EINSTEIN WITH SKEW-TORSION CONNECTIONS
ON BERGER SPHERES

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Abstract. The invariant metric affine connections on Berger spheres which are Einstein with skew torsion are determined in both Riemannian and Lorentzian signature. Expressions of such connections are explicitly given. In particular, every Berger sphere with Lorentzian signature admits invariant metric affine connections which are Einstein with skew-torsion up to $S^3$. For Riemannian signature, the existence of such connections strongly depends on the dimension of the sphere and on the scale of the deformation used for the Berger metric. In particular, there are Riemannian Berger spheres, not Einstein, which admit invariant Einstein with skew-torsion affine connections.

1. Introduction

Marcel Berger classified the simply connected normal homogeneous Riemannian manifolds with strictly positive sectional curvature [4] (the Berger list was incomplete, missing one example, nowadays called the Aloff-Walach space $\mathbb{M}_{1,1}$). In that paper, Berger found a family of 3-dimensional Riemannian manifolds diffeomorphic to the 3-sphere with strictly positive sectional curvature but not constant. These metrics can be obtained as a deformation of the round metric on $S^3$ by changing the relative scale of the fibers of the Hopf fibration (see details in Section 2). These metrics, called now the Berger metrics, provided counterexamples to several geometric conjectures. For example, Klingenberg proved that for an orientable even-dimensional manifold $M$ whose sectional curvatures $K_M$ lie in the interval $[k_1, k_2]$ with $k_1 > 0$, every closed geodesic in $M$ has length at least $2\pi/\sqrt{k_2}$, while Berger metrics show that such a result does not hold in general (see [10, Chap. 3 and 5] for details and more references). Similar metrics for all odd-dimensional spheres were constructed by I. Chavel [9], and A. Weinstein realized these metrics as distance spheres in the complex projective space $\mathbb{C}P^n$. On the other hand, the first nonzero eigenvalue of the Laplace operator of Berger spheres shows surprising properties which were first pointed out by H. Urakawa [16]. Later, a nice description and extension of these properties from the submersion theory perspective was obtained in [5], where the reader will find more details and explanations about these facts.

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From another point of view, the study of submanifolds immersed in Berger spheres has also been an active research field. Recall, for example, the works on Willmore surfaces [3] and minimal surfaces [15]. Moreover, for suitable scales of the fibers of the Hopf fibration, we can also consider the Berger spheres with Lorentzian signature (see Section 2 for details). Lorentzian Berger spheres were studied in [12], where several properties of the conjugate points along their lightlike geodesics were obtained. From the point of view of dynamical systems, we would like to mention the recent interest in the study of magnetic curves in Berger spheres [13].

The Berger spheres (both Riemannian or Lorentzian signature) are odd-dimensional spheres $S^{2n+1}$ endowed with a SU($n+1$)-invariant metric $g_\varepsilon$ and so with constant sectional curvature but not Einstein, in general. Among the generalizations of the Einstein condition, this paper will focus on the notion of Einstein manifold with skew-torsion as was introduced in [2].

An Einstein manifold with skew-torsion is a triple $(M, g, \nabla)$ where $(M, g)$ is a semi-Riemannian manifold endowed with a metric affine connection $\nabla$ which shares geodesics with the Levi-Civita connection and such that the symmetric part of the Ricci tensor of $\nabla$ is a multiple of the metric tensor $g$. In order not to be so long, we recommend the complete and nice survey [1], which explains the role of the affine connections with torsion in Mathematics and Physics and provides a wide family of examples of connections with torsion in different contexts. In this paper, we explicitly determine and describe the SU($n+1$)-invariant affine connections $\nabla$ on the Berger spheres in such a way that $(S^{2n+1}, g_\varepsilon, \nabla)$ is an Einstein manifold with skew-torsion. According to our conventions, for every $\varepsilon \neq 0$ the corresponding Berger sphere $(S^{2n+1}, g_\varepsilon)$ is a Riemannian manifold for $\varepsilon < 0$ and Lorentzian for $\varepsilon > 0$. This paper can be seen as a continuation of [11], which studies the case of the usual round Riemannian metric of constant sectional curvature 1, which corresponds to the value $\varepsilon = -1$. It is proved in [11] that, in this case, there are always Einstein connections with skew-torsion, although only for $n = 3$ and $n = 1$ there are nontrivial solutions, that is, different from the Levi-Civita connection. Now the obtained results are qualitatively different: take into account that the Levi-Civita connection is not longer an Einstein connection with skew-torsion (for $\varepsilon \neq -1$), so any solution is nontrivial, and moreover, there are solutions for all the odd dimensions except for $S^3$. In order to be precise, let us enunciate the main result.

**Theorem 1.** Let $S^{2n+1}$ be the odd-dimensional sphere viewed as homogeneous space $SU(n+1)/SU(n)$ and endowed with the invariant Berger metric $g_\varepsilon$ determined by Eq. (3). Take $\Phi, \eta, \psi, \Theta, \tilde{\Theta}, \hat{\psi}$ the invariant tensors given by Eqs. (5), (7) and (9). Let us denote by $\nabla^{g_\varepsilon}$ the Levi-Civita connection of $g_\varepsilon$. Then, for the different values of $n$, we have

\[
\nabla_X Y = \nabla^{g_\varepsilon}_X Y + s (\Phi(X, Y) \xi + \varepsilon (\eta(X) \psi(Y) - \eta(Y) \psi(X))
\]

for any $s \in \mathbb{R}$.

