Towards a Loop Representation of Connection Theories defined over a Super Lie Algebra

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Abstract. The purpose of this contribution is to review some aspects of the loop space formulation of pure gauge theories having the connection defined over a Lie algebra. The emphasis is focused on the discussion of the Mandelstam identities, which provide the basic constraints upon both the classical and the quantum degrees of freedom of the theory. In the case where the connection is extended to be valued on a super Lie algebra, some new results are presented which can be considered as first steps towards the construction of the Mandelstam identities in this situation, which encompasses such interesting cases as supergravity in 3 + 1 dimensions together with 2 + 1 super Chern-Simons theories, for example. Also, these ideas could be useful in the loop space formulation of fully supersymmetric theories.

1. INTRODUCTION

Gauge theories provide a successful framework to unify the strong, weak and electromagnetic interactions in nature. The recent introduction of the Ashtekar variables in general relativity has permitted to reformulate the gravitational interaction in the form of a standard gauge theory, plus additional constraints. Gauge theories where no matter is included, are fully described in terms of a single field: the connection $A_\mu$, which is the non-abelian generalization of the standard electromagnetic potentials. In general, gauge theories are characterized by being invariant under transformations generated by local symmetries, i.e. transformations of the connection which may be differently chosen at each point in space time. This freedom has the consequence of introducing arbitrary functions in the dynamics of the connection. In this way, only those specific fields constructed from the connection which are independent of these arbitrary functions, will be of physical relevance:
they are called gauge invariant objects. From this point of view one could say that taking $A_\mu$ as the basic field in a gauge theory is an unnecessary complication. This observation has prompted many proposals to describe gauge theories starting ab initio with only gauge invariant objects.

In this contribution we will consider the so called loop representation approach, to be described with more detail in the sequel. This method has been very successful in the description of non-perturbative aspects of gauge theories and also in providing spectacular advancements in the problem of quantizing Einstein gravity. The basic underlying idea of the loop representation is that the Hilbert space of gauge invariant states can be spanned by states which are labeled by loops. A main feature of the loop approach is the redundancy of the gauge invariant degrees of freedom: they constitute an overcomplete set which is restricted by the so called Mandelstam identities. These identities must be taken into account, either by imposing them over the physical states in a manner similar to the standard Dirac constraints, or by explicitly solving them in order to identify the corresponding reduced phase space. Even the preliminary problem of identifying the complete set of Mandelstam identities for a given physical situation is not completely solved in the general case for gauge theories over a Lie algebra.

In this work we will discuss only theories described by a connection field, i.e. those where matter fields are absent. These include pure Yang Mills theories, pure 2 + 1 dimensional Chern Simons theories and also gravity and supergravity in the Ashtekar formulation, for example. From the Hamiltonian point of view, these theories are basically on the same footing. Each one is characterized by a specific canonical Hamiltonian together with a given set of first-class constraints. We will be mostly concerned with the elucidation of the problem of the Mandelstam identities, both in the standard case and also when the connection of the gauge theory is defined over a super Lie algebra. Besides its own interest, the latter situation is also relevant as a first step in the application of loop representation methods to fully supersymmetric gauge theories, which involve matter fields as well. The discussion will be focused at the classical level, without considering in detail the construction of quantum states as functionals of loops. These notes do not have the pretense whatsoever of being a review on the subject and from the very beginning we apologize to those authors whose work we have involuntarily not cited.

The general organization of this paper is as follows: in Section 2 we provide a brief review of the standard formulation of Yang Mills theories in terms of the connection. We recall the Lagrangian formulation as well as the Hamiltonian approach, in which the loop representation is based. Section 3 contains the basic material necessary to extend the connection to a super Lie algebra object and includes the definition of a supermatrix together with those of the fundamental invariants under similarity transformations: the supertrace and the superdeterminant. An introduction to the loop space formulation of gauge theories is presented in Section 4, where the basic concepts are reviewed to end up with the introduction of the gauge invariant degrees of freedom: the Wilson loops. Also, the construction of the wave functionals of the system in the loop space is briefly sketched.
Section 5 is devoted to the discussion of the overcompleteness of the loop space variables and the Mandelstam identities are introduced to account for this redundancy, both in the classical and quantum case. The so called generic Mandelstam identities are subsequently derived from the identities arising from the application of the Cayley-Hamilton theorem in the specific representation where the connection lives. Section 6 deals with the extension of the previous ideas to the case where the connection is defined over a graded Lie algebra. To begin with, we present an extension of the Cayley-Hamilton theorem to the case of supermatrices. Besides its own interest, our motivation to deal with this mathematical problem is the claim that the algorithm developed for the standard case in Section 5 can be extended to this situation, thus providing the corresponding Mandelstam identities. The extension of the Cayley-Hamilton theorem proceeds in two steps: (i) the identification and definition of a characteristic polynomial for supermatrices and (ii) the proof that each supermatrix annihilates the null polynomial previously defined. Also, examples of characteristic polynomials for supermatrices are given in some simple cases. Finally, examples of Mandelstam identities constructed for some simple supermatrices, according to the previous ideas, are presented. We close with Section 7, which contains some conclusions and a brief outlook of some remaining open problems related to the topics discussed in this paper.

2. THE CONNECTION APPROACH TO YANG MILLS THEORIES

2.1. The Lagrangian Formulation

Let us consider the Yang Mills theory corresponding to the compact group $G$, which is characterized by its Lie algebra $\mathcal{A}$ having the antihermitian generators $T^A, A = 1, \ldots, n$. They satisfy the commutation relations $[T^A, T^B] = f^{ABC} T^C$, where $f^{ABC}$ are the structure constants of the group. A standard representation of these generators is in terms of ordinary matrices, with additional restrictions appropriate to the group under consideration.

The basic object in this formulation is the connection $A_\mu(x)$, which is a covariant vector field that lives in the Lie algebra of $G$, i.e. $A_\mu(x) = A_{\mu C}(x) T^C$, where $\mu = 0, 1, 2, 3$ is a world subindex. The infinitesimal local transformations $\delta \Theta(x)$ generated by the group can be written as $\delta \Theta(x) = \delta \Theta(x)_C T^C$, where $\delta \Theta(x)_C$ are arbitrary numerical functions. Under such rotation, the connection transforms as

$$\delta_\Theta A_\mu(x) = \partial_\mu \delta \Theta(x) + [A_\mu(x), \delta \Theta(x)] := D_\mu \delta \Theta(x),$$

which generalizes the gauge transformation of electrodynamics. Here we have introduced also the covariant derivative of any object $M_C$ with one subindex in the Lie algebra, in the form

$$D_\mu M = \partial_\mu M + [A_\mu(x), M],$$
where \( M = M_C T^C \). In components, the above expression reads

\[
(D_\mu M)_C = \partial_\mu M_C + f^{BD} C_{\mu B} M_D.
\]

(3)

The object which is covariant under the local group of transformations is the field strength \( F_{\mu\nu} = F_{\mu\nu C} T^C \) defined by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\]

(4)

which generalizes the electromagnetic field of electrodynamics. Under gauge rotations, \( F_{\mu\nu} \) transforms covariantly, i.e. \( \delta_\Theta F_{\mu\nu} = [F_{\mu\nu}, \delta \Theta] \). Besides, the field strength satisfies the Bianchi identity

\[
D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0.
\]

(5)

Let us emphasize the basic property that the covariant derivative (2) of any covariant object is itself a covariant object.

The dynamics of the pure Yang Mills is described by the action

\[
S = \int d^4 x \ L = \int d^4 x \frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu}),
\]

(6)

where \( L \) is the Lagrangian density and \( Tr(T^A T^B) = h^{AB} \) is an invariant symmetrical tensor in the corresponding representation of the Lie algebra. For a semisimple group (i.e. one having no \( U(1) \) invariant subgroups) it is possible to choose \( h^{AB} = -\delta^{AB} \). The world indices are raised and lowered by the Minkowsky metric \( \eta^\mu_\nu = \text{diag}(1, -1, -1, -1) \). The action (6) is invariant under the gauge transformations (1). Finally, the resulting equations of motion are

\[
D_\mu F^{\mu\nu} = 0.
\]

(7)

### 2.2 The Hamiltonian Formulation

This formulation corresponds to the \((1 + 3)\) splitting of Minkowsky space-time, where we select three-dimensional hypersurfaces of constant time. It is convenient to introduce the chromoelectric, \( E_i \), and chromomagnetic, \( B^i \), fields in the following way

\[
E_i = F_{0i} = A_i - D_i A_0, \quad B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}.
\]

(8)

Here, \( i = 1, 2, 3 \) labels the components of a vector living on the three-dimensional hypersurface and the dot denotes the time derivative evaluated on that hypersurface. The covariant derivative \( D_i \) corresponds to the definition (2) restricted again to the hypersurface, through the connection \( A_i \). Introducing the canonical momenta
\[
\Pi^i_C := \frac{\partial L}{\partial \dot{A}_{iC}} = -E^i_C = E_{iC} \tag{9}
\]

and defining the Hamiltonian density \( \mathcal{H} = \Pi^i_C \dot{A}_{iC} - L \), we obtain the action

\[
S = \int dt \int d^3 x \left( \Pi^i_C \dot{A}_{iC} - \frac{1}{2}(E^2 + B^2) + A_{0C}(\mathcal{D}_i E^i_C) \right), \tag{10}
\]

where \( E^2 = E^i_C E^i_C \geq 0 \) and analogously for \( B^2 \). The above expression arises after an additional integration by parts, which can be performed in virtue of the following properties of the covariant derivative: (i) \( \mathcal{D}_i (M N) = (\mathcal{D}_i M) N + M (\mathcal{D}_i N) \) and (ii) \( Tr(\mathcal{D}_i M) = \partial_i Tr(M) \). The action (10) tells us that the phase space variables are the coordinates \( A_{iC}(\vec{x}, t) \), together with their conjugated momenta \( \Pi^j_D(\vec{y}, t) = E^j_D(\vec{y}, t) \), which satisfy the standard Poisson brackets

\[
\{ A_{iC}(\vec{x}, t), \; \Pi^j_D(\vec{y}, t) \}_PB = \delta^3(\vec{x} - \vec{y}) \delta^j_i \delta_{CD}, \tag{11}
\]

at equal times. Also, we infer from (10) that \( A_{0C}(x) \) are Lagrange multipliers leading to the Gauss law constraints \( \mathcal{G}_C = (\mathcal{D}_i E^i)_C \approx 0 \), where \( \mathcal{G}_C \) is such that \( \mathcal{G} = \mathcal{D}_i E^i = \mathcal{G}_C T^C \). In particular

\[
\mathcal{G}_C = \partial_i E^i_C + f^{BD} C A_{iB} E^i_D. \tag{12}
\]

Applying the standard Dirac method for constrained systems \([1,2]\), we can verify that there are no secondary constraints. Also, one obtains that \( \mathcal{G}_C \) are first-class constraints, which generate gauge transformations on the three-dimensional hypersurface. A compact way to show these properties is by introducing the averaged constraints

\[
\mathcal{G}_A = \int d^3 x \; \Lambda^C \mathcal{G}_C, \tag{13}
\]

where \( \Lambda^C(\vec{x}) \) are arbitrary functions of position. In this way, the Poisson brackets of the constraints turn out to be

\[
\{ \mathcal{G}_A, \; \mathcal{G}_\Theta \}_PB = \mathcal{G}_{A \times \Theta}, \tag{14}
\]

where \( (\Lambda \times \Theta)_A = 2 f^{BC} A_{\Lambda B} \Theta_C \). The group indices are lowered and raised with the metric \( h^{BC} \) together with its inverse \( h_{BC} \). The fact that the constraints \( \mathcal{G}_A \) generate gauge transformations can be seen from the following calculation

\[
\delta_\Lambda A_{iC}(\vec{x}, t) := \{ A_{iC}(\vec{x}, t), \; \mathcal{G}_\delta \}_PB = -(\mathcal{D}_i \delta \Lambda)_C, \tag{15}
\]

which reproduces the spatial part of the transformation (1). The precise relation between the gauge symmetries obtained in the Hamiltonian formulation and those appearing in the Lagrangian formulation can be found in Ref. [3].
The canonical quantization of the Hamiltonian version of a gauge theory follows
the standard steps:

(i) The canonical variables are promoted to the range of hermitian operators:
\( A_i, E_i \to \hat{A}_i, \hat{E}_i \).

