New Orlicz-Hardy spaces associated with divergence form elliptic operators

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Dedicated to Professor Lizhong Peng in celebration of his 66th birthday

Abstract

Let $L$ be the divergence form elliptic operator with complex bounded measurable coefficients, $\omega$ the positive concave function on $(0, \infty)$ of strictly critical lower type $p_\omega \in (0, 1]$, and $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$ for $t \in (0, \infty)$. In this paper, the authors study the Orlicz-Hardy space $H_{\omega,L} (\mathbb{R}^n)$ and its dual space $\text{BMO}_{\omega,L^*} (\mathbb{R}^n)$, where $L^*$ denotes the adjoint operator of $L$ in $L^2(\mathbb{R}^n)$. Several characterizations of $H_{\omega,L} (\mathbb{R}^n)$, including the molecular characterization, the Lusin-area function characterization and the maximal function characterization, are established. The $\rho$-Carleson measure characterization and the John-Nirenberg inequality for the space $\text{BMO}_{\omega,L^*} (\mathbb{R}^n)$ are also given.

As applications, the authors show that the Riesz transform $\nabla L^{-1/2}$ and the Littlewood-Paley $g$-function $g_L$ map $H_{\omega,L} (\mathbb{R}^n)$ continuously into $L(\omega)$. The authors further show that the Riesz transform $\nabla L^{-1/2}$ maps $H_{\omega,L} (\mathbb{R}^n)$ into the classical Orlicz-Hardy space $H_{\omega,\tilde{\omega}} (\mathbb{R}^n)$ for $p_\omega \in (0, 1]$ and the corresponding fractional integral $L^{-\gamma}$ for certain $\gamma > 0$ maps $H_{\omega,L} (\mathbb{R}^n)$ continuously into $H_{\omega,\tilde{\omega}} (\mathbb{R}^n)$, where $\tilde{\omega}$ is determined by $\omega$ and $\gamma$, and satisfies the same property as $\omega$. All these results are new even when $\omega(t) = t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1)$.

Keywords: divergence form elliptic operator; Gaffney estimate; Orlicz-Hardy space; Lusin-area function; maximal function; molecule; Carleson measure; John-Nirenberg inequality; dual; BMO; Riesz transform; fractional integral

1 Introduction

Ever since Lebesgue’s theory of integration has taken a center stage in concrete problems of analysis, the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families

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naturally arose. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$ is a good substitute of $L^p(\mathbb{R}^n)$ when studying the boundedness of operators, for example, the Riesz operator is bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. The theory of Hardy spaces $H^p$ on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [39]. Later, Fefferman and Stein [18] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which now plays an important role in various fields of analysis and partial differential equations; see, for example, [38, 11, 20, 32, 36]. A key feature of the classical Hardy spaces is their atomic decomposition characterizations, which were obtained by Coifman [10] when $n = 1$ and Latter [30] when $n > 1$. On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum-Orlicz in [8] and Orlicz in [33], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [34, 35, 9, 31, 1, 26]. Moreover, the Orlicz-Hardy spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis, say, the boundedness of operators. In particular, Strömberg [40] and Janson [27] introduced generalized Hardy spaces $H_\omega(\mathbb{R}^n)$, via replacing the norm $\| \cdot \|_{L^p(\mathbb{R}^n)}$ by the Orlicz-norm $\| \cdot \|_{L(\omega)}$ in the definition of $H^p(\mathbb{R}^n)$, where $\omega$ is an Orlicz function on $[0, \infty)$ satisfying some control conditions. Viviani [42] further characterized these spaces $H_\omega$ on spaces of homogeneous type via atoms. The dual spaces of these spaces were also studied in [40, 27, 42, 25]. All theories of these spaces are intimately connected with properties of harmonic analysis and of the Laplacian operator on $\mathbb{R}^n$.

In recent years, function spaces, especially Hardy spaces and BMO spaces, associated with different operators inspire great interests; see, for example, [3, 6, 7, 15, 16, 17, 22, 43, 28, 21] and their references. In particular, Auscher, Duong and McIntosh [3] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator $L$ whose heat kernel satisfies a pointwise Poisson type upper bound by means of a corresponding variant of the Lusin-area function, and established its molecular characterization. Duong and Yan [16, 17] introduced its dual space $\text{BMO}_L(\mathbb{R}^n)$ and established the dual relation between $H^1_L(\mathbb{R}^n)$ and $\text{BMO}_L(\mathbb{R}^n)$. Yan [43] further generalized these results to the Hardy spaces $H^p_L(\mathbb{R}^n)$ with certain $p \leq 1$ and their dual spaces. Also, Auscher and Russ [7] studied the Hardy space $H^1_L$ on strongly Lipschitz domains associated with a divergence form elliptic operator $L$ whose heat kernels have the Gaussian upper bounds and regularity. Very recently, Auscher, McIntosh and Russ [6] treated the Hardy space $H^p$ with $p \in [1, \infty]$ associated to Hodge Laplacian on a Riemannian manifold with doubling measure, and Hofmann and Mayboroda [22] further studied the Hardy space $H^1_L(\mathbb{R}^n)$ and its dual space adapted to a second order divergence form elliptic operator $L$ on $\mathbb{R}^n$ with bounded complex coefficients and these operators may not have the pointwise heat kernel bounds.

Motivated by [22, 27, 42], in this paper, we study Orlicz-Hardy spaces $H_{\omega, L}(\mathbb{R}^n)$ associated to the divergence form elliptic operator $L$ in [22] and their dual space $\text{BMO}_{p, L^*}(\mathbb{R}^n)$, where $L^*$ denotes the adjoint operator of $L$ in $L^2(\mathbb{R}^n)$, the positive function $\omega$ on $(0, \infty)$ is concave and of strictly critical lower type $p_\omega \in (0, 1]$ and $\rho(t) = t^{-1}/\omega(t)^{-1}$ for all $t \in (0, \infty)$. A typical example of such Orlicz functions is $\omega(t) = t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1]$. As applications, we obtain the boundedness of the Riesz transform, the
Littlewood-Paley $g$-function and the fractional integral associated with $L$ on $H_{\omega,L}(\mathbb{R}^n)$, which may not be bounded on the classical Orlicz-Hardy space $H_\omega(\mathbb{R}^n)$ or the Orlicz space $L(\omega)$. Thus, it is necessary to introduce and study the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$.

Recall that the classical BMO ($\mathbb{R}^n$) was originally introduced and studied by John and Nirenberg [29] in the context of partial differential equations, which has been identified as the dual space of $H^1(\mathbb{R}^n)$ in the work by Fefferman and Stein [18]. Also, the generalized space $\text{BMO}_p(\mathbb{R}^n)$ was introduced and studied in [40, 27, 42, 25] and it was proved therein to be the dual space of $H_\omega(\mathbb{R}^n)$.

To state the main content of this paper, we first recall some notation and known facts on second order divergence form elliptic operators on $\mathbb{R}^n$ with bounded complex coefficients from [2, 22]. Let $A$ be an $n \times n$ matrix with entries \( \{a_{j,k}\}_{j,k=1}^n \subset L^\infty(\mathbb{R}^n, \mathbb{C}) \) satisfying the ellipticity conditions, namely, there exist constants $0 < \lambda_A \leq \Lambda_A < \infty$ such that for all $\xi, \zeta \in \mathbb{C}^n$,

\[
\lambda_A |\xi|^2 \leq \text{Re} \langle A\xi, \xi \rangle \quad \text{and} \quad |\langle A\xi, \zeta \rangle| \leq \Lambda_A |\xi||\zeta|.
\]  

(1.1)

Then the second order divergence form operator is given by

\[
Lf \equiv \text{div}(A\nabla f),
\]

interpreted in the weak sense via a sesquilinear form. Following [22], set

\[
p_L \equiv \inf \left\{ p \geq 1 : \sup_{t>0} \|e^{-tL}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \right\}
\]

and

\[
\tilde{p}_L \equiv \sup \left\{ p \leq \infty : \sup_{t>0} \|e^{-tL}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \right\}.
\]

It was proved by Auscher [2] that if $n = 1, 2$, then $p_L = 1$ and $\tilde{p}_L = \infty$, and if $n \geq 3$, then $p_L < 2n/(n+2)$ and $\tilde{p}_L > 2n/(n-2)$. Moreover, thanks to a counterexample given by Frehse [19], this range is also sharp, which was pointed out to us by Professor Pascal Auscher.

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

\[
S_Lf(x) \equiv \left( \int_{\Gamma(x)} |t^2 Le^{-t^2L}f(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}.
\]

(1.3)

The space $H_{\omega,L}(\mathbb{R}^n)$ is defined to be the completion of the set \( \{ f \in L^2(\mathbb{R}^n) : S_Lf \in L(\omega) \} \) with respect to the quasi-norm

\[
\|f\|_{H_{\omega,L}(\mathbb{R}^n)} \equiv \|S_Lf\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{S_Lf(x)}{\lambda} \right) \, dx \leq 1 \right\}.
\]

If $p \leq 1$ and $\omega(t) = t^p$ for all $t \in (0, \infty)$, we then denote the Hardy space $H_{\omega,L}(\mathbb{R}^n)$ by $H^2_\omega(\mathbb{R}^n)$. The Hardy space $H^1_\omega(\mathbb{R}^n)$ was studied by Hofmann and Mayboroda in [22] (see also [23] for a corrected version).
In this paper, we first obtain the molecular decomposition of the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$. Using this molecular decomposition, we then establish the dual relation between the spaces $H_{\omega,L}(\mathbb{R}^n)$ and $\text{BMO}_{\rho,L^*}(\mathbb{R}^n)$, and the molecular characterization of $H_{\omega,L}(\mathbb{R}^n)$. Characterizations via the Lusin-area function associated to the Poisson semigroup and the maximal functions are also obtained. We also establish the $\rho$-Carleson measure characterization and the John-Nirenberg inequality for the space $\text{BMO}_{\rho,L}(\mathbb{R}^n)$. As applications, we show that the Riesz transform $\nabla L^{-1/2}$ and the Littlewood-Paley $g$-function $g_L$ map $H_{\omega,L}(\mathbb{R}^n)$ continuously into $L(\omega)$; in particular, $\nabla L^{-1/2}$ maps $H_{\omega,L}(\mathbb{R}^n)$ into the classical Orlicz-Hardy space $H_\omega(\mathbb{R}^n)$ for $p_\omega \in (\frac{n}{n+1}, 1]$. Moreover, we show that the corresponding fractional integral $L^{-\gamma}$ for all $\gamma \in (0, \frac{1}{2}(\frac{1}{p_L} - \frac{1}{p_L})$ maps $H_{\omega,L}(\mathbb{R}^n)$ continuously into $H_{\omega,L}(\mathbb{R}^n)$, where $\omega$ is determined by $\omega$ and $\gamma$, and satisfies the same property as $\omega$. All these results are new even when these results are new even when $\omega(t) = t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1)$. When $p = 1$ and $\omega(t) = t$ for all $t \in (0, \infty)$, some of results are also new.

The key step of the above approach is to establish a molecular characterization of the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$. To this end, a main difficulty encountered is the convergence problem of the summation of molecules, i.e., in what sense does the molecular characterization hold? In Theorem 5.1 below, we prove that our molecular characterization holds in the dual of $\text{BMO}_{\rho,L^*}(\mathbb{R}^n)$. This is quite different from the cases for the Hardy space $H_L^1(\mathbb{R}^n)$ in [22] and the Hardy space $H^1(\Lambda T^*M)$ in [6], which only need that the molecular characterizations hold pointwise; see [22, (1.11)] (or its corrected version in [23]) and [6, Definition 6.1]. Recall that $M$ denotes a complete Riemannian manifold and

$$\Lambda T^*M \equiv \bigoplus_{0 \leq k \leq \dim M} \Lambda^k T^*M$$

the bundle over $M$ whose fibre at each $x \in M$ is given by $\Lambda^k T_x^*M$, the complex exterior algebra over the cotangent space $T_x^*M$; see [6, p. 194]. In this paper, to obtain the molecular characterization of $H_{\omega,L}(\mathbb{R}^n)$, we first need to show that the dual space of $H_{\omega,L}(\mathbb{R}^n)$ is $\text{BMO}_{\rho,L^*}(\mathbb{R}^n)$ in Theorem 4.1 below. The key ingredients used in the proof of Theorem 4.1 is the Calderón reproducing formula (Lemma 4.3 below) and the atomic decomposition of the tent space $T_\omega(\mathbb{R}^{n+1}_+)$ (Theorem 3.1 below). We point out that the dual space of $H_L^1(\mathbb{R}^n)$ was already obtained in [22, Theorems 8.2 and 8.6] by a different, but more complicated, approach, without invoking the atomic decomposition of the tent space. Also, the dual space of $H^1(\Lambda T^*M)$ was obtained in [6] as a direct corollary of the dual theorem on the corresponding tent space; see [6, Theorem 5.8].

Another key tool used in this paper to obtain the maximal function characterizations of $H_{\omega,L}(\mathbb{R}^n)$ and their applications in boundedness of operators is Lemma 5.1 below, which gives a sufficient condition for the boundedness of linear or non-negative sublinear operators from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$. Such a condition for the molecular Hardy space in $H_L^1(\mathbb{R}^n)$ case was also given in [22, Lemma 3.3], which is a direct corollary of the definition of the molecular Hardy space; see its corrected version in [23]. To obtain Lemma 5.1, we need the following important observation that for all $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, since $t^2 L e^{-t^2 L} f \in T_2^2(\mathbb{R}^{n+1}_+) \cap T_\omega(\mathbb{R}^{n+1}_+)$, by Proposition 3.1 below, the atomic decomposition of $t^2 L e^{-t^2 L} f$ holds in both $T_\omega(\mathbb{R}^{n+1}_+)$ and $T_2^2(\mathbb{R}^{n+1}_+)$ for all $p \in [1, 2]$. Then by the fact that
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the operator $\pi_{L,M}$, which is introduced in [17] and initially defined on $F \in L^2(\mathbb{R}_+^{n+1})$ with compact support by

$$\pi_{L,M}F \equiv C_M \int_0^{\infty} (t^2 L)^{M+1} e^{-t^2 L} F(\cdot, t) \frac{dt}{t}, \quad (1.4)$$

is bounded from $T^p_2(\mathbb{R}_+^{n+1})$ to $L^p(\mathbb{R}^n)$ for $p \in (p_L, \bar{p}_L)$ (see Proposition 4.1 below), we further obtain the $L^p(\mathbb{R}^n)$-convergence with $p \in (p_L, 2]$ and the $H_{\omega,L}(\mathbb{R}^n)$-convergence of the corresponding molecular decomposition for functions in $H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in Proposition 4.2 below. These convergences are necessary and play a fundamental role in the whole paper, which is totally different from the $\delta$-representation used in [22, 23, 21]. Here and in what follows, $M \in \mathbb{N}$ and

$$C_M \int_0^{\infty} t^{2(M+2)} e^{-2t^2} \frac{dt}{t} = 1.$$

We remark that the convergence of the atomic decomposition of the tent spaces was also already carefully dealt with in [6] (We thank Professor Pascal Auscher to point out this to us). To be precise, in [6, pp. 209-210], Auscher, McIntosh and Russ proved that for any functions $F$ in the intersection of the tent spaces $T^{1,2}(\Lambda T^* M)$ and $T^{2,2}(\Lambda T^* M)$ with the support $M \times [\epsilon, \infty)$ for some $\epsilon > 0$, $F_n \equiv F_{\chi_B(x_0, n) \times (1/n, n)}$ for any $x_0 \in M$ has an atomic decomposition which converges in both $T^{1,2}(\Lambda T^* M)$ and $T^{2,2}(\Lambda T^* M)$; see [6, (4.5)]. Observe that the compact support of $F_n$ plays an important role in establishing the convergence of its atomic decomposition in [6]. However, Proposition 3.1 below are true for all functions in $T_\omega(\mathbb{R}_+^{n+1}) \cap T^p_2(\mathbb{R}_+^{n+1})$ without assuming the compact supports. To obtain this proposition, we need to subtly use the construction of the supports of atoms in the atomic decomposition of tent spaces $T_\omega(\mathbb{R}_+^{n+1})$ in Theorem 3.1 below and the Lebesgue dominated convergence theorem.

This paper is organized as follows.

In Section 2, we recall some notions and known results concerning operators associated with $L$ and describe some basic assumptions on the Orlicz function $\omega$ considered in this paper. We point out that throughout the whole paper, we always assume that $\omega$ on $(0, \infty)$ is concave and of strictly critical lower type $p_\omega \in (0, 1]$. These restrictions are necessary for the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$ to have the molecular characterization; see Theorem 5.1 below. Thus, under these restrictions, the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$ behaves more closely like the classical Hardy space. We leave the study on the Orlicz-Hardy space with a Young function in a forthcoming paper, which may have some properties similar to those of the spaces $H^p(\Lambda T^* M)$ with $p \in (1, \infty]$ as in [6].

In Section 3, we introduce the tent spaces $T_\omega(\mathbb{R}_+^{n+1})$ associated to $\omega$ and establish its atomic characterization; see Theorem 3.1 below. By the proof of Theorem 3.1, we observe that if a function $F \in T_\omega(\mathbb{R}_+^{n+1}) \cap T^p_2(\mathbb{R}_+^{n+1})$, $p \in (0, \infty)$, then there exists an atomic decomposition of $F$ which converges in both $T_\omega(\mathbb{R}_+^{n+1})$ and $T^p_2(\mathbb{R}_+^{n+1})$; see Proposition 3.1 below. As a consequence, we prove that if $F \in T_\omega(\mathbb{R}_+^{n+1}) \cap T^2_2(\mathbb{R}_+^{n+1})$, then there exists an atomic decomposition of $F$ which converges in both $T_\omega(\mathbb{R}_+^{n+1})$ and $T^p_2(\mathbb{R}_+^{n+1})$ for all $p \in [1, 2]$; see Corollary 3.1 below.
In Section 4, we first introduce the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$, and then prove that the operator $\pi_{L,M}$ in (1.4) maps the tent space $T_{2}^{p}(\mathbb{R}^{n+1}_{\pm})$ continuously into $L^{p}(\mathbb{R}^n)$ for $p \in (p_{L}, \tilde{p}_{L})$ and $T_{\omega}^{p}(\mathbb{R}^{n+1}_{\pm})$ continuously into $H_{\omega,L}(\mathbb{R}^n)$ (see Proposition 4.1 below). Combined this with Corollary 3.1, we obtain a molecular decomposition for elements in $H_{\omega,L}(\mathbb{R}^n) \cap L^{2}(\mathbb{R}^n)$ which converges in $L^{p}(\mathbb{R}^n)$ for $p \in (p_{L}, 2]$; see Proposition 4.2 below. Via this molecular decomposition of $H_{\omega,L}(\mathbb{R}^n)$, we further obtain the duality between $H_{\omega,L}(\mathbb{R}^n)$ and $\text{BMO}_{\rho,L}^{\ast}(\mathbb{R}^n)$ (see Theorem 4.1 below). We also remark that the proof of Theorem 4.1 is much simpler than the proof of [22, Theorem 8.2].

In Section 5, we introduce the molecular Hardy space, where the summation of molecules converges in the space $(\text{BMO}_{\rho,L}^{\ast}(\mathbb{R}^n))^{\ast}$, the dual space of $\text{BMO}_{\rho,L}^{\ast}(\mathbb{R}^n)$. Then we show that the molecular Hardy space is equivalent to the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$ with equivalent norms; see Theorem 5.1 below. Furthermore, we characterize $H_{\omega,L}(\mathbb{R}^n)$ via the Lusin-area function associated to the Poisson semigroup, and the maximal functions; see Theorem 5.2 below. We also point out that a sufficient condition for the boundedness of linear or non-negative sublinear operators from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$ is also given in Lemma 5.1 below, which plays a key role in the proof of Theorem 5.2 and is very useful in applications (see Section 7 of this paper). This condition is also necessary if $\omega(t) = t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1]$.

Section 6 is devoted to establish the $\rho$-Carleson measure characterization (see Theorem 6.1 below) and the John-Nirenberg inequality (see Theorem 6.2 below) for the space $\text{BMO}_{\rho,L}(\mathbb{R}^n)$.

In Section 7, as an application, we give some sufficient conditions which guarantee the boundedness of linear or non-negative sublinear operators from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$; in particular, we show that the Riesz transform $\nabla L^{-1/2}$ and the Littlewood-Paley $g$-function $g_{L}$ map $H_{\omega,L}(\mathbb{R}^n)$ continuously into $L(\omega)$; see Theorem 7.1 below. A fractional variant of Theorem 7.1 is also given in this section; see Theorem 7.2 below. Using Theorem 7.2, we prove that the fractional integral $L^{-\gamma}$ for all $\gamma \in (0, \frac{n}{2}(\frac{1}{p_{L}} - \frac{1}{\tilde{p}_{L}}))$ maps $H_{\omega,L}(\mathbb{R}^n)$ continuously into $H_{\tilde{\omega},L}(\mathbb{R}^n)$, where $\tilde{\omega}$ is determined by $\omega$ and $\gamma$ and satisfies the same property as $\omega$; see Theorem 7.3 below. In particular, $L^{-\gamma}$ maps $H_{\omega,L}^{p}(\mathbb{R}^n)$ continuously into $H_{\tilde{\omega},L}^{q}(\mathbb{R}^n)$ for $0 < p \leq q \leq 1$ and $n/p - n/q = 2\gamma$; see Remark 7.3 below. Applying Theorems 7.1 and 7.3, we further show that $\nabla L^{-1/2}$ maps $H_{\omega,L}(\mathbb{R}^n)$ continuously into $H_{\omega}(\mathbb{R}^n)$ for $p_{\omega} \in (\frac{n}{n+1}, 1]$, and in particular, $H_{L}^{p}(\mathbb{R}^n)$ into the classical Hardy space $H^{p}(\mathbb{R}^n)$ for $p \in (\frac{2}{n+1}, 1]$; see Theorem 7.4 below. Moreover, we show that $H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega}(\mathbb{R}^n)$ for all $p_{\omega} \in (\frac{n}{n+1}, 1]$ in Remark 7.4 below. It was also pointed out by Hofmann and Mayboroda in [22] that $H_{L}^{1}(\mathbb{R}^n)$ is a proper subspace of $H^{1}(\mathbb{R}^n)$ for certain $L$ as in (1.2). We remark that if $L = -\Delta + V$ with $V \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ is the Schrödinger operator on $\mathbb{R}^n$, then it was proved in [21] that $\nabla L^{-1/2}$ maps $H_{L}^{1}(\mathbb{R}^n)$ into the classical Hardy space $H^{1}(\mathbb{R}^n)$.

We point out that the proof of this paper is strongly motivated by Hofmann and Mayboroda [22], and we also directly use some estimates from [22] which simplify the proofs of some theorems of this paper.

Finally, we make some conventions. Throughout the whole paper, $L$ always denotes the second order divergence form operator as in (1.2). We denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $X \lesssim Y$ means that there exists a positive constant $C$ such that $X \leq CY$; the
symbol $\lfloor \alpha \rfloor$ for $\alpha \in \mathbb{R}$ denotes the maximal integer no more than $\alpha$; $B(z_B, r_B)$ denotes an open ball with center $z_B$ and radius $r_B$ and $CB(z_B, r_B) \equiv B(z_B, C r_B)$. Set $\mathbb{N} \equiv \{1,2,\cdots\}$ and $\mathbb{Z}^+ \equiv \mathbb{N} \cup \{0\}$. For any subset $E$ of $\mathbb{R}^n$, we denote by $E^c$ the set $\mathbb{R}^n \setminus E$.

2 Preliminaries

In this section, we recall some notions and notation on the divergence form elliptic operator, and present some basic properties on Orlicz functions and also describe some basic assumptions on them.

2.1 Some notions on the divergence form elliptic operator $L$

In this subsection, we present some known facts about the operator $L$ considered in this paper.

A family $\{S_t\}_{t>0}$ of operators is said to satisfy the $L^2$ off-diagonal estimates, which is also called the Gaffney estimates (see [22]), if there exist positive constants $c, C$ and $\beta$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$,

$$\|S_t f\|_{L^2(F)} \leq C e^{-\left(\frac{\text{dist}(E,F)^2}{ct}\right)^\beta} \|f\|_{L^2(E)}$$

for every $t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in $E$. Here and in what follows, for any $p \in (0, \infty]$ and $E \subset \mathbb{R}^n$, $\|f\|_{L^p(E)} \equiv \|f \chi_E\|_{L^p(\mathbb{R}^n)}$; for any sets $E, F \subset \mathbb{R}^n$, $\text{dist}(E,F) \equiv \inf\{|x-y| : x \in E, y \in F\}$.

The following results were obtained in [2, 4, 22, 24].

Lemma 2.1 ([24]). If two families of operators, $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$, satisfy Gaffney estimates, then so does $\{S_t T_t\}_{t>0}$. Moreover, there exist positive constants $c, C$ and $\beta$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$,

$$\|S_t T_t f\|_{L^2(F)} \leq C e^{-\left(\frac{\text{dist}(E,F)^2}{ct}\right)^\beta} \|f\|_{L^2(E)}$$

for every $s, t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in $E$.

Lemma 2.2 ([4, 24]). The families,

$$\{e^{-tL}\}_{t>0}, \quad \{tLe^{-tL}\}_{t>0}, \quad \{t^{1/2}\nabla e^{-tL}\}_{t>0}, \quad (2.1)$$

as well as

$$\{(I + tL)^{-1}\}_{t>0}, \quad \{t^{1/2}\nabla (I + tL)^{-1}\}_{t>0}, \quad (2.2)$$

are bounded on $L^2(\mathbb{R}^n)$ uniformly in $t$ and satisfy the Gaffney estimates with positive constants $c, C$ depending on $n, \lambda_A, \Lambda_A$ as in (1.1) only. For the operators in (2.1), $\beta = 1$, while in (2.2), $\beta = 1/2$. 

