A comment on compactification of M-theory on an (almost) light-like circle

ADEL BILAL

CNRS - Laboratoire de Physique Théorique de l’École Normale Supérieure
24 rue Lhomond, 75231 Paris Cedex 05, France
bilal@physique.ens.fr

ABSTRACT

In perturbative quantum field theory the limit of compactification on an almost light-like circle has recently been shown to be plagued by divergences. We argue that the light-like limit for M-theory probably is free of such divergences due to, among others, the existence of the wrapping modes of the membranes. To illustrate this, we consider superstring theory compactified on an almost light-like circle.

Specifically, we compute a one-loop four-point amplitude in type II theory. As is well known, if the external states have vanishing momenta in the compact dimension, the divergence in the light-like limit is even stronger than in field theory. However, in the case of present interest, where these external momenta are non-vanishing, there is a subtle compensation and the resulting amplitude has a well-defined and finite light-like limit. The net effect of taking the light-like limit is to replace the integration over one of the moduli of the 4-punctured torus by a sum over a discrete modulus taking values in a finite lattice on the torus. The same result can also be obtained from a suitably “Wick rotated” amplitude computed directly with a compact light-like circle.

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1. Introduction and summary

1.1. Introduction

M-theory compactified on a circle of radius $R_{11}$ is type IIA superstring theory with coupling $g_s = R_{11}/\sqrt{\alpha'}$ [1]. Banks, Fischler, Shenker and Susskind [2] conjectured that when the theory is boosted to the infinite momentum frame in the $x^{11}$ direction, the only relevant degrees of freedom are D0-branes, and M-theory then is described by a ten-dimensional $U(N)$ super YM theory reduced to $0 + 1$ dimensions, i.e. matrix quantum mechanics (for a pedagogical review, see ref. 3). The momentum in the $x^{11}$ direction is $p_{11} = N R_{11} / \sqrt{\alpha'}$ so that the infinite momentum limit is obtained by letting $N \to \infty$. The low-energy sector for large $R_{11}$ should then describe eleven-dimensional supergravity [4].

A little later, Susskind [5] suggested to consider M-theory compactified on a light-like circle, identifying $x^{-} \equiv (x^0 - x^{11})/\sqrt{2}$ with $x^{-} + 2\pi R$. One should keep in mind that the proper length of this circle vanishes and that the parameter $R$ has no invariant meaning since it can be changed by a Lorentz boost. The momenta $p_{-} = \frac{N}{R_1}$ are discrete and the conjecture states that the discrete light-cone quantization (DLCQ) of M-theory in a sector of fixed total $p_{-}$ is again given by a $U(N)$ matrix quantum mechanics as obtained by reduction from the ten-dimensional super YM theory. This formalism has the advantage that various dualities are already manifest at finite $N$.

In an insightful paper, Seiberg [6] (see also Sen [7]) has related both approaches to M-theory by considering the light-like compactification on a (null) circle with radius $R_0$ as being obtained in the limit of a very large boost from a space-like compactification on a circle of very small radius $R_s = R_{11}$. More precisely, a space-like circle $x^{11} \simeq x^{11} + 2\pi R_s$ when subject to a very large boost becomes an almost light-like circle $x^{-} \simeq x^{-} + 2\pi R_0$, $x^{+} \simeq x^{+} + \pi R_s^2 / R_0$. In the limit $R_s \to 0$ with $R_0$ fixed, the boost becomes infinite and the latter circle is really light-like, while the space-like circle in the $x^{11}$ direction has shrunk to zero length. Using this very large boost combined with scaling arguments, Seiberg argued that the DLCQ of M-theory should be interpreted as being equivalent to M-theory compactified on a very small space-like circle. The latter is the IIA string at weak coupling and in a sector of non-zero D0-brane charge where only the open string ground states survive while the oscillators decouple. This must indeed be described by the matrix model.
1.2. Motivation

As appealing as this argument is, one might feel uneasy about infinite boosts, or else about approximating a light-like circle by an almost light-like, i.e. still space-like circle. To elucidate further whether one might trust such an approximation, Hellerman and Polchinski [8] have studied some loop diagrams in quantum field theory when compactified on an almost light-like circle. To do so, they introduce a parameter $\epsilon$ in the (flat) space-time metric such that for $\epsilon \to 0$ the circle is truly light-like ($\epsilon \approx R_s/R_0$ and $2\pi \epsilon R_0$ is the proper length of the compact direction). Specifically, start in ten-dimensional Minkowski space with metric $d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^i)^2$ ($i = 2, \ldots 9$) and compactify $x^1$ on a circle of radius $\epsilon R_0$:

$$x^1 \simeq x^1 + 2\pi \epsilon R_0 \quad , \quad x^0 \simeq x^0 . \quad (1.1)$$

(The “transverse” $x^i$ are unaffected in all what follows.) Next, one makes a large boost with parameter $\beta = (1 - \epsilon^2/2)/(1 + \epsilon^2/2)$ so that the boosted coordinates $\tilde{x}^0$ and $\tilde{x}^1$ satisfy $\tilde{x}^1 \simeq \tilde{x}^1 - (1+\epsilon^2/2)2\pi R_0/\sqrt{2}$, $\tilde{x}^0 \simeq \tilde{x}^0 + (1-\epsilon^2/2)2\pi R_0/\sqrt{2}$, the metric being unaffected. Introduce $x^\pm = (\tilde{x}^0 \pm \tilde{x}^1)/\sqrt{2}$. Then

$$x^- \simeq x^- + 2\pi R_0 \quad , \quad x^+ \simeq x^+ - \epsilon^2 \pi R_0 \quad , \quad ds^2 = -2dx^dx^- + (dx^i)^2 . \quad (1.2)$$

