Indefinite $q$-integrals from a method using $q$-Ricatti equations

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**ABSTRACT**

Earlier work introduced a method for obtaining indefinite $q$-integrals of $q$-special functions from the second-order linear $q$-difference equations that define them. In this paper, we reformulate the method in terms of $q$-Ricatti equations, which are nonlinear and first order. We derive $q$-integrals using fragments of these Ricatti equations, and here only two specific fragment types are examined in detail. The results presented here are for $q$-Airy function, Ramanujan function, Jackson $q$-Bessel functions, discrete $q$-Hermite polynomials, $q$-Laguerre polynomials, Stieltjes-Wigert polynomial, little $q$-Legendre, and big $q$-Legendre polynomials.

**KEYWORDS**

$q$-integrals, $q$-Bernoulli fragment, $q$-Linear fragment, Simple algebraic form, $q$-Airy function, Ramanujan function.

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1. Introduction and Preliminaries

In [11], we introduced a method to obtain indefinite $q$-integrals of the form

$$\int f(x)\left(\frac{1}{q}D_{q^{-1}}D_q h(x) + p(x)D_{q^{-1}}h(x) + r(x)h(x)\right)y(x)d_qx = f(x/q)\left(y(x/q)D_{q^{-1}}h(x) - h(x/q)D_{q^{-1}}y(x)\right) \quad (1.1)$$

or

$$\int F(x)\left(\frac{1}{q}D_{q^{-1}}D_q k(x) + p(x)D_qk(x) + r(x)k(x)\right)y(x)d_qx = F(x)\left(y(x)D_{q^{-1}}k(x) - k(x)D_{q^{-1}}y(x)\right), \quad (1.2)$$

where the functions $p(x)$ and $r(x)$ are continuous functions in an interval $I$. In (1.1) the function $y(x)$ is a solution of

$$\frac{1}{q}D_{q^{-1}}D_qy(x) + p(x)D_{q^{-1}}y(x) + r(x)y(x) = 0, \quad (1.3)$$
\( f(x) \) is a solution of
\[
\frac{1}{q} D_{q^{-1}} f(x) = p(x) f(x)
\] (1.4)
and \( h(x) \) is an arbitrary function. In (1.2) the function \( y(x) \) is a solution of
\[
\frac{1}{q} D_{q^{-1}} D_q y(x) + p(x) D_q y(x) + r(x) y(x) = 0,
\] (1.5)
\( F(x) \) is a solution of
\[
D_q F(x) = p(x) F(x)
\] (1.6)
and \( k(x) \) is an arbitrary function. The indefinite \( q \)-integral
\[
\int f(x) d_q x = F(x),
\] (1.7)
means that \( D_q F(x) = f(x) \), where \( D_q \) is Jackson’s \( q \)-difference operator which is defined in (1.13) below. The indefinite \( q \)-integrals in (1.1) and (1.2) generalize Conway’s indefinite integral
\[
\int f(x) (d^2 h + p(x) \frac{dh}{dx} + r(x) h(x)) y(x) dx = f(x) \left( \frac{dh}{dx} y(x) - h(x) \frac{dy}{dx} \right),
\] (1.8)
where \( y(x) \) is a solution of
\[
\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + r(x) y(x) = 0,
\] (1.9)
f(x) is a solution of \( f'(x) = p(x) f(x) \) and \( h(x) \) is an arbitrary function. See [2-7,9]. Conway in [8,10] reformulated (1.8) to take the form
\[
\int f(x) h(x) (u'(x) + u^2(x) + p(x) u(x) + r(x)) y(x) dx = f(x) h(x) \left( u(x) y(x) - y'(x) \right),
\] (1.10)
where
\[
h(x) = \exp(\int u(x) dx),
\]
and \( u(x) \) is an arbitrary function. Then he derived many indefinite integrals by considering fragments of the Ricatti equation
\[
u'(x) + u^2(x) + p(x) u(x) + r(x) = 0,
\]
of the form
\[
u'(x) + u^2(x) + p(x) u(x) = 0,
\] (1.11)
or
\[ u'(x) + p(x)u(x) + r(x) = 0. \]  
(1.12)

He called (1.11) the Bernoulli fragment, and (1.12) the linear fragment.

This paper is organized as follows. In the remaining of this section, we introduce the q-notations and notions needed in the sequel. In Section 2, we give a q-analogue of Conway’s indefinite integral formula in (1.10) to the q-setting. In Section 3, we introduce indefinite q-integrals by considering Bernoulli fragments of the q-Riccati equation. In Section 4, we introduce indefinite q-integrals by considering Linear fragments of the q-Riccati equation. Finally, in Section 5, we introduce new q-integrals by setting \( u(x) = \frac{a}{2} + b \), with appropriate choice of \( a \) and \( b \) in (5.2) and (5.4). Finally, we added an appendix for all q-special functions, we used in this paper. Throughout this paper, \( q \) is a positive number less than 1, \( \mathbb{N} \) is the set of positive integers, and \( \mathbb{N}_0 \) is the set of non-negative integers. We use \( I \) to denote an interval with zero or infinity or an accumulation point. We follow Gasper and Rahman [12] for the definitions of the q-shifted factorial, q-gamma, q-beta function, and q-hypergeometric series.

A q-natural number \( [n]_q \) is defined by \( [n]_q = \frac{1-q^n}{1-q}, \ n \in \mathbb{N}_0 \). The q-derivative \( D_q f(x) \) of a function \( f \) is defined by
\[ (D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ \text{if} \ \ x \neq 0, \]  
(1.13)
and \( (D_q f)(0) = f'(0) \) provided that \( f'(0) \) exists, see [14,15]. Jackson’s q-integral of a function \( f \) is defined by
\[ \int_0^a f(t) d_q t := (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \ a \in \mathbb{R}, \]  
(1.14)
provided that the corresponding series in (1.14) converges, see [16].

The fundamental theorem of q-calculus [1, Eq.(1.29)]
\[ \int_0^a D_q f(t) d_q t = f(a) - \lim_{n \to \infty} f(aq^n). \]  
(1.15)

If \( f \) is continuous at zero, then
\[ \int_0^a D_q f(t) d_q t = f(a) - f(0). \]

2. q-integrals from Riccati fragments

In this section, we extend Conway’s result (1.10) to functions satisfying homogenous second-order q-difference equation of the form (1.3) or (1.5).

**Theorem 2.1.** Let \( y(x) \) and \( f(x) \) be solutions of Equations (1.3) and (1.4) in an open interval \( I \), respectively. Let \( u(x) \) be a continuous function on \( I \) and \( h(x) \) be an
arbitrary function satisfying
\[ D_q h(x) = u(x) h(x) \quad (x \in I). \tag{2.1} \]

Then
\[
\int f(x) h(x/q) \left( \frac{1}{q} D_{q^{-1}} u(x) + \frac{1}{q} u(x) u(x/q) + A(x) u(x/q) + r(x) \right) y(x) d_q x
= f(x/q) h(x/q) \left( y(x/q) u(x/q) - D_{q^{-1}} y(x) \right), \tag{2.2}
\]
where the functions \( p(x), r(x) \) are defined as in (1.3) and
\[ A(x) = p(x) - \frac{1}{q} x (1 - q) r(x). \tag{2.3} \]

**Proof.** Equation (1.1) can be written as
\[
\int f(x) h(x/q) \left[ \frac{1}{q} D_{q^{-1}} D_q h(x) + p(x) \frac{D_{q^{-1}} h(x)}{h(x/q)} + \frac{r(x) h(x)}{h(x/q)} \right] y(x) d_q x
= f(x/q) h(x/q) \left[ y(x/q) \frac{D_{q^{-1}} h(x)}{h(x/q)} - D_{q^{-1}} y(x) \right]. \tag{2.4}
\]

Then from (2.1), we get
\[ \frac{D_{q^{-1}} D_q h(x)}{h(x/q)} = D_{q^{-1}} u(x) + \frac{D_{q^{-1}} h(x)}{h(x/q)} u(x) \tag{2.5} \]
\[ = D_{q^{-1}} u(x) + u(x) u \left( \frac{r}{q} \right). \tag{2.6} \]

Also,
\[ r(x) \frac{h(x)}{h(x/q)} = \frac{r(x)}{h(x/q)} \left( h(x/q) + (1 - \frac{1}{q}) x D_{q^{-1}} h(x) \right) \tag{2.7} \]
\[ = r(x) \left( 1 + (1 - \frac{1}{q}) x u \left( \frac{x}{q} \right) \right). \tag{2.8} \]

Substituting with (2.5) and (2.7) into (2.4), we get (2.2) and completes the proof. \( \square \)

**Theorem 2.2.** Let \( y(x) \) and \( F(x) \) be solutions of Equations (1.5) and (1.6) in an open interval \( I \), respectively. Let \( u(x) \) be a continuous function on \( I \) and \( k(x) \) be an arbitrary function satisfying
\[ D_{q^{-1}} k(x) = u(x) k(x) \quad x \in I. \tag{2.9} \]

Then
\[
\int F(x) k(qx) \left( D_q u(x) + u(x) u(qx) + \tilde{A}(x) u(qx) + r(x) \right) y(x) d_q x
= F(x) k(x) \left( y(x) u(x) - D_{q^{-1}} y(x) \right), \tag{2.10}
\]

where the functions p(x), r(x) are defined as in (1.3) and
\[ \tilde{A}(x) = p(x) + x(1 - q)r(x). \]  
(2.11)

**Proof.** The proof follows similarly as the proof of Theorem 2.1 and is omitted. \(\square\)

Consider the \(q\)-Ricatti equation
\[ \frac{1}{q} D_q u(x) + \frac{1}{q} u(x) u(x/q) + A(x) u(x/q) + r(x) = 0, \]  
(2.12)
and
\[ D_q u(x) + u(x) u(qx) + \tilde{A}(x) u(qx) + r(x) = 0, \]  
(2.13)
where \(A(x)\) and \(\tilde{A}(x)\) are defined as in (2.3) and (2.11), respectively. We can prove that Equation (2.12) (2.13) is equivalent to Equation (1.3) (1.5) by setting \(\frac{D_q u(x)}{u(x)} = u(x)\)
\(\left(\frac{D_q u(x)}{u(x)} = u(x)\right)\), respectively.

