Differential invariants for flows of fluids and gases

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1 Introduction

The paper is an extended overview of the papers [11–17]. The main extension is a detailed analysis of thermodynamic states, symmetries, and differential invariants. This analysis is based on consideration of Riemannian structure [8] naturally associated with Lagrangian manifolds that represent thermodynamic states. This approach radically changes the description of the thermodynamic part of the symmetry algebra as well as the field of differential invariants.

The paper is organized as follows.

In Section 2 we discuss thermodynamics in terms of contact and symplectic geometries. The main part of this approach is a presentation of thermodynamic states as Lagrangian manifolds equipped with an additional Riemannian structure. Application of this approach to fluid motion is new, though the relationship between contact geometry and thermodynamics was well-known since Gibbs [3] and Carathéodory [4]. For some modern studies see also [5] and [6].

In Section 3 the motion of inviscid media is considered. We discuss flows of inviscid fluids on different Riemannian manifolds: a plane, sphere, and a spherical layer. Such flows are governed by a generalization of the Euler equation system. For each of these cases, we find a Lie algebra of symmetries, provide a classification of symmetry algebras depending on a thermodynamic state admitted by media, and describe the field of differential invariants for the Euler system.

In Section 4 the motion of viscid media on Riemannian manifolds is studied. First, we discuss a generalization of the Navier–Stokes equations for an arbitrary oriented Riemannian manifold. Then for the cases of a plane, space, sphere, and a spherical layer, we provide classification of symmetry algebras with respect to possible thermodynamic states and give full description for the field of differential invariants.
2 Thermodynamics

Here we consider the media with thermodynamics described by two types of quantities. The first are extensive quantities: the specific entropy $s$, the specific volume $\rho^{-1}$, the specific internal energy $\varepsilon$; and the second are intensive quantities: the absolute temperature $T > 0$ and the pressure $p$.

A thermodynamic state of such media is a two-dimensional Legendrian manifold $L \subset \mathbb{R}^5(\varepsilon, \rho, p, T, s)$, a maximal integral manifold of the differential 1-form

$$\theta = d\varepsilon - T ds - p\rho^{-2} d\rho,$$

i.e. a manifold such that the first law of thermodynamics $\theta|_L = 0$ holds.

Following [8] a point $(\varepsilon, \rho, p, T, s)$ on the Legendrian manifold can be considered as a triplet: the expected value $(\varepsilon, \rho^{-1})$ of a stochastic process of measurement of internal energy and volume, the probabilistic measure corresponding to $(p, T)$ and the information $(-s)$, which is given up to a constant.

Since the information $I$ is a positive quantity, the entropy $s$ satisfies the inequality $s \leq s_0$ for a certain constant $s_0$, which, generally speaking, depends on the nature of a process under consideration.

Let us denote the variance of the stochastic process by $(-\kappa)$. In terms of the given probabilistic measure and expected values it has form [8]:

$$\kappa = d(T^{-1}) \cdot d\varepsilon - \rho^{-2} d(pT^{-1}) \cdot d\rho.$$

Thus, by a thermodynamic state we mean a two-dimensional Legendrian submanifold $L$ of the contact manifold $(\mathbb{R}^5, \theta)$, such that the quadratic differential form $\kappa$ on the surface $L$ is negative definite, i.e.

$$\kappa|_L < 0.$$

Because the energy can be excluded from the conservation laws that govern medium motion, we also eliminate it from our geometrical interpretation of the thermodynamics.

Consider the projection

$$\varphi : \mathbb{R}^5 \to \mathbb{R}^4, \quad \varphi : (\varepsilon, \rho, p, T, s) \mapsto (\rho, p, T, s).$$

The restriction of the map $\varphi$ on the state surface $L$ is a local diffeomorphism on the image $\tilde{L} = \varphi(L)$ and the surface $\tilde{L}$ is an immersed Lagrangian manifold in the symplectic space $\mathbb{R}^4$ equipped with the structure form

$$\Omega = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Therefore, the first law of thermodynamics is equivalent to the condition that $\tilde{L} \subset \mathbb{R}^4$ is a Lagrangian manifold.

The two-dimensional surface $\tilde{L}$ in the four-dimensional space can be defined by two equations

$$f(p, \rho, s, T) = 0, \quad g(p, \rho, s, T) = 0 \quad (1)$$

and the condition for the surface $\tilde{L}$ to be Lagrangian means vanishing of the Poisson bracket of these functions

$$[f, g]|_{\tilde{L}} = 0, \quad (2)$$
that in the coordinates \((p, \rho, s, T)\) takes the form

\[
[f, g] = \rho^2 (f_{\rho}g_{\rho} - f_{\rho}g_{\rho}) + f_{s}g_{T} - f_{T}g_{s}.
\]

Thus, the thermodynamic state can be defined as Lagrangian surface \((\text{1})\) in the four-dimensional symplectic space, such that the condition \((\text{2})\) holds and the symmetric differential form \(\kappa\) is negative definite on this surface.

Note, if the equation of state is given in the form \(\epsilon = \epsilon(\rho, s)\), then the two-dimensional Legendrian manifold \(L\) can be defined by the structure equations

\[
\epsilon = \epsilon(\rho, s), \quad T = \epsilon_s, \quad p = \rho^2\epsilon_{\rho},
\]

and the restriction of the form \(\kappa\) gives

\[
\kappa|_L = -\epsilon_s^{-1} ((\epsilon_{\rho\rho} + 2\rho^{-1}\epsilon_{\rho}) d\rho^2 + 2\epsilon_{\rho s}d\rho \cdot ds + \epsilon_{s s}ds^2).
\]

The condition of negative-definiteness this form leads us to the following additional relations

\[
\begin{cases}
\epsilon_{\rho\rho} + 2\rho^{-3}p > 0, \\
\epsilon_{s s} (\epsilon_{\rho\rho} + 2\rho^{-3}p) - \epsilon_{\rho s}^2 > 0
\end{cases}
\]

on the function \(\epsilon(\rho, s)\) or

\[
\begin{cases}
p_{\rho} > 0, \\
T_s p_{\rho} - \rho^2 T_{\rho}^2 > 0.
\end{cases}
\]
3 Compressible inviscid fluids or gases

In this section we study differential invariants of compressible inviscid fluids or gases.

The system of differential equations (the Euler system) describing flows on an oriented Riemannian manifold $(M, g)$ consists of the following equations (see [1] for details):

$$\begin{align*}
\rho (u_t + \nabla_u u) &= - \operatorname{grad} p + g\rho, \\
\frac{\partial (\rho \Omega_g)}{\partial t} + L_u (\rho \Omega_g) &= 0, \\
T (s_t + \nabla_s s) - \frac{k}{\rho} \Delta_g T &= 0,
\end{align*}$$

(4)

where the vector field $u$ is the flow velocity, $p$, $\rho$, $s$, $T$ are the pressure, density, entropy, temperature of the fluid respectively, $k$ is the thermal conductivity, which is supposed to be constant, and $g$ is the gravitational acceleration.

Here $\nabla_X$ is the directional covariant Levi–Civita derivative with respect to a vector field $X$, $L_X$ is the Lie derivative along a vector field $X$, $\Omega_g$ is the volume form on the manifold $M$, $\Delta_g$ is the Laplace–Beltrami operator corresponding to the metric $g$.

The first equation of system (4) represents the law of momentum conservation in the inviscid medium, the second is the continuity equation, and the third is the equation representing the effect of heat conduction in the medium.

We consider the following examples of manifold $M$: a plane, sphere and a spherical layer.

Note that in all these cases the number of unknown functions is greater than the number of system equations by 2, i.e. the system (4) is incomplete. We get two additional equations taking into account the thermodynamics of the medium.

Thus, by the Euler system of differential equations we mean the system (4) extended by two equations of state (1), such that the functions $f$ and $g$ satisfy the additional relation (2) and the form $\kappa$ is negative definite.

Geometrically, we represent this system in the following way. Consider the bundle of rank $(\dim M + 4)$

$$\pi : \mathbb{R} \times TM \times \mathbb{R}^4 \longrightarrow \mathbb{R} \times M,$$

where $(t, x, u, \rho, p, T, s) \rightarrow (t, \bar{x})$ and $t \in \mathbb{R}$, $x \in M$, $u \in T_x M$. Then the Euler system is a system of differential equations on sections of the bundle $\pi$.

Note that system (4) defines the zeroth order system $\mathcal{E}_0 \subset J^0 \pi$.

Denote by $\mathcal{E}_1 \subset J^1 \pi$ the system of order $\leq 1$ obtained by the first prolongation of the system $\mathcal{E}_0$ and by the first 2 equations of system (4) (Euler’s and the continuity equations).

Let also $\mathcal{E}_2 \subset J^2 \pi$ be the system of differential equations of order $\leq 2$ obtained by the first prolongation of the system $\mathcal{E}_1$ and the last equation of system (4).

For the case $k \geq 3$, we define $\mathcal{E}_k \subset J^k \pi$ to be the $(k - 2)$-th prolongation of the system $\mathcal{E}_2$.

Note that due to the relations (4) the system $\mathcal{E}_\infty = \lim \mathcal{E}_k$ is a formally integrable system of differential equations, which we also call the Euler system.

3.1 2D-flows

Consider Euler system (4) on a plane $M = \mathbb{R}^2$ equipped with the coordinates $(x, y)$ and the standard flat metric $g = dx^2 + dy^2$. 


The velocity field of the flow has the form \( u = u(t, x, y) \partial_x + v(t, x, y) \partial_y \), the pressure \( p \), the density \( \rho \), the temperature \( T \) and the entropy \( s \) are the functions of time and space with the coordinates \((t, x, y)\).

Here we consider the flow without any external force field, so \( g = 0 \).

### 3.1.1 Symmetry Lie algebra

The symmetry algebra of the Euler system has been found in \[11\], here we observe the main statements.

First of all, by a symmetry of the PDE system we mean a point symmetry, i.e. a vector field \( X \) on the jet space \( J^0\pi \) such that its second prolongation \( X^{(2)} \) is tangent to the submanifold \( \mathcal{E}_2 \subset J^2\pi \).

To describe the Lie algebra of symmetries of the Euler system, we consider the Lie algebra \( g \) generated by the following vector fields on the space \( J^0\pi \):

\[
\begin{align*}
X_1 &= \partial_x, & X_4 &= t \partial_x + \partial_u, \\
X_2 &= \partial_y, & X_5 &= t \partial_y + \partial_v, \\
X_3 &= y \partial_x - x \partial_y + v \partial_u - u \partial_v, & X_6 &= \partial_t, \\
X_7 &= \partial_s, & X_{10} &= t \partial_t + x \partial_x + y \partial_y - s \partial_s, \\
X_8 &= \partial_p, & X_{11} &= t \partial_t - u \partial_u - v \partial_v - 2p \partial_p + s \partial_s, \\
X_9 &= T \partial_T, & X_{12} &= p \partial_p + \rho \partial_\rho - s \partial_s.
\end{align*}
\]

Note that this symmetry algebra consists of pure geometric and thermodynamic parts.

The geometric part \( g_m \) is generated by the fields \( X_1, \ldots, X_6 \). Transformations corresponding to the elements of this algebra are generated by the motions, Galilean transformations and the time shift.

In order to describe the pure thermodynamic part of the system symmetry algebra, consider the Lie algebra \( \mathfrak{h} \) generated by the vector fields:

\[
\begin{align*}
Y_1 &= \partial_s, & Y_3 &= \rho \partial_\rho, & Y_5 &= p \partial_p, \\
Y_2 &= \partial_p, & Y_4 &= s \partial_s, & Y_6 &= T \partial_T.
\end{align*}
\]

Denote by \( \vartheta : g \mapsto \mathfrak{h} \) the following Lie algebras homomorphism

\[
\vartheta(X) = X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(T)\partial_T,
\]

(5)

where \( X \in g \).

Note that, the kernel of the homomorphism \( \vartheta \) is the ideal \( g_m \subset g \).

Let also \( \mathfrak{h}_t \) be the Lie subalgebra of the algebra \( \mathfrak{h} \) that preserves the thermodynamic state \([1\ldots\ldots]\).

**Theorem 1** \([1]\) A Lie algebra \( g_{\text{sym}} \) of symmetries of the Euler system of differential equations on a plane coincides with

\[
\vartheta^{-1}(\mathfrak{h}_t).
\]

Note that, for the general equation of state, the algebra \( \mathfrak{h}_t = 0 \), and the symmetry algebra coincides with the Lie algebra \( g_m \).

Observe that, usually, the equations of state are neglected, and vector fields like \( f(t) \partial_p \) and \( g(t)T \partial_T \), where \( f \) and \( g \) are arbitrary functions, are considered as symmetries of the Euler system.
3.1.2 Symmetry classification of states

In this section we classify the thermodynamic states or the Lagrangian surfaces \( \tilde{L} \) depending on the dimension of the symmetry algebra \( \mathfrak{h}_t \subset \mathfrak{h} \).

We consider one- and two-dimensional symmetry algebras only, because the requirement on the thermodynamic state to have a three or more dimensional symmetry algebra is very strict and leads to the trivial solutions.

States with a one-dimensional symmetry algebra

Let \( \dim \mathfrak{h}_t = 1 \) and let \( Z = \sum_{i=1}^{6} \lambda_i Y_i \) be a basis vector in this algebra.

The state \( \tilde{L} \subset \mathbb{R}^4 \) is Lagrangian, i.e. \( \Omega|_{\tilde{L}} = 0 \), and therefore the vector field \( Z \) is tangent to the surface \( \tilde{L} \), if and only if the differential 1-form

\[
\iota_Z \Omega = \frac{\lambda_3}{\rho} dp - \frac{\lambda_5 p + \lambda_2}{\rho^2} dp - \lambda_6 T ds + (\lambda_4 s + \lambda_1) dT
\]

vanishes on the surface \( \tilde{L} \).

In other words, the surface \( \tilde{L} \) is the solution of the following system of differential equations

\[
\begin{aligned}
\Omega|_{\tilde{L}} &= 0, \\
(\iota_Z \Omega)|_{\tilde{L}} &= 0.
\end{aligned}
\]

In terms of specific energy (3) the last system has the following form

\[
\begin{aligned}
\lambda_3 \rho \epsilon_{pp} + (\lambda_4 s + \lambda_1) \epsilon_{ps} + (2\lambda_3 - \lambda_5) \epsilon_p - \frac{\lambda_2}{\rho^2} &= 0, \\
(\lambda_4 s + \lambda_1) \epsilon_{ss} + \lambda_3 \rho \epsilon_{ps} - \lambda_6 \epsilon_s &= 0.
\end{aligned}
\]

It is easy to check that the bracket of these last two equations (see [9]) vanishes, and therefore the system is formally integrable and compatible.

