On instability of ground states in 2D $\mathbb{C}P^{N-1}$ and $O^N$ models at large $N$

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We consider properties of the inhomogeneous solution found recently for $\mathbb{C}P^{N-1}$ model. The solution was interpreted as a soliton. We reevaluate its energy in three different ways and find that it is negative contrary to the previous claims. Hence, instead of the solitonic interpretation it calls for reconsideration of the issue of the true ground state. While complete resolution is still absent we show that the energy density of the periodic elliptic solution is lower than the energy density of the homogeneous ground state. We also discuss similar solutions for the $O(N)$ model and for SUSY extensions.
I. INTRODUCTION

Two-dimensional $\mathbb{CP}^{N-1}$ sigma-model allows exact solution at large $N$ and represents such nonperturbative effects as gap generation, condensates, nontrivial $\theta$-dependence. It is an asymptotically free theory and in many respects serves as the laboratory for investigation of complicated nonperturbative phenomena in QCD. It was usually assumed that in the infinite volume the theory is in the confinement phase. However, more recently it was demonstrated that the phase transition from the confinement phase to the Higgs phase occurs if the model is perturbed by the twisted mass term, considered on $S^1$ or at the finite interval.

It was known for a while that in spite of many similar properties of 2D $\mathbb{CP}^{N-1}$ and QCD there is one notable difference – the signs of the nonperturbative vacuum energies in 2D $\mathbb{CP}^{N-1}$ sigma-model and QCD are opposite. In QCD the vacuum energy density is proportional to the gluon condensate,

$$\epsilon_{\text{vac}}^{\text{QCD}} = \frac{1}{4} \langle \theta^\mu_{\mu} \rangle = \left\langle M \frac{d\mathcal{L}^{\text{QCD}}}{dM} \right\rangle = \frac{1}{32g^4} M \frac{dg(M)}{dM} \langle \text{Tr} G^{\mu\nu} G_{\mu\nu} \rangle,$$

while in $\mathbb{CP}^{N-1}$ it is the $\langle -D_\mu \bar{n}_a D^\mu n^a \rangle$ condensate instead,

$$\epsilon_{\text{vac}}^{\text{CP}} = \frac{1}{2} \langle \theta^\mu_{\mu} \rangle = \left\langle M \frac{d\mathcal{L}^{\text{CP}}}{dM} \right\rangle = \frac{1}{4g^4} M \frac{dg(M)}{dM} \langle -D_\mu \bar{n}_a D^\mu n^a \rangle.$$

Both theories are asymptotically free, i.e. have $Mdg/dM < 0$, and both condensates $\langle \text{Tr} G^{\mu\nu} G_{\mu\nu} \rangle$ and $\langle -D_\mu \bar{n}_a D^\mu n^a \rangle$ are positively definite in the Euclidean signature. However, the gluon condensate is positive in its both perturbative and nonperturbative pieces while positivity of $\langle -D_\mu \bar{n}_a D^\mu n^a \rangle$ is due to perturbative part only – nonperturbative part is negative, see [3] for details.

The model can be also considered in the SUSY setting and it turns out that the observed similarity between $\mathbb{CP}^{N-1}$ model and QCD has very attractive explanation in the SUSY context. The SQCD allows the non-abelian strings and the SUSY–$\mathbb{CP}^{N-1}$ is just the world-sheet theory on the non-abelian string (see [12, 13] for the review). The degrees of freedom in $\mathbb{CP}^{N-1}$ model are identified with the orientational modes on the non-abelian string. A similar non-abelian string solution occurs also in the non-SUSY 4D gauge model which is essentially the bosonic part of the SQCD Lagrangian. In this case the worldsheet theory on the string is non-SUSY $\mathbb{CP}^{N-1}$ model.
There is 2D-4D correspondence between SQCD and the world-sheet theory on the defect. It claims that running of the coupling constant, spectrum of the stable particles, twisted superpotentials in 4D and 2D theories fit each other. The very 2D-4D correspondence reflects the property that the non-abelian string can exist on the top of the SQCD vacuum not destroying it as the electron can propagate at the top of the Cooper condensate. It just makes quantitative that properties of any object considered from the viewpoints of 2D and 4D observers should be the same.

Recently the new inhomogeneous solution to $\mathbb{C}P^{N-1}$ model has been found in Ref. [16]. The key tool for the derivation of the solution was the particular mapping of the $\mathbb{C}P^{N-1}$ model to the Gross-Neveu (GN) model. The new solution of $\mathbb{C}P^{N-1}$ model was obtained from the kink solution of GN model interpolating between two vacua with the different values of the fermion condensate. More general kink lattice configuration has been found as well using the elliptic solution to the GN model. This inhomogeneous solution and especially the lattice solution has some common properties with the inhomogeneous condensates in the GN and the chiral GN models [17, 18]. Note that there is also some analogy with the Peierls model of 1+1 superconductivity. In that case the electron propagates along some nontrivial profile of the lattice state and the integrability of the model allows to get its exact solution in some continuum [19] and discrete cases [20]. The fermions play the role of the eigenfunctions for the Lax operator for some integrable model and the spectral curve describing the finite-gap solution simultaneously plays the role of the dispersion law for the fermions. The ground state of the system strongly depends on the fermionic density and the temperature.

In this study we focus at some aspects of this new solution. We reevaluate accurately its energy and find that it is negative contrary to the statement made in [16]. Three different approaches of derivation of the ground state energy yield the same result. This raises the question concerning the true ground state of the model. We shall argue that the inhomogeneous solution and in particular the elliptic soliton lattice are the candidate ground state of the model. However, there are some reservations due to the IR properties of the solution.

Let us recall that the conventional viewpoint implies an existence of single homogeneous ground state separated by the small gaps of order $1/N$ from the set of the metastable vacua. The ground state of $\mathbb{C}P^1$ model becomes degenerate only at $\theta = \pi$ when kinks are allowed, and in SUSY case for $\mathbb{C}P^{N-1}$ when $N$ degenerate vacua exist. At one loop-level the
kinetic term for the photon is generated which yields the linear potential between charges. It was argued in [2] that the excitations of the model are identified as the singlet \( n^*n \) states. It was also noted in [2] that the \( n \)-particle corresponds to the kink between two vacua if the fermions are added to the Lagrangian. To some extent \( n^*n \) pair corresponds to the interpolation between the excited metastable vacuum and the true one. In this paper we question this standard picture.

The soliton solution in the \( \mathbb{CP}^{N-1} \) model obtained in [16] is the counterpart of the elementary kink solution in the GN model or the composite kink solution in the chiral GN model. In the GN model there are two vacua therefore the interpolating kink with the well-defined topological charge does exist. The topology guarantees its stability. Since it is this solution which gets mapped into \( \mathbb{CP}^{N-1} \) solution we could wonder if there is some topological reason which yields the stability of new solution in \( \mathbb{CP}^{N-1} \) case.

We also discuss the similar solution in \( O^N \) model and in the \( \mathcal{N}=1 \) SUSY extensions. Although the kinks in SUSY case are well defined BPS particles saturating the corresponding central charges the evaluation of their masses was the controversial issue for a while with several different answers. This puzzle has been resolved in [21, 22] where the effect of anomalies has been taken into account carefully. The finite effects of the anomalies in the mode counting has been also found in the non-SUSY \( \mathbb{CP}^{N-1} \) model in [23].

The paper is organized as follows. In Section II we recall the main features of the nonperturbative solution to the \( \mathbb{CP}^{N-1} \) model and the inhomogeneous solution is derived via the method of resolvent. Its energy is evaluated by three different approaches in Section III. Some remarks concerning the SUSY generalization of the solution are presented in Section IV while the elliptic kink crystal solution is considered in Section V. The results and open questions are summarized in the Discussion, Section VI, while some technical details are collected in the Appendices.
II. \( \mathbb{CP}^{N-1} \) MODEL

A. Saddle point equations

Let us remind the standard derivation of the saddle point approximation to the solution. Lagrangian of \( \mathbb{CP}^{N-1} \) model in Minkowski space is

\[
\mathcal{L} = D^\mu \bar{n}_a D_\mu n^a - \lambda (\bar{n}_a n^a - r)
\]  

where \( n^a, \ a = 1, \ldots, N \) are complex fields in the fundamental representation of SU(\(N\)), \( r = 1/g^2 \) defines the coupling constant, \( \bar{n}_a = (n^a)^* \) and \( \lambda \) is the Lagrange multiplier. Moreover, \( D_\mu n^a = (\partial_\mu + iA_\mu) n^a \) where \( A_\mu \) is a dummy field.

