Automorphisms of O’Grady’s Manifolds Acting Trivially on Cohomology

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Abstract

We determine the subgroup of automorphisms acting trivially on the second integral cohomology for hyperkähler manifolds which are deformation equivalent to O’Grady’s sporadic examples. In particular, we prove that this subgroup is trivial in the ten dimensional case and isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8$ in the six dimensional case.

Introduction

Automorphisms of irreducible symplectic or hyperkähler manifolds have recently been studied by numerous mathematicians pursuing varying objectives and using different techniques. A key to the understanding of the underlying geometry is usually played by understanding the induced action of an automorphism on the second integral cohomology: For any irreducible symplectic manifold $X$ the cohomology group $H^2(X, \mathbb{Z})$ carries a natural non-degenerate lattice structure and a weight-two Hodge structure. An automorphism of $X$ preserves both these structures and we obtain a homomorphism of groups:

$$\nu: \text{Aut}(X) \to O(H^2(X, \mathbb{Z})).$$

It is this homomorphism that allows us to study the geometry of $X$ and its automorphisms using lattice theory. This has been done very successfully and extensively in the case of K3 surfaces. From the strong Torelli theorem for K3 surfaces it follows that $\nu$ is injective in this case. Thus, by passing from the geometric picture on to the lattice side, we do not lose any information and we can classify automorphisms using lattice theory.

When constructing the first examples of higher dimensional irreducible symplectic manifolds, Beauville (cf. [Beau83][Prop. 10]) soon realised that the injectivity of $\nu$ holds also true in the case of Hilbert schemes of points on a K3. This fact was applied by Boissière–Sarti ([BST12]) to understand when an automorphism on such a Hilbert scheme is induced by an automorphism of the surface, constituting a first step in the classification of automorphisms of Hilbert schemes. These results should not lead to the idea that $\nu$ is injective in general. It was shown in [BNS11] that for generalised Kummer varieties of dimension $2n - 2$ the kernel of $\nu$ is generated by induced

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automorphisms coming from the underlying abelian surface, i.e. by translations by points of order \( n \) and by \(-\text{id}\). These automorphisms preserve the Albanese fibres of the Hilbert scheme of \( n \) points on the surface and they surely act trivially on cohomology. Thus, in general, \( \ker \nu \) is not trivial. But in the case of generalised Kummers we do not have an explicit construction of the automorphisms in \( \ker \nu \).

A fundamental step towards a better understanding of the kernel of \( \nu \) is the result by Hassett–Tschinkel stating that - as a group - this kernel is a deformation invariant of the manifold \( X \) (cf. [HT13] Thm. 2.1). It implies that we now know the group structure of \( \ker \nu \) for all manifolds of \( K3^{[n]} \)-type and of Kummer-type. Note, that for a general deformation of a generalised Kummer variety we do not have an explicit construction of the automorphisms in \( \ker \nu \).

There are two more known deformation types of hyperkähler manifolds. The first examples in both cases have been constructed by O'Grady (cf. [OGr99] for the ten dimensional example and [OGr03] for the example of dimension six). The main results of this article concern the kernel of the cohomological representation \( \nu \) for manifolds which are deformation equivalent to these manifolds.

In particular, we prove the following two theorems:

**Theorem (Theorem 2.1).** Let \( X \) be a manifold of \( Og_{10} \)-type. Then \( \nu \) is injective.

**Theorem (Theorem 3.2).** Let \( X \) be a manifold of \( Og_{6} \)-type. Then
\[
\ker \nu \cong (\mathbb{Z}/2\mathbb{Z})^8.
\]

Let us outline the idea of the proofs of the theorems. The geometry of O'Grady’s examples is more complicated and less understood compared to the case of Hilbert schemes of points or Generalised Kummers. Therefore a detailed analysis was needed. In the ten-dimensional case we consider a relative compactified Jacobian of degree four over a (five dimensional) linear system of genus five curves on a K3 surface. Its resolution of singularities is an irreducible symplectic manifold of \( Og_{10} \)-type. There are two main ingredients to the proof of the injectivity of \( \nu \). First we show that every automorphism in \( \ker \nu \) acts fibrewise on the Jacobian. Secondly we prove that the relative Theta divisor is rigid, thus must be preserved by any such automorphism. For the second step (rigidity of the relative Theta divisor) we first prove that the relative Theta divisor has the structure of a \( \mathbb{P}^1 \)-bundle. From this we deduce its rigidity using a criterion that should be known to the experts (cf. Lemma 1.5). We can then conclude that \( \ker \nu \) is trivial using the Torelli Theorem for Jacobians.