(ii) The Poisson brackets algebra is turned into a commutator (anticommutator)
algebra, according to the statistic of the involved fields. In our case the resulting
commutators are
\[
[\hat{A}_i(x,t), \hat{E}_{jD}(\vec{y},t)] = i\hbar\delta^3(\vec{x} - \vec{y})\delta_{ij}\delta_{CD},
\]
with all others been zero.

(iii) A possible way of realizing the above algebra is in the “coordinate” represent-

ation, where
\[
\hat{A}_i := A_i(x,t), \quad \hat{E}_i := i\hbar\frac{\delta}{\delta A_i(x,t)}.
\]

The wave function of the system is then a functional of the connec-
tion: \( \Psi(A) \) and we can calculate the action of any operator upon it. The basic
functional derivative is defined by
\[
\frac{\delta A_i(x,t)}{\delta A_j(y,t)} = \delta^3(\vec{x} - \vec{y})\delta_{ij}\delta_{CD}.
\]

(iv) The realization of the quantum operators may lead to ordering problems.
In particular, the first-class constraints \( \hat{G}_C \) must be realized as operators which
correctly close under commutation, in order to have a consistent theory. If it not
possible to do so, we say that an anomaly appears. The first-class constraints are
subsequently imposed as null operator conditions upon the wave functions repre-
senting the physical states
\[
\hat{G}_C \Psi(A) = 0.
\]

In our case, the operator expression for the Gauss law constraint is
\[
\hat{G}_\Theta = i\hbar \int d^3x (D_i \Theta)_C \frac{\delta}{\delta A_i},
\]
and the condition (19) simply means that the physical wave functions must be
gauge invariant functionals of the connection.

(v) There must be a well defined scalar product which allows for the realization
of hermitian operators together with the calculation of the transition probabilities
that permits us to make predictions regarding the observables of the theory.

(vi) Finally, the dynamics is contained in the wave function \( \Psi(A,t) \), which sat-

\[ i\hbar \frac{\partial \Psi(A,t)}{\partial t} = \left( \int d^3 x \frac{1}{2} \left( \hat{E}^2 + \hat{B}^2 \right) \right) \Psi(A,t). \] (21)

A corresponding Hamiltonian formulation can be also developed for Chern-Simons theories, which are connection theories defined in a space-time having an odd number of dimensions. These topological field theories are defined by a Lagrangian density given by the \(2n-1\) form \(\Omega_{CS}\) defined by \(d\Omega_{CS} = Tr(F^n)\). Here, \(F\) is the 2-form corresponding to \(F = \frac{1}{2} F_{\mu \nu} dx^\mu dx^\nu\), where \(F_{\mu \nu}\) is the field strength given in Eq. (4). In the case of \(2 + 1\) dimensions, the Lagrangian action for the Chern-Simons theory is

\[ S = \frac{1}{2} Tr \int_M \left( dA - \frac{2}{3} A \wedge A \right) \wedge A, \] (22)

where the integration is made over a three-dimensional manifold \(M\) and \(A = A_\mu dx^\mu\) is the connection 1-form.

3. YANG-MILLS THEORIES WITH A CONNECTION DEFINED OVER A SUPER LIE ALGEBRA

Super Lie algebras are a special case of graded Lie algebras, which appear in the description of supersymmetry. This new type of symmetry arises when attempting to provide a unified description of bosons and fermions. In this case, the basic object will be a multiplet incorporating both kind of fields. The allowed “rotations” of the theory will mix the components of such multiplet: i.e. bosons and fermions. In this section we will present only a brief review of some underlying ideas in supersymmetry, which are relevant to the construction of supersymmetric connection theories [4].

3.1. Grassmann Numbers

Let us start from the “classical” version of supersymmetric theories, which requires the introduction of Grassmann numbers to represent the algebraic properties of fermionic fields. The simplest way to motivate these numbers is by starting from the quantum description of two independent fermionic operators \(f\) and \(g\), defined by the following anticommutators

\[ \{f, f^\dagger\} = 1, \quad \{f, f\} = 2f^2 = 0, \quad \{f^\dagger, g^\dagger\} = 0, \quad \{f, g^\dagger\} = 0, \] (23)

with analogous expressions obtained interchanging \(f\) and \(g\). Consider a realization of the above anticommutation relations in terms of coordinates and derivatives in a manner analogous to the holomorphic representation for the standard harmonic
oscillator. To this end, let us introduce two independent coordinates $\theta$ and $\eta$ and define the following realization of the above operators

$$
<\theta \eta | f^\dagger = \theta <\theta \eta |, \quad <\theta \eta | f = \frac{d}{d\theta} <\theta \eta |, \\
<\theta \eta | g^\dagger = \eta <\eta \theta |, \quad <\theta \eta | g = \frac{d}{d\eta} <\theta \eta |.
$$

(24)

The third relation in Eq.(23) requires $\theta \eta + \eta \theta = 0$, while the second relation demands that each of the above numbers must have zero square, i.e. $\theta^2 = 0 = \eta^2$.

The last relation in Eq.(23) says that $\frac{d}{d\theta} \eta + \eta \frac{d}{d\theta} = 0$, which require that the derivative operator anticommutes with the coordinates.

The above realization of the fermionic operators must be supplemented with a scalar product which guarantees that in fact $\theta^\dagger = \frac{d}{d\theta}$. For the independent coordinates $\theta, \theta^\ast$, for example, such scalar product is given by

$$
(\Psi, \Delta) = \int d\theta d\theta^\ast \, \Psi^* e^{\theta^\ast \theta} \Delta,
$$

(25)

where the integration over the Grassmann variables is defined by

$$
\int d\theta = 0, \quad \int d\theta \, \theta = 1,
$$

(26)

and analogously for the complex conjugated variable $\theta^\ast$.

The “numbers” $\theta, \theta^\ast$ and $\eta, \eta^\ast$ that satisfy the above properties are called odd Grassmann numbers and provide a “classical” description of fermionic degrees of freedom, in the same way as complex numbers allow for the realization of bosonic degrees of freedom. The product $\theta \eta = -\eta \theta$ is called an even Grassmann number because it commutes either with $\theta$ or $\eta$ and also has zero square. Thus, the set of all commuting numbers is augmented from the complex numbers to the set of even Grassmann numbers. The reader is encouraged to look into Ref. [5] for a detailed description of the properties of Grassmann numbers.

In this way, a unified description of bosonic and fermionic degrees of freedom will start from a multiplet, say,

$$
|u> = [q^1, q^2, \ldots, q^m; \theta^1, \theta^2, \ldots, \theta^n]^T,
$$

(27)

which contains bosonic degrees of freedom $q^i$, $i = 1, \ldots m$, described by even Grassmann numbers (complex numbers in particular), and fermionic degrees of freedom $\theta^\alpha$, $\alpha = 1, \ldots, n$ described by odd Grassmann numbers. The superindex $T$ denotes standard transposition, in such a way that the array (27) is a column.

### 3.2. Supermatrices

The natural operators acting on the state $|u>$ are linear transformations $M$ that produce a “rotated” state $|v> = M|u>$, which components preserve the even/odd character of each entry in the multiplet. Such object is the $(m+n) \times (m+n)$ array
where the corresponding blocks have the following properties: (i) $A$ and $D$ are respectively $m \times m$ and $n \times n$ matrices with all entries been even Grassmann numbers. (ii) $B$ and $C$ are respectively $m \times n$ and $n \times m$ matrices with all entries been odd Grassmann numbers. The array $M$ is called a supermatrix and the corresponding action upon the state $|u>\rangle$ is called a supersymmetry transformation.

Let us observe that the addition and the multiplication of two supermatrices (with both operations defined in the standard way) is again a supermatrix. Since the unit matrix $I$ is also a supermatrix, the inverse of a supermatrix can also be defined in the usual manner.

An important topic in the study of supersymmetry is the construction of the supermatrix invariants under similarity transformations, which generalize the basic concepts of the trace and the determinant in classical linear algebra. The supertrace, denoted by $Str$, is defined by

$$Str M := Tr A - Tr D,$$

where the relative minus sign is required in order to guarantee the cyclic property $Str(M_1 M_2) = Str(M_2 M_1)$, under the presence of odd Grassmann numbers in each supermatrix. The definition of the superdeterminant, denoted by $Sdet$, follows the same pattern of the classical case and it is given by

$$Sdet M := \exp(\text{Str}(\ln M)).$$

This expression can be written in infinitesimal form as

$$\delta \ln(Sdet M) = \text{Str}(M^{-1}\delta M),$$

subject to the boundary condition $Sdet I = 1$. The equation (31) is a condensed way of writing the partial derivatives of the superdeterminant with respect to the supermatrix entries, in terms of the inverse supermatrix $M^{-1}$. The superdeterminant satisfies

$$Sdet(M_1 M_2) = Sdet(M_1) Sdet(M_2).$$

Using these properties one can find the general expressions

$$Sdet(M) = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)},$$

where we assume the existence of $A^{-1}$ and $D^{-1}$. The determinants appearing in Eq. (32) are defined in the usual way because they involve only even (commuting) Grassmann numbers. The equivalence among the two forms of writing the superdeterminant in Eq.(32) is proved in Ref. [6].
3.3. The Superconnection

Motivated by the above discussion, it is possible to construct connection theories where \( A_\mu \) is extended to be a \((m+n) \times (m+n)\) supermatrix

\[
A_\mu = A_{\mu C} T^C + \psi_{\mu c} S^c,
\]

which incorporates bosonic \((A_{\mu C})\) and fermionic \((\psi_{\mu c})\) gauge fields that are even and odd Grassmann functions of position, respectively. The generators \( T^C \) and \( S^c \) are purely numerical (complex) matrices which have the generic structure

\[
T = \begin{pmatrix}
a & 0 \\
0 & b \\
d & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & b \\
c & 0
\end{pmatrix},
\]

in the same block form of Eq.(28). They are labeled as even and odd generators, respectively. Under an infinitesimal transformation generated by the supermatrix \( \delta \Theta = \delta \Theta_C T^C + \delta \Theta_c S^c \), the connection transforms in the form given in Eq.(1). In order that this transformation be closed in the algebra generated by \( T^C, S^c \) it is necessary to require the following (anti)commutation relations among the generators

\[
[ T^A, T^B ] = f^{ABC} T^C, \quad [ T^A, S^b ] = g^{Abc} S^c, \quad \{ S^a, S^b \} = h^{abc} T^C.
\]

The structure (35) is called a super Lie algebra and it is characterized by the presence of anticommutators among the fermionic generators besides the standard commutators for the remaining cases. The anticommutators are induced from the basic commutator in Eq. (1) due to the anticommuting properties of the fermionic components of the multiplet under consideration. We must emphasize that the functions which multiply an odd generator in each multiplet , like \( \psi_{\mu c} \) or \( \delta \Theta_c \), for example, are fermionic in character , so that they anticommute among each other, being realized by odd Grassmann functions of position. The quantities \( f^{ABC}, g^{Abc}, h^{abc} \) are the numerical structure constants of the super Lie algebra.

