On Average Properties of Inhomogeneous Cosmologies

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Abstract

The present talk summarizes the recently progressed state of a systematic re-evaluation of cosmological models that respect the presence of inhomogeneities. Emphasis is given to identifying the basic steps towards an effective (i.e. spatially averaged) description of structural evolution, also unfolding the various facets of a “smoothed-out” cosmology. We shall highlight some results obtained within Newtonian cosmology, discuss expansion laws in general relativity within a covariant fluid approach, and put forward some promising directions of future research.

1 Motivation and Results in Newtonian Cosmology

1.1 Bridging the Gap

Does an inhomogeneous model of the Universe evolve on average like a homogeneous solution of Einstein’s or Newton’s laws of gravity? This question is not new, at least among relativists who think that the answer is certainly no, not only in view of the nonlinearity of the theories mentioned [26]. The problem was and still is the notion of averaging whose specification and unambiguous definition turned out to be an endeavor of high magnitude, mainly because it is not straightforward to give a unique meaning to the averaging of tensors, e.g., a given metric of spacetime. This problem seems to lie in the backyard of relativists who, from time to time, add another effort towards a solution of this problem. On the other hand, the community of cosmologists “should” locate exactly this problem at the basis of their evolutionary models of the Universe. Although there are many exceptions (e.g. [28], [3], [4], [29], [43], [37], [47], a certainly incomplete list), most researchers in this field are drawn back to the historical development of cosmologies starting with Friedmann, Einstein and de–Sitter at the beginning of the last century. Despite the drastic changes of our picture of structures in the Universe on large scales, still, the cosmologist’s thinking rests on the hegemony of the so–called “standard model” (i.e. the family of FLRW models for homogeneous and isotropic matter distributions). This standard model, up to the present state of knowledge, explains (or better is employed to explain) a wide variety of orthogonal observations, and it is therefore hard to beat due to its (suggestively) established status of resistance against observational tests. Therefore, most discussions in this field are based on the vocabulary of the standard model, aiming to constrain its “cosmological parameters”, often on the basis of observations of structure in the regional Universe that is very different from homogeneous and isotropic.

Bridging the gap between an involved mathematical problem of general relativity and the practical modeling of cosmological dynamics is possible, if ambiguities of averages could be removed, and results related to contemporary discussions in observational cosmology. In the sequel we shall follow a line of thought that will match this need and could intensify work directed to mastering an inhomogeneous spacetime.

1.2 Setting the Pace

Averaging procedures can be defined in a vast variety of ways (see, e.g., a recent summary by Stoeger et al. [45]). A smoothing–out operator can live on different foliations of spacetime; it can leave an averaged field space–dependent, thus operating on a given spatial scale; it can also smooth out all inhomogeneities.

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It could involve statistical ensemble averaging. It can act on scalars, vectors or tensors; it can smooth matter variables or, most elegantly, the geometry of spacetime itself \([2], [22]\). It can average spacetime variables and not merely spatial variables, etc. To remove ambiguity here, it is best to work within the framework of Newtonian cosmology as a first step of understanding, since there the choice of foliation is not a problem and spatial averaging can be simply put into practice by Euclidean volume integration. As a second step, we confine ourselves to scalar variables. This allows us to get at least some corresponding answer in general relativity, since the averaging of scalars is, for a given foliation, a covariant operation. We therefore propose to look at the simplest (mass–conserving) averager as follows.

### 1.3 A Newtonian Averager

Consider any simply–connected spatial domain in Newtonian spacetime. With \(\langle \cdot \rangle_\mathcal{D}\) we denote spatial averaging in Eulerian space, e.g., for a spatial tensor field \(\mathcal{A}(\mathbf{x}, t) = \{A_{ij}(\mathbf{x}, t)\}\) we simply have the Euclidean volume integral normalized by the volume of the domain:

\[
\langle \mathcal{A} \rangle_\mathcal{D} (t) = \frac{1}{V(t)} \int_{\mathcal{D}} d^3x \, \mathcal{A}(\mathbf{x}, t). \tag{1}
\]

(Here, \(\mathbf{x}\) are non–rotating Eulerian coordinates.) Since, with this averager, the averaged field is only time–dependent, the space–dependence is only implicit by a functional dependence on the domain’s morphology and position. We shall evolve the domain in time by preserving its mass content. This is a natural assumption, if we also want to extend the domain to the whole Universe.

Now, consider the volume of an Eulerian spatial domain \(\mathcal{D}\) at a given time, \(V = \int_{\mathcal{D}} d^3x\), and follow the position vectors \(\mathbf{x} = f(\mathbf{X}, t)\) of all fluid elements (indexed by the Lagrangian coordinates \(\mathbf{X}\)) within the domain. Then, the volume elements are deformed according to \(d^3x = Jd^3X\), where \(J\) is the Jacobian determinant of the transformation from Eulerian to Lagrangian coordinates. The total rate of change of the volume of the same collection of fluid elements may then be calculated as follows:

\[
\frac{dV}{V} = \frac{1}{V} \frac{dV}{dt} \int_{\mathcal{D}_i} d^3X \, J = \frac{1}{V} \int_{\mathcal{D}_i} d^3X \, dt J = \frac{1}{V} \int_{\mathcal{D}_i} d^3X \, \theta J = \langle \theta \rangle_\mathcal{D} = \frac{3}{H_\mathcal{D}}, \tag{2}
\]

where \(\frac{d}{dt}\) is the (Lagrangian) time–derivative, \(\theta = \nabla \cdot \mathbf{v} = \frac{\dot{a}}{a}\) the local expansion rate of a given velocity model \(\mathbf{v}(\mathbf{x}, t)\); \(\mathcal{D}_i\) denotes the initial domain found by mapping back \(\mathcal{D}\) with the help of \(f^{-1}\) (provided \(f^{-1}\) exists), and \(H_\mathcal{D} = a_\mathcal{D}/a_D\) naturally defines an effective Hubble–parameter on the domain.

### 1.4 Commuting Averaging and Evolution

From the point of view of the standard model of cosmology it makes no difference, if we smooth the initial inhomogeneities (say, at a time in the matter dominated epoch) and then evolve the smoothed data with a FLRW solution, or if we evolve these inhomogeneities until present and then smooth the distribution. The conjecture is held that both ways produce the same values for the characteristic parameters of a FLRW cosmology.

