Hard, firm, soft … Etherealware: Computing by Temporal Order of Clocking

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Abstract

We define Etherealware as the concept of implementing the functionality of an algorithm by means of the clocking scheme of a cellular automaton (CA). We show, which functions can be implemented in this way, and by which CAs.

Keywords: cellular automaton, etherealware, asynchronous, update rule, universality, temporal order, clocking-computable

Your task: Compute a lot of different functions on $n$-bit inputs.

Your device: A (fixed!) cellular automaton (CA) on the (fixed!) ring or torus topology with $n$ cells, is capable of holding one bit each.

You may not change the CA (its update rule) nor the topology. You may not enter additional information in the form of parameters (there would be no space to store them anyway) — and yet you are supposed to evaluate many different functions. The available degree of freedom is the clocking scheme of the cells, anything from synchronous to completely asynchronous is allowed.

Can you do it?

The perhaps surprising answer is: yes!

Every bijective function on the set $\{0, 1, ..., 2^n - 1\}$, which acts as an even permutation is clocking-computable, as well as many non-bijective functions.
1. Introduction

1.1. What is etherealware?

Computation takes place in dedicated hardware or on general-purpose hardware by dedicated software. Different functionality requires either changing the hardware (think ASIC, FPGA) or changing the software running on it. Firmware is an intermediate concept, where the hardware is modified by microprogramming a CPU or personalizing an FPGA.

Etherealware is the first way to use fixed hardware (certain cellular automata (CA) in this case), run fixed software (the same update rule for all cells, for all time, for all purposes), and still deliver diversity in the resulting function: by changing only the clocking scheme, the order in which the CA’s cells are updated.

1.2. State-of-the-art

The study of synchronous CAs starts with Wolfram [1]. We use asynchronous CAs as deterministic devices with a finite number of computation steps, which is a new point of view.

Previously, asynchronous CAs have been treated as dynamical systems, where infinite computations are considered, and the focus lies on concepts like orbits, fixed points, ergodicity, transients, cycles and their periods, and long-term behavior. Also, randomness can be introduced to average over many possible asynchronous schemes. Papers in this respect are:

Ingerson and Buvel [2], 1984, distinguish synchronous, random (completely asynchronous), and periodic clocking, which yield clearly distinguishable behavior.

Barrett et al. [3–6], 1999–2003, consider sequential dynamical systems (SDS), including CAs with arbitrary topology and neighborhoods. They cover random graphs as topology and dynamical systems topics such as fixed points and invertibility.

Siwak [7], 2002, gives an overview of simulating machines, including CAs and SDSs, and unifies them under the concept of “filtrons.”

Lee et al. [8], 2003, give an asynchronous CA on the two-dimensional grid $\mathbb{Z} \times \mathbb{Z}$, which is Turing universal.

Laubenbacher and Pareigis [9], 2006, build upon [3–6] and observe that not all $n!$ permutations of the cells lead to different temporal rules. Their equivalence classes coincide—for our setting, CAs on the torus—with our result ([10], Thm. 1]).

Fatès et al. [11], 2006, consider ECAs with quiescent states (000 $\mapsto$ 0, 111 $\mapsto$ 1, i.e., with even Wolfram rule $\geq 128$). They show that 9 ECAs diverge, while the other 55 converge to a random fixed point, in 4 clearly distinguishable time frames $\Theta(n \log(n)), \Theta(n^2), \Theta(n^3)$, or $\Theta(n^{2^n})$ with characteristic behavior per time frame.

Macauley, McCammond, and Mortveit [12, 13], 2007–2010, also treat SDSes, in particular ECAs. For each ECA, [13] gives the periodic states and the dynamics group. Conjecture 5.10 in [13] about Wolfram rule 57 coincides with our finding that ECA-57 generates the alternating groups...
on patterns of \( n \) bits. They verified this claim for \( n \) up to 8, while in [14] this is extended to \( n \leq 10 \) and in this paper up to \( n \leq 28 \).

Dennunzio et al. [15, 16], 2012–2013, consider ACAs, every turing machine can be simulated by an ACA, with quadratic slowdown. Introducing a certain fairness measure, they show that injectivity and surjectivity are equivalent (\( \mu \)-a.a.), and the existence of a diamond is equivalent to not \( \mu \)-a.a. injectivity.

Salo [17], 2014, shows that nonuniform CA generates all SFTs (subshifts of finite type) and several non-SFT sofic shifts.

2. Notations: ECA and update rules

Here, continuing the work in [10, 14], we again employ CAs as computing devices, whose work comes to an end, when the pattern transformation or function evaluation has been obtained. Also, the clocking, the temporal update rule, is completely deterministic and replaces the usual ways of representing an algorithm, either in software (initial data) or in hardware (choice of ECA and connecting graph). Thus, the algorithm resides exclusively in the clocking scheme. We therefore call functions computable in this way as “clocking-computable functions.” The main additional contribution of this paper is the introduction of unfair clocking schemes.

2.1. Cellular automata: Neighborhoods and local update rules

We consider cellular automata (CA) on a torus or ring of \( n \) cells, that is index set \( \mathbb{Z}/n\mathbb{Z} \), over the binary alphabet \( \{0, 1\} \). Cell index wraparound, that is, \( c_i = c_{i+j \mod n} \), and the canonical cell names are \( c_{n-1}, c_{n-2}, \ldots, c_0 \). We deal with elementary CAs (ECA) with three input cells, where the middle one is also the output cell.

The neighborhood \((c_{i+1}, c_i, c_{i-1})\) can have eight different values from \( \mathbb{F}_2^3 \). Let \( k = 4c_{i+1} + 2c_i + c_{i-1} \in \{0, \ldots, 7\} \). Then \( c_i \) is replaced by \( c_i^+ = p_k \in \{0, 1\} \), where \( p_0, \ldots, p_7 \) are defined via Wolfram’s rule [1] \( \sum_{k=0}^{7} 2^k p_k \in \{0, \ldots, 255\} \).

The \( 2^{2^3} = 256 \) ECAs can be arranged into 88 equivalence classes under the symbolic symmetry 0/1 and the chiral symmetry left/right \((c_{i+1} \leftrightarrow c_{i-1})\); see ([14], Appendix A). It is sufficient to consider one member per class.

We considered quad CAs (QCAs) with four inputs and nonstandard neighborhoods in ([10], Section 1.2).

Local bijectivity requires that ECA \((a, 0, c) = 1 \rightarrow \) ECA \((a, 1, c)\). This is equivalent to requiring that the hexadecimal digits of the rule be from 3, 6, 9, C.

