ONLINE AND LIGHTWEIGHT KERNEL-BASED APPROXIMATE POLICY ITERATION FOR DYNAMIC P-NORM LINEAR ADAPTIVE FILTERING

Yuki Akiyama  Minh Vu  Konstantinos Slavakis
Tokyo Institute of Technology, Japan
Department of Information and Communications Engineering
Emails: {akiyama.y.am, vu.d.aa, slavakis.k.aa}@m.titech.ac.jp

ABSTRACT

This paper introduces a solution to the problem of selecting dynamically (online) the “optimal” p-norm to combat outliers in linear adaptive filtering without any knowledge on the probability density function of the outliers. The proposed online and data-driven framework is built on kernel-based reinforcement learning (KBRL). To this end, novel Bellman mappings on reproducing kernel Hilbert spaces (RKHSs) are introduced. These mappings do not require any knowledge on transition probabilities of Markov decision processes, and are nonexpansive with respect to the underlying Hilbert space. The fixed-point sets of the proposed Bellman mappings are utilized to build an approximate policy-iteration (API) framework for the problem at hand. To address the “curse of dimensionality” in RKHSs, random Fourier features are utilized to bound the computational complexity of the API. Numerical tests on synthetic data for several outlier scenarios demonstrate the superior performance of the proposed API framework over several non-RL and KBRL schemes.

1. INTRODUCTION

The least-squares (LS) error/loss (between an observed value and its predicted one) plays a pivotal role in signal processing, e.g., adaptive filtering [1], and machine learning [2]. For example, the least-mean squares (LMS) and recursive (RLS) [1] are two celebrated algorithms in adaptive filtering and stochastic approximation based on the LS-error criterion. Notwithstanding, LS methods are notoriously sensitive to the presence of outliers within data [3], where outliers are defined as (sparsely) contaminating data that do not adhere to a nominal data generation model, and are often modeled as random variables (RVs) with non-Gaussian heavy tailed distributions, e.g., α-stable ones [4]. To combat outliers, several non-LS criteria, such as least mean p-power (LMP) [5–10] and maximum correlation [4], have been introduced. Nevertheless, it seems that an online and data-driven solution to the problem of dynamically selecting p, without any prior knowledge on the PDF of αn, is yet to be found.

This paper offers a solution to the aforementioned open problem via reinforcement learning (RL) [13]: a machine-learning paradigm where an “agent” interacts with the surrounding environment to identify iteratively the policy which minimizes the cost of its “actions.” More specifically, the well-known policy-iteration (PI) framework [13] of RL is adopted, because of its well-documented merits (e.g., [14–16]) over the alternative RL frameworks of temporal-difference (TD) and Q-learning [13], especially for continuous and high-dimensional state spaces. PI comprises two stages at every iteration n: policy evaluation and policy improvement. At policy evaluation, the current policy is evaluated by a Q-function [13], which represents, loosely speaking, the long-term cost that the agent would suffer had the current policy been chosen to determine the next state, whereas at the policy-improvement stage, the agent uses the Q-function value to update the policy. The underlying state space is considered to be continuous, due to the nature of (x,n, y,n), while the action space is considered to be discrete: an action is a value of p taken from a finite grid of the interval [1, 2].

Deep neural networks offer approximating spaces for Q-functions, e.g., [17], but they may require processing of batch data (even retraining) during online-mode operation, since they may face test data generated by PDFs different from those of the training ones (dynamic environments). Such batch processing inflicts large computational times and complexity, discouraging the application of deep neural networks to online modes of operation where a small computational footprint is desired. To meet such computational complexity requirements, this study builds an approximate (API) framework for online RL along the lines of kernel-based (KB)RL [14–16, 18–25].

Central to the proposed API is the construction of novel Bellman mappings [13, 26]. The proposed Bellman mappings are defined on a reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ [27, 28], which serves as the approximating space for the Q-functions. Unlike the classical Bellman operators, where information on transition probabilities in a Markov decision process is needed [13], the proposed Bellman mappings make no use of such information, and need neither training/offline data nor past policies, but sample and average the sample space on the fly, at each iteration n, to perform exploration of the surrounding environment. This suits the current adaptive-filtering setting, where the presence of outliers, with a possibly time-varying

$$\theta_{n+1} = \theta_n + \rho p|e_n|^{p-2} e_n x_n,$$ (1)

where $e_n := y_n - x_n^T \theta_n$, $\rho$ is the learning rate (step size), and $p$ is a fixed user-defined real-valued number within the interval [1, 2] to ensure that the p-norm loss $|y_n - x_n^T \theta|^p$ is a convex function of $\theta$ [5]. Notice that if $p = 1$ and 2, then (1) boils down to the classical sign-LMS and LMS, respectively [1].