However, averaging, as defined above, and evolving inhomogeneities are non–commuting operations. To see this we have to notice that the total (Lagrangian) time–derivative does not commute with spatial averaging in Eulerian space. For an arbitrary tensor field \(\mathcal{A}\) we can readily derive, with the help of the above definitions, the following Commutation Rule \([18]\):

\[
\frac{d}{dt} \langle \mathcal{A} \rangle_\mathcal{D} - \langle \frac{d}{dt} \mathcal{A} \rangle_\mathcal{D} = \langle \theta \mathcal{A} \rangle_\mathcal{D} - \langle \mathcal{A} \theta \rangle_\mathcal{D}. \tag{3}
\]

This tells us that exchanging the operators for averaging and time–evolution produces a source due to the presence of inhomogeneities (that we will discuss to consist of positive–definite fluctuations). As an example consider the expansion rate itself. Setting \(\mathcal{A} = \theta\), we get

\[
\frac{d}{dt} \langle \theta \rangle_\mathcal{D} - \langle \frac{d}{dt} \theta \rangle_\mathcal{D} = \langle \theta^2 \rangle_\mathcal{D} - \langle \theta \rangle_\mathcal{D}^2 = \langle \left(\theta - \langle \theta \rangle_\mathcal{D}\right)^2 \rangle_\mathcal{D} \geq 0, \tag{4}
\]

i.e., the source is the averaged mean square fluctuation of the expansion rate. It vanishes for the case where the local rate equals the global one, which is true for homogeneous–isotropic matter distributions.
1.5 Formulating the General Expansion Law of Newtonian Cosmology

As suggested by Eq. (2) we may measure the effective expansion of a portion of the Universe with the help of the rate of volume change. Let us introduce an effective dimensionless scale–factor $a_D$ via the domain’s volume $V(t) = |\mathcal{D}|$ and the initial volume $V_i = V(t_i) = |\mathcal{D}_i|$ (compare Fig. 1):

$$a_D(t) = \left( \frac{V(t)}{V_i} \right)^{\frac{1}{3}}, \quad \text{i.e.}, \quad H_D = \frac{\dot{a}_D}{a_D}. \quad (5)$$

For domains $\mathcal{D}$ with constant mass $M_D$, as for Lagrangian defined domains, the average mass density evolves as:

$$\langle \rho \rangle_{\mathcal{D}} = \frac{\langle \rho (t_i) \rangle_{\mathcal{D}_i}}{a^3_D} = \frac{M_D}{a_D^3 V_i}. \quad (6)$$

Looking at the Commutation Rule, Eq. (4), we appreciate that this rule furnishes a purely kinematical relation provided we give a velocity model. Dynamics enters by saying how this velocity model is generated by gravity. In Eq. (4) we can express most of the terms already through the scale–factor and its time–derivatives. The only unknown is the evolution equation for the local expansion rate. This is furnished by Raychaudhuri’s equation (which employs the Newtonian field equation $\nabla \cdot g = \Lambda - 4\pi G\rho$ for the gravitational field strength of dust matter $g = \dot{\mathbf{v}}$).

Inserting Raychaudhuri’s equation,

$$\dot{\theta} = \Lambda - 4\pi G\rho - \frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) ; \quad \sigma := \sqrt{\frac{1}{2} \sigma_{ij} \sigma_{ij}} ; \quad \omega := \sqrt{\frac{1}{2} \omega_{ij} \omega_{1ij}}, \quad (7)$$

into Eq. (4), we find that the scale–factor $a_D$ in general obeys the dynamical expansion law \cite{18}:

$$3\frac{\ddot{a}_D}{a_D} + 4\pi G \langle \rho \rangle_{\mathcal{D}} - \Lambda = Q_D, \quad (8)$$

with Newton’s gravitational constant $G$, the cosmological constant $\Lambda$, and the source term $Q_D$, which we may call “backreaction term”, since it measures the departure from the standard model due to the influence of inhomogeneities.

$Q_D$ depends on the kinematical scalars, the expansion rate $\theta$, the rate of shear $\sigma$, and the rate of vorticity $\omega$ featuring three positive–definite fluctuation terms:

$$Q_D = \frac{2}{3} \left( \langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 \right) + 2 \langle \omega^2 \rangle_{\mathcal{D}} - 2 \langle \sigma^2 \rangle_{\mathcal{D}}. \quad (9)$$
1.6 The Quintessence of Inhomogeneities and Anisotropies

It is easily verified that $Q_D = 0$ is a necessary and sufficient condition for having $a_D = a(t)$, with the global scale–factor $a(t)$ solving the standard Friedmann equations. Assuming for the moment that the averaged Universe would be described by the standard model, then the universal expansion rate would be determined by four components: the average mass density (commonly conceived to have baryonic and non–baryonic “dark” components), the value of the cosmological constant, and the global “curvature parameter” $k$, which arises as an integration constant by integrating the second–order Friedmann equation (Eq. (8) for $Q_D = 0$) in order to get the global expansion rate $3H(t) = \frac{3 \dot{a}}{a}$.

Upon integrating the actual expansion law, Eq. (8), we instead obtain ([13] used a different sign convention for $Q_D$):

$$3 \frac{\dot{a}^2}{a^2} + 3 \frac{k_D}{a^2} - 8\pi G \langle \varrho \rangle_D - \Lambda = \frac{1}{a^2} \int_{t_i}^{t} dt' Q_D \frac{da^2}{dt'}(t') ,$$

(10)

where $k_D$ enters as a domain–dependent integration constant. The effective Hubble–parameter $H_D = \frac{\dot{a}}{a}$ is now determined by the previous (now domain–dependent) components, but also by a fifth component due to the presence of inhomogeneities.

Since there are many unresolved conundrums of the standard model (see, e.g., [27], [46]), cosmologists are searching for a fifth parameter that may resolve them (see the talk by Prof. Fujii on “Quintessence” in the present Proceedings). The use of the letter $Q$ for the “backreaction” is coincidental, but it may possibly be a working coincidence.

1.7 Cosmic Triangle or Cosmic Quartett ?

The vocabulary of the standard model may be condensed into the picture of a “cosmic triangle” [2], consisting of three cosmological parameters, which are constructed by normalizing the mass density, the cosmological constant, and the “curvature parameter” by the square of the Hubble–parameter.

