Remote preparation of arbitrary ensembles and quantum bit commitment

Hans Halvorson
Department of Philosophy, Princeton University
hhalvors@princeton.edu

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Abstract

The Hughston-Jozsa-Wootters theorem shows that any finite ensemble of quantum states can be prepared “at a distance”, and it has been used to demonstrate the insecurity of all bit commitment protocols based on finite quantum systems without superselection rules. In this paper, we prove a generalized HJW theorem for arbitrary ensembles of states on a $C^*$-algebra. We then use this result to demonstrate the insecurity of bit commitment protocols based on infinite quantum systems, and quantum systems with Abelian superselection rules.

I. Introduction

Recent work in quantum cryptography has focused on questions of which sorts of information-transfer protocols are secure from attempts at cheating by an intruder or by one of the participants. As early as 1984, the question was raised whether quantum theory would permit a secure bit commitment protocol — i.e., a protocol in which a bit of information is committed by one party Alice to another party Bob, such that Alice cannot change her commitment, and such that Bob cannot determine Alice’s commitment until given further information by Alice. An initial protocol using pairs of polarized photons was proposed by Bennett and Brassard [2]; however, Bennett and Brassard showed that this protocol can be cheated by exploiting the nonlocal correlations of the EPR-Bohm state.
A number of other quantum bit commitment protocols have been proposed in the intervening years (see \[4, 6\] for reviews). Most of these protocols rely on the fact that a non-pure density operator corresponds to more than one ensemble of quantum states. In particular, two different ensembles on a composite system can induce the same density operator on a local system. Thus, if Alice encodes her bits into these two ensembles, then Bob cannot possibly determine Alice’s commitment until she provides further information about the composite system.

However, Lo and Chau \[15\] and Mayers \[16, 17\] show that, as a consequence of the Hughston-Jozsa-Wootters theorem, if a bit commitment protocol is concealing against Bob, then it is not binding against Alice. (Kent’s \[11\] relativistic bit commitment protocol does not rely on the existence of alternative decompositions of a density operator, and so its security is not challenged by the Mayers-Lo-Chau result.) That is, if the ensembles prepared in the protocol are indistinguishable to Bob (i.e., correspond to approximately the same local density operator), then Alice can “steer” between these ensembles after the Commit stage of the protocol.

**HJW Theorem (\[10\]).** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces, let $\{D_i\}_{i=1}^n$ be density operators on $\mathcal{H}_B$, and let $x$ be a unit vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\text{Tr}_A(P_x) = \sum_{i=1}^n \lambda_i D_i$. Then there are positive operators $\{A_i\}_{i=1}^n$ in $\mathcal{B}(\mathcal{H}_A) \otimes I$ such that

$$\left\langle A_i^{1/2}x \middle| BA_i^{1/2}x \right\rangle = \lambda_i \text{Tr}(D_i B),$$

for all $B \in I \otimes \mathcal{B}(\mathcal{H}_B)$.

Thus, the HJW theorem shows that any finite decomposition of $\text{Tr}_A(P_x)$ can be prepared from the state $P_x$ by a measurement operation on $\mathcal{H}_A$. However, all non-pure density operators have countably infinite convex decompositions, as well as uncountably infinite integral decompositions. For the case of countably infinite decompositions, Cassinelli et al. \[7\] have proved a generalized HJW theorem; but their results do not cover the case of integral decompositions. What is more, the HJW theorem and its generalization by Cassinelli et al. apply only to a very narrow class of quantum systems — namely those whose observables are represented by type I von Neumann factors. Thus, these results do not directly establish the insecurity of bit commitment protocols that employ systems with non-trivial superselection rules (represented by direct sums of type I von Neumann factors), or bit
commitment protocols that employ infinite quantum systems (represented by type II or type III von Neumann algebras).

In this paper, we prove a generalized HJW theorem for arbitrary ensembles of states on a $C^*$-algebra. We show first (in Section II) that each measure on the state space of a $C^*$-algebra $B$ gives rise to a positive-operator valued measure with range in the commutant $B'$ of $B$. (This first result is completely general, and does not impose any restrictions on the $C^*$-algebra $B$.) We then show that when $B'$ is a hyperfinite von Neumann algebra, there is a completely positive instrument that prepares the relevant ensemble of states on $B$. In Section III we apply our results to the question of the security of bit commitment protocols.

II. Generalized HJW theorem

Our first result (Theorem 1) shows that for any $C^*$-algebra $B$ of operators acting on a Hilbert space $H$, a measure on the state space of $B$ gives rise to a corresponding POV measure with values in the commutant $B' = \{ A \in B(H) : [A, B] = 0 \text{ for all } B \in B \}$. For the case that $B = I \otimes M_n$, where $M_n$ is the $C^*$-algebra of $n \times n$ matrices over $\mathbb{C}$, our result yields an alternate proof of the original HJW theorem.

Let $K$ denote the compact convex set of states of $B$ with the weak* topology. [A net $\{ \omega_a \}_{a \in A}$ of states of $B$ converges in the weak* topology to a state $\omega$ just in case, for each $B \in B$, $\lim_a \omega_a(B) = \omega(B)$. If $B = M_n$, then the weak* topology on states is equivalent to the standard topology on density operators.] In this paper, we consider positive regular measures on $(K, \Sigma)$, where $\Sigma$ is the Borel $\sigma$-algebra of $K$. When we say that $\mu$ is a measure, it can be assumed that $\mu$ is positive and regular.

**Definition ([11, p. 12]).** If $\mu$ is a measure on the state space $K$, then the state

$$\rho_\mu = \mu(K)^{-1} \int x \, d\mu(x), \quad (2)$$

is called the barycenter of $\mu$. Measures $\mu$ and $\nu$ on $K$ are said to be equivalent if they have the same barycenter.

Let $K$ be the convex set of density operators on $\mathbb{C}^n$. If $\rho$ is a density operator and $\mu$ is a finitely supported measure on $K$ with barycenter $\rho$, then Hughston et al. call $\mu$ a $\rho$-ensemble. So, the set of $\rho$-ensembles consists of
those measures on $K$ that have barycenter $\rho$, and that are supported on a finite set. In this paper, we consider all measures with barycenter $\rho$, and not just those with finite support.

**Notation.** If $x$ is a vector in $\mathcal{H}$, we let $\omega_x(A) = \langle x \mid Ax \rangle$, for all $A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{B}$ is a set of operators on $\mathcal{H}$, we let $\mathcal{B}x = \{ Bx : B \in \mathcal{B} \}$, and we let $[\mathcal{B}x]$ denote the closed linear span of $\mathcal{B}x$.

