Formation of Conic Cusps at the Surface of Liquid Metal in Electric Field

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The formation dynamics is studied for a singular profile of a surface of an ideal conducting fluid in an electric field. Self-similar solutions of electrohydrodynamic equations describing the fundamental process of formation of surface conic cusps with angles close to the Taylor cone angle 98.6° are obtained. The behavior of physical quantities (field strength, fluid velocity, surface curvature) near the singularity is established.

It is known [1, 2] that a flat boundary of liquid metal becomes unstable in a strong electric field. The development of instability results in conic cusp singularities, from which the strengthened field initiates emission processes [3, 4, 5, 6]. The description of these processes is a key problem of the electrohydrodynamics of conducting fluids with free surfaces; interest in this problem is largely caused by the practical use of liquid-metal sources of charged particles. The progress in this field is associated with Taylor’s work [7], where it was demonstrated that the surface electrostatic pressure \( P_E \) for a cone with angle 98.6° depends on the distance from its axis as \( r^{-1} \) and, hence, can be counterbalanced by the surface pressure \( P_S \sim r^{-1} \). Since the force balance is violated at the cone apex, Taylor’s solution cannot be treated as the exact solution of the problem of equilibrium configuration of a charged surface of conducting fluid and only represents the possible asymptotic form at \( r \to \infty \). At the same time, it turned out that Taylor’s solution nicely describes the experimentally observed surface shape before the instant of singularity formation. It was pointed out in [3, 4, 5, 6] that the angle of incipient conic formations is close to the Taylor cone angle.

What is the reason for such a coincidence? One may assume that the mechanism for the formation of conic cusps with an angle of 98.6° during a finite time is not directly associated with the static Taylor model. A high reproducibility of experimental results and a weak

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dependence of fluid behavior at the final instability stages on the geometry of the system suggest that the behavior of fluid near the singularity has a self-similar character.

Let us check the validity of this hypothesis. Consider the potential motion of an ideal fluid occupying the region bounded by free surface \( z = \eta(x, y, t) \). We will assume that the vector of an external electric field is directed along the \( z \) axis and equals \( E \). The velocity potential \( \Phi \) of fluid and the electric-field potential \( \varphi \) satisfy the Laplace equations

\[
\nabla^2 \Phi = 0, \quad \nabla^2 \varphi = 0
\]

with the following boundary conditions:

\[
\Phi_t + \frac{\left| \nabla \Phi \right|^2}{2} = \frac{\left| \nabla \varphi \right|^2}{8\pi \rho} + \frac{\alpha}{\rho} \frac{\nabla \cdot \nabla \eta}{\sqrt{1 + (\nabla \eta)^2}}, \quad z = \eta(x, y, t),
\]

\[
\eta_t = \Phi_z - \nabla \eta \cdot \nabla \Phi, \quad z = \eta(x, y, t),
\]

\[
\varphi = 0, \quad z = \eta(x, y, t),
\]

\[
\left| \nabla \Phi \right| \to 0, \quad z \to -\infty,
\]

\[
\varphi \to -Ez, \quad z \to \infty,
\]

(1)

where \( \alpha \) is the surface tension coefficient and \( \rho \) is the density of a medium.

We are interested in the dynamics of formation of a singular profile for a conducting fluid. It is natural to assume that the electric field near the cusp appreciably exceeds the external field; i.e., \( \left| \nabla \varphi \right| \gg E \). In this case, the interface evolution is fully determined by the intrinsic field, which decreases with distance from the singularity. One can thus use the condition

\[
\left| \nabla \varphi \right| \to 0, \quad z \to \infty
\]

(2)

instead of the field uniformity condition (1). This agrees with the assumption about the universal behavior of a fluid in the formation of a singular surface profile, because it allows the fluid motion near the singular point to be analyzed without regard for the particular geometry of the problem. The applicability of condition (2) will be discussed below in more detail after establishing some regularities for the dynamics of a conducting fluid near the singularity. Note that the possibility of secondary Taylor cones nucleating at the already formed cones counts in favor of the universal mechanism of formation of conic cusps [4]. It is
clear from this example that fluid "forgets" the boundary conditions at infinity at the stage of collapse.

