Very ample linear systems on abelian varieties
preliminary version

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1 Introduction

Let \(M\) be an ample line bundle on a abelian variety \(X\). (Throughout this paper, we shall work over the complex numbers \(\mathbb{C}\). Hence by abelian variety

*The first author is partially supported by N.S.F. grant DMS 92-03919 and the European Science Project “Geometry of Algebraic Varieties”, contract no. SCI-0398-C (A). The second and third author are supported by the DFG project “Schwerpunktprogramm komplexe Mannigfaltigkeiten”. 
we always mean complex abelian variety.) It is known that $M^n$ is very ample for $n \geq 3$ (Lefschetz theorem) and that $M^2$ is very ample if and only if $|M|$ has no fixed divisor (Ohbuchi theorem). Very little is known for line bundles on $X$ which are not non-trivial powers of ample line bundles.

In this article we consider exclusively the case of line bundles $L$ of type $(1, \ldots, 1, d)$, which are exactly the pullbacks of principal polarizations by cyclic isogenies of degree $d$.

When $X$ is an abelian surface, such an $L$ is base-point-free if and only if $d \geq 3$ and $(X, L)$ is not a product [2, lemma 10.1.2]; it is very ample if and only if $d \geq 5$ and $X$ contains no elliptic curve $E$ such that $L.E \leq 2$ [2, corollary 10.4.2].

In higher dimensions, we show that for $(X, L)$ generic of type $(1, \ldots, 1, d)$ and dimension $g$, the linear system $|L|$ is base-point-free if and only if $d \geq g+1$ (see proposition [2]). Moreover, the morphism $\phi_L : X \to \mathbb{P}^{d-1}$ that it defines is birational onto its image if and only if $d \geq g + 2$ (see proposition [6]).

These results are part of a general conjectural picture: for $d > g$, the morphism $\phi_L$ should be an embedding outside of a set of dimension $2g+1-d$. In particular, $L$ should be very ample if and only if $d \geq 2g + 2$. (For $g > 2$, Barth and Van de Ven have shown that $L$ cannot be very ample for $d \leq 2g+1$ [1, 16].)

We show that for $(X, L)$ generic of type $(1, \ldots, 1, d)$ and dimension $g$, the line bundle $L$ is very ample for $d > 2g$, by checking it on a rank-$(g-1)$ degeneration (see corollary [25]). The same result was proved in [3] by a different method for $g = 3$ and $d \geq 13$. For $g = 3$, this leaves only the case $d = 8$ open (the polarization is never very ample in that case for the degenerations we consider).

As shown by Kollár in [4], our result in dimension 3 implies the following version of a conjecture of Griffiths and Harris: for $d$ odd, $d \geq 9$, the degree of any curve on a very general hypersurface of degree $6d$ in $\mathbb{P}^4$ is divisible by $d$.

This paper was completed during a visit of the first author at the University of Hannover. The authors are grateful to the DFG for financial support which made this visit possible.
2 Linear systems on abelian varieties

In this section we focus on the behaviour of morphisms \( \phi_L : A \to \mathbb{P}(H^0(A, L)^*) \), where \( A \) is a generic abelian variety and \( L \) an ample line bundle on \( A \), such that \( h^0(A, L) = d \) or, equivalently, \( L^g = g!d \).

**Theorem 1** Let \( A \) be an abelian variety of dimension \( g \) and let \( \phi : A \to \mathbb{P}^{d-1} \) be a finite morphism. Then
(i) the ramification locus of \( \phi \) has dimension at least \( 2g - d \)
(ii) if \( F \) is a closed subset of \( A \) such that the restriction of \( \phi \) to \( A - F \) is an embedding, then \( \dim(F) \geq 2g + 1 - d \), except if \( (g, d) = (1, 3) \) or \( (2, 5) \).

**Proof:** The ramification of \( \phi \) is the locus where \( d\phi : TA \to \phi^*T\mathbb{P}^{d-1} \) has rank less than \( g \). Since \( \phi \) is finite, \( \phi^*T\mathbb{P}^{d-1} \) is ample, hence so is \( \phi^*T\mathbb{P}^{d-1} \otimes T^*A \). By [7, theorem 1.1], this locus is non-empty if \( g \geq d - 1 - (g - 1) = d - g \) and has dimension at least \( g - (d - g) \). This proves (i).

To prove (ii), we follow ideas of Van de Ven ([16]). Assume that \( \dim(F) \leq 2g - d \). The intersection of \( \phi(A) \) with \( s = 2g - d + 1 \) generic hyperplanes is a smooth irreducible \((g - s)\)-dimensional scheme \( S \) contained in \( \phi(A - F) \). Note that \( S \) sits in \( \mathbb{P}^{2(g-s)} \), so that we can do a Chern class computation as in [16].

Set \( l = c_1(O_S(1)) \). Then \( c(TS) = (1 + l)^{-s} \) and the exact sequence

\[
0 \to TS \to T\mathbb{P}^{2(g-s)}|_S \to N \to 0
\]

gives \( c(N) = (1 + l)^d \). It follows ([16, prop. 3]) that \( (\deg S)^2 = c_{g-s}(N) = \binom{d}{g-s}g^{-s} \). Since \( g^{-s} = \deg S = \deg A \), we finally get \( \deg A = \binom{d}{g+1} \). Since the degree of any ample line bundle on an abelian variety of dimension \( g \) is divisible by \( g! \), the conclusion is that \( \binom{d}{g+1} \) is divisible by \( g! \). We may assume that \( g < d \leq 2g + 1 \). We will let the reader check that this can only happen for \( g = 1 \) or \( 2 \). (For the case \( d = 2g + 1 \), see [16].) Cases where \( d = g + 1 \) are trivially excluded and so is \( (g, d) = (2, 4) \).

This finishes the proof of (ii). \( \Box \)

One might expect that there is equality in (i) and (ii) for a generic abelian variety, but the answer probably depends on the type of the polarization \( \phi^*O_F(1) \). From now on, we will restrict ourselves to polarizations of type \((1, \ldots, 1, d) \). We shall need the following facts.
Let $L$ be a line bundle of type $(1, \ldots, 1, d)$ on an abelian variety $A$. We define the groups $G(L)$ and $K(L)$ as in [2, chapter 6]. Then, $K(L)$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^2$, and there is a central extension:

$$1 \to \mathbb{C}^* \to G(L) \xrightarrow{\pi} K(L) \to 0.$$ 

The group $G(L)$ operates on $H^0(A, L)$ [11, p. 295]. If $\tilde{\epsilon}$ is an element of $G(L)$ of order $d$, it generates a maximal level subgroup of $G(L)$ (in the sense of [11, p. 291], hence there exists a non-zero section $s$ of $L$ unique up to multiplication by a non-zero scalar, such that $\tilde{\epsilon} \cdot s = s$ [11, prop. 3]. In particular, for $0 \leq \lambda < d$, there exists a non-zero section $s_\lambda$ such that $\tilde{\epsilon} \cdot s_\lambda = e^{2i\pi \lambda/d}s_\lambda$. Given $A$, $L$ and $\tilde{\epsilon}$, the ordered set $\{s_0, \ldots, s_{d-1}\}$ is well-defined up to multiplication of its elements by non-zero scalars; it is a basis for $H^0(A, L)$, which we will call a canonical basis.

**Proposition 2** Let $(A, L)$ be a generic polarized abelian variety of dimension $g$ and type $(1, \ldots, 1, d)$. Then $|L|$ is base-point-free if and only if $d > g$.

**Proof:** If $d \leq g$, then $d$ elements in $|L|$ always intersect, since $L$ is ample.

Now let $d > g$. It is enough to exhibit one example. The construction is the same as in [1]. Let $E_1, \ldots, E_g$ be elliptic curves, let $\epsilon_j$ be a point of order $d$ on $E_j$, for $j = 1, \ldots, g$ and let $\pi : E_1 \times \cdots \times E_g \to A$ be the quotient by the subgroup generated by $\epsilon_j - \epsilon_k$, for $1 \leq j < k \leq g$.

