Canonical Quantization of Massive Symmetric Rank-Two Tensor in String Theory

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The canonical quantization of a massive symmetric rank-two tensor in string theory, which contains two Stueckelberg fields, was studied. As a preliminary study, we performed a canonical quantization of the Proca model to describe a massive vector particle that shares common properties with the massive symmetric rank-two tensor model. By performing a canonical analysis of the Lagrangian, which describes the symmetric rank-two tensor, obtained by Siegel and Zwiebach (SZ) from string field theory, we deduced that the Lagrangian possesses only first class constraints that generate local gauge transformation. By explicit calculations, we show that the massive symmetric rank-two tensor theory is gauge invariant only in the critical dimension of open bosonic string theory, i.e., $d = 26$. This emphasizes that the origin of local symmetry is the nilpotency of the Becchi-Rouet-Stora-Tyutin (BRST) operator, which is valid only in the critical dimension. For a particular gauge imposed on the Stueckelberg fields, the gauge-invariant Lagrangian of the SZ model reduces to the Fierz-Pauli Lagrangian of a massive spin-two particle. Thus, the Fierz-Pauli Lagrangian is a gauge-fixed version of the gauge-invariant Lagrangian for a massive symmetric rank-two tensor. By noting that the Fierz-Pauli Lagrangian is not suitable for studying massive spin-two particles with small masses, we propose the transverse-traceless (TT) gauge to quantize the SZ model as an alternative gauge condition. In the TT gauge, the two Stueckelberg fields can be decoupled from the symmetric rank-two tensor and integrated trivially. The massive spin-two particle can be described by the SZ model in the TT gauge, where the propagator of the massive spin-two particle has a well-defined massless limit.

PACS numbers: 11.25.-w, 11.15.-q

I. INTRODUCTION

Ever since Dirac [1] attempted to quantize Einstein’s gravity in the framework of the canonical Hamiltonian formulation, the problem of quantization of the spin-two particle has remained unsolved. When we perform a canonical quantization of Einstein’s gravity, a number of difficulties may be encountered [2–5]: Among others, Einstein’s gravity is highly non-linear, and it is difficult to apply the usual perturbation theory. If we expand Einstein’s gravity in terms of perturbative metrics near the flat Minkowski space-time, we obtain an infinite number of interaction terms. Second, because Einstein’s gravity does not possess dimensionless coupling, the perturbative quantum theory of Einstein’s gravity cannot be renormalized to four dimensions. String theory is anticipated to provide reasonable measures to address such difficulties associated with the application of perturbation theory to Einstein’s gravity. In this respect, it is worth noting that the graviton scattering amplitudes of Einstein’s gravity [6–11] arise from the closed string scattering amplitude in the low-energy approximation where $\alpha' \to 0$. By replacing Einstein’s gravity with closed string theory in the high-energy region, it is expected that the problem of ultraviolet divergence in quantum gravity could be resolved.

In the present work, we approach the canonical quantization of the spin-two field from a different angle: we discuss the massive symmetric rank-two tensor, which is closely related to the linear theory of massive gravity [12], within the framework of open string theory. (For a review of massive gravity, see Refs. [13, 14].) In open string theory [15], the massive spin-two field is generated as a component of the massive symmetric rank-two tensor multiplet, which additionally contains a vector field and a scalar field. These additional fields may be considered as Stueckelberg fields [16] to ensure local gauge symmetry, as will be shown later. The massless limit of the massive spin-two field corresponds to the high energy limit in open string theory where $\alpha' \to \infty$. Thus, canonical quantization of the massive spin-two field is also intrinsically important when exploring the high energy limit of open string theory [17, 18].

The purpose of this study is twofold: The first is to perform a canonical quantization of the massive symmetric rank-two tensor theory established by Siegel and Zwiebach (SZ) using the BRST-invariant open string field theory. The second is to clarify its relation to the linear theory of massive gravity studied by Fierz and Pauli (FP) and to propose an alternative, yet equivalent model, that may not share the undesired features of the FP model in the massless limit.
We note some of the problems of the FP model in the massless limit from the point of view of canonical quantization and propose an alternative theory for the massive spin-two field, which can be obtained from the massive symmetric rank-two tensor theory, by choosing the transverse-traceless (TT) gauge [19–23]. The problematic features of the FP model, such as the discontinuity of the massless limit [24, 25], which is termed as the van Dam-Veltman-Zakharov (vDVZ) discontinuity, may be attributed to inadequate choice of gauge fixing for the massive symmetric rank-two tensor theory of Siegel and Zwiebach with small mass. By an explicit construction, we show that in the TT gauge, the propagator of the massive spin-two field smoothly reduces to that of a massless graviton in the massless limit without encountering any discontinuities.

II. PRELIMINARY: CANONICAL QUANTIZATION OF THE PROCA MODEL

As a preliminary study, it may be instructive to first discuss the Proca model [26] that describes a massive vector particle because it is simple and shares many similarities with the massive symmetric rank-two tensor. A massive spin-one field is described by the Proca model whose Lagrangian is given as follows:

$$L_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Throughout this paper, we will use $\eta_{\mu\nu} = diag(-, +, \ldots, +)$ as the $d$-dimensional space-time metric. Performing the Hamiltonian analysis of the Proca action, we find two second class constraints [27, 28], namely $\varphi_1$ and $\varphi_2$:

$$\varphi_1 = \Pi_0 = 0,$$

$$\varphi_2 = \partial_\mu \Pi_\mu - m^2 A_0 = 0,$$

$$[\varphi_1(x), \varphi_2(x')]_{PB} = m^2 \delta(x - x') \neq 0.$$

where $\Pi_\mu$, $\mu = 0, 1, \ldots, d - 1$ are canonical momenta that are conjugate to $A^\mu$. This pair of second class constraints compels us to use the Dirac brackets instead of the Poisson brackets in the Hamiltonian formulation of the model. Usually, canonical quantization with second class constraints often turns out to be cumbersome and complicated [1]. This problem may be resolved by introducing an auxiliary scalar field called Stueckelberg field [16]. With the help of a Stueckelberg field, $\phi$, we can recast the model into a more convenient form as follows:

$$L_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 (A_\mu + \partial_\mu \phi)^2.$$  

This Lagrangian Eq. (3) is now invariant under a local gauge transformation

$$\delta A^\mu = \partial^\mu \epsilon, \quad \delta \phi = -\epsilon.$$

By imposing a gauge fixing condition $\phi = 0$, the gauge-invariant Lagrangian Eq. (3) reduces to the Proca Lagrangian Eq. (1). Thus, the Proca Lagrangian Eq. (1) can be understood as a gauge-fixed version of the gauge-invariant Lagrangian Eq. (3).

The advantage of using the gauge-invariant Lagrangian is that we do not need to employ the Dirac brackets to perform the canonical analysis: The Hamiltonian system of the gauge-invariant model possesses only first class constraints, which generate a local gauge transformation. Because we can use the usual Poisson brackets to carry out the Hamiltonian formulation, quantization of the system is simpler and easier than that with the Dirac brackets. Defining the canonical momenta as

$$\Pi^\mu = \frac{\partial L}{\partial \dot{A}_\mu}, \quad \pi_\phi = \frac{\partial L}{\partial \dot{\phi}},$$

we find

$$\Pi^i = F_{0i}, \quad \pi_\phi = m^2 \left( \dot{\phi} + A_0 \right)$$

and a primary constraint

$$\varphi_1 = \Pi^0 = 0.$$
The Hamiltonian density for the system may be written as follows:

\[
\mathcal{H} = \pi_i A^i + \pi_\phi \dot{\phi} - \mathcal{L}
\]

\[
= \frac{1}{2} \Pi_i \Pi^i + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2m^2} \pi_\phi^2 + \frac{1}{2} m^2 (A_t + \partial_t \phi)^2 + A^0 (\partial_t \Pi^i + \pi_\phi).
\] (8)

It is clear from Eq. (8) that \(A^0\) is a Lagrangian multiplier. Taking a commutator between the primary constraint Eq. (7) and the Hamiltonian, we obtain a secondary constraint as follows:

\[
[\varphi_2, H]_{PB} = - (\partial_t \Pi^i + \pi_\phi) = 0.
\] (9)

