Algebraic reduction of certain almost Kähler manifolds *

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Abstract

We study almost Kähler manifolds whose curvature tensor satisfies the third curvature condition of Gray. We show that the study of manifolds within this class reduces to the study of a subclass having the property that the torsion of the first canonical Hermitian connection has the simplest possible algebraic form. This allows to understand the structure of the Kähler nullity of an almost Kähler manifold with parallel torsion.

Contents

1 Introduction 1
2 Preliminaries 3
3 The third Gray condition 5
4 An algebraic decomposition 8

1 Introduction

An almost Kähler manifold (shortly AK) is a Riemannian manifold \((M^{2n}, g)\), together with a compatible almost complex structure \(J\), such that the Kähler form \(\omega = g(J\cdot, \cdot)\) is closed. Hence, almost Kähler geometry is nothing else that symplectic geometry with a preferred metric and almost complex structure. Since symplectic manifolds often arise in this way is rather natural to ask under which conditions on the metric we get integrability of the almost complex \(J\). In this direction, a famous (still open) conjecture of S. I. Golberg asserts that every compact, Einstein, almost Kähler manifold is, in fact, Kähler. Nevertheless, they are a certain number of partial results, supporting this conjecture. First of all, K. Sekigawa proved [26] that the Goldberg conjecture is true when the scalar curvature is positive. We have to note that the Goldberg conjecture is definitively not true with the compactness assumption removed by the examples of [4] in complex dimension \(n \geq 3\) and those of [3, 9, 21] in (real) dimension 4. In [4], a potential source of compact almost Kähler, Einstein manifolds

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is considered, namely those compact Kähler manifolds whose Ricci tensor admits two distinct, constant eigenvalues; integrability is proven under certain positivity conditions. The rest of known results, most of them enforcing or replacing the Einstein condition with some other natural curvature assumption are mainly in dimension 4. To cite only a few of them, we mention the beautiful series of papers [5, 2, 6] giving a complete local and global classification of almost Kähler manifolds of 4 dimensions satisfying the second and third Gray condition on the Riemannian curvature tensor. Other recent results, again in 4-dimensions, are concerned with the study of local obstructions to the existence of Einstein metrics [9], *-Einstein metrics [22], etc. As for the class of almost Kähler manifolds satisfying the second curvature condition of Gray it was established recently [19, 20] that the it coincides with the class of almost Kähler manifolds whom torsion of the first canonical connection is parallel. In view of the known twistorial examples [11, 1], we have therefore a strong resemblance between this class and that of nearly Kähler manifolds.

In this paper our main object of study will be the class of almost Kähler manifolds satisfying the third curvature condition of Gray (shortly $\mathcal{AK}_3$). To the best of our knowledge, in dimension greater than 6 almost nothing is known about the structure of this class of manifolds. Our approach to the study of the class $\mathcal{AK}_3$-manifolds will be directed from the point of view of the canonical Hermitian connection. Actually, we are going to study the geometric as well as the algebraic effects of the third curvature condition of Gray over the torsion of the last mentioned connection.

**Theorem 1.1** Let $(M^{2n}, g, J)$ belong to the class $\mathcal{AK}_3$. Then, over some dense open subset, the manifold $M$ is locally the Riemannian product of a strict almost Kähler manifold with parallel torsion and a special $\mathcal{AK}_3$-manifold.

The precise definition of special $\mathcal{AK}_3$ is given in section 4. They are characterized by an algebraic feature of their Kähler nullity which generalizes, to some extent, properties of almost Kähler 4-manifolds. Finally, theorem 1.1 enables us to give a first structure result concerning almost Kähler manifolds with parallel torsion.

**Corollary 1.1** Let $(M^{2n}, g, J)$ be almost Kähler with parallel torsion. Then $M$ is locally the Riemannian product of a Kähler manifold, a strict $\mathcal{AK}$-manifold with parallel torsion and a locally 3-symmetric space of type $I$.

Here, and in view of further progress on the classification of almost Kähler manifolds with parallel torsion, we regrouped in the class I those special almost Kähler manifolds with parallel torsion.

The paper is organised as follows. In section 2 we review some classical facts and definitions from almost Hermitian geometry, mainly related to the first canonical Hermitian connection and Gray’s curvature conditions. The main differential geometry properties of the torsion of the canonical connection, such as the parallelism in directions orthogonal to the Kähler nullity are established in section 3. The last section is devoted to the proof of theorem 1.1. We start from a result used in the $\mathcal{AK}_2$-context (see [19]) and observe that it continues to hold in the present setting. Then, due to more flexibility of $\mathcal{AK}_3$ we need a finer analysis in order to produce the proof of theorem 1.1. One of main ingredients consists in using a suitably defined Hermitian
Ricci tensor in order to eliminate those parts of the tangent bundle of the manifold obstructing the decomposition in theorem 1.1.

2 Preliminaries

Let \((M^{2n}, g, J)\) be an almost Hermitian manifold. Let \(\nabla\) be the Levi-Civita connection of the metric \(g\). Then for all \(X\) in \(TM\) we have that \(\nabla_X J\) is a skew-symmetric, \(J\)-anti-invariant endomorphism of \(TM\). Imposing further algebraic constraints to the tensor \(\nabla J\) leads to the introduction of several classes of almost Hermitian manifolds. We will recall the definition of those that will be significant for this paper.

For instance, \((M^{2n}, g, J)\) is quasi-Kähler iff for all \(X, Y\) in \(TM\) we have:

\[
(\nabla_{JX} J)JY = - (\nabla_X J)Y.
\]

If \(\omega = g(J \cdot, \cdot)\) is the Kähler form of \((g, J)\) then we are in presence of an almost Kähler structure iff \(d\omega = 0\). It should be noted that almost Kähler structures are always quasi-Kähler.

