Multi-matrix models and Noncommutative Frobenius algebras obtained from symmetric groups and Brauer algebras

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Abstract

It has been understood that correlation functions of multi-trace operators in $\mathcal{N} = 4$ SYM can be neatly computed using the group algebra of symmetric groups or walled Brauer algebras. On the other hand such algebras have been known to construct 2D topological field theories (TFTs). After reviewing the construction of 2D TFTs based on symmetric groups, we construct 2D TFTs based on walled Brauer algebras. In the construction, the introduction of a dual basis manifests a similarity between the two theories. We next construct a class of 2D field theories whose physical operators have the same symmetry as multi-trace operators constructed from some matrices. Such field theories correspond to non-commutative Frobenius algebras. A matrix structure arises as a consequence of the noncommutativity. Correlation functions of the Gaussian complex multi-matrix models can be translated into correlation functions of the two-dimensional field theories.

1 Introduction

Motivated by the development of AdS/CFT correspondence [1], the study of $\mathcal{N} = 4$ super Yang-Mills theory has attracted lots of interests for these 15 years. The anomalous dimensions are of particular interest because they correspond to energies in the dual string theory side. If we restrict our attention to the planar limit, the problem to obtain the spectrum of anomalous dimensions is replaced with the conventional problem to diagonalise the Hamiltonian in an integrable system [2, 3]. Compared to the enormous development of the planar theory, we had only limited information on non-planar corrections. Recent development to study non-planar corrections, however, has brought us to obtain some concrete results of non-planar corrections [4]–[21].

Consider multi-trace gauge invariant operators constructed from a single complex matrix, relevant to the half-BPS sector in $\mathcal{N} = 4$ super Yang-Mills theory. They are classified by conjugacy classes of the symmetric group, and the two-point function can be expressed except the trivial space-time dependence as

$$\langle O_\tau O_\sigma \rangle = \sum_{\rho \in S_n} N^n \delta_n (\Omega_n \tau \rho \sigma \rho^{-1}).$$

(1)
The derivation will be given in section 2. It is interesting to find that the right-hand side is written completely in terms of symmetric group data, suggesting an effective role of the symmetric group in the problem. Behind the fact that the symmetric group shows up in the evaluation of the matrix integral, there is a mathematical structure that relates the general linear group and the symmetric group known as Schur-Weyl duality. It indeed plays a central role in our idea to employ group representation theory in the recent development of non-planar physics. We will review it in section 2 and some mathematical notions regarding Schur-Weyl duality are given in appendix A. By the way it has been known that symmetric groups give the description of coverings of two-dimensional Riemann surfaces, which played a central role in the string theoretic interpretation of the large $N$ expansion of two-dimensional Yang-Mills [23, 24, 25]. This is an example that symmetric groups are used to describe two-dimensional field theories. In fact, if we ignore the $\Omega$ factor in the right-hand side of (1), the right-hand side is nothing but the two-point function of a two-dimensional topological field theory. In section 3 we give a review of the construction of two-dimensional topological field theories associated with symmetric groups. From these facts, it is expected to be fruitful to learn four-dimensional theories from two-dimensional theories by means of symmetric groups [21].

Walled Brauer algebras [26, 27, 28, 29] are another convenient tool to organise multi-trace gauge invariant operators constructed from some matrices [9, 13]. In section 4 we will explicitly construct two-dimensional topological field theories based on walled Brauer algebras. The introduction of a dual basis manifests a similarity to the construction of topological field theories by symmetric groups. We will also show that correlation functions of the Brauer topological field theory can be decomposed into correlation functions of the topological field theories obtained from symmetric groups.

The central part of this paper is, based on these connections, devoted to the study of two-dimensional field theories that are closely related to the description of multi-trace operators built from some scalar fields in $\mathcal{N} = 4$ super Yang-Mills theory. As will be reviewed in section 2 when we organise gauge invariant operators that involve $p$ kinds of matrices in terms of symmetric groups or walled Brauer algebras, the conjugation under some permutation belonging to the symmetric group $S_{n_1} \times \cdots \times S_{n_p}$ is essentially important [9, 10, 11, 12, 13, 20]. In section 5 we will construct two-dimensional field theories whose physical operators are characterised by the same conjugation. The idea of the construction is given in [21]. It has been understood that two-dimensional topological quantum field theories are in one-to-one correspondence with commutative Frobenius algebras. What we describe in section 3 and 4 can be phrased mathematically that commutative Frobenius algebras are constructed from symmetric groups and walled Brauer algebras. By contrast, algebras playing a role in the theories we consider in section 5 are non-commutative Frobenius algebras. These theories have some properties owned by topological field theories such as that the two-point function is a projector. We will see that an effective matrix-like structure shows up as a consequence of the noncommutativity, where multiplicity indices on the restricted characters [7, 9, 11, 13] behave like matrix-indices. We will also discuss the connection between correlation functions of the two-dimensional field theories and correlation functions of $\mathcal{N} = 4$ super Yang-Mills theory in section 6 hoping to illuminate a new geometric correspondence between the two theories. In section 7, we discuss the counting problem of multi-traces from the point of view of the two-dimensional field theories. Section 8 is devoted to discussions. In some appendices we collect useful materials.
2 Correlation functions in multi-matrix models

In this section, we will review the recent approach to compute exact finite $N$ correlation functions of multi-trace operators in $\mathcal{N} = 4$ Super Yang-Mills theory. The mathematical background behind the approach is supplemented in appendix A.

We consider the free field theory of scalar fields, and we use the complex notation to denote them by $X, Y, Z$ or $X_a$ ($a = 1, 2, 3$). The two-point function is determined by conformal symmetry to take the following form

$$\langle O_i(x)O_j(y) \rangle_{\text{SYM}} = \frac{c_{ij}}{(x-y)^{n_i+n_j}} \delta_{n_i,n_j},$$

(2)

where $n_i$ is the number of fields involved in the operator $O_i$. General gauge invariant operators are given by a product of an arbitrary number of single trace operators built from the fundamental fields $X_a$ and $X_a^\dagger$. The non-trivial $N$-dependence of the correlator is encoded in $c_{ij}$, and it is obtained by solving the combinatorial problem of contractions of the free field captured by the matrix integral with the Gaussian weight $[22]

$$c_{ij} = \langle O_iO_j \rangle := \int \prod_a [dX_adX_a^\dagger] e^{-\frac{2}{a} tr(X_aX_a^\dagger)} O_iO_j, $$

(3)

where the measure is normalised to give

$$\langle (X_a)_{ij}(X_b^{\dagger})_{kl} \rangle = \delta_{il} \delta_{jk} \delta_{ab}. $$

The evaluation of the matrix integral will be neatly performed if we introduce an appropriate algebra as a tool to organise the multi-trace structure. Let us first consider the half BPS chiral primary operators described by a single holomorphic matrix $X$ [4, 5]. Any multi-trace operators built from $n$ copies of $X$ are conveniently labelled by an element of the symmetric group $S_n$ as

$$tr_n(\sigma X^\otimes n) = X_{i_1}^{i_1(1)} \cdot \cdot \cdot X_{i_n}^{i_n(n)} \quad (\sigma \in S_n).$$

(4)

Here $tr_n$ is a trace over the tensor space $V^\otimes n$, where $X$ is regarded as a linear map on $V$. Because two elements that are conjugate each other give the same multi-trace

$$tr_n(\sigma X^\otimes n) = tr_n(\rho \sigma \rho^{-1} X^\otimes n) \quad (\rho \in S_n),$$

(5)

the number of independent multi-traces is not equal to the number of elements in the symmetric group $S_n$. Taking the equivalence relation into account, the multi-trace operators are correctly classified by conjugacy classes. The matrix integral of the two-point function can be evaluated as

$$\langle tr_n(\tau X^\dagger \otimes n)tr_n(\sigma X^\otimes n) \rangle = \sum_{\rho \in S_n} tr_n(\tau \rho \sigma \rho^{-1})$$

$$= \sum_{\rho \in S_n} \sum_{R \vdash n} t_R \chi_R(\tau \rho \sigma \rho^{-1})$$

$$= \sum_{h \in S_n} N^n h(n) \delta_n(\Omega_n \tau \rho \sigma \rho^{-1}),$$

(6)

At the first equality the Wick-contractions are expressed by elements in $S_n$ [4]. The second and third step result from the Schur-Weyl duality, as is explained in [A.6] - [A.9]. The dimension of an irreducible representation $R$ of the symmetric group is denoted by $d_R$, while the dimension of an irreducible representation $R$ of the $SU(N)$ group is denoted by $t_R$. The $\Omega_n$ is a specific central element in the group algebra of the symmetric group,

$$\Omega_n = \sum_{\sigma \in S_n} \sigma N^{C_\sigma-n},$$

(7)

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where $C_\sigma$ is the number of cycles in the permutation $\sigma$. The last equality of (9) is valid for $N$ that is larger than $n$. The two-point functions can be diagonalised by the basis change of (A.4) as

$$\langle tr_n(p_R X^{\dagger \otimes n}) tr_n(p_S X^{\otimes n}) \rangle = n! d_R t_R \delta_{RS}.$$  \hspace{1cm} (8)

The diagonal basis is labelled by a Young diagram with $n$ boxes.

The idea to label multi-trace operators in terms of an element in the group algebra of the symmetric group can be applied to operators described by some kind of matrices [6, 11]. For a multi-trace constructed from $m$ copies of $X$ and $n$ copies of $Y$, using $\sigma \in S_{m+n}$ we have

$$tr_{m+n}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) = X_{i_1}^{(1)} \cdots X_{i_m}^{(m)} Y_{i_{m+1}}^{(m+1)} \cdots Y_{i_{m+n}}^{(m+n)}. \hspace{1cm} (9)$$

The difference from (4) is that equivalence classes for (9) are characterised by the equivalence relation determined by the subgroup $H = S_m \times S_n$. In this description the subgroup $H$ plays a role. Two-point function can be evaluated as

$$\langle tr_{m+n}(\tau X^{\dagger \otimes m} \otimes Y^{\dagger \otimes n}) tr_{m+n}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) \rangle = \sum_{h \in S_m \times S_n} tr_{m+n}(\tau h \sigma h^{-1}) = \sum_{h \in S_m \times S_n} N^{m+n} \delta_{m+n}(\Omega_{m+n} \tau h \sigma h^{-1}). \hspace{1cm} (10)$$

We emphasise that the free field Wick-contractions are expressed by elements of the subgroup $S_m \times S_n$. The diagonal two-point function can be obtained by a change of basis in (A.15) as

$$\langle tr_{m+n}(P_{A,\mu \nu}^R X^{\dagger \otimes m} \otimes Y^{\dagger \otimes n}) tr_{m+n}(P_{A,\mu \nu'}^S X^{\otimes m} \otimes Y^{\otimes n}) \rangle = m! n! d_A t_R \delta_{RS} \delta_{AA'} \delta_{\mu \nu} \delta_{\mu' \nu'}. \hspace{1cm} (11)$$

The diagonal operators are labelled by a set of three Young diagrams and two multiplicity labels.

We have another way to label multi-traces constructed from some fields using walled Brauer algebras [9, 13]. Walled Brauer algebras can be introduced as a Schur-Weyl dual to the $GL(N)$ groups (see (A.19)). Mainly consider multi-trace operators constructed from $m$ copies of $X$ and $n$ copies of $Y$, and let $B_N(m, n)$ be the walled Brauer algebra relevant for the description of such operators. The Brauer algebra contains the group algebra of $S_m \times S_n$ as a subalgebra. Gauge invariant operators are constructed by regarding $X^{\otimes m} \otimes Y^{\otimes n}$ as operators acting on the space $V^{\otimes m} \otimes V^{\otimes n}$, followed by the action of an element of the Brauer algebra and taking a trace:

$$tr_{m,n}(b X^{\otimes m} \otimes Y^{\otimes n}) \hspace{1cm} (b \in B_N(m, n)). \hspace{1cm} (12)$$

The equivalence relation is very similar to (10),

$$tr_{m,n}(b X^{\otimes m} \otimes Y^{\otimes n}) = tr_{m,n}(h h^{-1} X^{\otimes m} \otimes Y^{\otimes n}) \hspace{1cm} (h \in S_m \times S_n). \hspace{1cm} (13)$$

By expressing free field Wick-contractions in terms of elements in $H$, the two-point function is computed to give

$$\langle tr_{m,n}(b X^{\dagger \otimes m} \otimes Y^{\dagger \otimes n}) tr_{m,n}(c X^{\otimes m} \otimes Y^{\otimes n}) \rangle = \sum_{h \in S_m \times S_n} tr_{m,n}(bh^{-1} c) \hspace{1cm} = \sum_{h \in S_m \times S_n} \sum_{\gamma} t_{\gamma} \chi(bh^{-1}), \hspace{1cm} (14)$$
where $b, c$ are elements of the walled Brauer algebra. The dimension of an irreducible representation $\gamma$ of the walled Brauer algebra is denoted by $d_\gamma$, while the dimension of an irreducible representation $\gamma$ of the $GL(N)$ group is denoted by $t_\gamma$. The last equality in (15) is a consequence of the Schur-Weyl duality (A.19). By taking the linear combination in (A.28), we will obtain diagonal two-point functions [9].

Before closing this section, we will introduce some symbols to denote the equivalence classes characterising the multi-matrix structures. For a fixed element $\sigma \in S_n$ the following sum over a complete basis of $S_n$ gives a central element

$$[\sigma] = \frac{1}{n!} \sum_{\rho \in S_n} \rho \sigma \rho^{-1} \quad (\sigma \in S_n).$$

(17)

In symmetric group $S_{m+n}$ the sum over a complete basis of $S_m \times S_n$ gives an element which commutes with any elements in $S_m \times S_n$,

$$[\sigma]_H = \frac{1}{m!n!} \sum_{h \in H} h \sigma h^{-1} \quad (\sigma \in S_{m+n}),$$

(18)

where $H = S_m \times S_n$. Similarly an element in the Brauer algebra $B_N(m, n)$ which commutes with any elements in $S_m \times S_n$ can be constructed by

$$[b]_H = \frac{1}{m!n!} \sum_{h \in H} hh^{-1} \quad (b \in B_N(m, n)).$$

(19)

The number of the equivalence classes coincides with the number of independent multi-matrix operators at large $N$ (see also [36], [121] and [128]). If $N$ is small compared to the number of fields involved in the multi-matrix models, the multi-traces are in general linearly dependent. Linearly independent multi-traces are given by the Young diagram basis [4, 9, 10, 11, 30].

