Generalized $k$-core pruning process on directed networks

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The resilience of a complex interconnected system concerns the size of the macroscopic functioning node clusters after external perturbations based on a random or designed scheme. For a representation of the interconnected systems with directional or asymmetrical interactions among constituents, the directed network is a convenient choice. Yet how the interaction directions affect the network resilience still lacks thorough exploration. Here, we study the resilience of directed networks with a generalized $k$-core pruning process as a simple failure procedure based on both the in- and out-degrees of nodes, in which any node with an in-degree $< k_{in}$ or an out-degree $< k_{ou}$ is removed iteratively. With an explicitly analytical framework, we can predict the relative sizes of residual node clusters on uncorrelated directed random graphs. We show that the discontinuous transitions rise for cases with $k_{in} \geq 2$ or $k_{ou} \geq 2$, and the unidirectional interactions among nodes drive the networks more vulnerable against perturbations based on in- and out-degrees separately.

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I. INTRODUCTION

Resilience of the real interconnected systems (such as infrastructural, ecological, and financial systems) against external perturbations is among the fundamental themes in the understanding of complex systems, especially the phenomena of sharp transitions or tipping points in their structural and also dynamical regimes [1]. When it’s adopted as a representation of nodes as constituents and edges as interactions among them, the complex network theory [2–5] in many cases offers an analytical framework for the resilience problem as a percolation transition [6] which usually involves an emergence of macroscopic residual subgraph structure against a removal procedure of nodes and edges. Typical methods to explore the resilience of networks with undirected interactions are the shrinkage of the giant connected component against a random node removal [7–9], the $K$-core percolation based on a degree-constrained pruning process [10–13], the $K$-core percolation with an inducing effect on the intact (not failed) nodes exerted by the collapsed ones [14], the core percolation with a greedy leaf-removal procedure [15], and a node removal process based on articulation points each of which can disintegrate a network into multiple components after its removal [16]. When we consider the networks with directional interactions, we have examples as the emergence of giant strongly connected components against random node removals [17–19], the core percolation in specific contexts of defining leaves and node removals [15–20]. With the notion of multilayer networks [21–22], the effect of the coupling between nodes or node copies on the network resilience are studied through the stability of connected components under a random node removal [23] and a $K$-core-like pruning process [24]. The interaction direction is further incorporated into the multilayer networks to explore their resilience based on the size of the strongly connected components against a random node removal [22]. Besides the above research line of the network resilience in various contexts, there is another one on the implication of interaction directions on networks from the perspectives of the structural organization [26, 27] and the dynamical processes on them [28]. Yet the intersection of the two research lines, the effect of interaction directions on the network resilience, is still far from being fully discussed from an analytical perspective except for the above few cases, on which we lay our focus in this paper.

Here we study the resilience of directed networks through a node failure model which can be considered as a generalized $K$-core pruning process. In the failure scheme, a node fails once it has too small an in-degree or an out-degree. The motivation of the failure model is that, in systems with directed or asymmetrical interactions such as those involving information flows or purely with a principle of redundancy design, the proper functioning of each constituent can be assumed to be based on sufficient sizes of both its neighbors from which it receives interactions and to which it delivers influences. Formulating the failure model into a pruning process on a directed network, we randomly and iteratively remove any node with an in-degree smaller than an integer $k_{in}(\geq \ 0)$ or with an out-degree smaller than an integer $k_{ou}(\geq \ 0)$ along with all its adjacent arcs, and we study the relative size of the macroscopic residual subgraph. As we can see, the above process is basically an extension of $K$-core pruning process on undirected networks into the case with directed interactions. This pruning process is initially discussed in [29], which lists the sizes of all the non-trivial residual structures after the above pruning process on a real network as a tool for data analysis. Yet a detailed theoretical analysis of the related percolation problem, or the $(k_{in}, k_{ou})$-core percolation as we can simply put, is still missing. In this paper, we consider the generalized $k$-core pruning process and the related $(k_{in}, k_{ou})$-core percolation problem as a solvable model for the resilience study of the directed networks, and we mainly work on the derivation of an analytical framework and the analysis of its transition behaviors.

There are two parts of our main results. (1) The $(k_{in}, k_{ou})$-core percolation problem with $k_{in} \geq 2$ or $k_{ou} \geq 2$ shows abrupt transitions on infinitely large directed random networks, which can be proved with our analytical framework. (2) We compare the relative sizes of the $(k_{in}, k_{ou})$-core on a directed random graph with the $K$-core on its undirected counterpart when the arc directions are totally ignored. Based on the transition points of two percolation problems, we can see from an analytical perspective that when a macroscopic $(k_{in}, k_{ou})$-core is permitted in some case, a $K$-core with a much larger $K$ than $k_{in}$ and $k_{ou}$ is possible, indicating that the introduction of unidirectional interactions between nodes can drive the connected systems more vulnerable against external perturbations based on in-degrees and/or out-degrees of nodes.

Here is the structure of the paper. In Sec II, we explain the pruning process and the percolation problem in a general setup in networks with both undirected and directed interactions. In Sec III, we present an analytical theory for the problem on random networks with only unidirectional interactions. In Sec IV, we test the theory on model directed random networks and also real network data sets, in which we also examine the discontinuity and the scaling property of the hybrid transitions. In Sec V, we conclude the paper with a discussion.

II. MODEL

First we explain some notations for directed networks. We consider a directed network instance $D = \{V, A\}$ with a node set $V (|V| = N)$ and an arc set $A (|A| = M)$, correspondingly its arc density $c = M/N$. In the context of
and its adjacent arcs. The generalized in-degree can see that if there are some nodes remained after the removal process, any node $i$ along with their adjacent arcs will be removed in the pruning process.

Directed networks, we adopt the arc density (the arc-node ratio, or the number of connections divided by the number of nodes) to describe the density of connections among nodes for a network. We should mention that in the context of undirected networks, a conventional notation of connection densities is the mean degree or the mean connectivity (average number of connections adjacent to a node), which is two times of the edge-node ratio (also the number of connection divided by the number of nodes). In order to avoid the confusion of notations, we will specify the context of undirected networks when we mention the mean degree or the mean connectivity. A directional arc between two nodes in the network, say $i$ and $j$, is denoted as an ordered pair $(i, j)$ as an interaction or relation pointing from $i$ to $j$. For an arc $(i, j)$, $i$ is considered as the predecessor (an in-neighbor) of $j$, and $j$ is considered as the successor (an out-neighbor) of $i$. Correspondingly, an arc $(i, j)$ is considered as an out-going arc for node $i$ and an in-coming arc for node $j$. For any node $i$, all its in-neighbors constitute a set $\partial i^+$ with an in-degree $k_+ (\equiv |\partial i^+|)$, and all its out-neighbors constitute a set $\partial i^-$ with an out-degree $k_-(\equiv |\partial i^-|)$. We define the degree distribution $P(k_+, k_-)$ of a directed network $D$ as the probability that a randomly chosen node has $k_+$ in-neighbors and $k_-$ out-neighbors. The arc density is thus $c = \sum_{k_+, k_-} k_+ P(k_+, k_-) = \sum_{k_+, k_-} k_- P(k_+, k_-)$. We further define two excess degree distributions. For a randomly chosen arc $(i, j)$, from node $i$ following the arc direction to node $j$, the probability of node $j$ having $k_+$ in-neighbors and $k_-$ out-neighbors is $Q_+(k_+, k_-)$; from node $j$ following the opposite arc direction to node $i$, the probability of node $i$ having $k_+$ in-neighbors and $k_-$ out-neighbors is $Q_-(k_+, k_-)$. We simply have $Q_+(k_+, k_-) = k_+ P(k_+, k_-)/c$ and $Q_-(k_+, k_-) = k_- P(k_+, k_-)/c$.

