Decision-Making in Quantum State Discrimination

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Quantum state discrimination involves finding the best measurement to distinguish between quantum states in a given set. Typically, one wants the POVM which produces the minimal amount of error, or the maximal amount of confidence. We show that this task can easily be cast as an example of Bayesian experimental design, which makes explicit the decision process involved when discriminating between states. This highlights the common implicit assumption that the agent simply decides on the state indicated by the measurement outcome. This naive decision leads to anomalous results, which are resolved when using the optimal decision strategy. Further we generalise state discrimination to POVMs with an arbitrary number of elements, and generalise some well-known results.

I. INTRODUCTION

Quantum state discrimination [1–3] is a foundational task in quantum information theory in which an agent tries to distinguish between a collection of quantum states. If the states are not pairwise orthogonal, then the probability of successfully distinguishing between any pair of states is less than one even with perfect measurements.

Quantum state discrimination is typically thought to have two different primary concerns. In the first, known as minimal error state discrimination, the agent attempts to report the correct state while minimising the probability of an error. In the second, known as maximal confidence state discrimination, the agent tries to minimise the probability of a false positive. Both these cases involve the agent trying to best distinguish between quantum states; they differ in their respective notions of what constitutes ‘best’.

With this realisation, both concerns of quantum state discrimination can be cast as examples of Bayesian experimental design. In this framework, the agent chooses the experiment which maximises the average utility. We show that both minimal error and maximal confidence state discrimination use similar utility functions.

The framework of Bayesian experimental design makes explicit the decision an agent has to make when discriminating between states. This reveals the implicit assumption typical in the literature, that the decision is simply the state reported by the measurement device. We show that this assumption leads to some curious results, whereby trivial measurements are better at state discrimination than measurements providing some information. We use Bayesian experimental design to show how lifting this assumption resolves these anomalies, and provide generalisations of previous results, such as the necessary and sufficient conditions [4, 5] for a measurement to maximise the probability of success; and the conditions under which the agent will always choose the same state, no matter the measurement or measurement result [6]. Furthermore, it allows us to consider measurements whose number of outcomes are not limited by the number of quantum states the agent has to distinguish between; that is state discrimination is generalised to all quantum measurements.

This paper is organised as follows. In section II, we describe the task of state discrimination as it is normally presented in the literature and highlight some difficulties. We introduce Bayesian experimental design in section III and specialise it to quantum state discrimination. In section IV we explore the consequences of explicitly accounting for the decision process; and we conclude in section V.

II. FORMALISM OF STATE DISCRIMINATION

In this section, we present the task of state discrimination and describe some implicit assumptions that lead to counter-intuitive results. The task can be framed as a communication protocol between two parties. Suppose one party, Alice, wishes to send a classical message to another, Bob. Alice will choose one of n distinct messages (labelled ℓ = 1,...,n) and encode message ℓ as a quantum state with density operator ρℓ. Alice will then send ρℓ to Bob using a noiseless quantum communication channel. Bob knows that Alice intends to send message ℓ with probability qℓ (∑ℓ qℓ = 1). Bob’s task is to determine Alice’s message once he has received her state.

In order to determine Alice’s message, Bob must measure the received state. His choice of measurement can be represented with a positive operator-valued measure (POVM); that is, a set of m operators

\( \{ A_1, \ldots, A_m \} , \quad \forall \ell : A_\ell \geq 0, \quad \sum_\ell A_\ell = 1 \). (1)

The two main paradigms of quantum state discrimination evaluate Bob’s measurement depending upon whether he
values making as few mistakes as possible, or having the largest possible confidence in each measurement outcome. In the former case, Bob will choose an $n$-outcome measurement device which indicates the state sent by Alice. Bob wants the POVM that minimises the average probability of error, or equivalently, the POVM that maximises the average probability of success

$$\sum_{\ell=1}^{n} q_{\ell} \text{Tr} (A_{\ell} \rho) .$$

This priority is known as minimal error quantum state discrimination. Analytic solutions for minimal error discrimination are known only for a few special cases such as $n = 2$ [4] or when the alphabet exhibits specific symmetries [7, 8]. Necessary and sufficient conditions for Bob’s choice of POVM have been known for some time [4, 5]. Since the ‘best’ POVM here is the one that maximises (2), this function can be seen as measuring the quality of a POVM in performing this task. Clearly a POVM with a larger probability of success is better at discriminating between quantum states.

For certain alphabets Bob can choose a POVM for which he can have certainty in the measurement outcomes, that is $P(\rho_{y}|A_{\ell}) = 0$ for all $y$. This is known as unambiguous state discrimination [9–13]. For non-orthogonal alphabets, unambiguous state discrimination can only be done at the cost of adding an ‘inconclusive’ outcome, which when measured, Bob discards the state sent by Alice. Unambiguous state discrimination is possible for $n$ pure states if and only if they are linearly independent [14]. Solutions to this problem exist for up to three linearly independent pure states [15], and $n$ linearly independent symmetric states [16].