Intuition suggests that the choice of $p$ should be based on the probability density function (PDF) of the RV $\alpha_n$. For example, if $\alpha_n$ obeys a Gaussian PDF, then $p = 2$ should be chosen (recall the maximum-likelihood criterion). To enhance robustness against outliers, combination of adaptive filters with different forgetting factors, but with the same fixed $p$-norm, have been also introduced [8]. Nevertheless, it seems that an online and data-driven solution to the problem of dynamically selecting $p$, without any prior knowledge on the PDF of $\alpha_n$, is yet to be found.
PDF, may render the information obtained offline or from past policies outdated. As such, the proposed Bellman mappings fall closer to [15] than to studies which use training data collected beforehand (offline), e.g., [16, 29, 30].

Further, in contrast to the prevailing route in KBRL [14, 15, 19–24], which views Bellman mappings as contractions in \( \mathcal{L}_\infty \)-norm Banach spaces (by definition, no inner product available), this study introduces nonexpansive [31] Bellman operators on \( \mathcal{H} \) to capitalize on the reproducing property of the inner product of \( \mathcal{H} \) [27, 28], and to open the door to powerful Hilbertian tools [31]. A byproduct of this path is the additional flexibility offered to the user by the fact that the fixed-point set of a nonexpansive mapping is non-singleton in general, as opposed to the case of a contraction mapping which is known to have a unique fixed point. Supersets of those fixed-point sets are designed to build the proposed API framework.

It is worth stressing here that the proposed API framework, together with its complementary study [32], appear to be the first attempts to apply RL arguments to robust adaptive filtering. In contrast to [32], where the state space is the high-dimensional \( \mathbb{R}^L \) (\( L \) is the set of all real numbers), this study confines the state space to the low-dimensional \( \mathbb{R}^4 \). Moreover, this study constructs potentially infinite-dimensional hyperplanes as supersets of the fixed-point sets of a proposed Bellman mappings, as opposed to [32] where finite-dimensional affine sets are designed. To address the “curse of dimensionality,” which arises naturally in online learning in RKHSs (\( \mathcal{H} \) may be infinite dimensional), the proposed framework uses random Fourier features (RFF) [33, 34] to bound the computational complexity of the proposed API, while the approximate-linear-dependency criterion [35], which does not ensure a bounded computational complexity, is used in [32]. Finally, to robustify the proposed scheme, experience replay [36] is applied, whereas [32] employs rollout [13].

Numerical tests on synthetic data showcase the promising performance of the the advocated framework, which outperforms several RL and non-RL schemes. Due to space limitations, long proofs, experience replay [36] is applied, whereas [32] employs rollout [13].

2. NONEXPANSIVE BELLMAN MAPPINGS ON RKHSs

2.1. State-Action Space

Following the setting of (1), the state space \( \mathcal{S} \) is assumed to be continuous. In contrast to [32], where the state space is the high dimensional \( \mathbb{R}^{2L+3} \), this study considers the case where \( \mathcal{S} := \mathbb{R}^4 \), with the dimension of \( \mathcal{S} \) rendered independent of \( L \). Due to the streaming nature of \( (x_n,y_n)_{n\in\mathbb{N}} \), state vectors \( s_n := [s_1(n), s_2(n), s_3(n), s_4(n)] \) \( n \in \mathbb{N} \) are defined inductively by the following heuristic rules:

\[
\begin{align*}
    s_1(n) &:= \log_{10}|y_n - \theta_1^* x_n|, \\
    s_2(n) &:= \frac{1}{M_0} \sum_{k=1}^{M_0} \log_{10} \frac{|y_{n-k} - \theta_1^* x_{n-k}|}{\|x_{n-k}\|_2}, \\
    s_3(n) &:= \log_{10} \|x_n\|_2, \\
    s_4(n) &:= s_3(n-1) + \left(1 - \varpi \right) \frac{\|\theta_* - \theta_{n-1}\|_2}{\rho}.
\end{align*}
\]

