Improved Uniform Linear Array Fitting Scheme With Increased Lower Bound on Uniform Degrees of Freedom for DOA Estimation

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Abstract—Recently, a uniform linear array (ULA) fitting (UF) principle is proposed for sparse array (SA) design using pseudopolynomial equations. Typically, it is verified that the designed SAs via UF enjoy a lower bound on uniform degrees of freedom (uDOFs), which is \( \approx 0.5N^2 \) with \( N \) sensors. Herein, an improved UF (IUF) scheme is proposed to significantly increase the lower bound on uDOF, which is realized by introducing two sub-ULAs, with the name of adjoint transfer arrays (ATAs), on both sides of a transfer sub-ULA. The ATAs together with the transfer sub-ULA serve as a new layer to construct an adjoint transfer layer (ATL) that has improved aperture in comparison with the traditional transfer layer to improve the lower bound on uDOF, which is derived in detail. Furthermore, two novel SAs are developed based on the ATL to verify the effectiveness of the proposed IUF scheme to increase the uDOF. Numerical simulations verify the superiority of devised SAs for direction-of-arrival (DOA) estimations.

Index Terms—Difference coarray (DCA), direction-of-arrival (DOA) estimation, sparse array (SA), weight function.

I. INTRODUCTION

ARRAY signal processing is essential for many engineering applications, such as sonar [1], radar [2], communication [3], [4], and others. An important application and development for array signal processing is direction-of-arrival (DOA) estimation [5], in which estimating more targets than sensors is currently of great interest [6], [7].

Recently, extensive studies have demonstrated that estimating more targets than sensors is achievable when the difference coarray (DCA) concept is proposed and utilized [8], [9], [10], [11]. However, previous DCA techniques, such as the minimum hole array (MHA) [9] and minimum redundancy array (MRA) [10], cannot provide closed-form expression that limits the application of DCAs. Then, the nested arrays (NAs) [11] and coprime arrays (CAs) [12], [13] are proposed with good behavior and easy closed-form expressions. In these sparse array (SA) designs, NAs consist of two sub-uniform linear arrays (ULAs), namely, a dense sub-ULA and a sparse sub-ULA [11], while CAs consist of two sparse ULAs whose interelement spacing are Co-prime (CP) integers [12], [13]. Motivated by NAs and CAs, a great deal of SAs has been developed for various applications [14], [15], [16], [17], [18], [19], [20], [21]. For example, using the parallel arrangements, the linear SAs can be utilized to implement 2-D DOA estimation [19].

However, the NA-based SAs, such as improved NA [17] and multiple-level NA [21], are sensitive to mutual coupling (MC) that happened between elements in the arrays. Meanwhile, the CA-based SAs have low MC, but they are short of degrees of freedom (DOFs) [12], [13]. From the previous reports, we know that the MC is a primary problem for active sensing to achieve high performance in electromagnetic engineering applications [22], [23]. To solve this problem, super NAs (SNAs) [24], [25], [26], [27] have been developed, which inherit the DOF of NAs with decreased MC, which renew the study interest in designing SA geometries with high DOF and low MC.

In [26] and [27], it is clarified that the MC can be analyzed using weight function. Specifically, the first three values of weight function, namely, \( w(1) \), \( w(2) \), and \( w(3) \), have primary effects on the MC. Following this property, many SAs are proposed and discussed [28], [29], [30], [31] to analyze the SAs. For instance, the augmented NAs (ANAs) [28], the maximum interelement spacing constraint (MISC) arrays [30], the two-side extended NAs (TSENAs) [29], and the extended padded CAs (ePCAs) [31] have been designed to achieve reduced MC or improved DOF. In [32] and [33], a systematically SA design procedure, namely, the ULA fitting (UF) principle, is proposed, analyzed, and used for DOA estimations. UF allows to design SAs using a combination of several ULAs, by which the MC can be regulated. Using UF, several SAs with good properties are developed, such as UF using three-base layer (UF-3BL) and UF using four-base layer (UF-4BL) [33].

Typically, the achievable lower bound on uniform DOF (uDOF) is \( O(N^2/2) \) for the UF method with \( N \) sensors. In this article, an improved UF (IUF) scheme is proposed by introducing two sub-ULAs, termed adjacent transfer arrays (ATAs), into both sides of the transfer sub-ULA in traditional UF. Then, a new adjacent transfer layer (ATL) is formed using the ATAs and the transfer sub-ULA to provide an enlarged aperture, and hence, the lower bound on uDOF is improved. The proposed
IUF scheme can increase the achievable lower bound on uDOF to $O(2N^2/3)$, and two novel SAs are designed with significantly high uDOF accordingly. Computation simulations and experimental validation are set up to verify the superiority of the proposed IUF scheme for DOA estimation. The main contributions of this article are as follows.

1) A new general SA design method, the IUF scheme, is proposed, derived, and analyzed. On the basis of the traditional UF principle, the IUF scheme aims to significantly increase the lower bound on uDOF and, hence, provides a better balance between uDOF and MC.

2) The lower bound on uDOF for the IUF scheme is $O(2N^2/3)$, which is the highest in the state of the art.

3) The SA design problem is transferred to solve the UF principle, and the corresponding strategy for finding the particular solution is presented.

4) The IUF scheme enables to design SAs with closed-form expressions. Two novel geometries are devised accordingly, where one has significantly high uDOF, and the other one provides a good balance between uDOF and MC.

This article is organized as follows. Section II explains the fundamental concepts. Section III gives a brief review of the UF principle. In Section IV, the IUF scheme is proposed and analyzed in detail. Section IV-C analyzes the available lower bound on uDOF in terms of the IUF scheme. Section V presents the design examples, and Section VI gives simulation and experimental examples. Finally, Section VII concludes this article.

II. FUNDAMENTAL

A. Signal Model

Let $S$ be a normalized position set of an SA, giving by

$$S = \{p_l, l = 0, 1, \ldots, N - 1\}$$

where $N$ is the number of sensors. In this way, the physical position set $S$ of SA can be written as $Sλ/2$, where $λ$ denotes the wavelength of incoming signal. The steering vector of this SA in direction $θ$ is then expressed as

$$a(θ) = [e^{j(2π/λ)p_0/\sin(θ)}, \ldots, e^{j(2π/λ)p_{N−1}/\sin(θ)}]^T.$$  

Furthermore, assuming that there are $B$ narrowband signals, which are uncorrelated with each other with powers $σ^2_i, i = 1, \ldots, B$, impinging from azimuths $[θ_i, i = 1, \ldots, B]$. Then, the received data is given by

$$x_q = As_q + n_q, \quad q = 1, \ldots, Q$$

where $A = [a(θ_1), \ldots, a(θ_q)]$ represents the array manifold matrix, $s_q = [s_q(1), \ldots, s_q(B)]^T$ is the signal vector, $n_q$ denotes uncorrelated white noise, and $Q$ denotes the number of snapshots. The covariance matrix of the received data $x_q$ is

