On the Jaśkowski Models for Intuitionistic Propositional Logic

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Abstract

In 1936, Stanislaw Jaśkowski [1] gave a construction of an interesting sequence $J_0, J_1, \ldots$ of what he called “matrices”, which we would today call “finite Heyting Algebras”. He then gave a very brief sketch of a proof that if a propositional formula holds in every $J_i$ then it is provable in intuitionistic propositional logic (IPL). The sketch just describes a certain normal form for propositional formulas and gives a very terse outline of an inductive argument showing that an unprovable formula in the normal form can be refuted in one of the $J_i$. Unfortunately, it is far from clear how to recover a complete proof from this sketch.

In the early 1950s, Gene F. Rose [4] gave a detailed proof of Jaśkowski’s result, still using the notion of matrix rather than Heyting algebra, based on a normal form that is more restrictive than the one that Jaśkowski proposed. However, Rose’s paper refers to his thesis [3] for additional details, particularly concerning the normal form.

This note gives a proof of Jaśkowski’s result using modern terminology and a normal form more like Jaśkowski’s. We also prove a semantic property of the normal form enabling us to give an alternative proof of completeness of IPL for the Heyting algebra semantics. We outline a practical decision procedure for IPL based on our proofs and illustrate it in action on some simple examples.

Let $H = (H, f, t, \sqcap, \sqcup, \rightarrow)$ be a Heyting algebra. We will define $\Gamma(H)$ to be an extension of $H$ as a $(f, t, \sqcap, \sqcup, \rightarrow)$-algebra that adds a new co-atom (i.e., a new element $\ast$ such that $x < \ast < t$ for $x \in H \setminus \{t\}$) and preserves as many joins as possible. To do this, we choose some object $\ast = \ast_H$ that is not an element of $H$ and let $\Gamma(H) = (H \cup \{\ast\}, f, t, \sqcap, \sqcup, \rightarrow)$, where the operations $\sqcap, \sqcup$ and $\rightarrow$ are derived from those of $H$ as shown in the operation tables below, in which $x$ and $y$ range over $H \setminus \{t\}$ and where $\alpha : H \rightarrow (H \setminus \{t\}) \cup \{\ast\}$ satisfies $\alpha(x) = x$ for $x \neq t$ and $\alpha(t) = \ast$.

\[
\begin{array}{cccc|cccc|cccc}
\sqcap & y & \ast & t & \sqcup & y & \ast & t & \rightarrow & y & \ast & t \\
x & x \sqcap y & x & x & x & \alpha(x \sqcup y) & \ast & t & x & x \rightarrow y & t & t \\
\ast & y & \ast & \ast & \ast & \ast & \ast & t & \ast & y & t & t \\
t & y & \ast & t & t & t & t & t & t & y & \ast & t \\
\end{array}
\]
Let $\mathbb{B}$ be the two-element Heyting algebra and, as usual, let us write $H^i$ for the $i$-fold power of a Heyting algebra $H$. Then define a sequence $J_0, J_1, \ldots$ of finite Heyting algebras as follows:

$$J_0 = \mathbb{B}$$

$$J_{k+1} = \Gamma(J_k^{k+1})$$

We take the language $\mathcal{L}$ of intuitionistic propositional calculus, $\text{IPL}$, to be constructed from a set $\mathcal{V} = \{P_1, P_2, \ldots\}$ of variables, the constants $\perp, \top$, and the binary connectives $\land, \lor$ and $\Rightarrow$. We do not take negation as primitive: $\neg A$ is an abbreviation for $A \Rightarrow \perp$. The metavariables $A, B, \ldots, M$ (possibly with subscripts) range over formulas. $E$ and $F$ are reserved for formulas that are either variables or $\perp$. $P, Q, \ldots, Z$ range over variables. We assume known one of the many ways of defining the logic of $\text{IPL}$ and write $\text{IPL} \vdash A$, if $A$ is provable in $\text{IPL}$. $\text{IPL}$ has an algebraic semantics in which, given a Heyting algebra $H$ and an interpretation $I : \mathcal{V} \rightarrow H$, we extend $I$ to a mapping $v_I : \mathcal{L} \rightarrow H$ by interpreting $\perp, \top, \land, \lor$ and $\Rightarrow$ as $f, t, \sqcap, \sqcup$ and $\rightarrow$ respectively. As usual we write $I \models A$ if $v_I(A) = t$, $H \models A$ if $I \models A$ for every interpretation $I : \mathcal{V} \rightarrow H$ and $\models A$ if $H \models A$ for every Heyting algebra $H$. We assume known the fact that $\text{IPL}$ is sound with respect to this semantics in the sense that, if $\text{IPL} \vdash A$, then $\models A$. The converse statement, i.e., the completeness of $\text{IPL}$ with respect to the semantics is well-known, but we do not use it: in fact we will give an alternative to the usual proofs.

We write $A \iff B$ for $(A \Rightarrow B) \land (B \Rightarrow A)$ and $A[B/X]$ for the result of substituting $B$ for each occurrence of $X$ in $A$. We have the following substitution lemma:

**Lemma 1 (substitution)** For any formulas $A$, $B$ and $C$ and any variable $X$ we have:

(i) if $\text{IPL} \vdash C$, then $\text{IPL} \vdash C[A/X]$;

(ii) if $\text{IPL} \vdash A \iff B$, then $\text{IPL} \vdash C[A/X] \iff C[B/X]$;

**Proof:** (i) is proved by induction on a proof of $C$. (ii) is proved by induction on the structure of $C$. □

We say a formula $A$ is **reduced** if $\top$ does not appear in $A$ as the operand of any connective and $\perp$ does not appear in $A$ as the operand of any connective other than as the right-hand operand of $\Rightarrow$. Thus the only reduced formula containing $\top$ is $\top$ itself.

**Lemma 2** Any formula is equivalent to a reduced formula.

**Proof:** This follows by repeated use of the substitution lemma and the provable equivalences $\top \land A \iff A$, $\perp \land A \iff \perp$ etc. □

We define a formula to be **basic** if it is reduced and is either a variable or has one of the forms $P \Rightarrow A$ or $A \Rightarrow P$ where $P$ is a variable and $A$ contains at
most one connective. Thus a basic formula has one of the following forms:

\[
\begin{align*}
P & \Rightarrow Q \\
\neg P & \Rightarrow \neg Q \\
P & 
\Rightarrow \neg Q
\end{align*}
\]

Note that if \( A \) is a basic formula of a form other than \( P \), \( (P \Rightarrow Q) \Rightarrow R \) or \( \neg P \Rightarrow Q \), then \( V_I(A) = t \) in any Heyting algebra under the interpretation \( I \) that maps every variable to \( t \). Our convention for the metavariables \( E \) and \( F \) allows us to write, for example, \( (P \Rightarrow E) \Rightarrow R \) as a metanotation for the forms \( (P \Rightarrow Q) \Rightarrow R \) and \( \neg P \Rightarrow R \).

We say a formula is a **basic context** if it is reduced and is a conjunction of one or more pairwise distinct basic formulas. We say a formula is regular if it is an implication \( K \Rightarrow F \) where \( K \) is a basic context (and following our convention \( F \) is a variable or \( \bot \)).

We say \( A \) and \( B \) are equiprovable and write \( A \vdash B \) if \( \text{IPL} \vdash A \) iff \( \text{IPL} \vdash B \).

**Lemma 3** Every formula \( A \) is equiprovable with a regular formula \( M \Rightarrow Z \) such that if \( H \) is any Heyting algebra and \( I \) is an interpretation in \( H \) with \( V_I(M) = t \), then \( V_I(A) \leq V_I(Z) \).

