Minimax solutions for a problem with sign changing nonlinearity and lack of strict convexity

Paola Magrone
Dipartimento di Architettura, Università degli Studi Roma Tre
Via della Madonna dei Monti 40, Roma, Italia

Abstract
A result of existence of a nonnegative and a nontrivial solution is proved via critical point theorems for non smooth functionals. The equation considered presents a convex part and a nonlinearity which changes sign.

1 Introduction and main results
Let us consider the problem

\[
\begin{cases}
-\text{div}(\Psi'(\nabla u)) = \lambda u + b(x)|u|^{p-2}u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(P)

where \(\lambda\) is a real parameter, \(\Omega\) is a bounded open subset of \(\mathbb{R}^N\), \(N \geq 2\), \(b(x) \in \mathcal{C}(\Omega)\) changes sign in \(\Omega\). Finally \(2 < p < 2^* = \frac{2N}{N-2}\), and we will assume that \(\Psi : \mathbb{R}^N \rightarrow \mathbb{R}\) is a convex function of class \(C^1\) satisfying the following conditions:

\[\text{AMS Subject Classification 35J65, 58E05; keywords: non strict convexity, sign changing, Linking theorem}\]
\[
(\Psi_1) \quad \lim_{\xi \to 0} \frac{\Psi(\xi)}{|\xi|^2} = \frac{1}{2};
\]

\[
(\Psi_2) \quad \exists \mu > 0 : \mu |\xi|^2 \leq \Psi(\xi) \leq \frac{1}{\mu} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N;
\]

\[
(\Psi_3) \quad \lim_{|\xi| \to \infty} \frac{\Psi'(\xi) \cdot \xi - 2\Psi(\xi)}{|\xi|^2} = 0;
\]

Moreover the function \(b(x)\) has to be strictly positive in a non zero measure set, and the zero set must be "thin", in other words \(b(x)\) must satisfy the following conditions:

\[
(b_1) \quad \Omega^+ := \{ x \in \Omega : b(x) > 0 \} \quad \text{is a nonempty open set}
\]

\[
(b_2) \quad \Omega^0 := \{ x \in \Omega : b(x) = 0 \} \quad \text{has zero measure}
\]

Conditions \((b_1)\) and \((b_2)\) imply that \(b^+(x) = b(x) + b^-(x) \neq 0\) and that, since \(b\) is continuous, the set \(\Omega^0\) is closed in \(\Omega\).

Let us also denote by \((\lambda_k)\) the eigenvalues of \(-\Delta\) with homogeneous Dirichlet boundary condition.

In the model case \(\Psi(\xi) = \frac{1}{2} |\xi|^2\), there is a wide literature on problem \((P)\).

To cite only some of the existing results, in \([2]\) the authors found positive solutions to \((P)\) in case that \(\lambda_1 < \lambda < \Lambda^*\), with \(\Lambda^*\) suitably near to \(\lambda_1\). In the following many other papers (\([1], [2], [3], [5], [6]\)) were devoted to prove existence of (possibly infinitely many) solutions for \(\lambda \in [\lambda_1, \Lambda^*]\) or also for every \(\lambda\), in case the nonlinearity satisfies some oddness assumption. A result concerning all \(\lambda\) different from the eigenvalues of the Laplacian under some quite general assumptions can be found in \([11]\), while in \([8]\) the authors proved a result of existence of a nontrivial solution (possibly changing sign) for every \(\lambda\).

On the other hand, only a small literature is available when dealing with equations with a non strictly convex principal part. In this framework, in \([7]\) the author applies non smooth variational methods in presence of subcritical, positive, nonlinearities; while using similar techniques a nonlinearity with criti-
cal growth was considered in [9].

The aim of this paper is to extend to the setting of non strictly convex functionals some of the results contained in [2] (existence of a positive solution for $\lambda < \lambda_1$) and [8] (existence of a nontrivial solution for any $\lambda$).

Problem $(P)$ can be treated by variational techniques. Indeed, weak solutions $u$ of $(P)$ can be found as critical points of the $C^1$ functional $J : H^1_0(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \int_{\Omega} \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)|u|^p \, dx. \quad (1.1)$$

The key point here is that, although $\Psi$ shares some properties with this typical case, there is no assumption of strict convexity with respect to $\xi$.

