Semi-continuity of Automorphism Groups of Strongly Pseudoconvex Domains: the Low Differentiability Case

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Abstract: We study the semicontinuity of automorphism groups for perturbations of domains in complex space or in complex manifolds. We provide a new approach to the study of such results for domains having minimal boundary smoothness. The emphasis in this study is on the low differentiability assumption and the new methodology developed accordingly.

1 Introduction

It is a familiar perception of everyday life that symmetry is hard to create, but easily destroyed. To make the crooked straight requires some definite effort, but the slightest change can suffice to make the straight a little crooked and hence not straight at all. This perception is easily substantiated in precise form for geometric objects in Euclidean space. And it is natural to ask if something similar might apply for automorphism groups in complex analysis, that is, for the group of biholomorphic self-maps of, say, a bounded domain in complex Euclidean space.

In one complex variable, this idea does not yield much, at least in the topologically trivial case. Since all bounded domains that are topologically equivalent to the unit disc are biholomorphic to the unit disc (Riemann Mapping Theorem, of course), there is not much interest in discussing how the automorphism group varies with the domain: it does not vary at all.

But, in higher dimensions, the idea comes into its own. Domains near the unit ball can have no automorphisms whatever except the identity, and indeed domains with trivial automorphism group are dense in the set of \( C^\infty \) strongly pseudoconvex domains in the \( C^\infty \) topology (cf. [10] for detailed references to the literature): the proof of this in fact goes back really to Poincaré, in effect, since it depends essentially only on counting parameters rather than on the details of local invariant theory, at least once one knows that biholomorphic maps extend smoothly to the boundary [5]. It is also the case that domains near the unit ball have automorphism groups which are isomorphic to a subgroup of the automorphism group of the ball. Indeed, if a domain is \( C^\infty \) close enough to the ball, the domain is either biholomorphic to the ball or its automorphism group is isomorphic to a (closed) subgroup of the unitary group (10).

This kind of semicontinuity holds in greater generality (10). If a \( C^\infty \) strongly pseudoconvex domain is not biholomorphic to the ball, then there is a neighborhood of the domain in the \( C^\infty \) topology on the set of all \( C^\infty \) bounded domains with the property that the automorphism group of every domain in
the neighborhood is isomorphic to a subgroup of the automorphism group of the original domain. (The case of the fixed domain being biholomorphic to the ball is as in the previous paragraph).

The goal of this paper is to explore the possibility of reducing the level of differentiability required for this type of result, both for the fixed domain itself and for the varied domains and the topology upon them. We shall show in fact that $C^\infty$ can be reduced to $C^2$. This is optimal in the sense that $C^2$ is the natural setting for the discussion of strong pseudoconvexity and is the lowest level of regularity for which the definition is naturally given. (One can of course construct somewhat more intricate and to some extent artificial ideas of strong pseudoconvexity wherein the boundary need not have that much regularity, but these will not be explored here).

It will turn out that the particular complex analysis just discussed can in fact be treated by changing the whole context to manifolds and general group actions. The role of complex analysis becomes simply to guarantee a kind of uniform compactness discussed in Section 2 in detail and in general terms, momentarily.

To put this matter in perspective, it is desirable to recall in outline how the semicontinuity results in [10] were obtained. The starting point is the use of normal family arguments. In this context, the set-up is as follows. Fix a bounded domain $\Omega_0$. Then a sequence of bounded domains $\Omega_j$ is considered to converge to $\Omega_0$ if there is a sequence of maps $\Phi_j: \Omega_0 \to \Omega_j$ which converges to the identity in some appropriate topology. Now, in this situation, a sequence of automorphisms $f_j: \Omega_j \to \Omega_j$ always has a subsequence $f_{jk}$ such that the maps $\Phi_{jk}^{-1} \circ f_{jk} \circ \Phi_{jk}$ converge to some map of $\Omega_0$ to the closure of $\Omega_0$. Here convergence means uniform convergence on compact subsets of $\Omega_0$.

The closure may in fact be required. For example, if all the $\Omega_j$ are the same as $\Omega_0$ with the $\Phi_j$ being the identity, then the sequence $f_j$ could have a limit that had image in the boundary of $\Omega_0$, a familiar situation in one variable for the unit disc. For example, $\Phi_j(z) = (z - (1 - \frac{1}{j}))/((1 - (1 - \frac{1}{j})z)$ on the disc $\Delta$ in $\mathbb{C}$ would converge to the constant map $-1$.

However, it is relatively easy to show, and is in fact a classical result that, if the limit mapping is in fact interior, i.e., if its image lies in $\Omega_0$ itself, then in fact that limit is an automorphism of $\Omega_0$. (A detailed proof is given in [17]). Thus, in trying to relate the automorphisms of the $\Omega_j$’s to those of $\Omega_0$, one is interested in situations where it is guaranteed that the family of maps of the sort described always has “nondegenerate” limits, that is, the limits are necessarily the maps into $\Omega_0$ itself, with no boundary points in the image.

A natural first restriction, arising from looking at the examples where all the domains are the same, is to those domains $\Omega_0$ which have compact automorphism group. Then the orbits of the group are necessarily compact and the limit of any sequence of automorphisms which converges uniformly on compact sets to some limit will necessarily converge to an interior limit.

As it happens, every strongly pseudoconvex bounded domains that is not biholomorphic to the ball has a compact automorphism group. This was proved by B. Wong [25] in the mid 1970s and has been much generalized since, to
the point where the result is not only valid for $C^2$ domains but is localized completely. If a sequence of automorphisms has the property that, for some interior point the sequence of the images of the point converge to a $C^2$ strongly pseudoconvex boundary point of a domain in a general complex manifold, then the domain is biholomorphic to the ball ([3], [7]). This line of thought makes it natural to consider the whole normal families situation for bounded strongly pseudoconvex domains that are not biholomorphic to the ball, which will indeed be the main topic in this paper. However, certain aspects of the situation can be treated with no pseudoconvexity invoked at all. If one simply assumes the relevant kind of nondegeneracy of normal families as a hypothesis, then a semicontinuity result already follows. This matter is treated in Section 2.

It is natural to ask when that hypothesis is satisfied; that is, under what conditions of a more familiar sort the non-degeneracy condition (stably-interior) that is required in Section 2 is sure to hold. As we shall see, it in fact always holds under the hypothesis of $C^2$ strong pseudoconvexity of the boundary of $\Omega_0$ ($\Omega_0$ not biholomorphic to the ball) and the assumption that the $\Omega_j$ converge to $\Omega_0$ in the $C^2$ topology. How this arises requires some explanation.

Already, in [10], it was observed that non-degeneracy could be established by considering curvature invariants of the Bergman metric, at least in the $C^\infty$ case. The argument in outline was as follows: The Bergman metric of a strongly pseudoconvex domain is complete in the usual sense of Riemannian geometry ([11]). The well-known theorem of Lu Qi-Keng asserts that, if the Bergman metric had constant holomorphic sectional curvature, then the bounded domain would be biholomorphic to the ball. Thus, if it is assumed to be not biholomorphic to the ball, then the holomorphic sectional curvature is not constant.

On the other hand, according to a calculation by Klembeck [16] using the Fefferman asymptotic expansion of the Bergman kernel, the holomorphic sectional curvature approaches a negative constant at the boundary. (In the usual normalization, the constant is $-4/(n+1)$, where $n$ is the complex dimension.) Let $p$ be a point in the interior where some holomorphic sectional curvature is not $-4/(n+1)$. Then, since the holomorphic sectional curvature of the Bergman metric is a biholomorphic invariant, it follows that there is some positive $\epsilon$ such that the distance to the boundary of the orbit of $p$ under the automorphism group is greater than or equal to $\epsilon$. This gives a proof of Bun Wong’s theorem on the compactness of the automorphism group. But, more significantly from our viewpoint, it was shown in [9] that this $\epsilon$ can be chosen stably with respect to variation of the domain in the $C^\infty$ topology. This stability was established by combining interior stability of the Bergman metric with a (not so easily established) stability of the Fefferman expansion with respect to variation of the domain.

This program worked, but it was tied specifically to the $C^\infty$ situation, since the Fefferman expansion requires $C^\infty$ boundary (or at least a large, and rather difficult to determine, number of derivatives).

The semicontinuity of automorphism groups in the $C^2$ case will be obtained in this paper again by using curvature invariants to bound the distance of orbits from the boundary stably. But the stability of the asymptotic constancy of
holomorphic curvature of the Bergman metric will be obtained without using the Fefferman expansion, thus avoiding the need for a large number of derivatives. Instead, the behavior of the holomorphic sectional curvature of the Bergman metric will be analyzed using the “scaling method,” as explained in Section 3. The possibility of using the scaling method depends on noting that the holomorphic sectional curvature can be expressed in terms of a special basis for the Hilbert space of square integrable holomorphic functions (cf. [12] and [4] for the special basis concept in generality). This means that one can detour around the rather awkward formulas from Riemannian geometry that express the curvature tensor as a whole in terms of the metric and operate instead with more directly accessible aspects of the fundamental Bergman construction.

The mechanism by which the general normal family hypotheses introduced in Section 2 yield semi-continuity results in the sense of isomorphism to subgroups is the application of the corresponding result in Riemannian geometry for compact Riemannian manifolds, as established originally by Ebin [2]: If $g_j$ is a sequence of $C^\infty$ Riemannian metrics on a compact manifold $M$ converging in the $C^\infty$ topology to a $C^\infty$ limit $g_0$ then, for all sufficiently large $j$, the isometry group of $g_j$ is isomorphic to a subgroup of the isometry group of $g_0$ via an isomorphism obtained by conjugation by a diffeomorphism of $M$. This result is actually established in [2] for a finite degree of differentiability in the sense of Sobolev $H^s$ spaces, but the degree of differentiability depends on the dimension of $M$, as is typical in $H^s$-space arguments. Moreover, in the decades since Ebin’s paper [2], there have been alternative approaches developed and reductions in the number of derivatives needed. These improved results will be discussed in Section 5.

To relate this result for compact Riemannian manifolds to the noncompact case of automorphisms of complex domains, one proceeds as follows: In the non-degenerate normal family situation already indicated (to be discussed in detail in Section 2), there can be constructed group-invariant sub-domains by taking sub-level sets of group-invariant exhaustion functions. The exhaustion functions are obtained by averaging an arbitrary exhaustion function with respect to the group action, and the invariant sub-level set can be taken to have smooth boundary by choosing a sublevel set of a noncritical value of the invariant exhaustion. Since this invariant sublevel set is strictly interior, that is has compact closure contained in the domain itself, the sub-domain will have $C^\infty$ boundary. And the group action on it can thus be extended to the “double” of the sub-domain, regarded as a compact manifold with boundary. In this doubled situation Ebin’s result then applies directly.

In this setup, the regularity of the boundary of the domain itself plays no role. As soon as one has the non-degenerate normal family situation, via curvature invariants or otherwise, then all considerations occur strictly inside the domain where all mappings involved are holomorphic and hence $C^\infty$.

However, if one wants to extend to the noncompact case the part of Ebin’s result about diffeomorphism conjugation, then the regularity of the boundary and of the automorphisms up to the boundary becomes involved. In the $C^\infty$ case, it was shown in [10] that in fact one could form the double of the domain
itself and extend the group actions to the double, rather than forming the double of an invariant sub-domain. Thus the analogue of Ebin’s diffeomorphism conjugacy statement was obtained.

In the last section of this paper, a corresponding result involving diffeomorphism conjugacy will be obtained for strongly pseudoconvex domains with low boundary regularity. For technical reasons, the regularity cannot be quite reduced to the $C^2$ level which will be all that is needed for the subgroup semicontinuity. It may be possible that diffeomorphism conjugacy also applies in the $C^2$ case, but this result cannot be proved by the methods used here.

It is worth noting that the reference [11] established a version of the semicontinuity theorem for automorphism groups in the context of $C^2$ convergence. That paper was an important first step in the program we are developing here. The role of holomorphic curvature of the Bergman metric was replaced by the quotient invariant, that is the Carathéodory volume divided by the Kobayashi-Eisenmann volume. But the curvature methods here are of independent interest, and the needed stable uniformity of extension of automorphisms is checked here in more detail.

The present paper is in some respects a natural continuation of [11]. In [11], semicontinuity of automorphism groups in the sense of isomorphism to a subgroup was established in the $C^2$ strongly pseudoconvex category (with $C^2$ topology). Isomorphism via a conjugating diffeomorphism was established, however, only in the $C^\infty$ category ([11]), with a program outlined briefly in [11] to establish a conjugating diffeomorphism in the $C^k$ case, $k$ finite but (unspecified) large. This latter was to be based on the Ligocka’s extension results for biholomorphic maps. In all cases, the property called in this paper “stably-interior” was established using not curvature but rather the quotient of the Carathéodory and Kobayashi volume forms. This sufficed for the specific purpose, but it was a less geometrically illuminating biholomorphic invariant than is Bergman metric curvature. But at the time, curvature estimates were only able to be derived from the Fefferman expansion and were hence available only in the $C^\infty$ case.