Let $0_j$ be the origin of $E_j$, let $L_j = \mathcal{O}_{E_j}(d0_j)$ and let $M$ be the polarization $\bigotimes_{j=1}^g \text{pr}_j^*L_j$ on $E_1 \times \cdots \times E_g$. Pick a lift $\tilde{\epsilon}_j$ of $\epsilon_j$ of order $d$ in $G(L_j)$. Then the $\tilde{\epsilon}_j \tilde{\epsilon}_1^{-1}$, for $1 < j \leq g$, generate a level subgroup of $G(M)$ hence, by [1] prop. 1], there exists a polarization $L$ on $A$ of type $(1, \ldots, 1, d)$ such that $\pi^*L = M$. Moreover, if $\{s_{j,\lambda}\}_{0 \leq \lambda \leq d-1}$ is a canonical basis for $(E_j, L_j, \tilde{\epsilon}_j)$, then $\{s_{1,\lambda}s_{2,\lambda}\cdots s_{g,\lambda}\}_{0 \leq \lambda \leq d-1}$ is a basis for $\pi^*H^0(A, L)$.

If all these sections vanish at some point $(e_1, \ldots, e_g)$ and $d > g$, then at least two different $s_{j,\alpha}$, $s_{j,\beta}$ must vanish at $e_j$ for some $j$. But this cannot happen since $\text{div}(s_{j,\beta})$ and $\text{div}(s_{j,\alpha})$ are distinct and are both translates of the divisor $\sum_{i=1}^d (le_i)$. Hence $L$ is base-point-free.

**Remarks 3** (i) A similar argument shows that for $0 < d \leq g$, the generic base locus has dimension exactly $g - d$. 

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By an argument similar to the one used in [12], the following can be shown. Let $A$ be a moduli space of polarized abelian varieties of dimension $g$ and degree $g + 1$. Then the locus of polarized abelian varieties for which the corresponding linear system has a base point is either $A$ or a divisor. The proposition implies that it is a divisor when the type is $(1, \ldots, 1, g + 1)$. It may happen that this locus be everything (e.g. for type $(1, 2, 2)$ (cf. also [13])).

In view of theorem 1, it is tempting to make the following

**Conjecture 4** Let $(A, L)$ be a generic polarized abelian variety of dimension $g > 2$ and type $(1, \ldots, 1, d)$ with $d > g$ and let $\phi : A \to \mathbb{P}^{d-1}$ be the morphism associated with $|L|$ (cf. proposition 3). Then the ramification of $\phi$ has dimension $2g - d$ and there exists a closed subset $F$ of $A$ of dimension $2g + 1 - d$ such that the restriction of $\phi$ to $A - F$ is an embedding.

It is of course understood that a set of negative dimension is empty. In particular, the conjecture implies that for $d \geq g + 2$, the morphism $\phi$ should be birational onto its image. This is proven below (proposition 6). For $d \geq 2g + 2$, the line bundle $L$ should be very ample. In §5, we will prove that this is the case for $d > 2^g$ (see theorem 24).

The following proposition shows that, to prove the conjecture for given $d$ and $g$, it is enough to exhibit one polarized abelian variety, or a suitable degeneration, for which it holds.

**Proposition 5** Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
g \downarrow & & \downarrow h \\
W & \xrightarrow{g} & W,
\end{array}
\]

where $X$, $Y$, $W$ are analytic (resp. algebraic) varieties and $f$, $g$ and $h$ are proper. Let $p$ be a point in $W$, let $X_p$ be the fibre of $f$ over $p$ and assume that there is a closed (resp. Zariski closed) subset $F_p$ of $X_p$ such that the restriction of $f$ to $X_p - F_p$ is unramified (an embedding). Then, for all points $w$ in an open (resp. Zariski open) neighbourhood of $p$ in $W$, there exists a closed (resp. Zariski closed) subset $F_w$ of $X_w$ with $\dim(F_w) \leq \dim(F_p)$ such that the restriction of $f$ to $X_w - F_w$ is unramified (an embedding).
Proof: Let $G$ be the support of $\Omega_{X/Y}$. Since $\Omega_{X/Y} \otimes \mathcal{O}_{X_p} \cong \Omega_{X_p/Y_p}$ and $f|_{X_p-F_p}$ is unramified, $G \cap X_p$ is contained in $F_p$. By semicontinuity of the dimension of the fibres of a morphism [3] prop. 3.4, p. 134, this proves the first part of the proposition, since $f$ is unramified outside of $G$.

Assume now that $f|_{X_p-F_p}$ is an embedding. Let $Z$ be the union of the components of $X \times_Y X$ other than the diagonal $\Delta_X$, whose image in $W$ contains $p$. Then, by definition of $\Omega_{X/Y}$, the set $Z \cap \Delta_X$ is contained in $G \times G$; moreover, since $f|_{X_p-F_p}$ is injective, $Z_p - \Delta_{X_p}$ is contained in $F'_p \times F'_p$, where $F'_p = f^{-1}(f(F_p))$. Therefore, $Z_p$ is contained in $F'_p \times F'_p$. Since $F'_p - F_p$ embeds in $f(F_p)$, one has $\dim(F'_p) = \dim(F_p)$. This finishes the proof: one can take for $F_w$ the first projection of $Z_w$ in $X_w$.

Proposition 6 Let $(A, L)$ be a generic polarized abelian variety of dimension $g$ and type $(1, \ldots, 1, d)$ with $d \geq g+2$. Then the morphism $\phi : A \to \mathbb{P}^{d-1}$ associated with $|L|$ is birational onto its image.

Proof: Let $\mathcal{A}_{g,d,\theta}$ be the moduli space of abelian varieties $A$ of dimension $g$, with a polarization $L$ of type $(1, \ldots, 1, d)$ and a point $\tilde{\epsilon}$ of order $d$ in $G(L)$. Let $P = (A, L, \tilde{\epsilon})$ be a point in $\mathcal{A}_{g,d,\theta}$, and let $\{s_0, \ldots, s_{d-1}\}$ be a corresponding canonical basis for $H^0(A, L)$.

We first claim that for $P$ in a dense open set $U$ of $\mathcal{A}_{g,d,\theta}$, no $g+1$ distinct sections in the canonical basis have a common zero. Since $\mathcal{A}_{g,d,\theta}$ is irreducible [2] chapter 8, §3, it is enough to find one point $P$ for which this property holds. This follows directly from the proof of proposition 1: keeping the same notation, the point $\tilde{\epsilon}_1$ of $G(L_1)$ corresponds to a point $\tilde{\epsilon}$ of $G(L)$ of order $d$ [4] prop. 2], and the basis for $H^0(A, L)$ given in the proof is canonical for $(A, L, \tilde{\epsilon})$; no $g+1$ elements of this basis vanish simultaneously.

Now let $(A, L, \tilde{\epsilon})$ be an element of $U$ with $A$ simple (i.e. such that $A$ contains no non-trivial abelian subvarieties). The choice of a canonical basis defines a morphism

$$\phi : A \to \mathbb{P}^{d-1}.$$ 

Assume that $\phi$ is not birational over its image. Then there exists a component $D$ of $A \times_{\mathbb{P}^{d-1}} A$, distinct from the diagonal, such that the first projection $D \to A$ is surjective. In particular, $\dim(D) \geq g$. Let $m$ be the morphism

$$m : D \to A$$

$$(x, y) \mapsto x - y.$$
Let $a$ be a generic point in $m(D)$, let $F$ be an irreducible component of $m^{-1}(a)$ and let $G = \text{pr}_2(F)$. Then

$$F = \{(x + a, x); x \in G\}.$$  

This implies that $\phi(x) = \phi(x + a)$ for all $x \in G$, hence $L|_G \cong \tau_{-a}^*L|_G$. (For any $x \in A$, $\tau_x : A \to A$ denotes translation over $x$.) Therefore, $a$ lies in the kernel of

$$A \to \text{Pic}^0(\Gamma)$$

$$x \mapsto (\tau_x^* L \otimes L^{-1})|_\Gamma.$$  

Since $A$ is simple, it follows that either $a$ is torsion, in which case $m$ is constant with image $a$, or else $m^{-1}(a)$ is finite, in which case $m$ is surjective. In the first case, all elements of $|L|$ are invariant by translation by $a$, which implies $a = 0$ and contradicts our choice of $D$. In the second case, there exists $x \in A$ such that $\phi(x) = \phi(x + \epsilon)$, where $\epsilon$ is the image of $\tilde{\epsilon}$ in $K(L)$. Since

$$\phi(x + \epsilon) = (s_0(x), \omega s_1(x), \ldots, \omega^{d-1}s_{d-1}(x)),$$

where $\omega = e^{2i\pi/d}$, it follows that all of the $d$ sections in the canonical basis but one vanish at $x$. When $d \geq g + 2$, this contradicts the fact that $P \in U$. It follows that $\phi$ is birational onto its image.

\[\square\]

## 3 Degeneration of abelian varieties

### 3.1 Abelian varieties

We consider the Siegel space of degree $g$, i.e.

$$\mathcal{H}_g = \{\tau \in M(g \times g, \mathbb{C}); \tau = \tau, \text{Im } \tau > 0\}.$$  

Fix an integer $d \geq 1$. Then every point

$$\tau = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{pmatrix} \in \mathcal{H}_g$$
defines a \((1, \ldots, 1, d)\)-polarized abelian variety, namely

\[ A_\tau = \mathbb{C}^g / L_\tau, \]

where \(L_\tau\) is the lattice spanned by the rows of the period matrix

\[ \Omega_\tau = \begin{pmatrix} \tau \\ D \end{pmatrix}, \quad D = \text{diag}(1, \ldots, 1, d). \]

We are interested in certain degenerations of these abelian varieties, namely those which arise if \(\tau_{11}, \ldots, \tau_{g-1,g-1}\) go to \(i\infty\). We shall first treat the principally polarized case, i.e. \(d = 1\). The general case can then be derived easily from this.

