On the consistency theory of high dimensional variable screening

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Abstract

Variable screening is a fast dimension reduction technique for assisting high dimensional feature selection. As a preselection method, it selects a moderate size subset of candidate variables for further refining via feature selection to produce the final model. The performance of variable screening depends on both computational efficiency and the ability to dramatically reduce the number of variables without discarding the important ones. When the data dimension \( p \) is substantially larger than the sample size \( n \), variable screening becomes crucial as 1) Faster feature selection algorithms are needed; 2) Conditions guaranteeing selection consistency might fail to hold.

This article studies a class of linear screening methods and establishes consistency theory for this special class. In particular, we prove the weak diagonally dominant (WDD) condition is a necessary and sufficient condition for strong screening consistency. As concrete examples, we show two screening methods SIS and HOLP are both strong screening consistent (subject to additional constraints) with large probability if \( n > O((\rho s + \sigma/\tau)^2 \log p) \) under random designs. In addition, we relate the WDD condition to the irrepresentable condition, and highlight limitations of SIS.

1 Introduction

The rapidly growing data dimension has brought new challenges to statistical variable selection, a crucial technique for identifying important variables to facilitate interpretation and improve prediction accuracy. Recent decades have witnessed an explosion of research in variable selection and related fields such as compressed sensing (Donoho, 2006; Baraniuk, 2007), with a core focus on regularized methods (Tibshirani, 1996; Fan and Li, 2001; Candes and Tao, 2007; Bickel et al., 2009; Zhang, 2010). Regularized methods can consistently recover the support of coefficients, i.e., the non-zero signals, via optimizing regularized loss functions under certain conditions (Zhao and Yu, 2006; Wainwright, 2009;
Lee et al. (2013). However, in the big data era when $p$ far exceeds $n$, such regularized methods might fail due to two reasons. First, the conditions that guarantee variable selection consistency might fail to hold when $p >> n$; Second, the computation burden of the corresponding optimization problem increases dramatically with large $p$.

Bearing these concerns in mind, Fan and Lv (2008) propose the concept of “variable screening”, a fast technique that reduces data dimensionality from $p$ to a size comparable to $n$, with all predictors having non-zero coefficients preserved. They propose a marginal correlation based fast screening technique “Sure Independence Screening” (SIS) that can preserve signals with large probability. However, this method relies on a strong assumption that the marginal correlations between the response and the important predictors are high (Fan and Lv, 2008), which is easily violated in the practice. Li et al. (2012) extends the marginal correlation to the Spearman’s rank correlation, which is shown to gain certain robustness but is still limited by the same strong assumption. Wang (2009) and Cho and Fryzlewicz (2012) take a different approach to attack the screening problem. They both adopt variants of a forward selection type algorithm that includes one variable at a time for constructing a candidate variable set for further refining. These methods eliminate the strong marginal assumption in Fan and Lv (2008) and have been shown to achieve better empirical performance. However, such improvement is limited by the extra computational burden caused by their iterative framework, which is reported to be high when $p$ is large (Wang and Leng, 2013). To ameliorate concerns in both screening performance and computational efficiency, Wang and Leng (2013) develop a new type of screening method termed “High-dimensional ordinary least-square projection” (HOLP). This new screener relaxes the strong marginal assumption required by SIS and can be computed efficiently (complexity is $O(n^2p)$), thus scalable to ultra-high dimensionality.

This article focuses on linear models for tractability. As computation is one vital concern for designing a good screening method, we primarily focus on a class of linear screeners that can be efficiently computed, and study their theoretical properties. The main contributions of this article lie in three aspects.

1. We define the notion of strong screening consistency to provide a unified framework for analyzing screening methods. In particular, we show a necessary and sufficient condition for a screening method to be strong screening consistent is that the screening matrix is weak diagonally dominant (WDD). This condition gives insights into the design of screening matrices, while providing a framework to assess the effectiveness of screening methods.

2. We relate WDD to the irrepresentable condition (Zhao and Yu, 2006) that is necessary
and sufficient for sign consistency of lasso \cite{Tibshirani1996}. In contrast to the irrepresentable condition (IC) that is specific to the design matrix, WDD involves another ancillary matrix that can be chosen arbitrarily. Such flexibility allows WDD to hold even when IC fails if the ancillary matrix is carefully chosen (as in \textit{HOLP}). When the ancillary matrix is chosen as the design matrix, certain equivalence is shown between WDD and IC, revealing the difficulty for \textit{SIS} to achieve screening consistency.

3. We study the behavior of \textit{SIS} and \textit{HOLP} under random designs, and prove that a sample size of \( n = O((\rho s + \sigma/\tau)^2 \log p) \) is sufficient for \textit{SIS} and \textit{HOLP} to be screening consistent, where \( s \) is the sparsity, \( \rho \) measures the diversity of signals and \( \tau/\sigma \) evaluates the signal-to-noise ratio. This is to be compared to the sign consistency results in \textit{Wainwright} (2009) where the design matrix is fixed and assumed to follow the irrepresentable condition.

The article is organized as follows. In Section 1, we set up the basic problem and describe the framework of variable screening. In Section 2, we provide a deterministic necessary and sufficient condition for consistent screening. Its relationship with the irrepresentable condition is discussed in Section 3. In Section 4, we prove the consistency of \textit{SIS} and \textit{HOLP} under random designs by showing the WDD condition is satisfied with large probability, although the requirement on \textit{SIS} is much more restrictive.

## 2 Linear screening

Consider the usual linear regression

\[
Y = X \beta + \epsilon,
\]

where \( Y \) is the \( n \times 1 \) response vector, \( X \) is the \( n \times p \) design matrix and \( \epsilon \) is the noise. The regression task is to learn the coefficient vector \( \beta \). In the high dimensional setting where \( p >> n \), a sparsity assumption is often imposed on \( \beta \) so that only a small portion of the coordinates are non-zero. Such an assumption splits the task of learning \( \beta \) into two phases. The first is to recover the support of \( \beta \), i.e., the location of non-zero coefficients; The second is to estimate the value of these non-zero signals. This article mainly focuses on the first phase.

As pointed out in the introduction, when the dimensionality is too high, using regularization methods methods raises concerns both computationally and theoretically. To reduce the dimensionality, \textit{Fan and Lv} (2008) suggest a variable screening framework by finding a
The submodel

$$\mathcal{M}_d = \{i : |\hat{\beta}_i| \text{ is among the largest } d \text{ coordinates of } |\hat{\beta}| \} \quad \text{or} \quad \mathcal{M}_\gamma = \{i : |\hat{\beta}_i| > \gamma \}.$$  

Let $$Q = \{1, 2, \cdots, p\}$$ and define $$S$$ as the true model with $$s = |S|$$ being its cardinality. The hope is that the submodel size $$|\mathcal{M}_d|$$ or $$|\mathcal{M}_\gamma|$$ will be smaller or comparable to $$n$$, while $$S \subseteq \mathcal{M}_d$$ or $$S \subseteq \mathcal{M}_\gamma$$. To achieve this goal two steps are usually involved in the screening analysis. The first is to show there exists some $$\gamma$$ such that $$\min_{i \in S} |\hat{\beta}_i| > \gamma$$ and the second step is to bound the size of $$|\mathcal{M}_\gamma|$$ such that $$|\mathcal{M}_\gamma| = O(n)$$. To unify these steps for a more comprehensive theoretical framework, we put forward a slightly stronger definition of screening consistency in this article.

**Definition 2.1.** (Strong screening consistency) An estimator $$\hat{\beta}$$ (of $$\beta$$) is strong screening consistent if it satisfies that

$$\min_{i \in S} |\hat{\beta}_i| > \max_{i \notin S} |\hat{\beta}_i| \quad (1)$$

and

$$\text{sign}(\hat{\beta}_i) = \text{sign}(\beta_i), \quad \forall i \in S. \quad (2)$$

**Remark 2.1.** This definition does not differ much from the usual screening property studied in the literature, which requires $$\min_{i \in S} |\hat{\beta}_i| > \max_{i \notin S}^{(n-s)} |\hat{\beta}_i|$$, where $$\max^{(k)}$$ denotes the $$k$$th largest item.

The key of strong screening consistency is the property (1) that requires the estimator to preserve consistent ordering of the zero and non-zero coefficients. It is weaker than variable selection consistency in [Zhao and Yu (2006)]. The requirement in (2) can be seen as a relaxation of the sign consistency defined in [Zhao and Yu (2006)], as no requirement for $$\hat{\beta}_i, i \notin S$$ is needed. As shown later, such relaxation tremendously reduces the restriction on the design matrix, and allows screening methods to work for a broader choice of $$X$$.

The focus of this article is to study the theoretical properties of a special class of screeners that take the linear form as

$$\hat{\beta} = AY$$

for some $$p \times n$$ ancillary matrix $$A$$. Examples include sure independence screening ($$SIS$$) where $$A = X^T/n$$ and high-dimensional ordinary least-square projection ($$HOLP$$) where $$A = X^T(XX^T)^{-1}$$. We choose to study the class of linear estimators because linear screening
is computationally efficient and theoretically tractable. We note that the usual ordinary least-squares estimator is also a special case of linear estimators although it is not well defined for \( p > n \).

