UNBOUNDED NEGATIVITY ON RATIONAL SURFACES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We give explicit blowups of the projective plane in positive characteristic that contain smooth rational curves of arbitrarily negative self-intersection, showing that the Bounded Negativity Conjecture fails even for rational surfaces in positive characteristic.

INTRODUCTION

A smooth projective surface $X$ over an algebraically closed field is said to have Bounded Negativity if there exists a positive integer $b(X)$ such that $C^2 \geq -b(X)$ for any reduced curve $C \subset X$. A folklore conjecture, going back to Enriques and discussed in [Har10, Conjecture I.2.1] and [BHK+13, Conjecture 1.1], is the

Bounded Negativity Conjecture. — Any smooth projective surface in characteristic 0 has Bounded Negativity.

The assumption on the characteristic cannot be dropped: if $C$ is a curve over $\overline{\mathbb{F}}_p$, then the graph $\Gamma_{p^e} \subseteq C \times C$ of the $p^e$-th power Frobenius endomorphism has self-intersection $p^e(2 - 2g)$, which becomes arbitrarily negative as $e \to \infty$ when $g \geq 2$. Nonetheless, it is conceivable that certain geometric assumptions on the surface may still guarantee Bounded Negativity in positive characteristic. For instance, [BBC+12, discussion preceding Example 3.3.3] and [Har18, Conjecture 2.1.2] ask whether smooth rational surfaces over a field of positive characteristic have Bounded Negativity. We give a negative answer to this question:

Main Theorem. — Let $k$ be an algebraically closed field of characteristic $p > 0$, let $m$ be a positive integer invertible in $k$, and let $R_m$ be the blowup of $\mathbb{P}^2$ along

$$Z_m := \{ [x_0 : x_1 : x_2] \mid x_0^m = x_1^m = x_2^m \}.$$

Let $C_1 = V(x_0 + x_1 + x_2) \subseteq \mathbb{P}^2$, and for $d \geq 1$ invertible in $k$, write $C_d \subseteq \mathbb{P}^2$ for the image of

$$\phi_d : C_1 \to \mathbb{P}^2$$

$$[x_0 : x_1 : x_2] \mapsto [x_0^d : x_1^d : x_2^d].$$

If $dm = p^e - 1$ for some positive integer $e$, then the strict transform $\tilde{C}_d \subseteq R_m$ of $C_d$ is a smooth rational curve with $\tilde{C}_d^2 = d(3 - m) - 1$. Thus, if $m > 3$, the rational surface $R_m$ does not have Bounded Negativity over $k$. 

Date: 2 March 2021.

2010 Mathematics Subject Classification. 14C17 (primary); 14C20, 14G17, 14J26, 14E05 (secondary).

Key words and phrases. Bounded negativity conjecture, rational surfaces, positive characteristic, Fermat varieties, line configurations, Bogomolov–Miyaoka–Yau inequality.
Since $\mathbb{P}^2$ has Bounded Negativity, this shows that [BDRH+15, Problem 1.2] has a negative answer in positive characteristic:

**Corollary.** — Bounded Negativity is not a birational property of smooth projective surfaces in positive characteristic.

In fact, since every smooth projective surface $X$ admits a finite morphism $X \to \mathbb{P}^2$, pulling back the blowup $R_m \to \mathbb{P}^2$ gives a blowup $\tilde{X} \to X$ with a finite morphism $\tilde{X} \to R_m$. Pulling back the curves $\tilde{C}_d$ to $\tilde{X}$ shows:

**Corollary.** — If $X$ is a smooth projective surface over an algebraically closed field $k$ of positive characteristic, then there exists a blowup $\tilde{X} \to X$ such that $\tilde{X}$ does not have Bounded Negativity.