- If $n = 1$, they are Einstein with skew-torsion affine connections if and only if $\varepsilon = -1$. In this case, every choice of the parameter $s$ gives such a connection.
- If $n > 3$, they are Einstein with skew-torsion affine connections if and only if

\[
s^2 = \left( \frac{n+1}{n-1} \right) \left( \frac{\varepsilon + 1}{\varepsilon} \right).
\]
The invariant metric connections with skew-torsion are:

\[ \nabla_X Y = \nabla_X^{g_s} Y + s\left(\Phi(X, Y) \xi + \varepsilon(\eta(X)\psi(Y) - \eta(Y)\psi(X))\right) + s_1 \Theta(X, Y) + s_2 \Theta'(X, Y) \]

for any \( s, s_1, s_2 \in \mathbb{R} \). They are Einstein with skew-torsion affine connections if and only if

\[ \varepsilon s^2 + s_1^2 + s_2^2 = 2(\varepsilon + 1). \]

\( n = 3 \)
The invariant metric connections with skew-torsion are:

\[ \nabla_X Y = \nabla_X^{g_s} Y + s\left(\Phi(X, Y) \xi + \varepsilon(\eta(X)\psi(Y) - \eta(Y)\psi(X))\right) + s_3 \left(\Phi(\hat{\psi}(X), Y) \xi + \varepsilon(\eta(X)\hat{\psi}(Y)) - \eta(Y)\hat{\psi}(X)\right) + s_4 \left(-g_\varepsilon(\hat{\psi}(X), Y) \xi + \varepsilon(\eta(X)\hat{\psi}(Y)) - \eta(Y)\hat{\psi}(X)\right) \]

for any \( s, s_3, s_4 \in \mathbb{R} \). They are Einstein with skew-torsion affine connections if and only if

\[ s^2 + s_3^2 + s_4^2 = \frac{3\varepsilon + 1}{\varepsilon}. \]

In particular, we recover the above mentioned well-known result (see [6, 9.81] for a completely different proof), asserting that the unique Einstein Riemannian metric in the family \( g_\varepsilon \) with \( \varepsilon < 0 \) is the canonical round metric \( g_{-1} \) of constant sectional curvature 1. We also note that, up to \( S^5 \), the contraction of the 3-differential forms corresponding with the torsions (see Section 2) are exactly the closed 2-forms used in [13] to define magnetic curves.

As a direct corollary of Theorem 1 we can analyze the existence of Einstein with skew-torsion affine connections according to the values of \( \varepsilon \) and \( n \) (see Table 1).

For Lorentzian signature (\( \varepsilon > 0 \)), the Berger sphere \((S^{2n+1}, g_\varepsilon)\) always admits a SU\((n+1)\)-invariant Einstein with skew-torsion affine connection up to \( n = 1 \).

For \( n \in \{2, 3\} \), the set of such connections is parametrized by the points of an ellipsoid while reduces to just two connections for \( n \geq 4 \).

This is already interesting, since Lorentzian metrics \( g_\varepsilon \) are never Einstein (in the usual sense). Thus, there would be another natural choice for a distinguished affine connection in the Lorentzian odd-dimensional spheres of dimension at least 9.

For Riemannian signature (\( \varepsilon < 0 \)), the Berger sphere \((S^3, g_\varepsilon)\) admits SU\((2)\)-invariant Einstein with skew-torsion affine connections only for \( \varepsilon = -1 \), and the set of such connections is parametrized by a line. For \((S^5, g_\varepsilon)\), the situation depends on \( \varepsilon \); for \( \varepsilon > -1 \) there is no such connection, for \( \varepsilon = -1 \) there is only one and for \( \varepsilon < -1 \) the set of such connections is parametrized by an ellipsoid. For \((S^7, g_\varepsilon)\), the set of SU\((4)\)-invariant Einstein with skew-torsion affine connections is parametrized by a one-sheet hyperboloid if \( \varepsilon > -1 \), by a cone if \( \varepsilon = -1 \) and by a two-sheets hyperboloid if \( \varepsilon < -1 \). Finally, for \( n > 3 \), \((S^{2n+1}, g_\varepsilon)\) admits a SU\((n+1)\)-invariant Einstein with skew-torsion affine connection only for \( \varepsilon \leq -1 \); there is only one if \( \varepsilon = -1 \) and exactly two if \( \varepsilon < -1 \).

The sphere \( S^7 \) is exceptional (among other properties) from this point of view. In fact, \( S^7 \) is the unique sphere such that all Riemannian metrics \( g_\varepsilon \) admit invariant Einstein with skew-torsion affine connections. In the opposite position we find \( S^3 \), which admits invariant Einstein with skew-torsion affine connections only for the round metric.
In our view, we can divide the Riemannian Berger metrics with similar behaviors into three groups: the round metric (alone), those metrics with distort enlarging the vertical direction, and those ones shortening the vertical direction.

The structure of the paper is the following: We recall in Section 2 the algebraic tools to work with invariant affine connections in a reductive homogeneous space, mainly Nomizu’s Theorem. We also describe the sphere as SU(n+1)-homogeneous space and list the invariant tensors which will need for the concrete descriptions of the connections in Theorem 1. As this paper continues the work in [11], we have repeated only the necessary material to achieve and write down the results. Section 3 contains the proof of Theorem 1 in a schematic way.

2. Preliminaries.

2.1. The odd-dimensional spheres as homogeneous spaces for the special unitary group and algebraic tools. Throughout this paper we consider the sphere $M = S^{2n+1} \subset \mathbb{C}^{n+1}$ as homogeneous space under the action of the special unitary group $SU(n+1)$. Invariant affine connections on $M$ have been studied on [11], where those ones compatible with the round (Riemannian) metric have been explicitly described, and more precisely, being besides Einstein with skew-torsion in the sense of [2]. Part of the results of [11] are the starting point now. We briefly recall the situation.

