A Minimal Periods Algorithm with Applications

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Abstract

Kosaraju in “Computation of squares in a string” briefly described a linear-time algorithm for computing the minimal squares starting at each position in a word. Using the same construction of suffix trees, we generalize his result and describe in detail how to compute in $O(k|w|)$-time the minimal $k$th power, with period of length larger than $s$, starting at each position in a word $w$ for arbitrary exponent $k \geq 2$ and integer $s \geq 0$. We provide the complete proof of correctness of the algorithm, which is somehow not completely clear in Kosaraju’s original paper. The algorithm can be used as a sub-routine to detect certain types of pseudo-patterns in words, which is our original intention to study the generalization.

1 Introduction

A word of the form $ww$ is called a square, which is the simplest type of repetition. The study on repetitions in words has been started at least as early as Thue’s work [21] in the early 1900’s. Since then, there are many work in the literature on finding repetitions (periodicities), which is an important topic in combinatorics on words. In the early 1980’s, Slisenko [19] described a linear-time algorithm for finding all syntactically distinct maximal repetitions in a word. Crochemore [5], Main and Lorentz [15] described a linear-time algorithm for testing whether a word contains a square and thus testing whether a word contains any repetition. Since a word $w$ of length $n$ may have $\Omega(n^2)$ square factors (for example, let $w = 0^n$), usually only primitively-rooted or maximal repetitions are computed. Crochemore [4] described an $O(n \log n)$-time algorithm for finding all maximal primitively-rooted integer repetitions, where maximal means that a $k$th power cannot be extend by either direction to obtain a $(k+1)$th power. The $O(n \log n)$-time is optimal since a word $w$ of length $n$ may have $\Omega(n \log n)$ primitively-rooted repetitions (for example, let $w$ be a Fibonacci word). Apostolico and Preparata [1] described an $O(n \log n)$-time algorithm for finding all right-maximal repetitions, which means a repetition $x^k$ cannot be extend to the right to obtain a repetition $y^l = x^kz$ such that $|y| \leq |x|$. Main and Lorentz [14] described an $O(n \log n)$-time algorithm for finding all maximal repetitions. Gusfield and Stoye [20, 10] also described several algorithms on finding repetitions. We know that both the number of distinct squares [8] and the number of maximal repetitions (also called runs) [12] in a words are in $O(n)$. This fact suggests the existence of linear-time algorithms on repetitions that are distinct (respectively, maximal). Main [16] described a linear-time algorithm for finding all leftmost occurrences of distinct maximal repetitions. Kolpakov and Kucherov [12] described a linear-time algorithm for finding all occurrences of maximal repetitions. For a most-recently survey on the topic of repetitions in words, see the paper [6].

Instead of considering repetitions from a global point of view, there are works on a local point of view, which means repetitions at each positions in a word. Kosaraju in a five-pages extended abstract [13] briefly
described a linear-time algorithm for finding the minimal square starting at each position of a given word. His algorithm is based on an alternation of Weiner’s linear-time algorithm for suffix-tree construction. In the same flavor, Duval, Kolpakov, Kucherov, Lecroq, and Lefebvre [7] described a linear-time algorithm for finding the local periods (squares) centered at each position of a given word. There may be Ω(log n) primarily-rooted maximal repetitions starting at the same position (for example, consider the left-most position in Fibonacci words). So, neither of the two results can be obtained with the same efficiency by directly applying linear-time algorithms on finding maximal-repetitions.

In this paper, we generalize Kosaraju’s algorithm [13] for computing minimal squares. Instead of squares, we discuss arbitrary kth powers and show Kosaraju’s algorithm with proper modification can in fact compute minimal kth powers. Using the same construction of suffix trees, for arbitrary integers k ≥ 2 and s ≥ 0, we describe in details a O(k|w|)-time algorithm for finding the minimal square, with period of length larger than s, starting at each position of a given word w. “The absence of a complete proof prevents the comprehension of the algorithm (Kosaraju’s algorithm) in full details . . .” [2] In this paper, we provide a complete proof of correctness of the modified algorithm. At the end, we show how this O(k|w|)-time algorithm can be used as a sub-routine to detect certain types of pseudo-patterns in words, which is the original intention why we study this algorithm.

2 Preliminary

Let w = a1a2 ··· an be a word. The length |w| of w is n. A factor w[p .. q] of w is the word ap · · · an if 1 ≤ p ≤ q ≤ n; otherwise w[p .. q] is the empty word. In particular, w[1 .. q] and w[p .. n] are called prefix and suffix, respectively. The reverse of w is the word wR = an · · · a1. Word w is called a kth power for integer k ≥ 2 if w = xk for some non-empty word x, where k is called exponent and x is called period. The 2nd power and the 3rd power are called square and cube, respectively.

The minimal (local) period mpk(w) larger than s of word w with respect to exponent k is the smallest integer m > s such that w[1 .. km − 1] is a kth power, if there is such one, or otherwise +∞. For example, mp3 0100101001 = 3 and mp2 0100101001 = 5. The following results follow naturally by the definition of minimal period.

Lemma 1. Let k ≥ 2 and s ≥ 0 be two integers and u be a word. If mpk(u) ̸= +∞, then for any word v,

\[ mpk(uv) = mpk(u) \]

Proof. Suppose mpk(uv) < mpk(u). We can write uv = xk for some words x, y with |x| = mpk(uv) > s. Then |xk| = k · mpk(u) < k · mpk(u) ≤ |u| and thus xk is also a prefix of u. So mpk(u) ≤ |x| = mpk(uv), which contradicts to our hypothesis. So mpk(uv) ≥ mpk(u). On the other hand, if any word xk is a prefix of u, the word xk is also a prefix of uv. So mpk(uv) ≤ mpk(u). Therefore, mpk(uv) = mpk(u).

