THE $\overline{\partial}$-CAUCHY PROBLEM ON WEAKLY $q$-CONVEX DOMAINS IN $\mathbb{C}P^n$

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ABSTRACT. Let $D$ be a weakly $q$-convex domain in the complex projective space $\mathbb{C}P^n$. In this paper, the (weighted) $\overline{\partial}$-Cauchy problem with support conditions in $D$ is studied. Specifically, the modified weight function method is used to study the $L^2$ existence theorem for the $\overline{\partial}$-Neumann problem on $D$. The solutions are used to study function theory on weakly $q$-convex domains via the $\overline{\partial}$-Cauchy problem.

1. INTRODUCTION AND MAIN RESULTS

The $\overline{\partial}$-problem is one of the important central problems of complex variables. A classical result due to Hörmander tells us that the $\overline{\partial}$-problem is solvable in pseudoconvex domains, and hence, pseudoconvex domains has been widely accepted as the standard domain which we can solve the $\overline{\partial}$-problem. In [16], Ho extend this problem to weakly $q$-convex domains. In fact, Ho is the first person to study the $\overline{\partial}$-problem in $q$-convex domains in $\mathbb{C}^n$. This paper is devoted to studying the $L^2\overline{\partial}$ Cauchy problem and the $\overline{\partial}$-closed extension problem for forms on a weakly $q$-convex domain $D$ in the complex projective space $\mathbb{C}P^n$. These problems were first studied by Kohn and Rossi [20] (see also [12]). They proved the holomorphic extension of smooth $CR$ functions and the $\overline{\partial}$-closed extension of smooth forms from the boundary $bD$ of a strongly pseudoconvex domain to the whole domain $D$. The $L^2$ theory of these problems has been obtained for pseudoconvex domains in $\mathbb{C}^n$ or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [6] and the references therein). The $L^2\overline{\partial}$ Cauchy problem was considered by Derridj [8,9]. In [30,31] Shaw has obtained a solution to this problem on a pseudoconvex domain with $C^1$ boundary in $\mathbb{C}^n$. Also, in the setting of strictly

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q-convex (or q-concave) domains, this problem has been studied by Sambou in his thesis (see [29]). In [1], Abdelkader-Saber studied this problem on pseudoconvex manifolds satisfying property $B$. In [26,27], Saber studied this problem on a weakly q-convex domain with $C^1$-smooth boundary and on a $q$-pseudoconvex domain $D$ in $\mathbb{C}^n$, $1 \leq q \leq n$, with Lipschitz boundary. Recently, Saber [28] studied this result to a $q$-pseudoconvex domain $D$ in a Stein manifold. On a pseudoconvex domain in $\mathbb{C}P^n$, Cao-Shaw-Wang [4] (cf. also [5]) obtained the $L^2$ existence theorem for the $\overline{\partial}$-Neumann operator $N$ and obtained the (weighted) $L^2 \overline{\partial}$ Cauchy-problem on such domains. The aim of this paper is to extend this result to the situation in which the boundaries are assumed weakly q-convex domain $D$ in $\mathbb{C}P^n$. Moreover, the solutions are used to study function theory on such domains via the $\overline{\partial}$-Cauchy problem.

2. Notation and Preliminaries

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{C}P^n$. If $U_0$ is the open set in $\mathbb{C}P^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \ldots, z_n)$, where $z_i = x_i / x_0$, is the homogeneous coordinates of $U_0$, we assume that

$$
\omega = \frac{\sum_{i=1}^{n} |dz_i|^2}{1 + \sum_{i=1}^{n} |z_i|^2} - \frac{\sum_{i=1}^{n} z_i d\bar{z}_i|^2}{(1 + \sum_{i=1}^{n} |z_i|^2)^2} \text{ on } U_0.
$$

The Fubini-Study metric of $\mathbb{C}P^n$ determined by $(x_0, x_1, \ldots, x_n)$. This is well-known standard Kähler metric of $\mathbb{C}P^n$.

Let $D$ be a bounded domain in $\mathbb{C}P^n$ and let $C_{p,q}^\infty(D)$ be the space of complex-valued differential forms of class $C^\infty$ and of type $(p, q)$ on $D$. Denote by $L^2(D)$ the space of square integrable functions on $D$ with respect to the Lebesgue measure in $\mathbb{C}P^n$, $L^2_{p,q}(D)$ the space of $(p, q)$-forms with coefficients in $L^2(D)$ and $L^2_{p,q}(D, \phi)$ the space of $(p, q)$-forms with coefficients in $L^2(D)$ with respect to the weighted function $e^{-\phi}$. For $u, v \in L^2_{p,q}(D)$, the inner product $\langle u, v \rangle$ and the norm $\| u \|$ are denoted by:

$$
\langle u, v \rangle = \int_D u \wedge \star \overline{v} \quad \text{and} \quad \| u \|^2 = \langle u, u \rangle,
$$

where $\star$ is the Hodge star operator. Let $\text{dist}(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$ and let $\delta$ be a $C^2$ defining function for $D$ normalized by $|d\delta| = 1$ on $bD$ such that

$$
\delta = \delta(z) = \begin{cases}
- \text{dist}(z, bD), & \text{if } z \in D, \\
\text{dist}(z, bD), & \text{if } z \in \mathbb{C}P^n \setminus D.
\end{cases}
$$