**Proof:** By Lemma 2, we may assume \( A \) is reduced. If \( A \) is \( \top \), let \( Z \) be any variable and let \( M \equiv Z \), then \( A \) and \( M \Rightarrow Z \) are both provable and hence they are equiprovable. If \( A \) is \( \bot \), take \( M \) and \( Z \) to be distinct variables, then neither \( A \) nor \( M \Rightarrow Z \) is provable, and hence they are equiprovable. Otherwise, choose some variable \( Z \) that does not occur in \( A \). Then it is easy to see that \( A \vdash (A \Rightarrow Z) \Rightarrow Z \) (for the right-to-left direction, use the substitution lemma to substitute \( A \) for \( Z \)). Our plan is to replace \( K \equiv A \Rightarrow Z \) by a basic context by “unnesting” all its non-atomic subformulas. Assume \( K \) contains \( k \) non-atomic subformulas. Starting with \( K \equiv A_1 \equiv B_1 \circ_1 C_1 \), enumerate the \( k \) non-atomic sub-formulas, \( A_1 \equiv B_1 \circ_1 C_1, \ldots, A_k \equiv B_k \circ_k C_k \). Choose fresh variables \( P_i \), \( i = 1, \ldots, k \). Define atomic formulas, \( G_i, H_i \), for \( i = 1, \ldots, k \) as follows: \( G_i \) is \( B_i \) if \( B_i \) is atomic and is \( P_j \) if \( B_i \) is the \( j \)-th non-atomic subformula; \( H_i \) is \( C_i \) if \( C_i \) is atomic and is \( P_j \) if \( C_i \) is the \( j \)-th non-atomic subformula. Now define formulas \( L \) and \( M \) as follows:

\[
L \equiv \bigwedge_{i=1}^{k} (P_i \iff (G_i \circ_i H_i))
\]

\[
M \equiv P_1 \land L
\]

Recalling that \( B \iff C \) is just shorthand for \( (B \Rightarrow C) \land (C \Rightarrow B) \), and using the fact that \( A \) and hence \( K \) are reduced, we see that \( M \) is a basic context, so \( M \Rightarrow Z \) is regular.

We must show that \( K \Rightarrow Z \vdash M \Rightarrow Z \). To see this, first assume \( \text{IPL} \vdash K \Rightarrow Z \). By induction on the size of the \( A_i \), we have that \( \text{IPL} \vdash L \Rightarrow (P_i \iff A_i) \),
Lemma 4 If $B$ is a basic formula that is not of the form $P$ or $P \Rightarrow Q \land R$ and $P$ occurs in $B$, then $\text{IPL} \vdash P \land B \iff P \land C$ where $C$ has fewer connective occurrences than $B$ and is either a basic formula, an atom or a basic context comprising a conjunction of two variables.

Proof: Routine using the fact that $\text{IPL} \vdash P \land B \Rightarrow P \land B[\top/P]$ (which may be proved for arbitrary $B$ by induction on the structure of $B$).

Lemma 5 If $\text{IPL} \vdash K \land A \land (B \Rightarrow C) \Rightarrow B$, then

$\text{IPL} \vdash ((K \land ((A \Rightarrow B) \Rightarrow C)) \Rightarrow D) \iff (K \land C \Rightarrow D)$.

Proof: $\Rightarrow$: easy using $\text{IPL} \vdash C \Rightarrow ((A \Rightarrow B) \Rightarrow C)$.

$\Leftarrow$: the following gives the highlights of the natural deduction proof.

1. $K \land A \land (B \Rightarrow C) \Rightarrow B$ [Given] (1)
2. $K \land C \Rightarrow D$ [Assume] (2)
3. $K \land (B \Rightarrow C) \Rightarrow A \Rightarrow B$ [By $\text{(1)}$] (3)
4. $K \land ((A \Rightarrow B) \Rightarrow C) \Rightarrow A \Rightarrow B$ [By $\text{(3)}$] (4)
5. $K \land ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$ [By $\text{(2)}$] (5)
6. $K \land ((A \Rightarrow B) \Rightarrow C) \Rightarrow D$ [By $\text{(5)}$ and $\text{(2)}$] (6)
7. $(K \land C \Rightarrow D) \Rightarrow ((K \land ((A \Rightarrow B) \Rightarrow C)) \Rightarrow D)$ [By $\text{(6)}$, discharge $\text{(2)}$] (7)