Let us consider the most important case of the axially symmetric perturbation of the surface. Taking into account that, after substitutions

\[
\varphi \rightarrow \varphi 4\pi \alpha E^{-1}, \quad \Phi \rightarrow \Phi 2\pi \frac{1}{2} \rho^{-rac{1}{2}} \alpha E^{-1}, \quad \eta \rightarrow \eta 4\pi \alpha E^{-2},
\]

\[
r \rightarrow r 4\pi \alpha E^{-2}, \quad z \rightarrow z 4\pi \alpha E^{-2}, \quad t \rightarrow t 8\pi \frac{1}{2} \rho^{-rac{1}{2}} \alpha E^{-3},
\]

where \( r = \sqrt{x^2 + y^2} \), the equations of motion become dimensionless and do not contain any physical characteristics, one obtains

\[
\Phi_{\,rr} + r^{-1} \Phi_r + \Phi_{zz} = 0, \quad z < \eta(r, t), \quad (3)
\]

\[
\varphi_{\,rr} + r^{-1} \varphi_r + \varphi_{zz} = 0, \quad z > \eta(r, t), \quad (4)
\]

\[
\Phi_t + \frac{\Phi_r^2 + \Phi_z^2}{2} = \frac{\varphi_r^2 + \varphi_z^2}{2} + \frac{1}{1 + \eta_r^2} \left( \frac{\eta_r^2}{1 + \eta_r^2} + \frac{\eta_r}{r} \right), \quad z = \eta(r, t), \quad (5)
\]

\[
\eta_t = \Phi_z - \eta_r \Phi_r, \quad z = \eta(r, t), \quad (6)
\]

\[
\varphi = 0, \quad z = \eta(r, t), \quad (7)
\]

\[
\Phi_r^2 + \Phi_z^2 \rightarrow 0, \quad \varphi_r^2 + \varphi_z^2 \rightarrow 0, \quad r^2 + z^2 \rightarrow \infty, \quad (8)
\]

\[
\Phi_r = 0, \quad \varphi_r = 0, \quad \eta_r = 0, \quad r = 0. \quad (9)
\]

These equations allow the only self-similar substitution

\[
\Phi(x, y, z, t) = \bar{\Phi}(\bar{r}, \bar{z}) \bar{r}^{1/3}, \quad (10)
\]

\[
\varphi(x, y, z, t) = \bar{\varphi}(\bar{r}, \bar{z}) \bar{r}^{1/3}, \quad (11)
\]

\[
\eta(x, y, t) = \bar{\eta}(\bar{r}) \bar{r}^{2/3}, \quad (12)
\]
where \( \tau = t_c - t \) and \( t_c \) is the collapse time. This substitution occurs due to the fact that Eqs. (3)–(9) are invariant about dilatations

\[
\Phi \to \Phi c, \quad \varphi \to \varphi c, \quad \eta \to \eta c^2,
\]
\[
r \to rc^2, \quad z \to zc^2, \quad t \to tc^3,
\]
i.e., it occurs, in fact, from dimensional considerations (\( c \) is an arbitrary constant). Note that the initial electrohydrodynamic equations with condition (1) do not permit one to introduce any self-similar variables.

Substituting Eqs. (10)–(14) in Eqs. (3)–(9), one finds that the functions \( \tilde{\Phi}, \tilde{\varphi} \) and \( \tilde{\eta} \) obey the following set of partial differential equations:

\[
\tilde{\Phi}_{\tilde{r}\tilde{r}} + \tilde{r}^{-1}\tilde{\Phi}_\tilde{r} + \tilde{\Phi}_{\tilde{z}\tilde{z}} = 0, \quad \tilde{z} < \tilde{\eta}(\tilde{r}),
\] (15)

\[
\tilde{\varphi}_{\tilde{r}\tilde{r}} + \tilde{r}^{-1}\tilde{\varphi}_\tilde{r} + \tilde{\varphi}_{\tilde{z}\tilde{z}} = 0, \quad \tilde{z} > \tilde{\eta}(\tilde{r}),
\] (16)

\[
\frac{2\tilde{\Phi}_{\tilde{r}} + 2\tilde{\Phi}_{\tilde{z}} - \tilde{\Phi}}{3} + \frac{\tilde{\varphi}^2}{2} + \frac{\tilde{\eta}^2}{2} = \frac{\tilde{\varphi}^2}{2} + \frac{\tilde{\eta}^2}{2} + \frac{1}{1 + \tilde{\eta}_F^2} \left( \frac{\tilde{\eta}_{\tilde{r}\tilde{r}} + \tilde{\eta}_\tilde{r}}{\tilde{r}} \right), \quad \tilde{z} = \tilde{\eta}(\tilde{r}),
\] (17)

