TESTS FOR INJECTIVITY OF MODULES
OVER COMMUTATIVE RINGS

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ABSTRACT. It is proved that a module $M$ over a commutative noetherian ring $R$ is injective if $\text{Ext}^i_R((R/p)_p, M) = 0$ holds for every $i \geq 1$ and every prime ideal $p$ in $R$. This leads to the following characterization of injective modules: If $F$ is faithfully flat, then a module $M$ such that $\text{Hom}_R(F, M)$ is injective and $\text{Ext}^i_R(F, M) = 0$ for all $i \geq 1$ is injective. A limited version of this characterization is also proved for certain non-noetherian rings.

1. Introduction

Let $R$ be a commutative ring. In terms of cohomology, Baer's criterion asserts that an $R$-module $M$ is injective if (and only if) $\text{Ext}^1_R(R/\mathfrak{a}, M) = 0$ holds for every ideal $\mathfrak{a}$ in $R$. When $R$ is also noetherian, it suffices to test against prime ideals and locally, namely, $M$ is injective if either of the following conditions holds:

- $\text{Ext}^1_R(R/p, M) = 0$ for every prime ideal $p$ in $R$;
- $\text{Ext}^1_R(k(p), M_p) = 0$ for every prime ideal $p$ in $R$.

Here, and henceforth, $k(p)$ denotes the field $(R/p)_p$. The main result of this paper is that injectivity can be detected by vanishing of $\text{Ext}$ globally against these fields.

Theorem 1.1. Let $R$ be a commutative noetherian ring and let $M$ be an $R$-complex. If for some integer $d$, one has

$$\text{Ext}^i_R(k(p), M) = 0 \quad \text{for every prime ideal } p \text{ in } R \text{ and all } i > d,$$

then the injective dimension of $M$ is at most $d$.

As recalled in Example 2.2, the module $\text{Ext}^1_R(k(p), M)$ can be quite different from $\text{Ext}^1_R(R/p, M)$ and $\text{Ext}^1_R(k(p), M_p)$. Nevertheless the appearance of $\text{Ext}^1_R(k(p), -)$ in this context is not unexpected in the light of the recent work on cosupport of complexes in [3]; see also the discussion around Corollary 3.3.

The proof of the theorem above is given in Section 2 and applications are presented in Section 3. One such, discussed in Remark 3.2, is a characterization of injectivity of an $R$-module $M$ in terms of that of $\text{Hom}_R(F, M)$, where $F$ is a faithfully flat $R$-module. In Section 4 we establish a partial extension of this last result to certain non-noetherian rings.

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2. Proof of Theorem \[1.1\]

Our standard reference for basic definitions and constructions involving complexes is \[1\]. We first recall that as a consequence of Baer’s criterion, the injective dimension of an \(R\)-complex is detected by vanishing of \(\text{Ext}\) against cyclic modules.

**Baer’s criterion.** Let \(R\) be a commutative noetherian ring, \(M\) an \(R\)-complex, and \(d\) an integer. One has \(\text{inj} \dim_R M \leq d\) if and only if

\[
\text{Ext}_R^i(R/a, M) = 0 \quad \text{for every ideal } a \text{ in } R \text{ and all } i > d.
\]

This result is contained in \[1, \text{Theorem 2.4.I}\].

Consider the collection of ideals

\[
\mathcal{U} := \{a \subset R \mid \text{Ext}_R^i(R/a, M) \neq 0 \text{ for some } i > d\}.
\]

If this collection is empty, then the desired inequality, \(\text{inj} \dim_R M \leq d\), holds by Baer’s criterion. Thus, we assume that \(\mathcal{U}\) is non-empty and aim for a contradiction. It is achieved by establishing a sequence of claims, the first of which is standard but included for convenience.