Let us go to Euclidian signature and integrate over \( N-1 \) fields \( n^a, \ a = 1, \ldots, N-1 \), but not over \( n^N = n \). Due to gauge invariance the \( n^N \) field can be chosen to be real. Besides the field \( n \) the arising effective action depends on two more real fields: \( \lambda \) and \( A_\mu \). For \( A_\mu = 0 \) the Euclidian effective action takes the form

\[
S = (N-1) \text{Tr} \log (-\partial^2 + \lambda) + \int d^2x \left( (\partial n)^2 + \lambda (n^2 - r) \right)
\]  

Let us write now the saddle point equation implying that the fields \( \lambda \) and \( n \) are static, i.e., do not depend on time, but could depend on space coordinate \( x \). Variation over \( n(x) \) leads to

\[
\left( \partial_x^2 - \lambda(x) \right) n(x) = 0,
\]  

what allows to express \( \lambda \) in terms of \( n \),

\[
\lambda = \frac{\partial^2 n}{n}.
\]  

From variation over \( \lambda(x) \) we get (neglecting difference between \( N-1 \) and \( N \)),

\[
\int dt \left[ N \langle x, t \rangle \frac{1}{-\partial_t^2 - \partial_x^2 + \lambda} |x, t\rangle + n^2 - r \right] = 0,
\]  

what is equivalent to

\[
\frac{N}{2\pi} \int d\omega \langle x | \frac{1}{-\partial_x^2 + \omega^2 + \lambda} |x\rangle + n^2 (x) - r = 0.
\]  

For the homogenous solution with \( \lambda = m^2 \) the field \( n = 0 \) and

\[
r = \frac{N}{(2\pi)^2} \int d\omega dk \frac{1}{k^2 + \omega^2 + \lambda} = \frac{N}{4\pi} \int d\omega \frac{1}{\sqrt{\omega^2 + m^2}} = \frac{N}{2\pi} \log \frac{M}{m},
\]
where $M$ denotes the UV cut-off introduced via Pauli-Villars regularization (see part B in Sec. III for details).

For inhomogeneous solution we can then rewrite Eq. (8) as

$$n^2(x) = \frac{N}{2\pi} \int_{-\infty}^{\infty} d\omega \left[ \frac{1}{2\sqrt{\omega^2 + m^2}} - R_\omega(x) \right], \quad (10)$$

where $R_\omega$ denotes the resolvent,

$$R_\omega = \left\langle x \left| \frac{1}{-\partial_x^2 + \omega^2 + \lambda} \right| x \rightangle. \quad (11)$$

The equation (10) can be also written as a sum over eigenfunctions of the operator $-\partial_x^2 + \lambda$,

$$n^2 = r - N \sum \frac{|f_k(x)|^2}{2\omega_k}, \quad (-\partial_x^2 + \lambda(x))f_k(x) = \omega_k^2 f_k(x). \quad (12)$$

In finding a inhomogeneous solution the main idea is to use well-known fact that resolvent $R_\omega$ satisfies the Gelfand-Dikii equation

$$-2R_\omega \partial_x^2 R_\omega + (\partial_x R_\omega)^2 + 4 (\omega^2 + \lambda(x)) R_\omega^2 = 1 \quad (13)$$

If we use the relation (6) to substitute $\lambda$ and propose some ansatz for $R_\omega$ we obtain a differential equation for $n$ with parameter $\omega$. This equation must hold for all values of $\omega$ which is possible only for special choice of coefficients.

Assume that the spectrum of Schrodinger operator consists of one translational zero mode and continuum starting at eigenvalue $\omega^2 = m^2$. Hence we suppose that

$$R_\omega = a(\omega) + b(\omega) n^2(x) \quad (14)$$

This is the simplest choice which is consistent with (10). It is also reasonable to assume that

$$a(\omega) = \frac{1}{2\sqrt{\omega^2 + m^2}} \quad (15)$$

but for a moment we will not use this assumption. After substitution of (14) and (5) in (13) we obtain the equation

$$4a (a + bn^2) \partial_x^2 n + 4\omega^2 n (a + bn^2)^2 - 4abn (\partial_x n)^2 = n \quad (16)$$

If we use (15) and assume $b = Ca/\omega^2$ where $C$ is some constant we obtain that (16) is equivalent to two equations

$$n\partial_x^2 n + Cn^4 - (\partial_x n)^2 = 0 \quad (17)$$
\[ \partial^2_x n + 2C n^3 = m^2 n \quad (18) \]

From these equations we easily obtain that
\[ (\partial_x n)^2 = n^2 (m^2 - C n^2) \quad (19) \]

For \( C > 0 \) the solution is
\[ n(x) = \frac{m}{\sqrt{C}} \frac{1}{\cosh (m (x - x_0))} \quad (20) \]

where \( x_0 \) is the center of the soliton. Thus, the condensate \( \lambda \) is
\[ \lambda(x) = \frac{\partial^2_x n}{n} = m^2 \left[ 1 - \frac{2}{\cosh^2 (m (x - x_0))} \right] \quad (21) \]

This is the solution found in [16]. Eigenfunctions with given momentum at infinity may be found via supersymmetric quantum mechanics,
\[ (-\partial^2_x + \lambda(x)) f_k(x) = \omega_k^2 f_k(x), \]
\[ \omega_k^2 = m^2 + k^2, \quad f_k(x) = \frac{-ik + m \tanh mx}{\sqrt{m^2 + k^2}} \exp (ikx). \quad (22) \]

We put \( x_0 = 0 \) above. These functions are normalized as
\[ \int_{-\infty}^{+\infty} dx f_k(x) f_{k'}^*(x) = 2\pi \delta (k - k') \]

Thus, from Eq. (12) we get the same solution,
\[ n^2(x) = N \int \frac{dk}{2\pi} \left[ \frac{1}{2\sqrt{k^2 + m^2}} - \frac{|f_k(x)|^2}{2\sqrt{k^2 + m^2}} \right] = \frac{N}{4\pi} \int dk \frac{m^2 (1 - \tanh^2 mx)}{(k^2 + m^2)^{3/2}} = \frac{N}{2\pi} \frac{1}{\cosh^2 mx}. \quad (23) \]

Let us comment on the topological aspect of the solution. In the GN model the kink interpolates between two vacuum states and has the standard topological charge which is due to the difference of the field at two spatial infinities. Our soliton has no naive local topological charge since values of the fields at two space asymptotics are the same. The solution looks like the soliton solution in the KdV equation and in the integrability context one could say that selecting the soliton solution which has positive energy we select the topological sector of the theory and the topology can be read off only from the geometry of the spectral curve.

In our case if our solution would have the conserved topological charge and have the positive energy one could claim that it is just particular sector of excitations above the ground state. However there is no local conserved charge and its energy is negative hence we interpret it as the instability mode for the homogeneous ground state.
III. ENERGY OF THE SOLITON

In this Section we will provide three different ways of evaluation of energy for the solution obtained in the previous section. Firstly we will use simple regularization by introducing ultraviolet cut-off and taking into account the anomaly found in \[23\]. Then we obtain the same result using Pauli-Villars regularization. Finally, we calculate the average of energy-momentum tensor. A bit surprisingly in all calculations we obtain a negative value for the soliton energy

\[ E = -\frac{2Nm}{\pi} \]  

(24)

A. Regularized sum over the modes

We first use the expression from \[23\] that energy density for a static configuration of \(\lambda\) which satisfies the gap equation is

\[ \varepsilon(x) = \varepsilon_0 + \frac{N}{2\pi} \lambda(x), \quad \partial_x \varepsilon_0 = 0. \]  

(25)

Let us emphasize that this expression takes into account the anomalous contribution emerging from the regularization of the sum over the modes.

If we subtract the vacuum energy density \(\varepsilon_{\text{vac}}\) given the same expression with \(\lambda = m^2\) we obtain

\[ \varepsilon(x) - \varepsilon_{\text{vac}} = \text{const} + \frac{N}{2\pi} \left( \lambda(x) - m^2 \right). \]

It is reasonable to assume that at spacial infinity energy density is the same as in vacuum so \(\text{const} = 0\). After substitution of solution (21) into the energy density and integration we find

\[ E = \int_{-\infty}^{+\infty} dx \left( \varepsilon(x) - \varepsilon_{\text{vac}} \right) = -\frac{Nm^2}{\pi} \int_{-\infty}^{+\infty} dx \frac{1}{\cosh^2 mx} = -\frac{2Nm}{\pi}. \]  

(26)

Since the energy of the soliton derived in \[16\] is different and positive one could wonder what is the reason for the discrepancy. In \[16\] the following expression for the energy was used \(E = N \sum \omega_n - r \int dx \lambda + \text{b.t.}\) and the derived energy of soliton is positive and reads as \(E_{\text{sol}} - E_0 = r \int (\lambda_0 - \lambda_{\text{sol}}) = 4rm\) where the complete cancellation of the sum over the modes around the vacuum and soliton was assumed. The first point of concern is the presence of the bare coupling constant \(r\) in the expression for the quantum energy. The second point which is not correct is the complete cancellation of the modes at the top of the solution.
which was shown to be incomplete. Finally the anomaly for the energy due to the proper regularization procedure has not been taken into account.

