Abstract—This paper deals with the output regulation problem of a linear time-invariant system in the presence of sporadically available measurement streams. A regulator with a continuous intersample injection term is proposed, where the intersample injection is provided by a linear dynamical system and the state of which is reset with the arrival of every new measurement update. The resulting system is augmented with a timer triggering an instantaneous update of the new measurement and the overall system is then analyzed in a hybrid system framework. With the Lyapunov based stability analysis, we offer sufficient conditions to ensure the objectives of the output regulation problem are achieved under intermittency of the measurement streams. Then, from the solution to linear matrix inequalities, a numerically tractable regulator design procedure is presented. Finally, with the help of an illustrative example, the effectiveness of the theoretical results are validated.

I. INTRODUCTION

The objective of an output regulation problem is to control a given output of the plant to asymptotically track a prescribed reference trajectory and rejecting asymptotically undesired disturbances, both of which are generated by an exosystem, while keeping all the trajectories of the system bounded [1], [2]. In contrast to [1], where the measured output of the plant was assumed to be continuous, in this work we consider that the output measurement streams are only available sporadically. Owing to the intermittent availability of the measured plant output, the classical output regulation theory based on the internal model principle in [3] is not applicable.

Output regulation problem for linear networked control systems with measurement intermittency has been addressed in the works of [2], where the impulsive updates of the latest measurements from the plant are subjected to a zero-order holding device. With the assumption that the impulsive new measurement updates are held constant until the measurement arrives, the results of [2] are then extended to the case of minimum phase nonlinear systems in [4]. Output regulation problem with periodically sampled measurement updates for linear systems are studied by the authors of [5], [6]. The works in [5]–[7] do not take into account the intersampling behavior and phenomenon like uncertain time-varying transmission and scheduling are neglected. While these issues are addressed in the context of output regulation problems for linear networked control systems in [2], the proposed continuous-discrete regulator therein keeps the received plant measurement constant in between the sampling times, which is a restrictive requirement.

A. Contribution

In this paper, we consider the output regulation problem for linear time-invariant plant under sporadic measurements, i.e., we assume that the plant output to be sampled with an arbitrarily large bounded nonuniform sampling period. To address this problem, we propose an observer with continuous intersample injection and state resets. This intersample injection is provided by a linear dynamical system whose state is reset to the measured regulated output at each sampling time.

The contribution of this work is briefly highlighted as follows. First, we construct a hybrid model of the observer and the intersample injection dynamics for the proposed class of linear time-invariant system. Relying on the Lyapunov theory for hybrid systems [8], we pursue similar Lyapunov based analyses to ensure exponential convergence of the regulated error signal. The resulting conditions from this stability analysis are then turned into matrix inequalities, which are used to derive efficient design procedures of the proposed observer and regulator.

The remainder of the paper is organized in the following manner. First, in Section II we briefly present some preliminaries on classical output regulation problem, and hybrid systems theory. The construction of the hybrid linear regulator along with the hybrid modeling of overall closed loop system is proposed in Section III. The sufficient conditions concerning the stability of the overall closed loop system with intermittent measurements are given in Section IV. Next, we present in Section V an efficient numerical design procedure for the regulator with the help of LMIs. A numerical example illustrating the effectiveness of the proposed solution is given in [9]. Finally, some concluding remarks are provided in Section VII.

Notation. We will now introduce some notations which will be used throughout the text. The set \( \mathbb{N}_{>0} \) is the set of all strictly positive integers and \( \mathbb{N} = \mathbb{N}_{>0} \cup \{0\} \). The set of \( \mathbb{R}_{>0} \) (or \( \mathbb{R}_{>0} \)) represent the set of positive (or non-negative) real numbers. \( \mathbb{R}^{n \times m} \) represent the set of all real matrices with order \( n \times m \). I and 0 are respectively the identity and
null matrices of appropriate dimensions. A square, symmetric matrix \( A = A^T \) with \( A \geq 0 \) (or \( A > 0 \)) implies that the matrix \( A \) is a positive semi-definite (or a positive definite) matrix and equivalently \(-A\) is a negative semi-definite (or negative definite) matrix. Given square matrices \( A_1, A_2, \ldots , A_N \) of compatible dimensions, \( A = \text{blk diag}(A_1, A_2, \ldots , A_N) \) denotes a block diagonal matrix with the \( i^{th} \) diagonal element being \( A_i \) and \( \text{col}(A_1, A_2, \ldots , A_N) = [A_1^T, A_2^T, \ldots , A_N^T]^T \). For a square matrix \( A, \sigma(A) \) is the eigenspectrum of \( A \), \( \text{He}(A) = A + A^T, \det(A) \) is the determinant and product of all eigenvalues of \( A \). For \( x, y \in \mathbb{R}^N \), \( ||x|| \) denotes the Euclidean norm of vector \( x \), \( \text{col}(x,y) = [x^T, y^T]^T, \langle x, y \rangle \) is the inner product. Given a vector \( x \in \mathbb{R}^n \) and a closed set \( \mathcal{A} \), the distance of \( x \) to \( \mathcal{A} \) is defined as \( |x|_\mathcal{A} = \inf_{y \in \mathcal{A}} ||x - y|| \). Given a real valued function \( f(x) : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2} \), \( \text{dom} f \) denotes the domain \( \mathbb{R}^{n_1} \) of \( f(x) \).

