On Jump Measures of Optional Processes with Regulated Trajectories

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Abstract Starting from an iterative and hence numerically easily implementable representation of the thin set of jumps of a càdlàg adapted stochastic process \( X \) (including a few applications to the integration with respect to the jump measure of \( X \)), we develop similar representation techniques to describe the set of jumps of optional processes with regulated trajectories and introduce their induced jump measures with a view towards the framework of enlarged filtration in financial mathematics.

1 Preliminaries and Notation

In this section, we introduce the basic notation and terminology which we will use throughout in this paper. Most of our notation and definitions including those ones originating from the general theory of stochastic processes and stochastic analysis are standard. We refer the reader to the monographs [6], [10], [12] and [14].

Since at most countable unions of pairwise disjoint sets play an important role in this paper, we use a well-known symbolic abbreviation. For example, if \( A := \bigcup_{n=1}^{\infty} A_n \), where \( (A_n)_{n \in \mathbb{N}} \) is a sequence of sets such that \( A_i \cap A_j = \emptyset \) for all \( i \neq j \), we write shortly \( A := \bigcup_{n=1}^{\infty} A_n \).

Throughout this paper, \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) denotes a fixed probability space, together with a fixed filtration \( \mathbb{F} \). Even if it is not explicitly emphasized, the filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) always is supposed to satisfy the usual conditions.\(^1\) A real-valued (stochastic) process \( X : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R} \) (which may be identified with the family of

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\(^1\) \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets and \( \mathbb{F} \) is right-continuous.
random variables $(X_t)_{t \geq 0}$, where $X_t(\omega) := X(\omega, t)$ is called adapted (with respect to $\mathcal{F}$) if $X_t$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}^+$. $X$ is called right-continuous (respectively left-continuous) if for all $\omega \in \Omega$ the trajectory $X_t(\omega) : \mathbb{R}^+ \rightarrow \mathbb{R}, t \mapsto X_t(\omega)$ is a right-continuous (respectively left-continuous) real-valued function. If all trajectories of $X$ do have left-hand limits (respectively right-hand limits) everywhere on $\mathbb{R}^+$, the jump process $\Delta X = (\Delta X_t)_{t \geq 0}$ is well-defined on $\Omega \times \mathbb{R}^+$. It is given by $\Delta X := X^+ - X^-$. 

A right-continuous process whose trajectories do have left-hand limits everywhere on $\mathbb{R}^+$, is known as a càdlàg process. If $X$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, $X$ is said to be measurable. $X$ is said to be progressively measurable (or simply progressive) if for each $t \geq 0$, its restriction $X|_{\Omega \times [0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-measurable. Obviously, every progressive process is measurable and (thanks to Fubini) adapted.

A random variable $T : \Omega \rightarrow [0, \infty]$ is said to be a stopping time or optional time (with respect to $\mathcal{F}$) if for each $t \geq 0$, $\{T \leq t\} \in \mathcal{F}$. Let $\mathcal{S}$ denote the set of all stopping times, and let $S, T \in \mathcal{S}$ such that $S \leq T$. Then $[S, T] := \{(\omega, t) \in \Omega \times \mathbb{R}^+ : S(\omega) \leq t < T(\omega)\}$ is an example for a stochastic interval. Similarly, one defines the stochastic intervals $[S, T], ]S, T]$, and $[S, T]$. Note again that $[T] := [T, T] = \text{Gr}(T)|_{\Omega \times \mathbb{R}^+}$ is simply the graph of the stopping time $T : \Omega \rightarrow [0, \infty]$ restricted to $\Omega \times \mathbb{R}^+$. $\mathcal{G} = \sigma\{[T, \infty] : T \in \mathcal{S}\}$ denotes the optional $\sigma$-field which is generated by all càdlàg adapted processes. The predictable $\sigma$-field $\mathcal{P}$ is generated by all left-continuous adapted processes. An $\mathcal{G}$- (respectively $\mathcal{P}$-) measurable process is called optional or well-measurable (respectively predictable). All optional or predictable processes are adapted.