B. Pauli-Villars regularization

We calculate energy of the soliton by regularizing its effective action by Pauli-Villars method. In this calculation we follow ideas from [3]. The regularized action is

$$S = N \sum_{i=0}^{I} C_i \text{Tr} \log \left( -\partial^2 + m_i^2 + \lambda \right) + \int d^2x \left[ (\partial n)^2 + \lambda (n^2 - r) \right]$$

Following the Pauli - Villars procedure, we introduce in addition to each original field with $m_0 = 0$ a number $I$ of regulator fields with masses $m_i$, $i = 1, \ldots, I$, and constants $C_i$, $i = 0, 1, \ldots, I$, satisfying

$$\sum_{i=0}^{I} C_i = 0, \quad \sum_{i=0}^{I} C_i m_i^2 = 0, \quad C_0 = 1, \quad m_0 = 0.$$  

For our purposes it is sufficient to take $I = 2$. Then the constants $C_i$ are

$$C_1 = \frac{m_2^2}{m_1^2 - m_2^2}, \quad C_2 = -\frac{m_1^2}{m_1^2 - m_2^2}.$$  

At the end of calculation we will take a limit when all regulator masses $m_i (i = 1, \ldots, I)$ go to the UV cut-off $M$. The connection between effective action and energy is $S = E \cdot T$, where $T$ is a large time cut-off.

The general scheme of calculations is as follows. First, we find coupling constant $r$ in terms of regulator fields masses and mass scale of the theory from the gap equation for homogeneous solution $\lambda = m^2$. Next, we can show that terms with the $n$ field do not contribute to the energy because $n$ is proportional to zero mode:

$$\int_{-\infty}^{+\infty} dx \left[ (\partial_x n)^2 + \lambda n \right] = \int_{-\infty}^{+\infty} dx n \left( -\partial_x^2 n + \lambda n \right) = 0.$$  

After that we express the trace term of as a sum over eigenvalues and take into account the change in the density of states for inhomogeneous solution. Finally, we perform integration over eigenvalues and confirm the result. Details of the computation are presented in Appendix A.
C. Energy of soliton, explicit evaluation

In this section we are going to calculate the average of energy-momentum tensor for a soliton solution. We quantize the \( n \) fields canonically and introduce Pauli-Villars regulator fields to deal with divergences and take into account the conformal anomaly. The energy-momentum tensor in Minkowski space is

\[
\theta_{\mu\nu} = \sum C_i \theta^i_{\mu\nu}, \quad \theta^i_{\mu\nu} = \partial_\mu n_i \partial_\nu n_i^* + \partial_\mu n_i^* \partial_\nu n_i - g_{\mu\nu} \left( |\partial n_i|^2 - \lambda (|n_i|^2 - r) - m_i^2 |n_i|^2 \right).
\]

The components \( \theta_{00}, \theta_{11} \) are

\[
\theta_{00} = \sum C_i \left( |\partial_t n_i|^2 + |\partial_x n_i|^2 + \lambda |n_i|^2 + m_i^2 |n_i|^2 \right) - \lambda r ,
\]
\[
\theta_{11} = \sum C_i \left( |\partial_t n_i|^2 + |\partial_x n_i|^2 - \lambda |n_i|^2 - m_i^2 |n_i|^2 \right) + \lambda r ,
\]
\[
\theta_{01} = \sum C_i \left( \partial_t n_i \partial_x n_i^* + \partial_t n_i^* \partial_x n_i \right) .
\]

We consider field \( \lambda \) as classical and suppose that the \( n \) field has a classical component:

\[
\lambda = m^2 \left( 1 - \frac{2}{\cosh^2 mx} \right), \quad n_{cl} = \sqrt{\frac{N}{2\pi}} \frac{1}{\cosh mx} .
\]

The modes on the \( n \) field in continuum spectrum are given by Eq. (22). Also there is a zero mode

\[
\psi_0 = \sqrt{\frac{m}{2}} \frac{1}{\cosh mx}.
\]

Quantization of field \( n = n^N \) and regulator fields \( n_i, (i = 1, 2) \), are slightly different. The \( n \) field has classical component, proportional to zero mode, while the regulator field have additional component with frequency \( m_i \). The masses of auxiliary fields and coefficients \( C_i \) are the same as in the calculation of the determinant via Pauli-Villars regularization. The frequencies for regulator fields are \( \omega_{k,i} = \sqrt{\omega^2_k + m_i^2} \). In terms of creation and annihilation operators we have

\[
n^a \left( x, t \right) = \delta_N^a n_{cl} \left( x \right) + \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} \left( a^a_k f_k \left( x \right) e^{-i\omega_k t} + b^a_k f_k^* \left( x \right) e^{+i\omega_k t} \right) \]

for \( n^a \) field. For the the regulator fields \( n^a_i, i = 1, ..., I \), we have

\[
n^a_i = \frac{1}{\sqrt{2m_i}} \left( A^a_i e^{-im_i t} + B^a_i e^{+im_i t} \right) \psi_0 \left( x \right) + \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_{k,i}}} \left( a^a_{k,i} f_k \left( x \right) e^{-i\omega_{k,i} t} + b^a_{k,i} f_k^* \left( x \right) e^{+i\omega_{k,i} t} \right) ,
\]

(29)
The canonical commutation relations for $n$ field is modified by the presence of zero mode:

$$[n^a(x, t), \partial_t \bar{n}_b(y, t)] = i\delta^a_b (\delta(x-y) - i\delta^a_N \delta_N^b \psi_0(x) \psi_0(y)).$$

However, for regulator fields commutation relation is unchanged,

$$[n^a_i(x, t), \partial_t \bar{n}_{kb}(y, t)] = i\delta_{ik}\delta^a_b \delta(x-y).$$

We take average over the state, which is annihilated by all operators $a_k, a_{k,i}, A_i$ and $b_k, b_{k,i}, B_i$. For the product of two $n = n^N$ fields we get

$$\langle n(x_1, t_1) n^\dagger(x_2, t_2) \rangle = n_{cl}(x_1) n_{cl}(x_2) + N \int \frac{dk}{2\pi} \frac{1}{2\sqrt{k^2 + m^2}} e^{i\omega_k(t_1 - t_2)} f_k^*(x_1) f_k(x_2).$$

For corresponding regulators it gives

$$\langle n_i(x_1, t_1) n^\dagger_i(x_2, t_2) \rangle = N \psi_0(x_1) \psi_0(y) e^{im\omega_k(t_1 - t_2)} + N \int \frac{dk}{2\pi} \frac{1}{2\sqrt{k^2 + m^2 + m_i^2}} e^{i\omega_k,i(t_1 - t_2)} f_k^*(x_1) f_k(x_2).$$

The expression for the regularized square of the field is then,

$$\sum_{i=2}^{i=0} C_i \langle |n_i(x)|^2 \rangle = n_{cl}^2(x) + N \int \frac{dk}{2\pi} \frac{C_i |f_k(x)|^2}{2\sqrt{k^2 + m^2 + m_i^2}} + N \psi_0(x)^2 \sum_i \frac{C_i}{2m_i} = r.$$ 

This equality is equivalent to the gap equation, therefore the $r$ term in energy momentum tensor cancels by the $n^2$ term.

The calculation of other contributions to energy-momentum tensor is straightforward. Details are provided in Appendix B. The final answer is consistent with other methods:

$$\langle \theta_{00} \rangle = \frac{Nm^2}{4\pi} - \frac{N}{\pi} \frac{m^2}{\cosh^2 mx} = \frac{Nm^2}{4\pi} + \frac{N}{2\pi} (\lambda - m^2).$$

(30)

The other components of energy-momentum tensor are the same as ones of the homogeneous phase

$$\langle \theta_{11} \rangle = -\frac{Nm^2}{4\pi}, \quad \langle \theta_{01} \rangle = 0.$$ 

(31)

This can be compared with evaluation of the energy density of the homogeneous ground state via the conformal anomaly [3]. Since there is no scale at the classical level the trace of the energy stress tensor gets contribution from the running of the coupling constant only and therefore is proportional to the $\beta$-function, $\theta^\mu_\mu = N\lambda/2\pi$. Hence the vacuum energy density $\epsilon_{vac} = (1/2)\langle vac | \theta^\mu_\mu | vac \rangle = Nm^2/4\pi$. Similarly the mass of the particle can be evaluated from the matrix element of the $\theta^\mu_\mu$ over the corresponding state [3]. For instance we can use
the relation for the $\sigma$-particle mass, $2m^2 = \langle \sigma | \theta_\mu^\nu | \sigma \rangle$ and express it via the propagator of the $\lambda$-field $D(p^2)$ at zero momentum $D(0)$ and simple $\sigma \sigma \lambda$ vertex proportional to $2m^2/N$.