Example: The behavior of the ECA with Wolfram rule 57 = 00111001_2 = 39_{16}:

\[
\begin{array}{l}
111 \mapsto 0, \quad 110 \mapsto 0, \quad 101 \mapsto 1, \quad 100 \mapsto 1, \quad 011 \mapsto 1, \quad 010 \mapsto 0, \quad 001 \mapsto 0, \quad 000 \mapsto 1,
\end{array}
\]

in other words \( 0c_1 \mapsto c_i^+ = c_i \) for all other contexts we have \( 0c_i^0, 1c_1^0, 1c_1 \mapsto c_i^+ = \bar{c}_i \).
2.2. ECA: Global update rules on the torus

2.2.1. Fair update schemes

We repeat the definition of asynchronicity rules from ([10], Section 2).

The set $AS_n$ of asynchronicity rules over $\mathbb{Z}/n\mathbb{Z}$ consists in all words of length $n$ over the alphabet $\{<, \equiv, >\}$ such that both “<” and “>” occur at least once. We also include the word “$\equiv \ldots \equiv$,” the synchronous case, and have.

\[ AS_n = (\{<, \equiv, >\}^n \setminus \{<, \equiv, >\}^n) \cup \{\equiv^n\} \] with $|AS_n| = 3^n - 2^{n+1} + 2$.

A rule $AS = AS_{n-1} \ldots AS_0 \in AS_n$ defines the firing order as follows:

| $AS_i$ | Meaning |
|-------|---------|
| $<$ | Cell $c_i$ fires after cell $c_{i-1}$ |
| $>$ | Cell $c_i$ fires before cell $c_{i-1}$ |
| $\equiv$ | Cell $c_i$ fires simultaneously with cell $c_{i-1}$ |

To ensure bijectivity, we must first have a locally bijective CA, and, furthermore, no two adjacent cells may fire simultaneously. Why this is so will be dealt with in Chapter 5. There are exactly $2^n - 2$ bijective fair rules, those from $\{<, >\}^n \setminus \{<^n, >^n\}$; see also ([10], 4.1)).

A fair update step (bijective or not) can be decomposed into a sequence of elementary steps such that all cells fire exactly once during the execution of that sequence; see [3].

2.2.2. Unfair update schemes

We now include unfair updates, where some cells may fire less often than others (even not at all).

We start with elementary steps ($\mu$ steps in [10]). During one elementary step, any nonempty subset $I \subseteq \{n - 1, n - 2, \ldots, 1, 0\}$ of indices may define the active cells.

These cells fire simultaneously, hence $c_i^+$ = \( ECA(c_{i+1}, c_i, c_{i-1}), \quad i \in I, \)
\( c_i, \quad i \notin I. \)

We define elementary steps as words $s = s_{n-1} \ldots s_1 s_0 \in \{0, 1\}^n$, with the meaning $s_i = 1$, if cell $c_i$ fires, and $s_i = 0$ if cell $c_i$ is inactive in this step.

A sequence of such elementary steps is upper indexed by the time step $(t)$.

A fair update rule consists in a number of elementary steps such that every cell fires exactly once. The fair rule $as = "<><><>"$ for $n = 6$ can be decomposed into $(s^{(1)} = 010101, s^{(2)} = 101010)$, while the sequence $(s^{(1)} = 010001, s^{(2)} = 101010)$ is unfair, since cell $c_2$ does not fire at all.

The number of bijective elementary steps is the number of words of length $n$ over the alphabet $\{0, 1\}$, such that no adjacent 1’s occur to ensure bijectivity. For $n = 3, 4, 5, 6$ there are 3, 6, 10, 17 such steps, respectively. The sequence obeys the law $n_k = n_{k-1} + n_{k-2} + 1$. At least for $n = 3, \ldots, 7$, the sets are as follows: prepend a 0 to each pattern of length $k - 1$, prepend a 10 (a 01) to each pattern of size $k - 1$ terminating in 0 (in 1), and add the new pattern 10$^{k-1}$.
sequence is used again in Section 5 and has been verified to coincide with OEIS A001610 up to $n = 24$.

We also can define unfair bijective rules (full steps), where we first fix some subset of size $1 \leq k \leq n - 1$ of active cells by a word $a$ from $\{0, 1\}^n \setminus \{0^n, 1^n\}$, $a_i = 1$ meaning that cell $c_i$ is active.

We next order adjacent active cells by the usual $<, >$ signs. Hence, a run of $r$ consecutive 1’s (with wraparround) has $2^{r-1}$ ways to fix the internal firing order. This order is independent of the other 1-runs, since the cells are separated by at least an inactive cell with $a_i = 0$. The number of bijective unfair rules on a torus of size $n$ is

$$\sum_{p=1}^{2^e-1} \sum_{i=1}^{\text{runs}} 2^{r_i-1}$$

where the pattern $p$ avoids $0^n$ and $1^n$ and then the 1-runs in the pattern have lengths $r_1, r_2, \ldots$, considering wraparround.

For $n = 3, \ldots, 7$, we have $9 = 3 + 6$, $30 = 4 + 10 + 16$, $90 = 5 + 15 + 30 + 40$, $257 = 6 + 21 + 50 + 84 + 96$, and $714 = 7 + 28 + 77 + 154 + 224 + 224$, such unfair rules, respectively. The terms count how many patterns with $k = 1, 2, \ldots, n - 1$ active cells are feasible. These terms can be found in OEIS [18] as subsequences of A209697.

3. Patterns

We consider pattern conversions $\mathbb{F}_2^n \ni v \mapsto w \in \mathbb{F}_2^n$, where $\mathbb{F}_2^n$ can be identified with the set $\{0, 1, \ldots, 2^n - 1\}$. From Definition ([10], Def. 3), by $ECA_{AS}(v) = w$ we mean that the elementary CA with rule ECA maps $v \in \{0, 1\}^n$ to $w \in \{0, 1\}^n$ via the asynchronicity scheme AS. We define $ECA_{AS}^\tau(v) = ECA_{AS}(ECA_{AS}^{\tau-1}(v))$ recursively for $\tau \in \mathbb{N}$, starting with $ECA_{AS}^1(v) = ECA_{AS}(v)$.