The following simple algebraic relations confirm that the closure property is satisfied:

\[
[\varphi_1, H] = \varphi_2, \quad [\varphi_2, H] = 0, \quad [\varphi_1, \varphi_2] = 0.
\] (10)

Using this algebraic constraint, we may construct a gauge generator as follows:

\[
\Omega_\epsilon(t) = \int d^{d-1} x \left( i \varphi_1 + \epsilon \varphi_2 \right)
\]

\[
= \int d^{d-1} x \left\{ i \Pi^0 - \epsilon ( \partial_\mu \Pi^\mu + \pi_\phi ) \right\},
\] (11)

which then generates the local gauge transformation of Eq. (4) as

\[
\delta A^\mu = [A^\mu, \Omega_\epsilon] = \partial^\mu \epsilon, \quad \delta \phi = [\phi, \Omega_\epsilon] = -\epsilon.
\] (12)

Having performed the canonical analysis, we may choose a suitable gauge condition to fix the degrees of freedom of the local gauge. Various choices of the gauge fixing conditions are available: We may choose the usual gauge conditions such as the Lorenz gauge condition \(\partial_\mu A^\mu = 0\), Coulomb gauge condition \(\nabla \cdot A = 0\), or axial gauge condition \(\nabla \cdot A = 0\). Alternatively, we may choose a gauge fixing condition that can be imposed on the Stueckelberg field as follows:

\[
\chi = \phi = 0.
\] (13)

This gauge condition also fixes the gauge degrees of freedom appropriately and causes the gauge-invariant Lagrangian of Eq. (3) to reduce to the Proca model, i.e., Eq. (1). As \(\Delta \chi(x)/\Delta \chi(x') = -\delta(x - x')\), this gauge fixing does not produce a nontrivial Faddeev-Popov ghost term. It may thus be appropriate to call this gauge condition (Eq. (13)) as the Proca gauge.

In the Proca gauge \(\phi = 0\), the generating function may be written as

\[
Z = \int D[A] D[\phi] \exp \left\{ i \int d^d x \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 (A_\mu + \partial_\mu \phi)^2 - \frac{\lambda}{2} \phi^2 \right) \right\}
\] (14)

where \(\lambda\) is a gauge parameter [29]. If we are interested in constructing a propagator for a massive spin-one particle using the Stueckelberg field, it would be convenient to treat the vector field and the Stueckelberg field as components of a multiplet \(\Psi^4 = (A^\mu, \phi)\): In terms of the multiplet, the Proca action may be written as

\[
A = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \bar{\psi} \mathcal{O} \psi,
\] (15)

\[
\mathcal{O} = \left( \begin{array}{cc} -\eta_{\mu \nu} (p^2 + m^2) + p_\mu p_\nu & -i m^2 p_\mu \\ im^2 p_\nu & -\lambda - m^2 p^2 \end{array} \right).
\] (16)

Inverting the operator \(\mathcal{O}\), we obtain the propagator \(G\) for \(\Psi\) as follows:

\[
G = \frac{1}{i} \mathcal{O}^{-1} = \left( \begin{array}{cc} \eta^{\mu \nu} + \frac{p^\mu p^\nu}{m^2} & \frac{1}{i} p^\nu \\ -\frac{1}{i} p^\mu & \frac{1}{i} \lambda \end{array} \right).
\] (17)

Here, we use the block matrix inversion

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.
\] (18)
In the limit where $\lambda \to \infty$, the Stueckelberg field $\phi$ completely decouples from the gauge fields $A^\mu$, and the propagator reduces to

$$G = \begin{pmatrix} \frac{i}{p^2 + m^2} \left( \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) & 0 \\ 0 & 0 \end{pmatrix}. \quad (19)$$

Therefore, in the Proca gauge $\phi = 0$ and Eq. (1), we only need to consider the gauge fields $A^\mu$, whose propagator is given by

$$G_A^{\mu\nu} = \frac{i}{p^2 + m^2} \left( \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right). \quad (20)$$

One of the interesting points that we should note is the massless limit of the Proca model. From the Lagrangian of the Proca model Eq. (1), we expect that in the massless limit, $m \to 0$, and the Proca model reduces to the $U(1)$ gauge theory. However, in quantum theory, this limit may not be considered in a straightforward manner: The propagator in Eq. (19) is ill-defined in the massless limit. This difficulty may be attributed to the fact that the numbers of degrees of freedom in the two theories differ from each other: The Proca model describing a massive vector particle has $d$ degrees of freedom in $d$ dimensions, whereas the $U(1)$ gauge theory that describes a massless gauge particle possesses only two degrees of freedom.

In order to explore the massless limit of the Proca model, it may be more appropriate to choose the usual Lorenz gauge $\partial_\mu A^\mu = 0$ in this covariant gauge, the Proca action with the Stueckelberg field Eq. (3) may be read as

$$S = \int d^d x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right\}. \quad (22)$$

where we scale the Stueckelberg field as $\phi \to \phi/m$. As the vector field $A_\mu$ decouples from the Stueckelberg field, we can obtain their propagators separately as follows:

$$G_A^{\mu\nu} = \frac{i}{p^2 + m^2} \left( \eta^{\mu\nu} + \alpha \frac{p^\mu p^\nu}{p^2} \right), \quad G_\phi = \frac{i}{p^2}, \quad \alpha = \frac{p^2 (1 - \lambda)}{p^2 + m^2 - p^2 (1 - \lambda)}. \quad (23a, b)$$

In the limit where $\lambda \to \infty$, we find that the propagator for $A_\mu$ satisfies the Lorenz gauge condition:

$$G_A^{\mu\nu} = \frac{i}{p^2 + m^2} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right), \quad G_A^{\mu\nu} p_\mu = G_A^{\mu\nu} p_\nu = 0. \quad (24)$$

In the Lorenz gauge, the massive vector particle is described by $A_\mu$, and satisfies the gauge condition $\partial_\mu A^\mu = 0$ and massless scalar field $\phi$. It is worth noting the difference between the propagator for the massless $U(1)$ gauge field in the Lorenz gauge

$$G_{U(1)}^{\mu\nu} = \frac{i}{p^2} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \quad (25)$$

and the propagator for the massive vector of Eq. (24). In contrast to the model of Eq. (1) in the Proca gauge, where the propagator for $A^\mu$ becomes singular in the massless limit, the model in the Lorenz gauge has a well-defined propagator for $A^\mu$, i.e., Eq. (24), which reduces to the propagator of the massless $U(1)$ gauge particle in the massless limit. Thus, the model in the Lorenz gauge Eq. (22) appears to be more suitable than the conventional Proca model of Eq. (1) for exploring the small-mass limit of the massive vector particle. The two models, Eq. (1) and Eq. (22), are gauge equivalent to each other. If we further integrate the massless scalar Stueckelberg field $\phi$, we get the following Lagrangian for a massive vector field that satisfies the Lorenz gauge condition:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu - \frac{\lambda}{2} (\partial_\mu A^\mu)^2. \quad (26)$$
III. MASSIVE SYMMETRIC RANK-TWO TENSOR IN STRING THEORY

We are now ready to discuss the canonical quantization of the massive symmetric rank-two tensor. In string theory, massive spin-two fields arise in the spectrum of open strings rather than closed strings as a part of massive rank-two tensor multiplet, which may be described by the following Lagrangian \[15\]:

\[
\mathcal{L}_{SZ} = \frac{1}{4} h_{\mu \nu} (\partial^2 - m^2) h^{\mu \nu} + \frac{1}{2} B_{\mu} (\partial^2 - m^2) B^\mu - \frac{1}{2} \eta (\partial^2 - m^2) \eta \\
+ \frac{1}{2} (\partial^\nu h_{\mu \nu} + \partial_\mu \eta - m B_\mu)^2 + \frac{1}{2} (\frac{m}{4} h + \frac{3}{2} m \eta + \partial_\mu B^\mu)^2
\]  

(27)

where \( h = h^\sigma_\sigma \). Here, \( B^\mu \) and \( \eta \) are Stueckelberg fields that ensure the local gauge symmetry of \( \mathcal{L} \). By algebraic manipulation, we find that this Lagrangian is invariant under the following local gauge transformation:

\[
\delta h_{\mu \nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - \frac{1}{2} m \eta_{\mu \nu} \epsilon, \quad \delta B_\mu = \partial_\mu \epsilon + m \epsilon_\mu, \quad \delta \eta = -\partial_\mu \epsilon^\mu + \frac{3}{2} m \epsilon.
\]  

(28)

It is worth noting that the explicit calculations given in the Appendix show that the gauge invariance works only for the critical dimension \( d = 26 \). This reminds us that the origin of the gauge symmetry is the nilpotency of the Becchi-Rouet-Stora-Tyutin (BRST) charge \[30, 31\] of open bosonic string theory, which requires that \( d = d_{\text{critical}} = 26 \).