With these new notations, the two-point functions are rewritten as

$$\langle tr_n([\tau]X^\otimes n)tr_n([\sigma]X^\otimes n)\rangle = n!N^n \delta_{\tau}[\sigma]$$

$$\langle tr_{m+n}([\tau]_H X^\otimes n)tr_{m+n}([\sigma]_H X^\otimes n)\rangle = m!n!N^{m+n} \delta_{m+n+[\tau]_H}[\sigma]_H$$

$$\langle tr_{m,n}([b]_H X^\otimes n)\rangle = m!n! \sum_{\gamma} t_\gamma \chi^\gamma([b]_H[c]_H).$$

(20)

We will reconsider the meaning of these equations in section 3.

3 2D topological field theories and semisimple algebras

In the previous section we have reviewed that employing the group algebra of the symmetric group and the walled Brauer algebra is very effective for labelling multi-trace gauge invariant operators in the multi-matrix models. On the other hand it has been known that two-dimensional topological quantum field theories can be defined from such algebras. This fact would bring up
a new aspect on the role of the algebras, suggesting a new connection with two-dimensional field theories. In this section we will review the construction of two-dimensional topological quantum field theories. For our purpose it is convenient to define them in terms of lattice [31, 32, 33], where it was shown that semisimple algebras have a one-to-one correspondence with two-dimensional topological quantum field theories.

Consider the triangulation of a two-dimensional compact orientable surface. The structure of the surface is encoded in the way of gluing the triangles. To each edge we assign a colour index \( i \), and to each triangle with edges labelled by \( i, j, k \), we assign a complex number \( C_{ijk} \). We assume that \( C_{ijk} \) is invariant under cyclic permutations of the colour indices

\[
C_{ijk} = C_{jki} = C_{kji},
\]

(21)

but no relation is imposed between \( C_{ijk} \) and the orientation-reversed object \( C_{ikj} \). Two adjacent edges are identified by introducing a gluing operator \( g^{ij} \), which is assumed to be symmetric \( g^{ij} = g^{ji} \) and to have the inverse \( g_{ij} \). In the dual diagrams \( g^{ij} \) and \( C_{ikj} \) correspond to the propagator and the three-point vertex respectively. The partition function of a triangulated surface is given by the product of complex numbers \( C_{ijk} \) assigned to each triangle with each edge glued by the gluing operator \( g^{ij} \) and the summation over all possible triangulations.

Topological quantum field theories are characterised by the invariance under local deformations of the background. In the lattice construction, topological models are constructed by imposing the invariance of partition functions under any local change of the triangulations. It is known that two basic moves are sufficient to generate all topologically equivalent triangulations. We will use so called bubble move and 2-2 move (see figure 1). From the invariance of partition functions under the bubble moves, we obtain

\[
C_{ik} C_{jl} = g_{ij},
\]

(22)

where

\[
C_{ij} = C_{ij} g^{lk}.
\]

(23)

We use the usual summation convention that repeated indices are summed. From the invariance of partition functions under the 2-2 moves, we have

\[
C_{ij} C_{pk} = C_{ip} C_{jk}.\]

(24)

These two conditions are solved by introducing a semisimple associative algebra [31, 32]. Let a basis and the structure constant of the algebra be \( \phi_i \) and \( C_{ij} \), that is, \( \phi_i \phi_j = C_{ij} \phi_k \). It is easy to find that the condition (24) is derived from the associativity of the algebra; \( (\phi_i \phi_j) \phi_k = \phi_i (\phi_j \phi_k) \).
The condition (22) indicates that we can define a nondegenerate metric \(g_{ij}\) given in the equation. With these conditions, partition functions only depend on the genus of the manifold.

In order to make the construction more concrete, we will consider the group algebra of symmetric group \(S_n\) as an example of semisimple algebras [31]. Introducing the regular representation, the structure constant is given by

\[
C_{ijk} = \frac{1}{n!} \text{tr}(r(\sigma_k^{-1}\sigma_i \sigma_j)),
\]

where \(\text{tr}(r)\) is the trace of the regular representation. Defining the delta function over the group algebra of \(S_n\) by \(\delta_n(\sigma) = 1\) if \(\sigma = 1\) and 0 otherwise, the trace of the regular representation can be expressed by

\[
\text{tr}(r(\sigma)) = n!\delta_n(\sigma).
\]

It is also convenient to consider the expansion in terms of the character of the symmetric group as

\[
\text{tr}(r(\sigma)) = \sum_{R \vdash n} d_R \chi_R(\sigma),
\]

where \(R \vdash n\) means that \(R\) is a Young diagram with \(n\) boxes. From this formula we can verify

\[
\text{tr}(r(1)) = \sum_{R \vdash n} (d_R)^2,
\]

where \(d_R\) is the dimension of an irreducible representation \(R\) of \(S_n\).

From (22) and (23) we determine

\[\begin{align*}
g_{ij} &= \text{tr}(r(\sigma_i \sigma_j)) = n!\delta_n(\sigma_i \sigma_j) \\
C_{ijk} &= \text{tr}(r(\sigma_i \sigma_j \sigma_k)) = n!\delta_n(\sigma_i \sigma_j \sigma_k),
\end{align*}\]

and

\[
g^{ij} = \frac{1}{(n!)^2} \text{tr}(r(\sigma_i^{-1}\sigma_j^{-1})) = \frac{1}{n!} \delta(\sigma_i^{-1}\sigma_j^{-1}).
\]

Let us now introduce a dual basis defined by

\[
\sigma^i := g^{ij} \sigma_j = \frac{1}{n!} \sigma_i^{-1},
\]

which satisfies

\[
\sum_i \sigma^i \sigma_i = 1.
\]

If we use the dual basis, we do not have an extra factor arising from raising or lowering indices

\[\begin{align*}
g^{ij} &= \text{tr}(r(\sigma^i \sigma^j)) = n!\delta(\sigma^i \sigma^j), \\
C_{ij}^k &= \text{tr}(r(\sigma_i \sigma_j \sigma^k)) = n!\delta(\sigma_i \sigma_j \sigma^k).
\end{align*}\]

We next construct the two-point function on a sphere. A simple triangulation shown in figure [2] leads to

\[
\eta_{ij} = C_{ikjs} g^{js} g^{kt} = C_{ik}s C_{ij}^k.
\]
We can show that

\[ \eta_{ij} = \sum_{\mathcal{R} \vdash n} \chi_{\mathcal{R}}(\sigma_i) \chi_{\mathcal{R}}(\sigma_j) \]

\[ \eta^{ij} = \sum_{\mathcal{R} \vdash n} \chi_{\mathcal{R}}(\sigma^i) \chi_{\mathcal{R}}(\sigma^j) \]

\[ \eta^i_j = \sum_{\mathcal{R} \vdash n} \chi_{\mathcal{R}}(\sigma_i) \chi_{\mathcal{R}}(\sigma^j), \] (34)

or we have

\[ \eta_{ij} = \sum_k \text{tr}^{(r)}(\sigma_i \sigma_k \sigma_j \sigma^k) = \text{tr}^{(r)}([\sigma_i] [\sigma_j]) \]

\[ \eta^{ij} = \sum_k \text{tr}^{(r)}(\sigma^i \sigma_k \sigma^j \sigma^k) = \text{tr}^{(r)}([\sigma^i] [\sigma^j]) \]

\[ \eta^i_j = \sum_k \text{tr}^{(r)}(\sigma_i \sigma_k \sigma^i \sigma^k) = \text{tr}^{(r)}([\sigma_i] [\sigma^j]). \] (35)

One important property is that this is a projection operator, \( \eta^2 = \eta \). More explicitly, we have

\[ \eta_{ij} \eta^{jk} = \eta_i^k, \quad \eta^i_j \eta^{jk} = \eta_i^k, \quad \eta_{ij} \eta^j_k = \eta_{ik}, \] (36)

and so on. The fact that a cylinder is a projection operator is a general property of topological field theories [34]. Lower indices and upper indices correspond to different orientations. The difference comes from the difference between the basis and the dual basis. In order to distinguish them it is convenient to to call the two kinds of boundaries \textit{in-boundaries} and \textit{out-boundaries}. We can draw three kinds of cylinders as in figure 3 corresponding to \( \eta^{ij} \), \( \eta_{ij} \) and \( \eta^{ij} \). The readers can find the diagrammatic meaning of the relation (36) using the oriented cylinders in figure 3.

In this theory, the difference is just the factor of \( 1/n! \),

\[ \chi^R(\sigma^i) = \frac{1}{n!} \chi^R(\sigma_i^{-1}) = \frac{1}{n!} \chi^R(\sigma_i), \] (37)

because \( \sigma^{-1} \) is conjugate to \( \sigma \). The difference between different orientations will be more important in theories we will consider in what follows.
The projection operator determines the space of physical operators \[31\]. It can be shown that the \( \eta \) is the projection operator from the algebra onto the centre of the algebra,

\[
\eta^{i} \sigma_{j} = \sum_{R \vdash n} \frac{1}{d_{R}} \chi_{R}^{R}(\sigma_{i}) p_{R} = [\sigma_{i}],
\]

where \( p_{R} \) is a central element given in (\ref{4}), and \( [\sigma] \) is defined in (\ref{17}). In other words, physical states are invariant under time translation,

\[
\eta^{i} [\sigma_{j}] = [\sigma_{i}].
\]

We thus have a one-to-one correspondence between central elements and physical states. The number of the physical states is the number of conjugacy classes.

Because the cylinder is a projection operator onto the physical Hilbert space, the torus, which is obtained by gluing two boundaries of the cylinder, gives the dimension of the vector space \[35\] \[34\],

\[
Z_{G=1} = \eta^{i} = \sum_{R \vdash n} 1.
\]

This counts the number of Young diagrams built from \( n \) boxes. It is equivalent to the number of multi-matrices built from \( n \) copies of a matrix at large \( N \).

The three-point function \( N_{ijk} \) is obtained from the structure constant of the operator product of physical states,

\[
[\sigma_{i}][\sigma_{j}] = N_{ij}^{\ k} [\sigma_{k}].
\]

From this we find

\[
N_{ijk} = tr^{(r)}([\sigma_{i}][\sigma_{j}][\sigma_{k}]) = \sum_{R \vdash n} \frac{1}{d_{R}} \chi_{R}^{R}(\sigma_{i}) \chi_{R}^{R}(\sigma_{j}) \chi_{R}^{R}(\sigma_{k})
\]

and

\[
N_{ij}^{\ k} = tr^{(r)}([\sigma_{i}][\sigma_{j}][\sigma^{k}]) = \sum_{R \vdash n} \frac{1}{d_{R}} \chi_{R}^{R}(\sigma_{i}) \chi_{R}^{R}(\sigma_{j}) \chi_{R}^{R}(\sigma^{k}),
\]

where \( N_{ijk} = N_{ik}^{\ l} \eta_{lk} \). Because this operator algebra is commutative

\[
[\sigma_{i}][\sigma_{j}] = [\sigma_{j}][\sigma_{i}],
\]
we have $N_{ij}^k = N_{ji}^k$. We can also check the associativity,

$$N_{ij}^k N_{kl}^n = N_{ik}^n N_{jl}^k. \quad (45)$$

From $[\sigma_i] = \eta^p_i \sigma_p$, we find that the cylinder and the three-holed sphere are also obtained by acting with the projection operator on $g_{ij}$ and $C_{ijk}$ as

$$\eta^i_k \eta^j_l g_{ij} = \eta_{kl}$$

$$\eta^i_l \eta^m_j \eta^n_k C_{ijk} = N_{lmn}. \quad (46)$$

These are also obtained by a simple triangulation [31].

An associative algebra with a non-degenerate inner product is called Frobenius algebra. It has been understood that Frobenius algebras play a role in the algebraic and axiomatic formulation of topological quantum field theories, and commutative Frobenius algebras are in one-to-one correspondence with two-dimensional topological quantum field theories [36, 37]. In the example, the algebra of $[\sigma_i]$ with the metric $\eta_{ij}$ is a commutative Frobenius algebra. The $\eta_{ij}$ is called Frobenius pairing. The pairing also determines a linear map from the vector space to the ground field,

$$\eta_{\nu_i} = \sum_{R=0} d_R \chi_R(\sigma_i) = n! \delta(\sigma_i), \quad (47)$$

where $\sigma_0 = 1$. This is called Frobenius form. Formally it is the sphere with a hole. The three objects, $\eta_{ij}$, $N_{ijk}$ and $\eta_{k0}$ can be basic building blocks to build up all surfaces. There is a relation among them, $N_{ij}^k \eta_{k0} = \eta_{ij}$ (see figure 4).

All partition functions are built up from the building blocks $\eta_{ij}$ and $N_{ijk}$. The partition function of a Riemann surface of genus $G$ is

$$Z_G = \sum_{R=0} \frac{1}{(d_R)^{2G-2}}. \quad (48)$$

The correlation function on the manifold of genus $G$ with $B_1$ in-boundaries and $B_2$ out-boundaries is given by

$$Z_{G,B_1,B_2} = \sum_{R=0} \frac{1}{(d_R)^{2G+B_1+B_2-2}} \chi^R(\sigma_{i_1}) \cdots \chi^R(\sigma_{i_{B_1}}) \chi^R(\sigma_{j_1}) \cdots \chi^R(\sigma_{j_{B_2}}). \quad (49)$$

1 Important remarks are found in p.98 of [37]. A Frobenius algebra is defined by providing with its Frobenius structure (Frobenius pairing or Frobenius form). If we choose two different Frobenius structures in an algebra, we will obtain two different Frobenius algebras. Being a Frobenius algebra is not a property of an algebra, but it is a structure of the algebra.
The repeated use of (B.3) and (B.4) helps us to find an equivalent form

\[ Z_{G, B_1, B_2} = \sum_{k,l} tr^{(r)} \left( \left( \rho_{k_1} \tau_{i_1} \rho^k \tau^l \right) \cdots \left( \rho_{k_G} \tau_{i_G} \rho^{k_G} \tau^{l_G} \right) [\sigma_{i_1}] \cdots [\sigma_{i_{B_1}}] [\sigma^{j_1}] \cdots [\sigma^{j_{B_2}}] \right). \quad (50) \]

We note that \( \sum_{k,l} \rho_k \tau_l \rho^k \tau^l \) and [\( \sigma \)] are central elements of the group algebra of the symmetric group, so their positions inside the trace are irrelevant. The form shows that the correlation functions are also obtained by representing the surface as a polygone with edges properly identified [38].

