A note on split extension classifiers of perfect objects

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Abstract

We show that for a pointed protomodular category $C$ satisfying a certain condition on those Huq commutators which exist, if $X$ is a perfect object in $C$ such that the split extension classifier $\llbracket X \rrbracket$ exists, then the centralizer of the conjugation morphism $c_X : X \to \llbracket X \rrbracket$ is trivial and hence $\llbracket X \rrbracket$ has trivial center.

1 Introduction and preliminaries

Recall that the automorphisms of a group commuting with every inner automorphism are called central. For a group $X$ these automorphisms form a normal subgroup $\text{Aut}_c(X)$ of the automorphism group $\text{Aut}(X)$ which has been the subject of several papers whose study goes back at least to J. E. Adney and T. Yen [1]. One of the methods of studying these automorphisms is via the so called Adney-Yen map which is an injective map from $\text{Aut}_c(X)$ to the set of homomorphisms from $X$ to its center $Z(X)$. As a immediate consequence of the existence of this injective map one sees that if $X$ is perfect, then $\text{Aut}_c(X)$ is trivial.

Our main aim in this paper is to generalize this result to, in particular, arbitrary semi-abelian action representable categories satisfying a mild condition on commutators. The terminology will be recalled below.

As a special case we will recover the following known fact about Lie algebras. For a fixed commutative ring $R$ and a Lie algebra $X$ over $R$, the Lie algebra of derivations of $X$, $\text{Der}(X)$, has underlying abelian group the subgroup of $R$-linear maps $d : X \to X$ satisfying $d([x, y]) = [d(x), y] + [x, d(y)]$ for each $x, y$ in $X$, and has Lie bracket defined by $[d, e] = d \circ e - e \circ d$. For each $x$ in $X$, the map $y \mapsto [x, y]$ determines a derivation called an inner derivation. In analogy with the group theoretic case (in fact both are special cases of the same categorical construction) derivations which commute with (annihilate) every inner derivation are called central and form an ideal of $\text{Der}(X)$ which we denote by $\text{Der}_c(X)$. The analogous fact is that if $X$ is perfect, then $\text{Der}_c(X)$ is trivial.

We now recall the necessary background and introduce notation so that we can precisely state our main aim.
For a pointed category $C$ we will denote by 0 a chosen zero object (that is, an object which is both initial and terminal). For objects $A$ and $B$ in $C$ we will also denote by 0 the unique morphism from $A$ to $B$ which factors through the zero object 0. The product of a pair of objects $A$ and $B$ will be written $A \times B$ with first and second projection denoted by $\pi_1$ and $\pi_2$, respectively. The coproduct will be written $A + B$ with coproduct inclusions $\iota_1$ and $\iota_2$. For morphisms $u : W \to A$, $v : W \to B$, $f : A \to C$ and $g : B \to C$ we denote by $\langle u, v \rangle : W \to A \times B$ with first and second projection denoted by $\pi_1$ and $\pi_2$, respectively.

Recall that a split extension in a pointed category $C$ is a diagram

$$X \xrightarrow{\kappa} A \xrightarrow{\alpha} B \xleftarrow{\beta} X$$

in $C$ where $\alpha \beta = 1_B$ and $\kappa$ is the kernel of $\alpha$. A morphism of split extensions in $C$ is a diagram

$$X \xrightarrow{\kappa} A \xrightarrow{\alpha} B \xleftarrow{\beta} Z \xrightarrow{\delta} D$$

such that the top and bottom rows are split extensions (the domain and codomain, respectively) and $g\kappa = \sigma f$, $g\beta = \delta h$ and $h\alpha = \gamma g$. We denote by $\text{SplExt}(C)$ the category of split extensions in $C$, and by $K$ and $P$ the functors sending (1) to $X$ and $B$ respectively, and (2) to $f$ and $h$ respectively. We denote by $K^{-1}(X)$ the fiber of the functor $K$ above $X$, that is the subcategory of $\text{SplExt}(C)$ with objects and morphism mapped by $K$ to $X$ and $1_X$, respectively. Following F. Borceux, G. Janelidze and G. M. Kelly [5] we call a terminal object in $K^{-1}(X)$ a generic split extension with kernel $X$ which we will denote by