The field strength \( F_{\mu \nu} \) is now a supermatrix and together with the covariant derivative \( D_\mu \) retain their definitions given in Eqs. (4) and (2) respectively. Finally, the action for a pure Yang Mills supersymmetric theory is given by an equation analogous to (6),

\[
S = \int d^4 x \ \mathcal{L} = \int d^4 x \ \frac{1}{4} \text{Str}(F_{\mu \nu} F^{\mu \nu}),
\]

where the trace has been replaced by the supertrace. The canonical analysis and quantization of this theory follows analogous steps to the standard case, except that now fermions have been included in the connection.

This ideas can be extended to the construction of supersymmetric Chern-Simons theories. The formulation of gauge theories where the connection is defined over a super Lie algebra encompasses some interesting cases like 2 + 1 dimensional supergravity with cosmological constant [7] and also the standard supergravity in 3 + 1 dimensions, in terms of the Ashtekar variables [8]. The case of global supersymmetry is not included in this way.
4. THE LOOP APPROACH TO GAUGE THEORIES

The description of gauge theories only in terms of gauge invariant objects, like the Wilson loops for example, was initiated by the work of Mandelstam [9]. Other earlier references on the subject are [10], [11], [12], [13]. The loop representation, which is a quantum Hamiltonian representation of gauge theories in terms of closed curves (loops) was subsequently introduced in Refs. [14], [15]. Roughly speaking, the dynamical variables in this method are the generalization of the Aharonov-Bohm phase of electrodynamics to the general non-abelian case: the Wilson loops. In the abelian case, this phase is basically the integral of the connection around a closed loop and it is a gauge invariant object. In this way, one considers all possible Wilson loops in a three dimensional surface of constant time, as the degrees of freedom of the system. Thus, one shifts from the space of connections to the space of loops so that the wave function of the system, originally a functional of the connection, turns out to be a functional of loops. Besides allowing for the construction of gauge invariant objects, loops provide a natural geometric framework to describe gauge theories and gravitation. This approach is a non-local description of the dynamics and it is well suited to the discussion of non-perturbative effects. One of the central problems in this method is to find an independent set of loop-space degrees of freedom, because as we will see in the sequel, the loop-space variables form an overcomplete set.

A very nice example of the properties of this formulation in the case of standard gauge theories is given in a paper by Brugmann [16] where the lattice gauge theory for the group $SU(2)$ is completely solved within this scheme, and compared very favorably with alternative solutions. In the author’s words:

"the eigenvalue problem for the Hamiltonian of $SU(2)$ lattice gauge theory is formulated in the loop representation, which is based on the fact that the physical Hilbert space can be spanned by states which are labeled by loops. Since the inner product between loop states can be calculated analytically, the eigenvalue problem is expressed in terms of vector components and matrix elements with respect to the loop basis. A small-scale numerical computation in $2 + 1$ dimensions yields results which agree with results obtained from other methods."

Another success of the loop space approach can be found in its application to Quantum Gravity [17], [18]. The use of the Ashtekar variables [19], allows to rewrite Einstein gravity as a connection theory, in terms of a selfdual connection which lives in the Lie algebra of $SU(2)$, plus some reality conditions. The structure of the constraints in such formulation is that of the corresponding gauge theory plus some extra constraints related to the invariance of gravity under diffeomorphism in $3 + 1$ dimensions. Using the loop space approach, it has been possible for the first time to find explicit solutions to the diffeomorphism constraints of the complexified theory in terms of functionals of knots. The reality conditions, necessary to get back to real Einstein gravity, subsequently provide a definition of the required scalar product that leads to an interpretation of the quantized theory. As we will see in the sequel, loop states automatically solve the local $SU(2)$
gauge invariance of the theory. In this way, the three dimensional diffeomorphism invariance of Einstein gravity can be realized by labeling the states by knot classes, which are just the equivalence classes of loops under diffeomorphism. Finally, there remains only one constraint to solve, which is satisfied in terms of superposition of knot states.

In this work we present only a brief introduction to the loop space approach of gauge theories. Detailed versions, including applications to quantum gravity, can be found in Refs. [20], [21], [22], [23], [24].

4.1. Open paths, loops and basic dynamical variables

Definitions

Our discussion will start from a space-time of the form $\Sigma \times \mathcal{R}$, with $\Sigma$ being a three dimensional surface of constant time, where we have defined the Hamiltonian version of our gauge theory. All curves to be handled in the sequel will lay on this surface.

Let us consider a smooth and continuous function $\eta : s \rightarrow \Sigma$, where $s$ is a parameter in the real interval $s_1 \leq s \leq s_2$. An open path on the surface, joining the points $A$ (initial) and $B$ (final point) of it, is defined by $\eta_{AB}(s) = \eta$, with $\eta(s_1) = A$ and $\eta(s_2) = B$. We will use greek letters in the middle of the alphabet to denote open paths. The choice to move along the path, either from $A$ to $B$ or vice versa, defines its orientation. Given any oriented path $\eta_{AB}(s)$, its inverse $(\eta^{-1})_{BA}(t)$ is the original path traversed in the inverse sense, with $t$ being the corresponding parameter.

A loop, or a closed path, is an open path where the initial and final points coincide, i.e. such that $A = B$. They will be denoted by greek letters from the beginning of the alphabet: $\alpha(s), \beta(s)$, for example. Again, the sense of traversing the loop defines its orientation. In this way, the inverse of any loop is the same loop run over in the opposite sense. Finally, it will prove convenient in the sequel to introduce the idea of a multiloop $\tilde{\gamma}$ as a collection of loops $\alpha_1, \alpha_2, \ldots, \alpha_p$, which will be denoted by $\tilde{\gamma} = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_p$. A multiloop is a direct product of loops and does not involve any composition property.

Composition properties

It is convenient to define the multiplication of paths, which we denote by $\circ$. Two open paths $\eta_{AB}$ and $\sigma_{CD}$ can be multiplied only if either (i) $A = D$, or (ii) $C = B$. In the first case, the composed path, called $(\sigma \circ \eta)_{CD}$, is the result of going first through the path $\sigma$ and subsequently through the path $\eta$, in that order. In the second situation we obtain the path $(\eta \circ \sigma)_{AD}$. The multiplication $\eta_{AB} \circ (\eta^{-1})_{BA}$ produces what
is called a tree, which amounts to start from point $A$ and going back to it, without enclosing any area on $\Sigma$. Any two open paths or loops differing by a collection of trees, are considered to be equivalent.

Two oriented loops, $\alpha$ and $\beta$ can be multiplied only when they have a point of intersection, say $C$. Then, the loop $\alpha \circ \beta$ is obtained starting from the common point $C$, going first through the loop $\alpha$ to end up in $C$, subsequently going through the loop $\beta$, to finally end up again at the intersection point. In each case, the sense of travelling is defined by the orientation of the loops to be multiplied. From now on we will consider that all loops are based on a fixed point on $\Sigma$, so that they always have at least this point in common. This loop space has the structure of a semigroup.

**Parallel transport matrix**

To any open path $\eta \in \Sigma$, we associate the group element

$$U(\eta_{s_1}^{s_2}) = P \exp \int_{s_1}^{s_2} ds \dot{\eta}^i(s) A_i(s),$$

which is the non-abelian generalization of the Aharanov-Bohm phase factor. Here, $\dot{\eta}^i(s)$ denotes the tangent vector of the path $\eta$ and $P$ stands for the path-ordering operator defined as

$$P \exp \int_{s_1}^{s_2} dt \ M(t) = 1 + \int_{s_1}^{s_2} dt \ M(t) + \int_{s_1}^{s_2} dt \int_{t_1}^{s_2} dt_1 \ M(t) M(t_1)$$
$$+ \int_{s_1}^{s_2} dt \int_{t_1}^{s_2} dt_1 \int_{t_2}^{s_2} dt_2 \ldots \ M(t) M(t_1) M(t_2) \ldots \ := V(s_1, s_2),$$

where $M(t)$ denotes any element of the corresponding Lie algebra, valued along the path $\eta$. Let us emphasize that, in general, $[M(t_1), M(t_2)] \neq 0$ for arbitrary points in the path. The ordering in (38) is defined by $s_1 < t < t_1 < t_2 < \ldots < s_2$. An alternative way of writing the generic ordered integral is in the form

$$\int_{s_1}^{s_2} dt_2 \int_{s_1}^{t_2} dt_1 \int_{s_1}^{t_1} dt \ldots \ M(t) M(t_1) M(t_2) \ldots .$$

(39)

The parallel transport matrix (37) is also known as the integrated connection along the corresponding path. As a function of the end points, the group element $V(s_1, s_2)$ satisfies the properties

$$\frac{\partial V(s_1, s_2)}{\partial s_2} = V(s_1, s_2) M(s_2), \quad \frac{\partial V(s_1, s_2)}{\partial s_1} = -M(s_1) V(s_1, s_2),$$

(40)

which can be directly seen from Eq. (39) and Eq. (38) respectively. Also we have the composition property
provided the final point of the path $\eta$ coincides with the initial point of the path $\sigma$. In this way, both paths can be composed as indicated in the RHS of Eq. (41). The composition appearing in the LHS of Eq.(41) corresponds to the group multiplication. Finally, under non-abelian finite gauge transformations, (generated by the group element $g(x)$), $A_i \to \tilde{A}_i = g A_i g^{-1} - g \partial_i g^{-1}$, the integrated connection transforms as

$$U(\eta_{s_1}^{s_2}) \to \tilde{U}(\eta_{s_1}^{s_2}) = g(s_1)U(\eta_{s_1}^{s_2})g^{-1}(s_2).$$  \hfill (42)

