Sensitivity of string compressors and repetitiveness measures

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Abstract

The sensitivity of a string compression algorithm C asks how much the output size C(T) for an input string T can increase when a single character edit operation is performed on T. This notion enables one to measure the robustness of compression algorithms in terms of errors and/or dynamic changes occurring in the input string. In this paper, we analyze the worst-case multiplicative sensitivity of string compression algorithms, which is defined by \(\max_{T \in \Sigma^n} \{C(T')/C(T) : \text{ed}(T,T') = 1\}\), where \(\text{ed}(T,T')\) denotes the edit distance between T and T'. In particular, for the most common versions of the Lempel-Ziv 77 compressors, we prove that the worst-case multiplicative sensitivity is only a small constant (2 or 3, depending on the version of the Lempel-Ziv 77 and the edit operation type), i.e., the size of the Lempel-Ziv 77 factorizations can be larger by only a small constant factor. We strengthen our upper bound results by presenting matching lower bounds on the worst-case sensitivity for all these major versions of the Lempel-Ziv 77 factorizations. We generalize these results to the smallest bidirectional scheme \(b\). In addition, we show that the sensitivity of a grammar-based compressor called GCIS (Grammar Compression by Induced Sorting) is also a small constant. Further, we extend the notion of the worst-case sensitivity to string repetitiveness measures such as the smallest string attractor size \(\gamma\) and the substring complexity \(\delta\), and show that the worst-case sensitivity of \(\delta\) is also a small constant. These results contrast with the previously known related results such that the size \(z_{78}\) of the Lempel-Ziv 78 factorization can increase by a factor of \(\Omega(n^{1/4})\) [Lagarde and Perifel, 2018], and the number \(r\) of runs in the Burrows-Wheeler transform can increase by a factor of \(\Omega(\log n)\) [Giuliani et al., 2021] when a character is prepended to an input string of length \(n\). By applying our sensitivity bounds of \(\delta\) or the smallest grammar to known results (c.f. [Navarro, 2021]), some non-trivial upper bounds for the sensitivities of important string compressors and repetitiveness measures including \(\gamma\), \(r\), LZ-End, RePair, LongestMatch, and AVL-grammar, are derived. We also exhibit the worst-case additive sensitivity \(\max_{T \in \Sigma^n} \{C(T') - C(T) : \text{ed}(T,T') = 1\}\), which allows one to observe more details in the changes of the output sizes.

keywords: lossless data compression, Lempel-Ziv factorizations, run-length BWT, bidirectional scheme, string attractors, substring complexity, grammar compression, edit operations, sensitivity

1 Introduction

In this paper we introduce a new notion to quantify efficiency of (lossless) compression algorithms, which we call the sensitivity of compressors. Let \(C\) be a compression algorithm and let \(C(T)\)
denote the size of the output of $C$ applied to an input text (string) $T$. Roughly speaking, the sensitivity of $C$ measures how much the compressed size $C(T)$ can change when a single-character-wise edit operation is performed at an arbitrary position in $T$. Namely, the worst-case multiplicative sensitivity of $C$ is defined by

$$\max_{T \in \Sigma^n} \{C(T')/C(T) : \text{ed}(T, T') = 1\},$$

where $\text{ed}(T, T')$ denotes the edit distance between $T$ and $T'$. This new and natural notion enables one to measure the robustness of compression algorithms in terms of errors and/or dynamic changes occurring in the input string. Such errors and dynamic changes are commonly seen in real-world texts such as DNA sequences and versioned documents.

The so-called highly repetitive sequences, which are strings containing a lot of repeated fragments, are abundant today: Semi-automatically generated strings via M2M communications, and collections of individual genomes of the same/close species are typical examples. By intuition, such highly repetitive sequences should be highly compressible, however, statistical compressors are known to fail to capture repetitiveness in a string [37]. Therefore, other types of compressors, such as dictionary-based, grammar-based, and/or lex-based compressors are often used to compress highly repetitive sequences [41, 63, 38, 24, 48].

Let us recall two examples of well-known compressors: The run-length Burrows-Wheeler Transform (RLBWT) is one kind of compressor that is based on the lexicographically sorted rotations of the input string. The number $r$ of equal-character runs in the BWT of a string is known to be very small in practice: Indeed, BWT is used in the bzip2 compression format, and several compressed data structures which support efficient queries have been proposed [16, 3, 55, 56]. The Lempel-Ziv 78 compression (LZ78) [69] is one of the most fundamental dictionary based compressors that is a core of in the gif and tiff compression formats. While LZ78 only allows $\Omega(\sqrt{n})$ compression for any string of length $n$, its simple structure allows for designing efficient compressed pattern matching algorithms and compressed self-indices (c.f. [32] and references therein).

The recent work by Giuliani et al. [22], however, shows that the number $r$ of runs in the BWT of a string of length $n$ can grow by a multiplicative factor of $\Omega(\log n)$ when a single character is prepended to the input string. It is noteworthy that the family of strings discovered by Giuliani et al. [22] satisfies $r(T) = O(1)$ and $r(T') = \Omega(\log n)$, where $r(T)$ and $r(T')$ respectively denote the number of runs in the BWTs of $T$ and $T'$. The other work by Lagarde and Perifel [40] shows that the size of the dictionary of LZ78, which is equal to the number of factors in the respective LZ78 factorization, can grow by a multiplicative factor of $\Omega(n^{1/4})$, again when a single character is prepended to the input string. Letting the LZ78 dictionary size be $z_{78}$, this multiplicative increase can also be described as $\Omega(z_{78}^{3/2})$. Lagarde and Perifel call the aforementioned phenomenon on LZ78 as “one-bit catastrophe”. Based on these known results, here we introduce the three following classes of string compressors depending on their sensitivity.

(A) Those whose sensitivity is $O(1)$;

(B) Those whose sensitivity is $\text{polylog}(n)$;

(C) Those whose sensitivity is proportional to $n^c$ with some constant $0 < c \leq 1$.

By generalizing the work of Lagarde and Perifel [40], we say that Class (C) is catastrophic in terms of the sensitivity. Class (B) may not be catastrophic but the change in the compression size can

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1It is well known that if the string ends with a unique end-marker $\$, then the number $r$ of runs in the BWT increases additively by at most 2 after a character is prepended to the string. The work by Giuliani et al. [22], however, shows that this is not the case without $\$. 

2
still be quite large just for a mere single character edit operation to the input string. Class (A) is
the most robust against one-character edit operations among the three classes. Recall that LZ78
belongs to Class (C), while it is not clear yet whether RLBWT r belongs to Class (B) or (C)
(note that the work of Giuliani et al. [22] showed only a lower bound $\Omega(\log n)$). In this paper,
we show that the other major dictionary compressors, the Lempel-Ziv 77 compression family, belong to
Class (A), and thus such a catastrophe never happens with this family. The LZ77 compression [68],
which is the greedy parsing of the input string $T$ where each factor of length more than one refers
to a previous occurrence to its left, is the most important dictionary-based compressor both in
theory and in practice. The LZ77 compression without self-references (resp. with self-references)
can achieve $O(\log n)$ compression (resp. $O(1)$ compression) in the best case as opposed to the
$\Omega(\sqrt{n})$ compression by the LZ78 counterpart, and the LZ77 compression is a core of common
lossless compression formats including gzip, zip, and png. In addition, its famous version called
LZSS (Lempel-Ziv-Storer-Szymanski) [64], has numerous applications in string processing, including
finding repetitions [13, 36, 23, 4], approximation of the smallest grammar-based compression [62, 11],
and compressed self-indexing [7, 8, 47, 5], just to mention a few.
We show that the multiplicative sensitivity of LZ77 with/without self-references is at most 2,
namely, the number of factors in the respective LZ77 factorization can increase by at most a factor
of 2 for all types of edit operations (substitution, insertion, deletion of a character). Then, we prove
that the multiplicative sensitivity of LZSS with/without self-references is at most 3 for substitutions
and deletions, and that it is at most 2 for insertions. We also present matching lower bounds for the
multiplicative sensitivity of LZ77/LZSS with/without self-references for all types of edit operations
as well. In addition, the multiplicative sensitivity of RLBWT r turns out to be $O(\log r \log n)$, which
implies that r belongs to Class (B) [22]. These results suggest that, LZ77 and LZSS of Class (A) may
better capture the repetitiveness of strings than RLBWT of Class (B) and LZ78 of Class (C), since
a mere single character edit operation should not much influence the repetitiveness of a sufficiently
long string. We also consider the smallest bidirectional scheme [64] that is a generalization of the
LZ family where each factor can refer to its other occurrence to its left or right. It is shown that for
all types of edit operations, the multiplicative sensitivity of the size $b$ of the smallest bidirectional
scheme is at most 2, and that there exist strings for which the multiplicative sensitivity of $b$ is 2
with insertions and substitutions, and it is 1.5 with deletions. The smallest grammar problem [11]
is a famous NP-hard problem that asks to compute a grammar of the smallest size $g^*$ that derives
only the input string. We show that the multiplicative sensitivity of the smallest grammar size $g^*$
is at most 2. Further, we extend the notion of the worst-case multiplicative sensitivity to string
repetitiveness measures such as the size $\gamma$ of the smallest string attractor [30] and the substring
complexity $\delta$ [35], both receiving recent attention [29, 60, 39, 13, 12]. We prove that the value of $\delta$
can increase by at most a factor of 2 for substitutions and insertions, and by at most a factor of 1.5
for deletions. We show these upper bounds are also tight by presenting matching lower bounds for
the sensitivity of $\delta$. We also present non-trivial upper and lower bounds for the sensitivity of $\gamma$.
As is mentioned above, the work by Lagarde and Perifel [40] considered only the case of prepend-
ing a character to the string for the multiplicative sensitivity of LZ78. We show that the same lower
bounds hold for the multiplicative sensitivity of LZ78 in the case of substitutions and deletions, and
insertions inside the string, by using a completely different instance from the one used in [40].
Studying the relations between different string repetitiveness measures/string compressor output
sizes has attracted much attention in the last two decades (for details see the survey [48]). Combining
these known relations and our new sensitivity upper bounds mentioned above gives us a kind of

\[2\] This $O(\log r \log n)$ upper bound for the sensitivity of $r$ follows from our result on the sensitivity of $\delta$ and our
Lemma [1] and from the known results between $r$ and $\delta$ [23, 35].
“sandwich” argument, which is formalized in Lemma 1. Using this lemma, some non-trivial upper bounds for the sensitivity of other measures can be driven, including the LZ-End compressor [37] and grammar-based compressors RePair [41], Longest-Match [33], Greedy [2], Sequential [66], LZ78 [69], α-balanced grammars [11], AVL-grammars [62], and Simple [26]. These upper bound results are reported as corollaries in the following sections.

Moreover, we consider the sensitivity of other compressors and repetitiveness measures including Bisection [52], GCIS [58, 59], and CDAWGs [10].

Table 1 summarizes our results on the multiplicative sensitivity of the string compressors and repetitiveness measures.

| compressor/repetitiveness measure | edit type | upper bound | lower bound |
|-----------------------------------|-----------|-------------|-------------|
| Substring Complexity δ           | ins./subst. | 2           | 2           |
|                                   | deletion   | 1.5         | 1.5         |
| Smallest String Attractor γ       | all        | $O(\log n)^\dagger$ | 2           |
| RLBWT r                           | insertion  | $O(\log n \log r)^\dagger$ | $\Omega(\log n)$ [22] |
|                                   | del./subst.| -           | -           |
| Bidirectional Scheme b            | ins./subst.| 2           | 2           |
|                                   | deletion   | 2           | 1.5         |
| LZ77 $z_{77}$                     | all        | 2           | 2           |
| LZ77sr $z_{77sr}$                 | all        | 2           | 2           |
| LZSS $z_{SS}$                     | del./subst.| 3           | 3           |
| LZSSsr $z_{SSsr}$                 | insertion  | 2           | 2           |
| LZ78 $z_{78}$                     | insertion  | $O((n/\log n)^2)^\dagger$ | $\Omega(n^{\frac{2}{3}})$ [40] |
|                                   | del./subst.| $\Omega(n^{\frac{2}{3}})$ | $\Omega(n^{\frac{2}{3}})$ |
| LZ-End $z_{End}$                  | all        | $O(\log^2(n/\delta))^\dagger$ | 2           |
| Smallest grammar $g^*$            | all        | 2           | -           |
| Repair $g_{repair}$               | all        | 2           | -           |
| Longest match $g_{long}$          | all        | $O((n/\log n)^{\frac{2}{3}})^\dagger$ | -           |
| Greedy $g_{grdy}$                 | all        | -           | -           |
| Sequential $g_{seq}$              | all        | $O((n/\log n)^{\frac{2}{3}})^\dagger$ | -           |
| α-balanced grammar $g_{\alpha}$   | all        | $O(\log(n/g^*))^\dagger$ | -           |
| AVL grammar $g_{avl}$             | all        | -           | -           |
| Simple $g_{simple}$               | substitution | 2           | 2           |
|                                   | ins./del.  | $|\Sigma| + 1$ | $|\Sigma|$ |
| GCIS $g_{gs}$                     | all        | 4           | 4           |
| CDAWG $e$                         | all        | -           | 2           |

In addition to the afore-mentioned multiplicative sensitivity, we also introduce the worst-case
additive sensitivity, which is defined by
\[
\max_{T \in \Sigma^n} \{C(T') - C(T) : \text{ed}(T, T') = 1\},
\]
for all the string compressors/repetitiveness measures \( C \) dealt in this paper. We remark that the additive sensitivity allows one to observe and evaluate more details in the changes of the output sizes, as summarized in Table 2. For instance, we obtain strictly tight upper and lower bounds for the additive sensitivity of LZ77 with and without self-references in the case of substitutions and insertions. Studying the additive sensitivities of string compressors is motivated by approximation of the Kolmogorov complexity. Let \( K(T) \) denote the Kolmogorov complexity of string \( T \), that is the length of a shortest program that produces \( T \). While \( K(T) \) is known to be uncomputable, the additive sensitivity \( K(T') - K(T) \) for deletions is at most \( O(\log n) \) bits, since it suffices to add “Delete the \( i \)th character \( T[i] \) from \( T \)” at the end of the program. Similarly, the additive sensitivity of \( K \) for insertions and substitutions is at most \( O(\log n + \log \sigma) \) bits, where \( \sigma \) is the alphabet size. Therefore, a “good approximation” of the Kolmogorov complexity \( K \) should have small additive sensitivity.

1.1 Related work

1.1.1 String monotonicity

A string repetitiveness measure \( C \) is called monotone if, for any string \( T \) of length \( n \), \( C(T') \leq C(T) \) holds with any of its prefixes \( T' = T[1..i] \) and suffixes \( T' = T[j..n] \) [35]. Kociumaka et al [35] pointed out that \( \delta \) is monotone, and posed a question whether \( \gamma \) or the size \( b \) of the smallest bidirectional macro scheme [64] are monotone. This monotonicity for \( C \) can be seen as a special and extended case of our sensitivity for deletions, namely, if we restrict \( T' \) to be the string obtained by deleting either the first or the last character from \( T \), then it is equivalent to asking whether \( \max_{T \in \Sigma^n} \{C(T')/C(T) : T' \in \{T[1..n-1], T'[2..n]\} \} \leq 1 \). Mantaci et al. [13] proved that \( \gamma \) is not monotone, by showing a family of strings \( T \) such that \( \gamma(T) = 2 \) and \( \gamma(T') = 3 \) with \( T' = T[1..n-1] \), which immediately leads to a lower bound \( 3/2 = 1.5 \) for the multiplicative sensitivity of \( \gamma \). In this paper, we present a new lower bound for the multiplicative sensitivity of \( \gamma \), which is 2. Mitsuya et al. [15] considered the monotonicity of LZ77 without self-references \( z_{77} \) presented a family of strings \( T \) for which \( z_{77}(T')/z_{77}(T) \approx 4/3 \) with \( T' = [2..n] \). Again, our matching upper and lower bounds for the multiplicative sensitivity of \( z_{77} \), which are both 2, improve this 4/3 bound.

1.1.2 Comparison to sensitivity of other algorithms

The notion of the sensitivity of (general) algorithms was first introduced by Varma and Yoshida [65]. They studied the average sensitivity of well-known graph algorithms, and presented interesting lower and upper bounds on the expected number of changes in the output of an algorithm \( A \), when a randomly chosen edge is deleted from the input graph \( G \). The worst-case sensitivity of a graph algorithm for edge-deletions and vertex-deletions was considered by Yoshida and Zhou [67].

As opposed to these existing work on the sensitivity of graph algorithms, our notion of the sensitivity of string compressors focuses on the size of their compressed outputs and does not formulate the perturbation of their structural changes. This is because the primary task of data compression is to represent the input data with as little memory as possible, and the structural changes of the compressed outputs can be of secondary importance.

We remark that most instances of \( \Sigma^n \) are not compressible, or in other words, a randomly chosen string \( T \) from \( \Sigma^n \) is not compressible. Such a string \( T \) does not become highly compressible
Table 2: Additive sensitivity of the string compressors and string repetitiveness measures studied in this paper, where \( n \) is the input string length and \( \Sigma \) is the alphabet. Some upper/lower bounds are described in terms of both the measure and \( n \). In the table “sr” stands for “with self-references”. The upper bounds marked with “†” are obtained by applying known results [30, 35, 28, 37, 31, 11, 62, 26] and our results on the sensitivity of the substring complexity \( \delta \) or the smallest grammar \( g^* \) to Lemma 1.

| compressor/repetitiveness measure | edit type       | upper bound                           | lower bound                           |
|-----------------------------------|-----------------|---------------------------------------|---------------------------------------|
| Substring Complexity \( \delta \) | all             | \( O(\delta \log n) \)†             | \( \gamma - 3 \)                      |
| Smallest String Attractor \( \gamma \) | all             | \( O(r \log n \log r) \)†             | \( \Omega(\log n) \) [22]             |
| RLBWT \( r \)                     | insertion       | \( O(r \log n \log r) \)†             | -                                     |
|                                  | del./subst.     |                                       |                                       |
| Bidirectional Scheme \( b \)      | all             | \( b + 2 \)                           | \( b/2 - 3 \)                         | \( \Omega(\sqrt{n}) \)             |
| LZ77 \( z_{77} \)                 | subst./ins.     | \( z_{77} - 1 \)                      | \( z_{77} - 1 \)                      | \( \Omega(\sqrt{n}) \)             |
|                                  | deletion        | \( z_{77} - 2 \)                      | \( z_{77} - 2 \)                      | \( \Omega(\sqrt{n}) \)             |
| LZ77sr \( z_{77} \)               | subst./ins.     | \( z_{77sr} \)                        | \( z_{77sr} \)                        | \( \Omega(\sqrt{n}) \)             |
|                                  | deletion        | \( z_{77sr} - 2 \)                    | \( z_{77sr} - 2 \)                    | \( \Omega(\sqrt{n}) \)             |
| LZSS \( z_{SS} \)                 | del./subst.     | \( z_{SS} - 2 \)                      | \( z_{SS} - \Theta(\sqrt{\log n}) \)| \( \Omega(\sqrt{n}) \)             |
|                                  | insertion       | \( z_{SS} \)                         | \( z_{SS} - \Theta(\sqrt{\log n}) \)| \( \Omega(\sqrt{n}) \)             |
| LZSSsr \( z_{SSsr} \)             | del./subst.     | \( z_{SSsr} - 2 \)                    | \( z_{SSsr} - \Theta(\sqrt{\log n})\) | \( \Omega(\sqrt{n}) \)             |
|                                  | insertion       | \( z_{SSsr} + 1 \)                    | \( z_{SSsr} - \Theta(\sqrt{\log n})\) | \( \Omega(\sqrt{n}) \)             |
| LZ78 \( z_{78} \)                 | insertion       | \( g^* \cdot (n/\log n)^{2/3} \)†   | \( \Omega((z_{78})^{2/3}) \) [40]    | \( \Omega(n/\log n) \) [40]        |
|                                  | del./subst.     |                                       |                                       |                                       |
| LZ-End \( z_{End} \)              | all             | \( O(z_{End} \log^2 (n/\delta)) \)† | \( z_{End} - \Theta(\frac{\sqrt{n}}{z_{End}}) \) | \( \Omega(\sqrt{n}) \)             |
| Smallest grammar \( g^* \)        | all             | \( g^* \)                             |                                       |                                       |
| Repair \( g_{\text{pair}} \)      | all             | \( g_{\text{pair}} \)                | \( (\log_2 n) \)                     | \( g_{\text{pair}} - 4 \)          |
| Longest match \( g_{\text{long}} \) | all             | \( O(\log^2 n) \)†                   | -                                     |
| Greedy \( g_{\text{greedy}} \)    | all             | \( g_{\text{greedy}} \)              | \( (\log_2 n) \)                     | \( g_{\text{greedy}} - 4 \)        |
| Sequential \( g_{\text{seq}} \)   | all             | \( g_{\text{seq}} \)                 | \( (\log_2 n) \)                     | \( g_{\text{seq}} - 4 \)           |
| \( \alpha \)-balanced grammar \( g_{\alpha} \) | all             | \( g_{\alpha} \)                    | \( (\log_2 n) \)                     | \( g_{\alpha} - 4 \)               |
| AVL grammar \( g_{\text{avl}} \)  | all             | \( O(g^* \log(n/g^*)) \)†             | -                                     |
| Simple \( g_{\text{simple}} \)    | substitution    | \( g_{\text{bsc}} \)                 | \( g_{\text{bsc}} \)                 | \( 2 \log_2 n - 4 \)               |
|                                  | ins./del.       | \( \Omega((\log_2 n)^{2/3}) \)       | \( g_{\text{bsc}} \)                 | \( 2 \log_2 n - 4 \)               |
| GCIS \( g_{\text{gs}} \)          | all             | \( g_{\text{gs}} \)                  | \( g_{\text{gs}} - 29 \)              | \( (3/4)n + 1 \)                   |
| CDAWG \( e \)                     | all             | -                                     | \( e \)                              | \( n \)                            |
just after a one-character edit operation, and hence \( C(T) \) and \( C(T') \) are expected to be almost the same. Therefore, considering the average sensitivity of string compressors and repetitiveness measures does not seem worth discussing, and this is the reason why we focus on the worst-case sensitivity of string compressors and repetitiveness measures.

Still, our notion permits one to evaluate the worst-case size changes of several known compressed string data structures in the dynamic setting, as will be discussed in the following subsection.

### 1.1.3 Compressed string data structures

A compressed string data structure is built on a compressed representation of the string and supports efficient queries such as pattern matching and substring extraction within compressed space. Since the string compressors and string repetitiveness measures that we deal with in this paper are models for highly repetitive strings, we mention some compressed string indexing structures for highly repetitive sequences below.

The Block tree of a string of length \( n \) uses \( O(z_{SS} \log(n/z_{SS})) \) words of space and supports random access queries in \( O(\log(n/z_{SS})) \) time. Navarro [37] proposed an LZ-based indexing structure that uses \( O(z_{SS} \log(n/z_{SS})) \) words of space and counts the number of occurrences of a query pattern in the text string in \( O(m \log^{2+\epsilon} n) \) time, where \( m \) is the length of the pattern and \( \epsilon > 0 \) is any constant. An \( O(\log n) \)-time longest common extension (LCE) data structure that takes \( O(z_{SS} \log(n/z_{SS})) \) space and is based on Recompression [26] was proposed by I [25]. Nishimoto et al. [54] presented a dynamic \( O(\min\{z_{SS} \log n \log^2 n, n\}) \)-space compressed data structure that supports pattern matching and substring insertions/deletions in \( O(m \cdot \text{polylog}(n)) \) time, where \( m \) is the length of the pattern/substring. Kociumaka et al. [35] proposed a compressed indexing structure that uses \( O(\delta \log(n/\delta)) \) words of space, performs random access in \( O(\log(n/\delta)) \) time, and finds all the \text{occ} occurrences of a given pattern of length \( m \) in \( O(m \log n + \text{occ} \log^\epsilon n) \) time. Very recently, Kociumaka et al. [54] proposed an improved data structure of \( O(\delta \log(n/\delta))-\)space that supports pattern matching queries in \( O(m + (\text{occ} + 1) \log^\epsilon n) \) time. Two independent compressed indexing structures, which are based on grammar compression called GCIS (Grammar Compression by Induced Sorting) [28] have been proposed [1][14]. Our constant upper bounds on the multiplicative sensitivity for \( z_{SS}, \delta, \) and \( g_{is} \) imply that the afore-mentioned compressed data structures retain their asymptotic space complexity even after one-character edit operation at an arbitrary position, though they may incur a certain amount of structural changes.

The r-index [16], the refined r-index [3], and the OptBWTR [55] are efficient indexing structures which are built on the RLBWT and use \( O(r) \) words of space. The result by Giuliani et al. [22], which uses a family of strings of length \( n \) with \( r = O(1) \), shows that the space complexity of these indexing structures can grow from \( O(1) \) words of space to \( O(\log n) \) words of space, after appending a character to the string. In turn, our upper bound for the sensitivity of \( r \) implies that after a one-character edit operation, the space usage of these indexing structures is bounded by \( O(r \log r \log n) \) for any string of length \( n \).

There also exist compressed data structures based on other string compressors and/or repetitiveness measures: Kempa and Prezza [30] presented an \( O(\gamma \tau \log_{\omega}(n/\gamma)) \)-space data structure that allows for extracting substrings of length-\( \ell \) in \( O(\log_{\omega}(n/\gamma) + \ell \log(\sigma)/\omega) \) time, where \( \tau \geq 2 \) is an integer parameter, \( \sigma \) is the alphabet size, and \( \omega \) is the machine-word size in the RAM model. Navarro and Prezza [50] gave a data structure of size \( O(\gamma \log(n/\gamma)) \) that supports pattern matching queries in \( O(m \log n + \text{occ} \log^\epsilon n) \) time. Christiansen et al. [12] introduced a compressed indexing structure that occupies \( O(\gamma \log(n/\gamma) \log^\epsilon n) \) space and finds all the \text{occ} pattern occurrences in optimal \( O(m + \text{occ}) \) time (for other trade-offs between the space and the query time are also reported, see [12]). Gawrychowski et al. [21] presented a data structure for maintaining a dynamic set of
strings, which is based on Recompression by Jeż [26]. Kempa and Saha [31] developed a compressed data structure that occupies \( O(z_{\text{End}}) \) space and supports random access and LCE queries in \( O(\text{polylog}(n)) \) time. A compressed indexing structure that can be built directly from the LZ77-compressed text is also known [28, 27]. For other compressed string indexing structures, see this survey [49].

\[ \text{Section 2} \]

1.2 Paper organization

Section 2 introduces necessary notations. We then present the worst-case sensitivity of string compressors and repetitiveness measures in the increasing order of their respective sizes: from \( \delta \) to \( \gamma \), LZ77 family, LZ-End, and grammars: Section 3 deals with the substring complexity \( \delta \); Section 4 deals with the smallest string attractor \( \gamma \), Section 5 deals with the RLBWT \( r \), Section 6 deals with the smallest bidirectional scheme \( b \), Section 7 deals with the LZ77 with/without self-references \( z_77 \) and \( z_{77sr} \); Section 8 deals with the LZSS with/without self-references \( z_{SS} \) and \( z_{SSr} \). Section 9 deals with the LZ-End \( z_{\text{End}} \); Section 10 deals with the LZ78 \( z_{78} \); Section 11 deals with the smallest grammar \( g^* \), and its applications to practical and/or approximation grammars RePair \( g_{\text{pair}} \), LongestMatch \( g_{\text{long}} \), Greedy \( g_{\text{greedy}} \), Sequential \( g_{\text{seq}} \), LZ78 \( z_{78} \), \( \alpha \)-balanced grammar \( g_{\alpha} \), AVL-grammar \( g_{\text{avl}} \), and Simple grammar \( g_{\text{simple}} \). Section 12 deals with the Bisection grammar \( g_{\text{bsc}} \); Section 13 deals with the CDAWG size \( e \). In Section 15, we conclude the paper and list several open questions of interest.

