Minimal obstructions to 2-polar cographs

Pavol Hell, César Hernández-Cruz, Cláudia Linhares Sales

Abstract

A graph is a cograph if it is \( P_4 \)-free. A \( k \)-polar partition of a graph \( G \) is a partition of the set of vertices of \( G \) into parts \( A \) and \( B \) such that the subgraph induced by \( A \) is a complete multipartite graph with at most \( k \) parts, and the subgraph induced by \( B \) is a disjoint union of at most \( k \) cliques with no other edges.

It is known that \( k \)-polar cographs can be characterized by a finite family of forbidden induced subgraphs, for any fixed \( k \). A concrete family of such forbidden induced subgraphs is known for \( k = 1 \), since 1-polar graphs are precisely split graphs. For larger \( k \) such families are not known, and Ekim, Mahadev, and de Werra explicitly asked for the family for \( k = 2 \). In this paper we provide such a family, and show that the graphs can be obtained from four basic graphs by a natural operation that preserves 2-polarity and also preserves the condition of being a cograph. We do not know such an operation for \( k > 2 \), nevertheless we believe that the results and methods discussed here will also be useful for higher \( k \).

Keywords: Polar graph, cograph, forbidden sugraph characterization, \( k \)-polar graph, matrix partition, generalized colouring

2000 MSC: 05C 69, 05C70, 05C75

1 Introduction

All graphs in this paper are considered to be finite and simple. We refer the reader to [1] for basic terminology and notation. In particular, we use \( P_k \) and \( C_k \) to denote the path and cycle on \( k \) vertices, respectively. A graph is a cograph if it is \( P_4 \)-free.

This research was supported by a research grant from NSERC Canada for the first and second author, and by Projects CAPES/Brazil 99999.000458/2015-05 and CNPq/Brazil 307252/2013-2 for the third author.

Corresponding author

Email addresses: pavol@sfu.ca (Pavol Hell), cesar@matem.unam.mx (César Hernández-Cruz), linhares@lia.ufc.br (Cláudia Linhares Sales)
A polar partition of a graph $G$ is a partition of the vertices of $G$ into parts $A$ and $B$ in such a way that the subgraph induced by $A$ is a complete multipartite graph and the subgraph induced by $B$ is a disjoint union of cliques, with no other edges. A graph $G$ is polar, if it admits a polar partition, and is $(s,k)$-polar if it admits a polar partition $(A,B)$ in which $A$ has at most $s$ parts and $B$ at most $k$ parts. In particular, when $s = k$, we use the term $k$-polar partition and $k$-polar graph. Note that 1-polar graphs are precisely split graphs. It was shown by Foldes and Hammer \cite{9} that a graph is split if and only if it does not contain $2K_2$, $C_4$ or $C_5$ as an induced subgraph; as a consequence, testing whether a given graph is split can be done in polynomial time.

The concept of a matrix partition unifies many interesting graph partition problems, including $(s,k)$-partition. Given a symmetric $n \times n$ matrix $M$, with entries in $\{0, 1, *\}$, an $M$-partition of a graph $G$ is a partition\footnote{As it is usual in graph theory, we do not require every part of the partition to be non-empty.} $(V_1, \ldots, V_n)$ of $V(G)$ such that, for every $i, j \in \{1, \ldots, n\}$,

- $V_i$ is completely adjacent to $V_j$ if $M_{ij} = 1$,
- $V_i$ is completely non-adjacent to $V_j$ if $M_{ij} = 0$,
- There are no restrictions if $M_{ij} = *$.

It follows from the definition that, in particular, if $M_{ii} = 0$ ($M_{ii} = 1$), then $V_i$ is a stable set ($V_i$ is a clique). The $M$-partition problem asks whether or not an input graph $G$ admits an $M$-partition. It is easy to verify that, e.g., the $k$-colouring and split partition problems are matrix partition problems. See \cite{10} for a survey on the subject. It is also easy to see that an $(s,k)$-partition of $G$ is a matrix partition in which the matrix $M$ has $s + k$ rows and columns, the principal submatrix induced by the first $s$ rows is obtained from an identity matrix by exchanging 0’s and 1’s, the principal submatrix induced by the last $k$ rows is an identity matrix, and all other entries are *. Therefore, it follows from \cite{7} (as explicitely observed in \cite{5}), that for any fixed $s$ and $k$, the class of $(s,k)$-polar graphs can be recognized in polynomial time.

On the other hand, it was shown by Chernyak and Chernyak \cite{3} that the recognition of general polar graphs is $NP$-complete. Interestingly, the class of polar graphs that admit an $(s,k)$-partition with $s = 1$ or $k = 1$ (sometimes called monopolar graphs), is also $NP$-complete to recognize (as proved by Farrugia \cite{6}). It was shown recently that this remains true even in severely restricted graph classes, for instance Le and Nevries \cite{11} have shown that both $NP$-completeness results hold for triangle-free planar graphs of maximum degree 3.