The generalisation to all alphabets is known as maximal confidence measurements [17]. Here Bob chooses a POVM with $m = n + 1$ outcomes. For each of $\ell = 1, \ldots, n$, he chooses $A_{\ell}$ to maximise $P(\rho_{y}|A_{\ell})$ subject to the constraint $\sum_{\ell=1}^{n} A_{\ell} \leq 1$, so

$$A_{\ell} = \arg\max_{0 < A \leq 1} P(\rho_{y}|A_{\ell}) = \arg\max_{0 < A \leq 1} \frac{P(A|\rho_{y}) P(\rho_{y})}{P(A)} .$$

Then define the inconclusive outcome $A_{0} := I - \sum_{\ell=1}^{n} A_{\ell}$. Since $P(A|\rho_{y})$ and $P(A)$ are linear in $A$ (owing to the linearity of the trace function), $P(\rho_{y}|A)$ is unchanged by rescaling of $A$. Hence the elements $A_{\ell}$ can be rescaled to ensure that $A_{0}$ is positive semi-definite. The ability to rescale elements implies that there is a family of POVMs that maximise the confidence (3) for each outcome. Bob typically seeks a unique element of this family using a further optimisation: such as the POVM which minimises the probability of the inconclusive outcome. We can measure the quality of a POVM in performing maximal confidence state discrimination by the aggregate confidence

$$\sum_{\ell=1}^{n} P(\rho_{y}|A_{\ell}) ,$$

where if $A_{\ell} = 0$ then we define $P(\rho_{y}|A_{\ell}) = 0$.

These approaches to quantum state discrimination do not differentiate between the result reported by the measurement device and the decision Bob makes when he receives that information. It is typically assumed that Bob decides that Alice sent the state indicated by the measurement outcome, and if Bob receives an inconclusive outcome he simply discards the state. This naive decision leads to some counter intuitive results.

Consider the case in which Alice wishes to communicate a single bit to Bob. She encodes her bit as one of two orthogonal states $\rho_{1} = |+\rangle\langle+|$ or $\rho_{2} = |\cdot\cdot\rangle\langle\cdot\cdot|$. Clearly, Bob should choose the POVM $\{A_{1}, A_{2}\}$ with $A_{1} = \rho_{1}$ and $A_{2} = \rho_{2}$. But suppose Bob chooses $A_{1} = \rho_{2}$ and $A_{2} = \rho_{1}$; that is he has relabelled the measurement outcomes. Then the probability of successfully identifying Alice’s message is now 0: his outcome is anticorrelated with Alice’s message. Yet there is nothing wrong with his measurement, only his labelling of the measurement. Clearly, Bob should merely decide that Alice sent the opposite message to that indicated by the measurement device.

Consider another example in which Alice again wishes to send a single bit encoded in the same way as before. Bob knows she will send $|+\rangle\langle+|$ with probability $q_{1}$ and $|\cdot\cdot\rangle\langle\cdot\cdot|$ with probability $q_{2}$. But now Bob’s measurement apparatus is noisy: his POVM elements are $A_{1} = \frac{1}{2} (I + \epsilon X)$ and $A_{2} = \frac{1}{2} (I - \epsilon X)$, where $X$ is the usual Pauli-$X$ operator and $\epsilon > 0$ is small. The probability of success is $\frac{1}{2} (1 + \epsilon)$. Note that if $q_{1} > \frac{1}{2} (1 + \epsilon)$ Bob’s probability of success can be improved by ignoring the measurement outcome and always guessing Alice sent $|+\rangle\langle+|$. It is strange to consider no measurement to be more useful than even a noisy measurement.

This anomaly arises from Bob automatically reporting the outcome displayed by the measurement device. The measurement device represented by the POVM $\{A_{1}, A_{2}\}$ is providing some information, but not enough to overturn the bias reflected in the prior probabilities, that state $\rho_{1}$ was more likely to have been sent than state $\rho_{2}$. After both measurement outcomes $\rho_{1}$ is still more likely than $\rho_{2}$ to have been sent. These examples show the need to explicitly consider the decision after the measurement result in quantum state discrimination.

In [6], it is shown that there exist non-trivial alphabets for which the optimal strategy is to perform no measurement and simply guess the state with the largest prior probability (implemented via a POVM with identity on the most probable outcome and zero on all other states). Equivalently the optimal POVM has a single non-zero element. However, since all non-trivial measurements provide some information about the state it is curious to ever claim that a guessing strategy is optimal. In the section IV D we clarify this result by explicitly including Bob’s decision.