2 Preliminaries

2.1 Strings, factorizations, and grammars

Let \( \Sigma \) be an alphabet of size \( \sigma \). An element of \( \Sigma^* \) is called a string. For any non-negative integer \( n \), let \( \Sigma^n \) denote the set of strings of length \( n \) over \( \Sigma \). The length of a string \( T \) is denoted by \( |T| \). The empty string \( \varepsilon \) is the string of length 0, namely, \( |\varepsilon| = 0 \). The \( i \)-th character of a string \( T \) is denoted by \( T[i] \) for \( 1 \leq i \leq |T| \), and the substring of a string \( T \) that begins at position \( i \) and ends at position \( j \) is denoted by \( T[i..j] \) for \( 1 \leq i \leq j \leq |T| \). For convenience, let \( T[i..j] = \varepsilon \) if \( j < i \). Substrings \( T[1..j] \) and \( T[i..|T|] \) are respectively called a prefix and a suffix of \( T \).

A factorization of a non-empty string \( T \) is a sequence \( f_1, \ldots, f_x \) of non-empty substrings of \( T \) such that \( T = f_1 \cdots f_x \). Each \( f_i \) is called a factor. The size of the factorization is the number \( x \) of factors in the factorization.

A context-free grammar \( \mathcal{G} \) which generates only a single string \( T \) is called a grammar compression for \( T \). The size of \( \mathcal{G} \) is the total length of the right-hand sides of all the production rules in \( \mathcal{G} \). The height of \( \mathcal{G} \) is the height of the derivation tree of \( \mathcal{G} \).

2.2 Worst-case sensitivity of compressors and repetitiveness measures

For a string compression algorithm \( C \) and an input string \( T \), let \( C(T) \) denote the size of the compressed representation of \( T \) obtained by applying \( C \) to \( T \). For convenience, we use the same notation when \( C \) is a string repetitiveness measure, namely, \( C(T) \) is the value of the measure \( C \) for \( T \).

Let us consider the following edit operations on strings: character substitution (sub), character insertion (ins), and character deletion (del). For two strings \( T \) and \( S \), let \( \text{ed}(T,S) \) denote the edit distance between \( T \) and \( S \), namely, \( \text{ed}(T,S) \) is the minimum number of edit operations that transform \( T \) into \( S \).
Our interest in this paper is: “How much can the compression size or the repetitiveness measure size change when a single-character-wise edit operation is performed on a string?” To answer this question, for a given string length \( n \), we consider an arbitrarily fixed string \( T \) of length \( n \) and all strings \( T' \) that can be obtained by applying a single edit operation to \( T \), that is, \( \text{ed}(T, T') = 1 \). We define the worst-case multiplicative sensitivity of \( C \) w.r.t. a substitution, insertion, and deletion as follows:

\[
\begin{align*}
\text{MS}_{\text{sub}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T')/C(T): T' \in \Sigma^n, \text{ed}(T, T') = 1\}, \\
\text{MS}_{\text{ins}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T')/C(T): T' \in \Sigma^{n+1}, \text{ed}(T, T') = 1\}, \\
\text{MS}_{\text{del}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T')/C(T): T' \in \Sigma^{n-1}, \text{ed}(T, T') = 1\}.
\end{align*}
\]

We also consider the worst-case additive sensitivity of \( C \) w.r.t. a substitution, insertion, and deletion, as follows:

\[
\begin{align*}
\text{AS}_{\text{sub}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T') - C(T): T' \in \Sigma^n, \text{ed}(T, T') = 1\}, \\
\text{AS}_{\text{ins}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T') - C(T): T' \in \Sigma^{n+1}, \text{ed}(T, T') = 1\}, \\
\text{AS}_{\text{del}}(C, n) &= \max_{T' \in \Sigma^n} \{C(T') - C(T): T' \in \Sigma^{n-1}, \text{ed}(T, T') = 1\}.
\end{align*}
\]

We remark that, in general, \( C(T') \) can be larger than \( C(T) \) even when \( T' \) is obtained by a character deletion from \( T \) (i.e. \( |T'| = n - 1 \)). Such strings \( T \) are already known for the Lempel-Ziv 77 factorization size \( z \) when \( T' = T[2..n] \) [43], or for the smallest string attractor size \( \gamma \) when \( T' = T[1..n-1] \) [43].

The above remark implies that in general the multiplicative/additive sensitivity for insertions and deletions may not be symmetric and therefore they need to be discussed separately for some \( C \). Note, on the other hand, that the maximum difference between \( C(T') \) and \( C(T) \) when \( |T'| = n - 1 \) (deletion) and \( C(T') - C(T) < 0 \) is equivalent to \( \text{AS}_{\text{ins}}(C, n - 1) \), and symmetrically the maximum difference of \( C(T') \) and \( C(T) \) when \( |T'| = n + 1 \) (insertion) and \( C(T') - C(T) < 0 \) is equivalent to \( \text{AS}_{\text{del}}(C, n + 1) \), with the roles of \( T \) and \( T' \) exchanged. Similar arguments hold for the multiplicative sensitivity with insertions/deletions. Consequently, it suffices to consider \( \text{MS}_{\text{ins}}(C, n) \), \( \text{MS}_{\text{del}}(C, n) \), \( \text{AS}_{\text{ins}}(C, n) \), \( \text{AS}_{\text{del}}(C, n) \) for insertions/deletions.

Consider two measures \( \alpha \) and \( \beta \). An upper bound for the multiplicative sensitivity of \( \beta \) can readily be derived in the same cases, as follows:

**Lemma 1.** Let \( T \) be any string of length \( n \) and let \( T' \) be any string with \( \text{ed}(T, T') = 1 \). If the following conditions:

- \( \alpha(T')/\alpha(T) = O(1) \);
- \( \alpha(T) \leq \beta(T) \);
- \( \beta(T) = O(\alpha(T) \cdot f(n, \alpha(T))) \), where \( f \) is a function such that for any constant \( c \) there exists a constant \( c' \) satisfying \( f(n, c \cdot \alpha(T)) \leq c' \cdot f(n, \alpha(T)) \).

all hold, then we have the following upper bounds (1), (2), and (3) for the sensitivity of \( \beta \):

1. \( \text{MS}_{\text{sub}}(\beta, n) = O(f(n, \alpha)) \) and \( \text{AS}_{\text{sub}}(\beta, n) = O(\alpha \cdot f(n, \alpha)) \);
2. \( \text{MS}_{\text{ins}}(\beta, n) = O(f(n, \alpha)) \) and \( \text{AS}_{\text{ins}}(\beta, n) = O(\alpha \cdot f(n, \alpha)) \);
(3) $\text{MS}_{\text{del}}(\beta, n) = O(f(n, \alpha))$ and $\text{AS}_{\text{del}}(\beta, n) = O(\alpha \cdot f(n, \alpha))$.

**Proof.** Let $c = \alpha(T')/\alpha(T)$, where $c$ is a constant. Then we have
\[
\frac{\beta(T')}{\beta(T)} = O\left(\frac{\alpha(T') \cdot f(n, \alpha(T'))}{\alpha(T)}\right)
\]
\[
= O\left(\frac{\alpha(T') \cdot f(n, c \cdot \alpha(T))}{\alpha(T)}\right)
\]
\[
= O\left(\frac{\alpha(T') \cdot c' \cdot f(n, \alpha(T))}{\alpha(T)}\right)
\]
\[
= O\left((c' \cdot \alpha(T') - \alpha(T)) \cdot f(n, \alpha(T))\right)
\]
\[
= O\left((c' \cdot c \cdot \alpha(T) - \alpha(T)) \cdot f(n, \alpha(T))\right)
\]
\[
= O(\alpha(T) \cdot f(n, \alpha(T))).
\]

Also,
\[
\beta(T') - \beta(T) = O(\alpha(T') \cdot f(n, \alpha(T')) - \alpha(T) \cdot f(n, \alpha(T)))
\]
\[
= O(\alpha(T') \cdot f(n, c \cdot \alpha(T)) - \alpha(T) \cdot f(n, \alpha(T)))
\]
\[
= O(\alpha(T') \cdot c' \cdot f(n, \alpha(T)) - \alpha(T) \cdot f(n, \alpha(T)))
\]
\[
= O((c' \cdot \alpha(T') - \alpha(T)) \cdot f(n, \alpha(T)))
\]
\[
= O((c' \cdot c \cdot \alpha(T) - \alpha(T)) \cdot f(n, \alpha(T)))
\]
\[
= O(\alpha(T) \cdot f(n, \alpha(T))).
\]

The functions satisfying $f(n, c \cdot \alpha(T)) \leq c' \cdot f(n, \alpha(T))$ include functions $f$ which are polynomial, poly-logarithmic, or constant in terms of $\alpha(T)$.

### 3 Substring Complexity

In this section, we consider the worst-case sensitivity of the string repetitiveness measure $\delta$, which is the *substring complexity* of strings [35]. For any string $T$ of length $n$, the substring complexity $\delta(T)$ is defined as $\delta(T) = \max_{1 \leq k \leq n} (\text{Substr}(T, k)/k)$, where $\text{Substr}(T, k)$ is the number of distinct substrings of length $k$ in $T$. It is known that $\delta(T) \leq \gamma(T)$ holds for any $T$ [35].

In what follows, we present tight upper and lower bounds for the multiplicative sensitivity of $\delta$ for all cases of substitutions, insertions, and deletions. We also present the additive sensitivity of $\delta$.

#### 3.1 Lower bounds for the sensitivity of $\delta$

**Theorem 1.** The following lower bounds on the sensitivity of $\delta$ hold:

**Substitutions:** $\text{MS}_{\text{sub}}(\delta, n) \geq 2$. $\text{AS}_{\text{sub}}(\delta, n) \geq 1$.

**Insertions:** $\text{MS}_{\text{ins}}(\delta, n) \geq 2$. $\text{AS}_{\text{ins}}(\delta, n) \geq 1$.

**Deletions:** $\lim_{n \to \infty} \text{MS}_{\text{del}}(\delta, n) \geq 1.5$. $\lim_{n \to \infty} \text{AS}_{\text{del}}(\delta, n) \geq 1$.

**Proof.**

**Substitutions:** Consider strings $T = a^n$ and $T' = a^{n-1}b$. Then $\delta(T) = 1$ and $\delta(T') = 2$ hold. Thus we get $\text{MS}_{\text{sub}}(\delta, n) \geq 2$ and $\text{AS}_{\text{sub}}(\delta, n) \geq 1$.

**Insertions:** Consider strings $T = a^n$ and $T' = a^nb$. Then $\delta(T) = 1$ and $\delta(T') = 2$ hold. Thus we get $\text{MS}_{\text{ins}}(\delta, n) \geq 2$ and $\text{AS}_{\text{ins}}(\delta, n) \geq 1$.

**Deletions:** Consider string

$$T = (abb)^m a (bba)^{m+1} a^3 m (bba)^m$$
with a positive integer $m$. Let $n = 12m + 4 = |T|$. For the sake of exposition, let $w_1 = (abb)^m$, $w_2 = (bba)^{m+1}$, $w_3 = a^{3m}$, and $w_4 = (bba)^n$ such that $T = w_1w_2w_3w_4$. To analyze $\delta(T)$, we consider $\text{Substr}(T,k)$ for four different groups of $k$, as follows:

- For $1 \leq k \leq 2$: Since $T$ is a binary string, $\max_{1 \leq k \leq 2} (\text{Substr}(T,k)/k) = 2$.

- For $3 \leq k \leq 3m$: The prefix $w_1aw_2 = (abb)^{m}a(bba)^{m+1}$ and the suffix $w_4 = (bba)^{n}$ contain three distinct substrings $(abb)^{k/3}$, $(bba)^{k/3}$, and $(bab)^{k/3}$ for each length $k$, and the substring $w_3 = a^{3m}$ contains a unique substring $a^{k}$ for each length $k$. The remaining distinct substrings must contain the range $[6m + 4, 6m + 5]$ or $[9m + 4, 9m + 5]$, which are the left and right boundaries of $w_3$, respectively. There are $k - 1$ distinct substrings containing $[6m + 4, 6m + 5]$ of form:

$$\begin{align*}
(bba)^l_1a^{k-3l_1} & \quad \text{for } 1 \leq l_1 \leq \lfloor (k-1)/3 \rfloor; \\
a(bba)^l_2a^{k-3l_2+2} & \quad \text{for } 1 \leq l_2 \leq \lfloor (k+1)/3 \rfloor; \\
ba(bba)^l_3a^{k-3l_3+1} & \quad \text{for } 1 \leq l_3 \leq \lfloor k/3 \rfloor.
\end{align*}$$

Also, there are $k - 1$ distinct substrings containing $[9m + 4, 9m + 5]$ of form

$$a^{k-l_4}(bba)^{l_4/3} \quad \text{for } 1 \leq l_4 \leq k - 1.$$  

Notice however that the two substrings $a(bba)^{l_2-1}a^{k-3l_2+2} = a^{k}$ with $l_2 = 1$ and $a^{k-l_4}(bba)^{l_4/3} = (abb)^{k/3}$ with $l_4 = k - 1$ have already been counted in the other positions in $T$, and thus these duplicates should be removed. Summing up all these, we obtain $\text{Substr}(T,k) = 3 + 1 + 2(k - 1) - 2 = 2k$ for every $3 \leq k \leq 3m$, implying $\max_{3 \leq k \leq 3m} (\text{Substr}(T,k)/k) = 2$.

- For $3m < k \leq n$: The prefix $w_1aw_2$ contains at most three distinct substrings for every $k$ and the substrings $w_3$ and $w_4$ contain no substrings of length $k > 3m$. The remaining distinct substrings must again contain the positions in $[6m + 4, 6m + 5]$ or $[9m + 4, 9m + 5]$. These substrings can also be described in a similar way to the previous case for $3 \leq k \leq 3m$, except for how we should remove duplicates. We have the two following sub-cases:

- For $k = 3m + 1$: Since $a^k = a^{3m+1}$ has no occurrences in $T$ but $(abb)^{k/3}$ has other occurrences and it has already been counted, the number of such distinct substrings is at most $2(k-1) - 1$.

- For $k > 3m + 1$: There exists at least one substring which contains both $[6m + 4, 6m + 5]$ and $[9m + 4, 9m + 5]$. Therefore, the number of such distinct substrings is at most $2(k-1) - 1$.

Hence, $\text{Substr}(T,k) \leq 3 + 2(k - 1) - 1 = 2k$ for every $3m < k \leq n$, which implies that $\max_{3m < k \leq n} (\text{Substr}(T,k)/k) \leq 2$.

Consequently, we have that $\delta(T) = 2$.

Consider the string

$$T' = (abb)^{m}(bba)^{m+1}a^{3m}(bba)^{n} = w_1w_2w_3w_4$$

that can be obtained from $T$ by removing $T[3m + 1] = a$ between $w_1$ and $w_2$. We consider the number of distinct substrings of length $3m + 1$ in $T'$: Because of the lengths of $w_j$ with $j \in \{1, 2, 3, 4\}$, each substring of length $3m + 1$ is completely contained in $w_2$ or it contains some boundaries of $w_j$.

- The prefix $w_1(w_2[1..|w_2| - 3]) = (abb)^{m}(bba)^{n}$ contains $3m$ distinct substrings of length $3m + 1$.
• The substring $w_2$ contains 3 distinct substrings of length $3m + 1$.
• The substring $w_2[4..w_2]|w_3 = (bba)^m a^{3m}$ contains $3m$ distinct substrings of length $3m + 1$.
• The suffix $w_3,w_4 = a^{3m}(bba)^m$ contains $3m - 1$ distinct substrings of length $3m + 1$ (note that $a(bba)^m$ is a duplicate and is not counted here).

Hence,
\[
\delta(T') \geq \text{Substr}(T, 3m + 1)/(3m + 1) = \frac{9m + 2}{3m + 1} = 3 - \frac{1}{3m + 1}.
\]

Thus we obtain $\lim \inf_{n \to \infty} MS_{\text{del}}(\delta, n) \geq \lim \inf_{m \to \infty}((3 - 1/(3m + 1))/2) \geq 1.5$ and
\[
\lim \inf_{n \to \infty} AS_{\text{del}}(\delta, n) \geq \lim \inf_{m \to \infty}((3 - 1/(3m + 1)) - 2) = 1.
\]

\section*{3.2 Upper Bounds for the sensitivity of $\delta$}

\textbf{Theorem 2.} The following upper bounds on the sensitivity of $\delta$ hold:

\textbf{substitutions:} $MS_{\text{sub}}(\delta, n) \leq 2$. $AS_{\text{sub}}(\delta, n) \leq 1$.

\textbf{insertions:} $MS_{\text{ins}}(\delta, n) \leq 2$. $AS_{\text{ins}}(\delta, n) \leq 1$.

\textbf{deletions:} $\limsup_{n \to \infty} MS_{\text{del}}(\delta, n) \leq 1.5$. $\limsup_{n \to \infty} AS_{\text{del}}(\delta, n) \leq 1$.

\textbf{Proof.} First we consider the additive sensitivity for $\delta$. For each $k$, the number of substrings of length $k$ that contains the edited position $i$ is clearly at most $k$. Therefore, after a substitution or insertion, at most $k$ new distinct substrings of length $k$ can appear in $T'$. Hence, in the case of substitutions and insertions, $\delta(T') \leq \max_{1 \leq k \leq n}( \text{Substr}(T, k) + k)/k \leq \max_{1 \leq k \leq n}( \text{Substr}(T, k))/k + \max_{1 \leq k \leq n}(k/k) = \delta(T) + 1$ holds. Also, in the case of deletions, $\delta(T') \leq \max_{1 \leq k \leq n}(\text{Substr}(T, k) + k - 1)/k \leq \delta(T) + \max_{1 \leq k \leq n}(k - 1)/k$ holds. Thus we obtain $AS_{\text{sub}}(\delta, n) \leq 1$, $AS_{\text{ins}}(\delta, n) \leq 1$, and $\limsup_{n \to \infty} AS_{\text{del}}(\delta, n) \leq \limsup_{k \to \infty}(k - 1)/k = 1$.

Next we consider the multiplicative sensitivity for $\delta$. Note that $\delta(T') \geq 1$ for any non-empty string $T'$, since $\text{Substr}(T', 1) \geq 1$. Combining this with the afore-mentioned additive sensitivity, we obtain $MS_{\text{sub}}(\delta, n) \leq 2$ and $MS_{\text{ins}}(\delta, n) \leq 2$. For the case of deletions, observe that $\delta(T) = 1$ only if $T$ is a unary string. However $\delta(T')$ cannot increase after a deletion since $T'$ is also a unary string. Thus we can restrict ourselves to the case where $T$ contains at least two distinct characters. Then, we have $\limsup_{n \to \infty} MS_{\text{del}}(\delta, n) \leq 1.5$, which is achieved when $\delta(T) = 2$ and $\delta(T') = 2 + \frac{k - 1}{k}$ with $k \to \infty$. 

\section*{4 String Attractors}

In this section, we consider the worst-case sensitivity of the string repetitiveness measure $\gamma$, which is the size of the smallest string attractor [30]. A string attractor $\Gamma(T)$ for a string $T$ is a set of positions in $T$ such that any substring $T$ has an occurrence containing a position in $\Gamma(T)$. We denote the size of the smallest string attractor of $T$ by $\gamma(T)$. It is known that $\gamma(T)$ is upper bounded by any of $\gamma(T), r(T), e(T)$ for any string $T$ [30].

In what follows, we present lower bounds for the multiplicative sensitivity of $\gamma$ for all cases of substitutions, insertions, and deletions. We also present the additive sensitivity of $\gamma$.

\subsection*{4.1 Lower bounds for the sensitivity of $\gamma$}

\textbf{Theorem 3.} The following lower bounds on the sensitivity of $\gamma$ hold:

\textbf{substitutions:} $\lim \inf_{n \to \infty} MS_{\text{sub}}(\gamma, n) \geq 2$. $AS_{\text{sub}}(\gamma, n) \geq \gamma - 2$ and $AS_{\text{sub}}(\gamma, n) = \Omega(\sqrt{n})$. 

12
insertions: \( \liminf_{n \to \infty} \text{MS}_{\text{ins}}(\gamma, n) \geq 2 \). \( \text{AS}_{\text{ins}}(\gamma, n) \geq \gamma - 2 \) and \( \text{AS}_{\text{ins}}(\gamma, n) = \Omega(\sqrt{n}) \).
deletions: \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(\gamma, n) \geq 2 \). \( \text{AS}_{\text{del}}(\gamma, n) \geq \gamma - 3 \) and \( \text{AS}_{\text{del}}(\gamma, n) = \Omega(\sqrt{n}) \).

Proof. Consider string \( T = a^k x a^{k+1} \#_1 a^{k-1} x a^2 \#_2 a^{k-2} x a^3 \cdots \#_1 x a^k \), where \#_j for every \( 1 \leq j \leq k \) is a distinct character. The position where \#_j for each \( 1 \leq j \leq k \) occurs has to be an element of any string attractor for \( T \). Also, each of the intervals \([1, k + 1]\) and \([k + 2, 2k + 2]\) has to contain at least one element of any string attractor for \( T \), since each of the substrings \( T[1..k+1] = a^k x \) and \( T[k+2..2k+2] = a^{k+1} \) occurs only once in \( T \). Therefore, \( \gamma(T) \geq k + 2 \) holds. Consider the set \( S = \{k+1, k+2, 2k+3, 3k+5, \ldots, k+1+k(k+2)\} \) of \( k + 2 \) positions in \( T \) which contains all the positions required above. Since each substring \( a^{k-j} x a^j \) of length \( k + 1 \) immediately preceded by \#_j (\( 1 \leq j \leq k \)) occurs in the prefix \( a^k x a^{k+1} \) and contains the position \( k + 1 \), \( S \) is indeed a string attractor for \( T \), we get \( \gamma(T) = k + 2 \). In the following, we use this string \( T \) for the analysis of lower bounds for the sensitivity of \( \gamma \).

substitutions: Let \( T' \) be the string obtained by substituting the leftmost occurrence of \( x \) at position \( k + 1 \) in \( T \) with character \( b \), yielding the new prefix \( a^k b a^{k+1} \) right before \#_1. The size of the smallest string attractor for \( T' \) is as follows: Each occurrence position of \#_j for \( 1 \leq j \leq k \) still has to be an element of any string attractor for \( T' \). Also, each of the intervals \([k+1, k+2, 2k+2]\) has to contain at least one element of any string attractor for \( T' \). In addition, each of the intervals \([2k+4, 3k+4], [3k+6, 4k+6], \ldots, [k+2+k(k+2), 2k+2+k(k+2)]\) where are the occurrences of substrings \( a^{k-1} x a, a^{k-2} x a^2, \ldots, x a^k \) has to contain one string attractor, since we have lost the prefix \( a^{k+1} x a^{k+1} \). Therefore, \( \gamma(T') \geq 2k + 2 \) holds and the set \( \{k+1, k+2, 2k+3, 3k+5, \ldots, k+1+k(k+2)\} \) of \( k + 2 \) positions in \( T' \) is a string attractor for \( T' \), implying \( \gamma(T') = 2k + 2 \). Thus we get \( \liminf_{n \to \infty} \text{MS}_{\text{sub}}(\gamma, n) \geq \liminf_{k \to \infty}(2k + 2)/(k + 2) = 2 \) and \( \text{AS}_{\text{sub}}(\gamma, n) \geq \gamma - 2 \). Since \( n = k^2 + 4k + 2 \) and \( \gamma(T) = k + 2 \), \( \text{AS}_{\text{sub}}(\gamma, n) = \Omega(\sqrt{n}) \) holds.

insertions: Let \( T' \) be the string obtained by inserting \( b \) between \( T[k+1] = x \) and \( T[k+2] = a \), yielding the new prefix \( a^k x b a^{k+1} \) right before \#_1. The size of the smallest string attractor for \( T' \) is as follows, using a similar argument to the case of insertions: Each occurrence position of \#_j for \( 1 \leq j \leq k \) still has to be an element of any string attractor for \( T' \). Also, each of the intervals \([k+2, 2k+2] \) and \([k+3, 2k+3]\) have to contain at least one element of any string attractor for \( T' \). In addition, each of the intervals \([2k+5, 3k+5], [3k+7, 4k+7], \ldots, [k+3+k(k+2), 2k+3+k(k+2)]\) which are the occurrences of substrings \( \{a^{k-1} x a, a^{k-2} x a^2, \ldots, x a^k\} \) have to contain one string attractor. Therefore, \( \gamma(T') \geq 2k + 2 \) holds and the set \( \{k+2, k+3, 2k+4, 3k+6, \ldots, k+2+k(k+2), 2k+3+k(k+2)\} \) achieves \( \gamma(T') = 2k + 2 \). Thus we get \( \liminf_{n \to \infty} \text{MS}_{\text{ins}}(\gamma, n) \geq \liminf_{k \to \infty}(2k + 2)/(k + 2) = 2 \), \( \text{AS}_{\text{ins}}(\gamma, n) \geq \gamma - 3 \), and \( \text{AS}_{\text{ins}}(\gamma, n) = \Omega(\sqrt{n}) \).

deletions: Let \( T' \) be the string obtained by deleting \( T[k+1] = x \) from \( T \), yielding the new prefix \( a^{2k+1} \) right before \#_1. The size of the smallest string attractor for \( T' \) is as follows, using a similar argument to the cases of insertions and substitutions: Each occurrence position of \#_j for \( 1 \leq j \leq k \) still has to be an element of any string attractor for \( T' \). Also, the interval \([1, 2k+1]\) has to contain one element of any string attractor for \( T' \). In addition, each of the intervals \([2k+3, 3k+3], [3k+5, 4k+5], \ldots, [k+1+k(k+2), 2k+1+k(k+2)]\) has to contain one string attractor for \( T' \). Therefore, \( \gamma(T') \geq 2k+1 \) holds and the set \( \{1, 2k+2, 3k+4, \ldots, k+k(k+2), 2k+3, 3k+5, \ldots, k+1+k(k+2)\} \) achieves \( \gamma(T') = 2k + 1 \). Thus we get \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(\gamma, n) \geq \liminf_{k \to \infty}(2k+1)(k+2) = 2 \), \( \text{AS}_{\text{del}}(\gamma, n) \geq \gamma - 3 \), and \( \text{AS}_{\text{del}}(\gamma, n) = \Omega(\sqrt{n}) \).

