The PP-Wave Limits of Orbifolded $AdS_5 \times S^5$

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Abstract

Using the supergravity solution of $N_1$ D3-branes probing $A_{N_2-1}$ singularities we study the pp-wave limit of $AdS_5 \times S^5/Z_{N_2}$. We show that there are two different pp-wave limits. One is the orbifold of the pp-wave limit of $AdS_5 \times S^5$. In this case there is no symmetry enhancement. The other case is the same as the pp-wave limit of $AdS_5 \times S^5$ and therefore we again see the maximal supersymmetry. We will also identify operators in the four dimensional $\mathcal{N} = 2$ $SU(N_1)^{N_2}$ gauge theory which correspond to stringy excitations in the orbifolded pp-wave geometry. The existence of the maximal pp-wave geometry indicates that there is a subsector of the corresponding $\mathcal{N} = 2$ gauge theories which has enhanced $\mathcal{N} = 4$ supersymmetry. We also study the pp-wave limits of $AdS_7/A \times S^{4.7}/Z_N$. 
1 Introduction

Although the maximal supersymmetric backgrounds have been known for a while, until the strong motivation by AdS/CFT duality (see [1] for a review), string theory was only formulated on the simpler case of ten dimensional flat background. The AdS/CFT correspondence, in its simplest form, states that string theory (or in certain limits, supergravity) on the $AdS_5 \times S^5$ background is dual to the large $N$ $\mathcal{N} = 4$ four dimensional SYM [2, 3, 4]. According to this conjecture the spectrum of string theory on $AdS_5 \times S^5$ corresponds to spectrum of single trace operators of the $\mathcal{N} = 4$ gauge theory. However, this correspondence has only been studied for the supergravity modes on the $AdS_5 \times S^5$ which are in one-to-one correspondence with the chiral operators of the $\mathcal{N} = 4$ gauge theory.

Another supergravity background which admits the maximal supersymmetry is the pp-wave [5, 6, 7]. Soon after that the string theory on this background was formulated [8] and shown to be exactly solvable [9]. However, the observation that the supersymmetric pp-waves background can be understood as a certain limit of $AdS_5 \times S^5$ geometry led the authors of [10] to the conjecture that string theory on the maximally supersymmetric ten dimensional pp-waves has a description in terms of a certain subsector of the large $N$ four dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory at weak coupling. More precisely this subsector is parametrized by states with conformal weight $\Delta$ carrying $J$ units of charge under the $U(1)_R$ subgroup of the $SU(4)_R$ R-symmetry of the gauge theory, such that both $\Delta$ and $J$ are parametrically large (both scale as $\sqrt{N}$ in the large $N$ limit) while their difference $\Delta - J$ is finite. Then, as further evidence the perturbative string spectrum was worked out from gauge theory side.

This procedure has also been generalized for the pp-wave limit of $AdS_5 \times M^5$, where $M^5$ is smooth and preserves some supersymmetry, leading to the conjecture that for such background the pp-wave limit is universal and maximally supersymmetric [11, 12, 13]. In particular focusing on the $AdS_5 \times T^{1,1}$, it was shown that this pp-wave is identical to the one which appears in the $AdS_5 \times S^5$ case. Therefore there should be a subsector of the corresponding $\mathcal{N} = 1$ gauge theory describing the string theory on the pp-wave, which has enhanced $\mathcal{N} = 4$ supersymmetry.

In this letter we consider non-smooth backgrounds and study their pp-wave limits. We consider the $AdS_5 \times M^5$ cases, where $M^5$ is an orbifold of $S^5$, namely $S^5/Z_N$. The special case of $S^5/Z_2$ has been briefly considered in [12]. We start with supergravity solution of $N_1$ D3-branes in the $A_{N_2-1}$ singularities, the near horizon limit of which corresponds to geometry of the form $AdS_5 \times S^5/Z_{N_2}$ [14]. We show that the orbifolded case admits two different pp-wave limits. One is with half supersymmetries and is in fact the orbifolded version of maximally supersymmetric pp-wave. The other one is the maximal supersymmetric case. Therefore, the universality of the pp-waves limit for the cases which $M^5$ is a smooth manifold [12] is not maintaining in the non-smooth cases.

The article is organized as follows. In section 2, we review how to construct
the supergravity solution of $N_1$ D3-branes in the $A_{N_2-1}$ singularities. This solution can be obtained by supergravity solution of a M5-brane configuration in eleven dimensional supergravity with worldvolume of the form $R^4 \times \Sigma$ \cite{13}, where $\Sigma$ is the Seiberg-Witten holomorphic curve. In section 3, we shall perform the pp-wave limits of $AdS_5 \times S^5 / Z_{N_2}$ and show the existence of two pp-wave limits. In section 4, we will consider subsector of the $\mathcal{N} = 2$ gauge theory corresponding to the string on the orbifolded pp-wave. In section 5, we study the eleven dimensional $AdS_4 \times S^7 / Z_N$ and $AdS_7 \times S^4 / Z_N$ orbifolds and their pp-wave limits. We show that for the $AdS_4 \times S^7 / Z_N$ case there are two pp-wave limits, one is the orbifold version of eleven dimensional pp-waves of \cite{6, 10} and the other is the maximal supersymmetric case. However, for the $AdS_7 \times S^4 / Z_N$ the only possible pp-wave limit is the maximally supersymmetric case.