II. PROBLEM FORMULATION

Consider a linear time-invariant plant of the form

\[
\mathcal{P} \begin{cases}
\dot{x}_p = A_p x_p + B_p u + E_p w, \\
y_p = C_p x_p, \\
\dot{e}_p = y_p - y_w, \\
y_w = F_p w,
\end{cases}
\]

(1)

with \( x_p \in \mathbb{R}^{n_p}, u \in \mathbb{R}^{n_u}, y_p, e_p \in \mathbb{R}^p \) being respectively the state, control law to be designed, and regulated output of the plant which we aim to regulate to zero. The exogenous signal \( w \in \mathbb{R}^q \) is generated by an exosystem of the form

\[
\dot{w} = S w,
\]

(2)

where the exosystem matrix \( S \) is assumed to be neutrally stable, i.e. \( S \) has all eigenvalues on the imaginary axis. While \( S \) is perfectly known, the exosystem state \( w \) in (2) not directly available for feedback design. The matrices \( A_p, B_p, E_p, C_p \) and \( F_p \) in (1) are constant matrices of appropriate dimensions and such that the pair \((A_p, C_p)\) is detectable. The output \( y \) is available only at some isolated time instances \( t_k, k \in \mathbb{N}_{>0} \), not known a priori. We assume that the sequence \( \{t_k\}_{k=1}^\infty \) are strictly increasing, \( t_k \to \infty \) as \( k \to \infty \) and there exist two positive scalars \( T_1 \) and \( T_2 \) which uniformly bounds the consecutive intersampling intervals \( [t_k, t_{k+1}] \) of \( \{t_k\} \) as follows

\[
0 < t_1 \leq T_2, \quad T_1 \leq t_{k+1} - t_k \leq T_2, \quad \forall k \in \mathbb{N}_{>0}.
\]

(3)

As noted in [8], the strictly positive lower bound \( T_1 \) prevents the existence of accumulation points in the sequence \( \{t_k\}_{k=1}^\infty \) and thus avoids zero behavior. On the other hand, \( T_2 \) defines the maximum allowable transfer time (MATI) [9].

A. Preliminaries on Linear Output Regulation Problem

We now define the objectives of the output regulation problem. For system (1), the output regulation problem is said to be solved if the following two conditions are satisfied.

1) When \( w = 0 \), all the trajectories of the closed loop system exponentially converge to zero, i.e. the origin of the unperturbed closed loop system \( (w = 0) \) is exponentially stable.

2) When \( w \neq 0 \), the trajectories of the closed loop system (1) are internally stable and the regulated output signal \( e_p(t) \) exponentially converges to zero, i.e. \( \lim_{t \to \infty} e_p(t) = 0 \).

We now consider the following two assumptions which are required to guarantee the solvability of the classical output regulation problem [1], [3].

Assumption 1. The matrix pair \((A_p, B_p)\) is stabilizable and \((A_p, C_p)\) is detectable.

Assumption 2. The matrix \( \begin{bmatrix} A_p - \lambda I & B_p \\ C_p & 0 \end{bmatrix} \), \( \lambda \in \sigma(S) \) is of full rank or equivalently there exists a unique solution pair \((X_p, R)\) to the following linear regulator equation

\[
X_p S = A_p X_p + B_p R + E_p, \\
0 = C_p X_p - F_p,
\]

(4)

where the matrix \( X_p \) uniquely defines the steady state \( x_p = X_p w \) on which the regulated output \( e_p = 0 \). Additionally, the steady state input \( u = Rw \) renders the given manifold \( x_p = X_p w \) to be positively invariant.

As noted in [1], under Assumptions 1 and 2, the output regulation problem for the plant (1) is solvable by the dynamic error feedback control of the form

\[
\begin{bmatrix} u \\ \dot{z} \end{bmatrix} = K z + G_1 z + G_2 e_p,
\]

(5)

where \( z \in \mathbb{R}^{n_z} \) is the regulator state to be specified later and the constant controller gain matrices \( G_1 \) and \( G_2 \) of appropriate dimensions are defined as

\[
G_1 = T \begin{bmatrix} S_1 & S_2 \\ S_3 & G_1 \end{bmatrix} T^{-1}, \quad G_2 = T \begin{bmatrix} S_4 \\ G_2 \end{bmatrix},
\]

(6)

with \( \sigma(G_1) = \sigma(S) \) and \( \beta_1, \gamma_1 \) respectively being a constant square matrix and a column vector of dimension \( d_1 \in \mathbb{N}_{>0} \) such that the pair \((\beta_1, \gamma_1)\) is controllable and the matrix \( A = \begin{bmatrix} A & BK \end{bmatrix} \) is Hurwitz. The matrices \( S_1, S_2, S_3, S_4 \) are arbitrary constant matrices of appropriate dimensions and \( T \in \mathbb{R}^{n_z \times n_z} \) is any nonsingular matrix. Therefore, when the signal \( y_p \) is continuously measured, the requirements for the solvability of the output regulation problem are said to be satisfied with Assumptions 1 and 2.

