Observations on integral and continuous $U$-duality orbits in $\mathcal{N} = 8$ supergravity

L Borsten$^1$, D Dahanayake$^1$, M J Duff$^1$, S Ferrara$^{2,3,4}$, A Marrani$^5$ and W Rubens$^1$

$^1$ Theoretical Physics, Blackett Laboratory, Imperial College London, London SW7 2AZ, UK
$^2$ Physics Department, Theory Unit, CERN, CH 1211, Geneva 23, Switzerland
$^3$ INFN—Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy
$^4$ Department of Physics and Astronomy, University of California, Los Angeles, CA USA
$^5$ Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305-4060, USA

E-mail: leron.borsten@imperial.ac.uk, duminda.dahanayake@imperial.ac.uk, m.duff@imperial.ac.uk, sergio.ferrara@cern.ch, marrani@lnf.infn.it and william.rubens06@imperial.ac.uk

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Abstract

One would often like to know when two $a$ priori distinct extremal black $p$-brane solutions are in fact related by $U$-duality. In the classical supergravity limit the answer for a large class of theories has been known for some time now. However, in the full quantum theory the $U$-duality group is broken to a discrete subgroup, a consequence of the Dirac–Zwanziger–Schwinger charge quantization conditions. The question of $U$-duality orbits in this case is a nuanced matter. In the present work we address this issue in the context of $\mathcal{N} = 8$ supergravity in four, five and six dimensions. The purpose of this paper is to present and clarify what is currently known about these orbits while at the same time filling in some of the details not yet appearing in the literature. For the continuous case we present the cascade of relationships existing between the orbits, generated as one descends from six to four dimensions, together with the corresponding implications for the associated moduli spaces. In addressing the discrete case we exploit the mathematical framework of integral Jordan algebras, the integral Freudenthal triple system and, in particular, the work of Krutelevich. The charge vector of the dyonic black string in $D = 6$ is $SO(5, 5; \mathbb{Z})$ related to a two-charge reduced canonical form uniquely specified by a set of two arithmetic $U$-duality invariants. Similarly, the black hole (string) charge vectors in $D = 5$ are $E_{6(6)}(\mathbb{Z})$ equivalent to a three-charge canonical form, again uniquely fixed by a set of three arithmetic $U$-duality invariants. However, the situation in four dimensions is, perhaps predictably, less clear. While black holes preserving more than $1/8$ of the supersymmetries may be fully classified by the known arithmetic $E_{7(7)}(\mathbb{Z})$ invariants, $1/8$-BPS and non-BPS black holes yield increasingly subtle orbit structures, which remain to
be properly understood. However, for the very special subclass of *projective* black holes a complete classification is known. All projective black holes are $E_{7(7)}(\mathbb{Z})$ related to a four- or five-charge canonical form determined uniquely by the set of known arithmetic $U$-duality invariants. Moreover, $E_{7(7)}(\mathbb{Z})$ acts transitively on the charge vectors of projective black holes with a given leading-order entropy.

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1. Introduction

The extremal black $p$-brane solutions of supergravity have played, and continue to play, a key role in unravelling the non-perturbative aspects of M-theory. Evidently, understanding the structure of these solutions is of utmost importance. In particular, one would like to know how such solutions are interrelated by the set of global symmetries collectively known as $U$-duality. The electric/magnetic charge vectors of the asymptotically flat $p$-brane solutions form irreducible $U$-duality representations as shown in table 1. In many relevant cases the macroscopic leading-order black $p$-brane entropy is a function of these charges only, a result of the attractor mechanism [1–4]. Consequently, an important question is whether two *a priori* distinct black $p$-brane charge configurations are in fact related by $U$-duality. Mathematically this amounts to determining the distinct charge vector orbits under $U$-duality. In the classical limit the answer for a large class of theories has been known for some time now [5–8]. In particular, for the maximally supersymmetric theories, obtained by the toroidal compactification of $D = 11, \mathcal{N} = 1$ supergravity, a complete classification of all orbits in all dimensions $D \geq 4$ is known [5, 6]. However, in the full quantum theory the $U$-duality group is broken to a discrete subgroup, a consequence of the Dirac–Zwanziger–Schwinger charge quantization conditions [9]. Consequently, the $U$-duality orbits are furnished with a further level structural complexity, which, in some cases, is of particular mathematical significance [10, 11]. However, the question of discrete $U$-duality orbits is not only interesting in its own right, but it is also of physical importance with implications for a number of topics including the stringy origins of microscopic black hole entropy [12–20]. Moreover, following a conjecture of finiteness of $D = 4, \mathcal{N} = 8$ supergravity [21], it has recently been observed that some of the orbits of $E_{7(7)}(\mathbb{Z})$ should play an important role in counting microstates of this theory [20], even if it may differ from its superstring or M-theory completion [22].

In the present work we address this issue in the context of $\mathcal{N} = 8$ supergravity in four, five and six dimensions. To this end we exploit the mathematical framework of *integral* Jordan algebras and the *integral* Freudenthal triple system (FTS), both of which have at their basis the ring of *integral* split octonions [11, 23–25]. To a large extent this work is a continuation of the analysis used in studying the recently introduced black hole *Freudenthal duality* [26], which in turn has its provenance in recently established connections relating black hole entropy in M-theory to entanglement in quantum information theory [26–42].

It is well known that the black holes and strings appearing in the maximally supersymmetric six-, five- and four-dimensional classical theories are elegantly described by the exceptional Jordan algebras and the closely related FTS [5, 43–45].

In particular, the black string charge vectors of $D = 6, \mathcal{N} = 8$ supergravity may be represented as the elements of the exceptional Jordan algebra of $2 \times 2$ Hermitian matrices defined over the split octonions, denoted by $\mathcal{O}_2$. See appendix C.2 for details. The *reduced*
Table 1. Asymptotically flat p-brane U-duality representations.

| D  | G                | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|----|------------------|---|---|---|---|---|---|---|
| 10A | SL(2, R)         |   | 1 | 1 | 1 | 1 | 1 | 1 |
| 10B | SL(2, R) × R⁺    |   | 2 | 1 | 2 |   |   |   |
| 9  | SL(2, R) × SL(2, R) | 2 + 1 | 2 | 1 | 1 | 2 | 2 + 1 |   |
| 8  | SL(3, R)         |   | (3', 2) | (3, 1) | (1, 2) | (3', 1) | (3, 2) |   |
| 7  | SL(5, R)         |   | 10 | 5 | 5' | 10 |   |   |
| 6  | SO(5, 5; R)      |   | 16 | 10 | 16' |   |   |   |
| 5  | E₆(6)(R)         |   | 27 |   |   |   |   |   |
| 4  | E₇(7)(R)         |   | 56 |   |   |   |   |   |

structure group Str₀(3²⁺), defined as the set of invertible linear transformations preserving N₂, the quadratic norm (C, 15), is the D = 6 U-duality group SO(5, 5; R), under which the black string charges transform as the vector 10. Moreover, in this case the quadratic norm N₂ is nothing but I₅, the SO(5, 5; R) singlet in 10 × 10, which determines the black string entropy

\[ S_{D=6, BS} \sim |I₅| \]

See e.g. [5, 46, 47], and references therein. There are two distinct charge vector orbits under SO(5, 5; R), one consisting of 1/2-BPS states and other consisting of 1/4-BPS states, distinguished respectively by the vanishing or not of I₅ [6, 47]. Equivalently, these orbits may be distinguished by the rank of the Jordan algebra element representing the charge vector, rank 1 states being 1/2-BPS while rank 2 states 1/4-BPS. See section 2.1 for details.

Similarly, the black hole charge vectors of D = 5, N = 8 supergravity may be represented as elements of the exceptional Jordan algebra of 3 × 3 Hermitian matrices defined over the split octonions, denoted by 3³⁻ [5]. See appendix D.2 for details. The reduced structure group Str₀(3³⁻), defined as the set of invertible linear transformations preserving N₃, the cubic norm (D.1), and the symmetric bilinear trace form (D.3d), is the D = 5 U-duality group E₆(6)(R), under which the black hole charges transform as the fundamental 27. Moreover, in this case the cubic norm N₃ is in fact I₅, the E₆(6)(R) singlet in 27 × 27 × 27, which determines the black hole entropy

\[ S_{D=5, BH} = \pi \sqrt{|I₅|} \]

There are three distinct charge vector orbits under E₆(6)(R), 1/2-BPS, 1/4-BPS and 1/8-BPS, distinguished by the vanishing or not of I₅ and its derivatives [48]. See section 3.1 for details. Again, these orbits may also be distinguished by the rank of the Jordan algebra element representing the charge vector. Rank 1 states are 1/2-BPS, rank 2 states are 1/4-BPS while rank 3 states are 1/8-BPS [26, 38]. A directly analogous treatment goes through for the black string charges in D = 5, which transform as the contragredient 27' of E₆(6)(R).

Finally, the black hole charge vectors of D = 4, N = 8 supergravity may be represented as elements of the FTS denoted by Ω(3²⁻). See appendix E for details. The automorphism group Aut(Ω(3²⁻)), defined as the set of invertible linear transformations preserving Δ, the quartic norm (E.4b), and the antisymmetric bilinear form (E.4a), is nothing but E₇(7)(R), the D = 4 U-duality group under which the black hole charges transform as the fundamental 56. Moreover, the quartic norm Δ is exactly I₄, the unique E₇(7)(R) quartic invariant, which again determines the black hole entropy [49]

\[ S_{D=4, BH} = \pi \sqrt{|I₄|} \]
This is the first example exhibiting a non-BPS orbit. In total there are five distinct charge vector orbits under $E_7(\mathbb{R})$, three of which have vanishing $I_4$, 1/2-BPS, 1/4-BPS and 1/8-BPS, which are distinguished by the vanishing or not of the derivatives of $I_4$ [48]. The two orbits with non-vanishing $I_4$ are either 1/8-BPS or non-BPS according to whether $I_4 > 0$ or $I_4 < 0$ respectively. Again, these orbits may also be distinguished by the rank of the FTS element representing the charge vector. States of rank 1, 2 and 3 are 1/2-BPS, 1/4-BPS and 1/8-BPS respectively, all with vanishing $I_4$. Rank 4 states are split into 1/8-BPS and non-BPS as determined by the sign of $I_4$. See section 4.1 for details. For further details concerning these orbits and their defining U-invariant BPS conditions the reader is referred to the original works [5, 6, 48].

As one descends from $D = 6$ to $D = 4$, via spacelike dimensional reductions, a series of relationships connecting these U-duality orbits is generated. The U-invariant BPS conditions in $D$ dimensions are 'embedded' in those of $D - 1$ dimensions, as is best understood by decomposing the $(D - 1)$-dimensional U-duality group with respect to the $D$-dimensional U-duality group. Taking care of whether or not the charges of the Kaluza–Klein vector are vanishing, one is then able to understand how the various black $p$-brane solutions, their orbits and the associated moduli spaces are embedded under these spacelike reductions. Moreover, in this way we can also study the reverse situation and so understand which higher dimensional black $p$'-brane solutions a given black $p$-brane in a given dimension may be 'uplifted' to. It should be pointed out that the vanishing or not of the Kaluza–Klein vector charges is crucial in discriminating the various possible uplifts. A general result holding throughout the present treatment in $D = 4, 5, 6$ can be stated as follows: if the charges of the Kaluza–Klein vector are not switched on, the supersymmetry preserving features of the solution are unaffected by the dimensional reduction. These results are presented for $D = 6 \leftrightarrow D = 5$ and $D = 5 \leftrightarrow D = 4$ in sections 3.2 and 4.2 respectively.

This summarizes the classification of the U-duality orbits for real valued charges. However, as previously emphasized, the charges are actually quantized and the U-dualities are correspondingly broken to discrete subgroups as described in [9]. The integral charge vector orbits are a nuanced matter and a complete characterization is not known as yet. Despite these additional complications, the discrete orbit classification is made possible in certain cases by the introduction of new arithmetic U-duality invariants not appearing in the continuous case. These are typically given by the greatest common divisor (gcd) of U-duality representations built out of the basic charge vector representations [11, 13–19, 25, 26]. One purpose in this paper is to present and clarify what is currently known about these discrete orbits while at the same time filling in some of the details, not yet appearing in the literature.

Let us now briefly summarize the present situation. An important general observation is that, since the conditions separating the continuous orbits are manifestly invariant under the corresponding discrete U-dualities, those states unrelated in the continuous cases remain unrelated in the discrete case. Consequently, the discrete orbits fall into disjoint sets corresponding directly to the orbits of the classical theory. A second important observation, emphasized in [47], is that the gcd of a U-duality representation, built out of the relevant basic charge vector representation, is only well defined if that representation is non-vanishing. In practice this means first computing which class of orbits as defined by the continuous analysis a given state lies in. This, in turn, determines the subset of the arithmetic invariants that are well defined for this particular state. It is this subset that is then to be used in specifying the particular discrete orbit to which the state belongs, the remaining arithmetic invariants being ill-defined and contentless.

Beginning in $D = 6$ the integral charge vectors of the dyonic black strings may be represented as elements of the integral Jordan algebra of $2 \times 2$ Hermitian matrices defined
over the ring of integral split octonions, denoted by $\mathcal{O}_2$. See appendix C.3 for details. The discrete $U$-duality group $SO(5, 5; \mathbb{Z})$ is given by the set of invertible $\mathbb{Z}$-linear transformations preserving the quadratic norm (C.15). An arbitrary charge vector is $SO(5, 5; \mathbb{Z})$ related to a two-charge reduced canonical form (12) uniquely specified by a set of two arithmetic $U$-duality invariants (11). The two orbits of the continuous case now form two disjoint countably infinite sets of discrete orbits which may be parametrized using the arithmetic invariants.

Similarly, the black hole charge vectors in $D = 5$ may be represented as the elements of the integral Jordan algebra of $3 \times 3$ Hermitian matrices defined over the ring of integral split octonions, denoted by $\mathcal{O}_3$. See appendix D.3 for details. The discrete $U$-duality group $E_{6(6)}(\mathbb{Z})$ is given by the set of invertible $\mathbb{Z}$-linear transformations preserving the cubic norm (D.1) and the trace bilinear form (D.3d). An arbitrary charge vector is $E_{6(6)}(\mathbb{Z})$ equivalent to a three-charge canonical form (61), again uniquely fixed by a set of three arithmetic $U$-duality invariants (60). The three orbits of the continuous case now form three disjoint classes of discrete orbits which may be parametrized using arithmetic invariants [25, 26]. A directly analogous treatment goes through for the black string charges in $D = 5$, which transform as the contragredient $27'$ of $E_{6(6)}(\mathbb{Z})$.

However, the situation in four dimensions is, perhaps predictably, less clear. The black hole charge vectors may be represented as elements of the integral FTS defined over $\mathcal{O}_4$. See appendix E for details. The discrete $U$-duality group $E_{7(7)}(\mathbb{Z})$ is given by the set of invertible $\mathbb{Z}$-linear transformations preserving the quartic norm (E.4b) and the antisymmetric bilinear form (E.4d). An arbitrary charge vector is $E_{7(7)}(\mathbb{Z})$ equivalent to a five-charge canonical form (103). However, this canonical form is not uniquely fixed by the known set of arithmetic $U$-duality invariants (102). Despite this, for particular subcases more can be said. Indeed, the classes of discrete orbits corresponding to the 1/2-BPS and 1/4-BPS continuous orbits may be completely classified using the known arithmetic invariants [11], as described in section 4.3. For those black hole preserving less than 1/4 of the supersymmetries the orbit structure becomes more complicated and the orbit classification is not known. However, even in this case a full classification is possible for the subclass of projective black holes. See section 4.4 for details. The concept of projectivity was originally introduced in the number-theoretic context of [10] where such elements are mapped to invertible ideal classes of quadratic rings. This notion was later generalized by Krutelevich in [11] with a view to understanding the $E_{7(7)}(\mathbb{Z})$ orbit structure. Indeed, $E_{7(7)}(\mathbb{Z})$ acts transitively on the set of projective black holes of a given quartic norm. Moreover, they are $E_{7(7)}(\mathbb{Z})$ equivalent to a simplified four- or five-charge canonical form (109) depending on whether the quartic norm is even or odd respectively [11, 26].

This paper is organized as follows. In section 2 we begin by recalling the continuous $U$-duality orbits in $D = 6$, emphasizing the Jordan algebraic perspective, before presenting in detail the corresponding discrete treatment. In section 3 the same treatment is applied to continuous $U$-duality black hole charge orbits in $D = 5$. Subsequently, the intricate web of relations connecting the orbits and moduli spaces of the six-dimensional theory to those of the five-dimensional theory are presented. This analysis is concluded with the corresponding discrete $U$-duality treatment of the $D = 5$ integral black hole charge orbits. The same continuous analysis is undertaken for $D = 4$ in section 4, completing the cascade of relationships between the orbits and moduli spaces in six, five and four dimensions. This is followed by the discrete $U$-duality treatment of the integral black hole charge orbits. We conclude with a summary of open questions. For the most part the technical details are relegated to the appendices in an effort to avoid an oppressive number of formal definitions in the main body of the text. In appendix A we present the minimal background necessary
to introduce the ring of integral split octonions underlying this analysis. In appendix C we describe the continuous and integral quadratic Jordan algebras together with their application to black strings in $D = 6$. In appendix D we describe the continuous and integral cubic Jordan algebras together with their application to black holes (strings) in $D = 5$. In appendix E we describe the continuous and integral FTS, defined over the $D = 5$ Jordan algebra, together with its application to black holes in $D = 4$.

