A QUARTET OF FERMIONIC EXPRESSIONS FOR $M(k,2k\pm1)$ VIRASORO CHARACTERS VIA HALF-LATTICE PATHS

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Abstract. We derive new fermionic expressions for the characters of the Virasoro minimal models $M(k,2k\pm1)$ by analysing the recently introduced half-lattice paths. These fermionic expressions display a quasiparticle formulation characteristic of the \(\phi_{2,1}\) and \(\phi_{1,5}\) integrable perturbations. We find that they arise by imposing a simple restriction on the RSOS quasiparticle states of the unitary models $M(p,p+1)$. In fact, four fermionic expressions are obtained for each generating function of half-lattice paths of finite length $L$, and these lead to four distinct expressions for most characters $\chi_{r,s}^{k,2k\pm1}$. These are direct analogues of Melzer’s expressions for $M(p,p+1)$, and their proof entails revisiting, reworking and refining a proof of Melzer’s identities which used combinatorial transforms on lattice paths.

We also derive a bosonic version of the generating functions of length $L$ half-lattice paths, this expression being notable in that it involves $q$-trinomial coefficients. Taking the $L \to \infty$ limit shows that the generating functions for infinite length half-lattice paths are indeed the Virasoro characters $\chi_{r,s}^{k,2k\pm1}$.

1. Introduction

1.1. Fermionic expressions for Virasoro characters. The royal road for generating and proving fermionic expressions for Virasoro characters in minimal models is the path description of the states induced by an underlying statistical model \[1\]-\[8\]. The statistical model generally considered is the Forrester-Baxter RSOS model \[9\], which is defined for any pair of relatively prime integers $p$ and $p'$, and referred to here as the RSOS($p,p'$) model. The RSOS($p,p+1$) model is then the Andrews-Baxter-Forrester (ABF) model \[10\]. The combinatorial techniques for computing the generating functions of the paths lead to expressions that are manifestly positive, these expressions being termed fermionic. On the other hand, an inclusion-exclusion calculation shows that these generating functions are also given by

\[
\chi_{r,s}^{p,p'}(q) = \frac{1}{(q)_\infty} \sum_{\lambda \in \mathbb{Z}^-} (q^{\lambda^2 p'p+\lambda'(p'r-ps)} - q^{\lambda p+p+r(\lambda p'+s)}),
\]

for $1 \leq r < p$ and $1 \leq s < p'$, where $(q)_\infty = \prod_{i=1}^{\infty} (1-q^i)$. This bosonic type expression is the well-known Rocha-Caridi \[11\] expression for the (normalised) Virasoro character $\chi_{r,s}^{p,p'}$ from the minimal model $M(p,p')$.

A fermionic expression encodes a construction of the Hilbert space by a filling process in which each species of quasiparticle is subject to a generalized exclusion principle. This is expected to provide a formulation of the minimal model under study that is attuned to a particular integrable perturbation \[12\]-\[13\], in the sense that when the specified perturbation is switched on, the quasiparticles become genuine particles off-criticality.

Most fermionic expressions derived so far can be linked to the RSOS($p,p'$) model in a regime that corresponds to a $\phi_{1,3}$ perturbation \[14\]. The corresponding RSOS($p,p'$) paths are defined on an integer lattice, with path segments orientated either in the NE or SE direction.

The aim of this work is to highlight the use of a different type of paths for deriving fermionic characters. These so-called half-lattice paths are, as we argue below, associated with the $\phi_{2,1}/\phi_{1,5}$ perturbation. The half-lattice paths are refined versions of paths defined on an integer lattice. Informally, the half-lattice paths lie on a half-integer grid and also have path segments orientated in either the NE or SE direction. However, they are not simply rescaled versions of integer lattice paths since they are defined with an extra constraint on the position of the valleys: each valley must lie at an integer height. An example is given in Figure 1.

It will turn out that the generating functions for the half-lattice paths are the $M(k,2k\pm1)$ instances of \[11\]. Therefore, the fermionic expressions for the same generating functions are the fermionic expressions that we seek for the characters $\chi_{r,s}^{k,2k\pm1}$. It will also turn out that these expressions are intimately related to those for the unitary $M(p,p+1)$ models described by the ABF paths. From the path perspective, the reason for this is clear: the weight functions of the two types of paths are similar in that, in each case, the only contributing vertices are those that lie between two NE or two SE edges, with each such vertex contributing half its horizontal position. In fact, for the unitary models $M(p,p+1)$, there are four...
fermionic expressions for each character $\chi_{r,s}^{p,p+1}$. The identities arising from equating these with the corresponding instances of the fermionic characters are notable in that they are given instead in terms of $q$-binomials. The bosonic expressions that we derive for the finitized $M(k,2k+1)$ characters are notable in that they are given instead in terms of $q$-trinomials.

Each of the four fermionic expressions for the unitary characters $\chi_{r,s}^{p,p+1}$ has a finitized version which yields the former in the limit $L \to \infty$. On equating these expressions with the bosonic expressions for the finitized characters, bosonic-fermionic polynomial identities are obtained. These were also originally due to Melzer [1]. For the finitized $M(k,2k+1)$ characters, we also obtain four fermionic expressions for the finitized characters as generating functions of length $L$ half-lattice paths, and each of these also implies a novel bosonic-fermionic polynomial identity.

With this work, we thus initiate a systematic analysis of fermionic forms pertaining to the $\phi_{2,1}/\phi_{1,5}$ perturbation. Earlier work in this direction, which did not involve paths, was carried out in [15, 16].

1.2. Melzer identities in the unitary case. Given that these are the objects we intend to generalize, let us recall the four different fermionic expressions for the unitary minimal model irreducible characters $\chi_{r,s}^{p,p+1}$, where $1 \leq r < p$ and $1 \leq s \leq p$. These are written compactly using the following notation:

$$\tilde{n} = (n_2, n_3, \ldots, n_{p-1}), \quad m = (m_1, m_2, \ldots, m_{p-2}), \quad \text{and} \quad |i|^+ = \begin{cases} i & \text{if } i > 0, \\ 0 & \text{if } i \leq 0 \end{cases}$$

(2)

The indices of the components of $\tilde{n}$ start at 2 here to make simpler the correspondence with expressions that appear later. Each of the four expressions is a multiple summation over all non-negative integer vectors $\tilde{n}$:

$$\chi_{r,s}^{p,p+1} = q^{-\frac{1}{2}(s-r)(s-r-1)} \sum_{\tilde{n} \in \mathbb{Z}^{p-2}_{\geq 0}} q^{\frac{1}{2}mCm^T} \frac{1}{(q)_{m_1}} \prod_{i=2}^{p-2} \left[ \frac{|n_i + m_i|^+}{q} \right]_{m_i}$$

(3)

where $m_i = 2 \sum_{i < k < p} (k - i)n_k - \Delta_i$ for $1 \leq i < p$.

but they differ in the linear term in the exponent, indexed by $\ell$, and the value of $\Delta_i$ used to define $m_i$; these are tabulated in Table [1].

|  | $r$ | $s$ | $\ell$ | $\Delta_i$ |
|---|---|---|---|---|
| (a) | $\neq 1$ | $\neq 1$ | $s-1$ | $|s-1-i|^+ + |r-i|^+$ |
| (b) | $\neq 1$ | $p$ | $p-s$ | $|p-s-i|^+ + |r-i|^+ + p-1-i$ |
| (c) | $\neq p-1$ | $\neq p$ | $p-s$ | $|p-s-i|^+ + |p-r-i|^+$ |
| (d) | $\neq p-1$ | $\neq 1$ | $s-1$ | $|s-1-i|^+ + |p-r-i|^+ + p-1-i$ |

Table 1. Parameters for the four cases of the expressions.
The \( q \)-binomial appearing in (3) is defined by

\[
\begin{pmatrix} n + m \\ n \end{pmatrix}_q = \begin{cases} \frac{(q)_{n+m}}{(q)_n(q)_m} & \text{if } n, m \geq 0, \\ 1 & \text{otherwise}, \end{cases}
\]

where \( (q)_n = \prod_{i=1}^{n}(1 - q^i) \) with \( (q)_0 = 1 \).

Note that \( m_{p-1} = 0 \) in each case. The matrix \( C = C^{(p-2)} \), where \( C^{(n)} \) is the \( n \times n \) tri-diagonal matrix with entries \( C_{ij} \) for \( 1 \leq i, j \leq n \) given by, when the indices are in this range, \( C_{ii} = 2 \) and \( C_{i,i+1} = -1 \). Thus, \( C^{(n)} \) is the Cartan matrix of the finite-dimensional simple Lie algebra \( A_n \).

In Table 1 it is seen that certain cases of \( r \) and \( s \) are excluded. This is solely to prevent repetition of (essentially) identical expressions in the extreme cases where \( r = 1, r = p - 1, s = 1 \) and \( s = p \). For these values, the equivalence between the various cases of (3) is readily seen after, perhaps, shifting the value of \( n_{p-1} \). For (3) to be correct in the excluded cases where \( \ell = 0 \), it would also be necessary to set \( m_0 = 0 \). In this paper, we obtain various expressions of a form similar to (3), and corresponding comments apply (in these later expressions, however, care would need to be taken in the excluded cases for which \( \ell = 0 \), because \( m_0 \) might already refer to a non-zero value).

Expressions (3a) and (3c) (meaning (3) with entries (a) and (c) of Table 1 respectively) were first conjectured in [12], one being obtained from the other using the equivalence \( \chi_{r,s}^{p+1} = \chi_{p-r,p+1-s}^{p+1} \). The expressions (3b) and (3d) were conjectured in [1], with again one being obtained from the other using the above equivalence. In addition, [11] gave proofs for all four expressions in the cases of \( p = 3 \) and \( p = 4 \). A proof of (3a) and (3c) in the special case where \( s = 1 \) was given in [17] using the technique of “telescopic expansion”. A similar technique was used later in [18] to give a proof of all four expressions for each \( \chi_{r,s}^{p+1} \). Independently, a complete proof of (3a) and (3d) was given in [23] using a lattice path construction. A different lattice path proof of all four expressions was given in [1]. We rework and refine this latter proof here.

### 1.3. Non-unitary Melzer-type identities

To give our fermionic expressions for \( M(k; 2k \pm 1) \) in a uniform manner, we extend the definition (1) for \( \chi_{r,s}^{p'} \) so that it applies for \( p \in \frac{1}{2} \mathbb{Z} \). The symmetry relation

\[
\chi_{r,s}^{p,p'} = \chi_{s/2,2r}^{p'/2,2p},
\]

which is a direct consequence of (1), then transforms the cases where \( p \in \mathbb{Z} + \frac{1}{2} \) and \( s \) and \( p' \) are even, to genuine minimal-model characters.

The expressions make use of a modified version of the \( q \)-binomial defined by (compare with (1))

\[
\begin{pmatrix} n + m \\ n \end{pmatrix}'_q = \begin{cases} (q)_{n+m} \frac{1}{(q)_n(q)_m} & \text{if } n, m \geq 0, \\ 1 & \text{if } n = 0 \text{ and } m = -1, \\ 0 & \text{otherwise}. \end{cases}
\]

Also, they are expressed in terms of the parity vectors \( Q^{(c,j)} \) and \( R^{(c,j)} \), defined for \( 1 \leq c \leq j + 1 \) by setting

\[
Q^{(c,j)} = (Q_1^{(c,j)}, Q_2^{(c,j)}, \ldots, Q_j^{(c,j)}) \quad \text{with} \quad Q_i^{(c,j)} = \lfloor (c - 1 - i)^+ \text{ mod 2};
\]

and setting

\[
R^{(c,j)} = (R_1^{(c,j)}, R_2^{(c,j)}, \ldots, R_j^{(c,j)}) \quad \text{with} \quad R_i^{(c,j)} = \lfloor (i + c - j - 1)^+ + c + 1 \text{ mod 2}. \]

For example,

\[
Q^{(8,12)} = \langle 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0 \rangle, \quad R^{(7,12)} = \langle 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0 \rangle, \quad R^{(6,12)} = \langle 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 0 \rangle.
\]

In each of these cases, a tilde indicates that the first component should be dropped. Therefore, for example, \( \check{Q}^{(8,12)} = \langle 1, 0, 1, 0, 1, 0, 0, 0, 0 \rangle \).

**Theorem 1.** Let \( t \in \frac{1}{2} \mathbb{Z} \) and \( a, r \in \mathbb{Z} \) with \( 1 \leq a \leq t \) and \( 1 \leq r < t \), and set \( C = C^{(2t-3)} \). Then, we have four expressions for \( \chi_{r,2a}^{t,2t+1} \), distinguished by the values of \( \ell \), \( \Delta_i \), \( T^\ell = (T_1^\ell, \ldots, T_{2t-2}^\ell) \) and \( T^k = (T_1^k, \ldots, T_{2t-2}^k) \), and the type of \( q \)-binomial
\[ \chi_{r,2a}^{t,2t+1} = q^{-\frac{1}{4}(a-r)(a-r-\frac{1}{2})} \sum_{\vec{n} \in \mathbb{Z}_{\geq 0}^{2t-3}} q^{\frac{1}{2}mCmT^* - \frac{1}{4}m_i + \frac{1}{2}n_i} \frac{1}{(q)_{m_i}} \prod_{i=2}^{2t-3} \left[ \frac{n_i + \hat{m}_i}{n_i} \right]_q, \]

(10)

where \( m_i = 2 \sum_{i < k < 2t-1} (k - i)n_k - \Delta_i \) and \( \hat{m}_i = \frac{1}{2}(m_i - T_i^L - T_i^R) \) for \( 1 \leq i \leq 2t - 2 \).

Here, each sum is over all non-negative integer vectors \( \vec{n} = (n_2, n_3, \ldots, n_{2t-2}) \), with \( \vec{m} = (m_1, m_2, \ldots, m_{2t-3}) \) obtained from \( \vec{n} \) as indicated. Note that \( m_{2t-2} = 0 \) in each case, and for each \( i < 2t - 2 \), \( m_i \) is an integer equal to either \( \frac{1}{2}m_i \), \( \frac{1}{2}(m_i - 1) \) or \( \frac{1}{2}m_i - 1 \).

### Table 2. Parameters for the four cases of the expressions (10).

| Case | Parameter | Value | Parameters | Value | Parameters | Value | Parameters | Value |
|------|-----------|-------|------------|-------|------------|-------|------------|-------|
| (a)  | \( r \neq 1 \) | \( a \neq 1 \) | \( a = 2a - 2 \) | \( \Delta_i = 2a - 2 - i \) | \( \Delta_i = 2a - 2 - i \) | \( T_i^L = Q(2a - 2t - 2) \) | \( T_i^R = Q(2r, 2t - 2) \) | \( \|q \) |
| (b)  | \( r \neq 1 \) | \( a = t \) | \( 2t - 2a \) | \( \Delta_i = |2t - 2a - i| + |2t - 2a - i| + 2t - 2i \) | \( T_i^L = R(2a - 2t - 2) \) | \( T_i^R = Q(2r, 2t - 2) \) | \( \|q \) |
| (c)  | \( r = t - \frac{1}{2} \) | \( a = t \) | \( 2t - 2a \) | \( \Delta_i = |2t - 2a - i| + |2t - 2a - i| + 2t - 2i \) | \( T_i^L = R(2a - 2t - 2) \) | \( T_i^R = R(2r, 2t - 2) \) | \( \|q \) |
| (d)  | \( r = t - \frac{1}{2} \) | \( a \neq 1 \) | \( 2a - 2 \) | \( \Delta_i = |2a - 2 - i| + |2a - 2 - i| + 2t - 2i \) | \( T_i^L = Q(2a - 2t - 2) \) | \( T_i^R = R(2r, 2t - 2) \) | \( \|q \) |

In the above theorem, the \( t \in \mathbb{Z} \) cases encompass all the characters of the minimal models \( M(k, 2k+1) \), also giving four expressions for each character (except for the \( a = 1 \), \( a = t \) and \( r = 1 \) cases, for which there are one or two expressions). For the characters \( \chi_{r,s}^{k,2k+1} \) with \( s \) even, this is immediate. Because \( \chi_{r,s}^{k,2k+1} = \chi_{r-k,2k+1-s}^{k,2k+1} \), the cases of odd \( s \) are also covered. The \( t \in \mathbb{Z} + \frac{1}{2} \) cases, in fact, encompass all the characters of the minimal models \( M(k, 2k-1) \), giving four expressions for each character (except for the \( a = 1 \), \( a = t \), \( r = 1 \) and \( r = t - \frac{1}{2} \) cases). This follows because, from (5),

\[ \chi_{r,2a}^{t,2t+1} = \chi_{a,2t}^{t,2t+1} \]

and thus if \( k = t + \frac{1}{2} \), we obtain an \( M(k, 2k-1) \) character. Again, all characters are covered because \( \chi_{a,2r}^{k,2k+1} = \chi_{a-k,2k+1-2r}^{k,2k+1} \).