$$R_{xx}(q) \triangleq E[x_qx_q^H] = AR_{ss}(q)A^H + σ^2_nI$$

where $R_{ss}(q) \triangleq \text{diag}[σ^2_1, σ^2_2, \ldots, σ^2_B]$ and $σ^2_n$ are the covariance matrix of source signal and the power of noise, respectively. Then, one can obtain the following equation using the vector form of (3):

$$w \triangleq \text{vec}(R_{xx}) = (A^* \odot A)h + σ^2_n1_n$$

where $\odot$ is the Khatri–Rao product, $1_n = \text{vec}(I_n)$, and $h = [σ^2_1, \ldots, σ^2_B]^T$. The famous DCA is then originated from (4), where $h$ is referred to as the collected signal for a virtual array with manifold $(A^* \odot A)$. This virtual array is the so-called DCA of array $S$ (whose manifold is $A$). Therefore, the DCA of SA with position set (1) is given as

$$D = \{p_k - p_l, \quad l = 0, 1, \ldots, N - 1\}.$$  

Definition 1 (uDOF): The uDOF $U$ for an SA $S$ is the cardinality of the longest ULA in its DCA $D$.

Definition 2 (Weight Functions): The array weight functions, denoted by $w(n)$, stand for the number of element pairs with spacing $n$.

For SA $S$, its weight functions can be obtained using the following equation [11]:

$$w(n) = c(n) \oplus c(−n)$$

where $\oplus$ is convolution, and $c(n)$ represents the binary expression of $S$ with value 1 denoting a physical sensor, and 0 for otherwise. Fig. 1 gives an example of the binary expression for an SA.

B. Coupling Leakage

MC is nonnegligible in practical applications. Considering the MC, the received data given in (2) are rewritten as

$$x_q = CAs_q + n_q$$

where $C$ represents the MC matrix. For linear arrays, the MC matrix can be approximately expressed as [26], [28], [30]

$$C_{kl} = \begin{cases} \frac{|C_{pl}|}{|C_{p_l}|}, & |C_{pl}| ≤ D \\ 0, & \text{elsewhere} \end{cases}$$

where $p_k, p_l ∈ S$ and $c_d, d ∈ [0, D]$ are elements of $C$, which satisfy

$$|c_0| > |c_1| > |c_2| > \cdots > |c_D|$$

$$|c_h/c_g| = g/h, \quad h, g ∈ [0, D].$$

The parameter for evaluating the MC effect is the coupling leakage, which is given by [26], [28], [30]

$$L = \frac{||C - \text{diag}(C)||_F}{||C||_F}.$$  

(10)

Qualitatively, the higher the coupling leakage is, the heavier the MC will be. In addition, from (8)–(10), one can find out that the MC is dominated by $w(1)$, $w(2)$, and $w(3)$. 

![Image](image_url)
C. Polynomial Model for Arrays

Polynomial models for arrays have long been used for radiation pattern designs [6, 34, 35]. The polynomial model can be obtained through discrete Fourier transform (DFT) [33], which enables to transform the convolution in (6) to the multiplication of polynomials. The DFT of the binary expression \( c(n) \) is

\[
C(k) = \sum_{n=0}^{\max(S)} c(n)e^{-j2\pi kn/\max(S)}, \quad k = 0, \ldots, \max(S).
\]  

Setting \( x = e^{-j2\pi k/\max(S)+1} \), then (11) becomes

\[
P_{SA}(x) = \sum_{n=0}^{\max(S)} c(n)x^n.
\]  

The relationships among array position set \( S \), array binary expression, and array polynomial model are then established based on (12). According to (12), the polynomial expression for the SA shown in Fig. 1 can be expressed as

\[
P(x) = x^0 + x^1 + x^2 + x^4 + x^6 + x^{10}.
\]

In summary, the polynomial model of SA \( S_A \), whose physical sensor position set is given in (1), is established as

\[
P_S(x) = x^{P_0} + x^{P_1} + \ldots + x^{P_{N-1}}.
\]  

Then, based on (6), (12), and (13), the DCA of an SA with position set \( S \) can be formulated as

\[
P_S(x) = P_S(x) \times P_{S}(x^{-1})
\]  

where \( \times \) is multiplication.

In this article, a ULA is described as \( \alpha, \beta, \gamma \), where \( \alpha \) represents the initial position, \( \beta \) is the interspace (the distance between adjacent sensors), and \( \gamma \) denotes the number of elements. Besides, any ULA with parameter \( \alpha, \beta, \gamma \) can be expressed as a shifted version of a special type of ULA, termed prototype array in this article, giving by

\[
P_{ULA}(x) = x^{\alpha}P_{proto}(\beta, \gamma)
\]  

where \( P_{proto}(\beta, \gamma) = x^0 + x^\beta + \ldots + x^{\beta(i-1)} \) is the prototype array associated with this ULA. The initial position of prototype array is located at 0 point.

III. REVIEW OF THE UF PRINCIPLE

A. General Components

The UF principle is proposed in [32] and [33], which aims to design SAs from cascaded sub-ULAs, as shown in Fig. 2. The \( i \)th sub-ULA is expressed as \( \alpha_i, \beta_i, \gamma_i \) with polynomial expression \( P_{sub_i}(x) \), where \( i = 1, \ldots, n \). Besides, the distance between adjacent sub-ULAs is referred to as \( gap_{i,i+1} \).

In UF, sub-ULAs are divided into three layers, namely, the base layer, the addition layer, and the transfer layer, where each layer has specific functions.

In the base layer, the sub-ULAs have completely the same structure with \( N_{base} \) denotes the number of sub-ULAs, while \( S_b \) and \( N_b \) indicate interspace and number of sensors within each sub-ULA, respectively. Based on the properties of base layer, we have \( N_{base} = S_b \). Besides, the base layer is named by \( S_0 \). For instance, the base layer with \( S_b = 3 \) is referred to as the base layer.

