Abstract

This work presents a novel derivation of the expressions that describe the distortions of the CMB curve due to the interactions between photons and the electrons present in dilute ionized systems. In this approach, a simplified one-dimensional evolution equation for the photon number occupation is applied in order to describe the aforementioned interaction. This methodology allows to emphasize the physical features of the Sunyaev-Zeldovich effect and suggests the existence of links between basic statistical physics and complex applications involving radiative processes.

1 Introduction

One of the most relevant physical phenomena in modern cosmology is the Sunyaev-Zeldovich effect (SZE). Cosmic microwave background photons get Comptonized when they interact with electrons in systems such as the hot gas present between clusters of galaxies. The frequency shift associated with the Compton effect cause distortions to the Planck CMB curve ($T \simeq 2.725K$). These distortions were first quantified by means of the Kompaneets equation, which corresponds to a photon diffusion approach to the problem [1].

Two basic phenomena have been identified in the SZE in terms of the motion of the electrons. The first one is related to the bulk motion of the cluster (kinematic SZE), while the second one corresponds to the random motion of the particles present in the intracluster gas (thermal SZE). Both effects are important in order to determine cosmological parameters such as the Hubble constant, as well as primary anisotropies in the CMB spectrum [2, 3].

The photon diffusion approach was questioned by Rephaeli nearly 25 years ago, due to the low density of the gas. This author also noticed that mild-relativistic effects become
relevant in order to determine cluster velocities in several astrophysical scenarios [4].
The approach taken by Rephaeli to describe the thermal SZE was based in photon scattering techniques that involved convolution integrals that were evaluated numerically. After that work, there have been lots of contributions regarding alternative derivations of the SZE or the inclusion of other possible causes of additional CMB distortions such as magnetic fields or very large wavelength acoustic waves [5]. More recent work suggests a relation between the SZE and the characterization of dark matter particles related to high energy reactions [6].

The aim of the present paper is to present a derivation of the SZE based on a simple kinetic 1D model of the electron gas in order to show statistical properties that link the photon scattering approach to other branches of statistical physics. The formalism reproduces both the kinematic and thermal effects in the non relativistic regime and suggests extensions of the SZE formalism to interdisciplinary areas corresponding to low density limits of diffusive-type processes.

In order to accomplish this task, the paper has been divided as follows: In section two, the basic thermodynamic properties of black body radiation are reviewed. Section three is dedicated to the derivation of the kinematic SZE while section four is devoted to the analysis of the thermal SZE in which the present approach is compared with the original formalism based on the Kompaneets equation. Section five includes final remarks and a brief description of future work regarding this active area of research.

2 CMB Basics

The starting point of the formalism is the occupation number for a Planck distribution:

\[ n_{(0)}(\nu) = \frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{1} \]

where \( h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} \) is the Planck constant, \( k = 1.38 \times 10^{-23} \text{ J/K} \) is Boltzmann’s constant and \( T = 2.725 \text{K} \) is the CMB temperature. The internal energy density corresponding to the occupation number is given by

\[ u(\nu) = \frac{8\pi h\nu^3}{c^3} n_{(0)}(\nu) \tag{2} \]

The intensity associated to the internal energy density reads:

\[ I(\nu) = \frac{c}{4\pi} u(\nu) \tag{3} \]

In CMB physics it is customary to define the dimensionless frequency \( x \) as:

\[ x = \frac{h\nu}{kT} \tag{4} \]
Eq. (3) can then be rewritten as:

\[ I(x) = \frac{2(kT)^3}{(hc)^2} \frac{x^3}{e^x - 1} \]  
(5)

The maximum intensity value corresponding to Eq. (5) is \( x \simeq 2.821 \) which corresponds to the microwave frequency \( \nu \simeq 1.601 \times 10^{11} \text{Hz} \) at the CMB temperature.

