic vector potential in any dimension. The proof combined Thomas’ argument with a pseudodifferential technique.
3 Quantitative properties of one-body operators

3.1 Lieb-Thirring inequalities

One of the fundamental results about the standard Schrödinger operator \( -\Delta + V \) is the Lieb-Thirring bound \([119]\) on the moments of negative eigenvalues, \( E_j \), in terms of integral norms of the negative parts of the potential, \( V_- \)

\[
\text{tr} \left[ -\Delta + V \right]_\gamma^\gamma = \sum_j |E_j|_\gamma^\gamma \leq L_{d,\gamma} \int [V]_-^{d+\gamma/2}
\]

with a finite constant \( L_{d,\gamma} \) for \( d \geq 3, \gamma \geq 0 \); \( d = 2, \gamma > 0 \) or \( d = 1, \gamma \geq 1/2 \). This bound plays a crucial role in the proof of the stability of matter and it provides a basic a priori estimate for the semiclassical formulas and for justification of the Thomas-Fermi theory for the ground state energy of atoms and molecules.

By the diamagnetic inequality, the usual proof of the Lieb-Thirring (LT) bound for the non-magnetic operator, \( -\Delta + V \), applies directly to the magnetic Schrödinger operator, \((p+A)^2 + V\) as well. The same holds for the Cwikel-Lieb-Rozenblum (CLR) bound on the number of eigenvalues \((\gamma = 0)\). The presence of a magnetic field should, in principle, improve these estimates, but no such non-trivial result is available.

The systematic study of Lieb-Thirring bound and semiclassics for the Pauli operator has started with a series of seminal papers by Lieb, Solovej and Yngvason \([116, 117, 118]\). For the \( d = 3 \) dimensional Pauli operator, \( H = [\sigma \cdot (p+A)]^2 + V \), with a constant magnetic field, \( B \), and external potential, the following bound was proven for the sum of negative eigenvalues of \( H \) \([116]\)

\[
\sum_j |E_j| \leq \text{(const.)} \int [V]_-^{5/2} + \text{(const.)} \int |B| [V]_-^{3/2}
\]

where \( [V]_- = -\min\{0, V\} \). A similar bound holds in \( d = 2 \) dimensions as well \([118]\). The first term in \((3.6)\) is the corresponding Lieb-Thirring estimate for \( -\Delta + V \). Due to the paramagnetism, the Pauli energy may be below the non-magnetic energy and the additional term \( \int |B| [V]_-^{3/2} \) is indeed necessary. The number of eigenvalues can be infinite in \( d = 2, 3 \) dimensions, so there is no CLR-bound for the Pauli operator.

For non-homogeneous magnetic fields, the bound \((3.6)\) does not hold. Most importantly, the existence of the Loss-Yau zero modes shows that, in the perturbative regime, the lowest eigenvalue itself may scale linearly in \( [V]_- \). Moreover, the pointwise density of the Loss-Yau zero mode scales as \( \max |\psi(x)|^2 \sim B^{3/2} \). Therefore a general LT estimate in the strong field regime must contain a term that grows as the \( 3/2 \) power of \( B \).
To prove a LT estimate with the $B^{3/2}$ scaling, the spin-coupling term $\sigma \cdot B$ in (1.1) is treated as a potential and the diamagnetic inequality is used for $(p + A)^2$. Several papers \cite{36, 156, 145, 19} used this idea with different assumptions on the magnetic field. The most general result in this direction is due to Lieb, Loss and Solovej \cite{113} showing that

$$\sum_j |E_j| \leq (\text{const.}) \int [V]^{5/2} + (\text{const.}) \|B\|_2^{3/2} \|[V]_{-}\|_4. \quad (3.7)$$

The proof introduces the so-called running energy scale method. It consists of artificially scaling down the Pauli kinetic energy in an energy-dependent way to reduce the negative effect of $\sigma \cdot B$. The main advantage of this method is that it uses no other assumptions on $B$ apart from the finiteness of its $L^2$-norm. Note that $\int B^2$ is the field energy.

Although a term growing as $B^{3/2}$ (in the large field regime) is necessary for a general LT bound, a smaller power is sufficient if some control on $\nabla B$ is allowed. Especially, the linearity in $B$ of the bound (3.6) reflects the basic fact that the space with a magnetic field cannot be considered isotropic: the magnetic field affects only the quantum motion in the transversal directions.

