Weak limit theorem for a nonlinear quantum walk

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Abstract
This paper continues the study of large time behavior of a nonlinear quantum walk begun in Maeda et al. (Discrete Contin Dyn Syst 38:3687–3703, 2018). In this paper, we provide a weak limit theorem for the distribution of the nonlinear quantum walk. The proof is based on the scattering theory of the nonlinear quantum walk, and the limit distribution is obtained in terms of its asymptotic state.

Keywords Quantum walk · Nonlinear quantum walk · Scattering theory · Weak limit theorem

1 Introduction
This paper continues the study of a one-dimensional nonlinear quantum walk (NLQW) begun in [14], where we developed a scattering theory for NLQW. The model treated
there covers a nonlinear optical Galton board [1,16], a quantum walk with a feed-forward quantum coin [19], nonlinear discrete dynamics [12], and a model exhibiting topological phenomena [5]. For more details on earlier works, we refer to the previous paper [14]. In a forthcoming companion paper [15], we numerically study a solitonic behavior of NLQW. In this paper, we study a weak limit theorem (WLT) for NLQW. The WLT for the one-dimensional (linear) quantum walk (QW) was first found by Konno [9], proved in [10], and then generalized by several authors [2–4,6–8,11,13,18,20]. The WLT states that

$$\frac{X_t}{t}$$ converges in law to a random variable $V$ as $t \to \infty$,

where $X_t$ is a random variable denoting the position of a quantum walker at time $t = 0, 1, 2, \ldots$. Because $X_t/t$ is the average velocity of the walker, $V$ is interpreted as the asymptotic velocity of the walker and hence WLT well describes the asymptotic behavior of the walker. Here, the probability distribution of $X_t$ is naturally defined according to Born’s rule as

$$P(X_t = x) = \|\Psi_t(x)\|_C^2, \quad x \in \mathbb{Z},$$

where $\Psi_t$ is the state of the walker at time $t$, which is in the state space $\mathcal{H} := l^2(\mathbb{Z}; \mathbb{C}^2)$. The state evolution is governed by

$$\Psi_{t+1}(x) = P(x + 1)\Psi_t(x + 1) + Q(x - 1)\Psi_t(x - 1), \quad x \in \mathbb{Z}, \; t = 0, 1, 2, \ldots,$$

where $P(x)$ and $Q(x) \in M(2; \mathbb{C})$ satisfy $P(x) + Q(x) =: C(x) \in U(2)$. More precisely, the state at time $t$ is given by $\Psi_t = U_L^t\Psi_0$, where $\Psi_0$ is the initial state, which is a normalized vector in $\mathcal{H}$, and $U_L$ is the evolution operator defined as follows. Let $\hat{C}$ be the coin operator defined as the multiplication by $C(x)$ and $S$ be the shift operator, i.e., $(\hat{C}\Psi)(x) := C(x)\Psi(x)$ and $(S\Psi)(x) := (\Psi(1 + x), \Psi_2(x - 1))$ ($x \in \mathbb{Z}$) for $\Psi = (\Psi_1, \Psi_2) \in \mathcal{H}$. The evolution operator $U_L$ is a unitary operator defined as $U_L = S\hat{C}$. In this setting, we have

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$

As shown in [20], if $C(x) = C_0 + O(|x|^{-1-\epsilon})$ with some $C_0 \in U(2)$ and $\epsilon > 0$ independent of $x$, then WLT is proved and the limit distribution $\mu_V$ is expressed in terms of the wave operator $W_+ := s\lim_{t \to \infty} U_L^{-t}U_0^t\Pi_{ac}(U_0)$, where $U_0 = \hat{S}\hat{C}0$ and $\Pi_{ac}(U_0)$ is the projection onto the subspace of absolute continuity. See also [2,17,18] for anisotropic cases.