Notice that having an $M$-partition is a hereditary property, and hence, the family of $M$-partitionable graphs admits a characterization in terms of forbidden induced subgraphs. A minimal $M$-obstruction is a graph which does not admit an $M$-partition, but such that every proper induced subgraph does. Feder, Hell and Hochstättler proved in \cite{8} that, for any matrix $M$, there are only finitely many minimal $M$-obstructions which are cographs. (This
can also be derived from [4]. In other words, when we restrict the $M$-partition problem to the class of cographs, there are only finitely minimal $M$-obstructions, and, consequently, any $M$-partition problem is solvable in polynomial time for cographs.

Thus, in particular, for any $s$ and $k$, there are only finitely many minimal $(s,k)$-polar obstructions that are cographs. For $s = k = 1$, an explicit list follows from the result of Foldes and Hammer mentioned above: only $2K_2$ and its complement ($C_4$) are cograph minimal 1-polar obstructions. In this paper we provide a compact description of cograph minimal 2-polar obstructions. We believe the ideas generated might yield at least some kind of description of all cograph minimal $(s,k)$-polar obstructions, and thus for a fairly wide class of matrix partition problems. Moreover, we believe that knowing the minimal obstructions might lead to a certifying algorithm for the recognition of these graphs.

It is worth noticing that Ekim, Mahadev and de Werra proved in [5] that it is possible to recognize polar and monopolar graphs in polynomial time in the class of cographs. Moreover, they proved that there are only finitely many cograph minimal polar obstructions (eight), and cograph minimal monopolar obstructions (eighteen). In the same paper, they propose the problem of finding a characterization of 2-polar cographs by forbidden subgraphs as a natural continuation of their work.

We will denote the complement of $G$ by $\overline{G}$. Cographs can be characterized as those graphs $G$ such that they are either trivial, or one of $G$ or $\overline{G}$ is disconnected, and its components are cographs. It follows from this characterization that if $G$ is a cograph, then so is $\overline{G}$. Observe that $G$ is a $k$-polar cograph if and only if $\overline{G}$ is a $k$-polar cograph as well. Therefore, if $H$ is a cograph that is a minimal $k$-polar obstruction then so is $\overline{H}$. Hence, we can focus our attention in disconnected cograph minimal $k$-polar obstructions $H$. We denote the components of $H$ by $B_1, \ldots, B_m$. We say that a component of $H$ is trivial or an isolated vertex if it is isomorphic to $K_1$.

Given graphs $G$ and $H$, the disjoint union of $G$ and $H$ is denoted by $G + H$, and the join of $G$ and $H$ is denoted by $G \oplus H$.

Every pair of non-adjacent vertices of a $C_4$ are called antipodal vertices. A wheel $W_k$ is a $C_k$ together with a universal vertex.

The rest of the paper is organized as follows. In Section 2 we will prove some basic facts on the structure of $k$-polar obstructions for any positive integer $k$. In Section 3 we will introduce an operation that preserves the 2-polarity of a graph, proving some of its basic properties. Section 4 is devoted to prove our main result, exhibiting the complete list of cograph minimal 2-polar obstructions. Finally, in Section 5, we present our conclusions and future lines of research.

2 Preliminary results

A minimal $k$-polar obstruction is extremal if it has exactly $(k + 1)^2$ vertices; the reason for this name will be clear from Theorem 5 in Section 4. Our first lemma states the possible number of components of a minimal $k$-obstruction, as well as some general facts about their structure.
There is one argument that we will be using in many of our proofs. Let \( H \) be a minimal \( k \)-polar obstruction, and let \( v \) be a vertex in \( H \). Thus, \( H - v \) has a \( k \)-polar partition \( (V_1, \ldots, V_{2k}) \). We will assume that \( A = \bigcup_{i=1}^{k} V_i \) induces a multipartite graph with parts \( V_1, \ldots, V_k \). Notice that, if at least two of these parts are non-empty, then all the vertices in \( A \) are contained in a single component of \( H - v \). Otherwise, either \( A \) is empty, or only one of its parts is non-empty, but, since these two cases can be usually handled in a very similar way, we will often assume without loss of generality that one of these parts is non-empty.

**Lemma 1.** Let \( H \) be a minimal \( k \)-polar obstruction. The following statements are true:

1. \( H \) has at most \( k + 2 \) components;
2. \( H \) has at least one non trivial component;
3. \( H \) has at most \( k + 1 \) trivial components;
4. If \( H \) has at least one trivial component, then \( H \) has at most one non-complete component.
5. If \( H \) is not an extremal minimal \( k \)-polar obstruction, then every complete component is isomorphic to \( K_1 \) or \( K_2 \).

**Proof.** For 1., suppose, by contradiction, that \( H \) has more than \( k + 2 \) components. If there are isolated vertices in \( H \), consider \( v \), one of them. Thus, \( H - v \) has at least \( k + 2 \) components, and by the minimality of \( H \), it has a \( k \)-polar partition \( P = (V_1, \ldots, V_{2k}) \). If there is a unique non empty stable set in this partition, then we can assume without loss of generality that this set is \( V_1 \), and hence, \( (V_1 \cup \{v\}, V_2, \ldots, V_{2k}) \) is a \( k \)-polar partition of \( H \), a contradiction. Thus, the subgraph induced by \( \bigcup_{i=1}^{k} V_i \) is connected, and hence contained in a single component of \( H - v \). But in this case, the rest of the \( k + 1 \) components should be covered by \( k \) cliques, which is impossible.

