SHARP $\frac{1}{2}$-HÖLDER CONTINUITY OF THE LYAPUNOV EXPONENT AT THE BOTTOM OF THE SPECTRUM FOR A CLASS OF SCHRÖDINGER COCYCLES

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Abstract. We consider the setting for the disappearance of uniform hyperbolicity as in Bjerklöv and Saprykina (2008 Nonlinearity 21), where it was proved that the minimum distance between invariant stable and unstable bundles has a linear power law dependence on parameters. In this scenario we prove that the Lyapunov exponent is sharp $\frac{1}{2}$-Hölder continuous.

In particular, we show that the Lyapunov exponent of Schrödinger cocycles with a potential having a unique non-degenerate minimum is sharp $\frac{1}{2}$-Hölder continuous below the lowest energy of the spectrum, in the large coupling regime.

1. Introduction. Suppose that we are given a smooth family of smooth dynamical systems, say $\{F_t\}$, with corresponding (maximal) Lyapunov exponents $L(t)$. Understanding how the Lyapunov exponent $L(t)$ depends on $t$ is a classical problem, and one without a very satisfying answer. In general, the Lyapunov exponent can depend discontinuously on the parameter $t$, even when the family of systems is smooth itself (see for example [19, 23]).

In this paper, we exhibit a class of families (see Section 2.1 for its definition), for which the Lyapunov exponent is sharply $\frac{1}{2}$-Hölder continuous: it is Hölder with exponent $\frac{1}{2}$, but not $\frac{1}{2} + \delta$ for any $\delta > 0$. This behaviour occurs, for energies below the bottom of the spectrum, for Diophantine quasi-periodic $C^2$-Schrödinger cocycles having a unique non-degenerate minimum, for large couplings, making this condition open.

In order to define the class of functions we consider, recall that a quasi-periodic cocycle is a map of the form

$$
\begin{align*}
(\theta, v) &\mapsto (\theta + \omega, A(\theta)v)
\end{align*}
$$

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where $T = \mathbb{R}/\mathbb{Z}$, $\omega$ is an irrational number and $A \in C^0(\mathbb{T}, \text{SL}(2, \mathbb{R}))$. A cocycle is called uniformly hyperbolic if there exist two continuous maps $W_1, W_2 : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}^2$, that are invariant, meaning that $A(\theta)W_i(\theta) = \lambda_i(\theta)W_i(\theta + \omega)$, with $\lambda_i : \mathbb{T} \to \mathbb{R} - \{0\}$ and

$$\left| \int_{\mathbb{T}} \log |\lambda_i(\theta)| d\theta \right| > 0.$$  

(2)

Since we consider only $\text{SL}(2, \mathbb{R})$ cocycles, one of these integrals will be positive, and the other integral negative (simply minus the first one). Without loss of generality, we may assume that

$$\int_{\mathbb{T}} \log |\lambda_1(\theta)| d\theta > 0,$$

and we call this quantity the (maximal) Lyapunov exponent.

In this paper, we tackle the problem of Hölder continuity of the Lyapunov exponent of one-parametric families of quasi-periodic cocyles with transfer matrix of the form

$$A_E(\theta) = \begin{pmatrix} V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix},$$

where the function $V \in C^2(\mathbb{T}, \mathbb{R})$ has a non-degenerate unique global minimum (notice that this condition is open and dense in the $C^2$ category). These cocycles arise from the study of the spectrum of discrete Schrödinger operators $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ of the form

$$(H_\theta x)_n = x_{n+1} + V(\theta + n\omega)x_n + x_{n-1}.$$

It is known that the set of parameters $E$ for which the cocycle is not uniformly hyperbolic (the spectrum of $H$) is a compact set on the real line. From now on, we will denote by $E_0$ the bottom edge of this set.

We prove that, on the whole interval $(-\infty, E_0]$, the Lyapunov exponent is $\frac{1}{2}$-Hölder continuous, but not $\left(\frac{1}{2} + \delta\right)$-Hölder continuous for any positive $\delta$. Since the Lyapunov exponent is real analytic in the interval $(-\infty, E_0)$ it implies that its asymptotic behaviour at $E_0$ is square root like. This problem is part of the open conjectures appearing in [14] on the asymptotics of disappearance of normally hyperbolic invariant tori in quasi-periodically forced systems. See also [8, 9, 10] for further numerical studies in different contexts. There, the authors numerically study one-parameter families of quasi-periodic skew product systems, and conjectured that the normal dynamics around the invariant tori satisfy certain asymptotic laws. In particular, they conjectured that the minimum distance between the invariant bundles $W_1(E)$ and $W_2(E)$ (defined above), depending on the parameter $E$, satisfies