As suggested by Eq. (10) we are led to define an additional dimensionless “kinematical backreaction parameter” through

$$\Omega_Q^D := \frac{1}{3 a^2 H^2} \int_{t_i}^{t} dt' Q_D \frac{da^2(t')}{dt'},$$

(11)

in addition to the common cosmological parameters:

$$\Omega_m^D := \frac{8\pi G \langle \varrho \rangle}{3H^2} , \quad \Omega_\Lambda^D := \frac{\Lambda}{3H^2} , \quad \Omega_k^D := -\frac{k_D}{a^2 H^2} .$$

(12)

However, contrary to the standard model, all $\Omega^D$–parameters are now domain–dependent and transformed into fluctuating fields on the domain; for $Q_D = 0$ the “cosmic triangle” is undistorted and the parameters acquire their global standard values.

Comparing these definitions with Eq. (10) we have

$$\Omega_m^D + \Omega_\Lambda^D + \Omega_k^D + \Omega_Q^D = 1 ,$$

(13)

i.e., there are four players in the game (or five, respectively, if we split $\Omega_m^D$ into baryonic matter and non–baryonic “dark matter”).

In Friedmann–Lemaître cosmologies there is by definition no backreaction: $\Omega_Q^D = 0$. In this case a universe model with $\Omega_m = 1$ conserves the matter parameter. Also, the initial value of the “curvature parameter” $\Omega_k = 0$ remains so during the entire evolution. This changes, if “backreaction” is taken into account. $\Omega_Q^D$ itself may act as a “kinematical dark matter”, or as a “kinematical cosmological term”, respectively, depending on the relative strength of shear–, expansion– and vorticity–fluctuations.

As we have learned [19] and as we shall illustrate below, the parameters corresponding to the “cosmic triangle” can experience large changes, even in situations when the “backreaction parameter” is seemingly negligible, but non–zero.
1.8 Construction Principle for Evolution Models

In order to obtain quantitative results on the value and impact of the “backreaction parameter”, we have to employ evolution models for the inhomogeneities. Their implementation seems not to be straightforward, since cosmological evolution models (analytical or N-body simulations) are constructed in such a way that they evolve an initial power spectrum of density– and velocity–fluctuations on a given global background (a solution of the standard Friedmann equations), implying a vanishing “backreaction” by construction. However, the framework of Newtonian cosmology offers the possibility of making use of contemporary evolution schemes by assigning sense to a global background within the general expansion law of an averaged inhomogeneous model. We shall now elaborate on this fact by demonstrating the validity of the following Construction Principle:

*The average expansion of a generic inhomogeneous matter distribution on (topologically) closed Newtonian space sections is given by the solution of the standard Friedmann equations.*

This principle, although usually not explicitly stated, lies at the basis of any evolution model in Newtonian cosmology. It appears to be very restrictive in light of the general relativistic framework (see Subsection 2.5).

In order to proof this principle let us work with the invariants of the gradient of the velocity field. They are expressible in terms of kinematical scalars and can be written as total divergences of vector fields, which have been used and discussed in the context of perturbation solutions ([11], [23]):

\[
I(v_{i,j}) = \nabla \cdot v = \theta, \quad \Pi(v_{i,j}) = \frac{1}{2} \nabla \cdot \left( (\nabla \cdot v) \right) = \omega^2 - \sigma^2 + \frac{1}{3} \theta^2.
\]  

(14)

The “backreaction term” can be entirely expressed in terms of the first and second invariants:

\[
Q_D = 2 \langle \Pi(v_{i,j}) \rangle_D - \frac{2}{3} \langle I(v_{i,j}) \rangle_D^2.
\]

(15)

Now, let us formally assume that there exists a global, homogeneous and isotropic reference model with scale–factor \(a(t)\). We introduce the following variables with respect to this reference background. With the global Hubble–parameter \( H = \dot{a} / a \) we define comoving Eulerian coordinates \( q := x / a \) and peculiar–velocities \( u := v - H x \) as usual. Using the derivative \( \partial_q u_i \equiv \partial u_i / \partial q_j \) with respect to comoving coordinates we obtain for the first and second invariants:

\[
I(v_{i,j}) = 3H + \frac{1}{a} \Pi(\partial_q u_i), \quad \Pi(v_{i,j}) = 3H^2 + \frac{2H}{a} I(\partial_q u_i) + \frac{1}{a^2} \Pi(\partial_q u_i).
\]

(16)

The “backreaction term” remains form–invariant (note: \( \frac{1}{a} \partial_q u_i = u_{i,j} \)):

\[
Q_D = \frac{1}{a^2} \left( 2 \langle \Pi(\partial_q u_i) \rangle_D - \frac{2}{3} \langle I(\partial_q u_i) \rangle_D^2 \right).
\]

(17)

Thus, all terms corresponding to the background flow cancel in this expression, and only inhomogeneities contribute to “backreaction”.

Using Eq. (14) we write the “backreaction term” as a volume–average over divergences. Hence, using the theorem of Gauss we obtain:

\[
Q_D = \frac{1}{a^2} \left[ 2 \frac{1}{V_q} \int_{\partial D_q} dS \cdot (u(\nabla_q \cdot u) - (u \cdot \nabla_q)u) - \frac{2}{3} \left( \frac{1}{V_q} \int_{\partial D_q} dS \cdot u \right)^2 \right],
\]

with the surface \( \partial D_q \) bounding the comoving domain \( D_q \), the surface element \( dS \), and the comoving differential operator \( \nabla_q \).

From Eq. (18) we directly obtain \( Q_D = 0 \) for a domain with empty boundary, e.g., for toroidal space sections, or periodic peculiar–velocity fields, respectively. In turn, the assumed reference solution is a standard Hubble–flow and may serve as a global background. (For more details see [13].)

This establishes the *Construction Principle q.e.d.*

\[\]
1.9 Generalization of the Top–Hat Model

The Construction Principle provides room for employing the language of standard theories of structure formation. As an example we apply the general expansion law to domains within a global Hubble–flow.

Assuming an Einstein–de–Sitter background for this example we subtract a standard Friedmann equation, \( 3\dot{a}^2 + 4\pi G\varrho_H = 0 \), from Eq. (8), with the background density \( \varrho_H = \frac{\dot{H}^2}{8\pi G} \), and obtain the following differential equation for \( a_D(t) \) [19]:

\[
3 \left( \frac{\dot{a}_D}{a_D} - \frac{\ddot{a}}{a} \right) + \frac{3}{2} \left( \frac{\dot{a}}{a} \right)^2 \langle \delta \rangle_D = Q_D ,
\]

with \( \langle \delta \rangle_D = (\langle \varrho \rangle_D - \varrho_H) / \varrho_H \) specifying the averaged density contrast \( \delta \) in \( D \). For \( Q_D = 0 \) and \( \langle \delta \rangle_D = 0 \) this equation simply states that the time evolution of a domain follows the global expansion, \( a_D(t) = a(t) \).