We will employ the following notation

\[ e : \mathbb{C} \to \mathbb{C}^*, \quad z \mapsto e^{2\pi i z}. \]

Let

\[ t_{ij} = e^{\tau_{ij}}. \]

Let \(z_1, \ldots, z_g\) be the standard coordinates on \(\mathbb{C}^g\) and let

\[ w_i = e(z_i). \]

The abelian variety \(A_\tau\) can be written as a quotient

\[ A_\tau = \mathbb{Z}^g \backslash (\mathbb{C}^*)^g, \]

where \(l = (l_1, \ldots, l_g) \in \mathbb{Z}^g\) operates on \((\mathbb{C}^*)^g\) by

\[ l(w_1, \ldots, w_g) = (w'_1, \ldots, w'_g) \]

\[ w'_i = \prod_{j=1}^g t_{ij}^{l_j} w_i. \]

We are interested in what happens as \(t_{11}, \ldots, t_{g-1,g-1}\) go to zero.
3.2 Toroidal embedding

Recall that $\gamma \in \text{Sp}(2g, \mathbb{Z})$ operates on $\mathcal{H}_g$ by

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$  

(Here $A$, $B$, $C$, and $D$ are $(g \times g)$-matrices.) The quotient $\text{Sp}(2g, \mathbb{Z})\backslash \mathcal{H}_g$ is the moduli space of principally polarized abelian varieties of dimension $g$. The symplectic group $\text{Sp}(2g, \mathbb{Z})$ contains the lattice subgroup

$$\mathcal{P} = \left\{ \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} ; N \in \text{Sym}(g, \mathbb{Z}) \right\},$$

which acts on $\mathcal{H}_g$ by $\tau \mapsto \tau + N$. We consider the partial quotient

$$\mathcal{P}\backslash \mathcal{H}_g \subset \text{Sym}(g, \mathbb{Z})\backslash \text{Sym}(g, \mathbb{C}).$$

Using the coordinates $t_{ij}$, one can make the identification

$$\text{Sym}(g, \mathbb{Z})\backslash \text{Sym}(g, \mathbb{C}) \cong (\mathbb{C}^*)^{\frac{1}{2}g(g+1)}.$$  

We use the standard coordinates $T_{ij}$ on $\text{Sym}(g, \mathbb{C}) \cong \mathbb{C}^{\frac{1}{2}g(g+1)}$ and consider the embedding

$$\phi_0 : (\mathbb{C}^*)^{\frac{1}{2}g(g+1)} \to \mathbb{C}^{\frac{1}{2}g(g+1)}$$

given by

$$T_{ij} = \begin{cases} \prod_{k=1}^{g} t_{ki} & \text{if } i = j \\ t_{ij}^{-1} & \text{if } i \neq j. \end{cases} \quad (1)$$

The image of $(\mathbb{C}^*)^{\frac{1}{2}g(g+1)}$ under $\phi_0$ is the standard torus $(\mathbb{C}^*)^{\frac{1}{2}g(g+1)}$ in $\mathbb{C}^{\frac{1}{2}g(g+1)}$ and the inverse of $\phi_0$ is given by

$$t_{ij} = \begin{cases} \prod_{k=1}^{g} T_{ki} & \text{if } i = j \\ T_{ij}^{-1} & \text{if } i \neq j. \end{cases} \quad (2)$$

The reason why we are interested in the map $\phi_0$ is that it is closely related to the so-called principal cone (see [14, p. 93]), which plays an essential role in the reduction theory of quadratic forms. The embedding $\phi_0$ is also a central building block for toroidal compactifications of $\mathcal{A}_g$. In fact we have
Lemma 7 The embedding given by $\phi_0$ is the toroidal embedding corresponding to the principal cone.

Proof: Let $n_{ij} \in \text{Sym}(g, \mathbb{Z})$ be the matrix defined by
\[
(n_{ij})_{kl} = \begin{cases} 1 & \text{if } \{i, j\} = \{k, l\} \\ 0 & \text{otherwise}. \end{cases}
\]
The set $\{n_{ij}\}_{1 \leq i \leq j \leq g}$ is a basis of $\text{Sym}(g, \mathbb{Z})$. We call it the standard basis. We define another basis $\{n'_{ij}\}_{1 \leq i \leq j \leq g}$ by
\[
n'_{ij} = n_{ii} + n_{jj} - n_{ij}.
\]
This basis defines the principal cone. In the notation of [13, p. 5], we have
\[
T_{ij} = e(m'_{ij}) \quad \text{and} \quad t_{ij} = e(m_{ij}),
\]
where $\{m_{ij}\}$ is the dual of the standard basis and $\{m'_{ij}\}$ is the dual of the basis which defines the principal cone. The relation between the two bases for $\text{Sym}(g, \mathbb{Z})$ gives a relation between the two dual bases and this yields the required relation between $T_{ij}$ and $t_{ij}$. \qed

3.3 Mumford’s construction

Let
\[
X_0 = \phi_0(\mathcal{P} \setminus \mathcal{H}_g) \subset \mathbb{C}^{g(g+1)}.
\]
Recall from section 3.1 that $A_\tau$ is the quotient of $(\mathbb{C}^*)^g$ by the rank-$g$ lattice generated by
\[
r_i = (t_{1i}, \ldots, t_{gi}) = (T_{1i}^{-1}, \ldots, \prod_{k=1}^{g} T_{ki}, \ldots, T_{gi}^{-1}) \quad (i = 1, \ldots, g). \tag{3}
\]
This shows that there is a family of abelian varieties $\mathcal{A}_0 \to X_0$ such that for each point $[\tau] \in X_0$ the fibre $\mathcal{A}_{[\tau]}$ is isomorphic to $A_\tau$. We now want to add “boundary points”, i.e., we consider the set $X$, which is defined to be the
interior of the closure of \( X_0 \) in \( \mathbb{C}^{1/2g(g+1)} \). Mumford’s construction enables us to extend the family \( \mathcal{A}_0 \to X_0 \) to a family \( \mathcal{A} \to X \) by adding degenerate abelian varieties over the boundary points.

For this purpose we consider

\[
A = \mathbb{C}[T_{ij}], \quad I = (T_{ij}) \subset A.
\]

Let \( K \) be the quotient field of \( A \) and consider the torus

\[
\tilde{G} = \text{Spec} \mathbb{C}[w_1, \ldots, w_g, w_1^{-1}, \ldots, w_g^{-1}] = (\mathbb{C}^*)^g \times \text{Spec} A.
\]

By \( \tilde{G}(K) \) we denote the \( K \)-valued points of \( \tilde{G} \). The character group \( \mathfrak{X} = \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) \) is spanned by \( w_1, \ldots, w_g \). Finally, we consider the lattice \( \mathbb{Y} \subset \tilde{G}(K) \) which is spanned by the \( r_1, \ldots, r_g \) from (3). Following the terminology of [10], we shall call \( \mathbb{Y} \) the period lattice.