In [10, 14], we considered five universality properties $(o)$ to $(iv)$, where each $v \mapsto w$ makes use of a certain update rule AS applied several times. We only give a summary here. Property $(iv)$ is ruled out for any $n \in \mathbb{N}$, while properties $(o)$ to $(iv)$ have only been verified experimentally, for $n \leq 15$.

\begin{enumerate}
  \item $(o)$ \quad \exists v \in \mathbb{F}_2^n, \quad \forall w \in \mathbb{F}_2^n, \quad \exists \tau \in \mathbb{N}, \quad \exists AS \in AS_n : \quad ECA_{AS}^{\tau}(v) = w.
  \end{enumerate}

Some $v$ is mapped to every $w$ by varying the rule AS and the required number of time steps. There are 44 ECAs doing this.

\begin{enumerate}[resume]
  \item $(i)$ \quad \forall v \in \mathbb{F}_2^n, \quad \forall w \in \mathbb{F}_2^n, \quad \exists \tau \in \mathbb{N}, \quad \exists AS \in AS_n : \quad ECA_{AS}^{\tau}(v) = w.
  \end{enumerate}

All $v$ are mapped to all $w$ by varying the rule AS and the required number of time steps. There are 6 ECAs (rules 19, 23 (for $n \equiv 0 \text{ mod } 2$), 37 (for $n \equiv 0 \text{ mod } 3$), 41, 57, 105 (for $n \equiv 0 \text{ mod } 4$)) doing this (checked for $n \leq 15$).

\begin{enumerate}[resume]
  \item $(ii)$ \quad \forall v \in \mathbb{F}_2^n, \quad \exists \tau \in \mathbb{N}, \quad \forall w \in \mathbb{N}_2^n, \quad \exists AS \in AS_n : \quad ECA_{AS}^{\tau}(v) = w.
  \end{enumerate}

All $v$ are mapped to all $w$ at the same time, which time may vary for $v$ but not for $w$, for different rules AS.
ECA-57 realizes this for $n = 5, \ldots, 15$ (no further $n$ has been considered).

ECA-105 realizes this for odd $n = 7, 9, 11, 13, 15$ (no further $n$ has been considered).

$$(iii) \quad \exists \tau \in \mathbb{N}, \quad \forall v \in \mathbb{F}_2^n, \quad \forall w \in \mathbb{F}_2^n, \quad \exists AS \in AS_n : \quad ECA^\tau_{AS}(v) = w.$$ 

All $v$ are mapped to all $w$ at the same time, varying the rule $AS$. This is actually possible for the two survivors of property (ii); see Table 1. The required time roughly decreases with growing $n$, since we have $3^n - 2^{n+1} + 1$ rules to choose from for $2^n$ patterns $w$. Thus for higher $n$ the probability to meet the conversion early on increases.

$$(iv) \quad \exists \tau_0 \in \mathbb{N}, \quad \forall \tau \geq \tau_0, \quad \forall v, w \in \mathbb{F}_2^n, \quad \exists AS \in AS_n : \quad ECA^\tau_{AS}(v) = w.$$ 

Eventually, all conversions may happen at all times for some update scheme. This property cannot be satisfied, for no ECA; see ([14], Thm. 2).

For more details and results for QCAs consult Section 2 of [14] and Section 3 of [10].

### 4. Bijective functions

We first introduce the computer algebra system GAP and then give several examples.

#### 4.1. GAP: Graphs, algorithms, programming

#### 4.1.1. GAP and the alternating group $A_{2^n}$

GAP [19] is a system for computational discrete algebra, in particular computational group theory. We use GAP to decide, whether certain fair or unfair update rules generate the full symmetric or alternating group $S_{2^n}$ or $A_{2^n}$, respectively.

Our results so far:

**Theorem 1.**

i. The fair update rules for ECA-57 generate the full symmetric group $S_8$ for $n = 3$.

ii. The fair update rules for ECA-57 generate the full alternating group $A_{2^n}$ for $n = 4, \ldots, 11$.

| ECA  | $n = 5$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------|---------|---|---|---|---|----|----|----|----|----|----|
| 57   | 445     |   | 70| 242| 35| 13 | 13 | 13 | 13 | 13 | 10 |
| 105  |         |   | 570| 14 | 6 |   |   |   | 6 |   | 8  |

*Table 1.* Minimum time $\tau$ required to satisfy (iii).
iii. The (unfair) elementary update rules with exactly one active cell for ECA-57 generate the full alternating group \(A_{2^n}\) for \(n = 4, \ldots, 28\).

**Proof.**

(i) By exhaustive generation of all \(8! = 40320\) permutations on \{000, 001, \ldots, 111\}.

(ii, iii) First observe that all these rules consist of an even number of transpositions. Hence, we will at most obtain the alternating group \(A_{2^n}\). This group comprises those bijective functions on \{0, 1, \ldots, 2^n – 1\} which have positive sign as a permutation.

We checked (ii) with GAP for certain sets of 5 fair rules for each of these sizes \(n\), and GAP’s function `IsNaturalAlternatingGroup(G)` returned true.

For (iii) we have the canonical set of \(n\) elementary unfair update rules, with exactly one cell active in each rule. This set generates all fair and unfair update rules and hence is sufficient to decide on the group generated by all rules.

Again, GAP shows that indeed the alternating group is generated, for \(n = 4, \ldots, 28\). We used 144 GB of RAM, which was sufficient for \(n = 28\) but not so for \(n = 29\).

We believe that, apart from the special case \(n = 3\), we always obtain the alternating group.

**Conjecture.**

For every torus size \(n \in \mathbb{N}, n \geq 4\), both the \(2^n – 2\) fair update rules, as well as the \(n\) elementary unfair update rules with a single active cell, are a generating set for the full alternating group on \(2^n\) elements, using ECA-57.

4.1.2. Example for GAP usage with \(n = 4\)

We replaced 0 by \(2^n\), since GAP only uses numbers from \(\mathbb{N}\). P00 to P03 are the permutations generated by the elementary steps \(s = 0001, 0010, 0100, \) and 1000, respectively.

```
mjv@Panda ~/GAP $ gap -b
gap> P00 := (16,1) (2,3) (4,5) (6,7) (10,11) (14,15);
   (1,16) (2,3) (4,5) (6,7) (10,11) (14,15)
gap> P01 := (16,2) (4,6) (5,7) (8,10) (12,14) (13,15);
   (2,16) (4,6) (5,7) (8,10) (12,14) (13,15)
gap> P02 := (16,4) (1,5) (8,12) (9,13) (10,14) (11,15);
   (1,5) (4,16) (8,12) (9,13) (10,14) (11,15)
gap> P03 := (16,8) (1,9) (2,10) (3,11) (5,13) (7,15);
   (1,9) (2,10) (3,11) (5,13) (7,15) (8,16)
gap> G04 := Group(P00,P01,P02,P03);
gap> IsNaturalAlternatingGroup(G04);
true
gap> Size(G04);
10461394944000
```
4.1.3. Example for GAP usage with $n = 5$

mjv@Panda ~/GAP $cat G5
P00 := (32,1) (2,3) (4,5) (6,7) (8,9) (10,11) (12,13) (14,15)
(18,19) (22,23) (26,27) (30,31);
P01 := (32,2) (4,6) (5,7) (8,10) (12,14) (13,15) (16,18)
(20,22) (21,23) (24,26) (28,30) (29,31);
P02 := (32,4) (1,5) (8,12) (9,13) (10,14) (11,15) (16,20)
(17,21) (24,28) (25,29) (26,30) (27,31);
P03 := (32,8) (1,9) (2,10) (3,11) (16,24)
(17,25) (18,26) (19,27) (20,28) (21,29) (22,30) (23,31);
P04 := (32,16) (1,17) (2,18) (3,19) (4,20) (5,21) (6,22)
(7,23) (9,25) (11,27) (13,29) (15,31);
G05 := Group(P00, P01, P02, P03, P04);

mjv@Panda ~/GAP $gap -b
gap> Read("G5");
gap> IsNaturalAlternatingGroup(G05);
true
gap> Size(G05);
131565418466846765083609006080000000

gap> quit;

Here, $1.315\ldots \cdot 10^{35} = 32!/2$.

