SYMMETRIC AND ANTISYMMETRIC
VECTOR-VALUED JACK POLYNOMIALS

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Abstract. Polynomials with values in an irreducible module of the symmetric group can be given the structure of a module for the rational Cherednik algebra, called a standard module. This algebra has one free parameter and is generated by differential-difference (“Dunkl”) operators, multiplication by coordinate functions and the group algebra. By specializing Griffeth’s (arXiv:0707.0251) results for the $G(r,p,N)$ setting, one obtains norm formulae for symmetric and antisymmetric polynomials in the standard module. Such polynomials of minimum degree have norms which involve hook-lengths and generalize the norm of the alternating polynomial.

1. Introduction

Hook-lengths of nodes in Young tableaux appear in a variety of different settings. Griffeth [6] introduced Jack polynomials whose values lie in irreducible modules of the family $G(r,p,N)$ of complex reflection groups. This class of polynomials forms an orthogonal basis for the associated standard module of the rational Cherednik algebra. In this paper we specialize his results to the symmetric group and show how the norms of two special symmetric and antisymmetric polynomials in the standard module depend on the hook-lengths of the partition associated to the representation. These norm formulae prove a necessity condition for aspherical parameter values. This condition was first found by Gordon and Stafford [5].

For $N \geq 2$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$. For $a, b \in \mathbb{N}$ and $a \leq b$ let $[a, b] = \{a, a+1, \ldots, b\}$ (an interval of integers). The cardinality of a set $E$ is denoted by $\#E$. For $\alpha \in \mathbb{N}^N$ (a composition) let $|\alpha| := \sum_{i=1}^{N} \alpha_i$, $x^\alpha := \prod_{i=1}^{N} x_i^{\alpha_i}$, a monomial of degree $|\alpha|$. The spaces of polynomials, respectively homogeneous, polynomials

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are
\[ \mathcal{P} := \text{span}_F \{ x^\alpha : \alpha \in \mathbb{N}^N \}, \]
\[ \mathcal{P}_n := \text{span}_F \{ x^\alpha : \alpha \in \mathbb{N}^N, |\alpha| = n \}, \quad n \in \mathbb{N}, \]
where \( F \) is a field \( \supset \mathbb{Q} \). Consider the symmetric group \( S_N \) as the group of permutations of \([1, N]\). The group acts on polynomials by linear extension of \((xw)_i = x_{w(i)}, w \in S_N, 1 \leq i \leq N\), that is, \( w \cdot f(x) := f(xw), f \in \mathcal{P} \). For \( \alpha \in \mathbb{N}^N \) let \( (w\alpha)_i = \alpha_{w^{-1}(i)} \), then \( w(x^\alpha) = x^{w\alpha} \).

Also \( S_N \) is a finite reflection group whose reflections are the transpositions \((i, j); x(i, j) = (\ldots, \hat{i}, \ldots, \hat{j}, \ldots)\). The simple reflections \( s_i := (i, i + 1), 1 \leq i < N \), generate \( S_N \).

Say \( \lambda \in \mathbb{N}^N \) is a partition if \( \lambda_i \geq \lambda_{i+1} \) for all \( i \). Denote the set of partitions by \( \mathbb{N}^N_\geq \). Suppose \( \tau \) is a partition of \( N \), that is, \( |\tau| = N \); then there is an associated Ferrers diagram, namely the set of lattice points \( \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq \ell(\tau), 1 \leq j \leq \tau_i \} \), also denoted by \( \tau \); the length of \( \tau \) is \( \ell(\tau) := \max \{i : \tau_i > 0\} \). The conjugate partition \( \tau' \) is the partition whose diagram is the transpose of the diagram of \( \tau \); also denoted by \( \tau' \) (that is, \( \tau'_m = \# \{i : \tau_i \geq m\} \)). For a node (or point) \((i, j) \in \tau \) the arm-length is \( \text{arm}(i, j) := \tau_i - j \), the leg-length is \( \text{leg}(i, j) := \tau'_j - i \), and the hook-length is \( h(i, j) := \text{arm}(i, j) + \text{leg}(i, j) + 1 \). We will use \( \text{arm}(i, j; \tau) \) etc. if it is necessary to specify the partition.

To each partition \( \tau \) of \( N \) there is an associated irreducible \( S_N \)-module \( V_\tau \). We analyze the space \( M(\tau) \) of \( V_\tau \)-valued polynomials under the action of differential-difference ("Dunkl") operators. There is a canonical symmetric bilinear (the contravariant) form \( \langle \cdot, \cdot \rangle \) on this space. We will construct distinguished polynomials \( f^s_\tau, f^a_\tau \in M(\tau) \), with \( f^s_\tau \) being symmetric and \( f^a_\tau \) being antisymmetric, such that
\[
\langle f^s_\tau, f^s_\tau \rangle = c_0 \prod_{(i,j) \in \tau} (1 - h(i,j) \kappa)_{\text{leg}(i,j)},
\]
\[
\langle f^a_\tau, f^a_\tau \rangle = c_1 \prod_{(i,j) \in \tau} (1 + h(i,j) \kappa)_{\text{arm}(i,j)};
\]
and \( c_0, c_1 \in \mathbb{Q} \) are constants depending on \( \tau \), and the Pochhammer symbol is
\[
(t)_n := \prod_{i=1}^{n} (t + i - 1), \quad n \in \mathbb{N}.
\]
This result generalizes the situation of the trivial representation of \( S_N \); in this case \( \tau = (N), f^s_\tau = 1, f^a_\tau = \prod_{1 \leq i < j \leq N} (x_i - x_j) \) and \( \langle f^s_\tau, f^a_\tau \rangle = c_1 \prod_{i=2}^{N} (1 + i \kappa)_{i-1} \) (see Opdam [10]).
Section 2 collects the needed information about representations of $S_N$. The Dunkl operators and their action on monomials are discussed in Section 3. The (nonsymmetric) Jack polynomials are constructed in Section 4; this material is the specialization of Griffeth’s results for $G(r,p,N)$ to $S_N = G(1,1,N)$ and some of the proofs are in the Appendix. Our main results on symmetric and antisymmetric polynomials are contained in Section 5. There is the description of an orthogonal basis and the detailed exposition of the special polynomials whose norms involve the hook-lengths. In fact these are the symmetric and antisymmetric polynomials of minimum degree. In the standard module $M(\tau)$ the orthogonal basis of symmetric (respectively antisymmetric) Jack polynomials is labeled by the column-strict (respectively row-strict) tableaux of shape $\tau$.

2. Representations of $S_N$

Let $Y(\tau)$ be the set of reversed standard Young tableaux (RSYT) of shape $\tau$, namely, an assignment of the numbers $1, 2, \ldots, N$ to each node of $\tau$ such that entries decrease in each row and in each column. (Note: we use reversed tableaux because they provide a direct way of expressing eigenvalues of the operators defining the nonsymmetric Jack polynomials, see (4.1).) The node of $T$ containing $i$ is denoted $T(i)$ and the row and column of this node are denoted by $\text{rw}(i,T)$ and $\text{cm}(i,T)$ respectively, $i \in [1,N]$. The content of $T(i)$ is $c(i,T) := \text{cm}(i,T) - \text{rw}(i,T)$. Thus $c(N,T) = 0$ for each $T$. The well-known hook-length formula asserts that $\#Y(\tau) = N! / \prod_{(i,j) \in \tau} h(i,j)$. Following Murphy [8] define an action of $S_N$ on the $\#(Y(\tau))$-dimensional vector space $V_\tau := \operatorname{span}_F \{v_T : T \in Y(\tau)\}$ as follows:

**Proposition 1.** Suppose $T \in Y(\tau)$ and

$$b_i(T) := 1 / (c(i,T) - c(i+1,T))$$

for $1 \leq i < N$ then:

1. if $b_i(T) = 1$ (when $\text{rw}(i,T) = \text{rw}(i+1,T)$) then $s_i v_T = v_T$;
2. if $b_i(T) = -1$ (when $\text{cm}(i,T) = \text{cm}(i+1,T)$) then $s_i v_T = -v_T$;
3. if $0 < b_i(T) \leq \frac{1}{2}$ (when $\text{rw}(i,T) < \text{rw}(i+1,T)$ and $\text{cm}(i,T) > \text{cm}(i+1,T)$) then

$$s_i v_T = b_i(T) v_T + v_{s_i T},$$

$$s_i v_{s_i T} = (1 - b_i(T)^2) v_T - b_i(T) v_{s_i T};$$
(4) if $-\frac{1}{2} \leq b_i(T) < 0$ (when $\text{rw}(i, T) > \text{rw}(i+1, T)$ and $\text{cm}(i, T) < \text{cm}(i+1, T)$) then
\[
s_i v_T = b_i(T) v_T + (1 - b_i(T)^2) v_{s_i T},
\]
and
\[
s_i v_{s_i T} = v_T - b_i(T) v_{s_i T}.
\]
In cases (3) and (4) the tableau $s_i T$ is obtained by interchanging the entries $i, i+1$. Furthermore in case (3) $s_i f_0 = f_0$ and $s_i f_1 = -f_1$ for
\[
f_0 = (b_i(T) + 1) v_T + v_{s_i T},
f_1 = (b_i(T) - 1) v_T + v_{s_i T}.
\]
There is an ordering on tableaux such that $T > s_i T$ in case (4).

**Corollary 1.** Let $f = \sum_{T \in Y(\tau)} k_T v_T$ with the coefficients $k_T \in \mathbb{Q}$ and $s_i f = \pm f$ for some $i \in [1, N-1]$. Then
\begin{enumerate}
  \item $T, s_i T \in Y(\tau)$ implies $k_{s_i T} = r k_T$ for some $r \neq 0$;
  \item $s_i f = f$ and $\text{cm}(i, T) = \text{cm}(i+1, T)$ implies $k_T = 0$;
  \item $s_i f = -f$ and $\text{rw}(i, T) = \text{rw}(i+1, T)$ implies $k_T = 0$.
\end{enumerate}

Statement (1) means that $k_{s_i T}$ and $k_T$ are either both nonzero or both zero.

**Definition 1.** The Jucys-Murphy elements (in the group algebra $\mathbb{Q} S_N$) are
\[
\omega_i := \sum_{j=i+1}^{N} (i, j), \; i \in [1, N].
\]

There are commutation relations: $\omega_i \omega_j = \omega_j \omega_i$ for all $i, j$; $\omega_i s_j = s_j \omega_i$ for $j \neq i-1, i$; $s_i \omega_i - \omega_{i+1} s_i = 1$ (see Vershik and Okounkov [9, Sec. 4] for the representations of the algebra generated by $\{\omega_i, \omega_{i+1}, s_i\}$). Murphy proved the following:

**Theorem 1.** Suppose $T \in Y(\tau)$ and $i \in [1, N]$ then $\omega_i v_T = c(i, T) v_T$.

Let $\langle \cdot, \cdot \rangle_0$ be a $S_N$-invariant positive-definite bilinear form on $V_\tau$, (the form is unique up to a multiplicative constant) then each $\omega_i$ is self-adjoint and hence the vectors $v_T$ are pairwise orthogonal, being eigenvectors with different eigenvalues. Denote $\|v\|^2 = \langle v, v \rangle_0$. For given $T$ and $i$ as in case (4) we have $\|v_T\|^2 = b(i, T)^2 \|v_T\|^0 + \|v_{s_i T}\|^2$ (since $s_i$ is an isometry) and thus $\|v_{s_i T}\|^2 = (1 - b(i, T)^2) \|v_T\|^2$. There is one formula for $\|v_T\|^2$ in [8, Thm. 4.1]. The following is based on the content vector of $T$ (that is, $(c(1, T), \ldots, c(N, T)))$:
Lemma 1. Suppose by sequences of rows; then

Definition 2. For in the numerator cancel out, by Lemma 1. □

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\[ \sum_{1 \leq i < j \leq N} (c(i, T) - c(j, T))^2 \]

For example, let N = 4, \( \tau = (3, 1) \) and for convenience label tableaux by sequences of rows; then \( \|v_{[(4,2,1),(3)}\|_c^2 = 1, \|v_{[(4,3,1),(2)}\|_c^2 = \frac{3}{4}, \|v_{[(4,3,2),(1)}\|_c^2 = \frac{2}{3}. \)

Lemma 1. Suppose \( \{g_{ij}(T) : 1 \leq i < j \leq N\} \) is a collection of functions on Y (\( \tau \)) such that

1. \( g_{ij}(T) = g_{ij}(s_m T) \) for all \( i, j \) with \( \{i, j\} \cap \{m, m+1\} = \emptyset, \)
2. \( g_{i,m}(T) = g_{i,m+1}(s_m T) \) and \( g_{i,m+1}(T) = g_{i,m}(s_m T) \) for \( i < m, \)
3. \( g_{m,j}(T) = g_{m+1,j}(s_m T) \) and \( g_{m+1,j}(T) = g_{m,j}(s_m T) \) for \( j > m + 1, \) whenever \( m \in [1, N-1] \) and \( \{T, s_m T\} \subset Y(\tau) \), then

\[ \frac{\prod_{1 \leq i < j \leq N} g_{ij}(T)}{\prod_{1 \leq i < j \leq N} g_{ij}(s_m T)} = \frac{g_{m,m+1}(T)}{g_{m,m+1}(s_m T)}. \]

The proof is a straightforward calculation.

Proposition 2. Suppose \( 0 < b_i(T) \leq \frac{1}{2} \) for \( T \in Y(\tau) \) and some \( i \in [1, N-1] \) then \( \|v_{s_i T}\|_c^2 = (1 - b_i(T)^2) \|v_T\|_c^2. \) Thus \( \|\|_c \) is an \( S_N \) invariant norm.