Under a control on the $H^1$ norm of $B$ the LT bound in \cite{21} scales as the 17/12 power of the field. With more regularity on $B$ and $V$ the lower power 5/4 was obtained in \cite{44}. Finally, the correct linear behavior in the field strength under a stronger regularity assumption was proved in \cite{17} and \cite{47}. The proof in \cite{17} is shorter, but the estimate is not local: a large irregular magnetic field far away from the support of $[V]_{-}$ should not influence the eigenvalue sum too much, but the estimate in \cite{17} does not reflect this. A conceptually different proof was given in \cite{47} that relies on a much stronger localization and approximation technique.

The main difficulty behind these proofs is to control the density of Loss-Yau zero modes. It is amusing to note that it was a substantial endeavour to show that zero modes may exist at all (Section 2.3). On the other hand, it is quite difficult to prove an upper bound on their number in terms of the expected first power of the magnetic field \cite{48}.

### 3.2 Semiclassics and strong fields

We have seen that magnetic fields can cause surprising effects when the magnetic lengthscale is comparable with other lengthscales in the problem. However, in the semiclassical and/or in the strong field regimes, lengthscales are typically separated, rendering simpler formulas available in the limit. One studies the magnetic Schrödinger or Pauli operators with two parameters:

$$H(h, b) := (hp + bA)^2 + V \quad \text{or} \quad [\sigma \cdot (hp + bA)]^2 + V,$$
where $h \ll 1$ is the semiclassical parameter and $b$ is the field strength; in most cases $b \gg 1$ (assuming that $A$ and $V$ are fixed). The magnetic field is $bB(x) = b \text{curl } A(x)$. Under these scalings, the magnetic field can typically be approximated by a (locally) homogeneous one, since the magnetic lengthscale $(b/h)^{-1/2}$ is short. If, in addition, $hb \ll 1$, then the gap between (local) Landau levels shrinks to zero and magnetic effects usually do not contribute to the main term in the asymptotic regime. This is especially the case for the standard semiclassics when $h \to 0$ and $b$ is fixed \cite{25}. If, on the other hand, $hb \not\to 0$, then the main term is typically obtained by replacing the magnetic field by a locally constant one.

The key technical step is, therefore, to localize the problem to a sufficiently small scale, where the data (especially the magnetic field) can be considered homogeneous. Similarly to the non-magnetic theories, two basic techniques have been developed: pseudodifferential calculus and coherent states.

Note that the presence of a constant magnetic field changes structure of the phase space. For example, in three dimensions, the phase space is $\bigcup_{\nu=0}^{\infty} \mathbb{R}^4$, where $\nu$ labels the Landau levels. The phase space for each level is four dimensional; it consists of three position coordinates and only one momentum coordinate which represents the free motion parallel with the magnetic field. The momenta transversal to the field are not present due to the localization effect of the field. Accordingly, one either has to develop a pseudodifferential calculus that treats the harmonic oscillators on each Landau level exactly or one has to construct magnetic coherent states.

### 3.2.1 Results on individual eigenvalues

Semiclassical estimates on individual eigenvalues and eigenfunctions have mainly been carried out for the ground states. For the magnetic Schrödinger operator without potential the ground state is localized near the minimum of the field strength (“magnetic bottles”, see \cite{6} and \cite{22}). The basic observation is that due to the positivity of the Pauli operator and the Lichnerowicz formula \cite{14}, the magnetic Schrödinger operator can always be estimated from below by $B(x)$, at least in two dimensions and if $B \geq 0$. Similar result holds in higher dimensions, at least locally. This concentration of ground state is especially visible in the large field regime. Note that if external potential is not present, the semiclassical limit is formally identical to the large magnetic field limit, $h = 1/B$.

Precise analysis of this phenomenon was initiated by several groups with different methods. With the help of the Feynman-Kac formula, the magnetic ground state energy can be turned into a question about the rate of decay of an oscillatory Wiener integral, see \cite{126} and for a more precise bound \cite{35}, \cite{162}. Ueki has also explored the connection with the hypoellipticity of the $\overline{\partial_b}$ problem \cite{163}. Montgomery \cite{130} has analysed the case when the two dimensional magnetic field vanished along a curve. In this case $\min |B| = 0$, hence the leading term in the
large field asymptotics vanishes, and Montgomery obtained the subleading term that involved the curvature of the zero locus. This approach was later generalized in [73].