In §1, we give a direct proof of the Main Theorem. In §2, we realise the plane curves $C_d$ as norms of line configuration, thereby deriving equations for them. In §3, we view $R_m$ as an isotrivial family of diagonal curves over $C_d$ and relate the curves $\tilde{C}_d$ on $R_m$ to graphs of Frobenius morphisms on Fermat curves. Finally, we close in §4 with some questions and remarks towards characteristic zero.

Sections 2 and 3 each give alternative methods for computing the self-intersections of $\tilde{C}_d$. Given the simplicity of the formulas for $\tilde{C}_d$ and the many connections to other well-studied examples, it is surprising that these curves have not been found before.

**Notation**

Throughout the paper, $k$ will be an algebraically closed field of arbitrary characteristic, and $m$ and $d$ will denote positive integers invertible in $k$. We will use the notation of the Main Theorem throughout.

**1. Proof of Main Theorem**

In this section, fix $m$ and write $R := R_m$ for the blowup of $\mathbb{P}^2$ along $Z := Z_m$. The generators $s_0 = x_1^m - x_2^m$, $s_1 = x_2^m - x_0^m$, and $s_2 = x_0^m - x_1^m$ of the ideal of $Z$ give a closed immersion $R \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. Since $s_0 + s_1 + s_2 = 0$, one of the $s_i$ can be eliminated at the expense of breaking the symmetry in the computations below.

**1.1. Lemma.** — The embedding $R \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ realises $R$ as the complete intersection

$$\left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \bigg| \begin{array}{l}
  y_0 + y_1 + y_2 = 0 \\
  x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0
\end{array} \right\}$$

of degrees $(0,1)$ and $(m,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. In particular, $K_R = \mathcal{O}_R(m - 3, -1)$.

**Proof.** The generators $s_0, s_1, s_2$ of the ideal of $Z$ identify $R$ as

$$\left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \bigg| \begin{array}{l}
  y_0(x_2^m - x_0^m) = y_1(x_1^m - x_2^m) \\
  y_1(x_0^m - x_1^m) = y_2(x_2^m - x_0^m) \\
  y_2(x_1^m - x_2^m) = y_0(x_0^m - x_1^m)
\end{array} \right\}.$$
The relation \( s_0 + s_1 + s_2 = 0 \) shows that \( R \) is contained in the locus \( y_0 + y_1 + y_2 = 0 \). The equation \( y_0(x_2^m - x_0^m) = y_1(x_1^m - x_2^m) \) can be rewritten as \( (y_0 + y_1)x_2^m = x_0^m y_0 + x_1^m y_1 \), which is equivalent to \( x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0 \) since \( y_0 + y_1 + y_2 = 0 \). The same holds for the other two equations by symmetry. The final statement then follows from the adjunction formula since \( K_{P^1 : P^2} = O(-3, -3) \).

Alternatively, one can observe that the complete intersection of Lemma 1.1 maps birationally onto its first factor, where the fibres are points when \([x_0^m : x_1^m : x_2^m] \neq [1 : 1 : 1]\) and lines otherwise.

1.2. Lemma. — If char \( k = p > 0 \) and \( dm = p^e - 1 \) for some positive integer \( e \), then the map \( \tilde{\phi}_d : C_1 \to \mathbb{P}^2 \times \mathbb{P}^2 \) given by
\[
[x_0 : x_1 : x_2] \mapsto ([x_0^d : x_1^d : x_2^d], [x_0 : x_1 : x_2])
\]
lands in \( R \). In particular, it is the unique map lifting \( \phi_d : C_1 \to \mathbb{P}^2 \).

Proof. Since \( x_0 + x_1 + x_2 = 0 \), the image of \( \tilde{\phi}_d \) is contained in the locus \( y_0 + y_1 + y_2 = 0 \). Since \( dm = p^e - 1 \), the expression \( x_0^m y_0 + x_1^m y_1 + x_2^m y_2 \) pulls back to \( x_0^{p^e} + x_1^{p^e} + x_2^{p^e} \), which vanishes because the \( p^e \)-th power Frobenius is an endomorphism. Thus \( \tilde{\phi}_d \) is a lift of \( \phi_d \) to \( R \), and it is the unique lift since the first projection \( \pi_1 : R \to \mathbb{P}^2 \) is birational.