Proof. By hypothesis \( c(i, T) \geq c(i+1, T) + 2 \) and

\[ c(i, s_i T) = c(i+1, T), c(i+1, s_i T) = c(i, T). \]

All factors in the ratio \( \|v_{s_i T}\|_c^2 / \|v_T\|_c^2 \) except \( 1 - \left( \frac{1}{c(i+1, T) - c(i, T)} \right)^2 \)
in the numerator cancel out, by Lemma 1. □

Henceforth we drop the subscript “c” and use “0” for the form. Next we consider invariance properties for certain subgroups of \( S_N \), specifically the stabilizer subgroups of the monomials \( x^\lambda \), where \( \lambda \in \mathbb{N}^{N,+}. \)

Definition 3. For \( 1 \leq a < b \leq N \) let

\[ S_{[a,b]} := \{ w \in S_N : i \notin [a, b] \implies w(i) = i \}, \]

the subgroup of permutations of \( [a, b] \), generated by \( \{s_i : a \leq i < b\}. \)

We look for elements \( f \) of \( V \) which are symmetric or antisymmetric for a group \( S_{[a,b]} \), or the equivalent properties: \( s_i f = f \), respectively, \( s_i f = -f \), for \( a \leq i < b \). The idea is to find the expansions of \( \sum_{w \in S_{[a,b]}} w v_T \), or \( \sum_{w \in S_{[a,b]}} \text{sgn}(w) w v_T \), in the basis \( \{v_S : S \in Y(\tau)\} \) for given \( T \in Y(\tau) \).
Definition 4. For $T \in Y(\tau)$ and a subgroup $H$ of $S_N$ let $V_T(H) = \text{span}\{w_T : w \in H\}$ and let $Y(T; H) = \{T' \in Y(\tau) : v_{T'} \in V_T(H)\}$.

In the case $H = S_{[a,b]}$ there are two extremal elements of $Y(T; H)$, namely $T_0$ with the property $cm\,(i, T_0) \geq cm\,(i + 1, T_0)$ for $a \leq i < b$, and $T_1$ with the property $rw\,(i, T_0) \geq rw\,(i + 1, T_0)$ (it is possible that $T_0 = T_1$). To produce $T_0$ one applies a sequence of transformations of type (4) (in Proposition 1) (type (3) for $T_1$). If $cm\,(i_1, T) = cm\,(i_2, T)$ for some $i_1, i_2 \in [a, b]$ (suppose $i_1 > i_2$ then any entry $j$ in this column of $T$ between $i_1$ and $i_2$ has to satisfy $i_1 > j > i_2$) then $T_0$ has $cm\,(i, T_0) = cm\,(i + 1, T_0)$ for some $i \in [a, b - 1]$. Similarly if $rw\,(i_1, T) = rw\,(i_2, T)$ for some $i_1, i_2 \in [a, b]$ then $T_1$ has $rw\,(i, T_1) = rw\,(i + 1, T_1)$ for some $i \in [a, b - 1]$.

2.1. Subgroup symmetric vectors. First consider the invariant (symmetric) situation. Corollary 1 and the properties of $T_0$ imply the following necessary condition for $V_T(S_{[a,b]})$ to contain a nontrivial $S_{[a,b]}$-invariant.

Say $T$ satisfies condition $[a, b]_{cm}$ if the entries $a, a + 1, \ldots, b$ are in distinct columns of $T$, that is, $a \leq i < j \leq m$ implies $cm\,(i, T) \neq cm\,(j, T)$. Fix some $T$ satisfying this condition and consider the subspace $V_T(S_{[a,b]})$. Let $T_0 \in Y(T; S_{[a,b]})$ satisfy $cm\,(i, T) > cm\,(j, T)$ for $a \leq i < j \leq b$ (equality is ruled out by hypothesis). It is possible that $i$ and $i + 1$ are in the same row of $T_0$ for some $i \in [a, b]$ (in which case $\#Y(T; S_{[a,b]}) < (b - a + 1)! = \#S_{[a,b]}$). For $a \leq i < b$ we have $rw\,(i, T_0) \leq rw\,(i + 1, T_0)$, thus $a \leq i < j \leq b$ implies $c(j, T_0) - c(i, T_0) \leq -2$ or $j = i + 1$ and $rw\,(i + 1, T_0) = rw\,(i, T_0)$; indeed suppose the latter condition does not hold then if $j > i + 1$

$$c(j, T_0) - c(i, T_0) = (cm\,(j, T_0) - cm\,(i, T_0)) + (rw\,(i, T_0) - rw\,(j, T_0))$$

$$\leq cm\,(j, T_0) - cm\,(i, T_0) \leq i - j \leq -2,$$

or $j = i + 1$ and

$$c(i + 1, T_0) - c(i, T_0)$$

$$= (cm\,(i + 1, T_0) - cm\,(i, T_0)) + (rw\,(i, T_0) - rw\,(i + 1, T_0))$$

$$\leq -1 - 1 = -2.$$

Definition 5. Suppose $T \in Y(\tau)$ satisfies condition $[a, b]_{cm}$ then let

$$P_0(T; a, b) := \prod_{a \leq i < j \leq b, \; cm(i, T) < cm(j, T)} \frac{c(j, T) - c(i, T)}{1 + c(j, T) - c(i, T)}.$$

The denominator can not vanish, for suppose \( i < j \), \( \text{cm} (i, T) < \text{cm} (j, T) \), and \( T (i) = T_0 (i_1), T (j) = T_0 (i_2) \) with \( i_1 < i_2 \) (this follows from \( \text{cm} (i_2, T) < \text{cm} (i_1, T) \)) then \( c (i, T) - c (j, T) = c (i_2, T_0) - c (i_1, T_0) \leq -2 \), and \( \text{rw} (i, T) = \text{rw} (i_2, T_0) \neq \text{rw} (j, T) = \text{rw} (i_1, T_0) \).

For notational convenience we use the fact \( Y (T; S_{[a,b]}) = Y (T_0; S_{[a,b]}) \) (and let \( T \) be variable, henceforth).

**Proposition 3.** Let \( f = \sum_{T \in Y (T_0; S_{[a,b]})} P_0 (T; a, b) v_T \) then \( w f = f \) for all \( w \in S_{[a,b]} \).

**Proof.** Suppose \( a \leq i < b \) then let

\[
A = \{ T \in Y (T_0; S_{[a,b]}) : \text{rw} (i, T) = \text{rw} (i + 1, T) \}
\]

and

\[
B = \{ T \in Y (T_0; S_{[a,b]}) : \text{rw} (i, T) < \text{rw} (i + 1, T) \}.
\]

Then

\[
f = \sum_{T \in A} P_0 (T; a, b) v_T + \sum_{T \in B} (P_0 (T; a, b) v_T + P_0 (s_i T; a, b) v_{s_i T}).
\]

Fix \( T \in B \) and compute \( P_0 (T; a, b) / P_0 (s_i T; a, b) \) using Lemma 1; set \( g_{mn} (T) = 1 \) if \( \text{cm} (m, T) \geq \text{cm} (n, T) \) and \( g_{mn} (T) = \frac{c(n,T) - c(m,T)}{1 + c(n,T) - c(m,T)} \) if \( \text{cm} (m, T) < \text{cm} (n, T) \). Then \( g_{i,i+1} (T) = 1 \) and

\[
g_{i,i+1} (s_i T) = \frac{c (i + 1, s_i T) - c (i, s_i T)}{1 + c (i + 1, s_i T) - c (i, s_i T)} = \frac{1}{1 + b_i (s_i T)} = \frac{1}{1 + b_i (T)}.
\]

Thus \( P_0 (T; a, b) / P_0 (s_i T; a, b) = 1 + b_i (T) \) and \( s_i f = f \) by Proposition 1. \( \square \)

**Corollary 2.** Let \( n_0 = \# \{ w \in S_{[a,b]} : w v_{T_0} = v_{T_0} \} \), then

\[
\| f \|_0^2 = \frac{(b - a)!}{n_0} P_0 (T_1; a, b) \| v_{T_0} \|_0^2.
\]

**Proof.** If \( T, T' \in Y (\tau) \) and \( T' \) is obtained from \( T \) by a sequence of steps of type (3) in Proposition 1 then \( T' = w T \) for some \( w \in S_N \) and \( v_{T'} = w v_T + \sum_j b_j v_{S_j} \), where \( b_j \in \mathbb{Q} \) and \( [S_1 = T, S_2, \ldots] \) is the list of intermediate steps. Let \( f_1 = \sum_{w \in S_{[a,b]}} w v_{T_0} \) thus \( f_1 = c f \) for some constant \( c \). In the expansion of \( f_1 \) in the basis \( \{ v_T : T \in Y (T_0; S_{[a,b]}) \} \) the coefficient of \( v_{T_1} \) is \( n_0 \), because \( T_0, T_1 \) have the property described above and \( v_{T_1} \) is extremal in \( Y (T_0; S_{[a,b]}) \) (heuristically the “bubble sort” is used; first apply \( (b - 1, b) (b - 2, b - 1) \ldots (a, a + 1) \) to \( T_0 \); this moves \( b \) to the column with highest possible number; then repeat the process with \( S_{[a,b-1]} \), or \( S_{[a,b-k]} \) if \( b - k + 1, \ldots, b \) are now in the same
row, and so on). The coefficient of \( v_T \) in \( f \) is \( P_0(T; a, b) \). Thus \( c = \frac{n_0}{r_b(T_1; a, b)} \) in \( f \). Finally

\[
\langle f, f \rangle_0 = \frac{1}{c} \langle f_1, f \rangle_0 = \frac{1}{c} \sum_{w \in S_{[a,b]}} \langle w v_{T_0}, f \rangle_0
\]

\[
= \frac{(b - a)!}{c} \langle v_{T_0}, f \rangle_0 = \frac{(b - a)!}{c} \langle v_{T_0}, v_{T_0} \rangle.
\]

This completes the proof. \( \square \)

It is straightforward to extend these methods to the case \( H = S_{[a_1, b_1]} \times S_{[a_2, b_2]} \times \ldots \times S_{[a_n, b_n]} \) where \( 1 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \leq N \). This requires a tableau \( T_0 \in Y(\tau) \) satisfying condition \( [a_i, b_i]_{cm} \) for \( 1 \leq i \leq n \). Then

\[
f = \sum_{T \in Y(T_0; H)} \prod_{i=1}^{n} P_0(T; a_i, b_i) v_T
\]

is the unique \( H \)-invariant element of \( V_{T_0}(H) \).

2.2. Subgroup antisymmetric vectors. We turn to the problem of antisymmetric vectors in \( V_T(H) \). The previous arguments transfer almost directly by transposing tableaux and inserting minus signs at appropriate places.

Say \( T \) satisfies condition \( [a, b]_{rw} \) if the entries \( a, a+1, \ldots, b \) are in distinct rows of \( T \), that is, \( a \leq i < j \leq m \) implies \( rw(i, T) \neq rw(j, T) \). Fix some \( T \) satisfying this condition and consider the subspace \( V_T(S_{[a,b]}) \).

Let \( T_0 \in Y(T; S_{[a,b]}) \) satisfy \( cm(i, T) \geq cm(j, T) \) for \( a \leq i < j \leq b \).

**Definition 6.** Suppose \( T \in Y(\tau) \) satisfies condition \( [a, b]_{cm} \) then let

\[
P_1(T; a, b) := \prod_{a \leq i < j \leq b, \ cm(i,T) < cm(j,T)} \frac{c(j, T) - c(i, T)}{1 - c(j, T) + c(i, T)}.
\]

As before we use the basic set \( Y(T_0; S_{[a,b]}) \) to produce an antisymmetric vector. Note \( P_1(T_0; a, b) = 1 \).

**Proposition 4.** Let \( f = \sum_{T \in Y(T_0; S_{[a,b]})} P_1(T; a, b) v_T \) then \( s_i f = - f \) for \( a \leq i < b \) and \( w f = \text{sgn}(w) f \) for all \( w \in S_{[a,b]} \). Let \( n_0 = \# \{ w \in S_{[a,b]} : w v_{T_0} = \pm v_{T_0} \} \) then \( \| f \|^2 = \frac{(b-a)!}{n_0} |P_1(T_1; a, b)| \| v_{T_0} \|^2 \).

**Proof.** Suppose \( a \leq i < b \) then let

\[
A = \{ T \in Y(T_0; S_{[a,b]} : cm(i, T) = cm(i+1, T) \}
\]

and

\[
B = \{ T \in Y(T_0; S_{[a,b]} : cm(i, T) > cm(i+1, T) \}
\]
$T \in A$ implies $s_i v_T = -v_T$. Then
\[ f = \sum_{T \in A} P_1 (T; a, b) v_T + \sum_{T \in B} (P_1 (T; a, b) v_T + P_1 (s_i T; a, b) v_{s_i T}). \]

Fix $T \in B$ and compute $P_1 (T; a, b) / P_1 (s_i T; a, b)$ using Lemma 1; set $g_{mn} (T) = 1$ if $cm (m, T) \geq cm (n, T)$ and $g_{mn} (T) = \frac{c (n, T) - c (m, T)}{1 - c (n, T) + c (m, T)}$ if $cm (m, T) < cm (n, T)$. Then $g_{i, i+1} (T) = 1$ and
\[ g_{i, i+1} (s_i T) = \frac{c (i + 1, s_i T) - c (i, s_i T)}{1 - c (i + 1, s_i T) + c (i, s_i T)} = \frac{1}{-1 - b_i (s_i T)} = \frac{1}{b_i (T) - 1}. \]

Thus $P_0 (T; a, b) / P_0 (s_i T; a, b) = b_i (T) - 1$ and $s_i f = -f$ by Proposition 1. The norm formula follows from the proof of Corollary 2 with some small modifications to take care of sign-changes. \qed

There are corresponding statements for $H = S_{[a_1, b_1]} \times \cdots \times S_{[a_n, b_n]}$, using disjoint intervals. The branching theorem for the restriction of irreducible representations of $S_N$ to those of the parabolic subgroups (like $H$) implicitly appears in the previous discussion, in connection with the conditions $[a, b]_{cm}$ and $[a, b]_{rw}$.