Let $\phi_t = -t \log |\delta|$, $t \geq 0$, for $u, v \in L^2_{p,q}(D, \phi_t)$, the inner product $\langle u, v \rangle_{\phi_t}$ and the norm $\| u \|_{\phi_t}$ are denoted by:

$$
\langle u, v \rangle_{\phi_t} = \langle u, v \rangle_t = \int_D u \wedge \star(t) \overline{v},
$$

$$
\| u \|_{\phi_t}^2 = \| u \|^2_t = \langle u, u \rangle_t,
$$
where \( * \) = \( \delta^t * = \delta^t \delta^t \). Since \( \phi_t \) is bounded on \( \overline{D} \), the two norms \( \| \cdot \| \) and \( \| \cdot \|_t \) are equivalent. Let \( \overline{\partial} : \text{dom} \overline{\partial} \subset L^2_{p,q}(D, \phi_t) \to L^2_{p,q+1}(D, \phi_t) \) be the maximal closure of the Cauchy-Riemann operator and \( \overline{\partial}^* \) be its Hilbert space adjoint. Let \( \square_t = \overline{\partial}_t \overline{\partial}_t^* \overline{\partial}^* \overline{\partial}_t \) be the Laplace-Beltrami operator, where \( \overline{\partial}_t = \overline{\partial}^* \overline{\partial}_t^* \).

Denote by \( \nabla \) the Levi-Civita connection of \( \mathbb{C}P^n \) with the standard Fubini-Study metric \( \omega \). Let \( \{ e_i \} \) be an orthonormal basis of vector fields. For any two vector fields \( f, g \), the curvature operator of the connection \( \nabla \) is denoted by

\[
\mathcal{R}(f, g) = \nabla_f \nabla_g - \nabla_g \nabla_f - [f, g].
\]

By setting \( \mathcal{R}_{ijk} = \omega(\mathcal{R}(e_i, e_j)e_k, e_l) \), the Ricci tensor \( \mathcal{R}_{ij} \) is denoted by

\[
\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj},
\]

which turns out to be self-adjoint with respect to \( \omega \) and the scalar curvature

\[
(2.1) \quad \Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{iijj}
\]
as the trace of the Ricci tensor.

**Definition 2.1.** Let \( D \) be an open set in an \( n \)-dimensional complex manifold \( X \), let \( k \) be an integer with \( 1 \leq k \leq n - 1 \) and put \( E = X \setminus D \). The set \( D \) is said to be pseudoconvex of order \( k \) in \( X \) if, for every \( b \in E \) and for every coordinate neighborhood \( (U, (z_1, \ldots, z_n)) \) which contains \( b \) as the origin, the set

\[
\left\{ (z_1, \ldots, z_n) \in U : z_i = 0, 1 \leq i \leq k, 0 < \sum_{i=k+1}^n |z_i|^2 < t \right\}
\]
contains no points of \( E \) for some \( t > 0 \), then there exists \( \ell > 0 \) such that for each \( (z'_1, \ldots, z'_k) \) with \( |z'_i| < \ell, 1 \leq i \leq k \), the set

\[
\left\{ (z_1, \ldots, z_n) \in U : z_i = z'_i, 1 \leq i \leq k, \sum_{i=k+1}^n |z_i|^2 < t \right\}
\]
contains at least one point of \( E \).

**Definition 2.2.** Let \( D \) be an \( n \)-dimensional complex manifold and let \( q \) be an integer, \( 1 \leq q \leq n \). By Fujita ([13], Proposition 8) a \( C^2 \) function \( \phi : D \to \mathbb{R} \) is pseudoconvex of order \( n - q \), if and only if its Levi form \( \partial \overline{\partial} \phi \) has at least \( n - q + 1 \) non negative eigenvalues at each point of \( D \).