The special unitary group $G = SU(n+1) = \{ A \in \mathcal{M}_{n+1}(\mathbb{C}) : A^t A = I_{n+1}, \det(A) = 1\}$ acts smoothly on

$$M = S^{2n+1} = \{(z_1, \cdots, z_{n+1})^t \in \mathbb{C}^{n+1} : \sum_{i=1}^{n+1} z_i \bar{z}_i = 1\},$$

by matrix multiplication. As the action is transitive, and the isotropy group of the point $o = (0, \cdots, 0, 1)^t \in M$ is $H_o = \{(\begin{bmatrix} B & \bar{0} \\ 0 & 0 \end{bmatrix} : B \in SU(n)\}$, the sphere $S^{2n+1}$ is diffeomorphic to the homogeneous space $G/H = SU(n+1)/SU(n)$. Moreover, this homogeneous space is reductive, that is, if $g$ is the Lie algebra of $G$ and $h$ the Lie algebra of (the connected group) $H$, there is a complementary subspace $m$ of $h$ such that $[h, m] \subset m$. There is always a natural identification of $m$ with the tangent space at $o = \pi(e)$, provided by the differential of the projection $\pi : G \to M = G/H$, which gives a linear isomorphism $(\pi_*)_{e|m} : m \to T_oM$. For our precise example, $g = su(n+1)$, $h = \{\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} : B \in su(n)\}$ and the complementary $m$ is unique, namely,

$$m = \left\{\begin{bmatrix} -\frac{a I_n}{z} \\ z \end{bmatrix} \in \mathcal{M}_{n+1}((\mathbb{C}) : z \in \mathbb{C}^n, a \in \mathbb{R}i\right\}.\]$$

Under the natural identifications $h \cong su(n)$ and $m \cong \mathbb{C}^n \oplus \mathbb{R}i$, the action $[h, m] \subset m$ becomes $B \cdot (z, a) = (Bz, 0)$.

The study of invariant affine connections in reductive homogeneous spaces can be translated to an algebraic setting thanks to Nomizu’s Theorem (cf. [14]).

**Theorem 2.** Let $G/H$ be a reductive homogeneous space with a fixed reductive decomposition $g = h \oplus m$ and $H$ connected. Then, there is a bijective correspondence between the set of $G$-invariant affine connections on $G/H$ and the vector space of bilinear maps $\alpha : m \times m \to m$ such that

$$[h, \alpha(X, Y)] = \alpha([h, X], Y) + \alpha(X, [h, Y])$$
for all $X,Y \in \mathfrak{m}$ and $h \in \mathfrak{h}$.

Moreover, given a $G$-invariant affine connection $\nabla$ and $\alpha$ the associated bilinear map, the torsion and curvature tensors of $\nabla$ are given by

\begin{align*}
(1) \quad T_\alpha(X,Y) &= \alpha(X,Y) - \alpha(Y,X) - [X,Y]_\mathfrak{m}, \\
(2) \quad R_\alpha(X,Y,Z) &= \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_\mathfrak{m}, Z) - [[X,Y]_\mathfrak{h}, Z],
\end{align*}

for any $X,Y,Z \in \mathfrak{m}$, where $[\ , \ ]_\mathfrak{h}$ and $[\ , \ ]_\mathfrak{m}$ denote the composition of the bracket $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{g}$ with the projections of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ on each factor.

These expressions give the torsion $T$ and the curvature $R$ at the point $o$ (by means of the above identification of $\mathfrak{m}$ with $T_oM$ via $\pi_o$). The invariance permits to recover the whole tensors.

In particular, there is also a bijective correspondence among the set of $G$-invariant affine connections on $G/H$ and the vector space of homomorphisms of $\mathfrak{h}$-modules $\text{Hom}_\mathfrak{h}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, where the morphism related to $\alpha$ is given by $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, $X \otimes Y \mapsto \alpha(X,Y)$.

The vector space $\text{Hom}_\mathfrak{h}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ has been studied in [11] for our spheres. Concrete bases are exhibited in [11] Eqs. (25), (31), §6.1 and §7, and the dimension turns out to be 7, 9, 13 and 27 for $n \geq 4$, $n = 3$, $n = 2$ and $n = 1$ respectively.

If a reductive homogeneous space $G/H$ is endowed with a $G$-invariant metric $g$, there seems natural to ask for the invariant connections to be also compatible with the metric ($\nabla g = 0$). In such a case, the affine connection is determined by the torsion. Nomizu’s theorem can be also adapted to this setting. For instance from [11] Theorem 2.7, Remark 2.8], if we also denote by $g: \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ the $\mathfrak{h}$-invariant nondegenerate symmetric bilinear map induced by $g|_{T_oM}$, then we have

**Lemma 1.** A $G$-invariant affine connection is metric if and only if $\alpha(X, -) \in \mathfrak{so}(\mathfrak{m}, g)$ for all $X \in \mathfrak{m}$.

In other words, there is a bijective correspondence between the set of $G$-invariant affine connections compatible with the invariant metric $g$ on $M = G/H$ and the vector space $\text{Hom}_\mathfrak{h}(\mathfrak{m}, \wedge^2 \mathfrak{m})$, taking into account that $\wedge^2 \mathfrak{m}$ and $\mathfrak{so}(\mathfrak{m}, g)$ are isomorphic $\mathfrak{h}$-modules. In particular, the set of $G$-invariant metric affine connections is parametrized by a number of parameters which does not depend on the concrete choice of the invariant metric.