### Holonomies and Wilson loops

The final goal of this formulation is to define strictly gauge invariant objects, which will constitute the appropriate variables to formulate the theory. To this end, let us consider the holonomy $U(\alpha)$ associated to the loop $\alpha$, which is just the integrated connection (37) around the loop,

$$U(\alpha) = P \exp \oint_{\alpha} ds \dot{\alpha}^i(s) A_i(s).$$  \hfill (43)

This object is still not gauge invariant, but transforms according to (42) with the corresponding group elements evaluated at different values of the parameter, which nevertheless describe the same point on $\Sigma$: the chosen initial ($s = s_1$) and final point ($s = s_2$) of the loop. Group elements must be single valued on $\Sigma$, so that $g^{-1}(s_2) = (g(s_1))^{-1}$. In this way, $Tr U(\alpha)$ is indeed a gauge invariant complex number. This is the original Wilson loop, whose definition has been generalized to include insertions of the gauge-covariant chromoelectric field, as a way to incorporate the variable conjugated to the connection in a gauge invariant manner. Then, a tower of Wilson loops can be constructed as follows

$$T^0(\alpha) : = Tr U(\alpha) = h(\alpha, A),$$

$$T^i(\alpha) : = Tr \left[ U(\alpha^{s_0} s_1) E^i(s_1) U(\alpha s_1^{s_0}) \right],$$

$$T^{ij}(\alpha) : = Tr \left[ U(\alpha^{s_0} s_1) E^i(s_1) U(\alpha s_1^{s_2}) E^j(s_2) U(\alpha s_2^{s_0}) \right], \ldots,$$  \hfill (44)

where $s_0$ parametrizes the origin of the loop $\alpha$. Each of the numbers in (44) is a gauge invariant quantity and we will denote them generically by $T^0, T^1, T^2, \ldots$ according to the number of insertions introduced. These objects constitute the fundamental dynamical variables used to formulate the gauge theory in this approach.

The original symplectic structure (16) of the gauge theory will induce a Poisson brackets structure among the Wilson loops of Eq.(44). In a very sketchy form, these Poisson brackets will be of the form
\[ \{T^m, T^n\}_{PB} = \sum_{k=1}^{m+n-1} C_k T^k. \]  

The result in the RHS will involve some recomposition and rerouting of the initial loops appearing in the LHS. The details are given in Refs. [15], [17].

### 4.2. The loop space

The dynamical variables defined above, i.e. the Wilson loops, have support on the space of closed loops on \( \Sigma \), which now replaces the configuration space associated to the connection representation. To complete the formulation of the gauge theory in the loop space is still necessary to rewrite the Hamiltonian and the constraints in the new representation. This can be accomplished through the introduction of differential operators which act on loop dependent functions: the loop derivative and the connection derivative, for example. The reader is referred to Ref. [21] for the details.

Accordingly, when quantizing the theory, the corresponding operators \( T^n \rightarrow \hat{T}^n \) will have also support on the space of loops and they will be constructed as operators acting on a Hilbert space which now will be spanned by functionals of loops. An heuristic, but very illuminating way of looking at some properties of the loop space is via the loop-transform introduced in Refs. [17], [18]. This transforms provides the change of basis from the connection representation \( \{|A>\} \) to the loop representation \( \{\tilde{\alpha}>\} \) in the form

\[
|\tilde{\alpha}> = \int [dA] \ |A> <A|\tilde{\alpha}>, \\
< A|\tilde{\alpha} > = \mathcal{K}(\tilde{\alpha}, A) := TrU(\alpha_1)TrU(\alpha_2)\ldots TrU(\alpha_p),
\]  

where \( \tilde{\alpha} \) is a multiloop with components \( \alpha_1, \alpha_2, \ldots, \alpha_p \). The measure \([dA] \) in (46) is not presently known and this fact is what makes this transform only an heuristic tool. The wave function of the system \(|\Psi>\) can then be projected either into the connection representation \( <A|\Psi> \) or into the loop representation \( <\tilde{\alpha}|\Psi> \), with the two projections being related by the loop transform (46). The change of basis (46) will provide also the representation of the operators acting in the loop space, starting from their counterparts in the connection representation.

Nevertheless, there is an alternative way to this construction, based upon the formulation of the quantum theory directly in the loop space. To this end, one starts from the Poisson brackets relations (45) among the Wilson loops. Following the standard procedure, the classical variables are promoted into operators \( \hat{T}^n \) and their Poisson brackets are translated into commutators. A basis in the loop space is constructed by first defining the zero-loop state \(|0>\) such that \( \hat{T}^n|0> = 0, \ n \geq 1 \). An arbitrary multiloop state \(|\tilde{\alpha}>\) is subsequently constructed by the action \( \hat{T}^0(\alpha_1)\hat{T}^0(\alpha_2)\ldots\hat{T}^0(\alpha_p)|0> = |\tilde{\alpha}> \). Then, using the commutation relations it is

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]
possible to figure out the action of any operator upon the multiloop states. Ref. [16] contains a very nice and explicit application of these ideas.

Being gauge invariant, the Wilson loop variables automatically satisfy the Gauss law constraints of the theory. In this way, the arbitrary multiloop quantum states constructed by the action of the operators $T^0(\alpha_1)T^0(\alpha_2)\ldots T^0(\alpha_p)$ on the zero loop state, will be automatically annihilated by the quantum Gauss law constraints.

The equivalence between the connection representation and the loop representation of gauge theories defined over a Lie algebra was proved in Ref. [25] for the group $SU(N)$. There, it is given a procedure to reconstruct, up to a gauge transformation, the gauge field at each point (written as a complex matrix) starting from the knowledge of all the Wilson loops $T^0(\alpha)$ going through that point, together with the imposition of the so called Mandelstam identities, which set constraints among the Wilson loops. This construction demonstrates that the Mandelstam identities are a sufficient set of conditions on the Wilson loop variables which guarantee that such variables represent a local gauge theory over a Lie algebra.

5. OVERCOMPLETENESS OF THE LOOP SPACE VARIABLES

Let us summarize what has been done by saying that the loop space formulation of classical gauge theories starts from an infinite set \( \{T^n\} \) of numerical, non-local, gauge invariant degrees of freedom, defined over loops in the three-dimensional surface \( \Sigma \). These variables replace those of the standard local connection representation, \( E(\vec{x},t), A(\vec{x},t) \), which are not gauge invariant.

The loop space degrees of freedom are basically traces of group elements, which in the sequel will be thought of as traces of matrices of a given dimension. Now, it is well known that the Cayley-Hamilton theorem of linear algebra, for example, sets relations among the traces of powers of a given matrix, which depend on the dimension of the representation. As we will show in the sequel, a similar mechanism induces relations among the different Wilson loops $T^n(\alpha)$, with the consequence that \( \{T^n\} \) is an overcomplete set of dynamical variables. These relations among powers of traces of products of matrices are generically known as Mandelstam identities (MI). Even in the standard case (i.e. gauge theories over a Lie algebra) there is lacking a general procedure to obtain the full set of MI for arbitrary restricted groups. The situation in the case where the connection is valued on a super Lie algebra is still more open and we hope that some of the ideas presented here could provide a starting point to deal with these problems.

At the quantum level, these MI will generate, via the loop transform (46) for example, relations among the multiloop states, which will make this set also overcomplete.

In this way, any loop space formulation of a gauge theory will require the knowledge of the corresponding complete set of MI, in order that the true degrees of freedom of the system can be identified. The subsequent problem of imposing
and/or solving the MI is still highly involved and has been carried over, for example, in the following cases: (i) $SU(2)$ pure lattice gauge [26], [27]. (ii) $2 + 1$ dimensional Einstein gravity [28] and (iii) a specific sector of $2 + 1$ supergravity [29].

The observation made above concerning the relation between the MI and the Cayley-Hamilton identities will be made more explicit in what follows, by formulating an algorithm to go from one type of identity to the other, in the generic case of matrices over the complex numbers. This procedure will be next extended to generate the generic MI for supermatrices, after having also extended the Cayley-Hamilton theorem to this case.

5.1. The Mandelstam identities

The simplest MI (known as MI of the first class) is just the statement of the cyclic property of the trace. For any two complex square matrices $M$ and $N$, we have

$$Tr (MN) = Tr (NM).$$  \tag{47}

Let us consider the effect of this identity upon the loop space variable $T^0(\alpha \circ \beta)$ for example. We have $T^0(\alpha \circ \beta) = Tr U(\alpha \circ \beta) = Tr (U(\alpha)U(\beta)) = Tr (U(\beta)U(\alpha))$, where we have used the composition property (41) for the holonomies. The above chain of equalities leads to the conclusion

$$T^0(\alpha \circ \beta) = T^0(\beta \circ \alpha).$$  \tag{48}

When translated into an statement over the quantum loop states, Eq. (47) implies

$$|\alpha \circ \beta >= |\beta \circ \alpha >,$$  \tag{49}

which arises either from the loop transform (46), or from the method of creating the quantum loop space vectors starting from the zero loop state $|0 >$. The above constitutes a very simple example of the overcompleteness of both the classical loop space variables together with the corresponding quantum states.

All the remaining MI are known as Mandelstam identities of the second class and there is a hierarchy of them, according to the restrictions satisfied by the group under consideration. Suppose we are dealing with a group representation in terms of $n \times n$ matrices $M$. There will be what we call a generic MI, related to the specific dimension $n$. If we impose further restrictions upon the representation, like unit determinant or unitarity, for example, new MI reflecting them have to be generated. Most of what we will discuss in these notes has to do with the generic MI and their extension to the supersymmetric case.
The generic Mandelstam identities

Let us consider the group elements \( U(\alpha) \) realized as \( n \times n \) matrices with no further restrictions. The corresponding generic MI are obtained starting from the identity [25]

\[
0 = \epsilon_{i_1 j_2 \ldots i_{n+1} j_{n+2}} = \sum_{P \in S_{n+1}} (-1)^{\pi(P)} \delta_{i_1 P(j_1) \ldots \delta_{i_{n+1} P(j_{n+1})}};
\]

(50)

where \( i_k, j_k = 1, \ldots, n \), \( k = 1, \ldots, n, n+1 \). Here, the sum is over all permutations \( P \) of the symmetric group \( S_{n+1} \) and \( \pi(P) \) denotes the parity of the permutation. The zero in the LHS of Eq.(50) comes about because there must be a repetition when distributing \( n \) objects among \( (n+1) \) places. The second equality is just the determinantal expansion of the product of two \( \epsilon \)-symbols.

