Integral pentagon relations for 3d superconformal indices

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Abstract. The superconformal index of a three-dimensional supersymmetric field theory can be expressed in terms of basic hypergeometric integrals. By comparing the indices of dual theories, one can find new integral identities for basic hypergeometric integrals. Some of these integral identities have the form of the pentagon identity which can be interpreted as the 2–3 Pachner move for triangulated 3-manifolds.

1. Introduction

The superconformal index is one of the efficient tools in the study of non-perturbative aspects of supersymmetric field theory providing the most rigorous mathematical check of supersymmetric dualities. Recent progress in superconformal index computations have significant implications for mathematics.

A rather striking example is the observation made by Dolan and Osborn [DO] that the superconformal index of four-dimensional theories is expressible in terms of elliptic hypergeometric integrals [Ro, Ra, Sp]. The identification of superconformal indices of supersymmetric dual theories is given by the Weyl group symmetry transformations for certain elliptic hypergeometric functions on different root systems. The computations of the superconformal indices of supersymmetric dual theories in four dimensions have led to new non-trivial integral identities for elliptic hypergeometric functions [SV1, SV2, S, KL].

Superconformal indices of three-dimensional theories have attracted much attention recently [HKPP, KKP, GR, KSV, GV2]. Their exact computation

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yields new powerful verifications of various supersymmetric dualities as mirror symmetry, Seiberg-like duality etc. Since in three dimensions the superconformal index is expressed in terms of basic hypergeometric integrals \[ G, GaRa \], by studying supersymmetric dualities one can get new identities for this type of special functions. In this work we consider a special type of such identities, namely five term relations or the so-called pentagon identities. The pentagon relations are interesting from different aspects, see, for instance, [FK, GR, K, KL]. Here we present some examples of integral pentagon relations related to the three-dimensional superconformal index.

The rest of the paper is organized in the following way. In Section 2 we make a brief review of the superconformal index in three dimensions. We present some examples of the integral pentagon identities in Sections 3 and 4 and briefly discuss some open problems in Section 5.

2. The superconformal index

In this section we give a short introduction to a three-dimensional superconformal index and refer the reader to [IY, KSV] and references therein for more details.

Let us first roughly review the well-known Witten index. Consider a supersymmetric quantum mechanics

\[ \{Q, Q^\dagger\} = 2H, \]
\[ \{Q, (-1)^F\} = 0, \]

where \( Q, H \) and \( (-1)^F \) are the supersymmetric charge, the Hamiltonian and the fermion number operator respectively. In order to check whether the supersymmetry is broken or not, Witten introduced the topological invariant of a theory \[ W \]

\[ I_W = \text{Tr}(-1)^F e^{-\beta H}, \]

which tells us that supersymmetry is not spontaneously broken if \( I_W \neq 0 \). The sum in the definition \( (2.3) \) runs over all physical states of the theory. The Witten index is independent of the parameter \( \beta \) and counts the difference between the number of bosonic and fermionic ground states. It is an analogue of the Atiyah–Singer index [AS].

In the case of a supersymmetric field theory one can generalize the Witten index by including to the index global symmetries of a theory commuting with \( Q \) and \( Q^\dagger \) [KMMR, R1, R2]. For a \( d \)-dimensional supersymmetric theory the superconformal index is the following partition function defined on \( S^{d-1} \times S^1 \),

\[ I(\{t_i\}) = \text{Tr}(-1)^F e^{-\beta (Q, Q^\dagger)} \prod t_i^{F_i}, \]

where the trace is taken over the Hilbert space on \( S^{d-1} \), \( F_i \) are generators for global symmetries that commute with \( Q \) and \( Q^\dagger \), and \( t_i \) are additional regulators (fugacities) corresponding to the global symmetries. The superconformal index counts short BPS operators of the theory. We refer the reader to [R2, SV1, GV1] for more details and references.