It is possible to choose a diagonal basis so that the operator product is given by \( O_\alpha O_\beta = \delta_{\alpha\beta} O_\alpha \). This is realised by

\[ p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma^{-1}) |\sigma] = d_R \sum_i \chi_R(\sigma^i) [\sigma_i]. \quad (51) \]

The role of \( p_R \) is explained around (A.4).

4 2D topological field theories obtained from walled Brauer algebras

4.1 Construction of Brauer topological field theory

In this section we present the topological lattice field theory obtained from the walled Brauer algebra \( B_N(m, n) \), following the construction in the previous section. Walled Brauer algebras have a parameter \( N \), which is identified with the matrix size \( N \) when we use them to organise the multi-trace structure of \( N \times N \) matrices. We will assume that \( N \) is large enough so that \( m + n < N \) is satisfied. This large \( N \) condition secures the semisimplicity of the algebra, which we need to construct a non-degenerate metric. In this construction we will use the idea presented in [39] that general semisimple algebras can be analogously introduced to finite groups.

Let a basis of the algebra be \( b_i \). We first define a dual basis \( b^*_j \) with respect to the bilinear form given by the trace \( tr_{m,n} \) over the mixed tensor space \( V^\otimes m \otimes \bar{V}^\otimes n \) in (A.19),

\[ tr_{m,n}(b_i b^*_j) = \delta_{ij}. \quad (52) \]

This dual basis was exploited in [9].

The trace of the regular representation is introduced by

\[ tr^{(r)}(b) = \sum_i tr_{m,n}(b^*_i b b_i) = \sum_\gamma d_\gamma \chi_\gamma(b), \quad (53) \]

where (A.20) and (B.10) are helpful to confirm the second equality, and \( \gamma \) are irreducible representations of the walled Brauer algebra. An irreducible representation \( \gamma \) is labelled by a bi-partition (see below equation (A.19).) Here \( d_\gamma \) is the dimension of an irreducible representation \( \gamma \) of the walled Brauer algebra, while \( t_\gamma \) is that of the \( GL(N) \) group. The appearance of the \( GL(N) \) group is a consequence of the Schur-Weyl duality. The multiplication of the algebra, \( b_i b_j = C_{ij}^k b_k \), is determined by

\[ C_{ij}^k = tr_{m,n}(b^*_k b_i b_j). \quad (54) \]
Orthogonality relations in appendix B will give
\[ g_{ij} = tr^{(r)}(b_ib_j), \]
\[ C_{ijk} = tr^{(r)}(b_ib_jb_k), \] (55)
and
\[ g^{ij} = \sum_{\gamma} \frac{t^{\gamma}}{d^\gamma} \chi^\gamma(b^*_i b^*_j). \] (57)

Let us now introduce another dual basis by
\[ b^i = g^{ij} b_j, \] (58)
and we can confirm the following equation
\[ \sum_i b^i b_i = 1. \] (59)
The following equation is helpful to relate the first dual basis \( b_i^* \) and the second dual basis \( b^i \),
\[ D^\gamma(b^j) = g^{ij} D^\gamma(b_j) = \frac{t^{\gamma}}{d^\gamma} D^\gamma(b^*_i), \] (60)
where \( D^\gamma(b) \) is the representation matrix of \( b \) in \( \gamma \). From this, we find that \( b^i \) is a dual basis with respect to the bilinear form determined by the trace of the regular representation,
\[ tr^{(r)}(b_i b_j) = \delta_i^j. \] (61)

Defining a dual basis allows us to obtain the representation theory of general semisimple algebras in a very similar way to finite groups [39]. In fact, we will find that almost all formulae are very similar to the formulae used in the symmetric group. In terms of the dual basis, we will obtain a handy expression for quantities with upper indices,
\[ g^{ij} = tr^{(r)}(b^i b^j) \]
\[ C_{ijk} = tr^{(r)}(b_i b_j b_k). \] (62)
In what follows we will exploit the second dual basis \( b^i \).

The computation of the two-point function is the same as (33). We obtain
\[ \eta_{ij} = \sum_{\gamma} \chi^\gamma(b_i) \chi^\gamma(b_j) \]
\[ \eta^{ij} = \sum_{\gamma} \chi^\gamma(b^i) \chi^\gamma(b^j) \]
\[ \eta_{i^j} = \sum_{\gamma} \chi^\gamma(b_i) \chi^\gamma(b^j). \] (63)
If we use (57), we find another expression of the two-point function,
\[ \eta_{ij} = \sum_k tr^{(r)}(b_i b_k b_j b_k) \]
\[ \eta^{ij} = \sum_k tr^{(r)}(b^i b_k b^j b_k) \]
\[ \eta_{i^j} = \sum_k tr^{(r)}(b_i b_k b^j b_k). \] (64)
Because $\chi^\gamma(b_i)$ is not proportional to $\chi^\gamma(b^j)$, the difference between in-boundaries (corresponding to lower indices) and out-boundaries (corresponding to upper indices) is more important.

$\sum_{\gamma}$ denotes the sum over all irreducible representations of the walled Brauer algebra. An irreducible representation of $B_N(m, n)$ is labelled by a bi-partition $(\gamma_+^+, \gamma_-^-)$, where $\gamma_+^+ \vdash (m - k)$ and $\gamma_-^- \vdash (n - k)$, and $k$ is an integer in the range $0 \leq k \leq \min(m, n)$. The sum will be performed by

$$
\sum_{\gamma} = \sum_{k=0}^{\min(m,n)} \sum_{\gamma_+^+ \vdash (m-k)} \sum_{\gamma_-^- \vdash (n-k)}.
$$

(65)

The Hilbert space of physical states is determined by the projection operator $\eta$. It is the projection onto the centre of the algebra,

$$
\eta^ib_j = \sum_{\gamma} \chi^\gamma(b_i) \chi^\gamma(b^j)b_j = \sum_{\gamma} \frac{1}{d_\gamma} \chi^\gamma(b_i)P^\gamma.
$$

(66)

where $P^\gamma$, which is given in (A.23), is a central element in the Brauer algebra. If we define

$$
[[b]] = \sum_i b_ib^i,
$$

(67)

then we can show that

$$
\eta^ib_j = [[b_i]].
$$

(68)

Equivalently we have

$$
\eta^j[[b_j]] = [[b_i]].
$$

(69)

Physical operators are invariant under time evolution.

The three-point function is determined by the operator product of physical states. From

$$
[[b_i]][[b_j]] = N_{ij}^kB_k,[[b_k]],
$$

(70)

we obtain

$$
N_{ijk} = N_{ij}^k\theta_{lk} = tr^{(r)}([[b_i]][[b_j]][[b_k]]) = \sum_{\gamma} \frac{1}{d_\gamma} \chi^\gamma(b_i)\chi^\gamma(b_j)\chi^\gamma(b_k).
$$

(71)

We can also use (46) to derive this. The associativity can be confirmed by using (B.11). The diagonal operator product is realised by switching to the representation basis $P^\gamma$, that is, $P^\gamma P^{\gamma'} = \delta^{\gamma\gamma'}P^\gamma$. The algebra of $[[b_i]]$ with the bilinear form $\eta$ is a commutative Frobenius algebra.

The partition function of a Riemann surface of genus $G$ is

$$
Z_G = \sum_{\gamma} \frac{1}{(d\gamma)^{2G-2}},
$$

(72)

and this is equivalent to

$$
Z_G = \sum_{i,j} t_{r}^{(r)} \left( (c_i^1 d_j^1 e^{i_1} d^{i_1}) \cdots (c_i^G d_j^G e^{i_G} d^{i_G}) \right),
$$

(73)
where \( c, d \) are elements of the Brauer algebra. The partition function of \( G = 1 \) gives the dimension of the vector space

\[
Z_{G=1} = \sum_{i} \eta_{ii} = \sum_{\gamma} 1.
\]

(74)

This counts the number of all irreducible representations.

Likewise for a Riemann manifold of genus \( G \) with \( B_1 \) in-boundaries and \( B_2 \) out-boundaries the correlation function is

\[
Z_{G,B_1,B_2} = \sum_{\gamma} \chi_{\gamma}(b_{i1}) \cdots \chi_{\gamma}(b_{iB_1}) \chi_{\gamma}(b_{j1}) \cdots \chi_{\gamma}(b_{jB_2}).
\]

(75)

An equivalent form is

\[
Z_{G,B_1,B_2} = \sum_{k,l} \text{tr}(r) \left( (c_k d_k c_k d_k) \cdots (c_k d_k c_k d_k) \right) [b_{i1}] \cdots [b_{iB_1}] [b_{j1}] \cdots [b_{jB_2}].
\]

(76)

Both \( \sum_{k,l} (c_k d_k c_k d_k) \) and \([b]\) are central elements of the Brauer algebra. The indices are raised or lowered by \( \eta \). For example we can use \( \eta^{ij} \) to convert an in-boundary into an out-boundary, finding from

\[
\eta^{ij} \chi_{\gamma}(b_j) = \chi_{\gamma}(b_i).
\]

(77)

There is a formal correspondence between the symmetric group \( S_{m+n} \) and the walled Brauer algebra \( B_N(m,n) \), clarified thanks to the introduction of the dual basis. The correspondence is as follows

\[
(R, A) \leftrightarrow (\gamma, A),
\]

\[
t_R \leftrightarrow t_{\gamma},
\]

\[
d_R \leftrightarrow d_{\gamma},
\]

\[
tr^{(r)}(\sigma) \leftrightarrow tr^{(r)}(b),
\]

\[
s_i \leftrightarrow b_{i},
\]

\[
\sigma^i \leftrightarrow b^i.
\]

(78)

The only difference between them is that in the symmetric group \( \sigma_i \sigma^i \) is proportional to the unit element, while in the Brauer algebra \( b_i b^i \) is not proportional to the unit, where the repeated index \( i \) is not summed in each case.

4.2 Brauer topological field theory as a collection of symmetric group topological field theories

The partition functions obtained from the symmetric group in section 3 can be regarded as counting of \( n \)-fold coverings of a general Riemann manifold, and this idea plays a crucial role in the string theoretic description of large \( N \) two-dimensional Yang-Mills theory \[23, 24, 25\]. On the other hand, an interpretation in terms of coverings is less clear for walled Brauer algebras, but it is naturally expected to give two kinds of coverings because the walled Brauer algebra contains the group algebra of \( S_m \times S_n \) as a subalgebra. In fact, in \[23\] the coupled representations, which are irreducible representations of the Brauer algebra corresponding to \( k = 0 \), were considered to describe non-holomorphic maps from worldsheets to the target space. In \[40\] the full expansion of
two-dimensional Yang-Mills is described in terms of only holomorphic maps from a new formula for the couple representation of \(GL(N)\) derived with the help of the walled Brauer algebra.

In this subsection, we will show that the correlation functions of the Brauer topological field theory can be expressed in terms of correlation functions obtained from the symmetric group \(S_{m-k} \times S_{n-k}\), where \(k\) takes all integers in \(1 \leq k \leq \min(m, n)\).

The partition functions of a surface of genus \(G\) are easily expressed in terms of the symmetric group data, if we use the formula (A.22), as

\[
Z_{G}^{B_{N}(m,n)} = \sum_{\gamma} \frac{1}{(d^{+})^{2G-2}} \sum_{k=0}^{\min(m,n)} \sum_{\gamma_{+}^{+}(m-k)\gamma_{-}^{+}(n-k)} \frac{(m-k)!(n-k)!k!}{m!n!} \frac{1}{(d^{+})^{2G-2}} \frac{1}{(d^{-})^{2G-2}}
\]

where \(Z_{G}^{S_{m-k}}\) is the partition function (83) corresponding to \(S_{m-k}\),

\[
Z_{G}^{S_{m-k}} = \sum_{R^{\pm}(m-k)} \frac{1}{(d^{R})^{2G-2}} \tag{79}
\]

Because the right-hand side in (79) has a clear interpretation in terms of coverings of the Riemann surface of genus \(G\), the partition function of the Brauer topological field theory can have a meaning in terms of \((m-k)\)-coverings and \((n-k)\)-coverings of the Riemann surface. It is interesting that a smaller number of sheets than \(m, n\) also comes in the game.

We next consider correlation functions. For the purpose we will use a character formula of the walled Brauer algebra. The character of a general element \(b\) in the walled Brauer algebra can be expressed in terms of the character of an element of the form \(C^{\otimes h} \otimes b_{+} \otimes b_{-}\) as

\[
\chi^{\gamma}(b) = N^{\gamma(b)+h} \chi^{\gamma(C^{\otimes h} \otimes b_{+} \otimes b_{-})}, \tag{81}
\]

where \(b_{+}\) and \(b_{-}\) are elements in \(S_{m-h}\) and \(S_{n-h}\) respectively, and \(C\) is the contraction. The \(z(b)\) is a quantity that is read from the element \(b\). This formula is explained in (C.1). Then the two-point function can be

\[
\eta^{B_{N}(m,n)}(\tilde{b}, \tilde{b}) = \sum_{\gamma} \chi^{\gamma}(b) \chi^{\gamma}(\tilde{b})
\]

\[
= N^{z(b)+z(\tilde{b})-h-\gamma} \eta^{B_{N}(m,n)}(C^{\otimes h} \otimes b_{+} \otimes b_{-}, C^{\otimes h} \otimes b_{+} \otimes b_{-}). \tag{82}
\]

We will next use

\[
\chi^{\gamma}(C^{\otimes h} \otimes b_{+} \otimes b_{-}) = \frac{N^{h}}{(m-k)!(n-k)!} \sum_{\sigma_{2} \in S_{m-k}, \tau_{2} \in S_{n-k}} \Delta(b_{+}, b_{-}; \sigma_{2}, \tau_{2}) \chi^{\gamma}(\sigma_{2}^{1} \chi_{\gamma_{-}}(\tau_{2}^{-1}), \tag{83}
\]

where

\[
\Delta(b_{+}, b_{-}; \sigma_{2}, \tau_{2}) = \frac{1}{(k-h)!} \sum_{\sigma_{1}, \tau_{1} \in S_{k-h}} \eta^{S_{m-k}}(b_{+}, \sigma_{1} \circ \sigma_{2}) \eta^{S_{n-h}}(b_{-}, \tau_{1} \circ \tau_{2}) \eta^{S_{k-h}}(\sigma_{1}^{-1}, \tau_{1}^{-1}). \tag{84}
\]
This is derived in (C,6). The Δ is composed of three two-point functions defined in the topological field theory of the symmetric group (see figure 5).