where \( M_0 \in \mathbb{N}+, \alpha \in (0, 1) \) are user-defined parameters, while \( \rho \) comes from (1). The classical prior loss of adaptive filtering [1] is used in (2a), an \( M_0 \)-length sliding-window sampling average of the posterior loss [1] is provided in (2b), normalized by the norm of the input signal to remove as much as possible its effect on the error, the instantaneous norm of the input signal in (2c), and a smoothing autoregressive process in (2d) to monitor the consecutive displacement of the estimates \( (\theta_n)_{n \in \mathbb{N}} \). The reason for including \( \rho \) in (2d) is to remove \( \rho \)'s effect from \( s_4(n) \). Owing to (1), the initial value \( s_4(0) \) in (2d) is set equal to \( \log_{10} \frac{|\theta_1^* - \theta_0|}{2} = \log_{10} p_0 + (p_0 - 1)s_3(0) + s_3(0) \). The \( \log_{10}(\cdot) \) function is employed to decrease the dynamic range of the positive values in (2).

The action space \( \mathcal{A} \) is defined as any finite grid of the interval \([1, 2]\), so that an action \( a \in \mathcal{A} \) becomes any value of \( p \) taken from that finite grid. The state-action space is defined as \( S := \mathcal{S} \times \mathcal{A} \), and its element is denoted as \( s = (s, a) \).

Along the lines of the general notation in [13], consider now the set of all mappings \( \mathcal{M} := \{\mu(\cdot) \mid \mu(\cdot) : \mathcal{S} \to \mathcal{A}\} \to \mu(s) \). In other words, given a \( \mu \in \mathcal{M} \), \( \mu(s) \) denotes the action that the “system” may take at state \( s \) to “move to” the state \( s' \in \mathcal{S} \).

The one-step loss for this transition is denoted by \( g : S \to \mathbb{R} : (s, a) \to g(s,a) \). The set \( \Pi \) of policies is defined as \( \Pi := \{\mu^0 := \{\mu_0, \mu_1, \ldots, \mu_n, \ldots\} \mid \mu_n \in \mathcal{M}, n \in \mathbb{N}\} \). A policy will be denoted by \( \pi \in \Pi \). Given \( \mu \in \mathcal{M} \), the stationary policy \( \pi \in \Pi \) is defined as \( \pi_\mu := (\mu, \mu, \ldots, \mu, \ldots) \). It is customary for \( \mu \) to denote also the stationary policy \( \pi_\mu \). Function \( Q : S \to \mathbb{R} : (s, a) \to Q(s,a) \) quantifies the long-term cost that the agent would suffer had the action \( a \) been used to determine the next state of \( s \).

2.2. Novel Bellman Mappings

Central to dynamic programming and RL [13] is the concept of Bellman mappings, which operate on \( Q \)-functions. Typical definitions, e.g., [37], are as follows: \( \forall (s,a) \notin \mathbb{Z} \).

\[
\begin{align*}
    (T_\mu Q)(s, a) &:= g(s, a) + \alpha \mathbb{E}_{a'}[Q(s', \mu(s'))], \quad (3a) \\
    (T_\pi Q)(s, a) &:= g(s, a) + \alpha \mathbb{E}_{a'}[Q(s', a')] \min_{s' \in \mathbb{S}} Q(s', a'), \quad (3b)
\end{align*}
\]

where \( \mathbb{E}_{a'}[s'] \{ \cdot \} \) stands for the conditional expectation operator with respect to \( s' \) conditioned on \( (s, a) \), and \( \alpha \) is the discount factor with typical values in \([0, 1]\). In the case where \( Q \) is considered an element of the Banach space of all (essentially) bounded functions [38], equipped with the \( \mathcal{L}_\infty \)-norm \( \| \cdot \|_\infty \), then it can be shown that the mappings in (3) are contractions [13], and according to the Banach-Picard theorem [31], they possess unique fixed points \( Q_\mu^*, Q_\pi^* \), i.e., points which solve the Bellman equations \( T_\mu Q_\mu^* = Q_\mu^* \) and \( T_\pi Q_\pi^* = Q_\pi^* \), and which characterize “optimal” long-term losses [13].