On a domain with boundary, however, the Lichnerowicz formula (1.1) does not hold, unless Dirichlet boundary condition are imposed. In particular, the ground state energy of the magnetic Schrödinger operator with Neumann boundary condition is smaller than $B$ even for a constant magnetic field. In this latter case the ground state is localized near the boundary, more precisely near the point with largest curvature of the boundary. The second term in the semiclassical expansion of the ground state energy is determined by the curvature, similarly to Montgomery’s result. Similar phenomenon occurs in three dimensions as well with a proper definition of an effective curvature [74]. Recently a complete expansion for the energy of the low lying eigenvalues in $d = 2$ dimensions was carried out in [55]. We remark that the Neumann boundary problem naturally arises at the minimization of the Ginzburg Landau energy functional describing superconducting states. Several people have contributed to these results, see [74], [55] and references therein.

3.2.2 Results on cumulative spectral quantities

The first result on spectral statistics, where the magnetic field played a non-trivial role, is probably due to Colin de Verdière [22] and Tamura [160] (independently) who proved a Weyl-type asymptotics for the number of eigenvalues with a non-homogeneous magnetic field and a confining potential. The magnetic field increases at infinity, ensuring that magnetic effects contribute to the large energy asymptotics. Colin de Verdière used the magnetic extension of the classical Dirichlet-Neumann bracketing, while Tamura estimated the short time asymptotics of the magnetic heat kernel. Several authors extended these results, see [127] for references.

A technically somewhat similar problem is the rate of the eigenvalue accumulation near the Landau levels due to perturbation by a decaying potential. The basic idea is that the magnetic field strongly localizes the particle and its interaction with the potential can be computed fairly precisely. The accumulation rate is explicitly given by the decay rate of the external field, see [138] and references therein.

The next semiclassical question concerns the moments of negative eigenvalues in the spirit of [115]. Here we consider only the more interesting case of the Pauli operator, $H(h, b) := [\sigma \cdot (hp + bA)]^2 + V$. For simplicity, we work in $d = 3$ dimensions and we will approximate only the eigenvalue sum;

$$\Sigma(h, b) := \text{tr}[H(h, b)]_-. $$

The corresponding semiclassical expression is

$$E_{sc}(h, b) := -h^{-3} \int_{\mathbb{R}^3} P(hb|B(x)|, [V(x)]_-)dx$$
with
\[ P(B, W) := \frac{B}{3\pi^2} \left( W^{3/2} + \sum_{\nu=1}^{\infty} (2\nu B - W)^{3/2} \right). \] (3.8)

This formula can be simply deduced from the structure of the phase space outlined above.

For homogeneous magnetic field, the semiclassical limit
\[ \lim_{h \to 0} \frac{\Sigma(h, b)}{E_{sc}(h, b)} = 1 \] (3.9)
was proved uniformly in the field strength \( b \) \( [117] \) (the two-dimensional result was obtained in \( [118] \)). The main ingredients were the magnetic Lieb-Thirring inequality (3.6) and new magnetic coherent states. For the non-homogeneous case, a Lieb-Thirring inequality that scales as \( B^{3/2} \) (see Section 3.1) allows one to prove the semiclassical formula only up to \( \hbar b = O(1) \) \( [157] \). With the improved Lieb-Thirring inequality \( [11] \) and a new construction of coherent states, the proof can be extended to \( b \ll \hbar^{-3} \). This result is already sufficient to cover the full semiclassical regime of the large atoms \( [15] \), see Section 4.2. Uniform semiclassics can be obtained with the help of the uniform Lieb-Thirring inequalities \( [47, 48] \).

The development using pseudodifferential calculus has focused on obtaining precise spectral asymptotics for the local traces of the form \( \text{tr} \chi \varphi(H) \), where \( \chi \) is a spatial cutoff function. These efforts have culminated in the book of Ivrii \( [89] \) where precise remainder estimates were proven in a great generality. His recent work investigates the same questions with irregular data \( [90] \). A more concise result using these ideas is the proof of a certain local version of (3.9) for homogeneous field in \( [154] \), improved later in \( [155] \) to include Coulomb singularities. The microlocal technique gives also higher order corrections to the leading term. However, these methods require, in general, strong regularity assumptions on the data. Moreover, some non-asymptotic apriori estimate (Lieb-Thirring bound) is necessary to remove the cutoff \( \chi \).

