Tracy-Widom limit for the largest eigenvalue of high-dimensional covariance matrices in elliptical distributions

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Abstract: Let $X$ be an $M \times N$ random matrix consisting of independent $M$-variate elliptically distributed column vectors $x_1, \ldots, x_N$ with general population covariance matrix $\Sigma$. In the literature, the quantity $XX^*$ is referred to as the sample covariance matrix after scaling, where $X^*$ is the transpose of $X$. In this article, we prove that the limiting behavior of the scaled largest eigenvalue of $XX^*$ is universal for a wide class of elliptical distributions, namely, the scaled largest eigenvalue converges weakly to the same limit regardless of the distributions that $x_1, \ldots, x_N$ follow as $M, N \to \infty$ with $M/N \to \varphi_0 > 0$ if the weak fourth moment of the radius of $x_1$ exists. In particular, via comparing the Green function with that of the sample covariance matrix of multivariate normally distributed data, we conclude that the limiting distribution of the scaled largest eigenvalue is the celebrated Tracy-Widom law.

Keywords and phrases: Sample covariance matrices, Elliptical distributions, Edge universality, Tracy-Widom distribution, Tail probability.

1. Introduction

Suppose one observed independent and identically distributed (i.i.d.) data $x_1, \ldots, x_N$ with mean 0 from $\mathbb{R}^M$, where the positive integers $N$ and $M$ are the sample size and the dimension of data respectively. Define $W = N^{-1} \sum_{i=1}^{N} x_i x_i^*$, referred to as the sample covariance matrix of $x_1, \ldots, x_N$, where $*$ is the conjugate transpose of matrices throughout this article. A fundamental research question in statistics is to analyze the behavior of $W$. Let $X = (x_1, \ldots, x_N)$ be the $M \times N$ matrix, while $\Sigma = \mathbb{E} x_1 x_1^*$ is defined as the population covariance matrix. In recent decades, fruitful results exploring the asymptotic property of $W$ have been established by random matrix theory under the high-dimensional asymptotic regime. In contrast to the traditional low dimensional asymptotic regime where the dimension $M$ is usually fixed or small and the sample size $N$ is large, the high-dimensional asymptotic regime refers to that both $N, M$ are large and even of comparable magnitude. For a list of introductory materials on random matrix theory, see e.g. [3, 5, 10, 13, 19, 39]. Let $T$ be a matrix such that $TT^* = \Sigma$, It is worth noting that most works on the inference of high-dimensional covariance matrices using random matrix theory assume that $X = TY$ with the $M \times N$ matrix $Y$ consisting of i.i.d. entries with mean 0 and variance 1. This assumption excludes many practically useful statistical models, for instance, almost all members in the family of elliptical distributions. One exception is the case where $x_1, \ldots, x_N$ follow $M$-variate normal distribution with mean 0 and population covariance matrix $\Sigma$. If $x_1, \ldots, x_N$ are not normal, the entries in each column of $X$ are only guaranteed to be uncorrelated instead of independent, the latter being a much stronger notion than the former.
In this article, we consider the case where $x_1, \ldots, x_N$ follow elliptical distribution which is a family of probability distributions widely used in statistical modeling. See e.g. the technical report [1] for a comprehensive introduction to elliptical distributions. Generally, we say a random vector $y$ follows elliptical distribution if $y$ can be written as

$$y = \xi Au,$$  \hspace{1cm} (1.1)

where $A \in \mathbb{R}^{M \times M}$ is a nonrandom matrix with $\text{rank}(A) = M$, $\xi \geq 0$ is a scalar random variable representing the radius of $y$, and $u \in \mathbb{R}^M$ is the random direction, which is independent of $\xi$ and uniformly distributed on the $M - 1$ dimensional unit sphere $\mathbb{S}^{M-1}$ in $\mathbb{R}^M$, denoted as $u \sim U(\mathbb{S}^{M-1})$. See e.g. [15, 16, 24, 43] for some recent advances on statistical inference for elliptically distributed data. The current paper focuses on the problem involving the largest eigenvalue of sample covariance matrix with elliptically distributed data. Briefly speaking, we show the following result.

**Claim 1.** If $\Sigma$ satisfies some mild assumptions, then the rescaled largest eigenvalue $N^{2/3}(\lambda_1(W) - \lambda_+)$ converges weakly to the celebrated Tracy-Widom law ([17, 27, 35, 42]) if the radius satisfies the following tail probability condition:

$$\lim_{s \to \infty} \limsup_{N \to \infty} N^2 P(|N\xi^2 - M| \geq \sqrt{Ms}) = 0 \hspace{1cm} (1.2)$$

Our arguments are built upon the pioneering works [9, 14, 22, 30, 37, 32].

The eigenvalues of sample covariance matrix widely appear in statistical applications such as principal component analysis (PCA), factor analysis, hypothesis testing. As an instance, in PCA, the eigenvalue of covariance matrix represents the variance of each component of rotated vector. In many practical situations, such as financial asset pricing and signal processing (see, e.g. [2, 12]), the data observed are usually of high-dimension but are actually sparse in nature. A common practical act is to keep only the small portion of components of large variances suggested by the eigenvalues of covariance matrix with others discarded. This way of data manipulation often acts as an effective dimension reduction technique in practice. However, most of the time, the eigenvalues of population covariance matrix are unknown. At this time, the largest sample eigenvalue performs as a good candidate for the inference of properties of population eigenvalues. See e.g. [8, 28, 29, 36].

We summarize the contributions of this paper here. We first prove the local law and Tracy-Widom limit for the sample covariance matrices of elliptical high-dimensional random vectors. The weak correlations cross the coordinates differentiate the model from the existing studies, hence facilitate to the applications in more general scenarios. The correlations also bring new challenge to the technical proofs, such as calculating the large deviation bounds and fluctuation averaging errors. Corresponding lemmas in this paper can be of independent interest. Moreover, we relax the typical moment conditions in existing results, e.g. [24, 25], to the sharper tail probability assumption (1.2). We prove that the tail probability assumption is sufficient for the Tracy-Widom limit under elliptical distribution. This result will undoubtedly push forward the research on random matrix theory with elliptically distributed data.

This article is organized as follows. In Section 2, we introduce our notation and list the basic conditions. In Section 3, we present our main results and the sketch of proof. We first prove the deformed local law which is a bound of difference between the Stieltjes transform of empirical distribution of sample eigenvalues and that of its limiting counterpart under some
bounded restrictions. This result will be our starting point to derive the limiting distribution of the rescaled largest eigenvalue. Meanwhile, it may be of interest for its own right since a number of other useful consequences regarding sample covariance matrix can be obtained from it, such as eigenvector delocalization and eigenvalue spacing (see e.g. [10]). Taking the deformed local law as an input, the next step is to prove a strong average local law with some restrictions on bounded support and the four leading moments of \( \xi \). Finally, we show that the limiting distribution of the rescaled largest eigenvalue does not depend on the specific distribution of matrix entries under a tail probability condition. Comparison with the normally distributed data then indicates that the limiting distribution of the rescaled largest eigenvalue is the Tracy-Widom (TW) law. In Sections 4 to 8, we give the detailed proof of our main theorems, while several lemmas and results will be also put into the Appendices.

2. Notation and basic conditions

Throughout this article, we set \( C > 0 \) to be a constant whose value may be different from line to line. \( \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+, \mathbb{C}, \mathbb{C}^+ \) denote the sets of integers, positive integers, real numbers, positive real numbers, complex numbers and the upper half complex plane respectively. For \( a, b \in \mathbb{R} \), \( a \land b = \min(a, b) \) and \( a \lor b = \max(a, b) \). \( z = i = \sqrt{-1} \). For a complex number \( z \), \( \text{Re} z \) and \( \text{Im} z \) denote the real and imaginary parts of \( z \) respectively. For a matrix \( A = (A_{ij}) \), \( \text{Tr} A \) denotes the trace of \( A \), \( \| A \| \) denotes the spectral norm of \( A \) equal to the largest singular value of \( A \) (usually we use \( \| \cdot \|_2 \) as well) and \( \| A \|_F \) denotes the Frobenius norm of \( A \) equal to \( \sum_{ij} |A_{ij}|^2 / 2 \). \( M \in \mathbb{Z}_+ \), \( \text{diag}(a_1, \ldots, a_M) \) denote the diagonal matrix with \( a_1, \ldots, a_M \) as its diagonal elements. For two sequences of numbers \( \{a_N\}_{N=1}^\infty, \{b_N\}_{N=1}^\infty \), \( a_N \asymp b_N \) if there exist constants \( C_1, C_2 > 0 \) such that \( C_1 |b_N| \leq |a_N| \leq C_2 |b_N| \) and \( O(a_N) \) and \( o(a_N) \) denote the sequences such that \( |O(a_N)/a_N| \leq C \) with some constant \( C > 0 \) for all large \( N \) and \( \lim_{N \to \infty} o(a_N)/a_N = 0 \). \( I \) denotes the identity matrix of appropriate size. For a set \( A \), \( A^c \) denotes its complement (with respect to some whole set which is clear in the context). For some integer \( M \in \mathbb{Z}_+, \chi^2_M \) denotes the chi-square distribution with degrees of freedom \( M \). For a measure \( \varrho \), \( \text{supp}(\varrho) \) denotes its support. For any finite set \( T \), we let \( |T| \) denote the cardinality of \( T \). For any event \( \Xi \), \( 1(\Xi) \) denotes the indicator of the event \( \Xi \), equal to 1 if \( \Xi \) occurs and 0 if \( \Xi \) does not occur. For any \( a, b \in \mathbb{R} \) with \( a \leq b \), \( 1[a, b](x) \) is equal to 1 if \( x \in [a, b] \) and 0 if \( x \not\in [a, b] \).

We consider the \( M \times N \) data matrix \( X \) as follows. Let \( U = (u_1, \ldots, u_N) \) and \( \varrho = \text{diag}(\xi_1, \ldots, \xi_N) \). Then the corresponding column vectors and data matrices are

\[
X := (x_1, \ldots, x_N), \quad \Sigma^{1/2} U := (r_1, \ldots, r_N), \quad X = \Sigma^{1/2} U \varrho = \Sigma^{1/2} (\xi_1 u_1, \ldots, \xi_N u_N) = (\xi_1 r_1, \ldots, \xi_N r_N)
\]

where \( u_i \)'s are i.i.d. from \( U(\mathbb{S}^{M-1}) \) and \( \xi_i \)'s are i.i.d. nonnegative random variables independent with all \( u_i \)'s. Our sample covariance matrix is \( XX^* \). So \( \xi_i \) has absorbed the usual normalised factor \( 1/\sqrt{N} \) for all \( i \). For convenience we write \( \xi_i := \sqrt{N} \xi_i \).

The \( \Sigma^{1/2} \) in the above equation can be also replaced by some general \( M \times M \) matrix \( A \) with \( AA^* = \Sigma \). We claim that the technical proof is totally the same, using the singular value decomposition \( A = U_A D_A V_A \) and the observation that the distribution of \( u_1, \ldots, u_N \) is orthogonally invariant. For the same reason, without loss of generality we assume that \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_M) \), where \( \sigma_1, \ldots, \sigma_M \) denote the descending eigenvalues.
Denote the empirical spectral density of $\Sigma$ as

$$\pi := \frac{1}{M} \sum_{i=1}^{M} \delta_{\sigma_i}.$$  

Following the general assumptions on $\Sigma$ in the literature, we suppose that for a small enough constant $\tau > 0$,

$$\sigma_1 \leq \tau^{-1} \quad \pi([0, \tau]) \leq 1 - \tau. \quad (2.1)$$

Fix $0 < \tau < 1$, and define

$$D \equiv D(\tau, N) := \{ z = E + i\eta \in \mathbb{C}^+ : |z| \geq \tau, |E| \leq \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1} \}.$$  

For $z := E + i\eta \in \mathbb{C}^+$, further define the following quantities

$$W = X^*X, \quad W = XX^*,$$

while the respective Green functions of $W$ and $W$ are

$$G(z) = (W - zI)^{-1}, \quad \mathcal{G}(z) = (W - zI)^{-1}.$$  

We denote the respective empirical spectral density of $W$ and $W$ as

$$\rho_W := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(W)}, \quad \rho_W := \frac{1}{M} \sum_{i=1}^{M} \delta_{\lambda_i(W)}.$$  

The stieltjes’ transforms of $\rho_W$ and $\rho_W$ are given by

$$m_N^W := \int \frac{1}{x-z} \rho_W = \frac{1}{N} \text{Tr} G(z), \quad m_N^W := \int \frac{1}{x-z} \rho_W = \frac{1}{M} \text{Tr} \mathcal{G}(z).$$  

Throughout the rest of the paper, we denote $m_N(z) := m_N^W(z)$ for simplification.

It’s easy to see that the eigenvalues of $W$ and $W$ are the same up to $|M - N|$ number of 0s. We denote the descending eigenvalues of $W$ and $W$ in the unified manner as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{M \land N}$, where $\lambda_1, \ldots, \lambda_M$ and $\lambda_1, \ldots, \lambda_N$ are understood to be the eigenvalues of $W$ and $W$ respectively. In particular, $\lambda_{M \land N+1}, \ldots, \lambda_{M \land N}$ are all 0. Consequently,

$$\phi_N \rho_W = \rho_W + (1 - \phi_N) \delta_0 \quad (2.2)$$

and

$$\phi_N m_N^W(z) = \frac{1 - \phi_N}{z} + m_N(z), \quad (2.3)$$

where $\phi_N := M/N$. We may suppress the subscript $N$ and use $\phi$ directly hereafter.

Denote the index set $\mathcal{I} = \{1, \ldots, N\}$. For $T \subset \mathcal{I}$, we introduce the notation $X^{(T)}$ to denote the $M \times (N - |T|)$ minor of $X$ obtained from removing all the $i$th columns of $X$ for $i \in T$. In particular, $X^{(\emptyset)} = X$. For convenience, we briefly write ($\{i\}$, ($\{i, j\}$) and $\{i, j\} \cup T$ as $i$, ($i, j$) and ($ijT$) respectively. Correspondingly,

$$W^{(T)} = (X^{(T)})^*X^{(T)}, \quad W^{(T)} = X^{(T)}(X^{(T)})^*.$$
and
\[ G^{(T)}(z) = (W^{(T)} - zI)^{-1}, \quad G_\Omega^{(T)}(z) = (W^{(T)} - zI)^{-1}, \quad m_N^{(T)}(z) = \frac{1}{N} \text{Tr} G^{(T)}(z). \]

Throughout this article, we denote \( X_{ij} \) as the \((i,j)\)-th entry of a matrix \( X \). In particular, in the minor \( X_{ij} \) with \( i, j \notin T \), we keep the original indices of \( X \).

In the following, we present a notion introduced in [20]. It provides a simple way of systematizing and making precise statements for two families of random variables \( A, B \) of the form "\( A \) is bounded with high probability by \( B \) up to small powers of \( N \)."

**Definition 2.1 (Stochastic domination).**
(a) For two families of nonnegative random variables
\[ A = \{ A_N(t) : N \in \mathbb{Z}_+, t \in T_N \}, \quad B = \{ B_N(t) : N \in \mathbb{Z}_+, t \in T_N \}, \]
where \( T_N \) is a possibly \( N \)-dependent parameter set, we say that \( A \) is stochastically dominated by \( B \), uniformly in \( t \) if for all (small) \( \varepsilon > 0 \) and (large) \( D > 0 \) there exists \( N_0(\varepsilon, D) \in \mathbb{Z}_+ \) such that as \( N \geq N_0(\varepsilon, D) \),
\[ \sup_{t \in T_N} \mathbb{P} (A_N(t) > N^\varepsilon B_N(t)) \leq N^{-D}. \]

If \( A \) is stochastically dominated by \( B \), uniformly in \( t \), we use notation \( A \prec B \) or \( A = O_\infty(B) \). Moreover, for some complex family \( A \) if \( |A| \prec B \) we also write \( A = O_\infty(B) \).

(b) Let \( A \) be a family of random matrices and \( \zeta \) be a family of nonnegative random variables. Then we denote \( A = O_\infty(\zeta) \) if \( A \) is dominated under weak operator norm sense, i.e. \( |\langle v, Aw \rangle| \prec \zeta \|v\|_2 \|w\|_2 \) for any deterministic vectors \( v \) and \( w \).

(c) For two sequences of numbers \( \{ a_N \}_{N=1}^\infty, \{ b_N \}_{N=1}^\infty \), \( a_N \prec b_N \) if for all \( \varepsilon > 0 \), \( a_N \leq N^\varepsilon b_N \).

**Remark 2.2.** The stochastic domination throughout this article holds uniformly for the matrix indices and \( z \in D \) (or the set \( D^\varepsilon \) defined later). For simplicity, in the proof of each result, we omit the explicit indication of this uniformity.

The discussion in this paper highly relies on the following global definitions.

**Definition 2.3 (High probability event).** We say that an \( N \)-dependent event \( \Omega \) holds with overwhelming high probability if there exists constant \( c > 0 \) independent of \( N \), such that
\[ \mathbb{P}(\Omega) \geq 1 - \exp(-N^c), \quad (2.4) \]
for all sufficiently large \( N \).

**Definition 2.4 (Bounded support condition).** We say that an \( N \)-dependent random variable \( x := x(N) \) satisfies the bounded support condition with \( q \equiv q(N) \) if
\[ \mathbb{P}(|x| \leq q) \geq 1 - \exp(-N^c), \quad (2.5) \]
for some \( c > 0 \).

**Remark 2.5.** Note that if \( x \) satisfies Condition (2.5), it is equivalent to that the event \( \{|x| \leq q\} \) holds with high probability. Consequently, we can neglect the bad event \( \{|x| > q\} \). In other words, in our proof, we are in a high probability whole set \( \Omega \). For instance, under \( \Omega \), the entries of a data matrix satisfy the bounded support condition.
Throughout this article, we assume the following conditions.

**Condition 2.6.** \( N \to \infty \) with \( M \equiv M(N) \to \infty \) such that \( \phi := M/N \to \phi_0 \in [a, b] \) for all large \( N \) where \( a < b \) are two positive numbers.

**Condition 2.7.** \( \xi_1, \ldots, \xi_N \) are independent nonnegative random variables such that \( \mathbb{E} \xi_i^2 = \phi \) and

\[
\lim_{s \to \infty} \limsup_{N \to \infty} s^2 \mathbb{P}(|\hat{\xi}_i^2 - M| \geq \sqrt{Ms}) = 0,
\]

for all \( i \in \{1, \cdots, N\} \). Recall that \( \xi_i = \hat{\xi}_i/\sqrt{N} \) and \( \hat{\xi}_i^2 \) has left tight support i.e. \( \hat{\xi}_i^2 \geq 0 \).

We remark that in Condition 2.7, the general choice of \( s \) diverges with \( N \), e.g. \( s = N^{1/2 - \epsilon} \) in the proof of Theorem 3.6 below.

We put an alternative restriction on \( \xi_1, \ldots, \xi_N \).

**Condition 2.8.** \( \xi_1, \ldots, \xi_N \) are independent nonnegative random variables such that \( \mathbb{E} \xi_i^2 = \phi \) and \( \xi_i^2 - \phi \) has bounded support \( q \) in the sense of Definition 2.4 with

\[
N^{-1/2} \log N \leq q \leq N^{-c}
\]

for some \( c < 1/2 \) and all \( i \in \{1, \cdots, N\} \).

**Remark 2.9.** Condition 2.7 excludes some elliptical distributions, such as multivariate student- \( t \) distributions and normal scale mixtures. The limiting spectral distribution of sample covariance matrix from these distributions do not follow the Marchenko-Pastur equation (2.7), see ([18, 33]), and hence is out of scope of this article. Actually there are still a wide range of distributions satisfying Condition 2.7, including the multivariate Pearson type II distribution and the family of Kotz-type distributions, see the examples and Table 1 in [24]. In particular, if \( \xi^2 \) can be written as \( \xi^2 = N^{-1}(y_1^2 + \cdots + y_M^2) \) with \( y_1, \ldots, y_M \) being a positive i.i.d. sequence such that \( \mathbb{E}y_1 = 1 \) and \( \mathbb{E}y_1^4 < \infty \), then \( \xi^2 \) satisfies Condition 2.7.

**Remark 2.10.** (2.6) is weaker than

\[
\limsup_{N \to \infty} \frac{1}{M} \mathbb{E}|\hat{\xi}_i^2 - M|^2 < \infty
\]

but stronger than

\[
\limsup_{N \to \infty} \frac{1}{M} \mathbb{E}|\hat{\xi}_i^2 - M|^{2-\delta} < \infty
\]

for arbitrary \( \delta > 0 \).

One can check that Conditions 2.6 and 2.8 are sufficient for Theorem 1.1 of [6]. Hence we have the following result.

**Lemma 2.11.** Suppose, given Conditions 2.6 and 2.8, \( \pi \) converges weakly to a probability distribution \( \pi_0 \) and \( \phi \to \phi_0 \in (0, \infty) \). Then, almost surely, \( \rho_W \) converges weakly to a deterministic limiting probability distribution \( \rho_0 \) and for any \( z \in \mathbb{C}^+ \), almost surely, \( m_N(z) \) converges to the Stieltjes transform of \( \rho_0 \) which we denote as \( m_0(z) \). Moreover, for all \( z \in \mathbb{C}^+ \), \( m_0(z) \) is the unique value in \( \mathbb{C}^+ \) satisfying the equation

\[
z = -\frac{1}{m_0(z)} + \phi_0 \int \frac{x}{1 + x m_0(z)} \pi_0(dx).
\]
Remark 2.12. If we replace \( \pi_0 \) and \( \phi_0 \) by their finite sample counterparts \( \pi \) and \( \phi \) in (2.7) and solve for \( m \) for each \( z \in \mathbb{C}^+ \), we obtain a Stieltjes transform of a deterministic probability distribution. Throughout this article, we denote this deterministic probability distribution and its Stieltjes transform as \( \rho \) and \( m(z) \) respectively. By Lemma 2.11, when \( N \) is large \( \rho \) and \( m_N(z) \) are close to \( \rho \) and \( m(z) \). The aim of next section is to evaluate the bound of \( |m_N(z) - m(z)| \).

We define the function \( f : \mathbb{C} \to \mathbb{C} \),
\[ f(w) = -\frac{1}{w} + \phi \int \frac{x \pi(dx)}{1 + wx}, \tag{2.8} \]
and assume that
\[ f'(-c) = 0, \quad 0 < \liminf_{N \to \infty} \sigma_M \leq \limsup_{N \to \infty} \sigma_1 < \infty, \quad \limsup_{N \to \infty} \sigma_1 c < 1. \tag{2.9} \]
for \( c \in (0, \sigma_1^{-1}) \). Let \( \lambda_+ := f(-c) \), so it can be shown that \( \lambda_+ \) is the rightmost endpoint of \( \text{supp}(\rho) \) (see the discussion on page 4 of [9] or Lemma 2.4 of [30]), i.e., the edge of \( \rho \).

For \( \tau, \tau' \in (0, \infty) \), \( N \in \mathbb{Z}^+ \), define
\[ D = D_\tau(\tau, \tau', N) := \{ z \in D_\tau(N) : E \in [\lambda_+ - \tau', \lambda_+ + \tau'] \} \tag{2.10} \]
as the subset of \( D \) with the real part of \( z \) restricted to a small closed interval around \( \lambda_+ \).

Also we define the distance to the rightmost edge as
\[ \kappa = \kappa_E := |E - \lambda_+| \quad \text{for} \quad z = E + i\eta. \tag{2.11} \]

Now we introduce some definitions before presenting our main results.

**Definition 2.13** (Linearizing block matrix). For \( z \in \mathbb{C}_+ \), we define the \( (N + M) \times (N + M) \) block matrix (no commas in matrices)
\[ H := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \tag{2.12} \]
and
\[ \mathcal{H} := \begin{pmatrix} -I_{M \times M} & X \\ X^* & -zI_{N \times N} \end{pmatrix}^{-1} \]
\[ = \begin{pmatrix} zG & GX \\ (G X)^* & G \end{pmatrix} \tag{2.13} \]
\[ = \begin{pmatrix} zG & (XG)^* \\ (G X)^* & G \end{pmatrix}. \tag{2.14} \]

**Definition 2.14** (Deterministic limit of \( \mathcal{H} \)). We define the deterministic limit \( \Pi \) of \( \mathcal{H} \) as
\[ \Pi(z) := \begin{pmatrix} -(1 + m(z) \Sigma)^{-1} & 0 \\ 0 & m(z) I_{N \times N} \end{pmatrix}. \tag{2.15} \]

Define the control parameters by
\[ \Lambda \equiv \Lambda(z) := \max_{i,j \in \mathbb{Z}} |G_{ij}(z) - \delta_{ij} m(z)|, \quad \Lambda_o := \max_{i,j \in \mathbb{Z}, i \neq j} |G_{ij}(z)|, \]
\[ \Theta \equiv \Theta(z) := |m_N(z) - m(z)|, \quad \Psi \equiv \Psi(z) := \sqrt{\frac{3 \text{Im} m(z) + \Theta}{N \eta}}, \quad \Xi := \{ \Lambda \leq (\log N)^{-1} \}. \]
where \( \delta_{ij} \) denotes the Kronecker delta, i.e. \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) if \( i \neq j \) and \( \Xi \) is a \( z \)-dependent event. For simplicity of notation, we occasionally omit the variable \( z \) for those \( z \)-dependent quantities provided no ambiguity occurs.