Finally, as a result of this naive decision, quantum state discrimination is typically limited to POVMs with $n$ elements in the case of minimal error state discrimination,
or $n+1$ elements in the case of maximal confidence state discrimination. In fact, (2) and (4) are only defined in these cases. But for any POVM with any number of elements, Bob can decide on which of $n$ states he received. We develop new measures of the quality of a POVM for both tasks, which focuses on Bob’s decision rather than the measurement outcome. In this we are guided by the framework of Bayesian experimental design, which we outline in the next section.

III. BAYESIAN EXPERIMENTAL DESIGN

A. General Formalism

Here we show that quantum state discrimination can be described in terms of Bayesian experimental design, where the two different paradigms mentioned in the previous section correspond to small differences in the utility function. This formulation will also make explicit the typical assumption that Bob simply accepts the outcome displayed by his measurement device.

Bayesian experimental design is a framework for deriving the best experiment for a particular task, where the notion of ‘best’ is encoded into a utility function $U$. An intuitive introduction to Bayesian experimental design is through Bayesian decision theory [18], which involves both inference and optimisation. We begin with a problem that depends on unknown parameters $\theta$ from some set $S_\theta$. There will be some background information $\epsilon$ about the problem, and there will some relevant data $y$ about $\theta$ such as experimental or measurement results. Our knowledge of the parameters $\theta$ is then captured by the conditional probability distribution $P(\theta|y,\epsilon)$.

The experimenter will then make a decision $d$ of alternatives contained in the set $S_d$. We can quantify the utility of every decision we could make by the utility function $U(\epsilon, y, d, \theta)$, which depends on all the parameters for generality. The objective is then to pick the optimal decision $d^\star$. The optimisation process should clearly be averaged over the possible experimental outcomes:

$$
\epsilon^\star = \arg \max_{\epsilon \in S_\epsilon} \int_{S_y} dy \int_{S_\theta} d\theta P(y|\epsilon) \max_{d \in S_d} \int_{S_\theta} d\theta U(\epsilon, y, d, \theta) P(\theta|y,\epsilon).
$$

The last equation is in the form presented by the influential review [19], and is notable for explicitly including the intermediate decision optimisation. Obviously the integrals become summations for discrete sets.

Note that the utility function can be transformed by a linear function with positive slope and this will not affect the optimal choice. For convenience we will assume that $U(\epsilon, y, d, \theta) \geq 0$, so that the worst possible utility is represented by $U = 0$.

In cases where the utility does not depend explicitly on a decision the process simplifies to

$$
\epsilon^\star = \arg \max_{\epsilon \in S_\epsilon} \int_{S_y} dy \int_{S_\theta} d\theta P(y|\epsilon) U(\epsilon, y, \theta),
$$

which is a common presentation of Bayesian experimental design. Eq. (8) is also the consequence if a naive decision is made: one that is automatically the same as the data $y$ obtained, i.e. $U(\epsilon, y, d, \theta) = \delta_{d,y} U(\epsilon, y, \theta)$ where $\delta_{d,y}$ is either the Dirac or Kronecker delta function between the decision and data sets.

B. Application to State Discrimination

This framework can be specialised to both minimal error and maximal confidence quantum state discrimination; and in both cases the experiment is a quantum measurement. In quantum state discrimination the unknown parameter is the message sent by Alice, so $S_\theta = \{1, \ldots, n\}$, which she will encode into a quantum state. The set of experiments $S_d$ is the set of all POVMs, and the output data is the possible outcomes of the particular POVM $\epsilon$. If $m$ is the number of elements in the particular POVM $\epsilon$, then $S_y = \{1, \ldots, m\}$. So (7) reduces to

$$
\epsilon^\star = \arg \max_{\epsilon \in S_\epsilon} \sum_{y=1}^m P(y|\epsilon) \max_{d \in S_d} \sum_\theta \max_{y=1}^n U(\epsilon, y, d, \theta) P(\theta|y,\epsilon).
$$

In standard quantum state discrimination the set of decisions is equal to the set of measurement outcomes $S_d = S_y$.

In minimal error state discrimination, the number of measurement outcomes matches the number of states sent by Alice, $m = n$. The probability of success can be derived using the utility function

$$
U(\epsilon, y, d, \theta) = \delta_{d,\theta} \delta_{d,y}.
$$

This utility function rewards Bob when the measurement outcome correctly identifies the state sent by Alice, which
he will automatically report (the naive decision). With this utility function (9) becomes,
\[
\epsilon^* = \arg\max_{\epsilon \in \mathcal{S}_r} \sum_{y=1}^{n} P(y|\epsilon) P(\theta = y|y, \epsilon).
\] (11)

By inspection, this expression is equivalent to finding the POVM which maximises (2): Bob wants the POVM which maximises the average probability that the measurement correctly identifies the state sent by Alice.