As in [8] introduce $t = x^+ + \epsilon^2 x^-/2$ so that

$$x^- \simeq x^- + 2\pi R \quad , \quad t \simeq t \quad , \quad ds^2 = -2dtdx^- + \epsilon^2 dx^-dx^- + (dx^i)^2 . \quad (1.3)$$

Note that $g^{t-} = g^{-t} = -1$, $g^{tt} = -\epsilon^2$, $g^{-} = 0$, so that $p^t = -p_- - \epsilon^2 p_t$. Equations (1.2) and (1.3) then clearly show that in the $\epsilon \to 0$ limit, the circle becomes truly light-like with radius $R_0$, while the equivalent space-like circle (1.1) has shrunk to zero size. Note that $t = \epsilon x^0$ and $x^- = (x^0 - x^1)/\epsilon$. Obviously, the momentum $p_- = n/R_0$ is discretised, while $p_t$ is not. How do the momenta in the compact space-like circle ($p_0$) and compact light-like circle ($p_-$) compare? From $p_0 x^0 + p_1 x^1 = p_- x^- + p_t x^t = \frac{n}{R_0} \frac{x^0 - x^1}{\epsilon} + p_t \epsilon x^0$ we get

$$p_0 = \epsilon p_t + \frac{n}{\epsilon R_0} \quad , \quad -p_1 = \frac{n}{\epsilon R_0} . \quad (1.4)$$

The point we want to stress is that the integer $n$ that characterizes the compact light-like momentum $p_-$ is the same integer $^*$ as the $n$ that characterizes the space-like momentum $p_1$.

$^*$ up to a sign flip which we could have avoided by defining $x^-$ and hence $p_-$ with the opposite sign.
So for fixed light-like momentum $p_-$, we have a fixed $n$ and hence the momentum $p_1$ in the compact space-like dimension must be taken to diverge as $\epsilon \to 0$. This will be important below.

Hellerman and Polchinski find [8] that in a generic $D$-dimensional QFT, loop diagrams with vanishing $p_-$ exchange diverge as $\frac{1}{\epsilon}$ when $\epsilon \to 0$. This is due to the longitudinal zero-modes becoming strongly coupled. Indeed, concerning the zero-modes the theory effectively behaves as a $D - 1$ dimensional theory with effective coupling $\frac{g^2}{2\pi\epsilon R_0}$ which blows up as $\epsilon \to 0$. One should note that the treatment of the zero-modes in ref. 8 is different from what people usually do in DLCQ, e.g. of QCD, which consists in first solving the first-order equations of motion for the zero-modes and then plugging the solution back into the action, generating instantaneous Coulomb-like interactions. Hellerman and Polchinski also note that in certain supersymmetric QFTs the divergence of loop-diagrams when $\epsilon \to 0$ can be avoided. This raises the hope that M-theory might be well-behaved in this limit. They argue that, if this limit exists, it should be the only reasonable way to define what one means by the DLCQ of M-theory.

One might argue that M-theory certainly is not an ordinary QFT and the analysis of Hellerman and Polchinski need not be relevant in M-theory. This doubt is supported by the fact that M-theory contains extended objects - membranes and five-branes - that can wrap around the compact dimension. The existence of wrapping or winding modes is one of the crucial differences between e.g. string and ordinary field theory. Such winding states become very light (or of small effective tension in the $D - 1$ dimensional theory) as the compact circle shrinks. Said differently, as the radius goes to zero, more and more winding modes contribute up to a given energy (tension) scale, and as $\epsilon \to 0$ this may give rise to a new divergence not present in field theory. Although we do not know how to precisely evaluate this effect in M-theory, the presence of winding states is very familiar from string theory. Another crucial difference between field and string theory scattering amplitudes is the existence, in the latter, of the moduli of the punctured Riemann surface which is the string world-sheet, that have to be integrated over. These integrations that translate the non-point-like character of string theory tend to soften many of the field theory singularities.

So the best analogue for M-theory of the Hellerman-Polchinski one-loop amplitude probably is some closed superstring one-loop scattering amplitude with one spatial dimension com-
pactified on a circle of radius $R = \epsilon \sqrt{\alpha'} \equiv \epsilon l_s$, in the $\epsilon \to 0$ limit\(^\dagger\). It is such an amplitude we will study in some detail below and show that the $\epsilon \to 0$ limit is finite and well-defined provided at least one external state has *non-vanishing* momentum in the compact dimension. This is of course different from what one usually assumes in a string compactification. However, with view on the DLCQ of M-theory where the compact momenta are $p_r^c = n_r/R_0$, this is just the case we are interested in. Indeed, we showed above that the corresponding compact space-like momenta then are $-p_1^r = n_r/(\epsilon R_0)$ with the same (fixed) $n_r$.

### 1.3. Summary

Specifically, we will compute a four-point one-loop scattering amplitude in type II superstring theory compactified on a spatial circle of small radius $R = \epsilon \sqrt{\alpha'} = \epsilon l_s$. The external states are taken to have arbitrary momenta $n_r/R$ in the compact direction, but no windings. The absence of windings of the external states does not seem to be essential but simplifies the formulae. With view on M-theory, we found it convenient to continue to call the mass of a state its ten-dimensional mass\(^\ddagger\), i.e. $M_r^2 = -p_r \cdot p_r = -p_r^\mu p_r^\mu - l_r^2/R^2$. One of the simplest computations then is the one with all external states being mass-less and having factorized polarisations $\zeta_i^r \zeta_j^r$. This is the amplitude we will compute using the standard Green-Schwarz formalism\(^\S\). All conventions are as in Green-Schwarz-Witten [9].