For instance, one could consider

$$\Psi(\xi) = \psi(\xi_1) + \frac{1}{2} \sum_{j=2}^{N} \xi_j^2, \quad (1.2)$$

where

$$\psi(t) = \begin{cases} \frac{1}{2} t^2 & \text{if } |t| < 1, \\ |t| - \frac{1}{2} & \text{if } 1 \leq |t| \leq 2, \\ \frac{1}{2} |t|^2 - |t| + \frac{3}{2} & \text{if } |t| > 2. \end{cases}$$

If we look at the principal part of $J$ as the energy stored in the deformation $u$, this means that the material has a plastic behavior when $1 \leq |D_1 u| \leq 2$. We refer the reader to [13, Chapter 6] for a discussion of several models of plasticity.

As shown in [7, 9], it may happen that Palais Smale sequences, even if bounded in $H^1_0(\Omega)$-norm, do not admit any subsequence which converges strongly in this norm. And there is no way to prevent the interaction between the area where $\Psi$ loses strict convexity and the values of $\nabla u$. A possible strategy is to look for compactness in a weaker norm ($L^2$).

Let us introduce the following notations: let $k \geq 1$ be such that $\lambda_k \leq \lambda < \lambda_{k+1}$
and let $e_1, \ldots, e_k$ be eigenfunctions of $-\Delta$ associated to $\lambda_1, \ldots, \lambda_k$, respectively.

Finally, let $E_- = \text{span}\{e_1, \ldots, e_k\}$ and $E_+ = E_-^\perp$. The main result of this paper are the following:

**Theorem 1.1.** Let $N \geq 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_1), (\Psi_2)$. Moreover let the function $b(x)$ verify $(b_1)$. Then, for every $\lambda \in ]0, \lambda_1[$, problem $(P)$ admits a nontrivial and nonnegative weak solution $u \in H^1_0(\Omega)$.

**Theorem 1.2.** Let $N \geq 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_1), (\Psi_2)$ and let $\lambda \geq \lambda_1$. Moreover let the function $b(x)$ verify $(b_1)$, and the following assumptions:

$$\int_{\Omega} b(x)|v|^p \geq 0 \quad \forall v \in E_-.$$ (1.3)

$$\exists e \in E_-^\perp \setminus \{0\} : \int_{\Omega} b(x)|v|^p dx \geq C \int_{\Omega} |v|^p dx \quad \forall v \in E_- \oplus \text{span}\{e\}. \quad \text{ (1.4)}$$

Then problem $(P)$ admits a nontrivial weak solution $u \in H^1_0(\Omega)$.

**Remark 1.3.** Arguing as in section 2 of [9] we can deduce the following properties for $\Psi$, up to modifying the constant $\mu$:

$$\Psi'(\xi) \cdot \xi \geq \mu |\xi|^2 \quad \forall \xi \in \mathbb{R}^N,$$ (1.5)

$$|\Psi'(\xi)| \geq \mu |\xi| \quad \forall \xi \in \mathbb{R}^N \quad \text{ (1.6)}$$

$$|\Psi'(\xi)| \leq \frac{1}{\mu} |\xi| \quad \forall \xi \in \mathbb{R}^N \quad \text{ (1.7)}$$

Furthermore $(\Psi_3)$ yields that $\forall \sigma > 0, \exists M_\sigma \in \mathbb{R}$:

$$\Psi'(\xi)\xi - 2\Psi(\xi) \leq \sigma |\xi|^2 + M_\sigma \quad \text{ (1.8)}$$
2 The variational framework

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), with Lipschitz boundary and let \( \lambda \in \mathbb{R} \). Let us define the following functional \( J : H^1_0(\Omega) \to \mathbb{R} \)

\[
J(u) = \int_{\Omega} \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)|u|^p \, dx.
\]

By \((\Psi_1), (\Psi_2)\) the functional \( J \) is of class \( C^1 \) on \( H^1_0(\Omega) \). We wish to apply variational methods to functional \( J \), but, as already mentioned, it is well known that the Palais Smale (PS) condition for a functional which is not strictly convex is not satisfied on \( H^1_0(\Omega) \). So it is convenient to extend the functional \( J \) to \( L^2^* \) with value \(+\infty\) outside \( H^1_0(\Omega) \).