On a directed graph $D = \{V, A\}$, an initial fraction $p \in [0, 1]$ is defined in a starting step, and a fraction $1 - p$ of nodes are randomly chosen and further removed along with all their adjacent arcs. An iterative pruning process is then carried out as any node $i$ with an in-degree $k^+_i < k_{in}$ or an out-degree $k^-_i < k_{ou}$ is removed, along with all its adjacent arcs. The generalized $k$-core pruning process can be named as the $(k_{in}, k_{ou})$-core pruning process. We can see that if there are some nodes remained after the removal process, any node $i$ in this subgraph has both an in-degree $k^+_i \geq k_{in}$ and an out-degree $k^-_i \geq k_{ou}$. We simply call the collection of the residual nodes and arcs as the $(k_{in}, k_{ou})$-core or the core structure, and the emergence of a macroscopic residual subgraph as the $(k_{in}, k_{ou})$-core percolation. See figure 1 for some examples of the elementary pruning steps in the $(k_{in}, k_{ou})$-core percolation.

The above model is considered on networks with only directed arcs, yet it can be also defined in a generalized case on networks with both undirected edges and directed arcs. In a mapping procedure, each undirected edge is
considered as a degenerate connection, and is further split into two directed arcs in opposite directions. For example, an undirected edge as an unordered pair \( \{i,j\} \) between nodes \( i \) and \( j \) is split into two directed arcs \( (i,j) \) and \( (j,i) \). After this mapping procedure, the \((k_{in}, k_{ou})\)-core percolation can be well-defined on the new graph with only directed arcs. As a special example, the \(K\)-core percolation on an undirected network can be considered as the \((K,K)\)-core percolation on its directed network counterpart after the above mapping. In such a general setting, we can consider the \((k_{in}, k_{ou})\)-core percolation problem on a graph \( G = \{V, A^{un}, A^{di}\} \) with a node set \( V \), a set of undirected edges \( A^{un} \), and a set of directed arcs \( A^{di} \). The degree distribution \( P(k_{in}, k_{+, k_-}) \) for the graph \( G \) can be defined as the probability of a randomly chosen node with \( k_{in} \) undirected interacted neighbors, \( k_+ \) in-coming arcs, and \( k_- \) out-going arcs. A special instance with mixed connections can be generated from an undirected graph by assigning a fraction \( \rho \) of randomly chosen edges into directed arcs, in which an undirected edge between a node pair \( i \) and \( j \) is annotated as \((i,j)\) or \((j,i)\) with equal probabilities.

Yet, in order to keep our focus on the analytical solutions and avoid the extra complexity from mixed types of connections, here we consider the \((k_{in}, k_{ou})\)-core percolation problem on directed networks with only unidirectional connections among nodes, or at most one directed arc between any two nodes.

III. THEORY

The \((k_{in}, k_{ou})\)-pruning process can be applied on any network instance to reveal its core structure. Yet for the directed uncorrelated random graphs, we can derive a mean-field theory based on the cavity method \cite{31} to theoretically predict the relative sizes of their \((k_{in}, k_{ou})\)-cores. With the cavity method, we arrive at a set of self-consistent belief-propagation (BP) equations of cavity probabilities or messages, whose formalism has roots in both statistical mechanics \cite{32} and computer science \cite{33,34}. With the stable solutions of the BP equations, we can calculate the quantities related to the core structure.

For a directed random graph \( D = \{V,A\} \), we define two cavity probabilities to derive the BP equations. On a randomly chosen arc \((i,j)\), we start from the node \( j \) which is in the core structure, and arrive at the node \( i \) following the opposite arc direction; we define \( \alpha \) as the probability that \( i \) is in the core structure while \( j \) is not considered. In a similar sense, we start from the node \( i \) which is in the core structure, and arrive at the node \( j \) following the arc direction; we define \( \beta \) as the probability that \( j \) is in the core structure while \( i \) is not considered. We further assume that on an infinitely large random graph which is uncorrelated and sparse, a randomly chosen node \( i \) has a locally tree-like structure, or its neighbors show no correlation in states in the pruning process after \( i \) is removed. With this approximation \cite{31}, on a directed random graph with a degree distribution \( P(k_{+, k_-}) \), we have the self-consistent equations for \( \alpha \) and \( \beta \).

\[
\alpha = p \sum_{k_{+, k_-}} Q_- (k_{+, k_-}) \left[ \sum_{k_1=k_{in}}^{k_+} \left( \frac{k_+}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_- - k_1} \right] \sum_{k_2=k_{ou}}^{k_- - 1} \left( \frac{k_- - 1}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_- - 1 - k_2},
\]

\[
\beta = p \sum_{k_{+, k_-}} Q_+ (k_{+, k_-}) \left[ \sum_{k_1=k_{in}}^{k_- - 1} \left( \frac{k_- - 1}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_+ - 1 - k_1} \right] \sum_{k_2=k_{ou}}^{k_-} \left( \frac{k_-}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_- - k_2}.
\]

We briefly explain the derivation of the self-consistent equation of \( \alpha \), and the equation for \( \beta \) follows a rather similar logic. On a randomly chosen arc \((i,j)\) in a directed graph \( D \), from node \( j \) following the opposite arc direction to node \( i \), the node \( i \) has a probability \( p \) that it remains in the residual graph after the initial removal step. In order to further remain in the core structure after the pruning process, the node \( i \) should have both at least \( k_{in} \) in-neighbors in the core structure and \( k_{ou} - 1 \) out-neighbors in the core structure besides the node \( j \) which is already in the core structure. With the stable solutions of \( \alpha \) and \( \beta \), the relative size of nodes in the core structure \( n_{core} \) can be calculated.

\[
n_{core} = p \sum_{k_{+, k_-}} P(k_{+, k_-}) \left[ \sum_{k_1=k_{in}}^{k_+} \left( \frac{k_+}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_- - k_1} \right] \sum_{k_2=k_{ou}}^{k_-} \left( \frac{k_-}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_- - k_2}.
\]

Here is an explanation of the equation for \( n_{core} \): if a newly added node can be in the final core structure, it should have both at least \( k_{in} \) in-neighbors and \( k_{ou} \) out-neighbors in the core structure, provided that it is not removed in the initial step.

We can also derive the arc density \( c_{core} \) of the core structure as
\[ c_{\text{core}} = \sum_{k_1 \geq k_{\text{in}}, k_2 \geq k_{\text{out}}} k_1 P_{\text{core}}(k_1, k_2) = \sum_{k_1 \geq k_{\text{in}}, k_2 \geq k_{\text{out}}} k_2 P_{\text{core}}(k_1, k_2), \]  

(4)

while \( P_{\text{core}}(k_1, k_2) \) with \( k_1 \geq k_{\text{in}} \) and \( k_2 \geq k_{\text{out}} \) denotes the degree distribution of the core structure and has the form

\[ P_{\text{core}}(k_1, k_2) = \frac{1}{n_{\text{core}}} \sum_{k_+ \geq k_1, k_- \geq k_2} P(k_+, k_-)[(\frac{k_+}{k_1})^\alpha (1-\alpha)^{k_+ - k_1}](\frac{k_-}{k_2})^\beta (1-\beta)^{k_- - k_2}. \]  

(5)

We briefly explain Eq.\( 4 \) and then Eq.\( 3 \). If a node survives the initial removal and has an in-degree \( k_1 \) and an out-degree \( k_2 \) after the pruning process, surely it has \( k_1 \geq k_{\text{in}} \) and \( k_2 \geq k_{\text{out}} \), and it should have exactly \( k_1 \) in-neighbors and \( k_2 \) out-neighbors in the core structure, before the node is added into the original graph. The above probability is divided by \( n_{\text{core}} \) to derive the probability of the node with \( k_1 \) in-neighbors and \( k_2 \) out-neighbors in the core structure. Thus we arrive at Eq.\( 4 \). After the average on all the in-degrees or all the out-degrees in the core structure, we have the mean arc density of the core structure as shown in Eq.\( 3 \).

