Smoothness of moduli space of stable torsion-free sheaves with fixed determinant in mixed characteristic

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Abstract

Let $R$ be a complete discrete valuation ring with fraction field of characteristic 0 and algebraically closed residue field of characteristic $p > 0$. Let $X_R \to \text{Spec}(R)$ be a smooth projective morphism of relative dimension 1. We prove that, given a line bundle $L_R$ the moduli space of Gieseker stable torsion-free sheaves of rank $r \geq 2$ over $X_R$, with determinant $L_R$, is smooth over $R$.

1 Introduction

Notation 1.1. Let $R$ be a complete discrete valuation ring with maximal ideal $m$. Denote by $K$ its fraction field of characteristic 0 and by $k$ its residue field of characteristic $p > 0$. Assume $k$ is algebraically closed. Let $X_R \to \text{Spec}(R)$ be a smooth fibred surface and $X_k$ its special fibre. Fix a line bundle $L_R$ on $X_R$. Let $P$ be a fixed Hilbert polynomial. Throughout this note, semistability always refers to Gieseker semistability (see [6, Definition 1.2.4]).

In [8, Theorem 0.2], Langer proves that the moduli functor of semi(stable) torsion-free sheaves with fixed Hilbert polynomial $P$ on $X_R$ is uniformly (universally) corepresented by a $R$-scheme $M_{X_R}(P)$ (respectively $M_{X_R}^s(P)$). Recall the definition of the moduli functor of flat families of (semi)stable torsion free sheaves with fixed Hilbert polynomial $P$ and determinant $L_R$ on $X_R$ (see Definition 2.2). We denote this functor by $M_{X_R,L_R}^s$. In this note we prove the following:

Theorem 1.2 (see Proposition 2.3, Remark 2.4 and Theorem 4.5). We have the following:

1. The moduli functor $M_{X_R,L_R}$ is uniformly corepresented by a projective $R$-scheme of finite type denoted $M_{R,L_R}$. The open subfunctor $M_{X_R,L_R}^s$ for stable sheaves is universally corepresented by a $R$-scheme of finite type, denoted $M_{R,L_R}^s$.

2. The morphism $M_{R,L_R}^s \to \text{Spec}(R)$ is smooth.
Part 1 is proven analogously to [2, Theorem 3.1]. For part 2, we prove that the deformation functor at a point in the moduli space $M_{R,L,R}^s$ is unobstructed (see Theorem 3.19).

Note that Theorem 1.2 is proven by Langer in the case when $R$ is a $k$-algebra (see [7, Proposition 3.4]). However, the proof does not generalize to our setup. This is because it relies on [1, Proposition 1], the proof of which does not hold in mixed characteristic. The main difficulty is that even in the case of vector bundles it uses the structure of $R$ as a $k$-algebra in a fundamental way (see [1, Section 3]). We use the same philosophy as [1, Proposition 1] (of using Cech cohomology) but take a more direct approach since we are working on a family of curves.

The setup is as follows: in §2 we recall the basic definitions and results needed for this note. We also prove the existence of the moduli space of stable torsion free sheaves with fixed determinant over $\text{Spec}(R)$. In §3 we show that the deformation functor at a point in the moduli space $M_{R,L,R}^s$ is unobstructed. Finally in §4 we prove that this moduli space is smooth over $\text{Spec}(R)$.

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2 Basic Definitions and results

Keep Notations 1.1

In this section we define the moduli functor of (semi)stable sheaves with fixed determinant. We prove that it is uniformly corepresented by an $R$-scheme of finite type.

Definition 2.1. Let $X_R \to \text{Spec}(R)$ be as in Notation 1.1

1. Let $\mathcal{M}_{X_R/\text{Spec}(R)}(P)$ (as in [2, Theorem 3.1]) of pure Gieseker semistable sheaves. For simplicity we will denote this functor by $\mathcal{M}_R$ and the corresponding moduli space by $M_R$. Denote by $\mathcal{P}ic_{X_R}$ the moduli functor for line bundles. By assumption $X_R \to \text{Spec}(R)$ is flat, projective with integral fibres, therefore by [3, Theorem 9.4.8] the functor $\mathcal{P}ic_{X_R}$ is representable. We denote this moduli space by $\text{Pic}(X_R)$.

2. By assumption $X_R$ is smooth over $R$. By [1, Theorem 2.1.10], every coherent sheaf $\mathcal{E}$ on $X_R$ admits a locally free resolution

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{E} \to 0.$$  

Then $\det(\mathcal{E}) := \otimes \text{det}(\mathcal{E}_i)(-1)^i$.

Therefore we can define a natural transformation $\det : \mathcal{M}_R \to \mathcal{P}ic_{X_R}$. This induces a morphism between the schemes corepresenting these functors $M_R \to \text{Pic}(X_R)$. 

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Now we define the moduli functor for families of pure Gieseker semistable sheaves with fixed determinant.

**Definition 2.2.** Let $X_R \to \text{Spec}(R)$ be a smooth, projective morphism and $L_R$ a line bundle on $X_R$. For a fixed Hilbert polynomial, we define the moduli functor $\mathcal{M}_{X_R,L_R}^X(P)$, denoted $\mathcal{M}_{R,L_R}$ for simplicity, on $X_R$ of sheaves with fixed determinant $L_R$. Let $\mathcal{M}_{X_R,L_R}^X : (\text{Sch}/R)^o \to (\text{Sets})$ be such that for an $R$-scheme $T$,

$$
\mathcal{M}_{X_R,L_R}^X(T) := \left\{ \begin{array}{l}
\text{S-equivalence classes of families of pure Gieseker semistable sheaves } F \text{ on } X_T \\
\text{with the property that } \det(F) \simeq \pi_X^* L_R \otimes \pi_T^* Q,
\end{array} \right\} / \sim
$$

where $\pi_{X_R} : X_T \to X_R$ and $\pi_T : X_T \to T$ are the natural projection maps and $F \sim F'$, if and only if there exists a line bundle $L$ on $T$, such that $F \simeq F' \otimes \pi_T^* L$.