4.1.4. Example for GAP usage with $n = 28$

We define the permutations P00 to P27 via a C++ program; see Appendix A. Using it like gap.out 28 > G28, its output, the file G28, is then read in by GAP.

mjv@turing:~/var/GAP$ gap -b -m 140G
gap> Runtime(); Read("G28"); Runtime(); IsAlternatingGroup(G28); Runtime();
IsNaturalAlternatingGroup(G28); Runtime();
5592
1639756
true
8915688
true
8915688

gap>

Times are in milliseconds. Therefore, reading in the 56 GB of permutations P00 to P27 generating the group G28 took 1640 seconds, or about half an hour, while actually checking the resulting group for $A_{28}$ took another 7280 seconds or 2 hours. The same procedure for G29 resulted in lack of memory; 140 GB of RAM are not sufficient.
Recall that, according to Stirling’s formula, $A_{228}$ has $2^{228}/2 \approx \left(\frac{2^{228}}{e}\right) \approx 10^{109}$ elements, a number with about 1 billion (1000$^3$) digits. Amazingly, GAP gets it done!

4.2. Examples for bijective functions

4.2.1. Multiplication by 9 mod 16

We give two realizations of the function $byNine : x \mapsto 9x$ mod 16 in Table 2.

Observe that the first 23 steps only implement the permutation $(08)(2A)$, consisting of two transpositions. The final 24th step is almost identical to the whole function.

| First realization in hexadecimal | 1. Active cells | 2. Active cells | Second realization in hexadecimal |
|---------------------------------|----------------|----------------|----------------------------------|
| 0xFEDCBA9876543210              | 0101           | 0100           | 0xFEDCBA9876543210 (Id)          |
| 0xABA98EFD67012345              | 1010           | 1000           | 0xBA98EFD676103254               |
| 0x0312C57E4DA98B6F              | 0101           | 0001           | 0x32107E5CF698B4D                 |
| 0x524306B19FDCB7A               | 1000           | 1000           | 0x23016F4CE798ABD5               |
| 0x0DA8B0063175CEF2              | 0001           | 0001           | 0xBA89674CEF10235D               |
| 0x0B5A18729064C6E3              | 0010           | 1010           | 0xBA89765C6E01324D               |
| 0x0BF781A0509246EDC3            | 0100           | 0101           | 0x30214DFE5CA98B67               |
| 0x17F7C5E14D206A983             | 0100           | 0100           | 0x253491AB08FDEC75               |
| 0x07FCD94E5A86210               | 0001           | 0001           | 0x073691B2AD5FC85                |
| 0x26ECDF954B87301A              | 1010           | 1010           | 0x1627908A3BDCE4F5               |
| 0x08CE751F632DBA90              | 0101           | 0101           | 0x948DA2B3CE56F                  |
| 0xC18864047239EFD5              | 1010           | 1010           | 0xD1C94F3E268B07A                 |
| 0x0B9234A6D0B1C57F              | 0101           | 0101           | 0x79E165BFC8423AD0                |
| 0x0B3211F759CE4806A             | 1010           | 1010           | 0x6DB4708A8C132F95                |
| 0x17B895DF1E6C2A40              | 0100           | 0100           | 0x4736D4C0E98B51F                 |
| 0x37FCD19B5A862E04              | 1000           | 0010           | 0x07369E842ADFC15B                |
| 0x0BF7C5913D206A884             | 0100           | 0101           | 0x253491CA698DE17B                |
| 0x0FB78D539246EAC0              | 0010           | 1010           | 0x3021D8F759CA98B4E               |
| 0x0DB5A1F739064C8E2             | 0101           | 0001           | 0xBA89725DFE01364C               |
| 0x0DA8B0E629175C8F3             | 1000           | 1000           | 0xBA8963DE510275C                 |
| 0x5243E6A19FDC07B               | 0101           | 0001           | 0x23016B45E798AFDC                |
| 0x0312CB7F4DA9856E              | 1010           | 1000           | 0x32107A54F698B6DC                |
| 0x0AB983D567012D6C              | 0101           | 0100           | 0xBA89F20D761038C                 |
| 0x0FEDC829076543A18             | 1000           | 1000           | 0x0FEDC829076543A18 (08)(2A)      |
| 0x7E5C3A18F6D4B290              | 0101           | 0100           | 0x7E5C3A18F6D4B290                |

Table 2. Multiplication $x \mapsto 9x$ in $\mathbb{Z}/16\mathbb{Z}$.
Hence by $\text{Nine} = (19)(3B)(5D)(7F) = [(08)(2A)][(08)(19)(2A)(3B)(5D)(7F)]$, where the first bracket requires 23 steps, and the second is the elementary rule 1000 (only the leftmost cell is active). The difference between the two realizations (sequences $s^{(1,...,24)}$ in hex), where the outer parentheses are inverses of each other and the inner part is self-inverse, and their difference:

1.: $(5A581248)1(A5A5A)4(842185A5)8$

Diff 124–81A—-A18–421-

2.: $(48181A52)1(A5A5A)4(25A18184)8$

| Active cells | FED...210 → 62C...931 some patterns in binary | All values F...0 in hexadecimal |
|--------------|---------------------------------------------|---------------------------------|
| 0001         | 1111 1110 1101 ... 0010 0001 0000          | 0xFEDCBA9876543210              |
| 1010         | 1110 1111 1101 ... 0011 0000 0001          | 0xEFDCAB9867452301              |
| 0101         | 1100 0101 0111 ... 1011 1010 1001          | 0xC51E03124DF88BA9              |
| 1010         | 1000 0000 0110 ... 1110 1111 1011          | 0xB06E5243197ACEFD              |
| 0101         | 0010 1010 0100 ... 1100 0101 0111          | 0x2A43F86B91D0BC57              |
| 1000         | 0011 1111 0001 ... 1000 0000 0110          | 0x3F12AC76D495B806              |
| 0100         | 1011 0111 1001 ... 0000 1000 0110          | 0xB79A2CF6541D3086              |
| 1000         | 1111 0111 1101 ... 0100 1100 0110          | 0xF7DE28BA105934C6              |
| 0001         | 0111 1111 0101 ... 0100 1100 0110          | 0x7F5E0A3298D1B4C6              |
| 1000         | 0110 1110 0100 ... 0101 1100 0111          | 0x6E4F12398D05AC7               |
| 0101         | 0110 1110 0100 ... 1101 1100 1111          | 0x6E4739AB10582DCF              |
| 1010         | 0110 1110 0000 ... 1001 1000 1101          | 0x6A073DE541C298B               |
| 0100         | 0100 0000 1010 ... 0001 0010 0011          | 0x40ADB7C5F69E8123              |
| 1010         | 0000 0100 1110 ... 0101 0010 0011          | 0x04E9F781B6D7AC52              |
| 0001         | 1010 0110 1100 ... 1111 1000 1101          | 0xA6C15D293470EF8B              |
| 1000         | 0111 0111 1100 ... 1110 1000 1101          | 0x6B7C0D432561FE8A              |
| 0101         | 0011 1111 1100 ... 1110 0000 0010          | 0x3FC845B1AD697E02              |
| 1000         | 0010 1010 1000 ... 1011 0101 0111          | 0x2F81C4E975D6B53              |
| 0100         | 1010 0010 0000 ... 0011 1101 1011          | 0xA20C9B471F563DB              |
| 1000         | 1110 0010 0100 ... 0011 0101 1111          | 0xE248DCA075B1639F              |
| 0100         | 1110 1110 0000 ... 1111 0101 0111          | 0xEA405C28FD396B17              |
| 1000         | 0000 1100 1010 ... 0101 1111 1101          | 0xC06928E31B745FD              |
| 0100         | 0100 1000 1110 ... 0001 0111 1001          | 0x48E62C3A5F701B9              |
| 0100         | 0010 1110 1100 ... 1001 0011 0001          | 0x62C478E0BF5DA931              |

Table 3. Exponentiation $x \mapsto 3^x$ in $\mathbb{F}_{17}$.
4.2.2. Exponentiation and logarithm in $\mathbb{F}_{17}$

Identifying 16 with 0b0000, we can map $\mathbb{F}^\ast_{17}$ to \{0, 1\}$^4$. Here is the exponentiation $x \mapsto 3^x$; see Table 3.

$\text{FEDCBA9876543210} \rightarrow \ldots$

$s^{(1\ldots24)} = [0001, 1010, 0101, 1010, 0101, 1000, 0100, 1000, 0001, 1000, 0100, 1010, 0100, 1010, 0001, 1000, 0101, 1000, 0100, 1000, 0100, 1010, 0100, 1010, 0101, 1000, 0101, 1010, 0100, 1000, 0001]$.

The inverse function $x \mapsto \log_3(x)$ is therefore computed by the inverse update sequence $s^{(1\ldots24)} = [1010, 0100, 1010, 0100, 1000, 0100, 1000, 0100, 1010, 0100, 1010, 0001, 1000, 0100, 1000, 0101, 1000, 0001, 1000, 0100, 1000, 0101, 1010, 0101, 1010, 0001]$.

5. Non-bijective functions

5.1. Non-bijective global rules and in-degree distributions

In Section 2.2, we have said that, in order to ensure bijectivity, we must first have a locally bijective CA, and furthermore, no two adjacent cells may fire simultaneously. Here is why:

Example.

We start with the effect of $AS_i \equiv \equiv$ for the locally bijective ECA-57. We consider four adjacent cells and the effect of $\equiv$ between the two middle cells, which are thus updated simultaneously, $s = 0110$.

$v \mapsto \text{ECA-57}_{0\equiv}(v)$

| $v$  | $\text{ECA-57}_{0\equiv}(v)$ |
|------|-----------------------------|
| 0000 | 0110                        |
| 0010 | 0000                        |
| 0100 | 0101                        |
| 0110 | 0100                        |

We obtain the patterns 0011 and 0101 twice, while missing 0001 and 0111. Hence the image is smaller than the full $2^4$ by 2 or by a factor 7/8 in general.

We have the following in-degree distribution (loss pattern 4 of [10], Table 8):|

| #(domain) | #(range) | In-degree | Distribution $\varphi$ |
|-----------|----------|-----------|------------------------|
| 12        | 12       | 1         | $\varphi(1) = 12$      |
| 4         | 2        | 2         | $\varphi(2) = 2$       |
| 0         | 2        | 0         | $\varphi(0) = 2$       |

Twelve patterns map bijectively (1,1) to 12 patterns, 4 patterns map 2:1 to 2 patterns, while 2 patterns of the range are not met at all (0,1).
We have \( \#(\text{domain}) = \text{in-degree} \times \#(\text{range}) \) for every in-degree. Also, the overall sum is
\[
\sum \#(\text{domain}) = \sum \text{in-degree} \times \#(\text{range}) = 2^n.
\]

Any additional simultaneous firing may increase the losses. Additional in-degree distributions (loss patterns) for ECAs and QCAs may be found in ([10], Table 8).

We restate Theorem 3 (i) from [14]:

**Theorem 2.**

We assume an ECA that generates at least the alternating group \( A_{2^n} \), when using temporal rules from \( \{<, >\} \setminus \{<^n, >^n\} \). Let \( f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n \) be any non-bijective function on \( n \) symbols, where we require \( n \geq 4 \) for ECA.

Let \( \#(w) = |\{v | f(v) = w\}| \) be the number of configurations \( v \) leading to configuration \( w \). Then \( f \) is clocking-computable by an ECA with in-degree distribution \( \phi(0, 1, 2) = (12, 2, 2) \) or \( (24, 4, 4) \), if and only if
\[
\sum_{w \in \mathbb{Z}_2^n} \left\lfloor \frac{\#(w)}{2} \right\rfloor \geq \phi(2) \cdot 2^n / \sum_{k=0}^{n} \phi(k).
\]

**Proof.** See [14], Theorem 3. \qed

### 5.2. Algorithm for non-bijective functions

The computation of a non-bijective function can be decomposed into 3 steps:

**Step I.** We start in the middle: Shrink the \( 2^n \) singletons of the domain to the desired distribution on the range. Find a sequence of elementary steps, necessarily including non-bijective ones, which generates the same distribution of counts in the image space as for the original function. Usually there are more than one of these sequences with the same complexity. Values/patterns as such do not yet play a role; therefore the requirements are usually easy to meet in a variety of ways, and the sequence of necessary elementary steps is short.