Lemma 2.3 ([2, 22]). There exist \( p_L \in [1, \frac{2n}{n+2}) \), \( \tilde{p}_L \in (\frac{2n}{n-2}, \infty) \) and \( c, C \in (0, \infty) \) such that

(i) for every \( p \) and \( q \) with \( p_L < p \leq q < \tilde{p}_L \), the families \( \{e^{-tL}\}_{t>0} \) and \( \{tLe^{-tL}\}_{t>0} \) satisfy \( L^p - L^q \) off-diagonal estimates, i.e., for arbitrary closed sets \( E, F \subset \mathbb{R}^n \),

\[
\|e^{-tL}f\|_{L^q(F)} + \|tLe^{-tL}f\|_{L^q(F)} \leq C t^{\frac{q}{2} - \frac{1}{p}} e^{\frac{\text{dist}(E,F)}{ct}} \|f\|_{L^p(E)}
\]

for every \( t > 0 \) and every \( f \in L^p(\mathbb{R}^n) \) supported in \( E \). The operators \( \{e^{-tL}\}_{t>0} \) and \( \{tLe^{-tL}\}_{t>0} \) are bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) with the norm \( Ct^{\frac{q}{2} - \frac{1}{p}} \);

(ii) for every \( p \in (p_L, \tilde{p}_L) \), the family \( \{(I + tL)^{-1}\}_{t>0} \) satisfies \( L^p - L^p \) off-diagonal estimates, i.e., for arbitrary closed sets \( E, F \subset \mathbb{R}^n \),

\[
\|(I + tL)^{-1}f\|_{L^q(F)} \leq C t^{\frac{q}{2} - \frac{1}{p}} e^{\frac{\text{dist}(E,F)}{ct^{1/2}}} \|f\|_{L^p(E)}
\]

for every \( t > 0 \) and every \( f \in L^p(\mathbb{R}^n) \) supported in \( E \).

Lemma 2.4 ([22]). Let \( k \in \mathbb{N} \) and \( p \in (p_L, \tilde{p}_L) \). Then the operator given by for any \( f \in L^p(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
S^k_L f(x) = \left( \int_0^\infty \left( \int_{\Gamma(x)} |(t^2L)^k e^{-t^2L} f(y)|^2 \frac{dy}{\rho^{n+1}} \right)^{1/2} \right) ,
\]

is bounded on \( L^p(\mathbb{R}^n) \).

2.2 Orlicz functions

Let \( \omega \) be a positive function defined on \( \mathbb{R}_+ = (0, \infty) \). The function \( \omega \) is said to be of upper type \( p \) (resp. lower type \( p \)) for certain \( p \in [0, \infty) \), if there exists a positive constant \( C \) such that for all \( t \geq 1 \) (resp. \( t \in (0,1] \)) and \( s \in (0, \infty) \),

\[
\omega(st) \leq Ct^p \omega(s). \tag{2.3}
\]

Obviously, if \( \omega \) is of lower type \( p \) for certain \( p > 0 \), then \( \lim_{t \to 0^+} \omega(t) = 0 \). So for the sake of convenience, if it is necessary, we may assume that \( \omega(0) = 0 \). If \( \omega \) is of both upper type \( p_1 \) and lower type \( p_0 \), then \( \omega \) is said to be of type \( (p_0, p_1) \). Let

\[
p^+_\omega = \inf\{p > 0 : \text{there exists } C > 0 \text{ such that (2.3) holds for all } t \in [1, \infty), \ s \in (0, \infty)\},
\]

and

\[
p^-_\omega = \sup\{p > 0 : \text{there exists } C > 0 \text{ such that (2.3) holds for all } t \in (0,1], \ s \in (0, \infty)\}.
\]

The function \( \omega \) is said to be of strictly lower type \( p \) if for all \( t \in (0,1) \) and \( s \in (0, \infty) \), \( \omega(st) \leq t^p \omega(s) \), and define

\[
p_\omega = \sup\{p > 0 : \omega(st) \leq t^p \omega(s) \text{ holds for all } s \in (0, \infty) \text{ and } t \in (0,1)\}.
\]

It is easy to see that \( p_\omega \leq p^-_\omega \leq p^+_\omega \) for all \( \omega \). In what follows, \( p_\omega, p^-_\omega \) and \( p^+_\omega \) are called to be the strictly critical lower type index, the critical lower type index and the critical upper type index of \( \omega \), respectively.
Remark 2.1. We claim that if \( p_\omega \) is defined as above, then \( \omega \) is also of strictly lower type \( p_\omega \). In other words, \( p_\omega \) is attainable. In fact, if this is not the case, then there exist certain \( s \in (0, \infty) \) and \( t \in (0, 1) \) such that \( \omega(st) > t^{p_\omega}s \omega(s) \). Hence there exists \( \epsilon \in (0, p_\omega) \) small enough such that \( \omega(st) > t^{p_\omega-\epsilon}s \omega(s) \), which is contrary to the definition of \( p_\omega \). Thus, \( \omega \) is of strictly lower type \( p_\omega \).

Throughout the whole paper, we always assume that \( \omega \) satisfies the following assumption.

**Assumption (A).** Let \( p_\omega \) be defined as above. Suppose that \( \omega \) is a positive Orlicz function on \( \mathbb{R}_+ \) with \( p_\omega \in [0, 1] \), which is continuous, strictly increasing and concave.

Notice that if \( \omega \) satisfies Assumption (A), then \( \omega(0) = 0 \) and \( \omega \) is obviously of upper type 1. Since \( \omega \) is concave, it is subadditive. In fact, let \( 0 < s < t \), then

\[
\omega(s + t) \leq \frac{s + t}{t} \omega(t) \leq \omega(t) + \frac{s}{t} \omega(s) = \omega(s) + \omega(t).
\]

For any concave function \( \omega \) of strictly lower type \( p \), if we set \( \tilde{\omega}(t) \equiv \int_0^t \omega(s)/s \, ds \) for \( t \in [0, \infty) \), then by [42, Proposition 3.1], \( \tilde{\omega} \) is equivalent to \( \omega \), namely, there exists a positive constant \( C \) such that \( C^{-1} \omega(t) \leq \tilde{\omega}(t) \leq C \omega(t) \) for all \( t \in [0, \infty) \); moreover, \( \tilde{\omega} \) is strictly increasing, concave and continuous function of strictly lower type \( p \). Since all our results are invariant on equivalent functions, we always assume that \( \omega \) satisfies Assumption (A); otherwise, we may replace \( \omega \) by \( \tilde{\omega} \).

**Convention (C).** From Assumption (A), it follows that \( 0 < p_\omega \leq p_\omega^- \leq p_\omega^+ \leq 1 \). In what follows, if (2.3) holds for \( p_\omega^+ \) with \( t \in [1, \infty) \), then we choose \( \tilde{p}_\omega \equiv p_\omega^- \); otherwise \( p_\omega^- < 1 \) and we choose \( \tilde{p}_\omega \in (p_\omega^-, 1) \) to be close enough to \( p_\omega^- \).

For example, if \( \omega(t) = t^p \) with \( p \in (0, 1] \), then \( p_\omega = p_\omega^+ = \tilde{p}_\omega = p \); if \( \omega(t) = t^{1/2} \ln(e^4 + t) \), then \( p_\omega = p_\omega^+ = 1/2 \), but \( 1/2 < \tilde{p}_\omega < 1 \).

Let \( \omega \) satisfy Assumption (A). A measurable function \( f \) on \( \mathbb{R}^n \) is said to be in the Lebesgue type space \( L(\omega) \) if

\[
\int_{\mathbb{R}^n} \omega(|f(x)|) \, dx < \infty.
\]

Moreover, for any \( f \in L(\omega) \), define

\[
\|f\|_{L(\omega)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

Since \( \omega \) is strictly increasing, we define the function \( \rho(t) \) on \( \mathbb{R}_+ \) by setting, for all \( t \in (0, \infty) \),

\[
\rho(t) \equiv \frac{t^{-1}}{\omega^{-1}(t^{-1})}, \tag{2.4}
\]

where and in what follows, \( \omega^{-1} \) denotes the inverse function of \( \omega \). Then the types of \( \omega \) and \( \rho \) have the following relation; see [42] for its proof.

**Proposition 2.1.** Let \( 0 < p_0 \leq p_1 \leq 1 \) and \( \omega \) be an increasing function. Then \( \omega \) is of type \( (p_0, p_1) \) if and only if \( \rho \) is of type \( (p_1^{-1} - 1, p_0^{-1} - 1) \).
3 Tent spaces associated to Orlicz functions

In this section, we study the tent spaces associated to Orlicz functions. We first recall some notions.

For any $\nu > 0$ and $x \in \mathbb{R}^n$, let $\mathbb{R}^n_+ = \mathbb{R}^n \times (0, \infty)$ and
\[
\Gamma_\nu(x) \equiv \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < \nu t\}
\]
denoting the cone of aperture $\nu$ with vertex $x \in \mathbb{R}^n$. For any closed set $F$ of $\mathbb{R}^n$, denote by $R_\nu F$ the union of all cones with vertices in $F$, i.e., $R_\nu F \equiv \bigcup_{x \in F} \Gamma_\nu(x)$; and for any open set $O$ in $\mathbb{R}^n$, denote the tent over $O$ by $T_\nu(O)$, which is defined as $T_\nu(O) \equiv |R_\nu(O^c)|^\nu$. Notice that $T_\nu(O) = \{(x, t) \in \mathbb{R}^n \times (0, \infty) : \text{dist}(x, O^c) \geq \nu t\}$.

In what follows, we denote $\Gamma_1(x)$, $R_1(F)$ and $T_1(O)$ simply by $\Gamma(x)$, $R(F)$ and $\hat{O}$, respectively.

Let $F$ be a closed subset of $\mathbb{R}^n$ and $O \equiv F^c$. Assume that $|O| < \infty$. For any fixed $\gamma \in (0, 1)$, we say that $x \in \mathbb{R}^n$ has the global $\gamma$-density with respect to $F$ if
\[
\frac{|B(x, r) \cap F|}{|B(x, r)|} \geq \gamma
\]
for all $r > 0$. Denote by $F^*$ the set of all such $x$. Obviously, $F^*$ is a closed subset of $F$. Let $O^* \equiv (F^*)^c$. Then it is easy to see that $O \subset O^*$. In fact, we have
\[
O^* = \{x \in \mathbb{R}^n : \mathcal{M}(\chi_O)(x) > 1 - \gamma\},
\]
where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$. As a consequence, by the weak type $(1, 1)$ of $\mathcal{M}$, we have $|O^*| \leq C(\gamma)|O|$, where and in what follows, $C(\gamma)$ denotes a positive constant depending on $\gamma$.

The proof of the following lemma is similar to that of [12, Lemma 2]; we omit the details.

**Lemma 3.1.** Let $\nu$, $\eta \in (0, \infty)$. Then there exist positive constants $\gamma \in (0, 1)$ and $C(\gamma, \nu, \eta)$ such that for any closed subset $F$ of $\mathbb{R}^n$ whose complement has finite measure and any non-negative measurable function $H$ on $\mathbb{R}^{n+1}_+$, $\int \int_{R_\nu(F^*)} H(y, t)t^n \, dy \, dt \leq C(\gamma, \nu, \eta) \int_F \left\{ \int_{\Gamma_\nu(x)} H(y, t) \, dy \, dt \right\} \, dx,$
where $F^*$ denotes the set of points in $\mathbb{R}^n$ with global $\gamma$-density with respect to $F$.

Let $\nu \in (0, \infty)$. For all measurable functions $g$ on $\mathbb{R}^{n+1}_+$ and all $x \in \mathbb{R}^n$, let
\[
\mathcal{A}_\nu(g)(x) \equiv \left( \int_{\Gamma_\nu(x)} |g(y, t)|^2 \, \frac{dy \, dt}{t^{n+1}} \right)^{1/2},
\]
and denote $\mathcal{A}_1(g)$ simply by $\mathcal{A}(g)$.
Coifman, Meyer and Stein [12] introduced the tent space $T^p_2(\mathbb{R}^{n+1})$ for $p \in (0, \infty)$, which is defined as the space of all measurable functions $g$ such that $\|g\|_{T^p_2(\mathbb{R}^{n+1})} \equiv \|A(g)\|_{L^p(\mathbb{R}^n)} < \infty$.

On the other hand, let $\omega$ satisfy Assumption (A). Harboure, Salinas and Viviani [25] defined the tent space $T_\omega(\mathbb{R}^{n+1})$ associated to the function $\omega$ as the space of measurable functions $g$ on $\mathbb{R}^{n+1}$ such that $A(g) \in L(\omega)$ with the norm defined by

$$\|g\|_{T_\omega(\mathbb{R}^{n+1})} \equiv \|A(g)\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{A(g)(x)}{\lambda} \right) \, dx \leq 1 \right\}.$$

**Lemma 3.2.** Let $\eta, \nu \in (0, \infty)$. Then there exists a positive constant $C$, depending on $\eta$ and $\nu$, such that for all measurable functions $H$ on $\mathbb{R}^{n+1}$,

$$C^{-1} \int_{\mathbb{R}^n} \omega(A_\eta(H)(x)) \, dx \leq \int_{\mathbb{R}^n} \omega(A_\nu(H)(x)) \, dx \leq C \int_{\mathbb{R}^n} \omega(A_\eta(H)(x)) \, dx.$$  \hspace{1cm} (3.1)

**Proof.** By the symmetry, we only need to establish the first inequality in (3.1). To this end, let $\lambda \in (0, \infty)$ and $O_\lambda \equiv \{x \in \mathbb{R}^n : A_\nu(H)(x) > \lambda\}$. If $|O_\lambda| = \infty$, then $\int_{\mathbb{R}^n} \omega(A_\nu(H)(x)) \, dx = \infty$ and the inequality automatically holds. Now, assume that $|O_\lambda| < \infty$. Applying Lemma 3.1 with $F_\lambda \equiv (O_\lambda)^C$, we have

$$\int_{\mathbb{R}^n} \omega(A_\eta(H)(x)) \, dx \leq \int_{\mathbb{R}^n} \omega(A_\nu(H)(x)) \, dx \leq C \int_{\mathbb{R}^n} \omega(A_\eta(H)(x)) \, dx.$$  \hspace{1cm} (3.1)

Here and in what follows, we denote $(F_\lambda)^*$ and $(O_\lambda)^* = ((F_\lambda)^C)^C$ simply by $F_\lambda^*$ and $O_\lambda^*$, respectively. Observe that

$$\int_{F_\lambda^*} \int_{\Gamma_\eta(x)} |H(y,t)|^2 \frac{dy \, dt}{t} \leq \int_{F_\lambda} \int_{\Gamma_\nu(x)} |H(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \, dx \leq \int_{F_\lambda} |A_\nu(H)(x)|^2 \, dx,$$

which implies that

$$\int_{F_\lambda^*} |A_\eta(H)(x)|^2 \, dx \leq \int_{F_\lambda} |A_\nu(H)(x)|^2 \, dx.$$

Here and in what follows, for a measurable function $g$ on $\mathbb{R}^n$ and $\lambda > 0$, let $\sigma_g(\lambda)$ denote the distribution of $g$, namely, $\sigma_g(\lambda) = |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|$. Hence, we have

$$\sigma_{A_\eta(H)}(\lambda) \leq |O_\lambda^*| + \frac{1}{\lambda^2} \int_{(O_\lambda)^C} |A_\nu(H)(x)|^2 \, dx \leq |O_\lambda| + \frac{1}{\lambda^2} \int_t^\lambda t \sigma_{A_\nu(H)}(t) \, dt.$$

Since $\omega$ is of upper type $1$ and lower type $p_\omega \in (0, 1]$, we have

$$\omega(t) \sim \int_0^t \frac{\omega(u)}{u} \, du \quad \text{for each } t \in (0, \infty),$$  \hspace{1cm} (3.2)

which further implies that

$$\int_{\mathbb{R}^n} \omega(A_\eta(H)(x)) \, dx \sim \int_{\mathbb{R}^n} \int_0^{A_\eta(H)(x)} \frac{\omega(t)}{t} \, dt \, dx \sim \int_0^\infty \sigma_{A_\eta(H)}(t) \frac{\omega(t)}{t} \, dt.$$
A function $a$ on $\mathbb{R}_+^{n+1}$ is called an $(\omega, p)$-atom if

(i) there exists a ball $B \subset \mathbb{R}^n$ such that $\text{supp } a \subset \tilde{B}$;

(ii) $\|a\|_{T^p_{\omega}(\mathbb{R}_+^{n+1})} \leq |B|^{1/p-1} |p(|B|)|^{-1}$.

Since $\omega$ is concave, by the Jensen inequality, it is easy to see that for all $(\omega, p)$-atoms $a$, we have $\|a\|_{T^p_{\omega}(\mathbb{R}_+^{n+1})} \leq 1$.

Furthermore, if $a$ is an $(\omega, p)$-atom for all $p \in (1, \infty)$, we then call $a$ an $(\omega, \infty)$-atom.

**Theorem 3.1.** Let $\omega$ satisfy Assumption (A). Then for any $f \in T_\omega(\mathbb{R}_+^{n+1})$, there exist $(\omega, \infty)$-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that for almost every $(x, t) \in \mathbb{R}_+^{n+1}$,

$$f(x, t) = \sum_{j=1}^\infty \lambda_j a_j(x, t).$$

Moreover, there exists a positive constant $C$ such that for all $f \in T_\omega(\mathbb{R}_+^{n+1})$,

$$\Lambda(\{\lambda_j a_j\}_{j=1}^\infty) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty |B_j| |\frac{\lambda_j}{\lambda B_j |\rho(B_j)}| \leq 1 \right\} \leq C \|f\|_{T_\omega(\mathbb{R}_+^{n+1})},$$

where $\tilde{B}_j$ appears as the support of $a_j$.

**Proof.** We prove this theorem by borrowing some ideas from the proof of Theorem 1 in Coifman, Meyer and Stein [12]. Let $f \in T_\omega(\mathbb{R}_+^{n+1})$. For any $k \in \mathbb{Z}$, let $O_k \equiv \{ x \in \mathbb{R}^n : A(f)(x) > 2^k \}$ and $F_k \equiv (O_k)^c$. Since $f \in T_\omega(\mathbb{R}_+^{n+1})$, for each $k$, $O_k$ is an open set and $|O_k| < \infty$.

Since $\omega$ is of upper type 1, by Lemma 3.1, for $k \in \mathbb{Z}$ and $k \leq 0$, we have

$$\int \int_{\mathcal{R}(F_k)} |f(y, t)|^2 \frac{dy\, dt}{t} \leq \int_{F_k} \int_{\Gamma(x)} |f(y, t)|^2 \frac{dy\, dt}{t^{n+1}} \, dx$$

$$\leq \int_{F_k} A(f)(x)^2 \, dx \leq \int_{F_k} \omega(A(f)(x)) \, dx \to 0,$$

as $k \to -\infty$, which implies that $f = 0$ almost everywhere in $\cap_{k \in \mathbb{Z}} \mathcal{R}(F_k^*)$, and hence, supp$f \subset \{ \cup_{k \in \mathbb{Z}} \hat{O}_k^* \cup E \}$, where $E \subset \mathbb{R}^{n+1}$ and $\int_E \frac{dx\, dt}{t} = 0$. 
Thus, for each \( k \), by applying the Whitney decomposition to the set \( O_k^* \), we obtain a set \( I_k \) of indices and a family \( \{Q_{k,j}\}_{j \in I_k} \) of disjoint cubes such that

(i) \( \bigcup_{j \in I_k} Q_{k,j} = O_k^* \), and if \( i \neq j \), then \( Q_{k,i} \cap Q_{k,j} = \emptyset \),

(ii) \( \sqrt{n} \ell(Q_{k,j}) \leq \text{dist} (Q_{k,j}, (O_k^*)^c) \leq 4 \sqrt{n} \ell(Q_{k,j}) \), where \( \ell(Q_{k,j}) \) denotes the side-length of \( Q_{k,j} \).

Next, for each \( j \in I_k \), we choose a ball \( B_{k,j} \) with the same center as \( Q_{k,j} \) and with radius \( \frac{1}{2} \sqrt{n} \)-times \( \ell(Q_{k,j}) \). Let \( A_{k,j} \equiv \widetilde{B}_{k,j} \cap (Q_{k,j} \times (0, \infty)) \cap (\widehat{O}_k^* \setminus \widehat{O}_{k+1}^*) \),

\[
a_{k,j} = 2^{-k} |B_{k,j}|^{-1} [\rho(|B_{k,j}|)]^{-1} f \chi_{A_{k,j}}
\]

and \( \lambda_{k,j} = 2^k |B_{k,j}| \rho(|B_{k,j}|) \). Notice that \( \{(Q_{k,j} \times (0, \infty)) \cap (\widehat{O}_k^* \setminus \widehat{O}_{k+1}^*)\} \subset \widetilde{B}_{k,j} \). From this, we conclude that \( f = \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \lambda_{k,j} a_{k,j} \) almost everywhere.

Let us show that for each \( k \in \mathbb{Z} \) and \( j \in I_k \), \( a_{k,j} \) is an \((\omega, \infty)\)-atom supported in \( \widetilde{B}_{k,j} \). Let \( p \in (1, \infty) \), \( q \equiv p' \) be the conjugate index of \( p \), i.e., \( 1/p + 1/p = 1 \), and \( h \in T^q_2(\mathbb{R}^{n+1}_+) \) with \( \|h\|_{T^q_2(\mathbb{R}^{n+1}_+)} \leq 1 \). Since \( A_{k,j} \subset (\widehat{O}_{k+1}^*)^c = \mathcal{R}(F_{k+1}^*) \), by Lemma 3.1 and the Hölder inequality, we have

\[
|\langle a_{k,j}, h \rangle| \leq \int_{\mathbb{R}^{n+1}_+} |(a_{k,j} \chi_{A_{k,j}})(y,t)h(y,t)| \frac{dy \, dt}{t} \leq \int_{F_{k+1}} \int_{T^c_{\gamma(x)}} |a_{k,j}(y,t)h(y,t)| \frac{dy \, dt}{t^{n+1}} \, dx \leq \int_{(O_{k+1})^c} A(a_{k,j})(x) A(h)(x) \, dx \leq 2^{-k} |B_{k,j}|^{-1} [\rho(|B_{k,j}|)]^{-1} \left( \int_{B_{k,j} \cap (O_{k+1})^c} [A(f)(x)]^p \, dx \right)^{1/p} \|h\|_{T^q_2(\mathbb{R}^{n+1}_+)} \leq |B_{k,j}|^{1/p-1} [\rho(|B_{k,j}|)]^{-1},
\]

which implies that \( a_{k,j} \) is an \((\omega, p)\)-atom supported in \( \widetilde{B}_{k,j} \) for all \( p \in (1, \infty) \), hence, an \((\omega, \infty)\)-atom.

By (3.2), for any \( \lambda > 0 \), we further obtain

\[
\sum_{k \in \mathbb{Z}} \sum_{j \in I_k} |B_{k,j}| \omega \left( \frac{|\lambda a_{k,j}|}{|\lambda| B_{k,j} |\rho(|B_{k,j}|)|} \right) \leq \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} |Q_{k,j}| \omega \left( \frac{2^k}{\lambda} \right) \leq \sum_{k \in \mathbb{Z}} |O_k^*| \omega \left( \frac{2^k}{\lambda} \right) \leq \sum_{k \in \mathbb{Z}} |O_k| \omega \left( \frac{2^k}{\lambda} \right) \leq \int_{\mathbb{R}^n} \omega \left( \frac{2^k}{\lambda} \right) \, dx \leq \int_{\mathbb{R}^n} \sum_{k \leq \log_2 |A(f)(x)|} \omega \left( \frac{2^k}{\lambda} \right) \, dx \leq \int_{\mathbb{R}^n} \frac{2^{2A(f)(x)}}{\lambda} \omega \left( \frac{t}{\lambda} \right) \frac{dt}{t} \, dx \leq \int_{\mathbb{R}^n} \omega \left( \frac{A(f)(x)}{\lambda} \right) \, dx,
\]

which implies that (3.4) holds, and hence, completes the proof of Theorem 3.1. \( \square \)
Remark 3.1. (i) Notice that the definition $\Lambda(\{\lambda_j a_j\}_j)$ in (3.4) is different from [25, 42]. In fact, if $p \in (0, 1]$ and $\omega(t) = t^p$ for all $t \in (0, \infty)$, then $\Lambda(\{\lambda_j a_j\}_j)$ here coincides with $(\sum_j |\lambda_j|^p)^{1/p}$, which seems to be natural.

(ii) Let $\{\lambda_j^i\}_{i,j} \subset \mathbb{C}$ and $\{a_j^i\}_{i,j}$ be $(\omega, p)$-atoms for certain $p \in (1, \infty)$, where $i = 1, 2$. If $\sum_j \lambda_j^1 a_j^1, \sum_j \lambda_j^2 a_j^2 \in T_\omega(\mathbb{R}_+^{n+1})$, then by the fact that $\omega$ is subadditive and of strictly lower type $p_\omega$, we have

$$\left[\Lambda(\{\lambda_j^i a_j^i\}_{i,j})\right]^{p_\omega} \leq \sum_{i=1}^2 \left[\Lambda(\{\lambda_j^i a_j^i\}_j)\right]^{p_\omega}.$$ 

(iii) Since $\omega$ is concave, it is of upper type 1. Then, with the same notation as in Theorem 3.1, we have $\sum_{j=1}^\infty |\lambda_j| \leq C \Lambda(\{\lambda_j a_j\}_j) \leq C \|f\|_{T_\omega(\mathbb{R}_+^{n+1})}$.

Let $p \in (0, 1]$ and $q \in (p, \infty) \cap [1, \infty)$. Recall that a function $a$ on $\mathbb{R}_+^{n+1}$ is called a $(p, q)$-atom if

(i) there exists a ball $B \subset \mathbb{R}^n$ such that $\text{supp } a \subset \hat{B}$;

(ii) $\|a\|_{T_\omega^p(\mathbb{R}_+^{n+1})} \leq |B|^{1/q-1/p}$.

We have the following convergence result.

Proposition 3.1. Let $\omega$ satisfy Assumption (A) and $p \in (0, \infty)$. If $f \in (T_\omega(\mathbb{R}_+^{n+1}) \cap T_2^p(\mathbb{R}_+^{n+1}))$, then the decomposition (3.3) holds in both $T_\omega(\mathbb{R}_+^{n+1})$ and $T_2^p(\mathbb{R}_+^{n+1})$.

