A Method to Tackle First Order Ordinary Differential Equations with Liouvillian Functions in the Solution - II

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Abstract

We present a semi-decision procedure to tackle first order differential equations, with Liouvillian functions in the solution (LFOODEs). As in the case of the Prelle-Singer procedure, this method is based on the knowledge of the integrating factor structure.
1 Introduction

The problem of solving ordinary differential equations (ODEs) has led, over the years, to a wide range of different methods for their solution. Along with the many techniques for calculating tricky integrals, these often occupy a large part of the mathematics syllabuses of university courses in applied mathematics round the world.

For a first order differential equation (FOODE), finding the solution can be equated to determining an integrating factor. A remarkable method for finding such factors was developed, in 1983, by Prelle and Singer [1]. Their method is based on the knowledge of the general structure of the integrating factor for FOODEs of the type \( \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \), with \( M \) and \( N \) polynomials in their arguments, which present a solution that can be written in terms of elementary functions (EFOODEs)\(^1\). Their approach is very attractive due to the fact that it is non-classificatory and of a semi-decision nature. Therefore, it has motivated many extensions of the original idea [2, 3, 4, 5].

Using the results presented in [6], we presented (see [7]) a method which is an extension to the Prelle-Singer (PS) procedure allowing for the solution of some LFOODEs\(^2\) (\( \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \), with \( M \) and \( N \) polynomials in their arguments). The method is also based on the general structure of the integrating factor, that was concluded to be of the form: 

\[
R = e^{r_0(x,y)} \prod_{i=1}^{n} v_i(x,y)^{c_i},
\]

where \( r_0 \) is a rational function of \((x,y)\), the \( v_i \)'s are irreducible polynomials in \((x,y)\) and the \( c_i \)'s are constants.

The method presented on [7] used a conjecture about the nature of the \( v_i \)'s (proved on [8]) and was restricted to a class of LFOODEs, namely the ones to which \( r_0(x,y) \) is either \( f(x) \) or \( g(y) \) or \( f(x) + g(y) \), where \( f, g \) are rational functions. Here, we further detail the structure of the integrating factor thus allowing us to remove the above mentioned restriction about \( r_0 \), maintaining the semi-decision nature of the approach.

The paper is organized as follows: in section 2, we analyze the structure of the integrating factor for LFOODEs; in the following section, we show how to apply that knowledge to construct a semi-decision method to tackle LFOODEs and show some examples of the application of the method. Finally, we present our conclusions.

2 The Structure of the Integrating Factor

Based on earlier results [], we will, in this section, improve our knowledge of the structure of the integrating factor thus setting the stage for the presentation of a semi-decision method to deal with LFOODEs.

Let us then summarize these initial results:

\(^1\)For a formal definition of elementary function, see [10].

\(^2\)Liouvillian functions are an extension of elementary functions, see [10].
2.1 Previous Results

A seminal result on dealing with LFOODEs was obtained by Prelle and Singer in 1983 [1]. They have demonstrated that, for a LFOODE

\[ \frac{dy}{dx} = \frac{M(x, y)}{N(x, y)} \]  

where \( M \) and \( N \) are polynomials in \((x, y)\) with coefficients in the complex field \( \mathbb{C} \), if its solution can be written in terms of elementary functions, then there exists an integrating factor of the form \( R = \prod_i f_i^{n_i} \) where \( f_i \) are irreducible polynomials and \( n_i \) are non-zero rational numbers. Applying this to FOODEs of the type (1), we have:

\[ \frac{D[R]}{R} = \sum_i n_i D[f_i] = -\left( \partial_x N + \partial_y M \right), \]  

where \( D \equiv N\partial_x + M\partial_y \).

From (3), plus the fact that \( M \) and \( N \) are polynomials, they concluded that \( D[R]/R \) is a polynomial and that the \( v_i \)'s are eigenpolynomials of the \( D \) operator.

In [6, 7, 8], next steps were taken: it was shown that, for a LFOODE of type (1), the integrating factor is of the form:

\[ R = e^{r_0(x,y)} \prod_{i=1}^n v_i(x,y)^{c_i}. \]  

where \( r_0 \) is a rational function of \((x, y)\), the \( v_i \)'s are irreducible polynomials in \((x, y)\) and the \( c_i \)'s are constants.

From this, we could conclude (see [8]) that \( D[r_0] \) is a polynomial and that the \( v_i \)'s are eigenpolynomials of the \( D \) operator.