In other words we define the convex, lower semicontinuous functional (still denoted \( J \))

\[
J : L^2^*(\Omega) \to [0, +\infty]
\]

\[
J(u) = \begin{cases} 
\int_{\Omega} \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)|u|^p \, dx & \text{if } u \in H^1_0(\Omega), \\
+\infty & \text{if } u \in L^2^*(\Omega) \setminus H^1_0(\Omega)
\end{cases}
\tag{2.1}
\]

This setting will allow us to recover PS condition.

This functional can be written as \( J = J_0 + J_1 \), where

\[
J_0 = \int_{\Omega} \Psi(\nabla u) \, dx,
\]

is proper, convex and l.s.c., while

\[
J_1 = -\frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)|u|^p \, dx,
\]

is of class \( C^1 \). We will use the following definitions (\[12\], \[17\]) of critical point and PS sequence for functionals of the type \( J = J_0 + J_1 \):

**Definition 2.1.** Let \( X \) be a real Banach space, \( u \in X \) is a critical point for \( J \)
if \( J(u) \in \mathbb{R} \) and \(-J'(u) \in \partial J_0\), where \( \partial J_0 \) is the subdifferential of \( J_0 \) at \( u \).

**Definition 2.2.** Let \( X \) be a real Banach space and let \( c \in \mathbb{R} \). We say that \( u_k \) is a Palais Smale sequence at level \( c \) (\((PS)_c\) sequence for short) for \( J \) if \( J(u_k) \to 0 \)
and there exists \( \alpha_k \in \partial J_0 \) with \((\alpha_k + J'(u_k)) \to 0 \) in \( X^* \).

The following proposition (see [7]) assures that the critical points of the extendend functional already defined gives the solutions of our problem.

**Proposition 2.3.** Let \( u \in L^2(\Omega, \mathbb{R}^N) \). Then \( u \) is a critical point of \( J \) if and only if \( u \in H_0^1(\Omega) \) and \( u \) is a weak solution of \((P)\).

**Proof** Let \( v \in L^2 \). Then \( v \in \partial J_0 \), if and only if \( u \in H_0^1(\Omega) \) and

\[-\text{div}(\Psi'(\nabla u)) = v\]

that is a reformulation of definition 2.1.

\[\square\]

Moreover we will apply the compactness result contained in [7], which we recall.

Let us define the functional \( E : W_0^{1,2}(\Omega, \mathbb{R}^N) \to \mathbb{R} \) as

\[E(u) = \int_{\Omega} \Psi(\nabla u) \, dx\]

**Theorem 2.4.** Assume that \( \Omega \) is bounded. If \( \{u_h\} \) is weakly convergent to \( u \) in \( W_0^{1,2}(\Omega, \mathbb{R}^N) \) with \( E(\{u_h\}) \to E(\{u\}) \), then \( u \) is strongly convergent to \( u \) in \( L^2(\Omega) \).

### 3 Proof of main results

Since \( \Psi'(0) = 0 \), of course 0 is a solution of \((P)\). Therefore we are interested in nontrivial solutions. In order to find nonnegative solutions of \((P)\), we consider
the modified functional $\mathcal{J} : L^2(\Omega) \rightarrow ]-\infty, +\infty]$ defined as

$$
\mathcal{J}(u) = \begin{cases} 
\int_{\Omega} \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_{\Omega} (u^+)^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)(u^+)^p \, dx & \text{if } u \in H^1_0(\Omega), \\
+\infty & \text{if } u \in L^2(\Omega) \setminus H^1_0(\Omega)
\end{cases}
$$

Of course, $\mathcal{J}$ is also convex and lower semicontinuous.

**Proposition 3.1.** Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_2)$ with $\mu > 0$, and (1.6). Then each critical point $u \in L^2(\Omega)$ of $\mathcal{J}$ is a nonnegative solution of $(P)$.

