Translating solutions of the nonparametric mean curvature flow with nonzero Neumann boundary data in product manifold $M^n \times \mathbb{R}$

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Abstract

In this paper, we can prove the existence of translating solutions to the nonparametric mean curvature flow with nonzero Neumann boundary data in a prescribed product manifold $M^n \times \mathbb{R}$, where $M^n$ is an $n$-dimensional ($n \geq 2$) complete Riemannian manifold with nonnegative Ricci curvature, and $\mathbb{R}$ is the Euclidean 1-space.

1 Introduction

The mean curvature flow (MCF for short) is one of the most important extrinsic curvature flows and has many nice applications. For instance, by using the curve shortening flow (i.e., the lower-dimensional case of MCF), Topping [20] successfully gave an isoperimetric inequality on simply connected surfaces with Gaussian curvature satisfying some integral precondition. This result extends those isoperimetric inequalities (introduced in detail in, e.g., [6, 21]) obtained separately by Alexandrov, Fiala-Huber, Bol, and Bernstein-Schmidt. Applying the long-time existence and convergence conclusions of graphic MCF of any codimension in prescribed product manifolds (see [23]), Wang [24] showed that for a bounded $C^2$ convex domain $\mathbb{D}$ (with diameter $\delta$ and boundary $\partial \mathbb{D}$) in the Euclidean n-space $\mathbb{R}^n$ and $\phi : \partial \mathbb{D} \to \mathbb{R}^m$ a continuous map, there exists a map $\psi : \mathbb{D} \to \mathbb{R}^m$, with $\psi|_{\partial \mathbb{D}} = \phi$ and with the graph of $\psi$ a minimal submanifold in $\mathbb{R}^{n+m}$, provided $\psi|_{\partial \mathbb{D}}$ is a smooth map and $8n\delta \sup_{\partial \mathbb{D}} |D^2 \psi| + \sqrt{2} \sup_{\partial \mathbb{D}} |D \psi| < 1$. This conclusion provides classical solutions to the Dirichlet problem for minimal surface systems in arbitrary codimensions for a class of boundary maps. Specially, when $m = 1$, the existence of $\psi$ was obtained by Jenkins and Serrin [14] already. Inspired by Wang’s work mentioned above, by applying the spacelike MCF in the Minkowski space $\mathbb{R}^{n+m,n}$, Mao [17] can successfully get the existence of $\psi$ for maximal spacelike submanifolds (with index $n$) in $\mathbb{R}^{n+m,n}$.

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In the early study of the theory of MCF, a classical result from Huisken [11] says that a given compact strictly convex hypersurface $M^n$ in $\mathbb{R}^{n+1}$ evolving along the MCF would contract to a single point at finite time. More precisely, let $X(\cdot,t) = X_t$ be a one-parameter family of immersions $X_t : M^n \to \mathbb{R}^{n+1}$ whose images $M^n_t = X_t(M^n)$ satisfy

$$\frac{\partial}{\partial t} X(x,t) = \vec{H}, \quad \text{on } M^n \times [0,T), \quad (1.1)$$

for some $T > 0$, with the initial condition $X(x,0) = X_0(x)$ on $M^n$, where $\vec{H}$ is the mean curvature vector of the evolving hypersurface $M^n_t$, by using the method of $L^p$ estimates, Huisken [11] proved that if $M^n$ is a compact strictly convex hypersurface in $\mathbb{R}^{n+1}$, the MCF equation (1.1), with the initial condition, has a unique smooth solution on the finite time interval $[0,T_{\text{max}})$ with $T_{\text{max}} < \infty$, and the evolving hypersurfaces $M^n_t$ contract to a single point as $t \to T_{\text{max}}$. By imposing a pinching condition on the second fundamental form of the initial hypersurface, Huisken [12] has extended the above conclusion to a more general setting that the ambient space $\mathbb{R}^{n+1}$ was replaced by a class of smooth complete Riemannian manifolds $\mathbb{N}^{n+1}$ having some uniform bounds for curvatures and injectivity radius (of course, $\mathbb{N}^{n+1}$ covers $\mathbb{R}^{n+1}$ as a special case). From these two facts, one might know that generally the MCF would occur singularity at finite time. A natural question is:

**Problem 1. When does the MCF exist for all the time?**

That is to say, under specified settings, there is no singularity formed during the evolution of MCF.

If there exists a constant vector $V$ such that

$$\vec{H} = V^\perp,$$

then the evolving submanifold $X_t : M^n \to \mathbb{R}^{n+m}$ is called a translating soliton of the MCF equation (1.1). Here $(\cdot)^\perp$ denotes the normal projection of a prescribed vector to the normal bundle of $M^n_t$ in $\mathbb{R}^{n+m}$. It is easy to see that the translating soliton gives an eternal solution $X_t = X_0 + tV$ to (1.1), which is called the translating solution. Translating solitons play an important role in the study of type-II singularities of the MCF. For instance, Angenent and Velázquez [4, 5] gave some examples of convergence which implies that type-II singularities of the MCF there are modeled by translating surfaces. Clearly, the existence of translation solutions to the equation (1.1) can give a positive answer to Problem 1.

Huisken [13] considered the evolution of graphic hypersurfaces over a bounded domain (with smooth boundary) in $\mathbb{R}^n$ under the MCF with a vanishing Neumann boundary condition (NBC for short), and proved that the flow exists for all the time and evolving graphic hypersurfaces in $\mathbb{R}^{n+1}$ converge to the graph of a constant function as $t \to \infty$. The vanishing NBC here has strong geometric meaning, that is, the evolving graphic hypersurface is perpendicular with the parabolic boundary during the evolution (or the contact angle between the evolving graphic hypersurface and parabolic boundary is $\pi/2$). Is the vanishing NBC necessary? What about the non-vanishing case? There are many literatures working on this direction and we would like to mention some of them. When the dimension $n$ satisfies $n = 1$ or $n = 2$, Altschuler and Wu [2, 3] gave a positive answer to these questions. In fact, they proved:

- when $n = 1$, a graphic curve defined over an open bounded interval evolves along the flow given by a class of quasilinear parabolic equations (of course, including the MCF as a special
case), with arbitrary contact angle (i.e., with nonzero NBC), would exist for all the time, and the evolving curves converge as \( t \to \infty \) to a solution moving by translation with speed uniquely determined by the boundary data.

- when \( n = 2 \), a graphic surface defined over a compact strictly convex domain (with smooth boundary) in \( \mathbb{R}^2 \) evolves along the MCF, with arbitrary contact angle (i.e., with nonzero NBC), would exist for all the time, and the evolving surfaces converge as \( t \to \infty \) to a surface (unique up to translation) which moves at a constant speed (uniquely determined by the boundary data).

For the higher dimensional case, Guan \cite{10} have given a partial answer. In fact, he can get the long-time existence of the evolution of graphic hypersurfaces, defined over a bounded domain (with smooth boundary) in \( \mathbb{R}^n \), under a nonparametric mean curvature type flow (i.e., the MCF with a forcing term given by an admissible function defined therein) with nonzero NBC. However, the asymptotic behavior of the flow cannot be obtained in his setting. Zhou \cite{25} extended Altschuler-Wu’s conclusion \cite{3} to the situation that graphic surfaces were defined over a compact strictly convex domain (with smooth boundary) in \( 2 \)-dimensional Riemannian surfaces \( M^2 \) with nonnegative Ricci curvature, and extended Guan’s conclusion \cite{10} to the situation that graphic hypersurfaces were defined over a bounded domain (with smooth boundary) in \( n \)-dimensional (\( n \geq 2 \)) Riemannian manifolds \( M^n \). However, similar to Guan’s work \cite{10}, Zhou \cite{25} also cannot give the asymptotic behavior of the MCF with a forcing term (given by an admissible function) and with nonzero NBC in product manifolds \( M^n \times \mathbb{R} \). Recently, Ma, Wang and Wei \cite{16} improved Huisken’s work \cite{13} to a more general setting that the vanishing NBC therein can be replaced by a nonzero NBC of specialized type.

Our purpose here is trying to extend the main conclusion in \cite{16} to a more general case – the ambient space \( \mathbb{R}^{n+1} \) will be replaced by product manifolds of type \( M^n \times \mathbb{R} \), where \( M^n \) is a complete Riemannian manifold of nonnegative Ricci curvature.