1.3. Corollary. — The map \( \tilde{\phi}_d : C_1 \to R \) is a closed immersion, whose image \( \bar{C}_d \) is a smooth rational curve in \( R \) with \( \bar{C}_d^2 = d(3 - m) - 1 \).

Proof. The first two statements follow from the coordinate expression in Lemma 1.2, since \( \tilde{\phi}_d \) embeds \( C_1 \) linearly into the second factor. The same expression shows that \( \tilde{\phi}_d^* \mathcal{O}_R(a, b) = \mathcal{O}_{C_1}(da + b) \), so \( K_R \cdot \bar{C}_d = d(m - 3) - 1 \) by Lemma 1.1. Then the adjunction formula shows that
\[
\bar{C}_d^2 = -2 - K_R \cdot \bar{C}_d = d(3 - m) - 1.
\]

This completes the proof of the Main Theorem.

A consequence of Corollary 1.3 is that the singularities of \( C_d \) are contained in \( Z \). However, the individual multiplicities are not so easy to determine. For example, in Lemma 3.8 we will compute the multiplicity of \( C_d \) at \([1 : 1 : 1]\) in terms of point counts on Fermat curves.

2. Relation with line configurations

In this section, we observe that the \( d \)-th power maps \( \pi_d : \mathbb{P}^2 \to \mathbb{P}^2 \) are finite Galois morphisms such that \( \pi_d^* C_d \) is the union of the Galois translates of \( C_1 \). Thus the \( C_d \) are norms of line configurations, from which we derive in Corollary 2.3 a formal product formula for the equation of the plane curves \( C_d \). In the second half of this section, we observe that in characteristic \( p > 0 \) and for \( q \) a power of \( p \), the curve \( C_{q-1} \) comes from a subconfiguration of the set of all \( F_q \)-rational lines. This allows us to show in Corollary 2.7 that an equation of \( C_{q-1} \) in this case is the complete homogeneous polynomial of degree \( q - 1 \).

2.1. Power Maps. For any integer \( a \geq 1 \) invertible in \( k \), write \( \pi_a \) for the \( a \)-th power map \( \mathbb{P}^2 \to \mathbb{P}^2 \). Since \( \pi_a^* \mathbb{Z}_m = \mathbb{Z}_{am} \), the map \( \pi_a \) lifts to a finite morphism \( \overline{\pi}_a : R_{am} \to R_m \) given by
\[
([x_0 : x_1 : x_2], [y_0 : y_1, y_2]) \mapsto ([x_0^a : x_1^a : x_2^a], [y_0 : y_1, y_2]).
\]
Since $a$ is invertible in $k$, both $\pi_a$ and $\bar{\pi}_a$ are finite Galois with group $G = \mu_a^3/\mu_a$, where $(\zeta_0, \zeta_1, \zeta_2) \in G$ acts on $\mathbb{P}^2$ via

$$[x_0 : x_1 : x_2] \mapsto [\zeta_0 x_0 : \zeta_1 x_1 : \zeta_2 x_2].$$

This gives a tower of extensions

$$\begin{array}{c|c|c}
R_4 & R_6 & R_9 \\
\hline
\vdots & \vdots & \vdots \\
R_2 & R_3 & \vdots \\
\hline
R_1 & & \\
\end{array}$$

indexed by the poset of positive integers invertible in $k$ under the divisibility relation.

2.2. Lemma. — If $a, d \geq 1$ are invertible in $k$, then

(i) The map $\phi_d : C_1 \to \mathbb{P}^2$ is unramified and birational onto its image;

(ii) The inverse image $\pi_a^*C_{ad}$ is totally split into the $G$-translates of $C_d$.