Our computation below shows some very general features and mechanisms that are clearly not specific to a four-point amplitude, and we believe that they hold in a much more general context. There are two competing effects as the radius of the circle shrinks to zero. The first is due to the winding modes running around the loop becoming very light. This is the usual condensation of light winding states that make divergent the naive compactification of strings on a circle of vanishing size. Indeed there is one factor of $1/R = 1/(\epsilon l_s)$ just from

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\(^\dagger\) Up to now, we have called the radius of the space-like circle $R_s = \epsilon R_0$, so the present choice seems to imply $R_0 = l_s$. Since $R_0$ does not have an invariant meaning this is of no relevance. Else one can consider that we have rescaled $\epsilon$, which also does not matter since in the end we are only interested in the $\epsilon \to 0$ limit, anyhow.

\(^\ddagger\) Our convention is such that $\mu$ runs over the nine non-compact dimensions $\mu = 0, 2, \ldots, 8$, while $a \cdot b$ denotes a full ten-dimensional sum.

\(^\S\) This standard Green-Schwarz light-cone formalism should of course not be confused with the light-cone we are interested in here. In order to separate things as much as possible, note that the standard light-cone formalism eliminates two sets of oscillators, say in the 0 and 9 directions, while the light-cone and light-like limit we are interested in here concerns the 0 and 1 directions.
replacing the momentum integral by the discrete sum, and another from the condensation just mentioned, alltogether giving a $1/\epsilon^2$. One could still argue that after T-duality this is equivalent to an uncompactified theory, and hence in terms of the T-dual coupling the result is finite. However, we want to keep the original coupling constant fixed and then one cannot escape the presence of an $1/\epsilon^2$ factor. The second effect, for non-vanishing external momenta $n_r/R$ in the compact direction, is the presence of a zero-mode factor $\exp\left[-\pi Q(\nu_r, n, m)/\epsilon^2\right]$ in the amplitude where $Q$ is some positive definit complex quadratic form containing the moduli $\nu_r$ of the 4-punctured torus, as well as the external momentum quantum numbers $n_r$ and the loop “momentum and winding” quantum numbers $n$ and $m$. Generically, $Q \neq 0$ and as $\epsilon \to 0$ this exponential vanishes. The main point is that, in combination with the $1/\epsilon^2$ from the first effect, this precisely combines to give a complex $\delta^{(2)}(Q(\nu_r, n, m))$. The effect of this $\delta$-function is to eliminate the integration over one of the moduli, say $\nu_3$. However, one still has the sum over $n$ and $m$, and the net effect will turn out to be that $\nu_3$ can now only take finitely many discrete values on a lattice lying on the torus. So all that has happened in the light-like limit is to discretise one of the moduli! We then check that this does not bring about any new divergences and that the only singularities of the amplitude are those required by unitarity. In particular, the discrete nature of $\nu_3$ is just what is needed to produce the extra poles due to on-shell intermediate states with non-vanishing winding numbers. We will also show how the same amplitude can be obtained by working with a light-like compactification from the outset, after doing some suitable “Wick rotation” of the otherwise divergent DLCQ amplitude.

So we conclude that in a setting relevant to DLCQ of M-theory (not all momenta in the compact dimension vanishing) the one-loop scattering amplitude in type II superstring theory with four external massless states has a finite and well-defined limit as the radius of the space-like compact dimension shrinks to zero. This limit coincides with the result obtained directly from a light-like compactification. Since the mechanism just described seems to pertain not only to four-point amplitudes, we are confident that it is a feature of any one-loop type II superstring amplitude, and probably also of all higher genus ones as well. We take this as evidence that also in M-theory the light-like limit does exist and coincides with its DLCQ. Of course, a more M-theoretic investigation is called for.
2. The four-point one-loop amplitude for a space-like circle of vanishing radius

We will now describe the computation of the one-loop amplitude in some detail. We work within type II superstring theory with one space-like dimension, say $x^1$, compactified on a circle of radius $R = \ell_s$ which we will let go to zero in the end. The momenta of the four external states are denoted by $k_r$, $r = 1, \ldots, 4$, with $\sum_{r=1}^4 k_r = 0$, and their polarisations are taken to factorize as $\zeta^i_r \bar{\zeta}^{jr}$. The external states are massless in the ten-dimensional sense, i.e. $k_r \cdot k_r \equiv k_r^\mu k_r_\mu + n_r^2/R^2 = 0$ where $n_r/R$ are the components of their momenta in the compact dimension. At least one of the $n_r$ (and hence by momentum conservation actually at least two of them) are supposed to be non-vanishing. We suppose that the external states have vanishing winding quantum numbers. It will be clear from the computation below that non-vanishing winding numbers for the external states would not change the conclusion, but would slightly complicate the formulae. We will use the operator (Hamiltonian) approach as extensively described in [9] to compute the amplitude, but we will eventually arrive at a form that could also be directly derived from a path integral approach.

2.1. The bosonic zero-modes

The zero-mode part of the $L_0$ and $\overline{L}_0$ appearing in the string propagators are

$$L_0 = -\frac{\alpha'}{4} p_0^2 + \frac{\alpha'}{4} \left( \frac{n}{R} - \frac{m R}{\alpha'} \right)^2 + \frac{\alpha'}{4} p_i^2 + \text{oscillators} \quad (2.1)$$

and similarly for $\overline{L}_0$, with $m \rightarrow -m$. This form of $L_0$, $\overline{L}_0$ corresponds to the Minkowski metric (1.1). We can rewrite them in a way that corresponds to the Lorenz-equivalent choice (1.3) by letting $p_0 = \epsilon p_t + \frac{n}{R}$ (cf. (1.4) and remember $R = \ell_s$) so that

$$L_0 = -\frac{\alpha'}{2} \left( \frac{n}{l_s} + \epsilon^2 p_t \right) \left( p_t + \frac{m}{l_s} \right) + \frac{\alpha'}{4} \epsilon^2 \left( p_t + \frac{m}{l_s} \right)^2 + \frac{\alpha'}{4} p_i^2 + \text{oscillators} \quad (2.2)$$