The cases \( a = r = 1 \) of (10) are implicit in [15, eqn. (9.4)] (see [19, IVD] for a proof). The expressions obtained in [20] for the characters of \( M(k, 2k+1) \) are equivalent to the expressions (10) (however, the proof in [20] is incomplete). Expressions for the \( M(k, 2k-1) \) characters, also equivalent to (10), were obtained similarly in [21].

### 1.4. Remarks on the structure of these non-unitary Melzer-type identities.

A number of aspects of the four expressions (10) for \( M(k, 2k \pm 1) \) characters are particularly noteworthy when compared with previously obtained fermionic expressions for minimal model Virasoro characters.

(1). Our first observation is that each expression of (10) for \( M(k, 2k+1) \), when considered in terms of the excitations of quasiparticles, involves a restriction of the corresponding expression of (3) for the unitary model \( M(2k-1, 2k) \), with the quasiparticles permitted only to inhabit alternate excited states. Likewise, each expression of (10) for \( M(k, 2k-1) \) may be considered a similar restriction of the corresponding expression of (3) for the unitary model \( M(2k-2, 2k-1) \). Also, because each \( \hat{m}_i \) is an integer equal to either \( \frac{1}{2}m_i \), \( \frac{1}{2}(m_i - 1) \) or \( \frac{1}{2}m_i - 1 \), the number of moves that the ith quasiparticle can make is then about 1/2 that for the corresponding unitary case.

(2). These expressions (10) differ from the fermionic expressions for the same \( M(k, 2k \pm 1) \) characters obtained [4,8,22,23] via the RSOS statistical models. In particular, they encode the excitations of a different number of species of quasiparticles: the \( M(2k+1) \) and \( M(2k-1) \) expressions given in Theorem 1 involve \( 2k - 3 \) and \( 2k - 4 \) species of quasiparticles respectively (the number of species being counted by the number of entries of the vector \( \vec{n} \)), whereas the expressions derived from the corresponding RSOS models involve \( k - 1 \) and \( k - 2 \) species of quasiparticles respectively.

(3). The expressions (3) give four distinct expressions for each unitary character \( \chi_{r,s}^{p,p+1} \) (except for the exceptional cases where \( r = 1 \), \( r = p - 1 \), \( s = 1 \) or \( s = p \), where there are fewer than four expressions). These expressions are not trivially equivalent. However, the knowledge that \( \chi_{r,s}^{p,p'} = \chi_{p-r,p'-s}^{p,p'} \) shows that the expressions (3a) and (3b) are indeed equivalent. The expressions (3a) and (3b) are also equivalent for the same reason. In the unitary case, the equivalence \( \chi_{r,s}^{p,p+1} = \chi_{p-r,p+1-s}^{p,p+1} \) can be seen to arise from the up-down symmetry of the ABF paths.
In the case of the half-lattice paths, however, the up-down symmetry is lost because the valley restriction is not preserved under up-down reflection. Thus, the equivalence of expressions (10b) and (10d) cannot be attributed to the up-down symmetry that accounts for the corresponding equivalence in the unitary case. A similar statement is true for the expressions (10b) and (10d).

(4). The modified $q$-binomials need to be employed in some cases. Note [25] gives specific information, and indicates, in particular, why the usual $q$-binomial is sufficient in (10b).

(5). In addition to the four fermionic expressions for $\chi_{t,2a}^{r}$ given in Theorem 2, a further two fermionic expressions arise naturally from our analysis. These, however, are for the sum $\chi_{r,2a}^{t,2t+1} + q^{a-r} \chi_{r-1,2a}^{t,2t+1}$ of two characters.

(6). The key tool in the derivation of the fermionic formulae is a combinatorial transform that increases by one half-integer unit the vertical size of the path grid, thereby relating the character of a model with parameter $t$ to the one with $t + \frac{1}{2}$. It thus interpolates between the two classes of models $M(k, 2k - 1)$ and $M(k, 2k + 1)$. Specifically, it provides the following “combinatorial flow” between minimal models

$$M(2, 5) \longrightarrow M(3, 5) \longrightarrow M(3, 7) \longrightarrow M(4, 7) \longrightarrow M(4, 9) \longrightarrow M(5, 9) \longrightarrow \cdots .$$

This is discussed further in the following subsection.

1.5. Relation with integrable perturbations. Remark (2) in Section 1.4 concerning the number of quasiparticles, is a clear indication that these fermionic forms are associated with an integrable perturbation [24] other than $\phi_{1,3}$. The model $M(p, p')$ perturbed by the $\phi_{1,3}$ field is described by the restricted sine-Gordon model [25–27], whose lattice regularization is the RSOS($p, p'$) model. The underlying structure is that of the affine Lie algebra $A_{1}^{(1)}$, which manifests itself in the path description through there being two possible orientations for the edges, two being the dimension of the fundamental representation of $A_{1}^{(1)}$.

The other integrable perturbations, $\phi_{1,2}$ and the pair $\phi_{2,1}/\phi_{1,5}$ (depending upon which one is relevant), are associated with the affine Lie algebra $A_{2}^{(2)}$. The corresponding field theory is the restricted Bullough-Dodd-Zhiber-Mikhailov-Shabat model (with imaginary coupling constant) [28–29], a Toda-type model which comes in two versions due to the asymmetry between the two simple roots of $A_{2}^{(2)}$.

Statistical models associated with $A_{2}^{(2)}$ are described in [30–32], with [33] dealing more generally with the case of an arbitrary affine Lie algebra. For the $A_{2}^{(2)}$ cases, application of Baxter’s corner transfer matrix method [34] to these models gives rise to paths that have three possible directions at each step. In [33], their paths are shown to provide a realisation of the crystal graph of the level 1 representation of $A_{2}^{(2)}$, with the three directions corresponding to the three nodes of the relevant perfect crystal (and corresponding to the dimension of the defining representation of $A_{2}^{(2)}$). The half-lattice paths may be obtained from these paths by splitting each of the segments in half: each NE (resp. SE) segment is split into two NE (resp. SE) segments of half the length, while each E segment is split into a half-length NE segment followed by a half-length SE segment. This yields the three possible shapes on the half-lattice depicted on the left of Figure 2.

Note that the forbidden fourth shape there, a half-length SE segment followed by a half-length NE segment, does not arise: this corresponds to the restriction on valleys in our definition of the half-lattice paths.

Although the redefinition seems somewhat arbitrary from this point of view, the weighting for the half-lattice paths is much simpler, being similar to that for the original ABF paths (compare [15] and [56]). For this reason, the half-lattice paths can be analysed using combinatorial techniques similar to those developed for the ABF paths.

We also note that the ‘three edge-orientations’ pattern is manifest in the bosonic form for the finitized characters, which is expressed in terms of $q$-trinomial coefficients (see Section 1.2).

**Figure 2.** Allowed edge orientations between two integer heights ($c$ is integer)

[Diagram showing allowed and forbidden edge orientations]

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1. In this paper, we are interested in half-lattice paths whose heights are restricted: when recast back in terms of lattice paths with segments orientated in the NE, SE or E directions, such paths are sometimes referred to as Motzkin or Riordan paths (these correspond to whether, in Section 3, $t \in \mathbb{Z} + \frac{1}{2}$ or $t \in \mathbb{Z}$ respectively).
In order to pinpoint exactly the perturbation associated with these new paths, we rely on our last remark in Section 1.4. It displays a combinatorial flow that is precisely the reverse of the $\phi_{2,1}/\phi_{1,5}$ renormalization-group flow observed in [4] (see also [37,38]):

$$\cdots \xrightarrow{(1,5)} M(5,9) \xrightarrow{(2,1)} M(4,9) \xrightarrow{(1,5)} M(4,7) \xrightarrow{(2,1)} M(3,7) \xrightarrow{(1,5)} M(3,5) \xrightarrow{(2,1)} M(2,5),$$

where the label above each arrow indicates the perturbing field. The correspondence between the two types of flows selects thus the $\phi_{2,1}/\phi_{1,5}$ perturbation. That such a correspondence is reliable is supported by the fact that for ABF paths, the analogous combinatorial transform (called the $C$-transform and defined in Section 2 below) relates the paths of the RSOS($p-1, p$) model to those of RSOS($p, p+1$), which is the reverse of the $\phi_{1,3}$ renormalization-group flow between unitary models [39,40]. Moreover, the latter combinatorial transform is decomposed in [4] into a sequence 'DBD' of three transformations and also applied to non-unitary models. Their combined effect is to transform paths between the models in the following sequence:

$$\text{RSOS}(p, p') \xrightarrow{D} \text{RSOS}(p' - p, p') \xrightarrow{B} \text{RSOS}(p' - p, 2p' - p) \xrightarrow{D} \text{RSOS}(p', 2p' - p).$$

Correspondingly, the renormalization-group flow along the $\phi_{1,3}$ perturbation relates precisely the minimal models $M(p', 2p' - p)$ and $M(p, p')$ [41,42].

### 1.6. Relation to previous works.

Half-lattice paths were originally introduced in [20] as a description of the $M(p, 2p+1)$ minimal models. However, no precise a priori argument pointed toward the correspondence between these particular paths and the states of irreducible modules for these specific non-unitary models. We remedied this in [43] by providing a bijection between these half-integer paths and the paths for the RSOS($p, 2p + 1$) model.

In this original formulation of the half-integer paths [20,43], the peaks were forced to have integer heights. In the conclusion of [43], we pointed out that on modifying the restriction so that peaks are forced to have half-integer heights, one observes a correspondence with the states in the irreducible modules of the $M(p+1, 2p+1)$ minimal models. This observation was put on firm foundations in [44] by providing a bijection between these paths and the RSOS($p+1, 2p+1$) paths.

In [44], it was pointed out that half-integer paths pertaining to these two classes of models can be treated uniformly by redefining the paths to be the up-down reflections of the original ones. In this way, the restriction becomes identical for the two cases: the valleys must lie at integer heights. This is the description used in the present work.

### 1.7. Organization of the article.

Section 2 is devoted to reformulating the combinatorial transforms used in [4] to prove Melzer’s identities. In particular, three combinatorial transforms, $C_1$, $C_2$ and $C_3$, that act on ABF lattice paths, are introduced. These are combined to define the $C$-transform. In Section 3 it is shown how to use the $C$-transform to rederive the four fermionic expressions for finitized $M(p, p+1)$ characters. These immediately imply the expressions for $M(p, p+1)$ characters themselves. This novel rederivation of Melzer’s identities is an essential ingredient of this work because its half-lattice generalization proceeds along similar lines.

Half-lattice paths are defined in Section 4. In the same section, we also present bosonic and fermionic expressions for the (polynomial) generating functions for half-lattice paths of finite length. In the limit of infinite length paths, these yield the bosonic character formula for $M(k, 2k+1)$ characters, and the fermionic expressions given in Theorem 1. Section 5 is devoted to proving the fermionic expressions for the finite length half-lattice paths. Thereupon, a minor variant of the $C$-transform enables an approach parallel to that of Section 5 to be used. Finally, the bosonic expressions for finite length half-lattice paths are derived in Section 6 making use of various identities for $q$-trinomial coefficients given in the Appendix.

### 2. Combinatorial transforms for ABF paths

The proof of Melzer’s four expressions for the finitized ABF characters that was developed in [5], using ideas of Bressoud [15], employs combinatorial methods to transform between generating functions for different sets of paths. Thereby, the generating function for a trivial case (essentially the $p = 2$ case) is used to obtain Melzer’s expressions in the case of general $p$. In this section, we rework this proof with a view to refining it to apply to the case of half-lattice paths. Although the method presented in this section differs from the proof given in [5], it is, in essence, a dual version of that proof, obtained naturally through mapping $q \rightarrow q^{-1}$. 


2.1. ABF paths. We define an ABF path $h$ of length $L$ to be a finite sequence $h = (h_0, h_1, h_2, \ldots, h_L, h_{L+1})$ satisfying $h_i \in \mathbb{Z}$ and $|h_i - h_{i-1}| = 1$ for $0 \leq i \leq L + 1$. An ABF path $h$ is said to be $(f, g)$-restricted if $f \leq h_i \leq g$ for $0 \leq i \leq L$ (note that this doesn’t apply to $h_{-1}$ and $h_{L+1}$). For $a, b, p, L \in \mathbb{Z}$ and $e, f \in \{0, 1\}$, define $A^{e,f,p}_{a,b}(L)$ to be the set of all ABF paths $h$ of length $L$ that are $(1, p)$-restricted with $h_0 = a$, $h_L = b$, $h_{-1} = a + 1 - 2e$, $h_{L+1} = b + 1 - 2f$. These paths are a minor variant on those originally obtained in [10], and are special cases of the RSOS paths discussed in [44].

The path picture of an ABF path $h \in A^{e,f,p}_{a,b}(L)$ is obtained by linking the vertices $(0, h_0), (1, h_1), (2, h_2), \ldots, (L, h_L)$ on the plane. The values of $e$ and $f$ serve to specify a path presegment and postsegment respectively; the presegment extends between $(-1, h_{-1})$ and $(0, h_0)$, while the postsegment extends between $(L, h_L)$ and $(L + 1, h_{L+1})$. The presegment is then in the SE direction if $e = 0$, and in the NE direction if $e = 1$; the postsegment is in the NE direction if $f = 0$, and in the SE direction if $f = 1$.

A vertex $(i, h_i)$ for $0 \leq i \leq L$ is said to be a peak, a valley, straight-up or straight-down, depending on whether the pair of edges that neighbour $(i, h_i)$ in this path picture are in the NE-SE, SE-NE, NE-NE, or SE-SE directions respectively. Defining paths with presegments and postsegments is convenient because the manipulations of the ABF paths that we describe below depend on the nature of the vertices at $(0, a)$ and $(L, b)$ that are thus determined.

The weight $w^e(h)$ of a length $L$ ABF path $h$ is defined by
\begin{equation}
(15) \quad w^e(h) = \frac{1}{4} \sum_{i=1}^{L} i |h_{i+1} - h_{i-1}|.
\end{equation}

The weight $w^o(h)$ is thus half the sum of the $i \in \mathbb{Z}$ for which $(i, h_i)$ is a straight vertex.

**Lemma 2.** Let $1 \leq a, b \leq p$ and $L \geq 0$. Then define the four paths $h^{(0,0)}, h^{(1,0)}, h^{(0,1)}, h^{(1,1)}$ such that each $h^{(e,f)} \in A^{e,f,p}_{a,b}(L)$ and $h^{(e,f)}_i = h^{(e,f)}_i$ for $0 \leq i \leq L$. Then for $e, f \in \{0, 1\}$:
(1) $w^o(h^{(0,f)}) = w^o(h^{(1,f)})$;
(2) If $b = 1$ then $w^o(h^{(e,1)}) = w^o(h^{(e,0)}) + \frac{1}{2}L$;
(3) If $b = p$ then $w^o(h^{(1,e)}) = w^o(h^{(e,1)}) + \frac{1}{2}L$.

**Proof:** The first expression is immediate because, by [15], changing the direction of the presegment does not affect the weight. The other two expressions are immediate for $L = 0$. In the case where $L > 0$ and $b = 1$, necessarily $h^{(e,f)}_{L-1} = 2$ and so, in [15], the $i = L$ term contributes $L/2$ to the weight $w^o(h^{(e,1)})$, but nothing to the weight $w^o(h^{(e,0)})$. The first expression follows. In the case where $L > 0$ and $b = p$, necessarily $h^{(e,f)}_{L-1} = p - 1$, whence a similar argument gives the second expression. \[ \square \]

For $a, b, p, L \in \mathbb{Z}$ and $e, f \in \{0, 1\}$, we define the path generating functions
\begin{equation}
(16) \quad \hat{A}^{e,f,p}_{a,b}(L) = \hat{A}^{e,f,p}_{a,b}(L; q) = \sum_{h \in A^{e,f,p}_{a,b}(L)} q^{w^e(h)}.
\end{equation}

Note that the sets $A^{0,0,p}_{a,b}(L), A^{1,0,p}_{a,b}(L), A^{0,1,p}_{a,b}(L),$ and $A^{1,1,p}_{a,b}(L)$ are all trivially in bijection with one another, with, if $h$ belongs to one of these sets, the corresponding element $h'$ of another defined by $h_i = h'_i$ for $0 \leq i \leq L$. This observation, together with Lemma 2(1) and (10), implies that
\begin{equation}
(17) \quad \hat{A}^{0,1,p}_{a,b}(L) = \hat{A}^{1,0,p}_{a,b}(L)
\end{equation}
for $f \in \{0, 1\}$.

The following bosonic expression for $\hat{A}^{e,f,p}_{a,b}(L)$ was obtained in [10]:
\begin{equation}
(18) \quad \hat{A}^{e,f,p}_{a,b}(L) = q^{\frac{1}{4}(a-b)(a-b-1+2f)} \sum_{\lambda = -\infty}^{\infty} \left( q^{\lambda(p+1)(\lambda + b - f)} - q^{L+2} \right) \chi_{L-a-b}^{L+2} \chi_{b-f,a}^{p+1}
\end{equation}
for $e \in \{0, 1\}$. This immediately leads to
\begin{equation}
(19) \quad \lim_{L \to \infty} \hat{A}^{e,f,p}_{a,b}(L) = q^{\frac{1}{4}(a-b)(a-b-1+2f)} \chi_{b-f,a}^{p+1}
\end{equation}
for $e \in \{0,1\}$. The expression on the right may then be viewed as the generating function for infinite length ABF paths that are $(1,p)$-restricted and eventually oscillate between heights $b$ and $b+1-2f$.