Let us return to the odd-dimensional spheres $\mathbb{S}^{2n+1} = \frac{\text{SU}(n+1)}{\text{SU}(n)}$. The Berger metric $g_\varepsilon$ for each $0 \neq \varepsilon \in \mathbb{R}$ can be introduced as follows. The bilinear map

\begin{equation}
(3) \quad g_\varepsilon: \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad g_\varepsilon((z,a),(w,b)) := \text{Re}(z^t\bar{w}) + \varepsilon ab
\end{equation}

is an $\mathfrak{h}$-invariant nondegenerate symmetric bilinear map, so that it determines an invariant metric $g_\varepsilon \in \mathcal{T}^{0,2}(M)$. The round Riemannian metric used in [11] corresponds with $\varepsilon = -1$. In general, when $\varepsilon < 0$, the obtained metric $g_\varepsilon$ is Riemannian, while $g_\varepsilon$ is Lorentzian for $\varepsilon > 0$. Indeed, $g_\varepsilon|_{\mathfrak{c}^0}$ is positive definite, independently of $\varepsilon$, and the index is determined by $g_\varepsilon((0,i),(0,i)) = -\varepsilon$. Moreover this family $\{g_\varepsilon : \varepsilon \neq 0\}$ exhausts the set of all invariant metrics up to nonzero scalar multiple, as explained in [11] Remark 3.1.

Now we can use, for any of the metrics $g_\varepsilon$, the results in [11] Prop. 4.4, Prop. 5.3, Prop. 6.3, §7, according to which

$$\dim \text{Hom}_{\text{SU}(n)}(\mathfrak{m}, \wedge^2 \mathfrak{m}) = 3, 5, 7, 9$$
respectively for \( n \geq 4, n = 3, n = 2 \) and \( n = 1 \). Anyway, we will find a concrete description of the metric connections without using the previous knowledge of the dimension of the vector space.

We want to find connections as good as possible. At this point we recall that a semi-
Riemannian manifold \((M, g)\) endowed with a metric affine connection \( \nabla \) is called \textit{Einstein with skew-torsion} (see [2] for a variational justification of this notion) if: \( \nabla \) has totally skew-
symmetric torsion, that is, \( \omega(X, Y, Z) := g(T(X, Y), Z) \) is a 3-form on \( M \) for \( T \) the usual
torsion tensor; and the symmetric part of the Ricci tensor satisfies

\[
\text{Sym}(\text{Ric}^{\nabla}) = \frac{s^{\nabla}}{\dim M} g,
\]

where \( s^{\nabla} \) denotes the scalar curvature. We will also say that \( \nabla \) is an Einstein with skew-
torsion connection. In the above setting, the condition of having skew-symmetric torsion is
a linear condition on the parameters of the vector space \( \text{Hom}_{\text{SU}(n)}(m, \wedge^2 m) \), and the obtained
maps can be proved to be in one-to-one correspondence with \( \text{Hom}_0(\wedge^3 m, \mathbb{R}) \). On the contrary,
Eq. (4) provides a set of quadratic equations on these parameters, so that the set of solutions
could be empty.

In what follows, we occasionally write connections, metric connections and metric connections
with skew-torsion for referring to the bilinear maps related to those connections through
Nomizu’s Theorem.

### 2.2. Invariant tensors

We will derive concrete expressions for the \( \text{SU}(n+1) \)-invariant affine
connections using some invariant tensors on \( M = S^{2n+1} \). For most of the cases \( (n \neq 2, 3) \),
the connections will be described using only the usual Sasakian structure on \( S^{2n+1} \). We recall
briefly here the involved tensors (see [7]). For \( X \in \mathfrak{X}(S^{2n+1}) \), the decomposition of \( iX \)
in tangent and normal components defines the \((1,1)\)-tensor field \( \psi \) and the 1-differential form \( \eta \)
as follows,

\[
iX = \psi(X) + \eta(X) N,
\]

where \( N \) denotes the unit outward normal vector field to \( S^{2n+1} \). Let \( \xi \in \mathfrak{X}(S^{2n+1}) \) be the
Killing vector field (for all \( g_\varepsilon \)) defined by \( \xi_\varepsilon = -iz \) at any \( z \in S^{2n+1} \). The Sasakian form
\( \Phi \in \Omega^2(S^{2n+1}) \) is defined by \( \Phi(X, Y) = g_\varepsilon(X, \psi(Y)) \) (note that \( \Phi \) is independent of \( \varepsilon \)).

The tensor fields \( \psi, \eta \) and \( \Phi \) are \( \text{SU}(n+1) \)-invariant [11, Lemma 3.2], so that they are
determined by the value at the point \( o \in M \). Note that, if \( X = (z, a), Y = (w, b), Z = (u, c) \in m = T_oM \cong \mathbb{C}^n \oplus \mathbb{R}i \), then

\[
\psi(X) = (iz, 0) \in m, \quad \eta(X) = ia \in \mathbb{R},
\xi = (0, -i) \in m, \quad \Phi(X, Y) = g_\varepsilon(X, \psi(Y)) = -\text{Im}(\overline{z}w) \in \mathbb{R}.
\]