For \( Q_D = 0 \) and \( 1 + \langle \delta \rangle_D = \frac{\langle \varrho \rangle_D a^3}{\varrho_H a^3} \) the evolution of \( a_D \) is still of Friedmann type, but with a mass different from the background mass. An important subcase with \( Q_D = 0 \) and \( \langle \delta \rangle_D \neq 0 \) is the well–known spherical top–hat model [56]. In Eq. (19) there are two sources determining the deviations from the Friedmann acceleration, the over/under–density and the “backreaction term”. This shows that, in general, the evolution of a Newtonian portion of the Universe is triggered by an over/under–density and velocity–fluctuations.

1.10 An Averaged Lagrangian Perturbation Scheme

As a second example we apply the Construction Principle to the widely used and well–tested Lagrangian perturbation schemes (see: [33], [25] for reviews, and [12] for a tutorial).

The trajectory field in the mostly employed “Zel’dovich approximation” [49], [50] (which can be derived as a subcase of the first–order Lagrangian perturbation scheme [11]), is given by:

\[
f^Z(\mathbf{X}, t) = a(t) \left( \mathbf{X} + \xi(t) \nabla_0 \psi(\mathbf{X}) \right) .
\]

\( \psi(\mathbf{X}) \) is the initial displacement field, \( \nabla_0 \) the gradient with respect to Lagrangian coordinates and \( \xi(t) \) a global time–dependent function (given for all background models in [5]).

Given this trajectory field we can already approximate the rate of volume change by the volume deformation that is caused by the field. For the effective scale–factor we obtain [5]:

\[
(a_D^{\text{kin}})^3 = a^3 \left( 1 + \xi \langle \mathbf{I}_1 \rangle_{D_1} + \xi^2 \langle \mathbf{I}_1^2 \rangle_{D_1} + \xi^3 \langle \mathbf{I}_1^3 \rangle_{D_1} \right) .
\]

A better estimate is to calculate the “backreaction term” from the approximation of the velocity field \( \mathbf{v}^Z = \dot{f}^Z \) and solve the dynamical expansion law, Eq. (8), for \( a_D \). The “backreaction term” separates into its time–evolution given by \( \xi(t) \) and the spatial dependence on the initial displacement field given by averages over the invariants of the displacement gradient \( \mathbf{I}_1, \mathbf{I}_1, \) and \( \mathbf{I}_1^3 \):

\[
Q_D^Z = \frac{\xi^2}{\left( 1 + \xi \langle \mathbf{I}_1 \rangle_{D_1} + \xi^2 \langle \mathbf{I}_1^2 \rangle_{D_1} + \xi^3 \langle \mathbf{I}_1^3 \rangle_{D_1} \right)^2} \times \\
\left[ \left( 2 \langle \mathbf{I}_1 \rangle_{D_1} - \frac{2}{3} \langle \mathbf{I}_1^2 \rangle_{D_1} \right) + \xi \left( 6 \langle \mathbf{I}_1^3 \rangle_{D_1} - \frac{2}{3} \langle \mathbf{I}_1 \rangle_{D_1} \langle \mathbf{I}_1^2 \rangle_{D_1} \right) + \xi^2 \left( 2 \langle \mathbf{I}_1 \rangle_{D_1} \langle \mathbf{I}_1^2 \rangle_{D_1} - \frac{2}{3} \langle \mathbf{I}_1^3 \rangle_{D_1} \right) \right] .
\]

The numerator of the first term is global and corresponds to the damping factor that also arises in the Eulerian linear theory of gravitational instabiliy; in an Einstein–de–Sitter universe \( \xi^2 \propto a^{-1} \). The denominator of the first term is the volume effect discussed above, whereas the second term in brackets features the initial “backreaction” as a leading term and higher–order terms.

A property of this approximation that renders it very useful as an averaged evolution model should be emphasized: it is exact for two orthogonal symmetry assumptions: for the evolution of plane–symmetric inhomogeneities, which is a consequence of the known properties of the first–order Lagrangian approximation [9], and for the evolution of spherically symmetric inhomogeneities, as will be shown below.
Figure 2: The evolution of the “cosmological parameters” $\Omega^D_m$ (solid line), $\Omega^D_k$ (short dashed), and $\Omega^D_Q$ (long dashed) in an expanding (under–dense) domain with initial radius 0.5Mpc (a scaled radius of 100Mpc today). The dotted line is marking $\Omega^D_m + \Omega^D_k + \Omega^D_Q$. The upper left plot is for one–σ, the upper right plot is for three–σ fluctuations (assigned to the expected amplitudes of the initial invariants in a Gaussian random field of standard Cold–Dark–Matter fluctuations). The lower plots show the result for a collapsing domain. We see here that the “backreaction parameter” is quantitatively more important than in an expanding domain – an interpretation of this result will be offered below: a collapsing domain experiences more drastic changes in shape.
Employing the averaged Lagrangian scheme we can quantify the impact of backreaction on the domain-dependent “cosmological parameters”. For details the reader may look at a recent work [19]. Here, I would like to mention the remarkable result that for an accelerating, i.e., under-dense region, \( Q \) may be numerically negligible as seen in (Fig. 2, upper plots), but dramatic changes in the other parameters are observed. For a collapsing domain with a present-day radius of 100Mpc the mass parameter of the domain may even differ by more than 100% from the global mass parameter (Fig. 2, lower plots).

1.11 Newton’s Iron Spheres

Looking at the general expansion law, Eq. (8), the careful reader may object that Friedmann’s equations also hold, if we study the motion of a spherically symmetric domain, although the matter distribution inside the sphere may be inhomogeneous. This fact is known as Newton’s “Iron Sphere Theorem”. Let us show that the general expansion law respects this theorem.

We note that, for a spherically symmetric distribution of matter inside the domain, the invariants of the velocity gradient, averaged over a ball \( B_R \), obey relations resulting in [19]:

\[
\langle \Pi(v_{i,j}) \rangle_{B_R} = \frac{1}{3} \langle I(v_{i,j}) \rangle_{B_R}^2, \quad \text{and} \quad \langle III(v_{i,j}) \rangle_{B_R} = \frac{1}{27} \langle I(v_{i,j}) \rangle_{B_R}^3.
\] (23)

Inserting this into the “backreaction term”, Eq. (15), shows that \( Q_{spherical} = 0 \) in accordance with Newton’s theorem. The inhomogeneous model discussed in the last subsection also has this property: inserting the above expressions into Eq. (22) (using the proportionality of displacement gradient and initial velocity gradient), we also obtain \( Q_{Z} = 0 = Q_{spherical} \).

