A NOTE ON TOURNAMENTS AND NEGATIVE DEPENDENCE

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Abstract

Negative dependence of sequences of random variables is often an interesting characteristic of their distribution, as well as a useful tool for studying various asymptotic results, including central limit theorems, Poisson approximations, the rate of increase of the maximum, and more. In the study of probability models of tournaments, negative dependence of participants’ outcomes arises naturally with application to various asymptotic results. In particular, the property of negative orthant dependence was proved in several articles for different tournament models, with a special proof for each model. In this note we unify these results by proving a stronger property, negative association, a generalization leading to a very simple proof. We also present a natural example of a knockout tournament where the scores are negatively orthant dependent but not negatively associated. The proof requires a new result on a preservation property of negative orthant dependence that is of independent interest.

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1. Introduction

1.1. Tournaments

A tournament consists of competitions between several players where the final score or payoff of each player is determined by the sum of scores of the player’s matches. For a tournament with \( n \) players, let \( S = (S_1, \ldots, S_n) \) denote the vector of their final scores. Under natural probability models and in many kinds of tournaments, the components of \( S \) exhibit some type of negative dependence. We briefly define two concepts of dependence to be considered in this paper and then we discuss various tournaments where these concepts are relevant. We present a theorem on negative association that unifies and strengthens known results on negative dependence of tournament scores, and leads to new ones. Specifically, we prove negative association in various models. We also analyze a tournament in which, interestingly, negative association holds when the draw of matches is random, and otherwise only a weaker notion of negative dependence, negative orthant dependence, holds.

1.2. Two notions of negative dependence

We define the following negative dependence notions. See [6] and references therein for details. Throughout this paper increasing (decreasing) stands for nondecreasing (nonincreasing).

**Definition 1.** ([6], Definition 2.3) The random variables \( S_1, \ldots, S_n \) or the vector \( S = (S_1, \ldots, S_n) \) are said to be **negatively lower orthant dependent** (NLOD) if for all \( s_1, \ldots, s_n \in \mathbb{R} \),

\[
P(S_1 \leq s_1, \ldots, S_n \leq s_n) \leq P(S_1 \leq s_1) \cdots P(S_n \leq s_n),
\]

(1)

and **negatively upper orthant dependent** (NUOD) if

\[
P(S_1 > s_1, \ldots, S_n > s_n) \leq P(S_1 > s_1) \cdots P(S_n > s_n).
\]

(2)

**Negative orthant dependence** (NOD) is said to hold if both (1) and (2) hold.

**Definition 2.** ([6], Definition 2.1) The random variables \( S_1, \ldots, S_n \) or the vector \( S = (S_1, \ldots, S_n) \) are said to be **negatively associated** (NA) if for every pair of disjoint subsets
\( A_1, A_2 \) of \( \{1, 2, \ldots, n\} \),
\[
\text{Cov} \left( f_1(S_i, i \in A_1), f_2(S_j, j \in A_2) \right) \leq 0,
\]
whenever \( f_1 \) and \( f_2 \) are real-valued functions, increasing in all coordinates.

Clearly NA implies NOD (see [6]). In Section 3.2 we provide a natural example of a tournament where \( S_1, \ldots, S_n \) are NOD but not NA.

### 1.3. Round-robin tournaments

#### 1.3.1. A general round-robin tournament

We start with a general formulation of round-robin tournaments. See, e.g., [12] and [3]. Assume that each of \( n \) players competes against each of the other \( n - 1 \) players. When player \( i \) plays against \( j \), where \( i < j \), player \( i \)'s reward is a random variable \( X_{ij} \) having a distribution function \( F_{ij} \) with support on \([0, r_{ij}]\), and \( X_{ji} = r_{ij} - X_{ij} \); for \( i < j \) this determines \( F_{ji}(t) = 1 - F_{ij}(r_{ij} - t) \) for \( t \in [0, r_{ij}] \). Thus each pair of players compete for a share of a given reward. We assume that \( X_{ij} \) are independent for \( i < j \), and also that \( r_{ij} \geq 0 \). The case where \( r_{ij} = 0 \) has the interpretation that players \( i \) and \( j \) do not compete against each other.