and again similarly for $\overline{L}_0$, with $m \rightarrow -m$. This latter form is more convenient when starting directly with a compact light-like dimension because for $\epsilon = 0$ one simply gets $L_0 = -\frac{1}{2} n (l_s p_t + m) + \frac{\alpha'}{4} p_i^2 + \text{oscillators}$. This will be used lateron. But first we work with a space-like compactification, and hence with (2.1).
The amplitude contains a zero-mode piece

$$\prod_{r=1}^{4} \frac{\alpha' (p_R^r)^2}{x_r} \frac{\alpha' (p_L^r)^2}{x_r} \equiv F_1 F_2 \tag{2.3}$$

where the momenta in the loop are

$$p_r^R, p_r^L, p_r^R = p_r^L = p^\mu - k_1^\mu - \cdots - k_{r-1}^\mu$$

$$p_r^{R,1} = \frac{n}{R} - \frac{R}{\alpha'} m - \frac{n_1 + \cdots n_{r-1}}{R}$$

$$p_r^{L,1} = \frac{n}{R} + \frac{R}{\alpha'} m - \frac{n_1 + \cdots n_{r-1}}{R}. \tag{2.4}$$

The factors $F_1$ and $F_2$ respectively correspond to the contributions of the non-compact and of the compact dimensions. If we define as usual

$$x_1 \cdots x_r = \rho_r, \quad \rho_4 \equiv w, \quad w = e^{2\pi i \tau}, \quad p_r = e^{2\pi i \nu_r}$$

$$\nu_{sr} = \nu_s - \nu_r, \quad \nu_4 \equiv \tau \tag{2.5}$$

then $F_1$ and $F_2$ are given by

$$F_1 = \exp \left\{ -\pi \alpha' \sum_{s>r} k^\mu_s k^\mu_{r\mu} \left[ \frac{(\Im \nu_{sr})^2}{\Im \tau} - \Im \nu_{sr} \right] \right\} \exp \left\{ -\pi \alpha' \Im \tau \left( p^\mu + \sum_s k^\mu_s \frac{\Im \nu_{sr}}{\Im \tau} \right)^2 \right\}$$

$$F_2 = \exp \left\{ \frac{i\pi}{2} \alpha' \sum_{s>r} \frac{n_s n_r}{R^2} \left( \frac{\nu^2_{sr}}{\tau} - \nu_{sr} - \frac{\overline{\nu^2_{sr}}}{\overline{\tau}} + \overline{\nu}_{sr} \right) \right\} \times \exp \left\{ \frac{i\pi}{2} \tau \alpha' \left( \frac{n}{R} - \frac{Rm}{\alpha'} + \sum_s \frac{n_s \nu_s}{R \tau} \right)^2 - \frac{i\pi}{2} \tau \alpha' \left( \frac{n}{R} + \frac{Rm}{\alpha'} + \sum_s \frac{n_s \nu_s}{R \tau} \right)^2 \right\}. \tag{2.6}$$

(Sums over $s$ or $r$ always run from 1 to 4.) Note that in the usual string theory compactification with all $n_r$ vanishing the first exponential factor in $F_2$ is absent. One needs to compute $(\int d^9 p F_1) \left( \frac{1}{R} \sum_{m,n} F_2 \right)$. In the integral involving $F_1$ one simply shifts the integration variable to obtain as usual

$$\int d^9 p F_1 = (\alpha' \Im \tau)^{-9/2} \prod_{s>r} \left[ \exp \left\{ -\pi \left[ \frac{(\Im \nu_{sr})^2}{\Im \tau} - \Im \nu_{sr} \right] \right\} \right] \alpha' k^\mu_s k^\mu_{r\mu}. \tag{2.7}$$

The sum over $n, m$ of $F_2$ is more similar to the lattice sum for the heterotic string, except that here we have a $\tau, \nu$ and a $\overline{\tau}, \overline{\nu}$ part.
2.2. The Amplitude

The contributions of the fermionic zero-modes and of the non-zero modes, bosonic and fermionic, are the standard ones, see [9]. They give rise to the kinematic factor

\[ K_{\text{cl}} = K\left(\frac{k_r}{2}, \zeta_r\right) K\left(\frac{k_r}{2}, \overline{\zeta_r}\right) \]

with the \( R_{ij}^0 \) defined in [9], and the factors

\[ \prod_{s>r} \chi(\nu_{sr}, \tau)^{\alpha' k_r \cdot k_s} \]

where \( \chi(\nu, \tau) \) is defined by

\[ \chi(\nu, \tau) = 2\pi \exp \left\{ -\pi \left( \frac{(\text{Im}\nu)^2}{\text{Im}\tau} \right) \right\} \frac{\theta_1(\nu, \tau)}{\theta_1'(\nu, \tau)}. \]

In (2.9) each single factor includes a piece \( \exp \left\{ -\pi \left( \frac{(\text{Im}\nu_{sr})^2}{\text{Im}\tau} - \text{Im}\nu_{sr} \right) \right\} \alpha' n_{sr}/R^2 \) which should have come from (2.7) if we had no compact dimension, or else which would be equal to one if the \( n_r \) would vanish. Here we have included this piece by hand in order that full ten-dimensional scalar products \( k_r \cdot k_s \) appear in (2.9), so we have to divide this piece out again.