In the intervening quarter of a century(!) various developments made it possible to view the situation both more broadly and more precisely, the latter in the sense of obtaining specific (low) values for the degree of differentiability needed. These developments include more specific estimates of the differentiability needed in Ebin’s theorem ([13], [15]) and Lempert’s extension theorem ([19]), established here in stable form relative to the variation of the domain. Finally, as shown here (cf. also [8]) the asymptotic constancy of holomorphic sectional curvature can be analyzed by the scaling method, bypassing the Fefferman expansion and hence obviating the need for $C^\infty$, as already noted. These developments combined make possible a precise completion of the finite-differentiability program begun in [11], precise in particular in precise $k$ values.

Useful though the Carathéodory-Kobayashi volume quotient was in [11], it is our perception that the more detailed geometric information provided by the curvature analysis here has more potential for future further applications, as well as being, as we see it, geometrically satisfying in its own right. And the
stable Lempert extension estimates also seem to us to have potential for further use, as we hope.

2 Normal Families and General Semicontinuity of Groups of Mappings

In this section, some very general results will be discussed about groups of diffeomorphisms of open sets in Euclidean spaces. The fundamental idea is that, as far as semi-continuity of the groups is concerned, the noncompact case can be converted to the compact case. This is, more precisely, true as far as semi-continuity in the sense of isomorphism to a subgroup is concerned. We begin with a definition of an appropriate idea of convergence of the open sets. For convenience, and without any particular loss of generality, we restrict our attention to connected open sets, i.e., domains.

Definition 2.1 A sequence $\Omega_j$ of connected open sets, or domains, in a Euclidean space $\mathbb{R}^n$, is said to contain-converge to a limit domain $\Omega_0$ if, for every compact subset $K$ of $\Omega_0$, $K$ is contained in $\Omega_j$ for all sufficiently large $j$.

Definition 2.2 If the sequence $\{\Omega_j\}$ of domains containment-converges to a domain $\Omega_0$, then a sequence of $C^\infty$ mappings $f_j: \Omega_j \to \mathbb{R}^n$ is said to converge $C^\infty$ normally if, for each compact subset $K$ of $\Omega_0$, the mappings $f_j$ and their derivatives of all orders converge uniformly on $K$.

Note here that the $f_j$ are defined in a neighborhood of $K$, any compact set $K$, for all $j$ sufficiently large, so that the desired uniform convergence indeed makes sense.

For our next definition, we recall that there is a metric, to be denoted $g_K$, on the set of all $C^\infty$ mappings of a neighborhood of a compact subset $K$ to $\mathbb{R}^n$ such that convergence in this metric is equivalent to convergence of the mappings and their derivatives of all orders uniformly on the compact set $K$. (cf., e.g., [9])

Definition 2.3 Suppose that $\{\Omega_j\}$ is a sequence of domains which containment-converges to a domain $\Omega_0$ and also suppose that, for each $j$, $G_j$ is a group of diffeomorphisms of $\Omega_j$ and that $G_0$ is a group of diffeomorphisms of $\Omega_0$. We say that the sequence of groups $G_j$ converges normally to $G_0$ if, for each compact subset $K$ of $\Omega_0$ and for each $\epsilon > 0$, there is a $j_{\epsilon,K}$ such that, for each $j > j_{\epsilon,K}$ and each $\phi_j \in G_j$, the mapping $\phi_j\big|_K$ lies within $g_K$-distance $\epsilon$ of some element of $G_0$.

In case one has not domains, but compact manifolds and compact groups, then the situation is as follows:

Lemma 2.1 (from [2], cf. [15] and [8]) If $M$ is a compact manifold and if $G_j$ is a sequence of compact subgroups of the diffeomorphism group of $M$ [in the topology determined by the metric $\gamma_M$] such that $G_j$ converges to the compact
subgroup $G_0$ then, for all $j$ sufficiently large, $G_j$ is isomorphic to a subgroup of $G_0$. Moreover, the isomorphism can be obtained by conjugation by a diffeomorphism $\phi_j$ and the $\phi_j$ can be chosen to converge to the identity [again in the topology determined by the metric $\gamma_M$].

**Proof:** This result is implied by the result of D. Ebin already alluded to together with a classical result of Lie group theory. Ebin’s result in detail is that, if $\{g_j\}$ is a sequence of Riemannian metrics on a compact manifold $M$ which converge in the $C^\infty$ sense to a limit metric $g_0$ then, for all $j$ sufficiently large, the isometry group of $g_j$ is isomorphic to a subgroup of the isometry group of $g_0$. Now this result is related to the compact group situation as follows: With the groups $G_j$ and $G_0$ as above, there is a metric $g_0$ which is invariant under $G_0$. Then, because the elements of $G_j$, for $j$ large, are close to elements of $G_0$, the metric $g_0$ is in an obvious sense close to being invariant under $G_j$. In particular, averaging $g_0$ with respect to the action of $G_j$ in the usual fashion produces a metric $g_j$ that is close to $g_0$. The sequence $\{g_j\}$ converges to $g_0$ in the $C^\infty$ sense. Ebin’s theorem then gives that the isometry group of $g_j$, which of course includes $G_j$, is isomorphic to a subgroup of the isometry group of $g_0$. But the connection is not quite complete, since the isometry group of $g_0$ may in fact be larger than the group $G_0$. But this difficulty can be handled as follows: Part of Ebin’s result is that in fact the isomorphism to a subgroup can be obtained via conjugation by a diffeomorphism which can be taken to be close to the identity. So, for all $j$ sufficiently large, we can choose diffeomorphisms $\psi_j$ such that the group $\hat{G}_j := \psi_j \circ G_j \circ \psi_j^{-1}$ is a subgroup of the group $\text{Isom}(g_0)$ of isometries of $g_0$.

Now suppose that the diffeomorphisms $\psi_j$ converge in the metric $\gamma_M$ defined above to the identity, as they can certainly be chosen to do. Then the groups $\hat{G}_j$, which are subgroups of $\text{Isom}(g_0)$, converge to $G_0$ in the Lie group topology of $\text{Isom}(g_0)$: for every open set $U$ containing $G_0$, there is a $j_U$ such that, for any $j > j_U$, every element of $\hat{G}_j$ lies in $U$.

Now one can apply this theorem of Montgomery and Samelson ([20]): If $G$ is a compact Lie group and $H$ a closed subgroup, then there is an open neighborhood $U$ of $H$ such that every subgroup of $G$ lying in $U$ is conjugate to a subgroup of $H$.

This now gives the result on compact group actions that we were seeking.

\[\square\]

Our goal here is to show how to reduce the domain case to the compact manifold situation described in the Lemma. Specifically, we want to prove the following proposition:

**Proposition 2.1** Suppose that $\{\Omega_j\}$ is a sequence of bounded domains in $\mathbb{R}^N$ which containment-converges to $\Omega_0$ in the sense of Definition 2.1 and that, for each $j$, $G_j$ is a compact group of diffeomorphisms of $\text{cl}(\Omega_j)$ and that the sequence $\{G_j\}$ converges $C^\infty$ normally to a compact group $G_0$ of diffeomorphisms of $\text{cl}(\Omega_0)$ [convergence in the sense of Definition 2.3]. Here, of course, $\text{cl}$ denotes
the closure of the indicated set. Then, for all sufficiently large \( j \), the group \( G_j \) is isomorphic to a subgroup of \( G_0 \).

The essential tool is to use group-invariant exhaustion functions to find a smoothly bounded sub-domain of \( \Omega_0 \) that is taken to itself by each element of the group \( G_0 \) and then pass to the “double” of these sub-domains to form a compact manifold. Then one does a similar construction to nearby \( G_j \)-invariant sub-domains of \( \Omega_j \) and thus attains the situation of Ebin’s Theorem. We now describe this situation in more detail, following the arguments developed in [9]:

**Definition 2.4** A real-valued function \( \rho: \Omega \to \mathbb{R} \) on a domain \( \Omega \) is said to be an exhaustion function if, for every \( \alpha \in \mathbb{R} \), the set \( \rho^{-1}\left(\left(-\infty, \alpha\right]\right) \) is compact—that is, the sub-level sets of \( \rho \) are compact.

Exhaustion functions of course always exist on domains and indeed on manifolds in general. One for (not necessarily bounded) domains that frequently occurs in complex analysis is \( \max(\|z\|^2, -\log \text{dist}(z, \text{the complement of the domain})) \).

Exhaustion functions with special properties play an important role, for instance, in the study of Stein manifolds; these are of course more difficult to construct.

Now suppose that \( G \) is a compact group of diffeomorphisms on a domain \( \Omega \) and suppose that \( \rho \) is an exhaustion function on \( \Omega \). Then the function \( \hat{\rho} \) defined by \( \hat{\rho}(z) := \int_G \rho(g(z)) \, d\lambda(g) \), where \( d\lambda \) is the normalized Haar measure on \( G \), is also an exhaustion function, as one easily sees. This function is \( G \)-invariant in the sense that \( \hat{\rho}(g(z)) = \hat{\rho}(z) \). Thus its sub-level sets are invariant under the action of \( G \): a given sub-level set is mapped to itself by each element of \( G \).

If \( \rho \) is \( C^\infty \), then \( \hat{\rho} \) is also \( C^\infty \). In this case, for all sufficiently large \( \alpha \), except for a set of measure 0, the sub-level set \( \hat{\rho}^{-1}(\left(-\infty, \alpha\right]\) is a compact \( C^\infty \) manifold-with-boundary. This follows from Sard’s Theorem: one need only take \( \alpha \) so large that the sub-level set is nonempty and such that \( \alpha \) is a regular value for \( \hat{\rho} \).

Now we return to the situation of a sequence of compact groups \( G_j \) converging in our previous sense to a compact group \( G_0 \). As in the general setting above, we choose a \( C^\infty \) exhaustion function \( \rho_0 \) and average it over \( G_0 \) to get a \( G_0 \)-invariant, \( C^\infty \) exhaustion function \( \hat{\rho}_0 \).

Because \( G_j \) is defined on \( \Omega_j \) while \( \hat{\rho}_0 \) is defined on \( \Omega_0 \), we cannot average \( \hat{\rho}_0 \) to make it \( G_j \)-invariant. We can, however, perform the averaging on arbitrary compact subsets.

Specifically, choose \( \alpha \) as above, so that \( \hat{\rho}_0^{-1}(\left(-\infty, \alpha\right] \) is nonempty and of course is a compact subset of \( \Omega_0 \). Let \( L \) be a compact subset of \( \Omega_0 \) which contains \( \hat{\rho}_0^{-1}(\left(-\infty, \alpha\right] \) in its interior and let \( L_1 \) be a compact subset of \( \Omega_0 \) that contains \( L \) in its interior.

Because the sequence \( G_j \) converges to \( G_0 \), it follows easily that, for \( j \) sufficiently large, the images under \( G_j \) of points of \( L \) lie in \( L_1 \). It then follows in addition that one can average the function \( \rho_0 \) over the action of \( G_j \), as in the process of averaging to construct \( \hat{\rho} \). Denote this new function on \( L \) by \( \hat{\rho}_j \).
Note that, because the elements of $G_j$ are, for $j$ large, close to those of $G_0$, the function $\hat{\rho}_j$ is $C^\infty$-close (i.e., $\gamma_L$-close) to $\hat{\rho}_0$ on $L$. In particular, the sub-level set $L_1 \cap \hat{\rho}_j^{-1}(-\infty, \alpha]$ will be, for $j$ sufficiently large, a smooth manifold-with-boundary which is $C^\infty$ close to $\hat{\rho}_0^{-1}(-\infty, \alpha]$.

In particular, if we choose a regular value $\alpha$ for $\hat{\rho}_0$ with the sub-level set $M_0 := \hat{\rho}_0^{-1}(-\infty, \alpha]$ nonempty then, for all $j$ sufficiently large, the sub-level set $M_j := \hat{\rho}_j^{-1}(-\infty, \alpha]$ will be a nonempty $C^\infty$ manifold-with-boundary. Moreover it will be close to $\hat{\rho}_0^{-1}(-\infty, \alpha]$ in the $C^\infty$ sense. Namely, there will be a sequence of diffeomorphisms $\phi_j: M_0 \to M_j$ which converges in the $C^\infty$ sense to the identity on $M_0$.