2 Gravity solution of $AdS_5 \times S^5 / Z_{N_2}$

In this section we present the supergravity solution of $AdS_5 \times S^5 / Z_{N_2}$ starting from intersecting M5-branes solution. This supergravity solution has been studied in \cite{16, 17}. We denote the eleven dimensional space-time coordinates by $(x_{||}, \vec{x}, \vec{y}, \vec{z})$, where $x_{||}$ parameterize the $(0, 1, 2, 3)$ coordinates, $\vec{x} = (x_1, x_2)$ the $(4, 5)$ coordinates, $\vec{y} = (y_1, y_2)$ the $(6, 7)$ coordinates and $\vec{z} = (z_1, z_2, z_3)$ the $(8, 9, 10)$ coordinates.

Let us start with two sets of fivebranes in M theory: $N_1$ coinciding M5 branes and $N_2$ coincident M5' branes. Their worldvolume coordinates are $M_5 : (x_{||}, \vec{y})$ and $M_5' : (x_{||}, \vec{x})$. Such a configuration preserves eight supercharges. The eleven-dimensional supergravity background is given by \cite{18, 19, 20}

$$ds^2_{11} = (H_1 H_2)^{2/3}[(H_1 H_2)^{-1} dx_{||}^2 + H_2^{-1} d\vec{x}^2 + H_1^{-1} d\vec{y}^2 + d\vec{z}^2],$$

with the 4-form field strength $\mathcal{F}$

$$\mathcal{F} = 3 \left( * d(H_1^{-1}) \wedge dy^1 \wedge dy^2 + * d(H_2^{-1}) \wedge dx^1 \wedge dx^2 \right),$$

where $*$ defines the dual form in the three dimensional space ($z_1, z_2, z_3$). Consider the semi-localized case when the M5 branes are completely localized while the M5' branes are only localized along the overall transverse directions. When the branes are at the origin of $(\vec{x}, \vec{z})$ space the harmonic functions in the near core limit of the M5' branes take the form \cite{21, 22}

$$H_1 = 1 + \frac{4\pi l_p^4 N_1 N_2}{(|\vec{x}|^2 + 2l_p N_2 |\vec{z}|)^2}, \quad H_2 = \frac{l_p N_2}{2 |\vec{z}|}.$$

It is useful to make a change of coordinates $l_p z = (r^2 \sin^2 \alpha)/2 N_2, \ x = r \cos \alpha, 0 \leq \alpha \leq \pi/2$. The decoupling limit of the theory is defined by $l_p \rightarrow 0$ while keeping $U = r/l_p$ and $\vec{y} = \vec{y}/l_p$ fixed. In this limit the metric (11) is of the form of a warped
product of AdS$_5$ and a six dimensional manifold $\mathcal{M}_6$ \cite{17}

$$ds_{11}^2 = l_p^2 (4\pi N_1)^{-1/3} (\sin^{2/3} \alpha) \left( \frac{U^2}{N_2} dx_\parallel^2 + \frac{4\pi N_1}{U^2} dU^2 + d\mathcal{M}_6^2 \right), \quad (4)$$

with

$$d\mathcal{M}_6^2 = 4\pi N_1 (d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} d\Omega_2^2) + \frac{N_2}{\sin^2 \alpha} (d\hat{y}^2 + \hat{y}^2 dy^2)$$

$$d\Omega_2^2 = d\gamma^2 + \sin^2 \gamma d\delta^2. \quad (5)$$

The four form field in this limit is given by

$$F = -2\pi N_1 l_p^3 \sin^3 \alpha \cos \alpha \sin \gamma \ d\alpha \wedge d\theta \wedge d\gamma \wedge d\delta$$

$$-\frac{1}{2} N_2 l_p^3 \sin \gamma \ d\hat{y}_1 \wedge d\hat{y}_2 \wedge d\gamma \wedge d\delta. \quad (6)$$

The curvature of this metric behaves like $R \sim \frac{1}{l_p^2 N_1^{1/3} \sin^{2/3} \alpha}$. This brane configuration can be understood as uplifted elliptic brane system in type IIA. It consists of $N_2$ NS5-branes with the worldvolume directions $(0,1,2,3,4,5)$ periodically arranged in the 6-direction and $N_1$ D4-branes with worldvolume coordinates $(0,1,2,3,6)$ stretched between them. The four dimensional theory at low energies on the D4-branes worldvolume is a supersymmetric gauge theory with 8 supercharges. In this brane set-up the R-symmetry group $SU(2)_R \times U(1)_R$ is realized as the rotation group $SU(2)_{8910} \times U(1)_{45}$.