Putting everything together, we obtain for the full four-point one-loop amplitude

\[ A^{(4)}_{\text{cl}} = (\pi \kappa)^4 K_{\text{cl}} \int d^2 \tau d^2 \nu_1 d^2 \nu_2 d^2 \nu_3 I \]

\[ I = (\alpha' \text{Im}\tau)^{-9/2} \prod_{s>r} \chi(\nu_{sr}, \tau)^{\alpha' k_r \cdot k_s} J \]

\[ J = \exp \left\{ \pi \alpha' \sum_{s>r} \frac{n_s n_r}{R^2} \left[ \frac{(\text{Im}\nu_{sr})^2}{\text{Im}\tau} - \frac{\nu_{sr}^2}{2i\tau} + \frac{\overline{\nu}_{sr}^2}{2i\tau} \right] \right\} S \]

\[ S = \frac{1}{R} \sum_{n,m} \exp \left\{ \frac{i\pi \tau}{2} \alpha' \left( \frac{n}{R} - \frac{Rm}{\alpha'} + \sum_s \frac{n_s \nu_s}{R \tau} \right)^2 - \frac{i\pi \tau}{2} \alpha' \left( \frac{n}{R} + \frac{Rm}{\alpha'} + \sum_s \frac{n_s \overline{\nu}_s}{R \tau} \right) \right\}. \]

(2.11)

It will be useful to rewrite \( J \), using the identity \( \sum_{s>r} n_r n_s \nu_{sr}^2 = \frac{1}{2} \sum_{r,s} n_r n_s \nu_{sr}^2 = -(\sum_r n_r \nu_r)^2 \)
(since $\sum n_r = 0$), as

$$J = \exp \left\{ -\pi \alpha' \sum_{s,r} \frac{n_s n_r}{R^2} \left[ \frac{\Im \nu_s \Im \nu_r}{\Im \tau} - \frac{\Im \nu_s \nu_r}{\tau} \right] \right\} S$$

$$= \exp \left\{ -\pi \alpha' \sum_{s,r} \frac{n_s n_r}{R^2} \frac{\Im (\nu_s/\tau) \Im (\nu_r/\tau)}{\Im (-1/\tau)} \right\} S . \quad (2.12)$$

### 2.3. Modular invariance

It is easy to verify that the new factors involving the $n_r$ do not spoil modular invariance, thus providing a check of the above expression for the amplitude. First, invariance under $\nu_r \to \nu_r + 1$ and under $\nu_r \to \nu_r + \tau$ follows trivially from the standard properties of the $\chi(\nu, \tau)$. Invariance under $\tau \to \tau + 1$ no longer is manifest, but was evident initially in (2.3) because $\alpha'(p^R_r)^2 - \alpha'(p^L_r)^2$ is an integer. To check invariance under $\tau \to -1/\tau$, $\nu_r \to -\nu_r/\tau$ one has to perform a standard Poisson resummation of $S$:

$$S \left( -\frac{\nu_r}{\tau}, -\frac{1}{\tau} \right) = |\tau| \exp \left\{ \pi \alpha' \sum_{s,r} \frac{n_s n_r}{R^2} \frac{\Im \nu_s \nu_r}{\tau} \right\} S(\nu_r, \tau) \quad (2.13)$$

so that using the form (2.12) of $J$ it is obvious that $J(-\nu_r/\tau, -1/\tau) = |\tau| J(\nu_r, \tau)$ and thus $I(-\nu_r/\tau, -1/\tau) = |\tau|^{10} I(\nu_r, \tau)$ which proves modular invariance of the amplitude. Consequently, as usual, $\tau$ is to be integrated over the standard fundamental domain, and each of the $\nu_r$ over the parallelogram with corners $(0, 1, 1 + \tau, \tau)$. In other words, for each given $\tau$ determining the shape of the world-sheet torus, the $\nu_1, \nu_2, \nu_3$ are to be integrated over this torus (with $\nu_4$ being fixed at $\tau$ or equivalently at 0).

### 2.4. Path integral form of the amplitude

We now rewrite the sum $S$ by performing a partial Poisson resummation in $m$ only. This yields

$$S = \left( \frac{\alpha'}{\Im \tau} \right)^{1/2} \frac{1}{R^2} \exp \left\{ \pi \alpha' \sum_{s,r} \frac{n_s n_r}{R^2} \frac{\Im (\nu_s/\tau) \Im (n_r/\tau)}{\Im (-1/\tau)} \right\}$$

$$\times \sum_{n,m} \exp \left\{ -\pi \frac{\alpha'}{R^2} \left[ \Im \tau \left( n + \sum_s n_s \frac{\Im \nu_s}{\Im \tau} \right)^2 + \frac{1}{\Im \tau} \left( m + \Re \tau n + \sum_s n_s \Re \nu_s \right) \right]^2 \right\} . \quad (2.14)$$
The first exponential cancels against the one in $J$ and the second exponential can be rearranged so that

$$J = (\alpha' \text{Im}\tau)^{-1/2} \frac{\alpha'}{R^2} \sum_{n,m} \exp \left\{ -\pi \frac{\alpha'}{R^2 \text{Im}\tau} \left| m + n\tau + \sum_s n_s \nu_s \right|^2 \right\}$$

(2.15)

which is now quite simple and which is the form one would have gotten directly from a path-integral computation. Also, the modular properties of $J$ now are manifest. Inserting this form of $J$ into eq. (2.11) we finally get for the amplitude

$$A^{(4)}_{\text{cl}} = \frac{(\pi \kappa)^4}{\alpha'^6} K_{\text{cl}} \int \frac{d^2\tau}{(\text{Im}\tau)^2} \prod_{r=1}^{3} \frac{d^2\nu_r}{\text{Im}\tau} \prod_{s>r} \chi(\nu_{sr}, \tau)^{\alpha' k_r \cdot k_s} \times \sum_{n,m} \frac{\alpha'}{R^2} \exp \left\{ -\pi \frac{\alpha'}{R^2 \text{Im}\tau} \left| m + n\tau + \sum_s n_s \nu_s \right|^2 \right\}$$

(2.16)

2.5. The limit of the space-like circle of zero radius

So far, the radius $R$ of the compact space-like dimension was arbitrary. Now we will study what happens if we let

$$R^2/\alpha' = \epsilon^2 \to 0 .$$

(2.17)