The equations of motion for fields \( h_{\mu \nu}, B_\mu \), and \( \eta \) are respectively read from the Lagrangian as follows:

\[
\frac{1}{2} (\partial^2 - m^2) h_{\mu \nu} - \partial_\mu \partial^\lambda h_\nu^\lambda + m \partial_\mu \partial_\nu \eta + \eta_{\mu \nu} \left( \frac{m^2}{16} h + \frac{3}{8} m^2 \eta + \frac{m}{4} \partial_\lambda B^\lambda \right) = 0, \quad \text{(30a)}
\]

\[
\frac{1}{2} \partial^\nu B_\mu - \partial_\mu \partial_\nu B^\nu - m \partial^\nu h_{\mu \nu} - \frac{m}{4} \partial_\mu h - \frac{5}{2} m \partial_\mu \eta = 0, \quad \text{(30b)}
\]

\[
-2 \partial^\nu \eta + \frac{13}{4} m^2 \eta - \partial_\mu \partial_\nu h^{\mu \nu} + \frac{3}{8} m^2 h + \frac{5}{2} m \partial_\mu B^\mu = 0. \quad \text{(30c)}
\]

If we choose a gauge

\[ B^\mu = 0, \quad \eta = -\frac{1}{2} h^\mu_\mu, \quad \text{(30)} \]

to fix the gauge degrees of freedom of Eq. \[28\], the gauge-invariant Lagrangian \( \mathcal{L}_{SZ} \) of Eq. \[27\] reduces to the Fierz-Pauli Lagrangian \[12\], which was proposed to describe the massive spin-two particle

\[
\mathcal{L}_{FP} = \frac{1}{4} h_{\mu \nu} \partial^2 h^{\mu \nu} + \frac{1}{2} (\partial^\nu h_{\mu \nu})^2 - \frac{1}{2} \partial^\nu h_{\mu \nu} \partial^\mu h - \frac{1}{4} h \partial^2 h - \frac{1}{4} \left( m^2 h^{\mu \nu} h_{\mu \nu} - m^2 h^2 \right). \quad \text{(31)}
\]

For the massless case, \( m = 0 \), the Fierz-Pauli Lagrangian is identified as the linearized Lagrangian of Einstein’s gravity. Thus, the Fierz-Pauli Lagrangian corresponds to the gauge-invariant Lagrangian \( \mathcal{L}_{FP} \) for a particular gauge. We may call this gauge condition, Eq. \[30\], as the Fierz-Pauli gauge condition.

IV. CANONICAL QUANTIZATION OF THE FIERZ-PAULI LAGRANGIAN

Being a gauge fixed Lagrangian, the Fierz-Pauli Lagrangian is expected to possess second class constraints, similar to the Proca Lagrangian. Defining the canonical momenta for the Fierz-Pauli Lagrangian

\[
\Pi_{00} = \frac{\partial \mathcal{L}_{FP}}{\partial h_{00}}, \quad \Pi_{0i} = \frac{\partial \mathcal{L}_{FP}}{\partial h_{0i}}, \quad \Pi_{ij} = \frac{\partial \mathcal{L}_{FP}}{\partial h^{ij}}, \quad \text{we find that}
\]

\[
\Pi_{00} = 0, \quad \Pi_{0i} = \partial^j h_{ij} - \partial_i h^{jk} \eta_{jk}, \quad \text{(33a)}
\]

\[
\Pi_{ij} = \frac{1}{2} \delta_{ij} - \frac{1}{2} \eta_{ij} h^{kl} \eta_{kl}. \quad \text{(33b)}
\]

The first two equations of Eq. \[33a\], which do not contain time derivatives of the canonical variables, are identified as the primary constraints:

\[
\varphi_0 = \Pi_{00} = 0, \quad \varphi_i = \Pi_{0i} - \partial^j h_{ij} + \partial_i h^{jk} \eta_{jk}. \quad \text{(34a)}
\]

\[
\varphi_j = \Pi_{ij} - \partial^k h_{ijk} + \partial_i h^{jk} \eta_{jk}. \quad \text{(34b)}
\]
The Hamiltonian corresponding to the Fierz-Pauli Lagrangian is found as

\[
H = \Pi_0 \dot{h}^0 + \Pi_i \dot{h}^i - \mathcal{L}_{FP}
\]

\[
= (\Pi_0)^2 - \frac{1}{d-2}(\Pi_i \eta^{ij})^2 + V,
\]

(35a)

\[
V = -\frac{1}{2}(\partial_i h^{0j})^2 + \frac{1}{4}(\partial_k h^{ij})^2 + \frac{1}{2}(\partial_i h^{00})^2 - \frac{1}{2}(\partial_i h^{ij})^2
\]

\[
- \frac{1}{2}(\partial_j h^{00})^2 + \frac{1}{2}(\partial_i h^{00})^2 + \frac{1}{2}(\partial_i h^{ij})^2
\]

\[
+ \frac{1}{4} m^2 (-2(h^{0i})^2 + (h^{ij})^2 + 2h^{00}h^{ij} \eta^{ij} - (h^{ij})^2).
\]

(35b)

Commutators of the primary constraints with the Hamiltonian generate the secondary constraints as

\[
\chi_0 = [\varphi_0, H]_{PB} = -\frac{1}{2} \partial_i \partial_j h^{ij} + \frac{1}{2} \nabla^2 h^{ij} \eta^{ij} - \frac{1}{2} m^2 h^{ij} \eta^{ij} = 0,
\]

(36a)

\[
\chi_i = [\varphi_i, H]_{PB} = -\nabla^2 h^{0i} + \partial_i \partial_j h^{0j} + m^2 h^{0i} - 2 \partial_j \Pi^j_i = 0.
\]

(36b)

Using algebraic manipulations, we find that they are the second class constraints for \(m \neq 0\):

\[
[\varphi_0(x), \varphi_i(x')]_{PB} = 0, \quad [\varphi_0(x), \chi_0(x')]_{PB} = 0,
\]

(37a)

\[
[\varphi_0(x), \chi_i(x')]_{PB} = 0, \quad [\varphi_i(x), \varphi_j(x')]_{PB} = 0,
\]

(37b)

\[
[\varphi_i(x), \chi_0(x')]_{PB} = 0, \quad [\varphi_i(x), \chi_j(x')]_{PB} = -m^2 \delta_{ij} \delta(x - x'),
\]

(37c)

\[
[\chi_i(x), \chi_j(x')]_{PB} = 0, \quad [\chi_0(x), \chi_i(x')]_{PB} = -m^2 \partial_j \delta(x - x').
\]

(37d)

Therefore, we should employ the Dirac brackets to perform the canonical quantization of the Fierz-Pauli model, which is expected to be very complicated. Moreover, as we observed in the previous section on the discussion of the Proca model, the Dirac brackets may not be well defined in the massless limit. This leads us to conclude that the Fierz-Pauli Lagrangian may not be suitable for studying the small-mass limit of the massive spin-two particle. We should therefore look for an alternative Lagrangian that is well defined in the small-mass limit yet gauge equivalent to the Fierz-Pauli Lagrangian if we are interested in studying the spin-two particle with a small mass.