### 3 Deterministic guarantees

In this section, we derive the necessary and sufficient condition that guarantees \( \hat{\beta} = AY \) to be strong screening consistent. The design matrix \( X \) and the error \( \epsilon \) are treated as fixed in this section and we will investigate random designs later. We consider the set of sparse coefficient vectors defined by

\[
B(s, \rho) = \left\{ \beta \in \mathbb{R}^p : |\text{supp}(\beta)| \leq s, \quad \frac{\max_{i \in \text{supp}(\beta)} |\beta_i|}{\min_{i \in \text{supp}(\beta)} |\beta_i|} \leq \rho \right\}.
\]

The set \( B(s, \rho) \) contains vectors having at most \( s \) non-zero coordinates with the ratio of the largest and smallest coordinate bounded by \( \rho \). Before proceeding to the main result of this section, we introduce some terminology that helps to establish the theory.

**Definition 3.1.** (Weak diagonally dominant matrix) A \( p \times p \) symmetric matrix \( \Phi \) is weak diagonally dominant with sparsity \( s \) if for any \( I \subseteq Q, |I| \leq s - 1 \) and \( i \in Q \setminus I \)

\[
\Phi_{ii} > C_0 \max \left\{ \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}|, \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right\} + |\Phi_{ik}| \quad \forall k \neq i, k \in Q \setminus I,
\]

where \( C_0 \geq 1 \) is a constant.

Notice this definition implies that for \( i \in Q \setminus I \)

\[
\Phi_{ii} \geq C_0 \left( \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}| + \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right) / 2 \geq C_0 \sum_{j \in I} |\Phi_{ij}|,
\]

which is related to the usual diagonally dominant matrix. The weak diagonally dominant matrix provides a necessary and sufficient condition for any linear estimators \( \hat{\beta} = AY \) to be strong screening consistent. More precisely, we have the following result.

**Theorem 1.** For the noiseless case where \( \epsilon = 0 \), a linear estimator \( \hat{\beta} = AY \) is strong screening consistent for every \( \beta \in B(s, \rho) \), if and only if the screening matrix \( \Phi = AX \) is weak diagonally dominant with sparsity \( s \) and \( C_0 \geq \rho \).

The noiseless case is a good starting point to analyze \( \hat{\beta} \). Intuitively, in order to preserve the correct order of the coefficients in \( \hat{\beta} = AX \beta \), one needs \( AX \) to be close to a diagonally...
dominant matrix, so that \( \hat{\beta}_i, i \in \mathcal{M}_S \) will take advantage of the large diagonal terms in \( AX \) to dominate \( \hat{\beta}_i, i \notin \mathcal{M}_S \) that is just linear combinations of off-diagonal terms.

When noise is considered, the condition in Theorem 1 needs to be changed slightly to accommodate extra discrepancies. In addition, the smallest non-zero coefficient has to be lower bounded to ensure a certain level of signal-to-noise ratio. Thus, we augment our previous definition of \( \mathcal{B}(s, \rho) \) to have a signal strength control

\[
\mathcal{B}_\tau(s, \rho) = \{ \beta \in \mathcal{B}(s, \rho) | \min_{i \in \text{supp}(\beta)} |\beta_i| \geq \tau \}.
\]

Then we can obtain the following modified Theorem.

**Theorem 2.** With noise, the linear estimator \( \hat{\beta} = AY \) is strong screening consistent for every \( \beta \in \mathcal{B}_\tau(s, \rho) \) if \( \Phi = AX - 2\tau^{-1}\|A\epsilon\|_\infty I_p \) is weak diagonally dominant with sparsity \( s \) and \( C_0 \geq \rho \).

The condition in Theorem 2 can be further tailored to a necessary and sufficient version with extra manipulation on the noise term. Nevertheless, this might not be useful in practice due to the randomness in noise. In addition, the current version of Theorem 2 is already tight in the sense that there exists some noise vector \( \epsilon \) such that the condition in Theorem 2 is also necessary for strong screening consistency.

Theorems 1 and 2 establish ground rules for verifying consistency of a given screener and provide practical guidance for screening design. In Section 4, we consider some concrete examples of ancillary matrix \( A \) and prove that conditions in Theorems 1 and 2 are satisfied by the corresponding screeners with large probability under random designs.

## 4 Relationship with the irrepresentable condition

The weak diagonally dominant (WDD) condition is closely related to the strong irrepresentable condition (IC) proposed in [Zhao and Yu (2006)] as a necessary and sufficient condition for sign consistency of lasso. Assume each column of \( X \) is standardized to have mean zero. Letting \( C = X^TX/n \) and \( \beta \) be a given coefficient vector, the IC is expressed as

\[
\|C_{S^c, S} C_{S, S}^{-1} \cdot \text{sign}(\beta_S)\|_\infty \leq 1 - \theta
\]

for some \( \theta > 0 \), where \( C_{A,B} \) represents the sub-matrix of \( C \) with row indices in \( A \) and column indices in \( B \). The authors enumerate several scenarios of \( C \) such that IC is satisfied. Following result verifies some of these scenarios for screening matrix \( \Phi \).
Corollary 1. If $\Phi_{ii} = 1$, $\forall i$ and $|\Phi_{ij}| < c/(2s)$, $\forall i \neq j$ for some $0 \leq c < 1$ as defined in Corollary 1 and 2 in Zhao and Yu (2006), then $\Phi$ is a weak diagonally dominant matrix with sparsity $s$ and $C_0 \geq 1/c$.

If $|\Phi_{ij}| < r^{\|i-j\|}$, $\forall i, j$ for some $0 < r < 1$ as defined in Corollary 3 in Zhao and Yu (2006), then $\Phi$ is a weak diagonally dominant matrix with sparsity $s$ and $C_0 \geq (1-r)^2/(4r)$.

A more explicit but nontrivial relationship between IC and WDD is illustrated below when $|S| = 2$.

Theorem 3. Assume $\Phi_{ii} = 1$, $\forall i$ and $|\Phi_{ij}| < r$, $\forall i \neq j$. If $\Phi$ is weak diagonally dominant with sparsity 2 and $C_0 \geq \rho$, then $\Phi$ satisfies

$$\|\Phi_{S^c,S}^{-1} \cdot \Phi_{S,S}^{-1} \cdot \text{sign}(\beta_S)\|_\infty \leq \frac{\rho^{-1}}{1-r}$$

for all $\beta \in \mathcal{B}(2, \rho)$. On the other hand, if $\Phi$ satisfies the irrepresentable condition for all $\beta \in \mathcal{B}(2, \rho)$ for some $\theta$, then $\Phi$ is a weak diagonally dominant matrix with sparsity 2 and

$$C_0 \geq \frac{1}{1-\theta} \frac{1-r}{1+r}.$$

Theorem 3 demonstrates certain equivalence between IC and WDD. However, it is worth noting that IC is directly imposed on the covariance matrix $X^T X/n$. This makes IC a strong assumption that is easily violated; for example, when the predictors are highly correlated. In contrast to IC, WDD is imposed on matrix $AX$ where there is still flexibility for choosing $A$. As we show in Section 4, the ancillary matrix $A$ defined in HOLP satisfies WDD even when predictors are highly correlated and IC fails to hold. Therefore, WDD is a weaker constraint in some sense.

For sure independence screening, the screening matrix $\Phi = X^T X/n$ coincides with the covariance matrix, making WDD and IC effectively equivalent. The following theorem formalizes this.

Theorem 4. Let $A = X^T / n$ and standardize columns of $X$ to have sample variance one. Assume $X$ satisfies the sparse Riesz condition (Zhang and Huang, 2008), i.e,

$$\min_{\pi \subseteq Q, |\pi| \leq s} \lambda_{\min}(X_{\pi}^T X_{\pi} / n) \geq \mu,$$

for some $\mu > 0$. Now if $AX$ is weak diagonally dominant with sparsity $s + 1$ and $C_0 \geq \rho$ with $\rho > \sqrt{s}/\mu$, then $X$ satisfies the irrepresentable condition for any $\beta \in \mathcal{B}(s, \rho)$.

In other words, under the condition $\rho > \sqrt{s}/\mu$, the strong screening consistency of SIS for $\mathcal{B}(s + 1, \rho)$ implies the model selection consistency of lasso for $\mathcal{B}(s, \rho)$.
Theorem 4 illustrates the difficulty of SIS. The necessary condition that guarantees good screening performance of SIS also guarantees the model selection consistency of lasso. However, such a strong necessary condition does not mean that SIS should be avoided in practice given its substantial advantages in terms of simplicity and computational efficiency. The strong screening consistency defined in this article is stronger than conditions commonly used in justifying screening procedures as in Fan and Lv (2008).

5 Screening under random designs

In this section, we consider linear screening under random designs when \( X \) and \( \epsilon \) are Gaussian. The theory developed in this section can be easily extended to a broader family of distributions, for example, where \( \epsilon \) follows a sub-Gaussian distribution (Vershynin, 2010) and \( X \) follows an elliptical distribution (Fan and Lv, 2008; Wang and Leng, 2013). We focus on the Gaussian case for conciseness. Let \( \epsilon \sim N(0, \sigma^2) \) and \( X \sim N(0, \Sigma) \). We prove the screening consistency of SIS and HOLP by verifying the condition in Theorem 2. Recall the ancillary matrices for SIS and HOLP are defined respectively as

\[
A_{SIS} = X^T / n, \quad A_{HOLP} = X^T (XX^T)^{-1}.
\]

For simplicity, we assume \( \Sigma_{ii} = 1 \) for \( i = 1, 2, \ldots, p \). To verify the WDD condition, it is essential to quantify the magnitude of the entries of \( AX \) and \( A\epsilon \).