The next step of the proof is to form the doubles of the invariant sub-domains with smooth boundary and extend the compact group actions to them. This will make it possible to apply the lemma above to the present situation.

For this, suppose that $\Omega$ is a domain, $M$ a compact subset that is a (nonempty) smooth manifold with boundary and $H$ a compact group of diffeomorphisms of $\Omega$ that maps $M$ to itself. By the usual averaging process, similar to the construction of the invariant exhaustion functions as already discussed, there is a Riemannian metric $g$ on $\Omega$ for which the elements of $H$ act as isometries, i.e., $H$ is contained in $\text{Isom}(g)$. Now the metric $g$ restricted to $M$ can be modified so as to remain invariant under $H$ while being a product metric at and near the boundary of $M$ (see [10] for an early instance of this construction). This modification is obtained by first noting that, if $N$ is the inward unit normal (relative to $g$) along the boundary $\partial M$, then there is an $\epsilon > 0$ such that the $g$-exponential map $E: \partial M \times [0, \epsilon) \to M$ defined by $E(p, s) = \exp_p (sN(p))$ is a diffeomorphism for $|s| < \epsilon$ and moreover $E(p, s)$, $p \in \partial M$, $0 \leq s < \epsilon$, is a diffeomorphism of manifolds with boundary onto a neighborhood $V$ of $\partial M$ in $M$. This is the usual tubular neighborhood construction. Then one obtains a product metric $h$ on the neighborhood of the boundary as $h = ds^2 + dp^2$, where $dp^2$ is the metric on $\partial M$ and we push this metric over via $E$ to the neighborhood $V$ of $\partial M$ in $M$. This is clearly invariant under $H$. Then one can extend this metric to all of $M$ in an $H$-invariant way, by taking a function $\phi$ on $V$ that depends on $s$ alone and hence is invariant under the $H$-action. This function is to be 1 in a neighborhood of $s = 0$, and hence as a function on $M$, is equal to 1 in a neighborhood of $\partial M$. And it is to be equal to 0 when $s > \epsilon/2$. Then $\phi h + (1 - \phi)g$ will be a metric on $M$ as desired: it is smooth on all of $M$, is invariant under $H$, and is a product metric near $\partial M$.

This metric now extends smoothly to be a metric $\tilde{h}$ on the double $\tilde{M}$ of $M$ in an obvious way. And the group $H$ acts on $\tilde{M}$ as a subgroup of the isometry group of $\tilde{h}$. This subgroup of the isometry group of $\tilde{h}$ will be denoted by $\tilde{H}$.

Our construction can clearly be taken to be stable with respect to the original $H$-invariant metric $g$ on $M$ in the sense that, if $g_1$ is another $H$-invariant metric on $M$ which is $C^\infty$ close to $g$, then the corresponding metric $\tilde{h}_1$ on the double $\tilde{M}$ of $M$ will be $C^\infty$ close to $\tilde{h}$.

With these ideas in mind, we return to the convergence situation as before. Namely, we continue to denote by $\tilde{M}_j$ the doubles of the $G_j$-invariant sub-level
sets, and let $\hat{G}_j$ denote the extension of the $G_j$. Now, when $j$ is large, there are diffeomorphisms $\beta_j : \hat{M}_0 \to \hat{M}_j$ which have the property that the pullback to $\hat{M}_0$ of the $G_j$-action on $M_j$ via $\beta_j$ converges in the sense of Lemma 2.1 above.

In particular, $G_j$ is then isomorphic to a subgroup of $G_0$, for all sufficiently large $j$. Note that, as such, these isomorphisms apply not to $G_j$ itself but to the restriction of $G_j$ to $M_j$. But, since $M_j$ has nonempty interior, the restriction of $G_j$ to be an action on the ($G_j$-invariant) set $M_j$ is injective: two isometries of a connected manifold which are equal on a nonempty open set are equal. (This follows easily by a standard continuation argument.) Hence the original $G_j$ are indeed isomorphic to a subgroup of $G_0$ when $j$ is sufficiently large. Thus the proposition is established.

3 Bergman Metric and Curvature with $C^2$ Stability Near the Strongly Pseudoconvex Boundary

Let $n > 1$ throughout this section. Denote by $\mathcal{D}_n$ the collection of bounded domains in $\mathbb{C}^n$ with $C^2$ smooth, strongly pseudoconvex boundary, equipped with the $C^2$ topology via the $C^2$ topology on defining functions. The goal of this section is to establish the following result, which is Klembeck’s theorem [16] for domains in $\mathcal{D}_n$, with $C^2$ stability. In the statement below the notation $S_\Omega(p; \xi)$ denotes the holomorphic sectional curvature of the Bergman metric of the domain $\Omega$ at $p$ along the holomorphic section generated by the tangent vector $\xi$.

**Theorem 3.1** Let $\Omega_0 \in \mathcal{D}_n$. Then, for every $\epsilon > 0$, there exist $\delta > 0$ and an open neighborhood $\mathcal{U}$ of $\Omega_0$ in $\mathcal{D}_n$ such that, whenever $\Omega \in \mathcal{U}$, for any $p \in \Omega$ satisfying $\text{dis}(p, \mathbb{C}^n \setminus \Omega) < \delta$,

$$\sup_{\Omega \in \mathcal{U}, \xi \in \mathbb{C}^n \setminus \{0\}} \left| S_\Omega(p; \xi) - \left( -\frac{4}{n+1} \right) \right| < \epsilon$$

for any $p \in \Omega$.

**Proof.** It suffices to show that the following cannot hold:

(†) $\exists \epsilon_0 > 0, \exists \{\Omega_\nu\} \subset \mathcal{D}_n$ such that $\Omega_\nu \to \hat{\Omega}$ in the $C^2$ topology as $\nu \to \infty$ and $\exists$ a sequence $\{p_\nu \in \Omega_\nu\}$ with $\lim_{\nu \to \infty} \text{dis}(p_\nu, \partial \Omega_\nu) = 0$ such that

$$\left| S_{\Omega_\nu}(p_\nu; \xi_\nu) + \frac{4}{n+1} \right| \geq \epsilon_0,$$

for every $\nu$.

Let $\hat{\Omega}, \Omega_\nu, p_\nu$ be as in Section 1. Since the goal is to show that

$$\lim_{\nu \to \infty} \left| S_{\Omega_\nu}(p_\nu; \xi_\nu) + \frac{4}{n+1} \right| = 0,$$
we may assume without loss of generality that \( \lim_{\nu \to \infty} p_\nu \) exists. Denote this limit by \( \hat{p} \). Notice that \( \hat{p} \in \partial \hat{\Omega} \).

Let \( q_\nu \in \partial \Omega_\nu \) be the closest boundary point of \( \Omega_\nu \) to \( p_\nu \) for every \( \nu = 1, 2, \ldots \). Then consider a sequence \( R_\nu : \mathbb{C}^n \to \mathbb{C}^n \) of complex rigid motions (i.e., unitary maps followed by translations) in \( \mathbb{C}^n \) and another rigid motion \( \hat{R} \) satisfying:

1. \( \hat{R}(\hat{p}) = 0 \) and \( R_\nu(q_\nu) = 0 \) for every \( \nu \).
2. \( R_\nu(\partial \Omega_\nu) \) for every \( \nu \), and \( \hat{R}(\partial \hat{\Omega}) \) are tangent at 0 to the hyperplane defined by \( \text{Re} z_1 = 0 \).
3. \( \lim_{\nu \to \infty} \| R_\nu - \hat{R} \|_{C^2} = 0 \), where the norm here is the \( C^2 \)-norm of mappings on an open neighborhood of the closure of \( \hat{\Omega} \) in \( \mathbb{C}^n \).

Notice that \( R_\nu(\Omega_\nu) \) converges to \( \hat{R}(\hat{\Omega}) \) in the \( C^2 \) topology on bounded domains with smooth boundaries. Therefore, without loss of generality, we may also assume the following:

1. \( 0 \in \partial \hat{\Omega} \cap \left( \bigcap_{\nu = 1}^{\infty} \partial \Omega_\nu \right) \).
2. \( \partial \hat{\Omega} \) and \( \partial \Omega_\nu \) (for every \( \nu = 1, 2, \ldots \)) share the same outward normal vector \( n = (-1, 0, \ldots, 0) \) at the origin.
3. \( p_\nu = (r_\nu, 0, \ldots, 0) \) with \( r_\nu > 0 \) for every \( \nu \).

Now we need the following three lemmas for the proof. The first is

**Lemma 3.1 ([14], cf. [8, Ch. 10])** There exists an open neighborhood \( U \) of the origin in \( \mathbb{C}^n \) such that

\[
\lim_{\nu \to \infty} \sup_{0 \neq \xi \in \mathbb{C}^n} \left| \frac{2 - S_{\Omega_\nu \cap U}(p_\nu; \xi)}{2 - S_{\Omega_\nu}(p_\nu; \xi)} - 1 \right| = 0.
\]

Notice that this lemma implies: if \( \lim_{\nu \to \infty} S_{\Omega_\nu \cap U}(p_\nu; \xi) \) exists, it will coincide with \( \lim_{\nu \to \infty} S_{\Omega_\nu}(p_\nu; \xi) \).

The next two lemmas convert the problem of understanding the boundary asymptotic behavior of the Bergman curvature to that of the stability of the Bergman kernel function in the interior under perturbation of the boundary:

**Lemma 3.2 ([14]; cf. [8, Ch. 10])** Let the sequence \( \{(p_\nu; \xi_\nu) \in \Omega_\nu \times (\mathbb{C}^n \setminus \{0\})\} \) be chosen as above. Let \( B^n \) denote the open unit ball in \( \mathbb{C}^n \). Then there exists a sequence of injective holomorphic mappings \( \sigma_\nu : \Omega_\nu \cap U \to \mathbb{C}^n \) satisfying the following properties:
(i) \( \sigma_\nu(p_\nu) = 0 \) (the origin of \( \mathbb{C}^n \)).

(ii) For every \( r \) with \( 0 < r < 1 \), there exists \( N > 0 \) such that

\[
(1 - r)B^n \subset \sigma_\nu(\Omega_\nu \cap U) \subset (1 + r)B^n
\]

for every \( \nu > N \).

Proof of Lemma 3.2: In our case the situation is simple, because all the points in the sequence \( \{p_\nu\} \) under consideration are located on the \( \text{Re}\,z_1 \)-axis.

Let \( \rho \) be a constant with \( 0 < \rho < 1 \), to be chosen later (depending on \( r \)).

Let

\[
E_\rho = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \text{Re}\,z_1 > (1 - \rho)(|z_1|^2 + \ldots + |z_n|^2) \}
\]

and

\[
S_\rho = \{ z \in \mathbb{C}^n : \text{Re}\,z_1 > (1 + \rho)(|z_1|^2 + \ldots + |z_n|^2) \}.
\]

Since \( \hat{\Omega} \) is a domain with \( C^2 \) smooth, strongly pseudoconvex boundary, there exists an open neighborhood \( U \) of the origin in \( \mathbb{C}^n \) and a biholomorphism-into \( \Psi : U \rightarrow \mathbb{C}^n \) such that

\[
\Psi(\hat{\Omega} \cap U) = \{ z \in \Psi(U) : \text{Re}\,z_1 > |z_1|^2 + \ldots + |z_n|^2 + R_2(z) \},
\]

where \( R_2(z) = o(|z_1|^2 + \ldots + |z_n|^2) \). Let \( V = \Psi(U) \). Shrinking the neighborhood \( U \) if necessary, one obtains that

\[
S_\rho \cap V \subset \Psi(\hat{\Omega} \cap U) \subset E_\rho.
\]

Because of the \( C^2 \) convergence, and by (1')--(3'), one deduces that there exists \( N > 0 \) such that

\[
S_\rho \cap V \subset \Psi(\Omega_\nu \cap U) \subset E_\rho
\]

for every \( \nu > N \). Now let \( \lambda_\nu \equiv |\Psi(p_\nu)| \) for every \( \nu \). Consider the dilatation maps

\[
\Lambda_\nu(z_1, \ldots, z_n) = \left( \frac{z_1}{\lambda_\nu}, \frac{z_2}{\sqrt{\lambda_\nu}}, \ldots, \frac{z_n}{\sqrt{\lambda_\nu}} \right).
\]

Notice here that the point sequence \( \Psi(p_\nu) \) approaches the origin non-tangentially to the hypersurface defined by \( \text{Re}\,z_1 = 0 \), which is tangent to \( \Psi(\partial\hat{\Omega}) \) at the origin.