An important object related to the almost Hermitian structure \((g, J)\) is the first canonical Hermitian connection defined by

\[
\nabla_X Y = \nabla_X Y + \eta_X Y
\]

whenever \(X, Y\) are vector fields on \(M\) where, to save space, we setted \(\eta_X Y = \frac{1}{2} (\nabla_X J)JY\). We obtained a metric Hermitian connection on \(M\), that is \(\nabla g = 0\) and \(\nabla J = 0\). The torsion tensor of the canonical Hermitian canonical connection, to be denoted by \(T\) is given by

\[
T_X Y = \eta_X Y - \eta_Y X
\]

for all \(X, Y\) in \(TM\). Then by the torsion of the almost Hermitian manifold \((M^{2n}, g, J)\) we will mean simply the torsion tensor of the canonical Hermitian connection. Note that among connections respecting both the metric \(g\) and the almost complex structure \(J\) the first canonical Hermitian connection is the one whose torsion has minimal possible norm [13].

For the almost Hermitian \((M^{2n}, g, J)\) be almost Kähler one requires that the Kähler form \(\omega(X, Y) = < JX, Y >\) be closed. In our notations, this is equivalent to have

\[
<T_X Y, Z > = - < \eta_Z X, Y >
\]

for all \(X, Y, Z\) in \(TM\). When dealing with almost Kähler manifolds, this relation will be used almost implicitly in this paper.

We denote by \(R\) resp. \(\overline{R}\) the curvature tensor of the Levi-Civita connection \(\nabla\) resp. of the canonical Hermitian connection \(\overline{\nabla}\) with the convention that \(\overline{R}(X, Y) = - [\nabla_X, \nabla_Y] + \nabla_{[X,Y]}\) for all vector fields \(X, Y\) on \(M\). Now a standard calculation involving the definitions yields to

\[
\overline{R}(X, Y)Z = R(X, Y)Z + [\eta_X, \eta_Y]Z - \left[ d\nabla u(X, Y) \right] Z
\]
where
\[ [d_u(X, Y)] Z = (\nabla_X \eta)(Y, Z) - (\nabla_Y \eta)(X, Z) + \eta T_{XY} Z \]
for all vector fields \( X, Y, Z \) on \( M \). Note that \( d_u(X, Y) \) is a \( J \)-anticommuting endomorphism of \( TM \), whenever \( X, Y \) are tangent vectors to \( M \).

**Remark 2.1** In the formula (2.2), the notation \( u \) stands for the tensor \( \eta \), considered as a \( 1 \)-form with values in the bundle \( \Omega^2(M) \). Then \( d_u \), with its expression given below, is the twisted differential acting on twisted one forms, when considering the tangent bundle of \( M \) endowed with the connection \( \nabla \). Since our discussion is intended to be self-contained and at the elementary level, we will keep things at the level of the notation after (2.2).

The fact that \( \nabla \) is a Hermitian connection implies that \( R(X, Y, JZ, JU) = R(X, Y, Z, U) \). Using this in formula (2.2), together with the skew-symmetry of the \( J \)-anticommuting endomorphism \( \eta \) gives us:

\[ \begin{align*}
2.3 \quad R(X, Y, Z, U) - R(JX, JY, Z, U) &= 2\left[ (d_u(X, Y))(Z, U) \right](X, Y) \\
\end{align*} \]

Using the symmetry property of the Riemannian curvature we also deduce that

\[ \begin{align*}
2.4 \quad R(X, Y, Z, U) - R(JX, JY, Z, U) &= 2\left[ (d_u(Z, U))(X, Y) \right](X, Y)
\end{align*} \]

whenever \( X, Y, Z, U \) are in \( TM \). We are going to make an intensive use of the two formulas below in the next section.

The rest of the present section is destined to present some classes of almost Hermitian manifolds, the third of which will be the object of our attention in this paper. We begin by recalling how one can distinguish several classes of almost Hermitian manifolds by “the degree of resemblance” of their Riemannian curvature tensor with the curvature tensor of a Kähler manifold [15, 23]:

\[ \begin{align*}
(G_1) & : R(X, Y, JZ, JU) = R(X, Y, Z, U) \\
(G_2) & : R(X, Y, Z, U) - R(JX, JY, Z, U) = R(JX, Y, JZ, U) + R(JX, Y, Z, JU) \\
(G_3) & : R(JX, JY, JZ, JU) = R(X, Y, Z, U)
\end{align*} \]

Using the first Bianchi identity it is a simple exercise to see that \( G_1 \Rightarrow G_2 \Rightarrow G_3 \). It is also clear that a Kähler structure satisfies all the three conditions. Let us set now some notations.

Following [15], let \( \mathcal{AK} \) be the class of almost Kähler manifolds. Then the class \( \mathcal{AK}_i, 1 \leq i \leq 3 \) contains those almost Kähler manifolds whose curvature tensor satisfies the condition \( (G_i) \). Obviously, we have the inclusions:

\[ \mathcal{AK}_1 \subseteq \mathcal{AK}_2 \subseteq \mathcal{AK}_3. \]

Note that it was shown in [14] that locally \( \mathcal{AK}_1 = \mathcal{K} \), where \( \mathcal{K} \) denotes the class of Kähler manifolds. Let us give a very simple proof of this result.

**Proposition 2.1** Any almost Kähler manifold \( (M^{2n}, g, J) \) satisfying condition \( (G_1) \) is Kähler.

**Proof**:
From (2.3) we obtain that condition \( (G_1) \) is equivalent with

\[ (\nabla_X \eta)(Y, Z) - (\nabla_Y \eta)(X, Z) + \eta T_{XY} Z = 0 \]
for all $X, Y, Z$ in $TM$. Using the quasi-Kähler condition, one notices that the second term of this equation is $J$-anti-invariant in $X, Z$ whilst the last term is $J$-invariant in the same variables. Therefore we obtain

$$\nabla_{JX}\eta(Y, JZ) + \nabla_X\eta(Y, Z) = -2\eta_{TXYZ}.$$  

Using again the quasi-Kähler condition we find that left hand side of the equation below is $J$-invariant in $X, Y$ whilst the right hand one is $J$-anti-invariant in the same variables. We conclude that $\eta_{TXYZ} = 0$ for all $X, Y, Z$ in $TM$ and then the almost Kähler condition (2.1) implies the vanishing of $T$, and hence that of $\eta$. ■

The other inclusions between the previously defined classes are strict in dimensions $2n \geq 6$, as shows the examples of [11], multiplied by Kähler manifolds. In the same spirit, the class $\mathcal{AH}_i, 1 \leq i \leq 3$ contains those almost Hermitian manifolds whose Riemannian curvature tensor satisfies condition $(G_i)$. By contrast with the almost Kähler case, note that they are classes of almost complex manifolds where conditions $(G_2)$ and $(G_3)$ are known to be equivalent, such as the class of Hermitian manifolds [15] and the class of locally conformally Kähler manifolds [12].