In what follows, we obtain fermionic expressions for $A^{e,f,p}_{a,b}(L)$. Using (19), these then yield the fermionic expressions 3 for the characters $\chi_{p,n}^{e,f,p+1}$.

### 2.2. Vertex word.
Each ABF path $h \in A^{e,f,p}_{a,b}(L)$ is conveniently encoded by its vertex word $v(h)$ which is a word $v_0v_1v_2 \cdots v_L$ of length $L+1$ in the symbols $N$ and $S$ 2. The symbols in this word describe the sequence of vertices, non-straight or straight, of $h$ read from left to right, beginning with the vertex at $(0,a)$. Thus $v(h)$ depends on the direction of the presegment of $h$, and also its postsegment, and hence on the values of $e$ and $f$.

Given a word $v$ in the symbols $N$ and $S$, and values $a$ and $e$, the ABF path $h$ such that $v(h) = v$ and $h_0 = a$ and $h_{-1} = a + 1 - 2e$ is readily determined by working from left to right (the values of $b$ and $f$ are determined by $v$, $a$ and $e$). Thus $h$ is uniquely determined by $v(h)$.

**Lemma 3.** For $1 \leq a, b \leq p$ and $e, f \in \{0,1\}$ and $L \geq 0$, let $h \in A^{e,f,p}_{a,b}(L)$. Let the corresponding vertex word $v(h)$ have symbols $N$ at positions $j_0, j_1, j_2, \ldots, j_k$. Then:

\begin{equation}
(20) \quad w^o(h) = 2L(L + 1) - \frac{1}{2} \sum_{i=0}^{k} j_i.
\end{equation}

**Proof:** This follows because, by (13), $w^o(h)$ is half the sum of the positions of the $S$ symbols in $v(h)$. \hfill \square

For $h \in A^{e,f,p}_{a,b}(L)$, we define $m(h)$ to be the number of symbols $S$ in $v(h)$. We then define the generating functions

\begin{equation}
(21) \quad \hat{A}^{e,f,p}_{a,b}(L, m) = \hat{A}^{e,f,p}_{a,b}(L, m; q) = \sum_{h \in A^{e,f,p}_{a,b}(L), m(h)=m} q^{w^o(h)}.
\end{equation}

Note that if $h \in A^{e,f,p}_{a,b}(L)$ and $h' \in A^{e',f',p}_{a,b}(L)$ are such that $h_i = h'_i$ for $0 \leq i \leq L$ then $m(h)$ and $m(h')$ are not necessarily equal when $e \neq e'$ or $f \neq f'$. In particular, we have the following result:

**Lemma 4.** Let $1 \leq a, b \leq p$ and $L \geq 0$. Then define the four paths $h^{(0,0),f}$, $h^{(1,0),f}$, $h^{(0,1),f}$, $h^{(1,1),f}$, such that each $h^{(c,f)} \in A^{e,f,p}_{a,b}(L)$ and $h_i^{(c,f)} = h_i^{(c',f')}$ for $0 \leq i \leq L$. Then for $e, f \in \{0,1\}$:

\begin{enumerate}
\item $m(h^{(c,f)}) = L + e + f$;
\item If $a = 1$ then $m(h^{(0,1),f}) = m(h^{(1,1),f}) - 1$;
\item If $a = p$ then $m(h^{(1,1),f}) = m(h^{(0,1),f}) - 1$;
\item If $b = 1$ then $m(h^{(c,0)}) = m(h^{(c,1)}) - 1$;
\item If $b = p$ then $m(h^{(c,1)}) = m(h^{(c,0)}) - 1$.
\end{enumerate}

**Proof:** If $v(h^{(c,f)})$ contains $k+1$ symbols $N$ then $e + f \equiv k$ because the path changes direction at each $N$. The first case then follows because $L = k + m(h^{(c,f)})$. The other cases result from changing the direction of the presegment or postsegment of the appropriate $h^{(c,f)}$. \hfill \square

**Lemma 5.** Let $1 \leq a, b \leq p$ and $L \geq 0$ and $e, f \in \{0,1\}$. Then:

\begin{enumerate}
\item If $m \neq L + e + f$ then $\hat{A}^{e,f,p}_{a,b}(L, m) = 0$;
\item $\hat{A}^{e,f,p}_{a,b}(L) = \sum_{m \geq 0} \hat{A}^{e,f,p}_{a,b}(L, m)$;
\item $\hat{A}^{e,1,p}_{a,b}(L, m) = \hat{A}^{0,1,p}_{a,b}(L, m - 1)$;
\item $\hat{A}^{0,1,p}_{a,b}(L, m) = \hat{A}^{1,1,p}_{a,b}(L, m - 1)$;
\item $\hat{A}^{1,1,p}_{a,b}(L, m) = qL/2 \hat{A}^{0,0,p}_{a,b}(L, m - 1)$;
\item $\hat{A}^{e,0,p}_{a,b}(L, m) = qL/2 \hat{A}^{e,1,p}_{a,b}(L, m - 1)$.
\end{enumerate}

2This notion of vertex word for RSOS paths was originally described in 16.

3Throughout this paper, we use the symbol “≡” to denote congruence modulo 2.
Proof: The first case follows from Lemma 3(1) and the definition (21). The second case follows from the definitions (16) and (21). The other cases follow from Lemmas 2 and 4 and the definition (21).

Lemma 6. Let 1 ≤ a, b ≤ p and L, m ≥ 0 and e, f ∈ {0, 1}. Then:

1. \( \hat{A}_{a,b}^{e,f}(0,m) = \delta_{a,b} \delta_{m,|e-f|} \)
2. \( \hat{A}_{a,b}^{e,f,1}(L,m) = \delta_{L,0} \delta_{m,|e-f|} \)
3. \( \hat{A}_{a,b}^{e,f,0}(L,0) = \delta_{a-e,b-f} \delta_{L+(e+f),0} \mod 2,0 \)

Proof: If a ≠ b then \( \hat{A}_{a,b}^{e,f,0}(0) = \emptyset \). On the other hand, if a = b then \( \hat{A}_{a,b}^{e,f,0}(0) \) contains a single element \( h \), indicated in Figure 3. For this, \( v(h) = N \) if \( e = f \), and \( v(h) = S \) if \( e \neq f \). For these two cases, we then have \( m(h) = 0 \) and \( m(h) = 1 \) respectively. Then, after noting that \( w^{\alpha}(h) = 0 \) in both cases, the first result follows from the definition (21).

The second result follows from the first after it is noted that \( \hat{A}_{a,b}^{e,f,1}(L) = \emptyset \) for \( L > 0 \).

For the third result, first note that if \( m(h) = 0 \) then the segments of the path \( h \), together with its presegment and postsegment, necessarily alternate in direction. As indicated in Figure 4 there can be only one such path, for which \( e = f \) if and only if \( L \) is even. Furthermore, if \( L \) is even then necessarily \( a = b \), and if \( L \) is odd then necessarily \( |a-b| = 1 \), with \( b = a - 1 \) if \( e = 1 \) and \( b = a + 1 \) if \( e = 0 \). The third result follows.

Figure 3. Paths \( h \) of zero length

Figure 4. Paths without straight vertices

2.3. \( C_1 \)-transform. In this section, we specify a method of transforming a path in \( \mathcal{A}_{a,b}^{e,f,p}(L) \) to one in \( \mathcal{A}_{a,b}^{e,f,p+1}(L') \) for certain \( a', b', L' \). This transform is referred to as a \( C_1 \)-transform. It is closely related to the \( B_1 \)-transform of [4], which itself was inspired by the "volcanic uplift" of [15].

The \( C_1 \)-transform of \( h \in \mathcal{A}_{a,b}^{e,f,p}(L) \) depends on \( e, f \in \{0, 1\} \), and is readily described in terms of the corresponding vertex word \( v(h) \). Let this vertex word have \( k+1 \) symbols \( N \) (\( k \geq -1 \)), and let their positions be \( j_0, j_1, j_2, \ldots, j_k \). With \( a' = a + e \), the action of the \( C_1 \)-transform on \( h \) is then defined to result in the path \( h^{(0)} \in \mathcal{A}_{a',b'}^{e,f,p+1}(L+k) \) whose vertex word \( v(h^{(0)}) \) also has \( k+1 \) symbols \( N \), these being at positions \( j_0, j_1 + 1, j_2 + 2, \ldots, j_k + k \). An example of a \( C_1 \)-transform is given in Figure 5. In the case of a zero length path \( h \) for which \( e \neq f \), so that \( v(h) \) is \( S \), we choose not to define the action of the \( C_1 \)-transform (Lemma 7 below does not then apply to this case).

Figure 5. Example of \( C_1 \)-transform (here \( e = 0 \) and \( f = 1 \))

Note that for the path \( h^{(0)} \) that results from the \( C_1 \)-transform, \( v(h^{(0)}) \) has no adjacent \( NN \) pair of vertices. This fact will be useful later.
Lemma 7. Let $1 \leq a, b \leq p$ and $L \geq 0$ and $e, f \in \{0, 1\}$, with $L > 0$ if $e \neq f$. Then let $h^{(0)} \in A_{a, b}^e f p+1 (L')$ be obtained from the action of the $C_1$-transform on $h \in A_{a, b}^e f p (L)$. Set $m = m(h)$. Then:

1. $a' = a + e$ and $b' = b + f$;
2. $L' = 2L - m$;
3. $m(h^{(0)}) = L$;
4. $w^\circ (h^{(0)}) = w^\circ (h) + \frac{1}{2} L(L - m)$.

Proof: That $a' = a + e$ follows from the definition of the $C_1$-transform. Let $v(h)$ contain $k + 1$ symbols $N$, and let their positions be $j_0, j_1, \ldots, j_k$. Consideration of the distances between the non-straight vertices in $h$ shows that

$$b - a = (1 - 2e)(-j_0 + (j_1 - j_0) - (j_2 - j_1) + \cdots + (-1)^k(L - j_k))$$
$$= (1 - 2e)(2(-j_0 + j_1 - j_2 + \cdots + (-1)^k j_k) + (-1)^k L).$$

Comparing this with the analogous expression for $h^{(0)}$, and noting that $L' = L + k$, shows that $(b' - a') - (b - a) = 0$ if $k$ is even, and $(b' - a') - (b - a) = 1 - 2e$ if $k$ is odd. Using $a' = a + e$ then shows that $b' - b = e$ if $k$ is even, and $b' - b = 1 - e$ if $k$ is odd. Because $e + f = k$, it follows that $b' - b = f$, as required for the first result.

That $v(h)$ contains $k + 1$ symbols $N$ implies $m + k = L$. Then $L' = L + k = L + (L - m)$ gives the second result. Because $v(h^{(0)})$ also has $k + 1$ symbols $N$, $m(h^{(0)}) = L' - k = L$ gives the third.

Applying (22) to both $h$ and $h^{(0)}$, and noting that $L' - L = L - m$ gives

$$4w^\circ (h^{(0)}) - 4w^\circ (h) = L'(L' + 1) - L(L + 1) - 2 \sum_{i=0}^{L-m} i$$

$$= (L' + L + 1)(L' - L) - (L - m + 1)(L - m)$$
$$= (3L - m + 1)(L - m) - (L - m + 1)(L - m)$$
$$= 2L(L - m),$$

thereby yielding the final expression of the lemma.

2.4. Particle insertion. In this section, we specify a further method of transforming a path. This method simply extends the path to the right, by augmenting it with an alternating sequence of NE and SE segments. This insertion process depends on a value $f \in \{0, 1\}$ in that the first of these segments is in the same direction as the postsegment specified by $f$ (the process is not affected by the direction of the presegment).

Specifically, for $n \geq 0$ and a path $h^{(0)} \in A_{a, b}^e f p (L)$, we say that the path $h^{(n)} \in A_{a, b}^e f p (L + 2n)$ has been obtained by inserting $n$ particles into $h^{(0)}$ if the vertex word $v(h^{(n)})$ is obtained from $v(h^{(0)})$ by appending $2n$ symbols $N$. We refer to this process of inserting $n$ particles as a $C_2 (n)$-transform.

![Figure 6. Example of $C_2$-transform (here $f = 0$ and $n = 4$)](image)

![Figure 7. Example of $C_2$-transform (here $f = 1$ and $n = 4$)](image)

In comparing this process with the analogous process of $[5]$, we see that the insertion takes place at the right end of the path rather than at the left. This is in accord with the earlier claim that the process described here is the dual of that of $[5]$. 
Lemma 8. With \(1 \leq a, b \leq p\) and \(L(0) \geq 0\) and \(e, f \in \{0, 1\}\), let \(h^{(n)} \in \mathcal{A}^{e,f}_{a,b}(L')\) be obtained from the \(C_2(n)\)-transform of \(h^{(0)} \in \mathcal{A}^{e,f}_{a,b}(L(0))\). Then:

1. \(L' = L(0) + 2n\),
2. \(m(h^{(n)}) = m(h^{(0)})\),
3. \(w^o(h^{(n)}) = w^o(h^{(0)})\).

Proof: The first statement follows from the definition. The second follows because the number of \(S\) vertices in \(v(h^{(0)})\) and \(v(h^{(n)})\) are equal. The third statement follows from the definition \(11\), after noting that the insertion adds no straight vertices. \(\square\)

2.5. Particle moves. In this section, we specify two types of local deformation of a path. These deformations are known as moves. In each case, a particular sequence of four segments of a path is changed to a different sequence, the remainder of the path being unchanged. These two moves are depicted in Figure 8: the portion of the path to the left of the arrow is changed to the portion on the right. These moves apply whenever \(x > 0\) and \(x + 2 < L\). They also apply in the cases \(x = 0\) and \(x + 2 = L\) if the path's presegment and postsegment respectively accord with the corresponding segments in Figure 8. Specifically, for \(x = 0\), the first move applies if \(e = 0\), and the second applies if \(e = 1\). Similarly, for \(x + 2 = L\), the first move applies if \(f = 1\), and the second applies if \(f = 0\). (If both \(x = 0\) and \(L = 2\) then the first move applies if both \(e = 0\) and \(f = 1\), and the second applies if both \(e = 1\) and \(f = 0\).)

The particle moves described in Figure 8 above are easily described in terms of the vertex words: each merely exchanges one each. The first result then follows from \(20\). Because \(v(h)\) and \(v(h')\) have the same number of symbols \(S\), the second result follows. \(\square\)

If a particle, in the guise of an \(NN\) pair, having moved to the left beyond a single straight vertex \(S\), finds another vertex \(S\) to its left, it may move again beyond this \(S\). Having done so, it may do the same through subsequent vertices \(S\), until it encounters another \(N\). If this \(N\) has another \(N\) to its left, then the original particle is said to be blocked, and cannot move further. However, if instead it has an \(S\) to its left, the sequence \(SNNN\) of vertices appears. We then re-identify the particle with the leftmost pair \(NN\) therein, whereupon it can continue to move to the left. For example, the following sequence of moves is attributed to the movement of a single particle:

\[
SSNSSNSN\rightarrow SSNSSNNN\rightarrow SSNSSNSNS
\]
\[
\rightarrow SSNNSSSS\rightarrow SNNSNSS\rightarrow NNSSNSSSS.
\]

For each particle, the number of vertices \(S\) to its right in \(v(h)\) is said to be the particle's excitation. Each move a particle makes thus increases its excitation by 1. Within a sequence of exactly 2\(k\) adjacent symbols \(N\), there are precisely \(k\) particles, each having the same excitation. Only the particle on the left is not blocked by one of the others. However,

\footnote{This interpretation is the dual of that given in \(46\).}
that on the right can perform a backward move, decreasing its excitation and that of the path. There are also precisely \( k \) particles within a sequence of exactly \( 2k + 1 \) adjacent symbols \( N \), each having the same excitation. Here, however, when performing backward moves, different \( NN \) pairs are viewed as particles. Thus, for odd-length sequences of \( N \) vertices, which actual pairs correspond to the particles is ambiguous. Nonetheless, each move is well-defined, and is attributed to a specific particle whose excitation changes.

If \( h \) contains \( n \) particles, let the excitation of the \( i \)th, counting from the left, be \( \lambda_i \). Then \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \), and therefore \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a partition having at most \( n \) parts. We refer to it as the excitation partition of \( h \). As usual, the weight \( |\lambda| \) of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is defined by \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n \).