The near-horizon limit of the ten dimensional metric describing the elliptic type IIA brane configuration is \cite{17, 16}

$$ds_{10}^2 = l_s^2 \left( \frac{U^2}{R^2} dx_\parallel^2 + R^2 \frac{dU^2}{U^2} + d\mathcal{M}_5^2 \right), \quad (7)$$

where

$$d\mathcal{M}_5^2 = R^2 \left( d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} d\Omega_2^2 \right) + \frac{N_2^2}{R^2 \sin^2 \alpha} d\hat{y}^2 \quad (8)$$

and $R^2 = (4\pi g_s N_1 N_2)^{1/2}$. In this limit the dilaton and non-zero three and four form field strengths of the corresponding supergravity solution are given by

$$e^\phi = \frac{g_s N_2}{R \sin \alpha}, \quad H_{g\gamma\delta} = -\frac{N_2 l_s^2}{2} \sin \gamma,$$

$$F_{\alpha\theta\gamma\delta} = -2\pi g_s N_1 l_p^3 \sin^3 \alpha \cos \alpha \sin \gamma. \quad (9)$$

The NS-NS B field corresponding to the NS5-branes charge can be read from (9) as

$$B_{g\delta} = \frac{N_2 l_s^2}{2} (\cos \gamma - 1). \quad (10)$$
It is useful to make a T-duality and study these theories from Type IIB string theory point of view. Doing so, we obtain [16]

\[ ds^2_{10} = l_s^2 \left( \frac{U^2}{R^2} dx^2_{\parallel} + R^2 dU^2 + R^2 d\mathcal{M}^2 \right), \]  

(11)

with

\[ d\mathcal{M}^2 = d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} d\Omega^2_2 + \frac{\sin^2 \alpha}{N_2^2} [d\chi + \frac{N_2}{2} (\cos \gamma - 1) d\delta]^2 \]  

(12)

where \( \chi = \hat{y}/l_s \). The IIB self-dual five form is found to be [11]

\[ F_{\chi\alpha\theta\gamma\delta} = (\ast F)_{\chi\alpha\theta\gamma\delta} = \frac{3}{20 N_2} \alpha'^2 \cos \alpha \sin^3 \alpha \sin \gamma . \]  

(13)

The curvature of the above metric behaves like \( R \sim \frac{1}{R^2} \). Note that (12) is the metric of \( S^5/Z_{N_2} \) orbifold with \( Z_{N_2} \subset SU(2) \subset SU(4) \). Therefore the solution is nothing but D3-branes solution in the \( Z_{N_2} \) orbifold. This theory has been considered in [14] by orbifolding of \( AdS_5 \times S^5 \) in the context AdS/CFT correspondence. The theory is four dimensional \( N = 2 \) SYM theory with gauge group \( SU(N_1)^{N_2} \) with \( N_2 \) bifundamental matter hypermultiplets i.e. \( (Q_i, \tilde{Q}_i) \) in \( N = 1 \) notation. The \( Z_{N_2} \) acts as a permutation of the gauge factors. The dimensionless gauge coupling of each \( SU(N_1) \) part of the gauge group is \( g^2_{YM} \sim g_{s N_2} \).

3 PP-waves as limits of \( AdS_5 \times S^5/Z_{N_2} \)

Following [11], we obtain the pp-wave geometries arising as limits of \( AdS_5 \times S^5/Z_{N_2} \) and then study string theory on pp-wave background through the corresponding gauge theory. The strategy is to consider the trajectory of a particle which is moving very fast along the \( S^5/Z_{N_2} \) and focusing on the geometry near this trajectory. We note that the metric of \( AdS_5 \times S^5/Z_{N_2} \) background (11) can be recast to

\[ l_s^{-2} ds^2 = R^2 \left[ -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega^2_3 \right. \]

\[ + \left. d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} d\Omega^2_2 + \frac{\sin^2 \alpha}{N_2^2} [d\chi + \frac{N_2}{2} (\cos \gamma - 1) d\delta]^2 \right] \]  

(14)

Depending on the fact that particle trajectories we consider are near the singular point or far from that one can distinguish two different interesting pp-wave limits:

1) Trajectories near \( \alpha = 0 \)