By saturating the relation (50) with \( n+1 \) different matrices one obtains an identity among products of traces of products of matrices. The resulting Mandelstam identity can be written as [30], [31]

\[
\sum_{\text{Perm}(1,2,\ldots,n,n+1)} (-1)^{\pi(P)}W(M_1, M_2, \ldots, M_n, M_{n+1}) = 0,
\]

(51)

where

\[
W(M_1, M_2, \ldots, M_n, M_{n+1}) = Tr(M_{a(1)} \ldots M_{a(i)})Tr(M_{a(i+1)} \ldots),
\]

(52)

corresponds to the cycle decomposition \((a(1) \ldots a(i))(a(i+1) \ldots)\) of the permutation \( P \). The generic MI is a function of degree one in each of the \( n+1 \) different matrices involved. As an example, let us consider the \( 2 \times 2 \) case, which leads to the following MI among three matrices

\[
0 = Tr(M_1)Tr(M_2)Tr(M_3) + Tr(M_1 M_3 M_2) + Tr(M_1 M_2 M_3)
\]

\[
-T_1(M_1)Tr(M_2 M_3) - Tr(M_2)Tr(M_1 M_3) - Tr(M_3)Tr(M_1 M_2).
\]

(53)

The restriction implied among the classical loop variables is

\[
0 = T^0(\alpha_1) T^0(\alpha_2) T^0(\alpha_3) + T^0(\alpha_1 \circ \alpha_3 \circ \alpha_2) + T^0(\alpha_1 \circ \alpha_2 \circ \alpha_3)
\]

\[
-T^0(\alpha_1) T^0(\alpha_2 \circ \alpha_3) - T^0(\alpha_2) T^0(\alpha_1 \circ \alpha_3) - T^0(\alpha_3) T^0(\alpha_1 \circ \alpha_2),
\]

(54)

which has the following expression in terms of the quantum loop states

\[
0 = |\alpha_1 \cup \alpha_2 \cup \alpha_3 > + |\alpha_1 \circ \alpha_3 \circ \alpha_2 > + |\alpha_1 \circ \alpha_2 \circ \alpha_3 >
\]

\[
|\alpha_1 \cup \alpha_2 \circ \alpha_3 > + |\alpha_2 \cup \alpha_1 \circ \alpha_3 > + |\alpha_3 \cup \alpha_1 \circ \alpha_2 >.
\]

(55)
The restricted Mandelstam identities

To the author’s knowledge, there is no systematic procedure to construct the MI that reflect further restrictions on the group elements. In this section we give a brief description of the method used in Ref. [31] to implement the MI for the group SU\((N)\) realized in terms of \(N \times N\) matrices \(M_i\) in the adjoint representation. In this reference, the basic objects to be considered are the numbers \(W_k(M_1, M_2, \ldots M_k)\) which are defined through the following recurrence relation

\[
W_1(M_1) = \text{Tr}(M_1),
\]

\[
(k + 1)W_{k+1}(M_1, M_2, \ldots M_{k+1}) = W_1(M_{k+1})W_k(M_1, M_2, \ldots M_k) - W_k(M_1 M_{k+1}, M_2, \ldots M_k M_{k+1}).
\]

We have changed the notation with respect to Ref. [31], in order to be consistent with our own conventions. In this construction, the generic MI is written as

\[
W_{N+1}(M_1, M_2, \ldots M_{N+1}) = 0.
\]

Unitarity is imposed by demanding

\[
W_1(M^{-1}) = W_1(M)^*, \quad |W_1(M)| \leq N.
\]

Finally, the unit determinant condition can be written as \(W_N(M, M, \ldots M) = 1\). This, in turn, can be reexpressed in terms of \(N + 1\) matrices in the form

\[
W_N(M_1 M_{N+1}, M_2 M_{N+1}, \ldots M_N M_{N+1}) = W_N(M_1, M_2, \ldots M_N).
\]

This last form is useful to produce a geometric interpretation of the condition of unit determinant in the language associated with the loop description of this gauge theory.

5.2. The relation between the Mandelstam and the Cayley-Hamilton identities

In order to motivate the results of this section, let us consider an interesting particular case of Eq.(53), by setting \(M_1 = M_2 = M\), \(M_3 = X\), where \(X\) is an arbitrary \(2 \times 2\) matrix. Substituting in Eq. (53) and using the property that \(\text{Tr}(CX) = 0\), \(\forall X\), implies \(C = 0\), we obtain the matrix relation

\[
M^2 - Tr(M)M + \text{det}(M)I_2 = 0,
\]

where \(I_2\) is the \(2 \times 2\) identity matrix. In the above we have used the fact that \(\text{det}(M) = \frac{1}{2} \left(\text{Tr}(M)^2 - Tr(M^2)\right)\) for \(2 \times 2\) matrices. Next, we recognize that Eq.(60) is nothing but the statement of the Cayley-Hamilton theorem for the \(2 \times 2\) matrix \(M\). As we will show in the sequel, this construction can be generalized to \(n \times n\) matrices and can also be used in the reversed sense, that is to say, starting from the identities arising from the Cayley-Hamilton theorem, an algorithm to obtain the generic MI in the general case can be constructed.
The Cayley-Hamilton theorem in classical linear algebra. Let us provide a brief review of this important theorem. For an arbitrary $n \times n$ complex matrix $M$, the corresponding characteristic polynomial $P(x)$ is defined as

$$P(x) = \det(xI_n - M) = x^n + a_1x^{n-1} + \ldots + a_n.$$  \hfill (61)

The Newton equations provide a recursive method to calculate the coefficients $a_i$, in terms of traces of powers of $M$ [32]

$$a_{i+1} = -\frac{1}{i+1} \sum_{k=0}^{i} t_{i+1-k} a_k,$$  \hfill (62)

where $t_k = Tr(M^k)$. An explicit solution of the above recursion relations is

$$a_{i+1} = \sum_{\alpha_0 + \alpha_1 + \ldots + \alpha_s = i+1} \frac{(-1)^{s+1} t_{\alpha_0} t_{\alpha_1} \ldots t_{\alpha_s}}{(\alpha_0 + \alpha_1 + \ldots + \alpha_s)(\alpha_1 + \ldots + \alpha_s) \ldots (\alpha_s)},$$  \hfill (63)

where the summation is over all the unordered distinct partitions $(\alpha_0, \ldots, \alpha_s)$ of $i+1$, with $s+1$ being the total number of terms in each partition.

Extending the complex variable $x$ to a matrix-valued variable, the Cayley-Hamilton theorem states that each matrix annihilates its characteristic polynomial [33], i.e.

$$P(M) = M^n + a_1M^{n-1} + \ldots + a_nI_n = 0.$$  \hfill (64)

Let us emphasize that (64) is a matrix identity involving $n^2$ numerical relations. We will refer to the relations of the type (64) as the Cayley-Hamilton identities (CHI).

Now we describe the proposed algorithm to obtain the MI starting from the CHI [34]. The basic idea is to produce a sequence of CHI that leads to the MI, incorporating $n+1$ different matrices and recalling that the final MI must be an homogeneous function of degree one in each matrix. The procedure is as follows: starting from the identities $P_{M_1}(M_1) = 0$, $P_{M_2}(M_2) = 0$, $P_{M_1+M_2}(M_1+M_2) = 0$ we construct

$$T_2(M_1, M_2) := P_{M_1+M_2}(M_1 + M_2) - P_{M_1}(M_1) - P_{M_2}(M_2) = 0.$$  \hfill (65)

We are denoting by $P_M(x)$ the characteristic polynomial corresponding to the matrix $M$ and also we have set $P_M(M) := T_1(M)$. Every term in (65) is an homogeneous function of $M_1$ and $M_2$ and the subtractions are designed in such a way that both $M_1$ and $M_2$ appear at least once in every term of $T_2(M_1, M_2)$. In this way $T_2(M_1 = 0, M_2)$ and $T_2(M_1, M_2 = 0)$ are identically zero. Moreover, we consider that $T_2(M_1, M_2)$ can be fully expanded using the distributive property of both the trace and the matrix product with respect to matrix addition. The next step is to construct
\[ T_3(M_1, M_2, M_3) := P_{M_1+M_2+M_3}(M_1 + M_2 + M_3)_{\text{red}}, \]
\[ := P_{M_1+M_2+M_3}(M_1 + M_2 + M_3) - T_3(M_1, M_2) - T_2(M_1, M_3) \]
\[ - T_2(M_2, M_3) - T_1(M_1) - T_1(M_2) - T_1(M_3) = 0. \] 

(66)

Again, \( T_3 \) is identically zero whenever any of the \( M_i \) is set equal to zero. We have introduced the subscript \( |_{\text{red}} \) to indicate an identity which has been reduced in such a way that every matrix involved is present at least once in every term, after the identity is fully expanded. In other words, the expression \( P_{M_1+M_2+M_3}(M_1 + M_2 + M_3)_{\text{red}} \) in (66) can be directly constructed by expanding the corresponding characteristic polynomial and discarding all terms in which any one of the three matrices is missing.

Extending this idea, we construct reduced identities of always increasing order, where we subtract all the lower order identities at our disposal. This procedure leads to

\[ T_k(M_1, \ldots, M_k) := P_{M_1+\ldots+M_k}(M_1 + \ldots + M_k) \]
\[ - \sum_{i<k} T_i(M_{s_1}, \ldots, M_{s_i}), \] 

(67)

where the sum is carried over all subsets \( \{s_1, \ldots, s_i\} \) of \( \{1, \ldots, k\} \). The fact that the characteristic polynomial is of order \( n \) guarantees that \( T_k(M_1, \ldots, M_k) \) are identically zero for \( k \geq n + 1 \). The expression \( T_n(M_1, \ldots, M_n) \) is homogeneous of degree one in each of the \( n \) matrices involved.

The generic MI is obtained from \( \text{Tr} (T_n(M_1, \ldots, M_n)M_{n+1}) \) as shown with all detail in Ref. [34]. The converse statement, which amounts to recovering the CHI (64) starting from the MI (51), is also proved in general in this work.

Before closing this section let us observe that the immediate application of the above method to the construction of the MI in the case where further restrictions upon the group elements are required, will work only if such restrictions are preserved by matrix addition [35]. In particular, the restriction to matrices with unit determinant cannot be directly implemented in this way. It remains still an open problem to work out a general procedure to construct the MI corresponding to arbitrarily restricted groups.

Nevertheless, there are two interesting examples where a restricted MI is directly obtained from the characteristic polynomial. These correspond to the cases of the groups \( SL(2, \mathbb{R}) \) and \( SU(2) \), which are relevant in the study of \( 2 + 1 \) de Sitter gravity, and \( 3 + 1 \) Einstein gravity in the Ashtekar formulation, respectively. Let us consider two of such \( 2 \times 2 \) matrices, \( M_1 \) and \( M_2 \), with unit determinant. The CHI for the first matrix can be rewritten as

\[ M_1 - \text{Tr}(M_1)\mathcal{T}_2 + M_1^{-1} = 0, \] 

(68)
after the Eq. (60), with \( \text{det}(M_1) = 1 \), has been multiplied by \( M_1^{-1} \). Multiplying further Eq. (68) by \( M_2 \) and taking the trace, we end up with the restricted MI.
\[ Tr(M_1 M_2^{-1}) + Tr(M_1 M_2) = Tr(M_1) Tr(M_2), \]

which involves only two matrices, instead of three as in the generic case presented in Eq. (53).