Finally let

\[
\Phi(z_1, \ldots, z_n) = \left( \frac{z_1 - 1}{z_1 + 1}, \frac{2z_2}{z_1 + 1}, \ldots, \frac{2z_n}{z_1 + 1} \right)
\]

and

\[
\sigma_\nu = \Phi \circ \Lambda_\nu \circ \Psi
\]
for every $\nu$. Notice that the composition for each $\nu$ by the Möbius transformation $\Phi$ adjusts $\sigma_\nu(p_\nu)$ to the origin while preserving the unit ball. So, there exists an $\eta \in (0, 1)$ such that $\{\sigma_\nu\}$ yields a sequence of holomorphic maps satisfying the desired conclusion. 

The third and last lemma toward the proof of Theorem 3.1 is as follows:

**Lemma 3.3 ([24], [14]; cf. [8, Ch. 10])** Let $D$ be a bounded domain in $\mathbb{C}^n$ containing the origin 0. Let $\{D_\nu\}$ denote a sequence of bounded domains in $\mathbb{C}^n$ that satisfies the following convergence condition:

given $\epsilon > 0$, there exists $N > 0$ such that

$$(1 - \epsilon)D \subset D_\nu \subset (1 + \epsilon)D$$

for every $\nu > N$.

Then, for every compact subset $F$ of $D$, the sequence of Bergman kernel functions $K_{D_\nu}$ of $D_\nu$ converges uniformly to the Bergman kernel function $K_D$ of $D$ on $F \times F$.

Now we return to the proof of Theorem 3.1

Let $q_\nu, \xi_\nu, \widehat{\Omega}, \Omega_\nu$ be as above. Let $U$ be an open neighborhood of the origin as in Lemma 3.1. Taking a subsequence, we may assume that $q_\nu \in \Omega_\nu \cap U$ for every $\nu$. Select $\sigma_\nu$ as in Lemma 3.2.

Apply Lemma 3.3 (a theorem of Ramadanov [23]) to our setting, with $D_\nu = \sigma_\nu(\Omega_\nu \cap U)$ and $D = B^n$. The conclusion of Lemma 3.3 states that the sequence $K_{D_\nu}(z, \zeta)$ converges uniformly to $K_D(z, \zeta)$ on $F \times F$. This of course implies that the sequence $K_{D_\nu}(z, \zeta)$ converges to $K_D(z, \zeta)$. Notice that the functions now involved are holomorphic functions in the $z$ and $\zeta$ variables together. Therefore Cauchy estimates imply that $K_{D_\nu}(z, \zeta)$ converges uniformly to $K_D(z, \zeta)$ on $F \times F$ in the $C^k$ sense for any positive integer $k$. Since the holomorphic sectional curvature of the Bergman metric involves derivatives of the Bergman kernel function up to fourth order, we may conclude that $S_{\sigma_\nu(\Omega_\nu \cap U)}(0; \cdot)$ converges uniformly to $S_{B^n}(0; \cdot)$ on $\{\xi \in \mathbb{C}^n : \|\xi\| = 1\}$. Notice that the latter is the constant function with value $-4/(n + 1)$.

Combining this result with the localization lemma (Lemma 3.1), the conversion lemma (Lemma 3.2) and the fact that every biholomorphism is an isometry for the Bergman metric, we see that:

$$-\frac{4}{n + 1} = \lim_{\nu \to \infty} S_{\sigma_\nu(\Omega_\nu \cap U)}(0; d\sigma_\nu|_{q_\nu}(\xi_\nu))$$

$$= \lim_{\nu \to \infty} S_{\sigma_\nu(\Omega_\nu \cap U)}(\sigma_\nu(q_\nu); d\sigma_\nu|_{q_\nu}(\xi_\nu))$$

$$= \lim_{\nu \to \infty} S_{\Omega_\nu \cap U}(q_\nu; \xi_\nu)$$

$$= \lim_{\nu \to \infty} S_{\Omega_\nu}(q_\nu; \xi_\nu).$$
This completes the proof of Theorem 3.1.

Remark 3.1 (Completeness of the Bergman metric) The Bergman metric of a bounded strongly pseudoconvex domain is known to be complete (\cite{1}; for the more general case cf. \cite{23}). Since the scaled limit shown in the proof of Lemma 3.2 is the unit ball, a variation of that proof-argument also yields the same conclusion as \cite{1} regarding completeness also (see \cite[Section 10.1.7]{8}).

4 Stable $C^k$-Extension of Automorphisms

The purpose of this section is to establish the stability of the extension theorem for the automorphisms of a bounded strongly pseudoconvex domain under $C^k$ perturbation for finite $k$.

4.1 Convergence of Lempert’s Representative Map

Let $X, Y$ be complex Banach spaces. Let $\phi: U \to Y$ be a map from an open subset $U$ of $X$ into $Y$. The map $\phi$ is said to be differentiable at $x \in X$, if there exists a bounded linear map $D_x\phi: X \to Y$ such that

$$
\|\phi(x + h) - \phi(x) - (D_x\phi)(h)\|_Y = o(\|h\|_X)
$$

as $\|h\|_X \to 0$. Let $L(X,Y)$ denote the set of bounded linear maps from $X$ into $Y$. It is naturally equipped with the operator norm and hence becomes a Banach space. Then $\phi$ is said to be $C^1$ on $U$ if $D_x\phi$ exists for all $x \in U$ and $D\phi: x \in U \mapsto D_x\phi \in L(X,Y)$ is continuous.

It is also well established what it means for $\phi$ to belong to the class $C^k$ (cf., e.g., \cite{21}). To understand this point, consider the space $L(X \times \cdots \times X,Y)$ of bounded $k$-linear maps with values in $Y$. For an $S \in L(X \times \cdots \times X,Y)$, define its norm as follows:

$$
\|S\|_k = \sup\{\|S(h_1,\ldots,h_k)\|_Y : \|h_1\|_X \leq 1,\ldots,\|h_k\|_X \leq 1\}.
$$

One more piece of notation is necessary: for a $k$-linear map $S$, a $(k-1)$-linear map $[S](h)$ is defined by

$$
[S](h_1,\ldots,h_{k-1}) := S(h,h_1,\ldots,h_{k-1}).
$$

Now the idea of a map belonging to the class $C^k$ can be defined inductively: the map $\phi$ is said to be $C^k$ at $x \in X$, for $k = 1,2,\ldots$, if there exits a bounded $k$-linear map $D_x^k\phi: \underbrace{X \times \cdots \times X}_k \to Y$ such that

$$
\|D_{x+h}^{k-1}\phi - D_x^{k-1}\phi - [D_x^k\phi](h)\|_{k-1} = o(\|h\|_X)
$$

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as $h \to 0$ and $D^k\phi: x \in U \mapsto D^k_x\phi \in L(X \times \cdots \times X, Y)$ is continuous. It is also known that such a $D^k_x\phi$ is symmetric $k$-linear.

Similarly, we may define the concept of Hölder class. For an $\alpha$ with $0 < \alpha \leq 1$, a map $\phi$ is said to belong to the class $C^{k,\alpha}$ if $\phi$ is $C^k$ and

$$\sup_{x, y \in U, x \neq y} \frac{\|D^k_x\phi - D^k_y\phi\|_k}{\|x - y\|_X^\alpha} < \infty.$$ 

Throughout this section, we denote by $\Delta$ the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. We shall follow the terminology of [19] closely. Let $s$ be such that $0 < s < \alpha$ and set

$$X_n = \{f : \partial \Delta \to \mathbb{C}^n \mid f \in C^{0,s}\}$$
$$Y_n = \{f \in X_n : f \text{ admits a holomorphic continuation to } \text{cl}(\Delta)\}$$
$$Y_n^\perp = \{f \in X_n : f \text{ admits an anti-holomorphic continuation to } \text{cl}(\Delta) \text{ with } f(0) = 0\}.$$ 

Notice that $X_n = Y_n \oplus Y_n^\perp$.

Let $\Omega = \Omega_\rho$ be a bounded strictly convex domain defined by the $C^{k+1,\alpha}$ defining function $\rho$. Then there exists a convex open neighborhood $V$ of $\text{cl}(\Omega)$ such that $\Omega = \Omega_\rho = \{z \in V : \rho(z) < 0\}$, where the defining function $\rho : U \to \mathbb{R}$, defined on a convex open set $U$ with $\text{cl}(V) \subset U$, is of class $C^{k+1,\alpha}$ ($k \geq 1, 0 < \alpha < 1$) with $d\rho \neq 0$ at any point of $\partial \Omega$. We may further assume without loss of generality that

(1) $\rho : U \to \mathbb{R}$ is compactly supported

and

(2) the real Hessian of $\rho$ is strictly positive at every point of $\partial \Omega$.

Let $\mathcal{N}$ be a $C^{k+1,\alpha}$ neighborhood of $\rho$ chosen so small that every element of $\mathcal{N}$ has its real Hessian strictly positive at every point of $V$. We may require further that there exists a constant $R' > 0$ such that, if $\eta, \tau \in \mathcal{N}$, then $\|\eta - \tau\|_{C^{k+1,\alpha}(U)} < 1$ and $\|\eta\|_{C^{k+1,\alpha}(U)} < R'$.

Let $p$ be a point in $\Omega$ and let $W$ a neighborhood of $p$ in $\Omega$ such that $W \subset \Omega_\rho$ for all $\eta \in \mathcal{N}$. Define $\Theta : \mathcal{N} \oplus (\mathbb{C}^n \setminus \{0\}) \oplus W \to Y_\rho$ by $\Theta(\eta, \zeta, q) = e_{\eta, \zeta, q}$, where $e_{\eta, \zeta, q}$ is the stationary map (= extremal map) from $\text{cl}(\Delta)$ to $\text{cl}(\Omega_\eta)$ satisfying $e_{\eta, \zeta, q}(0) = q$ and $e_{\eta, \zeta, q}'(0) = \mu \zeta$ for some $\mu > 0$.

**Proposition 4.1** The map $\Theta$ is locally $C^{k,\alpha-s}$ for any $0 < s < \alpha$.

**Proof:** Let $(\eta, v, q) \in \mathcal{N} \oplus \mathbb{C}^n \setminus \{0\} \oplus \mathcal{W}$. We shall prove that $\Theta$ is $C^{k,\alpha-s}$ near $(\eta, v, q)$. Let $e = e_{\eta, v, q} = (e_1, \ldots, e_n) : \text{cl}(\Delta) \to \text{cl}(\Omega_\eta)$ and $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_n)$ be...
the dual map of $e$. (See [18] for the definition of the dual map and its basic properties.) Since $\tilde{e}$ has no zeros, there exist two components which do not vanish simultaneously by a generic linear change of coordinates. Hence we may assume without loss of generality that $\tilde{e}_1$ and $\tilde{e}_2$ do not vanish simultaneously on $\text{cl}(\Delta)$. It is also shown in [18] that $\tilde{e}$ extends to a $C^{k,\alpha}$ map up to the boundary, and that there exist functions $G_1, G_2 \in C^{k,\alpha}(\text{cl}(\Delta))$ that are holomorphic in $\Delta$ and satisfy $\tilde{e}_1 G_1 + \tilde{e}_2 G_2 \equiv 1$. Define the holomorphic matrix $H$ on $\Delta$ by

$$
H = \begin{pmatrix}
\tilde{e}'_1 & -\tilde{e}_2 & -G_1 \tilde{e}_3 & \cdots & -G_1 \tilde{e}_n \\
\tilde{e}'_2 & \tilde{e}_2 & -G_2 \tilde{e}_3 & \cdots & -G_2 \tilde{e}_n \\
\tilde{e}_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{e}_n & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

Notice that $H \in C^{0,\alpha}(\text{cl}(\Delta))$ and $\det(H) \neq 0$ on $\text{cl}(\Delta)$. Set

$$Y_{n}^{R,U} = \left\{ f \in Y_n : \|f\|_{C^k(\text{cl}(\Delta))} < R, f(\partial \Delta) \subset U \right\}$$

and define the map

$$\Phi : \mathcal{N} \oplus \mathbb{C}^n \setminus \{0\} \oplus \mathcal{W} \oplus Y_{n}^{R,U} \oplus \mathbb{R} \rightarrow T \oplus Y_{n-1}^+ \oplus \mathbb{C}^n \oplus \mathbb{C}^n$$

by

$$\Phi(r, v, f, \lambda) = \left( r \circ f, \pi \left( \left\langle H^t r_z \circ f \right\rangle, f(0) - q, f'(0) - \lambda v \right) \right),$$

where:

(i) $T = \{ g : \partial \Delta \rightarrow \mathbb{R} : g \in C^{0,\alpha} \}$,

(ii) $\pi : Y_{n-1} \rightarrow Y_{n-1}^+$ is defined by $\pi \left( \sum_{k=1}^{\infty} a_k z^k \right) = \sum_{k=1}^{\infty} a_k z^k$, and

(iii) $\langle H^t r_z \circ f \rangle_j$ denotes the $j$-th component of $H^t r_z \circ f$ and

$\langle H^t r_z \circ f \rangle = ((H^t r_z \circ f)_1, \ldots, (H^t r_z \circ f)_n)$.