Let us consider a particle moving along the \( \theta \) direction and sitting at \( \rho = 0 \) and \( \alpha = 0 \). We recall that \( \theta \) parametrizes the direction which is invariant under the

\[ ^1\text{Here we are using the conventions of [16].} \]
$Z_{N_2}$ action. The geometry close to this trajectory can be obtained by the following rescaling

$$t = x^+ + R^{-2} x^- , \quad \theta = x^+ - R^{-2} x^- , \quad \rho = \frac{r}{R} , \quad \alpha = \frac{v}{R} , \quad R \to \infty ,$$

and keeping $x^+, x^-, r$ and $v$ fixed. In this limit the above metric reads as

$$l_s^{-2} ds^2 = \left( -4 dx^+ dx^- - (r^2 + v^2) dx^{+2} + dr^2 + r^2 d\Omega^2_3 \right) + dv^2 + \frac{1}{4} v^2 \left[ d\Omega^2_2 + \frac{4}{N_2^2} [ d\chi + \frac{N_2}{2} (\cos \gamma - 1) d\delta]^2 \right] .$$

(16)

We also have a self-dual RR 5-form flux, $(\star F)_{+v\gamma\delta\chi} = F_{+v\gamma\delta\chi} \sim \frac{1}{N_2} d(Vol_4)$, where $d(Vol_4)$ is volume form for $v\gamma \delta \chi$ directions. Besides the explicit form of metric (16), the fact that the five form flux is proportional to $\frac{1}{N_2}$ justifies that our geometry contains the $Z_{N_2}$ orbifold. Since the curvature of the above metric is proportional to $\frac{1}{R^2}$, in the $R \to \infty$ limit curvature always remains small. This pp-wave limit of $AdS_5 \times S^5 / Z_{N_2}$ is $Z_{N_2}$ orbifold of the pp-wave limit of $AdS_5 \times S^5$ [10] and hence this background preserves only half of supersymmetry (16 supercharges).

2) Trajectories near $\alpha = \frac{\pi}{2}$

Besides the above limit there is another pp-wave metric considering the particles moving very fast along the $\chi$ direction, sitting at $\rho = 0$, $\gamma = 0$ and $\alpha = \frac{\pi}{2}$. Note that $\chi$ is the direction involved in the orbifolding. Let us consider the scaling

$$t = x^+ + R^{-2} x^- , \quad \frac{1}{N_2} \chi = x^+ - R^{-2} x^- ,$$

$$\rho = \frac{r}{R} , \quad \gamma = \frac{2x}{R} , \quad \alpha = \frac{\pi}{2} - \frac{y}{R} , \quad R \to \infty$$

(17)

while keeping $x^+, x^-, r$, $x$ and $y$ fixed. In this limit the metric (14) becomes

$$l_s^{-2} ds^2 = -4 dx^+ dx^- - \mu^2 z^0 dx^{+2} + dz^0 \ ,$$

(18)

and

$$F_{+1234} = F_{+4567} = \frac{3}{20} \mu .$$

(19)

To obtain the above metric we have redefined $\delta - x^+$ as the new coordinate $\delta$. Moreover the mass parameter $\mu$ has been introduced by reacaling $x^- \to x^-/\mu$ and $x^+ \to \mu x^+$. As we see the above metric, as well as the self dual five form field, do not depend on $N_2$ and it is exactly the pp-wave limit of $AdS_5 \times S^5$ discussed in [10]. This pp-wave limit corresponds to a maximally supersymmetric background. The very fact that the $N_2$ dependence has been removed in the pp-wave limit metric can also be understood from the four dimensional gauge field theory side. For that it is enough to recall that in the scaling limit (17) $x^+$ is related to $\chi$ by a factor of $N_2$ and hence the $U(1)_R$ charges of the states should now be measured in units of $N_2$. 

One can work out the pp-wave limit of the eleven dimensional solutions of (1), (3). This we have presented in the Appendix. We note that the maximally supersymmetric pp-wave do not have an eleven dimensional counterpart.

Therefore, for the $\text{AdS}_5 \times \mathcal{M}^5$ spaces, where $\mathcal{M}^5$ is an orbifold of $S^5$ which preserves some supersymmetry, depending on the scaling we use for going to pp-wave limit, one may find maximally supersymmetry case or the cases with (half) of supersymmetry broken.

Here we will mainly focus on the first pp-wave case. Following [10] we can write the light-cone momenta in terms of conformal weight $\Delta$ and the angular momentum, $J = -i \partial_\theta$, of the operator in the superconformal field theory as following

$$
2p^- = i \partial_{x^+} = i(\partial_t + \partial_\theta) = \Delta - J
$$

$$
2p^+ = i \frac{\partial_{x^-}}{R^2} = i \frac{(\partial_t - \partial_\theta)}{R^2} = \frac{\Delta + J}{R^2}.
$$

(20)

Configurations with fixed non-zero $p^+$ in the limit (15) correspond to the states in the $\text{AdS}$ with large angular momentum $J \sim R^2$. In the gauge theory side such configurations correspond to operators with R-charge $J \sim \sqrt{N_1}$ and $\Delta - J$ fixed, in the $N_1 \rightarrow \infty$ limit and keeping the gauge theory coupling $g^2_{YM}$, fixed and small. In next section we will study these operators and identify them with excited string states.