In the addition layer, \( N_{addition} \) denotes the number of sub-ULAs, while \( S_a(i) \) and \( N_a(i) \) \( (i = 1, \ldots, N_{addition}) \) are the interspace and number of sensors within each sub-ULA. Based on the properties of addition layer, one can get

\[
N_{addition} < S_b, \quad S_a(i) \leq S_b, \quad \text{and} \quad N_a(i) = 2, \quad (i = 1, \ldots, N_{addition})
\]

The transfer layer only contains one sub-ULA, which is also known as transfer sub-ULA, with \( S_t \) represents the interspace, and \( N_t \) is the number of sensors. Basically, \( S_t \) can be selected based on the following equations:

\[
S_t = N_{base} \cdot N_b + \sum_{i=1}^{N_{addition}} N_a(i) + 1
\]  

or

\[
S_t = N_{base} \cdot N_b + \sum_{i=1}^{N_{addition}} N_a(i).
\]

B. Array Division

In [33], it has been verified that in the UF scheme, the DCA of an SA is symmetric and can be divided into two partitions, namely, the self-DCAs (SDCAs) of each sub-ULA, and the inter-DCAs (IDCAs) between each sub-ULA pair. In this article, the SCA of sub-ULA \( i \) is denoted as \( SCA_i \), and the IDCA between sub-ULA \( i \) and sub-ULA \( j \) is denoted as \( IDC_{i,j} \). Based on [33], the SDCAs and IDCAs can be calculated using

\[
P_{SCA_i}(x) = P_{sub_i}(x) \times P_{sub_i}(x^{-1})
\]  

and

\[
P_{IDCA_{i,j}}(x) = P_{sub_{i}}(x) \times P_{sub_{j}}(x^{-1})
\]

where \( P_{sub_{i}}(x) \) and \( P_{sub_{j}}(x) \) are polynomial expressions for sub-ULA \( i \) and sub-ULA \( j \), respectively.

Based on the properties of SDCAs and IDCAs, one can get the following conclusions. The SCA (positive set) of a ULA has the same structure as its prototype array. The IDCAs can be regarded as a duplication and transfer procedure, where one ULA (referred to as the prototype sub-ULA) is duplicated and transferred, and the other ULA (referred to as the transfer sub-ULA) gives the number of duplication and the period of transfer. In UF, the sub-ULA within transfer layer is the only transfer sub-ULA that determine the number of duplication and the transfer period. Besides, the SCA of the transfer sub-ULA provides a periodic frame in DCA domain, and the

\[\text{Traditional UF requires } S_t(i) < S_b \] [33]. In the UF, this restriction is relaxed to \( S_t(i) \leq S_b \) for a more flexible design procedure.
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Fig. 3. Example of an SA designed via UF. (a) Physical SA, where sub-ULA 1, sub-ULA 4, and sub-ULA 6 are the sub-ULAs in the three-base layer, sub-ULA 2 and sub-ULA 5 belong to the addition layer, and sub-ULA 3 is the transfer sub-ULA. (b) TR in DCA domain, where the SDCA of transfer sub-ULA provides the periodic frame to be filled by the IDCAs related to the transfer sub-ULA.

Fig. 4. Illustration of array division and mapping relationships.

IDCAs between the transfer sub-ULA and other sub-ULAs are utilized to pad each period in the periodic frame. For better understanding, we provide the detailed duplication and transfer process in Fig. 3, where the physical SA is composed of six sub-ULAs.

In UF, sub-ULAs are divided into three groups. The sub-ULAs on the left-hand side of the transfer sub-ULA belong to left group (LG), sub-ULAs on the right-hand side of the transfer sub-ULA pertain to right group (RG), and the transfer sub-ULA itself belongs to one group, termed the transfer layer. The LG, transfer layer, and RG have the following polynomial expressions:

\[
P_{LG}(x) = \sum_{i=1}^{m-1} P_{sub_i}(x) \\
P_{TL}(x) = P_{sub_m}(x) \\
P_{RG}(x) = \sum_{i=m+1}^{n} P_{sub_i}(x).
\]  

By dividing the sub-ULAs into three groups, the SDCA of each sub-ULA and the IDCAs between sub-ULA pairs can be conveniently mapped into three ranges in DCA domain, namely, the near end range (NER), the transfer range (TR), and the far end range (FER). For a better understanding, the array division and the mapping relationships are demonstrated in Fig. 4, and polynomials for NER, TR, and FER are as follows:

\[
P_{NER}(x) = \left( \sum_{i=m+1}^{n} P_{SDCA_{i,m}} - P_{SDCA_{m}} \right) \\
+ \left( \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} P_{IDCA_{i,j}} \right) \\
+ \left( \sum_{i=m+1}^{n-1} \sum_{j=i+1}^{n} P_{IDCA_{i,j}} \right),
\]

\[
P_{TR}(x) = P_{SDCA_{m}} \mp \left( \sum_{i=m+1}^{n-1} \sum_{j=i+1}^{n} P_{IDCA_{i,j}} \right)
\]

\[
P_{FER}(x) = \sum_{i=1}^{m-1} \sum_{j=m+1}^{n} P_{IDCA_{i,j}}
\]  

(20)
where $\bar{+}$ is the simplified plus operation, which did not consider the coefficients when computing, i.e., $x \bar{+} x = x$. Besides, in this article, the operator $\bar{x}$ has the same feature with $\bar{+}$.

C. Design Procedure

The design procedure of the UF principle can be summarized as follows.

1) Determine parameters for each layer and the sequence of all sub-ULAs.
2) List the objective function.
3) Find a specific solution that satisfies the objective function, and further optimize the parameters that can maximize the uDOF.
4) If step 3) fails, then select another sequence and redo step 2).

The objective function in UF has the following form:

$$\text{cons.}\{P_{\text{NER}}(x) \bar{+} P_{\text{TR}}(x) \bar{+} P_{\text{TER}}(x)\} = P_{\text{proto}}[1, J + 1]$$

such that $J \in [AP_t, \max\{PS_A\}]$

$$\text{num}\{\text{gap}_{i,j} = 1\} = 1$$

where $\max\{\cdot\}$ gives the maximum exponent, $\text{cons.}\{\cdot\}$ gives the consecutive polynomial, $\text{num}\{\cdot\}$ represents the number of $x$, $J$ is the maximum consecutive position in DCA, and $AP_t$ is the aperture of transfer sub-ULA. Solving (22), one can get the designed SA with closed-form expression.

Generally, the UF method can regulate the MC of the SA by exactly adjusting the values of $\omega(1)$, $\omega(2)$, and $\omega(3)$. To design an SA with low MC, a base layer with a large interspace is always a good choice. More detailed instructions for the UF principle can be found in [33], and the examples provided therein.

IV. IUF SCHEME

A. Structure of ATL

Based on (22), the lower bound on uDOF provided by traditional UF principle is evaluated by $AP_t$, giving by [33]

$$\text{uDOF} \geq 2AP_t + 1.$$ (23)

With $N$ physical sensors, the lower bound on uDOF provided by traditional UF strategy is [33]

$$\text{uDOF}_{\text{lower}} = \frac{N^2 + \beta}{2} + 1$$ (24)

where

$$\beta = \begin{cases} -1, & N \text{ is odd} \\ 0, & N \text{ is even.} \end{cases}$$ (25)

To improve the lower bound on uDOF, two sub-ULAs on both left-hand and right-hand sides of the transfer sub-ULA are introduced, termed the left ATA (LATA) and the right ATA (RATA), where the ATA is defined as follows.