**Claim 1.** With respect to inclusion, \(\mathcal{U}\) is a poset and its maximal elements are prime ideals.

**Proof.** Let \(a\) be a maximal element in \(\mathcal{U}\). Choose a prime ideal \(p \supseteq a\) such that \(p/a\) is an associated prime of \(R/a\), and pick an element \(r \in R\) be such that \(p = (a : r)\). The ideal \(a + (r)\) properly contains \(a\) and hence is not in \(\mathcal{U}\). From the exact sequence of \(\text{Ext}\) modules associated to the standard exact sequence

\[
0 \to R/p \to R/a \to R/(a + (r)) \to 0
\]

it follows that \(p\) is in \(\mathcal{U}\). Since \(a\) is maximal in \(\mathcal{U}\), the equality \(a = p\) holds. \(\square\)

Fix a maximal element \(p\) in \(\mathcal{U}\); by Claim 1 it is a prime ideal. Set \(S := R/p\) and let \(Q\) be the field of fractions of the domain \(S\). We proceed to analyze the \(S\)-complex

\[
X := \mathbf{R}\text{Hom}_R(S, M).
\]

**Claim 2.** The natural map \(H^i(X) \to Q \otimes_S H^i(X)\) is an isomorphism for all \(i > d\).

**Proof.** Fix an element \(s \neq 0\) in \(S\). Let \(x\) be an element in \(R\) whose residue class mod \(p\) is \(s\). By the maximality of \(p\), the ideal \(p + (x)\) is not in \(\mathcal{U}\). As one has \(S/(s) \cong R/(p + (x))\), it follows that \(\text{Ext}_R^i(S/(s), M) = 0\) holds for all \(i > d\). Thus, applying \(\mathbf{R}\text{Hom}_R(-, M)\) to the exact sequence

\[
0 \to S \xrightarrow{s} S \to S/(s) \to 0,
\]

shows that multiplication \(H^i(X) \xrightarrow{s} H^i(X)\) is an isomorphism for \(i > d\). \(\square\)

In the derived category over \(S\), consider the triangle defining (soft) truncations \((2.1)\)

\[
\tau^{<d}X \to X \to \tau^{>d}X \to .
\]

**Claim 3.** There is an isomorphism \(\tau^{>d}X \cong H(\tau^{>d}X)\) in the derived category over \(S\), and the action of \(S\) on \(H(\tau^{>d}X)\) factors through the embedding \(S \to Q\).
Proof. It follows from Claim 2 that the canonical morphism \( \tau^{>d}X \to Q \otimes_S \tau^{>d}X \) yields an isomorphism in the derived category over \( S \). The right-hand complex is one of \( Q \)-vector spaces, so it is isomorphic to its homology, and another invocation of Claim 2 yields the claim. \( \square \)

Claim 4. One has inj dim\( _S(\tau^{<d}X) \leq d \).

Proof. By Baer’s criterion it suffices to show that \( \text{Ext}^i_S(S/\mathfrak{b}, \tau^{<d}X) \) vanishes for every ideal \( \mathfrak{b} \) in \( S \) and all \( i > d \). Notice first that we may assume that \( \mathfrak{b} \) is non-zero, because for \( i > d \) one has \( \text{Ext}^i_S(S, \tau^{<d}X) \cong H^i(\tau^{<d}X) = 0 \), where the vanishing is by construction. For \( \mathfrak{b} \neq 0 \) one has \( Q \otimes_S S/\mathfrak{b} = 0 \), and Claim 3 together with Hom-tensor adjunction yields

\[
\text{Ext}^*_S(S/\mathfrak{b}, \tau^{>d}X) \cong \text{Ext}^*_S(S/\mathfrak{b}, H(\tau^{>d}X)) \\
\cong \text{Ext}^*_Q(Q \otimes_S S/\mathfrak{b}, H(\tau^{>d}X)) = 0.
\]

For \( i > d \) the exact sequence in homology associated to (2.1) now gives the first isomorphism below

\[
\text{Ext}^i_S(S/\mathfrak{b}, \tau^{<d}X) \cong \text{Ext}^i_S(S/\mathfrak{b}, X) \\
\cong \text{Ext}^i_R(S/\mathfrak{b}, M) \\
\cong \text{Ext}^i_R(R/\mathfrak{a}, M) \\
= 0.
\]