Maximal confidence state discrimination has an additional inconclusive outcome, and so $m = n + 1$. The decision $d = 0$ represents Bob reporting this inconclusive outcome and discarding the state. In maximal confidence measurements, Bob values confidence in his measurement outcome, independent of its likelihood to occur. So we can derive it from the utility function
\[
U(\epsilon, y, d, \theta) = \delta_{d, 0} \delta_{d, y} \frac{P(\theta = y|y, \epsilon)}{P(y|\epsilon)}.
\] (12)
in which case (9) becomes
\[
\epsilon^* = \arg\max_{\epsilon \in \mathcal{S}_r} \sum_{y=1}^{n} P(\theta = y|y, \epsilon).
\] (13)

Here $\sum_{y=1}^{n} P(\theta = y|y, \epsilon)$ is equivalent to the aggregate confidence (4). This is clearly satisfied by a maximal confidence measurement, since
\[
\max_{\epsilon \in \mathcal{S}_r} \sum_{y=1}^{n} P(\theta = y|y, \epsilon) \leq \sum_{\epsilon \in \mathcal{S}_r} \max_{\epsilon \in \mathcal{S}_r} P(\theta = y|y, \epsilon),
\] (14)
and the maximal confidence measurement maximises the right hand side. As mentioned in section II, in general there will be a family of measurements which maximise the aggregate confidence. One can then prefer a member of this family by requiring an additional optimisation: such as the member which minimises the probability of the inconclusive outcome.

\section*{IV. CONSEQUENCES OF INCLUDING THE DECISION IN STATE DISCRIMINATION}

We now show that removing the Kronecker delta function in the utility function, which rewards Bob for always deciding on the state corresponding to the measurement outcome, resolves the anomalies mentioned in section II. Further it allows Bob to discriminate between states using POVMs with an arbitrary number of elements.

As usual, Alice will send one of the $n$ states in her alphabet
\[
\rho_1, \ldots, \rho_n,
\] (15)
to Bob. Bob makes a measurement represented by the POVM,
\[
A_1, \ldots, A_m.
\] (16)

Note that the number $m$ of measurement outcomes need not be equal to the number $n$ of states in Alice’s alphabet. Upon obtaining the measurement result he will decide on which of the $n$ states in Alice’s alphabet was sent.

\subsection*{A. Minimal Error State Discrimination}

The utility function for minimal error quantum state discrimination should reward Bob when his decision matches the state sent by Alice, that is
\[
U(\epsilon, y, d, \theta) = \delta_{d, \theta}.
\] (17)

With this utility function, the optimal measurement (7) becomes
\[
\epsilon^* = \arg\max_{\epsilon \in \mathcal{S}_r} \sum_{y=1}^{m} P(\theta = d^*|y, \epsilon) P(y|\epsilon),
\] (18)
where $d^*$ is defined as in (5):
\[
d^* = \arg\max_{d \in \mathcal{S}_d} P(\theta = d|y, \epsilon).
\]

We see that the decision $d^*$ that Bob will make is to guess the state $\theta$ with the greatest posterior probability conditioned on POVM outcome $y$. That is to say he chooses the state $\rho_{y^*}$ which maximises the posterior distribution,
\[
\rho_{y^*} := \arg\max_{\rho \in \{\rho_1, \ldots, \rho_n\}} P(\rho|A_y).
\] (19)

The optimal measurement $\epsilon^*$, that is the POVM which discriminates between states with the smallest error, is the POVM $\{A_1, \ldots, A_m\}$ that which maximises the average probability that Bob correctly decides on the state sent to him by Alice,
\[
\langle s_A \rangle := \sum_{y=1}^{m} P(\rho_{y^*}|A_y) P(A_y).
\] (20)

We should therefore regard $\langle s_A \rangle$ as the correct measure of the quality of POVM $\{A_1, \ldots, A_m\}$ in performing quantum state discrimination. We recover (2) when $m = n$, and $\rho_{y^*} = \rho_y$ for all outcomes $y$, but clearly this is not always going to be optimal. The function (2) will often undervalue the actual probability of success since the state corresponding to the measurement outcome may not be the the state with the largest posterior probability.

We can now calculate the probability of success for POVMs with arbitrary numbers of elements. For example, we can compute the probability of success for unambiguous state discrimination. In this task the measurement outcomes are represented by POVMs with $n + 1$ elements, including an inconclusive result $A_0$. If the outcome $y \neq 0$ occurs, we know with certainty that Alice sent $\rho_y$. However, if $y = 0$ occurs, we may now guess the most probable state $\rho_{y^*}$, though in this case we will...
be uncertain with regards to the outcome. This means
\[ P (\rho_y | A_y) = \delta_{\mu,y} \] and (20) becomes
\[ \sum_{y=1}^{n} P (A_y) + P (\rho_0^* | A_0) P (A_0). \] (21)

The first term is \( 1 - P (A_0) \), this means that one can improve the probability of success by minimising the probability of the inconclusive result. The second term is the probability of receiving the inconclusive outcome and then correctly deciding on the state sent by Alice. Since Bob typically makes no decision when receiving the inconclusive outcome, this term is usually fixed to be zero. But since the second term is positive, the probability of success can be further improved by deciding on the most likely state when the measurement reports the inconclusive result.