The formula enable us to compute

\[
\eta^{B_N(m,n)}(b,\tilde{b}) = \chi_{\gamma_+}^{\gamma_+}(C^{\otimes h} \otimes b_+ \otimes b_-, C^{\otimes \tilde{h}} \otimes \tilde{b}_+ \otimes \tilde{b}_-)
\]

\[
= Nz(b) + z(\tilde{b}) - h - \tilde{h} \sum_{\gamma}^{\gamma_+} \chi_{\gamma}^{\gamma_+}(C^{\otimes h} \otimes b_+ \otimes b_-)\chi_{\gamma}^{\gamma_+}(C^{\otimes \tilde{h}} \otimes \tilde{b}_+ \otimes \tilde{b}_-)
\]

\[
= Nz(b) + z(\tilde{b}) \sum_{m,n}^{\text{min}(m,n)} \sum_{k=\max(h,\tilde{h})}^{\gamma_+(m-k)} \sum_{\gamma_-(n-k)} \Delta(b_+, b_-; \sigma_2, \tau_2)\chi_{\gamma_+}^{\gamma_+}(\sigma_2^{-1})\chi_{\gamma_-}^{\gamma_-}(\tau_2^{-1})
\]

\[
\times \frac{1}{(m-k)!} \sum_{\sigma_2 \in S_{m-k}} \sum_{\tau_2 \in S_{n-k}} \Delta(\tilde{b}_+, \tilde{b}_-; \tilde{\sigma}_2, \tilde{\tau}_2)\chi_{\gamma_+}^{\gamma_+}(\tilde{\sigma}_2^{-1})\chi_{\gamma_-}^{\gamma_-}(\tilde{\tau}_2^{-1})
\]

\[
= Nz(b) + z(\tilde{b}) \sum_{m,n}^{\text{min}(m,n)} \sum_{k=\max(h,\tilde{h})}^{\gamma_+(m-k)} \sum_{\sigma_2 \in S_{m-k}} \sum_{\tau_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2)\Delta(\tilde{b}_+, \tilde{b}_-; \tilde{\sigma}_2, \tilde{\tau}_2)
\]

\[
\times \eta^{S_{m-k} \times S_{n-k}}(\sigma_2^{-1} \otimes \tau_2^{-1}, \tilde{\sigma}_2^{-1} \otimes \tilde{\tau}_2^{-1}),
\]

where \( \eta \) is the two-point function (54) corresponding to \( S_{m-k} \times S_{n-k} \),

\[
\eta^{S_{m-k} \times S_{n-k}}(\sigma_2^{-1} \otimes \tau_2^{-1}, \tilde{\sigma}_2^{-1} \otimes \tilde{\tau}_2^{-1}) = \eta^{S_{m-k}}(\sigma_2^{-1}, \tilde{\sigma}_2^{-1})\eta^{S_{n-k}}(\tau_2^{-1}, \tilde{\tau}_2^{-1}),
\]

and

\[
\eta^{S_{m-k}}(\sigma_2^{-1}, \tilde{\sigma}_2^{-1}) = \frac{1}{(m-k)!} \sum_{\gamma}^{\gamma_+(m-k)} \chi_{\gamma}^{\gamma_+}(\sigma_2^{-1})\chi_{\gamma}^{\gamma_+}(\tilde{\sigma}_2^{-1}).
\]

We have shown that the two-point function of the Brauer topological field theory can be expressed in terms of the two-point function of the topological field theory obtained from the group algebra of \( S_{m-k} \times S_{n-k} \), where \( k \) takes all integers in \( \max(h,\tilde{h}) \leq k \leq \min(m,n) \).
It is straightforward to extend this to any correlation functions. For example the three-holed sphere \((71)\) is as follows:

\[
Z_{G=0,B=3}^{B_N(m,n)}(b^1, b^2, b^3) = N^2(b^1+1b^2+1b^3)_{B_N(m,n)}Z_{G=0,B=3}(C^0 \otimes b^1_+ \otimes b^1_-, C^0 \otimes b^2_+ \otimes b^2_-, C^0 \otimes b^3_+ \otimes b^3_-),
\]

where \(Z_{G=0,B=3}^{S_{m-k} \times S_{n-k}}(\sigma_1^{-1} \otimes \tau_1^{-1}, \sigma_2^{-1} \otimes \tau_2^{-1}, \sigma_3^{-1} \otimes \tau_3^{-1})\) is the three-holed sphere given in \((12)\) associated with \(S_{m-k} \times S_{n-k}\). Thus we can express any correlation functions obtained from the Brauer algebra as a set of correlation functions obtained from the symmetric group \(S_{m-k} \times S_{n-k}\).

## 5 2D field theories with the restricted structure

As we have reviewed in section \(2\) gauge invariant operators in \(\mathcal{N} = 4\) super Yang-Mills built from \(p\) complex matrices can be labelled by an element of the form \([\sigma_i]_H\) or \([b_i]_H\), where \(H = S_{n_1} \times \cdots \times S_{n_p}\). With the motivation to associate the description of the matrix models with two-dimensional field theories, we will construct a class of two-dimensional field theories whose physical states are given by \([\sigma_i]_H\) or \([b_i]_H\). The idea of such theories is given in \(21\). For simplicity in what follows we consider the case \(H = S_m \times S_n\), but the generalisation to \(S_{n_1} \times \cdots \times S_{n_p}\) is straightforward.

We will first consider the symmetric group \(S_{m+n}\). As we have seen in the last two sections, physical operators are determined by the two-point function, because it is a projection operator onto the Hilbert space of physical operators. In order to obtain physical operators labelled by \([\sigma_i]_H\), we will consider the two-point function drawn in figure \(4\) in stead of the cylinder \((33)\). On the double line in the figure we restrict the sum to the subgroup \(H\) \((21)\), and triangulations should be done consistently with the restriction. A simple triangulation of such a cylinder is shown in the RHS of figure \(4\).

\[
\theta_{ij} = \sum_k \sum_a C_{ikb}C_{ajl} h^{ab} g^{kl},
\]

(89)

where \(a\) runs over a complete set of the subgroup \(H\). The \(g^{ij}\) and \(C_{ijk}\) are defined in \((28)\) and \((29)\). The \(h\) is the metric defined in \(H\) as

\[
h_{ab} = m!n! \delta_{m,n}(\sigma_a \sigma_b)
\]

\[
h^{ab} = m!n! \delta_{m,n}(\sigma^a \sigma^b),
\]

(90)

where \(\sigma^a\) is the dual basis of \(\sigma_a\), which is defined by \(\sigma^a = \frac{1}{m!n!} \delta^{-1}_{a \sigma a}\). The delta function defined over the group algebra of \(H\) is denoted by \(\delta_{m,n}(\sigma)\).
Figure 6: LHS - a cylinder with the restriction on the double line, RHS - a simple triangulation

We can show that

\[ \theta_{ij} = \sum_a \text{tr}^r(h_a^i \sigma_i h^a j) = \text{tr}^r([\sigma_i]_H[\sigma_j]_H) \]

\[ \theta^{ij} = \sum_a \text{tr}^r(h_a \sigma^i h^a j) = \text{tr}^r([\sigma^i]_H[\sigma^j]_H) \]

\[ \theta^{ji} = \sum_a \text{tr}^r(h_a \sigma_i h^a \sigma^j) = \text{tr}^r([\sigma_i]_H[\sigma^j]_H), \]

(91)

and the projector relation \( \theta^2 = \theta \) can be checked explicitly. The use of (B.9) leads to another expression:

\[ \theta_{ij} = \sum_{R,A,\mu,\nu} \frac{d_R}{d_A} \chi^R_{A,\mu\nu}(\sigma_i) \chi^R_{A,\nu\mu}(\sigma_j) \]

\[ \theta^{ij} = \sum_{R,A,\mu,\nu} \frac{d_R}{d_A} \chi^R_{A,\mu\nu}(\sigma^i) \chi^R_{A,\nu\mu}(\sigma^j) \]

\[ \theta^{ji} = \sum_{R,A,\mu,\nu} \frac{d_R}{d_A} \chi^R_{A,\nu\mu}(\sigma_i) \chi^R_{A,\mu\nu}(\sigma^j), \]

(92)

where \( d_R \) is the dimension of an irreducible representation \( R \) of the symmetric group \( S_{m+n} \), while \( d_A \) is the dimension of an irreducible representation \( A \) of the symmetric group \( S_m \times S_n \). The \( \chi^R_{A,\mu\nu}(\sigma) \) is the restricted character (see around (A.15)). The indices \( \mu, \nu \) run over 1, \( \cdots, M_A^R \), where \( M_A^R \) counts the number of times the representation \( A \) appears in the \( R \). This projector (92) was obtained in [21] from the invariance (10), which leads to \( \theta^{ij}_r \text{tr}_{m+n}(\sigma_j X^{\otimes m} \otimes Y^{\otimes n}) = \text{tr}_{m+n}(\sigma_i X^{\otimes m} \otimes Y^{\otimes n}) \).

From the cyclic property of the restricted character

\[ \chi^R_{\bar{A},\mu\nu}(h\sigma) = \chi^R_{\bar{A},\mu\nu}(\sigma h) \quad (h \in S_m \times S_n) \]

\[ \chi^R_{\bar{A},\mu\nu}(\rho\sigma) \neq \chi^R_{\bar{A},\mu\nu}(\sigma \rho) \quad (\rho \in S_{m+n}/S_m \times S_n), \]

(93)

boundary operators are more sensitive to the orientations corresponding to upper and lower indices because \( \sigma_i \) is not conjugate by some permutation in the subgroup \( S_m \times S_n \) to the dual element \( \sigma^i \).

Physical operators are determined by the projection operator

\[ \theta^{ij}_r \sigma_j = \sum_{R,A,\mu,\nu} \frac{1}{d_R} \chi^R_{\bar{A},\mu\nu}(\sigma_i) P^R_{\bar{A},\mu\nu} = [\sigma_i]_H, \]

(94)
where $[\sigma_i]_H$ is defined in (18), and $P^R_{A,\mu\nu}$ is a basis of elements commuting with any elements in $H = S_m \times S_n$ [11]. See (A.15). The equation (94) also means that physical states are invariant under time evolution,

$$\theta^j_i [\sigma_j]_H = [\sigma_i]_H.$$  

(95)

Because $\theta^j_i$ is a projector, we expect to obtain the the dimension of the vector space by taking a trace in the same way as the topological field theories in section 3 and 4.

$$Z_{G=1} = \sum_i \theta^i_i = \sum_i ( M^R_{A,i} )^2,$$  

(96)

which comes from [13,6]. This is the partition function of a torus. The dimension of the vector space gives the expected result of the number of multi-traces constructed from $m$ copies of $X$ and $n$ copies of $Y$ via the method of using the basis [9], as shown in [11,42]. The counting of multi-traces is associated with $Z_{G=1}$ by considering the counting of orbits of the group actions via Burnside’s lemma in [21,41].

The three-point function (three-holed sphere) $\Xi_{ijk}$ can be obtained as the structure constant of the algebra whose elements are given by $[\sigma_i]_H$. From

$$[\sigma_i]_H [\sigma_j]_H = \Xi_{ijk} [\sigma_k]_H,$$  

(97)

we find that

$$\Xi_{ij}^k = tr^{(r)}([\sigma_i]_H [\sigma_j]_H [\sigma^k]_H),$$  

$$\Xi_{ijk} = tr^{(r)}([\sigma_i]_H [\sigma_j]_H [\sigma_k]_H) = \Xi_{ij}^l \theta^{lk}.$$  

(98)

Note that the two-point function and the three-point function are also obtained by acting with the projection operator on $g_{ij}$ and $C_{ijk}$

$$\theta^i_j g_{kl} = \theta^i_{ij},$$  

$$\theta^i_j \theta^q_r C_{pqr} = \Xi^{ij}_k,$$  

(99)

which is analogous to [46]. It is also convenient to rewrite the expression in terms of the restricted characters:

$$\Xi_{ij}^k = \sum_{R,A,\mu,\nu,\lambda} \frac{d^R}{d_A} \chi^{R}_{A,\mu\nu}(\sigma_i) \chi^{R}_{A,\nu\lambda}(\sigma_j) \chi^{R}_{A,\lambda\mu}(\sigma^k).$$  

(100)

The associativity

$$\Xi_{ij}^k \Xi_{kl}^m = \Xi_{ik}^n \Xi_{jl}^k$$  

(101)

is verified by using orthogonality relations in appendix [3]

The vector space whose elements are $[\sigma]_H$, equipped with the bilinear form $\theta_{ij}$, is a Frobenius algebra. The Frobenius form is $\theta_{i0}$. One can check

$$\Xi_{ij}^k \theta_{k0} = \theta_{ij}.$$  

(102)

A big difference from the Frobenius algebras in section 3 and 4 is that the Frobenius algebra in this section is noncommutative, that is,

$$\Xi_{ij}^k \neq \Xi_{ij}^k.$$  

(103)
as a consequence of the non-commutative operator algebra

\[ [\sigma_i]_H [\sigma_j]_H \neq [\sigma_j]_H [\sigma_i]_H. \]  

(104)

By a change of basis, we will obtain the following (almost diagonal) operator product

\[ P_{A,\mu}^R P_{A',\nu}^{R'} = \delta_{AA'} \delta_{\nu\mu} P_{A,\mu}^R. \]  

(105)

The existence of the \( \mu, \nu \) indices reflects the noncommutativity, and we also find the \( \mu, \nu \) indices indeed behave like matrix-indices. (See also orthogonality relations for restricted characters in appendix B.) Due to the non-commutativity, this field theory is not topological.

