LEFT APP-RINGS OF SKEW GENERALIZED POWER SERIES *

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Abstract
A ring $R$ is called a left APP-ring if the left annihilator $l_R(Ra)$ is right $s$-unital as an ideal of $R$ for any $a \in R$. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. The skew generalized power series ring $[[R^S, \leq, \omega]]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Malcev-Neumann Laurent series rings. We study the left APP-property of the skew generalized power series ring $[[R^S, \leq, \omega]]$. It is shown that if $(S, \leq)$ is a strictly totally ordered monoid, $\omega : S \rightarrow \text{Aut}(R)$ a monoid homomorphism and $R$ a ring satisfying descending chain condition on right annihilators, then $[[R^S, \leq, \omega]]$ is left APP if and only if for any $S$-indexed subset $A$ of $R$, the ideal $l_R(\sum_{a \in A} \sum_{s \in S} R\omega_s(a))$ is right $s$-unital.

Key Words: left APP-ring, skew generalized power series ring.

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1. Introduction and preliminaries

Throughout this paper, $R$ denotes a ring (not necessarily commutative) with unity. For a nonempty subset $X$ of $R$, $l_R(X)$ and $r_R(X)$ denote the left and right annihilator of $X$ in $R$, respectively. We will denote by $\text{End}(R)$ the monoid of ring endomorphisms of $R$, and by $\text{Aut}(R)$ the group of ring automorphisms of $R$.

Recall that a ring $R$ is a right (resp. left) PP-ring if the right (resp. left) annihilator of an element of $R$ is generated by an idempotent. The ring $R$ is called a PP-ring if it is both right and left PP. A ring $R$ is called (quasi-) Baer if the left annihilator of every nonempty subset (every left ideal) of $R$ is generated by an idempotent of $R$. For more details and examples of PP-rings, Baer rings and

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quasi-Baer rings, see [2, 3, 5, 7, 8, 10]. As a generalization of quasi-Baer rings, G.F. Birkenmeier, J.Y. Kim and J.K. Park in [6] introduced the concept of left principally quasi-Baer rings. A ring \( R \) is called left principally quasi-Baer (or simply, left p.q.-Baer) if the left annihilator of a principal left ideal of \( R \) is generated by an idempotent. Similarly, right p.q.-Baer rings can be defined. A ring \( R \) is called p.q.-Baer if it is both right and left p.q.-Baer. Observe that biregular rings and quasi-Baer rings are left p.q.-Baer.

In recent years, many researches have carried out an extensive study of rings of (skew) generalized power series (for example, P. Ribenboim [26, 27, 28], Z.K. Liu [14, 16, 17, 19], H. Kim [11, 12], R. Mazurek and M. Ziembowski [22, 23, 24, 25], and the present author [31, 32], etc.). In particular, it was shown in [32, Corollary 3.8] that if \( (S, \leq) \) is a strictly totally ordered monoid and \( R \) a ring satisfying the condition that \( ab = 0 \iff a \omega_s(b) = 0 \) for any \( a, b \in R \) and any \( s \in S \), then \( [[R^{S, \leq}, \omega]] \) is a left p.q.-Baer ring if and only if for any \( S \)-indexed set \( A \) of \( R \), \( l_R(\sum_{a \in A} Ra) \) is generated by an idempotent of \( R \). In [31, Corollary 5.5], we proved that \( [[R^{S, \leq}]] \) is a reduced PP-ring if and only if for any two countable subsets \( A \) and \( B \) of \( R \) with \( A \subseteq \text{ann}_R(B) \), there exists \( r \in \text{ann}_R(B) \) such that \( ar = a \) for all \( a \in A \).

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a least upper bound in the set of all idempotents of $R$. H. Kim and T.I. Kwon proved in [12, Theorem 2.4] that if $(S, \leq)$ is a strictly totally ordered monoid, then $[[R^{S, \leq}]]$ is a PF-ring if and only if for any two $S$-indexed subsets $A$ and $B$ of $R$ with $A \subseteq \text{ann}_R(B)$, there exists $r \in \text{ann}_R(B)$ such that $ar = a$ for all $a \in A$.