**Definition 2.3.** Let \( D \) be an open subset of an \( n \)-dimensional complex manifold \( X \). \( D \) is said to have \( C^2 \) boundary in \( X \) if for all \( z \in bD \) there exist an open neighborhood \( U \) of \( z \) and a \( C^2 \) function \( \delta : U \to \mathbb{R} \), called a defining function of \( D \) at \( z \) such that \( d\delta(z) \neq 0 \) and \( D \cap U = \{ z \in U : \delta(z) < 0 \} \). Following Ho [16], \( D \) is said to be a
weakly \( q \)-convex \((q \geq 1)\) if at every point \( x_0 \in bD \) we have
\[
\sum |K| \sum_{j,k} \frac{\partial^2 \delta}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u}_{kK} \geq 0,
\]
for every \((0, q)\)-form, where
\[
u = \sum_{|J|=q} u_J d\overline{z}_J \text{ such that } \sum_{j=1}^n \frac{\partial \delta}{\partial z_j} u_{jK} = 0, \text{ for all } |K| = q - 1.
\]
Moreover, \( D \) is weakly \( q \)-convex if and only if for any \( z \in bD \) the sum of any \( q \) eigenvalues \( \delta_{i_1}, \ldots, \delta_{i_q} \), with distinct subscripts, of the Levi-form at \( z \) satisfies \( \sum_{j=1}^q \delta_{i_j} \geq 0 \) (cf. [15] and Lemma 4.7 in [34]).

**Definition 2.4.** Let \( D \) be a smooth domain in \( \mathbb{C}^n \), \( D \) is said to be a weakly \( q \)-concave if \( \overline{D}^c \) is weakly \( q \)-convex.

**Lemma 2.1** ([16]). Let \( D \) be a smooth domain in \( \mathbb{C}^n \) and \( \rho \) be its defining function. The following two conditions are equivalent.

1. \( D \) is weakly \( q \)-convex.
2. For any \( z \in bD \) the sum of any \( q \) eigenvalues \( \rho_{i_1}, \ldots, \rho_{i_q} \), with distinct subscripts, of the Levi-form at \( z \) satisfies \( \sum_{j=1}^q \rho_{i_j} \geq 0 \).

It follows from Lemma 2.1 that \( D \) is weakly \( q \)-concave if and only if for any \( q \) eigenvalues \( \rho_{i_1}, \ldots, \rho_{i_q} \) of the Levi-form at \( z \in bD \) with distinct subscripts we have \( \sum_{j=1}^q \rho_{i_j} \leq 0 \).

**Example 2.1.** Let \( D \) be an open subset of an \( n \)-dimensional complex manifold \( X \) and suppose that the boundary \( bD \) is a real hypersurface of class \( C^2 \) in \( X \), that is, there exist, for each \( z \in bD \), a neighborhood \( U \) of \( z \) and a \( C^2 \) function \( \rho : U \to \mathbb{R} \) such that \( d\rho(z) \neq 0 \) and \( D \cap U = \{ z \in U : \rho(z) < 0 \} \). Then \( D \) is pseudoconvex of order \( n - q \) in \( X \), if and only if the Levi form \( \partial \overline{\partial} \rho \) has at least \( n - q \) non-negative eigenvalues on \( T_z'(bD) \) for each defining function \( \rho \) of \( D \) near \( z \), where \( T_z'(bD)(\subset T_z(bD)) \) is the holomorphic tangent space of the real hypersurface \( bD \) at \( z \) (cf. [10,35] called such a subset \( D \) a \((q - 1)\)-pseudoconvex open subset with \( C^2 \) boundary).

**Theorem 2.1** ([23]). Let \( D \subset \mathbb{C}P^n \) be a pseudoconvex domain of order \( n - q \), \( 1 \leq q \leq n \). Let \( d(z, bD) \) be the Fubini distance from \( z \in D \) to the boundary \( bD \). Then the function \( -\log d(z, bD) \) is \((q - 1)\)-plurisubharmonic in \( D \).

**Lemma 2.2** ([17], Lemma 2.6). Let \( \phi \) be a real valued function of class \( C^2 \) defined in an \( n \)-dimensional complex manifold \( D \). Then \( \phi \) is \((q - 1)\)-plurisubharmonic, \( 1 \leq q \leq n \), in \( D \) if and only if \( \phi \) is weakly \( q \)-convex in \( D \).

**Remark 2.1.** Pseudoconvex open sets in the original sense are pseudoconvex of order \( n - 1 \).

**Remark 2.2.** The pseudoconvexity of order \( n - q \) of an open subset \( D \) in \( X \) is a local property of the boundary \( bD \subset X \) of \( D \). More precisely, \( D \) is pseudoconvex of order
n – q in X if, for each p ∈ bD, there exists a neighborhood U ⊂ X of p such that $D \cap U$ is pseudoconvex of order $n – q$ in U.

Remark 2.3. If an open set D in an n-dimensional complex manifold X is weakly $q$-convex, $1 \leq q \leq n$, then D is pseudoconvex of order $n – q$ in X. However, the converse is not valid even if $X = \mathbb{C}^n$ (see [10] and [22]). By Fujita [13], an open subset $D$ of $\mathbb{C}^n$ is pseudoconvex of order $n – q$ in $\mathbb{C}^n$, if and only if D has an exhaustion function which is pseudoconvex of order $n – q$ on D. Thus, by the approximation theorem of Bungart [3], an open subset D of X is pseudoconvex of order $n – q$ in X, if and only if D is locally $q$-complete with corners in X in the sense of Peternell [24].

