Composite operators from the operator product expansion: what can go wrong?

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The operator product expansion is used to compute the matrix elements of composite renormalized operators on the lattice. We study the product of two fundamental fields in the two-dimensional \(\sigma\)-model and discuss the possible sources of systematic errors. The key problem turns out to be the violation of asymptotic scaling.

The Operator Product Expansion

\[ A(x; \mu)B(-x; \mu) \sim \sum_C W_{AB}(x; \mu)C(0; \mu) \] (1)

is widely thought to hold beyond perturbation theory.

The use of Eq. (1) in lattice simulations is still in its infancy. In Ref. it was suggested to use Eq. (1) in order to compute renormalized matrix elements. The OPE approach consists in the following steps: one computes the (matrix element of the) l.h.s. of Eq. (1), then renormalizes \(A\) and \(B\) in some scheme, and finally obtains (the matrix element of) \(C\) through a fit, using some perturbative approximation of the Wilson coefficients.

The main sources of systematic errors in this approach are the following:

(a). finite-size effects and corrections to scaling, i.e. lattice artifacts;

(b). “power-correction effects” which are due to the fact that we truncate the expansion to some finite order in \(x^2\);

(c). corrections to asymptotic scaling which must be taken in account since the Wilson coefficients in Eq. (1) have to be substituted by the first few terms of their perturbative expansion.

Errors of type (a) are widely studied and do not need more explanations. Here we shall focus on errors of type (b) and (c).

The use of Eq. (1) in a continuum scheme, for which only a perturbative computation of the Wilson coefficients is available, poses a restriction on the operators which can be obtained in this approach. Only the operators of lowest dimension for each spin sector, i.e. those of lowest twist, can be computed using Eq. (1). Higher-twist operators give rise to systematic errors of order \(O(x^2)\). Moreover, because of statistical errors, only the operators of low dimension can be reliably extracted: the remaining ones are strongly subleading in the region of validity of the OPE.

Problem (c) can be stated in a cleaner way if we get rid of the scale dependence which is introduced in this approach somehow artificially through the Wilson coefficients. One can rewrite Eq. (1) by making use of renormalization-group invariant operators defined as follows:

\[ Q^{\text{RGI}}(x) \equiv Q(x; \mu)/F_Q(g(\mu)) \, , \] (2)

\[ F_Q(g) \equiv g^{\gamma Q/\beta_0} \exp \left\{ \int_0^g \left[ \frac{\gamma Q(x)}{\beta(x)} - \frac{\gamma Q}{\beta_0} \right] dx \right\} \, . \] (3)

The Wilson coefficients obviously become \(\mu\)-independent and their general perturbative form is

\[ W^{\text{RGI}}(\Lambda x) = g(\Lambda x) \sum_{k=0}^\infty c_k g(\Lambda x)^k \, , \] (4)

where \(\Lambda\) is the “lambda parameter” of the theory.
The use of a truncation of Eq. (4) introduces systematic errors of order $O(\log^2 (\Lambda x))$. Notice, however, that this approach allows to compute directly “infinite-energy” quantities (i.e. the renormalization-group invariant matrix elements) which are of interest in phenomenological applications. Errors of order $\log^k (\Lambda x)$ arise also in the widely used “non-perturbative renormalization method” in which perturbation theory is used to “evolve” the renormalization constants computed at some energy scale achievable on the lattice up to high energies.

We have considered several products of operators for the $O(N)$ nonlinear $\sigma$-model in two dimensions, with lattice action

$$S(\sigma) \equiv \frac{1}{2g_t} \sum_{x,\mu}(\partial_x \sigma)^2,$$

where $\sigma \in S^{N-1}$, $(\partial_x f)_x \equiv f_{x+\mu} - f_x$, and $N = 3$. Here we shall refer to the following (respectively scalar and symmetric) products of fundamental fields:

$$\sigma(x) \cdot \sigma(-x) \sim W_0(x) + O(x^2),$$

$$\sigma^a(x)\sigma^b(-x) + (a \leftrightarrow b) - \text{trace} \sim W_2(x) [\sigma^a \sigma^b - \text{trace}] (0) + O(x^2).$$

The Monte Carlo data presented refer to two lattices: the first one of size $L \times T = 128 \times 256$ and correlation length $(am)^{-1} = 13.632 (6)$ ($m$ is the mass gap); the second with $L \times T = 256 \times 512$ and correlation length $(am)^{-1} = 27.094 (43)$.