The partition functions can be computed from the building blocks \( \theta_{ij}, \Xi_{ijk} \). The partition function of a Riemann surface of genus \( G \) can be computed as

\[ Z_G = \sum_{R,A} \frac{1}{(d_R d_A)^{G-1}} (M_A^R)^{G+1}. \]  

(106)

Equations in appendix B will be helpful to derive this. The partition function depends only on \( G \).

If we set \( M_A^R = 1 \) and \( d_A = d_R \), we reproduce the result (108) as expected. The partition function has another form,

\[ Z_G = \sum_{a,j} \text{tr}^{(r)} \left( (h_{a_1} \sigma_j h^{a_1}) \cdots (h_{a_G} \sigma_j h^{a_G} \sigma^{jG}) \right), \]  

(107)

where \( h_a \) is a basis of the subgroup \( S_m \times S_n \) and \( h^a \) is the dual basis. The indices \( a_1, \cdots, a_G \) run over a complete set of \( S_m \times S_n \).

The next interest will be a Riemann surface with boundaries. This should be paid more attention because the Frobenius algebra is noncommutative. Correlation functions depend on the way of gluing the building blocks. As a simple example consider the case of \( G = 1 \) and \( B = 2 \). We have two possible ways, \( \Xi_{ijk} \Xi_k^{jkl} \) and \( \Xi_{ijk} \Xi^{kjl} \), giving different answers. The first one is computed to give

\[ \Xi_{ijk} \Xi_k^{jkl} = \sum_{R,A,\mu,\nu} \frac{1}{(d_A)^G} \text{tr}^{(r)} (\chi_{A,\mu}(\sigma_i) \chi_{A,\nu}(\sigma^l)), \]  

(108)

while the second one is

\[ \Xi_{ijk} \Xi^{kjl} = \sum_{R,A,\mu,\nu} \frac{1}{(d_A)^{G}} M_{A,\mu}^R \chi_{A,\nu}^R (\sigma_i) \chi_{A,\nu}^R (\sigma^l). \]  

(109)

The second one is the same as \( \Xi_{i,k} \Xi_{p}^{pk} \). If we set \( \sigma_i = 1 \) and \( \sigma^l = 1 \), both reduce to the case \( G = 1 \) in (106). The difference between the two correlation functions is how multiplicity indices \( \mu, \nu \) are contracted. Introducing \( M_A^R \times M_A^R \) matrices \( M_i \), the structure of \( tr(M_1) tr(M_2) \) arises effectively from the first one, while \( trtr(M_1 M_2) \) from the second one. The first one can also be written as

\[ \Xi_{ijk} \Xi_k^{jkl} = \sum_{a,p} \text{tr}^{(r)} (h_{a} \tau_p [\sigma_i]_H h^a \tau^p [\sigma^l]_H), \]  

(110)

while the second can be

\[ \Xi_{ijk} \Xi^{kjl} = \sum_{a,p} \text{tr}^{(r)} (h_{a} \tau_p h^a \tau^p [\sigma_i]_H [\sigma^l]_H). \]  

(111)
Note that
\[ [\sigma_i]_H h = h[\sigma_i]_H \quad (h \in S_m \times S_n), \] (112)
but
\[ [\sigma_i]_H \tau \neq \tau [\sigma_i]_H \quad (\tau \in S_{m+n}/S_m \times S_n). \] (113)

As we have seen in the example, the correlation functions are not uniquely determined when we specify \( G \) and \( B \), and they are sensitive to the positions of boundaries. Reflecting the matrix structure, they have cyclic symmetries acting on the boundary positions. For example the following correlation function, which is one of correlation functions of a surface of genus \( G \) with \( B \) boundaries,
\[
\sum_{R,A} \theta_{ij} = \sum_{\gamma,A,\mu,\nu} \frac{1}{d_A} \chi_{A,\mu}\gamma(b_i)\chi_{A,\nu}\gamma(b_j) = \text{tr}^{(r)}([b_i]_H[b_j]_H),
\] (114)
is invariant under cyclic permutations acting on the indices \( i_1, \ldots, i_B \). One may read the single trace structure \( tr(M_1 \cdots M_B) \) from this. The restricted characters were first introduced to describe open strings on giant gravitons [6, 7]. It is interesting to study if the origin of the non-commutativity can be explained in terms of D-branes in the context of two-dimensional theories.

So far we have considered a new kind of two-dimensional field theories based on the symmetric group, and the same construction can be applied to the walled Brauer algebra. The formal replacement (78) with
\[
[\sigma_i]_H \leftrightarrow [b_i]_H,
\]
\[
M_R^A \leftrightarrow M_A^R,
\] (115)
where \( M_A^\gamma \) is the number of times \( A \) appears in \( \gamma \), will work to obtain the new two-dimensional field theory based on the walled Brauer algebra. We will show some of them explicitly for convenience.

The two-point function is given by
\[
\theta_{ij} = \sum_{\gamma,A,\mu,\nu} \frac{1}{d_A} \chi_{A,\mu}\gamma(b_i)\chi_{A,\nu}\gamma(b_j) = \text{tr}^{(r)}([b_i]_H[b_j]_H),
\] (116)
and this determines the Hilbert space of physical operators,
\[
\theta_{ij} b_j = \sum_{\gamma,A,\mu,\nu} \frac{1}{d_A} \chi_{A,\mu}\gamma(b_i)Q_{A,\mu}\gamma = [b_i]_H.
\] (117)

Changing to the basis \( A.28 \), the almost diagonal operator product is obtained [9, 13]
\[
Q_{A,\mu}\gamma Q_{A',\mu'}\gamma' = \delta^{\gamma\gamma'} \delta_{A'A''} \delta_{\nu\nu'} Q_{A,\mu}\gamma,
\] (118)
where \( \mu, \nu \) are multiplicity labels running over \( 1, \cdots, M_A^\gamma \). The three-point function is the structure constant (i.e. \( [b_i]_H[b_j]_H = \Xi_{ijk}^k[b_k]_H \)),

\[
\Xi_{ijk} = \sum_{\gamma, A, \mu, \nu, \lambda} \frac{d^\gamma}{(d_A)^2} \chi^\gamma_{A,\mu\nu}(b_i) \chi^\gamma_{A,\nu\lambda}(b_j) \chi^\gamma_{A,\lambda\mu}(b_k) = tr^{(r)}([b_i]_H[b_j]_H[b_k]_H).
\]

(119)

The partition function of a Riemann surface of genus \( G \) is given by

\[
Z_G = \sum_{a,j} tr^{(r)} \left( (h_a^i b_j^i h^a h^{-1}) \cdots (h_a^i b_j^i h^a h^{-1}) \right)
\]

\[
= \sum_{\gamma, A} \frac{1}{(d^\gamma d_A)^{G-1}} (M_A^\gamma)^{G+1}.
\]

(120)

Furnished with the bilinear form (116), the algebra formed by \([b_i]_H\) is a noncommutative Frobenius algebra. The partition function of \( G = 1 \) gives the dimension of the vector space

\[
Z_{G=1} = \sum_i \theta_i = \sum_{\gamma, A} (M_A^\gamma)^2.
\]

(121)

This is equivalent to the counting of the number of multi-traces built from \( m \) copies of \( X \) and \( n \) copies of \( Y \) via the method of using the basis \( \Xi \), as shown in [9, 43].

### 6 2D theoretic interpretation of the multi-matrix models

In this section we will study a relation between correlators of the Gaussian matrix models and correlation functions of the two-dimensional field theories.

First consider the method of using symmetric group elements to describe multi-trace operators. Recall (11),

\[
S_{\sigma_i,\sigma_j} = \langle tr_{m+n}(\sigma_i X^{\otimes m} \otimes Y^{\otimes n}) tr_{m+n}(\sigma_j X^{\otimes m} \otimes Y^{\otimes n}) \rangle = \sum_{h \in H} tr_{m+n}(h \sigma_i h^{-1} \sigma_j)
\]

\[
= \frac{N_{m+n}}{(m+n)!} \sum_{h \in H} tr^{(r)}(\Omega_{m+n} h \sigma_i h^{-1} \sigma_j),
\]

(122)

where \( \sigma_i, \sigma_j \in S_{m+n} \) and \( H = S_m \times S_n \). With the notation (20), one may write it as

\[
S_{[\sigma_i]_H,[\sigma_j]_H} = N_{m+n} \frac{m! n!}{(m+n)!} tr^{(r)}(\Omega_{m+n}[\sigma_i]_H[\sigma_j]_H).
\]

(123)

Comparing to the three-point function (118), we find that the \( S_{\sigma_i,\sigma_j} \) can be interpreted to be the three-point function with the Omega factor put at one of the boundaries (see also [21]). More explicitly, we have

\[
S_{\sigma_i,\sigma_j} = N_{m+n} \frac{m! n!}{(m+n)!} \sum_k \Xi_{ijk} N^{C_{\sigma_k}-(m+n)}.
\]

(124)

Because \( \Omega_{m+n} \) is a central element in the group algebra of \( S_{m+n} \), the ordering of the three objects is not important.
Next consider the way of labelling multi-traces in terms of Brauer elements. Recall the two-point function (15):

\[ B_{b_i,b_j} = \langle tr_{m,n}(b_i X^{\otimes m} \otimes Y^{\otimes n})tr_{m,n}(b_j X^{\otimes m} \otimes Y^{\otimes n}) \rangle = \sum_{h \in S_m \times S_n} tr_{m,n}(hb_i h^{-1} b_j) \]

\[ = \sum_{\gamma} \sum_{h \in S_m \times S_n} t^\gamma \chi^\gamma(hb_i h^{-1} b_j). \] (125)

We can also write it as

\[ B_{[b_i]_H,[b_j]_H} = \frac{1}{mn!} \sum_{\gamma} t^\gamma \chi^\gamma([b_i]_H [b_j]_H). \] (126)

The sum is over all Young diagrams \(\gamma = (\gamma_+, \gamma_-)\) by means of (65), where \(\gamma_+\) and \(\gamma_-\) are partitions of \(m - k\) and \(n - k\) respectively. We now use the fact that the dimension of an irreducible representation \(\gamma\) of the \(GL(N)\) group can be expressed in terms of elements in \(S_{m-k} \times S_{n-k}\) as

\[ t_\gamma = \frac{N^{m+n-2k}}{(m-k)!(n-k)!}\chi_{(\gamma_+,-\gamma_-)}(\Omega_{m-k,n-k}). \] (127)

The character of \(S_{m-k} \times S_{n-k}\) is denoted by \(\chi_{(\gamma_+, -\gamma_-)}\), and \(\Omega_{m-k,n-k}\) is a central element in the group algebra of \(S_{m-k} \times S_{n-k}\) which is called coupled Omega factor [23]. We do not use an explicit form of \(\Omega_{m-k,n-k}\), but we should keep in mind that the \(N\)-dependence of \(t_\gamma\) is encoded in it, and it is not just the product of \(\Omega_{m-k} \times \Omega_{n-k}\). In order to combine \(t^\gamma\) and \(\chi^\gamma(hb_i h^{-1} b_j)\), we will express \(\chi^\gamma(hb_i h^{-1} b_j)\) in terms of characters in \(S_{m-k} \times S_{n-k}\) by using the formulae (81) and (83).

Let us first consider the simplest case that both \(b_i\) and \(b_j\) are elements in the group algebra of \(S_m \times S_n\) as an exercise before going to the general case. Suppose \(b_i = b_i^+ \otimes b_i^-\), \(b_j = b_j^+ \otimes b_j^-\), where \(b_i^+\) and \(b_j^+\) are elements in \(S_m\) and \(b_i^-\) and \(b_j^-\) are elements in \(S_n\). For \(h = \sigma \otimes \tau \in S_m \times S_n\),

\[ hb_i h^{-1} b_j = (\sigma b_i^+ \sigma^{-1} b_j^+) \otimes (\tau b_i^- \tau^{-1} b_j^-) \in S_m \times S_n. \] (128)

Applying the formula (83) to \(\chi^\gamma(hb_i h^{-1} b_j)\), we obtain

\[ \chi^\gamma(hb_i h^{-1} b_j) = \frac{1}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \]

\[ \times \chi_{(\gamma_+,-\gamma_-)}(\sigma_2^{-1} \otimes \tau_2^{-1}). \] (129)

where

\[ \chi_{(\gamma_+,-\gamma_-)}(\sigma_2^{-1} \otimes \tau_2^{-1}) = \chi_{\gamma_+}(\sigma_2^{-1})\chi_{\gamma_-}(\tau_2^{-1}). \] (130)
Then the two-point function can be written as

\[
B_{b_i, b_j} = \sum_{\gamma} \sum_{h \in S_m \times S_n} t^{\gamma} \chi^\gamma(h b_i h^{-1} b_j)
\]

\[
= \sum_{k=0}^{N_{m+n-2k}} \sum_{\gamma_+ \vdash (m-k), \gamma_- \vdash (n-k)} \sum_{\sigma \in S_m, \tau \in S_n} \frac{N_{m+n-2k}}{(m-k)!(n-k)!} \chi(\gamma_+, \gamma_-)(\Omega_{m-k, n-k}) \\
\times \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \chi(\gamma_+, \gamma_-)(\sigma_2^{-1} \otimes \tau_2^{-1})
\]

\[
= \sum_{k=0}^{N_{m+n-2k}} \sum_{\sigma \in S_m, \tau \in S_n} \frac{N_{m+n-2k}}{(m-k)!(n-k)!} \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \delta_{m-k, n-k}(\Omega_{m-k, n-k} \sigma_2^{-1} \otimes \tau_2^{-1}), \quad (131)
\]

where \(\delta_{m-k, n-k}(\sigma \otimes \tau)\) is the delta function defined over the group algebra of \(S_{m-k} \times S_{n-k}\).