The form (2.16) of the amplitude is particularly convenient to study this limit. First, let us make some preliminary remarks. Obviously, we have to study the $\epsilon \to 0$ limit of

$$\frac{1}{\epsilon^2} \exp \left\{ -\pi \frac{1}{\epsilon^2 \text{Im}\tau} \left| m + n\tau + \sum_s n_s \nu_s \right|^2 \right\} .$$

(2.18)

The $1/\epsilon^2$ in the exponential ensures that one can get a non-vanishing contribution only if $m + n\tau + \sum_s n_s \nu_s = 0$. If all $n_s$ vanish (the usual case studied in string compactification on a circle) then only $m = n = 0$ can contribute to the sum, in which case the exponential in (2.18) just gives 1 and we are left with the $1/\epsilon^2$ factor leading to the well-known divergence discussed in the introduction. For non-vanishing $n_s$ however, the argument of the exponential depends on the integration variables $\nu_r$ and things are more subtle.
In fact, the factors of \( \epsilon \) are precisely such that in the \( \epsilon \to 0 \) limit one obtains a delta-function:

\[
\frac{1}{\epsilon^2} \exp \left\{ -\frac{\pi}{\epsilon^2} \frac{1}{\text{Im}\tau} \left| m + n\tau + \sum_s n_s \nu_s \right|^2 \right\} \to \text{Im}\tau \delta^{(2)} \left( m + n\tau + \sum_s n_s \nu_s \right),
\]

so that we arrive at

\[
A^{(4)}_{\text{cl}} \bigg|_{\epsilon \to 0} = \frac{(\pi\kappa)^4}{\alpha^5} K_{\text{cl}} \int \frac{d^2\nu_3}{(\text{Im}\tau)^2} \prod_{r=1}^{3} \frac{d^2\nu_r}{\text{Im}\tau} \prod_{s>r} \chi(\nu_{sr}, \tau) \alpha^{'k_r} k_s \text{Im}\tau \sum_{m,n} \delta^{(2)} \left( m + n\tau + \sum_s n_s \nu_s \right)
\]

The delta-function suppresses one full complex integration over one modulus \( \nu_r \) provided not all \( n_r \) vanish. If all \( n_r \) vanish, one just gets \( \delta^{(2)}(m + m\tau) \) singling out \( m = n = 0 \) and giving a divergent \( \delta^{(2)}(0) \sim 1/\epsilon^2 \) as before. However, as already stressed, we are interested in the case where at least some \( n_r \neq 0 \). To be concrete, let’s assume \( n_3 \neq 0 \), otherwise relabel the external states. We then want to trade the delta-function against the \( \nu_3 \)-integration. While the the form (2.20) of the amplitude still was manifestly symmetric under exchange of the external particles, this will of course no longer be true in the following. In the sum, only those \( m, n \) can contribute that are such that \( \nu_3 \) is within its integration region, namely the parallelogram \((0, 1, 1 + \tau, \tau)\). Since

\[
m + n\tau + \sum_s n_s \nu_s = n_3 \left( \nu_3 - \frac{\nu_{41} n_1 + \nu_{42} n_2 - m - n\tau}{n_3} - \tau \right)
\]

there are precisely \( n_3 \) values of \( m \) and \( n_3 \) values of \( n \) that contribute, thus \( n_3^2 \) discrete values of \( \nu_3 \) that contribute to the sum. These \( n_3^2 \) values of \( \nu_3 \) fill out a finite regular lattice within the parallelogram \((0, 1, 1 + \tau, \tau)\). If we denote by \( \bar{\nu}_0 \) the point among them that is closest to the origin then for any function \( f \) of \( \nu_3 \)

\[
\int d^2\nu_3 \sum_{m,n=-\infty}^{\infty} \delta^{(2)} \left( m + n\tau + \sum_s n_s \nu_s \right) f(\nu_3) = \frac{1}{n_3^2} \sum_{m,n=0}^{n_3-1} f \left( \bar{\nu}_0 + \frac{m + n\tau}{n_3} \right).
\]

Of course, \( \bar{\nu}_0 \) depends on \( \nu_1, \nu_2, \tau \) as well as on the \( n_r \). A convenient way to characterise \( \bar{\nu}_0 \) is the following: denote by \( F[x] = x - E[x] \) the fractional part of the real number \( x \), and for
every complex number of the form $z = x + \tau y$ with real $x, y$, let $F_c[z] = F[x] + \tau F[y]$. Then one simply has

$$\tilde{\nu} = \frac{1}{n_3} F_c[\nu_4 n_1 + \nu_2 n_2].$$  \hspace{1cm} \text{(2.23)}$$

Putting everything together we find that the $\epsilon \to 0$ limit of the amplitude is

$$A_{\text{cl}}^{(4)} \bigg|_{\epsilon \to 0} = \frac{\left(\pi \kappa\right)^4}{\alpha'^5} K_{\text{cl}} \int \frac{d^2 \nu_1}{(\text{Im} \tau)^2} \frac{d^2 \nu_2}{\text{Im} \tau} \frac{1}{n_3^2} \sum_{m,n=0}^{n_3-1} \prod_{s>r} \chi(\nu_{sr}, \tau)^{\alpha'_k k_r} \bigg|_{\nu_3 = \tilde{\nu} + \frac{m+n \tau}{n_3}}. \hspace{1cm} \text{(2.24)}$$

We see that the only effect of the compactification on a space-like circle of vanishing size (with external momenta in the compact direction being $n_r/(\ell_s)$, $n_r$ being kept fixed) is to simply replace one of the $\nu_r$ integrations by a discrete sum over $n_r^2$ values on a regular lattice on the torus, i.e the parallelogram $(0, 1, 1 + \tau, \tau)$. This is quite striking. In particular, the amplitude (2.24) is perfectly finite. Of course one still has to check that the integrations over the remaining moduli do not induce any new divergences. It is however easy to verify that the only singularities of the amplitude (2.24) are those poles that are compatible with unitarity, corresponding to on-shell intermediate states. This will be discussed next.

