IDENTIFYING AF-ALGEBRAS THAT ARE GRAPH $C^*$-ALGEBRAS

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Abstract. We consider the problem of identifying exactly which AF-algebras are isomorphic to a graph $C^*$-algebra. We prove that any separable, unital, Type I $C^*$-algebra with finitely many ideals is isomorphic to a graph $C^*$-algebra. This result allows us to prove that a unital AF-algebra is isomorphic to a graph $C^*$-algebra if and only if it is a Type I $C^*$-algebra with finitely many ideals. We also consider nonunital AF-algebras that have a largest ideal with the property that the quotient by this ideal is the only unital quotient of the AF-algebra. We show that such an AF-algebra is isomorphic to a graph $C^*$-algebra if and only if its unital quotient is Type I, which occurs if and only if its unital quotient is isomorphic to $M_k$ for some natural number $k$. All of these results provide vast supporting evidence for the conjecture that an AF-algebra is isomorphic to a graph $C^*$-algebra if and only if each unital quotient of the AF-algebra is Type I with finitely many ideals, and bear relevance for the intriguing question of finding $K$-theoretical criteria for when an extension of two graph $C^*$-algebras is again a graph $C^*$-algebra.

1. Introduction

Since the introduction of graph $C^*$-algebras in the 1990s, it has been observed that graph $C^*$-algebras contain numerous AF-algebras. Indeed, Drinen proved that every AF-algebra is Morita equivalent to a graph $C^*$-algebra [4]. At the same time, it is easily seen that there are AF-algebras that are not isomorphic to any graph $C^*$-algebra. For example, the only commutative graph $C^*$-algebras that are AF-algebras are the direct sums of complex numbers, so any commutative AF-algebra that is not isomorphic to the direct sum of copies of $C$ (for instance, the $C^*$-algebra of continuous complex-valued functions on the Cantor set) is not isomorphic to a graph $C^*$-algebra. This has led to the natural question of determining exactly which AF-algebras are isomorphic to graph $C^*$-algebras.

An extensive exploration of this question was undertaken by Sims together with the second and fourth named authors in [10], where they not only investigated which AF-algebras are isomorphic to graph $C^*$-algebras, but also which AF-algebras are isomorphic to Exel-Laca $C^*$-algebras, and which AF-algebras are isomorphic to ultragraph $C^*$-algebras. A complete answer to this question for the class of graph $C^*$-algebras was not obtained in [10], although many useful partial results were deduced. In particular, if one restricts to

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the class of row-finite graphs with no sinks, the question has been completely answered: An AF-algebra is isomorphic to the $C^*$-algebra of a row-finite graph with no sinks if and only if it has no unital quotients [10, Theorem 4.7]. In addition, an interesting necessary condition for an AF-algebra to be isomorphic to a graph $C^*$-algebra was obtained in [10, Proposition 4.21], where it is shown that an AF graph $C^*$-algebra has the property that all of its unital quotients are Type I $C^*$-algebras with finitely many ideals. This naturally leads one to conjecture that the converse is true. We state this conjecture here, and we will refer to it throughout the paper.

**Conjecture:** An AF-algebra is isomorphic to a graph $C^*$-algebra if and only if every unital quotient of the AF-algebra is a Type I $C^*$-algebra with finitely many ideals.

As we have mentioned, [10, Proposition 4.21] establishes the “only if” direction of the conjecture, so the open question is to determine whether the “if” direction holds. Also, we observe that the result in [10, Theorem 4.7] is consistent with the conjecture, since it states that any AF-algebra with no unital quotients (which therefore vacuously satisfies the condition of the conjecture) is isomorphic to the $C^*$-algebra of a row-finite graph with no sinks.

In this paper we prove results that provide mounting evidence in support of this conjecture. After some preliminaries in Section 2, we consider Type I $C^*$-algebras in Section 3 and prove in Theorem 3.13 that any separable, unital, Type I $C^*$-algebra with finitely many ideals is isomorphic to a graph $C^*$-algebra. This allows us to give a complete description of the unital AF-algebras that are isomorphic to graph $C^*$-algebras, and in Corollary 3.14 we prove that a unital AF-algebra is isomorphic to a graph $C^*$-algebra if and only if it is a Type I $C^*$-algebra with finitely many ideals. This result supports the conjecture mentioned above, since all quotients of a unital Type I $C^*$-algebra with finitely many ideals are also unital Type I $C^*$-algebras with finitely many ideals.

In the remainder of the paper we consider nonunital AF-algebras that have a unital quotient. However, this situation here is much more difficult than the unital case. Indeed, we restrict our attention to nonunital AF-algebras that have a largest ideal (i.e., a proper ideal that contains all other ideals) with the property that the quotient by this ideal is the only unital quotient of the AF-algebra. Studying these nonunital AF-algebras requires a subtle analysis of Bratteli diagrams, and we spend Section 4 developing the needed technical lemmas. In Section 5 we prove in Theorem 5.7 that if $A$ is an AF-algebra with a largest ideal having the property that the quotient by this ideal is the only unital quotient, then $A$ is isomorphic to a graph $C^*$-algebra if and only if this unital quotient is a Type I $C^*$-algebra, which is also equivalent to the unital quotient being isomorphic to $M_k$ for some natural number $k$. This result provides additional support for the conjecture mentioned earlier, since these AF-algebras have exactly one unital quotient, and this unital quotient is simple. Moreover, unlike the result for unital $C^*$-algebras in Section 3, our result in Theorem 5.7 is entirely constructive, and shows exactly how to build the $C^*$-algebra from a Bratteli diagram for the AF-algebra.

Combining our results for unital and nonunital AF-algebras, we are also able to show in Theorem 5.9 that the conjecture from above holds for all AF-algebras with exactly one proper nonzero ideal. Finally, we end the paper with an alternate proof of [10, Theorem 4.7].
The original proof in [10] shows that an AF-algebra with no unital quotients is isomorphic to a graph $C^*$-algebra in an indirect way, through the use of ultragraphs. Our alternate proof in Theorem 5.10 shows exactly how to construct the necessary graph from a Bratteli diagram for the AF-algebra.

The results presented here bear relevance for the intriguing question of finding $K$-theoretical criteria for when an extension of AF graph $C^*$-algebra, and our main results confirming this in key cases may be used to close the gap (cf. [5]) in our present knowledge and complete the picture when both $C^*(E)$ and $C^*(F)$ are simple.

2. Background and Preliminaries

A graph $E = (E^0, E^1, r, s)$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges, and maps $r: E^1 \to E^0$ and $s: E^1 \to E^0$ identifying the range and source of each edge. A path in a graph $E = (E^0, E^1, r, s)$ is a finite sequence of edges $\alpha := e_1 \ldots e_n$ with $s(e_{i+1}) = r(e_i)$ for $1 \leq i \leq n - 1$. We say that $\alpha$ has length $n$, and we write $|\alpha|$ for the length of $\alpha$. We regard vertices as paths of length 0 and edges as paths of length 1, and we then extend our notation for the vertex set and the edge set by writing $E^n$ for the set of paths of length $n$ for all $n \geq 0$. We write $E^*$ for the set $\bigcup_{n=0}^{\infty} E^n$ of paths of finite length, and extend the maps $r$ and $s$ to $E^*$ by setting $r(v) = s(v) = v$ for $v \in E^0$, and $r(\alpha_1 \ldots \alpha_n) = r(\alpha_n)$ and $s(\alpha_1 \ldots \alpha_n) = s(\alpha_1)$.

If $\alpha$ and $\beta$ are elements of $E^*$ such that $r(\alpha) = s(\beta)$, then $\alpha \beta$ is the path of length $|\alpha| + |\beta|$ obtained by concatenating the two. Given $\alpha, \beta \in E^*$, and a subset $X \subseteq E^*$, we define

$$\alpha X \beta := \{ \gamma \in E^* : \gamma = \alpha \gamma' \beta \text{ for some } \gamma' \in X \}.$$ 

So when $v$ and $w$ are vertices, we have

$$vX = \{ \gamma \in X : s(\gamma) = v \},$$

$$Xw = \{ \gamma \in X : r(\gamma) = w \},$$

$$vXw = \{ \gamma \in X : s(\gamma) = v \text{ and } r(\gamma) = w \}.$$ 

In particular, $vE^1w$ denotes the set of edges from $v$ to $w$ and $|vE^1w|$ denotes the number of edges from $v$ to $w$. Furthermore, if $V \subseteq E^0$, $W \subseteq E^0$, and $X \subseteq E^*$, we define

$$VXW := \{ \alpha \in X : s(\alpha) \in V \text{ and } r(\alpha) \in W \}.$$ 

We say a vertex $v$ is a sink if $vE^1 = \emptyset$ and an infinite emitter if $vE^1$ is infinite. A singular vertex is a vertex that is either a sink or an infinite emitter. A graph is called row-finite if it has no infinite emitters.
Definition 2.1. If $E = (E^0, E^1, r, s)$ is a graph, then the graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

1. $s_e^*s_e = p_{r(e)}$ for all $e \in E^1$
2. $p_v = \sum_{e \in vE^1} s_es_e^*$ for all $v \in E^0$ such that $0 < |vE^1| < \infty$
3. $s_es_e^* \leq p_{s(e)}$ for all $e \in E^1$.

We write $v \geq w$ to mean that there is a path $\alpha \in E^*$ such that $s(\alpha) = v$ and $r(\alpha) = w$. A cycle in a graph $E$ is a path $\alpha \in E^*$ of nonzero length with $r(\alpha) = s(\alpha)$. A graph is called acyclic if it has no cycles. A graph $C^*$-algebra $C^*(E)$ is an AF-algebra if and only if $E$ is acyclic [11, Theorem 2.4].

Definition 2.2. If $E = (E^0, E^1, r, s)$ is a graph, a subset $H \subseteq E^0$ is called hereditary if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. A hereditary set $H$ is called saturated if whenever $v \in E^0$ is a vertex that is neither a sink nor an infinite emitter, then $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

If $H$ is a saturated hereditary subset of $E^0$, we define the graph $E_H := (E^0, E^1_H, r_{E_H}, s_{E_H})$ as follows: The vertex set is $E^0_H := E^0 \setminus H$, the edge set is $E^1_H := s^{-1}(H)$, and the range and source maps are $r_{E_H} := r|_{E^1_H}$ and $s_{E_H} := s|_{E^1_H}$, which are obtained by restricting $r$ and $s$ to $E^1_H$.

3. **Unital Type I $C^*$-algebras and unital AF-algebras**

In this section, we prove that any unital separable Type I $C^*$-algebra with finitely many ideals is isomorphic to a graph $C^*$-algebra. We obtain this result in two steps: First, we show that any unital, separable, Type I $C^*$-algebra with finitely many ideals is stably isomorphic to a the $C^*$-algebra of an amplified graph with finitely many vertices (see Definition 3.6 and Proposition 3.8). Second, we show that if $\overline{G}$ is an acyclic amplified graph with a finite number of vertices, then any full unital corner of the stabilization $C^*(\overline{G}) \otimes K$ is isomorphic to a graph $C^*$-algebra (see Proposition 3.11). To do this it will be convenient for us to apply theorems for $C^*$-algebras classified by their tempered primitive ideal space, so we begin by establishing the necessary terminology and preliminary results.

3.1. **The tempered primitive ideal space of a $C^*$-algebra**. Let $X$ be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of $X$ partially ordered by set inclusion. A subset $Y \subseteq X$ is called locally closed if $Y = U \setminus V$ where $U, V \in \mathcal{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathbb{L}(X)$.