Lemma 2. Let k ≥ 2 and s ≥ 0 be two integers and u be a word. For any word v,

\[ mpk(u) = \begin{cases} mpk(u), & \text{if } |u| \geq k \cdot mpk(u); \\ +∞, & \text{otherwise.} \end{cases} \]

Proof. Suppose mpk(u) ̸= +∞. By Lemma 1 it follows that mpk(uv) = mpk(u) and |u| ≥ k · mpk(u) = k · mpk(uv). So, by contraposition, mpk(uv) = +∞ and |u| < k · mpk(uv). On the other hand, when |u| ≥ k · mpk(uv), we can write uv = xk for some words x, w such that |x| = mpk(uv). Then xk is also a prefix of u and thus mpk(uv) ̸= +∞. So, by Lemma 1 mpk(uv) = mpk(uv).

The right minimal period array of word w with respect to exponent k and period larger than s is defined by \( rmpw[i] = mpk(w[i .. n]) \) for 1 ≤ i ≤ n and the left minimal period array of word w with respect to exponent k and period larger than s is defined by \( lmpw[i] = mpk(w[1 .. i]) \) for 1 ≤ i ≤ n. For example,

\[
\begin{align*}
2_0rmp_{0100101001} &= [3, +∞, 1, 2, 2, +∞, +∞, 1, +∞, +∞], \\
2_0lmp_{0100101001} &= [+∞, +∞, +∞, 1, +∞, 3, 2, 2, 1, 5].
\end{align*}
\]
Lemma 3. Let $T_w$ be the suffix tree of word $w$. If $\text{leaf}_i$ and $\text{leaf}_j$ are two leaves such that $i > j$, then the label on the edge from $p(\text{leaf}_i)$ to $\text{leaf}_i$ is not longer than the label on the edge from $p(\text{leaf}_j)$ to $\text{leaf}_j$.

Proof. Let $n = |w|$ and words $e_i, e_j$ be the labels on the edges from $p(\text{leaf}_i)$ to $\text{leaf}_i$ and from $p(\text{leaf}_j)$ to $\text{leaf}_j$, respectively. We now prove $|e_i| \leq |e_j|$. Since $i > j$, by definitions, we can write $\tau(\text{leaf}_j) = x\tau(\text{leaf}_i)$ for some word $x$ and thus

$$\tau(p(\text{leaf}_j))e_j = \tau(\text{leaf}_j) = x\tau(\text{leaf}_i) = x\tau(p(\text{leaf}_i))e_i.$$
If \(|e_j| \geq \delta(\text{leaf}_i)|y\rangle\), then \(|e_i| \leq \delta(\text{leaf}_i) \leq |e_j|\). Otherwise, we can write \(\tau(\text{leaf}_i) = ye_j\) for some word \(y\) and thus \(\tau(p(\text{leaf}_j)) = x\). Let \(\text{leaf}_k\) be another leaf that is a descendent of \(p(\text{leaf}_j)\). Then we can write \(\tau(\text{leaf}_k) = \tau(p(\text{leaf}_j))z = x\) for some word \(z\) such that \(z \neq e_j\) and \(e_j\) are different at the first letter. The word \(y\) is a suffix of \(w\) and the longest common prefix of the two words \(\tau(\text{leaf}_i) = ye_j\) and \(y\) is \(y\). So there is an ancestor \(v\) of \(\text{leaf}_i\) such that \(\tau(v) = y\) and thus \(\delta(p(\text{leaf}_i)) \geq |y|\). But \(\tau(p(\text{leaf}_i))e_i = \tau(\text{leaf}_i) = ye_j\). Therefore, \(|e_i| \leq |e_j|\).

A suffix tree for a given word \(w\) can be constructed in linear time \cite{23,17,22}. Both Kosaraju’s algorithm \cite{13} for computing \(k\)th\(rmp_w\) and our modification on his algorithm for computing \(k\)th\(lmp_w\) for arbitrary \(k \geq 2\) and \(s \geq 0\) are based on Weiner’s linear-time algorithm \cite{22} for constructing the suffix tree \(T_w\). So we briefly describe Weiner’s algorithm here.

Weiner’s algorithm extends the suffix tree by considering the suffix \(w[n..n], \ldots, w[2..n], w[1..n]\) and adding \(\text{leaf}_n, \ldots, \text{leaf}_2, \text{leaf}_1\) into the suffix tree incrementally. After each extension by \(w[i..n]\), the new tree is precisely the suffix tree \(T_{w[i..n]}\). The algorithm is outlined in Algorithm 1. By using indicator vectors and inter-node links, the total time to locate each proper position \(y\) at lines 9–10 can be in \(O(n)\). Since how to locate the \(y\) is not quite relevant to the algorithm we will present later, we omit the details here.

\begin{verbatim}
Input: a word \(w = w[1..n]\).
Output: the suffix tree \(T_w\).
1 begin function make_suffix_tree(w)
2    construct \(T_n = T_{w[n..n]}\);
3    for \(i\) from \(n - 1\) to \(1\) do
4        // assert: \(T_i = T_{w[i..n]}\)
5        \(T_i \leftarrow\) extend\((T_{i+1}, w[i..n])\);
6    end
7 return \(T_1\);
8 begin function extend\((tree, \text{word}[i..n])\)
9    // we assume \(\text{tree} = T_{\text{word}[i+1..n]}\)
10    find the proper position \(y\) in \(\text{tree}\) to insert the new node \(\text{leaf}_i\);
11    if needed, split an edge \(x \rightarrow z\) to two \(x \rightarrow y, y \rightarrow z\) by adding a new node \(y\);
12    create and label the edge \(y \rightarrow \text{leaf}_i\) by \(\text{word}[i + |\tau(y)|..n]\).
13 end
\end{verbatim}

Algorithm 1: Framework of Weiner’s algorithm for constructing suffix tree

Once a node \(v\) is created, although the node-depth \(|v|\) may change in later extensions by splitting on an edge in the path from the root to node \(v\), the depth \(\delta(v)\) will never change in later extensions in a suffix tree. So we assume the depth \(\delta(v)\) is also stored on the node \(v\) in the suffix tree and can be accessed in constant time. The update of \(\delta(v)\) only happens when \(v\) is created and can be computed by \(\delta(v) = \delta(p(v)) + |u|\), where \(u\) is the label on the edge from \(p(v)\) to \(v\). So computing and storing the information \(\delta\) will not increase the computational complexity of the construction of a suffix tree.