If there are no isolated vertices in \( H \), consider any vertex \( v \) of \( H \). Let \( P = (V_1, \ldots, V_{2k}) \) be a \( k \)-polar partition of \( H - v \). Note that \( H - v \) has at least \( k + 3 \) components of which at least \( k + 2 \) are not trivial. Hence, \( (H - v) - \bigcup_{i=1}^{k} V_i \) has at least \( k + 2 \) components that should be covered by \( k \) cliques, a contradiction.

Item 2. follows from the fact the any empty graph is trivially \( k \)-polar. Item 3 follows from 1. and 2.

For 4., suppose that \( H \) has one trivial component and let \( v \) be an isolated vertex of \( H \). By contradiction, suppose that \( B_1 \) and \( B_2 \) are two non-complete components of \( H \). Since a \( k \)-polar partition \( P = (V_1, \ldots, V_{2k}) \) of \( H - v \) has necessarily two non-empty stable sets (otherwise, if we add \( v \) to the unique non-empty stable set of \( P \), or to any stable set of \( P \) if all of them are empty, we would obtain a \( k \)-partition of \( H \), a contradiction), and \( B_1 \) and \( B_2 \) cannot be covered only by cliques, \( \bigcup_{i=1}^{k} V_i \) belongs to one of \( B_1 \) or \( B_2 \), let us say, \( B_1 \). Now, \( B_2 \) cannot be covered only by cliques, since it is connected. But it also has no vertex belonging to \( V_1, \ldots, V_k \). By consequence, \( H - v \) has no \( k \)-polar partition, a contradiction.

For 5., let \( B_1 \) be any complete component with more than 2 vertices. Let \( v \) be any vertex of \( B_1 \) and let \( P = (V_1, \ldots, V_{2k}) \) be a \( k \)-polar partition of \( H - v \). If \( V(B_1 - v) \cap V_i \neq \emptyset \) for some \( i \in \{k + 1, \ldots, 2k\} \), then \( (V_1, \ldots, V_i \cup \{v\}, \ldots, V_{2k}) \) is a \( k \)-partition for \( H \), contradicting \( H \) to be a minimal obstruction. Thus, \( V(B_1 - v) = \bigcup_{i=1}^{k} V_i \), with \( V_i \neq \emptyset \) for \( 1 \leq i \leq k \),
else, we could place \( v \) in one of the empty stable sets to obtain a \( k \)-polar partition of \( H \), a contradiction. Now, all the other components of \( H \) have to be complete, and cannot have less than \( k + 1 \) vertices, otherwise by covering \( B_1 \) by a clique and any smaller clique by \( k \) completely adjacent stable sets would lead to a \( k \)-partition of \( H \), a contradiction. As a conclusion, every other component is a complete graph with at least \( k + 1 \) vertices and there are at least \( k + 1 \) components, otherwise \( H \) would be \( k \)-polar. Therefore \( H \) is the extremal \( k \)-polar obstruction \( (k + 1)K_{k+1} \).

The following Lemma describes the family of graphs with exactly \( k + 2 \) components and at least one of them being trivial.

**Lemma 2.** Let \( \ell \) be an integer such that \( 1 \leq \ell \leq k + 1 \). Up to isomorphism, there is exactly one minimal \( k \)-polar obstruction with \( k + 2 \) components and precisely \( \ell \) of them trivial, and it is isomorphic to

\[
\ell K_1 + (k - \ell + 1)K_2 + K_{\ell,\ell}.
\]

Moreover, every minimal \( k \)-polar obstruction with \( k + 2 \) components has at least one trivial component.

**Proof.** Let us consider a \( k \)-polar minimal obstruction \( H \) satisfying the requirements of the Lemma. Let \( v \) be an isolated vertex of \( H \). The graph \( H - v \) admits a \( k \)-polar partition \( P = (V_1, \ldots, V_{2k}) \), such that, at least two of the stable sets are non-empty. Otherwise, if we add \( v \) to the only non-empty stable set of \( P \) (if any, otherwise place \( v \) in \( V_1 \)), then the resulting partition would be a \( k \)-polar partition for \( H \). Thus, all the stable sets of \( P \) are contained in the same component of \( H - v \). Now, the remaining \( k \) components of \( H - v \) should be covered by the \( k \) cliques in \( P \). But this means that the component containing the stable sets of \( P \) is a complete multipartite graph.

Thus, \( H \) is the disjoint union of \( \ell K_1 \), \((k - \ell + 1)\) non-trivial cliques, and a complete \( m \)-partite graph \( K \), with \( 2 \leq m \leq k \).