### 3. Dunkl Operators

Let $\kappa$ be a transcendental (formal parameter) and set $F = Q (\kappa)$. Consider the space $P \otimes V_\tau = span_F \{ x^\alpha v_T : \alpha \in \mathbb{N}^N, T \in Y (\tau) \}$, polynomials $p (x)$ on $\mathbb{R}^N$ with values in $V_\tau$. The space is an $S_N$-module with the action $w (x^\alpha v_T) = x^{w \alpha} (w v_T)$ for $w \in S_N$, extended to all of $P \otimes V_\tau$ by linearity. For $p \in P$ and $u \in V_\tau$ and $1 \leq i \leq N$ let
\[ (3.1) \quad D_i (p (x)) u := \frac{\partial}{\partial x_i} p (x) u + \kappa \sum_{j=1, j \neq i}^N \frac{p(x) - p(x(i,j))}{x_i - x_j} (i, j) u. \]

The definition is extended to $P \otimes V_\tau$ by linearity. Then $D_i D_j = D_j D_i$ for $1 \leq i, j \leq N$. The proof is a straightforward adaptation of the original proof for scalar polynomials $p (x)$ (see [3, Ch. 4]). There are important commutators (appearing in the definition of the rational Cherednik algebra, the algebra generated by $F S_N$ and $\{ x_i, D_i : i \in [1, N] \}$):
\[ (3.2) \quad D_i x_j - x_j D_i = -\kappa (i, j), i \neq j \]
\[ D_i x_i - x_i D_i = 1 + \kappa \sum_{j \neq i} (i, j). \]
Definition 7. The space \( \mathcal{P} \otimes V_\tau \) equipped with the action of \( \mathbb{F} S_N \) and \( \{x_i, D_i : i \in [1, N]\} \) is a standard module of the rational Cherednik algebra and is denoted by \( M(\tau) \). For \( n \in \mathbb{N} \) the linear subspace \( \mathcal{P}_n \otimes V_\tau \) is denoted by \( M_n(\tau) \).

The representation theory of rational Cherednik algebras is described in the survey \([11]\) by Rouquier. For \( p(x) \in \mathcal{P} \otimes V_\tau \), set

\[
\mathcal{U}_i p(x) = D_i (x) p(x) - \kappa \sum_{j=1}^{i-1} (i, j) p(x), \quad 1 \leq i \leq N.
\]

The operators \( \mathcal{U}_i \) also commute pairwise. They have a triangularity property (a special case of a result of Griffeth \([6]\) for the complex reflection groups \( G(r, p, N) \)). There is an important function on compositions:

Definition 8. For \( \alpha \in \mathbb{N}^N \) and \( 1 \leq i \leq N \) let

\[
r(\alpha, i) := \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \}
\]

be the rank function.

A consequence of the definition is that \( r(\alpha, i) < r(\alpha, j) \) is equivalent to \( \alpha_i > \alpha_j \), or \( \alpha_i = \alpha_j \) and \( i < j \). For any \( \alpha \) the function \( i \mapsto r(\alpha, i) \) is one-to-one on \( \{1, 2, \ldots, N\} \). Let \( w_\alpha \) denote the inverse function, thus \( r(\alpha, w_\alpha(i)) = i \). Further \( \alpha \) is a partition if and only if \( r(\alpha, i) = i \) for all \( i \). In general \( (w_\alpha^{-1})_i = \alpha_{w_\alpha(i)} \) for \( 1 \leq i \leq N \), and thus \( w_\alpha^{-1} \alpha \) is a partition, denoted by \( \alpha^+ \). For example, let \( \alpha = (1, 0, 4, 2, 4) \) then

\[
[r(\alpha, i)]_{i=1}^5 = [4, 5, 1, 3, 2],
\]

\[
[w_\alpha(i)]_{i=1}^5 = [3, 5, 4, 1, 2],
\]

\[
\alpha^+ = (4, 4, 2, 1, 0).
\]

The order on compositions is derived from the dominance order.

Definition 9. For \( \alpha, \beta \in \mathbb{N}^N \) the partial order \( \alpha \succeq \beta \) (\( \alpha \) dominates \( \beta \)) means that \( \alpha \neq \beta \) and \( \sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i \) for \( 1 \leq j \leq N \); and \( \alpha \triangleright \beta \) means that \( |\alpha| = |\beta| \) and either \( \alpha^+ \triangleright \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \triangleright \beta \).

For example \( (5, 1, 4) \triangleright (1, 5, 4) \triangleright (4, 3, 3) \), while \( (1, 5, 4) \) and \( (6, 2, 2) \) are not comparable in \( \triangleright \). There are some results useful in analyzing \( \mathcal{U}_i x^{\epsilon a} u \). Let \( \epsilon(i) \) be the \( i \)th standard basis vector in \( \mathbb{N}^N \), for \( 1 \leq i \leq N \). By \([3\), Lemma 8.2.3\] the following hold for \( \alpha \in \mathbb{N}^N \):

1. if \( \alpha_i > \alpha_j \) and \( i < j \) then \( (i, j) \alpha < \alpha \);
2. \( \alpha^+ \triangleright \alpha \);
3. if \( 1 \leq s < (\alpha_i - \alpha_j) \) then \( \alpha^+ \triangleright (\alpha - s(\epsilon(i) - \epsilon(j)))^+ \).
The following is a consequence of these relations and an easy computation.

**Lemma 2.** For \( \alpha \in \mathbb{N}^N \) and \( i \neq j \) let \( B_{ij}x^\alpha := \frac{x_i x^\alpha - x_j (i, j) x^\alpha}{x_i - x_j} \), then

1. If \( \alpha_i = \alpha_j \) then \( B_{ij}x^\alpha = x^\alpha \);
2. If \( \alpha_i > \alpha_j \) then

\[
B_{ij}x^\alpha = x^\alpha + (i, j)x^\alpha + \sum_{s=1}^{\alpha_i-\alpha_j-1} x^\alpha - s(\varepsilon(i) - \varepsilon(j))
\]

and \( \alpha^+ \triangleright (\alpha - s(\varepsilon(i) - \varepsilon(j)))^+ \) for \( 1 \leq s \leq \alpha_i - \alpha_j - 1 \);
3. If \( \alpha_i < \alpha_j \) then

\[
B_{ij}x^\alpha = -\sum_{s=1}^{\alpha_j-\alpha_i-1} x^\alpha - s(\varepsilon(j) - \varepsilon(i))
\]

and \( \alpha^+ \triangleright (\alpha - s(\varepsilon(j) - \varepsilon(i)))^+ \) for \( 1 \leq s \leq \alpha_j - \alpha_i - 1 \).

The following proposition can be elegantly stated in terms of conjugates of Jucys-Murphy elements. Recall the conjugation relation \( w(i, j) w^{-1} = (w(i), w(j)) \).

**Definition 10.** For \( \alpha \in \mathbb{N}^N \) and \( 1 \leq i \leq N \) let \( \omega_i^\alpha := w_\alpha r_{(\alpha, i)} w^{-1}_\alpha \), where \( w_\alpha \) is the inverse of \( r(\alpha, \cdot) \). Equivalently \( \omega_i^\alpha = \sum_{r(\alpha,j) > r(\alpha,i)} (i, j) \).

To justify the second equation observe that

\[
w_\alpha r_{(\alpha,i)} w^{-1}_\alpha = \sum_{r(\alpha,i)<j} (w_\alpha (r(\alpha,i)), w_\alpha (j)) = \sum_{r(\alpha,i)<j} (i, w_\alpha (j))
\]

and \( r(\alpha, w_\alpha (j)) = j \).

**Proposition 5.** Suppose \( \alpha \in \mathbb{N}^N \), \( u \in V_r \) and \( 1 \leq i \leq N \) then

\[
U_r x^\alpha u = x^\alpha [(\alpha_i + 1) u + \kappa \omega_i^\alpha u] + \kappa \sum_{\beta < \alpha} x^{\beta} u_\beta,
\]

where each \( u_\beta = 0 \) or \( \pm (i, j) u \) for some \( j \).

**Proof.** Let \( q_\alpha \) denote elements of span \( \{x^{\beta} : \beta \triangleleft \alpha \} \). In the case \( 1 \leq j < i \) the coefficient of \( \kappa (i, j) u \) is \( B_{ij}x^\alpha - (i, j)x^\alpha \) which equals (1) 0 if \( \alpha_i = \alpha_j \), (2) \( x^\alpha + q_\alpha \) if \( \alpha_i > \alpha_j \), (3) \( -(i, j)x^\alpha + q_\alpha \) if \( \alpha_j > \alpha_i \), so that \( (i, j) \alpha \triangleleft \alpha \). In the case \( i < j \leq N \) the coefficient of \( \kappa (i, j) u \) is \( B_{ij}x^\alpha \) which equals (1) \( x^\alpha \) if \( \alpha_i = \alpha_j \), (2) \( x^\alpha + (i, j)x^\alpha + q_\alpha \) if \( \alpha_i > \alpha_j \), so that \( (i, j) \alpha \triangleleft \alpha \). Thus \( \kappa x^\alpha (i, j) u \) appears in \( U_r x^\alpha u \) exactly when \( \alpha_i > \alpha_j \) or \( \alpha_i = \alpha_j \) and \( j > i \), that is, \( r(\alpha, j) > r(\alpha, i) \). \( \square \)
Following Griffeth we define an order on the pairs \( \{(\alpha, u) : \alpha \in \mathbb{N}^N\} : (\alpha, u_1) \triangleright (\beta, u_2) \) means that \( \alpha \triangleright \beta \). For this order the leading term of \( U_i x^\alpha u \) is \( x^\alpha (\alpha_i + 1 + \kappa \omega_i^\alpha) u \).

4. Nonsymmetric Jack polynomials

This section presents the structure of the simultaneous eigenvectors of \( \{U_i : 1 \leq i \leq N\} \) in \( M(\tau) \). These are vector-valued generalizations of the nonsymmetric Jack polynomials (see [3, Ch. 8]). The operators \( U_i \) are self-adjoint with respect to the contravariant form, which is described as follows:

The contravariant form \( \langle \cdot, \cdot \rangle \) on \( M(\tau) \) is the canonical symmetric \( S_N \)-invariant bilinear form, extending the form \( \langle \cdot, \cdot \rangle_0 \) on \( V_\tau \) such that

\[
\langle x_i f, g \rangle = \langle f, D_i g \rangle, \quad i \in [1, N], \quad f, g \in M(\tau).
\]

An existence proof can be based on the operator

\[
\sum_{i=1}^N x_i D_i + \kappa \sum_{1 \leq i < j \leq N} (i, j)
\]

and induction. The important properties of the form are:

1. if \( f \in \mathcal{P}_m \otimes V_\tau, g \in \mathcal{P}_n \otimes V_\tau \) and \( m \neq n \) then \( \langle f, g \rangle = 0 \);
2. if \( w \in S_N \) then \( \langle w f, w g \rangle = \langle f, g \rangle \) for all \( f, g \in M(\tau) \), if \( 1 \leq i < j \leq N \) then \( \langle (i, j) f, g \rangle = \langle f, (i, j) g \rangle \);
3. if \( i \in [1, N] \) and \( f, g \in M(\tau) \) then \( \langle D_i x_i f, g \rangle = \langle f, D_i x_i g \rangle \).

We use \( \|f\|^2 \) to denote \( \langle f, f \rangle \) although the form may not be positive-definite. For a specific value \( \kappa \in \mathbb{Q} \) the kernel of the form, that is, \( \{f : \langle g, f \rangle = 0, \forall g \in M(\tau)\} \), is called the radical of \( M(\tau) \) and denoted \( J_\kappa(\tau) \), and the quotient module \( M(\tau)/J_\kappa(\tau) \) is denoted \( L_\kappa(\tau) \). Values of \( \kappa \) such that \( J_\kappa(\tau) \neq 0 \) are called singular values.

If \( \lambda \in \mathbb{N}^{N,+} \) then the leading term in \( U_i x^\lambda u \) is \( x^\lambda (\lambda_i + 1 + \kappa \omega_i) u \); this suggests that eigenvectors of \( \omega_i \) have good properties under the action of \( U_i \). For compositions the coordinates have to be appropriately permuted. From (5) we see that for \( T \in Y(\tau) \) and \( \alpha \in \mathbb{N}^N \) the leading term in \( U_i x^\alpha w_\alpha v_T \) is \( (\alpha_i + 1 + \kappa c(\alpha_i, i), T) w_\alpha v_T \), because \( \omega_i^\alpha w_\alpha v_T = w_\alpha \omega_{r(\alpha, i)} v_T = c(\alpha, i), T) w_\alpha v_T. \) For any \( n \in \mathbb{N} \) the set \( \{x^\alpha w_\alpha v_T : \alpha \in \mathbb{N}^N, \|\alpha\| = n, T \in Y(\tau)\} \) is a basis of \( M_n(\tau) \) on which the operators \( U_i \) act in a triangular manner (with respect to \( \triangleright \)). For \( \alpha \in \mathbb{N}^N, T \in Y(\tau), \) let

\[
(4.1) \quad \xi_i(\alpha, T) = \alpha_i + 1 + \kappa c(\alpha_i, i), T), \quad i \in [1, N].
\]

For any \( \beta \neq \alpha \) and \( |\beta| = |\alpha| \) there is at least one \( i \) such that \( \alpha_i \neq 0 \) and \( \alpha_i \neq \beta_i \), thus \( \xi_i(\alpha, T) \neq \xi_i(\beta, T') \) for any \( T, T' \in Y(\tau) \) (and
generic $\kappa$). (The restriction to $\alpha_i \neq 0$ is needed in the next section; if $|\alpha| = |\beta|$ and $\alpha_i \neq 0$ implies $\alpha_i = \beta_i$ then $\alpha = \beta$.) Thus there exists a basis of simultaneous eigenvectors of $\{U_i : i \in [1, N]\}$. The following is the specialization to $\mathcal{S}_N$ of Griffeth’s construction [6, Thm. 5.2] of nonsymmetric Jack polynomials.