The \(q\)-integrals presented in the sequel are obtained by choosing the function \(u(x)\) to be a solution of a fragment of the \(q\)-Ricatti equations (2.12) or (2.13).

**Theorem 2.3.** Let \(n \in \mathbb{N}\) and \(c\) be a real number. If \(h_n(x; q)\) is the discrete \(q\)-Hermite I polynomial of degree \(n\) which is defined in (1.7), then
\[ \int (q^2 x^2; q^2)_\infty ((cq + x)[n]_q - x) h_n(x; q)d_q x = 
\[ q^{n-1}(1 - q)(x^2; q^2)_\infty \left( qh_n(x; q) - \frac{x}{q}_n q(h_n(x; q) - \frac{1 - q^n}{q} x h_n(x; q) \right), \]  
(2.14)

\[ \int x(q^2 x^2; q^2)_\infty h_n(x; q)d_q x = \frac{q^{n-1}(x^2; q^2)_\infty}{[n - 1]_q} \left( 1 - q x h_n(x; q) - \frac{1 - q^n}{q} x h_n(x; q) \right), \]  
(2.15)

\[ \int \frac{(q^2 x^2; q^2)_\infty}{(q^{-(n+1)}x^2; q^2)_\infty} h_n(x; q)d_q x = 
\[ \frac{(x^2; q^2)_\infty}{[n + 1]_q (q^{-(n+1)}x^2; q^2)_\infty} \left( x h_n(x; q) - q^n(1 - q^n) x h_n(x; q) \right), \]  
(2.16)

and
\[ \int x^{n-2}(q^2 x^2; q^2)_\infty h_n(x; q)d_q x = \frac{x^n (x^2; q^2)_\infty}{[n - 1]_q} \left( \frac{h_n(x; q)}{x} - \frac{1}{q} h_{n-1}(x; q) \right). \]  
(2.17)
**Proof.** The discrete $q$-Hermite I polynomial of degree $n$ is defined in (1.7) and satisfies the second order $q$-difference equation (1.8). By comparing (1.8) with (1.3), we get

\[ p(x) = -\frac{x}{1 - q}, \quad r(x) = \frac{q^{1-n} [n]_q}{1 - q}. \]  

(2.18)

Then

\[ f(x) = (q^2 x^2; q^2)_\infty \]  

is a solution of (1.4). Therefore Equation (2.2) becomes

\[
\int (q^2 x^2; q^2)_\infty h(x/q) \left( \frac{1}{q} D_{q^{-1}} u(x) + \frac{1}{q} u(x) u(x/q) - \frac{q^{-n} x}{1 - q} u(x/q) + \frac{q^{1-n} [n]_q}{(1 - q)} y(x) \right) d_q x \\
= (x^2; q^2)_\infty h(x/q) \left( y(x/q) u(x/q) - D_{q^{-1}} y(x) \right). 
\]

(2.20)

By taking the fragment

\[ D_{q^{-1}} u(x) + u(x) u(x/q) = 0, \]  

(2.21)

we get

\[ u(x) = \frac{1}{x + c}. \]  

(2.22)

Hence

\[ h(x) = \begin{cases} 
1 + \frac{x}{c}, & \text{if } c \neq 0; \\
x, & \text{if } c = 0
\end{cases} \]  

(2.23)

is a solution of (2.1). Substituting with $h(x) = 1 + \frac{x}{c}$ into (2.20) and using

\[ D_{q^{-1}} h_n(x; q) = [n]_q h_{n-1}(\frac{x}{q}; q), \]  

(2.24)

see [17, Eq.(3.28.7)], we get (2.14). Substituting with $h(x) = x$ into (2.20) and using (2.24) we get (2.15). To prove (2.16), we consider the fragment

\[ \frac{1}{q} u(x) u(x/q) - \frac{q^{-n} x}{1 - q} u(x/q) = 0, \]

then $u(x) = q^{1-n} x$ and $h(x) = \frac{1}{(q^{1-n} x^2; q^2)_\infty}$ is a solution of (2.1). Substituting with $h(x)$ and $u(x)$ into (2.20) and using (2.24), we get (2.16). Finally, the proof of (2.17) follows by taking the fragment

\[ -\frac{q^{-n} x}{1 - q} u(x/q) + \frac{q^{1-n} [n]_q}{(1 - q)} = 0. \]
In this case, \( u(x) = \frac{[n]_q}{x} \) and \( h(x) = x^n \) is a solution of (2.21). Substituting with \( h(x) \) and \( u(x) \) into (2.20) and using (2.24), we get (2.17).

**Theorem 2.4.** Let \( n \in \mathbb{N} \) and \( c \) be a real number. If \( \tilde{h}_n(x; q) \) is the discrete \( q \)-Hermite II polynomial of degree \( n \) which is defined in (1.9), then

\[
\int \frac{1}{(-x^2; q^2)_\infty} (qx[n - 1]_q + c[n]_q) \tilde{h}_n(x; q) d_q x = \frac{1 - q}{[n - 1]_q(-x^2; q^2)_\infty} \left( \frac{1}{q} \tilde{h}_n(x; q) - q^{-n}[n]_q x \tilde{h}_{n-1}(x; q) \right),
\]

(2.25)

\[
\int \frac{x}{(-x^2; q^2)_\infty} \tilde{h}_n(x; q) d_q x = \frac{1 - q}{[n - 1]_q(-x^2; q^2)_\infty} \left( \frac{1}{q} \tilde{h}_n(x; q) - q^{-n}[n]_q x \tilde{h}_{n-1}(x; q) \right).
\]

(2.26)

\[
\int \frac{(-q^{n+3}x^2; q^2)_\infty}{(-x^2; q^2)_\infty} \tilde{h}_n(x; q) d_q x = \frac{(-q^{n+1}x^2; q^2)_\infty}{[n + 1]_q(-x^2; q^2)_\infty} \left( q^n x \tilde{h}_n(x; q) - q^{-n}(1 - q^n) \tilde{h}_{n-1}(x; q) \right),
\]

(2.27)

and

\[
\int \frac{x^{n-2}}{(-x^2; q^2)_\infty} \tilde{h}_n(x; q) d_q x = \frac{x^n}{[n - 1]_q(-x^2; q^2)_\infty} \left( \frac{\tilde{h}_n(x; q)}{x} - \tilde{h}_{n-1}(x; q) \right).
\]

(2.28)

**Proof.** The discrete \( q \)-Hermite II polynomial of degree \( n \) is defined in (1.9) and satisfies the second order \( q \)-difference equation (1.10). By comparing (1.10) with (1.5), we get

\[
p(x) = -\frac{x}{1 - q}, \quad r(x) = \frac{[n]_q}{1 - q}.
\]

Then \( F(x) = \frac{1}{(-x^2; q^2)_\infty} \) is a solution of (1.6), and (2.10) becomes

\[
\int \frac{k(qx)}{(-x^2; q^2)_\infty} \left( D_q u(x) + u(x) u(qx) - \frac{q^n x}{(1 - q)} u(qx) + \frac{[n]_q}{(1 - q)} \right) y(x) d_q x = \frac{k(x)}{(-x^2; q^2)_\infty} \left( y(x) u(x) - D_{q^{-1}} y(x) \right).
\]

(2.29)

Consider the fragment

\[
D_q u(x) + u(x) u(qx) = 0.
\]

(2.30)

Hence,

\[
u(x) = \frac{1}{x + c}
\]

(2.31)
and

\[ k(x) = \begin{cases} 
1 + \frac{x}{c}, & \text{if } c \neq 0; \\
x, & \text{if } c = 0 
\end{cases} \]  \tag{2.32}

is a solution of (2.9). Substituting with \( u(x) = 1 + \frac{x}{c} \) into (2.29), and using [17, Eq.(3.29.7)] (with \( x \) is replaced by \( \frac{x}{q} \))

\[ D_{q^{-1}} \tilde{h}_n(x; q) = q^{1-n} [n]_q \tilde{h}_{n-1}(x; q), \quad \tag{2.33} \]

we get (2.25). Substituting with \( u(x) = x \) into (2.29), and using (2.33) we get (2.26). Also, by taking the fragment

\[ u(x)u(qx) - \frac{q^n x}{1 - q} u(qx) = 0, \]

which has the solution \( u(x) = \frac{q^n}{1 - q} x \). Hence \( k(x) = (-q^{n+1} x^2; q^2)_\infty \) is a solution of (2.9). Substituting with \( k(x) \) into (2.29), we get (2.27). Similarly, to prove (2.28), we consider the fragment

\[ -\frac{q^n x}{1 - q} u(qx) + \frac{[n]_q}{1 - q} = 0, \]

then we obtain \( u(x) = q^{1-n} [n]_q \frac{1}{x} \) and \( k(x) = x^n \). Substituting with \( u(x) \) and \( k(x) \) into Equation (2.29), we get (2.28). \( \square \)