To give a very simple interpretation of the third Gray condition on curvature, let us recall that we have a decomposition of the bundle of (real valued) two forms:

$$\Lambda^2(M) = \Lambda^{1,1}(M) \oplus [\Lambda^{2,0}(M)]$$  

into $J$-invariant and $J$-anti-invariant parts, where the action of $J$ on a two form $\alpha$ is given by $(J\alpha)(X, Y) = \alpha(JX, JY)$ for all $X, Y$ in $TM$. It is easily checked that $(g, J)$ belongs to the class $\mathcal{AH}_3$ iff the Riemannian curvature operator of $g$ preserves the decomposition (2.5).

3 The third Gray condition

This section is dedicated to give an interpretation, in terms of the torsion of the first canonical Hermitian connection of the third Gray condition on curvature. In the context of quasi-Kähler geometry we will deduce some important geometric consequences of the condition $(G_3)$ putting the basis of the discussion in the next section, where we will restrict finally to our main object of interest, the subclass of almost Kähler manifolds. We begin by the following very simple observation.

**Lemma 3.1** The almost Hermitian manifold $(M^{2n}, g, J)$ satisfies the condition $(G_3)$ iff

$$[\nabla u(Z, U)](X, Y) = \nabla u(X, Y)(Z, U).$$

Moreover, we have $\nabla u(JX, JY) + \nabla u(X, Y) = 0$, hence $\nabla u$ defines a symmetric endomorphism of $\Lambda^2(M)$.

**Proof**:
We change $Z$ and $U$ in $JZ$ and $JU$ respectively in (2.4) and take the sum with (2.3).
Using \((G_3)\), we get \[\left(\frac{d\varphi}{\varphi}(JZ, JU)\right)(X, Y) = -\left(\frac{d\varphi}{\varphi}(X, Y)\right)(Z, U).\] To get (3.1) we change again \(Z\) in \(JZ\) and \(U\) in \(JU\) and use that \(d\varphi(X, Y)\) is \(J\)-anticommuting. The rest is straightforward. ■

The previous lemma enable us to give a number of useful algebraic properties of the tensor \(\overline{R}\).

**Corollary 3.1** Let \((M^{2n}, g, J)\) be a quasi-Kähler manifold in the class \(\mathcal{AH}_3\) and let \(X, Y, Z, U\) be vector fields on \(M\). The following holds :

(i) \[\overline{R}(X, Y, Z, U) - \overline{R}(Z, U, X, Y) = \langle [\eta_X, \eta_Y]Z, U \rangle - \langle [\eta_Z, \eta_U]X, Y \rangle.\]

(ii) \[\overline{R}(JX, JY), Z, U) = \overline{R}(X, Y, Z, U).\]

**Proof:** Property (i) follows immediately from (3.1), the symmetry of Riemannian curvature operator and lemma 3.1. Since \(\overline{R}\) is a Hermitian connection we have \(\overline{R}(Z, U, JX, JY) = \overline{R}(Z, U, X, Y)\), hence (ii) follows from (i) and the quasi-Kähler condition on the tensor \(\eta\). ■

Note that the previous corollary is well known for nearly-Kähler manifolds (see [16] for instance). Also note that property (i) holds in fact for any almost Hermitian manifold in the class \(\mathcal{AH}_3\).

With this preliminaries in mind, we are now going to investigate some illuminating consequences of the third Gray condition.

**Proposition 3.1** Let \((M^{2n}, g, J)\) be a quasi-Kähler manifold. If \((M^{2n}, g, J)\) satisfies condition \((G_3)\) then :

\[3.2\] \[(\nabla_{JX}\eta)(JY, Z) + (\nabla_X\eta)(Y, Z) = 0.\]

and \[< \eta_{TX} Y Z, W >= < \eta_{TZ} W X, Y >\]

whenever \(X, Y, Z\) and \(U\) are in \(TM\).

**Proof:**

From lemma 3.1 we know that

\[< (\nabla_X\eta)(Y, Z) - (\nabla_Y\eta)(X, Z), U > + < \eta_{TX} Y Z, U >=\]

\[< (\nabla_Z\eta)(U, X) - (\nabla_U\eta)(Z, X), Y > + < \eta_{TZ} U X, Y >\]

for all \(X, Y, Z, U\) in \(TM\). The quasi-Kähler condition ensures that the second term of each side is \(J\)-anti-invariant in \(X\) and \(Z\). Therefore, identifying the invariant parts (in \(X\) and \(Z\)) of the previous equation we get

\[< (\nabla_{JX}\eta)(JY, Z) + (\nabla_X\eta)(Y, Z), U > + 2 < \eta_{TX} Y Z, U >=\]

\[< (\nabla_{JZ}\eta)(U, JX) + (\nabla_Z\eta)(U, X), Y > + 2 < \eta_{TZ} U X, Y >.\]
After rearranging terms and using the quasi-Kähler condition we find that
\[
< (\nabla_J X \eta) (Y, Z) + (\nabla_X \eta) (Y, Z), U > =
< (\nabla_J Z \eta) (JU, X) + (\nabla_Z \eta) (U, X), Y > + 2 \eta_{TX} X, Y > - 2 \eta_{TX} Y, Z, U > .
\]

But the left hand side of this equation is \( J \)-antinvariant in \( Z \) and \( U \) whilst the right hand is \( J \)-invariant (here one uses the quasi-Kähler condition on the last two terms) in the same variables. This leads to the proof of the proposition. ■

Remark 3.1 The algebraic constraint in proposition 3.1 is automatically satisfied in both nearly Kähler and almost Kähler cases.

We are now going to show that the torsion of \( \mathcal{AH}_3 \)-manifolds enjoys a particularly pleasant property.