In the case \( n = 3 \), the geometric structure of the sphere \( S^7 \) is richer. In fact, \( S^7 \) is a
3-Sasakian manifold as follows. If we realize \( S^7 \) as a hypersurface of \( \mathbb{H}^2 \) by means of
\( S^7 \subset \mathbb{C}^4 \cong \mathbb{H}^2, \quad z = (z_1, z_2, z_3, z_4) \mapsto (z_1 + z_2 j, z_3 + z_4 j) \),
we can consider the vector fields \( \xi_1, \xi_2, \xi_3 \in \mathfrak{X}(S^7) \) given by \( \xi_1(z) = -iz, \xi_2(z) = -jz \) and
\( \xi_3(z) = -kz \), respectively. For any vector field \( X \in \mathfrak{X}(S^7) \), the decompositions of \( iX, jX \) and
\( kX \) into tangent and normal components determine as above the \((1,1)\)-tensor fields \( \psi_1, \psi_2 \)
and \( \psi_3 \) and the differential 1-forms \( \eta_1, \eta_2 \) and \( \eta_3 \) on \( S^7 \), respectively. The three Sasakian structures \( (\xi, \eta, \psi_i) \) are compatible in the sense that (indices modulo 3)
\[
[\xi_i, \xi_{i+1}] = 2\xi_{i+2}, \quad \psi_{i+1} \circ \psi_i = -\psi_{i+1} + \eta_i \otimes \xi_{i+1}, \quad \psi_i \circ \psi_{i+1} = \psi_{i+2} + \eta_{i+1} \otimes \xi_i.
\]
The Sasakian structure \( (\xi, \eta, \psi_1) \) is SU(4)-invariant, but this is not the case for the other two indices. Fortunately, it is found in [11, Proposition 5.9] the following SU(4)-invariant linear map \( \hat{\psi} \) almost contact metric structure \( (\xi, \eta, \psi, \hat{J}) \) the is SU(3)-invariant (but not Sasakian). The sphere \( S^5 \) in \( \mathbb{R}^7 \) with unit outward normal vector field \( N \), and the almost complex structure \( J \) on \( S^6 \) defined by \( J(X) = N \times X \) for every \( X \in \mathfrak{X}(S^6) \), for \( x \) a cross product in \( \mathbb{R}^7 \). Thus, \( (S^6, g_{-1}, J) \) is a nearly Kähler manifold [7]. The sphere \( S^5 \) can be totally geodesic embedded in \( S^6 \) at the equator \( (x_7 = 0) \). For every \( X \in \mathfrak{X}(S^5) \), the decomposition of \( J(X) \) in tangent and normal components with respect to the above isometric embedding \( S^5 \subset S^6 \) determines the \( (1, 1) \)-tensor field \( \psi \) and (again) the \( 1 \)-differential form \( \eta \) on \( S^5 \) such that
\[
\Theta(X, Y, Z) = -\operatorname{Re}(\det(z, w, u)), \quad \Omega(X, Y, Z) = \operatorname{Im}(\det(z, w, u));
\]
that gives
\[
\Theta(X, Y) = -(\bar{z} \times \bar{w}, 0), \quad \widetilde{\Theta}(X, Y) = (i(\bar{z} \times \bar{w}), 0).
\]

Finally the sphere \( S^5 \) has a distinguished invariant tensor too. Let us consider the sphere \( S^6 \) in \( \mathbb{R}^7 \) with unit outward normal vector field \( N \), and the almost complex structure \( J \) on \( S^6 \) defined by \( J(X) = N \times X \) for every \( X \in \mathfrak{X}(S^6) \), for \( x \) a cross product in \( \mathbb{R}^7 \). Thus, \( (S^6, g_{-1}, J) \) is a nearly Kähler manifold [7]. The sphere \( S^5 \) can be totally geodesic embedded in \( S^6 \) at the equator \( (x_7 = 0) \). For every \( X \in \mathfrak{X}(S^5) \), the decomposition of \( J(X) \) in tangent and normal components with respect to the above isometric embedding \( S^5 \subset S^6 \) determines the \( (1, 1) \)-tensor field \( \psi \) and (again) the \( 1 \)-differential form \( \eta \) on \( S^5 \) such that
\[
J(X) = \hat{\psi}(X) + \eta(X)\nu,
\]
where \( \nu \) is the unit normal vector field along the above totally geodesic embedding [7]. The almost contact metric structure \( (\xi, \eta, \hat{\psi}) \) on \( S^5 \) is SU(3)-invariant (but not Sasakian). The linear map \( \hat{\psi}_0: T_0 S^5 \to T_0 S^5 \) corresponds under the identification \( \pi_+: m \to T_0 S^5 \) with the endomorphism of \( m \cong \mathbb{C}^2 \oplus \mathbb{R}i \) given by
\[
\hat{\psi}(z, a) = (\theta(z), 0),
\]
for \( \theta: \mathbb{C}^2 \to \mathbb{C}^2 \) the \( \mathfrak{h} = \mathfrak{su}(2) \)-homomorphism given by \( \theta \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c} -\bar{z}_2 \\ \bar{z}_1 \end{array} \right) \) [11].

3. Proof of the Main Theorem

For brevity, we take \( X = (z, a), Y = (w, b) \) and \( Z = (u, c) \) arbitrary elements in \( m \).

3.1. Case \( S^{2n+1} \cong \frac{\mathbb{SU}(n+1)}{\mathbb{SU}(n)} \) for \( n \geq 4 \). Recall from [11, Proposition 4.3] that an \( \mathbb{R} \)-bilinear map \( \alpha: m \times m \to m \) \( \mathfrak{h} \)-invariant if and only if there exist \( q_1, q_2, q_3 \in \mathbb{C} \) and \( t \in \mathbb{R} \) such that, for any \( X, Y \in m \),
\[
\alpha(X, Y) = (q_1 b z + q_2 a w, i (t a b + \text{Im}(q_3 z b w))).
\]

We easily check that \( \alpha \in \mathfrak{so}(m, g_\epsilon) \) when \( t = \text{Im}(q_2) = 0 \) and \( q_1 = \epsilon q_3 \), so that, by renaming, there are \( q \in \mathbb{C} \) and \( t \in \mathbb{R} \) such that
\[
\alpha(X, Y) = (-\epsilon q b z + t a w, -i \text{Im}(q \bar{z} w)).
\]
The torsion tensor as in Eq. (11) is

\[ T_\alpha(X,Y) = \left( -\varepsilon q - t - \frac{n+1}{n} \right) (bz - aw), (\text{Re}(q) - 1) (\overline{w^\prime}z - \overline{\xi} w) \] ,

which vanishes for \( q = 1, t = -\varepsilon - \frac{n+1}{n} \). This means that the Levi-Civita connection of \( g_\varepsilon \) corresponds to

\[ \alpha_{g_\varepsilon}(X,Y) = \left( -\varepsilon bz - (\varepsilon + \frac{n+1}{n}) aw, -i \text{Im}(\overline{\xi} w) \right) . \]