**Theorem 2.5.** Let \( \nu, c \) and \( q \) be real numbers such that \( \nu > 1 \). Let \( \alpha \in \mathbb{R} \) be such that \( \alpha = \frac{\ln(q^\nu - q^{-\nu} - q^{-1})}{\ln q} \). Then

\[ \int_{-\infty}^{\infty} \frac{x}{(-x^2(1-q^2); q^2)_\infty} \left( \frac{q^{1-\nu} [\nu]^2}{x^2} - \frac{cq^{\nu} [\nu]^2}{x^2} + c + qx \right) J^{(2)}_{\nu}(x|q^2) dx = \]

\[ \frac{x}{q(-x^2(1-q^2); q^2)_\infty} \left( J^{(2)}_{\nu}(x|q^2) - (c + x) D_{q^{-1}} J^{(2)}_{\nu}(x|q^2) \right), \quad \tag{2.34} \]

\[ \int \frac{x}{(-x^2(1-q^2); q^2)_\infty} \left( \frac{q^{1-\nu} [\nu]^2}{x} + qx \right) J^{(2)}_{\nu}(x|q^2) dx = \]

\[ \frac{x}{q(-x^2(1-q^2); q^2)_\infty} \left( J^{(2)}_{\nu}(x|q^2) - x D_{q^{-1}} J^{(2)}_{\nu}(x|q^2) \right), \quad \tag{2.35} \]

\[ \int \frac{x^{\alpha+1}}{(-x^2(1-q^2); q^2)_\infty} \left( q + \frac{1 - q^{1-\nu} [\nu]^2}{x^2} \right) J^{(2)}_{\nu}(x|q^2) dx = \]

\[ \frac{x^{\alpha+1}}{q^\alpha(-x^2(1-q^2); q^2)_\infty} \left( q^{1-\nu} (1-q) [\nu]^2 - 1 \right) J^{(2)}_{\nu}(x|q^2) - D_{q^{-1}} J^{(2)}_{\nu}(x|q^2) \). \tag{2.36} \]
**Proof.** By comparing (1.12) with (1.5), we obtain

\[
p(x) = \frac{1}{x} - q(1 - q)x, \quad r(x) = q - q^{1-\nu}[\nu]^2 \frac{1}{x^2}.
\]

Then \( F(x) = \frac{x}{(-x^2(1 - q)^2; q^2)_\infty} \) is a solution of (1.6), and (2.10) becomes

\[
\int \frac{xk(qx)y(x)}{(-x^2(1 - q)^2; q^2)_\infty} \left( D_qu(x) + u(x)u(qx) + \frac{1 - q^{1-\nu}(1 - q)[\nu]^2}{x} u(qx) + \frac{qx^2 - q^{1-\nu}[\nu]^2}{x^2} \right) dx
\]

\[
= \frac{xk(x)}{(-x^2(1 - q)^2; q^2)_\infty} \left( y(x)u(x) - D_q^{-1}y(x) \right).
\] (2.37)

Considering the fragment (2.30), we get \( u(x) \) and \( k(x) \) as in (2.31) and (2.32), respectively. Substituting with \( u(x) \) and \( k(x) = 1 + \frac{x}{c} \) into (2.37) to obtain (2.34). Substituting with \( u(x) \) and \( k(x) = x \) into (2.37) we obtain (2.35). To prove (2.36), consider the fragment

\[
u n\infty
(1)_{n=0} (-1)^n q^n (c - 1 + q^2 + q^n x) Ai_q(q^n x)
\]

which implies that \( u(x) = \frac{q^{1-\nu} + q^{1+\nu} - (1 + q)}{(1-q)x} \) and then \( k(x) = x^\alpha \) is a solution of (2.9). Substituting with \( u(x) \) and \( k(x) \) into (2.37) to obtain (2.36).

**Theorem 2.6.** Let \( c \) be a real number. If \( Ai_q(x) \) is the \( q \)-Airy function which is defined in (1.13), then

\[
\sum_{n=0}^{\infty} (-1)^n q^n (c - 1 + q^2 + q^n x) Ai_q(q^n x)
\]

\[
= \frac{qc + x}{q(1 + q)} \frac{1}{1} \phi_1(0; -q^2; q, -x) - (1 - q)Ai_q(x),
\] (2.38)

\[
\sum_{n=0}^{\infty} (-1)^n q^n (1 - q^2 - q^n x) Ai_q(q^n x)
\]

\[
= (1 - q)Ai_q(x) - \frac{x}{q(1 + q)} \frac{1}{1} \phi_1(0; -q^2; q, -x),
\] (2.39)

\[
\sum_{n=0}^{\infty} q^n (q - q^3 - q^n x)(-q^{-3}x; q)_n Ai_q(q^n x)
\]

\[
= \frac{x}{1 + q} \frac{1}{1} \phi_1(0; -q^2; q, -x) - \left( q(1 + q) + \frac{x}{q} \right) Ai_q(x).
\] (2.40)

**Proof.** The \( q \)-Airy function is defined in (1.13) and satisfies the second order \( q-
difference equation (1.14). By comparing (1.14) with (1.3), we get

\[
p(x) = -\frac{1 + q}{q(1 - q)x}, \quad r(x) = \frac{1}{q(1 - q)^2x}.
\]

(2.41)

By taking the fragment (2.21), we get \(u(x)\) and \(h(x)\) as in (2.22) and (2.23), respectively. Therefore (2.2) takes the form

\[
\int f(t)h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2t} \right) y(t) d_q t = f(x/q) h(x/q) \left( y(x/q) u(x/q) - D_{q^{-1}} y(x) \right).
\]

(2.42)

Denote the right hand side of Equation (2.42) by \(H(x)\). I.e

\[
H(x) = f(x/q) h(x/q) \left( y(x/q) u(x/q) - D_{q^{-1}} y(x) \right).
\]

Then from (1.15), we obtain

\[
\int_0^x f(t)h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2t} \right) y(t) d_q t = H(x) - \lim_{n \to \infty} H(q^n x).
\]

(2.43)

From (1.4), we obtain \(f(qx) = (-q)f(x)\). Consequently,

\[
f(q^n x) = (-1)^n q^n f(x) \quad (n \in \mathbb{N}_0).
\]

(2.44)

Since

\[
D_{q^{-1}}\Phi_q(x) = \frac{1}{1 - q^2} \Phi_1(0; -q^2; q, -x),
\]

(2.45)

and using \(h(x) = 1 + \frac{x}{2}\) we get

\[
H(x) = \frac{f(x)}{cq} \left( \frac{cq + x}{q(1 - q^2)} \Phi_1(0; -q^2; q, -x) - \Phi_q\left(\frac{x}{q}\right) \right).
\]

(2.46)

Hence \(\lim_{n \to \infty} H(q^n x) = 0\). From (1.14), (2.22), (2.23) with \(c \neq 0\) and (2.43), we obtain

\[
\int_0^x f(t)h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2t} \right) y(t) d_q t
\]

\[
= \frac{f(x)}{cq(1 - q)} \sum_{n=0}^{\infty} (-q)^n (c - 1 + q^2 + q^n x) y(q^n x) = H(x).
\]

(2.47)

Combining equations (2.46) and (2.47) yields (2.48). Substituting with \(h(x) = x\) we get

\[
H(x) = \frac{f(x)}{q} \left( \frac{x}{q(1 - q^2)} \Phi_1(0; -q^2; q, -x) - \Phi_q\left(\frac{x}{q}\right) \right).
\]

(2.48)
Hence \( \lim_{n \to \infty} H(q^n x) = 0 \). From (1.14), (2.22), (2.23) with \( c = 0 \) and (2.43), we obtain

\[
\int_0^x f(t) h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2 t} \right) y(t) \, d_q t
\]

\[
= \frac{f(x)}{q(1 - q)} \sum_{n=0}^{\infty} (-q)^n (-1 + q^2 + q^n x) y(q^n x) = H(x). \tag{2.49}
\]

Combining equations (2.48) and (2.49) yields (2.39). Similarly, we prove (2.40), by taking the fragment

\[
\frac{1}{q} u(x) u(x/q) - \frac{q(1 + q) + x}{q^2(1 - q)x} u(x/q) = 0,
\]

which implies that \( u(x) = \frac{q(1 + q) + x}{q(1 - q)x} \). Since \( h(x) \) satisfies (2.1), then

\[
h(q^n x) = (-q)^n(\frac{-x}{q^2}; q)_n h(x) \quad (n \in \mathbb{N}_0). \tag{2.50}
\]

Substituting with \( u(x) \) into (2.22) and using equations (2.41), (2.45) and (2.50), we get (2.40).