Proposition 2.1 (Bochner-Hörmander-Kohn-Morrey formula). Let D be a compact domain with $C^2$-smooth boundary bD and $\delta(x) = -d(x, bD)$. Suppose that $\Theta$ is the curvature term defined in (2.1) with respect to the Fubini-Study metric $\omega$. Then, for any $u \in C^\infty_p(D) \cap \text{dom}\overline{\partial}^*$ with $1 \leq q \leq n - 1$, and $\phi \in C^2(D)$, we have

$$
\overline{\partial}\|u\|_\phi^2 + \|\overline{\partial}^*u\|_\phi^2 = (\Theta u, \overline{u})_\phi + \|\frac{\partial u_{IJ}}{\partial z^k}\|_\phi^2 + \langle (i\overline{\partial}\partial\delta)u, \overline{u}\rangle_\phi + \int_{bD} ((i\overline{\partial}\partial\delta)u, \overline{u}) e^{-\phi} ds.
$$

This formula is known (cf. [2, 7, 15, 18, 19, 32, 36]) for some special cases, although it has not been stated in the literature in the form (2.2). If u has compact support in the interior of D, the (2.2) was proved in [2], Chapter 8 of [7] and (2.12) of [36]. The boundary term had been computed in [14], Chapter 3 by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of [15] and [37] with the interior formulae discussed above, one can prove that (2.2) holds for the general case with a weight function $e^{-\phi}$ and the curvature term. Specially, for $\phi = 0$, (2.2) was proved in [32].

Proposition 2.2. For any $(p, q)$-form $u$ of $D \subseteq \mathbb{C}P^n$ with $q \geq 1$,

$$
(\Theta u, \overline{u}) = q(2n + 1)|u|^2,
$$

when $u$ is a $(0, q)$-form,

$$
(\Theta u, \overline{u}) = 0,
$$

for any $(n, q)$-form $u$, and $u$ is a $(p, q)$-form.

The statement for $(0, q)$-forms and $(n, q)$-forms was computed in [32] and [36]. Also, following Lemma 3.3 of Henkin-Iordan [14] and its proof showed that the curvature operator $\Theta$ acting on $L^2_{p,q}(D)$ is a non-negative operator.

3. The $\overline{\partial}$-Cauchy Problem on Weakly $q$-Convex Domains

This section is devoted to showing the existence of the $\overline{\partial}$-Neumann operator on a weakly $q$-convex domain $D$ in $\mathbb{C}P^n$, $1 \leq q \leq n$, and by applying these existence to solve the $\overline{\partial}$ problem with support conditions on $D$. The boundary integral in (2.2) is
non-negative for \( q \geq 1 \) by the assumption on \( D \). Also, by taking \( \phi \equiv 0 \) in (2.2) and using Proposition 2.2, we find the fundamental estimate
\[
\|u\|^2 \leq c \left( \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \right).
\]
This means that \( \bar{\partial} \) has closed range and \( \ker \bar{\partial} = \{0\} \). Thus, one can establish the \( L^2 \)-existence theorem of the \( \bar{\partial} \)-Neumann operator \( N \).

**Theorem 3.1.** Let \( D \subseteq \mathbb{C}^n \) be a weakly \( q \)-convex domain with \( C^2 \) smooth boundary. Then, for each \( 0 \leq p \leq 1 \), \( 1 \leq q \leq n \), there exists a bounded linear operator \( N : L^2_{p,q}(D) \rightarrow L^2_{p,q}(D) \) with the following properties:

1. Range \( N \subseteq \text{dom} \, \bar{\partial}, \, \text{dom} \bar{\partial} = \text{Id} \) on \( \text{dom} \bar{\partial} \);
2. for \( f \in L^2_{p,q}(D) \), 
   \[
   f = \bar{\partial} \bar{\partial}^* Nf + \bar{\partial}^* \bar{\partial}Nf;
   \]
3.\( N \bar{\partial} = \bar{\partial} N \) on \( \text{dom} \bar{\partial}^* \), \( 1 \leq q \leq n - 1 \);
4. \( \bar{\partial}^* N = N \bar{\partial}^* \) on \( \text{dom} \bar{\partial}^* \), \( 2 \leq q \leq n \);
5. \( N, \bar{\partial}N \) and \( \bar{\partial}^* N \) are bounded linear operators on \( L^2_{p,q}(D) \).