For the convenience of the reader, we recall and summarise the precise relation between those different types of processes in the following

**Theorem 1.** Let $(\Omega, \mathcal{F}, \mathcal{G}, \mathbb{P})$ be a filtered probability space such that $\mathcal{F}$ satisfies the usual conditions. Let $X$ be a (real-valued) stochastic process on $\Omega \times \mathbb{R}^+$. Consider the following statements:

(i) $X$ is predictable;
(ii) $X$ is optional;
(iii) $X$ is progressive;
(iv) $X$ is adapted.

Then the following implications hold:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

If $X$ is right-continuous, then the following implications hold:

$$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv).$$

If $X$ is left-continuous, then all statements are equivalent.

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$\mathbb{R}^+ := [0, \infty).$
Proof. The general chain of implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ is well-known (for a detailed discussion cf. e.g. [6, Chapter 3]). If $X$ is left-continuous and adapted, then $X$ is predictable. Hence, in this case, all four statements are equivalent. If $X$ is right-continuous and adapted, then $X$ is optional (cf. e.g. [10, Theorem 4.32]). In particular, $X$ is progressive. □

Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be regulated on $\mathbb{R}^+$ if $f$ has right- and left-limits everywhere on $(0, \infty)$ and $f(0^+)$ exists (cf. e.g. [9, Ch. 7.6]).

Let us also commemorate the following

**Lemma 1.** Let $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a stochastic process such that its trajectories are regulated. Then all trajectories of the left limit process $X^-$ (respectively of the right limit process $X^+$) are left-continuous (respectively right-continuous). If in addition $X$ is optional, then $X^-$ is predictable and $X^+$ is adapted.

Given an optional process $X$ with regulated trajectories, we put

$$\{\Delta X \neq 0\} := \{(\omega, t) \in \Omega \times \mathbb{R}^+: \Delta X_t(\omega) \neq 0\}.$$

Recall the important fact that for any $\varepsilon > 0$ and any regulated function $f : \mathbb{R}^+ \to \mathbb{R}$ the set $J_f(\varepsilon) := \{t > 0 : |\Delta f(t)| > \varepsilon\}$ is at most countable, implying that

$$J_f := \{t > 0 : \Delta f(t) \neq 0\} = \{t > 0 : |\Delta f(t)| > 0\} = \bigcup_{n \in \mathbb{N}} J_f(\frac{1}{n})$$

is at most countable as well (cf. [11, p. 286-288] and [13, Theorem 1.3]).

### 2 Construction of Thin Sets of Jumps of Càdlàg Adapted Processes

In the general framework of semimartingales with jumps (such as e.g. Lévy processes) there are several ways to describe a stochastic integral with respect to a (random) jump measure $\lambda_X$ of a càdlàg adapted stochastic process $X = (X_t)_{t \geq 0}$. One approach is to implement the important subclass of “thin” subsets of $\Omega \times \mathbb{R}^+$ (cf. [12, Def. 1.30]) in order to analyse the set $\{\Delta X \neq 0\}$:

**Theorem 2 (Dellacherie, 1972).** Let $X = (X_t)_{t \geq 0}$ be an arbitrary $\mathcal{F}$-adapted càdlàg stochastic process on $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$. Then there exist a sequence $(T_n)_{n \in \mathbb{N}}$ of $\mathcal{F}$-stopping times such that $[T_n] \cap [T_k] = \emptyset$ for all $n \neq k$ and

$$\{\Delta X \neq 0\} = \bigcup_{n=1}^{\infty} [T_n].$$

In particular, $\Delta X_{T_n}(\omega)(\omega) \neq 0$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. 
A naturally appearing, iterative and hence implementable exhausting representation is given in the following important special case (cf. e. g. [14, p. 25] or the proof of [1] Lemma 2.3.4.]):

**Proposition 1** Let $X = (X_t)_{t \geq 0}$ be an arbitrary $\mathbf{F}$-adapted càdlàg stochastic process on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ and $A \in \mathcal{B}(\mathbb{R})$ such that $0 \notin A$. Put

$$T^A_0(\omega) := \inf \{ t > 0 : \Delta X_t(\omega) \in A \}$$

and

$$T^A_n(\omega) := \inf \{ t > T^A_{n-1}(\omega) : \Delta X_t(\omega) \in A \} \quad (n \geq 2).$$