**Theorem 2.7.** Let \( c \in \mathbb{R} \). If \( A_q(x) \) is the Ramanujan function which is defined in (1.15), then

\[
\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} x^n (1 - q + q^n + x) A_q(q^n x) = (1 - q) A_q(\frac{x}{q}) - (cq + x) A_q(qx), \tag{2.51}
\]

\[
\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} x^n (1 - q + q^n + x) A_q(q^n x) = (1 - q) A_q(\frac{x}{q}) - x A_q(qx), \tag{2.52}
\]

\[
\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} (1 - q^2 + q^n + x) (x; q)_n A_q(q^n x) = \frac{x(1 + q) - q}{x} A_q(\frac{x}{q}) - x A_q(qx). \tag{2.53}
\]

**Proof.** The Ramanujan function is defined in (1.15) and satisfies the second order \( q \)-difference equation (1.16). By comparing (1.16) with (1.3), we get

\[
p(x) = \frac{1 - qx}{q(1 - q)x^2}, \quad r(x) = \frac{1}{(1 - q)^2 x^2}. \tag{2.54}
\]

By taking the fragment (2.21), we get \( u(x) \) and \( h(x) \) as in (2.22) and (2.23), respectively. Therefore (2.2) takes the form

\[
\int f(t) h(t/q) \left( \frac{1 - t(1 + q)}{q(1 - q)t^2} u(t/q) + \frac{1}{(1 - q)^2 t^2} \right) y(t) \, d_q t
\]

\[
= f(x/q) h(x/q) \left( y(x/q) u(x/q) - D_{q^{-1}} y(x) \right). \tag{2.55}
\]
Denote the right hand side of Equation (2.55) by \( G(x) \). I.e
\[
G(x) = f(x/q)h(x/q) \left( y(x/q)u(x/q) - D_{q^{-1}}y(x) \right). 
\]

Then from (1.15), we get
\[
\int_0^x f(t)h(t/q) \left( \frac{1 - t(1 + q)}{q(1 - q)t^2} u(t/q) + \frac{1}{(1 - q)^2t^2} \right) y(t) dt = G(x) - \lim_{n \to \infty} G(q^n x). 
\] (2.56)

From (1.4), we obtain \( f(qx) = q^2 xf(x) \). Consequently,
\[
f(q^n x) = q \frac{n(n-1)}{2} x^n f(x) \quad (n \in \mathbb{N}_0). 
\] (2.57)

Since
\[
D_{q^{-1}} A_q(x) = \frac{q}{1-q} A_q(qx), 
\] (2.58)

substituting with \( h(x) = 1 + \frac{x}{c} \), then
\[
G(x) = \frac{f(x)}{cq} \left( A_q\left(\frac{x}{q}\right) - \frac{(cq + x)}{1-q} A_q(qx) \right). 
\] (2.59)

Hence \( \lim_{n \to \infty} G(q^n x) = 0 \). From (1.14), (2.22), (2.23) with \( c \neq 0 \) and (2.56), we obtain
\[
\int_0^x f(t)h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2t} \right) y(t) dt = \frac{f(x)}{cq(1 - q)} \sum_{n=0}^{\infty} q \frac{n(n-1)}{2} x^{n-1} \left( 1 - q^n + q^{n+2} x \right) y(q^n x) = G(x). 
\] (2.60)

Combining equations (2.59) and (2.60) yields (2.51). Substituting with \( h(x) = x \), then
\[
G(x) = \frac{f(x)}{qx} \left( A_q\left(\frac{x}{q}\right) - \frac{x}{1-q} A_q(qx) \right). 
\] (2.61)

Hence \( \lim_{n \to \infty} G(q^n x) = 0 \). From (1.14), (2.22), (2.23) with \( c = 0 \) and (2.56), we obtain
\[
\int_0^x f(t)h(t/q) \left( -\frac{q(1 + q) + t}{q^2(1 - q)t} u(t/q) + \frac{1}{q(1 - q)^2t} \right) y(t) dt = \frac{f(x)}{q(1 - q)} \sum_{n=0}^{\infty} q \frac{n(n-1)}{2} x^{n-1} \left( 1 - q^n + q^{n+2} x \right) y(q^n x) = G(x). 
\] (2.62)

Combining equations (2.61) and (2.62) yields (2.52). Similarly, we prove (2.53), by taking the fragment
\[
\frac{1}{q} u(x) u(x/q) + \frac{1 - x(1 + q)}{q(1 - q)x^2} u(x/q) = 0, 
\]
which implies that \( u(x) = \frac{x^{(1+q)−1}}{(1-q)x}. \) Since \( h(x) \) satisfies (2.1), then

\[
h(q^n x) = \frac{(qx; q)_n}{x^n} h(x) \quad (n \in \mathbb{N}_0).
\] (2.63)

Substituting with \( u(x) \) into (2.2) and using (2.57), (2.58) and (2.63), we get (2.53). \( \square \)

**Corollary 2.8.** Let \( Ai_q(x) \) and \( A_q(x) \) be the \( q \)-Airy function and the Ramanujan function which are defined in (1.13) and (1.15), respectively. Then

\[
\sum_{n=0}^{\infty} (-1)^n q^{2n} \frac{x^n}{q^{n-1}} Ai_q(q^n x) = \frac{q(1-q^2)+x}{x} \frac{1}{1-q}\varphi_1(0; -q^2; q, -x) - \frac{(1-q)}{x} A_i_q(x),
\]

\[
\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} x^n A_q(q^n x) = \frac{1-q}{q^2x} A_q(x) + \frac{1-q-x}{q^2x} A_q(qx).
\]

**Proof.** Substituting with \( c = 1-q^2 \) and \( c = \frac{1}{q}(1-q) \) in (2.58) and (2.51), respectively, we get the desired results. \( \square \)

### 3. \( q \)-integrals from Bernoulli fragments

In this section, we introduce indefinite \( q \)-integrals involving Bernoulli fragments of (2.12) or (2.13), of the form

\[
\int f(x) D_q^{-1} u(x) \, dq \, = \, -f(x/q) D_q^{-1} y(x).
\] (3.3)

Similarly, the trivial solution \( u(x) = 0 \) of (3.2) implies that \( k(x) = c \) is a solution of (2.9), where \( c \) is a non zero constant. Then (1.2) becomes

\[
\int F(x) D_q \, y(x) \, dq \, = \, -F(x) D_q \, y(x).
\] (3.4)

**Theorem 3.1.** Let \( n \in \mathbb{N} \). The following statements are true:

(a) If \( h_n(x; q) \) is the discrete \( q \)-Hermite I polynomial of degree \( n \) which is defined in (2.17), then

\[
\int (q^2 x^2; q^2) \infty h_n(x; q) \, dq \, = \, -q^{n-1}(1-q)(x^2; q^2) \infty h_{n-1}(x; q).
\] (3.5)
(b) If \( p_n(x; a, b; q) \) is the big \( q \)-Laguerre polynomial of degree \( n \) which is defined in (1.4), then
\[
\int \frac{(\frac{x}{a}, \frac{x}{b}; q)_\infty}{(x; q)_\infty} p_n(x; a, b; q) d_q x = \frac{abq^2(1 - q)}{(1 - aq)(1 - bq)} \frac{(\frac{x}{a}, \frac{x}{b}; q)_\infty}{(x; q)_\infty} p_{n-1}(x; aq, bq; q). \tag{3.6}
\]

Proof. The proof of (a) follows by substituting with \( r(x) \) and \( f(x) \) from (2.18) and (2.19), respectively, into (3.3). The proof of (b) follows by comparing (1.2) with (1.3) to get
\[
p(x) = \frac{x - q(a + b - qab)}{abq^2(1 - q)(1 - x)} , \quad r(x) = -\frac{q^{-n-1}[n]_q}{ab(1 - q)(1 - x)}.
\]

Hence \( f(x) = \frac{(\frac{x}{a}, \frac{x}{b}; q)_\infty}{(qx; q)_\infty} \) is a solution of (1.4). Substituting with \( r(x) \) and \( f(x) \) into Equation (3.3) and using
\[
D_{q^{-1}} p_n(x; a, b; q) = \frac{q^{1-n}[n]_q}{(1 - aq)(1 - bq)} p_{n-1}(x; aq, bq; q), \tag{3.8}
\]

see [17, Eq.(3.11.7)], we get (3.6). To prove (c), compare (1.4) with (1.3) to obtain
\[
p(x) = \frac{1 - q^{\alpha+1}(1 + x)}{q^{\alpha+1} x(1 + x)(1 - q)} , \quad r(x) = \frac{[n]_q}{x(1 - q)(1 + x)}.
\]

Hence \( f(x) = \frac{x^{\alpha+1}}{(-qx; q)_\infty} \) is a solution of (1.4). Finally, we prove (3.7) by substituting with \( r(x) \) and \( f(x) \) into (3.3) and using
\[
D_{q^{-1}} L_n^\alpha(x; q) = \frac{-q^{\alpha+1}}{(1 - q)} L_{n-1}^\alpha(x; q), \tag{3.9}
\]

see [17, Eq.(3.21.8)].