3 The algorithm for computing \(k\)th\(rmp_w\) and \(k\)th\(lmp_w\)

First we show that how the minimal period \(mp^k_w\) can be obtained from the suffix tree \(T_w\) in linear time \(O(|w|/\min\{s, mp^k_0(w)\})\). In particular, if \(s = \Omega(|w|)\) and \(w\) satisfies \(mp^k_0(w) = \Omega(|w|)\), then the algorithm computes \(mp^k_s(w)\) in constant time, which is one of the essential ideas in the computing of \(k\)th\(rmp_w\) and \(k\)th\(lmp_w\).

Lemma 4. Let \(k \geq 2\) and \(s \geq 0\) be two integers and \(T_w\) be the suffix tree of a word \(w\). Then \(mp^k_s(w)\) can be computed in \(O\left(|w|/\min\{s, mp^k_0(w)\}\right)\) time.
Proof. Let $n = |w|$. There is an $O\left(n/\min\{s, mp_k^s(w)\}\right)$-time algorithm to compute $mp_k^s(w)$. First along the path from the leaf $f_1$ to the root, we find the highest ancestor $h$ of leaf $f_1$ such that $\delta(h) \geq (k-1)(s+1)$. Since $\delta(\text{root}) = 0$, node $h$ always has a father and $\delta(p(h)) < (k-1)(s+1)$. Then we find the least common ancestor of leaf $f_1$ with any other leaf leaf $f_2$ that is a descendent of $h$ and check whether the equation

$$\delta(\text{lca}(\text{leaf}_f_1, \text{leaf}_f_2)) \geq (k-1)(i-1)$$

(1)

holds. If no leaf $f_i$ satisfies [1], then $mp_k^s(w) = +\infty$; otherwise, $mp_k^s(w) = i - 1$, where $i$ is the smallest $i$ that satisfies [1]. The algorithm is presented in Algorithm 2.

**Input**: a suffix tree $T_{w[1..n]}$ and two integers $s \geq 0, k \geq 2$.

**Output**: the minimal period $mp_k^s(w)$.

```
begin function compute_mp(tree, s, k)
  if $k(s+1) > n$ then return $+\infty$ else $h \leftarrow \text{leaf}_f_1$
  while $\delta(p(h)) \geq (k-1)(s+1)$ do $h \leftarrow p(h)$
  $mp \leftarrow +\infty$
  // linear-time preprocessing for constant-time finding lca
  preprocessing the tree rooted at $h$ for lca
  foreach leaf $\text{leaf}_f_i$ being a descendent of $h$ other than $\text{leaf}_f_1$ do
    if $\delta(\text{lca}(\text{leaf}_f_1, \text{leaf}_f_i)) \geq (k-1)(i-1)$ then
      // assert: $w[1..i-1]$ is a period of the word $w$
      if $mp > i-1$ then $mp \leftarrow i-1$
    end
  end
  return $mp$
end
```

**Algorithm 2**: Algorithm for computing $mp_k^s(w)$ by using the suffix tree $T_w$

Now we prove the correctness of this algorithm. First we observe that $w = x^ky$ for some non-empty word $x$, if and only if the common prefix of $w[1..n]$ and $w[|x|+1..n]$ is of length at least $(k-1)|x|$, which means the leaf leaf $f_{x|+1}$ satisfies [1]. Furthermore, $|x| > s$, if and only if leaf $f_{x|+1}$ satisfies $\delta(\text{lca}(\text{leaf}_f_1, \text{leaf}_f_{x|+1})) \geq (k-1)(s+1)$, which means that leaf $f_{x|+1}$ is a descendent of $h$. (Since $h$ has two descendents, $h$ is not a leaf and thus $h \neq \text{leaf}_f_{x|+1}$.) So each time line 8 is executed, if and only if there is a corresponding prefix of $w$ that is a $k$th power with period of length $i-1 > s$. The minimal length of such period, if any, is returned and the correctness is ensured.

Now we discuss the computational complexity of this algorithm. Let $T_h$ be the sub-tree rooted at $h$ and $l$ be the number of leaves in $T_h$. By the definition of suffix tree, each internal node has at least two children in $T_h$ and thus the number of internal nodes in $T_h$ is less than $l$. Furthermore, the node-depth of any leaf in $T_h$ is also less than $l$. So the computational time of the algorithm is linear in $l$. (For details on constant-time algorithm finding lowest common ancestor with linear-time preprocessor, see [11, 18].) In order to show the computation is in $O\left(n/\min\{s, mp_k^s(w)\}\right)$-time, it remains to see $l = O\left(n/\min\{s, mp_k^s(w)\}\right)$. We prove $l \leq n/\min\{s + 1, mp_k^s(w)\}$ by contradiction. Suppose $l > n/\min\{s + 1, mp_k^s(w)\}$. Since there are $l$ leaves $i_1, i_2, \ldots, i_l$ with the same ancestor $h$, there are $l$ factors of length $t = (k-1)(s+1)$ such that

$$w[i_1..i_1 + t - 1] = w[i_2..i_2 + t - 1] = \cdots = w[i_l..i_l + t - 1].$$

Since $1 \leq i_j \leq n$ for $1 \leq j \leq l$, by the pigeon hole principle, there are two indices, say $i_1$ and $i_2$, such that $0 \leq i_2 - i_1 < n/l < \min\{s+1, mp_k^s(w)\}$. Then the common prefix of $w[i_1..n]$ and $w[i_2..n]$ is of length at least $t = (k-1)(s+1) > (k-1)(i_2 - i_1)$, which means there is a prefix of $w[i_1..i_1 + t - 1] = w[i_2..i_2 + t - 1] = w[1..t-1]$ that is a $k$th power with period of length $i_2 - i_1$. Then $mp_k^s(w) \leq i_2 - i_1 < mp_k^s(w)$, a contradiction. So the number of leaves in $T_h$ is $\leq n/\min\{s + 1, mp_k^s(w)\}$ and thus the algorithm is in $O\left(n/\min\{s, mp_k^s(w)\}\right)$-time. □
For a word \( w = w[1..n] \), by definitions, the left minimal period array and the right minimal period array satisfy the equation

\[
^{k_s}lm_{mp_w}[i] = ^{k_s}rmp_{mp_a}[n + 1 - i], \quad \text{for} \ 1 \leq i \leq n.
\]

So the left minimal period array of \( w \) can be obtained by computing the right minimal period array of \( w \). Hence in what follows we only discuss the algorithm for computing the right minimal period array of \( w \); the algorithm for computing the left minimal period array of \( w \) follows immediately.