**Step II.** Bijectively map the input of the original function to the input of any of the results of Step I in such a way that the desired image counts are matched. Only the occurrence counts must be observed, while, within this restriction, pattern values can be permuted.

**Step III.** Bijectively map the output from Step I to the output of the desired function, considering the flow within Steps II and I in that order. Now all values matter and cannot be permuted. However, this step takes place on a subset of size \( |\text{Im}(f)| \) instead of \( 2^n \).

In Steps II and III we use the meet-in-the-middle approach, applying update rules starting both from the input (identity) and the desired function values to take advantage of the birthday paradox: In this way, \( 2 \cdot \sqrt{\#} \) patterns are sufficient to generate \( \sqrt{\#} \cdot \sqrt{\#} = \# \) potential matches in the middle (here \( \# = |A_{2^n}| \)), a considerable saving in both space and time.
5.3. Example of a non-bijective function
multiplication up to $3 \times 3 = 9$ on the torus of size $n = 4$

The 4-bit input is interpreted as a pair of numbers from the set $\{0, 1, 2, 3\}$, whose product, the output, lies within the set $\{0, 1, 2, 3, 4, 6, 9\}$.

| In  | Out | In  | Out | In  | Out | In  | Out |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 00.00 | 0000 | 01.00 | 0000 | 10.00 | 0000 | 11.00 | 0000 |
| 00.01 | 0000 | 01.01 | 0001 | 10.01 | 0010 | 11.01 | 0011 |
| 00.10 | 0000 | 01.10 | 0010 | 10.10 | 0100 | 11.10 | 0110 |
| 00.11 | 0000 | 01.11 | 0011 | 10.11 | 0110 | 11.11 | 1001 |

5.3.1. Step I

The image consists of 7 patterns with occurrence counts $7$ (0000), $2$ (0010,0011,0110), and $1$ (0001,0100,1001).

In Step I, we thus have to shrink the set of patterns present from 16 to 7 in such a way that (any) 7 patterns are mapped to the same one, and additionally 3 sets of two patterns are joined within each set. Finally, the three remaining input patterns stay on their own.

An exhaustive search over sequences with 4 update rules, starting and ending with a non-bijective one, yields 72 such sequences with the desired shrinking factor, one example is the sequence $s^{(1,\ldots,4)} = 0011, 1101, 0110, 0111$ of active cells.

| All patterns in hex | Active cells | Occurrence counts |
|---------------------|--------------|-------------------|
| 0123.4567.89AB.CDEF | 0011         | 0 0 0 0 0         |
| 3012.7654.A99A.EFDC | 1101         | 14 2 0 0 0        |
| ADCB.E781.7557.B218 | 0110         | 5 4 1 0 0        |
| CBAP.85E5.5335.F05E | 0111         | 5 3 0 1 0        |
| AED8.E292.2222.8729 |              | 3 3 0 0 1        |

5.3.2. Step II

After Step I, the output pattern 2 has frequency 7. Therefore, 2 has to match 0 in Step III, while 0,1,2,3,4,8,C are matched to 5,7,8,9,A,B,E in any order in Step II. Thus, we already have $7!$ different possibilities for Step II.

Similarly, we can map 0110 and 1001 to either $\{1, 4\}, \{3, C\}$, or $\{6, F\}$ and so forth. Let $c$ run through the occurrence counts, here $c = 7$, 2 and 1 are actually taken, and let then $p(c)$ be the number of output patterns with in-degree $c$, here $p(7) = 1$, $p(2) = p(1) = 3$. We get
\[
\prod_c c^{l(p(c))}p(c)! = 7! \cdot 1! \cdot (2!)^3! \cdot (1!)^3! = 1451520
\]

bijective functions consistent with the count distribution to choose from in Step II. One of these is given in the left column of Table 4.

5.3.3. Step III

The outcome of Steps I + II completely fixes the necessary permutation for Step III. However, we do not have to deal with \(2^n\) values, but only \(|\text{Im}(f)|\) are relevant, in our example 7 instead of 11.

| Step II | Step I | Step III |
|--------|--------|----------|
| 0xFEDCBA9876543210 | 0001 | 0x0123456789ABCDEF | 0011 | 0x2ABE879D | 0001 |
| 0xEFDCAB9876543201 | 1010 | 0x30127654A99AEFDC | 1101 | 0x3BFE869D | 0010 |
| 0xC57E03124D6F89A9 | 0100 | 0xADCB7817557B218 | 0110 | 0x3BDA49F | 0100 |
| 0x817A4352096BCFED | 0010 | 0xCBFA85E5315F05E | 0111 | 0x3F9E0DB | 1010 |
| 0xA15863702948EDCF | 0001 | 0xAED8E2922228729 | 0x851CA73 | 0101 |
| 0xB0487261395AFDCE | 1010 | 0xBE0F4862 | 0x2C6A4D4 | 0101 |
| 0x3A62D84981F057EC | 0100 | 0x3837F915 | 0x3A35D917 | 0101 |
| 0x3B629C0DF58417A8 | 0010 | 0x3F9E0DB | 0x7A81546 | 0100 |
| 0x3C409E2F7B6158A | 0100 | 0xF209D46 | 0x7ECS106 | 1010 |
| 0x3804DA2B97F651CE | 1010 | 0x2B3F690A854DE17C | 02DEFC5A9 | 1010 |
| 0xB2A670831D54F9EC | 0001 | 0x2F3B6D4EC109A578 | 0x3BDA49F | 0100 |
| 0xA15863702948EDCF | 0001 | 0xAED8E2922228729 | 0x851CA73 | 0101 |
| 0xB0487261395AFDCE | 1010 | 0xBE0F4862 | 0x2C6A4D4 | 0101 |
| 0x3A62D84981F057EC | 0100 | 0x3837F915 | 0x3A35D917 | 0101 |
| 0x3B629C0DF58417A8 | 0010 | 0x3F9E0DB | 0x7A81546 | 0100 |
| 0x3C409E2F7B6158A | 0100 | 0xF209D46 | 0x7ECS106 | 1010 |
| 0x3804DA2B97F651CE | 1010 | 0x2B3F690A854DE17C | 02DEFC5A9 | 1010 |
| 0xB2A670831D54F9EC | 0001 | 0x2F3B6D4EC109A578 | 0x3BDA49F | 0100 |

Table 4. Steps II, I, and III for multiplication up to \(3 \times 3 = 9\).

| IN | Step II | Step I | Step III | IN | Step II | Step I | Step III |
|----|---------|--------|----------|----|---------|--------|----------|
| 0010 | 5       | 2      | 0       | 0110 | 1       | E      | 2        |
| 0001 | 7       | 2      | 0       | 1001 | 4       | E      | 2        |
| 0000 | 8       | 2      | 0       | 1101 | 3       | 8      | 3        |
| 0100 | 9       | 2      | 0       | 0111 | C       | 8      | 3        |
| 0011 | A       | 2      | 0       | 1010 | D       | 7      | 4        |
| 1100 | B       | 2      | 0       | 1011 | 6       | 9      | 6        |
| 0100 | E       | 2      | 0       | 1110 | F       | 9      | 6        |
| 0101 | 0       | A      | 1       | 1111 | 2       | D      | 9        |

Table 5. Steps II, I, and III by output patterns.
16. Again, meet-in-the-middle yields a match. Be aware that the necessary effort does not drop from 16! to 7! but only to \( \frac{16!}{(16-7)!} = 11! \) or \( \frac{2^n}{(2^m-1)!} \) in general.