In order to solve the last system we reduce its order and get the equivalent system

\[
\begin{aligned}
\lambda_3 \rho \epsilon_p + (\lambda_4 s + \lambda_1) \epsilon_s + (\lambda_3 - \lambda_5) \epsilon + \frac{\lambda_2}{\rho} + f(s) &= 0, \\
\lambda_3 \rho \epsilon_p + (\lambda_4 s + \lambda_1) \epsilon_s - (\lambda_6 + \lambda_4) \epsilon + g(\rho) &= 0,
\end{aligned}
\]

where \( f(s) \) and \( g(\rho) \) are some differentiable functions.

Below we list solutions of the system under the assumption of parameters \( \lambda \) generality. The more detailed description can be found in [11].

In the general case, when \( \lambda_6 + \lambda_4 - \lambda_5 - \lambda_3 \neq 0 \), solving the last system we find

\[
p = C_1 \rho^\frac{\lambda_5}{\lambda_3} - \frac{\lambda_2}{\lambda_5}, \quad T = C_2 (\lambda_4 s + \lambda_1) \rho^\frac{\lambda_5}{\lambda_3},
\]

where \( C_1, C_2 \) are constants.

Moreover, the negative definiteness of the quadratic differential form \( \kappa \) on the surface \( \tilde{L} \) leads to the relations

\[
\frac{\lambda_4 s + \lambda_1}{\lambda_6} > 0, \quad \frac{C_1 \lambda_5}{\lambda_3} > 0
\]

for all \( s \in (-\infty, s_0] \).
Theorem 2 The thermodynamic states admitting a one-dimensional symmetry algebra have the form

\[ p = C_1 \rho^{\frac{\lambda_5}{\lambda_3}} - \frac{\lambda_2}{\lambda_3} \rho^{\frac{\lambda_4}{\lambda_3}}, \quad T = C_2 (\lambda_4 s + \lambda_1)^{\frac{\lambda_6}{\lambda_4}}, \]

where the constants defining the symmetry algebra satisfy inequalities

\[ s_0 < -\frac{\lambda_1}{\lambda_4}, \quad C_1 > 0, \quad \frac{\lambda_5}{\lambda_3} > 0, \quad \frac{\lambda_2}{\lambda_5} < 0, \]

and besides they must meet one of the following conditions:

1. if \( \frac{\lambda_5}{\lambda_4} \) is irrational, then \( \lambda_4 < 0, \lambda_6 > 0, C_2 > 0 \);
2. if \( \frac{\lambda_5}{\lambda_4} \) is rational, then \( \frac{\lambda_5}{\lambda_4} < 0 \) (i.e. \( \frac{\lambda_5}{\lambda_4} = -\frac{m}{n} \)) and
   (a) if \( k \) is even, then \( \lambda_4 < 0, C_2 > 0 \);
   (b) if \( k \) is odd and \( m \) is even, then \( C_2 > 0 \);
   (c) if \( k \) is odd and \( m \) is odd, then \( C_2 \lambda_4 < 0 \).

States with a two-dimensional non-commutative symmetry algebra

Let \( h_1 \subset h \) be a non-commutative two-dimensional Lie subalgebra. Then \([h, h] \supset [h_1, h_1] = \langle Y_1, Y_2 \rangle\). Therefore, any non-zero vector \( A = \alpha_0 Y_1 + \beta_0 Y_2 \in h_1 \) can be chosen as one of the basis vectors. The second basis vector \( B \) in the subalgebra may be chosen such that \([A, B] = A\). Let \( B = \sum_{i=1}^{6} \gamma_i Y_i \), then the condition \([A, B] = A\) gives two relations

\[ \alpha_0 (\gamma_4 - 1) = 0, \quad \beta_0 (\gamma_5 - 1) = 0. \]

Restriction of the forms \( \iota_A \Omega \) and \( \iota_B \Omega \) on the surface \( L \) leads us to the following system of differential equations:

\[
\begin{align*}
\alpha_0 \epsilon_{ss} &= 0, \\
\alpha_0 \rho^2 \epsilon_{ps} - \beta_0 &= 0, \\
\gamma_3 \rho \epsilon_{ps} + (\gamma_4 s + \gamma_1) \epsilon_{ss} - \gamma_6 \epsilon_s &= 0, \\
\gamma_3 (\rho \epsilon_{pp} + 2 \epsilon_{\rho}) + (\gamma_4 s + \gamma_1) \epsilon_{ps} - \gamma_5 \epsilon_{\rho \rho} - \frac{\gamma_2}{\rho^2} &= 0.
\end{align*}
\]

Note that from the first two equations of this system follows that \( \alpha_0 \neq 0 \). Computing brackets \( [9] \) we get that this system is integrable if

\[ \beta_0 (\gamma_4 - \gamma_5) = 0, \quad \beta_0 (\gamma_3 \gamma_5 + \gamma_4 \gamma_6) = 0. \]

Then solving this system for the case \( \beta_0 = 0 \) and \( \gamma_4 = 1 \) we have \( T = 0 \) which is not sensible from the physical point of view.

For the case \( \gamma_4 = 1 \) and \( \gamma_5 = 1 \) we get

\[ p = C \rho^{\frac{1}{\gamma_3}} + \frac{\beta_0}{\alpha_0} (s + \gamma_1) - \gamma_2, \quad T = -\frac{\beta_0}{\alpha_0 \rho}, \]

but the condition on the form \( \kappa \) gives

\[ \frac{C}{\gamma_3} > 0, \quad -\frac{1}{\rho^2} > 0. \]

So there are no thermodynamic states that admit a two-dimensional non-commutative symmetry algebra.
States with a two-dimensional commutative symmetry algebra

Let now $\mathfrak{h}_1 \subset \mathfrak{h}$ be a commutative two-dimensional Lie subalgebra, and let $A = \sum_{i=1}^{6} \alpha_i Y_i$, $B = \sum_{i=1}^{6} \beta_i Y_i$ be basis vectors in the algebra $\mathfrak{h}_1$.

Then the condition $[A, B] = 0$ gives the following relations on $\alpha$’s and $\beta$’s:

$$\alpha_1 \beta_4 - \alpha_4 \beta_1 = 0, \quad \alpha_2 \beta_5 - \alpha_5 \beta_2 = 0. \quad (6)$$

Then, as above, restriction of the forms $\iota_A \Omega$ and $\iota_B \Omega$ on the state surface $\tilde{L}$ leads us to the following system of differential equations:

$$\begin{align*}
\alpha_3 \rho \epsilon_{PP} + (\alpha_4 s + \alpha_1) \epsilon_{PS} + (2 \alpha_3 - \alpha_5) \epsilon_P - \frac{\alpha_2}{\rho^2} &= 0, \\
\beta_3 \rho \epsilon_{PP} + (\beta_4 s + \beta_1) \epsilon_{PS} + (2 \beta_3 - \beta_5) \epsilon_P - \frac{\beta_2}{\rho^2} &= 0, \\
(\alpha_4 s + \alpha_1) \epsilon_{SS} + \alpha_3 \rho \epsilon_{PS} - \alpha_6 \epsilon_S &= 0, \\
(\beta_4 s + \beta_1) \epsilon_{SS} + \beta_3 \rho \epsilon_{PS} - \beta_6 \epsilon_S &= 0.
\end{align*}$$

The formal integrability condition for this system has the form

$$(\beta_5 - 5 \beta_3)(\alpha_2 \beta_5 - \alpha_5 \beta_2) = 0,$$

which is satisfied due to relations (6).

Therefore, this system is integrable, and for all $\alpha$’s and $\beta$’s. In most of cases this system has the “nonphysical” solution of the form $\epsilon = C_1 \rho^{-1} + C_2$. For the special case, for example,

$$\alpha_1 = \frac{\alpha_4 \beta_1}{\beta_4}, \quad \alpha_2 = \frac{\alpha_5 \beta_2}{\beta_5}, \quad \text{and} \quad \left\{ \begin{array}{l}
\alpha_3 = \alpha_5 - \alpha_4 - \alpha_6, \\
\beta_3 = \beta_5 - \beta_4 - \beta_6
\end{array} \right. \quad (7)$$

we have the following expressions for the pressure and the temperature

$$p = C_1 \rho^{\frac{s_1}{1 + s_2}} (\beta_4 s + \beta_1)^{\frac{s_1 + s_2}{1 + s_2}} - \frac{\beta_2}{\beta_5}, \quad T = C_2 \rho^{\frac{s_1}{1 + s_2}} (\beta_4 s + \beta_1)^{\frac{s_1 + s_2}{1 + s_2}} \rho^{-1} (\beta_4 s + \beta_1)^{-1},$$

where

$$s_1 = \alpha_6 \beta_4 - \alpha_4 \beta_6, \quad s_2 = \alpha_4 \beta_5 - \alpha_5 \beta_4, \quad s_3 = \alpha_6 \beta_5 - \alpha_5 \beta_6.$$

And negative definiteness of the form $\kappa$ leads to the relations

$$\frac{s_2 (\beta_4 s + \beta_1)}{s_3 \beta_4} > 0, \quad \frac{-s_1}{(s_1 + s_2)(s_2 + s_3)} > 0.$$

**Theorem 3** In the general case, there are no physically applicable thermodynamic states, which admit a two-dimensional commutative symmetry algebra.

For the special case (7), the thermodynamic states admitting a two-dimensional commutative symmetry algebra have the form

$$p = C_1 \rho^{\frac{s_1}{1 + s_2}} (\beta_4 s + \beta_1)^{\frac{s_1 + s_2}{1 + s_2}} - \frac{\beta_2}{\beta_5}, \quad T = C_2 \rho^{\frac{s_1}{1 + s_2}} (\beta_4 s + \beta_1)^{\frac{s_1 + s_2}{1 + s_2}},$$

8
where the constants defining the symmetry algebra satisfy inequalities
\[
 s_0 < -\frac{\beta_1}{\beta_4}, \quad \frac{\beta_2}{\beta_5} < 0, \quad \frac{s_2}{s_3} < 0, \quad \frac{s_1}{(s_1 + s_2)(s_2 + s_3)} < 0,
\]
and besides they must meet one of the conditions:

1. if \( \frac{s_1 + s_2}{s_1 + s_2} \) is irrational, then \( \beta_4 < 0, \ C_1 > 0, \ C_2 > 0; \)

2. if \( \frac{s_1 + s_2}{s_1 + s_2} \) is rational (i.e. \( \frac{s_1 + s_2}{s_1 + s_2} = \pm \frac{m}{k} \)), then
   
   (a) if \( k \) is even, then \( \beta_4 < 0, \ C_1 > 0, \ C_2 > 0; \)
   
   (b) if \( k \) is odd and \( m \) is even, then \( C_1 \beta_4 < 0, \ C_2 > 0; \)
   
   (c) if \( k \) is odd and \( m \) is odd, then \( C_2 \beta_4 < 0, \ C_1 > 0. \)

3.1.3 Differential invariants

We consider two group actions on the Euler equation \( \mathcal{E} \). The first one is the prolonged action of the group generated by the action of the Lie algebra \( \mathfrak{g}_m \). The second action is the action generated by the prolongation of the action of the Lie algebra \( \mathfrak{g}_{\text{sym}} \).

First of all, observe that fibers of the projection \( \mathcal{E}_k \to \mathcal{E}_0 \) are irreducible algebraic manifolds.

Then we say that a function \( J \) on the manifold \( \mathcal{E}_k \) is a **kinematic differential invariant** of order \( \leq k \) if

1. \( J \) is a rational function along fibers of the projection \( \pi_{k,0} : \mathcal{E}_k \to \mathcal{E}_0, \)

2. \( J \) is invariant with respect to the prolonged action of the Lie algebra \( \mathfrak{g}_m \), i.e.
\[
X^{(k)}(J) = 0, \quad (8)
\]

for all \( X \in \mathfrak{g}_m \).

Here we denote by \( X^{(k)} \) the \( k \)-th prolongation of a vector field \( X \in \mathfrak{g}_m \).

We say also that the kinematic invariant is an **Euler invariant** if condition \( [ \text{S} ] \) holds for all \( X \in \mathfrak{g}_{\text{sym}} \).

We say that a point \( x_k \in \mathcal{E}_k \) and the corresponding orbit \( \mathcal{O}(x_k) \) (\( \mathfrak{g}_m \) or \( \mathfrak{g}_{\text{sym}} \)-orbit) are **regular**, if there are exactly \( m = \text{codim} \mathcal{O}(x_k) \) independent invariants (kinematic or Euler) in a neighborhood of this orbit.

Thus, the corresponding point on the quotient \( \mathcal{E}_k/\mathfrak{g}_m \) or \( \mathcal{E}_k/\mathfrak{g}_{\text{sym}} \) is smooth, and these independent invariants (kinematic or Euler) can serve as local coordinates in a neighborhood of this point.

Otherwise, we say that the point and the corresponding orbit are **singular**.

It is worth to note that the Euler system together with the symmetry algebras \( \mathfrak{g}_m \) or \( \mathfrak{g}_{\text{sym}} \) satisfies the conditions of Lie-Tresse theorem (see [10]), and therefore the kinematic and Euler differential invariants separate regular \( \mathfrak{g}_m \) and \( \mathfrak{g}_{\text{sym}} \) orbits on the Euler system \( \mathcal{E} \) correspondingly.

By a \( \mathfrak{g}_m \) or \( \mathfrak{g}_{\text{sym}} \)-invariant derivation we mean a total derivation
\[
\nabla = \frac{A}{dt} + B \frac{dx}{dt} + C \frac{dy}{dt}
\]
that commutes with prolonged action of algebra \( \mathfrak{g}_m \) or \( \mathfrak{g}_{\text{sym}} \). Here \( A, B, C \) are rational functions on the prolonged equation \( \mathcal{E}_k \) for some \( k \geq 0 \).
The field of kinematic invariants

First of all, observe that the functions

\[ \rho, \, s \]

(as well as \( p \) and \( T \)) on the equation \( E_0 \) are \( g_m \)-invariants.

Straightforward computations using DifferentialGeometry package by I. Anderson [7] in Maple show that the following functions are the first order kinematic invariants:

\[
\begin{align*}
J_1 &= u_x + v_y, & J_5 &= \rho_x s_y - \rho_y s_x, \\
J_2 &= u_y - v_x, & J_6 &= s_t + s_x u + s_y v, \\
J_3 &= \rho_x^2 + \rho_y^2, & J_7 &= \rho_x (\rho_x u_x + \rho_y u_y) + \rho_y (\rho_x v_x + \rho_y v_y), \\
J_4 &= s_x^2 + s_y^2, & J_8 &= s_x (\rho_x u_x + \rho_y u_y) + s_y (\rho_x v_x + \rho_y v_y).
\end{align*}
\]

It is easy to check that the codimension of the regular \( g_m \)-orbits on \( E_1 \) is equal to 10.