In addition to the energy, other physical quantities are also of interest. Fournais has studied the quantum current in a magnetic field and proved the corresponding semiclassical formula. Note that the current is a second order effect in the semiclassical expansion and it becomes a leading term only after non-trivial cancellations. Both microlocal techniques similar to \( [154] \) and coherent states methods similar to \( [117] \) have been tested, e.g. in \( [52] \) and \( [53] \).

### 3.3 Peierls substitution and corrections to the semiclassics

The Bloch decomposition for a single particle Schrödinger operator with a periodic external potential can be extended to include weak electromagnetic fields. The basic idea due to Peierls is to substitute the minimally coupled magnetic momentum \( p + A \) into the band functions, \( E_n(p) \), obtained from the non-magnetic Bloch decomposition. If the electromagnetic field...
varies on a much larger scale than the periodic background, then the problem is effectively
semiclassical with a scale-separation parameter $\varepsilon$ (for the general theory, see [135]). The
resulting pseudo-differential operator can be analyzed with well developed mathematical tools.
Algebraic methods were applied in [14], for a systematic presentation of the pseudo-differential
approach see [77]. For example, it was shown in [63] that near a fixed energy level the original
Hamiltonian is isospectral to a pseudodifferential operator with the same principal symbol as
the Peierls Hamiltonian has. The detailed behavior of the density of states, in particular the
dea Haas-van Alphen effect for the oscillation of the magnetization was shown in [78].

The de Haas-van Alphen oscillation is due to a subleading effect in a semiclassical type
expansion, however, it determines the current to leading order. The electromagnetic field is
weak, but on the long time scale, $t \sim \varepsilon^{-1}$, it yields an order one change in the dynamics. To
describe these effects on the dynamics correctly, Panati, Spohn and Teufel [136] have developed
a time dependent version of the Peierls substitution. The classical equations are corrected by
an the effective magnetic moment (Rammal-Wilkinson term) and an “anomalous velocity”
term due to the curvature of the Berry connection. This latter, in particular, provides a
simple semiclassical explanation of the quantum Hall current.

We remark that an oscillation similar to the de Haas-van Alphen effect is exhibited for
the Harper operator (2.4) for magnetic fluxes (per unit cells) that are near a rational number,
see [62] and references therein. The Harper operator itself can also be viewed as a Peierls
substitution by quantizing the classical symbol $H(\xi, \eta) = \cos \xi + \cos \eta$ with $[\xi, \eta] = 2\pi i \alpha$. In
this case the distance of $\alpha$ to a nearby rational number with small denominator plays the role
of the semiclassical parameter [76].

4 Many-body magnetic systems

4.1 Magnetic stability of matter

In a system of charged fermionic quantum particles subject to a Coulomb interaction the
ground state energy per particle is uniformly bounded, independently on the number of par-
ticles. This fundamental fact is called the stability of matter. For an excellent review of the
progress in the last 35 years, see [106].

The first proof of the stability of matter with the nonrelativistic kinetic energy, $-\Delta = p^2$,
is due to Dyson and Lenard. A simpler proof was given later by Lieb and Thirring using the
Lieb-Thirring inequality. Stability of matter also holds if the nonrelativistic kinetic energy
operator, $p^2$, is replaced by $|p| = \sqrt{-\Delta}$ (“relativistic” kinetic energy) and the fine structure
constant, $\alpha$, is sufficiently small. The first proof was given by Conlon, improved by Fefferman-
de la Llave and finally the optimal bound was obtained by Lieb and Yau; see also a recent
improvement and references in [112].

In the relativistic case, the kinetic energy and the potential energy both scale as \([\text{length}]^{-1}\), therefore the energy per particle can be bounded from below only if the Hamiltonian is non-negative, i.e.

\[
\sum_{j=1}^{N} |p_j| + \alpha V_c \geq 0 \tag{4.10}
\]

where \(V_c\) stands for the Coulomb potential of \(N\) electrons and \(K\) nuclei with charges \(Z\). This inequality was proven by Lieb and Yau in [120] (Theorem 2) for \(\alpha \leq 1/94\) and \(Z\alpha \leq 2/\pi\), the second condition being optimal. Theorem 1 of [120] has a weaker result (\(\alpha \leq 0.016, Z\alpha \leq 1/\pi\)) but its proof can easily be generalized to the magnetic case as well, where the kinetic energy \(|p|\) can is replaced by its magnetic counterpart \(|p + A|\). This follows from a simple application of the diamagnetic inequality as it was pointed out in [113].