Here in step $\text{(2)}$ we use $\text{IPL} \vdash ((A \Rightarrow B) \Rightarrow C) \Rightarrow (B \Rightarrow C)$. ■

Lemma 6 Let $B$ be a basic formula that is not a variable and let $I$ be an interpretation in a non-trivial Heyting algebra $H$ such that $V_I(B) = t$. Let $\alpha : H \rightarrow (H \setminus \{\top\}) \cup \{\ast_H\}$ be as in the definition of $\Gamma(H)$. Define an interpretation $J$ in $\Gamma(H)$ by $J = \alpha \circ I$. 
(i) If \( B \) does not have the form \((P \Rightarrow E) \Rightarrow R\) then \( V_I(B) = t \).
(ii) If \( B \) has the form \((P \Rightarrow E) \Rightarrow R\) and if in addition \( V_I(P) = V_I(E \Rightarrow R) = t \) while \( V_I(E) \neq t \), then also \( V_I(B) = t \).

**Proof:** (i): This is easily checked for the case \( P \Rightarrow E \) and for the cases \( P \circ Q \Rightarrow R \) and \( P \Rightarrow Q \circ R \) when \( \circ \in \{\wedge, \vee\} \). In the remaining case \( B \equiv P \Rightarrow Q \Rightarrow E \). As \( B \) is equivalent to \( P \wedge Q \Rightarrow E \), we have already covered the case when \( E \) is a variable, while if \( E \) is \( \perp \), \( V_I(B) = \alpha(p) \wedge \alpha(q) \Rightarrow f \), where \( p = I(P) \) and \( q = I(Q) \), but then, by inspection of the operation tables, we have \( \alpha(p) \wedge \alpha(q) = p \wedge q \) unless \( p = q = t \), but as \( H \) is non-trivial and \( V_I(B) = t \), the case \( p = q = t \) cannot arise.

(ii): we have \( V_I(B) = (\alpha(p) \rightarrow \alpha(e)) \rightarrow \alpha(r) \), where \( p = V_I(P), e = V_I(E) \) and \( r = V_I(R) \). By assumption, \( p = t \) and \( e \neq t \), so \( \alpha(p) = * \) and \( \alpha(e) = e \), hence \( \alpha(p) \rightarrow \alpha(e) = * \rightarrow e = e \), so that \( V_I(B) = e \rightarrow \alpha(r) \) which is \( e \rightarrow * = t \), if \( r = t \), and is \( e \rightarrow r \) otherwise, in which case, as we are given that \( V_I(E \Rightarrow R) = t \), we have \( e \rightarrow r = V_I(E \Rightarrow R) = t \).

To state our main theorem, we define an interpretation \( I \) to be a strong refutation of a formula of the form \( K \Rightarrow C \), if \( V_I(K) = t \) while \( V_I(C) \neq t \).

**Theorem 7** Let \( A \equiv K \Rightarrow F \) be a regular formula (so that \( F \) is either a variable or \( \perp \)), let \( K \equiv B_1 \wedge \ldots \wedge B_k \) display \( K \) as a disjunction of basic formulas and let \( d = d(A) \) be the number of \( B_i \) of the form \((P \Rightarrow E) \Rightarrow R\). Either \( \text{IPL} \vdash A \) or \( A \) has a strong refutation in \( J_d \).

**Proof:** The proof is by induction on the sum \( s(A) = c(A) + d(A) + v(A) \), where \( c(A) \) is the number of connective occurrences in \( K \), \( d(A) \) is as in the statement of the theorem and \( v(A) \) is the number of conjuncts of \( K \) comprising a single variable.