Let

\[
\Phi : \mathbb{Y} \to \mathfrak{X}, \quad r_i \mapsto w_i.
\]

**Lemma 8** The homomorphism \( \Phi \) is a polarization in the sense of Mumford.

**Proof:** Let \( y = \sum y_i r_i \) and \( z = \sum z_i r_i \in \mathbb{Y} \). Then the character \( X^{\Phi(y)} = \prod w_i^{y_i} \) given by \( \Phi(y) \) satisfies

\[
X^{\Phi(y)}(z) = \prod_{i=1}^g \left( \prod_{j \neq i} T_{ij}^{-z_j} \right) \left( \prod_{k=1}^g T_{ki} \right)^{z_i} \left( \prod_{i<j} T_{ij}^{(y_i-y_j)(z_i-z_j)} \right).
\]

Hence \( X^{\Phi(y)}(z) = X^{\Phi(z)}(y) \) and \( X^{\Phi(y)}(y) \in I \) unless \( y = 0 \). It follows that \( \Phi \) is a polarization in the sense of Mumford. \( \square \)

**Remark 9** In fact \( \Phi : \mathbb{Y} \to \mathfrak{X} \) is an isomorphism, hence it is a principal polarization in the sense of [3, chapter II].
Before we can explain Mumford’s construction, we still have to choose a star $\Sigma \subset X$. For $\alpha, \beta \in \mathbb{R}^g$, we say that $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for $i = 1, \ldots, g$. If furthermore $\alpha \neq \beta$, then we say that $\alpha > \beta$. Now set

$$\Sigma' = \{ \alpha \in \{ 0, \pm 1 \}^g; \alpha \geq 0 \text{ or } \alpha \leq 0 \}.$$ 

We identify this set with

$$\Sigma = \{ X^\alpha = \prod w_i^\alpha_i; \alpha \in \Sigma' \} \subset X.$$ 

This is a star in the sense of [14].

For technical reasons we note

**Lemma 10** For all $y \in Y$ and for all $\alpha \in \Sigma$, we have $X^{\Phi(y) + \alpha}(y) \in A$.

**Proof:** The calculations in the proof of lemma 8 show that

$$X^{\Phi(y) + \alpha_i}(y) = \prod_{i=1}^g T_{ii}^{y_i + \alpha_i} \cdot \prod_{i<j} T_{ij}^{(y_i - y_j)(y_i + y_j + \alpha_i - \alpha_j)}.$$ 

The claim now follows since $z(z + \beta) \geq 0$ if $z \in \mathbb{Z}$ and $\beta \in \{ 0, \pm 1 \}$ and $\alpha_i, \alpha_i - \alpha_j \in \{ 0, \pm 1 \}$ for all $\alpha \in \Sigma$. 

We are now ready to explain Mumford’s construction. As in [10, 3] we consider the graded ring

$$R = \sum_{k=0}^\infty \{ K[\ldots, X^\alpha, \ldots]_{\alpha \in \mathbb{Y}}/(X^{\alpha+\beta} - X^\alpha X^\beta, X^0 - 1) \} \theta^k,$$

where $\theta$ is an indeterminate of degree 1 and all other elements have degree 0. Let $R_{\Phi, \Sigma}$ be the subring of $R$ given by

$$R_{\Phi, \Sigma} = A[\ldots, X^{\Phi(y) + \alpha}(y) X^{2\Phi(y) + \alpha}, \ldots]_{\alpha \in \Sigma, y \in \mathbb{Y}}.$$ 

By lemma 10

$$R_{\Phi, \Sigma} \subset A[\ldots, X^\alpha \theta, \ldots]_{\alpha \in \mathbb{Y}}.$$ 

Let

$$\tilde{P} = \text{Proj} R_{\Phi, \Sigma}. $$

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This is a scheme over Spec $A$. The group $\mathcal{Y}$ acts on $\tilde{P}$ and the desired extension $A \to X$ of $A_0 \to X_0$ is given by

$$A = \mathcal{Y}\backslash(\tilde{P}|_X).$$

The scheme $\tilde{P}$ is covered by the affine open sets

$$U_{\alpha,y} = \text{Spec } R_{\alpha,y},$$

where

$$R_{\alpha,y} = A[\ldots, \frac{X^{\Phi(z)+\beta(z)}}{X^{\Phi(y)+\alpha(y)}}, \ldots]_{\beta \in \Sigma, z \in \mathcal{Y}},$$

as $\alpha$ runs through $\Sigma$ and $y$ runs through $\mathcal{Y}$. The action of $y \in \mathcal{Y}$ on $\tilde{P}$ identifies $U_{\alpha,0}$ with $U_{\alpha,y}$, so it suffices to calculate

$$U_\alpha = U_{\alpha,0} = \text{Spec } R_{\alpha,0}$$

$$R_\alpha = R_{\alpha,0} = A[\ldots, M_{\beta,z}, \ldots]_{\beta \in \Sigma, z \in \mathcal{Y}},$$

where

$$M_{\beta,z} = X^{\Phi(z)+\beta(z)}X^{2\Phi(z)+\beta-\alpha}$$

$$= \prod_{i=1}^{g} T_{ii}^{z_i+\beta_i} \cdot \prod_{i<j} \tau_{ij}(z_i-z_j)(z_i-z_j+\beta_i-\beta_j) \cdot \prod_{i=1}^{g} u_i^{z_i-\beta_i}. $$

We are not interested in all degenerations of abelian varieties arising from this construction, but only in those which correspond to $\tau_{11}, \ldots, \tau_{g-1,g-1} \to i\infty$. Hence we can fix the entries $\tau_{ij}$ for $i \neq j$ in the matrix $\tau$. This corresponds to fixing the coordinates $T_{ij} = t_{ij}^{-1}$ for $i \neq j$ and hence defines an affine subspace

$$L = L(\tau_{ij}) \subset \mathbb{C}^{\frac{1}{2}g(g+1)}.$$

We can use $T_{11}, \ldots, T_{gg}$ as coordinates on $L$. For the sake of simplicity, we introduce the notation

$$\tau_i = \tau_{ii}$$

$$T_i = T_{ii}, \quad i = 1, \ldots, g,$$

and consider the ring

$$A' = \mathbb{C}[T_1, \ldots, T_g].$$
By abuse of notation we shall denote the restriction of \( \tilde{P} \) (resp. \( U_\alpha \)) to \( L \) also by \( \tilde{P} \) (resp. \( U_\alpha \)). Then

\[
U_\alpha = \text{Spec} \, R_\alpha,
\]

where now

\[
R_\alpha = A'[\ldots, M_{\beta, z}, \ldots]_{\beta \in \Sigma, z \in Y},
\]

and

\[
M_{\beta, z} = \prod_{i=1}^{g} T_i^{z_i(\beta_i)} \cdot \prod_{i=1}^{g} w_i^{2z_i + \beta_i - \alpha_i}.
\]

**Proposition 11** For \( \alpha \in \Sigma \), we have \( R_\alpha = A'[X_1, \ldots, X_g, Y_1, \ldots, Y_g] \), where

\[
X_i = \begin{cases} 
T_i^{\alpha_i}w_i & \text{if } \alpha \geq 0 \\
w_i & \text{if } \alpha \leq 0
\end{cases}
\quad
Y_i = \begin{cases} 
w_i^{-1} & \text{if } \alpha \geq 0 \\
T_i^{-\alpha_i}w_i^{-1} & \text{if } \alpha \leq 0.
\end{cases}
\]

**Proof:** This follows easily from the observation that \( z(z + \beta) - (2z + \beta) \geq -1 \) if \( z \in \mathbb{Z} \) and \( \beta \in \{0, \pm 1\} \).

**Corollary 12** The scheme \( \tilde{P} \) is smooth.

**Proof:** It is enough to show that all \( U_\alpha \) are smooth. If \( \alpha = 0 \), then \( U_\alpha = \mathbb{C}^9 \times (\mathbb{C}^*)^g \). Now let \( \alpha \neq 0 \). We treat the case \( \alpha = (1, \ldots, 1) \), the other cases being similar. Then

\[
U_{(1, \ldots, 1)} = \text{Spec}(\mathbb{C}[T_1, \ldots, T_g, Z_1, \ldots, Z_{2g}]/(Z_iZ_{i+g} - T_i)).
\]

Projecting onto \( \text{Spec} \, \mathbb{C}[T_1, \ldots, T_g, Z_1, \ldots, Z_{2g}] \) shows that \( U_{(1, \ldots, 1)} \cong \mathbb{C}^{2g} \).