Then $f : \text{cl}(\Delta) \rightarrow \text{cl}(\Omega)$ is an extremal map satisfying $f(0) = q, f'(0) = \lambda v$ if and only if $\Phi(r, v, f, \lambda) = 0$. So, according to [19], we only need to prove that $\Phi$ is $C^{k,\alpha-\alpha}$. For this purpose define the map $\Psi : \mathcal{N} \oplus Y_{n}^{R,U} \rightarrow T$ by

$$\Psi(r, f) = r \circ f.$$ Then we pose the following

**Claim.** $\Psi$ is $C^{k,\alpha-\alpha}$.

We shall prove this claim by induction on $k$. We need some notation. For a domain $\Omega$, $k \in \mathbb{Z}^+$, and $0 < \alpha \leq 1$, denote by

$$\|g\|_{C^{k,\alpha}(\text{cl}(\Omega))} = \sup_{x \in \text{cl}(\Omega)} |D^\gamma g(x)| + \sup_{x, y \in \text{cl}(\Omega)} \frac{|D^\gamma g(x) - D^\gamma g(y)|}{|x - y|^\alpha}.$$
Moreover, $A \lesssim B$ will mean that $A \leq CB$ for some constant $C$. In turn, $A \lessgtr B$ will mean that $A \to 0$ whenever $B \to 0$.

Let $j \in \{0, \ldots, k\}$. Let $\mathcal{N}_j = \{r \in C^{j+1, \alpha}(U) : ||r||_{C^{j+1, \alpha}(U)} < R'\}$. Define $\Psi_j : \mathcal{N}_j \oplus Y_{n}^{R,U} \to T$ by $\Psi_j(r, f) = r \circ f$. Suppose that, for all $r, \tau \in \mathcal{N}_j$, we have $||r - \tau||_{C^{j, \alpha}(U)} < 1$.

In case $j = 0$, it suffices to show that

$$\|\Psi_0(r, f) - \Psi_0(\tau, g)\|_{C^{0, \alpha}(\partial \Delta)} \lesssim \left(||r - \tau||_{C^{0, \alpha}(U)} + \|f - g\|_{C^{0, \alpha}(\partial \Delta)}\right)^{\alpha-s}$$

For $x \in \partial \Delta$,

$$|r \circ f(x) - \tau \circ g(x)| \leq |r \circ f(x) - r \circ g(x)| + |r \circ g(x) - \tau \circ g(x)|$$

$$\lesssim |f(x) - g(x)|^\alpha + |(r - \tau) \circ g(x)|$$

$$\lesssim \left(||f - g||_{C^{0, \alpha}(\partial \Delta)} + ||r - \tau||_{C^{0, \alpha}(U)}\right)^{\alpha-s}.$$  

For $x, y \in \partial \Delta$, let $\delta(x, y) = r \circ f(x) - \tau \circ g(x) - r \circ f(y) + \tau \circ g(y)$. Then

$$|\delta(x, y)| \leq |r \circ f(x) - \tau \circ g(x)| + |r \circ g(x) - \tau \circ g(x)| + |r \circ f(y) - \tau \circ g(y)| + |r \circ g(y) - \tau \circ g(y)|$$

$$\leq 2R'||f(x) - g(x)||^\alpha + 2||r - \tau||_{C^{0, \alpha}(U)}$$

and

$$|\delta(x, y)| \leq |r \circ f(x) - f(y)|^\alpha + |\tau \circ g(x) - \tau \circ g(y)|$$

$$\leq R'|f(x) - f(y)|^\alpha + R'|g(x) - g(y)|^\alpha$$

$$\leq 2RR'|x - y|^\alpha.$$  

This implies that

$$|\delta(x, y)| \lesssim (||f - g||_{C^{0, \alpha}(\partial \Delta)} + ||r - \tau||_{C^{0, \alpha}(U)})^{\alpha-s}|x - y|^s,$$

which proves the case $j = 0$.

Let $j > 0$. Suppose that $\Psi_j : \mathcal{N}_j \oplus Y_{n}^{R,U} \to T$ is of class $C^{j, \alpha-s}(U)$. Then, since

$$D_{(r,f)}\Psi_{j+1}(\tau,g) = (r' \circ f)g + \tau \circ f = \Psi_j(r', f)g + \Psi_j(\tau, f),$$

it follows that $\Psi_{j+1}$ is of $C^{j+1, \alpha-s}(U)$. This proves the claim.

Since $\pi$ is a bounded linear map, the second component of $\Phi$ is also of class $C^{k, \alpha-s}(U)$. The proof of the proposition is now complete. \qed
Next, for \( r \in \mathcal{N}, q \in W \), consider Lempert’s representation map at \( q \) for the domain \( \Omega_r \). We have \( L_{r,q} : \text{cl}(\mathbb{B}^n) \to \text{cl}(\Omega_r) \) defined by \( L_{r,q}(\zeta) = \Theta(r, \zeta, q)(|\zeta|) = e_{r,\zeta,q}(|\zeta|) \). The following proposition discusses the convergence of these representation maps.

**Proposition 4.2** Let \( \rho_j, \rho \in \mathcal{N} \) and let \( p_j, p \in W \) be such that \( \| \rho_j - \rho \|_{C^{k+1,\alpha}(U)} \to 0 \), \( |p_j - p| \to 0 \) as \( j \to \infty \). Set the notation \( L_j := L_{\rho_j, p_j}, L := L_{\rho, p} \) and \( \mathbb{B}_\delta^n := \mathbb{B}^n \setminus \{ z \in \mathbb{C}^n : |z| < \delta \} \). Then, for \( 0 < \beta < \alpha \) and \( 0 < \delta < 1 \), Lempert’s representation maps \( L_j \) for \( \Omega_{p_j} \) converge to Lempert’s representation map \( L \) for \( \Omega_p \) on \( \mathbb{B}_\delta^n \) in the \( C^{k,\beta} \) norm, as \( j \to \infty \).

**Proof:** Let \( ev : Y_n \to \mathbb{C}^n \) be defined by \( ev(q) = g(1) \) (here “ev” stands for “evaluation” map). Since \( L(\zeta) = \Theta(\rho, \zeta, p)(1) = \Theta(\rho, \zeta, p) \) for \( \zeta \in \partial \mathbb{B}_\delta^n \), \( ev \) is bounded linear. Write \( D^\ell \Theta(\zeta)(x_1, \ldots, x_n)(1) = \left( \frac{\partial^{\ell}}{\partial x_1 \cdots \partial x_n} \right) \Theta(\zeta)(x_1, \ldots, x_n)(1) \). So \( \| L_j - L \|_{C^{k,\beta}(\mathbb{B}_\delta^n)} \to 0 \) as \( j \to \infty \).

Given \( v \in \mathbb{C}^n, |v| = 1, \xi \in \Delta \), denote by \( e \) the extremal map satisfying \( e(0) = p, e'(0) = \mu v \) for some \( \mu > 0 \). Then \( L(\xi v) = e(\xi |v|) = e(\xi) \). This implies that \( L(\xi v) \) is holomorphic with respect to \( \xi \). Now the Poisson integral formula for \( \Delta \) yields the desired conclusion. \( \square \)

### 4.2 A Simultaneous Extension Theorem for Automorphisms

The next goal is to establish the following theorem:

**Theorem 4.1** *(Uniform extension)* Let \( \Omega_j, \Omega \) be strongly pseudoconvex, bounded domains in \( \mathbb{C}^n \) with \( C^{k+1,\alpha}(k \in \mathbb{Z}, k \geq 2, 0 < \alpha \leq 1) \) boundaries such that \( \Omega_j \) converges to \( \Omega \) as \( j \to \infty \) in the \( C^{k+1,\alpha} \) topology. Let a sequence \( \{ f_j \in \text{Aut}(\Omega_j) : j = 1, 2, \ldots \} \) be given. Then, for any \( \beta \) with \( 0 < \beta < \alpha \), the sequence \( f_j \) (every one of which extends to a \( C^{k,\beta} \) diffeomorphism of the closure \( \text{cl}(\Omega_j) \)) by the ‘sharp extension theorem’ of Lempert [19] admits a subsequence \( \Omega_{j_\ell} \) and \( f_{j_\ell} \in \text{Aut}(\Omega_{j_\ell}) \) that converges to a \( C^{k,\beta} \)-diffeomorphism, the extension of \( f \in \text{Aut}(\Omega) \), in the \( C^{k,\beta} \) topology.

This indeed is a normal family theorem together with Hölder convergence up to the boundary. Of course precise definitions and terminology are in order, which will be presented here as the exposition progresses.

**Definition 4.1** Let \( \Omega_j \) and \( \Omega \) be bounded strongly pseudoconvex domains in \( \mathbb{C}^n \) with \( C^{k,\alpha}(k \in \mathbb{Z}, k \geq 2, 0 < \alpha \leq 1) \) boundaries. As \( j \to \infty \), the sequence of domains \( \Omega_j \) is said to converge to \( \Omega \) in the \( C^{k,\alpha} \) topology, if there exist an open neighborhood \( U \) of \( \text{cl}(\Omega) \), \( C^{k,\alpha} \) diffeomorphisms \( F_j : U \to U \), and a positive integer \( N \) such that:
• \( \text{cl}(\Omega) \subset U \),
• \( \text{cl}(\Omega_j) \subset U \) for all \( j > N \),
• each \( F_j \) maps \( \text{cl}(\Omega) \) onto \( \text{cl}(\Omega_j) \) as a \( C^{k,\alpha} \) diffeomorphisms, for every \( j > N \), and
• \( \| F_j - \text{id} \|_{C^{k,\alpha}(U)} \to 0 \) and \( \| F_j^{-1} - \text{id} \|_{C^{k,\alpha}(U)} \to 0 \), as \( j \to \infty \).

In a similar manner, we say that the sequence of maps \( f_j \in C^{k,\alpha}(\Omega_j, \mathbb{C}^m) \) converges to \( f \in C^{k,\alpha}(\Omega, \mathbb{C}^m) \) in the \( C^{k,\alpha} \) sense, if \( \lim_{j \to \infty} \| f_j \circ F_j - f \|_{C^{k,\alpha}(\Omega)} = 0 \).

We now present several technical lemmas.

**Lemma 4.1** Let \( \Omega_j \) be a domain in \( \mathbb{R}^{n_j} \) for each \( j = 1, 2, 3 \), respectively. If

(i) \( g, h : \Omega_1 \to \Omega_2 \) are \( C^{k',\alpha'} \) maps that are injective,

(ii) \( f : \Omega_2 \to \Omega_3 \) is a \( C^{k'',\alpha''} \) map,

and

(iii) \( (k, \alpha) \) is the pair of the positive integer \( k \) and the real number \( \alpha \) satisfying \( k + \alpha = \min\{k' + \alpha', k'' + \alpha''\} \) and \( 0 < \alpha \leq 1 \),

then

(1) \( f \circ g \in C^{k,\alpha}(\Omega_1, \Omega_3) \)

and

(2) \( \| f \circ g - f \circ h \|_{C^{k,\beta}(\Omega_1)} \lesssim \| g - h \|_{C^{k,\alpha}(\Omega_1)} \) for any \( \beta \) with \( 0 < \beta < \alpha \).

