CLIFFORD ALGEBRAS AND NEW SINGULAR RIEMANNIAN FOLIATIONS IN SPHERES

MARCO RADESCHI

Dedicated to the memory of Sergio Console

Abstract. Using representations of Clifford algebras we construct indecomposable singular Riemannian foliations on round spheres, most of which are non-homogeneous. This generalises the construction of non-homogeneous isoparametric hypersurfaces due to Ferus, Karcher and Münzner.

A singular Riemannian foliation on a Riemannian manifold $M$ is, roughly speaking, a partition of $M$ into connected complete submanifold, not necessarily of the same dimension, that locally stay at a constant distance from each other. Singular Riemannian foliations on round spheres provide local models of general singular Riemannian foliations around a point.

An example of singular Riemannian foliation on round spheres is given by the decomposition into the orbits of an isometric group action, and such a foliation is called homogeneous.

A different family of singular Riemannian foliations on spheres is induced by isoparametric hypersurfaces. A hypersurfaces of $S^n$ is called isoparametric if it has constant principal curvatures. Isoparametric hypersurfaces were first studied by Cartan who classified those with $g \leq 3$ distinct principal curvatures, and a lot of progress has been made (cf. for example the surveys [2, 12]), even though the complete classification is still an important open problem. Every isoparametric hypersurface partitions the sphere into parallel hypersurfaces, which are isoparametric as well, and this partition is a special example of a singular Riemannian foliation. For a long time all the known codimension 1 singular Riemannian foliations from isoparametric hypersurfaces appeared to be orbits of some isometric group action on $S^n$, so much so that Cartan asked [3] whether every isoparametric hypersurface arised in this way. The question was answered in the negative by Ferus, Karcher and Münzner [4], who found infinite families of isoparametric foliations with 4 distinct principal curvatures defined in terms
of representations of Clifford algebras (the FKM examples) and showed that most of these examples are not homogeneous. In fact it is conjectured that every isoparametric hypersurface is either homogeneous or of FKM type.

As in the isoparametric case, classifying non-homogeneous singular Riemannian foliations seems a very hard and complex problem. A trivial way to obtain new foliations from old ones is called spherical join. Given singular Riemannian foliations \((S^n_i, F_i)\), \(i = 1, 2\), the spherical join gives a new foliation \((S^{n_1+n_2+1}, F_1 \ast F_2)\). Any foliation that cannot be written as a spherical join is called indecomposable, and every foliation can be written in an essentially unique way as a spherical join of indecomposable ones. Because of this, our main interest lies in finding non-homogeneous, indecomposable singular Riemannian foliations.

The only known indecomposable non-homogeneous singular Riemannian foliation, other than the FKM examples mentioned above, is the foliation in \(S^{15}\) given by the fibers of the Hopf fibration \(S^{15} \rightarrow S^8\). Recently A. Lytchak and B. Wilking proved, using a previous result of Wilking [13] and Grove-Gromoll [5], that this is the only non-homogeneous regular foliation, i.e., with leaves of the same dimension [9].

In this paper, as in [4], we use Clifford systems, closely related to representations of Clifford algebras, to produce a large class of indecomposable, non-homogeneous singular Riemannian foliations of arbitrary codimension, which in particular include all the previously known examples. Before we state the result, recall that a Clifford system can be thought of as a family \(C = (P_0, \ldots, P_m)\) of symmetric matrices in \(\mathbb{R}^{2l}\), equipped with an inner product \(\langle , \rangle\), such that \(P_i^2 = Id\) for all \(i = 0, \ldots, m\) and \(P_i P_j = -P_j P_i\) for \(i \neq j\). We define the map

\[
\pi_C : S^{2l-1} \longrightarrow \mathbb{R}^{m+1} \\
x \mapsto \left(\langle P_0 x, x \rangle, \ldots, \langle P_m x, x \rangle\right).
\]

**Theorem A.** Let \(C = (P_0, \ldots, P_m)\) be a Clifford system on \(\mathbb{R}^{2l}\). Then the image of \(\pi_C\) is contained in the unit disk \(\mathbb{D}_C\) around the origin in \(\mathbb{R}^{m+1}\), and the following hold:

1. The preimages of \(\pi_C\) are connected if \(l \neq m + 1\) and in this case they define a singular Riemannian foliation \((S^{2l-1}, F_C)\) whose leaf space is either the \(m\)-sphere \(S_C = \partial \mathbb{D}_C\) (if \(l = m\)) or the disk \(\mathbb{D}_C\) (if \(l > m + 1\)). In either case the induced metric on the quotient is a round metric of constant sectional curvature 4.

2. The foliation \((S^{2l-1}, F_C)\) is homogeneous if and only if \(m = 1, 2\) or \(m = 4\) and \(P_0 \cdot P_1 \cdot P_2 \cdot P_3 \cdot P_4 = \pm Id\), in which cases it is spanned by the orbits of the diagonal action of \(SO(k)\) on \(\mathbb{R}^k \times \mathbb{R}^k\) (\(m = 1\)), \(SU(k)\) on \(\mathbb{C}^k \times \mathbb{C}^k\) (\(m = 2\)) or \(Sp(k)\) on \(\mathbb{H}^k \times \mathbb{H}^k\) (\(m = 4\)).
When the leaf space is a sphere one recovers the Hopf fibrations \( \pi_C : S^{2m-1} \to S^m \), \( m = 2, 4, 8 \). When the leaf space is \( \mathbb{D}_C \) with the round metric (also hemisphere metric) the \( \pi_C \)-preimages in \( S^{2l-1} \) of the concentric spheres in \( \mathbb{D}_C \) give rise to the FKM family associated to the Clifford system \( C \).

A singular Riemannian foliation \( \mathcal{F}_0 \) on the \( m \)-sphere \( S_C = \partial \mathbb{D}_C \subseteq \mathbb{R}^{m+1} \) extends by homotheties to a singular Riemannian foliation \( \mathcal{F}_0^h \) on \( \mathbb{D}_C \) (with the hemisphere metric) and the \( \pi_C \)-preimages of the leaves of \( \mathcal{F}_0^h \) define a new foliation \( \mathcal{F}_0 \circ \mathcal{F}_C \). This is a special case of a more general construction of Lytchak [8, Sect. 2.5].

**Theorem B.** Let \( C \) a Clifford system on \( \mathbb{R}^{2l} \) and let \( (S^{2l-1}, \mathcal{F}_C) \) be the associated Clifford foliation.

1. If \( \mathcal{F}_0 \) is any singular Riemannian foliation on \( S_C \), then the foliation \( (S^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C) \) is a singular Riemannian foliation as well.
2. Let \( C_{8,1} \) and \( C_{9,1} \) denote, respectively, the unique Clifford systems \((P_0, \ldots, P_8)\) on \( \mathbb{R}^{16} \) and \((P_0, \ldots, P_9)\) on \( \mathbb{R}^{32} \). If \( C \neq C_{8,1}, C_{9,1} \) then \((S^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)\) is homogeneous if and only if both \( \mathcal{F}_0 \) and \( \mathcal{F}_C \) are homogeneous. If \( C = C_{9,1} \) and \((S^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)\) is homogeneous, then \( \mathcal{F}_0 \) is homogeneous.

We call the foliations \( \mathcal{F}_C \) described above Clifford foliations, and the foliations \( \mathcal{F}_0 \circ \mathcal{F}_C \) composed foliations. Notice that in a Clifford foliation the set of singular leaves is a connected, smooth, non totally geodesic submanifold of \( S^{2l-1} \). This can be shown never to be the case for decomposable foliations, and therefore every Clifford foliation is indecomposable.

**Example 1.** If \( \mathcal{F}_0 \) is a trivial foliation whose leaves consist of points, \( \mathcal{F}_0 \circ \mathcal{F}_C = \mathcal{F}_C \) and in particular every Clifford foliation is a composed foliation as well. If \( \mathcal{F}_0 \) is the trivial foliation consisting of one leaf, \( \mathcal{F}_0 \circ \mathcal{F}_C \) is the codimension 1 FKM examples corresponding to the Clifford system \( C \). Since the foliation induced by the Hopf fibration \( S^{15} \to S^8 \) is of the form \( \mathcal{F}_C \) all previously known examples of indecomposable, non-homogeneous foliations are of the form \( \mathcal{F}_0 \circ \mathcal{F}_C \), with \( \mathcal{F}_0 \) trivial.

**Example 2.** Let \((S^{15}, \mathcal{F}_C)\) be the Clifford foliation with quotient \( S^8 \). The group \( \text{SO}(3) \times \text{SO}(3) \) acts on \( \mathbb{R}^9 = \mathbb{R}^3 \otimes \mathbb{R}^3 \) via the tensor product representation, and the restriction of this action on the unit sphere induces a (homogeneous) foliation \((S^8, \mathcal{F}_0)\) whose quotient space is a spherical triangle of curvature 1 with angles \( \pi/3, \pi/3, \pi/2 \). The composed foliation \( \mathcal{F}_0 \circ \mathcal{F}_C \) is thus a singular Riemannian foliation on \( S^{15} \) whose quotient is isometric to a spherical triangle of curvature 4, with angles \( \pi/3, \pi/3, \pi/2 \). Such a quotient does not appear as a quotient of an isometric group action of cohomogeneity 2 (see Straume classification [11, Table II]) and therefore \( \mathcal{F}_0 \circ \mathcal{F}_C \) is non-homogeneous. Moreover, such a triangle does not admit submetries onto a segment and therefore
$\mathcal{F}_0 \circ \mathcal{F}_C$ is not contained in any codimension 1 foliation. To our knowledge, this is the only known singular Riemannian foliation of codimension 2 with this property.

**Example 3.** Let $C$ be a Clifford system on $\mathbb{R}^{2l}$, and $(S_C, \mathcal{F}_0)$ be a singular Riemannian foliation without 0-dimensional leaves. Then the leaf space $S_C/\mathcal{F}_0$ has diameter $\leq \frac{\pi}{2}$, and the composed foliation $(S^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)$ has quotient of diameter strictly smaller than $\pi/2$. Such foliations are called irreducible, because in the homogeneous setting they appear as quotient of irreducible representations. Since decomposable foliations have quotient with diameter $\geq \pi/2$, it follows in particular that these examples are indecomposable.

Unlike the FKM examples, inequivalent Clifford system give rise to different Clifford foliations, see Proposition 4.2. Moreover, Clifford foliations can be geometrically characterized as the only singular Riemannian foliations on spheres whose quotient is a sphere or a hemisphere of curvature 4.

**Theorem C.** Let $(S^n, \mathcal{F})$ be a singular Riemannian foliation whose quotient is either a sphere or a hemisphere of constant curvature 4. Then $\mathcal{F}$ is congruent to a Clifford foliation $\mathcal{F}_C$, for some Clifford system $C$.