\( \square \)

Theorem 3.2. The following statements are true:

(a) If \( \tilde{h}_n(x; q) \) is the discrete \( q \)-Hermite II polynomial of degree \( n \) which is defined in (1.9), then
\[
\int \frac{\tilde{h}_n(x; q)}{(-x^2; q^2)_\infty} d_q x = -\frac{q^{1-n}(1 - q)}{(-x^2; q^2)_\infty} \tilde{h}_{n-1}(x; q). \tag{3.10}
\]
(b) If $\nu$ is a real number, $\nu > -1$, then
\[
\int \frac{qx^2 - q^{1-\nu}[\nu]^2}{x(-x^2(1-q^2); q^2)_\infty} J_\nu^{(2)}(x|q^2) d_q x = \frac{-x}{(-x^2(1-q)^2; q^2)_\infty} D_q^{-1} J_\nu^{(2)}(x|q^2).
\]

**Proof.** The proofs of (a) and (b) follow by substituting with $r(x)$ and $F(x)$ as in the proof of Theorems 2.4 and 2.5 into Equation (3.4), respectively.

**Theorem 3.3.** The following statements are true:

(a) If $\text{Ai}_q(x)$ is the $q$-Airy function which is defined in (1.13), then
\[
\sum_{k=0}^\infty (-q)^k \text{Ai}_q(q^k x) = \frac{1}{1+q} \phi_1(0; -q^2; q, -x). \tag{3.11}
\]

(b) If $A_q(x)$ is the Ramanujan function which is defined in (1.15), then
\[
\sum_{k=0}^\infty q^{k(k-3)/2} x^k A_q(q^k x) = -\phi_1(-; 0; q, -q^2 x). \tag{3.12}
\]

(c) If $S_n(x; q)$ is the Stieltjes-Wigert polynomial of degree $n$ $(n \in \mathbb{N})$ which is defined in (1.5), then
\[
\sum_{k=0}^\infty q^{k(k-3)/2} x^k S_n(q^k x; q) = \frac{1}{1-q^n} S_{n-1}(qx; q). \tag{3.13}
\]

**Proof.** The proof of (a) follows by substituting with $r(x)$ from (2.41) into (3.3) and using (2.44) and (2.45). The proof of (b) follows by substituting with $r(x)$ from (2.54) into (3.3) and using (2.57) and (2.58). To prove (c), compare Equation (1.6) with (1.3) to get
\[
p(x) = \frac{1 - q x}{qx^2(1-q)}, \quad r(x) = \frac{[n]_q}{x^2(1-q)}.
\]

From (1.4), we obtain $f(qx) = q^2 x f(x)$. Consequently,
\[
f(q^k x) = q^{k(k-1)/2} x^k f(x) \quad (k \in \mathbb{N}_0). \tag{3.14}
\]
Substituting with $r(x)$ into (3.3) and using (1.4), (3.14) and
\[
D_q^{-1} S_n(x; q) = \frac{-q}{1-q} S_{n-1}(qx; q), \tag{3.15}
\]
see [17, Eq.(3.27.7)], we get (3.13).

**Remark 1.**
The indefinite \( q \)-integral (3.6) is nothing else but [11, Eq.(42)] or [17, Eq.(3.11.9)] (with \( n \) is replaced by \( n - 1 \))

\[
D_q \left( w(x;aq,bq;q) p_{n-1}(x;aq,bq;q) \right) = \frac{(1-aq)(1-bq)}{abq^2(1-q)} w(x;a,b;q)p_n(x;a,b;q),
\]

where \( w(x;a,b;q) = \frac{(\frac{x}{q^a})}{(x;q)_\infty} \).

The indefinite \( q \)-integral (3.7) is equivalent to [11, Eq.(46)] (if \( m = n \)) and to [17, Eq.(3.21.10)] (if \( m = 0 \)) (with \( \alpha \) is replaced by \( \alpha + 1 \) and \( n \) is replaced by \( n - 1 \))

\[
D_q \left( w(x;\alpha+1;q)L_{n-1}^{\alpha+1}(x;q) \right) = [n]_q w(x;\alpha;q)L_{n}^{\alpha}(x;q),
\]

where \( w(x;\alpha;q) = \frac{x^\alpha}{(-x;q)_\infty} \).

The indefinite \( q \)-integral (3.10) is equivalent to [11, Eq.(68)] and to

\[
D_q \left( w(x;q)\bar{h}_{n-1}(x;q) \right) = \frac{q^{n-1}}{1-q} w(x;q)\bar{h}_n(x;q),
\]

where \( w(x;q) = \frac{1}{(x^2q^x)} \), see [17, Eq.(3.29.9)].

We need the following results to prove Theorem 3.6:

**Theorem 3.4.** Let \( I \) be an interval containing zero and \( g(x) \) is a solution of the first order \( q \)-difference equation

\[
\frac{1}{q} D_{q^{-1}} g(x) = A(x)g(x), \quad g(0) = 1,
\]

where \( A(x) \) is the function which is defined in (2.3). Assume that there exists \( \gamma, 0 \leq \gamma < 1 \) such that \( \frac{x^\gamma}{g(x)} \) is bounded on \( I \). Then the function

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u, \quad x \in I
\]

satisfies

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]

**Proof.** Multiplying both sides of Equation (3.18) by \( \frac{1}{g(x/q)} \) to obtain

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{qA(x)v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{D_{q^{-1}} g(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u,
\]

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]

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\[
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\]

where \( A(x) \) is the function which is defined in (2.3). Assume that there exists \( \gamma, 0 \leq \gamma < 1 \) such that \( \frac{x^\gamma}{g(x)} \) is bounded on \( I \). Then the function

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u, \quad x \in I
\]

satisfies

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]

**Proof.** Multiplying both sides of Equation (3.18) by \( \frac{1}{g(x/q)} \) to obtain

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{qA(x)v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{D_{q^{-1}} g(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u,
\]

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]

We need the following results to prove Theorem 3.6:

**Theorem 3.4.** Let \( I \) be an interval containing zero and \( g(x) \) is a solution of the first order \( q \)-difference equation

\[
\frac{1}{q} D_{q^{-1}} g(x) = A(x)g(x), \quad g(0) = 1,
\]

where \( A(x) \) is the function which is defined in (2.3). Assume that there exists \( \gamma, 0 \leq \gamma < 1 \) such that \( \frac{x^\gamma}{g(x)} \) is bounded on \( I \). Then the function

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u, \quad x \in I
\]

satisfies

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]

**Proof.** Multiplying both sides of Equation (3.18) by \( \frac{1}{g(x/q)} \) to obtain

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{qA(x)v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} - \frac{D_{q^{-1}} g(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
\frac{D_{q^{-1}} v(x)}{g(x/q)} = \frac{1}{g(x/q)},
\]

\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u,
\]

\[
D_{q^{-1}} v(x) - qA(x)v(x) = 1.
\]
or equivalently, \( D_q \left( \frac{v(x)}{g(x)} \right) = \frac{1}{g(x)} \). Hence, from (1.15), we get
\[
v(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u.
\]

**Theorem 3.5.** Assume that \( v(x) \) and \( g(x) \) are defined as in Theorem 3.4 in an interval \( I \) containing zero. Then
\[
\int \frac{f(x)}{g(x/q)} v(x/q) r(x) y(x) d_q x = \frac{f(x/q)}{g(x/q)} \left( y(x/q) - v(x/q) D^{-1}_q y(x) \right), \quad (3.19)
\]
where \( f(x) \) is a solution of (1.4) and \( y(x) \) is a solution of (1.3).

**Proof.** In (2.1) set \( u(x) = \frac{1}{v(x)} \), where \( v(x) \) is defined in Equation (3.17). Then
\[
\frac{D_q h(x)}{h(x)} = \frac{1}{v(x)} = \frac{1}{g(x)} \int_0^x \frac{1}{g(u)} d_q u.
\]
Hence,
\[
D_q \left( \frac{h(x)}{\int_0^x \frac{1}{g(u)} d_q u} \right) = 0.
\]
Therefore,
\[
h(x) = c \int_0^x \frac{1}{g(u)} d_q u,
\]
where \( c \) is a constant, we can choose \( c = 1. \) Hence,
\[
g(x) h(x) = g(x) \int_0^x \frac{1}{g(u)} d_q u = v(x).
\]
Substituting with \( u(x) \) and \( v(x) \) into Equation (2.2), we get Equation (3.19).

The solutions of the second-order \( q \)-difference equation, see [1],
\[
\frac{1}{q} D_{q-1} D_q y(x) - y(x) = 0 \quad (x \in \mathbb{R}) \quad (3.20)
\]
under the initial conditions
\[
y(0) = 1, \quad D_q y(0) = 0, \quad \text{and} \quad y(0) = 0, \quad D_q y(0) = 1
\]
are the functions \( \cos(x; q) \) and \( \sin(x; q) \) which are defined in Equation (1.22) and (1.21), respectively.
Theorem 3.6.

\[
\int \frac{x \cos(x; q)}{(-x^2/q)(1 - q)^2; q^2)} \phi_1\left(\frac{-x^2}{q} (1 - q)^2, q^2; 0; q^2, q) d_qx
\]

\[
= \frac{-\sqrt{q}}{(-x^2/q)(1 - q)^2; q^2)} \left( \frac{\sqrt{q} \cos(x; q)}{1 - q} + x \sin(q \frac{1}{2} x; q) \frac{\phi_1\left(\frac{-x^2}{q} (1 - q)^2, q^2; 0; q^2, q \right)}{1 - q} \right),
\]

(3.21)

and

\[
\int \frac{x \sin(x; q)}{(-x^2/q)(1 - q)^2; q^2)} \phi_1\left(\frac{-x^2}{q} (1 - q)^2, q^2; 0; q^2, q) d_qx
\]

\[
= \frac{1}{(-x^2/q)(1 - q)^2; q^2)} \left( x \cos(q \frac{1}{2} x; q) \frac{\phi_1\left(\frac{-x^2}{q} (1 - q)^2, q^2; 0; q^2, q \right)}{1 - q} - \frac{q \sin(x/q; q)}{1 - q} \right),
\]

(3.22)

where \(\sin(x; q)\) and \(\cos(x; q)\) are defined in (1.21) and (1.22), respectively.