3 Kinematic SZE

In the simplest model, all the electrons present in the dilute gas move with velocity \( u_k \) along the \( x \) direction. The dimensionless parameter \( \beta_k \) is given by:

\[ \beta_k = \frac{u_k}{c} \]  
(6)

The frequency shift \( \hat{\nu} \) for CMB photons reads:

\[ \hat{\nu} = \nu (1 \pm \beta_k) \]  
(7)

Now, if \( \tau \) represents the fraction of photons scattered by the electrons, the perturbed occupation number \( n_{(p)}(\nu) \) is given by:

\[ n_{(p)}(\nu) = n_{(0)}(\nu) - \tau n_{(0)}(\nu) + \tau n_{(0)}(\hat{\nu}) \]  
(8)

The last term in the right hand side of equation (8) corresponds to the number of photons with frequency \( \hat{\nu} \) after the photon-electron interactions. This term can be approximated using a Taylor series expansion as:

\[ n_{(0)}(\hat{\nu}) \approx n_{(0)}(\nu) \pm \beta_k \nu \frac{\partial n_{(0)}}{\partial \nu} \]  
(9)

so that Eq. (8) now reads:

\[ n_{(p)}(\nu) - n_{(0)}(\nu) = \Delta n = \pm \beta_k \tau \frac{\partial n_{(0)}}{\partial \nu} \]  
(10)

In order to compute the change in the intensity spectrum \( \Delta I = I_{(p)} - I_{(0)} \) due to the kinematic SZE one can apply the relation:

\[ \frac{\Delta I}{I_{(0)}} = \frac{\Delta n}{n_{(0)}} \]  
(11)

\( \frac{\Delta I}{I_{(0)}} \) can now be established using Eqs. (1), (4) and (11), thus obtaining

\[ \Delta I(x) = \mp \frac{2(kT)^3}{(hc)^2} \frac{x^4}{(e^x - 1)^2} \beta_k \tau \]  
(12)

Eq. (12) is the well known expression of the kinematic SZE [4]. Its shape, scaled in terms of the factor \( \beta \tau \) is shown in figure 1.
Figure 1: Kinematic SZE. The intracluster gas is assumed to move homogeneously with scaled speed $-\beta$ producing an intensity distortion. $\Delta I$ is expressed as a function of $x = \frac{b\nu}{kT}$ and in units of $\frac{2(kT)^3}{(hc)^2}$.

4 Thermal SZE

We now consider the case in which the electron velocities satisfy a given 1D distribution function $P = P(v)$. Photons of different frequencies contribute to the perturbed occupation number $n_{(p)}(\nu)$ in the SZE. Historically, $n_{(p)}(\nu)$ was first established using the Kompaneets equation approximation:

$$\frac{n_p - n_0}{\tau z} = \frac{\Delta n(\nu)}{\tau z} = 4x \frac{\partial n_{(0)}}{\partial x} + x^2 \frac{\partial^2 n_{(0)}}{\partial x^2}$$

(13)

where $z = \frac{kT_{el}}{mc^2}$ is the relativistic parameter for a free electron gas at temperature $T_{el}$. Eq. (13) can be established using a kinetic theory formalism in which the drift term usually present in the relativistic Boltzmann equation is neglected. In that case, the photon occupation number is modified due to the Compton interactions included in the corresponding collision kernel [7].

In the present formalism, the varying electron velocities are expressed as

$$\bar{\beta}_{Th} = \frac{v}{c}$$

(14)

The thermal SZE corresponds to a slight variation of Eq. (8) that reads:

$$n_{(p)}(\nu) = n_{(0)}(\nu) - \tau n_{(0)}(\nu) + \tau n_{(d)}(\nu)$$

(15)

where $n_{(d)}(\nu)$ is the occupation number of the photons that contribute to a fixed per-
turbed frequency band, which is given by:

\[ n_{(d)}(\nu) = \int_{-\infty}^{\infty} P(\bar{\beta}_{Th}) n(\nu + \bar{\beta}_{Th} \nu) d\bar{\beta}_{Th} \] (16)

The Taylor expansion of \( n(\nu + \bar{\beta}_{Th} \nu) \) up to second order in \( \Delta \nu = \bar{\beta}_{Th} \nu \) leads to the expression:

\[ n_d(\nu) \approx n_{(0)} \int_{-\infty}^{\infty} P(\bar{\beta}_{Th}) (n_{(0)} + \nu \bar{\beta}_{th} \frac{\partial n_{(0)}}{\partial \nu} + \frac{\nu^2 \bar{\beta}_{th}^2}{2} \frac{\partial^2 n_{(0)}}{\partial \nu^2}) d\bar{\beta}_{Th} \] (17)

or, using Eq. (4):

\[ n_d(x) \approx n_{(0)} \int_{-\infty}^{\infty} P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} + x \int_{-\infty}^{\infty} \bar{\beta}_{Th} P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} + \frac{x^2}{2} \int_{-\infty}^{\infty} \bar{\beta}_{Th}^2 P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} \] (18)