In total, 13 + 4 + 15 = 32 steps are required to compute this multiplication function (Table 5).

6. Efficiency

In the case of unfair bijective functions, any subset of cells may fire simultaneously, provided that no adjacent cells are contained in the set.

We define local efficiency of a bijective update sequence by two properties:

- No cell is active during two consecutive time steps (these two would cancel each other).
- No active cell can be moved to the previous time step.

Global efficiency—which shortest update sequence generates a certain function/permutation—is beyond the scope of this paper (it is dealt with implicitly by brute force in a breadth-first manner).

Lemma 3.

A sequence of rules is bijective and efficient, if \( s_k(t) = 1 \) (cell \( c_k \) active in step \( t \)) implies that.

i. \( s_{k-1}(t) = 0 \) and \( s_{k+1}(t) = 0 \) (both inactive),

ii. \( s_k(t-1) = 0 \) (inactive), and.

iii. at least one of \( s_{k-1}(t-1) \) and \( s_{k+1}(t-1) \) equals 1, active.

Proof.

(i) is required for and then ensures bijectivity.

If \( s_k(t-1) \) was active, both \( s_{k-1}(t-1) \) and \( s_k(t) \) could be removed, since each single-location action is an involution. Thus (ii) is required.

If \( s_{k-1}(t-1) \) and \( s_{k+1}(t-1) \) were both inactive, the active cell \( s_k(t) \) could be moved to \( s_k(t-1) \), maintaining bijectivity. Hence (iii) is required.

Using this lemma, we now can compute an upper bound on the number of bijective functions realizable with \( t \) update steps: We define a matrix \( A \) with rows and columns indexed by the bijective elementary update rules, here named \( s(1), s(2), \ldots \). Set \( a_{ij} := 1 \), if rule \( s(i) \) followed by rule \( s(j) \) is efficient, \( a_{ij} := 0 \) otherwise.

Let \( \lambda(n) \) be the largest eigenvalue of \( A \).
Example

\[
A_{ij} \text{ for } n = 5:
\]

| \(s(i) \setminus s(j)\) | \(s(1)\) | \(s(2)\) | \(s(3)\) | \(s(4)\) | \(s(5)\) | \(s(6)\) | \(s(7)\) | \(s(8)\) | \(s(9)\) | \(s(10)\) |
|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(s(1) = 00001\)    | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 0     |
| \(s(2) = 00010\)    | 1     | 0     | 1     | 1     | 0     | 0     | 0     | 0     | 0     | 0     |
| \(s(3) = 00100\)    | 0     | 1     | 0     | 0     | 1     | 0     | 1     | 0     | 0     | 0     |
| \(s(4) = 00101\)    | 0     | 1     | 0     | 0     | 1     | 0     | 1     | 1     | 1     | 0     |
| \(s(5) = 01000\)    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 1     | 0     | 1     |
| \(s(6) = 01001\)    | 0     | 1     | 1     | 0     | 0     | 0     | 0     | 1     | 1     | 1     |
| \(s(7) = 01010\)    | 1     | 0     | 1     | 1     | 0     | 0     | 0     | 1     | 0     | 1     |
| \(s(8) = 10,000\)   | 1     | 0     | 0     | 0     | 1     | 1     | 0     | 0     | 0     | 0     |
| \(s(9) = 10,010\)   | 1     | 0     | 1     | 1     | 1     | 1     | 0     | 0     | 0     | 0     |
| \(s(10) = 10,100\)  | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 0     | 0     |

Observe that 10,100 followed by 00001 is efficient, while the other way round 00001 leaves room to move \(s_2^{(2)}\) up to \(s_2^{(1)}\). Hence, \(A\) is not symmetrical for \(n \geq 5\).

In Table 6, we denote several figures describing the number of efficient bijective update schemes (including unfair ones). The number of rules is denoted by \(\#S(n)\); it can be found as A001610 in OEIS [18]; see also Section 2.2.2. The number of nonzero entries in the matrix \(A\) is \(\#A(n)\). The quotient \(\#A(n)/\#S(n)\) is the average number of 1’s in each row/column.

We denote by \(\#_n(t)\) the number of effective bijective update schemes of length \(t\) on a torus of size \(n\). Essentially, \(\#_n(t) = \Theta(\lambda(n)^t)\).

Let \(\rho(n) = \lim_{t \to \infty} \frac{\#_n(t)}{\lambda(n)^t}\). Then \(\rho(n)\) gives the constant hidden in the Landau symbol \(\Theta\).

We have \(\#_n(t)/(\lambda(n)^t \cdot \rho(n)) \to 1\).

Finally,

\[
T(n) = \log_{\lambda(n)}(2^n/(2 \cdot \rho(n)))
\]

is the number of steps necessary such that \(\#(T(n)) = |A_2| = 2^n/2\). Invoking the birthday paradox by the meet-in-the-middle approach, we therefore expect to require \(T(n)/2\) steps each, starting from the initial identity permutation and backwards from the desired function, to achieve a match yielding the function evaluation.

The asymptotic behavior, as far as we can infer from the range \(n = 3, \ldots, 24\), is

\[
\#S(n) = \Theta(q^n), q = 1.618…
\]

\[
\lambda(n) = \Theta(1.3^n)
\]
We can compare the asymptotic number $T(n)$ of elementary steps necessary for unfair update rules with the lower bound on the (full) steps for fair rules.

**Lemma 4 ([10], Section 5, Lemma 1(i)).**

There are clocking-computable bijective functions that require at least

$$[\log (2^n!)/\log (2^n - 2)] = 2^n + O(1)$$

steps or evaluations per cell.

Remarkably, as far as the experimental results for $n \leq 24$ indicate, the number $T(n)$ of elementary steps is only a constant factor ($\approx 2.4$) higher than the lower bound for full update steps.