In the case of NLQW, the dynamics is governed by

$$u(t + 1, x) = (\hat{P}u(t))(x + 1) + (\hat{Q}u(t))(x - 1), \quad x \in \mathbb{Z}, \; t = 0, 1, 2, \ldots, \quad (1.1)$$

where $t \mapsto u(t) := u(t, \cdot) \in \mathcal{H}$ is in $l^\infty(\mathbb{N} \cup \{0\}; \mathcal{H})$. $\hat{P}$ and $\hat{Q}$ are nonlinear maps on $\mathcal{H}$ and give a norm-preserving nonlinear map $\hat{C} : \mathcal{H} \ni u \mapsto \hat{C}u := \hat{P}u + \hat{Q}u$. © Springer
Although the dynamics is defined similarly to the linear quantum walk, it does not define a quantum system. However, 

\[ p_t(x) := \| u(t, x) \|_{C^2}^2, \quad x \in \mathbb{Z}, \]  

(1.2)
defines a probability distribution. Indeed, similarly to the linear quantum walk, the dynamics (1.1) is expressed as 

\[ u(t+1, \cdot) = U u(t, \cdot), t = 0, 1, 2, \ldots \]

where \( U := S\hat{C} \) is a nonlinear map on \( \mathcal{H} \). Because \( U \) preserves the norm, (1.2) defines the probability distribution provided that the initial state \( u(0, \cdot) = u_0 \in \mathcal{H} \) is a normalized vector. We use \( X_t \) to denote the random variable that follows (1.2), i.e., 

\[ P(X_t = x) = p_t(x). \]

Of course, \( X_t \) never describes the position of a walker that occupies any single position in \( \mathbb{Z} \), but we dare to call \( X_t \) the position of a nonlinear quantum walker in analogy with the linear quantum walk. It is mathematically more convenient to study the limit behavior of \( X_t \) than the distribution \( p_t(x) \) itself.

In this paper, we consider a nonlinear coin given by

\[ (\hat{C}u)(x) = C_N(g|u_1(x)|^2, g|u_2(x)|^2)u(x), \quad x \in \mathbb{Z} \text{ for } u = (u_1, u_2) \in \mathcal{H}, \]

where \( g > 0 \) controls the strength of the nonlinearity and \( C_N : [0, \infty) \times [0, \infty) \ni (s_1, s_2) \mapsto C_N(s_1, s_2) \in U(2) \) with \( C_N(0, 0) =: C_0 \in U(2) \). As shown in [14], in the weak nonlinear regime, \( U(t)u_0 \) scatters, i.e., 

\[ \lim_{t \to \infty} \| u(t, \cdot) - U_0^t u_+ \|_{\mathcal{H}} = 0 \]

with some asymptotic state \( u_+ \in \mathcal{H} \). The aim of this paper is to establish WLT for \( X_t \) that follows (1.2) and prove that the limit distribution is given by

\[ \mu_V(dv) = w(v) f_K(v; |a|) dv, \]

where \( w(v) \) is a function expressed in terms of \( u_+ \) and \( f_K(v; r) \) is the Konno function \( (r > 0) \).

The rest of this paper is organized as follows: Sect. 2 is devoted to reviewing the results of [14]. We state our main results and prove them in Sect. 3. We conclude the paper by discussing the results and possible future work.

### 2 Preliminaries

In this section, we review the definition of NLQW and results obtained in [14]. Throughout this paper, we set \( \mathcal{H} = l^2(\mathbb{Z}; \mathbb{C}^2) \) and drop the subscript \( \mathcal{H} \) in the norm and inner product when there is no ambiguity. Let

\[ C_N : [0, \infty) \times [0, \infty) \to U(2) \]

satisfy

\[ C_0 := C_N(0, 0) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } |a|^2 + |b|^2 = 1 \text{ and } 0 < |a| < 1. \]
We define a nonlinear coin operator $\hat{C}$ as

$$(\hat{C}u)(x) = C_N(g|u_1(x)|^2, g|u_2(x)|^2)u(x), \quad x \in \mathbb{Z} \quad \text{for} \quad u = (u_1, u_2) \in \mathcal{H}, \quad (2.1)$$

where $g > 0$ is a constant that controls the strength of the nonlinearity. Let $u_0 \in \mathcal{H}$ be the initial state of a walker with $\|u_0\| = 1$. The state $u(t)$ of the walker at time $t = 1, 2, \ldots$ is defined by induction as follows.

$$u(0) = u_0, \quad u(t + 1) = Uu(t), \quad t = 0, 1, 2, \ldots,$$

where $U = S\hat{C}$. We then define a nonlinear evolution operator $U(t)$ as

$$U(t)u_0 = u(t).$$

Similarly, we define a linear coin operator $\hat{C}_0$ as $(\hat{C}_0u)(x) = C_0u(x)$ and set $U_0 = S\hat{C}_0$. By scattering, we mean the following:

**Definition 2.1** We say $U(t)u_0$ scatters if there exists $u_+ \in \mathcal{H}$ such that

$$\lim_{t \to \infty} \|U(t)u_0 - U_0^t u_+\| = 0.$$

We use $U_{g=1}(t)$ to denote the evolution operator $U(t)$ that has the nonlinear coin $\hat{C}$ defined in (2.1) with $g = 1$. As mentioned in the previous paper [14], the smallness of $\|u_0\|_\mathcal{H}$ and $\|u_0\|_{l^1}$ corresponds to the smallness of $g$, because

$$U(t)u_0 = \frac{1}{\sqrt{g}}U_{g=1}(t)v_0 \quad \text{with} \quad v_0 := \sqrt{g}u_0.$$

Thus, the result in [14] is reformulated as follows. We use $\|A\|_{\mathbb{C}^2 \to \mathbb{C}^2}$ to denote the operator norm of the matrix $A$, i.e., $\|A\|_{\mathbb{C}^2 \to \mathbb{C}^2} := \sup_{v \in \mathbb{C}^2, \|v\|_{\mathbb{C}^2} = 1} \|Av\|_{\mathbb{C}^2}$.

**Theorem 2.2** ([14]) Assume that $C_N \in C^1(\Omega; U(2))$ with some domain $\Omega$ including $[0, \infty) \times [0, \infty)$ and there exist $c_0 > 0$ and $m \geq 2$ such that $\|C_N(s_1, s_2) - C_0\|_{\mathbb{C}^2 \to \mathbb{C}^2} \leq c_0(s_1 + s_2)^m$ and $\|\partial_{s_j}C_N(s_1, s_2)\|_{\mathbb{C}^2 \to \mathbb{C}^2} \leq c_0(s_1 + s_2)^{m-1}$ for $j = 1$ or 2. Let $u_0 \in \mathcal{H}$ be a normalized vector. Suppose in addition that either of the following conditions holds: (1) $m \geq 3$; (2) $m = 2$ and $u_0 \in l^1(\mathbb{Z}, \mathbb{C}^2)$. Then $U(t)u_0$ scatters if $g$ is sufficiently small.

**Theorem 2.2** says that in the weak nonlinear regime there exists an asymptotic state $u_+ \in \mathcal{H}$ such that $U(t)u_0$ behaves like $U_0^t u_+$ if the nonlinear coin satisfies

$$C_N(s_1, s_2) = C_0 + O((s_1 + s_2)^m), \quad m \geq 2,$$

and some technical condition.

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3 Weak limit theorem

Our aim is to establish the weak limit theorem for the position $X_t$ of a walker at time $t$ that follows the probability distribution

$$P(X_t = x) = \|u(t, x)\|_{C^2}^2, \quad x \in \mathbb{Z},$$

where $u(t, \cdot) := U(t)u_0$ with $\|u_0\|_{\mathcal{H}} = 1$. By Theorem 2.2, if $g$ is sufficiently small, then $U(t)u_0$ scatters, i.e., there exists $u_+ \in \mathcal{H}$ such that

$$\lim_{t \to \infty} \|U(t)u_0 - U_0^t u_+\| = 0.$$

Let $\hat{x}$ be the position operator. The asymptotic velocity operator $\hat{v}_0$ for $U_0 = S\hat{C}_0$ is a unique self-adjoint operator such that

$$e^{i\xi \hat{v}_0} = \lim_{t \to +\infty} e^{i\xi \hat{x}_0(t)/t}, \quad \xi \in \mathbb{R},$$

where $\hat{x}_0(t) = U_0^{-t} \hat{x} U_0$ is the Heisenberg operator of $\hat{x}$. To give the precise definition of $\hat{v}_0$, we introduce the Fourier transformation $F : \mathcal{H} \to L^2(\mathbb{T}; C^2; dk/2\pi)$ with $\mathbb{T} := \mathbb{R}/2\pi \mathbb{Z}$. Because $U_0$ is translation invariant, it can be decomposed by $F$ and the Fourier transform $F U_0 F^{-1}$ is the multiplication operator by

$$\hat{U}_0(k) = \left( \begin{array}{cc} e^{ik\bar{a}} & e^{ik\bar{b}} \\ -e^{-ik\bar{b}} & e^{-ik\bar{a}} \end{array} \right) \in U(2), \quad k \in \mathbb{T}.$$