A suffix tree with minimal periods \(^{k_s}T_w\) for a word \( w \) is a suffix tree \( T_w \) with a function \( \pi^k_s \), which is defined at each node \( v \) such that \( \pi^k_s(v) = mp^k_s(\tau(v)) \). By definitions, since \(^{k_s}T_w\) is created for a word \( w = w[1..n] \), the \(^{k_s}rmp_w\) can be obtained by reading the value \( \pi^k_s \) at each leaf in order as follows:

\[
^{k_s}rmp_w[1..n] = [\pi^k_s(leaf_1), \pi^k_s(leaf_2), \ldots, \pi^k_s(leaf_n)].
\]

The suffix tree with minimal periods satisfies the following property.

**Lemma 5.** Let \( k \geq 2 \) and \( s \geq 0 \) be two integers and \( w \) be a word. For any node \( v \) in the suffix tree with minimal periods \(^{k_s}T_w\) such that \( \pi^k_s(p(v)) = +\infty \), then either \( \pi^k_s(v) = +\infty \) or \( \pi^k_s(v) \) is between

\[
\frac{\delta(p(v))}{k} < \pi^k_s(v) \leq \frac{\delta(p(v))}{k - 1}.
\]

**Proof.** Let \( v \) be a node in \(^{k_s}T_w\) such that \( \pi^k_s(p(v)) = +\infty \). Since \( \tau(p(v)) \) is a prefix of \( \tau(v) \) and \( \pi^k_s(p(v)) = +\infty \), by Lemma 2, it follows that

\[
\delta(p(v)) = |\tau(p(v))| < k \cdot mp^k_s(\tau(v)) = k \cdot \pi^k_s(v).
\]

Suppose \( \pi^k_s(v) \neq +\infty \). The common prefix of \( \tau(v)[1..\delta(v)] \) and \( \tau(v)[\pi^k_s(v) + 1..\delta(v)] \) is of length at least \( (k - 1)\pi^k_s(v) \). Then \( (k - 1)\pi^k_s(v) \leq \delta(p(v)) \), since \( p(v) \) is the lowest ancestor of \( v \) in \(^{k_s}T_w\). Therefore, either \( \pi^k_s(v) = +\infty \) or \( \delta(p(v))/k < \pi^k_s(v) \leq \delta(p(v))/(k - 1) \).

In what follows, we will show how to construct the \(^{k_s}T_w\) for a word \( w \) with fixed \( k \) in linear time by a modified version of Kosaraju’s algorithm [13]. Kosaraju’s algorithm constructs only \(^{k_s}T_w\) but our modification can construct \(^{k_s}T_w\) for arbitrary \( s \geq 0 \) and \( k \geq 2 \). Both algorithms are based on the alternation of Weiner’s algorithm [23] for constructing suffix tree \( T_w \). The suffix tree with minimal periods satisfies the following property.

**Theorem 6.** Let \( k \geq 2 \) and \( s \geq 0 \) be two integers. Function \texttt{compute}_\texttt{rmp} in Algorithm 3 correctly computes the right minimal period array \(^{k_s}rmp_w\) for the word \( w \).

**Proof.** Since each element \(^{k_s}rmp_w[i]\) is assigned by the value \( \pi^k_s(leaf_i) \) on the leaves of suffix tree \( T_i \) with minimal periods, the correctness of the algorithm relies on the claim \( T_i = ^{k_s}T_{w[i..n]} \). The algorithm is based on Weiner’s algorithm and the only change is to update the \( \pi^k_s \) values. So the underlying suffix tree of \( T_i \) correctly presents the suffix tree \( T_{w[i..n]} \). The update to \( \pi^k_s(v) \) only happens when the node \( v \) is created in some \( T_{w[i..n]} \). By definitions, \( \pi^k_s(v) = mp^k_s(\tau(v)) \) in any expanded suffix tree \(^{k_s}T_{w[j..n]}\) for \( j < i \) is equal to
Input: a word $w = w[1..n]$ and two integers $s \geq 0$, $k \geq 2$.
Output: the right minimal period array $rmp_w$.