2.2 A Theorem Concerning the Structure of \( r_0 \)

In this section, we are going to demonstrate a result about the structure of \( r_0 \) that will allow us to generalize the method presented in [7]. In order to do so, we are going to use some earlier results [4].

For a LFOODE of the form \( \frac{dy}{dx} = \frac{M(x, y)}{N(x, y)} \), where \( M \) and \( N \) are polynomials in \((x, y)\), the integrating factor \( R \) is given by \( R = e^{r_0(x,y)} \prod_{i=1}^n v_i(x,y)^{c_i} \), where \( r_0 \) is rational function of \((x, y)\), \( v_i \) are irreducible eigenpolynomials of the \( D \) operator (where \( D \equiv N\partial_x + M\partial_y \)), \( c_i \) are constants and \( D[r_0] \) is a polynomial in \((x, y)\).

**Theorem 1:** Let the exponent \( r_0 \) be expressed as \( P(x,y)/Q(x,y) \), where \( P \) and \( Q \) are polynomials in \((x, y)\), with no common factor. Then we have that \( Q[D] \) (i.e., \( D[Q]/Q \) is a polynomial in \((x, y)\)).

**Proof:** Since \( D[r_0] \) is polynomial (see [8]), we can write it as:

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3In other words, \( f_i \) is a factor of \( D[f_i] \).

4These results can be found in [6, 7].
\[ D[r_0] = D \left[ \frac{P}{Q} \right] = \frac{Q D[P] - P D[Q]}{Q^2} = \Pi \tag{4} \]

where \( \Pi \) is polynomial in \((x, y)\). Multiplying (4) by \( Q \), one obtains:

\[ D[P] - \frac{P D[Q]}{Q} = \Pi Q. \tag{5} \]

Since \( D \) is a linear differential operator, with polynomial coefficients, and \( P \) is polynomial, \( D[P] \) is also polynomial. Therefore, since \( \Pi Q \) is also polynomial, we may conclude that \( \frac{P D[Q]}{Q} \) is polynomial either. Since, by hypothesis, \( P \) and \( Q \) have no common factor, we can infer that \( Q | D[Q] \) (i.e., \( D[Q]/Q \) is a polynomial), as we wanted to demonstrate.

This result, in turn, leads to the following corollary:

**Corollary 1:** We can write \( Q \) as \( \prod_{i=1}^{n} q_i(x,y)^{m_i} \), where the \( q_i \)'s are irreducible independent eigenpolynomials of the \( D \) operator and the \( m_i \)'s are positive integers.

**Proof:** \( Q \) is a polynomial so it can be written as \( \prod_{i=1}^{n} q_i(x,y)^{m_i} \), where the \( q_i \)'s are independent irreducible polynomials and the \( m_i \)'s are positive integers. So, we can write:

\[ \frac{D[Q]_i}{Q} = \sum_m D[q_i]/q_i \tag{6} \]

implying that \( q_i | D[q_i] \) (i.e., \( D[q_i]/q_i \) is a polynomial), as we wanted to show.

# 3 A Semi-Decision Method to Tackle LFOODEs

In this section, we will show how to construct, in a semi-decision way (as in the case of the PS-method for EFOODEs), a method to deal with LFOODEs of the form \( \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \), where \( M \) and \( N \) are polynomials in \((x,y)\).

## 3.1 Introduction

We now know that the integrating factor for these LFOODEs is of the form,

\[ R = e^{P(x,y)/Q(x,y)} S(x,y), \tag{7} \]

where we have that:

- \( P \) and \( Q \) are polynomials in \((x, y)\).
- \( Q \) can be written as \( \prod_i (v_{q_i})^{m_i} \), where the \( v_{q_i} \)'s are independent irreducible eigenpolynomials (of the \( D \) operator) and the \( m_i \)'s are positive integers.
- \( S \) can be written as \( \prod_j (v_{s_j})^{c_j} \), where the \( v_{s_j} \)'s are independent irreducible eigenpolynomials (of the \( D \) operator) and the \( c_j \)'s are positive integers.
In what follows, let us convert this knowledge into a form that will allow us to build a solving method.

