Dynamics of magnetized particles around Einstein-Æther black hole with uniform magnetic field

Javlon Rayimbaev,1,2,3, * Ahmadjon Abdujabbarov,1,2,3,4, † Mubasher Jamil,6,7,8, ‡ and Wenbiao Han5, §

1Ulugh Beg Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan
2National University of Uzbekistan, Tashkent 100174, Uzbekistan
3Institute of Nuclear Physics, Ulugbek 1, Tashkent 100214, Uzbekistan
4Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, Kori Niyoziy, 39, Tashkent 100000, Uzbekistan
5Shanghai Astronomical Observatory, 80 Nandan Road, Shanghai 200030, China
6Institute for Theoretical Physics and Cosmology, Zhejiang University of Technology, Hangzhou 310023, China
7School of Natural Sciences, National University of Sciences and Technology, Islamabad 44000, Pakistan
8Canadian Quantum Research Center 204-3002 32 Ave Vernon, BC V1T 2L7 Canada

(Dated: September 11, 2020)

This work is devoted to study the effects of Einstein-Æther gravity on magnetized particles moving around a static, spherically symmetric and uncharged black hole immersed in an external asymptotically uniform magnetic field. The analysis is carried out by varying the free parameters c13 and c14 of the Einstein-Æther theory and noticing their impacts on the particle trajectories and the amount of center-of-mass energy produced as a result of collision. The strength of magnetic field and the location of the circular orbits significantly changes by varying the above free parameters. We have also made a comparison between Einstein-Æther and the Kerr black hole and noticed that both black holes depict similar behaviour for suitable choices of c13, c14, spin and the magnetic field.

PACS numbers: 04.50.-h, 04.40.Dg, 97.60.Gb

I. INTRODUCTION

Lorentz invariance is a fundamental consequence and principle of the Einstein’s special relativity theory and hence of nature itself, which is lead to a further generalization as diffeomorphism invariance in the general relativity (GR). Due to fundamental limitations of GR to describe physics at Planck scale and least understood aspect of quantization of gravity, the assumptions of GR are required to be relaxed to study physics near the Planck scale. It is now well understood that the structure of space near the Planck regime is discrete, obeys non-commutative rules of geometry, violates the Lorentz symmetry and obeys some form of generalized uncertainty principle. Among few candidates of Lorentz symmetry violating theories is the Einstein-Æther theory which is a generally-covariant theory of gravity. In order to violate the Lorentz symmetry, an Æther field (a timelike vector field) is introduced which defines a preferred timelike direction at every point of space [1, 2]. In literature, numerous aspects of this theory have been explored such as cosmological perturbations [3, 4], the effects on the generation and propagation of gravitational waves [5? ], and shadow of black holes [6], etc. The theory involves several coupling parameters such as c1’s have been constrained via astrophysical data of the gravitational wave events GW170817 and GRB 170817A [7]. The theory predicts new gravitational wave polarizations, faster than light propagation speeds of gravitational waves without violating causality in some novel ways. Furthermore, by coupling the Æther field with the electromagnetic field, two static, electrically charged, and spherical symmetric black hole solutions have been found in the Einstein-Æther theory [8, 9]. An n-dimensional extension of charged, static and spherically symmetric black holes is also proposed in this theory [10]. Instead of Killing horizon, these black holes admit universal horizons. The laws of black hole thermodynamics and the analysis of cosmic censorship conjecture have been studied in [11]. Recently, one of us explored the phenomenology of the two charged black holes in Einstein-Æther theory. By investigating the dynamics of test particle around the black hole in near circular motion, the properties of quasi-periodic oscillations, epicyclic frequencies, gravitational lensing, periodic orbits, marginally bound orbits and innermost stable circular orbits (ISCO) were studied [12].

In this paper, we study the motion of test charged particles around an exact black hole in the Einstein-Æther theory in the presence of a test uniform magnetic field. From the astrophysical perspective, the nearby environment of black holes is filled with high energy particles. The evidence to support this claim comes from the electromagnetic spectrum of astrophysical black holes which mainly results from the radiation emitted by the particles in the accretion disk and outward collimated jets [13]. It is the spacetime geometry of black hole which determines the motion of these particles and the propagation of radiation. The effects of Doppler and gravitational red-
shift can be deduced from the electromagnetic spectrum. In this scenario, the study of circular orbits in general and the innermost stable circular orbit (ISCO) in particular give the maximum information about the nearby activity of the black hole. In addition to gravitational field, the dynamics of particles are moderately affected by the magnetic field in the black hole-accretion disk environment. The origin of the magnetic field around the black hole can be primordial i.e. a relic of the early universe or by the gravitational collapse of the progenitor star carrying the magnetic field [14]. In literature, the motion of test charged particles around various kinds of black holes with uniform magnetic field has been studied extensively [15–19]. Another important aspect of black hole-accretion disk environment is the particle collisions near the black hole. It was earlier shown that the particle collision near the black hole horizon could lead to production of arbitrary centre of mass energy [20]. Since this aspect has received considerable interest of researchers and several aspects of BSW mechanism have been explored [21, 22], and references therein. In order to see the effects of the AE parameters more clearly and considering the electrical neutrality of most astrophysical back holes, we shall ignore the electric charge parameter in the black hole spacetime.

The black hole cannot have its own magnetic field, however one may consider the external magnetic field near the black hole. The solution of electromagnetic field equation around Kerr black hole immersed in external asymptotically uniform magnetic field has been obtained first in [23]. In later papers, various properties of electromagnetic field around black hole in external magnetic field were studied [24–40]. In particular, the dynamics of magnetized particle around non-rotating and rotating black holes have already been studied in diverse gravitational theories [41–51].

This manuscript is organized as follows: In Sec. II, we start with a brief review of Einstein-Æther black hole immersed in an external magnetic field. Sec. III is devoted to study the magnetized particle motion around Einstein-æther black hole in the presence of external magnetic field. The acceleration process near the Einstein-æther black hole is considered in Sec. IV. We consider the astrophysical applications in Sec. V and conclude our results in Sec. VI. Throughout this work we use signature (−, +, +, +) for the space-time and geometrized unit system $G_N = c = 1$. Latin indices run from 1 to 3 (or 4 depending on the context) and Greek ones vary from 0 to 3.