1.12 Averaging and the Evolution of Form

The expansion law, Eq. (5), is built on the rate of change of a simple morphological quantity, the volume content of a domain. Although functionally it depends on other morphological characteristics of a domain, it does not explicitly provide information on their evolution. An evolution equation for the “backreaction term” is missing. This fact touches on the problem of closing the hierarchy of dynamical evolution equations considered as a set of coupled ordinary differential equations in Lagrangian space. The problem of closing such a hierarchy of equations is often considered in the literature and various closure conditions are formulated (e.g., [30]), one of them being the “Silent Universe Model” in general relativity, which assumes a vanishing magnetic part of the Weyl tensor [6]. Averaging such a hierarchy would result in evolution equations for the “backreaction term” and would, with some local closure condition, also close the system of averaged equations. Here, we will not pursue this problem further, but instead begin to develop the morphological point of view that eventually implies an alternative proposition of a morphological closure condition.

Let us focus our attention on the boundary of the spatial domain \( D \). A priori, the location of this boundary in space enjoys some freedom which we may constrain by saying that the boundary coincides with a velocity front of the fluid (hereby restricting attention to irrotational flows). This way we employ the Legendrian point of view of velocity fronts that is dual to the Lagrangian one of fluid trajectories. Let \( S(x, y, z, t) = s(t) \) define a velocity front at Newtonian time \( t \), \( v = \nabla S \). Below, we shall need that the three principal scalar invariants of the velocity gradient \( v_{i,j} = S_{ij} \) can be transformed into divergences of vector fields as written explicitly in Eqs. (14, 26).

Defining the unit normal vector \( n \) on the front, \( n = \pm \frac{\nabla S}{|\nabla S|} \) (the sign depends on the direction of its motion), the average expansion rate can be written as a flux integral using Gauss’ theorem:

\[
\langle \theta \rangle_D = \frac{1}{V} \int_D d^3x \nabla \cdot v = \frac{1}{V} \int_{\partial D} dS \cdot v,
\] (24)

and, with the surface element \( d\sigma \), \( dS = n d\sigma \), we obtain the intuitive result that the average expansion rate is related to another morphological quantity of the domain, the total area of the enclosing surface:

\[
\langle \theta \rangle_D = \frac{1}{V} \int_{\partial D} d\sigma |\nabla S|.
\] (25)
Inserting the velocity potential also into the other invariants, Eq. (14) and

\[ \mathbf{III}(\psi, \epsilon) = \frac{1}{3} \nabla \cdot \left( \frac{1}{2} \nabla \cdot \left( \mathbf{v} (\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{v} - \left( \mathbf{v} (\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{v} \right) \cdot \nabla \mathbf{v} \right) \right), \tag{26} \]

and performing the spatial average, we obtain [17]:

\[ \langle \mathbf{II} \rangle_D = \frac{1}{V} \int_D d^3x \mathbf{II} = \int_{\partial D} d\sigma |\nabla \mathbf{S}|^2 \mathbf{H}; \quad \langle \mathbf{III} \rangle_D = \frac{1}{V} \int_D d^3x \mathbf{III} = \int_{\partial D} d\sigma |\nabla \mathbf{S}|^3 \mathbf{K}, \tag{27} \]

where \( \mathbf{H} \) is the mean curvature and \( \mathbf{K} \) the Gaussian curvature of the 2–surface bounding the domain. \(|\nabla \mathbf{S}| = \frac{d\sigma}{d\mathbf{r}}\) equals 1, if the intrinsic arc–length \( s \) of the trajectories is used instead of the extrinsic Newtonian time \( t \). The averaged invariants comprise, together with the volume a complete set of morphological characteristics related to the Minkowski functionals of a body:

\[ \mathcal{W}_0(s) := \int_D d^3x = V; \quad \mathcal{W}_1(s) := \frac{1}{3} \int_{\partial D} d\sigma; \quad \mathcal{W}_2(s) := \frac{1}{3} \int_{\partial D} d\sigma \mathbf{H}; \quad \mathcal{W}_3(s) := \frac{1}{3} \int_{\partial D} d\sigma \mathbf{K} = \frac{4\pi}{3} \chi. \tag{28} \]

The Euler–characteristic \( \chi \) determines the topology of the domain and is assumed to be an integral of motion (\( \chi = 1 \)), if the domain should remain simply–connected (a morphological closure condition).

Thus, we have gained a morphological interpretation of the “backreaction term”: it can be entirely expressed through three of the four Minkowski functionals:

\[ Q_D(s) = 6 \left( \frac{\mathcal{W}_2}{\mathcal{W}_0} - \frac{\mathcal{W}_2^3}{\mathcal{W}_0^3} \right). \tag{29} \]

The \( \mathcal{W}_\alpha; \ \alpha = 0, 1, 2, 3 \) have been introduced as “Minkowski functionals” into cosmology by Mecke et al. [33] in order to statistically assess morphological properties of cosmic structure. Minkowski functionals proved to be useful tools to also incorporate information from higher–order correlations, e.g., in the distribution of galaxies, galaxy clusters, density fields or cosmic microwave background temperature maps [14, 34, 35, 40, 41; see the review by Kerscher [32 and ref. therein]. Related to the morphology of individual domains is the study of building blocks of large–scale cosmic structure [39, 42].

For a ball with radius \( R \) we have for the Minkowski functionals:

\[ \mathcal{W}_0^{B_R}(s) := \frac{4\pi}{3} R^3; \quad \mathcal{W}_1^{B_R}(s) := \frac{4\pi}{3} R^2; \quad \mathcal{W}_2^{B_R}(s) := \frac{4\pi}{3} R; \quad \mathcal{W}_3^{B_R}(s) := \frac{4\pi}{3}. \tag{30} \]

Inserting these expressions into the “backreaction term”, Eq. (29), shows that \( Q^{B_R}_D(s) = 0 \), and we have confirmed Newton’s “Iron Sphere Theorem” once more. Moreover, we can understand now that the “backreaction term” encodes the deviations of the domain’s morphology from that of a ball, a fact which we shall illustrate now with the help of Steiner’s formula of integral geometry (see also [35]).