Proof. We use the same notation as in the proof of Theorem 3.1. We first show that (3.3) holds in $T_\omega(\mathbb{R}_+^{n+1})$. In fact, since $\omega$ is concave and $\omega^{-1}$ is convex, by the Jensen inequality and the Hölder inequality, for each $k \in \mathbb{Z}$ and $j \in I_k$, we have

$$\omega^{-1}\left(\frac{1}{|B_{k,j}|} \int_{\mathbb{R}^n} \omega(A(\lambda_{k,j} a_{k,j})(x)) \, dx\right) \leq \frac{|\lambda_{k,j}|}{|B_{k,j}|} \int_{\mathbb{R}^n} A(\lambda_{k,j} a_{k,j})(x) \, dx \leq \frac{|\lambda_{k,j}|}{|B_{k,j}|^{1/2}} \|a_{k,j}\|_{T_\omega^p(\mathbb{R}_+^{n+1})} \leq \frac{|\lambda_{k,j}|}{|B_{k,j}| \rho(|B_{k,j}|)}.$$

From this and the continuity of $\omega$ together with the subadditive property of $\omega$ and $A$, it follows that

$$\int_{\mathbb{R}^n} \omega\left(A\left(f - \sum_{|k|+|j| \leq N} \lambda_{k,j} a_{k,j}\right)(x)\right) \, dx \leq \sum_{|k|+|j| > N} \int_{\mathbb{R}^n} \omega(A(\lambda_{k,j} a_{k,j})(x)) \, dx \leq \sum_{|k|+|j| > N} |B_{k,j}| \omega\left(\frac{|\lambda_{k,j}|}{|B_{k,j}| \rho(|B_{k,j}|)}\right) \to 0, \quad (3.6)$$

as $N \to \infty$, by (3.5). Now for any $\epsilon > 0$, by the fact that $\omega$ is of upper type 1 and (3.6), there exists $N_0 \in \mathbb{N}$ such that when $N > N_0$,

$$\int_{\mathbb{R}^n} \omega\left(\frac{1}{\epsilon} A\left(f - \sum_{|k|+|j| \leq N} \lambda_{k,j} a_{k,j}\right)(x)\right) \, dx \leq 1,$$
which implies that when \( N > N_0 \), \( \|f - \sum_{|k|+|j|\leq N} \lambda_{k,j}a_{k,j} \|_{T^p_1(\mathbb{R}^{n+1}_+)} \leq \epsilon \). Thus, (3.3) holds in \( T^p_1(\mathbb{R}^{n+1}_+) \).

We now prove that (3.3) holds in \( T^p_2(\mathbb{R}^{n+1}_+) \). For the case \( p \in (0,1) \), notice that \( \{A_{k,j}\}_{k \in \mathbb{Z}, j \in I_k} \) are independent of \( \omega \). In this case, letting \( \tilde{a}_{k,j} \equiv 2^{-k} |B_{k,j}|^{-1/p} f \chi_{A_{k,j}} \) and \( \tilde{\lambda}_{k,j} \equiv 2^k |B_{k,j}|^{1/p} \), we then have that \( \{a_{k,j}\}_{k \in \mathbb{Z}, j \in I_k} \) are \((p,q)\)-atoms, where \( q \in (p, \infty) \cap [1, \infty) \), and \( \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} |\tilde{\lambda}_{k,j}|^p \lesssim \|f\|_{T^p_2(\mathbb{R}^{n+1}_+)}^p \), which combined with the fact that \( \lambda_{k,j}a_{k,j} = \tilde{\lambda}_{k,j}\tilde{a}_{k,j} \) implies that (3.3) holds in \( T^p_2(\mathbb{R}^{n+1}_+) \) in this case.

Let us now consider the case \( p \in (1,\infty) \). To prove that (3.2) holds in \( T^p_2(\mathbb{R}^{n+1}_+) \), it suffices to show that for any \( \beta > 0 \), there exists \( N_0 \in \mathbb{N} \) such that if \( N > N_0 \), then

\[
\left\| \sum_{|k|+|j|>N} \lambda_{k,j}a_{k,j} \right\|_{T^p_2(\mathbb{R}^{n+1}_+)} = \left\| \sum_{|k|+|j|>N} f \chi_{A_{k,j}} \right\|_{T^p_2(\mathbb{R}^{n+1}_+)} < \beta. \tag{3.7}
\]

To see this, noticing that \( \{A_{k,j}\}_{k \in \mathbb{Z}, j \in I_k} \) are disjoint, hence, we have

\[
\sum_{k \in \mathbb{Z}} \sum_{j \in I_k} |f \chi_{A_{k,j}}| = |f|. \tag{3.8}
\]

Write \( H_{N,1} \equiv \sum_{k<-N, j \in I_k} f \chi_{A_{k,j}} \) and \( H_{N,2} \equiv \sum_{k>N, j \in I_k} f \chi_{A_{k,j}} \). To estimate the term \( H_{N,1} \), let \( q \) be the conjugate index of \( p \) and \( h \in T^p_2(\mathbb{R}^{n+1}_+) \) with \( \|h\|_{T^p_2(\mathbb{R}^{n+1}_+)} \leq 1 \). Notice that for each \( k < -N \), \( A_{k,j} \subset \hat{\mathcal{O}^*_N} \cap \Gamma(x) \), and hence \( \text{supp} H_{N,1} \subset \hat{\mathcal{O}^*_N} \cap \Gamma(x) = \mathcal{R}(F^*_N) \). From this, (3.8), Lemma 3.1 and the Hölder inequality, we deduce that

\[
|\langle H_{N,1}, h \rangle| \leq \int \int_{\mathcal{R}(F^*_N)} \left| \sum_{k<-N, j \in I_k} (f \chi_{A_{k,j}})(y,t)h(y,t) \right| dy \, dt \\
\lesssim \int_{F^{-N}} \int_{\Gamma(x)} \left| \sum_{k<-N, j \in I_k} (f \chi_{A_{k,j}})(y,t)h(y,t) \right| \frac{dy \, dt}{t^{n+1}} dx \\\n\lesssim \int_{F^{-N}} |A(f)(x)|^p dx \lesssim \left( \int_{F^{-N}} |A(f)(x)|^p dx \right)^{1/p},
\]

which implies that

\[
\|H_{N,1}\|_{T^p_2(\mathbb{R}^{n+1}_+)} \lesssim \left( \int_{F^{-N}} |A(f)(x)|^p dx \right)^{1/p}.
\]

Then by the Lebesgue dominated convergence theorem, we have

\[
\lim_{N \to \infty} \|H_{N,1}\|_{T^p_2(\mathbb{R}^{n+1}_+)} = 0,
\]

which implies that there exists \( N_1 \in \mathbb{N} \) such that if \( N \geq N_1 \), then \( \|H_{N,1}\|_{T^p_2(\mathbb{R}^{n+1}_+)} < \beta/3. \)
For the term $H_{N,2}$, notice that for each $k > N$, $A_{k,j} \subset \widehat{O_N^*}$ and hence, $\text{supp } H_{N,2} \subset \widehat{O_N^*}$, which together with (3.8) implies that

$$
\|H_{N,2}\|_{T_2^p(\mathbb{R}_n^+)}^p = \int_{\mathbb{R}^n} \left[ A \left( \sum_{k > N, j \in I_k} f \chi_{A_{k,j}} \right)(x) \right]^p \, dx \leq \int_{\text{supp } H_{N,2}} [A(f)(x)]^p \, dx.
$$

Since $|O_N^*| \lesssim |O_N| \to 0$ as $N \to \infty$, by the continuity of Lebesgue integrals (or the Lebesgue dominated convergence theorem in measures), we have

$$
\lim_{N \to \infty} \|H_{N,2}\|_{T_2^p(\mathbb{R}_n^+)} = 0,
$$

which implies that there exists $N_2 \in \mathbb{N}$ such that if $N \geq N_2$, then $\|H_{N,2}\|_{T_2^p(\mathbb{R}_n^+)} < \beta/3$.

Now let $H_{N,3} \equiv \sum_{-N_1 \leq k \leq N_2, |k|+|j|>N} f \chi_{A_{k,j}}$. Since $A_{k,j} \subset \widehat{B_{k,j}}$, by (3.8), we obtain

$$
\|H_{N,3}\|_{T_2^p(\mathbb{R}_n^+)}^p = \int_{\mathbb{R}^n} \left[ A \left( \sum_{-N_1 \leq k \leq N_2, |k|+|j|>N} f \chi_{A_{k,j}} \right)(x) \right]^p \, dx \leq \int_{\bigcup_{-N_1 \leq k \leq N_2, |k|+|j|>N} B_{k,j}} [A(f)(x)]^p \, dx.
$$

From the Whitney decomposition, it follows that for each fixed $k$,

$$
\sum_{j \in I_k} |B_{k,j}| \lesssim \sum_{j \in I_k} |Q_{k,j}| \lesssim |O_k^*| \lesssim |O_k| < \infty,
$$

and hence, $\lim_{N \to \infty} \sum_{\{j \in I_k : |j|>N\}} |B_{k,j}| = 0$, which implies that

$$
\lim_{N \to \infty} \left| \bigcup_{-N_1 \leq k \leq N_2, |k|+|j|>N} B_{k,j} \right| \leq \lim_{N \to \infty} \sum_{-N_1 \leq k \leq N_2} \sum_{|j|+|k|>N} |B_{k,j}| = 0.
$$

Applying the continuity of Lebesgue integrals (or the Lebesgue dominated convergence theorem in measures) again, we obtain

$$
\lim_{N \to \infty} \|H_{N,3}\|_{T_2^p(\mathbb{R}_n^+)} = 0,
$$

which implies that there exists $N_3 \in \mathbb{N}$ such that if $N \geq N_3$, then $\|H_{N,3}\|_{T_2^p(\mathbb{R}_n^+)} < \beta/3$.

Letting $N_0 \equiv \max\{N_1, N_2, N_3\}$ and noticing that when $N > N_0$,

$$
\left\| \sum_{|k|+|j|>N} f \chi_{A_{k,j}} \right\|_{T_2^p(\mathbb{R}_n^+)} \leq \sum_{i=1}^3 \|H_{N_i,i}\|_{T_2^p(\mathbb{R}_n^+)} < \beta,
$$

we then obtain (3.7), which completes the proof of Proposition 3.1.
As a consequence of Proposition 3.1, we have the following corollary which plays an important role in this paper.

**Corollary 3.1.** Let \( \omega \) satisfy Assumption (A). If \( f \in T_\omega(G) \cap T_\omega^2(G) \), then \( f \in T_\omega^2(G) \) for all \( p \in [1, 2] \), and hence, the decomposition (3.3) holds in \( T_\omega^2(G) \).

**Proof.** Observing that \( \omega \) is of upper type 1, we have

\[
\int_{\mathbb{R}^n} [A(f)(x)]^p dx \leq \int_{\{x \in \mathbb{R}^n : A(f)(x) < 1\}} A(f)(x) dx + \int_{\{x \in \mathbb{R}^n : A(f)(x) \geq 1\}} [A(f)(x)]^2 dx
\]

\[
\lesssim \int_{\{x \in \mathbb{R}^n : A(f)(x) < 1\}} \omega(A(f)(x)) dx + \|f\|_{T_\omega^2(G)}^2 < \infty,
\]

which implies that \( f \in T_\omega^2(G) \). Then by Proposition 3.1, we have that the decomposition (3.3) holds in \( T_\omega^2(G) \), which completes the proof of Corollary 3.1. \( \square \)

In what follows, let \( T_\omega^c(G) \) and \( T_\omega^p,c(G) \) denote the set of all functions in \( T_\omega(G) \) and \( T_\omega^2(G) \) with compact supports, respectively, where \( p \in (0, \infty) \).

**Lemma 3.3.** (i) For all \( p \in (0, \infty) \), \( T_\omega^p,c(G) \subset T_\omega^2,c(G) \). In particular, if \( p \in (0, 2] \), then \( T_\omega^p,c(G) \) coincides with \( T_\omega^2,c(G) \).

(ii) Let \( \omega \) satisfy Assumption (A). Then \( T_\omega^c(G) \) coincides with \( T_\omega^2,c(G) \).

**Proof.** By (1.3) in [12, p. 306], we have \( T_\omega^p,c(G) \subset T_\omega^2,c(G) \) for all \( p \in (0, \infty) \). If \( p \in (0, 2] \), then from the Hölder inequality, it is easy to follow that \( T_\omega^2,c(G) \subset T_\omega^p,c(G) \).

Thus, (i) holds.

Let us prove (ii). To prove \( T_\omega^c(G) \subset T_\omega^2,c(G) \), by (i), it suffices to show that \( T_\omega^c(G) \subset T_\omega^p,c(G) \) for certain \( p \in (0, \infty) \). Suppose that \( f \in T_\omega^c(G) \) and supp \( f \subset K \), where \( K \) is a compact set in \( \mathbb{R}^n \). Let \( B \) be a ball in \( \mathbb{R}^n \) such that \( K \subset \hat{B} \). Then supp \( A(f) \subset B \). This, together with the lower type property of \( \omega \), yields that

\[
\int_{\mathbb{R}^n} [A(f)(x)]^{p-\varepsilon} dx = \int_{\{x \in \mathbb{R}^n : A(f)(x) < 1\}} [A(f)(x)]^{p-\varepsilon} dx + \int_{\{x \in \mathbb{R}^n : A(f)(x) \geq 1\}} [A(f)(x)]^{p-\varepsilon} dx \leq |B| + \int_{\mathbb{R}^n} \omega(A(f)(x)) dx < \infty.
\]

That is, \( f \in T_\omega^{2\varepsilon,c}(G) \subset T_\omega^{2,c}(G) \).

Conversely, let \( f \in T_\omega^{2,c}(G) \) supporting in a compact set \( K \) in \( \mathbb{R}^n \). Then there exists a ball \( B \) such that \( K \subset B \) and supp \( A(f) \subset B \). This, together with the upper type property of \( \omega \), yields that

\[
\int_{\mathbb{R}^n} \omega(A(f)(x)) dx \leq \int_{\{x \in \mathbb{R}^n : A(f)(x) < 1\}} \omega(1) dx + \int_{\{x \in \mathbb{R}^n : A(f)(x) \geq 1\}} A(f)(x) dx \leq |B| + \|f\|_{T_\omega(G)} < \infty,
\]

which implies that \( f \in T_\omega^c(G) \), and hence, completes the proof of Lemma 3.3. \( \square \)
4 Orlicz-Hardy spaces and their dual spaces

In this section, we always assume that the Orlicz function \( \omega \) satisfies Assumption (A). We introduce the Orlicz-Hardy space associated to \( L \) via the Lusin-area function and establish its duality. Let us begin with some notions and notation.

Let \( S_L \) be the same as in (1.3). It follows from Lemma 2.4 that the operator \( S_L \) is bounded on \( L^p(\mathbb{R}^n) \) for \( p \in (p_L, \bar{p}_L) \). Hofmann and Mayboroda [22] introduced the Hardy space \( H^1_{L}(\mathbb{R}^n) \) associated to \( L \) as the completion of \( \{ f \in L^2(\mathbb{R}^n) : S_L f \in L^1(\mathbb{R}^n) \} \) with respect to the norm \( \| f \|_{H^1_{L}(\mathbb{R}^n)} \equiv \| S_L f \|_{L^1(\mathbb{R}^n)} \).

Using some ideas from [17, 22], we now introduce the Orlicz-Hardy space \( H_{\omega,L}(\mathbb{R}^n) \) associated to \( L \) and \( \omega \) as follows.

**Definition 4.1.** Let \( \omega \) satisfy Assumption (A). A function \( f \in L^2(\mathbb{R}^n) \) is said to be in \( \widetilde{H}_{\omega,L}(\mathbb{R}^n) \) if \( S_L f \in L(\omega) \); moreover, define

\[
\| f \|_{\widetilde{H}_{\omega,L}(\mathbb{R}^n)} \equiv \| S_L f \|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{S_L f(x)}{\lambda} \right) dx \leq 1 \right\}.
\]

The Orlicz-Hardy space \( H_{\omega,L}(\mathbb{R}^n) \) is defined to be the completion of \( \widetilde{H}_{\omega,L}(\mathbb{R}^n) \) in the norm \( \cdot \|_{H_{\omega,L}(\mathbb{R}^n)} \).

In what follows, for a ball \( B \equiv B(x_B, r_B) \), we let \( U_0(B) \equiv B \), and for \( j \in \mathbb{N} \), \( U_j(B) \equiv B(x_B, 2^j r_B) \setminus B(x_B, 2^{j-1} r_B) \).

**Definition 4.2.** Let \( q \in (p_L, \bar{p}_L) \), \( M \in \mathbb{N} \) and \( \epsilon \in (0, \infty) \). A function \( \alpha \in L^q(\mathbb{R}^n) \) is called an \((\omega, q, M, \epsilon)\)-molecule adapted to \( B \) if there exists a ball \( B \) such that

(i) \( \| \alpha \|_{L^q(U_j(B))} \leq 2^{-j^q} |2^j B|^{1/q-1} \rho(|2^j B|)^{-1} \), \( j \in \mathbb{Z}_+ \);

(ii) for every \( k = 1, \cdots, M \) and \( j \in \mathbb{Z}_+ \), there holds

\[
\| (r_B^{-2} L^{-1})^{k} \alpha \|_{L^q(U_j(B))} \leq 2^{-j^q} |2^j B|^{1/q-1} \rho(|2^j B|)^{-1}.
\]

Finally, if \( \alpha \) is an \((\omega, q, M, \epsilon)\)-molecule for all \( q \in (p_L, \bar{p}_L) \), then \( \alpha \) is called an \((\omega, \infty, M, \epsilon)\)-molecule.

**Remark 4.1.** (i) Since \( \omega \) is of strictly lower type \( p_\omega \), we have that for all \( f_1, f_2 \in H_{\omega,L}(\mathbb{R}^n) \),

\[
\| f_1 + f_2 \|^2_{H_{\omega,L}(\mathbb{R}^n)} \leq \| f_1 \|^2_{H_{\omega,L}(\mathbb{R}^n)} + \| f_2 \|^2_{H_{\omega,L}(\mathbb{R}^n)}.
\]

In fact, if letting \( \lambda_1 \equiv \| f_1 \|_{H_{\omega,L}(\mathbb{R}^n)} \) and \( \lambda_2 \equiv \| f_2 \|_{H_{\omega,L}(\mathbb{R}^n)} \), by the subadditivity, the continuity and the lower type \( p_\omega \) of \( \omega \), we have

\[
\int_{\mathbb{R}^n} \omega \left( \frac{S_L(f_1 + f_2)(x)}{(\lambda_1 + \lambda_2)^{1/p_\omega}} \right) dx \leq \sum_{i=1}^2 \int_{\mathbb{R}^n} \omega \left( \frac{S_L(f_i)(x)}{(\lambda_1 + \lambda_2)^{1/p_\omega}} \right) dx
\]

\[
\leq \sum_{i=1}^2 \frac{\lambda_i}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \omega \left( \frac{S_L(f_i)(x)}{\lambda_1^{1/p_\omega}} \right) dx \leq 1,
\]
which implies \( \|f_1 + f_2\|_{H_{\omega,L}(\mathbb{R}^n)} \leq (\|f_1\|_{H_{\omega,L}(\mathbb{R}^n)}^{p_\omega} + \|f_2\|_{H_{\omega,L}(\mathbb{R}^n)}^{p_\omega})^{1/p_\omega} \), and hence, the desired conclusion.

(ii) From the theorem of completion of Yosida [44, p. 56], it follows that \( \tilde{H}_{\omega,L}(\mathbb{R}^n) \) is dense in \( H_{\omega,L}(\mathbb{R}^n) \), namely, for any \( f \in H_{\omega,L}(\mathbb{R}^n) \), there exists a Cauchy sequence \( \{f_k\}_{k=1}^\infty \subset \tilde{H}_{\omega,L}(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} \|f_k - f\|_{H_{\omega,L}(\mathbb{R}^n)} = 0 \). Moreover, if \( \{f_k\}_{k=1}^\infty \) is a Cauchy sequence in \( \tilde{H}_{\omega,L}(\mathbb{R}^n) \), then there uniquely exists \( f \in H_{\omega,L}(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} \|f_k - f\|_{H_{\omega,L}(\mathbb{R}^n)} = 0 \).

(iii) If \( \omega(t) = t \), then the space \( H_{\omega,L}(\mathbb{R}^n) \) is just the space \( H_1^p(\mathbb{R}^n) \) introduced by Hofmann and Mayboroda [22]. Furthermore, when \( \omega(t) \equiv t^p \) for all \( t \in (0, \infty) \) with \( p \in (0, 1] \), we then denote the space \( H_{\omega,L}(\mathbb{R}^n) \) simply by \( H_L^p(\mathbb{R}^n) \).

4.1 Molecular decompositions of \( H_{\omega,L}(\mathbb{R}^n) \)

In what follows, let \( L_c^2(\mathbb{R}^{n+1}_+) \) denote the set of all functions in \( L^2(\mathbb{R}^{n+1}_+) \) with compact supports. Recall that \( \omega \) is a concave function of strictly lower type \( p_\omega \), where \( p_\omega \in (0, 1] \).

**Proposition 4.1.** Let \( \omega \) satisfy Assumption (A), \( M \in \mathbb{N} \) and \( M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{q}) \), and \( \pi_{L,M} \) be as in (1.4).

(i) The operator \( \pi_{L,M} \), initially defined on \( T_2^{p,c}(\mathbb{R}^{n+1}_+) \), extends to a bounded linear operator from \( T_2^{p,c}(\mathbb{R}^{n+1}_+) \) to \( L^p(\mathbb{R}^n) \), where \( p \in (p_L, \tilde{p}_L) \).

(ii) The operator \( \pi_{L,M} \), initially defined on \( T_2^{p,c}(\mathbb{R}^{n+1}_+) \), extends to a bounded linear operator from \( \omega \in L^p(\mathbb{R}^{n+1}_+) \) to \( H_{\omega,L}(\mathbb{R}^n) \).

**Proof.** Let \( k \in \mathbb{N} \). By Lemma 2.4 and a duality argument, we know that the operator \( S_{L^*}^k \) is bounded on \( L^p(\mathbb{R}^n) \) for \( p \in (p_L, \tilde{p}_L) \), where \( \frac{1}{p_L} + \frac{1}{\tilde{p}_L} = 1 = \frac{1}{p_L} + \frac{1}{\tilde{p}_L} \).

Let \( f \in T_2^{p,c}(\mathbb{R}^{n+1}_+) \), where \( p \in (p_L, \tilde{p}_L) \). For any \( g \in L^q(\mathbb{R}^n) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), by the Hölder inequality, we have

\[
\left| \int_{\mathbb{R}^n} \pi_{L,M}(f)(x)g(x) \, dx \right| \leq \int \int_{\mathbb{R}^{n+1}_+} f(y,t)(t^2L^*)^{M+1} e^{-t^2L^*} g(y) \frac{dy \, dt}{t} \leq \|A(f)\|_{L^p(\mathbb{R}^n)} \|S_{L^*}^{M+1} g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{T_2^{p,c}(\mathbb{R}^{n+1}_+)} \|g\|_{L^q(\mathbb{R}^n)},
\]

which implies that \( \pi_{L,M} \) maps \( T_2^{p,c}(\mathbb{R}^{n+1}_+) \) continuously into \( L^p(\mathbb{R}^n) \). Then by a density argument, we obtain that \( \pi_{L,M} \) is bounded from \( T_2^{p,c}(\mathbb{R}^{n+1}_+) \) to \( L^p(\mathbb{R}^n) \). This proves (i).

Let us prove (ii). Assume that \( f \in T_2^{p,c}(\mathbb{R}^{n+1}_+) \). By Theorem 3.1, we have \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) pointwise, where \( \{\lambda_j\}_{j=1}^\infty \) and \( \{a_j\}_{j=1}^\infty \) are as in Theorem 3.1 and \( \Lambda(\{\lambda_j a_j\}) \leq \|f\|_{T_2^{p,c}(\mathbb{R}^{n+1}_+)} \). From Lemma 3.3 (ii), it follows that \( f \in T_2^{p,c}(\mathbb{R}^{n+1}_+) \), which together with (i) and Corollary 3.1 further implies that

\[
\pi_{L,M}(f) = \sum_{j=1}^{\infty} \lambda_j \pi_{L,M}(a_j) = \sum_{j=1}^{\infty} \lambda_j \alpha_j
\]
in $L^p(\mathbb{R}^n)$ for $p \in (p_L, 2]$.

On the other hand, notice that the operator $\mathcal{S}_L$ is bound on $L^p(\mathbb{R}^n)$, which together with the subadditivity and the continuity of $\omega$ yields that

$$
\int_{\mathbb{R}^n} \omega(\mathcal{S}_L(\pi_{L,M}(f))(x)) \, dx \leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \omega(|\lambda_j|\mathcal{S}_L(\alpha_j)(x)) \, dx. \tag{4.1}
$$

We claim that for any fixed $\epsilon \in (0, \infty)$, $\alpha_j = \pi_{L,M}(a_j)$ is a multiple of an $(\omega, \infty, M, \epsilon)$-molecule adapted to $B_j$ for each $j$.

In fact, assume that $a$ is an $(\omega, \infty)$-atom supported in the ball $B \equiv B(x_B, r_B)$ and $q \in (p_L, p_L)$. Since for $q \in (p_L, 2)$, each $(\omega, 2, M, \epsilon)$-molecule is also an $(\omega, q, M, \epsilon)$-molecule, to prove the above claim, it suffices to show that $\alpha \equiv \pi_{L,M}(a)$ is a multiple of an $(\omega, q, M, \epsilon)$-molecule adapted to $B$ with $q \in [2, p_L)$.