Nevertheless, in most cases of practical interest, there is not sufficient information on the conditional probability distribution to compute the expectation operator in (3). To this end, this study proposes approximations of the Bellman mappings in (3) by assuming that losses \( g \) and \( Q \) belong to an RKHS \( \mathcal{H} \), i.e., a Hilbert space with inner product \( \langle \cdot | \cdot \rangle_{\mathcal{H}} \), norm \( \| \cdot \|_{\mathcal{H}} := \langle \cdot | \cdot \rangle_{\mathcal{H}}^{1/2} \), and a reproducing kernel \( \kappa(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathcal{H} \), such that \( \kappa(z) \in \mathcal{H}, \forall z \in \mathcal{S} \), and the reproducing property holds true: \( Q(z) := \langle Q | \kappa(z, \cdot) \rangle_{\mathcal{H}} \forall Q \in \mathcal{H}, \forall z \in \mathcal{S} \). Space \( \mathcal{H} \) may be infinite dimensional; e.g., \( \kappa(\cdot, \cdot) \) is a Gaussian kernel [27, 28]. For compact notations, let \( \varphi(z) := \kappa(z, \cdot) \), and \( Q^* := Q | \varphi(z) \).

Hereafter, losses \( g, Q \) are assumed to belong to \( \mathcal{H} \). The proposed Bellman mappings \( T_\mu \varphi : \mathcal{H} \to \mathcal{H} : Q \to T_\mu Q \) and \( T : \mathcal{H} \to \mathcal{H} : Q \to TQ \) are defined as:

\[
\begin{align*}
    T_\mu Q &:= g + \alpha \sum_{j=1}^{N_\mu} Q(s_j^w, \mu(s_j^w)) \cdot \psi_j, \quad (4a) \\
    T Q &:= g + \alpha \sum_{j=1}^{N_\mu} \inf_{a_j \in \mathcal{A}} Q(s_j^w, a_j) \cdot \psi_j, \quad (4b)
\end{align*}
\]
Algorithm 1  Approximate policy iteration for LMP.
1: Arbitrarily initialize \( Q_0, \mu_0 \in \mathcal{M} \), and \( \theta_0 \in \mathbb{R}^d \).
2: while \( n \in \mathbb{N} \) do
3: Data \( (x_n, y_n) \) become available. Let \( s_n \) as in (2).
4: **Policy improvement:** Update \( \alpha_n := \mu_n(s_n) \) by (5).
5: Update \( \theta_{n+1} \) by (1), where \( p := \alpha_n = \mu_n(s_n) \).
6: Define \( \{s^\nu_n[n]\}_{j=1}^{N_a[n]} \) (see Section 3).
7: Run experience replay on \( Q_n \) (see Section 3).
8: **Policy evaluation:** Update \( Q_{n+1} \) by (9).
9: Increase \( n \) by one, and go to Line 2.
10: end while

where \( \{\psi_j\}_{j=1}^{N_a[n]} \) are vectors in \( \mathcal{X} \), for a user-defined positive integer \( N_a[n] \), and \( \{s^\nu_n[n]\}_{j=1}^{N_a[n]} \) are state vectors chosen by the user for the summations in (4) to approximate the conditional expectations in (3). See for example [32], where \( \{s^\nu_n[n]\}_{j=1}^{N_a[n]} \) are drawn from a Gaussian distribution centered at a state of interest (the current state \( s_n \) in Section 4). For notational convenience, let \( \Psi := \{\psi_1, \ldots, \psi_{N_a[n]}\} \), and its \( N_a[n] \times N_a[n] \) kernel matrix \( K_{\psi} := \Psi^\top \Psi \) whose \((j, j')\) entry is equal to \( \langle \psi_j, \psi_{j'} \rangle \). Moreover, let \( \Phi_{\mu} := \Phi^\top_{\mu} = \Psi^\top \mu \), with kernel matrix \( K_{\mu} := \Phi_{\mu} \phi_{\mu} \).