**Lemma 1.** Let \( \Phi = A_{SIS} X \), then for any \( t > 0 \) and \( i \neq j \in Q \), we have

\[
P \left( |\Phi_{ii} - \Sigma_{ii}| \geq t \right) \leq 2 \exp \left\{ - \min \left( \frac{t^2 n}{8e^2 K}, \frac{tn}{2eK} \right) \right\},
\]

and

\[
P \left( |\Phi_{ij} - \Sigma_{ij}| \geq t \right) \leq 6 \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{tn}{6eK} \right) \right\},
\]

where \( K = \|X^2(1) - 1\|_{\psi_1} \) is a constant, \( X^2(1) \) is a chi-square random variable with one degree of freedom and the norm \( \| \cdot \|_{\psi_1} \) is defined in Vershynin (2010).

Lemma 1 states that the screening matrix \( \Phi = A_{SIS} X \) for SIS will eventually converge to the covariance matrix \( \Sigma \) in \( l_\infty \) when \( n \) tends to infinity and \( \log p = o(n) \). Thus, the screening performance of SIS strongly relies on the structure of \( \Sigma \). In particular, the (asymptotically) necessary and sufficient condition for SIS being strong screening consistent is \( \Sigma \) satisfying the WDD condition. For the noise term, we have the following lemma.
Lemma 2. Let \( \eta = A_{SIS} \epsilon \). For any \( t > 0 \) and \( i \in Q \), we have

\[
P(|\eta_i| \geq \sigma t) \leq 6 \exp \left\{ -\min \left( \frac{t^2 n}{72 e^2 K}, \frac{tn}{6eK} \right) \right\},
\]

where \( K \) is defined the same as in Lemma 1.

The proof of Lemma 2 is essentially the same as the proof of off-diagonal terms in Lemma 1 and is thus omitted. As indicated before, the necessary and sufficient condition for SIS to be strong screening consistent is that \( \Sigma \) follows WDD. As WDD is usually hard to verify, we consider a stronger sufficient condition inspired by Corollary 1.

Theorem 5. Let \( r = \max_{i \neq j} |\Sigma_{ij}| \). If \( r < \frac{1}{2ps} \), then for any \( \delta > 0 \), if the sample size satisfies

\[
n > 144K \left( \frac{1 + 2\rho s + 2\sigma / \tau}{1 - 2\rho sr} \right)^2 \log(3p/\delta),
\]

(5)

where \( K \) is defined in Lemma 1, then with probability at least \( 1 - \delta \), \( \Phi = A_{SIS} X - 2\tau^{-1} ||A_{SIS} \epsilon||_{\infty} I_p \) is weak diagonally dominant with sparsity \( s \) and \( C_0 \geq \rho \). In other words, SIS is screening consistent for any \( \beta \in B_{r}(s, \rho) \).

The requirement that \( \max_{i \neq j} |\Sigma_{ij}| < 1/(\rho sr) \) or the necessary and sufficient condition that \( \Sigma \) is WDD strictly constrains the correlation structure of \( X \), causing the difficulty for SIS to be strong screening consistent. Moreover, these constraints are independent of sample sizes and cannot be overcome by large \( n \). As shown below, HOLP relaxes these constraints by a constraint on the conditional number \( \kappa = \lambda_{\max}(\Sigma)/\lambda_{\min}(\Sigma) \). For HOLP we instead have the following result.

Lemma 3. Let \( \Phi = A_{HOLP} X \). Assume \( p > c_0 n \) for some \( c_0 > 1 \), then for any \( C > 0 \) there exists some \( 0 < c_1 < 1 < c_2 \) and \( c_3 > 0 \) such that for any \( t > 0 \) and any \( i \in Q, j \neq i \), we have

\[
P \left( |\Phi_{ii}| < c_1 \kappa^{-1} \frac{n}{p} \right) \leq 2e^{-Cn}, \quad P \left( |\Phi_{ii}| > c_2 \kappa \frac{n}{p} \right) \leq 2e^{-Cn}
\]

and

\[
P \left( |\Phi_{ij}| > c_4 \kappa t \frac{\sqrt{n}}{p} \right) \leq 5e^{-Cn} + 2e^{-t^2/2},
\]

where \( c_4 = \sqrt{\frac{c_2(c_0-c_1)}{c_3(c_0-1)}} \).

Lemma 3 quantifies the entries of the screening matrix for HOLP. As illustrated in the
lemma, regardless of the covariance $\Sigma$, diagonal terms of $\Phi$ are always $O(\frac{p}{n})$ and the off-diagonal terms are $O(\sqrt{\frac{p}{n}})$. Thus, with $n \geq O(s^2)$, $\Phi$ is likely to satisfy the WDD condition with large probability. For the noise vector we have the following result.

**Lemma 4.** Let $\eta = A_{HOLP} \epsilon$. Assume $p > c_0 n$ for some $c_0 > 1$, then for any $C > 0$ there exist the same $c_1, c_2, c_3$ as in Lemma 3 such that for any $t > 0$ and $i \in Q$,

$$P(|\eta_i| \geq \frac{2\sigma \sqrt{c_2 \kappa t \sqrt{n}}}{1 - c_0^{-1} \frac{1}{p}}) < 4e^{-Cn} + 2e^{-t^2/2},$$

if $n \geq 8C/(c_0 - 1)^2$.

The following theorem results after combining Lemma 3 and 4.

**Theorem 6.** Assume $p > c_0 n$ for some $c_0 > 1$. For any $\delta > 0$, if the sample size satisfies

$$n > \max \left\{ 2C' \kappa^4 (\rho s + \sigma/\tau)^2 \log(3p/\delta), \frac{8C}{(c_0 - 1)^2} \right\},$$

where $C' = \max\{4c_1^2, \frac{4c_2}{c_2^2(1-c_0^{-1})^2}\}$ and $c_1, c_2, c_3, c_4, C$ are the same constants defined in Lemma 3, then with probability at least $1 - \delta$, $\Phi = A_{HOLP} X - 2\tau^{-1}||A_{HOLP}\epsilon||_\infty I_p$ is weak diagonally dominant with sparsity $s$ and $C_0 \geq \rho$. This implies HOLP is screening consistent for any $\beta \in B_r(s, \rho)$.

There are several interesting observations on equation (5) and (6). First, $(\rho s + \sigma/\tau)^2$ appears in both expressions, suggesting the optimal screener might also possess a similar sample size requirement in the form of

$$n_{opt} = O((\rho s + \sigma/\tau)^2 \log(p)).$$

We note that $\rho s$ evaluates the sparsity and the diversity of the signal $\beta$ while $\sigma/\tau$ is closely related to the signal-to-noise ratio. Furthermore, HOLP relaxes the correlation constraint $r < 1/(2\rho s)$ or the covariance constraint ($\Sigma$ is WDD) with the conditional number constraint. Thus for any $\Sigma$, as long as the sample size is large enough, strong screening consistency is assured. Finally, HOLP provides an example to satisfy the WDD condition in answer to the question raised in Section 2.

### 6 Concluding remarks

This article studies the theoretical properties of a class of high dimensional variable screening methods. In particular, we establish a necessary and sufficient condition in the form of weak
diagonally dominant screening matrices for strong screening consistency of a linear screener. We verify the condition for both SIS and HOLP under random designs. In addition, we show a close relationship between WDD and the irrepresentable condition, highlighting the difficulty of using SIS in screening for arbitrarily correlated predictors.

For future work, it is of interest to see how linear screening can be adapted to compressed sensing (Xue and Zou, 2011) and how techniques such as preconditioning (Jia and Rohe, 2012) can improve the performance of marginal screening and variable selection.

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A Proofs for Section 3

In this section, we prove the two theorems in Section 3.
Proof of Theorem \[7\]. If $\Phi$ is weak diagonally dominant with sparsity $s$ and $C_0 \geq \rho$, we have for any $I \subseteq Q$ and $|I| \leq s - 1$,

$$\Phi_{ii} > \rho \max \left\{ \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}|, \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right\} + |\Phi_{ik}| \quad \forall k \neq i \in Q \setminus I.$$

Recall $\hat{\beta} = \Phi\beta$. Suppose $S$ is the index set of non-zero predictors. For any $i \in S, k \notin S$, of we fix $I = S \setminus \{i\}$, we have

$$|\hat{\beta}_i| = |\Phi_{ii}\beta_i + \sum_{j \in I} \Phi_{ij}\beta_j| \geq |\beta_i|(\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{ij})$$

$$= |\beta_i|(\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i}(\Phi_{ij} + \Phi_{kj} - \Phi_{ki}) - \sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{kj} + \Phi_{ki})$$

$$> -|\beta_i|(\sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{kj} + \Phi_{ki}) = -\frac{|\beta_i|}{\beta_i}(\sum_{j \in I} \beta_j\Phi_{kj} + \beta_i\Phi_{ki})$$

$$= -\text{sign}(\beta_i) \cdot \hat{\beta}_k.$$

Similarly we have

$$|\hat{\beta}_i| = |\Phi_{ii}\beta_i + \sum_{j \in I} \Phi_{ij}\beta_j| \geq |\beta_i|(\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{ij})$$

$$= |\beta_i|(\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i}(\Phi_{ij} - \Phi_{kj}) + \sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{kj} + \Phi_{ki})$$

$$> |\beta_i|(\sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{kj} + \Phi_{ki}) = \text{sign}(\beta_i) \cdot \hat{\beta}_k.$$

Therefore, whatever value \text{sign}(\beta_i) is, it always holds that $|\hat{\beta}_i| > |\hat{\beta}_k|$. Since this result is true for any $i \in S, k \notin S$, we have

$$\min_{i \in S} |\hat{\beta}_i| > \max_{k \notin S} |\hat{\beta}_k|.$$

To prove the sign consistency for non-zero coefficients, notice that for $i \in S$,

$$\Phi_{ii} > \rho \left( \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}| + \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right) / 2 \geq \rho \sum_{j \in I} |\Phi_{ij}|.$$

Thus,

$$\hat{\beta}_i\beta_i = \Phi_{ii}\beta_i^2 + \sum_{j \in I} \Phi_{ij}\beta_j\beta_i = \beta_i^2(\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i}\Phi_{ij}) > 0.$$
On the other hand, if \( \hat{\beta} \) is screening consistent, i.e., \( |\hat{\beta}_i| \geq |\hat{\beta}_k| \) and \( \hat{\beta}_i \beta_i \geq 0 \), we can construct \( S = I \cup \{i\} \) for any fixed \( i, k, I \). Without loss of generality, we assume \( \Phi_{ik} \geq 0 \). If we select \( \beta \) such that \( \beta_i > 0 \), then the strong screening consistency implies \( \hat{\beta}_i > \hat{\beta}_k \) and \( \hat{\beta}_i > -\hat{\beta}_k \).