We use $\varphi_j(k)$ to denote the normalized eigenvectors of $\hat{U}_0(k)$ corresponding to the eigenvalues

$$\lambda_j(k) = |a| \cos(k + \theta_a) + i (-1)^{j-1} \sqrt{|b|^2 + |a|^2} \sin(k + \theta_a), \quad j = 1, 2.$$

As shown in [6,20], the Fourier transform of the asymptotic velocity operator $\hat{v}_0$ is the multiplication operator by

$$\hat{v}_0(k) = \sum_{j=1,2} v_j(k) |\varphi_j(k)\rangle \langle \varphi_j(k)|, \quad k \in \mathbb{T},$$

where

$$v_j(k) := \frac{i}{\lambda_j(k)} \frac{d}{dk} \lambda_j(k) = \frac{(-1)^j |a| \sin(k + \theta_a)}{\sqrt{|b|^2 + \sin^2(k + \theta_a)}}.$$

We use $E_A(\cdot)$ to denote the spectral projection of a self-adjoint operator $A$. 
Theorem 3.1 (Weak limit theorem) Let $X_t$, $\hat{v}_0$, and $u_+$ be as above. Then there exists a random variable $V$ such that $X_t/t$ converges in law to $V$, whose distribution $\mu_V$ is given by

$$\mu_V(dv) = d\|E_{\hat{v}_0}(v)u_+\|^2.$$ 

Proof The proof proceeds along the same lines as that of [20, Corollary 2.4]. By (3.1), the characteristic function of $X_t/t$ is given by

$$\Re \ni \xi \mapsto \mathbb{E}\left(e^{i\xi X_t/t}U(t)u_0\right),$$

where $\mathbb{E}(X)$ denotes the expectation value of a random variable $X$. By (3.4) and Lemma 3.2 below,

$$\lim_{t \to \infty} E(e^{i\xi X_t/t}) = \langle u_+, e^{i\xi \hat{v}_0}u_+ \rangle = \int_{[-|a|,|a|]} e^{i\xi v} d\|E_{\hat{v}_0}(v)u_+\|^2,$$

where we have used the spectral theorem. The right-hand side in the above equation is equal to the characteristic function of a random variable $V$ following the probability distribution $\mu_V = \|E_{\hat{v}_0}(v)u_+\|^2$. This completes the proof. □

It remains to prove the following lemma.

Lemma 3.2

$$\lim_{t \to \infty} \langle U(t)u_0, e^{i\xi \hat{v}_0}U(t)u_0 \rangle = \langle u_+, e^{i\xi \hat{v}_0}u_+ \rangle.$$ 

Proof A direct calculation yields

$$\left| \langle U(t)u_0, e^{i\xi \hat{v}_0}U(t)u_0 \rangle - \langle u_+, e^{i\xi \hat{v}_0}u_+ \rangle \right| \leq \left| \langle U(t)u_0 - U'_0u_+, e^{i\xi \hat{v}_0}U(t)u_0 \rangle \right| + \left| \langle U'_0u_+, e^{i\xi \hat{v}_0}(U(t)u_0 - U'_0u_+) \rangle \right|$$

$$+ \left| \langle U'_0u_+, e^{i\xi \hat{v}_0}/U'_0u_+ \rangle - \langle u_+, e^{i\xi \hat{v}_0}u_+ \rangle \right|$$

$$=: I_1(t) + I_2(t) + I_3(t).$$

Because $e^{i\xi \hat{v}_0}$ and $U(t)$ preserve the norm and $U(t)u_0$ scatters, $\lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} I_2(t) = 0$. By (3.2), $\lim_{t \to \infty} I_3(t) = 0$. Hence, the proof is completed. □