**Proof:** We shall present the verification of (1) only, as our arguments are mostly by straightforward computation and the proof of (2) is similar. The chain rule implies that

\[
D^\ell (f \circ g)(x) = \sum (D^m f)(g(x))(D^{m_1} g(x))^{m_1} (D^{m_2} g(x))^{m_2} \cdots (D^{m_n} g(x))^{m_n},
\]

where \( \ell \) and \( m \) are multi-indices and \( m_j \) nonnegative integers satisfying \( |m| \leq |\ell| \) and \( \sum m_j' \leq |\ell| \). (We use the usual multi-index notation here; we omit detailed expressions as they are standard.) Note that

\[
\| f \circ g \|_{C^{k,\alpha}} = \sup_{x \in \Omega_1} |D^\gamma (f \circ g)(x)| + \sup_{x,y \in \Omega_1 \atop \gamma \neq \delta} \frac{|D^\gamma (f \circ g)(x) - D^\gamma (f \circ g)(y)|}{|x - y|^\alpha}.
\]

First, one sees immediately that

\[
\sup_{x \in \Omega_1 \atop \gamma \neq \delta} |D^\gamma (f \circ g)(x)| \lesssim \| f \|_{C^{k,\alpha}(\Omega_2)} \sum_{\gamma} \| g \|_{C^{k,\alpha}(\Omega_1)}^{m_1' + \cdots + m_n'} < \infty.
\]
On the other hand,

\[
|D^n(f \circ g)(x) - D^n(f \circ g)(y)| = \left| \sum \left\{ D^m f(g(x)) \cdot (D^{m_1} g(x))^{m_1} \cdot \ldots \cdot (D^{m_n} g(x))^{m_n} \right\} - D^m f(g(y)) \cdot (D^{m_1} g(y))^{m_1} \cdot \ldots \cdot (D^{m_n} g(y))^{m_n} \right| \leq \sum \left\{ \left| (D^m f(g(x)) - D^m f(g(y))) \cdot (D^{m_1} g(x))^{m_1} \cdot \ldots \cdot (D^{m_n} g(x))^{m_n} \right| + \left| D^m f(g(y)) \cdot ((D^{m_1} g(x))^{m_1} - (D^{m_1} g(y))^{m_1}) \cdot (D^{m_2} g(x))^{m_2} \cdot \ldots \cdot (D^{m_n} g(x))^{m_n} \right| + \ldots + \left| (D^m f(g(y))) \cdot (D^{m_1} g(y))^{m_1} \cdot \ldots \cdot (D^{m_n} g(y))^{m_n} - (D^{m_1} g(y))^{m_1} \cdot \ldots \cdot (D^{m_n} g(y))^{m_n} \right| \right\},
\]

where \( P \) is an appropriate polynomial with \( P(0,...0) = 0 \). Hence (1) follows. We omit the proof of (2).

\[ \square \]

**Lemma 4.2** Let \( k \geq 1 \). Assume that \( \Omega_1, \Omega_2 \) are bounded domains in \( \mathbb{R}^n \) admitting \( C^{k,\alpha} \) diffeomorphisms \( f_j, f : \text{cl}(\Omega_1) \rightarrow \text{cl}(\Omega_2) \) satisfying \( \|f_j - f\|_{C^{k,\alpha} (\text{cl}(\Omega_1))} \rightarrow 0 \) as \( j \rightarrow \infty \). If \( \lim_{j \rightarrow \infty} \sup_{x \in \text{cl}(\Omega_2)} |f_j^{-1}(x) - f^{-1}(x)| = 0 \), then \( \lim_{j \rightarrow \infty} \|f_j^{-1} - f^{-1}\|_{C^{k,\beta} (\text{cl}(\Omega_2))} = 0 \) for any \( 0 < \beta < \alpha \).

**Proof:** The inverse function theorem implies that \( df_j^{-1} \big|_{f_j(y)} = (df_j \big|_y)^{-1} \) and \( df^{-1} \big|_{f(y)} = (df \big|_y)^{-1} \). Since \( \text{cl}(\Omega_1) \) and \( \text{cl}(\Omega_2) \) are compact, there exist a constant \( C > 0 \) and a positive integer \( N \) such that \( |\det(df \big|_y)| > C \) and \( |\det(df_j \big|_y)| > C \) for any point \( y \in \Omega_1 \) and any integer \( j > N \). Lemma 4.1 and its proof-argument above now yield the desired conclusion.

\[ \square \]

**Lemma 4.3** Let \( k \) be an integer with \( k \geq 2 \) and \( \alpha \) a real number satisfying \( 0 < \alpha \leq 1 \). If \( \Omega \) is a bounded, strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^{k+1,\alpha} \) boundary then, for any \( \beta \) with \( 0 < \beta < \alpha \), there exist an open neighborhood \( U \) of \( \Omega \) and a constant \( C \) such that \( \|f\|_{C^{k,\beta}(\text{cl}(\Omega'))} < C \) for any \( \Omega' \in U \) and any \( f \in \text{Aut}(\Omega') \).

**Proof:** Assume the contrary. Then there exists a sequence of strongly pseudoconvex domains \( \Omega_j \) with \( C^{k+1,\alpha} \) boundary converging to \( \Omega \) in the \( C^{k+1,\alpha} \)
topology and a sequence $f_j \in \text{Aut}(\Omega_j)$ such that
\[
\lim_{j \to \infty} \|f_j\|_{C^k,\beta(\text{cl}(\Omega_j))} = \infty.
\]

Then either

1. there exists a sequence $\{x_j \in \Omega_j : j = 1, 2, \ldots\}$ such that $|D^\gamma f_j(x_j)| \to \infty$ as $j \to \infty$ for some multi-index $\gamma$ satisfying $0 \leq |\gamma| \leq k$;

or

2. there exist $x_j, y_j \in \Omega_j$ such that $\lim_{j \to \infty} \frac{|D^\gamma f_j(x_j) - D^\gamma f_j(y_j)|}{|x_j - y_j|^\beta} = \infty$ for some multi-index $\gamma, |\gamma| = k$.

Suppose that (1) holds. Then, since the sequence $f_j$ converges to $f$ in the $C^\infty(K)$ topology on every compact subset $K$ of $\Omega$, it must be the case that $\lim_{j \to \infty} x_j = p \in \partial \Omega$ (taking a subsequence if necessary).

We shall arrive at the desired contradiction to (1) by means of the following three steps:

**Step 1. Adjustments.**

Here $F_j$ denotes the same diffeomorphism of $\text{cl}(\Omega)$ onto $\text{cl}(\Omega_j)$ as in Definition 4.1. Set $F_j(p) = p_j$, $f_j(p_j) = q_j$, $f(p) = q$. Take the invertible affine $\mathbb{C}$-linear transformations $T, T_j, t, t_j : \mathbb{C}^n \to \mathbb{C}^n$ such that

- $T_j(p_j) = T(p) = t_j(q_j) = t(q) = (0, \ldots, 0)$;

- the outward normal vectors to the boundary of $T_j(\Omega_j), T(\Omega), t_j(\Omega_j)$ and $t(\Omega)$, respectively, at $(0, \ldots, 0)$ are equal to $(1, 0, ..., 0)$; and

- $\lim_{j \to \infty} T_j = T$ and $\lim_{j \to \infty} t_j = t$.

Then $T_j(\Omega_j)$ converges to $T(\Omega)$ in the $C^{k+1,\alpha}$ topology, and also $t_j(\Omega_j)$ converges to $t(\Omega)$. Replacing therefore $f$ and $f_j$, respectively, by $t \circ f \circ T^{-1}$ and $t_j \circ f_j \circ T_j^{-1}$, we may assume that

- $\Omega, \Omega_j, \hat{\Omega}, \hat{\Omega}_j$ are bounded strongly pseudoconvex domains with $C^{k+1,\alpha}$ boundaries such that $\Omega_j$ (and $\hat{\Omega}_j$, respectively) converges to $\Omega$ (and to $\hat{\Omega}$, respectively) in the $C^{k+1,\alpha}$ topology. More precisely, there exist a neighborhood $U$ (and $\hat{U}$, respectively) of $\text{cl}(\Omega)$ (and of $\text{cl}(\hat{\Omega})$, respectively) and diffeomorphisms $F_j : \text{cl}(\Omega) \to \text{cl}(\Omega_j)$ and $\hat{F}_j : \text{cl}(\hat{\Omega}) \to \text{cl}(\hat{\Omega}_j)$ such that $F_j(0) = \hat{F}_j(0) = 0$ and the maps $F_j, F_j^{-1}, \hat{F}_j$ and $\hat{F}_j^{-1}$ converge to the identity map in the $C^{k+1,\alpha}$ sense.
• \( \rho, \rho_j = \rho \circ F_j^{-1}, \hat{\rho}, \hat{\rho}_j = \hat{\rho} \circ \hat{F}_j^{-1} \) are defining functions of \( \Omega, \Omega_j, \hat{\Omega}, \hat{\Omega}_j \), respectively, such that \( \| \rho - \rho_j \|_{C^{k+1, \beta}(U)} \to 0 \) and \( \| \hat{\rho} - \hat{\rho}_j \|_{C^{k+1, \beta}(\hat{U})} \to 0 \) as \( j \to \infty \) and

\[
(1,0,\ldots,0) = \left( \frac{\partial \rho}{\partial z_1}(0), \ldots, \frac{\partial \rho}{\partial z_n}(0) \right) = \left( \frac{\partial \rho_j}{\partial z_1}(0), \ldots, \frac{\partial \rho_j}{\partial z_n}(0) \right) = \left( \frac{\partial \hat{\rho}}{\partial z_1}(0), \ldots, \frac{\partial \hat{\rho}}{\partial z_n}(0) \right) = \left( \frac{\partial \hat{\rho}_j}{\partial z_1}(0), \ldots, \frac{\partial \hat{\rho}_j}{\partial z_n}(0) \right).
\]

• There exist biholomorphisms \( f_j : \Omega_j \to \hat{\Omega}_j, f : \Omega \to \hat{\Omega} \) and a sequence \( x_j \in \Omega_j \) converging to \( 0 \in \partial \Omega \) as \( j \to \infty \) such that \( f_j \) converges to \( f \) uniformly on every compact subset \( K \) of \( \Omega \) while \( |D^\ell f_j(x_j)| \to \infty \) as \( j \to \infty \) for some multi-index \( \ell \) with \( 1 \leq |\ell| \leq k \).

**Step 2. Simultaneous convexification.**

The content of this step is from [6]. To the expansion of \( \rho \) at 0,

\[
\rho(z) = 2\text{Re } z_1 + \text{Re } \sum \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0)z_iz_j + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \overline{z}_j}(0)z_i\overline{z}_j + o(|z|^2),
\]

apply the local biholomorphic change \( \Upsilon = (w_1, w_2, \ldots, w_n) \) of holomorphic coordinate system at the origin 0 defined by

\[
w_i(z) = \begin{cases} 2z_1 + \sum \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0)z_iz_j, & i = 1, \\ z_i, & i = 2, \ldots, n. \end{cases}
\]

The new defining function (we continue to use \( \rho \), as there is little danger of confusion) takes the form

\[
\rho = \text{Re } w_1 + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \rho}{\partial w_i \partial \overline{w}_j}(0)w_i\overline{w}_j + \varepsilon(w),
\]

where \( \varepsilon(w) = o(|w|^2) \). Note that \( \Upsilon(\Omega) \) is strictly convex in a small neighborhood of 0. Furthermore, there exists a positive integer \( N \) such that \( \Upsilon(U' \cap \Omega_j) \) is strictly convex for any \( j > N \). Let \( \rho_j \) denote \( \hat{\rho}_j \circ \Upsilon \), where \( \hat{\rho}_j \) is strictly convex on \( V' \cap \Omega_j \) for all \( j > N \). Set \( \hat{\rho}(z) = \text{Re } z_1 + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \overline{z}_j}(0)z_i\overline{z}_j + \sigma(z) \). There exists a positive constant \( R \) sufficiently large so that the real Hessian forms of
\( \rho(z) - \frac{|z|^2}{2R} - \text{Re} z_1 \) and \( \hat{\rho}_j(z) - \frac{|z|^2}{2R} - \text{Re} z_1 \) are positive-definite at every \( z \in V' \). Choose \( h \in C^\infty(\mathbb{R}) \) such that

\[
\begin{align*}
    h(x) &= 0 & \text{if} & & x \geq 1, \\
    0 &\leq h(x) \leq 1 & \text{if} & & 0 \leq x \leq 1, \\
    h(x) &= 1 & \text{if} & & x \leq 0.
\end{align*}
\]

Taking a larger value for \( N \) if necessary, we may have that the real Hessian forms of

\[
\text{Re} z_1 + \frac{|z|^2}{2R} + \frac{1}{N} h\left(\frac{|z| - \eta}{\eta}\right) (\hat{\rho}(z) - \frac{|z|^2}{2R} - \text{Re} z_1)
\]

and

\[
\text{Re} z_1 + \frac{|z|^2}{2R} + \frac{1}{N} h\left(\frac{|z| - \eta}{\eta}\right) (\hat{\rho}_j(z) - \frac{|z|^2}{2R} - \text{Re} z_1)
\]

are both positive-definite real Hessian at every point of \( V_\delta := \{ z \in \mathbb{C}^n : |z| < \delta \} \subset \subset V' \) whenever \( \eta \) satisfies \( 0 < \eta < \frac{\delta}{3} \). Take \( \eta > 0 \) such that \( 2^{2N+2}\eta < \frac{\delta}{3} \) and set

\[
\tau(z) = \text{Re} z_1 + \frac{|z|^2}{2R} + \frac{1}{N} \sum_{m=1}^N h\left(\frac{|z| - 2^{2m}\eta}{2^{2m}\eta}\right) (\hat{\rho}(z) - \frac{|z|^2}{2R} - \text{Re} z_1)
\]

and

\[
\tau_j(z) = \text{Re} z_1 + \frac{|z|^2}{2R} + \frac{1}{N} \sum_{m=1}^N h\left(\frac{|z| - 2^{2m}\eta}{2^{2m}\eta}\right) (\hat{\rho}_j(z) - \frac{|z|^2}{2R} - \text{Re} z_1).
\]

We further let \( C = \{ z \in \mathbb{C}^n : \tau(z) < 0 \} \), \( C_j = \{ z \in \mathbb{C}^n : \tau_j(z) < 0 \} \) and \( U'' = W^{-1}(V'_2) \). Then \( C, C_j \) are bounded strictly convex domains such that the restricted mappings \( \Upsilon|_{U'' \cap \Omega} : U'' \cap \Omega \to V'_2 \cap C \) and \( \Upsilon|_{U'' \cap \Omega_j} : U'' \cap \Omega_j \to V'_2 \cap C_j \) are biholomorphisms, and \( \tau_j \) converges to \( \tau \) in the \( C^{k+1, \beta} \) norm, for every \( \beta, 0 < \beta < \alpha \).