B. Preliminaries on hybrid Systems

We consider a hybrid system with state \( x \in \mathbb{R}^n \) of the form

\[
\mathcal{H} \begin{cases}
\dot{x} = f(x), & x \in C \\
x^+ = G(x), & x \in D,
\end{cases}
\]

(7)

where the shorthand notation \( \mathcal{H} = \{C, f, D, G\} \) comprises of the flow set \( C \), flow map \( f \), jump set \( D \), and jump map \( G \). A set \( \{t, j\} \in E \subset \mathbb{R}_{>0} \times \mathbb{N} \) is said to be a hybrid
time domain if it is the union of finite or infinite sequence of intervals \([t_j, t_{j+1}) \times \{j\}\). A function \(\phi: \text{dom} \phi \rightarrow \mathbb{R}^n\) is a hybrid arc if dom \(\phi\) is a hybrid time domain with \(\phi(t^+, j)\) being locally absolutely continuous for each \(j\). A solution to \(\mathcal{H}\) is said to be complete if its domain is unbounded and maximal, and if it is not the truncation of another solution [10]. Given a hybrid system \(\mathcal{H}\) in [7], we say that \(\mathcal{H}\) satisfies hybrid basic conditions [10], [11], if \(C\) and \(D\) are closed in \(\mathbb{R}^n\), \(f: \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous and \(G \Rightarrow \mathbb{R}^n\) is locally bounded, nonempty, outer semicontinuous relatively in \(D\).

III. SOLUTION OUTLINE

A. Proposed Controller

Since the output of the plant is available sporadically, we here propose a control scheme, depicted in Fig. 1, constituted by a preprocessing internal model \(\mathcal{G}\) of the exosystem [12], stabilizing controller \(\mathcal{K}\), and a holding device \(\mathcal{J}\). In the proposed control scheme, the plant \(\mathcal{P}\) along with the internal model of the exosystem \(\mathcal{G}\), viewed together as an extended continuous plant \(\tilde{\mathcal{P}}\) is stabilized by a dynamic controller \(\mathcal{K}\) which relies on the continuous regulated error signal \(\tilde{\theta}\) generated by the holding device \(\mathcal{J}\). The hold device \(\mathcal{J}\) receives the intermittent regulated error signal \(e(t_k)\) available from the output of the sampler \(\mathcal{S}\) at every nonuniform time instants \(t_k\).

With a little abuse of notation from [5], the continuous-time internal model controller \(\mathcal{G}\) and its input to the plant \(\mathcal{P}\) is given as follows

\[
\mathcal{G} \left\{ u = Kz, \, \tilde{z} = G_1z + G_2v, \right. \tag{8}
\]

where \(K \in \mathbb{R}^{m_p \times n_x}\) is a controller gain matrix of the extended plant \(\tilde{\mathcal{P}}\), the internal model controller pair \((G_1, G_2)\) are defined in (6), and the continuous-time signal \(v(t) \in \mathbb{R}^{n_u}\) is the output of the stabilizer, defined next.

\[
\mathcal{K} \left\{ \dot{x}_c = A_c x_c + B_c \theta, \right. \tag{9}
\]

where \(x_c \in \mathbb{R}^{n_p + n_z}\) is the stabilizer state and \(\theta \in \mathbb{R}^p\) is the state of the holding device \(\mathcal{J}\). From the controller state \(x_c(t)\) and last received measurement of the regulated output \(\epsilon_p(t)\), the holding device \(\mathcal{J}\) generates an intersample signal to feed the stabilizer \(\mathcal{K}\). For all \(k \in \mathbb{N}_0\) The dynamics of the holding device \(\mathcal{J}\) is given as follows

\[
\mathcal{J} \left\{ \begin{array}{l}
\dot{\theta}(t) = H\theta + Ex_c, \; t \neq t_k, \\
\theta(t^+) = \epsilon_p(t), \quad t = t_k,
\end{array} \right. \tag{10}
\]

where \(R\) is a solution to the linear regulator equations from [4]. The arrival of new measurements instantaneously updates \(\theta(t)\) to \(\epsilon_p(t)\) and in between updates the holding state \(\theta\) evolves according to the continuous dynamics of \(\mathcal{J}\).

In the next section, we present the hybrid modeling of the overall closed loop system in the presence of sporadic measurements of the regulated output \(\epsilon_p(t)\). But, to simplify our analysis in the modeling stage, we first transform coordinates of the plant state \(x_p\), internal model state \(z\), and holding state \(\theta\) as follows.

\[
\hat{x} = \text{col}(\check{x}, \tilde{\theta}, \tau) \in \mathbb{R}^{2(n_p + n_z) + p + 1}, \quad f(\hat{x}) = \text{col}(A\check{x} + B\tilde{\theta}, \tilde{\theta} \check{x} + H\theta, -1) \quad \text{and jump map } G(\hat{x}) = \text{col}(\check{x}, 0, [T_1, T_2]). \tag{14}
\]

Fig. 1. Schematic representation of the closed loop system with sporadic measurements. Continuous-time signals are marked with solid arrows, while the sporadic measurements are with dashed arrows.

\[
\check{x}_p = x_p - X_p w, \quad \tilde{z} = z - Z_p w, \quad \tilde{\theta} = \theta - e_p, \tag{11}
\]

where \(X_p\) is a solution to the linear regulator equations [4], and \(Z_p \in \mathbb{R}^{q \times n_z}\) is a transformation matrix which by virtue of internal model principle [1] satisfies

\[
Z_p S = G_1 Z_p, \quad K Z_p = R \tag{12}
\]

with \(G_1\) and \(K\) given in (8).