For left APP-rings, it was proved in [21, Theorem 2] that if $M$ is an ordered monoid and $\phi : M \to \text{Aut}(R)$ is a monoid homomorphism, then the skew monoid ring $R \ast M$ is a left APP-ring if and only if for any $b \in R$, $l_R(\sum_{g \in M} R\phi(g)(b))$ is pure as a left ideal of $R$. It was noted in [18, Example 2.4] that there exists a commutative von Neumann regular ring $R$ (hence left APP), but the ring $R[[x]]$ is not APP. In [20, Theorem 2], it was shown that if $R$ is a ring satisfying descending chain condition on right annihilators then $R[[x, \alpha]]$ is a left APP-ring if and only if for any sequence $(b_0, b_1, \ldots)$ of elements of $R$ the ideal $l_R(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} R\alpha^k(b_j))$ is pure as a left ideal of $R$, where $\alpha \in \text{Aut}(R)$.

In this note, we will consider left APP-property of skew generalized power series rings. We will show that if $(S, \leq)$ is a strictly totally ordered monoid, $\omega : S \to \text{Aut}(R)$ a monoid homomorphism and $R$ is a ring satisfying descending chain condition on right annihilators, then $[[R^{S, \leq}, \omega]]$ is left APP if and only for any $S$-indexed subset $A$ of $R$, the ideal $l_R(\sum_{a \in A} \sum_{s \in S} R\omega_s(a))$ is pure as a left ideal of $R$.

In order to recall the skew generalized power series ring construction, we need some definitions. Let $(S, \leq)$ be a partially ordered set. Recalled that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Let $S$ be a commutative monoid. Unless stated otherwise, the operation of $S$ shall be denoted additively, and the neutral element by $0$. The following definition is due to [28, 19] and [25].

Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid such that if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and $\omega : S \to \text{End}(R)$ a monoid homomorphism. For any $s \in S$, let $\omega_s$ denote the image of $s$ under $\omega$, that is $\omega_s = \omega(s)$. Consider the set $A$ of all maps $f : S \to R$ whose support $\text{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$$

is finite. This fact allows to define the operation of convolution as follows:

$$(fg)(s) = \sum_{(u, v) \in X_s(f, g)} f(u)\omega_u(g(v)), \quad \text{if} \quad X_s(f, g) \neq \emptyset$$

and $(fg)(s) = 0$ if $X_s(f, g) = \emptyset$. With this operation and pointwise addition, $A$ becomes a ring, which is called the ring of skew generalized power series with coefficients in $R$ and exponents in $S$, and we denote by $[[R^{S, \leq}, \omega]]$.

The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Malcev-Neumann Laurent series rings and of course the “untwisted” versions of all of these.
If \((S, \leq)\) is a strictly totally ordered monoid and \(0 \neq f \in [[R^{S}, \leq}, \omega]],\) then \(\text{supp}(f)\) is a nonempty well-ordered subset of \(S\). We denote \(\pi(f)\) the smallest element of \(\text{supp}(f)\). To any \(r \in R\) and any \(s \in S\) we associated the maps \(\lambda^s_r \in [[R^{S}, \leq}, \omega]]\) defined by
\[
\lambda^s_r(t) = \begin{cases} 
  r, & t = s, \\
  0, & t \neq s,
\end{cases} \quad t \in S.
\]
In particular, denote \(c_r = \lambda^0_r, e_s = \lambda^s_s\). It is clear that \(r \mapsto c_r\) is a ring embedding of \(R\) into \([[R^{S}, \leq}, \omega]]\), \(s \mapsto e_s\) is a monoid embedding of \(S\) into the multiplicative monoid of ring \([[R^{S}, \leq}, \omega]]\), and \(\lambda^s_r e_s, e_r c_r = c_{\omega_s(r)} e_s\).

2. Main Results

An ideal \(I\) of \(R\) is said to be right \(s\)-unital if, for each \(a \in I\) there exists an element \(x \in I\) such that \(ax = a\). Note that if \(I\) and \(J\) are right \(s\)-unital ideals, then so is \(I \cap J\) (if \(a \in I \cap J\), then \(a \in aI \subseteq a(I \cap J)\)). It follows from [30, Theorem 1] that \(I\) is right \(s\)-unital if and only if for any finitely many elements \(a_1, a_2, \ldots, a_n \in I\) there exists an element \(x \in I\) such that \(a_i = ax, i = 1, 2, \ldots, n\). A submonoid \(N\) of a left \(R\)-module \(M\) is called a pure submodule if \(L \otimes_R N \rightarrow L \otimes_R M\) is a monomorphism for every right \(R\)-module \(L\). By [29, Proposition 11.3.13], an ideal \(I\) is right \(s\)-unital if and only if \(R/I\) is flat as a left \(R\)-module if and only if \(I\) is pure as a left ideal of \(R\).