2.6. Particle waves. Consider a path \( h^{(0)} \) whose vertex word \( v(h^{(0)}) \) has no adjacent pair \( NN \), and thus contains no particle. For \( n > 0 \), let \( h^{(n)} \) be obtained from \( h^{(0)} \) by acting on it with the \( C_2(n) \)-transform. Because there are no particles to its left, the leftmost particle in \( h^{(n)} \) is able to make as many moves as there are symbols \( S \) in \( v(h^{(n)}) \). This maximal number is \( m' = m(h^{(n)}) \). Similarly, the second leftmost particle in \( h^{(n)} \), if there is one, can move beyond the same vertices \( S \) as the first, until if it makes the same number of moves as the first, it is blocked and cannot move further. Continuing in this way, we see that if the \( i \)th leftmost particle in \( h^{(n)} \) makes \( \lambda_i \) moves then \( m' \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). Thus, by proceeding in this way, we see that every excitation partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) for which \( \lambda_1 \leq m' \) can be obtained by moving the particles in \( h^{(n)} \). If \( h' \) results from moving these particles according to the parts of a partition \( \lambda \), we say that it results from the action of the \( C_3(\lambda) \)-transform on \( h^{(n)} \).

**Lemma 10.** Let \( 1 \leq a, b \leq p \) and \( L^{(0)} \geq 0 \) and \( e, f \in \{0, 1\} \) and \( h^{(0)} \in \mathcal{A}_{a,b}^{e,f,p}(L^{(0)}) \). Then, for \( n \geq 0 \) and \( \lambda \) a partition having at most \( n \) parts with \( \lambda_1 \leq m(h^{(0)}) \), let \( h' \in \mathcal{A}_{a,b}^{e,f,p}(L') \) be obtained from \( h^{(0)} \) through the action of the \( C_2(n) \)-transform followed by the \( C_3(\lambda) \)-transform. Then:

1. \( L' = L^{(0)} + 2n \);
2. \( m(h') = m(h^{(0)}) \);
3. \( w^\circ(h') = w^\circ(h^{(0)}) + |\lambda| \).

**Proof:** This follows immediately from Lemmas 8 and 9. \( \square \)

**Note 11.** The path \( h' \) obtained from \( h^{(0)} \) as in the above lemma is easily obtained using the vertex words. Namely, \( v(h') \) is obtained from \( v(h^{(0)}) \) by, for each \( i \leq n \), inserting an \( NN \) pair so that it has exactly \( \lambda_i \) symbols \( S \) to its right.

The three aspects of path manipulation that are described above, are combined to define the \( C \)-transform. For \( h \in \mathcal{A}_{a,b}^{e,f,p}(L) \), the result of acting on \( h \) with the combined action of the \( C_1 \)-transform followed by the \( C_2(n) \)-transform followed by the \( C_3(\lambda) \)-transform for \( \lambda \) a partition with at most \( n \) parts, is termed the \( C(n, \lambda) \)-transform.

In what follows, we use a succession of \( C \)-transforms to obtain the generating function for certain sets of paths. The particles manipulated during different \( C \)-transforms are said to belong to different species. Only those particles of the same species are subject to the fermionic exclusion property, which results from the “blocking” described above. A path obtained from a succession of \( C \)-transforms is thus seen to contain a specific number of particles of various species. In our proofs below, these particle counts will be given by \( n_1, n_2, n_3, \ldots \), with \( n_1 \) the number of particles inserted during the final \( C \)-transform. This approach is valid for each of the four cases, these leading to the four cases of \( 3 \) and the four cases of Theorem 1. In \( 3 \), an alternative non-recursive (somewhat) approach to identifying particles in a path is given. Therein, particles are viewed as triangular deformations in the path, with the differently sized triangles corresponding to the different species. However, this approach can only be applied directly to two of the four cases (one instance was considered in \( 4 \)), a deficiency from which the current approach doesn’t suffer.

The \( C \)-transform is used in both Section 4 below, to obtain Melzer’s expressions for the generating functions of the ABF paths, and in Section 5 to obtain fermionic expressions for the generating functions of the half-lattice paths.

### 3. Revisiting Melzer’s identities: Finitized fermionic expressions for ABF paths

#### 3.1. Transforming the generating function.** In this subsection, we use the \( C \)-transform to describe a bijection involving different sets of ABF paths. To specify this, define \( P_{n,m} \) to be the set of all partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) with \( m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0 \). Below, we will use the fact that

\[
\sum_{\lambda \in P_{n,m}} q^{\lambda} = \left( \frac{n + m}{n} \right)_q.
\]
Lemma 12. Let \(1 \leq a, b \leq p\) and \(L, L' \geq 0\) and \(e, f \in \{0, 1\}\), with \(L > 0\) if \(e \neq f\). Then there is a bijection between the sets
\[
\mathcal{A}_{a+e, b+f}^c,f; p+1 (L', L) \leftrightarrow \bigcup_{n \geq 0} \mathcal{A}_{a, b}^c,f; p (L, 2L - L' + 2n) \times \mathcal{P}_{n,L},
\]
under which, if \(h' \in \mathcal{A}_{a+e, b+f}^c,f; p+1 (L', L)\) maps to the triple \((h, n, \lambda)\), where \(h \in \mathcal{A}_{a, b}^c,f; p (L, m)\) with \(m = 2L - L' + 2n\), and \(\lambda \in \mathcal{P}_{n,L}\), then \(h'\) is the result of the \(C(n, \lambda)\)-transform acting on \(h\), and
\[
w^\circ (h') = w^\circ (h) + \frac{1}{2}L(L-m) + |\lambda|.
\]

Proof: Let \(h \in \mathcal{A}_{a, b}^c,f; p (L, m), n \geq 0\) and \(\lambda \in \mathcal{P}_{n,L}\), and let \(h'\) be obtained from \(h\) by the action of the \(C(n, \lambda)\)-transform. Lemma 7 and Lemma 10 show that \(h' \in \mathcal{A}_{a+e, b+f}^c,f; p+1 (L', L)\) where \(L' = 2L - m + 2n\).

We now claim that for each \(h' \in \mathcal{A}_{a+e, b+f}^c,f; p+1 (L', L)\) there is a unique triple \((h, n, \lambda)\) with \(h \in \mathcal{A}_{a, b}^c,f; p (L, 2L - L' + 2n)\), such that \(h'\) arises from the action of the \(C(n, \lambda)\)-transform on \(h\). To see this, first note that the path \(h^{(0)}\) which would arise from the action of the \(C_1\)-transform on \(h\) has a vertex word with no adjacent pair \(NN\). Thus, in view of Note 11 \(h^{(0)}\) is determined uniquely by \(h'\), with its vertex word obtained from that of \(h'\) by repeatedly removing pairs \(NN\) from the latter until no such pairs remain. The value of \(n\) is the number of such pairs, while their excitation partition gives \(\lambda\).

Because \(h^{(0)}\) would be obtained from \(h\) through the action of the \(C_1\)-transform, if \(v(h^{(0)})\) has symbols \(N\) at positions \(j_0, j_1, j_2, \ldots, j_k\) \((k = L' - m \geq -1)\) then if \(k \geq 0\), the vertex word of \(v(h)\) has symbols at positions \(j_0, j_1 - 1, j_2 - 2, \ldots, j_k - k\) if \(k = -1\) then both \(v(h^{(0)})\) and \(v(h)\) comprise only symbols \(S\). In particular, in all cases, \(h\) is determined uniquely. The Lemma is then proved, with (27) following from Corollary 11(4) and Lemma 10(3).

Corollary 13. Let \(1 \leq a, b \leq p\) and \(L, L' \geq 0\) and \(e, f \in \{0, 1\}\). Then:
\[
\hat{A}_{a+e, b+f}^c,f; p+1 (L', L) = \sum_{n \geq 0} q^{\frac{n + L}{n}} \binom{n + L}{q} \hat{A}_{a, b}^c,f; p (L, m),
\]
where \(m\) is obtained from \(n\) via \(m = 2L - L' + 2n\).

Proof: In the cases other than where \(L = 0\) and \(e \neq f\), this follows from equating the generating functions of the two sides of (26), taking (27) into account, and using (25).

Now consider the case where \(L = 0\) and \(e \neq f\). For the LHS, Lemma 6(3) shows that \(\hat{A}_{a+e, b+f}^c,f; p+1 (L', 0) = 1\) if \(L'\) is odd and \(a = b\), and is zero otherwise. For the RHS, Lemma 11(1) shows that \(\hat{A}_{a, b}^c,f; p (0, m) = 1\) if \(m = 1\) and \(a = b\), and is zero otherwise. Because \(L' = 2L - m + 2n\), and thus \(L' = 2n - 1\) here, the RHS is also 1 if \(L'\) is odd and \(a = b\), and zero otherwise.

3.2. Melzer’s expressions. We now show how Melzer’s fermionic expressions for the generating functions \(\hat{A}_{a, b}^c,f; p (L)\) of the finite length AFB paths may be obtained by recursive use of Corollary 13.

Theorem 14. Let \(1 \leq a, b \leq p\) with \(p \geq 3\), and set \(C = C^{(p-2)}\). Then, for \(e, f \in \{0, 1\}\), we have the following expression for \(\hat{A}_{a, b}^c,f; p (L)\) in which the values of \(\ell\) and \(\Delta_i\) are as given in Table 3 below:
\[
\hat{A}_{a, b}^c,f; p (L) = \sum_{n \in \mathbb{Z}^n_{\geq 0} | m_0 = L} q^{\frac{1}{2}mCm^T - \frac{1}{2}m_i} \prod_{i=1}^{p-2} \left[ \binom{n_i + m_i}{q} \right],
\]
where \(m_i = 2 \sum_{i < k < p} (k - i)n_k - \Delta_i\) for \(0 \leq i < p\).

Here, the sum is over all non-negative integer vectors \(n = (n_1, n_2, \ldots, n_{p-1})\), with \(m^* = (m_0, m_1, \ldots, m_{p-2})\) obtained from \(n\) as indicated, and \(m = (m_1, m_2, \ldots, m_{p-2})\). Note that by virtue of the restriction \(m_0 = L\), the sum is, in effect, finite. Also note that \(m_{p-1} = 0\) in each case.
Table 3. Parameters for the four cases of the expressions (29).

| $\{e,f\}$ | $a$ | $b$ | $\ell$ | $\Delta_i$ |
|-----------|-----|-----|------|---------|
| (a) $(1,1)$ | $\neq 1$ | $\neq 1$ | $a - 1$ | $|a - 1 - i| + |b - 1 - i|$ |
| (b) $(0,1)$ | $\neq p$ | $\neq 1$ | $p - a$ | $|p - a - i| + |b - 1 - i| + p - 1 - i$ |
| (c) $(0,0)$ | $\neq p$ | $\neq p$ | $p - a$ | $|p - a - i| + |p - b - i|$ |
| (d) $(1,0)$ | $\neq 1$ | $\neq p$ | $a - 1$ | $|a - 1 - i| + |p - b - i| + p - 1 - i$ |

The four expressions (3) with Table 4 are an immediate corollary of these, following by taking the limit $L \rightarrow \infty$ through taking $n_1 \rightarrow \infty$, and using (19).

In what follows, each of the four expressions of Theorem 14 is proved by recursively applying Corollary 13 to express the generating function $A_{e,f}^{c_i} f^{p}(L,m)$, in terms of the trivial $A_{1,1}^{c_i} f^{p}(m_{p-1},m_p)$, which is given by Lemma 6. The four cases arise from the choices $e, f \in \{0,1\}$. By repeated application of Corollary 13 a succession of generating functions $A_{e,f}^{c_i} f^{p-1}(m_{i-1},m_i)$ is obtained for $i = 1, 2, \ldots, p$. This sequence is determined by the two sequences $e_1, e_2, \ldots, e_{p-1}$ and $f_1, f_2, \ldots, f_{p-1}$, each element of which is 0 or 1. However, within each of these sequences, the value may change only once (why this is so will become apparent below), and $a_p = 1$ and $b_p = 1$ will be achieved only if the first sequence contains exactly $(a - 1)$ 1s, and the second contains exactly $(b - 1)$ 1s. Thus in each case, only two sequences are possible. They are given explicitly in the second column of the following list (although, it isn’t necessary to specify $e_p$ and $f_p$, it is convenient to do so in order to treat uniformly the applications of Corollary 13 below):

(30a) Case L1: $e_i = \begin{cases} 1 & \text{for } 1 \leq i < a, \\ 0 & \text{for } a \leq i \leq p; \end{cases}$ $a_i = \begin{cases} a - i + 1 & \text{for } 1 \leq i \leq a, \\ 1 & \text{for } a \leq i \leq p; \end{cases}$

(30b) Case L0: $e_i = \begin{cases} 0 & \text{for } 1 \leq i \leq p - a, \\ 1 & \text{for } p - a < i \leq p; \end{cases}$ $a_i = \begin{cases} a & \text{for } 1 \leq i \leq p - a + 1, \\ p - i + 1 & \text{for } p - a + 1 \leq i \leq p; \end{cases}$

(30c) Case R1: $f_i = \begin{cases} 1 & \text{for } 1 \leq i < b, \\ 0 & \text{for } b \leq i \leq p; \end{cases}$ $b_i = \begin{cases} b - i + 1 & \text{for } 1 \leq i \leq b, \\ 1 & \text{for } b \leq i \leq p; \end{cases}$

(30d) Case R0: $f_i = \begin{cases} 0 & \text{for } 1 \leq i \leq p - b, \\ 1 & \text{for } p - b < i \leq p; \end{cases}$ $b_i = \begin{cases} b & \text{for } 1 \leq i \leq p - b + 1, \\ p - i + 1 & \text{for } p - b + 1 \leq i \leq p. \end{cases}$

The values $a_i$ and $b_i$ stated in this list give the startpoints and endpoints of the sequences of paths enumerated. As Corollary 13 indicates, these values are determined by the $e_i$ and $f_i$ using

$$a_i = a - \sum_{j=1}^{i-1} e_j, \quad b_i = b - \sum_{j=1}^{i-1} f_j$$

for $1 \leq i \leq p$. Note then that $a_i = a_{i+1} + e_i$ and $b_i = b_{i+1} + f_i$ for $1 \leq i < p$. Also note that $a_1 = a$, $b_1 = b$ and $a_p = b_p = 1$.

The four cases of Theorem 14 are now obtained by combining each of the Cases L1 and L0 with each of the Cases R1 and R0. Here we explicitly give proofs of all four cases (albeit briefly for all but the first). This contrasts with the presentation given in [5], where, because of the up-down symmetry, two of the cases could be obtained simply from the other two. Although this symmetry could be exploited likewise to prove Theorem 14 here, it no longer exists once we refine our approach to the cases of the half-lattice paths, and thus we require the details of all four cases.

3.2.1. System A. For $1 < a, b \leq p$, consider the sequence of C-transforms governed by Cases L1 and R1. Corollary 13 implies that for each $i = 1, 2, \ldots, p - 1$,

$$\hat{A}_{e_i f_i}^{c_i, i+1-i}(m_{i-1}, m_i) = \sum_{n_i \geq 0} q^{m_i(n_i-m)} \binom{n_i + m_i}{n_i} A_{e_i f_i}^{c_i, i+1-i}(m_{i}, m),$$

where $m = 2m_i + 2n_i - m_{i-1}$. In the $i$th case, we replace the variable $m$ in (32) with

$$m = m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}.$$
We now express \( \hat{A}^{c_i,f_i; p-i}(m_i, m) \) in terms of \( \hat{A}^{c_{i+1}, f_{i+1}; p-i}(m_i, m_{i+1}) \). Firstly, we obtain

\[
\hat{A}^{c_i,f_i; p-i}(m_i, m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) = \hat{A}^{c_{i+1}, f_{i+1}; p-i}(m_i, m_{i+1} + \delta_{i,b-1}),
\]

which follows from Lemma 5(3) in the \( i = a - 1 \) case because then \( a_{i+1} = 1, c_i = 1 \) and \( c_{i+1} = 0 \), and follows trivially in the \( i \neq a - 1 \) case because then \( c_{i+1} = c_i \). We then obtain

\[
\hat{A}^{c_{i+1}, f_{i+1}; p-i}(m_i, m_{i+1} + \delta_{i,b-1}) = \hat{A}^{c_{i+1}, f_{i+1}; p-i}(m_i, m_{i+1} + \delta_{i,b-1}),
\]

which follows from Lemma 5(5) in the \( i = b - 1 \) case because then \( b_{i+1} = 1, f_i = 1 \) and \( f_{i+1} = 0 \), and follows trivially in the \( i \neq b - 1 \) case because then \( f_{i+1} = f_i \).