Up to an evanescent set $(T^A_n)_{n \in \mathbb{N}}$ defines a sequence of strictly increasing $\mathbf{F}$-stopping times, satisfying

$$\{ \Delta X \in A \} = \bigcup_{n=1}^{\infty} \| S^A_n \|,$$

where

$$S^A_n := T^A_n \mathbb{1}_A(\Delta X_{T^A_n}) + (+\infty) \mathbb{1}_A(\Delta X_{T^A_n}).$$

**Proof.** In virtue of [14, Chapter 4, p. 25] each $T^A_n$ is a $\mathbf{F}$-stopping time and $\Omega_0 \times \mathbb{R}^+$ is an evanescent set, where $\Omega_0 := \{ \omega \in \Omega : \lim_{n \to \infty} T^A_n(\omega) < \infty \}$. Fix $(\omega, t) \notin \Omega_0 \times \mathbb{R}^+$. Assume by contradiction that $T^A_{m_0}(\omega) = T^A_{m_0+1}(\omega) =: t^*$ for some $m_0 \in \mathbb{N}$. By definition of $t^* = T^A_{m_0+1}(\omega)$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$\lim_{n \to \infty} t_n = t^*, \Delta X_{t_n}(\omega) \in A,$$

and $t^* = T^A_{m_0}(\omega) < t_{n+1} \leq t_n$. Consequently, since $X$ has right-continuous paths, it follows that $\Delta X_{t^*}(\omega) = \lim_{n \to \infty} \Delta X_{t_n}(\omega) \in A$, implying that

$$\Delta X_{t^*}(\omega) \neq 0 \quad (0 \notin A).$$

Thus $\lim_{n \to \infty} t_n = t^*$ is an accumulation point of the at most countable set $\{ t > 0 : \Delta X_t(\omega) \neq 0 \}$ - a contradiction.

To prove the set equality let firstly $\Delta X_t(\omega) \in A$. Assume by contradiction that for all $m \in \mathbb{N}$ $T^A_m(\omega) \neq t$. Since $\omega \notin \Omega_0$, there is some $m_0 \in \mathbb{N} \cap [2, \infty)$ such that $T^A_{m_0}(\omega) > t$. Choose $m_0$ small enough, so that $T^A_{m_0-1}(\omega) \leq t < T^A_{m_0}(\omega)$. Consequently, since $\Delta X_t(\omega) \in A$, we must have $t \leq T^A_{m_0-1}(\omega)$ and hence $T^A_{m_0-1}(\omega) = t$.

However, the latter contradicts our assumption. Thus, $\{ \Delta X \in A \} \subseteq \bigcup_{n=1}^{\infty} \| S^A_n \|$. The claim now follows from [10, Theorem 3.19].

**Remark 1** Note that $\{ S^A_n < +\infty \} \subseteq \{ \Delta X_{T^A_n} \in A \} \subseteq \{ S^A_n = T^A_n \}$. Hence,

$$\mathbb{1}_A(\Delta X_{T^A_n}) \mathbb{1}_{\{ T^A_n \leq t \}} = \mathbb{1}_{\{ S^A_n \leq t \}}$$

for all $n \in \mathbb{N}$.

Next, we recall and rewrite equivalently the construction of a random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ (cf. e. g. [12, Def. 1.3]):

**Definition 1**. A random measure on $\mathbb{R}^+ \times \mathbb{R}$ is a family $\mu \equiv (\mu(\omega; d(s, x)) : \omega \in \Omega)$ of non-negative measures on $(\mathbb{R}^+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}))$, satisfying $\mu(\omega; \{ 0 \} \times \mathbb{R}) = 0$ for all $\omega \in \Omega$. 

Given an adapted $\mathbb{R}$-valued càdlàg process $X$, a particular (integer-valued) random measure (cf. e.g. [12, Prop. 1.16]) is given by the jump measure of $X$, defined as

$$j_X(\omega, B) := \sum_{s \geq 0} \mathbb{1}_{\{\Delta X(s) \neq 0\}}(\omega, s) \mathbb{1}_{\{s, \Delta X(s)\}}(B)$$

$$= \sum_{s \geq 0} \mathbb{1}_{B}(s, \Delta X(s)) \mathbb{1}_{\mathbb{R}^+}(\Delta X(s))$$

$$= \#\{s > 0 : \Delta X(s) \neq 0 \text{ and } (s, \Delta X(s)) \in B\},$$

where $\delta_a$ denotes the Dirac measure at point $a$ and $B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$.