Here we consider the numerical method to derive the stable solutions of \( (\alpha, \beta) \) given \( P(k_+, k_-) \) and \( p \), with which we can calculate \( n_{\text{core}} \) and \( c_{\text{core}} \). For the ease of discussion, we denote the right-hand side of Eq.\( 1 \) as \( f(\alpha, \beta) \) while a function \( F(\alpha, \beta) \) is defined as \( F(\alpha, \beta) = -\alpha + f(\alpha, \beta) \), the right-hand size of Eq.\( 2 \) as \( g(\alpha, \beta) \) while a function \( G(\alpha, \beta) \) is defined as \( G(\alpha, \beta) = -\beta + g(\alpha, \beta) \). For an \( \alpha \in [0,1] \), we can calculate the corresponding stable \( \beta \) with Eq.\( 2 \). The calculation of stable \( \beta \) goes like this: for a \( \beta \in [0,1] \) given \( \alpha \), we can calculate \( G(\alpha, \beta) \) with Eq.\( 2 \) with an incremental procedure with small steps for \( \beta \), we can find all the fixed \( \beta \) in the range when \( G(\alpha, \beta) \) change its sign; the stable \( \beta \) is the largest one among its fixed solutions. With the stable \( \beta \) given \( \alpha \), we can calculate \( F(\alpha, \beta) \) with Eq.\( 1 \) With a similar procedure for calculating the stable \( \beta \) with given \( \alpha \), we can calculate all the fixed \( \alpha \), thus the stable \( \alpha \). With the stable \( \alpha \), we can derive its corresponding stable \( \beta \) with Eq.\( 2 \). Thus we have the stable solution for \( (\alpha, \beta) \).

We should mention that there are some special examples of the \((k_{\text{in}}, k_{\text{out}})\)-core percolation. When \( k_{\text{in}} = \{0,1\} \), the term from the in-neighbors on the right-hand side of Eq.\( 2 \) reduces to 1; when \( k_{\text{out}} = \{0,1\} \), the term from the out-neighbors on the right-hand side of Eq.\( 1 \) reduces to 1. In the case of \((k_{\text{in}}, k_{\text{out}}) = (1,0)\), with the substitution of \( 1 - \alpha \rightarrow x \), we arrive at the equation for the in-components of directed graphs; in the same sense for the case of \((k_{\text{in}}, k_{\text{out}}) = (0,1)\) with the mapping \( 1 - \beta \rightarrow y \), we have the theory for the out-components of directed graphs [35]. In the case of \((k_{\text{in}}, k_{\text{out}}) = (1,1)\), we further have the theory for the strongly connected components (SCC) as the intersection of in-components and out-components in directed random graphs [17–19]. Transition behaviors in the above three cases of the percolation problem in directed random graphs are continuous. Yet for the cases with \( k_{\text{in}} \geq 2 \) or \( k_{\text{out}} \geq 2 \), later we will see that a quite different scenario of percolation transitions happens.

IV. RESULTS

A. Random networks

Here we consider the \((k_{\text{in}}, k_{\text{out}})\)-cores of some model directed random networks. We leave the simplified equations for the \((k_{\text{in}}, k_{\text{out}})\)-core percolation on directed random networks in Appendix A. In the paper, we also compare the \((k_{\text{in}}, k_{\text{out}})\)-cores with the \( K \)-cores on the undirected version of directed random networks, and we left the theory of the \( K \)-core percolation problem on undirected networks in Appendix B.

First we consider the case of directed Erdős-Rényi (ER) random graphs [30–37]. To generate a directed ER random graph, we first generate an undirected ER random graph instance; then for each undirected edge between a node pair \( i \) and \( j \), we assign a direction to form a directed arc \((i,j)\) or \((j,i)\) with an equal probability. For a directed ER random graph with an arc density \( c \), the degree distribution is \( P(k_+, k_-) = e^{-c^{k_+}/k_+!} \times e^{-c^{k_-}/k_-!} \). In figure 2 we show the normalized sizes of \((k_{\text{in}}, k_{\text{out}})\)-cores with different initial fraction \( p \) and arc density \( c \). We can see that for \((k_{\text{in}}, k_{\text{out}}) = \{0,1\}\), the emergence of the core structure undergoes a continuous transition. While in the cases with \( k_{\text{in}} \geq 2 \) or \( k_{\text{out}} \geq 2 \), the emergence shows a first-order transition. This constitutes our first interesting result, as it is well-known that the \( K \)-core percolation with \( K \geq 3 \) on undirected ER graphs is discontinuous. An intuitive understanding of this result is that a node removal procedure, in which part of the neighbors of a node rather than all its residual neighbors can result in the failure of the node, usually leads to a more aggressive network breakdown process. A similar logic shows in the model with an extra inducing effect from the failed nodes on undirected networks [14] and the models with failure propagation through coupling or interdependency between nodes on multilayer structures [23–24]. A theoretical explanation of the discontinuous emergence of core structures can be understood from the behaviors of the stable solutions of \( \alpha \) and \( \beta \). We first take the \((0,2)\)-core percolation
problem as an example. For the (0, 2)-core percolation with \( p = 1 \), the iterative equations and the normalized core structure are \( \alpha = 1 - e^{-c\beta}, \beta = 1 - e^{-c\beta}(1 + c\beta) \), \( n_{\text{core}} = \beta \). Since it only depends on \( \beta \), the transition behavior of \( n_{\text{core}} \) can be explained by the stable \( \beta \). With the functions defined in the previous section, we show \( G(\alpha, \beta) \) in figure 3(a) as we can find all the fixed \( \beta \) with \( G(\alpha, \beta) = 0 \). For \( c < c^* \) as the critical arc density \( c^* \approx 3.35002 \), there is only one fixed and also the stable solution \( \beta = 0 \), correspondingly \( n_{\text{core}} = 0 \). When \( c = c^* \), a second fixed and also the stable solution \( \beta \approx 0.535 \) shows up abruptly, thus a sudden emergence of the core structure with a critical normalized size \( n_{\text{core}}^* = \beta \). As \( c > c^* \), the stable solution of \( \beta \) and the corresponding \( n_{\text{core}} \) increase further. For the
The core structure, a scaling property applies as a general proof of this discontinuity, where we also show that in the supercritical region of a discontinuous transition of graphs can be applied to other cases of similar transition behavior of stable solutions of an initial fraction of nontrivial stable solutions of transition pattern happens for the stable transition of percolation in (b) are calculated as fixed solutions of stable, and the corresponding abrupt behavior of stable $\alpha, \beta$ shows up besides the trivial fixed solution $\beta = 0$, thus both the stable $\beta$ and $n_{\text{core}}$ experience a sudden jump. In (b), a similar transition behavior of stable $\alpha$ happens at $c = c^*$ with $c^* \approx 3.81662$.

The second interesting result on directed ER random graphs is that a (0, 2)-core emerges around an arc density where a 5-core suddenly shows up on the undirected counterparts. As in figure (a), a 5-core on an undirected ER ensemble with a mean degree $d = 10.0$ emerges at an initial fraction $p \approx 0.680$, while a (0, 2)-core on the directed ER random graph ensemble with an arc density $c \equiv d/2 = 5.0$ shows up at an initial fraction $p \approx 0.671$; in figure (b), a 5-core on an undirected ER ensemble shows up at a mean degree $d \approx 6.800$, correspondingly an edge-node ratio 3.400, while a (0, 2)-core on the directed ER random graph ensemble emerges at an arc density $c \approx 3.351$. The above comparison means that for a directed ER random graph which permits a certain $(k_{\text{in}}, k_{\text{out}})$-core, a macroscopic K-core with a much larger parameter $K$ than $k_{\text{in}}$ and $k_{\text{out}}$ is possible when the arc directions are ignored. It is reasonable to expect that the $(k_{\text{in}}, k_{\text{out}})$-core pruning process is generally a more aggressive network breakdown procedure than the K-core pruning process after ignoring arc directions, for example in the case of $k_{\text{in}} + k_{\text{out}} = K$, yet here we provide an analytical ground for this intuitive understanding.