Combining the \( i = 1, 2, \ldots, p - 1 \) cases of (32), (34) and (35) results in

\[
\hat{A}^{1:1; p}(m_0, m_1) = \sum_{n \in \mathbb{Z}^+} q^{-1} m_0 - |2 - i| \sum_{i=1}^{p-1} m_i(m_i - m_{i+1}) \hat{A}^{0:1; 1}(m_0, m_1) \prod_{i=1}^{p-1} \left[ \frac{n_i + m_i}{n_i} \right]_q,
\]

where the values of \( m_i \) are obtained recursively from \( n = (n_1, n_2, \ldots, n_{p-1}) \) using

\[
m_{i+1} = 2m_i + 2n_i - m_i - 1 - \delta_{i,a-1} - \delta_{i,b-1} \quad (1 \leq i < p).
\]

Because \( \hat{A}^{0:1; 1}(m_{p-1}, m_p) = \delta_{m_{p-1}, 0} \delta_{m_p, 0} \) by Lemma 5 and \( \left[ n_{p-1} \right]_{q} = 1 \), this results in

\[
\hat{A}^{1:1; p}(m_0, m_1) = \sum_{n \in \mathbb{Z}^+} q^{-1} m_0 - |2 - i| \sum_{i=1}^{p-1} m_i(m_i - m_{i+1}) \hat{A}^{0:1; 1}(m_0, m_1) \prod_{i=1}^{p-1} \left[ \frac{n_i + m_i}{n_i} \right]_q,
\]

with \( n \) constrained such that (37) yields \( m_{p-1} = m_p = 0 \). The constraints (37) are readily solved to yield

\[
m_i = 2 \sum_{i<k<p} (k - i)n_k - |a - 1 - i| + |b - 1 - i| \quad (0 \leq i < p).
\]

Then, using Lemma 5(2) in the form

\[
\hat{A}^{1:1; p}(m_0, m_1) = \sum_{m_0 \geq 0} \hat{A}^{1:1; p}(m_0, m_1),
\]

proves (29).
3.2.3. System C. For \(1 \leq a, b < p\), consider the sequence of \(C\)-transforms governed by Cases L0 and R0.

Again proceeding as for system A, but using

\[
m = m_{i+1} + \delta_{i,p-a} + \delta_{i,p-b}
\]

here instead of (33), we obtain the following analogue of (36):

\[
\tilde{A}_{a,b}^{0,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}} q^{-\frac{1}{2}m_{a-1} + \frac{1}{2} \sum_{i=1}^{p-1} m_i(m_i - m_{i+1})} \tilde{A}_{1,1}^{1,1;1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ \frac{n_i + m_i}{n_i} \right],
\]

where the values of \(m_i\) are obtained recursively from \(n = (n_1, n_2, \ldots, n_{p-1})\) using

\[
m_{i+1} = 2m_i + 2n_i - m_{i-1} - \delta_{i,p-a} - \delta_{i,p-b} \quad (1 \leq i < p).
\]

Because \(\tilde{A}_{1,1}^{1,1;1}(m_{p-1}, m_p) = \delta_{m_{p-1},0} \delta_{m_p,0}\) by Lemma 6 and \(\left[\frac{n_{p-1}}{n_p} - 1\right] q = 1\), this results in

\[
m_i = 2 \sum_{i<k<p} (k-i) n_k - |p-a-i| + |p-b-i| \quad (0 \leq i < p).
\]

Then, summing (49) over \(m_1\), using Lemma 5(2), proves (29).

3.2.4. System D. For \(1 < a \leq p\) and \(1 \leq b < p\), consider the sequence of \(C\)-transforms governed by Cases L1 and R0.

Again proceeding as for system A, but using

\[
m = m_{i+1} + \delta_{i,a-1} + \delta_{i,p-b}
\]

here instead of (33), we obtain the following analogue of (36):

\[
\tilde{A}_{a,b}^{1,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}} q^{-\frac{1}{2}m_{a-1} + \frac{1}{2} \sum_{i=1}^{p-1} m_i(m_i - m_{i+1})} \tilde{A}_{1,1}^{0,1;1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ \frac{n_i + m_i}{n_i} \right],
\]

where the values of \(m_i\) are obtained recursively from \(n = (n_1, n_2, \ldots, n_{p-1})\) using

\[
m_{i+1} = 2m_i + 2n_i - m_{i-1} - \delta_{i,a-1} - \delta_{i,p-b} \quad (1 \leq i < p).
\]

Because \(\tilde{A}_{1,1}^{0,1;1}(m_{p-1}, m_p) = \delta_{m_{p-1},0} \delta_{m_p,1}\) by Lemma 6 and \(\left[\frac{n_{p-1}}{n_p} - 1\right] q = 1\), this results in

\[
m_i = 2 \sum_{i<k<p} (k-i) n_k - |a-1-i| + |p-b-i| + p + 1 \quad (0 \leq i < p).
\]

Then, summing (54) over \(m_1\), using Lemma 5(2), proves (29)
4. Half-lattice paths

4.1. Half-lattice path definition. A half-lattice path $h$ of length $L \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ is a finite sequence $h = (h_{-1/2}, h_{1/2}, h_1, h_{3/2}, \ldots, h_L, h_{L+1/2})$ satisfying $h_x \in \frac{1}{2} \mathbb{Z}$ and $|h_x - h_{x-1}| = \frac{1}{2}$ for each $x \in \frac{1}{2} \mathbb{Z}_{\geq 0}$, with the additional restriction that if $h_x = h_{x+1} \in \mathbb{Z}$, then $h_{x+1/2} = h_x + 1/2$. A half-lattice path $h$ is said to be $(f, g)$-restricted if $f \leq h_x \leq g$ for $0 \leq x \leq L$.

For $t, a, b, L \in \frac{1}{2} \mathbb{Z}$, define $\mathcal{H}_{a,b}^{t}(L)$ to be the set of all length $L$ half-lattice paths $h$ that are $(1, t)$-restricted with $h_0 = a$, $h_L = b$, $h_{1/2} = a + 1/2 - e$, $h_{L+1/2} = b + 1/2 - f$.

The path picture of a half-lattice path $h \in \mathcal{H}_{a,b}^{t}(L)$ is obtained by linking the vertices $(0, h_0), (1/2, h_{1/2}), (1, h_1), \ldots, (L, h_L)$ on the plane. The examples in Figures 1 and 9 pertain to the $t = 7/2$ and $t = 4$ cases respectively. When convenient, we also specify a path presegment linking $(-1/2, h_{-1/2})$ and $(0, h_0)$, and a path postsegment linking $(L, h_L)$ and $(L + 1/2, h_{L+1/2})$. The presegment is then in the SE direction if $e = 0$, and in the NE direction if $e = 1$; the postsegment is in the NE direction if $f = 0$, and in the SE direction if $f = 1$.

![Figure 9. Half-lattice path $h \in \mathcal{H}_{a,b}^{t}(L)$](image)

A vertex $(x, h_x)$, for $x \in \frac{1}{2} \mathbb{Z}$ with $0 \leq x \leq L$, is said to be a peak, a valley, straight-up or straight-down, depending on whether the pair of edges that neighbour $(x, h_x)$ in this path picture are in the NE-SE, SE-NE, NE-NE, or SE-SE directions respectively. The specification of the path’s presegment and postsegment then determines the nature of the vertices at the path’s startpoint $(0, a)$ and endpoint $(L, b)$, respectively. Note that the additional restriction above implies that valleys can occur only at integer heights.

The weight $\hat{w}^\circ(h)$ of a half-lattice path $h \in \mathcal{H}_{a,b}^{t}(L)$ is defined by

$$\hat{w}^\circ(h) = \frac{1}{2} \sum_{i=1}^{2L} i|h_{(i+1)/2} - h_{(i-1)/2}|,$$

Thus $\hat{w}^\circ(h)$ is half the sum of the $x \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ for which $(x, h_x)$ is a straight-vertex.

We then define the generating function

$$\hat{H}_{a,b}^{t}(L) = \hat{H}_{a,b}^{t}(L; q) = \sum_{h \in \mathcal{H}_{a,b}^{t}(L)} q^{\hat{w}^\circ(h)}.$$

It is useful to note that for $a \in \mathbb{Z}$,

$$\hat{H}_{a,b}^{0}(L) = \hat{H}_{a,b}^{1}(L) = \hat{H}_{a,b}^{0}(L).$$

This follows because for $a \in \mathbb{Z}$, the two sets $\mathcal{H}_{a,b}^{0}$ and $\mathcal{H}_{a,b}^{1}$ are in bijection with $h \in \mathcal{H}_{a,b}^{0}$ mapping to $h' \in \mathcal{H}_{a,b}^{1}$ with $h_x = h_x$ for $0 \leq x \leq L$; and then $\hat{w}^\circ(h) = \hat{w}^\circ(h')$. For $a \in \mathbb{Z} + \frac{1}{2}$, (25) doesn’t hold, in general.

In what follows, we obtain bosonic expressions and fermionic expressions for $\hat{H}_{a,b}^{t}(L; q)$, the former only for $a \in \mathbb{Z}$, this being sufficient for our purposes.

4.2. Bosonic expressions and $q$-trinomials. The polynomials $Y_{a,b}^{t}(L; q) = Y_{a,b}^{t}(L; q)$ are defined for $t \in \frac{1}{2} \mathbb{Z}$ and $L, a, b, n \in \mathbb{Z}$ by

$$Y_{a,b}^{t}(L; q) = \sum_{\lambda = -\infty}^{\infty} \left( q^{\lambda^2 t^2 + \lambda (t^2 + 2 t a)} U_n(L, a - b - t' \lambda) - q^{(t^2 + 2 a)} U_n(L, -a - b - t' \lambda) \right),$$

where $t' = 2t + 1$, and $U_n(L, d)$ are the $q$-analogues of trinomial coefficients defined by

$$U_n(L, d) = U_n(L, d; q) = \sum_{k=0}^{[(L-d)/2]} q^{k(d-n)} \frac{(q)_L}{(q)_k(q)_{k+d} q_{L-2k-d}}.$$
These \(q\)-trinomial coefficients \(U_n(L, d)\) were first introduced by Andrews and Baxter \(^{48}\). Some of their properties are given in Appendix \(A\).

The next theorem, which is proved in Section \(B\) below, shows that for \(a \in \mathbb{Z}\), the generating functions \(\hat{H}^{e,f,t}_{a,b}(L)\) are, up to normalisation, given by \(Y^{f,t}_{a,b}(L)\) when \(b \in \mathbb{Z}\) or \(f = 0\), and by a sum of two such terms when \(b \in \mathbb{Z} + \frac{1}{2}\) and \(f = 1\).

**Theorem 15.** Let \(t \in \frac{1}{2} \mathbb{Z}\) and \(a, b, L \in \mathbb{Z}\) with \(1 \leq a, b \leq t\). If \(e, f \in \{0, 1\}\), then

\[
\hat{H}^{e,f,t}_{a,b}(L) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2}) + \frac{1}{2}fL} Y^{f,t}_{a,b}(L).
\]

If \(b < t\) and \(e \in \{0, 1\}\), then

\[
\hat{H}^{e,1,t}_{a,b+1/2}(L + 1/2) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2})} Y^{0,1}_{a,b}(L) + q^{L+1+\frac{1}{2}(a-b)(a-b-\frac{1}{2})} Y^{1,1}_{a,b+1}(L),
\]

\[
\hat{H}^{e,0,t}_{a,b+1/2}(L + 1/2) = q^{\frac{1}{2}L+\frac{1}{2}+\frac{1}{2}(a-b)(a-b-\frac{1}{2})} Y^{0,0}_{a,b}(L).
\]

### 4.3. Infinite length half-lattice paths and Virasoro characters.

Applying the \(L \to \infty\) properties \((116)\) and \((118)\) of the \(q\)-trinomials to \((59)\) and comparing the result with \((11)\) leads to:

\[
\lim_{L \to \infty} Y^{0,1}_{a,b}(L) = \chi_{b,2a}^{t,2t+1},
\]

\[
\lim_{L \to \infty} Y^{1,1}_{a,b}(L) = \chi_{b,2a}^{t,2t+1} + q^{a-b} \chi_{b-1,2a}^{t,2t+1}.
\]

Theorem \(15\) then immediately implies:

**Corollary 16.** If \(t \in \frac{1}{2} \mathbb{Z}\), \(e \in \{0, 1\}\) and \(a, b \in \mathbb{Z}\) with \(1 \leq a \leq t\) and \(1 \leq b < t\) then

\[
\lim_{L \to \infty} \hat{H}^{e,0,t}_{a,b}(L) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2})} \chi_{b,2a}^{t,2t+1},
\]

\[
\lim_{L \to \infty} \hat{H}^{e,1,t}_{a,b+1/2}(L) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2})} \chi_{b,2a}^{t,2t+1},
\]

\[
\lim_{L \to \infty} q^{-\frac{1}{2}L} \hat{H}^{e,0,t}_{a,b+1/2}(L) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2})} \chi_{b,2a}^{t,2t+1},
\]

\[
\lim_{L \to \infty} q^{\frac{1}{2}L} \hat{H}^{e,1,t}_{a,b+1/2}(L) = q^{\frac{1}{2}(a-b)(a-b-\frac{1}{2})} \left( \chi_{b,2a}^{t,2t+1} + q^{a-b} \chi_{b-1,2a}^{t,2t+1} \right) \quad (b > 1).
\]

The functions on the left sides of \((65a)\) and \((65b)\) are readily interpreted as the generating functions of infinite-length half-lattice paths as follows. For \(t \in \frac{1}{2} \mathbb{Z}\) and \(a, b \in \mathbb{Z}\) with \(1 \leq a, b \leq t\), define \(\mathcal{H}^{e,f}_{a,b}\) to be the set of half-lattice paths \(h = (h_0, h_{1/2}, h_1, h_{3/2}, \ldots)\) that are \((1, t)\)-restricted with \(h_0 = a\), and are \(b\)-tailed in that for each \(h\), there exists \(L\) for which \(h_i \in \{b, b + \frac{1}{2}\}\) for all \(i \geq L\). Let \(L(h)\) be the smallest such \(L\), and define

\[
\hat{\omega}^e(h) = \frac{1}{2} \sum_{i=1}^{2L(h)} i |h_{(i+1)/2} - h_{(i-1)/2}|.
\]

Then define the generating function

\[
\hat{H}^{t}_{a,b}(q) = \sum_{h \in \mathcal{H}^{e,f}_{a,b}} q^{\hat{\omega}^e(h)}.
\]

**Theorem 17.** Let \(t \in \frac{1}{2} \mathbb{Z}\) and \(a, b \in \mathbb{Z}\) with \(1 \leq a \leq t\) and \(1 \leq b < t\). Then

\[
q^{-\frac{1}{2}(a-b)(a-b-\frac{1}{2})} \hat{H}^{t}_{a,b}(q) = \chi_{b,2a}^{t,2t+1}.
\]

**Proof:** From \((57)\) and \((66)\), we see that

\[
\hat{H}^{t}_{a,b}(q) = \lim_{L \to \infty} \hat{H}^{0,0,t}_{a,b}(L) = \lim_{L \to \infty} \hat{H}^{1,1,t}_{a,b+1/2}(L).
\]

for \(e \in \{0, 1\}\). Corollary \(16\) thus yields \((67)\). \(\square\)

This result, which shows that, up to normalisation, \(\chi_{b,2a}^{t,2t+1}\) is the generating function for infinite length half-lattice paths, was established in \(13\) by the alternative method of constructing a weight-preserving bijection between those paths and the RSOS paths \(10\) of \(M(p, 2p \pm 1)\).
By virtue of \((63a)\), \((64a)\) and \((61)\), the \(t \in \mathbb{Z}\) and \(t \in \mathbb{Z} + \frac{1}{2}\) cases of \(Y_{a,b}^{0,t}(L;q)\) are finitizations of characters \(\chi_{r,s}^{p,p'}\) for which \(p' = 2t + 1\) and \(p' = 2t\) respectively. This, in particular, accounts for the appearance in \((5)\) eqn. (9.4) of the two different finitizations of \(\chi_{r,s}^{p,p'}\) defined in \((3)\) eqns. (3.10) and (5.7).

It is intriguing to note the related fact that, for the \(\chi_{r,s}^{2p-1}\) characters, the startpoint and tail configuration of the half-lattice paths being enumerated are determined by \(r\) and \(s\) respectively. This is in contrast to the \(\chi_{r,s}^{2p+1}\) case, where they are determined by \(s\) and \(r\) respectively. The latter statement also applies to the RSOS paths \((4)\) for all \(\chi_{r,s}^{p,p'}\).

### 4.4. Fermionic expressions for half-lattice paths

In this section, we state fermionic expressions for the half-lattice path generating functions \(\hat{H}_{a,b}^{e;f}(L; q)\). These expressions will be proved in Section 5 below, by extending the combinatorial techniques of Sections 2 and 5.

**Theorem 18.** Let \(a, b, t, L \in \frac{1}{2}\mathbb{Z}\) with \(1 \leq a, b \leq t\) and \(L = a + b \in \mathbb{Z}\), and set \(C = C^{(2t-3)}\). Then, for \(e, f \in \{0, 1\}\), we have the following expression for \(\hat{H}_{a,b}^{e;f}(L)\) in which the values of \(\Delta_i, T^L = (T^L_1, \ldots, T^L_{2t-2})\) and \(T^R = (T^R_1, \ldots, T^R_{2t-2})\), are as given in Table 4 below:

\[
\hat{H}_{a,b}^{e;f}(L) = \sum_{n \in \mathbb{Z}^{2t-2}} q^{\frac{1}{2}m^T - \frac{1}{2}m} n^{T^L} \prod_{i=1}^{2t-3} \left( n_i + \hat{m}_i \right)^{\ell}
\]

where \(m_i = 2 \sum_{i<k<2t-1} (k-i) n_k - \Delta_i\) for \(0 \leq i \leq 2t - 2\) and \(\hat{m}_i = \frac{1}{2}(m_i - T^L_i - T^R_i)\) for \(1 \leq i \leq 2t - 2\).