2. Black strings in $D = 6$

2.1. U-duality orbits of $SO(5, 5; \mathbb{R})$

In the classical supergravity limit the $5 + 5$ electric/magnetic black string charges form an $SO(5, 5; \mathbb{R})$ vector $Q_r$ ($r = 1, \ldots, 10$ throughout). Under $SO(1, 1; \mathbb{R}) \times SO(4, 4; \mathbb{R})$ the vector breaks as

$$10 \rightarrow 1_2 + 1_{-2} + 8_{0,0},$$

(1)

where the singlets lie in the NS–NS sector and correspond to a fundamental string and an NS5-brane, while the $8_{0,0}$, is made up of R–R charges. In this basis the charges $Q_r$ may be conveniently represented as an element $Q$ of the Jordan algebra $J_{OS}^{2}$ of split octonionic $2 \times 2$ Hermitian matrices,

$$Q = \begin{pmatrix} p^0 & Q_v \\ Q_v^* & q^0 \end{pmatrix},$$

where $q^0, p^0 \in \mathbb{R}$ and $Q_v \in O'$. (2)

The set of linear invertible transformations leaving the quadratic norm,

$$N_2(Q) = \det(Q),$$

(3)

invariant is the $D = 6$ U-duality group $SO(5, 5; \mathbb{R})$. Using the dictionary (C.19) one finds

$$N_2(Q) = I_2(Q),$$

(4)

where

$$I_2(Q) = \eta^{rs} Q_r Q_s,$$

(5)

and $\eta^{rs}$ is the $SO(5, 5; \mathbb{R})$ metric,

$$\eta = \begin{pmatrix} 0 & \mathds{1} \\ \mathds{1} & 0 \end{pmatrix}.$$ (6)

The black string entropy is proportional to the quadratic norm

$$S_{D=6,BS} \sim |I_2(Q)| = |N_2(Q)|.$$ (7)

See e.g. [5, 46, 47], and references therein.

There are two U-duality orbits, one 1/2-BPS ‘small’ orbit and other 1/4-BPS ‘large’ orbit [6, 47, 48]. Note that an orbit is referred to as ‘small’ in the sense that the associated U-duality is invariant and, hence, Bekenstein–Hawking entropy are vanishing. Correspondingly, for ‘large’ orbits the U-duality invariant and Bekenstein–Hawking entropy are non-vanishing. These orbits may be distinguished by the Jordan rank of $Q$ as detailed in appendix C.1,

| Rank | Condition | BPS State |
|------|-----------|-----------|
| 1    | $Q \neq 0, N_2(Q) = 0$ | 1/2-BPS, |
| 2    | $N_2(Q) \neq 0$ | 1/4-BPS |

(8)

which is precisely the condition originally presented in [48]. The orbits, their rank conditions, dimensions and representative states are summarized in table 2.
2.2. U-duality orbits of $SO(5, 5; \mathbb{Z})$

For quantized charges the continuous U-duality is broken to an infinite discrete subgroup, which for $D = 6$ is given by $SO(5, 5; \mathbb{Z}) \subset SO(5, 5; \mathbb{R})$ [9]. The integral Jordan algebra, $\mathfrak{J}_O^Z$, of integral split octonionic $2 \times 2$ Hermitian matrices provides a natural model for $SO(5, 5; \mathbb{Z})$, which may be used to analyse the discrete $U$-duality orbits. The quantized black string charge vector is given by

$$Q = \left( p^0 \ O_v \ q^0 \right),$$

where $q^0, p^0 \in \mathbb{Z}$ and $Q_v \in O_2^Z$. (9)

The discrete group $SO(5, 5; \mathbb{Z})$ is defined by the set of norm-preserving invertible $\mathbb{Z}$-linear transformations:

$$\{ \sigma : \mathfrak{J}_O^Z \rightarrow \mathfrak{J}_O^Z \ | \ N_2(\sigma(Q)) = N_2(Q) \}. \hspace{1cm} (10)$$

It is with this framework that we shall study the discrete $U$-duality orbits.

The first important observation is that the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup $SO(5, 5; \mathbb{Z})$ and, hence, those states unrelated by $U$-duality in the classical theory remain unrelated in the quantum theory. There are two disjoint classes of orbits, 1/2-BPS and 1/4-BPS, corresponding to the two orbits of the continuous case. However, each of these classes is broken up into a countably infinite set of discrete orbits. To classify these orbits we use $SO(5, 5; \mathbb{Z})$ to bring an arbitrary charge vector into a diagonally reduced canonical form, which is uniquely defined by the following set of two discrete invariants:

$$b_1(Q) := \text{gcd}(Q),$$

$$b_2(Q) := N_2(Q). \hspace{1cm} (11)$$

See appendix C.3 for details.

2.2.1. $D = 6$ diagonally reduced canonical form. Every element $Q \in \mathfrak{J}_O^Z$ is $SO(5, 5; \mathbb{Z})$ equivalent to a diagonally reduced canonical form

$$Q_{\text{can}} = k \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix},$$

where $k > 0$, $|l| \geq 0$. (12)

The canonical form is uniquely determined by (11) since

$$b_1(Q_{\text{can}}) = k,$$

$$b_2(Q_{\text{can}}) = k^2 l, \hspace{1cm} (13)$$

so that for arbitrary $Q$ one obtains $k = b_1(Q)$ and $l = k^{-2} b_2(Q)$. 

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Table 2. $D = 6$ black string orbits, their corresponding rank conditions, dimensions and SUSY.

| Rank/orbit conditions | Representative state | Orbit | dim | SUSY |
|-----------------------|----------------------|-------|-----|------|
| Rank non-vanishing    | Vanishing            | Orbit dim | SUSY |
| 1 $Q$                 | $N_2(Q)$ diag(1, 0)  | $SO(5, 5; \mathbb{R})$ | 9 | 1/2  |
| 2 $N_2(Q)$            | diag(1, $k$)         | $SO(5, 5; \mathbb{R}) \rtimes R$ | 9 | 1/4  |
2.2.2. $D = 6$ Black string orbit classification.

(1) The complete set of distinct 1/2-BPS charge vector orbits is given by
\[
\left\{ \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}, \text{ where } k > 0 \right\}.
\] (14)

(2) The complete set of distinct 1/4-BPS charge vector orbits is given by
\[
\left\{ \begin{pmatrix} k \\ 0 \\ k_l \end{pmatrix}, \text{ where } k, |l| > 0 \right\}.
\] (15)

3. Black holes in $D = 5$

3.1. U-duality orbits of $E_{6(6)}(\mathbb{R})$

In the classical supergravity limit the 27 $Q_i$ ($i = 1, \ldots, 27$ throughout) electric black hole charges transform as the fundamental 27 of the continuous U-duality group $E_{6(6)}(\mathbb{R})$. Under $SO(1, 1; \mathbb{R}) \times SO(5, 5; \mathbb{R})$ the 27 breaks as
\[
27 \rightarrow 1_4 + 10_{-2} + 16_1,
\] (16)
where the singlet may be identified as the graviphoton charge descending from $D = 6$, the 10 as the remaining NS–NS sector charges and the 16 as the R–R sector charges. Further decomposing under $SO(4, 4; \mathbb{R})$ one obtains
\[
27 \rightarrow 1 + 1 + 1 + 8_e + 8_e + 8_e.
\] (17)

In this basis the charges $Q_i$ may be conveniently represented as an element $Q$ of the cubic Jordan algebra $\mathfrak{J}_O^3$ of split octonionic $3 \times 3$ Hermitian matrices,
\[
Q = \begin{pmatrix} q_1 & Q_e \\ Q_e & q_2 & Q_v \\ Q_v & Q_c & q_3 \end{pmatrix} , \quad \text{where } q_1, q_2, q_3 \in \mathbb{R} \text{ and } Q_{v,s,c} \in O^e.
\] (18)

The cubic norm $N_3$ (D.14) is then given by the determinant-like object:
\[
N_3(Q) = q_1 q_2 q_3 - q_1 Q_e Q_c - q_2 Q_e Q_v - q_3 Q_v Q_c + (Q_e Q_v) Q_c + (Q_e Q_c) Q_v.
\] (19)

The set of invertible linear transformations leaving the cubic norm and trace bilinear form invariant is nothing but the $D = 5$ U-duality group $E_{6(6)}(\mathbb{R})$. Moreover,
\[
N_3(Q) = I_3(Q),
\] (20)
where
\[
I_3(Q) = \frac{1}{3!} d^{ijk} Q_i Q_j Q_k,
\] (21)
and $d^{ijk}$ is the $E_{6(6)}(\mathbb{R})$ invariant tensor.

The black hole entropy is simply given by the cubic norm
\[
S_{BS} = \pi \sqrt{|N_3(Q)|} = \pi \sqrt{|I_3(Q)|}.
\] (22)

In this case there are three U-duality orbits: 1/2-BPS and 1/4-BPS ‘small’ orbits and a single 1/8-BPS ‘large’ orbit [5]. These orbits may be distinguished by the Jordan rank of $Q$:

| Rank | Condition | Type |
|------|-----------|------|
| 1    | $Q \neq 0$, $Q^2 = 0$ | 1/2-BPS, |
| 2    | $Q^2 \neq 0$, $N_3(Q) = 0$ | 1/4-BPS, |
| 3    | $N_3(Q) \neq 0$ | 1/8-BPS, |
manifestly transforms as a $27$ under $E_{6(6)}(\mathbb{R})$. Similarly

$$\partial_i I_3(Q) = d^{ijk} Q_j Q_k$$  \hspace{1cm} (24)

where $^z$ is the quadratic adjoint map (D.18). See appendix D.1 for details. Note that $Q^z$ transforms as a $27'$ under $E_{6(6)}(\mathbb{R})$. Similarly

$$\partial_i I_3(Q) = \frac{\partial I_3(Q)}{\partial Q_i} \sim d^{ijk} Q_j Q_k$$

manifestly transforms as a $27'$ under $E_{6(6)}(\mathbb{R})$ so that $Q^z \sim \partial_i I_3(Q)$. Hence, these conditions are entirely equivalent to the conditions originally presented in [48],

- Rank 1 $Q \neq 0$, $\partial_i I_3(Q) = 0$ $1/2$-BPS,
- Rank 2 $\partial_i I_3(Q) \neq 0$, $I_3(Q) = 0$ $1/4$-BPS,
- Rank 3 $I_3(Q) \neq 0$ $1/8$-BPS.

The orbits with their rank conditions, dimensions and representative states are summarized in table 3. The 27 magnetic black string charges $P$ form the contragredient $27'$ of $E_{6(6)}(\mathbb{R})$. The orbit classification is identical to the black hole case.

**3.2. $D = 5$ $\leftrightarrow$ $D = 6$ relations for charge orbits and moduli spaces**

Through the branching of the $27''$ irrep. of $D = 5$ U-duality $E_{6(6)}(\mathbb{R})$ with respect to $D = 6$ U-duality $SO(5, 5; \mathbb{R}) \times SO(1, 1; \mathbb{R})$ [20, 46, 48] ($r = 1, \ldots, 10$, and $\alpha = 1, \ldots, 16$ throughout; cf. equation (16))

\[
\begin{align*}
27 & \rightarrow I_4 + 10_{-2} + 16_1, \\
Q_i & \rightarrow (q_i, Q_i, q_0); \\
27' & \rightarrow 1_{-4} + 10_2 + 16'_{-1}, \\
P^\mu & \rightarrow (p^\mu, P^\mu, p^\alpha),
\end{align*}
\]

(26a)

the unique electric and magnetic invariants (77a) and (77b) of the 27 can be written in a manifestly $SO(5, 5; \mathbb{R})$-invariant way respectively as follows [48, 50]:

\[
\begin{align*}
I_3(Q) &= \frac{1}{2} [q_i, I_2(Q) + Q_\alpha (\gamma^\alpha)^{\gamma\beta} q_\gamma q_\beta], \\
I_3(P) &= \frac{1}{2} [p_i, I_2(P) + P^\alpha (\gamma^\alpha)^{\gamma\beta} p_\gamma p_\beta],
\end{align*}
\]

(27a)

where $\gamma^\alpha$'s are the $SO(5, 5; \mathbb{R})$-gamma matrices, $q_i$ and $p_i$ respectively are the electric and magnetic charge of the $D = 6 \rightarrow 5$ Kaluza–Klein vector and the unique quadratic invariant of the 10 of $SO(5, 5; \mathbb{R})$ is defined as follows (note the basis change compared to (5)−(6) $\eta_{rs} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$):

\[
I_2(Q) := \eta^{rs} Q_r Q_s,
\]

(28a)
In order to study the relations among the various charge orbits of maximal supergravity in $D = 5$ and $D = 6$ (and the consequences for the related moduli spaces\footnote{In all the treatment $M$, $O$ and $M$ respectively denote a scalar manifold, a charge orbit and a moduli space. $M$ is defined all along the scalar flow, from the near horizon geometry (if any, at the classical level) to asymptotically flat spatial infinity. Thus, if the corresponding $O$ is ‘large’, $M$ can be interpreted as the moduli space of attractor solutions (at the near horizon geometry) and the moduli space of the ADM mass (at spatial infinity). In the case of ‘small’ $O$, the attractor near horizon interpretation of corresponding $M$ breaks down.}), let us briefly recall the $U$-invariant classification of the charge orbits of black $p = 0, 2$-branes (black holes, respectively black membranes) in $\mathcal{N} = (2, 2)$, $D = 6$ supergravity \cite{6, 48}.

3.2.1. Résumé on $p = 0, 2$ black branes in $\mathcal{N} = (2, 2)$, $D = 6$. Without any loss of generality, let us consider a black hole ($p = 0$) corresponding to the $D = 6$ uplift of a $D = 5$ black hole. In other words, we only consider the orbits of chiral spinor $16$ of $SO (5, 5; R)$ determined by the branching \cite{26a} of $27$ of $E_{6(6)}(R)$.

(1) 1/4-BPS ‘small’ orbit (two-charge solution) \cite{6, 48}

$$\mathcal{O}_{1/4\text{-BPS, small, } p=0,2, D=6} = \frac{SO(5, 5; R)}{SO(3, 4; R) \times \mathbb{R}^8},$$

(29)

defined by

$$(\gamma^r)^{\alpha \beta} q_\alpha q_\beta \neq 0,$$

(30)

for at least some $r = 1, \ldots, 10$. Namely, the $SO(5, 5; R)$-invariant constraint defining such an orbit is the fact that $q_\alpha$ is not a pure chiral spinor of $SO (5, 5; R)$. The resulting Bekenstein–Hawking entropy is vanishing:

$$S_{\text{BH, } D=6} = 0.$$  

(31)

(2) 1/2-BPS ‘small’ orbit (one-charge solution) \cite{6, 48}

$$\mathcal{O}_{1/2\text{-BPS, small, } p=0,2, D=6} = \frac{SO(5, 5; R)}{SL (5, \mathbb{R}) \times \mathbb{R}^{16}},$$

(32)

defined by

$$(\gamma^r)^{\alpha \beta} q_\alpha q_\beta = 0, \forall r.$$  

(33)

In other words, the $SO(5, 5; R)$-invariant constraint defining such an orbit is the fact that $q_\alpha$ is a pure chiral spinor of $SO (5, 5; R)$. However, it is worth recalling that the non-triviality of the background implies that at least some $\alpha \in \{1, \ldots, 16\}$ exist such that

$$\frac{\partial}{\partial q_\beta} [(\gamma^r)^{\alpha \beta} q_\alpha q_\beta] = 2(\gamma^r)^{\delta \beta} q_\beta \neq 0.$$  

(34)

As for case 1, the resulting Bekenstein–Hawking entropy is vanishing (see (31)).
3.2.2. The 1/4-BPS black string orbit under $D = 6 \rightarrow 5$. A representative of the 1/4-BPS 'large' charge orbit

$$\mathcal{O}_{1/4-\text{BPS, large, } p=1, D=6} = \frac{SO(5, 5; \mathbb{R})}{SO(5, 4; \mathbb{R})}$$

(35)

of a dyonic black string in $\mathcal{N} = (2, 2)$, $D = 6$ supergravity is provided by

$$p^i = 0;$$
$$q_\nu = 0;$$
$$Q_\nu : I_2(\mathbb{Q}) \neq 0.$$ (36a)

By plugging (36a) into (27a), one obtains

$$I_3(Q) = \frac{1}{2} q_z I_2(Q) \neq 0.$$ (36b)

The treatment splits into two separate cases, depending on the vanishing or not of the magnetic charge of Kaluza–Klein vector:

(1) In the case

$$q_z = 0,$$ (36c)

equation (36b) yields

$$\begin{cases} I_3(Q) = 0; \\ \frac{\partial I_3(Q)}{\partial q_z} = \frac{1}{2} I_2(Q) \neq 0, \end{cases}$$ (36d)

corresponding to a 1/4-BPS 'small' black hole of $\mathcal{N} = 8$, $D = 5$ supergravity.