**Proposition 3.2** Let \((M^{2n}, g, J)\) be quasi-Kähler in the class \( \mathcal{AH}_3 \). Then:
\[
\nabla_{TX} Y \eta = 0
\]
for all \( X, Y \) in \( TM \).

**Proof**:
Derivating (3.2) we obtain:
\[
(\nabla^2_{X,Y \eta})(JZ, \cdot) + (\nabla^2_{X,Y \eta})(Z, \cdot) = 0.
\]
Replacing \( X \) with \( JY \) and \( Y \) with \( JX \) respectively we obtain also
\[
-(\nabla^2_{JY,X \eta})(JZ, \cdot) + (\nabla^2_{JY,X \eta})(Z, \cdot) = 0.
\]
After addition of these two equations and by making use of the Ricci identity (with respect of the connection \( \nabla \)) we obtain:
\[
(\overline{R}(X, JY). \eta)(JZ, \cdot) + \nabla^2_{TX(JY) \eta}(JZ, \cdot) = (\nabla^2_{X,Y \eta})(Z, \cdot) + (\nabla^2_{JY,JX \eta})(Z, \cdot).
\]
Here, the action \( G. \eta \) of an endomorphism \( G \) of \( TM \) on the tensor \( \eta \) is defined by
\[
(G. \eta)(X, Y) = G(\eta_X Y) - \eta_{GX} Y - \eta_X (GY) \text{ for all } X, Y \text{ in } TM.
\]
We antisymmetrize (3.4) in \( X \) and \( Y \) in order to get, after using twice the Ricci identity:
\[
\left[ \overline{R}(X, JY) + \overline{R}(JX, JY), \eta \right](JZ, \cdot) + (\nabla^2_{TX(JY) + T_JX \eta})(JZ, \cdot) =
\left[ \overline{R}(JX, JY) - \overline{R}(X, JY), \eta \right](Z, \cdot) + (\nabla^2_{T_JX(JY) - T_X \eta})(Z, \cdot).
\]
Using corollary 3.1, (ii) it follows that \( \nabla^2_{TX(JY) + T_JX \eta}(JZ, \cdot) = (\nabla^2_{T_JX(JY) - T_X \eta})(Z, \cdot) \) or further \( (\nabla^2_{TX(JY)})(JZ, \cdot) = (\nabla^2_{T_X \eta})(Z, \cdot) \) since \((M^{2n}, g, J)\) is quasi-Kähler. Our claimed result is implied now by (3.2). ■

We finish this section by showing that in the quasi-Kähler, \( \mathcal{AH}_3 \)-setting, one can get a simplified form of the first Bianchi identity for the first canonical Hermitian connection.
Lemma 3.2 For any quasi-Kähler, $AH_3$-manifold $(M^{2n}, g, J)$ the following holds:

$$\sigma_{X,Y,Z} \left[ R(X,Y)Z + T_{X,Y}Z \right] = 0$$

whenever $X, Y, Z$ are in $TM$.

Proof:
We know that

$$\langle (\nabla_X \eta)(Y, Z) - (\nabla_Y \eta)(X, Z), U \rangle = \langle (\nabla_Z \eta)(U, X) - (\nabla_U \eta)(Z, X), Y \rangle.$$

Using the almost Kähler condition $\langle \eta_U X, Y \rangle = -\langle T_X Y, U \rangle$ in the right hand side of the previous equation we are lead to

$$\langle (\nabla_X \eta)(Y, Z) - (\nabla_Y \eta)(X, Z), U \rangle = -\langle (\nabla_Z T)(X, Y) - (\nabla_U \eta)(Z, X), Y \rangle.$$

Taking the symmetric sum of the last identity yields to

$$2\sigma_{X,Y,Z} < (\nabla_X T)(Y, Z), U >= -\left[ < (\nabla_U \eta)(Z, X), Y > + < (\nabla_U \eta)(X, Y), Z > + < (\nabla_U \eta)(Y, Z), X > \right].$$

But $< (\nabla_U \eta)(Z, X), Y > + < (\nabla_U \eta)(X, Y), Z >= < (\nabla_U T)(Z, X), Y >= - < (\nabla_U \eta)(Y, Z), X >$ by the use of the almost Kähler condition $< T_Z X, Y > + < \eta_Y Z, X >$ and using the first Bianchi identity for the connection $\nabla$ finishes the proof of the lemma.

4 An algebraic decomposition

In this section we begin the geometric study of $AK_3$-manifolds. It will lead to the proof of the theorem 1.1, hence giving a simplified algebraic form of the torsion tensor of an almost Kähler structure satisfying the third Gray condition on curvature.

Throughout this section $(M^{2n}, g, J)$ will be an almost Kähler in the class $AK_3$. An important associated object is the Kähler nullity of $(g, J)$, the vector subbundle of $TM$ defined by $H = \{ v \in TM : \eta_v = 0 \}$. Note that, a priori, $H$ need not to have constant rank over $M$. However, this is true locally, in the following sense. Call a point $m$ of $M$ regular if the rank of $\eta$ attains a local maximum at $m$. Using standard continuity arguments, it follows that around each regular point, the rank of $\eta$, and hence that of $H$ is constant in some open subset. It is also easy to see that the set of regular point is dense in $M$, provided that the manifold is connected. As we are concerned with the local (in some neighbourhood of a regular point) structure of $AK_3$-manifolds we can assume, without loss of generality, that $H$ has constant rank over $M$. This assumption will be made in the whole rest of this paper.

We have therefore a $J$-invariant and orthogonal decomposition

$$TM = V \oplus H$$

where we define the distribution $V$ to be the orthogonal of $H$ in $TM$. Let us note that it is straightforward consequence of the definitions (in conjunction with the almost
Kähler condition (2.1)) to check that \( V = T(TM, TM) \). Then proposition 3.2 can be stated in the equivalent form:

4.1 \[ \nabla_V \eta = 0 \]

for all \( V \) in \( V \). We investigate below some of the most immediate consequences of (4.1).

**Lemma 4.1** (i) For all \( V, W \) in \( V \) and \( X \) we have that \( \nabla_V W \) belongs to \( V \) and \( \nabla_V X \) belongs to \( H \).

(ii) The distribution \( V \) is integrable.