From \( \hat{\beta}_i > \hat{\beta}_k \) we have

\[
\Phi_{ii}\beta_i + \sum_{j \in I} \Phi_{ij}\beta_j > \sum_{j \in I} \Phi_{kj}\beta_j + \Phi_{ki}\beta_i.
\]

By rearranging terms and selecting \( \beta \in B(s, \rho) \) as \( \beta_i = 1 \), \( \beta_j = -\rho \cdot \text{sign}(\Phi_{ij} - \Phi_{kj}), j \in S \) we have

\[
\Phi_{ii} > -\sum_{j \in I} (\Phi_{ij} - \Phi_{kj})\beta_j + \Phi_{ki} = \rho \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| + |\Phi_{ki}|.
\]

Following the same argument on \( \hat{\beta}_i \geq -\hat{\beta}_k \) with a choice of \( \beta_i = 1 \), \( \beta_j = -\rho \cdot \text{sign}(\Phi_{ij} + \Phi_{kj}), j \in S \) we have

\[
\Phi_{ii} > \rho \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}| + |\Phi_{ki}|.
\]

This concludes the proof. \( \square \)

**Proof of Theorem**

Proof of Lemma 3 follows almost the same as the sufficiency part of Theorem. Notice that now the definition of \( \hat{\beta} \) becomes

\[
\hat{\beta} = X^T (XX^T)^{-1}X \beta + X^T (XX^T)^{-1} \epsilon.
\]

If the condition holds, i.e., for any \( i \in S \), \( I = S \setminus \{i\} \) and \( k \notin S \), we have

\[
\Phi_{ii} > \rho \max \left\{ \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}|, \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right\} + |\Phi_{ik}| + 2\tau^{-1} \|X^T (XX^T)^{-1} \epsilon\|_\infty.
\]

Defining \( \eta = X^T (XX^T)^{-1} \epsilon \), we have for any \( i \in S \),

\[
|\hat{\beta}_i| = |\Phi_{ii}\beta_i + \sum_{j \in I} \Phi_{ij}\beta_j + \eta_i| \geq |\beta_i| (|\Phi_{ii}| + \sum_{j \in I} \frac{\beta_j}{\beta_i} (\Phi_{ij} + \Phi_{kj}) + \Phi_{ki} + \beta_i^{-1} \eta_i - \sum_{j \in I} \frac{\beta_j}{\beta_i} (\Phi_{kj} - \Phi_{ki} - \beta_i^{-1} \eta_k))
\]

\[
> -|\beta_i| (\sum_{j \in I} \frac{\beta_j}{\beta_i} (\Phi_{kj} + \Phi_{ki} + \beta_i^{-1} \eta_k)) = -|\beta_i| (\sum_{j \in I} \beta_j \Phi_{kj} + \beta_i \Phi_{ki} + \eta_k)
\]

\[
= -\text{sign}(\beta_i) \cdot \hat{\beta}_k,
\]
Similarly, we can prove \(|\hat{\beta}_i| > \text{sign}(\beta_i) \cdot \hat{\beta}_k\), and thus \(|\hat{\beta}_i| > |\hat{\beta}_k|\), which implies that

\[
\min_{i \in S} |\hat{\beta}_i| > \max_{k \notin S} |\hat{\beta}_k|.
\]

The weak sign consistency is established since

\[
\hat{\beta}_i \beta_i = \Phi_{ii} \beta_i^2 + \sum_{j \in I} \Phi_{ij} \beta_j \beta_i + \eta_i \beta_i = \beta_i^2 (\Phi_{ii} + \sum_{j \in I} \frac{\beta_j}{\beta_i} \Phi_{ij} + \beta_i^{-1} \eta_i) > 0,
\]

for any \(\beta_i \neq 0\).

The tightness of this theorem is given by the case when \(\epsilon = 0\), for which the condition has already been shown to be necessary and sufficient in Theorem 1.

\(\square\)

### B Proofs for Section 4

In this section, we prove results from Section 4.

**Proof of Corollary** \[\square\] Letting \(I \subseteq Q, |I| \leq s - 1\), we have for any \(i \neq k \in Q \setminus I\),

\[
\Phi_{ii} - \frac{1}{c} \max \left\{ \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}|, \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right\} + |\Phi_{ik}| \geq 1 - \frac{1}{c} \left( 2(s - 1) \frac{c}{2s} + \frac{c}{2s} \right) = \frac{1}{2s} > 0.
\]

This completes the proof for the first case.

Now for the second case, notice that the sum of an entire row (except the diagonal term) can be bounded by \(\sum_{j \neq i} |\Phi_{ij}| < 2 \sum_{k=1}^{\infty} r^k < \frac{2r}{1-r}\). Therefore, we have

\[
\Phi_{ii} - \frac{(1-r)^2}{4r} \max \left\{ \sum_{j \in I} |\Phi_{ij} + \Phi_{kj}|, \sum_{j \in I} |\Phi_{ij} - \Phi_{kj}| \right\} - |\Phi_{ik}| > 1 - \frac{(1-r)^2}{2r} \sum_{j \neq i} |\Phi_{ij}| - r = 0.
\]

\(\square\)

**Proof of Theorem** \[\square\] First, from WDD to IC: Without loss of generality, we assume \(S = \{1, 2\}\). For any \(k \in Q \setminus S\), we have

\[
|\Phi_{k1} \Phi_{k2} \Phi^{-1}_{S,S} \text{sign}(\beta_S)| = \left| \frac{\text{sign}(\beta_1)(\Phi_{k1} - \Phi_{12} \Phi_{k2}) + \text{sign}(\beta_2)(-\Phi_{12} \Phi_{k1} + \Phi_{k2})}{1 - \Phi^2_{12}} \right|.
\]

The r.h.s. becomes \(|\Phi_{k1} + \Phi_{k2}|(1 - \Phi_{12})/(1 - \Phi^2_{12})\) when \(\text{sign}(\beta_1) = \text{sign}(\beta_2)\) and \(|\Phi_{k1} - \Phi_{k2}|(1 - \Phi_{12})/(1 - \Phi^2_{12})\) when \(\text{sign}(\beta_1) \neq \text{sign}(\beta_2)\).
\[ \Phi_{k2}(1 + \Phi_{12})/(1 - \Phi_{12}^2) \text{ when } \text{sign}(\beta_1) = -\text{sign}(\beta_2). \] In either case we have

\[
\left| \Phi_{k1} \Phi_{k2} \Phi_{S}^{-1} s \text{sign}(\beta_S) \right| \leq \frac{\max \left\{ |\Phi_{1k} + \Phi_{2k}|, |\Phi_{1k} - \Phi_{2k}| \right\}}{1 - r} < \frac{\rho^{-1}}{1 - r}.
\]

Second, from IC to WDD: Let \( I \subseteq Q, |I| = 1 \) and \( i \neq k \in Q \setminus I \). Without loss of generality, we assume \( i = 1, k = 2 \), and we construct \( S = \{1, 2\} \). Now for any \( j \in I \), using the same formula as shown above, we have

\[
1 - \theta > \left| \Phi_{j1} \Phi_{j2} \Phi_{S}^{-1} s \text{sign}(\beta_S) \right| = \left| \frac{\text{sign}(\beta_1)(\Phi_{j1} - \Phi_{12}\Phi_{j2}) + \text{sign}(\beta_2)(-\Phi_{12}\Phi_{j1} + \Phi_{j2})}{1 - \Phi_{12}^2} \right|.
\]

Using the same result on the r.h.s., i.e., it becomes \( |\Phi_{k1} + \Phi_{k2}|(1 - \Phi_{12})/(1 - \Phi_{12}^2) \) when \( \text{sign}(\beta_1) = \text{sign}(\beta_2) \) and \( |\Phi_{k1} - \Phi_{k2}|(1 + \Phi_{12})/(1 - \Phi_{12}^2) \) when \( \text{sign}(\beta_1) = -\text{sign}(\beta_2) \), we have for any \( j \in I \) that

\[
\max \left\{ |\Phi_{1j} + \Phi_{2j}|, |\Phi_{1j} - \Phi_{2j}| \right\} \leq (1 - \theta)(1 + r).
\]