$$\text{dist}(E) \sim a(E - E_0),$$

(3)

where $E_0$ is some critical parameter for which the bundles collide, and that the (maximal) Lyapunov exponent $L(E)$ of $A_E$, satisfies

$$L(E) \sim L(E_0) + b\sqrt{E_0 - E},$$

(4)

where $a, b \neq 0$ are constants depending on the system and $L(E_0) > 0$. Their conjectures apply to the more general setting mentioned above, in which normally hyperbolic invariant tori disappear, with $E_0$ representing some parameter for which the invariant tori suddenly disappear, after having existed up to that parameter.
A step forward towards solving this problem was taken in [4], where the authors proved that the distance is asymptotically linear in the case of linear Hénon maps. These are one-parameter linear cocycles of the form (1) with transfer matrix
\[ A_E(\theta) = \begin{pmatrix} V(\theta) - E & 1 \\ 1 & 0 \end{pmatrix}, \]
with \( V \) being any \( C^2 \) function close to \( V(\theta) = \frac{1}{1 + \lambda \sin^2(\pi x)} \), \( \lambda \gg 1 \) and \( E \) being a parameter.

**Remark 1.** A similar result as in [4] with the same type of potential \( V \) as in this paper appears in [22].

In the literature there are plenty of results about Hölder continuity of the Lyapunov exponent. Among them, in [16] the authors prove that, if the cocycle is analytic, then the Lyapunov exponent is continuous with respect to the parameter \( E \). In [11] they established that it is Hölder continuous. For more references see also [2, 5, 6, 7, 12, 13, 18, 20].

Proving asymptotics for the distance and Lyapunov exponents when the potential \( V \) is close to zero is quite easy: both are square root. This can be proved by noticing that the collision of the invariant stable and unstable bundles is smooth, leading to a saddle-node bifurcation, which implies that both the distance between the invariant bundles and the Lyapunov exponent have square root asymptotics.

We prove, under the assumption that the minimum distance satisfies linear asymptotics, as in (3), together with some other general assumptions, that the Lyapunov exponent has (almost) square-root asymptotic behaviour, as in (4), but where the constant is allowed to fluctuate between two fixed, positive constants.

2. **Statement of the results.** Suppose that \((\omega, A_t(\theta))\) is an analytic family of quasi-periodic cocycles given by
\[ A_t(\theta) = A(\theta)e^{tw}, \tag{5} \]
where \( t \) is a real parameter, \( A \in C^2(T, SL(2, \mathbb{R})) \), and \( w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). The results in this paper hold for any such family of cocycles that satisfies the following list of assumptions. First, note that the (maximal) Lyapunov exponent of \( A_t \) is the limit
\[ L(t) = \lim_{n \to \infty} \frac{1}{n} \int_T \log \|A^n_t(\theta)\|d\theta \]
where \( A^n(\theta) = A(\theta + (n-1)\omega) \cdots A(\theta) \). If there are two continuous maps \( W_i : T \to \mathbb{R}^2 \) \( (i = 1, 2) \), that are invariant, \( A_t(\theta)W_i(\theta) = \lambda_i(\theta)W_i(\theta + \omega) \), as in (2), then \( L(t) \) equals the biggest of the expressions
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_T \log |\lambda_i(\theta + j\omega)|d\theta. \]

As in the introduction, we may assume that the biggest of them has \( i = 1 \). Then, it follows that
\[ L(t) = \int_T \log |\lambda_1(\theta)|d\theta. \]

From now on, we will assume that the cocycles considered in this paper satisfy the following assumptions:
We write the difference between the two functions $r^u_i$, $r^s_i$ and the growth of $D_i$ be the constant in A1. 

A0 There exists a parameter $t_0$ such that $(\omega, A_t)$ is uniformly hyperbolic for $t \in (-\infty, t_0)$ with orientable $W_i$.

A1 Suppose that, for some $\epsilon > 0$, and every $t \in (t_0 - \epsilon, t_0)$, there are two continuous functions $r^u_i, r^s_i : T \to [\frac{1}{C}, C]$, where $C > 0$ is independent of $t$, such that

$$ \begin{pmatrix} r^u_i(\theta) \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r^s_i(\theta) \\ 1 \end{pmatrix} $$

are invariant directions for $A_t$, and that they lie in the one dimensional subspaces spanned by $W_1(\theta)$ and $W_2(\theta)$, respectively.