With the Pauli kinetic energy, however, even the hydrogen atom is unstable because in a strong magnetic field the electron can be strongly localized around the nucleus without a penalty in the kinetic energy. The ground state energy of the hydrogen in a constant magnetic field \(B\) diverges as \((\log B)^2\), if \(B\) is large [6].

If the energy of the magnetic field is added to the total energy, then stability is restored:

\[
\sum_{j=1}^{N} \left[ \sigma \cdot (p + A)_j \right]^2 + V_c + \frac{1}{8\pi \alpha^2} \int B^2 \geq -C(Z)(K + N) \tag{4.11}
\]

where the constant depends only on the charges of the nuclei. The parameter \(\alpha\) (fine structure constant) must be sufficiently small in order (4.11) to hold. The existence of a Loss-Yau zero mode [124] shows that the total energy can be negative if \(\alpha\) is large. Actually, an absolute upper bound on \(\alpha\) and an upper bound on \((\max Z_j)\alpha^2\) are both necessary, where \(Z_j\) are the nuclear charges.

The bound (4.11) is called the stability of matter interacting with a classical magnetic field. It was first proven for atoms [57] and single electron molecules [107]. The general case was settled by Fefferman [50] for a very small \(\alpha\). Lieb, Loss and Solovej [113] gave a much shorter proof that also holds for the physical value of the fine structure constant. The backbone of this proof is the magnetic Lieb-Thirring inequality [37].

Using the Birman-Koplienko-Solomyak trace inequality and the magnetic version of the Lieb-Yau bound (4.10), Lieb, Siedentop and Solovej [114] also proved the stability of matter for the Dirac operator. The particles must be restricted to the positive energy subspace of the one-particle Dirac operator \(\alpha \cdot (p + A) + \beta\) (filling the “Dirac sea”). It is important to note that the Dirac sea must be defined via the gauge invariant Dirac operator. Restriction onto
the positive energy subspace of the free Dirac operator (as it is often done in perturbation theory) leads to instability of matter for any $\alpha$.

Ultimately, the electromagnetic field must also be quantized. Imposing and ultraviolet cutoff and using results from [113], Bugliaro, Fröhlich and Graf proved stability of matter for the Pauli operator with a quantized electromagnetic field [20]. The essential observation is that to restore the magnetic stability for the Pauli operator with a classical field, it is sufficient to add the field energy only near the nuclei. On a finite volume and with an ultraviolet cutoff, the classical and quantized field energies can be compared. Lieb and Loss [109] showed stability of matter for the Dirac operator with a quantized electromagnetic field and with a suitable one-body spectral projection, similar to [114].

The quantization of the electromagnetic field poses several complications, such as ultraviolet cutoff and mass renormalization, and little is known about how to rigorously include them into a fully consistent theory. However, one problem has been settled satisfactorily: the existence of atoms in nonrelativistic QED. Because of the quantized field, the ground state of the total system (atom and photons) is at the bottom of a continuous spectrum, and it is not at all obvious that it is an eigenvalue. Moreover, the so-called binding condition also has to be satisfied, that is the energy of a system of $N$ electrons is actually lower than that of a system with fewer electrons, otherwise the ground state may contain no electron at all. For small values of the parameters (ultraviolet cutoff parameter and fine structure constant) the existence of atoms was shown in [8] and for arbitrary parameter values in [70, 110]. No infrared cutoff was needed, unlike for the scattering problem (Section 4.3). More recently, the thermodynamic limit for non-relativistic Coulomb matter with quantized electromagnetic field was investigated and the lower bound was proved [111].

4.2 Large atoms

One of the main motivations to study semiclassical spectral asymptotics for $-\hbar^2 \Delta + V$ originates in the seminal paper of Lieb and Simon, [115], where the exactness of the Thomas-Fermi theory for the ground state energy of atoms with large nuclear charge, $Z \gg 1$, was proven with semiclassical methods.