(2) On the other hand, for

$$q_z \neq 0,$$ (36e)

then equation (36b) gives

$$I_3(q) \neq 0,$$ (36f)

corresponding to a 1/8-BPS 'large' black hole of $\mathcal{N} = 8$, $D = 5$ supergravity.

Thus, at the level of charge orbits, equations (36) correspond to the following picture:

$$\mathcal{O}_{1/4-\text{BPS, large, } p=1, D=6} = \frac{SO(5, 5; \mathbb{R})}{SO(5, 4; \mathbb{R})}$$ (37)

Indeed, at the level of the semi-simple part of the orbit stabilizers, the embedding$^7$

$$SO(5, 4; \mathbb{R}) \subseteq_{\text{symm}} F_{4(4)}(\mathbb{R})$$ (38)

holds. At the level of corresponding moduli spaces, (37) implies that (see also (89))

$$\mathcal{M}_{1/4-\text{BPS, large, } p=1, D=6} = \frac{SO(5, 4; \mathbb{R})}{SO(5, \mathbb{R}) \times SO(4, \mathbb{R})}$$ (39)

satisfies

$$\mathcal{M}_{1/4-\text{BPS, large, } p=1, D=6} \subseteq \mathcal{M}_{1/8-\text{BPS, } D=5}.$$ (40)

$^7$ ‘max’ and ‘symm’ respectively denote the maximality and symmetricity of the group embedding under consideration.
which is nothing but a part of the embedding (91).

Thus, it follows that a 'small' $1/4$-BPS black hole, as well as a 'large' $1/8$-BPS black hole, of $\mathcal{N} = 8$, $D = 5$ supergravity can be uplifted to a 'large' $1/4$-BPS dyonic black string of $\mathcal{N} = (2, 2)$, $D = 6$ supergravity.

It should be pointed out that the non-vanishing or not of charges of Kaluza–Klein vector is crucial in order to discriminate among the various possible uplifts. A general result holding throughout the present treatment in $D = 4, 5, 6$ can be stated as follows: if the charges of the Kaluza–Klein vector are not switched on, the supersymmetry-preserving features of the solution are unaffected by the dimensional reduction.

3.2.3. The $1/2$-BPS black string orbit under $D = 6 \rightarrow 5$. A representative of the $1/2$-BPS 'small' charge orbit

$$\mathcal{O}_{1/2\text{-BPS, small}, p=1, D=6} = \frac{SO(5, 5; \mathbb{R})}{SO(4, 4; \mathbb{R}) \ltimes \mathbb{R}^3},$$

of a dyonic black string in $\mathcal{N} = (2, 2)$, $D = 6$ supergravity is provided by

$$\begin{align*}
p^i &= 0; \\
q_{\alpha} &= 0; \\
Q_r : I_2(Q) &= 0.
\end{align*}$$

By plugging (42a) into (27a), one obtains

$$\begin{align*}
\begin{cases}
I_3(Q) &= 0; \\
\frac{\partial I_3(Q)}{\partial Q_r} &= \frac{1}{2} q_z \frac{\partial I_2(Q)}{\partial Q_r}.
\end{cases}
\end{align*}$$

(42b)

where $\frac{\partial I_3(Q)}{\partial Q_r}$ is the unique, possibly non-vanishing component of $\frac{\partial I_3(Q)}{\partial Q_r}$.

As above, depending on the vanishing or not of the magnetic charge of Kaluza–Klein vector, the treatment splits into two cases as follows:

1. In the case $q_z = 0$,

$$\begin{align*}
\begin{cases}
I_3(Q) &= 0; \\
\frac{\partial I_3(Q)}{\partial Q_r} &= 0,
\end{cases}
\end{align*}$$

(42d)

corresponding to a $1/2$-BPS 'small' black hole of $\mathcal{N} = 8$, $D = 5$ supergravity.

2. On the other hand, for $q_z \neq 0$,

$$\begin{align*}
\begin{cases}
I_3(Q) &= 0; \\
\frac{\partial I_3(Q)}{\partial Q_r} &\neq 0,
\end{cases}
\end{align*}$$

(42f)

corresponding to a $1/4$-BPS 'small' black hole of $\mathcal{N} = 8$, $D = 5$ supergravity.
Thus, at the level of charge orbits, equations (42) correspond to the following picture:

\[
\mathcal{O}_{1/2\text{-}BPS, p=1, D=6} = \frac{SO(5, 5; \mathbb{R})}{SO(4, 4; \mathbb{R}) \times \mathbb{R}^8}
\]

(43)

Indeed, at the level of the semi-simple part of the orbit stabilizers, the following embeddings trivially hold:

\[
SO(4, 4; \mathbb{R}) \subseteq \text{max}_{\text{symm}} SO(5, 4; \mathbb{R}) \subseteq \text{max}_{\text{symm}} SO(5, 5; \mathbb{R});
\]

SO(4, 4; \mathbb{R}) \times SO(1, 1; \mathbb{R}) \subseteq \text{max}_{\text{symm}} SO(5, 5; \mathbb{R}).

(44)

At the level of corresponding moduli spaces, (43) implies that

\[
\mathcal{M}_{1/2\text{-}BPS, p=1, D=6} = \frac{SO(4, 4; \mathbb{R})}{SO(4, \mathbb{R}) \times SO(4, \mathbb{R})} \times \mathbb{R}^8
\]

satisfies (the embedding in the second line being trivial; see also (89)) [1]

\[
\mathcal{M}_{1/2\text{-}BPS, p=1, D=6} \subseteq \mathcal{M}_{1/4\text{-}BPS, D=5} = \mathcal{M}_{1/4\text{-}BPS, p=1, D=6} \times \mathbb{R}^{16}
\]

\[
\subseteq \mathcal{M}_{1/2\text{-}BPS, D=5} = M_{D=5} \times \mathbb{R}^{16} = \frac{SO(5, 5; \mathbb{R})}{SO(5, \mathbb{R}) \times SO(5, \mathbb{R})} \times \mathbb{R}^{16}.
\]

Thus, it follows that a ‘small’ 1/2-BPS black hole, as well as a ‘small’ 1/4-BPS black hole, of \( \mathcal{N} = 8 \), \( D = 5 \) supergravity can be uplifted to a ‘small’ 1/2-BPS dyonic black string of \( \mathcal{N} = (2, 2) \), \( D = 6 \) supergravity.

3.2.4. The 1/4-BPS black hole orbit under \( D = 6 \rightarrow 5 \). A representative of the 1/4-BPS ‘small’ charge orbit (29) of a black hole in \( \mathcal{N} = (2, 2) \), \( D = 6 \) supergravity is provided by

\[
p^i = 0; \quad Q_r = 0
\]

(47a)

and

\[
q_\alpha : (y^r)^{\alpha \beta} q_\alpha q_\beta \neq 0,
\]

(47b)

for at least some \( r = 1, \ldots, 10 \). By plugging (47a) and (47b) into (27a), one obtains

\[
\begin{align*}
I_3(Q) &= 0; \\
\frac{\partial I_3(Q)}{\partial Q_r} &= \frac{1}{2} (y^r)^{\alpha \beta} q_\alpha q_\beta \neq 0.
\end{align*}
\]

(47c)

where \( \frac{\partial I_3(Q)}{\partial Q_r} \) is the unique non-vanishing component of \( \frac{\partial I_3(Q)}{\partial Q_r} \).

Thus, independently of the vanishing or not of \( q_\alpha \), this case corresponds to a 1/4-BPS ‘small’ black hole of \( \mathcal{N} = 8 \), \( D = 5 \) supergravity.

At the level of charge orbits, equations (47) depict the following situation:

\[
\mathcal{O}_{1/4\text{-}BPS, p=0, D=6} = \frac{SO(5, 5; \mathbb{R})}{SO(3, 4; \mathbb{R}) \times \mathbb{R}^8}
\]

(48)

\[
\mathcal{O}_{1/4\text{-}BPS, D=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 4; \mathbb{R}) \times \mathbb{R}^{16}}.
\]
Indeed, at the level of the semi-simple part of the orbit stabilizers, the following embedding trivially holds:

\[
SO(3, 4; \mathbb{R}) \times \left\{ \begin{array}{l}
SO(1, 1; \mathbb{R}) \\
SO(2, \mathbb{R})
\end{array} \right\} \subset_{\text{symm}}^{\text{max}} SO(5, 4; \mathbb{R}).
\]  

(49)

At the level of corresponding moduli spaces, (48) implies that

\[
\mathcal{M}_{1/4\text{-BPS, small, } p=0,2, \mathcal{D}=6} = \frac{SO(3, 4; \mathbb{R})}{SO(1, 1; \mathbb{R}) \times SO(4, \mathbb{R})} \times \mathbb{R}^8
\]

(50)

satisfies (see also (89)) [1]

\[
\mathcal{M}_{1/4\text{-BPS, small, } p=0,2, \mathcal{D}=6} \subseteq \mathcal{M}_{1/4\text{-BPS, large, } p=1, \mathcal{D}=6} = \frac{SO(5, 4; \mathbb{R})}{SO(5, \mathbb{R}) \times SO(4, \mathbb{R})} \times \mathbb{R}^{16}.
\]  

(51)

Thus, it follows that a ‘small’ 1/4-BPS black hole of \( \mathcal{N} = 8 \), \( \mathcal{D} = 5 \) supergravity can be uplifted to a ‘small’ 1/4-BPS black hole of \( \mathcal{N} = (2, 2) \), \( \mathcal{D} = 6 \) supergravity.

3.2.5. The 1/2-BPS black hole orbit under \( \mathcal{D} = 6 \rightarrow 5 \). A representative of the 1/2-BPS ‘small’ charge orbit (32) of a black hole in \( \mathcal{N} = (2, 2) \), \( \mathcal{D} = 6 \) supergravity is provided by

\[
P^i = 0;
\]

\[
\Omega_r = 0
\]

(52a)

and

\[
q_\alpha : (\gamma^r)^{\alpha\beta} q_\alpha q_\beta = 0, \quad \forall r.
\]

(52b)

By plugging (52a) and (52b) into (27a), one obtains

\[
\begin{cases}
I_3(Q) = 0; \\
\frac{\partial I_3(Q)}{\partial Q_r} = \frac{1}{2} (\gamma^r)^{\alpha\beta} q_\alpha q_\beta = 0 \iff \frac{\partial I_3(Q)}{\partial Q_t} = 0.
\end{cases}
\]

(52c)

Thus, independently on the vanishing or not of \( q_t \), this case corresponds to a 1/2-BPS ‘small’ black hole of \( \mathcal{N} = 8 \), \( \mathcal{D} = 5 \) supergravity.

At the level of charge orbits, equations (52) depict the following situation:

\[
\mathcal{O}_{1/2\text{-BPS, small, } p=0,2, \mathcal{D}=6} = \frac{SO(5, 5; \mathbb{R})}{SL(5, \mathbb{R}) \times \mathbb{R}^{16}} \downarrow
\]

\[
\mathcal{O}_{1/2\text{-BPS, } \mathcal{D}=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 5; \mathbb{R}) \times \mathbb{R}^{16}}.
\]

(53)

Indeed, at the level of semi-simple part of the stabilizers of orbits, the following embedding trivially holds:

\[
SL(5, \mathbb{R}) \times SO(1, 1; \mathbb{R}) \subset_{\text{symm}}^{\text{max}} SO(5, 5; \mathbb{R}).
\]

(54)

At the level of corresponding moduli spaces, (53) implies that

\[
\mathcal{M}_{1/2\text{-BPS, small, } p=0,2, \mathcal{D}=6} = \mathcal{M}_{\mathcal{D}=7} \times \mathbb{R}^{10} = \frac{SL(5, \mathbb{R})}{SO(5, \mathbb{R})} \times \mathbb{R}^{10}
\]

(55)

satisfies [1]

\[
\mathcal{M}_{1/2\text{-BPS, small, } p=0,2, \mathcal{D}=6} \subseteq \mathcal{M}_{1/2\text{-BPS, } \mathcal{D}=5} = \mathcal{M}_{\mathcal{D}=6} \times \mathbb{R}^{16},
\]

(56)
which is a trivial consequence of the embedding between the scalar manifolds of maximal supergravity in $D = 6$ and $D = 7$.

Thus, it follows that a ‘small’ $1/2$-BPS black hole of $\mathcal{N} = 8$, $D = 5$ supergravity can be uplifted to a ‘small’ $1/2$-BPS black hole of $\mathcal{N} = (2, 2)$, $D = 6$ supergravity.

Summarizing the embeddings of moduli spaces (40), (46), (51) and (56), related to the various $D = 6 \rightarrow 5$-dimensional reductions considered in sections 3.2.2–3.2.5, the following result is achieved [1]:

$$\mathcal{M}_{1/4\text{-BPS, small}, p=0, \mathcal{D}=6} \subseteq_{\text{max \ symm}} \mathcal{M}_{1/2\text{-BPS, small}, p=1, \mathcal{D}=6} \subset \mathcal{M}_{1/4\text{-BPS, D=5}}$$

$$\mathcal{M}_{1/8\text{-BPS, D=5}} \subseteq_{\text{max \ symm}} \mathcal{M}_{1/2\text{-BPS, small}, p=0, \mathcal{D}=6} \supset \mathcal{M}_{1/2\text{-BPS, small}, \mathcal{D}=6} \setminus \mathcal{M}_{1/8\text{-BPS, D=5}} \setminus \mathcal{M}_{1/2\text{-BPS, small}, p=0, \mathcal{D}=6}.$$  \hspace{1cm} (57)

### 3.3. U-duality orbits of $E_{6(6)}(\mathbb{Z})$

For quantized charges the continuous $U$-duality is broken to an infinite discrete subgroup, which for $D = 5$ is given by $E_{6(6)}(\mathbb{Z}) \subset E_{6(6)}(\mathbb{R})$ [9]. The integral Jordan algebra $3_{\mathbb{Z}}$ of integral split octonionic $3 \times 3$ Hermitian matrices provides a natural model for $E_{6(6)}(\mathbb{Z})$, which may used to analyse the discrete $U$-duality orbits. See [23–25] and appendix D for further details. The quantized black hole charge vector is given by

$$Q = \begin{pmatrix} q_1 & Q_s & Q_v \\ Q_v & q_2 & Q_s \\ Q_s & q_3 & Q_v \end{pmatrix}, \quad \text{where } q_1, q_2, q_3 \in \mathbb{Z} \quad \text{and} \quad Q_v, s, c \in O_{\mathbb{Z}}.$$  \hspace{1cm} (58)

The discrete group $E_{6(6)}(\mathbb{Z})$ is defined by the set of norm-preserving invertible $\mathbb{Z}$-linear transformations:

$$\left\{ \sigma : 3_{\mathbb{Z}} \rightarrow 3_{\mathbb{Z}} \left| N_3(\sigma(Q)) = N_3(Q) \right. \right\}.$$  \hspace{1cm} (59)

It is with this framework that we shall study the discrete $U$-duality orbits.

As was the case in $D = 6$, the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup $E_{6(6)}(\mathbb{Z})$ and, hence, those states unrelated by $U$-duality in the classical theory remain unrelated in the quantum theory. There are three disjoint classes of orbits, $1/2$-BPS, $1/4$-BPS and $1/8$-BPS, corresponding to the three continuous orbits. However, each of these classes is broken up into a countably infinite set of discrete orbits [26]. To classify these orbits Krutelevich used $E_{6(6)}(\mathbb{Z})$ to bring an arbitrary charge vector into a diagonally reduced canonical form, which is uniquely defined by the following set of three discrete invariants:

$$c_1(Q) := \gcd(Q),$$

$$c_2(Q) := \gcd(Q^2),$$

$$c_3(Q) := N_3(Q).$$  \hspace{1cm} (60)

See [25] and appendix D.3 for details. Note that $c_2$ is only well defined for black holes preserving less than $1/2$ the supersymmetries and hence only enters the discrete orbit.
classification for 1/4-BPS and 1/8-BPS states. Further, the invariants $c_i$ appear in the 1/8-BPS degeneracy formula, for primitive states, of [12].

$D = 5$ diagonally reduced canonical form (Krutelevich [25]). Every element $Q \in \mathcal{O}_s^3(Z)$ is $E_{6(6)}(Z)$ equivalent to a diagonally reduced canonical form

$$Q_{\text{can}} = k \begin{pmatrix} 1 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & lm \end{pmatrix}, \quad \text{where} \quad k > 0, \quad l \geq 0.$$ (61)

The canonical form is uniquely determined by (60) since

$$c_1(Q_{\text{can}}) = k,$$
$$c_2(Q_{\text{can}}) = k^2 l,$$
$$c_3(Q_{\text{can}}) = k^3 l^2 m.$$ (62)

So for arbitrary $Q$ one obtains $k = c_1(Q), l = k^{-2} c_2(Q)$ and $m = k^{-3} l^{-2} c_3(Q)$.