### 2.6. Finiteness of the amplitude in the light-like limit

Possible divergences of the amplitude (2.24) arise whenever two or more of the $\nu_r$ come close to each other, which now in particular also means $\nu_1$ or $\nu_2$ close to any of the discrete values of $\nu_3$, or $\nu_1$ and $\nu_2$ such that $\nu_3$ is close to $\tau \equiv \nu_4$. All these singularities can be easily studied using the asymptotic form of

$$\chi(\nu_{sr}, \tau) \sim 2\pi |\nu_{sr}| \quad \text{as} \quad \nu_{sr} \to 0. \hspace{1cm} \text{(2.25)}$$

We have checked that the only divergences that arise are the poles required by unitarity. Here we will only present one example which corresponds to the case studied in field theory in [8]. This is the case were there is vanishing momentum transfer in the compact direction between the two scattering particle. Here this corresponds to the four-point amplitude with e.g. $n_1 = -n_2 = \tilde{l}$ (first particle) and $n_3 = -n_4 = l$ (second particle). The dangerous field theory diagram then corresponds to the limit where $\nu_{21} \to 0$. Of course, the $\nu_{21} = 0$ limit of the string amplitude integrand is divergent, but what we must do is to carry out the integral and check whether the small $\nu_{21}$ region gives a divergence or not.
Note that with the present choice of \( n_r \) one has \( (\nu_{41} n_1 + \nu_{42} n_2) = \tilde{t} \nu_{21} \) so that in the region of interest where \( \nu_{21} \) is small, this quantity is small as well. Then

\[
\bar{v}_0 = \frac{1}{\tilde{t}} F_c[\tilde{t} \nu_{21}] = \frac{\tilde{t}}{l} \nu_{21} \quad \text{and} \quad \nu_3 = \frac{\tilde{t} \nu_{21} + m + m \tau}{l}.
\] (2.26)

First, for \( m \) and \( n \) not both zero, \( \nu_3 \) will not be close to any other \( \nu_{r} \) in general, and the discrete nature of \( \nu_3 \) plays no special role, so that one only gets divergences from

\[
\int d^2 \nu_{21} \chi(\nu_{21}, \tau) \alpha' k_1 \cdot k_2 \sim \int d^2 \nu_{21} (2\pi |\nu_{21}|) \alpha' k_1 \cdot k_2 \sim \frac{1}{2 - \alpha' t/2}
\] (2.27)

where we introduced the Mandelstam variable

\[
t = -(k_1 + k_2)^2 = -2k_1 \cdot k_2 = -2k_3 \cdot k_4.
\] (2.28)

Thus there are poles for states in the \( t \)-channel that have mass squared equal \( 4/\alpha' \). With the conventions used [9] for the closed string, this is just the first massive level of the uncompactified closed superstring. Thus this pole must well be there for unitarity.

However, the compactified closed superstring has more massive levels. We always refer to the ten-dimensional mass. The point is that in the limit we consider, the left-right level matching condition no longer is \( N_{\text{oscill}} = \overline{N}_{\text{oscill}} \) thus forcing the total level to be even, but rather

\[
N_{\text{oscill}} - nm/2 = \overline{N}_{\text{oscill}} + nm/2
\]

thus allowing any integer level number. In particular we expect the first massive pole to appear already at mass squared equal to \( 2/\alpha' \), i.e. at \( \alpha' t = 2 \) rather than \( \alpha' t = 4 \). We will now show that it is precisely the discrete nature of the moduli \( \nu_3 \) that gives rise to these new poles, and hence remembers that one dimension was compactified so that there were winding states running around the loop!

Above, we have looked at \( m \) and \( n \) not both zero. Now consider \( n = m = 0 \). Then \( \nu_3 = \tilde{t} \nu_{21} \). So as long as \( \nu_{21} \) is very small \( \nu_3 \) is also forced to be very small, meaning it is very close to zero, which by the periodicity of the \( \chi \) is equivalent to being very close to \( \tau \). Hence also \( \nu_{43} \) is very small. Rather than being an independent integration variable, \( \nu_{43} \) is driven to zero if \( \nu_{21} \) is taken to zero. This changes the nature of the singularity to be

\[
\int d^2 \nu_{21} \chi(\nu_{21}, \tau) \alpha' k_1 \cdot k_2 \chi(\nu_{43}, \tau) \alpha' k_3 \cdot k_4
\sim \int d^2 \nu_{21} (2\pi |\nu_{21}|)^{-\alpha' t/2} (2\pi |\tilde{t}/l| |\nu_{21}|)^{-\alpha' t/2} \sim \frac{1}{2 - \alpha' t}.
\] (2.29)

Thus we indeed see the desired pole at \( \alpha' t = 2 \).
3. Direct DLCQ computation of the four-point one-loop amplitude

In this section we will show how to compute the four-point one-loop amplitude directly with a compactified light-like circle. In the first place, the result will be very singular, however. The reason is that it is not clear how to naturally implement the Wick rotation that was implicitly made above when we computed $\int dp_0 F_1$. Very formally, we can then just replace the integration over the continuous light-cone momentum by a “Wick-rotated” one ($p \rightarrow ip$). The resulting expression can then be shown to coincide with the above amplitude $A_{cl1}|_{\epsilon \rightarrow 0}$ of eq. (2.24). The fact that both expressions coincide can be taken as an a posteriori justification of this “Wick rotation”. In this sense we can say that in type II string theory the DLCQ can indeed be viewed as the limit of an almost light-like compactification.