Here, the sum is over all non-negative integer vectors \(n = (n_1, n_2, \ldots, n_{2t-2})\), with \(m^b = (m_0, m_1, \ldots, m_{2t-3})\) obtained from \(n\) as indicated, and \(m = (m_1, m_2, \ldots, m_{2t-3})\). Note that by virtue of the restriction \(m_0 = 2L\), the sum is, in effect, finite. Also note that \(m_{2t-2} = 0\) in each case, and for each \(i < 2t - 2\), \(\hat{m}_i\) is an integer equal to either \(\frac{1}{2}m_i\), \(\frac{1}{2}(m_i - 1)\) or \(\frac{1}{2}m_i - 1\).

**Table 4.** Parameters for the four cases of the expressions \((69)\).

| \((e,f)\) | \(a\) | \(b\) | \(\ell\) | \(\Delta_i\) | \(T^L\) | \(T^R\) |
| --- | --- | --- | --- | --- | --- | --- |
| (a) | \((1,1)\) | \(\neq 1\) | \(\neq 1\) | \(2a - 2\) | \([2a - 2 - i]^+ + [2b - 2 - i]^+\) | \(Q^{(2a-1,2t-2)}\) | \(Q^{(2b-1,2t-2)}\) |
| (b) | \((0,1)\) | \(\neq t\) | \(\neq 1\) | \(2t - 2a\) | \([2t - 2a - i]^+ + [2b - 2 - i]^+ + 2t - 2 - i\) | \(R^{(2a-1,2t-2)}\) | \(Q^{(2b-1,2t-2)}\) |
| (c) | \((0,0)\) | \(\neq t\) | \(\neq t\) | \(2t - 2a\) | \([2t - 2a - i]^+ + [2b - 2 - i]^+ + 2t - 2 - i\) | \(R^{(2a-1,2t-2)}\) | \(R^{(2b-1,2t-2)}\) |
| (d) | \((1,0)\) | \(\neq 1\) | \(\neq 1\) | \(2a - 2\) | \([2a - 2 - i]^+ + [2t - 2b - i]^+ + 2t - 2 - i\) | \(Q^{(2a-1,2t-2)}\) | \(R^{(2b-1,2t-2)}\) |

Theorem 11 follows from Theorem 18 by taking the \(L \to \infty\) limit of \((69)\) through taking \(n_1 \to \infty\), and using \((64c)\) and \((61b)\). Using \((64c)\) in the same way leads to expressions equivalent to those already obtained. On the other hand, using \((64d)\) in the same way leads to fermionic expressions for the combination \(\chi_{b,2a}^{t,2t+1} + q^{a-b} \chi_{b,2a}^{t,2t+1}\) of two characters. We don’t give these expressions explicitly, but they are easily obtained. Note that for the cases \((64c)\) and \((64d)\), in taking the \(L \to \infty\) limit of \((69)\), the \(q^{-L/2}\) factor in the former is compensated for by a \(q^{L/2}\) factor arising from the 1st component of \(\frac{1}{2}n \cdot T^R\) in the exponent in \((69)\).

### 5. Non-unitary Melzer-type identities: Refining the combinatorial transforms

A refinement of the strategy used above is now used to obtain generating functions for the half-lattice paths. First note that because the half-lattice paths are defined on a half-integer lattice, they may be obtained by shrinking the ABF paths by a factor of 2, and then excluding all paths that have valleys at half-integer heights. We will use the equivalent approach of first excluding ABF paths that have valleys at even heights before shrinking them. We say that an ABF path is valley-restricted if it has no valley at an even height.

The key to obtaining the fermionic generating functions for the valley-restricted ABF paths is to note that for both of the particle moves depicted in Figure 8 a valley at an even or odd height is exchanged for one at a height of the opposite parity (the same is also true of a peak). Thus, to retain a valley-restricted path, it is necessary for each particle to make moves in steps of two. Therefore in the valley-restricted case, a particle can make, roughly, half as many moves as in the ABF case. A minor complication arises because the \(C_2(n)\)-transform sometimes inserts particles with their valleys lying
at an even height, as in the example of Figure 2. This requires their initial position to be excluded. Care must then be taken, however, when \( n = 0 \), for naively excluding the initial position in this case, would exclude a valid path. As will be seen, obviating this necessitates use of the modified \( q \)-binomial defined by (60).

5.1. Valley-restricted ABF paths. Let \( h \in \mathcal{A}_{a,b}^{e,f,p}(L,m) \), and let \( v(h) = v_0 v_1 \cdots v_L \) contain \( k + 1 \) symbols \( N \) (here \( k \geq -1 \)). Because \( v(h) \) contains \( m \) symbols \( S \) then \( m + k = L \). Let the indices of the symbols \( N \) in \( v(h) \) be \( j_0, j_1, j_2, \ldots, j_k \). For \( e = 0 \), the valleys are then at positions \( j_0, j_2, j_4, \ldots \); while for \( e = 1 \), the valleys are at positions \( j_1, j_3, j_5, \ldots \). Then the number of valleys in \( h \) that are at even height is given by \( \xi(h) \), where we define

\[
\xi(h) = \# \{ i \mid 0 \leq i \leq k, i \equiv e, j_i \neq a \}.
\]

We are, of course, particularly interested in the paths \( h \), for which \( \xi(h) = 0 \), these being the valley-restricted paths.

Thereupon, for \( e, f \in \{0, 1\} \), we define the generating functions

\[
\hat{R}_{a,b}^{e,f,p}(L) = \hat{R}_{a,b}^{e,f,p}(L; q) = \sum_{h \in \mathcal{A}_{a,b}^{e,f,p}(L,m)} q^{w^e(h)}
\]

and

\[
\hat{R}_{a,b}^{e,f,p}(L,m) = \hat{R}_{a,b}^{e,f,p}(L,m; q) = \sum_{h \in \mathcal{A}_{a,b}^{e,f,p}(L,m)} q^{w^e(h)}.
\]

In what follows, we require analogues of Lemmas 5 and 6 for these generating functions.

**Lemma 19.** Let \( 1 \leq a, b \leq p \) and \( L \geq 0 \) and \( e, f \in \{0, 1\} \). Then:

1. If \( m \neq L + e + f \) then \( \hat{R}_{a,b}^{e,f,p}(L,m) = 0 \);
2. \( \hat{R}_{a,b}^{e,f,p}(L) = \sum_{m \geq 0} \hat{R}_{a,b}^{e,f,p}(L,m) \);
3. \( \hat{R}_{1,b}^{e,f,p}(L,m) = R_{1,b}^{e,f,p}(L,m - 1) \);
4. \( \hat{R}_{0,b}^{e,f,p}(L,m) = \hat{R}_{0,b}^{e,f,p}(L,m - 1) \);
5. \( \hat{R}_{a,1}^{e,0,p}(L,m) = q^{L/2} \hat{R}_{a,1}^{e,0,p}(L,m - 1) \);
6. \( \hat{R}_{a,1}^{e,0,p}(L,m) = q^{L/2} \hat{R}_{a,1}^{e,0,p}(L,m - 1) \).

**Proof:** The first two cases follow from Lemma 1(1) and the definitions (16) and (21). The other cases follow from Lemmas 2 and 3 and the definition (24).

**Lemma 20.** Let \( 1 \leq a, b \leq p \) and \( L, m \geq 0 \) and \( e, f \in \{0, 1\} \). Then:

1. \( \hat{R}_{a,b}^{e,f,p}(0, m) = \begin{cases} 0 & \text{if } e = f = 0 \text{ and } a \text{ is even,} \\ \delta_{a,b} \delta_{m,|e-f|} & \text{otherwise;} \end{cases} \)
2. \( \hat{R}_{1,1}^{e,f,p}(L,m) = \delta_{L,0} \delta_{m,|e-f|}; \)
3. \( \hat{R}_{a,b}^{e,f,p}(L,0) = \begin{cases} 0 & \text{if } a \text{ is even and } e = 0, \\ 0 \delta_{a-c,b-f} \delta_{(L+e+f) \mod 2,0} & \text{if } a \text{ is odd and } e = 1 \text{ and } L > 0, \end{cases} \)

**Proof:** If \( a \neq b \) then \( \mathcal{A}_{a,b}^{e,f,p}(0) = \emptyset \). On the other hand, if \( a = b \) then \( \mathcal{A}_{a,b}^{e,f,p}(0) \) contains a single element \( h \) for which \( v(h) = N \) if \( e = f \), and \( v(h) = S \) if \( e \neq f \). For these two cases, we then have \( m(h) = 0 \) and \( m(h) = 1 \) respectively, with \( \xi(h) \neq 0 \) only if \( e = f = 0 \) and \( a \) even. Then, after noting that \( w^e(h) = 0 \) in both cases, the first result follows from the definition (22).

The second result follows from the first after it is noted that \( \mathcal{A}_{1,1}^{e,f,p}(L) = \emptyset \) for \( L > 0 \).

For the third result, first note that if \( m(h) = 0 \) then the segments of the path \( h \), together with its presegment and postsegment, necessarily alternate in direction. As indicated in Figure 4 there can only be one such path, for which \( e = f \) if and only if \( L \) is even. Furthermore, if \( L \) is even then necessarily \( a = b \), and if \( L \) is odd then necessarily \( |a - b| = 1 \), with \( b = a - 1 \) if \( e = 1 \) and \( b = a + 1 \) if \( e = 0 \). As seen in Figure 4 the valleys of such a path would all occur at height \( a - e \), yet if (and only if) \( L = 0, a \) is odd and \( e = 1 \), there are none at even height. The third result follows.
5.2. Refining the $C$-transform. Let the path $h^{(0)} \in A^{c,f:p+1}_{a+b+1}(L')$ result from the action of the $C_1$-transform on $h \in A^{c,f:p}_{a,b}(L)$. If the indices of the symbols $N$ in $v(h)$ are $j_0,j_1,j_2,\ldots,j_k$ then, by the definition in Section 2.3 those of $v(h^{(0)})$ are $j_0,j_1+1,j_2+2,\ldots,j_k+k$ (again we avoid considering the case where $L = 0$ and $e \neq f$). We see from (70) that $\xi(h') = \xi(h)$. In particular, if either $h$ or $h^{(0)}$ has no valley at an even height, then the same is true of the other.

So now assume that $h^{(0)}$ is valley-restricted. The $C_3(n)$-transform maps $h^{(0)}$ to a path $h^{(n)}$ by appending to the former an alternating sequence of peaks and valleys, $n$ of each. We see that the newly created valleys are each at height $b - f$ (with the newly created peaks each at height $b - f + 1$). Thus, if $b - f$ is even, the path $h^{(n)}$ itself should not be included when enumerating the valley-restricted paths. However, paths obtained from $h^{(n)}$ by moving particles should be included.

As indicated above, each particle move, although maintaining the number of valleys, changes the parity of the height of one valley. Therefore, if an ABF path is valley-restricted, then moving the particles in steps of two maintains the restriction. Thus for $h^{(n)}$ obtained above, in the case where $b - f$ is odd, the action of $C_3(\lambda)$-transform on $h^{(n)}$ results in a valley-restricted path $h'$ if each part $\lambda_i$ of $\lambda$ is even. On the other hand, if $b - f$ is even, then valley-restricted paths are obtained by moving each particle in $h^{(n)}$ by a single initial step, and thereafter in steps of two. Thus, in this case, the $C_3(\lambda)$-transform acting on $h^{(n)}$ results in a valley-restricted path $h'$ if each $\lambda_i$ is odd. Figures 6 and 7 provide examples for which $b - f$ is odd and even respectively.

5.3. Transforming the valley-restricted generating function. We now use this refined $C$-transform to describe a bijection involving sets of restricted paths.

Lemma 21. Let $1 \leq a, b \leq p$ and $L > 0$ and $L' \geq 0$ and $e, f \in \{0, 1\}$, and set $T^L = (a + 1) \mod 2$ and $T^R = (b + 1) \mod 2$. Then there is a bijection between the sets

$$
R^{c,f:p+1}_{a+e,b+1}(L', L) \longleftrightarrow \bigcup_{n \geq 0} R^{c,f:p}_{a,b}(L, 2L - L' + 2n) \times P_{n,(L - T^L - T^R)/2},
$$

under which, if $h' \in R^{c,f:p+1}_{a+e,b+1}(L', L)$ maps to $(h, n, \mu)$, where $h \in R^{c,f:p}_{a,b}(L, m)$ with $m = 2L - L' + 2n$, and $\mu \in P_{n,(L - T^L - T^R)/2}$, then

$$
w_f^{\mu}(h') = w_f^{\mu}(h) + \frac{1}{2} L(L - m) + 2|\mu| + nT^R.
$$

Proof: The sets $R^{c,f:p+1}_{a+e,b+1}(L', L)$ and $R^{c,f:p}_{a+e,b+1}(L', L)$ of paths are subsets of $A^{c,f:p+1}_{a+e,b+1}(L', L)$ and $A^{c,f:p}_{a,b}(L, m)$, respectively. We claim that the bijective map of Lemma 12 remains a bijection under restriction to these subsets with $\lambda \in P_n$, there obtained from $\mu \in P_{n,(L - T^L - T^R)/2}$ by setting $\lambda_i = 2\mu_i + T^R$ for $1 \leq i \leq n$. Note then that $\lambda_i \equiv T^R$ for $1 \leq i \leq n$.

Throughout this proof, let $h \in A^{c,f:p}_{a,b}(L, 2L - L' + 2n)$ and let $h' \in A^{c,f:p+1}_{a+e,b+1}(L', L)$ result from the action of the $C(n, \lambda)$-transform on $h$. Also, let the path $h^{(0)} \in A^{c,f:p+1}_{a+e,b+1}(L', L - 2n)$ be obtained by the action of the $C_1$-transform on $h$, and $h^{(n)} \in A^{c,f:p+1}_{a+e,b+1}(L')$ be obtained from the action of the $C_3(\lambda)$-transform on $h^{(n)}$. The path $h'$ is then obtained by the $C_3(\lambda)$-transform acting on $h^{(n)}$. We claim that $h' \in R^{c,f:p+1}_{a+e,b+1}(L', L)$ if and only if both $h \in R^{c,f:p}_{a,b}(L, 2L - L' + 2n)$ and $\lambda_i \equiv T^R$ for $1 \leq i \leq n$.

Assuming the latter, the above discussion shows that the path $h^{(0)}$ is valley-restricted. It also shows that each of the $n$ particles in $h^{(n)}$ has a valley at height $b$. Because each move switches the parity of a valley, if $\lambda_i \equiv T^R$ for $1 \leq i \leq n$, all the valleys in $h'$ are at odd height, thus establishing the ‘if’ part of the above claim.

For the ‘only if’ part, first consider $\lambda_i \not\equiv T^R$ for some $i$. The above discussion immediately shows that the $i$th particle in $h'$ has a valley at an even height. Secondly, if $h \not\in R^{c,f:p}_{a,b}(L, 2L - L' + 2n)$ then $h$ has a valley at even height, and thus so does $h^{(0)}$. In $h^{(0)}$, this valley has no adjacent non-straight vertex, and is thus not part of a particle. This feature remains in $h^{(n)}$. It also remains however the particles in $h^{(n)}$ are moved. Thus $h' \not\in R^{c,f:p+1}_{a+e,b+1}(L', L)$ and the ‘only if’ part of the above claim is established.

The Lemma is then proved, with (73) following from (27) after noting that $|\lambda| = 2|\mu| + nT^R$. \qed

Corollary 22. Let $1 \leq a, b \leq p$ and $L, L' \geq 0$ and $e, f \in \{0, 1\}$, and set $T^L = (a + 1) \mod 2$ and $T^R = (b + 1) \mod 2$. Then:

$$
\tilde{R}^{c,f:p+1}_{a+e,b+1}(L', L) = \sum_{n \geq 0} q^{\frac{1}{2} L(L - m) + nT^R} \left[ n + \frac{1}{2}(L - T^L - T^R) \right] q^2 R^{c,f:p}_{a,b}(L, m),
$$

where $m$ is obtained from $n$ via $m = 2L - L' + 2n$. \qed
Proof: In the $L > 0$ cases, this follows from equating the generating functions of the two sides of (73), using (74), and (75) in the form

\[
(76) \sum_{\mu \in \Phi_n(L - T^L - T^R)} q^{2|\mu|} = \left[ \frac{n + \frac{1}{2}(L - T^L - T^R)}{n} \right] q^2 .
\]

The $L = 0$ case is somewhat tricky and requires the consideration of various subcases. In each of these subcases, we use Lemma 20(3,1) to evaluate $\hat{R}_{a+e,b+f}(L',0)$ and $\hat{R}_{a,b}(0,m)$ on the two sides of (75). For $a \neq b$, we immediately see that both sides of (75) are zero. Thus we only consider the $a = b$ cases hereafter.