As a result, we have

\[
\sum_{j \in I} \max \left\{ |\Phi_{1j} + \Phi_{2j}|, |\Phi_{1j} - \Phi_{2j}| \right\} < (1 - \theta)(1 + r) < (1 - \theta) \frac{1 + r}{1 - r} \left( \Phi_{11} - |\Phi_{12}| \right),
\]

which implies

\[
\Phi_{11} > \frac{1}{1 - \theta} \frac{1 - r}{1 + r} \sum_{j \in I} \max \left\{ |\Phi_{1j} + \Phi_{2j}|, |\Phi_{1j} - \Phi_{2j}| \right\} + |\Phi_{12}|.
\]

**Proof of Theorem** We just need to check (3). We prove the absolute value of the first coordinate of \( C_{S^c, s} C_{S^c, s}^{-1} \cdot \text{sign}(\beta_S) \) is less than one, and the rest just follow the same argument. From the condition we know \( C = X^TX/n \) is weak diagonally dominant. Then equation (3) implies that for any \( I \subseteq Q \) with \( |I| = s \), we have for any \( k \notin I \),

\[
\rho \sum_{i \in I} |C_{ki}| < 1.
\]

Now for any \( S \subseteq Q \) with \( |S| = s \), we choose \( I = S \) and let \( \alpha^T \) be the first row of \( C_{S^c, s} = \)
we have
\[ |\alpha^T(X_S^TX_S/n)^{-1}\text{sign}(\beta_S)| \leq \|\alpha\|_2\|\text{sign}(\beta_S)\|_2 \mu^{-1}. \]

Because \( \rho \sum_{i=1}^s |\alpha_i| < 1 \), we have
\[ \rho^2 \sum_{i=1}^s \alpha_i^2 < \rho^2 (\sum_{i=1}^s |\alpha_i|)^2 < 1, \]

which implies that
\[ |\alpha^T(X_S^TX_S/n)^{-1}\text{sign}(\beta_S)| \leq \rho^{-1} \sqrt{s} \mu^{-1} = \frac{\sqrt{s}}{\rho \mu} < 1. \]

\[ \square \]

C Proofs for Section 5 (SIS)

Proofs in Section 6 are divided into two parts. In this section, we provide the proofs related to SIS, and leave those pertaining to HOLP to the next section. The proof requires the following proposition,

**Proposition 1.** Assume \( X_i \sim \chi^2(1), i = 1, 2, \ldots, n \), where \( \chi^2(1) \) is the chi-square distribution with one degree of freedom. Then for any \( t > 0 \), we have
\[ P(|\sum_{i=1}^n X_i/n - 1| \geq t) \leq 2 \exp \left\{ -\min \left( \frac{t^2 n}{8e^2 K}, \frac{tn}{2eK} \right) \right\}, \]

where \( K = \|\chi^2(1) - 1\|_{\psi_1} \). Alternatively, for any \( C > 0 \), there exists some \( 0 < c_3 < 1 < c_4 \) such that,
\[ P\left( \frac{\sum_{i=1}^n X_i}{n} \leq c_3 \right) \leq e^{-Cn}, \tag{7} \]

and
\[ P\left( \frac{\sum_{i=1}^n X_i}{n} \geq c_4 \right) \leq e^{-Cn}. \]

**Proof.** It is a direct application of Proposition 5.16 in Vershynin (2010). Notice that in the proof of Proposition 5.16 we have \( C = 2e^2 \) and \( c = e/2 \) for \( \chi^2(1) - 1 \).

\[ \square \]
Proof of Lemma 1. For diagonal term we have for any \( i \in \{1, 2, \ldots, p\} \)

\[
\Phi_{ii} - \Sigma_{ii} = \sum_{k=1}^{n} \frac{x_{ik}^2}{n} - 1,
\]

where \( x_{ik}, k = 1, 2, \ldots, n \)'s are \( n \) iid standard normal random variables. Using Proposition 1 we have for any \( t > 0 \),

\[
P\left( |\Phi_{ii} - \Sigma_{ii}| \geq t \right) \leq 2 \exp \left\{ - \min \left( \frac{t^2 n}{8e^2 K}, \frac{tn}{2eK} \right) \right\}. \tag{8}
\]

For the off-diagonal term, we have for any \( i \neq j \),

\[
\Phi_{ij} - \Sigma_{ij} = \frac{\sum_{k=1}^{n} x_{ik} x_{jk}}{n} - \Sigma_{ij} \\
= \frac{\sum_{k=1}^{n} (x_{ik} + x_{jk})^2}{2n} - \frac{\sum_{k=1}^{n} x_{ik}^2}{2n} - \frac{\sum_{k=1}^{n} x_{jk}^2}{2n} - \Sigma_{ij} \\
= \frac{1}{2} \left( \frac{\sum_{k=1}^{n} (x_{ik} + x_{jk})^2}{n} - (2 + 2\Sigma_{ij}) \right) - \frac{1}{2} \left( \frac{\sum_{k=1}^{n} x_{ik}^2}{n} - 1 \right) - \frac{1}{2} \left( \frac{\sum_{k=1}^{n} x_{jk}^2}{n} - 1 \right).
\]

Notice that \( x_{ik} + x_{jk} \sim N(0, 2 + 2\Sigma_{ij}) \). Hence the three terms in the above equation can be bounded using the same inequality before, i.e., for any \( t > 0 \),

\[
P\left( |\Phi_{ij} - \Sigma_{ij}| \geq (2 + \Sigma_{ij}) t \right) \leq 6 \exp \left\{ - \min \left( \frac{tn}{72e^2 K}, \frac{tn}{6eK} \right) \right\}.
\]

Clearly, we have \( \Sigma_{ij} \leq \sqrt{\Sigma_{ii}} \sqrt{\Sigma_{jj}} \leq 1 \). Therefore, we have

\[
P\left( |\Phi_{ij} - \Sigma_{ij}| \geq t \right) \leq 6 \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{tn}{6eK} \right) \right\}.
\]

\(\square\)

Proof of Lemma 2. The proof is essentially the same for proving the off diagonal terms of \( \Phi \) as in Lemma 1. The only difference is that \( E(\Phi_{ij}) = \Sigma_{ij} \) while \( E(X_\epsilon) = 0 \). Note

\[
\eta_i / \sigma = \frac{\sum_{k=1}^{n} x_{ik} \epsilon_k / \sigma}{n} = \frac{\sum_{k=1}^{n} (x_{ik} + \epsilon_k / \sigma)^2}{2n} - \frac{\sum_{k=1}^{n} x_{ik}^2}{2n} - \frac{\sum_{k=1}^{n} \epsilon_k^2 / \sigma^2}{2n}.
\]

Using Proposition 1 we have

\[
P\left( |\eta_i / \sigma| \geq t \right) \leq 6 \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{tn}{6eK} \right) \right\}.
\]

\(\square\)
Proof of Theorem 5. Taking union bound on the results from Lemma 1 and 2, we have for any $t > 0$,

$$P\left( \min_{i \in Q} \Phi_{ii} \leq 1 - t \right) \leq 2p \exp \left\{ - \min \left( \frac{t^2 n}{8e^2 K}, \frac{t n}{2eK} \right) \right\},$$

$$P\left( \max_{i \neq j} |\Phi_{ij}| \geq r + t \right) \leq 6(p^2 - p) \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{t n}{6eK} \right) \right\},$$

and

$$P\left( \max_{i \in Q} |\eta_i| \geq \sigma t \right) \leq 6p \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{t n}{6eK} \right) \right\}.$$

Thus, when $p > 2$ we have

$$P\left( \min_{i \in Q} \Phi_{ii} \leq 1 - t \text{ or } \max_{i \neq j} |\Phi_{ij}| \geq r + t \text{ or } \max_{i \in Q} |\eta_i| \geq \sigma t \right) \leq 7p^2 \exp \left\{ - \min \left( \frac{t^2 n}{72e^2 K}, \frac{t n}{6eK} \right) \right\}.$$

In other words, for any $\delta > 0$, when $n \geq K \log(7p^2/\delta)$, with probability at least $1 - \delta$, we have

$$\min_{i \in Q} \Phi_{ii} \geq 1 - 6\sqrt{2e} \sqrt{\frac{K \log(7p^2/\delta)}{n}} , \quad \max_{i \neq j} |\Phi_{ij}| \leq r + 6\sqrt{2e} \sqrt{\frac{K \log(7p^2/\delta)}{n}},$$

and

$$\max_{i \in Q} |\eta_i| \leq 6\sqrt{2e} \sigma \sqrt{\frac{K \log(7p^2/\delta)}{n}}.$$

A sufficient condition for $\Phi$ to be weak diagonally dominant is that

$$\min_i \Phi_{ii} > 2\rho s \max_{i \neq j} |\Phi_{ij}| + 2\tau^{-1} \max_i |\eta_i|.$$

Plugging in the values and solving the inequality, we have (notice that $7p^2/\delta < 9p^2/\delta^2$) $\Phi$ is WDD as long as

$$n > 144K \left( \frac{1 + 2\rho s + 2\sigma/\tau}{1 - 2\rho sr} \right)^2 \log(3p/\delta).$$

This completes the proof. \qed
In this section we prove Lemma 3, 4 and Theorem 5. Several propositions and lemmas are needed for establishing the whole theory. We list all prerequisite results without proofs but provide readers references for complete proofs.