(iii) \( \nabla(V_1, V_2) \eta = 0 \) for all \( V_i \) in \( V \), \( i = 1, 2 \).

**Proof**:

(i) We have \( (\nabla_V \eta)(X, U) = 0 \) for all \( U \) in \( TM \) or further, since \( H \) is the Kähler nullity of \((g, J)\), \( \eta_{\nabla_V X} U = 0 \). This guarantees that \( \nabla_V X \) belongs to \( H \) and since \( \nabla \) is a metric connection it follows that \( \nabla_V W \) is in \( V \).

(ii) follows now by (i) and the (easy to check) fact that \( T(V, V) \subseteq V \). To get (iii) it suffices to derivate (4.1) in the vertical directions. ■

Let us set a notational convention, to be used intensively in the present and the next section and intended to improve presentation. If \( E \) and \( F \) and vector subbundles of \( TM \) and \( Q \) is a tensor of type \((2, 1)\), we will denote by \( Q(E, F) \) (or \( Q_{E,F} \)) the subbundle of \( TM \) generated by elements of the form \( Q(u, v) \) where \( u \) belongs to \( E \) and \( v \) is in \( F \). We will also denote by \( < E, F > \) the product of two generic elements of \( E \) and \( F \) respectively.

The starting point of our discussion will be the following result which was one of the key ingredients leading to the algebraic reduction of the torsion for manifolds in the class \( \mathcal{AK}_2 \).

**Proposition 4.1** We have an orthogonal, \( J \)-invariant and \( \nabla \)-parallel (inside \( V \)) decomposition \( V = V_0 \oplus V_1 \) which further satisfies

4.2 \[ T(V_0, V_0) = V_0, \quad \eta_{V_1} V_1 \subseteq H \text{ and } \eta_{V_1} V_0 = 0. \]

**Proof**:

The proof is exactly the same as that given in [19] in the context of \( \mathcal{AK}_2 \), as the last mentioned proof uses in fact only conditions shared by the \( \mathcal{AK}_3 \) class. In fact, the result holds even more generally, namely for almost Kähler manifolds satisfying (4.1). ■

Since \( \mathcal{AK}_3 \)-geometry is much less rigid than the \( \mathcal{AK}_2 \)-one, we need a finer analysis to produce the proof of the theorem 1.1. Our discussion needs several preliminary lemmas. First of all, let we introduce the configuration tensor \( A : H \times H \to V \) by setting

\[ \nabla_X Y = \nabla_X Y + A_X Y \]

for all \( X, Y \) in \( H \). It is clear that \([A_X, J] = 0\) whenever \( X \) is in \( H \). We also define \( B : H \times V \to H \) by

\[ \nabla_X V = \nabla_X V + B_X V \]

for all \( X, V \) in \( H \) and \( V \) respectively. Obviously, \( < A_X Y, V > = - < B_X V, Y >. \)
Lemma 4.2 The distribution $\mathcal{V}_0$ is $\nabla$-parallel.

Proof:
Let us denote by $A^+$ and $A^-$ the symmetric resp. skew-symmetric components of $A$. If $V, W$ and $X, Y$ are in $\mathcal{V}$ and $H$ respectively we must have (cf. (3.1)):

$$ \left[ (d\nabla u)(X, V) \right] (W, Y) = \left[ (d\nabla u)(W, Y) \right] (X, V). $$

Taking into account the parallelism of $\eta$ over $\mathcal{V}$ (i.e. (4.1)) it follows that

$$ < (\nabla_X \eta)(V, W), Y > = - < (\nabla_Y \eta)(W, X), V > = < (\nabla_Y \eta)(W, V), X > . $$

We antisymmetrise in $V, W$ and we arrive at

$$ < (\nabla_X T)(V, W), Y > + < (\nabla_Y T)(V, W), X > = 0. $$

Since $T(V, W)$ belongs to $\mathcal{V}$ this is clearly equivalent with $< T_V A^X Y, W > = 0$ so as to obtain that $A^X Y$ is in $\mathcal{V}_1$. On the other side, we know that (cf. (3.1))

$$ (d\nabla u)(X, Y)(V, W) = (d\nabla u)(V, W)(X, Y) $$

and once again the parallelism of the torsion over $\mathcal{V}$ leads us to $< (\nabla_X \eta)(Y, V) - (\nabla_Y \eta)(X, V), W > = 0$. But this is to say that $< \eta_{A^X Y}, V, W > = 0$ and the almost Kähler condition ensures that $< A^X Y, T_V W > = 0$ hence $A^X Y$ belongs to $\mathcal{V}_1$. We showed that $A^X Y$ belongs to $\mathcal{V}_1$ and this implies that $B_X$ vanishes on $\mathcal{V}_0$ for all $X$ in $H$. It follows that $\nabla_X V$ is in $\mathcal{V}$ for all $X$ in $H$ and $V$ in $\mathcal{V}_0$. Therefore $(\nabla_X T)(V_0, W_0)$ belongs to $\mathcal{V}$ for all $V_0, W_0$ in $\mathcal{V}_0$, but since $< \nabla_X \eta)(V, W), U > = 0$ for all $V, W, U$ in $\mathcal{V}$ (again by (3.1) and (4.1)) we see that $(\nabla_X T)(V_0, W_0)$ is horizontal, hence equal to zero and the $\nabla$-parallelism of $\mathcal{V}_0$ is now immediate. ■

Getting closer to the geometric link between $\mathcal{V}_0$ and its orthogonal complement in $TM$ requires some curvature informations, and these are provided by the second Bianchi identity.

Lemma 4.3 Let $V_i, 1 \leq i \leq 3$ be in $\mathcal{V}$ and $X$ in $H$. We have:

(i) $\overline{\mathcal{R}}(V_1, V_2, V_3, X) = 0$

(ii) $\overline{\mathcal{R}}(X, V_1, V_2, V_3) = - [\eta_{V_2}, \eta_{V_3}] X, V_1$

(iii) $(\nabla_V \overline{\mathcal{R}})(X, V_1, V_2, V_3) = 0.$

Proof:
(i) follows directly from lemma 4.1, (i) and the integrability of $\mathcal{V}$. To obtain (ii) one uses the symmetry property of corollary 3.1, (i). Finally, (iii) follows by derivating (ii) and taking into account that $\nabla_V X$ belongs to $H$ for all $X$ in $H$ and $V$ in $\mathcal{V}$ and (4.1). ■

We will now use the second Bianchi identity for the canonical Hermitian connection in order to get more information about the algebraic properties of $\eta$ with respect to the decomposition $TM = \mathcal{V} \oplus H$. 