The second isomorphism follows from Hom-tensor adjunction and the definition of \( X \). The next isomorphism holds for any choice of an ideal \( \mathfrak{a} \) in \( R \) that reduces to \( \mathfrak{b} \) in \( S \), i.e. \( S/\mathfrak{b} \cong R/\mathfrak{a} \) as \( R \)-modules. Since \( \mathfrak{b} \subset S \) is non-zero, the ideal \( \mathfrak{a} \) properly contains \( \mathfrak{p} \) and hence it is not in \( \mathfrak{U} \). That explains the vanishing of Ext. \( \square \)

Claim 5. One has \( H(\tau^{>d}X) = 0 \).

Proof. By construction one has \( H^i(\tau^{>d}X) = 0 \) for \( i \leq d \). Apply \( R \text{Hom}_S(Q, -) \) to the exact triangle (2.1). By Claim 3, using that \( Q \)-vector spaces are injective \( S \)-modules, one has

\[
\text{Ext}^*_S(Q, \tau^{>d}X) \cong \text{Ext}^*_S(Q, H(\tau^{>d}X)) \\
\cong \text{Hom}_S(Q, H(\tau^{>d}X)) \\
\cong H(\tau^{>d}X).
\]

For \( i > d \), Claim 4 yields \( H^i(R \text{Hom}_S(Q, \tau^{<d}X)) = 0 \), and together with the computation above, this explains the first two isomorphisms in the next chain

\[
H^i(\tau^{>d}X) \cong \text{Ext}^i_S(Q, \tau^{>d}X) \\
\cong \text{Ext}^i_S(Q, X) \\
\cong \text{Ext}^i_R(k(\mathfrak{p}), M) \\
= 0.
\]
The third isomorphism follows from Hom-tensor adjunction, recalling that $Q = S(0)$ as an $R$-module is $(R/p)_p/p \cong k(p)$. The vanishing of Ext is by hypothesis. \qed

Finally, from Claim 5 and (2.1) one gets the second isomorphism below

$$\text{Ext}^i_R(R/p, M) \cong H^i(X) \cong H^i(\tau^{\leq d}X);$$

the first one holds by the definition of $X$. Thus one has $\text{Ext}^i_R(R/p, M) = 0$ for all $i > d$, and this contradicts the assumption that $p$ is in $U$.

This completes the proof of Theorem 1.1. \qed

To use Theorem 1.1 to verify injectivity of an $R$-module $M$ one would have to check vanishing of $\text{Ext}^i_R(k(p), M)$, not only for all prime ideals $p$ but also for all $i > 0$. However, building on this result, in recent work with Marley [8] we have been able to prove that it suffices to verify the vanishing for a single $i$, as long as $i$ is large enough. The example below illustrates that such a restriction is needed.

**Example 2.1.** If $R$ is a complete local ring with depth $R \geq 2$, then one has

$$\text{Ext}^1_R(k(p), R) = 0$$

for every prime ideal $p$ in $R$.

Indeed, if $p$ is the maximal ideal of $R$, then vanishing holds by the assumption depth $R \geq 2$, and for every non-maximal prime $p$ one has $\text{Ext}^1_R(k(p), R) = 0$ for all $i$; see [1] Example 4.20 and [3].

The next example illustrates that the vanishing of $\text{Ext}^i_R(k(p), R)$ does not imply that of $\text{Ext}^i_R(R/p, R)$ and $\text{Ext}^i_R(k(p), R_p)$, and vice versa. Thus Theorem 1.1 is not obviously a consequence of Baer’s criterion, nor does it subsume it.

**Example 2.2.** Let $R$ be as in Example 2.1 and $p$ a prime ideal minimal over $(r)$ where $r$ is not a zero divisor. In this case, both $\text{Ext}^1_R(R/p, R)$ and $\text{Ext}^1_R(k(p), R_p)$ are nonzero, whilst $\text{Ext}^1_R(k(p), R) = 0$. On the other hand, $\text{Ext}^1_R(Z, Z)$ is nonzero, whilst $\text{Ext}^1_R(Z, Z) = 0 = \text{Ext}^1_R(Q, Q)$.