1 A fermion number operator takes the value zero on bosons and one on fermions.
2 The original Witten index for supersymmetric gauge theories gives the dual Coxeter number for the gauge group.
We will restrict ourselves to three-dimensional theories with $\mathcal{N} = 2$ supersymmetry. These theories have four real supercharges which we organize into complex spinor $Q_\alpha$ and its conjugate $\bar{Q}_\alpha$ with $\alpha = 1, 2$. We focus on theories in the far IR, where the symmetry is enhanced to the superconformal symmetry and therefore there are four additional supercharges $S_\alpha$ and $\bar{S}_\alpha$ with $\alpha = 1, 2$. We choose one of the supercharges, say, $Q = Q_1$. For the theory on $S^2 \times S^1$ we have $Q_1^\dagger = S_1$ and the following commutation relation (for the full superconformal algebra, see [DI])

$$\{Q, Q_1^\dagger\} = \Delta - R - j_3,$$

where $\Delta, R, j_3$ are the energy, the generator of the $R$-symmetry and the third component of the angular momentum on $S^2$, respectively. Note that, because of conformal symmetry, $R$-symmetry appears explicitly in the commutation relations.

The superconformal index of this theory is defined as [BM]

$$\text{Tr} \left[ (-1)^F e^{-\beta (Q, Q_1^\dagger)} x^{\Delta + j_3} \prod_i t_i^{F_i} \right],$$

where $F_i$ are generators for flavor symmetries of the theory.

Using the localization technique [Pes] it was shown that the superconformal index of a three-dimensional theory has the form of matrix integral (see, for instance, [Kl, LY]) which is a basic hypergeometric integral [Gi, GaRa]. For instance, the chiral multiplet with components being a complex scalar field $\phi$, a Weyl fermion $\psi$ and an auxiliary complex scalar $F$, (2.7) $\Phi = \phi + \sqrt{2} \theta \psi + \theta^2 F$

($\theta$ is a Grassman coordinate), and with $R$-charge $r$ in the fundamental representation of the gauge group $U(N)$, contributes to the index as

$$\prod_{i=1}^{N_c} \frac{(x^{2-r+|m_i|} z_i^{-1}; x^2)_\infty}{(x^{r+|m_i|} z_i; x^2)_\infty},$$

where the $q$-Pochhammer symbol is defined as

$$ (z; q)_\infty = \prod_{i=0}^{\infty} (1 - z q^i) $$

and the integer parameters $m_i$ stand for the magnetic charges corresponding to the gauge group $U(N)$ and run over integers; the fugacities $z_i$ correspond to the gauge group.

One can also compute the superconformal index by using representations of the superconformal algebra [KSV], i.e. by the so-called Romelsberger prescription [R2].

From now on we will use the notation

$$x^2 = q,$$

where there are many interesting results in this direction also for theories with $\mathcal{N} = 4$ [KW], $\mathcal{N} = 6$ [BM, Kl] and $\mathcal{N} = 8$ [BK] supersymmetry.

In the case of $\mathcal{N} = 2$ supersymmetric theories without conformal symmetry, the $R$-symmetry is only an automorphism group and does not appear in a direct way as in (2.5). For details see, for instance, [AHSS].

In the literature it is also called $q$-hypergeometric integral.
and express the superconformal index via the so-called tetrahedron index (2.11)

\[ I_q[m, z] = \prod_{i=0}^{\infty} \frac{1 - q^{i+\frac{1}{2}m+1}z^{-1}}{1 - q^{i+\frac{1}{2}m}z}, \text{ with } |q| < 1 \text{ and } m \in \mathbb{Z}. \]

The contribution of the chiral multiplet (2.8) in terms of tetrahedron index has the following form

\[ \prod_{i=0}^{\infty} \frac{1 - q^{i+\frac{1}{2}|m|+1}z^{-1}}{1 - q^{i+\frac{1}{2}|m|}z} = (-q^{\frac{1}{2}})^{-\frac{1}{2}(m+|m|)}z^{\frac{1}{2}(m+|m|)}I_q[m, z]. \]

We defined the tetrahedron index for the free chiral with zero \( R \)-charge, but one can write the index for general \( R \)-charge by the shift \( z \to zq^{r/2} \).

The three-dimensional index can be factorized into vertex and anti-vertex partition function (KSV, Pas, HKP) and all results presented in the next section can be written in this fashion; however, this subject is beyond the scope of the present work.

It is worth mentioning here that we will consider theories with \( U(1) \) gauge group only in order to get the pentagon identities which we will study in the next section.