In the case of the group $SL(2, \mathbb{R})$ it is also possible to exhibit an example of the drastic reduction of the independent loop space variables produced by the MI. Let us consider the infinite set of Wilson loops of the form

\[ Tr(M_1^{p_1} M_2^{q_1} M_1^{p_2} M_2^{q_2} \ldots M_1^{p_n} M_2^{q_n} \ldots), \]

for any integer $p_i, q_i$. Using the MI (69) one can show that any of such Wilson loops can be expressed as a function of three traces only: $Tr(M_1), Tr(M_2)$ and $Tr(M_1 M_2)$ [36]. Thus, in the corresponding sector of the theory we will have only three independent degrees of freedom. A simple example of such reduction is to consider $Tr(M_1^2 M_2)$ for example. Here we apply the relation (69) with $M_1 \to M_1, M_2 \to M_1 M_2$ obtaining

\[ Tr(M_1^2 M_2) = Tr(M_1) Tr(M_1 M_2) - Tr(M_2). \]  

6. THE CAYLEY-HAMILTON THEOREM FOR SUPERMATRICES

Our main motivation for the work described in the previous section has been the possibility of extending these ideas to the loop space formulation of pure gauge theories having their connection valued on a super Lie algebra, as it is the case of supergravity in the Ashtekar variables formalism [8], for example. In this situation, the group elements are described by generic supermatrices as defined in Eq. (28). The knowledge of the MI for supermatrices will be also relevant to the loop space formulation of fully supersymmetric gauge theories.

It is well known that Grassmann numbers can be realized in terms of complex numerical matrices. From this point of view one could think that the Cayley-Hamilton theorem for supermatrices would be just a trivial extension of the standard case. Nevertheless, this is not the case for at least two reasons: (i) after a realization of the Grassmann numbers in terms of numerical matrices it would not be possible to recover a result in terms of the original Grassmann numbers. (ii) the matrix realization will effectively augment the size of the resulting supermatrix, which now would be completely numerical, in such a way that the resulting standard characteristic polynomial would be also of higher degree.

Let us give a precise meaning to the above observations using the simplest case of a $(1 + 1) \times (1 + 1)$ supermatrix

\[ M_2 = \begin{pmatrix} p & \eta \\ \theta & q \end{pmatrix}, \]

where $p, q$ are even Grassmann numbers while $\eta, \theta$ are odd Grassmann numbers such that $\eta^2 = 0 = \theta^2$, $\eta \theta = -\theta \eta$. The minimum size for the
realization of these Grassmann numbers corresponds to $4 \times 4$ gamma-matrices, like $\eta = \gamma_0 + \gamma_1$, $\theta = \gamma_2 + i\gamma_3$ in the signature $(+,-,-,-)$, for example. In this way, $M_2$ will be realized as an $8 \times 8$ numerical matrix and, consequently, the resulting characteristic polynomial will be a numerical polynomial of degree 8. Clearly, it will not be possible to rewrite the numerical coefficients of the polynomial in terms of the original Grassmann numbers. Nevertheless, it is possible to find a characteristic polynomial of degree 2 for this case, which is given by

$$P_2(x) = (q - p)x^2 - (q^2 - p^2 - 2\eta \theta)x + ((q - p)qp - (q + p)\eta \theta). \quad (72)$$

We can verify, by direct substitution, that the above polynomial satisfies $P_2(M_2) = 0$. In this simple case, the polynomial (72) can be constructed just by solving this condition. When $(q - p)^2 \neq 0$, the polynomial (72) can be redefined in the monic form of Eq.(61) with the choices $a_1 = q + p - \frac{2}{q-p}\eta \theta$ and $a_2 = qp - \frac{q + p}{q-p}\eta \theta$. Thus, $a_1$ generalizes $Tr(M_2) = p + q$, while $a_2$ generalizes $det(M_2) = qp - \eta \theta$ corresponding to the case where all entries in $M_2$ would be complex numbers. Here we also see that the standard determinant is not well defined for supermatrices: we would have to make a choice among all possibilities $qp - (A\eta \theta + (1 - A)\theta \eta)$, for arbitrary $A$. In this way, the heuristical result (72) certainly motivates the search for a general procedure to construct such null polynomials, leading to the construction of the Cayley-Hamilton theorem for supermatrices. This is an interesting problem in its own, besides the possible applications to the loop space formulation of gauge theories involving supersymmetry.

### 6.1. The characteristic and null polynomials for supermatrices

The first step in this direction is to provide a definition of the characteristic polynomial, which extends the standard one given in Eq. (61). This problem is certainly related to the eigenvalue problem of a supermatrix, which is discussed in Ref. [37]. The eigenvalues of an $(m+n) \times (m+n)$ supermatrix $M$ are even Grassmann numbers which can be of two types: (i) first-class eigenvalues, $\lambda_i$, $i = 1, \ldots, m$, whose eigenvectors are of the form $[E_1, \ldots, E_m, O_1, \ldots, O_n]^T$, where $E_k$ denotes even Grassmann numbers, while $O_l$ labels odd Grassmann numbers. (ii) second-class eigenvalues, $\bar{\lambda}_{\alpha}$, $\alpha = 1, \ldots, n$ with corresponding eigenvectors of the form $[O_1, \ldots, O_m, E_1, \ldots, E_n]^T$. The characteristic polynomial will certainly read

$$P(x) = (x - \lambda_1) \ldots (x - \lambda_m)(x - \bar{\lambda}_1) \ldots (x - \bar{\lambda}_n), \quad (73)$$

in terms of the eigenvalues of the supermatrix. When written in the form of Eq. (61), the explicit expressions for the coefficients $a_i$ will be given again by Eq. (63) with $t_k = \lambda_k^k + \ldots + \lambda_k^m + \bar{\lambda}_k^1 + \ldots + \bar{\lambda}_k^n$. A problem now arises if we want to rewrite such coefficients in terms of the supermatrix itself: the appropriate invariants for this case are not traces of powers of the supermatrix, but supertraces of powers of
the supermatrix instead. The latter are given by $Str(M^k) = \lambda_1^k + \ldots + \lambda_m^k - \tilde{\lambda}_1^k - \ldots - \tilde{\lambda}_n^k$. These expressions break the permutation symmetry among all eigenvalues, leaving only a permutation symmetry among the first-class eigenvalues together with an independent permutation symmetry among the second-class eigenvalues. The coefficients $a_i$ are nevertheless symmetric in the whole set of eigenvalues and this is precisely why it is a more involved task to rewrite them in terms of the corresponding supertraces.

It would seem that a reasonable starting point for the construction of null polynomials in the case of supermatrices is the natural extension of the definition (61) to $P(x) \to Sdet(xI - M)$. Nevertheless, this function is not a polynomial but a ratio of polynomials, as can be seen from its expression in terms of the eigenvalues $Sdet(xI - M) = (x - \lambda_1) \ldots (x - \lambda_m) (x - \tilde{\lambda}_1) \ldots (x - \tilde{\lambda}_n)$. (74)

As a preliminary step towards the definition of the null polynomial associated to the supermatrix (28), let us introduce the standard null polynomials corresponding to the even block-matrices $A$ and $D$

$$a(x) = \det(xI_m - A), \quad d(x) = \det(xI_n - D),$$

respectively. Next, let us consider the general expression for

$$h(x) := Sdet(xI_{m+n} - M),$$

which will be called the characteristic function in the sequel. From the two alternative expressions to calculate the superdeterminant, given in Eq. (32), we are able to write the characteristic function as

$$h(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{F(x)}{G(x)},$$

where the basic polynomials $\tilde{F}$, $\tilde{G}$, $F$ and $G$ are given by

$$\tilde{F}(x) = \det(d(x)(xI - A) - B\text{adj}(xI - D)C), \quad \tilde{G}(x) = (d(x))^{m+1},$$

$$F(x) = (a(x))^{n+1}, \quad G(x) = \det(a(x)(xI - D) - C\text{adj}(xI - A)B).$$

The above expressions are readily obtained from Eqs. (32), using the relation $(xI - F)^{-1} = [\det(xI - F)]^{-1}\text{adj}(xI - F)$ valid for any even matrix $F$. Notice that $\tilde{F}$ is expressed in terms of the determinant of a $m \times m$ even matrix, while $G(x)$ is the determinant of a $n \times n$ even matrix.

In order to motivate the basic idea of our definition for the characteristic polynomial of a supermatrix, let us consider the simple case of a block-diagonal supermatrix $M$ (i.e. $B = 0, C = 0$). Here $h(x) = a(x)/d(x)$ and clearly the characteristic polynomial is $P(x) = a(x)d(x)$, which is the product of the numerator and the denominator of the corresponding superdeterminant. In fact we have
\[ P(M) = \begin{pmatrix} a(A) & 0 \\ 0 & a(D) \end{pmatrix} \begin{pmatrix} d(A) & 0 \\ 0 & d(D) \end{pmatrix} \equiv 0, \] (79)

because \( a(A) = 0, d(D) = 0 \). An analogous statement is obtained for a supermatrix written in terms of its eigenvalues. In the general case, where \( h(x) \) is given by Eq.(77), the numerator of the superdeterminant is \( \tilde{F}(F) \) while the denominator is \( \tilde{G}(G) \). These observations lead to the following definition of the characteristic polynomial of a supermatrix

\[ \mathcal{P}(x) := \tilde{F}(x)\tilde{G}(x) = F(x)\tilde{G}(x), \] (80)

in terms of the basic polynomials \( \tilde{F}, \tilde{G}, F \) and \( G \), given in Eqs.(78). For notational simplicity we will not necessarily write explicitly the \( x \)-dependence on many of the polynomials considered in the sequel.

Let us consider again the block-diagonal case, this time when \( a(x) \) and \( d(x) \) have a common factor \( f(x) \), i.e.

\[ a(x) = f(x)a_1(x), \quad d(x) = f(x)d_1(x). \] (81)

In this example, a null polynomial is given by \( P(x) = f(x)a_1(x)d_1(x) \), which is a polynomial of degree lower than the product \( a(x)d(x) \). Motivated by this fact, together with the work of Ref. [38], we realize that there are some cases in which we can construct null polynomials of lower degree than \( \mathcal{P}(x) \), according to the factorization properties of the basic polynomials \( \tilde{F}, \tilde{G}, F, G \).

At this point it is important to observe that we do not have a unique factorization theorem for polynomials defined over a Grassmann algebra. This can be seen, for example, from the identity \( x^2 = (x + z\alpha)(x - z\alpha) \), where \( \alpha \) is an even Grassmann with \( \alpha^2 = 0 \) and \( z \) is an arbitrary complex number.