B. Hybrid modeling

The hybrid closed-loop system with state \(\check{x} = \text{col}(\check{x}_p, \tilde{z}, x_c) \in \mathbb{R}^{2(n_p + n_z)}\) and jumps in \(\theta(t)\), depicted in Fig. 1 is described as follows

\[
\begin{array}{l}
\dot{\check{x}} = A \check{x} + B \tilde{\theta}, \quad \dot{\tilde{\theta}} = H\check{\theta} + \mathcal{J} \check{x}, \quad t \neq t_k, \\
\check{x}(t^+) = \check{x}, \quad \tilde{\theta}(t^+) = 0, \quad t = t_k,
\end{array} \tag{13}
\]

where

\[
\begin{align*}
A &= \begin{bmatrix} A + BD_c C & BC_c \\
B_c C & A_c \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} H C - CA & E \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_p & 0 \end{bmatrix}.
\end{align*}
\]

Similar to [8], we now introduce a timer variable \(\tau\) which keeps track of the duration of flows and triggers a jump when certain condition is violated. Therefore, from [8], [10], \(\tau\) is made to decrease as ordinary time increases satisfying (3) and it resets to any point in \([T_1, T_2]\) when \(\tau\) reaches 0. The overall closed loop system composed of the states \(\check{x}, \tilde{\theta}, \tau\) can then be represented by the following hybrid system

\[
\mathcal{H}_{cl} \left\{ \begin{array}{l}
\check{x}, \tilde{\theta} \in C, \\
\check{x}, \tilde{\theta} \in D,
\end{array} \right. \tag{14}
\]
dynamical system $\Sigma_x$ and a hybrid time-dynamical system $\Sigma_{\dot{\theta}}$ as follows

$$
\Sigma_x \left\{ \begin{array}{l}
\dot{x} = A_x \dot{x} + B \dot{\theta} \\
\dot{\theta} = H \theta + J \dot{x}
\end{array} \right., \quad \theta \in \mathbb{R}^p, \dot{x} \in \mathbb{R}^{2(n_x+n_z)}
$$

$$
\Sigma_{\dot{\theta}} \left\{ \begin{array}{l}
\dot{\theta}^+ = P \chi_{\tau} \in \left[ \begin{array}{c}
0 \\
T_1, T_2 \end{array} \right], \quad \tau \in [0, T_2]
\end{array} \right.$$

(15)

C. Problem Statement

Global exponential stability of $H_{cl}$ is defined as follows.

**Definition 1.** (Global exponential stability [10]). Given the set $A = \{0\} \times \{0\} \times [0, T_2] \subseteq \mathbb{R}^{2(n_x+n_z)+p+1}$ be closed, the set $A$ is then globally exponentially stable (GES) for $H_{cl}$ if there exists strictly positive real numbers $\lambda_c$ and $k_c$ such that for any initial condition every maximal solution $\phi_{cl}$ to $H_{cl}$ is complete and satisfies for all $(t, j) \in \text{dom} \phi_{cl}$

$$
|\phi_{cl}(t, j)|_A \leq k_c \frac{e^{-\lambda_c(t+j)}}{|\phi_{cl}(0, 0)|_A}. \quad \text{(16)}
$$

The output regulation defined in Section II-A can be restated as follows.

**Problem 1.** Given the extended plant model $\hat{P}$ composed of $P$ in (1) and internal model $G$ in (8), design control and hold parameters

$$
\Delta_K = \left[ \begin{array}{c}
A_c \\
B_c \\
C_c \\
D_c
\end{array} \right], \quad \Delta_T = \left[ \begin{array}{c}
H \\
E
\end{array} \right]
$$

such that the set $A = \{0\} \times \{0\} \times [0, T_2]$ is GES for the hybrid system $H_{cl}$.}

IV. MAIN RESULTS

In this section we present the stability results of the overall closed-loop hybrid system $H_{cl}$, which is a reminiscent of the “input-to-state stability” small gain philosophy [10]. Inspired by the works of [10], we consider the Lyapunov function $V(\hat{x})$ for $H_{cl}$ in the form of $V(\hat{x}) = W_1(\hat{x}) + W_2(\hat{\theta}, \tau)$, where $W_1(\hat{x}) = \hat{x}^T P_1 \hat{x}$, $W_2(\hat{\theta}, \tau) = e^{\delta T} P_2 \hat{\theta}$ with $P_1 \in \mathbb{R}^{2(n_x+n_z) \times 2(n_x+n_z)}$, $P_2 \in \mathbb{R}^{p \times p}$ > 0 and $\tau \in [0, T_2]$. Take $\chi_1 = \min(\lambda_{\min}(P_1), \lambda_{\min}(P_2))$, $\chi_2 = \max(\lambda_{\max}(P_1), e^{\delta T} \lambda_{\max}(P_2))$, then

$$
\chi_1 |\hat{x}|^2_A \leq V(\hat{x}) \leq \chi_2 |\hat{x}|^2_A. \quad \text{(18)}
$$

To satisfy input-output stability conditions for both $\Sigma_x$ and $\Sigma_{\dot{\theta}}$, let us consider that the flow maps in (14) satisfy the following properties.