**Proposition 1** The singular points belong to the union of two sets:

\[
\begin{align*}
\varUpsilon_1 &= \{ u_x - v_y = 0, \, u_y + v_x = 0, \, u_t = v_t = \rho_x = \rho_y = s_x = s_y = 0 \}, \\
\varUpsilon_2 &= \{ J_3 J_5 (J_3 J_4 - J_5^2) = 0 \}.
\end{align*}
\]

The set \( \varUpsilon_1 \) contains singular points that have five-dimensional singular orbits. The set \( \varUpsilon_2 \) contains points where differential invariants \( J_1, J_2, \ldots, J_8 \) are dependent.

The proofs of the following theorems can be found in [11].

**Theorem 4** [11] The field of the first order kinematic invariants is generated by the invariants \( \rho, s, J_1, J_2, J_3, \ldots, J_8 \). These invariants separate the regular \( g_m \)-orbits.

**Theorem 5** [11] The derivations

\[
\nabla_1 = \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy}, \quad \nabla_2 = \rho_x \frac{d}{dx} + \rho_y \frac{d}{dy}, \quad \nabla_3 = s_x \frac{d}{dx} + s_y \frac{d}{dy}
\]

are \( g_m \)-invariant. They are linearly independent if

\[ \rho_x s_y - \rho_y s_x \neq 0. \]

The bundle \( \pi_{2,1} : E_2 \to E_1 \) has rank 14, and by applying the derivations \( \nabla_1, \nabla_2, \nabla_3 \) to the kinematic invariants \( J_1, J_2, \ldots, J_8 \) we get 24 kinematic invariants. Straightforward computations show that among these invariants 14 are always independent (see http://d-omega.org).

Moreover, starting with the order \( k = 1 \) dimensions of regular orbits are equal to \( \dim g_m = 6 \) and all equations \( E_k, k \geq 3 \), are the prolongations of \( E_2 \).

Therefore, if we denote by \( H(k) \) the Hilbert function of the \( g_m \)-invariants field, i.e. \( H(k) \) is the number of independent invariants of pure order \( k \) (see [10] for details), then \( H(k) = 5k + 4 \) for \( k \geq 2 \), and \( H(0) = 2, \, H(1) = 8 \).

The corresponding Poincaré function is equal to

\[ P(z) = \frac{2 + 4z - z^3}{(1 - z)^2}. \]

Summarizing, we get the following result.

**Theorem 6** [11] The field of the kinematic invariants is generated by the invariants \( \rho, s \) of order zero, by the invariants \( J_1, J_2, \ldots, J_8 \) of order one and by the invariant derivations \( \nabla_1, \nabla_2, \nabla_3 \). This field separates the regular orbits.
The field of Euler invariants

Let us consider the case when the thermodynamic state admit a one-dimensional symmetry algebra generated by the vector field

\[ A = \xi_1 X_7 + \xi_2 X_8 + \xi_3 X_9 + \xi_4 X_{10} + \xi_5 X_{11} + \xi_6 X_{12}. \]

Note that, the \( g_m \)-invariant derivations \( \nabla_1, \nabla_2, \nabla_3 \) do not commute with the thermodynamic symmetry \( A \).

Moreover, the action of the thermodynamic vector field \( A \) on the field of kinematic invariants is given by the following derivation

\[ \xi_6 \rho \partial_\rho + (\xi_1 - s(\xi_4 - \xi_5 + \xi_6)) \partial_s - J_1(\xi_4 + \xi_5) \partial_{J_1} - J_2(\xi_4 + \xi_5) \partial_{J_2} - 2J_3(\xi_4 - \xi_6) \partial_{J_3} - 2J_4(2\xi_4 + \xi_5 + \xi_6) \partial_{J_4} - J_5(3\xi_4 - \xi_5) \partial_{J_5} - J_6(2\xi_4 + \xi_5) \partial_{J_6} - J_7(3\xi_4 + \xi_5 - 2\xi_6) \partial_{J_7} - 4\xi_4 J_6 \partial_{J_6}. \]

Therefore, finding the first integrals of this vector field we get the basic Euler invariants of the first order.

**Theorem 7** \[11\] The field of the Euler differential invariants for thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

\[
\frac{J_1}{J_6} \left( s - \frac{\xi_1}{\xi_4 - \xi_5 + \xi_6} \right), \quad \frac{J_2}{J_1}, \quad \frac{J_3}{\rho^3 J_6}, \quad \frac{J_4}{\rho J_6^3}, \quad \frac{J_5}{J_8}, \quad \frac{J_6}{\rho^{\frac{3\xi_4 - \xi_6}{\xi_4}}}, \quad \frac{J_7}{J_3}, \quad \frac{J_8}{\rho^{\frac{3\xi_4 - \xi_6}{\xi_4} + 1}},
\]

of the first order and by the invariant derivations

\[
\rho^{-\frac{\xi_4 + \xi_6}{\xi_4}} \nabla_1, \quad \rho^{\frac{2\xi_4}{\xi_4}} \nabla_2, \quad \rho^{\frac{3\xi_4 - \xi_6}{\xi_4} + 1} \nabla_3.
\]

This field separates the regular orbits.

Now consider the case when the thermodynamic state admits a commutative two-dimensional symmetry algebra generated by the vector fields \( A = \sum_{i=1}^6 \mu_i X_{i+6} \) and \( B = \sum_{i=1}^6 \eta_i X_{i+6} \) such that \( \mu \)'s and \( \eta \)'s satisfy relations

\[
\begin{align*}
\eta_1 \mu_4 - \eta_4 \mu_1 - \eta_1 \mu_5 + \eta_5 \mu_1 + \eta_1 \mu_6 - \eta_6 \mu_1 &= 0, \\
2\eta_2 \mu_5 - 2\eta_5 \mu_2 - \eta_2 \mu_6 + \eta_6 \mu_2 &= 0.
\end{align*}
\]

Using similar computations we get the following result.

**Theorem 8** \[11\] The field of the Euler differential invariants for thermodynamic states admitting a commutative two-dimensional symmetry algebra is generated by differential invariants

\[
\frac{J_1 \rho^{\frac{\xi_4 + \xi_5 - 2\xi_4}{\xi_4 - \xi_1}} ((\mu_4 - \mu_5 + \mu_6)s - \mu_1)^{\frac{\xi_4 + \xi_5}{\xi_4 - \xi_1}}}{J_6}, \quad \frac{J_2}{J_1}, \quad \frac{J_3}{\rho^3 J_6}, \quad \frac{J_4}{\rho^3 J_1}, \quad \frac{J_5}{J_5}, \quad \frac{J_6}{J_3}, \quad \frac{J_7}{J_3}, \quad \frac{J_8}{J_1 J_3},
\]

\[
\rho^8 \frac{J_1 J_4}{J_3^3}, \quad \rho^4 J_1 J_5, \quad \rho^3 J_6 J_3, \quad \rho^4 J_1 J_7, \quad \rho^4 J_1 J_8.
\]
of the first order and by the invariant derivations

$$\rho \frac{\varsigma_1 + \varsigma_2 - 2\varsigma_3}{\varsigma_2 - \varsigma_3} (\mu_4 - \mu_5 + \mu_6) s - \mu_4 \frac{\varsigma_1 + \varsigma_2}{\varsigma_2 - \varsigma_3} \nabla_1, \quad \rho \frac{3\varsigma_1 - \varsigma_2 - 2\varsigma_3}{\varsigma_2 - \varsigma_3} (\mu_4 - \mu_5 + \mu_6) s - \mu_1 \frac{2\varsigma_1}{\varsigma_2 - \varsigma_3} \nabla_2,$$

$$\rho \frac{2\varsigma_3}{\varsigma_2 - \varsigma_3} (\mu_4 - \mu_5 + \mu_6) s - \mu_1 \frac{3\varsigma_1 - \varsigma_2 - 2\varsigma_3}{\varsigma_2 - \varsigma_3} \nabla_3,$$

where

$$\varsigma_1 = \eta_4 \mu_6 - \eta_6 \mu_4, \quad \varsigma_2 = \eta_5 \mu_6 - \eta_6 \mu_5, \quad \varsigma_3 = \eta_4 \mu_5 - \eta_5 \mu_4.$$

This field separates the regular orbits.

Note that these theorems are valid for general $\xi$'s. The special cases are considered in [1].

### 3.2 Flows on a sphere

In this section we consider Euler system (4) on a two-dimensional unit sphere $M = S^2$ with the metric $g = \sin^2 y \, dx^2 + dy^2$ in the spherical coordinates.

The velocity field of the flow has the form $u = u(t, x, y) \partial_t + v(t, x, y) \partial_y$, the pressure $p$, the density $\rho$, the temperature $T$ and the entropy $s$ are the functions of time and space with the coordinates $(t, x, y)$.

Here we consider the flow without any external force field, so $g = 0$.

#### 3.2.1 Symmetry Lie algebra

As in the previous section, to describe the Lie algebra of symmetries, we consider the Lie algebra $g$ generated by the following vector fields on the manifold $\mathcal{J}^0 \pi$:

$$X_1 = \partial_t, \quad X_2 = \partial_x,$$

$$X_3 = \frac{\cos x}{\tan y} \partial_x + \sin x \partial_y - \left( \frac{\sin x}{\tan y} u + \frac{\cos x}{\sin^2 y} v \right) \partial_u + u \cos x \partial_v,$$

$$X_4 = \frac{\sin x}{\tan y} \partial_x - \cos x \partial_y + \left( \frac{\cos x}{\tan y} u - \frac{\sin x}{\sin^2 y} v \right) \partial_u + u \sin x \partial_v,$$

$$X_5 = \partial_s, \quad X_6 = \partial_p, \quad X_7 = T \partial_T,$$

$$X_8 = t \partial_t - u \partial_u - v \partial_v + 2\rho \partial_p - s \partial_s,$$

$$X_9 = p \partial_p + \rho \partial_p - s \partial_s.$$

Consider the pure geometric and thermodynamic parts of this symmetry algebra.

The geometric part $g_m = \langle X_1, X_2, X_3, X_4 \rangle$ represents by the symmetries with respect to a group of sphere motions and time shifts, i.e. $g_m = \mathfrak{so}(3, \mathbb{R}) \oplus \mathbb{R}$, and $g_m = \ker \partial$.

To describe the thermodynamic part of the symmetry algebra, we denote by $h$ the Lie algebra generated by the vector fields

$$Y_1 = \partial_s, \quad Y_2 = \partial_p, \quad Y_4 = T \partial_T,$$

$$Y_4 = 2\rho \partial_p - s \partial_s, \quad Y_5 = p \partial_p - \rho \partial_p.$$

This is a solvable Lie algebra with the following structure

$$[Y_1, Y_4] = -Y_1, \quad [Y_2, Y_5] = Y_2.$$

As above, let $h_t$ be the Lie subalgebra of algebra $h$ that preserves thermodynamic state (1).
Theorem 9 The Lie algebra \( \mathfrak{g}_{\text{sym}} \) of point symmetries of the Euler system of differential equations on a two-dimensional unit sphere coincides with

\[ \vartheta^{-1}(\mathfrak{h}_1). \]

3.2.2 Symmetry classification of states

In this section we classify the thermodynamic states or Lagrangian surfaces \( \tilde{L} \) (compare with the previous section) depending on the dimension of the symmetry algebra \( \mathfrak{h}_1 \subset \mathfrak{h} \).

We consider one- and two-dimensional symmetry algebras only.

**States with a one-dimensional symmetry algebra**

Let \( \dim \mathfrak{h}_1 = 1 \) and let \( Z = \sum_{i=1}^{5} \lambda_i Y_i \) be a basis vector in this algebra, then the differential 1-form \( \iota_Z \Omega \) has the form

\[ \iota_Z \Omega = \frac{2\lambda_4 - \lambda_5}{\rho} dp - \frac{\lambda_5 p + \lambda_2}{\rho^2} dp - \lambda_3 T ds + (\lambda_1 - \lambda_4 s) dT, \]

and the surface \( \tilde{L} \) can be found from the following PDE system

\[
\begin{cases}
(2\lambda_4 - \lambda_5) \rho \epsilon_{\rho \rho} + (\lambda_1 - \lambda_4 s) \epsilon_{\rho s} + (4\lambda_4 - 3\lambda_5) \epsilon_{\rho} - \frac{\lambda_2}{\rho^2} = 0, \\
(\lambda_1 - \lambda_4 s) \epsilon_{ss} + (2\lambda_4 - \lambda_5) \rho \epsilon_{\rho s} - \lambda_3 \epsilon_s = 0.
\end{cases}
\]  

(9)

It is easy to check that the bracket of these two equations (see [9]) vanishes, and therefore the system is formally integrable and compatible.

Below we list solutions of this system under the assumption of parameters \( \lambda \) generality. A more detailed description may be found in [11], [12].

Solving the last system in case \( \lambda_3 + \lambda_4 - 2\lambda_5 \neq 0 \), we find the following expressions for the pressure and the temperature:

\[ p = C_1 \rho^{\frac{\lambda_4}{\lambda_5}} \frac{\lambda_3}{\lambda_5} - \frac{\lambda_2}{\lambda_5}, \quad T = C_2 (\lambda_1 - \lambda_4 s)^{-\frac{\lambda_3}{\lambda_4}}, \]

where \( C_1, C_2 \) are constants.

The admissibility conditions (the negative definiteness of the form \( \kappa \)) have the form

\[ \frac{\lambda_3}{\lambda_1 - \lambda_4 s} > 0, \quad \frac{\lambda_5 C_1}{2\lambda_4 - \lambda_5} > 0 \]

for all \( s \in (-\infty, s_0] \).