4.2 Upper Bounds for the sensitivity of \( \gamma \)

In this section, we present some upper bounds for the worst-case sensitivity of the smallest string attractor size \( \gamma \).

We use the following known results:

13
Theorem 4 (Lemma 3.7 of [30]). For any string $T$, $\gamma(T) \leq z_{SSsr}(T)$.

Theorem 5 (Lemma 1 of [35]). For any string $T$ of length $n$, $z_{SSsr}(T) = O(\delta(T) \log(n/\delta(T)))$.

Theorem 6 (Lemma 2 of [35]). For any string $T$, $\gamma(T) \geq \delta(T)$.

We are ready to show our results:

Corollary 1. The following upper bounds on the sensitivity of $\gamma$ hold:
- substitutions: $MS_{\text{sub}}(\gamma, n) = O(\log n)$. $AS_{\text{sub}}(\gamma, n) = O(\delta \log n)$.
- insertions: $MS_{\text{ins}}(\gamma, n) = O(\log n)$. $AS_{\text{ins}}(\gamma, n) = O(\delta \log n)$.
- deletions: $MS_{\text{del}}(\gamma, n) = O(\log n)$. $AS_{\text{del}}(\gamma, n) = O(\delta \log n)$.

Proof. Let $T$ be any string of length $n$, and let $T'$ be any string such that $ed(T, T') = 1$.

It follows from Theorem 4 and Theorem 5 that $\gamma(T') \leq z_{SSsr}(T') = O(\delta(T') \log n)$. Also, $\gamma(T) \geq \delta(T)$ by Theorem 6 and $\delta(T') = O(\delta(T))$ by Theorem 2. Then, Lemma 1 leads that

$$\frac{\gamma(T')}{\gamma(T)} = O\left(\frac{\delta(T') \log n}{\delta(T)}\right) = O(\log n).$$

Similarly, $\gamma(T') - \gamma(T) = O(\delta(T') \log n) \subseteq O(\delta(T) \log n)$ holds.

5 Run-Length Burrows-Wheeler Transform (RLBWT)

The Burrows-Wheeler transform (BWT) of a string $T$, denoted BWT$(T)$, is the string obtained by concatenating the last characters of the lexicographically sorted suffixes of $T$. The run-length BWT (RLBWT) of $T$ is the run-length encoding of BWT$(T)$ and $r(T)$ denotes its size, i.e., the number of maximal character runs in BWT$(T)$.

For example, for string $T = \text{abbaabababab}$, $r(T) = 4$ since BWT$(T) = \text{babbbbbaaaaa}$ consists in four maximal character runs $\text{ba}^4\text{a}^5\text{b}^5$.

Theorem 7 (Theorem 1 of [22]). There exists a family of strings $S$ such that $r(S) = 2$ and $r(S') = \Theta(\log n)$, where $n = |S|$ and $S'$ is a string obtained by prepending a character to $S$. The string $S$ is a reversed Fibonacci word.

Theorem 7 immediately leads to the following lower bound for the sensitivity of $r$:

Corollary 2. The following lower bound on the sensitivity of RLBWT with $|\Sigma| = 2$ hold:
- insertions: $MS_{\text{ins}}(r, n) = \Omega(\log n)$. $AS_{\text{ins}}(r, n) = \Omega(\log n)$.

To obtain a non-trivial upper bound for the sensitivity of $r$, we can use the following known result:

Theorem 8 (Theorem III.7 of [28]). For any string $T$ of length $n$,

$$r(T) = O\left(\delta(T) \max\left(1, \log \frac{n}{\delta(T) \log \delta(T)}\right) \log \delta(T)\right).$$

Corollary 3. The following upper bounds on the sensitivity of $r$ hold:
- substitutions: $MS_{\text{sub}}(r, n) = O(\log n \log r)$. $AS_{\text{sub}}(r, n) = O(r \log n \log r)$.
- insertions: $MS_{\text{ins}}(r, n) = O(\log n \log r)$. $AS_{\text{ins}}(r, n) = O(r \log n \log r)$.
- deletions: $MS_{\text{del}}(r, n) = O(\log n \log r)$. $AS_{\text{del}}(r, n) = O(r \log n \log r)$.
Lemma 1. This leads to the claimed upper bounds for the sensitivity for

\[ r(T) = O(\log n \log \log \log T) \] from Theorem 8 which always holds and is sufficient for our purpose.

Let \( T' \) be any string with \( ed(T, T') = 1 \). It follows from Theorem 2 that \( \delta(T') \leq 2\delta(T) \). Therefore, we obtain

\[ r(T') = O(\delta(T') \log n \log \delta(T')) = O(\delta(T) \log n \log \delta(T)) = O(r(T) \log n \log r(T)) \] by Lemma 1. This leads to the claimed upper bounds for the sensitivity for \( r \).

We remark that the lower bounds \( \text{MS}_{\text{ins}}(r, n) = \Omega(\log n) \) and \( \text{AS}_{\text{ins}}(r, n) = \Omega(\log n) \) from Theorem 7 and Corollary 2 are asymptotically tight when \( r = O(1) \), since \( \text{MS}_{\text{ins}}(r, n) = O(\log n \log r) = O(\log n) \) and \( \text{AS}_{\text{ins}}(r, n) = \Omega(\log n) \) in this case.

6 Bidirectional Scheme

In this section, we consider the worst-case sensitivity of the size of bidirectional scheme [64]. A factorization \( T = f_1 \cdots f_b \) for a string \( T \) of length \( n \) is a bidirectional scheme of \( T \) if each phrase \( f_j = T[p_j..p_j + \ell_j - 1] \) is either a single character or corresponding to another substring \( T[q_j..q_j + \ell_j - 1] \) where \( \ell_j = |f_j| \) such that \( p_j \neq q_j \). We denote \( f_j \) either a single character or the pair \((q_j, \ell_j)\). If \(|f_j| = 1\), then \( f_j \) is called a ground phrase. A bidirectional scheme \( B \) for \( T \) defines a function \( F_B : [1..n] \cup \{0\} \to [1..n] \cup \{0\} \), where

\[
\begin{align*}
F_B(p_j) &= 0, & \text{if } f_j \text{ is a ground phrase,} \\
F_B(p_j + k) &= q_j + k, & \text{if } f_j = (q_j, \ell_j) \text{ and } 0 \leq k < \ell_j, \\
F_B(0) &= 0.
\end{align*}
\]

Let \( F_B^0(p_j) = p_j \) and \( F_B^m(p_j) = F_B(F_B^{m-1}(p_j)) \) for any \( m \geq 1 \). A bidirectional scheme \( B \) is called valid if \( F_B \) has no cycles; namely, there exists an \( m \geq 1 \) such that \( F_B^m(x) = 0 \) for every \( x \in [1..n] \). The string \( T \) can be reconstructed from the bidirectional scheme if and only if it is valid. The size of a valid bidirectional scheme \( B \) is the number of phrases in \( B \). We denote by \( b(T) \) the size of a valid bidirectional scheme for \( T \) of the smallest size possible.

For example, for string \( T = \text{abaabababbbba} \), \( B \) shown below is a valid bidirectional scheme of the smallest size possible:

\[ B = (4, 3)(6, 4)\text{ab}(9, 3)\text{a}, \]

where its corresponding factorization is:

\[ B = \text{aba|abab|a|b|b|b|a}. \]

Here we have \( b(T) = 6 \).

In what follows, we present upper and lower bounds for the multiplicative/additive sensitivity of \( b \). It is noteworthy that our upper and lower bounds for the multiplicative sensitivity of \( b \) for substitutions and insertions are tight.

6.1 Lower bounds for the sensitivity of \( b \)

Theorem 9. The following lower bounds on the sensitivity of \( b \) hold:

- substitutions: \( \text{MS}_{\text{sub}}(b, n) \geq 2 \).
- insertions: \( \text{MS}_{\text{ins}}(b, n) \geq 2 \).
Proof. substitutions: Consider strings $T = a^n$ and $T' = a^{(n/2) - 1}ba^{(n/2)}$. Then $b(T) = 2$ and $b(T') = 4$ hold. Thus we get $\text{MS}_{\text{sub}}(b, n) \geq 2$.

insertions: Consider strings $T = a^n$ and $T' = a^{(n/2)}ba^{(n/2)}$. Then $b(T) = 2$ and $b(T') = 4$ hold. Thus we get $\text{MS}_{\text{ins}}(b, n) \geq 2$.\hfill $\Box$

The family of strings used in Theorem 9 gives us tight lower bounds for multiplicative sensitivities. However, this family of strings only provides us with weak lower bound 2 for the additive sensitivity of $b$. The following theorem will give us stronger lower bounds for the additive sensitivity for $b$. We remark that this theorem also leads us to a non-trivial lower bound for the multiplicative sensitivity of $b$ in the case of deletions.

Theorem 10. The following lower bounds on the sensitivity of $b$ hold:

substitutions: $\text{AS}_{\text{sub}}(b, n) \geq b/2 - 1$, and $\text{AS}_{\text{sub}}(b, n) = \Omega(\sqrt{n})$.

insertions: $\text{AS}_{\text{ins}}(b, n) \geq b/2 - 1$, and $\text{AS}_{\text{ins}}(b, n) = \Omega(\sqrt{n})$.

deletions: $\liminf_{n \to \infty} \text{MS}_{\text{del}}(b, n) \geq 1.5$, $\text{AS}_{\text{del}}(b, n) \geq b/2 - 3$, and $\text{AS}_{\text{del}}(b, n) = \Omega(\sqrt{n})$.

Proof. Consider string

$$T = a^kxa^{k+1}#_1a^kxa#_2a^{k-1}xa^2#_3\cdots #_kaxa^k,$$

where $#_j$ for every $1 \leq j \leq k$ is a distinct character. One of the valid bidirectional schemes $B$ for $T$ is

$$B = (k + 2, k)xa(k + 2, k)#_1(1, k + 2)#_2(2, k + 2)#_3\cdots #_k(k, k + 2).$$

The corresponding factorization of the above bidirectional scheme is as follows:

$$B = a^k|x|a|a^k|1|a^kxa|2|a^{k-1}xa^2|3|\cdots |k|axa^k|.$$  

The size of $B$ is $2k + 4$ and thus $b(T) \leq 2k + 4$.

As for substitutions, let $T'$ be the string obtained by substituting the leftmost occurrence of $x$ at position $k + 1$ in $T$ with a character $y$ such that $y \neq x$, that is,

$$T' = a^kya^{k+1}#_1a^kxa#_2a^{k-1}xa^2#_3\cdots #_kaxa^k.$$

Then, one of the valid bidirectional schemes $B'$ of $T'$ is:

$$B' = (k + 2, k)ya(k + 2, k)#_1(1, k)xa#_2(2k + 5, k)(1, 2)#_3\cdots #_k(3k + 4, 2)(1, k).$$

Also, the corresponding factorization for $B'$ is as follows:

$$B' = a^k|y|a|a^k|1|a^k|x|a|2|a^{k-1}x|a^2|3|\cdots |k|axa^k|.$$  

The size of $B'$ is $3k + 5$. We show that $B'$ is a valid bidirectional scheme for $T'$ of the smallest size possible, namely, $b(T') = 3k + 5$. Since $y$ and $#_j$ for every $1 \leq j \leq k$ are unique characters in $T'$, they have to be ground phrases. Also, since each substring $a^{k-j+1}xa^j$ of length $k + 2$ for all $1 \leq j \leq k$ and $a^{k+1}$ are unique in $T'$, each corresponding interval has to have at least one boundary of phrases. In addition, at least one occurrence of $x$ has to be a ground phrase. Then, $b(T') = 3k + 5$ holds. Since $|T| = n = k^2 + 5k + 2$, we have $k = \Theta(\sqrt{n})$. Hence, we get $\liminf_{n \to \infty} \text{MS}_{\text{sub}}(b, n) \geq 1.5$ and $\text{AS}_{\text{sub}}(b, n) \geq k + 1 = b/2 - 1 = \Omega(\sqrt{n})$.

Moreover, by considering the case where the character $T[k + 1]$ is deleted and the case where the character $y$ is inserted between positions $k + 1$ and $k + 2$, we obtain Theorem 10.\hfill $\Box$
6.2 Upper bounds for the sensitivity of $b$

**Theorem 11.** The following upper bounds on the sensitivity of $b$ hold:

- **substitutions:** $\limsup_{n \to \infty} MS_{\text{sub}}(b, n) \leq 2$. $AS_{\text{sub}}(b, n) \leq b + 2$.

- **insertions:** $MS_{\text{ins}}(b, n) \leq 2$. $AS_{\text{ins}}(b, n) \leq b$.

- **deletions:** $\limsup_{n \to \infty} MS_{\text{del}}(b, n) \leq 2$. $AS_{\text{del}}(b, n) \leq b + 1$.

**Proof.** In the following, we consider the case that $T[i] = a$ is substituted by a character # that does not occur in $T$. The other cases of insertions, deletions, and substitutions with another character $b$ ($\neq a$) occurring in $T$, can be proven similarly. We show how to construct a valid bidirectional scheme of $T'$ of the size $b' \geq b(T')$ by dividing each phrase of $B$ into some phrases, where $B$ is a valid bidirectional scheme for $T$ of the smallest size possible. We categorize each phrase $f_j = T[p_j..p_j + \ell_j - 1]$ of $B$ into one of the three following cases:

1. $i \in [p_j..p_j + \ell_j - 1]$;
2. $i \notin [p_j..p_j + \ell_j - 1]$ and $i \notin [q_j..q_j + \ell_j - 1]$;
3. $i \notin [p_j..p_j + \ell_j - 1]$ and $i \in [q_j..q_j + \ell_j - 1]$.

**Case (1):** Let $T[p_j..p_j + \ell_j - 1] = w_1aw_2$ and $T'[p_j..p_j + \ell_j - 1] = w_1\#w_2$, where $a \in \Sigma$ and $w_1, w_2 \in \Sigma^*$. If $i \notin [q_j..q_j + \ell_j - 1]$, then the phrase $f_j$ is divided into three phrases $w_1 = (q_j, |w_1|), \#, w_2 = (q_j + |w_1| + 1, |w_2|)$ in $T'$. See also the top of Figure 1. Otherwise, i.e., if $i \in [q_j..q_j + \ell_j - 1]$, intervals $[p_j..p_j + \ell_j - 1]$ and $[q_j..q_j + \ell_j - 1]$ are overlapping. We consider the case $p_j < q_j$. (Another case can be treated similarly.) Then $(q_j..q_j + 1)w_1]$ contains the edited position $i$. Let $T[p_j..p_j + \ell_j - 1] = w_1'aw_2'aw_2$, where $w_1', w_2' \in \Sigma^*$ and $q_j + |w_1'| = i$. We divide the phrase $f_j$ into at most five phrases $w_1' = (q_j, |w_1'|), a, w_2' = (q_j + |w_1'| + 1, |w_2'|), \#, w_2 = (q_j + |w_1| + 1, |w_2|)$. See also the middle of Figure 1.

**Case (2):** No changes are made to the phrase $f_j$ in this case, since $f_j$ can continue to refer to the same reference.

**Case (3):** Among all phrases in Case (3), let $f_k$ be the phrase whose ending position of the reference is the rightmost. Let $T[p_k..p_k + \ell_k - 1] = u_1au_2$, where $u_1, u_2 \in \Sigma^*$ and $q_k + |u_1| = i$. Then we divide the phrase $f_k$ into at most three phrases $u_1 = (q_k, |u_1|), a, u_2 = (q_k + |u_1| + 1, |u_2|)$ in $T'$. For the other phrases of Case (3), we divide $f_j = v_1av_2$, where $v_1, v_2 \in \Sigma^*$ and $q_j + |v_1| = i$, into at most two phrases $v_1 = (q_j, |v_1|)$ and $av_2 = (q_k + |u_1|, |v_2| + 1)$. From the above operations, the character that referred to position $i$ in $T$ becomes a ground phrase or refers to position $q_k + |u_1|$, which is a ground phrase, in $T'$. The other substrings refer to the original reference positions or to a subinterval of $[q_k + |u_1|..q_k + |f_k| - 1]$. The reference of the subinterval corresponds to the original reference of the substring. See also the bottom of Figure 1.

Then, the bidirectional scheme obtained from the above operations is ensured to be valid. The size of the bidirectional scheme $b'$ is maximized if exactly one phrase of Case (1) is divided into five phrases, and the remaining $b(T) - 1$ phrases belong to Case (3). Since at most one of the $b(T) - 1$ phrases of Case (3) can be divided into three phrases, and all the others can be divided into two phrases, $b'$ is at most $5 + 3 + 2(b(T) - 2) = 2b(T) + 4$. Furthermore, if $T$ is a unary string, then $b(T) = 2$ and the valid bidirectional scheme of size $4(= 2b(T))$ can be constructed easily. Otherwise, there are at least two ground phrases in $T$, and these phrases can not be divided into some phrases in $T'$. Then we get $b' \leq 2b(T) + 2$ and Theorem 11.

\[\square\]
Subcase of Case (1): $i \in [p_j..p_j + \ell_j - 1]$ and $i \notin [q_j..q_j + \ell_j - 1]$.

Subcase of Case (1): $i \in [p_j..p_j + \ell_j - 1]$ and $i \in [q_j..q_j + \ell_j - 1]$.

Case (3): $i \notin [p_j..p_j + \ell_j - 1]$ and $i \in [q_j..q_j + \ell_j - 1]$.

Figure 1: Illustration for changes of references in Case (1) and Case (3).
7 Lempel-Ziv 77 factorizations with/without self-references

In this section, we consider the worst-case sensitivity of the Lempel-Ziv 77 factorizations (LZ77) \[68\] with/without self-references.

For convenience, let \( f_0 = \varepsilon \). A factorization \( f_1 \cdots f_z \) for a string \( T \) of length \( n \) is the non self-referencing LZ77 factorization \( LZ77(T) \) of \( T \) if for each \( 1 \leq i < z \) the factor \( f_i \) is the shortest prefix of \( f_1 \cdots f_z \) that does not occur in \( f_0 f_1 \cdots f_{i-1} \) (or alternatively \( f_k [1..|f_k| - 1] \) is the longest prefix of \( f_i \cdots f_z \) that occurs in \( f_0 f_1 \cdots f_{i-1} \). Since \( f_k [1..|f_k| - 1] \) never overlaps with its previous occurrence, it is called non self-referencing. The last factor \( f_z \) is the suffix of \( T \) of length \( n - |f_1 \cdots f_{z-1}| \) and it may have multiple occurrences in \( f_1 \cdots f_z \).

A factorization \( f_1 \cdots f_z \) for a string \( T \) of length \( n \) is the self-referencing LZ77 factorization \( LZ77sr(T) \) of \( T \) if for each \( 1 \leq i < z \) the factor \( f_i \) is the shortest prefix of \( f_1 \cdots f_z \) that occurs exactly once in \( f_1 \cdots f_i \) as a suffix (or alternatively \( f_k [1..|f_k| - 1] \) is the longest prefix of \( f_i \cdots f_z \) which has a previous occurrence beginning at a position in range \([1..|f_1 \cdots f_{k-1}|] \). Since \( f_k [1..|f_k| - 1] \) may overlap with its previous occurrence, it is called self-referencing. The last factor \( f_z \) is the suffix of \( T \) of length \( n - |f_1 \cdots f_{z-1}| \) and it may have multiple occurrences in \( f_1 \cdots f_z \).

If we use a common convention that the string \( T \) terminates with a unique character $\$, then the last factor \( f_z \) satisfies the same properties as \( f_1, \ldots, f_{z-1} \), in both cases of (non) self-referencing LZ77 factorizations.

To avoid confusions, we use different notations to denote the sizes of these factorizations. For a string \( T \) let \( z_77(T) \) and \( z_{77sr}(T) \) denote the number \( z \) of factors in \( LZ77(T) \) and \( LZ77sr(T) \), respectively.

For example, for string \( T = abaabababababab\$, \)

\[
\begin{align*}
LZ77(T) &= a|b|aa|bab|ababa|bab\$, \\
LZ77sr(T) &= a|b|aa|bab|abababab\$,
\end{align*}
\]

where | denotes the right-end of each factor in the factorizations. Here we have \( z_{77}(T) = 6 \) and \( z_{77sr}(T) = 5 \).

In what follows, we present tight upper and lower bounds for the multiplicative sensitivity of \( z_{77} \) and \( z_{77sr} \) for all cases of substitutions, insertions, and deletions. We also present the additive sensitivity of \( z_{77} \) and \( z_{77sr} \).

7.1 Lower bounds for the sensitivity of \( z_{77} \)

**Theorem 12.** The following lower bounds on the sensitivity of non self-referencing LZ77 factorization hold:

- **substitutions:** \( \liminf_{n \to \infty} MS_{sub}(z_{77}, n) \geq 2 \). \( AS_{sub}(z_{77}, n) \geq z_{77} - 1 \).
- **insertions:** \( \liminf_{n \to \infty} MS_{ins}(z_{77}, n) \geq 2 \). \( AS_{ins}(z_{77}, n) \geq z_{77} - 1 \).
- **deletions:** \( \liminf_{n \to \infty} MS_{del}(z_{77}, n) \geq 2 \). \( AS_{del}(z_{77}, n) \geq z_{77} - 2 \).

**Proof.** Let \( p \geq 2 \) and \( \Sigma = \{0, 1, 2\} \). We use the following string \( T \) for our analysis in all cases of substitutions, insertions, and deletions.

Let \( Q_1 = 0 \) and \( Q_k = Q_1 \cdots Q_{k-1} 11 \) with \( 2 \leq k \leq p \). Let

\[
T = Q_1 Q_2 \cdots Q_p = 0 \cdot 01 \cdot 0011 \cdot 00100111 \cdot 0010011001001111 \cdots Q_p
\]
with $|T| = n = \Theta(2^p)$. Since $Q_k[1..|Q_k|-1] = T[1..|Q_k|-1]$, $Q_k||Q_k|| = 1$, and $T[|Q_k|] = 0$ for $2 \leq k \leq p$, each $Q_k$ forms a single factor in the non self-referencing LZ77 factorization of $T$. Namely,

$$\text{LZ77}(T) = Q_1|Q_2|\cdots|Q_p| = 0|01|0011|00100111|0010011001001111|\cdots|Q_p|$$

with $z_{77}(T) = p = \Theta(\log n)$.

**substitutions:** Consider the string

$$T' = 2 \cdot T[2..n] = 2 \cdot Q_2\cdots Q_p = 2 \cdot 01 \cdot 0011 \cdot 00100111 \cdot 0010011001001111 \cdots Q_p$$

which can be obtained from $T$ by substituting the first 0 with 2. Let us analyze the structure of the non self-referencing LZ77 factorization $\text{LZ77}(T')$ of $T'$. We prove by induction that $Q_k$ is divided into exactly two factors for every $2 \leq k \leq p$ in $\text{LZ77}(T')$. $Q_2$ is factorized as $0|1$ in $\text{LZ77}(T')$. Suppose that $Q_{k-1}$ is divided into exactly two factors in $\text{LZ77}(T')$, which means that the next factor is a prefix of $Q_k \cdots Q_p$. Since $T'[1] = 2$, each $Q_k[1..|Q_k|-1]$ cannot occur as a prefix of $T'$. The longest prefix of $Q_k = Q_1 \cdots Q_{k-1}$ that occurs in $T'[1..|Q_k|-1]$ is $Q_{k-1}[1..|Q_{k-1}| - 1] = Q_1 \cdots Q_{k-2}$. Thus, $Q_1 \cdots Q_{k-2}0$ is the shortest prefix of $T'[Q_1 \cdots Q_{k-1} + 1..n] = Q_k \cdots Q_p$ that does not occur in $T'[1..|Q_1 \cdots Q_{k-1}|] = Q_1 \cdots Q_{k-1}$. The remaining suffix of $Q_k$ is $Q_{k-1}[2..|Q_{k-1}|] = Q_2 \cdots Q_{k-1}$. Since $Q_k$ has 01$^{k-1}$ as a suffix and this is the leftmost occurrence of 1$^{k-1}$ in $T'$, the next factor is this remaining suffix $Q_2 \cdots Q_{k-2}11$ of $Q_k$. Thus, the non self-referencing LZ77 factorization of $T'$ is

$$\text{LZ77}(T') = 2|0|1|00|11|0010|0111|00100110|01001111|\cdots|Q_1 \cdots Q_{p-2}|Q_2 \cdots Q_{p-11}|Q_{p-2}11$$

with $z_{77}(T') = 2p - 1$, which leads to $\liminf_{n \to \infty} \text{MS}_{\text{sub}}(z_{77}, n) \geq \liminf_{p \to \infty}((2p - 1)/p) = 2$, $\text{AS}_{\text{sub}}(z_{77}, n) \geq 2p - 1 - p = p - 1 = z_{77} - 1 = \Omega(\log n)$.

**Insertions:** Let $T'$ be the string obtained by inserting 2 immediately after the first character $T[1] = 0$, namely,

$$T' = Q_1 \cdot 2 \cdot Q_2\cdots Q_p = 0 \cdot 2 \cdot 01 \cdot 0011 \cdot 00100111 \cdot 0010011001001111 \cdots Q_p.$$

Then, by similar arguments to the case of substitutions, we have

$$\text{LZ77}(T') = 0|2|01|00|11|0010|0111|00100110|01001111|\cdots|Q_1 \cdots Q_{p-2}|Q_2 \cdots Q_{p-11}|Q_p$$

with $z_{77}(T') = 2p - 1$, which leads to $\liminf_{n \to \infty} \text{MS}_{\text{ins}}(z_{77}, n) \geq \liminf_{p \to \infty}((2p - 1)/p) = 2$, $\text{AS}_{\text{ins}}(z_{77}, n) \geq 2p - 1 - p = p - 1 = z_{77} - 1 = \Omega(\log n)$.

**Deletions:** Let $T'$ be the string obtained by deleting the first character $T[1] = 0$, namely

$$T' = Q_2\cdots Q_p = 01 \cdot 0011 \cdot 00100111 \cdot 0010011001001111 \cdots Q_p.$$

Then, by similar arguments to the case of substitutions, we have

$$\text{LZ77}(T') = 0|1|00|11|0010|0111|00100110|01001111|\cdots|Q_1 \cdots Q_{p-2}|Q_2 \cdots Q_{p-11}|Q_p$$

with $z_{77}(T') = 2p - 2$, which leads to $\liminf_{n \to \infty} \text{MS}_{\text{del}}(z_{77}, n) \geq \liminf_{p \to \infty}((2p - 2)/p) = 2$, $\text{AS}_{\text{del}}(z_{77}, n) \geq (2p - 2) - p = p - 2 = z_{77} - 2 = \Omega(\log n)$.