Using the duality relations pertaining to the \( \bar{\partial} \)-Neumann problem, one solve the \( L^2 \) \( \bar{\partial} \) Cauchy problem on weakly \( q \)-convex domains in \( \mathbb{C}^n \), \( 1 \leq q \leq n \). This method was first used by Kohn-Rossi [20] for smooth forms on strongly pseudoconvex domains. More precisely, we prove the following \( L^2 \) Cauchy problem for \( \bar{\partial} \) in \( \mathbb{C}^n \):

**Theorem 3.2.** Let \( D \subseteq \mathbb{C}^n \) be a weakly \( q \)-convex domain, \( 1 \leq q \leq n \) with \( C^2 \) smooth boundary. Then, for \( f \in L^2_{p,q}(\mathbb{C}^n) \), \( \text{supp} f \subset \partial D \), \( 1 \leq q \leq n - 1 \), satisfying \( \bar{\partial} f = 0 \) in the distribution sense in \( \mathbb{C}^n \), there exists \( u \in L^2_{p,q-1}(\mathbb{C}^n) \), \( \text{supp} u \subset \partial D \) such that \( \bar{\partial} u = f \) in the distribution sense in \( \mathbb{C}^n \).

**Proof.** Let \( f \in L^2_{p,q}(\mathbb{C}^n) \), \( \text{supp} f \subset \partial D \), then \( f \in L^2_{p,q}(D) \). From Theorem 3.1, \( N_{n-p,n-q} \) exists for \( n-q \geq 1 \). Since \( N_{n-p,n-q} = \square^{-1}_{n-p,n-q} \) on \( \text{Range} \square_{n-p,n-q} \), then \( N_{n-p,n-q} \) exists for \( n-q \geq 1 \). Thus, we can define \( u \in L^2_{p,q-1}(D) \) by
\[
u = -\star \bar{\partial} N_{n-p,n-q} \star f.
\]
Thus \( \text{supp} u \subset \partial D \) and \( u \) vanishes on \( bD \). Now, we extend \( u \) to \( \mathbb{C}^n \) by defining \( u = 0 \) in \( \mathbb{C}^n \setminus D \). It follows from the same arguments of Theorem 9.1.2 in [6] and Theorem 2.2 in [1] that the form \( u \) satisfies the equation \( \bar{\partial} u = f \) in the distribution sense in \( \mathbb{C}^n \). Thus the proof follows.

4. The Weighted \( \bar{\partial} \)-Cauchy Problem

In this section, we assume that \( D \) is a weakly \( q \)-convex domain, \( 1 \leq q \leq n \), with \( C^2 \) smooth boundary in \( \mathbb{C}^n \). Also, we will choose \( \phi_t = -t \log |\delta|, \, \delta > 0 \) in (2.2), and using Remark 2.3 and by using Proposition 2.2, the inequality (2.2) implies the
weighted $L^2$-existence for the $\overline{\partial}$. Also, for $u \in \text{Dom}(\Box_t)$ of degree $q \geq 1$ and for $t > 0$, we have

$$t\|u\|_{t}^{2} \leq (\|\Box_t u\|_{t}^{2} + \|\Box_t^* u\|_{t}^{2})$$

$$= \langle \Box_t u, u \rangle_t$$

$$\leq \|\Box_t f\|_t \|u\|_t,$$

i.e.,

$$t\|u\|_t \leq \|\Box_t u\|_t.$$

Since $\Box_t$ is a linear closed densely defined operator, then, from [15, Theorem 1.1.1], $\text{Range}(\Box_t)$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_t$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(D, \phi_t) = \overline{\partial} \partial^* \text{dom}(\Box_t) \oplus \partial^* \overline{\partial} \text{dom}(\Box_t).$$

Since $\Box_t$ is one to one on $\text{dom}(\Box_t)$ from (1.5.3) in [15], then there exists a unique bounded inverse operator

$$N_t : \text{Ran}(\Box_t) \to \text{dom}(\Box_t) \cap (\ker(\Box_t))^\perp$$

such that $N_t \Box_t f = f$ on $\text{dom}(\Box_t)$. Therefore, we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\overline{\partial}$-Neumann operator $N_t$.

**Theorem 4.1.** For any $1 \leq q \leq n$ and $t > 0$, there exists a bounded linear operator $N_t : L^2_{p,q}(D, \phi_t) \to L^2_{p,q}(D, \phi_t)$ satisfies the following properties:

(i) $\text{Range}(N_t) \subset \text{dom}(\Box_t)$, $N_t \Box_t = I$ on $\text{dom}(\Box_t)$;

(ii) for $f \in L^2_{p,q}(D, \phi_t)$, we have $u = \overline{\partial} \partial^* N_t f + \partial^* \overline{\partial} N_t f$;

(iii) $\overline{\partial} N_t = N_t \overline{\partial}$, $1 \leq q \leq n - 1$;

(iv) $\Box_t N_t = N_t \Box_t$, $2 \leq q \leq n$;

(v) for all $f \in L^2_{p,q}(D, \phi_t)$, we have the estimates

$$t\|N_t f\|_t \leq \|f\|_t,$$

$$\sqrt{t}\|\overline{\partial} N_t f\|_t + \sqrt{t}\|\partial^* N_t f\|_t \leq \|f\|_t;$$

(vi) if $\overline{\partial} f = 0$, then $u_t = \partial^* N_t f$ solves the equation $\overline{\partial} u_t = f$.