**Theorem 10** The thermodynamic states admitting a one-dimensional symmetry algebra have the form

\[ p = C_1 \rho^{\frac{\lambda_4}{\lambda_5}} \frac{\lambda_3}{\lambda_5} - \frac{\lambda_2}{\lambda_5}, \quad T = C_2 (\lambda_1 - \lambda_4 s)^{-\frac{\lambda_3}{\lambda_4}}, \]

where the constants defining the symmetry algebra satisfy inequalities

\[ s_0 < \frac{\lambda_1}{\lambda_4}, \quad C_1 > 0, \quad \frac{\lambda_2}{\lambda_5} < 0, \quad \frac{\lambda_5}{2\lambda_4 - \lambda_5} > 0, \]

and besides they must meet one of the following conditions:
1. if $\frac{\lambda_3}{\lambda_4}$ is irrational, then $\lambda_3 > 0$, $\lambda_4 > 0$, $C_2 > 0$;

2. if $\frac{\lambda_3}{\lambda_4}$ is rational, then $\frac{\lambda_3}{\lambda_4} > 0$ (i.e. $\frac{\lambda_3}{\lambda_4} = \frac{m}{n}$) and
   
   (a) if $k$ is even, then $\lambda_4 > 0$, $C_2 > 0$;
   
   (b) if $k$ is odd and $m$ is even, then $C_2 > 0$;
   
   (c) if $k$ is odd and $m$ is odd, then $C_2\lambda_4 > 0$.

**States with a two-dimensional symmetry algebra**

As in the plane case there are no thermodynamic states that admit a two-dimensional non-commutative symmetry algebra.

**States with a two-dimensional commutative symmetry algebra**

Let now $\mathfrak{h}_1 \subset \mathfrak{h}$ be a commutative two-dimensional Lie subalgebra, and let $A = \sum_{i=1}^{5} \alpha_i Y_i$, $B = \sum_{i=1}^{5} \beta_i Y_i$ be basis vectors in this algebra.

Then the condition $[A, B] = 0$ gives the following relations on $\alpha$’s and $\beta$’s:

$$\alpha_1 \beta_4 - \alpha_4 \beta_1 = 0, \quad \alpha_2 \beta_5 - \alpha_5 \beta_2 = 0. \quad (10)$$

Then, as above, restriction of the forms $\iota_A \Omega$ and $\iota_B \Omega$ on the state surface $\tilde{L}$ leads us to the four differential equations of the form (9), and the formal integrability condition for obtained system has the form

$$(\alpha_2 \beta_5 - \alpha_5 \beta_2)(5 \beta_4 - 3 \beta_5) = 0,$$

which is satisfied due to relations (10). Solving this system for the general parameters $\alpha$ and $\beta$ we get only the “nonphysical” solution of the form $\epsilon = C_1 \rho^{-1} + C_2$.

For the special case, for example,

$$\alpha_3 = \frac{\alpha_4 \beta_3}{\beta_4}, \quad \alpha_5 = \frac{\alpha_4 \beta_5}{\beta_4},$$

we get

$$p = C_1 \rho^{\frac{s_3}{s_4 - s_5}} - \frac{\beta_2}{\beta_5}, \quad T = C_2 \left( s - \frac{\beta_1}{\beta_4} \right)^{\frac{s_3}{\beta_4}}.$$

And the admissibility condition leads to the relations

$$\frac{\beta_3}{\beta_1 - \beta_4 s} > 0, \quad \frac{C_1 \beta_5}{2 \beta_4 - \beta_5} > 0.$$

**Theorem 11** In the general case, there are no physically applicable thermodynamic states, which admit a two-dimensional commutative symmetry algebra.
For the special case $\alpha_3 = \frac{\alpha_3 \beta_3}{\beta_4}$ and $\alpha_5 = \frac{\alpha_4 \beta_5}{\beta_4}$, the thermodynamic states admitting a two-dimensional commutative symmetry algebra have the form

$$p = C_1 \rho \frac{\alpha_4}{\alpha_5} \beta_5 \beta_4 - \beta_5, \quad T = C_2 \left( s - \frac{\beta_5}{\beta_4} \right) \frac{\alpha_4}{\alpha_5},$$

where the constants defining the symmetry algebra satisfy inequalities

$$s_0 < \frac{\beta_1}{\beta_4}, \quad C_1 > 0, \quad \frac{\beta_2}{\beta_5} < 0, \quad \frac{\beta_3}{2\beta_4 - \beta_5} > 0, \quad \frac{\beta_3}{\beta_4} = \frac{m}{k} > 0,$$

i.e. $\frac{\beta_3}{\beta_4}$ is rational positive number, and the following cases are possible:

1. if $k$ is odd and $m$ is even, then $C_2 > 0$;
2. if $k$ is odd and $m$ is odd, then $C_2 \lambda_4 > 0$.

3.2.3 Differential invariants

As in the previous section, we consider two group actions on the Euler equation $E$, i.e. the prolonged action of the group generated by the action of the Lie algebra $g_{m}$ and the action generated by the prolongation of the action of the Lie algebra $g_{sym}$. So we get two types of differential invariants – the kinematic and the Euler invariants.

The field of kinematic invariants

First of all, the functions $\rho, s, g(u, u)$ (as well as $p$ and $T$) generate all $g_{m}$-invariants of order zero.

Consider two vector fields $u$ and $\tilde{u}$ such that $g(u, \tilde{u}) = 0$ and $g(u, u) = g(\tilde{u}, \tilde{u})$. Writing the covariant differential $d_{\tilde{u}}u$ with respect to the vectors $u$ and $\tilde{u}$ as the sum of its symmetric and antisymmetric parts we obtain the 4 invariants of the first order:

$$J_1 = u_x + v_y + v \cot y, \quad J_2 = u_y \sin y - \frac{v}{\sin y} + 2u \cos y,$$
$$J_3 = (u(v_x - u_x) + v(u_y - v_y)) \sin y + u \cos y(u^2 \sin^2 y + 2v^2), \quad J_4 = v(u_x v - v_x u) - u(u_y v - v_y u) \sin^2 y + v^3 \cot y. \quad (11)$$

The proof of the following theorem can be found in [12].

**Theorem 12** [12] The following derivations

$$\nabla_1 = \frac{d}{dt}, \quad \nabla_2 = \frac{\rho_x}{\sin^2 y} \frac{d}{dx} + \rho_y \frac{d}{dy}, \quad \nabla_3 = \frac{s_x}{\sin^2 y} \frac{d}{dx} + s_y \frac{d}{dy}$$

are $g_{m}$-invariant. They are linearly independent if

$$\rho_x s_y - \rho_y s_x \neq 0.$$

It is easy to check that the codimension of regular $g_{m}$-orbits is equal to 12. The Rosenlicht theorem [18] gives us the following result.
Theorem 13 [12] The field of the first order kinematic invariants is generated by the invariants $\rho, s, g(u, u)$ of order zero and by the invariants (11) and

$$\nabla_1 \rho, \nabla_1 s, \nabla_2 \rho, \nabla_2 s, \nabla_3 s$$

(12)
of order one. These invariants separate regular $g_m$-orbits.

The bundle $\pi_{2,1} : E_2 \rightarrow E_1$ has rank 14, and by applying the derivations $\nabla_1, \nabla_2, \nabla_3$ to the kinematic invariants (11) and (12) we get 27 kinematic invariants. Straightforward computations show that among these invariants 14 are always independent (see http://d-omega.org).

Therefore, starting with the order $k = 1$ dimensions of regular orbits are equal to $\dim g_m = 4$.

The Hilbert function (the number of independent invariants) of the $g_m$-invariants field has form $H(k) = 5k + 4$ for $k \geq 1$ and $H(0) = 3$, and the corresponding Poincaré function is equal to

$$P(z) = \frac{3 + 3z - z^3}{(1 - z)^2}.$$

Summarizing, we get the following result.

Theorem 14 [12] The field of the kinematic invariants is generated by the invariants $\rho, s, g(u, u)$ of order zero, by the invariants (11) and (12) of order one and by the invariant derivations $\nabla_1, \nabla_2, \nabla_3$. This field separates regular orbits.

The field of Euler invariants

Let us consider the case when the equations of thermodynamic state $\tilde{L}$ admit a one-dimensional symmetry algebra generated by the vector field

$$A = \xi_1 X_5 + \xi_2 X_6 + \xi_3 X_7 + \xi_4 X_8 + \xi_5 X_9.$$

Using a similar computations as in the plane case we get the following result.

Theorem 15 [12] The field of the Euler differential invariants on a sphere for thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

$$J_1 \rho \left( s - \frac{\xi_1}{\xi_4 + \xi_5} \right), \quad J_1 \rho \frac{\xi_4}{\xi_4 + \xi_5}, \quad \frac{g(u, u)}{J_1^2},$$

$$\frac{J_2}{J_1}, \quad \frac{J_3}{J_1^3}, \quad \frac{J_4}{J_1^2}, \quad \frac{\nabla_1 \rho}{J_1 \rho}, \quad \frac{\nabla_2 \rho}{\rho^2}, \quad \rho \nabla_1 s, \quad J_1 \nabla_2 s, \quad J_2 \rho^{\frac{1}{2}} \nabla_3 s$$
of the first order and by the invariant derivations

$$\rho \frac{\xi_4}{\xi_4 + \xi_5} \nabla_1, \quad \rho^{-1} \nabla_2, \quad \rho \frac{\xi_4 + \xi_5}{\xi_4 + \xi_5} \nabla_3.$$

This field separates regular orbits.

The last formulas are valid for general $\xi$’s. All details and the special cases are considered in [12].
Now let the thermodynamic state admit a commutative two-dimensional symmetry algebra generated by the vector fields $A = \sum_{i=1}^{5} \mu_i X_{i+4}$, $B = \sum_{i=1}^{5} \eta_i X_{i+4}$ such that $\mu$'s and $\eta$'s satisfy relations

$$\begin{align*}
\eta_1 \mu_4 - \eta_4 \mu_1 + \eta_5 \mu_5 - \eta_5 \mu_4 &= 0, \\
\eta_2 \mu_5 - \eta_5 \mu_2 &= 0.
\end{align*}$$

**Theorem 16** [12] The field of Euler differential invariants for the thermodynamic states admitting a commutative two-dimensional symmetry algebra is generated by differential invariants

$$J_1 \rho((\mu_4 + \mu_5)s - \mu_1), \quad \frac{\nabla_1 \rho}{J_1 \rho}, \quad \frac{\nabla_2 \rho}{\rho^2}, \quad J_1 \nabla_2 s, \quad J_1 \rho^2 \nabla_3 s$$

of the first order and by the invariant derivations

$$\rho((\mu_4 + \mu_5)s - \mu_1)\nabla_1, \quad \rho^{-1}\nabla_2, \quad ((\mu_4 + \mu_5)s - \mu_1)^{-1}\nabla_3.$$

This field separates regular orbits.

### 3.3 Flows on a spherical layer

Consider Euler system [4] on a spherical layer $M = S^2 \times \mathbb{R}$ with the coordinates $(x, y, z)$, where $(x, y)$ are the stereographic coordinates on the sphere, and the metric

$$g = \frac{4}{(x^2 + y^2 + 1)^2}(dx^2 + dy^2) + dz^2.$$

The velocity field of the flow has the form $u = u(t, x, y, z) \partial_x + v(t, x, y, z) \partial_y + w(t, x, y, z) \partial_z$, the pressure $p$, the density $\rho$, the temperature $T$ and the entropy $s$ are the functions of time and space with the coordinates $(t, x, y, z)$.

The vector of gravitational acceleration is of the form $g = (0, 0, g)$.

#### 3.3.1 Symmetry Lie algebra

Consider the Lie algebra $\mathfrak{g}$ generated by the following vector fields on the manifold $J^0 \pi$:

$$
\begin{align*}
X_1 &= \partial_t, & X_3 &= t \partial_z + \partial_w, \\
X_2 &= \partial_z, & X_4 &= y \partial_x - x \partial_y + v \partial_u - u \partial_v, \\
X_5 &= xy \partial_x - \frac{1}{2}(x^2 - y^2 - 1) \partial_y + (xv + yu) \partial_u - (xu - yv) \partial_v, \\
X_6 &= \frac{1}{2}(x^2 - y^2 + 1) \partial_x + xy \partial_y + (xu - yv) \partial_u + (xv + yu) \partial_v, \\
X_7 &= \partial_s, & X_{10} &= t \partial_t + gt^2 \partial_z - u \partial_u - v \partial_v + (2gt - w) \partial_w + 2\rho \partial_\rho - s \partial_s, \\
X_8 &= \partial_p, & X_{11} &= p \partial_p + \rho \partial_\rho - s \partial_s, \\
X_9 &= T \partial_T.
\end{align*}
$$

The pure geometric part $\mathfrak{g}_m$ generated by the vector fields $X_1, X_2, \ldots, X_6$. Transformations corresponding to the elements of the Lie group generated by the algebra $\mathfrak{g}_m$ are
compositions of sphere motions, Galilean transformations and shifts along the \( z \) direction, time shifts.

To describe thermodynamic part of the symmetry algebra, we consider the Lie algebra \( \mathfrak{h} \) generated by the vector fields

\[
Y_1 = \partial_s, \quad Y_2 = \partial_p, \quad Y_3 = T \partial_T, \quad Y_4 = 2\rho \partial_p - s \partial_s, \quad Y_5 = p \partial_p - \rho \partial_p.
\]

This is a solvable Lie algebra with the following structure

\[
[Y_1, Y_4] = -Y_1, \quad [Y_2, Y_5] = Y_2.
\]

Let also \( \mathfrak{h}_t \) be the Lie subalgebra of algebra \( \mathfrak{h} \) that preserves thermodynamic state (1). Then the following result is valid.

**Theorem 17** [13] The Lie algebra \( \mathfrak{g}_{\text{sym}} \) of point symmetries of the Euler system of differential equations on a spherical layer coincides with \( \vartheta^{-1}(\mathfrak{h}_t) \).

### 3.3.2 Symmetry classification of states

The Lie algebra generated by the vector fields \( Y_1, \ldots, Y_5 \) coincides with the Lie algebra of the thermodynamic symmetries of the Euler system on a sphere.

Thus the classification of the thermodynamic states or Lagrangian surfaces \( \tilde{L} \) depending on the dimension of the symmetry algebra \( \mathfrak{h}_t \subset \mathfrak{h} \) is the same as the classification presented in the previous section.

### 3.3.3 Differential invariants

The field of kinematic invariants

First of all, the functions \( \rho, s, g(u, u) - w^2 \) (as well as \( p \) and \( T \)) generate all \( \mathfrak{g}_m \)-invariants of order zero.

The proofs of the following theorems can be found in [13].