Proposition 4.2 Let $X, V_i, 1 \leq i \leq 4$ be vector fields on $H$ and $V$ respectively. We have:

(i) 
\[
\bar{R}(\eta_{V_2} X, V_1, V_3, V_4) - \bar{R}(\eta_{V_1} X, V_2, V_3, V_4) = -\langle [\eta_{V_2}, \eta_{V_4}] X, T_{V_1} V_2 \rangle.
\]

(ii) 
\[
(\nabla_X \bar{R})(V_1, V_2, V_3, V_4) = 0.
\]

Proof:
Using the second Bianchi identity we obtain
\[
(\nabla_X \bar{R})(V_1, V_2, V_3, V_4) + (\nabla_{V_1} \bar{R})(V_2, X, V_3, V_4) + (\nabla_{V_2} \bar{R})(X, V_1, V_3, V_4) + \bar{R}(T_X V_1, V_2, V_3, V_4) + \bar{R}(T_{V_1} V_2, X, V_3, V_4) + \bar{R}(T_{V_2} X, V_1, V_3, V_4) = 0.
\]

Now, the second and the third terms of this equation are vanishing by lemma 4.3, (iii). It is easy to see that the first term is $J$-invariant in $V_1$ and $V_2$ and that all the remaining terms are $J$-anti-invariant in the same variables. Therefore, (ii) is proven and we obtain:
\[
\bar{R}(T_X V_1, V_2, V_3, V_4) + \bar{R}(T_{V_1} V_2, X, V_3, V_4) + \bar{R}(T_{V_2} X, V_1, V_3, V_4) = 0.
\]

Since $T(V_1, V_2) \subseteq V$ it suffices now to use lemma 4.3, (ii) to conclude. ■

Lemma 4.4 Let $(M^{2n}, g, J)$ be an AK$_3$-manifold admitting an orthogonal and $J$-invariant decomposition $TM = D_1 \oplus D_2$ which is also $\nabla$-parallel. Then the following hold:

(i) $\eta_{D_1} T(D_2, D_2) = 0$.

(ii) 
\[
\bar{R}(V, W, X, Y) = -\langle T_{T_{V_1} W} X + T_{T_{X} V} W, Y \rangle.
\]

whenever $V, W$ are in $D_1$ and $X, Y$ in $D_2$ respectively.

Proof:
We will prove both assertions in the same time. Let $V, W$ be in $D_1$ and $X, Y$ belong to $TM$ and $D_2$ respectively. Then using the first Bianchi identity for $\nabla$ (see lemma 3.2) and the parallelism of $D_i, i = 1, 2$ we get:
\[
\bar{R}(V, W, X, Y) + \bar{R}(W, X, V, Y) + \bar{R}(X, V, W, Y) = -\langle T_{T_{V_1} W} X + T_{T_{W} X} V + T_{T_{X} V} W, Y \rangle.
\]

By the parallelism of $D_1$ the last two terms of the left hand side vanish leading to
\[
\bar{R}(V, W, X, Y) = -\langle T_{T_{V_1} W} X + T_{T_{W} X} V + T_{T_{X} V} W, Y \rangle.
\]

Now, the right hand side of this equation and the last two terms its the left hand side are $J$-invariant in $X$ and $Y$ while the rest is $J$-anti-invariant in the same variables. We conclude that $\langle T_{T_{V_1} W} X, Y \rangle = 0$ and since $X$ is arbitrary chosen in $TM$ the use of the almost Kähler condition (2.1) leads to the proof of (i). Now (ii) follows from (i) by a simple computation. ■

Corollary 4.1 We have $\eta_{V_1} H = V_1$ and $\eta_{V_0} V_1 = 0$.

Proof:
With the facts we’ve already discussed, the proof is completely analogous to that
of the corresponding result in the $\mathcal{A}K_2$-setting (see [19]). However, we reproduce it below for the convenience of the reader. Let us prove the first assertion.

Since $\eta_{\mathcal{V}_1}\mathcal{V}_0 = 0$ we have that $\eta_{\mathcal{V}_1}H \subseteq \mathcal{V}_1$. Consider the decomposition $\mathcal{V}_1 = E \oplus F$ with $\eta_{\mathcal{V}_1}H = E$ and $F$ the orthogonal complement of $E$ in $\mathcal{V}_1$. From the definition of $F$ it follows that $\eta_{\mathcal{V}_1}F$ is orthogonal to $H$ and hence it vanishes (recall that $\eta_{\mathcal{V}_1}\mathcal{V}_1 \subseteq H$).

Since $T(\mathcal{V}_1, \mathcal{V}_1) = 0$ it also follows that $\eta_F\mathcal{V}_1 = 0$. This implies that $\eta_FH$, which a subspace of $\mathcal{V}_1$, is orthogonal to $\mathcal{V}_1$, and hence $\eta_FH = 0$. Finally since $\eta_F\mathcal{V}_0 = 0$ ($F$ lies in $\mathcal{V}_1$) we get that $F$ is contained in the Kähler nullity of $(g, J)$ and then, of course, $F = 0$.

To prove the second assertion, we note that, by lemma 4.2 the orthogonal and $J$-invariant decomposition $TM = \mathcal{V}_0 \oplus (\mathcal{V}_1 \oplus H)$ is $\nabla$-parallel. Then by lemma 4.4, (i) we find

$$\eta_{\mathcal{V}_0}T(\mathcal{V}_1, H) = \eta_{\mathcal{V}_0}\eta_{\mathcal{V}_1}H = 0.$$ Combining this with $\eta_{\mathcal{V}_1}H = \mathcal{V}_1$ finishes the proof of the lemma. ■

We need now to state a partial version of lemma 4.4. The proof will be omitted since analogous to that of the previously mentioned lemma.