By the way in this case we know that the second line in (125) is factorised to give

\[
\sum_{h \in S_m \times S_n} tr_m(h b_i h^{-1} b_j) = \sum_{\sigma \in S_m, \tau \in S_n} tr_m(\sigma b_i^+ \sigma^{-1} b_j^+) tr_n(\sigma b_i^- \sigma^{-1} b_j^-) = \sum_{\sigma \in S_m, \tau \in S_n} N_{m+n} \delta_m(\Omega_{m} \sigma b_i^+ \sigma^{-1} b_j^+) \delta_n(\Omega_{n} \sigma b_i^- \sigma^{-1} b_j^-). \quad (132)
\]

Comparing (131) with (132) gives an identity

\[
\delta_{m,n}(\Omega_{m} \sigma \otimes \Omega_{n} \tau) = \sum_{k} \frac{N_{m+n-2k}}{(m-k)!(n-k)!} \Delta(\sigma, \tau; \sigma_2, \tau_2) \delta_{m-k, n-k}(\Omega_{m-k, n-k} \sigma_2^{-1} \otimes \tau_2^{-1}), \quad (133)
\]

where \(\sigma\) and \(\tau\) are elements in \(S_m\) and \(S_n\) respectively.

Let us next study a general situation, where \(b_i\) and \(b_j\) are general elements in the Brauer algebra. We define \(b_{ij}^\alpha := ab_i \alpha^{-1} b_j\). (We will use \(\alpha\) for elements in \(S_m \times S_n\) in stead of \(h\) not to get an extra confusion with the number of contractions \(h\).) Using the formula (81),

\[
\sum_{\alpha \in S_m \times S_n} \chi^\gamma(ab_i \alpha^{-1} b_j) = \sum_{\alpha \in S_m \times S_n} N^{z(b_{ij}^\alpha)-h_\alpha} \chi^\gamma(C \otimes h_\alpha \otimes b_{ij}^\alpha), \quad (134)
\]

where \(b_{ij}^\alpha \otimes b_{ij}^{\alpha^+}\) is an element in the group algebra of \(S_{m-h_\alpha} \times S_{n-h_\alpha}\), and \(z(b_{ij}^\alpha)\) is the number of zero-cycles in \(b_{ij}^\alpha\), and \(h_\alpha, b_{ij}^\alpha, b_{ij}^{\alpha^+}\) are defined by this equation. We have added the subscript \(\alpha\) to mean that they depend on \(\alpha\).
We will get an expression for $B_{b_i, b_j}$,

\begin{align*}
B_{b_i, b_j} &= \sum_{\gamma} \sum_{\alpha \in S_m \times S_n} \ell^\alpha \chi^\gamma (\alpha b_i \alpha^{-1} b_j) \\
&= \sum_{\gamma} \sum_{\alpha \in S_m \times S_n} \ell^\alpha N^{z(b_i^\alpha)} \chi^\gamma (C^{\otimes h_{\alpha}} \otimes b_i^{\alpha+} \otimes b_i^{\alpha-}) \\
&= \sum_{\alpha \in S_m \times S_n} \sum_{k = h_{\alpha}}^{\min(m,n)} N^{m+n-2k} N^{z(b_i^\alpha)} \frac{1}{(m-k)!(n-k)!} \\
&\times \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(b_i^{\alpha+}, b_i^{\alpha-}; \sigma_2, \tau_2) \delta_{m-k, n-k}(\Omega_{m-k, n-k} \sigma_2^{-1} \otimes \tau_2^{-1}). \tag{135}
\end{align*}

In this case we do not have a factorisation like (132). We can also write it as

\begin{align*}
B_{b_i, b_j} &= \sum_{\alpha \in S_m \times S_n} \sum_{k = h_{\alpha}}^{\min(m,n)} N^{m+n-2k} N^{z(b_i^\alpha)} \\
&\times \frac{1}{(k-h_{\alpha})!(m-k)!(n-k)!^2} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \\
&\times \sum_{\sigma_1, \tau_1 \in S_{k-h_{\alpha}}} \eta^{S_{m-h_{\alpha}}} (b_i^{\alpha+}, \sigma_1 \circ \sigma_2) \eta^{S_{n-h_{\alpha}}} (b_i^{\alpha-}, \tau_1 \circ \tau_2) \eta^{S_{k-h_{\alpha}}} (\sigma_1^{-1}, \tau_1^{-1}) \\
&\times \eta^{S_{m-k} \times S_{n-k}} (\Omega_{m-k, n-k} \sigma_2^{-1} \otimes \tau_2^{-1}). \tag{136}
\end{align*}

A pictorial drawing is presented in figure 7. Each piece in the above equation has an interpretation as a correlation function of the topological field theory based on the symmetric group. What we have done is almost the same as what we have done in subsection 4.2. Correlation functions expressed in terms of elements in the Brauer algebra have an interpretation as an assemblage of correlation functions in terms of symmetric group elements.
7 Some relations among the 2D quantum field theories

In this section we will explore some relations among the field theories we have shown, by exploiting a map $\Sigma$ between elements of the Brauer algebra and elements of the symmetric group, which was exploited in [9, 11, 40].

Under the map, elements of the Brauer algebra $B_N(m, n)$ are related to elements of the symmetric group $S_{m+n}$. Suppose that a basis in $B_N(m, n)$ is related to a basis in $S_{m+n}$ by

$$\sigma_i = \Sigma(b_i), \quad (137)$$

and the inverse map is denoted by

$$b_i = \Sigma^{-1}(\sigma_i). \quad (138)$$

When two bases are related by the map in the above way, we can show

$$tr^{(r)}(b_ib_j) = tr^{(r)}(\sigma_i\sigma^j). \quad (139)$$

Note that $b^j$ is the dual basis obtained from a basis $b_i$ and $\sigma^j$ is the dual basis obtained from a basis $\sigma_i$, and the two dual bases are not related by the map, i.e. $\sigma^j \neq \Sigma(b^i)$. In fact we find that the both sides are equal to $\delta^{ij}$, which come from the definition of the dual bases. Here we will give a proof of the equality (139) using the property of $\Sigma$,

$$tr^{(r)}(b_ib^j) = tr^{(r)}(\sigma_i\sigma^j), \quad (140)$$

where we have used the following equations found in [9],

$$\Sigma(b^*) = \Sigma(1^*)(\Sigma(b))^{-1} \quad \Sigma(1^*) = \frac{1}{N_{m+n} \Omega^{-1}_{m+n}}. \quad (141)$$

Note that neither $tr^{(r)}(b_ib_j) = tr^{(r)}(\sigma_i\sigma_j)$ nor $tr^{(r)}(b^ib^j) = tr^{(r)}(\sigma^i\sigma^j)$ is satisfied. The relation (139) can be generalised to

$$tr^{(r)}(\Sigma^{-1}(\tau)b^j) = tr^{(r)}(\tau\sigma^j) \quad (142)$$

for any element $\tau$ in the group algebra of $S_{m+n}$.

---

2 Elements of the symmetric group $S_{m+n}$ are expressed diagrammatically by $m+n$ vertical edges between two horizontal lines where each horizontal line has $m+n$ points. Similarly elements of the walled Brauer algebra $B_N(m, n)$ are expressed by $m+n$ lines between the two horizontal lines with a vertical barrier separating the $m$ points from the $n$ points. Vertical edges never cross the wall, and horizontal edges start and end on opposite sides of the wall. The map $\Sigma$ reflects the upper right segment and the lower right segment into each other. For more details see section 3.3 in [9].
We will consider the two-point function of the topological field theories obtained from the symmetric group \( S_{m+n} \) and the Brauer algebra \( B_N(m, n) \). The two correlation functions are related if boundary elements belong to the group algebra of \( S_m \times S_n \), as we will show below. For \( h_1 = h^+_1 \otimes h^-_1, h_2 = h^+_2 \otimes h^-_2 \in S_m \times S_n \), using (142),
\[
\eta^{S_{m+n}}(h_1, h_2) = \sum_i \text{tr}^{(r)}(h_1 \sigma, h_2 \sigma^i) = \sum_i \text{tr}^{(r)}(\Sigma^{-1}(h_1 \sigma, h_2)b^i)
\]
\[
= \sum_i \text{tr}^{(r)}((h^+_1 \circ (h^-_2)^{-1})\Sigma^{-1}(\sigma)(h^+_2 \circ (h^-_1)^{-1})b^i)
\]
\[
= \sum_i \text{tr}^{(r)}((h^+_1 \circ (h^-_2)^{-1})b_i(h^+_2 \circ (h^-_1)^{-1})b^i)
\]
\[
= \eta^{B_N(m,m)}(h^+_1 \circ (h^-_2)^{-1}, h^+_2 \circ (h^-_1)^{-1}),
\]
where we have used the following equation
\[
\Sigma^{-1}(h_1 \sigma h_2) = (h^+_1 \circ (h^-_2)^{-1})\Sigma^{-1}(\sigma)(h^+_2 \circ (h^-_1)^{-1}),
\]
which come from the definition of \( \Sigma \). Note that we have used the same symbol for the regular representation of \( S_{m+n} \) and that of \( B_N(m, n) \).

Gluing two boundaries of the cylinders, we have
\[
\sum_{h \in S_m \times S_n} \eta^{S_{m+n}}(h, h^{-1}) = \sum_{h^+ \in S_m, h^- \in S_n} \eta^{B_N(m,m)}(h^+ \circ h^-, (h^+)^{-1} \circ (h^-)^{-1})
\]
\[
= \sum_{h \in S_m \times S_n} \eta^{B_N(m,m)}(h, h^{-1}).
\]

One find that gluing two ends of a cylinder with boundary elements restricted to \( S_m \times S_n \) gives the torus given in section 5,
\[
\sum_{h \in S_m \times S_n} \eta^{S_{m+n}}(h, h^{-1}) = \theta^{S_{m+n}}_{i}
\]
\[
\sum_{h \in S_m \times S_n} \eta^{B_N(m,m)}(h, h^{-1}) = \theta^{B_N(m,m)}_{i}.
\]

Therefore (145) means
\[
\theta^{S_{m+n}}_{i} = \theta^{B_N(m,m)}_{i}.
\]
Taking into account that the partition function of a torus gives the dimension of the vector space, this equation states that the vector space of \([\sigma_i]_H\) and the vector space of \([h_i]_H\) have the same dimension.

In fact the result itself has been known from the counting of gauge invariant operators at large \( N \). Let the number of gauge invariant operators built from \( m \) copies of \( X \) and \( n \) copies of \( Y \) be \( N(m, n) \). There are several ways to obtain this. If we use the restricted Schur basis (12), we obtain \( N(m, n) = \sum_{R, A} (M^R_A)^2 \) (11) (12), which is equal to \( \theta^{S_{m+n}}_{i} \). If we use the Brauer basis (15), we have \( N(m, n) = \sum_{R, A} (M^R_A)^2 \) (9) (13), which is equal to \( \theta^{B_N(m,m)}_{i} \). Thus (147) indeed is an expected equation from the counting of multi-trace operators at large \( N \). It is interesting that
\[
\sum_{R, A} (M^R_A)^2 = \sum_{\gamma, A} (M^\gamma_A)^2
\]
has been derived as a consequence of the relation (143) between two topological field theories.
8 Discussions

Having studied two-dimensional (almost topological) field theories related to the multi-matrix models, it is now good to get back to the equations in (20). Let us first see the left-hand side. It is given in the $U(N)$ gauge theory language, and gauge invariant quantities are integrated. Multi-trace operators are characterised by the invariance under the gauge transformation,

$$X \rightarrow gXg^{-1}, \quad Y \rightarrow gYg^{-1}. \quad (149)$$

(Enhanced symmetries at the free theory of $\mathcal{N} = 4$ SYM were discussed in [13].) On the other hand, the equations in the right-hand side are completely expressed in terms of the symmetric group or the walled Brauer algebra. The quantities inside the trace, $[\sigma]$, $[\sigma]_H$ and $[b]_H$, are characterised by the invariance under the following gauge transformation,

$$\sigma \rightarrow h\sigma h^{-1}, \quad b \rightarrow hbh^{-1} \quad (h \in S_m \times S_n). \quad (150)$$

One might regard the equations in (20) as an analogue of the GKPW relation [44, 45]. Mathematical manipulations behind the equality can be understood from the Schur-Weyl duality. Frobenius algebras are algebras of these gauge invariant quantities, and the bilinear forms (the Frobenius forms) are projection operators onto gauge invariant quantities. These Frobenius algebras are noncommutative except for the case of the one-complex matrix model. The noncommutativity is related to the existence of the multiplicity indices $\mu, \nu$ on the restricted characters, which are originally introduced to describe open strings on giant gravitons [6, 7]. Noncommutative Frobenius algebras would play a role in the description of two-dimensional field theories with a certain structure relevant for the noncommutativity. It would be interesting to use axiomatic notions of noncommutative Frobenius algebras to understand what kind of field theories are described reflecting the noncommutative nature. For example see [46, 47, 48, 49] for references of the direction. Even apart from the connection to $\mathcal{N} = 4$ SYM, noncommutative Frobenius algebras themselves seem to be an interesting subject to learn.

In subsection 4.2 and section 6, we have decomposed correlation functions of two-dimensional quantum field theories obtained from walled Brauer algebras into correlation functions of two-dimensional quantum field theories obtained from symmetric groups, with exploiting the property that the character of the walled Brauer algebra can be expressed in terms of the character of the symmetric group $S_{m-k} \times S_{n-k}$. From the link between permutations and coverings, walled Brauer algebras can also be interesting mathematical tools incorporate two kinds of maps, i.e. a holomorphic map from $m$ worldsheets to the target and an anti-holomorphic map from $n$ worldsheets. The complete large $N$ expansion of two-dimensional Yang-Mills is a well-known example [23, 24, 25]. A non-holomorphic extension of [50, 51] exploiting walled Brauer algebras might be a possible future direction.