**Theorem 1.** Let \( \psi(z) \geq 0, \forall z \in \mathcal{X}, \forall j \in \{1, \ldots, N_a[n]\} \). If \( \alpha \leq \frac{\|K_{\psi}\|^{-1/2}(\sup_{\mu \in \mathcal{M}}\|K_{\mu}\|^{-1/2})}{|\mathcal{X}|} \), then \( \forall j \in \mathcal{M} \), the mapping \( T_{\mu} \) in (4a) is affine nonexpansive and \( T_{\mu} \) in (4b) is nonexpansive within the Hilbert space \( (\mathcal{X}, \langle \cdot, \cdot \rangle) \). Norms \( \|K_{\psi}\|, \|K_{\mu}\| \) are the spectral norms of \( K_{\psi}, K_{\mu} \).

Nonexpansivity for \( T_{\mu} \) in (a) (Euclidean) Hilbert space \( (\mathcal{X}, \langle \cdot, \cdot \rangle) \) means \( \|T_{\mu}Q - T_{\mu}Q'\| \leq \|Q - Q'\|, \forall Q, Q' \in \mathcal{X} \) [31]. Moreover, \( T_{\mu} : \mathcal{X} \rightarrow \mathcal{X} \) is affine iff \( T_{\mu} = \alpha T_{\mu}Q + (1 - \lambda)Q' = QT_{\mu} + (1 - \lambda)Q' \), \( \forall Q, Q' \in \mathcal{X}, \forall \lambda \in \mathbb{R} \).

Mappings (4) share similarities with those in [14, 15, 19–21, 24]. However, in [14, 15, 19–21, 24] as well as in the classical context of (3), Bellman mappings are viewed as contractions on the Banach space of (essentially) bounded functions with the \( L_\infty \)-norm [13], while no discussion on RKHSs is reported. Recall that, by definition, Banach spaces are not equipped with inner products. On the other hand, Theorem 1 opens the door not only to the rich toolbox of nonexpansive mappings in Hilbert spaces [31], but also to the reproducing property of the inner product in RKHSs [27, 28].

### 3. APPROXIMATE POLICY ITERATION

With the Bellman mappings (4) serving as approximations of the classical ones (3), Algorithm 1 offers an approximate policy iteration (API) framework. The framework operates sequentially, with its iteration index \( n \) coinciding with the time index of the streaming data \( (x_n, y_n)_{n \in \mathbb{N}} \) of (1). To this end, the arguments of Section 2 are adapted to include hereafter the extra time dimension \( n \), which will be indicated by the super-/sub-scripts \([n], (n)\) or \( n \) in notations.

Algorithm 1 follows the standard path of PI [13]. Policy improvement is performed in Line 4 of Algorithm 1 according to the standard greedy rule of [13]

\[
\mu_n(s_n) := \arg \min_{\mu_n} Q_n(s_n, \mu), \tag{5}
\]

A different way for policy improvement via rollout can be found in [32].

The following proposition constructs a superset for the fixed-point set of Fix \( T_{\mu_n}^{-1} \). The superset \( \mathcal{H}_n \) is a potentially infinite-dimensional hyperplane, in contrast to the superset in [32] which is a finite-dimensional affine set.

**Proposition 1.** The fixed-point set Fix \( T_{\mu_n}^{-1} \) is a subset of the hyperplane \( \mathcal{H}_n \). Whenever experience replay (ER) is utilized to allow re-use of past data. To this end, an experience-replay (ER) buffer is constructed to comprise information \( \{s^\nu_n[n], \mu_n(s^\nu_n[n])\} \) which is collected at instances \( \nu \) taken from \( \{1, \ldots, n\} \). Whenever experience replay

| 1. Arbitrarily initialize \( Q_0, \mu_0 \in \mathcal{M} \), and \( \theta_0 \in \mathbb{R}^d \). |
| 2. while \( n \in \mathbb{N} \) do |
| 3. Data \( (x_n, y_n) \) become available. Let \( s_n \) as in (2). |
| 4. **Policy improvement:** Update \( \alpha_n := \mu_n(s_n) \) by (5). |
| 5. Update \( \theta_{n+1} \) by (1), where \( p := \alpha_n = \mu_n(s_n) \). |
| 6. Define \( \{s^\nu_n[n]\}_{j=1}^{N_a[n]} \) (see Section 3). |
| 7. Run experience replay on \( Q_n \) (see Section 3). |
| 8. **Policy evaluation:** Update \( Q_{n+1} \) by (9). |
| 9. Increase \( n \) by one, and go to Line 2. |
| 10. end while |
is applied, data from the ER buffer are utilized. In short, the following route is followed at each $n$: $Q_n \rightarrow (9) \rightarrow \{\text{Re-use past data from the ER buffer}\} \rightarrow (9) \rightarrow Q_{n+1}$. Details on how to select information for the ER buffer and to utilize that information in the proposed API will be reported in the journal version of the paper.