In the presence of magnetic fields the Thomas-Fermi theory is more complex. Depending on the strength of the magnetic field compared with $Z$, there are five different regimes.

Within the classical Thomas-Fermi theory, the kinetic energy as a functional of the density is always given by the Legendre transform of the pressure [38]. This classical magnetic Thomas-Fermi theory, however, holds only for weak and moderate magnetic fields ($B \ll Z^3$). More precisely, for $B \ll Z^{4/3}$ the magnetic effects are absent in the leading term of the large $Z$ asymptotics. For $B \sim Z^{4/3}$ the full magnetic Thomas-Fermi theory is needed. If $Z^{4/3} \ll B \ll Z^3$, the usual Thomas-Fermi theory still applies, but only the first summand
in the pressure function (3.8) is needed. The atom is spherical in all these cases, but the energy is affected by the magnetic field. These results were proven in [117] with the help of the magnetic Lieb-Thirring inequality and the semiclassical result (3.9).

For a strong magnetic field, \( B \geq (\text{const}) Z^3 \) with a positive constant, the electrons are confined to the lowest Landau band and the shape of the atom is a long cylinder along the field. For \( B \sim Z^3 \) the atoms in the transversal direction have a non-trivial structure, that can be described by a new density functional theory relying on minimizing density matrices instead of density functions [116]. Finally, if \( B \gg Z^3 \), the atom becomes effectively one-dimensional and a one dimensional Thomas-Fermi caricature applies. Analogous results in \( d = 2 \) were obtained in [118].

If the magnetic field goes to infinity, but \( Z \) is fixed, then the ground state energy diverges as \(-\frac{1}{4} Z^2 (\log B/2)^2\). For one electron atoms this has been established in [6]. The energy of the many-body system, after factoring out the divergent \((\log B)^2\) term, is given by the ground state energy of an effective one dimensional bosonic Hamiltonian with Dirac delta interactions [12]. This idea has been extended to prove resolvent convergence and explore other effective Hamiltonians in [18] and references therein.

The correctness of the Thomas-Fermi theory in the semiclassical regime \((B \ll Z^3)\) for non-homogeneous magnetic fields were proven in [45] after extending the Lieb-Thirring inequality [44] and constructing appropriate coherent states. The uniform magnetic Lieb-Thirring inequality [47] should allow also the extension of the strong field regime from [116] to non-homogeneous magnetic fields.

The asymptotic behavior of the total magnetization and of the current for large atoms in homogeneous fields was obtained in [53, 54]. A bound on the maximal ionization is proven in [143].

### 4.3 Multiparticle scattering in a magnetic field

A detailed presentation of scattering theory and asymptotic completeness is given in the contribution of C. Gérard in this volume; here we just shortly mention the most important results involving magnetic fields.

\(N\)-body asymptotic completeness is well understood for non-relativistic particles without magnetic field. Scattering in the presence of a constant magnetic field is a much more delicate question since a classical charged particle moves on circles. Therefore charged subsystems can scatter only parallel with the field, while neutral systems may move out to infinity in all directions. The general theory has been developed and asymptotic completeness has been proved for the case when all possible subsystems are charged by Gérard and Laba (the best reference is their book [65]). The zero charge case in general is still open. The special case of three particles with Coulomb forces was solved in [64] and one charged particle was considered.
Constant electric field can also be included. Skibsted [153] reduced this problem to scattering of non-interacting subsystems all having the same charge/mass ratio.

In the presence of a quantized electromagnetic field, the scattering of photons on an bound electron (Rayleigh scattering) and the scattering of electron dressed with photons (Compton scattering) have been studied. The mathematical framework is non-relativistic quantum mechanics with a quantized field with ultraviolet cutoff to ensure that the Hamiltonian is well defined. For total energies below the electron-positron pair creation threshold, this non-relativistic caricature of QED is well justified. One also has to cope with the infrared (IR) problem (“soft bosons”): there could be infinitely many photons with a finite total energy. For technical simplicity, scalar photons are considered. This set of problems was initiated by Fröhlich who first investigated the infrared problem and constructed wave operators (without completeness) [56].