Now we consider the set

\[
V = X \cap L.
\]

One easily checks that \( V \) contains the lines where all \( T_i \) but one are zero. Let \( \tilde{P}_V \) be the restriction of \( \tilde{P} \) to \( V \).
Proposition 13  
(i) The group of periods $Y$ acts freely and properly discontinuously on $\tilde{P}_V$.
(ii) The quotient $A_V = Y \backslash \tilde{P}_V$ is smooth. Moreover, the family $A_V \rightarrow V$ is flat. It extends the family $A_0|_{V \cap X_0}$. In particular, the general fibre is a smooth abelian $g$-fold.

Proof:
(i) This can be done as in [3, theorem 3.14 (i)].
(ii) Smoothness follows from (i) and corollary [12]. The family is flat since $A_V$ is smooth of dimension $2g$ and every fibre has dimension $g$. It extends $A_0|_{V \cap X_0}$ by construction.

3.4 Description of the degenerate abelian varieties
We now want to describe the fibre of $A_V$ over a point $p = (0, \ldots, 0, T_g)$ with $T_g \neq 0$. We shall denote this fibre by $A_p$ and we shall denote the fibres of $\tilde{P}, U_\alpha$ and $U_{\alpha,y}$ over this point by $\tilde{P}_p, (U_\alpha)_p$ and $(U_{\alpha,y})_p$ respectively. Recall that

$$(U_\alpha)_p = \text{Spec}(R_\alpha)_p,$$

where

$$(R_\alpha)_p = \mathbb{C}[X_1, \ldots, X_g, Y_1, \ldots, Y_g] = \mathbb{C}[X_1, \ldots, X_{g-1}, Y_1, \ldots, Y_{g-1}, w_g, w_{g-1}]$$

with $X_i$ and $Y_i$ as in proposition [11]. We first consider $(U_\alpha)_p$. Clearly $(U_0)_p = (\mathbb{C}^*)^g$. In general, $(U_\alpha)_p$ consists of $2^h$ irreducible components, where

$$h = \#\{i; 1 \leq i \leq g - 1, \alpha_i \neq 0\}.$$ 

It is singular for $h \geq 1$. Its regular part is the disjoint union of $2^h$ tori. These tori can be described as follows. Consider

$$w_i = T_i^{-\beta_i}, \quad i = 1, \ldots, g,$$

where $\beta_i \in \{0, \alpha_i\}$ for $i = 1, \ldots, g - 1$. Outside the hyperplanes $\{T_i = 0\}$, this defines a section of the torus bundle $U_0$. Note that $U_0 = U_\alpha$ on $L - \cup_i\{T_i = 0\}$. This section can be extended to a section of $U_\alpha$ over $p,$
where it meets exactly one of the $2^k$ tori whose union is the smooth part of $(U_\alpha)_p$. This shows also that $(U_\alpha)_p \subset (U_\beta)_p$ if and only if $\beta \geq \alpha \geq 0$ or $\beta \leq \alpha \leq 0$.

The subgroup $\langle r_g \rangle$ of $\mathcal{Y}$ generated by $r_g$ acts on $U_0$ and hence also on $(U_p)_0$. It also acts on $\mathrm{Spec} \mathbb{C}[w_g, w_g^{-1}] = \mathbb{C}^*$ by

$$r_g(w_g) = e(\tau_g)w_g.$$  

The inclusion of rings

$$\mathbb{C}[w_g, w_g^{-1}] \subset (R_\alpha)_p$$  

defines a map

$$(U_0)_p \to \mathbb{C}^*,$$

which is equivariant with respect to the action of $\langle r_g \rangle$. In this way we get a semi-abelian variety of rank $g - 1$, i.e. an extension

$$0 \to (\mathbb{C}^*)^{g-1} \to \mathbb{Z}\backslash(U_0)_p \to E_{\tau_g,1} \to 0,$$

where $E_{\tau_g,1}$ is the elliptic curve

$$E_{\tau_g,1} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_g).$$

The closure of $(U_0)_p$ in $\tilde{P}_p$ has the structure of a $(\mathbb{P}^1)^{g-1}$-bundle over $\mathbb{C}^*$. Taking the quotient by $\langle r_g \rangle$ gives rise to a $(\mathbb{P}^1)^{g-1}$-bundle over the elliptic curve $E_{\tau_g,1}$.

We now return our attention to $\tilde{P}_p$. Recall that

$$\tilde{P}_p = \cup_{\alpha \in \Sigma, y \in \mathcal{Y}} (U_{\alpha,y})_p.$$  

It follows from our observations above that the regular part of $\tilde{P}_p$ is the union of countably many tori. These tori can be labelled in a natural way by elements $(l_1, \ldots, l_{g-1}) \in \mathbb{Z}^{g-1}$: the section given outside the union of the hyperplanes $\{T_i = 0\}$ by

$$w_i = T_i^{-l_i}$$

can be extended over the point $p$ where it meets exactly one of the tori contained in $\tilde{P}_p$. We shall label this torus by $(l_1, \ldots, l_{g-1})$. The element $r_i$ of $\mathcal{Y}$ ($i = 1, \ldots, g - 1$) then maps the torus $(l_1, \ldots, l_i, \ldots, l_{g-1})$ to the torus $(l_1, \ldots, l_i - 1, \ldots, l_{g-1})$, whereas $r_g$ maps each of these tori to itself.

We can now summarize our discussion in
Proposition 14 Let $A_p = \mathbb{Y} \setminus \hat{P}_p$ be the fibre of $A_V$ over the point $(0, \ldots, 0, T_g)$ of $V$ with $T_g \neq 0$. Then the following holds

(i) The regular part $A_p^{reg}$ of $A_p$ is a semi-abelian variety of rank $g - 1$. More precisely, there exists an extension
\[ 0 \rightarrow (\mathbb{C}^*)^{g-1} \rightarrow A_p^{reg} \rightarrow E_{\tau g, 1} \rightarrow 0.\]

(ii) The normalization of $A_p$ is a $(\mathbb{P}^1)^{g-1}$-bundle over the elliptic curve $E_{\tau g, 1}$. The identifications given by the normalization map are induced by the following identifications on $(\mathbb{P}^1)^{g-1} \times \mathbb{C}^*$
\[ r_i : (w_1, \ldots, w_{i-1}, \infty, w_i, \ldots, w_g) \mapsto (t_1 w_1, \ldots, t_{i-1} w_{i-1}, 0, t_i w_i, \ldots, w_g), \]
where $i$ runs through $\{1, \ldots, g - 1\}$.

Remarks 15 (i) Let
\[ a_i = [\tau_{ig}] \in E_{\tau g, 1} \quad i = 1, \ldots, g - 1.\]

Then the action of $r_i$ lies over the translation $x \mapsto x + a_i$ on the elliptic curve $E_{\tau g, 1}$.

(ii) The singularities of $A_p$ can be read off from proposition 11, but can also be understood in terms of the identifications described in proposition 14 (ii).
For every $h \in \{0, \ldots, g - 1\}$, there is a locally closed $(g - h)$-dimensional subset of $A_p$, where $2^h$ smooth branches meet and where the Zariski tangent space has dimension $g + h$. The “worst” singularities of $A_p$ occur along an elliptic curve isomorphic to $E_{\tau g, 1}$, where $2^{g-1}$ smooth branches meet.