**Lemma 1 ([13, Prop. 7.3.5]).** If $\mathcal{B}$ is a $C^*$-algebra of operators acting on the Hilbert space $\mathcal{H}$ and $\rho$ is a positive linear functional on $\mathcal{B}$ such that $\rho \leq \omega_x|_{[\mathcal{B}]}$ for some vector $x$ in $\mathcal{H}$, then there is a positive operator $H$ in the unit ball of $\mathcal{B}'$ such that $\rho(A) = \omega_x(HA) = \omega_{H^{1/2}}(A)$, for all $A \in \mathcal{B}$.

**Proof.** Define a conjugate-bilinear functional $\varphi$ on $\mathcal{B}x$ by setting $\varphi(Ax,Bx) = \rho(A^*B)$. Then,

$$|\varphi(Ax,Bx)|^2 = \rho(A^*B)^2 \leq \rho(A^*A)\rho(B^*B) \leq \|Ax\|^2\|Bx\|^2.$$

The first inequality follows from the Cauchy-Schwartz inequality for the inner product $\langle A \mid B \rangle_{\rho} = \rho(A^*B)$ on $\mathcal{B}$. Thus $\varphi$ is positive and bounded by 1. It follows that $\varphi$ has a unique extension to the subspace $[\mathcal{B}x]$. Moreover, the Riesz representation theorem entails that there is a positive operator $H$ on $[\mathcal{B}x]$ such that $\|H\| \leq 1$ and $\varphi(Ax,Bx) = \langle Ax \mid HBx \rangle$. In particular, $\rho(A) = \langle x \mid HAx \rangle = \omega_x(HA)$ for all $A \in \mathcal{B}$. Extend $H$ to the entire Hilbert space $\mathcal{H}$ by setting it to zero on $\mathcal{H} \ominus [\mathcal{B}x]$. Since

$$\langle Ax \mid HCBx \rangle = \rho(A^*CB) = \rho((C^*A)^*B)$$

$$\quad = \langle C^*Ax \mid H Bx \rangle = \langle Ax \mid CHBx \rangle$$

for all $C$ in $\mathcal{B}$, it follows that $[H,C] = 0$ on $[\mathcal{B}x]$. Since $[H,C] = 0$ on $\mathcal{H} \ominus [\mathcal{B}x]$, it follows that $[H,C] = 0$ on the entire Hilbert space. Therefore, $H \in \mathcal{B}'$. \hfill $\square$

The following result is a special case of a theorem proved by Tomita [25] in 1956 (compare with [5, Lemma 4.1.21, Prop. 4.1.22]).

**Theorem 1.** Let $\mathcal{B}$ be a $C^*$-algebra acting on the Hilbert space $\mathcal{H}$, and let $\mu$ be a probability measure on the state space of $\mathcal{B}$. If there is a unit vector $x$ in $\mathcal{H}$ such that $\omega_x|_{\mathcal{B}}$ is the barycenter of $\mu$, then there is a POV measure $A$ with range in $\mathcal{B}'$ such that

$$\langle A(S)^{1/2}x \mid BA(S)^{1/2}x \rangle = \int_S \omega(B)d\mu(\omega),$$

where $S$ is a compact subset of $\mathcal{B}'$. The result can be extended to the case where $\mu$ is not a probability measure by allowing for the possibility of a finite measure with a suitable definition of the barycenter.\hfill $\square$
for all $S \in \Sigma$ and $B \in \mathcal{B}$.

**Proof.** Let $S$ be a Borel subset of the state space of $\mathcal{B}$, and let $\rho_S = \int_S \omega d\mu(\omega)$. Then $\rho_S$ is a positive linear functional on $\mathcal{B}$ with $\rho_S \leq \omega_x|_B$. By Lemma 2 there is a positive operator $A(S)$ in the unit ball of $\mathcal{B}'$ such that $\rho_S(B) = \omega_x(A(S)B)$ for all $B \in \mathcal{B}$, and $A(S) = 0$ on $\mathcal{H} \ominus \{Bx\}$. In order to verify that $S \mapsto A(S)$ is countably additive, suppose that $\{S_i : i \in \mathbb{N}\}$ are disjoint Borel subsets, and let $S = \bigcup_{i=1}^{\infty} S_n$. Then for fixed $B \in \mathcal{B}$,

$$\sum_{i=1}^{\infty} \chi_{S_i}(\omega) \cdot \omega(B) = \chi_S(\omega) \cdot \omega(B), \quad (7)$$

and so the monotone convergence theorem entails that

$$\sum_{i=1}^{\infty} \left( \int_{S_i} \omega(B) d\mu(\omega) \right) = \int_S \omega(B) d\mu(\omega) = \langle x | A(S)Bx \rangle. \quad (8)$$

Furthermore, countable additivity of the map $Z \mapsto \langle x | ZBx \rangle$ entails that

$$\left\langle x \left| \left( \sum_{i=1}^{\infty} A(S_i) \right) Bx \right\rangle = \sum_{i=1}^{\infty} \langle x | A(S_i)Bx \rangle. \quad (9)$$

Replacing $B$ with $B^*C$, where $B, C \in \mathcal{B}$, we have

$$\left\langle Bx \left| \sum_{i=1}^{\infty} A(S_i)Cx \right\rangle = \langle Bx | A(S)Cx \rangle, \quad (10)$$

and therefore $(\sum_{i=1}^{\infty} A(S_i))y = A(S)y$, for all $y \in \{Bx\}$. Since $A(S) = 0$ on $\mathcal{H} \ominus \{Bx\}$, it follows that $\sum_{i=1}^{\infty} A(S_i) = A(S)$. \hfill $\square$

Thus, if $\mu$ is a measure on the state space of Bob’s algebra $\mathcal{B}$, Alice’s algebra $\mathcal{A} = \mathcal{B}'$ contains the range of a POV measure $A$ satisfying Eqn. 6. But this does not yet yield the conclusion that Alice can prepare the ensemble $\mu$ on Bob’s system — for that, we need to show that Alice has an “instrument” corresponding to the POV measure $A$.

**Definition** ([9, 19]). Let $(X, \Sigma)$ be a Borel space. A **completely positive (CP) instrument** on $\mathcal{B}(\mathcal{H})$ is a map $\mathcal{E} : \Sigma \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that:
1. For fixed $B \in \mathcal{B}(\mathcal{H})$, $\mathcal{E}[\cdot](B)$ is $\sigma$-additive in the weak-operator topology;

2. For fixed $S \in \Sigma$, $\mathcal{E}[S](\cdot)$ is a completely positive linear map such that $\mathcal{E}[S](I) \leq I$.

