Lattice formulation of chiral gauge theories

Werner Kerler

Institut für Physik, Humboldt-Universität, D-12489 Berlin, Germany

Abstract

We present a general formulation of chiral gauge theories, which admits Dirac operators with more general spectra, reveals considerably more possibilities for the structure of the chiral projections, and nevertheless allows appropriate realizations. In our analyses we use two forms of the correlation functions which both also apply in the presence of zero modes and for any value of the index. To account properly for the conditions on the bases the concept of equivalence classes of pairs of them is introduced. The behaviors under gauge transformations and under CP transformations are unambiguously derived.

1 Introduction

For the non-perturbative definition of the quantum field theories of particle physics only the lattice approach is available. While for QCD it is nowadays clear how to proceed, in the case of chiral gauge theories still quite some details remain to be clarified. To make progress within this respect is the aim of the present paper. In this context generalizations help to see which features are truly relevant.

Starting from the basic structure of chiral gauge theories on the lattice, which has been introduced in the overlap formalism of Narayanan and Neuberger [1] and in the formulation of Lüscher [2], we have recently worked out a generalization [3]. Our definitions of operators there have referred to a basic unitary and $\gamma_5$-Hermitian operator. While this has provided a guide to many detailed results, to base the formulation on this operator introduces unnecessary restrictions. Therefore we here develop a more general formulation which does not rely on this operator.

We further base our developments now on the concept of equivalence classes of pairs of bases which appears more appropriate than our previous use of separate classes since it exploits the respective freedom fully and also turns out to be more natural in view of the structures we find for the general chiral projections.
It appears worthwhile to emphasize that in contrast to other approaches we take care that our results also hold in the presence of zero modes of the Dirac operator and for any value of the index.

In Section 2 we start with relations for the Dirac operators, removing restrictions on their spectra, which have been inherent in all analytical forms so far. The resulting operators are seen to have still realizations with appropriate locality properties and methods of numerical evaluation. A discussion of the locations of the spectra illustrates the respective new possibilities. We also introduce the generalization of the unitary operator of previous formulations and see its connection to the index.

In Section 3 we derive the properties of the chiral projections for given Dirac operator, revealing considerably more possibilities for their structures. This derivation is based on the spectral representations of the operators and a careful consideration of details related to the Weyl degrees of freedom.

We further express the chiral projections in an alternative form which is particularly useful for the study of CP properties as well as with respect to applications of generalized chiral symmetries of the Dirac operator. We also see that there are appropriate realizations of the more general chiral projections.

In Section 4 we consider correlation functions and analyze the emerging conditions on the bases. We first formulate fermionic functions in terms of alternating multilinear forms. The requirement of invariance of these functions imposes restrictions on possible basis transformations. To account for this we introduce the concept of the decomposition into equivalence classes of pairs of bases and discuss its crucial significance.

The relations for the chiral projections, which we find in our analysis, imply corresponding relations for the bases. This allows us to obtain a form of the correlation functions which involves a determinant and separate zero mode terms. It has the virtue that the contributions of particular amplitudes become explicit also in the general case considered.

In Section 5 we give a general derivation of gauge-transformation properties closing a loophole in our previous derivation. The cases where both of the chiral projections are gauge-field dependent and where one of them is constant are treated separately. We add a discussion of perturbation theory showing the necessity of the anomaly cancelation condition in the continuum limit.

In Section 6 we similarly give an improved derivation of CP-transformation properties and confirm certain features for the more general structures here, too.

In Section 7 we consider some details of interest also in terms of gauge-field variations, which appears useful for making contact to the work of Lüscher [2].

In Section 8 we collect some conclusions.
2 Dirac operator

2.1 Spectral properties

We consider a finite lattice and require the Dirac operator to be normal, \([D^\dagger, D] = 0\), and \(\gamma_5\)-Hermitian, \(D^\dagger = \gamma_5 D \gamma_5\). It then has the spectral representation

\[
D = \sum_j \hat{\lambda}_j (P_j^+ + P_j^-) + \sum_k (\lambda_k P_k^I + \lambda_k^* P_k^II),
\]

where the eigenvalues are all different and satisfy \(\text{Im } \hat{\lambda}_j = 0\) and \(\text{Im } \lambda_k > 0\). For the projections the relations \(\gamma_5 P_j^\pm = P_j^\mp \gamma_5 = \pm P_j^\mp\) and \(\gamma_5 P_k^I = P_k^II \gamma_5\) hold. With this we have \(\text{Tr}(\gamma_5 P_k^I) = \text{Tr}(\gamma_5 P_k^II) = 0\), \(\text{Tr} P_k^I = \text{Tr} P_k^II = : N_k\) and \(\text{Tr} P_j^\pm = : N_j^\pm\).

Presence of zero modes of \(D\) means that one of the \(\hat{\lambda}_j\) is zero, which we take to be that with \(j = 0\). Since the zero-mode part of \(D\) commutes with \(\gamma_5\), the projector on the respective space is of form \(P_0^+ + P_0^-\) with \(P_0^\pm\) having the properties described for \(P_j^\pm\) above. Accordingly the index of \(D\) is given by \(I = N_0^+ - N_0^-\).

In terms of the introduced projections the identity operator can be represented by

\[
\mathbb{I} = \sum_j (P_j^+ + P_j^-) + \sum_k (P_k^I + P_k^II),
\]

which implies the relation

\[
\text{Tr}(\gamma_5 \mathbb{I}) = \sum_j (N_j^+ - N_j^-) = 0.
\]

It is to be noted that with (2.2) and (2.3) we also have

\[
\sum_j N_j^+ + \sum_k N_k = \sum_j N_j^- + \sum_k N_k = \frac{1}{2} \text{Tr} \mathbb{I} = : d.
\]

2.2 Associated unitary operator

In the absence of zero modes of \(D\) the operator \(V = -DD^\dagger\) is a well defined unitary and \(\gamma_5\)-Hermitian operator. To include the case with zero modes we require

\[
D + D^\dagger V = 0,
\]

in addition to unitarity and \(\gamma_5\)-Hermiticity, which fixes \(V\) up to the sign of the \(P_0^+ + P_0^-\) term in the spectral representation. Taking the positive one we get

\[
V = P_0^+ + P_0^- - \sum_{j \neq 0} (P_j^+ + P_j^-) - \sum_k \left( \frac{\lambda_k}{\lambda_k^*} P_k^I + \frac{\lambda_k}{\lambda_k^*} P_k^II \right).
\]
With this we obtain \( \text{Tr}(\gamma_5 V) = N_0^+ - N_0^- - \sum_{j \neq 0}(N_j^+ - N_j^-) \), which together with (2.3) gives for the index

\[
I = \frac{1}{2} \text{Tr}(\gamma_5 V),
\]

i.e. still the form which in Refs. [4, 3] has been seen to generalize earlier results [1, 5, 6].

The negative sign for the \( P_0^+ + P_0^- \) term instead leads to an operator \( \tilde{V} \) with \( \text{Tr}(\gamma_5 \tilde{V}) = 0 \).

In (2.6) the projector related to the eigenvalue \(-1\) decomposes into projections corresponding to those associated to the different real eigenvalues of \( D \) which occur in addition to zero. Similarly for complex eigenvalues \( \lambda_k = r_k e^{i\alpha_k} \) the associated eigenvalues of \( V \) do not differ for \( r_{k'} \neq r_k \) if \( \alpha_{k'} = \alpha_k \). Furthermore because of \( 0 < \alpha_k < \pi \) the factors \( -\frac{\lambda_k}{\lambda_k} = e^{i(2\alpha_k - \pi)} \) of form \( e^{i\beta_k} \) for \( 0 < \beta_k < \pi \) have contributions of the types \( \beta_{k'} = 2\alpha_{k'} - \pi \) and \( \beta_{k''} = \pi - 2\alpha_{k''} \) while \( \beta_k = 0 \) is obtained for \( \alpha_k = \pm ir_k \). Comparing all this with the spectral representation

\[
P_0^+ + P_0^- - P_1^+ - P_1^- + \sum_k \left( e^{\eta_k} P_{k}^I + e^{-\eta_k} P_{k}^{II} \right), \quad 0 < \eta_k < \pi,
\]

of the special case of the operator \( V \) in Refs. [4, 3] it becomes obvious that (2.6) resulting from \( D \) is within several respects a considerable generalization.

Since the Dirac operators in Refs. [4, 3] constructed on the basis of (2.8) are ones admitting only one real eigenvalue of \( D \) in addition to zero and, as the comparison with (2.6) also shows, with restrictions of the complex eigenvalues, too, we see that not to start from (2.8) as we do here opens much more general possibilities for \( D \).

The classes of Dirac operators with the indicated restrictions [4, 3] contain as the simplest case Ginsparg-Wilson (GW) fermions [7] for which \( D \) is of form \( D = \rho(1 - V) \) with a real constant \( \rho \). Further special cases are the ones proposed by Fujikawa [8], the extension of them [9] and the various examples constructed in Ref. [4].

