REPRESENTATIONS OF TAME QUIVERS AND AFFINE CANONICAL BASES

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Dedicated to Claus Michael Ringel on the occasion of his 60th birthday

Abstract. An integral PBW-basis of type $A_1^{(1)}$ has been constructed by Zhang [Z] and Chen [C] using the Auslander-Reiten quiver of the Kronecker quiver. We associate a geometric order to elements in this basis following an idea of Lusztig [L1] in the case of finite type. This leads to an algebraic realization of a bar-invariant basis of $U_q(\hat{sl}_2)$. For any affine symmetric type, we obtain an integral PBW-basis of the generic composition algebra, by using an algebraic construction of the integral basis for a tube in [DDX], an embedding of the module category of the Kronecker quiver into the module category of the tame quiver, and a list of the root vectors of indecomposable modules according to the preprojective, regular, and preinjective components of the Auslander-Reiten quiver of the tame quiver. When the basis elements are ordered to be compatible with the geometric order given by the dimensions of the orbit varieties and the extension varieties, we can show that the transition matrix between the PBW-basis and a monomial basis is triangular with diagonal entries equal to 1. Therefore we obtain a bar-invariant basis. By a orthogonalization for the PBW-basis with the inner product, we finally give an algebraic way to realize the canonical bases of the quantized enveloping algebras of all symmetric affine Kac-Moody Lie algebras.

0. Introduction

0.1 Let $U^+$ be the positive part of the quantized enveloping algebra of $U$ associated to a Cartan datum. For a finite type root system, Lusztig’s construction of the canonical basis of $U^+$ [L1] involves three ingredients. The first one can be understood as purely combinatorial. By applying Lusztig’s symmetries and the induced actions of the braid group on $U^+$, one may have a complete list of root vectors of $U^+$. Associated to each reduced expression of the longest element of the Weyl group, there is a PBW-basis of $U^+$ with a specific order and a monomial basis on the Chevalley generators such that the transition matrix between these two bases is triangular with diagonal entries equal to 1. (See [L1, 7.8-7.9].) The second is the quiver approach. Each isomorphism class of the Dynkin quiver corresponds to a PBW-type basis element $E^c$, $(c \in \mathbb{N}^{\Phi^+})$ of $U^+$. Now the representations of a fixed dimension vector of the quiver are the orbits of an algebraic group action on an affine variety. The geometric dimension of these orbits can be applied to give an order in $\{E^c|c \in \mathbb{N}^{\Phi^+}\}$. This ordered basis relates to a monomial basis by...
a triangular transition matrix with diagonal entries equal to 1. By a standard linear algebra method one can easily obtain the canonical basis. The third is the geometric approach by using perverse sheaves and intersection cohomology. There is also a different approach to construct the global crystal basis of $U^+$ in the Kashiwara’s work [K]. Now it is well known that Lusztig has generalized his geometric method to construct the canonical bases of $U^+$ for all infinite type (see [L2] and [L3]).

0.2 Although most knowledge on the canonical basis in finite type can be carried out in a pure combinatorial way, it is obvious to see that the definition of the canonical basis was introduced by Lusztig in a framework of representations of quivers. Specifically, Lusztig has extended the Gabriel’s theorem to build up a PBW type basis for $U^+$, which is ordered by the geometric properties of the corresponding orbit varieties. The representation category of a tame quiver has been completely described by a generalization of the Gabriel’s theorem and its Auslander-Reiten quiver (see [DR]). The objective of this paper is to provide a process to construct a PBW type basis and characterize the canonical basis of $U^+$ of affine type by using Ringel-Hall algebra and the knowledge of the representations of tame quivers. We hope that the approach we adopt here is closer to Lusztig’s original idea of [L1].

0.3 For infinite type root systems, there is no longest elements and the braid group action does not construct PBW-type basis. A natural question is to seek an algebraic construction of PBW type basis, monomial basis and the canonical basis, just like Lusztig did for finite type cases. For affine types, a PBW type basis was attained first by Beck, Chari and Pressley in [BCP] for the quantized enveloping algebra of untwisted affine type, and then was improved and extended by Beck and Nakajima in [BN] to all twisted and untwisted affine types. Their approach is to give the real root vectors by applying Lusztig’s symmetries on the generators and to construct the imaginary root vectors by using Schur functions in the Heisenberg generators; and then use these PBW-bases with the almost orthonormal property to obtain the crystal bases. However we like to point out that the order of the PBW-basis elements from the representations of tame quivers is different from theirs. A detailed analysis for this order enables us to construct the PBW-basis, also the monomial basis and a triangular transition matrix with diagonal entries equal to 1. Then we can use the standard linear method, which was used by Lusztig for finite type cases, to obtain the canonical basis.

0.4 In Section 1 we recall the definition of Hall algebras of quivers by Ringel and by Lusztig respectively, and point out that the two constructions coincide essentially for the representations of a quiver over a finite field. Section 2 presents the basic geometric properties of the orbit varieties and extension varieties for the representations of quivers. In Section 3 we construct an integral PBW basis of $A_1^{(1)}$ type by using the representations of the Kronecker quiver. Most results in this section are already known for some experts (see [Z] and [C]). The category mod $\Lambda$ of the Kronecker quiver has a strong representation-directed property [DR]. This enables us in Section 4 to arrange the positive roots in a special order. In addition, by the basic properties of the orbit varieties, we find a monomial basis whose transition matrix with the PBW basis is triangular with diagonal entries equal to 1. Section 5 is taken from [DDX], in which the integral basis and the canonical basis of $A_n^{(1)}$ type were given in terms of the nilpotent representations of the cyclic quivers. In Section 6 we consider the $\mathbb{Z}$-submodule of $U^+$ generated by $\langle u_M \rangle$ for $M$ being preprojective or preinjective. It is a $\mathbb{Z}$-subalgebra of $U^+$. An integral basis for this $\mathbb{Z}$-subalgebra can be listed in an order with respect to the representation-directed property of the
preprojective (resp. preinjective) component. We verify that the basis elements are products of images of Chevalley generators under the action of the sequences of Lusztig’s symmetries in an admissible order. So the situation in Section 6 resembles the construct of PBW-type basis in the finite type case. In Section 7, we show that the subalgebras corresponding to the preprojective component, preinjective component, non-homogeneous tubes, and an embedding of the module category of the Kronecker quiver can be put together, according to the representation-directed property of the tame quiver. This gives rise to an integral basis of $U^+$ over $\mathbb{Q}[v,v^{-1}]$. In Section 8, we again find a monomial basis, which has a unipotent triangular relation with the integral PBW type basis we obtained. But this needs a little more subtle analysis of the orbit varieties and the extension varieties. Finally, a bar-invariant basis $\{E^c|c \in \mathcal{M}\}$ of $U^+$ can be constructed in an elementary and algebraic way. The last section is new to an old version of the paper. By a detailed calculation of the inner product on the PBW-basis in the orthogonalization process using the properties of the Schur functions, we can answer Nakajima’s question in [N] affirmatively, that is, we show that the basis $\{E^c|c \in \mathcal{M}\}$, which is a modified form of the basis $\{E^c|c \in \mathcal{M}\}$, exactly equal to the canonical basis in [L2].

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1. Ringel-Hall algebras

1.1 A quiver $Q = (I, H, s, t)$ consists of a vertex set $I$, an arrow set $H$, and two maps $s, t : H \to I$ such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$.

Throughout the paper, $\mathbb{F}_q$ denotes a finite field with $q$ elements, $Q = (I, H, s, t)$ is a fixed connected quiver, and $\Lambda = \mathbb{F}_qQ$ is the path algebra of $Q$ over $\mathbb{F}_q$. By $\text{mod } \Lambda$ we denote the category of all finite dimensional left $\Lambda$-modules, or equivalently finite modules. It is well-known that mod $\Lambda$ is equivalent to the category of finite dimensional representations of $Q$ over $\mathbb{F}_q$. We shall simply identify $\Lambda$-modules with representations of $Q$.

1.2 Ringel-Hall algebra. Given three modules $L, M, N$ in mod $\Lambda$, let $g_{MN}^L$ denote the number of $\Lambda$-submodules $W$ of $L$ such that $W \simeq N$ and $L/W \simeq M$ in mod $\Lambda$. More generally, for $M_1, \cdots , M_t, L \in \text{mod } \Lambda$, let $g_{M_1 \cdots M_t}^L$ denote the number of the filtrations $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = L$ of $\Lambda$-submodules such that $L_i/L_{i-1} \simeq M_i$ for $i = 1, \cdots , t$. Let $v_q = \sqrt{q} \in \mathbb{C}$ and $\mathcal{P}$ be the set of isomorphism classes of finite dimensional nilpotent $\Lambda$-modules. Then the Ringel-Hall algebra $\mathcal{H}(\Lambda)$ of $\Lambda$ is by definition the $\mathbb{Q}(v)$-space with basis $\{u_{|M||M|} | [M] \in \mathcal{P}\}$ whose multiplication
is given by
\[ u[M]u[N] = \sum_{[L] \in \mathcal{P}} g^L_{MN} u[L]. \]

Note that \( g^L_{MN} \) depends only on the isomorphism classes of \( M, N \) and \( L \), and for fixed isomorphism classes of \( M, N \) there are only finitely many isomorphism classes \([L]\) such that \( g^L_{MN} \neq 0\). It is clear that \( \mathcal{H}(\Lambda) \) is associative \( \mathbb{Q}(v_q) \)-algebra with unit \( u_0 \), where 0 denotes the zero module.

The set of isomorphism classes of (nilpotent) simple \( \Lambda \)-modules is naturally indexed by the set \( I \) of vertices of \( Q \). Then the Grothendieck group \( G(\Lambda) \) of \( \text{mod} \Lambda \) is the free Abelian group \( \mathbb{Z}I \). For each nilpotent \( \Lambda \)-module \( M \), the dimension vector \( \dim M = \sum_{i \in I} (\dim M_i)i \) is an element of \( G(\Lambda) \). The Ringel-Hall algebra \( \mathcal{H}(\Lambda) \) is graded by \( \mathbb{N}I \), more precisely, by dimension vectors of modules.

The Euler form \( \langle -, - \rangle \) on \( G(\Lambda) = \mathbb{Z}I \) is defined by
\[
\langle \alpha, \beta \rangle = \sum_{i \in I} a_i b_i - \sum_{\rho \in H} a_{s(\rho)} b_{t(\rho)}
\]
for \( \alpha = \sum_{i \in I} a_i i \) and \( \beta = \sum_{i \in I} b_i i \) in \( \mathbb{Z}I \). For any nilpotent \( \Lambda \)-modules \( M \) and \( N \) one has
\[
\langle \dim M, \dim N \rangle = \dim_{\mathbb{Q}} \text{Hom}_\Lambda(M, N) - \dim_{\mathbb{Q}} \text{Ext}_\Lambda(M, N).
\]
The symmetric Euler form is defined as
\[
\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \quad \text{for} \quad \alpha, \beta \in \mathbb{Z}I.
\]
This gives rise to a symmetric generalized Cartan matrix \( C = (a_{ij})_{i,j \in I} \) with \( a_{ij} = (i, j) \). It is easy to see that \( C \) is independent of the field \( \mathbb{F}_q \) and the orientation of \( Q \).

The twisted Ringel-Hall algebra \( \mathcal{H}^*(\Lambda) \) is defined by setting \( \mathcal{H}^*(\Lambda) = \mathcal{H}(\Lambda) \) as \( \mathbb{Q}(v_q) \)-vector space, but the multiplication is defined by
\[
u[M] * u[N] = v^{\dim M \dim N} \sum_{[L] \in \mathcal{P}} g^L_{MN} u[L].
\]
Following \([R3]\), for any \( \Lambda \)-module \( M \), we denote \( \langle M \rangle = v^{-\dim M + \dim \text{End}_\Lambda(M)} u[M] \). Note that \( \{ \langle M \rangle \mid M \in \mathcal{P} \} \) a \( \mathbb{Q}(v_q) \)-basis of \( \mathcal{H}^*(\Lambda) \).

The \( \mathbb{Q}(v_q) \)-algebras \( \mathcal{H}^*(\Lambda) \) and \( \mathcal{H}(\Lambda) \) depends on \( q \). We will use \( \mathcal{H}^*_q(\Lambda) \) and \( \mathcal{H}_q(\Lambda) \) indicate the dependence on \( q \) when such a need arises.

### 1.3 A construction by Lusztig.

For any finite dimensional \( I \)-graded \( \mathbb{F}_q \)-vector space \( V = \bigoplus_{i \in I} V_i \), let \( \mathbb{E}_V \) be the subset of \( \oplus_{\rho \in H} \text{Hom}(V_{s(\rho)}, V_{t(\rho)}) \) defining nilpotent representations of \( Q \). Note that \( \mathbb{E}_V = \oplus_{\rho \in H} \text{Hom}(V_{s(\rho)}, V_{t(\rho)}) \) when \( Q \) has no oriented cycles. The group \( G_V = \prod_{i \in I} GL(V_i) \) acts naturally on \( \mathbb{E}_V \) by
\[
(g, x) \to g \bullet x = x' \quad \text{where} \quad x'_\rho = g_{t(\rho)} x_{\rho} g_{s(\rho)}^{-1} \quad \text{for all} \quad \rho \in H.
\]
Let \( \mathbb{C}_G(\mathbb{E}_V) \) be the space of \( G_V \)-invariant functions \( \mathbb{E}_V \to \mathbb{C} \). For \( \gamma \in \mathbb{N}I \), we fix a \( I \)-graded \( \mathbb{F}_q \)-vector space \( V_\gamma \) with \( \dim V_\gamma = \gamma \). There is no danger of confusion if we denote by \( \mathbb{E}_\gamma = \mathbb{E}_{V_\gamma} \) and \( G_\gamma = G_{V_\gamma} \). For \( \alpha, \beta \in \mathbb{N}I \) and \( \gamma = \alpha + \beta \), we consider the diagram
\[
\mathbb{E}_\alpha \times \mathbb{E}_\beta \xrightarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_\gamma.
\]
Here \( \mathbb{E}'' \) is the set of all pairs \( (x, W) \), consisting of \( x \in \mathbb{E}_\gamma \) and an \( x \)-stable \( I \)-graded subspace \( W \) of \( V_\gamma \) with \( \dim W = \beta \), and \( \mathbb{E}' \) is the set of all quadruples \( (x, W, W', W'') \), consisting of \( (x, W) \in \mathbb{E}'' \) and two invertible linear maps \( R': \mathbb{F}_q^\beta \to W \) and \( R'': \mathbb{F}_q^\alpha \to \mathbb{F}_q^\alpha / W \). The maps are defined in
obvious way as follows: \( p_2(x, W, R', R'') = (x, W) \), \( p_3(x, W) = x \), and \( p_1(x, W, R', R'') = (x', x'') \), where \( x_\rho R_\rho'(\rho) = R_\rho'(\rho) x_\rho' \) and \( x_\rho R_\rho''(\rho) = R_\rho''(\rho) x_\rho'' \) for all \( \rho \in H \).

For any map \( p : X \to Y \) of finite sets, \( p^* : \mathbb{C}(Y) \to \mathbb{C}(X) \) is defined by \( p^*(f)(x) = f(p(x)) \) and \( p_1 : \mathbb{C}(X) \to \mathbb{C}(Y) \) is defined by \( p_1(h)(y) = \sum_{x \in p^{-1}(y)} h(x) \) (integration along the fibers).

Given \( f \in C_G(E_\alpha) \) and \( g \in C_G(E_\beta) \), there is a unique \( h \in C_G(E^\prime) \) such that \( p_2(h) = p_1^*(f \times g) \). Then define

\[
 f \circ g = (p_3)_!(h) \in C_G(E_\gamma).
\]

Let

\[
 m(\alpha, \beta) = \sum_{i \in I} a_i b_i + \sum_{\rho \in H} a_{s(\rho)} b_{t(\rho)}.
\]

We again define the multiplication in the \( \mathbb{C} \)-space \( K = \oplus_{\alpha \in M} C_G(E_\alpha) \) by

\[
 f \ast g = v_q^\cdot m(\alpha, \beta) f \circ g
\]

for all \( f \in C_G(E_\alpha) \) and \( g \in C_G(E_\beta) \). Then \( (K, \ast) \) becomes an associative \( \mathbb{C} \)-algebra.

**Convention.** Although we are working over finite \( \mathbb{F}_q \), we will regularly use \( G_V \) and \( E_V \) for the algebraic group and the algebraic variety which are defined over \( \mathbb{F}_q \) and use the features of algebraic geometry without introducing extra notations, i.e., the set of \( \mathbb{F}_q \)-rational points and the algebraic variety are denoted by the same notation. This should not cause any confusion and in particular, the concept of \( G_V \)-orbits will be consistent in both cases due to Lang's theorem for this group \( G_V \) acting on \( E_V \). For \( M \in E_V \), we will use \( M \) to denote the representation of \( Q \) on \( V \) defined by \( M \).

For \( M \in E_\alpha \), let \( O_M \subset E_\alpha \) be the \( G_\alpha \)-orbit of \( M \). We take \( 1_{[M]} \in C_G(V_\alpha) \) to be the characteristic function of \( O_M \), and set \( f_{[M]} = v_q^{-\dim O_M} 1_{[M]} \). We consider the subalgebra \( (L, \ast) \) of \( (K, \ast) \) generated by \( f_{[M]} \) over \( \mathbb{Q}(v_q) \), for all \( M \in E_\alpha \) and all \( \alpha \in N \). In fact \( L \) has a \( \mathbb{Q}(v_q) \)-basis \( \{ f_{[M]} | M \in E_\alpha, \alpha \in N \} \), since we have the relation \( 1_{[M]} \ast 1_{[N]}(W) = g_{MN}^W \) for any \( W \in E_\gamma \).

**Proposition 1.1** The linear map \( \varphi : (L, \ast) \to \mathcal{H}^*(\Lambda) \) defined by

\[
 \varphi(f_{[M]}) = \langle M \rangle, \quad \text{for all } [M] \in \mathcal{P}
\]

is an isomorphism of the associative \( \mathbb{Q}(v_q) \)-algebras.

**Proof.** Note that \( \varphi \) is a linear isomorphism. For \( [M], [N] \in \mathcal{P} \) with \( \dim M = \alpha \) and \( \dim N = \beta \), since \( 1_{[M]} \circ 1_{[N]} = \sum_{[L] \in \mathcal{P}} g_{MN}^L 1_{[L]} \) in \( L \), we have

\[
 f_{[M]} \ast f_{[N]} = \sum_{[L] \in \mathcal{P}} v_q^{-\dim O_M - \dim O_N + \dim O_M \cdot \dim O_N \cdot m(\alpha, \beta) + \dim O_L} g_{MN}^L f_{[L]}.
\]

Note that \( \dim O_M = \dim G_\alpha - \dim \text{End}_\Lambda(M) \) and \( \dim G_\alpha + \beta - \dim G_\alpha - \dim G_\beta = \langle \alpha, \beta \rangle + m(\alpha, \beta) \). In \( \mathcal{H}^*(\Lambda) \) we have

\[
 \langle M \rangle \ast \langle N \rangle = v_q^{-\dim M + \dim \text{End}_\Lambda(M) - \dim N + \dim \text{End}_\Lambda(N) + \langle \alpha, \beta \rangle} u_{[M]} \circ u_{[N]}
\]

\[
 = \sum_L v_q^{-\dim \text{End}_\Lambda(M) + \dim \text{End}_\Lambda(N) - \dim \text{End}_\Lambda(L) + \langle \alpha, \beta \rangle} g_{MN}^L \langle L \rangle
\]

\[
 = \sum_L v_q^{-\dim \text{End}_\Lambda(M) + \dim \text{End}_\Lambda(N) - \dim \text{End}_\Lambda(L) + \langle \alpha, \beta \rangle} g_{MN}^L \langle L \rangle.
\]

\[\square\]
1.4 The free abelian group $G(\Lambda) = \mathbb{Z}I$ with the symmetric Euler form $(-,-)$ defined in 1.2 is a Cartan datum in the sense of Lusztig [L5]. Associated to $(\mathbb{Z}I, (\cdot, \cdot))$ is the Drinfeld-Jimbo quantized enveloping algebra $U = U^- \otimes U^0 \otimes U^+$ defined over $\mathbb{Q}(t)$, where $t$ is transcendental over $\mathbb{Q}$. It is generated by $E_i, \alpha_i, K_i^\pm (i \in I)$ with respect to the quantum Serre relations. Let $Z = \mathbb{Z}[t, t^{-1}]$. The Lusztig form $U^+_Z$ of $U^+$ is the $Z$-subalgebra in $U^+$ generated by $E_i^{(m)} = \frac{E_i^m}{m!}$ $(m \geq 0$ and $i \in I$). For $v = v_q \in \mathbb{C}$, let $Z_v$ be the subring of $C$ as the image of $Z$ under the map $Z \to \mathbb{C}$ with $t \mapsto v$. Let $C^*(\Lambda)_Z$ be the $Z_v$-subalgebra of $H^*(\Lambda)$ generated by $u_i^{(sm)} = \frac{u_i^{(m)} [Z]}{m!} (i \in I)$, where

$$[n] = \frac{t^n - t^{-n}}{t - t^{-1}}, \quad [n]! = \prod_{r=1}^{n}[r], \quad \left[ \frac{n}{r} \right] = \frac{[n]!}{[r]![n-r]!}$$

and $[n]_v \in Z_v$ is the image of $[n]$ in $Z_v$.

It follows from the works of Ringel [R1], Green [G], and Sevenhant-Van den Bergh [SV] that $C^*(\Lambda)_Z$ is isomorphic to $U^+_Z \otimes_Z Z_v$ by sending $u_i^{(sm)}$ to $E_i^{(m)}$.

We will denote $C^*(\Lambda)_Z$ for $U^+_Z$ and call it the integral generic composition algebra. In fact, following Ringel’s point of view, $Z$ can be identified with the subring of $\prod_q Z_{v_q}$ generated by $t^{+1} = (v_q^{+1})$ and $C^*(\Lambda)_Z$ as a $Z$-subalgebra of $\prod_q H^*_\Lambda$ generated by $(u_i^{(sm)} |_{S_i \otimes \mathbb{F}_q})$, $m \geq 1$. Here the product is taken over all $q \ (\text{though infinitely many will be enough}).$

In this paper, computations in $\prod_q H^*_\Lambda$ will performed in each component. When an expression in each component is written as an element of $\mathbb{Z}[u_q, v_q^{-1}]$ with coefficients in $\mathbb{Z}$ independent of the choice of the field $\mathbb{F}_q$, we say that the expression is invariant (or generic) as $\mathbb{F}_q$ varies. In this case replacing $v_q$ by $t$ will get a formula in $\prod_q H^*_\Lambda$. We will not repeat this replacement each time and simply write $v = v_q$ and call it generic in this expression. In stead of write $t$, we will also use $v$ and this will not cause any confusion.

There is bar involution $\overline{\cdot} : U^+ \to U^+$ (of $\mathbb{Z}$-algebras) defined by $\overline{t} = t^{-1}$, $\overline{E_i} = E_i$ and $\overline{E_i^{(m)}} = E_i^{(m)}$. Then $U^+_Z = \overline{U^+_Z}$.

1.5 In general, if we take a special value $v = \sqrt{q}$ for the finite field $\mathbb{F}_q$, it is easy to see that

**Lemma 1.2** Given any monomial $m$ of $u_i^{(m)}$, $i \in I, m \in \mathbb{N}$ we have $m = \sum_{M \in \mathcal{P}} f_{M,q}(M)$ in $H^*(\Lambda)$ with $f_{M,q} \in Z_v$. Then for each $M$, there is an integer $b$ such that $u^b f_{M,q} \in \mathbb{Z}[v]$ (the subring of algebraic integers) and $b$ is independent of $\mathbb{F}_q$. □

2. The variety of representations

We need slightly more knowledge about the geometry of representations of quivers over algebraically closed field $k = \mathbb{F}_q$. In this section we only consider finite quivers $Q$ without oriented cycles. Take $\Lambda = kQ$ and all Hom and Ext are taken in $\Lambda$-mod.