Since the map $\mathcal{E}[S](\cdot)$, ($S \in \Sigma$), is positive, it is automatically norm-continuous. If, in addition, each such map is weak-operator continuous on bounded sets, then $\mathcal{E}$ is said to be normal. [A net $\{A_a\}_{a \in A}$ of bounded operators on $\mathcal{H}$ converges in the weak-operator topology to an operator $A$ just in case $\lim_{a} \langle x | A_a y \rangle = \langle x | A y \rangle$ for all vectors $x, y$ in $\mathcal{H}$.] However, we do not require instruments to be normal, because continuous PV measures give rise to non-normal instruments [9, 23], and a continuous ensemble of states on $\mathcal{B}$ will give rise to a continuous PV measure in $\mathcal{B}'$.

Each instrument $\mathcal{E}$ determines a unique POV measure $A$ via the formula

$$A(S) \equiv \mathcal{E}[S](I), \quad (S \in \Sigma).$$

(11)

If Eqn. 11 holds for an instrument $\mathcal{E}$ and a POV measure $A$, then $\mathcal{E}$ and $A$ are said to be compatible. For any given POV measure $A$, there are many instruments that are compatible with $A$. In fact, if $\Phi$ is a CP projection of $\mathcal{B}(\mathcal{H})$ with $\text{ran}(\Phi) \subseteq \text{ran}(A)'$ then

$$\mathcal{E}[S](B) = A(S)\Phi(B), \quad (S \in \Sigma, B \in \mathcal{B}(\mathcal{H})),$$

(12)

is compatible with $A$. In particular, if $\rho$ is a state on $\mathcal{B}(\mathcal{H})$ then

$$\mathcal{E}[S](B) = A(S)\rho(B), \quad (S \in \Sigma, B \in \mathcal{B}(\mathcal{H})),$$

(13)

is compatible with $A$.

Thus, given a POV measure $A$ with range in Alice’s algebra $\mathcal{A} = \mathcal{B}'$, we can easily find an instrument $\mathcal{E}$ that is compatible with $A$. However, it does not follow that Alice can in any sense measure $A$ with the instrument $\mathcal{E}$, because $\mathcal{E}$ may not be “local” to Alice’s system. In particular, an instrument that is local to Alice’s system should not disturb the statistics of measurements of observables in Bob’s algebra $\mathcal{B} = \mathcal{A}'$. In other words, for any state $\rho$ on $\mathcal{B}$, the equation

$$\rho(\mathcal{E}[X](B)) = \rho(B),$$

(14)

should hold for all $B \in \mathcal{B}$. But Eqn. 11 holds for all states $\rho$ on $\mathcal{B}$ iff the CP map $\mathcal{E}[X](\cdot)$ is the identity on $\mathcal{B}$. Thus, we capture the locality requirement with the following definition.
**Definition.** An instrument $\mathcal{E}$ is *local* to an algebra $\mathcal{A}$ just in case $\mathcal{E}[S](B) = \mathcal{E}[S](I)B$, for all $B \in \mathcal{A}'$ and $S \in \Sigma$.

Of course, if $\mathcal{A}$ is a POV measure on $\mathbb{N}$, there is a canonical instrument $\mathcal{E}^\mathcal{A}$ that is compatible with $\mathcal{A}$ and local to any algebra containing $\text{ran}(\mathcal{A})$:

$$\mathcal{E}^\mathcal{A}[S](B) = \sum_{n \in S} A_n^{1/2} B A_n^{1/2}, \quad (S \subseteq \mathbb{N}, B \in \mathcal{B}(\mathcal{H})).$$  \hfill (15)

We wish to extend this result to show that for each POV measure $\mathcal{A}$ with range in a $C^*$-algebra $\mathcal{A}$, there is a CP instrument $\mathcal{E}^\mathcal{A}$ that is compatible with $\mathcal{A}$ and local to $\mathcal{A}$. In this paper, we prove this result for finite quantum systems (Theorem 2), and for infinite quantum systems that can be approximated, in an appropriate sense, by finite quantum systems (Theorem 3). While our proof for the finite case uses only elementary linear algebra, our proof for the infinite case is non-constructive (i.e., invokes the axiom of choice in the form of the Tychonoff product theorem), and uses tools from the theory of operator algebras.

We first show that if Alice has a finite quantum system, then she can perform a “maximally disturbing” local operation — i.e., an operation that maps all her states to the maximally mixed state.

**Lemma 2.** If $\mathcal{A}$ is finite-dimensional $C^*$-algebra on the Hilbert space $\mathcal{H}$ then there is a completely positive projection $\Phi$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$. In particular, $\Phi$ maps $\mathcal{A}$ onto $\mathbb{C}I$.

**Proof.** (Compare with [13, Prop. 8.3.11] and [22,]) Since $\mathcal{A}$ is finite-dimensional, $\mathcal{A} = \bigoplus_{i=1}^m \mathcal{M}_{n(i)}$ for some positive integers $n(1), \ldots, n(m)$. Consider the following statement:

(†) There is a projective unitary representation $g \mapsto W(g)$ of a finite group $G$ in $\mathcal{A}$ such that $\{W(g) : g \in G\}$ spans $\mathcal{A}$.

We first show that (†) holds when $\mathcal{A} = \mathcal{M}_n$. Let $\{|0\rangle, \ldots, |n-1\rangle\}$ be a basis for $\mathbb{C}^n$, and for each $g \in \mathbb{Z}_n \times \mathbb{Z}_n$ let $W(g)$ be the unitary operator on $\mathbb{C}^n$ defined by

$$W(g)|a\rangle = e^{ig_1 a}|a + g_2\rangle, \quad (a = 1, \ldots, n).$$  \hfill (16)

Then $g \mapsto W(g)$ is a projective representation of $\mathbb{Z}_n \times \mathbb{Z}_n$ with bicharacter $\xi(g,h) = e^{ih_1 h_2}$; i.e., $W(g)W(h) = e^{ig_1 h_2}W(g + h)$. Furthermore, $\{W(g) : g \in \mathbb{Z}_2 \times \mathbb{Z}_2\}$ is an orthonormal basis for $\mathcal{M}_n$ relative to the inner product...
Thus, we have established (†) for the case that $\mathcal{A} = \mathcal{M}_n$. We now show that (†) holds when $\mathcal{A} = \bigoplus_{i=1}^m \mathcal{M}_{n(i)}$. Indeed, let

$$G = \bigoplus_{i=1}^m \left[ \mathbb{Z}_{n(i)} \times \mathbb{Z}_{n(i)} \right],$$

and take the direct sum of the corresponding projective representations.