Let $P \in O(p)$ be a $p \times p$ orthogonal matrix from the orthogonal group $O(p)$. Let $H$ denote the first $n$ columns of $P$. Then $H$ is in the Stiefel manifold (Chikuse, 2003). In general, the Stiefel manifold $V_{n,p}$ is the space whose points are $n$-frames in $\mathbb{R}^p$ represented as the set of $p \times n$ matrices $X$ such that $X^T X = I_n$. Mathematically, we can write

$$V_{n,p} = \{ X \in \mathbb{R}^{p \times n} : X^T X = I_n \}.$$ 

There is a natural measure $(dX)$ called Haar measure on the Stiefel manifold, invariant under both right orthogonal and left orthogonal transformations. We standardize it to obtain a probability measure as 

$$[dX] = (dX)/V(n,p),$$ 

where $V(n,p) = 2^n n^{np/2}/\Gamma_n(1/2p)$.

**Lemma 5.** (Chikuse, 2003, Page 41-44) Supposed that a $p \times n$ random matrix $Z$ has the density function of the form

$$f_Z(Z) = |\Sigma|^{-n/2} g(Z^T \Sigma^{-1} Z),$$

which is invariant under the right-orthogonal transformation of $Z$, where $\Sigma$ is a $p \times p$ positive definite matrix. Then its orientation $H_z = Z(Z^T Z)^{-1/2}$ has the matrix angular central Gaussian distribution (MACG) with a probability density function

$$MACG(\Sigma) = |\Sigma|^{-n/2} |H_z^T \Sigma^{-1} H_z|^{-p/2}.$$ 

In particular, if $Z$ is a $p \times n$ matrix whose distribution is invariant under both the left- and right-orthogonal transformations, then $H_Y$, with $Y = BZ$ for $BB^T = \Sigma$, has the $MACG(\Sigma)$ distribution.

When $n = 1$, the MACG distribution becomes the angular central Gaussian distribution, a description of the multivariate Gaussian distribution on the unit sphere (Watson et al., 1983).

**Lemma 6.** (Chikuse, 2003, Page 70, Decomposition of the Stiefel manifold) Let $H$ be a $p \times n$ random matrix on $V_{n,p}$, and write

$$H = (H_1 \ H_2),$$
with $H_1$ being a $p \times q$ matrix where $0 < q < n$. Then we can write

$$H_2 = G(H_1)U_1,$$

where $G(H_1)$ is any matrix chosen so that $(H_1 G(H_1)) \in \mathcal{O}(p)$; as $H_2$ runs over $V_{n-q,p}$, $U_1$ runs over $V_{n-q,p-q}$ and the relationship is one to one. The differential form $[dH]$ for the normalized invariant measure on $V_{n,p}$ is decomposed as the product

$$[dH] = [dH_1][dU_1]$$

of those $[dH_1]$ and $[dU_1]$ on $V_{q,p}$ and $V_{n-p-q}$, respectively.

**Lemma 7.** [Lemma 4 in Fan and Lv (2008)] Let $U$ be uniformly distributed on the Stiefel manifold $V_{n,p}$. Then for any $C > 0$, there exist $c_1', c_2'$ with $0 < c_1' < 1 < c_2'$, such that

$$P\left( e_1^T U U^T e_1 < c_1' \frac{n}{p} \right) \leq 2e^{-Cn},$$

and

$$P\left( e_1^T U U^T e_1 > c_2' \frac{n}{p} \right) \leq 4e^{-Cn}.$$

Some of our proof requires concentration properties of a random Gaussian matrix and $X_i^2$ random variables. For a Wigner matrix, we have the following result.

**Lemma 8.** Assume $Z$ is a $n \times p$ matrix with $p > c_0n$ for some $c_0 > 1$. Each entry of $Z$ follows a Gaussian distribution with mean zero and variance one and are independent. Then for any $t > 0$, with probability at least $1 - 2\exp(-t^2/2)$, we have

$$(1 - c_0^{-1} - t/p)^2 \leq \lambda_{\min}(ZZ^T/p) < \lambda_{\max}(ZZ^T/p) \leq (1 + c_0^{-1} + t/p)^2.$$

For any $C > 0$, taking $t = \sqrt{2Cn}$, we have with probability $1 - 2\exp(-Cn/2)$,

$$(1 - c_0^{-1} - \frac{\sqrt{2C}}{c_0\sqrt{n}})^2 \leq \lambda_{\min}(ZZ^T/p) \leq (1 + c_0^{-1} + \frac{\sqrt{2C}}{c_0\sqrt{n}})^2.$$

**Proof.** This is essentially Corollary 5.35 in Vershynin (2010).

The conditional number of $\Sigma$ is controled by $\kappa$, which simultaneously controls the largest and the smallest eigenvalues.

**Proposition 2.** Assume the conditional number of $\Sigma$ is $\kappa$ and $\Sigma_{ii} = 1$ for $i = 1, 2, \cdots, p$,
then we have

\[ \lambda_{\min}(\Sigma) \geq \kappa^{-1} \quad \text{and} \quad \lambda_{\max}(\Sigma) \leq \kappa. \]

**Proof.** Notice that \( p = tr(\Sigma) = \sum_{i=1}^{p} \lambda_i \). Therefore, we have

\[ \frac{p}{\lambda_{\max}} \geq p\kappa^{-1} \quad \text{and} \quad \frac{p}{\lambda_{\min}(\Sigma)} \leq p\kappa, \]

which completes the proof. \(\square\)

Now we prove the main results for HOLP.

**Proof of Lemma** Consider a transformed \( n \times p \) random matrix \( Z = X\Sigma^{-1/2} \), which, by definition, follows standard multivariate Gaussian. Consider its SVD decomposition,

\[ Z = VDU^T, \]

where \( V \in \mathcal{O}(n) \), \( D \) is a diagonal matrix and \( U \) is a \( p \times n \) random matrix belonging to the Stiefel manifold \( V_{n,p} \). With such notion, we can rewrite the projection matrix as

\[ X^T(XX^T)^{-1}X = \Sigma^{1/2}U(U^T\Sigma U)^{-1}U^T\Sigma^{1/2} = HH^T, \]

where \( H = \Sigma^{1/2}U(U^T\Sigma U)^{-1/2} \) and \( H \in V_{n,p-1} \). Therefore, the two quantities that we are interested in are \( \Phi_{ii} = e_i^THH^Te_i \) (diagonal term) and \( \Phi_{ij} = e_i^THH^Te_j \) (off-diagonal term), where \( e_i^T \) is the \( p \)-dimensional unit vector with the \( i^{th} \) coordinate being one. The proof is divided into two parts, where in the first part we consider diagonal terms and the second part takes care of off-diagonal terms.

**Part I:** First, we consider the diagonal term \( e_i^THH^Te_i \). Recall the definition of \( H \) and

\[ e_i^THH^Te_i = e_i^T\Sigma^{1/2}U(U^T\Sigma U)^{-1}U^T\Sigma^{1/2}e_i. \]

There always exists some orthogonal matrix \( Q \) that rotates the vector \( \Sigma^{1/2}e_i \) to the direction of \( e_1 \), i.e,

\[ \Sigma^{1/2}v = \|\Sigma^{1/2}v\|Qe_1. \]
Then we have
\[ e^T_i H H^T e_i = \| \Sigma^{\frac{1}{2}} e_i \|^2 e^T_1 Q^T U (U^T \Sigma U)^{-1} U^T Q e_1 = \| \Sigma^{\frac{1}{2}} v \|^2 e^T_1 \tilde{U} (U^T \Sigma U)^{-1} \tilde{U} e_1, \]
where \( \tilde{U} = Q^T U \) is uniformly distributed on \( V_{n,p} \), because \( U \) is uniformly distributed on \( V_{n,p} \) (see discussion in the beginning). Now the magnitude of \( e^T_i H H^T e_i \) can be evaluated in two parts. For the norm of the vector \( \Sigma^{\frac{1}{2}} v \), we have
\[ \lambda_{\text{min}}(\Sigma) \leq e^T_i \Sigma e_i = \| \Sigma^{\frac{1}{2}} e_i \|^2 \leq \lambda_{\text{max}}(\Sigma), \quad (9) \]
and for the remaining part,
\[ e^T_1 \tilde{U} (U^T \Sigma U)^{-1} \tilde{U} e_1 \leq \lambda_{\text{max}}((U^T \Sigma U)^{-1}) \| \tilde{U} e_1 \|^2 \leq \lambda_{\text{min}}(\Sigma)^{-1} \| \tilde{U} e_1 \|^2, \]
and
\[ e^T_1 \tilde{U} (U^T \Sigma U)^{-1} \tilde{U} e_1 \geq \lambda_{\text{min}}((U^T \Sigma U)^{-1}) \| \tilde{U} e_1 \|^2 \geq \lambda_{\text{max}}(\Sigma)^{-1} \| \tilde{U} e_1 \|^2. \]
Consequently, we have
\[ e^T_i H H^T e_i \leq \frac{\lambda_{\text{max}}(\Sigma)}{\lambda_{\text{min}}(\Sigma)} e^T_1 \tilde{U} U^T e_1, \quad e^T_i H H^T e_i \geq \frac{\lambda_{\text{min}}(\Sigma)}{\lambda_{\text{max}}(\Sigma)} e^T_1 \tilde{U} U^T e_1. \quad (10) \]
Therefore, following Proposition 7 for any \( C > 0 \) we have
\[ P \left( e^T_i H H^T e_i < c'_1 c_4 \kappa^{-1} \frac{n}{p} \right) \leq 2e^{-C n}, \]
and
\[ P \left( e^T_i H H^T e_i > c'_2 c_4^{-1} \kappa \frac{n}{p} \right) \leq 2e^{-C n}. \]
Denoting \( c'_1 c_4 \) by \( c_1 \) and \( c'_2 c_4^{-1} \) by \( c_2 \), we obtain the equation in Lemma 3.