**Theorem 4.2.** For $f \in L^2_{p,q}(D, \phi_t)$, $1 \leq q \leq n - 1$, supp $f \subset D$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(D, \phi_t)$, supp $u \subset D$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$.

**Proof.** Following Theorem 4.1, $N_t$ exists for forms in $L^2_{n-p,n-\bar{q}}(D, \phi_t)$. Thus, one can defines $u_t \in L^2_{p,q-1}(D, \phi_t)$ by

$$u_{(t)} = - \star_{(t)} \overline{\partial} N_{n-p,n-q} \star_{(-t)} f.$$

Thus supp $u_t \subset D$ and $u_t$ vanishes on $bD$. Now, we extend $u_t$ to $\mathbb{C}P^n$ by defining $u_t = 0$ in $\mathbb{C}P^n \setminus D$. We want to prove that the extended form $u_t$ satisfies the equation.
$\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. For $\eta \in L^2_{n-p,n-q-1}(D, -\phi_t) \cap \text{dom} \overline{\partial}$, we have

$$
\langle \overline{\partial} \eta, \ast(t) f \rangle_D = \int_D \overline{\partial} \eta \wedge \ast(-t) (\ast(t) f) \\
= \int_D \overline{\partial} \eta \wedge \ast(-t) \ast(t) f \\
= (-1)^{p+q} \int_D \overline{\partial} \eta \wedge f \\
= (-1)^{p+q} \langle f, \ast(-t) \overline{\partial} \eta \rangle_D \\
= (-1)^{p+q} \langle f, \ast(-t) \overline{\partial} \eta \rangle_{\mathbb{C}P^n},
$$

because $\text{supp} f \subset \overline{\partial}$. Since $\vartheta|_D = \partial^\ast|_D$, when $\vartheta$ acts in the distribution sense (see [15]), then we obtain

$$
\langle \overline{\partial} \eta, \ast(t) f \rangle_D = \langle f, \partial \ast(-t) \eta \rangle_{\mathbb{C}P^n} \\
= \langle \overline{\partial} f, \ast(-t) \eta \rangle_{\mathbb{C}P^n} \\
= 0.
$$

It follows that $\overline{\partial}'(\ast(t) f) = 0$ on $D$. Using Theorem 4.1 (iv), we have

$$
(4.2) \quad \overline{\partial}' N_t(\ast(t) f) = N_t \overline{\partial}'(\ast(t) f) = 0.
$$

Thus, from (4.1) and (4.2), one obtains

$$
\overline{\partial} u_t = - \partial \ast_t \overline{\partial} N_{n-p,n-q} \ast_t \overline{f} \\
= (-1)^{p+q+1} \ast \partial \ast \overline{\partial} N_{n-p,n-q} \ast \overline{f} \\
= (-1)^{p+q} \overline{\partial} \overline{\partial} N_{n-p,n-q} \ast \overline{f} \\
= (-1)^{p+q} (\overline{\partial} \overline{\partial} + \overline{\partial} \overline{\partial}^\ast) N_{n-p,n-q} \ast \overline{f} \\
= (-1)^{p+q} \ast \overline{f} \\
= f,
$$

in the distribution sense in $D$. Since $u = 0$ in $\mathbb{C}P^n \setminus D$, then for $u \in L^2_{p,q}(\mathbb{C}P^n) \cap \text{dom} \overline{\partial}^\ast$, one obtains

$$
< u, \overline{\partial}^\ast u >_{\mathbb{C}P^n} = < u, \overline{\partial}^\ast u >_D \\
= < \ast \overline{\partial}^\ast u, \ast(-t) u >_D \\
= (-1)^{p+q} < \overline{\partial} \ast u, \ast(-t) u >_D \\
= (-1)^{p+q} < \ast u, \overline{\partial} \ast(-t) u >_D \\
= < \ast u, \ast(-t) \overline{\partial} u >_D \\
= < f, u >_D \\
= < f, u >_{\mathbb{C}P^n},
$$

where the third equality holds since $\ast u = (-1)^{q+1} \overline{\partial} N_{n-p,n-q} \ast f \in \text{dom} \overline{\partial}^\ast$. Thus $\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. □
As in [5], we prove the following results.