$$X \xrightarrow{k} [X] \times X \xleftarrow{i} [X].$$

If $X$ is an object in a pointed category $C$ admitting a generic split extension with kernel $X$, then we will denote by $c_X : X \to [X]$ the morphism forming part of the unique morphism

$$X \xrightarrow{(0,1)} X \times X \xrightarrow{\pi_1} X \xleftarrow{(1,0)} X \xrightarrow{i} [X]$$

in $K^{-1}(X)$. Recall that there is also morphism $p_2 : [X] \times X \to [X]$ which together with $p_1$ and $i$ form part of a canonical groupoid structure (see [4] or the preliminaries of [20]), and furthermore $c_X = p_2 k$. Note that if $C$ is the
category of groups, then \([X] = \text{Aut}(X)\) and \(c_X\) is the homomorphism sending each \(x\) in \(X\) to the inner automorphism determined by conjugation (that is \(x \mapsto (y \mapsto xyx^{-1})\)) \([6]\). On the other hand if \(\mathbb{C}\) is the category of Lie algebras over a commutative ring, then \([X] = \text{Der}(X)\) and \(c_X\) is the homomorphism sending each \(x\) in \(X\) to the inner derivation determined by \(x\) (that is \(x \mapsto (y \mapsto [x, y])\)) \([6]\).

Recall that a pointed finitely complete category can equivalently be defined to be (Bourn)-protomodular \([7]\) if it satisfies the split short five lemma, that is, for each morphism of split extensions \([2]\) if \(f\) and \(h\) are isomorphisms, then \(g\) is an isomorphism. A category is regular if it has all finite limits, regular epimorphisms are pullback stable in it, and the coequalizer of kernel pairs exist in it. A category is homological if it is a regular pointed protomodular category.

A pair of morphisms \(f : A \to C\) and \(g : B \to C\) in a pointed protomodular category \(\mathbb{C}\) with binary products (Huq)-commute \([22]\) if there exists a morphism \(\varphi : A \times B \to C\) making the diagram

\[
\begin{array}{ccc}
A & \stackrel{(1,0)}{\longrightarrow} & A \times B \\
\downarrow f & & \downarrow \varphi \\
C & \leftarrow & B \\
\downarrow g & & \downarrow \end{array}
\]

commute. For \(f\) and \(g\) as above we use the following (equivalent) definition of the Huq-commutator (originally defined in \([22]\)). The Huq commutator \(\gamma_{[f,g]} : [(A, f), (B, g)] \to C\) of \(f\) and \(g\) is initial amongst those normal monomorphisms \(n\) with codomain \(C\) such that \(\text{coker}(n)f\) and \(\text{coker}(n)g\) commute (where \(\text{coker}(n)\) is the cokernel of \(n\)). Note that we will write \([X, X]\) instead of \([(X, 1_X), (X, 1_X)]\).

The centralizer of a morphism \(f : A \to C\) is the terminal object in the category of morphisms which commute with \(f\) and will be denoted \(Z_f : Z_{C(A, f)} \to C\).

Recall also that an object \(X\) is commutative if \(1_X\) commutes with \(1_X\) and is perfect if the only morphisms from it to commutative objects are zero morphisms.

Recall that a category \(\mathbb{C}\) is (Barr)-exact \([2]\), if it is regular, and equivalence relations in \(\mathbb{C}\) are effective (i.e. are the kernel pair of some morphism). A pointed protomodular category is semi-abelian \([23]\) if it has binary coproducts and is Barr-exact. In agreement with the definitions in the semi-abelian context (see \([21]\) and \([14]\), respectively) we consider the following conditions on a pointed category \(\mathbb{C}\):

\(\text{(WNH)}\). For each normal monomorphism \(k : K \to X\) in \(\mathbb{C}\) if the commutator \(\gamma_{[1_K, 1_K]}\) exists, then the morphism \(k\gamma_{[1_K, 1_K]}\) is the commutator morphism of \(k\) with itself.

\(\text{(NH)}\). For monomorphisms \(m : X \to Y\) and \(k : K \to X\) and \(l : L \to X\) in \(\mathbb{C}\) if \(\gamma_{[k,l]}\) exists and \(mk\) and \(ml\) are normal monomorphisms, then \(m\gamma_{[k,l]}\) is the commutator morphism of \(mk\) and \(ml\).
It turns out that (NH) and (WNH) can be rephrased in terms of Higgins commutators in the appropriate context where those commutators exist (see [21] and [14] where, in fact, (WNH) and (NH) were defined in terms of Higgins commutators).