**Proof.** By comparing Equation (3.20) with Equation (1.3), we get \(p(x) = 0\) and \(r(x) = -1\). Then \(f(x) = 1\) is a solution of (1.4) and \(g(x) = (-q(1 - q)^2x^2; q^2)_{\infty} \) is a solution of (3.16) with \(A(x) = \frac{x}{q}(1 - q)\). By Theorem 3.4

\[
v(x) = x(1 - q) \phi_1\left(-q(1 - q)^2x^2, q^2; 0; q^2, q \right).
\]

Substituting with \(v(x)\) and \(g(x)\) into (3.19) and using the \(q\)-difference equation (1.25) and (1.26), respectively, we get the desired results.

We need the following results to prove Theorem 3.9 and Theorem 3.10.

**Theorem 3.7.** Let \(I\) be an interval containing zero and let \(f(x)\) be a solution of the first order \(q\)-difference equation (1.4) in \(I\). Assume that there exists \(\eta\), \(0 \leq \eta < 1\) such that \(\frac{x^\eta}{f(x)}\) is bounded on \(I\). Then the function

\[
v(x) = f(x) \int_0^x \frac{1}{f(u)} d_qu, \quad x \in I
\]

satisfies

\[
D_{q^{-1}}v(x) - qp(x)v(x) = 1.
\]

(3.23)

**Proof.** The proof follows similarly to the proof of Theorem 3.3 and is omitted. \(\Box\)
Theorem 3.8. Assume that $v(x)$ and $f(x)$ are defined as in Theorem 3.7 in an interval $I$ containing zero. Then

$$
\int \frac{f(x)}{f(x/q)} \left(v(x/q) - \frac{1}{q}x(1 - q)\right) r(x)y(x)dx = y(x/q) - v(x/q)D_{q^{-1}}y(x), \quad (3.24)
$$

where $y(x)$ is a solution of (1.3).

**Proof.** The proof follows similarly to the proof of Theorem 3.5 and is omitted. $\blacksquare$

Theorem 3.9.

$$
\int x\cos(x; q)d_qx = -q^{-\frac{1}{2}}x\sin(q^{-\frac{1}{2}}x; q) - \cos(x/q; q), \quad (3.25)
$$

$$
\int x\sin(x; q)d_qx = \frac{x}{q}\cos(q^{-\frac{1}{2}}x; q) - \sin(x/q; q), \quad (3.26)
$$

where $\sin(x; q)$ and $\cos(x; q)$ are defined in (1.21) and (1.22), respectively.

**Proof.** From (3.20), we have $p(x) = 0, r(x) = -1$ and $f(x) = 1$. By Theorem 3.7, we get $v(x) = x$ is a solution of Equation (3.23). Substituting with $v(x)$, $f(x)$ and using the $q$-difference equations (1.25) and (1.26), respectively, we get the desired results. $\blacksquare$

Theorem 3.10. Let $n \in \mathbb{N}$. If $p_n(x; -1; q)$ is the big $q$-Legendre polynomial which is defined in (1.17), then

$$
\int \frac{x p_n(x; -1; q)}{(q^2 - x^2)} \left(2\phi_1\left(\frac{x^2}{q^2}, q^2; x^2, q^2, q\right) - 1\right)d_qx = (q^{-n} - 1)[n + 1]_q\left(p_n\left(\frac{x}{q}; -1; q\right) + \frac{x}{q} 2\phi_1\left(\frac{x^2}{q^2}, x^2; q^2, q^2, q\right) 3\phi_2(q^{-n}, q^{n+1}, x; q; -q; q, q)\right). \quad (3.27)
$$

**Proof.** By comparing (1.18) with (1.3), we get

$$
p(x) = \frac{x(1 + q)}{q^2(x^2 - 1)}, \quad r(x) = \frac{[n]_q [n + 1]_q}{q^{1+n}(x^2 - 1)}.
$$

Then $f(x) = (1 - x^2)$ is a solution of (1.4). From Theorem 3.7, the function

$$
v(x) = x(1 - q) 2\phi_1\left(x^2, q^2; x^2, q^2, q\right)
$$

is a solution of (3.23). Substituting with $v(x)$ and $f(x)$ into (3.24) and using

$$
D_{q^{-1}}p_n(x; -1; q) = \frac{-1}{1 - q} 3\phi_2\left(q^{-n}, q^{n+1}, x; q; -q; q, q, q\right), \quad (3.28)
$$

we obtain (3.27). $\blacksquare$
4. \(q\)-integrals from linear fragments

In this section, we introduce indefinite \(q\)-integrals involving linear fragment of (2.12) or (2.13), of the form

\[
\frac{1}{q}D_{q^{-1}}u(x) + p(x)u(x/q) + r(x) = 0, \tag{4.1}
\]

or

\[
D_q u(x) + p(x)u(qx) + r(x) = 0, \tag{4.2}
\]

respectively.

**Lemma 4.1.** Let \(I\) be an interval containing zero. Let \(p(x)\) and \(r(x)\) be continuous functions at zero. If \(f(x)\) is a solution of Equation (1.4), then

\[
u(x) = -\frac{1}{f(x)} \int_0^{qx} f(t)r(t)d_qt \tag{4.3}
\]

is a solution of Equation (4.1) in \(I\).

**Proof.** Multiplying both sides of (1.11) by \(f(x)\), we obtain

\[
D_{q^{-1}} \left( f(x)u(x) \right) = -q f(x)r(x),
\]

or equivalently

\[
D_q \left( f(x)u(x) \right) = -q f(qx)r(qx).
\]

Hence, from (1.13), we get (4.3) and completes the proof. \(\square\)

If \(u(x)\) is a solution of the \(q\)-linear fragment (4.1), then Equation (2.12) becomes

\[
\int f(x)h(x/q) \left( \frac{1}{q}u(x)u(x/q) + \frac{1}{q}xr(x)(q-1)u(x/q) \right) y(x)d_qx = f(x/q)h(x/q) \left( \frac{y(x/q)u(x/q) - D_{q^{-1}}y(x)}{y(x/q)} \right) . \tag{4.4}
\]

**Theorem 4.2.**

\[
\int \frac{x^2}{(x^2 q^{-1}(1-q); q^2)_\infty} \cos(x; q)d_qx = \frac{q}{(x q^{-1} q^{-1}(1-q); q^2)_\infty} \left( x \cos \left( \frac{x q^{-1}}{q} \right) + \sqrt{q} \sin \left( \frac{x q^{-1}}{q} \right) - \cos \left( q \frac{x q^{-1}}{q} \right) \right) , \tag{4.5}
\]

\[
\int \frac{x^2}{(x^2 q^{-1}(1-q); q^2)_\infty} \sin(x; q)d_qx = \frac{q}{(x q^{-1} q^{-1}(1-q); q^2)_\infty} \left( x \sin \left( \frac{x q^{-1}}{q} \right) - \cos \left( q \frac{x q^{-1}}{q} \right) \right) . \tag{4.6}
\]
where \( \sin(x; q) \) and \( \cos(x; q) \) are defined in Equation (1.21) and (1.22), respectively.

**Proof.** From Equation (3.20), we have \( p(x) = 0, r(x) = -1 \) and \( f(x) = 1 \) is a solution of Equation (1.4). By Lemma 4.1 we get \( u(x) = qx \) satisfies Equation (4.1). Hence

\[
h(x) = \frac{1}{(qx^2(1 - q); q^2)_\infty}
\]

is a solution of Equation (2.1). By substituting with \( u(x) \) and \( h(x) \) into Equation (1.4) and using the \( q \)-difference equation (1.25) and (1.26), respectively, we get the desired results.

**Theorem 4.3.** Let \( n \in \mathbb{N} \). If \( p_n(x|q) \) is the little \( q \)-Legendre polynomials defined in (1.19), \( r_n = \frac{2 - q^{-n}q^{n+1}}{1 - q} \), then

\[
\int \frac{x(qx; q)^\infty}{(q^nx; q)^\infty} p_n(x|q) d_q x = \frac{q^n x^n(q^x; q)^\infty}{[n]_q [n + 1]_q} \left( \frac{1}{q} \right)_\infty^2 \phi_1(q^{-n+1}, q^{n+2}; q, q, x) - p_n(x|q) \quad (4.8)
\]

**Proof.** By comparing Equation (1.20) with (1.3), we get

\[
p(x) = \frac{qx + x - 1}{qx(qx - 1)}, \quad r(x) = \frac{[n]_q [n + 1]_q}{q^n(x - 1)}.
\]

Then \( f(x) = x(1 - qx) \) is a solution of (1.4). From (1.3), we get \( u(x) = \frac{2 - q^{-n}[n]_q[n + 1]_q}{(1 - qx)} \), \( h(x) \) satisfies the \( q \)-difference equation (2.1). Consequently, \( h(x) = \frac{(qx^2; q^2)^\infty}{(r_n q^x; q^2)^\infty} \). By substituting with \( u(x) \) and \( h(x) \) into (1.4) and using the \( q \)-difference equation

\[
D_{q^{-1}} p_n(x|q) = -q^{-n}[n]_q[n + 1]_q 2\phi_1(q^{-n+1}, q^{n+2}; q^2; q, x), \quad (4.9)
\]

we get (4.8).