\[
2\tilde{\eta}_{\tilde{r}} - 2\tilde{\eta} = 3\tilde{\Phi}_{\tilde{z}} - 3\tilde{\eta}_F \tilde{\Phi}_\tilde{r}, \quad \tilde{z} = \tilde{\eta}(\tilde{r}),
\] (18)

\[
\tilde{\varphi} = 0, \quad \tilde{z} = \tilde{\eta}(\tilde{r}),
\] (19)

\[
\tilde{\Phi}_\tilde{r} + \tilde{\Phi}_\tilde{z} \to 0, \quad \tilde{\varphi}_\tilde{r} + \tilde{\varphi}_\tilde{z} \to 0, \quad \tilde{r}^2 + \tilde{z}^2 \to \infty,
\] (20)

\[
\tilde{\Phi}_\tilde{r} = 0, \quad \tilde{\varphi}_\tilde{r} = 0, \quad \tilde{\eta}_F = 0, \quad \tilde{r} = 0.
\] (21)
For self-similar solutions (10)–(14), the surface profile forms first at the periphery and then extends to the center \( r = z = 0 \) (the scale decreases as \( \tau^{2/3} \)). This implies that the formation of conic cusps at \( t = t_c \) is described by those solutions to the set of Eqs. (15)–(21) which provide conic asymptotic shape of the surface. In such a situation, the presence of asymptotic solutions for which \( \tilde{\eta} \sim \tilde{r} \) at \( \tilde{r} \to \infty \) is the necessary condition for the validity of our assumption about the self-similar nature of conic formations.

Analysis of Eqs. (15)–(21) in the limit \( R = \sqrt{\tilde{r}^2 + \tilde{z}^2} \to \infty \) showed that they have an asymptotic solution of the form

\[
\tilde{\varphi}(\tilde{r}, \tilde{z}) = p^{-1}[2R(s_0 - s)]^{1/2}P_{1/2}(\cos \theta),
\]

\[
\tilde{\Phi}(\tilde{r}, \tilde{z}) = sR^{-1},
\]

\[
\tilde{\eta}(\tilde{r}) = -s_0\tilde{r},
\]

\[
P_{1/2}(\cos \theta_0) = 0,
\]

\[
p = \left[dP_{1/2}(\cos \theta)/d\theta\right]_{\theta=\theta_0},
\]

\[
s_0 = -\cot \theta_0,
\]

where \( \theta = \arctg(\tilde{r}/\tilde{z}) \) is the polar distance in spherical coordinates, \( P_{1/2} \) is the Legendre polynomial of order \( \frac{1}{2} \), and \( s \) is a constant satisfying inequality \( 0 < s < s_0 \). This solution describes a conic surface with an angle of \( 2\pi - 2\theta_0 \) that is equal to approximately 98.6°, i.e., to the Taylor cone angle. According to Eq. (23), the fluid motion is spherically symmetric, and fluid moves to the sink point \( R = 0 \) along the tangent to the surface (24). Since the self-similar solution assumes its asymptotic form at \( \tau \to 0 \), a conic cusp with Taylor angle forms at time \( t_c \), and Eqs. (22)–(24) are the exact analytic solution of the problem. The electric field at the cusp increases as \( \tau^{-1/3} \), the cusp growth velocity increases as \( \tau^{-1/3} \), and the cusp curvature increases as \( \tau^{-2/3} \); i.e., these quantities become infinite during a finite time. At an appreciable distance from the singularity, the field strength does not change, and the velocity of fluid linearly decreases with time and becomes zero at \( t = t_c \). The latter fact allows one to explain qualitatively the mechanism of transition to the stationary regime occurring for liquid-metal sources after the initiation of field ion evaporation from the cusp (stationary emitter models were developed in [8, 9]).
This analysis is only valid on the condition that the asymptotic solutions to the set of partial differential Eqs. (15)–(21) have the form of Eqs. (22)–(24). To prove this statement, one should construct the asymptotic expansion for the solutions at \( R \to \infty \) with leading terms given by Eqs. (22)–(24). Let us seek this expansion in the form

\[
\tilde{\varphi}(\tilde{r}, \tilde{z}) = \sum_{n=0}^{\infty} a_n \frac{\partial^{3n}}{\partial \tilde{z}^{3n}} \left[ R^{1/2} P_{1/2}(\cos \theta) \right],
\]