The total reward for player \( i \) is defined for all tournaments we consider by
\[
S_i = \sum_{j=1, j \neq i}^{n} X_{ij}, \quad i = 1, \ldots, n.
\]

We shall prove that \( S_1, \ldots, S_n \) are NA (Definition 2), extending and simplifying various results in the literature, to be specified below, and more generally, if \( u_i \) are increasing functions, it follows that that \( u_1(S_1), \ldots, u_n(S_n) \) are also NA. These functions can represent the utilities of the players. See Proposition 1 for a further generalization.

The case of the above round-robin tournament model with an integer support \( \{0, 1, \ldots, r_{ij}\} \) of \( F_{ij} \) was considered recently in Malinovsky and Moon [11]. Our results on negative dependence for the general round-robin tournament generalize the negative dependence results in [11]. Specifically, the NLOD property is proved in [11], and our general result yields the NA property with a simpler proof.

We next discuss further special cases of our general formulation that have appeared in the literature.

#### 1.3.2. A round-robin tournament with pairwise repeated games

Recently, Ross [16] considered a special case of the above two models where \( X_{ij} \sim Binomial(r_{ij}, p_{ij}) \)
independently for all \( i < j \), \( r_{ij} = r_{ji} \) and \( X_{ji} = r_{ij} - X_{ij} \sim \text{Binomial}(r_{ij}, 1 - p_{ij}) \).

As always, \( S_i = \sum_{j=1, j \neq i}^{n} X_{ij} \). This model arises if each pair of players \((i, j)\) plays \( r_{ij} \) independent games, and \( i \) wins with probability \( p_{ij} \). Ross \cite{16} obtained NOD-type results for general \( p_{ij} \) using log-concavity, conditioning, and Efron’s well-known theorem \cite{4}. Again we strengthen and simplify these results and prove the NA property.

Ross used his results to study expressions such as \( P(S_i > \max_{j \neq i} S_j) \) and related inequalities, under a special model for \( p_{ij} \), given, e.g., in \cite{18, 2}.

1.3.3. **A simple round-robin tournament** Huber \cite{5} considered the above general model where for any \( i \neq j \), \( X_{ij} + X_{ji} = 1 \), \( X_{ij} \in \{0, 1\} \) and \( P(X_{ij} = 1) = p_{ij} \), and proved that \( S_1, \ldots, S_n \) are NLOD by invoking coupling arguments. He used the latter fact to prove that if \( P(X_{1j} = 1) = p > 1/2 \), and \( P(X_{ij} = 1) = 1/2 \) for all \( 1 < i \neq j \leq n \), then \( \lim_{n \to \infty} P(s_1 > \max\{s_2, \ldots, s_n\}) \to 1 \); that is, Player 1 achieves the highest score with probability approaching 1.

1.3.4. **A chess round-robin tournament with draws** Malinovsky \cite{9} and \cite{10} considered the following round-robin tournament model: for \( i \neq j \), \( X_{ij} + X_{ji} = 1 \), \( X_{ij} \in \{0, 1/2, 1\} \); this can be seen as a special case of the general model where \( F_{ij} \) have the support \( \{0, 1/2, 1\} \). Malinovsky considered the case where all players are equally strong, i.e. \( P(X_{ij} = 1) = P(X_{ji} = 1) \), and where the probability of a draw, \( p = P(X_{ij} = 1/2) \) is common to all games. He proved NLOD for \( S_1, \ldots, S_n \) using log-concavity of the probability function of \( 2X_{ij} \), which requires restricting the range of \( p \). He then used NLOD to prove a Poisson approximation to the number of times \( S_i \) exceeds a certain threshold. We strengthen his result to NA for all \( p \) and more generally for all values of \( P(X_{ij} = 1) \) and \( P(X_{ij} = 1/2) \).