□
The strings $T$ and $T'$ used in Theorem 12 give us optimal additive lower bounds in terms of $z_{77}$, are highly compressible ($z_{77}(T) = O(\log n)$) and only use two or three distinct characters. By using more characters, we can obtain larger lower bounds for the additive sensitivity for the size of the non self-referencing LZ77 factorizations $LZ77$ in terms of the string length $n$, as follows:

**Theorem 13.** The following lower bounds on the sensitivity of non self-referencing LZ77 factorization $LZ77$ hold:

- **substitutions:** $AS_{\text{sub}}(z_{77}, n) = \Omega(\sqrt{n})$.
- **insertions:** $AS_{\text{ins}}(z_{77}, n) = \Omega(\sqrt{n})$.
- **deletions:** $AS_{\text{del}}(z_{77}, n) = \Omega(\sqrt{n})$.

**Proof.** In A.1

### 7.2 Upper bounds for the sensitivity of $z_{77}$

**Theorem 14.** The following upper bounds on the sensitivity of non self-referencing LZ77 factorization $LZ77$ hold:

- **substitutions:** $\limsup_{n \to \infty} MS_{\text{sub}}(z_{77}, n) \leq 2$. $AS_{\text{sub}}(z_{77}, n) \leq z_{77} - 1$.
- **insertions:** $\limsup_{n \to \infty} MS_{\text{ins}}(z_{77}, n) \leq 2$. $AS_{\text{ins}}(z_{77}, n) \leq z_{77} - 1$.
- **deletions:** $\limsup_{n \to \infty} MS_{\text{del}}(z_{77}, n) \leq 2$. $AS_{\text{del}}(z_{77}, n) \leq z_{77} - 2$.

**Proof.** In the following, we consider the case that $T[i] = a$ is substituted by a character $\#$ that does not occur in $T$. The other cases of insertions, deletions, and substitutions with another character $b$ ($\neq a$) occurring in $T$, can be proven similarly, which will be discussed at the end of the proof.

We denote the factorizations as $LZ77(T) = f_1 \cdots f_z$ and $LZ77(T') = f'_1 \cdots f'_{z'}$. We denote the interval of factor $f_j$ (resp. $f'_j$) by $[p_j, q_j]$ (resp. $[p'_j, q'_j]$).

Now we prove the following claim:

**Claim.** Each interval $[p_j, q_j]$ has at most two starting positions $p'_k$ and $p'_{k+1}$ of factors in $LZ77(T')$ for some $1 \leq k < z'$.

**Proof of claim.** There are the three following cases:

1. When the interval $[p_j, q_j]$ satisfies $q_j < i$: $f_j = f'_j$ holds for any such $j$. Therefore, in the interval $[p_j, q_j]$ there exists exactly one starting position $p'_j = p_j$ of a factor in $LZ77(T')$.

2. When the interval $[p_j, q_j]$ satisfies $p_j \leq i \leq q_j$: Let $T[p_j..q_j] = w_1aw_2c$ and $T'[p_j..q_j] = w_1\#w_2c$, where $a,c,\# \in \Sigma$ and $w_1,w_2 \in \Sigma^*$. By definition, $w_1aw_2$ has at least one previous occurrence in $f_1 \cdots f_{j-1}$. After the substitution, $w_1\#$ becomes a factor $f'_j$ of $LZ77(T')$ since $\#$ is a fresh character, and $w_2c$ becomes a prefix of the next factor $f'_{j+1}$ in $LZ77(T')$. This means that $p'_j = p_j$ and $q'_{j+1} \geq q_j$. Therefore, the interval $[p_j, q_j]$ has at most two starting positions $p'_j$ and $p'_{j+1}$ of factors in $LZ77(T')$.

3. When the interval $[p_j, q_j]$ satisfies $i < p_j$: There are the two following sub-cases:

   (3-A) When $T[p_j..q_j - 1]$ has a previous occurrence which does not contain the edited position $i$ in $T$: In this case, any suffix of $T[p_j..q_j - 1]$ has a previous occurrence in $T'$. Therefore, $[p'_k, q'_k]$ with $p_j \leq p'_k$ satisfies $q'_k \geq q_j$. Hence, the interval $[p_j..q_j]$ has at most one starting position $p'_k$ of a factor in $LZ77(T')$. 

21
The analysis for Case (2) and Case (3) is analogous for all these cases. Also, in the case of deletions, MS\text{LZ77sr} with the following lower bounds on the sensitivity of self-referencing LZ77 factorization

\[ f \]

\[ \liminf \]

This completes the proof for the claim.

By the above claim, \( z_{77}(T') \leq 2z_{77}(T) \) holds for any string \( T \) and any substitution operation. Since \( f_1 \) consists of a single character for any string and the interval \([1, 1]\) cannot have two starting positions of factors in LZ77\((T')\), \( z_{77}(T') \leq 2z_{77}(T) - 1\) holds. This completes the proof for the case of substitution with \#.

The above proof can be generalized to all the other cases, by replacing \# in \( T' \) as follows:

- \( \# \leftarrow b \) for substitutions with character \( b \) occurring in \( T \), where we have \( T'[p_j, q_j] = w_1bw_2c \) for Case (2);
- \( \# \leftarrow T[i]\# \) for insertions with \#, where we have \( T'[p_j, q_j] = w_1T[i]bw_2c \) for Case (2);
- \( \# \leftarrow T[i]b \) for insertions with character \( b \) occurring in \( T \), where we have \( T'[p_j, q_j] = w_1T[i]bw_2c \) for Case (2);
- \( \# \leftarrow \varepsilon \) for deletions, where we have \( T'[p_j, q_j] = w_1w_2c \) for Case (2).

The analysis for Case (2) and Case (3) is analogous for all these cases. Also, in the case of deletions, since \( |f_2| \leq 2 \) and the interval can have two starting positions of factors in LZ77\((T')\) only when \( f_1 = T[1] \) is deleted, \( z_{77}(T') \leq 2z_{77}(T) - 2 \) holds.

\[ \square \]

### 7.3 Lower bounds for the sensitivity of \( z_{77sr} \)

**Theorem 15.** The following lower bounds on the sensitivity of self-referencing LZ77 factorization LZ77sr with \( |\Sigma| = 3 \) hold:

- **Substitutions:** \( \text{MS}_{\text{sub}}(z_{77sr}, n) \geq 2 \). \( \text{AS}_{\text{sub}}(z_{77sr}, n) \geq z_{77sr} \).
- **Insertions:** \( \text{MS}_{\text{ins}}(z_{77sr}, n) \geq 2 \). \( \text{AS}_{\text{ins}}(z_{77sr}, n) \geq z_{77sr} \).
- **Deletions:** \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(z_{77sr}, n) \geq 2 \). \( \text{AS}_{\text{del}}(z_{77sr}, n) \geq z_{77sr} - 2 \).

**Proof.** **Substitutions:** Let \( p \geq 2 \) and \( \Sigma = \{0, 1, 2\} \). We use the following string \( T \) for our analysis.

Let \( R_1 = 00 \) and \( R_k = R_1 \cdots R_{k-1}1 \) with \( 2 \leq k \leq p \). Consider the following string \( T \) of length \( n = \Theta(2^p) \):

\[
T = R_1 \cdots R_p
= 00 \cdot 001 \cdot 000011 \cdot 000010000111 \cdots R_p
\]

with \( |T| = n = \Theta(2^p) \). It immediately follows from the definition of \( T \) that the self-referencing LZ77 factorization of \( T \) is

\[
\text{LZ77sr}(T) = 0|0001|R_3|R_4|\cdots|R_p|
= 0|0001|000011|000010000111|\cdots|R_p|
\]
with \( z_{77sr}(T) = p = \Theta(\log n) \). Note that the second factor 0001 is self-referencing.

As for substitution, we consider the string

\[
T' = T[1] \cdot 2 \cdot T[3..n] = 02 \cdot 001 \cdot 000011 \cdot 000010000111 \cdots R_p
\]

which can be obtained from \( T \) by substituting the second 0 with 2. Let us analyze the structure of the self-referencing LZ77 factorization of \( T' \). The second factor 0001 in \( LZ77sr(T) \) becomes 2001 in the edited string \( T' \), and this is divided into exactly three factors as \( 2[0001] \) in \( LZ77sr(T') \) because 2 is a fresh character, 00 is the shortest prefix of \( T[3..n] = 001R_3 \cdots R_p \) that does not occur in \( T[1..2] = 02 \), and 1 is a fresh character. Our claim is that each \( R_k \) with \( 3 \leq k \leq p \) is halved into two factors \( R_k[1..|R_k|/2] = R_1 \cdots R_{k-2}0 \) and \( R_k[|R_k|/2 + 1..|R_k|] = 0R_2 \cdots R_{k-2}11 \) of equal length in \( LZ77sr(T') \). Suppose that \( R_{k-1} \) is factorized as \( R_{k-1}[1..|R_{k-1}|/2] | R_{k-1}[|R_{k-1}|/2 + 1..|R_{k-1}|] \) in \( LZ77sr(T') \), which means that the next factor is a prefix of \( R_k \cdots R_p \). Since \( R_k[1..2] = 00 \) and \( T'[1..2] = 02 \), \( R_k[1..|R_k| - 1] \) does not have a previous occurrence as a prefix of \( T' \). Since \( R_k = R_1 \cdots R_{k-2}R_{k-1} \) and \( R_1 \cdots R_{k-2} = R_{k-1}[1..|R_{k-1}| - 1] \), the longest prefix of \( T'[1..|R_1 \cdots R_{k-1}| + 1..n] = R_k \cdots R_p \) that has a previous occurrence beginning in range \( [1..|R_1 \cdots R_{k-1}|] \) is \( R_1 \cdots R_{k-2} \), which implies \( R_1 \cdots R_{k-2}R_{k-1} = 1 = R_1 \cdots R_{k-2}0 \) is the next factor in \( LZ77sr(T') \). The remaining part of \( R_k \) is \( R_{k-1}[2..|R_{k-1}|] = 0R_2 \cdots R_{k-2}11 \). Since its prefix \( 0R_2 \cdots R_{k-2}1 \) has a previous occurrence and \( 0R_2 \cdots R_{k-2}11 \) has a suffix \( 01^{k-1} \) which is the leftmost occurrence of \( 1^{k-1} \) in \( T' \), this remaining part \( R_2 \cdots R_{k-2} \) becomes the next factor in \( LZ77sr(T') \). Thus, the self-referencing LZ77 factorization of \( T' \) is

\[
LZ77sr(T') = 0[2|00|1|000|011|000010|000111] \cdots |R_1 \cdots R_{p-2}0|0R_2 \cdots R_{p-2}11|
\]

with \( z_{77sr}(T') = 2p \), which leads to \( MS_{sub}(z_{77sr}, n) \geq 2p/p = 2 \) and \( AS_{sub}(z_{77sr}, n) \geq 2p - p = p = z_{77sr} = \Omega(\log n) \).

**insertions:** We use the same string \( T \) in the case of substitutions. Let \( T' \) be the string obtained by inserting 2 immediately after \( T[1] = 0 \), namely,

\[
T' = 0 \cdot 2 \cdot 0 \cdot R_2 \cdots R_p
\]

Then, by similar arguments to the case of substitutions, we have

\[
LZ77sr(T') = 0[2|00|01|0000|11|000010|000111] \cdots |R_1 \cdots R_{p-2}0|0R_2 \cdots R_{p-2}11|
\]

with \( z_{77sr}(T') = 2p \), which leads to \( MS_{ins}(z_{77sr}, n) \geq 2p/p = 2 \) and \( AS_{ins}(z_{77sr}, n) \geq 2p - p = p = z_{77sr} = \Omega(\log n) \).

**deletions:** As for deletions, we use the same strings \( T \) and \( T' \) from Theorem 12. This string and the deletion also achieve the same lower bound for the self-referencing LZ77 factorization in the case of deletions. Then, we obtain \( z_{77sr}(T) = p, z_{77sr}(T') = 2p - 2 \), which leads to \( \liminf_{n \to \infty} MS_{del}(z_{77sr}, n) \geq 2 \) and \( AS_{del}(z_{77sr}, n) \geq z_{77sr} - 2 = \Omega(\log n) \).

The strings \( T \) and \( T' \) used in Theorem 13 give us optimal additive lower bounds in terms \( z_{77sr} \), are highly compressible \((z_{77sr}(T) = O(\log n))\) and only use two or three distinct characters. By using more characters, we can obtain larger lower bounds for the additive sensitivity for the size of the self-referencing LZ77 factorizations in terms of the string length \( n \), as follows:
Theorem 16. The following lower bounds on the sensitivity of self-referencing LZ77 factorizations \( \text{LZ77sr} \) hold:
\begin{itemize}
  \item \text{substitutions:} \( A_{\text{sub}}(z_{77sr}, n) = \Omega(\sqrt{n}) \).
  \item \text{insertions:} \( A_{\text{ins}}(z_{77sr}, n) = \Omega(\sqrt{n}) \).
  \item \text{deletions:} \( A_{\text{del}}(z_{77sr}, n) = \Omega(\sqrt{n}) \).
\end{itemize}

\textbf{Proof.} In \( \text{A.2} \) \hfill \( \Box \)

7.4 Upper bounds for the sensitivity of \( z_{77sr} \)

Theorem 17. The following upper bounds on the sensitivity of self-referencing LZ77 factorizations \( \text{LZ77sr} \) hold:
\begin{itemize}
  \item \text{substitutions:} \( M_{\text{sub}}(z_{77sr}, n) \leq 2. \ A_{\text{sub}}(z_{77sr}, n) \leq z_{77sr} \).
  \item \text{insertions:} \( M_{\text{ins}}(z_{77sr}, n) \leq 2. \ A_{\text{ins}}(z_{77sr}, n) \leq z_{77sr} \).
  \item \text{deletions:} \( M_{\text{del}}(z_{77sr}, n) \leq 2. \ A_{\text{del}}(z_{77sr}, n) \leq z_{77sr} \).
\end{itemize}

\textbf{Proof.} We use the same notations as in Theorem \( \text{14} \) of Section 7.2. We consider the case where \( T[i] \) is substituted by a fresh character \( \# \), as in the proof for Theorem \( \text{14} \). We prove the following claim:

\textbf{Claim.} Each interval \( [p_j, q_j] \) has at most two starting positions \( p'_k \) and \( p'_{k+1} \) of factors in \( \text{LZ77sr}(T') \) for \( 1 \leq k < z' \), excluding the interval \( [p_I, q_I] \) that contains the edited position \( i \). The interval \( [p_I, q_I] \) has at most three starting positions of factors in \( \text{LZ77sr}(T') \).

\textbf{Proof of claim.} Cases (1) and (3) which correspond to the positions before and after \( i \) can be shown by the same discussions in the case of non self-referencing LZ factorizations (Theorem \( \text{14} \) in Section 7.2). Now we consider case (2):

(2) The interval \( [p_j, q_j] \) satisfies \( p_j \leq i \leq q_j \) (namely, \( f_j = f_I \)). If \( f_I \) is not self-referencing, then by the same argument to the proof for Theorem \( \text{14} \) in Section 7.2 the interval has at most two starting positions of factors in \( \text{LZ77sr}(T') \). Now we consider the case that \( f_I \) is self-referencing. For the string \( w_1aw_2c = T[p_I,q_I] \), only the substrings of \( w_2 \) can have a self-referencing previous occurrence that contains the edited position \( i \) in \( T \). Therefore, \( w_1 \) has a previous occurrence in \( T' \) not containing \( i \), which means that \( q'_k = i \) where \( T'[i] = \# \) is a fresh character. For the \( w_2c \) part, we can apply the same discussion of Case (3) in Theorem \( \text{14} \) of Section 7.2. Therefore, the \( w_1\# \) part of \( T'[p_I,q_I] = w_1\#w_2c \) can have at most one starting position, and the \( w_2c \) part can have at most two starting positions of a factor in \( \text{LZ77sr}(T') \).

This completes the proof for the claim. \( \Box \)

By the above claim, \( z_{77sr}(T') \leq 2z_{77sr}(T) + 1 \) holds for any string \( T \) and any substitution. Since again \( |f_I| = 1 \), we get \( z_{77sr}(T') \leq 2z_{77sr}(T) \).

Using the same character(s) as in the proof for Theorem \( \text{14} \) we can generalize this proof to the other types of edit operations. \( \Box \)

8 Lempel-Ziv-Storer-Szymanski factorizations with/without self-references

In this section, we consider the worst-case sensitivity of the Lempel-Ziv-Storer-Szymanski factorizations (LZSS) \( \text{[14]} \) with/without self-references, a.k.a. C-factorizations \( \text{[13]} \).

Given a factorization \( T = f_1 \cdots f_z \) for a string \( T \) of length \( n \):
• it is the non self-referencing LZSS factorization \( \text{LZSS}(T) \) of \( T \) if for each \( 1 \leq i \leq z \) the factor \( f_i \) is either the first occurrence of a character in \( T \), or the longest prefix of \( f_i \cdots f_{i-1} \).

• it is the self-referencing LZSS factorization \( \text{LZSSsr}(T) \) of \( T \) if for each \( 1 \leq i \leq z \) the factor \( f_i \) is either the first occurrence of a character in \( T \), or the longest prefix of \( f_i \cdots f_z \) occurs at least twice in \( f_1 \cdots f_i \).

To avoid confusions, we use different notations to denote the sizes of these factorizations. For a string \( T \) let \( z_{\text{SS}}(T) \) and \( z_{\text{SSsr}}(T) \) denote the number \( z \) of factors in the non self-referencing LZSS factorization and in the self-referencing LZSS factorization of \( T \), respectively.

For example, for string \( T = \text{abaababababab} \), we have 

\[
\text{LZSS}(T) = a|b|a|aba|baba|bab|b,
\]

\[
\text{LZSSsr}(T) = a|b|a|aba|babababab,
\]

where \( | \) denotes the right-end of each factor in the factorizations. Here we have \( z_{\text{SS}}(T) = 7 \) and \( z_{\text{SSsr}}(T) = 5 \).

### 8.1 Lower bounds for the sensitivity of \( z_{\text{SS}} \)

**Theorem 18.** The following lower bounds on the sensitivity of non self-referencing LZSS factorization LZSS hold:

**substitutions:** \( \liminf_{n \to \infty} \text{MS}_{\text{sub}}(z_{\text{SS}},n) \geq 3 \). \( \text{AS}_{\text{sub}}(z_{\text{SS}},n) \geq 2z_{\text{SS}} - \Theta(\sqrt{z_{\text{SS}}}) \) and

\( \text{AS}_{\text{sub}}(z_{\text{SS}},n) = \Omega(\sqrt{n}) \).

**insertions:** \( \liminf_{n \to \infty} \text{MS}_{\text{ins}}(z_{\text{SS}},n) \geq 2 \). \( \text{AS}_{\text{ins}}(z_{\text{SS}},n) \geq z_{\text{SS}} - \Theta(\sqrt{z_{\text{SS}}}) \) and

\( \text{AS}_{\text{ins}}(z_{\text{SS}},n) = \Omega(\sqrt{n}) \).

**deletions:** \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(z_{\text{SS}},n) \geq 3 \). \( \text{AS}_{\text{del}}(z_{\text{SS}},n) \geq 2z_{\text{SS}} - \Theta(\sqrt{z_{\text{SS}}}) \) and

\( \text{AS}_{\text{del}}(z_{\text{SS}},n) = \Omega(\sqrt{n}) \).

**Proof.** Let \( \Sigma = \{0, 1, a_1, \ldots, a_p, b_1, \ldots, b_p\} \). Let

\[
Q_1 = (a_1 \cdots a_p)(a_1 \cdots a_{p-1}) \cdots (a_1 a_2)(a_1),
\]

\[
Q_2 = (b_1)(b_1 b_2) \cdots (b_1 \cdots b_{p-1})(b_1 \cdots b_p),
\]

and \( m = |Q_1| = |Q_2| = p(p+1)/2 \). Consider the following string:

\[
T = (Q_1 a_1Q_2) \cdot (a_1 1Q_2[1]) \cdot (Q_1[m]a_1Q_2[1..2]) \cdot (Q_1[m-1..m]a_1Q_2[1..3])
\]

\[
\cdots (Q_1[m-k+2..m]a_1Q_2[1..k]) \cdots (Q_1[2..m]a_1Q_2)
\]

\[
= (Q_1 a_1Q_2)(a_1 1b_1)(a_1 a_1 1b_1 b_1) \cdots (Q_1[m-k+2..m]a_1Q_2[1..k]) \cdots (Q_1[2..m]a_1Q_2)
\]

with \( 1 \leq k \leq m \).

Let us analyze the structure of the non self-referencing LZSS factorization of \( T \). \( Q_1 \) consists of \( p \) characters \( a_1, \ldots, a_p \), and the prefix \( a_1 \cdots a_p \) of \( Q_1 \) forms \( p \) factors of length 1. The remaining part of \( Q_1 \) is divided into \( p-1 \) factors as \( (a_1 \cdots a_k) \) with \( p-1 \geq k \geq 1 \) because \( (a_1 \cdots a_k)a_1 \) does not occur before. Next, both \( T[m+1] = a_1 \) and \( T[m+2] = 1 \) become a factor of length 1. As for the prefix of \( Q_2, b_1 \) is a fresh character and becomes a factor of length 1. For each \( (b_1 \cdots b_k) \) with \( 2 \leq k \leq p \), \( b_1 \cdots b_{k-1} \) occurs previously, and \( b_1 \cdots b_k \) does not occur before. Therefore, each interval of \( (b_1 \cdots b_k) \) has two factors as \( b_1 \cdots b_{k-1}[b_k] \). Then, there are \( 4p \) factors in the interval \( [1..|Q_1a_1Q_2]| \). The substring \( T[|Q_1a_1Q_2|..|T|] \) is the sequence of \( m \) parts \( (Q_1[m-k+1] \cdots Q_1[m-k+2] \cdots Q_1[m-k+3] \cdots Q_1[m-k+4]) \).
2..m|a_1Q_2[1..k]) with 1 ≤ k ≤ m. Each part becomes a factor because \((Q_1[m-k+2..m]a_1Q_2[1..k])\) occurs at \(T[m-k+2..m+k+2]\), and \((Q_1[m-k+2..m]a_1Q_2[1..k])Q_1[m-k+1]\) does not occur before. Therefore, the factorization of \(T\) is:

\[
\text{LZSS}(T) = Q_1|a_1|1|Q_2|(e_{a_1b_1})(a_1a_1b_1b_1)|\cdots
\]

\[
|(Q_1[m-k+2..m]a_1Q_2[1..k])|\cdots|(Q_1[2..m]a_1Q_2)|,
\]

where

\[
\text{LZSS}(Q_1) = a_1|\cdots|a_p|(a_1\cdots a_{p-1})|\cdots|(a_1a_2)|(a_1)
\]

and

\[
\text{LZSS}(Q_2) = b_1|b_1|b_2|\cdots|b_{p-2}|b_{p-1}|b_1\cdots b_{p-1}|b_p|.
\]

Then \(z_{SS}(T) = 4p + (1/2)p(p+1)\) holds.

**Substitutions:** Let

\[
T' = (Q_1a_0Q_2)(e_{a_1b_1})(a_1a_1b_1b_1)|\cdots|(Q_1[m-k+2..m]a_1Q_2[1..k])|\cdots|(Q_1[2..m]a_1Q_2)
\]

be the string obtained from \(T\) by substituting the first 1 with 0. It is clear that the factorization of the interval \([1..Q_1a_0Q_2]\) is unchanged, and there are \(4p\) factors in. Next, \(m\) factors \((Q_1[m-k+2..m]a_1Q_2[1..k])\) with \(1 ≤ k ≤ m\) lose the position they refer to. Then, each factor \(Q_1[m-k+2..m]a_1Q_2[1..k]\) is divided into three factors as \(Q_1[m-k+2..m]a_1Q_2[1..k-1]|Q_2[k]\) because of their previous occurrences. Therefore, the factorization of \(T'\) is:

\[
\text{LZSS}(T') = Q_1|a_1|0|Q_2|a_1|1|b_1|a_1a_1|1|b_1|b_2|\cdots|Q_1[m-k+2..m]a_1Q_2[1..k-1]|Q_2[k]|\cdots
\]

\[
|Q_1[2..m]a_1Q_2[1..m-1]|Q_2[m],
\]

where

\[
\text{LZSS}(Q_1) = a_1|\cdots|a_p|(a_1\cdots a_{p-1})|\cdots|(a_1a_2)|(a_1)
\]

and

\[
\text{LZSS}(Q_2) = b_1|b_1|b_2|\cdots|b_{p-2}|b_{p-1}|b_1\cdots b_{p-1}|b_p|.
\]

Then, \(z_{SS}(T') = 4p + (3/2)p(p+1)\) holds. Also,

\[
|T| = p(p+1) + 2 + \sum_{k=1}^{p(p+1)} (2k+1)
\]

\[
= p(p+1) + 2 + \sum_{k=1}^{p(p+1)} \frac{k + p(p+1)}{2}
\]

\[
= p(p+1) + 2 + \frac{p^2(p+1)^2}{4} + p(p+1) = \Theta(p^4)
\]

holds. Hence, we obtain

\[
\liminf_{n \to \infty} \text{MS}_{sub}(z_{SS}, n) \geq \liminf_{p \to \infty} \left(\frac{4p + \frac{3(p(p+1))}{2}}{4p + \frac{p(p+1)}{2}}\right) = 3,
\]

\[
\text{AS}_{sub}(z_{SS}, n) \geq \frac{4p + \frac{3(p(p+1))}{2}}{2} - \left(4p + \frac{p(p+1)}{2}\right) = p(p+1)
\]

\[
= 2z_{SS} - \Theta(\sqrt{z_{SS}}) \in \Omega(\sqrt{n}).
\]
Proof of claim. Now we show the following claim:

where

\[
\limsup \text{MS} \quad \text{deletions:} \\
\limsup \text{SS} \quad \text{substitutions:} \\
\limsup \text{AS} \quad \text{insertions:}
\]

Theorem 19. The following upper bounds on the sensitivity of non self-referencing LZSS factorization of \(T\) is:

\[
\text{LZSS}(T') = Q_1[a_1|0|Q_2|a_1|b_1|a_1a_1|b_1|\cdots |Q_1|m-k+2.m|a_1Q_2[1..k]|\cdots |Q_1|2.m|a_1Q_2],
\]

where

\[
\text{LZSS}(Q_1) = a_1\cdots a_p|(a_1\cdots a_{p-1})|\cdots|(a_1a_2)|(a_1) |
\]

and

\[
\text{LZSS}(Q_2) = b_1|b_1|b_2|\cdots |b_1\cdots b_{p-2}|b_{p-1}|b_1\cdots b_{p-1}|b_p|.
\]

Then, \(z_{SS}(T') = 4p + p(p + 1)\) holds. Hence, we obtain \(\liminf_{n\to\infty} \text{MS}_{\text{ins}}(z_{SS}, n) \geq 2, \text{AS}_{\text{ins}}(z_{SS}, n) \geq z_{SS} - \Theta(\sqrt{z_{SS}}), \text{and AS}_{\text{del}}(z_{SS}, n) = \Omega(\sqrt{n})\).

deletions: As for deletions, by considering \(T'\) obtained from \(T\) by deleting the first 1, we get a similar decomposition to the case of substitutions. Thus, we also obtain \(\liminf_{n\to\infty} \text{MS}_{\text{del}}(z_{SS}, n) \geq 3, \text{AS}_{\text{del}}(z_{SS}, n) \geq 2z_{SS} - \Theta(\sqrt{z_{SS}}), \text{and AS}_{\text{del}}(z_{SS}, n) = \Omega(\sqrt{n})\).