We now show that if (†) holds, then there is a CP projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$. For each $g \in G$, define an automorphism $\alpha_g$ of $\mathcal{B}(\mathcal{H})$ by

$$\alpha_g(A) = W(g)^*AW(g), \quad (B \in \mathcal{B}(\mathcal{H})).$$

Then $G = \{\alpha_g : g \in G\}$ is a finite group of automorphisms of $\mathcal{B}(\mathcal{H})$. If $\alpha(A) = A$ for all $\alpha \in G$, then $AW(g) = W(g)A$ for all $g \in G$, and $A \in \mathcal{A}'$. Thus,

$$\Phi(A) = |G|^{-1} \sum_{\alpha \in G} \alpha(A), \quad (A \in \mathcal{B}(\mathcal{H})),
$$

is a CP projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$.

We are now prepared to prove a generalized HJW theorem, valid for all finite quantum systems (i.e., systems whose algebra of observables is finite-dimensional). In particular, if the $C^*$-algebra $\mathcal{B}'$ is finite-dimensional, then the product $\Phi \otimes \mathcal{A}$ of the maximally disturbing operation $\Phi$ (from Lemma 2) and the POV measure $\mathcal{A}$ (from Theorem 1) yields an instrument that prepares the ensemble $\mu$ on system $\mathcal{B}$.

**Theorem 2 (Generalized HJW Theorem).** Let $\mathcal{B}$ be a $C^*$-algebra acting on the Hilbert space $\mathcal{H}$, let $x$ be a unit vector in $\mathcal{H}$, and let $\mu$ be a measure on the state space of $\mathcal{B}$ such that $\omega_x|_B$ is the barycenter of $\mu$. If $\mathcal{B}'$ is finite-dimensional then there is a CP instrument $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$ that is local to $\mathcal{B}'$ and

$$\langle x | \mathcal{E}[S](B)x \rangle = \int_S \omega(B) d\mu(\omega),$$

for all $S \in \Sigma$ and $B \in \mathcal{B}$.

**Proof.** By Lemma 2 there is a CP projection $\Phi$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}'$. Let $\mathcal{E} = \Phi \otimes \mathcal{A}$, where $\mathcal{A}$ is the POV measure defined in Theorem 1. That is,

$$\mathcal{E}[S](B) = \Phi(B)A(S),$$

for all $S \in \Sigma$ and $B \in \mathcal{B}(\mathcal{H})$. \qed
This generalized HJW theorem applies to bit commitment protocols that employ continuous ensembles on finite quantum systems (e.g., continuous measures on the Bloch sphere), and to finite quantum systems with Abelian superselection rules (direct sums of matrix algebras). However, this first result leaves open the possibility of secure bit commitment protocols that employ infinite quantum systems. So, in the following subsection, we prove a more general HJW theorem that also applies to infinite quantum systems.

II.1 HJW theorem for hyperfinite algebras

Definition. A von Neumann algebra $\mathcal{R}$ is said to be hyperfinite just in case there is an upward directed family $\{\mathcal{R}_a\}_{a \in A}$ of finite-dimensional $C^*$-algebras such that $\mathcal{R}$ is the weak-operator closure of $\bigcup_{a \in A} \mathcal{R}_a$.

As in the finite case, an observer with a hyperfinite von Neumann algebra can perform a maximally disturbing measurement operation.

**Lemma 3.** If $\mathcal{R}$ is a hyperfinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$ then there is a completely positive projection $\Phi$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}'$. In particular, $\Phi$ maps $\mathcal{R}$ onto $\mathbb{C}I$.

**Notation.** For an arbitrary operator $B$ in $\mathcal{B}(\mathcal{H})$ we write $\text{co}_\mathcal{R}(B)^-$ for the weak-operator closed convex hull of $\{UBU^* : U \in \mathcal{R}, U \text{ unitary}\}$.

**Proof.** (Compare with [13, Prop. 8.3.11; Exercise 8.7.24] and [22].) Let $\{\mathcal{R}_a : a \in A\}$ be an increasing net of finite-dimensional $C^*$-algebras on $\mathcal{H}$ such that

$$(\bigcup_{a \in A} \mathcal{R}_a)^- = \mathcal{R},$$

(22)

where $\mathcal{X}^-$ denotes the weak-operator closure of $\mathcal{X}$. By Lemma [2] for each $a \in A$ there is a CP projection $\Phi_a$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}_a'$. Let

$$X = \prod_{B \in \mathcal{B}(\mathcal{H})} \text{co}_\mathcal{R}(B)^-$$

(23)

be the product topological space, where each factor is equipped with the weak-operator topology. For fixed $B \in \mathcal{B}(\mathcal{H})$, $\text{co}_\mathcal{R}(B)^-$ is weak-operator closed and bounded, and is therefore weak-operator compact [13, Thm. 5.1.3]. Thus, the Tychonoff product theorem entails that $X$ is compact. Let $\mathcal{M}$ be
the subset of $X$ consisting of mappings $\Phi$ that are positive, linear, normalized, and such that
\[ \Phi(R'_1 B R'_2) = R'_1 \Phi(B) R'_2, \]
for all $R'_1, R'_2 \in \mathcal{R}'$ and for all $B \in \mathcal{B}(\mathcal{H})$. Since $\mathcal{M}$ is closed in $X$, $\mathcal{M}$ is compact, and $\{\Phi_a : a \in \mathbb{A}\}$ has a limit point $\Phi \in \mathcal{M}$. Note that $\lim_a \Phi_a = \Phi$ iff, for each fixed $B \in \mathcal{B}(\mathcal{H})$, $\text{w-lim}_a \Phi_a(B) = \Phi(B)$. We claim that $\Phi(B) \in \mathcal{R}'$ for each $B \in \mathcal{B}(\mathcal{H})$. Let $A \in \bigcup_{a \in \mathbb{A}} \mathcal{R}_a$; that is, there is an $m \in \mathbb{A}$ such that $A \in \mathcal{R}_m$. Then, $A \Phi_a(B) = \Phi_a(B) A$, for all $a \geq m$. Since the maps $Z \mapsto A Z$ and $Z \mapsto Z A$ are weak-operator continuous,
\[
A \left[ \text{w-lim}_{a \geq m} \Phi_a(B) \right] = \text{w-lim}_{a \geq m} \left[ A \Phi_a(B) \right] = \text{w-lim}_{a \geq m} \left[ \Phi_a(B) A \right]
= \left[ \text{w-lim}_{a \geq m} \Phi_a(B) \right] A.
\]
Since $\Phi(B) = \text{w-lim}_a \Phi_a(B) = \text{w-lim}_{a \geq m} \Phi_a(B)$, it follows that $A \Phi(B) = \Phi(B) A$. Since $A$ was an arbitrary element of $\bigcup_{a \in \mathbb{A}} \mathcal{R}_a$, $\Phi(B) \in (\bigcup_{a \in \mathbb{A}} \mathcal{R}_a)' = \mathcal{R}'$. Therefore $\text{ran}(\Phi) = \mathcal{R}'$. Finally, since $\Phi$ is an $\mathcal{R}'$-bimodule mapping (i.e., Eqn. 24 holds), $\Phi$ is idempotent and completely positive [24, Cor. 3.4].