**Lemma 4.5** Let $\mathcal{V}_0 = D_1 \oplus D_2$ be a orthogonal and $J$-invariant decomposition which is also $\nabla$-parallel (inside $\mathcal{V}$). Then $\hat{\eta}_{D_1}T(D_2, D_2) = 0$, where by $\hat{\eta}$ we denote the $\mathcal{V}_0$ component of the tensor $\eta$.

With this preparatives in mind, we can make a decisive step to the proof of theorem 1.1, provided we notice first the following fundamental fact. We consider the decomposition

$$\mathcal{V}_0 = E_1 \oplus E_2$$

where $E_2 = \eta_{\mathcal{V}_0}H$ and $E_1$ is the orthogonal complement of $E_2$ in $\mathcal{V}_0$. It is clear that this is a $J$-invariant decomposition.

**Lemma 4.6** We have that $\eta_{E_2}E_1 = \eta_{E_1}E_2 = 0$. Moreover, $T(E_i, E_i) = E_i$, $i = 1, 2$.

**Proof:**

Using the definition of $E_1$ we get that $\eta_{\mathcal{V}_0}E_1$ is orthogonal to $H$ and since $T(E_1, \mathcal{V}_0)$ is contained in $\mathcal{V}_0$ it follows that $\eta_{E_1}\mathcal{V}_0$ is orthogonal to $H$. It follows that $\eta_{E_1}H$ is orthogonal to $\mathcal{V}_0$ and using that $\eta_{\mathcal{V}_0}\mathcal{V}_1 = 0$ we arrive at

$$\eta_{E_1}H = 0.$$ The definition of $E_2$ implies then $\eta_{E_2}H = E_2$. Also from their definition and the parallelism of $\eta$ in vertical directions it is easy to see that $E_1$ and $E_2$ are $\nabla$-parallel inside $\mathcal{V}$. Therefore, taking $V_3$ in $E_1$ and $V_4$ in $E_2$ in (4.3) we find that

$$< \eta_{\mathcal{V}_3}, \eta_{\mathcal{V}_4}X, T_{\mathcal{V}_1}V_2 > = 0$$

for all $V_i$ in $\mathcal{V}$, $i = 1, 2$ and $X$ in $H$. Taking into account that $T(\mathcal{V}_0, \mathcal{V}_0) = \mathcal{V}_0$ and (4.4) we get that $< \eta_{\mathcal{V}_3}, \eta_{\mathcal{V}_4}X, U > = 0$ for all $U$ in $\mathcal{V}_0$. Or $\eta_{E_2}H = E_2$ and then $< \eta_{E_2}E_2, \mathcal{V}_0 > = 0$. But $\eta_{E_1}E_2$ is orthogonal to $H$ by (4.4) and also to $\mathcal{V}_1$ by corollary 4.1. We arrive at

$$\eta_{E_1}E_2 = 0.$$
We have:

**Lemma 4.7** We have:

\[(\nabla_U R)(V_1, V_2, V_3, V_4) = 0\]

for all \(U \in TM\) and \(V_i, 1 \leq i \leq 4\) in \(E_2\).

**Proof**:

If \(U\) is in \(H\) the result follows by the \(\nabla\)-parallelism of \(V_0\) and proposition 4.2, (ii). Let us suppose now that \(U\) belongs to \(V\). Deriving (4.3) in the direction of \(U\) and taking into account the symmetry property of the corollary 3.1, (i) and the parallelism of the torsion over \(V\) we find that:

\[(\nabla_U R)(V_3, V_4, \eta_{V_2} X, V_1) = (\nabla_U R)(V_3, V_4, \eta_{V_1} X, V_2)\]

for all \(X\) in \(H\) and \(V_i\) in \(V\), \(1 \leq i \leq 4\). Now, by lemma 4.1, (iii), we know that

\[R(V_3, V_4)(\eta_{V_2} X) = \eta_{V_2} R(V_3, V_4)X + \eta_{R(V_3, V_4)V_2} X\]

and furthermore, since \(V_0\) is \(\nabla\)-parallel, we find that \(R(V_3, V_4)X = 0\) for all \(V_3, V_4\) in \(V_0\), accordingly to lemma 4.4, (ii). Therefore

\[R(V_3, V_4)(\eta_{V_2} X) = \eta_{V_2} R(V_3, V_4)X\]

for all \(V_2, V_3, V_4\) in \(V_0\). Deriving the last equation in the direction of \(U\) and invoking again the parallelism of \(\eta\) in the vertical direction we find after an easy computation (using at its end the almost Kähler condition):

\[(\nabla_U R)(V_3, V_4, \eta_{V_2} X, V_1) = - (\nabla_U R)(V_3, V_4, \eta_{V_2} X, V_1).\]

Together with (4.6) and the fact that \(\eta_{E_2} H = E_2\), this leads easily to the proof of the lemma. ■

Before proceeding to the proof of theorem 1.1 we note that making the statement of theorem 1.1 precise requires the following:

**Definition 4.1** Let \((M^{2n}, g, J)\) belong to the class \(\mathcal{AK}_3\). It is said to be special iff \(\eta_V V \subseteq H\).

Note that any 4-dimensional almost Kähler manifold naturally satisfies the algebraic condition stated below. We would also like to point out the difference with the definition of special \(\mathcal{AK}_2\)-manifolds (see [10]) where an easy argument allowed reduction
to the more pleasant condition $\eta_V H = H$. In the $\mathcal{AK}_3$-case, this reduction is no more an obvious fact.

Another definition required to explain the statement of theorem 1.1 is:

**Definition 4.2** Let $(M^{2n}, g, J)$ be an almost Kähler with parallel torsion. It is called strict if $\nabla_v J = 0$ implies $v = 0$ whenever $v$ belongs to $TM$.

In other words the Kähler nullity of of a strict almost Kähler manifold with parallel torsion vanishes identically. We are now in position to prove the main result of this paper.