3.4. $D = 5$ black hole orbit classification

1. The complete set of distinct 1/2-BPS charge vector orbits is given by

$$\left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{where} \quad k > 0 \right\}.$$ (63)

2. The complete set of distinct 1/4-BPS charge vector orbits is given by

$$\left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{where} \quad k, l > 0 \right\}.$$ (64)

3. The complete set of distinct 1/8-BPS charge vector orbits is given by

$$\left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & klm \end{pmatrix}, \text{where} \quad k, l, |m| > 0 \right\}.$$ (65)

4. Black holes in $D = 4$

4.1. U-duality orbits of $E_{7(7)}(\mathbb{R})$

In the classical supergravity limit the 28+28 electric/magnetic black hole charges $x_I$ ($I = 1, \ldots, 56$ throughout) transform as the fundamental 56 of the continuous U-duality group $E_{7(7)}(\mathbb{R})$. Under $SO(1, 1; \mathbb{R}) \times E_{6(6)}(\mathbb{R})$ the 56 breaks as

$$56 \rightarrow 1_3 + 1_{-3} + 27_1 + 27_{-1}.$$ (66)

where the singlets may be identified as the graviphoton charge and its electromagnetic dual descending from $D = 5$. In this basis the charges $x_I$ may be conveniently represented as an element $x$ of the FTS $\mathfrak{M}(\mathcal{O}_s^3)$,

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathcal{O}_s^3.$$ (67)
Here, $p^0, q_0$ are the graviphotons and $P, Q$ are the magnetic/electric $2^7$ and $27$ respectively. The quartic norm is given by
\[
\Delta(x) = -[p^0 q_0 + \text{Tr}(P, Q)]^2 + 4[q_0 N(P) - p^0 N(Q) + \text{Tr}(P^2, Q^2)].
\] (68)

See appendices D and E for the necessary definitions.

The set of invertible linear transformations leaving the quartic norm and the antisymmetric bilinear form invar (E.4a) invariant is nothing but the $D = 4 U$-duality group $E_{7(7)}(\mathbb{R})$. Moreover,
\[
\Delta(x) = I_4(x),
\] (69)
where
\[
I_4(Q) = d^{IJKL} x_I x_J x_K x_L
\] (70)
and $d^{IJKL}$ is the $E_{7(7)}(\mathbb{R})$ invariant tensor. The black hole entropy is simply given by the quartic norm
\[
S_{D=4, BH} = \pi \sqrt{\Delta(x)} = \pi \sqrt{|I_4(x)|}. 
\] (71)

In this case there are five $U$-duality orbits, three $1/2$-BPS, $1/4$-BPS, $1/8$-BPS (‘small’) orbits and two ‘large’ orbits, one $1/8$-BPS and other non-BPS depending on the sign of the unique quartic $E_{7(7)}(\mathbb{Z})$ invariant [5]. These orbits may be distinguished by the FTS rank of
\[
\begin{array}{c|c}
\text{Rank} & \text{Conditions} \\
1 & x \neq 0, \exists y \text{ s.t. } 3T(x, x, y) + x\{x, y\} = 0 \forall y, 2T(x, x, x) = 0 \quad 1/2-\text{BPS}, \\
2 & \Delta(x) = 0, 1/4-\text{BPS}, \\
3 & \Delta(x) > 0, 1/8-\text{BPS}, \\
4 & \Delta(x) < 0, \text{non-BPS}.
\end{array}
\] (72)

where $T(x, x, x)$ is the triple product (E.4a) and $\{x, y\}$ is the antisymmetric bilinear form (E.4d). See appendix E for details.

Note that $T(x, x, x)$ transforms as a $56$ under $E_{7(7)}$. Similarly
\[
\partial_I I_4(Q) = \frac{\partial I_4(Q)}{\partial x_I} = 2d^{IJKL} x_J x_K x_L
\] (73)
manifestly transforms as a $56$ under $E_{7(7)}$, so that $2T(x, x, x) = \partial_I I_4(Q)$. Moreover, $3T(x, x, y) + x\{x, y\}$ vanishes for all $y$ if and only if the $133$ in $56 \times 56$ vanishes. Hence, the FTS rank conditions are equivalent to those originally used in [48]:
\[
\begin{array}{c|c}
\text{Rank} & \text{Conditions} \\
1 & x \neq 0, \partial_I I_4(x)|_{133} = 0 \quad 1/2-\text{BPS}, \\
2 & \partial_I I_4(x)|_{133} = 0, \partial_I I_4(x) = 0 \quad 1/4-\text{BPS}, \\
3 & \partial_I I_4(x) = 0, I_4(x) = 0 \quad 1/8-\text{BPS}, \\
4 & I_4(x) > 0 \quad 1/8-\text{BPS}, \\
4 & I_4(x) < 0 \quad \text{non-BPS}.
\end{array}
\] (74)

The orbits with their rank conditions, dimensions and representative states are summarized in table 4.

4.2. $D = 4 \leftrightarrow D = 5$ relations for charge orbits and moduli spaces

The decomposition (66) of the $56$ irrep. of $D = 4 U$-duality $E_{7(7)}(\mathbb{R})$ with respect to $D = 5 U$-duality $E_{6(6)}(\mathbb{R}) \times SO(1, 1; \mathbb{R}))$ [5, 6, 20, 48, 51] corresponds to the following branching of the $D = 4$ charge vector $x_I$:
\[
x_I \rightarrow (q_0, Q^i, P^i, p^0).
\] (75)
Table 4. $D = 4$ black hole orbits, their corresponding rank conditions, dimensions and SUSY.

| Rank | Non-vanishing ($\exists y$ s.t.) | Vanishing ($\forall y$) | Rep state | Orbit dim | SUSY |
|------|----------------------------------|-------------------------|-----------|-----------|------|
| 1    | $x$                              | $3T(x, x, y) + x[x, y]$ | $1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $E_{27} \otimes 
abla_{R^4}$ | 28 1/2 |
| 2    | $3T(x, x, y) + x[x, y]$          | $T(x, x, x)$            | $1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $E_{27} \otimes 
abla_{R^4}$ | 45 1/4 |
| 3    | $T(x, x, x)$                     | $\Delta(x)$            | $0 \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 \end{pmatrix}$ | $E_{27} \otimes 
abla_{R^4}$ | 55 1/8 |
| 4    | $\Delta(x) > 0$                 | -                       | $1 \begin{pmatrix} 1 & 1 & k \\ 0 & 0 \end{pmatrix}$ | $E_{27} \otimes 
abla_{R^4}$ | 55 1/8 |
| 5    | $\Delta(x) < 0$                 | -                       | $1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & k \end{pmatrix}$ | $E_{27} \otimes 
abla_{R^4}$ | 55 0  |

Thus, the unique invariant $I_4$ of the 56 can be written in a manifestly $E_6(R)$-invariant way as follows [5] (cf equation (68)):\[
I_4 = -(p^0 q_0 + P^i Q_i)^2 + 4 \left[ q_0 I_3 (P) - p^0 I_3 (Q) + \frac{\partial I_3 (Q)}{\partial Q_i} \frac{\partial I_3 (P)}{\partial P^i} \right],\] (76)

where $q_0$ and $p^0$ respectively are the electric and magnetic charge of the $d = 5 \rightarrow 4$ Kaluza–Klein vector, and the unique cubic electric (magnetic) invariant of the $27 (27')$ irrep. of $E_6(\mathbb{R})$ is defined as follows [5, 48]:\[
I_3 (q) := \frac{1}{3!} \delta^{ijk} Q_i Q_j Q_k; \quad (77a)\]
\[
I_3 (p) := \frac{1}{3!} \delta^{ijk} P^i P^j P^k. \quad (77b)\]

By recalling the continuous $U$-invariant definitions of the ‘large’ and ‘small’ charge orbits of $\mathcal{N} = 8$ supergravity in $D = 4$ and $D = 5$ [5, 6, 52], we are now going to elucidate the various relations among them (and the consequences for the related moduli spaces), by performing a $D = 5 \rightarrow 4$ spacelike dimensional reduction, with vanishing or non-vanishing Kaluza–Klein charges. See also [20, 53, 54].

4.2.1. The 1/8-BPS orbit under $D = 5 \rightarrow 4$. A representative of a 1/8-BPS charge orbit of $\mathcal{N} = 8$, $D = 5$ supergravity [5, 6] $\mathcal{O}_{1/8\text{-BPS}, D=5} = \frac{E_{27}(\mathbb{R})}{F_{24}(\mathbb{R})}$ (78)
is provided by a black hole with (‘large’ three-charge solution)
\[
P^i = 0; \quad (79a)\]
\[
Q_i : I_3 (Q) \neq 0, \quad (79b)\]

with non-vanishing Bekenstein–Hawking entropy\footnote{Note that in $D = 5$ $\text{sgn} (I_3)$ is not relevant since it flips under CPT transformations. The relative signs of the three charges in the diagonally reduced form, and hence $\text{sgn} (I_3)$, play no role—all large black holes are 1/8-BPS [5, 48].} given by equation (22).
By setting
\[ q_0 = 0 \]  \hspace{1cm} \text{(79b)}
and plugging (79a) into (76), one obtains
\[ I_4 = -4p^0 I_3(Q). \]  \hspace{1cm} \text{(79c)}

Depending on the vanishing or not of the magnetic charge of the Kaluza–Klein vector, the treatment splits as follows:

1. In the case \( p^0 = 0 \),
equation (79c) yields
\[
\begin{align*}
I_4 &= 0; \\
\frac{\partial I_4}{\partial p^0} &= 4I_3(Q) \neq 0,
\end{align*}
\]  \hspace{1cm} \text{(79e)}
corresponding to a 1/8-BPS ‘small’ black hole of \( \mathcal{N} = 8, \ D = 4 \) supergravity.

2. On the other hand, if \( p^0 \neq 0 \),
then equation (79c) yields
\[
I_4 \begin{cases} > 0 & \text{if } \text{sgn}(p^0 I_3(Q)) = -1; \\ < 0 & \text{if } \text{sgn}(p^0 I_3(P)) = 1,
\end{cases}
\]  \hspace{1cm} \text{(79g)}
thus corresponding to a 1/8-BPS \( (I_4 > 0) \) or to a non-BPS \( Z_{AB} \neq 0 \) \( (I_4 < 0) \) ‘large’ black hole of \( \mathcal{N} = 8, \ D = 4 \) supergravity.

Therefore, at the level of charge orbits, equations (79) correspond to the following picture:

\[
\begin{array}{ccc}
\mathcal{O}_{1/8-\text{BPS, small, } D=4} & \uparrow & \mathcal{O}_{1/8-\text{BPS, small, } D=5} \\
\mathcal{O}_{1/8-\text{BPS, large, } D=4} & \searrow & \mathcal{O}_{1/8-\text{BPS, large, } D=5} \\
\mathcal{O}_{\text{non-BPS, } D=4} & \leftarrow & \mathcal{O}_{\text{non-BPS, } D=5}
\end{array}
\]  \hspace{1cm} \text{(80)}

Indeed, at the level of the semi-simple part of the orbit stabilizers, it holds that
\[
F_{4(4)}(\mathbb{R}) \subseteq_{\text{symm}}^\text{max} E_{6(2)}(\mathbb{R};) \\
E_{4(4)}(\mathbb{R}) \subseteq_{\text{symm}}^\text{max} E_{6(6)}(\mathbb{R}).
\]  \hspace{1cm} \text{(81)}

At the level of corresponding moduli spaces, (80) implies that
\[
\mathcal{M}_{1/8-\text{BPS, } D=5} = \frac{F_{4(4)}(\mathbb{R})}{USp(6, \mathbb{R}) \times USp(2, \mathbb{R})} \\
\mathcal{M}_{1/8-\text{BPS, } D=4} \subseteq \left\{ \begin{array}{c}
\mathcal{M}_{\text{non-BPS, } D=4} = M_{D=5} = \frac{E_{6(6)}(\mathbb{R})}{USp(8, \mathbb{R})}; \\
\mathcal{M}_{1/8-\text{BPS, large, } D=4} = \frac{E_{6(2)}(\mathbb{R})}{SU(6, \mathbb{R}) \times SU(2, \mathbb{R})};
\end{array} \right.
\]  \hspace{1cm} \text{(83)}

9 Note that \( \text{sgn}(I_4) \) is a U-duality invariant quantity which distinguishes between 1/8-BPS and non-BPS large black holes [5, 48].
yielding the following relation:

\[ \mathcal{M}_{1/8\text{-BPS, } D=5} \subset [M_{D=5} \cap \mathcal{M}_{1/8\text{-BPS, large, } D=4}] \tag{84} \]

Thus, it follows that a ‘small’ 1/8-BPS black hole, as well as a ‘large’ 1/8-BPS and a ‘large’ non-BPS black hole, of \( \mathcal{N} = 8, D = 4 \) supergravity can be uplifted to a ‘large’ 1/8-BPS black hole of \( \mathcal{N} = 8, D = 5 \) supergravity.

4.2.2. The 1/4-BPS orbit under \( D = 5 \rightarrow 4 \). A representative of 1/4-BPS charge orbit of \( \mathcal{N} = 8, D = 5 \) supergravity \([5, 6]\)

\[ O_{1/4\text{-BPS, } D=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 4; \mathbb{R}) \rtimes \mathbb{R}^{15}} \tag{85} \]

is provided by a black hole with (‘small’ two-charge solution)

\[ P^i = 0; \]

\[ Q_i : \begin{cases} I_3(Q) = 0; \\ \frac{\partial I_3(Q)}{\partial Q_i} = \frac{1}{2} \delta^{ij} Q_j Q_k \neq 0, \end{cases} \tag{86a} \]

thus with vanishing Bekenstein–Hawking entropy given by equation (22).

By setting (79b) and plugging (86a) into (76), one obtains

\[ \begin{cases} I_4 = 0; \\ \frac{\partial I_4}{\partial Q_i} = 4 p^0 \frac{\partial I_3(Q)}{\partial Q_i}, \end{cases} \tag{86b} \]

where \( \frac{\partial I_4}{\partial Q_i} \) is the unique possibly non-vanishing component of \( \frac{\partial I_4}{\partial Q_i} \).

Depending on the vanishing or not of the magnetic charge of the Kaluza–Klein vector, the treatment splits as follows:

(1) In the case \( p^0 = 0 \),

equation (86b) yields

\[ \begin{cases} I_4 = 0; \\ \frac{\partial I_4}{\partial x_i} = 0, \\ \frac{\partial^2 I_4}{\partial Q_i \partial p^0} = 4 \frac{\partial I_3(Q)}{\partial Q_i} \neq 0, \end{cases} \tag{86d} \]

where \( \frac{\partial^2 I_4}{\partial Q_i \partial p^0} \) is the unique non-vanishing component of \( \frac{\partial^2 I_4}{\partial Q_i \partial p^0} \). Therefore, this case corresponds to a 1/4-BPS ‘small’ black hole of \( \mathcal{N} = 8, D = 4 \) supergravity.

(2) On the other hand, for \( p^0 \neq 0 \),

equation (86b) yields

\[ \begin{cases} I_4 = 0; \\ \frac{\partial I_4}{\partial x_i} \neq 0, \end{cases} \tag{86f} \]

corresponding to a ‘small’ 1/8-BPS black hole of \( \mathcal{N} = 8, D = 4 \) supergravity.
Therefore, at the level of charge orbits, equations (86) correspond to the following picture:

\[
\mathcal{O}_{1/4\text{-}BPS,D=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 4; \mathbb{R}) \times \mathbb{R}^{16}} \\
\mathcal{O}_{1/4\text{-}BPS,D=4} = E_{7(7)}(\mathbb{R}) \\
\mathcal{O}_{1/8\text{-}BPS,small,D=4} = \frac{E_{7(7)}(\mathbb{R})}{F_{4(4)}(\mathbb{R}) \times \mathbb{R}^{26}}.
\]

Indeed, at the level of the semi-simple part of the orbit stabilizers, it holds that

\[
SO(5, 4; \mathbb{R}) \subseteq \text{max} \quad \text{symm} \quad F_{4(4)}(\mathbb{R}); \\
SO(5, 4; \mathbb{R}) \times \left\{ \begin{array}{l}
SO(2, \mathbb{R}) \quad \text{symm} \quad SO(6, 5; \mathbb{R}) \\
SO(1, 1; \mathbb{R}) \quad \text{symm} \quad SO(6, 5; \mathbb{R})
\end{array} \right\}.
\]

At the level of corresponding moduli spaces, (87) implies that

\[
\mathcal{M}_{1/4\text{-}BPS,D=5} = \frac{SO(5, 4; \mathbb{R})}{SO(5, \mathbb{R}) \times SO(4, \mathbb{R}) \times \mathbb{R}^{16}}
\]

satisfies

\[
\mathcal{M}_{1/4\text{-}BPS,D=5} \subseteq \left\{ \begin{array}{l}
\mathcal{M}_{1/4\text{-}BPS,D=4} = \frac{SO(6, 5; \mathbb{R})}{SO(6, \mathbb{R}) \times SO(5, \mathbb{R}) \times \mathbb{R}^{12} \times \mathbb{R}}; \\
\mathcal{M}_{1/8\text{-}BPS,small,D=4} = \mathcal{M}_{1/8\text{-}BPS,D=5} \times \mathbb{R}^{26} = \frac{E_{6(6)}(\mathbb{R})}{SU(6, \mathbb{R}) \times SU(3, \mathbb{R})} \times \mathbb{R}^{26},
\end{array} \right.
\]

yielding the following relation:

\[
\mathcal{M}_{1/4\text{-}BPS,D=5} \subset \left\{ \mathcal{M}_{1/4\text{-}BPS,D=4} \cap \mathcal{M}_{1/8\text{-}BPS,small,D=4} \right\}.
\]

Thus, it follows that a ‘small’ 1/4-BPS black hole, as well as a ‘small’ 1/8-BPS black hole, of \(N = 8, D = 4\) supergravity can be uplifted to a ‘small’ 1/4-BPS black hole of \(N = 8, D = 5\) supergravity.