3. Main results

3.1. Deformed local law

**Theorem 3.1** (Deformed strong local law). Given Conditions 2.6, 2.8 as well as (2.1) and (2.9), there exists a constant \( \tau' \) depending only on \( \tau \) such that

\[
\Lambda(z) < \sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta} + q, \tag{3.1}
\]

\[
|m_N(z) - m(z)| < \left( \min\{q, \frac{q^2}{\sqrt{\kappa N}}\} + \frac{1}{N\eta} \right), \tag{3.2}
\]

uniformly for \( z \in D^c(\tau, \tau', N) \).

**Remark 3.1.** Theorem 3.1 can be strengthened in a simultaneous sense for \( z \in D^c(\tau, \tau', N) \), using the Lipschitz continuity of \( G_{ij}(z), m(z), \Psi_\Lambda(z), \Psi_m(z) \) and the fact that \( \Psi_\Lambda(z), \Phi_m(z) \geq 1 \) on \( D^c(\tau, \tau', N) \), where

\[
\Phi_\Lambda(z) := \sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta} + q, \quad \Phi_m(z) := \min\{q, \frac{q^2}{\sqrt{\kappa N}}\} + \frac{1}{N\eta}.
\]

The proof is essentially the same as the one in (III.5) of Appendix III. We just put down the conclusions as follows,

\[
\sup_{z \in D^c} \max_{i,j} \frac{\Lambda(z)}{\Phi_\Lambda(z)} < 1, \quad \sup_{z \in D^c} \frac{|m_N(z) - m(z)|}{\Phi_m(z)} < 1, \tag{3.3}
\]

under the assumptions in Theorem 3.1.

A direct consequence is the following theorem.

**Theorem 3.2.** Under the assumptions in Theorem 3.1, we have

\[
||H||^2 \leq \lambda_+ + N^{r}(q^2 + N^{-2/3}). \tag{3.4}
\]

Furthermore, for any real numbers \( a, b \) such that \( a \leq b \), define \( n_N(a, b) = \int_a^b g_N(dx) \) and \( n(a, b) = \int_a^b g(dx) \). Then there exists a constant \( \tau' \) depending only on \( \tau \) such that for any \( E_1, E_2 \in \{ \Re z : z \in D^c(\tau, \tau', N) \} \),

\[
|n_N(E_1, E_2) - n(E_1, E_2)| \ll N^{-1} + q^3 + q^2(\sqrt{\kappa_{E_1}} - \sqrt{\kappa_{E_2}}). \tag{3.5}
\]

Consequently, we have for \( q \leq N^{-1/3} \),

\[
|\lambda_i - \gamma_i| \ll i^{-1/3} N^{-2/3} + q^2, \tag{3.6}
\]

uniformly in \( i \) such that \( \gamma_i \in [\lambda_+ - c, \lambda_+] \) for some \( c > 0 \), where

\[
\gamma_i := \sup_{x} \{ \int_x^\infty g(x)dx > \frac{i - 1}{N} \}.
\]
The proof of this theorem is the same as the one in [14], and we summarize the main arguments in Appendix III.

3.2. Edge universality with small support

**Theorem 3.3** (Edge universality with small support). Suppose $X^W$ and $X^V$ are two random matrices satisfying Conditions 2.6 and 2.8 with $q \leq N^{-5/12 + \xi}$ for some small $\xi > 0$. Then there exist some positive constants $\epsilon, \delta > 0$ such that for any $s \in \mathbb{R}$

$$P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s - N^{-\epsilon}) - N^{-\delta} \leq P^W(N^{2/3}(\lambda_1 - \lambda_+) \leq s) \leq P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s + N^{-\epsilon}) + N^{-\delta},$$

where $P^V$ and $P^W$ denote the laws of $X^V$ and $X^W$ respectively.

**Remark 3.2.** Theorem 3.3 can be extended to the case of joint distribution of the largest $k$ eigenvalues for any fixed positive integer $k$, that is, for any real numbers $s_1, \ldots, s_k$ which may depend on $N$, there exist some positive constants $\epsilon, \delta > 0$ such that for all large $N$

$$P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s_1 - N^{-\epsilon}, \ldots, N^{2/3}(\lambda_k - \lambda_+) \leq s_k - N^{-\epsilon}) - N^{-\delta} \leq P^W(N^{2/3}(\lambda_1 - \lambda_+) \leq s_1, \ldots, N^{2/3}(\lambda_k - \lambda_+) \leq s_k) \leq P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s_1 + N^{-\epsilon}, \ldots, N^{2/3}(\lambda_k - \lambda_+) \leq s_k + N^{-\epsilon}) + N^{-\delta}.$$

3.3. Edge universality with large support

**Theorem 3.4** (Rigidity of eigenvalues with large support). Suppose random matrix $X$ satisfies Conditions 2.6 and 2.8 with $q \leq N^{-c}$ for some constant $c > 0$ and suppose moreover that

$$E|\xi_i^2 - \phi|^2 \leq BN^{-1} \log N,$$

for some constant $B > 0$. Then there exists constant $c_1, \tau, \tau'$ such that

$$\sup_{z \in D^c} \frac{|m_N(z) - m(z)|}{(N\eta)^{-1}} < 1,$$

for sufficient large $N$. Moreover, (3.10) implies that with high probability

$$|\lambda_i - \gamma_i| \prec i^{-1/3} N^{-2/3},$$

uniformly in $i$ such that $\gamma_i \in [\lambda_1 - c_1, \lambda_1]$, and

$$\sup_{E \geq \lambda_1 - c_1} |n_N(E) - n(E)| \prec \frac{1}{N},$$

**Theorem 3.5** (Edge universality with large support). Suppose $X^W$ and $X^V$ are two random matrices satisfying the assumptions in Theorem 3.4. Then there exist some positive constants $\epsilon, \delta > 0$ such that for any $s \in \mathbb{R}$

$$P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s - N^{-\epsilon}) - N^{-\delta} \leq P^W(N^{2/3}(\lambda_1 - \lambda_+) \leq s) \leq P^V(N^{2/3}(\lambda_1 - \lambda_+) \leq s + N^{-\epsilon}) + N^{-\delta},$$

where $P^V$ and $P^W$ denote the laws of $X^V$ and $X^W$ respectively.
3.4. Edge universality

Let \(u_1, \ldots, u_N\) be from \(U(S^{M-1})\), and \(\xi_1, \ldots, \xi_N\) be i.i.d. non-negative random variables such that \(\xi_i^2\) follows \(\chi^2_{3}/N\) distribution. Assume the independence of \(\{u_1, \ldots, u_N\}\) and \(\{\xi_1, \ldots, \xi_N\}\). Let \(\tilde{X} := \Sigma^{1/2}(\xi_1 u_1, \ldots, \xi_N u_N)\), so \(\tilde{X}\) is a matrix whose columns are i.i.d. Gaussian random vectors. We claim in the following theorem that for elliptically distributed data \(X\) with \(\Sigma\) satisfying Condition 2.6, (2.1) and (2.9), the largest eigenvalue of its sample covariance matrix follows the same limiting distribution as the one with \(\tilde{X}\) if Condition 2.7 holds.

**Theorem 3.6 (Edge universality).** Let \(W = X^* X\) be an \((N \times N)\) sample covariance matrix with \(X\) satisfying Condition 2.6, (2.1) and (2.9). If Condition 2.7 holds, then we have for all \(s \in \mathbb{R}\)

\[
\lim_{N \to \infty} P(N^{2/3}(\lambda_1 - \lambda_+) \leq s) = \lim_{N \to \infty} P(N^{2/3}(\tilde{\lambda}_1 - \lambda_+) \leq s),
\]

where \(\tilde{\lambda}_1\) is the largest eigenvalue of \(\tilde{X}^* \tilde{X}\).

**Corollary 3.3 (Tracy-Widom law).** Under assumptions in Theorem 3.6, we have

\[
\lim_{N \to \infty} P(\gamma N^{2/3}(\lambda_1 - \lambda_+) \leq s) = F_1(s),
\]

where \(\gamma\) is defined by

\[
\frac{1}{\gamma^3} = \frac{1}{c^3} \left(1 + \phi \int \left(\frac{\lambda c}{1 - \lambda c}\right)^3 \pi(d\lambda)\right),
\]

and \(F_1(s)\) is the type-1 Tracy-Widom distribution [42].

**Remark 3.4.** Theorem 3.6 can be extended to the case of joint distribution of the largest \(k\) eigenvalues for any fixed positive integer \(k\), namely, for any real numbers \(s_1, \ldots, s_k\) which may depend on \(N\), there exist some positive constants \(\varepsilon, \delta > 0\) such that for all large \(N\)

\[
P(N^{2/3}(\tilde{\lambda}_1 - \lambda_+) \leq s_1 - N^{-\varepsilon}, \ldots, N^{2/3}(\tilde{\lambda}_k - \lambda_+) \leq s_k - N^{-\varepsilon}) - N^{-\delta} \\
\leq P(N^{2/3}(\lambda_1 - \lambda_+) \leq s_1, \ldots, N^{2/3}(\lambda_k - \lambda_+) \leq s_k) \\
\leq P(N^{2/3}(\tilde{\lambda}_1 - \lambda_+) \leq s_1 + N^{-\varepsilon}, \ldots, N^{2/3}(\tilde{\lambda}_k - \lambda_+) \leq s_k + N^{-\varepsilon}) + N^{-\delta}.
\]

Accordingly, Corollary 3.3 can be extended to the case of joint distribution as follows,

\[
(\gamma N^{2/3}(\lambda_1 - \lambda_+), \ldots, \gamma N^{2/3}(\lambda_k - \lambda_+))
\]

converges to the \(k\)-dimensional joint Tracy-Widom distribution. Here we use the term “joint Tracy-Widom distribution” as in Theorem 1 of [41]. The extension (3.15) can be proved by a similar argument to the one in [37]. Hence we do not reproduce the details.

3.5. Sketch of the proof

First, we show Theorems 3.1 which will serve as crucial inputs for the proof of Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6. The proof strategy essentially dates
back to [22, 30, 37]. We start by studying each entry of the Green function $G(z)$. The general target is to show that each diagonal element of $G(z)$ is close to $m(z)$ and the off-diagonal elements of $G(z)$ are close to 0 under the bounded support $q$. Before attaining the final goal, our first step is to obtain a weaker but still nontrivial version of the local law, i.e. $\Lambda(z) \prec (N\eta)^{-1/4} + q$. Compared to previous papers e.g. [8, 9, 30, 37] assuming i.i.d. entries in the data matrix, the main difficulty of our work is to deal with dependence among entries in each column $x_i$, $i = 1, \ldots, N$. Due to the dependence, the usual large deviation bounds for i.i.d. vectors in [8, 9, 30, 37] are no longer applicable. In Section 4, we present the large deviation inequalities (Lemma 4.4) for uniformly spherically distributed random vectors and give their proofs in the Appendix I. Moreover, the radius variable $\xi_i$ causes extra randomness which is the reason for the introduction of Condition 2.8 as to reduce the variation. Also due to dependence, the strategy in [30] to expand the matrix $X$ along both rows and columns cannot be applied. We tackle this issue by expanding $X$ only along columns and bounding the errors emerging from the finite sample approximation of the Marčenko-Pastur equation. Then the weak deformed local law can be achieved by a bootstrapping procedure. Next, the weaker bound is strengthened to

$$\Lambda(z) \prec \sqrt{\frac{\text{Im} m(z)}{N\eta}} + \frac{1}{N\eta} + q$$

via the self-improving steps utilizing a so-called fluctuation averaging argument. This procedure involves estimating the conditional expectation of $\frac{1}{N} \sum_{i \in I} x_i^* G(i) \Sigma (m_N(i) \Sigma + I)^{-1} x_i$. The difficulty lies in not only the dependence among each column but also randomness in $(m_N(i) \Sigma + I)^{-1}$. In order to handle these difficulties, we expand $x_i^* G(i)$ and $(m_N(i) \Sigma + I)^{-1}$ respectively. It turns out to be several weakly correlated monomials of quadratic forms with entries of $G(i)$ as coefficients. For these Green function entries, we further expand them by resolvent identities. One can refer to Appendix II for the details.

With (3.2) at hand, Theorem 3.2 follows from a standard argument similar to Proposition 9.1 of [10], and the Helffer-Sjöstrand argument, see e.g. Theorem 2.8 and Appendix C of [10] or (8.6) of [37]. For the readers’ convenience, we write down the details of the proof of Theorem 3.2 in Appendix III.

For Theorem 3.3, we use the Green function comparison method. The strategy follows [37] with a Lindeberg-type column by column replacement due to the dependence within each column of $X$. The details will be provided in Section 6.

The establishment of Theorem 3.4 and Theorem 3.5 is the key step to prove Theorem 3.6. Roughly speaking, we find that the strong average local law holds with larger support and some mild restrictions on the four leading moments of $X$. Such moment restrictions can be further relaxed to the tail probability Condition 2.7 using the truncation technique, which concludes Theorem 3.6. The main tool is still the Green function comparison method, while the details are put in Sections 7 and 8.

4. Preliminary results

In this section, we present some preliminary results that will be used in the derivation of our main theorems in Sections 5 and 6. Lemma 4.1 is by Shur’s complement formula, whose proof can be found in Lemma 4.2 of [21]. The proof of Lemmas 4.2 and 4.4 are given in Appendix I. Lemma 4.3 is by elementary linear algebra whose proof is omitted.
Lemma 4.1. Under the above notation, for any $T \subset I$
\[ G_{ii}^{(T)}(z) = -\frac{1}{z + z_i^2 G^{(iT)}(z)x_i}, \quad \forall i \in \mathcal{T}, \]
\[ G_{ij}^{(T)}(z) = z G_{ii}^{(T)}(z) G_{jj}^{(iT)}(z) x_i x_j, \quad \forall i, j \in \mathcal{T}, i \neq j, \]
\[ G_{ij}^{(T)}(z) = G_{ij}^{(iT)}(z) + \frac{G_{ik}^{(T)}(z) G_{kj}^{(T)}(z)}{G_{kk}^{(T)}(z)}, \quad \forall i, j, k \in \mathcal{T}, i, j \neq k. \]

Lemma 4.2. Let $\{X_N\}_{N=1}^{\infty}$ be a sequence of random variables and $\Phi_N$ be deterministic. Suppose $\Phi_N \geq N^{-C}$ holds for large $N$ with some $C > 0$, and that for all $p$ there exists a constant $C_p$ such that $\mathbb{E}|X_N|^p \leq N^{C_p}$. Then we have the equivalence

$$X_N \prec \Phi_N \iff \mathbb{E}X_N^p \prec \Phi_N^p \quad \text{for any fixed } p \in \mathbb{N}.\]$$

Lemma 4.3. Let $A, B$ be two matrices with $AB$ well-defined. Then
\[ |\text{Tr}(AB)| \leq \|A\|_F \|B\|_F, \]
\[ \|AB\| \leq \|A\|\|B\|, \]
\[ \|AB\|_F \leq \min\{\|A\|_F \|B\|, \|A\| \|B\|_F\} \leq \|A\|_F \|B\|_F, \]
\[ \|A + B\|_F \leq \|A\|_F + \|B\|_F, \]
\[ |\text{Tr}(AB)| \leq \|A\| \text{Tr}|B|. \]

Lemma 4.4. Let $u = (u_1, \ldots, u_M)^*$, $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_M)^*$ be $U(S^{M-1})$ random vectors, $A = (a_{ij})$ an $M \times M$ matrix and $b = (b_1, \ldots, b_M)^*$ an $M$-dimensional vector, where $A$ and $b$ may be complex-valued and $u, \tilde{u}, A, b$ are independent. Then as $M \to \infty$
\[ |b^* u| \prec \sqrt{\frac{\|b\|^2}{M}}, \quad (4.1) \]
\[ |u^* Au - \frac{1}{M} \text{Tr} A| \prec \frac{1}{M} \|A\|_F, \quad (4.2) \]
\[ |u^* A\tilde{u}| \prec \frac{1}{M} \|A\|_F. \quad (4.3) \]

Moreover, if $u, \tilde{u}, A, b$ depend on an index $t \in T$ for some set $T$, then the above domination bounds hold uniformly for $t \in T$.

Recalling the definition of $\kappa$, we then introduce the following two results whose proof can be found in Lemmas A.4 and A.5 of [30]. In particular, the edge regularity condition required in [30] is encompassed in (2.9).

Lemma 4.5. Fix $\tau > 0$. Given assumption (2.9), there exists $\tau' > 0$ such that for any $z \in \mathbb{D}^\epsilon(\tau', \tau', N)$ we have
\[ \text{Im} m(z) \geq \begin{cases} \sqrt{\kappa + \eta} & \text{if } E \in \text{supp}(\varrho), \\ \eta & \text{if } E \notin \text{supp}(\varrho), \end{cases} \]
\[ |1 + m(z) \sigma_i| \geq \tau, \quad \forall i \in \{1, \ldots, M\}. \quad (4.4) \]
Proposition 4.6. Fix $\tau > 0$. There exists a constant $\tau' > 0$ such that $z = f(m)$ is stable at the edge $D^c(\tau, \tau', N)$ in the following sense. Suppose $\delta : D^c \to (0, \infty)$ satisfies $N^{-2} \leq \delta(z) \leq \log^{-1} N$ for $z \in D^c$ and that $\delta$ is Lipschitz continuous with Lipschitz constant $N^2$. Suppose moreover that for each fixed $E$, the function $\eta \to \delta(E + \eta)$ is nonincreasing for $\eta > 0$. Suppose that $u : D^c \to \mathbb{C}$ is the Stieltjes transform of a probability measure supported on $[0, C]$. Let $z \in D^c$ and suppose that $|f(u(z)) - z| \leq \delta(z)$.

If $\text{Im} z < 1$, suppose also that $|u - m| \leq C\delta \sqrt{\kappa + \eta + \delta}$, \hspace{1cm} (4.5)

holds at $z + iN^{-5}$. Then (4.5) holds at $z$.

5. Proof of the local law

In this section, we prove Theorem 3.1. Theorem 3.2 follows from Theorem 3.1 directly by standard arguments, whose details are put in Appendix III. Firstly, we prove a weaker result.

Proposition 5.1 (Deformed weak local law). Suppose Conditions 2.6 and 2.8 as well as (2.1) and (2.9) hold. Then there exists a constant $\tau' > 0$ depending only on $\tau$ such that $\Lambda \preceq (N\eta)^{-1/4} + q$ uniformly for $z \in D^c(\tau, \tau', N)$ with high probability.

For $i \in I$, define $P_i$ as the operator of expectation conditioning on all $(u_1, \ldots, u_N)$ and $(\xi_1, \ldots, \xi_N)$ except $u_i$. Denote $Q_i = 1 - P_i$. Define

$$Z_i := Q_i(x_i^* G(i) x_i) = x_i^* G(i) x_i - \frac{\xi_i^2}{M} \text{Tr}(G(i) \Sigma).$$

We observe from Lemma 4.1 that,

$$\frac{1}{G_{ii}} = -z - z x_i^* G(i) x_i = -z - \frac{\xi_i^2}{M} 2 \text{Tr}(G(i) \Sigma) - z Z_i. \hspace{1cm} (5.1)$$

In the following, we denote

$$U_i = \frac{1}{M} \{\text{Tr}(G \Sigma) - \text{Tr}(G(i) \Sigma)\}, \hspace{1cm} i \in I,$$

$$\mathcal{V} = \frac{1}{M} \{\text{Tr}\{(-zm_N \Sigma - z I)^{-1} \Sigma\} - \text{Tr}(G \Sigma)\}.$$ 

Note that from (5.1) and the definitions of $U_i$ and $\mathcal{V}$, we have

$$\frac{1}{G_{ii}} = -z + z \xi_i^2 U_i + z \xi_i^2 \mathcal{V} - z \frac{\xi_i^2}{M} \text{Tr}\{(-zm_N \Sigma - z I)^{-1} \Sigma\} - z Z_i. \hspace{1cm} (5.2)$$

Before proceeding to prove Proposition 5.1, we provide the following useful lemmas and propositions 5.2 to 5.6, whose proofs are in Appendix II. Recall that $\Xi$ is the event $\{\Lambda \leq (\log N)^{-1}\}$.
Lemma 5.2.
\[ G - (z m_N \Sigma - z I)^{-1} = \sum_{i \in I} \frac{(m_N \Sigma + I)^{-1}}{z(1 + x_i^* G^{(i)} x_i)} (x_i x_i^* G^{(i)} - \frac{1}{N} \Sigma G^*). \]

Lemma 5.3 (Ward identity). Let \( T \subset I \) such that \( 0 \leq |T| \leq C \). Then \( \|G^{(T)}\|_F = \eta^{-1} \operatorname{Im} \text{Tr} G^{(T)} \).

Lemma 5.4. For any \( i \in I \)
\[ |\operatorname{Tr}(G^{(i)} - G)| \leq \eta^{-1}, \]
\[ |\operatorname{Tr}(G^{(i)} - G)| \leq |z|^{-1} + \eta^{-1}, \]
\[ |\operatorname{Im} \operatorname{Tr}(G^{(i)} - G)| \leq \eta|z|^{-2} + \eta^{-1}. \]

Proposition 5.5 (General properties of \( m \)). Fix \( \tau > 0 \). Given (2.1) and (2.9), there exists a constant \( C > 0 \) such that
\[ |m(z)| \geq 1, \quad \operatorname{Im} m(z) \geq C^{-1} \eta, \quad (5.3) \]
for all \( z \in \mathbb{C}^+ \) satisfying \( \tau \leq |z| \leq \tau^{-1} \).

Lemma 5.6. Let \( T \) be an index set such that \( 0 \leq |T| \leq C_1 \) for some constant \( C_1 \geq 0 \) (\( T \) may be empty set). Then
\[ \{1(\Xi) + 1(\eta \geq 1)\}|G^{(T)}_{ij}| + 1(\Xi) \left| \frac{1}{G^{(T)}_{ii}} \right| \leq C, \]
for some constant \( C > 0 \) uniformly for \( i, j \in I \) and \( z \in \mathbb{D} \).

Now we proceed to prove the weak local law. We start with the next lemma which provides a good control for the error when \( \eta \geq 1 \) or \( \Xi \) holds.

Lemma 5.7. Suppose Conditions 2.6, 2.8, (2.1) and (2.9) hold. Then
\[ \{1(\eta \geq 1) + 1(\Xi)\}|Z_i + \Lambda_o| \prec \Psi, \quad (5.4) \]
\[ \{1(\eta \geq 1) + 1(\Xi)\}|\mathcal{V}| + |\mathcal{U}| \prec \Psi, \quad (5.5) \]
uniformly for \( i \in I \) and \( z \in \mathbb{D} \).