8.2 Upper bounds for the sensitivity of \(z_{SS}\)

Theorem 19. The following upper bounds on the sensitivity of non self-referencing LZSS factorization LZSS hold:

substitutions: \(\limsup_{n\to\infty} \text{MS}_{\text{sub}}(z_{SS}, n) \leq 3, \text{AS}_{\text{sub}}(z_{SS}, n) \leq 2z_{SS} - 2\).

insertions: \(\text{MS}_{\text{ins}}(z_{SS}, n) \leq 2, \text{AS}_{\text{ins}}(z_{SS}, n) \leq z_{SS}\).

deletions: \(\limsup_{n\to\infty} \text{MS}_{\text{del}}(z_{SS}, n) \leq 3, \text{AS}_{\text{del}}(z_{SS}, n) \leq 2z_{SS} - 3\).

Proof. Let LZSS\((T) = f_1 \cdots f_j\) and LZSS\((T') = f'_1 \cdots f'_{j'}\). We denote the interval of the \(j\)th factor \(f_j\) (resp. \(f'_j\)) by \([p_j, q_j]\) (resp. \([p'_j, q'_j]\)), namely \(T[p_j..q_j] = f_j\) and \(T'[p'_j..q'_j] = f'_j\). Also, let \(f_{i1}\) be the factor of LZSS\((T)\) whose interval \([p_i, q_i]\) contains the edited position \(i\), namely \(p_i \leq i \leq q_i\).

substitutions: In the following, we consider the case that the \(i\)th character \(T[i] = a\) is substituted by a fresh character \# which does not occur in \(T\). The other cases can be proven similarly. Now we show the following claim:

Claim. After the substitution, each interval \([p_j, q_j]\) has at most three starting positions \(p'_{k}, p'_{k+1}\), and \(p'_{k+2}\) of factors in LZSS\((T')\) for \(1 \leq k \leq z' - 2\).

Proof of claim. There are the three following cases:

(i) When the interval \([p_j, q_j]\) satisfies \(q_j < i\): By the same argument to Case (1) for LZ77, the interval \([p_j, q_j]\) contains exactly one starting position \(p'_j = p_j\).

(ii) When the interval \([p_j, q_j]\) satisfies \(p_j \leq i \leq q_j\) (namely, \(f_j = f_{i1}\)): For the string \(w_1aw_2 = T[p_j..q_j]\), it is guaranteed that \(w_1aw_2\) has at least one occurrence in \(f_1 \cdots f_{j-1}\). After the substitution which gives \(T'[p_j..q_j] = w_1\#w_2, w_1\) and \# become factors as \(f'_j\) and \(f'_{j+1}\), and \(w_2\) becomes the prefix of factor \(f'_{j+2}\). This means that \(p'_j = p_j\) and \(q'_{j+2} \geq q_j\). Therefore, the interval \([p_j, q_j]\) contains at most three starting positions \(p'_j, p'_{j+1}\) and \(p'_{j+2}\) of factors in LZSS\((T')\).

(iii) When the interval \([p_j, q_j]\) satisfies \(i < p_j\): We consider the two following sub-cases:
(iii-B) All occurrences of $T[p_j..q_j]$ in $T$ contain the edited position $i$: Let $u_1au_2 = T[p_j, q_j]$ with $a \in \Sigma$ and $u_1, u_2 \in \Sigma^*$. $u_1$ and $u_2$ have previous occurrences in $T'[1..p_j - 1]$. Let $p'_k$ be the starting position of the leftmost factor of $LZSS(T')$ which begins in range $[p_j, q_j]$. If $p'_k$ is in $u_2$, then $q'_k \geq q_k$ and thus there is only one starting position of a factor of $LZSS(T')$ in the interval $[p_j..q_j]$. Suppose $p'_k$ is in $u_1$. If $a$ has no previous occurrences (which happens when $T[i]$ was the only previous occurrence of $a$), then $T'[p_k + |u_1|] = a$ is the first occurrence of $a$ in $T'$ and thus $q'_k = p_k + |u_1| - 1$, $p_{k+1} = q'_k + 1$ and $q_{k+1} = p_{k+1} + 1$. Otherwise, $q'_k \geq p_k + |u_1| - 1$, $p'_{k+1} \geq q'_k + 1$ and $q_{k+1} = p_{k+1} + 1$. In either case, since $u_2$ has a previous occurrence, $q_{k+2} \geq q_{k+1}$. Thus, there can exist at most three starting positions of factors of $LZSS(T')$ in the interval $[p_j..q_j]$.

This completes the proof for the claim.

It follows from the above claim that $z_{SS}(T') \leq 3z_{SS}(T)$ for any string $T$ and substitutions with #. Since $|f_1| = 1$, $z_{SS}(T') \leq 3z_{SS}(T) - 2$ holds. Hence, we obtain $\limsup_{n \to \infty} MS_{sub}(z_{SS}, n) \leq 3$ and $\mathbf{AS}_{sub}(z_{SS}, n) \leq 2z_{SS} - 2$.

**insertions:** In the following, we consider the case that # is inserted to between positions $i - 1$ and $i$. The other cases can be proven similarly. Now we show the following claim:

**Claim.** After the insertion, each interval $[p_j, q_j]$ contains at most two starting positions $p'_k$ and $p'_{k+1}$ of factors in $LZSS(T')$ for $1 \leq k \leq z' - 1$, excluding the interval $[p_1, q_1]$. Also, the interval $[p_I, q_I]$ contains at most three starting positions of factors in $LZSS(T')$.

**Proof of claim.** For Cases (i), (ii), and (iii-A), we can use the same discussions as in the case of substitutions. Now we consider Case (iii-B):

(iii-B) When all occurrences of $T[p_j..q_j]$ in $T$ contain the edited position $i$: Let $u_1au_2 = T[p_j, q_j]$ with $a \in \Sigma$ and $w_1, w_2 \in \Sigma^*$. It is guaranteed that $w_1a, w_2$ still have previous occurrences in $T'$. Therefore, each range of $w_1a$ and $w_2$ can contain at most one starting position of a factor in $LZSS(T')$.

It follows from the above claim that $z_{SS}(T') \leq 2z_{SS}(T) + 1$ holds any string $T$ and insertions with #. By using the same discussion as for $f_1$, we obtain $z_{SS}(T') \leq z_{SS}(T)$ holds. Then we have $MS_{ins}(z_{SS}, n) \leq 2$ and $\mathbf{AS}_{ins}(z_{SS}, n) \leq z_{SS}$.

**deletions:** In the following, we consider the case that $T[i] = a$ is deleted. Now we show the following claim:

**Claim.** After the deletion, each interval $[p_j, q_j]$ contains at most three starting positions $p'_k$, $p'_{k+1}$, and $p'_{k+2}$ of factors in $LZSS(T')$ for $1 \leq k \leq z' - 2$, excluding the interval $[p_I, q_I]$. The interval $[p_I, q_I]$ contains at most two starting positions of factors in $LZSS(T')$.

**Proof of claim.** For Cases (i) and (iii), we can use the same discussions as in the case of substitutions. Now we consider case (ii):

(ii) When the interval $[p_j, q_j]$ satisfies $p_j \leq i \leq q_j$ (namely, $f_j = f_I$): Let $w_1aw_2 = T[p_j..q_j]$ with $a \in \Sigma$ and $w_1, w_2 \in \Sigma^*$. It is guaranteed that $w_1aw_2$ has at least one previous occurrence in
\[ f_1 \cdots f_{j-1}. \] Therefore, after the deletion of a, each range of \( w_1 \) and \( w_2 \) can contain at most one starting position of a factor in \( \text{LZSS}(T') \).

It follows from the above claim that \( \delta_{SS}(T') \leq 3 \delta_{SS}(T) - 1 \) holds for any string \( T \) and deletions. By using the same discussion as for \( f_1 \), \( \delta_{SS}(T') \leq 3 \delta_{SS}(T) - 3 \) holds. Then we get
\[
\limsup_{n \to \infty} \text{MS}_{\text{del}}(\delta_{SS}, n) \leq 3 \quad \text{and} \quad \text{AS}_{\text{del}}(\delta_{SS}, n) \leq 2 \delta_{SS} - 3.
\]

\section{Upper bounds for the sensitivity of \( \delta_{SSr} \)}

\begin{theorem} \label{thm:upper bounds for \( \delta_{SSr} \)}
The following upper bounds on the sensitivity of self-referencing LZSS factorization \( \text{LZSSr} \) hold:

- **Substitutions:** \( \liminf_{n \to \infty} \text{MS}_{\text{sub}}(\delta_{SSr}, n) \leq 3 \). \( \text{AS}_{\text{sub}}(\delta_{SSr}, n) \leq 2 \delta_{SSr} - \Theta(\sqrt{\delta_{SSr}}) \) and \( \text{AS}_{\text{sub}}(\delta_{SSr}, n) = \Omega(\sqrt{n}) \).

- **Insertions:** \( \limsup_{n \to \infty} \text{MS}_{\text{ins}}(\delta_{SSr}, n) \leq 2 \). \( \text{AS}_{\text{ins}}(\delta_{SSr}, n) \leq \delta_{SSr} - \Theta(\sqrt{\delta_{SSr}}) \) and \( \text{AS}_{\text{ins}}(\delta_{SSr}, n) = \Omega(\sqrt{n}) \).

- **Deletions:** \( \limsup_{n \to \infty} \text{MS}_{\text{del}}(\delta_{SSr}, n) \leq 3 \). \( \text{AS}_{\text{del}}(\delta_{SSr}, n) \leq 2 \delta_{SSr} - \Theta(\sqrt{\delta_{SSr}}) \) and \( \text{AS}_{\text{del}}(\delta_{SSr}, n) = \Omega(\sqrt{n}) \).

\end{theorem}

\begin{proof}
We use the same strings \( T \) and \( T' \) as in the proof for Theorem \ref{thm:lower bounds for \( \delta_{SSr} \)} which shows the upper bounds of the sensitivity of the non self-referencing LZSS. For the string \( T \) and each edit operation, the self-referencing LZSS factorization is the same as the non self-referencing LZSS factorization. Hence, we obtain Theorem \ref{thm:upper bounds for \( \delta_{SSr} \)}.
\end{proof}

\section{LZ-End factorizations}

In this section, we consider the worst-case sensitivity of the \( \text{LZ-End factorizations} \) \[37\]. This is an LZ77-like compressor such that each factor \( f_i \) has a previous occurrence which corresponds to the ending position of a previous factor. This property allows for fast substring extraction in practice \[37\].

A factorization \( T = f_1 \cdots f_{\text{End}} \) for a string \( T \) of length \( n \) is the LZ-End factorization \( \text{LZEnd}(T) \) of \( T \) such that, for each \( 1 \leq i < \text{End} \), \( f_i[1..|f_i| - 1] \) is the longest prefix of \( f_1 \cdots f_{\text{End}} \) which has a previous occurrence in \( f_1 \cdots f_{i-1} \) as a suffix of some string in \( \{ \varepsilon, f_1 f_2, \ldots, f_1 \cdots f_{i-1} \} \). The last factor \( f_{\text{End}} \) is the suffix of \( T \) of length \( n - |f_1 \cdots f_{\text{End}} - 1| \). Again, if we use a common convention
that the string $T$ terminates with a unique character $\$, then the last factor $f_{\text{End}}$ satisfies the same properties as $f_1, \ldots, f_{z-1}$, in the cases of LZ-End factorizations. Let $z_{\text{End}}(T)$ denote the number of factors in the LZ-End factorization of string $T$.

For example, for string $T = \text{abaababababab}$, $z_{\text{End}}(T) = 6$.

9.1 Lower bounds for the sensitivity of $z_{\text{End}}$

**Theorem 22.** The following lower bounds on the sensitivity of $z_{\text{End}}$ hold:

**substitutions:** $\liminf_{n \to \infty} \text{MS}_{\text{sub}}(z_{\text{End}}, n) \geq 2$. $\text{AS}_{\text{sub}}(z_{\text{End}}, n) \geq z_{\text{End}} - \Theta(\sqrt{z_{\text{End}}})$ and $\text{AS}_{\text{sub}}(z_{\text{End}}, n) = \Omega(\sqrt{n})$.

**insertions:** $\liminf_{n \to \infty} \text{MS}_{\text{ins}}(z_{\text{End}}, n) \geq 2$. $\text{AS}_{\text{ins}}(z_{\text{End}}, n) \geq z_{\text{End}} - \Theta(\sqrt{z_{\text{End}}})$ and $\text{AS}_{\text{ins}}(z_{\text{End}}, n) = \Omega(\sqrt{n})$.

**deletions:** $\liminf_{n \to \infty} \text{MS}_{\text{del}}(z_{\text{End}}, n) \geq 2$. $\text{AS}_{\text{del}}(z_{\text{End}}, n) \geq z_{\text{End}} - \Theta(\sqrt{z_{\text{End}}})$ and $\text{AS}_{\text{del}}(z_{\text{End}}, n) = \Omega(\sqrt{n})$.

**Proof.** Let $\sigma_j$ denote the $j$th character in the alphabet $\Sigma$ for $1 \leq j \leq |\Sigma|$. For a positive integer $p$, consider the string $Q = \sigma_1 \cdot \sigma_1 \sigma_2 \cdots \sigma_1 \cdots \sigma_p$ of length $q = |Q| = p(p + 1)/2 = \Theta(p^2)$. Consider the string

$$T = Q \cdot \sigma_1 \sigma_{p+1} \cdot Q[q] \sigma_1 \sigma_{p+1} \sigma_{p+2} \cdot Q[q - 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+3} \cdots Q \sigma_1 \sigma_{p+1} \sigma_{p+q+1}$$

with $|T| = \Theta(p^2)$. As for the interval $[1, q]$ in $T$, LZEnd$(Q) = f_1, \ldots, f_p$ such that $f_k = \sigma_1 \cdots \sigma_k$ for every $1 \leq k \leq p$. Since $f_p$ has no occurrences in $f_1 \cdots f_{p-1}$, the decomposition is not changed by appending any character to $Q$. Hence, the next factor $f_{p+1}$ starts at position $q + 1$. Then $f_{p+1} = \sigma_1 \sigma_{p+1}$ holds, since $\sigma_{p+1}$ is a fresh character and $\sigma_1 = f_1$. As for the remaining interval, we show $f_{p+1+j} = Q[q - j + 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+j+1}$ holds for each $1 \leq j \leq q$. At first, for $j = 1$, $f_{p+2} = Q[q] \sigma_1 \sigma_{p+1} \sigma_{p+2}$ holds since $f_{p+2}$ starts with $Q[q]$. $Q[q] \sigma_1 \sigma_{p+1}$ has an occurrence as a suffix of $T[1..q+2]$, and $\sigma_{p+2}$ is a fresh character in the prefix. Next, we assume that $f_{p+1+j} = Q[q - j + 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+j+1}$ holds with $1 \leq j \leq k - 1$ for some integer $k$. Then we consider whether $f_{p+1+k} = Q[q - k + 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+k+1}$ holds or not. By the assumption, $f_{p+1+k}$ starts with $Q[q - k + 1]$. Also, $Q[q - k + 1..q] \sigma_1 \sigma_{p+1}$ has an occurrence as a suffix of $T[1..q+2]$, and $\sigma_{p+k+1}$ is a fresh character in the prefix. Therefore, the assumption is also valid for $k$. By the above argument, $f_{p+1+j} = Q[q - j + 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+j+1}$ holds for each $1 \leq j \leq q$ by induction. Therefore,

$$\text{LZEnd}(T) = \sigma_1 \sigma_2 \cdots \sigma_1 \sigma_{p+1} | Q[q] \sigma_1 \sigma_{p+1} \sigma_{p+2} | \cdots | Q \sigma_1 \sigma_{p+1} \sigma_{p+q+1}$$

with $z_{\text{End}}(T) = p + 1 + q = \Theta(q) = \Theta(\sqrt{n})$.

As for substitutions, consider the string

$$T' = Q \cdot \# \sigma_{p+1} \cdot Q[q] \sigma_1 \sigma_{p+1} \sigma_{p+2} \cdot Q[q - 1..q] \sigma_1 \sigma_{p+1} \sigma_{p+3} \cdots Q \sigma_1 \sigma_{p+1} \sigma_{p+q+1}$$

which can be obtained from $T$ by substituting $T[q+1] = \sigma_1$ with a character $\#$ which does not occur in $T$. Let us analyze the structure of the $\text{LZEnd}(T')$. As mentioned above, $f_1, \ldots, f_p$ are not changed after the substitution. The $(p + 1)$th factor in $\text{LZEnd}(T)$, namely, $\# \sigma_{p+1}$ is factorized as $\# | \sigma_{p+1}$ in $\text{LZEnd}(T')$ since both characters have no occurrence in $T'[1..q] = Q$. Then the next factor starts with $Q[q]$. Each of $Q[q]$ and $\sigma_1$ have some occurrence as a suffix of a previous factor. On the other
hand, each of \(Q[q]\sigma_1 \) and \(\sigma_{p+1}\sigma_{p+2} \) have no occurrences previously. Therefore, \(Q[q]\sigma_1\sigma_{p+1}\sigma_{p+2} \) is factorized as \(Q[q]\sigma_1\sigma_{p+1}\sigma_{p+2} \) in \(\text{LZEnd}(T')\). Similarly, by induction, each \((p + 1 + k)\)th factor in \(\text{LZEnd}(T)\) for every \(2 \leq k \leq q\), namely, \(Q[q - k + 1..q]\sigma_1\sigma_{p+1}\sigma_{p+k+1} \) is also factorized as \(Q[q - k + 1..q]\sigma_1\sigma_{p+1}\sigma_{p+k+1+1} \). Thus, the LZ-End factorization of \(T'\) is

\[
\text{LZEnd}(T') = \sigma_1\sigma_1\sigma_2 \cdots \sigma_1 \cdots \sigma_p \# \sigma_{p+1} \sigma_{p+2} \# \cdots \# \sigma_{p+q+1} \]

with \(z_{\text{End}}(T') = p + 2 + 2q\). Recall \(p = \Theta(\sqrt{q})\). Hence we get \(\liminf_{n \to \infty} \text{MS}_{\text{sub}}(z_{\text{End}}, n) \geq \liminf_{q \to \infty} (p + 2q + 2)/(p + q + 1) \geq 2\), \(\text{AS}_{\text{sub}}(z_{\text{End}}, n) \geq (p + 2q + 2) - (p + q + 1) = z_{\text{End}} - \Theta(\sqrt{z_{\text{End}}})\), and \(\text{AS}_{\text{sub}}(z_{\text{End}}, n) = \Omega(\sqrt{n})\).

Also, as for deletions (resp. insertions), we get Theorem 22 by considering the case where the character \(T[q + 1] \) is deleted (resp. \# \) is inserted between positions \(q \) and \(q + 1\)).

9.2 Upper bounds for the sensitivity of \(z_{\text{End}}\)

To show a non-trivial upper bound for the sensitivity of \(z_{\text{End}}\), we use the following known results:

**Theorem 23** (\[\text{[87]}\). For any string \(T\), \(z_{\text{SSsr}}(T) \leq z_{\text{End}}(T)\).

**Theorem 24** (Theorem 3.2 of \[\text{[31]}\). For any string \(T\) of length \(n\), \(z_{\text{End}}(T) = O(\delta(T) \log^2(n/\delta(T)))\).

For any string \(T\), \(\delta(T) \leq z_{\text{End}}(T)\) holds from Theorems 22 and 23. Let \(T'\) be any string with \(\delta(T, T') = 1\). It follows from Theorem 2 that \(\delta(T') \leq 2\delta(T)\). Now let \(c\) be the constant value such that \(\delta(T') = c\delta(T)\) holds. Then, \(\log^2(n/\delta(T')) = \log^2 n + \log^2 (\log\delta(T')) = \log^2 n + \log^2 (\log\delta(T) + 2\log n \log c\delta(T)) = \log^2 n + \log^2 (\log\delta(T) + 2\log n \log c\delta(T)) \leq \log^2 n + \log^2 (\log\delta(T) + 2\log n \log c) = O(\log^2(n/\delta(T)))\).

Following Lemma 1, we now obtain \(z_{\text{End}}(T') = O(\delta(T') \log^2(n/\delta(T'))) = O(\delta(T) \log^2(n/\delta(T))) = O(z_{\text{End}}(T) \log^2(n/\delta(T)))\), which leads to the claimed upper bounds for the sensitivity for \(z_{\text{End}}\).

10 Lempel-Ziv 78 factorizations

In this section, we consider the worst-case sensitivity of the Lempel-Ziv 78 factorizations (LZ78) \[\text{[69]}\).

For convenience, let \(f_0 = \varepsilon\). A factorization \(T = f_1 \cdots f_{z_7} \) for a string \(T\) of length \(n\) is the LZ78 factorization \(\text{LZ78}(T)\) of \(T\) if for each \(1 \leq i < z_7\) the factor \(f_i\) is the longest prefix of \(f_i \cdots f_{z_7}\) such that \(f_i[1..|f_i| - 1] = f_j\) for some \(0 \leq j < i\). The last factor \(f_{z_7}\) is the suffix of \(T\) of length \(n - |f_1 \cdots f_{z_7-1}|\) and it may be equal to some previous factor \(f_j\) (\(1 \leq j < z_7\)). Again, if we use a common convention that the string \(T\) terminates with a unique character \(\$\), then the last factor \(f_{z_7}\) can be defined analogously to the previous factors. Let \(z_7(T)\) denote the number of factors in the LZ78 factorization of string \(T\).

For example, for string \(T = \text{abaabababababab}\$\),

\[
\text{LZ78}(T) = a|b|aa|ba|bab|ab|aba|b$\],

where \(|\) denotes the right-end of each factor in the factorization. Here we have \(z_7(T) = 8\).
As for the sensitivity of LZ78, Lagarde and Perifel \[40\] showed that $\text{MS}_{\text{ins}}(z_{78}, n) = \Omega(n^{1/4})$, $\text{AS}_{\text{ins}}(z_{78}, n) = \Omega(n^{3/2})$, and $\text{AS}_{\text{del}}(z_{78}, n) = \Omega(n \log n)$ for insertions. In this section, we present lower bounds for the multiplicative/additive sensitivity of LZ78 for the remaining cases, i.e., for substitutions and deletions, by using a completely different string from \[40\].

10.1 Lower bounds for the sensitivity of $z_{78}$

**Theorem 25.** The following lower bounds on the sensitivity of $z_{78}$ hold:

**Substitutions:** $\text{MS}_{\text{sub}}(z_{78}, n) = \Omega(n^{1/4})$. $\text{AS}_{\text{sub}}(z_{78}, n) = \Omega(\ell_{78}^{3/2})$ and $\text{AS}_{\text{sub}}(z_{78}, n) = \Omega(n^{3/4})$.

**Deletions:** $\text{MS}_{\text{del}}(z_{78}, n) = \Omega(n^{1/4})$. $\text{AS}_{\text{del}}(z_{78}, n) = \Omega(\ell_{78}^{3/2})$ and $\text{AS}_{\text{del}}(z_{78}, n) = \Omega(n^{3/4})$.

**Proof.** Consider the string

$$T = (\sigma_{k+1} \cdots (\sigma_{2k}) \cdot (\sigma_1) \cdot (\sigma_1 \sigma_2) \cdots (\sigma_1 \cdots y_i \cdot \sigma_{k+1}) \cdots (\sigma_1 \cdots y_k \cdot \sigma_{2k})$$

where $\sigma_i$ for every $1 \leq i \leq 2k$ is a distinct character and $y_j$ for every $1 \leq j \leq k$ satisfies the following property: $y_j$ is the maximum integer at most $k$ such that $2 + j + \ell_j - 1 \equiv y_j \pmod{\ell_j}$ where $\ell_j$ is an integer satisfying $(1/2)\ell_j(\ell_j - 1) + 1 \leq j \leq (1/2)\ell_j(\ell_j + 1)$. We remark that the parentheses ( and ) in $T$ are shown only for the better visualization and exposition, and therefore they are not the characters in $T$.

Let $n$ be the length of $T$. Since $k + (1/2)k(k+1) < n < k + (1/2)k(k+1) + k(k+1)$, $n \in \Theta(k^2)$ holds. In the LZ78 factorization of $T$, for each substring $w$, its suffix $w[1..|w| - 1]$ has a previous occurrence as $w[1..|w| - 1]$, and $w$ is the leftmost occurrence of $w$ in the string $T$. Therefore, the LZ78 factorization of $T$ is

$$\text{LZ78}(T) = \sigma_{k+1} | \cdots | \sigma_{2k} | \sigma_1 | \sigma_1 \sigma_2 | \cdots | \sigma_1 \cdots \sigma_k | \sigma_1 \cdots y_j \cdot y_{k+1} | \cdots | \sigma_1 \cdots y_k \cdot \sigma_{2k}$$

with $\ell_{78}(T) = 3k$.