In the GW case with \( D = \rho(1 - V) \) the explicit realization of Neuberger [10] creates the unitary operator \( V \) by the normalization of another operator, namely of the Wilson-Dirac operator. This construction has been generalized in Ref. [4] and still more in Ref. [3] to apply to specific subclasses, respectively, of the general classes of Dirac operators there.

Another type of explicit construction of \( V \) in the GW case is contained in a definition proposed for \( D \) by Chiu [11]. With it, however, on the finite lattice a non-vanishing index is prevented by a sum rule [12], which is the GW special case of (2.3). In Ref. [13] we have pointed out that the respective \( V \) is of the Cayley-transform type and shown that on the finite lattice, this type generally does not allow a non-vanishing index, while in the continuum limit due to the unboundedness of \( D \) it does.

### 2.3 Particular realizations

The conditions of normality and \( \gamma_5 \)-Hermiticity have been seen here to lead already to several general relations for chiral fermions. Further restrictions result from the connection
of locality and chiral properties. A definite requirement which can be formulated within this respect is that locality of $D$ should imply appropriate properties of the propagator.

In the GW case from $\{\gamma_5, D\} = \rho^{-1}D\gamma_5 D$ one gets $\{\gamma_5, D^{-1}\} = \rho^{-1}\gamma_5$ provided that $D^{-1}$ exists, which means that the propagator is chiral up to a local contact term. This can also be expressed by $D^{-1} + D^{-1} = \rho^{-1}$. The generalization of the latter condition is $D^{-1} + D^{-1} = 2F$ where $F$ is a local operator. To obtain a condition which applies also in the presence of zero modes of $D$ we multiply this by $D$ and $D^\dagger$ getting $\frac{1}{2}(D + D^\dagger) = DF D^\dagger = D^\dagger FD$, which indicates that

$$[F, D] = 0, \quad F^\dagger = F, \quad [\gamma_5, F] = 0. \quad (2.9)$$

To account for this we require $F$ to be a non-singular function of $D$, in detail a Hermitian one of the Hermitian arguments $DD^\dagger$ and $\frac{1}{2}(D + D^\dagger)$, and impose the condition

$$\frac{1}{2}(D + D^\dagger) = DD^\dagger F(DD^\dagger, \frac{1}{2}(D + D^\dagger)), \quad (2.10)$$

in which $F$ must be local for local $D$ (with exponential locality being sufficient).

The operators in Ref. [9] correspond to the special choice of $F$ with the dependence on $D^\dagger D$ only and with a monotony requirement which implies restriction to only one real eigenvalue in addition to zero. To study the choice $F = F(DD^\dagger)$ without such a restriction we consider the special case of a polynomial $F = \sum_{\nu=0}^{M} C_{\nu} (DD^\dagger)^{\nu}$ with real coefficients $C_{\nu}$. The eigenvalues of $D$ then satisfy

$$\text{Re} \lambda = \sum_{\nu=0}^{M} C_{\nu} |\lambda|^{2(\nu+1)}, \quad (2.11)$$

which describes the location of the spectrum. From this it is obvious that $\lambda = 0$ is always included and that the other real eigenvalues are subject to $\sum_{\nu=0}^{M} C_{\nu} |\hat{\lambda}|^{2\nu+1} = 1$, which indicates that in this example one can have up to $2M + 1$ further real eigenvalues.

If only one of the coefficients $C_{\nu}$ differs from zero this gives the proposal of Fujikawa [8] (for $\nu = 0$ the GW case) with only one real eigenvalue in addition to zero. If only $C_0$ and $C_1$ are non-zero for $C_0 > 0$ and $C_1 > 0$ one gets an example given in Ref. [4] still with only one additional real eigenvalue. However, for $C_0^3/C_1 < -27/4$ three different real eigenvalues get possible in addition to zero. Then the location of the spectrum is described by two closed curves, one through zero and further one surrounding it.

An overview in the general case is obtained by putting $\lambda = re^{i\alpha}$ and noting that the spectral function $f$ associated to $F$ is a real function with the dependences $f(r, \cos \alpha)$ and that (2.10) in terms of spectral functions reads $r \cos \alpha = rf(r, \cos \alpha)$. Obviously $\lambda = 0$ is always included in this and the other values are subject to the equation $\cos \alpha = f(r, \cos \alpha)$.

For the more general operators $D$ here the constructions relying on the special case of $V$ with the spectral representation [2,8] are no longer available. Since one cannot count on

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1. This differs from the GW relation $\{\gamma_5, D\} = 2D\gamma_5 RD$ with $[R, D] \neq 0$ in which $D$ is not normal. Though then nevertheless no eigen-nilpotent spoils the subspace of zero modes [14], the effect in other terms remains obscure and the analysis of interest here gets not feasible.
the existence of explicit analytical forms, one has to find other methods which on the one hand side provide a theoretical description and on the other numerical approximations.

The extension of the method of chirally improved fermions [15] of the GW case, which is based on a systematic expansion of the Dirac operator, is applicable also in the case considered here. Indeed, the mapping of the GW equation to a system of coupled equations there can as well be done for the more general relation (2.10). Apart from providing the theoretical possibility, appropriate choices in (2.10) could even be advantageous in numerical work.

3 Chiral projections

3.1 Basic properties

We introduce chiral projections $P_{\pm}$ and $\bar{P}_{\pm}$ with $P_{\pm}^\dagger = P_{\pm}$, $\bar{P}_{\pm}^\dagger = \bar{P}_{\pm}$ and $P_{\pm} + P_{-} = \bar{P}_{\pm} + \bar{P}_{-} = 1$, requiring that they satisfy

$$\bar{P}_{\pm} D = D P_{\pm}. \quad (3.1)$$

With this we get the decomposition of the Dirac operator into Weyl operators,

$$D = \bar{P}_{\pm}DP_{-} + \bar{P}_{-}DP_{+}, \quad (3.2)$$

in which $\bar{P}_{\pm}DP_{\pm} = DP_{\pm} = \bar{P}_{\pm}D$. Furthermore, since with (3.1) also $D_{\dagger}\bar{P}_{\pm} = P_{\pm}D_{\dagger}$ holds, we obtain the relations

$$[P_{\pm}, D_{\dagger}D] = 0, \quad [\bar{P}_{\pm}, DD_{\dagger}] = 0. \quad (3.3)$$

3.2 Spectral structure

Because with (2.1) we have the representation

$$D_{\dagger}D = \sum_{j} \lambda_j^2 (P_{j}^+ + P_{j}^-) + \sum_{k} |\lambda_k|^2 (P_{k}^I + P_{k}^{II}), \quad (3.4)$$

according to (3.3) the chiral projections $P_{-}$ and $\bar{P}_{+}$ decompose as

$$P_{-} = \sum_{j} P_{j}^{(+-)} + \sum_{k} P_{k}^R, \quad \bar{P}_{+} = \sum_{j} \bar{P}_{j}^{(+,-)} + \sum_{k} \bar{P}_{k}^R, \quad (3.5)$$

where $P_{j}^{(+-)}$ and $\bar{P}_{j}^{(+,-)}$ project within the subspace on which $P_{j}^+ + P_{j}^-$ projects, while $P_{k}^R$ and $\bar{P}_{k}^R$ project within that on which $P_{k}^I + P_{k}^{II}$ projects.

Noting that $P_{k}^I$, $P_{k}^{II}$, $\gamma_5 P_{k}^I$, $\gamma_5 P_{k}^{II}$ commute with $P_{k}^I + P_{k}^{II}$, imposing the general conditions $P^2 = P$ and $P_{j}^\dagger = P$ and according to (3.1) requiring $\bar{P}_{k}^RD = D\bar{P}_{k}^R$ we obtain the relations

$$P_{k}^R = c_k P_{k}^I + (1 - c_k) P_{k}^{II} - \sqrt{c_k(1 - c_k)} \gamma_5 (e^{\imath \varphi_k} P_{k}^I + e^{-\imath \varphi_k} P_{k}^{II}),$$

$$\bar{P}_{k}^R = c_k P_{k}^I + (1 - c_k) P_{k}^{II} + \sqrt{c_k(1 - c_k)} \gamma_5 (e^{-\imath \varphi_k} P_{k}^I + e^{\imath \varphi_k} P_{k}^{II}), \quad (3.6)$$
with real coefficients $c_k$ satisfying $0 \leq c_k \leq 1$ and phases $\varphi_k, \bar{\varphi}_k$ being for $c_k(1 - c_k) > 0$ subject to
\[ e^{i(\varphi_k + \bar{\varphi}_k - 2\alpha_k)} = -1 \quad \text{with} \quad e^{i\alpha_k} = \lambda_k/|\lambda_k|, \quad 0 < \alpha_k < \pi, \quad (3.7) \]
and where we have for the dimensions
\[ \text{Tr} \ P_k^R = \text{Tr} \ \bar{P}_k^R = \text{Tr} \ P_k^I = \text{Tr} \ \bar{P}_k^H = N_k. \quad (3.8) \]