**Part II:** Second, for off-diagonal terms, although the proof is almost identical to the proof of Lemma 5 in Wang and Leng (2013), we still provide a complete version here due to the importance of this result.

The proof depends on the decomposition of Stiefel manifold. Without loss of generality, we prove the bound only for \( e^T_2 H H^T e_1 \), then the other off-diagonal terms should follow exactly the same argument. According to Lemma 6 we can decompose \( H = (T_1, H_2) \) with \( T_1 = G(H_2) H_1 \), where \( H_2 \) is a \( p \times (n-1) \) matrix, \( H_1 \) is a \( (p-n+1) \times 1 \) vector and \( G(H_2) \) is a matrix such that \((G(H_2), H_2) \in \mathcal{O}(p) \). The invariant measure on the Stiefel manifold can
be decomposed as

$$[H] = [H_1][H_2]$$

where $[H_1]$ and $[H_2]$ are Haar measures on $V_{1,n-p+1}, V_{n-1,p}$ (Notice that $q = n - 1$ in this decomposition) respectively. As pointed out before, $H$ has the $MACG(\Sigma)$ distribution, which possesses a density as

$$p(H) \propto |H^T \Sigma^{-1} H|^{-p/2}[dH].$$

Using the identity for matrix determinant

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B| = |D||A - BD^{-1}C|,$$

we have

$$P(H_1, H_2) \propto |H_2^T \Sigma^{-1} H_2|^{-p/2}(T_1^T \Sigma^{-1} T_1 - T_1^T \Sigma^{-1} H_2 (H_2^T \Sigma^{-1} H_2)^{-1} H_2^T \Sigma^{-1} T_1)^{-p/2}$$

$$= |H_2^T \Sigma^{-1} H_2|^{-p/2}(H_1^T G(H_2)^T (\Sigma^{-1} - \Sigma^{-1} H_2 (H_2^T \Sigma^{-1} H_2)^{-1} H_2^T \Sigma^{-1}) G(H_2) H_1)^{-p/2}$$

$$= |H_2^T \Sigma^{-1} H_2|^{-p/2}(H_1^T G(H_2)^T \Sigma^{-1/2} (I - T_2) \Sigma^{-1/2} G(H_2) H_1)^{-p/2},$$

where $T_2 = \Sigma^{-1/2} H_2 (H_2^T \Sigma^{-1} H_2)^{-1} H_2^T \Sigma^{-1/2}$ is an orthogonal projection onto the linear space spanned by the columns of $\Sigma^{-1/2} H_2$. It is easy to verify the following result by using the definition of $G(H_2)$,

$$\begin{bmatrix} \Sigma^{1/2} G(H_2) (G(H_2)^T \Sigma G(H_2))^{-1/2}, \Sigma^{-1/2} H_2 (H_2^T \Sigma^{-1} H_2)^{-1/2} \end{bmatrix} \in \mathcal{O}(p),$$

and therefore we have

$$I - T_2 = \Sigma^{1/2} G(H_2) (G(H_2)^T \Sigma G(H_2))^{-1/2} G(H_2)^T \Sigma^{1/2},$$

which simplifies the density function as

$$P(H_1, H_2) \propto |H_2^T \Sigma^{-1} H_2|^{-p/2}(H_1^T (G(H_2)^T \Sigma G(H_2))^{-1} H_1)^{-p/2}.$$
where
\[ \Sigma' = G(H_2)^T \Sigma G(H_2). \]

Next, we relate the target quantity \( e_1^T H H^T e_2 \) to the distribution of \( H_1 \). Notice that for any orthogonal matrix \( Q \in O(n) \), we have
\[ e_1^T H H^T e_2 = e_1^T H Q Q^T H^T e_2 = e_1^T H' H'^T e_2. \]

Write \( H' = HQ = (T'_1, H'_2) \), where \( T'_1 = [T_1'(1), T_1'(2), \ldots, T_1'(p)] \), \( H'_2 = [H_2'(i, j)] \). If we choose \( Q \) such that the first row of \( H'_2 \) are all zero (this is possible as we can choose the first column of \( Q \) being the first row of \( H \) upon normalizing), i.e.,
\[ e_1^T H' = [T_1'(1), 0, \ldots, 0] \quad \text{and} \quad e_2^T H' = [T_1'(2), H_2'(2, 1), \ldots, H_2'(2, n-1)], \]
then immediately we have \( e_1^T H H^T e_2 = e_1^T H' H'^T e_2 = T_1'(1) T_1'(2) \). This indicates that
\[ e_1^T H H^T e_2 \overset{(d)}{=} T_1'(1) T_1'(2) \mid e_1^T H_2 = 0. \]

As shown at the beginning, \( H_1 \) follows \( ACG(\Sigma') \) conditional on \( H_2 \). Let \( H_1 = (h_1, h_2, \ldots, h_p)^T \) and let \( x^T = (x_1, x_2, \ldots, x_{p-n+1}) \sim N(0, \Sigma') \), then we have
\[ h_i \overset{(d)}{=} \frac{x_i}{\sqrt{x_1^2 + \cdots + x_{p-n+1}^2}}. \]

Notice that \( T_1 = G(H_2)H_1 \), a linear transformation on \( H_1 \). Defining \( y = G(H_2)x \), we have
\[ T_1^{(i)} \overset{(d)}{=} \frac{y_i}{\sqrt{y_1^2 + \cdots + y_p^2}}, \tag{11} \]
where \( y \sim N(0, G(H)\Sigma'G(H)^T) \) is a degenerate Gaussian distribution. This degenerate distribution contains an interesting form. Letting \( z \sim N(0, \Sigma) \), we know \( y \) can be expressed as \( y = G(H)G(H)^T z \). Write \( G(H_2)^T \) as \( [g_1, g_2] \) where \( g_1 \) is a \( (p-n+1) \times 1 \) vector and \( g_2 \) is a \( (p-n+1) \times (p-1) \) matrix, then we have
\[ G(H_2)G(H_2)^T = \begin{pmatrix} g_1^T g_1 & g_1^T g_2 \\ g_2^T g_1 & g_2^T g_2 \end{pmatrix}. \]

We can also write \( H_2^T = [0_{n-1, 1}, h_2] \) where \( h_2 \) is a \( (n-1) \times (p-1) \) matrix, and using the
orthogonality, i.e., \([H_2 \, G(H_2)]^T = I_p\), we have
\[
g_1^T g_1 = 1, \quad g_1^T g_2 = 0_{1,p-1} \quad \text{and} \quad g_2^T g_2 = I_{p-1} - h_2 h_2^T.
\]

Because \(h_2\) is a set of orthogonal basis in the \(p-1\) dimensional space, \(g_2^T g_2\) is therefore an orthogonal projection onto the space \(\{h_2\}^\perp\) and \(g_2^T g_2 = AA^T\) where \(A = g_2^T (g_2 g_2^T)^{-1/2}\) is a \((p-1) \times (p-n)\) orientation matrix on \(\{h_2\}^\perp\). Together, we have
\[
y = \begin{pmatrix} 1 & 0 \\ 0 & AA^T \end{pmatrix} z.
\]

This relationship allows us to marginalize \(y_1\) out with \(y\) following a degenerate Gaussian distribution.

We now turn to transform the condition \(e_1^T H_2 = 0\) onto constraints on the distribution of \(T_1^{(i)}\). Letting \(t_1^2 = e_1^T H H^T e_1\), then \(e_1^T H_2 = 0\) is equivalent to \(T_1^{(1,2)} = e_1^T H H^T e_1 = t_1^2\), which implies that
\[
e_1^T H H^T e_2 \overset{(d)}{=} T_1^{(1)} T_1^{(2)} \quad \left| T_1^{(1,2)} = e_1^T H H^T e_1.\right.
\]

Because the magnitude of \(e_1^T H H^T e_1\) has been obtained in Part I, we can now condition on the value of \(e_1^T H H^T e_1\) to obtain the bound on \(T_1^{(2)}\). From \(T_1^{(1,2)} = t_1^2\), we obtain that,
\[
(1 - t_1^2) y_1^2 = t_1^2 (y_2^2 + y_3^2 + \cdots + y_p^2). \tag{12}
\]

Notice this constraint is imposed on the norm of \(\bar{y} = (y_2, y_3, \cdots, y_p)\) and is thus independent of \((y_2/\|\bar{y}\|, \cdots, y_p/\|\bar{y}\|)\). Equation (12) also implies that
\[
(1 - t_1^2) (y_1^2 + y_2^2 + \cdots + y_p^2) = y_2^2 + y_3^2 + \cdots + y_p^2. \tag{13}
\]

Therefore, combining (11) with (12), (13) and integrating \(y_1\) out, we have
\[
T_1^{(i)} \mid T_1^{(1)} = t_1 \overset{(d)}{=} \frac{\sqrt{1 - t_1^2} y_i}{\sqrt{y_2^2 + \cdots + y_p^2}}, \quad i = 2, 3, \cdots, p,
\]

where \((y_2, y_3, \cdots, y_p) \sim N(0, AA^T \Sigma_{22} AA^T)\) with \(\Sigma_{22}\) being the covariance matrix of \(z_2, \cdots, z_p\).