Case (i): \( v(A) = d(A) = 0 \): in this case, the interpretation in \( J_0 = \mathbb{B} \) that maps every variable to \( f \) is easily seen to be a strong refutation of \( A \) (which is therefore unprovable, by the soundness of \( \text{IPL} \)).

Case (ii): \( v(A) > 0 \): in this case at least one \( B_i \) is a variable. If all the \( B_i \) are variables and if \( B_i \neq F \) for any \( i \), then \( A \) has strong refutation such that \( I(B_i) = t \), \( i = 1, \ldots, k \) and \( I(F) = f \). Otherwise, rearranging the \( B_i \) if necessary, we may assume that \( K \equiv P \wedge L \) where \( P \) is a variable and \( L \equiv B_2 \wedge \ldots \wedge B_k \).

If \( P \equiv F \), we are done: \( F \wedge L \Rightarrow F \) is provable. If \( P \neq F \) and \( P \) does not occur in \( L \), then it is easy to see that \( A \vdash A' \) where \( A' : \equiv L \Rightarrow F \). As \( s(A') < s(A) \), by induction, if \( \text{IPL} \nvdash L \Rightarrow F \), we can find a strong refutation \( I \) of \( L \Rightarrow F \), but then, because \( P \) does not occur in \( L \Rightarrow F \), by adjusting \( I \) if necessary to map \( P \) to \( t \) we obtain a strong refutation of \( A \). If \( P \) occurs in \( L \), let us rearrange the \( B_i \) again so that \( K \equiv P \wedge B \wedge M \) where \( M \equiv B_3 \ldots, B_k \) and \( P \) occurs in \( B \).

If \( B \) does not have the form \( P \Rightarrow Q \vee R \), then, by Lemma \[5\] we may replace \( P \wedge B \) by an equivalent formula \( P \wedge C \) where \( C \) is either a basic formula, an atom or a basic context comprising a conjunction of two variables and contains fewer connectives then \( B \). If \( C \) is \( \perp \), \( A \) is provable and we are done. Otherwise, we may replace \( A \) by the equivalent regular formula \( A' : \equiv P \wedge C \wedge M \Rightarrow F \)
(or \( P \land M \Rightarrow F \), if \( C \) is \( \top \)) and we are done by induction, since \( s(A') < s(A) \).

If \( B \) has the form \( P \Rightarrow Q \lor R \), then \( \text{IPL} \vdash P \land B \land M \iff K' \lor K'' \) where \( K' \equiv P \land Q \land M \) and \( K'' \equiv P \land R \land M \), and hence \( \text{IPL} \vdash A \Leftrightarrow A' \land A'' \) where \( A' \equiv K' \Rightarrow F \) and \( A'' \equiv K'' \Rightarrow F \). If \( A \) is not provable, then one of \( A' \) and \( A'' \) is not provable, in which case, as \( s(A') < s(A) \) and \( s(A'') < s(A) \), by induction we have a strong refutation in \( J_d \) of either \( A' \) or \( A'' \) and this will also strongly refute \( A \).

Case (iii): \( v(A) = 0 \) and \( d = d(A) > 0 \): Let \( X = \{j_1, \ldots, j_d\} \) be the set of \( i \) such that \( B_i \) has the form \( (P \Rightarrow E) \Rightarrow R \). For each \( i \in X \), let \( K_i \equiv B_1 \land \ldots \land B_{i-1} \land B_{i+1} \land \ldots \land B_k \) and let \( P_i, E_i \) and \( R_i \) be such that \( B_i \equiv (P_i \Rightarrow E_i) \Rightarrow R_i \). We now have two subcases depending on the provability of the formulas \( C_i \equiv K_i \land P_i \land (E_i \Rightarrow R_i) \Rightarrow E_i \):

Subcase (iii)(a): for some \( i \in X \), \( \text{IPL} \not\vdash C_i \): By Lemma 3, \( A \), which is equivalent to \( K_i \land ((P_i \Rightarrow E_i) \Rightarrow R_i) \Rightarrow F \), is equivalent to \( A' \equiv K_i \land R_i \Rightarrow F \). As \( s(A') < s(A) \), we are done by induction.