### 3. Integral pentagon identities

Our main interest is the five-term relation for the superconformal index. Let us consider the \( d = 3 \) \( \mathcal{N} = 2 \) supersymmetric quantum electrodynamics with \( U(1) \) gauge group and one flavor. The superconformal index of this theory is

\[ I_e = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi iz} z^{-m} I_q[m; q^{1/6}z^{-1}] I_q[-m; q^{1/6}z], \]

where the integration is over the unit circle with positive orientation. For simplicity we switched off the topological symmetry \( U(1)_j \).

The dual theory is the free Wess–Zumino theory (Is, BHoy, AHIss) with three chiral multiplets \( q, \tilde{q}, S \) interacting through the superpotential \( W = \tilde{q}Sq \). The index of this theory has more simple form, since we do not need to integrate over the gauge group,

\[ I_m = \left( I_q[0; q^{1/3}] \right)^3. \]

These two theories are dual under the mirror symmetry, i.e. under exchange of the Higgs and the Coulomb branches. The mirror duality leads to the following integral pentagon identity

\[ \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi iz} z^{-m} I_q[m; q^{1/6}z^{-1}] I_q[-m; q^{1/6}z] = \left( I_q[0; q^{1/3}] \right)^3. \]

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6See, for instance, (KSV, Kw). We consider the influence of the topological \( U(1)_j \) symmetry to the index in the next chapter, where we define the so-called generalized superconformal index.

7In the literature this theory sometimes is called the XYZ model.

8The permutation symmetry of the superpotential fixes the \( R \)-charges, but one can write the index for more general \( R \)-charge like in (IY).

9In three-dimensional supersymmetric theories the Coulomb and the Higgs branch are both hyper-Kähler manifolds.
This is the first example of a pentagon identity for the tetrahedron index. The mathematical proof of the identity can be found in [KSV].

The tetrahedron index can be written in the following form:

\[ I_q[m, z] = \sum_{e \in \mathbb{Z}} I(m, e) z^e \] (3.4)

where

\[ I(m, e) = \sum_{n=\frac{1}{2}(e|e)}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)-(n+\frac{1}{2}e)m}}}{(q)_n(q)_{n+e}}. \] (3.5)

This index was introduced in [DGG1]. It is also interesting from a mathematical point of view [Gar, GHRS]. The index \( I(m, e) \) obeys the following pentagon identity

\[ I(m_1 - e_2, e_1)I(m_2 - e_1, e_2) = \sum_{e_3} q^{e_3}I(m_1, e_1 + e_3)I(m_2, e_2 + e_3)I(m_1 + m_2, e_3). \] (3.6)

The proof of the identity (3.6) is given in the Appendix of [Gar]. This pentagon relation is a counterpart of the integral pentagon identity (3.3). In order to distinguish between this type relation and the identity of the form (3.3) we use the terminology “the integral pentagon identity” for the latter one.

As another example, we consider the following three-dimensional duality. The electric theory is the \( d = 3 \mathcal{N} = 2 \) superconformal field theory with \( U(1) \) gauge symmetry and six chiral multiplets, half of them transforming in the fundamental representation of the gauge group and another half transforming in the anti-fundamental representation. Its mirror dual is a theory with nine chirals and without gauge degrees of freedom (the gauge symmetry is completely broken). The mirror symmetry leads to the following identity

\[ \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi iz} (-z)^{-3m} \prod_{i=1}^{3} I_q[-m, q^{\xi_i} z] I_q[m, q^{\eta_i} z^{-1}] = \prod_{i, j=1}^{3} I_q[0, q^{\xi_i \eta_j}] , \] (3.7)

where the fugacities \( \xi_i \) and \( \eta_i \) stand for the flavor symmetry \( SU(3) \times SU(3) \) and the balancing condition is \( \prod_{i=1}^{3} \xi_i = \prod_{i=1}^{3} \eta_i = 1 \). Note that we again dropped the topological symmetry \( U(1)_J \). The identity (3.7) was introduced in [GR], to where we refer the reader for the details and the mathematical proof of it.