The Rayleigh scattering was first tackled in [28] where the photons were massive to avoid the IR problem and the potential was confining. Later it was extended to the physically more realistic massless photons, but with an IR-cutoff, and instead of confining potential, the total energy was set below the ionization threshold which also guarantees spatial localization of the electrons [59]. During scattering, photons are absorbed by the bound electrons, lifting them to higher excited states, but no electron can escape. Thus, after some time, the electron cloud relaxes to the ground state and emits photons that propagate essentially freely to infinity in space. Note that the analysis of Bach et al. guarantees that all excited states are unstable [8, 9].

For Compton scattering [60], the total energy is assumed to be sufficiently small so that the speed of the massive electron is less than 1/3 of the speed of light. This technical assumption safely separates free photons from the dressed electron after long time evolution. The asymptotic completeness in this model means that the long time evolution of the state is a linear combination of asymptotic states consisting of a freely moving electron dressed by a photon cloud plus freely moving excess photons.

5 Random Schrödinger operators with magnetic fields

Since the proof of the Anderson localization with the powerful multiscale method of Fröhlich and Spencer [61], random Schrödinger operators have become one of the main research directions in mathematical physics. The typical problems concern the self-averaging properties (deterministic spectrum, existence of density of states), the asymptotics of the density of states (Lifshitz tail), and establishing the dense point spectrum in the localization regime. These questions have been recently studied with magnetic field as well; most results are in the most
relevant $d = 2$ dimensions.

5.1 Constant magnetic field

First we consider the problems where the magnetic field is constant and only the external potential is random, i.e. the random Landau Hamiltonian (for a recent survey, see [103] and references therein). In this case, the standard self-averaging techniques work to establish deterministic spectrum, the translation must simply be replaced by the magnetic translation. The Lifshitz tail in the low energy regime for Gaussian random potential is very universal and it is insensitive to the magnetic field (it even holds for certain random magnetic fields). For Gaussian randomness even the density of states is bounded and the Wegner estimate holds [87]. Localization with algebraically decaying eigenfunctions were proved in [51], exponential localization in [163]. These proofs are valid only at very low energies, using the deep wells of the Gaussian randomness, in particular this regime is far from the analogue of the band-edge localization.

For repulsive Poissonian obstacles the precise Lifshitz tail is more delicate, similarly to the classical vs. quantum dichotomy in the non-magnetic setup. If the single-site potential has a slow decay, then classical effects dominate and the Lifshitz tail can be computed from a simple mean-field argument. Otherwise the quantum localization energy competes with the entropy of the large domains free of impurities. For magnetic fields, the threshold decay is not algebraic but Gaussian. The classical regime was investigated in [17] (in three dimensions [86]), the main result in the quantum regime was obtained in [40] using Sznitman’s coarse-graining probabilistic method (see also [83], [41]).

Since in the Poisson model the energy is not monotone in the random variable, so Wegner estimates are much harder to obtain, localization for a constant magnetic fields were investigated for Anderson type i.i.d. random potentials. Wang has given a full asymptotic expansion of the density of states away from the Landau bands [165]. More explicit quantitative results are known on the Lifshitz tails, including the double logarithmic asymptotics at each band edge [96]. The pure point spectrum at a certain distance away from the Landau levels was proven independently by Wang [164], Combes and Hislop [24] and in a single-band approximation by Dorlas, Macris and Pulé [29]. Since $d = 2$ is the borderline dimension for the necessary large distance decay of the Green’s function of the non-magnetic Laplacian, an additional decay must be extracted from the presence of the magnetic field. This eventually required basic results from two-dimensional bond percolation theory. Precise estimates on the localization length and dynamical localization near the Landau band edges was obtained by Germinet and Klein [67] using their extension of the Fröhlich-Spencer multiscale analysis [66].

In contrast to the localization regime, the presumed regime of delocalization in Anderson-type models is poorly understood and there are almost no mathematical results. Apart from
the Bethe-lattice, there is only one model, where the existence of the mobility edge has been rigorously proven: the random Landau Hamiltonian [68]. More precisely, it is shown that there exists at least one energy \( E \) near each Landau band, so that the local transport exponent, \( \beta(E) \), is positive. The local transport exponent measures the extension of the wave packet in a suitable averaged sense for large times. (Dynamical) localization is characterized by \( \beta(E) = 0 \), moreover, there is an important dichotomy for \( \beta(E) \): it is either zero or at least \( 1/2d \).