To complete this Section let us make a comment concerning the spectrum of excitations. First note that the photon acquires finite inhomogeneous mass in the non-homogeneous vacuum. This implies that there is no linear confinement of charged degrees of freedom. According to the emerging picture the homogeneous state is metastable and the kink-antikink pair in the homogeneous state now yield the bounce configuration in the Euclidean space. We shall discuss the spectrum and the $\theta$-dependence in the inhomogeneous ground state in more details elsewhere.

IV. $\mathcal{N}=1$ SUPERSYMMETRIC MODELS

A. SUSY $O^N$ sigma model

First let us argue that $O(N)$ model admits the similar inhomogeneous solution and then consider its minimal SUSY extension. The Lagrangian of the model reads as

$$\mathcal{L} = \frac{1}{2} (\partial_\mu n_a)^2 - \frac{\lambda}{2} ((n_a)^2 - r)$$

(32)

There are $N$ real fields $n_a$ and Lagrange multiplier $\lambda$ leads to constraint $n_a n_a = r = 1/\gamma^2$. Similar to the case of $\mathbb{CP}^N - 1$ model, this model demonstrates dynamical mass generation, so in vacuum $\lambda = m^2$. It is simple issue to show that in the large $N$ limit model (32) possess a soliton solution similar to the one being discussed in case of $\mathbb{CP}^N - 1$ model. The difference is only in number of degrees of freedom.

Large $N$ effective action is obtained similarly to the case of $\mathbb{CP}^N - 1$ model by integration over fields $n_a$, $a = 1, 2, \ldots, N - 1$, but not over $n_N = n$. In the Euclidean signature the effective action is

$$S_{\text{eff}} = \frac{N - 1}{2} \text{Tr} \log (-\partial^2 + \lambda) + \frac{1}{2} \int d^2 x \left((\partial n)^2 - \lambda(n^2 - r)\right).$$

(33)

The actions (4) and (33) differ only by numerical factor of 1/2. Thus, their stationary points are the same and (21) is solution in $O^N$ model with energy

$$E = -\frac{Nm}{\pi}.$$
Let us turn now to the case of $N = 1$ supersymmetric $O(N)$ model. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_{\mu} n_a)^2 + \bar{\psi}_a i \partial \psi_a + \frac{1}{4r} (\bar{\psi}_a \psi_a)^2 \right].$$

Here $\psi_a$ are Majorana fermions, $\partial = \gamma^\mu \partial_\mu$, $\gamma^0 = \sigma_2$, $\gamma^1 = i \sigma_3$, $\gamma^5 = -\gamma^0 \gamma^1 = \sigma_1$. The constraints $n_a n_a = r$ and $n_a \psi_a = 0$ are taken into account by Lagrange multipliers $\lambda$ and $\chi$. Also we introduce auxiliary field $\sigma \sim \bar{\psi} \psi$,

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_{\mu} n_a)^2 + \bar{\psi}_a (i \partial - \sigma) \psi_a - r \sigma^2 - \lambda (n_a)^2 - r \bar{\chi} n_a - \bar{\psi}_a \chi n_a \right].$$

In order to obtain effective action, we have to integrate over all fermionic fields and all fields $n_a$ but $n_N = n$. To integrate over $\psi_a$ we make shift of variables

$$\psi_a \to \psi_a + \phi_a, \quad \phi_a = (i \partial - \sigma)^{-1} \chi n_a.$$

Then terms in action linear in $\psi_a$ are canceled, but we have additional term $n_a n_a \bar{\chi} (i \partial - \sigma)^{-1} \chi = r \bar{\chi} (i \partial - \sigma)^{-1} \chi$. Then integration over $\chi$ can also be performed. Integration over $\psi_a$ and $\chi$ yields determinant contributions to effective action,

$$-\frac{iN}{2} \text{Tr} \log (i \partial - \sigma) + \frac{i}{2} \text{Tr} \log (i \partial - \sigma),$$

hence, the field $\chi$ integration reduces the number of degrees of freedom by 1. Effective action is

$$S_{\text{eff}} = \frac{i}{2} (N-1) \left[ \text{Tr} \log (-\partial^2 - \lambda) - \text{Tr} \log (i \partial - \sigma) \right] + \frac{1}{2} \int d^2 x \left[ (\partial n)^2 - \lambda (n^2 - r)^2 \right]$$

(34)

Note that this action can be rewritten in slightly different way, making the situation more clear. Before integration over $n_a$ we can use constraint $n_a n_a = r$ to put a factor $n_a n_a$ before the $\sigma$ term in Lagrangian,

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_{\mu} n^a)^2 + \bar{\psi}_a (i \partial - \sigma) \psi_a - \sigma^2 (n_a)^2 - D ((n_a)^2 - r) - \bar{\chi} n_a - \bar{\psi}_a \chi n_a \right].$$

In this equation we rename the Lagrange multiplier $\lambda$ and call it $D$. Thus, mass of both bosons and fermions is given by vev of the same field $\sigma$ and in homogeneous vacuum state $D = 0$ corresponds to unbroken supersymmetry. The effective action

$$S_{\text{eff}} = \frac{i(N-1)}{2} \text{Tr} \log (-\partial^2 - D - \sigma^2) - \frac{i(N-1)}{2} \text{Tr} \log (i \partial - \sigma)$$

$$+ \frac{1}{2} \int d^2 x \left[ (\partial n)^2 - (\sigma^2 + D) n^2 + rD \right].$$

(35)
The first form of effective action shows that fermionic part of the model is nothing but the Gross-Neveu model (with the number of degrees of freedom reduced by factor 2 because Majorana fermions are used instead of Dirac ones).

From identity \[ \gamma^5 (i\partial - \sigma) \gamma^5 = - (i\partial + \sigma) \] we can obtain

\[
\text{Tr} \log (i\partial - \sigma) = \frac{1}{2} \text{Tr} \log (- (i\partial - \sigma)(i\partial + \sigma)) = \frac{1}{2} \text{Tr} \log (\partial^2 + \sigma^2 - i\gamma^\mu \partial_\mu \sigma)
\]

If \( \sigma \) does not depend on time we have

\[
\text{Tr} \log (i\partial - \sigma) = \frac{1}{2} \text{Tr} \log (\partial^2 + \sigma^2 + \partial_\sigma) + \frac{1}{2} \text{Tr} \log (\partial^2 + \sigma^2 - \partial_\sigma) \quad (36)
\]

If \( \sigma \) is a topologically non-trivial solution for the GN model, then \( \lambda = \sigma^2 \pm \partial_\sigma \) is solution to \( \mathbb{CP}^{N-1} \) model and, thus, to \( O_N \) model. In terms of \( D \) it means \( D = \pm \partial_\sigma \). For definiteness we set \( \lambda = \sigma^2 - \partial_\sigma \). Thus,

\[
S_{\text{eff}} = \frac{i(N-1)}{4} \text{Tr} \log (\partial^2 - \sigma^2 + \partial_\sigma) - \frac{i(N-1)}{4} \text{Tr} \log (\partial^2 - \sigma^2 - \partial_\sigma) + \frac{r}{2} \int d^2 x D
\]

Here we used the fact that \( n \) is zero mode and that overall sign of expression under the logarithm is unimportant because leads only to pure imaginary constant contribution. The simplest inhomogeneous solution

\[
\sigma = m \tanh mx
\]

leads to \( \lambda \) in form (21). For this solution \( \sigma^2 + \partial_\sigma = m^2 \), so we can see that one of two terms in (36) is just a vacuum determinant and does not change energy. It is consistent with the fact that the GN energy \( E = Nm/2\pi \) instead of \( E = Nm/\pi \) as in Ref. [24] because we consider Majorana fermions) kink is minus half of energy of \( O_N \) soliton. Difference of signs of energies can be formally explained by the different signs of logarithms of bosonic and fermionic determinants.