The analogue of Theorem 1.1 for flat dimension is well-known and easier to verify.

**Remark 2.3.** Let $M$ be an $R$-complex. For each prime ideal $p$ and integer $i$ there is a natural isomorphism

$$\text{Tor}^i_R(k(p), M) \cong \text{Tor}^i_{R_p}(k(p), M_p).$$

It thus follows from [1] Proposition 5.3.F] that if there exists an integer $d$ such that $\text{Tor}^i_R(k(p), M) = 0$ for $i > d$ and each prime $p$, then the flat dimension of $M$ is at most $d$. However, Theorem 1.1 does not follow, it seems, from this result by standard injective–flat duality.

3. Applications

We present some applications of Theorem 1.1. The first one improves [7, Theorem 2.2] in two directions: There is no assumption on the projective dimension of flat modules, and an extension ring is replaced by a module.
Corollary 3.1. For every $R$-complex $M$ and every faithfully flat $R$-module $F$ there is an equality
\[ \text{inj dim}_R R\text{Hom}_R(F, M) = \text{inj dim}_R M. \]
In particular, $M$ is acyclic if and only if $R\text{Hom}_R(F, M)$ is acyclic.

Proof. For every prime ideal $p$ in $R$ and every integer $i$ one has
\[ \text{Ext}^i_R(k(p), R\text{Hom}_R(F, M)) \cong \text{Ext}^i_R(F \otimes_R k(p), M) \]
by adjunction and flatness of $F$. Observe that as an $R$-module $F \otimes_R k(p)$ is a direct sum of copies of $k(p)$; it is non-zero because $F$ is faithfully flat. It follows that $\text{Ext}^i_R(k(p), R\text{Hom}_R(F, M))$ is zero if and only if $\text{Ext}^i_R(k(p), M)$ is zero. The equality of injective dimensions now follows from Theorem 1.1.

In view of the equality, the statement about acyclicity is trivial as $M$ is acyclic if and only if 0 is a semi-injective resolution of $M$ if and only if $\text{inj dim}_R M = -\infty$. □

Let $F$ be a flat $R$-module. A module $F \otimes_R M$ is flat if $M$ is flat, and the converse holds if $F$ is faithfully flat; this is standard. It is equally standard that the module $\text{Hom}_R(F, M)$ is injective if $M$ is injective. The next remark provides something close to a converse; Example 2.1 suggests that the hypotheses are optimal.

Remark 3.2. Let $F$ be a faithfully flat $R$-module. If $M$ is an $R$-module with $\text{Ext}^i_R(F, M) = 0$ for all $i > 0$, then $R\text{Hom}_R(F, M)$ is isomorphic to $\text{Hom}_R(F, M)$ in the derived category over $R$. Thus, for such a module Corollary 3.1 asserts that $\text{Hom}_R(F, M)$ is injective if and only if $M$ is injective. This improves the Main Theorem in [7]; see also Theorem 4.3.

The only other result in this direction we are aware of is the Main Theorem in [7]. It deals with the special case where $F$ is a faithfully flat $R$-algebra, and the proof relies heavily on [4, Theorem 4.5] in the form recovered by Corollary 3.3.

This points to our next application, which involves the notion of cosupport introduced in [4], in a form justified by [4, Proposition 4.4]. The cosupport of an $R$-complex $M$ is the subset of $\text{Spec } R$ given by
\begin{equation}
\text{cosupp}_R M = \{ p \in \text{Spec } R \mid H(R\text{Hom}_R(k(p), M)) \neq 0 \}.
\end{equation}

The next result is [4, Theorem 4.5] applied to the derived category over $R$. The proof of op. cit. builds on the techniques developed in [3, 4] to apply to triangulated categories equipped with ring actions.

Corollary 3.3. An $R$-complex $M$ has cosupp$_R M = \emptyset$ if and only if $H(M) = 0$.