2.1 For $\alpha \in NI$, the $I$-graded $k$-vector space $\oplus_{i \in I} k^{\alpha_i}$ defines the affine algebraic $k$-variety $E_\alpha$ on which the algebraic group $G_\alpha$ acts in a similar way as in 1.3. For any $x \in E_\alpha$, we have the corresponding representation $M(x)$ of $Q$ over $k$. The following properties are well-known (see[CB]).

**Lemma 2.1** For any $\alpha \in NI$ and $M \in E_\alpha$, we have

1. $\dim E_\alpha - \dim O_M = \dim \operatorname{End}(M) - (\alpha, \alpha)/2 = \dim \operatorname{Ext}^1(M, M)$.
2. $O_M$ is open in $E_\alpha$ if and only if $M$ has no self-extension.
Lemma 2.2 Given any \( \alpha, \beta \in \mathbb{N} \), if \( \mathcal{A} \subset \mathbb{E}_\alpha \) and \( \mathcal{B} \subset \mathbb{E}_\beta \) are irreducible algebraic varieties and are stable under the action of \( G_\alpha \) and \( G_\beta \) respectively, then \( \mathcal{A} \star \mathcal{B} \) is irreducible and stable under the action of \( G_{\alpha+\beta} \), too. Moreover,

\[
\text{codim} \mathcal{A} \star \mathcal{B} = \text{codim} \mathcal{A} + \text{codim} \mathcal{B} - \langle \beta, \alpha \rangle + r,
\]

where \( 0 \leq r \leq \min \{ \dim_k \text{Hom}(M(y), M(x)) | y \in \mathcal{B}, x \in \mathcal{A} \} \).

2.2 For any \( \alpha, \beta \in \mathbb{N} \), we consider the diagram if algebraic \( k \)-varieties

\[
\mathbb{E}_\alpha \times \mathbb{E}_\beta \xrightarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta}
\]

defined by a similar way as in 1.3. It follows from the definition that \( \mathcal{A} \star \mathcal{B} = p_3 p_2 (p_1^{-1}(\mathcal{A} \times \mathcal{B})) \). Thus we have \( \mathbb{A} \star \mathbb{B} \subset \mathbb{A} \times \mathbb{B} \) since \( p_1 \) is a locally trivial fibration (see Lemma 2.3). For any \( M \in \mathbb{E}_\alpha, N \in \mathbb{E}_\beta \) and \( L \in \mathbb{E}_{\alpha+\beta} \) we define

\[
\mathcal{Z} = p_2 p_1^{-1}(\mathcal{O}_M \times \mathcal{O}_N), \quad \mathcal{Z}_{L,M,N} = \mathcal{Z} \cap p_3^{-1}(L).
\]

Then it follows from \([\text{L1}]\) that

Lemma 2.3 For the diagram above and \( M \in \mathbb{E}_\alpha, N \in \mathbb{E}_\beta \) and \( L \in \mathbb{E}_{\alpha+\beta} \), we have the following properties.

1. The map \( p_2 \) is a principal \( G_\alpha \times G_\beta \) fibration.
2. The map \( p_1 \) is a locally trivial fibration with smooth connected fibres of dimension

\[
\sum_{i \in I} a_i^2 + \sum_{i \in I} b_i^2 + m(\alpha, \beta).
\]

3. The map \( p_3 \) is proper.
4. The variety \( \mathcal{Z} \) is smooth and irreducible of dimension

\[
\dim \mathcal{Z} = \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + m(\alpha, \beta).
\]

5. If \( L \) is an extension of \( M \) by \( N \), then

\[
\dim(\mathcal{O}_L) \leq \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + m(\alpha, \beta)
\]

6. If \( \mathcal{O}_L \) is dense in \( p_3 \mathcal{Z} \), then

\[
\dim(\mathcal{O}_L) = \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + m(\alpha, \beta) - \dim \mathcal{Z}_{L,M,N}.
\]

7. Assume that \( \text{Ext}(M, N) = 0 \) and \( \text{Hom}(N, M) = 0 \). If \( M' \in \mathcal{O}_M \) and \( N' \in \mathcal{O}_N \) such that either \( M' \in \mathcal{O}_M \setminus \mathcal{O}_M \) or \( N' \in \mathcal{O}_N \setminus \mathcal{O}_N \), then \( X \in \mathcal{O}_{M \oplus N} \setminus \mathcal{O}_{M \oplus N} \) for all \( X \in \mathcal{O}_{M'} \setminus \mathcal{O}_{N'} \). In particular, \( \dim \mathcal{O}_X < \dim \mathcal{O}_{M \oplus N} \).
As a consequence of Lemma 2.2 we have

**Lemma 2.4** Given any two representations $M$ and $N$ of $Q$ over $k$, if $\text{Ext}(M, N) = 0$, then $\overline{O}_M * \overline{O}_N = \overline{O}_{M \oplus N}$, i.e., $\overline{O}_{M \oplus N}$ is open and dense in $\overline{O}_M * \overline{O}_N$. □

**Lemma 2.5** Let $M, N, X \in \text{mod } \Lambda$. Then $\mathcal{O}_X$ is open in $\mathcal{O}_M * \mathcal{O}_N$ if and only if $\mathcal{O}_X$ is open in $\overline{O}_M * \overline{O}_N$. In that case for any $Y \in \overline{O}_M * \overline{O}_N$ we have $\dim \mathcal{O}_Y \leq \dim \mathcal{O}_X$.

**Proof.** This follows from $\mathcal{O}_X \subseteq \mathcal{O}_M * \mathcal{O}_N \subseteq \overline{O}_M * \overline{O}_N$ and Lemma 2.2. □

### 3. The integral bases from the Kronecker quiver

Most results in this section can be found in [Z] and [C] while others can be found in [BK]. For completeness, we give some proofs here.

**3.1** Let $\mathbb{F}_q$ be the finite field with $q$ elements and $Q$ the Kronecker quiver with $I = \{1, 2\}$ and $H = \{\rho_1, \rho_2\}$ such that $s(\rho_1) = s(\rho_2) = 2$ and $t(\rho_1) = t(\rho_2) = 1$. Let $\Lambda = \mathbb{F}_q Q$ be the path algebra. It is known that the structure of the preprojective and preinjective components of $\text{mod } \Lambda$ is different with that of $\text{mod } kQ$.

The set of dimension vectors of indecomposable representations is

$$\Phi^+ = \{(l + 1, l), (m, m), (n, n + 1) | l \geq 0, m \geq 1, n \geq 0\}.$$ 

The dimension vectors $(n + 1, n)$ and $(n, n + 1)$ correspond to preprojective and preinjective indecomposable representations respectively and are called real roots. For each real root $\alpha$, there is only one isoclass of indecomposable representation with dimension vector $\alpha$ which will be denoted by $V_\alpha$. Define a total order $\prec$ on $\Phi^+$ by

$$(1, 0) \prec \cdots \prec (m + 1, m) \prec (m + 2, m + 1) \prec \cdots \prec (k, k) \prec (k + 1, k + 1) \prec \cdots \prec (n + 1, n + 2) \prec (n, n + 1) \prec \cdots \prec (0, 1).$$

The strong representation-directed property implies that there is no non-zero homomorphism from an indecomposable module of dimension vector $\alpha$ to an indecomposable module of dimension vector $\beta$ if $\beta \prec \alpha$. This property will be used frequently in the computation.

Any $\Lambda$-module is given by the date $(V_1, V_2; \sigma, \tau)$, where $V_1$ and $V_2$ are finite dimensional vector space over $\mathbb{F}_q$, $\sigma$ and $\tau$ are $\mathbb{F}_q$-linear maps from $V_2$ to $V_1$.

**Proposition 3.1.** The isomorphism classes of the regular quasi-simple modules in $\text{mod } \Lambda$ are indexed by $\text{spec}(\mathbb{F}_q[x])$. That is, each regular quasi-simple module is isomorphic to $(V_1, V_2; \sigma, \tau)$, where $V_1 = V_2 = \mathbb{F}_q[x]/(p(x))$ for an irreducible polynomial $p(x)$ in $\mathbb{F}_q[x]$, $\sigma$ is the identity map and $\tau$ is given by the multiplication by $x$, except $(\mathbb{F}_q, \mathbb{F}_q; 0, 1)$ which corresponds to the zero ideal.

**3.2** In this section, let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional $\Lambda$-modules, $\mathcal{H} = \mathcal{H}_q$ be the Ringel-Hall algebra of $\Lambda$ over $\mathbb{Q}(v)$, where $v^2 = q$, and $\mathcal{H}^*$ be the twisted form of $\mathcal{H}$. If $d \in NI$ be a dimension vector, we set in $\mathcal{H}$

$$R_d = \sum_{\substack{[M] \in \mathcal{P}, M \text{ regular} \atop \dim M = d}} u_{[M]}.$$
For an element \( x = \sum_{[M] \in \mathcal{P}} c_{[M]} u_{[M]} \in \mathcal{H} \), we call \( u_{[M]} \) to be a (non-zero) term of \( x \) if \( c_{[M]} \neq 0 \). Furthermore,
\[
R(x) = \sum_{[M] \in \mathcal{P}, M \text{ regular}} c_{[M]} u_{[M]}
\]
is called the regular part of \( x \). According to our notation, we denote \( u_\alpha = u_{[V_\alpha]} \) for \( \alpha = (n-1,n) \) or \((n,n+1)\) being real roots.

Let \( \alpha_1 = (1,0) \) and \( \alpha_2 = (0,1) \) be the simple root vectors. The orientation of \( Q \) implies \( \langle \alpha_1, \alpha_2 \rangle = 0 \) and \( \langle \alpha_2, \alpha_1 \rangle = -2 \). Thus for \( \delta = (1,1) \) we have \( \langle \delta, \alpha_1 \rangle = -1, \langle \alpha_1, \delta \rangle = 1, \langle \delta, \alpha_2 \rangle = 1 \) and \( \langle \alpha_2, \delta \rangle = -1 \).

### Lemma 3.3

In this section, the multiplication in \( \mathcal{H} \) will be simply written as \( xy \) instead of \( x \circ y \). The following can be computed easily as in [Z].

**Lemma 3.2.** Let \( i \) and \( j \) be two positive integers. Then
\[
u_{(i-1,j)} u_{(i,i-1)} = R(u_{(j-1,j)} u_{(i,i-1)}) + q^{i+j-2} u_{(i,i-1)} u_{(j-1,j)}.
\]

**Lemma 3.3**

\[
R_\delta = u_{(0,1)} u_{(1,0)} - u_{(1,0)} u_{(0,1)};
\]

\[
u_{(n+1,n)} = \frac{1}{q+1} (R_\delta u_{(n,n-1)} - qu_{(n,n-1)} R_\delta),
\]

\[
u_{(n,n+1)} = \frac{1}{q+1} (u_{(n-1,n)} R_\delta - q R_\delta u_{(n-1,n)}).
\]

**Lemma 3.4** Let \( i \) and \( j \) be two positive integers and \( n = i + j - 1 \). Then
\[
R(u_{(j-1,j)} u_{(i,i-1)}) = R(u_{(n-1,n)} u_{(1,0)}) = R(u_{(0,1)} u_{(n,n-1)}).
\]

**Lemma 3.5** Let \( m, n \geq 1 \). Then
\[
u_{(m-1,m)} R_{n\delta} = \sum_{0 \leq i \leq n} \frac{q^i - q^{n+1}}{1 - q} R_{i\delta} u_{(m+n-i-1,m+n-i)},
\]

\[
R_{n\delta} u_{(m,m-1)} = \sum_{0 \leq i \leq n} \frac{q^i - q^{n+1}}{1 - q} u_{(m+n-i,m+n-i-1)} R_{i\delta}.
\]

### 3.4

We will introduce a new set of elements in \( \mathcal{H}^* \) to describe a basis that resembles PBW basis for enveloping algebra of a Lie algebra. We give here some quantum commutative relations in \( \mathcal{H} \) and in \( \mathcal{H}^* \). We define (cf. 1.2)

\[
E_{(n+1,n)} = \langle u_{(n+1,n)} \rangle = v^{-2n} u_{(n+1,n)},
\]

\[
E_{(n,n+1)} = \langle u_{(n,n+1)} \rangle = v^{-2n} u_{(n,n+1)}.
\]

We will call \( E_1 = E_{(1,0)}, E_2 = E_{(0,1)} \) the Chevalley generators. For \( n \geq 1 \), define in \( \mathcal{H}^* \)

\[
\tilde{E}_{n\delta} = E_{(n-1,n)} \ast E_1 - v^{-2} E_1 \ast E_{(n-1,n)}.
\]

In the following we give a sequence of computations we will need. Most of them are known.

**Lemma 3.6** \( \tilde{E}_{n\delta} = v^{-3n+1} R(u_{(n-1,n)} u_{(0,1)}) \).
Lemma 3.10 Using Lemma 3.2, Lemma 3.4, and Lemma 3.6, we have

Proof. By taking $u_1 = u_{(1,0)}$ we have

$$
\tilde{E}_{n\delta} = v^{-2(n-1)} v^{((n-1)\delta+\alpha_2,\alpha_1)} u_{(n-1,n)} u_1 - v^{-2} v^{(\alpha_1, (n-1)\delta+\alpha_2)} u_1 u_{(n-1,n)}
$$

$$
= v^{-3n+1} (u_{(n-1,n)} u_1 - v^{2(n-1)} u_1 u_{(n-1,n)})
$$

$$
= v^{-3n+1} R(u_{(n-1,n)} u_1) \quad \text{by Lemma 3.2.} \quad \square
$$

Lemma 3.7 In $\mathcal{H}^*$ we have

$$
[E_{(n+1,n)}, \tilde{E}_\delta] = [2] v E_{(n+2,n+1)},
$$

$$
[E_{(n,n+1)}, \tilde{E}_\delta] = [2] v E_{(n+1,n+2)}.
$$

Proof. We only check the first equation. By definition and Lemma 3.3, we have

$$
[\tilde{E}_\delta, E_{(n+1,n)}] = v^{-2(n+1)} v^{(\delta,\delta\delta+\alpha_1)} R_{\delta} u_{(n+1,n)} - v^{-2(n+1)} v^{(\delta+\alpha_2,\alpha_1)} u_{(n+1,n)} R_{\delta}
$$

$$
= v^{-2(n+1)} v^{-1} ((q+1) u_{(n+2,n+1)} + q u_{(n+1,n)} R_{\delta}) - v^{-2(n+1)} v u_{(n+1,n)} R_{\delta}
$$

$$
= v^{-2(n+1)} (v + v^{-1}) u_{(n+2,n+1)} = [2] v E_{(n+2,n+1)}. \quad \square
$$

Lemma 3.8 $E_{(2,1)} \ast E_1 = v^2 E_1 \ast E_{(2,1)}$ and $E_2 \ast E_{(1,2)} = v^2 E_{(1,2)} \ast E_2$.

Proof. Let $M = V_{(1,0)} \oplus V_{(2,1)}$. Then $E_{(2,1)} \ast E_1 = v^{-2} v^{(\delta+\alpha_1,\alpha_1)} u_{(2,1)} u_1 = v^2 u_\beta$ and $E_1 \ast E_{(2,1)} = v^{-2} v^{(\alpha_1,\delta+\alpha_1)} u_1 u_{(2,1)} = u_\beta$. This proves the first equality. The second equality follows from a similar computation. \quad \square

Lemma 3.9 For any non-negative integers $r$ and $s$, we have in $\mathcal{H}^*$

$$
\tilde{E}_{(r+s+1)\delta} = E_{(r,r+1)} \ast E_{(s+1,s)} - v^{-2} E_{(s+1,s)} \ast E_{(r,r+1)}.
$$

Proof. Using Lemma 3.2, Lemma 3.4, and Lemma 3.6, we have

$$
E_{(r,r+1)} \ast E_{(s+1,s)} - v^{-2} E_{(s+1,s)} \ast E_{(r,r+1)}
$$

$$
= v^{-3(r+s)-2} u_{(r,r+1)} u_{(s+1,s)} - v^{-r+s-2} u_{(s+1,s)} u_{(r,r+1)}
$$

$$
= v^{-3(r+s)-2} R(u_{(r,r+1)} u_{(s+1,s)}) + q^{r+s} u_{(s+1,s)} u_{(r,r+1)} - v^{-r+s-2} u_{(s+1,s)} u_{(r,r+1)}
$$

$$
= v^{-3(r+s)-2} R(u_{(r,r+1)} u_{(s+1,s)}) = v^{-3(r+s)-2} R(u_{(r,r+1)} u_{(s+1,s)}) = \tilde{E}_{(r+s+1)\delta}. \quad \square
$$

Lemma 3.10 There exist $a_h^{(r)}(t) \in \mathbb{Z}[t,t^{-1}]$ for all $r \in \mathbb{N} \setminus \{0\}$ and $h \in \{0,1,\ldots,\lfloor r/2 \rfloor\}$ such that for all $n > m$ in $\mathbb{N}$,

$$
E_{(n+1,n)} \ast E_{(m+1,m)} = \sum_{h=0}^{\lfloor n-m \rfloor} a_h^{(n-m)}(v) E_{(m+h+1,m+h)} \ast E_{(n-h+1,n-h)}
$$

$$
E_{(m,m+1)} \ast E_{(n,n+1)} = \sum_{h=0}^{\lfloor n-m \rfloor} a_h^{(n-m)}(v) E_{(n-h+1,n-h)} \ast E_{(m+h,m+h+1)}
$$
Proof. Using the strong representation-directed property, we have

\[ E_{(n+1,m)} \ast E_{(m+1,m)} = v^{-2(n+m)}p^{(n\delta+\alpha_1,m\delta+\alpha_1)}u_{(n+1,m)}u_{(m+1,m)} = v^{-3n-m+1} \sum_{h=0}^{[a-m]} g_{V_{(n+1,m)}V_{(m+1,m)}}^{M_h} u[M_h]. \]

where \( M_h = V_{(n-h+1,n-h)} \oplus V_{(n+h+1,n+h)}. \) Since \( n-h \geq m+h, \) the strong representation-directed property again implies

\[ E_{(m+h+1,m+h)} \ast E_{(n-h+1,n-h)} = v^{-3m-n-2h+1} g_{V_{(m+h+1)}}^{M_h} u[M_h]. \]

Thus \( g_{V_{(m+h+1)}}^{M_h} u[M_h] = v^{3m+n-1+2k} = E_{(m+h+1,m+h)} \ast E_{(n-h+1,n-h)}. \) Substitution implies that \( a_h^{(r)}(t) = t^{-2(r-h)} \in \mathbb{Z}[t,t^{-1}]. \) To verify the second identity, one uses the strong representation-directed property again and carry out similar computation. The computation will give the same \( a_h^{(r)}(t) \) works for both identities. \( \square \)

For \( k \geq 0, \) we inductively define

\[ E_{0\delta} = 1, \quad E_{k\delta} = \frac{1}{[k]} \sum_{s=1}^{k} v^{s-k} \tilde{E}_{s\delta} \ast E_{(k-s)\delta}. \]

Lemma 3.11 We have \( E_{k\delta} = v^{-2k}R_{k\delta}. \)

Proof. If \( k = 1, E_{\delta} = \tilde{E}_{\delta} = v^{-2}R_{\delta}. \) We assume that the assertion is true for all numbers \( t < k. \) Then using Lemma 3.6, and \([Z] \) (Lem 3.7, Thm 4.1, Lem 4.7), we have

\[ E_{k\delta} = \frac{1}{[k]} \sum_{s=1}^{k} v^{s-k} v^{-3s+1} R(u_{(s-1,s)}u_1) \ast v^{-2(k-s)}R_{(k-s)\delta} \]

\[ = \frac{1}{[k]} \sum_{s=1}^{k} v^{-3k+1} R(u_{(s-1,s)}u_1) \ast R_{(k-s)\delta} \]

\[ = \frac{1}{[k]} \sum_{s=1}^{k} v^{-3k+1} a_s(R_\delta, R_{2\delta}, \cdots, R_{s\delta}) R_{(k-s)\delta} \]

\[ = v^{-3k+1} \frac{1 - q^k}{1 - q} R_{k\delta} = v^{-2k}R_{k\delta}. \quad \square \]

Lemma 3.12 For \( m, n \in \mathbb{N} \) we have in \( \mathcal{H}^* \)

\[ E_{n\delta} \ast E_{(m+1,m)} = \sum_{k=0}^{n} [n+1-k] E_{(m+n+1-k,m+n-k)} \ast E_{k\delta}; \]

\[ E_{(m,m+1)} \ast E_{n\delta} = \sum_{k=0}^{n} [n+1-k] E_{k\delta} \ast E_{(m+n-k,m+n-k+1)}. \]
Remark. The formulae in the lemmas are unchanged when we vary relations in the above lemmas imply that the zero with the product taken with respect to the order given in 3.1 and there are only finitely many non-

It contains the divided powers $E_{m,n}^{(s,t)}$, $s,t \in \mathbb{N}$, of the Chevalley generators. We have obtained an integral $Z_v$-basis of $L_v$ consisting of the monomials

$$\{ \prod_{m \geq 0} E_{(m+1,m)}^{(s,m)} \prod_{k \geq 1} E_{k\delta}^{(r_k)} \prod_{n \geq 0} E_{(n+1,n)}^{(t_n)} \mid m \geq 0, n \geq 0, s_m \geq 0, t_n \geq 0, k \geq 1, r_k \geq 0 \}$$

with the product taken with respect to the order given in 3.1 and there are only finitely many non-zero $s_m, t_n$, and $r_k$ in each monomial. This follows easily from the facts: (1) the commutation relations in the above lemmas imply that the $Z_v$-span of the monomials above is closed under the multiplication in $\mathcal{H}^*$ and that $L$ contains all monomials we defined above; (2) those monomials are linearly independent over $\mathcal{Z}$ (even over $\mathbb{Q}(v)$) by the definition of Ringel-Hall algebras.

3.5 Let $L_v$ be the $Z_v = \mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{H}^*_q$ generated by the set

$$\{ E_{(m+1,m)}^{(s,m)}, E_{k\delta}^{(r_k)}, E_{(n+1,n)}^{(t_n)} \mid m \geq 0, n \geq 0, s \geq 1, t \geq 1, k \geq 1 \}.$$

The formulae in the lemmas are unchanged when we vary $v = \sqrt{q}$. The statement of the lemmas can be stated in $\prod_q \mathcal{H}^*_q$ with $v$ replaced by $t = (v_q)$ in $\prod_q \mathbb{Z}_q$ and $E_{s,*}$ replaced by $E_{(s,s),q}^{(s,s),q}$. We then denote $L$ as the $Z = \mathbb{Z}[t, t^{-1}]$-algebra with a $Z$-basis consisting of the monomials described above.

As remarked in 1.4, Lusztig’s integral $Z$-form $C^*_Z$, which we called the generic composition algebra, can be viewed as a $Z$-subalgebra of $\prod_q \mathcal{H}^*_q$ by the Ringel-Green theorem (see [G],[R1]). Using this identification, we can view $C^*_Z$ as a $Z$-subalgebra of $L$. In the rest of this section, we will construct a $Z$-basis of $C^*_Z$.

For any $n > m \geq 0$, let $P_{(n,m)}$ (resp. $I_{(n,m)}$) be an isomorphism class of preprojective (resp. preinjective) modules with $\dim P_{(n,m)} = (n, m)$ (resp. $\dim I_{(n,m)} = (m, n)$). In the following formulas, the summation is taken over all nonzero preprojective and preinjective representations of the indicated dimension vectors.