Using (7) into $D[R]/R = -(\partial_x N + \partial_y M)$, we get:

$$D\left[\begin{bmatrix} P \\ Q \end{bmatrix}\right] + \frac{D[S]}{S} = -(\partial_y M + \partial_x N),$$

leading to

$$Q \frac{D[P] - PD[Q]}{Q^2} + \sum_j c_j \frac{D[v_{s,j}]}{v_{s,j}} = -(\partial_y M + \partial_x N).$$

(8)

Remembering that the $v_s$’s are eigenpolynomials of the $D$ operator, we can write $D[v_{s,j}] = \lambda_{s,j} v_{s,j}$, where the $\lambda_{s}$’s are the polynomial ‘eigenvalues’ in $(x, y)$ associated with the $v_s$’s. Thus, we can write (8) as:

$$\frac{1}{Q} \left( D[P] - P \frac{D[Q]}{Q} \right) + \sum_j c_j \lambda_{s,j} = -(\partial_y M + \partial_x N).$$

(9)

If we then multiply both sides of (9) by $Q$, we obtain:

$$D[P] - P \frac{D[Q]}{Q} + Q \sum_j c_j \lambda_{s,j} = -Q (\partial_y M + \partial_x N).$$

(10)

Since $Q = \prod_i (v_{q,i})^{m_i}$, where the $v_{q}$’s are independent irreducible eigenpolynomials of the $D$ operator, we can write:

$$\frac{D[Q]}{Q} = \sum_i m_i \frac{D[v_{q,i}]}{v_{q,i}} = \sum_i m_i \lambda_{q,i},$$

(12)

where the $\lambda_{q}$’s are the polynomial ‘eigenvalues’ associated with the $v_{q}$’s. Using this into (10) and re-arranging we get:

$$D[P] - P \sum_i m_i \lambda_{q,i} = -\prod_i (v_{q,i})^{m_i} \left( \sum_j c_j \lambda_{s,j} + \partial_y M + \partial_x N \right).$$

(13)

This equation will prove to be very important to our method. So, let us now do some analysing of its structure: We don’t know the degree of the eigenpolynomials $v_s$ that build up the function $S$. Neither do we know the degree of the eigenpolynomials $v_{q}$ that appear on $Q$. However, once these degrees are set and choosing the values for the exponents $m_i$ (thus completing the determination of the degree of $Q$), we can determine the degree of the polynomial $P$. Let us see how it goes by analysing both sides of (13).

First, note that $D[P] = M \partial_y P + N \partial_x P$. Since the degree of $\partial_x P$ is at most equal to the degree of $P$ minus 1, and the same is valid for the degree of $\partial_y P$, we may conclude that (in what follows, we will denote the degree of any term by $d_{\text{term}}$):

$$d_{D[P]} \leq \max(d_M, d_N) + d_P - 1,$$

(14)

Looking now to the second term of the left hand size of (13) we may say that

$$d_P \sum_i m_i \lambda_{q,i} \leq \max(d_{\lambda_q}) + d_P.$$

(15)
But we know that $D[v_qi] = \lambda_qi v_qi$, leading to:

$$\max(d_{\lambda_qi}) + d_{v_qi} \leq \max(d_M, d_N) + d_{v_qi} - 1 \quad (16)$$

and, consequently, to

$$\max(d_{\lambda_qi}) \leq \max(d_M, d_N) - 1. \quad (17)$$

Using (17) into (15) we can write

$$d_P \sum_{m_i} \lambda_qi \leq \max(d_M, d_N) + d_P - 1. \quad (18)$$

Through (14) and (18) we see that the maximum degree of the left rand side of (13) is $\max(d_M, d_N) + d_P - 1$. Now, looking at the right hand size of (13), we have that its degree is at most equal to $d_Q + \max(d_{\lambda_qi}, d_{\partial_y M}, d_{\partial_x N})$. Taking (17) into account and the fact that $d_{\partial_y M} \leq d_M - 1$ and $d_{\partial_x N} \leq d_N - 1$, the maximum degree of the right hand size of (13) is $d_Q + \max(d_M, d_N) - 1$. Since (13) has to be satisfied, we see that the degree of $Q$ bounds that of $P$.

### 3.2 The Steps of the Method

Based on what has been explained on section (3.1), let us now establish the actual procedure to be followed. In equation (13) (the cornerstone of our method), we see terms involving the eigenpolynomials (and/or the associated ‘polynomial eigenvalues’) of the $D$ operator (present both on the definition for the function $S$ and on for the exponent ($Q$)). So, the first job at hand is to determine those. In order to do that we have to choose a degree. Let us, for simplicity sake, start with the most simple possibility. So, the first step is the

**Step 1 - determination of the eigenpolynomials and of the ‘polynomial eigenvalues’ (of degree 1) of the $D$ operator.**

When these polynomials are determined and put on equation (13), the next thing to do is to

**Step 2 - determine the degree for the $Q$ polynomial.**

Please note that the degree for this polynomial is not fully determined by the determination of the degrees for the eigenpolynomials of the $D$ operator, there are still the powers for these polynomials ($m_i$).