Proof. The Jacobian $d \cdot x_1^{d-1}, d \cdot x_2^{d-1}, d \cdot x_3^{d-1})$ of $C_d$ only vanishes when $x_0 = x_1 = x_2 = 0$, showing that $\phi_d$ is unramified. Then the map $C_1 \to C_d$ to the normalisation of $C_d$ is unramified, hence an isomorphism since it is an étale map of smooth projective rational curves, proving (i).

Since $\pi_a \circ \phi_d = \phi_{ad}$, part (i) shows that $\pi_a$ maps $C_d$ birationally onto its image. This shows that the decomposition group of $C_d$ is trivial, so no two $G$-translates $\zeta C_d$ of $C_d$ coincide and $C_{ad}$ is totally split under $\pi_a$.

2.3. Corollary. — If $d$ is invertible in $k$, then the homogeneous ideal of $C_d \subseteq \mathbb{P}^2$ is generated by

$$f_d := N_{\pi_d, 0} \bigoplus_{\nu_2} (x_0^{1/d} + x_1^{1/d} + x_2^{1/d}) = \prod_{\zeta, \zeta' \in \mu_d} (x_0^{1/d} + \zeta x_1^{1/d} + \zeta' x_2^{1/d}).$$

Proof. By Lemma 2.2 (ii), the inverse image $\pi_d^{-1}(C_d)$ is the union of lines $\bigcup_{\zeta \in \mu_d^3/\mu_d} \zeta C_1$. The result follows since $C_1$ is cut out by $x_0 + x_1 + x_2 = 0$.

2.4. In general, the $f_d$ are complicated symmetric polynomials. However, in Corollary 2.7 we will show that the coefficients of $f_q^{-1}$ are congruent to 1 modulo $p$ if $q$ is a power of a prime $p$. For example, for $q = 3$, we get

$$N(x_0^2 + x_1^2 + x_2^2) = \left(x_0^2 + x_1^2 + x_2^2\right)^3\left(x_0^2 + x_1^2 - x_2^2\right)^3\left(x_0^2 - x_1^2 - x_2^2\right)^3\left(x_0^2 - x_1^2 - x_2^2\right)^3$$

$$= x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_1x_2 - 2x_2x_0$$

$$\equiv x_0^2 + x_1^2 + x_2^2 + x_0x_1 + x_1x_2 + x_2x_0 \pmod{3}.$$ In the remainder of this section, assume $\text{char } k > 0$ and let $q$ be a power of $p$.

2.5. Finite Field Line Configurations. The configuration of $F_q$-rational lines in $\mathbb{P}^2$ is the union of the lines $L_a = \{aqx_0 + a_1x_1 + a_2x_2 = 0\}$ indexed by $a = [a_0 : a_1 : a_2] \in \mathbb{P}^2(F_q)$. Their union is the divisor in $\mathbb{P}^2$ cut out by the polynomial

$$\det \begin{pmatrix} x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q \\ x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q \\ x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q & x_0^q x_1^q x_2^q \end{pmatrix} = x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q - x_0^q x_1^q x_2^q,
since the three columns become linearly dependent when \(x_0, x_1, \) and \(x_2\) satisfy a linear relation over \(\mathbb{F}_q\), and the degree equals \(q^2 + q + 1 = |\mathbb{P}^2(\mathbb{F}_q)|\). Now Lemma 2.2(ii) shows that \(\pi_{q-1}^n C_{q-1}\) consists of the \(q^2 - 2q + 1\) lines \(L_a\) with all coordinates of \(a = [a_0 : a_1 : a_2]\) nonzero. We can thus derive an equation for \(C_{q-1}\) by extracting factors cutting out the lines \(L_a\) in which \(a\) has a vanishing coordinate. A neat description of the result comes from the following polynomial identity, also observed in [RVVZ01, p. 90]:

2.6. Lemma. — For any nonnegative integer \(n\), define the polynomials
\[
g_n := \sum_{n_0 + n_1 + n_2 = n} x_0^{n_0} x_1^{n_1} x_2^{n_2} \quad \text{and} \quad h_n := x_0 x_1^n - x_0 x_1 x_2 x_2^n + x_2 x_0 x_0^n - x_2 x_0 x_0 x_2
\]
in \(\mathbb{Z}[x_0, x_1, x_2]\). Then \(h_2 = (x_2 - x_1)(x_0 - x_2)(x_1 - x_0)\) and \(h_n = h_2 g_{n-2}\) for \(n \geq 3\).