II. EINSTEIN-ÆTHER BLACK HOLES

The action of Einstein-Æther theory contains the Einstein-Hilbert action with an addition of an action corresponding to a dynamical, unit timelike Æther field [1, 52–54] which cannot vanish anywhere and breaks local Lorentz symmetry. The complete action has the following form

$$S_x = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} (R + \mathcal{L}_x),$$  

(1)

where $g = |g_{\mu\nu}|$ is the determinant of the spacetime metric around a gravitational object in the Einstein-Æther gravity. The Lagrangian of Æther field in the action (1) has the following form:

$$\mathcal{L}_x = -M^{\alpha\beta}_{\mu\nu}(D_\alpha u^\mu)(D_\beta u^\nu) + \lambda(g_{\mu\nu} u^\mu u^\nu + 1),$$  

(2)

here $D_\alpha$ is the covariant derivative with respect to $x^\alpha$, $\lambda$ is the Lagrangian multiplier which is responsible for the Æther four-velocity $u^\alpha$ always to be timelike, and $M^{\alpha\beta}_{\mu\nu}$ is defined as

$$M^{\alpha\beta}_{\mu\nu} = c_1 g_{\mu\nu} g^{\alpha\beta} + c_2 \delta^\alpha_\mu \delta^\beta_\nu + c_3 \delta^\alpha_\nu \delta^\beta_\mu - c_4 u^\alpha u^\beta g_{\mu\nu},$$  

(3)

where $c_i$ ($i = 1, 2, 3, 4$) are dimensionless coupling constants.

Note, that Æther gravitational constant has the following form

$$G_x = \frac{G_N}{1 - \frac{1}{14} c_1},$$  

(4)

where $G_N$ is the Newtonian gravitational constant.

The solution of field equation within the theory (1) describing the non-rotating black holes has the following line element in the spherical polar coordinates [8]:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$  

(5)

where

$$f(r) = 1 - \frac{2M}{r} - \frac{2c_{13} - c_{14}}{8(1 - c_{13})} \left(\frac{2M}{r}\right)^2,$$  

(6)

and $c_{13} = c_1 + c_3$ and $c_{14} = c_1 + c_4$, are the new coupling constants of the theory.

Consider the Einstein-Æther black hole immersed in an external asymptotically uniform magnetic field. We assume that there exists a magnetic field in the black hole vicinity which is static, axi-symmetric and homogeneous at the spatial infinity where it has the strength $B_0 > 0$. The magnetic field is assumed to be weak such that its effect on the spacetime geometry outside black hole is negligible. In the case when the magnetic field is strong, one needs to modify the spacetime geometry to include the magnetic field. Since the spacetime metric (5) allows timelike and spacelike Killing vectors, one may use Wald’s method to find the electromagnetic four potential in the following form [23]

$$A_\phi = \frac{1}{2} B_0 r^2 \sin \theta,$$  

$$A_t = 0,$$  

(7)

The non-zero components of the electromagnetic tensor $(F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu})$ have the following form

$$F_{\phi t} = B_0 r^2 \sin \theta \cos \theta,$$  

$$F_{\theta t} = B_0 r^2 \sin \theta \sin \phi,$$  

(8)

(9)
A magnetic field is defined with respect to an observer whose 4-velocity is $u_{\mu}$ as follows:

$$B^\alpha = \frac{1}{2} \eta^{\alpha \beta \gamma \delta} F_{\beta \gamma} u_\mu ,$$

where $\eta_{\alpha \beta \gamma \delta}$ is the pseudo-tensorial form of the Levi-Civita symbol $\epsilon_{\alpha \beta \gamma}$:

$$\eta_{\alpha \beta \gamma \delta} = \sqrt{-g} \epsilon_{\alpha \beta \gamma \delta} , \quad \eta^{\alpha \beta \gamma \delta} = - \frac{1}{\sqrt{-g}} \epsilon^{\alpha \beta \gamma \delta} ,$$

and $g = -r^4 \sin^2 \theta$. In orthonormal basis, the magnetic field has the following non-zero components

$$B^r = B_0 \cos \theta , \quad B^\theta = \sqrt{f(r)} B_0 \sin \theta .$$

**FIG. 1:** Graph showing the radial profile of the normalized angular component of the magnetic field.

Figure 1 illustrates the radial profiles of the angular component of magnetic field for different values of the parameters $c_{13}$ and $c_{14}$. One can see that the component of the magnetic field increases (decreases) with the increase of parameter $c_{14}$ ($c_{13}$).

### III. MAGNETIZED PARTICLE MOTION IN SPHERICALLY SYMMETRIC SPACETIME

Now we consider the magnetized particle around Einstein-\(\text{\AE}\)ther black hole. The motion of magnetized particles around black hole immersed in external magnetic fields can be studied using the Hamilton-Jacobi equation

$$g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = - \left( m - \frac{1}{2} D^{\mu \nu} F_{\mu \nu} \right)^2 ,$$

where $m$ is the rest mass of the particle and $D^{\mu \nu} F_{\mu \nu}$ is the interaction term between magnetized particle and the external magnetic field. According to Ref. [41], $D^{\alpha \beta}$ can be expressed as

$$D^{\mu \nu} = \eta^{\mu \nu \alpha \beta} u_\alpha \mathcal{M}_\beta , \quad D^{\alpha \beta} u_\beta = 0 ,$$

where $\mathcal{M}^\alpha$ is the four-vector of magnetic dipole moment and $u^\beta$ is the four-velocity of the particle. The electromagnetic field tensor $F_{\alpha \beta}$ can be decomposed into electric $E_\alpha$ and magnetic $B^\alpha$ fields in the following form

$$F_{\alpha \beta} = u_\alpha E_\beta - E_\alpha u_\beta - \eta_{\alpha \beta \gamma \delta} B^\gamma ,$$

Now one may easily express the interaction term $D^{\mu \nu} F_{\mu \nu}$ in the following form

$$D^{\mu \nu} F_{\mu \nu} = 2MB_0 \mathcal{L}[\lambda_\alpha] ,$$

where $\mathcal{M}$ is the module of dipolar magnetic moment of the particle and $\mathcal{L}[\lambda_\alpha]$ is a function of the spacetime coordinates, as well as other parameters defining the tetrad $\lambda_\alpha$ attached to the comoving fiducial observer.