### B. Supersymmetric \( \mathbb{CP}^{N-1} \) model

Calculation of effective action in supersymmetric \( \mathbb{CP}^{N-1} \) model is similar to the case of supersymmetric \( O_N \) model. Supersymmetric modification of (3) is

\[
\mathcal{L} = D^\mu \bar{n}_a D_\mu n^a + \bar{\psi}_a i \partial \psi^a + \frac{1}{4r} (\bar{\psi}_a \psi^a)^2 + \frac{1}{4r} (\bar{\psi}_a i\gamma^5 \psi^a)^2
\]
where $\psi$ are Dirac spinors. The constraints are: $\bar{n}_a n^a = r$, $\bar{n}_a \psi^a = 0$. We introduce Lagrange multipliers $\lambda$ and $\chi$ and auxiliary fields $\sigma$ and $\pi$,

$$
L = D^\mu \bar{n}_a D_\mu n^a + \bar{\psi}_a (i \hat{D} - \sigma - i \gamma^5 \pi) \psi^a - r \sigma^2 - r \pi^2 - \lambda (\bar{n}_a n^a - r) - \chi \bar{\psi}_a n^a - \bar{\psi}_a \chi n^a.
$$

The effective action is (we again set $A_\mu = 0$)

$$
S_{eff} = i(N-1) \text{Tr} \log (-\partial^2 - \lambda) - i(N-1) \text{Tr} \log \left(i \hat{D} - \sigma - i \gamma^5 \pi\right)
$$

$$
+ \int d^2x \left(\partial^\mu \bar{n} \partial_\mu n - \lambda (|n|^2 - r) - \sigma^2 - \pi^2\right).
$$

Fermionic part of the action coincides with the chiral Gross-Neveu model. This model has a continuous $U(1)$ spontaneously broken symmetry, so does not possess topologically stable kinks. However, it has inhomogeneous solution (37) and $\pi = 0$ which is stabilized by trapped fermions. For this solution the bound state should be half-filled, see [26]. So we have found a solution of the same type as in case of $\mathbb{O}^N$ model.

V. PERIODIC INHOMOGENEOUS SOLUTION

In this Section we analyze periodic solution, which corresponds to the kink crystal in Gross-Neveu model. We explicitly check that the gap equation is true for this solution. However, the amplitude of the $n^2$ condensate has an infrared divergence. We calculate the energy of this solution and find that it is lower than for homogeneous solution.

A. Gap equation

In this section we check self-consistency of periodic solution. In this calculation we follow the ideas from [30] and use results from [31]. For this purpose we consider possible solution $\lambda = \sigma^2 - \partial_x \sigma$, where

$$
\sigma = \nu m \frac{\text{sn} (mx; \nu) \text{cn} (mx; \nu)}{\text{dn} (mx; \nu)}
$$

is proportional to $\bar{\psi} \psi$ condensate in the GN model. It is also possible to write this condensate in form

$$
\sigma = m \frac{2 \sqrt{\nu_1}}{1 + \sqrt{\nu_1}} \text{sn} \left(\frac{2mx}{1 + \sqrt{\nu_1}}; \nu_1\right),
$$

where parameters are connected as

$$
\nu = \frac{4 \sqrt{\nu_1}}{(1 + \sqrt{\nu_1})^2}.
$$
Note that solutions $\lambda = \sigma^2 \pm \partial_x \sigma$ are different only by shift on a half of period, so we do not need to consider the solution with plus sign. For simplicity we will use only form (38) and omit the second argument of elliptic functions. Standard calculation yields

$$
\lambda = m^2 \nu \left( 2 \text{sn}^2 (mx) - 1 \right).
$$

We need to find eigenfunctions of the operator $-\partial_x^2 + \lambda$. For the operator $-\partial_y^2 + 2\nu \text{sn}^2 y$ (where $y = mx$) eigenfunctions are found in [31]:

$$
\left( -\partial_y^2 + 2\nu \text{sn}^2 y \right) f = \mathcal{E} f;
$$

$$
f(y) = \frac{\theta_1 \left( \frac{\pi(y+\alpha)}{2K}, q \right)}{\theta_4 \left( \frac{\pi y}{2K}, q \right)} \exp \left( -y Z(\alpha) \right), \quad q = \exp \left( -\pi K'/K \right).
$$

Here and later $K$ and $E$ denote full elliptic integrals of the first and the second kinds with argument $\nu$, if it is not stated otherwise, and $K'(\nu) = K(1 - \nu)$. The parameter $\alpha = K + i\eta$ for the lower band with eigenvalues $\nu < \mathcal{E} < 1$ and $\alpha = i\eta$ for the band $\mathcal{E} > 1 + \nu$. The eigenvalue can be expressed via parameter $\alpha$ as

$$
\mathcal{E} = \nu + \frac{\omega^2}{m^2} = \text{dn}^2 \alpha + \nu
$$

For the states of the spectrum $Z(\alpha)$ is purely imaginary and does not change the absolute value of $f$. Using the identities for the product of two theta-functions we can obtain

$$
|f(x)|^2 = A^2 \left( 1 - \frac{\text{cn}^2 mx}{\text{cn}^2 \alpha} \right).
$$

We need to fix the normalization factor $A$. The normalization condition is that the average of the square of the eigenfunction is equal to 1,

$$
A^2 \int_0^{2K/m} \left( 1 - \frac{\text{cn}^2 mx}{\text{cn}^2 \alpha} \right) \, dx = \frac{2K}{m}.
$$

The integral can be readily computed and we find normalized eigenfunctions

$$
|f_k|^2 = \frac{\omega^2/m^2 - \text{dn}^2 mx}{\omega^2/m^2 - E(\nu) / K(\nu)}.
$$

Note that for upper band both numerator and denominator are negative.

It is convenient to integrate over the eigenvalue $\omega$ instead of momentum $k$. To change the variable of integration, we use the formula from [31]

$$
\frac{1}{m} \, \frac{dk}{d\mathcal{E}} = \frac{\nu + E/K - \mathcal{E}}{\sqrt{(1 - \mathcal{E})(\mathcal{E} - \nu)(1 + \nu - \mathcal{E})}}.
$$
Therefore,
\[ \frac{dk}{d\omega} = \frac{E/K - z^2}{\sqrt{(1 - \nu - z^2)(1 - z^2)}}, \quad z = \omega/m. \]

The gap equation can be rewritten as
\[ n^2 = r - \frac{N}{2\pi} \int \frac{dz}{z} \left| \frac{dk}{d\omega} \right| f_k^2. \]

Integration over $z$ is over both bands. Bare coupling constant can be expressed as
\[ r = \frac{N}{4\pi} \int dk \left\{ \frac{1}{\sqrt{k^2 + \Lambda^2}} - \frac{1}{\sqrt{k^2 + M^2}} \right\} = \frac{N}{2\pi} \log \frac{M}{\Lambda}, \]

where $\Lambda$ is the mass scale of the theory and $M$ is the Pauli-Villars UV cut-off. Explicit form of gap equation is
\[ n^2 = \frac{N}{2\pi} \log \frac{m}{\Lambda} + \frac{N}{2\pi} \int_1^\infty dz \left\{ \frac{1}{\sqrt{z^2 - 1}} - \frac{1}{z} \frac{z^2 - dn^2 mx}{\sqrt{(z^2 - 1 + \nu)(z^2 - 1)}} \right\} \]
\[ - \frac{N}{2\pi} \int_0^{\sqrt{1 - \nu}} dz \frac{dn^2 mx - z^2}{z \sqrt{(1 - \nu - z^2)(1 - z^2)}} = \frac{N}{2\pi} (a + b \cdot dn^2 mx). \]

Here we extracted the term, proportional to the square of the zero mode of potential $\lambda$
\[ \psi_0 \sim dn (mx). \]

The second gap equation is
\[ (-\partial_x^2 + \lambda) n = 0, \]

so $n$ must be proportional to zero mode. It means that $a = 0$ and this condition determines the parameter $m$.

From the expressions above we obtain
\[ a = \log \frac{m}{\Lambda} + \int_1^\infty dz \left\{ \frac{1}{\sqrt{z^2 - 1}} - \frac{z}{\sqrt{(z^2 - 1 + \nu)(z^2 - 1)}} \right\} + \int_0^{\sqrt{1 - \nu}} dz \frac{z}{\sqrt{(1 - \nu - z^2)(1 - z^2)}}, \quad (43) \]
\[ b = \int_1^\infty \frac{dz}{z} \frac{1}{\sqrt{(z^2 - 1 + \nu)(z^2 - 1)}} - \int_0^{\sqrt{1 - \nu}} \frac{dz}{z} \frac{1}{\sqrt{(1 - \nu - z^2)(1 - z^2)}}. \quad (44) \]

All the integrals are elementary functions and their calculation is straightforward. However, the last integral in expression for $b$ is divergent in infrared. So we introduce a very small cut-off $\epsilon = \omega_{\text{min}}/m$. Physically it corresponds to placing the system in a box of large but finite size $L$ and dropping out zero mode from the gap equation. Then,
\[ k_{\text{min}} = \frac{2\pi}{L}, \quad \omega_{\text{min}} = k_{\text{min}} \frac{d\omega}{dk} (\omega = 0) = \frac{2\pi}{L} \sqrt{1 - \nu} \frac{K}{E}. \]
The calculation yields
\[ a = \log \frac{m}{\Lambda} + \log (1 + \sqrt{1 - \nu}) = 0, \quad m = \frac{\Lambda}{1 + \sqrt{1 - \nu}}. \] (45)

Here we recall the transformation of elliptic parameter \([40]\) and return to the original parameter \(\nu\),
\[ \Lambda = \frac{2m}{1 + \sqrt{\nu}}. \]

Thus, the fermionic condensate can be written in the form \([39]\),
\[ \sigma = \sqrt{\nu_1} \Lambda \text{sn} (\Lambda x; \nu_1). \]

In terms of mass of particle in homogeneous phase this expression takes especially simple form. However, physical reason for this simplification is unclear.