1.3.5. **Random-sum n-player games** The following somewhat abstract description of a tournament will be shown in Section \cite{2} Corollary \cite{2} to be a generalization of all the above tournament constructions. Consider a sequence of \( K \) \( n \)-player games, where the random payoff to player \( i \in \{1, \ldots, n\} \) in game \( k \in \{1, \ldots, K\} \) is \( X_{ki} \) and the components of each of the payoff vectors \( X_k = (X_{k1}, \ldots, X_{kn}) \) are NA, with \( X_k \)'s being independent. In general, the sum of the components of each \( X_k \) is assumed to be a random variable. Constant-sum (or, equivalently, zero-sum) examples are formed
when the payoff vectors $X_k$ have the multinomial or Dirichlet distribution (see Section 3.1 for these and further examples). An example where the sum of the players’ payoffs in each game is random is the case where the vector $X_k$ is jointly normal with correlations $\leq 0$ (Section 3.4).

The total payoff to player $i$ in the $K$ games is $S_i = \sum_{k=1}^{K} X_{ki}$, $i = 1, \ldots, n$. We shall prove in Section 2 Proposition 2 that $S_1, \ldots, S_n$ are NA. More generally, one can take $S_i = u_i(X_{i1}, \ldots, X_{Ki})$ where $u_i$ is any increasing function of player $i$’s payoffs. Note that here, unlike in pairwise duels, several and even all players may compete in each round. The limiting distribution of the number of pure Nash equilibria in such random games was studied in Rinott and Scarsini [15].

Two sport examples Football (Soccer in the US) provides another example of a round-robin tournament; however, here the sum of scores in each duel is not constant. The winning team is awarded three points, and if the game ends in a tie, each team receives one point. For a single match the score possibilities for the two teams are $(3,0), (1,1)$ and $(0,3)$ with some probabilities, forming an NA distribution for any probabilities. Let the $n$-dimensional vectors $X_k$, for $k = (ij)$, consist of zeros except for two coordinates $i$ and $j$ corresponding to the playing teams $i$ and $j$, where one of the above three vectors appears. Then $S = \sum_{k=(i,j):1 \leq i < j \leq n} X_k$ represents the vector of total scores of $n$ teams after one round in which all teams play each other. It is easy to see that each vector $X_k$ is NA.

Under some assumption (which are an approximation to reality), the Association of Tennis Professionals (ATP) ranking is another example. It can be seen as a tournament in which the number of points awarded to the winner of each game depends on the tournament and the stage reached. Players’ ranks are increasing functions of their total scores.

1.4. Knockout tournaments

Consider a knockout tournament with $n = 2^\ell$ players of equal strength; that is, player $i$ defeats player $j$ independently of all other duels with probability $1/2$ for all $1 \leq i \neq j \leq n$ and the defeated player is eliminated from the tournament. Let $S_i$ denote the number of games won by player $i$. We could also replace $S_i$ by the prize money of player $i$, which in professional tournaments is usually an increasing
function of $S_i$. For a completely random schedule of matches (aka the draw; see [1]), we show in Section 3 that the vector $S = (S_1, \ldots, S_n)$ is NA. Note that in tennis tournaments such as Wimbledon the draw is not completely random as top-seeded players’ matches are drawn in a way that prevents them from playing against other top-seeded players in early rounds. For non-random draws we prove the NOD property via a new preservation result, and we provide a counterexample to the NA property; thus it need not hold for fixed, non-random draws.

2. Negative association and round-robin tournaments

The following theorem generalizes Application 3.2(c) of [3]; it implies that the scores $S_1, \ldots, S_n$ in the general round-robin model of Section 1.3.1 and therefore in all round-robin models of Section 1.3 are NA, and therefore NLOD and NUOD, and therefore NOD.

**Theorem 1.** Let $X_1, \ldots, X_n$ be independent random variables and let $g_i, i = 1, 2, \ldots, n$ be decreasing functions. Set $Y_1 = g_1(X_1), \ldots, Y_n = g_n(X_n)$, and for $j = 1, 2, \ldots, m$ set

$$S_j = f_j(\{X_i : i \in A_j\}, \{Y_i : i \in B_j\}),$$

where $f_j$ are coordinatewise increasing functions of $|A_j| + |B_j|$ variables, and the sets $A_1, \ldots, A_m$ are disjoint subsets of $\{1, 2, \ldots, n\}$, and so are $B_1, \ldots, B_m$. Then the random variables $S_1, \ldots, S_m$ are NA.