For simplicity here we consider the orbital motion of magnetized particles around the Einstein-\(\text{\AE}\)ther black hole in the weak interaction approximation implying $\left( D^{\mu \nu} F_{\mu \nu} \right)^2 \to 0$. We also concentrate our study on the motion of magnetized particles in the equatorial plane, $\theta = \pi/2$ where the angular component of the four-momentum of the particle $p_\theta = 0$. We also consider the magnetic dipole moment of the particle to be perpendicular to the equatorial plane. The existence of Killing vectors guarantee the two conserved quantities: $p_\rho = L = m u_\rho$ and $p_\nu = -E = m u_\nu$ corresponding to angular momentum and energy of the particle, respectively. Thus the Hamilton-Jacobi action for the magnetized particle at the equatorial plane can be written as

$$S = -Et + L\phi + S_c(r) ,$$

which allows to separate variables in the Hamilton-Jacobi equation (13).

One can now easily get the expression for radial motion of the magnetized particle at the equatorial plane by inserting Eq. (16) in Eq.(13) and using the form of the action (17) as

$$\dot{r}^2 = \mathcal{E}^2 - V_{\text{eff}}(r; c_{13}, c_{14}, l, B) ,$$

where $l = L/(mM)$ and $\mathcal{E} = E/m$ are specific angular momentum and specific energy of the particle, respectively. The effective potential for radial motion of magnetized particle has the form

$$V_{\text{eff}}(r; c_{13}, c_{14}, l, B) = f(r) \left( 1 + \frac{\dot{r}^2}{r^2} - \mathcal{B}\mathcal{L}[\lambda_\alpha] \right)$$

where $\mathcal{B} = 2MB_0/m$ is the magnetic coupling parameter responsible for interaction a magnetized particle and the external magnetic field. $\mathcal{B} > 0$ ($\mathcal{B} < 0$) implies the directions of the external magnetic field and magnetic dipole moment of the particle are the same (opposite), while $\mathcal{B} = 0$ is the case when there is no external magnetic field or/and the particle has not the magnetic dipole moment. Now we will consider the circular orbits using the conditions

$$\dot{r} = 0 , \quad \frac{\partial V_{\text{eff}}}{\partial r} = 0 ,$$
Using the expression (19) we get
\begin{equation}
B(r; \ell, \mathcal{E}, c_{13}, c_{14}) = \frac{1}{\mathcal{L}[\lambda_\alpha]} \left( 1 + \frac{l^2}{r^2} - \frac{\mathcal{E}^2}{f(r)} \right),
\tag{21}
\end{equation}

The interaction of the magnetized and the external magnetic field can be characterized through only angular components of magnetic field and dipole moment
\begin{equation}
D^{\mu\nu}F_{\mu\nu} = 2M\dot{\theta}B_\theta.
\tag{22}
\end{equation}

In a comoving frame of reference we have
\begin{equation}
B_\ell = B_\phi = 0, \quad B_\theta = B_0 f(r) e^\Psi,
\tag{23}
\end{equation}
with
\begin{equation}
e^{-2\Psi} = f(r) - \Omega^2 r^2,
\tag{24}
\end{equation}
where \(\Omega\) is angular velocity of the particle
\begin{equation}
\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{f(r) l}{r^2} \frac{\mathcal{E}}{2}.
\tag{25}
\end{equation}

Now one may calculate the exact form of the interaction part in Hamilton-Jacobi equation inserting Eq. (23) into Eq. (22) in the following form
\begin{equation}
D^{\mu\nu}F_{\mu\nu} = 2M B_0 f(r) e^\Psi.
\tag{26}
\end{equation}

One may find the unknown function \(\mathcal{L}[\lambda_\alpha]\) after comparing the Eqns. (16) and (26) which yield
\begin{equation}
\mathcal{L}[\lambda_\alpha] = e^\Psi f(r).
\tag{27}
\end{equation}
Finally, one may find the exact form of the magnetic coupling parameter \(B(r; \ell, \mathcal{E}, c_{13}, c_{14})\) by inserting Eqs. (27) and (24) in (21) to obtain
\begin{equation}
B(r; \ell, \mathcal{E}, c_{13}, c_{14}) = \sqrt{\frac{1}{f(r)} - \frac{l^2}{\mathcal{E}^2 r^2}} \left( 1 + \frac{l^2}{r^2} - \frac{\mathcal{E}^2}{f(r)} \right),
\tag{28}
\end{equation}

Eq. (28) has the following physical meaning: a magnetized particle with specific energy \(\mathcal{E}\) and angular momentum \(l\) can be in the circular orbit at a certain distance \(r\) from the central object with the corresponding value of the magnetic interaction parameter which can be calculated from Eq. (28).

The radial profile of magnetic coupling function \(B\) for the different values of \(c_{13}\) and \(c_{14}\) parameters is shown in Fig. 2. One can see from the figure that the increase of the parameter of the Einstein-Æther gravity \(c_{13}\) \((c_{14})\) causes to increase (decrease) the maximum value of the magnetic coupling parameter correspond to circular orbits and an increase of both the Einstein-Æther gravity parameters shifts the the distance where the coupling parameter is maximum to the observer at infinite.

We now start to analyze the value of the magnetic coupling parameter corresponding to the stable circular orbits. The conditions for the stable circular orbits for magnetized particles have the following form
\begin{equation}
B = B(r; \ell, \mathcal{E}, c_{13}, c_{14}), \quad \frac{\partial B(r; \ell, \mathcal{E}, c_{13}, c_{14})}{\partial r} = 0.
\tag{29}
\end{equation}

This is a system of two equations with six unknown quantities \(B, r, \ell, \mathcal{E}, c_{13}, c_{14}\), hence its solution can be parameterized in terms of any two of the five independent variables. Here we use the magnetic coupling and the orbital radius \(r\) as free parameters. Our aim is then to find the angular momentum \(l\) and the specific energy \(\mathcal{E}\) of the particle as functions of \(r\) and \(B\). First, one can find the minimum energy of the particle which correspond to minimum value of the magnetic interaction parameter using the second part of the condition (29) and solving it for the specific energy
\begin{equation}
\mathcal{E}_{\text{min}}(r; \ell, c_{13}, c_{14}) = \left[ \frac{c_{14} + c_{13}(r - 1)^2 + r(r - 2)}{\sqrt{(1 - c_{13}) (r - c_{13}(r - 1) - \frac{c_{14}}{2})}} \right].
\tag{30}
\end{equation}

The radial profile of the minimum value of specific energy of magnetized particle is shown in Fig. 3 for different values of the parameters of Einstein-Æther gravity. The increase of the parameter \(c_{13}\) \((c_{14})\) causes the increase (decrease) of the maximum value of the specific energy.
Moreover, the distances where stable orbits exist and the energy takes the maximum value shift to the central object due to the increase of the parameter $c_{13}$ ($c_{14}$).