Our main aim is to show that in a pointed protomodular category satisfying (WNH), if $X$ is perfect and the generic split extension with kernel $X$ exists, then the centralizer of $c_X$ is a zero morphism.

In addition we produce categorical explanations of the following facts:

(i) if $f : X \to Y$ is a central extension of groups, and $\cdot$ and $\ast$ are actions of a group $B$ on $X$ such that, for each $b$ in $B$ each $x$ in $X$ we have that $f(b \cdot x) = f(b \ast x)$, then both actions have the same restriction to $[X, X]$ (Proposition 2.5);

(ii) if $\theta : X \to X$ is a central automorphism, then for each $x$ and $y$ in $X$ we have that $\theta([x, y]) = [x, y]$ (Corollary 3.3).

2 Lifting extensions and properties of morphisms with kernel a central monomorphism

In this section we establish some results about when split extensions can be lifted and the uniqueness of such liftings, which we will need in final section.

We begin by recalling some known facts. It is well known and easy to prove for a central extension $f : X \to Y$ of groups and for $a, a', b$ and $b'$ in $X$ if $f(a) = f(a')$, and $f(b) = f(b')$, then $[a, b] = [a', b']$. Categorically this follows from the following fact (which is essentially the same as Proposition 3.1 of [9]).

**Proposition 2.1.** Let $C$ be a pointed protomodular category, $f : A \to C$ a morphism in $C$, and $k : K \to A$ the kernel of $f$. If $k$ is central, then

(a) the pair $\varphi, \pi_2 : K \times A \to A$ (where $\varphi : K \times A \to A$ the cooperator of $k$ and $1_A$) is the kernel pair of $f$, and the composite $(0, 1)\gamma_{[1_A, 1_A]}$ is the commutator morphism of $1_{K \times A}$ with itself (provided $\gamma_{[1_A, 1_A]}$ exists);

(b) For pair of morphisms $u, v : W \to A$ if $fu = fv$, then $u\gamma_{[1_W, 1_W]} = v\gamma_{[1_W, 1_W]}$ (provided $\gamma_{[1_W, 1_W]}$ exists).

We will also need the following slight strengthening of Lemma 2.6 of [13].

**Lemma 2.2.** Let $C$ be a pointed protomodular category. For each split extension as displayed in the bottom row of the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\kappa'} & A' \\
\downarrow u & & \downarrow \alpha' \\
X & \xrightarrow{\kappa} & A \\
\downarrow v & & \downarrow \beta \\
& B
\end{array}
\]

\[\frac{\alpha'}{\beta} \cong \frac{\alpha}{\beta}\]
there exists a unique (up to isomorphism) lifting along a monomorphism $u$ to a morphism of split extensions, as displayed with dotted arrows, as soon as $\kappa u$ is normal (or more generally Bourn-normal).

Proof. The existence of a lifting follows from Lemma 2.6 of [14]. Uniqueness follows from e.g. Lemma 4.1 of [19].

It is easy to check and well known that for a central extension $f : X \to Y$ of a group if $\cdot, \ast$ are actions of a group $B$ on $X$ such that for each $x \in X$ and for each $b \in B$ we have that $f(b \cdot x) = f(b \ast x)$, then both actions have the same restriction to $[X, X]$. We will see that this can be recovered from the following categorical fact (which is closely related to Lemma 3.10 of [21]):

**Proposition 2.3.** Let $\mathcal{C}$ be a pointed protomodular category satisfying (WNH). For a morphism $f : X \to Z$ with kernel a central monomorphism and for morphisms of split extensions displayed in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa_1} & A_1 & \xleftarrow{\alpha_1} & B \\
\downarrow f & & \downarrow g_1 & & \downarrow \beta_1 \\
Z & \xrightarrow{\sigma} & C & \xleftarrow{\gamma} & B \\
\downarrow f & & \downarrow g_2 & & \downarrow \delta \\
X & \xrightarrow{\kappa_2} & A_2 & \xleftarrow{\alpha_2} & B
\end{array}
\]  

if $\gamma|_{1_X, 1_X}$ exists, then the unique (up to isomorphism) liftings of the split extensions at the top and bottom of (5) along $\gamma|_{1_X, 1_X}$ are isomorphic.