Proof. Let \(G(t) := \sum_{n \geq 0} g_n t^n\) and \(H(t) := \sum_{n \geq 0} h_n t^n\) be the generating functions of the \(g_n\) and \(h_n\), respectively. A standard computation gives
\[
G(t) = \frac{1}{(1-x_0 t)(1-x_1 t)(1-x_2 t)}.
\]
On the other hand, writing \(h_n = (x_2 - x_1) x_0^n + (x_0 - x_2) x_1^n + (x_1 - x_0) x_2^n\) gives
\[
H(t) = x_2 - x_1 + \frac{x_0 - x_2}{1-x_0 t} + x_1 - x_0 \frac{x_2 - x_1}{1-x_2 t} + (x_0 - x_2) x_0^2 x_2^2 + (x_1 - x_0) x_0 x_1 x_2
\]
\[
= (1-x_0 t)(1-x_1 t)(1-x_2 t).
\]
The result follows by recognising the numerator as \(h_2\).

2.7. Corollary. — Suppose \(\text{char } k = p > 0\) and \(q\) is a power of \(p\). Then \(g_{q-1}\) generates the homogeneous ideal of \(C_{q-1} \subseteq \mathbb{P}^2\). In particular, \(f_{q-1} \equiv g_{q-1} \pmod{p}\).

Proof. Since \(C_1 = L_{[1:1:1]}\) is among the \(\mathbb{F}_q\)-rational lines of 2.5 and is not a coordinate axis, the points \([x_0 : x_1 : x_2]\) of \(C_1 = V(x_0 + x_1 + x_2)\) satisfy the equation there divided by \(x_0 x_1 x_2\):
\[
x_0^{q-1} x_1^{q-1} - x_0^{q-1} x_1 x_2^{q-1} + x_1^{q-1} x_2^{q-1} - x_1^{q-1} x_2 + x_2^{q-1} x_0^{q-1} - x_2^{q-1} x_0 x_1^{q-1} = 0.
\]
Since \(C_1 \to C_{q-1}\) is \([x_0 : x_1 : x_2] \mapsto [x_0^{q-1} : x_1^{q-1} : x_2^{q-1}]\), any \([x_0 : x_1 : x_2] \in C_{q-1}\) satisfies
\[
x_0 x_1 x_2^{q-1} - x_0^{q-1} x_1 + x_1 x_2^{q-1} - x_1^{q-1} x_2 + x_2 x_0^{q-1} - x_2^{q-1} x_0 = 0.
\]
So \(h_{q+1}\) vanishes on \(C_{q-1}\), which by Lemma 2.6 equals \((x_2 - x_1)(x_0 - x_2)(x_1 - x_0) g_{q-1}\). The result follows since \(C_{q-1}\) is not contained in any of the lines \(\{x_2 = x_1\}, \{x_0 = x_2\},\) or \(\{x_1 = x_0\}\), and \(\text{deg } g_{q-1} = q - 1 = \text{deg } C_{q-1}\).