In the $e \neq f$ cases, $\hat{R}_{a+e,b+f}(L',0) = 1$ if both $L'$ and $a$ are odd, and is zero otherwise. In the $e \neq f$ cases, $\hat{R}_{a,b}(0,m) = 1$ if $m = 1$, and is zero otherwise. Because $m = 2n - L'$ here, the RHS is non-zero only if $L'$ is odd. Then, for a odd, so that $T^L = T^R = 0$, we see that the RHS is 1, whereas for $a$ even, so that $T^L = T^R = 1$, we see that the RHS is $q^n \binom{n-1}{q^2}$. For $n = (L'+1)/2 > 0$, and is thus zero in this case.

For the $e = f$ cases of $L' = 0$ for $a$ and $b$ odd, $\hat{R}_{e,b}(0,m) = 1$ if $m = 0$ and $a$ is odd, and is zero otherwise. Because $m = 2n - L'$ here, if $L'$ is even and $a$ is odd, we see that the RHS of (75) is also 1 after noting that $T^L = T^R = 0$, and is zero otherwise.

For the cases $e = f = 1$, we consider separately the even and odd cases of $a$. For an odd, $\hat{R}_{a+e,b+f}(L',0) = 1$ if $L'$ is even, and if $L'$ is odd. For the cases $e = f = 1$ with $a$ odd, $\hat{R}_{a,b}(0,m) = 1$ if $m = 0$, and zero otherwise. Because $m = 2n - L'$ here, if $L'$ is even and $a$ is odd, we see that the RHS of (75) is also 1 after noting that $T^L = T^R = 0$, and is zero otherwise.

Finally, for $e = f = 1$ and $a$ even, $\hat{R}_{a+e,b+f}(L',0) = 1$ if $L' = 0$, and is zero otherwise. For $e = f = 1$ and $a$ even, $\hat{R}_{a,b}(0,m) = 1$ if $m = 0$, and zero otherwise. Because $m = 2n - L'$ here, we see that for the RHS of (75) to be non-zero requires $L'$ to be even. However, $a$ and $b$ being even yields $T^L = T^R = 1$, and therefore the $q$-binomial on the RHS of (75) takes the form $\left[ \frac{n-1}{q^2} \right]$. In this case, the definition of the modified $q$-binomial ensures that the $n = 0$ term contributes 1 if and only if $L' = 0$. The proof of Corollary 22 is then complete.

\[ \square \]

Note 23. Examination of the proof of the above Corollary shows that the use of the modified $q$-binomial in (75), is necessary when $L = 0$, $e = f = 1$ and $a = b$ are even (and thus $T^L = T^R = 1$), but the standard $q$-binomial suffices in all other cases. In terms of the action of the $C$-transform, the reason for this is exemplified in Figure 10 for $n > 0$, the resulting path is not valley-restricted and thus should be excluded, whereas the opposite is the case for $n = 0$.

\[
\text{FIGURE 10. Explaining the necessity of the modified } q \text{-binomial}
\]

\[
\begin{array}{lll}
\begin{array}{|c|c|c|}
\hline
\text{a} & \text{a} & \text{a} \\
\hline
\end{array} & \leq 3 & \hline
\begin{array}{|c|c|c|}
\hline
\text{a} & \text{a} & \text{a} \\
\hline
\end{array} & \leq 0 & \hline
\begin{array}{|c|c|c|}
\hline
\text{a} & \text{a} & \text{a} \\
\hline
\end{array} & \leq 0 & \hline
\end{array}
\]

5.4 Refining Melzer’s expressions. We now use Corollary 22, together with Lemmas 19 and 20, to produce fermionic expressions for $\hat{R}_{a,b}(L)$, for each pair $e, f \in \{0, 1\}$. These four expressions are stated in the following Theorem.

\[ \text{Theorem 24. Let } p, a, b, L \in \mathbb{Z} \text{ with } 1 \leq a, b \leq p \text{ and } L + a + b \in 2\mathbb{Z}, \text{ and set } C = C^{(p-2)}. \text{ Then, for } e, f \in \{0, 1\}, \text{ we have the following expression for } \hat{R}_{a,b}^{e,f}(L) \text{ in which the values of } \ell, \Delta_i, T^L = (T^L_1, \ldots, T^L_{p-1}) \text{ and } T^R = (T^R_1, \ldots, T^R_{p-1}), \text{ are as given in Table 1 below:}
\]

\[
(77) \hat{R}_{a,b}^{e,f}(L) = \sum_{n \in \mathbb{Z}_{\geq 0}^{p-1} \mid m_0 = L} q^{\frac{1}{2} m_0 C \ell - \frac{1}{2} m_{i+n} T^L \prod_{i=1}^{p-2} \left[ \frac{n_i + \hat{m}_i}{n} \right]} q^{2i},
\]

where $m_i = 2 \sum_{i < k < 2i-1} (k-i)n_k - \Delta_i$ for $0 \leq i < p$, and $\hat{m}_i = \frac{1}{2}(m_i - T^L_i - T^R_i)$ for $1 \leq i < p$.

Here, the sum is over all non-negative integer vectors $n = (n_1, n_2, \ldots, n_{p-1})$, with $m^k = (m_0, m_1, \ldots, m_{p-2})$ obtained from $n$ as indicated, and $m = (m_1, m_2, \ldots, m_{p-2})$. Note that by virtue of the restriction $m_0 = L$, the sum is, in effect,
finite. Also note that \( m_{p-1} = 0 \) in each case, and for each \( i < p - 1 \), \( \hat{m}_i \) is an integer equal to either \( \frac{1}{2}m_i \), \( \frac{1}{2}(m_i - 1) \) or \( \frac{1}{2}m_i - 1 \).

**Table 5.** Parameters for the four cases of the expressions (77).

| \((e, f)\) | \(a\) | \(b\) | \(\ell\) | \(\Delta_i\) | \(T^L\) | \(T^R\) |
|---|---|---|---|---|---|---|
| \((1,1)\) | \(\neq 1\) | \(\neq 1\) | \(a - 1\) | \(|a - 1 - i|^+ + |b - 1 - i|^+\) | \(Q^{(a,p-1)}\) | \(Q^{(b,p-1)}\) |
| \((0,1)\) | \(\neq p\) | \(\neq 1\) | \(p - a\) | \(|p - a - i|^+ + |b - 1 - i|^+ + p - 1 - i\) | \(R^{(a,p-1)}\) | \(Q^{(b,p-1)}\) |
| \((0,0)\) | \(\neq p\) | \(\neq p\) | \(p - a\) | \(|p - a - i|^+ + |p - b - i|^+\) | \(R^{(a,p-1)}\) | \(R^{(b,p-1)}\) |
| \((1,0)\) | \(\neq 1\) | \(\neq p\) | \(a - 1\) | \(|a - 1 - i|^+ + |p - b - i|^+ + p - 1 - i\) | \(Q^{(a,p-1)}\) | \(R^{(b,p-1)}\) |

Theorem 18 immediately follows from Theorem 24 because, by virtue of the definitions (77) and (71), for \( t, \hat{a}, \hat{b}, L \in \frac{1}{2} \mathbb{Z} \) with \( 1 \leq \hat{a}, \hat{b} \leq t \),

\[
(78) \quad \hat{H}_{\hat{a}, \hat{b}}^{c, f; L}(q) = \hat{R}_{\hat{a}, \hat{b}}^{c, f; 2t-1}(2L, q^{1/2}).
\]

The expressions in Theorem 24 are proved in a similar way to those of Theorem 14 in Section 3.2. In particular, expressions for \( \hat{R}_{\hat{a}, \hat{b}}^{c, f; (L, m)} \) are first obtained for the four cases of \( e, f \in \{0, 1\} \) using the Cases specified by (30), with Case L1 (resp. L0) used for the \( e = 1 \) (resp. \( e = 0 \)) cases, and Case R1 (resp. R0) used for the \( f = 1 \) (resp. \( f = 0 \)) cases. Thus, in accordance with Table 5 we use \( T^L = Q^{(a,p-1)} \) for \( e = 1 \), \( T^L = R^{(a,p-1)} \) for \( e = 0 \), \( T^R = Q^{(b,p-1)} \) for \( f = 1 \), and \( T^R = R^{(b,p-1)} \) for \( f = 0 \). Comparison with (30) then shows that, in each case,

\[
(79) \quad T^L_i = (a_{i+1} + 1) \mod 2, \quad T^R_i = (b_{i+1} + 1) \mod 2,
\]

for \( 1 \leq i < p \). In particular, note that \( T^L_{p-1} = T^R_{p-1} = 0 \) in each case.

5.4.1. **System A.** For \( 1 < a, b < p \), consider the sequence of \( C \)-transforms governed by Cases L1 and R1.

Proceeding as in Section 3.2.1 Corollary 22 implies that for each \( i = 1, 2, \ldots, p - 1 \),

\[
(80) \quad \hat{R}_{a, b}^{c, f; p+1-i}(m_{i-1}, m_i) = \sum_{n_i \geq 0} q^{2m_i(m_i - m_n) + n_i T^L_i} \left[ n_i + \frac{1}{2} (m_i - T^L_i - T^R_i) \right]^{1/2} \hat{R}_{a, b}^{c, f; p-i}(m_i, m),
\]

where \( m = 2m_i + 2n_i - m_{i-1} \), after noting (79). In the \( i \)th case, we replace the variable \( m \) in (80) with

\[
(81) \quad m = m_{i+1} + \delta_{a, i-1} + \delta_{b, i-1}.
\]

We now express \( \hat{R}_{a, b}^{c, f; p-i}(m_i, m) \) in terms of \( \hat{R}_{a, b}^{c, f; 1-i}(m_i, m_{i+1}) \). Firstly, we obtain

\[
(82) \quad \hat{R}_{a, b}^{c, f; p-i}(m_i, m_{i+1} + \delta_{a, i-1} + \delta_{b, i-1}) = \hat{R}_{a, b}^{c, f; 1-i}(m_i, m_{i+1} + \delta_{a, i-1} + \delta_{b, i-1}),
\]

which follows from Lemma 18(3) in the \( i = a - 1 \) case because then \( a_{i+1} = 1, e_i = 1 \) and \( e_{i+1} = 0 \), and follows trivially in the \( i \neq a - 1 \) case because then \( e_{i+1} = e_i \). Then we obtain

\[
(83) \quad \hat{R}_{a, b}^{c, f; p-i}(m_i, m_{i+1} + \delta_{b, i-1}) = q^{2m_i(m_i - m_{i+1})} \hat{R}_{a, b}^{c, f; 1-i}(m_i, m_{i+1}),
\]

which follows from Lemma 18(5) in the \( i = b - 1 \) case because then \( b_{i+1} = 1, f_i = 1 \) and \( f_{i+1} = 0 \), and follows trivially in the \( i \neq b - 1 \) case because then \( f_{i+1} = f_i \).

Combining the \( i = 1, 2, \ldots, p - 1 \) cases of (80), (82) and (83) results in

\[
(84) \quad \hat{R}_{a, b}^{c, f; p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^{p-1}} q^{\frac{1}{2} m_{a-1} + \sum_{i=1}^{p-1} \frac{1}{2} m_i(m_i - m_{i+1}) + n T^L_i} \hat{R}_{a, b}^{c, f; 0, 1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2} (m_i - T^L_i - T^R_i) \right]^{1/2} \left( q^{2m_i(m_i - m_{i+1})} \right),
\]

where the values of \( m_i \) are obtained recursively from \( n = (n_1, n_2, \ldots, n_{p-1}) \) using (37). Because \( \hat{R}_{a, b}^{c, f; 1}(m_{p-1}, m_p) = \delta_{m_{p-1}, 0} \delta_{m_p, 0} \), by Lemma 20 and \( \left[ n_{p-1} - \frac{1}{2} (T^L_{p-1} + T^R_{p-1}) \right]^{1/2} = 1 \) (because \( T^L_{p-1} = T^R_{p-1} = 0 \)), this results in

\[
(85) \quad \hat{R}_{a, b}^{c, f; p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^{p-1}} q^{\frac{1}{2} m_{a-1} + \sum_{i=1}^{p-1} \frac{1}{2} m_i(m_i - m_{i+1}) + n T^L_i} \prod_{i=1}^{p-2} \left[ n_i + \frac{1}{2} (m_i - T^L_i - T^R_i) \right]^{1/2},
\]
with \( n \) constrained such that (34) yields \( m_{p-1} = m_p = 0 \). These constraints again yield (39).

Then, using Lemma (19)(2) in the form
\[
(94) \hat{\mathcal{R}}_{a,b}^{1,1;p}(m_0) = \sum_{m_1 \geq 0} \hat{\mathcal{R}}_{a,b}^{1,1;p}(m_0, m_1),
\]
proves (77).

5.4.2. System B. For \( 1 \leq a < p \) and \( 1 < b \leq p \), consider the sequence of \( C \)-transforms governed by Cases L0 and R1.

Proceeding as for system A, but using
\[
m = m_{i+1} + \delta_{i,p-a} + \delta_{i,b-1}
\]
instead of (31), we obtain the following analogue of (34):
\[
(87) \hat{\mathcal{R}}_{a,b}^{0,1;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^+} q^{-\frac{1}{2}m_{p-a} + \sum_{i=1}^{p-1} \frac{1}{2}m_i(n_i - m_{i+1}) + n_i} \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2}(m_i - T_i^{+} - T_i^{-}) \right]^{\prime} q^2,
\]
where the values of \( m_i \) are obtained recursively from \( n = (n_1, n_2, \ldots, n_{p-1}) \) using (43). Because \( \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) = \delta_{m_{p-1},0} \delta_{m_p,1} \), by Lemma (20) and \( \left[ n_{p-1} - \frac{1}{2}(T_{p-1}^{+} + T_{p-1}^{-}) \right]^{\prime} q^2 = 1 \), this results in
\[
(91) \hat{\mathcal{R}}_{a,b}^{0,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^+} q^{-\frac{1}{2}m_{p-a} + \sum_{i=1}^{p-1} \frac{1}{2}m_i(n_i - m_{i+1}) + n_i} \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2}(m_i - T_i^{+} - T_i^{-}) \right]^{\prime} q^2,
\]
with \( n \) constrained such that (34) yields \( m_{p-1} = 0 \) and \( m_p = 1 \). These constraints again yield (50). Then, summing (91) over \( m_1 \), using Lemma (19)(2), proves (77).

5.4.3. System C. For \( 1 \leq a, b < p \), consider the sequence of \( C \)-transforms governed by Cases L0 and R0.

Proceeding as for system A, but using
\[
m = m_{i+1} + \delta_{i,p-a} + \delta_{i,b-p}
\]
instead of (31), we obtain the following analogue of (34):
\[
(93) \hat{\mathcal{R}}_{a,b}^{0,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^+} q^{-\frac{1}{2}m_{p-a} + \sum_{i=1}^{p-1} \frac{1}{2}m_i(n_i - m_{i+1}) + n_i} \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2}(m_i - T_i^{+} - T_i^{-}) \right]^{\prime} q^2,
\]
where the values of \( m_i \) are obtained recursively from \( n = (n_1, n_2, \ldots, n_{p-1}) \) using (43). Because \( \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) = \delta_{m_{p-1},0} \delta_{m_p,0} \), by Lemma (20) and \( \left[ n_{p-1} - \frac{1}{2}(T_{p-1}^{+} + T_{p-1}^{-}) \right]^{\prime} q^2 = 1 \), this results in
\[
(92) \hat{\mathcal{R}}_{a,b}^{0,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^+} q^{-\frac{1}{2}m_{p-a} + \sum_{i=1}^{p-1} \frac{1}{2}m_i(n_i - m_{i+1}) + n_i} \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2}(m_i - T_i^{+} - T_i^{-}) \right]^{\prime} q^2,
\]
with \( n \) constrained such that (34) yields \( m_{p-1} = m_p = 0 \). These constraints again yield (50). Then, summing (92) over \( m_1 \), using Lemma (19)(2), proves (77).

5.4.4. System D. For \( 1 < a, b < p \), consider the sequence of \( C \)-transforms governed by Cases L1 and R0.

Proceeding as for system A, but using
\[
m = m_{i+1} + \delta_{i,a-1} + \delta_{i,b-p}
\]
instead of (31), we obtain the following analogue of (34):
\[
(94) \hat{\mathcal{R}}_{a,b}^{1,0;p}(m_0, m_1) = \sum_{n \in \mathbb{Z}_{\geq 0}^+} q^{-\frac{1}{2}m_{a-1} + \sum_{i=1}^{p-1} \frac{1}{2}m_i(n_i - m_{i+1}) + n_i} \hat{\mathcal{R}}^{1,1;1}_{1,1}(m_{p-1}, m_p) \prod_{i=1}^{p-1} \left[ n_i + \frac{1}{2}(m_i - T_i^{+} - T_i^{-}) \right]^{\prime} q^2,
\]
where the values of $m_i$ are obtained recursively from $n = (n_1, n_2, \ldots, n_{p-1})$ using (88). Because $R_{1,1}^{0,1}(m_{p-1}, m_p) = \delta_{m_{p-1},0} \delta_{m_{p-1},1}$, by Lemma 20 and $[n_{p-1}-\frac{1}{n_{p-1}} + T_{p-1}]^\prime$, this results in

$$R_{1,0}^{1,0}(p,m_0, m_1) = \sum_{n \in Z_{2}^{p-1}} q^{-\frac{1}{2}m_{a-1} + \sum_{i=1}^{p-1}\frac{1}{2}m_i(m_i-m_{i+1})+2} \prod_{i=1}^{p-2} \left( m_i + \frac{1}{2}(m_i - T_i - T_i^\prime) \right)^\prime q^2,$$

with $n$ constrained such that (83) yields $m_{p-1} = 0$ and $m_p = 1$. These constraints again yield (84). Then, summing (84) over $m_1$, using Lemma 19 (2), proves (77).