The minimum value of the magnetic coupling parameter can be found substituting Eq. (30) to Eq. 28 we have

$$B_{\min}(r; l, c_{13}, c_{14}) = -\frac{2\sqrt{1 - c_{13}}\sqrt{c_{13}(3r^2 - 2r^2 + 3 + 2r) - c_{14} + r(r - 3)}}{r(2c_{13}(1 - r) - c_{14} + 2r)\left[c_{14} - 2c_{13}(1 - r)^2 + 2r(r - 2)\right]} \cdot \left\{2r^2\left[c_{14} - c_{13}\left(2 - 3r + r^2\right) + r(r - 3)\right] + r^2\left[2c_{13}(r - 1) + c_{14} - 2r\right]\right\}. \quad (31)$$

$$\partial B_{\min}/\partial r = 0 \text{ with respect to } l$$

$$l_{\min}(r; c_{13}, c_{14}) = \frac{r\left[c_{13}(r - 1) - r + \frac{c_{14}}{2}\right]}{\sqrt{c_{14} - c_{13}(r - 1)(r - 2) + r(r - 3)}} \times \left\{\frac{c_{14}}{2} - c_{13}\left(2r^2 - 3r + 1\right) - r\left(3 - 2r\right)\right\}^{-\frac{1}{2}} \quad (32)$$

Figure 4 shows the radial dependence of minimal value of magnetic coupling parameter of the magnetized particle for the different values of the parameters $c_{13}$ and $c_{14}$. One can see from the figure that the maximal value of the minimum magnetic coupling parameter increases (decreases) and the distance where it reaches the maximum decreases (increases) with the increase of the parameter $c_{14}$ ($c_{13}$).

The upper limits for stable circular orbits can be calculated using extreme value of the magnetic coupling parameter which correspond to a some minimum value of the specific angular momentum. One may obtain the expression for the angular momentum solving the equation

$$\frac{\partial}{\partial r}[\ell_{\min}(r; c_{13}, c_{14}) + r\sqrt{c_{14} - c_{13}(r - 1)(r - 2) + r(r - 3)} - \frac{c_{14}}{2}] = 0$$

The radial profile of minimal value of the specific angular momentum.

Figure 5 shows the radial dependence of the minimum value of the specific angular momentum for the different values of the Einstein-Æther black hole parameters $c_{13}$ and $c_{14}$. One can see from the figure that the maximum value of the minimal angular momentum increase (decrease) with the increase of the parameter $c_{13}$ ($c_{14}$). However, the distance where the specific angular momen-
tum takes the maximum value increases (decreases) with increasing of the parameter $c_{14}$ ($c_{13}$).

Now it is possible to obtain and analyze the extreme value for the magnetic interaction parameter substituting Eq. (32) in to Eq. (31) and we have

$$B_{\text{extr}}(r; c_{13}, c_{14}) = \frac{\sqrt{c_{13}(r - 2)(r - 1) - c_{14} - (r - 3)r}}{c_{13}(4r^2 - 6r + 2) - c_{14} + 2r(3 - 2r)} \times \sqrt{1 - c_{13}r}. \quad (33)$$

Figure 6 illustrates the radial dependence of the extreme value of the magnetic coupling parameter and the minimum value of the magnetic coupling parameter at $B_{\text{min}}(l = 0)$, for the different values of the parameters $c_{13}$ and $c_{14}$. The range where stable circular orbits allowed for the magnetized particle with the magnetic coupling parameter $B_{\text{extr}} < B < B_{\text{min}}(l = 0)$ is shown with the colored areas. One can see from the Fig. 6 that the minimum distance of the circular orbits increases (decreases) with the increase of the value of the parameter $c_{13}$ ($c_{14}$). The effect of the Einstein-Æther gravity parameters on the range where circular orbits are allowed ($\Delta r = r_{\text{max}} - r_{\text{min}}$) are shown in Table I.

Table I demonstrates the range of circular orbits of magnetized particle around the Einstein-Æther black hole for the value of the magnetic coupling parameter $B = 0.1$ and the parameter of Einstein-Æther gravity. The values of the range $\Delta r$ are given in the unit of $M/10^3$. One can see from the figure that the range $\Delta r$ increases (decrease) with increasing the parameter $c_{13}$ ($c_{14}$).

Figures 7 and 8 depict trajectories of the magnetized particles coming from infinity with initial energies $E_1$ and $E_2$. The range where stable circular orbits are allowed for the magnetized particle with the magnetic coupling parameter $B = 0.1$ is shown with the colored areas. One can see from the Figs. 7 and 8 that the bound orbits are larger in the absence of the parameter $c_{13}$ than the case of vanishing parameter $c_{14}$. Moreover, the increase of the parameter $c_{14}$ ($c_{13}$) causes the increase (decrease) in the radius of orbits. One may conclude from the comparisons of the orbits that by increasing the parameter $c_{13}$ ($c_{14}$) leads to increase (decrease) of the gravitational potential of the central object.

**IV. ACCELERATION OF PARTICLES NEAR THE EINSTEIN-ÆTHER BLACK HOLE**

In this section we study the collision of particles near the black hole in Einstein-Æther gravity. Particularly, we consider the center of mass energy of two particles near the black hole immersed in magnetic field. We consider the effect of black hole parameters and external magnetic field to the center-mass-energy of the two colliding particles coming from infinity with energies $E_1$ and $E_2$. The center of mass energy of two particles can be found using the expression [20]

$$E_{\text{cm}}^2 = \frac{E_{\text{cm}}^2}{2m_{\text{c}}c^2} = 1 - g_{\alpha\beta}v_1^\alpha v_2^\beta \quad (34)$$

where $v_1^\alpha$ and $v_2^\beta$ are four-velocities of the colliding particles. Below we investigate head-on collisions of magnetized particles with magnetized, charged and neutral particles at the equatorial plane where $\theta = \pi/2$ with the initial energies $E_1 = E_2 = 1$.