(25)

\[
\tilde{\Phi}(\tilde{r}, \tilde{z}) = \sum_{n=0}^{\infty} b_n \frac{\partial^{3n}}{\partial \tilde{z}^{3n}} \left[ R^{-1} \right],
\]

(26)

\[
\tilde{\eta}(\tilde{r}) = \sum_{n=0}^{\infty} c_n \tilde{r}^{1-3n},
\]

(27)

where it is taken into account that the derivative of a harmonic function of any order with respect to \( \tilde{z} \) is also a harmonic function. The zero-order coefficients are determined by Eqs. (22)–(24),

\[ a_0 = p^{-1} [2(s_0 - s)]^{1/2}, \quad b_0 = s, \quad c_0 = s_0. \]

It turns out that, to the first order, the surface is conic:

\[ a_1 = 0, \quad b_1 = -\frac{s^2(1 + s_0^2)^{3/2}}{18s_0(3 - 2s_0^2)}, \quad c_1 = 0. \]

The correction to Eq. (24) for the surface shape appears in the next order. One finds from kinematic boundary condition (18) that

\[ c_2 = -\frac{s^2(4s_0^2 - 1)}{8s_0^3(1 + s_0^2)^2(3 - 2s_0^2)}. \]

The coefficients \( a_2 \) and \( b_2 \) can be determined from Eq. (19) and, correspondingly, Eq. (17), where one should use the linear order of perturbation theory for small deviation of the surface from the cone. In turn, the coefficient \( c_3 \) is found from Eq. (18), where one should take into account the quadratic nonlinearity, etc. Thus, the expansion coefficients are uniquely determined by the zero-order coefficients, i.e., by the parameter \( s \) of the problem, confirming the existence of solutions with the desired asymptotic form for the equations of motion. Note that, if one sets \( s = 0 \) and, hence, \( \Phi = 0 \), then Eqs. (15)–(21) coincide with the Taylor equations in the problem of equilibrium configuration of a charged liquid-metal surface. However, one fails to construct asymptotic expansion (25)–(27) in this case.
Further, the solutions given by expansions (25)-(27) adequately describe the experimental data if the condition $\tilde{\eta} < -s_0 \tilde{r}$ is satisfied, i.e., if the fluid surface is positioned above the asymptotic cone (see figure 1). Otherwise, the surface velocity would be directed in opposition to the $z$ axis. Let us check how this condition is fulfilled in the limit of large $r$. It follows from the expansion obtained above for the surface shape that it deviates from the conic shape in the direction specified by the sign of the $c_2$ coefficient. In our case, $c_2 < 0$ ($c_2 > 0$ for cone angles smaller than 78.5° and larger than 126.9°), so that the amplitude of surface perturbation should increase. Indeed, the evolution of the fluid boundary away from the singularity is determined by the leading terms of the expansion in small $\tau$ value,

$$\eta(r, t) = -s_0 r - |c_2| \tau^4 r^{-5},$$

from whence it follows that, when forming a conic cusp, the fluid moves upwards, as is expected from physical considerations.

Let us now consider the surface geometry for small $\tilde{r}$, where expansion (27) diverges. For the function $\tilde{\eta}$ to satisfy condition (21), the surface near the cone apex must be "rounded off" (figure). Let us estimate the distance $|\tilde{\eta}(0)|$ from the cone apex to the fluid surface. Multiplying kinematic boundary condition (18) by $2\pi \tilde{r}/3$ and integrating it over $\tilde{r}$, one obtains after simple mathematics

$$2V = \int_S \partial_n \tilde{\Phi} dS,$$

where $S$ stands for the fluid surface $\tilde{z} = \tilde{\eta}(\tilde{r})$, $V$ is the volume of a region bounded from above by the conic surface $\tilde{z} = -s_0 \tilde{r}$ and from below by the $S$ surface, and $\partial_n$ denotes the
derivative along the normal to $S$. The integral on the right-hand side of this expression is equal to the fluid velocity flux through the surface. Since the function $\tilde{\Phi}$ is harmonic, the vector-field flux $\nabla \tilde{\Phi}$ through any closed surface is zero. This fact allows the flux through the surface $S$ to be determined using the asymptotic form of velocity potential at $R \to \infty$. Taking into account that the fluid flows into a solid angle $2\pi(1 + \cos \theta_0)$ at infinity, one has from Eq. (23)