If we compute the \((0,3)\)-tensor \( \omega_\alpha \) related to \( T_\alpha \) in (11), it turns out to be totally skew-symmetric if and only if \( q \in \mathbb{R} \) and \( t = \varepsilon q - \frac{n+1}{n} - 2\varepsilon \). In this case,

\[ (\alpha - \alpha_{g_\varepsilon})(X,Y) = (1 - q) (\varepsilon bz - aw), i \text{Im}(\overline{\xi} w) \]

and

\[ \omega_\alpha(X,Y,Z) = 2\varepsilon(1 - q) \text{Re}(aw^t\overline{\xi} + bz^t\overline{u} + cw^t\overline{z}). \]

Following Eq. (12), we tediously compute the curvature tensor of the map \( \alpha \) as in (13) to get

\[ R_\alpha(X,Y,Z) = \left( \frac{\varepsilon q^2}{2}(z(\overline{w}^t u - w^t \overline{u}) + w(z^t \overline{u} - \overline{z}^t u)) + z(\overline{w}^t u) - w(\overline{z}^t u) \right) \]

\[ + (-\varepsilon q + 2\varepsilon + 1)(u(\overline{z}^t u - \overline{z}^t u^t) + \varepsilon(q^2 - 2q)c(bz - aw), \]

\[ - \frac{1}{2}\varepsilon(q^2 - 2q)((\overline{z}^t u + z^t u^t)b - (\overline{w}^t u + w^t u^t)a) \].

This allows us to compute the corresponding Ricci tensor

\[ \text{Ric}_\alpha(X,Y) = 2(\varepsilon(q^2 - 2q + 2) + n + 1) \text{Re}(z^t\overline{\xi} + 2n\varepsilon(q^2 - 2q)ab, \]

which is always symmetric. Thus, the Ricci tensor is a scalar multiple of the metric \( g_\varepsilon \) if and only if

\[ \varepsilon(q^2 - 2q + 2) + n + 1 = n\varepsilon(q^2 - 2q), \]

or equivalently if

\[ (q - 1)^2 = \frac{2\varepsilon + n + 1}{\varepsilon(n - 1)} + 1 = \frac{(\varepsilon+1)(n+1)}{\varepsilon(n - 1)}. \]

This gives two solutions whenever \( \varepsilon(\varepsilon+1) > 0 \) and one solution for \( \varepsilon = -1 \). This proves item \( n > 3 \) in Theorem 11 simply by taking into account Eq. (15), which implies

\[ \Phi(X,Y) z + \varepsilon(\eta(X)\psi(Y) - \eta(Y)\psi(X)) = (\varepsilon(bz - aw), i \text{Im}(\overline{\xi} w)), \]

at \( o \in S^{2n+1} \) under the identification \( T_\varepsilon S^{2n+1} \cong \mathfrak{m} \cong \mathbb{C}^n \oplus \mathbb{R} i \).

3.2. Case \( S^7 \cong SU(4)/SU(3) \). Again recall from [11 Eq. (31)] that an \( \mathbb{R} \)-bilinear map \( \beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) is \( h \)-invariant if and only if there exist \( q_1, q_2, q_3, q_4 \in \mathbb{C} \) and \( t \in \mathbb{R} \) such that, for any \( X, Y \in \mathfrak{m}, \)

\[ \beta(X,Y) = (q_1 bz + q_2 aw + q_4 \overline{z} \times \overline{w}, i (tb + \text{Im}(q_3 \overline{\xi} w))) \].

The conditions to assure \( \beta \in \mathfrak{so}(m, g_\varepsilon) \) are \( t = \text{Im}(q_2) = 0 \) and \( q_1 = \varepsilon q_3 \) as in the previous case, so that, by renaming, there are \( q, p \in \mathbb{C} \) and \( t \in \mathbb{R} \) such that

\[ \beta(X,Y) = \left( -\varepsilon q bz + t aw + p \overline{z} \times \overline{w}, -i \text{Im}(q \overline{\xi} w) \right). \]

The torsion tensor \( T_\beta \) is related to that one in Eq. (11) by

\[ (T_\beta - T_\alpha)(X,Y) = (2p \overline{z} \times \overline{w}, 0), \]

which vanishes for \( q = 1, t = -\varepsilon - \frac{n+1}{n}, p = 0, \) and the Levi-Civita connection is given again by Eq. (12).
The conditions for having skew-torsion are still $q \in \mathbb{R}$ and $t = \varepsilon q - \frac{n+1}{n} - 2\varepsilon$, for arbitrary $p$. In this case,

$$
(\beta - \alpha_{g_\varepsilon})(X, Y) = s \left( \varepsilon (b z - a w), i \text{Im}(\overline{z} w) \right) + (p \overline{z} \times \overline{w}, 0),
$$

for $s = 1 - q \in \mathbb{R}$ and $p \in \mathbb{C}$, and the 3-form becomes

$$
\omega_\beta(X, Y, Z) = 2\varepsilon s \text{Re}(a u^t \bar{w} + b z^t \bar{u} + c w^t \bar{z}) + 2\text{Re}(\bar{p} \text{det}(z, w, u)).
$$