**A. Two magnetized particles**

First we consider the case of two magnetized particles collision. The four-velocity of the magnetized particle at
equatorial plane have the following nonzero components:
\[
\begin{align*}
\dot{t} &= \frac{1}{f(r)}, \\
\dot{r}^2 &= 1 - f(r) \left( 1 + \frac{l^2}{r^2} - \mathcal{L}[\lambda_\alpha]B \right), \\
\dot{\phi} &= \frac{l}{r^2}.
\end{align*}
\]

The expression for center of mass energy of the two magnetized particles can be expressed by substituting four-velocities Eqs. (35) into (34) in the following form
\[
\mathcal{E}_{cm}^2 = 1 + \frac{1}{f(r)} - \frac{l_1 l_2}{r^2}
- \sqrt{1 - f(r) \left( 1 + \frac{l_1^2}{r^2} - \mathcal{L}[\lambda_\alpha]B_1 \right)}
\times \sqrt{1 - f(r) \left( 1 + \frac{l_2^2}{r^2} - \mathcal{L}[\lambda_\alpha]B_2 \right)}.
\]

Figure 9 shows the radial dependence of center-of-mass energy of head-on collision of two magnetized particles with the magnetic coupling parameter $B_1 = B_2 = 0.1$ and specific angular momentum $l_1 = 2$ and $l_2 = -2$. It shows that a high center-of-mass energy is produced when $r$ is reduced.

B. Magnetized and charged particles

In this subsection we consider collision of magnetized and charged particles. Four-velocities for charged particles can be found using the Lagrangian for the charged particles with the electric charge $e$ and mass $m$ in the presence of electromagnetic field
\[
\mathcal{L} = \frac{1}{2} m g_{\mu\nu} u^\mu u^\nu + e u^\mu A_\mu.
\]

The conserved quantities: the energy and the angular momentum can be found as
\[
\begin{align*}
p_t &= \frac{\partial \mathcal{L}}{\partial \dot{t}} = m g_{tt} \dot{t}, \\
p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m g_{\phi\phi} \dot{\phi} + e A_\phi.
\end{align*}
\]
FIG. 8: The same figure with Fig. 7, but for $c_{14} = 0$.

and four-velocity of the charged particle at equatorial plane has the following nonzero components

$$
\begin{align*}
\dot{t} &= \frac{1}{f(r)}, \\
\dot{r}^2 &= 1 - f(r) \left[ 1 + \left( \frac{l}{r} - \omega_B r \right)^2 \right], \\
\dot{\phi} &= \frac{1}{r^2} - \omega_B,
\end{align*}
$$

(40)

where $\omega_B = eB/(2m)$ is the cyclotron frequency responsible for the interaction the external magnetic field and charged particle. One may get the expression for the center-of-mass energy of the colliding charged and magnetized particles from substituting four-velocities of the particles Eqs. (35) and (40) into Eq. (34) and it takes the

FIG. 9: The radial dependence of the center-of-mass energy of two magnetized particles.
form:

\[
\mathcal{E}_{cm}^2 = 1 + \frac{1}{f(r)} - \left( \frac{l_1}{r^2} - \omega_B \right) l_2 - \sqrt{1 - f(r) \left( 1 + \left( \frac{l_1}{r} - \omega_B r \right)^2 \right)} \times \sqrt{1 - f(r) \left( 1 + \frac{l_2^2}{r^2} - \mathcal{L}[\lambda_0] B \right)} \]  (41)

The dependence of center-of-mass energy of the colliding charged and magnetized particles with the specific angular momentum \( l_1 = -l_2 = 2 \) around the Einstein-Æther black hole on radial coordinate in the different values of the parameters \( c_{13} \) and \( c_{14} \).

C. Magnetized and neutral particles

Finally, here we will carry on the studies of the collision of magnetized particles with neutral particles. One may immediately write the four-velocities for neutral particles

\[
i = \frac{1}{f(r)},
\]
\[\dot{\varphi} = 1 - f(r) \left( 1 + \frac{l_2^2}{r^2} \right),
\]
\[\dot{\phi} = \frac{l}{r^2}, \]  (42)

and the expression for the center-of-mass energy of the collision

\[
\mathcal{E}_{cm}^2 = 1 + \frac{1}{f(r)} - \frac{l_1 l_2}{r^2} - \sqrt{1 - f(r) \left( 1 + \frac{l_1^2}{r^2} \right)} \times \sqrt{1 - f(r) \left( 1 + \frac{l_2^2}{r^2} - \mathcal{L}[\lambda_0] B \right)} \]  (43)

One may conclude from Figs. 9, 10 and 11 that the increase of the parameter \( c_{14} \) \((c_{13})\) causes to increase (decrease) the center-of-mass energy in all collisions which we considered above.

V. ASTROPYSICAL APPLICATIONS

By considering the particle motion around black holes one can test the properties of different theories of gravity. However, in most cases the effects of the different theories may overlap each other and becomes impossible to say the effect belong to particular theory or model. Indeed there are many parameters of the different gravity theories giving the same observational orbital parameters of the particle. Here we test the Einstein-Æther black hole immersed in an external asymptotically uniform magnetic field whether it can mimic rotation of Kerr black hole considering the magnetized particles motion and innermost stable circular orbits (ISCO). Particularly we consider magnetized particles motion around:

1. Kerr black hole,
2. Einstein-Æther black hole,
3. Einstein-Æther black hole immersed in the magnetic field.

ISCO radius for test particles around rotating Kerr black holes can be written in the following form

\[
r_{\text{isco}} = 3 + Z_2 \pm \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}, \]  (44)

with

\[
Z_1 = 1 + \left( \sqrt{1 + a} + \sqrt{1 - a} \right) \sqrt{1 - a^2},
\]
\[
Z_2 = \sqrt{3a^2 + Z_1^2},
\]

where ± correspond to prograde and retrograde orbits. One may find the ISCO radius of the magnetized particle.
using the standard conditions $V'_{\text{eff}} = 0$ and $V''_{\text{eff}} \geq 0$, where prime $'$ implies partial derivative with respect to radial coordinate.

Below, we make numerical analysis the cases when the ISCO radius of the magnetized particle with magnetic coupling parameters $B = 0.5$ and $B = -0.5$ around Einstein-Æther and Kerr black holes take the same values.