$$\int_S \partial_n \tilde{\Phi} dS = 2\pi s (1 + \cos \theta_0),$$

and, hence, $V = \pi s (1 + \cos \theta_0)$. Notice that the volume of a region bounded by the conic surface $\tilde{z} = -s_0 \tilde{r}$ and the plane $\tilde{z} = -h$ (a circular right cone of height $h$) is equal to $V$ at

$$h = h(s) = \left[3ss_0^2(1 + \cos \theta_0)\right]^{1/3}.$$

Clearly, if the volume $V$ is fixed and the conditions $\tilde{\eta}(\tilde{r}) + s_0 \tilde{r} < 0$ and $\tilde{\eta}(\tilde{r}) \leq 0$ are fulfilled for any $\tilde{r}$, the quantity $|\tilde{\eta}(0)|$ cannot exceed the cone height. That is, the inequality

$$|\tilde{\eta}(0)| \leq h(s),$$

connecting the characteristic spatial scale at small $R$ with the asymptotic parameter $s$ is satisfied. Since the most probable value of $h(s)$ corresponds to the maximum allowable value $s_0$ of the $s$ constant, the following estimate is also valid:

$$|\tilde{\eta}(0)| \leq h(s_0) = s_0(3 + 3 \cos \theta_0)^{1/3},$$

which does not involves the free parameter $s$.

Let us return to the question of the applicability of approximation (3)–(9) to the initial equations of motion. As was pointed out above, condition (2) can be used instead of Eq. (1) only if the external electric field is weaker than the intrinsic cusp field. This implies that the inequality $\varphi_r^2 + \varphi_z^2 \gg 1$ must be fulfilled. After the transition to the self-similar variables, it is recast as

$$\varphi_r^2 + \varphi_z^2 \gg \tau^{2/3}.$$

It is clear that for small $\tau$ (i.e., immediately before the collapse) this condition is fulfilled near the singularity in a natural way. Because one can write $|\nabla \varphi| \sim (r^2 + z^2)^{-1/4}$ at small $\tau$, one has $|\nabla \varphi| \gg 1$ in a rather close vicinity of the singularity. In this case, $R_0$ and $\tau_0$ values exist for which model (3)–(9) with $0 \leq r^2 + z^2 < R_0^2$ and $0 \leq \tau < \tau_0$ adequately
describes the strongly nonlinear stages of electrohydrodynamic instability development for the free surface of a conducting fluid in an external electric field. At \( r^2 + z^2 > R_0^2 \), the role of nonlinear processes is rather insignificant; the condition for field uniformity at infinity (1) should be used together with the corresponding conditions in the limit \( r \to \infty \), and, in particular, with the condition for spatial localization of surface perturbation: \( \eta \to 0 \) at \( r \to \infty \). In this region, the evolution of fluid surface is described by perturbation theory for small surface slope; this procedure was implemented, e.g., in [10, 11].

In conclusion, let us discuss the possibility to form stronger singularities-cuspidal points. It is known [12] that the field mainly increases as \( r^{-1} \) upon approaching the apex of a thin point and, hence, the electrostatic pressure \( P_E \) increases as \( r^{-2} \). Since the surface pressure changes as \( P_S \sim r^{-1} \), \( P_E \gg P_S \) near the singularity, and the capillary effects can be ignored. It was shown in [10] that in the absence of surface tension weak root singularities \( \eta \sim r^{3/2} \) form, for which the curvature is infinite, while the surface itself remains smooth. Therefore, when assuming that the cuspidal points may appear, one arrives at a contradiction. This gives grounds to assume that conic singularities are precisely those which are the generic singular solutions of the electrohydrodynamic equations, so that the behavior of a charged liquid-metal surface with cusps is described by self-similar solutions (10)–(21).

Note also that the results of this work can be extended to dielectric fluids, in which the conic cusps with angles depending on the dielectric constant can form in an electric field [13]. In addition, the approach developed in this work can be applied to the description of the evolution of dimples sharpening in a finite time at a liquid helium surface (see, e.g., experimental work [14]). In my preceding work [15] devoted to the construction of exact analytic solutions to the equations of motion for liquid helium in the presence of weak capillary effects, I proved that cuspidal points \( \eta \sim |x|^{2/3} \) appear at the surface in planar geometry. The question of the singularity type for axial symmetry has not been considered so far.

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