Thus, once the degree for $Q$ is set, we determine all possible values for $m_i$ to accomodate this choice with the choice made on the first step just described. So, the next step is the

**Step 3 - determination of all possible values for $m_i$.**

After that step is taken, one can see that, on equation (13), we still have $P$ to determine. But, from the results present on section (3.1), from the previous choices we have made, the maximum degree for this polynomial can be determined: we have that the maximum degree for the left hand side of (13) is $d_P + \max(d_M, d_N) - 1$ and the maximum degree for the right hand side of (13) is $d_Q + \max(d_M, d_N) - 1$. Therefore, once the degree of $Q$ is set, the maximum degree for $P$ is $d_Q + \max(d_M, d_N)$. So we
Step 4 - construct a generic polynomial of degree 1 ($P = a_0 + a_1 x + a_2 y$).

Equation (13) then becomes of the general form:

$$P = 0$$  \hspace{1cm} (19)

where $P$ is a polynomial. For this to be satisfied, we have to

Step 5 - equate the coefficients of the different powers of $(x, y)$ to zero. Thus generating a set of linear algebraic equations on the undetermined parameters ($a$’s and $c$’s).

Step 6 - Solve the set of linear algebraic equations thus determining the $a$’s and $c$’s.

If the solution can not be found, we

Step 7 - increase the degree of $P$ (up to its maximum degree) and repeat the procedure from Step 5.

If the solution can not be found, we

Step 8 - increase the degree of $Q$ and repeat the procedure from Step 3 onwards.

Of course, this procedure can go on indefinitely (remember we do not know the bound for the degree of $Q$). However, the degree of the eigenpolynomials (from which we construct $Q$ and $S$) can also be increased (in Step 1 above it was set to be 1). So, instead of trying all the possibilities for $Q$ (they are infinite!), one can interrupt this arduous loop and experiment starting all over (from Step 1) but with an increased value for the degree of the eigenpolynomials.

By doing that, we are covering all the possibilities and we may hope that we will find a solution within our lifetime. On a brighter tone, most of the examples we have come across are solved with low degree for the eigenpolynomials of the $D$ operator and for $Q$.

3.3 Examples and Results

In this section, we are going to present examples of application of our method and discuss its effectiveness.

First, in order to illustrate the steps of the method just presented, we are going to start with a simple LFOODE. This example was artificially manufactured by us according to two criteria: It is (in a way) simple and it is not solved, as far as we know, by any other method.

Example 1:

Consider the following LFOODE:

$$\frac{dy}{dx} = \frac{(x + 1)y}{x - xy - y^2 + x^2}$$  \hspace{1cm} (20)

For this equation, up to degree 1 (Step 1), we have that the eigenpolynomials (with the associated eigenvalues) are:
\[
\begin{align*}
\bullet \ v_1 &= y, \quad \lambda_1 = x + 1, \\
\bullet \ v_2 &= x + y, \quad \lambda_2 = 1 + x - y.
\end{align*}
\]

The next step (Step 2) is to choose the degree for the polynomial \(Q\). Starting with \(d_Q = 1\), since we are in the case where \(d_v = 1\) and \(d_s = 1\), the only possible values for \(m_i\) (Step 3) are \(\{m_1 = 1, m_2 = 0\}\) and \(\{m_1 = 0, m_2 = 1\}\). For this particular example, the maximum degree for \(P\) is 2. So, starting with \(d_P = 1\) we have \(P = a_1 + a_2 x + a_3 y\) (Step 4) and equation \((13)\) leads to:

\[
y^{m_1} (x + y)^{m_2} (n_1 (x + 1) + n_2 (1 + x - y) + 3 x + 2 - y) + \left(x - x y - y^2 + x^2\right) a_2 + (x + 1) y a_3 - (a_1 + a_2 x + a_3 y) (m_1 (x + 1) + m_2 (1 + x - y)) = 0. \tag{21}
\]