The paper is structured as follows. After preliminary Section 1 we provide the construction of the Clifford foliations in Section 2 and that of composed foliations in Section 3. In Section 4 we also prove that both Clifford and composed foliations are singular Riemannian foliations, thereby finishing the proofs of the first statements of Theorems A and B. In Sections 4 we prove Theorem C and finally in Section 5 we prove the homogeneity statements in Theorems A and B. The last Section 6 is devoted to pointing out some properties that make Clifford and composed foliations very different from homogeneous ones. With the exception of the Hopf fibration $S^{15} \to S^8$, the quotient of every other previously known indecomposable foliation was isometric to the orbit space of a group action, and therefore shared many properties of homogeneous foliations. The main goal of this last section is thus to provide evidence that homogeneous foliations are indeed very special. In Section 6 we also show how Clifford foliations and some composed foliations descend to foliations in complex and quaternionic projective spaces.

**Acknowledgements.** The author thanks Alexander Lytchak for many helpful conversations and for inspiring Section 6 and Wolfgang Ziller and Marcos Alexandrino for their interest and many comments on a preliminary version of this work. The author also thanks Luigi Vezzoni and the whole Mathematics Department of Università di Torino for the hospitality during his visit, where part of this work was produced.
1. Preliminaries

1.1. Singular Riemannian foliations.

Definition 1.1. Let $M$ be a Riemannian manifold, and $\mathcal{F}$ a partition of $M$ into complete, connected, injectively immersed submanifolds, called leaves. The pair $(M, \mathcal{F})$ is called:

- a **singular foliation** if there is a family of smooth vector fields $\{X_i\}$ that span the tangent space of the leaves at each point.
- a **transnormal system** if any geodesic starting perpendicular to a leaf, stays perpendicular to all the leaves it meets. Such geodesics are called horizontal geodesics.
- a **singular Riemannian foliation** if it is both a singular foliation and a transnormal system.

Given a singular foliation $(M, \mathcal{F})$, the **space of leaves**, denoted by $M/\mathcal{F}$, is the set of leaves of $\mathcal{F}$ endowed with the topology induced by the canonical projection $\pi : M \to M/\mathcal{F}$ that sends a point $p \in M$ to the leaf $L_p \in \mathcal{F}$ containing it.

On a singular Riemannian foliation $(M, \mathcal{F})$ it is possible to define a stratification, as follows. For each nonnegative integer $r$ define $\Sigma_r$ to be the union of leaves of dimension $r$. The connected components of each $\Sigma_r$ are (possibly noncomplete) submanifolds, and such connected components are called strata of $(M, \mathcal{F})$. We denote by $\dim \mathcal{F}$ the maximum dimension of the leaves in $\mathcal{F}$, and call regular leaf a leaf of maximal dimension and regular point a point in a regular leaf. The set of regular leaves $\Sigma_{\dim \mathcal{F}}$ is open, dense and connected, and therefore it defines a stratum which we call the regular stratum.

A singular Riemannian foliation $(M, \mathcal{F})$ is called **closed** if all the leaves of $\mathcal{F}$ are closed. If $(M, \mathcal{F})$ is a closed foliation then all the leaves are at a constant distance from each other, and the space of leaves $M/\mathcal{F}$ has the structure of a Hausdorff metric space. Moreover, the strata $\Sigma$ project to orbifolds in $M/\mathcal{F}$, and the restriction of $\pi : M \to M/\mathcal{F}$ to $\Sigma$ is a Riemannian submersion. In particular, $M/\mathcal{F}$ is stratified by orbifolds $\Sigma/\mathcal{F}$, and the regular stratum $\Sigma_{\dim \mathcal{F}}/\mathcal{F}$ is open and dense in $M/\mathcal{F}$.

A typical example of singular Riemannian foliation is provided by the orbit decomposition of a Riemannian manifold $M$ into the orbits of an isometric actions of a connected Lie group. Such foliations are called homogeneous.

Finally, we define two singular Riemannian foliations $(M, \mathcal{F})$, $(M', \mathcal{F}')$ congruent if there is an isometry of $M \to M'$ that takes leaves of $\mathcal{F}$ isometrically onto leaves of $\mathcal{F}'$. 
1.2. Clifford algebras and Clifford systems. In this section we recall the basic definitions and results on Clifford algebras and Clifford systems, which we will need later on, see reference [4, Section 3].

The Clifford algebra $Cl_m(\mathbb{R}) = Cl(V)$ is constructed from a (real) vector space $V$ of dimension $m$ with a positive definite inner product $\langle , \rangle$ and is defined by the quotient of the tensor algebra $T(V)$ by the ideal $x \otimes y + y \otimes x - 2 \langle x, y \rangle 1$, where $1$ is the unit element in $T(V)$. The vector space $V$ naturally embeds in $Cl(V)$, and every $x, y \in V$ satisfy the relation

$$x \otimes y + y \otimes x = 2 \langle x, y \rangle 1.$$

A representation of a Clifford algebra $Cl_m(\mathbb{R})$, or Clifford module, is an algebra homomorphism $\rho : Cl_m(\mathbb{R}) \rightarrow \text{End}(\mathbb{R}^n)$. Two representations $\rho, \rho'$ are said to be equivalent if there is an isomorphism $A \in \text{GL}(\mathbb{R}^n)$ such that $\rho' = A^{-1} \circ \rho \circ A$. The restriction

$$\rho|_V : V \rightarrow \text{End}(\mathbb{R}^n)$$

will be called Clifford system on $\mathbb{R}^n$, and denoted by $C$. We will also denote by $\mathbb{R}_C$ the image $\rho(V)$, and call $m = \text{dim}(V)$ the rank of $C$. Given an orthonormal basis $x_0, \ldots, x_{m-1}$ of $V$, the images $P_i = \rho(x_i) \in \mathbb{R}_C$ are matrices that satisfy the relations $P_i^2 = Id$ and $P_i P_j = -P_j P_i$ for $i \neq j$.

**Lemma 1.2.** It is possible to find an inner product $\langle , \rangle$ on $\mathbb{R}^n$ such that $\mathbb{R}_C$ consists of symmetric matrices.

**Proof.** Let $x_0, \ldots, x_{m-1}$ be an basis of $V$ and let $P_i = C(x_i)$, $i = 0, \ldots, m - 1$. Fix an inner product $\langle , \rangle$ on $\mathbb{R}^n$, and define a new inner product

$$\langle u, v \rangle_0 = \langle u, v \rangle + \langle P_0 u, P_0 v \rangle.$$

We can define new inner products $\langle , \rangle_i$, $i = 1, \ldots, m - 1$ inductively, by letting

$$\langle u, v \rangle_i = \langle u, v \rangle_{i+1} + \langle P_i u, P_i v \rangle_i,$$

and finally define $\langle , \rangle = \langle , \rangle_{m-1}$. It can be easily checked that $\langle u, v \rangle = \langle P_i u, P_i v \rangle$ for every $u, v \in \mathbb{R}^n$ and every $i = 0, \ldots, m - 1$. Then

$$\langle P_i u, v \rangle = \langle P_i u, P_i^2 v \rangle = \langle u, P_i v \rangle \quad \forall u, v \in \mathbb{R}^n, i = 0, \ldots, m - 1$$

and therefore the matrices $P_i$ are also symmetric with respect to $\langle , \rangle$. In particular, any element of $\mathbb{R}_C$ is symmetric. □

Given a Clifford system $C$ on $\mathbb{R}^n$, we will implicitly fix an inner product $\langle , \rangle$ on $\mathbb{R}^n$ such that $\mathbb{R}_C \subseteq \text{Sym}^2(\mathbb{R}^n)$. If one endows $\text{Sym}^2(\mathbb{R}^n)$ with the inner product $\langle A, B \rangle = \frac{1}{n} \text{tr}(AB)$, the map $C : V \rightarrow \mathbb{R}_C \subseteq \text{Sym}^2(\mathbb{R}^n)$ is an isometry, i.e., $\langle C(x), C(y) \rangle = \langle x, y \rangle$.

Let $S_C$ denote the unit sphere in $\mathbb{R}_C$. For any $P \in S_C$, $P^2 = Id$, and therefore $P$ has eigenvalues $\pm 1$, with eigenspaces $E_{\pm}(P)$. If $Q \perp P$, $PQ = -QP$ and therefore $Q$
takes the positive eigenspace $E_+(P)$ isomorphically into the negative eigenspace $E_-(P)$, and vice versa. In particular, $\dim E_+(P) = \dim E_-(P)$ and since $\mathbb{R}^n$ splits as a sum $E_+(P) \oplus E_-(P)$, $n$ is always even dimensional, and we will write $n = 2l$.

Given two Clifford systems $C : V \to \text{Sym}^2(\mathbb{R}^2l)$, $C' : V \to \text{Sym}^2(\mathbb{R}^2r)$ on the same Clifford algebra $C\ell(V)$, one can produce a new Clifford system $C \oplus C' : V \to \text{Sym}^2(\mathbb{R}^{2(l+r)})$ by letting $(C \oplus C')(x) = (C(x), C'(x))$. We call $C \oplus C'$ a reducible Clifford system. Any Clifford system that cannot be written as a non trivial sum is called irreducible. If $C$ is an irreducible Clifford system of rank $m + 1$ on $\mathbb{R}^{2l}$ then $l = \delta(m)$, where the function $\delta(m)$ is given as follows

\begin{equation}
\begin{array}{c|cccccccc}
m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 + n \\
\delta(m) & 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16\delta(n)
\end{array}
\end{equation}

Two Clifford systems $C, C' : V \to \text{Sym}^2(\mathbb{R}^n)$ are algebraically equivalent if there is an isometry $A \in O(\mathbb{R}^n)$ such that $C' = A^{-1} \circ C \circ A$, and geometrically equivalent if there is an isometry $A \in O(\mathbb{R}^n)$ such that $R_{C'} = R_{A^{-1} \circ C \circ A}$. If $m \not\equiv 0(\text{mod} 4)$ there is a unique irreducible Clifford system on $\mathbb{R}^n$ up to algebraic equivalence, and in particular geometric equivalence. For $m \equiv 0(\text{mod} 4)$, there are two algebraic equivalence classes of Clifford systems, such that if $(P_0, P_1, \ldots, P_m)$ is the basis for one such class then the other can be identified with $(-P_0, P_1, \ldots, P_m)$. In particular, there is one geometric class of irreducible Clifford systems for $m \equiv 0(\text{mod} 4)$ as well.

Any Clifford system is algebraically equivalent to a direct sum of irreducible ones. In particular if $C$ is a Clifford system of rank $m + 1$ on $\mathbb{R}^{2l}$ then $l = k\delta(m)$ for some $k > 0$ and if $m \not\equiv 0(\text{mod} 4)$ there is only one algebraic equivalence of Clifford systems for each $k$. For $m \equiv 0(\text{mod} 4)$, however, a Clifford system of rank $m + 1$ on $\mathbb{R}^{2l}$, $l = k\delta(m)$ can be obtained by taking combinations of the two algebraically distinct irreducible Clifford systems, resulting in $\left\lfloor \frac{k}{2} \right\rfloor + 1$ geometrically distinct Clifford systems. These can be told apart by the invariant $|\text{tr}(P_0 \cdot P_1 \cdots \cdot P_m)|$, where $(P_0, \ldots, P_m)$ is a basis of $C$, which takes exactly the $\left\lfloor \frac{k}{2} \right\rfloor + 1$ distinct values $k - 2j$, $2j \leq k$.