By [18], a ring \(R\) is called a left APP-ring if the left annihilator \(l_R(Ra)\) is right \(s\)-unital as an ideal of \(R\) for any element \(a \in R\).

Right APP-rings may be defined analogously. Clearly every left p.q.-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings). If \(R\) is a commutative ring, then \(R\) is APP if and only \(R\) is FP. From [18, Proposition 2.3] it follows that right PP-rings are left APP and left APP-rings are quasi-Armendariz in the sense that whenever \(f(x) = a_0 + a_1 x + \cdots + a_m x^n, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]\) satisfy \(f(x)R[x]g(x) = 0\), we have \(a_i Rb_j = 0\) for each \(i\) and \(j\) (see, for example [9]). For more details on left APP-rings, see [18, 9].

**Lemma 1.** Let \((S, \leq)\) be a strictly totally ordered monoid and \(\omega : S \rightarrow \text{Aut}(R)\) a monoid homomorphism. If \(l_R(\sum_{s \in S} R\omega_s(a))\) is right \(s\)-unital for any \(a \in R\), then for any \(f, g \in [[R^{S}, \leq}, \omega]]\) satisfy \(g([[R^{S}, \leq}, \omega]])f = 0, g(u)\omega_u(R\omega_s(f(v))) = 0\) for any \(u, v, s \in S\).

**Proof.** Let \(0 \neq f, g \in [[R^{S}, \leq}, \omega]]\) be such that \(g([[R^{S}, \leq}, \omega]])f = 0\). Assume that \(\pi(g) = u_0\) and \(\pi(f) = v_0\). Then for any \((u, v) \in X_{u_0 + v_0}\), \(u_0 \leq u, v_0 \leq v\). If \(u_0 < u\), since \(\leq\) is a strict order, \(u_0 + v_0 < u + v_0 \leq u + v = u_0 + v_0\), a contradiction. Thus \(u = u_0\). Similarly, \(v = v_0\). Hence, for any \(r \in R\) and any \(s \in S\),
\[
0 = (g\lambda^{s}_r f)(u_0 + s + v_0) = \sum_{(u, v) \in X_{u_0 + s + v_0}} g(u)\omega_u(r\omega_s(f(v)))
\]
\[
= g(u_0)\omega_{u_0}(r\omega_s(f(v_0))).
\]
Now let \(w \in S\) with \(u_0 + v_0 \leq w\). Assume that for any \(u \in \text{supp}(g)\) and any \(v \in \text{supp}(f)\), if \(u + v < w\), then \(g(u)\omega_u(R\omega_s(f(v))) = 0\) for any \(s \in S\). We will show
that \( g(u)\omega_u(R\omega_s(f(v))) = 0 \) for any \( s \in S \), any \( u \in \text{supp}(\phi) \) and any \( v \in \text{supp}(f) \) with \( u + v = w \). For convenience, we write

\[
X_u(g, f) = \{(u, v_i) \mid i = 1, 2, \ldots, n\}
\]

with \( v_1 < v_2 < \cdots < v_n \) (Note that if \( v_1 = v_2 \), then from \( u_1 + v_1 = u_2 + v_2 \) it follows that \( u_1 = u_2 \), and thus \( (u_1, v_1) = (u_2, v_2) \)). Then for any \( r \in R \) and any \( s \in S \),

\[
0 = (g\lambda^r sf)(s + w) = \sum_{(u, v) \in X_u(g, \lambda^r sf)} g(u)\omega_u(r\omega_s(f(v))) = \sum_{i=1}^{n} g(u_i)\omega_{u_i}(r\omega_s(f(v_i))). \tag{1}
\]