**Lemma 3.13** In the following formulas all $P$ and $I$ are non-zero.

1. $E_{2n}^{(s)} \ast E_{1}^{(s+n+1)} = E_{n+1,n} + \sum_{1 \leq l \leq n} v^{-l-1} E_{n-l+1,n-l} \ast E_{l\delta}$

$$+ \sum_{0 \leq l \leq n-1 \atop p \geq 1, s \geq 0, t \geq 0 \atop s+t+l+(p-1)=n} v^{-\dim \operatorname{End}(P)-\dim \operatorname{End}(I)} v^{-p(l+t)-(s+l)(p-1)} \langle P_{(s+p,s)} \rangle \ast E_{l\delta} \ast \langle I_{(t,t+p-1)} \rangle;$$
(2) $E_2^{(s(n+1))} * E_1^{(sn)} = E_{(n,n+1)} + \sum_{1 \leq l \leq n} v^{-l-1} E_l \delta * E_{(n-l,n-l+1)}$

\[+ \sum_{0 \leq l \leq n-1} (v - \dim \text{End}(P) - \dim \text{End}(I)) v^{-p(l+s)+(t+l)(p-1)} (P_{(s+p-1,s)}) * E_l \delta * \langle I(t,t+p) \rangle;\]

(3) $E_2^{(sn)} * E_1^{(sn)} = E_{n \delta} + \sum_{0 \leq l \leq n-1, p \geq 1} (v - \dim \text{End}(P) - \dim \text{End}(I)) v^{-p(s+2l+t)} (P_{(s+p,s)}) * E_l \delta * \langle I(t,t+p) \rangle,$

\[\text{Proof.} \text{ We only verify (1) and others can be verified in a similar way. We have the following relation in } \mathcal{H} \text{ (see [R3]).}\]

\[u_n^2 u_1^{n+1} = \psi_n(q) \psi_{n+1}(q)(u_{n+1,n}) + \sum_{1 \leq l \leq n} u_{(n-l+1,n-l)} R_l \delta + \sum_{0 \leq l \leq n-1, p \geq 1} u_{p} \delta u_{[l]}(p);\]

where $P$ is a non-zero preinjective module with $\dim P = (s + p, s)$ and $I$ is a non-zero preinjective module with $\dim I = (t, t + p - 1)$ and

\[\psi_n(q) = \frac{(1 - q) \cdots (1 - q^n)}{(1 - q)^n}.\]

Then by a routine calculation according to the relation in 3.2, we have the relation (1). \qed

3.6 Note that the dimensions of $P, I, \text{End} I$ and $\text{End} P$ over $\mathbb{F}_q$ are invariant as $\mathbb{F}_q$ varies. By induction using Lemma 3.13, the set

\[\{ E_{(m+1,m)}, E_{k \delta}, E_{(n,n+1)} | m \geq 0, n \geq 0, k \geq 1 \}\]

is contained in $\mathcal{C}_Z^*$. If $M$ is indecomposable preprojective or preinjective, then, by [R3],

\[\langle u_{|M|} \rangle^{(s)} = \langle u_{|M| \oplus s} \rangle \in \mathcal{C}^* (\Lambda) \] for any $s \geq 1$.

Using this and the strong representation-directed property on preprojectives and preinjectives, we have, for $0 \leq n_1 < n_2 < \cdots < n_t$ and $s_1, s_2, s_t \geq 1$, we have

\[E_{(n_1+1,n_1)}^{(s_1)} * \cdots * E_{(n_t+1,n_t)}^{(s_t)} = v^a [\oplus_{i=1}^{n_1} V_{(n_i+1,n_i)}^{|s_i|}],\]

\[E_{(n_1,n_1+1)}^{(s_1)} * \cdots * E_{(n_t,n_t+1)}^{(s_t)} = v^b [\oplus_{i=1}^{n_1} V_{(n_i,n_i+1)}^{|s_i|}],\]

where $a$ and $b$ are integers depends only on the sequences $n_1 < \cdots < n_t$ and $s_1, s_2, s_t \geq 1$. Hence the subset

\[\{ E_{(m+1,m)}, E_{k \delta}, E_{(n,n+1)} | m \geq 0, n \geq 0, s \geq 1, t \geq 1, k \geq 1 \}\]

is also contained in $\mathcal{C}_Z^*$. Therefore, $\mathcal{L} = \mathcal{C}_Z^*$. Let $\mathcal{P}(n)$ the set of all partitions of $n$. Recall that there are no nontrivial extensions between homogeneous regular representation. For any $w = (w_1, w_2, \cdots, w_m) \in \mathcal{P}(n)$, we define

\[E_{w \delta} = E_{w_1 \delta} * E_{w_2 \delta} * \cdots * E_{w_n \delta}.\]

Proposition 3.14 The set

\[\{ \langle P \rangle * E_{w \delta} \langle I \rangle | P \in \mathcal{P} \text{ preprojective}, w \in \mathcal{P}(n), I \in \mathcal{P} \text{ preinjective}, n \in \mathbb{N} \} \]
is a $\mathcal{Z}$-basis of $\mathcal{C}_Z^\ast$. \hfill \qed

Remarks. (1) It has been proved by Zhang in [Z] that these monomials are $\mathbb{Q}(v)$-bases of $U^+$, then improved by Chen in [C] that they are $\mathcal{Z}$-bases of $\mathcal{U}_Z^\ast$.

(2) It is not difficult to see that the root vectors provided here exactly correspond to the root vectors of $U_q(\hat{sl}_2)$ provided by Damiani in [Da] and by Beck in [Be].

(3) It can be proved that the set in Proposition 3.14 is an integral basis of $\mathcal{C}^\ast$ over $\mathcal{A} = \mathbb{Q}[v,v^{-1}]$ by an easier way, see the proofs of Proposition 7.2 and 7.3 below.

4. A bar-invariant basis from the Kronecker quiver

With the PBW type basis constructed for $\mathcal{C}_Z^\ast$ We now can construct a bar-invariant basis following the approach in [L1, 7.8-7.11].

4.1 In this section, we work in $\mathcal{C}^\ast = \mathcal{C}_Z^\ast$. Recall from 3.1 that $\Phi^+$ is the positive root system of $\hat{sl}_2$. A function $c : \Phi^+ \to \mathbb{N}$ is called support-finite, if $c(\alpha) \neq 0$ only for finitely many $\alpha \in \Phi^+$.

Let $\mathbb{N}^{\Phi^+}_f$ be the set of all support-finite $\mathbb{N}$-valued functions. We will use the order in $\Phi^+$ given in 3.1.

For $c \in \mathbb{N}^{\Phi^+}_f$, if $\{\alpha \in \Phi^+ | c(\alpha) \neq 0\} = \{\beta_1 \prec \beta_2 \prec \cdots \prec \beta_k\}$, we set

$$E^c = E^{(c(\beta_1))}_{\beta_1} \ast E^{(c(\beta_2))}_{\beta_2} \ast \cdots \ast E^{(c(\beta_k))}_{\beta_k},$$

where $E^{(c(\beta_i))}_{\beta_i} \triangleq E^{c(\beta_i)}_{\beta_i}$ if $\beta_i = m\delta$. Then Proposition 3.14 is equivalent to the statement:

The set $\{E^c | c \in \mathbb{N}^{\Phi^+}_f\}$ is a $\mathcal{Z}$-basis of $\mathcal{C}^\ast$.

For $d = (d_1, d_2) \in \mathbb{N}^2_f$, we denote

$$E(d) = E^{(sd_2)}_2 \ast E^{(sd_1)}_1.$$  

Similarly we defined

$$E(c) = E(c(\beta_1)\beta_1) \ast E(c(\beta_2)\beta_2) \ast \cdots \ast E(c(\beta_k)\beta_k).$$

Note that $E(c) \in \mathcal{C}^\ast$ since it is a monomial on the Chevalley generators $E_1$ and $E_2$ in the form of divided powers. Moreover, by definition, we $E(d) = E(d)$. Thus $E(c) = E(c)$.

4.2 The rest of this section is devoted to giving a triangular relation between the PBW-basis and the monomial basis.

For any $c \in \mathbb{N}^{\Phi^+}_f$ we assume that $E^c = \langle P \rangle \ast E_{w\delta} \ast \langle I \rangle$, where $P$ is a preprojective module and $I$ is a preinjective module. For any partition $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$, write

$$E_{w\delta} = E_{w_{1}\delta} \ast E_{w_{2}\delta} \ast \cdots \ast E_{w_{m}\delta} = \sum_V a^\omega_V \langle V \rangle$$

over a fixed field $\mathbb{F}_q$. We choose $\square V_{w\delta}$ to be a module such that

$$\dim \mathcal{O}_{V_{w\delta}} = \max \{\dim \mathcal{O}_V | a^\omega_V \neq 0\}. $$

Set

$$V_c = P \oplus V_{\omega\delta} \oplus I, \quad \mathcal{O}_c = \mathcal{O}_{P \oplus V_{\omega\delta} \oplus I}. $$

For any $c \in \mathbb{N}^{\Phi^+}_f$ and any real root $\alpha \in \Phi^+$, define $u_{c(\alpha)\alpha} = u_{[V_{\alpha} \oplus \cdots \oplus V_{\alpha}]}$, where $V_{\alpha}$ is the indecomposable representation with $\dim V_{\alpha} = \alpha$.

\footnote{This selection is not unique, in fact we may require that $V_{m\delta}$ is defined over the prime field and absolutely indecomposable in a homogeneous tube.}
Lemma 4.2 For any $c \in \mathbb{N}_f^+$ and any real root $\beta \in \Phi^+$, we have in $C^*$
\[ E(c(\beta)\beta) = \langle u_{c(\beta)\beta} \rangle + \sum_{\begin{subarray}{c} c' \in \mathbb{N}_f^+ \\ \dim O_{c'} < \dim O_{c(\beta)\beta} \end{subarray}} v^{-\lambda(c')} E_{c'}, \]
where $\lambda(c') \in \mathbb{N}$.

Proof. Let $c(\beta)\beta = (m, n)$. In $H_q$ (for any fixed $F_q$) we have
\[ u_2^m u_1^n = \psi_m(q)\psi_n(q) \sum_{\dim N = (m, n)} u_{[N]}, \]
By Lemma 2.1
\[ u_2^m u_1^n = \psi_m(q)\psi_n(q)u_{[V_\beta \oplus \cdots \oplus V_\beta]} + \psi_m(q)\psi_n(q) \sum_{\dim N = (m, n)} u_{[P]} R_{I\delta} u_{[I]}, \]
where $P$ is preprojective, $I$ is preinjective, $\dim P + I\delta + \dim I = (m, n)$, and $\dim O_{P \oplus V_\beta \oplus I} \subset \dim O_{c(\beta)\beta}$. Although the number of terms of $R_{I\delta} = \sum_{[M]} u_{[M]}$ in $H_q$ depends on $q$, Lemma 3.11 shows that $R_{I\delta}$ has a generic form in $C^*_Z$ with each component in $H_q$ being $R_{I\delta}$. Then in $C^*_Z$
\[ u_2^{(m)} u_1^{(n)} = \frac{v^m v^{n-2mn}}{m! n!} u_2^m u_1^n = v^{m^2 + n^2 - 2mn} u_{[V_\beta \oplus \cdots \oplus V_\beta]} + v^{m^2 + n^2 - 2mn} \sum_{\dim N = (m, n)} u_{[P]} R_{I\delta} u_{[I]} = \langle u_{c(\beta)\beta} \rangle + \sum_{\begin{subarray}{c} c' \in \mathbb{N}_f^+ \\ \dim O_{c'} < \dim O_{c(\beta)\beta} \end{subarray}} v^{-\lambda(c')} E_{c'}. \]

Lemma 4.3 Let $\alpha, \beta \in \Phi^+$. be real roots and $\alpha \prec \beta$. We have in $C^*$
\[ E(\alpha) \ast E(\beta) = \langle u_{[V_\alpha]} \rangle \ast \langle u_{[V_\beta]} \rangle + \sum_{\begin{subarray}{c} c \in \mathbb{N}_f^+ \\ \dim O_c < \dim O_{V_\alpha \oplus V_\beta} \end{subarray}} h_\alpha^\beta E_{c}, \]
where $h_\alpha^\beta \in Z$.

Proof. By Lemma 3.13 we have
\[ E(\alpha) = \langle u_\alpha \rangle + \sum_{\begin{subarray}{c} c' \in \mathbb{N}_f^+ \\ \dim O_{c'} < \dim O_{V_\alpha} \end{subarray}} v^{-\lambda(c')} E_{c'}, \]
\[ E(\beta) = \langle u_\beta \rangle + \sum_{\begin{subarray}{c} c'' \in \mathbb{N}_f^+ \\ \dim O_{c''} < \dim O_{V_\beta} \end{subarray}} v^{-\lambda(c'')} E_{c''}. \]
Since $\text{Hom}_A(V_\beta, V_\alpha) = \text{Ext}_A(V_\alpha, V_\beta) = 0$ and $\dim Z_{V_\alpha \oplus V_\beta, V_\alpha, V_\beta} = 0$, by Lemma 2.3(7),
\[ \dim O_c < \dim O_{V_\alpha \oplus V_\beta} \]
for any extension $V_c$ of $V_c'$ by $V_c''$ with the property:
\[ O_c \subseteq \overline{O_{V_\alpha \ominus V_\alpha}} \subseteq O_{V_\alpha}, \text{ or } O_c \subseteq \overline{O_{V_\beta \ominus V_\beta}} \subseteq O_{V_\beta}. \]
Therefore, the conclusion follows from Proposition 3.14. □
Lemma 4.4 Let $\alpha = (n+1, n)$, $\beta = (l, l) = l\delta$, and $\gamma = (m, m+1)$ be in $\Phi^+$. The for all $s \geq 1$
\begin{align}
E(s\alpha) \cdot E(\beta) &= \langle u_{[a\gamma]} \rangle \cdot E_{l\delta} + \sum_{c \in \mathbb{N}_1^+} \dim \mathcal{O}_c \cdot \dim \mathcal{O}_{[a\alpha] \oplus [b\delta]} \cdot h(c) E^c, \\
E(\beta) \cdot E(s\gamma) &= E_{l\delta} \cdot \langle u_{[a\gamma]} \rangle + \sum_{c' \in \mathbb{N}_1^+} \dim \mathcal{O}_{c'} \cdot \dim \mathcal{O}_{[a\gamma] \oplus [b\gamma]} \cdot h(c') E^{c'}. 
\end{align}
Here $h(c'), h(c) \in \mathbb{Z}$.

**Proof.** Using Lemma 2.3(7) and Lemma 3.13(3), the proof is same as that of Lemma 4.3. \hfill \Box

Lemma 4.5 Let $V$ be an indecomposable regular module with $\dim V = n\delta$. $M = P \oplus M' \oplus I$ with $P \neq 0, M', I \neq 0$ are respectively preprojective, regular and preinjective modules and $\dim M = n\delta$. Then $\dim \mathcal{O}_V > \dim \mathcal{O}_M$.

**Proof.** By Lemma 2.1(1), we only need to prove that $\dim \text{End}(V) < \dim \text{End}(M)$. By Proposition 3.1, we have $\dim \text{End}(V) = n$. Suppose
\[ P = P_1 \oplus P_2 \oplus \cdots \oplus P_r, \quad I = I_1 \oplus I_2 \oplus \cdots \oplus I_t, \]
where $P_j$ and $I_j (j \geq 1)$ are respectively indecomposable preprojective and preinjective modules with $\dim P_j = (n_j + 1, n_j)$, $\dim I_j = (m_j, m_j + 1)$, and $\dim M' = s\delta$. Thus $r = t$ and $n = \sum_{j=1}^{t} (n_j + 1) + s + \sum_{j=1}^{t} m_j$. Note that
\begin{align*}
\dim \text{End}(P) &\geq t, \quad \dim \text{End}(M') = s, \quad \dim \text{End}(I) \geq t, \\
\dim \text{Hom}(P, M') = (\dim P, \dim M') = st, \quad \dim \text{Hom}(M', I) = st, \\
\dim \text{Hom}(P, I) = (\dim P, \dim I) = t(\sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j).
\end{align*}
Using the direct sum decomposition of $M$, one computes
\[ \dim \text{End}(M) \geq t + s + t + st + t(\sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j) = 2t + \sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j > n. \]
This implies that $\dim \mathcal{O}_V > \dim \mathcal{O}_M$. \hfill \Box

Lemma 4.6 Let $n \geq 1, m \geq 1$. Then
\[ E(n\delta) \cdot E(m\delta) = E_{n\delta} \cdot E_{m\delta} + \sum_{c \in \mathbb{N}_1^+} \dim \mathcal{O}_c \cdot \dim \mathcal{O}_{(n, m)\delta} \cdot h(c) E^c, \]
where $V_{(n, m)\delta}$ is defined in 4.2 and $h(c) \in \mathbb{Z}$.

**Proof.** By Lemma 3.13(3), we have
\begin{align*}
E(n\delta) &= E_{n\delta} + \sum_{P \neq 0, I \neq 0} \nu(l(P) * E_{l\delta} * (I)) \langle P \rangle \cdot E_{l\delta} \cdot \langle I \rangle, \\
E(m\delta) &= E_{m\delta} + \sum_{P \neq 0, I \neq 0} \nu(l(P) * E_{l\delta} * (I)) \langle P \rangle \cdot E_{l\delta} \cdot \langle I \rangle,
\end{align*}
where $l(\langle P \rangle * E_{l\delta} * (I)) \in \mathbb{Z}$.
We then have

\[ E(n\delta) \ast E(m\delta) = E_{n\delta} \ast E_{m\delta} + \sum h(c)E^c. \]

To prove the lemma, it is sufficient to prove that \( V_c \), which is defined in 4.2, is decomposable. This is easy to see from the structure of the AR-quiver of Kronecker quiver. \( \square \)

**Remark.** By Lemma 4.6 we can get

\[ E(\omega \delta) = E(\omega_1 \delta) \ast \cdots \ast E(\omega_n \delta) = E_{\omega \delta} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\omega \delta}} h(c)E^c, \]

where \( h(c) \in \mathbb{Z} \).

Let \( \varphi : \mathbb{N}^\Phi_f \to \mathbb{N}^2 \) be defined by \( \varphi(c) = \sum_{\alpha \in \Phi} c(\alpha)\alpha \). Then for any \( d \in \mathbb{N}^2 \), \( \varphi^{-1}(d) \) is a finite set. We define a (geometric) order in \( \varphi^{-1}(d) \) as follows: \( c' \preceq c \) if and only if \( c' = c \) or \( c' \neq c \) but \( \dim \mathcal{O}_{c'} < \dim \mathcal{O}_c \).

From Lemma 2.3(7) and above lemmas, we may summarize our results of this subsection as follows resembling [L1, 7.8].

**Proposition 4.7** For any \( c \in \mathbb{N}_f^\Phi \), we have

\[ E(c) = \sum_{c' \in \varphi^{-1}(\varphi(c))} h_{c'}^c E^{c'}. \]

such that (1) \( h_{c'}^c \in \mathbb{Z} \), (2) \( h_{c}^c = 1 \), (3) if \( h_{c'}^c \neq 0 \) then \( c' \preceq c \), (4) \( h(E(c)) = E(c) \).

For any \( c, c' \in \mathbb{N}_f^\Phi \) we define \( \omega_{c'}^c \in \mathbb{Z} \) such that

\[ E^c = \sum_{c' \in \mathbb{N}_f^\Phi} \omega_{c'}^c E^{c'}. \]

The following Proposition resembles [L1, Prop. 7.9].

**Proposition 4.8** \( \omega_{c'}^c = 1 \) and, if \( \omega_{c'}^c \neq 0 \) and \( c' \neq c \) then \( c'' < c \).

**Proof.** Using \( E(c) = E(c) \) and the fact that \( \{ E^c \mid c \in \mathbb{N}_f^\Phi \} \) is a \( \mathbb{Z} \)-bases of \( C^* \), we have

\[ h_{c''}^{c'} = \sum_{c'} h_{c'}^{c''} \omega_{c''}, \quad \text{for } c, c'' \in \varphi^{-1}(d). \]

By Lemma 4.5, the matrices \( (h_{c''}^{c'}) \) as well as \( (h_{c''}^{c''}) \), where the index set is \( \varphi^{-1}(d) \), are triangular with 1 on diagonal. Hence, by the equation above, the matrix \( (\omega_{c''}^{c''}) \) has the same property. \( \square \)

Consider the bar involution \( (\quad) : C^* \to C^* \). For any \( c \in \mathbb{N}_f^\Phi \),

\[ E^c = \overline{E^c} = \sum_{c'} \omega_{c'}^c E^{c'} = \sum_{c', c''} \omega_{c''}^{c'} \omega_{c''}^{c'} E^{c''}. \]

implies the orthogonal relation

\[ \sum_{c'} \omega_{c'}^c \omega_{c'}^c = \delta_{c c''}. \]

Therefore one can solve uniquely the system of equations

\[ \zeta_{c}^e = \sum_{c' \leq c' < c} \omega_{c' c'}^{c''} \zeta_{c''}^c. \]

\(^2\)This is independent of the choices of \( V_{c'} \) and \( V_c \) as in 4.2 such that \( \mathcal{O}_{V_{c'}} \subseteq \mathcal{O}_{V_c} \setminus \mathcal{O}_{V_c}. \)
with unknowns $\zeta_e \in \mathbb{Z}[v^{-1}]$, $e' \preceq c$ and $e', c \in \varphi^{-1}(d)$, such that

$$\zeta_e = 1 \quad \text{and} \quad \zeta_e \in v^{-1}\mathbb{Z}[v^{-1}] \quad \text{for all} \quad e' < c.$$ 

For any $d \in \mathbb{N}^2$ and $c \in \varphi^{-1}(d)$, we set

$$\mathcal{E}^c = \sum_{e' \in \varphi^{-1}(d)} \zeta_e E^{e'} \quad \text{and} \quad \mathcal{J} = \{\mathcal{E}^c | c \in \varphi^{-1}(d), d \in \mathbb{N}^2\}.$$ 

Let

$$\mathcal{L} = \text{span}_{\mathbb{Z}[v^{-1}]}\{\mathcal{E}^c | c \in \mathbb{N}_f^+\}.$$ 

We verify the following two properties of $\mathcal{J}$. The first is

$$\overline{\mathcal{E}^c} = \sum_{e'} \zeta_e \overline{E^{e'}} = \sum_{e'} \zeta_e \sum_{e''} \omega_{e'e''} E^{e''} = \sum_{e''} \left(\sum_{e'} \zeta_e \omega_{e'e''}\right) E^{e''} = \sum_{e''} \zeta_e E^{e''} = \mathcal{E}^c.$$ 

So the elements $\mathcal{E}^c$ are bar-invariant. The second, obviously the set $\mathcal{J}$ is a $\mathbb{Z}[v^{-1}]$-basis of the lattice $\mathcal{L}$. Therefore we have

**Proposition 4.9** The set $\mathcal{J}$ is a basis of $\mathcal{C}_\mathbb{Z}^*$, which satisfies that $\overline{\mathcal{E}^c} = \mathcal{E}^c$ and $\pi(\mathcal{E}^c) = \pi(\mathcal{E}^c)$ for any $\mathcal{E}^c \in \mathcal{L}$, where $\pi$ is the canonical projection $\mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$.

### 5. The integral and canonical bases arising from a tube

The main results we present in this section are taken from [DDX], in which the canonical bases of $U_q(sl_n)$ and $U_q(gl_n)$ are constructed by a linear algebra method from the category of finite dimensional nilpotent representations of a cyclic quiver, i.e., from a tube. However in an preliminary version of the present paper we assumed the existence and the structure of Lusztig’s canonical basis for the composition algebra of a tube from [L3] and [VV].