On the other hand for the pp-wave with maximal supersymmetry (18) the light-cone momenta in terms of conformal weight $\Delta$ and the angular momentum $J = -i \partial_\chi$ can be written as

$$
2p^- = i \partial_{x^+} = i(\partial_t + N_2 \partial_\chi) = \Delta - N_2 J
$$

$$
2p^+ = i \frac{\partial_{x^-}}{R^2} = i \frac{(\partial_t - N_2 \partial_\chi)}{R^2} = \frac{\Delta + N_2 J}{R^2}.
$$

(21)

Therefore in this case we will be looking for spectrum of states with $\Delta - N_2 J$ finite in the limit $N_1 \rightarrow \infty$. In fact from pp-wave limit (18) we learn that there should also be a subsector of $\mathcal{N} = 2$ which exhibits the supersymmetry enhancement, similar to the conifold case [11, 12]. This subsector is parametrized by $\Delta - N_2 J$ eigenvalue. We would like to comment that, since the above maximal SUSY PP-wave limit do not depend on $N_2$, one can study the large $N_2$ limit. In particular equations (21) are very suggestive that in the large $N_2$ one has the chance to keep $J$ finite. However, here we will only consider the half SUSY case and in the next section we construct the gauge theory operators corresponding to the strings on PP-wave orbifolds. Giving the description of strings in PP-waves in terms of $\mathcal{N} = 2$ gauge theories is very interesting and important question which we will come back to, in the future works [28].
4 Strings in PP-wave orbifolds from $\mathcal{N} = 2$ SYM

The twisted and untwisted states of AdS orbifolds have been studied in [23, 24]. In fact the twisted states of $AdS_5 \times S^5/N_2$ orbifold can be identified with the chiral primary operators of $\mathcal{N} = 2 SU(N_1)N_2$ conformal SYM theory which are not invariant under exchange of the gauge group factors, while the untwisted states can be identified with those which are invariant. In our notation the untwisted operators can be considered as single trace operators which are symmetric under exchange of gauge group, i.e. $\sum_{i=1}^{N_2} \text{Tr}(\varphi_i^k)$, where $\varphi_i$'s are adjoint complex scalars in the vector-multiplet of each gauge factor and carry a unit of R-charge under $U(1)_R$ factor of R-symmetry group. This operator has conformal weight $k$.

On the other hand the twisted operators which are not invariant under exchange of the gauge factors may be written in the following form [24]

$$O(i) - O(i + 1) ,$$

where $O$ can be thought as a basis which is constructed from a certain combination of chiral fields. For example we can consider those operators which are given in terms of scalars in the vector-multiplet, i.e. $O(i) = \text{Tr}(\varphi_i^k)$.

We want to study those states which carry large R-charge $J \sim \sqrt{N_1}$ under the generator of $U(1)_R$ subgroup of the R-symmetry which acts on the adjoint scalars in the vector-multiplet while has no effect on the scalars of the bi-fundamental matters; and we are looking for the spectrum of states with $\Delta - J$ fixed. In the untwisted sector the operator with lowest value of $\Delta - J$, which is zero, is $\sum_{i=1}^{N_2} \text{Tr}(\varphi_i^1)$. This operator is chiral primary and therefore its dimension is protected by supersymmetry and, is associated to the untwisted vacuum state of the corresponding string theory in the light-cone gauge, much similar to the $\mathcal{N} = 4$ case [10]. Operators in the twisted sector with lowest value of $\Delta - J$ can be read from (22) by setting $O(i) = \text{Tr}(\varphi_i^1)$.

Now we can proceed with constructing other operators. These operators can be parameterized by their $\Delta - J$ eigenvalue. For example for $\Delta - J = 0$ the corresponding operators are just what we have presented above. According to the prescription given in [10] the other operators can be obtained by inserting new operators in the trace. In other words replacing some of $\varphi_i$ in the operator $\text{Tr}(\varphi_i^1)$ by other fields.