In the six-dimensional case we follow a similar idea, where this time the first step (fibrewise action) turns out to be easier to prove. Note that the automorphism induced by \(-\text{id}\) on the abelian surface acts trivially on \( Og_{6} \). The group \( \ker \nu \) is composed by automorphism induced by translation by two-torsion points on \( A \times A^* \), where \( A \) is the underlying abelian surface and \( A^* \) its dual.

We want to emphasise one more result – since it is a beautiful result on its own – which is a side product of the proof of the injectivity of \( \nu \) for \( Og_{10} \):

**Proposition.** Let \( S \) be a K3 surface being a double cover of \( \mathbb{P}^2 \) ramified along a sextic curve admitting a unique tritangent. Denote by \( H \) the pullback of \( O(1) \). Then the rational map \( |2H| \to \mathcal{M}_5 \) is injective.

See Proposition 2.3 for details.
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The motivation to prove these results and to write this article grew out of the attempt to detect (using lattice theory) certain induced automorphisms (on O’Grady’s manifolds) that have been constructed by the authors in [MW14]. Using the article at hand we managed to state a lattice theoretic criterion to detect induced automorphisms on O’Grady-type moduli spaces (cf. loc. cit. Prop. 4.5).

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1. Preliminaries

In this introductory section we will gather the most important background material and known results about the desingularised moduli spaces as introduced by O’Grady.

Let $S$ be a projective $K3$ or abelian surface. Mukai defined a lattice structure on $\tilde{H}^\ast (S, \mathbb{Z}) := H^{\ast, ev} (S, \mathbb{Z})$ by setting

$$(r_1, l_1, s_1) \cdot (r_2, l_2, s_2) := l_1 \cdot l_2 - r_1 s_2 - r_2 s_1,$$

where $r_1 \in H^0$, $l_i \in H^2$ and $s_i \in H^4$. This lattice is referred to as the Mukai lattice and we call vectors $v \in \tilde{H}^\ast (S, \mathbb{Z})$ Mukai vectors. The Mukai lattice is isometric to $U^4 \oplus E_8 (-1)^2$ if $S$ is a $K3$ and $U^4$ if $S$ is abelian.

Furthermore we may introduce a weight-two Hodge structure on $\tilde{H}^\ast (S, \mathbb{Z})$ by defining the $(1,1)$-part to be

$$H^{1,1}(S) \oplus H^0(S) \oplus H^4(S).$$

For an object $\mathcal{F} \in D^b(S)$ we define the Mukai vector of $\mathcal{F}$ by

$$v(\mathcal{F}) := \text{ch}(\mathcal{F}) \sqrt{\text{td}_S}.$$

The following theorem summarises the famous results about moduli spaces of stable sheaves on $K3$ surfaces.

**Theorem 1.1.** Let $S$ be a projective $K3$ and let $v$ be a primitive Mukai vector. Assume that $H$ is $v$-generic. Then the moduli space $M(v)$ of stable sheaves on $S$ with Mukai vector $v$ is a hyperkähler manifold which is deformation equivalent to the hilbert scheme of $n$ points, where $2n = v^2 + 2$. Furthermore we have an isometry of lattices

$$H^2(M(v), \mathbb{Z}) \cong v^\perp \subset \tilde{H}^\ast (S, \mathbb{Z})$$

(1.1)
which preserves the weight-two Hodge structures.

O’Grady studied a particular case of a non-primitive Mukai vector: Let \( v \in \tilde{H}^2(S, \mathbb{Z}) \) be a primitive Mukai vector of square 2.

**Theorem 1.2** O’Grady, Perego–Rapagnetta. *The moduli space \( M(2v) \) is a 2-factorial symplectic variety of dimension ten admitting a Beauville–Bogomolov form and a pure weight-two Hodge structure on \( H^2(M(2v), \mathbb{Z}) \) such that equation (1.1) holds. Furthermore it admits a symplectic resolution \( \tilde{M}(2v) \) which is a hyperkähler manifold that is not deformation equivalent to any of the known examples.*

In the case of abelian surfaces there is one more step to take:

Let \( A \) be an abelian surface and \( v \) a primitive Mukai vector. Then the moduli space \( M(v) \) is not simply connected. Indeed, for a sheaf \( \mathcal{F} \) we define \( \text{alb}(\mathcal{F}) := (\Sigma c_2(\mathcal{F}), \det(\mathcal{F})) \in A \times A^\ast \) yielding an isotrivial surjective map \( \text{alb} : M(v) \to A \times A^\ast \) which turns out to be the albanese map of \( M(v) \). The fibre \( K(v) := \text{alb}^{-1}(0, 0) \) is a hyperkähler manifold and equation (1.1) holds, if we compose the left hand side with the restriction to the fibre.