We then consider the percolation problem on the directed regular random (RR) graphs. We generate a directed RR graph instance from an undirected RR graph instance, in which each edge of the undirected instance is assigned randomly with a direction just like we do in the generation of directed ER random graphs. For an undirected RR graph with an integer degree $k_0$ (an undirected neighbor for each node), a corresponding directed RR graph instance with an arc density $c \equiv k_0/2$ can be generated. In figure 4, we show the result for the normalized core structures on directed RR graphs. We further consider the percolation problem on the scale-free (SF) networks. SF networks show power-law degree distributions, and are ubiquitous in the real world. We consider the case of SF networks with a form of degree distribution without degree correlation as $P(k_+, k_-) \approx P_+(k_+)P_-(k_-)$ with $P_\pm(k_\pm) \propto k_\pm^{-\gamma_\pm}$, while $\gamma_+$ and $\gamma_-$ are respectively the exponents for the in-degrees and the out-degrees.

We first consider the directed SF networks generated with the configurational model which are constructed based on sequences of in-degrees and out-degrees generated directly from the power-law distribution. To construct such a directed SF network with a node size $N$ with the degree distribution $P(k_+, k_-) \propto k_+^{-\gamma_+}k_-^{-\gamma_-}$, we need to further specify a minimal degree $k^{\text{min}}_+$ and a maximal degree $k^{\text{max}}_+$ for in-degrees, and a minimal degree $k^{\text{min}}_-$ and a

![Diagram](image-url)
maximal degree $k_{\text{max}}^+$ for out-degrees. With the parameters $(\gamma_+, k_{\text{min}}^+, k_{\text{max}}^+)$, we can construct an in-degree sequence with $P_+(k_+) \propto k_+^{\gamma_+}$ with $k_{\text{min}}^+ \leq k_+ \leq k_{\text{max}}^+$, while there are $NP_+(k_+)$ nodes each with $k_+$ in-coming half-arcs and in total $E_+ (= \sum k_+ NP_+(k_+))$ in-coming half-arcs. The same procedure is adopted to construct an out-degree sequence through $P_-(k_-) \propto k_-^{\gamma_-}$ with $k_{\text{min}}^- \leq k_- \leq k_{\text{max}}^-$, while there are $NP_-(k_-)$ nodes each with $k_-$ out-going half-arcs and in total $E_- (= \sum k_- NP_-(k_-))$ out-going half-arcs. We further make sure that the numbers of in-coming and out-going half-arcs are equal as $E = (E_+ + E_-)/2$ by removing and adding in-coming and out-going

FIG. 4. Normalized sizes of $(k_{in}, k_{ou})$-cores on directed RR graphs. The node fractions of some $(k_{in}, k_{ou})$-cores are calculated from the simulation and the analytical theory. In (a), the node fractions are calculated on directed RR graphs with an arc density $c = 5.0$ with different initial fraction $p$. For a comparison, the relative sizes of 5- and 6-cores are also calculated on infinitely large undirected RR graphs with a degree $k_0 = 10$ with the analytical theory. In (b), the node fractions are calculated on directed RR graphs with different arc densities with an initial fraction $p = 1$. Points are for the averaged results of simulation on 40 independently generated graph instances with a node size $N = 10^5$. The standard deviation for each data point from simulation is also shown. In (a), the solid lines are for the analytical results on infinitely large graphs, while the dashed lines are for the discontinuous transitions from the analytical theory. In (b), the points of intersection between solid line segments are for the analytical results on infinitely large graphs.
FIG. 5. Normalized sizes of \((k_{in}, k_{ou})\)-cores of directed SF networks. In (a), we calculate the node fractions of the \((k_{in}, k_{ou})\)-cores from the simulation and the analytical theory on a directed SF network instance generated by the configurational model. The SF network instance has a node size \(N = 10^6\) with an in-degree exponent \(\gamma_+ = 2.5\), an out-degree exponent \(\gamma_- = 3.0\), and a minimal degree \(k_{in}^{\min} = k_{ou}^{\min} = 4\) and a maximal degree \(k_{in}^{\max} = k_{ou}^{\max} = \sqrt{N}\) for both in-degrees and out-degrees. We also calculate the relative sizes of the 5- and 6-cores with simulation and analytical theory on the undirected network counterpart after ignoring the arc directions. Points are for the simulation results averaged from those on 40 independently generated initial configurations with a given initial fraction \(p\), while their standard deviations are also shown. Solid lines are for the analytical results based on the empirical degree distribution of the graph instance. Dashed lines indicate the abrupt transitions by the analytical theory. In (b), we calculate the node fractions of the \((k_{in}, k_{ou})\)-core structures from the simulation and the analytical theory on asymptotical SF networks generated with the static model. The SF networks have an in-degree exponent \(\gamma_+ = 2.5\) and an out-degree exponent \(\gamma_- = 3.0\). We also calculate the relative sizes of the 5- and 6-cores with simulation on the undirected network counterparts. Each data point is for the simulation result averaged on 40 independently generated directed SF instances with the node size \(N = 10^6\) or \(10^5\) as indicated in the legend. Standard deviations for all the simulation results are presented. Solid lines are for the analytical results on infinitely large graphs. Dashed lines indicate the discontinuous transitions from the analytical theory.
half-arcs correspondingly. Then an incoming half-arc and an outgoing half-arc are randomly chosen and paired to establish a genuine directed arc until there is no half-arc left. The graph instance generated with this procedure has a degree distribution \( P(k_+, k_-) \approx k_+^{\gamma_+} / \sum_{k_+} k_+^{-\gamma_+} \times k_-^{-\gamma_-} / \sum_{k_-} k_-^{-\gamma_-} \) and an arc density \( c = \sum_{k_+, k_-} k_+ P(k_+, k_-) = \sum_{k_+, k_-} k_- P(k_+, k_-) \), while in the summations \( k_+^{\min} \leq k_+ \leq k_+^{\max} \) and \( k_-^{\min} \leq k_- \leq k_-^{\max} \). In figure 3 (a), on a SF network instance with an in-degree exponent \( \gamma_+ = 2.5 \) and an out-degree exponent \( \gamma_- = 3.0 \), we show the sizes of core structures from both simulation and analytical theory with different initial fraction \( p \). Our analytical theory predicts well the birth points and the relative sizes of core structures on the graph instance even with a finite size.

Directed SF networks can also be generated with the static model [40, 41]. For a SF network instance with an in-degree exponent \( \gamma_+ \) and an out-degree exponent \( \gamma_- \), the construction procedure goes like this: for an empty graph with \( N \) nodes indexed as \( i \in \{1, 2, ..., N\} \), each node is assigned with an in-degree weight \( w_i^+ \propto i^{-\xi_+} \) and an out-degree weight \( w_i^- \propto i^{-\xi_-} \) as \( \xi_\pm \equiv 1/(\gamma_\pm - 1) \); in the arc establishment process, two distinct nodes, say nodes \( i \) and \( j \), are chosen proportionally to their weights \( w_i^+ \) and \( w_j^- \), respectively, and are connected into a directed arc \((i, j)\); with this process, we establish \( M = cN \) arcs with an arc density \( c \) in the graph. Based on the theory in [40, 41], the graph generated with this method has a degree distribution \( P(k_+, k_-) = P(k_+)P(k_-) \) as \( P(k_\pm) = \frac{1}{\xi_\pm} \frac{(c(1-\xi_\pm))^{\xi_\pm}}{k_\pm^{\xi_\pm}} \int_1^\infty dt e^{-t(1-\xi_\pm)/t} k_\pm^{\gamma_- - 1-1/\xi_\pm} \). With large \( k_+, k_- \), we have \( P(k_+, k_-) \propto k_+^{\gamma_+} k_-^{\gamma_-} \). In figure 5 (b), we calculate the core structures on approximate SF networks generated by the static model with an in-degree exponent \( \gamma_+ = 2.5 \) and an out-degree exponent \( \gamma_- = 3.0 \). We can see that for \((0, 2)\)-cores which are revealed by a pruning process based only on the out-degrees of nodes, results show quite small differences between the simulation results on instances with different node sizes and the analytical theory on infinitely large graphs. While thing is different in the case of \((2, 0)\)-cores. It is a well-known observation that when the degree exponent \( \gamma \geq 3.0 \), an undirected SF graph generated with the static model is becoming more like an ER random graph with an increasing degree exponent [40, 41]. We can say that this observation still holds in the pruning processes based on only \( k_\text{in} \) or \( k_\text{ou} \) for respectively the in- or out-degrees of the directed graphs generated with the static model.