**Proof.** The pair of variables $X, g(X)$ with $g$ decreasing is NA. This is well known; for completeness, here is a simple proof. Let $X^*$ be an independent copy of $X$. For increasing functions $f_1$ and $f_2$, we have

$$2\text{Cov}(f_1(X), f_2(g(X))) = E\{[f_1(X) - f_1(X^*)][f_2(g(X)) - f_2(g(X^*))]\} \leq 0,$$

since the expression in the expectation is $\leq 0$. The pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and each pair is NA. Property $P_7$ of [6] states that the union of independent sets of NA random variables is NA. Therefore the random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are NA. Property $P_6$ in [6] states that increasing functions defined on disjoint subsets of a set of NA random variables are NA. Therefore $S_1, \ldots, S_m$ are NA. \qed
We now apply Theorem 1 to show the NA property in the general round-robin model of Section 1.3.1.

**Proposition 1.** Let $X_{ij} \sim F_{ij}$ with support on $[0, r_{ij}]$ be independent for $1 \leq i < j \leq n$, where $r_{ij} \geq 0$, and let $X_{ji} = r_{ij} - X_{ij}$. Set $S_i = \sum_{j=1, j \neq i}^{n} X_{ij}$, $i = 1, \ldots, n$. Then $S_1, \ldots, S_n$ are NA. More generally, if we set $S_i = u_i(X_{i1}, \ldots, X_{i,i-1}, X_{i,i+1}, \ldots, X_{in}$), $i = 1, \ldots, n$, where the $u_i$’s are any increasing functions, we again have that the variables $S_1, \ldots, S_n$ are NA.

**Proof.** Instead of a single index we apply Theorem 1 to the independent doubly indexed random variables $X_{ij}$ for $i < j$. Let $g_{ij}(x) = r_{ij} - x$, so that $X_{ji} = g_{ij}(X_{ij}) = r_{ij} - X_{ij}$ with $X_{ji}$ playing the role of $Y$’s in Theorem 1. Since the $S_i$’s are sums of disjoint subsets of the variables defined above, the result follows by Theorem 1, and the same argument holds with the functions $u_i$ replacing the sums. □

Since all round-robin models of Section 1.3 except for the football example are special cases of the general round-robin model, we have:

**Corollary 1.** The NA property for $S_1, \ldots, S_n$ holds in all the round-robin models in Section 1.3. The NLOD results proved in the literature for these models follow; moreover, NUOD and hence NOD also follow.

The football example of Section 1.3.5 is not a special case of the general round-robin model; here the NA property follows by the argument proving Proposition 1 by replacing the functions $g_{ij}$ by $g$ defined by $g(3) = 0, g(1) = 1$, and $g(0) = 3$.

We now consider the random-sum $n$-player games tournament of Section 1.3.5.

**Proposition 2.** Consider a sequence of $K$ $n$-player games, where the random payoff to player $i \in \{1, \ldots, n\}$ in game $k \in \{1, \ldots, K\}$ is $X_{ki}$ and the components of each payoff vector $X_k = (X_{k1}, \ldots, X_{kn})$ are NA, distributed independently. Let $S_i = \sum_{k=1}^{K} X_{ki}$. Then $S_1, \ldots, S_n$ are NA. More generally, the variables $S_i = u_i(X_{1i}, \ldots, X_{Ki})$, $i = 1, \ldots, n$, where the $u_i$’s are any increasing functions, are NA.

The above result can be restated in the following lemma, which follows readily from properties of negative association given in [6]. The same result for positive association, with the same proof, is given in [7] Remark 4.2.
Lemma 1. The convolution of NA vectors is NA.

Proof. Let $X_k \in \mathbb{R}^n$ be independent NA vectors and let $S = (S_1, \ldots, S_n) = \sum_{k=1}^{K} X_k$. By Properties $P_7$ and then $P_6$ of [6], the union of all variables in these vectors is NA and hence $S_1, \ldots, S_n$ are NA since they are increasing functions of disjoint subsets of the above union.

The above argument holds also when $S_i = u_i(X_{i1}, \ldots, X_{iK})$, thus proving the last part of Proposition 2.