In what follows, we provide an explicit formula for the density function of $\mu_V$ obtained in Theorem 3.1. To this end, we proceed along the lines of [18]. Let $f_K$ be the Konno function defined for all $r > 0$ as

$$f_K(v; r) = \begin{cases} \frac{\sqrt{1-r^2}}{\pi(1-v^2)\sqrt{r^2-v^2}}, & |v| < r, \\ 0, & |v| \geq r. \end{cases}$$
Similarly to [18], we introduce operators

\[ K_{j,m} : \mathcal{H} \rightarrow \mathcal{G} := L^2([-|a|, |a|], f_K(v, |a|)dv/2), \quad j = 1, 2, m = 0, 1, \]

as follows. Let \( k_{k,m} : [-|a|, |a|] \rightarrow I_m := [\pi(m - 1/2) - \theta_a, \pi(m + 1/2) - \theta_a] \) be a function defined as

\[ k_{j,m}(v) = -\theta_a + m\pi + \arcsin\left( \frac{(-1)^{j+m}|b|v}{|a|\sqrt{1 - v^2}} \right), \quad j = 1, 2, m = 0, 1, \]

where \( \theta_a \in [0, 2\pi) \) is the argument of \( a \). By direct calculation, \( k_{j,m} \) is differentiable in \((-|a|, |a|)\) and

\[ \frac{d}{dv} k_{j,m} = (-1)^{j+m} \pi f_K(v, |a|). \]

As shown in [18], \( v_j : I_m \rightarrow [-|a|, |a|] \) is the inverse function of \( k_{j,m} \). We now define the operators \( K_{j,m} \) as

\[ (K_{j,m}u)(v) = \langle \varphi_j(k_{j,m}(v)), \hat{u}(k_{j,m}(v)) \rangle_{\mathbb{C}^2}, \quad v \in [-|a|, |a|], \]

where \( \hat{u} \) is the Fourier transform of \( u \in \mathcal{H} \). The following lemma is proved similarly to [18].

**Lemma 3.3** We use \( \hat{G} \) to denote the multiplication operator on \( \mathcal{G} \) by a Borel function 
\( G : [-|a|, |a|] \rightarrow \mathbb{C} \). Then

\[ G(\hat{v}_0) = \sum_{j=1,2} \sum_{m=0,1} K_{j,m}^* \hat{G} K_{j,m}. \]

We are now in a position to state our main result.

**Theorem 3.4** Let \( u_+ \) and \( V \) be as in Theorem 3.1. Then

\[ \mu_V(dv) = w(v) f_K(v, |a|)dv, \]

where

\[ w(v) = \frac{1}{2} \sum_{j=1,2} \sum_{m=0,1} |(K_{j,m}u_+)(v)|^2, \quad v \in [-|a|, |a|]. \]

**Proof** It suffices to prove

\[ \langle u_+, e^{i\xi \hat{v}_0} u_+ \rangle = \int_{[-|a|, |a|]} e^{i\xi v} w(v) f_K(v, |a|)dv, \quad \xi \in \mathbb{R}. \]
Let $G(v) = e^{i\xi v}$. By Lemma 3.3, the left-hand side of (3.6) is

$$
\langle u_+, e^{i\xi_0 u_+} u_+ \rangle = \sum_{j=1,2} \sum_{m=0,1} \left\langle K_{j,m} u_+, \hat{G} K_{j,m} u_+ \right\rangle_G.
$$

Because

$$
\left\langle K_{j,m} u_+, \hat{G} K_{j,m} u_+ \right\rangle_G = \int_{[-|a|, |a|]} e^{i\xi v} |(K_{j,m} u_+)(v)|^2 f_K(v; |a|) dv / 2,
$$

the proof of theorem is complete. $\Box$

## 4 Discussion

In this paper we have proved the weak limit theorem for the nonlinear quantum walk in the weak nonlinear regime, i.e., $g \ll 1$ (Theorem 3.1), and we have expressed the limit density in terms of the asymptotic state $u_+$ and the Konno function (Theorem 3.4). When $g = 0$, the walk becomes a linear quantum walk and our results cover that of the homogeneous coin case [6,9]. In this case, the asymptotic state $u_+$ remains the initial state $u_0$ and the limit density function can be calculated explicitly if $u_0$ is a simple function such as $u_0(x) = 0$ except at $x = 0$. For linear quantum walks with inhomogeneous coins, $u_+$ is expressed with the aid of wave operators [4,18,20]. However, in general, it would be difficult to obtain an explicit form of $u_+$. In the case $g \neq 0$, because of nonlinearity, it would be difficult to calculate $u_+$ explicitly even for such a simple initial state. It is interesting future work to obtain an asymptotic expansion of $u_+$ even for the linear quantum walk. An asymptotic expansion of the density function will be obtained by the expansion of $u_+$.

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