**Proposition 6.** For $\alpha \in \mathbb{N}^N$, $T \in Y(\tau)$ there exists a unique element $\zeta_{\alpha,T}$ of $M(\tau)$ such that $U_i \zeta_{\alpha,T} = \xi_i(\alpha,T) \zeta_{\alpha,T}$ for $1 \leq i \leq N$ and

$$
\zeta_{\alpha,T}(x) = x^n w_{\alpha,T} + \sum_{\beta < \alpha} x^{\beta} u_{\beta\alpha},
$$

where $u_{\beta\alpha} \in V_T$.

Existence of this set of simultaneous eigenvectors of $\{U_i : i \in [1, N]\}$ follows from the triangular property, the commutativity, and the separation properties of the eigenvalues $(\alpha, T) \mapsto |\xi_i(\alpha,T)|_{i=1}^N$.

Because each $U_i$ is self-adjoint for $\langle \cdot, \cdot \rangle$ we have $\langle \zeta_{\alpha,T}, \zeta_{\beta,T'} \rangle = 0$ when $\alpha \neq \beta$ or $T \neq T'$.

We consider the action of $\mathcal{S}_N$ on the polynomials $\zeta_{\alpha,T}$. As usual there are explicit formulae for the action of $s_i = (i, i + 1)$ based on the commutations $U_j s_i = s_i U_j$ for $j \neq i, i + 1$ and $s_i U_i = U_{i+1} + \kappa$. These are special cases of [6, Thm. 5.3], however we use the nonnormalized basis for $V_T$ rather than the orthonormal one used there (so coefficients in $\mathbb{Q}(\kappa)$ suffice). As in [6] let $\sigma_i$ denote the formal operator $s_i + \frac{\kappa}{U_{i+1} - U_i}$; suppose $f \in M(\tau)$ and $U_j f = \lambda_j f$ for $1 \leq j \leq N$ (with $\lambda_j \in \mathbb{Q}(\kappa)$ and $\lambda_i \neq \lambda_{i+1}$) then $U_j \sigma_i f = \lambda_j \sigma_i f$ for $j \neq i, i + 1$ and $U_i \sigma_i f = \lambda_{i+1} \sigma_i f$ (where $\sigma_i f = s_i f + \frac{\kappa}{\lambda_{i+1} - \lambda_i} f$. Specifically there are two main cases $\alpha_i \neq \alpha_{i+1}$ and $\alpha_i = \alpha_{i+1}$. For $\alpha \in \mathbb{N}^N$ and $T \in Y(\tau)$ let

$$
b_i(\alpha,T) = \frac{\kappa}{\xi_i(\alpha,T) - \xi_{i+1}(\alpha,T)} = \frac{\kappa}{\alpha_i - \alpha_{i+1} + \kappa (c(r(\alpha,i),T) - c(r(\alpha,i+1),T))}.
$$

The proof of the following is in the Appendix.

**Proposition 7.** Suppose $\alpha \in \mathbb{N}^N$ and $\alpha_i > \alpha_{i+1}$ for some $i < N$. Then

$$
s_i \zeta_{\alpha,T} = b_i(\alpha,T) \zeta_{\alpha,T} + (1 - b_i(\alpha,T)^2) \zeta_{s_i \alpha,T},
$$

$$
s_i \zeta_{s_i \alpha,T} = \zeta_{\alpha,T} - b_i(\alpha,T) \zeta_{s_i \alpha,T};
$$

$$
\|\zeta_{\alpha,T}\|^2 = (1 - b_i(\alpha,T)^2) \|\zeta_{s_i \alpha,T}\|^2.
$$

**Remark 1.** A necessary condition for the form $\langle \cdot, \cdot \rangle$ to be positive-definite now becomes apparent: $b_i(\alpha,T)^2 < 1$ for all $i, \alpha, T$. The
“trivial” cases are $\tau = (N)$ and $\tau = (1, \ldots, 1)$ for which $\kappa > -\frac{1}{N}$ and $\kappa < \frac{1}{N}$ are necessary and sufficient, respectively. Otherwise let $h_{\tau} := \tau_1 + \ell(\tau) - 1$, the maximum hook-length of $\tau$, then $-\frac{1}{h_{\tau}} < \kappa < \frac{1}{h_{\tau}}$ implies $b_i(\alpha, T)^2 < 1$ for all $i, \alpha, T$. Note that $1 \leq i, j \leq N, T \in Y(\tau)$ implies $|c(i, T) - c(j, T)| \leq h_{\tau} - 1$.

Etingof, Stoica and Griffeth [4, Thm. 5.5] found the complete description of the set of values of $\kappa$ for which $L_\kappa(\tau)$ provides a unitary representation of the rational Cherednik algebra. We can find an expression for $\|\zeta_{\alpha,T}\|_2$ in terms of $\|\zeta_{\alpha+1,T}\|_2$, following the approach used in [3, Thm. 8.5.8].

**Definition 11.** For $\alpha \in \mathbb{N}^N$, $T \in Y(\tau)$ and $\varepsilon = \pm$ let

$$\mathcal{E}_\varepsilon(\alpha, T) = \prod_{1 \leq i < j \leq N, \alpha_i < \alpha_j} \left(1 + \frac{\varepsilon \kappa}{\alpha_j - \alpha_i + \kappa(c(r(\alpha, j), T) - c(r(\alpha, i), T))}\right),$$

and let $\mathcal{E}_2(\alpha, T) = \mathcal{E}_+(\alpha, T) \mathcal{E}_-(\alpha, T)$.

**Definition 12.** For $\alpha \in \mathbb{N}^N$ let

$$\text{inv}(\alpha) := \# \{(i, j) : 1 \leq i < j \leq N, \alpha_i < \alpha_j\},$$

the number of inversions in $\alpha$.

**Proposition 8.** Suppose $\alpha \in \mathbb{N}^N$, $T \in Y(\tau)$, $\varepsilon = \pm$ and $\alpha_{i+1} > \alpha_i$ for some $i \in [1, N - 1]$ then $\mathcal{E}_\varepsilon(s_i \alpha, T)/\mathcal{E}_\varepsilon(\alpha, T) = 1 + \varepsilon b_i(\alpha, T)$.

**Proof.** Using an argument similar to that of Lemma 1 we have

$$\frac{\mathcal{E}_\varepsilon(s_i \alpha, T)}{\mathcal{E}_\varepsilon(\alpha, T)} = 1 + \frac{\varepsilon \kappa}{(s_i \alpha)_{i+1} - (s_i \alpha)_i + \kappa(c(r(s_i \alpha, i + 1), T) - c(r(s_i \alpha, i), T))}$$

$$= 1 + \varepsilon b_i(\alpha, T),$$

because $r(s_i \alpha, i + 1) = r(\alpha, i)$ and $r(s_i \alpha, i) = r(\alpha, i + 1)$. \qed

**Corollary 3.** Suppose $\alpha \in \mathbb{N}^N$, $T \in Y(\tau)$ then

$$\|\zeta_{\alpha,T}\|^2 = \mathcal{E}_2(\alpha, T)^{-1} \|\zeta_{\alpha+1,T}\|^2.$$

**Proof.** Argue by induction on $\text{inv}(\alpha)$. If the formula is valid for some $\alpha$ with $\alpha_i > \alpha_{i+1}$ then by Proposition 7

$$\|\zeta_{s_i \alpha,T}\|^2 = (1 - b_i(\alpha, T)^2)^{-1} \|\zeta_{\alpha,T}\|^2$$

$$= (1 - b_i(\alpha, T)^2)^{-1} \mathcal{E}_2(\alpha, T)^{-1} \|\zeta_{\alpha+1,T}\|^2$$

$$= \mathcal{E}_2(s_i \alpha, T)^{-1} \|\zeta_{\alpha+1,T}\|^2.$$
This completes the induction. □

Consider the case \( \alpha_i = \alpha_{i+1} \) and let \( I = r(\alpha, i) \) so that \( r(\alpha, i + 1) = I + 1 \) and \( b_i(\alpha, T) = (c(I, T) - c(I + 1, T))^{-1} = b_I(T) \) (see Proposition 1). Furthermore

\[
s_iw_\alpha = w_\alpha(w_{\alpha}^{-1}(i), w_{\alpha}^{-1}(i + 1)) = w_\alpha(I, I + 1) = w_\alpha s_I.
\]

The transformation properties depend on the positions of \( I \) and \( I + 1 \) in \( T \).

**Proposition 9.** Suppose \( \alpha \in \mathbb{N}^N, T \in Y(\tau) \) and \( \alpha_i = \alpha_{i+1} \) for some \( i < N \). For \( I = r(\alpha, T) \) the following hold:

1. If \( b_I(T) = 1 \) then \( s_i\zeta_{\alpha, T} = \zeta_{\alpha, T} \),
2. If \( b_I(T) = -1 \) then \( s_i\zeta_{\alpha, T} = -\zeta_{\alpha, T} \),
3. If \( -\frac{1}{2} \leq b_I(T) < 0 \) then

\[
s_i\zeta_{\alpha, T} = b_I(T)\zeta_{\alpha, T} + \left(1 - b_I(T)^2\right)\zeta_{\alpha, s_I T},
\]

4. If \( 0 < b_I(T) \leq \frac{1}{2} \) then \( s_i\zeta_{\alpha, T} = b_I(T)\zeta_{\alpha, T} + \zeta_{\alpha, s_I T} \).

**Proof.** It suffices to consider the action of \( s_i \) on the leading term of \( \zeta_{\alpha, T} \). Indeed \( s_i x^\alpha w_\alpha v_T = x^\alpha w_\alpha (s_I v_T) \) and we use the equations from Proposition 1. □

Note that in case (3) \( \|\zeta_{\alpha, T}\|^2 = (1 - b_I(T)^2)\|\zeta_{\alpha, s_I T}\|^2 \) (and the reciprocal in case (4)). The closed formula for the norm is proven by means of induction and a raising operator involving a cyclic shift and multiplication by \( x_N \). The details are in the Appendix.

The following result is due to Griffeth [6, Thm. 6.1] for the general setting of \( G(r, 1, N) \). There is a slight change due to our use of nonnormalized vectors \( v_T \) and reversed standard tableaux.

**Theorem 2.** Suppose \( \lambda \in \mathbb{N}^{N,+} \) and \( T \in Y(\tau) \) then

\[
\|\zeta_{\lambda, T}\|^2 = \|v_T\|^2 \prod_{i=1}^N (1 + \kappa c(i, T))_{\lambda_i} \times \prod_{1 \leq i < j \leq N} \prod_{l=1}^{\lambda_i - \lambda_j} \left(1 - \frac{\kappa^2}{(l + \kappa(c(i, T) - c(j, T)))^2}\right).
\]
5. Symmetric and Antisymmetric Polynomials

We consider symmetric and antisymmetric linear combinations of \( \{\zeta_{\alpha,T}\} \). Recall
\[
\begin{align*}
\lambda_i (\alpha, T) &= \frac{\kappa}{\alpha_i - \alpha_{i+1} + \kappa (c (r (\alpha, i), T) - c (r (\alpha, i + 1), T))}, \\
\lambda_i (T) &= \frac{1}{c(i, T) - c(i + 1, T)}.
\end{align*}
\]
for \( \alpha \in \mathbb{N}^N, T \in Y(\tau), i \in [1, N - 1] \). Here is a description of \( s_i \)-invariant polynomials for a given \( i \):
\[
\begin{align*}
(1) & \quad \zeta_{\alpha,T} + (1 - b_i (\alpha, T)) \zeta_{s_i\alpha,T}, \text{ for } \alpha_i > \alpha_{i+1}; \\
(2) & \quad (b_i (T) + 1) \zeta_{\alpha,T} + \zeta_{s_i\alpha,T}, \text{ for } \alpha_i = \alpha_{i+1}, I = r (\alpha, i) \text{ and } 0 < b_i (T) \leq \frac{1}{2}; \\
(3) & \quad \zeta_{\alpha,T}, \text{ for } \alpha_i = \alpha_{i+1}, I = r (\alpha, i) \text{ and } b_i (T) = 1 (rw (I, T) = rw (I + 1, T)).
\end{align*}
\]
The antisymmetric polynomials for \( s_i \) \((s_if = -f)\) are
\[
\begin{align*}
(1) & \quad \zeta_{\alpha,T} - (1 + b_i (\alpha, T)) \zeta_{s_i\alpha,T}, \text{ for } \alpha_i > \alpha_{i+1}; \\
(2) & \quad (b_i (T) - 1) \zeta_{\alpha,T} + \zeta_{s_i\alpha,T}, \text{ for } \alpha_i = \alpha_{i+1}, I = r (\alpha, i) \text{ and } 0 < b_i (T) \leq \frac{1}{2}; \\
(3) & \quad \zeta_{\alpha,T}, \text{ for } \alpha_i = \alpha_{i+1}, I = r (\alpha, i) \text{ and } b_i (T) = -1 (cm (I, T) = cm (I + 1, T)).
\end{align*}
\]

Now we construct invariants. In any orbit span \( \{w\zeta_{\alpha,T} : w \in S_N\} \) there must be a polynomial with leading term \( x^{\alpha^+} \) so it suffices to consider the situation \( \zeta_{\lambda,T} \) for partitions \( \lambda \). We collect concepts for use in the sequel.