**Theorem 4.4.** Let \( n \in \mathbb{N} \). If \( p_n(x; -1; q) \) is the big \( q \)-Legendre polynomials defined in (1.17), \( r_n = \frac{2 - q^{-n}q^{n+1}}{1 - q} \), then

\[
\int \frac{x^2(x^2; q^2)^\infty}{(r_n x^2; q^2)^\infty} p_n(x; -1; q) d_q x = \frac{q^{n+2}(x^2; q^2)^\infty}{[n]_q [n + 1]_q [r_n x^2; q^2)^\infty} \left( \frac{q^{n+1}(1 - \frac{x^2}{q^2})}{[n + 1]_q (1 - q^n)} \right) 3\phi_2(q^{-n}, q^{n+1}, x; q; -q; q, q) - xp_n(x; -1; q) \quad (4.10)
\]

**Proof.** From Equation (1.18), we have

\[
p(x) = \frac{x(1 + q)}{q^2(x^2 - 1)}, \quad r(x) = \frac{[n]_q [n + 1]_q}{q^{1+n}(x^2 - 1)}.
\]
Then \( f(x) = (1 - x^2) \) is a solution of \((1.4)\). From \((4.3)\), we have
\[
u(x) = -q^{-n}[n]_q [n + 1]_q x \frac{x}{1 - x^2},
\]
and \( h(x) = \frac{(x^2;q^2)_\infty}{(n,x^2;q^2)_\infty} \) is a solution of \((2.1)\). By substituting with \( u(x) \) and \( h(x) \) into \((1.4)\) and using the \(q\)-difference equation \((3.28)\), we get \((4.10)\).

5. \( q \)-integrals from substitution of simple algebraic forms

In this section, we substitute into Equation \((2.2)\) with simple algebraic forms for \( u(x) \) which involve arbitrary constants, such as
\[
u(x) = \frac{a}{x} + b, \tag{5.1}
\]
to derive indefinite \( q \)-integrals. Set
\[
S_q(x) := \frac{1}{q} D_{q^{-1}} u(x) + \frac{1}{q} u(x) u(x/q) + A(x) u(x/q) + r(x). \tag{5.2}
\]
Then \((2.2)\) will be
\[
\int f(x) h(x/q) S_q(x) y(x) d_q x = f(x/q) h(x/q) (y(x/q) u(x/q) - D_{q^{-1}} y(x)), \tag{5.3}
\]
where the constants \( a \) and \( b \) in Equation \((5.1)\) are chosen so that \( S_q(x) \) has a simple form. Also, we define
\[
T_q(x) := D_q u(x) + u(x) u(qx) + \bar{A}(x) u(qx) + r(x). \tag{5.4}
\]
Then \((2.10)\) will be
\[
\int F(x) k(qx) T_q(x) y(x) d_q x = F(x) k(x) \left( y(x) u(x) - D_{q^{-1}} y(x) \right). \tag{5.5}
\]

**Theorem 5.1.** Let \( n \in \mathbb{N}, n \geq 2 \). Let \( h_n(x; q) \) be the discrete \( q \)-Hermite I polynomial of degree \( n \) which is defined in \((1.7)\). Then
\[
\int x(q^2 x^2; q^2)_\infty h_n(x; q) d_q x = \frac{(1 - q)x(x^2; q^2)_\infty}{[n]_q - 1} \left( \frac{q^n}{x} h_n(x/q; q) - q^{n-1}[n]_q h_{n-1}(x/q; q) \right), \tag{5.6}
\]
and
\[
\int x^{n-2}(q^2 x^2; q^2)_\infty h_n(x; q) d_q x = \frac{x^n (x^2; q^2)_\infty}{[n - 1]_q} \left( \frac{h_n(x/q; q)}{x} - \frac{1}{q} h_{n-1}(x/q; q) \right). \tag{5.7}
\]
Proof. From (I.8),

\[ p(x) = -\frac{x}{1-q}, \quad r(x) = \frac{q^{1-n}[n]_q}{1-q}. \]

Hence \( f(x) = (q^2 x^2; q^2)_\infty \) is a solution of Equation (1.4). Set \( u(x) \) as in (5.1). Then

\[ S_q(x) = \frac{a(a-1)}{x^2} + \frac{ab(1+q)}{qx} + \frac{q^{1-n}[n]_q - a - q^{-n}b x}{1-q} + \frac{b^2}{q}. \]

(5.8)

If \( a = 1 \) and \( b = 0 \) in (5.8), then

\[ S_q(x) = \frac{q^{2-n}[n-1]_q}{1-q}, \]

and \( h(x) = x \) is a solution of (2.1). By substituting with \( u(x) \), \( S_q(x) \) and \( h(x) \) into (5.3) and using (2.24), we get (5.6). If \( a = [n]_q \) and \( b = 0 \) in (5.8), then

\[ S_q(x) = q[n]_q [n-1]_q \frac{1}{x^2}. \]

Hence \( h(x) = x^n \) is a solution of (2.1). Substituting with \( u(x) \), \( S_q(x) \) and \( h(x) \) into (5.3) and using (2.24), we get (5.7).

Theorem 5.2. Let \( n \in \mathbb{N}, n \geq 2 \). Let \( \tilde{h}_n(x; q) \) be the discrete \( q \)-Hermite II polynomial of degree \( n \) which is defined in (1.9). Then

\[ \int \frac{x}{(-x^2; q^2)_\infty} \tilde{h}_n(x; q) d_q x = \frac{(1-q)x}{[n-1]_q (-x^2; q^2)_\infty} \left( \frac{\tilde{h}_n(x; q)}{qx} - q^{-n}[n]_q \tilde{h}_{n-1}(x; q) \right), \]

(5.9)

and

\[ \int \frac{x^{n-2}}{(-x^2; q^2)_\infty} \tilde{h}_n(x; q) d_q x = \frac{x^n}{[n-1]_q (-x^2; q^2)_\infty} \left( \frac{\tilde{h}_n(x; q)}{x} - \tilde{h}_{n-1}(x; q) \right). \]

(5.10)

Proof. From (I.10),

\[ p(x) = -\frac{x}{1-q}, \quad r(x) = \frac{[n]_q}{1-q}. \]

Then \( F(x) = \frac{1}{(-x^2; q^2)_\infty} \) is a solution of (1.6). Set \( u(x) \) as in (5.1), we get

\[ T_q(x) = \frac{a(a-1)}{qx^2} + \frac{ab(1+q)}{qx} + \frac{[n]_q - q^n(q^{-1}a + bx)}{1-q} + b^2. \]

(5.11)

If \( a = 1 \) and \( b = 0 \) in (5.11), then

\[ T_q(x) = \frac{[n-1]_q}{1-q}. \]

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Therefore \( k(x) = x \) is a solution of (2.9). By substituting with \( u(x) \), \( T_q(x) \) and \( k(x) \) into (5.5) and using Equation (2.33), we get (5.9). If \( a = q^{1-n}[n]_q \) and \( b = 0 \) in (5.11), then

\[
T_q(x) = q^{1-2n}[n]_q[n-1]_q \frac{1}{x^2}.
\]

Therefore \( k(x) = x^n \) is a solution of (2.9). By substituting with \( u(x) \), \( T_q(x) \) and \( k(x) \) into (5.5) and using (2.33), we get (5.10).

**Theorem 5.3.** Let \( n \in \mathbb{N} \). If \( S_n(x; q) \) is the Stieltjes-Wigert polynomial of degree \( n \) defined in (I.5), then

\[
\sum_{k=0}^{\infty} q^{(k-1)n} k S_n(q^k x; q) = S_n(\frac{x}{q} ; q) + \frac{x}{1-q^n} S_n-1(q x; q),
\]

(5.12)

and

\[
\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} x^k \left( 1 + q^{2+k}[n-1]_q x \right) S_n(q^k x; q) = S_n(\frac{x}{q} ; q) + \frac{x}{1-q} S_n-1(q x; q).
\]

(5.13)

**Proof.** By comparing (I.6) with (1.3), we get

\[
p(x) = \frac{1 - qx}{qx^2(1-q)}, \\
r(x) = \frac{[n]_q}{x^2(1-q)}.
\]

Set \( u(x) \) as in (5.1). Then

\[
S_q(x) = \frac{b + q[n]_q}{q(1-q)x^2} + \frac{a(a-[n]_q)}{x^2} + \frac{ab(1+q) - q[n-1]_q}{qx} + \frac{a(1-x)}{(1-q)x^3} - \frac{b}{q(1-q)x} + \frac{b^2}{q}.
\]

(5.14)

We set \( a = [n]_q \) and \( b = 0 \) in (5.14) then

\[
S_q(x) = \frac{[n]_q}{(1-q)x^3}.
\]

Therefore, \( h(x) = x^n \) is a solution of (2.1). By substituting with \( h(x) \), \( S_q(x) \) and \( u(x) \) into (5.3), using (2.57) and (3.15), we get (5.12).