We write the difference between the two functions $r^u_i$ and $r^s_i$ at the point $\theta$ as

$$ d(\theta) = r^u_i(\theta) - r^s_i(\theta). $$

Because of the fibred structure of the cocycle, the minimum of $d(\theta)$ is the minimum distance between the invariant bundles (in projective coordinates). Whenever $i \leq j$, there is a unique $D_{i,j}(\theta)$ such that

$$ d(\theta + j\omega) = d(\theta + i\omega) \cdot D_{i,j}(\theta). $$

That is, $D_{i,j}(\theta)$ measures how the distance changes from step $i$ to $j$, where the base step 0 is taken to be at $\theta$. We will make some assumptions about the function $d(\theta)$, and the growth of $D_{i,j}(\theta)$. From now on, let $\epsilon$ be the constant in A1.

A2 Suppose that, for every $t \in (t_0 - \epsilon, t_0)$, there is a distinguished interval $I = I(t) \subset T$, and a distinguished point $\theta_c = \theta_c(t) \in I$, satisfying the following conditions.

(a) Suppose that we are given stopping times $\sigma^\pm = \sigma^\pm(t, \theta) > 0$, such that, for every $\theta \in I$ and $0 < j \leq \sigma^+, 0 < k \leq \sigma^-$, we have

$$ D_{0,j}(\theta) \geq e^{aj}, \quad \text{and} $$

$$ D_{-k,0}(\theta) \leq e^{-ak}, $$

where $a > 0$ is some constant that is independent of both $t$ and $\theta$.

(b) There is a unique (global) minimum distance $d(\theta)$, which is attained at $\theta_c$, and satisfies the linear asymptotics

$$ d(\theta_c) = \min_{\theta \in I} d(\theta) = C_0|t - t_0| + o(|t - t_0|), $$

for some constant $C_0 > 0$ that is independent of $t$.

(c) There is a constant $C_1 > 0$, independent of both $t$ and $\theta$, such that for every $\theta \in I$, the distance is approximately quadratic around $\theta_c$:

$$ \frac{1}{C_1} (\theta - \theta_c)^2 \leq d(\theta) - d(\theta_c) \leq C_1 (\theta - \theta_c)^2. $$

(d) The length of the interval satisfies

$$ |I| \geq 2C_2 \cdot \sqrt{d(\theta_c)}, $$

where $C_2 > 0$ is some constant that is independent of $t$.

(e) For every $\theta \in T \setminus \{\theta + k\omega : \theta \in I, \sigma^{-}(\theta) \leq k \leq \sigma^{+}(\theta)\}$, the distance satisfies

$$ d(\theta) \geq \sqrt{d(\theta_c)}. $$
It is important to stress that (8) implies that \( \min_{\theta \in T} d(\theta) \to 0 \) as \( t \nearrow t_0 \), and that \( t_0 \) is indeed a bifurcation point, at which uniform hyperbolicity is lost. That is, we have a torus collision at the critical parameter \( t_0 \).

Lastly, we impose a continuity condition on the Lyapunov exponent.

**A3** Suppose that \( L(t) \) is continuous on the interval \( (t_0 - \epsilon, t_0] \). That is,
\[
\lim_{t \to t_0} L(t) = L(t_0).
\]
Since the parameter dependence is analytic, the Lyapunov exponent is real-analytic at parameters \( t \) satisfying that \( A_t \) is uniformly hyperbolic. That is, we only impose in this last assumption that \( L(t) \) is left-continuous at \( t_0 \).

We will discuss these assumptions in Section 5.

### 2.1. Main results

In this paper we prove

**Theorem 2.1.** Given a one-parameter family of cocycles \( A_t \) of the form in (5), and satisfying the assumptions A0, A1 and A2, for some \( \epsilon > 0 \) and \( t_0 \), there exist two positive constants \( K_1 \) and \( K_2 \), such that
\[
\frac{K_1}{\sqrt{|t - t_0|}} \leq \frac{d}{dt} L(t) \leq \frac{K_2}{\sqrt{|t - t_0|}},
\]
for every \( t \in (t_0 - \epsilon, t_0) \), where \( \epsilon' \) satisfies \( 0 < \epsilon' \leq \epsilon \).

A direct result of this is the following.