We will use the notation $C_{m,k}$ to denote a Clifford system of rank $m + 1$ on $\mathbb{R}^{2k\delta(m)}$. By the discussion above, when $m \not\equiv 0(\text{mod} 4)$ or $k = 1$ the notation $C_{m,k}$ uniquely determines the Clifford system up to geometric equivalence.

Finally, we we recall that if $C$ is a Clifford system and $P, Q$ are elements in $\mathbb{R}_C$, then $\langle Px, Qx \rangle = \langle P, Q \rangle \|x\|^2$.

1.3. The construction of the FKM examples. In \[4\], the authors use Clifford system to produce new examples of isoparametric hypersurfaces in spheres with 4 principal
curvatures. In the following we will refer to them as the FKM examples. Given a Clifford system $C$ of rank $m + 1$ on $\mathbb{R}^l$, $l = k\delta(m)$ and fixing a basis $P_0, \ldots, P_m$ of $C$, they define a polynomial $F : \mathbb{R}^l \to \mathbb{R}$ by

$$F(x) = \langle x, x \rangle - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2$$

This polynomial restricts to a map $F_0 : S^{2l-1} \to [-1, 1]$, such that the preimages of the level sets are smooth, closed submanifolds of $S^{2l-1}$. These submanifolds depend on the values of $m$ and $l$.

- If $l > m + 1$, $F_0$ is surjective, and the level sets are connected. The regular level sets form a family of isoparametric submanifolds, while the preimages $M_\pm = F_0^{-1}(\pm 1)$ are the focal submanifolds.
- If $l = m + 1$ (which can only happen for $(m, k) \in \{(1, 2), (3, 1), (7, 1)\}$) then $F_0$ is still surjective, but the fibers of $F_0$ are disconnected except for $M_-$, which is a hypersurface.
- If $l = m$ (which can only happen for $(m, k) \in \{(2, 1), (4, 1), (8, 1)\}$) then $F_0 \equiv -1$, and $M_- = S^{2l-1}$.

The map $F_0$ restricts to a submersion in the regular part $S^{2l-1} \setminus (M_+ \cup M_-)$. This map is not a Riemannian submersion, nevertheless it can be modified to become one, and its quotient is an interval of length $\pi/4$.

Restricting to the generic case $l > m + 1$, most of these examples are non homogeneous. More specifically, given a Clifford system $C$ of rank $m + 1$ on $\mathbb{R}^l$, $l = k\delta(m)$, the corresponding FKM example is homogeneous only for the following values of $(m, k)$ (cf. [4, Section 4.4], [7, Table F]):

\[
\begin{array}{c|c|c|c|c}
(m, k) & (1, k) & (2, k) & (4, k) & (9, 1) \\
condition & k \geq 2 & k \geq 1 & k \geq 1 & P_0 P_1 P_2 P_3 P_4 = \pm \text{Id} \\
\end{array}
\]

Where $(P_0, \ldots, P_4)$ is a basis of $C$.

2. The construction

We now proceed to define the new examples of singular Riemannian foliations of higher codimension. Let $C$ be a Clifford system of rank $m + 1$ on $\mathbb{R}^l$, $l = k\delta(m)$. On the unit sphere $S^{2l-1} \subseteq \mathbb{R}^l$ (endowed with the canonical inner product which we also denote by $\langle \cdot, \cdot \rangle$), consider the function

$$\pi_C : S^{2l-1} \longrightarrow \mathbb{R}^C = \mathbb{R}^{m+1}$$
that takes $x \in S^{2l-1}$ to the unique element $\pi_C(x) \in \mathbb{R}_C$ defined by the property
\begin{equation}
\langle \pi_C(x), P \rangle = \langle Px, x \rangle \quad \forall P \in \mathbb{R}_C
\end{equation}
Fixing a basis $(P_0, \ldots, P_m)$ of $\mathbb{R}_C$, the map $\pi_C$ can be rewritten as
\[
\pi_C(x) = \left( \langle P_0 x, x \rangle, \ldots, \langle P_m x, x \rangle \right).
\]

**Lemma 2.1.** The image of $\pi_C$ is contained in the unit disk $\mathbb{D}_C$ of $\mathbb{R}_C$.

**Proof.** Let $x_0 \in S^{2l-1}$ and $P = \pi_C(x_0)$. It is enough to show that $\|P\| \leq 1$. By the defining equation (2.1) we have
\begin{equation}
\|P\|^2 = \langle P, P \rangle = \langle Px_0, x_0 \rangle \leq \|P\| \cdot \|x_0\|^2 = \|P\|
\end{equation}
Hence $\|P\| \leq 1$ as we wanted. \qed

The relation with the polynomial $F_0$ of Ferus, Karcher, Munzner is explicit, as $F_0$ factors through $\pi_C$ as $F_0 = f \circ \pi_C$, where $f : \mathbb{R}_C \to \mathbb{R}$ is the polynomial
\begin{equation}
f(P) = 1 - 2\|P\|^2.
\end{equation}

We endow $\mathbb{D}_C$ with a hemisphere metric of constant sectional curvature 4, so that the boundary $S_C = \partial \mathbb{D}_C$ is totally geodesic. From now on, we will always assume that the metric on $\mathbb{D}_C$ is the round one.

**Remark 2.2.** By equation (2.3), the preimages under $\pi_C$ of the concentric spheres in $\mathbb{D}_C$ give back the FKM family associated to the Clifford system $C$. In particular the preimage of the origin is the focal manifold $M_+$ and the preimage of the boundary is $M_-$.

**Remark 2.3.** If $C$ and $C'$ are algebraically equivalent Clifford systems, by definition there exists an orthogonal map $A \in O(\mathbb{R}^{2l})$ such that $\pi_{C'} = \pi_C \circ A$. In particular, up to orthogonal transformation $\pi_C$ only depends on the algebraic equivalence class of $C$. We will see in Section 4.2 that the converse is also true, namely the geometric equivalence class of $C$ is uniquely determined by $\pi_C$.

**Proposition 2.4.** Given a Clifford system $C$ of rank $m+1$ on $\mathbb{R}^{2l}$, $l = k\delta(m)$, the corresponding map
\begin{equation}
\pi_C : S^{2l-1} \to \mathbb{D}_C
\end{equation}
satisfies:

1. The preimage of $P \in S_C = \partial \mathbb{D}_C$ is the unit sphere $E^1_+(P)$ in the positive eigenspace $E_+(P)$. Moreover, the restriction $\pi|_{M_-} : M_- \to S_C$ is a submersion.
2. If $l = m$, the image of $\pi_C$ is $S_C$. 

(3) If \( l \geq m + 1 \), the map \( \pi_C \) is surjective onto \( \mathbb{D}_C \) and its restriction to the regular part is a submersion.

(4) If \( l > m + 1 \), the fibers of \( \pi_C \) are connected.

(5) If \( l = m + 1 \), \( C \) can be extended to a Clifford system \( C' \) of rank \( m + 2 \), the image of \( \pi_{C'} \) is \( \mathbb{S}^{m+1} \) and \( \pi_C \) factors as \( \pi_C = Pr \circ \pi_{C'} \), where \( Pr : \mathbb{S}_{C'} \to \mathbb{D}_C \) is given by

\[
Pr(x_1, \ldots, x_m, x_{m+1}) = (x_1, \ldots, x_m).
\]

In particular, the fibers of \( \mathcal{F}_C \) are not connected.

Proof. Let \( x_0 \in \mathbb{S}^{2l-1} \) and \( P = \pi_C(x_0) \). \( P \) lies in \( \mathbb{S}_C \) and is only if \( ||P|| = 1 \). The inequality (2.2) is then an equality, and in particular \( \langle Px_0, x_0 \rangle = ||Px_0|| \cdot ||x_0|| \), which implies that \( x_0 \) is an eigenvector for \( P \). Since \( P \) has eigenvalues \( \pm 1 \), \( Px_0 = \pm x_0 \), and again from \( \langle Px_0, x_0 \rangle = 1 \) is must be \( Px_0 = x_0 \).

On the other hand, if \( x_0 \in E_+^1(P) \) for some \( x_0 \in \mathbb{S}^{2l-1} \), by (2.1)

\[
\langle P, \pi_C(x_0) \rangle = \langle Px_0, x_0 \rangle = 1
\]

and therefore \( P = \pi_C(x_0) \). Thus the whole unit sphere \( E_+^1(P) \) projects to \( P \). In particular, \( M_- \) embeds in \( \mathbb{S}^{2l-1} \times \mathbb{S}_C \) as \( M_- = \{(x, P) \in \mathbb{S}^{2l-1} \times \mathbb{S}_C|Px = x\} \) and \( \pi_C \) is just the projection onto the second factor, which can easily be checked to be a submersion.

Fix an orthonormal basis \( (P_0, \ldots, P_m) \) of \( \mathbb{R}_C \). Given \( x \in \mathbb{S}^{2m-1} \), let \( x = ax_+ + bx_- \) where \( x_\pm \in E_\pm(P_0) \) are unit vectors, and \( a^2 + b^2 = 1 \). We want to prove that

\[
\sum_{i=0}^{m} \langle P_i x, x \rangle^2 = 1.
\]

For \( i = 0 \) we have \( \langle P_0 x, x \rangle = a^2 - b^2 \), while for \( i = 1, \ldots, m \) we compute

\[
\langle P_i x, x \rangle = a^2 \langle P_i x_+, x_+ \rangle + b^2 \langle P_i x_-, x_- \rangle + 2ab \langle P_i x_+, x_- \rangle
\]

On the one hand, since \( P_i x_\pm \in E_+(P_0) \) for \( i = 1, \ldots, m \), the equation above simplifies as \( \langle P_i x, x \rangle = 2ab \langle P_i x_+, x_- \rangle \). On the other hand, since \( m = l = \dim E_-(P_0) \), the vectors \( P_1 x_+, \ldots, P_m x_+ \) form an orthonormal basis of \( E_-(P_0) \) and thus

\[
\sum_{i=1}^{m} \langle P_i x, x \rangle^2 = 4a^2b^2 \sum_{i=1}^{m} \langle P_i x_+, x_- \rangle^2 = 4a^2b^2.
\]

Therefore

\[
\sum_{i=0}^{m} \langle P_i x, x \rangle^2 = \langle P_0 x, x \rangle^2 + \sum_{i=1}^{m} \langle P_i x, x \rangle^2 = (a^2 - b^2)^2 + 4a^2b^2 = 1
\]

as we wanted. Moreover, since the preimage of any \( P \in \mathbb{S}_C \) consists of the unit sphere in \( E_+(P) \), it is non empty and thus the image of \( \pi_C \) is \( \mathbb{S}_C \).
Fix an orthonormal basis \((P_0, \ldots, P_m)\) of \(\mathbb{R}_C\) and let \(x_+ \in E^1_+(P_0)\). As \(P_0\) anticommutes with \(P_i\), \(i > 1\), we have \(P_ix_+ \in E_-(P_0)\). If \(l \geq m + 1\) there is a unit vector \(x_- \in E_-(P_0)\) which is perpendicular to \(P_1x_+, \ldots, P_mx_+, \) and let \(x = \sqrt{\frac{2}{l}}(x_+ + x_-) \in \mathbb{S}^{2l-1}\). It is easy to check that \(\pi_C(x) = 0\), and therefore the preimage of the origin (which is the manifold \(M_+\) as observed in Remark 2.2) is nonempty in this case. Moreover, the set

\[ M_{(Q, t)} = \{\cos(t)x + \sin(t)Qx, \ x \in M_+\}, \quad Q \in \mathbb{S}_C, \ t \in [0, \pi/4] \]

is contained in (and by dimensional reasons it coincides with) the preimage of the point \(\sin(2t)Q\). Since any point in \(\mathbb{D}_C\) can be written in this way, it follows that \(\pi_C\) is surjective onto \(\mathbb{D}_C\). Moreover for any \(P \in \mathbb{S}_C\) the gradient of \(x \mapsto \langle Px, x \rangle\) in \(\mathbb{S}^{2l-1}\) is \(X_P(x) = 2Px - 2\langle Px, x \rangle x\). If \(x\) is a \(\pi_C\)-regular point, the set \(\{X_{P_1}(x), \ldots, X_{P_m}(x)\}\) is linearly independent and it spans a \(m + 1\)-dimensional subspace of \(T_x\mathbb{S}^{2l-1}\) orthogonal to the fibers of \(\pi_C\), thus projecting onto \(T_{\pi_C(x)}\mathbb{D}_C\). Therefore \(\pi_C\) is a submersion.