Throughout this paper, let \((M^n, \sigma)\) be a complete \( n \)-manifold (\( n \geq 2 \)) with the Riemannian metric \( \sigma \), and let \( \Omega \subset M^n \) be a compact strictly convex domain with smooth boundary \( \partial \Omega \). Denote by \( \{U_1; w_{A}^1, w_{A}^2, \ldots, w_{A}^n\} \) the local coordinate coverings of \( M \), and \( \frac{\partial}{\partial w_A^i} \), \( i = 1, 2, \ldots, n \), the corresponding coordinate vector fields, where \( A \in I \subset N \) with \( N \) the set of all positive integers. For simplicity, we just write \( \{w_{A}^1, w_{A}^2, \ldots, w_{A}^n\} \) as \( \{w^1, w^2, \ldots, w^n\} \) to represent the local coordinates on \( M \), and write \( \frac{\partial}{\partial w_A^i} \) as \( \frac{\partial}{\partial w^i} \) or \( \partial_i \). In this setting, the metric \( \sigma \) should be \( \sigma = \sum_{i,j=1}^n \sigma_{ij} dw^i \otimes dw^j \) with \( \sigma_{ij} = \sigma(\partial_i, \partial_j) \). Denote by \( D, D^2 \Omega \) the covariant derivatives on \( \Omega \) and \( \partial \Omega \) respectively. Now, we would like to consider, along the MCF \cite{11} with nonzero NBC, the evolution of graphic hypersurfaces, defined over \( \Omega \), in product manifold \( M^n \times \mathbb{R} \) with the product metric \( \overline{g} = \sigma_{ij} dw^i \otimes dw^j + ds \otimes ds \). More precisely, given a smooth \( \Gamma \) graphic hypersurface \( \mathcal{G} \subset M^n \times \mathbb{R} \) defined over \( \Omega \), then there exists a smooth function \( u_0 \in C^\infty(\overline{\Omega}) \) such that \( \mathcal{G} \) can be represented by \( \mathcal{G} := \{(x, u_0(x)) | x \in \Omega \} \).

It is not hard to know that the metric of \( \mathcal{G} \) is given by \( g = i^* \overline{g} \), where \( i^* \) is the pullback mapping of the immersion \( i : \mathcal{G} \hookrightarrow M^n \times \mathbb{R} \), tangent vectors are given by

\[
\overline{e}_i = \partial_i + D_1 u \partial_x, \quad i = 1, 2, \ldots, n,
\]

\(^{1}\)In fact, it is not necessary to impose smoothness assumption on the initial hypersurface \( \mathcal{G} \). The \( C^{2,\alpha} \)-regularity for \( \mathcal{G} \) is enough to get all the estimates in the sequel. However, in order to avoid the boring regularity arguments, which is not necessary, here we assume \( \mathcal{G} \) is smooth.
and the corresponding upward unit normal vector is given by
\[ \nabla_{\vec{e}} \nabla x = \frac{\sum_{i=1}^{n} D^i u \partial_i - \partial_x}{\sqrt{1 + |Du|^2}}, \]
where \( D^i u = \sum_{i=1}^{n} \sigma^{ij} D_i u \). Denote by \( \nabla \) the covariant derivative operator on \( M^n \times \mathbb{R} \), and then the second fundamental form \( h_{ij} \omega^i \otimes \omega^j \) of \( \mathcal{G} \) is given by
\[ h_{ij} = -\langle \nabla_{\vec{e}} \nabla x, \nabla_{\vec{e}} \nabla x \rangle = -\frac{D_i D_j u}{\sqrt{1 + |Du|^2}}. \]
Moreover, the scalar mean curvature of \( \mathcal{G} \) is
\[ H = \sum_{i=1}^{n} h^i = -\sum_{i,k=1}^{n} g^{ik} D_i D_k u \frac{\sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}}{\sqrt{1 + |Du|^2}}. \]
(1.2)

Hence, in our situation here, the evolution of \( \mathcal{G} \) under the MCF with nonzero NBC in \( M^n \times \mathbb{R} \) with the metric \( \bar{g} \) can be reduced to solvability of the following initial-boundary value problem (IBVP for short)
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{n} \left( \sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) D_i D_j u & \text{in } \Omega \times [0, T), \\
D_{\bar{g}} u &= \phi (x) & \text{on } \partial \Omega \times [0, T), \\
u (\cdot, 0) &= u_0 (\cdot) & \text{on } \Omega_0,
\end{aligned}
\]
(1.3)
where \( \bar{v} \) is the inward unit normal vector of \( \partial \Omega \), \( \Omega_t = \Omega \times \{ t \} \) is a slice in \( \Omega \times [0, T) \), \( u_0 (x) \in C^\infty (\overline{\Omega}) \) and \( \phi (x) \in C^\infty (\overline{\Omega}) \) are smooth function satisfying
\[ u_0, \bar{v} = \phi (x) \quad \text{on } \partial \Omega. \]

Here (1.3) is called compatibility condition of the system (\( \sharp \)), and a comma “,” in the subscript means doing covariant derivative w.r.t. a prescribed tensor. This convention will also be used in the sequel. For the IBVP (\( \sharp \)), we can prove:

**Theorem 1.1.** If the Ricci curvature of \( M^n \) is nonnegative, then for the IBVP (\( \sharp \)), we have

1. the IBVP (\( \sharp \)) has a smooth solution \( u(x, t) \) on \( \overline{\Omega} \times [0, \infty) \);
2. the smooth solution \( u(x, t) \) converges as \( t \to \infty \) to \( \lambda t + w(x) \), i.e.,
\[ \lim_{t \to \infty} \| u(x, t) - (\lambda t + w(x)) \|_{C^0 (\overline{\Omega})} = 0, \]
where \( \lambda \in \mathbb{R} \) and \( w \in C^{2, \alpha} (\overline{\Omega}) \) (unique up to a constant) solving the following boundary value problem (BVP for short)
\[
\begin{aligned}
\sum_{i,j=1}^{n} \left( \sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) D_i D_j u &= \lambda & \text{in } \Omega, \\
D_{\bar{g}} u &= \phi (x) & \text{on } \partial \Omega.
\end{aligned}
\]
(\#)

Here \( 0 < \alpha < 1 \) and \( \lambda \) is called the additive eigenvalue of the BVP (\( \sharp \)).
Remark 1.2. (I) By (1.2), it is easy to know that
\[ \sum_{i,j=1}^{n} \left( \sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) D_i D_j u = H \cdot \sqrt{1 + |Du|^2} = \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) \cdot \sqrt{1 + |Du|^2}, \]
which, substituting into the first equation of (‡), implies
\[ \text{div} \left( \frac{Dw}{\sqrt{1 + |Dw|^2}} \right) = \frac{\lambda}{\sqrt{1 + |Dw|^2}}, \]
where \( u = w(x) \) is the solution to the BVP (‡). Integrating the above equality and using the divergence theorem, one can get
\[ \lambda = -\frac{\int_{\partial \Omega} \frac{\phi(x)}{\sqrt{1 + |Dw|^2}}}{\int_{\Omega} \frac{1}{1 + |Dw|^2} \cdot \frac{1}{2}}. \]
Clearly, if \( \phi(x) \equiv 0 \), then \( \lambda = 0 \). Moreover, in this setting, for the IBVP (‡), as \( t \to \infty \), its smooth solution \( u(x,t) \) would converge to a constant function defined over \( \Omega \subset M^n \).
(II) We would like to mention one thing, that is, if \( M^n = \mathbb{R}^n \) and \( \phi(x) \equiv 0 \), then Theorem [1.1] here degenerates into Huisken’s main conclusion in [13]; if \( M^n = \mathbb{R}^n \), our main conclusion here becomes exactly [16, Theorems 1.1 and 1.2].
(III) Recent years, the study of submanifolds of constant curvature in product manifolds attracts many geometers’ attention. For instance, Hopf in 1955 discovered that the complexity of the traceless part of the second fundamental form of an immersed surface \( \Sigma^2 \), with constant mean curvature (CMC for short) \( H \), in \( \mathbb{R}^3 \) is a holomorphic quadratic differential \( Q \) on \( \Sigma^2 \), and then he used this observation to get his well-known conclusion that any immersed CMC sphere \( S^2 \hookrightarrow \mathbb{R}^3 \) is a standard distance sphere with radius \( 1/H \). By introducing a generalized quadratic differential \( \tilde{Q} \) for immersed surfaces \( \Sigma^2 \) in product spaces \( S^2 \times \mathbb{R} \) and \( H^2 \times \mathbb{R} \), with \( S^2 \), \( H^2 \) the 2-dimensional sphere and hyperbolic surface respectively, Abresch and Rosenberg [11] can extend Hopf’s result to CMC spheres in these target spaces. Meeks and Rosenberg [18] successfully classified stable properly embedded orientable minimal surfaces in the product space \( M \times \mathbb{R} \), where \( M \) is a closed orientable Riemannian surface. In fact, they proved that such a surface must be a product of a stable embedded geodesic on \( M \) with \( \mathbb{R} \), a minimal graph over a region of \( M \) bounded by stable geodesics, \( M \times \{ t \} \) for some \( t \in \mathbb{R} \), or in a moduli space of periodic multigraphs parameterized by \( P \times \mathbb{R}^+ \), where \( P \) is the set of primitive (non-multiple) homology classes in \( H_1(M) \). Mazet, Rodríguez and Rosenberg [15] analyzed properties of periodic minimal or constant mean curvature surfaces in the product manifold \( H^2 \times \mathbb{R} \), and they also construct examples of periodic minimal surfaces in \( H^2 \times \mathbb{R} \). In [19], Rosenberg, Schulze and Spruck showed that a properly immersed minimal hypersurface in \( M \times \mathbb{R}^+ \) equals some slice \( M \times \{ c \} \) when \( M \) is a complete, recurrent \( n \)-dimensional Riemannian manifold with bounded curvature. Of course, for more information, readers can check references therein of these papers. Hence, it is interesting and important to consider submanifolds of constant curvature in the product manifold of type \( M^n \times \mathbb{R} \). Based on this reason, in our setting here, it should be interesting and important to consider the following CMC equation with nonzero NBC
\[ (\ddagger) \begin{cases} H = \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \lambda & \text{in } \Omega, \\ D_\nu u = \phi(x) & \text{on } \partial \Omega. \end{cases} \]
Of course, all the symbols in the above system have the same meaning as those in (‡). The existence and uniqueness of solution to the BVP (‡) have been obtained recently – see [9] for details.