**Corollary 1.** Under the same assumptions as in Theorem 2.1, together with the assumption A3, the Lyapunov exponent in the range \( (t_0 - \epsilon, t_0] \) is \( \frac{1}{2} \)-Hölder continuous, but not \( (\frac{1}{2} + \delta) \)-Hölder continuous, at \( t_0 \) for any \( \delta > 0 \). More specifically, we have the asymptotics
\[
K_1 \sqrt{|t - t_0|} \leq L(t) - L(t_0) \leq K_2 \sqrt{|t - t_0|},
\]
for every \( t \in (t_0 - \epsilon, t_0) \), and some positive constants \( K_1 \) and \( K_2 \).

**Proof.** Since the system is uniformly hyperbolic in \( (t_0 - \epsilon, t_0) \), the Lyapunov exponent is analytic there. The only point of interest is \( t_0 \). Let \( t \in (t_0 - \epsilon, t_0) \). Since the Lyapunov exponent is left-continuous at \( t_0 \), by assumption A3, we may simply integrate the derivative from \( t \) to \( t_0 \) to get the result. \( \Box \)

Recall that an irrational \( \omega \) is called Diophantine if there are constants \( \kappa > 0 \) and \( \tau \geq 1 \) such that
\[
\inf_{p \in \mathbb{Z}} |n\omega - p| \geq \frac{\kappa}{|n|^\tau},
\]
for every \( n \in \mathbb{Z} \setminus \{0\} \).

As an interesting application of Theorem 2.1 (in particular Corollary 1), we have the following corollary for Schrödinger cocycles.

**Corollary 2.** Suppose that \((\omega, A_E)\) is a family of Schrödinger cocycles where \( \omega \) is a Diophantine irrational,
\[
A_E = \begin{pmatrix} \lambda V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix},
\]

and \( V(\theta) \in C^2(\mathbb{T}, \mathbb{R}) \) has a unique, non-degenerate minimum. Then there is a \( \lambda_0 \) (depending on \( \omega \)) such that if \( \lambda \geq \lambda_0 \), the Lyapunov exponent satisfies the asymptotics

\[
K_1 \sqrt{|E - E_0|} \leq L(E) - L(E_0) \leq K_2 \sqrt{|E - E_0|},
\]

as \( E \nearrow E_0 \), where \( K_1 \leq K_2 \) are some positive constants and \( E_0 \) is the lowest energy of the spectrum.

**Remark 2.** We expect the same type of result to hold when \( E_0 \) is at the edge of a spectral gap, and \( E \) approaches \( E_0 \) from within that spectral gap. We discuss this in more detail in Section 5.

**Proof.** In this setting, the Lyapunov exponent is continuous up to the bottom edge of the spectrum, that is, \( A_3 \) holds. This follows from [24, Corollary 2].

The assumption \( A_0 \) is self-evident, while the assumptions \( A_1 \) and \( A_2 \) are proved in [22], for \( t_0 = E_0 \), and \( \epsilon = 1 \). In particular, there is an interval \( I(E) \) for each energy \( E < E_0 \), where \( E_0 \) is the lowest energy of the spectrum, for which assumption \( A_2 \) holds. The functions \( r_E^{(b)} \) and \( r_E^{(e)} \) are the functions \( \psi_E^{(b)} \) and \( \psi_E^{(e)} \) in that paper, respectively. Our assumptions appear also in that paper, with \((A_1)\) corresponding to the assumption \( A_1 \), and \((A_2)\) corresponding to the assumptions \( A_2b \) to \( A_2d \).

Moreover, the first estimate in \((A_3)\) (b) is the same one as in (6) for \( \theta \in I(E) = I \). However, note that the definition of \( D_{i,j}^{(b)} \) is slightly different in that paper, satisfying \( d(\theta + j\omega) = d(\theta + i\omega)D_{i,j-1}^{(b)}(\theta) \). That is, the last index is shifted by 1 in this paper. Similarly, the first estimate in \((A_3)\) (c) is the one in (7). That shows \( A_2a \). The last assumption \( A_2e \) corresponds to [22, Lemma 7.6].

This shows that all of the assumptions are satisfied, and the statement follows directly. \( \square \)

The proof of Theorem 2.1 appears in Section 4, but first we prove a small lemma.