Similarly since $P_j^+$ and $P_j^-$ commute with $P_j^+ + P_j^-$ in accordance with (3.1) we arrive at
\[ P_j^{(+,-)} = \bar{P}_j^{(+,-)} \quad \text{for} \quad j \neq 0. \quad (3.9) \]
For the numbers of anti-Weyl and Weyl degrees of freedom $\bar{N} = \text{Tr} \ \bar{P}_+^I$ and $N = \text{Tr} \ P_-^I$ we therefore obtain
\[ \bar{N} - N = \text{Tr} \ \bar{P}^{(+,-)}_0 - \text{Tr} \ P^{(+,-)}_0, \quad (3.10) \]
which requiring $\bar{N} - N = I$ leads to
\[ \bar{P}^{(+,-)}_0 = P_0^+, \quad P^{(+,-)}_0 = P_0^-; \quad (3.11) \]
We next note that we now have
\[ \bar{N} + N = N_0^+ + N_0^- + 2 \sum_{j \neq 0} \text{Tr} \ P_j^{(+,-)} + 2 \sum_k N_k \quad \text{for} \quad I = 0, \quad (3.12) \]
so that in view of (2.4) to get $\bar{N} + N = \text{Tr} \ 1 = 2d$ for $I = 0$ we must put
\[ P_j^{(+,-)} = P_j^+ \quad \text{for} \quad j \neq 0 \quad \text{or} \quad P_j^{(+,-)} = P_j^- \quad \text{for} \quad j \neq 0. \quad (3.13) \]
For these choices we get in the general case
\[ \bar{N} = d, \quad N = d - I \quad \text{or} \quad \bar{N} = d + I, \quad N = d, \quad (3.14) \]
respectively.

Inserting (3.11) and (3.13) into (3.5) we have
\[ P_- = P_0^- + \sum_{j \neq 0} P_j^\pm + \sum_k P_k^R, \quad \bar{P}_+ = P_0^+ + \sum_{j \neq 0} P_j^\pm + \sum_k \bar{P}_k^R, \quad (3.15) \]
which taking the traces gives for the dimensions
\[ N = N_0^- + L, \quad \bar{N} = N_0^+ + L, \quad L = \sum_{j \neq 0} N_j^\pm + \sum_k N_k. \quad (3.16) \]
Relation (3.16) shows that there is a $L \times L$ submatrix $\tilde{M}$ of the chiral matrix $M = \bar{u}^T D u$ from which the zero-mode parts are removed and that solely the latter can make $M$ non-quadratic.
We see now that for given Dirac operator there is still freedom in the details of the chiral projections, which consists in the possible two choices in (3.13) and furthermore in that of the coefficients $c_k$ and the phases $\varphi_k$ and $\bar{\varphi}_k$ in (3.6).

The index introduced in the Atiyah-Singer framework \[16\] on the basis of the Weyl operator corresponds to the one defined here for the Dirac operator. Since the non-zero modes there come in chiral pairs, our relation $\bar{\gamma} - N = I$ has the appearance of a transcription to the finite case of what one has there. However, the effects we observe for $\tilde{N} + N$ for different values of $I$ here, have no counterpart there. The sum rule \[2.3\] for the index of $D$ reflects the fundamental structural difference between the two approaches \[17\]. While in the Atiyah-Singer case the respective effects are accommodated by the space structure, in lattice theory (and thus in the quantized theory it is to define) the space structure is independent of the index.

### 3.3 Alternative form

To see further properties of the chiral projections we express them by

$$ P_- = \frac{1}{2}(\mathbb{1} - \gamma_5 G), \quad \bar{P}_+ = \frac{1}{2}(\mathbb{1} + \bar{G} \gamma_5), \quad (3.17) $$

which implies that $G$ and $\bar{G}$ are unitary and $\gamma_5$-Hermitian operators. According to \(3.1\) they satisfy

$$ D + \bar{G} D^\dagger G = 0. \quad (3.18) $$

Using the relations for $P_-$ and $\bar{P}_+$ derived before we obtain the spectral representations

$$ G = P_0^+ + P_0^- + \sum_{j \neq 0} (P_j^+ + P_j^-) + \sum_{k \ (0<\phi_k<\pi)} (e^{i\phi_k} P_k^A + e^{-i\phi_k} P_k^B), $$

$$ \bar{G} = P_0^+ + P_0^- \pm \sum_{j \neq 0} (P_j^+ + P_j^-) + \sum_{k \ (0<\phi_k<\pi)} (e^{i\phi_k} \bar{P}_k^A + e^{-i\phi_k} \bar{P}_k^B), \quad (3.19) $$

in which the new quantities are related to ones introduced before by

$$ \cos \phi_k = a_k \cos \varphi_k, \quad \sin \phi_k = \sqrt{1 - a_k^2 \cos^2 \varphi_k}, $$

$$ \cos \bar{\phi}_k = a_k \cos \bar{\varphi}_k, \quad \sin \bar{\phi}_k = \sqrt{1 - a_k^2 \cos^2 \bar{\varphi}_k}, \quad (3.20) $$

where $a_k = 2\sqrt{c_k(1 - c_k)}$,

$$ P^A_k = (h_k^2 P^I_k + b_k^2 P^H_k - i b_k h_k \gamma_5 (P^I_k - P^H_k)) / (h_k^2 + b_k^2), $$

$$ P^B_k = (b_k^2 P^I_k + h_k^2 P^H_k + i b_k h_k \gamma_5 (P^I_k - P^H_k)) / (h_k^2 + b_k^2), $$

$$ \bar{P}^A_k = (\bar{h}_k^2 P^I_k + \bar{b}_k^2 P^H_k - i \bar{b}_k \bar{h}_k \gamma_5 (P^I_k - P^H_k)) / (\bar{h}_k^2 + \bar{b}_k^2), $$

$$ \bar{P}^B_k = (\bar{b}_k^2 P^I_k + \bar{h}_k^2 P^H_k + i \bar{b}_k \bar{h}_k \gamma_5 (P^I_k - P^H_k)) / (\bar{h}_k^2 + \bar{b}_k^2), \quad (3.21) $$
where \( b_k = 1 - 2c_k \) and

\[
\begin{align*}
    h_k &= a_k \sin \varphi_k + \sin \phi_k, \\
    \bar{h}_k &= a_k \sin \bar{\varphi}_k + \sin \bar{\phi}_k.
\end{align*}
\]  

(3.22)

From (3.19) it is seen that \( G = 1 \) can be obtained by choosing the lower sign of the \( j \)-sums and putting \( \phi_k = 0 \). The latter according to (3.20) implies that one must have \( c_k = \frac{1}{2} \) and \( \varphi_k = 0 \). Because of (3.17) \( \varphi_k = 0 \) requires that \( \bar{\varphi}_k \) satisfies \( e^{i(\bar{\varphi}_k - 2\alpha_k)} = -1 \).

This and the opposite sign of the \( j \)-sum of \( \bar{G} \) in (3.19) show that one then necessarily obtains \( \bar{G} \neq 1 \). Analogously in the particular case \( \bar{G} = 1 \) one finds that one gets \( G \neq 1 \).

It becomes also obvious from (3.19) that one has always \( \bar{G} \neq G \). This is so because of the opposite signs of the respective \( j \)-sums there, which to allow for a non-vanishing index according to (2.3) must not vanish. (The \( k \)-sums in (3.19) can be made equal by putting \( \bar{\varphi}_k = \varphi_k \), in which case condition (3.7) gets \( e^{2i(\varphi_k - \alpha_k)} = -1 \).)

### 3.4 Special realizations

If one puts \( c_k = \frac{1}{2} \) the operators \( G \) and \( \bar{G} \) commute with \( D \), as can be seen from (3.21). Then one also gets \( \bar{G}G = V \), where \( V \) is the general operator in (2.6). This becomes obvious comparing (2.5) and (3.18) and noting the sign resulting according to (3.19) for the \( P_0^+ + P_0^- \) term. The operators \( G \) and \( \bar{G} \) then nevertheless remain still more general than those in Ref. [3].

The formulations of Refs. [1, 2] use GW fermions, in Ref. [1] in the explicit form of the Neuberger operator [10]. The chiral projections in these approaches in our notation correspond to the special choice \( G = V, \bar{G} = 1 \). Also in the GW case a generalization of this has been proposed by Hasenfratz [18], which in our notation is

\[
G = \left((1 - s)1 + sV\right)/N, \quad \bar{G} = \left(s1 + (1 - s)V\right)/N,
\]  

(3.23)

with a real parameter \( s \neq \frac{1}{2} \) and \( N = \sqrt{1 - 2s(1 - s)(1 - \frac{1}{2}V + V^\dagger)} \). This is also the choice in Ref. [9] with the \( D \) introduced there, as is seen switching to the related \( V \) which has been determined in Ref. [4]. It should be noted that for \( \bar{G} \) and \( G \) satisfying (3.23) one generally has \( \bar{G}G = G\bar{G} = V \).