**Theorem 18** [13] The following derivations

\[
\nabla_1 = \frac{d}{dz}, \quad \nabla_2 = \frac{d}{dt} + \frac{d}{dz}, \quad \nabla_3 = \frac{d}{dx} + v \frac{d}{dy}, \quad \nabla_4 = \frac{d}{dx} - u \frac{d}{dy}
\]

are \( \mathfrak{g}_m \)-invariant. They are linearly independent if

\[u^2 + v^2 \neq 0.\]

**Theorem 19** [13] The field of the first order kinematic invariants is generated by the invariants \( \rho, s, g(u, u) - w^2 \) of order zero and by the invariants

\[
\begin{align*}
\nabla_1 \rho, \quad \nabla_2 \rho, \quad \nabla_3 \rho, \quad \nabla_4 \rho, \quad \nabla_1 s, \quad \nabla_2 s, \quad \nabla_3 s, \quad \nabla_4 s, \\
\n\nabla_1 (g(u, u) - w^2), \quad \nabla_2 (g(u, u) - w^2), \quad \nabla_3 (g(u, u) - w^2), \quad \nabla_4 (g(u, u) - w^2), \\
\n\nabla_1 w, \quad \nabla_2 w, \quad \nabla_3 w, \quad \nabla_4 w, \quad J_1 = u_z w_x + v_z w_y, \quad J_2 = \frac{u_t v_x - u_x v_t}{u_x^2 + v_x^2}
\end{align*}
\]

of order one. These invariants separate regular \( \mathfrak{g}_m \)-orbits.
The bundle $\pi_{2,1} : \mathcal{E}_2 \to \mathcal{E}_1$ has rank 33, and by applying the derivations $\nabla_i, i = 1, \ldots, 4$ to the first order kinematic invariants we get 64 kinematic invariants. Straightforward computations show that among these invariants 33 are always independent.

Therefore, starting with the order $k = 1$ dimensions of the regular orbits are equal to $\dim g_0 = 6$.

Moreover, the number of independent invariants (the Hilbert function) is equal to $H(k) = 3k^2 + 8k + 5$ for $k \geq 1$ and $H(0) = 3$.

The corresponding Poincaré function has the form

$$P(z) = \frac{3 + 7z - 6z^2 + 2z^3}{(1 - z)^3}.$$ 

**Theorem 20** [13] The field of the kinematic invariants is generated by the invariants $\rho, s, g(u, u) - w^2$ of order zero, by the invariants (13) of order one and by the invariant derivations $\nabla_i, i = 1, \ldots, 4$. This field separates regular orbits.

**The field of Euler invariants**

At first we consider the case when the thermodynamic state $\hat{L}$ admits a one-dimensional symmetry algebra generated by the vector field

$$A = \xi_1 X_7 + \xi_2 X_8 + \xi_3 X_9 + \xi_4 X_{10} + \xi_5 X_{11}.$$ 

Then for general values of the parameters $\xi$’s we have the following result. The special cases are considered in [13].

**Theorem 21** [13] The field of the Euler differential invariants for thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

\[
\begin{align*}
&\nabla_1 \rho, \quad \nabla_2 \rho, \quad \nabla_3 \rho, \quad \nabla_4 \rho, \\
&w_1 \nabla s, \quad w_2 \nabla s, \quad w_3 \nabla s, \quad w_4 \nabla s, \\
&w_5 \nabla (g(u, u) - w^2), \quad w_6 \nabla (g(u, u) - w^2), \\
&w_7 \nabla (g(u, u) - w^2), \quad w_8 \nabla (g(u, u) - w^2), \\
&w_9 \nabla (g(u, u) - w^2), \quad w_{10} \nabla (g(u, u) - w^2), \\
&w_1 \nabla \xi_1, \quad w_2 \nabla \xi_2, \quad w_3 \nabla \xi_3, \quad w_4 \nabla \xi_4, \quad w_5 \nabla \xi_5,
\end{align*}
\]

of the first order and by the invariant derivatives

$$\nabla_1, \quad w_1^{-1} \nabla_2, \quad w_3^{-1} \nabla_3, \quad w_4^{-1} \nabla_4.$$ 

This field separates regular orbits.

Now, let the thermodynamic state admit a commutative two-dimensional symmetry algebra generated by the vector fields $A = \sum_{i=1}^{5} \mu_i X_{i+6}$, $B = \sum_{i=1}^{5} \eta_i X_{i+6}$, then $\mu$’s and $\eta$’s satisfy relations

\[
\begin{align*}
\eta_1 \mu_4 - \eta_4 \mu_1 + \eta_4 \mu_5 - \eta_5 \mu_1 &= 0, \\
\eta_2 \mu_5 - \eta_5 \mu_2 &= 0.
\end{align*}
\]
Theorem 22 [13] The field of Euler differential invariants for thermodynamic states admitting a commutative two-dimensional symmetry algebra is generated by differential invariants

\[ w_z^{-2} (g(u, u) - w^2), \]
\[ \nabla_1 \rho, \, \nabla_2 \rho, \, \nabla_3 \rho, \, \nabla_4 \rho, \]
\[ w_z \rho \nabla_1 s, \, \rho \nabla_2 s, \, \rho \nabla_3 s, \, \rho \nabla_4 s, \]
\[ w_z^{-2} \nabla_1 (g(u, u) - w^2), \, w_z^{-3} \nabla_3 (g(u, u) - w^2), \, w_z^{-3} \nabla_4 (g(u, u) - w^2), \]
\[ w_z \rho ((\mu_4 + \mu_5) s - \mu_1), \, w_z^{-2} \nabla_3 w, \, w_z^{-2} \nabla_4 w, \, w_z^{-2} J_1, \, w_z^{-1} J_2 \]

of the first order and by the invariant derivatives

\[ \nabla_1, \, w_z^{-1} \nabla_2, \, w_z^{-1} \nabla_3, \, w_z^{-1} \nabla_4. \]

This field separates regular orbits.
4 Compressible viscid fluids or gases

In this section we study differential invariants of compressible viscid fluids or gases.

The system of differential equations (the Navier–Stokes system) describing flows on an oriented Riemannian manifold \((M, g)\) consists of the following equations (see [1] for details):

\[
\begin{aligned}
\rho (u_t + \nabla_u u) - \text{div} \sigma - g\rho &= 0, \\
\partial_t (\rho \Omega_g) + \mathcal{L}_u (\rho \Omega_g) &= 0, \\
\rho T (s_t + \nabla_u s) - \Phi + k(\Delta_g T) &= 0.
\end{aligned}
\]

(14)

Here the divergence operator \(\text{div} : \mathcal{S}^2 T^* M \rightarrow TM\) is given by

\[
(\text{div} \sigma)_l = (d\nabla \sigma)_{ijk} g_{jk} g_{il},
\]

where \(d\nabla\) is the covariant differential.

The fluid under consideration is assumed to be newtonian and isotropic. Therefore, the fluid stress tensor \(\sigma\) is symmetric, and it depends on the rate of deformation tensor \(D = \frac{1}{2} \mathcal{L}_u (g)\) linearly. These two conditions give the following form of the stress tensor:

\[
\sigma = -pg + \sigma',
\]

where the viscous stress tensor \(\sigma'\) is given by

\[
\sigma' = 2\eta \left( D - \frac{\langle D, g \rangle_g}{\langle g, g \rangle_g} g \right) + \zeta \langle D, g \rangle_g g.
\]

The quantity \(\Phi = \langle \sigma', D \rangle_g\) represents the rate of dissipation of mechanical energy [1].

The first equation of system (14) is the Navier–Stokes equation, the second one is the continuity equation and the third one is the general equation of heat transfer.

In this section we consider the following examples of manifold \(M\): a plane, a three-dimensional space, a sphere and a spherical layer.

Note that in all these cases the number of unknown functions is greater than the number of system equations by 2, i.e. the system (14) is incomplete. As above we get two additional equations using the thermodynamics of the medium.

Thus, by the Navier–Stokes system of differential equations we mean the system (14) extended by two equations of state (1), where functions \(f\) and \(g\) satisfy the additional relation (2) and the form \(\kappa\) is negative definite.

Geometrically, we represent this system in the following way. Consider the bundle

\[
\pi : \mathbb{R} \times TM \times \mathbb{R}^4 \rightarrow \mathbb{R} \times M
\]

of rank \((\dim M + 4)\).

Then the Navier–Stokes system is a system of differential equations on sections of the bundle \(\pi\).

Note that system (11) defines the zeroth order system \(\mathcal{E}_0 \subset J^0 \pi\).

Denote by \(\mathcal{E}_1 \subset J^1 \pi\) the system of order \(\leq 1\) obtained by the first prolongation of the system \(\mathcal{E}_0\) and by the continuity equation of system (14).

Let also \(\mathcal{E}_2 \subset J^2 \pi\) be the system of differential equations of order \(\leq 2\) obtained by the first prolongation of the system \(\mathcal{E}_1\) and all equations of system (14).

For the case \(k \geq 3\), we define \(\mathcal{E}_k \subset J^k \pi\) to be the \((k-2)\)-th prolongation of the system \(\mathcal{E}_2\).

Note that the system \(\mathcal{E}_\infty = \lim \mathcal{E}_k\) is a formally integrable system of differential equations, which we also call the Navier–Stokes system.
4.1 2D-flows

Consider Navier–Stokes system (14) on a plane $M = \mathbb{R}^2$ equipped with the coordinates $(x, y)$ and the standard flat metric $g = dx^2 + dy^2$.

The velocity field of the flow has the form $u = u(t, x, y) \partial_x + v(t, x, y) \partial_y$, the pressure $p$, the density $\rho$, the temperature $T$ and the entropy $s$ are the functions of time and space with the coordinates $(t, x, y)$.

Here we also consider the flow without any external force field, so $g = 0$.

4.1.1 Symmetry Lie algebra

To describe the Lie algebra of symmetries of the Navier–Stokes system we consider a Lie algebra $\mathfrak{g}$ generated by the following vector fields on space $\mathcal{J}^0\pi$:

\[ X_1 = \partial_x, \quad X_4 = t \partial_x + \partial_u, \]
\[ X_2 = \partial_y, \quad X_5 = t \partial_y + \partial_v, \]
\[ X_3 = y \partial_x - x \partial_y + v \partial_u - u \partial_v, \quad X_6 = \partial_t, \]
\[ X_7 = \partial_s, \quad X_8 = \partial_p, \]
\[ X_9 = x \partial_x + y \partial_y + u \partial_u + v \partial_v - 2\rho \partial_\rho + 2T \partial_T, \]
\[ X_{10} = t \partial_t - u \partial_u - v \partial_v + \rho \partial_\rho - p \partial_p - 2T \partial_T. \]

In general the symmetry algebra of system (14) consists of pure geometric and thermodynamic parts.

The geometric part is represented by the algebra $\mathfrak{g}_m = \langle X_1, X_2, \ldots, X_6 \rangle$ with respect to the group of motions, Galilean transformations and time shifts.

Moreover, the kernel of homomorphism $\vartheta (5)$ is an ideal $\mathfrak{g}_m$ in the Lie algebra $\mathfrak{g}$.

The thermodynamic part strongly depends on the symmetries of the thermodynamic state. In order to describe it, denote by $\mathfrak{h}$ a Lie algebra generated by the vector fields

\[ Y_1 = \partial_\rho, \quad Y_2 = \partial_p, \quad Y_3 = \rho \partial_\rho - T \partial_T, \quad Y_4 = p \partial_p + T \partial_T. \]

Let also $\mathfrak{h}_t$ be a Lie subalgebra of the algebra $\mathfrak{h}$ which preserves the thermodynamic state (1).

**Theorem 23** (14) A Lie algebra $\mathfrak{g}_{\text{sym}}$ of symmetries of the Navier–Stokes system of differential equations on a plane coincides with $\vartheta^{-1}(\mathfrak{h}_t)$.

Note that, usually, the equations of state are neglected and the vector fields like $f(t) \partial_p$, where $f$ is an arbitrary function, considered as symmetries of the Navier–Stokes system.

For the general equation of state $\mathfrak{h}_t = 0$ and the symmetry algebra coincides with the algebra $\mathfrak{g}_m$.

4.1.2 Symmetry classification of states

In this section we classify thermodynamic states or Lagrangian surfaces $\tilde{L}$ depending on the dimension of the symmetry algebra $\mathfrak{h}_t \subset \mathfrak{h}$.

We consider one- and two-dimensional symmetry algebras only. One can easily check that there are no physically valuable thermodynamic states with three or more dimensional symmetry algebras.
States with a one-dimensional symmetry algebra

Let \( \dim \mathfrak{h}_1 = 1 \) and let \( Z = \sum_{i=1}^{4} \lambda_i Y_i \) be a basis vector in this algebra, then the differential 1-form \( \iota_Z \Omega \) has the form

\[
\iota_Z \Omega = -\frac{\lambda_3}{\rho} \, dp + \frac{\lambda_4 \rho + \lambda_2}{\rho^2} \, d\rho + (\lambda_3 - \lambda_4) T \, ds - \lambda_1 dT,
\]

and, in terms of specific energy \( \epsilon(\rho, s) \), the Lagrangian surface \( \tilde{L} \) can be found as a solution of the following PDE system

\[
\begin{align*}
\lambda_1 \epsilon_{ss} + \lambda_3 \rho \epsilon_{s\rho} + (\lambda_3 - \lambda_4) \epsilon_s &= 0, \\
\lambda_3 \rho \epsilon_{\rho \rho} + \lambda_1 \epsilon_{s\rho} + (2\lambda_3 - \lambda_4) \epsilon_\rho - \frac{\lambda_2}{\rho^2} &= 0.
\end{align*}
\]

It is easy to check that the bracket of these two equations (see [9]) vanishes and therefore the system is formally integrable and compatible.

Solving this system for general values of parameters \( \lambda \), all special cases are considered in [14], we get expressions for the presser and the temperature

\[
T = \rho^{\frac{\lambda_3}{\lambda_3}} F', \quad p = \rho^{\frac{\lambda_3}{\lambda_3}} \left( \frac{\lambda_4}{\lambda_3} - 1 \right) F - \frac{\lambda_1}{\lambda_3} F' - \frac{\lambda_2}{\lambda_4}, \quad F = F \left( s - \frac{\lambda_1}{\lambda_3} \ln \rho \right),
\]

where \( F \) is a smooth function.