Subcase (iii)(b): for every \( i \in X \), \( \text{IPL} \not\vdash C_i \): By induction, as \( s(C_i) < s(A) \) and \( d(C_i) = d - 1 \), for each \( i \in X \) there is an interpretation \( I_i \) in \( J_{d-1} \) that strongly refutes \( C_i \), i.e., \( K_i \land P_i \land (E_i \Rightarrow R_i) \Rightarrow E_i \). Now define an interpretation \( I \) in \( J_{d-1} \), by \( I(U) = (I_{j_1}(U), \ldots, I_{j_d}(U)) \). Then \( V_I(B_i) = t \) for \( i = 1, \ldots, k \) (because, for \( i \in X \), \( V_I(P_i) = V_I(E_i \Rightarrow R_i) = t \) and \( B_i \equiv (P_i \Rightarrow E_i) \Rightarrow R_i \)). But then applying Lemma 3 to \( I \) gives us an interpretation \( J \) in \( J_d = \Gamma(J_{d-1}) \) that strongly refutes \( A \).

**Corollary 8** Let \( A \equiv K \Rightarrow F \) be a regular formula and let \( d \) be the number of conjuncts of \( K \) of the form \( (P \Rightarrow E) \Rightarrow R \). Then \( \text{IPL} \vdash K \) iff \( J_d \models K \).

**Proof:** Immediate from the theorem given the soundness of \( \text{IPL} \) for the Heyting algebra semantics.

**Corollary 9** \( \text{IPL} \) is complete for the Heyting algebra semantics.

**Proof:** Assume \( \models A \). We have to show that \( \text{IPL} \vdash A \). Consider the regular formula \( A' \equiv M \Rightarrow Z \) such that \( A \vdash A' \) whose existence is given by Lemma 3. If \( \text{IPL} \not\vdash A' \), then \( \text{IPL} \not\vdash \text{IPL}' \), whence by the theorem, \( A' \) has a strong refutation in \( J_k \) for some \( k \), i.e., an interpretation \( I \) in \( J_k \) such that \( V_I(M) = t \), but \( V_I(Z) < t \). But then Lemma 3 gives us that \( V_I(A) \leq V_I(Z) < t \), so \( I \not\models A \) contradicting our assumption that \( \models A \).

**Corollary 10** \( \text{IPL} \) has the finite model property.

**Proof:** It is immediate from the theorem and soundness that a refutable regular formula has a refutation in a finite model. Argue as in the proof of Corollary 9 to reduce the general case to the case of regular formulas.

If \( H_0, H_1, \ldots \) is a sequence of Heyting algebras, let us define \( \bigoplus_k H_k \) to be the subalgebra of \( \prod_k H_k \) comprising sequences \( (p_0, p_1, \ldots) \) such that for all sufficiently large \( k \), the \( p_k \) are either all \( f \) or all \( t \). Our final corollary shows that there is countably infinite Heyting algebra \( J \), such that for any formula \( \phi \), \( J \models \phi \) iff \( \text{IPL} \vdash \phi \).
Corollary 11  For any formula $A$, $\text{IPL} \vdash A$ iff $J \models A$, where $J = \bigoplus J_k$.

Proof:  The left-to-right direction is just the soundness of $\text{IPL}$ for Heyting algebras. For the right-to-left direction argue as in the proof of Corollary 9 and note that a refutation in $J_d$ gives a refutation in the subalgebra of $J$ comprising the sequences $(p_0, p_1, \ldots)$ such that $p_i$ is constant for $i > d$.