To obtain realizations of the more general chiral projections here the choice \( c_k = \frac{1}{2} \) is convenient. Then in particular the form (3.23) can be used inserting the general operators (2.6). Comparing (2.5) and (2.10) one gets the more detailed form

\[
V = 1 - 2DF(DD^\dagger, \frac{1}{2}(D + D^\dagger))
\]  

(3.24)

for this, which also has appropriate locality properties.
3.5 Generalized chiral symmetries

In Ref. [6] Lüscher has pointed out that in the GW case there is a generalized chiral symmetry of the Dirac operator. The results here give precise informations about the respective possibilities in the general case.

To see this in detail we note that invariance of the action \( \bar{\psi} D \psi \) under transformations

\[
\bar{\psi}' = \bar{\psi} e^{i \varepsilon \bar{\Gamma}}, \quad \psi' = e^{i \varepsilon \Gamma} \psi
\]

with parameter \( \varepsilon \) requires that the operators \( \bar{\Gamma} \) and \( \Gamma \) satisfy

\[
\bar{\Gamma} D + D \Gamma = 0. \tag{3.25}
\]

Furthermore for a chiral transformations we must have

\[
\bar{\Gamma}^\dagger = \bar{\Gamma} = \bar{\Gamma}^{-1}, \quad \Gamma^\dagger = \Gamma = \Gamma^{-1}. \tag{3.26}
\]

Now putting

\[
\bar{\Gamma} = \bar{G} \gamma_5, \quad \Gamma = \gamma_5 G \tag{3.27}
\]

we see that \( \bar{G} \) and \( G \) must be unitary and \( \gamma_5 \)-Hermitian operators which satisfy \( (3.18) \), i.e. which are identical to the quantities \( \bar{G} \) and \( G \) introduced before.

It thus becomes obvious that for given Dirac operator we get all the possibilities for generalized chiral transformations described by the forms of operators \( \bar{G} \) and \( G \) derived before. It is to be emphasized in this context that one then also generally gets \( \bar{G} \neq G \) as we have seen in Section 3.3. We add here that one then also has \( \bar{\Gamma} \neq \Gamma \), again because of the opposite signs of the respective \( j \)-sums in \( (3.19) \), which to allow for a non-vanishing index according to \( (2.3) \) must not vanish.

4 Correlation functions and bases

4.1 Basic fermionic functions

In terms of Grassmann variables non-vanishing fermionic correlation functions for the Weyl degrees of freedom are given by

\[
\langle x_{i_{r+1}} \ldots x_{i_N} \bar{x}_{j_{r+1}} \ldots \bar{x}_{j_N} \rangle_f = s_r \int d\bar{x}_N \ldots d\bar{x}_1 d\chi_N \ldots d\chi_1 \ e^{-\bar{x}Mx} \ x_{i_{r+1}} \ldots x_{i_N} \bar{x}_{j_{r+1}} \ldots \bar{x}_{j_N}, \tag{4.1}
\]

so that putting \( s_r = (-1)^{rN-r(r+1)/2} \) we have

\[
\langle x_{i_{r+1}} \ldots x_{i_N} \bar{x}_{j_{r+1}} \ldots \bar{x}_{j_N} \rangle_f = \frac{1}{r!} \sum_{j_1, \ldots, j_r=1}^{\bar{N}} \sum_{i_1, \ldots, i_r=1}^{N} \epsilon_{j_1, \ldots, j_N} \epsilon_{i_1, \ldots, i_N} M_{j_{i_{r+1}}} \ldots M_{j_{i_r}}. \tag{4.2}
\]

The fermion field variables \( \bar{\psi}_\sigma \) and \( \psi_\sigma \) are given by \( \bar{\psi} = \bar{\chi} \bar{u}^\dagger \) and \( \psi = u \chi \) with bases \( \bar{u}_\sigma \) and \( u_\sigma \) which satisfy

\[
P_- = uu^\dagger, \quad u^\dagger u = 1_w, \quad \bar{P}_+ = \bar{u} \bar{u}^\dagger, \quad \bar{u}^\dagger \bar{u} = 1_w. \tag{4.3}
\]
where $1_w$ and $1_{\bar{w}}$ are the identity operators in the spaces of the Weyl and anti-Weyl degrees of freedom, respectively. Now with the fermion action $\bar{\chi} M \chi = \bar{\psi} D \psi$, in which one gets $M = \bar{u} \dagger D u$, we obtain from (4.2) for fermionic correlation functions

$$\langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_{N}} \bar{\psi}_{\bar{\sigma}_{r+1}} \cdots \bar{\psi}_{\bar{\sigma}_{N}} \rangle_f = \frac{1}{r!} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_r} \sum_{\sigma_1, \ldots, \sigma_r} \bar{\Upsilon}^{*}_{\bar{\sigma}_1 \ldots \bar{\sigma}_N} \Upsilon_{\sigma_1 \ldots \sigma_N} D_{\sigma_1 \sigma_1} \cdots D_{\sigma_r \sigma_r}$$

(4.4)

with the alternating multilinear forms

$$\Upsilon_{\sigma_1 \ldots \sigma_N} = \sum_{i_1, \ldots, i_N=1}^{N} \epsilon_{i_1, \ldots, i_N} u_{\sigma_1 i_1} \cdots u_{\sigma_N i_N},$$

(4.5)

$$\bar{\Upsilon}^{*}_{\bar{\sigma}_1 \ldots \bar{\sigma}_N} = \sum_{j_1, \ldots, j_N=1}^{\bar{N}} \epsilon_{j_1, \ldots, j_N} \bar{u}_{\bar{\sigma}_1 j_1} \cdots \bar{u}_{\bar{\sigma}_N j_N}.$$

(4.6)

General fermionic correlation functions can be constructed as linear combinations of the particular non-vanishing functions (4.4). Having the fermionic correlation functions, the inclusion of the gauge fields and the definition of full correlation functions is straightforward, at least for vanishing index $I = 0$. For $I \neq 0$ in Ref. [2] the question of $I$-dependent complex factors multiplying the fermionic correlation functions has been raised. In Ref. [18] the importance of such factors for the magnitude of fermion number violating processes has been stressed. However, there has been no theoretical principle for deciding about them. In Refs. [19, 9] it has been suggested that the modulus of them could possibly be one. This is supported by our observation that for the multilinear forms in (4.4) we have

$$\frac{1}{N!} \sum_{\sigma_1, \ldots, \sigma_N} |\Upsilon_{\sigma_1 \ldots \sigma_N}|^2 = \frac{1}{\bar{N}!} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_\bar{N}} |\bar{\Upsilon}^{*}_{\bar{\sigma}_1 \ldots \bar{\sigma}_\bar{N}}|^2 = 1,$$

(4.7)

which means that the averages of $|\Upsilon_{\sigma_1 \ldots \sigma_N}|^2$ and of $|\bar{\Upsilon}^{*}_{\bar{\sigma}_1 \ldots \bar{\sigma}_\bar{N}}|^2$ are equal to 1 independently of the particular values of $N$ and of $\bar{N}$.

4.2 Subsets of bases

By (4.3) the bases are only fixed up to unitary transformations, $u(S) = u S$, $\bar{u}(\bar{S}) = \bar{u} \bar{S}$. While the chiral projections remain invariant under such transformations, the forms $\Upsilon_{\sigma_1 \ldots \sigma_N}$ and $\bar{\Upsilon}^{*}_{\bar{\sigma}_1 \ldots \bar{\sigma}_\bar{N}}$ get multiplied by factors $\det_w S$ and $\det_w \bar{S}$, respectively. Therefore in order that general expectations remain invariant, we have to impose

$$\det_w S \cdot \det_w \bar{S}^\dagger = 1.$$

(4.8)

This is so since firstly in full correlation functions only a phase factor independent of the gauge field can be tolerated. Secondly this factor must be 1 in order that in functions
with more than one contribution individual basis transformations in its parts leave the interference terms in the moduli of the amplitudes invariant. It should be noted that in practice reactions involving more that one contribution are indeed of interest.

Condition (4.8) has important consequences. Without it all bases related to a chiral projection are connected by unitary transformations. With it the total set of pairs of bases \( u \) and \( \bar{u} \) is decomposed into inequivalent subsets, beyond which legitimate transformations do not connect. These subsets of pairs of bases obviously are equivalence classes. Because the formulation of the theory must be restricted to one of such classes, the question arises which choice is appropriate for the description of physics.

Different ones of the indicated equivalence classes are related by pairs of basis transformations \( S, \bar{S} \) for which

\[
\det_w S \cdot \det_w \bar{S} = e^{i\Theta} \quad \text{with} \quad \Theta \neq 0 \quad (4.9)
\]

holds. The phase factor \( e^{i\Theta} \) then determines how the results of the formulation of the theory with one class differ from the results of the formulation with the other class.