As extensively discussed in the introduction, doing a DLCQ computation amounts to taking $\epsilon = 0$ from the outset, in particular before doing the zero-mode integration $\int dp_0$. We will here simply use the formalism of the preceding section and examine how to set $\epsilon = 0$ from the start. The reader should be warned that most of this is very formal, since e.g. we used the representation of the $r$th string propagator $\frac{1}{L_0}$ as $\int_0^1 dx_r x_r^{L_0-1}$ which can be justified only after a Wick rotation of $p_0$ that makes $L_0$ positive. On the other hand, in DLCQ, $L_0$ is indefinite, and hence the whole procedure remains formal. Nevertheless it is interesting to see that in the end, after these formal manipulation one obtains exactly the same amplitude as was derived quite rigorously in the preceding section.

As noted in the beginning of section 2, the DLCQ form of $L_0$ is obtained from (2.2) with $\epsilon = 0$. Therefore one needed to change the momentum variable from $p_0$ to $p_t$ by $p_0 = \epsilon p_t + \frac{\mathbb{P}}{\epsilon}$. The resulting $L_0$ and $\overline{L}_0$ are

\begin{align*}
L_0 &= -\frac{1}{2} n(l_s p_t + m) + \frac{\alpha'}{4} p_t^2 + \text{oscillators} \\
\overline{L}_0 &= -\frac{1}{2} n(l_s p_t - m) + \frac{\alpha'}{4} p_t^2 + \text{oscillators}
\end{align*}

(3.1)

The bosonic zero-mode piece $F_1 F_2$ coming from (2.3) then is modified. In practice, the easiest
way to obtain it is to rewrite (2.6) as

\[ F_1 F_2 = \exp \left\{ -\pi \alpha' \sum_{s>r} k_s \cdot k_r \left[ \frac{(\text{Im} \nu_{sr})^2}{\text{Im} \tau} - \text{Im} \nu_{sr} \right] \right\} \exp \left\{ -2\pi i m \text{Re} \left( n \tau + \sum_s n_s \nu_s \right) \right\} \]

\[ \times \exp \left\{ -\pi \text{Im} \tau \left[ \alpha' \left( p^\mu + \sum_s k_s^\mu \nu_s \right)^2 + \frac{1}{\epsilon^2} \left( n + \sum_s n_s \nu_s \right)^2 + \epsilon^2 m^2 \right] \right\} \]

and to change the loop momentum variable from \( p_0 \) to

\[ \epsilon p_t = p_0 - \frac{n}{\ell_s} + \sum_s \left( k_s^0 - \frac{n_s}{\ell_s} \right) \frac{\text{Im} \nu_s}{\text{Im} \tau} \]

so that one can now safely set \( \epsilon = 0 \) and obtain

\[ F_1 F_2 \bigg|_{\epsilon=0} = \exp \left\{ -\pi \alpha' \sum_{s>r} k_s \cdot k_r \left[ \frac{(\text{Im} \nu_{sr})^2}{\text{Im} \tau} - \text{Im} \nu_{sr} \right] \right\} \times \exp \left\{ 2\pi l_s p_t \text{Im} \left( n \tau + \sum_s n_s \nu_s \right) - 2\pi i m \text{Re} \left( n \tau + \sum_s n_s \nu_s \right) \right\} \]

\[ \times \exp \left\{ -\pi \alpha' \text{Im} \tau \left( p^j + \sum_s k_s^j \frac{\text{Im} \nu_s}{\text{Im} \tau} \right)^2 \right\} \]

where the indices \( j \) only run over the eight transverse dimensions, \( j = 2, \ldots 9 \). On the other hand, the integration and sum are replaced by

\[ d p_0 \frac{1}{\ell_s} \sum_{n,m} = \frac{1}{l_s} d p_t \sum_{n,m} \]

which also is independent of \( \epsilon \).

Adding as before the contributions (2.9) of the non-zero modes, as well as of the fermionic zero-modes (2.8), the amplitude in the DLCQ reads (we write \( p = l_s p_t \))

\[ A^{(4)}_{\text{cl, DLCQ}} = \frac{(\pi \kappa)^4}{\alpha'^5} K_{\text{cl}} \int d^2 \tau d^2 \nu_1 d^2 \nu_2 d^2 \nu_3 (\text{Im} \tau)^{-4} \prod_{s>r} \chi(\nu_{sr}, \tau)^{\alpha' k_r \cdot k_s} S_{\text{DLCQ}} \]

\[ S_{\text{DLCQ}} = \int d p \sum_{n,m} \exp \left\{ 2\pi p \text{Im} \left( n \tau + \sum_s n_s \nu_s \right) - 2\pi i m \text{Re} \left( n \tau + \sum_s n_s \nu_s \right) \right\} \]

The quantity \( S_{\text{DLCQ}} \) looks awfully divergent due to the absence of a Wick rotation as discussed above. Comparing with the amplitude (2.20) computed in the previous section, we see that we
want to interpret $S_{DLCQ}$ as $\sum_{m,n} \delta^{(2)}(m + n\tau + \sum_s n_s \nu_s)$. But this can easily be achieved. All one has to do is to “Wick rotate” the light-cone $p$ as $p \rightarrow ip$. Then

$$S_{DLCQ} \rightarrow \sum_n \int dp \exp \left\{ 2\pi i p \Im \left( n\tau + \sum_s n_s \nu_s \right) \right\} \sum_m \exp \left\{ -2\pi i m \Re \left( n\tau + \sum_s n_s \nu_s \right) \right\}$$

While the integral over $p$ of the first exponential just gives $\delta(\Im (n\tau + \sum_s n_s \nu_s))$, the sum over $m$ of the second exponential gives a periodic delta-function so that

$$S^{\text{"Wick}}_{DLCQ} = \sum_n \delta \left( \Im \left( n\tau + \sum_s n_s \nu_s \right) \right) \sum_{\tilde{m}} \delta \left( \tilde{m} + \Re \left( n\tau + \sum_s n_s \nu_s \right) \right)$$

The resulting four-point one-loop amplitude then is identical with the one derived in the previous section in the light-like limit, $\epsilon \rightarrow 0$, i.e. with (2.20) or (2.24).

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