Although it is cumbersome to carry out canonical quantizations of the Fierz-Pauli Lagrangian, it is not difficult to find the propagator of the massive spin-two particle described by the Fierz-Pauli Lagrangian: In momentum space, we may rewrite the Fierz-Pauli Lagrangian of Eq. \(31\) as

\[
\mathcal{L}_{FP} = \frac{1}{2} h^{\mu \nu} \mathcal{O}_{\mu \nu, \alpha \beta}^{FP} h_{\alpha \beta},
\]

(38a)

\[
\mathcal{O}_{\mu \nu, \alpha \beta}^{FP} = -\frac{(p^2 + m^2)}{4} (\eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} - 2 \eta_{\mu \nu} \eta_{\alpha \beta}) - \frac{1}{2} (\eta_{\mu \nu} p^2 p_{\alpha \beta} + \eta_{\alpha \beta} p^2 p_{\mu \nu})
\]

\[
+ \frac{1}{4} (\eta_{\mu \alpha} p^2 p_{\nu \beta} + \eta_{\mu \beta} p^2 p_{\nu \alpha} + \eta_{\nu \alpha} p^2 p_{\mu \beta} + \eta_{\nu \beta} p^2 p_{\mu \alpha}).
\]

(38b)

The propagator of the spin-two field is defined by

\[
\mathcal{G}_{\mu \nu, \alpha \beta}^{FP} G_{\mu \nu, \alpha \beta}^{FP, \sigma \lambda} = \frac{i}{2} (\delta_{\mu}^{\sigma} \delta_{\nu}^{\lambda} + \delta_{\nu}^{\sigma} \delta_{\mu}^{\lambda}).
\]

(39)

Using algebraic manipulations, we obtain

\[
G_{\mu \nu, \alpha \beta}^{FP} = \frac{i}{p^2 + m^2} \left\{ \frac{2}{d-1} \left( \eta_{\alpha \beta} + \frac{p_{\alpha} p_{\beta}}{m^2} \right) \left( \eta_{\sigma \lambda} + \frac{p_{\sigma} p_{\lambda}}{m^2} \right)
\]

\[
- \left( \eta_{\alpha \sigma} + \frac{p_{\alpha} p_{\sigma}}{m^2} \right) \left( \eta_{\beta \lambda} + \frac{p_{\beta} p_{\lambda}}{m^2} \right) - \left( \eta_{\alpha \lambda} + \frac{p_{\alpha} p_{\lambda}}{m^2} \right) \left( \eta_{\beta \sigma} + \frac{p_{\beta} p_{\sigma}}{m^2} \right) \right\}.
\]

(40)

This propagator of the spin-two field for the Fierz-Pauli Lagrangian and the propagator of the vector field Eq. \(21\) for the Proca model are similar in that both diverge in the massless limit. Therefore, the Fierz-Pauli Lagrangian may not be useful for studying the spin-two particle with small mass.
V. CANONICAL QUANTIZATION OF MASSIVE SYMMETRIC RANK-TWO TENSOR

Returning to the gauge-invariant Lagrangian of Siegel and Zwiebach, we begin the canonical quantization of the massive symmetric rank-two tensor by defining the canonical momenta: We may rewrite $\mathcal{L}_{SZ}$ of Eq. (27) as follows

$$\mathcal{L}_{SZ} = \frac{1}{4} (\dot{h}^{00})^2 + h^{00} (\partial_t h^{0i} + \dot{\eta} + mB^0) + \dot{h}^{0i}(\partial^i h_{ij} + \partial_i \eta - mB_i)$$
$$+ \frac{1}{4} (\dot{h}^{ij})^2 + \dot{B}^0 (\partial_t B^i + \frac{1}{4} m(-h^{00} + h^{ij} \eta_{ij}) + \frac{5}{2} m\eta) + \frac{1}{2} (\dot{B}^i)^2$$
$$- \dot{\eta}^2 + \partial_i h^{0i} \dot{\eta} - V. \tag{41}$$

Here, $V$ denotes a potential term that does not contain time derivatives of the fields,

$$V = \frac{1}{4} (\partial_t h^{00})^2 + \frac{7}{32} m^2 (h^{00})^2 - \frac{1}{2} (\partial_t h^{0i})^2 + \frac{1}{2} (\partial_i h^{0i})^2 - \frac{1}{2} m^2 (h^{0i})^2$$
$$+ \frac{1}{4} (\partial_t h^{ij})^2 - \frac{1}{2} (\partial_t h^{ij})^2 + \frac{1}{4} m^2 (h^{ij})^2 - \frac{1}{32} m^2 (h^{ij} \eta_{ij})^2 - \frac{1}{2} (\partial_i B^0)^2$$
$$+ \frac{1}{2} (\partial_i B^i)^2 - \frac{1}{2} (\partial_i B^i)^2 - (\partial_i \eta)^2 - \frac{13}{8} m^2 \eta^2 + \frac{1}{4} m h^{00} \partial_i B^i + \frac{1}{16} m^2 h^{00} h^{ij} \eta_{ij}$$
$$+ \frac{3}{8} m^2 \eta h^{00} - \partial_i h^{0i} B^0 - \partial_i h^{ij} \partial_j \eta + m \partial_i h^{ij} B_j - \frac{1}{4} m h^{ij} \eta_{ij} \partial_i B^k$$
$$- \frac{3}{8} m^2 \eta h^{ij} \eta_{ij} - \frac{5}{2} m \eta \partial_i B^i. \tag{42}$$

The canonical momenta that are conjugate to the fields are defined as

$$\Pi_{00} = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{h}^{00}}, \quad \Pi_{0i} = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{h}^{0i}}, \quad \Pi_{ij} = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{h}^{ij}},$$
$$\Pi_0^B = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{B}^0}, \quad \Pi_i^B = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{B}^i}, \quad \pi_\eta = \frac{\partial \mathcal{L}_{SZ}}{\partial \dot{\eta}}. \tag{43a}$$

From Eqs. (41) (42) (43a) (43b) we obtain

$$\Pi_{00} = \frac{1}{2} \dot{h}^{00} - \partial_t h^{00} + \dot{\eta} + mB^0, \quad \Pi_{0i} = \partial^i h_{ij} + \partial_i \eta - mB_i, \tag{44a}$$
$$\Pi_{ij} = \frac{1}{2} \dot{h}_{ij}, \quad \Pi_0^B = \partial_t B^i + \frac{1}{4} m(-h^{00} + h^{ij} \eta_{ij}) + \frac{5}{2} m\eta, \tag{44b}$$
$$\Pi_i^B = \dot{B}_i, \quad \pi_\eta = \dot{h}^{00} - 2\dot{\eta} + \partial_i h^{0i} \tag{44c}$$

and the Hamiltonian density

$$\mathcal{H}_{SZ} = -\Pi_{00}^2 + \Pi_{ij}^2 + \frac{1}{2} (\Pi_i^B)^2 - \pi_\eta (mB^0 - \partial_t h^{00})$$
$$+ mB^0 (mB^0 - \partial_t h^{00}) + V. \tag{45}$$

We may manufacture the primary constraints by taking the linear combination of equations Eqs.(44a) (44b) (44c):

$$\varphi_0 = 2\Pi_{00} + \pi_\eta + \partial_t h^{0i} - 2mB^0 = 0, \tag{46a}$$
$$\varphi_i = \Pi_{0i} - \partial^i h_{ij} - \partial_i \eta + mB_i = 0, \tag{46b}$$
$$\varphi^B = \Pi_0^B - \partial_t B^i + \frac{1}{4} m(h^{00} - h^{ij} \eta_{ij}) - \frac{5}{2} m\eta = 0. \tag{46c}$$

Commutators of these primary constraints with the Hamiltonian yield the secondary constraints as follows:

$$\chi_0 = [\varphi_0, \mathcal{H}_{SZ}]_{PB} = \nabla^2 h^{00} - m^2 h^{00} + m\pi_{B0} - \partial^i \pi_{0i} - \nabla^2 \eta, \tag{47a}$$
$$\chi_i = [\varphi_i, \mathcal{H}_{SZ}]_{PB} = \partial_i \pi_\eta - m\partial_i B^0 - \nabla^2 h^{0i} + m^2 h^{0i} + m\pi_{Bi} - 2\partial^i \pi_{ij}, \tag{47b}$$
$$\chi^B = [\varphi^B, \mathcal{H}_{SZ}]_{PB} = \frac{1}{2} m^2 \pi_{B0} + \frac{3}{2} m\pi_\eta - \partial^i \pi_{B1} - \frac{1}{2} m\eta \pi_{ij} - \frac{1}{2} m^2 B^0 - \nabla^2 B^0. \tag{47c}$$
where \( H_{SZ} = \int d^{d-1} x \mathcal{H}_{SZ}(x) \). The primary constraints, Eqs. (46a, 46b, 46c), and the secondary constraints, Eqs. (47a, 47b, 47c), form a set of first class constraints, which generate the local gauge transformation in Eq. (28)

\[ \delta h_{\mu\nu} = [h_{\mu\nu}, \Omega(e^0, e^i, e)]_{PB}, \quad \delta B_{\mu} = [B_{\mu}, \Omega(e^0, e^i, e)]_{PB}, \quad \delta \eta = [\eta, \Omega(e^0, e^i, e)]_{PB} \]  