### 4.3 Relations for bases

The relations between the chiral projections as well the as the decompositions of them which we have found lead to corresponding properties of the bases. To work this out we note that with (3.4) and (3.5) we have \( DD^\dagger \bar{P}_k^R = |\lambda_k|^2 \bar{P}_k^R \), which using \( D^\dagger \bar{P}_k^R = P_k^R D^\dagger \) (obtained from (3.1)) becomes

\[
\bar{P}_k^R = |\lambda_k|^{-2} D P_k^R D^\dagger. \quad (4.10)
\]

Putting \( P_k^R = \sum_{l=1}^{N_k} u_i^{[k]} u_i^{[k]\dagger} \) in this we see that

\[
u_i^{[k]} = e^{-i\Theta_k} |\lambda_k|^{-1} D u_i^{[k]} \quad (4.11)
\]

with phases \( \Theta_k \) gives the representation \( \bar{P}_k^R = \sum_{l=1}^{N_k} \bar{u}_i^{[k]} u_i^{[k]\dagger} \). Furthermore, for \( P_j^\pm = \sum_{l=1}^{N_j^\pm} u_i^{\pm[j]} u_i^{\pm[j]\dagger} = \sum_{l=1}^{N_j^\pm} \hat{u}_i^{\pm[j]} \hat{u}_i^{\pm[j]\dagger} \) with \( j \neq 0 \) using (2.1) we have with phases \( \Theta_j^\pm \)

\[
\bar{u}_i^{\pm[j]} = e^{-i\Theta_j^\pm} |\hat{\lambda}_j|^{-1} D u_i^{\mp[j]} \quad (4.12)
\]

With (4.11) and (4.12) it becomes obvious that the \( L \times L \) submatrix \( \hat{M} \) of the chiral matrix \( M = \bar{u}^\dagger D u \), from which according to (3.16) the zero modes are removed, has the eigenvalues

\[
e^{i\Theta_k} |\lambda_k|, \quad e^{i\Theta_j^\pm} |\hat{\lambda}_j| \quad (4.13)
\]

with multiplicities \( N_k \) and \( N_j^\pm \), respectively. Its determinant in the subspace thus is

\[
\det_L \hat{M} = \prod_{j \neq 0} (e^{i\Theta_j^\pm} |\hat{\lambda}_j|)^{N_j^\pm} \prod_k (e^{i\Theta_k} |\lambda_k|)^{N_k}. \quad (4.14)
\]

The zero mode parts are described by \( P_0^- = \sum_{l=L+1}^N u_l u_l^\dagger \) and \( P_0^+ = \sum_{l=L+1}^N \bar{u}_l \bar{u}_l^\dagger \), where the numberings with \( l > L \) are chosen for later notational convenience. Because of \( DP_0^- = DP_0^+ = 0 \) these bases satisfy \( D u_l = 0 \) and \( D \bar{u}_l = 0 \).
4.4 Correlation functions with determinant

Using the bases of the preceding Subsection to work out the combinatorics in (4.2) and denoting the eigenvalues of $\hat{M}$ by $\Lambda_l$ we obtain

$$\langle \chi_{i_{r+1}} \cdots \chi_{i_N} \bar{\chi}_{j_{r+1}} \cdots \bar{\chi}_{j_N} \rangle_t =$$

$$\frac{1}{(L-r)!} \sum_{l_{r+1}, \ldots, l_L=1}^L \Lambda_{l_{r+1}}^{-1} \cdots \Lambda_{l_L}^{-1} \epsilon^{j_{r+1} \cdots j_N}_{l_{r+1} \cdots l_L, L+1, \ldots, N} \epsilon^{i_{r+1} \cdots i_N}_{l_{r+1} \cdots l_L, L+1, \ldots, N} \det_L \hat{M} \quad (4.15)$$

for $L \geq r$ (where for $L = r$ the $\Lambda$ factors and the sum are absent), while for $L < r$ the function vanishes. With this we find for the correlation functions (4.4)

$$\langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \cdots \bar{\psi}_{\bar{\sigma}_N} \rangle_t = \sum_{\sigma'_{r+1}, \ldots, \sigma'_N} \epsilon_{\sigma_{r+1} \cdots \sigma_N}^{\sigma'_{r+1} \cdots \sigma'_N} \sum_{\bar{\sigma}'_{r+1}, \ldots, \bar{\sigma}'_N} \epsilon_{\bar{\sigma}'_{r+1} \cdots \bar{\sigma}'_N}^{\sigma'_{r+1} \cdots \sigma'_N} \frac{1}{(L-r)!} \mathcal{G}_{\sigma'_{r+1} \bar{\sigma}'_{r+1} \cdots} \cdot \cdot \cdot \mathcal{G}_{\sigma'_N \bar{\sigma}'_N} \cdot \cdot \cdot e^{-i\bar{\phi}_{L+1} \bar{u}_{L+1} \cdots u_{N} } e^{i\bar{\phi}_{L+1} \bar{u}_{L+1} \cdots u_{N} } \det_L \hat{M} \quad (4.16)$$

for $L \geq r$ (where for $L = r$ the $\mathcal{G}$ factors, for $L = N$ the $u$ factors and for $L = \bar{N}$ the $\bar{u}$ factors are absent), while for $L < r$ the function vanishes. In $\mathcal{G} = \tilde{P} - \tilde{D}^{-1} \tilde{P}_+ = \tilde{P} - \tilde{D}^{-1} = \tilde{D}^{-1} \tilde{P}_+$ the operators $\tilde{D}$, $\tilde{P}_-$, $\tilde{P}_+$ are the restrictions of $D$, $P_-$, $P_+$, respectively, to the subspace on which $\mathds{1} - P_0^+ - P_0^-$ projects. It should be noticed that for given numbers of $\psi$ and $\bar{\psi}$ fields, the numbers of zero modes decide which types of contributions do occur.

The equivalence class to which the chosen pair of bases belongs is characterized by the value of

$$\sum_k N_k \Theta_k + \sum_{j \neq 0} N_j^+ \Theta_j^+ + \theta_z^+ - \theta_z^- , \quad (4.17)$$

where $\theta_z^+$ and $\theta_z^-$ are the phases related to the zero modes which we have introduced to keep (4.16) general. The introduction of phases suffices in this context since a general unitary matrix $S$ in $N$ dimensions with $\det S = e^{i\theta}$ can be expressed by the product of the matrix $e^{i\theta/N} \mathds{1}$ and of the unimodular matrix $Se^{-i\theta/N}$ which is irrelevant here.

In the absence of zero modes of $D$, where $\bar{N} = N$ and $I = 0$, the general form (4.16) simplifies to

$$\langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \cdots \bar{\psi}_{\bar{\sigma}_N} \rangle_t = \sum_{\sigma'_{r+1}, \ldots, \sigma'_N} \epsilon_{\sigma_{r+1} \cdots \sigma_N}^{\sigma'_{r+1} \cdots \sigma'_N} \mathcal{G}_{\sigma'_{r+1} \bar{\sigma}'_{r+1} \cdots} \cdot \cdot \cdot \mathcal{G}_{\sigma'_N \bar{\sigma}'_N} \cdot \cdot \cdot \det_N M \quad (4.18)$$

with $\mathcal{G} = P_- D^{-1} \tilde{P}_+$ and $\det_N M = \langle 1 \rangle_t$.

Note that $\epsilon_{j_1, \ldots, j_r} = 1, -1$ or 0 if $j_1, \ldots, j_r$ is an even, an odd or no permutation of $j_1, \ldots, j_r$, respectively, with the special case $\epsilon_{j_1, \ldots, j_N} \equiv \epsilon_{j_1, \ldots, j_N}^{1, \ldots, N}$.
5 Gauge transformations

5.1 Non-constant chiral projections

A gauge transformation \( D' = TDT^\dagger \) of the Dirac operator by (3.1), or by (3.18) and (3.17), implies the corresponding transformations

\[
P'_- = TP_- T^\dagger, \quad \bar{P}'_+ = T\bar{P}_+ T^\dagger
\]

of the chiral projections. We first consider the case where \([T, P_-] \neq 0\) and \([T, \bar{P}_+] \neq 0\), i.e. where \(G \neq 1\) and \(\bar{G} \neq 1\).

To get the behavior of the bases it is to be noted that the conditions (4.3) must be satisfied such that the relations (5.1) hold. It is obvious that given a solution \(u\) of the conditions (4.3), then \(Tu\) is a solution of the transformed conditions (4.3). All solutions are then obtained by performing basis transformations.