Definition 3 (ATA): The ATAs are the sub-ULAs adjacent to both left-hand and right-hand sides of the transfer sub-ULA, where the interspace of these two sub-ULAs is determined by $S_t$.

Fig. 5. ATL example 1. (a) ATL with $S_t = 7$, $S_{\text{LATA}} = 3$, $S_{\text{RATA}} = 4$, and $N_{\text{LATA}} = N_{\text{RATA}} = 2$. (b) Influence in TR.

The number of sensors and the interspace of LATA and RATA are denoted as $N_{\text{LATA}}$, $S_{\text{LATA}}$, $N_{\text{RATA}}$, and $S_{\text{RATA}}$, respectively. The LATA, transfer sub-ULA, and the RATA will be regarded as a new layer, termed the ATL, which has enlarged aperture in comparison with $AP_t$ and, hence, increases the lower bound on uDOF. The following properties hold for the ATL.

1) $N_{\text{LATA}} = N_{\text{RATA}}$.
2) $S_{\text{LATA}}$ and $S_{\text{RATA}}$ are determined by $S_t$ using the following equation:

$$\begin{cases} S_{\text{LATA}} = \left\lfloor \frac{S_t - 1}{2} \right\rfloor \\ S_{\text{RATA}} = \left\lceil \frac{S_t + 1}{2} \right\rceil \end{cases}$$ (26)

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceil operations, respectively.

3) The gap between LATA and the transfer sub-ULA is $S_{\text{LATA}}$. Similarly, the gap between RATA and the transfer sub-ULA is $S_{\text{RATA}}$.

As all the necessary components and parameters for ATL have been declared, we present an example of ATL and its influence in TR in Fig. 5.

The polynomial model of the ATL can be written as

$$\begin{align*}
\text{POL}_t(x) &= P_{\text{proto}}[S_{\text{LATA}}, N_{\text{LATA}}] \\
&+ x^{S_{\text{LATA}}N_{\text{LATA}}} P_{\text{proto}}[S_t, N_t] \\
&+ x^3 P_{\text{proto}}[S_{\text{RATA}}, N_{\text{RATA}}]
\end{align*}$$ (27)
When the ATL is used to design SAs, the selection function follows in this article, the sequence for the sub-ULAs is assigned as distinguished using superscripts, e.g., $B_{LATA}$, and the RATA. Additionally, the two base layers are the sub-ULA within the addition layer with interspace 1, the for each layer inherit UF in [33]. Here, need two-base layers to pad. In this article, the expressions layer, just as the traditional UF did. Note that, two holes can be padded using the base layer and the addition to be padded, as shown in Fig. 5(b). These regularly distributed into two parts, and each part contains certain consecutive holes divide the periods (not every period, but most periods) in TR, and the SDCA of transfer sub-ULA can be written as

$$\delta = S_{LATA}N_{LATA} + S_{RATA} + AP_1.$$  

(28)

Based on (14), (18), and (19), the summation of the IDCA between the LATA and the transfer sub-ULA (IDCA-LT), the IDCA between the RATA and the transfer sub-ULA (IDCA-RT), and the SDCA of transfer sub-ULA can be written as

$$\left[ x^0 + x^{S_{LATA}} P_{proto} \{S_{LATA}, N_{LATA}\} + x^{S_{RATA}} P_{proto} \{S_{RATA}, N_{RATA}\} \right] P_{proto} \{S_t, N_t\}.$$  

(29)

One can see from (29) that IDCA-LT and IDCA-RL can divide the periods (not every period, but most periods) in TR into two parts, and each part contains certain consecutive holes to be padded, as shown in Fig. 5(b). These regularly distributed holes can be padded using the base layer and the addition layer, just as the traditional UF did. Note that, two holes need two-base layers to pad. In this article, the expressions for each layer inherit UF in [33]. Here, $B_2$, $T$, $A_1$, $LAT$, and $RAT$ are represented as the two-base layer, the transfer layer, the sub-ULA within the addition layer with interspace 1, the LATA, and the RATA. Additionally, the two base layers are distinguished using superscripts, e.g., $B^1_2$ and $B^2_2$. Therefore, in this article, the sequence for the sub-ULAs is assigned as follows:

$$B^1_2 \rightarrow LAT \rightarrow T \rightarrow RAT \rightarrow A_1 \rightarrow B^2_2.$$  

When the ATL is used to design SAs, the selection function for $S_t$ will change from (16) and (17) to

$$S_t = \sum_{i=1}^{2} N^i_{base} \cdot N^i_0 + N_{add} + N_{LATA} + N_{LATA} + 1.$$  

(30)

and

$$S_t = \sum_{i=1}^{2} N^i_{base} \cdot N^i_0 + N_{add} + N_{LATA} + N_{LATA}.$$  

(31)

respectively, where

$$N_{add} = \sum_{i=1}^{N_{add}} N_a(i).$$  

(32)

Besides, the following relationship is established:

$$N = \sum_{i=1}^{2} N^i_{base} \cdot N^i_0 + N_{add} + N_1 + N_{LATA} + N_{LATA}.$$  

(33)

To date, the basic structure of ATL is presented. The IUF scheme is composed of the base layer, the ATL, and the addition layer. One should note that, in IUF, the array division in both physical domain and DCA domain follows (20) and (21). Moreover, the rest design procedure for the IUF scheme is the same as the traditional UF principle provided in Section III-C.

B. Influence of $N_{LATA}$ and $N_{RATA}$

In ATL, the number of sensors in LATA and RATA has great influence on TR. Based on (29), it can be seen that with the increasing of $N_{LATA}$ and $N_{RATA}$, the regularity within the periods in periodic frame will be destroyed. To better explain this point, two examples are presented in Figs. 5 and 6 with $N_{LATA} = N_{RATA} = 2$ and $N_{LATA} = N_{RATA} = 4$, respectively. One can obtain from Fig. 5 that when $N_{LATA} = N_{RATA} = 2$, only period 1 in TR is different from other periods. However, when $N_{LATA} = N_{RATA} = 4$, one can find in Fig. 6 that both period 1 and period 2 are different from other periods. In other words, the regularity in period 2 is destroyed when $N_{LATA} = N_{RATA} = 4$. In the same way, when $N_{LATA} = N_{RATA} = 6,
the regularity in period 3 will be destroyed. Note that in both Figs. 5 and 6, the values of \( S_t \) are odd numbers. When \( S_t \) is even, the situation is very similar and, hence, is omitted for conciseness.

It is obvious that using IUF, the design difficulty and achievable uDOF increase with the increase in \( N_{\text{LATA}} \) and \( N_{\text{RATA}} \). Therefore, it is always a trade-off to choose the values for \( N_{\text{LATA}} \) and \( N_{\text{RATA}} \). In this article, we consider the case when the values of \( N_{\text{LATA}} = N_{\text{RATA}} \) are increasing with the total number of sensors to pursue a longer achievable uDOF.