The next corollary shows that the NA property of the general round-robin model of Section 1.3.1 and hence in all the models of 1.3 follows also from Proposition 2.

Corollary 2. The scores $S_1, \ldots, S_n$ of the general round-robin models in Section 1.3.1 are NA.

Proof. For clarity we start with the simple case of $n = 3$. Define the vectors $Y^{12} = (X_{12}, r_{12} - X_{12}, 0)$, $Y^{13} = (X_{13}, 0, r_{13} - X_{13})$, and $Y^{23} = (0, X_{23}, r_{23} - X_{23})$ with $X_{ij}$ of the general round-robin model. It is easy to see that $S_i = \sum_{1 \leq k < \ell \leq 3} Y_i^{kl}$.

In general, starting with the rewards $X_{ij}$ of the general round-robin model, form the $K =: n(n-1)/2$ vectors $Y^{ij} \in \mathbb{R}^n, 1 \leq i < j \leq n$, whose $i$th component, $Y_i^{ij}$, equals $X_{ij}$, its $j$th component, $Y_j^{ij}$, equals $r_{ij} - X_{ij}$, and the remaining components equal zero. The components $(Y_1^{ij}, \ldots, Y_n^{ij})$ of each of the $K$ vectors $Y^{ij}$ are obviously NA. Setting

$$S_i = \sum_{1 \leq k < \ell \leq n} Y_i^{kl}, \quad i = 1, \ldots, n,$$

it is easy to see that these $S_i$ coincide with those of the general round-robin model. Proposition 2 applied to the $K$ vectors $Y^{kl}$ implies that the variables $S_i$ are NA.

3. Knockout tournaments

3.1. Knockout tournaments with a random draw

Proposition 3. Consider a knockout tournament starting with $n = 2^\ell$ players, where player $i$ defeats player $j$ independently of all other duels with probability $1/2$ for all $1 \leq i \neq j \leq n$; the defeated player is eliminated from the tournament. Let $S_i$ denote
the number of games won by player $i$. Assume a completely random schedule (draw) of the matches. Then $S_1, \ldots, S_n$ are NA.

Proof. First note that for a given $\ell$, the vector $S = (S_1, \ldots, S_n)$ contains the components $i = 0, \ldots, \ell$ with $i < \ell$ appearing $2^{\ell-1-i}$ times, and $\ell$ appearing once. For example, if $n = 4$ ($\ell = 2$) then there are 2 players with 0 wins, 1 player (the losing finalist) with 1 win, and 1 player (the champion) with 2 wins. Thus, the vector $S$ is a permutation of the vector $(0, 0, 1, 2)$. If $n = 8$ ($\ell = 3$) then $S$ is a permutation of the vector $(0, 0, 0, 0, 1, 1, 2, 3)$. Under the assumption of a random draw, all permutations are equally likely as all players play a symmetric role. Theorem 2.11 of [6] states that if $X = (X_1, \ldots, X_n)$ is a random permutation of a given list of real numbers, then $X$ is NA, and the result follows. □

3.2. Knockout tournaments with a non-random draw

This section provides a counterexample showing that for knockout tournaments with a given non-random draw, the scores $S_1, \ldots, S_n$ need not be NA; however, we prove that they are NOD. To obtain the latter result we prove a result on NOD (and NLOD and NUOD) of independent interest.

A counterexample to NA Consider a knockout tournament with $n = 4$ players and a draw where in the first round player 1 plays against 2, and 3 against 4. In this case only 8 permutations of $(0, 0, 1, 2)$ are possible and one of the first two coordinates must be positive and so $(0, 0, 1, 2)$ itself is is not a possible outcome. Consider the functions $f_1(s_1, s_3)$ taking the value 0 everywhere, except that $f_1(0, 1) = f_1(0, 2) = 1$, and $f_2(s_2, s_4)$ which is 0 everywhere, except for $f_2(2, 0) = 1$. We have $E[f_1(s_1, s_3)f_2(s_2, s_4)] = 1/8$, $E[f_1(s_1, s_3)] = 2/8$, and $Ef_2(s_2, s_4) = 1/8$, and (3) does not hold. □

In a tennis tournament, the above arrangement of matches occurs if Players 1 and 3 are top-seeded and the draw prevents them from being matched against each other in the first round.