The partially ordered set $(\mathcal{O}(X), \subseteq)$ is a lattice with meet and join given by $Y_1 \cap Y_2 := Y_1 \cap Y_2$ and $Y_1 \vee Y_2 := Y_1 \cup Y_2$, respectively. For a $C^*$-algebra $\mathfrak{A}$, we let $l(\mathfrak{A})$ denote the set of closed ideals of $\mathfrak{A}$. The partially ordered set $(l(\mathfrak{A}), \subseteq)$ is also a lattice with meet and join given by $I_1 \cap I_2 := I_1 \cap I_2$ and $I_1 \vee I_2 := \overline{I_1 + I_2}$. If $\mathfrak{A}$ is a $C^*$-algebra, we let $\text{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$ equipped with the usual hull-kernel topology. For any $C^*$-algebra $\mathfrak{A}$, the lattices $\mathcal{O}(\text{Prim}(\mathfrak{A}))$ and $l(\mathfrak{A})$ are isomorphic via the lattice isomorphism

$$U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$$
We shall frequently identify $O(\text{Prim}(A))$ and $\mathcal{L}(A)$ in this way.

**Definition 3.1.** Let $X$ be a topological space. A $C^*$-algebra over $X$ is a pair $(A,\psi)$ consisting of a $C^*$-algebra $A$ and a continuous map $\psi: \text{Prim}(A) \to X$.

If $(A,\psi)$ is a $C^*$-algebra over $X$, we have a map $\psi^*: O(X) \to O(\text{Prim}(A))$ defined by

$$U \mapsto \{p \in \text{Prim}(A) : \psi(p) \in U\}.$$  

Using the isomorphism $O(\text{Prim}(A)) \cong \mathcal{L}(A)$, we obtain a map from $O(X)$ to $\mathcal{L}(A)$ given by $U \mapsto A[U]$, where

$$A[U] := \bigcap \{p \in \text{Prim}(A) : \psi(p) \notin U\}.$$  

If $Y \in \mathcal{L}(X)$, we may write $Y = U \setminus V$ for open sets $U,V \subseteq X$ with $V \subseteq U$, and we define $A[Y] := A[U]/\mathcal{A}[V]$. It follows from [12, Lemma 2.15] that $A[Y]$ is independent of the choice of $U$ and $V$.

**Remark 3.2.** Any $C^*$-algebra $A$ can be viewed as a $C^*$-algebra over $\text{Prim}(A)$ by taking $\psi := \text{id} : \text{Prim}(A) \to \text{Prim}(A)$. In this case we shall simply write $A$ in place of $(A,\text{id})$.

**Definition 3.3 (The Tempered Primitive Ideal Space).** Let $A$ be a $C^*$-algebra, and view $A$ as a $C^*$-algebra over $\text{Prim}(A)$. Define $\tau_A : \text{Prim}(A) \to \mathbb{Z} \cup \{-\infty, \infty\}$ by

$$\tau_A(x) := \begin{cases} -\text{rank}(K_0(A[x])) & \text{if } K_0(A[x])_+ \neq K_0(A[x]) \\ \text{rank}(K_0(A[x])) & \text{if } K_0(A[x])_+ = K_0(A[x]). \end{cases}$$

The **tempered primitive ideal space** of $A$ is defined to be the pair

$$\text{Prim}^\tau(A) := (\text{Prim}(A), \tau_A).$$

If $A$ and $B$ are $C^*$-algebras, we say that $\text{Prim}^\tau(A)$ and $\text{Prim}^\tau(B)$ are isomorphic, denoted by $\text{Prim}^\tau(A) \cong \text{Prim}^\tau(B)$, if there exists a homeomorphism $\alpha : \text{Prim}(A) \to \text{Prim}(B)$ such that $\tau_B \circ \alpha = \tau_A$.

**Definition 3.4** (Definition 6.1 of [8]). Let $C$ be the class of separable, nuclear, simple, purely infinite $C^*$-algebras satisfying the UCT that have free $K_0$-group and zero $K_1$-group.

Let $C_\text{free}$ be class of all $C^*$-algebras $A$ satisfying all of the following four properties:

1. $\text{Prim}(A)$ is finite.
2. For each $x \in \text{Prim}(A)$, the subquotient $A[x]$ is either unital or stable.
3. For each $x \in \text{Prim}(A)$, the subquotient $A[x]$ is either in $C$ or stably isomorphic to $K$.
4. For each $x \in \text{Prim}(A)$, if the subquotient $A[x]$ is unital, then there exists an isomorphism from $K_0(A[x])$ onto $\bigoplus_n \mathbb{Z}$ that takes $[1_{A[x]}]$ to an element of the form $(1, \lambda) \in \bigoplus_n \mathbb{Z}$ for some $\lambda$.

It turns out that $C^*$-algebras in $C_\text{free}$ are classified up to stable isomorphism by the tempered primitive ideal space.

**Theorem 3.5.** (Theorem 6.17 of [8]) If $A,B \in C_\text{free}$, then $A \otimes K \cong B \otimes K$ if and only if $\text{Prim}^\tau(A) \cong \text{Prim}^\tau(B)$. 

3.2. Realizing unital Type I $C^*$-algebras as graph $C^*$-algebras.

**Definition 3.6.** If $E = (E^0, E^1, r, s)$ is graph, we say that $E$ is an **amplified graph** if for all $e \in E^1$ the number of edges from $s(e)$ to $r(e)$ is countably infinite.

If $G = (G^0, G^1, r, s)$ is a graph, the **amplification of $G$** is defined to be the graph $\overline{G} = (\overline{G}^0, \overline{G}^1, r_{\overline{G}}, s_{\overline{G}})$ defined by $\overline{G}^0 := G^0$,
\[ \overline{G}^1 := \{ (e, w)^n : v, w \in G^0, n \in \mathbb{N}, \text{and there exists an edge from } v \text{ to } w \}, \]
$s_{\overline{G}}((v, w)^n) := v$, and $r_{\overline{G}}((v, w)^n) := w$.

Note that a graph $E$ is an amplified graph if and only if $E = \overline{G}$ for some graph $G$, and in this case the graph $G$ may always be chosen to be row-finite with the same number of vertices as $E$. Because of this, we shall often write an amplified graph as $\overline{G}$ for a row-finite graph $G$.

**Lemma 3.7.** If $\mathfrak{A}$ is a separable Type I $C^*$-algebra, then every simple subquotient of $\mathfrak{A}$ is isomorphic to either $\mathbb{K}$ or $M_n$ for some $n \in \mathbb{N}$. If, in addition, $\mathfrak{A}$ has finitely many ideals, then $\mathfrak{A} \otimes \mathbb{K} \in C_{\text{free}}$.

**Proof.** Since $\mathfrak{A}$ is a separable Type I $C^*$-algebra, every simple subquotient of $\mathfrak{A}$ is a separable Type I $C^*$-algebra, and hence isomorphic to either $\mathbb{K}$ or $M_n$ for some $n \in \mathbb{N}$. Moreover, since every simple subquotient of $\mathfrak{A}$ is isomorphic to $\mathbb{K}$ or $M_n$ for some $n \in \mathbb{N}$, we have that $(\mathfrak{A} \otimes \mathbb{K})[x]$ is isomorphic to $\mathbb{K}$ for all $x \in \text{Prim}(\mathfrak{A} \otimes \mathbb{K})$. Thus Property 2, Property 3, and Property 4 of Definition 3.4 hold for $\mathfrak{A} \otimes \mathbb{K}$ (with Property 4 holding vacuously). In addition, if $\mathfrak{A}$ has finitely many ideals, then $\mathfrak{A} \otimes \mathbb{K}$ has finitely many ideals, so that $\text{Prim}(\mathfrak{A})$ is finite, Property 1 of Definition 3.4 is satisfied by $\mathfrak{A} \otimes \mathbb{K}$, and $\mathfrak{A} \otimes \mathbb{K} \in C_{\text{free}}$. \qed

**Proposition 3.8.** If $\mathfrak{A}$ is a unital separable Type I $C^*$-algebra with finitely many ideals, then there exists a finite graph $G$ such that $\mathfrak{A} \otimes \mathbb{K} \cong C^*(\overline{G}) \otimes \mathbb{K}$.

**Proof.** Lemma 3.7 shows that $\mathfrak{A} \otimes \mathbb{K} \in C_{\text{free}}$ and every simple subquotient of $\mathfrak{A} \otimes \mathbb{K}$ is isomorphic to $\mathbb{K}$. Since every simple subquotient of $\mathfrak{A} \otimes \mathbb{K}$ has $K_0$-group isomorphic to $\mathbb{Z}$ and $\mathfrak{A} \otimes \mathbb{K}$ has finitely many ideals, it follows that
\[ K_0(\mathfrak{A} \otimes \mathbb{K}) \cong \bigoplus_{x \in \text{Prim}(\mathfrak{A} \otimes \mathbb{K})} K_0((\mathfrak{A} \otimes \mathbb{K})[x]) \cong \bigoplus_{x \in \text{Prim}(\mathfrak{A} \otimes \mathbb{K})} \mathbb{Z}. \]

For any $x \in \text{Prim}(\mathfrak{A} \otimes \mathbb{K})$ we have $(\mathfrak{A} \otimes \mathbb{K})[x] \cong \mathbb{K}$ and hence $\tau_{\mathfrak{A} \otimes \mathbb{K}}(x) = \{-1\} \subseteq \{-1\} \cup \mathbb{N}$. Therefore [8, Theorem 7.3] shows there exists a finite graph $G$ such that
\[ \text{Prim}^\tau(\mathfrak{A} \otimes \mathbb{K}) \cong \text{Prim}^\tau(\overline{G} \otimes \mathbb{K}). \quad (3.1) \]

Since $\overline{G}$ is an amplified graph, every vertex of $\overline{G}$ is a singular vertex and $\overline{G}$ has no breaking vertices. It follows from [8, Proposition 6.10] that $C^*(\overline{G}) \otimes \mathbb{K} \in C_{\text{free}}$. In addition, Lemma 3.7 implies $\mathfrak{A} \otimes \mathbb{K} \in C_{\text{free}}$. By Theorem 3.5, (3.1) implies $\mathfrak{A} \otimes \mathbb{K} \cong C^*(\overline{G}) \otimes \mathbb{K}$. \qed

Proposition 3.8 shows to establish that $\mathfrak{A} \otimes \mathbb{K}$ is isomorphic to a graph $C^*$-algebra, it suffices to show any full unital corner of $C^*(\overline{G}) \otimes \mathbb{K}$ is isomorphic to a graph $C^*$-algebra. We shall accomplish this by examining the range of the order unit of $C^*(\overline{G})$. A more systematic study of hereditary subalgebras of graph $C^*$-algebras will appear in work in preparation by the third named author together with Sara Arklint and James Gabe [1].
Lemma 3.9. Let $\overline{G}$ be an acyclic amplified graph with a finite number of vertices, and let $S = \{ v \in \overline{G}^0 : v \text{ is a source in } \overline{G} \}$. Let $H \subseteq \bigoplus_{v \in \overline{G}^0} \mathbb{Z}$ be the monoid generated by

$$\left\{ \delta_v : v \in \overline{G}^0 \right\} \cup \left\{ \delta_v - \sum_{e \in T} \delta_{\gamma(e)} : v \in \overline{G}^0 \text{ and } T \text{ is a finite subset of } s^{-1}(v) \right\}.$$

If $(n_v)_{v \in \overline{G}^0} \in H$ with $n_v \geq 1$ for all $v \in S$, then there exists $(m_v)_{v \in \overline{G}^0} \in H$ with $m_v \geq 1$ for all $v \in \overline{G}^0$ and there exists an isomorphism $\alpha : \bigoplus_{v \in \overline{G}^0} \mathbb{Z} \to \bigoplus_{v \in \overline{G}^0} \mathbb{Z}$ such that $\alpha(H) = H$ and $\alpha \left( (n_v)_{v \in \overline{G}^0} \right) = (m_v)_{v \in \overline{G}^0}$.