We denote by $\mathcal{M}_{X_R,L_R}^s$ the subfunctor for the stable sheaves.

We note that the moduli space $M_{R,L_R}^s$ is a projective $R$-scheme.

**Proposition 2.3.** The functor $\mathcal{M}_{R,L_R}^s$ is universally corepresented by a $R$-scheme of finite type. We denote this scheme by $M_{R,L_R}^s$.

**Proof.** We know from the proof of [2, Theorem 3.1], there exists a subset of the Quot scheme denoted $R^s$, such that $M_R^s$ is a universal categorical quotient of this subset by the action of a certain general linear group. Let $\alpha : R^s \to M_R^s$ denote this quotient.

The natural transformation $\mathcal{M}_{R,L_R}^s \to \text{Pic}(X_R)$ which induces the determinant morphism $\det : M_R^s \to \text{Pic}(X_R)$. By composing the morphism det with $\alpha$ we obtain, a morphism $\det R^s : R^s \to M_R^s$. Since the quotients $R^s \to M_R^s$ and $R^s \to M_R^s$ are $\text{PGL}(V)$-bundles in the fppf topology (see [3, Lemma 6.3]), it implies $\det R^s$ is an isomorphism. Therefore, we have the following diagram,

$$
\begin{array}{ccc}
M_R^s & \simeq & N_{R,L_R} \\
\downarrow \text{det} & & \downarrow \text{det} \\
\text{Spec}(R) & \to & \text{Pic}(X_R)
\end{array}
$$

Finally by [3, Theorem 4.3.1] we conclude that the functor $\mathcal{M}_{R,L_R}^s$ is universally corepresented by the $R$-scheme $M_{R,L_R}^s$.  \qed
Remark 2.4. Note that the functor \( \mathcal{M}_{R,L_R} \) is corepresented by a projective \( R \)-scheme, denoted \( M_{R,L_R} \) of finite type. Recall the proof of [2, Theorem 3.1]. Since \( X_R \) is smooth, using [6, Theorem 2.1.10], we can define a morphism \( \det' : \text{Quot}_{X_R}(\mathcal{H}, P) \to \text{Pic}(X_R) \) mapping a coherent sheaf on \( X_R \) to its determinant bundle. Denote by \( A \) the (scheme-theoretic) intersection of \( \det^{-1}(L_R) \) and \( Q \), where \( Q \) as in the proof of [2, Theorem 3.1]. Then the statement follows after replacing \( Q \) by \( A \) in the proof of [2, Theorem 3.1].

3 Deformation of moduli spaces with fixed determinant

Keep Notations 1.1. We have seen in the proof of Proposition 2.3 how \( M_{R,L_R} \) can be considered as the fiber of the determinant morphism \( \det : \mathcal{M}_{R} \to \text{Pic}(X_R) \) over the point corresponding to \( L_R \). Using the trace map (see Definition 3.13), we relate the obstruction theory of the deformation functor at a point in the moduli space \( M_{R}^* \) to the obstruction theory of the deformation functor at a point in the moduli space \( \text{Pic}(X_R) \). We use this (see Theorem 3.19) to show that the deformation functor at a point in the moduli space \( M_{R,L_R}^* \) is unobstructed.

We begin by recalling some basic definitions.

Notation 3.1. We denote by \( \text{Art}/R \) the category of local artinian \( R \)-algebras with residue field \( k \). Denote by \( X_k := X_R \times_{\text{Spec}(R)} \text{Spec}(k) \) and \( X_A := X_R \times_{\text{Spec}(R)} \text{Spec}(A) \). Let \( [F_k] \) denote a closed point of \( M_{R}^* \). As \( M_{R}^* \to \text{Spec}(R) \) is a morphism of finite type, the closed points of the moduli space \( M_{R}^* \) are \( k \)-points. Since \( k \) is algebraically closed, by [2, Theorem 3.1] we have a bijection

\[
\theta(k) : \mathcal{M}_R(k) \to \text{Hom}_R(k, M_R).
\]

Therefore to a closed point of \( M_{R}^* \) say \([F_k]\), we can associate a Gieseker stable sheaf \( \mathcal{F}_k \) on the curve \( X_k \). Since the curve \( X_k \) is smooth, the torsion-free sheaf is in fact locally free.

We define a covariant functor at the point \([F_k]\) in \( M_{R}^* \).

Definition 3.2. We define the deformation functor \( \mathcal{D}_{[F_k]} : \text{Art}/R \to (\text{Sets}) \), such that for \( A \in \text{Art}/R \)

\[
\mathcal{D}_{[F_k]}(A) := \left\{ \begin{array}{l} \text{coherent sheaves } \mathcal{F}_A \text{ with Hilbert polynomial } P \\
\text{on } X_A \text{ flat over } A \text{ such that its pull-back to } X_k \\
\text{is isomorphic to } \mathcal{F}_k. \end{array} \right\}
\]

Similarly, we define a covariant functor at the point \([\text{det}(\mathcal{F}_k)]\) of the moduli space \( \text{Pic}(X_R) \).
Definition 3.3. Let $D_{\det(F_k)} : \text{Art}/R \to (\text{Sets})$ be a covariant functor such that for $A \in \text{Art}/R$

$$D_{\det(F_k)}(A) := \left\{ \text{coherent sheaves } F_A \text{ with Hilbert polynomial the same as } \det(F_k) \text{ on } X_A \text{ flat over } A \text{ such that its pull-back to } X_k \text{ is isomorphic to } \det(F_k) \right\}$$

The following theorem gives the obstruction theories of $D_{[F_k]}$ and $D_{\det(F_k)}$.

Using this we prove the following corollary.