Let \( d\sigma_0 \) be the surface element on the unit sphere, then (according to the Gaussian map) \( d\sigma = R_1 R_2 d\sigma_0 \) is the surface element of a 2–surface with radii of curvature \( R_1 \) and \( R_2 \). Moving the surface a distance \( \epsilon \) along its normal we get for the surface element of the parallel velocity front:

\[ d\sigma_\epsilon = (R_1 + \epsilon)(R_2 + \epsilon)d\sigma_0 = \frac{R_1 R_2 + \epsilon(R_1 + R_2) + \epsilon^2}{R_1 R_2} d\sigma = (1 + \epsilon^2 \mathbf{H} + \epsilon^2 \mathbf{K}) d\sigma, \tag{31} \]

where

\[ \mathbf{H} = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \quad \mathbf{K} = \frac{1}{R_1 R_2}, \tag{32} \]

are the mean curvature and Gaussian curvature of the front as before.

Integrating Eq. (31) over the whole front we arrive at a relation between the total surface area \( A \) of the front and \( A_\epsilon \) of its parallel front. The gain in volume may then be expressed by an integral of the resulting relation with respect to \( \epsilon \) (which is known as Steiner’s formula defining the Minkowski functionals of a (convex) body in three spatial dimensions):

\[ V_\epsilon = V + \int_0^\epsilon d\epsilon A_\epsilon = V + \epsilon A + \epsilon^2 \int_{\partial D} d\sigma \mathbf{H} + \epsilon^3 \int_{\partial D} d\sigma \mathbf{K}. \tag{33} \]
2 Average Models in General Relativity

2.1 An Averager for Scalars

The key to reducing ambiguity of choosing an averager in general relativity is to confine ourselves to scalars, since spatially averaging a scalar field \( \Psi \) is a covariant operation. However, we have to make sure that we understand the word “spatial” physically. Geometrically, we shall introduce a foliation of spacetime, but we shall encounter ambiguity in choosing it. Suppose for the moment that we insist on spatial averaging and consider for all what follows a foliation of spacetime as given. The simplest averager is then immediately written down as follows:

\[
\langle \Psi(X^i, t) \rangle_D := \frac{1}{V} \int_D J d^3X \Psi(X^i, t) ; \quad V = \int_D J d^3X ,
\]

with \( J := \sqrt{\det(g_{ij})} \), where \( g_{ij} \) is the metric of the spatial hypersurfaces, and \( X^i \) are coordinates in the hypersurfaces, conveniently chosen such that they are constant along flow lines.

2.2 Foliating Spacetime: Covariant Fluid Approach

We wish to conserve mass inside a spatial domain of spacetime. Therefore, let us first consider the (conserved) restmass flux vector

\[
M^\mu := \rho u^\mu ; \quad M^\mu ;_\mu = 0 ; \quad \rho > 0 ,
\]

where \( \rho \) is the restmass density and the flow lines are integral curves of the 4–velocity \( u^\mu \). We shall confine ourselves to irrotational perfect fluids with energy density \( \varepsilon \) and pressure \( p \), which allows us to simplify the splitting of spacetime, e.g. the spatial hypersurfaces can be chosen flow–orthogonal in this case, the unit normal on the hypersurfaces coincides with the 4–velocity and the shift vector vanishes. Irrotationality guarantees the existence of a scalar function \( S \), such that

\[
u^\mu = -\frac{\partial^\mu S}{h} ,
\]

In general, we identify the magnitude \( h \) with the “injection energy per fluid element and unit restmass”,

\[
h := \frac{\varepsilon + p}{\rho} ,
\]

which is related to the relativistic enthalpy \( \eta := \frac{\varepsilon + p}{\rho} \) by \( h = \eta/m \) with \( m \) the unit restmass of a fluid element, and \( n \) the baryon density. Note that \( d\varepsilon = h d\rho \). For a barotropic fluid we can easily see that \( \varepsilon \) is a function of the restmass density only and, hence, \( h \) is a function of \( \rho \). \( h \) is identical to 1 in the case of dust. The magnitude \( h \) normalizes the 4–gradient \( \partial^\mu S \) so that \( u^\mu u_\mu = -1 \),

\[
h = \sqrt{-\partial^\alpha S \partial_\alpha S} = u^\mu \partial_\mu S = \dot{S} > 0 .
\]

The overdot stands for the material derivative operator along the flow lines of any tensor field \( F \) as defined covariantly by \( \dot{F} := u^\mu F ;_\mu \). It reduces in the case of a scalar function in a flow–orthogonal slicing (which we want to envisage) to the partial time–derivative multiplied by \( \frac{1}{N} \), where \( N \) is the lapse function (for more details see: \[10\]).

It can be shown that \( S \) is spatially homogeneous. Thus, \( S(t) \) and \( h(X^i, t) \) play the role of “phase” and “amplitude” of the fluid’s wave fronts. Foliating the spacetime into hypersurfaces \( S(t) = \text{const} \) is a gauge condition that is naturally adapted to the fluid itself and, therefore, this foliation can be given a covariant meaning (\[11\], \[8\] and ref. therein).

\[3\] Greek indices run through 0...3, while latin indices run through 1...3 as before; summation over repeated indices is understood. A semicolon will denote covariant derivative with respect to the 4–metric with signature \((-+,+,+); \) the units are such that \( c = 1 \).
2.3 A Commutation Rule in General Relativity

The rate of change of the volume $V(t)$ in the hypersurfaces $S(t) = \text{const.}$ is evaluated by taking the partial time–derivative of the volume and dividing by the volume. Since $\partial_t$ and $d^3X$ commute (but not $\frac{d}{d\tau} := \frac{\partial_t}{N}$ and $d^3X$ !) we obtain:

$$\frac{\partial_t V}{V} = \frac{1}{V} \int_D d^3X \partial_t J = \frac{1}{V} \int_D d^3X N \dot{J} = \frac{1}{V} \int_D d^3X N \theta \dot{J} = \langle \dot{N} \theta \rangle_D .$$  \hspace{1cm} (39)

Introducing the scaled (t–)expansion $\dot{\theta} := N \theta$ we define an effective (t–)Hubble function in the hypersurfaces by

$$\langle \dot{\theta} \rangle_D = \frac{\partial_t V}{V} = 3 \frac{\partial_t a_D}{a_D} =: 3 \dot{H}_D .$$  \hspace{1cm} (40)

With the help of these definitions we readily derive the Commutation Rule:

$$\partial_t \langle \dot{\Psi} \rangle_D - \langle \partial_t \dot{\Psi} \rangle_D = \langle \dot{\Psi} \dot{\theta} \rangle_D - \langle \dot{\Psi} \rangle_D \langle \dot{\theta} \rangle_D .$$  \hspace{1cm} (41)

The lapse–weighted quantities have to be introduced only in the case of non–vanishing pressure, since the pressure gradient induces deviations from a geodesic flow implying an inhomogeneous lapse function. In the much simpler case of a dust matter model, the lapse function is homogeneous and can be chosen equal to 1, and the covariant fluid gauge is identical to the comoving and synchronous gauge, which makes the correspondence to the Newtonian investigation most transparent.