From the results on the directed RR and SF networks, we can see again the discontinuity with \( k_\text{in} \geq 2 \) or \( k_\text{ou} \geq 2 \) and a relatively large arc density for the discontinuous emergence of \((k_\text{in}, k_\text{ou})\)-cores compared with the \( K \)-cores on undirected network counterparts, just like we see in the results on the directed ER random graphs.

Apart from the major observations between the model parameters \((k_\text{in} \text{ and } k_\text{ou})\) and the core sizes, we also have two observations related to degree distributions and the core sizes. (1) For directed ER graphs, RR graphs, and SF networks with large enough degree exponents, the analytical theory predicts nearly the exact relative sizes of the core structures even for finite-size graphs based only on their degree distributions, which means that the size of the \((k_\text{in}, k_\text{ou})\)-core of a random graph is much coded in its degree distribution. (2) For networks with different distributions of in-degrees and out-degrees like SF networks we consider above, \((k_\text{in}, k_\text{ou})\) and \((k_\text{ou}, k_\text{in})\)-cores with \( k_\text{in} \neq k_\text{ou} \) show a difference. From these two aspects, later we will discuss the core structures on real network data sets which need a more comprehensive structural characterization than the one for random networks.

### B. Real networks

We consider here the \((k_\text{in}, k_\text{ou})\)-cores on the real networks. In Tab. 1 we list the names, the node sizes, and the arc sizes for the 19 real network instances we will consider. For these networks, self-connections are removed, yet the multiple directional connections between nodes are permitted.

In figure 6 we show the sizes of the \((0, 2)\)- and \((2, 0)\)-cores from the simulation on networks instances and from the analytical theory based on the empirical degree distributions of network data sets. We should mention that, the analytical result derived with empirical degree distributions is just like averaging the sizes of core structures on instances when the in-neighbors and the out-neighbors for each node are fully randomized separately (for example, by switching corresponding predecessors or successors of two randomly chosen arcs). We can see that, for most of these real networks, the results from the analytical theory show a considerable discrepancy from those derived from the simulation. This discrepancy comes from a fact that a sufficient description of a real network is far from a degree distribution without degree correlation. Possible reasons are the degree correlation [50], the community structure [57, 58], the hierarchical structure [59], and so on.

In figure 7 for a protein-protein interaction network [45] with a node size \( N = 6,339 \) and an arc size \( M = 34,814 \), we show the shrinkage of the \((k_\text{in}, k_\text{ou})\)-cores and also the \( K \)-cores on its undirected network counterpart against a random node damage process (see from right to left). The random node damage is carried out through the initial fraction \( p \). This network shows a significant robustness against random node damages in quite a range of \( p \). We can further see a clear difference between the sizes of \((k_\text{in}, k_\text{ou})\)- and \((k_\text{ou}, k_\text{in})\)-cores when \( k_\text{in} \neq k_\text{ou} \). It’s reasonable to understand that the real networks are usually embedded with some information processing tasks, in which the roles of in-coming and out-going arcs (directed interactions) have some intrinsic differences, expressing themselves in the
TABLE I. Real directed networks. For each real network, Type and Name list the general category and the name, Description a brief description, N the size of nodes, and M the size of directed arcs.

| Type and Name | Description | N  | M     |
|---------------|-------------|----|-------|
| Regulatory    |             |    |       |
| EGFR [42]     | Signal transduction network of EGF receptor. | 61 | 112   |
| E. coli [43]  | Transcriptional regulatory network of E. coli. | 418| 519   |
| S. cerevisiae [44] | Transcriptional regulatory network of S. cerevisiae. | 688| 1,079 |
| PPI [45]      | Protein-protein interaction network of human. | 6,339| 34,814 |
| Metabolic     |             |    |       |
| C. elegans [46] | Metabolic network of C. elegans. | 1,469| 3,447 |
| S. cerevisiae [46] | Metabolic network of S. cerevisiae. | 1,511| 3,833 |
| E. coli [46]  | Metabolic network of E. coli. | 2,275| 5,763 |
| Neuronal      |             |    |       |
| C. elegans [47] | Neuronal network of C. elegans. | 297| 2,359 |
| Ecosystems    |             |    |       |
| Chesapeake [48] | Ecosystem in Chesapeake Bay. | 39| 176   |
| St. Marks [49] | Ecosystem in St. Marks River Estuary. | 54| 353   |
| Florida [50]  | Ecosystem in Florida Bay. | 128| 2,106 |
| Electric circuits |         |    |       |
| s208 [44]     | Electronic sequential logic circuit. | 122| 189   |
| s420 [44]     | Same as above. | 252| 399   |
| s838 [44]     | Same as above. | 512| 819   |
| Ownership     |             |    |       |
| USCorp [51]   | Ownership network of US corporations. | 7,253| 6,724 |
| Internet p2p  |             |    |       |
| Gnutella04 [52, 53] | Gnutella peer-to-peer file sharing network. | 10,876| 39,994|
| Gnutella30 [52, 53] | Same as above (at different time). | 36,682| 88,328|
| Gnutella31 [52, 53] | Same as above (at different time). | 62,586| 147,892|
| Social        |             |    |       |
| WiKi-Vote [54, 55] | Wikipedia who-votes-on-whom network. | 7,115| 103,689|

FIG. 6. Normalized sizes of \((k_{in}, k_{ou})\)-cores on real network instances. We calculate the relative sizes of the \((0, 2)\)-cores in (a) and the \((2, 0)\)-cores in (b) for the 19 real network instances by the simulation \(n_{real}\) and the analytical theory \(n_{theory}\) with only their empirical degree distributions as inputs. The initial fraction \(p = 1.0\).
FIG. 7. Normalized sizes of \((k_{in}, k_{ou})\)-cores and \(K\)-cores of a protein-protein interaction network against random node damages. \(K\)-cores are derived on the undirected version of the network instance. All the data points are averaged from the results of pruning processes performed on 40 independently generated configurations with a given initial fraction \(p\). The standard deviation for each data point is also shown.

degree distributions and the higher-order structures. The \((k_{in}, k_{ou})\)-core pruning process can thus be adopted as an intermediate method to reveal this structural subtlety in real directed networks.

C. Hybrid transitions

Here we consider the discontinuity and the scaling behavior relating to the hybrid transitions in the \((k_{in}, k_{ou})\)-core percolation problem. We prove that no matter the network type, the degree constraint parameters with \(k_{in} \geq 2\) or \(k_{ou} \geq 2\) lead to a discontinuous transition for the emergence of the core structures on infinitely large graphs, and the scaling property in the supercritical region of the hybrid transitions has an exponent \(1/2\). Other hybrid transitions can be found in the percolation problems as [12, 14–16, 24].

1. Discontinuity

We prove that the transition behavior in the \((k_{in}, k_{ou})\)-core percolation problem is discontinuous when \(k_{in} \geq 2\) or \(k_{ou} \geq 2\). We focus on the case with \(k_{in} \geq 2\) here, and the case with \(k_{ou} \geq 2\) has a similar analysis.