Remark 3.4. By [5, Theorem 7.3] the functors $D_{[F_k]}$ and $D_{\det(F_k)}$ have obstruction theories in the groups $H^2(\text{Hom}_{X_k}(F_k, F_k) \otimes_k I)$ and $H^2(\text{Hom}_{X_k}((\det(F_k), \det(F_k)) \otimes_k I)$ respectively. For $X_k$ a curve, by Grothendieck vanishing theorem, $H^2(\text{Hom}_{X_k}(F_k, F_k) \otimes_k I)$ and $H^2(\text{Hom}_{X_k}((\det(F_k), \det(F_k)) \otimes_k I)$ vanish. Therefore, $D_{[F_k]}$ and $D_{\det(F_k)}$ are unobstructed.

Now we define a natural transformation between the two deformation functors.

Definition 3.5. By assumption $F_k$ is a locally-free $O_{X_k}$ module. Moreover, by [5, Exercise 7.1] any coherent sheaf $F_A$ on $X_A$ satisfying the property $F_A \otimes_{O_{X_k}} O_{X_k} \simeq F_k$ is a locally free $O_{X_A}$-module. Therefore, the notion of determinant is well-defined for any coherent sheaf on $X_A$ which pulls back to $F_k$.

We define a natural transformation $\text{Det} : D_{[F_k]} \to D_{\det(F_k)}$ such that for $A \in \text{Art}/R$,

$$\text{Det}_A : D_{[F_k]}(A) \to D_{\det(F_k)}(A), \quad E_A \mapsto \det(E_A).$$

Using this we define a deformation functor at a point in the moduli space $M^k_{R, \mathcal{L}_R}$.

Definition 3.6. Let $\mathcal{L}_R$ be as in Notation [11]. For $A$ a $R$-algebra, denote by $\mathcal{L}_A$ the pullback $p_A^* \mathcal{L}_R$ under the natural morphism $p_A : X_A \to X_R$.

We define a functor $D_{[F_k], \det(F_k)} : \text{Art}/R \to (\text{Sets})$, such that for $A \in \text{Art}/R$,

$$D_{[F_k], \det(F_k)}(A) := \text{Det}_A^{-1}(\mathcal{L}_A).$$

3.7. Group action on the torsors: By [5, Theorem 7.3], the set $D_{[F_k]}(A')$ (respectively $D_{\det(F_k)}(A')$) is a torsor under the action of $H^1(\text{Hom}_{X_k}(F_k, F_k) \otimes_k I)$ (respectively $H^1(\text{Hom}_{X_k}((\det(F_k), \det(F_k)) \otimes_k I)$).

Since $X_k$ is noetherian, we can identify the sheaf cohomology $H^1(X_k, \text{Hom}(F_k, F_k) \otimes_k I)$ with the Cech cohomology $\check{H}^1(\mathcal{U}, \text{Hom}(F_k, F_k) \otimes_k I)$, where $\mathcal{U}$ is an affine open covering of $X_k$. Then an element, say $\xi$ of the cohomology group $H^1(\text{Hom}(F_k, F_k) \otimes_k I)$ can be seen as a collection of elements $\{\phi_{ij}^k\} \in \Gamma(U_i \cap U_j, \text{Hom}(F_k^i, F_k^j))$ satisfying the cocycle condition i.e. for any $i, j, k$, we have $\phi_{ij}^k|_{U_i \cap U_j} = \phi_{ij}^k|_{U_i \cap U_k} + \phi_{ij}^k|_{U_i \cap U_k}$. Since $I$ is a $k$-vector space, $\check{H}^1(\mathcal{U}, \text{Hom}(F_k, F_k) \otimes_k I) \simeq \check{H}^1(\mathcal{U}, \text{Hom}(F_k, F_k) \otimes_k I)$. Therefore, $\{\phi_{ij}^k\}_{i,j}$
is of the form \( \{ \phi_{ij}'' \otimes a \}_{i,j} \) for \( a \in I \) not depending on \( i, j \) and \( \phi_{ij}'' \in \Gamma(U_i \cap U_j, \text{Hom}(\mathcal{F}_k, \mathcal{F}_k)) \) satisfying \( \phi_{ik}''|_{U_{ij}k} = \phi_{jk}''|_{U_{ij}k} + \phi_{ij}''|_{U_{ij}k} \).

Let \( \mathcal{F}_A' \) be an extension of \( \mathcal{F}_A \) on \( X_A' \) i.e. an element of \( \mathcal{D}_{\mathcal{F}_A}(A') \). Since it is locally free, there exists a covering \( \mathcal{U}' = \{ U'_i \} \) of \( X_A \) by such that \( \mathcal{F}_A'|_{U'_i} \) is \( \mathcal{O}_{U'_i} \)-free. Denote by \( \mathcal{U} := \{ U_i \} \) the cover of \( X_k \) where \( U_i := U'_i \cap X_k \). We know from the proof of [5, Theorem 7.3] that \( \mathcal{F}_A(\xi) \) is given by a collection of sheaves \( \mathcal{F}_i := \mathcal{F}_A'|_{U'_i} \) and isomorphisms \( \phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j} \) such that

\[
\phi_{ii} = \text{Id}, \quad \phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j'} = \mathcal{F}_A'|_{U'_i \cap U'_j} \xrightarrow{\text{Id} + (a \otimes \phi''_{ij}) \circ \pi} \mathcal{F}_A'|_{U'_i \cap U'_j} = \mathcal{F}_j|_{U'_i \cap U'_j}
\]

where \( a, \phi''_{ij} \) are as above and \( \pi \) is the natural restriction morphism \( \mathcal{F}_A' \to \mathcal{F}_k \). Then by [4, Ex. II.1.22], \( \mathcal{F}_A(\xi) \) glues to a sheaf if the morphisms \( \{ \phi_{ij} \} \) satisfy the cocycle condition. In the following lemma we prove that this is indeed the case.

**Lemma 3.8.** Let \( \mathcal{F}_i' \) and \( \phi_{ij} \) be as above. The morphisms \( \{ \phi_{ij} \} \) satisfy the cocycle condition i.e. for any \( i, j, k \) \( \phi_{ik} = \phi_{jk} \circ \phi_{ij} \).