The second coefficient
\[ b = \frac{1}{\sqrt{1 - \nu}} \log \left( \frac{1 + \sqrt{1 - \nu}}{\pi K} \right). \]

Note that this coefficient has logarithmic divergence and is negative at sufficiently large length. It implies the inequality \(n^2 < 0\).

**B. Energy density**

If we ignore the infrared divergence, average energy density can be calculated in much similar way to the calculation of the energy of soliton. Omitting rather tricky technical details we give here the final result is
\[ \epsilon = \frac{N\Lambda^2}{4\pi} - \frac{E(\nu)}{K(\nu)} \frac{Nm^2}{\pi}. \] (46)

Now we discuss some arguments connected with calculation of energy-momentum tensor \([27]\). Due to conservation of momentum \(\partial_\mu \theta^\mu_\nu = 0\) we have \(\partial_\nu \langle \theta_{11} \rangle = 0\). The \(r\) term and \(n^2\) term cancel each other similarly to the case of soliton. The mass term contribution
\[ \sum_i C_i m_i^2 |n_i|^2 = N \int \frac{dk}{2\pi} \sum_i \frac{C_i m_i^2}{2\sqrt{\omega_k^2 + m_i^2}} |f_k|^2 = \frac{N}{2\pi} (\alpha + \beta \text{dn}^2 m x), \]
where the square of the mode is given by \( \langle \theta_{11} \rangle = \text{const} \). We are going to calculate only the coefficient \( \beta \),

\[
\beta = - \int_1^{\infty} dz \sum_i \frac{C_i m_i^2}{\sqrt{(z^2 + a_i^2)(z^2 - 1)(z^2 - 1 + \nu)}} + \int_0^{\sqrt{1-\nu}} dz \sum_i \frac{C_i m_i^2}{\sqrt{(z^2 + a_i^2)(z^2 - 1)(z^2 - 1 + \nu)}} = -m^2.
\]

We are not able to calculate derivative terms in energy-momentum tensor but the fact that \( \langle \theta_{11} \rangle = \text{const} \) suggests that

\[
\sum_i C_i (|\partial_t n_i|^2 + |\partial_x n_i|^2) = \frac{N}{2\pi} (\alpha_1 + \beta \, \text{dn}^2mx)
\]

with the same coefficient \( \beta \) but different coefficient \( \alpha_1 \). Therefore energy density is

\[
\epsilon(x) = \langle \theta_{00} \rangle = -\frac{Nm^2}{\pi} \, \text{dn}^2mx + \text{const}.
\]

This result is consistent with the formula

\[
\epsilon(x) = \frac{N}{2\pi} \lambda(x) + \text{const}.
\]

The value of the constant can be determined from the average energy density

\[
\epsilon(x) = \frac{N}{2\pi} \lambda(x) - \frac{N\Lambda^2}{4\pi} \left( \frac{1 - \sqrt{1-\nu}}{1 + \sqrt{1-\nu}} \right).
\]

The obtained energy is lower than the one of homogeneous solution. However, due to infrared divergence this solution can possibly be considered on a finite part of a plane only.

C. \( n \lambda \) cross-term correction

In the subsection we compute effective action more carefully, taking into account the quadratic quantum \( n_q \lambda_q \) terms in the Lagrangian. Such term is absent in the standard analysis in the confinement phase. For simplicity we consider \( \mathbb{O}^N \) model and suppose that similar results are valid for the \( \mathbb{C}P^{N-1} \) model. We find out that additional term is a \( 1/N \) correction to the effective action and therefore should not be taken into consideration to the leading order.

The partition function is

\[
Z = \int DnD\lambda \exp \{-S\}, \quad (47)
\]
where the action is obtained from \([32]\) after proper rescaling

\[
S = \frac{1}{2} \int d^2x \left( [\partial n]^2 + \lambda (n^2 - r) \right). \tag{48}
\]

We separate the fields into classical and quantum components,

\[n = n_{cl} + n_q, \quad n_{cl} = (n_0, 0, \ldots, 0), \quad \lambda = \lambda_0 + \lambda_q, \tag{49}\]

and perform functional integration over the quantum components in the Gaussian approximation. The action in terms of quantum and classical components

\[S = \frac{1}{2} \int d^2x \left[ (\partial n_0)^2 + \lambda_0 (n_0^2 - r) + (\partial n_q)^2 + (\lambda_0 + \lambda_q) n_q^2 + 2n_0 \lambda_q n_1 \right]. \tag{50}\]

After integration over all but the first components of the \(n_q\) fields we obtain effective action,

\[S_{eff}^{(1)} = \frac{N-1}{2} \text{Tr} \log (-\partial^2 + \lambda_0 + \lambda_q) + \frac{1}{2} \int d^2x \left[ (\partial n_0)^2 + \lambda_0 (n_0^2 - r) + n_1 (-\partial^2 + \lambda_0) n_1 + 2n_0 \lambda_q n_1 \right]. \tag{51}\]

To deal with the cross term \(\lambda n\) we shift the variable of functional integration and obtain Gaussian integrals for \(n\) and \(\lambda\)

\[n_1 \rightarrow n_1 + \chi, \quad \chi = -\frac{1}{-\partial^2 + \lambda} n_0 \lambda_q, \tag{52}\]

\[n_1 (-\partial^2 + \lambda_0) n_1 + 2n_0 \lambda_q n_1 \rightarrow n_1 (-\partial^2 + \lambda_0) n_1 - n_0 \lambda_q \frac{1}{-\partial^2 + \lambda_0} n_0 \lambda_q. \]

Integration over \(n_{1q}\) is trivial. However, effective action for \(\lambda\) contains a complicated integral operator \(K\) with the kernel \(K(x, y)\). This kernel can be expressed in terms of the Green function of the \(n\) field in the \(\lambda_0\) background, \(G(x, y) = \langle x \rangle (-\partial^2 + \lambda_0)^{-1} |y\rangle\),

\[S_{eff}^{(2)} = \frac{N-1}{2} \text{Tr} \log (-\partial^2 + \lambda_0) - \frac{N-1}{4} \text{Tr} \left( \frac{1}{-\partial^2 + \lambda_0} \right)^2 + \frac{1}{2} \int d^2x \left[ [\partial n_0]^2 + \lambda_0 (n_0^2 - r) - n_0 \lambda_q \frac{1}{-\partial^2 + \lambda_0} n_0 \lambda_q \right]. \tag{53}\]

The action for the \(\lambda_q\) reads as

\[S_\lambda = -\frac{N-1}{4} \int d^2x d^2y G(x, y) G(y, x) \lambda_q(x) \lambda_q(y) - \frac{1}{2} \int d^2x d^2y \lambda_q(x) \lambda_q(y) n_0(x) n_0(y) G(x, y), \tag{54}\]

\[S_\lambda = -\frac{1}{2} \int d^2x d^2y \lambda_q(x) \lambda_q(y) K(x, y), \]

where the kernel is

\[K(x, y) = \frac{N-1}{2} G(x, y)^2 + n_0(x) n_0(y) G(x, y). \tag{55}\]
The final answer for the effective action is

\[ S_{\text{eff}} = \frac{N-1}{2} \text{Tr} \log \left( -\partial^2 + \lambda_0 \right) + \frac{1}{2} \text{Tr} \log K + \frac{1}{2} \int d^2x \left[ (\partial n_0)^2 + \lambda_0 (n_0^2 - r) \right]. \]  

(56)

The second term in the effective action is the correction we have calculated. In this expression all terms but the second contain a large \( N \) factor. So the correction is suppressed in large \( N \) limit.