(IV) The evolution of space-like surfaces in the Lorentz 3-manifold $M^2 \times \mathbb{R}$ under the MCF with arbitrary contact angle (of course, in this situation, the NBC is nonzero) has been investigated in [7], and the long-time existence and the existence of translating solutions to the flow have been obtained.

(V) As we know, if the warping function was chosen to be a constant function, then warped product manifolds would degenerate into product manifolds. Hence, one might ask “whether one could expect to get a similar conclusion to Theorem 1.1 in warped product manifolds or not?”. By constructing an interesting graphic hypersurface example in a prescribed warped product (see [25, Appendix A]), Zhou gave a negative answer to this question. Speaking in other words, he showed that the MCF with nonzero NBC in warped product manifolds would form singularities within finite time.

(VI). In fact, Huisken [13] considered the following IBVP

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{n} \left( \delta^{ij} - \frac{\delta^{ij}D^{ij}u}{1+|Du|^2} \right) D_{\eta}D_{\eta}u \\
D_{\nu}u &= 0 \\
u(\cdot, 0) &= u_0(\cdot)
\end{aligned}
\]

in $\Omega \times [0, T)$, on $\partial \Omega \times [0, T)$, and on $\Omega_0$, which, as mentioned before, describes the evolution of graphic hypersurfaces over $\Omega \subset \mathbb{R}^n$ under the MCF with a zero NBC, and obtained the long-time existence, i.e., $T = \infty$. The vanishing NBC here means that

\[
\langle \bar{\gamma}, \bar{\nu} \rangle_g = \frac{D_{\nu}u}{\sqrt{1+|Du|^2}} = 0,
\]

which is to say $\bar{\gamma} \perp \bar{\nu}$, i.e., the contact angle between $\bar{\gamma}$ and $\bar{\nu}$ is $\pi/2$. If the contact angle is arbitrary, then the corresponding NBC should have the form $D_{\nu}u |_{\partial \Omega} = \phi(x) \cdot \sqrt{1+|Du|^2}$ for some $\phi(x) \in C^\infty(\overline{\Omega})$, $|\phi(x)| \leq 1$ on $\partial \Omega$, and $\phi(x) = u_0, \bar{\nu}$ on $\partial \Omega$. Based on this reason, we can say that although the IBVP (‡) has nonzero NBC, the geometric meaning of the NBC in (‡) is not sufficient. Can we deal with the IBVP (‡) if the RHS of the nonzero NBC therein contains $Du$ also? Inspired by a recent work [22], Gao and Mao [8] considered a generalization of the IBVP (‡) where the NBC can be replaced by

\[
D_{\nu}u = \phi(x) \cdot \left( \sqrt{1+|Du|^2} \right)^{1-q}
\]

for any $q > 0$, and similar conclusion to Theorem [11] could be derived.

This paper is organized as follows. In Section 2, the time-derivative estimate, the gradient estimate, and the estimate for higher-order derivatives of $u$ will be shown in detail, which naturally lead to the long-time existence of the IBVP (‡). In section 3, by using an approximating approach, the solvability of the BVP (‡) can be given first, which will be used later to get the asymptotic behavior of solutions $u$ to the IBVP (‡).
2 The long-time existence

For convenience, we use several notations as follows:

\[
V = \sqrt{1 + |Du|^2},
\]

\[
g_{ij} = \sigma_{ij} + D_i u D_j u,
\]

\[
g^{ij} = \sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2},
\]

\[
u_t = \frac{\partial u}{\partial t}.
\]

For vectors, \(V, W\) or matrices \(A, B\), we shall use the shorthand as follows:

\[
\langle V, W \rangle_g = \sum_{i,j=1}^n g^{ij}V_i W_j, \quad \langle V, W \rangle_\sigma = \sum_{i,j=1}^n \sigma^{ij}V_i W_j, \quad \langle A, B \rangle_g,\sigma = \sum_{i,j,k,l=1}^n g^{ij} \sigma^{kl} A_{ik} B_{jl}.
\]

First, by applying a similar method to that in the proof of [3, Lemma 2.2], we would like to show the time-derivative estimate for \(u\).

**Lemma 2.1.** For the IBVP (\(\xi\)), we have

\[
\sup_{\Omega \times [0,T]} |u_t|^2 = \sup_{\Omega_0} |u_t|^2.
\]

That is to say, there exists some positive constant \(c_0 = c_0(u_0) \in \mathbb{R}^+\) such that for any \((x,t) \in \overline{\Omega} \times [0,T]\), we have

\[
|u_t|^2(x,t) \leq c_0.
\]

**Proof.** We first show that the maximum of \(u_t\) must occur on \((\partial \Omega \times [0,T]) \cup \Omega_0\). By a direct computation, we have

\[
\frac{\partial}{\partial t} |u_t|^2 = 2u_t \frac{\partial u_t}{\partial t} = \sum_{i,j=1}^n 2u_t \left( \frac{\partial g^{ij}}{\partial t} D_i D_j u + g^{ij} D_i D_j u_t \right)
\]

\[
= \sum_{i,j,k=1}^n 2u_t \left( \frac{\partial g^{ij}}{\partial D^k u} \frac{\partial D^k u}{\partial t} D_i D_j u + \sum_{i,j=1}^n g^{ij} (D_i D_j |u_t|^2 - 2D_i u D_j u_t) \right)
\]

\[
= \sum_{i,j,k,l=1}^n 2u_t \left( \frac{\partial g^{ij}}{\partial D^k u} \frac{\partial \sigma^{kl} D^l D_j u}{\partial t} D_i D_j u + \sum_{i,j=1}^n g^{ij} D_i D_j |u_t|^2 - 2\langle Du_t, Du_t \rangle_\sigma \right)
\]

\[
= \sum_{i,j,k,l=1}^n \frac{\partial g^{ij}}{\partial D^k u} \sigma^{kl} D_i D_j u D_l |u_t|^2 + \sum_{i,j=1}^n g^{ij} D_i D_j |u_t|^2 - 2\langle Du_t, Du_t \rangle_\sigma.
\]
which implies
\[
\sup_{O \times [0,T]} |u_t|^2 = \sup_{(\partial O \times [0,T]) \cup \Omega_0} |u_t|^2
\]
by directly applying the weak maximum principle.

Next, we expel the possibility that the maximum occurs at \(\partial O \times [0,T]\). Assume that
\[
\max_{O \times \{t\}} |u_t|^2 = |u_t|^2(\xi, \tau) > 0
\]
for some \((\xi, \tau) \in \partial O \times [0,T]\). By the Hopf Lemma, it follows that \(\frac{\partial |u_t|^2}{\partial \nu}|_{(\xi, \tau)} < 0\). But by the boundary condition of the IBVP (2), one has
\[
\frac{\partial |u_t|^2}{\partial \nu}|_{(\xi, \tau)} = \frac{\partial}{\partial \nu} (D_\nu u|_{(\xi, \tau)}) = \frac{\partial}{\partial \nu} (\phi(x)) = 0.
\]
It is a contradiction. Therefore, the maximum cannot be achieved at \(\partial O \times [0,T]\). The conclusion of Lemma 2.1 follows.

We know that if \(O\) is a strictly convex domain with smooth boundary \(\partial O\), then there exists a smooth function \(\beta\) on \(O\) such that \(\beta|_O < 0\), \(\beta|_{\partial O} = 0\), \(\sup O |D\beta| \leq 1\),
\[
(\beta_{ij})_{n \times n} \geq k_0 (\delta_{ij})_{n \times n}
\]
for some positive constant \(k_0 > 0\), \(\beta_\nu = D_\nu \beta = -1\) and \(|D\beta| = 1\) on \(\partial O\). Besides, since \(O\) is strictly convex, we have
\[
(h_{ij}^{\partial O})_{(n-1) \times (n-1)} \geq \kappa_1 (\delta_{ij})_{(n-1) \times (n-1)},
\]
where \(h_{ij}^{\partial O}, 1 \leq i, j \leq n - 1\), is the second fundamental form of the boundary \(\partial O\), and \(\kappa_1 > 0\) is the minimal principal curvature of \(\partial O\).