### 4.2.3. The 1/2-BPS orbit under \(D = 5 \to 4\).

A representative of a 1/2-BPS charge orbit of \(N = 8, D = 5\) supergravity [5, 6]

\[
\mathcal{O}_{1/2\text{-}BPS,D=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 5; \mathbb{R}) \times \mathbb{R}^{16}}
\]

is provided by a black hole with (‘small’ one-charge solution)

\[
P^i = 0; \\
Q_i : \begin{cases}
I_3(Q) = 0; \\
\frac{\partial I_3(Q)}{\partial Q_i} = \frac{1}{2} \delta^{ijk} Q_j Q_k = 0,
\end{cases}
\]

with

\[
\frac{\partial^2 I_3(Q)}{\partial Q_i \partial Q_j} = \delta^{ijk} Q_k \neq 0
\]

for at least some \(i \in \{1, \ldots, 27\}\), due to the non-triviality of the background under consideration. As given by equation (22), the resulting Bekenstein–Hawking entropy vanishes.
By plugging (93a) into (76), one obtains
\[
\begin{align*}
I_4 &= 0; \\
\frac{\partial I_4}{\partial Q_i} &= 0; \\
\frac{\partial^2 I_4}{\partial y^0 \partial Q_i} &= 4 p^0 \frac{\partial^2 I_3(Q)}{\partial Q_0 \partial Q_i}, \\
\end{align*}
\]
where \(4 p^0 \frac{\partial^2 I_3(Q)}{\partial Q_0 \partial Q_i}\) is the unique possibly non-vanishing component of \(\frac{\partial^2 I_4}{\partial y^0 \partial y^0} \bigg|_{133}\).

As above, depending on the vanishing or not of the magnetic charge of Kaluza–Klein vector, the treatment splits as follows:

1. In the case \(p^0 = 0\), equation (93c) yields
\[
\begin{align*}
I_4 &= 0; \\
\frac{\partial I_4}{\partial x_I} &= 0, \\
\frac{\partial^2 I_4}{\partial x_I \partial x_J} \bigg|_{133} &= 0.
\end{align*}
\]
Therefore, this case corresponds to a \(1/2\)-BPS ‘small’ black hole of \(N = 8, D = 4\) supergravity.

2. On the other hand, for \(p^0 \neq 0\), equation (93c) gives
\[
\begin{align*}
I_4 &= 0; \\
\frac{\partial I_4}{\partial x_I} &= 0, \\
\frac{\partial^2 I_4}{\partial x_I \partial x_J} \bigg|_{133} &\neq 0,
\end{align*}
\]
corresponding to a ‘small’ \(1/4\)-BPS black hole of \(N = 8, D = 4\) supergravity.

Thus, at the level of charge orbits, equations (93) correspond to the following picture:
\[
\mathcal{O}_{1/2\text{-BPS}, D=5} = \frac{E_{6(6)}(\mathbb{R})}{SO(5, 5; \mathbb{R}) \rtimes \mathbb{R}^{15}}
\]
\[
\mathcal{O}_{1/2\text{-BPS}, D=4} = \frac{E_{7(7)}(\mathbb{R})}{E_{6(6)}(\mathbb{R}) \rtimes \mathbb{R}^{27}}
\]
\[
\mathcal{O}_{1/4\text{-BPS}, D=4} = \frac{E_{7(7)}(\mathbb{R})}{SO(6, 5; \mathbb{R}) \rtimes \mathbb{R}^{15} \times \mathbb{R}}.
\]

Indeed, at the level of the semi-simple part of the orbit stabilizers, it holds that
\[
SO(5, 5; \mathbb{R}) \subset_{\text{symm}}^{\text{max}} SO(6, 5; \mathbb{R}); \\
SO(5, 5; \mathbb{R}) \times SO(1, 1; \mathbb{R}) \subset_{\text{symm}}^{\text{max}} E_{6(6)}(\mathbb{R}).
\]

At the level of corresponding moduli spaces, (87) implies that [1]
\[
\mathcal{M}_{1/2\text{-BPS}, D=5} = M_{D=6} \rtimes \mathbb{R}^{16} = \frac{SO(5, 5; \mathbb{R})}{SO(5, \mathbb{R}) \times SO(5, \mathbb{R}) \rtimes \mathbb{R}^{16}}
\]
\[
\mathcal{M}_{1/4\text{-BPS}, D=4} = \frac{E_{7(7)}(\mathbb{R})}{SO(6, 5; \mathbb{R}) \rtimes \mathbb{R}^{15} \times \mathbb{R}}.
\]
satisfies (the first embedding being trivial)

\[
\mathcal{M}_{1/2\text{-BPS},D=4} \subseteq \left\{ \begin{array}{l}
\mathcal{M}_{1/2\text{-BPS},D=4} = \mathcal{M}_{0\text{on-BPS},D=4} \times \mathbb{R}^{27} = M_{D=5} \times \mathbb{R}^{27} = \frac{E_6(\mathbb{R})}{U_{SP(6,\mathbb{R})}} \times \mathbb{R}^{27}, \\
\mathcal{M}_{1/4\text{-BPS},D=4}.
\end{array} \right.
\]

yielding the following relation:

\[
\mathcal{M}_{1/2\text{-BPS},D=5} \subseteq [M_{D=5} \cap \mathcal{M}_{1/4\text{-BPS},D=4}].
\]

Furthermore, equations (90) and (97) yield

\[
\mathcal{M}_{1/4\text{-BPS},D=5} \subseteq \max \mathcal{M}_{1/2\text{-BPS},D=5} \subseteq \mathcal{M}_{1/4\text{-BPS},D=4}.
\]

Thus, it follows that a ‘small’ 1/2-BPS black hole, as well as a ‘small’ 1/4-BPS black hole, of \( N = 8 \), \( D = 4 \) supergravity can be uplifted to a ‘small’ 1/2-BPS black hole of \( N = 8 \), \( D = 5 \) supergravity.

4.3. U-duality orbits of \( E_{7(7)}(\mathbb{Z}) \)

For quantized charges the continuous U-duality is broken to an infinite discrete subgroup, which for \( D = 4 \) is given by \( E_{7(7)}(\mathbb{Z}) \subseteq E_{7(7)}(\mathbb{R}) \) [9]. The integral FTS \( \mathfrak{M}(\mathbb{Z}_3^{O_7}) \) provides a natural model for \( E_{7(7)}(\mathbb{Z}) \), which may used to analyse the discrete U-duality orbits [11]. The quantized black hole charge vector is given by

\[
x = \begin{pmatrix} -q_0 \\
Q \\
p_0 \end{pmatrix}, \quad \text{where } q_0, p_0 \in \mathbb{Z} \quad \text{and} \quad Q, P \in \mathbb{Z}_3^{O_7}.
\]

Note that the quartic norm and, hence, the entropy squared are quantized. In fact, \( \Delta(x) \) is equal to either \( 4n \) or \( 4n + 1 \) for some \( n \in \mathbb{Z} \). The discrete group \( E_{7(7)}(\mathbb{Z}) \) is defined by the set of invertible \( \mathbb{Z} \)-linear transformations \( \sigma : \mathfrak{M}(\mathbb{Z}_3^{O_7}) \rightarrow \mathfrak{M}(\mathbb{Z}_3^{O_7}) \) preserving both the antisymmetric bilinear form and the quartic norm invariant,

\[
[\sigma(x), \sigma(y)] = [x, y], \quad \Delta(\sigma(x), \sigma(y), \sigma(z), \sigma(w)) = \Delta(x, y, z, w).
\]

Like the previous examples in \( D = 5, 6 \), the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup \( E_{7(7)}(\mathbb{Z}) \) and, hence, those states unrelated by U-duality in the classical theory remain unrelated in the quantum theory. There are five disjoint classes of orbits corresponding to the five continuous orbits. Three of which are the small 1/2-BPS, 1/4-BPS and 1/8-BPS classes, with vanishing \( I_4(x) \). There are two large classes of orbits, 1/8-BPS and non-BPS, as determined by the sign of \( I_4(x) \). However, each of these classes is broken up into a countably infinite set of discrete orbits. To classify these orbits Krutelevich used \( E_{7(7)}(\mathbb{Z}) \) to bring an arbitrary charge vector into a diagonally reduced canonical form. See [11] and appendix E. However, unlike the previous case this canonical form is not uniquely defined.

A partial classification of the orbits is achieved via the set of four discrete invariants:

\[
\begin{aligned}
d_1(x) &:= \gcd(x), \\
d_2(x) &:= \gcd(3T(x, x, y) + x \{x, y\}) \forall y, \\
d_3(x) &:= \gcd(T(x, x, x)), \\
d_4(x) &:= \Delta(x).
\end{aligned}
\]

Note that there are two further arithmetic invariants appearing in the literature, but are not used here, see appendix E.1.
4.3.1. $D = 4$ diagonally reduced canonical form (Krutelevich [11]). Every element $x \in \mathfrak{M}(J_{OS}^{3})$ is $E_{7/17}(\mathbb{Z})$ equivalent to a diagonally reduced canonical form
\[ x_{\text{can}} = \alpha \begin{pmatrix} 1 & k \text{ diag}(1, l, l m) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad \alpha > 0. \quad (103) \]
Note that $k \text{ diag}(1, l, l m)$ is the $D = 5$ diagonally reduced canonical form (61). From here on in we will often use $(a, b, c)$ to mean $\text{diag}(a, b, c)$.

While the $D = 4$ canonical form for a generic charge vector is not uniquely determined by the discrete invariants (102) it is uniquely specified for the subclass of black holes preserving more than $1/8$ of the supersymmetries, i.e. rank 1 and rank 2 charge vectors [11]. In this case the canonical form is simplified.

4.3.2. $D = 4$ Rank $<3$ diagonally reduced canonical form (Krutelevich 2004 [11]). Every element $x \in \mathfrak{M}(J_{OS}^{3})$ rank $< 3$ and so preserving more than $1/8$ of the supersymmetries is $E_{7/17}(\mathbb{Z})$ equivalent to a diagonally reduced canonical form
\[ \alpha \begin{pmatrix} 1 & k(1, 0, 0) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad \alpha > 0. \quad (104) \]
The simplified canonical form is uniquely determined by the two well-defined arithmetic invariants from (102), since
\[ d_{1}(Q_{\text{can}}) = \alpha, \]
\[ d_{2}(Q_{\text{can}}) = 2\alpha^{2}k, \quad (105) \]
so that for arbitrary rank 1 or 2 one obtains $\alpha = d_{1}(x)$ and $k = (\sqrt{2}\alpha)^{-2}d_{2}(x)$. This facilitates the orbit classification for such states as is described below.

4.3.3. $D = 4$ Rank $< 3$ black hole orbit classification (Krutelevich [11]).

(1) The complete set of distinct 1/2-BPS charge vector orbits is given by
\[ \left\{ \frac{\alpha}{0}, \frac{0}{0}, \text{where} \quad \alpha > 0 \right\}. \quad (106) \]

(2) The complete set of distinct 1/4-BPS charge vector orbits is given by
\[ \alpha \begin{pmatrix} 1 & k(1, 0, 0) \\ 0 & j \end{pmatrix}, \text{where} \quad \alpha, k > 0 \right\}. \quad (107) \]

4.4. Projective black holes

For black holes preserving less than $1/4$ of the supersymmetries the analysis becomes increasing complex and the orbit classification for generic charge vectors is not known. However, for a subclass of such black holes, satisfying particular arithmetic conditions, the orbit classification is known. These black holes are referred to as projective.

A black hole charge vector $x$ is said to be projective if its $U$-duality orbit contains a diagonally reduced element (103) satisfying [11, 26]
\[ \gcd(ak, a j, (ak)^{2}lm) = 1; \]
\[ \gcd(a kl, a j, (ak)^{2}lm) = 1; \]
\[ \gcd(aklm, a j, (ak)^{2}l) = 1. \quad (108) \]
One immediately sees that projectivity implies $\alpha = 1$ in the canonical form (103) and therefore $\gcd(x) = 1$. Black holes satisfying $\gcd(x) = 1$ are conventionally referred to as primitive.

While the general treatment of orbits in $D = 4$ is lacking, the orbit representatives of projective black holes have been fully classified in [11, 26].

4.4.1. $D = 4$ Projective black hole orbit classification (Krutelich 2004 [11]). Any projective black hole charge vector $x$ is $U$-duality equivalent to an element,

$$
\begin{pmatrix}
1 & (1, 1, m) \\
0 & j
\end{pmatrix},
$$

where $j \in [0, 1]$. (109)

The values of $m$ and $j$ are uniquely determined by $I_4(x)$. Further,

- $E_{7(7)}(Z)$ acts transitively on projective elements of a given norm $I_4(x)$.
- If $I_4(x)$ is a squarefree integer equal to $1$ (mod 4) or if $I_4(x) = 4n$, where $n$ is squarefree and equal to 2 or 3 (mod 4), then $x$ is projective and hence $U$-duality acts transitively.

In the projective case all black holes with the same quartic norm and hence lowest order entropy are $U$-duality related.

As already emphasized the generic case of not necessarily projective black holes is not fully understood.

5. Remark on the uplift of 1/2-BPS $D = 4$ ‘small’ black holes

Within the analysis of maximal supergravity in $D = 4, 5, 6$ in the continuous $U$-duality regime performed in sections 4.2 and 3.2, 1/2-BPS $D = 4$ ‘small’ black holes have been uplifted to 1/2-BPS $D = 4$ 5 ‘small’ black holes (section 4.2.3), which in turn have been shown to uplift to 1/2-BPS ‘small’ dyonic black strings (section 3.2.3) and 1/2-BPS ‘small’ black holes/black 2-branes (section 3.2.5) in $D = 6$.

However, within present treatment 1/2-BPS $D = 4$ ‘small’ black holes have not been related to ‘large’ solutions of Einstein equations coupled to maximal local supersymmetry. This is due to the fact that such a class of asymptotically flat solutions of $\mathcal{N} = 8$, $D = 4$ supergravity admits an uplift to a ‘large’ solution only starting from maximal $\mathcal{N} = 2$, $D = 7$ supergravity. See e.g. [47] and references therein.

On the other hand, as given by the general analysis of [48], in order to have an asymptotically flat ‘large’ solution of $\mathcal{N} = 2$, $D = 7$ supergravity, an intersecting configuration of black branes must be considered. See also [52]. Some of these configurations, with simple, factorized near-horizon geometries, have been considered in section 7 of [47].

By analysing the spectrum of asymptotically flat black $p$-branes allowed in $D$ space-time dimensions by the bound $p \leq D - 4$ [48, 55], the lowest dimensional ‘large’ solution of maximal supergravity to which 1/2-BPS $D = 4$ ‘small’ black holes can be uplifted should be provided by 1/2-BPS dyonic black 2-branes in $\mathcal{N} = 2$, $D = 8$ supergravity. See e.g. section 8 of [47] and references therein. The following one should then be given by 1/2-BPS dyonic black 3-branes (D3-branes) of type IIB maximal supergravity in $D = 10$. See also the treatment given in section 9 of [47]. We leave the interesting issue of these uplifts for future investigation.

It is also worth remarking here that, since $\mathcal{N} = 8$, $D = 4$ supergravity might be expected to be all-loop UV finite [21, 56], ‘small’ solutions within such a theory might not be expected to have their horizon stretched by quantum corrections, simply because these latter might not

10 An integer is squarefree if its prime decomposition contains no repetition.
be there. Thus, such solutions would correspond to ‘small’ states in the spectrum of the theory at the full quantum level\(^\text{11}\). However, the same does not hold for (maximal) supergravity theories in \(D \neq 4\). In particular, as mentioned in [20], the regular solutions in \(D = 10\) and \(D = 11\) are expected to receive corrections at the quantum level, because a consistent UV completion of supergravities in such space-time dimensions would be provided by superstrings and \(M\)-theory, respectively.

6. Conclusion

We have summarized our current understanding of the black hole/string charge vector orbits under the discrete \(U\)-dualities of \(\mathcal{N} = 8\) supergravity in six, five and four dimensions. The discrete orbits of both the black strings in \(D = 6\) and the black holes/strings in \(D = 5\) [25] admit a complete classification. Two distinct technical elements made this analysis tractable. First, the discrete \(U\)-duality groups, \(SO(5, 5; \mathbb{Z})\) in \(D = 6\) and \(E_6^6(\mathbb{Z})\) in \(D = 5\), may be modelled, in the sense of [23], by the integral exceptional quadratic and cubic Jordan algebras, respectively. These explicit representations, which both fundamentally rely upon the ring of integral split octonions, yielded diagonally reduced canonical forms for the charge vectors, from which the orbit representatives could, in principle, be obtained. Second, a complete list of independent arithmetic invariants, typically given by the gcd of irreps built out of the basic charge vector representations, is known. These invariants are sufficient to uniquely fix the canonical form for a given charge vector. These two features together allow for the complete classification of the discrete orbits.