Fix an orthonormal basis \((P_0, \ldots, P_m)\) of \(\mathbb{R}_C\) and take \(x_+ \in E^1_+(P_0)\). On \(E_-(P_0)\), consider the orthogonal complement \(V^1_{x_+}\) of \(\text{span}(P_i x_+, \ldots, P_m x_+)\), and take its unit sphere \(V^1_{x_+} \subset \mathbb{S}^{2l-1}\). The dimension of \(V^1_{x_+}\) is \(l - m - 1\) and for every \(x_- \in V^1_{x_+}\) the element \(x = \sqrt{\frac{2}{l}}(x_+ + x_-) \in \mathbb{S}^{2l-1}\) satisfies \(\pi_C(x) = 0\) and thus \(x\) belongs to \(M_+\).

Taking the union of all \(V^1_x\) as \(x\) varies in \(E^1_+(P_0)\), we obtain a sphere bundle \(V^1 \to E^1_+(P_0)\) whose fiber has dimension \(l - m - 1\). In particular, if \(l > m + 1\) the fiber is connected, and so is \(V^1\). As we have a surjective map \(V^1 \to M_+\) sending \(y \in V^1_x\) to \(\sqrt{\frac{2}{l}}(x + y)\), \(M_+\) is connected as well. Finally, since all the regular fibers of \(\pi_C\) are homeomorphic to each other (and, in particular, to \(M_+\)), every fiber is connected.

If \(l = m + 1\), by table 1.1 it follows that \(m = 1, 3, 7\) and for all cases \(m\) is not a multiple of 4. Given a Clifford system \(C'\) of rank \(m + 2\) in \(\mathbb{R}^{2l}\), by the uniqueness of Clifford systems for \(m \not\equiv 0(\text{mod } 4)\) it follows that \(C\) is algebraically equivalent to a sub-Clifford system of \(C'\). We can thus find an orthonormal basis \((P_0, \ldots, P_{m+1})\) of \(\mathbb{R}_{C'}\) such that \((P_0, \ldots, P_m)\) is a basis for \(\mathbb{R}_C\). Since we can express \(\pi_C(x)\) as \(\langle P_0x, x \rangle, \ldots, \langle P_{m}x, x \rangle\) and similarly for \(\pi_C\), \(\pi_C\) factors as \(\pi_C = Pr \circ \pi_{C'}\), where \(Pr : \mathbb{S}_C \to \mathbb{D}_C\) is given by \((x_0, \ldots, x_m, x_{m+1}) \mapsto (x_0, \ldots, x_m)\).

Remark 2.5. Since we are interested in having connected fibers, we will not consider from now on the Clifford systems with \(l = m + 1\).

Proposition 2.6. Let \(C\) be a Clifford system of rank \(m + 1\) on \(\mathbb{R}^{2l}\). The fibers of \(\pi_C\) define a transnormal system on \(\mathbb{S}^{2l-1}\), whose leaf space is \(\mathbb{D}_C\) (if \(l > m + 1\)) or \(\mathbb{S}_C\) (if \(l = m\)) with a round metric of curvature 4.
Proof. We prove the proposition when the quotient is $\mathbb{D}_C$, the other case immediately follows from this. In order to prove the proposition, we consider the family $\mathfrak{F}$ of geodesics in $S^{2l-1}$ given by

$$\mathfrak{F} = \{ \gamma(t) = \cos(t)x_+ + \sin(t)x_+ | P \in S_C, x_\pm \in E^1_\pm(P) \}$$

and we show the following properties hold:

1. Every geodesic in $\mathfrak{F}$ is orthogonal to the fibers of $\pi_C$ at all points.
2. For every point $x \in S^{2l-1}$ and vector $z$ normal to the fiber of $\pi_C$ through $x$, there is a geodesic in $\mathfrak{F}$ passing through $x$ and tangent to $z$.
3. Every geodesic in $\mathfrak{F}$ projects to a unit speed geodesic in $\mathbb{D}_C$.

1) The normal space of a regular fiber at a point $x$ is spanned by the vectors $X_P(x) = P_x - \langle P_x, x \rangle x$, while the normal space of a singular fiber $E^1_+(P)$ is just $E_-(P)$. Any geodesic $\gamma \in \mathfrak{F}$, $\gamma(t) = \cos(t)x_+ + \sin(t)x_-$ for some $x_\pm \in E^1_\pm(P)$ is by definition perpendicular in $x_+, x_-$ to the corresponding fibers. In a regular fiber, we have $P\gamma(t) = -\cos(t)x_- + \sin(t)x_+$, $(P\gamma(t), \gamma(t)) = -\cos(2t)$ and it is just matter of computations to show that $\gamma'(t) = \frac{1}{\sin(2t)} \cdot X_P(\gamma(t))$:

$$X_P(\gamma(t)) = P\gamma(t) - (P\gamma(t), \gamma(t))\gamma(t)$$

$$= (-\cos(t)x_- + \sin(t)x_+ + \cos(2t)(\cos(t)x_- + \sin(t)x_+))$$

$$= -\cos(t)x_- + \sin(t)x_+ + \cos(2t)(\cos(t)x_- + \sin(t)x_+)$$

$$= \sin(2t)(-\sin(t)x_- + \cos(t)x_+)$$

$$= \sin(2t)\gamma'(t)$$

2) If $x$ is a singular point, then it belongs to the positive eigenspace $E^1_+(P)$ of some $P \in S_C$, and if $z$ is perpendicular to the fiber through $x$ it belongs to $E_-(P)$. Therefore, $\gamma(t) = \cos(t)x_+ + \sin(t)z$ belongs to $\mathfrak{F}$ and it satisfies $\gamma(0) = x$, $\gamma'(0) = z$. If $x$ is regular, any $z$ is normal to the fiber through $x$ is of the form $z = X_P(x)$ for some $P \in S_C$. Such a $P$ gives a splitting $\mathbb{R}^{2l} = E_+(P) \oplus E_-(P)$, and $x$ can be written as $x = \cos(t_0)x_- + \sin(t_0)x_+$ for some $x_\pm \in E^1_\pm(P)$. Equation (2.5) says that $z$ is parallel to $\gamma'(t_0)$, where $\gamma(t) = \cos(t)x_+ + \sin(t)x_+$ is in $\mathfrak{F}$.

3) Notice first that the unit speed geodesics in $\mathbb{D}_C$ with the round metric of constant curvature $4$ are of the form $\cos(2t)P + \sin(2t)Q$ where $P, Q \in \mathbb{D}_C$ satisfy $\langle P, Q \rangle = 0$. Given $\gamma(x) = \cos(t)x_+ + \sin(t)x_+$ for some $x_\pm \in E^1_\pm(P)$, $P \in S_C$, let $Q = \sum\langle P, x_+, x_- \rangle P$. Again it is just a computation to check that $\|Q\|^2 \leq 1$ and thus $Q \in \mathbb{D}_C$, $\langle -P, Q \rangle = 0$, and $\pi_C(\gamma(t)) = -\cos(2t)P + \sin(2t)Q$. Therefore $\pi_C(\gamma(t))$ is a geodesic in $\mathbb{D}_C$, as we wanted to show.

\[\Box\]

Definition 2.7. Given a Clifford system $C$ of rank $m+1$ in $\mathbb{R}^{2l}$, with $l \neq m+1$, we define the Clifford foliation $\mathcal{F}_C$ to be the foliation on $S^{2l-1}$ given by the fibers of $\pi_C$. 

Remark 2.8. By Proposition 2.6 any Clifford foliation $\mathcal{F}_C$ is a transnormal system. In fact, we will prove that $\mathcal{F}_C$ is a singular Riemannian foliation. This requires proving the existence of smooth vector fields spanning the leaves of the foliation. We will prove this in Proposition 3.3 for a larger class of foliations that includes the Clifford foliations.

Corollary 2.9. If $C$ is a Clifford system of rank $m+1$ on $\mathbb{R}^{2l}$ and $l = m$, $\pi_C : S^{2m-1} \to S_C$ is a Hopf fibration.

Proof. In this particular case $\pi_C$ is a submersion, and by Proposition 2.6 it is Riemannian. By the work of Grove and Gromoll [5] and Wilking [13], the submersion $\pi_C$ must be in fact a Hopf fibration. □

3. Composed foliations

Let $(M, \mathcal{F})$ be a transnormal system with closed leaves and leaf space $\Delta$, and let $\pi : M \to \Delta$ be the canonical projection. The metric on $\Delta$ is defined so that $\pi$ is a submetry, i.e., it sends $R$-balls in $M$ to $R$-balls in $\Delta$ for any $R > 0$. Given a metric space $\Delta'$ and a submetry $s : \Delta \to \Delta'$, the composition $s \circ \pi : M \to \Delta'$ is a submetry as well, and the preimages of the composition define a new transnormal system on $M$.

The goal of this section is to employ this method, introduced by A. Lytchak, to produce new singular Riemannian foliations on spheres out of the Clifford foliations.

Fix a Clifford system $C$ of rank $m+1$ on $\mathbb{R}^{2l}$. From Proposition 2.6 we know that the leaf space of a Clifford foliation is isometric to either $S_C$ with a metric of constant curvature $4$ ($m=2, 4, 8$) or $D_C$ with a hemisphere metric of curvature $4$. With such a metric, $D_C$ can be described metrically as a spherical join $\frac{1}{2}(S_C(1) \star \{pt\})$, where $S_C(1)$ denotes $S_C$ with the round metric of curvature $1$ and the factor $\frac{1}{2}$ in front denotes a rescaling of the metric by a factor $\frac{1}{2}$.

Let $(S_C(1), \mathcal{F}_0)$ be a closed transnormal system on $S_C(1)$, with leaf space $\Delta$ and projection $\pi_0 : S_C(1) \to \Delta$. If $l = m$ the composition $\pi_0 \circ \pi_C$ gives a submetry $S^{2l-1} \to \frac{1}{2}\Delta$.