Table 6. Efficiency related values for $n = 3, \ldots, 24$.

| $n$ | $\#S(n)$ | $\#A(n)$ | $|A(n)|_{180/0}$ | $\lambda(n)$ | $\rho(n)$ | $[\log(1A_{2^1})]$ | $[T(n)]$ |
|-----|--------|--------|----------------|-------------|---------|----------------|--------|
| 3   | 3      | 6      | 2              | 2.000       | 1.500   | 10             | 14     |
| 4   | 6      | 18     | 3              | 3.000       | 2.000   | 30             | 27     |
| 5   | 10     | 40     | 4              | 3.732       | 2.887   | 81             | 61     |
| 6   | 17     | 98     | 5              | 5.000       | 4.000   | 204            | 126    |
| 7   | 28     | 224    | 8              | 6.236       | 6.261   | 495            | 270    |
| 8   | 46     | 514    | 11             | 8.522       | 7.044   | 1166           | 544    |
| 9   | 75     | 1158   | 15             | 10.697      | 11.581  | 2685           | 1132   |
| 10  | 122    | 2602   | 21             | 14.599      | 11.142  | 6077           | 2266   |
| 11  | 198    | 5808   | 29             | 18.288      | 20.464  | 13,571         | 4668   |
| 12  | 321    | 12,930 | 40             | 24.941      | 17.141  | 29,978         | 9319   |
| 13  | 520    | 28,704 | 55             | 31.253      | 35.107  | 65,630         | 19,065 |
| 14  | 842    | 63,618 | 75             | 42.634      | 25.696  | 142,612        | 38,002 |
| 15  | 1363   | 140,806| 103            | 53.426      | 58.716  | 307,933        | 77,402 |
| 16  | 2206   | 311,362| 141            | 72.878      | 73.982  | 661,287        | 154,189|
| 17  | 3570   | 688,024| 192            | 91.321      | 96.473  | 1,413e + 06    | 313,092|
| 18  | 5777   | 1,519,586| 263              | 124.571     | 55.699  | 3.008e + 06    | 623,547|
| 19  | 9348   | 3,354,944| 358             | 156.098     | 156.314 | 6.380e + 06    | 1.26e + 06|
| 20  | 15,126 | 7,405,058| 489             | 212.932     | 81.313  | 1.348e + 07    | 2.51e + 06|
| 21  | 24,475 | 16,341,254| 667             | 266.822     | 250.502 | 2.842e + 07    | 5.08e + 06|
| 22  | 39,602 | 36,056,154| 910             | 363.969     | 118.336 | 5.976e + 07    | 1.01e + 07|
| 23  | 64,078 | 79,547,616| 1241            | 456.085     | 397.758 | 1.253e + 08    | 2.04e + 07|
| 24  | 103,681| 175,485,442| 1692           | 622.138     | 171.942 | 2.623e + 08    | 4.07e + 07|

Table 6. Efficiency related values for $n = 3, \ldots, 24$.

$$T(n) = 2.4 \cdot 2^n \cdot (1 + o(1)).$$

We can compare the asymptotic number $T(n)$ of elementary steps necessary for unfair update rules with the lower bound on the (full) steps for fair rules.

**Lemma 4 ([10], Section 5, Lemma 1(i)).**

There are clocking-computable bijective functions that require at least

$$[\log (2^n!)/\log (2^n - 2)] = 2^n + O(1)$$

steps or evaluations per cell.

Remarkably, as far as the experimental results for $n \leq 24$ indicate, the number $T(n)$ of elementary steps is only a constant factor ($\approx 2.4$) higher than the lower bound for full update steps.
If that result remains valid for all $n \in \mathbb{N}$, and the reported values strongly suggest this, this would mean that the typical full step can be replaced by no more than 2..3 elementary unfair steps, independent of $n$.

As can be appreciated, $n = 5$ might be feasible for an attack by meet-in-the-middle, but $n = 6$ and above is certainly no candidate for this brute-force approach. Dealing with these sizes will require a more intelligent approach, for instance using group-theoretic techniques like representation theory, applied to the alternating group.

7. Conclusion

We have seen that many functions are clocking-computable, namely, the even bijective ones and the non-bijective ones with enough “loss” in their image.

The elementary cellular automaton ECA-57 can be used to implement an etherealware computing device: The computed function is a result only of the clocking order.

Temporal order of activating the CA cells is thus a new way to encode algorithms, a “volatilization of information”.

We increased the size for example programs from $n = 3$ in [10, 14] to $n = 4$ and are confident to also be able to tackle the case $n = 5$.

From $n = 6$ onwards, we shall need more mathematical concepts, e.g., from group theory.

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Appendix A. gap.cc

```c
#include <stdlib.h>
#include <iostream>
#include <fstream>
#include <iomanip>
using namespace std;

int main(int argc, char** args) {
    long long N = atoi(args[1]); // size of torus
    const long long EN = 1LL<<N; // $2^N$
```
long long ix;

cout << "P00 := (" << (1LL<<N) << ",1)";
ix = 0; // rightmost cell active, n=0
for (long long i = 1; i < EN; i++) {
    if (((i ^ 1LL) > i) (((5LL<<(N-1LL))
    & (i ^ (i<<N))) != (1LL<<(N-1LL)))) {
        cout << "(" << i << "," << (i^1LL) << ")";
        ix++;
    if ((ix%6LL) == 0) {cout << endl ;}
    }
} cout << ";" << endl;

for (long long n = 1; n < N; n++) { // active cell/bit
    long long En = 1LL<<n;
    ix = 0;
    cout << "P"<<(char) (0x30+(n/10))
    << (char) (0x30+(n%10)) << ",="
    << (1LL<<N) << "," << (1LL<<n) << ");";
    for (long long i = 1; i < EN; i++) // all 2^N patterns
        if (((i ^ En) > i) && (((0x5LL<<(n-1LL)) & (i ^ (i<<N)))
            != (1LL<<(n-1LL)))) {
            cout << "(" << i << "," << (i^En) << ")";
            ix++;
            if ((ix%6LL) == 0) {cout << endl ;}
    }
} cout << ";" << endl;

for (int n = 1; n < N; n++) {
    cout << ",P"<<(char) (0x30+(n/10)) << (char) (0x30+(n%10));
}

//remainder, what to put interactively in GAP, no use as part of file
cout << "#IsNaturalAlternatingGroup(G"<<(char) (0x30+(N/10)
    << (char) (0x30+(N/10)%10)) << ") ;" << endl;
cout << "#IsNaturalSymmetricGroup(G"<<(char) (0x30+(N/10) %10))
    << (char) (0x30+(N%10)) << ");" << endl;
cout << "#Size(G"<<(char) (0x30+(N/10)%10)
    << (char) (0x30+(N%10)) << ");" << endl;
}
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