**3. The derivative of the Lyapunov exponent and Avila’s lemma.** One of the key tools for proving Theorem 2.1 is expressing the derivative of the Lyapunov exponent as an integral with respect to the stable and unstable directions, and the distance between them. Since, for parameters below \( t_0 \), the cocycle is uniformly hyperbolic, it means that there exists a \( B : \mathbb{T} \rightarrow \text{SL}(2, \mathbb{R}) \), with the same regularity as the cocycle,

\[
B_t(\theta) = \begin{pmatrix} \alpha_t(\theta) & \beta_t(\theta) \\ \gamma_t(\theta) & \delta_t(\theta) \end{pmatrix}
\]

such that \( B(\theta + \omega)^{-1}A_t(\theta)B(\theta) \) is the diagonal transfer matrix

\[
D_t(\theta) = \begin{pmatrix} d_{1,1}(\theta) & 0 \\ 0 & d_{2,2}(\theta) \end{pmatrix},
\]

where \( d_{1,1}(\theta) \geq d_{2,2}(\theta) \). In fact, since the dynamics on the base is irrational, the Lyapunov exponent is given by

\[
L(t) = \int_T \log|d_{1,1}(\theta)|d\theta.
\]
Lemma 3.1 (Avila’s lemma [1]). Given a one-parameter family of uniformly hyperbolic quasi-periodic cocycles $(\omega, A_t)$, of the form (1), with $A_t \in C^1(\mathbb{R} \times \mathbb{T}, \text{SL}(2, \mathbb{R}))$, and $A_t(\theta) = A_0(\theta)e^{i\omega(\theta)}$, where $w(\theta) \in \text{sl}(2, \mathbb{R})$ is given by

$$w(\theta) = \begin{pmatrix} w_1(\theta) & w_2(\theta) \\ w_3(\theta) & -w_1(\theta) \end{pmatrix},$$

then

$$\frac{d}{dt} L|_{t=0} = \sum_{i=1}^{3} q_i(\theta) w_i(\theta) d\theta,$$

where

$$q_1(\theta) = \alpha_0(\theta) \delta_0(\theta) + \beta_0(\theta) \gamma_0(\theta), \quad q_2(\theta) = \gamma_0(\theta) \delta_0(\theta), \quad \text{and} \quad q_3(\theta) = -\beta_0(\theta) \alpha_0(\theta).$$

This lemma appears in [1] in the case that the transfer matrix is analytic. For completeness sake, we include a slightly different proof of this lemma.

**Remark 3.** The proof of Lemma 3.1 can be generalized mutatis mutandis for one-parameter families of uniformly hyperbolic cocycles acting on $M \times \mathbb{R}^n$, where $M$ is a compact manifold and $n \geq 2$, with base dynamics on $M$ satisfying that its jacobian at any point has determinant 1.

**Proof.** Under the assumptions of the lemma,

$$B(\theta + \omega)^{-1} A(\theta) e^{i\omega(\theta)} B(\theta) = D(\theta),$$

(10)

where the matrix-valued maps $B$ and $D$ also depend on the parameter $t$.

First notice that

$$\frac{d}{dt} L|_{t=0} = \int_{\mathbb{T}} \frac{d_1(\theta)}{d_1(\theta)} d\theta.$$

So, by differentiating (10) with respect $t$ and setting $t = 0$ we obtain

$$(B(\theta + \omega)^{-1})^t A(\theta) B(\theta) + B(\theta + \omega)^{-1} A(\theta) w(\theta) B(\theta) + B(\theta + \omega)^{-1} A(\theta) B'(\theta) = D'(\theta).$$

Since $(B(\theta)^{-1})^t = -B(\theta)^{-1} B'(\theta) B(\theta)^{-1}$ (which can be seen by differentiating $B(\theta)^{-1} B(\theta) = \text{Id}$ with respect to $t$), we can use identity (10) to transform the preceding equation into

$$-B(\theta + \omega)^{-1} B'(\theta + \omega) D(\theta) + D(\theta) B(\theta)^{-1} w(\theta) B(\theta) + D(\theta) B(\theta)^{-1} B'(\theta) = D'(\theta).$$

Finally, by dividing both sides by $d_1(\theta)$ in the last equation, considering the average of the $(1, 1)$ entry, and using the fact that the first and third monomials of the left-hand side cancel out, we get the desired result. 