Proof. For each $v \in \overline{G}^0$, set $T_v := \{ w \in S : w \geq v \}$. Note that $T_v$ is a finite set because $\overline{G}$ has a finite number of vertices, and $T_v$ is nonempty since $v \in T_v$. For each $v \in \overline{G}^0$ and $w \in T_v$, let $k_{v,w}$ denote the smallest element of $\mathbb{N} \cup \{0\}$ such that $n_w k_{v,w} + n_v \geq 1$. (Note that if $v \in S$, then $T_v = \{ v \}$ and $k_{v,v} = 0$ since $n_v \geq 1$ by hypothesis.)

Define $(m_v)_{v \in \overline{G}^0} \in H$ by $m_v := n_v + \sum_{w \in T_v} n_w k_{v,w}$ for $v \in \overline{G}^0$. Observe that $m_v \geq 1$ for all $v \in \overline{G}^0$ and $m_v = n_v$ for all $v \in S$. Also define a homomorphism $\alpha : \bigoplus_{v \in \overline{G}^0} \mathbb{Z} \to \bigoplus_{v \in \overline{G}^0} \mathbb{Z}$ by

$$\alpha(\delta_v) = \begin{cases} \delta_v + \sum_{w \in \overline{G}^0 \setminus T_v} k_{v,w}\delta_w & \text{if } v \in S \\ \delta_v & \text{if } v \in \overline{G}^0 \setminus S \end{cases}$$

One can verify that $\alpha$ is an isomorphism with inverse given by the homomorphism $\beta : \bigoplus_{v \in \overline{G}^0} \mathbb{Z} \to \bigoplus_{v \in \overline{G}^0} \mathbb{Z}$ with

$$\beta(\delta_v) = \begin{cases} \delta_v - \sum_{w \in \overline{G}^0 \setminus T_v} k_{v,w}\delta_w & \text{if } v \in S \\ \delta_v & \text{if } v \in \overline{G}^0 \setminus S \end{cases}.$$

In addition,

$$\alpha \left( (n_v)_{v \in \overline{G}^0} \right) = \alpha \left( \sum_{v \in \overline{G}^0} n_v \delta_v \right) = \alpha \left( \sum_{v \in S} n_v \delta_v \right) + \alpha \left( \sum_{w \in \overline{G}^0 \setminus S} n_w \delta_w \right)$$

$$= \sum_{v \in S} n_v \left( \delta_v + \sum_{w \in \overline{G}^0 \setminus T_v} k_{v,w} \delta_w \right) + \sum_{w \in \overline{G}^0 \setminus S} n_w \delta_w$$

$$= \sum_{v \in S} n_v \delta_v + \sum_{w \in \overline{G}^0 \setminus S} \left( n_w + \sum_{v \in T_w} n_v k_{w,v} \right) \delta_w = \sum_{v \in S} m_v \delta_v + \sum_{w \in \overline{G}^0 \setminus S} m_w \delta_w = (m_v)_{v \in \overline{G}^0}.$$
fact that \( T_w \) is finite for all \( w \in \mathcal{G}^0 \), we see that \( \alpha(\delta_v) \in H \). Next, let \( v \in \mathcal{G}^0 \) and let \( T \) be a finite subset of \( s^{-1}(v) \).

If \( v \notin S \), then \( r(e) \notin S \) for all \( e \in T \), and \( \alpha \left( \delta_v - \sum_{e \in T} \delta_r(e) \right) = \delta_v - \sum_{e \in T} \delta_r(e) \in H \). If \( v \in S \), then \( r(e) \notin S \) for all \( e \in T \), and

\[
\alpha \left( \delta_v - \sum_{e \in T} \delta_r(e) \right) = \left( \delta_v - \sum_{e \in T} \delta_r(e) \right) + \sum_{w \in \mathcal{G}^0 \text{ with } v \in T_w} k_{v,w} \delta_v \in H.
\]

Since these elements generate \( H \) and \( \alpha \) is a homomorphism, we may conclude that \( \alpha(H) \subseteq H \). A nearly identical argument shows that \( \beta(H) \subseteq H \). Hence \( \alpha(H) = H \). \( \square \)

**Definition 3.10.** An element \( a \) in a \( C^* \)-algebra \( \mathfrak{A} \) is said to be full if \( a \) is not contained in a proper ideal of \( \mathfrak{A} \).

**Proposition 3.11.** Let \( \mathcal{G} \) be an acyclic amplified graph with a finite number of vertices, and let \( p \) be a full projection in \( C^*(\mathcal{G}) \otimes \mathbb{K} \). Then there exists a graph \( E \) with finitely many vertices such that \( p(C^*(\mathcal{G}) \otimes \mathbb{K})p \cong C^*(E) \).

**Proof.** It follows from [13, Theorem 2.2] that \( K_0(C^*(\mathcal{G})) \cong \bigoplus_{v \in \mathcal{G}} \mathbb{Z} \) via an isomorphism taking \( K_0(C^*(\mathcal{G}))_+ \) onto the monoid \( H \subseteq \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \) generated by

\[
\left\{ \delta_v : v \in \mathcal{G}^0 \right\} \cup \left\{ \delta_v - \sum_{e \in T} \delta_{r(e)} : v \in \mathcal{G}^0_{\text{inf}} \text{ and } T \text{ is a finite subset of } s^{-1}(v) \right\}.
\]

Denote this isomorphism from \( K_0(C^*(\mathcal{G})) \) to \( \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \) by \( \phi \).

Let \( p \) be a full projection in \( C^*(\mathcal{G}) \otimes \mathbb{K} \). Let \( S = \left\{ v \in \mathcal{G}^0 : v \text{ is a source in } \mathcal{G} \right\} \). Then \( \phi([p]_0) = (n_v)_{v \in \mathcal{G}} \) such that \( n_v \geq 0 \) for all \( v \in \mathcal{G}^0 \). Since \( p \) is a full projection, we must have that \( n_v \geq 1 \) for all \( v \in S \). By Lemma 3.9, there exists \( (m_v)_{v \in \mathcal{G}^0} \in H \) with \( m_v \geq 1 \) for all \( v \in \mathcal{G}^0 \) and there exists an isomorphism \( \alpha : \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \to \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \) such that \( \alpha(H) = H \) and \( \alpha \left( (n_v)_{v \in \mathcal{G}^0} \right) = (m_v)_{v \in \mathcal{G}^0} \).

Define a directed graph as follows: Set

\[
E^0 := \mathcal{G}^0 \cup \left\{ w_{k,v} : v \in \mathcal{G}^0 \text{ and } 1 \leq k \leq m_v - 1 \right\}
\]

and

\[
E^1 := \mathcal{G}^1 \cup \left\{ e_{k,v} : v \in \mathcal{G}^0 \text{ and } 1 \leq k \leq m_v - 1 \right\}.
\]

Define \( s_{E|\mathcal{G}} = s_{\mathcal{G}} \) and \( r_{E|\mathcal{G}} = r_{\mathcal{G}} \). Also define \( s_{E}(e_{k,v}) = w_{k,v} \) for all \( k \) and \( v \), and define \( r_{E}(e_{k,v}) = w_{k+1,v} \) when \( 1 \leq k \leq m_v - 2 \) and \( r_{E}(e_{m_v-1,v}) = v \). Using [13, Theorem 2.2] and the fact that \( E \) is obtained by adding finite heads at the vertices of \( \mathcal{G}^0 \), we have that \( K_0(C^*(E)) \) is isomorphic to \( \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \) via an isomorphism taking \( K_0(C^*(E))_+ \) onto the monoid \( H \subseteq \bigoplus_{v \in \mathcal{G}^0} \mathbb{Z} \) generated by

\[
\left\{ \delta_v : v \in \mathcal{G}^0 \right\} \cup \left\{ \delta_v - \sum_{e \in T} \delta_{r(e)} : v \in \mathcal{G}^0_{\text{inf}} \text{ and } T \text{ is a finite subset of } s^{-1}(v) \right\},
\]
and furthermore, this isomorphism takes $[1_{C^*(E)}]_0$ to $(m_v)_{v \in G \cup}$. Denote this isomorphism from $K_0(C^*(E))$ to $\bigoplus_{v \in \overline{G}} \mathbb{Z}$ by $\gamma$.

Let $\iota: p(C^*(\overline{G}) \otimes \mathbb{K})p \hookrightarrow C^*(\overline{G}) \otimes \mathbb{K}$ be the inclusion map. Then

$$K_0(\iota): K_0(p(C^*(\overline{G}) \otimes \mathbb{K})p) \to K_0(C^*(\overline{G}))$$

is an order isomorphism with $K_0(\iota) \left( [1_{p(C^*(\overline{G}) \otimes \mathbb{K})p}]_0 \right) = [p]_0$. Hence $\gamma^{-1} \circ \alpha \circ \phi \circ K_0(\iota)$ is an order isomorphism from $K_0(p(C^*(\overline{G}) \otimes \mathbb{K})p)$ to $K_0(C^*(E))$ with

$$(\gamma \circ \alpha \circ \phi \circ K_0(\iota)) \left( [1_{p(C^*(\overline{G}) \otimes \mathbb{K})p}]_0 \right) = (\gamma \circ \alpha \circ \phi)([p]_0) = (\gamma \circ \alpha) \left( (m_v)_{v \in \overline{G}} \right)
= \gamma \left( (m_v)_{v \in \overline{G}} \right) = [1_{C^*(E)}]_0.$$

Since $\overline{G}$ and $E$ are acyclic graphs with finitely many vertices, $p(C^*(\overline{G}) \otimes \mathbb{K})p$ and $C^*(E)$ are unital AF-algebras. Hence, by Elliott’s classification theorem for AF-algebras [9], we have $p(C^*(\overline{G}) \otimes \mathbb{K})p \cong C^*(E)$.

**Example 3.12.** The proof of Proposition 3.11 actually shows how to construct the graph $E$ from the graph $G$ in the proposition’s statement, provided we know the class of $[p]_0$ in $K_0(C^*(\overline{G}))$. For example, suppose $G$ is the graph

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\end{array}
\]

and suppose $p$ is full projection in $C^*(\overline{G}) \otimes \mathbb{K}$ such that $[p]_0$ is identified with $(3, 2)$ in $K_0(C^*(\overline{G})) \cong \mathbb{Z} \oplus \mathbb{Z}$. If we define $E$ to be the graph

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\end{array}
\]

then the proof of Proposition 3.11 shows that $p(C^*(\overline{G}) \otimes \mathbb{K})p \cong C^*(E)$.

**Theorem 3.13.** If $\mathcal{A}$ is a separable, unital, Type I $C^*$-algebra with finitely many ideals, then $\mathcal{A}$ is isomorphic to a graph $C^*$-algebra.