In the sequel we shall, for simplicity, discuss the case of a pressure–less fluid only and resume the discussion of the more general case thereafter.

2.4 Expansion Law of General Relativity: Dust Models

Employing the averaging procedure outlined above and following the line of thought of the Newtonian investigation in Section 1, we derive the general expansion law for dust matter in the synchronous and comoving gauge. The result will be covariant with respect to this foliation.

I here give the result derived in [15]: the spatially averaged equations for the scale–factor $a_D$, respecting mass conservation, read:
The averaged Raychaudhuri equation:
\[ 3 \frac{\ddot{a}}{a} + 4\pi G \frac{M_D}{V_i a_D^3} - \Lambda = Q_D \; ; \] (42)

The averaged Hamiltonian constraint:
\[ \left( \frac{\dot{a}}{a} \right)^2 - \frac{8\pi G M_D}{3 V_i a_D^3} + \frac{\langle R \rangle_D}{6} - \frac{\Lambda}{3} = -\frac{Q_D}{6} \; , \] (43)

where the mass \( M_D \), the averaged spatial Ricci scalar \( \langle R \rangle_D \) and the “backreaction term” \( Q_D \) are domain–dependent and, except the mass, time–dependent functions. The backreaction source term is given by \( Q_D := 2 \langle I \rangle_D - \frac{2}{3} \langle \mathbf{I} \rangle_D^2 = 2 \left( \frac{\theta}{H_D^2} - \langle \theta \rangle_D \right)^2 - \frac{2}{3} \langle \mathbf{I} \rangle_D^2 \). (44)

Here, \( I \) and \( \mathbf{I} \) denote the invariants of the extrinsic curvature tensor that correspond to the kinematical invariants we employed earlier. The same expression (except for the vorticity) as in Eq. (9) follows by introducing the split of the extrinsic curvature into the kinematical variables shear and expansion (second equality above).

We appreciate an intimate correspondence of the GR equations with their Newtonian counterparts (Eq. (8) and Eq. (10)). The first equation is formally identical to the Newtonian one, while the second delivers an additional relation between the averaged curvature and the “backreaction term” that has no Newtonian analogue. This implies an important difference that becomes manifest by looking at the time–derivative of Eq. (43). The integrability condition that this time–derivative agrees with Eq. (42) is non–trivial in the GR context and reads:
\[ \partial_t Q_D + 6 \frac{\dot{a}}{a} Q_D + \partial_t \langle R \rangle_D + 2 \frac{\dot{a}}{a} \langle R \rangle_D = 0 \; . \] (45)

The correspondence between the Newtonian \( k_D \)–parameter and the averaged spatial Ricci curvature is more involved in the presence of a “backreaction term”:
\[ k_D = a_D^2 \int_0^t d\tau \frac{Q_D}{a_D^2(\tau)} = \frac{1}{6} (\langle R \rangle_D + Q_D) \; . \] (46)

The time–derivative of Eq. (46) is equivalent to the integrability condition Eq. (45). Eq. (45) shows that averaged curvature and “backreaction term” are directly coupled unlike in the Newtonian case, where the domain–dependent \( k_D \)–parameter is fixed by the initial conditions.

For dust models in general relativity we may therefore introduce dimensionless average characteristics that are slightly different from those in Newtonian cosmology, the difference being that the “backreaction parameter” is now directly expressible in terms of the “backreaction source term”, and not just as an integral expression:
\[ \Omega_m^p := \frac{8\pi G M_D}{3 V_i a_D^3 H_D^2} ; \Omega_m^\Lambda := \frac{\Lambda}{3 H_D^2} ; \Omega_m^p := -\frac{\langle R \rangle_D}{6 H_D^2} ; \Omega_m^Q := -\frac{Q_D}{6 H_D^2} \; , \] (47)

which also obey Eq. (13).

The evolution of these parameters is intimately related, unlike the situation in the standard model, as may be illustrated by the following equation [13]:
\[ \dot{\Omega}_m^p + 6H_D \Omega_m^p (1 - \Omega_k^P - \Omega_Q^P) + \dot{\Omega}_m^\Lambda + 2H_D \Omega_m^\Lambda (1 - \Omega_k^P - \Omega_Q^P) - 3H_D (1 - \Omega_k^P - \Omega_Q^P) (\Omega_k^p + \Omega_Q^P) = 0 \; . \] (48)

It is, e.g., a good exercise to show with the help of this equation that an inhomogeneous model (including “backreaction”), in which the matter parameter stays 1 like in the standard Einstein–de Sitter cosmos, does not exist.
2.5 Liberation from Strict Meter

Let us now hold in for a moment and sort out what the Newtonian and the relativistic expansion laws for dust matter distinguish. Their close correspondence bears the temptation of overlooking a crucial conceptual challenge for the modeling of inhomogeneities in general relativity.

Given the Newtonian average model, Eq. (8), and its quantitative consequences (elaborated in detail in [19]), we can draw the conclusion that we only have to consider spatial scales that are large enough to have a negligible influence from the “backreaction term”. This term, which brought the higher voltage of having mastered a generic inhomogeneous Newtonian cosmology, shows no global relevance, and it seems that we are drawn back to the previous state of low visibility of the standard cosmological models.

The key reason for this outcome is the validity of the Construction Principle. We force a globally vanishing “backreaction” by the way we construct inhomogeneous evolution models. (Remember that the majority of cosmological N–body simulations and analytical approximations are Newtonian and assume periodic boundary conditions for inhomogeneities on a FLRW background.) The apparent global irrelevance of “backreaction” should not be a surprise, since our evaluations have taken place within a periodic cube into which the universe model was forced: the scale of this cube introduces a “strict meter” governed by the scale–factor of a standard FLRW universe, the action of fluctuations of inhomogeneities is confined to the interior of this cube.