(48)

with a gauge generator

\[ \Omega(e^0, e^i, e) = -e^0 \varphi_0 - e^i \varphi_i + e^0 \chi_0 + e^i \chi_i. \]  

(49)

VI. MASSIVE SYMMETRIC RANK-TWO TENSOR IN THE TRANSVERSE-TRACELESS GAUGE

We have shown that the gauge degrees of freedom of the Siegel-Zwiebach Lagrangian can be fixed by the Fierz-Pauli gauge condition, and the Lagrangian reduces to the well-known Fierz-Pauli Lagrangian, which describes a massive spin-two particle. However, the Lagrangian in the Fierz-Pauli gauge may not be useful for studying a spin-two particle with small mass because the propagator for the spin-two field may be singular in the massless limit. From the study of the Proca model, we deduced that an appropriate gauge condition may be the transverse-traceless (TT) gauge \[19, 23\]

\[ \partial_\mu h^\mu_\nu = 0, \quad h = 0. \]  

(50)

In the TT gauge, \( \mathcal{L}_{SZ} \) reduces to \( \mathcal{L}_{TT} \):

\[ \mathcal{L}_{TT} = \frac{1}{4} h_{\mu\nu}(\partial^2 - m^2)h^{\mu\nu} + \frac{1}{2} B_{\mu}(\partial^2 - m^2)B^{\mu} - \frac{1}{2} \eta(\partial^2 - m^2)\eta 

+ \frac{1}{2} (\partial_{\mu}\eta - m B_{\mu})^2 + \frac{1}{2} m \eta + \partial_{\mu}B^{\mu})^2 - \frac{\lambda}{2} (\partial_{\mu}h^{\mu\nu})^2 - \frac{\sigma}{2} h^2 \]  

(51)

where \( \lambda \) and \( \sigma \) are Lagrangian multipliers. We note that the symmetric rank-two tensor field \( h_{\mu\nu} \) decouples from the Stueckelberg fields \( B_{\mu} \) and \( \eta \). Thus, we can separate the propagator for \( h_{\mu\nu} \) from those of \( B_{\mu} \) and \( \eta \).

It takes some algebraic manipulation to obtain the propagator for \( h_{\mu\nu} \). In momentum space, the Lagrangian for \( h_{\mu\nu} \) in the TT gauge may be written as

\[ \mathcal{L}_{TT}^h = \frac{1}{2} h_{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h^{\alpha\beta}, \]  

(52a)

\[ \mathcal{O}_{\mu\nu,\alpha\beta} = \frac{(p^2 + m^2)}{4} (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}) - \sigma \eta_{\mu\nu}\eta_{\alpha\beta} \]  

(52b)

and the propagator for \( h_{\mu\nu} \), \( G^{\alpha\beta,\sigma\lambda} \), which satisfies

\[ \mathcal{O}_{\mu\nu,\alpha\beta} G^{\alpha\beta,\sigma\lambda} = \frac{i}{2} (\delta^\sigma_\beta \delta^\lambda_\mu + \delta^\sigma_\mu \delta^\lambda_\beta). \]  

(53)

Some straightforward but tedious algebra yields an explicit form of the propagator for \( h_{\mu\nu} \) in the TT gauge:

\[ G^{\alpha\beta,\sigma\lambda} = \frac{i}{p^2 + m^2} \left\{ A \eta^{\alpha\beta} \eta^{\sigma\lambda} - (\eta^{\alpha\sigma} \eta^{\beta\lambda} + \eta^{\alpha\lambda} \eta^{\beta\sigma}) + \frac{C}{p^2} (\eta^{\beta\sigma} p^\sigma p^\lambda + \eta^{\sigma\lambda} p^\alpha p^\beta) \right\} + \frac{D}{p^2} (\eta^{\sigma\lambda} p^\alpha p^\beta + \eta^{\alpha\lambda} p^\beta p^\sigma + \eta^{\beta\sigma} p^\alpha p^\lambda + \eta^{\alpha\beta} p^\sigma p^\lambda) + \frac{E}{p^4} p^\alpha p^\beta p^\sigma p^\lambda. \]  

(54a)

\[ A = \frac{4p^2 (p^2 + m^2) \lambda + 8p^2 \lambda \sigma}{(p^2 + m^2)^2 + 2 (p^2 + m^2) (p^2 \lambda + d \sigma) + 4(d - 1)p^2 \lambda \sigma}, \]  

(54b)

\[ C = -\frac{8p^2 \lambda \sigma}{(p^2 + m^2)^2 + 2 (p^2 + m^2) (p^2 \lambda + d \sigma) + 4(d - 1)p^2 \lambda \sigma}, \]  

(54c)

\[ D = \frac{p^2 \lambda}{(p^2 + m^2)^2 + 2 (p^2 + m^2) (p^2 \lambda + d \sigma) + 4(d - 1)p^2 \lambda \sigma}. \]  

(54d)

\[ E = -\frac{4p^2 \lambda}{(p^2 + m^2)^2 + 2 (p^2 + m^2) (p^2 \lambda + d \sigma) + 4(d - 1)p^2 \lambda \sigma}. \]  

(54e)
In the limit where \( \lambda, \sigma \to \infty \), we find
\[
A \to \frac{2}{d-1}, \quad C \to -\frac{2}{d-1}, \quad D \to 1, \quad E \to -\frac{2(d-2)}{d-1}. \tag{55}
\]

Hence, we obtain the propagator, \( G_{TT}^{\alpha \beta, \sigma \lambda}(m) \) in the TT gauge, which reduces to
\[
\lim_{\lambda, \sigma \to \infty} G_{TT}^{\alpha \beta, \sigma \lambda}(m) = \frac{i}{p^2 + m^2} \left\{ \frac{2}{d-1} \left( \eta^\alpha - \frac{p^\alpha p^\beta}{p^2} \right) \left( \eta^\beta - \frac{p^\sigma p^\lambda}{p^2} \right) \right.
\]
\[
- \left( \eta^\alpha - \frac{p^\alpha p^\sigma}{p^2} \right) \left( \eta^\beta - \frac{p^\sigma p^\lambda}{p^2} \right) - \left( \eta^\lambda - \frac{p^\alpha p^\beta}{p^2} \right) \left( \eta^\sigma - \frac{p^\beta p^\sigma}{p^2} \right) \right\}. \tag{56}
\]

in the limit \( \lambda, \sigma \to \infty \). It is easy to check that \( G_{TT}^{\alpha \beta, \sigma \lambda}(m) \) satisfies the transverse-traceless gauge condition:
\[
G_{TT}^{\alpha \beta, \sigma \lambda}(m)_{\alpha} = G_{TT}^{\alpha \beta, \sigma \lambda}(m)_{\beta} = G_{TT}^{\alpha \beta, \sigma \lambda}(m)_{\sigma} = G_{TT}^{\alpha \beta, \sigma \lambda}(m)_{\lambda} = 0, \tag{57a}
\]
\[
\eta_{\alpha \beta} G_{TT}^{\alpha \beta, \sigma \lambda}(m) = \eta_{\sigma \lambda} G_{TT}^{\alpha \beta, \sigma \lambda}(m) = 0. \tag{57b}
\]

Apparently, the propagator for the massive spin-two field in the TT gauge has a well-defined massless limit:
\[
\lim_{m \to 0} G_{TT}^{\alpha \beta, \sigma \lambda}(m) = G_{TT}^{\alpha \beta, \sigma \lambda}(0) = \frac{i}{p^2} \left\{ \frac{2}{d-1} \left( \eta^\alpha - \frac{p^\alpha p^\beta}{p^2} \right) \left( \eta^\beta - \frac{p^\sigma p^\lambda}{p^2} \right) \right.
\]
\[
- \left( \eta^\alpha - \frac{p^\alpha p^\sigma}{p^2} \right) \left( \eta^\beta - \frac{p^\sigma p^\lambda}{p^2} \right) - \left( \eta^\lambda - \frac{p^\alpha p^\beta}{p^2} \right) \left( \eta^\sigma - \frac{p^\beta p^\sigma}{p^2} \right) \right\}. \tag{58}
\]

Here, \( G_{TT}^{\alpha \beta, \sigma \lambda}(0) \) is identified as the propagator for the massless graviton in the TT gauge \([19]\).