Throughout this paper, we have assumed that $N$ is large compared to the number of fields involved in multi-traces (i.e. $n < N$ for the one-matrix model, and $m + n < N$ for the two-matrix model - we call this large $N$). If the number of fields exceed the bound $N$ (i.e. $n > N$ or $m + n > N$ - we call this small $N$), we would have to give a different interpretation because some mathematical equations do not work at small $N$. For example the manipulation from the second line to the third line in [6] works only at large $N$. In the AdS/CFT correspondence, gauge invariant operators are considered to be dual to D-branes (called giant gravitons) or geometries if the number of fields are comparable to $N$ or $N^2$, and considering representation bases clarifies the correspondence between gauge theory operators and string states. It would be interesting to study a two-dimensional interpretation of representation bases, with focusing the role of Frobenius algebras.
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A Symmetric groups, Brauer algebras and Schur-Weyl duality

In this section we will make a brief introduction of symmetric groups and walled Brauer algebras, focusing on the role of Schur-Weyl duality in the description of multi-traces.

Let $V$ be an $N$-dimensional vector space over $\mathbb{C}$ on which an $N \times N$ matrix $X$ is supposed to act. The tensor product $X^\otimes n = X \otimes \cdots \otimes X$ can be viewed as an operator acting on the tensor space $V^\otimes n$. We define an action of the symmetric group $S_n$ as permuting $n$ vector spaces,

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \tag{A.1}$$

In addition to this the $GL(N)$ group acts in the standard way on it by the simultaneous matrix multiplication,

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n. \tag{A.2}$$

These two actions can be shown to commute, and the Schur-Weyl duality says that the tensor space is decomposed into the direct sum of irreducible representations for these groups as

$$V^\otimes n = \bigoplus_{R \vdash n} V^{GL(N)}_R \otimes V^S_n R. \tag{A.3}$$

The sum is taken for all Young diagrams with $n$ boxes satisfying $c_1(R) \leq N$, where $c_1(R)$ is the number of rows in $R$. The projection operator $p_R$ of an irreducible representation $R$ can be introduced as an element in the group algebra of $S_n$,

$$p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma^{-1})\sigma, \tag{A.4}$$

which acts on the tensor space as

$$p_R V^\otimes n = V^{GL(N)}_R \otimes V^S_n R. \tag{A.5}$$

We have denoted by $d_R$ the dimension of an irreducible representation $R$ of the symmetric group, and let $t_R$ be the dimension of an irreducible representation $R$ of the $GL(N)$ group.

Let $tr_n$ be a trace over the tensor space $V^\otimes n$. From the Schur-Weyl duality (A.3), we have

$$tr_n(\sigma) = \sum_R t_R \chi_R(\sigma) \tag{A.6}$$

for an element $\sigma$ in the symmetric group. If we introduce $C_\sigma$, the number of cycles in the permutation $\sigma$, we have $tr_n(\sigma) = N^{C_\sigma}$. The orthogonality relation of the characters leads to

$$t_R = \frac{N^n}{n!} \chi_R(\Omega_n), \tag{A.7}$$

where $\Omega_n$ is a central element in the group algebra of $S_n$ called Omega factor

$$\Omega_n = \sum_{\sigma \in S_n} \sigma N^{C_\sigma - n}. \tag{A.8}$$
By substituting (A.7) back to (A.6), the trace of an element $\tau$ in the group algebra of $S_n$ can be written as

$$tr_n(\tau) = \frac{N^n}{n!} \sum_{\tilde{R} \subseteq n} \chi_R(\Omega)\chi_R(\tau)$$

$$= \frac{N^n}{n!} \sum_{\tilde{R} \subseteq n} d_R \chi_R(\Omega \tau)$$

$$= N^n \delta_n(\Omega \tau),$$

where we have introduced the delta function defined over the group algebra of $S_n$ by

$$\delta_n(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{otherwise} \end{cases}$$

(A.9)

In order to derive (A.9), we have assumed $N > n$. When this is not satisfied, only irreducible representations satisfying the constraint $c_1(R) \leq N$ are summed in the first line and the second line of (A.9). On the other hand all Young diagrams with $n$ boxes are summed in (A.10).

We next consider the vector space that are relevant for the description of multi-trace operators made from two matrices $X$ and $Y$. When $m$ copies of $X$ and $n$ copies of $Y$ are considered, the tensor space we will consider is $V^\otimes (m+n)$, and the Schur-Weyl duality claims

$$V^\otimes (m+n) = \bigoplus_{R \subseteq (m+n)} V_R^{GL(N)} \otimes V_R^{S_{m+n}}.$$  

(A.11)

We now consider the decomposition of the irreducible representation $R$ of $S_{m+n}$ into irreducible representations of the subgroup $S_m \times S_n$,

$$V_R^{S_{m+n}} = \bigoplus_A M_A^R V_A^{S_m \times S_n}.$$  

(A.12)

On restricting to the subgroup, some copies of $A$ appear. The number of times $A$ appears in $R$ is denoted by $M_A^R$, which is given by the Littlewood-Richardson coefficient

$$M_A^R = g(\alpha, \beta; R) \quad A = (\alpha, \beta).$$  

(A.13)

We now define an operator $P_{A,\mu\nu}^R$ playing a role under the decomposition. Here $\mu, \nu$ run over $1, \cdots, M_A^R$, labelling which copy of $A$ we are using. If the multiplicity is trivial, $P_{A}^R$ is the projection operator onto the irreducible representation $A$ inside the $R$. If the multiplicity is non-trivial, $P_{A,\mu\nu}^R$ is an intertwiner mapping the $\nu$-th copy of the representations $A$ to the $\mu$-th copy of the representations $A$. It satisfies

$$P_{A,\mu\nu}^R P_{A',\mu'\nu'}^R = \delta_{AA'}^{\mu\mu'} \delta_{\nu\nu'} P_{A,\mu\nu}^R.$$  

(A.14)

In terms of elements in the symmetric group $S_{m+n}$, the operator $P_{A,\mu\nu}^R$ can be explicitly written as

$$P_{A,\mu\nu}^R = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi_{A,\mu\nu}^R(\sigma)^{-1}.$$  

(A.15)
where $\chi^{R}_{A,\mu}(\sigma)$ is a quantity called restricted character \[6, 7, 11\]. This can be computed by the trace of the matrix $\sigma$ in the representation $R$, but the trace is only over the subspace $A$ appearing in the $(\mu, \nu)$ component of the $M^{R}_{A} \times M^{R}_{A}$ matrix. The sum of the diagonal copies of all possible irreducible representations in $S_{m} \times S_{n}$ inside the representation $R$ gives rise to the usual character $\chi^{R}(\sigma) = \sum_{A,\mu} \chi^{R}_{A,\mu}(\sigma)$, and the projector of an irreducible representation $R$ is given by

$$P^{R} = \sum_{A,\mu} P^{R}_{A,\mu}. \quad (A.16)$$

The intertwiner operator has the symmetry,

$$h P^{R}_{A,\mu} = P^{R}_{A,\mu} h \quad (h \in S_{m} \times S_{n}). \quad (A.17)$$

But elements in $S_{m+n}/S_{m} \times S_{n}$ do not commute with it. Due to this, the cyclicity of the restricted character works for elements in the subalgebra,

$$\chi^{R}_{A,\mu}(h \sigma) = \chi^{R}_{A,\mu}(\sigma h) \quad (h \in S_{m} \times S_{n}). \quad (A.18)$$

Let us next consider the mixed tensor space $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ by including the complex conjugate space $\bar{V}$. This is relevant for the description of multi-trace operators made out of $X$ and $X^{T}$, or $X$ and $Y^{T}$. Similar to the previous cases, we can consider two commuting actions on this space, resulting in the following Schur-Weyl duality,

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{GL(N)} \otimes V_{\gamma}^{B_{N}(m, n)}. \quad (A.19)$$

Here $B_{N}(m, n)$ is the walled Brauer algebra \[26, 27, 28, 29\]. (This algebra is sensitive to $N$, which is a big difference from the group algebra of the symmetric group.) The $\gamma$ is a set of two Young diagrams $(\gamma_{+}, \gamma_{-})$, where $\gamma_{+}$ is a Young diagram with $m-k$ boxes and $\gamma_{-}$ is a Young diagram with $n-k$ boxes. The $k$ is an integer in $0 \leq k \leq \min(m, n)$. The sum over $\gamma$ in (A.19) is constrained by $c_{1}(\gamma_{+}) + c_{1}(\gamma_{-}) \leq N$.

From the Schur-Weyl duality, we have

$$tr_{m,n}(b) = \sum_{\gamma} t_{\gamma} \chi^{\gamma}(b) \quad (b \in B_{N}(m, n)) \quad (A.20)$$

where $t_{\gamma}$ is the dimension of an irreducible representation $\gamma$ of the $GL(N)$ group, and $\chi^{\gamma}(b)$ is the character of an irreducible representation $\gamma$ of the Brauer algebra. The $tr_{m,n}$ denotes a trace over the mixed tensor space. Let $d_{\gamma}$ be the dimension of $\gamma$ in the Brauer algebra. We have a formula of $t_{\gamma}$ using elements in $S_{m} \times S_{n}$,

$$t_{\gamma} = \frac{N^{m+n-2k}}{(m-k)!(n-k)!} \chi(\gamma_{+}, \gamma_{-})^{(\gamma_{+}, \gamma_{-})} (\Omega_{m-k,n-k}), \quad (A.21)$$

where $\chi(\gamma_{+}, \gamma_{-})(\sigma \otimes \tau)$ is the character of $S_{m-k} \times S_{n-k}$, and $\Omega_{m-k,n-k}$ is a central element in the group algebra of $S_{m-k} \times S_{n-k}$ called coupled Omega factor \[23\]. A formula to express $\Omega_{m,n}^{-1}$ in terms of $\Omega_{m+n}^{-1}$ is given in \[40\]. We have the following formula of $d^{\gamma}$,

$$d^{\gamma} = \frac{m!n!}{(m-k)!(n-k)!k!} d_{\gamma_{+}} d_{\gamma_{-}}, \quad (A.22)$$
where \( d_{\gamma^+} \) and \( d_{\gamma^-} \) are the dimensions of \( S_{m-k} \) and \( S_{n-k} \) respectively.

We can construct the projection operator \( P^\gamma \) as

\[
P^\gamma = d^\gamma \sum_i \chi^\gamma(b^i)b_i,
\]

where \( b^i \) is the dual basis in \([30]\). The Schur-Weyl duality asserts

\[
P^\gamma V^\otimes m \otimes \bar{V}^\otimes n = V_{\gamma}^{GL(N)} \otimes V_{\gamma}^{B_N(m,n)}.
\]

Because the walled Brauer algebra \( B_N(m, n) \) contains the group algebra of \( S_m \times S_n \), which we denote by \( \mathbb{C}(S_m \times S_n) \), we have a decomposition similar to \([A.12]\)

\[
V_{\gamma}^{B_N(m,n)} = \bigoplus_A M^\gamma_A V^C(S_m \times S_n),
\]

Here the multiplicity associated with the decomposition is given by

\[
M^\gamma_A = \sum_{\delta \vdash k} g(\gamma^+, \delta; \alpha) g(\gamma^-, \delta; \beta), \quad A = (\alpha, \beta).
\]

We can introduce an operator \( Q^\gamma_{A,\mu\nu} \) as an element in the Brauer algebra \([9, 13]\) that satisfies

\[
Q^\gamma_{A,\mu\nu} Q^\gamma_{A',\mu'\nu'} = \delta^\gamma \gamma' \delta_{AA'} \delta_{\nu\nu'} Q^\gamma_{A,\mu\nu}'.
\]

The role of the operator \( Q^\gamma_{A,\mu\nu} \) is completely the same as \( P^R_{A,\mu\nu} \). Introducing the restricted character of the walled Brauer algebra, we have

\[
Q^\gamma_{A,\mu\nu} = d^\gamma \sum_i \chi^\gamma_{A,\nu\mu}(b^i)b_i.
\]

The relation between \( P^\gamma \) and \( Q^\gamma_{A,\mu\nu} \) is

\[
P^\gamma = \sum_{A,\mu} Q^\gamma_{A,\mu\nu}.
\]

The intertwiner and the restricted character have the symmetry

\[
h Q^\gamma_{A,\mu\nu} h^{-1} = Q^\gamma_{A,\mu\nu},
\]

\[
\chi^\gamma_{A,\mu\nu}(h h^{-1}) = \chi^\gamma_{A,\mu\nu}(b) \quad (h \in S_m \times S_n).
\]

B Orthogonality relations

In this section we summarise orthogonality relations of representations and characters of the symmetric group and the walled Brauer algebra.

Orthogonality relations of the symmetric group \( S_n \) are

\[
\frac{1}{n!} \sum_{\sigma \in S_n} D^R(\sigma)_{ij} D^S(\sigma^{-1})_{kl} = \frac{1}{d_R} \delta_{il} \delta_{jk} \delta_{RS}.
\]

\(^3\) The expression is valid for \( m + n \leq N \).
\[
\frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \chi_S(\sigma^{-1}) = \delta_{RS},
\] (B.2)

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \chi_S(\sigma^{-1}\tau) = \frac{1}{d_R} \chi_R(\tau) \delta_{RS},
\] (B.3)

\[
\frac{1}{d_R} \chi_R(\sigma) \chi_R(\tau) = \frac{1}{n!} \sum_{\rho \in S_n} \chi_R(\rho \sigma \rho^{-1}\tau),
\] (B.4)

\[
\sum_{R \vdash n} \chi_R(\sigma) \chi_R(\tau) = \sum_{\rho \in S_n} \delta_n(\rho \sigma \rho^{-1}\tau).
\] (B.5)

We have similar formulae for the restricted characters. For the symmetric group \(S_{m+n}\) with the restriction to \(S_m \times S_n\) considered, we have

\[
\frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi_{R,A,\mu\nu}(\sigma) \chi_{R',A',\mu'\nu'}(\sigma^{-1}) = \frac{d_A}{d_R} \delta_{RR'} \delta_{AA'} \delta_{\mu\mu'} \delta_{\nu\nu'},
\] (B.6)

This formula is consistent with (B.2) due to \(d_R = \sum_A d_A M^R_A\). We also have

\[
\frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi_{A,\mu\nu}(\sigma) \chi_{A',\mu'\nu'}(\sigma^{-1}\tau) = \frac{1}{d_R} \chi_{A,\mu\nu}(\tau) \delta_{RR'} \delta_{AA'} \delta_{\mu\mu'},
\] (B.7)

\[
\frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi_{A,\mu\nu}(\sigma) \chi_{A',\mu'\nu'}(\tau \sigma^{-1}) = \frac{1}{d_R} \chi_{A,\mu\nu}(\tau) \delta_{RR'} \delta_{AA'} \delta_{\mu\mu'},
\] (B.8)

\[
\sum_{A,\mu,\nu} \frac{1}{d_A} \chi_{A,\mu\nu}(\sigma_1) \chi_{A,\nu\mu}(\sigma_2) = \frac{1}{m! n!} \sum_{h \in S_m \times S_n} \chi_R(h \sigma_1 h^{-1} \sigma_2).
\] (B.9)

If we introduce the dual basis \(\sigma^i = \frac{1}{m!} \sigma_i^{-1}\), factorials disappear from these formulae.