Note that \( u_i + v_i < u_i + v_i = w \) for each \( i = 2, \ldots, n \). Then by induction hypothesis, \( g(u_i)\omega_{u_i}(R\omega_i(f(v_i))) = 0 \) for any \( t \in S \) and each \( i = 2, \ldots, n \). Thus \( \omega^{-1}_{u_i}(g(u_i)) \in l_R(\sum_{t \in S} R\omega(t(f(v_i)))) \) since \( \omega_{u_i} \in \text{Aut}(R) \) for any \( i = 2, \ldots, n \). Hence there exists \( e_1 \in l_R(\sum_{t \in S} R\omega(t(f(v_i)))) \) such that \( g(u_i) = g(u_i)\omega_{u_i}(e_1) \) for \( i = 2, \ldots, n \) by the hypothesis. Let \( r^t \in R \), take \( r = e_1 r^t \) in the equation (1), we have

\[
0 = \sum_{i=1}^{n} g(u_i)\omega_{u_i}(e_1 r^t \omega_s(f(v_i))) = \sum_{i=2}^{n} g(u_i)\omega_{u_i}(r^t \omega_s(f(v_i))). \tag{2}
\]

Since \( u_i + v_2 < u_i + v_i = w \) for any \( i = 3, \ldots, n \), by hypothesis, there exists \( e_2 \in l_R(\sum_{t \in S} R\omega(t(f(v_i)))) \) such that \( g(u_i) = g(u_i)\omega_{u_i}(e_2) \) for \( i = 2, \ldots, n \). Hence take \( r' = e_2 r'' \) in (2) where \( r'' \in R \), we deduced that

\[
\sum_{i=3}^{n} g(u_i)\omega_{u_i}(r'' \omega_s(f(v_i))) = 0.
\]

Continuing in this manner yields that \( g(u_n)\omega_{u_n}(R\omega_s(f(v_n))) = 0 \) for any \( s \in S \). Consequently, for any \( s \in S \),

\[
g(u_{n-1})\omega_{u_{n-1}}(R\omega_s(f(v_{n-1}))) = 0, \ldots, g(u_1)\omega_{u_1}(R\omega_s(f(v_1))) = 0.
\]

Therefore, by transfinite induction, we have shown that \( g(u)\omega_u(R\omega_s(f(v))) = 0 \) for any \( u, v, s \in S \). \( \square \)

**Lemma 2.** Let \( (S, \leq) \) be a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. If \( [[R^{S, \leq}, \omega]] \) is a left APP-ring and \( S \) is cancellative, then \( l_R(\sum_{s \in S} R\omega_s(a)) \) is right \( s \)-unital for any \( a \in R \).

**Proof.** Let \( a \in R \) and \( b \in l_R(\sum_{s \in S} R\omega_s(a)) \). Then \( c_b[[R^{S, \leq}, \omega]]c_a = 0 \). Since \( [[R^{S, \leq}, \omega]] \) is left APP, there exists an \( h \in l_{[[R^{S, \leq}, \omega]]}([[R^{S, \leq}, \omega]]c_a) \) such that \( cb = cbh \).

Then \( b = c_b(0) = (cbh)(0) = bh(0) \) and, for any \( r \in R \), any \( s \in S \),

\[
0 = (h\lambda^s c_a)(s) = h(0)r\omega_s(a),
\]

which imply that \( l_R(\sum_{s \in S} R\omega_s(a)) \) is right \( s \)-unital for any \( a \in R \). \( \square \)
Let \((S, \leq)\) be a strictly ordered monoid and \(A\) a nonempty subset of \(R\). We will say \(A\) is \(S\)-indexed, if there exists an artinian and narrow subset \(I\) of \(S\) such that \(A\) is indexed by \(I\).

**Theorem 3.** Let \((S, \leq)\) be a strictly totally ordered monoid and \(\omega : S \to \text{Aut}(R)\) a monoid homomorphism. If \(R\) satisfies descending chain condition on right annihilators, then the following conditions are equivalent:

1. \([R^{S, \leq}, \omega]\) is a left APP-ring.
2. For any \(S\)-indexed subset \(A\) of \(R\), \(l_R\left(\sum_{a \in A} \sum_{s \in S} R\omega_s(a)\right)\) is right \(s\)-unital.