C. Lower Bound on uDOF Achievable by IUF

When IUF is used to design SAs, the corresponding objective function should have the following form:

\[
\left\{ P_{\text{NER}}(x) + P_{\text{TR}}(x) + P_{\text{FER}}(x) \right\} = P_{\text{proto}}(1, J + 1)
\]

such that \( J \in \{ A_P, \text{max}[P_{\text{SA}}] \} \)

\[
\text{num}\{g_{i,j} = 1\} = 1
\]  

(34)

where \( A_P \) is the aperture of the ATL. Based on (34), the lower bound on uDOF achievable by the IUF scheme is analyzed through \( A_P \). with the following equation:

\[
uDOF_{\text{IUF}} \geq 2 A_P + 1.
\]  

(35)

When the values of \( N_{\text{LATA}} = N_{\text{RATA}} \) are increasing with \( N \), it is important to obtain the values of \( N_{\text{LATA}} \) and \( N_{\text{RATA}} \) first. Hence, we present the following assumption, where the number of sensors in two base layers is the same. Using this assumption, experimentally, using ATL can design SAs successfully when \( N_{\text{LATA}} = N_{\text{RATA}} \) satisfies

\[
N_{\text{LATA}} = N_{\text{RATA}} = N_{\text{base}}^i - N_b^1 - 1, \quad i = 1 \text{ or } 2.
\]  

(36)

Following (30), (33), and (36), one can get:

\[
N = 2 \sum_{i=1}^{N_{\text{base}}^i} N_b^i + N_{\text{add}} + N_t + 2 N_{\text{LATA}}
\]  

(37)

and

\[
S_t = N - N_b + 1.
\]  

(38)

The aperture of ATL can be written as

\[
A_P = A_P + N_{\text{LATA}} S_t + N_{\text{RATA}} S_t
\]

\[
= (N_t - 1 + N_{\text{LATA}}) S_t.
\]  

(39)

Then, substituting (36)–(38) to (39) yields

\[
A_P = \frac{1}{4} \left[ -3 N_t^2 + (2 N + 7 + N_{\text{add}}) N_t + N_{\text{add}}^2 - (3 + N_{\text{add}}) N - 4 - N_{\text{add}} \right].
\]  

(40)

Maximizing (40) yields

\[
A_P = \frac{N_{\text{add}}}{3} + \frac{N_{\text{add}} + 1}{6} + \frac{(N_{\text{add}} + 1)^2}{48}.
\]  

Hence, based on (35) and (41), the lower bound on uDOF is

\[
uDOF_{\text{lower}} = \frac{2 N^2}{3} - \frac{N(N_{\text{add}} + 1)}{3} + \frac{(N_{\text{add}} + 1)^2}{24} + 1.
\]  

(42)

Equation (42) shows significant improvement for the lower bound on the uDOF, which is larger than the uDOF of most existing SAs. One should note that (42) is derived based on (30). When (31) is utilized, the achievable lower bound on uDOF will become smaller; meanwhile, the design process will become easier.

D. Steps for Solving the Objective Function

The IUF scheme provides a quite flexible SA design procedure shown in Section III-C, and the parameters for each layer are selected based on their properties. After determining the sequence of all sub-ULAs, the following problem is to list and solve the objective function.

For solving objective functions similar to (34), the strategy provided in [33] still works. After completing the first two steps in the design procedure, the following parameters, namely, \( S_t(i) \), \( N_{\text{add}}(i) \), \( S_b(i) \), \( S_t \), and \( N_{\text{base}}(i) \), are obtained, and \( S_b(i) \) is obtained from (30) or (31). The optimal selection for \( N_b(i) \) and \( N_t(i) \) is obtained via maximizing \( J \). Therefore, the solution of the objective function is the gaps of adjacent sub-ULAs. For a specific sequence, the strategy for solving the objective function obeys the following steps.

1) Initialization: First, set \( N_b = 2 \), and obtain a particular solution to guarantee that all the periods (except period 1 and period 2) are continuous in the TR [see Fig. 3(b)].

2) Verification: Verify if the particular solution obtained in the initial step satisfies the constraints in the objective function.

3) Induction: If the verification step succeeds, then get the particular solution for \( N_b > 2 \), and obtain and check if \( J \) satisfies the objective function.

4) If the verification step or induction step fails, then redo the initial step and try to obtain a different particular solution.

It is worthy to point out that the strategy provided earlier is similar to the Mathematical Induction process. Nevertheless, some particular solutions fail to generalize for \( N_b > 2 \). From the abovementioned discussions, we can see that the IUF scheme enables the designer to determine the parameters and sequence of all sub-ULAs flexibly based on the properties of each layer. Though the desired solution may not exist for some setups, the value of \( S_t \) can be reduced to get an easier design process.

V. SA DESIGN USING IUF

In this section, two SAs are designed using IUF as examples. Both are designed based on ATL with increasing number of sensors.

A. Design Example 1: \( S_t \) Is Odd

The first SA is designed via ATL with increasing number of sensors in ATAs and two one-base layers (ATLI-1BL). In this regard, the parameters satisfy

\[
\begin{align*}
N_{\text{base}}^1 &= N_{\text{base}}^2 = 1 \\
N_b^1 &= N_b^2 \\
S_b^1 &= S_b^2 = 1.
\end{align*}
\]  

(43)
Here, we set \( N_b = N^1_b = N^2_b \) and follow (36)–(38), which yields:
\[
\begin{align*}
N_LATA &= N_{RATA} = N_b - 1 \\
N &= 4N_b + N_t - 2 \\
S_i &= 4N_b - 1.
\end{align*}
\] (44)

In ATLI-1BL, there are five sub-ULAs, where the sequence is selected as
\[
B^i_1 \rightarrow LAT \rightarrow T \rightarrow RAT \rightarrow B^i_1.
\] (45)

To pursue a high uDOF, the target function of ATLI-1BL can be written as
\[
\text{cons.}\{ P_{\text{NER}}(x) = P_{\text{TR}}(x) = P_{\text{FER}}(x) \} = P_{\text{proto}}[1, J_{\text{ATLI-1BL}} + 1] \] such that \( J_{\text{ATLI-1BL}} \in \{ A_{\text{ATL}}, \max [P_{\text{SA}}] \} \). (46)

Based on the properties of ATL, one can simply get
\[
\begin{cases}
gap_{2,3} = 2N_b - 1 \\
gap_{3,4} = 2N_b.
\end{cases}
\] (47)