- **\(D = 6\):** the black string charge vector \(Q \in \mathfrak{A}_2^{O_2}\) is \(SO(5, 5; \mathbb{Z}) := \text{Str}_0(\mathfrak{A}_2^{O_2})\) equivalent to a two-charge diagonally reduced canonical form

\[
Q_{\text{can}} = \begin{pmatrix} k & 0 \\ 0 & kl \end{pmatrix}, \quad k > 0,
\]

which is uniquely determined by the two following arithmetic invariants:

\[
\begin{align*}
\quad b_1(Q) & := \gcd(Q), \\
\quad b_2(Q) & := N_2(Q) = \det Q.
\end{align*}
\]

- **\(D = 5\):** the black hole (string) charge vector \(Q \in \mathfrak{A}_3^{O_3}\) is \(E_{6(6)}(\mathbb{Z}) := \text{Str}_0(\mathfrak{A}_3^{O_3})\) equivalent to a three-charge diagonally reduced canonical form

\[
Q_{\text{can}} = \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & klm \end{pmatrix}, \quad k > 0, l \geq 0,
\]

which is uniquely determined by the three following arithmetic invariants:

\[
\begin{align*}
\quad c_1(Q) & := \gcd(Q), \\
\quad c_2(Q) & := \gcd(Q)^5, \\
\quad c_3(Q) & := N_3(Q).
\end{align*}
\]

The analogous treatment of the four-dimensional black hole is not so transparent. The integral FTS does indeed provide an elegant and natural representation of the discrete \(U\)-duality group \(E_7(7)(\mathbb{Z})\), which again yields a diagonally reduced canonical charge vector. However, this canonical form is not uniquely determined by the known set of arithmetic \(U\)-duality

\(^{11}\) Consistence issues related to such states have been recently addressed in [20, 57].
invariants. The complete classification is known for two subcases: (1) black holes preserving more than 1/8 of the supersymmetries and (2) black holes satisfying the projectivity condition.

- $D = 4 > 1/8$-BPS: the black hole charge vector $x \in \mathfrak{M}(\mathfrak{O}^O_3)$ is $E_{8(7)}(\mathbb{Z}) := \text{Aut}(\mathfrak{M}^{\mathfrak{O}^O_3})$ equivalent to a two-charge diagonally reduced canonical form

$$x_{\text{can}} = \alpha \begin{pmatrix} 1 & (k, 0, 0) \\ 0 & 0 \end{pmatrix}, \quad \alpha > 0,$$

which is uniquely determined by the two following arithmetic invariants:

$$d_1(x) := \gcd(x),$$

$$d_2(x) := \gcd(3T(x, x, y) + \{x, y\}x), \quad \forall y \in \mathfrak{M}(\mathfrak{O}^O_3).$$

- $D = 4$ projective: the black hole charge vector $x \in \mathfrak{M}(\mathfrak{O}^O_3)$ is $E_{7(7)}(\mathbb{Z}) := \text{Aut}(\mathfrak{M}(\mathfrak{O}^O_3))$ equivalent to a four- or five-charge diagonally reduced canonical form

$$x_{\text{projcan}} = \begin{pmatrix} 1 & (1, 1, m) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad j \in \{0, 1\}.$$

The values of $m$ and $j$ are uniquely determined by the quartic $E_{7(7)}(\mathbb{R})$ invariant, $I_4(x)$.

The orbit structure for generic 1/8-BPS and non-BPS charge vectors is far more complex. For example, it was shown in [11] that the sub-example given by the FTS defined over the cubic Jordan algebra of split complex $3 \times 3$ Hermitian matrices is equivalent to the $SL(6, \mathbb{Z})$-orbits in $\wedge^4(\mathbb{Z}^2)$. This is an example that appeared in [10], in which it was shown to be equivalent to the structure of balanced triples of ideal classes in quadratic rings. It would be interesting to understand what role, if any, these essentially number-theoretic observations might play in the physics of stringy black holes.

Evidently, there are a number of open questions. Chiefly, is it possible that the full space of four-dimensional orbits could be resolved if the complete list of independent arithmetic invariants was known? For example, we have thus far used gcd of the $133$ appearing in $56 \times 56$. What about the $1463$? Following [38, 58], we may truncate to the eight charges of the STU model [59–62], which transform as a $(2, 2, 2) \times SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. Using this truncation, the $1463$ in $56 \times 56$ reduces to the $(3, 3, 3)$ in $(2, 2, 2) \times (2, 2, 2)$. Computing the gcd of this $(3, 3, 3)$ gives the square of $d_1(x_{\text{can}})$ and, therefore, adds no additional information. To proceed further, it would serve us well to have a full classification of the independent $E_{7(7)}(\mathbb{Z})$ arithmetic invariants.

It is tempting to extend this analysis to the various supergravity theories not considered here, but none-the-less have a Jordan algebraic underpinning. In particular, one might wish to consider the series of $\mathcal{N} = 2$ ‘magic’ supergravities in $D = 5$ and $D = 4$, which have at their basis the cubic Jordan algebras defined over the four division algebras and the corresponding FTSs, respectively. However, this analysis is perhaps less well motivated, from both a physical and mathematical perspective. Physically, while the maximally supersymmetric theories considered here are expected to be protected from quantum corrections, these arguments do not in general hold for the less than maximal theories and, hence, the classical $U$-dualities would typically be destroyed by quantum anomalies. Mathematically, it is known that not every element of the Jordan algebra of integral octonionic $3 \times 3$ Hermitian matrices, corresponding to the exceptional magic supergravities, is diagonalizable [24]. Hence, the diagonally reduced canonical form used in the split octonion $\mathcal{N} = 8$ case will not generalize to these exceptional magic supergravities. Finally, returning to the maximally supersymmetric theory in six dimensions, the black hole and membrane charges transform as the spinors $16$ and $16'$ of $SO(5, 5; \mathbb{R})$, both of which may be represented as a pair of split octonions [63].
An integral structure could then be induced, as was for the string, by using the integral split octonions, again providing a natural framework with which to study the discrete $U$-duality orbits of $SO(5, 5; \mathbb{Z})$.

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Appendix A. The integral split octonions

A.1. Composition algebras

An algebra $\mathbb{A}$ defined over the reals $\mathbb{R}$ is said to be composition if it has a non-degenerate quadratic form $|\cdot| : \mathbb{A} \to \mathbb{R}$ such that for $a, b \in \mathbb{A}$,

$$|ab|^2 = |a|^2 |b|^2, \quad \forall \ a, b \in \mathbb{A}, \quad (A.1)$$

where we denote multiplicative product of the algebra by juxtaposition.

By considering $\mathbb{R} \subset \mathbb{A}$ as the scalar multiples of the identity we may decompose $\mathbb{A}$ into its real and imaginary parts $\mathbb{A} = \mathbb{R} \oplus \mathbb{A}'$, where $\mathbb{A}' \subset \mathbb{A}$ is the subspace orthogonal to $\mathbb{R}$. An arbitrary element $a \in \mathbb{A}$ may be written as $a = \Re(a) + \Im(a)$, where $\Re(a) \in \mathbb{R}$ and $\Im(a) \in \mathbb{A}'$. The conjugation operation $a \mapsto \bar{a}$ defined by scalar multiplying all elements of $\mathbb{A}'$ by $-1$ while leaving the all elements of $\mathbb{R}$ invariant satisfies

$$\bar{ab} = \bar{b}a, \quad \bar{a}a = |a|^2. \quad (A.2)$$

The natural inner product defined by $2\langle a, b \rangle = |a + b|^2 - |a|^2 - |b|^2$ is given, in terms of conjugation, by

$$\langle a, b \rangle = \Re(ab) = \Re(\bar{a}b). \quad (A.3)$$

We denote respectively the commutator and associator by $[a, b]$ and $[a, b, c]$,

$$[a, b] = ab - ba, \quad (A.4)$$

$$[a, b, c] = (ab)c - a(bc).$$

A composition algebra is said to be associative if the associator vanishes and commutative if the commutator vanishes. If the associator is an alternating function of its arguments then the algebra is said to be alternative.

A division algebra is a composition algebra satisfying the further requirement that it contains no zero-divisors,

$$ab = 0 \implies a = 0 \quad \text{or} \quad b = 0.$$

Hurwitz’s celebrated theorem states that there are only four division algebras: the reals, complexes, quaternions and octonions denoted respectively by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. These algebras are obtained by the Cayley–Dickson process. With a slight modification one can also generate their split signature cousins $\mathbb{C}^s$, $\mathbb{H}^s$ and $\mathbb{O}^s$, which are no longer division. We assume familiarity with $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and will only discuss here the most relevant case of the split octonions, $\mathbb{O}^s$. For a detailed account covering our many omissions the reader is referred to the review by Baez [65].
A.2. The (integral) split octonions

The split octonions are an eight-dimensional (non-division) composition algebra. They are both non-commutative and non-associative but are alternative. They may be generated from the split quaternions via the Cayley–Dickson process. The split quaternions are a four-dimensional (non-division) composition algebra. The three imaginary units $i, j, k$ obey the following multiplication rules:

\[ i^2 = 1, \quad j^2 = 1, \quad k^2 = -1, \]
\[ ij = -ji = k, \quad ik = -ki = j, \quad jk = -kj = i. \]  

\[ \text{(A.5)} \]

There is a convenient matrix representation of this algebra given by

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

\[ \text{(A.6)} \]

such that an arbitrary quaternion $a \in \mathbb{H}$ may be written as

\[ a = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}. \]  

\[ \text{(A.7)} \]

The norm, real part and conjugation are given by

\[ |a|^2 = \det(a), \quad 2\Re(a) = \text{tr}(a), \quad \overline{a} = -ja^Tj, \]  

\[ \text{(A.8)} \]

where $^T$ denotes the matrix transpose.

The split octonions $\mathbb{O}$ may then be defined by introducing a fourth imaginary unit $\nu$,

\[ a + b\nu, \quad a, b \in \mathbb{H}. \]  

\[ \text{(A.9)} \]

The octonionic multiplication rules are defined as per the Cayley–Dickson process,

\[ (a + b\nu)(c + d\nu) := (ac - db) + (da + b\nu)v. \]  

\[ \text{(A.10)} \]

The norm, real part and conjugation are given by

\[ |a + b\nu|^2 = \det(a) + \det(b), \quad 2\Re(a + b\nu) = \text{tr}(a), \quad \overline{a + b\nu} = \overline{a} - b\nu \]  

\[ \text{(A.11)} \]

The ring of integral split quaternions $\mathbb{H}_Z^4$ is defined as the ring of $2 \times 2$ matrices with entries in $\mathbb{Z}$. The norm and trace have integral values and the ring is closed under conjugation.

The ring of integral split octonions $\mathbb{O}_Z^8$ is then defined in the obvious manner and we may write an arbitrary integral split octonions as

\[ a + b\nu, \quad a, b \in \mathbb{H}_Z^4. \]  

\[ \text{(A.12)} \]

The norm, the trace and conjugation, as defined in (A.11), are well-defined functions taking their values in $\mathbb{Z}$ and, moreover, $\mathbb{O}_Z^8$ is a maximal order [66, 67].

Appendix B. Jordan algebras

A Jordan algebra $\mathcal{J}$ is vector space defined over a ground field $F$ (we assume $\text{char} F \neq 2, 3$ throughout) equipped with a bilinear product satisfying

\[ X \circ Y = Y \circ X, \quad X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y), \quad \forall X, Y \in \mathcal{J}. \]  

\[ \text{(B.1)} \]

While originally introduced with a view to generalizing the axiomatic basis of quantum mechanics [68–70] Jordan algebras are now largely studied in their own right and have
connections to numerous branches of mathematics, in particular exceptional Lie algebras, a fact we exploit here. For a detailed exposition of Jordan algebras and their historical development the reader is referred to [71, 72].

Using the nomenclature of [72] the subset of Jordan algebras relevant to supergravity may be divided into three types: spin factors, quadratic factors and cubic factors. All three occupy important positions in various supergravity theories [44, 45, 73]. In particular, the quadratic factors are relevant to the $D = 6, \mathcal{N} = 8$ theory and the cubic factors are relevant to the $D = 5, \mathcal{N} = 8$ theory and, accordingly, we focus on these two examples here.

While the formal description of these algebras is not strictly necessary for our purposes, we could just as well dive straight in with the relevant explicit examples, we give a brief description in the following for completeness and, more importantly, as it facilitates the definition of the FTS required for the black holes in $D = 4$.

Appendix C. Jordan algebras and 6D black strings

C.1. Quadratic Jordan algebras

A quadratic form\(^{12}\) $N_2$ on a vector space $V$ defined over a field $F$ is a homogeneous mapping from $V$ to $F$ of degree 2,

$$N_2 : V \to F \text{ s.t. } N_2(\alpha X) = \alpha^2 N_2(X) \quad \forall \alpha \in F, \ X \in V,$$

such that its linearization

$$N_2(X, Y) := N_2(X + Y) - N_2(X) - N_2(Y)$$

is bilinear. A base point is then defined as an element $c \in V$ satisfying $N_2(c) = 1$. Given a space equipped with a quadratic form and possessing a base point we can define the trace form as

$$\text{Tr}(X) := N_2(X, c).$$

A quadratic Jordan algebra $J_2$ may derived from such a space by setting the identity $1 = c$ and defining the Jordan product as

$$X \circ Y := \frac{1}{2}(\text{Tr}(X)Y + \text{Tr}(Y)X - N_2(X, Y)1).$$

On setting $X = Y$ one obtains

$$X^2 - \text{Tr}(X)X + N_2(X)1 = 0, \quad \forall X \in J_2,$$

and $J_2$ is said to be of degree 2 [72]. Moreover, on taking the trace of (C.5) one finds

$$N_2(X) = \frac{1}{2}[\text{Tr}(X^2) - \text{Tr}(X^2)],$$

which is suggestively the form of the determinant of a $2 \times 2$ matrix written in terms of the trace of powers and powers of the trace.

There are three groups of particular importance associated with such quadratic Jordan algebras.

1. The automorphism group $\text{Aut}(J_2)$ defined by the set of invertible $F$-linear transformations $\sigma$ preserving the Jordan product,

$$\sigma(X \circ Y) = \sigma(X) \circ \sigma(Y).$$

The corresponding Lie algebra is given by the set of derivations $\text{der}(J_2)$

$$D(X \circ Y) = D(X) \circ Y + X \circ D(Y), \quad \forall D \in \text{der}(J_2).$$

\(^{12}\) We avoid using the conventional notation $Q$ for the quadratic form due to the plethora of $Q$’s representing electric charges.
Table C1. \( J_2 \) ranks.

| Condition | \( X \) | \( N_2(X) \) |
|-----------|--------|-------------|
| 0         | 0      | 0           |
| 1         | \( \neq 0 \) | 0           |
| 2         | \( \neq 0 \) | \( \neq 0 \) |

(2) The structure group \( \text{Str}(J_2) \) defined by the set of invertible \( \mathbb{F} \)-linear transformations \( \sigma \) preserving the quadratic norm up to a scalar factor,

\[
N_2(\sigma(X)) = \alpha N_2(X), \quad \alpha \in \mathbb{F}.
\] (C.9)

The corresponding Lie algebra \( \mathcal{G}\text{Str}(J_2) \) is given by

\[
\mathcal{G}\text{Str}(J_2) = L(J_2) \oplus \text{det}(J_2),
\] (C.10)

where \( L(J_2) \) denotes the set of left Jordan products \( L(X) = X \circ Y \).

(3) The reduced structure group \( \text{Str}_0(J_2) \) defined by the set of invertible \( \mathbb{F} \)-linear transformations \( \sigma \) preserving the quadratic norm,

\[
N_2(\sigma(X)) = N_2(X).
\] (C.11)

The corresponding Lie algebra \( \mathcal{G}\text{Str}_0(J_2) \) is given by factoring out scalar multiples of the identity in \( L(J_2) \),

\[
\mathcal{G}\text{Str}_0(J_2) = L'(J_2) \oplus \text{det}(J_2),
\] (C.12)

where \( L'(J_2) \) denotes the set of left Jordan products by traceless elements, \( L(X) = X \circ Y \) where \( \text{Tr}(X) = 0 \).

A \( \text{Str}_0(J_2) \) invariant rank may be assigned to elements in \( J_2 \) as in table C1.

C.2. The Jordan algebra of split octonionic \( 2 \times 2 \) Hermitian matrices and black strings

Let us now focus our attention on the specific example relevant to the dyonic black strings of \( D = 6, \mathcal{N} = 8 \) supergravity. We denote by \( J_2^\mathbb{A} \) the Jordan algebra of \( 2 \times 2 \) Hermitian matrices with entries in a composition algebra \( \mathbb{A} \) defined over the field \( \mathbb{F} \). We will assume \( \mathbb{F} = \mathbb{R} \) here.