begin function compute_rmp($w$, $s$, $k$)
construct $T_n$ by constructing $T_{w[n..n]}$ with $\pi(root), \pi(leaf_n) \leftarrow +\infty$;
$A \leftarrow$ empty, $j \leftarrow n$, and $d \leftarrow 0$;
for $i$ from $n - 1$ to 1 do
    find the proper position $y$ in $T_{i+1}$ to insert the new node $leaf_i$;
    if needed then
        split an edge $x \rightarrow z$ to two $x \rightarrow y, y \rightarrow z$ by adding a new node $y$;
        if $\delta(y) \geq k\pi(z)$ then $\pi(y) \leftarrow \pi(z)$ else $\pi(y) \leftarrow +\infty$;
    end
    create and label the edge $y \rightarrow leaf_i$ by $w[i + |\tau(y)|..n]$ · $;
    // assert: suffix tree part of $T_i$ is $T_{w[i..n]}$
    if $j - i + 1 > 2kd/(k - 1)$ or $\delta(y) < d/2$ then $A \leftarrow$ empty;
    // assert: $A = $ empty or ($A = T_{w[i..j]}$ and $d/2 \leq \delta(p(leaf_i)) \leq 2d$)
    if $\pi(y) \neq +\infty$ then
        $\pi(leaf_i) \leftarrow \pi(y)$;
        if $A = $ empty then continue;
        else $A \leftarrow$ extend($A, w[i..j]$);
    else
        $d \leftarrow \delta(y)$ and $j \leftarrow i + (k + 1)d/(k - 1) - 1$;
        $A \leftarrow$ make_suffix_tree($w[i..j]$);
    end
    if $A = $ empty then
        $A \leftarrow$ extend($A, w[i..j]$);
    else
        $A \leftarrow$ extend($A, w[i..j]$);
    end
    $\pi(leaf_i) \leftarrow$ compute_mp($A, \max\{s, \delta(y)/k\}, k$);
end
// assert: $\forall v \in T_i : \pi(v) = mp^k_s(\tau(v))$ and thus $T_i = k_sT_{w[i..n]}$
rmp[$i$] $\leftarrow \pi(leaf_i)$;
end
$rmp[n] \leftarrow +\infty$ and return $rmp$;
end

Algorithm 3: Algorithm for computing $rmp_w$
the total cost of those underlined statements is in constant time. So the total time of computing

\[ T_n = k \frac{T_{w[i..n]}}{s} \]

is valid and \( T_n \) in \( s \).

Suppose it is true that \( T_{i+1} = k \frac{T_{w[i+1..n]}}{s} \) for some \( i, 1 \leq i \leq n-1 \), at the beginning of the execution of lines 5–25. Then on the next execution within the loop at lines 5–25, there are at most two nodes being created. One possible new node is \( y \), the father of \( lea_f_i \), and the other is the \( lea_f_i \).

For \( \pi(y) \) on line 8: if some split happens on an edge from \( x \) to \( z \) by adding a new node \( y \) and two new edges from \( x \) to \( y \), from \( y \) to \( z \), respectively, then we have \( \tau(z) = \pi(y)u \) for some \( u \neq \epsilon \). By Lemma 2 \( mp^k(\pi(y)) = mp^k(\tau(z)) \), if \( |\tau(y)| \geq k \cdot mp^k(\tau(z)) \); otherwise \( mp^k(\tau(y)) = +\infty \). So the assignments on line 8 of Algorithm 3 is valid.

For \( \pi(lea_f_i) \) on line 23: consider the value \( \pi^k \) on the new leaf \( lea_f_i \). Since \( y = p(lea_f_i) \), we have \( \tau(lea_f_i) = \pi(y)v \) for some \( v \neq \epsilon \). If \( mp^k(\pi(y)) \neq +\infty \), by Lemma 1 it follows that \( mp^k(\tau(lea_f_i)) = mp^k(\pi(y)) \) and thus the assignment in line 13 of Algorithm 3 is valid. If \( mp^k(\pi(y)) = +\infty \), then \( mp^k(\tau(lea_f_i)) = mp^k(w[i..n]) \) is computed with the assistant of the auxiliary suffix tree \( A = T_{w[i..j]} \) by the function \( compute_mp \) in Algorithm 2. Since \( y = p(lea_f_i) \), by Lemma 2 \( mp^k(\tau(lea_f_i)) > \delta(y)/k \) and thus the arguments in calling \( compute_mp \) is valid. To show the assignment on line 23 of Algorithm 3 is valid, the only thing remains to prove is that \( mp^k(w[i..n]) = mp^k(w[i..j]) \).

First we claim that \( \delta(pr_i(lea_f_i)) \leq \delta(pr_{i+1}(lea_f_{i+1})) + 1 \), where the subscript of \( p \) specifies in which tree the parent is discussed. If \( pr_i(lea_f_{i+1}) \neq pr_{i+1}(lea_f_{i+1}) \), then there is a split on the edge from \( pr_{i+1}(lea_f_{i+1}) \) to \( lea_f_{i+1} \) and leaves \( lea_f_i \). If \( pr_i(lea_f_{i+1}) = pr_{i+1}(lea_f_{i+1}) \), then by Lemma 3 it follows that \( \delta(pr_i(lea_f_i)) \leq \delta(pr_i(lea_f_{i+1})) + 1 = \delta(pr_{i+1}(lea_f_{i+1})) + 1 \).

Then we claim \( \delta(y) \leq j - i + 1 - 2d/(k-1) \) holds right before line 23, where \( y = p(lea_f_i) \). Consider the last created suffix tree \( A \), then \( A \neq empty \). If \( A \) is newly created, then \( \delta(p(lea_f_i)) = d \) and \( i = j + 1 - (k+1)d/(k-1) \). So \( \delta(p(lea_f_i)) = j - i + 1 - 2d/(k-1) \). Now we assume \( A \) extends from a previous one. In the procedure of extending \( A \), both \( j \) and \( d \) remain the same, exponent \( k \) is a constant, the index \( i \) increase by 1, and the depth \( \delta(pr_i(lea_f_i)) \) increases at most by 1. So \( \delta(pr_i(lea_f_i)) \leq j - (i+1) + 1 - 2d/(k-1) \) still holds.

Now we prove \( mp^k(w[i..n]) = mp^k(w[i..j]) \). If \( mp^k(w[i..n]) = +\infty \), by Lemma 2 it follows that \( mp^k(w[i..j]) = +\infty = mp^k(w[i..n]) \). Now we assume \( mp^k(w[i..n]) \neq +\infty \). By Lemma 2 it follows that \( mp^k(w[i..n]) = mp^k(\tau(lea_f_i)) \leq \delta(y)/(k-1) \). In addition, \( j - i + 1 \leq 2kd/(k-1) \) always holds when \( A \neq empty \). So the following holds

\[
k \cdot mp^k(w[i..n]) \leq k \frac{1 - (j - i + 1 - 2d/k-1)}{1 - 1/(k-1)} = (j - i + 1) + \frac{1}{k-1}(j - i + 1) - \frac{2kd}{k-1} \leq |w[i..j]|,
\]

and thus by Lemma 2 again \( mp^k(w[i..j]) = mp^k(w[i..n]) \). This finishes the proof \( T_i = k \frac{T_{w[i..n]}}{s} \). \( \square \)

**Theorem 7.** Let \( k \geq 2 \) and \( s \geq 0 \) be two integers. The time complexity of computing the right minimal period array \( s \cdot rmp \) for input word \( w \) in Algorithm 3 is \( O(k|w|) \).