The construction of the null polynomials of lower degree starts from finding the divisors of the pairs \( \tilde{F}, \tilde{G}, (F, G) \) which we denote by \( R, (S) \) respectively. This means that one is able to write

\[ \tilde{F} = R\tilde{f}, \quad \tilde{G} = R\tilde{g}, \]
\[ F = Sf, \quad G = Sg, \] (82)

where all polynomials are monic and also \( \tilde{f}, \tilde{g}, f, g \) are of lower degree that their parents \( \tilde{F}, \tilde{G}, F, G \), by construction. They must satisfy

\[ \tilde{f}/\tilde{g} = f/g, \] (83)

because of Eq. (77). The expressions in (82) might not be unique.

Let us emphasize that in the case of polynomials over the complex numbers, when \( R \) and \( S \) are of maximum degree, Eq. (83) would imply at most \( \tilde{f} = \lambda f, \tilde{g} = \lambda g \) with \( \lambda \) being a constant. Since we are considering polynomials over a Grassmann algebra, this is not necessarily true as can be seen again in the above mentioned identity \( x/(x - z\alpha) = (x + z\alpha)/x \), which we have rewritten in a convenient way.
The above discussion leads to the following definition: given an arbitrary \((m + n) \times (m + n)\) supermatrix \(M\), with a characteristic function \(h(x)\) such that the polynomials \(\tilde{F}, \tilde{G}\) have a common factor \(R\) and the polynomials \(F, G\) have a common factor \(S\), satisfying Eqs. (82) and (83), then a null polynomial of \(M\) is given by

\[
P(x) := \tilde{f}(x)g(x) = f(x)\tilde{g}(x). \tag{84}
\]

The polynomial (84) is clearly of lower degree than \(P(x)\), which is just a particular case of the null polynomials (84) when \(R = S = 1\). We will concentrate mostly on (84) in the sequel.

6.2. Proof of the Cayley-Hamilton theorem for supermatrices

In this section we show that the polynomial defined in Eq.(84) does in fact annihilates the supermatrix \(M\). To this end, we first extend a lemma often used to prove the Cayley-Hamilton theorem for ordinary matrices [33]. We briefly recall such lemma and emphasize that it is independent of the matrix considered being a standard matrix or a supermatrix. It goes as follows: let \(M\), \((xI - M)\) and \(N(x)\) be \((m + n) \times (m + n)\) supermatrices, where \(M\) is independent of \(x\). Let \(N(x)\) be a polynomial supermatrix of degree \((p - 1)\) in \(x\), i.e. \(N(x) = N_0 x^{p-1} + N_1 x^{p-2} + \ldots + N_{p-1} x^0\), (where each \(N_k\), \(k = 0, \ldots, p-1\), is a \((m + n) \times (m + n)\) supermatrix independent of \(x\)), such that

\[
(xI_{m+n} - M)N(x) = P(x)I_{m+n}, \tag{85}
\]

where \(P(x) = a_0 x^p + a_1 x^{p-1} + \cdots + a_n x^0\) is a numerical polynomial of degree \(p\). Then, one can prove that \(P(M) := a_0 M^p + a_1 M^{p-1} + \cdots + a_n I_{m+n} \equiv 0\). The proof follows by comparing the independent powers of \(x\) in Eq. (85) and then explicitly computing \(P(M)\) [33].

In the standard case, the polynomial matrix \(N(x)\) is just given by \(N(x) = \text{adj}(xI - M) = \text{det}(xI - M)(xI - M)^{-1}\), and \(P(x) = \text{det}(xI - M)\). In the case of a supermatrix we do not have an obvious generalization either of the matrix \(\text{adj}(xI - M)\) or of \(\text{det}(xI - M)\). Nevertheless, following the analogy as close as possible we define

\[
N(x) := P(x)(xI - M)^{-1}, \tag{86}
\]

where \(P(x)\) is the polynomial introduced in Eq.(84) of the previous section.

The challenge now is to prove that \(N(x)\), which trivially satisfies the Eq. (85), is indeed a polynomial matrix. In this way we would have proved that \(P(M) = 0\) according to the property stated after Eq.(85).

To begin with, we show that the blocks corresponding to the inverse supermatrix \((xI_{m+n} - M)^{-1}\) can be written in a compact form as
\[ (x \mathcal{I}_{m+n} - M)^{-1} = -\frac{1}{F} \frac{\partial \tilde{F}}{\partial A_{ji}}, \quad (x \mathcal{I}_{m+n} - M)^{-1} = \frac{1}{G} \frac{\partial G}{\partial C_{\alpha_i}} \] 
\[ (x \mathcal{I}_{m+n} - M)^{-1} = \frac{1}{F} \frac{\partial \tilde{F}}{\partial B_{j_\alpha}}, \quad (x \mathcal{I}_{m+n} - M)^{-1} = \frac{1}{G} \frac{\partial G}{\partial D_{\beta\alpha}}, \]

where $A_{ij}, B_{j\alpha}, C_{\alpha j}$ and $D_{\alpha\beta}$ are the entries of the supermatrix $M$ defined in Eq. (28) and $\tilde{F}, G,$ are the polynomials given in Eqs. (78). The derivative with respect to an odd Grassmann number is taken to be a left derivative, defined such that $\delta \tilde{F} \equiv \delta B_{j\alpha} \frac{\partial \tilde{F}}{\partial B_{j\alpha}}$. The proof of the above equations begins with the calculation of $(x \mathcal{I}_{m+n} - M)^{-1}$ in block form, with the results

\[
(x \mathcal{I}_{m+n} - M)_{11}^{-1} = ((x \mathcal{I}_m - A) - B(x \mathcal{I}_n - D)^{-1}C)^{-1},
(x \mathcal{I}_{m+n} - M)_{12}^{-1} = -(x \mathcal{I}_m - A)^{-1}B((x \mathcal{I}_n - D) - C(x \mathcal{I}_m - A)^{-1}B)^{-1},
(x \mathcal{I}_{m+n} - M)_{21}^{-1} = -(x \mathcal{I}_n - D)^{-1}C((x \mathcal{I}_m - A) - B(x \mathcal{I}_n - D)^{-1}C)^{-1},
(x \mathcal{I}_{m+n} - M)_{22}^{-1} = ((x \mathcal{I}_n - D) - C(x \mathcal{I}_m - A)^{-1}B)^{-1}.
\] 

(89)

Here, the subindices 11, 12, 21 and 22 denote the corresponding $m \times m, m \times n, n \times m,$ and $n \times n$ blocks. The above block form in Eq. (89) has the same structure as in the classical case. Let us concentrate now in the 11 block. Rewriting all the inverse matrices of the first Eq.(89) in terms of their adjoints, together with the corresponding determinants, we obtain

\[ (x \mathcal{I}_{m+n} - M)_{11}^{-1} = \frac{d}{F} \text{adj}((x \mathcal{I}_m - A)d - \text{Badj}(x \mathcal{I}_n - D)C). \] 

(90)

On the other hand, using the basic property

\[ \delta \text{det}\, Q = \text{Tr}(\text{adj}\, Q \delta Q), \]

(91)
valid for any even matrix $Q$, we can calculate the change of $\tilde{F}$ with respect to $A_{ij}$, keeping constant all other entries, obtaining

\[ \delta \tilde{F} = -d [\text{adj}((x \mathcal{I}_m - A)d - \text{Badj}(x \mathcal{I}_n - D)C)]_{ij} \delta A_{ji}, \]

(92)
which can be written as

\[ \frac{\partial \tilde{F}}{\partial A_{ji}} = -d [\text{adj}((x \mathcal{I}_m - A)d - \text{Badj}(x \mathcal{I}_n - D)C)]_{ij}. \]

(93)

The comparison of Eq. (93) with Eq. (90) completes the proof of the first relation in Eq. (87). The proof for the remaining Eqs. (87-88) is carried along similar steps.

We observe that the conditions for the existence of $(x \mathcal{I}_{m+n} - M)^{-1}$ are the same as those for the existence of $S\text{det}(x \mathcal{I}_{m+n} - M)$ which read $\text{det}(x \mathcal{I}_m - A) \neq 0$ and $\text{det}(x \mathcal{I}_n - D) \neq 0$. Since $x$ is a generic even Grassmann variable, we will assume that this is always the case. In this way, the term $((x \mathcal{I}_m - A) - B(x \mathcal{I}_n - D)^{-1}C)^{-1}$, for
example, can always be calculated as 

\[(I_m - (xI_m - A)^{-1}B(xI_n - D^{-1})C)^{-1}(xI_m - A)^{-1}.\]

The factor on the left can be thought as a series expansion of the form

\[
\frac{1}{1-z} = 1 + z + z^2 + \cdots, \quad \text{with } z = (xI_m - A)^{-1}B(xI_n - D^{-1})C. \]

Moreover, the series will stop at some power because \(z\) is a nilpotent matrix.

Now we are in position to show the principal result of this section which is that

\[N(x) := P(x)(xI_m + n - M)^{-1}, \]

with \(P(x)\) given in Eq.(84), is a polynomial supermatrix. Let us consider the block-element \(11\) of \(N(x)\) to begin with. According to Eqs. (87-88) together with Eq. (82), this block can be written as

\[N_{ij} = -g \frac{\partial \tilde{f}}{\partial A_{ji}} - \frac{g \tilde{f}}{R} \frac{\partial R}{\partial A_{ji}}, \quad (94)\]

The first term of the RHS in Eq.(94) is clearly of polynomial character. In order to see that the second term is also polynomial, we make use of the property

\[
\frac{\partial \ln \tilde{G}}{\partial A_{ji}} = 0 = \frac{\partial \ln R}{\partial A_{ji}} + \frac{\partial \ln \tilde{g}}{\partial A_{ji}}, \quad (95)
\]

which follows from the factorization \(\tilde{G} = R\tilde{g}\), together with the fact that \(\tilde{G}\) is just a function of \(D_{\alpha\beta}\), according to the first Eq. (78). In this way, and using also the Eq.(83), we obtain

\[N_{ij} = f \frac{\partial \tilde{g}}{\partial A_{ji}} - g \frac{\partial \tilde{f}}{\partial A_{ji}}, \quad (96)\]

which leads to the conclusion that the block-matrix \(N_{ij}\) is indeed polynomial. The proof for \(N_{ai}\) runs along the same lines, except that now the derivatives are taken with respect to \(B_{ia}\) and that we have to use \(\frac{\partial \ln \tilde{G}}{\partial B_{ia}} = 0\), instead of Eq. (95). The remaining terms \(N_{ia}\) and \(N_{a\beta}\) can be dealt with in analogous manner, by considering the derivatives of \(G = Sg\) with respect to \(C_{ai}\) and \(D_{\beta\alpha}\), and by replacing the condition (95) by \(\frac{\partial \ln F}{\partial C_{ai}} = 0\) and \(\frac{\partial \ln F}{\partial D_{\beta\alpha}} = 0\) respectively. The results are again of the form (96), the only difference been the variables with respect to which the derivatives are taken.