Now, let \( u \) be a vertex in \( K \). Again, \( H - u \) has a \( k \)-polar partition \( P' = (W_1, \ldots, W_{2k}) \). Since \( H - u \) has at least \( k + 2 \) components, and the cliques of \( P' \) can cover at most \( k \) different components, it must be the case that only one of the stable sets of \( P' \) is non-empty, say \( W_1 \), and contains all the isolated vertices of \( H \). Hence, \( K - W_1 \) must be a disjoint union of complete graphs, because it should be covered by the cliques of \( P' \). But this means that \( K - W_1 \) is an independent set with at most \( k - (k - \ell + 1) = \ell - 1 \) vertices. Thus, \( K \) is a complete bipartite graph. It is easy to observe that if \( K \) is smaller than \( K_{\ell,\ell} \), then \( H \) admits a \( k \)-polar partition. Finally, it follows from Lemma 1 that the remaining \((k - \ell + 1)\) non-trivial complete components are copies of \( K_2 \).

For the final statement, it is easy to verify that \( K_1 + (k + 1)K_2 \) is a minimal \( k \)-polar obstruction. Thus, any graph with \( k + 2 \) non-trivial components properly contains this obstruction.

\[\square\]
3 Switching and partial complementation

As we have mentioned in the introduction, \(k\)-polar cographs are a very convenient class of \((k,l)\)-polar cographs in terms of forbidden induced subgraph characterization; in order to find all the cograph minimal \(k\)-polar obstructions, it suffices to find only the disconnected ones. The family of 2-polar cographs enjoys an additional property not shared by \(k\)-polar cographs with \(k > 2\). Specifically, there are two very natural operations preserving the 2-polarity of a graph, which lead to a much more compact list of minimal obstructions, cf. Theorem 3.

Given a graph \(H\) and one of its vertices \(v\), a graph \(H'\) can be obtained from \(H\) by a switching on \(v\), that is, by making \(N_{H'}(v) = V(H) - N_H(v)\), while the rest of the graph remains unaltered. A partial complement of \(H\) is a graph obtained by splitting the components of \(H\) into two graphs, \(H'\) and \(H''\), and taking separately the complement of each of them. Notice that if \(H\) is connected, then one of \(H'\) or \(H''\) is empty, and the other one is \(H\); in this case, the partial complement coincides with the complement. Observe that a disconnected graph \(H\) with three or more components has several different ways of taking partial complementation, but, as long as both \(H'\) and \(H''\) are non-empty the resulting graph will always be disconnected.

Notice that partial complementation can be defined in terms of switching and regular complementation in the following way. Consider a disconnected graph \(H\), and split its components into two graphs \(H'\) and \(H''\). Now, perform switches on every vertex of \(H'\) (this will leave us with a graph which has the same edges as \(H\), plus all the edges between \(H'\) and \(H''\)), and then, take the complement of the resulting graph. Clearly, this procedure yields the same result as taking a partial complement with \(H'\) and \(H''\).

Lemma 3. If \(H\) is a 2-polar graph, and \(v\) is a vertex in \(H\), then the graph obtained from \(H\) by switching on \(v\) is also 2-polar. If additionally \(H\) is a disconnected cograph, then any partial complement of \(H\) is again a disconnected 2-polar cograph.

Proof. Let \((V_1, V_2, V_3, V_4)\) be a 2-polar partition of \(H\). We will assume that \(v \in V_1\), the remaining cases can be dealt similarly. Since \(V_1 \cup V_2\) induces a complete bipartite graph (where \(V_2\) is possibly empty), \(v\) is adjacent to every vertex in \(V_2\) and non-adjacent to every vertex in \(V_1\). Thus, after switching on \(v\), it is clear that \((V_1 \setminus \{v\}, V_2 \cup \{v\}, V_3, V_4)\) is a 2-polar partition of the resulting graph.

For the second statement, split the components of \(H\) into \(H'\) and \(H''\). From the remark previous to this lemma, and the previous paragraph, it is clear that taking the partial complement of \(H\) with \(H'\) and \(H''\) yields a 2-polar graph. Since \(H\) is a cograph, \(H'\) and \(H''\) are also cographs, as well as their complements. Thus, the partial complement of \(H\) is a disjoint union of cographs, which is again a disconnected cograph.

Since in general switching does not preserve the property of being a cograph, but partial complementation does, we will restrict ourselves to the use of the latter. It follows from Lemma 3 that if \(H\) is a cograph minimal 2-polar obstruction, then any partial complement of \(H\) is also a cograph minimal 2-polar obstruction. Since partial complements are reversible,
if we define two graphs to be related if one can be obtained by a sequence of partial comple-
mentations from the other, then this defines an equivalence relation. In particular, it follows
by the previous remark that the family of cograph minimal 2-polar obstructions admits a
partition into equivalence classes under this relation.

Let \( \mathcal{H}_7, \mathcal{H}_{8A}, \mathcal{H}_{8B} \) and \( \mathcal{H}_9 \) be the families of graphs depicted in Figures 2, 3, 4, and 5,
respectively, and together with their respective complements.

**Lemma 4.** The families \( \mathcal{H}_7, \mathcal{H}_{8A}, \mathcal{H}_{8B} \) and \( \mathcal{H}_9 \) are families of cograph minimal 2-polar ob-
structions closed under partial complementation.

**Proof.** It is a simple exercise to verify that each of the depicted graphs is a cograph minimal
2-polar cograph, and, although it takes a while, it is simple as well to verify that every
possible partial complement of every member of each of the families, belongs again to the
same family. We will mention how to obtain the rest of the disconnected members of \( \mathcal{H}_7 \) by
a sequence of partial complementations from \( F_1 \), the rest of the families can be dealt in a
similar way.