**Proof.** It suffices to prove this equality for the basis elements, say \( s_1^i, \ldots, s_{r'}^i \) generating \( \mathcal{F}_i'_{U'_i \cap U'_j \cap U'_k} \). For any basis element \( s_i^j \),

\[
\phi_{jk} \circ \phi_{ij}(s_i^j) = \phi_{jk}(\text{Id} + (a \otimes \phi''_{ij})(\pi(s_i^j))) = \phi_{jk}(\pi(s_i^j) + a\phi''_{ij}(\pi(s_i^j))) = (\text{Id} + a \otimes \phi''_{jk})(\pi(s_i^j)) + a\phi''_{ij}(\pi(s_i^j)) + a\phi''_{ij}(\pi(s_i^j)) = \pi(s_i^j) + \phi_{jk}(\pi(s_i^j)) + a\phi''_{ij}(\pi(s_i^j)) + a\phi''_{jk}(\pi(s_i^j)) = 0
\]

because \( a^2 = 0 \) in \( A' \). Since \( \phi''_{ik} = \phi''_{ij} + \phi''_{jk} \), we have

\[
\phi_{jk} \circ \phi_{ij}(s_i^j) = \pi(s_i^j) + a(\phi''_{ik}(\pi(s_i^j)) = \phi_{ik}(s_i^j).
\]

This shows that \( \{ \phi_{ij} \}_{i,j} \) satisfy the cocycle condition. \( \Box \)

Using this we conclude that \( \mathcal{F}_A(\xi) \), obtained by glueing the sheaves \( \mathcal{F}_i' \) along the isomorphism \( \phi_{ij} \) is a sheaf.

Similarly, an element say \( \xi' \) in \( H^1(\text{Hom}_{X_k}(\det(\mathcal{F}_k), \det(\mathcal{F}_k)) \otimes_k I) \) acts on an element in \( \mathcal{D}_{\text{det}(\mathcal{F}_k)}(A') \), say \( \det(\mathcal{F}_A') \) to produce a line bundle \( \det(\mathcal{F}_A')(\xi') \) given by a family of sheaves \( \{ \mathcal{L}_i := \mathcal{L}_A'|_{U'_i} \} \) and isomorphisms

\[
\phi_{ij} : \mathcal{L}_i|_{U'_i} \xrightarrow{\text{Id} + (a \otimes \phi''_{ij}) \circ \pi} \mathcal{L}_j|_{U'_j}
\]

where \( \phi''_{ij} \in \Gamma(U_i \cap U_j, \text{Hom}(\det(\mathcal{F}_k), \det(\mathcal{F}_k)) \otimes_k I) \). Again by Lemma 3.8 \( \det(\mathcal{F}_A')(\xi') \) is a sheaf.

**Definition 3.9.** We have the following definitions.

6.
1. We define a map
\[ \phi_1 : H^1(\text{Hom}_X(\mathcal{F}_k, \mathcal{F}_k) \otimes I_1) \to \mathcal{D}_{\mathcal{F}_k}(A'), \quad \xi \mapsto \mathcal{F}_A'(\xi) \]
which uniquely associates an extension \( \mathcal{F}_A'(\xi) \) of \( \mathcal{F}_A' \) (using Lemma 3.8) to an element \( \xi \) of \( H^1(\text{Hom}_X(\mathcal{F}_k, \mathcal{F}_k) \otimes I_1) \).

2. Replacing \( \mathcal{F}_A' \) by \( \text{det}(\mathcal{F}_A') \) and starting with \( \text{det}(\mathcal{F}_A') \) we associate an extension say \( (\text{det}(\mathcal{F}_A'))(\xi') \) to an element \( \xi' \) of \( H^1(\text{Hom}_X(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k)) \otimes I_1) \). Hence we define a map
\[ \phi_2 : H^1(\text{Hom}_X(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k)) \otimes I_1) \to \mathcal{D}_{\text{det}(\mathcal{F}_k)}(A'), \quad \xi' \mapsto \text{det}(\mathcal{F}_A')(\xi') \]

Remark 3.10. Note that by Corollary 3.4 there exist surjective morphisms \( r_1 : \mathcal{D}_{\mathcal{F}_k}(A') \to \mathcal{D}_{\mathcal{F}_k}(A) \) and \( r_2 : \mathcal{D}_{\text{det}(\mathcal{F}_k)}(A') \to \mathcal{D}_{\text{det}(\mathcal{F}_k)}(A) \). By [5] Theorem 7.3, \( r_1^{-1}(\mathcal{F}_A) = \text{Im}(\phi_1) \), \( r_2^{-1}(\text{det}(\mathcal{F}_A)) = \text{Im}(\phi_2) \).

The following lemma tells us that taking the determinant commutes with glueing of the sheaf.

Lemma 3.11. The determinant of the sheaf \( \mathcal{F}_A'(\xi) \) is the line bundle obtained by glueing \( \{ \text{det}(\mathcal{F}_i') \} \) along the isomorphisms
\[ \overline{\phi}_{ij} : \text{det}(\mathcal{F}_i'|_{U_i \cap U_j}) \to \text{det}(\mathcal{F}_j'|_{U_i \cap U_j}), \quad s_1^{(i)} \wedge \cdots \wedge s_r^{(i)} \mapsto \phi_{ij}(s_1^{(i)}) \wedge \cdots \wedge \phi_{ij}(s_r^{(i)}) \]
where \( s_1^{(i)}, \ldots, s_r^{(i)} \) are the basis elements of \( \mathcal{F}_i'|_{U_i \cap U_j} \).