**5.1** Let $\Delta = \Delta(n)$ be the cyclic quiver with vertex set $\Delta_0 = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$ and arrow set $\Delta_1 = \{i \to i + 1 | i \in \mathbb{Z}/n\mathbb{Z}\}$. We consider the category $\mathcal{T} = \mathcal{T}(n)$ of finite dimensional nilpotent representations of $\Delta(n)$ over $\mathbb{F}_q$. For the reason of the shape of its Auslander-Reiten quiver, $\mathcal{T}(n)$ is called a tube of rank $n$. Let $S_i, i \in \Delta_0$ be the irreducible objects in $\mathcal{T}(n)$ and $S_i[l]$ the (unique) absolutely indecomposable object in $\mathcal{T}(n)$ with top $S_i$ and length $l$. Note that $S_i[l]$ is independent of $q$. Again in this section, we let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathcal{T}(n)$, $\mathcal{H}$ the Ringel-Hall algebra of $\mathcal{T}(n)$, $\mathcal{H}^*$ the twisted Ringel-Hall algebra, and $\mathcal{L}$ the Lusztig form of the Hall algebra of $\mathcal{T}(n)$ (cf. 1.3). Because the Hall polynomials always exist in this case (see [R2]), we may regard the algebras $\mathcal{H}$, $\mathcal{H}^*$ and $\mathcal{L}$ in their generic form. So they all are defined generically over $\mathbb{Q}(t)$, where $t$ is an indeterminate. By Proposition 1.1, we may identify $\mathcal{L}$ with $\mathcal{H}^*$ via the morphism $\varphi$.

In this section, all properties we obtain are generic and independent of the base field $\mathbb{F}_q$, although the computations will be performed over $\mathbb{F}_q$ (for each $q$). We will omit the subscript $q$ for simplicity. Since the number $n$ is fixed, sometimes it is omitted too, e.g., $\mathcal{T} = \mathcal{T}(n)$.

**5.2** Let $\Pi$ be the set of $n$-tuples of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)})$ with each component $\pi^{(i)} = (\pi_1^{(i)} \geq \pi_2^{(i)} \geq \ldots)$ being a partition of an integer. For each $\pi \in \Pi$, we define an object in
In this way we obtain a bijection between Π and the set P. We will simply denote by \( u_\pi, \pi \in \Pi \) for \( u_{\pi} \) in H.

An \( n \)-tuple \( \pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}) \) of partition in \( \Pi \) is called aperiodic (in the sense of Lusztig [L3]), or separated (in the sense of Ringel [R2]), if for each \( l \geq 1 \) there is some \( i = i(l) \in \Delta_0 \) such that \( \pi^{(l)}_j \neq l \) for all \( j \geq 1 \). By \( \Pi^a \) we denote the set of aperiodic \( n \)-tuples of partitions. An object \( M \) in \( T \) is called aperiodic if \( M \simeq M(\pi) \) for some \( \pi \in \Pi^a \). For any dimension vector \( \alpha \in \mathbb{N}^n (= NI) \), we let

\[
\Pi_\alpha = \{ \lambda \in \Pi | \dim M(\lambda) = \alpha \} \quad \text{and} \quad \Pi_\alpha^a = \Pi^a \cap \Pi_\alpha.
\]

Given any two modules \( M, N \) in \( T \), there exists a unique (up to isomorphism) extension \( L \) of \( M \) by \( N \) with minimal \( \dim \text{End}(L) \) [Re]. This extension \( L \) is called the generic extension of \( M \) by \( N \) and is denoted by \( L = M \triangleleft N \). If we define the operation in \( P \) by \([M] \triangleleft [N] = [M \triangleleft N]\), then \((P, \odot)\) is a monoid with identity \([0]\).

Let \( \Omega \) be the set of all words on the alphabet \( \Delta_0 \). For each \( w = i_1i_2 \cdots i_m \in \Omega \), we set

\[
M(w) = s_{i_1} \odot s_{i_2} \cdots \odot s_{i_m}.
\]

Then there is a unique \( \pi \in \Pi \) such that \( M(\pi) \simeq M(w) \), we define \( \varphi(w) = \pi \). It has been proved in [R2] that \( \pi = \varphi(w) \in \Pi^a \) and \( \varphi \) induces a surjection \( \varphi : \Omega \to \Pi^a \).

We have a (geometric) partial order on \( \mathcal{P} \), or equivalently in \( \Pi \), as follows: for \( \mu, \lambda \in \Pi \), \( \mu \preceq \lambda \) if and only if \( \mathcal{O}_{M(\mu)} \subseteq \mathcal{O}_{M(\lambda)} \), or equivalently, \( \dim \text{Hom}(M, M(\lambda)) \leq \dim \text{Hom}(M, M(\mu)) \) for all modules \( M \) in \( T \).

For each module \( M \) in \( T \) and integer \( s \geq 1 \), we denote by \( sM \) the direct sum of \( s \) copies of \( M \). For \( w \in \Omega \), write \( w \) in tight form \( w = j_{1t}^{e_1} j_{2t}^{e_2} \cdots j_{lt}^{e_l} \in \Omega \) with \( j_{r-1} \neq j_r \) for all \( r \) and define \( \mu_r \in \Pi \) such that \( M(\mu_r) = e_{r}S_{j_r} \). For any \( \lambda \in \Pi \), write \( g_\lambda^w \) for the Hall polynomial \( g_{M(\mu_1), \ldots, M(\mu_l)}^w \). A word \( w \) is called distinguished if the Hall polynomial \( g_\lambda^w = 1 \). This means that \( M(\varphi(w)) \) has a unique reduced filtration of type \( w \), i.e., a filtration

\[
M(\varphi(w)) = M_0 \supset M_1 \supset \cdots \supset M_{l-1} \supset M_l = 0
\]

with \( M_{r-1}/M_r \simeq e_{r}S_{j_r} \) for all \( r \).

**Proposition 5.1** For any \( \pi \in \Pi^a \), there exists a distinguished word \( w_\pi = j_1^{\pi_1} j_2^{\pi_2} \cdots j_l^{\pi_l} \in \varphi^{-1}(\pi) \) in tight form.

In \( \mathcal{H}^* \), let \( u_i^{(sm)} = E_i^{(sm)} = \frac{u_i^m}{m!} \), \( i \in \Delta_0, m \geq 1 \). The Z-subalgebra \( C^* = C^*_Z \) of \( \prod_q \mathcal{H}_q^* \) generated by \( u_i^{(sm)}, i \in \Delta_0, m \geq 1 \), is the twisted composition algebra of \( T \) (cf. 1.4).

**5.3** For each \( w = j_1^{e_1} j_2^{e_2} \cdots j_l^{e_l} \in \Omega \) in tight form, define in \( C^* \) the monomial

\[
m^{(w)} = E_{j_1}^{(w_{e_1})} \ast \cdots \ast E_{j_l}^{(w_{e_l})}.
\]

For each \( \pi \in \Pi^a \), we from now on fix a distinguished word \( w_\pi \in \varphi^{-1}(\pi) \). Thus we have a section \( D = \{ w_\pi | \pi \in \Pi^a \} \) of \( \varphi \) over \( \Pi^a \). \( D \) is called a section of distinguished words in [DDX].

For each \( \pi \in \Pi^a \) with the fixed distinguished word \( w_\pi = j_1^{e_1} j_2^{e_2} \cdots j_l^{e_l} \) in tight form, define \( L_0 = e_{j_1}S_{j_1}, L_1 = e_{j_1}S_{j_1} \odot e_{j_2}S_{j_2}, L_2 = L_1 \odot e_{j_3}S_{j_3}, \ldots, L_{l-1} = L_{l-2} \odot e_{j_l}S_{j_l} \). Set \( \alpha = \dim L_{l-1} \).
As \( L_i \) is the generic extension of \( L_{i-1} \) by \( e_{j_i+1}S_{j_i+1} \) and thus \( \dim \text{End}(L_i) \) is minimal, we have \( M(\pi) \simeq L_{i-1} \). Since
\[
1 = g_{w_n}^\pi = g_{e_{j_1}S_{j_1}, e_{j_2}S_{j_2}g_{L_1, e_{j_3}S_{j_3}} \cdots g_{L_{t-2}, e_{j_t}S_{j_t}}},
\]
we get \( g_{L_{i-1}, e_{j_i+1}S_{j_i+1}}^L = 1, 1 \leq i \leq t - 2 \). Furthermore, by Lemma 2.3(6) and Proposition 1.1, we have
\[
\langle L_{i-1} \rangle \ast \langle e_{j_i+1}S_{j_i+1} \rangle = \langle L_i \rangle + \sum_{X, \dim \mathcal{O}_X < \dim \mathcal{O}_{L_i}} a_X \langle X \rangle,
\]
with \( a_X \in \mathbb{Z} \). Recall from 1.2 that \( \langle M \rangle = v^{-\dim \text{End}(M)}u[M] \). Thus
\[
\mathfrak{m}^{(w_n)} = \langle M(\pi) \rangle + \sum_{\lambda \prec \pi} \xi^\lambda \langle M(\lambda) \rangle,
\]
where \( \xi^\lambda \in \mathbb{Z} \). Note that \( \xi^\lambda \neq 0 \) implies \( \dim M(\lambda) = \dim M(\pi) = \alpha \). Although \( \langle M \rangle \) are in \( \mathcal{H}^* \), they are not necessarily in \( \mathcal{C}^* \). Define \( E_\pi \) inductively by the relation (noting that \( v^2 = q \) in each component)
\[
E_\pi = \mathfrak{m}^{(w_n)} - \sum_{\lambda \prec \pi, \lambda \in \Pi^a_\alpha} v^{-\dim M(\pi)+\dim \text{End} M(\pi)+\dim M(\lambda)-\dim \text{End} M(\lambda)} g_{w_n}^\lambda (v^2)E_\lambda.
\]
if \( \pi \in \Pi^a_\alpha \) is minimal, then \( E_\pi = \mathfrak{m}^{(w_n)} \in \mathcal{C}^* \). By inductions on the partial order, we have \( E_\lambda \in \mathcal{C}^* \) for all \( \lambda \in \Pi^a \). Therefore we have the relations
\[
E_\pi = \langle M(\pi) \rangle + \sum_{\lambda \in \Pi^a_\alpha \setminus \Pi^a_\lambda, \lambda \prec \pi} \eta^\lambda_\pi \langle M(\lambda) \rangle
\]
with \( \eta^\lambda_\pi \in \mathbb{Z} \).

**Proposition 5.2** Let \( \mathcal{D} = \{ w_\pi | \pi \in \Pi^a \} \) be a section of distinguished words of \( \Omega \) over \( \Pi^a \). Then both \( \mathfrak{m}^{(w_\pi)} | \pi \in \Pi^a \} \) and \( \{ E_\pi | \pi \in \Pi^a \} \) are \( \mathbb{Z} \)-bases of \( \mathcal{C}^*_\mathbb{Z} \). Furthermore, for any \( \pi \in \Pi^a_\alpha \),
\[
\mathfrak{m}^{(w_\pi)} = E_\pi + \sum_{\lambda \in \Pi^a_\alpha, \lambda \prec \pi} v^{-\dim M(\pi)+\dim \text{End} M(\pi)+\dim M(\lambda)-\dim \text{End} M(\lambda)} g_{w_\pi}^\lambda (v^2)E_\lambda.
\]

**Remark.** The definition of the basis \( \{ E_\pi | \pi \in \Pi^a \} \) depends on the choice of the section \( \mathcal{D} \) of distinguished words, but eventually it has been proved in [DDX] that it is independent of the selection of the sections of distinguished words.

We will call \( \mathfrak{m}^{(w_\pi)} | \pi \in \Pi^a \} \) a monomial \( \mathbb{Z} \)-basis of \( \mathcal{C}^*_\mathbb{Z} \) and \( \{ E_\pi | \pi \in \Pi^a \} \) as a “PBW”-basis of \( \mathcal{C}^*_\mathbb{Z} \). With the triangular relation between the two bases, we can follow the approach of Lusztig [L1, 7.8-7.11], as we did in Section 4, to obtain the canonical bases \( \{ \mathcal{E}_\pi | \pi \in \Pi^a \} \) of \( \mathcal{C}^*_\mathbb{Z} \) in the sense of [L1, 3.1] by
\[
\mathcal{E}_\pi = \sum_{\lambda \leq \pi, \lambda \in \Pi^a_\alpha} p_{\lambda \pi} E_\lambda, \quad \text{for} \quad \pi \in \Pi^a_\alpha, \alpha \in \mathbb{N}^a,
\]
with \( p_{\lambda \pi} = 1 \) and \( p_{\lambda \pi} \in v^{-1}\mathbb{Z}[v^{-1}] \) for \( \lambda < \pi \).

6. Integral bases arising from preprojective and preinjective components

In this section consider a connected tame quiver \( Q \) without oriented cycles. For the preprojective and preinjective components, the argument in this section is essentially as same as in the case of finite type.
6.1 Let $U$ be the quantized affine enveloping algebra associated to the quiver $Q$, with the Chevalley generators: $E_i, F_i$ and $K_i^\pm$. Lusztig in [L5] introduced the symmetries $T_{i,1}'' : U \to U$ for $i \in I$, as algebra automorphisms of $U$ defined by relations:

$$
T_{i,1}''(K_i) = K_{s_i}^{-1}(K_i) = K_i, \quad T_{i,1}''(E_i) = -F_iK_i, \quad T_{i,1}''(F_i) = -K_iE_i,
$$

$$
T_{i,1}''(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r}E_j^{(s)}E_i^{(r)} \quad \text{for} \ j \neq i \in I,
$$

$$
T_{i,1}''(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^sF_i^{(r)}F_j^{(s)} \quad \text{for} \ j \neq i \in I.
$$

Here $a_{ij} = (i, j)$ for $i, j \in I$, and $\beta \in \mathbb{Z}I$ and $s_i(\beta) = \beta - (\beta, i)i$. For each $i \in I$, define

$$
U^+[i] = \{x \in U^+ | T_{i,1}''(x) \in U^+\},
$$

which is subalgebra $U^+$. Then $T_{i,1}'' : U^+[i] \to U^+[i]$ is an automorphism. Moreover, if we consider the Lusztig form $U^\pm_\mathbb{Z}$ and let $U^\pm_\mathbb{Z}[i] = U^\pm_\mathbb{Z} \cap U^+[i]$, then $T_{i,1}'' : U^\pm_\mathbb{Z}[i] \to U^\pm_\mathbb{Z}[i]$ is an automorphism.

6.2 We define $\sigma_i Q$ to be the quiver obtained from $Q$ by reversing the direction of every arrow connected to the vertex $i$. If $i$ is a sink of $Q$, one may define the BGP reflection functor (see [BGP] or [DR]):

$$
\sigma_i^+ : \text{mod}\Lambda \longrightarrow \text{mod}\sigma_i\Lambda
$$

where $\Lambda = \mathbb{F}_q(Q)$ and $\sigma_i\Lambda = \mathbb{F}_q(\sigma_i Q)$ are path algebras. Therefore we have an algebra homomorphism:

$$
\sigma_i : \mathcal{H}^*(\Lambda)[i] \longrightarrow \mathcal{H}^*(\sigma_i\Lambda)[i]
$$

defined by

$$
\sigma_i(u_M) = u_{[\sigma_i^+(M)]} \text{ for any } M \in \text{mod}\Lambda[i].
$$

Here $\text{mod}\Lambda[i]$ is the subcategory of all representations which do not have $S_i$ as a direct summand and $\mathcal{H}^*(\Lambda)[i]$ is the subalgebra of $\mathcal{H}^*(\Lambda)$ generated by $u_M$ with $M \in \text{mod}\Lambda[i]$. Note that $\mathcal{C}^*(\Lambda)_\mathbb{Z}$ is canonically isomorphic to $\mathcal{C}^*(\sigma_i\Lambda)_\mathbb{Z}$ by fixing the Chevalley generators which correspond to the simple modules of $\Lambda$ and $\sigma_i\Lambda$ respectively. Furthermore, we may regard that the functor $\sigma_i^+$ induces the homomorphism:

$$
\sigma_i : \mathcal{C}^*(\Lambda)_\mathbb{Z}[i] \longrightarrow \mathcal{C}^*(\Lambda)_\mathbb{Z}[i],
$$

where $\mathcal{C}^*(\Lambda)_\mathbb{Z}[i] = \{x \in \mathcal{C}^*(\Lambda)_\mathbb{Z} | \sigma_i(x) \in \mathcal{C}^*(\Lambda)_\mathbb{Z}\}$. It is known that $\sigma_i = T_{i,1}''$ under the identification $\mathcal{C}^*(\Lambda) = U^+$ (for example, see [XY]).

Dually, if $i$ is a source of $Q$, we have the similar results.

We call an indecomposable $\Lambda$-module $M$ exceptional if $\text{Ext}^1_\Lambda(M, M) = 0$. It is proved in [CX] that

$$
\langle sM \rangle \in \mathcal{C}^*(\Lambda)_\mathbb{Z} \text{ for any } s \geq 1
$$

if $M$ is exceptional. In fact,

$$
\langle M \rangle^{(ss)} = \frac{1}{[s]!} v^{-s \dim \mathcal{C}^*(\Lambda)_\mathbb{Z} + s^2} u_M^{ss} = \frac{1}{[s]!} v^{-s \dim \mathcal{C}^*(\Lambda)_\mathbb{Z} + s^{(s)}(\psi_s(q))} u_M = v^{-s \dim \mathcal{C}^*(\Lambda)_\mathbb{Z} + s^2} u_M = \langle sM \rangle,
$$

where $\langle \rangle$ denotes the isomorphism class. We denote by $\text{Prep}$ and $\text{Prei}$, respectively, the isomorphism classes of indecomposable preprojective and preinjective $\Lambda$-modules. In particular, $\mathcal{C}^*_\mathbb{Z}$ contains the set

$$
\{\langle u_M \rangle | M \text{ is indecomposable in } \text{Prep or Prei and } s \geq 1\}.
$$
6.3 Let \( i_m, \ldots, i_1 \) be an admissible sink sequence of \( Q \), i.e., \( i_m \) is a sink of \( Q \) and for any \( 1 \leq t < m \), the vertex \( i_t \) is a sink for the orientation \( \sigma_{i_{t+1}} \cdots \sigma_{i_m} Q \). Let \( M \) be in \( \text{Prei} \). Then there exists an admissible sink sequence of \( Q \) such that

\[ M = \sigma_{i_1}^+ \cdots \sigma_{i_m}^+ (S_{i_{m+1}}) , \]

where \( S_{i_{m+1}} \) is a simple representation in \( \text{mod}\ \sigma_{i_m} \cdots \sigma_{i_1} \Lambda \).

**Lemma 6.1** Let \( M \) be an indecomposable preinjective representation. Then

\[ \langle M \rangle = T_{i_1,1}^\nu \cdots T_{i_m,1}^\nu (E_{i_{m+1}}) , \]

where \( M = \sigma_{i_1}^+ \cdots \sigma_{i_m}^+ (S_{i_{m+1}}) \), for an admissible sink sequence \( i_m, \ldots, i_1 \) of \( Q \).

**Proof.** See [R3].

Since \( \text{Prei} \) is representation-directed, we can total order on the set

\[ \Phi_{\text{Prei}}^+ = \{ \cdots, \beta_3, \beta_2, \beta_1 \} \]

of all positive real roots appearing in \( \text{Prei} \) with \( \{ \cdots, M(\beta_3), M(\beta_2), M(\beta_1) \} \) being the corresponding indecomposable \( \Lambda \)-modules such that

\[ \text{Hom}(M(\beta_i), M(\beta_j)) \neq 0 \] implies \( \beta_i \preceq \beta_j \) and \( i \geq j \).

Then such an ordering has the property

\[ \langle \beta_i, \beta_j \rangle > 0 \] implies \( \beta_i \preceq \beta_j \) and \( i \geq j \)

and

\[ \langle \beta_i, \beta_j \rangle < 0 \] implies \( \beta_j \prec \beta_i \) and \( i < j \)

and

\[ \text{Ext}(M(\beta_i), M(\beta_j)) = 0 \] for \( i \geq j \).

Therefore \( \beta_i \preceq \beta_j \) if and only if \( i \geq j \). Similarly, since \( \text{Prep} \) is representation-directed, we define a total ordering on the set

\[ \Phi_{\text{Prep}}^+ = \{ \alpha_1, \alpha_2, \alpha_3, \cdots \} \]

of all positive real roots appearing in \( \text{Prep} \) with \( \{ \alpha_1, M(\alpha_2), M(\alpha_3), \cdots \} \) be the corresponding indecomposable modules in \( \text{Prep} \) such that

\[ \text{Hom}(M(\alpha_i), M(\alpha_j)) \neq 0 \] implies \( \alpha_i \preceq \alpha_j \) and \( i \leq j \).

Then such an ordering has the property

\[ \langle \alpha_i, \alpha_j \rangle > 0 \] implies \( \alpha_i \preceq \alpha_j \) and \( i \leq j \)

and

\[ \langle \alpha_i, \alpha_j \rangle < 0 \] implies \( \alpha_j \prec \alpha_i \) and \( j < i \)

and

\[ \text{Ext}(M(\alpha_i), M(\alpha_j)) = 0 \] for \( i \leq j \).

We denote by \( \mathbb{N}_{\text{Prei}}^f \) the set of all support-finite functions \( b : \Phi_{\text{Prei}}^+ \to \mathbb{N} \). Each \( b \in \mathbb{N}_{\text{Prei}}^f \) defines a preinjective representation

\[ M(b) = \bigoplus_{\beta_i \in \Phi_{\text{Prei}}^+} b(\beta_i) M(\beta_i) \]

and any preinjective representation is isomorphic to one of the form. By Ringel (Proposition 1’ of [R3]) we have
Lemma 6.2 For any \( b \in \mathbb{N}^P_{\text{Prei}} \),
\[
\langle M(b) \rangle = \langle b(\beta_{m})M(\beta_{m}) \rangle \star \cdots \star \langle b(\beta_{1})M(\beta_{1}) \rangle,
\]
where \( \{ \beta_{m} < \beta_{m-1} \cdots < \beta_{1} \} \) are those \( \beta \in \Phi^+_{\text{Prei}} \) such that \( b(\beta) \neq 0 \).

Thus, by 6.2, \( \langle M(b) \rangle \in \mathcal{C}_Z^* \) for all \( b \in \mathbb{N}^P_{\text{Prei}} \). We now define \( \mathcal{C}^*(\text{Prei}) \) to be the \( \mathbb{Z} \)-submodule of \( \mathcal{C}_Z^* \) generated by
\[
\{ \langle M(b) \rangle | b \in \mathbb{N}^P_{\text{Prei}} \}.
\]

Lemma 6.3 The \( \mathbb{Z} \)-submodule \( \mathcal{C}^*(\text{Prei}) \) is an subalgebra of \( \mathcal{C}_Z^* \) and \( \{ \langle M(b) \rangle | b \in \mathbb{N}^P_{\text{Prei}} \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{C}^*(\text{Prei}) \).

Proof. If \( b, b_1, b_2 \in \mathbb{N}^P_{\text{Prei}} \), then the Hall polynomial \( g^M_{M(b_1)M(b_2)} \) always exists (see Ringel [R5]). Then it is easy to see that \( \mathcal{C}^*(\text{Prei}) \) is closed under the multiplication \( \star \).

With similar definitions for \( \text{Prep} \), we have

Lemma 6.4 For any \( a \in \mathbb{N}_{\text{Pre}}^P \), \( M(a) = \oplus_{\alpha_i \in \Phi^+_{\text{Pre}}_a} a(\alpha_i)M(\alpha_i) \), then
\[
\langle M(a) \rangle = \langle a(\alpha_{i_1})M(\alpha_{i_1}) \rangle \star \cdots \star \langle a(\alpha_{i_m})M(\alpha_{i_m}) \rangle,
\]
where \( \{ \alpha_{i_1} < \alpha_{i_2} \cdots < \alpha_{i_m} \} \) is the support of \( a \).

Lemma 6.5 Let \( \mathcal{C}^*(\text{Prep}) \) be the \( \mathbb{Z} \)-submodule of \( \mathcal{C}_Z^* \) generated by
\[
\{ \langle M(a) \rangle | a \in \mathbb{N}_{\text{Pre}}^P \}.
\]

Then \( \mathcal{C}^*(\text{Prep}) \) is an subalgebra of \( \mathcal{C}_Z^* \) and \( \{ \langle M(a) \rangle | a \in \mathbb{N}_{\text{Pre}}^P \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{C}^*(\text{Prep}) \).