**Proposition 4.1.** Let $D$ be the same as in Theorem 3.1. Put $\Omega = \mathbb{C}P^n \setminus \overline{D}$. Then, for any $f \in W^{1+\varepsilon}_{p,q}(\Omega)$, $\overline{\partial}f = 0$, $0 \leq \varepsilon < \frac{1}{2}$, there exists $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ such that $F|_{\Omega} = f$ and $\overline{\partial}F = 0$ in $\mathbb{C}P^n$.

**Proof.** Since $D$ has $C^2$ smooth boundary, there exists a bounded extension operator from $W^s_{p,q}(\Omega)$ to $W^s_{p,q}(\mathbb{C}P^n)$ for all $s \geq 0$ (cf. e.g. [33]). Let $f \in W^{1+\varepsilon}_{p,q}(\mathbb{C}P^n)$ be the extension of $f$ so that $\tilde{f}|_{\Omega} = f$ with

$$\|\tilde{f}\|_{W^{1+\varepsilon}_{p,q}(\mathbb{C}P^n)} \leq C\|f\|_{W^{1+\varepsilon}_{p,q}(\Omega)}.$$  

Furthermore, we can choose an extension such that $\overline{\partial}\tilde{f} \in W^\varepsilon(D) \cap L^2(D,\phi_{2\varepsilon})$.

One defines $T\tilde{f}$ by $T\tilde{f} = -\star_{2\varepsilon} \overline{\partial}N_{2\varepsilon}(\star_{-2\varepsilon}\overline{\partial}\tilde{f})$ in $\Omega$. As in Theorem 4.2, $T\tilde{f} \in L^2(D,\phi_{2\varepsilon})$. But for a $C^2$-smooth domain, we have that $T\tilde{f} \in L^2(D,\phi_{2\varepsilon})$ is comparable to $W^\varepsilon(\Omega)$ for $0 \leq \varepsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^\varepsilon_p(\Omega)$ and $Tf$ satisfies $\overline{\partial}Tf = \overline{\partial}f$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside $\Omega$.

Since $0 \leq \varepsilon < \frac{1}{2}$, the extension by 0 outside $\Omega$ is a continuous operator from $W^\varepsilon(\Omega)$ to $W^\varepsilon(\mathbb{C}P^n)$ (cf. e.g. [21]). Thus we have $T\tilde{f} \in W^\varepsilon(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & \text{if } z \in \overline{D}, \\ \tilde{f} - T\tilde{f}, & \text{if } z \in \Omega. \end{cases}$$

Then $F \in W^\varepsilon_{p,q}(\mathbb{C}P^n)$ and $F$ is $\overline{\partial}$-closed extension of $f$ to $\mathbb{C}P^n$.

**Corollary 4.1.** Let $D \subseteq \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geq 2$ with $C^2$ smooth boundary. Then $W^{1+\varepsilon}_{p,0}(D) \cap \ker \overline{\partial} = \{0\}$, $1 \leq p \leq n$ and $W^{1+\varepsilon}_{0,0}(D) \cap \ker \overline{\partial} = \mathbb{C}$.

**Proof.** Using Proposition 4.1 for $q = 0$, we have that any holomorphic $(p,0)$-form on $D$ extends to be a holomorphic $(p,0)$ in $\mathbb{C}P^n$, which are zero (when $p > 0$) or constants (when $p = 0$).

**Corollary 4.2.** Let $D \subseteq \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geq 2$ with $C^2$ smooth boundary. Then, for any $f \in W^{1+\varepsilon}_{p,q}(D)$, where $0 \leq p \leq n$, $1 \leq q \leq n-2$, $p \neq q$, and $0 \leq \varepsilon < \frac{1}{2}$, such that $\overline{\partial}f = 0$ in $D$, there exists $u \in W^{1+\varepsilon}_{p,q-1}(D)$ such that $\overline{\partial}u = f$ in $D$.

**Proof.** If $p \neq q$, we have that $F = \overline{\partial}u$ for some $U \in W^1_{p,q-1}(\mathbb{C}P^n)$. Let $u = U$ on $D$, we have $u \in W^1_{p,q-1}(D)$ satisfying $\overline{\partial}u = f$ in $D$.

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**References**

[1] O. Abdelkader and S. Saber, Solution to $\overline{\partial}$-equations with exact support on pseudoconvex manifolds, Int. J. Geom. Methods Mod. Phys. 4 (2007), 339–348.

[2] A. Andreotti and E. Vesentini, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, Publ. Math. Inst. Hautes Etudes Sci. (1965), 81–150.
[3] L. Bungart, *Piecewise smooth approximations to q-plurisubharmonic functions*, Pacific J. Math. 142 (1990), 227–244.

[4] J. Cao, M. C.-Shaw and L. Wang, *Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of $C^2$ Levi-flat hypersurfaces in $X$*, Math. Z. 248 (2004), 183–221.