Keeping the above representation of the jump measure $j_X$ in mind, we now are going to consider an important special case of a Borel set $B$ on $\mathbb{R}^+ \times \mathbb{R}$, leading to the construction of “stochastic” integrals with respect to the jump measure $j_X$ including the construction of stochastic jump processes which play a fundamental role in the theory and application of Lévy processes. To this end, let us consider all Borel sets $B$ on $\mathbb{R}^+ \times \mathbb{R}$ of type $B = [0, t] \times A$, where $t \geq 0$ and

$$A \in \mathcal{B}^* := \{A : A \in \mathcal{B}(\mathbb{R}), 0 \notin A\}.$$

Obviously, $A \subseteq \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ for all $\varepsilon > 0$, implying in particular that $A \in \mathcal{B}^*$ is bounded from below. Let us recall the following

**Lemma 2.** Let $X = (X_t)_{t \geq 0}$ be a càdlàg process. Let $A \in \mathcal{B}^*$ and $t > 0$. Then $N^A(t) := j_X(\cdot, [0, t] \times A) < \infty$ a.s.

**Proof.** This is [4, Lemma 2.3.4.]. \(\square\)

**Proposition 2** Let $X = (X_t)_{t \geq 0}$ be a càdlàg process and $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable. Let $A \in \mathcal{B}^*$ and $t > 0$. Then for all $\omega \in \Omega$ the function $\mathbb{1}_{[0,t] \times A} f$ is a.s. integrable with respect to the jump measure $j_X(\omega, d(s, x))$, and

$$\int_{[0,t] \times A} f(s, x) j_X(\omega, d(s, x))$$

$$= \sum_{0 < s \leq t} f(s, \Delta X(s)) \mathbb{1}_{A}(\Delta X(s))$$

$$= \sum_{n=1}^{\infty} f(T_n(\omega), \Delta X_{T_n(\omega)}(\omega)) \mathbb{1}_{A}(\Delta X_{T_n(\omega)}(\omega)) \mathbb{1}_{\{T_n \leq t\}}(\omega).$$

Moreover, given $\omega \in \Omega$ there exists $c^A_\omega(\omega) \in \mathbb{R}^+$ such that

$$\int_{[0,t] \times A} |f(s, x)| j_X(\omega, d(s, x)) \leq c^A_\omega(\omega) j_X(\omega, [0, t] \times A).$$

**Proof.** Fix $\omega \in \Omega$ and consider the measurable function $g^A_\omega := \mathbb{1}_{[0,t] \times A} f$. Then $\mathbb{R}^+ \times \mathbb{R} = B_1(\omega) \cup B_2(\omega)$, where $B_1(\omega) := \{(s, \Delta X(s) : s > 0\}$ and $B_2(\omega) := \mathbb{R}^+ \times \mathbb{R} \setminus B_1(\omega)$. Obviously, we have
\[ j_X(\omega, B_2(\omega)) = \sum_{s > 0} \mathbb{1}_{B_2(\omega)}(s, \Delta X_s(\omega)) \mathbb{1}_{\mathbb{R}^+}(\Delta X_s(\omega)) = 0, \]

implying that \( I_2 := \int_{B_2(\omega)} |g^A_t(s, x)| j_X(\omega, d(s, x)) = 0 \). Put \( I_1 := \int_{B_1(\omega)} |g^A_t(s, x)| j_X(\omega, d(s, x)). \)

Since on \([0, t]\) the càdlàg path \( s \mapsto X_s(\omega) \) has only finitely many jumps in \( A \in \mathcal{B}^* \) there exist finitely many elements \((s_1, \Delta X_{s_1}(\omega)), \ldots, (s_N, \Delta X_{s_N}(\omega))\) which all are elements of \([0, t] \times A\) (for some \( N = N(\omega, t, A) \in \mathbb{N} \)). Put

\[ 0 \leq c^A_t(\omega) := \max_{1 \leq k \leq N} |f(s_k, \Delta X_{s_k}(\omega))| < \infty. \]