For our analysis of the sensitivity of $z_{78}$ for substitutions, consider the string

$$T' = (\sigma_{k+1} \cdots (\sigma_{2k}) \cdot (\sigma_1) \cdot (\sigma_1 \sigma_2) \cdots (\sigma_1 \cdots y_i \cdot \sigma_{k+1}) \cdots (\sigma_1 \cdots y_k \cdot \sigma_{2k})$$

which can be obtained from $T$ by substituting the first character $\sigma_1$ of the string in the $2k + 1$th paring parentheses with a fresh character $\#$, which does not occur in $T$. Let us analyze the structure of the LZ78 factorization of $T'$. Clearly, the first $2k$ factors are unchanged after the substitution. Next, we consider $(\#\sigma_2 \cdots y_i \cdot \sigma_{k+1})$. First, the prefix $\#\sigma_2 \cdots y_i$ is decomposed into $y_i$ factors of length 1. The next factor is $\sigma_{k+1} \sigma_1$ since $\sigma_{k+1} \sigma_1$ has an occurrence as a previous factor and $\sigma_{k+1} \sigma_1$ has no occurrences as a previous factor. Now we show in each interval of the $2k + j$th paring parentheses for $2 \leq j \leq k$ (i.e., the interval of $\sigma_1 \cdots y_j \sigma_{k+j}$) there appear the right-ends $|$ of factors in $\text{LZ78}(T')$ as follows:

$$\sigma_1 | \sigma_2 \cdots \sigma_{j+1} | \sigma_{j+2} \cdots \sigma_{j+\ell_j+1} | \cdots | \sigma_{\ell_j-\ell_j+1} \cdots \sigma_{y_j} | \sigma_{k+j}.$$ (1)

Namely, the interval is decomposed into $3 + ((y_j - j - \ell_j - 1)/\ell_j) + 1 = 3 + (y_j - j - 1)/\ell_j$ pieces $d_1, \ldots, d_{3+(y_j-j-1)/\ell_j}$, where $|d_1| = |d_{3+(y_j-j-1)/\ell_j}| = 1$, $|d_2| = j$, and each of the others is of length $\ell_j$. At first, we show the partition \[1\] is valid for $j = 2$. As mentioned above, there is a factor $\sigma_{k+1} \sigma_1$ constructed with the immediately preceded character and the first character of the interval of $\sigma_1 \cdots y_2 \sigma_{k+2}$. And then, $\sigma_2 \cdots y_2$ is decomposed into $\sigma_2 \sigma_3 | \sigma_4 \sigma_5 | \cdots | \sigma_{y_2-1} \sigma_{y_2}$

\[\text{In the restricted case of appending a character to the top of a string or deleting the first character of a string, they showed upper bounds that the ratio is } O(n^{1/4}) \text{ and the increase is } O(\ell_{78}^{3/2}).\]
since \(y_2\) is the maximum odd value less than or equal to \(k\) and each \(\sigma_{2i}\) for \(1 \leq i \leq (y_2 - 1)/2\) is the longest prefix as some previous factor. Since \(\ell_2 = 2\) holds, the partitions of the interval are \(\sigma_1|\sigma_2|\sigma_3|\sigma_4|\sigma_5|\cdots|\sigma_{y_2-1}|\sigma_{y_2}\), and this satisfies the partition (1). Next, we assume the partition (1) is valid for \(j \leq h - 1\) for some integer \(h\), and we consider whether the partition (1) is valid or not for \(j = h\). From the assumption and the same discussion as the above, there is a factor \(\sigma_{k+h-1}\) constructed with the immediately preceded character and the first character of the interval of \(\sigma_1 \cdots \sigma_{y_h}\). Since the set of previous factors starting with the character \(\sigma_2\) is \(\{(\sigma_2), (\sigma_2, \sigma_3), \ldots, (\sigma_2, \cdots, \sigma_h)\}\), the next factor becomes \(\sigma_2 \cdots \sigma_{h+1}\). In addition, it is guaranteed that the set of previous factors starting with the character \(\sigma_i\) for every \(h + 2 \leq i \leq y_h - \ell_h + 1\) is equal to \(\{(\sigma_i), (\sigma_i, \sigma_{i+1}), \ldots, (\sigma_i, \cdots, \sigma_{i+\ell_i} - 2\}\). Since \(y_h\) can be described as \(h + \ell_h + 1 + t\ell_h\) for some integer \(t\), the decomposition of the interval becomes \(\sigma_1|\sigma_2 \cdots \sigma_{h+1}|\sigma_{h+2} \cdots \sigma_{h+\ell_h+1} \cdots |\sigma_{y_h - \ell_h+1} \cdots \sigma_{y_h}|\sigma_{k+h}\), and this satisfies the partition (1). From the above, the partition (1) is valid for \(2 \leq j \leq k\) by induction.

Thus, the LZ78 factorization of \(T'\) is

\[
\text{LZ78}(T') = \sigma_{k+1} \cdots |\sigma_{2k} |\sigma_1|\sigma_1 \sigma_2| \cdots |\sigma_{y_1} |\sigma_{k+1} \sigma_1 |\sigma_{2} \sigma_3| \cdots |\sigma_{y_2-1} \sigma_{y_2} |
\]

\[
\sigma_{k+2} \sigma_1 \cdots |\sigma_{2} \cdots \sigma_{j+1}| \sigma_{j+2} \cdots \sigma_{j+\ell_j+1} \cdots |\sigma_{y_j - \ell_j+1} \cdots \sigma_{y_j} |\sigma_{k+3} \sigma_1 \cdots | \cdots |\sigma_{y_k} |\sigma_{2k} .
\]

See also Figure 2 for a concrete example.

The size of LZ78\((T')\) is \(z_{78}(T') = 2k + y_1 + \sum_{j=2}^{k} \left(2 + \frac{y_j - j - 1}{\ell_j}\right) + 1 = 5k - 1 + \sum_{j=2}^{k} ((y_j - j - 1)/\ell_j)\). The total number of factors of length \(\ell_j\) for \(1 \leq j \leq k\) is equal to \(\sum_{j=2}^{k} ((y_j - j - 1)/\ell_j)\). The total number of factors of length \(L\) in \(L \in \{\ell_1, \ldots, \ell_k\}\). For all \(j\) such that \((1/2)L(L-1)+1 \leq j \leq (1/2)L(L+1)\), the total number of factors of length in \(L \in \sum_{j=2}^{k} ((y_j - j - 1)/L)\), where \(j_{\text{min}}^{L} = (1/2)L(L-1)+1\) and \(j_{\text{max}}^{L} = (1/2)L(L+1)\). From the definition of \(y_j\), \(\sum_{j} y_j = k + (k - 1) + \cdots + (k - L + 1)\) holds. Therefore, \(\sum_{j=1}^{L} j_{\text{min}}^{L} = (k + (k - 1) + \cdots + (k - L + 1) - j_{\text{min}}^{L} - j_{\text{max}}^{L} - L)/L = (L(k - 1)/L) - L = k - (1/2)L(L+1) - 1\). Let \(\ell_k = m\). Then the total number of factors of length \(\ell_j\) for \(1 \leq j \leq k\) is \(\sum_{j=2}^{m} \left(k + (k - 1) - (1/2)L(L+1) - 1\right) = (m - 1)k - (1/2)m(m+1)(2m+1) - (1/4)m(m+1) - m + 2\). Consider the case of \(m = \sqrt{k}\), then \(z_{78}(T') \in \Omega(k\sqrt{k})\). Thus we obtain \(M_{\text{sub}}(z_{78}, n) = \Omega(n^{1/4})\), \(A_{\text{sub}}(z_{78}, n) = \Omega(n^{3/2})\), and \(A_{\text{del}}(z_{78}, n) = \Omega(n^{3/4})\) also hold.

As for deletions, by considering \(T'\) obtained from \(T\) by deleting the first character of the \(2k+1\)th factor in \(\text{LZ78}(T)\), we obtain a similar decomposition as the above. Thus, \(M_{\text{del}}(z_{78}, n) = \Omega(n^{1/4})\), \(A_{\text{del}}(z_{78}, n) = \Omega(n^{3/2})\), and \(A_{\text{del}}(z_{78}, n) = \Omega(n^{3/4})\) also hold.

\[
\text{LZ78}(T') = \sigma_{51} | \cdots | \sigma_{100} | \sigma_1 | \sigma_1 \sigma_2 | \cdots | \sigma_1 \cdots \sigma_{50} |
\]

\[
\# | \sigma_2 | \sigma_3 | \cdots | \sigma_{50} | \sigma_1 \sigma_2 | \sigma_2 | \sigma_3 | \sigma_4 | \cdots | \sigma_{48} | \sigma_{49} | \sigma_{50} | \sigma_{51} | \sigma_{52} | \sigma_5 \sigma_6 | \cdots | \sigma_{48} | \sigma_{49} | \sigma_{50} | \sigma_{51} |
\]

\[
\sigma_2 | \sigma_3 | \sigma_4 | \sigma_5 | \sigma_6 | \cdots | \sigma_{48} | \sigma_{49} | \sigma_{50} | \sigma_{51} | \sigma_{52} | \sigma_5 | \sigma_6 | \cdots \]

\[
\cdots
\]

Figure 2: Illustration for \(\text{LZ78}(T')\) for the string \(T'\) of Theorem 25 with \(k = 50\).

We remark that our string also achieves \(M_{\text{ins}}(z_{78}, n) = \Omega(n^{1/4})\), \(A_{\text{ins}}(z_{78}, n) = \Omega(n^{3/2})\), and
Theorem 26. Let $T'$ be any string of length $n$, and let $G^*(T')$ be a grammar of size $g^*(T')$ that only generates $T'$.

We describe the case of substitutions. Let $T'$ be the string that can be obtained by substituting a character $c$ for the $i$th character $T[i]$ of $T$, where $c \neq T[i]$. Let $X$ be a non-terminal of $G^*(T)$ in the path $P$ from the root to the leaf for the $i$th character in the derivation tree of $G^*(T)$. Let $X \rightarrow Y_1 \cdots Y_k$ be the production from $X$, and let $Y_j$ ($1 \leq j \leq k$) be the non-terminal that is the child of $X$ in the path $P$. Then, we introduce a new non-terminal $X'$ and a new production $X' \rightarrow Y_1 \cdots Y_{j-1} Y_j' Y_{j+1} \cdots Y_k$, where $Y_j'$ will be the new non-terminal at the next depth in the path $P$. By applying this operation in a top-down manner on $P$, we can obtain a grammar $G(T')$ of size $g(T') \leq 2g^*(T)$ that generates $T'$. Since $g^*(T') \leq g(T')$, we have the claimed bounds. The cases with insertions and deletions are analogous.

Proof. Let $T$ be any string of length $n$, and let $G^*(T)$ be a grammar of size $g^*(T)$ that only generates $T$.

We consider the case of substitutions. Let $T'$ be the string that can be obtained by substituting a character $c$ for the $i$th character $T[i]$ of $T$, where $c \neq T[i]$. Let $X$ be a non-terminal of $G^*(T)$ in the path $P$ from the root to the leaf for the $i$th character in the derivation tree of $G^*(T)$. Let $X \rightarrow Y_1 \cdots Y_k$ be the production from $X$, and let $Y_j$ ($1 \leq j \leq k$) be the non-terminal that is the child of $X$ in the path $P$. Then, we introduce a new non-terminal $X'$ and a new production $X' \rightarrow Y_1 \cdots Y_{j-1} Y_j' Y_{j+1} \cdots Y_k$, where $Y_j'$ will be the new non-terminal at the next depth in the path $P$. By applying this operation in a top-down manner on $P$, we can obtain a grammar $G(T')$ of size $g(T') \leq 2g^*(T)$ that generates $T'$. Since $g^*(T') \leq g(T')$, we have the claimed bounds. The cases with insertions and deletions are analogous.

11.2 Practical grammars

Since computing a smallest grammar of size $g^*(T)$ is NP-hard, a number of practical grammar-based compressors have been proposed, including RePair [41], Longest-Match [33], Greedy [2], Sequential [66, 4], and LZ78 [69]. Charikar et al. [11] analyzed the approximation ratios of these grammar compressors to the smallest grammar. Let $g_{\text{pair}}, g_{\text{long}}, g_{\text{grdy}}, g_{\text{seq}}, g_{78}$ denote the sizes of the aforementioned compressors, respectively. It is known that for any $g \in \{g_{\text{pair}}, g_{\text{long}}, g_{\text{grdy}}, g_{78}\}$, $g(T) = O(g^*(T)(n/\log n)^{3/5})$ holds, and $g_{\text{seq}} = O(g^*(T)(n/\log n)^{3/5})$ holds [11]. By combining these results with Lemma [1] and Theorem [26], we obtain the following bounds:

$\text{AS}_{\text{ins}}(z_{78}, n) = \Omega(n^{3/4})$ for insertions, if we consider the string $T'$ obtained from $T$ by inserting $\#$ between the first and second characters of the $2k + 1$th factor of LZ78($T$).

In Section 11, we will present an $O((n/\log n)^{3/5})$ upper bound for the multiplicative sensitivity for LZ78.
Corollary 5. The following upper bounds for the sensitivity of $g \in \{g_{pair}, g_{long}, g_{grdy}, \tau_\sigma\}$ hold:

- **Substitutions:** $\text{MS}_{\text{sub}}(g, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{sub}}(g, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.
- **Insertions:** $\text{MS}_{\text{ins}}(g, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{ins}}(g, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.
- **Deletions:** $\text{MS}_{\text{del}}(g, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{del}}(g, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.

Corollary 6. The following upper bounds for the sensitivity of $g_{\text{seq}}$ hold:

- **Substitutions:** $\text{MS}_{\text{sub}}(g_{\text{seq}}, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{sub}}(g_{\text{seq}}, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.
- **Insertions:** $\text{MS}_{\text{ins}}(g_{\text{seq}}, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{ins}}(g_{\text{seq}}, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.
- **Deletions:** $\text{MS}_{\text{del}}(g_{\text{seq}}, n) = O((n / \log n)^{\frac{3}{4}})$. $\text{AS}_{\text{del}}(g_{\text{seq}}, n) = O(g^* \cdot (n / \log n)^{\frac{3}{4}})$.

11.3 Approximation grammars

There also exist (better) approximation algorithms in terms of the smallest grammar size $g^*$.

It is known that $\alpha$-balanced grammar compressor [11], the AVL-grammar compressor [62], and the really-simple grammar compressor [26] all achieve $O(\log(n/g^*))$-approximation ratios to $g^*$. Let $g_\alpha$, $g_{avl}$ and $g_{simple}$ denote the sizes of these compressors, respectively. Namely, for every $g \in \{g_\alpha, g_{avl}, g_{simple}\}$, $g = O(g^* \log(n/g^*))$ holds. Since $\log(n/g^*)$ satisfies the conditions for the function $f(n, g^*)$ in Lemma [1] and since $g^*$ satisfies the conditions Lemma [1] by Theorem [26] we obtain the following:

Corollary 7. The following upper bounds for the sensitivity of $g \in \{g_\alpha, g_{avl}, g_{simple}\}$ hold:

- **Substitutions:** $\text{MS}_{\text{sub}}(g, n) = O(\log(n/g^*))$. $\text{AS}_{\text{sub}}(g, n) = O(g^* \log(n/g^*))$.
- **Insertions:** $\text{MS}_{\text{ins}}(g, n) = O(\log(n/g^*))$. $\text{AS}_{\text{ins}}(g, n) = O(g^* \log(n/g^*))$.
- **Deletions:** $\text{MS}_{\text{del}}(g, n) = O(\log(n/g^*))$. $\text{AS}_{\text{del}}(g, n) = O(g^* \log(n/g^*))$.

12 Grammar compression by induced sorting (GCIS)

In this section, we consider the worst-case sensitivity of the grammar compression by induced sorting (GCIS) [58, 59]. GCIS is based on the idea from the famous SAIS algorithm [57] that builds the suffix array of an input string in linear time. Recently, it is shown that GCIS has a locally consistent parsing property similar to the ESP-index [44] and the SE-index [54], and grammar-based indexing structures based on GCIS have been proposed [11, 14].

Let $T$ be the string of length $n$ over an integer alphabet $\Sigma = \{1, \ldots, \sigma\}$. Let $\Pi = \{\sigma + 1, \ldots, \sigma + |\Pi|\}$ be the set of non-terminal symbols. For strings $x, y$ over $\Sigma$ or $\Pi$, we write $x < y$ iff $x$ is lexicographically smaller than $y$.

First we explain how the GCIS algorithm constructs its grammar from the input string. For any text position $1 \leq i \leq |T|$, position $i$ is of type $L$ if $T[i..|T|]$ is lexicographically larger than $T[i+1..|T|]$, and it is of type $S$ otherwise. For any $2 < i < |T|$, we call position $i$ an LMS (LeftMost S) position if $i$ is of type $S$ and $i - 1$ is of type $L$. For convenience, we append a special character $\$ to $T$ which does not occur elsewhere in $T$, and assume that positions 1 and $|T\$|$\$ are LMS positions.

Let $i_1, \ldots, i_{z+1}$ be the sequence of the LMS positions in $T$ sorted in increasing order. Let $D_j = T[i_j..i_{j+1} - 1]$ for any $1 \leq j \leq z$. When $z \geq 2$, then $T = D_1, \ldots, D_z$ is called the GCIS-parsing of $T$.

Next, we create new non-terminal symbols $R_1, \ldots, R_z$ such that $R_i = 1 + \sigma + \{|D_j : D_j < D_i : 1 \leq j \leq z\}$ for each $i$. Intuitively, we pick the least unused character from $\Pi$ and assign it to $R_i$. Then, $G_1 = R_1 \cdots R_z$ is called the GCIS-string of $T$. Let $G_1$ the set of all $z$ symbols in $G_1$, and $P_1 = \{R_i \rightarrow D_i : 1 \leq i \leq z\}$ is the set of production rules. Let $D_1 = \{D_1, \ldots, D_z\}$ be the set of all distinct factors. Let $G_0 = T$, then we define GCIS recursively, as follows:
**Definition 1.** For \( k \geq 0 \), let the sequence \( i_1, i_2, \ldots, i_{z_k+1} \) be all LMS positions sorted in increasing order, and \( D_j = G_k[i_j \ldots i_{j+1} - 1] \) for any \( 1 \leq j \leq z_k \). \( G_k = D_1, D_2, \ldots, D_{z_k} \) is the GCIS-parsing of \( G_k \). For all \( i \) in \( 1 \leq i \leq z_k \), we define \( R_i \) to satisfy:

\[
R_i = |\{D_j : D_j < D_i : 1 \leq j \leq z_k\}| + \sum_{t=1}^{k-1} |P_t| + \sigma + 1.
\]

Then, \( G_{k+1} = R_1 \ldots R_{z_k} \) is the GCIS-string of \( G_k \). \( G_{k+1} \) is the set of non-terminals, \( P_k = \{R_i \rightarrow D_i : 1 \leq i \leq z_k\} \) is the set of production rules. \( D_k = \{D_1, \ldots, D_{z_k}\} \) is the set of all distinct factors in the GCIS-parsing of \( G_k \).

Again, each \( R_i \) is chosen to be the least unused character from \( \Pi \). \( G_{k+1} \) is not defined if there are no LMS positions in \( G_k[2..|G_k|] \). Then, the GCIS grammar of \( T \) is \( (\Sigma, \bigcup_{t=1}^{k} \mathcal{G}_t, \bigcup_{t=1}^{k-1} P_t, G_k) \). \( T \) is derived from the recursive application of the rules \( \bigcup_{t=1}^{k-1} P_t \), which is the third argument, to the fourth argument \( G_k \), which is the start string, until there are no non-terminal characters, which is in the second argument \( \bigcup_{t=1}^{k} \mathcal{G}_t = \Pi \), in the string. Let \( r = k \) be the height of GCIS, in other words how many times we applied this GCIS method recursively to \( T \). Let \( g_{bs}(T) \) be the size of GCIS grammar of \( T \). Then, if \( r = 0 \), \( g_{bs}(T) = |T| \), and if \( r \geq 1 \), \( g_{bs}(T) = |D_1| + \cdots + |D_r| + G_r \), where \( ||S|| \) for a set of strings denotes the total length of the strings in \( S \).

Figure 3 shows an example on how GCIS is constructed from an input string.

**Figure 3:** Construction of GCIS from string \( T = G_0 = 21121211212112128 \). In this case, there are 8 LMS positions \( i_1, \ldots, i_8 \) in \( T \) and 7 factors \( D_1, \ldots, D_7 \). \( \mathcal{D}_1 = \{112, 12, 2\} \) is the set of distinct factors of the GCIS-parsing for \( T \), and \( G_1 = R_1 \cdots R_7 = 6454545 \) is the GCIS-string of \( T \). Recursively, \( \mathcal{D}_2 = \{6, 45\}, G_2 = 8777 \), and the start string of the GCIS for \( T \) is \( G_2 \) because the number of factors of the GCIS-parsing of \( G_2 \) is 1 (excluding \$), in other words there are no LMS positions in \( G_2[2..|G_2|] \). The size of the GCIS grammar of \( T \) is \( g_{bs}(T) = ||\mathcal{D}_1|| + ||\mathcal{D}_2|| + |G_2| = 6 + 3 + 4 = 13 \).

From now on, we consider to perform an edit operation to the input string \( T \) and will consider how the GCIS changes after the edit.

**Definition 2.** Let \( S \) and \( S' \) be strings. If \( S' \) is obtained from \( S \) by deleting the substring of length \( a \) starting from a position \( c \) in \( S \) and by inserting a string of length \( b \) to the same position \( c \), then we write \( F(S, S') = (a, b) \).

Our single-character edit operation performed to \( T \) can be described as \( F(T, T') = (1, 1) \) for substitution, \( F(T, T') = (0, 1) \) for insertion, and \( F(T, T') = (1, 0) \) for deletion. We will use this notation \( F \) to the GCIS-strings for \( T \) and \( T' \), in which case \( a, b \) can be larger than 1. Still, we will prove that \( a, b \) are small constants for the GCIS-strings.
As with the definitions for $T$, $T' = D'_1, \ldots, D'_z$ is the GCIS-parsing of $T'$, $G'_1 = R'_1 \cdots R'_z$ is the GCIS-string of $T'$, $G'_1$ is the set of non-terminals for $T'$, $D_1 = \{D'_1, \ldots, D'_z\}$ is the set of all distinct factors of the GCIS-parsing of $T'$, $P'_i = \{R_i \rightarrow D_i : 1 \leq i \leq z'\}$ is the set of production rules. Let $G'_0 = T$, then we can recursively define $G'_1, G'_2, \ldots, G'_r$, similarly to $T$, where $r'$ is the height of the GCIS for $T'$.

### 12.1 Upper bounds for the sensitivity of $g_{ls}$

This section presents the following upper bounds for the sensitivity of GCIS.

**Theorem 27.** The following upper bounds on the sensitivity of GCIS hold:

- **substitutions:** $M_{sub}(g_{ls}, n) \leq 4$. $A_{sub}(g_{ls}, n) \leq 3g_{ls}$.
- **insertions:** $M_{ins}(g_{ls}, n) \leq 4$. $A_{ins}(g_{ls}, n) \leq 3g_{ls}$.
- **deletions:** $M_{del}(g_{ls}, n) \leq 4$. $A_{del}(g_{ls}, n) \leq 3g_{ls}$.

We will prove this theorem as follows: We unify substitutions, insertions, and deletions by using the $F$ function in Definition 2. First, we prove that edit operations do not affect the size of the GCIS grammar. Second, we divide the size of GCIS grammar $g_{ls}(T)$ into $\|D_1\|$ and $g_{ls}(G_1)$, and prove that $\|D'_1\| \leq 4\|D_1\| + O(1)$. Then, $g_{ls}(T') = \|D'_1\| + g_{ls}(G'_1) \leq 4\|D_1\| + g_{ls}(G'_1) + O(1)$ holds. The essence is to find the two special strings $\hat{G}_1$ and $\hat{G}'_1$ which satisfy:

- $\hat{G}'_1$ can be obtained from $\hat{G}_1$ by some substitutions, insertions, and deletions.
- $g_{ls}(G_1) = g_{ls}(\hat{G}_1)$, and $g_{ls}(G'_1) = g_{ls}(\hat{G}'_1)$.

Then, we can apply the method to each height. The extra additive $O(1)$ factor can be charged to the process of the GCIS compression, which is to be proved in Lemma 12. Finally, we will obtain $g_{ls}(T') \leq 4g_{ls}(T)$.

**Lemma 2 ([57]).** The type of $T[k]$ is $S$ if $T[k] < T[k+1]$ and $L$ if $T[k] > T[k+1]$. If $T[k] = T[k+1]$, the type of $T[k]$ equals to the type of $T[k + 1]$.

Let $\text{rank}_T[i]$ be the lexicographical rank of the character $T[i]$ at position $i$ in $T$. Let $\hat{T}$ be any string of length $|\hat{T}| = |T|$ such that $\text{rank}_{\hat{T}}[i] = \text{rank}_T[i]$ for every $1 \leq i \leq |T|$.

**Lemma 3.** Let $G_1$ and $\hat{G}_1$ denote the GCIS-strings of $T$ and $\hat{T}$, respectively. Then $\hat{G}_1$ is the string that can be obtained by replacing the characters in $G_1$ without changing the ranks of any characters in $G_1$, and $g_{ls}(\hat{T}) = g_{ls}(T)$.

**Proof.** The lemma immediately follows from Lemma 2 and that $\text{rank}_{\hat{T}}[i] = \text{rank}_T[i]$ for every $1 \leq i \leq |T|$. 

Figure 4 shows a concrete example for Lemma 3.

A natural consequence of Lemma 3 is that edit operations which do not change the relative order of the characters in $T$ do not affect the size of the grammar.

From now on, we analyze how the size of the GCIS of the string $T$ can increase after the edit operation in the string $T'$. In the following lemmas, let $1 \leq h \leq r$, where $r$ is the height of the GCIS grammar for $T$.

**Lemma 4.** If $F(G_h, G'_h) = (x, y)$, then $|D_{h+1} \setminus D'_{h+1}| \leq 2 + \lceil (x + 1)/2 \rceil$. 

37
and we cannot create holds. See Figure 5 for illustration. However, in that case, $G$ at the same time. Therefore, it is impossible to create $i$ in $G$ where $h$ are not LMS positions in $G$. The numbers of factors in the GCIS-parsing of $G$ holds, and $G_1$ is recursively the string that can be obtained by replacing some characters in $G_1$ without changing the relative order of any characters. Therefore, we can consider the size of GCIS of $T$ using such a string $\hat{T}$ instead of $T$ itself.

**Proof.** First, let $c$ be the position such that $G'_h$ can be obtained from $G_h$ by deleting a substring of length $x$ from position $c$ and inserting a substring of length $y$ to position $c$. Let $z$ and $z'$ be the numbers of factors in the GCIS-parsing of $G_h$ and $G'_h$, respectively.

Considering $k$ where $i_k < c < i_{k+1}$ in $G_h$, the LMS positions $i_1, \ldots, i_{k-1}$ are also the LMS positions in $G'_h$, and for all $j$ where $1 \leq j \leq k - 2$, $D_j = D'_j$ holds. Similarly, for $l$ where $i_{z-l-1} \leq c + x < i_{z-l}$ in $G_h$, the positions $i_{z-l}, \ldots, i_z$ and corresponding positions $i_{z'-l}, \ldots, i_{z'}$ in $G'_h$ are also LMS positions. Therefore, for $l \leq j \leq z$ and $j' = j + (z' - z)$, $D_j = D'_j$. Note that $i_{z-l-1} - i_{k+1} < x$. Since $|D_j| \geq 2$, $|D'_j| \geq 2$ with $2 \leq j \leq z$, we obtain $|D_{h+1} \setminus D'_{h+1}| \leq 2 + \lfloor (x + 1)/2 \rfloor$.

**Lemma 5.** If $F(G_h, G'_h) = (x, y)$, $|D'_h| \leq 4|D_h| - x - y$.

**Proof.** Considering $k$ where $i_k < c < i_{k+1}$ in $G_h$ and $l$ where $i_{z-l-1} \leq c + x < i_{z-l}$, the total length of new factors to be added in $D'_h$, is at most $i_{z-l} - x + y - i_{k-2} \leq 2|D_h| - x - y$.