Again, a maximally disturbing operation $\Phi$ can be tensored with the POV measure $A$ to yield an instrument that prepares the ensemble $\mu$ on Bob’s system.

**Theorem 3 (Generalized HJW Theorem).** Let $\mathcal{B}$ be a $C^*$-algebra acting on the Hilbert space $\mathcal{H}$, let $x$ be a unit vector in $\mathcal{H}$, and let $\mu$ be a measure on the state space of $\mathcal{B}$ such that $\omega_x|_\mathcal{B}$ is the barycenter of $\mu$. If $\mathcal{B}'$ is hyperfinite then there is a CP instrument $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$ that is local to $\mathcal{B}'$ and
\[
\langle x | \mathcal{E}[S](B)x \rangle = \int_S \omega(B) d\mu(\omega),
\]
for all $S \in \Sigma$ and $B \in \mathcal{B}$.

**Proof.** The proof is identical to the proof of Theorem 2 with Lemma 3 replacing Lemma 2. \qed
III. Application to bit commitment

The Mayers-Lo-Chau theorem shows that when \( A = M_n \otimes I \) and \( B = A' \), and when bits are encoded in finite ensembles, then \((A, B)\) cannot be used to implement a secure bit commitment protocol. The generalized HJW theorem allows us to extend this result to the case where \( A (= B') \) is an arbitrary hyperfinite von Neumann algebra, and to encodings that employ arbitrary ensembles of states on \( B \). In particular, the generalized HJW theorem entails that there can be no secure bit commitment protocol using infinite (hyperfinite) quantum systems, or quantum systems with Abelian superselection rules.

III.1 Bit commitment with infinite quantum systems

The quantum bit commitment protocols that have been proposed to date employ finite quantum systems. In this subsection, we describe a bit commitment protocol that employs continuous ensembles of states on infinite qubit lattices. Since this protocol does not fall within the range of validity of the HJW theorem, it is immune to current no-go theorems against bit commitment. However, we show that this protocol can be cheated by exploiting the non-local correlations of an “infinitely entangled” EPR state (see \([12]\)).

Let \(|0, 0\rangle\) and \(|0, 1\rangle\) be orthogonal unit eigenvectors of \(\sigma_x\), and let \(|1, 0\rangle\) and \(|1, 1\rangle\) be orthogonal unit eigenvectors of \(\sigma_y\). Then, heuristically, the states of a one-dimensional infinite qubit lattice include vectors of the form

\[
\|b, s\rangle =_{\text{def}} \otimes_{i=1}^{\infty} |b, s(i)\rangle, \quad (s \in (\mathbb{Z}_2)^\omega),
\]  

(28)

with \(b = 0\) or \(b = 1\). (We provide a rigorous definition of these states below.)

During the Commit stage of the protocol, Alice performs operations on a composite \((A, B)\) consisting of two lattice systems \(A\) and \(B\), and she then sends system \(B\) to Bob. During the Unveil stage, Alice makes measurements on \(A\), and sends classical information to Bob, who then makes measurements on \(B\).
Commit: For \( b = 0,1 \), Alice chooses a random sequence \( s \in (\mathbb{Z}_2)^\omega \), and prepares the state
\[
\|b, s\rangle \rangle_A \otimes \|b, s\rangle \rangle_B.
\]
Alice holds part \( A \), and sends part \( B \) to Bob. (So, the ensemble Bob receives is an equal mixture over \( \|b, s\rangle \rangle \), for \( s \in (\mathbb{Z}_2)^\omega \).)

Unveil: Alice measures the observable
\[
A_b = \sum_{i=1}^{\infty} 2^{3i} P^{(i)}_b,
\]
on her systems, where \( P_b = \frac{1}{2}(I + \sigma_b) \) and
\[
P^{(i)}_b = I \otimes \cdots \otimes I \otimes P_b \otimes I \otimes \cdots.
\]
(Each state \( \|b, s\rangle \rangle \) is an eigenstate of \( A_b \), and when \( s_1 \neq s_2, \|b, s_1\rangle \rangle \) and \( \|b, s_2\rangle \rangle \) assign different values to \( A_b \).) Alice sends the results of her measurements (a list of numbers in the Cantor set) to Bob. Bob measures \( A_b \) on his systems and compares his numbers with Alice’s. Bob accepts if the two lists agree, and rejects if the two lists disagree.

Let \( \rho_b \) be the state that Bob receives. It is intuitively clear that if Alice follows the protocol honestly then \( \rho_0 = \rho_1 \), and so Bob cannot cheat. (We prove this fact below.)

We now tighten up the mathematical description of the systems involved in the protocol. The observables of a one-dimensional qubit lattice are represented by the \( C^* \)-algebraic infinite direct product
\[
\mathcal{A} = \bigotimes_{i \in \mathbb{N}} \mathcal{M}_{n(i)},
\]
where \( n(i) = 2 \) for each \( i \in \mathbb{N} \). For each \( i \in \mathbb{N} \) and \( A \in \mathcal{M}_2 \), let
\[
A^{(i)} = I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots,
\]
where \( A \) is in the \( i \)-th position. If for each \( i \in \mathbb{N}, \omega_i \) is a state of \( \mathcal{M}_2 \), then there is a unique state \( \otimes_{i=1}^{\infty} \omega_i \) of \( \mathcal{A} \) defined by
\[
(\otimes_{i=1}^{\infty} \omega_i) \left( A^{(i)} \right) = \omega_i(A).
\]
Furthermore, $\otimes_{i=1}^{\infty} \omega_i$ is pure iff each $\omega_i$ is pure, and is a trace iff each $\omega_i$ is a trace [13, Prop. 11.4.7]. Thus, if $\{|i\rangle : i \in \mathbb{N}\}$ are unit vectors in $\mathbb{C}^2$, then $\otimes_{i=1}^{\infty} |i\rangle$ can be used to denote the corresponding pure state of $\mathcal{A}$. In particular, for any $s \in (\mathbb{Z}_2)^\omega$, $\|b, s\|$ does in fact correspond to a pure state of $\mathcal{A}$.