Proof. By Lemma 3.8 for all \( t = 1, \ldots, r \), we have \( \phi_{ik}(s_t^{(i)}) = \phi_{jk}(s_t^{(i)}) \circ \phi_{ij}(s_t^{(i)}) \). Then,
\[ \overline{\phi}_{jk} \circ \overline{\phi}_{ij}(s_1^{(i)} \wedge \cdots \wedge s_r^{(i)}) = \overline{\phi}_{jk}(\phi_{ij}(s_1^{(i)}) \wedge \cdots \wedge \phi_{ij}(s_r^{(i)})) \]
\[ = (\phi_{jk} \circ \phi_{ij}(s_1^{(i)})) \wedge \cdots \wedge (\phi_{jk} \circ \phi_{ij}(s_r^{(i)})) \]
\[ = \phi_{jk}(s_1^{(i)}) \wedge \cdots \wedge \phi_{jk}(s_r^{(i)}) \]
\[ = \overline{\phi}_{jk}(s_1^{(i)} \wedge \cdots \wedge s_r^{(i)}) \]
Hence the morphisms \( \{ \overline{\phi}_{ij} \} \) satisfy the cocycle condition i.e \( \overline{\phi}_{jk} = \overline{\phi}_{jk} \circ \overline{\phi}_{ij} \).

By Lemma 3.8, there exist isomorphisms \( \psi_i : \mathcal{F}_A'(\xi)|_{U_i} \simeq \mathcal{F}_i' \) satisfying \( \psi_i|_{U_{ij}} = \phi_{ij} \circ \psi_i|_{U_{ij}} \). We define \( \overline{\psi}_i : \text{det}(\mathcal{F}_A'(\xi))|_{U_i} \simeq \text{det}(\mathcal{F}_i') \) as follows. Let \( s_1^{(i)}, \ldots, s_r^{(i)} \) be the basis of \( \mathcal{F}_A'(\xi)|_{U_i} \). Then \( \overline{\psi}_i(s_1^{(i)} \wedge \cdots \wedge s_r^{(i)}) := \psi_i(s_1^{(i)}) \wedge \cdots \wedge \psi_i(s_r^{(i)}) \). Therefore
\[ \overline{\phi}_{ij} \circ \overline{\psi}_i(s_1^{(i)} \wedge \cdots \wedge s_r^{(i)}) = \overline{\phi}_{ij}(\psi_i(s_1^{(i)}) \wedge \cdots \wedge \psi_i(s_r^{(i)})) \]
\[ = \phi_{ij}(\psi_i(s_1^{(i)})) \wedge \cdots \wedge \phi_{ij}(\psi_i(s_r^{(i)})) \]
\[ = \psi_j(s_1^{(i)}) \wedge \cdots \wedge \psi_j(s_r^{(i)}) \]
Then by the uniqueness of glueing mentioned in [4] Ex. II.1.22, \( \{ \text{det}(\mathcal{F}_i) \} \) glues along the isomorphisms \( \{ \overline{\phi}_{ij} \}_{i,j} \) to \( \text{det}(\mathcal{F}_A'(\xi)) \). \( \square \)
Lemma 3.15. The morphism

\[ \text{trace map} \]

\[ \phi \mapsto \text{tr}_U(\phi) := \left( s_1 \wedge \ldots \wedge s_r \mapsto \sum_j s_1 \wedge \ldots \wedge \phi(s_j) \wedge \ldots \wedge s_r \right). \]

Let \( U := \{ U_i \} \) be a small enough open cover of \( X_k \) such that \( \mathcal{F}_k \) is free on each \( U_i \). Then the trace map is given by

\[ \text{tr} : \text{Hom}_{X_k}(\mathcal{F}_k, \mathcal{F}_k) \to \text{Hom}_{X_k}(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k)) \]

such that \( \text{tr}|_{U_i} = \text{tr}_{U_i} \) for any affine open set \( U_i \) of \( X_k \).

Remark 3.14. Note that the morphism \( \text{tr}_U \) is \( O_{X_k} \)-linear. Let \( f \in O_{X_k}(U) \). Then

\[ \text{tr}_U(f \phi) = s_1 \wedge \ldots \wedge s_r \mapsto \sum_j s_1 \wedge \ldots \wedge f(\phi(s_j)) \wedge \ldots \wedge s_r \]

\[ = \sum_j f(s_1 \wedge \ldots \wedge \phi(s_j) \wedge \ldots \wedge s_r) \]

\[ = f \sum_j s_1 \wedge \ldots \wedge \phi(s_j) \wedge \ldots \wedge s_r \]

\[ = f \text{tr}_U(\phi). \]

Lemma 3.15. The morphism \( \text{tr} \) is surjective.

Proof. It suffices to prove surjectivity on the level of stalks. Let \( x \in X_k \) be a closed point. Consider the induced morphism

\[ \text{tr}_x : \text{Hom}_{X_k}(\mathcal{F}_k_x, \mathcal{F}_k_x) \to \text{Hom}_{X_k}(\text{det}(\mathcal{F}_k_x), \text{det}(\mathcal{F}_k_x)) \]

and basis \( s_1, \ldots, s_r \in \mathcal{F}_k_x \). Since the map \( \text{tr}_x \) is \( O_{X_k,x} \)-linear and \( \text{Hom}_{O_{X_k}}(\text{det}(\mathcal{F}_k_x), \text{det}(\mathcal{F}_k_x)) \cong O_{X_k,x} \), it suffices to show that \( \text{Id} \in \text{Im}(\text{tr}_x) \). Let \( \phi \in \text{Hom}_{X_k}(\mathcal{F}_k_x, \mathcal{F}_k_x) \) defined as \( \phi(s_i) = s_i \) for \( i = 1 \) and \( 0 \) otherwise. This concludes the proof.

We can define the trace map cohomologically as follows:

Definition 3.16. Let \( U := \{ U_i \} \) be a small enough open affine cover of \( X_k \) such that \( \mathcal{F}_k \) is free on each \( U_i \). Using \([4, \text{III. Theorem 4.5}]\) we define \( \check{C} \)-Cocycle \( C^p(U, \text{Hom}(\mathcal{F}_k, \mathcal{F}_k)) \) (resp \( \check{C}^p(U, \text{Hom}(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k))) \)), such that the corresponding \( \check{C} \)-cohomology coincides with the sheaf cohomology \( H^i(X_k, \text{Hom}(\mathcal{F}_k, \mathcal{F}_k)) \) (resp \( H^i(X_k, \text{Hom}(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k)))) \). The morphism \( (*) \) of Definition induces a morphism on cohomologies

\[ \text{tr}^i : H^i(X_k, \text{Hom}(\mathcal{F}_k, \mathcal{F}_k)) \to H^i(X_k, \text{Hom}(\text{det}(\mathcal{F}_k), \text{det}(\mathcal{F}_k))) \cong H^i(X_k, O_{X_k}). \]
As a corollary to Lemma 3.15 we have:

**Corollary 3.17.** The morphism induced on cohomology

\[
\text{tr}^1 : H^1(X_k, \text{Hom}_{X_k}(F_k, F_k)) \to H^1(X_k, \text{Hom}_{X_k}(\text{det}(F_k), \text{det}(F_k)))
\]

is surjective.