2.8. Negative Curves via Equations. If \(m > 3\) and \(q\) is a power of \(p\) congruent to 1 modulo \(m\), then the curves \(\tilde{C}_d \subseteq R_m\) with \(dm = q^2 - 1\) of the Main Theorem can therefore be obtained by starting with the very explicit equations
\[
C_{q-1} = V \left( \sum_{n_0 + n_1 + n_2 = q-1} x_0^{n_0} x_1^{n_1} x_2^{n_2} \right) \subseteq \mathbb{P}^2,
\]
blowing up at \([1 : 1 : 1]\), pulling back along \(\tilde{\pi}_m : R_m \to R_1\), and taking one of the \(m^2\) isomorphic components \(\zeta \tilde{C}_d\) for \(\zeta \in \mu^2_m/\mu_m\). From this point of view, the self-intersection may be computed as
\[
\tilde{C}_d^2 = \tilde{C}_d \cdot \tilde{\pi}_m^* (\tilde{C}_{q-1}^2) - \sum_{\zeta \neq 1} \tilde{C}_d \cdot (\zeta \tilde{C}_d) = (2dm - 1) - 3(m - 1)d = d(3 - m) - 1,
\]
Corollary 3.6

Thus, 

3.2. Generalized Power Maps.

3.1. Intermediate Surfaces.

\[ \text{consist of } m \]

\[ \text{R} \]

Note that \( m \)

It is smooth if and only if \( R \)

\[ \text{morphism of } C \]

\[ \text{curves. Pulling back along the Frobenius morphism of } C \]

\[ \text{e} \]

\[ \text{since the intersection number between } \tilde{C}_d \text{ and a Galois translate by } \zeta = (\zeta_0, \zeta_1, \zeta_2) \in G \setminus \{1\} \]

\[ \tilde{C}_d \cdot (\zeta \tilde{C}_d) = \begin{cases} d, & \text{if } \zeta_0 = \zeta_1 \text{ or } \zeta_1 = \zeta_2 \text{ or } \zeta_2 = \zeta_0, \\ 0, & \text{otherwise.} \end{cases} \]

Indeed, \( \zeta \tilde{C}_d \) is the image of the morphism \( \zeta \circ \tilde{\phi}_d \) given by

\[ [x_0 : x_1 : x_2] \mapsto ([\zeta_0 x_0^d : \zeta_1 x_1^d : \zeta_2 x_2^d], [x_0 : x_1 : x_2]). \]

Thus, \( \tilde{C}_d \) and \( \zeta \tilde{C}_d \) only intersect when \( \zeta \tilde{\phi}_d([x_0 : x_1 : x_2]) = \tilde{\phi}_d([x_0 : x_1 : x_2]) \). At most one of the \( x_i \) can vanish since \( x_0 + x_1 + x_2 = 0 \), so there are no intersections when \( \zeta_i \neq \zeta_j \) for \( i \neq j \), and a single intersection with multiplicity \( d \) at \( V(x_k) \) when \( \zeta_i = \zeta_j \) and \( \{i, j, k\} = \{0, 1, 2\} \).

3. Relation with Fermat Varieties and Frobenius Morphisms

By Lemma 1.1, the second projection \( \text{pr}_2 : R_m \to V(y_0 + y_1 + y_2) \) realises \( R_m \) as the family of diagonal degree \( m \) curves over \( C_1 \cong \mathbb{P}^1 \) given by

\[ x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0. \]

If \( \text{char } k = p > 0 \) and \( m \) is invertible in \( k \), then the curves \( \tilde{C}_d \subseteq R_m \) for \( dm = p^r - 1 \) are given by sections \( \tilde{\phi}_d : C_1 \to R_m \) of \( \text{pr}_2 \). In this section, we pull back the family \( R_m \to C_1 \) and the sections \( \tilde{\phi}_d \) along finite covers of \( C_1 \). Pulling back along covers by Fermat curves allows us to relate the \( \tilde{C}_d \) in Corollary 3.4 with graphs of Frobenius on products of Fermat curves. Pulling back along the Frobenius morphism of \( C_1 \) allows us to realise the \( \tilde{C}_d \) in Corollary 3.6 as pullbacks of a constant section \( \tilde{C}_0 \) under powers of a horizontal Frobenius morphism of \( R_m \) over \( C_1 \).