The bosonic operators which can be inserted in the trace are as following. We have four scalars in the directions that are not rotated by $J$. They are, in fact, the scalars in the hypermultiplet which correspond to the directions in which the $Z_{N_2}$ orbifold is defined. On the other hand there are four other fields which are made out of $\varphi_i$. These fields are derivative of $\varphi_i$, $D_a \varphi_i + [A_a, \varphi_i]$, where $a = 1, 2, 3, 4$ are the directions in $R^4$ parameterized by $\vec{r}$ in (10). Since these directions are not affected by the $Z_{N_2}$ orbifold, they should lead to the same excitation as those in $AdS_5 \times S^5$. These are operators with $\Delta - J = 1$.

\[\text{Here we shall only consider the bosonic operators. Their superpartners can be easily obtained using the } \mathcal{N} = 2 \text{ supersymmetry.}\]
The next bosonic operators we shall consider are those with $\Delta - J = 2$. A basis for these operators can be obtained by inserting the scalars in the hypermultiplet i.e. $(Q^\mu_i, \bar{Q}^\mu_i)$ in trace $\text{Tr}(\varphi_i^J)$ for $\mu = 1, 2$. These scalars which correspond to the directions where the orbifold is defined, are in the bi-fundamental representation of the gauge group. This means that under a gauge transformation they transform as $Q^\mu_i \rightarrow U_i Q^\mu_i U_i^{-1}$ and $\bar{Q}^\mu_i \rightarrow U_{i+1} \bar{Q}^\mu_i U_i^{-1}$. Being bi-fundamental scalars, we need to insert at least two of them in the trace to make a gauge invariant operator. Since these scalars are affected by the $Z_{N_2}$ orbifold, we can construct operators both in the untwisted (and twisted) sectors by using a combination of these basis which are invariant (and non-invariant) under exchange of the gauge group factor. The basis for these gauge invariant operators are given by

$$
O_1^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l Q_j^\mu \varphi_{j+1}^{i-l} \bar{Q}_j^\nu \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_2^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l Q_j^\mu \varphi_{j+1}^{i-l} \bar{Q}_j^\nu \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_3^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l \bar{Q}_j^\mu \varphi_{j+1}^{i-l} \bar{Q}_j^\nu \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_4^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l \bar{Q}_j^\mu \varphi_{j+1}^{i-l} \bar{Q}_j^\nu \right) e^{\frac{2\pi i n l}{N_2}}. \quad (23)
$$

Analogous to [10] these operators with $n$ being an integer multiple of $N_2$ correspond to the excited string states obtained by applying the creation operators $a^\mu_{-n/N_2}(j)a^\nu_{-n/N_2}(j+1)$ on the untwisted or twisted light-cone vacuum. For $n$ not an integer multiple of $N_2$ we have just states in twisted sector.

In general one can construct gauge invariant operators with $\Delta - J = 2(1+k)$ for $0 \leq k \leq N_2 - 1$. These operators are obtained by inserting $k+1$ scalars in the hypermultiplet into the trace in such a way that the whole combination is gauge invariant. For example one can consider the following operator with $\Delta - J = 2(1+k)$

$$
O_1^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l Q_j^{\mu_0} \cdots Q_j^{\mu_k} \varphi_{j+k+1}^{i-l} \bar{Q}_j^{\nu_k} \cdots \bar{Q}_j^{\nu_0} \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_2^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l Q_j^{\mu_0} \cdots Q_j^{\mu_k} \varphi_{j+k+1}^{i-l} \bar{Q}_j^{\nu_k} \cdots \bar{Q}_j^{\nu_0} \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_3^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l \bar{Q}_j^{\mu_0} \cdots \bar{Q}_j^{\mu_k} \varphi_{j+k+1}^{i-l} \bar{Q}_j^{\nu_k} \cdots \bar{Q}_j^{\nu_0} \right) e^{\frac{2\pi i n l}{N_2}},
$$

$$
O_4^n(j) = \sum_{l=1}^{J} \text{Tr} \left( \varphi_j^l \bar{Q}_j^{\mu_0} \cdots \bar{Q}_j^{\mu_k} \varphi_{j+k+1}^{i-l} \bar{Q}_j^{\nu_k} \cdots \bar{Q}_j^{\nu_0} \right) e^{\frac{2\pi i n l}{N_2}}. \quad (24)
$$

The corresponding string state can be obtained by applying creation operators like $a^\mu_{-n/N_2}(j) \cdots a^\mu_{-n/N_2}(j+k) a^{\nu_0}_{-n/N_2}(j) \cdots a^{\nu_k}_{-n/N_2}(j+k)$ on the light-cone vacuum.
5 PP-wave limits of orbifolds of $AdS_{4,7} \times S^{7,4}$

In section 3, studying the pp-wave limit of the $AdS_5 \times S^5/Z_N$ orbifold we showed that depending on how we take the pp-wave limit, we can find maximal and half supersymmetric pp-wave backgrounds. In this section we study the similar limits for eleven dimensional orbifold cases. However, first we need to work out the metric for such orbifold solutions. These solutions have been studied in [25].