### B. Necessary and Sufficient Conditions

In [5], the authors prove that a POVM \( \{ A_1, \ldots, A_n \} \) maximises the probability of success (2) if and only if
\[ \sum_{j=1}^{n} q_j \rho_j A_j - q_k \rho_k \geq 0, \quad \text{for all } k. \] (22)

This assumes that the number of POVM elements \( m \), is equal to the number of states \( n \), and that Bob always decides on the measurement outcome that his device reports. Given that we can now apply a wider class of measurements to the task one would hope that the probability of success could be improved. We show next that this is not the case. More specifically the conditions are still sufficient but no longer necessary.

**Proposition 1.** Let \( \{ A_1, \ldots, A_n \} \) satisfy (22) and \( \{ B_1, \ldots, B_m \} \) be any POVM; then \( \langle s_A \rangle \geq \langle s_B \rangle \).

**Proof.** Suppose \( \{ A_1, \ldots, A_n \} \) is a POVM satisfying the conditions in (22), and \( \{ B_1, \ldots, B_m \} \) is any POVM. Note that (22) implies that for any positive semi-definite operator \( B_\nu \),
\[ \text{Tr} \left( \sum_{j=1}^{n} q_j \rho_j A_j - q_k \rho_k \right) B_\nu \geq 0 \] (23)

holds for all \( k \). We also have
\[ \langle s_B \rangle = \sum_{\nu=1}^{m} P (B_\nu | \rho_\nu^*) P (\rho_\nu^*) \]
\[ = \sum_{\nu=1}^{m} \text{Tr} (B_\nu \rho_\nu^*) q_\nu^*. \] (24)

Now (23) together with (24) implies
\[ \langle s_B \rangle \leq \sum_{j=1}^{n} \sum_{\nu=1}^{m} \text{Tr} (\rho_j A_j B_\nu) q_j \]
\[ = \sum_{j=1}^{n} \text{Tr} (\rho_j A_j) q_j \leq \langle s_A \rangle. \] (25)

Hence \( \langle s_A \rangle \geq \langle s_B \rangle. \)

So (22) is still sufficient, however it is clearly no longer necessary since, for example, \( \{ A_1, \ldots, \frac{1}{2} A_n, \frac{1}{2} A_n \} \) has the same probability of success as \( \{ A_1, \ldots, A_n \} \).

This proposition implies that for the task of minimal error discrimination, one can always find an optimal measurement consisting of only \( n \)-elements, and where \( \rho_{\nu^*} = \rho_\nu \) for all \( y \). We can generalise the conditions (22) to allow for the freedom of POVMs with arbitrary numbers of elements, which are both sufficient and necessary.

**Proposition 2.** A POVM \( \{ A_1, \ldots, A_n \} \) has the optimal probability of success; that is it maximises (20) if and only if
\[ \sum_{j=1}^{m} q_j \rho_j A_j - q_k \rho_k \geq 0, \quad \text{for all } k. \] (26)

**Proof.** The proof that (26) is sufficient follows identically to that in proposition 1. That it is necessary follows very similarly to the proof presented in [5]. We include it here for completeness.

Suppose that \( \{ A_1, \ldots, A_n \} \) has the optimal probability of success and consider the operators
\[ G_k = \frac{1}{2} \sum_{j=1}^{m} q_j^* (\rho_j A_j^* + \rho_j^* A_j^\dagger) - q_k \rho_k. \] (27)

These operators are almost of the form as (26), therefore we first prove that the each operator \( G_k \) is positive semi-definite and then that this is the same condition as (26). Assuming they are not all positive semi-definite then there exists at least one \( G_k \) with a negative eigenvalue (let it be \( G_1 \)),
\[ G_1 |\lambda\rangle = -\lambda |\lambda\rangle. \] (28)

Then we can construct the POVM \( \{ A'_1, \ldots, A'_n \} \) with elements
\[ A'_j = (1 - \epsilon |\lambda\rangle \langle \lambda|) A_j (1 - \epsilon |\lambda\rangle \langle \lambda|) + \epsilon (2 + \epsilon) |\lambda\rangle \langle \lambda| \delta_{1,j}. \] (29)

where \( \epsilon \ll 1 \). Now if we temporarily use the notation \( \rho_\nu^A \) for \( \rho_{\nu^*} \) in the probability of success (20)
\[ \langle s_A \rangle = \sum_{j=1}^{m} q_j^A \text{Tr} (\rho_j^A A_j), \] (30)
then we can immediately demonstrate that \( \{A_s', \ldots, A_m'\} \) has a strictly larger probability of success. For the probability of success of the modified POVM \( A' \) we have

\[
\langle s_{A'} \rangle = \sum_{j=1}^{m} q_j' A_j' \sum_{j=1}^{m} q_j' A_j' \quad \text{(31)}
\]

where the inequality is obtained by definition of \( \rho_j' \). Evaluating the final term gives

\[
\langle s_{A} \rangle - \epsilon \sum_{j=1}^{m} q_j' (\lambda | A_j \rho_j' + \rho_j' A_j \lambda) + 2\epsilon q_j' (\lambda | \rho_j' | \lambda) + \mathcal{O} (\epsilon^2).
\]