Finally we can also consider non-primitive Mukai vectors as above. So let us fix \( v \) with \( v^2 = 2 \). Then we have a commuting diagram of resolutions and albanese fibres:

\[
\begin{array}{ccc}
\tilde{K}(2v) & \xrightarrow{\text{alb}} & \tilde{M}(2v) \\
\downarrow & & \downarrow \\
K(2v) & \xrightarrow{\text{alb}} & M(2v),
\end{array}
\]

where \( \tilde{K}(2v) \) is a six dimensional hyperkähler manifold.

Next, we state two fundamental results concerning automorphisms of irreducible symplectic manifolds acting trivially on cohomology. Let \( X \) be an irreducible symplectic. We let \( \nu : \text{Aut}(X) \to O(H^2(X, \mathbb{Z})) \) be the cohomological representation.

**Proposition 1.3** Huybrechts. *The kernel of \( \nu \) is finite.*

**Proof.** [Huy99][Prop. 9.1] \qed

**Theorem 1.4** Hassett–Tschinkel. *The kernel of \( \nu \) is a deformation invariant of the manifold \( X \).*

**Proof.** [HT13][Thm. 2.1] \qed

Finally we include a result concerning rigid divisors on symplectic varieties.

**Lemma 1.5.** Let \( X \) be a \( \mathbb{Q} \)-factorial symplectic variety and let \( D \subset X \) be a prime divisor admitting a fibration \( g : D \to Z \) with generic fibre isomorphic to \( \mathbb{P}^1 \). Then \( D \) is rigid, i.e. \( h^0(X, \mathcal{O}(D)) = 1 \).

**Proof.** Since \( X \) has trivial canonical bundle, it is enough to proof that \( H^0(D, K_D) = 0 \). But the latter is isomorphic to \( H^0(Z, g_* K_D) \) and the restriction of \( K_D \) to a general fibre of \( g \) is \( \mathcal{O}_{\mathbb{P}^1}(-1) \), thus \( g_* K_D \) is 0. \qed
2. The ten dimensional case

In this chapter we will prove the following theorem:

**Theorem 2.1.** Let \( X \) be a manifold of \( \text{Og}_{10} \)-type. Then the cohomological representation

\[
\nu: \text{Aut}(X) \to O(H^2(X, \mathbb{Z}))
\]

is injective.

**Proof.** By **Theorem 1.4** the kernel of \( \nu \) is a deformation invariant, thus we may assume that \( X \) is the desingularisation of the moduli space \( M(2v) \), \( v = (0, H, 2) \) on a K3 surface \( S \) which is a double cover of \( \mathbb{P}^2 \) branched along a sextic curve \( \Gamma \) and where \( H \) denotes the pullback of \( O(1) \). That is, \( X \) is the desingularisation of the relative compactified Jacobian \( M(0, 2H, 4) = J^4(2H) \) of degree four over \( |2H| \) and it comes with a lagrangian fibration \( X \to |2H| \) which factors as the blow down followed by the map \( \pi: J^4(|2H|) \to |2H| \) assigning to a sheaf its support. We may choose the sextic \( \Gamma \) as follows. Let \( \Gamma^* \) be a plane quartic with an ordinary triple point. Its dual curve \( \Gamma \) is a sextic curve with a unique tritangent. Now, let \( \psi \) be an automorphism of \( X \) acting trivially on \( H^2 \). Let us prove that \( \psi = \text{id} \). First of all, \( \psi \) fixes the class of the exceptional divisor of the blow up \( X \to J^4(|2H|) \). Thus the automorphism descends to an automorphism \( \psi' \) of the singular relative Jacobian \( J^4(2H) \) still acting trivially on cohomology.

**Lemma 2.2.** The relative theta divisor \( \Theta_{|2H|} \) is an effective rigid divisor on \( J^4(|2H|) \).

**Proof.** We will use **Lemma 1.5** Let \( C \) be a general curve in \( |2H| \). The fibre \( \pi^{-1}(C) \) is isomorphic to the Jacobian \( J^4(C) \). The Theta divisor \( \Theta_C \) is given as

\[
\{O(p_1 + \cdots + p_4) \mid p_1, \ldots, p_4 \in C\}.
\]

We can therefore define a rational map

\[
\Theta_{|2H|} \to S\text{ym}^4 S,
\]

\[
O(p_1 + \cdots + p_4) \mapsto p_1 + \cdots + p_4.
\]

The general fibre of this map can be identified with the set of curves in \( |2H| \) that pass through four given points. These are four linear conditions cutting out a line.