The curvature tensor is related to that one in Eq. (14) by

$$
(R_\beta - R_\alpha)(X, Y, Z) = \left( (2\varepsilon q - 4\varepsilon - 4)p(a \bar{w} \times \bar{u}) - b(z \times u) \right) + 2\varepsilon q \text{Re}(z^t \bar{w}) + \bar{p} \text{Re}(z^t \bar{w} + \bar{w} \times (z \times u)),
$$

and the Ricci tensor

$$
(\text{Ric}_\beta - \text{Ric}_\alpha)(X, Y) = -4\bar{p} \text{Re}(z^t \bar{w}),
$$

so that Eq. (15) gives

$$
\text{Ric}_\beta(X, Y) = 2(\varepsilon (q^2 - 2q + 2) + 4 - 2\bar{p}\bar{p})\text{Re}(z^t \bar{w}) + 6\varepsilon^2 (q^2 - 2q)ab,
$$

which is always symmetric. Thus, the Ricci tensor is a scalar multiple of the metric $g_\varepsilon$ if and only if

$$
\varepsilon(s^2 + 1) + 4 - 2\bar{p}\bar{p} = 3\varepsilon(s^2 - 1),
$$

or equivalently $\varepsilon s^2 + \bar{p}\bar{p} = 2(\varepsilon + 1)$, which gives item $n = 3$ in Theorem 1 for the choice $s_1 = -\text{Re}(p)$ and $s_2 = \text{Im}(p)$, by taking into account Eq. (3).

3.3. Case $S^5 \cong SU(3)/SU(2)$. Now an arbitrary $h$-invariant bilinear map $\gamma: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by

$$
\gamma(X, Y) = \left( q_1 b z + q_2 a w + p_1 b\theta(z) + p_2 a\theta(w), i t a b + i \text{Im}(q_3 \bar{z} w + p_3 \theta(z)^t) \right)
$$

for some $q_1, q_2, q_3, p_1, p_2, p_3 \in \mathbb{C}$ and $t \in \mathbb{R}$ [11, §6.1], where the map $\theta: \mathbb{C}^2 \to \mathbb{C}^2$ is defined after Eq. (10).

Those ones compatible with $g_\varepsilon$ form a 7-dimensional vector space, given by the relations $t = \text{Im}(q_2) = 0$, $q_1 = \varepsilon q_3$, $p_1 = \varepsilon p_3$, so that, by renaming, there are $q, p, p_2 \in \mathbb{C}$ and $t \in \mathbb{R}$ such that

$$
\gamma(X, Y) = \left( -\varepsilon q b z + t a w - \varepsilon p b\theta(z) + p_2 a\theta(w), -i \text{Im}(q \bar{z} w + \bar{p} \theta(z)^t) \right).
$$

The torsion tensor $T_\gamma$ is given by

$$
(T_\gamma - T_\alpha)(X, Y) = \left( (-\varepsilon q - p_2)(b\theta(z) - a\theta(w)), -2i \text{Im}(\bar{p} \theta(z)^t) \right),
$$

which vanishes for $q = 1, t = -\varepsilon - \frac{3}{2}, p = p_2 = 0$. This means that the Levi-Civita connection corresponds again with Eq. (12). Now $\omega_\gamma$ is a 3-form (equivalently, $\gamma - \alpha_{g_\varepsilon}$ is a skew-symmetric map) if and only if $q \in \mathbb{R}$, $t = \varepsilon q - \frac{3}{2} - 2\varepsilon$ and $p = \varepsilon p$. Hence, an arbitrary $h$-invariant bilinear map $\gamma$, metric with skew-torsion, is given by

$$
(\gamma - \alpha_{g_\varepsilon})(X, Y) = s \left( \varepsilon (b z - a w), i \text{Im}(\overline{z} w) \right) - \left( \varepsilon p(b\theta(z) - a\theta(w)), i \text{Im}(\bar{p} \theta(z)^t) \right).
$$
for \( s = 1 - q \in \mathbb{R} \) and \( p \in \mathbb{C} \). For computing the symmetrized Ricci tensor, we can save the computation of the curvature tensor\(^1\) by using the formula in [11] Appendix:

\[
(17) \quad \text{Sym}(\text{Ric}_\alpha) = \text{Ric}_{\alpha_{ge}} - \frac{1}{4} S,
\]

where the tensor \( S \in T^{0,2}(M) \) at \( p \in M \) is defined by

\[
S(X, Y)_p = \sum_{j=1}^{5} g_\varepsilon(T^\nabla(e_j, X_p), T^\nabla(e_j, Y_p)) g_\varepsilon(e_j, e_j),
\]

for \( \{e_1, ..., e_5\} \) any orthonormal basis of \( T_p M \). Again, by invariance, we work at \( p = o \), chosing, for instance, the following orthonormal basis of \( m \),

\[
e_1 = ((1, 0), 0), \quad e_2 = ((i, 0), 0), \quad e_3 = ((0, 1), 0), \quad e_4 = ((0, i), 0), \quad e_5 = \frac{1}{\sqrt{|\varepsilon|}}((0, 0), i).
\]

Now we check that

\[
S(X, Y) = -8\varepsilon(s^2 + pp\bar{p})\text{Re}(z^t \bar{w}) - 16\varepsilon^2(s^2 + pp\bar{p})ab.
\]

By extracting from Eq. (15) the Ricci tensor of the Levi-Civita connection (making \( q = 1, n = 2 \)),

\[
\text{Ric}_{\alpha_{ge}}(X, Y) = 2(\varepsilon + n + 1)\text{Re}(z^t \bar{w}) - 2n\varepsilon^2 ab = 2(\varepsilon + 3)\text{Re}(z^t \bar{w}) - 4\varepsilon^2 ab,
\]

we can substitute in Eq. (17) to get

\[
\text{Sym}(\text{Ric}_\alpha) = (2\varepsilon + 6 + 2\varepsilon(s^2 + pp\bar{p}))\text{Re}(z^t \bar{w}) + (s^2 + pp\bar{p} - 1)4\varepsilon^2 ab.
\]