Returning to the Lagrangian in the TT gauge of Eq. \([51]\), we may write the part of the Lagrangian containing the Stueckelberg fields \( B_\mu \) and \( \eta \) as
\[
L_{TT}^{B, \eta} = \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + 5 m_\eta \partial_\mu B^\mu - \frac{1}{2} \eta (\partial^2 - m^2) \eta. \tag{59}
\]

We note that the kinetic term for \( B_\mu \) is invariant under \( U(1) \) gauge transformation
\[
B_\mu \to B_\mu + \partial_\mu \Lambda. \tag{60}
\]

Decomposing the vector field \( B_\nu \) into a transverse part \( B_\nu^T \) and a longitudinal part \( B_\nu^L \)
\[
B_\nu = B_\nu^T + B_\nu^L = B_\mu \left( \eta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right) + B^\mu \left( \frac{p_\mu p_\nu}{p^2} \right), \tag{61}
\]

we find that \( \eta \) couples to the longitudinal part of the \( B_\nu \) field, which may be written with a scalar field \( \phi \) as \( B_\nu^L = \partial_\nu \phi \). As \( B^L \), equivalently \( \phi \), appears only in the Lagrangian Eq. \([59]\) through the linear coupling to \( \eta, \frac{1}{2} m \partial^2 \eta \phi \), \( \phi \) plays the role of a Lagrangian multiplier. Integrating \( \phi \) in the path integral imposes a condition \( \partial^2 \eta = 0 \); therefore, \( \eta \) is an auxiliary field, which can be trivially integrated. Hence, integrating \( B_\mu^L \) and \( \eta \), we get \( L_{TT}^{B, \eta} \to L_{TT}^B \), which may be written by
\[
L_{TT}^B = \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2, \quad \partial_\mu B^\mu = 0. \tag{62}
\]

The propagator for \( B_\mu^T \) is the bona-fide propagator for a massless \( U(1) \) gauge field in the Lorenz gauge
\[
G_{\mu \nu}^B = \frac{i}{p^2} \left( \eta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right), \quad p^\mu G_{\mu \nu}^B = p^\nu G_{\mu \nu}^B = 0. \tag{63}
\]
Collecting $\mathcal{L}_{TT}^h$ and $\mathcal{L}_{TT}^B$, we find that the Siegel-Zwiebach model for a massive symmetric rank-two tensor is described in the TT gauge by a massive spin-two field, satisfying the TT gauge condition and the Stueckelberg vector field $B_\mu$, which becomes a massless $U(1)$ gauge field, satisfying the Lorenz gauge condition

$$\mathcal{L}_{TT} = \frac{1}{4} h_{\mu\nu} (\partial^2 - m^2) h^{\mu\nu} + \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \frac{\lambda}{2} (\partial_\mu h^{\mu\nu})^2 - \frac{\sigma}{2} h^2 - \frac{\alpha}{2} (\partial_\mu B^\mu)^2. \quad (64)$$

It may be reasonable to integrate the Stueckelberg vector field $B_\mu$ further because it does not couple to physical degrees of freedom: in conclusion, we may describe the massive spin-two particle by $\mathcal{L}_{TT}^h$

$$\mathcal{L}_{TT}^h = \frac{1}{4} h_{\mu\nu} (\partial^2 - m^2) h^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu h^{\mu\nu})^2 - \frac{\sigma}{2} h^2 \quad (65)$$

and the propagator $G_{TT,\alpha\beta,\sigma\lambda}^\alpha(m)$ in the TT gauge given by Eq. (56). Fig. 1 summarizes the relations between the theories of the spin-two field.

**FIG. 1:** Relations between theories of the spin-two field.

**VII. CONCLUSIONS**

We have studied the massive symmetric rank-two tensor in open string theory in the framework of canonical quantization that contains two Stueckelberg fields. Performing canonical analysis, we find that the massive symmetric rank-two tensor theory of Siegel and Zwiebach contain only first class constraints. By explicit calculation, we have shown that the rank-two tensor theory is invariant under the local gauge transformation generated by the first class constraints in the critical dimension of bosonic string theory. In a particular gauge, the Stueckelberg fields the rank-two tensor theory reduces to the massive spin-two theory of Fierz and Pauli. Then, we pointed out that the massive spin-two theory of Fierz and Pauli possesses second class constraints, and the Dirac brackets are not well defined in the massless limit. Accordingly, the propagator for the massive spin-two field of Fierz-Pauli theory also diverges and does not reduce to that of the massless graviton of Einstein’s gravity in the massless limit: this gives rise to the vDVZ discontinuity [24, 25], which has been studied in detail in Refs. [32–35].

Because the massive symmetric rank-two tensor theory is gauge invariant, we may choose a gauge condition alternative to the Fierz-Pauli gauge condition. We may impose the gauge fixing condition on the spin-two field $h_{\mu\nu}$ instead of the Stueckelberg fields: the most plausible choice may be the transverse-traceless (TT) gauge condition by which the
spin-two field completely decouples from the two Stueckelberg fields. Hence, in the TT gauge, we may integrate the Stueckelberg fields, which may not couple to physical sources, to obtain a simple Lagrangian $L_{TT}^b$, Eq. (65) describing the massive spin-two field satisfying the TT gauge condition. By some algebra, we constructed a propagator for massive spin-two field, $G_{TT}^{ab,\sigma\lambda}(m)$ in the TT gauge. In contrast to the propagator in the massive spin-two theory of Fierz and Pauli, $G_{FP}^{ab,\sigma\lambda}(m)$, Eq.(40), the propagator in the TT gauge, $G_{TT}^{ab,\sigma\lambda}(m)$ smoothly reduces in the zero-mass limit to the graviton propagator of Einstein’s gravity in the TT gauge without encountering any discontinuity. It may be interesting to study this massive rank-two tensor theory further to understand how the Vainshtein mechanism [32] is realized in the TT gauge. It may also be worthwhile to investigate the propagation of the gravitational wave using the propagator in the TT gauge in details to explore the possibility of finite mass for graviton. It is apparent that the massive rank-two tensor theory of Siegel and Zwiebach is free of Boulware-Deser ghost [36–38], because this gauge-invariant theory reduces to the Fierz-Pauli theory in a particular gauge, where the Boulware-Deser ghost is absent.

As we have shown by explicit calculation, the symmetric rank-two tensor theory of Siegel and Zwiebach possesses local gauge invariance only at the critical dimension. In order to apply this symmetric rank-two tensor theory on four-dimensional space-time, we need to define the open string theory on a D3-brane. In this case we expect to have superfluous vector and scalar fields. The present work may be extended further to symmetric rank-two tensor theories on various dimensions by defining open string theories on general $Dp$-branes. The symmetric rank-two tensor theory based on string theory is also useful to construct a consistent interacting spin-two field theory [39][43], which is one of most challenging problems in theoretical physics. We can obtain cubic coupling terms directly from three open string scattering amplitudes, evaluating Polyakov string path integrals [44] in the proper-time gauge [45–50]. Choosing external string states as a composition of vector gauge field and spin-two field, we may fix the cubic interaction terms between vector gauge field and spin-two field. In principle, string scattering amplitudes will guide us to construct a consistent interacting spin-two field theory and we may resolve the Velo-Zwanziger problem [39–41] in the framework of string theory on $Dp$-branes. Extensions of the present work along these directions will be presented elsewhere.

Acknowledgments

This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2017R1D1A1A02017805) and also by the 2018 Research Grant (PoINT) from Kangwon National University. TL acknowledges and is thankful for the hospitality at APCTP, where a part of this work was done.