If $l > m+1$, the submetry $\pi_0 : S_C \to \Delta$ induces a submetry

$$\tilde{\pi}_0 : \frac{1}{2}(S_C(1) \star \{pt\}) \to \frac{1}{2}(\Delta \star \{pt\}).$$

Composing $\tilde{\pi}_0$ with $\pi_C : S^{2l-1} \to \frac{1}{2}(S_C(1) \star \{pt\})$, we again obtain a submetry $\tilde{\pi}_0 \circ \pi_C : S^{2l-1} \to \frac{1}{2}(\Delta \star \{pt\})$.

In either case, we obtain a submetry $S^{2l-1} \to \Delta$, where $\Delta = \frac{1}{2}\Delta$ or $\frac{1}{2}(\Delta \star \{pt\})$, and the fibers of this submetry are by construction the leaves of a transnormal system on $S^{2l-1}$, which we denote by $\mathcal{F}_0 \circ \mathcal{F}_C$.

The goal of this section is to prove the following result.
Proposition 3.1. If \((S^{2l-1}, \mathcal{F}_C)\) is a Clifford foliation and \((S_C, \mathcal{F}_0)\) is a singular Riemannian foliation, then the foliation \(\mathcal{F}_0 \circ \mathcal{F}_C\) is a singular Riemannian foliation as well.

Once again we prove the result in the case where the quotient is \(\mathbb{D}_C\), the other case being essentially contained in this one.

The foliation \(\mathcal{F}_0 \circ \mathcal{F}_C\) can be described in the following equivalent way. The singular Riemannian foliation \((S_C, \mathcal{F}_0)\) can be extended to singular Riemannian foliation \(\mathcal{F}_h\) on \(\mathbb{D}_C\), by defining the leaf \(L_tP\) through \(tP\), where \(P \in S_C\) and \(t \in [0,1]\). The foliation \(\mathcal{F}_0 \circ \mathcal{F}_C\) is then given by the preimages under \(\pi_C: S^{2l-1} \to \mathbb{D}_C\) of the leaves in \(\mathcal{F}_0^h\). From this, we obtain the following explicit description of the leaves of \(\mathcal{F}_0 \circ \mathcal{F}_C\).

Lemma 3.2. Any regular point \(x_0\) of \(\mathcal{F}_C\) can be written as \(x_0 = \cos(\pi t) x + \sin(\pi t) P x\) for some \(x \in M_+\) and \(P \in S_C\), and the leaf of \(\mathcal{F}_0 \circ \mathcal{F}_C\) through \(x_0\) is

\[
M_{t,P} = \{\cos(\pi t) y + \sin(\pi t) Q y \mid y \in M_+, Q \in L_P\}
\]

where \(L_P\) is the leaf of \(\mathcal{F}_0\) through \(P\). If \(x_0 \in M_-\), \(x_0\) belongs to some positive eigenspace \(E^1_+(P)\) and the leaf of \(\mathcal{F}_0 \circ \mathcal{F}_C\) through \(x_0\) is given by

\[
\bigcup_{Q \in L_P} E^1_+(Q).
\]

□

Let \(\mathring{\mathbb{D}}_C\) denote the interior of \(\mathbb{D}_C\). We now prove that \(\mathcal{F} = \mathcal{F}_0 \circ \mathcal{F}_C\) is indeed a singular foliation.

Proposition 3.3. (1) The map

\[
\phi : M_+ \times \mathring{\mathbb{D}}_C \to S^{2l-1} \setminus M_-
\]

\[(x, tP) \mapsto \cos \left(\frac{\pi t}{4}\right)x + \sin \left(\frac{\pi t}{4}\right)Py, \quad P \in S_C\]

is a diffeomorphism on \(S^{2l-1} \setminus M_-\) and \(\phi^*(\mathcal{F}) = M_+ \times \mathcal{F}_h^0\). In particular, \(\mathcal{F}\) is a singular foliation on \(S^m \setminus M_-\).

(2) Consider a singular leaf \(L = E^1_+(P)\) of \(\mathcal{F}_C\) in \(M_-\). The map

\[
\psi : L \times \text{Tub}_2(P \subseteq S_C) \to \text{Tub}_2(L \subseteq M_-)
\]

\[(x, \cos(2t)P + \sin(2t)Q) \mapsto \cos(t)x + \sin(t)Qx\]

is a diffeomorphism, and \(\psi^*(\mathcal{F}|_{M_-}) = L \times \mathcal{F}_0\). In particular, \(\mathcal{F}|_{M_}\) is a singular foliation.
(3) Consider a neighborhood $U \subseteq M_-$ of a point $x \in M_-$, such that $\nu(M_-)|_U$ admits an orthonormal frame $\{\xi_1, \ldots, \xi_r\}$, $r = \text{codim}(M_-)$. Then the trivialization

$$\rho : U \times \mathbb{D}^r(\varepsilon) \longrightarrow \text{Tub}_x(U)$$

$$(x, (a_1, \ldots, a_r)) \longmapsto \exp_x \left( \sum a_i \xi_i(x) \right)$$

is a diffeomorphism, and $\rho^*(\mathcal{F}) = \mathcal{F}|_U \times \mathcal{F}_{\mathbb{D}^r}$, where $(\mathbb{D}^r(\varepsilon), \mathcal{F}_{\mathbb{D}^r})$ is the foliation by concentric spheres around the origin. In particular, $\mathcal{F}$ is a singular foliation around $M_-$.

**Proof.** (1) It is an easy computation to check that $\phi$ is indeed a diffeomorphism. Moreover, by Lemma 3.2 the preimage of a general leaf $\{\cos(t)y + \sin(t)Qy \, | \, y \in M_+, Q \in L_P\}$ is

$$\{(y, tQ) | y \in M_+, Q \in L_P\} = M_+ \times t \cdot L_P$$

as we wanted to show.

(2) It is enough to prove that

$$\psi(L \times \{\cos(2t)P + \sin(2t)Q\}) = E_+(\cos(t)P + \sin(t)Q)$$

so that $\pi_C \circ \psi : L \times \text{Tub}_2(P \subseteq S_C) \rightarrow \text{Tub}_2(P \subseteq S_C)$ corresponds to the projection onto the second factor. But in fact, given $x \in L$, we have

$$\psi(x, \cos(2t)P + \sin(2t)Q) = \cos(t)x + \sin(t)Qx,$$

and

$$\left(\cos(2t)P + \sin(2t)Q\right)\left(\cos(t)x + \sin(t)Qx\right) =$$

$$= (\cos(2t)\cos(t) + \sin(2t)\sin(t))x + (\sin(2t)\cos(t) - \cos(2t)\sin(t))x$$

$$= \cos(t)x + \sin(t)Qx.$$

(3) Again, it is clear that $\rho$ is a diffeomorphism. We will now show that $\rho$ induces a bijection among the leaf spaces.

Consider $U \times [0, \varepsilon]$, together with the foliation $\mathcal{F}|_U \times \{\text{pts}\}$. The map

$$(id, r) : (U \times \mathbb{D}^r(\varepsilon), \mathcal{F}|_U \times \mathcal{F}_{\mathbb{D}^r}) \longrightarrow (U \times [0, \varepsilon], \mathcal{F}|_U \times \{\text{pts}\})$$

$$(u, v) \longmapsto (u, \|v\|)$$

clearly induces a bijection among leaf spaces. Moreover, let $p : \text{Tub}_x(U) \rightarrow U$ denote the metric projection, and consider the map

$$(p, d_U) : (\text{Tub}_x(U), \mathcal{F}) \longrightarrow (U \times [0, \varepsilon], \mathcal{F}|_U \times \{\text{pts}\})$$

$$x \longmapsto (p(x), \text{dist}(x, U))$$
This map takes the leaf \( M_{([P], t)} = \{ \cos(t)x + \sin(t)Qx \mid x \in M_+, Q \in L_P \} \) of \( \mathcal{F} \) to the point \((P, \pi/4 - t)\). Since every leaf of \( \mathcal{F} \) in \( \text{Tub}_x(U) \) is uniquely determined by \([P] \in U/\mathcal{F}|_U \) and \( t \in [\pi/4 - \epsilon, \pi/4] \), it follows that \((p, d_U)\) induces a bijection between the leaf spaces as well.

Finally, we claim that \((p, d_U) \circ \rho = (id, r)\).

\[
\begin{align*}
(p, d_U) \left( \rho \left( u, (a_1, \ldots, a_r) \right) \right) &= (p, d_U) \left( \exp_u \sum a_i \xi_i(u) \right) \\
&= \left( p \left( \exp_u \sum a_i \xi_i(u) \right), \text{dist} \left( \exp_u \sum a_i \xi_i(u), U \right) \right) \\
&= \left( u, \left\| (a_1, \ldots, a_r) \right\| \right)
\end{align*}
\]

In particular, \( \rho \) induces a bijection between the leaf spaces as well, and this finished the proof.

**Remark 3.4.** If \( \mathcal{F}_0 \) is a trivial foliation whose leaves consist of points, \( \mathcal{F}_0 \circ \mathcal{F}_C = \mathcal{F}_C \) and in particular, \( \mathcal{F}_C \) is a singular Riemannian foliation. Moreover, when \( C \) is a Clifford system of rank \( m + 1 \) on \( \mathbb{R}^2l \) and \( l = m + 1 \), the foliation \( \mathcal{F}_C \) given by the (non connected) fibers of \( \pi_C \) is still a *singular Riemannian foliation with disconnected fibers* as defined in [1, Sect. 3].

\section*{4. Rigidity of Clifford foliations}

In the FKM families, the map that associates an isoparametric foliation to each (geometric equivalence class of) Clifford system is neither injective, nor surjective. In fact, on the one hand there are examples of geometrically distinct Clifford systems giving rise to the same isoparametric foliation. On the other hand, there are isoparametric foliations that do not come from a Clifford algebra, whose quotient is isometric to that of the FKM examples.

The goal of this section is to prove that in our case, the map \( C \mapsto \mathcal{F}_C \) described in the previous sections does determine a bijection between the geometric equivalence classes of Clifford system, and the congruence classes of singular Riemannian foliations in spheres whose quotient is an upper hemisphere of curvature 4. We will prove this in the next two propositions.

**Proposition 4.1.** Suppose \((\mathbb{S}^n, \mathcal{F})\) is a singular Riemannian foliation such that the quotient space is a hemisphere \( \frac{1}{2}\mathbb{S}^{m+1}_+ \) of constant curvature 4. Then \( \mathcal{F} = \mathcal{F}_C \) for some Clifford system \( C \).