In addition (4.8) is to be satisfied, i.e. these considerations are to be restricted to an equivalence class of pairs of bases. Accordingly the original class \(uS\), \(\bar{u}\bar{S}\) and the transformed one \(u'S'\), \(\bar{u}'\bar{S}'\) are related by

\[
u'S' = TuS, \quad \bar{u}'\bar{S}' = T\bar{u}\bar{S}, \quad (5.2)
\]

where \(u, \bar{u}, S, \bar{S}\) satisfy (4.3) and (4.8), respectively, and \(u'\), \(\bar{u}'\), \(S'\), \(\bar{S}'\) their transformed versions. For full generality we have also introduced the unitary transformations \(S(T, U)\) and \(\bar{S}(T, U)\) obeying

\[
\det_w S(1, U)(\det_w \bar{S}(1, U))^* = 1, \quad (5.3)
\]

\[
\det_w \left( S(T_a, U)S(T_b, T_a U T_a^\dagger) \right) \left( \det_w \left( S(T_a, U)S(T_b, T_a U T_a^\dagger) \right) \right)^* = \det_w S(T_b T_a, T_b T_a U T_a^\dagger T_b^\dagger) \left( \det_w \bar{S}(T_b T_a, T_b T_a U T_a^\dagger T_b^\dagger) \right)^*, \quad (5.4)
\]

for which

\[
\det_w S \cdot \det_w \bar{S}^\dagger = e^{i\vartheta_T} \quad (5.5)
\]

with \(\vartheta_T \neq 0\) for \(T \neq 1\) is admitted. Obviously for \(\vartheta_T \neq 0\) (5.5) has just the form (4.9) corresponding to the transformation to an arbitrary inequivalent subset of pairs of bases, so that it ultimately cannot be tolerated. Such transformations are, on the other hand, also excluded by the covariance requirement for Lüscher's current, as will be shown in Section 7.

Inserting (5.2) into (4.3) we get for the correlation functions

\[
\langle \psi'_{\sigma_1} \ldots \psi'_{\sigma_R} \bar{\psi}'_{\bar{\sigma}_1} \ldots \bar{\psi}'_{\bar{\sigma}_R} \rangle_l = e^{i\vartheta_T} \sum_{\sigma_1, \ldots, \sigma_R} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_R} T_{\sigma_1 l} \cdots T_{\sigma_R l} \langle \psi_{\sigma_1} \ldots \psi_{\sigma_R} \bar{\psi}_{\bar{\sigma}_1} \ldots \bar{\psi}_{\bar{\sigma}_R} \rangle_l T_{\bar{\sigma}_1 l}^\dagger \cdots T_{\bar{\sigma}_R l}^\dagger, \quad (5.6)
\]

indicating that they transform gauge-covariantly for \(\vartheta_T = 0\).
5.2 One constant chiral projection

In the special case \( G \neq 1 \), \( \bar{G} = 1 \), where \( \bar{P}_+ \) is constant, the equivalence class of pairs of bases always contains members where \( \bar{P}_+ \) is represented by constant bases. Indeed, given the pair \( u, \bar{u} \), one can introduce a constant basis \( \bar{u}_c \) for which \( \bar{u} = \bar{u}_c \bar{S}_y \) holds. Then transforming \( u \) as \( u = u_y S_y \), where \( S_y \) is subject to \( \det_w S_y = \det_w \bar{S}_y \), according to (4.8) the pair \( u_y, \bar{u}_c \) is in the same equivalence class as the pair \( u, \bar{u} \).

For a transformed pair \( u', \bar{u}' \) we analogously get the equivalent pair \( u_y', \bar{u}_c \). Then instead of (5.2) we have

\[
\bar{u}_c = \mathcal{T} \bar{u}_c S_T
\]

where \( S_T \) is unitary. Insertion of (5.7) and (5.8) into (4.4) observing (4.8) gives again the form (5.6), however, with

\[
\det_w S_T^\dagger = \det_w (\bar{u}_c^\dagger T \bar{u}_c)
\]

where in detail

\[
\text{Tr} B = 4i \sum_{n, \ell} b_n^\ell \text{tr}_g T^\ell
\]

with constants \( b_n^\ell \) and group generators \( T^\ell \).

5.3 Perturbation theory

Since in continuum perturbation theory the anomaly cancelation condition is needed to get gauge invariance of the chiral determinant, it is to be checked whether this holds in the continuum limit for the lattice approach, too. A respective analysis has been presented in Ref. [3] of which we here briefly repeat some main points.

Putting \( M = M_0 + M_1 \) we get on the lattice the expansion

\[
\det_w M = \left( 1 + \sum_{\ell=1}^\infty z_\ell \right) \det_w M_0,
\]
implies for $P_n^\mu$ in which $T$ denotes transposition and where $W$ with the charge conjugation matrix

\[ C P \text{ transformations} \]

6 CP transformations

With the charge conjugation matrix$^3$ $C$ and with $P_{n'n} = \delta^{4}_{n'n}$, $U^{\text{CP}}_{4n} = U^{*}_{4n}$ and $U^{\text{CP}}_{kn} = U^{*}_{k,\bar{n} - k}$ for $k = 1, 2, 3$, where $\bar{n} = (-\bar{n}, n_4)$, we have

\[ D(U^{\text{CP}}) = WD^T(U)W^\dagger, \quad W = P\gamma_4C^\dagger, \]  

(6.1)

in which $T$ denotes transposition and where $W^\dagger = W^{-1}$. The behavior of $D$ by (3.1) implies for $P_-$ and $\bar{P}_+$ the relations

\[ P_+^{\text{CP}}(U^{\text{CP}}) = WP_+^{\text{T}}(U)W^\dagger, \quad \bar{P}_+^{\text{CP}}(U^{\text{CP}}) = WP_-^{\text{T}}(U)W^\dagger. \]  

(6.2)

$^3C$ satisfies $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ and $C^T = -C$. Using Hermitian $\gamma$-matrices with $\gamma^T_\mu = (-1)^\mu\gamma_\mu$ for $\mu = 1, \ldots, 4$ we choose $C = \gamma_2\gamma_4$ and get $\gamma_5^T = \gamma_5$ and $[\gamma_5, C] = 0$ for $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$.  

\[ z_\ell = \sum_{r=1}^\ell \frac{(-1)^\ell + r}{r!} \sum_{\rho_1=1}^{\ell-r+1} \cdots \sum_{\rho_r=1}^{\ell-r+1} \delta_{\ell, \rho_1 + \cdots + \rho_r} \frac{t_{\rho_1}}{\rho_1} \cdots \frac{t_{\rho_r}}{\rho_r}, \]  

(5.13)

\[ t_\rho = \text{Tr}((D_0^{-1}\mathcal{M})^\rho), \quad \mathcal{M} = \bar{u}_0M_1u_0^\dagger, \]  

(5.14)

with fermion loops $t_\rho$, free propagators $D_0^{-1}$ and vertices $\mathcal{M}$. With $D = D_0 + D_1$, $u = u_0 + u_1$ and $\bar{u} = \bar{u}_0 + \bar{u}_1$ the vertices decompose as

\[ \mathcal{M} = \bar{P}_+D_1P_- + \bar{u}_0\bar{u}_1^\dagger Du_1u_0^\dagger + \bar{u}_0\bar{u}_1^\dagger DP_- + \bar{P}_+Du_1u_0^\dagger. \]  

(5.15)

In the detailed discussion of the limit the survival of terms only at zero and at the corners of the Brillouin zone plays a central rôle. It turns out that in the limit $\bar{P}_+$ and $P_-$ of the first term on the r.h.s. of (5.15) can be replaced by $\frac{1}{2}(1 + \gamma_5)$ and $\frac{1}{2}(1 - \gamma_5)$, respectively. The other terms relying on $u_1$ and $\bar{u}_1$ are found to vanish because the related projections get constant.

Since in the limit the terms vanish, the compensating effect of which on the finite lattice provides gauge invariance of the chiral determinant, in any case this invariance gets lost. Furthermore, then obviously also the particular cases with one constant chiral projection are no longer distinct.

For the surviving contributions the agreement with usual perturbation theory is obvious at lower order. Considering higher orders not all Dirac operators can provide the appropriate results, as an example in Ref. [20] shows. Since the operator of this example is non-local, it can be expected that with the locality imposed in (2.10) the usual expansion is reproduced to any order, a proof of which remains, however, to be given.

For appropriate Dirac operators in the limit arriving at the usual structure of the expansion, clearly the anomaly cancelation condition is needed in order that a gauge-invariant continuum limit can exist.
Using \( I = \text{Tr} \, \bar{P}_+ - \text{Tr} \, P_- \) one gets \( I^{\text{CP}} = -I \) for the index.

To see more details we consider the form (3.17),

\[
\bar{P}_+(U) = \frac{1}{2}(1 + G(U)\gamma_5), \quad P_-(U) = \frac{1}{2}(1 - \gamma_5 G(U)),
\]

which inserted into (6.2) using \( \{\gamma_5, W\} = 0 \) gives

\[
P_{-\text{CP}}(U^{\text{CP}}) = \frac{1}{2}(1 - \gamma_5 W\bar{G}^T(U)W), \quad \bar{P}_{+\text{CP}}(U^{\text{CP}}) = \frac{1}{2}(1 + W\bar{G}^T(U)W^\dagger \gamma_5).
\]

From (6.1) by (3.18) one gets \( W\bar{G}^T(U)W^\dagger = \bar{G}(U^{\text{CP}}) \) and \( W\bar{G}^T(U)W = G(U^{\text{CP}}) \), so that (6.4) becomes

\[
P_{-\text{CP}}(U^{\text{CP}}) = \frac{1}{2}(1 - \gamma_5 \bar{G}(U^{\text{CP}})), \quad \bar{P}_{+\text{CP}}(U^{\text{CP}}) = \frac{1}{2}(1 + G(U^{\text{CP}})\gamma_5).
\]

Obviously this differs from the untransformed relation (6.3) by an interchange of \( G \) and \( \bar{G} \). Because generally \( G \neq \bar{G} \) holds, as we have shown in Section 3, one cannot get the symmetric situation of continuum theory.