To obtain the supergravity solution corresponding to $AdS_4 \times S^7/Z_N$ supersymmetric orbifolds, one may start with M2 or M5 -branes probing the orbifold. For that it is enough to replace the metric for the transverse direction by the proper orbifold solution. Since the orbifolds are Einstein manifolds, it is guaranteed that solutions found in this way are supergravity solutions. As we have shown in section 3, the metric of a supersymmetric $S^3$ orbifold, i.e. $S^3/Z_N$ is given by

$$d\tilde{\Omega}_3^2 = \frac{1}{4} d\Omega_2^2 + \frac{1}{N^2} \left[ d\chi + \frac{N}{2} (\cos \gamma - 1) d\delta \right]^2,$$

(25)

where $d\Omega_2^2 = d\gamma^2 + \sin^2 \gamma d\delta^2$. Here $0 \leq \gamma \leq \pi$ and $\delta$ and $\chi$ are periodic with period $2\pi$. In fact the metric (25) parameterizes the unit 3-sphere where $\chi$ ranges from 0 to $2\pi N$. The $Z_N$ action which leads to an orbifold is given by $\chi \equiv \chi + 2\pi$. With this information, we are ready to write down the eleven dimensional $AdS$ orbifold solutions.

5.1 The pp-wave limit of $AdS_4 \times S^7/Z_N$

Let us consider the $AdS_4 \times S^7$ metric\footnote{The corresponding field theory in these cases are $\mathcal{N} = (0, 1)$ six dimensional SCFT for $AdS_7 \times S^4$ orbifold and three dimensional $\mathcal{N} = 2$ SCFT in the Large $N$ limit for the $AdS_4 \times S^7$ orbifold [29].}

$$l_p^{-2} ds^2 = R^2 \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sin^2 \rho d\Omega_2^2 + 4 d\Omega_7^2 \right).$$

(26)

Now we split the transverse direction to two parts

$$d\Omega_7^2 = d\alpha^2 + \cos^2 \alpha d\Omega_3^2 + \sin^2 \alpha d\tilde{\Omega}_3^2.$$

(27)

Then replacing $d\Omega_3^2$ with $d\Omega_7^2$, we obtain

$$d\tilde{\Omega}_3^2 = d\alpha^2 + \cos^2 \alpha d\Omega_3^2 + \sin^2 \alpha d\tilde{\Omega}_3^2,$$

(28)

which is the metric for $S^7/Z_N$. Finally the $AdS_4 \times S^7/Z_N$ metric is obtained by replacing the the $S^7$ part of (26) with (28). The above can also be understood as the near horizon geometry of the supergravity solution of $N_1 N$ M2-branes in the\footnote{In our notations $R$ is the radius of the $AdS$ piece which is half of the radius of the $S^7$ for $AdS_4$ case and is twice bigger than the radius of $S^4$ for the $AdS_7$ case.}
orbifold background, where \( R = (\pi^2 N_1 N/2)^{1/3} \). The above solution can also be understood as T-dual of the near horizon limit of semi-localised intersection of the D3/D5/NS5 system \([27]\).

To obtain the pp-wave limits we note that the orbifold action on \( S^7 \) leaves a \( S^3 \) part invariant. Now, we can proceed in the same lines of section 3. Again there are two possibilities corresponding to two following choices:

1) \( t = x^+ + R^{-2} x^- \quad \theta = x^+ - R^{-2} x^- \), \( \beta = \frac{w}{R} \)

where \( \theta \) and \( \beta \) are directions along the invariant \( S^3 \). The other coordinates are scaled as \([15]\). This case, will correspond to the orbifold of the maximally supersymmetric pp-wave discussed in \([4, 11]\) and hence half of supersymmetric is broken.

2) \( t = x^+ + R^{-2} x^- \quad \chi = x^+ - R^{-2} x^- \), \( R \to \infty \),

and the other limits as defined in \([17]\). In this case we will end up with the maximal supersymmetric eleven dimensional pp-wave background.

### 5.2 The pp-wave limit of \( AdS_7 \times S^4/Z_N \)

One can repeat the above discussions for the \( AdS_7 \times S^4/Z_N \) case (or equivalently the near horizon geometry of M5-brane probing the orbifold). The metric for \( S^4/Z_N \) is

\[
d\Omega_4^2 = d\alpha^2 + \sin^2 \alpha \, d\tilde{\Omega}_3^2 .
\]

Hence, the supergravity metric for the \( AdS_7 \times S^4/Z_N \) will be given by

\[
L_p^{-2} ds^2 = R^2 \left\{ - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\tilde{\Omega}_3^2 \\
+ \frac{1}{4} d\alpha^2 + \frac{1}{16} \sin^2 \alpha \left( d\Omega_2^2 + \frac{4}{N^2} [d\chi + \frac{N}{2} (\cos \gamma - 1)d\delta]^2 \right) \right\} .
\]

where \( R = 2(\pi N_1 N)^{1/3} \). In this case, unlike the previous cases, there is no invariant direction in the \( S^4 \) part of the metric. Therefore, the only pp-wave limit will correspond to the maximally supersymmetric case \([3]\) which is obtained through

\[
t = x^+ + R^{-2} x^- \quad \chi = x^+ - R^{-2} x^- , \quad R \to \infty ,
\]

limit (the other coordinates scaled similar to \([17]\)).