**Property 1.** Consider positive definite continuously differentiable functions $W_1(\hat{x})$ and $W_2(\dot{\theta}, \tau)$ with flow maps in (14). There exist positive definite functions $\rho_1 \in \mathbb{R}^{2(n_x+n_z)} \to \mathbb{R}$, $\rho_2 \in \mathbb{R}^{p \times p} > 0$, $\omega_1 \in \mathbb{R}^{p \times p} \to \mathbb{R}$, $\omega_2 \in \mathbb{R}^{2(n_x+n_z)} \to \mathbb{R}$, positive scalars $k_1$, $k_2$ such that $\forall (\hat{x}, v_1) \in \mathbb{R}^{2(n_x+n_z)+p}$, $(\dot{\theta}, \tau, v_2) \in \mathbb{R}^{p \times [0, T_2]} \times \mathbb{R}^{2(n_x+n_z)}$ we have

$$
\langle \nabla W_1(\hat{x}), \hat{x} \rangle + \rho_1(\hat{x}) + \rho_2(v_1), \quad \text{(19)}
$$

$$
\langle \nabla W_2(\dot{\theta}, \tau), H \dot{\theta} + J v_2 \rangle \leq -\omega_1(\dot{\theta}) + \omega_2(v_2), \quad \text{(20)}
$$

$$
-\rho_1(\hat{x}) + \omega_2(\hat{x}) \leq -k_1 |\hat{x}|^2, \quad \text{(21)}
$$

$$
-\omega_1(\dot{\theta}) + \rho_2(\dot{\theta}) \leq -k_2 |\dot{\theta}|^2. \quad \text{(22)}
$$

**Theorem 1.** Let Property 1 hold. Then the set $A$ in Definition 1 is globally exponentially stable with respect to $H_{cl}$.

**Proof.** From (18), the Lyapunov function $V(\hat{x})$ is bounded between two monotonically increasing functions. Next, for each $\hat{x} \in D$ and a scalar $v_3 \in [0, T_2]$ with $\hat{x}(\hat{x}) = (\hat{x}, v_3) \in G(\hat{x})$ we have

$$
V(\hat{g}) - V(\hat{x}) = W_2(0, v_3) - W_2(\hat{\theta}, 0) \leq -\lambda_{\max}(P_2) |\dot{\theta}|^2. \quad \text{(23)}
$$

On the other hand, by evaluating $\dot{V} = \langle \nabla V(\hat{x}), f(\hat{x}) \rangle$ along flow directions in (14) and by virtue of equations (19) - (22), we obtain

$$
\langle \nabla V(\hat{x}), f(\hat{x}) \rangle = \langle \nabla W_1, A \hat{x} + B \dot{\theta} \rangle + \langle \nabla W_2, J \hat{x} + H \dot{\theta} \rangle
$$

$$
\leq -\rho_1(\hat{x}) + \rho_2(\dot{\theta}) - \omega_1(\dot{\theta}) + \omega_2(\hat{x})
$$

$$
\leq -k_1 |\dot{x}|^2 - k_2 |\dot{\theta}|^2 \leq -\chi_3 |\hat{x}|^2_A, \quad \text{(24)}
$$

where $\chi_3 = \min(k_1, k_2)$. From (18), $\dot{V}$ in (24) yields $V(\hat{x}) \leq -\chi_2 |\hat{x}|^2$ for all $\hat{x} \in C$, and therefore, thanks to (23), $\forall (t, j) \in \text{dom} \phi_{cl}$

$$
V(\phi_{cl}(t, j)) \leq e^{-\chi_2} V(\phi_{cl}(0, 0)),
$$

or equivalently,

$$
|\phi_{cl}(t, j)|_A \leq e^{\frac{-\chi_3 T_1}{2\chi_2}} e^{\frac{-\chi_3 T_1}{2\chi_2}} |\phi_{cl}(0, 0)|_A. \quad \text{(25)}
$$

As a result, the conditions of global exponential stability (16) are satisfied with $\lambda = \left( 0, \frac{-\chi_3 T_1}{2\chi_2 (1+T_1)} \right)$ and $k_c = \sqrt{\frac{-\chi_3 T_1}{2\chi_2}}, \omega \geq \lambda$ and hence the set $A$ is globally exponentially stable with respect to $H_{cl}$. □

We have observed in the results on Theorem 1 that, if the conditions in Property 1 hold, then the set $A$ is globally exponentially stable with respect to $H_{cl}$. In what follows, we derive sufficient stability conditions for $H_{cl}$ so that the requirements in Property 1 are met with the given choice of our Lyapunov function $V(\hat{x})$.