In the discussion of CP properties in Ref. [18], introducing the special form (3.23) in the GW case, it has been noted that this form gets singular for \( s = \frac{1}{2} \) so that the symmetric situation cannot be obtained. In the investigations of CP properties in Ref. [9], using the form (3.23) together with some more general \( D \), a singularity has been encountered if a symmetric situation has been enforced. In view of our general result that always \( \bar{G} \neq G \) this does not come as a surprise. The interchange of parameters under CP transformations in Ref. [9] corresponds to the interchange of \( G \) and \( \bar{G} \) in the general case here.

With the basic conditions (4.3) and (4.8) being satisfied by \( u, \bar{u}, S, \bar{S} \) as well as by \( u^{\text{CP}}, \bar{u}^{\text{CP}}, S^{\text{CP}}, \bar{S}^{\text{CP}} \), the equivalence classes of pairs of bases transform as

\[
u^{\text{CP}} S^{\text{CP}} = \mathcal{W}^* u^* \bar{S}^* S^{\text{CP}}, \quad \bar{u}^{\text{CP}} S^{\text{CP}} = \mathcal{W}^* u^* S^{\text{CP}} \bar{S}^{\text{CP}},
\]

where the additional unitary operators \( S_\xi \) and \( \bar{S}_\xi \) have been introduced for full generality. Inserting (6.6) into (6.1) gives for the correlation functions

\[
\langle \psi_{\sigma_1}^{\text{CP}} \ldots \psi_{\sigma_R}^{\text{CP}} \bar{\psi}_{\sigma_1}^{\text{CP}} \ldots \bar{\psi}_{\sigma_R}^{\text{CP}} \rangle^{\text{CP}} =
\]

\[
e^{i\theta_{\text{CP}}} \sum_{\sigma_1, \ldots, \sigma_R} \sum_{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_R} \mathcal{W}_{\sigma_1, \tilde{\sigma}_1}^{\dagger} \ldots \mathcal{W}_{\sigma_R, \tilde{\sigma}_R}^{\dagger} \langle \psi_{\sigma_1} \ldots \psi_{\sigma_R} \bar{\psi}_{\tilde{\sigma}_1} \ldots \bar{\psi}_{\tilde{\sigma}_R} \rangle \mathcal{W}_{\tilde{\sigma}_1, \sigma_1} \ldots \mathcal{W}_{\tilde{\sigma}_R, \sigma_R}. \tag{6.7}
\]

where

\[
e^{i\theta_{\text{CP}}} = \det_{\omega} S_\xi \cdot \det_{\omega} \bar{S}_{\xi}^\dagger. \tag{6.8}
\]

This factor is subject to the condition that repetition of the transformation must lead back, which is satisfied by restricting \( S_\xi \) and \( \bar{S}_{\xi} \) to choices for which \( \theta_{\text{CP}} \) is a universal constant. Then the factor \( e^{i\theta_{\text{CP}}} \) gets irrelevant in full correlation functions so that, without restricting generality, one may put \( \theta_{\text{CP}} = 0 \).

The discussion in Ref. [9] has been based on a generating functional the content of which is similar to the respective special case of (4.16). It does not account for the restrictions due to the number of zero modes explicit in (4.4). The non-unimodular transformation applied to it is not appropriate [3]. Instead of (6.6) a respective relation without the basis transformations has been used.
7 Variational approach

7.1 General relations

We define general gauge-field variations for a function \( \phi(U) \) by

\[
\delta \phi(U) = \frac{d \phi(U(t))}{dt} \bigg|_{t=0}, \quad U_\mu(t) = e^{iB_\mu(t) \cdot U_\mu(t)^\dagger},
\]

(7.1)

where \((U_\mu)_{n'n} = U_{\mu n'} \delta_{n'n+\hat{\mu}}\) and \((B_\mu \left/ \right. \mu_{n'n} \delta_{n'n} = B_{\mu n'} \delta_{n'n}^4\). The special case of gauge transformations is then described by

\[
B_\mu = B_\mu = B.
\]

(7.2)

To see the consequence of the general condition (4.8) we vary its logarithm which gives

\[
\text{Tr}(\delta S) - \text{Tr}(\delta \bar{S}) = 0.
\]

(7.3)

Instead of \(\det w S \cdot \det \bar{w} \bar{S}^\dagger = 1\), as needed for reactions with more than one contribution, (7.3) reflects the weaker condition \(\det w S \cdot \det \bar{w} \bar{S}^\dagger = \text{const}\). Relation (7.3) can also be expressed in terms of bases as

\[
\text{Tr}(\delta(uS)(uS)^\dagger) - \text{Tr}(\delta(\bar{u}\bar{S})(\bar{u}\bar{S})^\dagger) = \text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger),
\]

(7.4)

which indicates that \(\text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger)\) remains invariant within the extended equivalence class of pairs of bases specified by \(\det w S \cdot \det \bar{w} \bar{S}^\dagger = \text{const}\).

Applying variations to the basic conditions (4.3) one can derive many relations. All of such relations are weaker conditions than the original ones. A particular example, considered in Ref. [2], is the relation

\[
\text{Tr}(P_-[\delta_1 P_-, \delta_2 P_-]) = \delta_1 \text{Tr}(\delta_2 u u^\dagger) - \delta_2 \text{Tr}(\delta_1 u u^\dagger) + \text{Tr}(\delta [2,1] u u^\dagger),
\]

(7.5)

for which we have generators \(B_{\mu(1)}^{\text{left}}, B_{\mu(1)}^{\text{right}}\) and \(B_{\mu(2)}^{\text{left}}, B_{\mu(2)}^{\text{right}}\) and \([B_{\mu(1)}^{\text{left}}, B_{\mu(1)}^{\text{right}}], [B_{\mu(2)}^{\text{left}}, B_{\mu(1)}^{\text{left}}], [B_{\mu(2)}^{\text{right}}, B_{\mu(1)}^{\text{left}}]\). We emphasize that (7.5) follows solely from \(P_- = uu^\dagger, u^\dagger u = \mathbb{I}_w\) and is not subject to further restrictions.

In the special case where zero modes of \(D\) are absent and where its index is zero one can consider the effective action and obtains for its variation

\[
\delta \ln \det w w M = \text{Tr}(P_- D^{-1} \delta D) + \text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger).
\]

(7.6)

Because of (7.4) this is invariant within the respective extended equivalence class of pairs of bases. It is to be noted that in the presence of zero modes no longer only variational terms of type (7.4) occur for the bases, as is obvious from (4.16).
7.2 Gauge transformations

In the special case of gauge transformations we can use the definition (7.1) and the finite transformation relations to get the related variations. For operators with \( O(U(t)) = T(t) O(U(0)) T^\dagger(t) \) and \( T(t) = e^{iB} \) this gives

\[
\delta^g O = [B, O].
\]  

(7.7)

For the bases in the case \([T, P_-] \neq 0, [T, \bar{P}^+] \neq 0\) according to (5.2) we have \( u(t) = T(t) u(0) S_x(t), \bar{u}(t) = T(t) \bar{u}(0) \bar{S}_x(t) \) where \( S_x = SS^\dagger, \bar{S}_x = \bar{S}\bar{S}^\dagger \) and obtain

\[
\delta^g u = B u + u S_x^\dagger \delta^g S_x, \quad \delta^g \bar{u} = \bar{B} \bar{u} + \bar{u} \bar{S}_x^\dagger \delta^g \bar{S}_x.
\]  

(7.8)

With these relations the terms in the effective action become

\[
\text{Tr}(P_- D^{-1} \delta^g D) = \text{Tr}(B \bar{P}^+) - \text{Tr}(B P_-),
\]  

(7.9)

\[
\text{Tr}(\delta^g u u^\dagger) = \text{Tr}(B \bar{P}^+) + \text{Tr}_w(S_x^\dagger \delta^g S_x), \quad \text{Tr}(\delta^g \bar{u} \bar{u}^\dagger) = \text{Tr}(B P^+) + \text{Tr}_w(\bar{S}_x^\dagger \delta^g \bar{S}_x),
\]  

(7.10)

so that using (7.3) we obtain

\[
\delta^g \ln \det_{ww} M = \text{Tr}_w(S^\dagger \delta^g S) - \text{Tr}_w(\bar{S}_x^\dagger \delta^g \bar{S}_x).
\]  

(7.11)

7.3 Special case of Lüscher

Lüscher \cite{Luscher:1983gl} considers the variation of the effective action and assumes \( \delta \bar{P}^+ = 0 \) and \( \delta \bar{u} = 0 \) so that the last term in (7.10) is absent and condition (7.3) reduces to \( \text{Tr}(S^\dagger \delta^g S) = 0 \). With the term \( \text{Tr}(\delta u u^\dagger) \) he defines a current \( j_{\mu n} \) by

\[
\text{Tr}(\delta u u^\dagger) = -i \sum_{\mu, n} \text{tr}_g(\eta_{\mu n} j_{\mu n}), \quad \delta U_{\mu n} = \eta_{\mu n} U_{\mu n},
\]  

(7.14)
and requires it to transform gauge-covariantly.