Proof. The “if” is trivial, and the converse holds by Theorem 1.1 when one recalls that $H^i(M) \neq 0$ implies $\text{inj dim}_R M \geq i$. □

Remark 3.4. One can deduce the preceding corollary also from Neeman’s classification [11, Theorem 2.8] of the localizing subcategories of the derived category over $R$. Indeed, the subcategory of the derived category consisting of $R$-complexes $X$ with $\text{Ext}^i_R(X, M) = 0$ is localizing. Thus, if it contains $k(p)$ for each $p$ in $\text{Spec } R$, then it must contain $R$, by op. cit., that is to say, $H(M) = 0$. 

Conversely, Corollary 3.3 can be used to deduce Neeman’s classification, by mimicking the proof of [5, Theorem 6.1]. The crucial additional observation needed to do so is that for $R$-complexes $M$ and $N$, there is an equality

$$\text{cosupp}_R \mathbf{R}\text{Hom}_R(M, N) = \text{supp}_R M \cap \text{cosupp}_R N.$$  

It follows from two applications of the standard adjunction:

$$H(\mathbf{R}\text{Hom}_R(k(p), \mathbf{R}\text{Hom}_R(M, N)))$$  

$$\cong H(\mathbf{R}\text{Hom}_k(k(p) \otimes_R^L M, \mathbf{R}\text{Hom}_R(k(p), N)))$$

$$\cong \text{Hom}_k(k(p) \otimes_R^L M), H(\mathbf{R}\text{Hom}_R(k(p), N))).$$

4. Non-noetherian rings

In this section we establish, over certain not necessarily noetherian rings, a characterization of injective modules in the vein of [7]; see also Remark 3.2. This involves the following invariant:

$$\text{splf } R = \sup\{\text{proj dim}_R F \mid F \text{ is a flat } R\text{-module}\}.$$  

A direct sum of flat modules is flat with $\text{proj dim}(\bigoplus_{i \in I} F_i) = \sup_{i \in I} \{\text{proj dim } F_i\}$, so the invariant $\text{splf } R$ is finite if and only if every flat $R$-module has finite projective dimension. With a nod to Bass’ [2, Theorem P], a ring with $\text{splf } R \leq d$ is also called a $d$-perfect ring. If $R$ has cardinality at most $n$, for some natural number $n$, then one has $\text{splf } R \leq n + 1$ by a result of Gruson and Jensen [9, Theorem 7.10]. Osofsky [3, 3.1] has examples of rings for which the $\text{splf}$ invariant is infinite.

**Lemma 4.1.** Let $R$ be a commutative ring with $\text{splf } R < \infty$ and let $S$ be a faithfully flat $R$-algebra. An $R$-complex $M$ with $H^i(M) = 0$ for all $i \gg 0$ is acyclic if and only if $\mathbf{R}\text{Hom}_R(S, M)$ is acyclic.

**Proof.** The “only if” is trivial, so assume that $\mathbf{R}\text{Hom}_R(S, M)$ is acyclic. As $H(M)$ is bounded above, we may assume that $H^i(M) = 0$ holds for all $i > 0$, and it suffices to prove that also $H^0(M) = 0$. Set $d := \text{splf } R$.

Application of $\mathbf{R}\text{Hom}_R(-, M)$ to the exact sequence $0 \to R \to S \to S/R \to 0$ yields $M \cong \Sigma \mathbf{R}\text{Hom}_R(S/R, M)$ in the derived category over $R$. Repeated use of this isomorphism and adjunction yields $M \cong \Sigma^{d+1} \mathbf{R}\text{Hom}_R((S/R)^{\otimes d+1}, M)$. As $S$ is faithfully flat over $R$, the module $S/R$ is flat, and hence so are its tensor powers. Thus, the module $(S/R)^{\otimes d+1}$ has projective dimension at most $d$ and, therefore, $H^i(\mathbf{R}\text{Hom}_R((S/R)^{\otimes d+1}, M)) = 0$ holds for all $i > d$. In particular,

$$H^0(M) \cong H^0(\Sigma^{d+1} \mathbf{R}\text{Hom}_R((S/R)^{\otimes d+1}, M))$$

$$= H^{d+1}(\mathbf{R}\text{Hom}_R((S/R)^{\otimes d+1}, M))$$

$$= 0. \quad \square$$

**Proposition 4.2.** Let $R$ be a commutative ring with $\text{splf } R < \infty$ and let $S$ be a faithfully flat $R$-algebra of projective dimension at most 1. An $R$-complex $M$ is acyclic if and only if $\mathbf{R}\text{Hom}_R(S, M)$ is acyclic.