Proof. We begin by forming the pullback

\[
\begin{array}{ccc}
X \times Z & \xrightarrow{\kappa_1 \times \kappa_2} & A_1 \times C & \xleftarrow{\alpha_1 \pi_1} & B \\
\downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \beta_1 \beta_2 \\
X & \xrightarrow{\kappa_2} & A_2 & \xleftarrow{\alpha_2} & B \\
\downarrow f & & \downarrow g_2 & & \downarrow \delta \\
Z & \xrightarrow{\sigma} & C & \xleftarrow{\gamma} & B
\end{array}
\]  

in $\text{SplExt}(\mathcal{C})$. Since by Proposition 2.1 $\langle \gamma|_{1_X, 1_X}, \gamma|_{1_X, 1_X} \rangle$ is the commutator morphism of $1_{X \times Z}$ and itself, it follows from (WNH) that $(\kappa_1 \times \kappa_2)(\gamma|_{1_X, 1_X}, \gamma|_{1_X, 1_X})$
is normal. Therefore, Lemma 2.2 produces a morphism

\[
\begin{array}{c}
[X, X] \xrightarrow{\kappa} A \xrightarrow{\alpha} B \\
\downarrow \downarrow \\
X \times Z \xrightarrow{\kappa_1 \times \kappa_2} A_1 \times C A_2 \xrightarrow{\alpha_1 \pi_1} B,
\end{array}
\]

which after composing with the morphisms forming the upper and back faces of (6) produce liftings of the split extensions at the top and bottom of (5) with same domain, as desired.

For a semi-abelian category \( C \) and an object \( B \) in \( C \) let us recall from [10] that the functor sending a split extension in \( P^{-1}(B) \) (the fiber of \( P : \text{SplExt}(C) \to C \) above \( B \)) to its kernel turns out to be monadic, and the algebras of this monad, whose underlying functor we denote by \( B♭- \), are called internal \( B \)-actions. The induced equivalence associates to each split extension (1) the pair \((X, \chi)\), where \( \chi \) is the unique morphism making the diagram

\[
\begin{array}{c}
\text{B}♭X \xrightarrow{k} B + X \xrightarrow{[1, 0]} B \\
\chi \downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{\kappa} A \xrightarrow{\alpha} B,
\end{array}
\]

in which \( k \) is the kernel of \([1, 0]\), commute. In the case where \( C \) is the category of groups, it turns out that \( B♭X = \coprod_{b \in B} X \) and the internal \( B \)-actions are in one to one correspondence with the usual \( B \)-actions. Indeed, given \( \chi \) as above for each \( b \) in \( B \) and each \( x \) in \( X \) defining \( b \cdot x = \chi(\iota_b(x)) \) - where \( \iota_b : X \to \coprod_{b \in B} X \) is a coproduct injection - produces a \( B \)-action of \( B \) on \( X \). Translating Proposition 2.3 via this equivalence we obtain:

**Proposition 2.4.** Let \( C \) be a semi-abelian category satisfying (WNH). If \( f : X \to Z \) is a morphism in \( C \) with kernel central, and \( \chi_1, \chi_2 : B♭X \to X \) and \( \zeta : B♭Z \to Z \) are internal \( B \)-actions such that \( f \) is an internal \( B \)-action morphism from both \((X, \chi_1)\) and \((X, \chi_2)\) to \((Z, \zeta)\), then \( \chi_1 \) and \( \chi_2 \) restrict to the same internal \( B \)-action on \([X, X]\).

Recall that a normal monomorphism \( k : K \to X \) in a pointed category \( C \) is characteristic (in agreement with the definition in [15]) if for each normal monomorphism \( \eta : X \to Y \) the composite \( nk \) is a normal monomorphism. Recalling also that in the category of groups the center of each object exists and is characteristic we can recover the group theoretical fact mentioned just above Proposition 2.3 as a special case of the following proposition.