**Proof.** Let \( n = |w| \). Each assignment to elements in array \( rmp \) at lines 25,27 of Algorithm 3 can be done in constant time. So the total time of computing \( rmp = k \frac{s \cdot rmp}{s} \) from the suffix tree \( T_1 = k \frac{T_{w[i..n]}}{s} \) with minimal periods is in \( O(n) \).

The lines 2,5,7,10 of Algorithm 3 constitute exactly the Weiner’s algorithm for constructing the suffix tree \( T_w \), which is in \( O(n) \)-time.

Most of the underlined statements, except lines 15,19,21,23, in Algorithm 3 can be done in constant time in a unit-cost model, where we assume the arithmetic operations, comparison and assignment of integers with \( O(\log n) \)-bit can be done in constant time. The number of executions of lines 5–25 is \( n - 1 \) and thus the total cost of those underlined statements is in \( O(n) \).
Now we consider the computation of line 23. By Lemma 4, since \( A = T_{w[i..j]} \), the cost of each calling to compute_mp in Algorithm 2 is in time linear in

\[
\frac{|w[i..j]|}{\min \{ \max \{ s, \delta(y)/k \}, mp_0^p(w[i..j]) \}} \leq \frac{j - i + 1}{\min \{ \delta(y)/k, mp_0^p(w[i..j]) \}}.
\]

We already showed in the proof of Theorem 6 that \( mp_0^p(w[i..n]) > \delta(y)/k \). In addition, \( j - i + 1 \leq 2kd/(k - 1) \) and \( \delta(y) \geq d/2 \) always hold when \( A \neq empty \). So we have

\[
\frac{j - i + 1}{\min \{ \delta(y)/k, mp_0^p(w[i..j]) \}} \leq \frac{2kd/(k - 1)}{\min \{ d/2k, d/2k \}} = \frac{4k^2}{k - 1}.
\]

The number of executions of lines 5–25 is \( n - 1 \) and thus the total cost on line 23 is \( O(kn) \).

Now we consider the computation of lines 15,19,21. Those statements construct a series of suffix trees \( A = T_{w[i..j]} \) by calling to make_suffix_tree and extend in Algorithm 1. Each suffix tree is initialized at line 19, extended at lines 15,21, and destroyed at line 11. Suppose there are in total \( l \) such trees, and suppose, for \( 1 \leq m \leq l \), they are initialized by \( A = T_{w[i_m..j_m]} \) with \( d_m = \delta(p_{T_{i_m}}(leaf_{i_m})) \) and destroyed when \( A = T_{w[i_m..j_m]} \) such that \( j_m - (i_m - 1) + 1 > 2kd_m/(k - 1) \) or \( \delta(p_{T_{i_m-1}}(leaf_{i_m-1})) < d_m/2 \). In addition, when \( A \neq empty \), the inequality \( j_m - i_m + 1 \leq 2kd/(k - 1) \) always holds for \( i_m \leq i \leq i_m \). Since the construction of suffix tree in Algorithm 1 is in linear time, the total cost on lines 15,19,21 is in time linear in

\[
\sum_{m=1}^{l} |w[i_m..j_m]| = \sum_{m=1}^{l} (j_m - i_m + 1) \leq \sum_{m=1}^{l} \frac{2k^2}{k - 1} d_m.
\]

First, we consider those trees \( A \) destroyed by the condition \( j_m - (i_m - 1) + 1 > 2kd_m/(k - 1) \). Then \( j_m - i_m + 1 = 2kd_m/(k - 1) \) and \( j_m = i_m + (k + 1)d_m/(k - 1) - 1 \) hold, and thus the decrease of \( i \) is \( i_m - i_m = (j_m - (k + 1)d_m/(k - 1) - 1) - (j_m + 1 - 2kd_m/(k - 1)) = d_m \). Hence the total cost in this case is

\[
\sum_{j_m-(i_m-1)+1>2kd_m} \frac{2k}{k} d_m = \frac{2k}{k - 1} \sum_{i_m-i_m>2kd_m} (i_m - i_m - 1) = O(n).
\]

Second, we consider those trees \( A \) destroyed by the condition \( \delta(p_{T_{i_m-1}}(leaf_{i_m-1})) < d_m/2 \). Then \( \delta(p_{T_{i_m-1}}(leaf_{i_m-1})) = \delta(p_{T_{i_m}}(leaf_{i_m})) \leq d_m/2 \). In the proof of Theorem 4 we showed \( \delta(p_{T_{i_1}}(leaf_{i_1})) - \delta(p_{T_{i_1+1}}(leaf_{i_1+1})) \leq 1 \). Since \( \delta(p_{T_{i_1}}(leaf_{i_1})) \geq 0 \), it follows that the total cost in this case is

\[
\sum_{\delta(p_{T_{i_m-1}}(leaf_{i_m-1}))<d_m/2} \frac{2k}{k - 1} d_m < \frac{2k}{k - 1} \sum_{\delta(p_{T_{i_m}}(leaf_{i_m})) = \delta(p_{T_{i_m-1}}(leaf_{i_m-1}))} \frac{\delta(p_{T_{i_m}}(leaf_{i_m})) - \delta(p_{T_{i_m-1}}(leaf_{i_m-1}))}{2} \leq \frac{k}{k - 1} (n - 1) = O(n).
\]

The only remaining case is that the suffix tree \( A \) is not destroyed even after the construction of \( T_1 \). This can be avoided by adding a special character \( \ell \) not in the alphabet of \( w \) at the beginning of \( w \). Then for \( i = 1 \) the father of the \( leaf_f \) is the root and thus \( A \) is destroyed by the condition \( \delta(y) < d/2 \). In addition, \( mp_0^p(\ell \cdot w) = +\infty \) and thus this modification do not change the computational complexity of this algorithm. So, the total cost on lines 15,19,21 is \( O(n) \).