The statement of Theorem 7 leads to a decision procedure for $\text{IPL}$ that involves a search through all interpretations of a formula in one of the $J_d$ for a certain $d$. As Rose [4] observes, the size of the $J_k$ grows very rapidly with $k$, so this decision procedure is impractical. However, the proof of the theorem leads to a much better algorithm: given any formula $A$, we first apply the algorithm of Lemma 3 if necessary to convert $A$ into an equiprovable regular formula and then follow the case analysis of the proof: if we are in Case (i), $A$ is unprovable and we are done; if we are in Case (ii), the proof shows us how to produce one or two simpler formulas whose conjunction is equivalent to $A$ and we may proceed recursively to decide these formulas; if we are in Case (iii), we can derive the formulas $C_i$ described in the proof and decide them recursively; if any $C_i$ is provable, we are in Subcase (iii)(a) and we may replace $A$ by an equivalent and simpler formula that we can decide recursively; if no $C_i$ is provable, we are in Subcase (iii)(b) and $A$ is unprovable. In the appendix, we show some example calculations using this decision procedure.

The proof of Theorem 7 and in particular its use of Lemma 3 is largely due to Rose [3, 4]. Rose’s analogue of our basic formulas admits only 6 forms: $P$, $\neg P$, $P \Rightarrow Q$, $P \Rightarrow Q \lor R$, $P \land Q \Rightarrow R$ and $(P \Rightarrow Q) \Rightarrow R$. To prove his analogue of our Lemma 3 involves a lengthy case analysis, whereas our more liberal notion of basic formula admits the simpler and more intuitive proof given here. As far as I know, the observations that the main theorem leads to an alternative proof of the completeness of $\text{IPL}$ and that its proof leads to a practical decision procedure for $\text{IPL}$ are new.

References

[1] S. Jaśkowski. Recherches sur le système de la logique intuitionistique. In Actes du Congrès International de Philosophie Scientifique 6, pages 58–61. Paris, 1936. http://gallica.bnf.fr/ark:/12148/bpt6k383699 (Also available in an English translation in [2] pp. 259–263).

[2] Storrs McCall, editor. Polish Logic 1920–1939. Oxford University Press, 1967.

[3] Gene F. Rose. Jaśkowski’s Truth-Tables and Realizability. PhD thesis, University of Wisconsin, 1952.

[4] Gene F. Rose. Propositional calculus and realizability. Trans. Am. Math. Soc., 75:1–19, 1953.
Appendix: examples of the decision procedure

Throughout the examples “Case” and “Subcase” refer to the proof of Theorem 7.

We use the following tabular format for regular formulas $A \equiv B_1 \land \ldots B_k \Rightarrow F$ occurring as the goals we are trying to decide:

\[
\begin{array}{c}
B_1, \ldots, B_k \\
F
\end{array}
\]

**Example 1:** $A \equiv (P \lor Q) \land \lnot Q \Rightarrow P$

Noting that $A$ already has the form $B \Rightarrow Q$, we can skip the first step in the algorithm of Lemma 3 and simply “unnest” $B$. Listing the subformulas of $(P \lor Q) \land 1 \lnot Q$ as shown by the subscripts, our initial goal is:

\[
\begin{array}{c}
P_1, P_1 \leftrightarrow P_2 \land P_3, P_2 \leftrightarrow P \lor Q, P_3 \leftrightarrow \lnot Q \\
P
\end{array}
\]

We are in Case (ii) and we replace the occurrence of $P_1$ in $P_1 \leftrightarrow P_2 \land P_3$ by $\top$ and simplify giving:

\[
\begin{array}{c}
P_1, P_2, P_3, P_2 \leftrightarrow P \lor Q, P_3 \leftrightarrow \lnot Q \\
P
\end{array}
\]

We are again in Case (ii), but now $P_2$ appears in a subformula of the form $P_2 \Rightarrow P \lor Q$ and replacing $P_2$ by $\top$ in that formula gives us two subgoals:

\[
\begin{array}{c}
P_1, P_2, P_3, P_5 \leftrightarrow \lnot Q \\
P
\end{array}
\]
\[
\begin{array}{c}
P_1, P_2, P_3, Q, P_3 \leftrightarrow \lnot Q \\
P
\end{array}
\]

Both subgoals are in Case (ii). In the first, the succedent of the goal appears in the antecedent while in the second, replacing first $P_3$ and then $Q$ by $\top$ in $P_3 \leftrightarrow \lnot Q$ and simplifying gives the antecedent $\bot$. So in both cases, the subgoals and hence our original formula are provable.