Furthermore, we begin to solve the target function (46). According to the strategy provided in Section IV-D and [33], first, we do the initial step and set \( N_b = 2 \). In this case, the expression of the TR is
\[
P_{\text{TR}}(x) = \begin{cases} \sum x^0 + x^{N_b} P_{\text{proto}}[1, N_b] + x^2 N_b - 1 & \text{gap}_{2,3} = 2N_b - 1 \\
\sum x^{N_b} P_{\text{proto}}[1, N_b] + x^{2N_b} - 1 & \text{gap}_{3,4} = 2N_b
\end{cases}
\] (48)

Based on (48), a particular solution can be obtained
\[
\begin{cases}
gap_{1,2} = 2, & \text{gap}_{2,3} = 3 \\
gap_{3,4} = 4, & \text{gap}_{4,5} = 4.
\end{cases}
\] (49)

Using (49), we can easily check that \( A_{\text{ATL}} = \max [P_{\text{SA}}] \); i.e., the constraints in (46) are satisfied. Since the verification step succeeds, then the induction step should be carried out and generalize the particular solution to \( N_b > 2 \). By setting \( N_b = 3 \), the particular solution for (46) becomes
\[
\begin{cases}
gap_{1,2} = 3, & \text{gap}_{2,3} = 5 \\
gap_{3,4} = 6, & \text{gap}_{4,5} = 6.
\end{cases}
\] (50)

Therefore, the final generalized solution for (46) can be obtained and given by
\[
\begin{cases}
gap_{1,2} = N_b, & \text{gap}_{2,3} = 2N_b - 1 \\
gap_{3,4} = 2N_b, & \text{gap}_{4,5} = 2N_b.
\end{cases}
\] (51)

Based on (21) and (51), the polynomial expressions for NER, TR, and FER are
\[
P_{\text{NER}}(x) = P_{\text{proto}}[2, N_b] + x^0 + x^{N_b} P_{\text{proto}}[1, N_b] P_{\text{proto}}[2N_b - 1, N_b - 1] + x^{2N_b} P_{\text{proto}}[1, N_b] P_{\text{proto}}[2N_b, N_b - 1] \] (52)

and
\[
P_{\text{FER}}(x) = x^{2N_b^2 + 4N_b} N_b \cdot N_t - 2N_b + 4 N_b + 1 + x^{4N_b^2 + 4N_b} N_b - 4N_b + 2 + x^{4N_b^2 + 4N_b} N_b - 2N_b P_{\text{proto}}[2N_b, N_b - 1] + x^{2N_b^2 + 4N_b} N_b - 4N_b P_{\text{proto}}[2N_b, N_b - 1, N_b - 1] P_{\text{proto}}[1, N_b] \] (53)

Using (52)–(54), one can get
\[
\text{cons.}\{ P_{\text{NER}}(x) = P_{\text{TR}}(x) = P_{\text{FER}}(x) \} = P_{\text{proto}}[1, 4N_b N_t + 4N^2_b - N_t + 4N_b + 1].
\] (54)

Hence
\[
J_{\text{ATLI-1BL}} = 4N_b N_t + 4N^2_b - N_t - 4N_b.
\] (56)

Following the steps for solving the objective function, we now need to check if the obtained \( J_{\text{ATLI-1BL}} \) satisfies the constraints in (46). In this case, the aperture of the ATL is \( A_{\text{ATL}} = 4N_b N_t + 4N^2_b - N_t - 9N_b + 2 \). Obviously, we have \( J_{\text{ATLI-1BL}} > A_{\text{ATL}} \), and therefore, (51) is the correct solution for (46).

Maximizing (56) under the condition of (44), one can get the optimal parameter selection for ATLI-1BL as
\[
\begin{cases}
N_b = \left[ \frac{N + 4}{6} \right], & N \geq 14 \\
N_t = N - 4N_b + 2
\end{cases}
\] (57)

and \( N_b = 2 \) when \( 14 > N \geq 7 \). The uDOF expression for ATLI-1BL is
\[
uDOF = \begin{cases} 2N^2 + 2N - 3 & N \equiv 0 \pmod{6}, 2 \\
2N^2 + 2N - 1 & N \equiv 1 \pmod{6}, 5 \\
2N^2 + 2N - 1 & N \equiv 2 \pmod{6}, 4 \\
2N^2 + 2N - 4 & N \equiv 3 \pmod{6}, 3 \\
2N^2 - 5 & N \equiv 4 \pmod{6}, 5 \\
2N^2 - 6 & N \equiv 5 \pmod{6}, 3
\end{cases}
\] (58)

when \( N \geq 14 \) and
\[
uDOF = 14N - 67
\] (59)

when \( 14 > N \geq 7 \). The final structure for ATLI-1BL is
\[
\begin{cases}
\text{part } 1 : [0, 1, N_b] \\
\text{part } 2 : [2N_b - 1, 2N_b - 1, N_b - 1] \\
\text{part } 3 : [2N^2_b - N_b, 4N_b - 1, N_b] \\
\text{part } 4 : [4N_b N_t + 2N^2_b - N_t - 3N_b + 1, 2N_b, N_b - 1] \\
\text{part } 5 : [4N_b N_t + 4N^2_b - N_t - 5N_b + 1, 1, N_b]
\end{cases}
\] (60)

where each part represents a sub-ULA. Weight functions for the ATLI-1BL satisfy
\[
w(1) = 2N_b - 2, w(2) = 2N_b - 4, N_b \geq 3
\] (1)

and
\[
w(3) = 1, N_b = 2
\] (2)
B. Case 2: \( S_i \) Is Even

The second SA is designed via ATL with increasing number of sensors in ATAs, one two-base layer and one addition layer with \( S_i = 2 \) (termed ATL-2BL). In this case, one can get

\[
\begin{align*}
N_LATA &= N_{RATA} = N_b - 1 \\
N &= 4N_b + N_t \\
S_i &= 4N_b.
\end{align*}
\]

In ATL-2BL, there are six sub-ULAs, where the sequence is selected as

\[
B_2 \rightarrow A_2 \rightarrow LAT \rightarrow T \rightarrow RAT \rightarrow B_2.
\]

To pursue a trade-off between uDOF and MC, the objective function of ATL-2BL is written as

\[
\text{cons.}\left\{ P_{\text{NER}}(x) + P_{\text{TR}}(x) + P_{\text{FER}}(x) \right\} = P_{\text{proto}}(1, J + 1)
\]

such that \( \nu \left\{ \text{gap}_{i,j} = 1 \right\} = 1 \)

\( J \in [AP_{\text{ATL}}, \text{max}\{PA_{\text{SA}}\}] \).