**Lemma 2.2.** Assume that \(u(x,t) \in C^{3,2}(O \times [0,T])\) is a solution to the IBVP (2), and the Ricci curvature of \(M^n\) is nonnegative. Then there exists a constant \(c_1 := c_1(n, O, u_0, \phi(x))\) such that
\[
\sup_{O \times [0,T]} |Du| \leq c_1.
\]

**Proof.** To reach the conclusion of this lemma, we only need to prove that for \(0 < T' < T\), we can bound \(|Du|\) on \(O \times [0,T']\) independent of \(T'\) and then take a limit argument.

Let
\[
\Phi(x) := \log |D\omega|^2 + f(\beta),
\]
where
\[
\omega = u + \phi(x)\beta, \quad f = \zeta \beta,
\]
and \(\zeta\) is a positive constant which will be determined later. For convenience, denote by \(G = -\phi(x)\beta\).
We firstly show that the maximum of \( \Phi(x) \) on \( \overline{\Omega} \times [0, T'] \) cannot be achieved at the boundary \( \partial \Omega \times [0, T'] \).

Choose a suitable local coordinates around a point \( x_0 \in \overline{\Omega} \) such that \( \tau_n \) is the inward unit normal vector of \( \partial \Omega \), and \( \tau_i, i = 1, 2, \cdots, n - 1 \), are the unit smooth tangent vectors of \( \partial \Omega \). Denote by \( D_i u := u_i, D_{ij} u := u_{ij}, D_i D_j u := u_{ij} \) for \( 1 \leq i, j \leq n \). By the boundary condition, one has

\[
D_{\tau_n} \omega \mid_{\partial \Omega} = \omega_n \mid_{\partial \Omega} = u_n \mid_{\partial \Omega} + (\phi_{i\beta} \beta + \beta_n \phi) \mid_{\partial \Omega} = 0.
\]

If \( \Phi(x, t) \) attains its maximum at \( (x_0, t_0) \in \partial \Omega \times [0, T'] \), then at \( (x_0, t_0) \), we have

\[
0 \geq \Phi_n = \frac{|D\omega|_n^2}{|D\omega|^2} - \zeta
= \sum_{k=1}^{n-1} \frac{2 \omega^k D_{\tau_k} \tau_{\omega}}{|D\omega|^2} - \zeta
= \sum_{k=1}^{n-1} \frac{2 \omega^k [\tau_k (\tau_n (\omega)) - (D_\omega \tau_n) \omega] - \zeta}{|D\omega|^2}
= - \sum_{k=1}^{n-1} \frac{2 \omega^k (D_\omega \tau_n) \omega}{|D\omega|^2} - \zeta
= - \sum_{k=1}^{n-1} \frac{2 \omega^k \omega_j (D_\omega \tau_n) \tau_j}{|D\omega|^2} - \zeta
= \sum_{k,j=1}^{n-1} \frac{2 \omega^k \omega_j h_{k\omega}^{\rho \omega}}{|D\omega|^2} - \zeta
\geq 2 \kappa_1 - \zeta.
\]

Hence, by taking \( 0 < \zeta < 2 \kappa_1 \), the maximum of \( \Phi \) can only be achieved in \( \Omega \times [0, T'] \). BTW, there is one thing we would like to mention here, that is , in (2.1), the relation

\[
w^k = \sum_{l=1}^{n} \sigma^{k\ell} w_l = \sum_{l=1}^{n-1} \sigma^{k\ell} w_l
\]

holds. Here we have used the convention in Riemannian Geometry to deal with the subscripts and superscripts, and this convention will also be used in the sequel.

Assume that \( \Phi(x, t) \) attains its maximum at \( (x_0, t_0) \in \Omega \times [0, T'] \). By direct calculation, we have

\[
\Phi_t(x_0, t_0) = \frac{|D\omega|_i^2}{|D\omega|^2}, \quad \Phi_i(x_0, t_0) = \frac{|D\omega|_i^2}{|D\omega|^2} + \zeta \beta_i = 0
\]

\(^2\) Covariant derivatives of other tensors can be simplified similarly. For instance, one has \( \omega_i = D_{\tau_i} \omega, \omega_{ij} = D_{\tau_i} D_{\tau_j} \omega. \)
Since $\Phi_{ij}(x_0,t_0)$, we have

$$\Phi_{ij}(x_0,t_0) = \frac{|D\omega|_{ij}^2}{|D\omega|^2} - \frac{|D\omega|_{ij}^2|D\omega|_j^2}{|D\omega|^4} + \zeta \beta_{ij}$$

$$= \frac{|D\omega|_{ij}^2}{|D\omega|^2} + \zeta \beta_{ij} - \zeta^2 \beta_i \beta_j. \quad (2.3)$$

Since $g^{ij} = \sigma^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}$, we have

$$0 \geq \sum_{i,j=1}^n g^{ij} \Phi_{ij} - \Phi_t$$

$$= \sum_{i,j=1}^n g^{ij} \frac{|D\omega|_{ij}^2}{|D\omega|^2} - \frac{|D\omega|_{ij}^2|D\omega|_j^2}{|D\omega|^4} + \zeta \sum_{i,j=1}^n g^{ij} \beta_{ij} - \zeta^2 \sum_{i,j=1}^n g^{ij} \beta_i \beta_j$$

$$\triangleq I_1 + I_2, \quad (2.4)$$

where

$$I_1 = \sum_{i,j=1}^n g^{ij} \frac{|D\omega|_{ij}^2}{|D\omega|^2} - \frac{|D\omega|_{ij}^2}{|D\omega|^2}$$

and

$$I_2 = \sum_{i,j=1}^n (\zeta g^{ij} \beta_{ij} - \zeta^2 g^{ij} \beta_i \beta_j).$$

At $(x_0, t_0)$, one can make a suitable change to the coordinate vector fields $\{\tau_1, \tau_2, \cdots, \tau_n\}$ such that $|Du| = u_1$, $(u_{ij})_{2 \leq i,j \leq n}$ is diagonal, and $(\sigma_{ij})_{2 \leq i,j \leq n}$ is diagonal. Clearly, in this setting, $\sigma^{11} = 1$. Besides, we Then

$$g^{11} = \frac{1}{v^2}, \quad g^{ij} = 0 \text{ for } 2 \leq i,j \leq n, i \neq j, \text{ and } g^{ii} = \sigma^{ii} \text{ for } i \geq 2,$$

$$(g^{ij}) = \left( \sigma^{ij} - \frac{D^i u D^j u}{v^2} \right) = - \left( \frac{2u^i u^j}{v^2} - \sum_{m=1}^n \frac{2u^m u_m u^i u^j}{v^4} \right),$$

where $v = \sqrt{1 + |Du|^2} = \sqrt{1 + u_1^2}$.

Assume that $u_1$ is big enough such that $u_1, \omega_1, \omega^1, |D\omega|$ and $v$ are equivalent with each other at $(x_0, t_0)$. Otherwise, the conclusion of Lemma 2.2 is proved. It’s also noticeable that $|\omega_i| \leq c_2$.

---

3 This change can always be found. In fact, one can firstly rotate $\tau_i, i = 1, 2, \cdots, n$, such that the gradient vector $Du$ lies in the same or the opposite direction with $\tau_1$. Denote by the hyperplane, which is orthogonal with $\tau_1$, by $\Pi$. Then rotate $\tau_2, \tau_3, \cdots, \tau_n$ in $\Pi$, corresponding to an orthogonal matrix, such that the real symmetric matrices $(u_{ij})_{2 \leq i,j \leq n}$, $(\sigma_{ij})_{2 \leq i,j \leq n}$ change into diagonal matrices.
\( i = 2, \cdots, n \), for some nonnegative constant \( c_2 \). Here, in the proof, \( c_2 \) is denoted to be a nonnegative constant which may change in different places but has nothing to do with \( T \). Since \( \left( \beta_{ij} \right)_{n \times n} \geq k_0 \left( \delta_{ij} \right)_{n \times n} \), one can easily get

\[
I_2 = \sum_{i,j=1}^{n} (\xi g^{ij} \beta_{ij} - \xi^2 g^{ij} \beta_i \beta_j)
\]

\[
\geq \zeta \left[ \sum_{i=2}^{n} \sigma^i k_0 + \frac{k_0}{v^2} \right] - \zeta^2 \left( \frac{\beta_1^2}{v^2} + \sum_{i=2}^{n} \sigma^i \beta_i^2 \right). \tag{2.5}
\]