Let $\mathcal{B}$ be an isomorphic copy of $\mathcal{A}$. Since $\mathcal{A}$ is a uniform limit of an increasing sequence of finite-dimensional algebras, it is nuclear; i.e., there is a unique norm on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ whose completion is a $C^*$-algebra. We denote this $C^*$-algebra by $\mathcal{A} \otimes \mathcal{B}$. We now establish the existence of the ensembles described in the protocol, and we show that they give rise to the same quantum state (namely, the “maximally mixed” tracial state) on system $\mathcal{B}$.

**Proposition 4.** If $\mu$ is the normalized Haar measure on $(\mathbb{Z}_2)^\omega$ then there is a probability measure $\mu_b$ on the state space of $\mathcal{A} \otimes \mathcal{B}$ such that
\[
\mu_b(\{\|b, s\rangle_A \otimes \|b, s\rangle_B : s \in S\}) = \mu(S),
\] (32)
for every Borel subset $S$ of $(\mathbb{Z}_2)^\omega$. Furthermore, if $\rho_b$ is the barycenter of $\mu_b$ then $\rho_b|_{\mathcal{I} \otimes \mathcal{B}}$ is the tracial state.

To establish the first part of Proposition 4 it will suffice to show that
\[
s \mapsto \|b, s\rangle_A \otimes \|b, s\rangle_B,
\] (33)
is a continuous mapping of $(\mathbb{Z}_2)^\omega$ into the state space of $\mathcal{A} \otimes \mathcal{B}$ (with the weak* topology). For then the induced measure $\mu_b = \mu \circ \varphi^{-1}$ will satisfy Eqn. (32).

Let $G = \sum_{i \in \mathbb{N}} (\mathbb{Z}_2 \oplus \mathbb{Z}_2)_i$ be the direct sum of a countable number of copies of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Elements of $G$ are sequences with values in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ that differ from the identity $(0, 0)$ in only finitely many positions. Let $V_{(0,0)} = I$, $V_{(0,1)} = \sigma_x$, $V_{(1,0)} = \sigma_y$, and $V_{(1,1)} = \sigma_z$, and for any $s \in G$, let
\[
U(s) = \text{def} \ V_{s(1)} \otimes V_{s(2)} \otimes V_{s(3)} \otimes \cdots \in \mathcal{A}.
\] (34)
Then the set $\{U(s) : s \in G\}$ is linearly dense in $\mathcal{A}$. Let $H_b$ denote the subgroup of $G$ generated by those sequences $s$ with the property that $s(i) = (0, 0)$ or $s(i) = (b, b \oplus 1)$ for all $i \in \mathbb{N}$. Then, $\{U(s) : s \in H_b\}$ generates an Abelian subalgebra of $\mathcal{A}$; namely, the algebra generated by the spin operator $V_{(b,b\oplus 1)}$ at each lattice site.
Lemma 5. If \( \mathcal{C} \) is the Abelian subalgebra of \( \mathcal{A} \) generated by \( \{U(s) : s \in H_b\} \) then the pure state space of \( \mathcal{C} \) is homeomorphic to \( (Z_2)^\omega \).

Proof. The Abelian \( C^* \)-algebra \( \mathcal{C} \) is isomorphic to the \( C^* \)-algebra \( C(X) \) of continuous complex-valued functions on \( X \), where \( X \) is the pure state space of \( \mathcal{C} \) equipped with the weak* topology. Furthermore, if \( C(X) \) and \( C(Y) \) are isomorphic then \( X \) and \( Y \) are homeomorphic. Thus, if \( \mathcal{C} \simeq C(Y) \) then the space of pure states of \( \mathcal{C} \) is homeomorphic to \( Y \). Now, \( \mathcal{C} \simeq \bigotimes_{i=1}^\infty \mathcal{N}_i \), where \( \mathcal{N}_i \) is the Abelian algebra generated by \( \sigma_\mathcal{b} \). Since \( \mathcal{N}_i \) is isomorphic to \( C(Z_2) \),

\[
\bigotimes_{i=1}^\infty \mathcal{N}_i \simeq \bigotimes_{i=1}^\infty C(Z_2) \simeq C((Z_2)^\omega),
\]

where \( (Z_2)^\omega \) is equipped with the product topology (see [13, pp. 910–911; Prop. 11.4.3]). Therefore the pure state space of \( \mathcal{C} \) is homeomorphic to \( (Z_2)^\omega \).

Since \( \mathcal{C} \) is isomorphic to \( C((Z_2)^\omega) \), there is (by the Riesz representation theorem) a one-to-one correspondence between positive normalized measures on \( (Z_2)^\omega \) and states on \( \mathcal{C} \).

Lemma 6. If \( \mu \) is the Haar measure on \( (Z_2)^\omega \) then the barycenter of \( \mu \) is \( \tau|_{\mathcal{C}} \), where \( \tau \) is the trace on \( \mathcal{A} \).

Proof. Let \( \sigma(\mathcal{C}) \) denote the pure state space of \( \mathcal{C} \), and let \( \rho = \int_{\sigma(\mathcal{C})} \omega d\mu(\omega) \) be the barycenter of \( \mu \). To show that \( \rho = \tau \), it will suffice to show that \( \rho(U(s)) = 0 \) whenever \( s \in H_b - \{e\} \). Indeed, if \( s \neq e \) then there is an \( i \in \mathbb{N} \) such that \( s(i) = (b, b \oplus 1) \). Let \( s' \) be the element of \( \oplus_{i=1}^\infty (Z_2 \oplus Z_2) \) such that \( s'(j) = s(j) \) when \( j \neq i \), and \( s'(i) = (b \oplus 1, b) \). Then \( U(s')^* U(s) U(s') = -U(s) \). Since \( \mu \) is translation-invariant, \( \rho(U(s)) = -\rho(U(s)) \). Therefore \( \rho(U(s)) = 0 \).