An arbitrary element may be written as

\[
X = \begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix}, \quad \text{where} \quad \alpha, \beta \in \mathbb{R} \quad \text{and} \quad a \in \mathbb{A}.
\] (C.13)

The Jordan product (C.4) is given by

\[
X \circ Y = \frac{1}{2}(XY + YX), \quad X, Y \in J_2^\mathbb{A},
\] (C.14)

where juxtaposition denotes the conventional matrix product. The quadratic norm is simply given by the determinant

\[
N_2(X) = \det X = \alpha \beta - |a|^2.
\] (C.15)

The reduced structure group is \( SO(\text{dim } \mathbb{A} + 1, 1; \mathbb{R}) \) for \( \mathbb{A} \) one of the division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \) [63, 74]. However, setting \( \mathbb{A} = \mathbb{O}^\ast \) the reduced structure becomes \( SO(5, 5; \mathbb{R}) \), the \( D = 6, \mathcal{N} = 8 \) U-duality group for real valued charges. An element of \( J_2^{\mathbb{O}^\ast} \) transforms as the vector 10 of \( SO(5, 5; \mathbb{R}) \) in 10 × 10 which we denote as \( I_5 \). The quadratic norm in this case is nothing but the quadratic singlet of \( SO(5, 5; \mathbb{R}) \) in 10 × 10 which we denote as \( I_2 \).
The 5 + 5 electric/magnetic $D = 6$ black string charges form a 10 of $SO(5, 5; \mathbb{R})$. Denoting them as $Q_r$ ($r = 1, \ldots, 10$ throughout) the quadratic invariant may be written as
\[ \eta^{rs} Q_r Q_s, \quad (C.16) \]
where $\eta^{rs}$ is the $SO(5, 5; \mathbb{R})$ metric,
\[ \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (C.17) \]

We may associate $Q_r = (p^r, q_v)$, $v = 0, \ldots, 4$, with an element $Q \in \mathbb{J}^O_{\mathbb{Z}^2}$ in the following way:
\[ Q = \begin{pmatrix} p^0 & Q_v \\ Q_v & q_0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{R} \quad \text{and} \quad Q_v \in O^5, \quad (C.18) \]

\[ Q_v = \frac{1}{2} [(p^1 + q_1)e_0 + (p^2 + q_2)e_1 + (p^3 + q_3)e_2 + (p^4 + q_4)e_3 + (p^1 - q_1)e_4 + (p^2 - q_2)e_5 + (p^3 - q_3)e_6 + (p^4 - q_4)e_7]. \quad (C.19) \]

so that
\[ N_2(Q) = \det Q = I_2(Q). \quad (C.20) \]
The leading-order black string entropy is given by
\[ S_{D=6, BS} \sim |I_2(Q)| = |N_2(Q)|. \quad (C.21) \]

C.3. Integral Jordan algebra and black strings in $D = 6, N = 8$

The corresponding integral Jordan algebra $\mathbb{J}^O_{\mathbb{Z}^2}$ is defined as the set of $2 \times 2$ Hermitian matrices defined over the ring of integral split octonions defined in appendix A.2. An arbitrary element may be written as
\[ X = \begin{pmatrix} \alpha & a \\ \overline{a} & \beta \end{pmatrix}, \quad \text{where} \quad \alpha, \beta \in \mathbb{Z} \quad \text{and} \quad a \in \mathbb{O}_Z. \quad (C.22) \]

Evidently, $\mathbb{J}^O_{\mathbb{Z}^2}$ is not a linear Jordan algebra as it is not closed under the Jordan product. It is, however, a well-defined quadratic Jordan algebra [72]. Crucially, the quadratic norm and trace form take values in $\mathbb{Z}$.

The group $SO(5, 5; \mathbb{Z})$ is defined as the set of invertible $\mathbb{Z}$-linear transformations leaving the quadratic norm invariant. Under the action of $SO(5, 5; \mathbb{Z})$ every element of $\mathbb{J}^O_{\mathbb{Z}^2}$ is related to a diagonally reduced canonical form,
\[ X_{\text{can}} = \begin{pmatrix} k & 0 \\ 0 & kl \end{pmatrix}, \quad k > 0. \quad (C.23) \]

It is not difficult to verify that that the set of invertible $\mathbb{Z}$-linear transformations $\sigma^b_{st}$ given by
\[ \sigma^b_{st}(X) = (I + bE_{st})X(I + bE_{st}), \quad (C.24) \]
where $b \in \mathbb{O}_Z$ and $E_{st}$ is a $2 \times 2$ matrix with a single non-zero unit entry in the $st$ position, leave $N_2(X)$ invariant and so belong to $SO(5, 5; \mathbb{Z})$. Explicitly, the action of $\sigma^b_{12}$ on $X$ is given by
\[ \sigma^b_{12}(X) = \begin{pmatrix} \alpha + \text{tr}(b\overline{a}) + \beta |b|^2 & a + \beta a \\ \overline{a} + \beta \overline{b} & \beta \end{pmatrix}, \quad \text{and} \quad \sigma^b_{21}(X) = \begin{pmatrix} \alpha & a + \alpha \overline{b} \\ \overline{a} + \alpha \overline{b} & \alpha + \text{tr}(b\overline{a}) + |b|^2 \end{pmatrix}. \quad (C.25) \]
Through the successive application of these transformations a generic $X$ may be put into the canonical form (C.23) by suitably modifying the iterative procedure presented in [25].

For an element $X$ of an integral Jordan algebra, an integer $d$ divides $X$, denoted by $d | X$, if $X = dX'$ with $X'$ integral. By taking the gcd of the the rank conditions (table C1) we may define the following set of arithmetic $SO(5, 5; \mathbb{Z})$ invariants:

$$b_1(X) := \gcd(X),$$

$$b_2(X) := N_2(X).$$

These are sufficient to fix the canonical form uniquely. Note that $b_1$, unlike $b_2$, is not an invariant of $SO(5, 5; \mathbb{R})$.

**Appendix D. Jordan algebras and 5D black holes**

**D.1. Cubic Jordan algebras**

There is a general prescription for constructing cubic Jordan algebras, due to Freudenthal, Springer and Tits [72, 75, 76], for which all the properties of the Jordan algebra are essentially determined by the cubic form. We sketch this construction here, following closely the conventions of [11, 72].

Let $V$ be a vector space equipped with a cubic norm, i.e. a homogeneous map of degree 3:

$$N_3 : V \to \mathbb{F}, \quad \text{s.t.} \quad N_3(\alpha X) = \lambda^3 N_3(X), \quad \forall \alpha \in \mathbb{F}, \quad X \in V$$

such that its linearization

$$N_3(X, Y, Z) := N_3(X + Y + Z) - N_3(X + Y) - N_3(X + Z)$$

$$- N_3(Y + Z) + N_3(X) + N_3(Y) + N_3(Z)$$

is trilinear. If $V$ further contains a base point $N_3(c) = 1, c \in V$, one may define the following four maps:

1. the trace,

$$\text{Tr}(X) = N_3(c, c, X), \quad \text{(D.3a)}$$

2. a quadratic map,

$$S(X) = N_3(X, X, c), \quad \text{(D.3b)}$$

3. a bilinear map,

$$S(X, Y) = N_3(X, Y, c), \quad \text{(D.3c)}$$

4. a trace bilinear form,

$$\text{Tr}(X, Y) = \text{Tr}(X) \text{Tr}(Y) - S(X, Y). \quad \text{(D.3d)}$$

A cubic Jordan algebra $\mathfrak{J}$ with multiplicative identity $\mathbb{1} = c$ may be derived from any such vector space if $N_3$ is *Jordan cubic*.

1. The trace bilinear form (D.3d) is non-degenerate.
2. The quadratic adjoint map, $\sharp : \mathfrak{J} \to \mathfrak{J}$, uniquely defined by $\text{Tr}(X^\sharp, Y) = N(X, X, Y)$, satisfies

$$\text{(X}^\sharp)^2 = N_3(X)X, \quad \forall X \in \mathfrak{J}.$$

$$\text{(D.4)}$$
Table D1. \( \mathfrak{J}_3 \) ranks.

| Condition | Rank | \( X \) | \( X^\sharp \) | \( N_3(X) \) |
|-----------|------|--------|--------|---------|
| 0         | 0    | 0      | 0      | 0       |
| 1         | ≠ 0  | 0      | 0      | 0       |
| 2         | ≠ 0  | ≠ 0    | 0      | 0       |
| 3         | ≠ 0  | ≠ 0    | ≠ 0    | 0       |

The Jordan product is then defined using

\[
X \circ Y = \frac{1}{2} \left( X \times Y + \text{Tr}(X)Y + \text{Tr}(Y)X - S(X, Y) \mathbb{1} \right),
\]

where \( X \times Y \) is the linearization of the quadratic adjoint

\[
X \times Y = (X + Y)^\sharp - X^\sharp - Y^\sharp.
\]

Finally, the Jordan triple product is defined as

\[
\{X, Y, Z\} = (X \circ Y) \circ Z + X \circ (Y \circ Z) - (X \circ Z) \circ Y.
\]

There are three groups of particular importance associated with cubic Jordan algebras.

1. The automorphism group \( \text{Aut}(\mathfrak{J}_3) \) defined by the set of invertible \( \mathbb{F} \)-linear transformations \( \sigma \) preserving the Jordan product

\[
\sigma(X \circ Y) = \sigma(X) \circ \sigma(Y).
\]

The corresponding Lie algebra is given by the set of derivations \( \text{der}(\mathfrak{J}_3) \),

\[
D(X \circ Y) = D(X) \circ Y + X \circ D(Y), \quad \forall D \in \text{der}(\mathfrak{J}_3).
\]

2. The structure group \( \text{Str}(\mathfrak{J}_3) \) defined by the set of invertible \( \mathbb{F} \)-linear transformations \( \sigma \) preserving the quadratic norm up to a scalar factor,

\[
N_3(\sigma(X)) = \alpha N_3(X), \quad \alpha \in \mathbb{F}.
\]

The corresponding Lie algebra \( \text{Str}(\mathfrak{J}_3) \) is given by

\[
\text{Str}(\mathfrak{J}_3) = L(\mathfrak{J}_3) \oplus \text{der}(\mathfrak{J}_3),
\]

where \( L(\mathfrak{J}_3) \) denotes the set of left Jordan products \( L_X(Y) = X \circ Y \).

3. The reduced structure group \( \text{Str}_0(\mathfrak{J}_3) \) defined by the set of invertible \( \mathbb{F} \)-linear transformations \( \sigma \) preserving the quadratic norm,

\[
N_3(\sigma(X)) = N_3(X).
\]

The corresponding Lie algebra \( \text{Str}_0(\mathfrak{J}_3) \) is given by factoring out scalar multiples of the identity in \( L(\mathfrak{J}_3) \),

\[
\text{Str}_0(\mathfrak{J}_3) = L'(\mathfrak{J}_3) \oplus \text{der}(\mathfrak{J}_3),
\]

where \( L'(\mathfrak{J}_3) \) denotes the set of left Jordan products by traceless elements, \( L_X(Y) = X \circ Y \) where \( \text{Tr}(X) = 0 \).

A \( \text{Str}_0(\mathfrak{J}_3) \) invariant rank may be assigned to elements in \( \mathfrak{J}_3 \) as in table D1.
D.2. The Jordan algebra of split octonionic $3 \times 3$ Hermitian matrices and black holes (strings)

Let us now focus our attention on the specific example relevant to the black holes (strings) of $D = 5, \mathcal{N} = 8$ supergravity. We denote by $\mathfrak{J}_A^3$ the cubic Jordan algebra of $3 \times 3$ Hermitian matrices with entries in a composition algebra $A$ defined over the field $F$. We will assume $F = \mathbb{R}$ here.

An arbitrary element may be written as

$$X = \begin{pmatrix} \alpha & c & \overline{b} \\ \overline{c} & \beta & a \\ b & \overline{a} & \gamma \end{pmatrix}, \quad \text{where} \quad \alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad a, b, c \in A. \quad (D.14)$$

The Jordan product (D.5) is given by

$$X \circ Y = \frac{1}{2}(XY + YX), \quad X, Y \in \mathfrak{J}_A^3, \quad (D.15)$$

where juxtaposition denotes the conventional matrix product. The cubic norm (D.1) is given by the determinant like object

$$N_3(X) = \alpha \beta \gamma - \alpha a \overline{a} - \beta b \overline{b} - \gamma c \overline{c} + (ab)c + \overline{c}(\overline{b} \overline{a}). \quad (D.16)$$

The trace bilinear form (D.3d) is given by the conventional matrix trace

$$\text{Tr}(X, Y) = \text{tr}(X \circ Y). \quad (D.17)$$

The quadratic adjoint (D.4) is given by

$$X^\sharp = \begin{pmatrix} \beta \gamma - |a|^2 & \overline{b} \overline{a} - \gamma c & ca - \beta \overline{b} \\ ab - \gamma \overline{c} & \alpha \gamma - |b|^2 & \overline{c} \overline{b} - \alpha a \\ \overline{a} \overline{c} - \beta b & bz - a \overline{a} & \beta \alpha - |c|^2 \end{pmatrix}. \quad (D.18)$$

The elements $X \in \mathfrak{J}_A^3$ transform as the $(3 \dim A + 3)$-dimensional representation of the reduced structure group, $\text{Str}^0(\mathfrak{J}_A^3)$. For $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, $X \in \mathfrak{J}_A^3$ transforms as the $6, 9, 15, 27$ of $\text{SL}(3, \mathbb{R}), \text{SL}(3, \mathbb{C}), \text{SU}^*(6), E_{6(-26)}(\mathbb{R})$, respectively. These are the symmetries of the magic $\mathcal{N} = 2, D = 5$ supergravities [43–45] and the electric black hole charges fall into the corresponding representations.

Setting $A = \mathbb{O}'$ the reduced structure group becomes $E_{6(6)}(\mathbb{R})$, the $D = 5, \mathcal{N} = 8$ U-duality group for real valued charges. Elements of $\mathfrak{J}^{O'}_3$ transform as the fundamental 27 of $E_{6(6)}(\mathbb{R})$. The cubic norm in this case is nothing but the cubic singlet of $E_{6(6)}(\mathbb{R})$ in $27 \times 27 \times 27$ which we denote as $I_3$. The quadratic adjoint (D.6) gives the contragradient representation 27$'$ in $27 \times 27 \times 27$ (or equally the $27$ in $27 \times 27 \times 27$). The trace bilinear form (D.3d) gives the singlet in $27 \times 27 \times 27$.

We may associate the 27 electric black hole charges with an element $Q \in \mathfrak{J}^{O'}_3$ in the following way:

$$Q = \begin{pmatrix} q_1 & Q_v & \overline{Q}_v \\ \overline{Q}_v & q_2 & Q_c \\ Q & \overline{Q}_c & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{R} \quad \text{and} \quad Q_{v,c} \in \mathbb{O}', \quad (D.19)$$

so that

$$N_3(Q) = I_3(Q). \quad (D.20)$$

The leading-order black hole entropy is given by

$$S_{D=5\text{-BH}} = \pi \sqrt{|I_3(Q)|} = \pi \sqrt{|N_3(Q)|}. \quad (D.21)$$
Similarly, we may associate the 27 magnetic black string charges with an element \( P \in J_3^{O'} \) in the following way:

\[
P = \begin{pmatrix}
p_1 & P_e & P_v \\
P_e & p_2 & P_c \\
P_v & P_c & p_3 \\
\end{pmatrix},
\]

where \( p_1, p_2, p_3 \in \mathbb{R} \) and \( P_{v,c} \in O' \). (D.22)

so that

\[
N_3(P) = I_3(P). \quad \text{(D.23)}
\]

The leading-order black string entropy is given by

\[
S_{D=5,BS} = \pi \sqrt{|I_3(P)|} = \pi \sqrt{|N_3(P)|}. \quad \text{(D.24)}
\]

**D.3. Integral cubic Jordan algebras and black holes (strings) in \( D = 5, N = 8 \)**

The corresponding integral Jordan algebra \( J_3^{O_Z} \) is defined as the set of 3 \( \times \) 3 Hermitian matrices defined over the ring of integral split octonions defined in appendix A.2 [25]. An arbitrary element may be written as

\[
X = \begin{pmatrix}
\alpha & a & b \\
\bar{a} & \beta & c \\
b & \bar{c} & \gamma \\
\end{pmatrix}, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{Z} \quad \text{and} \quad a, b, c \in O^Z_3. \quad \text{(D.25)}
\]

\( J_3^{O_Z} \) is not closed under the Jordan product, however, the cubic norm and trace bilinear form are integer valued, which are the crucial properties for our purposes. Moreover, \( J_3^{O_Z} \) is closed under the quadratic adjoint map and its linearization as required.