6.4 Since \( Q \) is a tame quiver without oriented cycles, we can order a complete set \( \{ S_1, S_2, \cdots, S_n \} \) of non-isomorphic nilpotent simple modules of mod \( \Lambda \) such that
\[
\text{Ext}^1(S_i, S_j) = 0 \quad \text{for} \quad i \geq j.
\]

We can now identify \( I = \{ 1, 2, \cdots, n \} \) and \( \mathbb{N} I = \mathbb{N}^n \) such that \( S_i \) is the simple module at the vertex \( i \in I \). Any module \( M \) with dimension vector \( d = (d_1, d_2, \cdots, d_n) \) has a unique filtration
\[
M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0
\]
with factors \( M_{i-1}/M_i \) isomorphic to \( d_iS_i \), since \( \text{Ext}(S_i, S_j) = 0 \) for \( i \geq j \). This shows that the Hall polynomial \( g^M_{M_{d_1S_1} \cdots d_nS_n} = 1 \). Then we have, in \( \mathcal{H}_q \) and \( \mathcal{H}_q^* \) respectively,
\[
u d_1 \cdots d_n S_1 \cdots S_n = \psi d_1(q) \psi d_2(q) \cdots \psi d_n(q) \sum u[M(a) \oplus M(t) \oplus M(b)],
\]
\[
u(s) \cdots (s) d_1 \cdots d_n S_1 \cdots S_n = v^{-(d_1 + d_2 + \cdots + d_n)}(d, d) \sum u[M(a) \oplus M(t) \oplus M(b)],
\]
where the summation is over the triples \( (M(a), M(t), M(b)) \) with \( M(a) \) preprojective, \( M(t) \) regular, \( M(b) \) preinjective, and \( \text{dim} \ M(a) + \text{dim} \ M(t) + \text{dim} \ M(b) = (d_1, \cdots, d_n) = d \).

For any \( a \in \mathbb{N}_{\text{Pre}}^P \), let \( \{ \alpha_{i_1} < \alpha_{i_2} < \cdots < \alpha_{i_m} \} \) be the support of \( a \) and, for \( 1 \leq t \leq m \), define
\[
a_t = a(\alpha_{i_t})a_{i_t} = (a_{1t}, a_{2t}, \cdots, a_{nt}) \in \mathbb{N}^n;
\]
\[
a_{ai} = u(s) a_{ai} = u[s_{ai}] \cdots u[s_{ai}],
\]
\[
a_{mi} = a_{mi} = a_m \cdot a_{m2} \cdots a_{m1}.
\]
Similarly for \( b \in \mathbb{N}_f^{Prei} \) define
\[
\begin{align*}
\text{m}_{b_t} &= u_{[S_1]}^{(s_{b_{2t}})} \ast u_{[S_2]}^{(s_{b_{2t}})} \cdots \ast u_{[S_n]}^{(s_{b_{2t}})}; \\
\text{m}_b &= \text{m}_{b_n} \ast \text{m}_{b_{n-1}} \cdots \ast \text{m}_{b_1}.
\end{align*}
\]

Lemma 6.6 For any \( a \in \mathbb{N}_f^{Prei} \) and \( b \in \mathbb{N}_f^{Prei} \), in \( \mathcal{H}^* \), we have
\[
(1) \quad \text{m}_a = \langle M(a) \rangle + \sum_{\dim \text{O}_{M(a')} \leq \dim \text{O}_{M(a)}} c_{a't'b'q}^a u_{[M(a') \oplus M(t') \oplus M(b')]},
\]
where the sum ranges over all triples \( M(a'), M(t'), M(b') \) with \( M(a') \) preprojective, \( M(t') \) regular, \( M(b') \) preinjective, and \( \dim M(a') + \dim M(t') + \dim M(b') = \sum_{\alpha \in \text{Prei} a(\alpha) \alpha, \text{and}} c_{a't'b'q}^a \in \mathbb{Z}[v, v^{-1}];
\]
\[
(2) \quad \text{m}_b = \langle M(b) \rangle + \sum_{\dim \text{O}_{M(a')} \leq \dim \text{O}_{M(b)}} d_{a't'b'q}^b u_{[M(a') \oplus M(t') \oplus M(b')]},
\]
where the sum is over all triples \( M(a''), M(t''), M(b'') \) with \( M(a'') \) preprojective, \( M(t'') \) regular, \( M(b'') \) preinjective, and \( \dim M(a'') + \dim M(t'') + \dim M(b'') = \sum_{\beta \in \text{Prei} b(\beta) \beta, \text{and}} d_{a't'b'q}^b \in \mathbb{Z}[v, v^{-1}].
\]

Proof. (1) Since \( M(\alpha_{i_t}) \) is exceptional, then by Lemma 2.1, \( \text{O}_{a(a_{i_t})} M(\alpha_{i_t}) \) is a unique orbit of maximal dimension in \( \mathcal{E}_{a(\alpha_{i_t})} \alpha_{i_t} \). Note that all simple modules are exceptional. We have
\[
\text{m}_{a_t} = u_{[S_1]}^{(s_{a_{2t}})} \ast u_{[S_2]}^{(s_{a_{2t}})} \cdots \ast u_{[S_n]}^{(s_{a_{2t}})}
= \langle a_{1_t} S_1 \rangle \ast \langle a_{2_t} S_2 \rangle \cdots \ast \langle a_{n_t} S_n \rangle
= v^{-\dim a(\alpha_{i_t}) M(\alpha_{i_t}) + \dim \text{End}(a(\alpha_{i_t}) M(\alpha_{i_t}))} \sum_{\dim M = a(\alpha_{i_t}) \alpha_{i_t}} u_{[M]}
= \langle a(\alpha_{i_t}) M(\alpha_{i_t}) \rangle + \sum_{\dim \mathcal{O}_M \leq \dim \mathcal{O}_{a(\alpha_{i_t})} M(\alpha_{i_t})} v^{-\dim \text{Ext}(M, M)}(M).
\]

Because \( \text{Ext}(M(\alpha_{i_t}), M(\alpha_{i_t})) = 0 \) and \( \text{Hom}(M(\alpha_{i_t}), M(\alpha_{i_t})) = 0 \) for \( i_t \leq i_s \), by Lemma 2.3(7) and Lemma 6.4, we have
\[
\text{m}_a = \text{m}_{a_1} \ast \text{m}_{a_2} \ast \cdots \ast \text{m}_{a_n}
= \langle M(a) \rangle + \sum_{\dim \mathcal{O}_{M(a')} \leq \dim \mathcal{O}_{M(t')} \leq \dim \mathcal{O}_{M(b')} \leq \dim \mathcal{O}_{M(a)}} c_{a't'b'q}^a u_{[M(a') \oplus M(t') \oplus M(b')]},
\]
which satisfies the condition. The proof for (2) is dual, so the proof is completed. \( \square \)

Remark. In Lemma 6.6, the element \( v \) is equal to \( \sqrt{q} \), but the degree of \( v^{-1} \) in \( c_{a't'b'q}^a \) or in \( d_{a't'b'q}^b \) is bounded and independent of \( \mathbb{F}_q \). (See Lemma 1.2.)

7. Integral bases for the generic composition algebras

7.1 In this section, we still assume that \( Q \) is connected tame quiver without oriented cycles. We first consider the embedding of the representation category of the Kronecker quiver into the representation category of \( Q \).

Let \( e \) be an extending vertex of \( Q \) and \( \Lambda = \mathbb{F}_q Q \) : the path algebra of \( Q \) over \( \mathbb{F}_q \). Let \( P = P(e) \) be projective module cover of the simple module \( S_e \). Set \( P = \text{dim} P(e). \) Clearly \( \langle p, p \rangle = 1 = \langle p, \delta \rangle \)
and there exists a unique indecomposable preprojective module $L$ with $\dim L = p + \delta$. Moreover we have $\text{Hom}_\Lambda(L, P) = 0$ and $\text{Ext}_\Lambda(L, P) = 0$. This means that $(P, L)$ is an exceptional pair. Let $\mathcal{C}(P, L)$ be the smallest full subcategory of mod $\Lambda$ which contains $P$ and $L$ and is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. Also we have $\dim_{F_q} \text{Hom}_\Lambda(P, L) = 2$, therefore $\mathcal{C}(P, L)$ is equivalent to the module category of the Kronecker quiver over $F_q$. Thus it induces an exact embedding $F : \text{mod} K \leftrightarrow \text{mod} \Lambda$, where $K$ is the path algebra of the Kronecker quiver over $F_q$. We note here that the embedding functor $F$ is essentially independent of the field $F_q$. This gives rise to an injective homomorphism of algebras, still denoted by $F : \mathcal{H}^*(K) \hookrightarrow \mathcal{H}^*(\Lambda)$. In $\mathcal{H}^*(K)$ we have defined the element $E_{m\delta m}$ for $m \geq 1$. Denote $E_{m\delta m} = F(E_{m\delta m})$. Since $E_{m\delta m} \in \mathcal{C}^*(K)$, and $\langle L, (P) \rangle \in \mathcal{C}^*(\Lambda)$, so $E_{m\delta m}$ is in $\mathcal{C}^*(\Lambda)$ and even in $\mathcal{C}^*(\Lambda)_Z$. Let $K$ be the subalgebra of $\mathcal{C}^*(\Lambda)$ generalized by $E_{m\delta m}$ for $m \in \mathbb{N}$, it is a polynomial ring on infinitely many variables $\{E_{m\delta m} | m \geq 1\}$, and its integral form is the polynomial ring on variables $\{E_{m\delta m} | m \geq 1\}$ over $Z$.

**Proposition 7.2** We may list all non-homogeneous tubes $T_1, T_2, \cdots, T_s$ in mod $\Lambda$ (in fact, $s \leq 3$). For each $T_i$, let $r_i = r(T_i)$ be the period of $T_i$, i.e., the number of quasi-simple modules in $T_i$. Then $r_i > 1$. It is well-known that (for example see [CB])

**Lemma 7.1** We have the equation $\sum_{i=1}^s (r_i - 1) = |I| - 2$ and the multiplicity of each imaginary root $m\delta$ is equal to $|I| - 1$, where $|I|$ is the number of vertices of $Q$. \qed

**7.3** For each non-homogeneous tube $T_i$, as we did in Section 5, we have the generic composition algebra $\mathcal{C}^*(T_i)$ of $T_i$ and its integral form $\mathcal{C}^*(T_i)_Z$. For each $T_i$ we have the set $\Pi_i^\circ$ of aperiodic $r_i$-tuples of partitions such that for any $\pi_i \in \Pi_i^\circ$, $M_i(\pi_i)$ is an aperiodic module in $T_i$. We have constructed in 5.3 the element

$$E_{\pi_i} = \langle M_i(\pi_i) \rangle + \sum_{\lambda_i \in \Pi_i \setminus \Pi_i^\circ, \lambda_i \prec \pi_i} \eta^{\pi_i}_{\lambda_i} \langle M_i(\lambda_i) \rangle.$$ 

Then $\{E_{\pi_i} | \pi_i \in \Pi_i^\circ\}$ is a $Z$-basis of $\mathcal{C}^*(T_i)_Z$.

Let $M$ be the set of quadruples $c = (a_c, b_c, \pi_c, w_c)$ such that $a_c, b_c \in \mathbb{N}_{\text{Prep}}$, $b_c \in \mathbb{N}^{\text{Prei}}$, $\pi_c = (\pi_{1c}, \cdots, \pi_{sc}) \in \Pi_1^c \times \cdots \times \Pi_s^c$, and $w_c = (w_1 \geq w_2 \geq \cdots \geq w_s)$ is a partition.

Then for each $c \in M$ we define

$$E^c = \langle M(a_c) \rangle * E_{\pi_{1c}} * E_{\pi_{2c}} * \cdots * E_{\pi_{sc}} * E_{w_c\delta} * \langle M(b_c) \rangle,$$

where $\langle M(a_c) \rangle$ and $\langle M(b_c) \rangle$ are defined in 6.3, $E_{\pi_{sc}}$ is defined above and $E_{w_c\delta}$ is defined in 3.5. Obviously, $\{E^c | c \in M\}$ lies in $\mathcal{C}^*(\Lambda)$, in fact in $\mathcal{C}^*(\Lambda)_Z$, and are linearly independent over $\mathbb{Q}(v)$.

**Proposition 7.2** The set $\{E^c | c \in M\}$ is a $\mathbb{Q}(t)$-basis of $\mathcal{C}^*(\Lambda)$.

The proof of Proposition 7.2 will be given in 7.4. We first need some preparation.

**Lemma 7.3** Let $\{S_j | 1 \leq j \leq r_i\}$ be a complete set of non-isomorphic quasi-simple modules of a non-homogeneous tube $T_i$ such that $S_j = \tau^{-1} S_j$ and let $\mathcal{H}^*(T_i)$ be the generic integral form of the twisted Ringel-Hall algebra of $T_i$ over $Z = \mathbb{Z}[t, t^{-1}]$. For any $l \in \mathbb{N}$ and $1 \leq j \leq r_i$, let $\pi, \pi' \in \Pi_i^c$ such that $S_j[l] = M(\pi)$ and $S_{j+1}[l] = M(\pi')$. Then

$$(1) \quad u_{[S_j[l]]} = \sum_{\lambda \geq \pi, \lambda \in \Pi_i^c} a_\lambda E_\lambda \pmod{(t-1)\mathcal{H}^*(T_i)} \quad \text{if } r_i \nmid l,$$

$$u_{[S_j[l]]} - u_{[S_{j+1}[l]]} = \sum_{\lambda \geq \pi \cap \pi', \lambda \in \Pi_i^c} a_\lambda E_\lambda \pmod{(t-1)\mathcal{H}^*(T_i)} \quad \text{if } r_i \mid l.$$
Here \( a_\lambda \in \mathbb{Q} \).

**Proof.** Without loss of generality, we may take \( j = 1 \). When \( l = 1 \), we have \( u_{[S_1]} = E_{S_1} \). The conclusion follows. We suppose that the conclusion is true when \( 1 \leq l \leq r_i - 2 \). Then the assumption

\[
 u_{[S_1[l]]} \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)) \]

and \( u_{[S_1[l]]} u_{[S_1[l+1]]} - u_{[S_1[l]]} u_{[S_1[l]]} \equiv u_{[S_1[l+1]]} \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)) \) imply

\[
 u_{[S_1[l+1]]} \equiv (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) E_{S_{l+1}} - E_{S_{l+1}} (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)) \]

\[
 \equiv (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) E_{S_{l+1}} - E_{S_{l+1}} (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) = \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a'_\lambda E_\lambda \]

since \( \{ E_\lambda | \lambda \in \Pi_i^q \} \) is a basis of \( T_i \). Thus the conclusion is true for \( l + 1 \). For \( l = r_i \), by assumption, we have

\[
 u_{[S_2[l-1]]} \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)). \]

Now we consider the general case. Let \( l = kr_i + m, 0 < m \leq r_i - 1 \), if \( m = 1 \), by assumption, we have

\[
 u_{[S_1[l-1]]} - u_{[S_1[l-1]]} \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)). \]

Hence

\[
 u_{[S_1[l]]} \equiv (u_{[S_1[l-1]]} - u_{[S_2[l-1]]})u_{[S_1]} - u_{[S_1]} (u_{[S_1[l-1]]} - u_{[S_2[l-1]])} \]

\[
 \equiv (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) E_{S_1} - E_{S_1} (\sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda) \]

\[
 \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a'_\lambda E_\lambda \quad \text{(mod (t - 1) } \mathcal{H}^*(T_i)) \].

If \( 2 \leq m \leq r_i - 1 \), by assumption,

\[
 u_{[S_1[l-1]]} \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^q} a_\lambda E_\lambda \quad \text{(mod } (t - 1) \mathcal{H}^*(T_i)). \]
Hence

\[ u[S_1[t]] \equiv u[S_1[t-1]]u[S_1] - u[S_1]u[S_1[t-1]] \]
\[ \equiv \left( \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^\ast} a_{\lambda}E_{\lambda} \right)E_{S_1} - E_{S_1}\left( \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^\ast} a_{\lambda}E_{\lambda} \right) \]
\[ \equiv \sum_{\lambda \leq \pi_1, \lambda \in \Pi_i^\ast} a_{\lambda}E_{\lambda} \pmod{(t-1)\mathcal{H}^\ast(T_i)}. \]

Then the conclusion is true. When \( r_i \mid l \), it can be proved by a similar method for \( l = r_i \). \( \square \)

**Remark.** Of course we can replace \( \mathcal{H}^\ast(T) \) in Lemma 7.3 by \( \mathcal{C}^\ast(T) \) using the natural embedding \( \mathcal{C}^\ast(T)/(t-1)\mathcal{C}^\ast(T) \) into \( \mathcal{H}^\ast(T)/(t-1)\mathcal{H}^\ast(T) \), here we consider the integral forms over \( \mathcal{Z} \).

**Lemma 7.4** In \( \mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \),

\[ \tilde{E}_{n\delta} = \sum_{m_1 \leq \cdots \leq m_s, m_1 + \cdots + m_s = n} b_{m_1, \ldots, m_s}E_{m_1\delta} \cdots E_{m_s\delta}, \]

where \( b_{m_1, \ldots, m_s} \in \mathcal{Z} \).

**Proof.** By the relation

\[ E_{0\delta} = 1, \quad E_{k\delta} = \frac{1}{[k]} \sum_{s=1}^{k} v^{s-k} \tilde{E}_{s\delta} \ast E_{(k-s)\delta}, \]

we can solve the equation inductively to get the relation in the lemma. \( \square \)

It is known from Ringel that the Lie subalgebra \( \mathfrak{n}^+ \subseteq \mathcal{C}^\ast(\Lambda)_{\mathcal{Z}}/(t-1)\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \) generated by \( u[S_i] \) \((i \in I)\) over \( \mathbb{Q} \) is the positive part of the corresponding affine Kac-Moody Lie algebra over \( \mathbb{Q} \), and \( \mathcal{C}^\ast(\Lambda)_{\mathcal{Z}}/(t-1)\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \) is the universal enveloping algebra of \( \mathfrak{n}^+ \).

For each non-homogeneous tube \( T_i \) of rank \( r_i \), we denote \( u_{\alpha, i} = u[S_i[l]] \) where \( S_i[l] \) is indecomposable in \( T_i \) and \( \dim S_i[l] = \alpha \) is a real root; and \( u_{j,m\delta, i} = u[S_i[l]] - u_{S_i[l]} - u[S_i+1[l]] \) where \( S_i[l] \) is indecomposable in \( T_i \) and \( \dim S_i[l] = m\delta \) an imaginary root. Let \( \Psi : \mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \rightarrow \mathcal{C}^\ast(\Lambda)/(t-1)\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \) be the canonical projection. Then one of the main results in [FMV] is the following of which the proof depends on Lemma 7.1.

**Proposition 7.5** The vectors \( \Psi(u_{[M(\alpha)]}) \) for \( \alpha \in \Phi^+_\text{Prei} \); \( \Psi(u_{\alpha, i}) \) for \( \alpha \in T_i \) real root, \( i = 1, \cdots, s; \)
\( \Psi(u_{j,m\delta, i} - u_{j+1,m\delta, i}), m \geq 1, 1 \leq j \leq r_i, i = 1, \cdots, s; \)
\( \Psi(\tilde{E}_{n\delta}), n \geq 1 \) and \( \Psi(u_{[M(\beta)]}) \) for \( \beta \in \Phi^+_\text{Prei} \)
form a \( \mathcal{Z} \)-basis of \( \mathfrak{n}^+ \).

Note that it is easy to see that all vectors in Proposition 7.5 belong to the Lie algebra \( \mathfrak{n}^+ \), and they are linearly independent over \( \mathbb{Q} \). For example, \( \Psi(\tilde{E}_{n\delta}), n \geq 1 \), lie in \( \mathfrak{n}^+ \) by Lemma 3.9. Then by Lemma 7.1, one can prove that those vectors give rise to a \( \mathcal{Z} \)-basis of \( \mathfrak{n}^+ \).

**7.4 Proof of Proposition 7.2.** By the definition of \( \{E^c | c \in \mathcal{M}\} \), we see that they are linearly independent over \( \mathbb{Q}(t) \). For any weight ( or, dimension vector ) \( w \in \mathcal{N} \), we define the \( \mathbb{Q}(t) \)-space \( V_w \) to be spanned by those \( E^c, c \in \mathcal{M} \), such that \( E^c \in \mathcal{C}^\ast(\Lambda)_w \). It is well-known from Lusztig that

\[ \dim_{\mathbb{Q}(t)} \mathcal{C}^\ast(\Lambda)_w = \dim_{\mathbb{Q}}(\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}}/(t-1)\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}})_w \]

and the monomials in a fixed order on the basis elements of \( \mathfrak{n}^+ \) in Proposition 7.5 form a PBW basis of \( \mathcal{C}^\ast(\Lambda)_{\mathcal{Z}}/(t-1)\mathcal{C}^\ast(\Lambda)_{\mathcal{Z}} \) over \( \mathbb{Q} \). However, Lemma 7.3 and 7.4 implies that those PBW basis
The set $\mathcal{E}(\mathcal{P}, \mathcal{L})$ is an isomorphism of $\mathbb{Q}(t)$-spaces.

**Proposition 7.6** The set $\{E^c | c \in \mathcal{M}\}$ is an $\mathcal{A}$-basis of $\mathcal{C}^*(\Lambda)_A$.

**Proof.** For any monomial $m$ on the divided powers of $u_{[s]}$ (in $\mathcal{P}$) by Proposition 7.2,

$$m = \sum_{c \in \mathcal{M}} f_{m,c}(t)E^c \text{ (finite sum)}$$

in $\mathcal{C}^*(\Lambda)$, where $f_{m,c}(t) \in \mathbb{Q}(t)$ and $v$ is an indeterminate. Note that $E_{x_{ie}}$ in the definition of $E^c$ has the form (cf. 5.3)

$$E_x = \langle M(\pi) \rangle + \sum_{\lambda \in \Pi_a \setminus \Pi_\Lambda} \eta_\lambda^x \langle M(\lambda) \rangle$$

with $\eta_\lambda^x \in \mathbb{Z}$. The formula $m = \sum_{c \in \mathcal{M}} f_{m,c}(v)E^c$ still holds in $\mathcal{H}^*$ for taking $v = \sqrt{q}$. Thus, by Lemma 1.2, for each $c \in \mathcal{M}$, there exists $N(c) \in \mathbb{N}$ such that $(\sqrt{q})^{N(c)}f_{m,c}(\sqrt{q}) \in \mathbb{Z}$ for all $q = p^l$ with $p$ a prime number and $l \geq 1$ in $\mathbb{N}$. It is easily seen that $\eta_\lambda^x f_{m,c}(t)$ is a polynomial in $\mathbb{Q}[t]$. Therefore $f_{m,c}(v) \in \mathbb{Q}[t, t^{-1}]$. □

**Corollary 7.7** The multiplication map

$$\varphi : \mathcal{C}^*(\mathcal{P})A \otimes_A \mathcal{C}^*(\mathcal{T}_1)A \otimes_A \cdots \otimes_A \mathcal{C}^*(\mathcal{T}_s)A \otimes_A \mathcal{K}_A \otimes_A \mathcal{C}^*(\mathcal{P}e)A \to \mathcal{C}^*(\Lambda)_A$$

is an isomorphism of $\mathcal{A}$-modules. □

**8. A bar-invariant basis of $\mathcal{C}^*(\Lambda)_A$**

To simplify the notations, in the next rest of the paper, we will use $v$ for the indeterminate $t$. However, we will perform computations over $\mathbb{F}_q$ with $v = q^{1/2}$. It should be clear from the context that formulae are independent of $q$ and one can obtain the same formulation as in the generic case as discussed in 1.4.