Proof. We firstly show (5.4). Applying Lemmas 4.1, 4.3 and (4.3), we obtain that uniformly for \( z \in \mathbb{D} \) and \( i, j \in I \) with \( i \neq j \),
\[ 1(\Xi)|G_{ij}| \leq 1(\Xi)|z||G_{ii} G^{(ij)}_{jj} |x_i^* G^{(ij)} x_j| \prec 1(\Xi)|G_{ii} G^{(ij)}_{jj} |\zeta_i \zeta_j \frac{1}{M} \|\Sigma\| \|G^{(ij)}\|_F. \quad (5.6) \]

Using Lemma 4.1, we obtain that for any \( k \in \mathcal{I}\{i,j\}, \)
\[ G^{(ij)}_{kk} = G^{(i)} - \frac{G_{kk} G^{(ij)}_{jk}}{G^{(jj)}_{jj}} = G^{(i)} - \frac{G_{kk} G_{ii} G_{jk} G_{ij}}{G^{(jj)}_{jj}} = G^{(i)} - \frac{G_{kk} G_{ii} G_{jk} G_{ij}}{G^{(jj)}_{jj}} = G^{(i)} - \frac{G_{kk} G_{ii} G_{jk} G_{ij}}{G^{(jj)}_{jj}}. \]
Then we have from Lemma 5.6 that
\[
1(\Xi)|G^{(ij)}_{kk} - G_{kk}| 
\leq 1(\Xi) \left( \frac{|G_{kk}G_{ik}|}{|G_{ii}|} + \frac{|G_{kj}G_{jk}|}{|G^{(ij)}_{jj}|} + \frac{|G_{ki}G_{ij}G_{jk}|}{|G^{(ij)}_{ij}|} + \frac{|G_{ki}G_{ij}G_{jk}G_{ik}|}{|G^{(ij)}_{ij}G^{(ij)}_{ii}|} \right) 
\leq 1(\Xi) C(\Lambda^2_o + \Lambda^3_o + \Lambda^4_o) \leq 1(\Xi) C\Lambda^2_o, \tag{5.7}
\]
where the last inequality holds because \(\Lambda^3_o + \Lambda^4_o \leq \Lambda^2_o\) for large \(N\) given \(\Xi\). Then it follows from (5.7) and Lemma 5.6 that
\[
1(\Xi) |\text{Im} \, \text{Tr} G^{(ij)}_{ij} - \text{Im} \, \text{Tr} G_{ij}| = 1(\Xi) \left| \sum_{k \in I \setminus \{i,j\}} \text{Im} G^{(ij)}_{kk} - \sum_{k \in I} \text{Im} G_{kk} \right| 
\leq 1(\Xi) \left| \sum_{k \in I \setminus \{i,j\}} (G^{(ij)}_{kk} - G_{kk}) \right| + 1(\Xi) |\text{Im} G_{ii} + \text{Im} G_{jj}| 
\leq 1(\Xi) C N \Lambda^2_o + 1(\Xi) 2 \text{Im} m(z) + 2 \log N. \tag{5.8}
\]
We note that \(\text{Tr} G^{(ij)}_{ij} = \frac{(N - 2 - M)}{z} + \text{Tr} G^{(ij)}_{ij}\). \tag{5.9}

Applying Lemma 5.3 and (5.9), we have
\[
1(\Xi) \|G^{(ij)}\|_F^2 \leq 1(\Xi) \frac{\text{Im} \, \text{Tr} G^{(ij)}_{ij}}{M^2 \eta} = 1(\Xi) \left\{ \frac{\text{Im} \, \text{Tr} G^{(ij)}_{ij}}{M^2 \eta} - \frac{(N - 2 - M)}{M^2 |z|^2} \right\}. \tag{5.10}
\]
It then follows from (5.3), (5.8), (5.10) and \(MN^{-1} \asymp 1\) that
\[
1(\Xi) \frac{1}{M^2} \|G^{(ij)}\|_F^2 \leq 1(\Xi) C \frac{\text{Im} m(z) + \Theta + \Lambda^2_o}{N \eta}. \tag{5.11}
\]
Using Lemma 5.6, (5.6), (5.11) and the fact \(\xi_i \prec 1\) uniformly for all \(i \in I\), we have
\[
1(\Xi) |G_{ij}| \prec 1(\Xi) \left( \frac{\text{Im} m + \Theta + \Lambda^2_o}{N \eta} \right)^{1/2}. \tag{5.12}
\]
Therefore, by the definition of stochastic domination,
\[
1(\Xi) |\Lambda_o| \prec 1(\Xi) \sqrt{\frac{\text{Im} m + \Theta}{N \eta}} + 1(\Xi) \frac{\Lambda_o}{(N \eta)^{1/2}} \Rightarrow 1(\Xi) |\Lambda_o| \prec 1(\Xi) \Psi. \tag{5.13}
\]
Now we evaluate the bound for \(Z_i\). Similarly to (5.11), we can easily derive that uniformly for any \(i \in I\),
\[
1(\Xi) \frac{1}{M^2} \|G^{(i)}\|_F^2 \prec 1(\Xi) \frac{\text{Im} m(z) + \Theta + \Lambda^2_o}{N \eta}. \tag{5.14}
\]
It follows from Lemmas 4.4, 5.3, (5.12) and $\xi_i^2 \asymp 1$ by bounded support assumption for $i \in I$ that

$$1(\Xi)Z_i = 1(\Xi)\{z|x_i^iG^{(i)}x_i - z\frac{\xi_i^2}{M}\Tr(G^{(i)}\Sigma)\} \asymp 1(\Xi)z|\xi_i^2\frac{1}{M}\|G^{(i)}\Sigma\|_F$$

$$\leq 1(\Xi)z|\xi_i^2\frac{\sigma_i^2}{M}\|G^{(i)}\|_F \asymp 1(\Xi)\sqrt{\frac{\Im m + \Theta + \Lambda_i^2}{N\eta}}.$$  

Using the bound $1(\Xi)\Lambda_o \asymp 1(\Xi)\Psi_{\Theta}$, we obtain that

$$1(\Xi)Z_i \asymp 1(\Xi)\left(\sqrt{\frac{\Im m + \Theta}{N\eta}} + \sqrt{\frac{\Im m + \Theta}{N\eta}}\right) \asymp 1(\Xi)\sqrt{\frac{\Im m + \Theta}{N\eta}}.$$

Now we show the result when $\eta \geq 1$. Let $i, j \in I$ such that $i \neq j$. It follows from Lemma 5.6, (5.10) and $\xi_i \asymp 1$ for $i \in I$ that

$$1(\eta \geq 1)|G_{ij}| \leq 1(\eta \geq 1)|G_{ii}G_{jj}^{(i)}||x_i^iG^{(i)}x_j|$$

$$\leq 1(\eta \geq 1)\frac{1}{M}||\Sigma||\|G^{(i)}\|_F$$

$$\leq 1(\eta \geq 1)||\Sigma||\left(\frac{\Im \Tr(G^{(i)})}{M^2\eta}\right)^{1/2}$$

$$= 1(\eta \geq 1)||\Sigma||\left(\frac{\Im \Tr(G^{(i)})}{M^2\eta} - \frac{N - 2 - M}{M^2|z|^2}\right)^{1/2}.$$  

Let $T$ be a subset of $I$ such that $|T| \leq C$ for all large $N$. From Lemma 5.4, we know that

$$|\Tr G^{(T)} - \Tr G| \leq C\eta^{-1}. \quad (5.13)$$

It then follows from Proposition 5.5 and (5.13) that

$$1(\eta \geq 1)\frac{\Im \Tr G^{(T)}}{M^2\eta} = 1(\eta \geq 1)\left(\frac{\Im \Tr G^{(T)}}{M^2\eta} - \frac{N - |T| - M}{M^2|z|^2}\right)$$

$$\leq 1(\eta \geq 1)\left(\frac{\Im \Tr G^{(T)}}{M^2\eta} + \frac{C\eta^{-1}}{M^2\eta} - \frac{N - |T| - M}{M^2|z|^2}\right) \asymp 1(\eta \geq 1)\frac{\Im m + \Theta}{N\eta}.$$  

Consequently

$$1(\eta \geq 1)\Lambda_o = 1(\eta \geq 1)\max_{i,j \in I, i \neq j} |G_{ij}| \asymp 1(\eta \geq 1)\sqrt{\frac{\Im m + \Theta}{N\eta}}.$$  

For $Z_i$, using Lemma 4.4, (5.14) and $\xi_i^2 \asymp 1$ for $i \in I$, we have

$$1(\eta \geq 1)Z_i = 1(\eta \geq 1)\{z|x_i^iG^{(i)}x_i - z\frac{\xi_i^2}{M}\Tr(G^{(i)}\Sigma)\} \asymp 1(\eta \geq 1)z|\xi_i^2\frac{1}{M}\|G^{(i)}\Sigma\|_F$$

$$\leq 1(\eta \geq 1)z|\xi_i^2\|\Sigma\|\frac{1}{M}\|G^{(i)}\|_F \asymp 1(\eta \geq 1)\sqrt{\frac{\Im m + \Theta}{N\eta}}.$$
Hence (5.4) follows. Next, we will show (5.5). Under \( \Xi \), applying Lemmas 4.1, 4.3, 4.4, 5.3, (5.12) and \( \xi_i^2 \prec 1 \), we get, for any \( i \in \mathcal{I} \),

\[
|U_i| = \frac{1}{M} \left| \text{Tr}(G^{(i)} - G) \right| \leq \frac{1}{M} \left| \frac{x_i^* G^{(i)} \Sigma G^{(i)} x_i}{1 + x_i^* G^{(i)} x_i} \right|
\]

\[
= \frac{1}{M} \left| z G_{ii} x_i^* G^{(i)} \Sigma G^{(i)} x_i \right| \leq \frac{1}{M} \left| z G_{ii} \right| \left( \frac{1}{M} \left| \text{Tr}(G^{(i)} \Sigma G^{(i)} \Sigma) \right| + \frac{1}{M} \|G^{(i)} \Sigma G^{(i)} \Sigma\|_F \right)
\]

\[
\leq \frac{2}{M^2} |z G_{ii}| \|G^{(i)} \Sigma\|_F^2 \leq \frac{2}{M^2} |z G_{ii}| \|G^{(i)} \Sigma\|_F^2 \lesssim \Psi_\Theta.
\]

Similarly, under \( \Xi \),

\[
1(\Xi)|V| = 1(\Xi) \left| \frac{1}{M} \left( \text{Tr}(z m_N \Sigma - z I)^{-1} \Sigma - \text{Tr} G \Sigma \right) \right|
\]

\[
= 1(\Xi) \left| \frac{1}{M} \text{Tr} \left( \sum_{i \in \mathcal{I}} \frac{(m_N \Sigma + I)^{-1}}{z(1 + x_i^* G^{(i)} x_i)} (x_i x_i^* G^{(i)} \Sigma - \frac{1}{N} \Sigma G \Sigma + \frac{1}{N} \Sigma G^{(i)} \Sigma - \frac{1}{N} \Sigma G^{(i)} \Sigma) \right) \right|
\]

\[
\leq 1(\Xi) \left( \frac{1}{M} \sum_{i \in \mathcal{I}} \frac{\xi_i^2}{M} \|\Sigma\| \|(m_N \Sigma + I)^{-1} \Sigma G^{(i)} \|_F + \frac{q}{\sqrt{N}} + \Psi_\Theta^2 \right)
\]

\[
\leq 1(\Xi) \left( \frac{1}{M} \sum_{i \in \mathcal{I}} \frac{1}{M} \|\Sigma G^{(i)}\|_F + \frac{q}{\sqrt{N}} + \Psi_\Theta^2 \right).
\]

Since by assumption (2.9)

\[
|m \sigma_i + 1| \geq \tau,
\]

we have

\[
1(\Xi)|1 + m_N \sigma| \geq 1(\Xi)(|1 + m \sigma_i| - |m - m_N \sigma_i|) \geq \tau' > 0.
\]

Combining (5.15) we have

\[
1(\Xi)|V| < \frac{1}{M} \sum_{i \in \mathcal{I}} \frac{1}{M} \|\Sigma G^{(i)}\|_F + \frac{q}{\sqrt{N}} + \Psi_\Theta^2 < \Psi_\Theta.
\]

(5.16)

For \( \eta \geq 1 \), the procedure is similar and we omit the details. Then the lemma holds. \( \square \)

**Remark 5.8.** In the following proof we will use two relations several times,

\[
\{1(\eta \geq 1) + 1(\Xi)\} \frac{1}{M} \|G^{(i)}\|_F < \Psi_\Theta, \quad \{1(\eta \geq 1) + 1(\Xi)\} \frac{1}{M} \|G\|_F < \Psi_\Theta,
\]

(5.17)

so we summarize them here.

With the above results, we can further prove the next lemma.

**Lemma 5.9.** Under the assumptions in Lemma 5.7, one has

\[
\{1(\eta \geq 1) + 1(\Xi)\} |G_{ii} - G_{jj}| \prec \Psi_\Theta + q.
\]

(5.18)

uniformly for \( i, j \in \mathcal{I} \) and \( z \in \mathcal{D} \).
Proof. We observe from \((5.1)\) that
\[
|G_{ii} - G_{jj}| = |G_{ii}G_{jj}\left(\frac{1}{G_{jj}} - \frac{1}{G_{ii}}\right)|
\]
\[
\leq |G_{ii}G_{jj}| |Z_i - Z_j| + |G_{ii}G_{jj}z\left(\frac{\xi_i^2}{M} \text{Tr}(G^{(i)} \Sigma) - \frac{\xi_j^2}{M} \text{Tr}(G^{(j)} \Sigma)\right) |
\]
\[
\leq |G_{ii}G_{jj}| |Z_i - Z_j| + |G_{ii}G_{jj}||z\left(\frac{\xi_i^2}{M} \text{Tr}(G^{(i)} \Sigma) + \frac{\xi_j^2}{M} \text{Tr}(G^{(j)} \Sigma)\right) |
\]
\[
< |Z_i - Z_j| + \frac{\xi_i^2}{M} \text{Tr}(|G^{(i)} \Sigma - G^{(j)} \Sigma|) + \frac{\xi_j^2}{M} \text{Tr}(|G^{(j)} \Sigma|)
\]
\[
< |\Psi_\Theta + q + \Psi_\Theta^2|,
\]
where we used Condition \(2.8\) and the fact that under \(\Xi\), \(|g^{(i)}_{kk}| \asymp \sigma_k\). \(\square\)

Now we can complete the proof of Proposition \(5.1\).

Proof of Proposition \(5.1\). We observe from \((5.18)\) that
\[
\{1(\Xi) + 1(\eta \geq 1)\} \left\{\frac{1}{N} \sum_{i \in I} \frac{1}{G_{ii}} - \frac{1}{m_N}\right\}
\]
\[
= \{1(\Xi) + 1(\eta \geq 1)\} \frac{1}{N} \sum_{i \in I} \left(- \frac{G_{ii} - m_N}{m_N^2} + \frac{(G_{ii} - m_N)^2}{G_{ii}m_N^2}\right)
\]
\[
= \{1(\Xi) + 1(\eta \geq 1)\} \frac{1}{N} \sum_{i \in I} \frac{(G_{ii} - m_N)^2}{G_{ii}m_N^2}
\]
\[
< |\Psi_\Theta^2| + q^2.
\]
It then follows from \((5.2)\), Condition \(2.8\) and \((5.20)\) that
\[
\{1(\Xi) + 1(\eta \geq 1)\} \frac{1}{m_N} = \{1(\Xi) + 1(\eta \geq 1)\} \frac{1}{N} \sum_{i \in I} \frac{1}{G_{ii}} + O_\prec(\Psi_\Theta^2) + O_\prec(q^2)
\]
\[
= \{1(\Xi) + 1(\eta \geq 1)\} \left(- 1 + \frac{1}{N} \sum_{i \in I} \xi_i^2 U_i + \frac{1}{N} \nu \sum_{i \in I} \xi_i^2\right)
\]
\[
+ \frac{1}{M} \sum_{i \in I} \xi_i^2 \frac{1}{N} \text{Tr}\{(m_N \Sigma + I)^{-1} \Sigma\} + O_\prec(\Psi_\Theta^2) + O_\prec(q^2)
\]
\[
= \{1(\Xi) + 1(\eta \geq 1)\} \left(- z + \frac{1}{N} \text{Tr}\{(m_N \Sigma + I)^{-1} \Sigma\} + O_\prec(\Psi_\Theta) + O_\prec(q^2)\right).
\]
Since
\[
\text{Tr}\{(m_N \Sigma + I)^{-1} \Sigma\} = \sum_{i \in I} \frac{\sigma_i}{m_N \sigma_i + 1},
\]
it follows from the definition of \(f(x)\) in \((2.8)\) that
\[
\{1(\Xi) + 1(\eta \geq 1)\} \{f(m_N) - z\} < \Psi_\Theta + q^2.
\]
Applying Proposition 4.6, for any $\varepsilon > 0$ we have
\[
1(\eta \geq 1)|m_N - m| < \frac{\Psi_\Theta + q^2}{\sqrt{\kappa + \eta} + \sqrt{N^2(\Psi_\Theta + q^2)}} < \sqrt{\Psi_\Theta + q^2}. \tag{5.23}
\]
Therefore, it follows from (5.18), (5.23) and Lemma 5.7 that
\[
1(\eta \geq 1)\Lambda(z) \leq 1(\eta \geq 1)\{\max_i |G_{ii} - m_N| + |m_N - m| + \Lambda_0\} \prec N^{-1/2} + q. \tag{5.24}
\]
The rest proof of Proposition 5.1 follows from a standard bootstrapping step which we summarize into Appendix II-vi and omit further details here.

Now we can prove Theorem 3.1. Note that for $i \in I$,
\[
Q_i \frac{1}{G_{ii}} = Q_i\{-z - z\frac{\epsilon^2}{M} \text{Tr}(G^{(i)}\Sigma) - zZ_i\} = -zZ_i, \tag{5.25}
\]
and we write
\[
\mathcal{V} = \frac{1}{M} \text{Tr}\left(\sum_{i \in I} \frac{(m_N \Sigma + I)^{-1}}{z(1 + x_i^*G^{(i)}x_i)}(x_i x_i^*G^{(i)}\Sigma - \frac{1}{N} \Sigma G^{(i)}\Sigma + \frac{1}{N} \Sigma G^{(i)}\Sigma - \frac{1}{N} \Sigma G\Sigma)\right)
\]
\[
= \frac{1}{M} \sum_{i \in I} G_{ii} \text{Tr}(\mathcal{V}_i) + \frac{1}{M} \sum_{i \in I} G_{ii} \frac{1}{N} \text{Tr}((m_N \Sigma + I)^{-1} \Sigma(G^{(i)} - G)\Sigma), \tag{5.26}
\]
where
\[
\mathcal{V}_i := (m_N \Sigma + I)^{-1}(x_i x_i^*G^{(i)}\Sigma - \frac{1}{N} \Sigma G^{(i)}\Sigma). \tag{5.27}
\]
For the second term in (5.26),
\[
\left|\frac{1}{M} \sum_{i \in I} G_{ii} \frac{1}{N} \text{Tr}((m_N \Sigma + I)^{-1} \Sigma(G^{(i)} - G)\Sigma)\right|
\]
\[
= \left|\frac{1}{M} \sum_{i \in I} G_{ii} \frac{1}{N} \text{Tr}((m_N \Sigma + I)^{-1} \Sigma \frac{G^{(i)}x_i x_i^*G^{(i)}\Sigma}{1 + x_i^*G^{(i)}x_i})\right| \tag{5.28}
\]
\[
\leq \frac{1}{M} \sum_{i \in I} |G_{ii}|^2 \frac{1}{N} |x_i^*G^{(i)}(m_N \Sigma + I)^{-1}G^{(i)}x_i|,
\]
which can be bounded by $\Psi_\Theta^2$ by Lemma 4.3, Lemma 4.4, Lemma 5.6 and (5.17).

Furthermore, using the same methods one can easily verify that
\[
\frac{1}{M} \sum_{i \in I} G_{ii} \text{Tr}(\mathcal{V}_i) = \frac{1}{M} \sum_{i \in I} G_{ii} \text{Tr}(\mathcal{V}_i^{(i)}) + \frac{1}{M} \sum_{i \in I} G_{ii} \text{Tr}(\mathcal{V}_i - \mathcal{V}_i^{(i)})
\]
\[
\prec \frac{1}{M} \sum_{i \in I} G_{ii} \text{Tr}(\mathcal{V}_i^{(i)}) + \Psi_\Theta^2, \tag{5.29}
\]
where $\mathcal{V}_i^{(i)} := (m_N \Sigma + I)^{-1}(x_i x_i^*G^{(i)}\Sigma - \frac{1}{N} \Sigma G^{(i)}\Sigma)$.

From Proposition 5.1, we know that $\Xi$ is true with high probability, i.e. $1 \prec 1(\Xi)$. So from now on, we can drop the factor $1(\Xi)$ in all $\Xi$ dependent results without affecting their validity. To improve the deformed weak local law to the strong local law, a key input is Proposition 5.10 below whose proof we postpone to Appendix II-vii.
Proposition 5.10 (Fluctuation averaging). Let \( \nu \in [1/4, 1] \) and \( \tau' \) be defined in Proposition 5.1. Denote \( \Phi_\nu = \sqrt{\text{Im} m + (N\eta)^{-\nu + q} \over N \eta} \). Suppose moreover that \( \Theta \prec (N\eta)^{-\nu} \) uniformly for \( z \in \mathbb{D}^e(\tau, \tau', N) \). Then we have

\[
\frac{1}{N} \sum_{i \in I} Q_i \frac{1}{G_{ii}} \prec \Phi_\nu^2,
\]

and

\[
\frac{1}{N} \sum_{i \in I} Q_i \gamma_i' \prec \Phi_\nu^2,
\]

uniformly for \( z \in \mathbb{D}^e(\tau, \tau', N) \), where \( \gamma_i' \) is defined as

\[
\gamma_i' := x^*_i G^{(i)} \Sigma(m^{(i)} + I)^{-1} x_i.
\]

Proof of Theorem 3.1. Let \( \varepsilon > 0 \) be an arbitrary small number. Suppose \( \Theta \leq N^\varepsilon(q^{1/2} + (N\eta)^{-\nu}) \) holds with high probability for some \( \nu \in [1/4, 1] \) uniformly for \( z \in \mathbb{D}^e \). The idea is to update \( \nu \) by applying Proposition 5.10 iteratively.

Let \( \Phi_\nu \) be defined in Proposition 5.10. Given that \( \Theta \leq N^\varepsilon(q^{1/2} + (N\eta)^{-\nu}) \) holds with high probability, it follows from (5.25), (5.28), Proposition 5.10 and (5.21) that

\[
|f(m_N) - z| \leq N^\varepsilon \{ \Phi_\nu^2 + q^2 \} \leq N^\varepsilon \{ q^2 + {1 \over (N\eta)^{\nu + 1}} + \text{Im} m \over N \eta \},
\]

holds with high probability uniformly for \( z \in \mathbb{D}^e \).

Then we observe from Lemma 4.5 and Proposition 4.6 that

\[
\Theta \leq N^\varepsilon \left( {q^2 + {1 \over (N\eta)^{\nu + 1}} + \text{Im} m \over N \eta} \right)^{1/2} + \sqrt{\text{Im} m \over N \eta (\kappa + \eta)} + q \leq CN^\varepsilon \left( {\text{Im} m \over N \eta \sqrt{\kappa + \eta}} + \sqrt{q^2 + {1 \over (N\eta)^{\nu + 1}} + \text{Im} m \over N \eta} \right)
\]

holds with high probability uniformly for \( z \in \mathbb{D}^e \). Then using Lemma 5.7 and Lemma 5.9, it is easy to check

\[
\Lambda \leq CN^\varepsilon (\Theta + q) + \Theta \leq CN^\varepsilon \left( \sqrt{\text{Im} m \over N \eta} + q + {1 \over (N\eta)^{(\nu + 1)/2}} \right).
\]

One can see that after the self-improving arguments, the error bound of \( \Lambda \) improves from \( 1/(N\eta)^{\nu} \) to \( 1/(N\eta)^{(\nu + 1)/2} \). Hence implementing the arguments a finite number (depending only on \( \varepsilon \)) of times, we obtain that

\[
\Lambda \leq CN^\varepsilon (q + {1 \over N \eta} + \sqrt{\text{Im} m \over N \eta})
\]

holds with high probability uniformly for \( z \in \mathbb{D}^e \). Applying (5.34) in Lemma 4.5, Proposition 5.10 and Proposition 4.6, we conclude Theorem 3.1. \( \square \)
6. Proof of the edge universality with small support

Once the following Green function comparison Theorem 6.1 holds, Theorem 3.3 will follow from a standard procedure. We only prove Theorem 6.1 in this section while the complete proof of Theorem 3.3 is put in Appendix III.

**Theorem 6.1 (Green function comparison on the edge).** Let $X^V$ and $X^W$ be defined in Theorem 3.3. Let $F: \mathbb{R} \to \mathbb{R}$ be a function whose derivatives satisfy

$$\sup_{x \in \mathbb{R}} |F^{(k)}(x)|(1 + |x|)^{-C_1} \leq C_1, \quad k = 1, 2, 3, 4,$$

(6.1)

with some constants $C_1 > 0$. Then there exist $\varepsilon_0 > 0$, $N_0 \in \mathbb{Z}^+$ depending on $C_1$ such that for any $\varepsilon < \varepsilon_0$ and $N \geq N_0$ and for any real numbers $E, E_1$ and $E_2$ satisfying

$$|E - \lambda_+|, |E_1 - \lambda_+|, |E_2 - \lambda_+| \leq N^{-2/3+\varepsilon}$$

and $\eta = N^{-2/3-\varepsilon}$, we have

$$|\mathbb{E}(N\eta \Im m_N^V(z)) - \mathbb{E}(N\eta \Im m_N^W(z))| \leq C N^{-1/6+C_1}, \quad z = E + i\eta,$$

(6.2)

and

$$|\mathbb{E} F \left( \int_{E_1}^{E_2} N \Im m_N^V(y + i\eta)dy \right) - \mathbb{E} F \left( \int_{E_1}^{E_2} N \Im m_N^W(y + i\eta)dy \right) | \leq C N^{-1/6+C_1},$$

(6.3)

where $m_N^V(z) = N^{-1}\text{Tr}((X^V)^*X^V - zI)^{-1}$, $C_1$ is a constant which tends to 0 as $\varepsilon \to 0$.

**Proof.** Let $\gamma \in \{1, \ldots, N + 1\}$ and set $X_\gamma$ to be the matrix whose first $\gamma - 1$ columns are the same as those of $X^W$ and the remaining $N - \gamma + 1$ columns are the same as those of $X^V$. Then we note that since $X_\gamma$ and $X_{\gamma + 1}$ only differ in the $\gamma$-th column,

$$X_\gamma^{(\gamma)} = X_{\gamma+1}^{(\gamma)}.$$ 

We define $m_{N,\gamma}(z)$ and $m_{N,\gamma+1}(z)$ to be the analogs of $m_N(z)$ with the matrix $X$ replaced by $X_\gamma$ and $X_{\gamma+1}$ respectively. Similarly, for $i \in \mathcal{I}$, define $m_{N,i}^{(i)}(z)$ and $m_{N,\gamma+1}^{(i)}(z)$ to be the analogs of $m_N^{(i)}(z)$ with the matrix $X^{(i)}$ replaced by $X_\gamma^{(i)}$ and $X_{\gamma+1}^{(i)}$ respectively. Then we have

$$\mathbb{E}^V F(N\eta \Im m_N^V(z)) - \mathbb{E}^W F(N\eta \Im m_N^W(z)) = \sum_{\gamma=1}^{N} \left\{ \mathbb{E}(N\eta \Im m_{N,\gamma}(z)) - \mathbb{E}(N\eta \Im m_{N,\gamma+1}(z)) \right\}.$$ 

So (6.2) follows from Lemma 6.1 below. (6.3) follows from an analogous argument. Hence we omit its proof. \qed

**Lemma 6.1.** Let $F$ be a function satisfying (6.1) and $z = E + i\eta$. If $|E - \lambda_+| \leq N^{-2/3+\varepsilon}$ and $N^{-2/3-\varepsilon} \leq \eta \leq N^{-2/3}$ for some $\varepsilon > 0$, there exists some positive constant $C$ independent of $\varepsilon$ such that

$$|\mathbb{E}(N\eta \Im m_{N,\gamma}(z)) - \mathbb{E}(N\eta \Im m_{N,\gamma+1}(z))| \leq C N^{-7/6+C_1},$$

(6.4)

uniformly for $\gamma \in \{1, \ldots, N + 1\}$. 