**Proof.** By Proposition 3.8 there exists a finite graph $G$ such that $\mathcal{A} \otimes \mathbb{K} \cong C^*(\overline{G}) \otimes \mathbb{K}$. Let $\{e_{ij}\}$ be a system of matrix units for $\mathbb{K}$. Since $\mathcal{A} \cong (1_{\mathcal{A}} \otimes e_{11})(\mathcal{A} \otimes \mathbb{K})(1_{\mathcal{A}} \otimes e_{11})$ and $1_{\mathcal{A}} \otimes e_{11}$ is a full projection in $\mathcal{A} \otimes \mathbb{K}$, we have that $\mathcal{A} \cong p(C^*(\overline{G}) \otimes \mathbb{K})p$ for some full projection $p$ in $C^*(\overline{G}) \otimes \mathbb{K}$. By Proposition 3.11 $p(C^*(\overline{G}) \otimes \mathbb{K})p \cong C^*(E)$ for some graph $E$. Hence $\mathcal{A} \cong C^*(E)$.

We conclude this section by deducing necessary and sufficient conditions for a unital AF-algebra to be a graph $C^*$-algebra, and observe that this result provides further support for the conjecture from the introduction.

**Corollary 3.14.** A unital AF-algebra is isomorphic to a graph $C^*$-algebra if and only if it is a Type I $C^*$-algebra with finitely many ideals.

**Proof.** Suppose $\mathcal{A}$ is a unital AF-algebra. If $\mathcal{A}$ is a Type I $C^*$-algebra with finitely many ideals, then $\mathcal{A}$ is isomorphic to a graph $C^*$-algebra by Theorem 3.13. Conversely, if $\mathcal{A}$ is
isomorphic to a graph $C^*$-algebra, then [10, Proposition 4.21] shows that $\mathfrak{A}$ is a Type I $C^*$-algebra with a finite number of ideals.

4. Technical lemmas for Bratteli diagrams

In Section 5 we shall need a number of technical results about Bratteli diagrams for certain AF-algebras. This section is devoted to proving these lemmas, with our primary goal being the proof of Lemma 4.12 at the end of the section.

**Definition 4.1.** A Bratteli diagram $(E, d_E)$ consists of a graph $E = (E^0, E^1, r_E, s_E)$ and a degree function $d_E : E^0 \to \mathbb{N}$ such that

1. $E$ has no sinks;
2. $E^0$ is partitioned as disjoint sets $E^0 = \bigsqcup_{n=1}^\infty W_n$ with each $W_n$ a finite set;
3. for each $e \in E^1$, there exists $n \in \mathbb{N}$ such that $s_E(e) \in W_n$ and $r_E(e) \in W_{n+1}$;
4. for each $v \in E^0$,
   \[ d_E(v) \geq \sum_{e \in E^1 v} d_E(s_E(e)). \]

We call $W_n$ the $n$th level of the Bratteli diagram, and when we write $E^0 = \bigsqcup_{n=1}^\infty W_n$, we say $E^0$ is partitioned into levels by the $W_n$.

**Definition 4.2.** Let $(E, d_E)$ with $E^0$ be a Bratteli diagram partitioned into levels as $E^0 = \bigsqcup_{n=1}^\infty W_n$. For any increasing subsequence $\{n_m\}_{m=1}^\infty$ of $\mathbb{N}$, we define a new Bratteli diagram $(F, d_F)$ as follows:

1. The set of vertices is partitioned into levels as $F^0 := \bigsqcup_{m=1}^\infty W_{n_m}$;
2. The set of edges is $F^1 := \bigcup_{m=1}^\infty W_{n_m} E^* W_{n_{m+1}}$ with the range and source map as defined on the paths of $E$; and
3. $d_F = d_E|_{F^0}$.

We call $(F, d_F)$ a telescope of $(E, d_E)$.

We say that two Bratteli diagrams $(E, d_E)$ and $(F, d_F)$ are equivalent (sometimes also called telescope equivalent) if there is a finite sequence of Bratteli diagrams $(E_1, d_{E_1}), \ldots, (E_n, d_{E_n})$ such that $(E_1, d_{E_1}) = (E, d_E)$, $(E_n, d_{E_n}) = (F, d_F)$, and for each $1 \leq i \leq n - 1$, one of $(E_i, d_{E_i})$ and $(E_{i+1}, d_{E_{i+1}})$ is a telescope of the other. Bratteli proved in [2] that two Bratteli diagrams give rise to isomorphic AF-algebras if and only if the diagrams are equivalent (see [2, §1.8 and Theorem 2.7]).

The following lemma is contained implicitly in the proof of [10, Lemma 3.2]. For the convenience of the reader, we provide an explicit proof here.

**Lemma 4.3.** Let $(E, d_E)$ be a Bratteli diagram for the $C^*$-algebra $\mathfrak{A}$ with $E^0$ partitioned into levels as $E^0 = \bigsqcup_{n=1}^\infty W_n$. Suppose $v \in W_n$ and $k \leq n$. If $w \in E^0$ with $v \geq w$ and

\[ d_E(v) = \sum_{\alpha \in W_k E^* w} d_E(s_E(\alpha)), \]

then

\[ d_E(v) = \sum_{\alpha \in W_k E^* v} d_E(s_E(\alpha)). \]
In addition, if for every \( v \in E^0 \) there exists \( w \in E^0 \) with \( v \geq w \) and
\[
d_E(w) = \sum_{\alpha \in W_1 E^* w} d_E(s_E(\alpha)),
\]
then \( \mathfrak{A} \) is a unital AF-algebra.

Proof. Note that
\[
d_E(w) = \sum_{\alpha \in W_1 E^* w} d_E(s_E(\alpha)) = \sum_{\beta \in W_n E^* w} \left( \sum_{\gamma \in W_k E^* s_E(\beta)} d_E(s_E(\gamma)) \right)
\]
\[
\leq \sum_{\beta \in W_n E^* w} d_E(s_E(\beta)) \leq d_E(w)
\]
and hence we have equality throughout. In particular, we deduce
\[
d_E(s_E(\beta)) = \sum_{\gamma \in W_k E^* s_E(\beta)} d_E(s_E(\gamma)).
\]
for each \( \beta \in W_n E^* w \). Since \( v \geq w \), there exists \( \beta \in W_n E^* w \) such that \( s_E(\beta) = v \). Thus
\[
d_E(v) = \sum_{\gamma \in W_k E^* v} d_E(s_E(\gamma)).
\]

For the second part of the lemma, suppose that for every \( v \in F^0 \) there exists \( w \in F^0 \) with \( v \geq w \) and
\[
d_E(w) = \sum_{\alpha \in W_1 E^* w} d_E(s_E(\alpha)),
\]
To show that \( \mathfrak{A} \) is unital, it is enough to show that for each \( v \in \bigcup_{n=2}^\infty W_n \) we have
\[
d_E(v) = \sum_{e \in E^1_v} d_E(s_E(e)). \tag{4.1}
\]
We shall obtain this fact by induction on \( n \). For the base case of \( n = 2 \), we suppose \( v \in W_2 \). By hypothesis there exists \( w \in F^0 \) such that \( v \geq w \) and
\[
d_E(w) = \sum_{\alpha \in W_1 E^* w} d_E(s_E(\alpha)).
\]
It follows from the first part of this lemma, and the fact that \( v \in W_2 \), that
\[
d_E(v) = \sum_{\alpha \in W_1 E^* v} d_E(s_E(\alpha)) = \sum_{e \in E^1_v} d_E(s_E(e)).
\]
For the inductive step suppose \( n \geq 2 \) and (4.1) holds for all vertices in \( W_n \). Let \( v \in W_{n+1} \). By hypothesis, there exists \( w \in F^0 \) with \( v \geq w \) and
\[
d_E(w) = \sum_{\alpha \in W_1 E^* w} d_E(s_E(\alpha)).
\]
Thus
\[ d_E(v) = \sum_{\beta \in W_n}^{\beta \in W_n} \left( \sum_{\gamma \in W_1}^{\gamma \in W_1} d_E(s Е(\gamma)) \right) \leq \sum_{\beta \in W_n}^{\beta \in W_n} d_E(s Е(\beta)) \]
\[ = \sum_{e \in E^1 v} d_E(s Е(e)) \leq d_E(v). \]
Therefore, we have equality throughout and \( d_E(v) = \sum_{e \in E^1 v} d_E(s Е(e)). \) By induction (4.1) holds for all \( v \in \bigcup_{n=2}^{\infty} W_n. \)

**Lemma 4.4.** Let \((E, d_E)\) be a Bratteli diagram for \( M_k \) with \( E^0 \) partitioned into levels as \( E^0 = \bigcup_{n=1}^{\infty} W_n. \) Then there exists \( m \in \mathbb{N} \) such that for each \( n \geq m, W_n = \{w_n\} \) is a singleton set and \( d_E(w_n) = k. \)

**Proof.** Writing \( M_k \) as the direct limit coming from the Bratteli diagram, there exists an increasing sequence of finite-dimensional \( C^*\)-subalgebras \( \mathfrak{A}_n \) of \( M_k \), and hence there exists \( N \in \mathbb{N} \) such that \( 1_{\mathfrak{A}_n} = 1_{M_k} \) for all \( n \geq N \). Thus, for each \( n \geq N, k = \sum_{v \in W_n} d_E(v). \) Since \( M_k = \bigcup_{n=1}^{\infty} \mathfrak{A}_n \), there exists \( m \geq N \) such that for each \( n \geq m \), \( \dim_{\mathbb{C}}(M_k) = \dim_{\mathbb{C}}(\mathfrak{A}_n) \). Therefore, for each \( n \geq m, \)
\[ \left( \sum_{v \in W_n} d_E(v) \right)^2 = k^2 = \dim_{\mathbb{C}}(M_k) = \dim_{\mathbb{C}}(\mathfrak{A}_n) = \sum_{v \in W_n} d_E(v)^2. \]
Since the \( d_E(v) \) are non-negative integers, it follows that \( W_n = \{w_n\} \) is a singleton set and \( d_E(w_n) = k. \)

**Definition 4.5.** Let \((E, d_E)\) be a Bratteli diagram. A saturated, hereditary subset \( H \) of \( E^0 \) is a **largest saturated, hereditary subset of \( E^0 \)** if whenever \( Y \) is a saturated, hereditary subset of \( E^0 \), then either \( Y \subseteq H \) or \( Y = E^0 \).

**Definition 4.6.** Let \( \mathfrak{A} \) be a \( C^*\)-algebra. An ideal \( \mathfrak{J} \) of \( \mathfrak{A} \) is **essential** if for every nonzero ideal \( \mathfrak{K} \) of \( \mathfrak{A}, \mathfrak{J} \cap \mathfrak{K} \neq 0. \) An ideal \( \mathfrak{J} \) of \( \mathfrak{A} \) is a **largest ideal** of \( \mathfrak{A} \) if whenever \( \mathfrak{K} \) is an ideal of \( \mathfrak{A}, \) then either \( \mathfrak{K} \subseteq \mathfrak{J} \) or \( \mathfrak{K} = \mathfrak{A}. \)

**Remark 4.7.** Note that if \( \mathfrak{J} \) is a largest ideal of \( \mathfrak{A}, \) then \( \mathfrak{J} \) is unique, \( \mathfrak{J} \) is an essential ideal of \( \mathfrak{A}, \) and \( \mathfrak{A}/\mathfrak{J} \) is a simple \( C^*\)-algebra.