3.1. Intermediate Surfaces. For positive integers \( m \) and \( n \) invertible in \( k \) and \( r \in \mathbb{N} \), denote by \( R_{m,n,r} \) the normal surface

\[ R_{m,n,r} = \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \middle| \begin{array}{c} y_0^n + y_1^n + y_2^n = 0 \\ x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0 \end{array} \right\}. \]

It is smooth if and only if \( m = 1 \) or \( r \in \{0, 1\} \); in all other cases, the singular locus \( V(x_0 y_0, x_1 y_1, x_2 y_2) \) consists of the \( 3n \) points

\[ \left\{ \left( [1 : 0 : 0], [0 : s : t] \right), \left( [0 : 1 : 0], [s : 0 : t] \right), \left( [0 : 0 : 1], [s : t : 0] \right) \right\} \big| s^n + t^n = 0 \}

Note that \( R_{m,1,1} \) is none other than the surface \( R_m \) of Lemma 1.1. If \( X_n \) denotes the Fermat curve \( V(y_0^n + y_1^n + y_2^n) \subseteq \mathbb{P}^2 \) of degree \( n \), then \( R_{m,n,0} \) coincides with \( X_m \times X_n \). The surfaces \( R_{m,n,r} \) for \( r > 0 \) come with a projection

\[ \text{pr}_2 : R_{m,n,r} \to X_n \]

that is smooth away from the \( 3n \) fibres above \( V(y_0 y_1 y_2) \subseteq X_n \), and whose singular fibres consist of \( m \) lines meeting at a point.

3.2. Generalized Power Maps. For positive integers \( a \) and \( b \) invertible in \( k \), define the finite morphism

\[ \pi_{a,b} : R_{m,a,b,r} \to R_{m,n,r} \]

\[ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_0^a : x_1^a : x_2^a], [y_0^b : y_1^b : y_2^b]). \]
For $b = 1$ and $n = r = 1$, it coincides with the morphism $\bar{\pi}_a$ from 2.1. When $a = 1$, these fit into pullback squares

$$
\begin{array}{ccc}
R_{m,bn,br} & \xrightarrow{\pi_{1,b}} & R_{m,n,r} \\
pr_2 & & pr_2 \\
X_{bn} & \longrightarrow & X_n
\end{array}
$$

If $F^e : X_n \to X_n$ is the $p^e$-th power Frobenius morphism of $X_n$, there are pullback squares

$$
\begin{array}{ccc}
R_{m,n,p^er} & \longrightarrow & R_{m,n,r} \\
pr_2 & & pr_2 \\
X_n & \xrightarrow{F^e} & X_n
\end{array}
$$

so $R_{m,n,p^er}$ is the Frobenius twist $R_{m,n,r}^{(e)}$ of $R_{m,n,r}$ over $X_n$. We denote the top map by $\pi^{(e)}$.

**3.3. Lemma.** — Let $m$ and $n$ be positive integers invertible in $k$, let $a$, $r$ and $r + am$ be nonnegative. Then the map

$$
\psi_a : \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2 \times \mathbb{P}^2
$$

maps $R_{m,n,r+am}$ birationally onto $R_{m,n,r}$.

**Proof.** Note that $\psi_a$ is a birational map with rational inverse $\psi_{-a}$. The result follows since $\psi_a$ takes $R_{m,n,r+am}$ into $R_{m,n,r}$ and $\psi_{-a}$ does the opposite, and neither surface is contained in the locus where $\psi_a$ or $\psi_{-a}$ is undefined.