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Proof. Recall the relationship between the eigenvalues of $G$ and $G$, 

$$m_N = \frac{1}{N} \text{Tr} G = \frac{1}{N} \text{Tr} G - \frac{1 - \phi}{z},$$

with 

$$| \frac{1}{N} \text{Tr} G - \frac{1}{N} \text{Tr} G^{(\gamma)} | = \frac{zG_{\gamma}}{N} x_{\gamma}^*(G^{(\gamma)})^2 x_{\gamma}.$$

Here we can assume $|1 - \phi|$ to be 1 after introducing a multiplicative constant. Then 

$$\mathbb{E} f(N \eta \text{Im } m_{N,\gamma}(z)) = \mathbb{E} f(N \eta \text{Im } m_{N,\gamma}^{(\gamma)}(z) - \frac{1}{Nz} + \frac{zG_{\gamma}}{N} x_{\gamma}^*(G^{(\gamma)})^2 x_{\gamma}).$$

Denoting 

$$y^V = \eta z G_{\gamma} x_{\gamma}^*(G^{(\gamma)})^2 x_{\gamma},$$

by the Taylor expansion we obtain that 

$$\mathbb{E} f(N \eta \text{Im } m_{N,\gamma}(z)) = \mathbb{E} f(N \eta \text{Im } m_{N,\gamma}^{(\gamma)}(z) - \frac{\eta^2}{|z|^2}) + \text{O}((N^{-1/3 + C_1}.$$ 

Then the left-hand side of (6.4) reads 

$$| \mathbb{E} f(N \eta \text{Im } m_{N,\gamma}(z)) - \mathbb{E} f(N \eta \text{Im } m_{N,\gamma+1}(z)) |$$

$$= \frac{1}{k!} \mathbb{E} \left[ (N \eta \text{Im } m_{N,\gamma}^{(\gamma)}(z)) - \frac{\eta^2}{|z|^2}) (\text{Im } y^V)^k - \frac{1}{k!} (N \eta \text{Im } m_{N,\gamma+1}^{(\gamma)}(z) - \frac{\eta^2}{|z|^2}) (\text{Im } y^W)^k + \text{O}((N^{-1/3 + C_1}.$$ 

One can check that by Theorem 3.1 and Lemma 4.5 as well as the choice of $\eta$,

$$N \eta \text{Im } m_{N,\gamma}^{(\gamma)}(z) - \frac{\eta^2}{|z|^2} = N \eta \text{Im } m(z) + \text{O}(1) \prec 1.$$

Then for $k = 1, 2, 3, 4$, there exists $c > 0$ such that with high probability 

$$F^{(k)}(N \eta \text{Im } m_{N,\gamma}^{(\gamma)}(z) - \frac{\eta^2}{|z|^2}) \leq N^c.$$ 

(6.8)
Now the proof of (6.4) is reduced to showing
\[ |(y^V)^k - (y^W)^k| \prec N^{-7/6+C_*}, \]
for \( k = 1, 2, 3 \). Let
\[ B := \frac{(m - G_{\gamma \gamma})^2}{m^2 G_{\gamma \gamma}} = \frac{1}{G_{\gamma \gamma}} + \frac{G_{\gamma \gamma} - 2m}{m^2}, \]
so by Lemma 5.6 and Theorem 3.1 we have
\[ |B| \prec \frac{1}{(N\eta)^2} \leq N^{-2/3+\epsilon}. \]

On the other hand, we may write
\[ G_{\gamma \gamma} = \frac{m^2/(2m - G_{\gamma \gamma})}{m^2/(2m - G_{\gamma \gamma})B + 1} = \frac{m^2}{2m - G_{\gamma \gamma}} \sum_{k \geq 0} \left( \frac{-m^2}{2m - G_{\gamma \gamma}} B \right)^k. \]
Consequently, we obtain
\[ y = \sum_{k \geq 0} \eta^2 \frac{m^2}{2m - G_{\gamma \gamma}} \left( \frac{-m^2}{2m - G_{\gamma \gamma}} B \right)^k x^{\gamma \gamma}_y (G^{(1)})^2 x_\gamma = \sum_{k \geq 0} y_k, \]
where
\[ y_k := \eta^2 \frac{m^2}{2m - G_{\gamma \gamma}} \left( \frac{-m^2}{2m - G_{\gamma \gamma}} B \right)^k x^{\gamma \gamma}_y (G^{(1)})^2 x_\gamma. \]
Then one can check that
\[ |y_k| \prec N^{-2/3}N^{-2k/3+\epsilon}N^{1/3+2\epsilon} \leq N^{-1/3-2k/3+C_*}. \]
Therefore it suffices to prove
\[ \left| \sum_{k=1}^3 \left( \sum_{j \geq 0} y^{(V)}_j k - \sum_{k \geq 0} y^{(W)}_j k \right) \right| \prec N^{-7/6+C_*}. \]
We note that for \( j \geq 2 \), \( |y_j| \) is sufficiently small, hence it suffices to consider \( y_0, y_1 \) for the following three cases. Now let \( E_\gamma \) be the conditional expectation with respect to \( \xi^V_\gamma \) and \( \xi^W_\gamma \).

a) \( k = 1 \).
It suffices to bound
\[ |\text{Im}(y^V_0 + y^V_1) - \text{Im}(y^W_0 + y^W_1)|. \]
We observe that
\[ E_\gamma(y^V_0 - y^W_0) = 0; \]
\[ |E_\gamma(y^V_1 - y^W_1)| = \eta^2 \frac{1}{C^2} E_\gamma \left( \left( \frac{1}{G_{\gamma \gamma}} + C \right)x^{\gamma \gamma}_y (G^{(1)})^2 x_\gamma - \left( \frac{1}{G_{\gamma \gamma}} + C \right)x^{\gamma \gamma}_W (G^{(1)})^2 x^W_\gamma \right) \]
\[ = \eta^2 \frac{2}{C^2} E_\gamma \left( \left( \xi^V_\gamma \right)^4 - \left( \xi^W_\gamma \right)^4 \right)(\Sigma^{1/2} u^* \gamma^\gamma u, \Sigma u^* \gamma^\gamma (G^{(1)})^2 u, \Sigma^{1/2}) \]
\[ \prec \eta^2 \frac{2}{C^2} N^{-10/12+C_*} \Sigma^{1/2} u^* \gamma^\gamma u, \Sigma u^* \gamma^\gamma (G^{(1)})^2 u, \Sigma^{1/2} \]
\[ \prec N^{-7/6+C_*}, \]
where we have used Condition 2.8 and the fact that second moments of $\xi^V$, $\xi^W$ match.

b) $k = 2$.
In this case we only need to consider
$$|(\text{Im } y_0^V)^2 - (\text{Im } y_0^W)^2|.$$ Similarly, we observe that
$$\left| E \gamma \left( (\text{Im } y_0^V)^2 - (\text{Im } y_0^W)^2 \right) \right| = \eta^2 z^2 C e \gamma \left( (\xi^V)^2 - (\xi^W)^2 \right) \left( \Sigma^{1/2} u^*_\gamma (G(\gamma))^2 u_\gamma \Sigma^{1/2} \right) \ll N^{-3/2+C_*}.$$ (6.18)

c) $k = 3$.
In this case we need to bound
$$|(\text{Im } y_0^V)^3 - (\text{Im } y_0^W)^3|.$$ We observe that
$$\left| E \gamma \left( (\text{Im } y_0^V)^3 - (\text{Im } y_0^W)^3 \right) \right| = \eta^3 z^3 C e \gamma \left( (\xi^V)^6 - (\xi^W)^6 \right) \left( \Sigma^{1/2} u^*_\gamma (G(\gamma))^2 u_\gamma \Sigma^{1/2} \right) \ll N^{-11/6+C_*}.$$ (6.19)

Finally, combining all the results, we see that (6.4) holds. Thus we complete the proof of Lemma 6.1.  

7. Proof of Theorem 3.4 and Theorem 3.5

7.1. Proof of Theorem 3.4

We need the next lemma to prove Theorem 3.4.

**Lemma 7.1.** Suppose $\xi_i$’s satisfy the assumptions in Theorem 3.4. Then there exists one matrix $\tilde{X} = (\tilde{x}_{ij})$, such that the elements $\tilde{\xi}_i$’s satisfy Condition 2.8 with $q = O(N^{-1/2} \log N)$, and the first four moments of $\xi_i$ and $\tilde{\xi}_i$ match for all $i$, that is
$$E \xi_i^k = \tilde{E} \xi_i^k, \quad k = 1, 2, 3, 4.$$ (7.1)

The proof of this lemma can be found in [32]. We note that $\tilde{X}$ satisfies the conditions of Theorem 3.4. Now we process to prove Theorem 3.4.

**Proof.** Note that from Theorem 3.1, $\tilde{X}$ satisfies (3.10). We use the Green function comparison idea to show that (3.10) also holds for $X$. Since we have the trivial bound
$$\max_{ij} G_{ij} \leq C \eta^{-1} \leq N,$$
for any $X$ with $q \leq N^{-c}$. Then by Lemma 4.2, it suffices to show that
\[ E|m_N - m|^p \prec (N\eta)^{-p}, \tag{7.2} \]
for $X$ with $q \leq N^{-c}$.

Firstly, recall the relationship (2.3). For simplification, we denote $m_M := N^{-1} \text{Tr} \mathcal{G}$ and $m := m + (1 - \phi)z^{-1}$, so
\[ m_M = m_N + (1 - \phi)z^{-1}. \]

Then it is equivalent to showing that
\[ E|m_M - m|^p \prec (N\eta)^{-p}. \tag{7.3} \]

For $\gamma = 0, \cdots, N$, let $X_\gamma$ be the matrix whose first $\gamma$ columns are the same as those of $X$ and the remaining $N - \gamma$ columns are the same as those of $\bar{X}$ with entries $\xi_j u_{ij}$, where $\xi_j$'s satisfy the assumptions in Lemma 7.1. Then $X_0 = \bar{X}$ and $X_N = X$. Denote $\mathcal{G}_\gamma, \mathcal{G}_\gamma^*$ as the Green functions of $X_\gamma^* X_\gamma$ and $X_\gamma^* X_\gamma^*$ respectively, and $m_{M,\gamma} = N^{-1} \text{Tr} \mathcal{G}_\gamma, \mathcal{G}_\gamma^{(\gamma)}$ and $m_{M,\gamma}^{(\gamma)}$ are defined similarly with $X_\gamma^{(\gamma)}$.

The resolvent expansion gives
\[ \mathcal{G}_\gamma = \mathcal{G}^{(\gamma)}_\gamma - \mathcal{G}_\gamma x_\gamma x_\gamma^* \mathcal{G}_\gamma^* \gamma. \]

Consequently, by $m_{M,\gamma} = \frac{1}{N} \text{Tr} \mathcal{G}_\gamma$, we may write
\[ m_{M,\gamma}^{(\gamma)} - m_{M,\gamma}^{(\gamma)} = -\frac{1}{N} \xi_\gamma^2 r_\gamma^* \mathcal{G}^{(\gamma)}_\gamma \mathcal{G}_\gamma \gamma. \tag{7.4} \]

Similarly,
\[ m_{M,\gamma-1}^{(\gamma)} - m_{M,\gamma-1}^{(\gamma)} = -\frac{1}{N} \xi_\gamma^2 r_\gamma^* \mathcal{G}^{(\gamma)}_\gamma \mathcal{G}_{\gamma-1} \gamma. \]

We note that $|m_M - m|^p = (m_M - m)^p/2 (m_M^{(\gamma)} - m^{(\gamma)})^{p/2}$ for any even integer $p > 0$. In the following of this proof, we slightly abuse the notation by ignoring the conjugate $*$ in $m_M$ and $m$ for simplicity. We shall see that this will not affect the validity of our result.

When $\gamma = 0$, from Theorem 3.1 and the assumptions on $\bar{X}$, it is clear that
\[ |m_{M,0} - m| \prec (N\eta)^{-1}, \quad E|m_{M,0} - m|^p \prec (N\eta)^{-p}. \tag{7.5} \]

The target is to show that $|E(m_{M,N} - m_{M,N})|^p \prec (N\eta)^p$. Actually, in the proof below, we use the deterministic form of the bound in (7.3), that is, we choose $\epsilon > 0$ such that $|E(m_{M,N} - m_{M,N})|^p \leq (N\eta)^p N^\epsilon$.

Note that $\mathcal{G}^{(\gamma)}_\gamma = \mathcal{G}^{(\gamma)}_{\gamma-1}$, and
\[ r_\gamma^* \mathcal{G}^{(\gamma)}_\gamma \gamma r_\gamma = \frac{r_\gamma^* \mathcal{G}^{(\gamma)}_\gamma \mathcal{G}^{(\gamma)}_\gamma^* r_\gamma}{1 + \xi_\gamma^2 r_\gamma^* \mathcal{G}^{(\gamma)}_\gamma \gamma r_\gamma}. \]
It’s not hard to see that the local law also holds for $G_\gamma$, then by large deviations bounds
\begin{align*}
|r_\gamma^* G_\gamma(\gamma) r_\gamma| & \leq \sigma_1^2 |u_\gamma^* G_\gamma(\gamma) u_\gamma| < \left| \frac{1}{N} \text{Tr}(G_\gamma(\gamma)) \right| + \frac{1}{M} \text{Tr}(G_\gamma(\gamma))^2 \leq C, \\
\left| \frac{1}{N} r_\gamma^* G_\gamma(\gamma) G_\gamma(\gamma) r_\gamma \right| & \leq \sigma_1^2 \left| u_\gamma^* G_\gamma(\gamma) G_\gamma(\gamma) u_\gamma \right| < \frac{1}{N} \left( \left| \frac{1}{M} \text{Tr}(G_\gamma(\gamma) G_\gamma(\gamma)) \right| + \frac{1}{M} \text{Tr}(G_\gamma(\gamma) G_\gamma(\gamma))^2 \right) \\
& \leq \frac{1}{N} M \| G_\gamma(\gamma) \|_F^2 \times \frac{1}{N \eta},
\end{align*}
where we use
\begin{align*}
\frac{1}{N} \frac{1}{M} \| G_\gamma(\gamma) \|_F^2 = \frac{1}{N} M \left( \frac{\text{Im} \text{Tr} G_\gamma(\gamma)}{M^2 \eta} \right) = \frac{1}{N} M \left( \frac{N \text{Im} m + N \Theta}{M^2 \eta} - \frac{N - M}{M^2 |z|^2} \right) \times \frac{q + \sqrt{\eta}}{N \eta} \leq \frac{1}{N \eta}.
\end{align*}
Using Taylor’s expansion
\begin{align*}
\frac{1}{1 + \xi_\gamma^2 r_\gamma^* G_\gamma(\gamma) r_\gamma} = \sum_{k \geq 0} \left( \frac{1}{1 + \phi \xi_\gamma^2 G_\gamma(\gamma) r_\gamma} \right)^{k+1} \left( - (\xi_\gamma^2 - \phi) r_\gamma^* G_\gamma(\gamma) r_\gamma \right)^k \frac{1}{k!},
\end{align*}
and the fact that $|1 + \phi \xi_\gamma^2 G_\gamma(\gamma) r_\gamma|^{-1} \leq C$, $\xi_\gamma^2 = \xi_\gamma^2 - \phi + \phi$, the RHS of (7.4) can be written as
\begin{align*}
\sum_{k \geq 0} (\xi_\gamma^2 - \phi)^k A_{k,\gamma},
\end{align*}
where $A_{k,\gamma}$ is independent of $\xi_\gamma$ and for any $k \geq 0$,
\begin{align*}
|\mathbb{E}(A_{k,\gamma})| \leq (N \eta)^{-p} N^e.
\end{align*}
Similarly, we can write
\begin{align*}
m_{M,\gamma-1} - m_{M,\gamma-1}^{(\gamma)} = \sum_{k \geq 0} (\xi_\gamma^2 - \phi)^k A_{k,\gamma-1}, \quad \text{while} \quad A_{k,\gamma-1} = A_{k,\gamma}. \quad (7.6)
\end{align*}
Hence, we can write
\begin{align*}
\mathbb{E}(m_{M,\gamma} - m)^p
= & \mathbb{E}(m_{M,\gamma}^{(\gamma)} - m)^p + \mathbb{E} \sum_{k=1}^p \binom{p}{k} (m_{M,\gamma}^{(\gamma)} - m)^{p-k} \left( \sum_{k_1 \geq 0} (\xi_\gamma^2 - \phi)^{k_1} A_{k_1,\gamma} \right)^k, \quad (7.7)
\end{align*}
and similarly,
\begin{align*}
\mathbb{E}(m_{M,\gamma-1} - m)^p
= & \mathbb{E}(m_{M,\gamma-1}^{(\gamma)} - m)^p + \mathbb{E} \sum_{k=1}^p \binom{p}{k} (m_{M,\gamma-1}^{(\gamma)} - m)^{p-k} \left( \sum_{k_1 \geq 0} (\xi_\gamma^2 - \phi)^{k_1} A_{k_1,\gamma} \right)^k. \quad (7.8)
\end{align*}
We claim the fact that $m_{M,\gamma-1}^{(\gamma)} = m_{M,\gamma}^{(\gamma)}$, $m_{M,\gamma}^{(\gamma)}$ is independent of $\xi_\gamma$ and $\xi_\gamma$, $\mathbb{E}(\xi_\gamma^2 - \phi)^k = \mathbb{E}(\xi_\gamma^2 - \phi)^k$ for $k \leq 2$, and for any $k \geq 3$,
\begin{align*}
\mathbb{E}(\xi_\gamma^2 - \phi)^k \leq N^{-1} \log N q^{k-2}, \quad \mathbb{E}(\xi_\gamma^2 - \phi)^k \leq N^{-1} \log N q^{k-2}.
\end{align*}
Then, comparing (7.7) and (7.8), we infer from the Cauchy-Schwartz inequality that
\[ |\mathbb{E}(m_{M,\gamma} - \bar{m})^p| \leq |\mathbb{E}(m_{M,\gamma-1} - \bar{m})^p| + \sum_{k=1}^{p} \binom{p}{k} \left( \frac{N\eta}{1+\epsilon}\right)^{-k} \mathbb{E}(m_{M,\gamma-1} - \bar{m})^{2(p-k)-1/2}. \tag{7.9} \]

Moreover, we know that
\[ |\mathbb{E}(m_{M,\gamma-1} - \bar{m})^{2(p-k)}| = |\mathbb{E}(m_{M,\gamma-1} - m_{M,\gamma-1} + m_{M,\gamma-1} - \bar{m})^{2(p-k)}| \]
\[ = \sum_{l=0}^{2(p-k)} \binom{2(p-k)}{l} |\mathbb{E}((m_{M,\gamma-1} - m_{M,\gamma-1})^l(m_{M,\gamma-1} - \bar{m}))^{2(p-k)-l}| \]
\[ \leq \sum_{l=0}^{2(p-k)} \binom{2(p-k)}{l} \left( |\mathbb{E}(m_{M,\gamma-1} - m_{M,\gamma-1})^{2l}| \right)^{1/2} \left( |\mathbb{E}(m_{M,\gamma-1} - \bar{m})^{4(p-k)-2l}| \right)^{1/2} \tag{7.10} \]
\[ \leq C_P N^{c_p \epsilon} \sum_{l=0}^{2(p-k)} (N\eta)^{-l} \left( |\mathbb{E}(m_{M,\gamma-1} - \bar{m})^{4(p-k)-2l}| \right)^{1/2} \]
for some constants \(C_p\) and \(c_p\), where the last inequality is by (7.6).

We then use (7.9) and (7.10) to complete the induction. For \(\gamma = 0\), we already know that
\[ |\mathbb{E}(m_{M,0} - \bar{m})^p| \leq (N\eta)^{-p} N^\epsilon, \quad |\mathbb{E}(m_{M,0}^{(1)} - \bar{m})^p| \leq (N\eta)^{-p} N^\epsilon. \]

Then by (7.9) it’s easy to see that for \(\gamma = 1\),
\[ |\mathbb{E}(m_{M,1} - \bar{m})^p| \leq \left( 1 + \frac{1}{N^{1+\epsilon/2}} \right) (N\eta)^{-p} N^\epsilon. \]

Now assume that for some \(\gamma \geq 1\), there exists constant \(a > 0\) such that \(|\mathbb{E}(m_{M,\gamma-1} - \bar{m})^p| \leq (1 + N^{-1-c/2})^a (N\eta)^{-p} N^\epsilon\) for any fixed \(p\). By (7.10),
\[ |\mathbb{E}(m_{M,\gamma-1}^{(1)} - \bar{m})^{2(p-k)}| \leq C_p \left( 1 + \frac{1}{N^{1+c/2}} \right)^a (N\eta)^{-2(p-k)} N^{c_p \epsilon} \tag{7.11} \]
for some constants \(C_p\) and \(c_p\). Note that \(\epsilon\) is arbitrary small, so plug (7.11) into (7.9) to obtain
\[ |\mathbb{E}(m_{M,\gamma} - \bar{m})^p| \leq \left( 1 + \frac{1}{N^{1+c/2}} \right)^{a+1} (N\eta)^{-p} N^\epsilon. \]

Then by induction,
\[ |\mathbb{E}(m_{M,N} - \bar{m})^p| \leq \left( 1 + \frac{1}{N^{1+c/2}} \right)^N (N\eta)^{-p} N^\epsilon \leq (N\eta)^{-p} N^{2\epsilon}, \]
for arbitrary small \(\epsilon > 0\).

A similar but more complicated procedure can lead to
\[ |\mathbb{E}(m_{M,N} - \bar{m})^p| \leq (N\eta)^{-p} N^\epsilon, \]
and the theorem follows from Chebyshev’s inequality. The other conclusions in Theorem 3.4 can be obtained by the standard procedure used in the proof of Theorem 3.2. So we omit details.
7.2. Proof of Theorem 3.5

We note that $\tilde{X}$ in Lemma 7.1 satisfies the desired edge universality according to Theorem 3.3. Thus if we can prove the following lemma, then Theorem 3.5 follows immediately.

**Lemma 7.2.** Let $X$ and $\tilde{X}$ be two matrices in Lemma 7.1. Then there exist constants $\epsilon, \delta > 0$ such that, for any $s \in \mathbb{R}$

\[
\mathbb{P}^{\tilde{X}}(N^{2/3}(\lambda_1 - \lambda_+) \leq s - N^{-\epsilon} - N^{-\delta}) \leq \mathbb{P}^X(N^{2/3}(\lambda_1 - \lambda_+) \leq s) \leq \mathbb{P}^{\tilde{X}}(N^{2/3}(\lambda_1 - \lambda_+) \leq s + N^{-\epsilon} + N^{-\delta})
\]

where $\mathbb{P}^X$ and $\mathbb{P}^{\tilde{X}}$ are the laws of $X$ and $\tilde{X}$, respectively.

Most of the proof of Lemma 7.2 is the same as the one of Theorem 3.3. We only write down the Green function comparison part, which is slightly different from before but simpler since we have the first four moments matching at this time.

**Theorem 7.1.** Let $X$ and $\tilde{X}$ be two matrices in Lemma 7.1. Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ is a function whose derivatives satisfy

\[
\sup_{x \in \mathbb{R}} |F^{(l)}(x)|(1 + |x|)^{-C_2} \leq C_2, \quad l = 1, 2, 3
\]

with some constant $C_2 > 0$. Then for any sufficiently small constant $\epsilon > 0$ and for any real numbers $E, E_1$ and $E_2$ satisfying

\[
|E - \lambda_+|, |E_1 - \lambda_+|, |E_2 - \lambda_+| \leq N^{-2/3+\epsilon}
\]

and $\eta = N^{-2/3+\epsilon}$, we have

\[
|\mathbb{E}F(N\eta \text{Im} m_N(z)) - \mathbb{E}F(N\eta \text{Im} \tilde{m}_N(z))| \leq N^{-c_1+C_\epsilon}, \quad z = E + \eta,
\]

and

\[
\left| \mathbb{E} \left( \int_{E_1}^{E_2} \text{Im} m_N(y + \eta)dy \right) - \mathbb{E} \left( \int_{E_1}^{E_2} \text{Im} \tilde{m}_N(y + \eta)dy \right) \right| \leq N^{-c_1+C_\epsilon},
\]

where $c_1$ is a positive constant and $C_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$.