**Proof of theorem 1.1**:

We are going to show that $E_2 = 0$, in other words $\eta_{V_0} H = 0$. To this effect we consider the partial Hermitian Ricci curvature tensor $\bar{\rho}: V_0 \to V_0$ defined by

$$\bar{\rho} = \sum_{v_k \in E_2} \overline{R}(v_k, Jv_k)$$

where $\{v_k\}$ is an arbitrary orthonormal basis in $E_2$. Using lemma 4.4, (i) we obtain that

$$R(V, W, X, Y) = 0$$

for all $V, W$ in $V_0$ and $X, Y$ in $H$. Then the parallelism of the torsion over $V$ leads easily to $\bar{\rho}(\eta_V X) = \eta_{R(V)} X$ for all $V$ in $V_0$ and $X$ in $H$. We write now $\bar{\rho} = SJ$ where $S$ is a symmetric, $J$-invariant endomorphism of $V_0$. Then the previous relation reads:

$$4.7 S(\eta_V X) = -\eta_S V X$$

whenever $V$ is in $V_0$ and $X$ in $H$. Using lemma 4.7, it is easy to see that the restriction of $S$ to $E_2$ is $\nabla$-parallel, hence has a globally defined spectral decomposition and constant eigenvalues. We are now going to show that $S = 0$. By contradiction, let us assume that the restriction of $S$ to $E_2$ has a non-zero eigenvalue $\lambda$, and denote the corresponding eigenspace by $\mathcal{W}_1$. Set $\mathcal{W}_2 = \eta_{\mathcal{W}_1} H$ and note that by (4.7) we have that $\mathcal{W}_2 \subseteq \text{Ker}(S + \lambda)$. Then $\mathcal{W}_1$ and $\mathcal{W}_2$ are orthogonal, and again by (4.7) it follows that $\eta_{\mathcal{W}_2} H \subseteq \mathcal{W}_1$. Let $\mathcal{W}$ be the orthogonal complement of $\mathcal{W}$ in $V_0$. Then by (4.7) we get that $\eta_{\mathcal{W}} H \subseteq \mathcal{W}$. Now $\eta_{E_2} H = E_2$, hence a dimension argument shows that we must have

$$4.8 \eta_{\mathcal{W}_2} H = \mathcal{W}_1 \text{ and } \eta_{\mathcal{W}} H = \mathcal{W}.$$ 

From the $\nabla$-parallelism of $S$ it follows immediately that the distributions $\mathcal{W}, i = 1, 2$ and $\mathcal{W}$ are equally $\nabla$-parallel. With this fact in mind we will now make use of proposition 4.2, (i). Taking $V_3$ in $W$ and $V_4$ in $W_1$ in the relation (4.3) we obtain

$$< [\eta_{V_3}, \eta_{V_4}] X, T(V_1, V_2) >= 0$$

for all $V_1, V_2$ in $V$ and $X$ in $H$. Since $T(V_0, V_0) = V_0$ it follows further that $< [\eta_{V_3}, \eta_{V_4}] X, U >= 0$ for all $U$ in $V_0$. Now an invariance (with respect to $J$) argument in the variables $V_3, X$ for example yields to $< \eta_{V_3} \eta_{V_4} X, U >= < \eta_{V_4} \eta_{V_3} X, U >= 0$. Using (4.8), we get that $\eta_{\mathcal{W}_1}$ and $\eta_{\mathcal{W}_2}$ are orthogonal to $V_0$. But these spaces are orthogonal to $V_1$ (see corollary 4.1) and also to $H$ by (4.8). Therefore

14
4.9 \[ \eta_V W_1 = \eta_{V_1} W = 0 \]

and in a completely analogous manner one arrives to

4.10 \[ \eta_{V_2} W_1 = \eta_{V_1} W_2 = \eta_V W_2 = \eta_{V_2} W = 0. \]

Using the almost Kähler condition (2.1) it is easy to derive from these properties that 

\[ T(W, W) \subseteq W \text{ and } T(W_i, W_i) \subseteq W_i, i = 1, 2 \]

as well as \[ T(E_2, E_2) = E_2 \text{ and } E_2 = W_1 \oplus W_2 \oplus W \]

hence a dimension argument shows that 

\[ T(W_i, W_i) = W_i, i = 1, 2 \text{ and } T(W, W) \subseteq W. \]

In particular:

4.11 \[ T(W_1, W_1) = W_1. \]

Now, the space \[ \eta_{V_1} W_1 \] is orthogonal to \[ W_2 \oplus W \] by (4.9) and (4.10), to \[ E_1 \] by lemma 4.6, to \[ V_1 \] by corollary 4.1 and finally to \[ H \] by (4.8). We are lead to:

4.12 \[ \eta_{V_1} W_1 \subseteq W_1. \]

Using again the parallelism of the torsion over \( \mathcal{V} \) we find that

\[ \overline{\rho}(\eta_v w) = \eta_{\overline{\rho} v} w + \eta_v \overline{\rho} w \]

for all \( v, w \) in \( W_1 \). In view of (4.12) and of the fact \( \overline{\rho} \) acts on \( W_1 \) as \( \lambda J \) we obtain that \( \eta_{V_1} W_1 \) vanishes and then \( W_1 \) must be vanishing too by (4.11). We obtained a contradiction leading to the fact that \( \overline{\rho} = 0 \) on \( E_2 \). But this means that any integral manifold of \( E_2 \) is an almost Kähler manifold with parallel torsion and vanishing Hermitian Ricci tensor. As shown in [19] such a manifold has to be Kähler hence the fact that \( T(E_2, E_2) = E_2 \) ensures the vanishing of \( E_2 \). Therefore we have that \( \eta_{V_0} H = 0 \) and this implies immediately \( \eta_{V_0} V_0 \subseteq V_0 \). Since \( V_0 \) is a \( \nabla \)-parallel distribution, the last conditions are in fact telling us that \( V_0 \) is \( \nabla \)-parallel and some straightforward considerations are now finishing the proof. ■

To prove the corollary 1.1 one notices first that if the torsion is parallel, then the Kähler nullity is parallel with respect to the first Hermitian connection and hence the decomposition of theorem 1.1 holds globally. Moreover, in [20] it was shown that a special (in the sense of definition 4.1) \( \mathcal{AK}_2 \)-manifold is locally the product of a Kähler manifold and a space of type I, hence finishing the proof of the corollary.

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