With the eigenvalue equation (28) this becomes

\[
\langle s_{A'} \rangle \geq \langle s_{A} \rangle + 2\epsilon \lambda + \mathcal{O} (\epsilon^2) > \langle s_{A} \rangle.
\]

Since \( \epsilon \) can be always be chosen such that \( 2\epsilon \lambda + \mathcal{O} (\epsilon^2) \) is strictly positive, this contradicts the assumption that \( \{A_1, \ldots, A_m\} \) optimises the probability of success, and therefore all \( G_k \) operators must be positive semi-definite. We can show that this condition reduces to (26) by showing that \( \sum_{j=1}^{m} q_j \rho_j A_j \) is Hermitian. This follows from the fact that

\[
\sum_{k=1}^{m} G_k A_k = \frac{1}{2} \sum_{j=1}^{m} q_j (A_j \rho_j + \rho_j A_j) A_k - \sum_{k=1}^{m} q_k \rho_k A_k
\]

Since \( G_k \) and \( A_k \) are positive semi-definite, and the trace of the right hand side is clearly zero, we must have that \( G_k A_k = 0 \) for all \( k \). This means

\[
\frac{1}{2} \sum_{j=1}^{m} q_j (A_j \rho_j - \rho_j A_j) = 0,
\]

and hence \( \sum_{j=1}^{m} q_j \rho_j A_j \) is Hermitian.

**C. Maximal Confidence State Discrimination**

We can similarly generalise maximal confidence measurements by removing the restriction that Bob's decision matches the measurement outcome. That is, we use following utility function,

\[
U (\epsilon, y, d, \theta) = \delta_{d, \theta} P (y | \epsilon).
\]

With this utility function (7), becomes

\[
\epsilon^* = \arg \max_{\epsilon \in \mathcal{S}_c} \sum_{y=1}^{n} P (\theta = d^* | y, \epsilon),
\]

and so the measure of success in this framework is the aggregate confidence

\[
\sum_{y=1}^{m} P (\rho_y | A_y),
\]

and so the measure of success in this framework is the aggregate confidence

\[
\sum_{y=1}^{m} P (\rho_y | A_y),
\]

which is greater than the aggregate confidence for the POVM \( \{A_1, A_2\}. \) Since we can allow POVMs with an arbitrary number of elements, we can create POVMs with an arbitrarily large aggregate confidence. This can be avoided by restricting the number of allowed measurement outcomes in a POVM (as in the case of standard maximal confidence state discrimination).

Nevertheless this framework allows us to calculate the probability of success and aggregate confidence for a wider range of POVMs. For example, we can construct a maximal confidence measurement with extra 'inconclusive' outcomes, upon which we make a decision as to which state was sent, but which also achieves the maximal probability of success. Thus including a decision can unify the two branches of state discrimination, which could not be done previously due to the rigid requirement of the number of POVM elements.

**Proposition 3.** For a given alphabet and prior probabilities, let \( \{A_1, \ldots, A_n\} \) satisfy (22), and let \( \{E_1, \ldots, E_n, E_0\} \) be the maximal confidence measurement described in section II, where \( E_i \) maximises \( P (\rho_i | E_i) \) as in (3). We then construct

\[
\{k_1 E_1, \ldots, k_n E_n, A_1 - k_1 E_1, \ldots, A_n - k_n E_n\}
\]

where the constants \( k_j \geq 0 \) are chosen such that \( A_j - k_j E_j \) is positive semi-definite. This POVM has the same probability of success as \( \{A_1, \ldots, A_n\} \); that is it has the maximal probability of success.

**Proof.** Since \( A_j = k_j E_j + (A_j - k_j E_j) \),

\[
\langle s_{A} \rangle = \sum_{j=1}^{n} P (A_j | \rho_j) P (\rho_j)
\]

\[
= \sum_{j=1}^{n} (P (k_j E_j | \rho_j) + P (A_j - k_j E_j | \rho_j)) P (\rho_j)
\]

\[
\leq \sum_{j=1}^{n} P (k_j E_j | \rho_j) q_j + P (A_j - k_j E_j | \rho_j) q_j
\]

\[
=(s_{AE}).
\]
Since \( \{A_1, \ldots, A_n\} \) satisfies (3), then \( \rho_j^{\ast} = \rho_j \) in the first equality. The inequality arises by replacing the state \( \rho_j \) with the state \( \rho_j^{\ast} \) which maximises the posterior distribution for each measurement outcome. If \( k_j = 0 \), then we define \( P(k_j E_j | \rho_j^{\ast}) q_j^{\ast} = 0 \). Since \( \{A_1, \ldots, A_n\} \) satisfies (22), proposition 1 implies that \( \langle s_A \rangle \geq \langle s_{AE} \rangle \), hence \( \langle s_A \rangle = \langle s_{AE} \rangle \). 