**Proof.** Consider the short exact sequence,

\[
0 \to \ker \text{tr} \to \text{Hom}_{X_k}(F_k, F_k) \overset{\text{tr}}{\to} \text{Hom}_{X_k}(\text{det}(F_k), \text{det}(F_k)) \to 0.
\]

We get the following terms in the associated long exact sequence,

\[
\ldots \to H^1(X_k, \text{Hom}_{X_k}(F_k, F_k)) \overset{\text{tr}}{\to} H^1(X_k, \text{Hom}_{X_k}(\text{det}(F_k), \text{det}(F_k))) \to H^2(\ker \text{tr}) \to \ldots
\]

Since \(X_k\) is a curve, by Grothendieck’s vanishing theorem, \(H^2(\ker(\text{tr})) = 0\). Therefore, the morphism \(\text{tr}^1\) is surjective. \( \square \)

The following proposition tells us that the determinant map ‘commutes’ with the trace map.

**Proposition 3.18.** Notation as in 3.7. Let

\[
\text{det}_{ij} : \Gamma(U_i \cap U_j, \text{Hom}(F'_i, F'_i)) \to \Gamma(U_i \cap U_j, \text{Hom}(\text{det}(F'_i), \text{det}(F'_i)))
\]

be a morphism defined by

\[
\phi_{ij} \in \Gamma(U_i \cap U_j, \text{Hom}(F'_i, F'_i)) \mapsto \text{det}_{ij}(\phi_{ij}) := (s^1_i \wedge \ldots \wedge s^r_i \mapsto \phi_{ij}(s^1_i) \wedge \ldots \wedge \phi_{ij}(s^r_i))
\]

where \(s^1_i, \ldots, s^r_i\) are the basis elements of \(F'_i|_{U_i \cap U_j}\). Then for any pair \(i \neq j\), we have

\[
\text{det} \circ (\text{Id} + (\phi''_{ij} \otimes a) \circ \pi) = \text{Id} + (\text{tr}_{U_{ij}}(\phi''_{ij}) \otimes a) \circ \pi.
\]

In other words, the following diagram is commutative:

\[
\begin{array}{ccc}
H^1(\text{Hom}_{X_k}(F_k, F_k) \otimes_k I) & \xrightarrow{\phi_1} & D_{[F_k]}(A') \\
\text{tr}^1 \otimes \text{Id} & & \circ \text{Det}_{A'} \\
H^1(\text{Hom}_{X_k}(\text{det}(F_k), \text{det}(F_k)) \otimes_k I) & \xrightarrow{\phi_2} & D_{[\text{det}(F_k)]}(A')
\end{array}
\]

**Proof.** Let \(s^1_i, \ldots, s^r_i\) be the sections generating \(F'_i|_{U_i \cap U_j}\). Any section of \(\text{Hom}(\text{det}(F'_i), \text{det}(F'_i))\) is (uniquely) defined by the image of \(s^1_i \wedge \ldots \wedge s^r_i\). Hence it suffices to prove

\[
(\text{det}_{ij} \circ (\text{Id} + (\phi''_{ij} \otimes a) \circ \pi))(s^1_i \wedge \ldots \wedge s^r_i) = (\text{Id} + (\text{tr}_{U_{ij}}(\phi''_{ij}) \otimes a) \circ \pi)(s^1_i \wedge \ldots \wedge s^r_i).
\]
For $1 \leq t \leq r$, $(\text{Id} + (\phi''_{ij} \otimes a) \circ \pi)(s^t_1) = s^1_t + a\phi''_{ij}(\pi(s^t_1))$ and since $I.m_{A'} = 0$, $a' = 0$ for $t > 1$. Hence,

$$(\det_{ij} \circ (\text{Id} + (\phi''_{ij} \otimes a) \circ \pi))(s^t_1 \wedge ... \wedge s^t_r) = (s^1_t + a\phi''_{ij}(\pi(s^t_1))) \wedge ... \wedge (s^1_r + a\phi''_{ij}(\pi(s^t_r))) = s^t_1 \wedge ... \wedge s^t_r + a \sum_k s^1_k \wedge ... \wedge \phi''_{ij}(\pi(s^t_1)) \wedge ... \wedge s^t_r = (\text{Id} + (\text{tr}_{U_{ij}}(\phi''_{ij}) \otimes a) \circ \pi)(s^1_1 \wedge ... \wedge s^t_r).$$

This completes the proof of the proposition. \hfill \Box

We end this section with the following theorem.