As we shall see, with the values \(m_1 = 1, m_2 = 0\) is possible to find a solution. Substituting those into \((21)\) we get:

\[
y (n_1 (x + 1) + n_2 (1 + x - y) + 3 x + 2 - y) + \left(x - x y - y^2 + x^2\right) a_2 + (x + 1) y a_3 - (a_1 + a_2 x + a_3 y) (x + 1) = 0. \tag{22}
\]

In order to solve the above equation, the coefficients for different powers of \((x, y)\) have to be zero. Thus leading to the following set of equations (Step5):

\[
\begin{align*}
n_1 + n_2 + 2 &= 0 \\
- n_2 - a_2 - 1 &= 0 \\
- a_1 &= 0 \\
n_1 + n_2 + 3 - a_2 &= 0
\end{align*} \tag{23}
\]

Leading to the solution for the coefficients (Step6):

\[
a_1 = 0, a_3 = a_3, n_1 = 0, n_2 = -2, a_2 = 1 \tag{24}
\]

So, the integrating factor for this LFODE becomes (choosing \(a_3 = 0\)):

\[
R = \frac{e^{x/y}}{(x + y)^2} \tag{25}
\]

Then the solution is:

\[
C = \frac{y (-1 + y) e^{x/y}}{(x + y)} - e^{-1} Ei\left(1, -\frac{x + y}{y}\right) \tag{26}
\]

Our method is designed to deal with LFODEs, so, in order to analyze its effectiveness, we are going to use as our testing ground the LFODEs found on the book by Kamke [9], a traditional testing arena for ODEs. The LFODEs are the equations I.18, I.20, I.27, I.28, I.129, I.133, I.146, I.169 and I.235, as they are numerated in the book by Kamke. These equations and their corresponding Integrating Factors can be found on table ??.. The method deals with all these examples successfully. In order not to make it too cumbrous for the reader, we are going to present in detail the calculations for just one of those cases, actually, the
most involved one. This choice for the second example was made aiming to show that, even for complex cases, our method is contained and manageable.

**Example 2:**

Consider the following LFOODE (1.169 from the book by Kamke):

$$
(ax + b)^2 \frac{dy}{dx} + (ax + b)(y)^3 + c(y)^2
$$

(27)

For this equation, up to degree 1 (Step 1), we have that the eigenpolynomials (with the associated eigenvalues) are:

- \( v_1 = y, \quad \lambda_1 = -y c - b y^2 - a x y^2, \)
- \( v_2 = (ax + b)/a, \quad \lambda_2 = a b + a^2 x. \)

The next step (Step 2) is to choose the degree for the polynomial \( Q \). For this particular example, we will see that \( d_Q = 4 \) is needed. Since we are in the case where \( d_y = 1 \) and \( d_b = 1 \), the possible values for \( m_i \) (Step 3) are \( \{m_1 = 4, m_2 = 0\}, \{m_1 = 3, m_2 = 1\}, \{m_1 = 2, m_2 = 2\}, \{m_1 = 1, m_2 = 3\} \) and \( \{m_1 = 0, m_2 = 4\} \).

For this case the maximum degree for \( P \) is 8 and the lowest degree to which a solution can be found is 4. So, letting \( P = a_1 y^4 + a_2 x y^2 + a_3 y^3 + a_4 x y^2 + a_5 x + a_6 x^2 + a_7 x^3 + a_8 x^2 + a_9 y^3 + a_{10} x^3 y + a_{11} x y^3 + a_{12} x + a_{13} x^2 y^2 + a_{14} y + a_{15} \) (Step 4), equation (24) becomes:

\[
y^{m_1} \left( \frac{ax + b}{a} \right)^{m_2} \left( n_1 \left( -y c - by^2 - a x y^2 \right) + n_2 \left( ab + a^2 x \right) - 2 y (axy + by + c) - y^2 (ax + b) + 2 a^2 x + 2 ab \right) + \left( a^2 x^2 + 2 axb + b^2 \right) \left( 2 a_2 xy + a_4 y^2 + a_5 y + 4 a_6 x^3 + 3 a_7 x^2 + 2 a_8 x + 3 a_10 x^2 y + a_11 y^3 + a_12 + 2 a_{13} x y^2 \right) \nonumber
\]

\[
- y^2 (axy + by + c) \left( 4 a_1 y^3 + 2 a_2 x^2 + 2 a_3 y + 2 a_4 xy + 5 a_5 x + 3 a_9 y^2 + a_{10} x^3 y + 2 a_{13} x^2 y + a_{14} \right) - \left( a_1 y^4 + a_2 x y^2 + a_3 y^2 + a_4 x y^2 + a_5 xy + a_6 x^4 + a_7 x^3 + a_8 x^2 + a_9 y^3 + a_{10} x^3 y + a_{11} x y^3 + a_{12} x + a_{13} x^2 y^2 + a_{14} y + a_{15} \right) \left( m_1 \left( -y c - by^2 - a x y^2 \right) + m_2 \left( ab + a^2 x \right) \right) \nonumber
\]