Apply the same process to \( \hat{\Omega} \) and to \( \hat{\Omega}_j \) at 0. Denote by \( \hat{C}, \hat{C}_j \) the respective strictly convex domains with defining functions \( \hat{\tau}, \hat{\tau}_j \) and \( \hat{W} : 
\hat{U} \to \hat{V} \) produced by the same procedures.

**Step 3. Estimates.**

Let \( \omega \in C \cap V' \cap \left( \bigcap_{j=1}^\infty C_j \right) \) be a point that admits an extremal map \( e : \text{cl}(\Delta) \to \text{cl}(C) \) satisfying

\[
e(0) = \omega, \ e(1) = 0, \ \text{and} \ e(\text{cl}(\Delta)) \subset \text{cl}(C) \cap V'.
\]

Let \( e'(0) = \mu v \) where \( |v| = 1 \). Let \( L : \text{cl}(\mathbb{B}^n) \to \text{cl}(C) \) ( \( L_j : \text{cl}(\mathbb{B}^n) \to \text{cl}(C_j) \), respectively) be the Lempert’s representative map of \( C \) (\( C_j \), respectively) at \( \omega \).
By Proposition 4.2 there exists a $\epsilon > 0$ such that $\lim_{j \to \infty} \| L_j - L \|_{C^{k,\beta}(\cl(B^n))} = 0$ for any $\beta$ with $0 < \beta < \alpha$. Let $\Gamma$ be a closed cone containing $v$ in $\cl(B^n)$ so that $L(\Gamma) \subset \cl(C) \cap V_\delta$ and $L_j(\Gamma) \subset \cl(C_j) \cap V_\delta$ for all $j > N$. Let

$$\Upsilon^{-1}(\omega) = \zeta, \quad f(\zeta) = \zeta', \quad f_j(\zeta) = \zeta_j, \quad \hat{\Upsilon}(\zeta) = \hat{\omega}, \quad \hat{\Upsilon}(\zeta_j) = \hat{\omega}_j$$

and let $\hat{L} : \cl(B^n) \to \cl(\hat{C})$ and $\hat{L}_j : \cl(B^n) \to \cl(\hat{C}_j)$, respectively, denote the Lempert representative map of $\hat{C}$ at the point $\hat{\omega}$ and the Lempert representative map of $\hat{C}_j$ at the point $\hat{\omega}_j$.

Consider now the composite maps $\hat{L}^{-1} \circ \hat{\Upsilon} \circ f \circ \Upsilon^{-1} \circ L : \Gamma \to B^n$ and $\hat{L}_j^{-1} \circ \hat{\Upsilon} \circ f_j \circ \Upsilon^{-1} \circ L_j : \Gamma \to B^n$. Denote by $h : \cl(D) \to \cl(C)$ the extremal map satisfying $h(0) = \omega$, $h'(0) = \lambda \zeta$, for some $\lambda > 0$, and by $\hat{h} = \hat{\Upsilon} \circ f \circ \Upsilon^{-1} \circ h : \cl(D) \to \cl(\hat{C})$ the extremal map satisfying

$$\hat{W}_1 \circ \hat{L} \left( \frac{|d(\hat{W} \circ f \circ W^{-1})|_\omega(\zeta)}{|d(\hat{W} \circ f \circ W^{-1})|_\omega(\zeta)} \right) = f \circ W^{-1} \circ L(\zeta).$$

By the same reasoning we also have

$$\hat{W}_j^{-1} \circ \hat{L} \left( \frac{|d(\hat{W} \circ f_j \circ W^{-1})|_\omega(\zeta)}{|d(\hat{W} \circ f \circ W^{-1})|_\omega(\zeta)} \right) = f_j \circ W^{-1} \circ L_j(\zeta).$$

Considering the left-hand sides of the preceding identities, for any $\beta, 0 < \beta < \alpha$ we obtain

$$\lim_{j \to \infty} \| f \circ W^{-1} \circ L - f_j \circ W^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)} = 0,$$

where $\Gamma_\delta = \Gamma \setminus \{ z \in \Gamma : |z| < \delta \}$. Therefore

$$\lim_{j \to \infty} \| f \circ \Upsilon^{-1} \circ L - f \circ F_j^{-1} \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)} = \lim_{j \to \infty} \| f \circ F_j^{-1} \circ \Upsilon^{-1} \circ L_j - f_j \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)}.$$

Hence

$$\| f \circ \Upsilon^{-1} \circ L - f \circ F_j^{-1} \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)}$$

$$\lesssim \| \Upsilon^{-1} \circ L - F_j^{-1} \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)}$$

$$\lesssim \| \Upsilon^{-1} \circ L - \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)} + \| \Upsilon^{-1} \circ L_j - F_j^{-1} \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)}$$

$$\lesssim \| L - L_j \|_{C^{k,\beta}(\Gamma_\delta)} + \| (\id - F_j^{-1}) \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\delta)}$$

$$\to 0 \text{ as } j \to \infty.$$
On the other hand, by the proof-argument of Lemma 4.1, it holds that
\[
\| f \circ F_j^{-1} \circ \Upsilon^{-1} \circ L_j - f_j \circ \Upsilon^{-1} \circ L_j \|_{C^{k,\beta}(\Gamma_\varepsilon)}
\]
\[
= \|(f - f_j \circ F_j) \circ F_j^{-1} \circ \Upsilon^{-1} \circ L_j\|_{C^{k,\beta}(\Gamma_\varepsilon)}
\]
\[
\leq \|(f - f_j \circ F_j)\|_{C^{k,\beta}(\sigma)},
\]
on a sufficiently small neighborhood \( \sigma \) of \( p \). This contradicts (1).

To complete the proof let us now suppose that (2) holds. If \( |x_j - y_j| > \kappa \) for some positive constant \( \kappa \), then
\[
\frac{|D^\ell f_j(x_j) - D^\ell f_j(y_j)|}{|x_j - y_j|^\beta} \leq \frac{2C}{\kappa^\beta}
\]
holds for some constant \( C \). Without loss of generality, we may assume that \( x_j \to p \in \partial \Omega \) and \( |x_j - y_j| < \kappa \). Suppose that there exist sequences \( x_j, y_j \in \Omega_j \) and a positive constant \( \nu \) such that \( x_j \to 0 \in \partial \Omega \) as \( j \to \infty \) and \( |x_j - y_j| < \nu \) so that
\[
\frac{|D^\ell f_j(x_j) - D^\ell f_j(y_j)|}{|x_j - y_j|^\beta} \to \infty
\]
as \( j \to \infty \) for some multi-index \( \ell \) where \( |\ell| = k \). Repeating Steps 1, 2 and 3 above, we again arrive at a contradiction. Hence the proof of Lemma 4.3 is complete.

Now we present

**Proof of Theorem 4.1.** Throughout the proof, we shall take subsequences from the \( \{f_j\} \) several times. But we denote them by the same notation \( f_j \), since there is little danger of any confusion.

By Cauchy estimates and the standard normal family theorem, for any compact subset \( K \) of \( \Omega \) we have
\[
\lim_{j \to \infty} \|f_j - f\|_{C^{k,\beta}(K)} = 0.
\]
Denote by \( K_\eta = \{z \in \Omega \mid \text{dist}(\partial \Omega, z) \geq \eta\} \). Then there exist \( N > 0 \) and \( \eta > 0 \) such that \( F_j(K) \subset K_\eta \subset \subset \Omega \) for all \( j > N \). So
\[
\|f_j \circ F_j - f\|_{C^{k,\beta}(K)} \leq \|f_j \circ F_j - f_j\|_{C^{k,\beta}(K)} + \|f_j - f\|_{C^{k,\beta}(K)} \to 0
\]
as \( j \to \infty \) for all \( \beta < \alpha \) by the proof of Lemma 4.2.

Let \( \lambda > 0 \). For \( x \in \text{cl}(\Omega) - K_\epsilon \), there exists \( y \in K_\epsilon \) such that \( |x - y| < \epsilon \). By Lemma 4.3, we have
\[
|D^\ell (f_j \circ F_j)(x) - D^\ell f(x)| \leq |D^\ell (f_j \circ F_j)(x) - D^\ell (f_j \circ F_j)(y)|
\]
\[
+ |D^\ell (f_j \circ F_j)(y) - D^\ell f(y)| + |D^\ell f(y) - D^\ell f(x)|
\]
\[
\lesssim 2|x - y|^\beta + \epsilon
\]
\[
\lesssim 2\epsilon^\beta + \epsilon.
\]
Since
\[ \sup_{x \in \text{cl}(\Omega)} \left| D^\ell(f_j \circ F_j)(x) - D^\ell f(x) \right| \]
\[ \leq \max \left\{ \sup_{x \in K_\alpha} \left| D^\ell(f_j \circ F_j)(x) - D^\ell f(x) \right|, \sup_{x \in \text{cl}(\Omega), K_\alpha} \left| D^\ell(f_j \circ F_j)(x) - D^\ell f(x) \right| \right\}, \]
there exist \( N > 0 \) and \( \epsilon \) such that
\[ \sup_{x \in \text{cl}(\Omega)} \left| D^\ell(f_j \circ F_j)(x) - D^\ell f(x) \right| < \lambda \]
for all \( j > N \).

Let \( \delta_\ell(x, y) := \frac{\left| D^\ell(f_j \circ F_j)(x) - D^\ell f(x) - D^\ell(f_j \circ F_j)(y) + D^\ell f(y) \right|}{|x - y|^{3}} \). Then
\[ \sup_{x, y \in \text{cl}(\Omega), |\ell| = k} \delta_\ell(x, y) \leq \max \left( \sup_{x \in \text{cl}(\Omega), y \in K_\alpha, |\ell| = k} \delta_\ell(x, y), \sup_{x, y \in \text{cl}(\Omega) \setminus K_\alpha, |\ell| = k} \delta_\ell(x, y) \right). \]  \hspace{1cm} (1)

Consider the first supremum in the right-hand side of (1). For \( x \in \text{cl}(\Omega), y \in K_\alpha \), there exists \( z \in K_\epsilon \) such that \( \text{dist}(K_\epsilon, x) = |x - z| \). Therefore we see that
\[ \delta_\ell(x, y) \leq \frac{|D^\ell(f_j \circ F_j)(x) - D^\ell f(x) - D^\ell(f_j \circ F_j)(z) + D^\ell f(z)|}{|x - y|^3} \]
\[ + \frac{|D^\ell(f_j \circ F_j)(z) - D^\ell f(z) - D^\ell(f_j \circ F_j)(y) + D^\ell f(y)|}{|x - y|^3} \]
\[ \lesssim \delta_\ell(x, z) + \delta_\ell(z, y), \]
because \( |x - y| \geq |x - z| \) and \( |y - z| \leq |y - x| + |x - z| \leq 2|x - y| \). Notice now that, for \( \mu \) satisfying \( \beta + \mu < \alpha \), we have that \( \delta_\ell(x, z) \lesssim |x - z|^\mu < \epsilon^\mu \). So
\[ \sup_{x \in \text{cl}(\Omega), y \in K_\alpha, |\ell| = k} \delta_\ell(x, y) < \lambda \]
for any \( j > N \). (For this last, one needs to adjust the sizes of \( N \) and \( \epsilon \) if necessary.)