D. Comment on GN model at zero density

For comparison let us briefly comment on the periodic solution in Gross-Neveu model with the Minkowski Lagrangian

\[ \mathcal{L} = \bar{\psi} (i\partial - \sigma) \psi - r \sigma^2. \]

The similar problem was considered in [30]. For more similarity, in this section we consider the theory with Dirac fermions. Generically the period of the elliptic solution to the GN model is fixed by the chemical potential however for the zero density case we do not have the Fermi momentum parameter, the period of the solution remains a free parameter.

The effective action is

\[ S_{\text{eff}} = -i N \text{Tr} \log (i\partial - \sigma) - r \int d^2x \sigma^2. \]

We look for the solution in the form [38]. The mass parameter \( m \) of this solution is connected to the mass scale \( \Lambda \) of the theory through the gap equation that reads as

\[ \sigma (x) = \frac{N}{2r} \int \frac{dk}{2\pi} \frac{\bar{\psi}_k \psi_k}{\omega - m^2 E/K} \sigma (x), \]

where eigenfunctions

\[ \bar{\psi}_k \psi_k = \frac{\omega}{\omega^2 - m^2 E/K} \sigma (x). \]

Therefore gap equation reduces to

\[ 1 = \frac{N}{2r} \int \frac{dk}{2\pi} \frac{\omega (k)}{\omega^2 (k) - m^2 E/K}. \]

The fermionic gap equation leads to the same formula [45] for mass as bosonic one. Note that there is no infrared divergence. The energy of this solution can be calculated from the relation [36] between bosonic and fermionic determinants. Using the fact that the potentials
\[ \sigma^2 \pm \partial_x \sigma \] we find that energy density for fermionic case is different from bosonic only by sign,

\[ \epsilon_{GN} = -\epsilon = -\frac{N \Lambda^2}{4\pi} + \frac{E(\nu) N m^2}{K(\nu) \pi}. \]

Thus, the energy is minimal for homogeneous solution which is the correct ground state. However, the non-vanishing chemical potential modifies the ground state which becomes inhomogeneous.

\section*{VI. DISCUSSION}

In this paper we considered the properties of the inhomogeneous solutions \cite{16} found recently for $\mathbb{C}P^{N-1}$ sigma-model at large $N$. We focused at the soliton-like solution and the elliptic solution to the quantum gap equation. The careful analysis shows that the energy of the soliton is lower than the energy of the homogeneous ground state. This clearly makes questionable the common viewpoint that the ground state of the $\mathbb{C}P^{N-1}$ sigma-model at large $N$ is homogeneous.

The answer to the question about the true ground state of the model does not look simple. The naïve conjecture would be that the periodic elliptic kink crystal solution yields the true ground state and vacuum is in FFLO-like phase as in GN model with non-vanishing chemical potential. The energy for kink crystal solution can be evaluated and indeed it is lower than energy of the homogeneous state. However there are two points of concern which provide the difficulties with such immediate identification. First, the kink crystal solution suffers from the IR divergence at the infinite plane and deserves some IR regularization, for instance by introducing a box. Secondly the kink crystal solution has the free massive parameter which fixes the period whose interpretation is not completely clear in non-SUSY case. It is counterpart of the chemical potential in the GN model.

It is instructive to look at the massive deformations of the large $N$ sigma-models. It has been discussed in \cite{4} for $O^N$ and in \cite{5} for $\mathbb{C}P^{N-1}$. The mass provides the IR regularization of the models, at large masses the theory can be treated perturbatively and is proven to be in the Higgs-like phase. In both models there is a clear-cut phase transition at the value of the mass of order of nonperturbatively generated scale $\Lambda$. Moreover, it is demonstrated in \cite{4} that at the phase transition point two states become massless: the bound state of two n-particles and the soliton.
For masses below \( \Lambda \) these light states could hint at existence of a dual, more suitable, description. This is similar to the Sine-Gordon model transition from the bosonic description at weak coupling to the fermionic one at strong coupling. We did not explore this opportunity. Instead, in our analysis we suggest that the ground state of these models is a small mass deformation of the FFLO-like kink crystal solution. The (twisted) mass parameter fixes the period of the elliptic solution to the gap equation and provides the IR regularization hence everything is well defined in this case. We hope to investigate this issue elsewhere.

The massive deformations of the 2D theories have the clear-cut 4D counterparts – these are the gauge theories with flavor and masses of fundamental matter play the similar role. Instead of the kinks in 2D the domain walls in 4D are considered and the nontrivial mass dependence of their tensions are of interest. We would like to mention two examples: QCD at \( \theta = \pi \) and softly broken \( \mathcal{N} = 2 \) SQCD. In both cases there are domain walls with mass dependent tensions. In QCD case it was proved in [32] that the 3D theory on the domain wall is deconfined. However, the approach of [32] does not give exactly the critical value of the quark mass when the domain wall tension vanishes. On the other hand, in softly broken \( \mathcal{N} = 2 \) SQCD at \( N_f = 1 \) the critical value of the mass at the Argyres-Douglas point when the domain wall tension vanishes has been found exactly [33]. At the critical mass the whole 4D theory turns out in the deconfinement phase [33] and this fits with the deconfinement in 3D theory on the domain wall observed in [32]. Indeed, when the domain wall tension is small it becomes wide and finally the deconfined phase occupies the whole space-time at the Argyres-Douglas point.

One more comment is in order. Recently, it was recognized that the discrete anomaly matching provides the powerful tool for the analysis of the phase diagram of the strongly coupled theories. In particular this approach has been applied to the discussion of the ground state in the spin systems with the SU\((N)\) structure group in some representation [35]. As was known for a while [36] that the low-energy action for the SU\((2)\) group case gets identified with the \( \mathbb{CP}^1 \) model with the \( \theta \) term which depends on the spin representation. If \( \theta = \pi(2k + 1) \) the ground state turns out to be gapless and can be thought of as the the condensate of dimers. More recent analysis [35] suggests that the similar gapless phases for higher spin chains could occur at \( \theta = 2\pi/N \). For instance, in SU\((3)\) case at proper value of \( \theta \) the ground state is gapless and presumably is a kind of condensate of trimers.
We could speculate that gapless ground state we have found could be some analogue of the Haldane’s gapless phase and our periodic kink crystal is the generalization of the dimer and trimer condensates ground states for low rank spin systems. Indeed our soliton-like solution from the chiral GN viewpoint can be considered as the superposition of $N$ elementary kinks in the hedgehog shape. In our case we have $\theta = 0$ but presumably it can be reasonable approximation of $\theta = 2\pi/N$ at large $N$.

We have touched a bit the SUSY generalization of the new solution postponing the detailed analysis for the separate study. The immediate question concerns the BPS property of the solution. The SUSY picture implies also the several questions concerning its brane interpretation. Let us make a few remarks

- The nontrivial profile of the n-field corresponds to the pulling of D2 brane in particular direction by D2-D4 string. Hence to some extent the soliton is represented by the profile of F1 D2-D4 string. It would be interesting to get the interpretation of the soliton solution from the F1 worldsheet viewpoint

- The brane picture for the GN model [29] tells that the kink corresponds to the interpolation between two possible intersections of D4 and D6 branes. This resembles the appearance of the second vacuum in the $\mathbb{CP}^{N-1}$ model coupled to 4D degrees of freedom [28]. Hence it is natural to expect that the brane configuration responsible for the soliton and soliton lattice configurations involves D6 branes.

- The local negative energy contribution is typical for boojoums [12] when the magnetic non-abelian string is attached to the domain wall. The negative energy is localized on the domain wall near the intersection point. One could conjecture that the soliton solution corresponds to the region of the intersection of the D6 domain wall and D2 brane representing non-abelian string in 4D gauge theory.

- Recently the so-called negative branes with the negative tensions have been found [37]. These objects are identified both for extended branes and for point-like particles with negative mass. For some of them the supergravity solutions have been found and it was argued that they obey the fermion statistics. It is unclear if our finding is related with this issue.

Several questions concerns the IR properties of the periodic solution.
• Connection between infrared divergences in the solution and Coleman’s theorem deserves the careful study. There are some example of models in which 2D continuous symmetry can be broken (chiral GN and $\mathbb{CP}^{N-1}$ on a circle at large $N$, SUSY $\mathbb{CP}^{N-1}$ due to mixing of $\pi$ and $A_\mu$ propagators). Could something similar happen in our case?

• Our study imply that the homogeneous solution for $\mathbb{CP}^{N-1}$ model certainly is not the true ground state contrary to the standard viewpoint. Therefore it is necessary to clarify if it the the metastable minimum of just local extremum. If it is the metastable state the kink-antikink configuration usually considered as the excitation could be treated as the bounce responsible for the decay of the metastable vacuum.

• Even if periodic solutions do not exist on a plane, they can change phase structure on a circle. There are possible phase transitions when $n^2 = 0$.