By (i), for $i = 0, 1, 2$, we have

$$
\|\alpha\|_{L^q(U_i(B))} = \|\pi_{L,M}(a)\|_{L^q(U_i(B))} \lesssim \|a\|_{T^q_{2}(\mathbb{R}^{n+1})} \lesssim |B|^{1/q-1}[\rho(|B|)]^{-1}.
$$

For $i \geq 3$, let $q' \in (1, 2]$ being the conjugate number of $q$ and $h \in L^{q'}(\mathbb{R}^n)$ satisfying $\|h\|_{L^{q'}(\mathbb{R}^n)} \leq 1$ and $\supp h \subset U_i(B)$. By the Hölder inequality and Lemmas 2.1 and 2.3, we have

$$
|\langle \pi_{L,M}(a), h \rangle| 
\lesssim \int_0^{r_B} \int_B |a(x, t)|(t^2 L^*)^{M+1} e^{-t^2 L^*} (h)(x) \, dx \, dt \lesssim \|A(a)\|_{L^q(\mathbb{R}^n)} \|A(\chi_B(t^2 L^*)^{M+1} e^{-t^2 L^*} (h))\|_{L^{q'}(\mathbb{R}^n)}
\lesssim \|a\|_{T^q_{2}(\mathbb{R}^{n+1})} |B|^{1/q-1} \left( \int_B |(t^2 L^*)^{M+1} e^{-t^2 L^*} (h)(x, t)|^2 \, dx \, dt \right)^{1/2}
\lesssim \|a\|_{T^q_{2}(\mathbb{R}^{n+1})} |B|^{1/q-1} \left( \int_0^{r_B} \left[ t^{n(2-1/q')} \exp \left\{ -\frac{\text{dist} (B, U_i(B))^2}{ct^2} \right\} \right] t^{n(1/2-1/q')} \, dt \right)^{1/2}
\lesssim |B|^{-1/2} \rho(|B|)^{-1} \left( \int_0^{r_B} t^{n(1-2/q')} \left[ \frac{t}{2^i r_B} \right]^{\frac{2\epsilon+n/p}{2|\omega|}} \, dt \right)^{1/2}
\lesssim 2^{-i\epsilon} |2^i B|^{1/q-1}[\rho(|2^i B|)]^{-1}, \tag{4.2}
$$
which implies that $\alpha$ satisfies Definition 4.2 (i).

We now show that $\alpha$ also satisfies Definition 4.2 (ii). Let $k \in \{1, \cdots, M\}$. If $i = 0, 1, 2$, let $h$ be the same as in the proof of (4.2); similarly to the proof of (4.1), we have

$$
|\langle r_B^{-2} L^{-1} \rangle^k \pi_{L,M}(a), h \rangle| 
\lesssim \int_0^{r_B} \int_B \left( \frac{t}{r_B} \right)^{2k} |a(x, t)|(t^2 L^*)^{M+1-k} e^{-t^2 L^*} (h)(x) \, dx \, dt
\lesssim \|A(a)\|_{L^q(\mathbb{R}^n)} \|S_{L^*}^{M+1-k}(h)\|_{L^{q'}(\mathbb{R}^n)}
\lesssim \|a\|_{T^q_{2}(\mathbb{R}^{n+1})} \lesssim |B|^{1/q-1}[\rho(|B|)]^{-1},
$$
which is the desired estimate, where we used the Hölder inequality and Lemma 2.4 by noticing that \( q' \in (p_L, 2] \). If \( i \geq 3 \), an argument similar to that used in the estimate of (4.2) also yields the desired estimate. Thus, \( \alpha = \pi_{L,M}(a) \) is a multiples of an \( (\omega, q, M, \epsilon) \)-molecule adapted to \( B \) with \( q \in [2, \bar{p}_L) \), and the claim is proved.

Let \( q \in (p_L, \bar{p}_L) \) and \( \epsilon > n (\frac{1}{p_\omega} - \frac{1}{p_\omega'}) \), where \( \bar{p}_\omega \) is as in Convention (C). We now claim that for all \( (\omega, q, M, \epsilon) \)-molecules \( \alpha \) adapted to the ball \( B \equiv B(x, r_B) \) and \( \lambda \in \mathbb{C} \),

\[
\int_{\mathbb{R}^n} \omega(|\lambda|S_L(\alpha)(x)) \, dx \lesssim |B|\omega \left( \frac{|\lambda|}{|B|\rho(|B|)} \right). \tag{4.3}
\]

Once this is proved, then we have \( \|\alpha\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim 1 \), which together with (4.1) further implies that for all \( f \in T_{\omega}^c(\mathbb{R}^n_{+}) \),

\[
\int_{\mathbb{R}^n} \omega(S_L(\pi_{L,M}(f))(x)) \, dx \lesssim \sum_{j=1}^{\infty} |B_j|\omega \left( \frac{|\lambda_j|}{|B_j|\rho(|B_j|)} \right).
\]

Thus, for all \( f \in T_{\omega}^c(\mathbb{R}^n_{+}) \), we have

\[
\|\pi_{L,M}(f)\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j a_j\}) \lesssim \|f\|_{T_{\omega}^c(\mathbb{R}^n_{+})},
\]

which combined with a density argument implies (ii).

Now, let us prove the claim (4.3). Observe that if \( q > 2 \), then an \( (\omega, q, M, \epsilon) \)-molecule is also an \( (\omega, 2, M, \epsilon) \)-molecule. Thus, to prove the claim (4.3), it suffices to show (4.3) for \( q \in (p_L, 2] \). To this end, write

\[
\int_{\mathbb{R}^n} \omega(|\lambda|S_L(\alpha)(x)) \, dx \\
\leq \int_{\mathbb{R}^n} \omega(|\lambda|S_L([I - e^{-r_B^2 L^2}M] \alpha)(x)) \, dx + \int_{\mathbb{R}^n} \omega(|\lambda|S_L([I - e^{-r_B^2 L^2}M] \alpha)(x)) \, dx \\
\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \omega(|\lambda|S_L([I - e^{-r_B^2 L^2}M] \alpha \chi_{U_j(B)})(x)) \, dx \\
+ \sum_{j=0}^{\infty} \sup_{1 \leq k \leq M} \int_{\mathbb{R}^n} \omega \left( |\lambda|S_L \left\{ \left\lceil \frac{k}{M} r_B^2 L e^{-\frac{k r_B^2 L}{M}} \right\rceil^M \chi_{U_j(B)}(r_B^2 L^{-1} M \alpha) \right\} \right) \, dx \\
\equiv \sum_{j=0}^{\infty} H_j + \sum_{j=0}^{\infty} I_j.
\]

For each \( j \geq 0 \), let \( B_j \equiv 2^j B \). Since \( \omega \) is concave, by the Jensen inequality and the Hölder inequality, we obtain

\[
H_j \lesssim \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(|\lambda|S_L([I - e^{-r_B^2 L^2}M] \alpha \chi_{U_j(B)})(x)) \, dx \\
\sim \sum_{k=0}^{\infty} \int_{2^k B_j} \omega(|\lambda|\chi_{U_k(B_j)}(x)S_L([I - e^{-r_B^2 L^2}M] \alpha \chi_{U_j(B)})(x)) \, dx
\]
Similarly, we have
\[
\|S_L(I - e^{-r_0L})^M(\alpha \chi_{U_k(B_j)})(x)\|_{L^q(U_k(B_j))} \lesssim \|\alpha\|_{L^q(U_j(B))},
\]
and for \(k \geq 3\),
\[
\|S_L(I - e^{-r_0L})^M(\alpha \chi_{U_j(B_j)})\|_{L^q(U_k(B_j))}^2 \lesssim k \left( \frac{1}{2^{k+j}} \right)^{4M+2(n/2-n/q)} \|\alpha\|_{L^q(U_j(B))}^2,
\]
which, together with Definition 4.2, \(2M \rho_0 > n(1 - p_\omega/2)\) and Assumption (A), implies that
\[
H_j \lesssim |B_j| \omega \left( \frac{|\lambda|}{|B_j|} \right) + \sum_{k=3}^\infty |B_j| \omega \left( \frac{|\lambda|}{|B_j|} \right) \left( \frac{|\lambda|}{|B_j|} \right)
\]
\[
\lesssim 2^{-j_0 \omega} \left\{ 1 + \sum_{k=3}^\infty \sqrt{2^k \omega} \left( \frac{2^{-j_0 \omega} \omega}{2^{-j_0 \omega} \omega} \right) \right\} |B_j| \omega \left( \frac{|\lambda|}{|B_j|} \right)
\]
\[
\lesssim 2^{-j_0 \omega} |B_j| \omega \left( \frac{|\lambda|}{|B_j|} \right).
\]
Since \(\rho\) is of lower type \(1/\bar{\rho}_\omega - 1\) and \(\epsilon > n(1 + \bar{\rho}_\omega - 1),\) we further have
\[
\sum_{j=0}^\infty H_j \lesssim \sum_{j=0}^\infty 2^{-j_0 \omega} |B_j| \left\{ \frac{|B| \rho(|B_j|)}{|B_j| \rho(|B_j|)} \right\}^{\rho_0} \omega \left( \frac{|\lambda|}{|B_j|} \right)
\]
\[
\lesssim \sum_{j=0}^\infty 2^{-j_0 \omega} |B_j| \left\{ \frac{|B|}{|B_j|} \right\}^{\rho_0/\bar{\rho}_\omega} \omega \left( \frac{|\lambda|}{|B_j|} \right)
\]
\[
\lesssim \sum_{j=0}^\infty 2^{-j_0 \omega} 2^{jn(1 - \rho_0/\bar{\rho}_\omega)} |B_j| \omega \left( \frac{|\lambda|}{|B_j|} \right) \lesssim |B| \omega \left( \frac{|\lambda|}{|B|} \right).
\]
Similarly, we have
\[
\sum_{j=0}^\infty I_j \lesssim |B| \omega \left( \frac{|\lambda|}{|B|} \right),
\]
which completes the proof of (4.3), and hence, the proof of Proposition 4.1.

\begin{proposition}
Let \(\omega\) satisfy Assumption (A), \(\epsilon > n(1 + \bar{\rho}_\omega - 1/\bar{\rho}_\omega)\) and \(M > \frac{3}{2} (1/\bar{\rho}_\omega - 1/2).\)
If \(f \in H_{\omega, L} (\mathbb{R}^n) \cap L^2 (\mathbb{R}^n),\) then \(f \in L^p (\mathbb{R}^n)\) for all \(p \in (p_\omega, 2]\) and there exist \((\omega, \infty, M, \epsilon)\)-molecules \(\{\alpha_j\}_{j=1}^\infty\) and numbers \(\{\lambda_j\}_{j=1}^\infty\subset C\) such that
\[
f = \sum_{j=1}^\infty \lambda_j \alpha_j
\]
\end{proposition}
in both $H_{\omega,L}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ for all $p \in (p_L, 2]$. Moreover, there exists a positive constant $C$ independent of $f$ such that for all $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$
\Lambda(\{\lambda_j \alpha_j\}) = \inf \left\{ \lambda > 0 : \sum_{j=1}^{\infty} |B_j|\omega \left( \frac{|\lambda_j|}{\lambda |B_j| \rho(|B_j|)} \right) \leq 1 \right\} \leq C \|f\|_{H_{\omega,L}(\mathbb{R}^n)}, \quad (4.6)
$$

where for each $j$, $\alpha_j$ is adapted to the ball $B_j$.

**Proof.** Let $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. For each $N \in \mathbb{N}$, define $O_N \equiv \{(x,t) \in \mathbb{R}^{n+1} : |x| < N, 1/N < t < N\}$. Then by the $L^2(\mathbb{R}^n)$-functional calculi for $L$, we have

$$
f = C_M \int_0^\infty (t^2 L)^{M+2} e^{-2t^2 L} f \frac{dt}{t} = \lim_{N \to \infty} \pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))
$$
in $L^2(\mathbb{R}^n)$, where $M \in \mathbb{N}$, $\pi_{L,M}$ and $C_M$ are as in (1.4).

On the other hand, by Definition 4.1 and Lemma 2.4, we have $t^2 L e^{-t^2 L} f \in T_2^p(\mathbb{R}^{n+1}) \cap T_\omega(\mathbb{R}^{n+1})$. An application of Corollary 3.1 shows that $t^2 L e^{-t^2 L} f \in T_2^p(\mathbb{R}^{n+1})$, which together with Proposition 4.1 (i) implies that $\{\pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))\}_N$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. Then via taking subsequence, we have

$$
f = \lim_{N \to \infty} \pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))
$$
in $L^p(\mathbb{R}^n)$.

Now applying Theorem 3.1 and Proposition 3.1 to $t^2 L e^{-t^2 L} f$, we obtain $(\omega, \infty)$-atoms $\{a_j\}_{j=1}^{\infty}$ and numbers $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ such that $t^2 L e^{-t^2 L} f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $T_2^p(\mathbb{R}^{n+1})$ and

$$
\Lambda(\{\lambda_j a_j\}) \lesssim \|t^2 L e^{-t^2 L} f\|_{T_\omega(\mathbb{R}^{n+1})},
$$
which combined with Proposition 4.1 (i) further yields that

$$
f = \pi_{L,M}(t^2 L e^{-t^2 L} f) = \sum_{j=1}^{\infty} \lambda_j \pi_{L,M}(a_j) \equiv \sum_{j=1}^{\infty} \lambda_j \alpha_j \quad (4.7)
$$
in $L^p(\mathbb{R}^n)$ for $p \in (p_L, 2]$. By the proof of Proposition 4.1, we know that $\alpha_j$ is a multiple of an $(\omega, \infty, M, \epsilon)$-molecule for any $\epsilon > 0$, and $M > \frac{n}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right)$. Notice that $\Lambda(\{\lambda_j \alpha_j\}) = \Lambda(\{\lambda_j a_j\})$. We therefore obtain (4.6).

To finish the proof of Proposition 4.2, it remains to show that (4.5) holds in $H_{\omega,L}(\mathbb{R}^n)$. In fact, by Lemma 2.4, (3.5), (4.3) and (4.7) together with the continuity and the subadditivity of $\omega$, we have

$$
\int_{\mathbb{R}^n} \omega \left( S_L \left( f - \sum_{j=1}^{N} \lambda_j \alpha_j \right)(x) \right) dx \leq \sum_{j=N+1}^{\infty} \int_{\mathbb{R}^n} \omega(\mathcal{S}_L(\lambda_j \alpha_j)(x)) dx \lesssim \sum_{j=N+1}^{\infty} |B_j|\omega \left( \frac{|\lambda_j|}{|B_j| \rho(|B_j|)} \right) \to 0,
$$
as $N \to \infty$. We point out that here, in the last inequality, to use (4.3), we need to choose $\tilde{p}_\omega$ as in Convention (C) such that $\epsilon > n(1/p_\omega - 1/\tilde{p}_\omega)$, which is guaranteed by the
assumption $\epsilon > n(1/p_\omega - 1/p_\omega^+)$). This combined with an argument similar to the proof of Proposition 3.1 yields that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $H_{\omega,L}(\mathbb{R}^n)$, which completes the proof of Proposition 4.2.

Corollary 4.1. Let $\omega$ satisfy Assumption (A), $\epsilon > n(1/p_\omega - 1/p_\omega^+)$, $q \in (p_L, \bar{p}_L)$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then for every $f \in H_{\omega,L}(\mathbb{R}^n)$, there exist $(\omega, q, M, \epsilon)$-molecules $\{\alpha_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $H_{\omega,L}(\mathbb{R}^n)$. Furthermore, if letting $\Lambda(\{\lambda_j \alpha_j\})$ be as in (4.6), then there exists a positive constant $C$ independent of $f$ such that $\Lambda(\{\lambda_j \alpha_j\}) \leq C\|f\|_{H_{\omega,L}(\mathbb{R}^n)}$.

Proof. If $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then it immediately follows from Proposition 4.2 that all results hold.

Otherwise, there exist $\{f_k\}_{k=1}^\infty \subset (H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ such that for all $k \in \mathbb{N}$,

$$\|f - f_k\|_{H_{\omega,L}(\mathbb{R}^n)} \leq 2^{-k}\|f\|_{H_{\omega,L}(\mathbb{R}^n)}.$$

Set $f_0 \equiv 0$. Then $f = \sum_{k=1}^\infty (f_k - f_{k-1})$ in $H_{\omega,L}(\mathbb{R}^n)$. By Proposition 4.2, we have that for all $k \in \mathbb{N}$, $f_k - f_{k-1} = \sum_{j=1}^\infty \lambda_j^k \alpha_j^k$ in $H_{\omega,L}(\mathbb{R}^n)$ and $\Lambda(\{\lambda_j^k \alpha_j^k\}) \lesssim \|f_k - f_{k-1}\|_{H_{\omega,L}(\mathbb{R}^n)}$, where for all $j$ and $k$, $\alpha_j^k$ is an $(\omega, q, M, \epsilon)$-molecule. Thus, $f = \sum_{k,j=1}^\infty \lambda_j^k \alpha_j^k$ in $H_{\omega,L}(\mathbb{R}^n)$, and it further follows from Remark 3.1 (ii) that

$$[\Lambda(\{\lambda_j^k \alpha_j^k\})_{k,j}]^{\mathbb{R}^n} \lesssim \sum_{k=1}^\infty [\Lambda(\{\lambda_j^k \alpha_j^k\})_{j}]^{\mathbb{R}^n} \lesssim \sum_{k=1}^\infty \|f_k - f_{k-1}\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)},$$

which completes the proof of Corollary 4.1.

Let $H^{q,M,\epsilon}_{\omega,\text{fin}}(\mathbb{R}^n)$ denote the set of all finite combinations of $(\omega, q, M, \epsilon)$-molecules. From Corollary 4.1, we immediately deduce the following density result.

Corollary 4.2. Let $\omega$ satisfy Assumption (A), $\epsilon > n(1/p_\omega - 1/p_\omega^+)$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then the space $H^{q,M,\epsilon}_{\omega,\text{fin}}(\mathbb{R}^n)$ is dense in the space $H_{\omega,L}(\mathbb{R}^n)$.

4.2 Dual spaces of $H_{\omega,L}(\mathbb{R}^n)$

In this subsection, we study the dual space of the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$. We begin with some notions.

Following [22], for $\epsilon > 0$ and $M \in \mathbb{N}$, we introduce the space

$$\mathcal{M}^{M,\epsilon}_\omega(L) \equiv \{\mu \in L^2(\mathbb{R}^n) : \|\mu\|_{\mathcal{M}^{M,\epsilon}_\omega(L)} < \infty\},$$

where

$$\|\mu\|_{\mathcal{M}^{M,\epsilon}_\omega(L)} \equiv \sup_{j \geq 0} \left\{2^j|B(0, 2^j)|^{1/2}\rho(|B(0, 2^j)|) \sum_{k=0}^M \|L^{-k}\mu\|_{L^2(U_j(B(0,1)))}\right\},$$

Notice that if $\phi \in \mathcal{M}^{M,\epsilon}_\omega(L)$ with norm 1, then $\phi$ is an $(\omega, 2, M, \epsilon)$-molecule adapted to $B(0,1)$. Conversely, if $\alpha$ is an $(\omega, 2, M, \epsilon)$-molecule adapted to certain ball, then $\alpha \in \mathcal{M}^{M,\epsilon}_\omega(L)$. 

Let $A_t$ denote either $(I + i^2L)^{-1}$ or $e^{-t^2L}$ and $f \in (\mathcal{M}_L^M)^*$, the dual of $\mathcal{M}_L^M$. We claim that $(I - A_t^*)^M f \in L^2_{\text{loc}}(\mathbb{R}^n)$ in the sense of distributions. In fact, for any ball $B$, if $\psi \in L^2(B)$, then it follows from the Gaffney estimates via Lemmas 2.1 and 2.2 that $(I - A_t^*)^M \psi \in \mathcal{M}_L^M$ for all $\epsilon > 0$ and any fixed $t \in (0, \infty)$. Thus,

$$|\langle (I - A_t^*)^M f, \psi \rangle| \equiv |\langle (I - A_t)^M \psi \rangle| \leq C(t, r_B, \text{dist} (B, 0))\|f\|_{(\mathcal{M}_L^M)^*} \|\psi\|_{L^2(B)},$$

which implies that $(I - A_t^*)^M f \in L^2_{\text{loc}}(\mathbb{R}^n)$ in the sense of distributions.

Finally, for any $M \in \mathbb{N}$, define

$$\mathcal{M}_{\omega, L}^M(\mathbb{R}^n) = \bigcap_{\epsilon > n(1/p_\omega - 1/p_\omega^*)} \mathcal{M}_{\omega}^M(\mathbb{R}^n)^*.$$}

**Definition 4.3.** Let $q \in (p_L, \bar{p}_L)$, $\omega$ satisfy Assumption (A), $\rho$ be as in (2.4) and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. A functional $f \in \mathcal{M}_{\omega, L}^M(\mathbb{R}^n)$ is said to be in $\text{BMO}_{\omega, L}^q(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}_{\omega, L}^q(\mathbb{R}^n)} \equiv \sup_{B \subset \mathbb{R}^n} \frac{1}{\rho(|B|)} \left[ \frac{1}{|B|} \int_{B} |(I - e^{-r_B^2L})^M f(x)|^q \, dx \right]^{1/q} < \infty,$$

where the supremum is taken over all balls $B$ of $\mathbb{R}^n$.

In what follows, when $q = 2$, we denote $\text{BMO}_{\omega, L}^q(\mathbb{R}^n)$ simply by $\text{BMO}_{\omega, L}^M(\mathbb{R}^n)$. The proofs of following Lemmas 4.1 and 4.2 are similar to those of Lemmas 8.1 and 8.3 of [22], respectively; we omit the details.

**Lemma 4.1.** Let $\omega$, $\rho$, $q$ and $M$ be as in Definition 4.3. A functional $f \in \mathcal{M}_{\omega, L}^M(\mathbb{R}^n) \subset \text{BMO}_{\omega, L}^q(\mathbb{R}^n)$ if and only if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\rho(|B|)} \left[ \frac{1}{|B|} \int_{B} |(I - (I + r_B^2L)^{-1})^M f(x)|^q \, dx \right]^{1/q} < \infty.$$

Moreover, the quantity appeared in the left-hand side of the above formula is equivalent to $\|f\|_{\text{BMO}_{\omega, L}^q(\mathbb{R}^n)}$.

**Lemma 4.2.** Let $\omega$, $\rho$ and $M$ be as in Definition 4.3. Then there exists a positive constant $C$ such that for all $f \in \text{BMO}_{\omega, L}^M(\mathbb{R}^n)$,

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\rho(|B|)} \left[ \frac{1}{|B|} \int_{B} \left| (t^2L)^M e^{-t^2L} f(x) \right|^2 \frac{dx \, dt}{t} \right]^{1/2} \leq C \|f\|_{\text{BMO}_{\omega, L}^M(\mathbb{R}^n)}.$$
Then for every \((\omega, q', \tilde{M}, \epsilon)\)-molecule \(\alpha\),
\[
\langle f, \alpha \rangle = \tilde{C}_M \int_{\mathbb{R}^{n+1}_+} (t^2 L^*)^M e^{-t^2 L^*} f(x) t^2 L e^{-t^2 L} f(x) dx dt,
\]
where \(q' \in [2, \infty)\) satisfying \(1/q + 1/q' = 1\) and \(\tilde{C}_M\) is a positive constant satisfying
\[
\tilde{C}_M \int_0^\infty t^{2(M+1)} e^{-t^2} dt = 1.
\]

**Proof.** If we let \(\omega(t) \equiv t\) for all \(t \in (0, \infty)\), then this lemma are just Lemma 8.4 and Remark of Section 9 in [22].

Otherwise, let \(\alpha\) be an \((\omega, q', \tilde{M}, \epsilon)\)-molecule adapted to a ball \(B\). Then from Definition 4.2, it is easy to see that \(\rho(|B|)\alpha\) is an \((\tilde{\omega}, q', \tilde{M}, \epsilon)\)-molecule, where \(\tilde{\omega}(t) \equiv t\) for all \(t \in (0, \infty)\), and hence Lemma 4.3 holds for \(\rho(|B|)\alpha\), which implies the desired conclusion and hence, completes the proof of Lemma 4.3.

From Lemma 4.1, it is easy to follow that all \(f \in \text{BMO}_{q,M}^\epsilon(\mathbb{R}^n)\) satisfy (4.8) for all \(\epsilon_1 \in (0, \infty)\), and hence, Lemma 4.3 holds for all \(f \in \text{BMO}_{\rho,L}^\epsilon(\mathbb{R}^n)\).

Now, let us give the main result of this section.

**Theorem 4.1.** Let \(\omega\) satisfy Assumption (A), \(\rho\) be as in (2.4), \(\epsilon > n(1/p_\omega - 1/p_\omega^+)\), \(M > \frac{2}{p_\omega(1 - \frac{1}{p_\omega^+})}\) and \(\tilde{M} > M + \frac{4}{q}\). Then \((H_{\omega,L}(\mathbb{R}^n))^*\), the dual space of \(H_{\omega,L}(\mathbb{R}^n)\), coincides with \(\text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)\) in the following sense:

(i) Let \(g \in \text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)\). Then the linear functional \(\ell\), which is initially defined on \(H_{\omega,\text{fin}}^{2,\tilde{M},\epsilon}(\mathbb{R}^n)\) by
\[
\ell(f) \equiv \langle g, f \rangle,
\]
has a unique extension to \(H_{\omega,L}(\mathbb{R}^n)\) with \(\|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*} \leq C\|g\|_{\text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)}\), where \(C\) is a positive constant independent of \(g\).

(ii) Conversely, for any \(\ell \in (H_{\omega,L}(\mathbb{R}^n))^*\), then \(\ell \in \text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)\), (4.9) holds for all \(f \in H_{\omega,\text{fin}}^{2,\tilde{M},\epsilon}(\mathbb{R}^n)\) and \(\|\ell\|_{\text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)} \leq C\|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}\), where \(C\) is a positive constant independent of \(\ell\).

**Proof.** Let \(g \in \text{BMO}_{\rho,L}^{M*}(\mathbb{R}^n)\). For any \(f \in H_{\omega,\text{fin}}^{2,\tilde{M},\epsilon}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n)\), we have that \(f \in L^2(\mathbb{R}^n)\) and hence, \(t^2 L e^{-t^2 L} f \in (T_{\omega}(\mathbb{R}^n)^{n+1} \cap T_3^2(\mathbb{R}^n)^{n+1})\) by Lemma 2.4. By Theorem 3.1, there exist \(\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}\) and \((\omega, \epsilon)\)-atoms \(\{a_j\}_{j=1}^\infty\) supported in \(\{\hat{B}_j\}_{j=1}^\infty\) such that (3.4) holds. Notice that \(g\) satisfies (4.8) with \(q = 2\) (by Lemma 4.1), which, together with Lemmas 4.2 and 4.3, the Hölder inequality and Remark 3.1 (iii), yields that
\[
|\langle g, f \rangle| \leq \sum_{j=1}^\infty |\lambda_j| \int_{\mathbb{R}^{n+1}} |(t^2 L^*)^M e^{-t^2 L^*} g(x) a_j(x, t)| dx dt
\]
\[
\leq \sum_{j=1}^\infty |\lambda_j| \int_{\mathbb{R}^{n+1}} |(t^2 L^*)^M e^{-t^2 L^*} g(x) a_j(x, t)| dx dt.
\]
\[
\begin{align*}
&\lesssim \sum_{j=1}^{\infty} |\lambda_j| \|a_j\|_{T_2^{n}(\mathbb{R}^{n+1}_+)} \left( \iint_{B_j} \left| (t^2 L^*)^M e^{-t^2 L^*} g(x) \right|^2 \frac{dx dt}{t} \right)^{1/2} \\
&\lesssim \sum_{j=1}^{\infty} |\lambda_j| \|g\|_{\text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)} \lesssim \|t^2 L e^{-t^2 L} f\|_{L_\text{loc}(\mathbb{R}^{n+1})} \|g\|_{\text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)} \\
&\sim \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \|g\|_{\text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)}. \\
&\text{(4.10)}
\end{align*}
\]

Then by a density argument via Corollary 4.2, we obtain (i).