Finally, we prove that in a knockout tournament with a non-random schedule, $S = (S_1, \ldots, S_n)$ is NOD. We need the following theorem, which may be of independent interest.
Theorem 2. Let $X_k = (X_{k1}, \ldots, X_{kn}) \in \mathbb{R}^n$, $k = 1, \ldots, K$ satisfy the following two assumptions.

(i) For all $k = 1, \ldots, K$: $X_k \mid X_{k-1} + \ldots + X_1$ is NLOD, and

(ii) For all $k$ and $i$: $X_{ki} \mid X_{k-1} + \ldots + X_1 \overset{d}{=} X_{ki} \mid X_{k-1i} + \ldots + X_{1i}$;

that is, the conditional distribution of $X_{ki}$ depends only on the $i$th coordinate of the sum of its predecessors. Then $X_1 + \ldots + X_K$ is NLOD. Moreover, the result holds if we replace NLOD by NUOD and hence also by NOD.

Proof. It is well known that a random vector $Z = (Z_1, \ldots, Z_n)$ is NLOD if and only if

$$E \prod_{i=1}^n \phi_i(Z_i) \leq \prod_{i=1}^n E \phi_i(Z_i)$$

for any nonnegative decreasing functions $\phi_i$ (Theorem 6.G.1 (b) in [17] or Theorem 3.3.16 in [14]). The proof proceeds by induction, and it is easy to see that it suffices to prove it for $K = 2$. Set $X := X_1$ and $Y := X_2$. We have

$$E \prod_{i=1}^n \phi_i(X_i + Y_i) = E\{E[\prod_{i=1}^n \phi_i(X_i + Y_i) \mid X]\} \leq E \prod_{i=1}^n E[\phi_i(X_i + Y_i) \mid X] = E \prod_{i=1}^n E g_i(X_i),$$

where $g_i(X_i) = E[\phi_i(X_i + Y_i) \mid X]$, and the inequality holds by Assumption (i). By (ii) we have that $g_i(X_i)$ indeed depends only on $X_i$, and it is obviously nonnegative and decreasing. By the NLOD property of $X$ we have

$$E \prod_{i=1}^n g_i(X_i) \leq \prod_{i=1}^n E g_i(X_i) = \prod_{i=1}^n E \phi_i(X_i + Y_i),$$

and the result follows. The same proof holds for NUOD with the functions $\phi_i$ taken to be increasing. \qed

A special case of Theorem 2 is the following corollary that for nonnegative vectors follows from Theorem 6.G.19 of [17] and can be obtained from Theorem 1 of [8] (for vectors in $\mathbb{R}^2$) and from Theorem 4.2 (c) of [13].

Corollary 3. The sum of independent NOD (NLOD, NUOD) vectors is NOD (NLOD, NUOD).

Proposition 4. For the knockout tournament with a non-random draw, the vector $S = (S_1, \ldots, S_n)$ is NOD.

Proof. Without loss of generality assume that in the first round player $2i - 1$ plays against $2i$ for $i = 1, \ldots, n/2$. Let $X_{ij} = 0$ (1) if player $j$ loses (wins) the first round,
$j = 1, \ldots, n$. The pairs of variables $X_{1,2i-1}, X_{1,2i}$ are independent and NOD (in fact they are NA), taking the values $(0, 1)$ or $(1, 0)$ with probability $1/2$. It follows readily that the 0-1 vector $\mathbf{X}_1 = (X_{11}, \ldots, X_{1n})$, whose $j$th coordinate indicates a win or a loss of player $j$ in the first round, is NOD. Now the second round is similar with only half the players, those who won the first round, where the value 0 is set for players who lost in the first round. Continuing this way, we see that the vector $(S_1, \ldots, S_n)$ is the sum of the 0-1 vectors of all the rounds. These vectors are not independent because the value of 0 in a coordinate of a vector pertaining to a given round must by followed by a zero there in the next round. However, (i) and (ii) of Theorem 2 are easily seen to hold, and the NOD property follows. □

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