Since each \( E_j \) satisfies the maximal confidence requirements (3), \( k_j E_j \) does as well when \( k_j > 0 \), so (40) can be thought of a kind of maximal confidence measurement. So by extending the number of ‘inconclusive’ outcomes and adding a decision strategy, we have designed a maximal confidence measurement with the optimal probability of success. This means that we can address both concerns with the same measurement apparatus, just different data processing. Note that this POVM does not have the minimal probability of obtaining the inconclusive outcomes, this had to be traded off to obtain a POVM with the maximal probability of success.

\section{Measurements May Not Change the Decision}

In [6], the author uses the conditions in (22) to derive the following result: if there exists a \( \rho_m \) in Alice’s alphabet which satisfies

\[ q_m \rho_m - q_k \rho_k \geq 0, \quad \text{for all } k, \tag{42} \]

then the optimal measurement is represented by the POVM, whose \( j \)th element is \( \delta_{j,m} \mathbb{1} \). In other words, the optimal measurement is to perform no measurement and simply guess the state \( \rho_m \).

This appears counter-intuitive: measurements which provide some information should always be better (as we need not act on it). But we can provide an interpretation of this result using Bayesian experimental design — the state \( \rho_m \) will always be the state with the largest posterior probability, no matter the measurement or the measurement result, if and only if (42) holds. Therefore, regardless of what measurement Bob does, his best decision is to always choose that Alice sent state \( \rho_m \).

Each non-trivial measurement still provides information about Alice’s state though and further measurements may lead to Bob changing his decision. We can generalise the condition (42) to the case of repeated messages.

\textbf{Proposition 4.} Suppose Alice sends \( d \) copies of the same state. After Bob performs a local measurement on each individual state sent, then \( \rho_m \) will always have the largest posterior probability if and only if

\[ \sqrt{q_m} \rho_m - \sqrt{q_k} \rho_k \geq 0 \quad \text{for all } k. \tag{43} \]

\textit{Proof.} Suppose (43) holds; then for any positive semi-definite operator \( A_j \),

\[ \sqrt{q_m} \ Tr (\rho_m A_j) \geq \sqrt{q_k} \ Tr (\rho_k A_j), \tag{44} \]

for all \( k \). This inequality holds if we take the product of \( d \) such terms. Let \( A_j^{(j)} \) denote that result \( A_j \) was measured on the \( j \)th system.

\[ \prod_{j=1}^{d} \sqrt{q_m} \ Tr (\rho_m A_j) \geq \prod_{j=1}^{d} \sqrt{q_k} \ Tr (\rho_k A_j), \]

\[ \Rightarrow q_m \prod_{j=1}^{d} \ Tr (\rho_m A_j) \geq q_k \prod_{j=1}^{d} \ Tr (\rho_k A_j), \tag{45} \]

so \( P (\rho_m | A_j^{(1)} \ldots A_j^{(d)}) \geq P (\rho_k | A_j^{(1)} \ldots A_j^{(d)}) \), for all \( k \). If (43) doesn’t hold, then there exists a positive semi-definite operator \( B \) and state \( \rho_k \) such that

\[ \sqrt{q_m} \ Tr (\rho_m B) < \sqrt{q_k} \ Tr (\rho_k B). \tag{46} \]

Taking the \( d \)th power of both sides,

\[ q_m \ Tr (\rho_m B)^d < q_k \ Tr (\rho_k B)^d, \tag{47} \]

and so \( P (\rho_m | B^{(1)} \ldots B^{(d)}) < P (\rho_k | B^{(1)} \ldots B^{(d)}) \); there exists a measurement record which would cause Bob to decide that Alice sent \( \rho_k \) where \( k \neq m \). 

Clearly, (42) is the case where \( d = 1 \), which if it holds implies

\[ P (\rho_m | A_j) \geq P (\rho_k | A_j), \tag{48} \]

for all \( k \), and for all measurements; that is, for all positive semi-definite operators \( A_j \leq 1 \). Under this condition, the probability of success for correctly predicting the state Alice sent with \textit{any} measurement is

\[ \sum_{y=1}^{m} P (\rho_y | A_y) P (A_y) = \sum_{y=1}^{m} P (\rho_m | A_y) P (A_y) = q_m. \]

Thus, all measurements are optimal since all have the same probability of success. The conditions (42) can be derived from our new conditions (26) for the optimal measurement by realising \( q_y^\ast \rho_y^{\ast} = q_m \rho_m \) for all outcomes \( y \).