A direct application of (9) may lead to memory and computational complications, since at each $n$, (9) potentially adds new kernel functions into the representation of $Q_{n+1}$ via $h_n$. This unpleasant phenomenon is fueled by the potential infinite dimensionality of $\mathcal{F}$; see for example the case where the kernel of $\mathcal{F}$ is the Gaussian [28] $\kappa_G(z, z')$, $(z, z') \in \mathcal{F}^2$, as in Section 4. To address this “curse of dimensionality,” this work employs the methodology of RFF [33]. Avoiding most of the details due to space limitations, $\kappa_G(z, z')$ is approximated by the following inner product $\tilde{\varphi}(z)\tilde{\varphi}(z')$, where the Euclidean feature vector

$$\tilde{\varphi}(z) := (\frac{z}{D})^{1/2}[\cos (v_1^T z + b_1), \ldots, \cos (v_D^T z + b_D)]^T,$$

with $D \in \mathbb{N}$ being a user-defined dimension, while $\{v_i\}_{i=1}^D$ and $\{b_i\}_{i=1}^D$ are RVs following the Gaussian and uniform distributions, respectively. The feature mapping (10) is used instead of $\varphi(\cdot)$ throughout this work to transfer learning from the infinite dimensional $(\mathcal{F}, \kappa_G)$ to the $D$-dimensional $\mathbb{R}^D$. Mapping (10) together with the low-complexity iteration (9) yield an API with bounded computational complexity.

4. NUMERICAL TESTS

Algorithm 1 is tested against (i) (1), for the values $p \in \mathbb{R} : \{1, 1.25, 1.5, 1.75, 2\}$, which are kept fixed throughout all iterations, (ii) [8], which uses a combination of adaptive filters with different forgetting factors but with the same fixed $p$-norm, (iii) the kernel-based TD(0) [39], equipped with RFF and experience replay, and (iv) the kernel-based (K)LSPI [16]; see Figures 1 and 2. Tests were also run to examine the effect of several of Algorithm 1’s parameters on performance; see Figure 3. The metric of performance is the normalized deviation from the desired $\theta^*$, see the vertical axes in all figures. The Gaussian kernel [28] was used, approximated by RFF as described in Section 3. The dimension $L$ of $x_n, \theta^*$ in (1) is 100, with a learning rate $\rho = 10^{-3}$. Both $x_n$ and $\theta^*$ are generated from the Gaussian distribution $\mathcal{N}(0, \Sigma_L)$, with $(x_n)_{n \in \mathbb{N}}$ designed to be IID. Moreover, $M_{av} = 300$ and $\Sigma = 0.3$ in (2), and $\eta = 0.5$ in (9).

Two types of outliers were considered. First, $\alpha$-stable outliers, generated by [40]. Parameters $\alpha_{stable} = 1$, $\beta_{stable} = 0.5$, $\sigma_{stable} = 1$ were used, which yield a considerably heavy-tailed distribution for the outliers. Second, “sparse” outliers were generated, with values taken from the interval $[-100, 100]$ via the uniform distribution. Sparse outliers appear in 10% percent of the data, whereas in the rest 90% of the data, Gaussian noise with SNR = 30dB appears. As it is customary in adaptive filtering, system $\theta^*$ is changed at time 20,000 to test the tracking ability of Algorithm 1. Each test is repeated independently for 100 times, and uniformly averaged curves are reported.

As it can be verified by Figures 1 to 3, Algorithm 1 outperforms the competing methods. KLSPI [16] fails to provide fast convergence and performance close to the levels of the rest of the methods. The kernel-based TD(0) [39] converges fast, but with a subpar performance with regards to that of the proposed framework. More tests on several other scenarios, together with the results of [32], will be reported in the journal version of the paper.

Fig. 1: $\bigcirc$ Algorithm 1 w/ $N_{av} = 10, \alpha = 0.75$. Marks $\times, \triangle, \square$. $\bigcirc$ correspond to (1) w/ $p = 1, 1.25, 1.5, 1.75, 2$, respectively. Mark $\bigcirc$ denotes an algorithm which randomly chooses $p, \forall n$.