When \(k_{in} \geq 2\), the terms from the in-degrees on the right-hand sides of Eqs 1 and 2 both reduces to 0 when \(\alpha = 0\). Thus \((\alpha, \beta) = (0, 0)\) is always a fixed solution. Correspondingly, there is a trivial core size as \(n_{core} = 0\). We then analyze how the nontrivial stable solutions of \(\alpha\) and \(\beta\) emerge.

As in Sec III we define \(F(\alpha, \beta) \equiv -\alpha + f(\alpha, \beta)\), with \(f(\alpha, \beta)\) as the right-hand side of Eq 1. It’s easy to see that \(F(\alpha, \beta)|_{\alpha=0} = 0\). We further consider the derivative of \(F(\alpha, \beta)\) with respect to \(\alpha\) at \(\alpha = 0\).

\[
\frac{\partial F(\alpha, \beta)}{\partial \alpha} = -1 - p \sum_{k_+, k_-} Q_-(k_+, k_-) \sum_{k_1=0}^{k_{in}-1} \left( \frac{k_+}{k_1} \right) \left[ k_1 \alpha^{k_1-1}(1-\alpha)^{k_+ - k_1} - \alpha k_1 (k_+ - k_1)(1-\alpha)^{k_+ - k_1 - 1} \right]
\left[ 1 - \sum_{k_2=0}^{k_{ou}-2} \left( \frac{k_- - 1}{k_2} \right) \beta^{k_2} (1-\beta)^{k_- - k_2} \right].
\] (6)
When $\alpha = 0$, after some calculation, the second term of the right-hand side of the above equation reduces to 0, thus
\[
\frac{\partial F(\alpha, \beta)}{\partial \alpha}|_{\alpha=0} = -1.
\]

Summing the above results, at $\alpha = 0$ we have $F(\alpha, \beta)|_{\alpha=0} = 0$ and $\frac{\partial F(\alpha, \beta)}{\partial \alpha}|_{\alpha=0} < 0$. If a second and also the fixed solution $\alpha^* \neq 0$ for $F(\alpha, \beta) = 0$ emerges at a critical arc density $c^*$ or initial fraction $p^*$, $\alpha^*$ surely shows a gap from 0. Correspondingly we have a strictly positive stable $\beta^*$ from Eq.2 and finally a strictly positive $n_{\text{core}}^*$, or equivalently a discontinuous emergence of core structure. An example of the behavior of stable $\alpha$ and $\beta$ on directed ER random graphs is in figure 3.

2. Scaling property

We further consider the scaling property above the transition point after the discontinuity happens on directed random graphs. For the cases with $k_{in} \geq 2$ or $k_{ou} \geq 2$, the transition is hybrid with the scaling form $n_{\text{core}} - n_{\text{core}}^* \propto (c - c^*)^{1/2}$ with the arc density $c$ or $n_{\text{core}} - n_{\text{core}}^* \propto (p - p^*)^{1/2}$ with the initial fraction $p$. We first consider the scaling property of the stable $\alpha$ respective to the arc density $c$ above the discontinuous transition point. The scaling properties of $n_{\text{core}}$ respective to $c$ and $p$ follow the same procedure of proof.

For the ease of notation, we denote the right-hand side of Eq.1 as simply as $f(\alpha)$ instead of $f(\alpha, \beta)$, along with $F(\alpha)$ instead of $F(\alpha, \beta)$. At the transition point of the arc density $c^*$, there is an excess degree distribution $Q_+^*(k_+, k_-)$ which can be denoted as $Q_+^*$. Correspondingly, there is a stable (or degenerate) solution $(\alpha, \beta) = (\alpha^*, \beta^*)$ besides the trivial fixed solution $(\alpha, \beta) = (0, 0)$. At the critical point with $\alpha^*$ and $Q_+^*$, we have the relations $F(\alpha)|_{\alpha=\alpha^*, Q_-=Q_-^*} = 0$ and $\frac{\partial f(\alpha)}{\partial \alpha}|_{\alpha=\alpha^*, Q_-=Q_-^*} = 0$. Correspondingly, we have $f(\alpha^*)|_{\alpha=\alpha^*, Q_-=Q_-^*} = \alpha^*$ and $\frac{\partial f(\alpha)}{\partial \alpha}|_{\alpha=\alpha^*, Q_-=Q_-^*} = 1$. We further expand $f(\alpha)$ slightly above the transition point $\alpha^*$ and the corresponding $Q_+^*$.
Ignoring terms with orders higher than the quadratic ones, we have

\[ \alpha^* + \delta \alpha = f(\alpha^*)|_{\alpha=\alpha^*, Q_-=Q_-^*} \]

\[ + \frac{\partial f}{\partial \alpha}|_{\alpha=\alpha^*, Q_-=Q_-^*}\delta \alpha + \sum_{k_+, k_-} \frac{\partial f}{\partial Q_-^{-}(k_+, k_-)}|_{\alpha=\alpha^*, Q_-=Q_-^*} \delta Q_-^{-}(k_+, k_-) \]

\[ + \frac{1}{2} \frac{\partial^2 f}{\partial \alpha^2}|_{\alpha=\alpha^*, Q_-=Q_-^*}(\delta \alpha)^2 + \sum_{k_+, k_-} \frac{\partial^2 f}{\partial \alpha \partial Q_-^{-}(k_+, k_-)}|_{\alpha=\alpha^*, Q_-=Q_-^*} \delta \alpha \delta Q_-^{-}(k_+, k_-) \]

\[ + \frac{1}{2} \sum_{k_+, k_-} \frac{\partial^2 f}{\partial Q_-^{-}(k_+, k_-)\partial Q_-^{-}(k'_+, k'_-)}|_{\alpha=\alpha^*, Q_-=Q_-^*, Q'_-=Q'_-^*} \delta Q_-^{-}(k_+, k_-) \delta Q_-^{-}(k'_+, k'_-) \]

+ higher orders. \hspace{1cm} (7)

Then we have

\[ \delta \alpha \approx \left| \sum_{k_+, k_-} \frac{\partial f}{\partial Q_-^{-}(k_+, k_-)}|_{\alpha=\alpha^*, Q_-=Q_-^*} \delta Q_-^{-}(k_+, k_-) \right|^{1/2}. \hspace{1cm} (9) \]

For graphs at the critical point \( Q_-^* \) with the corresponding critical arc density \( c^* \), we have

\[ \delta c \equiv c - c^* \approx \sum_{k_+, k_-} k_- \delta P(k_+, k_-) \approx c^* \sum_{k_+, k_-} \delta Q_-^{-}(k_+, k_-). \hspace{1cm} (10) \]

Combining the above two equations, we have \( \delta \alpha \propto (\delta c)^{1/2} \) with \( \delta c \ll 1 \).

In figure 8 we show some results above the discontinuous transition points with the analytical theory on the directed random networks.

V. CONCLUSION

In this paper, we analytically study a generalized \( k \)-core pruning process as a node failure model based on both the in-degrees and the out-degrees of nodes to explore the effect of interaction directions on the resilience of directed networks. We test our theory on uncorrelated directed random networks as well as on real networks, and we show analytically that the introduction of unidirectional interactions between nodes can drive the networks more prone to abrupt collapses against a degree-based scheme of failures or damages which distinguishes in-degrees and out-degrees for nodes.