**Theorem 2.** If there exist $P_3, P_4 \in \mathbb{R}^{2(n_x+n_z) \times 2(n_x+n_z)} > 0$, $P_5 \in \mathbb{R}^{p \times p}$ > 0, and matrices $A_c \in \mathbb{R}^{(n_x+n_z) \times (n_x+n_z)}$, $B_c \in \mathbb{R}^{(n_x+n_z) \times p}$, $C_c \in \mathbb{R}^{n_x \times (n_x+n_z)}$, $D_c \in \mathbb{R}^{n_z \times p}$, $H \in \mathbb{R}^{p \times p}$, $E \in \mathbb{R}^{p \times (n_x+n_z)}$ be such that

$$
P_3 - P_4 < 0, \quad \text{(26)}
$$

$$
P_5 - P_6 < 0, \quad \text{(27)}
$$

$$
M_1 = \begin{bmatrix} 
\text{He}(P_1 A) + P_4 & P_3 B \\
E^T P_1 & -P_5 
\end{bmatrix} \leq 0, \quad \text{(28)}
$$

$$
M_2 = \begin{bmatrix} 
\text{He}(P_1 A) + P_4 & P_3 B \\
E^T P_1 & -P_5 
\end{bmatrix} \leq 0, \quad \text{(29)}
$$

where $M_2(\tau) = \begin{bmatrix} 
\text{He}(P_2 H) - \delta P_2 & P_6 e^{\delta T} P_2 \\
E^T P_1 & -P_5 
\end{bmatrix}$ with $\tau \in [0, T_2]$, then Property 1 holds.
Proof. Define $\rho_1(\hat{x}) = \hat{x}^T P_4 \hat{x}$, $\rho_2(\hat{\theta}) = \hat{\theta}^T P_5 \hat{\theta}$, $\omega_1(\hat{\theta}) = \hat{\theta}^T P_6 \hat{\theta}$, $\omega_2(\hat{x}) = \hat{x}^T P_3 \hat{x}$. Now we evaluate $\hat{W}_1$ as

$$
\hat{W}_1(\hat{x}) = (\nabla W_1(\hat{x}), A\hat{x} + B\hat{\theta}) = \hat{x}^T P_1 \hat{x} + \hat{\theta}^T P_2 \hat{\theta} = \hat{x}^T \text{He}(P_1 A) \hat{x} + \hat{\theta}^T \text{Bl} B \hat{\theta}.
$$

(30)

By virtue of (28), we thus obtain $\Omega_1(\hat{x}, \hat{\theta}) = \hat{W}_1 + \hat{x}^T P_4 \hat{x} - \hat{\theta}^T P_5 \hat{\theta} = [\hat{x}^T \hat{\theta}^T] M_1 [\hat{x} \hat{\theta}] \leq 0$, which as a consequence yields (19). On the other hand,

$$
\hat{W}_2 = e^{\hat{\theta} \tau \hat{\theta}} \hat{P}_4 \hat{x} + \hat{\theta}^T \hat{P}_3 \hat{x} - \delta e^{\hat{\theta} \tau \hat{\theta}} P_2 \hat{\theta} = \hat{x}^T \left[ \text{He}(P_2 H) - \delta P_2 \hat{\theta} + \hat{x}^T \hat{\theta} \hat{P}_3 \hat{x} + \hat{x}^T \hat{\theta} \hat{P}_2 \hat{\theta} + \hat{\theta}^T \hat{P}_2 \hat{\theta} \right].
$$

(31)

Then, $\Omega_2(\hat{x}, \hat{\theta}) = \hat{W}_2 + \hat{\theta}^T P_6 \hat{x} - \hat{x}^T P_3 \hat{x} = [\hat{x}^T \hat{\theta}^T] M_2(\tau) [\hat{x} \hat{\theta}], \text{ which is a convex expression with respect to each value of } \tau \in [0, T_2] \text{ and therefore, for each } \tau, \text{ there exists } (a, \tau) \text{ such that } M_2(\tau) = a(\tau) M_2(0) + (1 - a(\tau)) M_2(T_2). \text{ By virtue of } M_2(0), M_2(T_2) \leq 0, \text{ we thus obtain } M_2(\tau) \leq 0 \text{ and consequently } \Omega_2(\hat{x}, \hat{\theta}) \leq 0, \text{ which in turn yields (20). Since } \hat{\Omega}_1(\hat{x}, \hat{\theta}), \hat{\Omega}_2(\hat{x}, \hat{\theta}) \leq 0, \text{ we further obtain from Equations (26) and (27)}$

$$
\hat{V} = \hat{W}_1 + \hat{W}_2 - \hat{\theta}^T P_3 \hat{x} + \hat{\theta}^T P_5 \hat{\theta} \leq \lambda_{\max}(P_3 - P_5) \| \hat{x} \|^2 + \lambda_{\max}(P_5 - P_6) \| \hat{\theta} \|^2 - k_1 \| \hat{x} \|^2 - k_2 \| \hat{\theta} \|^2 \leq -C_1 \| \hat{x} \|^2 < 0,
$$

(32)

where $k_1 = -\lambda_{\max}(P_3 - P_5)$, $k_2 = -\lambda_{\max}(P_5 - P_6)$, and as a result, the last two conditions of Property 1 are evident.

In Theorem 2 we derived sufficient conditions to guarantee the exponential stability of $\mathcal{H}_{cl}$ with respect to $A$. However, these conditions in (26) - (29) cannot be directly used for designing the decision variables $P_1, P_2, A_c, B_c, C_c, D_c,$ and $\delta$. Therefore, some matrix manipulations are required to turn these conditions into an LMI feasibility problem.