Fig. 2: $\bigcirc$ Algorithm 1 w/ $N_{av} = 10, \alpha = 0.75$. $\times$ Algorithm 1 w/ $N_{av} = 1, \alpha = 0.75$. $\bigtriangleup$, $\bigtriangleup$: Kernel-based TD(0) w/ $\alpha = 0.9$ [39]. $\square$ [8] w/ $p = 1, \gamma_1 = 0.9, \gamma_2 = 0.99$. $\bigcirc$: KLSPI w/ $\alpha = 0.9$ [16]

Fig. 3: Algorithm 1 w/ several parameters. $\bigcirc$: $N_{av} = 10, \alpha = 0.9$. $\times$: $N_{av} = 10, \alpha = 0.75$. $\bigtriangleup$: $\alpha = 0$. $\square$: $N_{av} = 1, \alpha = 0.9$. $\bigcirc$: $N_{av} = 1, \alpha = 0.75$. S
5. REFERENCES

[1] A. H. Sayed, *Adaptive Filters*. Wiley, 2011.
[2] S. Theodoridis, *Machine Learning—A Bayesian and Optimization Perspective*. Elsevier, 2nd ed., 2020.
[3] P. J. Rousseeuw and A. Leroy, *Robust Regression and Outlier Detection*. Wiley, 1987.
[4] M. Shao and C. L. Nikias, “Signal processing with fractional lower order moments: Stable processes and their applications,” *Proc. IEEE*, vol. 81, no. 7, pp. 986–1010, 1993.
[5] S.-C. Pei and C.-C. Tseng, “Least mean p-power error criterion for adaptive FIR filter,” *IEEE Journal on Selected Areas in Communications*, vol. 12, no. 9, pp. 1540–1547, 1994.
[6] Y. Xiao, Y. Tadokoro, and K. Shida, “Adaptive algorithm based on least mean p-power error criterion for Fourier analysis in additive noise,” *IEEE Trans. Signal Process.*, vol. 47, no. 4, pp. 1172–1181, 1999.
[7] E. E. Kuruoğlu, “Nonlinear least $\ell_p$-norm filters for nonlinear autoregressive $\alpha$-stable processes,” *Digital Signal Processing*, vol. 12, no. 1, pp. 119–142, 2002.
[8] A. Navia-Vazquez and J. Arenas-Garcia, “Combination of recursive least p-norm algorithms for robust adaptive filtering in alpha-stable noise,” *IEEE Trans. Signal Process.*, vol. 40, no. 3, pp. 1478–1482, 2012.
[9] B. Chen, L. Xing, Z. Wu, J. Liang, J. C. Príncipe, and N. Zheng, “Smoothed least mean p-power error criterion for adaptive filtering,” *Digital Signal Processing*, vol. 40, pp. 154–163, May 2015.
[10] K. Slavakis and M. Yukawa, “Outlier-robust kernel hierarchical-optimization RLS on a budget with affine constraints,” in *Proc. IEEE ICASSP*, pp. 5335–5339, 2021.
[11] A. Singh and J. C. Príncipe, “Using correntropy as a cost function in linear adaptive filters,” in *Proc. International Joint Conference on Neural Networks*, pp. 2950–2955, 2009.
[12] C. Gentile, “The robustness of the p-norm algorithms,” *Machine Learning*, vol. 53, pp. 265–299, 2003.
[13] D. Bertsekas, *Reinforcement Learning and Optimal Control*. Athena Scientific, 2017.
[14] D. Ormoneit and S. Sen, “Kernel-based reinforcement learning,” *Machine Learning*, vol. 49, pp. 161–178, 2002.
[15] D. Ormoneit and P. Glynn, “Kernel-based reinforcement learning in average-cost problems,” *IEEE Transactions on Automatic Control*, vol. 47, pp. 1624–1636, Oct. 2002.
[16] X. Xu, D. Hu, and X. Lu, “Kernel-based least squares policy iteration for reinforcement learning,” *IEEE Transactions on Neural Networks*, vol. 18, no. 4, pp. 973–992, 2007.
[17] H. Van Hasselt, A. Guez, and D. Silver, “Deep reinforcement learning with double Q-learning,” in *Proc. AAAI conference on Artificial Intelligence*, vol. 30, 2016.
[18] J. Bae, L. S. Giraldo, P. Chhatbar, J. Francis, J. Sanchez, and J. Príncipe, “Stochastic kernel temporal difference for reinforcement learning,” in *Proc. IEEE MLSP*, pp. 1–6, 2011.