In Eq. (18) \( P(\bar{\beta}_{Th}) \) is assumed to be normalized, so that

\[ \int_{-\infty}^{\infty} P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} = 1 \] (19)

The first two moments of the distribution function lead to the Kompaneets equation structure. The coefficients corresponding to the right hand side of Eq. (13) are:

\[ \int_{-\infty}^{\infty} \bar{\beta}_{Th} P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} = 4z \] (20)

\[ \int_{-\infty}^{\infty} \bar{\beta}_{Th}^2 P(\bar{\beta}_{Th}) d\bar{\beta}_{Th} = 2z \] (21)

The distribution function in the 1D model is then given by:

\[ P(\bar{\beta}_{Th}) = \frac{1}{2\sqrt{\pi}z} e^{-\frac{\bar{\beta}_{Th}^2}{4z}} + \frac{\beta}{\sqrt{\pi}z} e^{-\frac{\beta^2}{4z}} \] (22)

At this point it is possible to apply Eq. (11) in order to establish the intensity distortion curve corresponding to the thermal SZE. The final result reads:

\[ \frac{\Delta I(x)}{y} = \frac{2(kT)^3}{(hc)^2} \frac{x e^x + x}{(e^x - 1)^2 (e^x - 1 - 4)} \] (23)

Eq. (24) is the well-known expression for the thermal non-relativistic SZE [4]. The distortion, scaled in terms of the factor \( y = \tau z \), is shown in figure 2.

5 Final Remarks

This work has been devoted to the analysis of the SZE in terms of a simplified 1D kinetic model. One of the results here obtained is the establishment of a Kompaneets-type equation in which the coefficients of the derivative terms correspond to the first moments of
the distribution function of the scatterers. Parity is a relevant feature to be considered in the structure of the distribution function, as was noticed in earlier work related to the SZE [8].

It is interesting to notice that the kinematic SZE is also compatible with the simplified kinetic approach here proposed. Indeed, if \( P(\bar{\beta}) = \delta(\bar{\beta} - \beta) \) Eq.(16) becomes:

\[
    n_d(x) = x \frac{\partial n(0)}{\partial x} \int_{-\infty}^{\infty} \bar{\beta} \delta(\bar{\beta} - \beta) d\bar{\beta}
\]

which immediately leads to Eq. (10).

The SZE is a 3D phenomenon which involves several physical processes present in ionized gases. In contrast, the present approach corresponds to a simple kinetic model that suggests the existence of a direct link between the diffusive-type formalisms and the use of scattering kernels. This type of formalism allows the direct calculation of perturbed occupation numbers for other physical systems. Further applications of the type of approach here presented will include relativistic effects, as well as multiple scattering scenarios in dense systems.

**Acknowledgements**

The author wishes to thank A. Sandoval-Rubalcava and A.R. Sagaceta-Mejía for their contributions to this article. This work has been supported by the Institute of Technology
and Applied Research (INIAT) of U. Iberoamericana, Mexico.

References

[1] Y.B. Zel’dovich and R.A. Sunyaev, ” The Interaction of Matter and Radiation in a Hot-Model Universe”, Astrophys. Space Sci. 4, 301 (1969).

[2] S. Colafrancesco, P. Marchegiani and E. Palladino,”The non-thermal Sunyaev-Zel’dovich effect in clusters of galaxies”, Astronomy and Astrophysics 397, 27-52 (2003).

[3] S. Colafrancesco, P. Marchegiani, M.S. Emritte, ”Probing the physics and history of cosmic reionization with the Sunyaev-Zel’dovich Effect”, Astronomy and Astrophysics 595, A21 (2016).

[4] Y. Rephaeli,”Cosmic microwave background comptonization by hot intracluster gas”, Astrophys. J. 445, 33 (1995).

[5] A.Sandoval-Villalbazo and R. Maartens, ”Brillouin scattering and the CMB”, General Relativity and Gravitation,37,1137-1143 (2005).

[6] Lavalle, J., Boehm, C., Barthe, J. "On the Sunyaev-Zel’dovich effect from dark matter annihilation or decay in galaxy clusters”, Journal of Cosmology and Astroparticle Physics, 2010, 005 (2010).

[7] J. Bernstein and S. Dodelson, "Aspects of the Zel’dovich-Sunyaev mechanism”, Phys. Rev. D41, 354 (1990).

[8] A. Sandoval-Villalbazo and L.S. García-Colín, "On a parity property in the thermal Sunyaev-Zel’dovich effect”, AIP Conf. Proc. 758:170-175, 2005.