**Example 2:** Peirce’s law: $A \equiv ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

$A$ is already regular, so we take it as our initial goal:

\[
\begin{array}{c}
(P \Rightarrow Q) \Rightarrow P \\
P
\end{array}
\]

We are in Case (iii) and our next step is to decide the goal:

\[
\begin{array}{c}
P, Q \Rightarrow P \\
Q
\end{array}
\]
This is in Case (ii) and replacing $P$ by $\top$ in $Q \Rightarrow P$ and simplifying leads to

\[
\begin{array}{c}
P \\
Q
\end{array}
\]

This is again in Case (ii) and is refuted by the interpretation $\{P \mapsto t, Q \mapsto f\}$. Following Lemma 8, this lifts to the refutation $\{P \mapsto \ast, Q \mapsto f\}$ of Peirce’s law in $J_1 = \mathbb{B} \cup \{\ast\}$.

**Example 3: prelinearity:** $A \equiv (P \Rightarrow Q) \lor (Q \Rightarrow P)$

Following the first part of Lemma 9, we replace $A$ by the equiprovable formula $((P \Rightarrow Z) \Rightarrow Z$ and list its subformulas as indicated by the subscripts in $((P \Rightarrow_3 Q) \lor_2 (Q \Rightarrow_4 P) \Rightarrow_1 Z) \Rightarrow Z$. This gives us the following initial goal:

\[
\begin{array}{c}
P_1, P_2 \iff P_3 \lor P_4, P_3 \iff P \Rightarrow Q, P_4 \iff (Q \Rightarrow P)
\end{array}
\]

This is in Case (ii) and replacing $P_1$ by $\top$ in $P_1 \iff P_2 \Rightarrow Z$ and simplifying we get:

\[
\begin{array}{c}
P_1, P_2 \Rightarrow Z, P_2 \iff P_3 \lor P_4, P_3 \iff P \Rightarrow Q, P_4 \iff (Q \Rightarrow P)
\end{array}
\]

This is now in Case (iii) with $d = 2$. This leads to two subgoals:

**$C_1$:**

\[
\begin{array}{c}
P_1, P_2 \Rightarrow Z, P_2 \iff P_3 \lor P_4, P_3 \iff P \Rightarrow Q, P_4 \iff (Q \Rightarrow P), P_1 \Rightarrow Z
\end{array}
\]

**$C_2$:**

\[
\begin{array}{c}
P_1, P_2 \Rightarrow Z, P_2 \iff P_3 \lor P_4, P_5 \iff P \Rightarrow Q, P_4 \Rightarrow (Q \Rightarrow P), Q, P \Rightarrow P
\end{array}
\]

Either continuing to follow Theorem 7 or by inspection, we find the following strong refutations of these subgoals in $\mathbb{B}$.

\[
\begin{align*}
C_1: & \quad ((P, P_1, P_2, P_3, Z) \times \{t\}) \cup (\{Q, P_3\} \times \{f\}) \\
C_2: & \quad ((Q, P_1, P_2, P_3, Z) \times \{t\}) \cup (\{P, P_4\} \times \{f\})
\end{align*}
\]

Combining these we should obtain a refutation $I = \{P \mapsto (t, f), Q \mapsto (f, t)\}$ of $A$ in $\Gamma(\mathbb{B}^2) \subseteq J_2$. And, indeed, in $\Gamma(\mathbb{B}^2)$ we have:

\[
\begin{align*}
(t, f) \lor (f, t) & = (f, t) \lor (t, f) \\
& = \alpha((f, t) \lor_{\mathbb{B}^2} (t, f)) \\
& = \alpha((t, t)) = \ast \neq t.
\end{align*}
\]