Let us remark that the lattice studies of the $\mathbb{CP}^{N-1}$ model also shows a unexpected structure of the ground state [34] which has in the Euclidean space the crystal-like double-layer structure. The distribution of the topological charge density has the dipole-like structure and vacuum was interpreted as a kind of condensate of the Wilson loops. It is unclear if the kink crystal solution we considered in this study with minimal energy has something to do with these lattice observations.

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Appendices

A. EFFECTIVE ACTION CALCULATION FOR SOLITON

Here we provide the technical details of computation of energy of the soliton. The coupling constant can be found from the gap equation for the homogeneous solution in space of large volume $V$,

$$ r \cdot V = \sum_{i=0}^{I} C_i \text{Tr} \frac{1}{-\partial^2 + m_i^2 + m^2} = V \cdot \int \frac{d^2 k}{4\pi^2} \sum_{i=0}^{I} C_i \frac{1}{k^2 + m_i^2 + m^2}, $$

$$ r = -\frac{N}{4\pi} \sum_{i=0}^{I} C_i \log (m^2 + m_i^2). $$

(57)

The trace of the operator can be written as a sum over the eigenvalues,

$$ \text{Tr} \log (-\partial^2 + m_i^2 + \lambda) = T \int \frac{d\omega}{2\pi} \sum_n \log (\omega^2 + \omega_n^2 + m_i^2). $$

Here $T$ stands for a large time cut-off and summation is over all eigenvalues $\omega_n^2$ of the operator $-\partial_x^2 + \lambda$. Therefore, we obtain the following expression for energy:

$$ E_1 = N \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_n \sum_{i=0}^{I} C_i \log (\omega^2 + \omega_n^2 + m_i^2) - \int_{-\infty}^{+\infty} dx \lambda. $$

(58)

The same expression can be written for energy of vacuum $E_{\text{vac}}$ when $\lambda = m^2$ and eigenvalues are $\omega_{0n}^2$.

We use expression (58) for the energy and subtract vacuum contribution:

$$ E = E_1 - E_{\text{vac}} = N \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{i=0}^{I} C_i \log (\omega^2 + m_i^2) + $$

$$ + N \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_n \sum_{i=0}^{I} C_i \log \frac{\omega^2 + \omega_n^2 + m_i^2}{\omega^2 + \omega_{0n}^2 + m_i^2} - \int_{-\infty}^{+\infty} dx (\lambda - m^2) r. $$

Here the first term is contribution from the zero mode and the second is contribution from the continuum. If we use integral

$$ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \log \left( 1 + \frac{a^2}{\omega^2} \right) = a $$

and integrate over $\omega$ in the first and second term, and over coordinate in the third we arrive at

$$ E = N \sum_{i=0}^{I} C_i m_i + N \sum_n \sum_{i=0}^{I} C_i \left( \sqrt{\omega_n^2 + m_i^2} - \sqrt{\omega_{0n}^2 + m_i^2} \right) + 4mr. $$
Summation over all eigenvalues can be replaced with integration over all momenta,
\[ \sum_n \rightarrow \int dk \rho(k), \quad \omega_n^2 \rightarrow k^2 + m^2, \]
where difference of densities of states for homogeneous and inhomogeneous states is
\[ \rho(k) = \frac{1}{\pi} \frac{d\delta(k)}{dk} = -\frac{2m}{\pi (k^2 + m^2)}. \]

Here \( \delta(k) = \pi - 2 \arctan(k/m) \) is phase shift for eigenfunctions (22). Therefore energy is
\[ E = N \sum_{i=0}^I C_i m_i - \frac{2Nm}{\pi} \int_0^{+\infty} dk \sum_{i=0}^I C_i \sqrt{k^2 + m^2 + m_i^2} \frac{\pi}{k^2 + m^2} + 4mr. \]

We use integral
\[ \int dk \frac{\sqrt{k^2 + m^2 + M^2}}{k^2 + m^2} = \frac{M}{m} \arctan \frac{Mk}{m \sqrt{k^2 + m^2 + M^2}} + \log \left( k + \sqrt{k^2 + m^2 + M^2} \right) \]
and obtain
\[ E = N \sum_{i=0}^I C_i m_i - \frac{2Nm}{\pi} \left[ \sum_{i=1}^I C_i \frac{m_i}{m} \arctan \frac{m_i}{m} - \frac{1}{2} \sum_{i=0}^I C_i \log \left( m^2 + m_i^2 \right) \right] + 4mr. \]

If we apply the expression (57) for \( r \) and assume that \( m_i \gg m \) and thus \( \arctan (m_i/m) = \pi/2 - m/m_i \) we obtain
\[ E = N \sum_{i=0}^I C_i m_i - \frac{2Nm}{\pi} \sum_{i=1}^I C_i \frac{m_i \pi}{2m} \frac{1}{2} \sum_{i=1}^I C_i + \]
\[ + \frac{Nm}{\pi} \sum_{i=0}^I C_i \log \left( m^2 + m_i^2 \right) - \frac{Nm}{\pi} \sum_{i=0}^I C_i \log \left( m^2 + m_i^2 \right). \]

We see that all terms except the third cancel. The sum in the third term is \( \sum_{i=1}^I C_i = -C_0 = -1 \) and we find the expression (24).

**B. ENERGY-MOMENTUM TENSOR OF THE SOLITON**

To calculate the average of energy-momentum tensor components (27) and we need following combinations
\[ \sum_i C_i m_i^2 |n_i(x)|^2 = N \int \frac{dk}{2\pi} \frac{m_i^2 |f_k(x)|^2}{2\sqrt{k^2 + m^2 + m_i^2}} + N\psi_0(x)^2 \sum_i \frac{C_i m_i}{2}, \quad (59) \]
\[ \sum_i C_i \langle |\partial_x n_i (x)|^2 \rangle = (\partial_x n_{cl} (x))^2 + N \int \frac{dk}{2\pi} \sum_i \frac{|\partial_x f_k (x)|^2}{2\sqrt{k^2 + m^2 + m_i^2}} + N\psi_0 (x)^2 \sum_i \frac{C_i}{2m_i}, \quad \text{(60)} \]

\[ \sum_i C_i \langle |\partial_t n_i (x)|^2 \rangle = N \int \frac{dk}{4\pi} \sum_i C_i \sqrt{k^2 + m^2 + m_i^2} |f_k (x)|^2 + N\psi_0 (x)^2 \sum_i \frac{C_i m_i}{2}. \quad \text{(61)} \]

The expressions for modes and their derivatives are

\[ |f_k (x)|^2 = \frac{k^2 + m^2 \tanh^2 mx}{k^2 + m^2} = 1 - \frac{m^2}{k^2 + m^2 \cosh^2 mx}; \]

\[ |\partial_x f_k (x)|^2 = k^2 + \frac{m^2}{\cosh^2 mx} + \frac{m^4}{k^2 + m^2} \left( \frac{1}{\cosh^4 mx} - \frac{1}{\cosh^2 mx} \right). \quad \text{(62)} \]

We consider mass term (59) and terms with derivatives (60) and (61) separately

\[ \sum_i C_i m_i^2 \langle |n_i (x)|^2 \rangle = N \int \frac{dk}{2\pi} \sum_i \frac{C_i m_i^2}{2\sqrt{k^2 + m^2 + m_i^2}} + \]

\[ + N \frac{m^2}{\cosh^2 mx} \left( - \int \frac{dk}{4\pi} \sum_i \frac{C_i m_i^2}{(k^2 + m^2) \sqrt{k^2 + m^2 + m_i^2}} + \sum_i \frac{C_i m_i}{4m} \right). \]

The first term yields the energy density of homogeneous state. Note that in the expression (60) for the spacial derivative the term with derivative of classical component cancels with the convergent part of the integral, which is a contribution from the third term in (62). So we can write down the remaining contributions

\[ \sum_i C_i \langle |\partial_x n_i (x)|^2 + |\partial_t n_i (x)|^2 \rangle = \]

\[ = N \frac{m^2}{\cosh^2 mx} \left( \int \frac{dk}{4\pi} \sum_i C_i \left( \frac{1}{\sqrt{k^2 + m^2 + m_i^2}} - \frac{\sqrt{k^2 + m^2 + m_i^2}}{k^2 + m^2} \right) + \sum_i \frac{C_i m_i}{4m} \right). \]

All integrals can be computed elementary. Thus we find that contribution to the inhomogeneous part of energy density from derivative terms (61) and (60) and term (59) with are equal. Therefore, corresponding contributions in the momentum flaw \(\theta_{11}\) in (27) cancel and this component does not depend on the coordinate. Combining the results, we obtain (30) and (31).
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