Conversely, let \(\ell \in (H_{\omega,L}(\mathbb{R}^n))^*\). For any \((\omega, 2, M, \epsilon)\)-molecule \(\alpha\), it follows from 4.3 that \(\|\alpha\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim 1\). Thus \(|\ell(\alpha)| \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}\), which implies that \(\ell \in \mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n)\).

To finish the proof of (ii), we still need to show that \(\ell \in \text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)\). To this end, for any ball \(B\), let \(\phi \in L^2(B)\) with \(\|\phi\|_{L^2(B)} \leq \frac{1}{|B|^{1/2} \rho(|B|)}\) and \(\tilde{\alpha} \equiv (I - [I + r_B^2 L]^{-1})^M \phi\). Then from Lemma 2.3, we deduce that for each \(j \in \mathbb{Z}_+\) and \(k = 0, 1, \ldots, M\),

\[
\begin{align*}
\| (r_B^2 L)^{-k} \tilde{\alpha} \|_{L^2(U_j(B))} &= \| (I - [I + r_B^2 L]^{-1})^{M-k} (I + r_B^2 L)^{-k} \phi \|_{L^2(U_j(B))} \\
&\lesssim \exp \left\{ - \frac{\text{dist}(B, U_j(B))}{c r_B} \right\} \|\phi\|_{L^2(B)} \\
&\lesssim 2^{-2j(M+\epsilon)} 2^{jn(1/p_\omega-1/2)} \frac{1}{|2^j B|^{1/2} \rho(|2^j B|)} \lesssim 2^{-2j\epsilon} \frac{1}{|2^j B|^{1/2} \rho(|2^j B|)},
\end{align*}
\]

where \(c\) is as in Lemma 2.3 and \(2M > n(1/p_\omega-1/2)\). Thus \(\tilde{\alpha}\) is a multiple of an \((\omega, 2, M, \epsilon)\)-molecule. Since \((I - ([I + t^2 L]^{-1})^*)^M \ell\) is well defined and belongs to \(L^2_{\text{loc}}(\mathbb{R}^n)\) for any fixed \(t > 0\). Thus, we have

\[
\|((I - [I + r_B^2 L]^{-1})^* M \ell, \phi)\| = \|\ell, (I - [I + r_B^2 L]^{-1})^* M \phi)\| = \|\ell, \tilde{\alpha}\| \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*},
\]

which further implies that

\[
\begin{align*}
&\frac{1}{\rho(|B|)} \left( \frac{1}{|B|} \int_B \| (I - [I + r_B^2 L]^{-1})^* M \ell(x) \|^2 dx \right)^{1/2} \\
= &\sup_{\|\phi\|_{L^2(B)} \leq 1} \left\langle \ell, (I - [I + r_B^2 L]^{-1})^* M \frac{\phi}{|B|^{1/2} \rho(|B|)} \right\rangle \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*},
\end{align*}
\]

Thus, \(\ell \in \text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)\) and \(\|\ell\|_{\text{BMO}^M_{\rho,L^*}(\mathbb{R}^n)} \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}\), which completes the proof of Theorem 4.1. \(\square\)

**Remark 4.2.** It follows from Theorem 4.1 that the spaces \(\text{BMO}^M_{\rho,L}(\mathbb{R}^n)\) for all \(M > \frac{\omega}{2} (\frac{1}{p_\omega} - \frac{1}{2})\) coincide with equivalent norms. Thus, in what follows, we denote \(\text{BMO}^M_{\rho,L}(\mathbb{R}^n)\) simply by \(\text{BMO}_{\rho,L}(\mathbb{R}^n)\).

## 5 Several equivalent characterizations of \(H_{\omega,L}(\mathbb{R}^n)\)

In this section, we establish several equivalent characterizations of the Orlicz-Hardy spaces. Let us begin with some notions.
Definition 5.1. Let \( q \in (p_L, \bar{p}_L) \), \( \omega \) satisfy Assumption (A), \( M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2}) \) and \( \epsilon > n(1/p_\omega - 1/p^+_\omega) \). A distribution \( f \in (\text{BMO}_{p,L}^*(\mathbb{R}^n))^* \) is said to be in the space \( H_{\omega,m}^{q,M,\epsilon}(\mathbb{R}^n) \) if there exist \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) and \( (\omega,q,M,\epsilon) \)-molecules \( \{\alpha_j\}_{j=1}^\infty \) such that

\[
\sum_{j=1}^\infty \lambda_j |\alpha_j| \leq 1
\]

in \( (\text{BMO}_{p,L}^*(\mathbb{R}^n))^* \) and

\[
\Lambda(\{\lambda_j\alpha_j\}_j) = \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty \frac{|\lambda_j|}{\lambda_j |B_j|} \leq 1 \right\} < \infty,
\]

where for each \( j \), \( \alpha_j \) is adapted to the ball \( B_j \).

If \( f \in H_{\omega,m}^{q,M,\epsilon}(\mathbb{R}^n) \), then define its norm by

\[
\|f\|_{H_{\omega,m}^{q,M,\epsilon}(\mathbb{R}^n)} \equiv \inf \Lambda(\{\lambda_j\alpha_j\}_j),
\]

where the infimum is taken over all possible decompositions of \( f \) as above.

For any \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), define the Lusin-area function associated to the Poisson semigroup as follows,

\[
S_P f(x) \equiv \left( \int_{\Gamma(x)} t |\nabla e^{-t \sqrt{L}} f(y)|^2 \frac{dy dt}{t} \right)^{1/2}.
\] (5.1)

Let \( \beta \in (0, \infty) \). Following [22], we define nontangential the maximal operators by setting, for all \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
N^\beta_h g(x) \equiv \sup_{(y,t) \in \Gamma^\beta(x)} \left( \frac{1}{\beta t^n} \int_{B(y,\beta t)} |e^{-t \sqrt{L}} g(z)|^2 dz \right)^{1/2},
\] (5.2)

and

\[
N^\beta_P g(x) \equiv \sup_{(y,t) \in \Gamma^\beta(x)} \left( \frac{1}{\beta t^n} \int_{B(y,\beta t)} |e^{-t \sqrt{L}} g(z)|^2 dz \right)^{1/2}.
\] (5.3)

In what follows, we denote \( N^\beta_h \) and \( N^\beta_P \) simply by \( N_h \) and \( N_P \).

We also define the radial maximal functions by setting, for all \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
R_h f(x) \equiv \sup_{t>0} \left( \frac{1}{t^n} \int_{B(x,t)} |e^{-t \sqrt{L}} f(y)|^2 dy \right)^{1/2},
\] (5.4)

and

\[
R_P f(x) \equiv \sup_{t>0} \left( \frac{1}{t^n} \int_{B(x,t)} |e^{-t \sqrt{L}} f(y)|^2 dy \right)^{1/2}.
\] (5.5)

Similarly to Definition 4.1, we define the space \( H_{\omega,S_P}(\mathbb{R}^n) \) as follows.
**Definition 5.2.** Let $\omega$ satisfy Assumption (A). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\tilde{H}_{\omega, S_p}(\mathbb{R}^n)$ if $S_p(f) \in L(\omega)$; moreover, define

$$\|f\|_{\tilde{H}_{\omega, S_p}(\mathbb{R}^n)} \equiv \|S_p(f)\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{S_p(f)(x)}{\lambda} \right) dx \leq 1 \right\}.$$  

The Orlicz-Hardy space $H_{\omega, S_p}(\mathbb{R}^n)$ is defined to be the completion of $\tilde{H}_{\omega, S_p}(\mathbb{R}^n)$ in the norm $\| \cdot \|_{H_{\omega, S_p}(\mathbb{R}^n)}$.

The spaces $H_{\omega, N_p}(\mathbb{R}^n)$, $H_{\omega, N_q}(\mathbb{R}^n)$, $H_{\omega, R_a}(\mathbb{R}^n)$ and $H_{\omega, R_p}(\mathbb{R}^n)$ are defined in a similar way. We now show that all the spaces $H_{\omega, L}(\mathbb{R}^n)$, $H_{\omega, S_p}(\mathbb{R}^n)$, $H_{\omega, N_p}(\mathbb{R}^n)$, $H_{\omega, R_a}(\mathbb{R}^n)$, $H_{\omega, N_q}(\mathbb{R}^n)$, $H_{\omega, R_p}(\mathbb{R}^n)$ coincide with equivalent norms.

### 5.1 The molecular characterization

In this subsection, we establish the molecular characterization of the Orlicz-Hardy spaces, which gives some understanding of the “distributions” in $H_{\omega, L}(\mathbb{R}^n)$ as elements of the dual of $\text{BMO}_{\rho, L^*}(\mathbb{R}^n)$. We start with the following auxiliary result.

**Proposition 5.1.** Let $\omega$ satisfy Assumption (A). Fix $t \in (0, \infty)$ and $\tilde{B} \equiv B(x_0, R)$ for some $x_0 \in \mathbb{R}^n$ and $R > 0$. Then there exists a positive constant $C(t, R)$ such that for all $\phi \in L^2(\mathbb{R}^n)$ supported in $\tilde{B}$, $t^2 Le^{-t^2L} \phi \in \text{BMO}_{\rho, L}(\mathbb{R}^n)$ and

$$\|t^2 Le^{-t^2L} \phi\|_{\text{BMO}_{\rho, L}(\mathbb{R}^n)} \leq C(t, R) \|\phi\|_{L^2(\tilde{B})}.$$  

**Proof.** Let $M > \frac{n}{2}(\frac{1}{p_0} - \frac{1}{2})$. For any ball $B \equiv B(x_B, r_B)$, let

$$H \equiv \frac{1}{\rho(|B|)} \left( \frac{1}{|B|} \int_B |(I - e^{-r_B^2L})^M t^2 Le^{-t^2L} \phi(x)|^2 dx \right)^{1/2}.$$  

For the case when $r_B \geq R$, from the $L^2(\mathbb{R}^n)$-boundedness of the operators $e^{-r_B^2L}$ and $t^2 Le^{-t^2L}$ (Lemma 2.2), it follows that

$$H \lesssim \frac{1}{|B|^{1/2} \rho(|B|)} \|\phi\|_{L^2(\tilde{B})}.$$  

Let us consider the case when $r_B < R$. It follows from the upper type property that

$$|\tilde{B}|^{1/2} \rho(|\tilde{B}|) \lesssim \left( \frac{R}{r_B} \right)^{n(1/p_0 - 1/2)} \|\phi\|_{L^2(\tilde{B})}.$$  

On the other hand, noticing that $I - e^{-r_B^2L} = \int_0^{r_B^2} Le^{-rL} dr$, thus by the Minkowski inequality and the $L^2(\mathbb{R}^n)$-boundedness of $t^2 Le^{-t^2L}$ (Lemma 2.2), we have

$$\left( \int_B |(I - e^{-r_B^2L})^M t^2 Le^{-t^2L} \phi(x)|^2 dx \right)^{1/2} \lesssim \|\phi\|_{L^2(\tilde{B})}.$$
By Corollary 4.1, for all $n$

$$
\| t^2 L \|_{L^2(\mathbb{B})} \| \phi \|_{L^2(\mathbb{B})} \| \phi \|_{L^2(\mathbb{B})} \leq \left( \frac{r_B}{t} \right)^2 \| \phi \|_{L^2(\mathbb{B})}. \tag{5.7}
$$

By the fact that $M > \frac{n}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right)$ and the estimates (5.6) and (5.7), we obtain

$$
H \lesssim \left( \frac{R}{t} \right)^{2M} \frac{\| \phi \|_{L^2(\mathbb{B})}}{|\mathbb{B}|^{1/2} \rho(\| \mathbb{B} \|)}.
$$

Thus $\| t^2 L e^{-t^2 L} \|_{\text{BMO}_{p_L}(\mathbb{R}^n)} \leq C(t, R) \| \phi \|_{L^2(\mathbb{B})}$, which completes the proof of Proposition 5.1.

\[\square\]

**Theorem 5.1.** Let $q \in (p_L, \frac{n}{p_\omega} - \frac{n}{2}]$, $\omega$ satisfy Assumption $(A)$, $M > \frac{n}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right)$ and $\epsilon > n(1/p_\omega - 1/p_\omega^+)$. Then the spaces $H_{\omega, L}(\mathbb{R}^n)$ and $H_{\omega, L}^q(\mathbb{R}^n)$ coincide with equivalent norms.

**Proof.** By Corollary 4.1, for all $f \in H_{\omega, L}(\mathbb{R}^n)$, there exist $(\omega, q, M, \epsilon)$-molecules $\{\alpha_j\}_{j=1}^\infty$ adapted to balls $B_j$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $H_{\omega, L}(\mathbb{R}^n)$ and $\Lambda(\{\lambda_j \alpha_j\}_{j=1}^\infty) \leq \| f \|_{H_{\omega, L}(\mathbb{R}^n)}$. Then Theorem 4.1 implies that the decomposition also holds in $(\text{BMO}_{p_L}^*(\mathbb{R}^n))^*$, and hence, $H_{\omega, L}(\mathbb{R}^n) \subset H_{\omega, L}^q(\mathbb{R}^n)$.

Conversely, let $f \in H_{\omega, L}^q(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\omega, q, M, \epsilon)$-molecules $\{\alpha_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\text{BMO}_{p_L}^*(\mathbb{R}^n))^*$ and $\Lambda(\{\lambda_j \alpha_j\}_{j=1}^\infty) < \infty$, where for each $j$, $\alpha_j$ is adapted to the ball $B_j$.

For all $x \in \mathbb{R}^n$, by Proposition 5.1, we have

$$
S_L f(x) = \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{dt}{t^{n+1}} \right)^{1/2} \right\}^{1/2} \leq \sum_{j=1}^\infty S_L(\lambda_j \alpha_j)(x).
$$

Then from (4.3) together with the continuity and the subadditivity of $\omega$, it follows that

$$
\int_{\mathbb{R}^n} \omega(S_L f(x)) \, dx \leq \sum_{j=1}^\infty \int_{\mathbb{R}^n} \omega(S_L(\lambda_j \alpha_j)(x)) \, dx \leq \sum_{j=1}^\infty \int_{\mathbb{R}^n} |B_j| \omega \left( \frac{|\lambda_j|}{|B_j| \rho(\| B_j \|)} \right).
$$
which implies that \( \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j \alpha_j\}_j) \). By taking the infimum over all decompositions of \( f \) as above, we obtain that \( \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\omega,M}^{q,q}(\mathbb{R}^n)} \), which completes the proof of Theorem 5.1.

\[ \square \]

5.2 Characterizations by the maximal functions

In this subsection, we characterize the Orlicz-Hardy space via the Lusin-area function \( \mathcal{S}_p \) and the maximal functions \( \mathcal{N}_h, \mathcal{N}_P, \mathcal{R}_h \) and \( \mathcal{R}_P \). Let us begin with the following very useful auxiliary result on the boundedness of linear or non-negative sublinear operators from \( H_{\omega,L}(\mathbb{R}^n) \) to \( L(\omega) \).

**Lemma 5.1.** Let \( q \in (p_L, 2] \), \( \omega \) satisfy Assumption (A), \( M > \frac{n}{2}\left(\frac{1}{p_\omega} - \frac{1}{2}\right) \) and \( \epsilon > n(1/p_\omega - 1/p_\omega^+) \). Suppose that \( T \) is a non-negative sublinear (resp. linear) operator which maps \( L^q(\mathbb{R}^n) \) continuously into weak-\( L^q(\mathbb{R}^n) \). If there exists a positive constant \( C \) such that for all \( (\omega, \infty, M, \epsilon) \)-molecules \( \alpha \) adapted to balls \( B \) and \( \lambda \in \mathbb{C} \),

\[
\int_{\mathbb{R}^n} \omega(T(\lambda\alpha)(x)) \, dx \leq C |B| \omega \left( \frac{|\lambda|}{|B| p(|B|)} \right),
\]

(5.8)

then \( T \) extends to a bounded sublinear (resp. linear) operator from \( H_{\omega,L}(\mathbb{R}^n) \) to \( L(\omega) \); moreover, there exists a positive constant \( \hat{C} \) such that for all \( f \in H_{\omega,L}(\mathbb{R}^n) \), \( \|Tf\|_{L(\omega)} \leq \hat{C} \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \).

**Proof.** It follows from Proposition 4.2 that for every \( f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( f \in L^q(\mathbb{R}^n) \) with \( q \in (p_L, 2] \) and there exists \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) and \( (\omega, \infty, M, \epsilon) \)-molecules \( \{\alpha_j\}_{j=1}^\infty \) such that \( f = \sum_{j=1}^\infty \lambda_j \alpha_j \) in both \( H_{\omega,L}(\mathbb{R}^n) \) and \( L^q(\mathbb{R}^n) \); moreover, \( \Lambda(\{\lambda_j \alpha_j\}_j) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \).

Thus if \( T \) is linear, then it follows from the fact that \( T \) is of weak type \((q, q)\) that \( T(f) = \sum_{j=1}^\infty T(\lambda_j \alpha_j) \) almost everywhere.

If \( T \) is a non-negative sublinear operator, then

\[
\sup_{t>0} t^{1/q} \left\{ x \in \mathbb{R}^n : \left| T(f)(x) - T \left( \sum_{j=1}^N \lambda_j \alpha_j \right)(x) \right| > t \right\} \lesssim \left\| f - \sum_{j=1}^N \lambda_j \alpha_j \right\|_{L^q(\mathbb{R}^n)} \to 0,
\]

as \( N \to \infty \). Thus there exists a subsequence \( \{N_k\}_k \subset \mathbb{N} \) such that

\[
T \left( \sum_{j=1}^{N_k} \lambda_j \alpha_j \right) \to T(f)
\]

almost everywhere, as \( k \to \infty \), which together with the non-negativity and the sublinearity of \( T \) further implies that

\[
T(f) - \sum_{j=1}^\infty T(\lambda_j \alpha_j) = T(f) - T \left( \sum_{j=1}^{N_k} \lambda_j \alpha_j \right) + T \left( \sum_{j=1}^{N_k} \lambda_j \alpha_j \right) - \sum_{j=1}^\infty T(\lambda_j \alpha_j)
\]
Moreover, since \( H \) implies that \( H \equiv \), which implies that \( H \equiv \) almost everywhere. Thus by the subadditivity and the continuity of \( \omega \) and (5.8), we finally obtain

\[
\int_{\mathbb{R}^n} \omega(T(f)(x)) \, dx \lesssim \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \omega(T(\lambda_j \alpha_j)(x)) \, dx \lesssim \sum_{j=1}^{\infty} |B_j| \omega \left( \frac{|\lambda_j|}{|B_j| \rho(|B_j|)} \right),
\]

which implies that \( \|T(f)\|_{L(\omega)} \lesssim \Lambda(\{\lambda_j \alpha_j\}_j) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \). This, combined with the density of \( H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) in \( H_{\omega,L}(\mathbb{R}^n) \), then finishes the proof of Lemma 5.1.

**Remark 5.1.** Let \( p \in (0,1] \). We point out that the condition (5.8) is also necessary, if \( \omega(t) \equiv t^p \) for all \( t \in (0,\infty) \). However, for a general \( \omega \) as in Lemma 5.1, it is still unclear whether (5.8) is necessary or not.

**Theorem 5.2.** Let \( \omega \) satisfy Assumption (A). Then the spaces \( H_{\omega,L}(\mathbb{R}^n) \), \( H_{\omega,S_p}(\mathbb{R}^n) \), \( H_{\omega,N_0}(\mathbb{R}^n) \) and \( H_{\omega,N_p}(\mathbb{R}^n) \) coincide with equivalent norms.

Before we prove Theorem 5.2, we recall some auxiliary operators introduced in [22]. Let \( \beta \in (0,\infty) \). For any \( g \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), let

\[
\tilde{S}_p^\beta g(x) \equiv \left( \int_{\Gamma_\beta(x)} |t^2 \rho e^{-t \rho^{1/\nu}} g(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}
\]

and

\[
\tilde{S}_h^\beta g(x) \equiv \left( \int_{\Gamma_\beta(x)} |t \rho e^{-t \rho^{1/\nu}} g(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}.
\]

We denote \( \tilde{S}_p^1 g \) and \( \tilde{S}_h^1 g \) simply by \( \tilde{S}_p g \) and \( \tilde{S}_h g \), respectively.

The proof of the following lemma is similar to that of [22, Lemma 5.4]. We omit the details.

**Lemma 5.2.** There exists a positive constant \( C \) such that for all \( g \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
\tilde{S}_p g(x) \leq C S_p g(x)
\]

and \( S_{Lg}(x) \leq C \tilde{S}_h g(x) \).

**Equivalence of** \( H_{\omega,L}(\mathbb{R}^n) \) **and** \( H_{\omega,S_p}(\mathbb{R}^n) \). Let \( \epsilon > n \left( \frac{1}{\nu_\omega} - \frac{1}{p_\omega} \right) \) and \( M > \frac{n}{\epsilon} \left( \frac{1}{\nu_\omega} - \frac{1}{p_\omega} \right) \). Suppose that \( f \in H_{\omega,S_p}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). It follows from (5.9) that \( \|\tilde{S}_p f\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,S_p}(\mathbb{R}^n)} \). Moreover, since \( S_p \) is bounded on \( L^2(\mathbb{R}^n) \) (see (5.15) in [22]), by (5.9), we have

\[
\|\tilde{S}_p f\|_{L^2(\mathbb{R}^n)} \lesssim \|S_p f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.
\]
Thus we obtain $t^2L e^{-t\sqrt{L}}f \in (T_\omega(\mathbb{R}^{n+1}) \cap T_2^2(\mathbb{R}^{n+1}))$. Let $\tilde{C}$ be a positive constant such that $\tilde{C} \int_0^\infty t^{2(s+1)}e^{-t^2}t^{\frac{d}{4}}e^{-t} \frac{dt}{t} = 1$. Then by the $L^2(\mathbb{R}^n)$-functional calculi, we have

$$f = \frac{\tilde{C}}{C_M}(t^2L e^{-t\sqrt{L}}f)$$

in $L^2(\mathbb{R}^n)$, where $C_M$ is the same as in (1.4).

Since $t^2L e^{-t\sqrt{L}}f \in T_\omega(\mathbb{R}^{n+1})$, by Proposition 4.1, we obtain that $f \in H_{\omega,L}(\mathbb{R}^n)$ and

$$\|f\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \|t^2L e^{-t\sqrt{L}}f\|_{T_\omega(\mathbb{R}^{n+1})} \sim \|S_Pf\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,S_P}(\mathbb{R}^n)}.$$  

Then a density argument yields that $H_{\omega,S_P}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n)$.

Conversely, similarly to the proof of (4.3), by using the estimates in the proof of [22, Theorem 5.3], we have

$$\int_{\mathbb{R}^n} \omega(\lambda|S_P(\alpha)(x)|) \, dx \lesssim |B|\omega\left(\frac{|\lambda|}{|B|\rho(|B|)}\right),$$

where $\alpha$ is an $(\omega,2,M,\epsilon)$-molecule adapted to the ball $B$ and $\lambda \in \mathbb{C}$. By the $L^2(\mathbb{R}^n)$-boundedness of $S_P$ and Lemma 5.1, we have $\|f\|_{H_{\omega,S_P}(\mathbb{R}^n)} = \|S_Pf\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}$, which implies that $H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega,S_P}(\mathbb{R}^n)$. Thus, $H_{\omega,L}(\mathbb{R}^n)$ and $H_{\omega,S_P}(\mathbb{R}^n)$ coincide with equivalent norms.

In what follows, the operators $N^\beta_h$ and $N^\beta_P$ are as in (5.2) and (5.3), respectively.

**Lemma 5.3.** Let $0 < \beta < \gamma < \infty$. Then there exists a positive constant $C$, depending on $\beta$ and $\gamma$, such that for all $g \in L^2(\mathbb{R}^n)$,

$$C^{-1}\|N^\beta_h g\|_{L(\omega)} \leq \|N^\gamma_h g\|_{L(\omega)} \leq C\|N^\beta_h g\|_{L(\omega)} \quad (5.10)$$

and

$$C^{-1}\|N^\beta_P g\|_{L(\omega)} \leq \|N^\gamma_P g\|_{L(\omega)} \leq C\|N^\beta_P g\|_{L(\omega)}. \quad (5.11)$$

**Proof.** We only prove (5.10); the proof of (5.11) is similar.

Since $\beta < \gamma$, for any $x \in \mathbb{R}^n$, it is easy to see that $N^\beta_h g(x) \leq (\frac{\gamma}{\beta})^n N^\gamma_h g(x)$, which implies the first inequality.