**Proof.** The “only if” is trivial, so assume that $\mathbf{R}\text{Hom}_R(S, M)$ is acyclic. To prove that $M$ is acyclic, we show that $H^0(\Sigma^n M) = 0$ holds for all $n \in \mathbb{Z}$. Fix $n$ and let
\[ \Sigma^n M \to I \] be a semi-injective resolution; the assumption is now \( H(\text{Hom}_R(S, I)) = 0 \)
and the goal is to prove \( H^0(I) = 0 \).

The soft truncation
\[ \tau^{\leq 1} \text{Hom}_R(S, I) = \cdots \to \text{Hom}_R(S, I)^{-1} \to \text{Hom}_R(S, I)^0 \to \text{Z}^1(\text{Hom}_R(S, I)) \to 0 \]
is acyclic, and by left-exactness of \( \text{Hom} \) one has \( \tau^{\leq 1} \text{Hom}_R(S, I) = \text{Hom}_R(S, \tau^{\leq 1} I) \).
Further, still by acyclicity of \( \text{Hom}_R(S, I) \), there is an equality
\[ \text{B}^2(\text{Hom}_R(S, I)) = \text{Hom}_R(S, \text{B}^2(I)). \]
Thus, the functor \( \text{Hom}_R(S, -) \) leaves the sequence
\[ 0 \to \text{Z}^1(I) \to I^1 \to \text{B}^2(I) \to 0 \]
exact, and that implies vanishing of \( \text{Ext}^1_R(S, \text{Z}^1(I)). \)

Let \( \pi: P \to S \) be a projective resolution over \( R \) with \( P_i = 0 \) for \( i > 1 \). Consider its mapping cone
\[ A = 0 \to P_1 \to P_0 \to S \to 0. \]
As \( \text{Hom}_R(A, I^n) \) is exact for every \( n \) and \( \text{Hom}_R(A, \text{Z}^1(I)) \) is exact by vanishing of \( \text{Ext}^1_R(S, \text{Z}^1(I)) \), it follows from [6, Lemma (2.5)] that \( \text{Hom}_R(A, \tau^{\leq 1} I) \) is acyclic.
Thus, \( \text{Hom}_R(\pi, \tau^{\leq 1} I) \) yields an isomorphism \( \text{RHom}_R(S, \tau^{\leq 1} I) \cong \text{Hom}_R(S, \tau^{\leq 1} I) \)
in the derived category, and the latter complex is acyclic. Now Lemma 4.1 yields \( H(\tau^{\leq 1} I) = 0 \), in particular \( H^0(I) = H^0(\tau^{\leq 1} I) = 0 \). \( \square \)

**Theorem 4.3.** Let \( R \) be a commutative ring with \( \text{splf} R < \infty \), let \( S \) be a faithfully flat \( R \)-algebra of projective dimension at most 1, and let \( M \) be an \( R \)-module. If \( \text{Ext}^1_R(S, M) = 0 \) and the \( S \)-module \( \text{Hom}_R(S, M) \) is injective, then \( M \) is injective.

**Proof.** The proof of [7, Theorem 1.7] applies with one modification: in place of [7, 1.5]—at heart a reference to [4, Theorem 4.5]—one invokes Proposition 4.2. \( \square \)

**Remark 4.4.** The assumption in Theorem 4.3 that the flat \( R \)-algebra \( S \) has projective dimension at most 1 is satisfied if

- \( R \) is countable; see [9, Theorem 7.10].
- \( S \) is countably related; in particular, if every ideal in \( R \) is countably generated, and \( S \) is countably generated as an \( R \)-module; see Osofsky [12, Lemma 1.2] and Jensen [10, Lemma 2].

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