The key quantity in the proof in [68] is the Hall conductance. In the regime of dynamical localization, the Hall conductance is constant in the mobility gap (see [11], strengthened later in [31]). On the other hand, the Hall conductivity jumps by one at each Landau level for the free Landau Hamiltonian and it is also constant, as a function of the disorder parameter, in the gaps. Therefore it must jump somewhere inside the bands, at least for sufficiently small disorder, but then the complete band cannot belong to the dynamically localized regime, completing the argument of delocalization in [68].

The Hall conductance for quantum Hall systems (two dimensional disordered samples subject to a constant perpendicular magnetic field) at energies falling into the mobility gap, \( \Delta \), can actually be defined in two different ways. The bulk conductance is defined on \( \mathbb{R}^2 \) by the Kubo-Streda formula (see [4])

\[
\sigma_B(\lambda) = -i \text{tr} P_\lambda \left[ \left[ P_\lambda, \Lambda_1 \right], \left[ P_\lambda, \Lambda_2 \right] \right]
\]

where \( P_\lambda \) is the spectral projection onto \((-\infty, \lambda] \) and \( \Lambda_i \) is the characteristic function of \( \{x_i < 0\} \), \( i = 1, 2 \).

The edge conductance is defined in a half plane sample \( x_2 \geq -a \), eventually \( a \to \infty \) by

\[
\sigma_E = -i \text{tr} \varrho'(H)[H, \Lambda_1]
\]

(modulo nontrivial technicalities), where \( \varrho \) is a smooth spectral cutoff function that is one below \( \Delta \) and zero above \( \Delta \). This quantity gives the derivative of the current flowing through the line \( x_1 = 0 \) with respect to lowering the chemical potential along the edge. These two conductances are the same. For intervals \( \Delta \) falling into the spectral gap, this was proven in [12] with K-theoretical methods and later in [30] by basic functional analysis. The proof given recently in [31] is valid for intervals \( \Delta \) that contain strongly localized spectrum as well, i.e. \( \Delta \) can be chosen the so-called mobility gap.

### 5.2 Random magnetic field

Since a magnetic field itself enhances localization, one expects that a random magnetic field is localizing even stronger than a random potential. Technically, however, random magnetic
fields are harder to fit into the multiscale analysis; mainly because the vector potential is non-local. There is a substantial difference between the zero and non-zero flux cases, the former being easier. In particular, stationary random vector potentials always generate magnetic fields that have zero flux on average.

The existence of the integrated density of states (IDS), and its independence of the boundary conditions in the thermodynamic limit (uniqueness) has first been shown by Nakamura for both the discrete and continuous Schrödinger operator with a random magnetic field with non-zero flux \cite{133,134}. In both cases Lifshitz tail was also obtained. Recently, Hundertmark and Simon gave a short proof for the existence and uniqueness of the IDS \cite{85}.

Anderson localization has been shown for a Gaussian vector potential by Ueki in \cite{163}. The Germinet-Klein multiscale analysis has been extended to include very general random magnetic fields, but the Wegner estimate requires random vector potential (which implies zero average flux) in addition to other technical conditions.

For the discrete random magnetic Schrödinger operator, the method of Nakamura \cite{133} has been extended to obtain Wegner estimate and localization \cite{95}. However, the zero flux condition is enforced in a strong sense: neighboring cells are paired and the magnetic flux is opposite in these pairs. A deterministic background magnetic field atop of the local random vector potential is allowed in \cite{81} where Wegner estimate was proved in the continuous model. A small stationary random vector-potential was included in a Schrödinger operator with a periodic background potential in \cite{69}, where Lifshitz tail was proven under a special non-resonance condition.

Band-edge localization for Schrödinger operators with random magnetic field is widely believed to hold in the most general case. The additional assumptions on the zero flux (which, in one form or another, is present in all papers so far) seems to be only technical. However, there is no agreement in the physics literature about the possible existence of the continuous spectrum for such operators, unlike in the non-magnetic case, where the existence of the extended states is universally accepted by physicists and “only” the mathematical proof is missing. The Landau orbits and their quantum counterparts, the strongly localized magnetic eigenfunctions are characteristic only to the constant magnetic field and they are not universal. There is an undecided competition between a possible weaker form of the Landau localization, that may still hold for random fields, and the resonance effects that enhance delocalization.

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