**Proof.** (2) \(\implies\) (1). Assume that \(f, g \in [R^{S, \leq}, \omega]\) are such that \(g[[R^{S, \leq}, \omega]]f = 0\). Then, by the hypothesis and Lemma 1, \(g(u)\omega_u(R\omega_s(f(v))) = 0\) for any \(u, v, s \in S\). Since \(\omega_u \in \text{Aut}(R)\), \(\omega_u^{-1}(g(u))R\omega_s(f(v)) = 0\) for any \(u, v, s \in S\). Thus for any \(u \in \text{supp}(g)\),

\[\omega_u^{-1}(g(u)) \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right).\]

Let

\[\mathcal{D} = \{r_R(Y)|Y \subseteq \{\omega_u^{-1}(g(u))|u \in \text{supp}(g)\}, |Y| < \infty\}.\]

Then \(\mathcal{D}\) is a nonempty set of right annihilators. Since \(R\) satisfies descending chain condition on right annihilators, \(\mathcal{D}\) has a minimal element, say \(r_R(Y_0)\). Assume that \(Y_0 = \{\omega_{u_1}^{-1}(g(u_1)), \omega_{u_2}^{-1}(g(u_2)), \ldots, \omega_{u_n}^{-1}(g(u_n))\}\). Then

\[\omega_{u_i}^{-1}(g(u_i)) \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right), \quad i = 1, 2, \ldots, n.\]

Thus, by (2), there exists \(e \in l_R\left(\sum_{v \in \text{supp}(f)} \sum_{s \in S} R\omega_s(f(v))\right)\) such that

\[\omega_{u_i}^{-1}(g(u_i)) = \omega_{u_i}^{-1}(g(u_i))e, \quad i = 1, 2, \ldots, n.\]

If \(\text{supp}(g) = \{u_1, u_2, \ldots, u_n\}\), then for all \(u \in \text{supp}(g), \omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e\). Now assume that \(u \in \text{supp}(g) \setminus \{u_1, u_2, \ldots, u_n\}\). Then, by the minimality of \(r_R(Y_0)\),

\[r_R(\omega_{u_1}^{-1}(g(u_1)), \ldots, \omega_{u_n}^{-1}(g(u_n)), \omega_u^{-1}(g(u))) = r_R(\omega_{u_1}^{-1}(g(u_1)), \ldots, \omega_{u_n}^{-1}(g(u_n))).\]

Thus \(\omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e\). This implies that \(\omega_u^{-1}(g(u)) = \omega_u^{-1}(g(u))e\) for any \(u \in \text{supp}(g)\). Thus for any \(h \in [R^{S, \leq}, \omega]\) and any \(t \in S\),

\[(c_e hf)(t) = \sum_{(s, v) \in X_t(h, f)} eh(s)\omega_s(f(v)) = 0,\]

and

\[(gc_e)(t) = g(t)\omega_t(e) = \omega_t(\omega_t^{-1}(g(t)))e = \omega_t(\omega_t^{-1}(g(t))) = g(t),\]

which imply that \(c_e \in l_{[R^{S, \leq}, \omega]}([R^{S, \leq}, \omega]]f)\) and \(g = gc_e\). Hence \([R^{S, \leq}, \omega]\) is a left APP-ring.
(1) $\implies$ (2). Let $A = \{a_t | t \in I\}$ be an $S$-indexed subset of $R$. Define $f \in [\mathbb{R}^{S \leq}, \omega]$ via

$$f(t) = \begin{cases} a_t, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Let $b \in l_R \left( \sum_{t \in I} \sum_{s \in S} R \omega_s(a_t) \right)$. Then $c_b \left[ [\mathbb{R}^{S \leq}, \omega] \right] f = 0$. Since $[\mathbb{R}^{S \leq}, \omega]$ is left APP, there exists an $h \in l_{[\mathbb{R}^{S \leq}, \omega]} \left( \left[ [\mathbb{R}^{S \leq}, \omega] \right] f \right)$ such that $c_b = c_b h$. Thus $b = c_b(0) = (c_b h)(0) = bh(0)$. By (1), Lemma 2 and Lemma 1, $h(u) \omega_b \left( R \omega_b(f(t)) \right) = 0$ for any $u, s, t \in S$. In particular, $h(0) R \omega_b(f(t)) = 0$ for any $s, t \in S$. This implies that $h(0) \in l_R \left( \sum_{t \in I} \sum_{s \in S} R \omega_s(f(t)) \right)$. Thus (2) holds.

**Corollary 4.** (20, Theorem 2) Let $R$ be a ring satisfying descending chain condition on right annihilators and $\alpha \in \text{Aut}(R)$. Then the following conditions are equivalent:

1. $R[[x; \alpha]]$ is a left APP-ring.
2. For any countable subset $A$ of $R$, $l_R \left( \sum_{a \in A} \sum_{i=0}^{\infty} R \alpha^i(a) \right)$ is right $s$-unital.