Lemma 7. There is a completely positive projection \( \Phi \) from \( \mathcal{A} \) onto \( \mathcal{C} \) such that \( \tau(\Phi(A)) = \tau(A) \) for all \( A \) in \( \mathcal{A} \).

Proof. For each \( t \in \mathbb{R} \), define an automorphism \( \alpha_t \) of \( \mathcal{A} \) by

\[
\alpha_t(B) = e^{-itA_b}Be^{itA_b}, \quad (B \in \mathcal{A}).
\]
Since $A_b$ is bounded, the map $t \mapsto \alpha_t(B)$ is norm-continuous. If $\nu$ is an invariant mean on $\mathbb{R}$, then

$$\Phi(B) = \int_{\mathbb{R}} \alpha_t(B) \, d\nu(t), \quad (B \in \mathcal{A}),$$

is a positive linear map on $\mathcal{A}$ \[20\] Lemma 7.4.4. (To show that $\Phi$ is completely positive, it will suffice to show that the range of $\Phi$ is Abelian.) Clearly $\Phi(C_1 B C_2) = C_1 \Phi(B) C_2$ for all $C_1, C_2 \in \mathcal{C}$, and $B \in \mathcal{A}$. In particular, $\Phi(C) = C$ for all $C \in \mathcal{C}$. To see that the image of $\Phi$ lies in $\mathcal{C}$, let $s$ be an element of $G = \sum_{i \in \mathbb{N}} (\mathbb{Z}_2 \oplus \mathbb{Z}_2)i$. If $s \in H_b$ then $U(s) \in \mathcal{C}$ and $\Phi(U(s)) = U(s)$. Suppose then that $s \not\in H_b$; that is, there is an $i \in \mathbb{N}$ such that either $s(i) = (b \oplus 1, b)$ or $s(i) = (1, 1)$. (It will suffice to consider the first case; the second case follows by symmetry.) Then

$$U(s) = V_{s(1)} \otimes \cdots \otimes V_{s(i-1)} \otimes V_{s(i+1)} \otimes \cdots,$$  \hspace{1cm} (38)

and

$$\Phi(U(s)) = V_{s(1)} \otimes \cdots \otimes V_{s(i-1)} \otimes B_i \otimes V_{s(i+1)} \otimes \cdots,$$  \hspace{1cm} (39)

where

$$B_i = \int_{\mathbb{R}} e^{-itP_b} \left[ V_{(b \oplus 1, b)} \right] e^{itP_b} d\nu(t) = 0.$$  \hspace{1cm} (40)

Thus, $\Phi(U(s)) = 0$. Since $\{U(s) : s \in G\}$ spans $\mathcal{A}$, it follows that $\text{ran}(\Phi) = \mathcal{C}$. To see that $\tau = \tau \circ \Phi$, note that every non-identity element of $\{U(s) : s \in G\}$ is trace-free. If $s \in H_b$, then $\Phi(U(s)) = U(s)$ and therefore $\tau(\Phi(U(s))) = \tau(U(s))$. If $s \not\in H_b$, then $\Phi(U(s)) = 0$ and $\tau(U(s)) = 0$. Since $\tau$ and $\tau \circ \Phi$ are continuous linear functionals, $\tau = \tau \circ \Phi$. \hfill $\Box$

The mapping $\Psi = \Phi \otimes \Phi$ is a CP projection from $\mathcal{A} \otimes \mathcal{B}$ onto $\mathcal{C} \otimes \mathcal{C}$, and its adjoint $\Psi^*$ is a weak* continuous mapping from the state space of $\mathcal{C} \otimes \mathcal{C}$ into the state space of $\mathcal{A} \otimes \mathcal{B}$. Let $\sigma(\mathcal{C} \otimes \mathcal{C})$ denote the pure state space of $\mathcal{C} \otimes \mathcal{C}$, and let $\sigma(\mathcal{A} \otimes \mathcal{B})$ denote the pure state space of $\mathcal{A} \otimes \mathcal{B}$. Using $\Psi^*$ again to denote the restriction of $\Psi^*$ to $\sigma(\mathcal{C} \otimes \mathcal{C})$, and identifying $\sigma(\mathcal{C} \otimes \mathcal{C})$ with $(\mathbb{Z}_2)^{\omega} \times (\mathbb{Z}_2)^{\omega}$, it follows that $\Psi^*$ is a continuous injection of $(\mathbb{Z}_2)^{\omega} \times (\mathbb{Z}_2)^{\omega}$ into $\sigma(\mathcal{A} \otimes \mathcal{B})$. Note that

$$\Psi^*[\langle s, s \rangle] = \langle b, s \rangle_A \otimes \langle b, s \rangle_B$$

and so the mapping

$$s \mapsto \langle b, s \rangle_A \otimes \langle b, s \rangle_B = (\Psi^* \circ \Delta)(s),$$  \hspace{1cm} (41)

and so the mapping

$$s \mapsto \langle b, s \rangle_A \otimes \langle b, s \rangle_B = (\Psi^* \circ \Delta)(s),$$  \hspace{1cm} (42)
where $\Delta(s) = (s, s)$, is continuous, which establishes the first part of Proposition 4.

Now let $\rho_b$ denote the barycenter of $\mu_b$, and let $\nu_b = \mu \circ (\Phi^*)^{-1}$ denote the measure on $\sigma(B)$ induced by $\Phi^*$ from the measure $\mu$ on $\sigma(C)$. Then for any $B \in \mathcal{B}$,

$$\rho_b(I \otimes B) = \int_{\sigma(A \otimes B)} \omega(I \otimes B) d\mu_b(\omega) = \int_{\sigma(B)} \omega(B) d\nu_b(\omega)$$

(43)

$$= \int_{\sigma(C)} \omega(\Phi(B)) d\mu(\omega) = \tau(\Phi(B)) = \tau(B).$$

(44)

This establishes the second part of Proposition 4. Thus, $\mu_0$ and $\mu_1$ are the ensembles prepared by Alice if she follows the protocol honestly.

Finally, we show that Alice can cheat by preparing an entangled state during the Commit stage rather than $\mu_0$ or $\mu_1$. In particular, if for each $i \in \mathbb{N}$, $\psi_i = \psi$ is the Bohm-EPR state of $M_2 \otimes M_2$, then $\omega = \mu \otimes \Phi \otimes \psi_1$ is a pure state of $\otimes_{i=1}^{\infty} (M_i \otimes M_i) = A \otimes B$ [12]. It is not difficult to see, then, that if Alice performs a nonselective measurement of $A_b$ (represented by the CP map in Eqn. 36) when $A \otimes B$ is in state $\omega$, then the posterior state is the ensemble $\mu_b$. Therefore, if Alice prepares $\omega$ during the Commit stage, then she can unveil either 0 or 1.