**Definition 4.8.** For any \( k \in \mathbb{N}, \) we say a Bratteli diagram \((E, d_E)\) is **\( M_k \)-separated** if it satisfies the following five properties:

1. \( E^0 = \bigcup_{n=1}^{\infty} W_n \) is partitioned into levels with \( W_n = H_n \cup \{y_n\}. \)
2. \( H_n E^0 y_{n+1} = \emptyset \) for all \( n \in \mathbb{N}. \)
3. \( d_E(y_n) = k \) for all \( n \in \mathbb{N}. \)
4. \( |y_n E^0 y_{n+1}| = 1 \) for all \( n \in \mathbb{N}. \)
5. \( y_n E^0 H_{n+1} \neq \emptyset \) for all \( n \in \mathbb{N}. \)

In addition, we say that a Bratteli diagram \((E, d_E)\) is **properly \( M_k \)-separated** if it is \( M_k \)-separated and satisfies the additional property:

6. For each \( n \in \mathbb{N} \) and \( v \in H_n \) we have
\[ d_E(v) > \sum_{e \in E^1 v} d_E(s Е(e)). \]
Remark 4.9. Note that if \((E, d_E)\) is an \(M_k\)-separated Bratteli diagram, the set \(H := \bigsqcup_{n=1}^{\infty} H_n\) is a largest saturated hereditary subset of \(E^0\). In addition, if \(\mathfrak{A}\) is the \(C^*\)-algebra associated with \((E, d_E)\), and \(\mathfrak{I}\) is the ideal in \(\mathfrak{A}\) associated with \(H\), then \(\mathfrak{I}\) is an essential ideal of \(\mathfrak{A}\), and the quotient \(\mathfrak{A}/\mathfrak{I}\) is an AF-algebra with Bratteli diagram \(k \to k \to k \to \ldots\), so that \(\mathfrak{A}/\mathfrak{I} \cong M_k\). The following lemma shows that, conversely, any AF-algebra with \(M_k\) as a quotient by an essential ideal has an \(M_k\)-separated Bratteli diagram.

Lemma 4.10. Let \(\mathfrak{A}\) be an AF-algebra with an essential ideal \(\mathfrak{I}\) such that \(\mathfrak{A}/\mathfrak{I} \cong M_k\). Then any Bratteli diagram for \(\mathfrak{A}\) can be telescoped to an \(M_k\)-separated Bratteli diagram.

Proof. Let \((E, d_E)\) be a Bratteli diagram of \(\mathfrak{A}\) with \(E^0 = \bigsqcup_{n=1}^\infty V_n\) partitioned into levels, and let \(S\) be the hereditary saturated subset of \(E^0\) that corresponds to the ideal \(\mathfrak{I}\). Then the Bratteli diagram obtained by restricting to \(E^0 \setminus S = \bigsqcup_{n=1}^\infty (V_n \setminus S)\) is a Bratteli diagram for \(\mathfrak{A}/\mathfrak{I}\). Since \(\mathfrak{A}/\mathfrak{I} \cong M_k\), by Lemma 4.4 there exists \(m \in \mathbb{N}\) such that for each \(n \geq m\), \(|V_n \setminus S| = 1\) and \(d_E(x_n) = k\), where \(\{x_n\} = V_n \setminus S\). Note that for each \(n \geq m\), \(V_n = (V_n \setminus S) \cup \{x_n\}\). To obtain the result, we shall establish two claims.

Claim 1: There exists infinitely many \(n\) such that \(x_n E^1 (V_{n+1} \cap S) \neq \emptyset\). Suppose not. Then there exists \(N \geq m\) such that for all \(n \geq N\), \(x_n E^1 (V_{n+1} \cap S) = \emptyset\) and \(x_n E^1 (V_{N+1} \cap S) = \emptyset\). Then \(\bigcup_{n=N+1}^\infty \{x_n\}\) is a saturated, hereditary subset of \(E^0\) disjoint from \(S\), which corresponds to a nonzero ideal \(\mathfrak{K}\) of \(\mathfrak{A}\) such that \(\mathfrak{I} \cap \mathfrak{K} = \emptyset\), contradicting the fact that \(\mathfrak{I}\) is an essential ideal of \(\mathfrak{A}\).

Claim 2: There exists \(m_1 \geq m\) such that for all \(n \geq m_1\) we have \((V_n \cap S) E^1 x_{n+1} = \emptyset\). Suppose not. Then for each \(m_1 \geq m\), there exists \(n \geq m_1\) such that \((V_n \cap S) E^1 x_{n+1} \neq \emptyset\). Since \(S\) is a saturated, hereditary subset of \(E^0\), this would imply that \(\bigsqcup_{n=m_1}^\infty V_n \subseteq S\). Hence, \(\mathfrak{I} = \mathfrak{A}\) contradicting that fact that \(\mathfrak{A}/\mathfrak{I} \cong M_k\).

By Claim 1 and Claim 2, there exists a subsequence \(\{k(n)\}_{n=1}^\infty\) of \(\{n \in \mathbb{N} : n \geq m_1\}\) such that \(x_{k(n)} E^* (V_{k(n+1)} \cap S) \neq \emptyset\) and \((V_{k(n)} \cap S) E^* x_{k(n+1)} = \emptyset\). Telescope \((E, d_E)\) to \(\bigsqcup_{n=1}^\infty V_{k(n)}\). Then we get a Bratteli diagram \((F, d_F)\) with \(F^0 = \bigsqcup_{n=1}^\infty W_n\) and \(W_n = U_n \cup \{t_n\}\), where \(U_n := V_{k(n)} \cap S\), and \(t_n := x_{k(n)}\). We see that \((F, d_F)\) satisfies properties (1)–(5) of Definition 4.8. □

Lemma 4.11. Let \(\mathfrak{A}\) be an AF-algebra with a largest ideal \(\mathfrak{I}\) such that \(\mathfrak{A}/\mathfrak{I} \cong M_k\) and \(\mathfrak{A}/\mathfrak{I}\) is the only unital quotient of \(\mathfrak{A}\). Then there exists an \(M_k\)-separated Bratteli diagram \((F, d_F)\) for \(\mathfrak{A}\) with \(F^0 := \bigsqcup_{n=1}^\infty W_n\) partitioned into levels for which \(W_n = U_n \cup \{t_n\}\) satisfies Properties (1)–(5) of Definition 4.8 and also satisfies the additional property:

\[ (6') \quad \text{For every } n \geq 2 \text{ and for every } v \in U_n \text{ either} \]
\[ d_F(v) > \sum_{\alpha \in W_n F^* v} d_F(s_F(\alpha)) \quad \text{or} \quad d_F(v) = \sum_{\alpha \in t_n F^* v} d_F(t_m). \]

Proof. Note that it suffices to show that \(\mathfrak{A}\) has an \(M_k\)-separated Bratteli diagram satisfying the following condition:

\[ (6)'' \quad \text{For every } m \in \mathbb{N}, \text{ there exists } n \geq m \text{ such that for every } v \in U_n \text{ either} \]
\[ d_F(v) > \sum_{\alpha \in W_m F^* v} d_F(s_F(\alpha)) \quad \text{or} \quad d_F(v) = \sum_{\alpha \in t_m F^* v} d_F(t_m) \]

since any such Bratteli diagram can be telescoped to an \(M_k\)-separated Bratteli diagram satisfying Property (6') in the statement of the lemma.
Since any largest ideal is also an essential ideal, Lemma 4.10 implies that there is an \( \mathbb{M}_k \)-separated Bratteli diagram \((F, d_F)\) for \( \mathfrak{A} \). Suppose \( F^0 := \bigcup_{n=1}^{\infty} W_n \) is partitioned into levels with \( W_n = U_n \cup \{ t_n \} \). We shall show that \((F, d_F)\) satisfies Property (6') of Lemma 4.11. We establish this through proof by contradiction. To this end, suppose there exists \( m \in \mathbb{N} \) such that for each \( n \geq m \), the set

\[
Y_n := \left\{ x \in W_n : d_F(x) = \sum_{\alpha \in W_n F^* x} d_F(s_F(\alpha)) \text{ and } d_F(x) \neq \sum_{\alpha \in t_n F^* x} d_F(t_m) \right\}
\]

is nonempty. Without loss of generality, we may assume that \( m = 1 \). Set

\[
T = \{ w \in F^0 : \text{there exist infinitely many } n \in \mathbb{N} \text{ such that } w \geq Y_n \}.
\]

Then \( F^0 \setminus T \) is a saturated hereditary subset of \( F^0 \). In addition, since \( \mathfrak{A} \) contains a largest ideal, and \( U := \bigcup_{n=1}^{\infty} U_n \) is a saturated hereditary subset of \( F^0 \) not contained in any proper saturated hereditary subset of \( F^0 \), it follows that \( U \) is the saturated hereditary subset corresponding to \( \mathcal{J} \) and \( U \) is a largest saturated hereditary subset of \( F^0 \). Therefore, \( F^0 \setminus T \subseteq U \) or \( F^0 \setminus T = F^0 \). We shall show that it must be the case that \( F^0 \setminus T \subseteq U \) by proving that \( T \neq \emptyset \).

We claim that \( T \cap U \neq \emptyset \). Suppose \( T \cap U = \emptyset \). Then, \( U \subseteq F^0 \setminus T \). Therefore, for every \( v \in U \), there exists \( n_v \in \mathbb{N} \) such that for each \( n \geq n_v \), there are no paths from \( v \) to \( Y_n \). Set \( N := \max \{ n_v : v \in U_1 \} \). Then for each \( n \geq N \) we have \( U_1 F^* Y_n = \emptyset \).

Let \( x \in Y_N \). Then \( d_F(x) = \sum_{\alpha \in W_1 F^* x} d_F(s_F(\alpha)) \) and \( d_F(x) \neq \sum_{\alpha \in t_1 F^* x} d_F(t_1) \). Note that \( x \neq t_N \) since \( \sum_{\alpha \in t_1 F^* t_N} d_F(t_1) = d_F(t_1) = k = d_F(x) \). Therefore, \( x \in U_N \). Since \( 0 \neq d_F(x) = \sum_{\alpha \in W_1 F^* x} d_F(s_F(\alpha)) \), we have that \( W_1 F^* x \neq \emptyset \). Since \( U_1 F^* Y_N = \emptyset \), it follows that \( U_1 F^* x = \emptyset \). Hence

\[
d_F(x) = \sum_{\alpha \in W_1 F^* x} d_F(s_F(\alpha)) = \sum_{\alpha \in t_1 F^* x} d_F(t_1),
\]

which contradicts the assumption that \( x \in Y_N \). Therefore, \( T \cap U \neq \emptyset \).

Since \( T \cap H \neq \emptyset \), we have that \( T \neq \emptyset \) and \( F^0 \setminus T \neq F^0 \). Hence it must be the case that \( F^0 \setminus T \subseteq H \). Since \( T \cap H \neq \emptyset \), we have that \( F^0 \setminus T \neq H \). Let \( \mathfrak{R} \) be the ideal of \( \mathfrak{A} \) corresponding to \( F^0 \setminus T \). Then \( \mathfrak{R} \neq \mathcal{J} \) and \( \mathfrak{A}/\mathfrak{R} \) has a Bratteli diagram obtained by restricting to the vertices of \( T \). By Lemma 4.3, \( \mathfrak{A}/\mathfrak{R} \) is a unital \( C^* \)-algebra, and since \( \mathfrak{R} \neq \mathcal{J} \), this contradicts the fact that \( \mathfrak{A}/\mathfrak{J} \) is the only unital quotient of \( \mathfrak{A} \). Hence the lemma holds.