**Notation 1.** For \( \lambda \in \mathbb{N}^{N,+} \) let \( W_\lambda = \{w \in S_N : w\lambda = \lambda\} \), the stabilizer subgroup of \( \lambda \). Thus
\[
W_\lambda = S_{[a_1,b_1]} \times S_{[a_2,b_2]} \times \ldots \times S_{[a_n,b_n]},
\]
where \( 1 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \leq N \) (this means \( \lambda_{a_i} = \lambda_{b_i} > \lambda_{b_i+1} \) and so forth). These intervals depend on \( \lambda \) but we will not incorporate this into the notation. Let \( \lambda^R = (\lambda_N, \lambda_{N-1}, \ldots, \lambda_1) \in \mathbb{N}^N \), the reverse of \( \lambda \). The permutation \( w_{\lambda^R} \) is defined by \( (w_{\lambda^R})^{-1} (i) = r (\lambda^R, i), i \in [1, N] \) (Definition 10).

Generally \( w_{\lambda^R} \neq w_0 \) where \( w_0 \) is the longest permutation given by \( w_0 (i) = N + 1 - i \) (example: \( \lambda = (3, 2, 2, 1) \) then \( [w_{\lambda^R} (i)]_{i=1}^4 = [4, 2, 3, 1] \)). The composition \( \lambda^R \) is the unique minimum for the order \( \succ \) on \( \{\alpha : \alpha^+ = \lambda\} \). For \( \alpha^+ = \lambda \) and \( T \in Y(\tau) \) the leading term of \( \zeta_{\alpha,T} \) is \( x^\alpha w_{\alpha^+T} \) (where \( w_{\alpha^+} (i) = r (\alpha, i) \)) and the minimality of \( \lambda^R \) implies that the expansion of \( \zeta_{\lambda^R,T} \) has no term of the form \( x^\alpha u \) with
u ∈ Vτ when α ≠ λR; α+ = λ. From the expressions in (2) above we see that the subgroup $W_λ$ is an important part of the analysis. The formulae developed in Section 2 will be used. The method is an analog of the one for scalar nonsymmetric Jack polynomials, introduced by Baker and Forrester [1].

**Definition 13.** For $λ ∈ \mathbb{N}^N$ and $T ∈ Y(τ)$ define the tableau $[λ, T]$ to be the assignment of $λ_1, λ_2, \ldots, λ_N$ to the nodes of the Ferrers diagram of $τ$ so that the entry at $T(i)$ is $λ_i, i ∈ [1, N]$. Thus the entries of $[λ, T]$ are weakly increasing ($≤$) in each row and in each column. The set of $T'$ satisfying $[λ, T'] = [λ, T]$ is exactly $Y(T; W_λ)$.

**5.1. Case:** $#Y(T; W_λ) = 1$. We begin with the situation of symmetrizing $ζ_{λ,T}$ when $#Y(T; W_λ) = 1$, that is, $v_T$ is $W_λ$-invariant so that each interval $[a_i, b_i]$ is contained in a row of $T,( 1 ≤ i ≤ n$ (rw $(j, T) =$ rw $(b_i, T)$ for $1 ≤ i ≤ n$ and $a_i ≤ j ≤ b_i)$. Then $\sum_{w ∈ S_N} wζ_{λ,T} = \sum_{α^+ = λ} A_αζ_{α,T}$ with coefficients to be determined.

**Theorem 3.** Suppose $λ ∈ \mathbb{N}^N$ and $T ∈ Y(τ)$ such that $w ∈ W_λ$ implies $wv_T = v_T$ then the polynomial $f_{λ,T}$ defined by

$$f_{λ,T} = \sum_{α^+ = λ} E_−(α, T) ζ_{α,T},$$

is $S_N$-invariant and

$$\|f_{λ,T}\|^2 = \frac{N!}{#W_λ} \frac{1}{E_+(λR, T)} \|ζ_{λ,T}\|^2.$$

**Proof.** Fix $i ∈ [1, N − 1]$ and let

$$A = \{α : α^+ = λ, α_i = α_{i+1}\},$$

$$B = \{α : α^+ = λ, α_i > α_{i+1}\}.$$

Write

$$f_{λ,T} = \sum_{α ∈ A} E_−(α, T) ζ_{α,T} + \sum_{α ∈ B} (E_−(α, T) ζ_{α,T} + E_−(s_iα, T) ζ_{s_iα,T}).$$

Suppose $α ∈ A$ then $r(i + 1, α) = r(i, α) + 1$ thus the values $r(i, α)$ and $r(i + 1, α)$ belong to some interval $[a_j, b_j]$ (where $S_{[a_j, b_j]}$ is a factor of $W_λ$) and are adjacent entries in some row of $T$, hence $s_iζ_{α,T} = ζ_{α,T}$. Next let $α ∈ B$ then the corresponding term in the sum is $E_−(α, T) \left(ζ_{α,T} + \frac{E_−(s_iα, T)}{E_−(α, T)}ζ_{s_iα,T}\right)$. Using the techniques of Lemma 1 we
find that
\[
\mathcal{E}_- (s_i \alpha, T) - \mathcal{E}_- (\alpha, T) = 1 - \kappa \left( (s_i \alpha)_{i+1} - (s_i \alpha)_i + \kappa (r (s_i \alpha, i+1), T) - c (r (s_i \alpha, i), T) \right)
\]
\[
= 1 - \kappa \left( \alpha_i - \alpha_{i+1} + \kappa (c (r (\alpha, i), T) - c (r (\alpha, i+1), T)) \right)
\]
and thus the term for \( \alpha \) in the sum over \( B \) is \( s_i \)-invariant. Consider
\[
g = \sum_{w \in S_N} w \zeta_{\lambda R, T} \zeta_{\lambda, T}
\]
since \( g \) is \( S_N \)-invariant it must equal a constant multiple \( \gamma \) of \( f_{s_{\lambda, T}} \). To find \( \gamma \) consider the coefficients of \( x^{\lambda} v_T \) in \( f_{s_{\lambda, T}} \) and \( g \). The leading term of \( \zeta_{\lambda R, T} \) is \( x^{\lambda} v_T \). The coefficient in \( f_{s_{\lambda, T}} \) is 1 (by definition of \( \zeta_{\lambda, T} \)). The term \( x^{\lambda} v_T \) appears in \( w \zeta_{\lambda R, T} \zeta_{\lambda, T} \) with coefficient 1 exactly when \( w = w_1 w_1^{-1} \) for \( w_1 \in W_\lambda \). Thus \( g = (\#W_\lambda) f_{s_{\lambda, T}} \) and
\[
\| f_{s_{\lambda, T}} \|^2 = \frac{1}{\#W_\lambda} \langle g, f_{s_{\lambda, T}} \rangle = \frac{1}{\#W_\lambda} \sum_{w \in S_N} \langle w \zeta_{\lambda R, T}, f_{s_{\lambda, T}} \rangle
\]
\[
= \frac{N!}{\#W_\lambda} \langle \zeta_{\lambda R, T}, f_{s_{\lambda, T}} \rangle = \frac{N!}{\#W_\lambda} \mathcal{E}_- (\lambda R, T) \| \zeta_{\lambda R, T} \|^2
\]
\[
= \frac{N! \mathcal{E}_- (\lambda R, T)}{(\#W_\lambda) \mathcal{E}_2 (\lambda R, T)} \| \zeta_{\lambda, T} \|^2.
\]
This completes the proof. \( \square \)

Continuing with the case \( \#_Y (T; W_\lambda) = 1 \) we turn to the corresponding antisymmetric function involving \( \zeta_{\lambda, T} \) such that \( v_T \) is antisymmetric for \( W_\lambda \). That is each interval \([a_i, b_i]\) (appearing in \( W_\lambda \)) is contained in a column of \( T \), \( a_i \leq j \leq b_i \) implies \( \text{cm} (j, T) = \text{cm} (b_i, T) \). The number of inversions \( \text{inv} (\alpha) \) takes the place of the sign of a permutation in order to allow \( \lambda \) to have some repeated values.

**Theorem 4.** Suppose \( \lambda \in \mathbb{N}^{N,+} \) and \( T \in Y (\tau) \) such that \( s_i \in W_\lambda \) implies \( s_i v_T = -v_T \) then the polynomial \( f_{s_{\lambda, T}} \) defined by
\[
f_{s_{\lambda, T}} = \sum_{\alpha^+ = \lambda} (-1)^{\text{inv} (\alpha)} \mathcal{E}_+ (\alpha, T) \zeta_{\alpha, T},
\]
is \( S_N \)-alternating, and
\[
\| f_{s_{\lambda, T}} \|^2 = \frac{N!}{\#W_\lambda} \mathcal{E}_- (\lambda R, T) \| \zeta_{\lambda, T} \|^2.
\]
Proof. Fix $i \in [1, N - 1]$ and let
\[
A = \{ \alpha : \alpha^+ = \lambda, \alpha_i = \alpha_{i+1} \}, \quad B = \{ \alpha : \alpha^+ = \lambda, \alpha_i > \alpha_{i+1} \}.
\]
Note $\alpha \in B$ implies $\text{inv}(s_i \alpha) = \text{inv}(\alpha) + 1$. Write
\[
f_{\lambda,T}^a = \sum_{\alpha \in A} (-1)^{\text{inv}(\alpha)} \mathcal{E}_+ (\alpha, T) \zeta_{\alpha,T}
\]
\[
+ \sum_{\alpha \in B} (-1)^{\text{inv}(\alpha)} (\mathcal{E}_+ (\alpha, T) \zeta_{\alpha,T} - \mathcal{E}_+ (s_i \alpha, T) \zeta_{s_i \alpha,T}).
\]
Suppose $\alpha \in A$ then $r(i + 1, \alpha) = r(i, \alpha) + 1$ thus the values $r(i, \alpha)$ and $r(i + 1, \alpha)$ belong to some interval $[a_j, b_j]$ (where $\mathcal{S}_{[a_j, b_j]}$ is a factor of $W_\lambda$) and are adjacent entries in some column of $T$, hence $s_i \zeta_{\alpha,T} = -\zeta_{\alpha,T}$. Next let $\alpha \in B$ then the corresponding term in the sum is
\[
(-1)^{\text{inv}(\alpha)} \mathcal{E}_+ (\alpha, T) \left( \zeta_{\alpha,T} - \frac{\mathcal{E}_+(s_i \alpha,T)}{\mathcal{E}^+(\alpha,T)} \zeta_{s_i \alpha,T} \right),
\]
which is a scalar multiple of $\zeta_{\alpha,T} - (1 + b_i(\alpha,T)) \zeta_{s_i \alpha,T}$, by an argument similar to the previous theorem. This term satisfies $s_i f = -f$. Thus $s_i f_{\lambda,T}^a = -f_{\lambda,T}^a$. Consider $g = \sum_{\omega \in S_N} \text{sgn}(w) w \lambda_{R,T}$; since $g$ is $S_N$-alternating it must equal a constant multiple $\gamma$ of $f_{\lambda,T}^a$. To find $\gamma$ consider the coefficients of $x^\lambda v_T$ in $f_{\lambda,T}^a$ and $g$. The coefficient in $f_{\lambda,T}^a$ is 1 (by definition of $\lambda_{\lambda,T}$). The term $x^\lambda v_T$ appears in $w \lambda_{R,T}$ exactly when $w = w_1 w_{\lambda,R}^{-1}$ for $w_1 \in W_\lambda$. Let $\varepsilon = \text{sgn}(w_{\lambda,R}) = (-1)^{\text{inv}(\lambda_R)}$, because the length of $w_{\lambda,R}$ is $\text{inv}(\lambda_R)$. Furthermore
\[
\text{sgn}(w_1 w_{\lambda,R}^{-1}) w_1 w_{\lambda,R}^{-1} \zeta_{\lambda_R,T} = \text{sgn}(w_1 w_{\lambda,R}^{-1}) w_1 w_{\lambda,R}^{-1} \left( x^\lambda w_{\lambda,R} v_T \right) + h_1
\]
\[
= \varepsilon \text{sgn}(w_1) w_1 \left( x^\lambda v_T \right) + h_2
\]
\[
= \varepsilon x^\lambda v_T + h_2,
\]
where $h_1$ and $h_2$ are terms of lower order, that is, of the form $\sum_{\beta < \lambda} x^\beta u_\beta$ with $u_\beta \in V_\tau$. Thus $g = \varepsilon (\# W_\lambda) f_{\lambda,T}$ and
\[
\| f_{\lambda,T}^a \| = \frac{\varepsilon}{\# W_\lambda} \left< g, f_{\lambda,T}^a \right> = \frac{\varepsilon}{\# W_\lambda} \sum_{\omega \in S_N} \text{sgn}(w) \left< w \lambda_{R,T}, f_{\lambda,T}^a \right>
\]
\[
= \frac{\varepsilon}{\# W_\lambda} \sum_{\omega \in S_N} \text{sgn}(w) \left< \lambda_{R,T}, w^{-1} f_{\lambda,T}^a \right> = \frac{\varepsilon N!}{\# W_\lambda} \left< \lambda_{R,T}, f_{\lambda,T}^a \right>
\]
\[
= \frac{\varepsilon N!}{\# W_\lambda} (-1)^{\text{inv}(\lambda_R)} \mathcal{E}_+(\lambda^R,T) \| \lambda_{\lambda,T} \|^2
\]
\[
= \frac{N! \mathcal{E}_+(\lambda^R,T)}{(\# W_\lambda) \mathcal{E}_2(\lambda^R,T)} \| \lambda_{\lambda,T} \|^2.
\]
This completes the proof. 