3.5 The $(1, \ldots, 1, d)$-polarized case.

We now turn to the case of general $d \geq 1$, i.e. we consider the period matrix
\[ \Omega_\tau = \begin{pmatrix} \tau \\ D \end{pmatrix}, \quad D = \text{diag}(1, \ldots, 1, d). \]

Dividing out by the last $g$ rows of this period matrix gives a torus $(\mathbb{C}^*)^g$ with coordinates
\[ w_i = \begin{cases} e(z_i) & i = 1, \ldots, g - 1 \\ e(z_g/d) & i = g. \end{cases} \]
Let
\[ t_{ij} = \begin{cases} e(\tau_{ij}) & \text{if } i, j = 1, \ldots, g - 1 \\ e(\tau_{ig}/d) & \text{if } i = g \text{ or } j = g. \end{cases} \]

Then the first \(g\) rows of \(\Omega_\tau\) act on \((\mathbb{C}^*)^g\) by multiplication by
\[ (t_{i1}, \ldots, t_{ig}) \quad \text{for the } i\text{-th row, where } i = 1, \ldots, g - 1 \]
\[ (t_{1g}^d, \ldots, t_{gg}^d) \quad \text{for the } g\text{-th row.} \]

Changing the polarization from a principal one to a polarization of type \((1, \ldots, 1, d)\) corresponds to changing the group of periods \(Y\) to the subgroup
\[ Y' = \langle r_1, \ldots, r_{g-1}, r_g^d \rangle. \]

We shall, therefore, consider the family
\[ \mathcal{A}_V = Y' \setminus \tilde{P}_V. \]

Now the general element is a smooth abelian variety with a polarization of type \((1, \ldots, 1, d)\). Proposition 14 remains unchanged with the one exception that the base curve \(E_{\tau_g, 1}\) has to be replaced with the elliptic curve
\[ E_{\tau_g, d} = \mathbb{C}/(zd + Z\tau_g). \]

## 4 Degeneration of the polarization

Here we shall always consider the case of polarizations of type \((1, \ldots, 1, d)\). What we have done in the previous section was to extend the family \(\mathcal{A}_0|_{V \cap X_0}\) to a family \(\mathcal{A}_V\) over \(V\). We would now like to construct a relative polarization on \(\mathcal{A}_0|_{V \cap X_0}\) which extends to the family \(\mathcal{A}_V\). Although this can be done, we shall actually do slightly less: since we are only interested in degenerations belonging to points \((0, \ldots, 0, T_g)\) with \(T_g \neq 0\), we shall restrict ourselves to small neighbourhoods of such points.

For each \(m \in \mathbb{R}^g\), consider the theta-function
\[ \theta_{m, 0} : \mathcal{H}_g \times \mathbb{C}^g \to \mathbb{C} \]
\[ (\tau, z) \mapsto \sum_{q \in \mathbb{Z}^g} e\left(\frac{1}{2}(q + m)\tau(q + m) + (q + m)^t z\right). \]
Let
\[ s(\tau) = \left( -\frac{1}{2}\tau_1, \ldots, -\frac{1}{2}\tau_{g-1}, 0 \right) \]
and
\[ r = (0, \ldots, 0, \frac{1}{d}). \]
For \( k \in \mathbb{Z} \), we can then consider the functions
\[ \theta_k : \mathcal{H}_g \times \mathbb{C}^g \to \mathbb{C} \]
\[ (\tau, z) \mapsto \theta_{kr,0}(\tau, z + s(\tau)). \]
Note that this depends only on the class of \( k \) in \( \mathbb{Z}/d\mathbb{Z} \). These functions all have the same automorphy factor and hence are sections of a line bundle \( L_\tau \) on \( A_\tau \). In fact, \( L_\tau \) represents the \((1, \ldots, 1, d)\)-polarization on \( A_\tau \) and the \( \theta_k \) define a basis of the space of sections of this line bundle [8, p. 75]. Let
\[
\tau' = \begin{pmatrix}
0 & \tau_{12} & \ldots & \ldots & \tau_{1,g-1} \\
\tau_{21} & 0 & \ldots & \ldots & \\
\vdots & \vdots & \ddots & \ddots & \\
\tau_{g-1,1} & \ldots & \ldots & 0 & \tau_{g-2,g-1} \\
\tau_{g-1,1} & \ldots & \ldots & \tau_{g-1,g-2} & 0
\end{pmatrix}
\]
and set
\[ \tau'' = ' (\tau_{1g}, \ldots, \tau_{g-1,g}). \]
Finally we consider the analogue of the functions \( \theta_k \) in one variable, i.e. the functions
\[ \vartheta_k : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C} \]
\[ (\tau, z) \mapsto \sum_{q \in \mathbb{Z}} e\left( \frac{1}{2}(q + k/d)^2 \tau + (q + k/d)z \right), \]
which also depend only on \( k \in \mathbb{Z}/d\mathbb{Z} \).

**Proposition 16** With the notation of §3, the functions \( \theta_k \) can be written in the form
\[
\theta_k(\tau, z) = \sum_{q \in \mathbb{Z}^{g-1}} c_q(\tau') \vartheta_k(\tau_g, z_g + q\tau'') \prod_{i=1}^{g-1} t_i^\frac{1}{2} q_i \eta(q_i - 1) u_i q_i,
\]
\[ c_q(\tau') = \prod_{0 < i < j < g} t_{ij}^{q_{ij}}. \]

Proof: This follows from a straightforward computation.

Remark 17 The shift \( z \mapsto z + s(\tau) \) was introduced in order to obtain integer exponents of the variables \( t_i \) in the above description.

From now on, we fix \( \tau' \) and \( \tau'' \). Let \( p = (0, \ldots, 0, T_g) \) be an element of \( V \) with \( T_g \neq 0 \). For small neighbourhoods \( W \) of \( p \) in \( V \), it follows from proposition 16 that we may consider the \( \theta_k \) as holomorphic functions on \( W \times (\mathbb{C}^*)^{g-1} \times \mathbb{C} \) (with coordinates \((T_1, \ldots, T_g; w_1, \ldots, w_{g-1}, z_g)\)).

Let \( W_0 = W \cap X_0 \) and let \( \mathcal{A}_{W_0} \), resp. \( \mathcal{A}_W \) be the restriction of the family \( \mathcal{A} \) to \( W_0 \), resp. \( W \). Since the automorphy factors of the functions \( \theta_k \) do not depend on \( k \), there exists a line bundle \( \mathcal{L}_0 \) on \( \mathcal{A}_{W_0} \) such that the functions \( \theta_k \) are sections of this line bundle. The line bundle \( \mathcal{L}_0 \) defines a relative polarization on \( \mathcal{A}_{W_0} \). Our aim is to extend the line bundle \( \mathcal{L}_0 \) and its sections \( \theta_k \) to \( \mathcal{A}_W \).

Proposition 18 The line bundle \( \mathcal{L}_0 \) on \( \mathcal{A}_{W_0} \) can be extended to a line bundle \( \mathcal{L} \) on \( \mathcal{A}_W \). Moreover, the sections of \( \mathcal{L}_0 \) defined by the functions \( \theta_k \) can be extended to sections of \( \mathcal{L} \).

Proof: Using proposition 16, this can be done in the same way as in [3, prop. II.5.13] or as in [4, prop. 4.1.3]. Therefore, we shall not give all technical details, but only an outline of the proof. We first consider the open part \( \mathcal{U} \) of \( \tilde{P}_W \) given by the union of the open sets \((U_{0,y})_W\), where \( y \in \mathbb{Y} \). The codimension of the complement of \( \mathcal{U} \) in \( \tilde{P} \) is 2. Each open set \((U_{0,y})_W\) is a trivial torus of rank \( g \) over \( W \). For \( y = 0 \), we can use coordinates \((T_1, \ldots, T_g; w_1, \ldots, w_{g-1}, w_g)\) to identify \((U_0)_W\) with \( W \times (\mathbb{C}^*)^g \). Since the function \( \vartheta_k(\tau_g, z_g + q\tau''') \) can be expressed in terms of the coordinate \( w_g = e(z_g) \), we can view the functions \( \theta_k \) as functions on \((U_0)_W\). Similarly, using the action of \( \mathbb{Y} \), we can choose coordinates for \((U_{0,y})_W\) for every \( y \in \mathbb{Y} \), such that this open set is also identified with \( W \times (\mathbb{C}^*)^g \). In this way we can consider the \( \theta_k \) also as functions on \((U_{0,y})_W\). We can think of \( \mathcal{U} \) as a complex manifold which is obtained by glueing the open sets \((U_{0,y})_W\). For every
\[ y \in \mathcal{Y}, \text{ we consider the } \theta_k \text{ as sections of the trivial line bundle on } (U_{0,y})_W. \]

Using the automorphy factors of the functions \( \theta_k \), we can glue these trivial line bundles and obtain a line bundle \( \mathcal{L}_U \) on \( U \). This can be done in such a way that the action of \( \mathcal{Y} \) on \( U \) lifts to an action of \( \mathcal{Y} \) on \( \mathcal{L}_U \). Hence this line bundle descends to a line bundle on \( \mathcal{Y} \backslash U \). By construction, the functions \( \theta_k \) define sections of this line bundle. We have now extended the line bundle \( \mathcal{L}_0 \) to an open set of \( \mathcal{A}_W \), whose complement has codimension 2. To extend the bundle to the whole of \( \mathcal{A}_W \), one can either use the Remmert-Stein extension theorem (cf. \[3, 4\]) or one can perform a similar construction using all sets \((U_{\alpha,y})_W \) \( \alpha \in \Sigma, y \in \mathcal{Y} \).