[5] J. Cao and M.-C. Shaw, *The $\overline{\partial}$-Cauchy problem and nonexistence of Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$*, Math. Z. 256 (2007), 175–192.

[6] S.-C. Chen and M.-C. Shaw, *Partial Differential Equations in Several Complex Variables*, AMS/IP Stud. Adv. Math. 19, AMS, Providence, Rhode Island, 2001.

[7] J.-P. Demailly, *Complex analytic and differential geometry*, American Math. Society (to appear).

[8] M. Derridj, *Regularité pour $\overline{\partial}$ dans quelques domaines faiblement pseudo-convexes*, J. Differential Geom. 13 (1978), 559–576.

[9] M. Derridj, *Inégalités de Carleman et extension locale des fonctions holomorphes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 15 (1981), 645–669.

[10] K. Diederich and J. E. Fornaess, *Smoothing $q$-convex functions and vanishing theorems*, Invent. Math. 82 (1985), 291–305.

[11] M. G. Eastwood and G. V. Suria, *Cohomologically complete and pseudoconvex domains*, Comment. Math. Helv. 55 (1980), 413–426.

[12] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Princeton Math. Ser. 75 (1972).

[13] O. Fujita, *Domaines pseudoconvexes d’ordre général et fonctions pseudoconvexes d’ordre général*, Kyoto J. Math. 30 (1990), 637–649.

[14] G. M. Henkin and A. Iordan, *Regularity of $\overline{\partial}$ on pseudococave compacts and applications*, Asian J. Math. 4 (2000), 855–884.

[15] L. Hörmander, *$L^2$-estimates and existence theorems for the $\overline{\partial}$-operator*, Acta Math. 113 (1965), 89–152.

[16] L. Ho, *$\overline{\partial}$-problem on weakly $q$-convex domains*, Math. Ann. 290 (1991), 3–18.

[17] L. R. Hunt and J. J. Murray, *$q$-plurisubharmonic functions and a generalized Dirichlet problem*, Michigan Math. J. 25 (1978), 299–316.

[18] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds I*, Ann. of Math. 78 (1963), 112–148.

[19] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds II*, Ann. of Math. 79 (1964), 450–472.

[20] J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. 81 (1965), 451–472.

[21] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin Heidelberg, New York, 1972.

[22] K. Matsumoto, *Pseudoconvex domains of general order in Stein manifolds*, Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics 43(2) (1989), 67–76.

[23] K. Matsumoto, *Pseudoconvex domains of general order and $q$-convex domains in the complex projective space*, Kyoto J. Math. 33 (1993), 685–695.

[24] M. Peternell, *Continuous $q$-convex exhaustion functions*, Invent. Math. 85 (1986), 249–262.

[25] R. M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer, Berlin, Heidelberg, New York, 1986.

[26] S. Saber, *Solution to $\overline{\partial}$ problem with exact support and regularity for the $\overline{\partial}$-Neumann operator on weakly $q$-convex domains*, Int. J. Geom. Methods Mod. Phys. 7(1) (2010), 135–142.

[27] S. Saber, *The $\overline{\partial}$ problem on $q$-pseudoconvex domains with applications*, Math. Slovaca 63(3) (2013), 521–530.

[28] S. Saber, *The $L^2$ $\overline{\partial}$-Cauchy problem on weakly $q$-pseudoconvex domains in Stein manifolds*, Czechoslovak Math. J. 65(3) (2015), 739–745.
[29] S. Sambou, Réolution du $\bar{\partial}$ pour les courants prolongeables définis dans un anneau, Annales de la Faculté des sciences de Toulouse: Mathématiques 11(1) (2002), 105–129.

[30] M. C. Shaw, Local existence theorems with estimates for $\bar{\partial}_b$ on weakly pseudo-convex boundaries, Math. Ann. 294 (1992), 677–700.

[31] M. C. Shaw, The closed range property for $\bar{\partial}$ on domains with pseudoconcave boundary, Complex Analysis Trends in Mathematics (2010), 307–320.

[32] Y. T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, J. Differential Geom. 17 (1982), 55–138.

[33] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Math. Series 30, Princeton University Press, Princeton, New Jersey, 1970.

[34] E. J. Straube, Lectures on the $L^2$-Sobolev Theory of the $\bar{\partial}$-Neumann Problem, ESI Lectures in Mathematics and Physics, Freiburg, Germany, 2010.

[35] G. V. Suria, q-pseudoconvex and q-complete domains, Compos. Math. 53 (1984), 105–111.

[36] H. H. Wu, The Bochner Technique in Differential Geometry, Harwood Academic, New York, 1988.

[37] G. Zampier, Complex Analysis and CR Geometry, AMS 43, Providence, Rhode Island, 2008.

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