Negative definiteness of the quadratic form \( \kappa \) gives the following relations on the function \( F \) and the parameters \( \lambda \):

\[
\begin{align*}
\lambda_1^2 \rho^{\frac{\lambda_3}{\lambda_3}} F'' + \lambda_1 (\lambda_3 - \lambda_4) \rho T + \lambda_3 (\lambda_4 \rho + \lambda_2) > 0, \\
\rho^{\frac{\lambda_1-2\lambda_3}{\lambda_3}} F'' (\lambda_1 (\lambda_3 - \lambda_4) \rho T - \lambda_3 (\lambda_4 \rho + \lambda_2)) + T^2 (\lambda_3 - \lambda_4)^2 < 0.
\end{align*}
\]

**Theorem 24** The thermodynamic states admitting a one-dimensional symmetry algebra have the form

\[
T = \rho^{\frac{\lambda_3}{\lambda_3}} F', \quad p = \rho^{\frac{\lambda_3}{\lambda_3}} \left( \frac{\lambda_4}{\lambda_3} - 1 \right) F - \frac{\lambda_1}{\lambda_3} F' - \frac{\lambda_2}{\lambda_4}, \quad F = F \left( s - \frac{\lambda_1}{\lambda_3} \ln \rho \right),
\]

where \( F \) is a smooth function, \( F' \) is positive and

\[
\begin{align*}
\lambda_1^2 F'' + \lambda_1 (\lambda_3 - 2\lambda_4) F' + \lambda_4 (\lambda_4 - \lambda_3) F > 0, \\
F'' (\lambda_4 (\lambda_4 - \lambda_3) F - \lambda_1 \lambda_3 F') - (F')^2 (\lambda_4 - \lambda_3)^2 > 0.
\end{align*}
\]

States with a two-dimensional non-commutative symmetry algebra

Let \( \mathfrak{h}_1 \subset \mathfrak{h} \) be a non-commutative two-dimensional Lie subalgebra. It is easy to check that two vectors of the form \( A = Y_2, \ B = \alpha Y_1 + \beta Y_3 + Y_4 \) are the basis vectors in the non-commutative algebra \( \mathfrak{h}_1 \).

Then, as above, the restrictions of forms \( \iota_A \Omega \) and \( \iota_B \Omega \) on the state surface \( \tilde{L} \) lead us to the solution \( \rho = \text{const} \).

Since we consider thermodynamic states such that the variables \( \rho \) and \( s \) are local coordinates then we do not consider the case of the non-commutative subalgebra.
States with a two-dimensional commutative symmetry algebra

Let now \( \mathfrak{h}_t \subset \mathfrak{h} \) be a commutative two-dimensional Lie subalgebra, and let \( A = \sum_{i=1}^{4} \alpha_i Y_i, \) \( B = \sum_{i=1}^{4} \beta_i Y_i \) be the basis vectors in the algebra \( \mathfrak{h}_t. \)

Then condition \( [A, B] = 0 \) gives the following relations on \( \alpha \)'s and \( \beta \)'s:

\[
\alpha_2 \beta_4 - \alpha_4 \beta_2 = 0. \tag{15}
\]

Then the restrictions of forms \( \iota_A \Omega \) and \( \iota_B \Omega \) on the state surface \( \tilde{L} \) lead us to the following system of differential equations:

\[
\begin{align*}
\alpha_1 \epsilon_{ss} + \alpha_3 \rho \epsilon_{sp} + (\alpha_3 - \alpha_4) \epsilon_s &= 0, \\
\alpha_3 \rho \epsilon_{rp} + \alpha_1 \epsilon_{sp} + (2 \alpha_3 - \alpha_4) \epsilon_p - \frac{\alpha_2}{\rho^2} &= 0, \\
\beta_1 \epsilon_{ss} + \beta_3 \rho \epsilon_{sp} + (\beta_3 - \beta_4) \epsilon_s &= 0, \\
\beta_3 \rho \epsilon_{rp} + \beta_1 \epsilon_{sp} + (2 \beta_3 - \beta_4) \epsilon_p - \frac{\beta_2}{\rho^2} &= 0.
\end{align*}
\]

The formal integrability condition for this system has the form

\[
(5\beta_3 - \beta_4)(\alpha_2 \beta_4 - \alpha_4 \beta_2) = 0,
\]

which is satisfied due to relations (15).

Solving this PDE system we get the following expressions for the pressure and the temperature

\[
p = C(\beta - 1)e^{\alpha s} \rho^\beta - \frac{\beta_2}{\beta_4}, \quad T = C\alpha e^{\alpha s} \rho^{\beta - 1}, \tag{16}
\]

where

\[
\alpha = \frac{\alpha_4 \beta_3 - \alpha_3 \beta_4}{\alpha_1 \beta_3 - \alpha_3 \beta_1}, \quad \beta = \frac{\alpha_1 \beta_4 - \beta_1 \alpha_4}{\alpha_1 \beta_3 - \alpha_3 \beta_1},
\]

and the admissibility conditions have the form \( \alpha > 0, \beta > 1. \)

**Theorem 25** The thermodynamic states admitting a two-dimensional commutative symmetry algebra have the form

\[
p = C(\beta - 1)e^{\alpha s} \rho^\beta - \frac{\beta_2}{\beta_4}, \quad T = C\alpha e^{\alpha s} \rho^{\beta - 1},
\]

where

\[
\alpha = \frac{\alpha_4 \beta_3 - \alpha_3 \beta_4}{\alpha_1 \beta_3 - \alpha_3 \beta_1} > 0, \quad \beta = \frac{\alpha_1 \beta_4 - \beta_1 \alpha_4}{\alpha_1 \beta_3 - \alpha_3 \beta_1} > 1, \quad C > 0, \quad \frac{\beta_2}{\beta_4} < 0.
\]

Observe that, the expressions for the temperature and the pressure for an ideal gas

\[
T = \frac{1}{\gamma} \rho^k e^{\frac{\rho}{\gamma}}, \quad p = k \rho^{k+1} e^{\frac{\rho}{\gamma}},
\]

where \( k \) and \( \gamma \) are constant depending on a gas, can be obtained from the equations (16) by choosing appropriate values of the constants.
4.1.3 Differential invariants

As in the case of compressible inviscid fluids or gases (the Euler system), we consider two group actions on the Navier–Stokes equation \( \mathcal{E} \).

The first is the prolonged action of the group generated by the action of Lie algebra \( \mathfrak{g}_m \) and the differential invariants with respect this action we call \textit{kinematic differential invariants}.

The second action is the action generated by prolongation of the action Lie algebra \( \mathfrak{g}_{sym} \), and the differential invariants with respect second action we call \textit{Navier–Stokes invariants}.

Also we say that a point \( x_k \in \mathcal{E}_k \) and the corresponding orbit \( \mathcal{O}(x_k) \) (\( \mathfrak{g}_m \), or \( \mathfrak{g}_{sym} \)-orbit) are \textit{regular}, if there are exactly \( m = \text{codim} \mathcal{O}(x_k) \) independent invariants (kinematic or Navier–Stokes) in a neighbourhood of this orbit.

Thus, the corresponding point on the quotient space \( \mathcal{E}_k/\mathfrak{g}_m \) or \( \mathcal{E}_k/\mathfrak{g}_{sym} \) is smooth, and these independent invariants (kinematic or Navier–Stokes) can serve as local coordinates in a neighbourhood of this point.

Otherwise, we say that the point and the corresponding orbit are \textit{singular}.

It is worth to note that the Navier–Stokes system together with the symmetry algebras \( \mathfrak{g}_m \) or \( \mathfrak{g}_{sym} \) satisfies the conditions of the Lie–Tresse theorem (see [10]), and therefore the above differential invariants separate regular \( \mathfrak{g}_m \) or \( \mathfrak{g}_{sym} \) orbits on the Navier–Stokes system \( \mathcal{E} \).

The field of kinematic invariants

First of all observe that the density \( \rho \) and the entropy \( s \) (as well as the pressure \( p \) and the temperature \( T \)) on the equation \( \mathcal{E}_0 \) are \( \mathfrak{g}_m \)-invariants.

Moreover, the following functions are the kinematic invariants of the first order (see [13]):

\[
\begin{align*}
J_1 &= u_x + v_y, & J_5 &= \rho_x s_y - \rho_y s_x, \\
J_2 &= u_y - v_x, & J_6 &= s_t + s_x u + s_y v, \\
J_3 &= \rho_x^2 + \rho_y^2, & J_7 &= \rho_x (\rho_x u_x + \rho_y u_y) + \rho_y (\rho_x v_x + \rho_y v_y), \\
J_4 &= s_x^2 + s_y^2, & J_8 &= s_x (\rho_x u_x + \rho_y u_y) + s_y (\rho_x v_x + \rho_y v_y), \\
J_9 &= s_x (u_t + uu_x + vu_y) + s_y (v_t + uv_x + vv_y), & J_{10} &= \rho_x (u_t + uu_x + vu_y) + \rho_y (v_t + uv_x + vv_y).
\end{align*}
\]

**Proposition 2** The singular points belong to the union of two sets:

\[
\begin{align*}
\Upsilon_1 &= \{ u_x - v_y = 0, \ u_y + v_x = 0, \ u_t = v_t = \rho_x = \rho_y = s_x = s_y = 0 \}, \\
\Upsilon_2 &= \{ J_3 J_5^2 (J_5 J_4 - J_5^2) = 0 \}.
\end{align*}
\]

The set \( \Upsilon_1 \) contains singular points that have five-dimensional orbits. The set \( \Upsilon_2 \) contains points where differential invariants \( J_1, J_4, \ldots, J_{10} \) are dependent.

It is easy to check that codimension of regular \( \mathfrak{g}_m \)-orbits is equal to 12. The proofs of the following theorems can be found in [14].

**Theorem 26** [14] The field of the first order kinematic invariants is generated by invariants \( \rho, s, J_1, \ldots, J_{10} \). These invariants separate regular \( \mathfrak{g}_m \)-orbits.
Theorem 27 [14] The following derivations
\[ \nabla_1 = \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy}, \quad \nabla_2 = \rho_x \frac{d}{dx} + \rho_y \frac{d}{dy}, \quad \nabla_3 = s_x \frac{d}{dx} + s_y \frac{d}{dy} \]
are \( g_m \)-invariant. They are linear independent if
\[ \rho_y s_x - \rho_x s_y \neq 0. \]

The bundle \( \pi_{2,1} : \mathcal{E}_2 \to \mathcal{E}_1 \) has rank 18 and by applying derivations \( \nabla_1, \nabla_2, \nabla_3 \) to the kinematic invariants \( J_1, J_2, \ldots, J_{10} \) we get 30 kinematic invariants. Straightforward computations show that among these invariants 18 are always independent (see http://d-omega.org).

Therefore, beginning with order \( k = 1 \) dimensions of regular orbits are equal to \( \dim g_m = 6 \).

Moreover, the number of independent invariants of pure order \( k \) (the Hilbert function) is equal to \( H(k) = 7k + 4 \) for \( k \geq 2 \), and \( H(0) = 2 \), \( H(1) = 10 \).

The corresponding Poincaré function is equal to
\[ P(z) = \frac{2 + 6z - z^3}{(1-z)^2}. \]

Theorem 28 [14] The field of kinematic invariants is generated by the invariants \( \rho, s \) of order zero, invariants \( J_1, J_2, \ldots, J_{10} \) of order one and by the invariant derivations \( \nabla_1, \nabla_2, \nabla_3 \). This field separates regular orbits.

The field of Navier–Stokes invariants
Here we consider the case when the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field
\[ A = \xi_1 X_7 + \xi_2 X_8 + \xi_3 X_9 + \xi_4 X_{10}. \]

Note that, the \( g_m \) invariant derivations \( \nabla_1, \nabla_2, \nabla_3 \) do not commute with the thermodynamic symmetry \( A \). Moreover, the action of the thermodynamic vector field \( A \) on the field of kinematic invariants is given by the following derivation
\[
(\xi_4 - 2\xi_3)\rho \partial_\rho + \xi_1 \partial_s - \xi_4 J_1 \partial_{J_1} - \xi_4 J_2 \partial_{J_2} - 2J_3 (3\xi_3 - \xi_4) \partial_{J_3} - \\
-2\xi_3 J_3 \partial_{J_4} + J_5 (\xi_4 - 4\xi_3) \partial_{J_5} - \xi_4 J_6 \partial_{J_6} - \\
-4\xi_3 J_7 \partial_{J_7} + J_8 (\xi_4 - 6\xi_3) \partial_{J_8} - 2\xi_4 J_9 \partial_{J_9} - (\xi_4 + 2\xi_3) J_{10} \partial_{J_{10}}.
\]

Finding the first integrals of this vector field we get the basic Navier-Stokes invariants of the first order. The following result is valid for general \( \xi \)'s and the special cases can be found in [14].

Theorem 29 [14] The field of the Navier–Stokes differential invariants for the thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants
\[ \frac{\xi_1}{2\xi_3 - \xi_4} \ln \rho + s, \quad J_1 \rho \frac{\xi_4}{2\xi_3 - \xi_4}, \]

26
of the first order and by the invariant derivations
\[ \rho^{\xi_4 - 2\xi_5} \nabla_1, \rho^{2\xi_4 - 2\xi_3} \nabla_2, \rho^{2\xi_4} \nabla_3. \]

This field separates regular orbits.

Consider the case when the thermodynamic state admits a commutative two-dimensional symmetry algebra generated by the vector fields
\[ A = \sum_{i=1}^{6} \mu_i X_i, \quad B = \sum_{i=1}^{6} \eta_i X_i, \]
then \( \mu_i \)'s and \( \eta_i \)'s satisfy relation
\[ \eta_2 \mu_4 - \eta_4 \mu_2 = 0. \]

Using similar computations we get.

**Theorem 30** [14] The field of Navier–Stokes differential invariants for the thermodynamic states admitting a commutative two-dimensional symmetry algebra is generated by the differential invariants
\[ \frac{J_1 \rho^{\xi_1} e^{\varsigma_2}}{J_1^3}, \frac{J_2}{\rho^2 J_1}, \frac{J_3}{\rho J_1}, \frac{J_4}{\rho^3 J_1}, \frac{J_5}{\rho^2 J_1}, \frac{J_6}{J_1}, \frac{J_7}{\rho^3 J_1^2}, \frac{J_8}{\rho^2 J_1^2}, \frac{J_9}{J_1^2}, \frac{J_{10}}{\rho J_1^2} \]
of the first order and by the invariant derivations
\[ \rho^{\xi_1} e^{\varsigma_2} \nabla_1, \rho^{\xi_1 - 2\xi_2} e^{\varsigma_2} \nabla_2, \rho^{\xi_1 - 1} e^{\varsigma_2} \nabla_3, \]
where
\[ \varsigma_1 = \frac{\eta_4 \mu_1 - \eta_1 \mu_4}{2(\eta_1 \mu_3 - \eta_3 \mu_1) + \eta_4 \mu_1 - \eta_1 \mu_4}, \quad \varsigma_2 = \frac{(\eta_4 \mu_3 - \eta_3 \mu_4)}{2(\eta_1 \mu_3 - \eta_3 \mu_1) + \eta_4 \mu_1 - \eta_1 \mu_4}. \]

This field separates regular orbits.