**8.1** The first part of this section is devoted to finding a monomial basis and a triangular relation with the basis $\{E^c | c \in \mathcal{M}\}$.

We first define the variety $\mathcal{O}_c$

$$\mathcal{O}_c = \mathcal{O}_{M(a_c)} \ast \mathcal{O}_{M_{x_{1e}}} \ast \mathcal{O}_{M_{x_{2e}}} \ast \cdots \ast \mathcal{O}_{M_{x_{se}}} \ast \mathcal{N}_{w_{1\delta}} \ast \cdots \ast \mathcal{N}_{w_{l\delta}}$$

for any $c \in \mathcal{M}$, where $\mathcal{N}_{w_{1\delta}} = \mathcal{N}_{w_{2\delta}} \ast \cdots \ast \mathcal{N}_{w_{l\delta}}$ if $w_c = (w_1, w_2, \cdots, w_l)$ and $\mathcal{N}_{w_{i\delta}}$ are the union of orbits of regular modules of $\mathcal{E}(P, L)$ with dimension vector $w_{i\delta}$.

Then by Proposition 7.6, Lemma 6.6 can be rewritten as follows:

---

"Note that the definition of $\mathcal{O}_c$ here is different with that in 4.2"
Lemma 8.1 For any \( a \in \mathbb{N}_f^{Prep} \) and \( b \in \mathbb{N}_f^{Prei} \), in \( C^*(\Lambda) \) we have

\[
\text{1)} \quad m_a = \langle M(a) \rangle + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_a} f^a_c E^c,
\]

\[
\text{2)} \quad m_b = \langle M(b) \rangle + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_b} g^b_c E^c,
\]

where \( f^a_c, g^b_c \in \mathbb{Q}[v, v^{-1}] \) and \( c \in \mathcal{M} \).

Remark. The conclusion is also true in Lemma 8.1 if we take \( M(a) \) to be finitely many copies of an exceptional module.

Lemma 8.2 Let \( \pi \in \Phi^i \) for some \( T_i \), then there exists a monomial \( m_\pi \) on the divided powers of \( u_{[s_i]} \) (\( i \in I \)) such that

\[
m_\pi = E^{\pi} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_e} f^\pi_c E^c,
\]

where \( f^\pi_c \in \mathbb{Q}[v, v^{-1}] \).

Proof. We set \( \{\theta_1, \theta_2, \ldots, \theta_n\} \) to be a complete set of non-isomorphic quasi-simple modules of \( T_i \) in the natural order (see Section 5). By Proposition 5.2, we then have

\[
m^{(w_\pi)} = E^{\pi} + \sum_{\lambda \in \Pi^\pi, \lambda < \pi} v^{-\dim M(\pi) + \dim \text{End} M(\pi) + \dim \text{End} M(\lambda)} g^{E_\lambda}(v^2) E^\lambda,
\]

where \( m^{(w_\pi)} = \theta^{(e_1)}_{\lambda_1} \cdots \theta^{(e_t)}_{\lambda_t} \). Since each \( \theta_j \) is an exceptional module, we have \( \langle u_{[\theta_j]} \rangle^{(e_p)} = \langle e_p \theta_j \rangle \) (see the proof in 6.2).

Let \( \pi_{j_p} \in \Phi^i \) such that \( M(\pi_{j_p}) = e_p \theta_{j_p} \) and \( \dim M(\pi_{j_p}) = (d_1, \ldots, d_n) \) with \( I \) ordered as in 6.4. By Lemma 8.1 and its remark, we define a monomial \( m_{j_p} \) such that

\[
m_{j_p} = \langle S_1 \rangle^{(sd_1)} \cdots \langle S_n \rangle^{(sd_n)} = \langle M(\pi_{j_p}) \rangle + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\pi_{j_p}}} f^\pi_{j_p} E^c
\]

where \( f^\pi_{j_p} \in \mathbb{Q}[v, v^{-1}] \).

Let \( L_0 = e_1 \theta_{j_1}, L_1 = e_1 \theta_{j_1} \circ e_2 \theta_{j_2}, L_2 = L_1 \circ e_3 \theta_{j_3}, \ldots, L_{t-1} = L_{t-2} \circ e_t \theta_{j_t} \). By Lemma 2.3(6), We have \( M(\pi) \simeq L_{t-1} \). Similar to the argument as in 5.3, we have \( g^{L_{p-1}}_{L_{p-1}, e_{j_p+1} \theta_{j_{p+1}}} = 1 \), for \( 1 \leq p \leq t - 2 \). Define \( \alpha_p = \dim L_{p-1} \) and \( \beta_p = \dim M(\pi_{j_p}) \). By Lemma 2.3(6), we have

\[
\dim \mathcal{O}_{L_p} = \dim \mathcal{O}_{L_{p-1}} + \dim \mathcal{O}_{e_{j_p+1} \theta_{j_{p+1}}} + m(\alpha_p, \beta_p)
\]

or

\[
codim \mathcal{O}_{L_p} = \dim \mathcal{O}_{L_{p-1}} + \dim \mathcal{O}_{e_{j_p+1} \theta_{j_{p+1}}} - \langle \beta_p, \alpha_p \rangle.
\]

Thus

\[
\dim \mathcal{O}_{M(\pi)} = \dim \mathcal{O}_{L_{t-1}} = \sum_{p=1}^{t-1} \dim \mathcal{O}_{e_{j_p} \theta_{j_p}} + \sum_{p=1}^{t-1} m(\alpha_p, \beta_p).
\]

For any \( c \in \mathcal{M} \) with \( \dim \mathcal{O}_c < \dim \mathcal{O}_{e_{j_p+1} \theta_{j_{p+1}}} \), by Lemma 2.2, we have

\[
codim(\mathcal{O}_{L_{p-1}} - c) = \dim(\mathcal{O}_{L_{p-1}}) + \dim(\mathcal{O}_c) - \langle \beta_p, \alpha_p \rangle + r
\]

\[
> \dim(\mathcal{O}_{L_{p-1}}) + \dim(\mathcal{O}_{e_{j_p+1} \theta_{j_{p+1}}}) - \langle \beta_p, \alpha_p \rangle
\]

\[
= \dim(\mathcal{O}_{L_p}),
\]
Thereby, if we take $m_\pi = m_{\pi_1} \cdots m_{\pi_l}$, then

$$m_\pi = (\langle \pi_1 \rangle^{(s_1*)} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\pi_1}} f_{c_1}^{\pi_1} E_c^*) \cdots (\langle \pi_{l*} \rangle^{(s_{l*})} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\pi_{l*}}} f_{c_{l*}}^{\pi_{l*}} E_c^*)$$

$$= (\langle M(\pi_1) \rangle + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\pi_1}} f_{c_1}^{\pi_1} E_c^*) \cdots (\langle M(\pi_{l*}) \rangle + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{\pi_{l*}}} f_{c_{l*}}^{\pi_{l*}} E_c^*)$$

$$= E_\pi + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_n} f_c^* E_c,$$

where $f_c^* \in \mathbb{Q}[v, v^{-1}]$. The proof is finished. \(\square\)

**Lemma 8.3** Let $E_{n\delta}$ be the image embedded in $C^*(\Lambda)$ of the element $E_{n\delta}$ in $\mathcal{K}$, then there exists a monomial $m_{n\delta}$ on the divided powers of $u_{[S]}$ ($i \in I$) such that

$$m_{n\delta} = E_{n\delta} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\delta}} h_{c\delta}^* E_c,$$

where $h_{c\delta}^* \in \mathbb{Q}[v, v^{-1}]$.

**Proof.** We let $\theta_1, \theta_2$ be the two simple objects of $\mathcal{C}(P, L)$. By Lemma 3.13(3), we then have

$$\langle \theta_2 \rangle^{(s\theta_2*)} \langle \theta_1 \rangle^{(s\theta_1*)} = E_{n\delta} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\delta}} f_c^{n\delta} E_c^*, \text{ with } f_c^{n\delta} \in \mathbb{Q}[v, v^{-1}].$$

Suppose that $\dim n\theta_1 = d' = (d'_1, \cdots, d'_n)$ and $\dim n\theta_2 = d'' = (d''_1, \cdots, d''_n)$ in $\mathbb{Z}I$. Since $\theta_1, \theta_2$ are the exceptional modules, by the remark of Lemma 8.1, we then have

$$m_1 = \langle S_1 \rangle^{(s\theta_1*)} \langle S_2 \rangle^{(s\theta_2*)} \cdots \langle S_n \rangle^{(s\theta_n*)}$$

$$= \langle \theta_1 \rangle^{(s\theta_1*)} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\theta_1}} f_c^{n\theta_1} E_c^*;$$

and

$$m_2 = \langle S_1 \rangle^{(s\theta_1*)} \langle S_2 \rangle^{(s\theta_2*)} \cdots \langle S_n \rangle^{(s\theta_n*)}$$

$$= \langle \theta_2 \rangle^{(s\theta_2*)} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\theta_2}} g_c^{n\theta_2} E_c^*, \text{,}$$

where $f_c^{n\theta_1}, g_c^{n\theta_2} \in \mathbb{Q}[v, v^{-1}]$. By representations of the Kronecker quiver, we know that $N_{n\delta}$ is open in $\mathcal{O}_{n\theta_2} \times \mathcal{O}_{n\theta_1}$. Moreover, $N_{n\delta}$ is open, then dense in $\overline{\mathcal{O}_{n\theta_2}} \times \mathcal{O}_{n\theta_1}$, that is, $N_{n\delta}$ is of maximum dimension, $G$-stable, irreducible and open subvariety of $\overline{\mathcal{O}_{n\theta_2}} \times \mathcal{O}_{n\theta_1}$. Since Hom($\overline{\mathcal{O}_{n\theta_1}}, \overline{\mathcal{O}_{n\theta_2}}$) = 0, we then obtain

$$\codim \overline{\mathcal{O}_{n\theta_2}} \times \mathcal{O}_{n\theta_1} = \codim \overline{\mathcal{O}_{n\theta_2}} + \codim \mathcal{O}_{n\theta_1} - \langle d', d'' \rangle$$

by Lemma 2.2. If either $\mathcal{O}_c \subset \mathcal{O}_{n\theta_2} \setminus \mathcal{O}_{n\theta_2}$ or $\mathcal{O}_c' \subset \mathcal{O}_{n\theta_1} \setminus \mathcal{O}_{n\theta_1}$, then

$$\codim \overline{\mathcal{O}_c} \times \overline{\mathcal{O}_{c'}} = \codim \overline{\mathcal{O}_c} + \codim \overline{\mathcal{O}_{c'}} - \langle d', d'' \rangle + r$$

$$> \codim \overline{\mathcal{O}_{n\theta_2}} \times \mathcal{O}_{n\theta_1} = \codim N_{n\delta}.$$
We now take $m_{n\delta} = m_2 \ast m_1$, then

$$m_{n\delta} = (\langle \theta_2 \rangle^{(sn)}) + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\theta_2}} g^\delta_{c} E^e \ast (\langle \theta_1 \rangle^{(sn)}) + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\theta_1}} f^\delta_{c} E^e'$$

$$= E_{n\delta} + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{n\delta}} h^\delta_{c} E^e,$$

where $h^\delta_{c} \in \mathbb{Q}[v, v^{-1}]$. \hfill \Box

**Proposition 8.4** For any $E^e, c \in \mathcal{M}$, there exists a monomial $m_c$ on the divided powers of $u_{S_i}, i \in I$, such that

$$m_c = E^e + \sum_{c' \in \mathcal{M}, \dim \mathcal{O}_{c'} < \dim \mathcal{O}_c} h^e_{c'} E^e'$$

where $h^e_{c'} \in \mathbb{Q}[v, v^{-1}]$.

**Proof.** According to the structure of the Auslander-Reiten quiver of a tame quiver, if $P \in \text{Prep}, I \in \text{Prei}$ and $R$ is a regular module, we then know that $\mathcal{O}_P \ast \mathcal{O}_R \ast \mathcal{O}_I$ by Lemma 2.3(7). So, we need to prove the same property for $E^e \ast E_{n\delta}$ where $\pi \in \Pi_i^\alpha$. By Lemma 8.2 and 8.3, there exist $m_{\pi}$ and $m_{n\delta}$ such that

$$m_{\pi} = E^e + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_e} f^\pi_{e} E^e,$$

and

$$m_{n\delta} = E_{n\delta} + \sum_{\dim \mathcal{O}_{c'} < \dim \mathcal{O}_{n\delta}} g^\delta_{c'} E^e'$$

where $f^\pi_{e}, g^\delta_{c'} \in \mathbb{Q}[v, v^{-1}]$.

Since we can find smooth points $A \in \mathcal{O}_\pi$ and $B \in \mathcal{O}_{n\delta}$ such that $\text{Hom}(B, A) = 0$, we have $\text{Hom}(\mathcal{O}_{n\delta}, \mathcal{O}_\pi) = 0$. Then,

$$\text{codim} \mathcal{O}_\pi \ast \mathcal{O}_{n\delta} = \text{codim} \mathcal{O}_\pi + \text{codim} \mathcal{O}_{n\delta} - \langle n\delta, \alpha \rangle.$$

If either $\mathcal{O}_c \subset \mathcal{O}_\pi \setminus \mathcal{O}_e$ or $\mathcal{O}_{c'} \subset \mathcal{O}_{n\delta} \setminus \mathcal{O}_{n\delta}$, we have again that

$$\text{codim} \mathcal{O}_\pi \ast \mathcal{O}_{c'} > \text{codim} \mathcal{O}_\pi \ast \mathcal{O}_{n\delta} = \text{codim} \mathcal{O}_\pi \ast \mathcal{O}_{n\delta}.$$

So, we get

$$m_c = m_{\pi} \ast m_{n\delta}$$

$$= (E^e + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_e} f^\pi_{e} E^e) \ast (E_{n\delta} + \sum_{\dim \mathcal{O}_{c'} < \dim \mathcal{O}_{n\delta}} g^\delta_{c'} E^e')$$

$$= E^e + \sum_{c' \in \mathcal{M}, \dim \mathcal{O}_{c'} < \dim \mathcal{O}_c} h^e_{c'} E^e'$$

where $h^e_{c'} \in \mathbb{Q}[v, v^{-1}]$. \hfill \Box

**8.2** Let $A = \mathbb{Q}[v, v^{-1}]$. We define the lattice $\mathcal{L}'$ to be the $\mathbb{Q}[v^{-1}]$-submodule of $\mathcal{C}^* (\Lambda)_A$ with the basis $\{E^e | c \in \mathcal{M}\}$. By the argument similar to that in Section 4, we can use the standard linear algebra method by Lusztig to get the relation:

$$\overline{E^e} = \sum_{c' \in \mathcal{M}} \omega_{c'} E^e'$$

for any $c \in \mathcal{M}$.
with \( \omega_{\mathcal{C}} \in \mathcal{A} \) such that \( \omega_{\mathcal{C}} = 1 \) and if \( \omega_{\mathcal{C}} \neq 0 \) and \( \mathcal{C} \neq \mathcal{C} \) then \( \dim \mathcal{O}_{\mathcal{C}} < \dim \mathcal{O}_{\mathcal{C}} \). Thus we can solve the system of equations

\[
\zeta_{\mathcal{C}} = \sum_{\dim \mathcal{O}_{\mathcal{C}} \leq \dim \mathcal{O}_{\mathcal{C}} \leq \dim \mathcal{O}_{\mathcal{C}}} \omega_{\mathcal{C}''} \zeta_{\mathcal{C}''}
\]

to get a unique solution such that

\[
\zeta_{\mathcal{C}} = 1 \quad \text{and} \quad \zeta_{\mathcal{C}} \in v^{-1}Q[v^{-1}] \quad \text{if} \quad \dim \mathcal{O}_{\mathcal{C}} < \dim \mathcal{O}_{\mathcal{C}}.
\]

Let

\[
\mathcal{E}_{\mathcal{C}} = \sum_{\mathcal{C} \in \mathcal{M}} \zeta_{\mathcal{C}} \mathcal{E}_{\mathcal{C}''} \quad \text{for any} \quad \mathcal{C} \in \mathcal{M}.
\]

Note that this is a finite sum. Then we have the following result.

**Theorem 8.5** The set \( \{\mathcal{E}_{\mathcal{C}} | \mathcal{C} \in \mathcal{M}\} \) provides a basis of \( \mathcal{C}^*(\Lambda)_{\mathcal{A}} \), which is characterized by the two properties: (a) \( \mathcal{E}_{\mathcal{C}} = \mathcal{E}_{\mathcal{C}} \) for all \( \mathcal{C} \in \mathcal{M} \). (b) \( \pi(\mathcal{E}_{\mathcal{C}}) = \pi(\mathcal{E}_{\mathcal{C}}) \), where \( \pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L} \) is the canonical projection. \( \square \)

### 9. Affine canonical bases

9.1 The Ringel-Hall algebra \( \mathcal{H}^*(\Lambda) \) is an associative \( Q(v) \)-algebra with the basis \( \{\langle M \rangle | M \in \mathcal{P}\} \). Note that \( \langle M \rangle = v^{-\dim M + \dim \text{End}_{\mathcal{A}}(M)}u_{[M]} \). According to [G], the inner product on \( \mathcal{H}^*(\Lambda) \) is given by the formula

\[
\langle \langle M \rangle, \langle N \rangle \rangle = \delta_{MN} \frac{v^{2\dim \text{End}(M)}}{a_{M}}
\]

where \( a_{M} = |\text{Aut}(M)| \). It is known that there exists a polynomial, still denoted by \( a_{M} \), such that \( a_{M}(E) = |\text{Aut}_{\mathcal{A}E} E^{M}| \) for any finite extension \( E \) of \( k \). Following Green [G] and [R1], we can define a linear operation

\[
r(u_{[L]}) = \sum_{[M],[N]} v^{(\dim M, \dim N)} g_{M,N} L a_{M} a_{M} u_{M} \otimes u_{N}.
\]

We have the following property:

\[
(x, y \ast z) = (r(x), y \otimes z).
\]

for any \( x, y \) and \( z \in \mathcal{H}^*(\Lambda) \).

It is not difficult to see the following:

**Proposition 9.1** For any preprojective \( \Lambda \)-modules \( P, P' \in \mathcal{P} \), regular \( \Lambda \)-modules \( R, R' \in \mathcal{P} \) and preinjective \( \Lambda \)-modules \( I, I' \in \Lambda \), in \( \mathcal{H}^*(\Lambda) \) we have

\[
\langle \langle P \rangle, \langle R \rangle, \langle I \rangle \rangle, \langle P' \rangle, \langle R' \rangle, \langle I' \rangle \rangle = \delta_{PP'} \delta_{RR'} \delta_{II} \frac{v^{2(\dim \text{End}(P) + \dim \text{End}(R) + \dim \text{End}(I))}}{a_{PP} a_{RR} a_{II}}.
\]

This inner product is also well-defined on \( \mathcal{C}^*(\Lambda) \), which coincides with the paring defined by Lusztig in [L5].

9.2 Let \( \Lambda = kQ \), \( k \) a finite field. Consider regular \( \Lambda \)-modules \( M_{1}, \ldots, M_{t} \) and \( L \), it is well known that there exists the Hall polynomial \( \varphi_{M_{1}, \ldots, M_{t}}^{L} \in \mathbb{Z}[T] \) such that \( \varphi_{M_{1}, \ldots, M_{t}}^{L}(q^{n}) = g_{M_{1} \otimes_{k} K, \ldots, M_{t} \otimes_{k} K}^{L} \).
where $K$ is a finite extension of $k$ with $[K:k] = n$ and $M \otimes_k K$ is naturally a $\Lambda \otimes_k K$-module. Similarly, we have the polynomial $a_M$ such that $a_M(q^n) = |\text{Aut}_{\Lambda \otimes_k K}(M \otimes_k K)|$.

Now we try to calculate the inner product on elements in the PBW-basis $\{E^c|c \in M\}$. First we consider $E_{n\delta}$, $m \geq 1$, which are defined in Section 7.1. We have obtained the following fundamental relations:

$$\tilde{E}_{n\delta} = E_{(n-1,n)} * E_1 - v^{-2} E_1 * E_{(n-1,n)}.$$  
$$E_{0\delta} = 1, \quad E_{n\delta} = \frac{1}{|n|} \sum_{s=1}^{n} v^{s-n} \tilde{E}_{s\delta} * E_{(n-s)\delta}.$$  

According to the calculations in [BCP], $E_{n\delta}$ correspond to the complete symmetric functions $h_{(n)}$ in $[M]$ and $E_{\omega_c \delta}$ correspond to the complete symmetric functions $h_{\omega_c}$ in $M$ (see Section 1 and Section 3 in [BCP]). Then

$$(E_{n\delta}, E_{n\delta}) \equiv 1 \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}$$  
$$(E_{n\delta}, E_{\omega_c \delta}) \in \mathbb{N}^* + v^{-1}Q[[v^{-1}]] \cap Q(v)$$  
$$(E_{\omega_c \delta}, E_{\omega_c \delta}) \in \mathbb{N}^* + v^{-1}Q[[v^{-1}]] \cap Q(v)$$

for any $n \geq 0$ and any partition $\omega_c$ of $n$.

Let $F : \mathcal{H}^s(K) \to \mathcal{H}^s(\Lambda)$ be the embedding and $\mathfrak{C}(P,L)$ be the full subcategory of $\text{mod} \Lambda$ with two relative simple objects $S_1, S_2$ as in Section 7.1. We denote by $\mathfrak{C}_0$ (resp. $\mathfrak{C}_1$) the full subcategory of $\mathfrak{C}(P,L)$ consisting of the $\Lambda$-modules which belong to homogeneous (resp. non-homogeneous) tubes of $\text{mod} \Lambda$. It is easy to see that this embedding preserves the inner product. According to the Auslander-Reiten quiver of regular $\Lambda$-modules, we have non-homogeneous tubes $T_i, i = 1, \cdots, s, s \leq 3$ and others are homogeneous tubes. Now we have the decomposition

$$E_{n\delta} = E_{n\delta,1} + E_{n\delta,2} + E_{n\delta,3}$$

where

$$E_{n\delta,1} = v^{-n \dim S_1 - n \dim S_2} \sum_{[M], M \in \mathfrak{C}_1, \dim M = n\delta} u[M],$$  
$$E_{n\delta,2} = v^{-n \dim S_1 - n \dim S_2} \sum_{[M] \dim M = n\delta \atop M = M_1 \oplus M_2, 0 \neq M_1 \in \mathfrak{C}_1, 0 \neq M_2 \in \mathfrak{C}_0} u[M],$$  
$$E_{n\delta,3} = v^{-n \dim S_1 - n \dim S_2} \sum_{[M], M \in \mathfrak{C}_0, \dim M = n\delta} u[M].$$

Note that $\dim S_i = \dim_k S_i, i = 1, 2$, but the values are independent of the choice of finite field $k$. It is easy to see that $(E_{n\delta,i}, E_{n\delta,j}) = 0$ for all $i \neq j$.