His generator is given by $\eta_{\mu n} = B_{\mu n, \nu} - U_{\mu n} B_{\nu}^\dagger U_{\mu n}$ in terms of our left and right generators. We get explicitly

$$j_{\mu n} = i(U_{\mu n} \rho_{\mu n} + \rho_{\mu n}^J U_{\mu n}^\dagger), \quad \rho_{\mu n, \alpha' \alpha} = \sum_{j, \sigma} u_{\sigma}^J \frac{\partial u_{\sigma j}}{\partial U_{\mu n, \alpha' \alpha}}. \quad (7.15)$$

The requirement of gauge-covariance $j'_{\mu n} = e^{B_{\mu + \hat{\hat{\mu}}} j_{\mu n} e^{-B_{\mu + \hat{\hat{\mu}}}}$ because of $U_{\mu n}^J = e^{B_{\mu + \hat{\hat{\mu}}} U_{\mu n} e^{-B_{\mu}}}$ implies that one must have $$\rho'_{\mu n} = e^{B_{n}} \rho_{\mu n} e^{-B_{n}}, \quad (7.16)$$

which with $u' = T u S \tilde{S} S'$, $\text{Tr}_w(S^\dagger \delta S) = 0$ and $\text{Tr}_w(S^\dagger \delta S') = 0$ leads to the condition

$$\sum_{j, k} S_{kj} \frac{\partial S_{jk}}{\partial U_{\mu n, \alpha' \alpha}} = 0. \quad (7.17)$$

From this and $S^{-1} = S^\dagger$ it follows that

$$\text{Tr}(S^\dagger \delta^2 S) = 0. \quad (7.18)$$

Thus with (7.13) in the special case considered by L"uscher one obtains the definite result

$$\delta^2 \ln \det_\omega M = \frac{1}{2} \text{Tr}(\gamma_5 B), \quad (7.19)$$

which leaves no freedom for changing gauge-transformation properties by a particular construction.

Relation (7.18) shows that a transformation to an inequivalent subset of bases is also excluded by the covariance requirement for L"uscher’s current. This extends to the case where both chiral projections are non-constant, too, since also introducing a current $\tilde{j}_{\mu n}$ related to $\tilde{u}$ the covariance of $j_{\mu n} - \tilde{j}_{\mu n}$ leads to $\text{Tr}(S^\dagger \delta^2 S) - \text{Tr}(\tilde{S}^\dagger \delta^2 \tilde{S}) = 0$.

8 Conclusions

To make progress with the non-perturbative definition of quantized chiral gauge theories we have generalized previous formulations and investigated which features are truly relevant and which properties are indeed there.

Starting with relations for the Dirac operators we have removed the restriction to one real eigenvalue in addition to zero and similar restrictions on the complex eigenvalues, which have been inherent in all analytical forms so far. A discussion of the locations of the spectra has illustrated the respective new possibilities. The generalization of the unitary and $\gamma_5$-Hermition operator of previous formulations has turned out to be again connected to the index.

The more general Dirac operators have been seen to have still realizations with appropriate locality properties. For their numerical evaluation an extension of the method of
chirally improved fermions is suitable. The additional freedom of these operators could possibly be advantageous for numerical work.

We have derived the properties of the chiral projections for given Dirac operator using the spectral representations of the operators and carefully considering the requirements related to the Weyl degrees of freedom. It has turned out that there are considerable possibilities for their structure. Nevertheless generally definite relations between Weyl and anti-Weyl projections and a decomposition into subspaces revealing the special rôle of zero modes have been found.

Expressing the chiral projections in an alternative form it has become obvious that the symmetry between the Weyl and anti-Weyl cases known in continuum theory can generally not be there. This, in particular, affects the behavior under CP transformations. Using the alternative form it has also been seen that there are appropriate realizations of the more general chiral projections.

We have further pointed out that the operators occurring in the alternative form of the chiral projections on the other hand provide generalized chiral symmetries of the Dirac operator. Thus there is generally a whole family of such symmetries. Furthermore accordingly the respective left and right transformations are generally different.

We have considered fermionic correlation functions in terms of alternating multilinear forms in order that our results also apply in the presence of zero modes of the Dirac operator and for any value of the index. The requirement of invariance of these functions imposes restrictions on possible basis transformations. To account properly for this we have introduced the concept of the decomposition into equivalence classes of pairs of Weyl and anti-Weyl bases.

The indicated concept of pairs not only exploits the respective freedom fully but is also natural in view of the relations between the Weyl and anti-Weyl projections which we find in our analysis and which imply corresponding relations for the bases. We have stressed that to describe physics one of the equivalence classes of pairs of bases is to be chosen and that the questions arises which choice is appropriate.

The relations between the Weyl and anti-Weyl bases together with the decomposition into subspaces we have found has allowed us to obtain a further form of the correlation functions which applies also in the presence of zero modes and for any value of the index. It involves a determinant and separate zero mode terms and has the virtue that the contributions of particular amplitudes become explicit also in the general case considered.

We have given a completely unambiguous derivation of the gauge-transformation properties of the correlation functions. In the case where both of the chiral projections are gauge-field dependent the exclusion of switching to an arbitrary inequivalent subset of pairs of bases leads to gauge covariance. In the cases where one of the chiral projections is constant a factor depending on the particular gauge transformation remains.

It has been noted that switching to an arbitrary inequivalent subset is also excluded by the covariance requirement for Lüscher’s current. Thus obviously gauge-transformation properties on the finite lattice are fully determined. It has been emphasized that in the continuum limit one nevertheless arrives at the usual situation where the anomaly
cancelation condition is needed for gauge invariance of the chiral determinant, pointing out that in the limit the compensating effects of the bases are no longer there.

We have similarly given an unambiguous derivation of CP-transformation properties of the correlations functions. It has been seen that also for the more general chiral projections one cannot get the symmetric situation with respect to CP transformations known in continuum theory.

Finally we have considered some issues of interest also in terms of gauge-field variations. After making certain relations precise we have turned to the variation of the effective action. We have then shown that requiring gauge covariance of Lüscher’s current prevents switching to inequivalent subsets of pairs of bases. Since thus there is no freedom for changing gauge-transformation properties by a particular construction, the respective efforts in literature actually cannot work.

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References

[1] R. Narayanan, H. Neuberger, Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B412 (1994) 574; Nucl. Phys. B443 (1995) 305.

[2] M. Lüscher, Nucl. Phys. B549 (1999) 295; Nucl. Phys. B568 (2000) 162.

[3] W. Kerler, Nucl. Phys. B680 (2004) 51.

[4] W. Kerler, Nucl. Phys. B646 (2002) 201.

[5] P. Hasenfratz, V. Laliena, F. Niedermayer, Phys. Lett. B427 (1998) 125.

[6] M. Lüscher, Phys. Lett. B428 (1998) 342.

[7] P.H. Ginsparg, K.G. Wilson, Phys. Rev. D25 (1982) 2649.

[8] K. Fujikawa, Nucl. Phys. B589 (2000) 487.

[9] K. Fujikawa, M. Ishibashi, H. Suzuki, Phys. Lett. B538 (2002) 197; JHEP 0204 (2002) 046.

[10] H. Neuberger, Phys. Lett. B417 (1998) 141; Phys. Lett. B427 (1998) 353.

[11] T.-W. Chiu, Phys. Lett. B521 (2001) 429.

[12] T.-W. Chiu, Phys. Rev. D58 (1998) 074511.
[13] W. Kerler, JHEP 0210 (2002) 019.

[14] W. Kerler, Phys. Lett. B510 (2001) 325.

[15] C. Gattringer, Phys. Rev. D63 (2001) 114501; C. Gattringer, I. Hip, C.B. Lang, Nucl. Phys. B597 (2001) 451.

[16] M.F. Atiyah, I.M. Singer, Ann. of Math. 87 (1968) 546, Section 5.

[17] For more details see the recent review: W. Kerler, Int. J. Mod. Phys. A18 (2003) 2565.

[18] P. Hasenfratz, Nucl. Phys. (Proc. Suppl.) 106 (2002) 159.

[19] H. Suzuki, JHEP 0010 (2000) 039.

[20] L.H. Karsten, J. Smit, Nucl. Phys. B144 (1978) 536.