The group \( E_{6(6)}(\mathbb{Z}) \) is defined as the set of invertible \( \mathbb{Z} \)-linear transformations leaving the cubic norm invariant. It was shown in [25] that under the successive application of such discrete transformations every element of \( J_3^{O_Z} \) is related to a diagonally reduced canonical form

\[
X_{\text{can}} = \begin{pmatrix}
k & 0 & 0 \\
0 & kl & 0 \\
0 & 0 & klm
\end{pmatrix}, \quad k > 0, \quad l \geq 0. \quad \text{(D.26)}
\]

By taking the gcd of the the rank conditions (table D1) we may define the following set of independent arithmetic \( E_{6(6)}(\mathbb{Z}) \) invariants:

\[
c_1(X) := \gcd(X), \\
c_2(X) := \gcd(X^2), \\
c_3(X) := N_3(X). \quad \text{(D.27)}
\]

These are sufficient to fix the canonical form uniquely. Note that \( c_1 \) and \( c_2 \), unlike \( c_3 \), are not invariants of \( E_{6(6)}(\mathbb{R}) \).

**Appendix E. The freudenthal triple system and 4D black holes**

Given a cubic Jordan algebra \( J_3 \) defined over \( \mathbb{R} \), there exists a corresponding FTS given by the vector space \( \mathcal{M}(J_3) \),

\[
\mathcal{M}(J_3) = \mathbb{R} \oplus \mathbb{R} \oplus J_3 \oplus \bar{J}_3.
\]

(E.1)
An arbitrary element \( x \in \mathfrak{M}(\mathfrak{J}_3) \) may be written as a ‘2 \times 2 matrix’,
\[
x = \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{R} \quad \text{and} \quad X, Y \in \mathfrak{J}_3.
\] (E.2)

For convenience we identify the quantity
\[
\kappa(x) := \frac{1}{2}(\alpha \beta - \text{Tr}(X, Y)). \tag{E.3}
\]

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product \([11, 77–80]\).

1. Quadratic form \([\bullet, \bullet] : \mathfrak{M}(\mathfrak{J}_3) \times \mathfrak{M}(\mathfrak{J}_3) \to \mathbb{R}\)
\[
\{x, y\} = \alpha \delta - \beta \gamma + \text{Tr}(X, Z) - \text{Tr}(Y, W),
\]
where \( x = \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix}, \quad y = \begin{pmatrix} \gamma & W \\ Z & \delta \end{pmatrix}. \tag{E.4a}
\]

2. Quartic form \(\Delta : \mathfrak{M}(\mathfrak{J}_3) \to \mathbb{R}\)
\[
\Delta(x) = -4[\kappa(x) x^2 + (\alpha N(X) + \beta N(Y) - \text{Tr}(X^2, Y^2))]. \tag{E.4b}
\]

3. Triple product \(T : \mathfrak{M}(\mathfrak{J}_3) \times \mathfrak{M}(\mathfrak{J}_3) \times \mathfrak{M}(\mathfrak{J}_3) \to \mathfrak{M}(\mathfrak{J}_3)\) which is uniquely defined by
\[
\{T(x, y, w), z\} = 2\Delta(x, y, w, z), \tag{E.4c}
\]
where \(\Delta(x, y, w, z)\) is the full linearization of \(\Delta(x)\) normalized such that \(\Delta(x, x, x, x) = \Delta(x)\). The explicit form of \(T(x) = T(x, x, x)\) is given as
\[
T(x) = \begin{pmatrix} T_a & T_x \\ T_y & T_b \end{pmatrix} = 2 \begin{pmatrix} -\alpha \kappa(x) - N(Y) & -\beta \gamma^2 - \text{Tr}(X^2, Y^2) + \kappa(x)X \\ \alpha X^2 - X \times Y^2 - \kappa(x)Y & \beta \kappa(x) + N(X) \end{pmatrix}. \tag{E.4d}
\]

Note that all the necessary definitions, such as the cubic and trace bilinear forms, are inherited from the underlying Jordan algebra \(\mathfrak{J}_3\).

The automorphism group \(\text{Aut}(\mathfrak{M}(\mathfrak{J}_3))\) is given by the set of all invertible \(\mathbb{R}\)-linear transformations which leave both \([x, y]\) and \(\Delta(x, y, w, z)\) invariant \([78]\). Note that for any transformation \(\sigma \in \text{Aut}(\mathfrak{M}(\mathfrak{J}_3))\) we have
\[
\sigma(T(x, y, w)) = \sigma(T(x, y, w)). \tag{E.5}
\]

The corresponding Lie algebra is given by \([81]\)
\[
\mathfrak{Aut}(\mathfrak{M}(\mathfrak{J}_3)) = \mathfrak{J}_3 \oplus \mathfrak{J}_3 \oplus \mathfrak{str}(\mathfrak{J}_3). \tag{E.6}
\]

The FTSs, defined over various Jordan algebras, and their associated automorphism groups are summarized in table \(E1\). This table covers a number supergravities of interest: \(\mathcal{N} = 2\) \(STU\), \(\mathcal{N} = 2\) coupled to \(n\) vector multiplets; magic \(\mathcal{N} = 2\) and \(\mathcal{N} = 8\). The heterotic string with \(\mathcal{N} = 4\) supersymmetry and \(SL(2, \mathbb{R}) \times SO(6, 22; \mathbb{R})\) \(U\)-duality may also be included by using the Jordan algebra \(\mathbb{R} \oplus Q_{5,21}\) \([82, 83]\).

The conventional concept of matrix rank may be generalized to FTSs in a natural and \(\text{Aut}(\mathfrak{M}(\mathfrak{J}_3))\) invariant manner as in table \(E2\) \([11, 80]\).

Let us now focus our attention on the specific example relevant to the black holes of \(D = 4, \mathcal{N} = 8\) supergravity. We denote by \(\mathfrak{M}(\mathfrak{J}_3)\) the FTS defined over the cubic Jordan algebra of \(3 \times 3\) Hermitian matrices with entries in split octonions. Elements of \(\mathfrak{M}(\mathfrak{J}_3)\) transform as the fundamental 56 of \(E_7(\mathbb{R})\). The quartic norm in this case is nothing but the unique quartic singlet of \(E_7(\mathbb{R})\) in \(56 \times 56 \times 56 \times 56\) which we denote as \(I_4\). The triple product \((E.4d)\) gives the fundamental 56 in \(56 \times 56 \times 56\). The antisymmetric bilinear form \((E.4a)\) gives the singlet in \(56 \times 56 \times 56\).
The quartic norm and antisymmetric bilinear form are both integer valued and consequently \( J \) to place an integral structure on the FTS we use, following [11], the integral Jordan algebra \( E \). The integral FTS and 4D black holes

where \( p \) \( \in \mathbb{R} \) and define \( \mathfrak{M}(\mathfrak{J}_3^O) := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J}_3^O \oplus \mathfrak{J}_3^O \). An arbitrary element may be written as

\[
\begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{Z} \quad \text{and} \quad X, Y \in \mathfrak{J}_3^O.
\]

The quartic norm and antisymmetric bilinear form are both integer valued and consequently \( T(x, x, x) \in \mathfrak{M}(\mathfrak{J}_3^O) \). The quartic norm \( \Delta(x) \) is either \( 4n \) or \( 4n + 1 \) for some \( n \in \mathbb{Z} \).

The discrete U-duality group \( E(\mathfrak{J}_3^O) \) is defined as the set of invertible \( \mathbb{Z} \)-linear transformation preserving the quartic norm and antisymmetric bilinear form. It is generated by the following three maps [11, 78]:

\[
\begin{align*}
\phi(W) & : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + (Y, W) + (X, W^2) + \beta N(W) & X + \beta W \\ Y + X \times \beta W & \beta \end{pmatrix}, \\
\psi(Z) & : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & X + Y \times Z + \alpha Z^2 \\ Y + \alpha Z & \beta + (X, Z) + (Y, Z^2) + \alpha N(Z) \end{pmatrix}.
\end{align*}
\]

### Table E1.

| Jordan algebra \( \mathfrak{J}_3 \) | Reduced structure group \( \text{Str}(\mathfrak{J}_3) \) | \( \dim \mathfrak{J}_3 \) | \( \text{Aut}(\mathfrak{M}(\mathfrak{J}_3)) \) | \( \dim \mathfrak{M}(\mathfrak{J}_3) \) |
|---|---|---|---|---|
| \( \mathbb{R} \) | \( \mathbb{R} \) | 1 | \( SL(2, \mathbb{R}) \) | 4 |
| \( \mathbb{R} \oplus \mathbb{R} \) | \( SO(1;1;\mathbb{R}) \) | 2 | \( SL(2,\mathbb{R}) \times SL(2, \mathbb{R}) \) | 6 |
| \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \) | \( SO(1;1;\mathbb{R}) \times SO(1;1;\mathbb{R}) \) | 3 | \( SL(2,\mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) | 8 |
| \( \mathbb{R} \oplus \Gamma^* \) | \( SO(1, 1; \mathbb{R}) \) | \( n+1 \) | \( SL(2, \mathbb{R}) \times SO(2, n; \mathbb{R}) \) | \( 2n+4 \) |

We may associate the 28 + 28 electric/magnetic black hole charges with an element \( x \in \mathfrak{M}(\mathfrak{J}_3^O) \) in the following way:

\[
x = \begin{pmatrix} q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where } q_0, p^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^O.
\]

The leading-order black hole entropy is given by

\[
S = \pi \sqrt{|I_4(x)|} = \pi \sqrt{|\Delta(x)|}.
\]

#### E.1. The integral FTS and 4D black holes

To place an integral structure on the FTS we use, following [11], the integral Jordan algebra \( \mathfrak{J}_3^O \) and define \( \mathfrak{M}(\mathfrak{J}_3^O) := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J}_3^O \oplus \mathfrak{J}_3^O \).

An arbitrary element may be written as

\[
\begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{Z} \quad \text{and} \quad X, Y \in \mathfrak{J}_3^O.
\]
Table E2. $\mathcal{M}(J_3)$ ranks.

| Condition | Rank | $3T(x, x, y)$ | $T(x, x, x)$ | $\Delta(x)$ |
|-----------|------|---------------|---------------|-------------|
| $x \neq 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0$ | 0 | 0 | 0 | 0 |
| $\neq 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0$ | 1 | 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 |
| $\neq 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0$ | 2 | 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 |
| $\neq 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0$ | 3 | 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 |
| $\neq 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0$ | 4 | 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 = 0 |

\[ T(s) : \left( \begin{array}{c} \alpha \\ X \\ Y \\ \beta \end{array} \right) \mapsto \left( \begin{array}{c} \alpha \\ \Delta^{-1}(Y) \\ Y \\ \beta \end{array} \right), \quad (E.12c) \]

where $s \in \text{Str}_0(J_3^{G_2})$ and $\Delta$ is its adjoint defined with respect to the trace bilinear form, $\text{Tr}(s(X), s'(Y)) = \text{Tr}(X, Y)$.

Using these transformations it was shown in [11] that every element $x \in \mathcal{M}(J_3^{G_2})$ is equivalent to a diagonally reduced canonical form

\[ x_{\text{can}} = \alpha \left( \begin{array}{c} 1 \\ X_{\text{can}} \\ j \end{array} \right), \quad \text{where} \quad \alpha > 0. \quad (E.13) \]

Here $X_{\text{can}} = k(1, l, lm)$ is the diagonally reduced canonical form of elements in $J_3^{G_2}$. However, the uniqueness of this canonical form is not guaranteed.

For an element $x$ of an integral FTS, an integer $d$ divides $x$, denoted by $d | x$, if $x = d x'$ with $x'$ integral. By taking the gcd of the the rank conditions (table E2) we may define the following set of independent arithmetic $E_{7(7)}(\mathbb{Z})$ invariants:

\[ d_1(x) := \gcd(x), \]
\[ d_2(x) := \gcd(3T(x, x, y) + \{x, y\} x), \quad \forall y \]
\[ d_3(x) := \gcd(T(x, x, x)), \]
\[ d_4(x) := \Delta(x). \quad (E.14) \]

Note that $d_1, d_2$ and $d_3$, unlike $d_4$, are not invariants of $E_{7(7)}(\mathbb{R})$. Additionally, we may also define

\[ d'_2(x) := \gcd(P(x), Q(x), R(x)) \]
\[ d'_4(x) := \gcd(x \wedge T(x)), \quad (E.15) \]

where $\wedge$ denotes the antisymmetric tensor product. $P(x) = Y^2 - \alpha X$ and $Q(x) = X^2 - \beta Y$ are the charge combinations appearing in the 4D/5D lift [26, 84] and $R(x) : J_3 \to J_3$ is a Jordan algebra endomorphism given by

\[ R(x)(Z) = 2x'(x)Z + 2[Z, X, Y, Z], \quad (E.16) \]

where $[X, Y, Z]$ is the Jordan triple product (D.7). Taken together, $(P(x), Q(x), R(x))$ form the adjoint representation of the four-dimensional $U$-duality: $133$ in the case of $E_{7(7)}(\mathbb{Z})$. Under the five-dimensional $U$-duality, they transform as the fundamental, contragredient fundamental and adjoint representations, respectively: $27$, $27'$ and $1 + 78$ in the case of $E_{6(6)}(\mathbb{Z})$. Moreover, after subtracting the symplectic trace, $x \wedge T(x)$ transforms as the $1539$ in $56 \times_a 56$.  


Evaluated on the canonical form (E.13) one obtains
\[ \begin{align*}
\delta_1(x_{\text{can}}) &= \alpha \\
\delta_2(x_{\text{can}}) &= \alpha^2 \gcd(j, 2k) \\
\delta_2'(x_{\text{can}}) &= \alpha^2 \gcd(j, k) \\
\delta_3(x_{\text{can}}) &= \alpha^3 \gcd(j, 2k^2l) \\
\delta_4(x_{\text{can}}) &= \alpha^4 (j^2 + 4k^3l^2m) \\
\delta_4'(x_{\text{can}}) &= \alpha^4 \gcd(j, k^2l).
\end{align*} \] (E.17)

Note that \( \delta_2'(x_{\text{can}}) \) is refined compared to \( \delta_2(x_{\text{can}}) \). Unlike the \( D = 5 \) case the invariants (E.17) are insufficient to determine uniquely \( j, k, l, m \), as can be seen by taking any example with \( j = 1 \). Note, however, that \( \alpha \) is clearly fixed by \( \delta_1(x) \). Consequently, the reduced canonical form (E.13) of any given black hole is not necessarily unique and, to the best of our knowledge, there is no complete classification of the \( U \)-duality orbits. For example,
\[ \begin{align*}
x &= \alpha \begin{pmatrix} 1 & (0, 0, 0) \\ (0, 0, 0) & j \end{pmatrix}, \\
x' &= \alpha \begin{pmatrix} 1 & (j, 0, 0) \\ (0, 0, 0) & j \end{pmatrix}
\end{align*} \] (E.18)
are both in canonical form and \( U \)-duality related using \( \phi(W) \) in (E.12a) with \( W = (1, 0, 0) \).

E.2. Projective elements

An element \( x \) is said to be projective if its \( U \)-duality orbit contains a diagonally reduced element [11],
\[ \begin{align*}
x &= \begin{pmatrix} \alpha & (X_1, X_2, X_3) \\ 0 & \beta \end{pmatrix},
\end{align*} \] (E.19)
satisfying
\[ \begin{align*}
\gcd(\alpha X_1, \alpha \beta, X_2 X_3) &= 1; \\
\gcd(\alpha X_2, \alpha \beta, X_1 X_3) &= 1; \\
\gcd(\alpha X_3, \alpha \beta, X_1 X_2) &= 1.
\end{align*} \] (E.20)

The concept of a projective element was originally introduced for the case \( J_3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) along with certain generalizations central to the new view on Gauss composition and its extension as expounded in [10].

The class of projective FTS elements is of particular relevance to recent developments in number theory [10, 11]. Note that
- If \( x \) is projective, then \( \delta_2'(x) = 1 \).
- If \( \delta_3(x) = 1 \), then \( x \) is projective.
- If \( \delta_3(x) \geq 3 \) or \( T(x) = 0 \) then, \( x \) is not projective.
- When \( \Delta \) is odd, \( \delta_3(x) = 1 \) iff \( x \) is projective.

While the general treatment of the \( E_{7(7)}(\mathbb{Z}) \) orbits is lacking, the orbit representatives of projective elements have been fully classified in [11], at least for \( \mathfrak{g} = 3^\Lambda \) where \( \Lambda \) is one of the three integral split composition algebras \( \mathbb{C}' \), \( \mathbb{H}' \) or \( \mathbb{O}' \). Any projective element \( x \) is \( U \)-duality equivalent to an element [11]:

\[ \begin{align*}
x &= \alpha \begin{pmatrix} 1 & (0, 0, 0) \\ (0, 0, 0) & j \end{pmatrix}, \\
x' &= \alpha \begin{pmatrix} 1 & (j, 0, 0) \\ (0, 0, 0) & j \end{pmatrix}
\end{align*} \] (E.18)
are both in canonical form and \( U \)-duality related using \( \phi(W) \) in (E.12a) with \( W = (1, 0, 0) \).
\[
\begin{pmatrix}
1 \\
(0, 0, 0) \\
(1, 1, m) \\
j
\end{pmatrix},
\]
where the values of \( m \) and \( j \) are uniquely determined by \( \Delta(x) \).

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