### 4.2 3D-flows

Consider the Navier–Stokes system [14] in a space \( M = \mathbb{R}^3 \) equipped with the coordinates \((x, y, z)\) and the standard metric \( g = dx^2 + dy^2 + dz^2 \).

The velocity field of the flow has the form \( \mathbf{u} = u(t, x, y, z) \partial_x + v(t, x, y, z) \partial_y + w(t, x, y, z) \partial_z \), the pressure \( p \), the density \( \rho \), the temperature \( T \) and the entropy \( s \) are the functions of time and space with the coordinates \((t, x, y, z)\).

The vector of gravitational acceleration is of the form \( \mathbf{g} = g \partial_z \).
4.2.1 Symmetry Lie algebra

First of all we consider the Lie algebra \( \mathfrak{g} \) generated by the following vector fields on the manifold \( J^0 \pi \)

\[
X_1 = \partial_x, \quad X_4 = -y \partial_x + x \partial_y - v \partial_u + u \partial_v,
\]

\[
X_2 = \partial_y, \quad X_5 = \left(\frac{gt^2}{2} - z\right) \partial_x + x \partial_z + (gt - w) \partial_u + u \partial_w,
\]

\[
X_3 = \partial_z, \quad X_6 = \left(\frac{gt^2}{2} - z\right) \partial_y + y \partial_z + (gt - w) \partial_v + v \partial_w,
\]

\[
X_7 = t \partial_x + \partial_u, \quad X_{10} = \partial_t,
\]

\[
X_8 = t \partial_y + \partial_v, \quad X_{11} = \partial_s,
\]

\[
X_9 = t \partial_z + \partial_w, \quad X_{12} = \partial_p,
\]

\[
X_{13} = x \partial_x + y \partial_y - \left(\frac{gt^2}{2} - z\right) \partial_z + u \partial_u + v \partial_v - (gt - w) \partial_w - 2\rho \partial_\rho + 2T \partial_T,
\]

\[
X_{14} = t \partial_t + gt^2 \partial_z - u \partial_u - v \partial_v + (2gt - w) \partial_w + \rho \partial_\rho - p \partial_p - 2T \partial_T
\]

and the Lie algebra \( \mathfrak{h} \) generated by the vector fields

\[
Y_1 = \partial_s, \quad Y_2 = \partial_p, \quad Y_3 = \rho \partial_\rho - T \partial_T, \quad Y_4 = p \partial_p + T \partial_T.
\]

The pure geometric part is represented by the algebra \( \mathfrak{g}_m = \langle X_1, X_2, \ldots, X_{10} \rangle \) with respect to the group of motions, Galilean transformations and time shifts.

In order to describe the pure thermodynamic part, we consider the Lie subalgebra \( \mathfrak{h}_t \) of the algebra \( \mathfrak{h} \) that preserves the thermodynamic state \( \mathfrak{h}_t \).

**Theorem 31** \([15]\) A Lie algebra \( \mathfrak{g}_{sym} \) of symmetries of the Navier–Stokes system of differential equations in 3-dimensional space coincides with

\[ \vartheta^{-1}(\mathfrak{h}_t). \]

4.2.2 Symmetry classification of states

The Lie algebra generated by the vector fields \( Y_1, \ldots, Y_4 \) coincides with the Lie algebra of the thermodynamic symmetries of the Navier–Stokes system on a plane.

Thus the classification of the thermodynamic states or Lagrangian surfaces \( \tilde{L} \) depending on the dimension of the symmetry algebra \( \mathfrak{h}_t \subset \mathfrak{h} \) is the same as the classification presented in the previous section (2D-flows).

4.2.3 Differential invariants

**The field of kinematic invariants**

First of all, we observe that the functions \( \rho \) and \( s \) (as well as \( p \) and \( T \)) generate all \( \mathfrak{g}_m \)-invariants of order zero.

Let fix the 0 point with coordinates \((0, \ldots, 0) \in J^0 \pi\) and consider the isotropy group of this point. It is easy to check that this group is isomorphic to the rotation group \( SO(3) \).
Then consider the following elements
\[ a_g = \begin{pmatrix} \frac{u_t}{v_t} & \frac{v_t}{w_t - g} \end{pmatrix}, \quad \nabla \rho = \begin{pmatrix} \rho_x \\ \rho_y \\ \rho_z \end{pmatrix}, \quad \nabla s = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}, \quad V = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \]
and suppose that first three vectors are linearly independent.

Note that, the group \( SO(3) \) acts on the matrix \( V \) by conjugacy: \( V \rightarrow RVR^{-1} \), where \( R \in SO(3) \).

Moreover, the action of the rotation group \( SO(3) \) preserves the dot products of the vectors \( a_g, \nabla \rho \) and \( \nabla s \).

Let \( H = (a_g, \nabla \rho, \nabla s) \) be a matrix with \( \det H \neq 0 \), then the elements of the product \( H^{-1}VH \) are 9 functions, which are invariant under the action of the rotation group.

Therefore, we have 15 independent invariants of the first order at the point \( (0, \ldots, 0) \).

Denote by \( \tau \) the following transformation:
\[
\begin{align*}
    t & \rightarrow t - t_0, & x & \rightarrow x - x_0 - u_0(t - t_0), & u & \rightarrow u - u_0, \\
    \rho & \rightarrow \rho, & y & \rightarrow y - y_0 - v_0(t - t_0), & v & \rightarrow v - v_0, \\
    s & \rightarrow s, & z & \rightarrow z - z_0 - w_0(t - t_0), & w & \rightarrow w - w_0.
\end{align*}
\]

Obviously, \( \tau \) is a symmetry of the equation \( \mathcal{E} \), which maps the point \( (t_0, x_0, y_0, z_0, u_0, v_0, w_0, \rho_0, s_0) \) to the point \( 0 \).

Applying the prolongation of \( \tau \) to the invariants (to the dot products and the elements of the matrix \( H^{-1}VH \)) we get 15 kinematic invariants of the first order.

The proofs of the following two theorems can be found in [15].

**Theorem 32** [15] The field of the first order kinematic invariants is generated by the invariants \( \rho, s \) and by the invariants
\[
\begin{align*}
    & s_1 + s_x u + s_y v + s_z w, \\
    & (\nabla \rho)^2, \quad (\nabla s)^2, \quad \nabla \rho \cdot \nabla s, \quad (a_g)^2, \quad \nabla \rho \cdot a_g, \quad \nabla s \cdot a_g, \quad (H^{-1}VH)_{ij},
\end{align*}
\]
transformed by \( \tau \), if \( \det H \neq 0 \). These invariants separate regular \( \mathfrak{g}_m \)-orbits.

**Theorem 33** [15] The following derivations
\[
\begin{align*}
    \nabla_1 &= \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz}, \\
    \nabla_2 &= \rho_x \frac{d}{dx} + \rho_y \frac{d}{dy} + \rho_z \frac{d}{dz}, \\
    \nabla_3 &= s_x \frac{d}{dx} + s_y \frac{d}{dy} + s_z \frac{d}{dz}, \\
    \nabla_4 &= (\rho_y s_z - \rho_z s_y) \frac{d}{dx} - (\rho_x s_z - \rho_z s_x) \frac{d}{dy} + (\rho_z s_y - \rho_y s_x) \frac{d}{dz}
\end{align*}
\]
are \( \mathfrak{g}_m \)-invariant. They are linearly independent if
\[
\begin{vmatrix}
    \rho_x & \rho_y & \rho_z \\
    s_x & s_y & s_z \\
    \rho_y s_z - \rho_x s_y & \rho_z s_z - \rho_x s_z & \rho_x s_y - \rho_y s_x
\end{vmatrix} \neq 0.
\]
The bundle $\pi_{2,1} : \mathcal{E}_2 \to \mathcal{E}_1$ has rank 42 and by applying the derivations $\nabla_i$, $i = 1, \ldots, 4$ to the kinematic invariants we get 64 kinematic invariants. Straightforward computations show that among these invariants 42 are always independent (see http://d-omega.org).

Therefore, starting with the order $k = 1$ dimensions of regular orbits are equal to $\dim g_m = 10$.

The Hilbert function of the $g_m$-invariants field (the number of independent invariants of pure order $k$) is equal to

$$H(k) = \frac{9}{2}k^2 + \frac{19}{2}k + 5$$

for $k \geq 2$, and $H(0) = 2, H(1) = 16$.

The corresponding Poincaré function has the form

$$P(z) = \frac{2 + 10z - 6z^3 + 3z^4}{(1 - z)^3}.$$

Summarizing, we get the following result.

**Theorem 34** [15] The field of kinematic invariants is generated by the invariants $\rho, s$ of order zero, by the invariants of order one (with transformation $\tau$) and by the invariant derivations $\nabla_i, i = 1, \ldots, 4$. This field separates the regular orbits.

**The field of Navier–Stokes invariants**

Consider the case when the equations of thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$A = \xi_1 X_{11} + \xi_2 X_{12} + \xi_3 X_{13} + \xi_4 X_{14}.$$ 

For general $\xi$’s we have the following result. The particular cases are considered in [15].

**Theorem 35** [15] The field of the Navier–Stokes differential invariants for the thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

$$\frac{\xi_1}{2\xi_3 - \xi_4} \ln \rho + s, \quad \rho \xi_2 \nabla_1 s, \quad \frac{(\nabla \rho)^2}{\rho \nabla_1 s}, \quad \frac{(\nabla s)^2}{\rho \nabla_1 s}, \quad \frac{\nabla \rho \cdot \nabla s}{\rho \nabla_1 s},$$

$$\rho (a_g)^2, \quad \nabla \rho \cdot a_g, \quad \nabla s \cdot a_g, \quad \frac{J_{11}}{\rho^2}, \quad \frac{J_{12}}{\rho}, \quad \frac{J_{13}}{\rho^2}, \quad \frac{J_{21}}{\rho}, \quad \frac{J_{22}}{\rho^2}, \quad \frac{J_{23}}{\rho}, \quad \frac{J_{24}}{\rho^2}, \quad \frac{J_{31}}{\rho}, \quad \frac{J_{32}}{\rho^2}, \quad \frac{J_{33}}{\rho}, \quad \frac{J_{34}}{\rho^2}, \quad \frac{J_{41}}{\rho}, \quad \frac{J_{42}}{\rho^2}, \quad \frac{J_{43}}{\rho}, \quad \frac{J_{44}}{\rho^2},$$

of the first order and by the invariant derivations

$$\rho \xi_1 \nabla_1, \quad \rho \frac{\xi_1^4 - 4\xi_4}{\xi_4^4 - 2\xi_3} \nabla_2, \quad \rho \frac{2\xi_3}{\xi_4^4 - 2\xi_3} \nabla_3, \quad \rho \frac{\xi_4^4 - 5\xi_3}{\xi_4^4 - 2\xi_3} \nabla_4,$$

here we denote by $J_{ij}$ the elements of the matrix $H^{-1}VH$. This field separates the regular orbits.
Consider the case when the thermodynamic state admits a commutative two-dimensional symmetry algebra generated by the vector fields $A = \sum_{i=1}^{6} \mu_i X_{i+10}, B = \sum_{i=1}^{6} \eta_i X_{i+10}$.

**Theorem 36** \([15]\) The field of the Navier–Stokes differential invariants for the thermodynamic states admitting a commutative two-dimensional symmetry algebra is generated by the differential invariants

\[
\begin{align*}
\rho e^{2s} \nabla &_{1s}, \\
\frac{\nabla^{2}}{\rho^{2}} & \nabla_{1s}, \\
\frac{\nabla^{2}}{\rho} & \nabla_{1s}, \\
\frac{\nabla^{2}}{\rho^{2}} & \nabla_{1s}, \\
\frac{\rho(a)}{\rho} & \nabla_{1s}, \\
\frac{\rho(a)}{\nabla^{2}} & \nabla_{1s}, \\
\frac{\rho(a)}{\rho} & \nabla_{1s}, \\
\frac{\rho(a)}{\rho} & \nabla_{1s}, \\
\rho J_{21} & \nabla_{1s}, \\
\rho J_{22} & \nabla_{1s}, \\
\rho J_{23} & \nabla_{1s}, \\
\rho J_{31} & \nabla_{1s}, \\
\rho J_{32} & \nabla_{1s}, \\
\rho J_{33} & \nabla_{1s},
\end{align*}
\]

of the first order and by the invariant derivations

\[\rho e^{2s} \nabla_{1}, \quad \rho e^{2s} \nabla_{2}, \quad \rho e^{2s} \nabla_{3}, \quad \rho e^{2s} \nabla_{4}\]

where $J_{ij}$ are the elements of the matrix $H^{-1}VH$ and $\varsigma_1 = \frac{\eta_4 \mu_1 - \eta_1 \mu_4}{2(\eta_1 \mu_3 - \eta_3 \mu_1) + \eta_4 \mu_1 - \eta_1 \mu_4}$, $\varsigma_2 = \frac{2(\eta_4 \mu_3 - \eta_3 \mu_4)}{2(\eta_1 \mu_3 - \eta_3 \mu_1) + \eta_4 \mu_1 - \eta_1 \mu_4}$.

This field separates the regular orbits.

### 4.3 Flows on a sphere

Consider Navier–Stokes system \([1]\) on a two-dimensional unit sphere $M = S^2$ with the metric $g = \sin^2 y \, dx^2 + dy^2$ in the spherical coordinates.

The velocity field of the flow has the form $u = u(t, x, y) \partial_x + v(t, x, y) \partial_y$, the pressure $p$, the density $\rho$, the temperature $T$ and the entropy $s$ are the functions of time and space with the coordinates $(t, x, y)$.

Here we consider the flow without any external force field, so $g = 0$.

#### 4.3.1 Symmetry Lie algebra

To describe the Lie algebra of symmetries of the Navier–Stokes system we consider the Lie algebra $\mathfrak{g}$ generated by the following vector fields on the manifold $J^0 \pi$:

\[
\begin{align*}
X_1 &= \partial_t, \\
X_2 &= \partial_x, \\
X_3 &= \frac{\cos x}{\tan y} \partial_x + \sin x \partial_y - \left( \frac{\sin x}{\tan y} u + \frac{\cos x}{\sin^2 y} v \right) \partial_u + u \cos x \partial_v, \\
X_4 &= \frac{\sin x}{\tan y} \partial_x - \cos x \partial_y + \left( \frac{\cos x}{\tan y} u - \frac{\sin x}{\sin^2 y} v \right) \partial_u + u \sin x \partial_v, \\
X_5 &= \partial_s, \\
X_6 &= \partial_p, \\
X_7 &= t \partial_t - u \partial_u - v \partial_v - p \partial_p + \rho \partial_p - 2T \partial_T,
\end{align*}
\]

and denote by $\mathfrak{h}$ the Lie algebra generated by the vector fields

\[Y_1 = \partial_s, \quad Y_2 = \partial_p, \quad Y_3 = p \partial_p - \rho \partial_p + 2T \partial_T.\]
Transformations corresponding to elements of the algebra $\mathfrak{g}_m = \langle X_1, X_2, X_3, X_4 \rangle$ (the pure geometric part) are generated by sphere motions and time shifts: $\mathfrak{g}_m = \mathfrak{so}(3, \mathbb{R}) \oplus \mathbb{R}$.