Consider now the second supremum in the right-hand side of (1). Let \( x, y \in \text{cl}(\Omega) - K_\alpha \). If \( |x - y| < \epsilon \), then for \( \mu \) satisfying \( \beta + \mu < \alpha \), \( \delta_\ell(x, y) < |x - y|^{\mu} < \epsilon^\mu \). If \( |x - y| \geq \epsilon \), let \( z \) be a point in \( K_\epsilon \) satisfying \( |x - z| = \text{dist}(K_\epsilon, x) \). Then \( \delta_\ell(x, y) \lesssim \delta_\ell(x, z) + \delta_\ell(z, y) \), since \( |x - z| < \epsilon < |x - y| \) and \( |y - z| < 2|x - y| \). So
\[ \sup_{x, y \in \text{cl}(\Omega) \setminus K_\alpha, |\ell| = k} \delta_\ell(x, y) < \lambda. \]

Since \( \lambda > 0 \) is arbitrary, we see that
\[ \lim_{j \to \infty} \sup_{x, y \in \text{cl}(\Omega), |\ell| = k} \delta_\ell(x, y) = 0 \]
for any \( \beta < \alpha \). This completes the proof of Theorem 4.1.$\square$
5 Conjugation by Diffeomorphism

For isometries of compact Riemannian manifolds, semicontinuity involves not just that nearby metrics have isometry groups which are isomorphic to subgroups of the unperturbed metric, but that the isomorphisms are obtainable via conjugation by diffeomorphism (cf. [2] and [13]). This conjugation by diffeomorphism actually applies in the case of bounded $C^\infty$ strongly pseudoconvex domains as well, e.g. [9, 8]. Naturally, the $C^\infty$ hypothesis used in these references is, as usually happens, replaceable by a finite differentiability hypotheses simply by tracing through the arguments and checking how many derivatives are needed.

In this section, the subject will be investigated of the finite differentiability version of the conjugation by diffeomorphism results already shown in the references indicated in the $C^\infty$ case. These results are of active interest because, by this time, quite precise results are known about extension to the boundary with finite smoothness of automorphisms of bounded strongly pseudoconvex domains with boundaries of finite smoothness. In particular, the results of the previous sections give motivation to study the issues discussed in the present section.

In the $C^\infty$ version presented in [10] and [8], the basic technique was to pass to the double in the topologist’s sense of the domain, thus creating a situation to which the compact manifold results could be applied. This technique can still be applied in the present case. The difference is that we need now to keep track of how many derivatives are lost in the passage to the double. For the manifold with boundary itself, no derivatives are lost. It is shown in [22] that a $C^k$ manifold with boundary, $k \geq 1$, has a $C^k$ double that is unique up to $C^k$ diffeomorphism.

But, in our case, the need to make the group act on the double requires that the doubling construction be invariant under the group. And this will turn out to reduce the guaranteed differentiability of the conjugating diffeomorphism.

To facilitate the discussion, we introduce a definition (similar to one given in Section 2) of the sense in which a sequence of groups of diffeomorphisms might converge to a limit group:

Suppose $M$ is a compact $C^k$ manifold with boundary, $k$ a positive integer. Suppose $G_0$ is a compact Lie group of $C^k$ diffeomorphisms of $M$ and that moreover $G_j$, $j = 1, 2, \ldots$ are a sequence of compact Lie groups of $C^k$ diffeomorphisms. Then we say that the sequence $G_j$ converges to $G_0$ in the $C^k$ sense if for each $\epsilon > 0$ there is a number $j_0$ such that, if $j > j_0$ and $g \in G_j$, then there is an element $g_0 \in G_0$ such that the distance from $g$ to $g_0$ is less than $\epsilon$.

Here the distance means relative to any metric on the set of $C^k$ mappings which gives the usual $C^k$ topology on $C^k$ maps from $M$ to $M$.

In these terms, we can now formulate the general real-differentiable result we shall use in the complex case:
Theorem 5.1  Suppose that $M$ is a compact $C^3$ manifold with boundary and that $r > 2$ is an integer, that $G_0$ is a compact Lie group of $C^r$ diffeomorphisms of $M$, and that $G_j$, $j = 1, 2, \ldots$, is a sequence of compact groups of $C^3$ diffeomorphisms which converge in the $C^3$ sense to $G_0$. Then, for all $j$ sufficiently large there is a $C^{r-2}$ diffeomorphism $F_j$ of $M$ to itself such that $F_j \circ G_j \circ F_j^{-1}$ is a subgroup of $G_0$, i.e., $F_j$ conjugates the elements of $G_j$ into elements of $G_0$.

The proof of this theorem follows almost precisely the pattern of the proof of Theorem 0.1 in [10] (cf. Theorem 4.4.1 [8]). The only difference is that we must here keep some track of the number of derivatives involved: Ebin’s theorem concerned the $C^\infty$ case so that loss of a derivative or two or indeed of any finite number was irrelevant.

As in Section 2, the essential method is to pass to the double of $M$ and extend the action of the groups to the double. Then one can use Ebin’s result in the form presented in [13], where only $C^1$ is required for the closeness of the group actions. But here we have to keep track of degrees of differentiability as opposed to the $C^\infty$ situation of Section 2.

The most natural way to form the equivariant double is via metric construction as already explained in Section 2 (cf. [10]): As before one takes a metric $g$ on the manifold with boundary that is invariant under the group $G$. Then one defines charts in neighborhoods of boundary points $p$ using the normal field to the boundary. Specifically, let $N(q)$ be the $g$-metric normal to the tangent space to the boundary $\partial M$ at the point $q$ in $\partial M$. Then one defines charts in a neighborhood of points $p$ in the boundary as follows: Map $\partial M \times (-\epsilon, \epsilon) \to M$ by $(q, t) \mapsto \exp_q(tN(q))$, where $\exp$ is the geodesic exponential map of the Riemannian metric $g$ and $N(q)$ is the inward pointing normal at $q$. Choosing a chart around $p$ in $\partial M$ then gives a chart in a neighborhood of $p$ in the double of $M$ if we interpret $\exp_q(tN(q))$ to be in the second copy of $M$ when $t < 0$.

In terms of derivative loss, the choice of the normal vector $N$ loses one derivative, since it is an algebraic process using $g$ and the tangent space of the boundary and the latter is not $C^r$ but $C^{r-1}$. But an additional loss of derivative, so that two derivatives are lost, occurs because the exponential map is defined by the geodesic equation and that equation involves the Christoffel symbols, which involves the first derivative of the metric $g$. And the metric $g$ has already lost one derivative in the averaging over the action of the group $G$.

Thus one obtains a $G$-equivariant construction of the double $\tilde{M}$ of $M$ and by construction the action of $G$ on $M$ extends to be an action of $G$ on $\tilde{M}$. This extended group action is $C^{r-2}$. Associate to the group $G$ a group $\tilde{G}$ defined to be $G \oplus \mathbb{Z}_2$. Then $\tilde{G}$ acts on $\tilde{M}$ in a natural way. Namely, we label the elements of $\tilde{M}$ by $(m, a)$ where $m \in M$ and $a \in \{0, 1\}$ with 0 corresponding to the original of $M$ and 1 corresponding to the second copy of $M$. Then we let $(g, b)$ acting on $(m, a)$ be $(g(m), a + b)$ where the addition $a + b$ is in $\mathbb{Z}_2$, i.e., mod 2. For example $(\text{id}_G, 1)$ acts on $\tilde{M}$ as the “flip” map that interchanges the two copies of $M$.  

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Note that the fixed point set of \((\text{id}_G, 1)\) is exactly \(\partial M\). And, for any element \(g \in G\), the fixed point set of \((g, 1)\) is contained in \(\partial M\), though it need not be all of it, and can indeed be empty if the action of \(g\) on \(\partial M\) has no fixed point. These observations will be important later.

Now we turn to the explicit situation of Theorem 5.1. We choose a sequence of \(G_j\)-invariant \(C^{r-1}\) metrics on \(M\), which can clearly be taken to converge in the \(C^{r-1}\) sense to a \(G_0\)-invariant \(C^{r-1}\) metric on \(M\). Passing to the double \(\tilde{M}\) gives a sequence of \(\tilde{G}_j\) group actions on \(\tilde{M}\). We can form a sequence of \(\tilde{G}_j\) invariant metrics by combining, via a partition of unity, a product metric structure near the boundary with the \(G_j\)-invariant metric on the interior of \(M\). Namely, as similar to before, let \(E_j\) be the exponential map of the metric \(g_j\), \(j = 0, 1, 2, \ldots\), acting on the normal bundle of the boundary \(\partial M\) of \(M\) in \(M\) to give maps also to be denoted by \(E_j: \partial M \times (-a, a) \to \tilde{M}\) of the boundary \(\partial M\) of \(M\) producted with an open interval \((-a, a)\) into \(\tilde{M}\). The size of \(a\) can, by the \(C^{r-2}\) convergence of the \(E_j\) to \(E_0\), be chosen uniformly so that these \(E_j\) are diffeomorphisms onto their images in \(\partial M\), which themselves converge in the \(C^{r-2}\) sense to the limit \(C^{r-2}\) diffeomorphism \(E_0\).

Via this diffeomorphism, we transfer the product metrics on \(\partial M \times [0, \epsilon)\), namely \(H_j \times dt^2\), to the associated tubular neighborhoods of \(\partial M\) in \(M\). This transfer gives a \(\tilde{G}_j\)-invariant metric for each \(j\) and these metrics converge \(C^{r-2}\) to the limit \(\tilde{G}_0\)-invariant metric. Now we can combine, using a \(\tilde{G}_j\)-equivariant partition of unity, these product metrics with the \(G_j\)-invariant metric \(g_j\) on \(M\) to obtain a \(\tilde{G}_j\)-invariant metric on \(\tilde{M}\), to be denoted \(\tilde{g}_j\). This metric is \(C^{r-2}\). And it converges in the \(C^{r-2}\) topology to the corresponding \(\tilde{G}_0\)-invariant metric \(\tilde{g}_0\) on \(\tilde{M}\). (The \(G_j\)-equivariant partition of unity is obtained by taking the partition of unity function to depend on \(t\) alone, \(t\) as above).

Now we can apply Ebin’s Theorem, in the form given in [13] and [15], for the \(C^{r-2}\) case to get \(C^{r-2}\) diffeomorphisms \(F_j: \tilde{M} \to \tilde{M}\) which conjugate \(G_j\) into a subgroup of \(\tilde{G}_0\). (Here we are reasoning as follows: There is a diffeomorphism that conjugates \(\text{Isom}\ (\tilde{g}_j)\) into a subgroup of \(\text{Isom}\ (\tilde{g}_0)\) and hence conjugates \(\tilde{G}_j\) into a subgroup of \(\text{Isom}\ (\tilde{g}_0)\) and these diffeomorphisms can be taken to converge to the identity map. So the image of \(\tilde{G}_j\) under this conjugation is close to \(\tilde{G}_0\) for large \(j\) in the sense of \(C^{r-2}\) convergence. By the classical theorem of [20], this conjugation image is in fact itself conjugate in \(\text{Isom}\ (\tilde{g}_0)\) to a subgroup of \(G_0\) by an element close to the identity. (cf., e.g., [8], Ch. 4, for more detail.)

Now we need to know that in fact the conjugation image of \(G_j\) lies in \(G_0\), not just in \(\tilde{G}_0\). For this, we need only show that the diffeomorphism that is conjugating takes \(\partial M\) to itself. This can be deduced as follows: Let us denote by \(\text{Fix}\ (\psi)\) the fixed point set of \(\psi\). Then conjugation takes fixed points to fixed points in the sense that \(\text{Fix}\ (f \circ \psi \circ f^{-1}) = f(\text{Fix}\ (\psi))\). Now consider the case of \(\psi = \text{the flip map which interchanges the two copies of } M\ in \tilde{M}\). When \(f\) is close to the identity, \(f \circ \psi \circ f^{-1}\) has to belong to the part of the group that interchanges the two components. So its fixed point set cannot be larger than \(\partial M\). Thus \(f(\partial M)\) lies in \(\partial M\) and hence equals \(\partial M\) (since \(f\) is a diffeomorphism
of $\partial M$ onto its image).

This completes the proof of the theorem. \hfill \Box

Note that these considerations of fixed points of the interchange map did not arise in Section 2, since we were concerned there only with isomorphism, not with the existence of a conjugating diffeomorphism of the manifolds with boundary.

The application to the strongly pseudoconvex case now follows:

**Theorem 5.2** Let $\Omega_0$ be a bounded strongly pseudoconvex domain with a $C^{4,\alpha}$ boundary in $\mathbb{C}^n$, not biholomorphic to the unit ball. Then there is a $C^{4,\alpha}$ neighborhood $\mathcal{N}$ of $\Omega_0$ such that, for any $\Omega \in \mathcal{N}$, there is a $C^3$ diffeomorphism $f : \Omega \to \Omega_0$ with the property that $f \circ \text{Aut}(\Omega) \circ f^{-1} \subset \text{Aut}(\Omega_0)$. 
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