**Lemma 6.** If $F(G_h, G'_h) = (x, y)$, $|G'_{h+1}| \leq |G_{h+1}| + 1 + \lfloor y/2 \rfloor$.

**Proof.** Assume $|G'_{h+1}| > |G_{h+1}| + 1 + \lfloor y/2 \rfloor$. In other words, there are at least $2 + \lfloor y/2 \rfloor$ positions which are not LMS positions in $G_h$ but are LMS positions in $G'_h$. Let $i$ be the right-most position where $G_h[i] \neq G_h[c]$ and $i < c$. For each $i \leq k \leq i$, $G_h[k]$ and $G'_h[k]$ are of the same type.

For all $k$ with $c < k \leq |G_h| - x$, $G_h[k + x]$ and $G'_h[k + y]$ are of the same type. Therefore, only $G_h[i..c + x + 1]$ and $G'_h[i..c + y + 1]$ can introduce new factors. Note that $G_h[i+1..c-1]$ and $G'_h[i+1..c-1]$ are of the same types by Lemma 3. There are $y + 2$ positions $i+1, c, c+1, \ldots, c+y$ that can be new LMS positions in $G_h$. Since any LMS position must be the left-most position of consecutive type $S$ positions, two possible positions adjacent each other cannot be LMS positions at the same time. Therefore, it is impossible to create $2 + \lfloor y/2 \rfloor$ new LMS positions if $y$ is even. If $y$ is odd, we can make $2 + \lfloor y/2 \rfloor$ new LMS positions in $G'_h$ by selecting $i+1, c+1, c+3, \ldots, c+y$. However, in that case, $G_h[i+1]$ must be of type $L$, and $G_h[c+x]$ must be of type $S$ since $G_h[c+x]$ and $G'_h[c+y]$ are of the same type. Then, there is at least an LMS position between $T[c]$ and $T[c+y]$, and we cannot create $2 + \lfloor y/2 \rfloor$ new LMS positions in $G'_h$, or there is at least an LMS position in $G_h[c..c+x]$ that is not in $G'_h$. Therefore, whichever $y$ is even or odd, $|G'_{h+1}| \leq |G_{h+1}| + 1 + \lfloor y/2 \rfloor$ holds. See Figure 5 for illustration.

\[
T = \begin{array}{c}
2 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 3 & 1 & 3 & $ \\
\end{array}
\]

\[
G_1 = \begin{array}{c}
6 & 4 & 5 & 4 & 5 & 4 & 5 & $ \\
\end{array}
\]

\[
\hat{T} = \begin{array}{c}
2 & 1 & 4 & 5 & 1 & 5 & 1 & 4 & 5 & 1 & 5 & 1 & 4 & 5 & 1 & 5 & $ \\
\end{array}
\]

\[
\hat{G}_1 = \begin{array}{c}
8 & 6 & 7 & 6 & 7 & 6 & 7 & $ \\
\end{array}
\]

Figure 4: Two strings $T$ and $\hat{T}$ that can be obtained by replacing some characters in $T$ without changing the relative order of any characters, result in the same number of factors in the GCIS-parsing, and each length exactly matches in both of the strings. Therefore, $|D_1| = |\hat{D}_1|$ and $|G_1| = |\hat{G}_1|$ holds, and $G_1$ is recursively the string that can be obtained by replacing some characters in $G_1$ without changing the relative order of any characters. Therefore, we can consider the size of GCIS of $T$ using such a string $\hat{T}$ instead of $T$ itself.
Lemma 7. If $F(G_h,G'_h) = (x,y)$, let $a = |D_{h+1}\setminus D'_{h+1}|$, $b = |D'_{h+1}\setminus D_{h+1}|$. Then $a \leq 2 + \lceil (x+1)/2 \rceil$, $b \leq 2 + \lceil (y+1)/2 \rceil$, $a + b \leq 4 + \lceil (x+y)/2 \rceil$ hold.

Proof. We immediately get $a \leq 2 + \lceil (x+1)/2 \rceil$, $b \leq 2 + \lceil (y+1)/2 \rceil$ by a direct application of Lemma 4. Assume $y \mod 2 = 1$, $b \leq 2 + \lceil (y+1)/2 \rceil$. Then, Lemma 6 shows that there is only one possible combination of new $b$ LMS positions $i+1, c+1, c+3, \ldots, c+y$ in $G'_h$. For that, neither $i+1$ nor $c+y$ can be LMS positions in $G_h$ in this case since they must be new LMS positions in $G'_h$. Therefore, $a \leq 2 + \lceil (x+1)/2 \rceil - 1$ since there are no possible combination of $a+1$ LMS positions in $G_h$. Assume $x \mod 2 = 1$ and $a \leq 2 + \lceil (x+1)/2 \rceil$. Then, Lemma 6 shows that there is only one possible combination of $a$ disappearing LMS positions $i+1, c+1, c+3, \ldots, c+x$ in $G_h$. For that, neither $i+1$ nor $c+x$ can be LMS positions in $G'_h$ in this case since they must be disappearing LMS positions in $G_h$. Therefore, $a \leq 2 + \lceil (x+1)/2 \rceil - 1$ since there are no possible combination of new $b+1$ LMS positions in $G'_h$.

Lemma 8. If $F(G_h,G'_h) = (x,y)$, there are two strings $\hat{G}_{h+1}, \hat{G}'_{h+1}$ such that $\hat{G}_{h+1}, \hat{G}'_{h+1}$ can be obtained by replacing some characters in $G_{h+1}, G'_{h+1}$ without changing the relative order of any characters in $G_{h+1}, G'_{h+1}$, respectively, and $F(\hat{G}_{h+1}, \hat{G}'_{h+1}) = (a,b)$, where $a \leq 2 + \lceil (x+1)/2 \rceil$, $b \leq 2 + \lceil (y+1)/2 \rceil$, and $a + b \leq 4 + \lceil (x+y)/2 \rceil$.

Proof. Assume $G_h = D_1, \ldots, D_z$ and $G'_h = D'_1, \ldots, D'_z$ are the GCIS-parsings of $G_h$ and $G'_h$, respectively. By Lemma 4, there are at most $j = 2 + \lceil (x+1)/2 \rceil$ consecutive factors $D_{i}, \ldots, D_{i+j-1}$ in $D_{h+1}\setminus D'_{h+1}$, and at most $\hat{j} = 2 + \lceil (y+1)/2 \rceil$ consecutive factors $D_{i}, \ldots, D_{i+\hat{j}-1}$ in $D'_{h+1}\setminus D_{h+1}$ and $D_k = D'_k$ for all $1 \leq k \leq \max(i,\hat{i})$, and $D_z-k = D'_{z'-k}$ for all $0 \leq k \leq \max(z-i-j-1, z'-\hat{i}-\hat{j}-1)$. By Lemma 7, $j + \hat{j} \leq 4 + \lceil (x+y)/2 \rceil$. Let

\[
\hat{S}_p = |\{D_s : D_s \prec D_p(1 \leq s \leq z)\}| + |\{D'_s : D'_s \prec D'_p(1 \leq s \leq z')\}|, \\
\hat{S}'_p = |\{D_s : D_s \prec D'_p(1 \leq s \leq z)\}| + |\{D'_s : D'_s \prec D'_p(1 \leq s \leq z')\}|
\]

Then, the string $\hat{G}_{h+1} = \hat{S}_1 \ldots \hat{S}_z$ can be obtained from $G_{h+1}$ by replacing some characters in $G_{h+1}, G'_h$ without changing the relative order of any characters, and $G'_{h+1}$ as well. In addition, $F(\hat{G}_{h+1}, \hat{G}'_{h+1}) = (j,\hat{j})$ holds because $R_k = R'_k$ for all $1 \leq k \leq \max(i,\hat{i})$, and $R_{z-k} = R'_{z'-k}$ for all $0 \leq k \leq \max(z-i-j-1, z'-\hat{i}-\hat{j}-1)$. See Figure 6.\[\square\]
Figure 6: Examples of $G_{h+1}^*$ and $\hat{G}_{h+1}^*$ for strings $G_h$ and $G_h'$, where $G_h'$ can be obtained from $G_h$ by substituting a 1 with a 4. The box under the 2 other boxes shows the “common” productions common to $G_{h+1}^*$ and $\hat{G}_{h+1}^*$, e.g., by applying $1 \rightarrow 123$ to the occurrences of 1 in $G_{h+1}^*$ and $\hat{G}_{h+1}^*$, we obtain the corresponding substrings 123 in $G_h$ and $G_h'$. Since the common number is assigned to each equal factors in $G_h$ and $G_h'$, the size of the symmetric difference of $\hat{G}_{h+1}$ and $\hat{G}_{h+1}'$ equals to the number of factors changed by substitution, insertion, or deletion from $G_h$ to $G_h'$, which is four in this example.

Lemma 9. If $F(T, T') \in \{(1, 1), (1, 0), (0, 1)\}$, then there are two strings $\hat{G}_1, \hat{G}_1'$ such that $\hat{G}_1, \hat{G}_1'$ can be obtained by replacing some characters in $G_1, G_1'$ without changing the relative order of any characters in $G_1, G_1'$, respectively, and $F(\hat{G}_1, \hat{G}_1') = (a, b)$, where $a \leq 4, b \leq 4, a + b \leq 7$.

Proof. Immediately follows from Lemma 8.

Lemma 10. If $F(G_h, G_h') = (x, y)$ and $r = h + 1$, then $g_{is}(G_h') \leq g_{is}(G_h) + 2(1 + y - |G_{h+1}|)$.

Proof. By construction of GCIS, $g_{is}(G_{h+1}') \leq 2|G_{h+1}'|$. Remembering $||D'_h|| \leq 4||D_h|| - x + y$,

$$g_{is}(G_h') = ||D'_h|| + g_{is}(G_{h+1}')$$

$$\leq 4||D_{h+1}|| - x + y + 2|G_{h+1}'|$$

$$\leq 4||D_{h+1}|| - x + y + 2(|G_{h+1} + 1 + [y/2])$$

$$= 4||D_{h+1}|| - x + y + 2|G_{h+1}| + 2 + y$$

$$= 4(||D_{h+1}|| + |G_{h+1}|) - 2|G_{h+1}| + 2 - x + 2y$$

$$= 4g_{is}(G_h) - 2|G_{h+1}| + 2 - x + 2y$$

$$\leq 4g_{is}(G_h) + 2(1 + y - |G_{h+1}|).$$

Lemma 11. If $F(T, T') = (x, y)$, $x \leq 1, y \leq 1$, and $r = 1$, then $g_{is}(T') \leq 4g_{is}(T)$ holds.
Lemma 3. Furthermore, Lemma 12 shows that
\[ g(T') = |D'_1| + g_a(G'_1) \]
\[ \leq 4|D_1| - x + y + 2|G'_1| \]
\[ \leq 4|D_1| - x + y + 2(|G_1| + 1 + \lceil y/2 \rceil) \]
\[ \leq 4|D_1| + 2|G_1| + 3 \]
\[ = 4|D_1| + 4|G_1| + 3 - 2|G_1| \]
\[ = 4g_a(T) + 3 - 2|G_1|. \]

Therefore, if \(|G_1| \geq 2\), then \(g_a(T') \leq 4g_a(T)\). If \(|G_1| = 1\), which is a special case, then \(g_a(T') \leq |D_1| - x + y + 2|G'_1| \leq 4|D_1| + 4|G_1| = 4g_a(T)\).

\[ \square \]

Lemma 12. If \(F(G_h, G'_h) = (x, y), \ |D_h| \geq 2 \text{ and } |D'_h| \leq 4(|D_h| - 2) - x + y\).

Proof. If \(|D_h| = 1\), then \(G_{h+1}\) must be a unary string, and therefore no \(G_{h+2}\) is constructed. If \(|D_h| = 2\) and there is a factor of length 1 in \(D_h\), then \(G_{h+1}\) is still a unary string except for the first position, and therefore no \(G_{h+2}\) is constructed. Therefore, \(G_{h+2}\) is constructed only if \(|D_h| \geq 2\) and there are at least two factors of length at least 2, and hence \(||D'_h|| \leq 4(|D_h| - 2) - x + y\) holds.

If \(F(T, T') = (x, y) \in \{(1, 1), (1, 0), (0, 1)\}\), then \(|D_h| - 4(|D_h| - 2) - x + y = 8 + x - y \geq 7\). It means that \(D'_1\) can afford to 7 character room to charge. Lemma 13 shows that we can use the room to charge the extra additive factor of 7 in \(|G'_r|\), and leads us to the desired upper bound \(g_a(T') \leq 4g_a(T)\), as follows:

Lemma 13. If \(F(T, T') \in \{(1, 1), (1, 0), (0, 1)\}\), then \(g_a(T') \leq 4g_a(T)\).

Proof. By Lemma 11, the lemma holds when \(r = 1\). If \(r \geq 2\), Lemma 9 shows that there are two strings \(G_1, G'_1\), and \(F(G_1, G'_1) = (a, b)\), where \(a \leq 4, b \leq 4, a + b \leq 7\) and \(g_a(G_1) = g_a(G'_1)\) by Lemma 3. Additionally, Lemma 12 shows that \(||D'_1|| = |D'_1| \leq 4||D_1|| - 7 = 4||D_1|| - 7\). Since \(F(G_1, G'_1) = (a, b)\), there are also two strings \(G_h, G'_h\), and \(F(G_h, G'_h) = (a, b)\), where \(a \leq 4, b \leq 4, a + b \leq 7\) and \(g_a(G_h) = g_a(G'_h)\), \(g_a(G_h) = g_a(G'_h)\) with \(2 \leq h \leq r - 1\) by Lemma 3. Furthermore, Lemma 12 shows that \(||D'_1|| = |D'_1| \leq 4||D_1|| - 4 = 4||D_1||\). Noting that \(g_a(G'_r) \leq 2|G'_r| = 2|G_r| = 2g_a(G_r)\) and \(|G_k| \geq 1\), we obtain:

\[ g_a(T') = ||D'_1|| + g_a(G'_1) \]
\[ \leq (4||D_1|| - 7) + g_a(G'_1) \]
\[ \leq (4||D_1|| - 7) + (4||D_2|| - 4) + g_a(G'_2) \]
\[ \leq \sum_{i=1}^{r-1} (4||D_i||) - 7 + g_a(G'_r) \]
\[ \leq \sum_{i=1}^{r-1} (4||D_i||) - 7 + 2|G_r| + 8 \]
\[ = \sum_{i=1}^{r-1} (4||D_i||) + 2|G_r| + 1 \]
\[ < \sum_{i=1}^{r-1} (4||D_i||) + 4|G_r| \]
\[ = 4g_a(T). \]
12.2 Lower bounds for the sensitivity of \( g_s \)

**Theorem 28.** The following lower bounds on the sensitivity of GCIS hold:
- **substitutions:** \( \liminf_{n \to \infty} \text{MS}_{\text{sub}}(g_s, n) \geq 4 \). \( \text{AS}_{\text{sub}}(g_s, n) \geq 3g_s - 13 = \Omega(n) \).
- **insertions:** \( \liminf_{n \to \infty} \text{MS}_{\text{ins}}(g_s, n) \geq 4 \). \( \text{AS}_{\text{ins}}(g_s, n) \geq 3g_s - 24 = \Omega(n) \).
- **deletions:** \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(g_s, n) \geq 4 \). \( \text{AS}_{\text{del}}(g_s, n) \geq 3g_s - 29 = \Omega(n) \).

**Proof.** Assume \( p > 1 \).

**substitutions:** Consider the following string of length \( n = 4p + 4 \in \Theta(p) \):

\[
T = 2^p 32^p 32^p 32^p 3
\]

By the construction of the GCIS grammar of \( T \), we obtain \( D_1 = \{2^p 3\}, G_1 = 4444, g_s(T) = \|D_1\| + |G_1| = p + 5 \). The following string

\[
T' = 2^p 32^p 32^p 12^p 3
\]

can be obtained from \( T \) by substituting the third \( 3 \) with \( 1 \). By the construction of GCIS grammar of \( T \), we obtain \( G' = 564, D' = \{12^p 3, 2^p 3, 2^p 32^p\}, g_s(T') = \|D'\| + |G'| = 4p + 7 \), which leads to \( \liminf_{n \to \infty} \text{MS}_{\text{sub}}(g_s, n) \geq \liminf_{p \to \infty} (4p + 7)/(p + 5) = 4 \), and \( \text{AS}_{\text{sub}}(g_s, n) = (4p + 7) - (p + 5) = 3p + 2 = 3g_s - 13 = \Omega(n) \).

**insertions:** Consider the following string of length \( n = 8p + 12 \in \Theta(p) \):

\[
T = (12)^p 122(12)^p 122(12)^p 122(12)^p 122
\]

By the construction of the GCIS grammar, we obtain \( D_1 = \{12, 122\}, G_1 = 3^p 43^p 43^p 43^p 4, D_2 = \{3^p 4\}, G_2 = 5555, g_s(T) = \|D_1\| + \|D_2\| + |G_2| = 5 + (p + 1) + 4 = p + 10 \).

The string

\[
T' = (12)^p 122(12)^p 122(12)^p 1122(12)^p 122
\]

can be obtained from \( T \) by inserting 0 to just before the third \( 1 \). By the construction of GCIS grammar of \( T' \), we obtain \( D_1 = \{112, 12, 1122\}, G_1' = 4^p 54^p 4^p 54^p 34^p 5, G_2' = 786, D_2' = \{34^p 5, 4^p 5, 4^p 54^p\}, g_s(T') = \|D_1\| + \|D_2\| + |G_2'| = 9 + (4p + 4) + 3 = 4p + 16 \), which leads to \( \liminf_{n \to \infty} \text{MS}_{\text{ins}}(g_s, n) \geq \liminf_{p \to \infty} (4p + 16)/(4p + 10) = 4 \), and \( \text{AS}_{\text{ins}}(g_s, n) = (4p + 16) - (p + 10) = 3p + 6 = 3g_s - 24 = \Omega(n) \).

**deletions:** Consider the following string of length \( n = 12p + 12 \in \Theta(p) \):

\[
T = (122)^p 132(122)^p 132(122)^p 132(122)^p 132
\]

By the construction of the GCIS grammar of \( T \), we obtain \( D_1 = \{122, 132\}, G_1 = 4^p 54^p 54^p 54^p 5, D_2 = \{4^p 5\}, G_2 = 6666, g_s(T) = \|D_1\| + \|D_2\| + |G_2| = 6 + (p + 1) + 4 = p + 11 \).

The string

\[
T' = (122)^p 132(122)^p 132(122)^p 132(122)^p 132
\]

can be obtained from \( T \) by deleting the third \( 3 \). By the construction of GCIS grammar of \( T' \), we obtain \( D_1 = \{01, 011, 021\}, G_1' = 5^p 65^p 65^p 45^p 6, D_2' = \{45^p 6, 5^p 6, 5^p 65^p\}, G_2' = 897, g_s(T') = \|D_1\| + \|D_2\| + |G_2'| = 8 + (4p + 4) + 3 = 4p + 15 \), which leads to \( \liminf_{n \to \infty} \text{MS}_{\text{del}}(g_s, n) \geq \liminf_{p \to \infty} (4p + 15)/(p + 11) = 4 \), and \( \text{AS}_{\text{del}}(g_s, n) = (4p + 15) - (p + 11) = 3p + 4 = 3g_s - 29 = \Omega(n) \).
13 Bisection

In this section, we consider the worst-case sensitivity of the compression algorithm Bisection, which is a kind of grammar-based compression that has a tight connection to BDDs.

Given a string $T$ of length $n$, the bisection algorithm builds a grammar generating $T$ as follows. We consider a binary tree $T$ whose root corresponds to $T$. The left and right children of the root correspond to $T_1 = T[1..2^j]$ and $T_2 = T[2^j + 1..n]$, respectively, where $j$ is the largest integer such that $2^j < n$. We apply the same rule to $T_1$ and to $T_2$ recursively, until obtaining single characters which are the leaves of $T$. After $T$ is built, we assign a label (non-terminal) to each node of $T$. If there are multiple nodes such that the leaves of their subtrees are the same substrings of $T$, we label the same non-terminal to all these nodes. The labeled tree $T$ is the derivation tree of the bisection grammar for $T$. We denote by $g_{bsc}(T)$ the size of the bisection grammar for $T$. Recall that $\Sigma$ is the alphabet.

Let us briefly consider the case of unary alphabet $\Sigma = \{a\}$. Let $h(T)$ denote the height of the derivation tree $T$ for $T = a^n$. After obtaining $T' = a^{n+1}$ for insertion or $T' = a^{n-1}$ for deletion, at most $h(T) - 1$ new productions are added (note that $X \to a$ exists both for $T$ and for $T'$). Thus the additive sensitivity of Bisection for unary alphabets is at most $h(T) - 1$. This bound is almost tight, e.g. deleting a single $a$ from $T = a^{2k}$ adds new $k - 2 = h(T) - 2$ non-terminals to the existing $k = h(T)$ non-terminals (note that the production $X \to a$ remains and the existing root of $T$ is replaced with the new one). The multiplicative sensitivity for Bisection is thus asymptotically $2 = |\Sigma_1| + 1$.

In what follows, let us consider the case of multi-character alphabets, where at least one of $T$ and $T'$ contains two or more distinct characters.

13.1 Lower bounds for the sensitivity of $g_{bsc}$

**Theorem 29.** The following lower bounds on the sensitivity of $g_{bsc}$ hold:

- **substitutions:** $\liminf_{n \to \infty} MS_{sub}(g_{bsc}, n) \geq 2$. $AS_{sub}(g_{bsc}, n) \geq g_{bsc} - 4$ and $AS_{sub}(g_{bsc}, n) \geq 2 \log_2 n - 4$.

- **insertions:** $\liminf_{n \to \infty} MS_{ins}(g_{bsc}, n) \geq |\Sigma|$. $AS_{ins}(g_{bsc}, n) \in \Omega(|\Sigma| g_{bsc})$ and $AS_{ins}(g_{bsc}, n) \in \Omega \left( |\Sigma|^2 \log \frac{n}{\log n} \right)$.

- **deletions:** $\liminf_{n \to \infty} MS_{del}(g_{bsc}, n) \geq |\Sigma|$. $AS_{del}(g_{bsc}, n) \in \Omega(|\Sigma| g_{bsc})$ and $AS_{del}(g_{bsc}, n) \in \Omega \left( |\Sigma|^2 \log \frac{n}{\log n} \right)$.

**Proof. substitutions:** Consider a unary string $T = a^n$ with $n = 2^k$. The set of productions for $T$ is

- $X_1 = a$ (generating $a$),
- $X_2 = X_1X_1$ (generating $aa$),
- $X_3 = X_2X_2$ (generating $aaaa$),
- ...
- $X_k = X_{k-1}X_{k-1}$ (generating $a^{2^k}$),

with $g_{bsc}(T) = 2k - 1$. Let $T' = a^{n-1}b$ that can be obtained by replacing the last $a$ in $T$ with $b$. 

43
The set of productions for $T'$ is

$$X_1 = a \quad \text{(generating a),}$$

$$X_2 = X_1X_1 \quad \text{(generating aa),}$$

$$X_3 = X_2X_2 \quad \text{(generating aaaa),}$$

$$\ldots$$

$$X_{k-1} = X_{k-2}X_{k-2} \quad \text{(generating a}^{2k-1}) ,$$

$$Y_1 = b \quad \text{(generating b),}$$

$$Y_2 = X_1Y_1 \quad \text{(generating ab),}$$

$$Y_3 = X_2Y_2 \quad \text{(generating aaab),}$$

$$\ldots$$

$$Y_k = X_{k-1}Y_{k-1} \quad \text{(generating a}^{2k-1}b).$$

with $g_{bsc}(T') = 2k-1+2(k-1)-1 = 4k-4$. Thus $\liminf_{n \to \infty} MS_{\text{sub}}(g_{bsc}, n) \geq \liminf_{k \to \infty} \frac{4k-4}{2k-1} \geq 2$.

Also, $AS_{\text{sub}}(g_{bsc}, n) \geq (4k-4) - (2k-1) = 2k-5 = g_{bsc}(T) - 4$ and $AS_{\text{sub}}(g_{bsc}, n) \geq 2 \log_2 n - 4$ as $k = \log_2 n$.

**deletions:** Assume that $|\Sigma| = 2^i$ with a positive integer $i \geq 1$. Let $Q$ be a string that contains $t = |\Sigma|^2$ distinct bigrams and $|Q| = |\Sigma|^2 + 1$. Let $Q' = Q[2..|Q|]$. Let $\sigma_i$ denote the lexicographically $i$th character in $\Sigma$. We consider the string

$$T = Q'[1]^{2p} \cdots Q'[|Q'|]^{2p}.$$

Note that $p = \log(n/\sigma)$. The set of productions for $T$ from depth 1 to $p$ is:

$$X_i \rightarrow \sigma_i \sigma_i \quad (1 \leq i \leq p),$$

$$X_{p|\Sigma|+i} \rightarrow X_{(p-1)|\Sigma|+i} X_{(p-1)|\Sigma|+i} \quad (1 \leq i \leq |\Sigma|, 2 \leq k \leq p).$$

Thus, the derivation tree $T$ has $p|\Sigma|$ internal nodes with distinct labels. Additionally, after height $|\Sigma|$, the string consists of $t-1$ distinct bigrams, and there is no run of length 2. Then the derivation tree $T$ has $t-1$ internal nodes with distinct labels in height above $p$. Finally, $g_{bsc}(T) = p|\Sigma| + t - 1$.

We consider the string $T'$ where $T[1]$ is removed, namely,

$$T' = T[2..|T|] = Q'[1]^{2p-1}Q'[2]^{2p} \cdots Q'[|Q'|]^{2p}.$$  

The set of productions for $T'$ of height 1 is:

$$X_{(i-1)|\Sigma|+j} \rightarrow \sigma_i \sigma_j \quad (1 \leq i \leq |\Sigma|, 1 \leq j \leq |\Sigma|).$$

Thus, the derivation tree $T'$ for string $T'$ has $t = |\Sigma|^2$ internal nodes with distinct labels at height one. Because of this, the number of internal nodes of the derivation tree $T'$ in each height $2 \leq p' \leq p$ is also at least $t = |\Sigma|^2$. After that, the string of height $p$ consists of $t$ distinct bigrams, and there is no run of length 2, which is the same condition of $T$. Then the derivation tree $T$ has additional $t-1$ internal nodes with distinct labels in height above $p$. Finally, $g_{bsc}(T') = tp + t$. Then, we obtain:

$$MS_{\text{del}}(g_{bsc}, n) \geq \lim_{n \to \infty} \frac{tp+t}{p|\Sigma| + t - 1} = \lim_{p \to \infty} \frac{tp+t}{p|\Sigma| + t - 1} = \frac{t}{|\Sigma|} \geq |\Sigma|,$$

$$AS_{\text{del}}(g_{bsc}, n) \geq (tp+t) - (p|\Sigma| + t - 1) = (t - |\Sigma|)p + 1 \in \Omega(|\Sigma|^2p),$$

44
where \( \Omega(|\Sigma|^2p) = \Omega \left( |\Sigma|^2 \log \frac{n}{|\Sigma|^2p} \right) \) and \( \Omega(|\Sigma|^2p) = \Omega(|\Sigma| |g_{bsc}(T)|) \).