\[ \square \]

5.2. Case: \#Y (T; W_\lambda) > 1. Let T_0 \in Y (\tau) such that T_0 satisfies condition \([a_i, b_i]_{cm}\) for each factor \(S_{[a_i, b_i]}\) of \(W_\lambda\) and \(a_i \leq j_1 < j_2 \leq b_i\) implies \(cm (j_1, T_0) > cm (j_2, T_0)\). This condition is equivalent to the tableau \([\lambda, T_0]\) being column-strict (the entries strictly increase in each column, see [7, p. 5], such tableaux are also called semistandard Young tableaux) and \(T_0\) has a certain extremal property among all \(T \in Y (T_0; W_\lambda)\). Let

\[
\begin{align*}
    f_{\lambda, T_0}^s &= \sum_{\alpha^+ = \lambda} \sum_{T \in Y (T_0; W_\lambda)} \prod_{j=1}^{n} P_0 (T; a_j, b_j) \mathcal{E}_{-} (\alpha, T) \zeta_{\alpha, T}, \\
    u_{\lambda, T_0} &= \sum_{T \in Y (T_0; W_\lambda)} \prod_{j=1}^{n} P_0 (T; a_j, b_j) v_T \in V_T.
\end{align*}
\]

The term involving \(x^\lambda\) is \(h_0 = \sum_{T \in Y (T_0; W_\lambda)} \prod_{j=1}^{n} P_0 (T; a_j, b_j) \zeta_{\lambda, T}\), thus the leading term in \(f_{\lambda, T_0}^s\) is \(x^\lambda u_{\lambda, T_0}\). From the transformation rules in Proposition 9 it follows that \(\|h_0\|^2 = \|\zeta_{\lambda, T_0}\|^2 \|u_{\lambda, T_0}\|^2 / \|v_T\|^2\) (see Corollary 2). Also \(h_0\) is \(W_\lambda\)-invariant. In the symbol \(f_{\lambda, T_0}^s\) one could replace \(T_0\) by any \(T \in Y (T_0; W_\lambda)\); then \(\prod_{j=1}^{n} P_0 (T; a_j, b_j) = 1\) and \(T \in Y (T_0; W_\lambda)\) implies \(T = T_0\).

Theorem 5. \(wf_{\lambda, T_0}^s = f_{\lambda, T_0}^s\) for all \(w \in \mathcal{S}_N\) and

\[
\|f_{\lambda, T_0}^s\|^2 = \frac{N!!}{\#W_\lambda \mathcal{E}_+ (\lambda, T_0)} \frac{\|u_{\lambda, T_0}\|^2}{\|v_T\|^2} \|\zeta_{\lambda, T_0}\|^2.
\]

Proof. Let \(\mathcal{F} (\alpha, T) = \prod_{j=1}^{n} P_0 (T; a_j, b_j) \mathcal{E}_{-} (\alpha, T)\). Fix \(i \in [1, N - 1]\) and collect the terms of \(f_{\lambda, T_0}^s\) into three parts. Let

\[
\begin{align*}
    L &= \{ (\alpha, T) : \alpha^+ = \lambda, T \in Y (T_0; W_\lambda) \} \\
    A &= \{ (\alpha, T) \in L : \alpha_i = \alpha_{i+1}, \text{rw} (r (\alpha, i), T) = \text{rw} (r (\alpha, i) + 1, T) \} \\
    B &= \{ (\alpha, T) \in L : \alpha_i > \alpha_{i+1} \} \\
    C &= \{ (\alpha, T) \in L : \alpha_i = \alpha_{i+1}, \text{rw} (r (\alpha, i), T) < \text{rw} (r (\alpha, i) + 1, T) \}.
\end{align*}
\]
The first part is \( \sum_{(\alpha, T) \in A} F(\alpha, T) \zeta_{\alpha, T} \) and in this case \( s_i \zeta_{\alpha, T} = \zeta_{\alpha, T} \).

The second part is

\[
\sum_{(\alpha, T) \in B} (F(\alpha, T) \zeta_{\alpha, T} + F(s_i \alpha, T) \zeta_{s_i \alpha, T}) = \sum_{(\alpha, T) \in B} F(\alpha, T) \left( \zeta_{\alpha, T} + \frac{F(s_i \alpha, T)}{F(\alpha, T)} \zeta_{s_i \alpha, T} \right).
\]

Just as in Proposition 1 \( \frac{F(s_i \alpha, T)}{F(\alpha, T)} = 1 - b_i (\alpha, T) \), and hence this sum is \( s_i \)-invariant. For use in \( C \) let \( I(\alpha) = rw(\alpha, i) \). Then the third part is

\[
\sum_{(\alpha, T) \in C} \left( F(\alpha, T) \zeta_{\alpha, T} + F(\alpha, s_I(\alpha) T) \zeta_{\alpha, s_I(\alpha) T} \right) = \sum_{(\alpha, T) \in C} F(\alpha, s_I(\alpha) T) \left( \frac{F(\alpha, T)}{F(\alpha, s_I(\alpha) T)} \zeta_{\alpha, T} + \zeta_{\alpha, s_I(\alpha) T} \right).
\]

To show that each term is \( s_i \)-invariant we must show

\[
\frac{F(\alpha, T)}{F(\alpha, s_I(\alpha) T)} = b_{I(\alpha)} + 1.
\]

Fix such a term. The equality \( \alpha_i = \alpha_{i+1} \) implies \( [I(\alpha), I(\alpha) + 1] \subset [a_i, b_i] \) for some \( i \). Thus

\[
\prod_{j=1}^n P_0(T; a_j, b_j) = \prod_{j=1}^n P_0(s_i(T); a_j, b_j) = 1 + b_{I(\alpha)}(T).
\]

Finally consider \( E_-(\alpha, T) / E_-(\alpha, s_I(\alpha) T) \); let

\[
g_{ij}(T) = 1 - \frac{\kappa}{\alpha_j - \alpha_l + \kappa (c(r(\alpha, j), T) - c(r(\alpha, l), T))}
\]

if \( l < j \) and \( \alpha_l < \alpha_j \), and \( g_{ij}(T) = 1 \) otherwise. Then \( r(\alpha, j) \notin \{I(\alpha), I(\alpha) + 1\} \) implies that

\[
c(r(\alpha, j), T) = c(r(\alpha, j), s_I(\alpha) T),
\]

also

\[
c(r(\alpha, i), T) = c(r(\alpha, i + 1), s_I(\alpha) T)
\]

and

\[
c(r(\alpha, i + 1), T) = c(r(\alpha, i), s_I(\alpha) T).
\]

Thus \( g_{ij}(T) = g_{i+1,j}(s_I(\alpha) T) \) and \( g_{ij}(s_I(\alpha) T) = g_{i+1,j}(T) \) for \( 1 \leq l < i \) with similar relations for \( g_{ij} \) and \( g_{i+1,j} \) when \( i + 1 < j \leq N \). Also

\[
g_{i,i+1}(T) = 1 = g_{i,i+1}(s_I(\alpha) T).
\]
thus
\[ \mathcal{E}_- (\alpha, T) = \prod_{1 \leq i < j \leq N} g_{ij}(T) = \mathcal{E}_- (\alpha, s_{\ell(\alpha)} T). \]

Hence \( s_1 f_{\lambda, T_0}^s = f_{\lambda, T_0}^s \).

To compute \( \| f_{\lambda, T_0}^s \|_2 \) consider \( \sum_{\omega \in S_N} \omega h_R \) where
\[ h_R := \sum_{T \in Y(T_0; W_\lambda)} \prod_{j=1}^n P_0(T; a_j, b_j) \zeta_{\lambda^R, T}. \]

By the argument used above for type (4), we have
\[ \mathcal{E}_- (\lambda^R, T) = \mathcal{E}_- (\lambda^R, T_0) \]
for all \( T \in Y(T_0; W_\lambda) \). Thus the term for \( \alpha = \lambda^R \) in \( f_{\lambda, T_0}^s \) is \( \mathcal{E}_- (\lambda^R, T_0) h_R \) and the leading term in \( h_R \) is \( x_{\lambda^R}^w u_{\lambda, T_0} \). Similarly to the proof of Theorem 3 we conclude \( \sum_{\omega \in S_N} \omega h_R = (\#W_\lambda) f_{\lambda, T_0}^s \) and
\[ (\#W_\lambda) \| f_{\lambda, T_0}^s \|_2^2 = N! \langle h_R, f_{\lambda, T_0}^s \rangle = N! \mathcal{E}_- (\lambda^R, T_0) \| g \|_2^2; \]

thus
\[ \| g \|_2^2 = \frac{\| \zeta_{\lambda^R, T_0} \|_2 \| u_{\lambda, T_0} \|_0^2}{\| v_{T_0} \|_0^2} = \frac{\| \zeta_{\lambda, T_0} \|_2 \| u_{\lambda, T_0} \|_0^2}{\mathcal{E}_2 (\lambda^R, T_0) \| v_{T_0} \|_0^2}. \]

\[ \square \]

**Corollary 4.** Suppose \( \lambda, \mu \in \mathbb{N}^{N,+} \) and \( T_1, T_2 \in Y(\tau) \) such that \([\lambda, T_1]\) and \([\mu, T_2]\) are column-strict. If \( \lambda \neq \mu \) or \( T_2 \notin Y(T_1; W_\lambda) \) then \( \langle f_{\lambda, T_1}^s, f_{\mu, T_2}^s \rangle = 0 \).

Let \( T_0 \in Y(\tau) \) such that \( T_0 \) satisfies condition \([a_i, b_i]_{rw}\) for each factor \( S_{[a_i, b_i]} \) of \( W_\lambda \) and \( a_i \leq j_1 < j_2 \leq b_i \) implies \( c (j_1, T_0) \leq c (j_2, T_0) \). This condition is equivalent to the tableau \([\lambda, T]\) being row-strict (the entries strictly increase in each row), and \( T_0 \) having a certain extremal property. Let
\[ f_{\lambda, T_0}^o = \sum_{\alpha^+ = \lambda} (-1)^{\text{inv}(\alpha)} \sum_{T \in Y(T_0; W_\lambda)} \prod_{j=1}^n P_1(T; a_j, b_j) \mathcal{E}_+ (\alpha, T) \zeta_{\alpha, T}, \]
\[ u_{\lambda, T_0} = \sum_{T \in Y(T_0; W_\lambda)} \prod_{j=1}^n P_1(T; a_j, b_j) v_T \in V_\tau. \]
The term involving \( x^\lambda \) is \( h_0 = \sum_{T \in Y(T_0; W_\lambda)} \prod_{j=1}^n P_1(T; a_j, b_j) \zeta_{\lambda,T} \), thus the leading term in \( f^a_{\lambda,T_0} \) is \( x^\lambda u_{\lambda,T_0} \). From the transformation rules in Proposition 9 it follows that \( \|h_0\|^2 = \|\zeta_{\lambda,T_0}\|^2 \|u_{\lambda,T_0}\|^2 / \|v_{T_0}\|^2 \) (see Proposition 4). Also \( h_0 \) is \( W_\lambda \)-antisymmetric.

**Theorem 6.** \( w f^a_{\lambda,T_0} = \text{sgn}(w) f^a_{\lambda,T_0} \) for all \( w \in S_N \) and

\[
\|f^a_{\lambda,T_0}\|^2 = \frac{N!}{\#W_\lambda \text{E}_- (\lambda^R, T_0)} \|u_{\lambda,T_0}\|^2 \|\zeta_{\lambda,T_0}\|^2 .
\]

**Proof.** Let \( \mathcal{F}(\alpha,T) = \prod_{j=1}^n P_1(T; a_j, b_j) \text{E}_+ (\alpha, T) \). Fix \( i \in [1, N-1] \) and collect the terms of \( f^a_{\lambda,T_0} \) into three parts. Let

\[
L = \{ (\alpha, T) : \alpha^+ = \lambda, T \in Y(T_0; W_\lambda) \},
\]

\[
A = \{ (\alpha, T) \in L : \alpha_i = \alpha_{i+1}, \text{cm} (r(\alpha, i), T) = \text{cm} (r(\alpha, i+1), T) \},
\]

\[
B = \{ (\alpha, T) \in L : \alpha_i > \alpha_{i+1} \},
\]

\[
C = \{ (\alpha, T) \in L : \alpha_i = \alpha_{i+1}, \text{cm} (r(\alpha, i), T) < \text{cm} (r(\alpha, i+1), T) \} .
\]

The proof that each of the following satisfies \( s_i f = -f \) is analogous to the proof of the previous theorem:

\[
\sum_{(\alpha, T) \in A} (-1)^{\text{inv}(\alpha)} \mathcal{F}(\alpha, T) \zeta_{\alpha,T},
\]

\[
\sum_{(\alpha, T) \in B} (-1)^{\text{inv}(\alpha)} \left( \mathcal{F}(\alpha, T) \zeta_{\alpha,T} - \mathcal{F}(s_i \alpha, T) \zeta_{s_i \alpha,T} \right)
\]

\[
= \sum_{(\alpha, T) \in B} (-1)^{\text{inv}(\alpha)} \mathcal{F}(\alpha, T) \left( \zeta_{\alpha,T} - \frac{\mathcal{F}(s_i \alpha, T)}{\mathcal{F}(\alpha, T)} \zeta_{s_i \alpha,T} \right),
\]

\[
\sum_{(\alpha, T) \in C} (-1)^{\text{inv}(\alpha)} \left( \mathcal{F}(\alpha, T) \zeta_{\alpha,T} + \mathcal{F}(\alpha, s_{I(\alpha)} T) \zeta_{\alpha,s_{I(\alpha)}T} \right)
\]

\[
= \sum_{(\alpha, T) \in C} (-1)^{\text{inv}(\alpha)} \mathcal{F}(\alpha, s_{I(\alpha)} T) \left( \frac{\mathcal{F}(\alpha, T)}{\mathcal{F}(\alpha, s_{I(\alpha)} T)} \zeta_{\alpha,T} + \zeta_{\alpha,s_{I(\alpha)}T} \right) .
\]

In the second equation \( \frac{\mathcal{F}(s_i \alpha, T)}{\mathcal{F}(\alpha, T)} = 1 + b_i(\alpha, T) \). In the third equation \( I = r(\alpha, i) \) and \( \frac{\mathcal{F}(\alpha, T)}{\mathcal{F}(\alpha, s_{I(\alpha)} T)} = b_{I(\alpha)} - 1 \). The proof for the norm formula is also analogous, based on \( \sum_{w \in S_N} w h_R \) where

\[
h_R = \sum_{T \in Y(T_0; W_\lambda)} \prod_{j=1}^n P_1(T; a_j, b_j) \zeta_{\lambda^R,T} .
\]
Note $\text{sgn}(w_{\lambda R}) = (-1)^{\text{inv}(\lambda R)}$.