**Lemma 4.12.** Let \( \mathfrak{A} \) be an AF-algebra with a largest ideal \( \mathcal{J} \) such that \( \mathfrak{A}/\mathcal{J} \cong \mathbb{M}_k \) and \( \mathfrak{A}/\mathcal{J} \) is the only unital quotient of \( \mathfrak{A} \). Then there exists a proper \( \mathbb{M}_k \)-separated Bratteli diagram for \( \mathfrak{A} \).

**Example 4.13.** By Lemma 4.11 it suffices to show that an \( \mathbb{M}_k \)-separated Bratteli diagram that satisfies Property (6') of Lemma 4.11 is equivalent to a proper \( \mathbb{M}_k \)-separated Bratteli diagram for \( \mathfrak{A} \). To help the reader follow the proof of Lemma 4.12, we give an example to illustrate how the telescoping constructions in the proof are performed.

Below are four Bratteli diagrams: \((F, d_F)\), \((A, d_A)\), \((B, d_B)\), and \((E, d_E)\). The Bratteli diagram \((F, d_F)\) is an \( \mathbb{M}_1 \)-separated Bratteli diagram that satisfies Property (6') of Lemma 4.11. In addition, \((E, d_E)\) is a proper \( \mathbb{M}_1 \)-separated Bratteli diagram. Telescoping
the Bratteli diagrams \((F,d_F)\) and \((B,d_B)\) at the odd levels, we obtain the Bratteli diagram \((A,d_A)\). Telescoping the Bratteli diagram \((B,d_B)\) at the even levels gives the Bratteli diagram \((E,d_E)\). Thus \((F,d_F)\) is equivalent to \((E,d_E)\).

\[
(F,d_F) \quad \begin{array}{cccccccc}
1 & 2 & 2 & 2 & 2 & \ldots & 2 & \ldots \\
 & 4 & 8 & 12 & 16 & \ldots & 4(n-1) & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots & 1 & \ldots \\
\end{array}
\]

\[
(A,d_A) \quad \begin{array}{cccccccc}
1 & 2 & 2 & 2 & 2 & \ldots & 2 & \ldots \\
 & 8 & 16 & 24 & 32 & \ldots & 8(n-1) & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots & 1 & \ldots \\
\end{array}
\]

\[
(B,d_B) \quad \begin{array}{cccccccc}
1 & 2 & 2 & \ldots & 2 & \ldots \\
 & 4 & 8 & 12 & 16 & \ldots & 4(n-1) & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots & 1 & \ldots \\
\end{array}
\]

\[
(E,d_E) \quad \begin{array}{cccccccc}
4 & 12 & 20 & 28 & 36 & \ldots & 4 + 8(n-1) & \ldots \\
6 & 6 & 6 & 6 & 6 & \ldots & 1 & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots & 1 & \ldots \\
\end{array}
\]

**Proof of Lemma 4.12.** By Lemma 4.11 \(\mathfrak{A}\) has an \(\mathbb{M}_k\)-separated Bratteli diagram \((F,d_F)\) that satisfies Property (6’) of Lemma 4.11. We shall prove that \((F,d_F)\) is equivalent to a proper \(\mathbb{M}_k\)-separated Bratteli diagram for \(\mathfrak{A}\).

For each \(n \geq 2\), set

\[
A_n := \left\{ v \in U_n : d_F(v) = \sum_{e \in \ell_{n-1}F^1v} d_F(t_{n-1}) \right\}.
\]

For each \(v \in A_n\) and for each \(w \in U_{n+1}\), let

\[
p(v,w) := |\{ ef \in F^2 : s_F(e) = t_{n-1}, r_F(e) = s_F(f) = v, \text{ and } r_F(f) = w \}|
\]

denote the number of paths in \(F\) from \(t_{n-1}\) to \(w\) that go through \(v\).
Define a Bratteli diagram \((B, d_B)\) as follows:

\[
B^0 = F^0 \setminus \left( \bigcup_{n=1}^{\infty} A_{2n} \right)
\]

\[
B^1 = \left( F^1 \setminus \left( \bigcup_{n=1}^{\infty} \{ e \in F^1 : r_F(e) \in A_{2n} \text{ or } s_F(e) \in A_{2n} \} \right) \right) \\
\cup \left( \bigcup_{n=1}^{\infty} \{ e_i(v, w, n) : v \in A_{2n}, w \in U_{2n+1}, \text{ and } 1 \leq i \leq p(v, w) \} \right)
\]

with range and source maps defined by

\[
r_B(e) = \begin{cases} 
  r_F(e), & \text{if } e \in F^1 \\
  w, & \text{if } e = e_i(v, w, n)
\end{cases}
\]

and

\[
s_B(e) = \begin{cases} 
  s_F(e), & \text{if } e \in F^1 \\
  t_{2n}, & \text{if } e = e_i(v, w, n)
\end{cases}
\]

and the degree function defined by \(d_B = d_{F|B^0}\).

Note that \(B^0 = \bigsqcup_{n=1}^{\infty} (V_n \cup \{ t_n \})\) with \(V_{2n-1} = U_{2n-1}\) and \(V_{2n} = U_{2n} \setminus A_{2n}\). To show that \((B, d_B)\) is a Bratteli diagram, we must show that

\[
d_B(v) \geq \sum_{e \in B^1 v} d_B(s_B(e)).
\]

for all \(v \in B^0\). To do this it suffices to show that for each \(w \in V_n \cup \{ t_n \}\) we have

\[
\sum_{e \in B^1 w} d_B(s_B(e)) = \sum_{e \in F^1 w} d_F(s_F(w)).
\]

To this end, let \(n \in \mathbb{N}\) and first suppose \(w \in V_{2n}\). Then \(w \in U_{2n} \setminus A_{2n}\). By the construction of \(B\), we have that \(B^1 w = F^1 w\). Hence

\[
\sum_{e \in B^1 w} d_B(s_B(e)) = \sum_{e \in F^1 w} d_F(s_F(e)).
\]

Next, suppose \(w \in V_{2n-1}\). If \(w \in A_{2n-1}\) with \(n \geq 2\) (the case when \(n = 1\) is clear), then \(B^1 w = F^1 w\) and

\[
\sum_{e \in B^1 w} d_B(s_B(e)) = \sum_{e \in F^1 w} d_F(s_F(e)).
\]
Next, suppose \( w \in U_{2n-1} \setminus A_{2n-1} \). Note that \( d_F(t_{2n-3}) = d_F(t_{2n-2}) \). Thus
\[
\sum_{e \in B^1 w} d_B(s_B(e)) = \sum_{e \in V_{2n-2} B^1 w} d_B(s_B(e)) + \sum_{e \in t_{2n-2} B^1 w} d_B(s_B(e))
\]
\[
= \sum_{e \in V_{2n-2} F^1 w} d_F(s_F(e)) + \sum_{e \in t_{2n-2} F^1 w} d_F(s_F(t_{2n-2})) + \sum_{e \in A_{2n-2}} s(v, w) d_F(t_{2n-2})
\]
\[
= \sum_{e \in V_{2n-2} F^1 w} d_F(s_F(e)) + \sum_{e \in t_{2n-2} F^1 w} d_F(s_F(t_{2n-2})) + \sum_{e \in A_{2n-2}} |t_{2n-3} F^1 v| v F^1 w d_F(t_{2n-2})
\]
\[
= \sum_{e \in V_{2n-2} F^1 w} d_F(s_F(e)) + \sum_{e \in t_{2n-2} F^1 w} d_F(s_F(t_{2n-2})) + \sum_{e \in A_{2n-2}} |v F^1 w| d_F(v)
\]
\[
= \sum_{e \in V_{2n-2} F^1 w} d_E(s_E(e)) + \sum_{e \in t_{2n-2} F^1 w} d_E(s_E(t_{2n-2})) + \sum_{e \in A_{2n-2}} d_F(s_F(e))
\]
\[
= \sum_{e \in F^1 w} d_F(s_F(e)).
\]
Since
\[
\sum_{e \in B^1 t_n} d_B(s_B(e)) = d_F(t_{n-1}) = \sum_{e \in F^1 t_n} d_F(s_F(e)),
\]
it follows that for each \( w \in B^0 \) we have
\[
\sum_{e \in B^1 w} d_B(s_B(e)) = \sum_{e \in F^1 w} d_F(s_F(e)) \leq d_F(w) = d_B(w).
\]
Thus \((B, d_B)\) is a Bratteli diagram.

Set \( Y_n = V_n \cup \{ t_n \} \). By the construction of \((B, d_B)\), for each \( w \in A_{2n+1} \) and for each \( v \in A_{2n-1} \) we have
\[
|v B^* w| = |v F^* w|.
\]
Let \((A, d_A)\) be the the Bratteli diagram obtained by telescoping \((B, d_B)\) at the levels \( \bigsqcup_{n=1}^{\infty} Y_{2n-1} \). Since \( Y_{2n-1} = W_{2n-1} \) for each \( n \in \mathbb{N} \), we see that the Bratteli diagram obtained by telescoping \((F, d_F)\) at the levels \( \bigsqcup_{n=1}^{\infty} W_{2n-1} \) is also equal to \((A, d_A)\). Thus \((F, d_F)\) and \((B, d_B)\) are equivalent Bratteli diagrams.

We now show that for each \( n \geq 2 \) and \( w \in Y_{2n} \) we have
\[
\sum_{e \in Y_{2n-2} B^* w} d_B(s_B(e)) = \sum_{e \in W_{2n-2} F^* w} d_F(s_F(e)).
\]
Let \( w \in V_{2n} \). First, suppose \( w = t_{2n} \). Then
\[
\sum_{e \in Y_{2n-2} B^* w} d_B(s_B(e)) = d_F(t_{2n}) = \sum_{e \in W_{2n-2} F^* w} d_F(s_F(e)).
\]
Next, suppose \( w \in V_{2n} \). Then \( w \in U_{2n} \setminus A_{2n} \) and \( B^1w = F^1w \). Therefore
\[
\sum_{e \in W_{2n-2}F^*w} d_F(s_F(e)) = \sum_{e \in W_{2n-2}F^*w} \sum_{f \in F^1s_F(e)} d_F(s_F(e)) = \sum_{e \in B^1w} \sum_{f \in F^1s_B(e)} d_F(s_F(e)) = \sum_{e \in Y_{2n-2}B^*w} d_B(s_B(e)) = \sum_{e \in W_{2n-2}B^*w} d_B(s_B(e)).
\]
Hence, for each \( n \geq 2 \) and \( w \in Y_{2n} \) we have
\[
\sum_{e \in Y_{2n-2}B^*w} d_B(s_B(e)) = \sum_{e \in W_{2n-2}B^*w} d_F(s_F(e)).
\]
In particular, for each \( n \geq 2 \) and \( w \in V_{2n} = U_{2n} \setminus A_{2n} \) we have
\[
\sum_{e \in Y_{2n-2}B^*w} d_B(s_B(e)) = \sum_{e \in W_{2n-2}B^*w} d_F(s_F(e)) < d_F(w) = d_B(w).
\]
Let \((E,d_E)\) be the Bratteli diagram obtained by telescoping \((B,d_B)\) to \( \bigsqcup_{n=1}^{\infty} Y_{2n} \). Set \( H_n := U_{2n} \) and \( y_n := t_{2n} \). Then \((E,d_E)\) is a proper \( \mathbb{M}_k \)-separated Bratteli diagram equivalent to \((F,d_F)\).

\[\square\]

5. Nonunital AF-algebras with a unique unital quotient

We begin by describing a way to construct a graph from a proper \( \mathbb{M}_k \)-separated Bratteli diagram.