4. Proof of main theorem.

**Proof of Theorem 2.1.** Suppose that all the assumptions are satisfied, and fix a $t \in (t_0 - \epsilon, t_0)$. Let us drop $t$ from the notation and simply write $r_u$ and $r_s$ instead of $r_u^t$ and $r_s^t$, respectively. In our setting,

$$B_t(\theta) = \begin{pmatrix} r_u(\theta) & r_s(\theta) \\ \sqrt{r_u(\theta) - r_s(\theta)} & \sqrt{r_u(\theta) - r_s(\theta)} \end{pmatrix}, \quad \text{and} \quad w(\theta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Lemma 3.1 can be used to express the derivative of $L(t)$ at any $t$, simply by shifting the parameter, since the formula doesn’t depend on $A_0$. That is, we can simply
apply the lemma to $A_{t+\tau} = (A_0e^{tw})e^{\tau w}$, and get the derivative at the parameter value $t$. The derivative of the Lyapunov exponent is therefore equal to

$$-\int_T \frac{r_u(\theta)r_s(\theta)}{r_u(\theta) - r_s(\theta)} d\theta.$$  

Recall the notation

$$d(\theta) = r_u(\theta) - r_s(\theta)$$

for the distance between the invariant directions, and $D_{i,j}(\theta)$, for $i \leq j$, that was given by the relations

$$d(\theta + j\omega) = d(\theta + i\omega) \cdot D_{i,j}(\theta).$$

First of all, for every $t \in (t_0 - \epsilon, t_0)$, we have the uniform inequalities

$$\int_T \frac{1}{C^2} \frac{1}{d(\theta)} d\theta \leq \int_T \frac{r_u(\theta)r_s(\theta)}{r_u(\theta) - r_s(\theta)} d\theta \leq C^2 \int_T \frac{1}{C^2} \frac{1}{d(\theta)} d\theta,$$

since $r_s(\theta), r_u(\theta) \in [\frac{1}{C}, C]$ for every $\theta$, where $C > 0$ is independent of $t$ and $\theta$, by assumption A1. In particular, this means that the result follows, if we can show that the inequalities

$$\frac{\tilde{K}_1}{\sqrt{|t - t_0|}} \leq \int_T \frac{1}{d(\theta)} d\theta \leq \frac{\tilde{K}_2}{\sqrt{|t - t_0|}}$$

hold, for some positive constants $\tilde{K}_1, \tilde{K}_2$. Recall the interval $I = I(t) \subseteq T$, and consider the transformation

$$\int_{I+k\omega} \frac{1}{d(\theta)} d\theta = \int_{I} \frac{1}{d(\theta + k\omega)} d\theta.$$  

Thus the integral over $I + k\omega$, where $k > 0$, becomes

$$\int_{I} \frac{1}{d(\theta)D_{0,k}(\theta)} d\theta,$$

and over $I - k\omega$, where again $k > 0$, it becomes

$$\int_{I} \frac{D_{-k,0}(\theta)}{d(\theta)} d\theta.$$  

Recall the stopping times $\sigma^{\pm}(\theta) > 0$ from the assumptions, where $\theta \in I$. Set

$$C^+ = \{\theta + k\omega : \theta \in I, 0 < k \leq \sigma^+(\theta)\},$$

$$C^- = \{\theta - k\omega : \theta \in I, 0 < k \leq \sigma^-(\theta)\},$$

$$C = I \cup C^+ \cup C^-.$$  

Using the relations above, we conclude that

$$\int_{I} \frac{1}{d(\theta)} d\theta \leq \int_{C} \frac{1}{d(\theta)} d\theta \leq \int_{I} \frac{1}{d(\theta)} \left(1 + \sum_{k=1}^{\sigma^+(\theta)} \frac{1}{D_{0,k}(\theta)} + \sum_{k=1}^{\sigma^-(\theta)} \frac{1}{D_{-k,0}(\theta)}\right) d\theta.$$  

From assumption A2a, it follows that

$$\int_{C} \frac{1}{d(\theta)} d\theta \leq \left(1 + \frac{2}{1 - e^{-a}}\right) \int_{I} \frac{1}{d(\theta)} d\theta.$$
Indeed, if we set
\[ S^+(\theta) = \sum_{k=1}^{\sigma^+(\theta)} D_{0,k}(\theta), \quad \text{and} \quad S^-(\theta) = \sum_{k=1}^{\sigma^-(\theta)} D_{-k,0}(\theta), \]
then the bounds in (6) and (7) lead to the inequalities
\[ S^\pm(\theta) \leq \sum_{k=1}^{\infty} e^{-ak} = \frac{1}{1 - e^{-a}}, \]
which immediately implies the above bound. Assumption A2c implies that
\[
\int_I \frac{1}{d(\theta_c)} d\theta_c + C_1(\theta - \theta_c)^2 d\theta_c \leq \int_I \frac{1}{d(\theta)} d\theta \leq \int_I \frac{1}{d(\theta_c)} + \frac{1}{C_1}(\theta - \theta_c)^2 d\theta.
\] (11)