**Remark 2.** The polynomials in Theorems 5 and 6 form orthogonal bases for the symmetric and antisymmetric polynomials, respectively, in $M(\tau)$.

5.3. Minimum degree polynomials. For a given partition $\tau$ of $N$ there are unique symmetric and antisymmetric polynomials of minimum degree in the standard module $M(\tau)$. We now establish the key results concerning the norms of these polynomials. It is obvious that the column-strict tableau $[\lambda, T]$ with minimum $|\lambda|$ has the entries 0 in row #1, 1 in row #2 and so on (consider the minimum entries in each column). Denote this partition by $\delta^s(\tau)$ and the unique possible $T$ by $T^s$ (the entries $N, N-1, \ldots, 2, 1$ are entered row-by-row in the Ferrers diagram of $\tau$). Example: let $\tau = (5, 3, 2)$ then

$$T^s = \begin{array}{cccccc}
10 & 9 & 8 & 7 & 6 & 0 & 0 & 0 & 0 \\
5 & 4 & 3 & 0 & 0 & 1 & 1 & 1 & 2 & 2
\end{array},$$

and $\delta^s(\tau) = (2, 2, 1, 1, 0, 0, 0, 0, 0, 0)$.

Similarly the row-strict tableau $[\lambda, T]$ with minimum $|\lambda|$ has the entries 0 in column #1, 1 in column #2 and so on (consider the minimum entries in each row). Denote this partition by $\delta^a(\tau)$ and the unique possible $T$ by $T^a$ (the entries $N, N-1, \ldots, 2, 1$ are entered column-by-column in the Ferrers diagram of $\tau$). Example: let $\tau = (5, 3, 2)$ then

$$T^a = \begin{array}{cccccc}
10 & 7 & 4 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\
9 & 6 & 3 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
8 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array},$$

and $\delta^a(\tau) = (4, 3, 2, 1, 1, 0, 0, 0, 0, 0)$. The sum of the hook-lengths of $\tau$ equals $|\delta^s(\tau)| + |\delta^a(\tau)| + N$ (see [7, Ex. 2, p. 11]).

Let $f^s_\tau = f^s_{\delta^s(\tau), T^s}$ and $f^a_\tau = f^a_{\delta^a(\tau), T^a}$. These polynomials are actually independent of $\kappa$; there is no composition $\alpha$ such that $\alpha < \delta^s(\tau)$ and $\alpha^+ \neq \delta^s(\tau)$ which can occur in a symmetric polynomial, due to the minimality of $\delta^s(\tau)$. A similar argument applies to $\delta^a(\tau)$. To compute the norms $\|f^s_\tau\|^2$ and $\|f^a_\tau\|^2$ we use the special properties of $\delta^s(\tau)$ to write simplified formulae. To use the formulae in Theorems 2 and 3 note that $\delta_a(\tau)_j = i - 1$ when $j$ appears in row #i of $T^s$, and the
corresponding contents of $T^s$ are $1 - i, \ldots, \tau_i - i$. Let $L = \ell(\tau)$ and

\[
P_1(\tau) = \prod_{i=2}^{L} \prod_{j=1}^{\tau_i} (1 + \kappa (j - i))_{i-1},
\]

\[
P_2(\tau) = \prod_{1 \leq i < j \leq L} \prod_{j_1=1, j_2=1}^{\tau_i} \prod_{r=1}^{j-\tau_j} \left( 1 - \frac{\kappa^2}{(r + \kappa (j_2 - j_1 - r + i))^2} \right),
\]

\[
P_3(\tau) = \prod_{1 \leq i < j \leq L} \prod_{j_1=1, j_2=1}^{\tau_i} \prod_{j=1}^{j-\tau_j} \left( 1 + \kappa \frac{j - i + \kappa (j_2 - j_1 - r + i)}{j_1} \right);
\]

\[
H^s(\tau) = \prod_{(i,j) \in \tau} (1 - \kappa h(i,j))_{\text{leg}(i,j)}.
\]

Then $\|z_{\delta^s(\tau), T^s}\|^2 = \|v_{T^s}\|^2 P_1(\tau) P_2(\tau)$ and $E_+ \left( \delta^s(\tau)^R, T^s \right) = P_3(\tau)$.

**Theorem 7.** Suppose $\tau$ is a partition then $P_1(\tau) P_2(\tau) P_3(\tau) = H^s(\tau)$.

**Proof.** We use induction on the last part $\tau_L$. The induction begins with $\tau = (N)$ where each product equals 1. Let $\sigma = (\tau_1, \tau_2, \ldots, \tau_{L-1})$ and assume the formula is valid for $\sigma$. (It is possible that $\tau_L = 1$ and $\ell(\sigma) = L-1$). The nodes in $\sigma$ and $\tau$ have the same hook-lengths except for the nodes $(i, L)$ with $1 \leq i < L$ and $(L, j)$ with $1 \leq j \leq \tau_L$. The latter have zero leg-length and do not contribute to $H^s(\sigma)$ or $H^s(\tau)$. Then for $1 \leq i < L$

\[
\text{arm}(i, L; \sigma) = \text{arm}(i, L; \tau) = \tau_i - \tau_L
\]

\[
\text{leg}(i, L; \sigma) + 1 = \text{leg}(i, L; \tau) = L - i
\]

\[
h(i, L; \sigma) + 1 = h(i, L; \tau) = 1 + \tau_i - \tau_L + L - i,
\]

and thus

\[
\frac{H^s(\tau)}{H^s(\sigma)} = \prod_{i=1}^{L-1} \frac{(1 - \kappa (\tau_i - \tau_L + L - i + 1))_{L-i}}{(1 - \kappa (\tau_i - \tau_L + L - i))_{L-i-1}}.
\]

Firstly,

\[
\frac{P_1(\tau)}{P_1(\sigma)} = (1 + \kappa (\tau_L - L))_{L-1};
\]

secondly

\[
\frac{P_2(\tau)}{P_2(\sigma)} = \prod_{i=1}^{L-1} p_i(\tau),
\]
where

\[ p_i(\tau) = \prod_{i=1}^{L-i} \prod_{j=1}^{\tau_i} \frac{(r + \kappa (\tau_L - L + i) - (j - 1) \kappa)}{(r + \kappa (\tau_L - L + i) - j \kappa)} \]

\[ \cdot \frac{(r + \kappa (\tau_L - L + i) - (j + 1) \kappa)}{(r + \kappa (\tau_L - L + i) - j \kappa)} \]

\[ = \prod_{i=1}^{L-i} \frac{(r + \kappa (\tau_L - L + i)) (r + \kappa (\tau_L - L + i) - (\tau_i + 1) \kappa)}{(r + \kappa (\tau_L - L + i) - \kappa) (r + \kappa (\tau_L - L + i) - \tau_i \kappa)} \]

\[ = \frac{(1 + \kappa (\tau_L - L + i))_{L-i}}{(1 + \kappa (\tau_L - L + i))_{L-i}} \frac{(1 + \kappa (\tau_L - \tau_i - L + i - 1))_{L-i}}{(1 + \kappa (\tau_L - \tau_i - L + i))_{L-i}} ; \]

a telescoping product argument is used to produce the third line from the second.

Thirdly,

\[ \frac{P_3(\sigma)}{P_3(\tau)} = \prod_{i=1}^{L-1} \prod_{j=1}^{\tau_i} \frac{L - i + \kappa (\tau_L - j_1 - L + i)}{L - i + \kappa (\tau_L + 1 - j_1 - L + i)} \]

\[ = \prod_{i=1}^{L-1} \frac{L - i + \kappa (\tau_L - \tau_i - L + i)}{L - i + \kappa (\tau_L - L + i)} . \]

Combining these products and by use of

\[ \frac{L - i + \kappa (\tau_L - \tau_i - L + i)}{(1 + \kappa (\tau_L - \tau_i - L + i))_{L-i}} \frac{1}{(1 + \kappa (\tau_L - L + i))_{L-i}} , \]

\[ \frac{L - i + \kappa (\tau_L - L + i)}{(1 + \kappa (\tau_L - L + i))_{L-i}} = (1 + \kappa (\tau_L - L + i))_{L-i-1} , \]

we obtain

\[ \frac{P_1(\tau) P_2(\tau) P_3(\sigma) H^s(\sigma)}{P_1(\sigma) P_2(\sigma) P_3(\tau) H^s(\tau)} \]

\[ = (1 + \kappa (\tau_L - L))_{L-1} \prod_{i=1}^{L-1} \frac{(1 + \kappa (\tau_L - L + i))_{L-i-1}}{(1 + \kappa (\tau_L - L + i - 1))_{L-i}} \]

\[ = 1 . \]

The last step is easy: replace i by i - 1 in the numerator (and now 2 \leq i \leq L) and cancel. This completes the induction. \qed
Theorem 8. Suppose $\tau$ is a partition of $N$ then
\[ \|f_\tau^s\|^2 = \frac{N!}{\prod_{i=1}^{\ell(\tau)} \tau_i} \|v_{\tau^s}\|^2 \prod_{(i,j) \in \tau} (1 - \kappa h(i,j))_{\text{leg}(i,j)}. \]

Proof. The formulae of Theorem 5 and the previous theorem imply this result. The stabilizer subgroup $W_{\delta_s(\tau)}$ acts on the rows of $\tau$ and has order $\tau_1! \tau_2! \ldots$.

Theorem 9. Suppose $\tau$ is a partition of $N$ then
\[ \|f_\tau^a\|^2 = \frac{N!}{\prod_{i=1}^{r'_1} \tau'_i} \|v_{\tau^a}\|^2 \prod_{(i,j) \in \tau} (1 + \kappa h(i,j))_{\text{arm}(i,j)}. \]

Proof. Apply Theorems 3, 6 and the formula in Theorem 8 to the conjugate $\tau'$ of $\tau$ and with $\kappa$ replaced by $-\kappa$. Then $\text{leg}(j,i; \tau') = \text{arm}(i,j; \tau)$ for $(i,j) \in \tau$. Note however that $\|v_{\tau^s}\|^2 / \|v_{\tau^a}\|^2$ is computed by use of Proposition 4.

As example we use $\tau = (5,3,2)$ again. The hook-lengths and norms are
\[
\begin{array}{cccc}
7 & 6 & 4 & 2 & 1 \\
4 & 3 & 1 & & \\
2 & 1 & & & \\
\end{array}
\]

\[
\|f_\tau^s\|^2 = c_0 (1 - 7\kappa)_2 (1 - 6\kappa)_2 (1 - 4\kappa)_2 (1 - 3\kappa),
\|f_\tau^a\|^2 = c_1 (1 + 7\kappa)_4 (1 + 6\kappa)_3 (1 + 4\kappa)_2 (1 + 3\kappa) (1 + 2\kappa)^2.
\]

Analogously to the $M((N))$ (trivial representation) result, each hook-length $m$ appears in $m - 1$ factors $(m\kappa + r)$ involving each nonzero residue class mod $m$. In the example for $m = 6$ we obtain $6\kappa - 2, 6\kappa - 1, 6\kappa + 1, 6\kappa + 2, 6\kappa + 3$. We conjecture that the singular values for $M(\tau)$ form a subset of \( \left\{ \frac{m}{n} : m = h(i,j), (i,j) \in \tau, \frac{n}{m} \notin \mathbb{Z} \right\} \). The condition is equivalent to $J_{\kappa_0}(\tau) \neq 0$ for all $i \in [1, N]$, that is, the generic $\kappa$ is specialized to $\kappa_0$.