For describing the pure thermodynamic part we consider the Lie subalgebra $\mathfrak{h}_t$ of algebra $\mathfrak{h}$ that preserves the thermodynamic state (1).

**Theorem 37** [16] The Lie algebra $\mathfrak{g}_{sym}$ of point symmetries of the Navier–Stokes system of differential equations on a sphere coincides with $\vartheta^{-1}(\mathfrak{h}_t)$.

### 4.3.2 Symmetry classification of states

Here we consider the thermodynamic states or Lagrangian surfaces $\tilde{L}$ (compare with the plane case) with a one-dimensional symmetry algebra $\mathfrak{h}_t \subset \mathfrak{h}$.

Cases when the thermodynamic states admit two or three-dimensional symmetry algebra are not interesting or have no physical meaning.

Let $\dim \mathfrak{h}_t = 1$ and let $Z = \sum_{i=1}^{3} \lambda_i Y_i$ be a basis vector in this algebra.

Then the thermodynamic state or the surface $\tilde{L}$ is the solution of the PDE system

\[
\begin{align*}
\lambda_3 \rho \varepsilon_{\rho\rho} - \lambda_1 \varepsilon_{\rho s} + \frac{\lambda_2}{\rho^2} &= 0, \\
\lambda_1 \varepsilon_{ss} - \lambda_3 \rho \varepsilon_{\rho s} - 2 \lambda_3 \varepsilon_s &= 0,
\end{align*}
\]

that is formally integrable and compatible.

Solving this system for general parameters $\lambda$ (some special cases can be found in [16]) we find expressions for the pressure and the temperature. Adding the admissibility conditions for this case we get the following result.

**Theorem 38** The thermodynamic states admitting a one-dimensional symmetry algebra have the form

\[
p = \frac{1}{\rho} \left( \frac{\lambda_1}{\lambda_3} F' - 2F \right) - \frac{\lambda_2}{\lambda_3}, \quad T = \frac{F'}{\rho^2}, \quad F = F \left( s + \frac{\lambda_1}{\lambda_3} \ln \rho \right),
\]

where $F$ is an arbitrary function and

\[
F' > 0, \quad \left( \frac{\lambda_1}{\lambda_3} \right)^2 F'' - 3 \frac{\lambda_1}{\lambda_3} F' + 2F > 0, \quad F'' \left( \frac{\lambda_1}{\lambda_3} F' + 2F \right) - 4(F')^2 > 0.
\]

### 4.3.3 Differential invariants

The field of kinematic invariants

First of all, the pressure $\rho$, the entropy $s$ and $g(u, u)$ (as well as $p$ and $T$) generate all $\mathfrak{g}_m$-invariants of order zero.

Consider two vector fields $u$ and $\tilde{u}$ such that $g(u, \tilde{u}) = 0$ and $g(u, u) = g(\tilde{u}, \tilde{u})$. Writing the acceleration vector with respect to the vectors $u$ and $\tilde{u}$ we obtain two invariants of the first order. Further writing the operator $d\varphi u$ with respect to these vectors as the
sum of its symmetric and antisymmetric parts we obtain another four invariants of the first order. Thus we get six invariants:

\[ J_1 = (uv_t - vu_t) \sin y, \quad J_2 = uu_t \sin^2 y + vv_t, \]
\[ J_3 = u_x + v_y + v \cot y, \quad J_4 = u_y \sin y - \frac{v_x}{\sin y} + 2u \cos y, \]
\[ J_5 = (u(u_xv - v_xu) + v(u_yv - v_yu)) \sin y + u \cos y(u^2 \sin^2 y + 2v^2), \]
\[ J_6 = v(u_xv - v_xu) - u(u_yv - v_yu) \sin^2 y + v^3 \cot y. \]

The proofs of the following theorems can be found in \[16\].

**Theorem 39** \[16\] The following derivations

\[ \nabla_1 = \frac{d}{dt}, \quad \nabla_2 = \frac{\rho_y}{\sin^2 y} \frac{d}{dx} + \frac{\rho_y}{\sin y} \frac{d}{dy}, \quad \nabla_3 = \frac{s_x}{\sin^2 y} \frac{d}{dx} + \frac{s_y}{\sin y} \frac{d}{dy} \]

are \(g_m\)-invariant. They are linearly independent if

\[ \rho_x s_y - \rho_y s_x \neq 0. \]

**Theorem 40** \[16\] The field of the first order kinematic invariants is generated by the invariants \(\rho, s, g(u, u)\), \[18\] and

\[ \nabla_1 \rho, \quad \nabla_1 s, \quad \nabla_2 \rho, \quad \nabla_2 s, \quad \nabla_3 s. \]

These invariants separate regular \(g_m\)-orbits.

The bundle \(\pi_{2,1} : \mathcal{E}_2 \to \mathcal{E}_1\) has rank 18, and by applying derivations \(\nabla_1, \nabla_2, \nabla_3\) to the kinematic invariants \[18\] and \[19\] we get 33 kinematic invariants. Straightforward computations show that among these invariants 18 are independent (see [http://d-omega.org](http://d-omega.org)).

Therefore, starting with the order \(k = 1\) dimensions of regular orbits are equal to \(\dim g_m = 4\).

Moreover, the number of independent invariants of pure order \(k\) (the Hilbert function) is equal \(H(k) = 7k + 4\) for \(k \geq 1\), and \(H(0) = 3\).

The corresponding Poincaré function is

\[ P(z) = \frac{3 + 5z - z^2}{(1 - z)^2}. \]

**Theorem 41** \[16\] The field of the kinematic invariants is generated by the invariants \(\rho, s, g(u, u)\) of order zero, by the invariants \[18\] and \[19\] of order one and by the invariant derivations \(\nabla_1, \nabla_2, \nabla_3\). This field separates regular orbits.

**The field of Navier–Stokes invariants**

Now, we find differential invariants of the Navier–Stokes system in the case when the thermodynamic state \(\tilde{L}\) admits a one-dimensional symmetry algebra generated by the vector field

\[ A = \xi_1 X_5 + \xi_2 X_6 + \xi_3 X_7. \]
Theorem 42 [16] The field of the Navier–Stokes differential invariants for thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

\[ s - \frac{\xi_1}{\xi_3} \ln \rho, \quad \rho^2 g(u, u), \quad \rho^3 J_1, \quad \rho^3 J_2, \quad \rho J_3, \quad \rho J_4, \]

\[ \rho^3 J_5, \quad \rho^3 J_6, \quad \nabla_1 \rho, \quad \rho \nabla_1 s, \quad \frac{\nabla_2 \rho}{\rho^2}, \quad \frac{\nabla_2 s}{\rho}, \quad \nabla_3 s \]

of the first order and by the invariant derivations

\[ \rho \nabla_1, \quad \rho^{-1} \nabla_2, \quad \nabla_3. \]

This field separates regular orbits.

This theorem is valid for general \( \xi \)'s. For special cases see [16].

4.4 Flows on a spherical layer

Consider the Navier–Stokes system (14) on a spherical layer \( M = S^2 \times \mathbb{R} \) with the coordinates \((x, y, z)\) and the metric

\[ g = \frac{4}{(x^2 + y^2 + 1)^2} (dx^2 + dy^2) + dz^2. \]

The velocity field of the flow has the form \( u = u(t, x, y, z) \partial_x + v(t, x, y, z) \partial_y + w(t, x, y, z) \partial_z \), the pressure \( p \), the density \( \rho \), the temperature \( T \) and the entropy \( s \) are the functions of time and space with the coordinates \((t, x, y, z)\).

The vector of gravitational acceleration is of the form \( g = g \partial_z \).

4.4.1 Symmetry Lie algebra

Consider the Lie algebra \( \mathfrak{g} \) generated by the following vector fields on the manifold \( J^0 \pi \):

\[ X_1 = \partial_t, \quad X_2 = \partial_z, \quad X_3 = t \partial_z + \partial_w, \]
\[ X_4 = y \partial_x - x \partial_y + v \partial_u - u \partial_v, \]
\[ X_5 = xy \partial_x - \frac{1}{2} (x^2 - y^2 - 1) \partial_y + (xy + yu) \partial_u - (xu - yv) \partial_v, \]
\[ X_6 = \frac{1}{2} (x^2 - y^2 + 1) \partial_x + xy \partial_y + (xu - yv) \partial_u + (xv + yu) \partial_v, \]
\[ X_7 = \partial_s, \quad X_8 = \partial_p, \]
\[ X_9 = t \partial_t + gt^2 \partial_z - u \partial_u - v \partial_v + (2gt - w) \partial_u - p \partial_p + \rho \partial_p - 2T \partial_T, \]

and denote by \( \mathfrak{h} \) the Lie algebra generated by the vector fields

\[ Y_1 = \partial_s, \quad Y_2 = \partial_p, \quad Y_3 = p \partial_p - \rho \partial_p + 2T \partial_T. \]

Transformations corresponding to the elements of the algebra \( \mathfrak{g}_m = \langle X_1, \ldots, X_9 \rangle \) (the pure geometric part) are compositions of sphere motions, Galilean transformations along the \( z \) direction and time shifts.

Let also \( \mathfrak{h}_1 \) be the Lie subalgebra of algebra \( \mathfrak{h} \) that preserves the thermodynamic state \( \Pi \).

Theorem 43 [17] The Lie algebra \( \mathfrak{g}_{sym} \) of point symmetries of the Navier–Stokes system of differential equations on a spherical layer coincides with

\[ \vartheta^{-1}(\mathfrak{h}_1). \]
4.4.2 Symmetry classification of states

The Lie algebra generated by the vector fields $Y_1, Y_2, Y_3$ coincides with the Lie algebra of thermodynamic symmetries of the Navier–Stokes system on a sphere.

Thus the classification of thermodynamic states or Lagrangian surfaces $\tilde{L}$ depending on the dimension of the symmetry algebra $\mathfrak{h}_t \subset \mathfrak{h}$ is the same as the classification presented in the previous section.

4.4.3 Differential invariants

The field of kinematic invariants

First of all, the following functions $\rho, s, g(u, u) - w^2$ (as well as $p$ and $T$) generate all $\mathfrak{g}_m$-invariants of order zero.

The proofs of the following theorems can be found in [17].

**Theorem 44** [17] The following derivations

$$\nabla_1 = \frac{d}{dz}, \quad \nabla_2 = \frac{d}{dt} + w \frac{d}{dz}, \quad \nabla_3 = u \frac{d}{dx} + v \frac{d}{dy}, \quad \nabla_4 = v \frac{d}{dx} - u \frac{d}{dy}$$

are $\mathfrak{g}_m$-invariant. They are linearly independent if $u^2 + v^2 \neq 0$.

**Theorem 45** [17] The field of the first order kinematic invariants is generated by the invariants $\rho, s, g(u, u) - w^2$ of order zero and by the invariants

$$J_1 = u_x w_x + v_x w_y, \quad J_2 = (u_x - v_y)^2 + (u_y + v_x)^2, \quad J_3 = \frac{u_t v_x - u_x v_t}{(x^2 + y^2 + 1)^2}$$

of order one, where $i = 1, \ldots, 4$. These invariants separate regular $\mathfrak{g}_m$-orbits.

The bundle $\pi_{2,1} : \mathcal{E}_2 \to \mathcal{E}_1$ has rank 42, and by applying derivations $\nabla_i, i = 1, \ldots, 4$ to the kinematic invariants (20) we get 88 kinematic invariants. Straightforward computations show that among these invariants 42 are independent (see http://d-omega.org).

Therefore, starting with the order $k = 1$ dimensions of regular orbits are equal to $\dim \mathfrak{g}_m = 6$.

The number of independent invariants of pure order $k$ (the Hilbert function) is equal to

$$H(k) = 5 + \frac{19}{2} k + \frac{9}{2} k^2$$

for $k \geq 1$, and $H(0) = 3$.

The corresponding Poincaré function has the form

$$P(z) = \frac{3 + 10z - 6z^2 + 2z^3}{(1 - z)^3}.$$

**Theorem 46** [17] The field of the kinematic invariants is generated by the invariants $\rho, s, g(u, u) - w^2$ of order zero, by the invariants (20) of order one and by the invariant derivations $\nabla_i, i = 1, \ldots, 4$. This field separates regular orbits.
4.5 The field of Navier–Stokes invariants

Consider the case when the thermodynamic state $\tilde{L}$ admits a one-dimensional symmetry algebra generated by the vector field

$$A = \xi_1 X_7 + \xi_2 X_8 + \xi_3 X_9.$$ 

We do not consider cases of a two- or three-dimensional symmetry algebra because they are not interesting from the physical point of view.

For general $\xi$’s we have the following theorem. For the case $\xi_3 = 0$ we have basic invariants $\rho, g(u, u) - w^2$, (20) and invariant derivatives $\nabla_i, i = 1, \ldots, 4$.

**Theorem 47** [17] The field of the Navier–Stokes differential invariants for thermodynamic states admitting a one-dimensional symmetry algebra is generated by the differential invariants

$$s - \frac{\xi_1}{\xi_3} \ln \rho, \quad \rho^2 (g(u, u) - w^2), \quad \frac{\nabla_1 \rho}{\rho}, \quad \nabla_j \rho, \quad \nabla_1 s, \quad \rho \nabla_j s,$$

$$\rho^2 \nabla_1 (g(u, u) - w^2), \quad \rho^3 \nabla_j (g(u, u) - w^2),$$

$$\rho \nabla_1 w, \quad \rho^2 (\nabla_2 w - g), \quad \rho^2 \nabla_3 w, \quad \rho^2 \nabla_4 w,$$

$$\rho^2 J_1, \quad \rho^2 J_2, \quad \rho^3 J_3$$

of the first order, here $j = 2, 3, 4$, and by the invariant derivations

$$\nabla_1, \quad \rho \nabla_2, \quad \rho \nabla_3, \quad \rho \nabla_4.$$

This field separates regular orbits.

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