If \( a = 1 \) and \( b = 0 \) in (5.14), then

\[
S_q(x) = \frac{1 + q^2[n-1]_q x}{(1-q)x^3}.
\]

Therefore, \( h(x) = x \) is a solution of (2.1). By substituting with \( h(x) \), \( S_q(x) \) and \( u(x) \) into (5.3) and using (2.57) and (3.15), we get (5.13).
Remark 2.

- The indefinite $q$-integral (5.7) is equivalent to [11, Eq.(67)] (if $m = n$) and to (2.17) in Theorem 2.3.

- The indefinite $q$-integral (5.10) is equivalent to [11, Eq.(69)] (if $m = n$) and to (2.28) in Theorem 2.4.

Appendix

The big $q$-Laguerre polynomial

\[ p_n(x; a, b; q) := \phi_2 \left( \frac{q^{-n}, 0, x}{aq, bq} \mid q; q \right) \]  

satisfies the second-order $q$-difference equation, see [17, Eq.(3.11.5)],

\[ \frac{1}{q} D_q^{-1} D_q y(x) + \frac{x - q(a + b - qab)}{abq^2(1 - q)(1 - x)} D_q^{-1} y(x) - \frac{q^{-n - 1}[n]_q}{ab(1 - q)(1 - x)} y(x) = 0. \]  

The $q$-Laguerre polynomial of degree $n$

\[ L_n^\alpha(x; q) := \frac{1}{(q; q)_n} \phi_1 \left( \frac{q^{-n}, -x}{0} \mid q; q^{n + \alpha + 1} \right), \quad \alpha > -1, n \in \mathbb{N} \]  

satisfies the second-order $q$-difference equation, see [17, Eq.(3.21.6)],

\[ \frac{1}{q} D_q^{-1} D_q y(x) + \frac{1 - q^{\alpha + 1}(1 + x)}{q^{\alpha + 1}x(1 + x)(1 - q)} D_q^{-1} y(x) + \frac{[n]_q}{x(1 - q)(1 + x)} y(x) = 0. \]  

The Stieltjes-Wigert polynomials

\[ S_n(x; q) := \frac{1}{(q; q)_n} \phi_1 \left( \frac{q^{-n}}{0} \mid q; -q^{n + 1}x \right), \quad (n \in \mathbb{N}_0) \]  

satisfies the second-order $q$-difference equation, see [17, Eq.(3.27.5)],

\[ \frac{1}{q} D_q^{-1} D_q y(x) + \frac{1 - qx}{qx^2(1 - q)} D_q^{-1} y(x) + \frac{[n]_q}{x^2(1 - q)} y(x) = 0. \]  

The discrete $q$-Hermite I polynomial of degree $n$

\[ h_n(x; q) := q^{\binom{n}{2}} \phi_1 \left( \frac{q^{-n}, x^{-1}}{0} \mid q; -qx \right), \quad n \in \mathbb{N}_0 \]  

satisfies the second-order $q$-difference equation, see [17, Eq.(3.28.5)],

\[ \frac{1}{q} D_q^{-1} D_q y(x) - \frac{x}{1 - q} D_q^{-1} y(x) + \frac{q^{1 - n}[n]_q}{1 - q} y(x) = 0. \]  

The discrete $q$-Hermite II polynomials of degree $n$

\[ \tilde{h}_n(x; q) := x^n \phi_1 \left( \frac{q^{-n}, q^{-n + 1}}{0} \mid q^2; -\frac{q^2}{x^2} \right), \quad n \in \mathbb{N}_0 \]  

satisfies the second-order $q$-difference equation, see [17, Eq.(3.29.5)],

\[ \frac{1}{q} D_q^{-1} D_q y(x) - \frac{x}{1 - q} D_q y(x) + \frac{[n]_q}{1 - q} y(x) = 0. \]
The second Jackson $q$-Bessel function

\[ J^{(2)}_{\nu}(x|q^2) := J^{(2)}_{\nu}(2x(1-q); q^2) \]  

satisfies the second-order $q$-difference equation \[18\]

\[
\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{1 - q x^2 (1 - q)}{x} D_q y(x) + \frac{q x^2 - q^{1-\nu} [\nu]^2}{x^2} y(x) = 0. \tag{1.12}
\]

The $q$-Airy function

\[ Ai_q(x) := {}_1\phi_1(0; -q; q, -x) \]  

satisfies the second-order $q$-difference equation, see \[19\], Eq.(4),

\[
\frac{1}{q} D_{q^{-1}} D_q y(x) - \frac{1 + q}{qx(1-q)} D_{q^{-1}} y(x) + \frac{1}{qx(1-q)^2} y(x) = 0. \tag{1.14}
\]

The Ramanujan function

\[ A_q(x) := {}_0\phi_1(-; 0; q, -q x) \]  

satisfies the second-order $q$-difference Equation, see \[19\], Eq.(5),

\[
\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{1 - qx}{q x^2(1-q)} D_{q^{-1}} y(x) + \frac{1}{x^2(1-q)^2} y(x) = 0. \tag{1.16}
\]

The big $q$-Legendre polynomials

\[ p_n(x; -1; q) := {}_3\phi_2 \left( \frac{q^{-n}, q^{n+1}, x}{q, q^{-1}} \middle| q; q \right) \]  

satisfies the second-order $q$-difference equation, see \[17\], Eq.(3.5.17),

\[
\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{x(1+q)}{q^2(x^2-1)} D_{q^{-1}} y(x) - \frac{[n]_q [n+1]_q}{q^{2n}(x^2-1)} y(x) = 0. \tag{1.18}
\]

The little $q$-Legendre polynomials

\[ p_n(x|q) := {}_2\phi_1 \left( \frac{q^{-n}, q^{n+1}}{q} \middle| q; qx \right) \]  

satisfies the second-order $q$-difference equation, see \[17\], Eq.(3.12.16),

\[
\frac{1}{q} D_{q^{-1}} D_q y(x) + \frac{qx + x - 1}{qx(x-1)} D_{q^{-1}} y(x) + \frac{[n]_q [n+1]_q}{q^n x(1-qx)} y(x) = 0. \tag{1.20}
\]

Jackson also introduced three $q$-analogs of Bessel functions, \[12, 16\], they are defined by

\[
J^{(1)}_{\nu}(z; q) := \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(q, q^{n+1}; q)_n} (z/2)^{2n+\nu}, \quad |z| < 2,
\]

\[
J^{(2)}_{\nu}(z; q) := \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^n (n+\nu)}{(q, q^{n+1}; q)_n} (z/2)^{2n+\nu}, \quad z \in \mathbb{C},
\]

\[
J^{(3)}_{\nu}(z; q) := \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}}{(q, q^{n+1}; q)_n} (z)^{2n+\nu}, \quad z \in \mathbb{C}.
\]

There are three known $q$-analogs of the trigonometric functions, \{\sin_q z, \cos_q z\}, \{\sin_q z, \cos_q z\}, and \{\sin(z; q), \cos(z; q)\}. Each set of $q$-analogs is related to one of the three $q$-analogs of Bessel
functions.
The functions $\sin_q z$ and $\cos_q z$ are defined for $|z| < \frac{1}{1-q}$ by

$$\sin_q z := \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} (z)^{1/2} J^{(1)}_{1/2}(2z; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)_q!},$$

$$\cos_q z := \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} (z)^{1/2} J^{(1)}_{1/2}(2z; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)_q!}.$$  

The functions $\Sin_q z$ and $\Cos_q z$ are defined for $z \in \mathbb{C}$ by

$$\Sin_q z := \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} (z)^{1/2} J^{(2)}_{1/2}(2z; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1} z^{2n+1}}{(2n+1)_q!},$$

$$\Cos_q z := \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} (z)^{1/2} J^{(2)}_{1/2}(2z; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n} z^{2n}}{(2n)_q!}.$$  

Finally, the functions $\sin(z; q)$ and $\cos(z; q)$ are defined for $z \in \mathbb{C}$ by

$$\sin(z; q) := \frac{(q; q)_\infty}{(q; q^2)_\infty} z^{1/2} J^{(3)}_{1/2}(z(1-q); q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2+n+2} z^{2n+1}}{(2n+1)_q(2n+2)},$$  

$$\cos(z; q) := \frac{(q; q)_\infty}{(q; q^2)_\infty} z^{1/2} J^{(3)}_{1/2}(z(1-q)/\sqrt{q}; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2} z^{2n}}{(2n+1)_q(2n+1)}.$$  

The $q$-trigonometric functions satisfy the $q$-difference equations

$$D^{-1}_q \sin_q z = \cos_q \left(\frac{z}{q}\right), \quad D^{-1}_q \cos_q z = -\sin_q \left(\frac{z}{q}\right).$$  

$$D^{-1}_q \Sin_q z = \Cos_q (z), \quad D^{-1}_q \Cos_q z = -\Sin_q (z).$$  

$$D^{-1}_q \sin(z; q) = \cos(q^{1/2} z; q),$$  

$$D^{-1}_q \cos(z; q) = -q^{1/2} \sin(q^{1/2} z; q).$$  

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