5.4. Aspherical values. S. Griffeth (personal communication, gratefully acknowledged) points out that Theorem 8 provides a new proof for one of the parts of the Gordon-Stafford Theorem [5, Cor. 3.13]; another proof was found by Bezrukavnikov and Etingof [2, Cor. 4.2]; note that these papers use $c = -\kappa$ as parameter. An *aspherical* module of the rational Cherednik algebra is one containing no nonzero $S_N$-invariant. If some quotient module of a standard module is aspherical
for a numerical value $\kappa_0$ of $\kappa$ then $\kappa_0$ is called an aspherical value. Theorem 8 shows that any aspherical value is in $\{\frac{m}{n}: 1 \leq m < n \leq N\}$ (this is one component of the Gordon-Stafford theorem, which deals with the problem of Morita equivalence of rational Cherednik algebras for parameters $\kappa$ and $\kappa - 1$). Suppose $M_0$ is a proper submodule of $M(\tau)$ for $\kappa = \kappa_0 \in \mathbb{Q}$ (that is, a specific numerical value). This means that $M_0$ is closed under multiplication by $x_i$ and the action of $D_i$ for $i \in [1, N]$ and under the action of $S_N$. Then $f \in M_0$ implies $\langle g, f \rangle = 0$ for all $g \in M(\tau)$ ($M_0$ is a submodule of the radical $J_{\kappa_0}(\tau)$, the maximal submodule.). Indeed, by the definition of the contravariant form, $\langle x^\alpha u, f \rangle = \langle u, D^\alpha f(x) \rangle$ for $\alpha \in \mathbb{N}^N, u \in V_\tau$ (and $D^\alpha = \prod_{i=1}^N D_i^{\alpha_i}$). If $f \in P_n \otimes V_\tau$ and $|\alpha| = n$ then $D^\alpha f(x) \in V_\tau$. If also $f \in M_0$ then $D^\alpha f(x) = 0$, or else $M_0 = M(\tau)$. If $M(\tau)/M_0$ is aspherical then $f_\tau \in M_0$ and $\kappa = \kappa_0$ is a zero of $\prod_{(i,j) \in \tau} (1 - \kappa h(i,j))_{\text{reg}(i,j)}$.

6. Appendix

This contains proofs of the specializations of Griffith’s results in the $G(r,1,N)$-context, which were stated in Section 3. Here is the restatement and proof of Proposition 7

Proposition 10. Suppose $\alpha \in \mathbb{N}^N$ and $\alpha_i > \alpha_{i+1}$ for some $i < N$. Then $s_i \zeta_{\alpha,T} = b_i(\alpha,T) \zeta_{\alpha,T} + (1 - b_i(\alpha,T)^2) \zeta_{s_i\alpha,T} = \zeta_{\alpha,T} - b_i(\alpha,T) \zeta_{s_i\alpha,T};$ and $\|\zeta_{\alpha,T}\|^2 = (1 - b_i(\alpha,T)^2) \|\zeta_{s_i\alpha,T}\|^2$.

Proof. The condition $\alpha_i \neq \alpha_{i+1}$ implies $r(s_i \alpha, i) = r(\alpha, i + 1)$ and $r(s_i \alpha, i + 1) = r(\alpha, i)$, thus $\xi_i(s_i \alpha, T) = \xi_{i+1}(\alpha, T)$ and $\xi_{i+1}(s_i \alpha, T) = \xi_i(\alpha, T)$ (and $\xi_j(s_i \alpha, T) = \xi_j(\alpha, T)$ for $j \neq i, i + 1$). Since the eigenvalues determine the eigenvectors uniquely we have that

$$s_i \zeta_{\alpha,T} - b_i(\alpha,T) \zeta_{\alpha,T} = a \zeta_{s_i\alpha,T},$$

$$s_i \zeta_{s_i\alpha,T} + b_i(\alpha,T) \zeta_{s_i\alpha,T} = a' \zeta_{\alpha,T},$$

for some scalars $a, a'$. The fact that $s_i^2 = 1$ implies $aa' = 1 - b_i(\alpha,T)^2$. We show that $a' = 1$ by finding the leading term in $s_i \zeta_{s_i\alpha,T}$, namely $x^\alpha s_i w_{s_i \alpha}v_T$. It remains to show that $w_{s_i \alpha} = s_i w_\alpha$, that is, $r(s_i \alpha, s_i w_\alpha(j)) = j$ for all $j$. If $w_{\alpha}^{-1}(j) \neq i, i+1$ then $r(s_i \alpha, s_i w_\alpha(j)) = r(\alpha, w_\alpha(j)) = j$. If $w_{\alpha}^{-1}(j) = i$ then $r(s_i \alpha, s_i w_\alpha(j)) = r(s_i \alpha, i + 1) = r(\alpha, i) = j$. The case $w_{\alpha}^{-1}(j) = i + 1$ follows similarly. The second displayed equation shows that $\|s_i \zeta_{s_i\alpha,T}\|^2 = \|\zeta_{s_i\alpha,T}\|^2 = \|a' \zeta_{\alpha,T}\|^2 - b_i(\alpha,T)^2 \|\zeta_{s_i\alpha,T}\|^2$.

$\square$
The proof of the norm formula of Theorem 2 uses a raising operator. From the commutators 3.2 we obtain:
\[
\mathcal{U}_i x_N f = x_N (\mathcal{U}_i - \kappa (i, N)) f, \quad 1 \leq i < N, \\
\mathcal{U}_N x_N f = x_N (1 + \mathcal{D}_N x_N) f.
\]
Let \( \theta_N = s_1 s_2 \ldots s_{N-1} \) thus \( \theta_N (N) = 1 \) and \( \theta_N (i) = i+1 \) for \( 1 \leq i < N \) (a cyclic shift). Then
\[
\mathcal{U}_i x_N f = x_N (\theta_N^{-1} \mathcal{U}_{i+1} \theta_N) f, \quad 1 \leq i < N, \\
\mathcal{U}_N x_N f = x_N (1 + \theta_N^{-1} \mathcal{U}_i \theta_N) f.
\]
If \( f \) satisfies \( \mathcal{U}_i f = \lambda_i f \) for \( 1 \leq i \leq N \) then
\[
\mathcal{U}_i (x_N \theta_N^{-1} f) = \lambda_{i+1} (x_N \theta_N^{-1} f)
\]
for \( 1 \leq i < N \) and
\[
\mathcal{U}_N (x_N \theta_N^{-1} f) = (\lambda_1 + 1) (x_N \theta_N^{-1} f).
\]
For \( \alpha \in \mathbb{N}^N \) let \( \phi (\alpha) := (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1) \), then \( x_N \theta_N^{-1} x^\alpha = x^{\phi(\alpha)} \).

**Proposition 11.** Suppose \( \alpha \in \mathbb{N}^N, T \in \mathcal{Y} (\tau) \), then
\[
\zeta_{\phi(\alpha), T} = x_N \theta_N^{-1} \zeta_{\alpha, T}.
\]

**Proof.** By straightforward arguments it follows that \( r (\phi (\alpha), i) = r (\alpha, i + 1) \) for \( 1 \leq i < N \) and \( r (\phi (\alpha), N) = r (\alpha, 1) \), that is, \( r (\phi (\alpha), i) = r (\alpha, \theta_N (i)) \) for all \( i \). This is equivalent to \( r (\phi (\alpha), \theta_N^{-1} w_\alpha (j)) = r (\alpha, w_\alpha (j)) = j \) for all \( j \), or \( w_\phi (\alpha) = \theta_N^{-1} w_\alpha \). The leading term \( x^\alpha w_\alpha v_T \zeta_{\alpha, T} \) is mapped to \( x^{\phi(\alpha)} w^{\phi(\alpha)} v_T \) by \( f \mapsto x_N \theta_N^{-1} f \). Note that \( \mathcal{U}_i \zeta_{\phi(\alpha), T} = (\alpha_{i+1} + 1 + \kappa c (r (\phi (\alpha), i), T)) \zeta_{\phi(\alpha), T} \) for \( 1 \leq i < N \) and \( \mathcal{U}_N \zeta_{\phi(\alpha), T} = (\alpha_1 + 2 + \kappa c (r (\phi (\alpha), N), T)) \zeta_{\phi(\alpha), T} \). Thus \( x_N \theta_N^{-1} \zeta_{\alpha, T} \) and \( \zeta_{\phi(\alpha), T} \) have the same eigenvalues for \( \{ \mathcal{U}_i \} \) and the same coefficient of \( x^{\phi(\alpha)} \). Hence \( x_N \theta_N^{-1} \zeta_{\alpha, T} = \zeta_{\phi(\alpha), T} \). \( \square \)

**Corollary 5.** \( \| \zeta_{\phi(\alpha), T} \|^2 = (\alpha_1 + 1 + \kappa c (r (\alpha, 1), T)) \| \zeta_{\alpha, T} \|^2. \)

**Proof.** Indeed
\[
\| \zeta_{\phi(\alpha), T} \|^2 = \langle \theta_N^{-1} \zeta_{\alpha, T}, \mathcal{D}_N x_N \theta_N^{-1} \zeta_{\alpha, T} \rangle \\
= \langle \theta_N^{-1} \zeta_{\alpha, T}, \theta_N^{-1} \mathcal{D}_1 x_1 \zeta_{\alpha, T} \rangle = \langle \zeta_{\alpha, T}, \mathcal{U}_1 \zeta_{\alpha, T} \rangle \\
= \xi_1 (\alpha, T) \| \zeta_{\alpha, T} \|^2.
\]

The following is a restatement and proof of Theorem (2) (Griffeth [6, Thm. 6.1]).
Theorem 10. Suppose \( \lambda \in \mathbb{N}^{N,+} \) and \( T \in Y(\tau) \) then

\[
\| \zeta_{\lambda,T} \|^2 = \| v_T \|^2 \prod_{i=1}^N (1 + \kappa c(i,T))_{\lambda_i} \\
\times \prod_{1 \leq i < j \leq N} \prod_{l=1}^{\lambda_i - \lambda_j} \left( 1 - \frac{\kappa^2}{(l + \kappa(c(i,T) - c(j,T)))^2} \right).
\]

Proof. Argue by induction. Suppose \( \lambda_1 = \lambda_2 = \ldots = \lambda_m > \lambda_{m+1} \). Let

\[
\beta = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m, \lambda_{m+1}, \ldots, \lambda_N),
\]

\[
\alpha = (\lambda_1 - 1, \lambda_1, \ldots, \lambda_{m-1}, \lambda_m, \lambda_{m+1}, \ldots, \lambda_N),
\]

\[
\mu = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_1 - 1, \lambda_{m+1}, \ldots, \lambda_N).
\]

Thus \( \beta = \phi(\alpha) \) and

\[
\| \zeta_{\beta,T} \|^2 = (\lambda_1 + \kappa c(m,T)) \| \zeta_{\alpha,T} \|^2
\]

\[
= (\lambda_1 + \kappa c(m,T)) \mathcal{E}_2(\alpha,T)^{-1} \| \zeta_{\mu,T} \|^2,
\]

\[
\| \zeta_{\lambda,T} \|^2 = \mathcal{E}_2(\beta,T) \| \zeta_{\beta,T} \|^2.
\]

We have

\[
\mathcal{E}_\epsilon(\alpha,T) = \prod_{j=2}^m \left( 1 + \frac{\epsilon \kappa}{1 + \kappa(c(j-1,T) - c(m,T))} \right),
\]

\[
\mathcal{E}_\epsilon(\beta,T) = \prod_{j=m+1}^N \left( 1 + \frac{\epsilon \kappa}{\lambda_1 - \lambda_j + \kappa(c(m,T) - c(j,T))} \right).
\]

The validity of the formula for \( \| \zeta_{\mu,T} \|^2 \) thus implies the validity for \( \| \zeta_{\lambda,T} \|^2 \) (that is, the value of \( \| \zeta_{\lambda,T} \|^2 / \| \zeta_{\mu,T} \|^2 \) from the formula agrees with \( (\lambda_1 + \kappa c(m,T)) \mathcal{E}_2(\beta,T) / \mathcal{E}_2(\alpha,T) \)). \( \Box \)

References

[1] T. Baker and P. Forrester, Symmetric Jack polynomials from non-symmetric theory, *Ann. Comb.* 3 (1999), 159–170.

[2] R. Bezrukavnikov and P. Etingof, Parabolic induction and restriction functors for rational Cherednik algebras, *Selecta Math. (N.S.)* 14 (2009), 397–425; arXiv:0803.3639v6, 28 Oct. 2009.

[3] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encycl. of Math. and its Applications 81, Cambridge University Press, Cambridge, 2001.

[4] P. Etingof and E. Stoica, with an appendix by S. Griffeth, Unitary representations of rational Cherednik algebras, *Represent. Theory* 13 (2009), 349–370; arXiv:0901.4595v3, 20 Mar. 2009.
[5] I. Gordon and J. Stafford, Rational Cherednik algebras and Hilbert schemes, 
   Adv. Math. **198** (2005), 222–274.

[6] S. Griffeth, Orthogonal functions generalizing Jack polynomials, Trans. Amer. 
   Math. Soc. **362** (2010), 6131–6157; arXiv:0707.0251v3, 9 Nov. 2008.

[7] I. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Clarendon 
   Press, Oxford, 1995.

[8] G. Murphy, A new construction of Young’s seminormal representation of the 
   symmetric groups, J. Algebra **69** (1981), 287–297.

[9] A. Okounkov and A. Vershik, A new approach to representation theory of 
   the symmetric groups, II, arXiv:math/0503040v3, 20 April 2005, (translation 
   from Russian) J. Math. Sci. (N.Y.) **131** (2005), 5471–5494.

[10] E. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite 
    Coxeter group, Compositio Math. **85** (1993), 333–373.

[11] R. Rouquier, Representations of rational Cherednik algebras, Infinite-
    dimensional aspects of representation theory and applications, 103–131, Contemp. 
    Math., **392**, Amer. Math. Soc., Providence, RI, 2005.

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