In ATL-2BL, one can simply get

\[
\begin{align*}
gap_{3,4} &= 2N_b - 1 \\
gap_{4,5} &= 2N_b + 1.
\end{align*}
\]

Then, based on the strategy provided in Section IV-D and [33], the final solution for (64) is obtained as

\[
\begin{align*}
gap_{1,2} &= 1, \quad \text{gap}_{2,3} = 2N_b - 3 \\
gap_{3,4} &= 2N_b - 1, \quad \text{gap}_{4,5} = 2N_b + 1 \\
gap_{5,6} &= 2N_b + 1.
\end{align*}
\]

In ATL-2BL, the polynomial expressions for NER, TR, and FER are

\[
P_{\text{NER}}(x) = (x^1 + x^3) P_{\text{proto}}(2, N_b)
+ (x^{2N_b-3} + x^{2N_b-1}) P_{\text{proto}}(2N_b - 1, N_b - 1)
+ x^{2N_b} P_{\text{proto}}(2, N_b) P_{\text{proto}}(2N_b - 1, N_b - 1)
+ x^{2N_b+1} P_{\text{proto}}(2, N_b) P_{\text{proto}}(2N_b + 1, N_b - 1)
\]

\[
P_{\text{TR}}(x) = P_{\text{proto}}(4N_b, N_t)
\times \left\{ x^0 + x^{2N_b-1} P_{\text{proto}}(2N_b - 1, N_b - 1) \right. \\
+ x^{2N_b+1} P_{\text{proto}}(2N_b + 1, N_b - 1) \\
+ x^{2N_b-2} (x^0 + x^2) \\
\left. + \left(x^{2N_b-2} - x^{2N_b} + x^{2N_b + N_b} \right) P_{\text{proto}}(2, N_b) \right\}
\]

and

\[
P_{\text{FER}}(x) = x^{2N_b-2} + 4N_bN_t - 3N_b + 2
+ x^{4N_bN_t} P_{\text{proto}}(2, N_b) P_{\text{proto}}(2N_b + 1, N_b - 1)
+ x^{4N_bN_t} P_{\text{proto}}(2, N_b - 1, N_b - 1) P_{\text{proto}}(2N_b + 1, N_b - 1)
+ x^{4N_bN_t} P_{\text{proto}}(2, N_b - 1, N_b - 1) P_{\text{proto}}(2N_b + 1, N_b - 1)
\]

Based on (67)–(69), one can get

\[
\text{cons.}\left\{ P_{\text{NER}}(x) + P_{\text{TR}}(x) + P_{\text{FER}}(x) \right\} = P_{\text{proto}}(1, 4N_bN_t + 4N_b^2 - 2N_b).
\]

Hence

\[
J_{\text{ATL-2BL}} = 4N_bN_t + 4N_b^2 - 2N_b - 1.
\]

Maximizing (71) under the condition of (62), one can get the optimal parameter selection and the uDOF expression for ATL-2BL as

\[
\begin{align*}
N_b &= \left\lfloor \frac{N + 2}{6} \right\rfloor, \quad N \geq 10 \\
N_t &= N - 4N_b.
\end{align*}
\]

The final structure for ATL-2BL is

\[
\begin{align*}
\text{part 1} : & (0, 2, N_b) \\
\text{part 2} : & (2N_b - 1, 2, 2) \\
\text{part 3} : & (4N_b - 2, 2N_b - 1, N_b - 1) \\
\text{part 4} : & (2N_b^2 + N_b - 1, 4N_b, N_t) \\
\text{part 5} : & (4N_bN_t + 2N_b^2 - N_b, 2N_b + 1, N_b - 1) \\
\text{part 6} : & (4N_bN_t + 4N_b^2 - 2N_b - 1, 2, N_b).
\end{align*}
\]

Particularly, ATL-2BL has low MC among sensors with

\[
\begin{align*}
\omega(1) &= \begin{cases} 
2, & N_b = 2 \\
1, & N_b \geq 3 
\end{cases} \\
\omega(3) &= \begin{cases} 
3, & N_b = 3 \\
4, & N_b = 2
\end{cases} \\
\omega(2) &= 2N_b - 1.
\end{align*}
\]

VI. PERFORMANCE EVALUATION

The performance of the designed ATL-1BL and ATL-2BL is evaluated using simulation and experimental examples. Herein, SAs considered in comparison are the TSENA [29], SNA [26], [27], ePCA [31], CP [14], NA [11], ANA [28], MISC [30], UF-3BL [33], and UF-4BL [33]. Parameters for each SA can be found in related references.

In all simulations, DOA estimation results are calculated using the spatial smoothing-multiple signal classification (SS-MUSIC) algorithm [11]. Unless otherwise specified, the
number of snapshots using in all examples is 1000. Coupling parameters chosen in all simulations satisfy $c_i = c_1 e^{-j(\theta_{i-1} - \theta_i) \pi / 3}$, $i = 2, \ldots, 100$. Therefore, in this article, the coupling parameters are regulated through $|c_1|$.

**A. uDOF and Coupling Leakage Versus Number of Sensors**

For the first test, we show the uDOF and coupling leakage versus the number of sensors of the related SAs, and relevant results are given in Fig. 7. As shown in Fig. 7(a), the proposed ATLI-1BL and ATLI-2BL have much higher uDOF than the existing structures. Besides, the larger the number of sensors is, the larger the advantage in uDOF of ATLI-1BL and ATLI-2BL will get. Additionally, SAs designed via IUF (ATLI-2BL and ATLI-1BL) have much higher uDOF than the SAs designed using traditional UF (UF-3BL and UF-4BL).

In terms of the coupling leakage, as shown in Fig. 7(b), the ATLI-2BL has good performance. The coupling leakage of the ATLI-1BL is increasing with the number of sensors, while the reverse is true for the ATLI-2BL. Besides, the coupling leakage of the ATLI-2BL is less than other SAs tested except the UF-3BL and UF-4BL. One can find clearly from Fig. 7(b) that, in the UF principle, the coupling leakage is inversely proportional to the interspace of the base layer. Therefore, in our proposed IUF scheme, one can further reduce the coupling leakage using base layers with larger interspace, which, however, belongs to future work.

**B. Target Identification and Angle Resolution**

In the second example, the target identification performance and the angle resolution for all the SAs tested are studied. Both non-coupling environment and high coupling scenario are considered. First, in Fig. 8(a), the targets identification in non-coupling environment is illustrated. In this case, all the SAs are composed of 20 sensors, and 80 targets are uniformly located in $[-60^\circ, 60^\circ]$ with SNR = 0 dB. One can see that ATLI-1BL, ATLI-2BL, and MISC can identify all the targets, due to the high uDOF they possess. In heavy coupling scenario, the number of sensors is chosen as 30, the number of targets is 45 (distributed uniformly in $[-60^\circ, 60^\circ]$), SNR = 0 dB, and $c_1 = 0.5 e^{j \pi / 3}$. One can conclude from Fig. 8(b) that only ATLI-2BL can find all the sources, while other SAs have spurious and missing targets.