A. $c_{14} = 0$

ISCO profiles of the magnetized particle around Kerr and Einstein-Æther black holes in the presence and absence of the external magnetic field at $c_{14} = 0$ are shown in Fig. 12. One can see that the parameter $c_{13}$ can mimic the innermost counter rotating orbits of the particle around Kerr black hole up to $c_{13} = 0.7596$ ($v_{12}$ vertical line) for $B = -0.5$, $c_{13} = 0.7262$ ($v_{11}$ vertical line) for $B = 0$ and $c_{13} = 0.4954$ ($v_{9}$ vertical line) for $B = 0.5$ in the range of ISCO radius $5.7243 \leq r_{\text{ISCO}} \leq 9$ ($h_1$ and $h_7$ horizontal lines).

ISCO radius is the same at the values of the spin and the Einstein-Æther parameter $c_{13} = a = 0.6266$ ($v_9$ vertical line) in the absence of the external magnetic field, $c_{13} = a = 0.6875$ ($v_{10}$ vertical line) for $B = -0.5$ and $c_{13} = a = 0.0244$ ($v_1$ vertical line) at $r_{\text{ISCO}} = 8.0501$ ($h_3$ horizontal line), $r_{\text{ISCO}} = 8.2077$ ($h_6$ horizontal line) and $r_{\text{ISCO}} = 5.7605$ ($h_8$ horizontal line), respectively.

ISCO radius of the magnetized particle with the coupling parameter $B = 0.5$ around Einstein-Æther black hole in the presence of the external magnetic field can be the same when it is around Kerr black hole in the absence of the magnetic field provided the spin parameter $0.4081 \leq a \leq 1$ ($v_2$ vertical line) with the radius $7.463 \leq r_{\text{ISCO}} \leq 9$ ($h_1$ and $h_4$ horizontal lines).

Moreover, one can observe the effect of the external magnetic field looking at the red, blue and grey lines from Fig. 12. One can see that it is not possible distinguish the existence of the external magnetic field at $c_{13} \geq 0.5314$ ($v_2$ vertical line) because of the ISCO is the same for the particle with $B = 0.5$. One may distinguish the orientation of the external magnetic field with respect to the direction of the magnetized particle’s dipole moment at $c_{13} < 0.5906$ ($v_8$ vertical line) and $r_{\text{ISCO}} > 7.463$.

B. $c_{13} = 0$

In this subsection we make similar discussions for the case of $c_{13} = 0$. It is not difficult to see that Fig. 13 is similar to Fig. 12 just for the Einstein-Æther parameter $c_{13} = 0$. One can see from the figure that in this case the effect of the another Einstein-Æther gravity parameter $c_{14}$ can mimic spin of Kerr black hole giving the same innermost co-rotating orbits.

One can see that in the cases, when the external magnetic field is present at the values of the spin parameter $a > 0.5388$ ($v_1$ vertical line) and $a < 0.0312$ ($v_4$ vertical line), can not mimic the Einstein-Æther gravity parameter for the magnetized particle with the coupling parameter $B = -0.5$. The parameter $c_{14}$ mimic spin of Kerr black hole in the range $6 > r_{\text{ISCO}} > 5.1623$ ($h_1$ and $h_3$ horizontal lines) at $2 \leq c_{14} \geq 1.4119$ ($v_7$ vertical line) for the magnetized particle with the parameter $B = 0.5$.

Further in the absence of the external magnetic field, the spin parameter can not mimic the parameter $c_{14}$ when $a > 0.4799$ ($v_3$ vertical line).

Now again looking for the effect of magnetic field, we will see at red, blue and grey lines. One can notice that the magnetic interaction can not mimic the effect of the parameter at the the range $c_{14} \leq 1.4119$ ($v_7$ vertical line) for $B = 0.5$ and $c_{14} \geq 1.8426$ for $B = -0.5$ ($v_9$ vertical line).

The orientation of the external magnetic field can be distinguishable when the Einstein-Æther gravity parameter $c_{14} < 1.6241$ ($v_4$ vertical line) for $B = 0.5$ and $c_{5}$ > 0.72 for $B = -0.5$ in the range of ISCO radius $5.7243 \leq r_{\text{ISCO}} \leq 5.1623$ ($h_2$ and $h_3$ horizontal lines).

C. $c_{13}=0$ solution vs $c_{14} = 0$ one

In this subsection we analyse ISCO radius of magnetized particles with the magnetic coupling parameter $B = 0.5$ and $B = -0.5$ around Einstein-Æther black hole in the presence and absence of the external parameters that in with cases the solution $c_{14} = 0$ can mimic the solution $c_{13} = 0$.

Figure 14 illustrates ISCO profiles of the magnetized particle around Einstein-Æther black holes with solutions $c_{13} = 0$ and $c_{14} = 0$ in the presence $|B| = 0.5$ and absence of the external magnetic field. One can see from the figure that ISCO radius of the particle is the same in cases of: a) the particle with coupling parameter $B = 0.5$ and $B = -0.5$ at $c_{13} = c_{14} = 0.502$ ($v_6$ vertical line) with the radius $r_{\text{ISCO}} = 6.9954$ ($h_9$ horizontal line); b) $B = 0.5$ and $B = 0$ at $c_{13} = c_{14} = 0.4407$ ($v_5$ vertical line) with the radius $r_{\text{ISCO}} = 7.0549$ ($h_2$ horizontal line); c) $B = -0.5$ and $B = 0$ at $c_{13} = c_{14} = 0.1166$ ($v_1$ vertical line) with the radius $r_{\text{ISCO}} = 5.9116$ ($h_6$ horizontal line).