Then

\[ |g^A_t| = \mathbb{1}_{[0, t] \times A} |f| \leq c^A_t(\omega) \mathbb{1}_{[0, t] \times A} \text{ on } B_1, \]

and it follows that \( I_2 \leq c^A_t(\omega) j_X(\omega, [0, t] \times A). \) A standard monotone class argument finishes the proof. □

**Remark 2** Note that in terms of the previously discussed stopping times \( S^A_n \) we may write

\[ \int_{[0, t] \times A} f(s, x) j_X(\omega, d(s, x)) = \sum_{n=1}^{\infty} f(S^A_n(\omega), \Delta X_{S^A_n(\omega)}(\omega)) \mathbb{1}_{\{S^A_n \leq t\}(\omega)}. \]

In the case of a Lévy process \( X \) the following important special cases \( f(s, x) := 1 \) and \( f(s, x) := x \) are embedded in the following crucial result (cf. e.g. [3]):

**Theorem 1** Let \( X = (X_t)_{t \geq 0} \) be a (càdlàg) Lévy process and \( A \in \mathcal{B}^* \).

(i) Given \( t \geq 0 \)

\[ N^A_X(t) = \int_A N^A_X(t) := j_X(\cdot, [0, t] \times A) = \int_{[0, t] \times A} j_X(\cdot, d(s, x)) \]

\[ = \sum_{0 < s \leq t} \mathbb{1}_{A}(\Delta X_s) = \sum_{n=1}^{\infty} \mathbb{1}_{A}(\Delta X_{S^A_n}) \mathbb{1}_{\{S^A_n \leq t\}} = \sum_{n=1}^{\infty} \mathbb{1}_{\{S^A_n \leq t\}} \]

induces a Poisson process \( N^A_X = (N^A_X(t))_{t \geq 0} \) with intensity measure \( \nu_X(A) := \mathbb{E}[N^A_X(1)] < \infty. \)

(ii) Given \( t \geq 0 \) and a Borel measurable function \( g : \mathbb{R} \rightarrow \mathbb{R} \)

\[ Z^A_X(t) := \int_A g(x) N^A_X(t) = \int_{[0, t] \times A} g(x) j_X(\cdot, d(s, x)) \]

\[ = \sum_{0 < s \leq t} g(\Delta X_s) \mathbb{1}_{A}(\Delta X_s) = \sum_{n=1}^{\infty} g(\Delta X_{S^A_n}) \mathbb{1}_{A}(\Delta X_{S^A_n}) \mathbb{1}_{\{S^A_n \leq t\}} \]

\[ = \sum_{n=1}^{\infty} g(\Delta X_{S^A_n}) \mathbb{1}_{\{S^A_n \leq t\}} = \sum_{n=1}^{N^A_X(t)} g(\Delta X_{S^A_n}) \]
induces a compound Poisson process $Z_X^n = (Z_X^n(t))_{t \geq 0}$. Moreover, if $g \in L^1(A, \nu_X)$ then $E[Z_X^n(t)] = t\nu_X(A)E[g(\Delta X_{S^n})]$. 

3 Jump Measures of Optional Processes with Regulated Trajectories

One of the aims of our paper is to transfer particularly Theorem 2 to the class of optional processes with regulated trajectories in order to construct a well-defined jump measure of such optional processes.

As we have seen the right-continuity of the paths of $X$ plays a significant role in the proof of Proposition 1. We will see that a similar result holds for optional processes with regulated trajectories. However, it seems that we cannot simply implement the above sequence $(S_{n}^{A})_{n \in \mathbb{N}}$ if the paths of $X$ are not right-continuous.

Our next contribution shows that we are not working with “abstract nonsense” only:

**Example 1** Optional processes which do not necessarily have right-continuous paths have emerged as naturally appearing candidates in the framework of enlarged filtration in financial mathematics (formally either describing “insider trading information” or “extended information by inclusion of the default time of a counterparty”) including the investigation of the problem whether the no-arbitrage conditions are stable with respect to a progressive enlargement of filtration and how an arbitrage-free semimartingale model is affected when stopped at a random horizon (cf. [1], [2] and [3]).