The relativistic average model not only has a somewhat richer tone by linking the averaged spatial Ricci curvature to the “backreaction term” as a result of the Hamiltonian constraint, it also places an equal stress on both of them in the following sense. Suppose we let the self–gravitating fluid evolve freely and link its structure to the curvature of space sections through Einstein’s equations. Let now the inhomogeneities evolve out of an almost homogeneous and isotropic state on almost flat space sections. Naturally, the inhomogeneities grow in the course of structure formation. The average Ricci curvature of the space sections, however, is dictated by the fluid structure unlike in the standard model which would suggest an evolution inversely proportional to the “strict meter” \( a(t) \). In contrast, the value of the averaged curvature is an outcome rather than a model assumption. Thus, a genuine property of the relativistic model, also globally, is a change of the average curvature that does not follow a predesigned global law; a priori it is not constrained on some large scale, because the link to the “backreaction term” makes sure that, at any time, the “interior” of a universe model (below this scale) also determines the global curvature. In other words: there is no analogue of the Construction Principle. This comment is not entirely correct, since further study may reveal that it could be possible to formulate an analogous principle on the basis of topological constraints that may be imposed on the space sections. In any case, research should not lack commitment to the challenge of constructing inhomogeneous evolution models on curved space sections, since this is evidently the generic case. We note that existing work on general relativistic evolution models and their averages relies in most cases on assumptions that are closely designed after the Newtonian case in order to rescue standard procedures like periodic boundary conditions, decomposition of inhomogeneities into plane waves and set–ups of initial conditions on locally flat space sections (compare [37] and comments in [15] and [14]).

2.6 What Einstein Wanted

In the previous subsection we advocated a viewpoint that concentrates on the physics of fluids, which (actively) determines the geometry of spacetime, in particular its average properties. As a showcase we learned in the Newtonian framework that the morphological properties of spatial domains are determined by the averages over fluctuations in the kinematical variables (Subsection 1.12). A similar view may also apply to the relativistic context and, most interestingly, also to the global morphology of space sections.

Turning this around, we may alternatively take the (passive) viewpoint of understanding the global world structure as given in terms of geometrical (and topological) conjectures (which may themselves be subjected to observational falsification; an example is the possibility of falsifying possible topological spaceforms on the basis of microwave background observations (\[24\]; see also \[15\] and ref. therein).

Einstein’s vision of a globally static universe model stands out as a famous example. His “prejudice” that the Universe as a whole should be static and all evolution (assigning sense to the notion of time) should take place in the “interior” of this world model entails a strong determination of the model’s average properties. Only his invention of the cosmological term made this vision possible resulting in the beautiful
structure of a spherical space in which, apart from topological degeneracies, many characteristics (like the radius of the universe model) are practically fixed. Hence, according to what we have learned about average models, the fluid’s fluctuations are “slaved” to the global predetermined structure. This global model even guarantees strong evolution, because fluctuations evolve exponentially in a static cosmos.

The problem that his model is unstable (within the class of FLRW models) applies also to the other expanding candidates, if the class is widened to include inhomogeneities. Einstein’s example so serves as a guide to think about the average model we have discussed in a geometrical way. The assumption of a static model (the general expansion law allows for a static cosmos even without the cosmological term) should be taken as an example for the possible slaving of fluctuations to global structural assumptions. Of course, other such assumptions are possible. This illustrates how global geometrical and topological constraints on spaceforms could provide “boundary conditions” for the evolution of fluctuations. Since, in Newtonian cosmology, we are always working on a toroidal space without further questioning this assumption, this point of view is not exotic, but plants the seed for possibly fruitful research directions. That these directions are mathematically very involved can be made obvious by pointing out the relation between the average curvature and its compatibility with topological spaceforms (a spaceform with globally negative spatial curvature is not compatible with simply–connected space sections). Due to the process of structure formation the boundary of the domain may break, or it may self–intersect and split into two domains (these events are topologically classified in the framework of Legendrian singularities, and ref. therein). Since the spatial metric is linked to the fluid, a singular dynamics could induce topology changes. Contrary to the existence of such natural metamorphoses, the average we have defined relies on the assumption that the domain remains simply–connected (Fig. 3).

2.7 Expansion Law of General Relativity: Perfect Fluid Models

As an outlook I briefly give one of the results obtained by averaging a perfect fluid cosmology on hypersurfaces $S(t) = \text{const.}$ as explained above.

The averaged equations can be summarized in the following way [16]:

Let us define effective densities as sources of an expansion law,

$$
\varepsilon_{\text{eff}} := \langle \tilde{\varepsilon} \rangle_D - \frac{\tilde{Q}_D}{16\pi G}, \quad p_{\text{eff}} := \langle \tilde{p} \rangle_D - \frac{\tilde{P}_D}{12\pi G},
$$

with the scaled matter sources $\tilde{\varepsilon} := N^2 \varepsilon$ and $\tilde{p} := N^2 p$. Then, the averaged equations can be cast into a form similar to the standard Friedmann equations:

$$
3 \frac{\partial^2 a_D}{a_D} + 4\pi G (\varepsilon_{\text{eff}} + 3 p_{\text{eff}}) = 0 ;
$$

$$
6 \dot{H}_D + \langle \tilde{R} \rangle_D - 16\pi G \varepsilon_{\text{eff}} = 0 ,
$$

and the integrability condition of Eq. (50) to yield Eq. (51) has the form of a balance equation between the effective sources and the averaged spatial (t–)Ricci scalar $\tilde{R} := N^2 \mathcal{R}$:

$$
\partial_t \varepsilon_{\text{eff}} + 3 \dot{H}_D (\varepsilon_{\text{eff}} + p_{\text{eff}}) = \frac{1}{16\pi G} \left( \partial_t \langle \tilde{R} \rangle_D + 2 \dot{H}_D \langle \tilde{R} \rangle_D \right) .
$$

The effective densities obey a conservation law, if the domains’ curvature evolves like in a “small” FLRW cosmology, $\langle \tilde{R} \rangle_D = 0$, or $\langle \tilde{R} \rangle_D \propto a_D^{-2}$, respectively.

The expressions $\tilde{Q}_D$ and $\tilde{P}_D$ are given in [16]. As can be seen above, $\tilde{Q}_D$ (the “kinematical backreaction”) acts like a fluid component with “stiff” equation of state, similar to the action of a minimally coupled scalar field, while $\tilde{P}_D$ (the “dynamical backreaction”) is due to the non–vanishing pressure gradient in the hypersurfaces.

It is interesting that the “kinematical backreaction” of the inhomogeneities could play the role of a free scalar field component that is commonly introduced when early evolutionary stages of the Universe are studied. It is also interesting that the expansion law in the case of a minimally coupled scalar field source gets particularly simple. This case will be investigated in a forthcoming work [20].
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