**Proof** Since by Proposition 2.3 we already know that the critical points of $J$ are solutions of our problem, it is only left to prove that the modified functional will give nonnegative solutions. By $(\Psi_2)$ one has

$$
\mu \int_{\Omega} |\nabla u^-|^2 \, dx \leq \int_{\Omega} \Psi'(\nabla u) \cdot (-\nabla u^-) \, dx = \\
= \lambda \int_{\Omega} u^+ (-u^-) \, dx + \int_{\Omega} (u^+)^{p-1} (-u^-) \, dx = 0,
$$

whence the assertion.

$\square$

**Remark 3.2.** From now on, to simplify notations, we will keep on using the functional $J$ instead of $\mathcal{J}$, since it is understood what has been proved in Proposition 3.1.

**Proof of Theorem 1.1**

We aim to apply to $J$ a nonsmooth version of Mountain Pass Theorem [12]. First of all, let us observe that, by $(\Psi_1)$, we have

$$
\frac{\int_{\Omega} \Psi(\nabla u) \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} \rightarrow \frac{1}{2} \quad \text{as } u \rightarrow 0 \text{ in } L^2.
$$

Then, as in the case $\Psi(\xi) = \frac{1}{2} |\xi|^2$ treated in [2] [8], we deduce that there exist $\varrho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ whenever $\|u\| = \varrho$. On the other hand,
there exists \( e \in L^2 \) with \( e \geq 0 \) a.e. in \( \Omega \) such that

\[
\lim_{t \to +\infty} J(te) = -\infty,
\]

again, this is proved in [2] in the case \( \Psi(\xi) = \frac{1}{2}|\xi|^2 \), but by \( (\Psi_2) \) the assertion is true also in our case.

By the Mountain Pass theorem, there exist a sequence \( (u_k) \) in \( L^2 \) and a sequence \( (w_k) \) in \( L^{(2^*)'}(\Omega) \) strongly convergent to 0 such that (see definition 2.2)

\[
\int \Omega \Psi'(\nabla u_k)(\nabla v - \nabla u_k) \, dx \geq \lambda \int \Omega u_k(v - u_k) \, dx + \int \Omega b(x)|u_k|^{p-1}(v - u_k) \, dx + \int \Omega w_k(v - u_k) \, dx \quad \forall v \in L^{(2^*)'}.
\]

Taking \( v = 0 \) and \( v = 2u_k \) as tests in the previous inequality yield

\[
\int \Omega \Psi'(\nabla u_k)\nabla u_k \, dx = \lambda \int \Omega (u_k)^2 \, dx + \int \Omega b(x)(u_k)^p \, dx + \int \Omega w_k u_k \, dx \quad \forall v \in L^{(2^*)'}.
\]

Furthermore also the following relation holds:

\[
\lim_{k \to \infty} \left( \int \Omega \Psi(\nabla u_k) \, dx - \frac{\lambda}{2} \int \Omega (u_k)^2 \, dx - \frac{1}{p} \int \Omega b(x)(u_k)^p \, dx \right) = c > \alpha. \tag{3.3}
\]

Let us write the expression \( pJ(u_k) - J'(u_k)u_k \), which is boundend by assumptions (3.2), (3.3):

\[
p \int \Omega \Psi(\nabla u_k) \, dx - \frac{p}{2} \lambda \int \Omega (u_k)^2 \, dx - \int \Omega b(x)(u_k)^p \, dx - \int \Omega \Psi'(\nabla u_k) \cdot \nabla u_k \, dx \\
+ \lambda \int \Omega (u_k)^2 \, dx + \int \Omega b(x)(u_k)^p \, dx = \\
\int \Omega (p-2)\Psi(\nabla u_k) \, dx + \int \Omega [2\Psi(\nabla u_k) - \Psi'(\nabla u_k) \cdot \nabla u_k] \, dx - \lambda \left( \frac{p}{2} - 1 \right) \int \Omega (u_k)^2 \, dx = \\
(p-2)c - \int \Omega w_k u_k \, dx + C. \tag{3.4}
\]
By (3.8) and (Ψ₂) one gets