**Corollary 5.** Let $R$ be a ring satisfying descending chain condition on right annihilators and $\alpha \in \text{Aut}(R)$. Then the following conditions are equivalent:

1. $R[[x, x^{-1}; \alpha]]$ is a left APP-ring.
2. For any countable subset $A$ of $R$, $l_R \left( \sum_{a \in A} \sum_{i=-\infty}^{\infty} R \alpha^i(a) \right)$ is right $s$-unital.

Let $\alpha$ and $\beta$ be ring automorphisms of $R$ such that $\alpha \beta = \beta \alpha$. Let $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ (resp. $\mathbb{Z} \times \mathbb{Z}$) be endowed the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$ (resp. $\mathbb{Z}$), and define $\omega : S \to \text{Aut}(R)$ via $\omega(m, n) = \alpha^m \beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$ (resp. $m, n \in \mathbb{Z}$). Then $[\mathbb{R}^{S \leq}, \omega] = R[[x, y; \alpha, \beta]]$ (resp. $[R[[x, x^{-1}, y^{-1}; \alpha, \beta]]] \omega$), in which

$\omega \alpha^m \beta^n(b) x^m y^n = \alpha^m \beta^n(b) x^{m+p} y^{n+q}$ for any $m, n, p, q \in \mathbb{N} \cup \{0\}$ (resp. $m, n, p, q \in \mathbb{Z}$).

**Corollary 6.** Let $R$ be a ring satisfying descending chain condition on right annihilators, $\alpha$ and $\beta$ be ring automorphisms of $R$ such that $\alpha \beta = \beta \alpha$. Then the following conditions are equivalent:

1. $R[[x, y; \alpha, \beta]]$ (resp. $R[[x, y, x^{-1}, y^{-1}; \alpha, \beta]]$) is a left APP-ring.
2. For any countable subset $A$ of $R$, $l_R \left( \sum_{a \in A} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} R \alpha^i \beta^j(a) \right)$ (resp. $l_R \left( \sum_{a \in A} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} R \alpha^i \beta^j(a) \right)$) is right $s$-unital.

If $S$ the multiplicative monoid $(\mathbb{N}, \cdot)$, endowed with the usual order $\leq$, then $[\mathbb{R}^{(\mathbb{N}, \cdot), \leq}]$ is the ring of arithmetical functions with values in $R$, endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d) g(n/d), \quad \text{for each } n \geq 1.$$  

**Corollary 7.** Let $R$ be a ring satisfying descending chain condition on right annihilators. Then the following conditions are equivalent:

1. $[\mathbb{R}^{(\mathbb{N}, \cdot), \leq}]$ is a left APP-ring.
2. For any countable subset $A$ of $R$, $l_R \left( \sum_{a \in A} Ra \right)$ is right $s$-unital.
Let \((S, \leq)\) be a strictly totally ordered monoid which is also artinian. Then the set \(X_s = \{(u, v) | u + v = s, u, v \in S\}\) is finite for any \(s \in S\). Let \(V\) be a free Abelian additive group with the base consisting of elements of \(S\). It was noted in \([13]\) that \(V\) is a coalgebra over \(\mathbb{Z}\) with the comultiplication map and the counit map as follows:

\[
\Delta(s) = \sum_{(u, v) \in X_s} u \otimes v, \quad \epsilon(s) = \begin{cases} 
1, & s = 0; \\
0, & s \neq 0,
\end{cases}
\]

and \([R^{S, \leq}] \cong \text{Hom}(V, R)\), the dual algebra with multiplication

\[
f \ast g = (f \otimes g) \Delta \quad \forall f, g \in \text{Hom}(V, R).
\]

**Corollary 8.** Let \((S, \leq)\) be a strictly totally ordered monoid which is also artinian, \(R\) a ring satisfying descending chain condition on right annihilators and \(\text{Hom}(V, R)\) defined as above. Then the following conditions are equivalent:

1. \(\text{Hom}(V, R)\) is a left APP-ring.
2. For any \(S\)-indexed subset \(A\) of \(R\), \(l_R(\sum_{a \in A} Ra)\) is right \(s\)-unital.

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