**Proof.** Consider the boundary \( \frac{1}{2}\mathbb{S}^m \) of \( \frac{1}{2}\mathbb{S}^{m+1}_+ \). Take an orthonormal basis of \( \frac{1}{2}\mathbb{S}^m \), i.e. \( m + 1 \) points \( p_0, \ldots, p_m \in \frac{1}{2}\mathbb{S}^m \) mutually at distance \( \pi/4 \).
Given a point \( p_0 \) and its antipodal \(-p_0\), we can think of them as the north and south pole of the half-sphere \( \frac{1}{2}S^{m+1}_+ \). The preimages under \( \pi : S^n \to \frac{1}{2}S^{m+1}_+ \) of the parallels of \( \frac{1}{2}S^{m+1}_+ \) form a codimension 1 foliation \( \mathcal{F}^* \) of \( S^n \) whose quotient is an interval of length \( \pi/2 \). By Cartan’s classification of these types of foliations, it follows that the singular leaves of \( \mathcal{F}^* \), i.e. the leaves of \( \mathcal{F} \) corresponding to \( \pm p_0 \), are totally geodesic subspaces of \( S^n \), and since they lie on the same stratum they must have the same dimension, call it \( l \). In particular, \( n = 2l - 1 \) and \( \mathbb{R}^{n+1} = \mathbb{R}^{2l} \) splits orthogonally as \( V_+(p_0) \oplus V_-(p_0) \), where \( V_\pm(p_0) \) is the space corresponding to the great sphere \( \pi^{-1}(\pm p_0) \). Define a linear map \( P_0 \in \text{Sym}^2(\mathbb{R}^{2l}) \) by
\[
P_0|_{V_+(p_0)} = id, \quad P_0|_{V_-(p_0)} = -id.
\]
Notice that by definition \( P_0^2 = id \) and \( E_{\pm}(P_0) = V_\pm(p_0) \). Of course we can repeat the same procedure for the other points \( p_1, \ldots, p_m \), and this produces maps \( (P_0, \ldots, P_m) \in \text{Sym}^2(\mathbb{R}^{2l}) \). In order to conclude the proof, it will be enough to prove that \( P_i P_j = -P_j P_i \) for \( i \neq j \), or equivalently, that \( P_i(E_+(P_j)) = E_+(P_j) \).

We will now show that \( P_0(E_+(P_1)) = E_-(P_1) \), and by the arbitrariness of \( P_0, P_1 \) the result will follow. Take a point \( x \in E_+(P_0) \) in the preimage of \( p_0 \), and take a horizontal geodesic \( \gamma \) starting at \( x \) and tangent to the singular stratum, such that \( \pi(\gamma) \) passes through \( p_1 \). Since \( \pi(\gamma)(\pi/2) = p_0 \), the point \( y = \gamma(\pi/2) \) belongs to \( E_-(P_0) \) and we can write \( \gamma(t) = \cos(t)x + \sin(t)y \). Moreover, \( w = \gamma(\pi/4) = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \) belongs to \( E_+(P_1) \) by construction of \( \gamma \). Then
\[
P_0(w) = P_0 \left( \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \right) = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = \gamma(-\pi/4)
\]
But \( \pi(\gamma)(-\pi/4) = p_1 \), that means \( P_0(w) \in E_-(P_1) \).

Since any \( w \in E_1^+(P_0) \) can be written as \( \gamma(\pi/4) \) for some horizontal geodesic \( \gamma \) from \( E_1^+(P_0) \) and \( E_1^+(P_0) \), we obtain that \( P_0(E_+(P_1)) \subseteq E_-(P_1) \). Since \( P_0 \) is nonsingular, by dimensional reasons is must be \( P_0(E_+(P_1)) = E_-(P_1) \) and this finishes the proof.

**Proposition 4.2.** The Clifford foliations \((S^{2l-1}, \mathcal{F}_C)\) distinguish the geometric equivalence classes of Clifford systems. In other words, if \( C \) and \( C' \) are geometrically inequivalent Clifford systems on \( \mathbb{R}^{2l} \) and \( \mathbb{R}^{2l'} \) respectively, then there are no foliated isometries between \((S^{2l-1}, \mathcal{F}_C)\) and \((S^{2l'-1}, \mathcal{F}_{C'})\).

**Proof.** Let \( (P_0, \ldots, P_m) \) be an orthonormal basis of \( \mathbb{R}_C \) and \( (Q_0, \ldots, Q_m') \) an orthonormal basis of \( \mathbb{R}_{C'} \). Since the leaf spaces of \( \mathcal{F}_C, \mathcal{F}_{C'} \) have dimension \( m+1, m'+1 \) respectively, it follows immediately that \( \mathcal{F}_C \neq \mathcal{F}_{C'} \) unless \( m = m' \). If \( m = m' \), we have \( l = k\delta(m) \), \( l' = k'\delta(m') = k'\delta(m) \) and therefore \( \mathcal{F}_C \neq \mathcal{F}_{C'} \) unless \( k = k' \) as well.

Assume now that \( m = m' \) and \( k = k' \). As we recalled in Section [12], if \( m \neq 0(\text{mod } 4) \) there is only one geometric class of Clifford systems for each \( k \), and therefore
$\mathcal{F}_C = \mathcal{F}_{C'}$. If $m \equiv 0 (\text{mod } 4)$ then the geometric class of $C$ is uniquely determined by the non-negative integer $|\text{tr}(P_0 \cdots P_m)|$. Therefore the last thing remained to prove is that $\mathcal{F}_C$ and $\mathcal{F}_{C'}$ are not congruent unless $|\text{tr}(P_0 \cdots P_m)| = |\text{tr}(Q_0 \cdots Q_m)|$. This was already established in [4, page 486], as they showed that the invariant $|\text{tr}(P_0 \cdots P_m)|$ represents a characteristic number of the vector bundle $E \to S_C$ whose sphere bundle is $\pi_C|_{M_-} : M_- \to S_C$. For the sake of completeness, we show such result here.

Consider the vector bundle $E \to S_C$ such that the fiber at $P$ is the positive eigenspace $E_+(P)$. Clearly, $M_-$ the unit sphere bundle of $E$, and $E$ can be thought of a subbundle in $S_C \times \mathbb{R}^2 \to S_C$. The flat connection $\nabla$ on $S_C \times \mathbb{R}^2$ induces a connection on $E$, that has a very simple form: if $(P(t), X(t))$ is a curve in $E \subseteq S_C \times \mathbb{R}^2$ such that $(P(0), X(0)) = (P, x)$, $(P'(0), X'(0)) = (Q, x_1)$ then

$$
\nabla_Q X = \frac{1}{2}(x_1 + Px_1).
$$

It is easy to see from this, that given a section $X$ of $E$ around $P_0$ and $P_i, P_j \in C$ perpendicular to $P_0$ (so that we can think of $P_i, P_j$ as elements of $T_{P_0}S_C$) we have $\nabla_{P_i P_j} X = \frac{1}{4}P_i P_j X$ and thus $R^E_{P_0}(P_i, P_j) = \frac{1}{2}P_i P_j |_{E_+(P_0)}$. Again by identifying span($P_1, \ldots P_m$) with the tangent space of $S_C$ at $P_0$, we define the form $\chi_E$ of $E$ at $P_0$ as

$$
\chi_E(P_0) = \text{tr}(\text{Pf}(R^E_{P_0}))
$$

where $m!! = m(m - 2)(m - 4) \cdots 2 = 2^{m/2}(m/2)!$, and Pf is the Pfaffian. We can compute the form $\chi_E$ explicitly:

$$
\chi_E(P_0) = \frac{1}{m!!} \text{tr} \left( \sum_{\sigma \in S_m} (-1)^{||\sigma||} \frac{1}{2} P_{\sigma(1)} P_{\sigma(2)} \bigg|_{E_+(P_0)} \cdots \frac{1}{2} P_{\sigma(m-1)} P_{\sigma(m)} \bigg|_{E_+(P_0)} \right)
$$

$$
= \frac{1}{m!!} \text{tr} \left( \frac{1}{2^{m/2}} \sum_{\sigma \in S_m} P_1 \cdots P_m \bigg|_{E_+(P_0)} \right)
$$

$$
= \frac{m!}{m!!} \text{tr} \left( P_1 \cdots P_m \bigg|_{E_+(P_0)} \right)
$$

Since the last expression is now independent of the point $P_0$, $\chi_E$ is constant and equal to $\text{tr}(P_0 \cdots P_m)$ up to a constant factor independent of $E$. This class is independent of the connection used to define it, and therefore only depends on the bundle $E$. \qed
5. Homogeneous foliations

In this section we investigate the Clifford and composed foliations that are homogeneous.

5.1. Clifford foliations. When $S^{2l-1}/\mathcal{F}_C$ is a sphere $S^m(4)$, it is known that the foliation is homogeneous if and only if $m = 2$ or $4$. Therefore we can restrict our attention to the case where the quotient is a hemisphere. Our first result restricts the list of possible homogeneous Clifford foliations.

**Proposition 5.1.** Let $C$ be a Clifford system of rank $m+1$ on $\mathbb{R}^{2l}$ such that $l > m+1$. On $S^{2l-1}$ consider the Clifford foliation $\mathcal{F}_C$ and the FKM isoparametric family $\mathcal{F}_{C'}$ associated to $C$. If $\mathcal{F}_C$ is homogeneous, then $\mathcal{F}_{C'}$ is homogeneous as well.

**Proof.** Let $P$ be an element of $S_C$. for any $x, y \in S^{2l-1}$, we have $\langle Px, Py \rangle = \langle P^2 x, y \rangle = \langle x, y \rangle$ and therefore the elements of $S_C$ are also orthogonal maps on $\mathbb{R}^{2l}$. Moreover, by definition of $\pi_C$ we have

$$\langle \pi_C(Px), Q \rangle = \langle QPx, Px \rangle \quad \forall Q \in S_C.$$

Since $QP = -PQ + 2\langle P, Q \rangle Id$, the equation before becomes

$$\langle \pi_C(Px), Q \rangle = -\langle PQx, Px \rangle + 2\langle P, Q \rangle \langle x, Px \rangle$$

$$= -\langle \pi_C(x), Q \rangle + 2\langle P, Q \rangle \langle \pi_C(x), P \rangle$$

$$= \langle -\pi_C(x) + 2\pi_C(x), P \rangle P, Q \rangle$$

Therefore, $\pi_C(Px) = -\pi_C(x) + 2\langle \pi_C(x), P \rangle P$ and therefore there is a commutative diagram

$$\begin{array}{ccc}
S^{2l-1} & \xrightarrow{P} & S^{2l-1} \\
\downarrow{\pi_C} & & \downarrow{\pi_C} \\
\mathbb{D}_C & \xrightarrow{\rho_P} & \mathbb{D}_C
\end{array}$$

where $\rho_P$ is the reflection of $\mathbb{D}_C$ along the segment through $P$. The group generated by the elements $P \in S_C$ is usually denoted $Pin(m+1)$. Its subgroup generated by the products $PQ, P, Q \in S_C$, is $Spin(m+1)$. Since any element $P \in S_C$ can be thought of a foliated isometry of $(S^{2l-1}, \mathcal{F}_C)$, there is an inclusion $\eta : Spin(m+1) \to SO(2l)$ whose induced action on $\mathbb{D}_C$ is isometric and has cohomogeneity 1. The north pole $\pi_C(M_+)$ is the only singular orbit of this action, the equator $\pi_C(M_-)$ consists of one orbit, and the quotient $\mathbb{D}_C/Spin(m+1)$ is isometric to $[0, \pi/4] = S^{2l-1}/\mathcal{F}_{C'}$.