Non-trivial measurements still provide Bob with information, modifying his posterior knowledge of what Alice sent; however no measurement exists which could overturn his bias that \( \rho_m \) is the most probable state, as reflected in the prior probabilities. Indeed it can be argued that performing no measurement is the least useful, since if Alice were to send multiple copies, non-trivial measurements exist which could indeed cause Bob to change his mind, as illustrated in fig 1.

We can relate this to the anomalies we highlighted in section II. In the first example, Bob tries to discriminate the alphabet \( \rho_1 = |+\rangle \langle +| \) and \( \rho_2 = |\rangle \langle -| \) with a perfect measurement device known to be incorrectly labelled, represented by \( A_1 = |\rangle \langle -| \), \( A_2 = |+\rangle \langle +| \). Choosing the state with the largest posterior probability is equivalent to correcting the labels of the measurement device. This
Bob will be more confident about choosing $A$. The updated posterior probabilities do give Bob some information about which state Alice is in. If Alice’s alphabet consists of states $\{0\}, \{1\}$ and $\rho_2 = |+\rangle\langle +|$, the prior probabilities are $q_1 = 0.85$ and $q_2 = 0.15$. The posterior probabilities for $q_1$ (dark) and $q_2$ (light) over the course of three repeated measurements using POVM $\{A_1 = |0\rangle\langle 0|, A_2 = |1\rangle\langle 1|\}$. We have $q_1 \rho_1 - q_2 \rho_2 \geq 0$, so no single measurement could cause Bob to choose $\rho_2$ over $\rho_1$. However, since $\sqrt{q_1} \rho_1 - \sqrt{q_2} \rho_2 \geq 0$, there exists a measurement record (in this case, receiving the outcome $A_2$ twice) which can cause Bob to choose $\rho_2$.

In the second example, Bob is trying to discriminate between the same states $\rho_1$ and $\rho_2$ using his weak measurement, represented by $A_1 = \frac{1}{2} (I + \epsilon X), A_2 = \frac{1}{2} (I - \epsilon X)$. We showed that if $q_1 > \frac{1}{2} (1 + \epsilon)$ then by (2), this has a lower probability of success than the trivial measurement $A_1' = I, A_2' = 0$. Since the measurement is noisy, if $q_1 > \frac{1}{2} (1 + \epsilon)$ the measurement is too weak for $\rho_2$ to have the larger posterior probability; therefore regardless of the outcome, Bob will always choose the state $\rho_1$. Correcting for this, the probability of success (20) using $\{A_1, A_2\}$ is $q_1$, the same as for $\{A_1', A_2'\}$. However, unlike this trivial measurement, the POVM $\{A_1, A_2\}$ does give Bob some information about which state Alice sent. The updated posterior probabilities are

$$P(\rho_1|A_1) = q_1 \frac{1 + \epsilon}{1 + \epsilon (q_1 - q_2)} > q_1, \quad (49a)$$
$$P(\rho_1|A_2) = q_1 \frac{1 - \epsilon}{1 - \epsilon (q_1 - q_2)} < q_1. \quad (49b)$$

So if the measurement device reports outcome $A_1$, then Bob will be more confident about choosing $\rho_1$ than he will be prior to the measurement; and if the device reports $A_2$, he will be less confident. In this way the POVM $\{A_1, A_2\}$ can be seen to be more useful than $\{A_1', A_2'\}$, even though they share the same probability of success within our framework.

V. CONCLUSION

When Bob receives a quantum state from Alice with the intent of discriminating between $n$ possible states, he always makes a decision as to what state was sent. The majority of the quantum state discrimination literature implicitly assumes that Bob simply decides on the state corresponding to the measurement outcome. But the state corresponding to the measurement outcome may not be the most likely state that was sent; indeed the measurement result may exclude that state. We showed several examples where this naive decision strategy increases the probability of error, and one can get a performance enhancement by using a trivial measurement, which provides no information about the system at all.

These implicit assumptions become obvious when quantum state discrimination is cast as an example of Bayesian experimental design. We derived minimal error and maximal confidence state discrimination from this framework with similar utility functions. We saw that when we removed the requirement that Bob’s decision match the measurement outcome, these anomalies were resolved. Further, we could use any quantum measurement with any number of outcomes, to discriminate between states, which broadens the toolkit that can be applied to state discrimination problems. This also lead to some generalisation of key previous results.

Finally, we saw in section II that an entire family of POVMs satisfies the requirements to be a maximal confidence measurement. Selecting a unique POVM from this family requires a further optimisation; typically this is done by selecting the POVM from this family with the smallest probability of yielding the inconclusive outcome. In proposition 3 we constructed a maximal confidence measurement which also maximised the probability of success. Both of these constructions can be viewed as multi-objective optimisations, hence an interesting extension would be to consider state discrimination within Bayesian experimental design with a multi-objective optimisation.

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