III.2 Bit commitment and superselection rules

It has recently been argued by Mayers, Kitaev, and Preskill [14, 18], in response to a question raised by Popescu [21], that the no-go theorem for bit commitment extends to the case of quantum systems with superselection rules. The generalized HJW theorem provides another route to this result, at least for systems whose superselection rules are Abelian. In the case of Abelian superselection rules, $A = B'$; that is, Alice can perform any operation that commutes with Bob’s measurement operations. And the generalized HJW theorem shows that an observer with algebra $B'$ can steer system $B$ into any ensemble consistent with $\omega_x|_B$. Thus, a bit commitment protocol is perfectly concealing against Bob only if it is not binding against Alice. However, the generalized HJW theorem has nothing to say (directly) about Alice’s ability to cheat when both systems are governed by non-Abelian superselection rules (in which case $A \subset B'$).

Mayers et al. [14] claim that — HJW theorem aside — Alice can always steer Bob’s system into the state of her choice by adding, if necessary, an
appropriate ancilla to her system. Their argument is based on a more general claim that restrictions imposed by superselection rules on a local system can always be effectively removed by embedding the local system in a larger system (in particular, by adding an ancilla).

The formalism of elementary quantum mechanics imposes no restriction on adding ancillae. However, in the setting of algebraic quantum field theory, an observer can measure only those observables that correspond to her spacetime region. As a result, adding ancillae is not permitted — at least if “adding an ancilla” is interpreted to mean that Alice can measure observables that are not in her local observable algebra $\mathcal{R}(O_A)$. Thus, in this richer theoretical framework, Alice is subject to further constraints on her ability to simulate any operation that commutes with Bob’s measurement operations, and these constraints could — it seems theoretically possible — prevent Alice from cheating in a bit commitment protocol. (It would be interesting to explore connections between the formal condition $A \subset B'$ and relativistic constraints of the sort exploited by Kent’s bit commitment protocol.)

III.3 Limitations on the generalized HJW theorem

Let us say that a bit commitment protocol employs a quantum encoding just in case Alice encodes her choice of a bit 0 or 1 in two ensembles $\mu_0$ or $\mu_1$ of quantum states. Then, even in the case of bit commitment schemes that employ quantum encodings, there is one further assumption of the generalized HJW theorem that is not prima facie guaranteed to hold in any bit commitment protocol: the assumption that the barycenter of $\mu_b$ is a vector state. (Let us call this latter assumption the vector state assumption.)

First, it is not difficult to find pairs of $C^*$-algebras $(\mathcal{A}, \mathcal{B})$, and measures $\mu_b$ on the state space of $\mathcal{B}$ such that the vector state assumption does not hold: e.g., let $\mathcal{B} = \mathcal{M}_2$, and let $\mu_b$ be the measure on the state space of $\mathcal{B}$ that assigns $\frac{1}{2}$ to each of $\frac{1}{2}(I + \sigma_b)$ and $\frac{1}{2}(I - \sigma_b)$. (Of course, this trivial example could not be used to construct a secure bit commitment protocol, since Alice could not perform any non-trivial measurements to verify her commitment to Bob.) However, the vector state assumption does hold when $\mathcal{B}$ has a separating vector in $\mathcal{H}$.

**Definition.** A vector $x$ in the Hilbert space $\mathcal{H}$ is said to be separating for the $C^*$-algebra $\mathcal{B}$ just in case $Bx = 0$ only if $B = 0$ for all $B \in \mathcal{B}$.
Proposition 8 ([13, Thm. 7.3.8]). If $\mathcal{B}$ is a $C^*$-algebra acting on the Hilbert space $\mathcal{H}$ and if $\mathcal{B}$ has a separating vector $x$ in $\mathcal{H}$, then each state of $\mathcal{B}$ is implemented by some vector in $\mathcal{H}$.

Thus, if $\mathcal{B}$ has a separating vector in the Hilbert space $\mathcal{H}$ (and if $\mathcal{A} = \mathcal{B}'$) then any ensemble of states on $\mathcal{B}$ corresponds to a state $\omega_x|_{\mathcal{B}}$ induced by a vector $x$ in $\mathcal{H}$, and the generalized HJW theorem entails that any two equivalent ensembles can be prepared at a distance (from a common state). For example, $\mathcal{B} = I_A \otimes \mathcal{B}(\mathcal{H}_B)$ has a separating vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if $\text{dim}(\mathcal{H}_B) \leq \text{dim}(\mathcal{H}_A)$ [8]. So, in the case of elementary quantum systems, by adding an ancilla, Alice can “make her Hilbert space as large as Bob’s”, which ensures that their joint Hilbert space $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_A) \otimes \mathcal{H}_B$ contains a vector representative of each of Bob’s states.

Nonetheless, there are $C^*$-algebras that do not have — and could not have, in any faithful representation — a separating vector, e.g., $C^*$-algebras which contain an uncountable family of mutually orthogonal projection operators. But if $\mathcal{B}$ does not have a separating vector, then the HJW theorem doesn’t show that an observer with algebra $\mathcal{B}'$ could perform operations that prepare any one of two equivalent measures on the state space of $\mathcal{B}$ (from a common ancestor state). Until the HJW theorem is generalized to cover such cases, there remains a small, but theoretically crucial, loophole in current proofs of the impossibility of secure bit commitment.

IV. Conclusion

We have shown, subject to a mild constraint (viz., that the systems involved are “hyperfinite”), that any two equivalent measures on the state space of a $C^*$-algebra can be prepared “at a distance”. This result generalizes the Hughston-Jozsa-Wootters theorem, and so can be used to extend the Mayers-Lo-Chau argument against the security of quantum bit commitment protocols.

However, the results proved to date — including the results in this paper — are not yet sufficient to rule out the security of any conceivable quantum bit commitment protocol. First, it remains an open question whether an analogue of the HJW theorem holds for any system whose observables can be represented by self-adjoint operators in some abstract (not necessarily nuclear) $C^*$-algebra. Second, in order to invoke the HJW theorem in an
argument against bit commitment, one must make further physical assumptions — e.g., that the states on Bob’s system correspond to vector states of some larger system $S$, and that Alice can perform any operation on $S$ that commutes with Bob’s measurement operations — that have yet to be justified in a fully general context.

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