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It may be instructive to check the local gauge invariance of the massive symmetric rank-two tensor theory $\mathcal{L}_{SZ}$ of Siegel and Zwiebach by explicit calculations. The massive symmetric rank-two tensor theory of Siegel and Zwiebach is described by the following Lagrangian:

$$\mathcal{L}_{SZ} = \frac{1}{4} h_{\mu\nu}(\partial^2 - m^2)h^{\mu\nu} + \frac{1}{2} B_\mu(\partial^2 - m^2)B^\mu - \frac{1}{2} \eta(\partial^2 - m^2)\eta$$

$$+ \frac{1}{2} (\partial' h_{\mu\nu} + \partial_\mu \eta - m B_\mu) + \frac{1}{2} (\frac{m}{4} h + \frac{3}{2} m\eta + \partial_\mu B^\mu)^2$$  \hspace{1cm} (A1)

where $h = h^\sigma \sigma$. From the BRST invariance \[15\], we deduce the gauge transformation as

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - \frac{1}{2} m_\nu \epsilon_\mu, \quad \delta B_\mu = \partial_\mu \epsilon + m \epsilon_\mu, \quad \delta \eta = -\partial_\mu \epsilon^\mu + \frac{3}{2} m \epsilon. \quad (A2)$$

Under gauge transformation, each term in $\mathcal{L}_{SZ}$ is transformed as follows:

$$\frac{1}{4} h_{\mu\nu}\partial^2 h^{\mu\nu} \rightarrow \frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \epsilon + \frac{1}{2} \partial_\nu \epsilon_\mu \partial^2 \epsilon + \frac{1}{16} m^2 d\epsilon \partial^2 \epsilon$$

$$+ h_{\mu\nu}\partial^2 \epsilon^\mu \epsilon - \frac{1}{4} m \partial_\nu \epsilon^\mu \partial^2 \epsilon,$$  \hspace{1cm} (A3a)

$$- \frac{m^2}{4} h_{\mu\nu}h^{\mu\nu} \rightarrow - \frac{1}{2} m^2 \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu - \frac{1}{2} m^2 \partial_\nu \epsilon_\mu \partial^2 \epsilon^\mu - \frac{1}{16} m^4 d\epsilon$$

$$- m^2 h_{\mu\nu} \partial^2 \epsilon^\mu \epsilon + \frac{1}{4} m^3 \partial_\nu \epsilon^\mu \partial^2 \epsilon,$$  \hspace{1cm} (A3b)

$$\frac{1}{2} B_\mu \partial^2 B^\mu \rightarrow \frac{1}{2} \partial_\mu \epsilon \partial^2 \epsilon + \frac{1}{2} m^2 \epsilon_\mu \partial^2 \epsilon + B_\mu \partial^2 \epsilon^\mu$$

$$+ m_\mu \partial^2 \epsilon^\mu + m \partial_\mu \epsilon \partial^2 \epsilon,$$  \hspace{1cm} (A3c)

$$- \eta \partial^2 \eta \rightarrow - \eta \partial^2 \epsilon \partial_\nu \epsilon^\nu - \frac{9}{4} m^2 \epsilon \partial^2 \epsilon + 2 \eta \partial^2 \epsilon \partial_\nu \epsilon^\nu$$

$$- 3 m \eta \partial^2 \epsilon + 3 m \partial_\nu \epsilon \partial^2 \epsilon,$$  \hspace{1cm} (A3d)
\begin{align}
\frac{13}{8} m^2 \eta^2 & \rightarrow \frac{13}{8} m^2 \partial_\mu \epsilon^\mu \partial_\nu \epsilon^\nu + \frac{117}{32} m^4 \epsilon^2 - \frac{13}{4} m^2 \partial_\mu \epsilon^\mu \eta \\
& + \frac{39}{8} m^3 \eta \epsilon - \frac{39}{8} m^3 \partial_\mu \epsilon^\mu \epsilon, \\
\frac{1}{2} \partial_\nu h_{\mu \nu} \partial_\chi \eta h^{\mu \lambda} & \rightarrow - \frac{1}{2} \partial_\mu \epsilon_\nu \partial_\nu \partial_\alpha \epsilon^\lambda - \frac{1}{2} \partial_\nu \epsilon_\mu \partial_\nu \partial_\sigma \epsilon^\mu - \frac{1}{8} m^2 \epsilon \partial^2 \epsilon \\
& - h_{\mu \nu} \partial_\nu \partial_\lambda \epsilon^\lambda - h_{\mu \nu} \partial_\nu \partial^2 \epsilon^\mu + \frac{1}{2} m h_{\mu \nu} \partial_\nu \partial_\sigma \epsilon^\mu \\
& - \partial_\mu \epsilon_\nu \partial_\nu \partial^2 \epsilon^\mu + m \epsilon \partial^2 \partial_\mu \epsilon^\mu, \\
\partial_\nu h_{\mu \nu} \partial_\chi \epsilon & \rightarrow h_{\mu \nu} \partial_\nu \partial_\lambda \epsilon^\lambda - \frac{3}{2} m h_{\mu \nu} \partial_\nu \epsilon^\epsilon - \partial_\nu \epsilon_\mu \partial_\mu \epsilon \eta \\
& + \partial_\mu \epsilon_\nu \partial_\nu \partial^2 \epsilon^\mu - \frac{3}{2} m \partial_\nu \epsilon_\nu \partial_\mu \partial^\nu \epsilon^\nu - \partial_\nu \epsilon_\mu \partial_\mu \partial^2 \epsilon^\nu \\
& + \partial_\nu \epsilon_\mu \partial_\nu \partial^2 \epsilon^\mu + \frac{1}{2} m \epsilon \partial^2 \eta \\
& - \frac{1}{2} m \epsilon \partial^2 \partial_\mu \epsilon^\mu + \frac{3}{4} m^2 \epsilon \partial^2 \epsilon, \\
-m \partial_\nu h_{\mu \nu} B^\mu & \rightarrow m h_{\mu \nu} \partial_\nu \partial^\nu \epsilon + m^2 h_{\mu \nu} \partial_\nu \epsilon^\nu \epsilon + m \partial_\nu \epsilon_\nu \partial_\mu B^\mu \\
& + m \partial_\nu \epsilon_\nu \partial^\nu \epsilon^\mu + m^2 \partial_\nu \epsilon_\nu \partial^\nu \epsilon^\nu + m \partial_\nu \epsilon_\nu \partial^\nu \epsilon_\mu - \frac{1}{2} m^2 \epsilon \partial^2 \epsilon \\
& - \frac{1}{2} m^2 \epsilon \partial^2 \epsilon - \frac{1}{2} m^2 \partial_\rho \epsilon^\rho, \\
m \eta \partial_\mu B^\mu & \rightarrow m \eta \partial^\nu \epsilon + m^2 \eta \partial_\mu \epsilon^\mu - m \partial_\nu \epsilon_\nu \partial_\mu B^\mu \\
& - m \partial_\nu \epsilon_\nu \partial^\nu \epsilon^\rho + m^2 \partial_\nu \epsilon_\nu \partial^\nu \epsilon_\rho + \frac{3}{2} m^2 \epsilon \partial_\mu B^\mu \\
& + \frac{3}{2} m^2 \epsilon \partial^2 \epsilon + \frac{3}{2} m^2 \partial_\mu \epsilon^\mu, \\
\frac{m^2}{32} h_\nu h^\nu & \rightarrow \frac{1}{8} m^2 \partial_\mu \epsilon_\nu \partial_\sigma \epsilon^\nu + \frac{1}{128} m^4 d^2 e^2 + \frac{1}{8} m^2 h \partial_\mu \epsilon^\nu \\
& - \frac{d}{32} m^3 h \epsilon - \frac{1}{16} m^3 \partial_\mu \epsilon^\mu \epsilon, \\
\frac{1}{2} \partial_\mu B^\mu \partial_\nu B^\nu & \rightarrow - \frac{1}{2} \partial_\nu \epsilon_\rho \partial_\rho \epsilon^\nu - \frac{1}{2} m^2 \epsilon^\nu \partial_\nu \partial_\sigma \epsilon^\sigma - B^\nu \partial_\mu \partial^2 \epsilon \\
& - m B^\nu \partial_\mu \partial_\nu \epsilon^\nu - m \epsilon \partial_\nu \partial^2 \epsilon, \\
\frac{3}{8} m^2 h_\nu \eta & \rightarrow \frac{3}{8} m^2 h \partial_\nu \epsilon^\nu + \frac{9}{16} m^3 h \epsilon + \frac{3}{4} m^2 \partial_\rho \epsilon^\rho \eta \\
& - \frac{3}{4} m^2 \partial_\rho \epsilon^\rho \partial_\nu \epsilon^\nu + \frac{9}{8} m^3 \partial_\mu \epsilon^\mu \epsilon - \frac{3 d}{16} m^3 \eta \\
& + \frac{3 d}{16} m^3 \partial_\rho \epsilon^\rho \epsilon^\nu - \frac{9}{32} m^4 d^2 \epsilon^\nu, \\
\frac{1}{4} m h_\nu \partial_\nu B^\nu & \rightarrow \frac{1}{4} m h \partial^2 \epsilon + \frac{1}{4} m^2 h \partial_\nu \epsilon^\nu + \frac{1}{2} m \partial_\nu \epsilon^\nu \partial_\rho B^\rho \\
& + \frac{1}{2} m \partial_\nu \epsilon^\nu \partial^2 \epsilon + \frac{1}{2} m \partial_\nu \epsilon^\nu \partial_\sigma \epsilon^\nu - \frac{d}{8} m^2 \epsilon \partial_\mu B^\mu \\
& - \frac{d}{8} m^2 \epsilon \partial^2 \epsilon - \frac{d}{8} m^3 \epsilon \partial_\mu \epsilon^\mu, \\
\frac{3}{2} m \eta \partial_\mu B^\mu & \rightarrow \frac{3}{2} m \eta \partial^2 \epsilon + \frac{3}{2} m \eta \partial_\nu \epsilon_\nu - \frac{3}{2} m \partial_\mu B^\mu \partial_\nu B^\nu \\
& - \frac{3}{2} m \partial_\mu \epsilon^\nu \partial^2 \epsilon - \frac{3}{2} m \partial_\mu \epsilon^\nu \partial_\sigma \epsilon^\nu + \frac{9}{4} m^2 \epsilon \partial_\mu B^\mu \\
& + \frac{9}{4} m^2 \epsilon \partial^2 \epsilon + \frac{9}{4} m^3 \epsilon \partial_\mu \epsilon^\mu, \\
\end{align}
It may be useful to rearrange the variations of $\mathcal{L}_{SZ}$ under the local gauge transformation in terms of powers of $m$ to explicitly show that they cancel out to the boundary terms.