\[ \mu(p - 2 - \sigma) \int_{\Omega} |\nabla u_k|^2 \, dx - \lambda \left( \frac{p}{2} - 1 \right) \lambda \int_{\Omega} (u_k)^2 \, dx \leq pc - \int_{\Omega} w_k u_k + C \]  (3.5)

so

\[ \mu(p - 2 - \sigma) \int_{\Omega} |\nabla u_k|^2 \, dx \leq \lambda \left( \frac{p}{2} - 1 \right) \int_{\Omega} (u_k)^2 \, dx + C \]  (3.6)

where the quantity \((p - 2 - \sigma)\) is strictly positive since \(\sigma\) is arbitrarily small. Our aim is to prove the boundedness of the \(H^1_0\) norm of the Palais Smale sequences, so arguing by contradiction, let us assume that

\[ ||u_k|| \to \infty \quad \text{as} \quad k \to +\infty. \]

Dividing (3.3) by \(||u_k||^p\) yields

\[ \liminf \left\{ \frac{p}{2} \int_{\Omega} \frac{\Psi(\nabla u_k)}{||u_k||^p} \, dx + \frac{\lambda p}{2} \int_{\Omega} (u_k)^2 \, dx - \frac{1}{p} \int_{\Omega} b(x) \left( \frac{u_k}{||u_k||} \right)^p \, dx \right\} = 0. \]

Since \(p > 2\) and (Ψ₂) holds, the first two terms go to zero. So

\[ \limsup \left( \int_{\Omega} b(x) \left( \frac{u_k}{||u_k||} \right)^p \, dx \right) = 0. \]  (3.7)

Since \(b\) is bounded, by Lebesgue dominated convergence Theorem we can take the limit and deduce that

\[ \lim b(x) \left( \frac{u_k}{||u_k||} \right)^p = 0. \]  (3.8)

This yields that

\[ \left( \frac{u_k}{||u_k||} \right) \to u_0 \]

strongly in \(L^p\) and weakly in \(H^1_0(\Omega)\). Arguing by contradiction let us suppose
that \( u_0 \equiv 0 \). Dividing (3.6) by \( ||u_k||^2 \) yields

\[
\mu (p - 2 - 2\sigma) \leq \lambda \left( \frac{p}{2} - 1 \right) \frac{1}{||u_k||^2} \int_\Omega (u_k)^2 \, dx + \frac{C}{||u_k||^2} \quad (3.9)
\]

the right hand side goes to zero, which leads to a contradiction since \( p - 2 - 2\sigma > 0 \) and \( \mu > 0 \), so \( u_0 \) must not be identically zero.

Now let \( \phi \in C^\infty_0(\Omega^+) \) be a compact support function, \( \phi \geq 0 \) and \( \phi \not\equiv 0 \). Let us use the function \( t\phi v, v \in H^1_0(\Omega) \) as a test in 3.1:

\[
\int_{\Omega^+} \Psi'(\nabla u_k)(t\phi \nabla v + tv \nabla \phi - \nabla u_k) \geq \lambda \int_{\Omega^+} u_k(tv\phi - u_k) + \int_{\Omega^+} b(x)(u_k)^{p-1}(tv\phi - u_k)
\]

\[
\int_{\Omega^+} w_k(tv\phi - u_k) \quad \forall v \in H^1_0(\Omega)
\]

Then let us divide the previous inequality by \( t \) and then let \( t \to +\infty \):

\[
\int_{\Omega^+} \Psi'(\nabla u_k)(\phi \nabla v) + \Psi'(\nabla u_k)v \nabla \phi \geq + \lambda \int_{\Omega^+} (u_k)^2 v\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1} v\phi +
\]

\[
+ \int_{\Omega} w_k v\phi \quad \forall v \in H^1_0(\Omega) \quad (3.10)
\]

On the other hand, if \( t \to -\infty \), one gets the opposite inequality, so we can deduce that the equality holds in the last expression, that is

\[
\int_{\Omega^+} \Psi'(\nabla u_k)(\phi \nabla v) + \Psi'(\nabla u_k)v \nabla \phi =
\]

\[
+ \lambda \int_{\Omega^+} (u_k)^2 v\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1} v\phi + \int_{\Omega} w_k v\phi \quad \forall v \in H^1_0(\Omega). \quad (3.11)
\]

Now let us choose \( v = u_k \) and divide both handsides of (3.11) by \( ||u_k||^p \). It is easily seen that the terms containing \( \lambda \) and \( w_k \) go to 0 as \( k \to +\infty \). Then
\[ \int_{\Omega^+} \frac{\Psi'(\nabla u_k) \nabla u_k \phi}{||u_k||^p} \]
graves 0 since \( p > 2 \) and (1.7) holds.