Since the class of the pullback \( \pi^* O(1) \) is fixed by \( \psi' \), we see that \( \pi \) is \( \psi' \)-equivariant, that is, \( \psi' \) maps fibres of \( \pi \) to fibres. Since generically these fibres are Jacobians of smooth curves and – by the above lemma – the classes of the respective theta divisors are mapped to each other, the Torelli theorem for Jacobians yields an isomorphism of the underlying curves. We continue by showing that this already implies that \( \psi' \) acts fibrewise. We will therefore prove the following result which is interesting on its own.

**Proposition 2.3.** Let \( S \) be a K3 surface being a double cover of \( \mathbb{P}^2 \) ramified along a sextic curve that admits a unique tritangent. Denote by \( H \) the pullback of \( O(1) \). Then the rational map \( \varphi:|2H| \to \overline{M}_5 \) is injective.

**Proof.** First we note that the differential of the map is injective. This can be seen as follows: Let \( C \in |2H| \) be a stable curve. The differential of \( \varphi \) at the point corresponding to \( C \) is given as the coboundary map

\[
H^0(N_C|S) \to H^1(T_C)
\]
in the long exact cohomology sequence associated with the normal bundle sequence
\[ 0 \to T_C \to T_S|C \to N_C|S \to 0. \]

Thus it is enough to prove \( h^0(T_S|C) = 0 \). This can be done using the same method as in the second half of the proof of Proposition 1.2 in [CK13]. The rational map \( \varphi \) has an indeterminacy locus of codimension at least two and can be extended to a morphism from a suitable blow up of \( |2H| \). Thus it is enough to prove injectivity along a divisor which is saturated in the fibres (i.e. a divisor \( D \) such that \( \varphi^{-1}(\varphi(D)) = D \)). We will choose this divisor to be the symmetric square \( S^2|H| \subset |2H| \) corresponding to reducible curves. The rational map \( \varphi|_{S^2|H|} \) is given as the symmetric square of the map \( |H| \to \overline{M}_2 \). Thus we have reduced the problem to showing that the latter map is injective. Again, by blowing up \( |H| \) we can extend this map to a proper morphism \( \overline{H} \to \overline{M}_2 \). Furthermore, the discussion in Chapter 3C of [HIM98] shows that the exceptional locus in \( \overline{H} \) is mapped to the locus in \( \overline{M}_2 \) of curves having an elliptic tail and thus its image is disjoint from the image of the locus of stable curves in \( |H| \). Hence it is enough to prove injectivity in a single point (inside the stable locus). Since we assumed that \( \Gamma \) is a sextic with a unique triple tangent, we will choose this single point to correspond to the double cover \( C_0 \) of this triple tangent. But now the uniqueness of this triple tangent ensures that \( C_0 \) (as a member of \( |H| \)) is unique in its isomorphism class.

Thus \( \psi' \) acts fibrewise and fixes the class of the theta divisor. Since the divisor \( \Theta_{|2H|} \) is rigid, \( \psi' \) cannot be given by translations on the fibres. Thus, again by the Torelli theorem for Jacobians, we see that \( \psi' \) acts trivially on all fibres corresponding to smooth curves, that is, \( \psi' \) is the identity.

3. The six dimensional case

In the case of abelian surfaces the situation is slightly different and more complicated. First we show that there are, in fact, automorphisms acting trivially on the second cohomology:

**Lemma 3.1.** Let \( G_0 \) be the group generated by points of order 2 in \( A \times A^* \). Then we have an induced action of \( G_0 \) on \( \overline{K}(2v) \) which acts trivially on \( H^2 \).