[19] A. Barreto, D. Precup, and J. Pineau, “Reinforcement learning using kernel-based stochastic factorization,” in *Proc. NIPS*, vol. 24, 2011.
[20] A. Barreto, D. Precup, and J. Pineau, “On-line reinforcement learning using incremental kernel-based stochastic factorization,” in *Proc. NIPS*, vol. 25, 2012.
[21] B. Kveton and G. Theodorou, “Structured kernel-based reinforcement learning,” in *Proc. AAAI Conference on Artificial Intelligence*, vol. 27, pp. 569–575, June 2013.
[22] W. Sun and J. A. Bagnell, “Online Bellman residual and temporal difference algorithms with predictive error guarantees,” in *Proc. International Joint Conference on Artificial Intelligence*, pp. 4213–4217, 2016.
[23] A.-M. Farahmand, M. Ghavamzadeh, C. Szepesvári, and S. Mannor, “Regularized policy iteration with nonparametric function spaces,” *J. Machine Learning Research*, vol. 17, no. 1, pp. 4809–4874, 2016.
[24] B. Kveton and G. Theodorou, “Kernel-based reinforcement learning on representative states,” *Proc. AAAI Conference on Artificial Intelligence*, vol. 26, pp. 977–983, Sept. 2021.
[25] Y. Wang and J. C. Príncipe, “Reinforcement learning in reproducing kernel Hilbert spaces,” *IEEE Signal Processing Magazine*, vol. 38, no. 4, pp. 34–45, 2021.
[26] R. E. Bellman, *Dynamic Programming*. Dover Publications, 2003.
[27] N. Aronszajn, “Theory of reproducing kernels,” *Transactions of the American Mathematical Society*, vol. 68, pp. 337–404, 1950.
[28] B. Schökopf and A. J. Smola, *Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond*. Adaptive computation and machine learning, MIT Press, 2002.
[29] M. G. Lagoudakis and R. Parr, “Least-squares policy iteration,” *J. Mach. Learn. Res.*, vol. 4, pp. 1107–1149, Dec. 2003.
[30] K. Panaganti, Z. Xu, D. Kalathil, and M. Ghavamzadeh, “Robust reinforcement learning using offline data,” arXiv, 2022. abs/2208.05129.
[31] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. New York: Springer, 2011.
[32] M. Yu, Y. Akiyama, and K. Slavakis, “Dynamic selection of p-norm in linear adaptive filtering via online kernel-based reinforcement learning,” Submitted for publication to arXiv, Oct. 2022.
[33] A. Rahimi and B. Recht, “Random features for large-scale kernel machines,” in *Proc. NIPS*, vol. 20, 2007.
[34] G. Konidaris, S. Osentoski, and P. Thomas, “Value function approximation in reinforcement learning using the Fourier basis,” in *Proc. AAAI Conference on Artificial Intelligence*, pp. 380–385, 2011.
[35] Y. Engel, S. Mannor, and R. Meir, “The kernel recursive least-squares algorithm,” *IEEE Trans. Signal Process.*, vol. 52, no. 8, pp. 2275–2285, 2004.
[36] T. Schaull, J. Quan, I. Antonoglou, and D. Silver, “Prioritized experience replay,” in *Proc. International Conference on Learning Representations*, 2016.
[37] M. G. Bellemare, G. Ostrovski, A. Guez, P. Thomas, and R. Munos, “Increasing the action gap: New operators for reinforcement learning,” *Proc. AAAI Conference on Artificial Intelligence*, vol. 30, no. 1, 2016.
[38] R. G. Bartle, *The Elements of Integration and Lebesque Measure*. John Wiley & Sons, 1995.
[39] J. Bae, P. Chhatbar, J. T. Francis, J. C. Sanchez, and J. C. Príncipe, “Reinforcement learning via kernel temporal difference,” in *Proc. IEEE EMBS*, pp. 5662–5665, 2011.
[40] J. M. Miotto, “Pylevy.” https://github.com/josemiotto/pylevy, 2020.