**Proof.** We only prove the first inequality and the second one follows from similar arguments. The beginning part is the same as before. We split

\[
\mathbb{E}F(N\eta \text{Im} m_N(z)) - \mathbb{E}F(N\eta \text{Im} \tilde{m}_N(z)) = \sum_{\gamma=1}^N \left\{ \mathbb{E}F(N\eta \text{Im} m_{N,\gamma}(z)) - \mathbb{E}F(N\eta \text{Im} m_{N,\gamma-1}(z)) \right\}.
\]

We will prove that

\[
|\mathbb{E}F(N\eta \text{Im} m_{N,\gamma}(z)) - \mathbb{E}F(N\eta \text{Im} m_{N,\gamma-1}(z))| \prec N^{-1-\epsilon+C_\epsilon}.
\]

\[(7.15)\]
Use the resolvent expansion that $G_\gamma = G_\gamma^{(\gamma)} - G_\gamma x_\gamma x_\gamma^{*} G_\gamma^{(\gamma)}$ and the relationship $m_{M, \gamma} = m_N + (1 - \phi)z^{-1}$, we have

$$N \eta \text{Im } m_{N, \gamma}(z) = N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } \frac{\eta x_\gamma^{*} G_\gamma^{(\gamma)} x_\gamma}{1 + x_\gamma^{*} G_\gamma^{(\gamma)} x_\gamma}.$$  

We further expand the last term (ignoring Im) of the last identity to

$$\eta(z^2 - \phi + \phi) r_\gamma^{*} G_\gamma^{(\gamma)} G_\gamma^{(\gamma)} r_\gamma \sum_{k \geq 0} \left( \frac{1}{1 + \phi r_\gamma^{*} G_\gamma^{(\gamma)} r_\gamma} \right)^{k+1} \left( \frac{-(\xi_\gamma^2 - \phi) r_\gamma^{*} G_\gamma^{(\gamma)} r_\gamma}{k!} \right)^k$$  

$$= \frac{\eta \phi r_\gamma^{*} G_\gamma^{(\gamma)} G_\gamma^{(\gamma)} r_\gamma}{1 + \phi r_\gamma^{*} G_\gamma^{(\gamma)} r_\gamma^2} + \frac{\eta (\xi_\gamma^2 - \phi) r_\gamma^{*} G_\gamma^{(\gamma)} G_\gamma^{(\gamma)} r_\gamma}{1 + \phi r_\gamma^{*} G_\gamma^{(\gamma)} r_\gamma^2} + R_\gamma$$  

$$= A_\gamma + B_\gamma + C_\gamma + R_\gamma,$$

where $A_\gamma$ is independent of $\xi_\gamma$, $E B_\gamma = E C_\gamma = 0$. We have already known that

$$\eta r_\gamma^{*} G_\gamma^{(\gamma)} G_\gamma^{(\gamma)} r_\gamma < q, \quad r_\gamma^{*} G_\gamma^{(\gamma)} G_\gamma^{(\gamma)} r_\gamma \leq C, \quad |1 + \phi r_\gamma^{*} G_\gamma^{(\gamma)} r_\gamma|^{-1} \leq C,$$

$$E(\xi_\gamma^2 - \phi)^{2k} \leq N^{-1} \log N, k \geq 1.$$  

Hence, $|E R_i| \leq q \times N^{-1} \log N \times N^\epsilon \leq N^{-1-c+2\epsilon}$, $i = 1, 2$. Then

$$F(N \eta \text{Im } m_{N, \gamma}(z)) - F\left(N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } A_\gamma \right)$$  

$$= - F^{(1)}\left(N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } A_\gamma \right) \times \text{Im } (B_\gamma + C_\gamma + R_\gamma)$$  

$$+ \frac{1}{2} F^{(2)}(\psi) \text{Im}^2(B_\gamma + C_\gamma + R_\gamma),$$

where $\psi$ is some number between $N \eta m_{N, \gamma}(z)$ and $N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } A_\gamma$. By the local law and large deviation bounds, $A_\gamma \sim q + \sqrt{\eta} \to 0$. Furthermore, we observe that from Theorem 3.4,

$$N \eta \text{Im } m_{N, \gamma}(z) \prec N \eta (\text{Im } m(z) + (N \eta)^{-1}) \leq C,$$

which implies

$$\left| F^{(1)}\left(N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } A_\gamma \right) \right| \prec 1, \quad |F^{(2)}(\psi)| \prec 1.$$  

Therefore,

$$\left| E \left(F(N \eta \text{Im } m_{N, \gamma}(z)) - F(N \eta \text{Im } m_{M, \gamma}^{(\gamma)}(z) + (1 - \phi) N^{-1/3 - \epsilon} - \text{Im } A_\gamma) \right) \right|$$  

$$\leq C(E |R_\gamma| + E |R_\gamma|^2 + E |B_\gamma|^2 + E |C_\gamma|^2)$$  

$$\leq q \times N^{-1} \log N \times N^\epsilon \leq N^{-1-c+2\epsilon}$$.  

Similarly, we can prove that
\[
\left| \mathbb{E} \left( F(N \eta \text{Im} m_{N, \gamma - 1}(z)) - F(N \eta \text{Im} m^{(\gamma)}_{M, \gamma - 1}(z) + (1 - \phi)N^{-1/3 - \epsilon} - \text{Im} A_{\gamma - 1}) \right) \right| \leq \frac{1}{N^{1+\epsilon - 2\epsilon}}.
\]
Note that \(m^{(\gamma)}_{M, \gamma - 1}(z) = m^{(\gamma)}_{M, \gamma}(z)\) and \(A_{\gamma - 1} = A_{\gamma}\), which conclude (7.15) and the Theorem holds.

8. Proof of Theorem 3.6

Suppose the matrix \(X\) satisfies Condition 2.6 and Condition 2.7. We can write the sample covariance matrix as
\[
W = XX^* = \sum_{i=1}^{N} \xi_i^2 r_i r_i^*, \tag{8.1}
\]
where \(r_i = \Sigma^{1/2} u_i\).

For any fixed \(\epsilon > 0\), define
\[
\alpha_N : = \mathbb{P}(|\hat{\xi}_i^2 - M| > N^{1-\epsilon}). \tag{8.2}
\]

Using Condition 2.7, we can see that for any \(\delta > 0\) and large enough \(N\),
\[
\alpha_N \leq \delta N^{-1+2\epsilon}. \tag{8.3}
\]

Let \(\rho(x)\) be the distribution of \(\xi_i^2\). Then we define independent random variables \(\zeta_i^s, \zeta_i^l\) and \(c_i, 1 \leq i \leq N\) in the following ways:

1. \(\zeta_i^s\) has distribution density \(\rho_s(x)\), where
   \[
   \rho_s(x) := 1(\left| x - \phi \right| \leq N^{-\epsilon}) \frac{\rho(x)}{1 - \alpha_N}; \tag{8.4}
   \]

2. \(\zeta_i^l\) has distribution density \(\rho_l(x)\), where
   \[
   \rho_l(x) := 1(\left| x - \phi \right| > N^{-\epsilon}) \frac{\rho(x)}{\alpha_N}; \tag{8.5}
   \]

3. \(c_i\) is a Bernoulli \(0 - 1\) random variable with \(\mathbb{P}(c_i = 1) = \alpha_N\) and \(\mathbb{P}(c_i = 0) = 1 - \alpha_N\).

It is easy to check
\[
\xi_i^2 \overset{d}{=} \zeta_i^s (1 - c_i) + \zeta_i^l c_i, \tag{8.6}
\]
therefore we may write
\[
W = \sum_{i=1}^{N} \xi_i^2 r_i r_i^* = \sum_{i=1}^{N} \left( \zeta_i^s (1 - c_i) + \zeta_i^l c_i \right) r_i r_i^* \tag{8.7}
\]
We observe that
\[
\mathbb{E} |\zeta_i^s - \phi|^2 = O(N^{-1} \log N), \tag{8.8}
\]
so $\zeta^*_i$ satisfies the assumptions in Theorem 3.5. We conclude that for the matrix

\[ \tilde{W} := \sum_{i=1}^{N} \zeta^*_i r_i r_i^*, \]

there exist constants $\epsilon, \delta > 0$ such that for any $s \in \mathbb{R},$

\[ \mathbb{P}^G(N^{2/3}(\lambda_1 - \lambda_+) \leq s - N^{-\epsilon}) - N^{-\delta} \leq \mathbb{P}^{\tilde{W}}(N^{2/3}(\lambda_1 - \lambda_+) \leq s) \]
\[ \leq \mathbb{P}^G(N^{2/3}(\lambda_1 - \lambda_+) \leq s + N^{-\epsilon}) + N^{-\delta}, \quad (8.9) \]

where $\mathbb{P}^G$ denotes the law for a Gaussian covariance matrix and $\mathbb{P}^{\tilde{W}}$ denotes the law for $\tilde{W}$.

Now we write the right-hand side of (8.7) as

\[ \mathcal{W} = \sum_{i=1}^{N} (\zeta^*_i + (\zeta^*_i - \zeta^{\dagger}_i)c_i)r_i r_i^* := \sum_{i=1}^{N} (\zeta^*_i + R_i c_i)r_i r_i^*, \]

where $R_i := \zeta^*_i - \zeta^{\dagger}_i$. We aim to show that the $R_i c_i$ terms have negligible effects on $\lambda_1$. Define the corresponding matrix as

\[ R^c := \sum_{i=1}^{N} R_i c_i r_i r_i^*. \]

Note that $c_i$ is independent of $\zeta^*_i$ and $\zeta^{\dagger}_i$. In order to understand the spectral behavior of this matrix, we first introduce the following event

\[ A := \{ \sharp\{i : c_i = 1\} \leq N^{5\epsilon}\}. \]

Since $c_i$’s are independent and identically distributed Bernoulli random variables, by Bernstein’s inequality it is easy to check

\[ \mathbb{P}(A) \geq 1 - \exp(-N^\epsilon). \quad (8.10) \]

Without loss of generality, we will assume that $c_i = 0$ for $i > N^{5\epsilon}$ and $c_i = 1$ for $i \leq N^{5\epsilon}$. On the other hand, by Condition 2.7, we have

\[ \mathbb{P}(|R_i| \geq \omega) \leq \mathbb{P}(|\tilde{\xi}_i^2 - \tilde{M}| \geq \omega) = \mathbb{P}(|\hat{\xi}_i^2 - M| \geq \omega N) = o(N^{-2}), \quad (8.11) \]

for any fixed constant $\omega > 0$. Hence, the event

\[ A \cap \{ \max_i |R_i| \leq \omega \} \]

happens with probability approaching to 1. Hereafter, we will focus on this event. Define

\[ \mathcal{W}_t(\lambda) = \lambda I - \left( \tilde{W} + t \sum_{i=1}^{N^{5\epsilon}} R_i r_i r_i^* \right), \quad t \in [0, 1]. \]
In fact, by taking $\omega$ sufficiently small, the eigenvalues of $\tilde{\mathcal{W}} + t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^*$ are continuous in $t$. Next, we aim to prove that for $\lambda = \mu := \lambda_1(\mathcal{W}) \pm N^{-3/4}$,\[ \mathbb{P}\left( \det(W_t(\mu)) \neq 0, \forall t \in [0,1] \right) = 1 - o(1). \quad (8.12)\]

If (8.12) holds, by continuity we know that the largest eigenvalue of $\tilde{\mathcal{W}} + t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^*$ will not cross the boundary $\lambda_1(\tilde{\mathcal{W}}) \pm N^{-3/4}$. Hence $\lambda_1(\mathcal{W})$ is sticking to $\lambda_1(\tilde{\mathcal{W}})$ with a rate smaller than $N^{-3/4}$, which concludes the theorem.

Now we prove (8.12). We know that the eigenvalues of GOE are separated at the scale of $N^{-2/3}$, so by (8.9),\[ \mathbb{P}(|\lambda_k(\tilde{\mathcal{W}}) - \mu| \geq N^{-3/4}) = 1 - o(1). \]

Therefore, $\mu$ is not an eigenvalue of $\mathcal{W}$, and\[ \det(W_t(\mu)) = \det(\mu - \tilde{\mathcal{W}}) \det \left( 1 - t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^* G^*(\mu) \right), \]

where $G^*(z) = (\tilde{\mathcal{W}} - z)^{-1}$ is the Green function. Hereafter, we ignore the superscript $s$ in $G^*(z)$ for simplicity. Let $z = \lambda_+ + i N^{-1+\delta}$ for some $\delta > 0$. Then\[ 1 - t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^* G(\mu) = 1 - t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^* (G(\mu) - G(z)) - t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^* G(z). \quad (8.13)\]

Note that for each $i$,$$ r_i r_i^* G(z) = r_i r_i^* G^{(i)}(z) + \frac{\zeta_i^* r_i r_i^* G^{(i)}(z) r_i r_i^* G^{(i)}(z)}{1 + \zeta_i^* r_i r_i^* G^{(i)}(z) r_i} ,$$

while$$ \left| r_i^* G^{(i)}(z) r_i - \frac{1}{M} m \text{Tr} \Sigma \right| \leq \left| r_i^* G^{(i)}(z) r_i - \frac{1}{M} \text{Tr} G^{(i)}(z) \Sigma \right| + \left| \frac{1}{M} \text{Tr}(G^{(i)}(z) - m) \Sigma \right| \prec N^{-1/6 + \varepsilon}. $$

Therefore, we can replace $r_i^* G^{(i)}(z) r_i$ with $M^{-1} m \text{Tr} \Sigma$ and write$$ \left\| t \sum_{i=1}^{N^{5\varepsilon}} R_i r_i r_i^* G(z) \right\| \leq \max |R_i| \left\| \sum_{i=1}^{N^{5\varepsilon}} r_i r_i^* \right\| \| G(z) \| \leq C \omega \left\| \sum_i r_i r_i^* \right\| \leq \frac{1}{10}, \quad (8.14) $$

where we have used the fact that $w$ can be sufficiently small and $\| G(z) \| \leq C$ by Theorem 3.2. Then, it remains to consider the second sum in (8.13).

Let $\beta_\alpha$ be the eigenvector of $\mathcal{W}$ corresponding to the $\alpha$-th eigenvalue $\lambda_\alpha$. Note that for any $\lambda_\alpha > \tau$ and $z^* = \lambda_\alpha + i N^{-1+\delta}$, we have $|r_i^* G(z^*) r_i| \leq C$ with high probability. Moreover,\[ |\text{Im} r_i^* G(z^*) r_i| = (\text{Im} z^*) \sum_{j} \frac{< r_i, \beta_j >^2}{|\lambda_j - z^*|^2} \geq \frac{|\text{Im} z^*|}{|\lambda_\alpha - z^*|^2} < r_i, \beta_\alpha >^2 = (\text{Im} z^*)^{-1} < r_i, \beta_\alpha >^2. \]
Therefore, we have $<r_i, \beta_\alpha>^2 < N^{-1}$ for any $\alpha$ satisfying $\lambda_\alpha > \tau$. Let $\alpha^*$ be the largest $\alpha$ satisfying this condition, so by the eigenvalue rigidity we have $\alpha^* \approx kN$ for some constant $k$. The eigenvalue rigidity also implies that $\lambda_\alpha - \lambda_\alpha^* \approx (\alpha/N)^{2/3}$ for any $\alpha \geq N^\epsilon$.

Recall $z = \lambda_\alpha + \imath N^{-2/3}$, so for each $i$,

$$
\left| r_i^* (G(\mu) - G(z)) r_i \right| = \sum_{\alpha} <r_i, \beta_\alpha>^2 \frac{1}{|\mu - \lambda_\alpha|} \left( \frac{1}{|z - \lambda_\alpha|} \right)
$$

$$
\leq \sum_{\alpha} <r_i, \beta_\alpha>^2 \left( \frac{\eta}{(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2} + \frac{(1 + o(1))\eta^2}{|\lambda_\alpha - \mu||(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2|} \right).
$$

Firstly,

$$
\sum_{\alpha \leq N^\epsilon} <r_i, \beta_\alpha>^2 \left( \frac{\eta}{(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2} + \frac{(1 + o(1))\eta^2}{|\lambda_\alpha - \mu||(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2|} \right)
< N^{\epsilon} N^{-1}(N^{2/3+\epsilon} + N^{3/4+\epsilon}) \leq N^{-1/4+2\epsilon}.
$$

Secondly,

$$
\sum_{N^{t} \leq \alpha \leq N^{\epsilon}} <r_i, \beta_\alpha>^2 \left( \frac{\eta}{(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2} + \frac{(1 + o(1))\eta^2}{|\lambda_\alpha - \mu||(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2|} \right)
< \sum_{N^{t} \leq \alpha \leq N^{\epsilon}} N^{-5/3+\epsilon} \left( \frac{\alpha}{N} \right)^{-4/3} \leq N^{-1/3+\epsilon} \sum_{1 \leq \alpha \leq kN} \alpha^{-4/3} \leq N^{-1/3+2\epsilon}.
$$

Lastly,

$$
\sum_{\alpha > \alpha^*} <r_i, \beta_\alpha>^2 \left( \frac{\eta}{(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2} + \frac{(1 + o(1))\eta^2}{|\lambda_\alpha - \mu||(\lambda_\alpha - \lambda_\alpha^*)^2 + \eta^2|} \right)
\leq \frac{\eta}{(\lambda_\alpha - \tau)^2} \sum_{\alpha} <r_i, \beta_\alpha>^2 \leq \frac{\eta}{(\lambda_\alpha - \tau)^2} \|r_i\|^2 \leq N^{-2/3+\epsilon}.
$$

Therefore, we have

$$
\left\| \sum_{i=1}^{N^{\epsilon}} R_i r_i^*(G(\mu) - G(z)) \right\| < N^{-1/4+7\epsilon}. \quad (8.15)
$$

Combining (8.13), (8.14) and (8.15), with probability approaching to 1 we have

$$
\det \left( 1 - t \sum_{i=1}^{N^{\epsilon}} R_i r_i^* G(z) \right) \neq 0, \forall t \in [0, 1],
$$

which concludes the theorem.
In the following appendices, we provide the proofs of some lemmas and results which are omitted in the main text.

Appendix I: Proof of results in Section 4.

i. Proof of Lemma 4.2

Proof. If $E X_N \prec \Phi_N$, then for any $\varepsilon > 0$ we get from Markov’s inequality that
\[
\mathbb{P}(|X_N| > N^\varepsilon \Phi) \leq \frac{E|X_N|^p}{N^{\varepsilon p} \Phi^p} \leq \frac{1}{N^{\varepsilon (p-1)}}.
\]
Choosing $p$ large enough (depending on $\varepsilon$) proves the “$\Leftarrow$” part. Conversely, if $X_N \prec \Phi_N$, then for any $D > 0$ we get
\[
|EX_N| \leq E|X_N|1(|X_N| \leq N^\varepsilon \Phi_N) + E|X_N|1(|X_N| > N^\varepsilon \Phi_N) \\
\leq N^\varepsilon \Phi_N + \sqrt{E|X_N|^2} \sqrt{\mathbb{P}(|X_N| > N^\varepsilon \Phi_N)} \leq N^\varepsilon \Phi_N + N^{C_2/2-D/2}.
\]
Using $\Phi_N \geq N^{-C}$ and choosing $D$ large enough, we obtain the “$\Rightarrow$” part for $p = 1$. The same implication for arbitrary $p$ follows from the fact that $X_N \prec \Phi_N$ implies $X_N^p \prec \Phi_N^p$ for any fixed $p$.

ii. Proof of large deviation bounds in Lemma 4.4

Proof. Let $\mathcal{F}_k = \sigma\{u_1, \ldots, u_k\}$ be the $\sigma$-algebra generated by $u_1, \ldots, u_k$. In particular, $\mathcal{F}_0$ is the trivial $\sigma$-algebra, i.e., $\mathbb{E}(\cdot|\mathcal{F}_0)$ is the unconditional expectation. For $k = 1, \ldots, M - 1$, we see by symmetry that conditioned on $\mathcal{F}_k$, $(u_{k+1}, \ldots, u_M)'$ follows the uniform distribution on the $(M - k)$-dimensional sphere with radius $\sqrt{1 - \sum_{i=1}^{M-k} u_i^2}$, namely,
\[
(u_{k+1}, \ldots, u_M)'|\mathcal{F}_k \sim U\left((1 - \sum_{i=1}^{k} u_i^2)^{1/2}\mathbb{S}^{M-k}\right).
\] (I.1)

Define the martingale difference sequence
\[
s_k = \sum_{i=1}^{M} b_i\{E(u_i|\mathcal{F}_k) - E(u_i|\mathcal{F}_{k-1})\}.
\]
A direct observation from (I.1) is that $s_k = b_k u_k$. Then it follows from Theorem V.1 in Appendix V and the Burkholder inequality [11] that for any positive integer $q$, there exists a constant $C_q > 0$
such that
\[
E|b^*u|^{2q} = E\left|\sum_{k=1}^{M} s_k\right|^{2q} \leq C_q E\left(\sum_{k=1}^{M} |s_k|^2\right)^q \leq C_q E\left(\sum_{k=1}^{M} |b_k|^2 u_k^2\right)^q
\]
\[
= C_q \left\{ \sum_{1 \leq k_1, \ldots, k_q \leq M} |b_{k_1}|^2 \cdots |b_{k_q}|^2 E(u_{k_1}^2 \cdots u_{k_q}^2) \right\}
\]
\[
\leq \frac{C_q}{M^q} \sum_{1 \leq k_1, \ldots, k_q \leq M} |b_{k_1}|^2 \cdots |b_{k_q}|^2 = C_q \left(\sum_{k=1}^{M} |b_k|^2\right)^q = C_q \left(\frac{\|b\|^2}{M}\right)^q.
\]

Then (4.1) follows from that for any \(q \in \mathbb{Z}_+\),
\[
P(|b^*u| > M^q \sqrt{\frac{\|b\|^2}{M}}) \leq \frac{E|b^*u|^{2q}}{M^{2eq} \left(\frac{\|b\|^2}{M}\right)^q} \leq C_q \frac{M^{2e^{-q}}}{M^{2eq}}.
\]

To show (4.2), we first show that
\[
\left|\sum_{k=1}^{M} a_{kk} (u_k^2 - \frac{1}{M})\right| \prec \frac{1}{M} \sqrt{\sum_{k=1}^{M} |a_{kk}|^2}.
\]

We construct the martingale difference sequence as
\[
N_k := \sum_{i=1}^{M} a_{i} \left( E(u_i^2 | \mathcal{F}_k) - E(u_i^2 | \mathcal{F}_{k-1}) \right).
\]

Note that \(\mathcal{F}_M = \mathcal{F}_{M-1}\), and
\[
E(u_i^2 | \mathcal{F}_k) = \frac{1}{M-k} (1 - \sum_{l=1}^{k} u_l^2), \forall i \geq k + 1.
\]

Therefore,
\[
N_k = a_{kk} u_k^2 + \sum_{i=k+1}^{M} a_{ii} E(u_i^2 | \mathcal{F}_k) - \sum_{i=k}^{M} a_{ii} E(u_i^2 | \mathcal{F}_{k-1})
\]
\[
= \left( a_{kk} - \sum_{i=k+1}^{M} \frac{a_{ii}}{M-k} \right) u_k^2 - \frac{1}{M-k+1} (1 - \sum_{l=1}^{k-1} u_l^2)
\]
\[
= \left( a_{kk} - \sum_{i=k+1}^{M} \frac{a_{ii}}{M-k} \right) u_k^2 - M^{-1} - \frac{1}{M-k+1} \sum_{l=k}^{M} (u_l^2 - M^{-1})
\]
\[
= \left( a_{kk} - \sum_{i=k+1}^{M} \frac{a_{ii}}{M-k} \right) u_k^2 - M^{-1} + \frac{1}{M-k+1} \sum_{l=1}^{k-1} (u_l^2 - M^{-1})
\]
\[
:= A_k \nu_k.
\]
Use the Burkholder inequality to obtain that
\[
E|\sum_{k=1}^{M} a_{kk}(u_k^2 - \frac{1}{M})^{2q}| = E|\sum_{k=1}^{M} a_k^2|^{2q} \leq C_q E\left(\sum_{k} a_k^2\right)^q \\
\leq C_q \sum_{1 \leq k_1, \ldots, k_q \leq M} A_{k_1}^2 \cdots A_{k_q}^2 \mathbb{E}\left(\nu_{k_1}^2 \cdots \nu_{k_q}^2\right) \\
\leq C_q \frac{M^{2q}}{M^{2q}} \left(\sum_{k} a_k^2\right)^q,
\]
where the third line is by Theorem V.1. So (1.2) holds.