For future reference we also note

**Proposition 19** Let \( S = \{0, 1\}^{g-1} \). Then

\[
\lim_{t_1, \ldots, t_{g-1} \rightarrow 0} \theta_k(\tau, z) = \sum_{q \in S} c_q(\tau')(\tau, z_g + q\tau')w^q,
\]

where

\[ w^q = \prod_{i=1}^{g-1} w_i^{q_i}. \]

*Proof:* This follows from proposition 16. \( \square \)

**Remark 20** We denote the restriction of the line bundle \( \mathcal{L} \) to \( \mathcal{A}_p \) by \( \mathcal{L}_p \). The functions from proposition 19 give \( d \) sections of \( \mathcal{L}_p \).

### 5 Very ampleness in the case \( d > 2^g \)

Let \( d \geq 1 \). Given a symmetric matrix

\[
\tau' = \begin{pmatrix}
0 & \tau_{12} & \cdots & \cdots & \tau_{1,g-1} \\
\tau_{21} & 0 & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \cdots & 0 & \tau_{g-2,g-1} \\
\tau_{g-1,1} & \cdots & \cdots & \tau_{g-1,g-2} & 0
\end{pmatrix},
\]

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an element $\tau'' = (\tau_1, \ldots, \tau_{g-1})$ of $\mathbb{C}^{g-1}$ and an element $\tau_g$ of $\mathcal{H}_1$, we have constructed in §3 a degenerate abelian variety $A_p$ of dimension $g$, whose normalization is a $(\mathbb{P}^1)^{g-1}$-bundle over the elliptic curve $E = E_{\tau_g}$, $d = \mathbb{C}/(\mathbb{Z}d + Z\tau_g)$. Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
E & \subset & A_p \\
\uparrow & & \uparrow \phi \\
\mathbb{C} \times \{0, \ldots, 0\} & \subset & \mathbb{C} \times (\mathbb{P}^1)^{g-1} \\
\uparrow \rho & & \uparrow \Phi \\
& & \mathbb{C}^d,
\end{array}
$$

where the map $\Phi$ is defined (with the notation of §4) by

$$
\phi_k(z; w_1, \ldots, w_{g-1}) = \sum_{q \in S} c_q(\tau') \vartheta_k(\tau_g, z + q\tau'') w_q
$$

(cf. proposition 19). We want to study the rational map $\phi$. Recall that when $\phi$ is a morphism, $\phi^* \mathcal{O}_{\mathbb{P}^{d-1}}(1)$ is the line bundle $\mathcal{L}_p$ on $A_p$ defined in remark 20 and note that $\phi(E)$ is a normal elliptic curve of degree $d$ in $\mathbb{P}^{d-1}$.

For any $z \in \mathbb{C}$, we let $[z]$ be the image of $z$ in $E$ and we set $a_i = [\tau_i]$ for $i = 1, \ldots, g - 1$, and $a = \tau'(a_1, \ldots, a_{g-1})$. Recall that $S = \{0, 1\}^{g-1}$.

From now on, we assume that $\tau''$ is generic. More precisely, it suffices that

\[ \text{the points } a_1, \ldots, a_{g-1} \text{ of } E \text{ are independent over } \mathbb{Z}. \quad (4) \]

Then, for any $x \in E$, the subset

$$
I(x) = \{x + qa; q \in S\}
$$

of $E$ has $2^{g-1}$ elements.

The following lemma is a consequence of the Riemann-Roch theorem.

**Lemma 21** Any set of at most $d - 1$ points on $\phi(E)$ is linearly independent.

**Proposition 22** Let $\tau''$ be generic. If $d > 2^{g-1}$, then $\phi_0, \ldots, \phi_{d-1}$ have no common zeroes. In particular, $\phi$ is a morphism and $\mathcal{L}_p$ is base-point-free.

**Proof:** For any $Z = (z; w_1, \ldots, w_{g-1})$ in $\mathbb{C} \times (\mathbb{P}^1)^{g-1}$, the vector $\Phi(Z)$ is a linear combination of the vectors $(\vartheta_k(\tau_g, z + q\tau''))_{k \in \mathbb{Z}/d\mathbb{Z}}$, whose coefficients are not all zero. The proposition then follows from lemma 21. \qed

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From now on, we assume that \( d > 2g - 1 \). Let \( z_g \in C \); the morphism \( \rho \) induces an isomorphism between \( \{ z_g \} \times (\mathbb{P}^1)^{g-1} \) and a closed subscheme of \( A_p \), which depends only on \( x = [z_g] \). We will denote it by \( F_x \). Note that \( I(x) \subset F_x \).

Since \( d > 2g - 1 \), the points of \( \phi(I(x)) \) are linearly independent. It follows from proposition 19, that the restriction of \( \phi \) to \( F_x \) is a Segre embedding.

**Proposition 23** Let \( \tau'' \) be generic. If \( d > 2g \), then \( \phi \) is injective.

**Proof:** From proposition 14, we see that the restriction of \( \rho \) to \( C \times (\mathbb{P}^1 - \{\infty\})^{g-1} \) induces a bijection
\[
B_p \to A_p,
\]
where \( B_p \) is an open subset of the normalization of \( A_p \), fibred over \( E \) with fibres isomorphic to \( (\mathbb{P}^1 - \{\infty\})^{g-1} \). For \( x \in E \), we let \( F^0_x \) be the image in \( A_p \) of the fibre of \( x \). It is a subset of \( F_x \).

Let \( x, y \) be two points of \( E \). Since \( d > 2g \), the points of \( \phi(I(x) \cup I(y)) \) are linearly independent. It follows that if \( \phi(F^0_x) \) and \( \phi(F^0_y) \) meet, then \( x \in I(y) \) and \( y \in I(x) \). Condition (11) then implies \( x = y \). Hence, for \( x \neq y \), the sets \( \phi(F^0_x) \) and \( \phi(F^0_y) \) do not meet. Since \( \phi|_{F_x} \) is an embedding, the lemma is proved.

**Theorem 24** Let \( \tau'' \) be generic. For \( d > 2g \), the morphism \( \phi \) is an embedding. In particular, \( L_p \) is very ample.

**Proof:** It remains to prove that the differential is injective on the Zariski tangent spaces.

After reordering the coordinates, we may assume that we are at a point
\[
P = \rho(z_g; 0, \ldots, 0, v_{h+1}, \ldots, v_{g-1}),
\]
where \( z_g \in C, v_{h+1}, \ldots, v_{g-1} \in \mathbb{P}^1 - \{0, \infty\} \). The Zariski tangent space of \( A_p \) at \( P \) has dimension \( g + h \) (see remark 13). Moreover, \( A_p \) has \( 2^h \) smooth branches at \( P \), which are indexed by subsets \( K \) of \( \{1, \ldots, h\} \). The branch corresponding to \( K \) is
\[
\rho(\{z_g - \tau''_K\} \times (\mathbb{P}^1)^{g-1}),
\]
where $\tau''_K = \sum_{i \in K} \tau_{ig}$ and in this branch, the point $P_K$ above $P$ is

$$\rho(z_g - \tau''_K; v'_1, \ldots, v'_{g-1}),$$

where

$$v'_l = \begin{cases} \infty & \text{if } l \in K \\ 0 & \text{if } l \in K' = \{1, \ldots, h\} - K \\ \prod_{i \in K} t_{il}^{-1} v_l & \text{for } h < l < g \end{cases}$$

(see proposition 14 (ii)).

We change the coordinates around $P_K$ by setting

$$w''_i = \begin{cases} (w'_i)^{-1} & \text{if } i \in K \\ w'_i & \text{otherwise.} \end{cases}$$

In these coordinates, $\Phi$ is given by

$$\Phi(z_g; w''_1, \ldots, w''_{g-1}) = \left( \sum_{q \in S} c_q \vartheta_k(\tau_g, z_g + q\tau''(w'')^{1-q_k(w'')^{q_{K'} + q_L}}) \right)_{k \in \mathbb{Z}/d\mathbb{Z}},$$

where

$$1 = (1, \ldots, 1) \in S$$

$$L = \{h + 1, \ldots, g - 1\}$$

and, for $M \subset \{1, \ldots, g - 1\}$ and $q \in S$, we define $q_M \in S$ by

$$(q_M)_i = \begin{cases} q_i & \text{if } i \in M \\ 0 & \text{otherwise.} \end{cases}$$

We need to calculate the corresponding Jacobian matrix at $P_K$ (whose new coordinates are $(z_g - \tau''_K; 0, \ldots, 0, v'_{h+1}, \ldots, v'_{g-1})$).