The probability of angle resolution versus SNR for all the SAs tested is given in Fig. 9, where two targets are closely located at $1.5^\circ$ and $2.5^\circ$. The estimated $\hat{\theta}_1$ and $\hat{\theta}_2$ are denoted as $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. The two targets are resolved when the following equation is satisfied [6]:

$$|\theta_i - \hat{\theta}_i| \leq \frac{\theta_2 - \theta_1}{2}, \quad i = 1, 2. \quad (76)$$

Fig. 9(a) shows the probability of angle resolution in non-coupling environment, where all SAs tested are composed of 30 sensors. It can be seen that the proposed ATLI-1BL and ATLI-2BL have much higher angle resolution than other competitors. The probability of angle resolution in high-coupling scenario is demonstrated in Fig. 9(b), where the number of sensors is 35 and $|c_1| = 0.3$. Clearly, the proposed ATLI-2BL has the highest probability of angle resolution. Note that since other competitors fail to reach such high angle resolution in this case, hence, their results are omitted to keep the figure concise and clear.

**C. RMSE Performance**

In the third example, the root-mean-square error (RMSE) performance in different conditions is analyzed.
Fig. 9. Angle resolution in different conditions. (a) Angle resolution in non-coupling scenario with 30 sensors. (b) Angle resolution with 35 sensors and $|c_1| = 0.3$.

Fig. 10. RMSE performance versus SNR. (a) DOA estimation in non-coupling scenario with 30 sensors. (b) DOA estimation with 30 sensors, $|c_1| = 0.3$.

1) RMSE Versus SNR: In the first case, the RMSE performance versus SNR in non-coupling environment is presented. All the SAs have 36 sensors, and there are 47 sources in $[-60^\circ, 60^\circ]$. From Fig. 10(a), it is shown that the ATLI-1BL and ATLI-2BL have better performance due to their high uDOF. In non-coupling environment, the RMSE performance is basically based on uDOF of all the SAs tested. In the second case, we consider the RMSE performance versus SNR in high coupling environment. Here, the number of sensors is chosen as 35, the number of targets is set as 45 with azimuth $[-50^\circ, 50^\circ]$, and $c_1 = 0.3e^{j\pi/3}$. In Fig. 10(b), it is shown that the ATLI-2BL has considerable performance improvement compared with other SAs tested.

2) RMSE Versus $|c_1|$: In this case, the influence of $|c_1|$ is presented. Here, all the SAs are consisted of 40 sensors, the number of sources is set as 57 within azimuth $[-60^\circ, 60^\circ]$, SNR = 0 dB, and $|c_1|$ varies from 0 to 1. As shown in Fig. 11, when $|c_1| < 0.2$, ATLI-1BL performs the best, while when $|c_1| > 0.2$, ATLI-2BL performs the best.
Fig. 13. Experimental validation. (a) Tracking result using ULA with 80 hydrophones and the CBF algorithm. (b) Tracking result using ATLI-1BL (proposed). (c) Tracking result using ATLI-2BL (proposed). (d) Tracking result using MISC. (e) Tracking result using ANA12. (f) Tracking result using NA. (g) Tracking result using ULA.

0.55 > |c₁| > 0.2, the ATLI-2BL has the best performance due to its good balance between uDOF and coupling leakage. When |c₁| > 0.55, UF-4BL contributes the best performance owing to its lowest coupling leakage.

D. Beamforming Performance

In the fourth example, the beam patterns of relevant SAs are investigated and compared. Beam patterns for MISC, ANA1-2, ePCA, nested, ATLI-1BL, and ATLI-2BL arrays are demonstrated in Fig. 12(a), where all the SAs are composed of 16 sensors. Clearly, the proposed ATLI-1BL and ATLI-2BL have sharper main lobes than the competitors.

However, one can find that the beam patterns for all the SAs tested have a high sidelobe level. To address this problem, the min processor is a good choice for the IUF-based SAs [33], [36]. In Fig. 12(b), the beam patterns for ATLI-2BL using the min processor beamformer and the conventional beamformer are presented. It is shown that compared with the conventional method, the min processor beamformer enables to reduce the sidelobe level significantly at the expense of main lobe width. Using only 42 physical sensors, the ATLI-2BL can realize a sharper main lobe as compared with the beam pattern obtained using a 180-sensor ULA and the conventional beamformer, with almost the same peak sidelobe level. In fact, the IUF scheme can apply the sub-ULA-based techniques conveniently, which can be regarded as another merit of the IUF scheme [33], [36], [37].

E. Experimental Validation

To better evaluate the proposed SAs, an experiment is carried out in the South China Sea. The receiving array has 80 uniformly located hydrophones, where the interspace is 6.25 m, and the deployment depth is 120 m. The observed space is [−90°, 90°], and the direction of the receiving ship (fixed above the receiving array) is 65°, while some other ships are working around. The analyzed frequency ranges from 50 to 120 Hz, and the total observe duration is 500 s.

First, the conventional beamforming (CBF) algorithm is utilized to track the targets within the observed space [38]. Here, all the data collected by the receiving array are used, and the observed result is shown in Fig. 13(a) to serve as the reference. Based on the observed results shown in Fig. 13(a), we can find two moving targets within [−35°, −25°]. Besides, several other targets are moving around −60° with very strong energy, resulting in blurred trajectories. Therefore, we mainly focus on the two targets within [−35°, −25°]. Then, the tracking results for SAs with 15 hydrophones are presented in Fig. 13(b)–(g), where the SS-MUSIC algorithm is utilized
to obtain the tracking results. Array configurations selected in this experiment are the MISC, nested, ANA12, ULA, and the proposed ATLI-1BL and ATLI-2BL. Overall, the tracking results of the ATLI-1BL and ATLI-2BL have fewer missing and spurious targets. As shown in Fig. 13(b), one can tell that the ATLI-1BL has the best tracking performance, which is consistent with the tracking result shown in Fig. 13(a). Besides, the ATLI-1BL, ATLI-2BL, and MISC can track the two two closely located targets within the range $[-35^\circ, -25^\circ]$, whereas the ANA12, NA, and ULA fail.

In summary, the proposed ATLI-1BL and ATLI-2BL have better resolution and tracking performance than most existing SAs. Besides, it is also interesting to further design algorithms that are suitable for the SAs to obtain better DOA estimation behavior in the future work.

VII. CONCLUSION

In this article, the IUF scheme is proposed to increase the lower bound on uDOF available by SAs designed via UF. The detailed structure of the ATL is introduced and analyzed in detail; two novel SAs, ATAF2-2BL and ATA1-1BL, with closed-form expressions, are designed accordingly. ATAI-1BL is designed to pursue a high uDOF, where ATAF2-2BL is trying to get a good balance between coupling leakage and uDOF. Simulation results verify that the proposed SAs both have high uDOF. Additionally, in low coupling scenarios, the proposed ATAF2-2BL and ATA1-1BL have better DOA estimation accuracy over the existing SAs.

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