**Definition 5.1.** Let \((E,d_E)\) be a proper \( \mathbb{M}_k \)-separated Bratteli diagram such that \( E^0 = \bigsqcup_{n=1}^{\infty} V_n \) is partitioned into levels with \( V_n = H_n \cup \{y_n\} \). Then \( H := \bigsqcup_{n=1}^{\infty} H_n \) is a saturated hereditary subset of \( E^0 \), and we construct a graph \( G = (G^0,G^1,r_G,s_G) \) from \((E,d_E)\) as follows: For each \( n \in \mathbb{N} \) and \( v \in H_n \), set
\[
\delta(v) := d_E(v) - \sum_{e \in r^{-1}(v)} d_E(s_E(e)) - 1 \quad \text{and} \quad m(v) := |y_{n-1}E^*v|.
\]
Let
\[
G^0 := E^0_H \cup \{z_i : i = 1, \ldots, k\} \cup \{x_i^v : v \in H, 1 \leq i \leq \delta(v)\}
\]
\[
G^1 := E^1_H \cup \{e_i : i = 1, \ldots, k-1\} \cup \{f_i^v : v \in H, 1 \leq i \leq m(v)\}
\]
\[
\sqcup \{g_i^v : v \in H, 1 \leq i \leq \delta(v)\}
\]
be the vertex and edge sets of \( G \), respectively, with
\[
s_G(e) = \begin{cases} s_E(e), & \text{if } e \in E^1_H \\ z_i, & \text{if } e = e_i \\ z_k, & \text{if } e = f_i^v \\ x_i, & \text{if } e = g_i^v \end{cases}
\]
and
\[
r_G(e) = \begin{cases} r_E(e), & \text{if } e \in E^1_H \\ z_k, & \text{if } e = e_i \\ v, & \text{if } e = f_i^v \text{ or } e = g_i^v \end{cases}
\]
as the range and source functions.

**Remark 5.2.** In Definition 5.1 the fact that \((E,d_E)\) is a proper \( \mathbb{M}_k \)-separated Bratteli diagram is needed to assure us that \( \delta(v) \geq 0 \) for all \( v \in H \).
Remark 5.3. In the graph $G$ of Definition 5.1 the vertex $z_k$ is an infinite emitter, and all other vertices of $G$ are regular vertices.

Example 5.4. We give an example to illustrate the construction of the graph in Definition 5.1. Consider the following proper $M_3$-separated Bratteli diagram. In the top row the values at the first three levels are 4, 24, 43, and then for level $n \geq 4$ the value is $20n + 4$.

We see that $H$ consists of the vertices labeled 4, 24, 43, 64, 84, ... and that each $H_n$ consists of a single vertex, which we shall denote $v_n$. Then $H = \{v_1, v_2, v_3, \ldots\}$, and $\delta(v_1) = 3$, $\delta(v_2) = 1$, $\delta(v_3) = 0$, $\delta(v_4) = 2$, ... The graph $G = (G^0, G^1, r_G, s_G)$ constructed from $(E, d_E)$, as described in Definition 5.1, is given by the following:

Note that the vertex $z_3$ is an infinite emitter.

Lemma 5.5. Let $(E, d_E)$ be a a proper $M_k$-separated Bratteli diagram such that $E^0 = \bigsqcup_{n=1}^\infty V_n$ is partitioned into levels with $V_n = H_n \cup \{y_n\}$. If we let $H := \bigsqcup_{n=1}^\infty H_n$ and let $G = (G^0, G^1, s_G, r_G)$ the graph constructed from $(E, d_E)$ as described in Definition 5.1, then

$$d_E(v) = |\{\alpha \in G^* : r_G(\alpha) = v\}| \quad \text{for all } v \in H.$$
and the base case holds. For the inductive step, assume that for a particular value of \( n \in \mathbb{N} \) we have \( d_E(v) = |\{ \alpha \in G^* : r_G(\alpha) = v \}| \) for all \( v \in H_n \). Choose \( v \in H_{n+1} \). Then
\[
|\{ \alpha \in G^* : r_G(\alpha) = v \}| = 1 + |\{ g^v_i : 1 \leq i \leq \delta(v) \}|
\]
\[
+ \sum_{e \in r_E^{-1}(v) \cap s_E^{-1}(H)} |\{ \alpha \in G^* : r_G(\alpha) = s_G(e) \}|
\]
\[
+ k |\{ f^v_i : 1 \leq i \leq m(v) \}|
\]
\[
= 1 + \delta(v) + \sum_{e \in r_E^{-1}(v) \cap s_E^{-1}(H)} d_E(s_G(e)) + km(v)
\]
\[
= 1 + \delta(v) + \sum_{e \in r_E^{-1}(v)} d_E(s_G(e))
\]
\[
= d_E(v)
\]
and our lemma holds for all vertices in \( H_{n+1} \). It follows from induction that the lemma holds.

Proposition 5.6. Let \( (E,d_E) \) be a proper \( \mathbb{M}_k \)-separated Bratteli diagram, and let \( G \) be the graph constructed from \( (E,d_E) \) as described in Definition 5.1. If \( \mathfrak{A} \) is the AF-algebra associated with \( (E,d_E) \), then \( \mathfrak{A} \cong C^*(G) \).

Proof. Let \( \{ S_e, P_v : e \in G^1, v \in G^0 \} \) be a universal Cuntz-Krieger \( G \)-family in \( C^*(G) \). Using the notation of Definition 5.1 set \( V_n := H_n \sqcup \{ y_n \} \) and define
\[
Q_n := P_{z_k} - \sum_{1 \leq i \leq n} \sum_{v \in H_i} \sum_{1 \leq j \leq m(v)} S_{f^v_i} S_{f^v_j}^*
\]
and
\[
\mathfrak{A}_n := C^*(\{ S_\alpha : \alpha \in E^*, r_G(\alpha) \in H_n \} \cup \{ S_e Q_n : i = 1, \ldots, k - 1 \}).
\]
We will prove the following:

1. \( \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \) and there exists an isomorphism
\[
\phi_n : \mathfrak{A}_n \to \left( \bigoplus_{v \in H_n} \mathbb{M}_{d_E(v)} \right) \oplus \mathbb{M}_k
\]
such that the induced homomorphism
\[
\psi_{n,n+1} : \left( \bigoplus_{v \in H_n} \mathbb{M}_{d_E(v)} \right) \oplus \mathbb{M}_k \to \left( \bigoplus_{v \in H_{n+1}} \mathbb{M}_{d_E(v)} \right) \oplus \mathbb{M}_k
\]
that makes the diagram
\[
\begin{array}{ccc}
\mathfrak{A}_n & \xrightarrow{\phi_n} & \mathfrak{A}_{n+1} \\
\psi_{n,n+1} \downarrow & & \downarrow \phi_{n+1} \\
( \bigoplus_{v \in H_n} \mathbb{M}_{d_E(v)} ) \oplus \mathbb{M}_k & \xrightarrow{\psi_{n,n+1}} & ( \bigoplus_{v \in H_{n+1}} \mathbb{M}_{d_E(v)} ) \oplus \mathbb{M}_k
\end{array}
\]
commutative has multiplicity matrix \( (|wE^1v|)_{w \in V_n, v \in V_{n+1}} \).
(2) \( \{ S_e, P_v : e \in G^1, v \in G^0 \} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{A}_n \).

Note that (1) implies that \( \mathfrak{A} \cong \bigcup_{n=1}^{\infty} \mathfrak{A}_n \) and (2) implies that \( \bigcup_{n=1}^{\infty} \mathfrak{A}_n = C^*(G) \), from which it follows that \( \mathfrak{A} \cong C^*(G) \). Thus establishing (1) and (2) will prove the theorem.

We first prove (1). Let \( \alpha \in G^* \) such that \( r_G(\alpha) = v \in H_n \). Note that
\[
S_\alpha = S_\alpha P_v = S_\alpha \sum_{e \in s^{-1}_G(v)} S_e S_e^* = \sum_{e \in s^{-1}_G(v)} S_\alpha S_e S_e^*.
\]

Since \( v \in H_n \), it follows that \( r_G(s^{-1}_G(v)) \subseteq H_{n+1} \). Thus, \( S_\alpha S_e S_e^* \in \mathfrak{A}_{n+1} \) for all \( e \in s^{-1}_G(v) \).

Note that
\[
S_{ei} Q_n = S_{ei} \left( Q_{n+1} - \sum_{v \in H_{n+1}} \sum_{1 \leq j \leq m(v)} S_{fj} S_{fj}^* \right) = S_{ei} Q_{n+1} - \sum_{v \in H_{n+1}} \sum_{1 \leq j \leq m(v)} S_{ei} S_{fj} S_{fj}^*.
\]

Since \( S_{ei} Q_{n+1} \in \mathfrak{A}_{n+1} \) and \( S_{ei} S_{fj} S_{fj}^* \in \mathfrak{A}_{n+1} \), we have that \( S_{ei} Q_n \in \mathfrak{A}_{n+1} \). Thus \( \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \).

For each \( v \in H_n \), set
\[
\mathfrak{B}_{n,v} := C^*\{ S_\alpha : r_G(\alpha) = v \} \quad \text{and} \quad \mathfrak{C}_n := C^*\{ S_{ei} Q_n : 1 \leq i \leq k - 1 \}.
\]

Define \( \alpha_n : \left( \bigoplus_{v \in H_n} \mathfrak{B}_{n,v} \right) \oplus \mathfrak{C}_n \rightarrow \mathfrak{A}_n \) by
\[
\alpha_n \left( \{ (x_v)_{v \in H_n}, y \} \right) := \sum_{v \in H_n} x_v + y.
\]

One can verify that \( \alpha_n \) is an isomorphism. In addition, for each \( v \in H_n \), we have \( d_{E}(v) = | \{ \alpha \in G^* : r_G(\alpha) = v \} | \), and hence \( \mathfrak{B}_{n,v} \cong M_{d_{E}(v)} \). Also, \( \mathfrak{C}_n \cong M_k \). Let \( \phi_n : \mathfrak{A}_n \rightarrow \left( \bigoplus_{v \in H_n} M_{d_{E}(v)} \right) \oplus M_k \) be the composition
\[
\mathfrak{A}_n \xrightarrow{\alpha_n^{-1}} \left( \bigoplus_{v \in H_n} \mathfrak{B}_{n,v} \right) \oplus \mathfrak{C}_n \cong \left( \bigoplus_{v \in H_n} M_{d_{E}(v)} \right) \oplus M_k.
\]

Let \( \iota_{n,n+1} : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1} \) be the inclusion map. Define \( \psi_{n,n+1} : \left( \bigoplus_{v \in H_n} M_{d_{E}(v)} \right) \oplus M_k \rightarrow \left( \bigoplus_{v \in H_n} M_{d_{E}(v)} \right) \oplus M_k \) to be the composition \( \phi_{n+1} \circ \iota_{n,n+1} \circ \phi_n^{-1} \).