**Derivative with respect to $x$.**

In the sum, we need only consider indices $q$ such that

$$q_i = \begin{cases} 1 & \text{if } i \in K \\ 0 & \text{if } i \in K'. \end{cases}$$
After setting \( r = q - 1_K \), we get
\[
\left( \sum_{r \in S} c_{1_K + r} \vartheta_k'(\tau_g, z_g + r\tau'')(v')^r \right)_{k \in \mathbb{Z}/d\mathbb{Z}}.
\]

Since
\[
c_{1_K + r} = \left( \prod_{i<j, i,j \in K} t_{ij} \right) \left( \prod_{i \in K, j \in L} t_{ij}^r \right) c_r
\]
and
\[
(v')^r_L = \prod_{i \in K, l \in L} t_{il}^{-r_i} v_{l}^r,
\]
we get a non-zero multiple of the vector
\[
\left( \sum_{r \in S} c_r \vartheta_k'(\tau_g, z_g + r\tau'')(v^r) \right)_{k \in \mathbb{Z}/d\mathbb{Z}}.
\]

**Derivative with respect to** \( w''_\beta, \beta \in K \).

In the sum, we need only consider indices \( q \) such that
\[
q_i = \begin{cases} 1 & \text{if } i \in K - \{\beta\} \\ 0 & \text{if } i \in K' \cup \{\beta\}. \end{cases}
\]

A similar calculation yields (after setting \( r = q - 1_K - \{\beta\} \)) a non-zero multiple of the vector
\[
\left( \sum_{r \in S} c_r \vartheta_k'(\tau_g, z_g - \tau_\beta g + r\tau'')(v^r) \left( \prod_{j \in L} t_{ij}^{-r_j} \right) \right)_{k \in \mathbb{Z}/d\mathbb{Z}}.
\]

**Derivative with respect to** \( w''_\beta, \beta \in K' \).

We get a non-zero multiple of
\[
\left( \sum_{r \in S} c_r \vartheta_k'(\tau_g, z_g + \tau_\beta g + r\tau'')(v^r) \left( \prod_{j \in L} t_{ij}^{r_j} \right) \right)_{k \in \mathbb{Z}/d\mathbb{Z}}.
\]

**Derivative with respect to** \( w''_\beta, \beta \in L \).

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We get a non-zero multiple of
\[ \left( \sum_{\substack{r \in S \\ \ \ r_1 = \cdots = r_h = 0 \ \ \ \ r_\beta = 1}} c_r \vartheta_k(\tau_g, z_g + r \tau'') v^r \right)_{k \in \mathbb{Z}/d\mathbb{Z}}. \]

Altogether, letting \( K \) vary among all subsets of \( \{1, \ldots, h\} \), we see that the closure of \( d\phi(T_P A_p) \) in \( \mathbb{P}^{d-1} \) is spanned by the following \((g+h+1)\) points
\[
\left( \sum_{\substack{r \in S \\ \ \ r_1 = \cdots = r_h = 0 \ \ \ \ r_\beta = 1}} c_r \vartheta_k(\tau_g, z_g + r \tau'') v^r \right)_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
\[
\left( \sum_{\substack{r \in S \\ \ \ r_1 = \cdots = r_h = 0 \ \ \ \ r_\beta = 1}} c_r \vartheta'_k(\tau_g, z_g + r \tau'') v^r \right)_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
\[
\left( \sum_{\substack{r \in S \\ \ \ r_1 = \cdots = r_h = 0 \ \ \ \ r_\beta = 1}} c_r \vartheta_k(\tau_g, z_g + \epsilon r \tau'' + r \tau') (\prod_{j \in L} \mathbb{I}^{\epsilon q_j \beta_j} \right)_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
for all \((\beta, \epsilon) \in \{1, \ldots, h\} \times \{-1, 1\}\), and
\[
\left( \sum_{\substack{r \in S \\ \ \ r_1 = \cdots = r_h = 0 \ \ \ \ r_\beta = 1}} c_r \vartheta_k(\tau_g, z_g + r \tau'') v^r \right)_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
for all \( \beta \in \{h+1, \ldots, g-1\} \). Since \( d > 2^g = 2^{g-h}(h+1) \), it follows from lemma 21 that the \( 2^{g-h}(h+1) \) vectors
\[
(\vartheta_k(\tau_g, z_g + r \tau''))_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
\[
(\vartheta'_k(\tau_g, z_g + r \tau''))_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
\[
(\vartheta_k(\tau_g, z_g + \epsilon r \tau'' + r \tau'))_{k \in \mathbb{Z}/d\mathbb{Z}}
\]
for \( r \in S \) with \( r_1 = \cdots = r_h = 0 \), \( \epsilon = \pm 1 \) and \( \beta \in \{1, \ldots, h\} \), are linearly independent in \( \mathbb{C}^d \).

This implies that the \((g + h + 1)\) vectors above are linearly independent in \( \mathbb{C}^d \) and proves that the image of the differential of \( \phi \) at \( P \) has dimension \( g + h \). The differential of \( \phi \) is therefore everywhere injective, which proves the theorem. \( \square \)
Corollary 25  Let \((A, L)\) be a generic polarized abelian variety of dimension \(g\) and type \((1, \ldots, 1, d)\). For \(d > 2^g\), the line bundle \(L\) is very ample.

Remark 26  Similar calculations show that for given \(\tau_g\) and \(\tau''\) satisfying condition (4), and for a generic choice of the matrix \(\tau'\), the morphism \(\phi\) is unramified for \(d \geq 2^g - g(g - 3)/2\) and is an embedding for \(d > 2^g - g(g - 3)/2\). For \(g \geq 4\), this improves slightly on the bound in theorem \(^{24}\). However, \(\phi\) is never an embedding for \(g = 3\) and \(d = 8\): for a generic choice of \(\tau'\), it is unramified and identifies (transversally) a finite number of pairs of smooth points of \(A_p\).

References

[1] W. Barth, Transplanting cohomology classes in complex-projective space, Am. J. of Math. 92 (1970), 951–967.

[2] Ch. Birkenhake, H. Lange, Complex abelian varieties, Springer-Verlag, Berlin (1992).

[3] K. Hulek, C. Kahn, S.H. Weintraub, Moduli spaces of polarized abelian surfaces: compactification, degenerations, and theta functions, to appear: de Gruyter (1993).

[4] K. Hulek, S.H. Weintraub, The principal degenerations of abelian surfaces and their polarizations, Math. Ann. 286 (1990), 281–307.

[5] Ch. Birkenhake, H. Lange, S Ramanan, Primitive line bundles on abelian threefolds, Preprint Erlangen (1993).

[6] G. Fischer, Complex analytic geometry, Lecture Notes in Math. 538, Springer-Verlag, Berlin (1976). (1984).

[7] W. Fulton, R. Lazarsfeld, On the connectedness of degeneracy loci and special divisors, Acta Math. 146 (1981), 271–283

[8] J. Igusa, Theta functions, Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 194, Springer-Verlag, Berlin (1967).
[9] J. Kollár, Trento Examples, in: Classification of irregular varieties, Lecture Notes in Math. 1515, Springer-Verlag, Berlin (1991).

[10] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings, Comp. Math. 24 (1972), 239–272.

[11] D. Mumford, On the equations defining abelian varieties, I, Invent. Math. 1 (1966), 287–354.

[12] D. Mumford, On the Kodaira dimension of the Siegel modular variety, Lecture Notes in Math. 997, Proceedings of the Ravello conference 1982, 348–376, Springer-Verlag, Berlin (1983).

[13] D. Nagaraj, S. Ramanan, Abelian varieties with polarisation of type $1, 2, \ldots, 2$, to appear.

[14] Y. Namikawa, Toroidal compactification of Siegel spaces, Lecture Notes in Math. 812, Springer-Verlag, Berlin (1980).

[15] T. Oda, Convex bodies and algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 15, Springer-Verlag, Berlin (1980).

[16] A. Van de Ven, On the embeddings of abelian varieties in projective spaces, Annali di Matematica 103 (1975), 127–129.