Set \( J := \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2 - |D\omega|_{i}^2 \), one has

\[
J = \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2 - |D\omega|_{i}^2
\]

\[
= \sum_{i,j,k=1}^{n} g^{ij} [2 \omega^k \omega_{ki} + 2 \omega^k \omega_{ikj} - 2 \sum_{k=1}^{n} \omega^k \omega_k]
\]

\[
= \sum_{i,j,k=1}^{n} g^{ij} [2 \omega^k \omega_{ki} + 2 \omega^k (u_{ijk} + R^l_{ikj} u_l - G_{ikj})] - 2 \sum_{k=1}^{n} \omega^k \omega_k
\]

\[
= \sum_{i,j,k=1}^{n} g^{ij} [2 \omega^k \omega_{ki} + 2 \omega^k (u_{ijk} + R^l_{ikj} u_l - G_{ikj})] - 2 \sum_{k=1}^{n} \omega^k \omega_k
\]

\[
= 2 \sum_{k=1}^{n} \omega^k \left[ \sum_{i,j=1}^{n} g^{ij} (u_{ijk} + R^l_{ikj} u_l - G_{ikj}) - \omega_k \right] + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega_{ki} \omega^k
\]

\[
= 2 \sum_{k=1}^{n} \omega^k \left[ \sum_{i,j=1}^{n} g^{ij} (u_{ijk} + R^l_{ikj} u_l - G_{ikj}) - \omega_k \right] + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega_{ki} \omega^k
\]

\[
= -2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k G_{ijk} + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k R^l_{ikj} u_l - 2 \sum_{i,j,k=1}^{n} \omega^k (g^{ij}) u_{ijk} + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega_{ki} \omega^k
\]

\[
\triangleq J_1 + J_2 + J_3 + J_4,
\]

where \( R^l_{ikj} \), \( 1 \leq i, j, k, l \leq n \), are coefficients of the curvature tensor on \( M^n \). It is obvious that

\[
J_1 = -2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k G_{ijk}
\]

\[
\geq -c_2 v. \tag{2.6}
\]
Next we deal with $J_2, J_3$. In fact, using the nonnegativity of the Ricci curvature on $M^n$, we have

$$J_2 = 2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k R^1_{ikj} u_1$$

$$= 2 \sum_{i=2}^{n} \sigma_{ii} R^{1}_{i11} v^2 + 2 \sum_{i,k=2}^{n} \sigma_{ii} \omega^k R^1_{ikj} v + 2 R^1_{i11} + 2 \sum_{k=2}^{n} \frac{\omega^k R^1_{kk1}}{v}$$

$$\geq c_2 v^2 + O(1) v + 2 R^1_{i11}$$

$$\geq c_2 v^2 + O(1) v, \quad (2.7)$$

and

$$J_3 = -2 \sum_{i,j,k=1}^{n} \omega^k (g^{ij}) u_{ij}$$

$$= \sum_{i,j,k=1}^{n} 4 \omega^k u^i_k u^j_i u_{ij} - \sum_{m,i,j,k=1}^{n} 4 \omega^k \sigma_{ij} u^m_i u^j_i u_{mik} u_{ij}$$

$$= \sum_{i,j,k=1}^{n} 4 \omega^k \sigma_{ij} u^i_1 u^j_1 u_{ij} - 4 \sum_{k=1}^{n} 4 \omega^k u^3_i u^j_1 u_{ij}$$

$$= 4 \frac{\omega^1 (u_{11})^2 u_1}{v^2} + 4 \sum_{i=2}^{n} \frac{\omega^1 u^1_i u_{11} u_{1i}}{v^4} + 4 \sum_{i=2}^{n} \frac{\omega^1 u^1_i \sigma_{ii} (u_{11})^2}{v^2} + 4 \sum_{i=2}^{n} \frac{\omega^1 \sigma_{ii} u^1_i u_{1i} u_{11}}{v^2}$$

$$\triangleq J_{31} + J_{32} + J_{33} + J_{34}.$$ 

By (2.2) and (2.3), for $2 \leq i \leq n$, we have

$$u_{1i} - G_{1i} = \frac{-\zeta \beta |D\omega|^2 - 2 \omega^i u_{ii} + 2 \sum_{k=2}^{n} \omega^k G_{ik}}{2 \omega^1} \quad (2.8)$$

and

$$\sum_{i=2}^{n} 2 \omega^i (u_{1i} - G_{1i}) \frac{|D\omega|^2}{|D\omega|^2} = -\zeta \beta \frac{2 \omega^1 (u_{11} - G_{11})}{|D\omega|^2}. \quad (2.9)$$

By (2.8), for $2 \leq i \leq n$, it follows that

$$u_{1i} = \frac{-\zeta \beta |D\omega|^2 - 2 \omega^i u_{ii} + 2 \sum_{k=2}^{n} \omega^k G_{ik}}{2 \omega^1} + G_{1i}$$

$$= -\frac{1}{2} \zeta \beta v - \frac{\omega^i u_{ii}}{\omega^1} + O(1). \quad (2.10)$$

By (2.9), we have

$$\sum_{i=2}^{n} \frac{2 \omega^i (u_{1i} - G_{1i})}{|D\omega|^2} = \sum_{i=2}^{n} \frac{2 \omega^i}{|D\omega|^2} \left( \frac{-\zeta \beta |D\omega|^2 - 2 \omega^i u_{ii} + 2 \sum_{k=2}^{n} \omega^k G_{ik}}{2 \omega^1} \right)$$

$$= \sum_{i=2}^{n} O\left( \frac{|\zeta \beta|}{v} \right) - \sum_{i=2}^{n} \frac{2 \omega^i u_{ii}}{|D\omega|^2 \omega^1} + \sum_{i,k=2}^{n} \frac{2 \omega^i \omega^k G_{ki}}{|D\omega|^2 \omega^1}. \quad (2.11)$$
By \((2.10)\) and \((2.11)\), we have

\[
-\zeta \beta_1 - \frac{2\omega^1 (u_{11} - G_{11})}{|D\omega|^2} = \sum_{i=2}^{n} O\left(\frac{|\zeta \beta_i|}{v}\right) - \sum_{i=2}^{n} \frac{2(\omega^i)^2 u_{ii}}{|D\omega|^2 \omega^1} + \sum_{i, k=2}^{n} \frac{2\omega^i \omega^k G_{ki}}{|D\omega|^2 \omega^1}.
\]

\[
-\zeta \beta_1 - \frac{2v (u_{11} - G_{11})}{v^2} = O\left(\frac{1}{v}\right) + O\left(\frac{1}{v^3}\right) u_{ii} + O\left(\frac{1}{v^3}\right).
\]

So,

\[
u_{11} = -\frac{1}{2} \zeta \beta_1 v + \sum_{i=2}^{n} O\left(\frac{1}{v^2}\right) u_{ii} + O(1) \tag{2.12}
\]

Now, we deal with \(J_{31}, J_{32}, J_{33}, J_{34}\) respectively. It is obvious that

\[
J_{31} + J_{33} \geq 0. \tag{2.13}
\]

For the term \(J_{32}\),

\[
J_{32} = 4 \sum_{i=2}^{n} \frac{\omega^i u_{1i} u_{1i}}{v^4}
\]

\[
= 4 \sum_{i=2}^{n} \frac{\omega^i u_{1i}}{v^4} \left(-\frac{1}{2} \zeta \beta_1 v + \sum_{i=2}^{n} O\left(\frac{1}{v^2}\right) u_{ii} + O(1)\right) \cdot \left(-\frac{1}{2} \zeta \beta_i v - \frac{\omega^i u_{ii}}{\omega^1} + O(1)\right)
\]

\[
= O\left(\frac{\zeta^2 |\beta_1 \beta_1|}{v}\right) + O\left(\frac{1}{v^2}\right) + \sum_{i=2}^{n} \left[O\left(\frac{|\zeta \beta_1|}{v^2}\right) + O\left(\frac{|\zeta \beta_i|}{v^4}\right)\right] u_{ii} + \sum_{i=2}^{n} O\left(\frac{1}{v^6}\right) u_{ii}^2. \tag{2.14}
\]

Besides, we have

\[
J_{34} = 4 \sum_{i=2}^{n} \frac{\omega^i \sigma^i u_{1i} u_{1i}}{v^2}
\]

\[
= 4 \sum_{i=2}^{n} \frac{\omega^i u_{1i} \sigma^i u_{ii}}{v^2} \left(-\frac{1}{2} \zeta \beta_i v - \frac{\omega^i u_{ii}}{\omega^1} + O(1)\right)
\]

\[
= \sum_{i=2}^{n} O\left(\frac{1}{v^2}\right) u_{ii}^2 + \sum_{i=2}^{n} O(\zeta |\beta_i|) u_{ii}. \tag{2.15}
\]