To show the second inequality in (5.10), without loss of generality, we may assume that $\|N^\beta_h g\|_{L(\omega)} < \infty$. Let $\sigma \in (0, \infty),\n
$$E_\sigma \equiv \{x \in \mathbb{R}^n : N^\beta_h g(x) > \sigma\} \quad \text{and} \quad E^*_\sigma \equiv \left\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{E_\sigma})(x) > \left(\frac{\beta}{3\gamma}\right)^n\right\}. \quad (5.12)$$

Suppose that $x \notin E^*_\sigma$. Thus for any $(y,t) \in \Gamma_{2\gamma}(x)$, we have $B(y,\beta t) \notin E_\sigma$; otherwise,

$$\mathcal{M}(\chi_{E_\sigma})(x) > \frac{|B(y,\beta t)|}{|B(x,3\gamma t)|} = \left(\frac{\beta}{3\gamma}\right)^n,$$
which contradicts with $x \notin E^*_\sigma$. Thus there exists $z \in (B(y, \beta t) \cap (E^*_\sigma)^C)$, which further implies that
\[
\left(\frac{1}{(\beta t)^n} \int_{B(y, \beta t)} |e^{-t^2/2} g(u)|^2 du \right)^{1/2} \leq N^\sigma_{h} g(z) \leq \sigma. \tag{5.13}
\]
For every $(w, t) \in \Gamma(x)$, we cover the ball $B(w, \gamma t)$ by no more that $N(n, \beta, \gamma)$ balls
\[
\{B(y_i, \beta t)\}_{i=1}^{N(n, \beta, \gamma)}, 
\]
where $(y_i, t) \in \Gamma_2(x)$ and $N(n, \beta, \gamma)$ depends only on $n, \beta, \gamma$. Thus, by (5.13), we obtain
\[
\left(\frac{1}{(\gamma t)^n} \int_{B(w, \gamma t)} |e^{-t^2/2} g(z)|^2 dz \right)^{1/2} \leq \left(\frac{\beta}{\gamma}\right)^{n/2} \sum_{i=1}^{N(n, \beta, \gamma)} \left(\frac{1}{(\beta t)^n} \int_{B(y_i, \beta t)} |e^{-t^2/2} g(z)|^2 dz \right)^{1/2} \leq C(n, \beta, \gamma) \sigma,
\]
where $C(n, \beta, \gamma)$ is a positive constant depending on $n, \beta, \gamma$. From this, it follows that for all $\sigma > 0$, \{x \in \mathbb{R}^n : N^\sigma_{h} g(x) > C(n, \beta, \gamma) \sigma\} \subset E^*_\sigma$. This combined (3.2) yields that
\[
\int_{\mathbb{R}^n} \omega(N^\sigma_{h} g(x)) \, dx \sim \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{\omega(t)}{t} \, dt \, dx \sim \int_{0}^{\infty} \frac{\omega(t)}{t} \, \{x \in \mathbb{R}^n : N^\sigma_{h} g(x) > t\} \, dt \sim \int_{0}^{\infty} \frac{\omega(t)}{t} \, E^*_t \, dt \sim \int_{\mathbb{R}^n} \omega(N^\sigma_{h} g(x)) \, dx,
\]
which further implies that $\|N^\sigma_{h} g\|_{L(\omega)} \lesssim \|N^\sigma_{h} g\|_{L(\omega)}$, and hence, completes the proof of Lemma 5.3.

\[\square\]

**Equivalence of $H_{\omega,L}(\mathbb{R}^n)$ and $H_{\omega,N_h}(\mathbb{R}^n)$**. By (3.2) and Lemmas 5.2 and 3.2, we have
\[
\|S_L f\|_{L(\omega)} \lesssim \|\tilde{S}_h f\|_{L(\omega)} \lesssim \|\tilde{S}_h^{1/2} f\|_{L(\omega)}. \tag{5.14}
\]
Recall that $\sigma_g$ denote the distribution function of a function $g$. The estimate (6.36) of [22] says that for any $\lambda \in (0, \infty),
\[
\sigma_{\tilde{S}_h^{1/2} f}(\lambda) \leq \frac{1}{\lambda^2} \int_{0}^{\lambda} t \sigma_{N_h^\sigma f}(t) \, dt + \sigma_{N_h^\sigma f}(\lambda), \tag{5.15}
\]
where $\beta \in (0, \infty)$ is large enough.

Since $\omega$ is of upper type 1, by (5.14), (3.2), (5.15) and Lemma 5.3, we obtain that
\[
\int_{\mathbb{R}^n} \omega(S_L f(x)) \, dx \lesssim \int_{\mathbb{R}^n} \omega(\tilde{S}_h^{1/2} f(x)) \, dx \sim \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{\omega(u)}{u} \, du \, dx \sim \int_{0}^{\infty} \frac{\omega(u)}{u} \sigma_{\tilde{S}_h^{1/2} f}(u) \, du.
\]
of the molecule, we have

\[ j > \frac{10}{n^2} \left( \int_0^u \frac{t \sigma_{N^0_h f}(t) dt}{u^2} + \sigma_{N^0_h f}(u) \right) du \]

\[ \leq \int_0^u t \sigma_{N^0_h f}(t) dt \int_1^\infty \frac{\omega(t)}{u^2} dt + \int_{\mathbb{R}^n} \omega(N^0_h f(x)) dx \]

\[ \leq \int_{\mathbb{R}^n} \omega(N^0_h f(x)) dx \leq \int_{\mathbb{R}^n} \omega(N_h f(x)) dx, \]

which implies that \( \|f\|_{H_{\omega,N}(\mathbb{R}^n)} \leq \|f\|_{H_{\omega,N_h}(\mathbb{R}^n)} \), and hence, \( H_{\omega,N_h}(\mathbb{R}^n) \subset H_{\omega,N}(\mathbb{R}^n) \).

Conversely, let \( R_h \) be as in (5.4). For all \( g \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we also define

\[ R_h^M g(x) \equiv \sup_{t > 0} \left( \frac{1}{t^n} \int_{B(x,t)} |(t^2 L)^M e^{-t^2 L} g(y)|^2 dy \right)^{1/2}. \]

By the proof of [22, Theorem 6.3], we know that the operators \( R_h \) and \( R_h^M \) are bounded on \( L^2(\mathbb{R}^n) \).

Since Lemma 5.3 implies that for all \( f \in L^2(\mathbb{R}^n) \cap H_{\omega,L}(\mathbb{R}^n) \),

\[ \|N_h f\|_{L(\omega)} \lesssim \|N^0_h f\|_{L(\omega)} \lesssim \|R_h f\|_{L(\omega)}, \]

by Lemma 5.1 and a density argument, to show \( H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega,N_h}(\mathbb{R}^n) \), it suffices to prove that for all \( (\omega, 2, M, \epsilon) \)-molecules \( \alpha \) adapted to balls \( B \) and \( \lambda \in \mathbb{C} \),

\[ \int_{\mathbb{R}^n} \omega(R_h(\lambda\alpha)(x)) dx \lesssim |B| \omega \left( \frac{|\lambda|}{|B|^{1/2}} \right). \] (5.16)

Since \( \omega \) is concave, by the Jensen inequality and the Hölder inequality, we obtain

\[ \int_{\mathbb{R}^n} \omega(R_h(\lambda\alpha)(x)) dx \leq \sum_{j=0}^{\infty} \int_{U_j(B)} \omega(R_h(\lambda\alpha)(x)) dx \]

\[ \lesssim \sum_{j=0}^{\infty} |2^j| \omega \left( \frac{\|R_h(\lambda\alpha)\|_{L^2(U_j(B))}}{|2^j|^{1/2}} \right), \]

For \( j \in \mathbb{Z}_+ \) and \( j \leq 10 \), by the \( L^2(\mathbb{R}^n) \)-boundedness of the operator \( R_h \) and the definition of the molecule, we have

\[ \sum_{j=0}^{10} |2^j| \omega \left( \frac{\|R_h(\lambda\alpha)\|_{L^2(U_j(B))}}{|2^j|^{1/2}} \right) \leq |B| \omega \left( \frac{|\lambda|}{|B|^{1/2}} \right). \]

Since \( M > \frac{n}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right) \), we let \( a \in (0, 1) \) such that \( a(2M + n/2) > n/p_\omega \). For \( j \in \mathbb{N} \) and \( j > 10 \), write

\[ R_h(\lambda\alpha)(x) \leq \sup_{t \leq 2^{j-2} r_B} \left( \frac{1}{t^n} \int_{B(x,t)} |e^{-t^2 L}(\lambda\alpha)(y)|^2 dy \right)^{1/2} \]
For the case \( t \leq 2^{n-j-2}r_B \), let

\[
V_j(B) = 2^{j+3}B \setminus 2^{j-3}B, \quad R_j(B) = 2^{j+5}B \setminus 2^{j-5}B \quad \text{and} \quad E_j(B) = (R_j(B))^c. \tag{5.17}
\]

If \( x \in U_j(B) \) and \( |x - y| < t \), then we have \( y \in V_j(B) \) and \( (V_j(B), E_j(B)) \sim 2^j r_B \), which together with Lemma 2.3 yields that

\[
\|H_j\|_{L^2(U_j(B))} \leq \left\| \sup_{t \leq 2^{n-j-2}r_B} \left( \frac{1}{t^n} \int_{B(t)} |e^{-t^2L}(\lambda \alpha)(y)|^2 \, dy \right) \right\|_{L^2(U_j(B))}^{1/2} \leq \left\| \sup_{t \leq 2^{n-j-2}r_B} \left( \frac{1}{t^n} \int_{B(t)} |e^{-t^2L}(\lambda \alpha\chi_{R_j(B)})(y)|^2 \, dy \right) \right\|_{L^2(U_j(B))}^{1/2} + \left\| \sup_{t \leq 2^{n-j-2}r_B} \left( \frac{1}{t^n} \int_{B(t)} |e^{-t^2L}(\lambda \alpha\chi_{E_j(B)})(y)|^2 \, dy \right) \right\|_{L^2(U_j(B))}^{1/2}
\]

\[
\lesssim \|\mathcal{R}_h(\lambda \alpha\chi_{R_j(B)})\|_{L^2(\mathbb{R}^n)} + |U_j(B)|^{1/2} \sup_{t \leq 2^{n-j-2}r_B} t^{-n/2} e^{-\frac{j^2 r_B^2}{\alpha^2}} \|\lambda\alpha\|_{L^2(E_j(B))}
\]

\[
\lesssim \|\lambda\alpha\|_{L^2(R_j(B))} + |U_j(B)|^{1/2} \sup_{t \leq 2^{n-j-2}r_B} t^{-n/2} \left( \frac{t}{2^j r_B} \right)^N \|\lambda\alpha\|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim |\lambda|^{2^{-j} e[\rho([2^j B])]}^{-1} |2^j B|^{-1/2} + |\lambda|^{2^{j(1-\alpha)(n/2-N)} [\rho([B])]^{-1} |B|^{-1/2},
\]

where \( c \) is a positive constant as in Lemma 2.3 and \( N \in \mathbb{N} \) is large enough such that \((1-\alpha)(N-n/2)p_\omega > n(1-p_\omega/2)\). Then by an argument similar to the proof of (4.4) and the fact that \( \omega \) is of lower type \( p_\omega \), we have

\[
\sum_{j=1}^{\infty} |2^j B|^{\omega} \left( \frac{\|H_j\|_{L^2(U_j(B))}}{|2^j B|^{1/2}} \right) \leq \sum_{j=1}^{\infty} |2^j B|^{\omega} \left( \frac{|\lambda|^{2^{-j} e}}{|2^j B|^{\rho([2^j B])}} \right) + \sum_{j=1}^{\infty} |2^j B|^{\omega} \left( \frac{|\lambda|^{2^{j(1-\alpha)(n/2-N)}}}{|2^j B|^{1/2} \rho([B]) |B|^{1/2}} \right)
\]

\[
\lesssim |B|^{\omega} \left( \frac{|\lambda|}{|B|^{\rho([B])}} \right) + \sum_{j=1}^{\infty} |B|^{2^{j(n-1-p_\omega/2)} 2^{j(1-\alpha)(n/2-N)} p_\omega} \left( \frac{|\lambda|}{\rho([B]) |B|} \right)
\]

\[
\lesssim |B|^{\omega} \left( \frac{|\lambda|}{|B|^{\rho([B])}} \right), \quad (5.18)
\]

which is a desired estimate.

For the term \( I_j \), by the \( L^2(\mathbb{R}^n) \)-boundedness of the operator \( \mathcal{R}_h^M \), we have

\[
\|I_j\|_{L^2(U_j(B))} \lesssim \left\| \sup_{t > 2^{n-j-2}r_B} \left( \frac{1}{t^n} \int_{B(t)} \left( \frac{r_B}{t} \right)^M \left| (t^2L)^M e^{-t^2L(\lambda(r_B^2L)^{-M} \alpha)(y)} \right|^2 \, dy \right) \right\|_{L^2(U_j(B))}^{1/2}
\]
\[ \lesssim 2^{-2aMj} \| R_h^M (\lambda (r_B^L - M \alpha)) \|_{L^2(U_j(L))} \lesssim |\lambda| 2^{-2aMj} |\rho(|B|)|^{-1/2} |B|^{-1/2}, \]

which together with the fact that \( a_\omega (2M + n/2) > n \) implies that

\[
\sum_{j=1}^\infty 2^j |B| |\omega| \left( \frac{\| I_j \|_{L^2(U_j(B))}}{|2^j B|^{1/2}} \right) \lesssim \sum_{j=1}^\infty 2^j |B| |\omega| \left( \frac{|\lambda| 2^{-2aMj}}{|2^j B|^{1/2} \rho(|B|)|B|^{1/2}} \right) \\
\lesssim \sum_{j=1}^\infty 2^{jn(1-\rho/2)} 2^{-2aMj \rho} |B| |\omega| \left( \frac{|\lambda|}{|B| \rho(|B|)} \right)
\]

\[
\lesssim |B| |\omega| \left( \frac{|\lambda|}{|B| \rho(|B|)} \right),
\]

which is a desired estimate.

Combining the estimates (5.18) and (5.19) yields (5.16), and hence, completes the proof of that \( H(\omega, L^p) \subset H(\omega, \mathcal{N}_h) \). Therefore, \( H(\omega, L^p) \) and \( H(\omega, \mathcal{N}_h) \) coincide with equivalent norms.

**Equivalence of** \( H(\omega, L^p) \) **and** \( H(\omega, \mathcal{N}_h) \). The proof of the equivalence of \( H(\omega, L^p) \) and \( H(\omega, \mathcal{N}_h) \) is similar to that of the equivalence of \( H(\omega, L^p) \) and \( H(\omega, \mathcal{N}_h) \); we omit the details.

This finishes the proof of Theorem 5.2.

From Theorem 5.2, it is easy to deduce the following radial maximal function characterizations of \( H(\omega, L^p) \). Recall that \( \mathcal{R}_h \) and \( \mathcal{R}_P \) are defined in (5.4) and (5.5), respectively.

**Corollary 5.1.** Let \( \omega \) satisfy Assumption (A). Then the spaces \( H(\omega, L^p) \), \( H(\omega, \mathcal{R}_h) \) and \( H(\omega, \mathcal{R}_P) \) coincide with equivalent norms.

**Proof.** We only give the proof of the equivalence between \( H(\omega, \mathcal{R}_h) \) and \( H(\omega, L^p) \), since the proof of the equivalence between \( H(\omega, \mathcal{R}_P) \) and \( H(\omega, L^p) \) is similar.

For any \( f \in (H(\omega, L^p) \cap L^2(\mathbb{R}^n)) \), by (5.2) and (5.4), we obviously have \( \mathcal{R}_h f \leq \mathcal{N}_h f \), which implies that \( H(\omega, \mathcal{N}_h) \subset H(\omega, \mathcal{R}_h) \).

Conversely, since for all \( f \in L^2(\mathbb{R}^n) \), we have \( \mathcal{N}_h^{1/2} f \leq \mathcal{R}_h f \), where \( \mathcal{N}_h^{1/2} \) is as in (5.2). Then by Lemma 5.3, we obtain that for all \( f \in L^2(\mathbb{R}^n) \cap H(\omega, \mathcal{R}_h) \),

\[ \| \mathcal{N}_h f \|_{L(\omega)} \lesssim \| \mathcal{N}_h^{1/2} f \|_{L(\omega)} \lesssim \| \mathcal{R}_h f \|_{L(\omega)}, \]

which implies that \( H(\omega, \mathcal{R}_h) \subset H(\omega, \mathcal{N}_h) \), and hence, completes the proof of Corollary 5.1.

**6 The Carleson measure and the John-Nirenberg inequality**

In this section, we characterize the space \( \text{BMO}_{\rho, L^p}(\mathbb{R}^n) \) via the \( \rho \)-Carleson measure and establish the John-Nirenberg inequality for elements in \( \text{BMO}_{\rho, L^p}(\mathbb{R}^n) \), where \( L^p \) denotes the conjugate operator of \( L \) in \( L^2(\mathbb{R}^n) \).
Recall that a measure $d\mu$ on $\mathbb{R}^{n+1}_+$ is called a $\rho$-Carleson measure if
\[
\|d\mu\|_\rho \equiv \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \left( \frac{1}{\rho(|B|)} \right)^2 \int_B |d\mu| \right\}^{1/2} < \infty,
\]
where the supremum is taken over all balls $B$ of $\mathbb{R}^n$ and $\hat{B}$ denotes the tent over $B$; see [25].

**Theorem 6.1.** Let $\omega$ satisfy Assumption (A), $\rho$ be as in (2.4) and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$.

(i) If $f \in \text{BMO}_{\rho,L^*}(\mathbb{R}^n)$, then $d\mu_f$ is a $\rho$-Carleson measure and there exists a positive constant $C$ independent of $f$ such that $\|d\mu_f\|_\rho \leq C\|f\|_{\text{BMO}_{\rho,L^*}(\mathbb{R}^n)}^2$, where
\[
d\mu_f \equiv \left( (t^2 L^*)_M e^{-t^2 L^*} f(x) \right)^2 \frac{dx \, dt}{t}.
\]

(ii) Conversely, if $f \in \mathcal{M}_{q',L^*}(\mathbb{R}^n)$ satisfies (4.8) with certain $q \in (p_{L^*}, 2]$ and $\epsilon_1 > 0$, and $d\mu_f$ is a $\rho$-Carleson measure, then $f \in \text{BMO}_{\rho,L^*}(\mathbb{R}^n)$ and there exists a positive constant $C$ independent of $f$ such that $\|f\|_{\text{BMO}_{\rho,L^*}(\mathbb{R}^n)} \leq C\|d\mu_f\|_\rho$, where $d\mu_f$ is as in (6.1).

**Proof.** It follows from Lemma 4.2 that (i) holds.

To show (ii), let $\bar{M} > M + \epsilon_1 + \frac{n}{4}$ and $\epsilon > n(\frac{1}{p_\omega} - \frac{1}{p_2})$. By Lemma 4.3, we have
\[
\langle f, g \rangle = \bar{C}_M \int_{\mathbb{R}^{n+1}} (t^2 L^*)_M e^{-t^2 L^*} f(x) t^2 L e^{-t^2 L} g(x) \frac{dx \, dt}{t},
\]
where $g$ is a finite combination of $(\omega, q', \bar{M}, \epsilon)$-molecules and $q' = \frac{q}{q-1}$. Then by (4.10), we obtain that
\[
|\langle f, g \rangle| \lesssim \|d\mu_f\|_\rho \|g\|_{H_{\omega,L}(\mathbb{R}^n)}.
\]
Since $H_{q',L^*}^{\bar{M},\epsilon}$ is dense in $H_{\omega,L}(\mathbb{R}^n)$, we obtain that $f \in (H_{\omega,L}(\mathbb{R}^n))^*$, which combined with Theorem 4.1 implies that $f \in \text{BMO}_{\rho,L^*}(\mathbb{R}^n)$ and $\|f\|_{\text{BMO}_{\rho,L^*}(\mathbb{R}^n)} \lesssim \|d\mu_f\|_\rho$. This finishes the proof of Theorem 6.1. 

Recall that for every cube $Q$, $\ell(Q)$ denotes its side-length.

**Lemma 6.1.** Let $F \in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Suppose that there exist $\beta \in (0,1)$ and $N \in (0, \infty)$ such that for certain $a \in \left(\frac{n}{2N}, \infty\right)$ and all cubes $Q \subset \mathbb{R}^n$,
\[
\left\{ x \in Q : \left( \int_0^{\ell(Q)} \int_{B(x,3at)} |F(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2} > N\rho(|Q|) \right\} \leq \beta|Q|.
\]

Then
\[
\sup_{\text{cubes } Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \left( \int_0^{\ell(Q)} \int_{B(x,at)} |F(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{p/2} dx \leq \frac{2N^p}{1 - \beta}
\]
for all $p \in (1, \infty)$.
Proof. Let \(\Omega \equiv \{ x \in Q : (\int_0^{\ell(Q)} \int_{B(x,3at)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}})^{1/2} > N \rho(|Q|) \}. \) Applying the Whitney decomposition to \(\Omega\), we obtain a family \(\{Q_j\}_j\) of disjoint cubes such that \((\cup_j Q_j) = \Omega\) and dist \((Q_j, Q \setminus \Omega) \in (\sqrt{n} \ell(Q_j), 4\sqrt{n} \ell(Q_j))\); see the proof of [22, Lemma 10.1]. For \(\delta \in (0, \ell(Q))\), define

\[
M(\delta) \equiv \sup_{\text{cubes } Q \subset Q} \left( \frac{1}{|Q|} \int_Q \left( \int_{J_\delta} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx, \right.
\]

where \(B(x,a(t-\delta)) \equiv \emptyset\) if \(\delta \geq t\). Now, observe that

\[
\int_Q \left( \int_{J_\delta} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx
\]

\[
\leq \int_{Q \setminus \Omega} \left( \int_0^{\ell(Q)} \int_{B(x,3at)} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx
\]

\[
+ \sum_j \int_{Q_j} \left( \int_{J_\delta} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx
\]

\[
+ \sum_j \int_{Q_j} \left( \int_{\max\{\ell(Q_j),\delta\}}^{\ell(Q)} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx
\]

\[
\leq N^p|Q| + \beta|Q|M(\delta) + \sum_j \int_{Q_j} \left( \int_{\max\{\ell(Q_j),\delta\}}^{\ell(Q)} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx
\]

\[
\equiv N^p|Q| + \beta|Q|M(\delta) + I.
\]

Since dist \((Q_j, Q \setminus \Omega) \in (\sqrt{n} \ell(Q_j), 4\sqrt{n} \ell(Q_j))\), there exists \(\bar{x} \in (Q \setminus \Omega)\) such that for all \(x \in Q_j\),

\[
|x - \bar{x}| \leq |x - x_{Q_j}| + |x_{Q_j} - \bar{x}| \leq 5\sqrt{n} \ell(Q_j).
\]

Then by the fact that \(a \geq 5\sqrt{n}/2\), we obtain

\[
\{(y,t) : y \in B(x,a(t-\delta)), \max\{\ell(Q_j),\delta\} < t < \ell(Q)\} \subset \{(y,t) : y \in B(\bar{x},3at), t < \ell(Q)\},
\]

which implies that

\[
I \leq \sum_j \int_{Q_j} \sup_{\bar{x} \in Q \setminus \Omega} \left( \int_0^{\ell(Q)} \int_{B(\bar{x},3at)} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx \leq N^p|Q|.
\]

For every cube \(\bar{Q} \subset Q\), let \(\bar{\Omega} \equiv \{ x \in \bar{Q} : (\int_0^{\ell(Q)} \int_{B(x,3at)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}})^{1/2} > N \rho(|Q|) \}. \) Then

\[
|\bar{\Omega}| \leq \left\{ x \in \bar{Q} : \left( \int_0^{\ell(Q)} \int_{B(x,3at)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} > N \rho(\bar{Q}) \right\} \leq \beta|\bar{Q}|.
\]
Repeating the above estimates, we obtain
\[
\int_{Q} \left( \int_{0}^{\ell(Q)} \int_{B(x,a(t-\delta))} \left| \frac{F(y,t)}{\rho(|Q|)} \right|^{2} \frac{dy}{t^{n+1}} dt \right)^{p/2} dx \leq 2^{Np}|Q| + \beta |Q|M(\delta),
\]
which via taking the supremum on \( \bar{Q} \) implies that \((1 - \beta)M(\delta) \leq 2^{Np}\). Letting \( \delta \to 0 \), we finally obtain
\[
\frac{1}{|Q|\rho(|Q|)^p} \int_{Q} \left( \int_{0}^{\ell(Q)} \int_{B(x,a(t))} |F(y,t)|^{2} \frac{dy}{t^{n+1}} dt \right)^{p/2} dx \leq \lim_{\delta \to 0} M(\delta) \leq \frac{2^{Np}}{1 - \beta},
\]
which implies (6.2), and hence, completes the proof of Lemma 6.1.

\[\Box\]

**Theorem 6.2.** Let \( \omega \) satisfy Assumption (A), \( \rho \) be as in (2.4) and \( M \geq \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2}) \). Then the spaces \( \text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n}) \) for all \( q \in (p_{L^{*}}, p_{L^{*}}) \) coincide with equivalent norms.

**Proof.** It follows from the Hölder inequality that
\[
\|f\|_{\text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n})} \leq \|f\|_{\text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n})} \leq \|f\|_{\text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n})},
\]
where \( p_{L^{*}} < p < 2 < q < p_{L^{*}} \).