Finally, we can state the following extension of the Cayley-Hamilton theorem to the case of supermatrices [39], [40], [41].

**Theorem** (Extended Cayley-Hamilton Theorem) Let \(M\) and \((xI - M)\) be \((m + n) \times (m + n)\) supermatrices, with \(x\) being a generic even Grassmann variable. Let also \(Sdet(xI_{m+n} - M) = \tilde{F}/\tilde{G} = F/G\), where the polynomials \(\tilde{F}, \tilde{G}, F\) and \(G\) are given in Eqs.(78). Then, for any common factor \(R\) such that \(\tilde{F} = R\tilde{f}, \tilde{G} = R\tilde{g}\) and \(S\) such that \(F = Sf, G = Sg\), where \(f/g = \tilde{f}/\tilde{g}\), the polynomial \(P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x)\) annihilates \(M\), i.e. \(P(M) = 0\).

### 6.3. Examples of null polynomials for supermatrices

Here we present two simple examples of null polynomials, constructed according to the procedure stated in the last section.
The case of \((1+1) \times (1+1)\) supermatrices. Let us consider the supermatrix (71) with \(\bar{d} \neq \bar{q}\). Here the bar denotes the complex component of an even Grassmann number, called the body in the literature. From Eqs. (78) we obtain the following basic polynomials

\[
\tilde{F} = (x - q)(x - p) - \eta\theta, \quad \tilde{G} = (x - q)^2, \\
F = (x - p)^2, \quad G = (x - q)(x - p) + \eta\theta,
\]

(97)

The above functions can be rewritten as follows

\[
\tilde{F} = \left(x - p + \frac{\eta\theta}{q - p}\right) \left(x - q - \frac{\eta\theta}{q - p}\right), \\
\tilde{G} = \left(x - q + \frac{\eta\theta}{q - p}\right) \left(x - q - \frac{\eta\theta}{q - p}\right), \\
F = \left(x - p + \frac{\eta\theta}{q - p}\right) \left(x - p - \frac{\eta\theta}{q - p}\right), \\
G = \left(x - q + \frac{\eta\theta}{q - p}\right) \left(x - p - \frac{\eta\theta}{q - p}\right).
\]

(98)

Thus, the factorization properties of Eq.(82) are realized with

\[
R = \left(x - q - \frac{\eta\theta}{q - p}\right), \quad S = \left(x - p - \frac{\eta\theta}{q - p}\right), \\
\tilde{f} = f = \left(x - p + \frac{\eta\theta}{q - p}\right), \quad \tilde{g} = g = \left(x - q + \frac{\eta\theta}{q - p}\right).
\]

(99)

In this way, using the definition (84), we recover the null polynomial of minimum degree given in Eq.(72).

The case of \(Osp(1|2; C)\) supermatrices. Another simple example corresponds to the case of supermatrices belonging to the supergroup \(Osp(1|2; C)\), which are relevant in the description of de Sitter supergravity in 2 + 1 dimensions [29]. We consider a \((2 + 1) \times (2 + 1)\) realization of this supergroup defined by the set of all supermatrices \(M\) which leave invariant the supersymplectic form \(H\)

\[
M^T H M = H, \quad H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(100)

where \(T\) denotes the supertransposed. The supermatrices in Eq.(100) can be parametrized in the following way

\[
M = \begin{pmatrix} A & \xi \\ \chi^T & a \end{pmatrix}, \quad \xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

(101)
where $A$ is a $(2 \times 2)$ even matrix; $x_1, x_2$ are arbitrary odd Grassmann numbers and the superindex $T$ denotes standard transposition. The condition (100) translates into the following relations among the elements of Eq. (101)

$$\chi^T = \xi^T EA, \quad a = 1 + x_1x_2, \quad \det(A) = 1 - x_1x_2,$$

(102)

where $E$ denotes the $2 \times 2$ antisymmetric block of $H$ in Eq. (100). Using the algorithm of the previous section, we conclude that the unique irreducible expression for the characteristic function is

$$h(x) = \frac{x^2 - (1 + \text{Str}(M))x + 1}{x - 1},$$

(103)

which means $f = \tilde{f}, g = \tilde{g}$. The null polynomial of minimum degree is then [29]

$$P(x) = fg = x^3 - (2 + \text{Str}(M))(x^2 - x) - 1.$$

(104)

6.4. Two examples of Mandelstam identities for supermatrices

As a first step towards the search of an algorithm to produce the MI for supermatrices, one may try to directly extend the procedure developed in section 5.3 for the case of ordinary matrices. The starting point now will be the null polynomials constructed in section 6.1, which can be rewritten in terms of a finite number of supertraces. In relation to this, we still suffer a main drawback which is the lack of knowledge of a recurrence that would allow to obtain a closed expression for the coefficients of the null polynomial in terms of supertraces, in a manner similar to the standard case. Another new feature is that the null polynomials are not monic any more, in such a way that $a_0$ would now be a function of the supertraces. This property will effectively raise the degree of homogeneity of the supermatrices in the corresponding CHI. Nevertheless, such identities will be homogeneous of some degree, say $t$ for example. This will allow us to make the following definition of the corresponding generic MI

$$\text{Str} \left( P_{M_1+M_2+\ldots+M_t}(M_1 + M_2 + \ldots + M_t)\mid_{\text{red}M_{t+1}} \right) = 0.$$

(105)

The case of $(1 + 1) \times (1 + 1)$ supermatrices. In this situation, the null polynomial (72) can be rewritten as

$$\text{Str}(M)M^2 - (\text{Str}(M^2))M + \frac{1}{3} \left( \text{Str}(M^3) - \text{Str}(M)^3 \right) I_{2+1} = 0,$$

(106)

in terms of supertraces, with $t = 3$. Using the definition (105) we obtain
\[ \text{Str}(A) (\text{Str}(BCD) + \text{Str}(CBD)) + \text{Str}(B) (\text{Str}(ACD) + \text{Str}(CAD)) \]
\[ \text{Str}(C) (\text{Str}(ABD) + \text{Str}(BAD)) + \text{Str}(D) (\text{Str}(ABC) + \text{Str}(BCA)) \]
\[ -2\text{Str}(AB)\text{Str}(CD) - 2\text{Str}(BC)\text{Str}(AD) - 2\text{Str}(AC)\text{Str}(BD) \]
\[ -2\text{Str}(A)\text{Str}(B)\text{Str}(C)\text{Str}(D) = 0, \quad (107) \]

which corresponds to a symmetric MI of order four. We have verified this identity using Mathematica.

The case of $Osp(1|2; C)$ supermatrices. This is an example of a restricted MI, which is the supersymmetric analogue of the identity (69), valid for the group $SU(2)$. Starting from the null polynomial (104) for the supermatrix $M_1$, multiplying this equation by $M_1^{-1}M_2$ and taking the supertrace we are left with

\[ \text{Str}(M_2M_2^2) - (2 + \text{Str}(M_1))(\text{Str}(M_1M_2) - \text{Str}(M_2)) - \text{Str}(M_2M_2^{-1}) = 0. \quad (108) \]

The above identity has been useful in the identification of the true degrees of freedom on one sector of $2 + 1$ super de Sitter gravity. For one genus of the generic spatial surface, the most general Wilson loop variables are the infinite set of supertraces:

\[ \text{Str}(M_1^{p_1} M_2^{q_1} M_1^{p_2} M_2^{q_2} \ldots M_1^{p_n} M_2^{q_n} \ldots), \]

for any integer $p_i, q_i$. Using the MI (108), it is possible to reduce the above infinite set of observables to only five complex quantities, which are $\text{Str}(M_1), \text{Str}(M_2), \text{Str}(M_1M_2), \text{Str}(M_1M_2^2)$ and $\text{Str}(M_1M_2M_2^2)$ [29], in complete analogy with $2 + 1$ de Sitter gravity [36].

## 7. SUMMARY AND OPEN PROBLEMS

We have presented a rather sketchy review of the loop space formulation of gauge theories, which does not make full justice to all the numerous achievements and applications that this method has produced so far. We have tried to incorporate a reasonable, but certainly not exhaustive, list of references which remedy this situation, offering the reader a detailed version in each situation.

Our major emphasis has been in the discussion of the formulation of the Mandelstam identities, which appear as unavoidable constraints either among the Wilson loop variables or among the quantum loop states, that constitute the natural degrees of freedom of the method in the classical and quantum situation, respectively. In the case of pure gauge theories over a Lie algebra, we have shown the equivalence between the generic Mandelstam identities, for a given dimension of the representation, and the identities arising from the application of the Cayley-Hamilton theorem to the matrices of such representation. We have provided an algorithm to go in either direction. The main thrust of this development has been its extension to provide a general procedure to construct the generic Mandelstam identities in the case of pure gauge theories defined over a super Lie algebra, as a first step to deal with fully supersymmetric theories. A previous step in this direction has been the formulation and proof of the Cayley-Hamilton theorem in the case of supermatrices. The extension of this theorem proceeded in two steps: (i) the identification and definition of a characteristic polynomial for supermatrices and (ii) the proof that each
supermatrix annihilates the polynomial previously defined. Furthermore, starting from the characteristic function (76), we have described a systematic method for constructing null polynomials for supermatrices. The resulting Cayley-Hamilton identities can be subsequently used to derive the corresponding Mandelstam identities. Two simple examples were presented. The construction of the Mandelstam identities for arbitrary restricted groups or supergroups remains still an open problem.

Another very interesting question in the loop space formulation of gauge theories is the inclusion of fermions, which has been the subject of recent investigations. There are at least three ways of approaching this problem: (i) one is to consider the fermions as standard matter to be coupled to the gauge theory, (ii) other possibility is to consider them as pieces of a superconnection, as has been done in these work and (iii) the final possibility is to introduce the fermions in a fully supersymmetric theory, as partners of integer spin fields. In the first case, several matter fields like electrons [42] and quarks [43] have been taken onto account in the loop space picture. Also, the introduction of fermions in Einstein gravity has been considered in Ref. [44]. The basic idea in these works is to define additional gauge invariant variables, besides the Wilson loops, represented by open paths which start and end up in the fermions. As mentioned previously in the text, the possibility (ii) has been already considered in the case of 2 + 1 super Chern-Simons theories and also in the recent discussion of 3 + 1 canonical supergravity in the loop space approach. From a general point of view, there is still lacking a proof of the equivalence between the proposed loop space representation of these theories and the standard connection-matter formulation of them. This will require, among many other developments, the construction of generalized Mandelstam identities including the open path variables together with the generalization of Giles theorem to the superconnection case. Finally, the alternative (iii) has recently being explored in Ref. [45], where the fully supersymmetric Wilson loop has been constructed in terms of chiral superfields and supercurrents in superspace.

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