5.5. The modified q-binomials are not always necessary. Note 23 indicates that in the iterations (80), the modified form of the q-binomial is necessary only if $e_i = f_i = 1$ and $a_i + 1 = b_i + 1$ with these even. Thus, (77) requires the modified q-binomial only for some values of $a$ and $b$, and then only for certain $i$. On inspecting (80a) and (80b), we find that for $(e, f) = (1, 1)$, they are only required if $a = b > 2$, and then only for those $i$ for which $i < 2a - 1$. For (80e), (80f), (80i) and (80m), shows that for $(e, f) = (0, 1)$, they are only required if $a > 1$ and $b = p$, and then only for those $i$ for which $i > 2a - 1$. For (80k), (80f), (80g), shows that for $(e, f) = (0, 0)$, they are only required if $a > 1$ and $b > 1$, and then only for those $i$ for which $i > \max(p - a, p - b)$ and $i \equiv p$. Inspection of (80a) and (80m), shows that for $(e, f) = (1, 0)$, they are only required if $a = p$ and $b > 1$, and then only for those $i$ for which $i > p - b$ and $i \equiv p$.

Using (78), we can transfer this observation to Theorems 1 and 18.

Note 25. In Theorems 1 and 18 the modified q-binomial can be replaced by the standard binomial in many cases. Here, we list those cases where the modified q-binomial is required. For (82a), it is required only if $a = b \geq 2$, and then only for those $i < 2a - 1$ for which $i \neq 2a$. Consequently, for (10a), it is never required. For (82d), it is required only if $a + 1 > 0$ and $b = 1$, and then only for those $i > 2(t - a)$ for which $i \neq 2t$. Consequently, for (10b), it is required only if $a > 1$ and $r = t - \frac{1}{2}$, and then only for those $i > 2(t - a)$ for which $i \neq 2t$. For (82e), it is required only if $a > 1$ and $b > 1$, and then only for those $i > \max(2(t - a), 2(t - b))$ for which $i \neq 2t$. Consequently, for (10c), it is required only if $a > 1$ and $r > 0$, and then only for those $i > \max(2(t - a), 2(t - r))$ for which $i \neq 2t$. For (82k), it is required only if $a = t$ and $b > 1$, and then only for those $i > 2(t - b)$ for which $i \neq 2t$. Consequently, for (10d), it is required only if $a = t$ and $r > 1$, and then only for those $i > 2(t - r)$ for which $i \neq 2t$.

6. Proving the bosonic generating function for half-lattice paths

In this section, we prove the bosonic expressions for the generating functions $\hat{H}_{a,b}^{c,f,t}(L)$ for finite length half-lattice paths, that were stated in Theorem 15.

Let $t = \frac{1}{2} Z$ and $a, b, L \in Z_{\geq 0}$ with $1 \leq a \leq \lfloor t \rfloor$. In view of (85), the following results are independent of $e \in \{0, 1\}$. For $1 < b < \lfloor t \rfloor$, after noting that for each path $h \in \mathcal{H}_{a,b}^{c,f,t}(L)$, the pair $(h_{L-1}, h_{L-1/2})$ must be one of $(b - 1, b - 1/2)$, $(b - 1/2, b)$ or $(b + 1, b + 1/2)$, we obtain the recurrence relations

$$\hat{H}_{a,b}^{c,0,t}(L) = q^{-\frac{1}{4}} \hat{H}_{a,b-1}^{c,0,t}(L - 1) + \hat{H}_{a,b}^{c,0,t}(L - 1) + q^{\frac{1}{4}} \hat{H}_{a,b+1}^{c,1,t}(L - 1),$$

$$\hat{H}_{a,b}^{c,1,t}(L) = q^{\frac{1}{4}} \hat{H}_{a,b-1}^{c,0,t}(L - 1) + q^{\frac{1}{4}} \hat{H}_{a,b}^{c,0,t}(L - 1) + q^{-\frac{1}{4}} \hat{H}_{a,b+1}^{c,1,t}(L - 1).$$

These expressions also apply in the cases $b = 1$ and $b = \lfloor t \rfloor$ on imposing the boundary conditions

$$\hat{H}_{a,0}^{c,0,t}(L) = 0,$$

$$\hat{H}_{a,t+1/2}^{c,1,t}(L) = 0 \quad \text{for } t \in \mathbb{Z} + \frac{1}{2},$$

$$\hat{H}_{a,t}^{c,0,t}(L) + q^{\frac{1}{4}} \hat{H}_{a,t+1}^{c,1,t}(L) = 0 \quad \text{for } t \in \mathbb{Z}.$$

In addition, for $L = 0$, we have the initial condition

$$\hat{H}_{a,b}^{c,0,0}(0) = \hat{H}_{a,b}^{c,1,0}(0) = \delta_{a,b}.$$

Our strategy for proving (61a) is to show that the polynomials on its right side satisfy the relations (96) - (101). So, for $e, f \in \{0, 1\}$ and all $a, b, L \in Z$, define $\overline{H}_{a,b}^{c,f,t}(L)$ by

$$\overline{H}_{a,b}^{c,f,t}(L) = q^{\frac{1}{2}(a-b)(a-b+\frac{1}{2})+\frac{1}{2}fL} Y_{a,b}^{c,f,t}(L).$$
The recurrence relations (112a) and (112b) for the $q$-trinomial coefficients $U_n(L,d)$ imply that for $L > 0$,

\begin{align}
Y^{n,t}_{a,b}(L) &= q^{L+a-b-n}Y^{n,t}_{a,b-1}(L-1) + Y^{n,t}_{a,b}(L-1) + q^{L-a+b}Y^{n+1,t}_{a,b+1}(L-1), \\
Y^{n,t}_{a,b}(L) &= q^{a-b-n-1}Y^{n-1,t}_{a,b-1}(L-1) + Y^{n-1,t}_{a,b}(L-1) + q^{L-a+b}Y^{n,t}_{a,b+1}(L-1).
\end{align}

These imply that for all $a,b,L \in \mathbb{Z}$ with $L > 0$, eqns. (96) and (97) hold with $\tilde{H}^{c,f,t}_{a,b}(L)$ in place of $\tilde{H}^{c,f,t}_{a,b}(L)$.

From the definition (59),

\begin{align}
Y^{0,t}_{a,0}(L) &= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t-2a)}U_0(L,a-t\lambda) - q^{\lambda t(\lambda t-2a)}U_0(L,-a-t\lambda) \\
&= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t-2a)}U_0(L,a-t\lambda) - q^{\lambda t(\lambda t-2a)}U_0(L,a-t\lambda) \\
&= 0,
\end{align}

after, in the second term of the first line, changing $\lambda \to -\lambda$ and then using (111). Thus $\tilde{H}^{c,0,t}_{a,0}(L) = 0$.

Changing $\lambda \to -(\lambda + 1)$ in the second term of (59) and using (111) yields

\begin{align}
Y^{n,t}_{a,b}(L) &= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t+2a)}U_n(L,a-b-t\lambda) \\
&= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t+2a)}U_n(L,a-b-t\lambda) \\
&= 0,
\end{align}

In the case $t \in \mathbb{Z} + \frac{1}{2}$, after recalling that $t' = 2t + 1$, we immediately obtain $\tilde{H}^{c,1,t}_{a,t+1/2}(L) = 0$.

Making use of (106) for $b = t + n$ in both cases $n = 0, 1$, we obtain

\begin{align}
Y^{0,t}_{a,t}(L) &= q^{L+1-a+t}Y^{1,t}_{a,t+1}(L) \\
&= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t+2a)}U_0(L,a-t-t\lambda) - U_0(L,a-t-1-t\lambda) \\
&= \sum_{\lambda = -\infty}^{\infty} q^{\lambda t(\lambda t+2a)}U_1(L,a-t-1-t\lambda) - q^tU_1(L,a-t-t\lambda) \\
&= 0,
\end{align}

having, for the final line, made use of the $n = 0$ case of (113). It follows that $\tilde{H}^{c,0,t}_{a,t}(L) + q^{t+1/2}\tilde{H}^{c,1,t}_{a,t+1}(L) = 0$.

For $1 \leq a,b \leq \lfloor t \rfloor$, necessarily $|a \pm b| < t'$. The definition (59) then implies that $Y^{n,t}_{a,b}(0) = \delta_{a,b}$ and therefore $\tilde{H}^{c,0,t}_{a,b}(0) = \tilde{H}^{c,1,t}_{a,b}(0) = \delta_{a,b}$.

Expression (61a) is now proved because $\tilde{H}^{c,f,t}_{a,b}(L)$ is determined uniquely by (96)–(101) for $f \in \{0,1\}$, and $\tilde{H}^{c,f,t}_{a,b}(L)$ satisfies the same relations.

To prove (61b) and (61c), let $t \in \mathbb{Z} + \frac{1}{2}$ and $a,b,L \in \mathbb{Z}_{\geq 0}$ with $1 \leq a \leq \lfloor t \rfloor$. For $1 \leq b < \lfloor t \rfloor$, after noting that for each path $h \in \mathcal{H}_{a,b+1/2}(L + 1/2)$, either $h_L = b$ or $h_L = b + 1$, we obtain

\begin{align}
\tilde{H}^{c,1,t}_{a,b+1/2}(L + 1/2) &= \tilde{H}^{c,0,t}_{a,b}(L) + q^{1/2+1/2} \tilde{H}^{c,1,t}_{a,b+1}(L).
\end{align}

This relation also applies in the case $b = t - 1/2$ when $t \in \mathbb{Z} + \frac{1}{2}$, by virtue of (99). Expression (61b) then results from applying (61a) to each of the terms on the right side.

For $1 \leq b \leq \lfloor t \rfloor$, each path $h \in \mathcal{H}_{a,b+1/2}(L + 1/2)$ necessarily has $h_L = b$ because of the restriction on valley positions. Therefore

\begin{align}
\tilde{H}^{c,0,t}_{a,b+1/2}(L + 1/2) &= q^{t+1/2} \tilde{H}^{c,0,t}_{a,b}(L).
\end{align}

Expression (61c) then results from (61a).
7. Discussion

The half-lattice paths analysed in this paper provide a combinatorial model for the characters of the conformal minimal models $M(k, 2k \pm 1)$ that is compatible with the $\phi_{2,1}$ and $\phi_{1,5}$ perturbations of these theories. The similarity of these paths to those of the ABF models enable techniques used in those cases to be applied here to yield novel fermionic expressions for the generating functions of both finite and infinite length paths, the latter generating functions being the characters themselves. In particular, the four known fermionic expressions for the characters of $M(p, p+1)$ that are compatible with the $\phi_{1,3}$ perturbation have analogues for these $M(k, 2k \pm 1)$ characters. It is especially interesting to note that, as revealed by the form of these fermionic expressions, the characters of the latter are obtained from those of the former on restricting the excitations of the quasi-particles to alternate states.

In [20], it was shown that half-lattice paths, with a weighting function dual to that used here, may be used to provide a combinatorial model for the graded parafermion models $Z_k$ [49]. After mapping $q \to q^{-1}$, the $t \in \mathbb{Z}$ cases of Theorem 15 then apply to these models, yielding fermionic expressions that extend those given in [20].

In future work, we will also explore extending the half-lattice path combinatorial models for the $\phi_{2,1}$ and $\phi_{1,5}$ perturbations to other $M(p, p')$. Such an extension may be expected to admit a generalisation of the combinatorial $C$-transform which is analogous to that for the $\phi_{1,3}$ perturbation described by [14], and whose action corresponds to the renormalisation group flows described in [38] eqns. (3.6)–(3.8)]. We anticipate that this extension will involve a banding structure for the half-lattice paths akin to that introduced in [4] for the RSOS paths of [9]. However, our initial investigations seem to indicate, curiously, that such an extension is not possible for all pairs of $p$ and $p'$.

Finally, it is fair to stress that the description of the $M(k, 2k \pm 1)$ states by half-lattice paths is somewhat ad hoc. This link should eventually be framed in a broader context by exhibiting a direct relationship between the $M(k, 2k \pm 1)$ minimal models and an integrable lattice model with an underlying $A_2^{(2)}$ structure. With this in mind, the non-compact nature of the continuum limit of regime III of the Izergin-Korepin model [20] unravelled in [50] is intriguing.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Appendix A. Identities Involving $q$-Trinomials

A.1. Definition. The $q$-trinomials $U_n(L, d)$ may be defined by:

$$U_n(L, d) = \frac{(q)_L}{(q)_k(q)_{k+d}(q)_{L-2k-d}} \sum_{k=0}^{\lfloor (L-d)/2 \rfloor} q^{k(d-n)} (q)_L (q)_{k+d} (q)_{L-2k-d}$$

$$= q^{-\frac{1}{2}k(d-n)} \sum_{r=0}^{L-d} q^{(L-n-r)^2/2} (q)_L (q)_{L-r} (q)_{L+r}/2(q)_r,$$

where the final expression follows from the previous by setting $k = (L-d-r)/2$ so that $4k(k+2d-n) = (L-n-r)^2 - (d-n)^2$.

It follows from (111) that

$$U_n(L, -d) = q^{-nd} U_n(L, d).$$

A.2. Recurrence relations. Of the many recurrence relations enjoyed by the $q$-trinomial coefficients $U_n(L, d)$, the following four prove useful in the current work: for $L > 0$,

$$U_n(L, d) = q^{L-d} U_{n+1}(L, d-1) + U_n(L-1, d) + q^{d-n+1} U_n(L-1, d+1)$$

$$= q^{L-d} U_{n-1}(L, d-1) + U_{n+1}(L-1, d) + q^{d-n+1} U_{n+1}(L-1, d+1)$$

$$= q^{L-d} U_{n-1}(L, d-1) + U_n(L, d-1) + q^{L-d-n} U_{n+1}(L-1, d+1)$$

$$= U_{n-1}(L, d-1) + q^d U_{n-1}(L, 1-d) + q^{L-d} U_n(L, 1, d+1).$$

The first two of these are obtained after substituting $(q)_L$ in (110a) for, respectively,

$$(q)_L = (q)_{L-1}(1 - q^{2L-2k-d}) + q^{2k-d}(1 - q^{d+1} + q^{L-k}(1 - q^k))$$

and

$$(q)_L = (q)_{L-1}(1 - q^k) + q^k (1 - q^{L-2k-d}) + q^{L-k}(1 - q^{d+1}).$$

Identities (112c) and (112d) are obtained from (112a) and (112b) respectively, on exchanging $d \to -d$, and using (111).
The first of the next pair of identities results from combining (112b) and (112c) (or from exchanging $d \rightarrow -d$ in the first):

\[(114a) \quad q^{L+1-d} U_{n+1}(L, d-1) + U_n(L, d) = U_{n-1}(L, d-1) + q^{d} U_{n-1}(L, d),
\]

\[(114b) \quad q^{L-n} U_{n+1}(L, d+1) + U_n(L, d) = q^{d-n+1} U_{n-1}(L, d+1) + U_{n-1}(L, d).
\]

Exchanging $d \rightarrow d + 1$ in (114a) and subtracting (114b) multiplied by $q^n$ yields:

\[(115) \quad U_n(L, d + 1) + q^{L-d} U_{n+1}(L, d) = q^L U_{n+1}(L, d + 1) + q^n U_n(L, d) + (1 - q^n) U_{n-1}(L, d).
\]

A.3. $q$-trinomial limits. In the $n = 0$ case of (110a), we have the important $L \rightarrow \infty$ limit

\[(116) \quad \lim_{L \rightarrow \infty} U_0(L, d) = \sum_{k=0}^{\infty} q^{k(k+d)} \frac{1}{(q)_k(q)_{k+d}} = \frac{1}{(q)_\infty},
\]

having used the Durfee rectangle identity [51] eqn. following (1.6.4)].

To obtain $\lim_{L \rightarrow \infty} U_n(L, d)$ for $n > 0$, first take $L \rightarrow \infty$ in (112d). This gives:

\[(117) \quad \lim_{L \rightarrow \infty} U_{n+1}(L, d) = \lim_{L \rightarrow \infty} U_n(L, d-1) + q^d \lim_{L \rightarrow \infty} U_n(L, d).
\]

In particular, the $n = 0$ case yields:

\[(118) \quad \lim_{L \rightarrow \infty} U_1(L, d) = \frac{1 + q^d}{(q)_\infty}.
\]

A.4. $q$-trinomial references. The $q$-trinomials $U_n(L, d)$ were first defined in [48]. Identities (111), (112a), (112c), (116) and (115) also appear in [48] (as eqns. (2.15), (2.29), (2.28), (2.48), and (2.49) resp.). The $n = 0$ case of (115) (the only case of which we make use) appears in [52].

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