For the walled Brauer algebra \(B_N(m,n)\), we have

\[
\sum_{b \in B_N(m,n)} D^\gamma(b)_{ij} D'^\gamma'(b^*)_{kl} = \frac{1}{t^\gamma} \delta_{il} \delta_{jk} \delta^{\gamma\gamma'},
\] (B.10)

\[
\sum_{b \in B_N(m,n)} \chi^\gamma(b) \chi'^\gamma'(b^*) = \frac{d^\gamma}{t^\gamma} \delta^{\gamma\gamma'},
\] (B.11)

\[
\sum_{b \in B_N(m,n)} \chi^\gamma(b) \chi'^\gamma'(b^* c) = \frac{1}{t^\gamma} \delta^{\gamma\gamma'} \chi^\gamma(c),
\] (B.12)

\[
\sum_{\gamma} \chi^\gamma(b) \chi^\gamma(c) = \sum_{d \in B_N(m,n)} t^\gamma \chi^\gamma(d b d^* c),
\] (B.13)
\[ \sum_{b \in B_{N(m,n)}} \chi_{A,\mu\nu}^\gamma(b) \chi_{A',\mu'\nu'}^\gamma(b^*) = \frac{d_A}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \delta_{\mu\mu'} \delta_{\nu\nu'}, \quad (B.14) \]

\[ \sum_{b \in B_{N(m,n)}} \chi_{A,\mu\nu}^\gamma(b) \chi_{A',\mu'\nu'}^\gamma(b^*c) = \frac{1}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \chi_{A,\mu'\nu'}^\gamma(c) \delta_{\mu\nu'}, \quad (B.15) \]

\[ \sum_{b \in B_{N(m,n)}} \chi_{A,\mu\nu}^\gamma(b) \chi_{A',\mu'\nu'}^\gamma(cb^*) = \frac{1}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \chi_{A,\mu'\nu'}^\gamma(c) \delta_{\mu\nu'}, \quad (B.16) \]

\[ \sum_{A,\mu,\nu} \chi_{A,\mu\nu}^\gamma(b_1) \chi_{A',\mu'\nu'}^\gamma(b_2) = \frac{1}{m!n!} \sum_{h \in S_m \times S_n} \chi^\gamma(hb_1 h^{-1} b_2). \quad (B.17) \]

The orthogonality relations in terms of the dual basis \( b^i = g^i j b_j \) are as follows:

\[ \sum_i D^\gamma(b_i)_{ij} D^{\gamma'}(b_i')_{kl} = \frac{1}{d^\gamma} \delta_{ij} \delta_{jk} \delta^{\gamma\gamma'}, \quad (B.18) \]

\[ \sum_i \chi^\gamma(b_i) \chi^{\gamma'}(b_i') = \delta^{\gamma\gamma'}, \quad (B.19) \]

\[ \sum_i \chi^\gamma(b_i) \chi^{\gamma'}(b_i c) = \frac{1}{d^\gamma} \delta^{\gamma\gamma'} \chi^\gamma(c), \quad (B.20) \]

\[ \sum_\gamma \chi^\gamma(b) \chi^{\gamma}(c) = \sum_i d^\gamma \chi^\gamma(b_i b^i c), \quad (B.21) \]

\[ \sum_i \chi_{A,\mu\nu}^\gamma(b_i) \chi_{A',\mu'\nu'}^\gamma(b_i^*) = \frac{d_A}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \delta_{\mu\mu'} \delta_{\nu\nu'}, \quad (B.22) \]

\[ \sum_i \chi_{A,\mu\nu}^\gamma(b_i) \chi_{A',\mu'\nu'}^\gamma(b_i^*c) = \frac{1}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \chi_{A,\mu'\nu'}^\gamma(c) \delta_{\mu\nu'}, \quad (B.23) \]

\[ \sum_i \chi_{A,\mu\nu}^\gamma(b_i) \chi_{A',\mu'\nu'}^\gamma(cb^i) = \frac{1}{d^\gamma} \delta_{\gamma\gamma'} \delta_{AA'} \chi_{A,\mu'\nu'}^\gamma(c) \delta_{\mu\nu'}. \quad (B.24) \]

The rewriting of orthogonality relations of the symmetric group using the dual basis \( \sigma^i = \frac{1}{m!} \sigma_i^{-1} \) allows us to find the perfect similarity between the symmetric group and the Brauer algebra (see also [39]).
A character formula of the walled Brauer algebra

In this section we will derive (33).

The character of an element $b$ in the walled Brauer algebra $B_N(m, n)$ is related to the character of an element of the form $C^h \otimes b_+ \otimes b_-$ as

$$
\chi^\gamma(b) = N z(b)^{-h} \chi^\gamma(C^h \otimes b_+ \otimes b_-),
$$

where $b_+$ is an element in $S_m \times S_n$ and $b_-$ is an element in $S_{n-h}$, and $h$ is the number of contractions. The $z(b)$ denotes the number of zero cycles involved in $b$. This formula is proved in theorem 3.1 of [52] or in theorem 5.13 of [33]. Note that $h$ is the number of contractions in $C^h \otimes b_+ \otimes b_-$, not the number of contractions in $b$.

Concretely, we will derive the formula for some elements in $B_N(3, 3)$. Elements $C_{11} C_{22}$ and $C_{22}(13)(1\bar{3})$ are already of the form $C^h \otimes b_+ \otimes b_-$. On the other hand, $C_{11} C_{22}(23)$ and $C_{22}(123)(\bar{1}3)$ are not of the form, which have $z = 1, 0$ respectively. Using $C^2 = NC$ and the cyclicity of the character, we can show

$$
\chi^\gamma(C_{11} C_{22}(23)) = \frac{1}{N} \chi^\gamma(C_{11} C_{22}(23) C_{22} C_{11}) = \frac{1}{N} \chi^\gamma(C_{11} C_{22}),
$$

$$
\chi^\gamma(C_{22}(123)(\bar{1}3)) = \frac{1}{N} \chi^\gamma(C_{22}(123)(\bar{1}3) C_{22}) = \frac{1}{N} \chi^\gamma(C_{22}(13)(\bar{1}3)),
$$

which have reproduced the formula (C.1).

For an element which is conjugate by some permutation in $S_m \times S_n$ to the form $C^h \otimes b_+ \otimes b_-$, the character of the element can be expanded in terms of the character in $S_m \times S_n$ as

$$
\chi^\gamma(C^h \otimes b_+ \otimes b_-) = N^h \sum_{\lambda \vdash (m-h)} \sum_{\pi \vdash (n-h)} \sum_{\delta \vdash (k-h)} g(\delta, \gamma_+; \lambda) g(\delta, \gamma_-; \pi) \chi_\lambda(b_+) \chi_\pi(b_-),
$$

where $\gamma = (\gamma_+, \gamma_-)$, $\gamma_+ \vdash (m-k)$, $\gamma_- \vdash (n-k)$. Note that the character vanishes if $k-h < 0$. This formula is found in theorem 7.20 of [53]. If we set $h = 0$, it becomes a familiar formula

$$
\chi^\gamma(\sigma \otimes \tau) = \sum_{R \vdash m} \sum_{S \vdash n} \sum_{\delta \vdash k} g(\delta, \gamma_+; \lambda) g(\delta, \gamma_-; \pi) \chi_R(\sigma) \chi_S(\tau)
= \sum_{\lambda \vdash m} \sum_{\pi \vdash n} M^\gamma_{(R,S)} \chi_R(\sigma) \chi_S(\tau),
$$

where $M^\gamma_{(R,S)}$ is the multiplicity of the representation $(R, S)$ inside the representation $\gamma$.

In order to rewrite (C.3) further, we will use a formula of Littlewood-Richardson coefficients,

$$
g(R, S; T) = \frac{1}{n_1! n_2!} \sum_{\sigma_1 \in S_{n_1}} \sum_{\sigma_2 \in S_{n_2}} \chi_R(\sigma_1^{-1}) \chi_S(\sigma_2^{-1}) \chi_T(\sigma_1 \circ \sigma_2),
$$

where we write the conventional diagram of a cycle of an element $b$ in the walled Brauer algebra (for example see [52, 53]), if the number of vertical edges on the left side of the wall minus the number of vertical edges on the right side of the wall is zero, we say that the cycle type of the cycle is zero, or it is a zero-cycle.
where $R \vdash n_1$, $S \vdash n_2$ and $T \vdash (n_1 + n_2)$. Substituting this into (C.3), we obtain

$$\chi^\gamma(C^\otimes h \otimes b_+ \otimes b_-) = N^h \sum_{\lambda \vdash (m-h), \pi \vdash (n-h), \delta \vdash (k-h)} \frac{1}{(k-h)!(m-k)!} \sum_{\sigma_1 \in S_{k-h}, \sigma_2 \in S_{m-k}} \chi_\delta(\sigma^{-1}_1) \chi_{\gamma}(\sigma^{-1}_2) \chi_\lambda(\sigma_1 \circ \sigma_2)$$

$$\times \frac{1}{(k-h)!(n-k)!} \sum_{\tau_1 \in S_{k-h}, \tau_2 \in S_{n-k}} \chi_\delta(\tau^{-1}_1) \chi_{\gamma}(\tau^{-1}_2)$$

$$\times \chi_\lambda(b_+) \chi_\pi(b_-)$$

$$= N^h \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \frac{1}{(m-h)! (n-h)!} \sum_{\sigma_1, \tau_1 \in S_{k-h}} \delta_{m-h}(b_+ [\sigma_1 \circ \sigma_2]) \delta_{n-h}(b_- [\tau_1 \circ \tau_2]) \delta_{k-h}(\sigma^{-1}_1 \tau^{-1}_1)$$

$$\times \chi_{\gamma}(\sigma^{-1}_2) \chi_{\gamma}(\tau^{-1}_2)$$

$$= N^h \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2) \chi_{\gamma}(\sigma^{-1}_2) \chi_{\gamma}(\tau^{-1}_2), \quad (C.6)$$

where we have defined

$$\Delta(b_+, b_-; \sigma_2, \tau_2) = \frac{1}{(k-h)!} \sum_{\sigma_1, \tau_1 \in S_{k-h}} \delta_{m-h}(b_+ [\sigma_1 \circ \sigma_2]) \delta_{n-h}(b_- [\tau_1 \circ \tau_2]) \delta_{k-h}(\sigma^{-1}_1 \tau^{-1}_1)$$

$$= \frac{1}{(k-h)!^2} \sum_{\sigma_1, \tau_1 \in S_{k-h}} tr_{m-h}^{(r)}(b_+ [\sigma_1 \circ \sigma_2]) tr_{n-h}^{(r)}(b_- [\tau_1 \circ \tau_2]) tr_{k-h}^{(r)}(\sigma^{-1}_1 \tau^{-1}_1). \quad (C.7)$$

Note again that $b_+ \in S_{m-h}$ and $b_- \in S_{n-h}$, while $\sigma_2 \in S_{m-k}$ and $\tau_2 \in S_{n-k}$. $\delta_\sigma(\sigma)$ is the delta function defined over the group algebra of $S_l$. To obtain the second equality in (C.6) we have used the formula (B.5), and $[\sigma]$ is defined in (17). We have introduced the trace of the regular representation of $S_n$ as

$$tr_{\sigma}^{(r)}(\sigma) = n! \delta_\sigma(\sigma) \quad (\sigma \in S_n). \quad (C.8)$$

If we use the two-point function of the topological field theory given in section 3, we have

$$\Delta(b_+, b_-; \sigma_2, \tau_2) = \frac{1}{(k-h)!^2} \sum_{\sigma_1, \tau_1 \in S_{k-h}} \eta^{S-h}(b_+, \sigma_1 \circ \sigma_2) \eta^{S-h}(b_-, \tau_1 \circ \tau_2) \eta^{S-h}(\sigma^{-1}_1 \tau^{-1}_1). \quad (C.9)$$

It is convenient to keep in our mind that $\sigma^{-1} \in S_n$ always appears with $1/n!$. Using the dual basis $\sigma^i = \frac{1}{n!} \sigma^{-1}_i$, we get tidy expressions without factorials;

$$\chi^\gamma(C^\otimes h \otimes b_+ \otimes b_-) = N^h \sum_{i \in S_{m-k}, j \in S_{n-k}} \Delta(b_+, b_-; \sigma_i, \tau_j) \chi_{\gamma}(\sigma^i) \chi_{\gamma}(\tau^j), \quad (C.10)$$

and

$$\Delta(b_+, b_-; \sigma_i, \tau_j) = \sum_{k, l \in S_{k-h}} \eta^{S-h}(b_+, \sigma_k \circ \sigma_i) \eta^{S-h}(b_-, \tau_l \circ \tau_j) \eta^{S-h}(\sigma^{-1}_i \tau^{-1}_j) \quad (C.11)$$

These look like pleasant but having too many indices, which might confuse the readers, so we do not use these expressions in the main text.
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