Here we briefly discuss some related problems worthy of further exploration. (1) Alternative definitions of node failures and pruning processes. In the model we consider here, an intact node has large enough both an in-degree and an out-degree. Based on this idea, we define the \((k_{in}, k_{ou})\)-core pruning process on directed networks. Yet other definitions of node intactness and the corresponding node removal processes are possible in specific contexts. For example, the paper [20] considers a generalized way of greedy leaf removal in which only in-coming or out-going arcs of a node are involved in a basic removal step; the paper [60] presents a more relaxed definition of node intactness to explain the adoption of node interdependency in real connected systems. (2) Dynamical significance of nested \((k_{in}, k_{ou})\)-core structures. On a directed graph instance, a \((k_{in}, k_{ou})\)-core can be derived with the pruning process from a \((k'_{in}, k'_{ou})\)-core while \( k_{in} \geq k'_{in} \) and \( k_{ou} \geq k'_{ou} \). Thus we can derive a procedure to reveal the nested structure of a directed network with increasing \( k_{in} \) and \( k_{ou} \). The dynamical significance of this nested structure, for example, in the spreading or the information processing, can be studied on real networks based on some dynamical models. A similar study line for the \( K \)-cores in undirected networks can be found in [61, 62]. (3) Network resilience as an optimization problem. Here we only consider the prediction problem on the sizes of the residual structure after
network failures or damages. Yet an optimization version of the problem, removing a minimal number of nodes along with their adjacent arcs or simply only arcs to disrupt a \((k_{in},k_{ou})\)-core in a directed network, is a totally different and probably an NP-hard problem \[63\]. Previous studies as \[64-66\] try to optimally remove the strongly connected components, or equivalently \((1,1)\)-cores, of directed networks, yet a general statistical-physical framework leading to an optimal destruction scheme of any core structure is still lacking. A possible framework can be an extension into the case of directed networks based on a study of dismantling undirected networks \[67\], in which a static description for a node removal process to reveal 2-cores is derived and is further combined with the cavity method \[31\].

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VII. APPENDIX A. THE \((k_{in},k_{ou})\)-CORE PERCOLATION ON DIRECTED RANDOM GRAPHS

We present here the simplified forms of the iterative equations for the cavity messages \(\alpha\) and \(\beta\), the relative sizes \(n_{core}\), and the arc densities \(c_{core}\) for the \((k_{in},k_{ou})\)-core percolation on directed Erdös-Rényi (ER) random graphs, directed regular random (RR) graphs, and directed scale-free (SF) networks generated both with the configurational model \[35, 39\] and the static model \[10, 41\].

The directed ER random graph with an arc density \(c\) generated with the method in the main text has a degree distribution \(P(k_{+,k_{-}}) = P_{+}(k_{+})P_{-}(k_{-})\), while \(P_{\pm}(k_{\pm}) = e^{-c k_{\pm}}/k_{\pm}!\). The excess degree distributions are \(Q_{+}(k_{+},k_{-}) = Q_{+}(k_{+})P_{-}(k_{-})\) and \(Q_{-}(k_{+},k_{-}) = P_{+}(k_{+})Q_{-}(k_{-})\), while \(Q_{\pm}(k_{\pm}) = e^{-c k_{\pm}}/(k_{\pm} - 1)!\). The iterative equations for \(\alpha\) and \(\beta\), \(n_{core}\), and \(c_{core}\) can be simplified as below.

\[
\alpha = p[1 - e^{-\alpha} \sum_{k_{1}=0}^{k_{n}-1} \frac{(c\alpha)_{k_{1}}}{k_{1}!}[1 - e^{-\beta} \sum_{k_{2}=0}^{k_{n}-2} \frac{(c\beta)_{k_{2}}}{k_{2}!}],
\]

\[
\beta = p[1 - e^{-\alpha} \sum_{k_{1}=0}^{k_{n}-2} \frac{(c\alpha)_{k_{1}}}{k_{1}!}[1 - e^{-\beta} \sum_{k_{2}=0}^{k_{n}-1} \frac{(c\beta)_{k_{2}}}{k_{2}!}],
\]

\[
n_{core} = p[1 - e^{-\alpha} \sum_{k_{1}=0}^{k_{n}-1} \frac{(c\alpha)_{k_{1}}}{k_{1}!}[1 - e^{-\beta} \sum_{k_{2}=0}^{k_{n}-1} \frac{(c\beta)_{k_{2}}}{k_{2}!}],
\]

\[
c_{core} = \frac{c\alpha\beta}{n_{core}}.
\]

A directed RR graph generated from an undirected RR graph instance with an integer degree \(k_{0}\) as in the main text has an arc density \(c(\equiv k_{0}/2)\) and a degree distribution \(P(k_{+,k_{-}}) = \binom{k_{0}}{k_{+}}/2^{k_{0}}\). Correspondingly, its excess degree distributions are \(Q_{+}(k_{+},k_{0} - k_{+}) = \binom{k_{0} - 1}{k_{+} - 1}/2^{k_{0} - 1}\) and \(Q_{-}(k_{+},k_{0} - k_{+}) = \binom{k_{0} - 1}{k_{0} - k_{+} - 1}/2^{k_{0} - 1}\). We can simplify the iterative equations for \(\alpha\) and \(\beta\), and \(n_{core}\) as below.
\begin{align}
\alpha &= p \sum_{k_0=0}^{k_0} \frac{1}{2k_0-1} \left( \frac{k_0 - 1}{k_0 - k_+ - 1} \right) \\
&\quad \left[ 1 - \sum_{k_1=0}^{k_1} \left( \frac{k_+}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_+ - k_1} \right] \left[ 1 - \sum_{k_2=0}^{k_2} \left( \frac{k_0 - k_+ - 1}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_0 - k_+ - 1 - k_2} \right], \\
\beta &= p \sum_{k_0=0}^{k_0} \frac{1}{2k_0-1} \left( \frac{k_0 - 1}{k_+ - 1} \right) \\
&\quad \left[ 1 - \sum_{k_1=0}^{k_1} \left( \frac{k_+ - 1}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_+ - 1 - k_1} \right] \left[ 1 - \sum_{k_2=0}^{k_2} \left( \frac{k_0 - k_+}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_0 - k_+ - k_2} \right], \\
n_{\text{core}} &= p \sum_{k_0=0}^{k_0} \frac{1}{2k_0} \left( \frac{k_0}{k_+} \right) \\
&\quad \left[ 1 - \sum_{k_1=0}^{k_1} \left( \frac{k_+}{k_1} \right) \alpha^{k_1} (1 - \alpha)^{k_+ - k_1} \right] \left[ 1 - \sum_{k_2=0}^{k_2} \left( \frac{k_0 - k_+}{k_2} \right) \beta^{k_2} (1 - \beta)^{k_0 - k_+ - k_2} \right].
\end{align}

A directed SF network instance generated with the configurational model with an in-degree exponent \( \gamma_+ \) and an out-degree exponent \( \gamma_- \) has a degree distribution

\begin{equation}
P(k_+, k_-) = \frac{k_+^{\gamma_+}}{\sum_{k_+} k_+^{\gamma_+}} \times \frac{k_-^{\gamma_-}}{\sum_{k_-} k_-^{\gamma_-}},
\end{equation}

while the summations in the denominators are carried out as \( k_+^{\min} \leq k_+ \leq k_+^{\max} \), while \( k_-^{\min} \) and \( k_-^{\max} \) are respectively the minimal and maximal in-degrees, and \( k_+^{\min} \) and \( k_-^{\max} \) are respectively the minimal and maximal out-degrees permitted in the graph instance. This summation rule for \( k_+ \) and \( k_- \) applies in the following equations for the directed SF networks generated with the configurational model. Correspondingly, its mean arc density \( c = \sum_{k_+} k_+ P(k_+, k_-) = \sum_{k_+} k_- P(k_+, k_-) \). The excess degree distributions can be calculated directly via \( Q_\pm(k_+, k_-) = k_\pm P(k_+, k_-)/c \). We can further simplify the equations for \( \alpha \), \( \beta \), and \( n_{\text{core}} \).