- **Terms containing $h_{\mu\nu}$:**
  - Order of $m^0$
    \[
    h_{\mu\nu} \partial^2 \partial^\mu \epsilon^\nu - h_{\mu\nu} \partial^\nu \partial^2 \epsilon^\mu + h_{\mu\nu} \partial^\mu \partial^\nu \partial_\lambda \epsilon^\lambda - h_{\mu\nu} \partial^\mu \partial^\nu \partial_\lambda \epsilon^\lambda = 0. \]
  - (A4a)

  - Order of $m^1$
    \[
    -\frac{1}{4} m h \partial^2 \epsilon + \frac{1}{4} m \partial^2 \epsilon + \frac{1}{2} m h_{\mu\nu} \partial^\mu \partial^\nu \epsilon - \frac{3}{2} m h_{\mu\nu} \partial^\mu \partial^\nu \epsilon + m h_{\mu\nu} \partial^\mu \partial^\nu \epsilon = 0. \]
  - (A4b)

  - Order of $m^2$
    \[
    -m^2 h_{\mu\nu} \partial^\mu \epsilon^\nu + m^2 h_{\mu\nu} \partial^\nu \epsilon^\mu + \frac{1}{8} m^2 h \partial_\mu \epsilon^\nu - \frac{3}{8} m^2 h \partial_\mu \epsilon^\nu + \frac{1}{4} m^2 h \partial_\mu \epsilon^\nu = 0. \]
  - (A4c)

  - Order of $m^3$
    \[
    \frac{1}{4} m^3 h - \frac{d}{16} m^3 h + \frac{9}{16} m^3 h = \left(\frac{26 - d}{32}\right) m^3 h = 0. \]
  - (A4d)

- **Terms containing $B_\mu$:**
  - Order of $m^0$
    \[
    B_\mu \partial^2 \partial^\mu \epsilon - B^\mu \partial_\mu \partial^2 \epsilon = 0. \]
  - (A5a)

  - Order of $m^1$
    \[
    m B_\mu \partial^2 \epsilon^\mu + m \partial_\mu \epsilon \partial^\nu B^\mu + m \partial_\nu \epsilon \partial^\mu B^\mu - m \partial_\nu \epsilon \partial_\mu B^\mu - m B^\mu \partial_\mu \partial_\nu \epsilon^\nu \\
    \frac{1}{2} m \partial_\mu \epsilon \partial_\nu B^\nu - \frac{3}{2} m \partial_\mu B^\mu \partial_\nu \epsilon^\nu = 0. \]
  - (A5b)

  - Order of $m^2$
    \[
    -\frac{1}{2} m^2 \epsilon \partial_\mu B^\mu + \frac{3}{2} m^2 \epsilon \partial_\mu B^\mu - \frac{d}{8} m^2 \epsilon \partial_\mu B^\mu + \frac{9}{4} m^2 \epsilon \partial_\mu B^\mu = \left(\frac{26 - d}{8}\right) m^2 \epsilon \partial_\mu B^\mu = 0. \]
  - (A5c)

- **Terms containing $\eta$:**
  - Order of $m^0$
    \[
    2 \eta \partial^2 \partial_\mu \epsilon^\mu - \partial_\mu \epsilon \partial^\nu \eta - \partial_\nu \epsilon \partial^\mu \partial^\nu \eta = 0. \]
  - (A6a)

  - Order of $m^1$
    \[
    -3 m \eta \partial^2 \epsilon + \frac{1}{2} m \epsilon \partial^2 \eta + m \eta \partial^2 \epsilon + \frac{3}{2} m \eta \partial^2 \epsilon = 0. \]
  - (A6b)

  - Order of $m^2$
    \[
    -\frac{13}{4} m^2 \epsilon \partial_\mu \epsilon \eta + m^2 \eta \partial_\mu \epsilon^\mu + \frac{3}{4} m^2 \partial_\mu \epsilon \eta + \frac{3}{2} m^2 \eta \partial_\mu \epsilon^\mu = 0. \]
  - (A6c)

  - Order of $m^3$
    \[
    \frac{39}{8} m^3 \eta \epsilon - \frac{3 d}{16} m^3 \epsilon \eta = \frac{3 (26 - d)}{16} m^3 \eta \epsilon = 0. \]
  - (A6d)

- **Terms not containing fields:**
\(- \text{Order of } m^0\)

\[
\frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu + \frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu - \frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \partial^\mu \epsilon - \partial_\mu \epsilon_\nu \partial^2 \partial^\nu \epsilon + \frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \partial^\nu \epsilon + \frac{1}{2} \partial_\mu \epsilon_\nu \partial^2 \partial^\nu \epsilon = 0.
\]

\(- \text{Order of } m^1\)

\[
- \frac{1}{2} m \partial_\mu \epsilon_\nu \partial^2 \epsilon + m \partial_\mu \epsilon_\nu \partial^2 \epsilon + 3 \partial_\mu \epsilon_\nu \partial^2 \epsilon + m \epsilon \partial^2 \partial_\mu \epsilon - \frac{3}{2} m \partial_\mu \epsilon_\nu \partial^2 \epsilon = 0.
\]

\(- \text{Order of } m^2\)

\[
\frac{d}{16} m^2 \epsilon \partial^2 \epsilon - \frac{1}{2} m^2 \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu + \frac{1}{2} m^2 \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu - \frac{1}{2} m^2 \partial_\mu \epsilon_\nu \partial^2 \epsilon^\nu - \frac{1}{2} m^2 \epsilon \partial^2 \epsilon = 0.
\]

\(- \text{Order of } m^3\)

\[
\left( \frac{41}{4} - \frac{41}{4} \right) \partial_\mu \epsilon_\nu \epsilon = 0.
\]

\(- \text{Order of } m^4\)

\[
\frac{(d - 26)(d - 18)}{128} m^4 \epsilon^2 = 0.
\]

We find that the cancellations are highly non-trivial and that some terms cancel out only if the space-time dimension \(d\) is equal to the critical value \(d_{\text{critical}} = 26\). We recall here that this local gauge symmetry originates in the nilpotency of the BRST operator, which is applicable only in the critical dimension condition. It is interesting to note that even for the classical considerations, the local gauge invariance is valid only when \(d = 26\).