In order to complete the proof of (4.2), we will prove that
\[
\left|\sum_{j<k}^{M} a_{jk} u_j u_k\right| < \frac{1}{M} \sqrt{\sum_{j \neq k} |a_{jk}|^2}. \tag{1.3}
\]

We begin with the martingale difference sequence
\[
\mathcal{Y}_l := \sum_{j<k} a_{jk} \left(\mathbb{E}(u_j u_k | \mathcal{F}_l) - \mathbb{E}(u_j u_k | \mathcal{F}_{l-1})\right).
\]

Note that
\[
\sum_{j<k} \mathbb{E}(u_j u_k | \mathcal{F}_l) = \sum_{j<k \leq l} \mathbb{E}(u_j u_k | \mathcal{F}_l) + \sum_{j<k} \mathbb{E}(u_j u_k | \mathcal{F}_l) + \sum_{l<j<k} \mathbb{E}(u_j u_k | \mathcal{F}_l) \\
= \sum_{j<k \leq l} \mathbb{E}(u_j u_k | \mathcal{F}_l) = \sum_{j<k \leq l} u_j u_k.
\]

Then,
\[
\mathcal{Y}_l = \sum_{j<l} a_{jl} u_j u_l.
\]

By the Burkholder inequality,
\[
E\left|\sum_{j<k}^{M} a_{jk} u_j u_k\right|^{2q} \leq C_q E\left(\sum_{l} \mathcal{Y}_l^2\right)^q \leq C_q E\left(\sum_{l} u_l^2 (\sum_{j<l} a_{jl} u_j)^2\right)^q \\
\leq C_q \sum_{k} \sum_{i<k \leq j<k} \sum_{k} \sum_{i<k \leq j<k} \sum_{k} a_{i_1 k_1} a_{j_1 k_1} \cdots a_{i_q k_q} a_{j_q k_q} \mathbb{E}\left(u_{k_1}^2 u_{i_1}^2 u_{j_1}^2 \cdots u_{k_q}^2 u_{i_q}^2 u_{j_q}^2\right) \\
\leq C_q \frac{M^{3q}}{M^{2q}} \left(\sum_{k} (\sum_{j<k} a_{jk})^2\right)^q \leq C_q \frac{M^{2q}}{M^{2q}} \left(\sum_{j \neq k} a_{jk}^2\right)^q.
\]
which concludes (I.3). Therefore, (4.2) follows from
\[ |u^* Au - \frac{1}{M} \text{tr} A| = \left| \sum_{k=1}^{M} a_{kk} \left( u_k^2 - \frac{1}{M} \right) + \sum_{j \neq k} a_{jk} u_j u_k \right| \]
\[ < \frac{1}{\sqrt{2M}} \sqrt{\sum_{k=1}^{M} |a_{kk}|^2 + \frac{1}{\sqrt{2M}} \sum_{j \neq k} |a_{jk}|^2} \]
\[ \leq \frac{1}{M} \sqrt{\sum_{k=1}^{M} |a_{kk}|^2 + \sum_{j \neq k} |a_{jk}|^2} \]
\[ = \frac{1}{M} \| A \|_F. \]

Denote the $i$th column of $A$ as $A_{\cdot i}$. Finally (4.3) follows from (4.1) conditioned on $u$,
\[ |u^* A \hat{u}| \prec \sqrt{\frac{\| u^* A \|^2}{M}}, \]
and
\[ \| u^* A \|^2 = \sum_{i=1}^{M} |u^* A_{\cdot i}|^2 \prec \sum_{i=1}^{M} \| A_{\cdot i} \|^2 = \frac{1}{M} \sum_{i,j} a_{ij}^2 = \frac{1}{M} \| A \|_F^2, \]
which concludes the lemma.

\[ \square \]

Appendix II: Proof of the results in Section 5.

i. Proof of Lemma 5.2

Proof. We recall that
\[ G = \left( \sum_{i \in I} x_i x_i^* - zI \right)^{-1}. \]

It follows from the resolvent identity that
\[ G - (-zm_N \Sigma - zI)^{-1} = (-zm_N \Sigma - zI)^{-1} \left( -zm_N \Sigma - \sum_{i \in I} x_i x_i^* \right) G. \]

Using the Sherman-Morrison formula (see, e.g., (2.2) of [40] or Lemma V.1 in Appendix V, we have
\[ x_i x_i^* G = \frac{x_i x_i^* G^{(i)}}{1 + x_i^* G^{(i)} x_i}. \]
Using (II.1), (II.2) and (5.1), we have

\[
G - (-zm \Sigma - zI)^{-1} = \frac{1}{N} \sum_{i \in \mathcal{I}} \frac{(-zm \Sigma - zI)^{-1} \Sigma G}{1 + x_i^* G^{(i)} x_i} - \sum_{i \in \mathcal{I}} \frac{(-zm \Sigma - zI)^{-1} x_i x_i^* G^{(i)}}{1 + x_i^* G^{(i)} x_i} = \sum_{i \in \mathcal{I}} \frac{(m \Sigma + I)^{-1}}{z(1 + x_i^* G^{(i)} x_i)} (x_i x_i^* G^{(i)} - \frac{1}{N} \Sigma G),
\]

which concludes the lemma.

\[\square\]

**ii. Proof of Lemma 5.3**

**Proof.** Let \(\tilde{\lambda}_k\) and \(\tilde{\mathbf{v}}_k\) be the \(k\)-th largest eigenvalue of \(W^{(T)}\) and the eigenvector corresponding to \(\tilde{\lambda}_k\) respectively for \(k = 1, \ldots, M\). Denote \(\tilde{\mathbf{v}}_k(i)\) as the \(i\)-th entry of \(\tilde{\mathbf{v}}_k\). We observe that

\[
\sum_{i,j \in \{1, \ldots, M\}} |G_{ij}^{(T)}|^2 = \sum_{i \in \{1, \ldots, M\}} (G^{(T)}(G^{(T)})^*)_{ii} = \sum_{i=1}^{M} \left\{ \left( \sum_{k=1}^{M} \tilde{\mathbf{v}}_{k_1}(i) \tilde{\mathbf{v}}_{k_1}^* \right) \left( \sum_{k=1}^{M} \tilde{\mathbf{v}}_{k_2}(i) \tilde{\mathbf{v}}_{k_2}^* \right) \right\} = \sum_{i=1}^{M} \sum_{k=1}^{M} \frac{\tilde{\mathbf{v}}_k(i) \tilde{\mathbf{v}}_k^*(i)}{\tilde{\lambda}_k - z} = \sum_{k=1}^{M} \eta^{-1} \text{Im} \left( \frac{1}{\tilde{\lambda}_k - z} \right) = \eta^{-1} \text{Im} \text{Tr}G^{(T)},
\]

which concludes the lemma.

\[\square\]

**iii. Proof of Lemma 5.4**

**Proof.** The first inequality follows from Theorem A.6 of [5]. To show the second and the third inequalities, we observe that

\[
\text{Tr}(G^{(i)} - G) = \frac{N - 1 - M}{z} - \frac{N - M}{z} + \text{Tr}(G^{(i)} - G) = -\frac{1}{z} + \text{Tr}(G^{(i)} - G),
\]

so the results follow.

\[\square\]

**iv. Proof of Proposition 5.5**

**Proof.** Following Lemma 1 of [26], taking imaginary part and multiplying \((\text{Im } m)^{-1}\) on both sides of \(z = f(m)\), we get

\[
\frac{1}{|m|^2} - \phi \int \frac{x \pi(dx)}{|1 + xm|^2} = \frac{\text{Im } z}{\text{Im } m} > 0.
\]
Hence
\[ \frac{1}{|m|^2} > \phi \int \frac{x \pi(dx)}{1 + xm^2}. \] (II.3)

By (II.3) and the Cauchy-Schwartz inequality, we obtain from \( z = f(m) \) that
\[
|m| = \left| \frac{1}{z} + \frac{\phi}{z} \int \frac{\pi(dx)}{1 + xm} \right|
< \frac{|1 - \phi|}{|z|} + \frac{\phi}{|z|} \left( \int \frac{x^2 \pi(dx)}{|1 + xm|^2} \right)^{1/2} \left( \int x^{-2} \pi(dx) \right)^{1/2}
< \frac{|1 - \phi|}{|z|} + \sqrt{\phi} \left( \int x^{-2} \pi(dx) \right)^{1/2}. \] (II.4)

This implies that
\[
|m|^2 - \frac{|1 - \phi||m|}{|z|} - \frac{\sqrt{\phi} \left( \int x^{-2} \pi(dx) \right)^{1/2}}{2|z|} < 0.
\]

Some basic calculations yield
\[
|m| \leq \frac{|1 - \phi| + \sqrt{|z||1 - \phi| + 4z|\sqrt{\phi} \left( \int x^{-2} \pi(dx) \right)^{1/2}}}{2|z|}.
\]

Since \( \text{supp}(\pi) \) is uniformly bounded away from 0 for all \( N \), \( \int x^{-2} \pi(dx) \) is uniformly bounded. Then from (II.4), we have
\[
\sup_{|z| \in [\tau, \tau^{-1}]} \sup_{N} |m| \leq C,
\]
for some constant \( C > 0 \).

Suppose \( \inf_{|z| \in [\tau, \tau^{-1}]} \inf_{N} |m| = 0 \). Then we can choose a sequence \( \{z_N\}_{N=1}^{\infty} \subset \{\tau \leq |z| \leq \tau^{-1}\} \) such that \( m(z_N) \to 0 \) as \( N \to \infty \). From \( z = f(m) \), we have for all \( N \)
\[
z_N m(z_N) + 1 = \phi \int \frac{x m(z_N) \pi(dx)}{1 + xm(z_N)},
\]
which implies that with probability 1, as \( N \to \infty \),
\[
\phi \int \frac{x m(z_N) \pi(dx)}{1 + m(z_N)} \to 1.
\]

However one can see
\[
\left| \phi \int \frac{x m(z_N) \pi(dx)}{1 + m(z_N)} \right| \leq |m(z_N)| \frac{\phi \int x \pi(dx)}{1 + m(z_N)} \to 0,
\]
which is a contradiction. Therefore we have
\[
\inf_{|z| \in [\tau, \tau^{-1}]} \inf_{N} |m| > 0.
\]

Finally,
\[
\text{Im} m = \frac{\text{Im} z}{1 + \phi \int \frac{x \pi(dx)}{1 + xm^2}} \geq |m^2| \text{Im} z \geq C^{-1} \eta,
\]
which concludes the lemma.
v. Proof of Lemma 5.6

Proof. Given the event $\Xi$, $G_{ii}$ is within $\log^{-1} N$ distance to $m$ uniformly for $i \in I$. Since $|m| \asymp 1$, it then follows that

$$1(\Xi)G_{ii} \asymp 1.$$ 

Next, it follows from Lemma 4.1 and the definition of $\Xi$ that

$$1(\Xi)|G_{ij}^{(k)}| \leq |G_{ij}| + \left| \frac{G_{ik}G_{kj}}{G_{kk}} \right| \leq \delta_{ij}|m| + \frac{\log^{-2} N}{|m| - \log^{-1} N},$$

$$1(\Xi)|G_{ij}^{(k)}| \geq |G_{ij}| - \left| \frac{G_{ik}G_{kj}}{G_{kk}} \right| \geq \delta_{ij}|m| - \frac{\log^{-2} N}{|m| - \log^{-1} N},$$

which implies that given $\Xi$, $G_{ij}^{(k)}$ is within $2\log^{-1} N$ distance to $m$ uniformly for all $i,j,k \in I$ such that $i,j \neq k$.

Applying this argument inductively, we conclude that for any index set $T$ such that $|T| \leq C_1$, given $\Xi$, there exists a constant $C_2 > 0$ such that

$$1(\Xi)|G_{ij}^{(T)} - \delta_{ij}m| \leq C_2 \log^{-1} N,$$  \hfill (II.5)

uniformly for $i,j \in I \setminus T$. Consequently, it follows from Proposition 5.5 that there exists some constant $C > 0$ such that

$$1(\Xi)|G_{ij}^{(T)}| + 1(\Xi)\frac{1}{|G_{ii}^{(T)}|} \leq C.$$ \hfill (II.6)

Let $G^{(T)} = V(L^{(T)} - zI)^{-1}V^*$ be the eigen-decomposition of $G^{(T)}$ where $V = (v_1, \ldots, v_N)$ with orthonormal columns $v_1, \ldots, v_N$ is an orthogonal matrix and $L^{(T)} = \text{diag}(\lambda_1^{(T)}, \ldots, \lambda_{N - |T|}^{(T)})$. Then we see that for any $i,j \in I$,

$$|G_{ij}^{(T)}| \leq \|G^{(T)}\| = \sup_{\|w\|=1} \left| \sum_{k=1}^{N - |T|} w^*v_kv_k^*w \right| \leq \sup_{\|w\|=1} \sum_{k=1}^{N - |T|} \frac{w^*v_kv_k^*w}{\lambda_k^{(T)} - z} = \eta^{-1}.$$ \hfill (II.7)

Therefore the desired result follows from (II.6) and (II.7).

vi. Complement of the proof of Proposition 5.1

Proof. Let $\omega_1, \omega_2 \in \mathbb{C}^+$. Some basic calculations yield that

$$|G_{ij}(w_1) - G_{ij}(w_2)| \leq (\text{Im } w_1)^{-1}(\text{Im } w_2)^{-1}|w_1 - w_2|, \quad i,j \in I.$$ \hfill (II.8)

Let $z = E + i\eta \in \mathbb{D}^c$. We construct a lattice as follows. Let $z_0 = E + i$. Fix $\varepsilon \in (0, \tau/8)$. For $k = 0, 1, 2, \ldots, N^3 - N^{4+\varepsilon}$, define

$$\eta_k = 1 - kN^{-5}, \quad z_k = E + i\eta_k,$$

$$\delta_k = (\eta_k)^{-1/2} + \frac{\varepsilon^2}{8}, \quad \Xi_k = \{\lambda \leq N^\varepsilon \sqrt{\delta_k}\}.$$
Let $C > 0$ be a fixed constant. We show by induction for $k = 1, \ldots, N^5 - N^{4+\tau}$ that if the two events
\[
\Theta(z_{k-1}) \leq \frac{CN^\epsilon \delta_{k-1}}{\sqrt{\kappa + \eta_{k-1} + N^\epsilon \delta_{k-1}}}, \quad 1(\Xi_{k-1}) = 1,
\] (II.9)
hold with high probability, then
\[
\Theta(z_k) \leq \frac{N^\epsilon \delta_k}{\sqrt{\kappa + \eta_k + N^\epsilon \delta_k}}, \quad 1(\Xi_k) = 1,
\] hold with high probability.

It is clear that (II.9) for $k = 1$ follows from (5.23) and (5.24). We verify that if $1(\Xi_{k-1}) = 1$, then $\Lambda(z_k) \leq \log^{-1} N$, $k = 1, \ldots, N^5 - N^{4+\tau}$. Using the Lipschitz condition (II.8), we have
\[
1(\Xi_{k-1})\Lambda(z_k) \leq 1(\Xi_{k-1})|\Lambda(z_k) - \Lambda(z_{k-1})| + 1(\Xi_{k-1})\Lambda(z_{k-1})
\leq \max_{i,j} |G_{ij}(z_k) - G_{ij}(z_{k-1})| + N^\epsilon \sqrt{\delta_{k-1}}
\leq |z_k - z_{k-1}| \eta_k^{-1} \eta_{k-1}^{-1} + N^\epsilon [(N\eta_{k-1})^{-1/4} + q]
\leq N^{-3 - 2\tau} + N^\epsilon [N^{-\tau/4} + q]
\leq \log^{-1} N.
\]

Let $D > 0$ be an arbitrarily large number. Therefore, by (5.18), (5.4) and (5.22), we can choose $N_0 \in \mathbb{Z}_+$ such that as $N \geq N_0$,
\[
\sup_{k \in \{1, \ldots, N^5 - N^{4+\tau}\}} \mathbb{P} \left( 1(\Xi_{k-1})\Lambda_\sigma(z_k) + \max_{i \in \mathbb{Z}} |G_{ii}(z_k) - m_N(z_k)| > \frac{1}{2} N^\epsilon \delta_k \right) \leq N^{-D}, \quad (II.10)
\]
and
\[
\sup_{k \in \{1, \ldots, N^5 - N^{4+\tau}\}} \mathbb{P} \left( 1(\Xi_{k-1})|f(m_N(z_k)) - z_k| > \delta_k \right) \leq N^{-D}. \quad (II.11)
\]

Then, applying Proposition 4.6, we obtain from the induction hypothesis (II.9) and (II.11) that
\[
\mathbb{P} \left( 1(\Xi_{k-1})\Theta(z_k) > CN^\epsilon / \sqrt{\delta_k} \right) \leq \mathbb{P} \left( 1(\Xi_{k-1})\Theta(z_k) > \frac{CN^\epsilon \delta_k}{\sqrt{\kappa + \eta_k + N^\epsilon \delta_k}} \right) \leq N^{-D}. \quad (II.12)
\]

Using (II.10), (II.12) and the fact that $\delta_k < \sqrt{\delta_k}$ for all $k = 1, \ldots, N^5 - N^{4+\tau}$, we get that as $N \geq N_0$,
\[
\mathbb{P}(\Xi_{k-1} \cap \Xi_k)
\leq \mathbb{P} \left( 1(\Xi_{k-1}) \left( \max_{i \in \mathbb{Z}} |G_{ii}(z_k) - m_N(z_k)| + \Theta(z_k) + \Lambda_\sigma(z_k) > N^\epsilon \sqrt{\delta_k} \right) \right)
\leq \mathbb{P} \left( 1(\Xi_{k-1}) \left( \max_{i \in \mathbb{Z}} |G_{ii}(z_k) - m_N(z_k)| + \Lambda_\sigma(z_k) > \frac{1}{2} N^\epsilon \sqrt{\delta_k} \right) \right)
+ \mathbb{P} \left( 1(\Xi_{k-1})\Theta(z_k) > \frac{1}{2} N^\epsilon \sqrt{\delta_k} \right) \leq 2N^{-D}.
\]
Then we see that for any \( k \in \{ 1, \ldots, N^5 - N^{4+\tau} \} \), as \( N \geq N_0 \),
\[
P(\Xi_k^c) = 1 - P(\Xi_k) = \sum_{i=1}^{k} P(\Xi_{i-1} \cap \Xi_i^c) + P(\Xi_0^c) \leq 2N^{5-D}.
\]
This shows that \( I \prec \Xi_k \) or equivalently \( \Lambda(z_k) = \sqrt{c_k} \) uniformly for all \( k \in \{ 0, \ldots, N^5 - N^{4+\tau} \} \).

Finally, by choosing \( \hat{k} \in \{ 1, \ldots, N^5 - N^{4+\tau} \} \) such that \( -\varepsilon(\hat{z} - z_k) \leq N^{-5} \), we have
\[
\Lambda(z) \leq |\Lambda(z) - \Lambda(z_k)| + \Lambda(z_k) \leq \max_{i,j} |G_{ij}(\hat{z}) - G_{ij}(z_k)| + \Lambda(z_k)
\]
\[
\leq N^{-3-2\tau} + \Lambda(z_k) \prec (N\eta)^{-1/4} + q.
\]
The proof of Proposition 5.1 is now complete.

\[\square\]

\textit{vii. Proof of Proposition 5.10.}

\textit{Proof.} We omit the proof of (5.30), since it is similar to that of (5.31) (actually it is also simpler than (5.31) since we only need to expand the \( G \) terms using the third identity of Lemma 4.1).

In the following, we give the proof of (5.31).

For simplicity of notation, denote \( \Sigma_0 = \Sigma(\eta_N^{(i)} + I)^{-1} \) and write \( \mathcal{V}_i = x_i^{(i)} G^{(i)} \Sigma_0 x_i \). In the following, we bound the quantity
\[
\left| \frac{1}{N} \sum_{i=1}^{N} Q_i \mathcal{V}_i \right|
\]
Let \( p \) be an even integer. Denote \( V_{i_s} := Q_{i_s} \mathcal{V}_{i_s} \) for \( s \leq p/2 \) and \( V_{i_s} := Q_{i_s} \mathcal{V}_{i_s}^* \) for \( s > p/2 \). We bound \( \mathbb{E}\left| \frac{1}{N} \sum_{i=1}^{N} V_i \right|^p \).

We see that
\[
\mathbb{E}\left| \frac{1}{N} \sum_{i=1}^{N} Q_i \mathcal{V}_i \right|^p = \frac{1}{N^p} \sum_{i_1, \ldots, i_p} \mathbb{E} \prod_{s=1}^{p} V_{i_s} = \frac{1}{N^p} \sum_{i_1, \ldots, i_p} \mathbb{E} \prod_{s=1}^{p} \left( \prod_{r=1}^{p} (P_{i_r} + Q_{i_r})V_{i_s} \right).
\]

Introducing the notation \( i = (i_1, \ldots, i_p) \), \( |i| = \{i_1, \ldots, i_p\} \), \( P_A = \prod_{i \in A} P_i \) and \( Q_A = \prod_{i \in A} Q_i \) for some index set \( A \), we get
\[
\mathbb{E}\left| \frac{1}{N} \sum_{i=1}^{N} Q_i \mathcal{V}_i \right|^p = \frac{1}{N^p} \sum_{i} \sum_{A_1, \ldots, A_p \subset |i|} \mathbb{E} \prod_{s=1}^{p} (P_{A_s} Q_{A_s} V_{i_s}). \tag{II.13}
\]

By definition of \( V_i \), we have that \( V_{i_s} = Q_{i_s} V_{i_s} \) and \( P_{i_s} V_{i_s} = 0 \), which imply that \( P_{A_s^c} V_{i_s} = 0 \) if \( i_s \notin A_s \). Hence we may restrict the summation to \( A_s \) satisfying
\[
i_s \in A_s \tag{II.14}
\]
for all $s$. Moreover, we see that if $i_s \in \cap_{q \neq s} A^q_s$ for some $s$, say $s = 1$, then $P_{A^q_s} Q_{A^q_s} V_1$ is $X^{(s)}$-measurable for each $q = 2, \ldots, p$. Thus, we have

$$
\mathbb{E} \prod_{s=1}^p (P_{A^q_s} Q_{A^q_s} V_1) = \mathbb{E} (P_{A^q_s} Q_{A^q_s} Q_{i_1} V_1) \prod_{s=2}^p (P_{A^q_s} Q_{A^q_s} V_1)
$$

$$
= \mathbb{E} Q_{i_1} \left\{ (P_{A^q_s} Q_{A^q_s} V_1) \prod_{s=2}^p (P_{A^q_s} Q_{A^q_s} V_1) \right\} = 0. \tag{II.15}
$$

(II.14) and (II.15) show that each index $i_s$ must belong to at least two different sets: $A_s$ and $A_q$ for some $q \neq s$. Hence

$$
\sum_{s=1}^p |A_s| \geq 2 ||i||. \tag{II.16}
$$

In the following, a crucial step is to show that for $i \in A$

$$
|Q_A V_i| \sim \Phi^{|A|}. \tag{II.17}
$$

When $|A| = 1$ (corresponding to the case $A = \{i\}$), it follows straightforward from Lemma 4.4. Suppose $|A| \geq 2$. For ease of presentation, we assume without loss of generality that $i = 1$ and $A = \{1, 2, \ldots, \nu\}$ for some $\nu \geq 2$. Before we proceed, we note the following equality that for any $i \neq j$ and $T \subseteq \mathbb{T}$ with $i, j \notin T$,

$$
x_i^* G^{(iT)} = x_i^* (G^{(iT)} - \frac{G^{(ijT)} x_j^* G^{(ijT)}}{1 + x_j^* G^{(ijT)} x_j})
$$

$$
= x_i^* G^{(ijT)} + z G^{(ijT)} x_i^* G^{(ijT)}
$$

$$
= x_i^* G^{(ijT)} + \frac{G^{(ijT)}}{G^{(iT)}} x_i^* G^{(ijT)}, \tag{II.18}
$$

and

$$
\Sigma_0^{(T)} = \Sigma_0^{(iT)} + \frac{1}{N} \sum_{j \in \mathbb{T} \setminus \{i, j\}} \frac{G_{j1}^{(iT)} G_{ij}^{(iT)}}{G_{ii}^{(iT)}} \Sigma_0^{(iT)} \Sigma_0^{(iT)}. \tag{II.19}
$$

We show an example of the expansion. It follows from Lemma V.1 and Lemma 4.1 that

$$
Q_2 \mathcal{F}_1 = Q_2 (x_1^* G^{(1)} \Sigma_0^{(1)} x_1)
$$

$$
= Q_2 (x_1^* G^{(1)} \Sigma_0^{(1)} x_1) + \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{j2}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} \right) x_1^* G^{(1)} \Sigma_0^{(1)} \Sigma_0^{(1)} x_1
$$

$$
= Q_2 (x_1^* G^{(12)} \Sigma_0^{(12)} x_1) + Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{j2}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} x_1^* G^{(12)} \Sigma_0^{(12)} x_1 \right)
$$

$$
+ Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{j2}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} x_1^* G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 \right)
$$

$$
= Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{j2}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} x_1^* G^{(12)} \Sigma_0^{(12)} x_1 \right) + Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{j2}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} x_1^* G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 \right).
$$
We note that
\[
\left| \frac{G_{12}}{G_{11}} x_2^* G^{(12)} \Sigma_0^{(12)} x_1 \right| \prec \frac{\Phi_\nu}{M} \| G^{(12)} \Sigma_0^{(12)} \|_F \leq \frac{\Phi_\nu}{M} \| G^{(12)} \|_F \| \Sigma_0^{(12)} \| \prec \Phi_\nu^2,
\]
and
\[
Q_1(\left( \frac{1}{N} \sum_{j \notin \{1,2\}} \frac{G_{12}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} \right) x_2^* G^{(1)} \Sigma_0^{(12)} \Sigma \Sigma_0^{(1)} x_1) 
= \left( \frac{1}{N} \sum_{j \notin \{1,2\}} \frac{G_{12}^{(1)} G_{22}^{(1)}}{G_{22}^{(1)}} \right) Q_1(\left( x_2^* G^{(1)} \Sigma_0^{(12)} \Sigma \Sigma_0^{(1)} x_1) 
\prec \frac{\Phi_\nu^2}{M} \| G^{(1)} \Sigma_0^{(12)} \Sigma \Sigma_0^{(1)} \|_F 
\prec \Phi_\nu^3.
\]

We see that
\[ Q_3 Q_2 \psi_1 \]
\[ = Q_3 Q_2 (x_1^0 G^{(1)}(1) \Sigma_0^{(1)} x_1) \]
\[ = Q_3 Q_2 (x_1^0 G^{(1)}(1) \Sigma_0^{(12)} x_1 + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(1)}(1) \Sigma_0^{(12)} \Sigma_0^{(1)} x_1) \]
\[ = Q_3 Q_2 (x_1^0 G^{(1)}(1) \Sigma_0^{(123)} x_1 + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(1)}(1) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1) \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(1)}(1) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]
\[ = Q_3 Q_2 (x_1^0 G^{(12)}(12) \Sigma_0^{(123)} x_1 + \frac{G^{(1)}_{12} G^{(12)}_{12}}{G^{(1)}_{11}} x_1^0 G^{(12)}(12) \Sigma_0^{(123)} x_1) \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(1)}(12) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(1)}(12) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]
\[ = Q_3 Q_2 (\frac{G^{(1)}_{12} G^{(12)}_{12}}{G^{(1)}_{11}} x_1^0 G^{(12)}(123) \Sigma_0^{(123)} x_1 + \frac{G^{(12)}_{12} G^{(12)}_{12}}{G^{(12)}_{11} G^{(12)}_{22}} x_1^0 G^{(123)}(123) \Sigma_0^{(123)} x_1) \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(12)}(123) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]
\[ = Q_3 Q_2 (\frac{G^{(1)}_{12} G^{(12)}_{12}}{G^{(1)}_{11}} x_1^0 G^{(12)}(123) \Sigma_0^{(123)} x_1 + \frac{G^{(12)}_{12} G^{(12)}_{12}}{G^{(12)}_{11} G^{(12)}_{22}} x_1^0 G^{(123)}(123) \Sigma_0^{(123)} x_1) \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(12)}(123) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2,3\}} G^{(123)}_{j3} G^{(123)}_{j3} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(12)} x_1 \]
\[ + \left( \frac{1}{N} \sum_{j \neq \{1,2\}} G^{(12)}_{j2} G^{(12)}_{j2} \right) x_1^0 G^{(123)}(123) \Sigma_0^{(123)} \Sigma_0^{(1)} x_1 \]

(II.20)
We observe that the first two terms in (II.20) are \( \prec \Phi^3 \nu \), and we continue the expansion procedures for \( G^{(T)} \) and \( \Sigma_0^{(T)} \) for which \( (T) \) is not maximally expanded. Thus, the third term can be written as

\[
Q_4 Q_2 (\frac{1}{N} \sum_{j \notin \{1,2,3\}} \frac{G_{jj}^{(12)} - G_{33}^{(12)}}{G_{33}^{(12)}}) G_{11}^2 \Sigma_0^{(123)} \Sigma_0^{(12)} x_1) \]

\[
= Q_4 Q_2 (\frac{1}{N} \sum_{j \notin \{1,2,3\}} \frac{G_{jj}^{(12)} - G_{33}^{(12)}}{G_{33}^{(12)}}) G_{11}^2 \Sigma_0^{(123)} \Sigma_0^{(12)} x_1) \preceq \Phi^3 \nu.
\]

Similar results can be obtained for the fourth term. Actually, one can observe that the first term of \( Q_4 V_i \) is the leading term, so we can only clarify the bounds for the first term.