**Proposition 2.5.** Let \( C \) be a semi-abelian category satisfying (WNH) in which the center of each object exists and is characteristic. If \( f : X \to Z \) is a morphism in \( C \) with kernel central and \( \chi_1, \chi_2 : B♭X \to X \) are internal \( B \)-actions such that \( f \chi_1 = f \chi_2 \), then \( \chi_1 \) and \( \chi_2 \) restrict to the same internal \( B \)-action on \([X, X]\).
Proof. Without loss of generality we may assume that \( f \) is a normal epimorphism (since otherwise factorizing \( f \) as \( me \) where \( m \) is a monomorphism and \( e \) is a normal epimorphism produces a normal epimorphism \( e \) with same properties as \( f \)). Since the kernel of \( f \) is central it follows that it factors through \( z_X \), the center of \( X \), and hence, \( q : X \to Y \), the cokernel of \( z_X \), factors through \( f \). Since \( z_X \) is characteristic, it follows by Proposition 2.7 of [15] that there is a unique morphism \( \zeta : B \to Y \) such that \( q \) is an internal \( B \)-action morphism from \( (X, \chi_1) \) to \( (Y, \zeta) \). Since \( f \chi_1 = f \chi_2 \), it follows that \( q \chi_1 = q \chi_2 \) and so \( q \) is also a morphism from \( (X, \chi_2) \) to \( (Y, \zeta) \). We conclude via the Proposition 2.4.

Remark 2.6. It seems worth mentioning that if a category \( C \) has centralizers of normal monomorphisms which are normal, then centers exist and are characteristic. In such a category one easily proves that if \( k : X \to Y \) is normal monomorphism, then \( k \wedge z_X \cong kz_X \).

3 Split extension classifiers of perfect objects

In this section we state and prove our main results.

Recall that a reflexive graph \((A, B, \alpha_1, \alpha_2, \beta)\) in a category \( C \) is a quadruple where \( A \) and \( B \) are objects in \( C \), and \( \alpha_1, \alpha_2 : A \to B \) and \( \beta : B \to A \) are morphisms in \( C \) such that \( \alpha_1 \beta = 1_B = \alpha_2 \beta \). Recall also that a split extension \([1]\) is called faithful if there is at most one morphism in \( K^{-1}(X) \) from any split extension in \( K^{-1}(X) \). We have:

**Theorem 3.1.** Let \( C \) be a pointed protomodular category satisfying (WNH), let \((A, B, \alpha_1, \alpha_2, \beta)\) be a reflexive graph in \( C \), and let \( \kappa : X \to A \) be the kernel of \( \alpha_1 \).

(a) If a morphism \( g : Y \to B \) commutes with \( \alpha_2 \kappa \) and \( \gamma_{[1, X, 1_X]} \) exists, then \( \kappa \gamma_{[1, X, 1_X]} \) and \( \beta g \) commute.

(b) If \( X \) is perfect and the split extension

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa} & A \\
& \searrow{\alpha_1} & \swarrow{\beta} \\
& & B
\end{array}
\]

is faithful, then the centralizer of \( \alpha_2 \kappa \) is trivial.

**Proof.** Suppose \( \gamma_{[1, X, 1_X]} \) exists and that \( g : Y \to B \) is a morphism commuting
with \( f = \alpha_2 \kappa \). Consider the diagram

\[
\begin{array}{cccccc}
\gamma(1_X,1_X) & \gamma(1_X,1_X) & 1 \times \gamma(1_X,1_X) & 1 \times \gamma(1_X,1_X) & 1 \times \gamma(1_X,1_X) & 1 \times \gamma(1_X,1_X) \\
X & A & Y \times [X,X] & Y \times [X,X] & Y \times [X,X] & Y \\
\bar{f} & \bar{h} & \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\delta} \\
\bar{f} & \bar{h} & \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\delta} \\
X & A & Y \times [X,X] & Y \times [X,X] & Y \times [X,X] & Y \\
\bar{f} & \bar{h} & \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\delta} \\
B & B \times B & B \times B & B \times B & B \times B & B \\
\end{array}
\]

in which

- \( \varphi \) is the cooperator of \( f \) and \( g \);
- the object \((\bar{A}, h, \bar{\alpha})\) is the pullback of \( \alpha_1 \) and \( g \);
- the morphisms \( \bar{k} \) and \( \bar{\beta} \) are the unique morphisms such that \( h\bar{k} = \kappa, \alpha_1 \bar{k} = 0, h\bar{\beta} = \beta g \) and \( \bar{\alpha} \bar{\beta} = 1_Y \).