**Theorem 3.19.** The functor $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}$ is unobstructed.

**Proof.** Let $A' \to A$ be a small extension in $\text{Art}/R$ and $\phi_1, \phi_2$ be as in Definition 3.8. Recall the surjective morphisms $r_1, r_2$ from Remark 3.10. Then we have the following diagram.

\[
\begin{array}{ccc}
\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}(A') & \xrightarrow{\psi} & \mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}(A) \\
\downarrow & & \downarrow \\
H^1(\text{Hom}_{X_k}(\mathcal{F}_k, \mathcal{F}_k) \otimes_k I) & \xrightarrow{\phi_1} & \mathcal{D}_{[\mathcal{F}_k]}(A') & \xrightarrow{r} & \mathcal{D}_{[\mathcal{F}_k]}(A) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\text{Hom}_{X_k}((\mathcal{F}_k), \det(\mathcal{F}_k)) \otimes_k I) & \xrightarrow{\phi_2} & \mathcal{D}_{[\mathcal{F}_k]}(A') & \xrightarrow{r} & \mathcal{D}_{[\det(\mathcal{F}_k)]}(A) \\
\downarrow & & \downarrow & & \downarrow \\
\text{tr}^1 \otimes \text{Id} & & \text{Det}_A' & & \text{Det}_A \\
\end{array}
\]

where the upper right square and the lower right square are commutative by definition and the lower left square is commutative by Proposition 3.18. To prove that $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}$ is unobstructed, we need to show that $\psi$ is surjective. Let $\mathcal{L}_A$ be the unique pull-back of $\mathcal{L}_R$ under the morphism $X_A \to X_R$ and $\mathcal{F}_A$ be an element in $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}(A)$. Since $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]]}(A') = \det(\mathcal{L}'_A)$ where $\mathcal{L}'_A$ is $\pi^*\mathcal{L}_R$ for $\pi : X_A' \to X_R$, we need to prove there exists a sheaf $\mathcal{F}_A'$ on $X_A'$ with determinant $\mathcal{L}_A'$ which is an extension of $\mathcal{F}_A$.

By definition $\mathcal{F}_2(\mathcal{L}_A') = \mathcal{L}_A$. Since $\phi_1$ and $\phi_2$ are injective, $r_1^{-1}(\mathcal{F}_A) = \text{Im}(\phi_1)$ and $r_2^{-1}(\mathcal{L}_A) = \text{Im}(\phi_2)$. Therefore, there exists $t \in H^1(\text{Hom}_{X_k}(\det(\mathcal{F}_k), \det(\mathcal{F}_k)) \otimes_k I)$ such that $\phi_2(t) = \mathcal{L}_A$. By Corollary 3.17, $\text{tr}^1 \otimes \text{Id}$ is surjective. Hence there exists $t' \in H^1(\text{Hom}_{X_k}(\mathcal{F}_k, \mathcal{F}_k) \otimes_k I)$ such that $\text{tr}^1 \otimes \text{Id}(t') = t$. Denote by $\mathcal{F}_A' := \phi_1(t')$. By commutativity of the lower left square, $\det(\mathcal{F}_A') = \mathcal{L}_A'$. This concludes the proof of the theorem. \hfill \Box
4 Main results

In Theorem [3.19] we showed that the deformation functor $D_{[\mathcal{F}_k], [\det(\mathcal{F}_k)]}$ is unobstructed for any closed point $[\mathcal{F}_k]$ of the moduli space $M^s_{R,L_R}$. In this section we prove that this functor is in fact pro-represented by the completion of the local ring at the point $[\mathcal{F}_k]$ (see Proposition 4.4). Using this we prove that the moduli space $M^s_{R,L_R}$ of pure stable sheaves with fixed determinant $L$ over $X_R$ is smooth over $\text{Spec}(R)$.

**Notation 4.1.** Keep Notations 1.1 and 3.1. Let $[\mathcal{F}_k]$ be a $k$-rational point of $M^s_{R,L_R}$ and denote by $\Lambda'' := \hat{\mathcal{O}}_{M^s_{R,L_R}}$ the completion of the local ring $\mathcal{O}_{M^s_{R,L_R}}$. Under the determinant morphism $\det : M^s_{R} \rightarrow \text{Pic}(X_R)$, the line bundle $\det([\mathcal{F}_k])$ is a $k$-point of $\text{Pic}(X_R)$. Denote by $\Lambda' := \hat{\mathcal{O}}_{\text{Pic}(X_R), [\det(\mathcal{F}_k)]}$ and by $\Lambda := \hat{\mathcal{O}}_{M^s_{R,L_R}, [\mathcal{F}_k]}$.

**Definition 4.2.** By $\hat{\mathcal{O}}_{M^s_{R}, [\mathcal{F}_k]}$ we denote the covariant functor $\text{Hom}(\Lambda'', -) : \text{Art}/R \rightarrow \text{Sets}, \ A \mapsto \text{Hom}_{R-\text{alg}}(\Lambda'', A)$.

We define the functors $\hat{\mathcal{O}}_{\text{Pic}(X_R), [\det(\mathcal{F}_k)]}$ and $\hat{\mathcal{O}}_{M^s_{R,L_R}, [\mathcal{F}_k]}$ similarly.

**Lemma 4.3.** The deformation functor $D_{[\mathcal{F}_k]}$ (resp. $D_{[\mathcal{L}_k]}$) are pro-representable by $\hat{\mathcal{O}}_{M^s_{R}, [\mathcal{F}_k]}$ (resp. $\hat{\mathcal{O}}_{\text{Pic}(X_R), [\det(\mathcal{F}_k)]}$).

**Proof.** Recall from the proof of [2] Theorem 3.1, that for $m$ sufficiently large, $\mathcal{R}^s$ is the open subset of $\text{Quot}(\mathcal{H}; P)$ where $\mathcal{H} := \mathcal{O}_{X_R}(-m)^P(m)$ parametrizing stable quotients. By [3] Lemma 6.3, $\phi : \mathcal{R}^s \rightarrow M^s_{R}$ is an etale $\text{PGL}(V)$-principal bundle. Therefore, $\hat{\mathcal{O}}_{\mathcal{R}^s, [\mathcal{F}_k]} \cong \hat{\mathcal{O}}_{M^s_{R}, [\mathcal{F}_k]}$.