**Lemma 9.2** We have the relations:

$$(E_{n\delta,1}, E_{n\delta,1}) \equiv 0 \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)},$$  
$$(E_{n\delta,2}, E_{n\delta,2}) \equiv 0 \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}$$

and

$$(E_{n\delta,3}, E_{n\delta,3}) \equiv 1 \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}$$

**Proof** Since
\[ E_{n\delta,1} = v^{-n \dim S_1 - n \dim S_2} \sum_{[M], M \in \mathcal{C}_1, \dim M = n\delta} u_{[M]} \]
\[ = \sum_{[M], M \in \mathcal{C}_1, \dim M = n\delta} v^{-\dim \text{End}(M)} \langle M \rangle \]
we have
\[ (E_{n\delta,1}, E_{n\delta,1}) = \sum_{[M], M \in \mathcal{C}_1, \dim M = n\delta} v^{-2\dim \text{End}(M)} \langle \langle M \rangle, \langle M \rangle \rangle. \]
Note that \( \langle \langle M \rangle, \langle M \rangle \rangle = \frac{|\text{End}(M)|}{a_M} \in \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v), \) \( |\text{End}(M)| = v^{2\dim \text{End}(M)} \) and \( a_M \) is a polynomial on \( v \) with the leading term \( v^{2\dim \text{End}(M)} \), then
\[ \langle \langle M \rangle, \langle M \rangle \rangle \geq 1 + v^{-1} \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v) \]
and
\[ (E_{n\delta,1}, E_{n\delta,1}) \in v^{-1} \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v). \]

Obviously, the result is true for \( n = 1 \). We assume now that it is also true for all \( m \) with \( m < n \). Since
\[ E_{n\delta,2} = v^{-n \dim S_1 - n \dim S_2} \sum_{[M], \dim M = n\delta} u_{[M]} \]
\[ = \sum_{[M], M_1 \in \mathcal{C}_1} v^{-\dim \text{End}(M_1)} \langle M_1 \rangle \dim_{M_1,3} \]
We have
\[ (E_{n\delta,2}, E_{n\delta,2}) = \sum_{[M], M_1 \in \mathcal{C}_1} v^{-2\dim \text{End}(M_1)} \langle \langle M_1 \rangle, \langle M_1 \rangle \rangle (E_{n\delta-\dim M_1,3}, E_{n\delta-\dim M_1,3}) \]
Since \( \dim \text{End}(M_1) \geq 1 \), by the inductive assumption, we have
\[ (E_{n\delta,2}, E_{n\delta,2}) \equiv 0 \ (v^{-1} \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)), \]
for all \( n > 0 \). Since
\[ (E_{n\delta}, E_{n\delta}) \equiv 1 \ (v^{-1} \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)) \]
and
\[ (E_{n\delta}, E_{n\delta}) = (E_{n\delta,1} + E_{n\delta,2} + E_{n\delta,3}, E_{n\delta,1} + E_{n\delta,2} + E_{n\delta,3}), \]
then
\[ (E_{n\delta,3}, E_{n\delta,3}) \equiv 1 \ (\text{mod } v^{-1} \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)). \]
The result is true for all \( n \). \( \square \)

9.3 In the following, we will define a decomposition of regular part of \( C^*(\Lambda) \) with respect to the inner product \( \langle -, - \rangle \).

**Lemma 9.3.1** Let \( M \) be a regular module with \( \dim M = n\delta \), and \( M = \bigoplus_{i=1}^s M_i, M_i \in \mathcal{T}_i \) for \( i = 1, \ldots, s \), then \( \dim M_i = n_i\delta \) and \( \sum_{i=1}^s n_i = n \).

**Proof** Let \( M = M_1 \oplus M'_1, M_1 \in \mathcal{T}_1 \) and \( M'_1 \) has no nonzero direct summand in \( \mathcal{T}_1 \). Otherwise, we may assume \( \dim M_1 = m_1\delta + \beta_1 \) with \( 0 < \beta_1 < \delta \) and \( \dim M'_1 = m'_1\delta + \beta'_1 \) with \( 0 < \beta'_1 < \delta \),
then $\beta_1 + \beta'_1 = \delta$. Since $0 = (m_1 \delta + \beta_1, m'_1 \delta + \beta'_1) = (\beta_1, \beta'_1) = (\delta - \beta'_1, \beta'_1) = -(\beta'_1, \beta'_1)$, we get $\beta'_1 = k \delta, k \in \mathbb{N}$. This is a contradiction. \qed

In Section 7, we have constructed the $\mathbb{Q}(v)$-basis $\{E^c | c \in \mathcal{M}\}$ of $\mathcal{C}^*(\Lambda)$. Let $\mathcal{R}(\mathcal{C}^*(\Lambda))$ be the $\mathbb{Q}(v)$-subspace of $\mathcal{C}^*(\Lambda)$ with the basis $\{E_{\pi_1 c} E_{\pi_2 c} \cdots E_{\pi_s c}, E_w \delta | \pi_1 c, \ldots, \pi_s c, w, w_c = (w_1, w_2, \ldots, w_t) \text{ is a partition}\}$. Obviously, it is a subalgebra of $\mathcal{C}^*(\Lambda)$. Naturally, we take $E_w \delta = 1$ if $w_c = 0$.

Let $\mathcal{R}^a(\mathcal{C}^*(\Lambda))$ be the subalgebra of $\mathcal{R}(\mathcal{C}^*(\Lambda))$ with the basis $\{E_{\pi_1 c} E_{\pi_2 c} \cdots E_{\pi_s c} | \pi_1 c, \ldots, \pi_s c = (\pi_1 c, \ldots, \pi_s c) \in \Pi_1 \times \cdots \times \Pi_s\}$. For $\alpha, \beta \in \mathbb{N}[I]$, we denote $\alpha \leq \beta$ if $\beta - \alpha \in \mathbb{N}[I]$. It follows that $\mathcal{R}^a(\mathcal{C}^*(\Lambda))_\beta = \mathcal{R}(\mathcal{C}^*(\Lambda))_\beta$ provided $\beta < \delta$. We now define $\mathcal{F}_\delta = \{x | (x, \mathcal{R}^a(\mathcal{C}^*(\Lambda))_\delta) = 0\}$.

By Proposition 9.1, $(-,-)$ is nondegenerate on the $\mathcal{R}(\mathcal{C}^*(\Lambda))$. According to Lemma 9.3.1, we get

$\mathcal{R}(\mathcal{C}^*(\Lambda))_\delta = \mathcal{R}^a(\mathcal{C}^*(\Lambda))_\delta \oplus \mathcal{F}_\delta$ and $\dim(\mathcal{F}_\delta) = 1$.

By the method of Schmidt orthogonalization, we may set $E'_\delta = E_\delta - \sum_{M(\pi_i c) \dim M(\pi_i c) = \delta, 1 \leq i \leq s} a_{\pi_i c} E_{\pi_i c}$ satisfying $\mathcal{F}_\delta = \mathbb{Q}(v)E'_\delta$. Now let $\mathcal{R}(\mathcal{C}^*(\Lambda))(1)$ be the subalgebra of $\mathcal{R}(\mathcal{C}^*(\Lambda))$ generated by $\mathcal{R}^a(\mathcal{C}^*(\Lambda))$ and $\mathcal{F}_\delta$. We have $\mathcal{R}(\mathcal{C}^*(\Lambda))(1)_\beta = \mathcal{R}(\mathcal{C}^*(\Lambda))_\beta$ if $\beta < 2\delta$. Define $\mathcal{F}_{2\delta} = \{x | (x, \mathcal{R}(\mathcal{C}^*(\Lambda))(1)_{2\delta}) = 0\}$.

Then $\dim \mathcal{F}_{2\delta} = 1$ and $\mathcal{R}(\mathcal{C}^*(\Lambda))_{2\delta} = \mathcal{R}(\mathcal{C}^*(\Lambda))(1)_{2\delta} \oplus \mathcal{F}_{2\delta}$.

In general, define $\mathcal{F}_{n\delta} = \{x \in \mathcal{R}(\mathcal{C}^*(\Lambda))_{n\delta} | (x, \mathcal{R}(\mathcal{C}^*(\Lambda))(n - 1)_{n\delta}) = 0\}$. Let $\mathcal{R}(\mathcal{C}^*(\Lambda))(n)$ be the subalgebra of $\mathcal{R}(\mathcal{C}^*(\Lambda))$ generated by $\mathcal{R}(\mathcal{C}^*(\Lambda))(n - 1)$ and $\mathcal{F}_{n\delta}$. We have $\mathcal{R}(\mathcal{C}^*(\Lambda))_{n\delta} = \mathcal{R}(\mathcal{C}^*(\Lambda))(n - 1)_{n\delta} \oplus \mathcal{F}_{n\delta}, \dim \mathcal{F}_{n\delta} = 1$. Also, we can choose $E'_{n\delta}$ such that $E_{n\delta} - E'_{n\delta} \in \mathcal{R}(\mathcal{C}^*(\Lambda))(n - 1)_{n\delta}$ and $\mathcal{F}_{n\delta} = \mathbb{Q}(v)E'_{n\delta}$ for all $n > 0$.

We shall need the following facts:

**Lemma 9.3.2** Let $M, N, M_i$ be regular $\Lambda$-modules with $\dim M, \dim N, \dim M_i \in \mathbb{N}\delta$, Then the degree of the Hall polynomial $\varphi^L_{MN}$ is no more than $\dim \text{End}(L) - (\dim \text{End}(M) + \dim \text{End}(N))$.

**Proof** We know the formula $g^L_{MN} = |\text{Ext}^1(M, N)L| / |\text{Aut} L|$. We have the Hall polynomial $\varphi^L_{MN}$ for $g^L_{MN}$, the polynomials $a_M, a_N, a_L$ and the polynomial for $|\text{Hom}(M, N)|$. Therefore, we have the rational function, denoted by $f$, such that $f(K) = |\text{Ext}^1(M \otimes_k K, N \otimes_k L)_{\otimes_k L}|$ for any finite extension $K$ of $k$. Since $f(K)$ is an integer for any finite extension $K$ of $k$, $f$ is a polynomial with coefficients in $\mathbb{Q}$. Since $\dim M, \dim N = 0$, $\dim \text{Ext}^1(M, N) = \dim \text{Hom}(M, N)$. The degree of the polynomial $f$ is no more than $\dim \text{Ext}^1(M, N)$. So,

$$\deg(f) \leq \deg(a_L) - (\deg(a_M) + \deg(a_N))$$

It is also known $\deg(a_X) = \dim_k \text{End}(X)$ for any $\Lambda$-module $X$. The proof is finished. \qed

Let $w_c = (w_1, \ldots, w_t)$ be a partition of $n$, then

$$E_w \delta = E_{w_1} \delta \cdots E_{w_t} \delta = (E_{w_1} \delta, 1 + E_{w_1} \delta, 2 + E_{w_1} \delta, 3) \cdots (E_{w_t} \delta, 1 + E_{w_t} \delta, 2 + E_{w_t} \delta, 3).$$
We set $E_{w_3,3} = E_{w_1,3} \cdot \cdots \cdot E_{w_t,3}$

**Lemma 9.3.3** Let $w_c$ be a partition of $n$, then

$$(E_{n,3}, E_{w_c,3}) \equiv (E_{n,3}, E_{w_c,3}) \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}$$

**Proof** When $t = 1$, the result is true by Lemma 9.2. Suppose

$$(E_{n,3}, E_{w_c,3}) \equiv (E_{n,3}, E_{w_c,3}) \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}$$

for all $|w_c| = s$. Now let $|w_c| = s$. Since $E_{k,3} = E_{k,1} + E_{k,2} + E_{k,3}$ for any $k \in \mathbb{N}$, we get by Section 9.2:

$$E_{m,3} = v^{-m \dim S_1 - m \dim S_2} \sum_{[M], M = M_1 \oplus M_2 \in \mathbb{C} \{P, L \}, \dim M = m} u[M]$$

$$= \sum_{[M], M = M_1 \in \mathbb{C}, \dim M = m} v^{-\dim \text{End}(M)} \langle M \rangle \ast E_{m,3} \ast \dim M, 3$$

for all $m > 0$. Then

$$E_{w_c,3} = (E_{w_1,3} \ast E_{w_2,3} \ast E_{w_3,3}) \ast \cdots \ast (E_{w_1,3} \ast E_{w_2,3} \ast E_{w_3,3})$$

$$= E_{w_1,3} \ast \cdots \ast E_{w_1,3} + E_{w_2,3} \ast \cdots \ast E_{w_2,3} + \text{“the rest part”}$$

Here

$$E_{w_1,3} \ast \cdots \ast E_{w_1,3}$$

$$= \left( v^{-w_1 \dim S_1 - w_1 \dim S_2} \sum_{[M], M_1 \in \mathbb{C}, \dim M_1 = w_1} u[M_1] \right) \ast \cdots \ast \left( v^{-w_1 \dim S_1 - w_1 \dim S_2} \sum_{[M], M_1 \in \mathbb{C}, \dim M_1 = w_1} u[M_1] \right)$$

$$= v^{-n \dim S_1 - n \dim S_2} \sum_{[M], M_1 \in \mathbb{C}, \dim M_1 = w_1} \ldots \sum_{[M], M_1 \in \mathbb{C}, \dim M_1 = w_1} \varphi_{M_1 \cdots M_t} (v^2) u[L]$$

$$= v^{-\dim \text{End} L} \varphi_{M_1 \cdots M_t} (v^2) \langle L \rangle$$

Using the above expression of $E_{m,3}$, we have

“the rest part” $= \sum_{j=1}^{s} \sum_{\dim M_j = \ldots = \dim M_t = n} v^{-\dim \text{End} M \varphi_{M_1 \cdots M_t} (v^2) \langle M \rangle} = E_{w_c,3}$. 

Applying the above expansion of $E_{w_c,3}$ to $(E_{n,3}, E_{w_c,3})$, and by Proposition 9.1.1 and Lemma 9.3.2, we have

$$(E_{n,3}, E_{w_c,3}) \equiv (E_{n,3}, E_{w_c,3}) \pmod{v^{-1}Q[[v^{-1}]] \cap Q(v)}.$$
Corollary 9.3.5

\[(E'_{\rho_0} * E'_{q_0}, E'_{q_0} * E'_{\rho_0}) = (E'_{\rho_0}, E'_{q_0}) (E'_{q_0}, E'_{\rho_0})\]

(3) \[(E'_{m_\delta} * E'_{n_\delta}, E'_{n_\delta} * E'_{m_\delta}) = \delta_{mn} (E'_{m_\delta}, E'_{n_\delta})(E'_{n_\delta}, E'_{m_\delta}).\]

(4) \[(E'_{m_\delta})^n, (E'_{n_\delta})^n = n! (E'_{m_\delta}, E'_{n_\delta})^n.\]

**Proof** We remark that if \(0 \to R \to M \to I \to 0\) is exact for a regular module \(R\) and a preinjective module \(I\), then \(M\) contains a nonzero direct summand being preinjective and contains no nonzero summand being preprojective. Dually, if \(0 \to P \to N \to R \to 0\) is exact for a preprojective module \(P\) and a regular module \(R\), then \(N\) contains a nonzero direct summand being preprojective and contains no nonzero summand being preinjective.

We may assume that \(p \geq m\), then \(n \geq q\). As we know \(\mathcal{R}(C^*(\Lambda))_{k^\delta} = \mathcal{R}(C^*(\Lambda))(k-1)_{k^\delta} \oplus \mathbb{Q}(v)E_{k^\delta}\), then

\[r(E'_{k^\delta}) \in E'_{k^\delta} \otimes 1 + 1 \otimes E'_{k^\delta} + \sum a_{ki} \otimes b_{ki} + \sum x_{ki} \otimes \langle I_{ki} \rangle \otimes \langle P_{ki} \rangle \otimes y_{ki},\]

where \(a_{ki}, b_{ki}, x_{ki}, y_{ki} \in \mathcal{R}(C^*(\Lambda))(j)\) and \(j < k\), \(I_{ki}\) is a nonzero preinjective module and \(P_{ki}\) is a nonzero preprojective module.

We have

\[(E'_{m_\delta} * E'_{n_\delta}, E'_{p_\delta} * E'_{q_\delta}) = (r(E'_{m_\delta}), E'_{p_\delta} \otimes E'_{q_\delta}) = (r(E'_{n_\delta}), E'_{p_\delta} \otimes E'_{q_\delta})\]

By the above formula of \(r(E'_{k^\delta})\) and the above remark, we have

\[(E'_{m_\delta} * E'_{n_\delta}, E'_{p_\delta} * E'_{q_\delta}) = ((E'_{m_\delta} \otimes 1 + 1 \otimes E'_{m_\delta}) + \sum a_{mi} \otimes b_{mi} (E'_{n_\delta} \otimes 1 + 1 \otimes E'_{n_\delta} + \sum a_{ni} \otimes b_{mi}), E'_{p_\delta} \otimes E'_{q_\delta})\]

If \(p > m\), it is easy to see that it vanishes. If \(p = m > q = n\), it is easy to see that it equals to \((E'_{p_\delta}, E'_{p_\delta})(E'_{q_\delta}, E'_{q_\delta})\). If \(p = q = m = n\), it equals to \(2(E'_{p_\delta}, E'_{p_\delta})^2\). In general, we have \((E'_{p_\delta})^l, (E'_{p_\delta})^l = l!(E'_{p_\delta}, E'_{p_\delta})^l\) for \(l \geq 0\).

Similarly, we can prove \((E'_{p_\delta} * E'_{q_\delta}, E'_{p_\delta} * E'_{q_\delta}) = (E'_{p_\delta}, E'_{p_\delta})(E'_{q_\delta}, E'_{q_\delta})\) for \(p > q\).

**Corollary 9.3.5** For \(m_i, n_i \in \mathbb{N}\) \((i = 1, \cdots, t)\) satisfying \(m_1 > \cdots > m_t\) and \(l_i, k_i \in \mathbb{N}\) \((i = 1, \cdots, j)\) satisfying \(l_1 > \cdots > l_j\), we have

\[((E'_{m_1})^{n_1} \cdots (E'_{m_t})^{n_t}, (E'_{l_1})^{k_1} \cdots (E'_{l_j})^{k_j}) = ((E'_{m_1})^{n_1}, (E'_{m_1})^{n_1}) \cdots ((E'_{m_t})^{n_t}, (E'_{m_t})^{n_t})\]

if \(t = j, m_i = l_i\) and \(n_i = k_i\) for all \(i = 1, \cdots, t\);

\[((E'_{m_1})^{n_1} \cdots (E'_{m_t})^{n_t}, (E'_{l_1})^{k_1} \cdots (E'_{l_j})^{k_j}) = 0\]

otherwise.

For a partition \(w = (w_1 \geq w_2 \geq \cdots \geq w_t)\), we define

\[E'_w = E'_{w_1} \cdots E'_{w_t}\]

**Lemma 9.3.6** Let \(\{E_\pi | \pi \in \Pi^*_i\}\) be the \(Z\)-basis of \(C^*(\mathcal{T}_i)_Z\) defined in Section 7.3. We have

\[(E_\pi * E'_m, E'_n) = \delta_{mn} (E_\pi, E'_m)(E'_n, E'_n)\]

and

\[(E_\pi, E'_w) = 0\]

**Proof** We may assume that \(m \leq n\). It follows
\[ r(E_π) = E_π ⊗ 1 + 1 ⊗ E_π + \sum_{π_1, π_2} c_{π_1, π_2} E_{π_1} ⊗ E_{π_2} + \sum_{π_1, π_2} d_{π_1, π_2} E_{π_1} * (I_{π_1, π_2}) ⊗ (P_{π_1, π_2}) * E_{π_2}, \]

where \( c_{π_1, π_2}, d_{π_1, π_2} \in Z \) and \( M(π), M(π_1), M(π_2) \) belong to \( T_τ \), \( I_{π_1, π_2} \) is a nonzero preinjective module and \( P_{π_1, π_2} \) is a nonzero preprojective module for all \( i \). As we know

\[ r(E_π') = E_π' ⊗ 1 + 1 ⊗ E_π' + \sum_{i} a_{mi} ⊗ b_{mi} + \sum_{i} x_{mi} * (I_{mi}) ⊗ (P_{mi}) * y_{mi}, \]

where \( a_{mi}, b_{mi}, x_{mi}, y_{mi} \in \mathcal{R}(C^*(Λ))(j), j < m \), and \( I_{mi} \) is a nonzero preinjective module and \( P_{mi} \) is a nonzero preprojective module for all \( i \).

The same calculation as that in the proof of Lemma 9.3.4 tells us

\[ (E_π * E_π' δ, E_π' * E_π' δ) = δ_{mn}(E_π, E_π')(E_π', E_π'). \]

The proof of the second identity is the same. \( \square \)

In Section 7.3, the set \( \{(M(\mathbf{a}_c)) * E_{π_1 c} * E_{π_2 c} * \cdots * E_{π_s c}(M(\mathbf{b}_c))\} \) is the basis of \( C^*(Λ) \). In the same way, we obtain

**Lemma 9.3.7** The following equalities hold

1. \( (\langle M(\mathbf{a}_c) \rangle * E_{π_1 c} * E_{π_2 c} * \cdots * E_{π_s c}, E'_{π_1 c} δ) = 0 \) for \( \mathbf{a}_c \neq 0 \) and partition \( w_{c'} \neq 0 \).

2. \( (E_π, E_{π_1 c} * \cdots * E_{π_s c}, E'_{π_1 c} δ) = 0 \) for \( w_{c} \neq 0 \).

3. \( (E_{π_1 c} * \cdots * E_{π_s c}, E_{π_1 c'} * \cdots * E_{π_s c'}, E'_{π_1 c'} δ) = 0 \) for \( w_{c'} \neq 0 \).

Based on Lemma 9.3.6 and Lemma 9.3.7, we obtain

**Lemma 9.3.8** The following holds

\[ (E_{π_{ic}} * E'_{π_{ic} c}, E_{π_{jc}} * E'_{π_{jc} c}) = (E_{π_{ic}}, E_{π_{jc}}, E'_{π_{ic}}, E'_{π_{jc}}), 1 \leq i, j \leq s. \]

**Proof** We know

\[ r(E_{π_{ic}}) = E_{π_{ic}} ⊗ 1 + 1 ⊗ E_{π_{ic}} + \sum_{π_1, π_2} c_{π_1, π_2} E_{π_1} ⊗ E_{π_2} + \sum_{π_1, π_2} d_{π_1, π_2} E_{π_1} * (I_{π_1, π_2}) ⊗ (P_{π_1, π_2}) * E_{π_2}, \]

where \( c_{π_1, π_2}, d_{π_1, π_2} \in Z \) and \( M(π), M(π_1), M(π_2) \) belong to \( T_τ \), \( I_{π_1, π_2} \) is a nonzero preinjective module and \( P_{π_1, π_2} \) is a nonzero preprojective module for all \( i \). Let

\[ r^0(E_{π_{ic}}) = E_{π_{ic}} ⊗ 1 + 1 ⊗ E_{π_{ic}} + \sum_{π_1, π_2} c_{π_1, π_2} E_{π_1} ⊗ E_{π_2} \]

and

\[ r^1(E_{π_{ic}}) = E_{π_{ic}} ⊗ 1 + 1 ⊗ E_{π_{ic}} \]
Also for $w_c = (w_1, \cdots, w_t)$

$$r(E'_{w_c\delta}) = r(E'_{w_1\delta}) \cdots \cdots r(E'_{w_t\delta})$$

$$= \left( E'_{w_1\delta} \otimes 1 + 1 \otimes E'_{w_1\delta} + \sum_i a_{w_i} \otimes b_{w_i} + \sum_i x_{w_i} \cdot \langle I_{w_i} \rangle \otimes \langle P_{w_i} \rangle \cdot y_{w_i} \right)$$

$$\cdots \cdots \left( E'_{w_t\delta} \otimes 1 + 1 \otimes E'_{w_t\delta} + \sum_i a_{w_i} \otimes b_{w_i} + \sum_i x_{w_i} \cdot \langle I_{w_i} \rangle \otimes \langle P_{w_i} \rangle \cdot y_{w_i} \right)$$

Let

$$r^0(E'_{w_c\delta}) = (E'_{w_1\delta} \otimes 1 + 1 \otimes E'_{w_1\delta}) \cdots \cdots (E'_{w_t\delta} \otimes 1 + 1 \otimes E'_{w_t\delta})$$

and

$$r^1(E'_{w_c\delta}) = (E'_{w_1\delta} \otimes 1 + 1 \otimes E'_{w_1\delta}) \cdots \cdots (E'_{w_t\delta} \otimes 1 + 1 \otimes E'_{w_t\delta})$$

It is clear that

$$(E_{\pi_{ce}} \ast E'_{w_c\delta}, E_{\pi_{je'}} \ast E'_{w_c\delta}) = (r(E_{\pi_{ce}}) \ast r(E'_{w_c\delta}), E_{\pi_{je'}} \otimes E'_{w_c\delta}) = (r^0(E_{\pi_{ce}}) \ast r^0(E'_{w_c\delta}), E_{\pi_{je'}} \otimes E'_{w_c\delta})$$

Based on Lemma 9.3.6 and Lemma 9.3.7, we use an induction on $|w_c\delta|$ to obtain that

$$r^0(E_{\pi_{ce}}) \ast r^0(E'_{w_c\delta}), E_{\pi_{je'}} \otimes E'_{w_c\delta}) = (r^1(E_{\pi_{ce}}) \ast r^1(E'_{w_c\delta}), E_{\pi_{je'}} \otimes E'_{w_c\delta})$$

$$= ((E_{\pi_{ce}} \otimes 1 + 1 \otimes E_{\pi_{ce}}) \ast (E_{w_{c\delta}} \otimes 1 + 1 \otimes E_{w_{c\delta}}), E_{\pi_{je'}} \otimes E'_{w_{c\delta}}) = (E_{\pi_{ce}}, E_{\pi_{je'}})(E'_{w_{c\delta}}, E'_{w_{c\delta}})$$

The proof is finished. \(\square\)

Finally, we have

**Theorem 9.3.9** With the same notations as above, the following holds

$$(E_{\pi_{1ce}} \ast \cdots \ast E_{\pi_{sc}} \ast E'_{w_{c\delta}}, E_{\pi_{1ce'}} \ast \cdots \ast E_{\pi_{se'}} \ast E'_{w_{c\delta}}) = (E_{\pi_{1ce}}, E_{\pi_{1ce'}})(E_{\pi_{se}}, E_{\pi_{se'}}) \cdots (E'_{w_{c\delta}}, E'_{w_{c\delta}}).$$

### 9.4

In this subsection, we come to construct the canonical basis. Let $\mathcal{T}$ be the isomorphism classes of indecomposable objects in the non-homogeneous tubes $\mathcal{T}_1, \ldots, \mathcal{T}_s$ and $\text{add}(\mathcal{T})$ be the objects that are direct sums of objects in $\mathcal{T}$.