We see that

\[
Q_4 Q_3 Q_2 Q_1 V_1 \]

\[
= Q_4 Q_3 Q_2 (x_1^* G^{(1)} \Sigma_0^{(123)} x_1) + (\frac{1}{N} \sum_{j \notin \{1,2,3\}} \frac{G_{jj}^{(12)} - G_{33}^{(12)}}{G_{33}^{(12)}}) x_1^* G^{(1)} \Sigma_0^{(123)} \Sigma_0^{(12)} x_1)
\]

\[
+ (\frac{1}{N} \sum_{j \notin \{1,2\}} \frac{G_{jj}^{(1)} G_{jj}^{(2)}}{G_{22}^{(12)}}) x_1^* G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(1)} x_1)
\]

\[
= Q_4 Q_3 Q_2 (x_1^* G^{(1)} \Sigma_0^{(123)} x_1)
\]

\[
+ (\frac{1}{N} \sum_{j \notin \{1,2,3,4\}} G_{jj}^{(12)} G_{jj}^{(123)} G_{jj}^{(1234)} x_1^* G^{(1)} \Sigma_0^{(1234)} \Sigma_0^{(123)} x_1)
\]

\[
+ (\frac{1}{N} \sum_{j \notin \{1,2,3\}} G_{jj}^{(12)} G_{jj}^{(123)} G_{jj}^{(1234)} x_1^* G^{(1)} \Sigma_0^{(123)} \Sigma_0^{(12)} x_1)
\]

\[
+ (\frac{1}{N} \sum_{j \notin \{1,2\}} G_{jj}^{(1)} G_{jj}^{(12)} G_{jj}^{(123)} x_1^* G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(1)} x_1)
\]

\[
= Q_4 Q_3 Q_2 (x_1^* G^{(12)} \Sigma_0^{(1234)} x_1) + (\frac{G_{12} G_{11}}{G_{11}^2} x_1^* G^{(12)} \Sigma_0^{(1234)} x_1).
\]
We see that the term $G$ and the remaining terms all contain three off-diagonal can be bounded by carrying out the following expansion

$$Q_1 Q_4 Q_3 Q_2 (x_1^* G^{(1)} \Sigma_0^{(1234)} x_1)$$

$$= Q_1 Q_4 Q_3 Q_2 (x_1^* G^{(12)} \Sigma_0^{(1234)} x_1 + \frac{G_{12}}{G_{11}} x_2^* G^{(12)} \Sigma_0^{(1234)} x_1)$$

$$= Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{12}}{G_{11}} x_2^* G^{(123)} \Sigma_0^{(134)} x_1 + \frac{G_{12} G^{(1)}_{12}}{G_{11} G^{(12)}_{12}} x_3^* G^{(123)} \Sigma_0^{(1234)} x_1 \right)$$

$$= Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{13} G^{(3)}_{13}}{G_{33} G^{(3)}_{11}} + \frac{G_{12} G^{(1)}_{12}}{G_{11} G^{(12)}_{12}} \right) \Sigma_0^{(134)} x_1 + \frac{G_{12} G^{(1)}_{23} G^{(12)}_{23}}{G_{11} G^{(12)}_{23}} x_3^* G^{(123)} \Sigma_0^{(1234)} x_1$$

$$= Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{13} G^{(3)}_{32}}{G_{33} G^{(3)}_{11}} + \frac{G_{12} G^{(1)}_{12}}{G_{11} G^{(12)}_{12}} \right) \Sigma_0^{(134)} x_1 + \frac{G_{12} G^{(1)}_{23} G^{(12)}_{23}}{G_{11} G^{(12)}_{23}} x_3^* G^{(123)} \Sigma_0^{(1234)} x_1$$

$$= Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{13} G^{(3)}_{32}}{G_{33} G^{(3)}_{11}} + \frac{G_{12} G^{(1)}_{12}}{G_{11} G^{(12)}_{12}} \right) \Sigma_0^{(134)} x_1 + \frac{G_{12} G^{(1)}_{23} G^{(12)}_{23}}{G_{11} G^{(12)}_{23}} x_3^* G^{(123)} \Sigma_0^{(1234)} x_1.$$

We see that the term

$$Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{13} G^{(3)}_{32}}{G_{33} G^{(3)}_{11}} + \frac{G_{12} G^{(1)}_{12}}{G_{11} G^{(12)}_{12}} \right) \Sigma_0^{(134)} x_1 + \frac{G_{12} G^{(1)}_{23} G^{(12)}_{23}}{G_{11} G^{(12)}_{23}} x_3^* G^{(123)} \Sigma_0^{(1234)} x_1$$

can be bounded by carrying out the following expansion

$$\frac{G_{13} G^{(3)}_{32}}{G_{33} G^{(3)}_{11}} = \left( G^{(1)}_{13} + G^{(4)}_{14} G^{(4)}_{43} \right) \left( G^{(2)}_{43} + G^{(4)}_{44} \right) \left( \frac{1}{G^{(3)}_{33}} - \frac{G^{(4)}_{34}}{G^{(3)}_{33} G^{(3)}_{34} G^{(4)}_{44}} \right) \left( \frac{1}{G^{(3)}_{11}} - \frac{G^{(4)}_{11} G^{(4)}_{43} G^{(4)}_{31}}{G^{(3)}_{11} G^{(3)}_{31} G^{(3)}_{31}} \right).$$

Thus the first term resulting from the expansion yields that

$$Q_1 Q_4 Q_3 Q_2 \left( \frac{G^{(4)}_{14} G^{(4)}_{32}}{G^{(3)}_{33} G^{(3)}_{11}} x_3^* G^{(1234)} \Sigma_0^{(1234)} x_1 \right) = 0,$$

and the remaining terms all contain three off-diagonal $G$ terms in the numerators. Consequently,

$$Q_1 Q_4 Q_3 Q_2 \left( \frac{G^{(4)}_{14} G^{(4)}_{32}}{G^{(3)}_{33} G^{(3)}_{11}} x_3^* G^{(1234)} \Sigma_0^{(1234)} x_1 \right) < \Phi^4.$$
Similarly, the term
\[ Q_1 Q_4 Q_3 Q_2 \left( \frac{G_{12} G_{23}^{(1)}}{G_{11} G_{22}^{(1)}} \right) x_1^i G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 \]
can be bounded by \( \Phi_4^{1} \) via expanding the \( G \) terms into those with (4) added to the superscripts. So the remaining term is
\[ Q_1 Q_4 Q_3 Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{12}^{(1)} G_{23}^{(1)}}{G_{22}^{(1)}} \right) x_1^i G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 \]

We note that
\[ Q_1 Q_4 Q_3 Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{12}^{(1)} G_{23}^{(1)}}{G_{22}^{(1)}} \right) x_1^i G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 \]

\[ = Q_1 Q_4 Q_3 Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2\}} \frac{G_{12}^{(1)} G_{23}^{(1)}}{G_{22}^{(1)}} \right) x_1^i G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 + O(\Phi_4^{1}) \]

\[ = Q_1 Q_4 Q_3 Q_2 \left( \frac{1}{N} \sum_{j \notin \{1, 2, 3\}} \frac{G_{12}^{(1)} G_{23}^{(1)}}{G_{22}^{(1)}} \right) x_1^i G^{(1)} \Sigma_0^{(12)} \Sigma_0^{(12)} x_1 + O(\Phi_4^{1}) \]

\[ \prec \Phi_4^{1}, \]

where \( \mathcal{T} \) is the term with at least three off-diagonal \( G \) terms in the numerator which can be bounded by \( \Phi_4^{1} \) via expanding the \( G \) terms.

Now we summarise the expansion steps as follows. Consider
\[ Q_A \mathcal{V}_1 := Q_\nu \cdots Q_2 Q_1 \mathcal{V}_1 = Q_\nu \cdots Q_2 Q_1 x_1^i G^{(1)} \Sigma_0^{(1)} x_1 \]

a) We first expand the term \( x_1^i G^{(1)} \) to \( x_1^i G^{(A)} \) from the smallest index to the largest index. We will get a sequence of monomials with the maximally expanded \( A_i := x_1^i G^{(A)} \) where \( i \in A \). The coefficients of \( A_i \)'s are of the pattern
\[ \prod_{(a, b) \in P_i} \frac{G_{ab}^{(T_{ab})}}{G_{aa}^{(T_{aa})}}, \]

where \( P_i \) is an ordered paired pattern subset of \( \{1, \cdots, i\} \subset A \), for example, \( P_i = \{\{1, 3\}, \{3, 4\}, \{4, i\}\} \) and \( T_{ab} = \{1, \cdots, b\} \setminus \{a, b\} \) or \( T_{ab} = \emptyset \). And these coefficients can be handled by the resolvent extension just like the same procedures in \( (1/N) \sum_{i=1}^{j} Q_i \frac{1}{G_{ii}} \).

One can observe that the only remaining terms after taking \( Q_A \) (we imprecisely ignore the term \( \Sigma_0 \) here) are those monomials whose lower indexes in the numerator contain all elements of \( A \). Since there are only off-diagonal entries in the numerator, by the paired pattern \( P_i \), we remark that for those monomials, the number of off-diagonal entries is \( |A| \).

b) We expand \( \Sigma_0 \) to match the upper index with \( G^{(T)} \) which ensures that several undesired terms vanish after taking conditional expectation. Actually, we use the same strategy as in step a), adding the upper index of \( \Sigma_0 \) from the smallest value to the largest value of \( A \). We remark that except the leading term with coefficient 1, other terms give us more off-diagonal entries as coefficients.
c) We expand the Green function of coefficients after steps a) and b) to a maximal extent, or have at least $|A|$ off-diagonal entries. Then it follows that for $\nu \geq 2$

$$Q_AV_i = Q_\nu \cdots Q_2Q_1\mathcal{Y}_i \prec \Phi_\nu^{\nu+1}.$$  

This completes the proof of (II.17). By (II.16) and (II.17), it follows from (II.13) that there exists some constant $C_p > 0$ depending on $p$ only such that

$$\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} Q_i \mathcal{Y}_i \right]^p = \frac{1}{N^p} \sum_i \sum_{A_1, \ldots, A_p \subset [i]} \mathbb{E}\left[ P_{A_1}^c Q_1 \cdots Q_p^c \right] \prec \mathcal{P} \sum_{s=1}^{p} \Phi_\nu^{2s} N^{s-p} \leq C_p \Phi_\nu + C_p N^{-1/2} 2^p \leq C_p \Phi_\nu^{2p}, \quad (\text{II.21})$$

where the second last step follows from the elementary inequality $a^n b^m \leq (a+b)^{n+m}$ for positive $a, b$ and the last step follows from the fact that $C N^{-1/2} \leq \Phi_\nu$. (II.21) shows that

$$\frac{1}{N} \sum_{i=1}^{N} Q_i \mathcal{Y}_i \prec \Phi_\nu^{2},$$

which concludes (5.31).

### Appendix III: Proof of Theorem 3.2.

Firstly we show (3.4).

**Proof of (3.4).** Using Proposition 5.10, we get from (5.22) that

$$|f(m_N) - z| \prec N^\varepsilon \left( q^2 + \frac{1}{(N\eta)^2} + \frac{\text{Im} m}{N\eta} \right),$$

uniformly for $z \in D^\varepsilon$. Note here we assume $q < N^{-1/3}$. Then, we obtain from Proposition 4.6 that, for any $\varepsilon, D > 0$, as $N$ is sufficiently large,

$$\sup_{z \in D^\varepsilon} \mathbb{P}\left(|m_N - m| \prec \frac{N^\varepsilon}{\sqrt{\kappa + \eta}} \left( \frac{\text{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + q^2 \right) \right)$$

$$\leq \sup_{z \in D^\varepsilon} \mathbb{P}\left(|m_N - m| \prec \frac{N^\varepsilon (\frac{\text{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + q^2)}{\sqrt{\kappa + \eta} + \sqrt{N^{\varepsilon/2}(\frac{\text{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + q^2)}} \right) \leq N^{-D},$$

so uniformly for $z \in D^\varepsilon$,

$$|m_N - m| \prec \frac{1}{\sqrt{\kappa + \eta}} \left( \frac{\text{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + q^2 \right). \quad (\text{III.1})$$

Denote

$$\tilde{\Psi}(z) = \frac{1}{\sqrt{\kappa + \eta}} \left( \frac{\text{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + q^2 \right).$$
Next, we show that
\[ \lambda_1 = \lambda_+ + O_{\prec}(N^{-2/3} + q^2). \] (III.2)

We know from Lemma V.2 there exists some constant \( C > 0 \) such that \( \lambda_1 \leq C \) with high probability. Therefore, it remains to show that for any fixed \( \varepsilon > 0 \), there is no eigenvalue of \( W \) in the interval
\[ I := [\lambda_+ + N^{-2/3+4\varepsilon} + N^{4\varepsilon}q^2, C] \] (III.3)
with high probability. The idea of the proof is to choose, for each \( E \in I \), a scale \( \eta(E) \) such that \( \Im m_N(E + \eta(E)) \leq \frac{N^{-\varepsilon}}{N\eta(E)} \) with high probability. First, we need a simultaneous version of (III.1), i.e.
\[ \bigcap_{z \in \mathbb{D}^*} \left\{ |m_N(z) - m(z)| \leq \hat{\Psi}(z) \right\} \text{ holds with high probability.} \] (III.4)

It suffices to show that
\[ \sup_{z \in \mathbb{D}^*} \frac{|m_N(z) - m(z)|}{\hat{\Psi}(z)} < 1. \] (III.5)

Let for \( i = 1, 2, z_i \equiv E_i + \eta_i \in \mathbb{D}^e \) and \( \kappa_i = |E_i - \lambda_+| \). Elementary calculation yields that there exists some constant \( C_1 > 0 \) such that
\[
\begin{align*}
|m_N(z_1) - m_N(z_2)| &\leq C_1 N^2 |z_1 - z_2|, \\
|m(z_1) - m(z_2)| &\leq C_1 N^2 |z_1 - z_2|, \\
|\hat{\Psi}(z_1) - \hat{\Psi}(z_2)| &\leq C_1 N^{5/2} |z_1 - z_2|, \\
\inf_{z \in \mathbb{D}^e} \hat{\Psi}(z) &\geq (C_1 N)^{-1}. \quad \text{(III.6)}
\end{align*}
\]

We define the \( N^{-4} \)-net \( \tilde{\mathbb{D}}^e = (N^{-4}Z^2) \cap \mathbb{D}^e \). Hence, \( |\tilde{\mathbb{D}}^e| \leq CN^8 \) and for any \( z \in \mathbb{D}^e \), there exists a \( w \in \tilde{\mathbb{D}}^e \) such that \( |z - w| \leq 2N^{-4} \). Then using a simple union bound and (III.6), we can deduce (III.5).

Let \( \varepsilon \) be as in (III.3). Then we observe from (III.4) and Lemma 4.5 that
\[ \bigcap_{z \in \mathbb{D}^*, E \geq \lambda_+} \left\{ |m_N(z) - m(z)| \leq N^{\varepsilon} \left( \frac{1}{\kappa N \eta} + \frac{1}{\sqrt{\kappa}} \left( \frac{1}{N \eta} + q^2 \right) \right) \right\} \] (III.7)
holds with high probability.

For each \( E \in I \), we define
\[ \eta(E) = q^{-1} N^{-1} \kappa(E)^{1/2}, \quad z(E) = E + \eta(E). \] (III.8)

Using Lemma 4.5, we find that there exists a constant \( C_2 > 0 \) such that for all \( E \in I \)
\[ \Im m(z(E)) \leq \frac{C_2 \eta(E)}{\sqrt{\kappa(E)}} \leq \frac{C_2 N^{-\varepsilon}}{N \eta(E)}. \] (III.9)

With the choice \( \eta(E) \) in (III.8), we obtain from (III.7) that
\[ \bigcap_{E \in I} \left\{ |m_N(z) - m(z)| \leq \frac{2N^{-\varepsilon}}{N \eta(E)} \right\} \text{ holds with high probability.} \] (III.10)
From (III.9) and (III.10) we conclude that

\[ \bigcap_{E \in \mathbf{I}} \left\{ \text{Im } m_N(z) \leq \frac{(2 + C_2)N^{-\frac{\varepsilon}{2}}}{N\eta(E)} \right\} \] holds with high probability. \hfill (III.11)

Now suppose that there is an eigenvalue, say \( \lambda_i \) of \( W \) in \( \mathbf{I} \). Then we find that

\[ \text{Im } m_N(z(\lambda_i)) = \frac{1}{N} \sum_j \frac{\eta(\lambda_i)}{(\lambda_i - \lambda_j)^2 + \eta(\lambda_j)^2} \geq \frac{1}{N\eta(\lambda_i)} \]

which contradicts with the inequality in (III.11). Therefore, we conclude that with high probability, there is no eigenvalue in \( \mathbf{I} \). Since \( \varepsilon > 0 \) in (III.3) is arbitrary, (III.2) follows. \( \square \)

Now we show (3.5). First we show the following lemma.

**Lemma III.1.** Let \( a_1, a_2 \) be two numbers with \( a_1 \leq a_2 \) and \( |a_1| + |a_2| = O(1) \). For any \( E_1, E_2 \in [a_1, a_2] \) and \( \eta = N^{-1} \), let \( \psi(\lambda) := \psi_{E_1, E_2, \eta}(\lambda) \) be a \( C^2(\mathbb{R}) \) function such that \( \psi(x) = 1 \) for \( x \in [E_1 + \eta, E_2 - \eta] \), \( \psi(x) = 0 \) for \( x \in \mathbb{R} \setminus [E_1, E_2] \) and the first two derivatives of \( \psi \) satisfy \( |\psi^{(1)}(x)| \leq C_N^{-1}, |\psi^{(2)}(x)| \leq C_N \eta^{-2} \) for all \( x \in \mathbb{R} \). Let \( g^\Delta \) be a signed measure on the real line and \( m^\Delta \) be the Stieltjes transform of \( g^\Delta \). Suppose, for some positive number \( c_N \) depending on \( N \), we have

\[ |m^\Delta(x + iy)| \leq c_N \left( \frac{1}{N + \frac{q^2}{\sqrt{\kappa + y}}} \right) \quad \forall y < 1, x \in [a_1, a_2], \] \hfill (III.12)

then

\[ \left| \int \psi(\lambda)g^\Delta(d\lambda) \right| \leq c_N \left( \frac{1}{N} + \frac{q^2}{\sqrt{\kappa + 1/2}} \right) \leq c_N \left( \frac{1}{N} + q^2 \sqrt{E_2 - E_1 + \eta} \right) \leq c_N \left( \frac{1}{N} + q^3 + q^2 \sqrt{E_1}. \right) \] \hfill (III.13)

**Proof.** By (3.4), it suffices to show the case where \( E_2 = \lambda_+ + N^\varepsilon(q^2 + N^{-2/3}) \). Let \( \chi(y) \) be a smooth cutoff function with support \([-1, 1]\) such that \( \chi(y) = 1 \) for \(|y| \leq 1/2 \) and \( \chi(y) \) has bounded derivatives otherwise. Define:

\[ g^\Delta(x) := \rho_W(x) - g(x), \quad m^\Delta(z) := m_N(z) - m(z). \]

Using the Helffer-Sjöstrand formula (setting \( \chi(x + iy) = \chi(y) \) in Proposition C.1 of [10]), we get that

\[ \psi(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y\psi^{(2)}(x)\chi(y) + \{\psi(x) + iy\psi^{(1)}(x)\} \chi^{(1)}(y)}{\lambda - x - iy} \, dx \, dy. \]
Integrating with respect to $q^\Delta$ and using the fact that $\psi$ and $\chi$ are real, we obtain that
\[
\left| \int \psi(\lambda)q^\Delta(d\lambda) \right| = \left| \frac{i}{2\pi} \int_{\mathbb{R}^2} [y\psi^{(2)}(x)(y) + \{\psi(x) + iy\psi^{(1)}(x)\}\chi^{(1)}(y)]m^\Delta(x + iy)dxdy \right|
\leq C \int_{\mathbb{R}^2} \{||\psi(x)|| + |y||\psi^{(1)}(x)||\}||\chi^{(1)}(y)||m^\Delta(x + iy)dxdy
+ C\int_{|y|\leq \eta} \int_{\mathbb{R}} y\psi^{(2)}(x)(y)\Im m^\Delta(x + iy)dxdy
+ C\int_{|y|> \eta} \int_{\mathbb{R}} y\psi^{(2)}(x)(y)\Im m^\Delta(x + iy)dxdy.
\] (III.14)

With (III.12), the first term in (III.14) can be estimated as
\[
C \int_{\mathbb{R}^2} \{||\psi(x)|| + |y||\psi^{(1)}(x)||\}||\chi^{(1)}(y)||m^\Delta(x + iy)dxdy
= C \int_{[-1,1] \setminus [-1/2,1/2]} \int_{E_1}^{E_2} \{||\psi(x)||\}||\chi^{(1)}(y)||m^\Delta(x + iy)dxdy
+ C \int_{[-1,1] \setminus [-1/2,1/2]} \int_{[E_1,E_2] \setminus [E_1+\eta,E_2-\eta]} |y||\psi^{(1)}(x)||\chi^{(1)}(y)||m^\Delta(x + iy)dxdy
\leq C\eta N \left( \frac{1}{N} + \frac{q^2}{\sqrt{\kappa + 1/2}} \right).
\]

The second term in (III.14) can be estimated as
\[
C \left| \int_{|y|\leq \eta} \int_{\mathbb{R}} y\psi^{(2)}(x)(y)\Im m^\Delta(x + iy)dxdy \right|
= C \left| \int_{|y|\leq \eta} \int_{[E_1,E_2] \setminus [E_1+\eta,E_2-\eta]} y\psi^{(2)}(x)(y)\Im m^\Delta(x + iy)dxdy \right|
\leq C\eta N \cdot \frac{C\eta N}{N}.
\]

For the third term in (III.14), we note that
\[
\frac{\partial}{\partial x} \Im m^\Delta(x + iy) = \Im \left\{ \frac{\partial}{\partial x} m^\Delta(x + iy) \right\} = \Im \left\{ -i \frac{\partial}{\partial y} m^\Delta(x + iy) \right\} = -\frac{\partial}{\partial y} \Re m^\Delta(x + iy).
\]