Recall that \( F_1 \) is isomorphic to \( 3K_2 + K_1 \). Notice that \( F_2 \) is isomorphic to \( 2K_2 + K_2 + K_1 \),
\( F_3 \) is isomorphic to \( 3K_2 + K_1 \), and \( F_4 \) is isomorphic to \( K_2 + 2K_2 + K_1 \). Thus, \( F_2, F_3 \) and \( F_5 \)
can be obtained from \( F_1 \) by a single partial complementation. Now, observe that the \( K_2 \) in
\( F_4 \) is just a \( 2K_1 \), so we can get \( F_3 \) as \( 2K_1 + K_1 + K_1 + K_1 \).

The following simple observation will be very useful in the next section. If \( H \) is a minimal
\((s,k)\)-polar obstruction, then it should contain a minimal \((n,m)\)-polar obstruction for every
\( n \leq s \) and every \( m \leq k \). Otherwise \( H \) would admit an \((n,m)\)-polar partition, which is
also an \((s,k)\)-polar partition. In particular, each minimal 2-polar obstruction should contain
a polar split obstruction (a \( 2K_2 \) or a \( C_4 \)), a minimal \((2,1)\)-polar cograph obstruction or a
minimal \((1,2)\)-polar cograph obstruction. Hence, it will be useful to reproduce, in Figure 1,
the complete list of cograph minimal \((2,1)\)-polar obstructions obtained by Bravo et al. in
\cite{2}.

## 4 2-polar cographs

The following theorem, giving an upper bound on the number of vertices of a cograph
minimal \(k\)-polar obstruction, is implicitly proved in \cite{8} by Feder, Hell and Hochstättler.

**Theorem 5.** Let \( H \) be a cograph minimal \((s,k)\)-polar obstruction. Then, \( H \) has at most
\((s+1)(k+1)\) vertices.

It follows from Theorem 5 that cograph minimal \(k\)-polar obstructions have at most \((k+1)^2\)
vertices, and thus, obstructions attaining this upper bound are called extremal. In particular,
cograph minimal 2-polar obstructions have at most nine vertices. The following lemma gives
a lower bound on the number of vertices of a minimal 2-polar obstruction (not necessarily
a cograph), as well as a structural property about the minimal obstructions attaining this
bound.
Lemma 6. Let $H$ be a minimal 2-polar obstruction.

1. $H$ has at least seven vertices.
2. If $H$ has seven vertices and three connected components, then at least one of them is an isolated vertex.

Proof. Let $H$ be a graph on at most 6 vertices. If $H$ is a split graph, then it is 2-polar. So, suppose that $H$ contains one of the minimal split obstructions as an induced subgraph. If $H$ contains an induced $C_5$ and, if (provided it exists) the remaining vertex is adjacent to two of its consecutive vertices, then we can find a $(2,1)$-polar partition of $H$, consisting of a $P_3$ and a $K_3$. On the other hand, if the remaining vertex is non-adjacent to two of its consecutive vertices, we can also find a 2-polar partition consisting of a $P_3$, a $K_1$ and a $K_2$. Now suppose that $H$ contains an induced $C_4$ and the remaining two vertices, if they exist, are mutually adjacent. Then we can find a $(2,1)$-polar partition consisting in the $C_4$ and a $K_2$. On the other hand, if the two remaining vertices are non-adjacent, we can find a 2-polar partition consisting of the $C_4$ and $2K_1$. The case when $H$ contains an induced $2K_2$ is analogous to the previous one.

For the second statement, let $H$ be a graph on 7 vertices with three connected components and without isolated vertices. It is easy to observe that two components of $H$ are $K_2$ and the remaining one is either $P_3$ or $K_3$. In either case, it is immediate to verify that $H$ admits a 2-polar partition.

Figure 1: Cograph $(2,1)$-polar minimal obstructions.
Lemma 7. The disconnected cograph minimal 2-polar obstructions on 7 vertices are exactly $F_1, \ldots, F_5$, see Figure 2.

Proof. Let $H$ be a cograph minimal 2-polar obstruction on 7 vertices. If $H$ has four components, then, according to Lemma 2, it must be $F_1$.

It follows from Lemma 4 that if $H$ can be transformed into a graph with four components through a sequence of partial complementations, then it is one of $F_i$ for $i \in \{1, \ldots, 5\}$. So, let us suppose that none of the graphs that can be obtained from $H$ by partial complementations has more than three components. Notice that any graph with two components can be transformed into a graph with at least three components using partial complementation. Thus, let us suppose without loss of generality that $H$ itself has three components $B_1, B_2, B_3$. Then, by Lemma 6, $H$ has an isolated vertex. Let us suppose that $B_3$ is the trivial component of $H$. By taking the partial complementation $B_3 + B_1 + B_2$, we obtain a graph with two components, one of them being an isolated vertex. Again, let us suppose that $H$ is such graph.