ISCO radius of the magnetized particle with the magnetic coupling parameter $B = 0.5$ and the parameter $c_{14}$ $\in (0, 1)$ can be measured the same as the ISCO of the particle with $B = -0.5$ and $B = 0$ with the parameter $c_{13} \in (0.3626, 0.5905)$ ($v_4$ and $v_8$ vertical lines) and $c_{13} \in (0.2528, 0.5318)$ ($v_3$ and $v_7$ vertical lines) in the range of the radius $7.463 \geq r_{\text{ISCO}} \geq 6.4809$ ($h_1$ and $h_4$ horizontal lines). Moreover, ISCO radius of the particle with the parameter $B = 0$, at the values of the parameter $c_{14} \in (0, 0.3625)$ ($v_4$ vertical line) can be the same with the ISCO radius of a magnetized particle with $B = -0.5$, at the values of the parameter $c_{13} \in (0, 0.1641)$ ($v_2$ vertical line) for the range of ISCO radius of $6 \geq r_{\text{ISCO}} \geq 5.7243$ ($h_3$ and $h_7$ horizontal lines).
FIG. 12: The dependence of ISCO radius on spin and Einstein-Æther parameter $c_{13}$ when $c_{14} = 0$. Black large dashed and grey solid lines correspond to the ISCO of magnetized particles around Kerr and the Einstein-Æther black holes in the absence of the magnetic field, respectively. Blue dot-dashed and red dashed lines correspond to ISCO of the magnetized particle around the Einstein-Æther black hole immersed in the magnetic field for the values of the magnetic coupling parameter $B_1 = -0.5$ and $B_2 = 0.5$, respectively. The black large dashed line corresponds to ISCO radius as a function of spin parameter; red, blue and gray lines correspond to ISCO radius as a function of the parameter $c_{13}$ for different values of $B$. Vertical ($v_i$, $i = 1 \div 12$) and horizontal ($h_i$, $i = 1 \div 7$) lines imply the important values for the spin of Kerr and Einstein-Æther black hole parameters and the values of the ISCO radius where the lines intersect (see the text for discussion).

FIG. 13: The same figure as Fig. 12, but for the case $c_{13} = 0$. 
VI. SUMMARY AND DISCUSSIONS

This work is devoted to study the effects of Einstein-æther gravity on magnetized particles motion around a static and spherically symmetric uncharged black hole immersed in an external asymptotically uniform magnetic field. The following major results have been obtained:

- The electromagnetic field solution has been obtained using Wald's method and shown that the existence of the parameter $c_{13}$ ($c_{14}$) causes the decrease (increase) of the magnetic field near the black hole.

- The studies of circular motion of the magnetized particles show that inner circular (bounded) orbits comes closer to (goes far from) the central object and the range where circular orbits are allowed increases (decreases) in the presence of the parameter $c_{14}$ ($c_{13}$), it implies that the parameter $c_{13}$ plays role as an additional gravity effect (in other word at the existence of the parameter $c_{13}$ the effective mass of central black hole increases).

- The analysis of the head-on collisions of magnetized particles around the æther black hole showed that the parameter $c_{13}$ ($c_{14}$) causes the increase (decrease) of center-of-mass energy.

- We have shown that the effects of Einstein-æther gravity and magnetic interaction can mimic the effects of rotation of Kerr black hole giving the same ISCO radius. We have analyzed in detail for the magnetized particles with the coupling parameter $|B| = 0.5$. It was also shown that the magnetized particles may have the same ISCO radius around rotating Kerr and Einstein-æther black holes at the range of $r_{\text{isco}} \in (5.7243, 9)$ at the parameter $c_{13} \leq 0.7556$ (0.7262 and 0.4954) for the particles with $B = 0.5$ ($B = 0.5$ and $B = 0$) when $c_{14} = 0$ and the spin parameter can not mimic the parameter $c_{14}$ when $a > 0.4799$.

- ISCO radius is the same for the values of the spin and the Einstein-æther parameter $c_{13} = a = 0.6266$ in the absence of the external magnetic field, $c_{13} = a = 0.6875$ for $B = -0.5$ and $c_{13} = a = 0.0244$ at $r_{\text{isco}} = 8.0501$, $r_{\text{isco}} = 8.2077$ and $r_{\text{isco}} = 5.7605$, respectively.

- ISCO radius of the particle is the same in cases of the particle with coupling parameter $B = 0.5$ and $B = -0.5$ at $c_{13} = c_{14} = 0.502$ with the radius $r_{\text{isco}} = 6.9954$. For $B = 0.5$ and $B = 0$ at $c_{13} = c_{14} = 0.4407$ with the radius $r_{\text{isco}} = 7.0549$. For $B = -0.5$ and $B = 0$ at $c_{13} = c_{14} = 0.1166$ with the radius $r_{\text{isco}} = 5.9116$.

Acknowledgement

The research work of AA is supported by postdoc fund PIFI of Chinese academy of sciences. This research is supported by Grants No. VA-FA-F-2-008, No.MRB-AN-2019-29 of the Uzbekistan Ministry for Innovative Development. JR an AA thank Silesian University in Opava.
for the hospitality during their visit.