If \( w \in H_n \), then
\[
P_w = \sum_{e \in s^{-1}_G(w)} S_e S_e^* = \sum_{v \in H_{n+1}} \sum_{e \in w E^* v} S_e S_e^* \subseteq \sum_{v \in H_{n+1}} \mathfrak{B}_{n+1,v}.
\]

Note that
\[
Q_n = Q_{n+1} - \sum_{v \in H_{n+1}} \sum_{1 \leq j \leq m(v)} S_{fj} S_{fj}^* \in \mathfrak{C}_{n+1} + \sum_{v \in H_{n+1}} \mathfrak{B}_{n+1,v}.
\]

Therefore, the multiplicity matrix \( \phi_{n,n+1} \) is given by \( (|w E^* v|)_{w \in V_n, v \in V_{n+1}} \). This establishes (1), which implies \( \mathfrak{A} \cong \bigcup_{n=1}^{\infty} \mathfrak{A}_n \).

We now prove (2). Note that for each \( v \in H_n \), we have \( S_{g_i}, S_e \in \mathfrak{A}_n \) for all \( e \in r_{G}^{-1}(v) \) and for all \( 1 \leq i \leq \delta(v) \). Since \( S_{g_i} S_{g_i}^* = P_{g_i} \) and \( S_e S_e^* = P_v \) for each \( e \in r_{G}^{-1}(v) \), we have that \( S_{g_i}, S_e, P_v, P_{g_i} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{A}_n \) for all \( n \in \mathbb{N}, v \in H_n \), and \( e \in r_{G}^{-1}(v) \).

All that remains is to show that \( S_{g_i} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{A}_n \) for \( i = 1, \ldots, k - 1 \) and \( P_{g_i} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{A}_n \) for \( i = 1, \ldots, k \) are in \( \bigcup_{n=1}^{\infty} \mathfrak{A}_n \). We shall actually show that all these elements are in \( \mathfrak{A}_1 \). To
do this, we see that for each \( v \in H_1 \) we have \( m(v) = 0 \) so that \( Q_1 = P_{z_k} \). Thus \( P_{z_k} \in \mathfrak{A}_1 \).

In addition, for all \( 1 \leq i \leq k - 1 \) we have \( S_{e_i} = S_{e_i} P_{z_k} = S_{e_i} Q_1 \in \mathfrak{A}_1 \). Moreover, it follows that for all \( 1 \leq i \leq k - 1 \) we have \( P_{z_i} = S_{e_i} S_{e_i}^* \in \mathfrak{A}_1 \).

The previous two paragraphs show that \( \{ S_{e_i}, g_v : e \in G^1, v \in G^0 \} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{A}_n \), which establishes (2). Since each \( \mathfrak{A}_n \) is a \( C^* \)-subalgebra of \( C^*(G) \), and the elements of the set \( \{ S_{e_i}, P_{v} : e \in G^1, v \in G^0 \} \) generate \( C^*(G) \), it follows that \( C^*(G) \cong \bigcup_{n=1}^{\infty} \mathfrak{A}_n \). \( \square \)

The following theorem shows that the conjecture from the introduction holds whenever \( \mathfrak{A} \) is an AF-algebra with a largest ideal \( \mathfrak{I} \) such that \( \mathfrak{A}/\mathfrak{I} \) is the only unital quotient of \( \mathfrak{A} \).

**Theorem 5.7.** Let \( \mathfrak{A} \) be a nonunital AF-algebra with a largest ideal \( \mathfrak{I} \). If \( \mathfrak{A}/\mathfrak{I} \) is the only unital quotient of \( \mathfrak{A} \), then the following are equivalent:

1. \( \mathfrak{A} \) is isomorphic to a graph \( C^* \)-algebra.
2. \( \mathfrak{A}/\mathfrak{I} \) is a Type I \( C^* \)-algebra with finitely many ideals.
3. \( \mathfrak{A}/\mathfrak{I} \cong \mathcal{M}_k \) for some \( k \in \mathbb{N} \).

**Proof.** If (1) holds, then it follows from [10, Proposition 4.21] that the unital quotient \( \mathfrak{A}/\mathfrak{I} \) is a Type I \( C^* \)-algebra with finitely many ideals. Hence (1) implies (2).

If (2) holds, then since \( \mathfrak{I} \) is a largest ideal of \( \mathfrak{A} \), the quotient \( \mathfrak{A}/\mathfrak{I} \) is simple. In addition, since \( \mathfrak{A} \) is AF, and hence Type I, it follows that \( \mathfrak{A}/\mathfrak{I} \) is Type I. Since any unital, simple, Type I \( C^* \)-algebra is isomorphic to \( \mathcal{M}_k \) for some \( k \in \mathbb{N} \), (3) holds. Thus (2) implies (3).

If (3) holds, then Lemma 4.12 implies there exists a proper \( \mathcal{M}_k \)-separated Bratteli diagram for \( \mathfrak{A} \). If \( G \) is the graph constructed from \((E, d_E)\) as described in Definition 5.1, then Proposition 5.6 implies that \( A \cong C^*(G) \). Hence (3) implies (1). \( \square \)

**Remark 5.8.** Recall that [10, Theorem 4.7] implies that an AF-algebra is isomorphic to the \( C^* \)-algebra of a row-finite graph with no sinks if and only if the AF-algebra has no unital quotients. If the conditions of Theorem 5.7 hold, then since \( \mathfrak{A}/\mathfrak{I} \) is a unital quotient of \( \mathfrak{A} \), we know that \( \mathfrak{A} \) is not isomorphic to the \( C^* \)-algebra of a row-finite graph with no sinks. The construction of Definition 5.1 shows that \( \mathfrak{A} \) is, however, isomorphic to the \( C^* \)-algebra of a graph with no sinks and exactly one infinite emitter.

It is easy to see that the conjecture from the introduction holds for simple AF-algebras: If a simple AF-algebra is unital, Corollary 3.14 shows it is isomorphic to a graph \( C^* \)-algebra if and only if it is a Type I \( C^* \)-algebra. If a simple AF-algebra is nonunital, [10, Theorem 4.7] shows it is always isomorphic to a graph \( C^* \)-algebra.

Combining our result for unital AF-algebras in Corollary 3.14 and for nonunital AF-algebras in Theorem 5.7 allows us to show that the conjecture from the introduction also holds for the class of AF-algebras with exactly one ideal.

**Theorem 5.9.** Let \( \mathfrak{A} \) be an AF-algebra with exactly one proper nonzero ideal. Then \( \mathfrak{A} \) is isomorphic to a graph \( C^* \)-algebra if and only if every unital quotient of \( \mathfrak{A} \) is a Type I \( C^* \)-algebra.

**Proof.** Necessity follows from [10, Proposition 4.21]. To see sufficiency, let \( \mathfrak{I} \) be the unique proper nonzero ideal of \( \mathfrak{A} \), and consider three cases.

**Case I:** \( \mathfrak{A} \) is unital. Then by hypothesis \( \mathfrak{A} \) is Type I with finitely many ideals, so by Corollary 3.14 we have \( \mathfrak{A} \) is isomorphic to a graph \( C^* \)-algebra.
Case II: \( \mathfrak{A} \) is nonunital, and \( \mathfrak{A}/\mathcal{I} \) is unital. Then \( \mathfrak{A}/\mathcal{I} \) is the only unital quotient of \( \mathfrak{A} \), and this quotient is simple and Type I by hypothesis. Hence Theorem 5.7 implies \( \mathfrak{A} \) is isomorphic to a graph \( C^* \)-algebra.

Case III: \( \mathfrak{A} \) is nonunital and \( \mathfrak{A}/\mathcal{I} \) is nonunital. Then \( \mathfrak{A} \) has no unital quotients, and [10, Theorem 4.7] implies \( \mathfrak{A} \) is isomorphic to a graph \( C^* \)-algebra.

It was proven in [10, Theorem 4.7] that an AF-algebra with no unital quotients is isomorphic to a graph \( C^* \)-algebra. However, the argument uses ultragraphs, and it is difficult to determine the required graph from the proof. Here we provide an alternate proof that shows explicitly how to construct a graph from a Bratteli diagram whose \( C^* \)-algebra is isomorphic to the AF-algebra.

**Theorem 5.10** (cf. Theorem 4.7 of [10]). Let \( A \) be an AF-algebra that has no nonunital quotients. Then \( A \) has a Bratteli diagram \( (E,d_E) \) such that \( d_E(v) \geq 2 \) and \( d_E(v) > \sum_{e \in r_E^{-1}(v)} d_E(s_E(e)) \) for all \( v \in E^0 \). For any such Bratteli diagram, construct a graph \( G \) from \( (E,d_E) \) as follows: Set \( \delta(v) = d_E(v) - \sum_{e \in r_E^{-1}(v)} d_E(s_E(e)) - 1 \) for each \( v \in E^0 \). Define the vertex set and edge set of \( G \) as

\[
G^0 = E^0 \cup \{ x_i^v : v \in E^0, 1 \leq i \leq \delta(v) \}
\]

and

\[
G^1 = E^1 \cup \{ e_i^v : v \in E^0, 1 \leq i \leq \delta(v) \},
\]

respectively. Also define the range and source maps of \( G \) as

\[
r_{G|E^1} = r_{E^1} \quad \text{and} \quad r_G(e_i^v) = v
\]

and

\[
s_{G|E^1} = r_{E^1} \quad \text{and} \quad s_G(e_i^v) = x_i^v,
\]

respectively. Then \( \mathfrak{A} \cong C^*(G) \).

**Proof.** It follows from [10, Lemma 3.5] that \( A \) has a Bratteli diagram \( (E,d_E) \) such that \( d_E(v) \geq 2 \) and \( d_E(v) > \sum_{e \in r_E^{-1}(v)} d_E(s_E(e)) \) for all \( v \in E^0 \).

Let \( E^0 \) be partitioned into levels as \( E^0 = \bigsqcup_{n=1}^{\infty} V_n \), let \( \{ S_e, P_v \}_{e \in G^1, v \in G^0} \) be a universal Cuntz-Krieger \( G \)-family for \( C^*(G) \), and let \( \mathfrak{A} \) be the AF-algebra associated to \( (E,d_E) \). Set

\[
\mathfrak{A}_n = C^*(\{ S_{e} : r_G(e) \in V_n \}).
\]

Using an argument similar to the one in the proof of Proposition 5.6, we obtain that \( \mathfrak{A}_n \) is a \( C^* \)-subalgebra of \( C^*(G) \) for each \( n \in \mathbb{N} \), and the following statements hold:

1. \( \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \) for all \( n \in \mathbb{N} \);
2. \( \mathfrak{A}_n \cong \bigoplus_{v \in V_n} M_{d_E(v)} \) for all \( n \in \mathbb{N} \);
3. \( \bigoplus_{n=1}^{\infty} \mathfrak{A}_n = C^*(G) \); and
4. the homomorphism \( \phi_{n,n+1} : \bigoplus_{v \in V_n} M_{d_E(v)} \to \bigoplus_{v \in V_{n+1}} M_{d_E(v)} \) given by

\[
\bigoplus_{v \in V_n} M_{d_E(v)} \cong \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \cong \bigoplus_{v \in V_{n+1}} M_{d_E(v)}
\]

has multiplicity matrix \( (|vE^*w|)_{v \in V_n, w \in V_{n+1}} \).
Hence,

\[ C^*(G) = \bigcup_{n=1}^{\infty} \mathcal{A}_n \cong \lim_{\rightarrow} \left( \bigoplus_{v \in V_n} M_{d_E(v)}, \phi_{n,n+1} \right) \cong \mathcal{A}. \]

\[ \Box \]

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