Suppose now that $(S^{2l-1}, \mathcal{F}_C)$ is given by an isometric action of some Lie group $H \subseteq SO(2l)$. Let $G \subseteq SO(2l)$ be the closure of the group generated by $H$ and $Spin(m+1)$. By the previous discussion, $G$ acts on $S^{2l-1}$ preserving the foliation $\mathcal{F}_C$ and it descends
to a cohomogeneity 1 action on $D$. In particular, the $G$-orbits in $S^{2l-1}$ correspond to the leaves of $\mathcal{F}_C$, and therefore $\mathcal{F}_C$ is homogeneous.

From Proposition 5.1 above and the table in section 1.3 it follows that the only possible homogeneous Clifford foliations with $l > m + 1$ come from Clifford systems with $(m, k) = (1, k), (2, k), (9, 1), \text{ or } m = 4 \text{ and } P_1 \cdots P_4 = \pm \text{Id}.

**Proposition 5.2.** Let $C$ be a Clifford system of rank $m + 1$ on $\mathbb{R}^{2l}$, $l = k\delta(m)$. Then:

- If $m = 1$, $\mathcal{F}_C$ is given by the orbits of the diagonal $SO(k)$-action on $S^{2k-1} \subseteq \mathbb{R}^k \oplus \mathbb{R}^k$.
- If $m = 2$, $\mathcal{F}_C$ is given by the orbits of the diagonal $SU(k)$-action on $S^{4k-1} \subseteq \mathbb{C}^k \oplus \mathbb{C}^k$.
- If $m = 4$ and $P_0 \cdots P_4 = \pm \text{Id}$, $\mathcal{F}_C$ is given by the orbits of the diagonal $Sp(k)$-action on $S^{8k-1} \subseteq \mathbb{H}^k \oplus \mathbb{H}^k$.

**Proof.** This proof is essentially a version of [4, Theorem 6.1], adapted to our situation. The Clifford systems with $m = 1, 2$ or $m = 4$ and $P_0 \cdots P_4 = \pm \text{Id}$ can be obtained in the following way: let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be the division algebra such that $\dim_{\mathbb{R}} \mathbb{F} = m$, and let $j_1, \ldots, j_{m-1}$ the canonical imaginary units of $\mathbb{F}$. For $q = q_0 + q_1 j_1 + \ldots + q_{m-1} j_{m-1} \in \mathbb{F}$, $q_i \in \mathbb{R}$, we define the real part of $q$ by $\Re(q) = q_0$ and the $r$-th imaginary part of $q$ by $\Im_r(q) = q_r = \Re(q \cdot j_r)$, $r = 1, \ldots, m-1$.

On $\mathbb{R}^{2\delta(m)} = \mathbb{R}^k \times \mathbb{R}^k$, let $C = (P_0, \ldots, P_m)$ be the Clifford system given by

$$P_0(u, v) = (u, -v), \quad P_1(u, v) = (v, u), \quad P_{r+1}(u, v) = (-j_r \cdot v, j_r \cdot u)$$

$r \in \{1, m-1\}$

where $u, v \in \mathbb{R}^k$, $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$. The projection $\pi_C$ is determined by the functions

$$\begin{cases}
\langle P_0(u, v), (u, v) \rangle &= \|u\|^2 - \|v\|^2 \\
\langle P_1(u, v), (u, v) \rangle &= 2\Re(\sum_i u_i \cdot \bar{v}_i) \\
\langle P_{r+1}(u, v), (u, v) \rangle &= 2\Re(\sum_i u_i \cdot \bar{v}_i \cdot j_r) = 2\Im_r(\sum_i u_i \cdot \bar{v}_i)
\end{cases}$$

and thus we can write

$$\pi_C(u, v) = (\|u\|^2 - \|v\|^2, 2\sum_i u_i \cdot \bar{v}_i) \in \mathbb{R} \oplus \mathbb{F}.$$
point \((u,v) \in S^{2mk-1} \subseteq \mathbb{R}^k \times \mathbb{R}^k\) can be moved by the \(U(\mathbb{F}, k)\)-action to a point of the form
\[
(u_1 e_1, v_1 e_1 + v_2 e_2)
\]
where \(e_1, e_2\) are elements of the canonical basis on \(\mathbb{R}^k\), \(v_1, v_2 \in \mathbb{R}_{\geq 0}\) and \(u_1^2 + |v_1|^2 + v_2^2 = 1\). It is easy to see that such \(u_1, v_1, v_2\) are uniquely determined by the functions \(f_i\), and therefore there is only one such point for each fiber of \(\pi_C\). In particular, every point in a fiber of \(\pi_C\) can be moved to a specific point via the action of \(U(\mathbb{F}, k)\), and therefore the orbits of \(U(\mathbb{F}, k)\) coincide with the leaves of \(\mathcal{F}_C\).  

On the other hand, the remaining foliation \(\mathcal{F}_C, C = C_{9,1}\) on \(\mathbb{R}^{32}\), is not homogeneous.

**Proposition 5.3.** The Clifford foliation induced by the Clifford system \(C = C_{9,1}\) on \(\mathbb{R}^{32}\) is not homogeneous.

Proof. Suppose \((S^{31}, \mathcal{F}_C)\) is homogeneous, given by the orbits of a group \(G \subseteq \text{SO}(32)\).

First of all, we prove that the principal isotropy group \(H\) must be trivial. If not, consider the subsphere \(S^h = \text{Fix}(H)\), and take \(G' = N(H)/H\), where \(N(H)\) is the normalizer of \(H\) in \(G\). The identity component \(G'_0\) of \(G'\) acts on \(S^h\) with trivial principal groups and there is an orbifold cover \(S^h/G'_0 \rightarrow S^{31}/G\), where \(S^{31}/G\) is isometric to the hemisphere \(\frac{1}{2}S^{10}\). The quotient \(S^h/G'_0\) cannot be \(\frac{1}{2}S^{10}\) (the only spheres that can arise as such quotient must have dimension 2, 4 or 8, see for example the introduction of [9]) and since \(\pi_1^{orb}(\frac{1}{2}S^{10}) = \mathbb{Z}/2\mathbb{Z}\), it must be \(S^h/G'_0 = \frac{1}{2}S^{10}\). By Proposition 4.1 it follows that \((S^h, G'_0)\) is itself a Clifford foliation, with respect to some Clifford system \((Q_0, \ldots, Q_9)\). In particular \(h \geq 31\), and therefore it must be \(S^h = S^{31}\) and \(H = \{1\}\).

Since the leaf \(E^1_+(P_0)\) is a totally geodesic sphere of dimension 15, \(G\) acts transitively on \(S^{15}\) by isometries. On the other hand, since the \(G\)-action has trivial principal isotropy groups, it must have \(\dim G = 21\), and this gives a contradiction since there are no groups of dimension 21 that act transitively on \(S^{15}\) (see for example [7], Table C).

Finally, we determine the homogeneity of a big fraction of the composed foliations \(\mathcal{F}_0 \circ \mathcal{F}_C\), in terms of the homogeneity of \(\mathcal{F}_0\) and and \(\mathcal{F}_C\).

**Proposition 5.4.** Let \(C, \mathcal{F}_C, \mathcal{F}_C'\) be defined as in proposition 5.3 and let \((S_C, \mathcal{F}_0)\) be a singular Riemannian foliation. If the leaf space of \(\mathcal{F}_C\) is a hemisphere and the composed foliation \(\mathcal{F}_0 \circ \mathcal{F}_C\) is homogeneous, then \(\mathcal{F}_0\) and \(\mathcal{F}_C'\) are homogeneous. On the other hand, if \(\mathcal{F}_C\) and \(\mathcal{F}_0\) are homogeneous, so is \(\mathcal{F}_0 \circ \mathcal{F}_C\).

Proof. Suppose first that \((S^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)\) is homogeneous, given by the orbits of a \(G\)-action. Remember that \(M_+\) is a leaf for both \(\mathcal{F}_C\) and \(\mathcal{F}_0 \circ \mathcal{F}_C\). For any point \(x \in M_+\),
Moreover, (the identity component of) the isotropy group $G_x$ acts on $\nu^1_x M_x$ via the slice representation, whose orbits get mapped to the leaves of $F_0$ via the same map $\pi_C \circ \exp_x^{\perp}$ and therefore $F_0$ is homogeneous as well. Moreover, as in Proposition 5.4 above, we can consider the group $G' \subseteq \SO(2l)$ generated by $G$ and $\eta(\Spin(m+1))$, and the orbits of $G'$ are, once again, the leaves of $F'_C$, which is then homogeneous.

Suppose now that $(S^m, F_0)$ is homogeneous and it is given by the orbits of a representation $\rho : H \to \SO(m+1)$. Up to a double cover $H' \to H$ we can lift $\rho$ to $\rho' : H' \to \Spin(m+1)$, and via the embedding $\eta : \Spin(m+1) \to \SO(2l)$ defined in Proposition homogeneous we have a representation $\rho'' : H' \to \SO(2l)$. By the way we defined $\eta$ it is clear that the $\rho''(H')$-orbits on $S^{2l-1}$ get projected, via $\pi_C$, to $\rho(H)$-orbits on $\DD_C$. In particular, if $F_C$ is homogeneous given by some $K$-action, the (closure of the) group $K' \subseteq \SO(2l)$ generated by $K$ and $\rho''(H')$ acts on $S^{2l-1}$ isometrically and the orbits are precisely the leaves of $F_0 \circ F_C$. \hfill \Box

**Corollary 5.5.** If $(S^{2l-1}, F_C)$ is a Clifford foliation with quotient $S^2$ or $S^4$, then for every singular Riemannian foliation $(S_C, F_0)$ the composed foliation $F_0 \circ F_C$ is homogeneous.

**Proof.** In this case $F_0$ is a singular Riemannian foliation on $S^2$ or $S^4$, and therefore either $\dim F_0 \leq 3$ or $F_0$ is the trivial foliation in $S^4$ consisting of one leaf. In the first case, $F_0$ is homogeneous by [10], in the second is trivially homogeneous. Since $F_C$ itself is homogeneous, $F_0 \circ F_C$ is homogeneous by Proposition 5.4. \hfill \Box

The result of this sections allow us to prove the second part of Theorem 5.2

**Proposition 5.6.** Let $C$ be a Clifford system on $\RR^{2l}$ and $(S_C, F_0)$ a singular Riemannian foliation. If $C \neq C_{8,1}, C_{9,1}$ then $(S^{2l-1}, F_0 \circ F_C)$ is homogeneous if and only if $F_0$ and $F_C$ are homogeneous. If $C = C_{9,1}$ then $(S^{2l-1}, F_0 \circ F_C)$ is homogeneous only if $F_0$ is homogeneous.

**Proof.** If $F_C$ and $F_0$ are homogeneous, then $F_0 \circ F_C$ is homogeneous by 5.4. If $F_0 \circ F_C$ is homogeneous then there are two cases to consider:

- If the leaf space of $F_C$ is $S_C$, then $C = C_{2,1}$ or $C_{4,1}$. In both cases $F_C$ is homogeneous by Proposition 5.2 and $F_0$ is homogeneous by Corollary 5.5.
- If the leaf space of $F_C$ is $\DD_C$, then by Proposition 5.4 both $F_C, F'_C$ are homogeneous. Moreover, if $C \neq C_{9,1}$ then $F_C$ is homogeneous as well by Table (1.2) and Proposition 5.2.