It is clear that $H$ contains an induced copy of $E_i$ for some $i \in \{1, \ldots, 9\}$ (see Figure 1). Since $H$ has two components, one of which is an isolated vertex, $i \notin \{2, 8, 9\}$. If $i = 4$, then $G$ is $F_5$. If $i = 5$, then $H$ is (1, 2)-polar: take the middle non-adjacent vertices of $E_5$ together with the isolated vertex in a stable set, and a $2K_2$.

If $i = 3$, since $H$ has only two components and $E_3$ has three components, then the vertex of $H$ which is not in the copy of $E_3$ should be adjacent to one of the isolated vertices of $E_3$. 

Figure 2: Cograph minimal 2-polar obstructions on 7 vertices.
The resulting $K_2$, together with the isolated vertex and the $C_4$ contained in $E_3$ conform a 2-polar partition of $H$, contradicting the assumption that it is an obstruction.

If $i = 6$, again, since $H$ has two components, the vertex of $H$, let us say, $x$, not in $E_6$, should be adjacent to the 4-wheel contained in $E_6$. If $x$ is adjacent to the middle vertex of the wheel, let us say $y$, then the resulting $K_2$, together with the isolated vertex of $H$ and the $C_4$ contained in $E_6$, conform a 2-polar partition of $H$, a contradiction. So, $x$ must be adjacent to some vertex in the $C_4$, let us say $w_1$, and thus, in order to not contain an induced $P_4$, it should be adjacent to a pair of antipodal vertices. If $x$ is only adjacent to a pair of antipodal vertices, then $H$ admits a 2-polar partition, which is a $K_{2,3}$ and a $2K_1$. Else, if $x$ is adjacent all the vertices of the $C_4$ but one, let us say $w_2$, then $xw_1yw_2$ is an induced $P_4$, a contradiction. Then, $x$ is adjacent to every vertex of the $C_4$ and so, $H$ is $F_5$.

If $i = 7$, as in the previous case, the vertex of $H$ not in $E_7$, let us say $x$, cannot be adjacent to the vertex in the center of the $C_4$, let us say, $y$. Also, we have a case similar to the previous one when $x$ is adjacent to only two antipodal vertices. So $x$ is adjacent to at least three vertices of the $C_4$. Hence, $x$ and $y$ have at least two common neighbors in the $C_4$. Let us call $w_1$ the non-neighbor of $y$ in the $C_4$. If $x$ is adjacent to $y$, then $yw_2w_1x$, where $w_2$ is any common neighbor of $x$ and $y$, is an induced $P_4$, contradiction. So, $x$ and $y$ have the same set of neighbors in the $C_4$. Therefore, $H$ admits a 2-polar partition consisting of a $P_3$, a $K_1$ and a $K_3$, a contradiction.

Finally, if $i = 1$, there are two new vertices besides the vertices from $E_1$, say $u$ and $v$. Since $G$ has two connected components, and recalling that there are not induced copies of $P_4$ in $G$, it can be observed that one of these two vertices, say $v$, is completely adjacent to the $2K_2$ in $E_1$. If $u$ is only adjacent to $v$, then $G$ is $F_3$. Otherwise, it follows from the fact that $G$ is a cograph that $u$ should be adjacent to $v$ and the two vertices of one of the $K_2$. But these four vertices induce a $K_4$, which together with the isolated vertex and the remaining $K_2$, conform a 2-polar partition of $G$, a contradiction.

Since the cases are exhaustive, the result follows. \qed

Although it may look a bit odd, we will deal with the cograph minimal 2-polar obstructions on 9 vertices before dealing with the ones on 8 vertices. This is because we will use the same proof strategy for both cases, which is easier to explain in the case of nine vertices. We consider $H$ a cograph minimal 2-polar obstruction. As in the proof of Lemma 7, we may assume that $H$ has three components, one of which is an isolated vertex $v$. From the minimality of $H$, $H - v$ has a 2-polar partition $P$. Analyzing the cases for the parts of $P$, it can be proved that one of the remaining components of $H$ is a clique, and the other one is a $(2,1)$-polar graph which is not a split graph. Until now, we have that one component contains an induced copy of either $2K_2$ or $C_4$, and there is at least one vertex in each of the remaining components of $H$, i.e., six vertices are completely determined. The rest of the proof is an analysis of cases for the remaining vertices. Since in the case when $H$ has nine vertices there are three remaining vertices, it has a more complex analysis, which actually, “includes” the case where there are only to vertices remaining.

**Lemma 8.** The disconnected cograph minimal 2-polar obstructions on 9 vertices are exactly $F_{21}, \ldots, F_{24}$, see Figure 3.
Proof. Let $H$ be a disconnected cograph minimal 2-polar obstruction on 9 vertices. If $H$ can be transformed by means of partial complementation into a graph with four components, then it follows from Lemma 4 that $H$ is one of $F_{21}, \ldots, F_{24}$. Otherwise, notice that we can obtain from $H$, through a sequence of partial complementations, a graph with three components, one of which is an isolated vertex. Thus, we may assume that $H$ has three components $B_1, B_2, B_3$, and $B_3$ is an isolated vertex.