Given a random time $\tau$, one can construct the smallest right-continuous filtration $\mathbb{G}$ which contains the given filtration $\mathbb{F}$ and makes $\tau$ a $\mathbb{G}$-stopping time (known as progressive enlargement of $\mathbb{F}$ with $\tau$). Then one can associate to $\tau$ the two $\mathbb{F}$-supermartingales $Z$ and $\tilde{Z}$, defined through

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) \quad \text{and} \quad \tilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t).$$

$Z$ is càdlàg, while $\tilde{Z}$ is an optional process with regulated trajectories only.

A first step towards the construction of a similar iterative and implementable exhausting representation of the set $\{\Delta X \neq 0\}$ for optional processes is encoded in the following

**Proposition 3** Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be an arbitrary regulated function. Then

$$J_f = \bigcup_{n=1}^{\infty} D_n,$$

where each $D_n$ is a finite set.

**Proof.** Since $(0, \infty) = \bigcup_{n=1}^{\infty} (n-1, n]$ it follows that $J_f = \bigcup_{n=1}^{\infty} J_{f_n}$, where $f_n := f|_{(n-1, n]}$ denotes the restriction of $f$ to the interval $(n-1, n]$. Fix $n \in \mathbb{N}$. Since every
bounded infinite set of real numbers has a limit point (by Bolzano-Weierstrass) the at most countable set

\[ J_{f_n}(\frac{1}{m}) = \{ t : n - 1 < t \leq n \text{ and } |\Delta f(t)| > \frac{1}{m} \} \]

must be already finite for each \( m \in \mathbb{N} \) (cf. [5] Theorem 2.6 and [11, p. 286-288]). Moreover, \( J_{f_n}(\frac{1}{m}) \subseteq J_{f_n}(\frac{1}{m+1}) \) for all \( m \in \mathbb{N} \). Consequently, we have

\[ J_{f_n} = \bigcup_{m=1}^{\infty} J_{f_n}(\frac{1}{m}) = \bigcup_{m=1}^{\infty} A_{m,n}, \]

where \( A_{1,n} := J_{f_n}(1) = \{ |\Delta f_n| > 1 \} \) and \( A_{m+1,n} := \{ \Delta f_n \in (\frac{1}{m+1}, \frac{1}{m}] \} \) for all \( m \in \mathbb{N} \), and hence

\[ J_f = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{m,n}. \]

Since \( A_{m,n} \subseteq J_{f_n}(\frac{1}{m}) \) for all \( m \in \mathbb{N} \), each set \( A_{m,n} \) consists of finitely many elements only.

\[ \square \]

**Lemma 3.** Let \( \emptyset \neq D \) be a finite subset of \( \mathbb{R} \), consisting of \( \kappa_D \) elements. Consider

\[ s^D_1 := \min(D) \]

and, if \( \kappa_D \geq 2 \),

\[ s^D_n := \min(D \cap (s^D_{n-1}, \infty)) = \min\{ t > s^D_{n-1} : t \in D \}, \]

where \( n \in \{2, 3, \ldots, \kappa_D\} \). Then \( D \cap (s^D_{n-1}, \infty) \neq \emptyset \) and \( s^D_{n-1} < s^D_n \) for all \( n \in \{2, 3, \ldots, \kappa_D\} \). Moreover, we have

\[ D = \{ s^D_1, s^D_2, \ldots, s^D_{\kappa_D} \}. \]

**Proof.** Let \( \kappa_D \geq 2 \). Obviously, it follows that \( D \cap (s^D_1, \infty) \neq \emptyset \). Now assume by contradiction that there exists \( n \in \{2, 3, \ldots, \kappa_D - 1\} \) such that \( D \cap (s^D_{n-1}, \infty) = \emptyset \). Let \( m^* \) be the smallest \( m \in \{2, \ldots, \kappa_D - 1\} \) such that \( D \cap (s^D_{m^*}, \infty) = \emptyset \). Then \( s^D_n := \min(D \cap (s^D_{m^*}, \infty)) \in D \) is well-defined for all \( k \in \{2, \ldots, m^*\} \), and we obviously have \( s^D_1 < s^D_2 < \ldots < s^D_{m^*} \). Moreover, by construction of \( m^* \), it follows that

\[ s \leq s^D_m, \quad \text{for all } s \in D. \tag{1} \]