By \((2.13)-(2.15)\), it follows that

\[
J_3 \geq \sum_{i=2}^{n} \left[O\left(\frac{1}{v^6}\right) + O\left(\frac{1}{v^2}\right)\right] u_{ii}^2 + \sum_{i=2}^{n} \left[O\left(\frac{|\zeta \beta_1|}{v^2}\right) + O\left(\frac{|\zeta \beta_i|}{v^4}\right) + O(\zeta |\beta_i|)\right] u_{ii}
\]

\[
+ O\left(\frac{\zeta^2 |\beta_1 \beta_1|}{v}\right) + O\left(\frac{1}{v^2}\right). \tag{2.16}
\]
Then, for $J_4$, we have

$$J_4 = 2 \sum_{i,j,k=1}^{n} g^{ij} \omega_{ki} \omega_{kj}$$

$$= 2 \sum_{i,j,k=1}^{n} g^{ij} \sigma^{kl} \omega_{ki} \omega_{lj}$$

$$= 2 \sum_{i=1}^{n} \frac{\sigma^{ii}}{v^2} (\omega_{ii})^2 + 2 \sum_{i=2}^{n} \sigma^{ii} (\omega_{ii})^2 + 2 \sum_{i=2}^{n} (\sigma^{ii})^2 (\omega_{ii})^2$$

$$\geq \sum_{i=2}^{n} (1 + \frac{1}{v^2}) \sigma^{ii} u_{ii}^2 + \sum_{i=2}^{n} (\sigma^{ii})^2 u_{ii}^2 - c_2. \quad (2.17)$$

By (2.6), (2.7), (2.16) and (2.18), we write all the terms containing $u_{ii}$ in $J$ as below

$$\sum_{i=2}^{n} \left[ O\left(\frac{1}{v^2}\right) + O\left(1\right) + (\sigma^{ii})^2 \right] u_{ii}^2 + \sum_{i=2}^{n} \left[ O\left(\frac{\zeta \beta_i}{v}\right) + O\left(\frac{1}{v^2}\right) + O\left(\frac{1}{v^2}\right) \right] u_{ii}$$

$$\geq -\sum_{i=2}^{n} \frac{O\left(\frac{\zeta |\beta_i|^2}{(\sigma^{ii})^2}\right)}{\left(\sigma^{ii}\right)^2} u_{ii}^2 - c_2 v + O\left(1\right) v + c_2 v^2. \quad (2.18)$$

Therefore, we can obtain

$$J = J_1 + J_2 + J_3 + J_4$$

$$\geq -\sum_{i=2}^{n} \frac{O\left(\frac{\zeta |\beta_i|^2}{(\sigma^{ii})^2}\right)}{\left(\sigma^{ii}\right)^2} u_{ii}^2 - c_2 v + O\left(1\right) v + c_2 v^2. \quad (2.18)$$

Hence,

$$I_1 = \frac{J}{|D\omega|^2}$$

$$\geq -\sum_{i=2}^{n} \frac{O\left(\frac{\zeta |\beta_i|^2}{(\sigma^{ii})^2}\right)}{\left(\sigma^{ii}\right)^2} u_{ii}^2 - c_2 v + O\left(1\right) v - c_2 v^2$$

$$\geq -\frac{\sum_{i=2}^{n} \frac{O\left(\frac{\zeta |\beta_i|^2}{(\sigma^{ii})^2}\right)}{\left(\sigma^{ii}\right)^2} u_{ii}^2 - c_2 v + O\left(1\right) v - c_2 v^2}{|D\omega|^2}. \quad (2.19)$$

By (2.4), (2.5) and (2.19), at the maximum point $(x_0, t_0)$, we can get

$$0 \geq \sum_{i,j=1}^{n} g^{ij} \Phi_{ij} - \Phi_t$$

$$\geq -\sum_{i=2}^{n} \frac{O\left(\frac{\zeta |\beta_i|^2}{(\sigma^{ii})^2}\right)}{\left(\sigma^{ii}\right)^2} u_{ii}^2 - c_2 v + O\left(1\right) v - c_2 v^2$$

$$\geq -\zeta \left[ \sum_{i=2}^{n} \sigma^{ii} k_0 + \frac{k_0}{v^2} \right] - \zeta^2 \left( \frac{1}{v^2} + \sum_{i=2}^{n} \sigma^{ii} \beta_i^2 \right)$$

$$\geq -\zeta \left[ \sum_{i=2}^{n} \sigma^{ii} k_0 + \frac{k_0}{v^2} \right] - \zeta^2 \left( \frac{1}{v^2} + \sum_{i=2}^{n} \sigma^{ii} \beta_i^2 \right)$$
Let \( \lambda = \min(\sigma^{ii}), \Lambda = \max(\sigma^{ii}), i \geq 2 \). Taking \( 0 < \zeta < \min\{\frac{k(1-n)n}{N}, 2\kappa_1\} \), we can obtain
\[
v(x_0, t_0) \leq c_3,
\]
where \( c_3 \) is independent of \( T' \). Then the conclusion of Lemma \( 2.2 \) follows immediately.

By Lemmas \( 2.1 \) and \( 2.2 \), together with the Schauder estimate for parabolic PDEs, we can get uniform estimates in any \( C^k \)-norm for the derivatives of \( u \), and locally (in time) uniform bounds for the \( C^0 \)-norm, which leads to the long-time existence, with uniform bounds on all higher derivatives of \( u \), to the IBVP \( \sharp \). This finishes the proof of (1) of Theorem \( 1.1 \).

3 Asymptotic behavior

In order to study the asymptotic behavior of the solution to the IVBP \( \sharp \), we need the following two conclusions.

**Lemma 3.1.** Let \( \Omega \) be a strictly convex bounded domain in \( \mathbb{M}^n \), \( n \geq 2 \), and \( \partial \Omega \in C^3 \). Assume that \( \varepsilon > 0 \), the Ricci curvature of \( \mathbb{M}^n \) is nonnegative, \( \phi \) is a function defined on \( \Omega \), and there exists a positive constant \( L > 0 \) such that
\[
|\phi|_{C^3(\overline{\Omega})} \leq L.
\]

Let \( u \in C^2(\overline{\Omega}) \cap C^3(\Omega) \) be a solution to the following BVP
\[
\begin{align*}
\varepsilon u &= \sum_{i=1}^{n} \left( \sigma^{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_i D_j u \quad \text{in } \Omega, \\
D_\nu u &= \phi(x) \quad \text{on } \partial \Omega,
\end{align*}
\]
then there exists a constant \( c_4 := c_4(n, \Omega, L) > 0 \) such that
\[
\sup_{\Omega} |Du| \leq c_4.
\]

**Proof.** Let \( \Phi(x) = \log |D\omega|^2 + \zeta \beta \), where \( \omega = u + \phi(x) \beta \), and \( \zeta \) will be determined later. Denote by \( G = -\phi(x) \beta \).

If one chooses \( 0 < \zeta < 2\kappa_1 \), using an almost same procedure as that in \( (2.1) \), it is easy to show that the maximum of \( \Phi \) can only be achieved in the interior of \( \Omega \).

Assume that \( \Phi(x) \) attains its maximum at \( x_0 \in \Omega \), then we have at this point that
\[
\Phi_i(x_0) = \frac{|D\omega|^2}{|D\omega|^2} + \zeta \beta_i = 0
\]
and
\[
0 \geq \Phi_{ij}(x_0) = \frac{|D\omega|^2}{|D\omega|^2} - \frac{|D\omega|^2 |D\omega|^2}{|D\omega|^4} + \zeta \beta_{ij} = \frac{|D\omega|^2}{|D\omega|^2} + \zeta \beta_{ij} - \zeta^2 \beta_i \beta_j.
\]
It follows that
\[
0 \geq \sum_{i,j=1}^{n} g^{ij} \Phi_{ij}
= \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2 + \sum_{i,j=1}^{n} \xi g^{ij} \beta_{ij} - \sum_{i,j=1}^{n} \xi^2 g^{ij} \beta_i \beta_j
\]
\[
\triangleq I_1 + I_2
\]
(3.2)

where
\[
I_1 = \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2
\]
and
\[
I_2 = \sum_{i,j=1}^{n} \xi g^{ij} \beta_{ij} - \sum_{i,j=1}^{n} \xi^2 g^{ij} \beta_i \beta_j.
\]

As in Lemma 2.2, one can choose suitable local coordinates around \(x_0\) such that \(|Du| = u_1\), \((u_{ij})_{2 \leq i,j \leq n}\) is diagonal, and \((\sigma_{ij})_{2 \leq i,j \leq n}\) is diagonal. Similarly, for the term \(I_2\), at \(x_0\), we have
\[
I_2 \geq \xi \left[ \sum_{i=2}^{n} \sigma^{i2} k_0 + \frac{k_0}{v} \right] - \xi^2 \left( \frac{\beta_1^2}{v^2} + \sum_{i=2}^{n} \sigma^{i2} \beta_i^2 \right).
\]