Let $v$ be the isolated vertex of $H$. It follows from the minimality of $H$ that $H - v$ has a 2-polar partition $P = (V_1, V_2, V_3, V_4)$. Notice that $V_2 \neq \emptyset$, else, $(V_1 \cup \{v\}, V_2, V_3, V_4)$ is a 2-polar partition of $H$. Analogously, $V_1 \neq \emptyset$. Thus, $H[V_1 \cup V_2]$ is connected, and it should be contained in one of the two non-trivial components of $H$, say, $B_1$. Thus, $B_2$ is covered by one of the cliques of $P$, without loss of generality suppose that $V_3 = V(B_2)$. Note that $V_4 \neq \emptyset$, otherwise $(V_1, V_2, V_3, \{v\})$ is a 2-polar partition of $H$. Hence, $B_1$ is a $(2,1)$-polar graph, which is not a split graph, because $V_1$ and $V_2$ are both non empty.

Suppose first that $|V_3| \geq 2$. Since $B_1$ is not a split graph, it should contain an induced copy of $2K_2$ or an induced copy of $C_4$. The former case cannot occur, since such copy of $2K_2$ together with two vertices in $V_3$ and the vertex $v$ would induce a copy of $F_1$, contradicting the minimality of $H$. For the latter case, notice that $B_1$ has at least five vertices, because $V_4 \neq \emptyset$. Let $u$ be the fifth vertex of $B_1$ (not in $C_4$). Since $G$ is a cograph, $u$ should be adjacent to two antipodal vertices, three vertices, or four vertices in $C_4$. If it is adjacent to three or four vertices, then $H$ contains $F_7$ or $F_4$ as an induced subgraph, respectively. In the remaining case, if $|V_3| = 3$, then $B_1$ is complete bipartite, and $G$ admits a 2-polar partition.
If $|V_3| = 2$, then there is an additional vertex $u'$ in $B_1$. By the same argument as above, $u'$ should be adjacent to two antipodal vertices of $C_4$. If $u$ and $u'$ are adjacent to the same pair of vertices in $C_4$, and $u$ is not adjacent to $u'$, then $B_1$ is again a complete bipartite graph. If $uu' \in E(G)$, then $H$ contains an induced copy of $F_7$. Thus, $u$ and $u'$ should be adjacent to different pairs of vertices in $C_4$. Again, in order for $H$ to be a cograph we need $u$ to be adjacent to $u'$. But now, $B_1$ is isomorphic to $K_{3,3}$.

Consider now the case $|V_3| = 1$. Since $B_1$ is a connected cograph, it should be a join of two smaller cographs $T_1$ and $T_2$. If $T_i$ is a complete graph on at least two vertices for some $1 \leq i \leq 2$, then $\overline{B_1} + \overline{B_2} + B_3$ has at least four components, contradicting the choice of $H$. Thus, either $T_1$ and $T_2$ both contain an induced $P_3$, or we assume without loss of generality that $T_1$ consists of a single vertex. In the former case, we may assume without loss of generality that $T_1$ is isomorphic to $P_3$, and thus, $\overline{B_1} + \overline{B_2} + B_3$ has at least four components. In the latter case, $\overline{B_1} + \overline{B_2} + B_3$ has three components, one of them isomorphic to $K_2$, and one of them an isolated vertex, so we are in the case $|V_3| = 2$.

**Lemma 9.** The disconnected cograph minimal 2-polar obstructions on 8 vertices are exactly $F_6, \ldots, F_{20}$, see Figures 4 and 5.

**Proof.** Let $H$ be a disconnected cograph minimal 2-polar obstruction on 8 vertices. If $H$ can be transformed by means of partial complementation into a graph with four components, then $H$ is one of $F_{13}, \ldots, F_{20}$.

Otherwise, an argument analogous to the one used in Lemma 8 shows that $H$ can be transformed through a sequence of partial complementations into $F_7$ and hence it is one of $F_6, \ldots, F_{12}$.

We are now ready to state our two main results.

**Theorem 10.** There are exactly 48 cograph minimal 2-polar obstructions. All the disconnected cograph minimal 2-polar obstructions are $F_1, \ldots, F_{24}$, see Figures 2, 3, 4, and 5.

**Proof.** The result follows directly from Lemmas 7, 8, and 9.

**Theorem 11.** All cograph minimal 2-polar obstructions are $F_1, F_6, F_{13}, F_{21}$ and every graph obtained from these by partial complementation.

**Proof.** The result follows directly from Theorem 10 and Lemma 4.

The existence of the partial complementation operation substantially reduces the number of minimal obstructions we need to consider in order to characterize 2-polar cographs. It would be great to find natural operations preserving $k$-polarity for values of $k$ greater than 2.
Figure 4: Family A of cograph minimal 2-polar obstructions on 8 vertices.
Figure 5: Family B of cograph minimal 2-polar obstructions on 8 vertices.
5 Conclusions