### 6 Discussions

In this note we have studied the Penrose limit of orbifolded \( AdS \) geometries. Considering \( AdS_p \times S^q/Z_N \) cases we have shown that for \( q > 4 \) in general there are two different pp-wave limits, our result can be summarized as follows:

\[\text{Here we define } d\Omega_3^2 = \cos^2 \beta \, d\theta^2 + d\beta^2 + \sin^2 \beta \, d\eta^2 \]
Consider $\mathcal{M}/\mathbb{Z}_N$ orbifold, which preserves half supersymmetries. In general the orbifolding acts only on some subgroup of the isometries of $\mathcal{M}$ and leaving some subgroups invariant. Now, if we study the geometries near the trajectories along the invariant directions, we will find half supersymmetries. However, if we consider the directions which are not invariant under the orbifold direction we may find the maximal supersymmetric case.

In particular for the $AdS_7 \times S^4$ case, the isometries are $SO(5)$ and the $\mathbb{Z}_N$ acts of a $SU(2)$ subset, there is no invariant direction. Therefore, we do not find half supersymmetric case.

We have also discussed how to obtain the strings on the $\mathbb{Z}_{N_2}$ orbifold of pp-wave limit of $AdS_5 \times S^5$ from the corresponding gauge theory, which in this case is large $N_1 \mathcal{N} = 2$, $D = 4 \ SU(N_1)^{N_2}$ theory at weak Yang-Mills (but large ’t Hooft) coupling. The relevant operators in the gauge theory are those with large conformal weight $\Delta$ and large R-charge $J$, with $\Delta - J$ fixed. In particular we have identified $\Delta - J = 0$ (the corresponding string theory vacuum) and $\Delta - J = 1, 2$ operators, which correspond to higher order stringy oscillations. The twisted and untwisted states of the stringy excitations correspond to a subsector of $\mathcal{N} = 2$ operators which are non-invariant and invariant under the exchange of the gauge group factors, respectively. The question of constructing the strings in (maximal SUSY) PP-waves in terms of the $\mathcal{N} = 2$ operators is an interesting and important question and will be studied in future works \[28\].

It would be very interesting to study M-theory on the pp-wave limit of $AdS_4 \times S^7/\mathbb{Z}_N$ which preserves 16 supersymmetries. Similar to the maximal supersymmetric case, we expect that to have a Matrix description. We postpone this to future works.

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Appendix

In the same spirit, one can also study the pp-wave limits of eleven dimensional solution (4), (5). For this purpose it is more convenient to write the $AdS_5$ piece in the global $AdS$ coordinates:

\[
l_p^{-2}ds^2 = \left(\frac{R^4 \sin \alpha}{N_2}\right)^{2/3} \left[ -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 
+ d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} d\Omega_2^2 + \frac{N_2^2}{R^4 \sin^2 \alpha} (dy^2 + \dot{y}^2 d\psi^2) \right]
\]
where \( R^4 = 4\pi N_1 N_2 \). In the limit

\[
\begin{align*}
    t &= x^+ + R^{-2} x^- , \\
    \rho &= \frac{r}{R} , \\
    \alpha &= \frac{\nu}{R} , \quad R \to \infty , \\
    \theta &= \frac{x^- - R^{-2} x^+}{\sqrt{2}} , \\
    \rho &= \frac{r}{R} , \\
    \alpha &= \frac{\nu}{R} , \quad R \to \infty , \\
\end{align*}
\]

and keeping \( x^+ , x^- , r \) and \( v \) fixed. In this limit the metric becomes

\[
l_p^{-2} ds^2 = \left( \frac{v}{N_2} \right)^{2/3} \left[ -4 dx^+ dx^- - (r^2 + v^2) (dx^+)^2 + 4 r^2 \\
+ (dv^2 + \frac{1}{4} v^2 d\Omega^2_2) + \frac{N_2^2}{v^2} (d\hat{y}^2 + \hat{y}^2 dv^2) \right].
\]

The four form field in the above limits becomes

\[
\begin{align*}
l_p^{-3} F_+^{\gamma \delta} &= - \frac{1}{N_2} v^3 \sin \gamma , \\
l_p^{-3} F_{\hat{y}_1 \hat{y}_2 \gamma \delta} &= - \frac{N_2}{2} \sin \gamma.
\end{align*}
\]

In this limit the curvature reads as \( l_p^2 R \sim N_2^{2/3} / v^{8/3} \). Therefore one can trust the gravity for large \( v \). The singularity at \( v = 0 \) can be interpreted as a signal that some degrees of freedom have been effectively integrated and are needed in order to resolve the singularity. The (DLCQ) of M-theory on the above background should correspond to a Matrix-theory. However, we do not study that Matrix theory here.

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