Let us now show that \( \|f\|_{\text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n})} \sim \|f\|_{\text{BMO}_{p,\ell}^{q,M}(\mathbb{R}^{n})} \) for all \( p \in (p_{L^{*}}, 2) \). Write
\[
\left\{ \int_{Q} \left[ \int_{0}^{\ell(Q)} \int_{B(x,9\sqrt{m}t)} |(t^{2}L^{*})^{M} e^{-t^{2}L^{*}} f(y)|^{2} \frac{dy}{t^{n+1}} dt \right]^{p/2} dx \right\}^{1/p} \leq \left\{ \int_{Q} \left[ \int_{0}^{\ell(Q)} \int_{B(x,9\sqrt{m}t)} |(t^{2}L^{*})^{M} e^{-t^{2}L^{*}} \times[I - (I + [9\sqrt{m}\ell(Q)]^{2}L^{*})^{-1}]^{M} f(y)|^{2} \frac{dy}{t^{n+1}} dt \right]^{p/2} dx \right\}^{1/p} + \left\{ \int_{Q} \left[ \int_{0}^{\ell(Q)} \int_{B(x,9\sqrt{m}t)} |(t^{2}L^{*})^{M} e^{-t^{2}L^{*}} \times[I - (I - (I + [9\sqrt{m}\ell(Q)]^{2}L^{*})^{-1})^{M}] f(y)|^{2} \frac{dy}{t^{n+1}} dt \right]^{p/2} dx \right\}^{1/p} \equiv H + I.
\]

Let \( 9\sqrt{m}Q \) denote the cube with the same center as \( Q \) and side-length \( 9\sqrt{m} \) times \( \ell(Q) \). Then by the Hölder inequality, Lemmas 2.1 and 2.3, we have
\[
H \leq \sum_{j=0}^{2} \left\{ \int_{Q} \left[ \int_{0}^{\ell(Q)} \int_{B(x,9\sqrt{m}t)} |(t^{2}L^{*})^{M} e^{-t^{2}L^{*}}
\right. \right. \]
\[
\times [\chi_{U_j(9\sqrt{n}Q)}(I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f](y)\right) \left[ \frac{dy \, dt}{t^{n+1}} \right]^{p/2} \, dx \right]^{1/p} + \sum_{j=3}^{\infty} \cdots \\
\equiv H_1 + H_2.
\]

By Lemma 2.4 and Proposition 4 in [12], we obtain
\[
H_1 \leq \sum_{j=0}^{2} \left\| A_{9\sqrt{n}} \left\{ (t^2L^*)^M e^{-t^2L^*} [\chi_{U_j(9\sqrt{n}Q)}(I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f] \right\} \right\|_{L^p(\mathbb{R}^n)}
\leq \sum_{j=0}^{2} \left\| A \left\{ (t^2L^*)^M e^{-t^2L^*} [\chi_{U_j(9\sqrt{n}Q)}(I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f] \right\} \right\|_{L^p(\mathbb{R}^n)}
\leq \sum_{j=0}^{2} \left\| (I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f \right\|_{L^p(\chi_{U_j(9\sqrt{n}Q)})}
\leq \rho(|Q|)|Q|^{1/p} \| f \|_{\text{BMO}^{p,M}_{\rho,L^*}(\mathbb{R}^n)}.
\]

For the term $H_2$, noticing that $U_j(9\sqrt{n}Q)$ can be covered by $2^j$ cubes of side-length $9\sqrt{n}\ell(Q)$, which together with Lemma 2.3 and the Hölder inequality implies that
\[
H_2 \leq \sum_{j=3}^{\infty} |Q|^{1/p-1/2} \left\{ \int_{0}^{\ell(Q)} \int_{10\sqrt{n}Q} (t^2L^*)^M e^{-t^2L^*} \times [\chi_{U_j(9\sqrt{n}Q)}(I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f](x)\right\} \left( \frac{dx \, dt}{t} \right)^{1/2}
\leq |Q|^{1/p-1/2} \sum_{j=3}^{\infty} \left\{ \int_{0}^{\ell(Q)} e^{-\left(\frac{2j\sqrt{n}\ell(Q)}{dt}\right)^2} \times \left\| (I - (I + [9\sqrt{n}\ell(Q)]^2L^*)^{-1})^M f \right\|_{L^p(U_j(9\sqrt{n}Q))} \frac{dt}{t^{1+n/p-n/2}} \right\}^{1/2}
\leq \rho(|Q|)|Q|^{1/p} \| f \|_{\text{BMO}^{p,M}_{\rho,L^*}(\mathbb{R}^n)}
\times \left\{ \sum_{j=3}^{\infty} \left[ \int_{0}^{\ell(Q)} \left( \frac{t}{2j\ell(Q)} \right)^N 2^{2jn} \frac{dt}{t^{1+n/p-n/2}} \right]^{1/2} \right\}
\leq \rho(|Q|)|Q|^{1/p} \| f \|_{\text{BMO}^{p,M}_{\rho,L^*}(\mathbb{R}^n)},
\]
where $c$ is the positive constant as in Lemma 2.3 and $N \in \mathbb{N}$ is large enough. Thus,
\[
H \leq H_1 + H_2 \leq \rho(|Q|)|Q|^{1/p} \| f \|_{\text{BMO}^{p,M}_{\rho,L^*}(\mathbb{R}^n)}.
\]

Applying the formula that
\[
\left[ I - (I - (I + (9\sqrt{n}\ell(Q))^2L^*)^{-1})^M \right] f
\]
where \( C_M^k \) denotes the combinatorial number, by a way similar to the estimate of \( H \), we also have \( I \lesssim \rho(|Q|)|Q|^{1/p} \| f \|_{\text{BMO}^{p,M}_{p,L^*}(\mathbb{R}^n)} \).

Combing the estimates of \( H \) and \( I \) yields that
\[
\sup_{B \subset \mathbb{R}^n} \frac{1}{\rho(|B|)} \left[ \frac{1}{|B|} \int_B \left| \int_Q |(t^2 L^*)^M e^{-t^2 L^*} f(x)|^2 \frac{dy}{t} \right|^2 \frac{dx}{t} \right]^{1/2} \\
\leq \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \rho(|Q|)^2 \int_Q \int_0^{\ell(Q)} |(t^2 L^*)^M e^{-t^2 L^*} f(x)|^2 \frac{dy}{t} \frac{dx}{t} \right)^{1/2} \\
\leq \| f \|_{\text{BMO}^{p,M}_{p,L^*}(\mathbb{R}^n)}.
\]

Then by Theorem 6.1, we obtain \( \| f \|_{\text{BMO}^{2,M}_{p,L^*}(\mathbb{R}^n)} \lesssim \| f \|_{\text{BMO}^{p,M}_{p,L^*}(\mathbb{R}^n)} \).

Finally, let us show that \( \| f \|_{\text{BMO}^{p,M}_{p,L^*}(\mathbb{R}^n)} \lesssim \| f \|_{\text{BMO}^{2,M}_{p,L^*}(\mathbb{R}^n)} \) for all \( q \in (2, \tilde{p}_L^*) \). Let \( q' \) be the conjugate index \( q \). For any ball \( B \), let \( h \in L^2(B) \subset L^{q'}(B) \) such that \( \| h \|_{L^{q'}(B)} \leq \frac{1}{|B|^{1/q' - 1/p} \rho(|B|)} \). From Lemma 2.3, similarly to the proof of Theorem 4.1, it is easy to follow that \( (I - e^{-t^2 L})^M h \) is a multiple of an \((\omega, q', M, \epsilon)\)-molecule, and hence,
\[
\|(I - e^{-t^2 L})^M h\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim 1.
\]

Now let \( f \in \text{BMO}^{2,M}_{p,L^*}(\mathbb{R}^n) \). By Theorem 4.1, \( f \in (H_{\omega,L}(\mathbb{R}^n))^* \), and hence,
\[
\|\langle (I - e^{-t^2 L})^M f, h \rangle\| = \|\langle f, (I - e^{-t^2 L})^M h \rangle\| \lesssim \| f \|_{\text{BMO}^{2,M}_{p,L^*}(\mathbb{R}^n)}.
\]

Taking supremum over all such \( h \) yields that \( \| f \|_{\text{BMO}^{p,M}_{p,L^*}(\mathbb{R}^n)} \lesssim \| f \|_{\text{BMO}^{2,M}_{p,L^*}(\mathbb{R}^n)} \), which completes the proof of Theorem 6.2.

7 Some applications

In this section, we establish the boundedness on Orlicz-Hardy spaces of the Riesz transform and the fractional integral associated with the operator \( L \) as in (1.2).
Recall that the Littlewood-Paley g-function $g_L$ is defined by setting, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$g_L f(x) \equiv \left( \int_0^\infty |t^2 Le^{-t^2 L} f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$ 

By the proof of Theorem 3.4 in [22], we know that $g_L$ is bounded on $L^2(\mathbb{R}^n)$.

Similarly to Theorems 3.2 and 3.4 in [22], we have the following conclusion.

**Theorem 7.1.** Let $\omega$ satisfy Assumption (A) and $p \in (p_L, 2]$. Suppose that the non-negative sublinear operator or linear operator $T$ is bounded on $L^p(\mathbb{R}^n)$ and there exist $C > 0$, $M \in \mathbb{N}$ and $M > \frac{9}{4} \left( \frac{1}{p_L} - \frac{1}{2} \right)$ such that for all closed sets $E, F$ in $\mathbb{R}^n$ with $\dist(E, F) > 0$ and all $f \in L^p(\mathbb{R}^n)$ supported in $E$,

$$\|T(I - e^{-tL})^M f\|_{L^p(F)} \leq C \left( \frac{t}{\dist(E, F)^2} \right)^M \|f\|_{L^p(E)} \quad (7.1)$$

and

$$\|T(tLe^{-tL})^M f\|_{L^p(F)} \leq C \left( \frac{t}{\dist(E, F)^2} \right)^M \|f\|_{L^p(E)} \quad (7.2)$$

for all $t > 0$. Then $T$ extends to a bounded sublinear or linear operator from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$. In particular, the Riesz transform $\nabla L^{-1/2}$ and the Littlewood-Paley $g$-function $g_L$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$.

**Proof.** Let $\epsilon > n(1/p_\omega - 1/\tilde{p}_\omega)$, where $\tilde{p}_\omega$ is as in Convention (C). Since $T$ is bounded on $L^p(\mathbb{R}^n)$, by Lemma 5.1, to show that $T$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$, it suffices to show that for all $\lambda \in \mathbb{C}$ and $(\omega, \infty, M, \epsilon)$-molecules $\alpha$ adapted to balls $B$,

$$\int_{\mathbb{R}^n} \omega |T(\lambda \alpha)(x)| \, dx \lesssim |B| \omega \left( \frac{|\lambda|}{|B| \rho(|B|)} \right). \quad (7.3)$$

To prove (7.3), we write

$$\int_{\mathbb{R}^n} \omega(T(\lambda \alpha)(x)) \, dx$$

$$\leq \int_{\mathbb{R}^n} \omega(|\lambda|T([I - e^{-r(L)}]^{M} \alpha)(x)) \, dx + \int_{\mathbb{R}^n} \omega(|\lambda|T((I - [I - e^{-r(L)}]^{M}) \alpha)(x)) \, dx$$

$$\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \omega(|\lambda|T([I - e^{-r(L)}]^M \alpha \chi_{U_j(B)}) \alpha)(x)) \, dx$$

$$+ \sum_{j=0}^{\infty} \sup_{1 \leq k \leq M} \int_{\mathbb{R}^n} \omega \left( |\lambda| \left\{ \frac{k}{M} T^2 Le^{-\frac{k}{M} r(L)} \right\}^{M} \chi_{U_j(B)}(r(L) - 1)^{M} \alpha \right) \, dx$$

$$\equiv \sum_{j=0}^{\infty} H_j + \sum_{j=0}^{\infty} I_j.$$
For each $j \geq 0$, let $B_j \equiv 2^j B$. Since $\omega$ is concave, by the Jensen inequality and the Hölder inequality, we obtain
\[
H_j \lesssim \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(|\lambda| T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))(x)) \, dx \\
\lesssim \sum_{k=0}^{\infty} \int_{2^k B_j} \omega(|\lambda| \chi_{U_k(B_j)}(x) T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))(x)) \, dx \\
\lesssim \sum_{k=0}^{\infty} |2^k B_j| \omega \left( \frac{|\lambda|}{|2^k B_j|} \int_{U_k(B_j)} T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))(x) \, dx \right) \\
\lesssim \sum_{k=0}^{\infty} |2^k B_j| \omega \left( \frac{|\lambda|}{|2^k B_j|^{1/p} |B_j|^{1/p'}} \|T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))\|_{L^p(U_k(B_j))} \right).
\]
By the $L^p(\mathbb{R}^n)$-boundedness of $T$, Lemma 2.3 and (7.1), we have that for $k = 0, 1, 2$,
\[
\|T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))\|_{L^p(U_k(B_j))} \lesssim \|\alpha\|_{L^p(U_j(B))},
\]
and that for $k \geq 3$,
\[
\|T([I - e^{-r^2_h} L]^M(\alpha \chi_{U_j(B)}))\|_{L^p(U_k(B_j))} \lesssim \left( \frac{1}{2k+j} \right)^{2M} \|\alpha\|_{L^p(U_j(B))},
\]
which, together with Definition 4.2, $2Mp_\omega > n(1 - p_\omega/2)$ and Assumption (A), implies that
\[
H_j \lesssim |B_j| \omega \left( \frac{|\lambda|^{2-j^2}}{|B_j|^{p_\omega(|B_j|)}} \right) + \sum_{k=3}^{\infty} |2^k B_j| \omega \left( \frac{|\lambda|^{2-(2M)(j+k)-j^2}}{|2^k B_j|^{1/p} |B_j|^{1-1/p} \rho(|B_j|)} \right) \\
\lesssim 2^{-jp_\omega} \left( 1 + \sum_{k=3}^{\infty} 2^{kn(1-p_\omega/p)} 2^{-2Mp_\omega(j+k)} \right) |B_j| \omega \left( \frac{|\lambda|}{|B_j|^{p_\omega(|B_j|)}} \right) \\
\lesssim 2^{-jp_\omega} |B_j| \omega \left( \frac{|\lambda|}{|B_j|^{p_\omega(|B_j|)}} \right).
\]
Since $\rho$ is of lower type $1/\bar{p}_\omega - 1$ and $\epsilon > n(1/p_\omega - 1/\bar{p}_\omega)$, we further have
\[
\sum_{j=0}^{\infty} H_j \lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega} |B_j| \left\{ \frac{|B| \rho(|B|)}{|B_j| \rho(|B_j|)} \right\}^{p_\omega} \omega \left( \frac{|\lambda|}{|B|^{p_\omega(|B|)}} \right) \\
\lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega} \left\{ \frac{|B|}{|B_j|} \right\}^{p_\omega/\bar{p}_\omega} \omega \left( \frac{|\lambda|}{|B|^{p_\omega(|B|)}} \right) \\
\lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega} 2^{jn(1-p_\omega/\bar{p}_\omega)} |B_j| \omega \left( \frac{|\lambda|}{|B|^{p_\omega(|B|)}} \right) \lesssim |B| \omega \left( \frac{|\lambda|}{|B|^{p_\omega(|B|)}} \right).
\]
Similarly, we have
\[
\sum_{j=0}^{\infty} I_j \lesssim |B|\omega \left( \frac{\lambda}{|B|\rho(|B|)} \right).
\]

Thus, (7.3) holds, and hence, \( T \) is bounded from \( H_{\omega,L}(\mathbb{R}^n) \) to \( L(\omega) \).

It was proved in [22, Theorem 3.4] that operators \( g_L \) and \( \nabla L^{-1/2} \) satisfy (7.1) and (7.2); thus \( g_L \) and \( \nabla L^{-1/2} \) are bounded from \( H_{\omega,L}(\mathbb{R}^n) \) to \( L(\omega) \), which completes the proof of Theorem 7.1. \( \square \)

We now give a fractional variant of Theorem 7.1. To this end, we first make some assumptions.

Let \( \omega \) and \( p_\omega \) satisfy Assumption (A) and \( q \in [p_\omega,1] \). In what follows, for all \( t \in (0,\infty) \), define
\[
v(t) \equiv \omega^{-1}(t)t^{1/q-1/p_\omega}.
\]

**Assumption (B).** Let \( \omega \) satisfy Assumption (A), \( q \in [p_\omega,1] \) and \( p_L < r_1 \leq \min\{2,r_2\} \leq r_2 < \tilde{p}_L \) satisfying that \( 1/p_\omega - 1/q = 1/r_1 - 1/r_2 \). Suppose that \( v \) as in (7.4) is convex and \( v(0) \equiv \lim_{t \to 0^+} v(t) = 0 \). Then for all \( t \in (0,\infty) \), let \( \tilde{\omega}(t) \equiv v^{-1}(t) \) and \( \tilde{\rho}(t) \equiv \frac{t^{-1}}{\omega^{-1}(t^{-1})} \).

**Remark 7.1.** (i) It is easy to see that if \( \tilde{\omega} \) is as in Assumption (B), then \( \tilde{\omega} \) also satisfies Assumption (A) with \( p_{\tilde{\omega}} = q \) and \( p^+_\tilde{\omega} = \frac{1}{1/p_\omega + 1/q - 1/p_\omega} \). Moreover,
\[
\tilde{\rho}(t) \equiv \frac{t^{-1}}{\omega^{-1}(t^{-1})} = \frac{t^{-1}}{\omega^{-1}(t^{-1})t^{1/p_\omega-1/q}} = \rho(t)t^{-1/p_\omega+1/q}.
\]

(ii) Let \( p \in (0,1] \) and \( \omega(t) \equiv tp \) for all \( t \in (0,\infty) \). In this case, \( p_\omega = p = p^+_\omega, q \in [p,1] \) and \( \tilde{\omega}(t) \equiv t^p \) for all \( t \in (0,\infty) \).

**Theorem 7.2.** Let \( q, r_1, r_2, \omega \) and \( \tilde{\omega} \) satisfy Assumption (B). Suppose that the linear operator \( T \) is bounded from \( L^{r_1}(\mathbb{R}^n) \) to \( L^{r_2}(\mathbb{R}^n) \) and there exist \( C > 0, M \in \mathbb{N} \) and \( M > \frac{9}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right) + n \left( \frac{1}{p_\omega} - \frac{1}{p^+_\omega} \right) \) satisfying that for all closed sets \( E, F \) in \( \mathbb{R}^n \) with \( \text{dist}(E,F) > 0 \), all \( f \in L^{r_1}(\mathbb{R}^n) \) supported in \( E \), and all \( t > 0 \),
\[
\|T(I-e^{-tL})^Mf\|_{L^{r_2}(F)} \leq C \left( \frac{t}{\text{dist}(E,F)^2} \right)^M \|f\|_{L^{r_1}(E)} \tag{7.5}
\]
and
\[
\|T(tLe^{-tL})^Mf\|_{L^{r_2}(F)} \leq C \left( \frac{t}{\text{dist}(E,F)^2} \right)^M \|f\|_{L^{r_1}(E)}. \tag{7.6}
\]
If \( T \) is commutative with \( L \), then \( T \) extends to a bounded linear operator from \( H_{\omega,L}(\mathbb{R}^n) \) to \( H_{\tilde{\omega},L}(\mathbb{R}^n) \).

**Proof.** Let \( \epsilon \in (n(\frac{1}{p_\omega} - \frac{1}{p^+_\omega}), M - \frac{9}{2} (\frac{1}{p_\omega} - \frac{1}{2}) \) \). It follows from Proposition 4.2 that for every \( f \in (H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)), f \in L^{r_1}(\mathbb{R}^n) \) and there exist \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) and \( (\omega, \infty, 2M, \epsilon) \)-molecules \( \{B_j\}_{j=1}^{\infty} \) adapted to balls \( \{B_j\}_{j=1}^{\infty} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \) holds in both \( H_{\omega,L}(\mathbb{R}^n) \) and \( L^{r_1}(\mathbb{R}^n) \); moreover, \( \Lambda(\{\lambda_j \alpha_j\}) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \).
We first show that $T$ maps each $(\omega, \infty, 2M, \epsilon)$-molecule into a multiple of an $(\bar{\omega}, r_2, M, \epsilon)$-molecule. To this end, assume that $\alpha$ is an $(\omega, \infty, 2M, \epsilon)$-molecule adapted to a ball $B \equiv B(x_j, r_B)$. By the boundedness of $T$ from $L^{r_1}(\mathbb{R}^n)$ to $L^{r_2}(\mathbb{R}^n)$ and Remark 7.1, we have $T\alpha \in L^{r_2}(\mathbb{R}^n)$ and for any $k \in \{0, \cdots, M\}$ and $j \in \{0, \cdots, 10\}$,
\[
\| (r_B^2 L)^{-k} T\alpha \|_{L^{r_2}(U_j(B))} \lesssim \| (r_B^2 L)^{-k} \alpha \|_{L^{r_1}(\mathbb{R}^n)} \lesssim 2^{-j} |2^j B|^{1/r_1-1} |\rho(|2^j B|)|^{-1},
\]
\[
\sim 2^{-j} |2^j B|^{1/r_2-1} |\tilde{\rho}(|2^j B|)|^{-1}.
\]
For $j \geq 11$, let $W_j(B) \equiv (2^{j+3} B \setminus 2^{j-3} B)$ and $E_j(B) \equiv (W_j(B))^c$. Thus,
\[
\| (r_B^2 L)^{-k} T\alpha \|_{L^{r_2}(U_j(B))} \leq \| T(I - e^{-r_B^2 L} M) [r_B^2 L]^{-k} \alpha \|_{L^{r_2}(U_j(B))}
\]
\[
+ \| T[I - (I - e^{-r_B^2 L} M)] (r_B^2 L)^{-k} \alpha \|_{L^{r_2}(U_j(B))} \lesssim (r_B^2 L)^{-k} \alpha \|_{L^{r_1}(E_j(B))}
\]
\[
\leq 2^{-j} |2^j B|^{1/r_1-1} |\rho(|2^j B|)|^{-1} + 2^{-2j} |B|^{1/r_1-1} |\rho(|B|)|^{-1} \lesssim 2^{-j} |2^j B|^{1/r_2-1} |\tilde{\rho}(|2^j B|)|^{-1}.
\]
By the boundedness from $L^{r_1}(\mathbb{R}^n)$ to $L^{r_2}(\mathbb{R}^n)$ of $T$, Lemma 2.3, (7.5), the choice of $\epsilon$ and Remark 7.1, we have
\[
H \lesssim \| (r_B^2 L)^{-k} \alpha \|_{L^{r_1}(W_j(B))} + \left( \frac{r_B^2}{\text{dist} (U_j(B), E_j(B))^2} \right) \| (r_B^2 L)^{-k} \alpha \|_{L^{r_1}(E_j(B))} \lesssim 2^{-j} |2^j B|^{1/r_1-1} |\rho(|2^j B|)|^{-1} + 2^{-2j} |B|^{1/r_1-1} |\rho(|B|)|^{-1} \lesssim 2^{-j} |2^j B|^{1/r_2-1} |\tilde{\rho}(|2^j B|)|^{-1}.
\]
Similarly, by (7.6), we have
\[
I \lesssim \sup_{1 \leq k \leq M} \left\| T \left( \frac{kr_B^2}{M} L e^{-kr_B^2 L} \right)^M [\chi_{W_j(B)} (r_B^2 L)^{-k-M} \alpha] \right\|_{L^{r_2}(U_j(B))}
\]
\[
+ \sup_{1 \leq k \leq M} \left\| T \left( \frac{kr_B^2}{M} L e^{-kr_B^2 L} \right)^M [\chi_{E_j(B)} (r_B^2 L)^{-k-M} \alpha] \right\|_{L^{r_2}(U_j(B))} \lesssim \| (r_B^2 L)^{-k-M} \alpha \|_{L^{r_1}(W_j(B))} + \left( \frac{r_B^2}{\text{dist} (U_j(B), E_j(B))^2} \right)^M \| (r_B^2 L)^{-k-M} \alpha \|_{L^{r_1}(E_j(B))} \lesssim 2^{-j} |2^j B|^{1/r_2-1} |\tilde{\rho}(|2^j B|)|^{-1}.
\]
Combining the above estimates, we finally obtain that $T\alpha$ is a multiple of an $(\bar{\omega}, r_2, M, \epsilon)$-molecule.

Since $T$ is bounded from $L^{r_1}(\mathbb{R}^n)$ to $L^{r_2}(\mathbb{R}^n)$, we have $Tf = \sum_{j=1}^{\infty} \lambda_j T(\alpha_j)$ in $L^{r_2}(\mathbb{R}^n)$. To finish the proof, it remains to show that $\| Tf \|_{H_\omega L(\mathbb{R}^n)} \lesssim \| f \|_{H_\omega L(\mathbb{R}^n)}$. To this end, by Lemma 2.4, the subadditivity and the continuity of $\omega$ and (4.3) with $\omega$ and $\rho$ replaced respectively by $\bar{\omega}$ and $\tilde{\rho}$, we obtain
\[
\int_{\mathbb{R}^n} \bar{\omega}(S_L(Tf)(x)) \, dx \leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \bar{\omega}(|\lambda_j| S_L(T\alpha_j)(x)) \, dx \lesssim \sum_{j=1}^{\infty} |B_j| \bar{\omega} \left( \frac{|\lambda_j|}{|B_j| \tilde{\rho}(|B_j|)} \right), \quad (7.7)
\]
Choose $\gamma \in \Lambda(\{\lambda_j \alpha_j\}_j), 2\Lambda(\{\lambda_j \alpha_j\}_j)$. Then for each $j \in \mathbb{N}$, we have $\gamma \geq |\lambda_j|$; otherwise, there exists $i \in \mathbb{N}$ such that $\gamma < |\lambda_i|$, which together with the strictly increasing property of $\omega$ further implies that

$$\sum_{j=1}^{\infty} |B_j| \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \geq |B_i| \omega \left( \frac{|\lambda_i|}{\gamma|B_i|\rho(|B_i|)} \right) > |B_i| \omega \left( \frac{1}{|B_i|\rho(|B_i|)} \right) = 1.$$  

This contradicts to the assumption $\lambda > \Lambda(\{\lambda_j \alpha_j\}_j)$. Thus, the claim is true. Therefore, by this claim and the strictly increasing property of $\omega$, for each $j \in \mathbb{N}$, we have

$$\left[ |B_j| \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \right]^{1/p_{\omega} - 1/q} \leq \left[ |B_j| \omega \left( \omega^{-1}(|B_j|^{-1}) \right) \right]^{1/p_{\omega} - 1/q} \leq 1,$$

which implies that

$$\frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} = \frac{|\lambda_j|}{\gamma} \omega^{-1}(|B_j|^{-1}) = \frac{|\lambda_j|}{\gamma} \omega^{-1}(|B_j|^{-1}) |B_j|^{1/p_{\omega} - 1/q} \leq \frac{|\lambda_j|}{\gamma} \omega^{-1}(|B_j|^{-1}) \left[ \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \right]^{1/q-1/p_{\omega}} = \tilde{\omega}^{-1} \left[ \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \right].$$

Since $\tilde{\omega}$ satisfies Assumption (B), we further obtain

$$\tilde{\omega} \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \leq \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right),$$

and hence, by (7.7),

$$\int_{\mathbb{R}^n} \tilde{\omega} \left( \frac{S_L(Tf)(x)}{\gamma} \right) dx \lesssim \sum_{j=1}^{\infty} |B_j| \tilde{\omega} \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \lesssim \sum_{j=1}^{\infty} |B_j| \omega \left( \frac{|\lambda_j|}{\gamma|B_j|\rho(|B_j|)} \right) \lesssim 1.$$  

Thus $\|Tf\|_{H_{\tilde{\omega},L}(\mathbb{R}^n)} \lesssim \gamma \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}$, which together with a standard density argument completes the proof of Theorem 7.2. \hfill \Box

**Remark 7.2.** If we let $p \in (0, 1]$ and $\omega(t) \equiv t^p$ for all $t \in (0, \infty)$, by Remark 7.1(ii) and Theorem 7.2, we know that the operator $T$ of Theorem 7.2 in this case extends to a bounded linear operator from $H^p_{\omega,L}(\mathbb{R}^n)$ to $H^p_{\omega,L}(\mathbb{R}^n)$.