Thus, the symmetric part of the Ricci tensor is a scalar multiple of the metric \( g_\varepsilon \) if and only if

\[
2\varepsilon + 6 + 2\varepsilon(s^2 + pp\bar{p}) = 4(\varepsilon^2 + pp\bar{p} - 1),
\]

or equivalently \( s^2 + pp\bar{p} = 3\varepsilon + 1 \). This equation has solutions whenever \( (\varepsilon + 1)\varepsilon > 0 \), which gives item \( n = 2 \) in Theorem [1] by taking \( s_3 = -\text{Re}(p) \) and \( s_4 = \text{Im}(p) \), taking into account Eqs. (10), (9) and noting that

\[
\Phi(\psi(X), Y) = (0, i \text{Im}(\theta(z) w)) \quad \eta(X)\psi(\psi(Y)) - \eta(Y)\psi(\psi(X)) = (b\theta(z) - a\theta(w), 0),
\]

\[
\Phi(\psi(\psi(X)), Y) = (0, -i \text{Re}(\theta(z) w)), \quad \eta(X)\psi(Y) - \eta(Y)\psi(X) = (-i (b\theta(z) - a\theta(w)), 0).
\]

We also observe \( \Phi(\psi(\psi(X)), Y) = g_\varepsilon(\psi(X), Y) \) (a not closed invariant 2-form).

3.4. Case \( S^3 \cong \text{SU}(2) \). We know from Eq. (13) that the map \( \alpha \) defined by

\[
(18) \quad (\alpha - \alpha_{ge})(X, Y) = (1 - q) (\varepsilon(bz - aw), i \text{Im}(\bar{w}) w),
\]

provides a metric with skew-torsion connection for any \( q \in \mathbb{R} \), where recall that the Levi-Civita connection is given through Nomizu’s theorem by

\[
\alpha_{ge}(X, Y) = (-\varepsilon bz - (\varepsilon + 2) aw, -i \text{Im}(\bar{w} w)).
\]

Now, taking into account that, in this case, \( \wedge^3 m \cong \mathbb{R} \), and hence \( \text{Hom}(\wedge^3 m, \mathbb{R}) \) is one-dimensional \((h = 0)\), the family in Eq. (15) exhausts the set of maps related to metric with skew-torsion invariant affine connections.

\(^1\) In fact, a tedious computation of the curvature shows that in this case, Ricci tensor is not necessarily symmetric. Alternatively, it is possible to check that \( \text{div}(T_\alpha) \neq 0 \), as done in [11] Remark 6.10.
Then Eq. (15) provides the Ricci tensor
\[
\text{Ric}_{\alpha}(X,Y) = 2(\varepsilon(q^2 - 2q + 2) + 2)\text{Re}(z^I \bar{w}) + 2\varepsilon^2(q^2 - 2q)ab,
\]
which is a scalar multiple of the metric \(g_{\varepsilon}\) if and only if \(\varepsilon(q^2 - 2q + 2) + 2 = \varepsilon(q^2 - 2q)\), that is, if and only if \(\varepsilon = -1\).

We have finished the proof of Theorem 1. The results are compiled in the following table.

| \varepsilon \in (-\infty, -1) | n \geq 4 | \varepsilon = -1 | n = 3 | \varepsilon \in (-1, 0) | n = 2 | \varepsilon > 0 | n = 1 |
|-----------------------------|---------|-----------------|------|--------------------------|-----|------------|------|
| Riemann                      | 2 pt.   | hyperboloid 2-sheets | ellipsoid | 0                       |     | hyperboloid 1-sheet | 0    |

| Lorentz                      | 2 pt.   | ellipsoid        | ellipsoid | 0                       |     | 0           |

Table 1. Varieties parametrizing invariant Einstein with skew-torsion connections.

Remark 1. As a consequence of Eq. (4), and with the notations used in Theorem 1, the scalar curvature of an Einstein with skew-torsion invariant connection becomes
\[
s^{\nabla} = \begin{cases} 
2n(2n + 1)\varepsilon(s^2 - 1) & \text{if } n \neq 2, \\
20\varepsilon(s^2 + s_3^2 + s_4^2 - 1) & \text{if } n = 2.
\end{cases}
\]

Hence, by taking into consideration the relationship among the parameters,
\[
s^{\nabla} = \begin{cases} 
2n(2n + 1)\frac{2\varepsilon + n + 1}{n - 1} & \text{if either } n = 2 \text{ or } n \geq 4, \\
6(1 - s^2) & \text{if } n = 1, \\
42(\varepsilon + 2 - s_1^2 - s_2^2) & \text{if } n = 3.
\end{cases}
\]

Thus, there are Ricci-flat invariant Einstein with skew-torsion connections for any odd-dimensional sphere for a suitable Berger metric, which are precisely,

- For \(n = 1\), the round metric \((\varepsilon = -1)\) and \(s = \pm 1\);
- For \(n = 2\), the metric associated to \(\varepsilon = -3/2\) and \(s^2 + s_3^2 + s_4^2 = 1\);
- For \(n = 3\), any metric \(g_{\varepsilon}\) with \(0 \neq \varepsilon \geq -2\), and \(s = \pm 1, s_1^2 + s_2^2 = \varepsilon + 2\);
- For \(n \geq 4\), the metric associated to \(\varepsilon = -\frac{n + 1}{2}\) and \(s = \pm 1\).

In particular, only in \(S^7\) there are Lorentzian Berger spheres satisfying such properties. In fact, in such a case every Lorentzian sphere does.

Also, we can conclude from Eq. (14) that, for \(n \geq 4\), there do not exist any flat invariant affine connection with skew-torsion (well-known from the work by Cartan and Schouten [8]), while for \(n = 3\), Eq. (16) implies that there are flat SU(4)-invariant affine connections with skew-torsion in \(S^7\) only for the round metric, and, in that case, the flat invariant connections correspond to \(s = 1\) and arbitrary \(s_1, s_2\) such that \(s_1^2 + s_2^2 = 1\).

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