Therefore, the total cost of the algorithm is \( O(n) + O(n) + O(n) + O(kn) + O(n) \) and thus is in time \( O(kn) \). The algorithm is in linear time when exponent \( k \) is fixed.

4 Applications — detecting special pseudo-powers

In this section, we discuss how the linear algorithm for computing \( k \cdot mp_0 \) and \( k \cdot s \cdot mp_0 \) for fixed exponent \( k \) can be applied to test whether a word \( w \) contains a particular type of repetition, called pseudo-powers.

Let \( \Sigma \) be the alphabet. A function \( \phi : \Sigma^* \to \Sigma^* \) is called an involution if \( \phi(\phi(w)) = w \) for all \( w \in \Sigma^* \) and called an antimorphism if \( \phi(wv) = \phi(v)\phi(u) \) for all \( u, v \in \Sigma^* \). We call \( \phi \) an antimorphic involution if
All maximal palindromes can be found in linear time (for example, see [9, pages 197–198]). In exactly

Lemma 8. Let \( \phi \) be an antimorphic involution. The centralized maximal pseudo-palindrome array \( ^c\text{cmp}_w \) of word \( w \) with respect to an antimorphic involution \( \phi \) is defined by

\[
^c\text{cmp}_w[i] = \max\{m : 0 \leq m \leq \min\{i, |w| - i\}, \phi(w[i - m + 1..i]) = w[i + 1..i + m]\} \quad \text{for} \quad 0 \leq i \leq |w|.
\]

For example, \( ^c\text{cmp}_{0100101001} = [0, 0, 0, 3, 0, 0, 0, 0, 2, 0, 0, 0] \).

Lemma 8. Let \( \phi \) be an antimorphic involution. The centralized maximal pseudo-palindrome array \( ^c\text{cmp}_w \) of word \( w \) with respect to \( \phi \) can be computed in \( O(|w|) \) time.

Proof. All maximal palindromes can be found in linear time (for example, see [3] pages 197–198). In exactly

the same manner, by constructing suffix tree \( T_w^L\phi(w) \), where \( L \) is a special character not in the alphabet of

w, the array \( ^c\text{cmp}_w \) can be computed in linear time. More precisely, the algorithm is outlined in Algorithm [4].

Now we prove the correctness of Algorithm [3]. Let \( n = |w| \) and \( \overline{w} = wL\phi(w) \). Then \( |\overline{w}| = 2n + 1 \). By

the definition of suffix tree \( T_{\overline{w}} \), word \( \tau(\text{lca}(\text{leaf}(f_{i+1}), \text{leaf}(f_{2n-i+2}))) \) is the longest common prefix of \( \tau(\text{leaf}(f_{i+1})) = \overline{w}[i + 1..2n + 1] \) and \( \tau(\text{leaf}(f_{2n-i+2})) = \overline{w}[2n - i + 2..2n + 1] \). Since the character \( L \) does not appear in word

\( \tau(\text{leaf}(f_{2n-i+2})) \) and \( \overline{w}[1..i] = \phi(\tau(\text{leaf}(f_{2n-i+2}))) \), it follows that \( \tau(\text{leaf}(f_{i+1}), \text{leaf}(f_{2n-i+2})) \) is the longest word

such that \( u \) is a prefix of \( w[i + 1..n] \) and \( \phi(u) \) is a suffix of \( w[1..i] \). (Here \( \phi \) is an antimorphism, so when

apply \( \phi \), suffix and prefix relations exchange each other.) This proves the correctness.

Both the construction of suffix tree \( T_{\overline{w}} \) and the preprocessing for fast finding lca is in linear time. In addition,

the computation of lca for any pair of leaves is constant after the preprocessing. So the total running time of Algorithm [4] is in \( O(|w|) \).
Input: a word $w = w[1..n]$ and an antimorphic involution $\phi$.
Output: the centralized maximal pseudo-palindrome array $\phi cmp_w$.

begin function compute_cmp($w, \phi$)
  $T \leftarrow \text{make_suffix_tree}(w, L\phi(w))$ ; // $L$ is a character not in $w$
  linear-time preprocessing the tree $T$ for constant-time finding lca ;
  for $i$ from 1 to $n - 1$ do
    $cmp[i] \leftarrow \delta(\text{lca}(\text{leaf}_{i+1}, \text{leaf}_{2n-i+2}))$ ;
  end
  $cmp[0] \leftarrow 0$ and $cmp[n] \leftarrow 0$ ;
  return $cmp$ ;
end

Algorithm 4: Algorithm for computing $\phi cmp_w$

Theorem 9. Let $k \geq 2$ and $s \geq 0$ be integers and $\phi$ be an antimorphic involution. Whether a word $w$ contains any factor of the form $x^{k-1}\phi(x)$ (respectively, $\phi(x)x^{k-1}$) with $|x| > s$ can be tested in $O(k|w|)$ time.

Proof. The main idea is first to compute $k-1 lmp_w$ (respectively, $k-1 rmp_w$) and $\phi cmp_w$, and then to compare the two arrays. There is a factor of the form $x^{k-1}\phi(x)$ (respectively, $\phi(x)x^{k-1}$) with $|x| > s$ if and only if there is an index $i$ such that $k-1 lmp_w[i] \leq \phi cmp[i]$ (respectively, $k-1 rmp_w[i] \leq \phi cmp[i-1]$). More details of detecting $x^{k-1}\phi(x)$ is given in Algorithm 5 and the case of $\phi(x)x^{k-1}$ is similar.