Then it follows from integration by parts first with respect to $x$ then $y$ that

$$C \left| \int_{|y|>\eta} y\psi^{(2)}(x)\chi(y) \text{Im} m^\Delta(x+iy) \, dx \, dy \right|$$

$$= C \left| \int_{|y|>\eta} \int_{\mathbb{R}} y\psi^{(1)}(x)\chi(y) \frac{\partial}{\partial x} \text{Im} m^\Delta(x+iy) \, dx \, dy \right|$$

$$= C \left| - \int_{\mathbb{R}} \int_{|y|>\eta} y\psi^{(1)}(x)\chi(y) \frac{\partial}{\partial y} \text{Re} m^\Delta(x+iy) \, dy \, dx \right|$$

$$= C \left| - \int_{\mathbb{R}} \left[ y\psi^{(1)}(x)\chi(y) \text{Re} m^\Delta(x+iy) \right]_{-\eta}^{\eta} \, dx \right.$$  

$$+ C \int_{\mathbb{R}} \int_{|y|>\eta} \psi^{(1)}(x)\{\chi(y) + y\chi^{(1)}(y)\} \text{Re} m^\Delta(x+iy) \, dy \, dx \bigg|$$

$$= C \left| 2 \int_{\mathbb{R}} \eta\psi^{(1)}(x) \text{Re} m^\Delta(x+i\eta) \, dx \right.$$  

$$+ C \int_{\mathbb{R}} \int_{|y|>\eta} \psi^{(1)}(x)\{\chi(y) + y\chi^{(1)}(y)\} \text{Re} m^\Delta(x+iy) \, dy \, dx \bigg|$$

$$\leq C \int_{\mathbb{R}} \eta|\psi^{(1)}(x)|| \text{Re} m^\Delta(x+i\eta)| \, dx$$

$$+ C \int_{|y|<\eta} \left| \int_{|y|\leq 1} \int_{[E_1,E_2] \setminus [E_1+\eta,E_2-\eta]} y\psi^{(1)}(x)\chi^{(1)}(y) \text{Re} m^\Delta(x+iy) \, dx \, dy \right|$$

$$\leq C \int_{[E_1,E_2] \setminus [E_1+\eta,E_2-\eta]} \frac{c_N}{\eta} \, dx + C \int_{|y|<\eta} \frac{c_N}{\eta} \left( \frac{1}{N|y|} + \frac{q^2}{\sqrt{\kappa + |y|}} \right) \, dy$$

$$+ C \int_{|y|<\eta} \left( \frac{1}{N} + \frac{q^2}{\sqrt{\kappa + 1/2}} \right) \, dy$$

$$\leq \frac{Cc_Nq^2}{\sqrt{\kappa + 1/2}} + \frac{Cc_N}{N} \log \eta \leq Cc_N \left( \frac{\log N}{N} + \frac{q^2}{\sqrt{\kappa + 1/2}} \right).$$

Then we summarise that

$$\left| \int \psi(\lambda) \rho^\Delta(\,d\lambda) \right| \leq Cc_N \left( \frac{1}{N} + \frac{q^2}{\sqrt{\kappa + 1/2}} \right)$$

$$\leq Cc_N \left( \frac{1}{N} + q^2\sqrt{E_2 - E_1 + \eta} \right) \quad \text{(III.15)}$$

$$\leq Cc_N \left( \frac{1}{N} + q^3 + q^2\sqrt{E_1} \right).$$

Next, let $\varrho^\Delta$ be the signed measure $\varrho_N - \varrho$. If $y \geq y_0 = N^{-1+r}$, the condition (III.12) in Lemma III.1 holds for the difference $m^\Delta = m_N - m$ and $c_N = N^{e}$ for any small $\varepsilon > 0$ with high
probability due to Theorem 3.1. For \( y \leq y_0 \), set \( z = x + iy \), \( z_0 = x + iy_0 \) and estimate

\[
|m_N(z) - m(z)| \leq |m_N(z_0) - m(z_0)| + \int_{y_0}^{y_0} \left| \frac{\partial}{\partial \eta} \{m_N(x + \eta) - m(x + \eta)\} \right| d\eta. \tag{III.16}
\]

Note that

\[
\left| \frac{\partial}{\partial \eta} m_N(x + i\eta) \right| = \left| \frac{\partial}{\partial \eta} \int \frac{1}{\lambda - x - i\eta} q_N(d\lambda) \right| \leq \int \frac{1}{|\lambda - x - i\eta|^2} q_N(d\lambda) = \eta^{-1} \text{Im } m_N(x + i\eta).
\]

The same bound applies to \( |\frac{\partial}{\partial \eta} m(x + i\eta)| \) with \( m_N \) replaced by \( m \).

Then using Theorem 3.1 and the fact that the functions \( y \to y \text{Im } m_N(x + iy) \) and \( y \to y \text{Im } m(x + iy) \) are both monotone increasing for any \( y > 0 \) since both are Stieltjes transforms of a positive measure, we obtain that

\[
\int_{y_0}^{y_0} \left| \frac{\partial}{\partial \eta} \{m_N(x + \eta) - m(x + \eta)\} \right| d\eta \leq \int_{y_0}^{y_0} \frac{1}{\eta} \{\text{Im } m_N(x + \eta) + \text{Im } m(x + \eta)\} d\eta \leq y_0 \{\text{Im } m_N(z_0) + \text{Im } m(z_0)\} \int_{y_0}^{y_0} \frac{1}{\eta^2} d\eta = y_0 \{\text{Im } m_N(z_0) + \text{Im } m(z_0)\} \left( \frac{1}{y} - \frac{1}{y_0} \right) = \{\text{Im } m_N(z_0) + \text{Im } m(z_0)\} \frac{y_0 - y}{y} \leq 2 \text{Im } m(z_0) + (Ny_0)^{-1}.
\]

Hence we have from (III.16) that

\[
|m_N(z) - m(z)| < 2 \text{Im } m(z_0) + (Ny_0)^{-1} + q \leq \frac{CNy + 1}{Ny} + q \leq \frac{CNy}{Ny} + q. \tag{III.17}
\]

Let \( \psi_{E_1,E_2,\eta} \) be the function in Lemma III.1. Applying Lemma III.1 below with \( c_N = N^{\tau} \), we obtain that for any \( \eta = N^{-1} \)

\[
\left| \int_{\mathbb{R}} \psi_{E_1,E_2,\eta}(\lambda) q_N(d\lambda) - \int_{\mathbb{R}} \psi_{E_1,E_2,\eta}(\lambda) q(d\lambda) \right| \prec N^{-1+\tau}.
\]

Integrating with respect to \( q_N(d\lambda) \) and \( q(d\lambda) \) on both sides of the following elementary inequality

\[
1_{[x-\eta,x+\eta]}(\lambda) \leq \frac{2\eta^2}{(\lambda - x)^2 + \eta^2} \quad \forall x, \lambda \in \mathbb{R},
\]

and using (III.17), Lemma 4.5 and the definitions of \( y_0 \) and \( q \), we get that for some constant \( C > 0 \)

\[
n_N(x - \eta, x + \eta) \leq C\eta \text{Im } m_N(x + i\eta) \leq Cy_0 \text{Im } m_N(x + iy_0) \prec N^{-1+\tau},
\]

and

\[
n(x - \eta, x + \eta) \leq C\eta \text{Im } m(x + i\eta) \leq Cy_0 \text{Im } m(x + iy_0) \prec N^{-1+\tau},
\]
uniformly for \( x \) in a small neighborhood of \( \lambda_+ \).

We note that

\[
|n_N(E_1, E_2) - n(E_1, E_2)| = \left| \int_{E_1}^{E_2} \varrho_N(d\lambda) - \int_{E_1}^{E_2} \varrho(d\lambda) \right|
\leq \left| \int_{E_1+\eta}^{E_2-\eta} \psi_{E_1, E_2, \eta}(\lambda) \varrho_N(d\lambda) - \int_{E_1+\eta}^{E_2-\eta} \psi_{E_1, E_2, \eta}(\lambda) \varrho(d\lambda) \right|
+ \left| \int_{E_1-\eta}^{E_2+\eta} \varrho_N(d\lambda) \right| + \left| \int_{E_2-\eta}^{E_2+\eta} \varrho(d\lambda) \right|
\prec \frac{1}{N^{1+\tau}} + q^3 + q^2(\sqrt{\kappa_{E_1}} - \sqrt{\kappa_{E_2}}),
\]

Since \( \tau \) is arbitrary, the desired result follows.

Finally, we are ready to show (3.6).

Proof of (3.6). With the choice of \( q < N^{-1/3} \) by Lemma 3.5 we have that if \( \lambda_i, \gamma_i \geq \lambda_+ - N^c N^{-2/3} \) for some \( c > 0 \), then, with high probability

\[
|\lambda_i - \gamma_i| \leq N^{-c} N^{-2/3},
\]

for some \( \epsilon > 0 \). By the square root behavior of \( \varrho \), we have \( n(x) \sim (\lambda_1 - x)^{3/2} \) when \( x \) is near the edge. That is

\[
n(\gamma_j) = \frac{j}{N} \sim (\lambda_1 - \gamma_j)^{3/2}.
\]

Thus we have proved the case where \( j \leq N^c \) for small \( c \). Together with (III.18), we conclude (3.6). For the rest of \( j \)'s, one can refer to [14], so we omit the details since we only need the result near the right edge. \( \square \)

Appendix IV: Complete the proof of Theorem 3.3.

We introduce the notation of functional calculus. Specifically, for a function \( f(\cdot) \) and a matrix \( H \), \( f(H) \) denotes the matrix whose eigenvectors are those of \( H \) and eigenvalues are the values of \( f \) applied to each eigenvalue of \( H \).

First, we present a lemma for the approximation of the eigenvalue counting function. For any \( \eta > 0 \), define

\[
\vartheta_\eta(x) = \frac{\eta}{\pi(x^2 + \eta^2)} = \frac{1}{\pi} \text{Im} \frac{1}{x - \eta}.
\]

We notice that for any \( a, b \in \mathbb{R} \) with \( a \leq b \), the convolution of \( 1_{[a,b]} \) and \( \vartheta_\eta \) applied to the eigenvalues \( \lambda_i, i = 1, \ldots, N \) yields that

\[
\sum_{i=1}^{N} 1_{[a,b]} * \vartheta_\eta(\lambda_i) = \frac{N}{\pi} \int_{a}^{b} \text{Im} m_N(x + \eta)dx.
\]
In terms of the functional calculus notation, we have

\[ \sum_{i=1}^{N} 1_{[a,b]}(\lambda_i) = \text{Tr} 1_{[a,b]}(W), \quad \sum_{i=1}^{N} 1_{[a,b]} \ast \vartheta_\eta(\lambda_i) = \text{Tr} 1_{[a,b]} \ast \vartheta_\eta(W). \]

For \( a, b \in \mathbb{R} \cup \{-\infty, \infty\} \), define \( N(a,b) = N \int_a^b \vartheta_N(dx) \) as the number of eigenvalues of \( W \) in \([a,b]\).

The following lemma shows that \( \text{Tr} 1_{[a,b]}(W) \) can be well approximated by its smoothed version \( \text{Tr} 1_{[a,b]} \ast \vartheta_\eta(W) \) for \( a, b \) around the edge \( \lambda_+ \) so that the problem can be converted to comparison of the Stieltjes transform.

**Lemma IV.1.** Let \( \varepsilon > 0 \) be an arbitrarily small number. Set \( E_{\varepsilon} = \lambda_+ + N^{-2/3+\varepsilon} \), \( \ell_1 = N^{-2/3-3\varepsilon} \) and \( \eta_1 = N^{-2/3-9\varepsilon} \). Then for any \( E \) satisfying \( |E-\lambda_+| \leq \frac{3}{2} N^{-2/3+\varepsilon} \), it holds with high probability that

\[ |\text{Tr} 1_{[E,E_\varepsilon]}(W) - \text{Tr} 1_{[E,E_\varepsilon]} \ast \vartheta_\eta(W)| \leq C(N^{-2\varepsilon} + N(E - \ell_1, E + \ell_1)). \]

**Proof.** See Lemma 4.1 of [37] or Lemma 6.1 of [22].

Let \( q : \mathbb{R} \to \mathbb{R}_+ \) be a smooth cutoff function such that

\[ q(x) = \begin{cases} 1 & \text{if } |x| \leq 1/9, \\ 0 & \text{if } |x| \geq 2/9, \end{cases} \]

so \( q(x) \) is decreasing for \( x \geq 0 \). Then we have the following corollary.

**Corollary IV.2.** Let \( \varepsilon, \ell_1, \eta_1, E_\varepsilon \) be defined in Lemma IV.1. Set \( \ell = \ell_1 N^{2\varepsilon}/2 = N^{-2/3-\varepsilon}/2 \). Then for all \( E \) such that

\[ |E-\lambda_+| \leq N^{-2/3+\varepsilon}, \tag{IV.1} \]

the inequality

\[ \text{Tr} 1_{[E-\ell,E_\varepsilon]} \ast \vartheta_\eta(W) - N^{-\varepsilon} \leq N(E, \infty) \leq \text{Tr} 1_{[E-\ell,E_\varepsilon]} \ast \vartheta_\eta(W) + N^{-\varepsilon} \tag{IV.2} \]

holds with high probability. Furthermore, for any \( D > 0 \), there exists \( N_0 \in \mathbb{N} \) independent of \( E \) such that for all \( N \geq N_0 \),

\[ \mathbb{E} q\left\{ \text{Tr} 1_{[E-\ell,E_\varepsilon]} \ast \vartheta_\eta(W) \right\} \leq \mathbb{P}(N(E, \infty) = 0) \leq \mathbb{E} q\left\{ \text{Tr} 1_{[E+\ell,E_\varepsilon]} \ast \vartheta_\eta(W) \right\} + N^{-D}. \tag{IV.3} \]

**Proof.** Notice that for \( E \) satisfying \( |E-\lambda_+| \leq N^{-2/3+\varepsilon} \), we have \( |E-\ell-\lambda_+| \leq |E-\lambda_+| + \ell \leq \frac{3}{2} N^{-2/3+\varepsilon} \). Therefore, Lemma IV.1 holds with \( E \) replaced by any \( x \in [E-\ell,E] \). By the mean
the Markov’s inequality proves the upper bound of (\( IV.2 \)). For the lower bound, by using the upper bound of (\( IV.2 \)) and the fact that \( N(E, \infty) \) is an integer, we see that

\[
\mathbb{P}(\mathcal{N}(E, \infty) = 0) \leq \mathbb{P}(\text{Tr}1_{[E+\ell,E]} * \vartheta_{\eta_1}(W) \leq 2/9) + N^{-D},
\]

which together with Markov’s inequality proves the upper bound of (\( IV.3 \)). For the lower bound, using the upper bound of (\( IV.2 \)) and the fact that \( N(E, \infty) \) is an integer, we see that

\[
\mathbb{P}(\text{Tr}1_{[E-\ell,E]} * \vartheta_{\eta_1}(W) \leq 2/9) \leq \mathbb{P}(\mathcal{N}(E, \infty) \leq 2/9 + N^{-\varepsilon}) = \mathbb{P}(\mathcal{N}(E, \infty) = 0).
\]

This completes the proof of Corollary IV.2.
Proof of Theorem 3.3. Let $\varepsilon > 0$ be an arbitrary small number. Let $E = \lambda + sN^{-2/3}$ for some $|s| \leq N^\varepsilon$. Define $E_x = \lambda + N^{-2/3+\varepsilon}, \ell = N^{-2/3-\varepsilon}/2$ and $\eta_1 = N^{-2/3-3\varepsilon}$. Define $\tilde{W}$, $\tilde{N}$ to be the analogs of $W$, $N$ but with $\tilde{X}$ in place of $X$.

Using Corollary IV.2, we have
\[
\mathbb{E}(\text{Tr} \mathbf{1}_{[E-\ell,E]} \ast \partial \eta_1(\tilde{W})) \leq \mathbb{P}(\tilde{N}(E, \infty) = 0). \tag{IV.4}
\]

Recall that by definition
\[
\text{Tr} \mathbf{1}_{[E-\ell,E]} \ast \partial \eta_1(W) = \frac{N}{\pi} \int_{E-\ell}^{E+\ell} \text{Im} m_N(x + \eta_1) dx.
\]

Theorem 6.1 applied to the case where $E_1 = E - \ell$ and $E_2 = E_2$ shows that there exists $\delta > 0$ such that
\[
\mathbb{E}q(\text{Tr} \mathbf{1}_{[E-\ell,E]} \ast \partial \eta_1(W)) \leq \mathbb{E}q(\text{Tr} \mathbf{1}_{[E-\ell,E]} \ast \partial \eta_1(\tilde{W})) + N^{-\delta}. \tag{IV.5}
\]

Then applying Corollary IV.2 to the left-hand side of (IV.5), we have for arbitrarily large $D > 0$
\[
\mathbb{P}(N(E - 2\ell, \infty) = 0) \leq \mathbb{E}q(\text{Tr} \mathbf{1}_{[E-\ell,E]} \ast \partial \eta_1(W)) + N^{-D} \tag{IV.6}
\]
as $N$ is sufficiently large.

Using the bounds (IV.4), (IV.5) and (IV.6), we get that
\[
\mathbb{P}(N(E - 2\ell, \infty) = 0) \leq \mathbb{P}(\tilde{N}(E, \infty) = 0) + 2N^{-\delta}
\]
for sufficiently small $\varepsilon > 0$ and sufficiently large $N$. Recall that $E = \lambda + sN^{-2/3}$. The proof of the first inequality of Theorem 3.3 is thus complete. By switching the roles of $X$ and $\tilde{X}$, the second inequality follows. The proof is done. \hfill \Box

Appendix V: Some useful results.

Lemma V.1 (Sherman-Morrison formula). Let $A$ be an invertible matrix and $\mathbf{x}$ be a column vector such that $\mathbf{x}\mathbf{x}^*$ has the same size as $A$. Then
\[
(A + \mathbf{x}\mathbf{x}^*)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{x}\mathbf{x}^*A^{-1}}{1 + \mathbf{x}^*A^{-1}\mathbf{x}}. \tag{V.1}
\]

Proof. Multiplying $A + \mathbf{x}\mathbf{x}^*$ on both sides of (V.1), we get the identity $I = I$. \hfill \Box

Theorem V.1 (Moments of uniform spherical distribution). Let $\mathbf{u} = (u_1, \ldots, u_M)'$ be an $M$-dimensional random vector of spherical uniform distribution. Let $n \in \{1, \ldots, M\}$, $i_1, \ldots, i_n \in \{1, \ldots, M\}$ and $k_0, \ldots, k_n$ be positive integers. Defining $k_0 = k_1 + \cdots + k_n$, we have
\[
\mathbb{E}|u_{i_1}^{k_1} \cdots u_{i_n}^{k_n}| = \frac{\Gamma(M/2) \prod_{k=1}^n \Gamma((k+1)/2)}{\pi^{n/2} \Gamma((k_0+M)/2)} \leq C_{k_0,n} M^{-k_0/2},
\]
where $C_{k_0,n} > 0$ is a constant depending on $k_0$ and $n$ only. Moreover, if for some $j \in \{1, \ldots, n\}$, $k_j$ is odd. Then
\[
\mathbb{E}(u_{i_1}^{k_1} \cdots u_{i_n}^{k_n}) = 0.
\]
Proof of Theorem V.1. Write \( u = z/\|z\| \) for some \( N_M(0, I) \) random vector \( z \). Then \( \|z\| \) follows a half normal distribution with scale parameter \( M \) and \( u \) is independent with \( \|z\| \) (see e.g. Page 37 of [34]).

\[
\mathbb{E}(\|z\|^k | u_{i_1}^{k_1} \cdots u_{i_n}^{k_n} ) = \mathbb{E}|z_{i_1}^{k_1} \cdots z_{i_n}^{k_n} | = \frac{2^{k_0/2} \Gamma \left( \frac{k_0 + M + 2}{2} \right)}{\Gamma \left( \frac{M + 2}{2} \right)}.
\]

We see that

\[
\mathbb{E}(\|z\|^k) = \frac{2^{k_0/2} \Gamma \left( \frac{k_0 + M + 2}{2} \right)}{\Gamma \left( \frac{M + 2}{2} \right)}
\]

\[
= \begin{cases} 
2^{k_0/2} \Gamma \left( \frac{k_0 + M + 2}{2} \right) & \text{if } k_0 \text{ is even}, \\
2^{k_0/2} \Gamma \left( \frac{k_0 - 1}{2} \right) & \text{if } k_0 \text{ is odd},
\end{cases}
\]

\[
\geq \begin{cases} 
M^{k_0/2} & \text{if } k_0 \text{ is even}, \\
\sqrt{2} M^{k_0/2} & \text{if } k_0 \text{ is odd},
\end{cases}
\]

where for odd \( k_0 \), we have used Wendel’s inequality (see e.g. [38]) that

\[
\frac{\Gamma(x + s)}{\Gamma(x)} \geq x^s \left( \frac{x}{x + s} \right)^{1-s} \quad \forall x > 0, \ 0 < s < 1.
\]

The second result simply follows from the fact that

\[
\mathbb{E}(u_{i_1}^{k_1} \cdots u_{i_j}^{k_j} \cdots u_{i_n}^{k_n}) = \mathbb{E}(u_{i_1}^{k_1} \cdots (-u_{i_j}^{k_j}) \cdots u_{i_n}^{k_n}) = \mathbb{E}(u_{i_1}^{k_1} \cdots (-u_{i_j}^{k_j}) \cdots u_{i_n}^{k_n}) = 0.
\]

\( \square \)

Lemma V.2. Under Conditions 2.6 and 2.8, there exists a constant \( C > 0 \) such that \( \|XX^*\| \leq C \) with high probability.

Proof. Recall that \( XX^* = \Sigma^{1/2} U \mathcal{G}^2 U^* \Sigma^{1/2} \). We observe that

\[
\max_{i} \xi_i^2 = \max_{i} (\xi_i^2 - MN^{-1}) + MN^{-1} = MN^{-1} + O_{\prec}(N^{-1/2}). \quad (V.2)
\]

From (V.2), the assumption that \( \|\Sigma\| \) is bounded and the inequality \( \|XX^*\| \leq \|\Sigma\| \|UU^*\| \|\mathcal{G}\| \), we see that it suffices to show that there exists a constant \( C > 0 \) such that \( \|UU^*\| \leq C \) with high probability. Write the \( i, j \)-th entry of \( U \) as \( U_{ij} \). We see that \( \mathbb{E} \sqrt{M} U_{ij} = 0 \), \( \mathbb{E} (\sqrt{M} U_{ij})^2 = 1 \) and \( \mathbb{E} (\sqrt{M} U_{ij})^k < \infty \) for all \( k \geq 3 \). The desired result then follows from the arguments in [44] applied to the matrix \( MN^{-1} U \mathcal{G} \) which shows that for any \( k \in \{1, 2, \cdots \} \) and \( c > (1 + \sqrt{M/N})^2 \), \( \mathbb{E} \text{Tr}(MN^{-1}U \mathcal{G})^k \leq c^k \) for all large \( N \). Indeed, the matrix \( U \) violates the assumption in [44] that all entries of \( U \) are mutually independent. However, this will not invalidate the proof because the strategy of [44] is to bound the probability \( \mathbb{P}(\text{Tr}(MN^{-1}U \mathcal{G})^k > c^k) \), for which the only inputs needed are the bounds on

\[
|\mathbb{E} U_{i_1 j_1} U_{i_2 j_2} \cdots U_{i_k j_k} U_{i_1 j_1}|.
\]

We note that the only consequence that the violation of independence leads to is that the expectation of products of the \( U \) entries from the same column do not factor into products of
expectation. However, this is not a problem since the expectation of the product of dependent $U$ entries can be bounded by product of individual expectations. To be specific, we consider, without loss of generality, the $U$ entries from the first column. Let $m \leq M$ be a positive integer, $i_1, \ldots, i_m \in \{1, \ldots, M\}$ be distinct $m$ integers and $a_1, \ldots, a_m$ be $m$ positive integers. Denote $a_0 = a_1 + \cdots + a_m$. Then we claim that

$$E(U_{i_1}^{a_1} \cdots U_{i_m}^{a_m}) \leq E(U_{i_1}^{a_1}) \cdots E(U_{i_m}^{a_m}).$$

(V.3)

If one of $a_1, \ldots, a_m$ is odd, (V.3) is true because by symmetry of the $U$ entries, both $E(U_{i_1}^{a_1} \cdots U_{i_m}^{a_m})$ and $E(U_{i_1}^{a_1}) \cdots E(U_{i_m}^{a_m})$ are 0. Hence from now on, we assume that all of $a_1, \ldots, a_m$ are even.

Let $z \equiv (z_1, \ldots, z_M)^\ast$ be a real-valued $M$ dimensional standard normal random vector. Then we have that $z/\|z\| \sim U(S^{M-1})$ and $z/\|z\|$ are independent with $\|z\|$. It thus follows that

$$E(U_{i_1}^{a_1} \cdots U_{i_m}^{a_m})E(\|z\|^{a_0}) = E(z_{i_1}^{a_1}) \cdots E(z_{i_m}^{a_m}).$$

Therefore,

$$\frac{E(U_{i_1}^{a_1} \cdots U_{i_m}^{a_m})}{E(U_{i_1}^{a_1}) \cdots E(U_{i_m}^{a_m})} = \frac{E(\|z\|^{a_1}) \cdots E(\|z\|^{a_m})}{E(\|z\|^{a_0})} \leq \frac{E(\|z\|^{a_1}) \cdots E(\|z\|^{a_m})}{E(\|z\|^{a_0}) E(\|z\|^{|a_1-a_0|})} \leq \cdots \leq \frac{E(\|z\|^{a_1}) \cdots E(\|z\|^{a_m})}{E(\|z\|^{a_1}) \cdots E(\|z\|^{a_m})} = 1,$$

where the first to the last inequalities follow from the fact that $\|z\|^{a_k}$ and $\|z\|^{a_0-\sum_{j=1}^{k} a_j}$ are positive correlated for all $k = 1, \ldots, m-1$. Now the proof of the claim is complete and this lemma is shown. 

\[\square\]

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