Assume now that there exists \( s \in D \) such that \( s \not\in \{ s^D_1, s^D_2, \ldots, s^D_{m^*} \} \). Then, by (1), there must exist \( l \in \{1, 2, \ldots, m^* - 1\} \) such that \( s^D_l < s < s^D_{l+1} \), which is a contradiction, due to the definition of \( s^D_{l+1} \). Hence, \( s \) cannot exist, and it consequently follows that \( D = \{ s^D_1, s^D_2, \ldots, s^D_{m^*} \} \). But then \( m^* = \#(D) \leq \kappa_D - 1 < \kappa_D \), which is a contradiction. Hence, \( D \cap (s^D_{m^*}, \infty) \neq \emptyset \) for any \( n \in \{2, \ldots, \kappa_D - 1\} \), implying that \( s^D_n \in D \) is well-defined and \( s^D_n < s^D_{n+1} \) for all \( n \in \{1, 2, \ldots, \kappa_D - 1\} \). Clearly, we must have

\[ D = \{ s^D_1, s^D_2, \ldots, s^D_{\kappa_D} \}. \]
Let $A \subseteq \Omega \times \mathbb{R}_+$ and $\omega \in \Omega$. Consider
$$D_A(\omega) := \inf\{t \in \mathbb{R}_+: (\omega, t) \in A\} \in [0, \infty]$$
$D_A$ is said to be the début of $A$. Recall that $\inf(\emptyset) = +\infty$ by convention. $A$ is called a progressive set if $1_A$ is a progressively measurable process. For a better understanding of the main ideas in the proof of Theorem 4, we need the following non-trivial result (a detailed proof of this statement can be found in e.g. [10]):

**Theorem 3.** Let $A \subseteq \Omega \times \mathbb{R}_+$. If $A$ is a progressive set, then $D_A$ is a stopping time.

Next, we reveal how these results enable a transfer of the jump measure for càdlàg and adapted processes to optional processes with infinitely many jumps and regulated trajectories which need not necessarily be right-continuous. To this end, we firstly generalise Theorem 2 in the following sense:

**Theorem 4.** Let $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an optional process such that all trajectories of $X$ are regulated and $\Delta X_0 = 0$. Then $\Delta X$ is also optional. If for each trajectory of $X$ its set of jumps is not finite, then there exists a sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ such that $(T_n(\omega))_{n \in \mathbb{N}}$ is a strictly increasing sequence in $(0, \infty)$ for all $\omega \in \Omega$ and

$$\Delta X(\omega) = \bigcup_{n=1}^{\infty} \{ T_n(\omega) \} \text{ for all } \omega \in \Omega,$$

or equivalently,

$$\{ \Delta X \neq 0 \} = \bigcup_{n=1}^{\infty} \| T_n \|.$$

In particular $\{ \Delta X \neq 0 \}$ is a thin set.

**Proof.** Due to the assumption on $X$ and Lemma $X^-$ is predictable, $X^+$ is adapted and all trajectories of $X^+$ are right-continuous on $\mathbb{R}_+$. Hence, by Theorem 1 both, $X^-$ and $X^+$ are optional processes, implying that the jump process $\Delta X = X^+ - X^-$ is optional as well.

Fix $\omega \in \Omega$. Consider the trajectory $f := X_\bullet(\omega)$. Due to Proposition 3 we may represent $J_f$ as

$$J_f = \bigcup_{m=1}^{\infty} D_m(\omega),$$

where $\kappa_m(\omega) := \#(D_m(\omega)) < +\infty$ for all $m \in \mathbb{N}$. Let $M(\omega) := \{ m \in \mathbb{N} : D_m(\omega) \neq \emptyset \}$. Fix an arbitrary $m \in M(\omega)$. Consider