\hfill \Box
6. Properties of Clifford and composed foliations

The new examples exhibit some behaviors that either do not appear in the homogeneous case, or have not been shown to appear.

6.1. Orbifold quotient. Recall that a singular Riemannian foliation \((M, \mathcal{F})\) is polar if, for every point \(p \in M\), there is a totally geodesic submanifold of dimension equal to the codimension of \(\mathcal{F}\) that passes through \(p\) and is perpendicular to all the leaves it meets. The quotient of a closed polar foliation \((S^n, \mathcal{F})\) has constant curvature 1. If \((S^{2l-1}, \mathcal{F}_C)\) is a Clifford foliation with hemispherical quotient and \((S_C, \mathcal{F}_0)\) is a polar foliation, then the quotient of \(\mathcal{F}_0 \circ \mathcal{F}_C\) is isometric to an orbifold of curvature 4. In particular there is a large variety of non polar homogeneous foliations whose leaf space is an orbifold of constant curvature 4, of any dimension. This should be compared with a recent result of C. Gorodski and A. Lytchak [6], who show that if the leaf space is an orbifold of constant curvature 4, of any dimension. In particular, there is a finite number of non-polar homogeneous foliations whose quotient is a good orbifold.

6.2. Strata on the leaf space. Let \((S^n, \mathcal{F})\) be a homogeneous foliation, induced by the action of a compact group \(G \subseteq SO(n+1)\). If \(x \in S^n\) is a singular point with isotropy group \(G_x\), the connected component of \(\text{Fix}(G_x)\) through \(x\) is a totally geodesic sphere \(S^k \subseteq S^n\) that projects via \(\pi : S^n \to S^n/G\) to the stratum \(\Sigma_x\) containing \(x\). Moreover, the group \(G' = N(G_x)/G_x\) acts effectively on \(S^k\) by isometries, and there is a map \(S^k/G' \to \Sigma_x\). If we let \(G'_0\) be the identity component of \(G'\), \(G'_0\) induces a homogeneous singular Riemannian foliation \(\mathcal{F}'\) on \(S^k\), and the stratum \(\Sigma_x\) is, up to a cover, the quotient of the singular Riemannian foliation \((S^k, \mathcal{F}')\).

This is no longer true in the case of (non homogeneous) Clifford foliations. In fact, the only singular stratum is \(S_C\), which is already simply connected but it cannot be the quotient of any singular Riemannian foliation \((S^k, \mathcal{F})\), unless \(m = 2, 4, 8\).

6.3. Iterated foliations. Fix an integer \(m_0\) and consider the sequence of integers \(m_0, m_1, \ldots, m_n\) such that \(m_j = 2\delta(m_{j-1}) - 1, j = 1, \ldots, n\). For each \(m_j, j = 0, \ldots, n\), consider a Clifford system \(C_j\) of rank \(m_j + 1\) on \(\mathbb{R}^{2\delta(m_j)} = \mathbb{R}^{m_j+1}\) and the corresponding Clifford foliation \(\mathcal{F}_{C_j}\) on \(S^{2\delta(m_j)-1} = S^{m_j+1}\). By the construction of the integers \(m_j\), it makes sense to define the iterated composed foliations

\[\mathcal{F}_k = ((\mathcal{F}_{C_1} \circ \mathcal{F}_{C_2}) \circ \ldots) \circ \mathcal{F}_{C_k}\]

on \(S^{m_j+1}\). We want to study the curvature of the quotients \(X_j = S^{m_j+1}/\mathcal{F}_j\) inductively, using the observation that \(X_{j+1} = \frac{1}{2}(X_j \ast \{pt\})\). The first foliation \(\mathcal{F}_1 = \mathcal{F}_{C_1}\) is just a Clifford foliation, and thus \(X_1\) has constant curvature 4. The spherical join \(X_1 \ast \{pt\}\)
contains $X_1$ as a totally geodesic subset, and thus the supremum of the curvatures of $X_1 \ast \{pt\}$ is at least 4. Since $X_2 = \frac{1}{2}(X_1 \ast \{pt\})$ and dividing the dimension by 2 increases the curvature by 4, the supremum of the curvatures of $X_2$ is at least $4^2$. In the same way, if the supremum of the curvatures of $X_j$ is at least $c$ then the supremum of the curvatures of $X_{j+1} = \frac{1}{2}(X_j \ast \{pt\})$ is at least $4^c$. By induction, the supremum of the curvature of $X_k$ is $\geq 4^k$, and therefore the sequence of singular Riemannian foliations $(\mathbb{S}^{m_j+1}, F_j)$ admits quotients whose maximum curvature grows unbounded.

6.4. Isometric quotients. Consider geometrically inequivalent Clifford system $C = (P_0, \ldots, P_m)$, $C' = (Q_0, \ldots, Q_m)$ on $\mathbb{R}^{2\delta(m)}$ ($m$ must necessarily be a multiple of 4).

By Proposition 4.2 the foliations $F_C, F_{C'}$ on $S^{2\delta(m)-1}$ are not congruent but there is an isometry $\phi : S^{2\delta(m)-1}/F_C \to S^{2\delta(m)-1}/F_{C'}$ that preserves the codimension of the leaves. If $(S^m, F_0)$ is a non-polar singular Riemannian foliation, the foliations $F_0 \circ F_C, F_0 \circ F_{C'}$ on $S^{2\delta(m)-1}$ are not infinitesimally polar. They are also not congruent, but the isometry $\phi$ induces an isometry

$$S^{2\delta(m)-1}/(F_0 \circ F_C) \to S^{2\delta(m)-1}/(F_0 \circ F_{C'})$$

that preserves the codimension of the leaves. This behaviour contrast with the homogeneous case, since to the best of the author’s knowledge there are no known examples of non-orbit equivalent representations whose orbit spaces admit an isometry that preserves the codimension of the orbits (cf. [9, Question 1.2]). By [1], the sheaf of smooth $F_0 \circ F_C$-basic functions (i.e. smooth functions that are constant along the leaves of $F_0 \circ F_C$) is isomorphic to the sheaf of smooth $F_0 \circ F_{C'}$-basic functions.

6.5. Induced foliations on projective spaces. Let $C$ be a Clifford algebra of rank $m + 1$ on $\mathbb{R}^l$ with $m \geq 1$ and let $(S^{2l-1}, F_C)$ be the associated Clifford foliation. If we define $i = P_0 P_1 \in \mathfrak{so}(2l)$, the flow of $i$ defines an isometric $S^1$ action on $S^{2l-1}$. This action preserves $F_C$, and thus it induces an isometric action on the quotient $\mathbb{S}^m(4)$ or $\mathbb{D}_C = \frac{1}{2}(S_C(1) \ast \{pt\})$ that acts transitively on the circle containing $P_0, P_1$ while fixing the other elements $P_i$, and the north pole. Therefore, the foliation $(S^{2l-1}, F_C)$ projects to a foliation $F_C^C$ on $S^{2l-1}/S^1 = \mathbb{CP}^{l-1}$ with quotient isometric to either $\frac{1}{2}(S^{m-2} \ast \{pt\})$ (a hemisphere of $S^{m-1}(4)$) or $\frac{1}{2}(S^{m-2} \ast [0,\pi/2])$ (a half hemisphere of $S^m(4)$).

Similarly, if $m \geq 2$ the Lie algebra $J$ generated by $\{i = P_0 P_1, j = P_1 P_2, k = P_0 P_2\} \subseteq \mathfrak{so}(2l)$ generates a subgroup $S^3 \subseteq \mathfrak{SO}(2l)$ which acts on $S^{2l-1}$. This action preserves $F_C$, and it induces an isometric action on the quotient $\mathbb{S}^m(4)$ or $\mathbb{D}_C = \frac{1}{2}(S^m \ast \{pt\})$, that acts transitively on the 2-sphere containing $P_0, P_1, P_2$ while fixing the other elements $P_i$ and the north pole. Once again, $F_C$ projects to a foliation $F_C^C$ on $S^{2l-1}/S^3 = \mathbb{H}\mathbb{P}^{l-2}$ with quotient isometric to either $\frac{1}{4}(S^{m-3} \ast \{pt\})$ (a hemisphere of $S^{m-2}(4)$) or $\frac{1}{2}(S^{m-3} \ast [0,\pi/2])$ (a half hemisphere of $S^{m-1}(4)$).
As in the spherical case, given a singular Riemannian foliation $\mathcal{F}_0$ on $S^{m-2}$ (resp. on $S^{m-3}$) we can define new foliations $(\mathbb{C}P^{l-1}, \mathcal{F}_0 \circ \mathcal{F}_C^l)$ (resp. $(\mathbb{H}P^{l/2-1}, \mathcal{F}_0 \circ \mathcal{F}_C^l)$).

**References**

1. Alexandrino, M. M. and Radeschi, M., *Isometries between leaf spaces*, preprint arXiv:1111.6178v1 [math.DG].
2. Cecil, T. E., *Isoparametric and Dupin hypersurfaces*. SIGMA, Symmetry Integrability Geom. Methods Appl. **4** (2008), Paper 062.
3. Cartan É., *Sur quelque familles remarquables d’hypersurfaces*, C.R. Congrès Math. Liège (1939), 30–41.
4. Ferus, D., Karcher, H. and Munzner, H. F., *Cliffordalgebren und neue isoparametrische Hyperachen* Math. Z. **177** (1981), no. 4, 479–502.
5. Gromoll, D. and Grove, K., *The low-dimensional metric foliations of Euclidean spheres*, J. Differential Geom. **28** (1988), no. 1, 143–156.
6. Gorodski, C. and Lytchak, A., *On orbit spaces of representations of compact Lie groups*, arXiv:1109.1739v2.
7. Grove, K., Wilking, B. and Ziller, W., *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Geom. Phys. **78** (2008), no. 1, 33–111.
8. Lytchak, A., *Polar foliations on symmetric spaces*, preprint arXiv:1204.2923v2 [math.DG].
9. Lytchak, A. and Wilking, B., *Riemannian foliations of spheres*, preprint arXiv:1309.7884v1 [math.DG].
10. Radeschi, M., *Low dimensional singular Riemannian foliations on spheres*, Ph.D. thesis, University of Pennsylvania (2012).
11. Straume, E., *On the invariant theory and geometry of compact linear groups of cohomogeneity < 3*, Diff. Geom. Appl. **4** (1994), 1–23.
12. Thorbergsson G., *A survey on isoparametric hypersurfaces and their generalizations*, Handbook of Differential Geometry, Vol. I, Elsevier Science, Amsterdam,(2000), 963–995.
13. Wilking, B., *Index parity of closed geodesics and rigidity of Hopf fibrations*, Inventiones Mathematicae **144** (2001), no. 2, 281–295.

(Radeschi) Mathematisches Institut, WWU Münster, Germany.

E-mail address: mrade02@uni-muenster.de