For any \( b > 0 \), we compute the indefinite integral
\[
\int_I \frac{1}{d(\theta_c)} + b \cdot (\theta - \theta_c)^2 d\theta = \frac{1}{\sqrt{b \cdot d(\theta_c)}} \arctan \left( (\theta - \theta_c) \sqrt{\frac{b}{d(\theta_c)}} \right).
\]
By assumption A2d, it follows that either \( I \supseteq [\theta_c, \theta_c + C_2 \sqrt{d(\theta_c)}] \) or \( I \supseteq [\theta_c - C_2 \sqrt{d(\theta_c)}, \theta_c] \). Since the problem is symmetric, it doesn’t matter which inclusion holds. Computing the integral over the interval \( I \), we obtain the inequality
\[
\frac{1}{\sqrt{b \cdot d(\theta_c)}} \arctan(C_2 \sqrt{b}) \leq \int_I \frac{1}{d(\theta)} d\theta \leq \frac{\pi}{\sqrt{b \cdot d(\theta_c)}}.
\] (12)

Using the inequality in (11), and plugging in the constants \( C_1 \) and \( \frac{1}{C_1} \) instead of \( b \), results in the bounds
\[
\frac{1}{\sqrt{C_1 \cdot d(\theta_c)}} \arctan(C_2 \cdot \sqrt{C_1}) \leq \int_{\mathcal{T}\setminus \mathcal{C}} \frac{1}{d(\theta)} d\theta \leq \left(1 + \frac{2}{1 - e^{-a}}\right) \frac{\pi \cdot \sqrt{C_1}}{\sqrt{d(\theta_c)}}.
\]
Finally, using the assumption A2e, the remainder of the integral can be computed as
\[
0 \leq \int_{\mathcal{T}\setminus \mathcal{C}} \frac{1}{d(\theta)} \leq \frac{1}{\sqrt{d(\theta_c)}}.
\]

In conclusion, there is a constant \( K > 0 \) (uniform in \( t \)) such that
\[
\frac{1}{K \sqrt{d(\theta_c)}} \leq \int_{\mathcal{T}} \frac{1}{d(\theta)} d\theta \leq \frac{K}{\sqrt{d(\theta_c)}}.
\]
Since \( d(\theta_c) = C_0 \cdot |t - t_0| + o(|t - t_0|) \), by assumption (A2b), it follows that
\[
\frac{K_1}{\sqrt{|t - t_0|}} \leq \frac{d}{dt} L(t) \leq \frac{K_2}{\sqrt{|t - t_0|}},
\]
where \( K_2 \geq K_1 > 0 \) are independent of \( t \), provided \( t \) is close enough to \( t_0 \). \( \square \)
5. Discussion and future directions. The first assumption A1 holds for Schrödinger cocycles that are homotopic to the identity, for energies below and above the lowest and highest energies of the spectrum, respectively. For a proof, see [15]. The important implication of this is that there is zero fibre rotation, and the assumption should apply to many classes of cocycles. It should be possible to extend the methods to include more general cases, where non-zero fibre rotation is allowed.

It is clear that the bounds
\[
\frac{1}{C \sqrt{d(\theta_c)}} \leq \int I d(\theta) \leq \frac{C}{\sqrt{d(\theta_c)}},
\]
critically depend on the assumptions A2c and A2d. (see the computations leading up to (12)). In some computed examples (see [21, 22]), the length of the interval actually satisfies much better bounds ($C_2$ can be chosen arbitrarily large as $t \to t_0$). Different local behaviour of the distance, that is replacing the square in A2c by another exponent, should produce different types of asymptotics.

The linear asymptotics for the distance in assumption A2b enters in the last step, and also affects the final asymptotics. As we mentioned in the introduction (see equation (3)), the minimum angle between the invariant directions is conjectured to behave asymptotically linearly as a system loses uniform hyperbolicity (see for instance [14]).

Remark 4. The assumptions A0, A1 and A2b together imply that the angle between the invariant directions satisfies linear asymptotics. Indeed, this angle is simply the difference between \(\arccot(r_s^E)\) and \(\arccot(r_u^E)\), and the mean-value theorem gives the linear asymptotics since, \(r_s^E, r_u^E \in [\frac{1}{C}, C]\).