$$0 < S_1^{(m)}(\omega) := \min(D_m(\omega))$$

and, if $\kappa_m(\omega) \geq 2$,

$$0 < S_{n+1}^{(m)}(\omega) := \min(D_m(\omega) \cap (S_n^{(m)}(\omega), \infty)),$$
where \( n \in \{1, 2, \ldots, \kappa_m(\omega) - 1\} \). Since \( \Delta X \) is optional, it follows that \( \{\Delta X \in B\} \) is optional for all Borel sets \( B \in \mathcal{B}(\mathbb{R}) \). Moreover, since \( \Delta f(0) = \Delta X_0(\omega) := 0 \) (by assumption), it actually follows that \( \{ s \in \mathbb{R}^+ : (\omega, s) \in \{\Delta X \in C\} \} = \{ s \in (0, \infty) : (\omega, s) \in \{\Delta X \in C\} \} \) for all Borel sets \( C \in \mathcal{B}(\mathbb{R}) \) which do not contain \( 0 \). Hence, as the construction of the sets \( D_m(\omega) \) in the proof of Proposition 3 clearly shows, \( S^{(m)}_1 \) is the début of an optional set. Consequently, due to Theorem 3, it follows that \( S^{(m)}_1 \) is a stopping time. If \( S^{(m)}_n \) is a stopping time, the stochastic interval \( [S^{(m)}_n, \infty) \) is optional too (cf. [10], Theorem 3.16). Thus, by construction, \( S^{(m)}_{n+1} \) is the début of an optional set and hence a stopping time. Due to Lemma 1, we have

\[
J_f = \bigcup_{m \in \mathcal{M}(\omega)}^{} D_m(\omega) = \bigcup_{m \in \mathcal{M}(\omega) \cap \mathbb{N}}^{} \bigcup_{n=1}^{} \{ S^{(m)}_n(\omega) \}.
\]

Hence, since for each trajectory of \( X \) its set of jumps is not finite, the at most countable set \( \mathcal{M}(\omega) \) is not finite, hence countable, and a simple relabeling of the stopping times \( S^{(m)}_n \) finishes the proof. \( \square \)

**Theorem 5.** Let \( X : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be an optional process such that all trajectories of \( X \) are regulated, \( \Delta X_0 = 0 \) and the set of jumps of each trajectory of \( X \) is not finite. Then the function

\[
j_X : \Omega \times \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{Z}^+ \cup \{ +\infty \}
\]

\[
(\omega, G) \mapsto \sum_{s > 0}^{} I_G(s, \Delta X(\omega)) I_{\{\Delta X \neq 0\}}(\omega, s)
\]

is an integer-valued random measure.

**Proof.** We only have to combine Theorem 4 and [10], Theorem 11.13. \( \square \)

Implementing the exhausting series of stopping times \( (T_n)_{n \in \mathbb{N}} \) of the thin set \( \{\Delta X \neq 0\} \) from Theorem 4 we immediately obtain

**Corollary 1.** Let \( B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}) \) and \( \omega \in \Omega \). Then

\[
j_X(\omega, B) = \int_{\mathbb{R}^+ \times \mathbb{R}} I_B(s, x) j_X(\omega, d(s, x))
\]

\[
= \sum_{n=1}^\infty I_B(T_n(\omega), \Delta X_{T_n(\omega)}(\omega))
\]

\[
= \# \{ n \in \mathbb{N} : \{ T_n(\omega, \Delta X_{T_n(\omega)}(\omega)) \} \in B \}.
\]

**Proof.** Since \( I_{\{\Delta X \neq 0\}}(\omega, s) = \sum_{n=1}^\infty I_{T_n(\omega)}(s, s) = \sum_{n=1}^\infty I_{T_n(\omega)}(s, s) \), we just have to permute the two sums. \( \square \)

We finish our paper with the following two natural questions:

**Problem 1** Let \( X : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be an optional process such that all trajectories of \( X \) are regulated and \( \Delta X_0 = 0 \). Does Lemma 2 hold for \( X \)?
Problem 2 Let $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be an optional process such that all trajectories of $X$ are regulated and $\Delta X_0 = 0$. Does Proposition 2 hold for $X$?

Acknowledgements The author would like to thank Monique Jeanblanc for the indication of the very valuable references [1], [2] and [3].

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