In what follows, let $\gamma \in (0, \frac{n}{p_\omega}(\frac{1}{p_L} - \frac{1}{p_\omega}))$. Recall that the generalized fractional integral $L^{-\gamma}$ is given by setting, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$L^{-\gamma}f(x) \equiv \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma - 1} e^{-tL} f(x) \frac{dt}{t}.$$  

Applying Theorem 7.2, we obtain the boundedness of $L^{-\gamma}$ from $H_{\omega,L}(\mathbb{R}^n)$ to $H_{\tilde{\omega},L}(\mathbb{R}^n)$ as follows.
Theorem 7.3. Let \( q, r_1, r_2, \omega \) and \( \bar{\omega} \) satisfy Assumption (B) and \( \gamma \in (0, \frac{q}{2} (\frac{1}{p_1} - \frac{1}{p_2})) \) satisfying that \( n(1/p_\omega - 1/q) = 2 \gamma \). Then the operator \( L^{-\gamma} \) satisfies (7.5) and (7.6) and hence, is bounded from \( H_{\omega,L}(\mathbb{R}^n) \) to \( H_{\bar{\omega},L}(\mathbb{R}^n) \).

**Proof.** Let \( M \in \mathbb{N} \) and \( M > \frac{q}{2} (\frac{1}{p_1} - \frac{1}{p_2}) + n(\frac{1}{p_2} - \frac{1}{p_1}) \). By [2, Proposition 5.3], the operator \( L^{-\gamma} \) is bounded from \( L^r_\Omega(\mathbb{R}^n) \) to \( L^s(\mathbb{R}^n) \). Thus, by Theorem 7.2, to show Theorem 7.3, we only need to prove that \( L^{-\gamma} \) satisfies (7.5) and (7.6). We only give the proof of the former one, since (7.6) can be proved in a similar way.

Let \( E, F \) be closed sets in \( \mathbb{R}^n \) with \( \text{dist} (E,F) > 0 \) and \( f \in L^r_\Omega(\mathbb{R}^n) \) supported in \( E \). Write

\[
\| L^{-\gamma} (I - e^{-tL})^{M} f \|_{L^2(\mathbb{R}^n)} = \frac{1}{\Gamma(\gamma)} \left\| \int_0^\infty s^{\gamma-1} e^{-sL} (I - e^{-tL})^{M} f ds \right\|_{L^2(\mathbb{R}^n)} \\
\lesssim \left\| \int_0^t s^{\gamma-1} e^{-sL} (I - e^{-tL})^{M} f ds \right\|_{L^2(\mathbb{R}^n)} + \left\| \int_t^\infty \cdots \right\|_{L^2(\mathbb{R}^n)} \\
\equiv H_1 + H_2.
\]

It follows from Lemma 2.3 that

\[
H_1 \lesssim \int_0^t \left\| s^{\gamma-1} e^{-sL} f \|_{L^2(\mathbb{R}^n)} ds + \sup_{1 \leq k \leq M} \int_0^t \left\| s^{\gamma-1} e^{-sL} e^{-ktL} f \right\|_{L^2(\mathbb{R}^n)} ds \\
\lesssim \left\{ \int_0^t s^{\gamma-1} s^{\frac{q}{2} (\frac{1}{p_1} - \frac{1}{p_2})} e^{-\text{dist} (E,F)^2} s + \int_0^t s^{\gamma-1} s^{\frac{q}{2} (\frac{1}{p_1} - \frac{1}{p_2})} e^{-\text{dist} (E,F)^2} s \right\} \| f \|_{L^1(\mathbb{R}^n)} \\
\lesssim \left( \frac{t}{\text{dist} (E,F)^2} \right)^M \| f \|_{L^1(\mathbb{R}^n)},
\]

here and in what follows, \( c \) is the positive constant as in Lemma 2.3.

For the term \( H_2 \), since \( I - e^{-tL} = \int_0^t Le^{-rL} dr \), by Lemmas 2.1 and 2.3, and the Minkowski inequality, we obtain

\[
H_2 \lesssim \int_0^\infty \left\| \int_0^t \left( \int_0^t \left( \int_0^t \cdots \int_0^t \right) \right) \right\|_{L^2(\mathbb{R}^n)} ds \\
\lesssim \int_0^\infty s^{\gamma-1} s^{\frac{q}{2} (\frac{1}{p_1} - \frac{1}{p_2})} t^M e^{-\text{dist} (E,F)^2} ds \left\| f \right\|_{L^1(\mathbb{R}^n)} \\
\lesssim \int_0^\infty \left( \frac{t}{\text{dist} (E,F)^2} \right)^M e^{-\frac{t}{r}} \| f \|_{L^1(\mathbb{R}^n)} \lesssim \left( \frac{t}{\text{dist} (E,F)^2} \right)^M \| f \|_{L^1(\mathbb{R}^n)},
\]

which implies that (7.5) holds for the operator \( L^{-\gamma} \), and hence, completes the proof of Theorem 7.3. \( \square \)

**Remark 7.3.** Similarly to Remark 7.2, as a special case of Theorem 7.3, we know that the operator \( L^{-\gamma} \) maps \( H^r_\Omega(\mathbb{R}^n) \) continuously into \( H^s_\Omega(\mathbb{R}^n) \), where \( \gamma, p, q \) satisfy \( 0 < p \leq q \leq 1 \) and \( n(1/p - 1/q) = 2 \gamma \).
Using Theorems 7.1 and 7.3, we further obtain the following boundedness of the Riesz transform $\nabla L^{-1/2}$ from $H_{\omega,L}(\mathbb{R}^n)$ to $H_{\omega}(\mathbb{R}^n)$. We first recall some notions; see [42, 37, 25].

In what follows, let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ the space of all Schwartz distributions.

**Definition 7.1.** Let $\omega$ satisfy Assumption (A) and $p_\omega \in (\frac{n}{n+1}, 1]$. A function $a$ is called a $(\rho,2)$-atom if

(i) $\text{supp } a \subseteq B$, where $B$ is a ball of $\mathbb{R}^n$;

(ii) $\|a\|_{L^2(\mathbb{R}^n)} \leq |B|^{-1/2}|\rho(|B|)|^{-1}$;

(iii) $\int_{\mathbb{R}^n} a(x) \, dx = 0$.

**Definition 7.2.** Let $\omega$ satisfy Assumption (A) and $p_\omega \in (\frac{n}{n+1}, 1]$. The Orlicz-Hardy space $H_{\omega}(\mathbb{R}^n)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be written as $f = \sum_{j=1}^\infty b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{b_j\}_{j=1}^\infty$ is a sequence of multiples of $(\rho,2)$-atoms such that

$$\sum_{j=1}^\infty |B_j| \omega\left(\frac{\|b_j\|_{L^2(\mathbb{R}^n)}}{|B_j|^{1/2}}\right) < \infty,$$

where $\text{supp } b_j \subset B_j$. Moreover, define

$$\|f\|_{H_{\omega}(\mathbb{R}^n)} \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty |B_j| \omega\left(\frac{\|b_j\|_{L^2(\mathbb{R}^n)}}{\lambda |B_j|^{1/2}}\right) \leq 1 \right\},$$

where the infimum is taken over all decompositions of $f$ as above.

It is well known that the classical Orlicz-Hardy space defined by using grand maximal functions is equivalent to the above atomic Orlicz-Hardy space $H_{\omega}(\mathbb{R}^n)$ as in Definition 7.2; see [42, 37]. Based on this fact, in what follows, we denote both spaces by the same notation. Recall that $H_{\omega}(\mathbb{R}^n)$ is complete.

**Theorem 7.4.** Let $\omega$ satisfy Assumption (A) and $p_\omega \in (\frac{n}{n+1}, 1]$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $H_{\omega}(\mathbb{R}^n)$. In particular, $\nabla L^{-1/2}$ is bounded from $H^p_{\omega}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+1}, 1]$

**Proof.** Let $\epsilon > 1 + n(\frac{1}{p_\omega} - \frac{1}{p_\omega})$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2}) + \epsilon$, where $p_\omega$ is as in Convention (C). Suppose that $\alpha$ is an $(\omega,\infty,M,\epsilon)$-molecule associated to a ball $B \equiv B(x_B,r_B)$. We first show that $\int_{\mathbb{R}^n} \nabla L^{-1/2} \alpha(x) \, dx = 0$.

From the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$ (see [2, Theorem 4.1]), it follows that for $j = 0, 1, \cdot \cdot \cdot, 10$,

$$\|\nabla L^{-1/2} \alpha\|_{L^2(U_j(B))} \leq \|\nabla L^{-1/2} \alpha\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(\mathbb{R}^n)} \lesssim |B|^{-1/2} \rho(|B|)^{-1}. \quad (7.8)$$

For $j \geq 11$, let $W_j(B) \equiv (2^{j+3}B \setminus 2^{j-3}B)$ and $E_j(B) \equiv (W_j(B))^\epsilon$. By the fact that $\nabla L^{-1/2}$ satisfies (7.1) and (7.2) (see Theorem 3.4 in [22]) together with the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$ and Lemma 2.2, we have

$$\|\nabla L^{-1/2} \alpha\|_{L^2(U_j(B))}$$
\[
\begin{align*}
&\leq \left\| \nabla L^{-1/2} (I - e^{-r_B^2 L})^M \alpha \right\|_{L^2(U_j(B))}^2 + \left\| \nabla L^{-1/2} \left[ I - (I - e^{-r_B^2 L})^M \right] \alpha \right\|_{L^2(U_j(B))}^2 \\
&\lesssim \left\| \nabla L^{-1/2} (I - e^{-r_B^2 L})^M \left[ (\chi_{W_j(B)} + \chi_{E_j(B)}) \alpha \right] \right\|_{L^2(U_j(B))}^2 \\
&\quad + \sup_{1 \leq k \leq M} \left\| \nabla L^{-1/2} \left[ \left( \frac{kr_B^2 L}{M} \right)^M \right] \left[ (\chi_{W_j(B)} + \chi_{E_j(B)}) (r_B^2 L)^{-M} \alpha \right] \right\|_{L^2(U_j(B))}^2 \\
&\lesssim \|\alpha\|_{L^2(W_j(B))} + 2^{-2jM} \|\alpha\|_{L^2(\mathbb{R}^n)} \\
&\quad + \|(r_B^2 L)^{-M} \alpha\|_{L^2(W_j(B))} + 2^{-2jM} \|(r_B^2 L)^{-M} \alpha\|_{L^2(\mathbb{R}^n)} \\
&\lesssim [2^{-j} + 2^{-j(2M-n/p_\omega+n/2)}]|2j B|^{-1/2}[\rho(|2j B|)]^{-1}. \\
&\text{(7.9)}
\end{align*}
\]

Combining the above estimates and using the Hölder inequality, we see that \( \nabla L^{-1/2} \alpha \in L^1(\mathbb{R}^n) \).

We now choose \( p_L < s \leq \min\{t, 2\} \leq t < \tilde{p}_L \) such that \( 1/s - 1/t = 1/n \). Since an \((\omega, \infty, M, \epsilon)\)-molecule is also an \((\omega, s, M, \epsilon)\)-molecule, by the fact that \( L^{-1/2} \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \) (see [2, Proposition 5.3]) and the Hölder inequality, we have that for \( j = 0, 1, \ldots, 10, \)

\[
\|L^{-1/2} \alpha\|_{L^1(U_j(B))} \leq |U_j(B)|^{1-1/t} \|L^{-1/2} \alpha\|_{L^t(\mathbb{R}^n)} \lesssim (2j r_B)^n (1-1/t) \|\alpha\|_{L^s(\mathbb{R}^n)} \\
\lesssim |B|^{1-1/t+1-s-1} [\rho(|B|)]^{-1} \sim |B|^{1/n} [\rho(|B|)]^{-1}.
\]

For \( j \geq 11, \) let \( W_j(B) \equiv (2j+3 B \setminus 2j^3 B) \) and \( E_j(B) \equiv (W_j(B))^C \). By Theorem 7.3, we have that \( L^{-1/2} \) satisfies (7.5) and (7.6), which together with Lemma 2.3 and the Hölder inequality yields that

\[
\begin{align*}
&\|L^{-1/2} \alpha\|_{L^1(U_j(B))} \\
&\leq |U_j(B)|^{1-1/t} \left\{ \left\| L^{-1/2} (I - e^{-r_B^2 L})^M \alpha \right\|_{L^t(U_j(B))}^2 \\
&\quad + \left\| L^{-1/2} \left[ I - (I - e^{-r_B^2 L})^M \right] \alpha \right\|_{L^t(U_j(B))}^2 \right\} \\
&\lesssim |U_j(B)|^{1-1/t} \left\{ \left\| L^{-1/2} (I - e^{-r_B^2 L})^M \left[ (\chi_{W_j(B)} + \chi_{E_j(B)}) \alpha \right] \right\|_{L^t(U_j(B))}^2 \\
&\quad + \sup_{1 \leq k \leq M} \left\| L^{-1/2} \left[ \left( \frac{kr_B^2 L}{M} \right)^M \right] \left[ (\chi_{W_j(B)} + \chi_{E_j(B)}) (r_B^2 L)^{-M} \alpha \right] \right\|_{L^t(U_j(B))}^2 \right\} \\
&\lesssim |U_j(B)|^{1-1/t} \left\{ \|\alpha\|_{L^s(W_j(B))} + 2^{-2jM} \|\alpha\|_{L^s(\mathbb{R}^n)} \\
&\quad + \|(r_B^2 L)^{-M} \alpha\|_{L^s(W_j(B))} + 2^{-2jM} \|(r_B^2 L)^{-M} \alpha\|_{L^s(\mathbb{R}^n)} \right\} \\
&\lesssim [2^{-j} + 2^{-j(2M-n(1-1/t))}] |B|^{1/n} [\rho(|B|)]^{-1}.
\end{align*}
\]

Since \( \epsilon > 1 + n(1/p_\omega - \tilde{p}_\omega) \) and \( M > \frac{n}{2p_\omega}, \) we obtain that \( L^{-1/2} \alpha \in L^1(\mathbb{R}^n) \).

Now we choose \( \{\varphi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^n) \) such that
(i) \( \sum_{j=0}^{\infty} \varphi_j(x) = 1 \) for almost every \( x \in \mathbb{R}^n \);
(ii) for each \( j \in \mathbb{Z}_+ \), supp \( \varphi_j \subset 2B_j \), \( \varphi_j = 1 \) on \( B_j \) and \( 0 \leq \varphi_j \leq 1 \);
(iii) there exists \( C_\varphi > 0 \) such that for all \( j \in \mathbb{Z}_+ \) and \( x \in \mathbb{R}^n \), \( \varphi_j(x) + |\nabla \varphi_j(x)| \leq C_\varphi \);
(iv) there exists \( N_\varphi \in \mathbb{N} \) such that \( \sum_{j=0}^{\infty} \chi_{2B_j} \leq N_\varphi \).

Using the properties of \( \{\varphi_j\}_{j=0}^{\infty} \) and the facts that \( L^{-1/2} \alpha, \nabla L^{-1/2} \alpha \in L^1(\mathbb{R}^n) \), we obtain
\[
\int_{\mathbb{R}^n} \nabla L^{-1/2} \alpha(x) \, dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \nabla(\varphi_j L^{-1/2} \alpha)(x) \, dx.
\]

For each \( j \), let \( \eta_j \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta_j = 1 \) on \( 2B_j \) and supp \( \eta_j \subset 4B_j \). Then for each \( i = 1, 2, \cdots, n \), we have
\[
\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} (\varphi_j L^{-1/2} \alpha)(x) \, dx = \int_{\mathbb{R}^n} \eta_j(x) \frac{\partial}{\partial x_i} (\varphi_j L^{-1/2} \alpha)(x) \, dx
\]
\[
= - \int_{\mathbb{R}^n} \varphi_j(L^{-1/2} \alpha)(x) \frac{\partial}{\partial x_i} \eta_j(x) \, dx = 0,
\]
which implies that \( \int_{\mathbb{R}^n} \nabla L^{-1/2} \alpha(x) \, dx = 0 \).

To finish the proof, we borrow some ideas from [41]. For \( k \in \mathbb{Z}_+ \), let \( \chi_k \equiv \chi_{U_k(B)} \), \( \tilde{\chi}_k \equiv [U_k(B)]^{-1} \chi_k \), \( m_k \equiv \int_{U_k(B)} \nabla L^{-1/2} \alpha(x) \, dx \) and \( M_k \equiv \nabla L^{-1/2} \alpha \chi_k - m_k \tilde{\chi}_k \). Then we have
\[
\nabla L^{-1/2} \alpha = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} m_k \tilde{\chi}_k
\]
in \( L^2(\mathbb{R}^n) \). Now let \( N_j \equiv \sum_{k=j}^{\infty} m_k \). Since \( \int_{\mathbb{R}^n} \nabla L^{-1/2} \alpha(x) \, dx = 0 \), we have
\[
\nabla L^{-1/2} \alpha = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k).
\] (7.10)

Obviously, for all \( k \in \mathbb{Z}_+ \), \( \int_{\mathbb{R}^n} M_k(x) \, dx = 0 \). Furthermore, by (7.8), (7.9) and the H"{o}lder inequality, we have that for all \( k \in \mathbb{Z}_+ \),
\[
\|M_k\|_{L^2(\mathbb{R}^n)} \lesssim \|\nabla L^{-1/2} \alpha\|_{L^2(U_k(B))} \lesssim 2^{-k\epsilon} |2^k B|^{-1/2} [\rho(|2^k B|)]^{-1},
\]
which implies that \( \{2^{k\epsilon} M_k\}_{k \in \mathbb{Z}_+} \) is a family of \( (\rho, 2) \)-atoms up to a harmless constant.

To deal with the second sum in (7.10), by (7.8), (7.9), \( |\tilde{\chi}_{k+1} - \tilde{\chi}_k| \lesssim |2^k B|^{-1} \) and the H"{o}lder inequality, we have that for all \( k \in \mathbb{Z}_+ \),
\[
\|N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k)\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{j=k}^{\infty} \frac{|2^j B|^{1/2}}{|2^k B|^{1/2}} \|\nabla L^{-1/2} \alpha\|_{L^2(U_j(B))}
\]
\[
\lesssim 2^{-k\epsilon} |2^k B|^{-1/2} [\rho(|2^k B|)]^{-1}.
\]

This, together with \( \int_{\mathbb{R}^n} [\tilde{\chi}_{k+1} - \tilde{\chi}_k] \, dx = 0 \), implies that for each \( k \in \mathbb{Z}_+ \), \( 2^{k\epsilon} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k) \) is a \( (\rho, 2) \)-atom up to a harmless constant.
By Assumption (A) and Convention (C), \( \rho \) is of lower type \( 1/p_\omega - 1 \), which implies that
\[
\sum_{j=0}^\infty |2^j B| \omega \left( \frac{\|M_j\|_{L^2(\mathbb{R}^n)}}{\lambda|2^j B|^{1/2}} \right) + \sum_{j=0}^\infty |2^j B| \omega \left( \frac{\|N_{j+1}(\xi_j+1 - \xi_j)\|_{L^2(\mathbb{R}^n)}}{\lambda|2^j B|^{1/2}} \right) 
\leq \sum_{j=0}^\infty 2^{-jp_\omega + jn(1-p_\omega/p_\omega)} |B| \omega \left( \frac{1}{\lambda|B| \rho(|B|)} \right) \sim |B| \omega \left( \frac{1}{\lambda|B| \rho(|B|)} \right).
\]
(7.11)

Now, suppose that \( f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). By Proposition 4.2, there exist \((\omega, \infty, M, \epsilon)\)-molecules \( \{\alpha_k\}_{k=1}^\infty \) adapted to balls \( \{B_k\}_{k=1}^\infty \) and numbers \( \{\lambda_k\}_{k=1}^\infty \subset \mathbb{C} \) such that \( f = \sum_{k=1}^\infty \lambda_k \alpha_k \) in both \( H_{\omega,L}(\mathbb{R}^n) \) and \( L^2(\mathbb{R}^n) \) with \( \Lambda(\{\lambda_k \alpha_k\}_{k}) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)} \).

For each \((\omega, \infty, M, \epsilon)\)-molecule \( \alpha_k \), by the above argument, we decompose \( \nabla L^{-1/2} \alpha_k \) into a summation of multiples of \((\rho, 2)\)-atoms with harmless constants, which converges in \( L^2(\mathbb{R}^n) \). For simplicity, we write it as \( \nabla L^{-1/2} \alpha_k = \sum_{j=1}^\infty b_{k,j} \), where \( b_{k,j} \) is a multiple of a \((\rho, 2)\)-atom supported in \( B_{k,j} \) with a harmless constant. Thus, by (7.11), we obtain
\[
\|\nabla L^{-1/2} f\|_{H_{\omega,L}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty \sum_{j=1}^\infty |B_{k,j}| \omega \left( \frac{\|\lambda_k b_{k,j}\|_{L^2(\mathbb{R}^n)}}{\lambda |B_{k,j}|^{1/2}} \right) \leq 1 \right\} 
\leq \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty |B_k| \omega \left( \frac{\|\lambda_k\|_{L^2(\mathbb{R}^n)}}{\lambda |B_k| \rho(|B_k|)} \right) \leq 1 \right\} 
\sim \Lambda(\{\lambda_k \alpha_k\}_{k}) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}.
\]

Then, by a standard density argument, we see that \( \nabla L^{-1/2} \) extends to a bounded linear operator from \( H_{\omega,L}(\mathbb{R}^n) \) to \( H_{\omega}(\mathbb{R}^n) \). This finishes the proof of Theorem 7.4. \( \square \)

**Remark 7.4.** Let \( \omega \) satisfy Assumption (A) and \( p_\omega \in (\frac{n}{n+1}, 1) \). We claim that the Orlicz-Hardy spaces \( H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega}(\mathbb{R}^n) \). In particular, \( H^p_{\omega}(\mathbb{R}^n) \subset H^p(\mathbb{R}^n) \) for all \( p \in (\frac{n}{n+1}, 1) \).

Let \( \epsilon \in (n/(1 - p_\omega^{-1} - 1)/\omega), M \in \mathbb{N} \) and \( M > \frac{n}{2} (\frac{1}{\rho_\omega} - \frac{1}{2}) \). For all \((\omega, \infty, M, \epsilon)\)-molecules \( \alpha \), we claim that \( \int_{\mathbb{R}^n} \alpha(x) \, dx = 0 \). To show this, write
\[
\alpha = \text{div}(A \nabla L^{-1} \alpha) = r_B \left\{ \text{div}(A[r_B \nabla(I + r_B^2 L)^{-1} 2_B^2 L^{-1} \alpha + r_B \nabla(I + r_B^2 L)^{-1} \alpha]) \right\},
\]
where \( \alpha \) is adapted to the ball \( B \equiv B(x_B, r_B) \).

From the Hölder inequality, Lemma 2.2 and Definition 4.2, it follows that for \( j = 0, 1, \cdots, 10, \)
\[
|r_B \nabla(I + r_B^2 L)^{-1} 2_B^2 L^{-1} \alpha|_{L^1(U_j(B))} \leq |U_j(B)|^{1/2} |r_B \nabla(I + r_B^2 L)^{-1} 2_B^2 L^{-1} \alpha|_{L^2(\mathbb{R}^n)} 
\lesssim |B|^{1/2} |(r_B^2 L^{-1} 2_B^2 L^{-1} \alpha|_{L^2(\mathbb{R}^n)} \lesssim |\rho(|B|)|^{-1}.
\]
For \( j \geq 11 \), let \( W_j(B) \equiv (2^{j+3} B \setminus 2^{j-3} B) \) and \( E_j(B) \equiv (W_j(B))^\circ \). By Lemma 2.2 and the Hölder inequality, we have
\[
|r_B \nabla(I + r_B^2 L)^{-1} 2_B^2 L^{-1} \alpha|_{L^1(U_j(B))}
\]
\[ |U_j(B)|^{1/2} \| r_B \nabla (I + r_B^2 L)^{-1} \left[ (\chi_{W_j(B)} + \chi_{E_j(B)}) (r_B^2 L)^{-1} \alpha \right] \|_{L^2(U_j(B))} \]

\[ \lesssim |U_j(B)|^{1/2} \left\{ \|(r_B^2 L)^{-1} \alpha\|_{L^2(W_j(B))} + \exp \left\{ -\frac{\text{dist} \left( U_j(B), E_j(B) \right)}{cr_B} \right\} \| \alpha \|_{L^\alpha(\mathbb{R}^n)} \right\} \]

\[ \lesssim 2^{-J_0\nu(B)} \| \rho(2^j B) \|^{-1} + 2^{j/n} \left( \frac{r_B}{2^j r_B} \right)^{n/2 + \nu} \| \rho(\|B\|) \|^{-1} \lesssim 2^{-J_0\nu(\|B\|)}^{-1}. \]

The above two estimates imply that \( r_B \nabla (I + r_B^2 L)^{-1} (r_B^2 L)^{-1} \alpha \in L^1(\mathbb{R}^n). \)

Similarly, we have that \( r_B \nabla (I + r_B^2 L)^{-1} \alpha \in L^1(\mathbb{R}^n), \) and hence, \( \nabla L^{-1} \alpha \in L^1(\mathbb{R}^n). \)

Let \( \{ \varphi_j \}_{j=0}^{\infty} \) be as in the proof of Theorem 7.4. Using the properties of \( \{ \varphi_j \}_{j=0}^{\infty} \) and the facts that \( \alpha, \nabla L^{-1} \alpha \in L^1(\mathbb{R}^n) \) together with the divergence theorem, we obtain

\[ \int_{\mathbb{R}^n} \alpha(x) \, dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \text{div}(\varphi_j A \nabla L^{-1} \alpha)(x) \, dx \]

\[ = \sum_{j=0}^{\infty} \int_{\partial(2B_j)} \langle \varphi_j(x) \vec{n}_{2B_j}(x), A \nabla L^{-1} \alpha(x) \rangle \, d\sigma_{2B_j} x = 0, \]

where \( \vec{n}_{2B_j} \) denotes the outward unit norm vector to \( 2B_j \) and \( \sigma_{2B_j} \) the surface measure over \( \partial(2B_j). \)

Then following the proof of Theorem 7.4, we obtain that for all \( f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \)

\[ \| f \|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \| f \|_{H_{\omega,L}(\mathbb{R}^n)}. \]

By a density argument, we obtain that \( H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n), \)

which completes the proof of the above claim.

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