To bound the numerator, we use the classical tail bound on the normal distribution as
for any $t > 0$, $(\sigma_i = \sqrt{\text{var}(y_i)} \leq \sqrt{\lambda_{\text{max}}(AA^T \Sigma_{22} AA^T)} \leq \lambda_{\text{max}}(\Sigma)^{1/2})$,

$$P(|y_i| > t \sigma_i) = P(|y_i| > t \lambda_{\text{max}}(\Sigma)) \leq 2e^{-t^2/2}. \quad (14)$$

For the denominator, letting $\tilde{z} \sim N(0, I)\_{p-1}$, we have

$$\tilde{y} = AA^T \Sigma_{22}^{1/2} \tilde{z} \quad \text{and} \quad \tilde{y}^T \tilde{y} = \tilde{z}^T \Sigma_{22}^{1/2} \Sigma_{22}^{1/2} \tilde{z} = \sum_{i=1}^{p-n} \lambda_i X_i^2(1),$$

where $X_i^2(1)$ are iid chi-square random variables and $\lambda_i$ are non-zero eigenvalues of matrix $\Sigma_{22}^{1/2} AA^T \Sigma_{22}^{1/2}$. Here $\lambda_i$’s are naturally upper bounded by $\lambda_{\text{max}}(\Sigma)$. To give a lower bound, notice that $\Sigma_{22}^{1/2} AA^T \Sigma_{22}^{1/2}$ and $A \Sigma_{22} A^T$ possess the same set of non-zero eigenvalues, thus

$$\min_i \lambda_i \geq \lambda_{\text{min}}(A \Sigma_{22} A^T) \geq \lambda_{\text{min}}(\Sigma).$$

Therefore,

$$\lambda_{\text{min}}(\Sigma) \sum_{i=1}^{p-n} \frac{X_i^2(1)}{p-n} \leq \frac{\tilde{y}^T \tilde{y}}{p-n} \leq \lambda_{\text{max}}(\Sigma) \sum_{i=1}^{p-n} \frac{X_i^2(1)}{p-n}.$$

The quantity $\sum_{i=1}^{p-n} \frac{X_i^2(1)}{p-n}$ can be bounded by Proposition 1. Combining with Proposition 2, we have for any $C > 0$, there exists some $c_3 > 0$ such that

$$P\left(\frac{\tilde{y}^T \tilde{y}}{(p-n)} < c_3 \lambda_{\text{max}}(\Sigma)\right) \leq e^{-C(p-n)}.$$

Therefore, noticing that $\lambda_{\text{max}}^{1/2}(\Sigma)/\lambda_{\text{min}}^{1/2}(\Sigma) = \kappa^{1/2}$, $T_1^{(2)}$ can be bounded as

$$P\left(|T_1^{(2)}| > \sqrt{\frac{1 - t_1^2 \kappa^2}{c_3 \sqrt{p-n}}} |T_1^{(1)} = t_1\right) \leq e^{-C(p-n)} + 2e^{-t^2/2}.$$

Using the results from the diagonal term, we have

$$P\left(t_2 > c_2 \kappa \frac{n}{p}\right) \leq 2e^{-Cn}. \quad \text{and} \quad P\left(t_2 < c_1 \kappa^{-1} \frac{n}{p}\right) \leq 2e^{-Cn}.$$
Consequently, we have
\[
P\left( |e_i^T H H^T e_2| > c_4 \kappa t \sqrt{\frac{n}{p}} \right) = P\left( |T_1^{(1)} T_1^{(2)}| > c_4 \kappa t \sqrt{\frac{n}{p}} \mid T_1^{(1)} = t_1 \right)
\]
\[
\leq P\left( T_1^{(1)^2} > c_2 \kappa \frac{n}{p} \mid T_1^{(1)} = t_1 \right) + P\left( |T_1^{(2)}| > \frac{\kappa^2 t \sqrt{1 - c_1 n/p}}{\sqrt{c_3 \sqrt{p - n}}} \mid T_1^{(1)} = t_1 \right)
\]
\[
\leq 5e^{-Cn} + 2e^{-t^2/2},
\]
where \( c_4 = \frac{\sqrt{c_2 (c_0 - 1)}}{\sqrt{c_3 (c_0 - 1)}} \).

Proof of Lemma 4. Notice that conditioning on \( X \), for any fixed index \( i \), \( e_i^T X^T (X X^T)^{-1} \epsilon \)
follows a normal distribution with mean zero and variance \( \sigma^2 ||e_i^T X^T (X X^T)^{-1}||^2 \). We can first bound the variance and then apply the normal tail bound (14) again to obtain an upper bound for the error term.

The variance term follows
\[
\sigma^2 ||e_i^T X^T (X X^T)^{-2} X e_i|| \leq \sigma^2 \lambda_{\max} ( (X X^T)^{-1}) e_i^T H H^T e_i.
\]

The \( e_i^T H H^T e_i \) part can be bounded according to Lemma 3, while the first part follows
\[
\lambda_{\max} ( (X X^T)^{-1}) = \lambda_{\max} ( (Z \Sigma Z^T)^{-1}) \leq \lambda_{\min}^{-1} (Z \Sigma Z^T) \lambda_{\min}^{-1} (\Sigma) = \frac{\kappa}{p} \lambda_{\min}^{-1} (p^{-1} Z Z^T).
\]

Thus, using Lemma 8 and 3, we have
\[
\sigma^2 ||e_i^T X^T (X X^T)^{-1}||^2 \leq \frac{4 \sigma^2 c_2}{(1 - c_0^{-1})^2} \frac{n \kappa^2}{p^2},
\]
with probability at least \( 1 - 4 \exp(-Cn) \) if \( n > 8C/(c_0 - 1)^2 \). Now combining (15) and (14) we have for any \( t > 0 \),
\[
P\left( |e_i^T X^T (X X^T)^{-1} \epsilon| \geq \frac{2 \sigma \sqrt{c_2 \kappa t} \sqrt{n}}{1 - c_0^{-1}} p \right) < 4e^{-Cn} + 2e^{-t^2/2}.
\]

Proof of Theorem 6. The proof depends on Lemma 3 and 4 and a careful choice of the value of \( t \) in these two lemmas. We first take union bounds of the two lemmas to obtain
\[
P(\min_{i \in Q} |\Phi_{ii}| < c_1 \kappa^{-1} \frac{n}{p}) \leq 2p e^{-Cn},
\]
\[ P\left( \max_{i \neq j} |\Phi_{ij}| > c_4 k t \frac{\sqrt{n}}{p} \right) \leq 5(p^2 - p)e^{-Cn} + 2(p^2 - p)e^{-\ell^2/2}, \]

and

\[ P\left( \|X^T (XX)^{-1}\epsilon\|_\infty \geq \frac{2\sigma \sqrt{c_2 K t} \sqrt{n}}{1 - c_0^{-1}} \frac{\sqrt{n}}{p^2} \right) < 4pe^{-Cn} + 2pe^{-\ell^2/2}. \]

Notice that once we have

\[ \min_i |\Phi_{ii}| > 2s \rho \max_{ij} |\Phi_{ij}| + 2\tau^{-1} \|X^T (XX)^{-1}\epsilon\|_\infty, \quad (16) \]

then the proof is complete because \( \Phi - 2\tau^{-1} \|X^T (XX)^{-1}\epsilon\|_\infty \) is already a weak diagonally dominant matrix. Let \( t = \sqrt{Cn/\nu} \). The above equation then requires

\[
c_1 k^{-1} n p - \frac{2c_4 \sqrt{C} k s \rho n}{\nu} - \frac{2\sigma \sqrt{c_2 C k t} n}{(1 - c_0^{-1}) \nu p} = (c_1 k^{-1} - \frac{2c_4 \sqrt{C} k s \rho}{\nu} - \frac{2\sigma \sqrt{c_2 C k}}{(1 - c_0^{-1}) \nu}) n > 0,
\]

which implies that

\[ \nu > \frac{2c_4 \sqrt{C} k^2 \rho s}{c_1} + \frac{2\sigma \sqrt{c_2 C k^2}}{c_1(1 - c_0^{-1}) \tau} = C_1 k^2 \rho s + C_2 k^2 \tau^{-1} \sigma > 1, \quad (17) \]

where \( C_1 = \frac{2c_4 \sqrt{C}}{c_1}, \quad C_2 = \frac{2\sqrt{c_2 C}}{c_1(1 - c_0^{-1})} \). Therefore, the probability that (16) does not hold is

\[ P\left( \{ \text{(16) does not hold}\} \right) < (p + 5p^2)e^{-Cn} + 2p^2 e^{-Cn/\nu} < (\frac{7}{n} + \frac{1}{n})p^2 e^{-Cn/\nu^2}, \]

where the second inequality is due to the fact that \( p > n \) and \( \nu > 1 \). Now for any \( \delta > 0 \), (16) holds with probability at least \( 1 - \delta \) requires that

\[ n \geq \frac{\nu^2}{C} \left( \log(7 + 1/n) + 2 \log p - \log \delta \right), \]

which is certainly satisfied if (notice that \( \sqrt{8} < 3 \)),

\[ n \geq \frac{2\nu^2}{C} \log \frac{3p}{\delta}. \]
Now pushing $\nu$ to the limit as shown in (17) gives the precise condition we need, i.e.

$$n > 2C'\kappa^4(\rho s + \tau^{-1}\sigma)^2 \log \frac{3p}{\delta},$$

where $C' = \max\left\{\frac{4c^2}{c^2}, \frac{4c^2}{c^2(1-c_0)^2}\right\}$. \qed