Set \(J := \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2\). By direct calculation, one has
\[
J = \sum_{i,j=1}^{n} g^{ij} |D\omega|_{ij}^2
\]
\[
= \sum_{i,j,k,l=1}^{n} g^{ij} \left[ 2 \omega^k \omega_{ki} + 2 \omega^k (u_{ijk} + R_{ikj}^{l} u_l - G_{ijk}) \right]
\]
\[
= 2 \sum_{i,j,k,l=1}^{n} g^{ij} \omega^k (u_{ijk} + R_{ikj}^{l} u_l - G_{ijk}) + 2 \sum_{i,j,k,l=1}^{n} g^{ij} \omega^k \omega_{ki}
\]
\[
= -2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k G_{ijk} + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega^k R_{ikj}^{l} u_l - 2 \sum_{i,j,k=1}^{n} \omega^k (g^{ij})_{k} u_{ij}
\]
\[
+ 2 \sum_{k=1}^{n} \omega^k (\varepsilon u_k) + 2 \sum_{i,j,k=1}^{n} g^{ij} \omega_{ki} \omega^k
\]
\[
\triangleq J_1 + J_2 + J_3 + J_4 + J_5.
\]

Without loss of generality, one may assume that \(u_1\) is big enough, then
\[
J_4 = 2 \varepsilon v^2 \geq 0.
\]
Otherwise, the conclusion of Lemma 3.1 follows.

As in Lemma 3.2, the other terms $J_1$, $J_2$, $J_3$, $J_5$ can be controlled similarly. In (3.2), taking $0 < \zeta < \min\left\{\frac{\lambda(n-1)\kappa_0}{\Lambda}, 2\kappa_1\right\}$, we can obtain

$$v(x_0) \leq c_5$$

for some positive constant $c_5 := c_5(n, \Omega, L)$, which is independent of $\epsilon$. Then the assertion of Lemma 3.1 follows.

\begin{proof}

Let $f$ be a smooth function on $\overline{\Omega}$ satisfying $D_\phi f < -\sup_{\Omega}|\phi(x)|$. Let $\rho$ be a point where $f - u_\epsilon$ achieves its minimum. Denote by $T$ the tangent vector to $\partial \Omega$. If $\rho \in \partial \Omega$, then $D_T f(\rho) = D_T u_\epsilon(\rho)$ and $D_\phi f(\rho) \geq D_\phi u_\epsilon(\rho) = \phi(\rho)$, which is a contradiction with the choice of $f$. So, $\epsilon \in \Omega$, and then $D f(\rho) = D u_\epsilon(\rho)$ and $D^2 f(\rho) \geq D^2 u_\epsilon(\rho)$. This gives the existence of a constant $c_6 := c_6(f)$ such that

$$c_6 \geq \sum_{i,j=1}^n g^{ij}(Df) f_{ij}(\rho) \geq \sum_{i,j=1}^n g^{ij}(Du_\epsilon)(u_\epsilon)_{ij}(\rho) = \epsilon u_\epsilon(\rho)$$

Together with the fact $f(x) - u_\epsilon(x) \geq f(\rho) - u_\epsilon(\rho)$ for $x \in \Omega$, we have

$$\epsilon u_\epsilon(x) \leq \epsilon f(x) - \epsilon f(\rho) + c_6.$$ 

Similarly, one can get a lower bound for $\epsilon u_\epsilon(x)$. Therefore, $\sup_{\Omega}|\epsilon u_\epsilon| \leq c_7$ holds for some nonnegative constant $c_7$. By the standard theory of second-order elliptic PDEs, one can get the existence of the solution to the BVP (3.1).

Set $\omega_\epsilon := u_\epsilon - \frac{1}{|\Omega|} \int_{\Omega} u_\epsilon dx$. It is easy to check that $\omega_\epsilon$ satisfies

$$\begin{cases}
\sum_{i,j=1}^n \left(\sigma^{ij} - \frac{(\omega_\epsilon)_{ij}(\omega_\epsilon)}{1 + |D\omega_\epsilon|^2}\right)(\omega_\epsilon)_{ij} = \epsilon \omega_\epsilon + \epsilon \frac{1}{|\Omega|} \int_{\Omega} u_\epsilon dx & \text{in } \Omega, \\
D_\phi \omega_\epsilon = (\omega_\epsilon)_{\phi} = \phi(x) & \text{on } \partial \Omega.
\end{cases}$$

By

$$\sup_{\Omega} |D\omega_\epsilon| = \sup_{\Omega} |Du_\epsilon| \leq c_4$$

(see Lemma 3.1) and the fact that $\omega_\epsilon$ has at least one zero point, we have $|\omega_\epsilon| \leq c_8$ for some nonnegative constant $c_8 := c_8(c_4, c_7)$, which also gives the boundedness of $\frac{1}{|\Omega|} \int_{\Omega} (\epsilon u_\epsilon) dx$. By the Schauder theory for second-order elliptic PDEs, one has $|\omega_\epsilon|_{C^{2,\alpha}(\Omega)} \leq c_9$ for some nonnegative
Hence, one has

\[ \lambda \]

Proof. Assume that there exist two pairs \((\lambda_1, u_1)\) and \((\lambda_2, u_2)\) solving the BVP (‡). Without loss of generality, we may assume that \(\lambda_1 \leq \lambda_2\). Let \(\omega = u_1 - u_2\), and by the linearization process for the quasilinear elliptic PDEs, it is clear that \(\omega\) satisfies

\[
\begin{aligned}
\sum_{i,j=1}^{n} g^{ij} \omega_{ij} + \sum_{i} b_i \omega_i &= \lambda_1 - \lambda_2 \leq 0 \\
D_{\nu} \omega &= 0 \\
\end{aligned}
\]

in \(\Omega\),

on \(\partial \Omega\),

(3.3)

where \(g^{ij} = g^{ij}(D\omega)\) and \(b_i = \sum_{k,l=1}^{n} (u_2)_{kl} \int_{0}^{1} s_{,pl}^{kl}(\eta Du_1 + (1 - \eta)Du_2) d\eta\). By Hopf’s lemma, \(\omega\) must be a constant, which gives the uniqueness (up to a constant) of the solution to the BVP (‡). Consequently, we have \(\lambda_1 = \lambda_2\). This completes the proof of Theorem 3.2.

Let

\[
\omega(x, t) := \omega + \lambda t,
\]

where \((\lambda, \omega)\) is the solution to the BVP (‡). It’s easy to check that \(\omega\) solves the following IBVP

\[
\begin{aligned}
u_t &= \sum_{i,j=1}^{n} \left( \sigma^{ij} - \frac{D^{ij} \omega}{1 + |D\omega|^2} \right) D_i D_j u \quad \text{on } \Omega \times (0, \infty), \\
D_{\nu} u &= \phi(x) \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) &= \omega(x) \quad \text{on } \Omega.
\end{aligned}
\]

(3.5)

As mentioned at the end of Section 2, by Lemmas 2.1 and 2.2, the Schauder theory for parabolic PDEs, one can obtain the long-time existence for the IBVP (‡), i.e., \(T = \infty\).

Corollary 3.3. For a solution \(u = u(x, t)\) to the IBVP (‡), there exists a positive constant \(c_{10}\), independent of \(t\), such that

\[ |u(x, t) - \lambda t| \leq c_{10}. \]

Proof. Set \(z(x, t) := u(x, t) - \omega(x, t)\). By the linearization process, it is easy to check that \(z(x, t)\) satisfies

\[
\begin{aligned}
\sum_{i,j=1}^{n} g^{ij} z_{ij} + \sum_{i=1}^{n} b_i z_i &= 0 \quad \text{in } \Omega \times (0, \infty), \\
D_{\nu} z &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
z(x, 0) &= u_0(x) - \omega(x) \quad \text{on } \Omega,
\end{aligned}
\]

where \(g^{ij} = g^{ij}(D\omega)\) and \(b_i = \sum_{k,l=1}^{n} (\omega)_{kl} \int_{0}^{1} s_{,pl}^{kl}(\eta Du + (1 - \eta)D\omega) d\eta\). By the maximum principle of second-order parabolic PDEs, we know that \(z\) attains its maximum and minimum on \(\Omega \times \{0\}\). Hence, one has

\[
\sup_{\Omega \times (0, \infty)} |u - \lambda t| \leq \sup_{\Omega} |\omega| + \sup_{\Omega} |u_0 - \omega|,
\]

which implies the conclusion of Corollary 3.3.
Lemma 3.4. Let \( u_1 \) and \( u_2 \) be any two solutions to the IBVP \((\bar{u})\) with initial data \( u_{0,1} \) and \( u_{0,2} \) respectively. Let \( u = u_1 - u_2 \), then \( u \) converges to a constant function as \( t \to \infty \). In particular, the limit of any solution to the IBVP \((\bar{u})\) is \( \omega \) up to a constant.