We present a complete list (up to complementation) of cograph minimal 2-polar obstructions. As mentioned in the introduction, it is interesting to have this list for at least two reasons. First, now we have a list of no-certificates in the case we would like to obtain a certifying algorithm for recognition of 2-polar cographs. Second, now the complete list of cograph minimal obstructions are known for (1,1)-polarity, (2,1)-polarity, (1,2)-polarity, and (2,2)-polarity. From here, some observations can be made regarding the structure of cograph minimal \((s,k)\)-polar obstructions, e.g., it is often the case that adding disjoint copies of \(K_1\) or \(K_2\), or adding universal vertices in some components of a cograph minimal \((s,t)\)-polar obstruction, we obtain a “higher order” minimal obstruction. In fact, we were able to generalize each of our 24 disconnected cograph minimal 2-polar obstruction to a cograph minimal \(k\)-polar obstruction for any positive integer \(k\). This results in 24 families of graphs, each of which has as members precisely a cograph minimal \(k\)-polar obstruction for every \(k \geq 2\). Although even for \(k = 3\) this list fails to produce all the cograph minimal \(k\)-polar obstructions, we give it here because we think it is interesting to look at how these families grow.

Lemma 12. For every positive integer \(k \geq 2\), the corresponding element of each of the following families is a cograph minimal \(k\)-polar obstruction.

\[
\begin{align*}
\mathcal{F}_1 &= \{K_1 + (k+1)K_2: \ k \geq 2\}. \\
\mathcal{F}_2 &= \{C_4 + P_3 + (k-2)K_2: \ k \geq 2\}. \\
\mathcal{F}_3 &= \{F_3 + (k-2)K_2: \ k \geq 2\}. \\
\mathcal{F}_4 &= \{2K_2 + (k-1)K_1 + kK_1: \ k \geq 2\}. \\
\mathcal{F}_5 &= \{(k+1)K_2 + (k+1)K_1: \ k \geq 2\}. \\
\mathcal{F}_6 &= \{2P_3 + (k-1)K_2: \ k \geq 2\}. \\
\mathcal{F}_7 &= \{P_3 + K_2 + (k-1)K_2 + K_1: \ k \geq 2\}. \\
\mathcal{F}_8 &= \{2P_3 + kK_1: \ k \geq 2\}. \\
\mathcal{F}_9 &= \{(P_3 + K_2) \oplus K_1 + kK_1: \ k \geq 2\}. \\
\mathcal{F}_{10} &= \{(K_2 + (k-1)K_1 + K_2) \oplus (k-1)K_1: \ k \geq 2\}. \\
\mathcal{F}_{11} &= \{(2P_3 \oplus K_1) + (k-1)K_1: \ k \geq 2\}. \\
\mathcal{F}_{12} &= \{(K_1 \oplus (K_1 + (2K_1 \oplus (K_2 + K_1)))) + (k-1)K_1: \ k \geq 2\}. \\
\mathcal{F}_{13} &= \{C_4 + (k-1)K_2 + 2K_1: \ k \geq 2\}. \\
\mathcal{F}_{14} &= \{K_2 + kK_1 + kK_2: \ k \geq 2\}. 
\end{align*}
\]
In a work in progress, we analyze the structure of disconnected cograph minimal $k$-polar obstructions for any positive integer $k$. As one would expect, the number of cograph minimal $k$-polar obstructions grows fast in terms of $k$, so it is increasingly difficult to provide complete lists of minimal obstructions. Nonetheless, it looks possible to describe a few families of minimal obstructions that would completely classify all the cograph minimal $k$-polar obstructions; this is our next step.

References

[1] J.A. Bondy and U.S.R Murty, Graph Theory, Springer, Berlin, 2008.

[2] R.S.F. Bravo, L.T. Nogueira, F. Protti and C. Vianna, Minimal obstructions of $(2, 1)$-cographs with external restrictions, in: Annals of I ETC - Encontro de Teoria da Computação (CSBC 2016) (2016) Porto Alegre, [http://www.pucrs.br/edipucrs/] ISSN 2175-2761 (In Portuguese).

[3] Z.A. Chernyak and A.A. Chernyak, About recognizing $(\alpha, \beta)$-classes of polar graphs, Discrete Math. 62 (1986) 133–138.

[4] P. Damaschke, Induced subgraphs and well-quasi-ordering, Journal of Graph Theory 14(4) (1990) 427–435.

[5] T. Ekim, N.V.R. Mahadev and D. de Werra, Polar cographs, Discrete Applied Mathematics 156 (2008) 1652–1660.

[6] A. Farrugia, Vertex-partitioning into fixed additive induced-hereditary properties is $NP$-hard, Electron. J. Combin. 11 (2004) #R46.
[7] T. Feder, P. Hell, S. Klein and R. Motwani, List Partitions, SIAM J. Discrete Math. 16(3) (2003) 449–478.

[8] T. Feder, P. Hell and W. Hochstädtler, Generalized Colourings (Matrix Partitions) of Cographs, in: Graph Theory in Paris, Birkhauser, 2006, 149–167.

[9] S. Foldes and P.L. Hammer, Split graphs, in: Proc. 8th Sout-Eastern Conf. on Combinatorics, Graph Theory and Computing, 1977, 311–315.

[10] P. Hell, Graph partitions with prescribed patterns, European Journal of Combinatorics 35 (2014) 335–353.

[11] V.B. Le, R. Nevries, Complexity and algorithms for recognizing polar and monopolar graphs, Theoretical Computer Science 528 (2014) 1–11.