[1] T. Jacobson and D. Mattingly, Phys. Rev. D 64, 024028 (2001), arXiv:gr-qc/0007031 [gr-qc].
[2] C. Eling, T. Jacobson, and D. Mattingly, arXiv e-prints, gr-qc/0410001 [gr-qc].
[3] B. Li, D. F. Mota, and J. D. Barrow, Phys. Rev. D 77, 024032 (2008), arXiv:0709.4581 [astro-ph].
[4] R. A. Battye, F. Pace, and D. Trinh, Phys. Rev. D 96, 064041 (2017), arXiv:1707.06508 [astro-ph.CO].
[5] C. Zhang, X. Zhao, A. Wang, B. Wang, K. Yagi, N. Yunes, W. Zhao, and T. Zhu, Phys. Rev. D 101, 044002 (2020), arXiv:1911.10278 [gr-qc].
[6] T. Zhu, Q. Wu, M. Jamil, and K. Justufi, Phys. Rev. D 100, 044055 (2019), arXiv:1906.05673 [gr-qc].
[7] J. Oost, S. Mukohyama, and A. Wang, Phys. Rev. D 97, 124023 (2018), arXiv:1802.04303 [gr-qc].
[8] C. Zhang, X. Zhao, K. Lin, S. Zhang, W. Zhao, and A. Wang, arXiv e-prints, arXiv:2004.06115 (2020), arXiv:2004.06115 [gr-qc].
[9] K. Lin, F.-H. Ho, and W.-L. Qian, International Journal of Modern Physics D 28, 1950049-308 (2019).
[10] M. Meiers, M. Saravani, and N. Ashradi, Phys. Rev. D 93, 104008 (2016), arXiv:1511.08969 [gr-qc].
[11] M. Azreg-Ainou, Z. Chen, B. Deng, M. Jamil, T. Zhu, Q. Wu, and Y.-K. Lim, arXiv e-prints, arXiv:2004.02602 (2020), arXiv:2004.02602 [gr-qc].
[12] C. Bambi, Black Holes: A Laboratory for Testing Strong Gravity (Springer, Singapore, 2017).
[13] K. Subramanian, Reports on Progress in Physics 79, 076901 (2016), arXiv:1504.02311 [astro-ph.CO].
[14] A. Jawad, F. Ali, M. Jamil, and U. Debnath, Communications in Theoretical Physics 66, 509 (2016), arXiv:1610.07411 [gr-qc].
[15] S. Hussain and M. Jamil, Phys. Rev. D 92, 043008 (2015), arXiv:1508.02123 [gr-qc].
[16] M. Jamil, S. Hussain, and B. Majeeed, European Physical Journal C 75, 24 (2015), arXiv:1404.7123 [gr-qc].
[17] Hussain, S., Hussain, I., and Jamil, M., Eur. Phys. J. C 74, 210 (2014).
[18] G. Z. Babar, M. Jamil, and Y.-K. Lim, International Journal of Modern Physics D 25, 1650024 (2016), arXiv:1504.00072 [gr-qc].
[19] M. Baniadhos, J. Silk, and S. M. West, Physical Review Letters 103, 111102 (2009).
[20] B. Majeeed and M. Jamil, International Journal of Modern Physics D 26, 1741017 (2017), arXiv:1705.04167 [gr-qc].
[21] A. Zakaria and M. Jamil, Journal of High Energy Physics 2015, 147 (2015), arXiv:1501.06306 [gr-qc].
[22] R. M. Wald, Phys. Rev. D. 10, 1680 (1974).
[23] A. N. Aliev, D. V. Galtsov, and V. I. Petukhov, Astrophys. Space Sci. 124, 137 (1986).
[24] A. N. Aliev and D. V. Galtsov, Soviet Physics Uspekhi 32, 75 (1989).
[25] A. N. Aliev and N. Özdemir, Mon. Not. R. Astron. Soc. 336, 241 (2002), gr-qc/0208025.
[26] V. P. Frolov and P. Krtouš, Phys. Rev. D 83, 024016 (2011), arXiv:1010.2266 [hep-th].
[27] V. P. Frolov, Phys. Rev. D. 85, 024020 (2012), arXiv:1110.6274 [gr-qc].
[28] C. A. Benavides-Gallego, A. Abdujabbarov, D. Malafarina, B. Ahmedov, and C. Bambi, Phys. Rev. D 99, 044012 (2019), arXiv:1812.04846 [gr-qc].
[29] S. Shyamratov, B. Ahmedov, Z. Stuchlik, and A. Abdujabbarov, International Journal of Modern Physics D 27, 1850088 (2018).
[30] Z. Stuchlik, J. Schee, and A. Abdujabbarov, Phys. Rev. D 89, 104048 (2014).
[31] A. Abdujabbarov and B. Ahmedov, Phys. Rev. D 81, 044022 (2010), arXiv:0905.2730 [gr-qc].
[32] A. Abdujabbarov, B. Ahmedov, and A. Hakimov, Phys.Rev. D, 83, 044053 (2011), arXiv:1101.4741 [gr-qc].
[33] A. A. Abdujabbarov, B. J. Ahmedov, S. R. Shyamratov, and A. S. Rahmatov, Astrophysics Space Sci 334, 237 (2011), arXiv:1105.1910 [astro-ph.SR].
[34] A. A. Abdujabbarov, B. J. Ahmedov, and V. G. Gramanovano, General Relativity and Gravitation 40, 2515 (2008), arXiv:0802.3439 [gr-qc].
[35] V. Karas, J. Kovař, O. Kopáček, Y. Kojima, P. Slany, and Z. Stuchlik in American Astronomical Society Meeting Abstracts #220, American Astronomical Society Meeting Abstracts, Vol. 220 (2012) p. 430.07.
[36] Z. Stuchlik and M. Kołos, European Physical Journal C 76, 32 (2016), arXiv:1511.02906 [gr-qc].
[37] J. Kovař, O. Kopáček, V. Karas, and Z. Stuchlik, Classical and Quantum Gravity 27, 135006 (2010), arXiv:1005.3270 [astro-ph.HE].
[38] J. Kovař, P. Slaný, C. Cremaschin, Z. Stuchlik, V. Karas, and A. Trova, Phys. Rev. D 90, 044029 (2014), arXiv:1409.0418 [gr-qc].
[39] M. Kołos, A. Tursunov, and Z. Stuchlik, Eur. Phys. J. C 77, 860 (2017), arXiv:1707.02224 [astro-ph.HE].
[40] F. de Felice and F. Sorge, Classical and Quantum Gravity 20, 469 (2003).
[41] F. de Felice, F. Sorge, and S. Zilio, Classical and Quantum Gravity 21, 961 (2004).
[42] J. R. Rayimbaev, Astrophysics Space Sc 361, 288 (2016).
[43] T. Oteev, A. Abdujabbarov, Z. Stuchlik, and B. Ahmedov, Astrophys. Space Sci. 361, 269 (2016).
[44] B. Toshmatov, A. Abdujabbarov, B. Ahmedov, and Z. Stuchlik, Astrophys Space Sci 360, 19 (2015).
[45] A. Abdujabbarov, B. Ahmedov, O. Rahimov, and U. Salikhbaev, Physica Scripta 89, 084008 (2014).
[46] O. G. Rahimov, A. A. Abdujabbarov, and B. J. Ahmedov, Astrophysics and Space Science 335, 499 (2011), arXiv:1105.4543 [astro-ph.SR].
[47] O. G. Rahimov, Modern Physics Letters A 26, 399 (2011), arXiv:1012.1481 [gr-qc].
[48] K. Haydarov, A. Abdujabbarov, J. Rayimbaev, and B. Ahmedov, Universe 6 (2020), 10.3390/universe6030044.
[49] K. Haydarov, J. Rayimbaev, A. Abdujabbarov, S. Palvanov, and D. Begmatova, arXiv e-prints, arXiv:2004.14868 (2020), arXiv:arXiv:2004.14868 [gr-qc].
[51] J. Rayimbaev, A. Abdjabbarov, B. Turimov, and F. Atamurotov, arXiv e-prints, arXiv:2004.10031 (2020), arXiv:2004.10031 [gr-qc].

[52] T. Jacobson, arXiv e-prints, arXiv:0801.1547 (2008), arXiv:0801.1547 [gr-qc].

[53] B. Z. Foster, Phys. Rev. D 73, 104012 (2006), arXiv:gr-qc/0602004 [gr-qc].

[54] D. Garfinkle, C. Eling, and T. Jacobson, Phys. Rev. D 76, 024003 (2007), arXiv:gr-qc/0703093 [gr-qc].