Following [GR] we introduce a new function

\[ B[m; a, b] = \frac{I_q[m, a] I_q[-m, b]}{I_q[0, ab]} , \] (3.8)

and rewrite the equality (3.7) in terms of this function. The final result is a new integral pentagon identity in terms of \( B[m; a, b] \) functions

\[ \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi iz} (-z)^{-3m} \prod_{i=1}^{3} B[m; \xi_i z^{-1}, \eta_i z] = B[0; \xi_1 \eta_2, \xi_3 \eta_1] B[0; \xi_2 \eta_1, \xi_3 \eta_2] \] (3.9)
where we have redefined the flavor fugacities $\xi_i \to q^{-1/6} \xi_i$ and $\eta_i \to q^{-1/6} \eta_i$ and the new balancing condition\[^{[6]}\] is $\prod_{i=1}^{3} \xi_i = \prod_{i=1}^{2} \eta_i = q$.

4. Generalized superconformal index

One can also find a similar pentagon relation for the generalized superconformal index\[^{[KW]}\]. Unlike four-dimensional gauge theories, in three dimensions there are no chiral anomalies, therefore there is no obstruction for considering a theory in a non-trivial background gauge field coupled to the global symmetries. Then one gets new discrete parameters for global symmetries in the expression of the superconformal index. The index with new integer parameters corresponding to the global symmetries is called the generalized superconformal index.

One can apply the above techniques similarly to the generalized superconformal index and obtain more general integral pentagon identities.

The expression (3.7) in terms of the generalized index has the following form

$$\sum_{m \in \mathbb{Z}} \int \frac{dz}{2\pi i z} \prod_{i=1}^{3} (-1)^{|m|} z^{-3|m|} \left( (\xi_i z)^{-1} q^{1+m/2}; q \right)_{\infty} (z/\eta_i q^{-m/2}; q)_{\infty}$$

$$= \prod_{i,j=1}^{3} \left( \frac{q/\xi_i \eta_j; q_{\infty}}{(\xi_i \eta_j q^{M_i + N_j}; q_{\infty})} \right)$$

(4.1)

where we switched on background fields coupled to the flavor symmetry and therefore the index has additional integer parameters $M_i$ and $N_i$ with the condition $\sum_i M_i = \sum_i N_i = 0$. The new discrete parameters are analogous to the magnetic charge $m$ for the gauge symmetry.

The analogue of the first pentagon identity\[^{[GS]}\] in terms of the generalized superconformal index is the following pentagon identity

$$\sum_{s \in \mathbb{Z}} \int \frac{dz}{2\pi i z} (-1)^{m-n} \omega^{-m+|s+n|} \alpha^{-m} q^{3m} I_q[s + m; q^{4} \alpha^{-1}]I_q[s - m; \alpha q^{4}]$$

$$= (-1)^{|s+n|} \omega^{-m} \alpha^{n+2m} q^{3m} I_q[m; q^{4} \alpha^{-1} \omega^{-1}]I_q[-m; q^{4} \alpha^{-1} \omega]I_q[2m; q^{4} \alpha^{2}],$$

where we switched on the background gauge field coupled to the topological $U(1)$ symmetry. Here $\alpha$ and $m$ denote the parameters for the axial $U(1)$ symmetry, $\omega$ and $n$ denote the parameters for the topological $U(1)$ symmetry and the discrete parameter $s$ stands for magnetic charge. Note that this identity was proven only for the case $m = 0$\[^{[KW]}\].

5. Concluding remarks

There is a recently proposed relation called 3d–3d correspondence\[^{[DGG1, DGG2]}\] (see also\[^{[TM1, TM2, TM3, GMMS, DGaGo, GKLP, CDGS]}\]) that connects $d = 3$ $\mathcal{N} = 2$ supersymmetric theories and triangulated 3-manifolds. Namely, the independence of the invariant of the corresponding 3-manifold on the choice of triangulation corresponds to the equality of superconformal indices of mirror dual theories\[^{[DGG1, GHR]}\]. In this context the interpretation of the integral pentagon identities discussed here is the 2–3 Pachner move\[^{[Pa1, Pa2]}\] for triangulated 3-manifolds, which relates different decompositions of a polyhedron.

\[^{[6]}\]We have a misprint in our previous paper\[^{[GR]}\].
with five ideal vertices into ideal tetrahedra. Much work remains to be done in this direction.

Note that one can write such pentagon identities also for partition functions on $S^3$ \[KLV\], i.e. for hyperbolic hypergeometric integrals \[Bu\].

As an aside comment, we would like to mention that the pentagon identity (3.9) may represent the star-triangle relation for some integrable model.

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INTEGRAL PENTAGON RELATIONS FOR 3D SUPERCONFORMAL INDICES

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