**Proof.** For any Mukai vector \( v = (r, l, a) \) we have an induced action of \( A \times A^* \) on \( M(v) \). It has been described by Yoshioka in [Yos99] (cf. diagram (1.8)). First he defines the following map

\[ \tau_v: A \times A^* \to A \times A^*, \]

\[ (x, L) \mapsto (x', L') := (rx - \hat{\phi}_l(L), -\phi_l(x) - aL), \]

where \( \phi_l: A \to A^* \) and \( \hat{\phi}_l: A^* \to A \) are defined as usual (e.g. \( \phi_l(x) := t^*_xN \otimes N^\vee \) for some \( N \) with \( c_1(N) = l \)). Yoshioka then defines an action on \( M(v) \) as follows:

\[ \Phi: A \times A^* \times M(v) \to M(v), \]

\[ (x, L, F) \mapsto t^*_xF \otimes L'. \]

The introduction of \( \tau_v \) has the advantage that

\[ \text{alb}(\Phi(x, L, F)) = \text{alb}(F) + (nx, L^\otimes n), \]

where \( n := l^2/2 - ra = v^2/2 \).

Now, the crucial point in our situation is that since our Mukai vector is non-primitive, we have \( \tau_{2v} = 2\tau_v \) and we define an action \( \Phi' \) as above on \( M(2v) \) using \( \tau_v \) instead of \( \tau_{2v} \). Since \( (2v)^2/2 = 4 \),
we see that \( alb(\Phi(x, L, F)) = alb(F) + (2x, L^{\otimes 2}) \) and can immediately deduce that the action of \( G_0 \) preserves the Albanese fibres if \( x \) and \( L \) are two-torsion. The action on \( H^2(K(2v), \mathbb{Z}) \) can be computed via Lemma 1.34 of [MW14] and is easily seen to be trivial. The group \( G_0 \) certainly preserves the singular locus of both \( M(2v) \) and \( K(2v) \) and the action extends naturally to an action on the desingularisation (cf. the description of the normal bundle of the singular locus in [MW14] Prop. 4.3].

Thus we have \( G_0 \subseteq \ker \nu \). The converse is also true:

**Theorem 3.2.** Let \( X \) be a manifold of \( O_{g_0} \)-type. Then the kernel of the cohomological representation

\[
\nu: \text{Aut}(X) \to O(H^2(X, \mathbb{Z}))
\]

is isomorphic to \( G_0 := (A[2], A^*[2]) \cong (\mathbb{Z}/2\mathbb{Z})^8 \).

**Proof.** Let \( \Gamma \) be a generic curve of genus two and denote by \( (A, H) \) its Jacobian together with its principal polarisation given by a symmetric theta divisor. For the Mukai vector \( v = (0, H, 2) \) on \( A \), the moduli space \( M(2v) \) is the relative compactified Jacobian of degree 4 over the continuous (5 dimensional) system \( \{2H\} \). It comes with the fibration map \( \pi: M(2v) \to \{2H\} \). The \( A^* \) component of the Albanese map factors via \( \pi \) and the natural isotrivial fibration \( \{2H\} \to A^* \) with fibre \( \{2H\} \cong \mathbb{P}^3 \). Now, let us consider the Albanese fibre \( K(2v) \) and denote the restriction \( K(2v) \to \{2H\} \) of \( \pi \) with the same symbol. If \( C \in \{2H\} \) is a smooth curve (of genus 5), then its fibre \( \pi^{-1}(C) \) is the kernel \( K^4(C) \) of the natural summation map

\[
\mathcal{J}^4(C) \to A,
\]

\[
\mathcal{O}(\Sigma n_i p_i) \to \Sigma n_i p_i.
\]

We denote the Theta divisor on \( \mathcal{J}^4(C) \) by \( \Theta_C \), the relative Theta divisor on \( M(2v) \) by \( \Theta \) and by \( i \) the embedding \( K^4(C) \to \mathcal{J}^4(C) \). Note that if two curves \( C_1 \) and \( C_2 \) in \( \{2H\} \) have different singularities, then also \( K^4(C_1) \) and \( K^4(C_2) \) cannot be isomorphic.

The linear system \( |2H| \) on \( A \) defines a degree two map \( f: A \to |2H| \cong \mathbb{P}^3 \). Let \( C \in |2H| \) be a smooth curve (of genus 5), then its image \( f(C) \) is a smooth quartic curve of genus 3. Denote by \( i \) the covering involution on \( C \).

Now, let \( \psi \) be an automorphism of \( \tilde{K}(2v) \) acting trivially on cohomology. Let us prove \( \psi \in G_0 \).

Again, \( \psi \) descends along the blow down to an automorphism of \( K(2v) \) which we will denote by the same symbol. Also, the fibration \( \pi \) is \( \psi \)-equivariant and this time we can prove directly that (up to the action of \( A[2] \)) \( \psi \) is, in fact, preserving the fibres of \( \pi \):}

**Lemma 3.3.** Any automorphism of \( |2H| \cong \mathbb{P}^3 \) preserving its stratification (by analytical type of the singularities of the curves) is, in fact, induced by translation of a point in \( A[2] \).