**insertions:** We use the same string \( T \) as in the case of deletions. We consider the string \( T' \) that is obtained by prepending \( Q[1] \) to \( T \), namely,

\[
T' = Q[1]T = Q[1]Q'[1]^{2p} \cdots Q'[|Q'|]^{2p}.
\]

The set of productions for \( T' \) of height 1 is:

\[
X_{(i-1)|\Sigma|+j} \rightarrow \sigma_i \sigma_j, \quad (1 \leq i \leq |\Sigma|, 1 \leq j \leq |\Sigma|) \\
X_{|\Sigma|^2+1} \rightarrow Q[1].
\]

Thus, the derivation tree \( T' \) has \( t + 1 \) internal nodes with distinct labels at height one. Because of this, the number of internal nodes of derivation tree \( T' \) of each height \( 2 \leq p' \leq p \) is also at least \( t = |\Sigma|^2 \) nodes. After that, the string of height \( p \) consists of \( t \) distinct bigrams, and there is no run of length 2, which is the same condition of \( T \). Then derivation tree \( T \) has additional \( t - 1 \) internal nodes with distinct labels in height above \( p \). Finally, \( g_{bsc}(T') = (t + 1)p + t \). Then, we obtain:

\[
MS_{ins}(g_{bsc}, n) \geq \lim_{n \to \infty} \frac{(t + 1)p + t}{|p| |\Sigma|^2 + t - 1} = \lim_{p \to \infty} \frac{(t + 1)p + t}{|p| |\Sigma|^2 + t - 1} = \frac{(t + 1)}{|\Sigma|^2} \geq |\Sigma|,
\]

\[
AS_{ins}(g_{bsc}, n) \geq ((t + 1)p + t) - (|p| |\Sigma|^2 + t - 1) = (t + 1 - |\Sigma|)p + 1 \in \Omega(|\Sigma|^2p),
\]

where \( \Omega(|\Sigma|^2p) = \Omega \left( |\Sigma|^2 \log \frac{n}{|\Sigma|^2p} \right) \) and \( \Omega(|\Sigma|^2p) = \Omega(|\Sigma| |g_{bsc}(T)|) \).

We show a concrete example of how the derivation tree of Bisection changes by an insertion in Figure 7.

### 13.2 Upper bounds for the sensitivity of \( g_{bsc} \)

**Theorem 30.** The following upper bounds on the sensitivity of \( g_{bsc} \) hold:

**substitutions:** \( MS_{sub}(g_{bsc}, n) \leq 2 \). \( AS_{sub}(g_{bsc}, n) \leq 2[\log_2 n] \leq 2g_{bsc} \).

**insertions:** \( MS_{ins}(g_{bsc}, n) \leq |\Sigma| + 1 \). \( AS_{ins}(g_{bsc}, n) \leq |\Sigma| g_{bsc} \).

**deletions:** \( MS_{del}(g_{bsc}, n) \leq |\Sigma| + 1 \). \( AS_{del}(g_{bsc}, n) \leq |\Sigma| g_{bsc} \).

**Proof.** **substitutions:** Let \( i \) be the position where we substitute the character \( T[i] \). We consider the path \( P \) from the root of \( T \) to the \( i \)th leaf of \( T \) that corresponds to \( T[i] \). We only need to change the labels of the nodes in the path \( P \), since any other nodes do not contain the \( i \)th leaf. Since \( T \) is a balanced binary tree, the height \( h \) of \( T \) is \( \lfloor \log_2 n \rfloor \) and hence \( |P| = h = \lfloor \log_2 n \rfloor \). Since \( h \leq g_{bsc} \), we get \( MS_{sub}(g_{bsc}, n) \leq 2 \). Since each non-terminal is in the Chomsky normal form and since \( \lfloor \log_2 n \rfloor \leq g_{bsc} \), \( AS_{sub}(g_{bsc}, n) \leq 2 \lfloor \log_2 n \rfloor \leq 2g_{bsc} \).

**insertions:** Let \( i \) be the position where we insert a new character \( a \) to \( T \), and let \( T \) and \( T' \) be the derivation trees for the strings \( T \) and \( T' \) before and after the insertion, respectively. For any node \( v \) in the derivation tree \( T \), let \( T(v) \) denote the subtree rooted at \( v \). Let \( \ell(v) \) and \( r(v) \) denote the text positions that respectively correspond to the leftmost and rightmost leaves in \( T(v) \). We use the same analysis for the left children of the nodes in the path \( P \) from the root to the \( i \)th leaf which corresponds to the inserted character \( a \). Let \( v' \) denote a node in \( T' \). From now on let us focus on the subtrees \( T'(v') \) of \( T' \) such that \( \ell(v') > i \) and \( v' \) is not in the rightmost path from the root of \( T' \). Let \( str(v') \) denote the string that is derived from the non-terminal for \( v' \), and let \( v \) be the node in \( T \) which corresponds to \( v' \). Observe that \( str(v') = T'[\ell(v')..r(v')] = T[\ell(v) - 1..r(v) - 1] \), namely,
Figure 7: An example of insertion for Bisection, where $p = 4$ and $\sigma = |\Sigma|$ in this figure. There are nodes $X_1, X_{\sigma+1}, X_{2\sigma+1}, X_{3\sigma+1}$ in the leftmost path in the derivation tree of $T = a^2b^4b^2\cdots$ (upper). After a $z$ is prepended to $T$ (yielding $T'$), new internal nodes $X'_1, X'_{\sigma+1}, X'_{2\sigma+1}, X'_{3\sigma+1}$ that correspond to $za, za^3, za^7, za^{15}$ occur in the derivation tree for $T'$ (lower). This propagates to the other $\sigma - 1$ bigrams $ab, bc, \ldots$, which consist of distinct characters.
str(v') has been shifted by one position in the string due to the new character a inserted at position i. Since $T[ℓ(v)..r(v)]$ is represented by the node v in T, there exist at most $g_{bsc}$ distinct substrings of T that can be the “seed” of the strings represented by the nodes v' of T' with $ℓ(v') > i$. Since the number of left-contexts of each $T[ℓ(v)..r(v)]$ is at most $|Σ|$, there can be at most $|Σ|$ distinct shifts from the seed $T[ℓ(v)..r(v)]$. Since the rightmost paths from the roots of T and T' are all distinct except the root, and since inserting the character can increase the length of the rightmost path by at most 1, overall, we have that

$$g_{bsc}(T') \leq |Σ|g_{bsc}(T) + \lceil \log_2 n \rceil + 1 \leq |Σ|g_{bsc}(T) + h(T) + 1,$$

where $h(T)$ is the height of T. For the case of multi-character alphabets $g_{bsc}(T) \geq h(T) + 1$ holds, and hence $g_{bsc}(T') \leq (|Σ| + 1)g_{bsc}(T)$ follows from formula (2). Hence we get $MS_{ins}(g_{bsc}, n) \leq |Σ| + 1$ and $AS_{ins}(g_{bsc}, n) \leq |Σ|g_{bsc}$.

**deletions:** By similar arguments to the case of insertions, we get $MS_{del}(g_{bsc}, n) \leq |Σ| + 1$ and $AS_{del}(g_{bsc}, n) \leq |Σ|g_{bsc}$.

### 14 Compact Directed Acyclic Word Graphs (CDAWGs)

In this section, we consider the worst-case sensitivity of the size of *Compact Directed Acyclic Word Graphs (CDAWGs)* [10]. The CDAWG of a string T, denoted CDAWG(T), is a string data structure that represents the set of suffixes of T, such that the number v of internal nodes in CDAWG(T) is equal to the number of distinct maximal repeats in T, and the number e of edges in CDAWG(T) is equal to the number of right-extensions of maximal repeats occurring in T. Therefore, the smaller CDAWG(T) is, the more repetitive T is. Since $v \leq e$ always holds, we simply use the number e of edges in the CDAWG as the size of CDAWG(T), and denote it by $e(T)$. It is known (c.f. [6]) that CDAWG(T) induces a grammar-based compression of size e for T.

#### 14.1 Lower bounds for the sensitivity of e

**Theorem 31.** The following lower bounds on the sensitivity of e hold:

- **deletions:** $\liminf_{n \to \infty} MS_{del}(e, n) \geq 2$. $AS_{del}(e, n) \geq e - 4$ and $AS_{del}(e, n) \geq n - 4$.
- **substitutions:** $\liminf_{n \to \infty} MS_{sub}(e, n) \geq 2$. $AS_{sub}(e, n) \geq e - 2$ and $AS_{sub}(e, n) \geq n - 2$.
- **insertions:** $\liminf_{n \to \infty} MS_{ins}(e, n) \geq 2$. $AS_{ins}(e, n) \geq e - 2$ and $AS_{ins}(e, n) \geq e - 2$.

**Proof. deletions:** Consider string $T = a^m b a^m b$ of length $n = 2m + 2$. All the maximal repeats of T are either of form (1) $a^h$ with $1 \leq h < m$ or (2) $a^m b$. Each of those in group (1) has exactly two out-going edges labeled with a and b, and the one in (3) has exactly one out-going edge labeled $a^m b$. Summing up these edges together with the two out-going edges from the source, the total number of edges in CDAWG(T) is $2m + 1 = n - 1$ (see also the left diagram of Figure 8). Consider string $T' = a^{2m} b$ of length $n - 1 = 2m + 1$ that can be obtained by removing the middle b from T. CDAWG(T) has $2m$ internal nodes each of which represents maximal repeat $a^k$ for $1 \leq k < 2m$ and has two out-going edges labeled with a and b. Thus, CDAWG(T') has exactly 4m = 2n - 4 edges, including the two out-going edges from the source (see also the right diagram of Figure 8). Thus we have $e(T')/e(T) = \frac{4m}{2m+2} = \frac{2m-2}{n}$ which tends to 2, and $e(T') - e(T) = 2m - 2 = n - 4 = e(T) - 4$. This gives us $\liminf_{n \to \infty} MS_{del}(e, n) \geq 2$. $AS_{del}(e, n) \geq n - 4$ and $AS_{del}(e, n) \geq e - 4$.

- **substitutions:** By replacing the middle b of T with a, we obtain string $T'' = a^{2m+1} b$, which gives us similar bounds $\liminf_{n \to \infty} MS_{del}(e, n) \geq 2$, $AS_{del}(e, n) \geq n - 2$ and $AS_{del}(e, n) \geq e - 2$. 

47
**insertions**: Consider string $S = a^n$ of length $n$. The maximal repeats of $\text{CDAWG}(S)$ are all of form $a^h$ with $1 \leq h < n$ and each of them has exactly one out-going edge labeled by $a$. The total number of edges in $\text{CDAWG}(S)$ is thus $n$ including the one from the source. Consider string $S' = a^n b$ of length $n + 1$. The set of maximal repeats does not change from $S$, but $b$ is a right-extension of $a^h$ for each $1 \leq h < n$. Thus, $\text{CDAWG}(S')$ has a total of $2n - 2$ edges, including the two out-going edges from the source. Thus we have $e(S')/e(S) = \frac{2n - 2}{n}$ and $e(S') - e(S) = n - 2$. This gives us $\liminf_{n \to \infty} \text{MS}_{\text{ins}}(e, n) \geq 2 \text{AS}_{\text{ins}}(e, n) \geq n - 2$ and $\text{AS}_{\text{ins}}(e, n) \geq e - 2$.

![Figure 8: The CDAWGs for strings $T = a^4ba^4b$ (left) and $T' = a^8b$ (right).](image)

### 15 Concluding remarks and future work

In the seminal paper by Varma and Yoshida [65] which first introduced the notion of sensitivity for (general) algorithms and studied the sensitivity of graph algorithms, the authors wrote:

> "Although we focus on graphs here, we note that our definition can also be extended to the study of combinatorial objects other than graphs such as strings and constraint satisfaction problems."

Our study was inspired by the afore-quoted suggestion, and our sensitivity for string compressors and repetitiveness measures enables one to evaluate the robustness and stability of compressors and repetitiveness measures.

The major technical contributions of this paper are the **tight and constant upper and lower bounds** for the multiplicative sensitivity of the LZ77 family, the smallest bidirectional scheme $b$, and the substring complexity $\delta$. We also presented tight and constant upper and lower bounds for the multiplicative sensitivity of the recently proposed grammar compressor GCIS, which is based on the idea of the Induced Sorting algorithm for suffix sorting. We also reported non-trivial upper and/or lower bounds for other string compressors, including RLBWT, LZ-End, LZ78, AVL-grammar, $\alpha$-balanced grammar, RePair, LongestMatch, Greedy, Bisection, and CDAWG. Some of the upper bounds reported here follow from previous important work [30, 35, 28, 37, 31, 11, 62, 26].

Apparent future work is to complete Tables 1 and 2 by filling the missing pieces and closing the gaps between the upper and lower bounds which are not tight there.

While we dealt with a number of string compressors and repetitiveness measures, it has to be noted that our list is far from being comprehensive: It is intriguing to analyze the sensitivity of other important and useful compressors and repetitiveness measures including the size $\nu$ of the smallest NU-systems [51], the sizes of the other locally-consistent compressed indices such as ESP-index [44] and SE-index [54].

Our notion of the sensitivity for string compressors/repetitiveness measures can naturally be extended to labeled tree compressors/repetitiveness measures. It would be interesting to analyze the sensitivity for the smallest tree attractor [61], the run-length XBWT [61], the tree LZ77 factorization [20], tree grammars [42, 17], and top-tree compression of trees [9].
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A Omitted proofs

In this section, we present omitted proofs.

A.1 Proof for Theorem 13 \((\Omega(\sqrt{n}) \text{ additive sensitivity for } z_T)\)

**Proof.** Let \(p = 2^h\) where \(h \geq 1\).

**substitutions:** Consider the following string \(T\) of length \(n = \Theta(p^2)\):

\[
T = a^{2p-2}b \cdot a^p b \#_1 \cdot a^{p+1} b \#_2 \cdot a^{p+2} b \#_3 \cdots a^{2p-2} b \#_{p-1},
\]

where \(#_j\) for every \(1 \leq j \leq p - 1\) is a distinct character. The non self-referencing LZ77 factorization of \(T\) is

\[
\text{LZ77}(T) = a|a^2|a^4| \cdots |a^{2h-1}|a^{p-1}b|a^p b \#_1|a^{p+1} b \#_2|a^{p+2} b \#_3| \cdots |a^{2p-2} b \#_{p-1}|
\]

with \(z_T(T) = h + p\). Then, we consider the string

\[
T' = a^{p-1} c a^{p-2} b \cdot a^p b \#_1 \cdot a^{p+1} b \#_2 \cdot a^{p+2} b \#_3 \cdots a^{2p-2} b \#_{p-1},
\]

which can be obtained from \(T\) by substituting the \(p\)-th \(a\) with \(c\). Let us analyze the structure of the non self-referencing LZ77 factorization of \(T'\). It is clear that \(h\) factors in the interval \([1..p - 1]\) are unchanged. Since \(c\) is a fresh character, it becomes a factor of length 1. Also, \(a^{p-2} b\) becomes a factor. The following each factor \(a^{p+k-2} b \#_{k-1}\) with \(2 \leq k \leq p\) is divided into two factors \(a^{p+k-2}\) and \(b \#_{k-1}\), since there are no previous occurrences of \(a^{p+k-2}\) and \(#_{k-1}\). Thus, the non self-referencing LZ77 factorization of \(T'\) is

\[
\text{LZ77}(T') = a|a^2|a^4| \cdots |a^{2h-1}|c|a^{p-2} b|a^p b \#_1|a^{p+1} b \#_2|a^{p+2} b \#_3| \cdots |a^{2p-2} b \#_{p-1}|
\]

with \(z_T(T') = h + 2p\), which leads to \(\liminf_{n \to \infty} \text{MS}_{\text{sub}}(z_T, n) \geq \liminf_{p \to \infty} (h + 2p)/(h + p) = 2\).

**insertions:** As for the same string \(T\), we consider the string

\[
T' = a^{p-1} c a^{p-1} b \cdot a^p b \#_1 \cdot a^{p+1} b \#_2 \cdot a^{p+2} b \#_3 \cdots a^{2p-2} b \#_{p-1},
\]

which can be obtained from \(T\) by inserting \(c\) between position \(p - 1\) and position \(p\) in \(T\). Then, by similar arguments to the case of substitutions, the non self-referencing LZ77 factorization of \(T'\) is

\[
\text{LZ77}(T') = a|a^2|a^4| \cdots |a^{2h-1}|c|a^{p-1} b|a^p b \#_1|a^{p+1} b \#_2|a^{p+2} b \#_3| \cdots |a^{2p-2} b \#_{p-1}|
\]

with \(z_T(T') = h + 2p\), which leads to \(\liminf_{n \to \infty} \text{MS}_{\text{ins}}(z_T, n) \geq \liminf_{p \to \infty} (h + 2p)/(h + p) = 2\).

**deletions:** Consider the following string \(T\) of length \(n = \Theta(p^2)\):

\[
T = a^{p-1} c b \cdot a c b \#_1 \cdot a^2 c b \#_2 \cdot a^3 c b \#_3 \cdots a^{p-1} c b \#_{p-1}.
\]

The non self-referencing LZ77 factorization of \(T\) is

\[
\text{LZ77}(T) = a|a^2|a^4| \cdots |a^{2h-1}|c|b|a c b \#_1|a^2 c b \#_2|a^3 c b \#_3| \cdots |a^{p-1} c b \#_{p-1}|
\]

with \(z_T(T) = h + p + 1\). Then, we consider the string

\[
T' = a^{p-1} b \cdot a c b \#_1 \cdot a^2 c b \#_2 \cdot a^3 c b \#_3 \cdots a^{p-1} c b \#_{p-1},
\]
which can be obtained from $T$ by deleting the first $c$ in $T$. Let us analyze the structure of the non self-referencing LZ77 factorization of $T'$. It is clear that $h$ factors in the interval $[1..p-1]$ are unchanged. The next factor is $b$ of length 1. The following each factor $a^kcb\#_k$ with $1 \leq k \leq p-1$ is divided into two factors $a^k$ and $b\#_k$, since there are no previous occurrences of $a^k$. Thus, the non self-referencing LZ77 factorization of $T'$ is

$$\text{LZ77}(T') = a|a^2|a^3|\cdots|a^{p-1}|b|ac|b\#_1|a^2|b\#_2|a^3|b\#_3|\cdots|a^{p-1}|b\#_{p-1}$$

with $z_{77}(T') = h + 1 + 2(p-1) = h + 2p - 1$, which leads to $\lim_{n \to \infty} MS_{\text{del}}(z_{77}, n) \geq \lim_{p \to \infty}(h + 2p - 1)/(h + p + 1) = 1$, $AS_{\text{del}}(z_{77}, n) \geq (h + 2p - 1) - (h + p + 1) = p - 2 = \Omega(\sqrt{n})$.

A.2 Proof for Theorem [16] ($\Omega(\sqrt{n})$ additive sensitivity of $z_{77sr}$)

**Proof.** Consider the following string $T$ of length $n = \Theta(p^2)$:

$$T = a^{p-1}a^p b \cdot a^{p+1}b\#_1 \cdot a^{p+2}b\#_2 \cdot \cdots \cdot a^{2p-1}b\#_{p-1}$$

which consists of $p + 1$ components. The self-referencing LZ77 factorization of $T$ is

$$\text{LZ77sr}(T) = a|a^2|a^3|\cdots|a^{p-1}|b|a^{p+1}b\#_1|a^{p+2}b\#_2|\cdots|a^{2p-1}b\#_{p-1}$$

with $z_{77sr}(T) = p + 1$. Notice that the second factor $a^{2p-1}1$ is self-referencing.

Consider the string $T'$

$$T' = a^{p-1}c \cdot a^p b \cdot a^{p+1}b\#_1 \cdot a^{p+2}b\#_2 \cdot \cdots \cdot a^{2p-1}b\#_{p-1}$$

that can be obtained from $T$ by substituting the $p$-th $a$ with $c$. The self-referencing LZ77 factorization of $T'$ is

$$\text{LZ77sr}(T') = a|a^{p-2}c|a^p|b|a^{p+1}b\#_1|a^{p+2}b\#_2|\cdots|a^{2p-1}b\#_{p-1}$$

with $z_{77sr}(T') = 2p + 2$, which leads to $MS_{\text{sub}}(z_{77sr}, n) \geq (2p + 2)/(p + 1) = 2$, $AS_{\text{sub}}(z_{77sr}, n) \geq (2p + 2) - (p + 1) = p + 1 = z_{77sr}$, and $MS_{\text{sub}}(z_{77sr}, n) = \Omega(\sqrt{n})$.

**Insertions:** Consider the following string $T$ of length $n = \Theta(p^2)$:

$$T = a^{p-1} \cdot a^p b \cdot a^{p+1}b\#_1 \cdot a^{p+2}b\#_2 \cdot \cdots \cdot a^{2p-1}b\#_{p-1}$$

The self-referencing LZ77 factorization of $T$ is

$$\text{LZ77sr}(T) = a|a^{p-2}b|a^{p+1}b\#_1|a^{p+2}b\#_2|\cdots|a^{2p-1}b\#_{p-1}$$

with $z_{77sr}(T) = p + 1$. Notice that the second factor $a^{2p-1}1$ is self-referencing.

Consider the string $T'$

$$T' = a^{p-1}c \cdot a^p b \cdot a^{p+1}b\#_1 \cdot a^{p+2}b\#_2 \cdot \cdots \cdot a^{2p-1}b\#_{p-1}$$

that can be obtained from $T$ by inserting $c$ between position $p - 1$ and position $p$. The self-referencing LZ77 factorization of $T'$ is

$$\text{LZ77sr}(T') = a|a^{p-2}c|a^p|b|a^{p+1}b\#_1|a^{p+2}b\#_2|\cdots|a^{2p-1}b\#_{p-1}$$

with $z_{77sr}(T') = 2p + 2$, which leads to $MS_{\text{ins}}(z_{77sr}, n) \geq 2$, $AS_{\text{ins}}(z_{77sr}, n) \geq p + 1 = z_{77sr}$, and $MS_{\text{ins}}(z_{77sr}, n) = \Omega(\sqrt{n})$. 

51
The self-referencing LZ77 factorization of $T$ is obtained by substituting $a$ with $z$.

Consider the following string $T$ of length $n = \Theta(p^2)$:

$$T = a^pbc \cdot abc\#_1 \cdot a^2bc\#_2 \cdots a^pbc\#_p.$$ 

The self-referencing LZ77 factorization of $T$ is

$$\text{LZ77sr}(T) = a|a^{p-1}b|c|abc\#_1|a^2bc\#_2| \cdots |a^pbc\#_p|$$

with $z_{77sr}(T) = p + 3$. Notice that the second factor $a^{p-2}1$ is self-referencing.

Consider the string $T'$

$$T' = a^pb \cdot abc\#_1 \cdot a^2bc\#_2 \cdots a^pbc\#_p,$$

that can be obtained from $T$ by deleting the first $c$ of position $p + 2$. Let us analyze the structure of the self-referencing LZ77 factorization of $T'$. The first two factors are unchanged. The third factor $c$ of $\text{LZ77sr}(T)$ is removed, and each of the remaining factors of form $a^kbc\#_k$ in $\text{LZ77sr}(T)$ is divided into two factors as $a^kbc\#_k$. Thus the self-referencing LZ77 factorization of $T'$ is

$$\text{LZ77sr}(T') = a|a^{p-1}b|abc\#_1|a^2bc\#_2| \cdots |a^pbc\#_p|$$

with $z_{77sr}(T') = 2p + 2$, which leads to $\lim \inf_{n \to \infty} MS_{del}(z_{77sr}, n) \geq \lim \inf_{p \to \infty} (2p + 2)/(p + 3) = 2$,$\ AS_{del}(z_{77sr}, n) \geq 2p + 2 - (p + 3) = p - 1 = \Omega(\sqrt{n})$.

It is also possible to binarize the strings $T$ and $T'$ in the above proof for the cases of substitutions and insertions, while retaining the same lower bounds:

**Corollary 8.** For the self-referencing LZ77 factorization, there are binary strings of length $n$ that satisfy $MS_{sub}(z_{77sr}, n) \geq 2$, $MS_{ins}(z_{77sr}, n) \geq 2$, respectively.

**Proof.** Let $p \geq 2$.

**Substitutions:** Consider the following string $T$ of length $n = \Theta(p^2)$:

$$T = 0^{p-1} \cdot 0 \cdot 02^p1 \cdot 02^p+1 101 \cdot 02^p+2 10^21 \cdots 03^p10^p1.$$ 

The self-referencing LZ77 factorization of $T$ is:

$$\text{LZ77sr}(T) = 0|03^{p-1}1|02^p1 101|02^p+2 10^21| \cdots |03^p10^p1|$$

with $p + 2$ factors. Then, we consider the string $T'$

$$T' = 0^{p-1} \cdot 1 \cdot 02^p1 \cdot 02^p+1 101 \cdot 02^p+2 10^21 \cdots 03^p10^p1$$

is obtained by substituting $p$-th $0$ with $1$. The self-referencing LZ77 factorization of $T'$ is:

$$\text{LZ77sr}(T') = 0|0^{p-2}1|0^p|0^p1|02^p+1 101|02^p+2 10^21| \cdots |03^p|10^p1|$$

with $2p + 4$ factors. Then we obtain $MS_{sub}(z_{77sr}, n) \geq (2p + 4)/(p + 2) = 2$.

**Insertions:** Consider the following string $T$ of length $n = \Theta(p^2)$:

$$T = 0^{p-1} \cdot 02^p1 \cdot 02^p+1 101 \cdot 02^p+2 10^21 \cdots 03^p10^p1.$$ 

The self-referencing LZ77 factorization of $T$ is:

$$\text{LZ77sr}(T) = 0|03^{p-2}1|02^p+1 101|02^p+2 10^21| \cdots |03^p10^p1|$$

52
with $p + 2$ factors. Consider the string

$$T' = 0^{p-1} \cdot 1 \cdot 0^{2p+1} \cdot 0^{2p+1} 101 \cdot 0^{2p+2} 10^2 1 \cdots 0^{3p} 10^p 1$$

is obtained by inserting 1 between $p - 1$ and $p$. The self-referencing LZ77 factorization of $T'$ is:

$$\text{LZ77sr}(T') = 0|0^{p-2}1|0^p1|0^{2p+1}101|0^{2p+2}10^21|\cdots|0^{3p}10^p1$$

with $2p + 4$ factors. Then we get $\text{MS}_{\text{ins}}(z_{77sr}, n) \geq (2p + 4)/(p + 2) = 2$. 

$\square$
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