The expectation value of the product (6) between states of momentum $p$ is shown in Fig. 1 for the two different lattices: corrections to scaling, i.e. errors of type (a), are completely under control in our simulations.

Our general procedure consists in choosing a truncation of the expansion (4) and in using it to fit the data in the region $\rho < |x| < R$. The results are independent of $\rho$ for $1.5 a \leq \rho \leq 3 a$. We use the stability of the fit with respect to the truncation and to the choice of $R$ as a criterion to distinguish whether errors of type (b) and (c) are relevant or not.

The quality of the fits obtained is well represented by the curves shown in Fig. 1, where the two-loop expression was used for the first term in the expansion (4) and the tree-level form for the terms of order $O(x^2)$.

An additional conclusion can be drawn from Fig. 1 in order to describe the symmetric product (6) up to distances $2x \sim m^{-1}, p^{-1}$, it is necessary (and almost sufficient) to include terms of order $O(x^2)$. However this does not mean that the terms of order $x^2$ with the same symmetry of the leading one — the higher-twist terms — can be obtained from the fit. Indeed, their matrix elements extracted from the fits are very unstable with respect to changes of $R$.

In order to test the stability of the fit, we considered the vacuum expectation value of the product (4), that is the two-point function. Here the renormalization constant of the field is a fit parameter. The results are reported in Fig. 1 and refer to the lattice with correlation length

Figure 1. The product (6) on the slice $x_0 = 0$. Circles are the data at $(ma)^{-1} \simeq 13.632$ and stars at $(ma)^{-1} \simeq 27.094$ (properly rescaled). Continuous and dashed lines are the fitting curves respectively with and without $O(x^2)$ terms.
Figure 2. The field-renormalization constant as computed with different truncations of the OPE. From top to bottom: one loop, two loops, three loops, three loops plus $O(x^2)$ corrections.

$\left(\frac{m}{a}\right)^{-1} \approx 27.094$; we used $\overline{m}/a \approx 10$. As a manifestation of asymptotic freedom, the various curves shrink when $R \to 0$. Their $R$-dependence becomes weaker as more terms of the perturbative expansion are included. Nevertheless, even if we use the three-loop Wilson coefficient, we do not obtain a value of $Z$ independent of $R$ within the statistical errors, as it should be in the asymptotic-scaling regime. Two remarks are in order here: first, it is notoriously difficult to reach asymptotic scaling in the $O(3)$ nonlinear $\sigma$-model \footnote{G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa, and A. Vladikas, Nucl. Phys. B445 (1995) 81}; second, we are studying a small effect which could be negligible in QCD applications with respect to other errors.

Note that the addition of $O(x^2)$ terms, which, in this case, have spin 0 and are therefore higher twists, seems to improve the situation, making the curve flatter. However, this must be regarded as a spurious effect. Higher twists are simply mimicking the contribution of higher orders in perturbation theory; otherwise, the curves obtained without including them should show a $(mR)^2$ behavior. From Fig. 2 we can estimate the systematic error due to the use of two- or three-loop Wilson coefficients to fit the data in this range of $mR$: the error is approximately 5%.

We report in Fig. 3 the results of the fit presented in Fig. 1. Each type of truncation, including or not $O(x^2)$ corrections, gives a reasonable, that is $p$ independent and (almost) $R$ independent, answer. The difference between them should be interpreted as a violation of asymptotic scaling and, indeed, it is of the same magnitude of the systematic error estimated for the renormalization constant of the field.

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