According to Proposition 2.3 (via Lemma 2.2) there must be a morphism \( u : Y \times [X,X] \to \bar{A} \) such that Diagram (7) consists of morphisms of split extensions. The proof of (iii) is completed by noting that \( hu \) produces a cooperator for \( \kappa\gamma(1_X,1_X) \) and \( \beta g \). To prove (i), we note that \( \alpha \) tells us that for any \( g \) commuting with \( f \), the morphisms \( \kappa\gamma(1_X,1_X) \) and \( \beta g \) commute. Therefore, when \( X \) is perfect \( \kappa \) and \( \beta g \) commute, since \( \gamma(1_X,1_X) \) is an isomorphism. But according to Proposition 5.2 of [11] (see also Lemma 2.4 of [13]) this means that \( g = 0 \).

Remark 3.2. Recall that a split extension \([11]\) can be equivalently defined to be eccentric \([8]\), if every morphism \( g : Y \to B \), such that \( \beta g \) and \( \kappa \) commute, is a zero morphism. Noting that, the above mentioned, Proposition 5.2 of [11] which, using this language, shows that “faithful” implies “eccentric”, one can generalize (b) of Theorem 3.1 by replacing “faithful” with “eccentric”.

Suppose that \( X \) is an object, in a pointed protomodular category, such that \([X]\) exists. Since the generic split extension \([3]\) is necessarily faithful and (as recalled above) forms part of a groupoid with underlying reflexive graph \([X] \times X, p_1, p_2, i\) such that \( c_X = p_2 k \), the previous theorem produces the following corollaries.
**Corollary 3.3.** Let $\mathcal{C}$ be a pointed protomodular category in satisfying (WNH). If $X$ is an object such that $\gamma([1_X,1_X])$, $[X]$ and $z_{c_X}$ exists, then the morphisms $k\gamma([1_X,1_X])$ and $iz_{c_X}$ commute.

**Corollary 3.4.** Let $\mathcal{C}$ be a pointed protomodular category satisfying (WNH). If $X$ is a perfect object in $\mathcal{C}$ such that $[X]$ exists, then $z_{c_X} = 0$ and hence $z_{[X]} = 0$.

**Remark 3.5.** Corollary 3.3 applied to the category of groups shows that if $\theta : X \rightarrow X$ is a central automorphism, then $\theta([x,y]) = [x,y]$. Applied to the category of Lie algebras it shows that if $d$ is a central derivation, then $d([x,y]) = 0$.

As an immediate consequence of Corollary 3.4 we obtain:

**Corollary 3.6.** Let $\mathcal{C}$ be a pointed protomodular category admitting all generic split extensions and satisfying (WNH). If $X$ is a perfect object in $\mathcal{C}$, then $z_{c_X} = 0$ and hence $z_{[X]} = 0$.

**Example 3.7.** Recall from [12] that algebraic coherence implies (NH) which is stronger than (WNH). Recall also that there are plenty of algebraically coherent categories, including all categories of interest in the sense of G. Orzech [24]. It follows that the above results apply to all action representable algebraically coherent categories. These include the categories of groups, Lie algebras, boolean rings and cocommutative Hopf algebras (for action representability of the first three categories see [6] and for the last one see [18], for algebraic coherence see [10]). Further examples can be obtained by taking a functor category of, or the category of internal groupoids in, a semi-abelian, algebraically coherent category with split extension classifiers and normalizers (see [19] and [17]). This produces, amongst many others, crossed modules of groups and Lie algebras as further examples.

**Proposition 3.8.** If $\mathcal{V}$ is a variety of semi-abelian algebras satisfying (WNH), then the category $\mathcal{V}(\text{Top})$ of internal $\mathcal{V}$ algebras in the category of topological spaces satisfies (WNH).

**Proof.** Recall that coequalizers and equalizers are calculated as in $\mathcal{V}$, equipped with the quotient and subspace topology, respectively (see [3]). This means that $\gamma([1_X,1_X]) = \ker(\text{coker}(\langle 1_X,1_X \rangle))/\langle 1,0 \rangle$ is calculated as in $\mathcal{V}$ with the subspace topology. Since a subspace of a subspace is a subspace of the whole space, this also implies that a normal monomorphism in $\mathcal{V}(\text{Top})$ is characteristic if and only if it is characteristic in $\mathcal{V}$. \hfill $\square$

**Example 3.9.** The category of topological groups has been shown to be action representable [12] and satisfies (WNH) by the previous proposition. It follows that our results hold for the category of topological groups.
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