(28)

As we shall see, with the values \( m_1 = 2, m_2 = 2 \) is possible to find a solution. Substituting those into (24) we get:

\[
\begin{align*}
4 b^2 a_6 + 6 a b a_7 + 2 a^2 a_8 &= 0, 3 b^2 a_7 + a^2 a_{12} + 4 a b a_8 = 0, \\
2 a b a_5 + a_{12} c + n_2 a^2 + 2 b^2 a_2 + 2 a^2 &= 0, -3 c a_1 - 2 b a_9 = 0, \\
b^2 a_{12} &= 0, -n_1 c + a_{15} b + b^2 a_4 - 2 c = 0, 2 a^2 a_2 + a_7 c + 6 a b a_{10} = 0, \\
8 a b a_6 + 3 a^2 a_7 &= 0, 2 b^2 a_8 + 2 a b a_9 = 0, 4 a^2 a_6 = 0, \\
-3 b - n_1 b + b^2 a_{11} - ca_3 &= 0, 2 a^2 a_5 + 3 b^2 a_{10} + a_8 c + 4 a b a_2 = 0, \\
b a_{12} + a_{15} a + 2 b^2 a_{13} + 2 a b a_4 &= 0, -n_1 a - 3 a - ca_4 + 2 a b a_{11} = 0, \\
b a_8 + a^2 a_4 + a a_{12} + 4 a b a_{13} &= 0, n_2 a b + 2 a b + b^2 a_5 + a_{15} c = 0, \\
a^2 a_{11} - ca_{13} &= 0, -b a_3 - 3 c a_9 = 0, a b a = 0, 3 a^2 a_{10} + a_6 c = 0, \\
-2 b a_{11} - 2 a b a_9 &= 0, -3 a a_1 = 0, -3 b a_1 = 0, -a a_{13} = 0, -2 a a_{11} = 0, \\
-b a_{13} - a a_4 &= 0, -2 c a_{11} - a a_3 - b a_4 = 0, \\
a a_8 + b a_7 + 2 a^2 a_{13} &= 0, a a_7 + b a_6 = 0
\end{align*}
\]
Leading to the solution for the coefficients:

\[
\begin{align*}
\{a6 = 0, a11 = 0, a10 = 0, a1 = 0, n2 = -1, a2 = 0, a7 = 0, a9 = 0, \\
a3 = -1/2 e^2 - 2 ab^2 a13, n1 = -3, a14 = -cb/a^2, \\
a15 = -1/2 b^2, a5 = -c/a, a12 = -b, a4 = 2, \\
a3/2, a13 = a13, a13 = -1/2 a\}\tag{30}
\end{align*}
\]

So, the integrating factor for this LFOODE becomes (choosing \(a_3 = 0\)):

\[
R = a e^{-(yc + ab)^2/(2 ay^2(ax+b)^2)} y^5(ax+b)
\]

Then the solution is:

\[
C = \int - (axy + by + c) e^{1/2 (ax+b)^2/(ax+b)^2} dx
\]

\[
(32)
\]

4 Conclusion

In this paper we have present a semi-decision procedure to tackle first order differential equations, with Liouvillian functions in the solution (LFOODEs). As in the case of the Prelle-Singer procedure, this method is based on the knowledge of the integrating factor structure that we now have shown to be:

\[
R = e^{r_0(x,y)} \prod_{i=1}^{n} v_i(x,y) c_i.
\]

where \(r_0 = P/Q, P \) and \(Q \) are polynomials in \((x,y), v_i \) are irreducible eigenpolynomials of the \( D \) operator (where \( D \equiv \hat{N}_x + M\hat{y} \)), \( c_i \) are constants, \( D[P/Q] \) is a polynomial in \((x,y) \) and \( Q[D[Q]] \) (i.e., \( D[Q]/Q \) is a polynomial in \((x,y)\)).

References

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