Proof. We use a similar method to that of the proof of \([16, \text{Lemma 2.5}]\).

As shown in Corollary 3.3, it is easy to know that \( u \) satisfies

\[
\begin{aligned}
    z_t &= n_{i,j=1} \bar{g}^{ij}z_{ij} + \sum_i b_i z_i &\text{in } \Omega \times (0, \infty),
    \\
    z_v &= 0 &\text{on } \partial \Omega \times (0, \infty),
    \\
    z(x,0) &= u_{0,1}(x) - u_{0,2}(x) &\text{on } \Omega,
\end{aligned}
\]

(3.6)

where \( \bar{g}^{ij} = g^{ij}(Du_1) \) and \( b_i = \sum_{k,l} (u_2)_{kl} \int_0^1 g^{kl}_{\eta}(\eta Du_1 + (1 + \eta)Du_2)\,d\eta \). Set

\[
\text{osc}(u)(t) = \max_{\Omega} u(x,t) - \min_{\Omega} u(x,t).
\]

By the strong maximum principle of second-order parabolic PDEs and Hopf’s lemma, one knows that \( \text{osc}(u)(t) \) is a strictly decreasing function unless \( u \) is a constant.

Now, we claim that

\[
\lim_{t \to \infty} \text{osc}(u)(t) = 0.
\]

Otherwise, one has \( \lim_{t \to \infty} \text{osc}(u)(t) = \chi \) for some \( \chi > 0 \). In fact, given a sequence \( t_n \to +\infty \), define

\[
u_{1,n}(\cdot,t) := u_1(\cdot,t + t_n) - \lambda t_n
\]

and

\[
u_{2,n}(\cdot,t) := u_2(\cdot,t + t_n) - \lambda t_n
\]

By Corollary 3.3, for \( i = 1,2 \), we have \( |u_{i,n} - \lambda t_n| \leq c_{10} \). By Lemmas 2.1, 2.2, and the Schauder theory of second-order parabolic PDEs, it follows that for any \( k \), \( u_{1,n}(\cdot,t) \) and \( u_{2,n}(\cdot,t) \) are locally (in time) \( C^k \) uniformly bounded with respect to \( n \). Therefore, there exists a subsequence (still denoted by \( t_n \)) such that \( u_{1,n}(\cdot,t) \) and \( u_{2,n}(\cdot,t) \) converge locally uniformly in any \( C^k \) to \( u_1^*(\cdot,t) \) and \( u_2^*(\cdot,t) \) respectively, i.e.,

\[
u_1^*(\cdot,t) = \lim_{n \to \infty} u_{1,n}(\cdot,t), \quad u_2^*(\cdot,t) = \lim_{n \to \infty} u_{2,n}(\cdot,t).
\]

Set \( u^* := u_1^* - u_2^* \), and then we have

\[
\begin{aligned}
    \text{osc}(u^*)(t) &= \text{osc}(u_1^* - u_2^*)
    \\
    &= \lim_{n \to \infty} \text{osc}(u_1(x,t + t_n) - \lambda t_n - u_2(x,t + t_n) + \lambda t_n)
    \\
    &= \lim_{n \to \infty} \text{osc}(u_1(x,t + t_n) - u_2(x,t + t_n))
    \\
    &= \lim_{n \to \infty} \text{osc}(u)(t + t_n)
    \\
    &= \chi.
\end{aligned}
\]

(3.7)
The second equality in (3.7) holds since \( u_{1,n}(\cdot,t) \) and \( u_{1,n}(\cdot,t) \) are uniformly convergent. Besides, it is easy to check that \( u^* \) satisfies
\[
\begin{aligned}
    z_i &= \sum_{i,j=1}^{n} g^{ij} z_{ij} + \sum_{i=1}^{n} b_i z_i & \quad & \text{in } \Omega \times (-\infty, \infty), \\
    D_{\nu} z &= 0 & \quad & \text{on } \partial \Omega \times (-\infty, \infty),
\end{aligned}
\]
where \( \tilde{g}^{ij} = g^{ij}(Du^*) \) and \( b_i = \sum_{k,l=1}^{n} (u^*)_{kl} \int_0^1 \tilde{g}^{kl}_{\eta}(\eta Du^*_i + (1 - \eta) Du^*_j) d\eta \). By the strong maximum principle of second-order parabolic PDEs and Hopf’s lemma, we know \( u^* \) is a constant. This is contradict with \( \text{osc}(u^*)(t) \equiv \chi \). Our claim follows. So, one has \( \lim_{t \to \infty} \max_{\Omega} u = \lim_{t \to \infty} \min_{\Omega} u = c_{11} \) for some constant \( c_{11} \), which implies \( \lim_{t \to \infty} |u - c_{11}| = 0 \). This finishes the proof of Lemma 3.4.

Clearly, by Corollary 3.3 and Lemma 3.4, we know that the limit of any solution to the IBVP \((\#)\) is \( \tilde{w} = w + \lambda t \) up to a constant, where \((\lambda, \omega)\) is the solution to the BVP \((\ddagger)\). This completes the proof of (2) of Theorem 1.1.

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References

[1] U. Abresch, H. Rosenberg, A Hopf differential for constant mean curvature surfaces in \( \mathbb{S}^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \), Acta Math. 193 (2004) 141–174.

[2] S.-J. Altschuler, L.-F. Wu, Convergence to translating solitons for a class of quasilnear parabolic equations with fixed angle of contact to a boundary, Math. Ann. 295 (1993) 761–765.

[3] S.-J. Altschuler, L.-F. Wu, Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle, Calc. Var. Partial Differential Equations 2 (1994) 101–111.

[4] S.-B. Angenent and J.-J.-L. Velázquez, Asymptotic shape of cusp singularities in curve shortening, Duke Math. J. 77 (1995) 71–110.

[5] S.-B. Angenent and J.-J.-L. Velázquez, Degenerate neckpinches in mean curvature flow, J. Reine Angew. Math. 482 (1997) 15–66.

[6] C. Bandle, Isoperimetric inequalities and applications, Pitman, 1980.

[7] L. Chen, D.-D. Hu, J. Mao, N. Xiang, Translating surfaces of the nonparametric mean curvature flow in Lorentz manifold \( M^2 \times \mathbb{R} \), available online at arXiv: 1804.10864v3.
[8] Y. Gao, J. Mao, *Translating solutions of the nonparametric mean curvature flow with nonzero Neumann boundary data in product manifold $M^n \times \mathbb{R}$, II*, preprint.

[9] Y. Gao, J. Mao, C.-L. Song, *Existence and uniqueness of solutions to the constant mean curvature equation with nonzero Neumann boundary data in product manifold $M^n \times \mathbb{R}$*, preprint.

[10] B. Guan, *Mean curvature motion of nonparametric hypersurfaces with contact angle condition*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), pages 47–56, 1996.

[11] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. 20 (1984) 237–266.

[12] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. 84 (1986) 463–480.

[13] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equat. 77 (1989) 369–378.

[14] H. Jenkins, J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. reine angew. Math. 229 (1968) 170–187.

[15] L. Mazet, M.-M. Rodríguez, H. Rosenberg, *Periodic constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Asian J. Math. 18 (2014) 829–858.

[16] X.-N. Ma, P.-H. Wang, W. Wei, *Constant mean curvature surfaces and mean curvature flow with nonzero Neumann boundary conditions on strictly convex domains*, J. Funct. Anal. 274 (2018) 252–277.

[17] J. Mao, *A new way to Dirichlet problem for minimal surface system in arbitrary dimensions and codimensions*, Kyushu J. Math. 69(1) (2015) 1–9.

[18] W.-H. Meeks III, H. Rosenberg, *Stable minimal surfaces in $M \times \mathbb{R}$*, J. Differential Geom. 68 (2004) 515–534.

[19] H. Rosenberg, F. Schulze, J. Spruck, *The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$*, J. Differential Geom. 95 (2013) 321–336.

[20] P. Topping, *Mean curvature flow and geometric inequalities*, J. reine angew. Math. 503 (1998) 47–61.

[21] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. 84 (1978) 1182–1238.

[22] J. Wang, W. Wei, J.-J. Xu, *Translating solutions of non-parametric mean curvature flows with capillary-type boundary value problems*, Commun. Pure Appl. Anal. 18(6) (2019) 3243–3265.

[23] M.-T. Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, Invent. Math. 148(3) (2002) 525–543.
[24] M.-T. Wang, *The Dirichlet problem for the minimal surface system in arbitrary codimension*, Commun. Pure. Appl. Math. **57**(2) (2004) 267–281.

[25] H.-Y. Zhou, *Nonparametric mean curvature type flows of graphs with contact angle conditions*, Int. Math. Res. Not. **19** (2018) 6026–6069.