On the other hand, by (1.7), since \( p > 2 \) and \( \phi \) is of class \( C^\infty \) in \( \Omega^+ \) bounded,

\[ \frac{1}{||u_k||^p} \int_{\Omega^+} \Psi'(\nabla u_k) u_k \nabla \phi \leq C \frac{||u_k||}{||u_k||_{L^2}} \frac{||u_k||_{L^2}}{||u_k||} \]

The term \( \frac{||u_k||_{L^2}}{||u_k||} \) is bounded, while \( \frac{||u_k||_{L^2}}{||u_k||} \) converges to 0.

By (3.11) We can conclude that

\[ \int_{\Omega^+} \frac{1}{||u_k||^p} b^+(x)(u_k)^p \phi \to 0 \text{ as } k \to \infty. \]

Applying Fatou’s Lemma yields

\[ \liminf \int_{\Omega^+} \frac{1}{||u_k||^p} b^+(x)(u_k)^p \phi \leq 0 \]

and since the integrand is nonnegative, this means that \( \frac{u_k}{||u_k||^p} \) must tend to 0 in \( \Omega^+ \) as \( k \to \infty \), but this is in contradiction with the fact that it was already proved that it converges to a nonzero function \( u_0 \).

Arguing in the same way, choosing now a compact support function \( \eta \in C_0^\infty (\Omega^-) \), yields that \( \frac{u_k}{||u_k||^p} \to 0 \) as \( k \to \infty \), in \( \Omega^- \).

This proves that \( ||u_k|| \) is bounded in \( H_0^1(\Omega)(\Omega^+ \cup \Omega^-) \), and since \( \Omega^0 \) is negligible, this concludes this part of the proof. Then \( u_k \) admits a subsequence weakly converging in \( L^2 \).

According to (3.1) and taking \( v = u \) as a test function yields

\[ \int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \geq \lambda \int_{\Omega} u_k (u - u_k) \, dx + \int_{\Omega} b(x)(u_k)^{p-1}(u - u_k) \, dx + o(1) \]

(3.12)
so as $k \to \infty$ the right hand-side terms go to zero, and we obtain

$$\liminf \int_\Omega \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \geq 0.$$  \hfill (3.13)

On the other hand, by convexity

$$\int_\Omega \Psi(\nabla u) \, dx \geq \int_\Omega \Psi(\nabla u_k) \, dx + \int_\Omega \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx$$  \hfill (3.14)

So by (3.13) and (3.14)

$$\limsup \int_\Omega \Psi(\nabla u_k) \, dx \leq \limsup \left( \int_\Omega \Psi(\nabla u) \, dx - \int_\Omega \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \right)$$

$$\leq \int_\Omega \Psi(\nabla u) \, dx - \liminf \int_\Omega \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \leq \int_\Omega \Psi(\nabla u) \, dx$$  \hfill (3.15)

By lower semicontinuity and convexity

$$\liminf \int_\Omega \Psi(\nabla u_k) \, dx \geq \int_\Omega \Psi(\nabla u) \, dx$$  \hfill (3.16)

We can conclude that

$$\int_\Omega \Psi(\nabla u_k) \, dx \to \int_\Omega \Psi(\nabla u) \, dx.$$

By Theorem 3.1, $u_k$ admits a subsequence strongly converging in $L^2$, which concludes the proof of PS condition and of Theorem 1.1.

\[ \square \]

**Proof of Theorem 1.2**

We are now concerned with the existence of (possibly sign-changing) nontrivial solutions $u$ of (P). Let $(\lambda_k)$ denote the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet condition, repeated according to multiplicity.