The assumption A2a is simply a statement of exponential growth when the invariant directions are very close to each other. It can be relaxed somewhat, as long as their sums
\[
S^+(\theta) = \sum_{k=1}^{\sigma^+(\theta)} \frac{1}{D_{0,k-1}^+}, \quad S^-(\theta) = \sum_{k=1}^{\sigma^-(-1)} \frac{1}{D_{-(k-1),0}^-},
\]
are uniformly bounded.

The assumption A2c simply ensures that the directions are not too close to each other outside some critical region. In several known examples satisfying these assumptions, the bound in (9) can in fact be replaced with $d(\theta) \gg d(\theta_c)$ (see [21, 22]).

As for the last assumption A3, we refer to [17] for the case of Schrödinger cocycles. There it is proved to be continuous in the open spectral gaps, for the continuous case, but it applies also to the discrete case. The Lyapunov exponent for Schrödinger cocycles is continuous in $E$ for analytic potentials (see [16]), but may fail to be so for non-analytic potentials. To the best of our knowledge, there is no example in the literature, of a Schrödinger cocycle that is discontinuous with respect to the energy $E$. However, there are explicit examples that are discontinuous in the space of potentials $v$ (see for instance [23]).

5.1. Extension to spectral gaps for Schrödinger cocycles. In Corollary 2, we apply Theorem 2.1 to show square root like behaviour (and in particular $\frac{1}{2}$-Hölder continuity) of the Lyapunov exponent up to the lowest (and by extension the highest) energy of the spectrum, under a generic condition.
Unfortunately, this doesn’t say much about the regularity of the Lyapunov exponent on the closures of spectral gaps, or even the question of Hölder continuity inside the spectrum. However, this should not be understood as an inherent limitation on the methods outlined in this paper. Rather, this is due to a lack of proper quantitative estimates of the invariant directions, for general energies. In the following paragraphs, we roughly outline the steps required to extend these methods to those cases we have not covered here.

The first step that is required is a more precise description of the invariant directions for general energy parameters $E$. The references [3] and [24] contain multi-scale estimates of the Lyapunov exponent, and the invariant directions, for so-called cosine like potentials ($C^2$ with unique non-degenerate minimum and maximum), in the large coupling regime. These are the exact same type of estimates that were further built on in [22] to obtain detailed quantitative information about the invariant directions. Both those papers suggest that, in the spectral gaps, the distance between the invariant directions, considered as a function on the circle $\mathbb{T}$, satisfies the following property:

Every local minimum, for which the distance is sufficiently small, is non-degenerate, or locally approximately quadratic. Moreover, the distance between them, following along forward and backward orbits starting at the global minimum, expands exponentially fast, until they separate.

These properties correspond to the Assumptions A2a and A2c. With a bit of work, this could be made rigorous. The estimates should also be sufficient to prove Assumption A2b, or the asymptotic linearity of the minimum distance. While these assumptions are a convenient way of capturing these properties, the key observation is that the difference between the invariant directions behaves locally quadratically when they are close to each other, that they approach linearly, and that there is enough expansion to separate them along orbits.

The other assumptions in Assumption A2 ensure that this local picture is valid for large enough intervals, and that the invariant curves do separate at some point. In fact, some more general form of assumption A2d should be a consequence of this exponential separation.

Assumption A1 is simply an artifact of the estimates in [22] being expressed in projective coordinates. This assumption allows us to ignore certain annoying technicalities. We mention that [3] uses projective coordinates (and shows how to deal with these technicalities), whereas [24] does not.

This does not prove that the result can be immediately extended to the closures of the spectral gaps, but rather outlines the steps that would have to be taken to do so. We also mention that our method does not crucially rely on Lemma 3.1 (Avila’s lemma). Indeed, a similar formula can be shown directly from the definition of the Lyapunov exponent along an invariant curve, and the proof would be independent of any cocycle structure. However, the formula obtained in that way will contain a few more technicalities to deal with, whereas Lemma 3.1 immediately gives a very convenient formula to work with.

Using a similar formula for finite-time approximations of the Lyapunov exponent, and limiting arguments, the method should be possible to extend to non-uniformly hyperbolic cocycles. This should allow the method to be extended to cover the whole of the spectrum. Again, the references [3] and [24] contain the ingredients for showing finite-time “locally quadratic” approximations of the distance between the invariant directions, that we discussed above. Of course, the constants appearing
in the estimates would likely not be uniform for all energies, which may result in 
\((\frac{1}{2} + \varepsilon)\)-Hölder continuity, rather than a sharp global exponent of \(\frac{1}{2}\).

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