Denote by $Q := \text{Quot}(\mathcal{H}; P)$ and by $D_{Q,[\mathcal{F}_k]}$ the deformation functor corresponding to the Quot-scheme at the point $[\mathcal{F}_k]$. Recall that for any local Artin ring $A$, Pic(Spec(A)) = 0, hence $D_{[\mathcal{F}_k]} = D_{Q,[\mathcal{F}_k]}$. Since the functor Quot is representable, the deformation functor $D_{Q,[\mathcal{F}_k]}$ is pro-representable by $\hat{\mathcal{O}}_{Q,[\mathcal{F}_k]}$ i.e.,

$D_{Q,[\mathcal{F}_k]} \cong \hat{\mathcal{O}}_{Q,[\mathcal{F}_k]} \cong \hat{\mathcal{O}}_{\mathcal{R}^s, [\mathcal{F}_k]}$,

where the second isomorphism follows from the fact that $\mathcal{R}^s$ is an open subset of $Q$. Therefore, $D_{[\mathcal{F}_k]}$ is isomorphic to $\hat{\mathcal{O}}_{M^s_{R}, [\mathcal{F}_k]}$.

Using the same argument we can show that $D_{[\det(\mathcal{F}_k)]} \cong \hat{\mathcal{O}}_{\text{Pic}(X_R), [\det(\mathcal{F}_k)]}$. This proves the lemma.

Using this lemma we prove the following proposition.

**Proposition 4.4.** The deformation functor $D_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]}$ is pro-represented by the completion of the local ring $\mathcal{O}_{M^s_{R,L_R}, [\mathcal{F}_k]}$.
Proof. By Lemma 4.3, \( D[\mathcal{F}_k] \) (respectively \( D[\det(\mathcal{F}_k)] \)) is pro-represented by \( \hat{\mathcal{O}}_{\text{det}(X_R),\{\det(\mathcal{F}_k)\}} \) (respectively \( \hat{\mathcal{O}}_{\text{Pic}(X_R),\{\det(\mathcal{F}_k)\}} \)). We have a natural transformation
\[
det : \hat{\mathcal{O}}_{M^*_R,\{\mathcal{F}_k\}} \to \hat{\mathcal{O}}_{\text{Pic}(X_R),\{\det(\mathcal{F}_k)\}}
\]
induced by the determinant morphism, \( \det : M^*_R \to \text{Pic}(X_R) \) localized at the point \([\mathcal{F}_k]\). Let \( A \in \text{Art}/R \) and \( \mathcal{L}_A \) be the pullback of the line bundle \( \mathcal{L}_R \) under the morphism \( X_A \to X_R \). Recall the natural transformation \( \text{Det}_A \) defined in Definition 3.5. We have the following commutative diagram
\[
\begin{array}{ccc}
D[\mathcal{F}_k](A) & \xrightarrow{\sim} & \hat{\mathcal{O}}_{M^*_R,\{\mathcal{F}_k\}}(A) \\
\text{Det}_A \downarrow & \circlearrowleft & \downarrow \text{det}_A \\
D[\det(\mathcal{F}_k)](A) & \xrightarrow{\sim} & \hat{\mathcal{O}}_{\text{Pic}(X_R),\{\det(\mathcal{F}_k)\}}(A)
\end{array}
\]
Hence the deformation functor \( D[\mathcal{F}_k],\{\det(\mathcal{F}_k)\}(A) \cong \text{det}_A^{-1}(\phi_{\mathcal{L}_A}) \), where \( \phi_{\mathcal{L}_A} := \sigma(\mathcal{L}_A) \). Therefore to prove that \( D[\mathcal{F}_k],\{\det(\mathcal{F}_k)\} \) is pro-represented by \( \hat{\mathcal{O}}_{M^*_R,\mathcal{L}_R,\{\mathcal{F}_k\}} \), we need to show that for any \( A \in \text{Art}/R \),
\[
\text{det}_A^{-1}(\phi_{\mathcal{L}_A}) \cong \text{Hom}_R(\Lambda, A). \tag{1}
\]
By Lemma 4.3, \( D[\det(\mathcal{F}_k)](A) \cong \text{Hom}_R(\Lambda', A) \). Hence for a fixed element \( \mathcal{L}_A \in D[\det(\mathcal{F}_k)](A) \), the corresponding morphism from \( \text{Spec}(A) \to \text{Spec}(\Lambda') \) is unique and this is the morphism \( \phi_{\mathcal{L}_A} \). This implies the commutativity of the following diagram
\[
\begin{array}{ccc}
\text{Spec}(A) & \xrightarrow{\phi_{\mathcal{L}_A}} & \text{Spec}(\Lambda') \\
\downarrow & \circlearrowleft & \downarrow \\
\text{Spec}(R) & \xrightarrow{\phi_{\mathcal{L}_A}} & \text{Spec}(\Lambda')
\end{array}
\]
where the morphism \( \text{Spec}(R) \to \text{Spec}(\Lambda') \) is the morphism corresponding to the line bundle \( \mathcal{L}_R \). Then the bijection in (1) follows from the property of fibre product and the following diagram.
Since $A$ was arbitrary, (1) holds for any $A \in \text{Art}/R$. Hence $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]}$ is pro-represented by $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]}$. 

Using this we prove the following theorem.

**Theorem 4.5.** The morphism $M_{R,L,R}^* \to \text{Spec}(R)$ is smooth.

**Proof.** Since the scheme $M_{R,L,R}^*$ is noetherian and smoothness is an open condition, it suffices to check that the morphism $M_{R,L,R}^* \to \text{Spec}(R)$ is smooth at closed points. Let $[\mathcal{F}_k]$ be a closed point of $M_{R,L,R}^*$. Since the morphism $M_{R,L,R}^* \to \text{Spec}(R)$ is of finite type, to prove that it is smooth at the point $[\mathcal{F}_k]$, we need to show that the functor $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]}$ is unobstructed.

By Proposition 4.4, the completion of the local ring $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]}$ pro-represents the functor $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]}$, i.e $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]} \simeq \mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]}$. By Theorem 3.19 the deformation functor $\mathcal{D}_{[\mathcal{F}_k],[\det(\mathcal{F}_k)]}$ is unobstructed. Hence the functor $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]}$ is unobstructed. This implies $\mathcal{O}_{M_{R,L,R}^*,[\mathcal{F}_k]}$ is unobstructed. Hence, the morphism $M_{R,L,R}^* \to \text{Spec}(R)$ is smooth at the point $[\mathcal{F}_k]$. 

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