V. LMI BASED REGULATOR DESIGN

In this section, we perform matrix and variable manipulations to turn conditions (26) - (29) into a tractable LMI based controller design procedure. First, we find the Schur complement of (28) as

$$
\hat{\mathcal{M}}_1 = \begin{bmatrix} \text{He}(P_1 A) & P_3 B & I \\ P_3^T P_1 & -P_5 & 0 \\ I & 0 & -P_8 \end{bmatrix} \leq 0
$$

(33)

where $P_8 = P_4^{-1}$. Therefore, (26) now becomes

$$
P_3 - P_8^{-1} < 0,
$$

(34)

which is not an LMI with respect to $P_8$. To transform this into an LMI, we need to find an upper bound of $P_3$ in (34) in terms of $P_8$. From Lemma 1 of [10], for all $\alpha \in \mathbb{R}$, $P_8^{-1}$ and $P_8$ are related by the following inequality

$$
P_8^{-1} > 2\alpha I - \alpha^2 P_8.
$$

(35)

Therefore, from (35), the matrix conditions in (34) are met if we assume that the following LMI holds:

$$
P_3 - 2\alpha I + \alpha^2 P_8 < 0, \quad \alpha \in \mathbb{R}.
$$

(36)

Next, in (33), we observe that the nonlinear terms are associated with the decision variable $P_1$. Let us now characterize the structure of $P_1$ as

$$
P_1 = \begin{bmatrix} X & U \\ U^T & W \end{bmatrix}, \quad P_1^{-1} = \begin{bmatrix} Y & V \\ V^T & Z \end{bmatrix},
$$

(37)

where $X, Y, Z, W \in \mathbb{R}^{n_p+n_c}$. From (37), we obtain $W = -V^{-1}YU = -V^{-1} Y (I - XY)V^{-1}$ since $XY + UV = I$, and therefore $P_1$ becomes

$$
P_1 = \begin{bmatrix} X & U \\ U^T & -V^{-1}(Y - XY)V^{-T} \end{bmatrix}.
$$

(38)

Next, define a matrix $\Psi = \begin{bmatrix} Y & I \\ V^T & 0 \end{bmatrix}$ which is nonsingular as $\det(V) \neq 0$. Since $P_1 > 0$, we also obtain

$$
\tilde{P}_1 = \Psi^T P_1 \Psi = \begin{bmatrix} Y & I \\ V^T & X \end{bmatrix} > 0,
$$

(39)

and then by congruence transformation on $\mathcal{M}_1$, (33) yields

$$
\hat{\mathcal{M}}_1 = \text{blkdiag}(\Psi^T, I, I) \tilde{\mathcal{M}}_1 \text{blk diag}(\Psi, I, I)
$$

$$
= \begin{bmatrix} \Psi^T \text{He}(P_1 A) \Psi & \Psi^T P_1 B & \Psi^T \\ \text{Bl} P_1^T \Psi & -P_5 & 0 \\ 0 & \Psi & -P_8 \end{bmatrix} 
$$

$$
= \text{He} \begin{bmatrix} \Psi^T \tilde{P}_1 A \Psi & \Psi^T \tilde{P}_1 B & 0 \\ 0 & -P_5 & 0 \\ \Psi & 0 & -P_8 \end{bmatrix} \leq 0,
$$

(40)

which is still not an LMI. By performing the change of variables as in [10] we define $K_1 \in \mathbb{R}^{n_u \times (n_p+n_c)}$, $K_2 \in \mathbb{R}^{n_u \times p}$, $K_3 \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $K_4 \in \mathbb{R}^{n_u \times p}$ such that

$$
\begin{bmatrix} K_1 & K_2 \\ K_3 & XAY \end{bmatrix} = \begin{bmatrix} I & 0 \\ XB \ U & [B_c A_c] \end{bmatrix} \begin{bmatrix} D_c C_c \ & CY \ I \end{bmatrix},
$$

(41)

$$
\Delta_K = \begin{bmatrix} U^{-1} & -U^{-1} XB \\ 0 & I \end{bmatrix}, \quad \Gamma_K \begin{bmatrix} V^{-T} & 0 \\ -CYV^{-T} & I \end{bmatrix}
$$

where $\Gamma_K = \begin{bmatrix} K_3 - XAY & K_4 \\ K_1 & K_2 \end{bmatrix}$. Then, by substituting the results of (41) in (40) we obtain

$$
\hat{\mathcal{M}}_1 = \text{He} \begin{bmatrix} \Pi_1 & \Pi_2 & 0 \\ 0 & -P_5 & 0 \\ \Psi & 0 & -P_8 \end{bmatrix} \leq 0,
$$

(42)

$$
\Pi_1 = \begin{bmatrix} AY + BK_1 & A_0 + BK_2 C \\ K_3 & XA + K_4 C \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} BK_1 \\ K_4 \end{bmatrix},
$$

which is an LMI with respect to $X, Y, V, K_i$, $i = 1, 2, 3, 4$. Next, by defining $Z_1 = P_2 H$ and $Z_2 = P_2 E$, it can be easily shown that the equation (29) can be turned into an LMI with respect to the decision variables $P_2, Z_1, Z_2, P_6$. The hold gain $\Delta_J$ thus becomes