Since the case $0 < \lambda < \lambda_1$ is already contained in Theorem 1.1, we may assume...
that $\lambda \geq \lambda_1$. Let $k \geq 1$ be such that $\lambda_k \leq \lambda < \lambda_{k+1}$, $e_1, \ldots, e_k$ are eigenfunctions of $-\Delta$, as defined in the introduction. Finally, let $E_- = \text{span}\{e_1, \ldots, e_k\}$ and $E_+ = E_-^\perp$.

Consider the functional $J$ defined in (2.1). We aim to apply the version of the Linking Theorem for convex functional presented by Szulkin in [12]. Since

$$\frac{\int_{\Omega} \Psi(\nabla u) \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} \to \frac{1}{2} \quad \text{as} \ u \to 0 \ \text{in} \ H_0^1(\Omega),$$

as in the case $\Psi(\xi) = \frac{1}{2} |\xi|^2$ treated in [8], we deduce that there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ whenever $u \in E_+$ with $\|u\| = \rho$. On the other hand, there exists $e \in H_0^1(\Omega) \setminus E_-$ such that

$$\lim_{\|u\| \to \infty, \ u \in \text{Re} \oplus E_-} J(u) = -\infty,$$

Again, this is proved in [8] when $\Psi(\xi) = \frac{1}{2} |\xi|^2$, but by $(\Psi_2)$ the assertion is true also in our case. Finally, it is clear that $J(u) \leq 0$ for every $u \in E_-$. By the Linking type theorem in [12] (Theorem 3.4), there exist a PS sequence $(u_k)$ in $H_0^1(\Omega)$ and we can continue, up to minor changes, as in the proof of Theorem 1.1 to prove that there exists a subsequence of $(u_k)$ strongly converging in $L^{2^*}$. This concludes the proof of Theorem 1.2, since the nontriviality of the solution comes directly from the characterization of the critical level of the solution.

\[\square\]

**Acknowledgment** The author thanks Prof. Marco Degiovanni for very helpful conversations.
References

[1] S. Alama and M. del Pino, Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking, Ann. I.H.P. Analyse non linéaire, Vol. 13 (1996), 95–115.

[2] S. Alama and G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, Calc. Var., Vol 1, (1993), 439–475.

[3] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, J. Funct. An. Vol. 141, (1996), 159–215.

[4] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal., Vol. 14 (1973), 349–381.

[5] M. Badiale, Infinitely many solutions for some indefinite nonlinear elliptic problems, Comm. on Appl. Non. Anal., Vol. 3 (1996), 61–76.

[6] M. Badiale and E. Nabana, A remark on multiplicity solutions for semilinear elliptic problems with indefinite nonlinearities, C.R. Acad. Sc. Paris, Vol. 323, serie 1 (1996), 151–156.

[7] M. Degiovanni, Variational methods for functionals with lack of strict convexity, in "Nonlinear Equations: Methods, Models and Applications" (Bergamo, 2001), D. Lupo, C. Pagani and B. Ruf, eds., 127–139, Progress in Nonlinear Differential Equations and their Applications, Vol. 54, Birkhäuser, Boston, Inc., Boston, Ma, 2003.

[8] M.Grossi, P.Magrone, M.Matzeu, Linking type solutions for elliptic equations with indefinite nonlinearities up to the critical growth, Discrete Contin. Dynam. Systems , Vol. 7, no 4, (2001), 703-718.

[9] P.Magrone, An existence result for a problem with critical growth and lack of strict convexity, NoDEA Nonlinear Differential Equations Appl. Vol. 15, no 6, (2008), 717–728.

[10] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations", "CBMS Regional Conference Series in
[11] M. Ramos, S. Terracini and C. Troestler, Superlinear indefinite elliptic problems and Pohozaev type identities, J. Funct. Anal., Vol 159, (1998), 596–628.

[12] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems., Ann. Inst. H. Poincaré Anal. Non Linéaire, Vol. 3, no. 2, (1986), 77-109.

[13] H.-C. Wu, Continuum mechanics and plasticity, CRC Series: Modern Mechanics and Mathematics, Chapman & Hall/CRC, Boca Raton, FL., (2005).