To see the correctness of Algorithm 5, we prove that word $w$ contains any factor of the form $x^{k-1}\phi(x)$ with $|x| > s$ if and only if $k-1 lmp_w[i] \leq \phi cmp[i]$ holds for some $i, 1 \leq i \leq n$, where $n = |w|$. Suppose the inequality $m = k-1 lmp_w[i] \leq \phi cmp[i]$ holds for some $i$. Then $w$ contains word $w[i-(k-1)m+1..i+m]$ of the form $x^{k-1}\phi(x)$ as a factor and $|x| > s$. Now suppose $w$ contains a factor $w[j..j+kp-1]$ of the form $x^{k-1}\phi(x)$ for $p = |x| > s$. Then by definitions, $k-1 lmp_w[j+(k-1)p-1] \leq p$ and $\phi cmp[j+(k-1)p-1] \geq p$. So $m = k-1 lmp_w[i] \leq \phi cmp[i]$ holds for $i = j+(k-1)p-1$.

The computation of $k-1 lmp_w$ is $O(k|w|)$-time and the computation of $\phi cmp$ is $O(|w|)$-time. There are $O(|w|)$ comparisons of integers. So the total running time of Algorithm 5 is in $O(k|w|)$.

Input: a word $w = w[1..n]$, an antimorphic involution $\phi$, and two integers $s, k \geq 0$.
Output: “NO” if $w$ contains a factor of the form $x^{k-1}\phi(x)$ with $|x| > s$; “YES” otherwise.

begin
  $lmp \leftarrow \text{compute_lmp}(w, s, k-1)$ ; // $rmp \leftarrow \text{compute_rmp}(w, s, k-1)$ for $\phi(x)x^{k-1}$
  $cmp \leftarrow \text{compute_cmp}(w, \phi)$ ;
  for $i$ from 1 to $n$ do
    if $lmp[i] \leq cmp[i]$ then return “NO” ; // $rmp[i] \leq cmp[i-1]$ for $\phi(x)x^{k-1}$
  end
  return “YES” ;
end

Algorithm 5: Algorithm for testing whether $w$ contains a factor of the form $x^{k-1}\phi(x)$ with $|x| > s$

Theorem 10. Let $k \geq 2$ and $s \geq 0$ be integers and $\phi$ be an antimorphic involution. Whether a word $w$ contains any factor of the form $(x\phi(x))^\frac{k}{2}$ (or $(x\phi(x))^\frac{k}{2}$ if $k$ is odd) with $|x| > s$ can be tested in $O(|w|^2/k)$ time.

Proof. The main idea is first to compute $\phi cmp_w$ and then to enumerate all possible indices and periods. There is a factor of the specified form as in the theorem if and only if there are $k-1$ consecutive terms greater than $s$ in $\phi cmp_w$ with indices being arithmetic progression with difference greater than $s$. The algorithm is given in Algorithm 6.
To see the correctness of Algorithm 6 we observe that $w$ contains a factor of the form $w[i..j + kp - 1] = xφ(x)ϕ(x) ⋯$ with $p = |x| > s$ if and only if there are $k$ consecutive terms $ϕ(cmp_w[i + p - 1], ϕ(cmp_w[i + 2p - 1], ⋯, ϕ(cmp_w[i + (k - 1)p - 1])$ that are $≥ p > s$.

The computation of $ϕ(cmp_w$ is $O(|w|)$-time and obviously the remaining part is $O(|w|^2/k)$-time. So the total running time of Algorithm 6 is in $O(|w|^2/k)$.

```
Input: a word $w = w[1..n]$, an antimorphic involution $ϕ$, and two integers $s ≥ 0$, $k ≥ 0$.
Output: “NO” if $w$ contains a factor of the form $(xϕ(x))^k$ (or $(xϕ(x))^{k/2}$ if $k$ is odd) with $|x| > s$; “YES” otherwise.
1 $cmp ← compute_cmp(w, ϕ)$ ;
2 for $d$ from $s + 1$ to $|n/k|$ do
3     for $i$ from $0$ to $d - 1$ do
4         consecutive ← $0$ ;
5            for $j$ from $1$ to $|(n - i)/d| - 1$ do
6                if $cmp[i + jd] ≥ d$ then consecutive ← consecutive + 1 ;
7                else consecutive ← $0$ ;
8            if consecutive $≥ k - 1$ then return “NO” ;
9         end
10     end
11 return “YES” ;
```

Algorithm 6: Algorithm for testing whether $w$ contains a factor of the form $(xϕ(x))^k$ with $|x| > s$

5 Conclusion

We generalized Kosaraju’s linear-time algorithm for computing minimal squares that start at each position in a word, which by our definition is denoted by the array $lmp_w$. We showed a modified version of his algorithm that can compute, for arbitrary integers $k ≥ 2, s ≥ 0$, the minimal $k$th powers, with period larger than $s$, that starts at each position (to the left and to the right) in a word, which by our definition is denoted by the right minimal period array $rmp_w$ and the left minimal period array $lmp_w$. The algorithm is in $O(k|w|)$-time.

The algorithm is based on the frame of Weiner’s suffix tree construction. Although there are other linear-time suffix tree construction algorithms, such as McCreight’s algorithm and Ukkonen’s algorithm, none of the two can be altered to compute minimal period arrays with the same efficiency, due to the special requirements that the suffices of the given word are added from the short to the long and $π_k(v)$ is only updated when $v$ is created.

We showed the $O(k|w|)$-time algorithm for computing minimal period arrays can be used to test whether a given word $w$ contains any factor of the form $x^kϕ(x)$ (respectively, $ϕ(x)x^k$) with $|x| > s$. We also discussed an $O(|w|^2/k)$-time algorithm for testing whether a given word $w$ contains any factor of the form $(xϕ(x))^k$ (or $(xϕ(x))^{k/2}$ if $k$ is odd) with $|x| > s$. All the word $xx⋯xϕ(x)$, $ϕ(x)x⋯xx$, $xϕ(x)xϕ(x)⋯$ are pseudo-powers. There are possibilities that some particular type of pseudo-powers other than the ones we discussed can also be detected faster than the known $O(|w|^2 \log |w|)$-time algorithm.

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