\begin{align}
\alpha &= p[1 - \frac{1}{c \sum_{k_-} k_-^{\gamma_-}} \sum_{k_-} k_-^{\gamma_-+1} - \sum_{k_2=0}^{k_2} \beta^{k_2} \sum_{k_-} k_-^{\gamma_-+1} \left( \frac{k_- - 1}{k_2} \right) (1 - \beta)^{k_- - 1 - k_2}], \\
\beta &= p \frac{1}{c \sum_{k_+} k_+^{\gamma_+}} \sum_{k_+} k_+^{\gamma_++1} - \sum_{k_1=0}^{k_1} \alpha^{k_1} \sum_{k_+} k_+^{\gamma_++1} \left( \frac{k_+ - 1}{k_1} \right) (1 - \alpha)^{k_+ - 1 - k_1}, \\
n_{\text{core}} &= p[1 - \frac{1}{c \sum_{k_-} k_-^{\gamma_-}} \sum_{k_-} k_-^{\gamma_-} \left( \frac{k_-}{k_2} \right) (1 - \beta)^{k_- - k_2}],
\end{align}

A directed SF network instance generated by the static model can be specified by an arc density \( c \), an in-degree exponent \( \gamma_+ \), and an out-degree exponent \( \gamma_- \). Defining the parameters \( \xi_\pm \equiv 1/(\gamma_\pm - 1) \), we have the degree distribution \( P(k_+, k_-) = P(k_+) P(k_-) \) as
while the general exponential integral function $E_{\alpha}(x) \equiv \int_{1}^{\infty} dt e^{-x} t^{-\alpha}$. The excess degree distributions are $Q_+(k_+, k_-) = Q_+(k_+) P_-(k_-)$ and $Q_-(k_+, k_-) = P_+(k_+) Q_-(k_-)$, while

$$Q_\pm(k_\pm) = \left(\frac{1}{\xi_\pm} - 1\right) \frac{(c(1 - \xi_\pm))^{k_\pm-1}}{(k_\pm - 1)!} E_{-k_\pm+1+1/\xi_\pm}(c(1 - \xi_\pm)).$$

We thus have the simplified iterative equations for $\alpha$, $\beta$, and $n_{\text{core}}$.

$$\alpha = p[1 - \sum_{k_1=0}^{k_{i,n}-1} \frac{1}{\xi_+} \frac{(c(1 - \xi_+))^{k_1}}{k_1!} E_{-k_1+1+\frac{1}{\xi_+}}(c(1 - \xi_+))]$$

$$[1 - \sum_{k_2=0}^{k_{j,n}-2} \frac{1}{\xi_-} (1 - 1) \frac{(c(1 - \xi_-))^{k_2}}{k_2!} E_{-k_2+1+\frac{1}{\xi_-}}(c(1 - \xi_-))]$$

$$\beta = p[1 - \sum_{k_1=0}^{k_{i,n}-2} \frac{1}{\xi_+} (1 - 1) \frac{(c(1 - \xi_+))^{k_1}}{k_1!} E_{-k_1+1+\frac{1}{\xi_+}}(c(1 - \xi_+))]$$

$$[1 - \sum_{k_2=0}^{k_{j,n}-1} \frac{1}{\xi_-} \frac{(c(1 - \xi_-))^{k_2}}{k_2!} E_{-k_2+1+\frac{1}{\xi_-}}(c(1 - \xi_-))]$$

$$n_{\text{core}} = p[1 - \sum_{k_1=0}^{k_{i,n}-1} \frac{1}{\xi_+} \frac{(c(1 - \xi_+))^{k_1}}{k_1!} E_{-k_1+1+\frac{1}{\xi_+}}(c(1 - \xi_+))]$$

$$[1 - \sum_{k_2=0}^{k_{j,n}-1} \frac{1}{\xi_-} \frac{(c(1 - \xi_-))^{k_2}}{k_2!} E_{-k_2+1+\frac{1}{\xi_-}}(c(1 - \xi_-))]$$

VIII. APPENDIX B. THE $K$-CORE PERCOLATION ON UNDIRECTED GRAPHS

An undirected graph $G = \{V, E\}$ has a node set $V$ ($|V| = N$) and an undirected edge set $E$ ($|E| = M$), correspondingly a mean degree $d \equiv 2M/N$. Before the $K$-core pruning process, an initial fraction $1 - p$ of nodes are randomly chosen and removed along with all their adjacent edges. We further apply the pruning process on the network as any node with a degree $< K$ is iteratively removed, and the $K$-core structure is the residual graph structure. A detailed analysis of the mean-field theory of the $K$-core percolation on undirected graphs can be found in papers like [12]. Yet in order to show an intrinsic similarity between the analytical frameworks of the $K$-core percolation on undirected networks and the $(k_{i,n}, k_{on})$-core percolation on directed networks, we present here the main equations for calculating the relative size and the mean degree of $K$-cores on random graphs, and also the simplified equations on undirected ER random and RR networks.

For an undirected graph $G$, the degree distribution $P(k)$ denotes the probability that a randomly chosen node has $k$ adjacent nearest neighbors. Following a randomly chosen edge to a node, for example node $i$, the excess degree distribution $Q(k)$ is the probability that $i$ has $k$ nearest neighbors. On the graph $G$, we randomly choose an edge $(i, j)$. From the node $i$ which is the $K$-core following the undirected edge $\{i, j\}$ to node $j$, we define $\alpha$ as the probability that $j$ is also in $K$-core when $i$ is not considered. With the assumption of local tree-like structures on sparse graphs [31], the self-consistent equation of $\alpha$ can be derived as

$$\alpha = p \sum_{k=K}^{\infty} \sum_{s=K-1}^{k-1} \binom{k-1}{s} \alpha^s (1 - \alpha)^{k-1-s}.$$
The explanation of the self-consistent equation is quite like the one for $\alpha$ in the $(k_{in}, k_{out})$-core percolation problem in the main text. We denote $n_{core}$ as the normalized size of nodes in the $K$-core structure. With the stable solution of $\alpha$, we can derive the formula for $n_{core}$ as

$$n_{core} = p \sum_{k=K}^{\infty} P(k) \sum_{s=K}^{k} \binom{k}{s} \alpha^s (1 - \alpha)^{k-s}. \quad (28)$$

The degree distribution $P_{core}(k)$ for the $K$-core subgraph can also be calculated with the stable $\alpha$ as

$$P_{core}(k) = \frac{1}{n_{core}} p \sum_{s=K}^{\infty} P(s) \binom{s}{k} \alpha^s (1 - \alpha)^{s-k}. \quad (29)$$

The mean degree of the $K$-core can thus be calculated as

$$c_{core} = \sum_{k=K}^{\infty} k P_{core}(k). \quad (30)$$

On undirected ER random graphs with a mean degree $d$, we have $P(k) = e^{-d}d^k/k!$ and $Q(k) = e^{-d}d^{k-1}/(k-1)!$. We then have the simplified equations for $\alpha$, $n_{core}$, and $c_{core}$ as

$$\alpha = p[1 - e^{-d\alpha} \sum_{k=0}^{K-2} \binom{K}{k} \alpha^k (1 - \alpha)^{K-2-k}], \quad (31)$$

$$n_{core} = p[1 - e^{-d\alpha} \sum_{k=0}^{K-1} \binom{K}{k} \alpha^k (1 - \alpha)^{K-1-k}], \quad (32)$$

$$c_{core} = \frac{d\alpha^2}{n_{core}}. \quad (33)$$

On undirected RR graphs with an integer degree $k_0$, we have $P(k) = Q(k) = \delta(k - k_0)$. We can further have the simplified equations for $\alpha$ and $n_{core}$.

$$\alpha = p[1 - \sum_{k=0}^{K-2} \binom{k_0 - 1}{k} \alpha^k (1 - \alpha)^{k_0 - 1 - k}], \quad (34)$$

$$n_{core} = p[1 - \sum_{k=0}^{K-1} \binom{k_0}{k} \alpha^k (1 - \alpha)^{k_0 - k}]. \quad (35)$$
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