Theorem 9.3.9 and Corollary 9.3.5 imply that $(E_{n_\delta} \ast E'_{n_\delta} - E'_{n_\delta} \ast E''_{n_\delta}, x) = 0$ for all $x \in \mathcal{R}(\mathcal{C}^*(\Lambda))$. Thus $E_{n_\delta} \ast E'_{n_\delta} = E'_{n_\delta} \ast E''_{n_\delta}$ by the nondegeneracy of $(\langle \cdot, \cdot \rangle)$ on the $\mathcal{R}(\mathcal{C}^*(\Lambda))$.

**Lemma 9.4.1** Assume $\sum_{i=1}^s \dim M(\pi_{ic}) + |w_c|\delta = n\delta$. Then

1. $(E_{n_\delta}, E_{\pi_{1c}} \ast \cdots \ast E_{\pi_{sc}} \ast E_{w_{c\delta}}) \in \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)$,
2. moreover, if $|w_c|\delta < n\delta$, then $(E_{n_\delta}, E_{\pi_{1c}} \ast \cdots \ast E_{\pi_{sc}} \ast E_{w_{c\delta}}) \in v^{-1}\mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)$,
3. $(E_{\pi_{1c}} \ast \cdots \ast E_{\pi_{sc}} \ast E_{w_{c\delta}}, E_{\pi_{1c}} \ast \cdots \ast E_{\pi_{se}} \ast E_{w_{c\delta}}) \in v^h(N + v^{-1}\mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v))$ for some $h \geq 0$.

**Proof** By the proof of Lemma 9.3.3, we have

$$E_{w_{c\delta}} = E_{w_{1\delta}} \ast \cdots \ast E_{w_{1\delta}} + E_{w_{1\delta}} \ast \cdots \ast E_{w_{1\delta}} + \text{"the rest part"},$$

where

$$E_{w_{1\delta}} \ast \cdots \ast E_{w_{1\delta}} = \sum_{[L], 0 \neq L \in \mathcal{C}} v^{-\dim \text{End} L} \varphi_{M_1 \cdots M_t}(v^2)(L)$$
and 

\[ \text{"the rest part"} = \sum_{[M_1], \ldots, [M_s]} \sum_{\lambda \in \Pi \setminus \Pi_{\gamma}, \lambda < \pi_{ie}} \sum_{j=1}^s \dim M_j = 0, \lambda_c \leq (w_e) \langle w_e \rangle] \]

If \( M(\pi_{ie}) = 0 \) for all \( 1 \leq i \leq s \), then both (1) and (3) are true by the property of the complete symmetric function with respect to the pair \((-,-)\), see subsection 9.2.

Suppose \( M(\pi_{ie}) \neq 0 \) for some \( i \). By [DDX],

\[ E_{\pi_{ie}} = \langle M(\pi_{ie}) \rangle + \sum_{\lambda \in \Pi \setminus \Pi_{\gamma}, \lambda < \pi_{ie}} \eta_\lambda \langle M(\lambda) \rangle \]

where \( \eta_\lambda \in v^{-1} \mathbb{Z}[v^{-1}] \).

\[ E_{\pi_{ie}} \cdots \cdots E_{\pi_{se}} = \left( \bigoplus_{i=1}^s M(\pi_{ie}) \right) + \sum_{N \in \text{add}(T_{ij} \mid i = 1, \ldots, s)} \eta_N \langle N \rangle \]

where \( \eta_N \in v^{-1} \mathbb{Z}[v^{-1}] \).

\[ E_{\pi_{ie}} \cdots \cdots E_{\pi_{se}} \ast E_{w_1 \delta_1} \ast \cdots \ast E_{w_t \delta_t} = \sum_{[L], 0 \neq L \in \mathcal{C}_i} v^{-\dim \text{End}_{L} \varphi_{M_1 \cdots M_t}(v^2) \left( \bigoplus_{i=1}^s M(\pi_{ie}) \right) \ast \langle L \rangle} \]

\[ + \sum_{N \in \text{add}(IT), [L], 0 \neq L \in \mathcal{C}_i} v^{-\dim \text{End}_{L} \varphi_{M_1 \cdots M_t}(v^2) \eta_N \langle N \rangle \ast \langle L \rangle} \]

Here,

\[ \left( \bigoplus_{i=1}^s M(\pi_{ie}) \right) \ast \langle L \rangle = \sum_{[U]} v^{\dim \text{End}_{U} \bigoplus_{i=1}^s M(\pi_{ie}) + \dim \text{End}_{L} - \dim \text{End}_{U} \varphi_{\bigoplus_{i=1}^s M(\pi_{ie}), L}(v^2) \langle U \rangle} \]

and

\[ \langle N \rangle \ast \langle L \rangle = \sum_{[V]} v^{\dim \text{End}_{N} + \dim \text{End}_{L} - \dim \text{End}_{V} \varphi_{N,L}(v^2) \langle V \rangle} \]

By Lemma 9.3.2, we know

\[ \deg_{q} \varphi_{\bigoplus_{i=1}^s M(\pi_{ie}), L} \leq \dim \text{End}_{U} - (\dim \text{End}_{L} + \dim \text{End}_{\bigoplus_{i=1}^s M(\pi_{ie})}) \]

\[ \deg_{q} \varphi_{M_1 \cdots M_t} \leq \dim \text{End}_{L} - \sum_{i=1}^t \dim \text{End}_M \]

and

\[ \deg_{q} \varphi_{N,L} \leq \dim \text{End}_{V} - (\dim \text{End}_{N} + \dim \text{End}_{L}) \]

Hence,

\[ E_{\pi_{ie}} \ast \cdots \ast E_{\pi_{se}} \ast E_{w_1 \delta_1} \ast \cdots \ast E_{w_t \delta_t} \]

\[ = \sum_{0 \neq [U] \in IT} v^{\dim \text{End}_{U} - \dim \text{End}_{\bigoplus_{i=1}^s M(\pi_{ie})} - 2 \sum_{i=1}^t \dim \text{End}_M} f_U(v^{-1}) \langle U \rangle \]

\[ + \sum_{0 \neq [V] \in IT} v^{\dim \text{End}_{V} - \dim \text{End}_{N} - 2 \sum_{i=1}^t \dim \text{End}_M} f_V(v^{-1}) \langle V \rangle \]

where \( f_U(v^{-1}), f_V(v^{-1}) \in \mathbb{Q}[v^{-1}] \).
In general,
\[ E_{\pi_1c} \ast \cdots \ast E_{\pi_sc} \ast E_{w_c, \delta} = \sum_{0 \neq [L] \in \mathcal{I}T} f_L(L) + \sum_{0 \neq [M] \in \mathcal{I}T, 1 \leq |(w'_c)| < |w_c|} f_M(M) \ast E_{w'_c, \delta, 3} \]
\[ + \langle M(\pi_1c) \oplus \cdots \oplus M(\pi_sc) \rangle \ast E_{w_c, \delta, 3} + \sum_{N \neq 0 \in \mathcal{I}T} f_N(N) \ast E_{w_c, \delta, 3}, \]
where \( v^{-\dim \text{End}(L)} f_L, v^{-\dim \text{End}(M)} f_M, f_N \in v^{-1}Q[v^{-1}]. \)

Using expressions of \( E_{n\delta, 1}, E_{n\delta, 2}, E_{n\delta, 3} \), it is easy to see that
\[ (E_{n\delta}, E_{\pi_1c} \ast \cdots \ast E_{\pi_sc} \ast E_{w_c, \delta}) \in v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v). \]

Then the conclusion (2) is proved.

By Lemma 9.3.9 and the property of complete symmetric function, we have
\[ ((M(\pi_1c) \oplus \cdots \oplus M(\pi_sc)) \ast E_{w_c, \delta, 3}) \ast M(\pi_1c) \oplus \cdots \oplus M(\pi_sc) \ast E_{w_c, \delta, 3}) \in \mathbb{N}^s + v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v) \]

Thus
\[ (E_{\pi_1c} \ast \cdots \ast E_{\pi_sc} \ast E_{w_c, \delta}, E_{\pi_1c} \ast \cdots \ast E_{\pi_sc} \ast E_{w_c, \delta}) \in v^h(N + v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v)) \]
for some \( h \geq 0. \) Then the conclusion (3) is proved. \( \square \)

Let \( \mathcal{A} = \mathcal{Q}[v, v^{-1}]. \) The lattice \( \mathcal{L} \) defined in [L5] is the \( \mathcal{Q}[v^{-1}] \)-submodule of \( \mathcal{C}^*(\Lambda)_{\mathcal{A}} \) which is characterized by
\[ \mathcal{L} = \{ x \in \mathcal{C}^*(\Lambda) \mid \langle x, x \rangle \in \mathcal{Q}[v^{-1}] \cap \mathcal{Q}(v) \} \]

Lemma 9.4.2 We have \( E'_{n\delta} \in \mathcal{L} \) and \( (E'_{n\delta}, E'_{n\delta}) \equiv 1/n \pmod{v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v)}. \)

Proof We know \( \{ E_{\pi_sc} \mid \dim M(\pi_ic) = \delta, 1 \leq i \leq s \} \) is the basis of \( \mathcal{R}^s(\mathcal{C}^*(\Lambda))_\delta. \) Via Schmidt orthogonalization, we define \( E'_{\pi_sc} \) to be the orthogonal element corresponding to \( E_{\pi_sc} \) for \( i = 1, \cdots, s. \)

It is easy to see that \( (E'_{\pi_sc}, E'_{\pi_sc}) \equiv 1 \pmod{v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v)}. \) Thus
\[ E'_\delta = E_\delta - \sum_{\pi} (E_{\pi_SC}, E'_{\pi_SC}) E'_{\pi_SC} \]

Hence, \( (E'_\delta, E'_\delta) \equiv 1 \pmod{v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v)}). \)

Now suppose \( E'_{n\delta} \in \mathcal{L} \) and \( (E'_{n\delta}, E'_{n\delta}) \equiv 1/n \pmod{v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v)} \) for all \( n \) with \( n < k. \)

By the definition of \( E'_{k\delta} \), we also have \( \{ E_{\pi_sc} \ast \cdots \ast E_{\pi_sc} \ast E'_{w_c, \delta}, E_{n\delta}, \sum_{1 \leq i \leq s} \dim M(\pi_ic) + |w_c|\delta = n\delta \}, \) for some \( \dim M(\pi_ic) \neq 0 \) is a basis of \( \mathcal{R}(\mathcal{C}^*(\Lambda))_{n\delta}. \)

By Lemma 9.3.9 and the induction hypothesis, \( \{ E_{\pi_sc} \ast \cdots \ast E_{\pi_sc} \ast E'_{w_c, \delta}, E_{n\delta} \} \subset \mathcal{L}. \) Similar as in Lemma 9.4.1, we may get \( (E_{n\delta}, E_{\pi_sc} \ast \cdots \ast E_{\pi_sc} \ast E'_{w_c, \delta}) \in v^{-1}Q[[v^{-1}]] \cap \mathcal{Q}(v) \) if there exists \( i \) such that \( M(\pi_ic) \neq 0. \)

Thus
\[ E'_{n\delta} \equiv E_{n\delta} - \sum_{w_c = n, w_c \neq (n)} (E_{n\delta}, E'_{w_c, \delta}) (E'_{w_c, \delta}) \pmod{v^{-1}\mathcal{L}}. \]

In addition, \( (E'_{n\delta}, E'_{n\delta}) = (E_{n\delta}, E'_{n\delta}) \) by Schmidt orthogonalization.

We now claim that
\[ (E_{n\delta}, E'_{w_c, \delta}) = (E'_{w_1}, E'_{w_1})^{k_1} (E'_{w_2}, E'_{w_2})^{k_2} \cdots (E'_{w_s}, E'_{w_s})^{k_s} \]
if \( w_c = (w_1^{k_1}, w_2^{k_2}, \ldots, w_t^{k_t}) \), \( w_1 > w_2 > \cdots > w_t \).

As we known
\[
   r(E_{n\delta}) = \sum_{0 \leq i \leq n} E_{i\delta} \otimes E_{(n-i)\delta} + \text{"the rest part"}.\]

Let \( r^0(E_{n\delta}) = \sum_{0 \leq i \leq n} E_{i\delta} \otimes E_{(n-i)\delta} \) and \( w'_c = (w_1^{k_1-1}, w_2^{k_2}, \ldots, w_t^{k_t}) \). Then
\[
   (E_{n\delta}, E'_{w_c\delta}) = (E_{n\delta}, E_{w'_1\delta} \otimes E'_{w_2\delta}) = (r^0(E_{n\delta}), E'_{w_1\delta} \otimes E'_{w'_2\delta}).\]

Based on the definition of \( E'_{w_1\delta} \) and Lemma 9.3.9, we have
\[
   (E_{n\delta}, E'_{w_1\delta}) = (E_{w_1\delta}, E'_{w_1\delta})(E_{(n-w_1)\delta}, E'_{w'_2\delta}).\]

By the induction hypothesis, we obtain
\[
   (E_{n\delta}, E'_{w_1\delta}) = (E'_{w_1\delta})^{k_1}(E'_{w_2\delta})^{k_2} \cdots (E'_{w_t\delta})^{k_t}.\]

Since \( n! = \sum \sum_{(1^r, 2^r \ldots) - n} \prod_{i \geq 1} n_i / r_i ! \), we have
\[
   (E'_{n\delta}, E'_{n\delta}) \equiv (E_{n\delta}, E_{n\delta}) - \sum_{w_c \neq n} \frac{(E_{n\delta}, E'_{w_c\delta})^2}{(E_{w_1\delta}, E'_{w_1\delta})} \equiv 1 - \sum_{(n) \neq (1^r, 2^r \ldots) - n} \prod_{i \geq 1} n_i / r_i ! \bmod (n-1)! \equiv 1/n \mod v^{-1}Q[[v^{-1}]] \cap Q(v).\]

So the proof is completed.

Let \( P_{n\delta} = nE'_{n\delta} \). For a partition \( w_c = (1^r, 2^r \ldots) \), let \( z_{w_c} = \prod_{i \geq 1} i^{r_i} r_i ! \) and \( P_{w_c\delta} = P_{s_{r_1}} \cdots P_{s_{r_t}} \).

**Corollary 9.4.3** Let \( w_c = (1^r, 2^r \ldots) \), \( w_{c'} = (1^{r'}, 2^{r'} \ldots) \) be partitions. Then

1. \[
   (E_{\pi_{1e}} \cdots E_{\pi_{se}} \ast E'_{w_c\delta}, E_{\pi_{1e}}' \cdots E_{\pi_{se}}' \ast E'_{w_{c'}\delta}) \equiv \delta_{\pi_{1e}, \pi_{1e}'} \cdots \delta_{\pi_{1e}, \pi_{1e}'} \delta_{w_c, w_{c'}} \prod_i r_i ! (E_{i\delta}, E_{i\delta}') \bmod v^{-1}Q[[v^{-1}]] \cap Q(v)\]
2. \[
   (P_{n\delta}, P_{n\delta}) \equiv n \mod v^{-1}Q[[v^{-1}]] \cap Q(v)\]

By this property of \( P_{w_c\delta} \), it is easy to see that \( P_{w_c\delta} \) corresponds to Newton symmetric functions.

Let \( S_{w_c\delta} \) be the Schur functions corresponding to \( P_{w_c\delta} \), and \( e_c = \langle M(a_c) \rangle \ast E_{\pi_{1e}} \ast E_{\pi_{2e}} \cdots \ast E_{\pi_{se}} \ast S_{w_c\delta} \ast \langle M(b_c) \rangle \) for \( c \in M \).

By Theorem 9.3.9, Lemma 9.4.2 Corollary 9.4.3 and the Nakayama Lemma, we have the following corollary:

**Corollary 9.4.4** \( \{e^c | c \in M\} \) is an almost orthonormal basis of \( L \), that is, \( (e^c, e^{c'}) \in \delta_{c, c'} + v^{-1}Q[[v^{-1}]] \cap Q(v) \) for \( c \) and \( c' \in M \).

We have defined the constructible set in Section 8.1
\[
   \mathcal{O}_c = O_{M(a_c)} \ast O_{M_{s_{1e}}} \ast O_{M_{s_{2e}}} \cdots \ast O_{M_{s_{se}}} \ast N_{w_c\delta} \ast O_{M(b_c)}
\]
for any $c \in M$.

Now we define a new partial order $\prec$ for those $e^c, c \in M$ with the same weight (dimension vector) as follow:

1. $e^c \prec e^{c'}$ if $\dim \mathcal{O}_c < \dim \mathcal{O}_{c'}$.
2. $e^c \prec e^{c'}$ if $\dim \mathcal{O}_c = \dim \mathcal{O}_{c'}$ but $w_c > w_{c'}$.

Base on the definition of $E'_{n\delta}$, we have

$$E'_{n\delta} = E_{n\delta} - \sum_{w_c \succ n, w_c \neq (n)} \left( \frac{E_{n\delta}}{E'}_{w_c \delta} \right) E'_{w_c \delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} a_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta$$

where $a_{n\delta, \delta'} \in \mathbb{Q}(v)$, in fact, by Corollary 9.4.4, we have $a_{n\delta, \delta'} \in \mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)$. Thus

$$E_{n\delta} = E'_{n\delta} + \sum_{w_c \succ n, w_c \neq (n)} \left( \frac{E_{n\delta}}{E'}_{w_c \delta} \right) E'_{w_c \delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} a_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta.$$

Let $H_{n\delta}$ be the nth complete symmetric function corresponding to $P_{n\delta}$. From [M], p25, we have

$$E_{n\delta} = H_{n\delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} a_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta.$$

Let $w_c$ be a partition of $n$, according to Lemma 2.2 and the above formula, we have

$$E_{w_c \delta} = H_{w_c \delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{w_c \delta}} a_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta.$$

We have the monomial $m_{w_c \delta}$ on the divided powers of $u_{S_i}, i \in I$, in Proposition 8.4, corresponding to $E_{w_c \delta}$, such that

$$m_{w_c \delta} = H_{w_c \delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} b_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} c_{n\delta, \delta} e^{e'}\delta \varepsilon$$

$$= S_{w_c \delta} + \sum_{w_{c'} > w_c} K_{w_c \delta} S_{w_{c'} \delta} + \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} b_{n\delta, \delta} E_{\pi_1} \ast E_{\pi_2} \cdots \ast E_{\pi_{se'}} \ast S_{w_{c'}} \delta$$

$$+ \sum_{\dim \mathcal{O}_{\delta} < \dim \mathcal{O}_{n\delta}} c_{n\delta, \delta} e^{e'} \varepsilon$$

where $K_{\mu \lambda}$ are Kostka numbers and $b_{n\delta, \delta}, c_{n\delta, \delta} \in \mathbb{Q}(v)$. Furthermore, for $c \in M$ and the monomials $m_c$ given in Proposition 8.4, we have

$$m_c = e^c + \sum_{e' \prec e} a_{e' c} e^{e'}$$

where $a_{e' c} \in \mathbb{Q}(v)$. However Proposition 8.4 and the above formulae tell us that the transition matrix between $\{E^c | c \in M\}$ and $\{e^c | c \in M\}$ is triangular with diagonal entries equal to 1, and $\{E^c | c \in M\}$ is an $A$-basis of $\mathcal{C}^*(\Lambda) \mathcal{A}$, $\{e^c | c \in M\} \subset \mathcal{L}$ and $\{m_c | c \in M\} \subset \mathcal{C}^*(\Lambda) \mathcal{A}$. Then the constants $a_{e' c}$ in the above formulae must lie in $\mathcal{A}$.
By applying the same argument as in Section 8 to \( \{e^c | c \in \mathcal{M}\} \), we obtain an \( \mathcal{A} \)-basis of \( \mathcal{C}(\Lambda)_{\mathcal{A}} \) which is denoted by \( \{E^{\mathcal{E}c} | c \in \mathcal{M}\} \) satisfying that
\[
E^{\mathcal{E}c} = \sum_{c' \in \mathcal{M}} \zeta_{c} c' e^{\mathcal{E}c'} \quad \text{for any } c \in \mathcal{M}
\]
where \( \zeta_{c} = 1 \) and \( \zeta_{c} \in v^{-1} \mathbb{Q}[v^{-1}] \) if \( e^{\mathcal{E}c'} \prec e^{c} \).

Finally, we have following theorem:

**Theorem 9.4.5** The set \( \{E^{\mathcal{E}c} | c \in \mathcal{M}\} \subset \mathcal{L} \) provides an \( \mathcal{A} \)-basis of \( \mathcal{C}(\Lambda)_{\mathcal{A}} \) which is characterized by the following three properties:

(a) \( \mathcal{E} \mathcal{E}c = \mathcal{E} \mathcal{E}c \) for all \( c \in \mathcal{M} \).

(b) \( \pi(E^{\mathcal{E}c}) = \pi(e^{c}) \), where \( \pi : \mathcal{L} \rightarrow \mathcal{L}/v^{-1}\mathcal{L} \) is the canonical projection.

(c) \( (E^{\mathcal{E}c}, E^{\mathcal{E}c'}) \equiv \delta_{cc'} \pmod{v^{-1}\mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)} \).

According to Lusztig [L5], we obtain the signed canonical basis \( (E^{\mathcal{E}c}) \) of \( \mathcal{L} \). From the above formulae, we have the relations
\[
m_{c} = E^{\mathcal{E}c} + \sum_{c' \prec c} d_{c'c} E^{\mathcal{E}c'}
\]
where \( d_{c'c} \in \mathcal{A} \). By the total positivity of the canonical bases, we have

**Theorem 9.4.7** The set \( \{E^{\mathcal{E}c} | c \in \mathcal{M}\} \) is the canonical basis of \( \mathcal{L} \) in the sense of Lusztig.

This answers a question raised by Nakajima in [N].

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