**Proof.** Following Rapagnetta (cf. [Rap07] Prop. 2.1.3]) we define the strata

\[
R(1) := \{ C = H_x \cup H_{-x} \mid H_x \cap H_{-x} \text{ consists of two distinct points} \}
\]

and

\[
N := \{ C = 2H_x \mid x \in A[2] \}.
\]

The detailed analysis of Lemma 2.1.2 in loc. cit. shows that the (closure of) \( R(1) \) is, in fact, given as the projective dual to the (singular) Kummer surface \( Kum_s := A/(−\text{id}) \) which is, in fact, self-dual. Thus \( \overline{R(1)} \) is isomorphic to \( Kum_s \) and \( N \) corresponds to its 16 nodes. Thus we deduce
that we get an automorphism of $\text{Kum}_s$ preserving the set of nodes. Such an automorphisms can be lifted to the abelian surface $A$ and has to act there as translation by a two-torsion point.

Composing with the translation of an appropriate element in $A[2]$, we may thus assume that the action on $|2H|$ is trivial, that is, for all generic smooth $C \in |2H|$ we obtain an automorphism of $\mathcal{K}^4(C)$.

**Lemma 3.4.** The restriction $D := \Theta \cap K(2\nu)$ of the relative Theta divisor to $K(2\nu)$ is an effective rigid divisor.

**Proof.** On a general fibre $\mathcal{K}^4(C)$, the divisor $D$ is given by

$$D_C = \iota^* \Theta_C = \{ \mathcal{O}(p + \iota(p) + q + \iota(q)) \mid p, q \in C \}.$$

We can thus define a rational map

$$D \to \text{Sym}^2(\text{Kum}_s),$$

$$\mathcal{O}(p + \iota(p) + q + \iota(q)) \mapsto p + q.$$

The fibre over $p + q$ consists of curves in $|2H|$ that pass through $p$ and $q$, hence is a $\mathbb{P}^1$. Thus, by Lemma [L.5] we deduce that $D$ is, in fact, rigid.

**Remark 3.5.** A divisor similar to $D$ above appears in the Main Theorem of [Nag13].

Thus (up to the action of $A[2]$) any automorphism $\varphi$ in $\ker \nu$ induces a non-trivial automorphism $\varphi$ of $\mathcal{K}^4(C)$ preserving a divisor in $i^*|\Theta_C|$. Following Chapter 12 in [BL92] we will analyse this situation more in detail. We have the following sequence of abelian varieties

$$\mathcal{K}^4(C) \xrightarrow{\iota} \mathcal{J}^4(C) \xrightarrow{q} A.$$

The image of $\mathcal{K}^4(C)$ is given by $\iota(\mathcal{K}^4(C)) = \{ x \in \mathcal{J}^4(C) \mid \iota^* x = x \}$. Furthermore, we can embed $A$ into $\mathcal{J}^4(C)$ via $j := \phi_{\Theta_C} \circ \hat{q} \circ \phi_H$. Its image is given by $j(A) = \{ x \in \mathcal{J}^4(C) \mid \iota^* x = -x \}$. In this notation the induced action of $A^*$ on $\mathcal{J}^4(C)$ is simply given by translations by elements in $j(A)$.

Note that our automorphism $\varphi$ cannot be given by $-\text{id}$ on a generic fibre $\mathcal{K}^4(C)$ because this would yield a non-symplectic automorphism of $\overline{K}(2\nu)$ (which would not act trivially on the second cohomology). Furthermore, since $\varphi$ is of finite order (Proposition [L.3]) and generically $\mathcal{K}^4(C)$ is a simple abelian variety, it then has to be given by the translation $t_x$ by a point $x \in \mathcal{K}^4(C)$ of finite order. By the lemma above, $t_x$ preserves a divisor in $|i^*\Theta_C|$. Hence $t_x^* \mathcal{O}(i^*\Theta_C) \cong \mathcal{O}(i^*\Theta_C)$ and therefore $x$ is in the kernel of the map $\varphi_{i^*\Theta_C}$ associated with the polarisation (usually denoted by $K(i^*\Theta_C)$). But it is well-known (cf. [LP09], Sect. 2) that

$$K(i^*\Theta_C) \cong A \cap \mathcal{K}^4(C) = \{ x = -x = t_x^* x = -t_x^* x \} = A[2].$$

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