$$
\Delta_J = \begin{bmatrix} P_2^{-1} Z_1 & P_2^{-1} E \end{bmatrix}.
$$

(43)
Proposition 1. Given a plant $\mathcal{P}$ in (1) with internal model controller $\mathcal{G}$ in (8), real scalars $\alpha, \delta > 0$, $X,Y \in \mathbb{R}^{(n_p+n_s)\times(n_p+n_s)} > 0$, $P_3,P_8 \in \mathbb{R}^{2(n_p+n_s)\times2(n_p+n_s)} > 0$, $P_5,P_6 \in \mathbb{R}^{p\times p} > 0$, $V \in \mathbb{R}^{(n_p+n_s)\times(n_p+n_s)}$ with $\det(V) \neq 0$, $K_1 \in \mathbb{R}^{n_s\times(n_p+n_s)}$, $K_2 \in \mathbb{R}^{n_s\times p}$, $K_3 \in \mathbb{R}^{(n_p+n_s)\times(n_p+n_s)}$, $K_4 \in \mathbb{R}^{(n_p+n_s)\times p}$, $Z_1 \in \mathbb{R}^{p\times p}$, $Z_2 \in \mathbb{R}^{p\times(n_p+n_s)}$ such that
$$
\bar{P}_1 = \begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0,
$$
$$
\bar{M}_1 = \text{He} \left( \begin{bmatrix} \Pi_1 & \Pi_2 & 0 \\ 0 & -P_5 & 0 \\ \Psi & 0 & -P_8 \end{bmatrix} \right) \leq 0
$$
$$
P_3 - 2\alpha I + \alpha^2 P_8 < 0,
$$
P_5 - P_6 < 0,
$$
\bar{M}_2(0) \leq 0 \quad \text{and} \quad \bar{M}_2(T_2) \leq 0,
$$
where $\bar{M}_2(\tau) = \frac{e^{\tau \Delta_1} + P_6}{e^{\tau \Delta_2} - P_3}$. $\Lambda_1 = \text{He}(Z_1) - \delta P_2$, $\Lambda_2 = [Z_1C_0 - P_2C_0A_0, Z_2]$, $\Psi = \begin{bmatrix} \Psi & 0 \\ V & I \end{bmatrix}$, $\Pi_1, \Pi_2$ are given in (42). Let $U \in \mathbb{R}^{(n_p+n_s)\times(n_p+n_s)}$ be any nonsingular matrix such that
$$
XY + UV^T = I.
$$
Then the conditions in Theorem 2 and subsequently those of Property 1 are satisfied. With the selection of control and hold gains $\Delta_K, \Delta_T$ as in (41) and (43), the solution to Problem 1 is obtained.

VI. NUMERICAL EXAMPLE

In this section, we present a numerical example to illustrate the effectiveness of our designed stabilizer for a double integrator plant (1) given as follows
$$
A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_p = [1 \ 0],
$$
$$
C_p = 10E_p, \quad F_p = 20B_p^T,
$$
and exosystem (2) is a single frequency harmonic oscillator of the form
$$
S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma(S) = \pm 1i.
$$
It is easy to verify that the Assumptions 1, 2 are satisfied. Then, according to [1], we select 1- copy internal model of the exosystem (8) as
$$
G_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$
With the above system parameters of the extended plant model $\mathcal{P}$ in Fig. 1, we then proceed to obtain a numerical solution to the set of LMIs in [1]. Numerical solutions to LMIs is obtained through a YALMIP toolbox [14] in Matlab and the SDPT3 solver [15]. To enforce the nonsingularity of matrix $V$ in Proposition 1 and avoid ill-conditioned controller matrix $\Delta_K$ (41), following [10], we respectively consider two additional constraints
$$
V + V^T > 0, \quad -50 \cdot \bar{P}_1 \leq \Pi_1 + \Pi_1^T \leq -0.2 \cdot \bar{P}_1,
$$
where matrices $\Pi_1$ and $\bar{P}_1$ are defined in (42), and (44). With our proposed design methodology, we find a feasible solution to the LMIs in Proposition 1 for $T_2 = 0.3$, $\alpha = 4.35$, $\delta = 3.5$ and the stabilizer and hold gain matrices are given as follows
$$
A_c = \begin{bmatrix} -134.8 & -2.71 & 1.23 & 1.15 \\ -121.38 & -27.87 & 15.25 & 10.68 \\ -101.35 & -23.71 & 10.69 & 9.69 \\ -112.5 & -30.24 & 2.82 & -11.37 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.09 \\ 0.83 \\ 0.7 \\ 0.78 \end{bmatrix},
$$
$$
C_c = \begin{bmatrix} 433.9 & -9.39 & -46.65 & -113.4 \end{bmatrix}, \quad D_c = -0.27, \quad H = -0.398, \quad E = [-0.398 \ 0.036 \ 0.465 \ 0.031],
$$
(54)
With these controller parameters, and minimum dwell time $T_1 = 0.1$, we observe from Figure 2 that the plant output successfully tracks a reference exosystem trajectory $F_p w(t)$ and thus $\lim_{t \to \infty} e(t) = 0$.

VII. CONCLUSION

In this paper, we studied the output regulation problem of a linear plant with sporadic output measurements. With the postprocessing internal model architecture we have turned the output regulation problem into a stabilization problem. Then, by designing hybrid stabilizer and hold devices, and with the help of Lyapunov analysis along with the “input-to-state-stability small gain” philosophy, we show that the overall closed loop system is globally exponentially stable. The sufficient stability conditions are turned into LMIs and the solution to these LMI is used to design the stabilizer and hold gains. With the help of a numerical example we have illustrated the effectiveness of our theoretical contribution.
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