This allows us to relate $R_{m,m,0}$ and $X_m \times X_m$:

**3.4. Corollary.** — The surfaces $X_m \times X_m \cong R_{m,m,0}$ and $R_{m,m,m}$ are birational via

$$
\psi : X_m \times X_m \xrightarrow{\sim} R_{m,m,m}
$$

$$(\{x_0 : x_1 : x_2\}, \{y_0 : y_1 : y_2\}) \mapsto \left(\left\{\frac{x_0}{y_0} : \frac{x_1}{y_1} : \frac{x_2}{y_2}\right\}, \{y_0 : y_1 : y_2\}\right).
$$

The composition $\rho : X_m \times X_m \to R_m$ of $\psi$ with $\pi_{1,m}$ is given by

$$(\{x_0 : x_1 : x_2\}, \{y_0 : y_1 : y_2\}) \mapsto \left(\left\{\frac{x_0}{y_0} : \frac{x_1}{y_1} : \frac{x_2}{y_2}\right\}, \{y_0^m : y_1^m : y_2^m\}\right).
$$

If $\text{char } k = p > 0$, $m$ is invertible in $k$, and $dm = p^e - 1$ for some positive integer $e$, then the strict transform of $\pi_{1,m}^e \tilde{C}_d$ under $\psi$ is the transpose $\Gamma_{F^e}^\top$ of the graph of the $p^e$-power Frobenius.

**Proof.** The first statement follows by applying Lemma 3.3 to $m = n = r$ and $a = -1$, and the second is immediate from the definitions. For the final statement, recall that $\Gamma_{F^e}^\top$ is given by the section $s : X_m \to X_m \times X_m$ of $pr_2$ given by

$$
\{y_0 : y_1 : y_2\} \mapsto \left(\left\{y_0^{p^e} : y_1^{p^e} : y_2^{p^e}\right\}, \{y_0 : y_1 : y_2\}\right).
$$

By the first pullback square of 3.2, the curve $\pi_{1,m}^e \tilde{C}_d$ is the image of the section $X_m \to R_{m,m,m}$ given by

$$
\{y_0 : y_1 : y_2\} \mapsto \left(\left\{y_0^m : y_1^m : y_2^m\right\}, \{y_0 : y_1 : y_2\}\right),
$$

which agrees with $\psi \circ s$. 

■
3.5. The curves $\Gamma_p \subseteq X_m \times X_m$ are the standard example of curves with unbounded negative self-intersection: the condition $m > 3$ of the Main Theorem is exactly the condition $g(X_m) > 1$ that makes $\Gamma_p^g = p^g(2 - 2g)$ negative. In fact, since $\Gamma_p^\top$ passes through $3m$ of the $3m^2$ points of indeterminacy of $\psi$, resolving the map shows that $m^2 C_d^2 = \Gamma_p^2 - 3m$.

On the other hand, $R_m \to C_1$ is an isotrivial family of diagonal degree $m$ curves that becomes rationally trivialised over the $m$-th power cover $X_m \to C_1$. Thus, we can also look directly at the pullback $\pi^e: R_m^e \to R_m$ of the Frobenius $F^e: C_1 \to C_1$. Note that $R_m^e = R_{m,p^e}$ by 3.2, so we get:

3.6. Corollary. — If $p^e = dm + 1$, then $R_m^e$ is birational to $R_m$ via

$$
\psi: R_m \sim \to R_m^e \quad \left( [x_0: x_1 : x_2], [y_0: y_1 : y_2] \mapsto \left( \frac{x_0}{y_0}, \frac{x_1}{y_1}, \frac{x_2}{y_2} \right), [y_0: y_1 : y_2] \right).
$$

If $\tilde{\phi}_d: C_1 \to R_m$ denotes the constant section $[y_0: y_1 : y_2] \mapsto \left( [1 : 1 : 1], [y_0: y_1 : y_2] \right)$ and $\tilde{C}_0 \subseteq R_m$ denotes its image, then $\tilde{C}_d$ is the strict transform of $\pi^e \cdot \tilde{C}_0$ under $\psi$.

Proof. The first statement follows from Lemma 3.3 applied to $n = 1$, $r = p^e$, and $a = -d$. For the second, by the second pullback square of 3.2, the curve $\pi^e \cdot \tilde{C}_0$ is the image of the constant section $C_1 \to R_m^e$ given by

$$
[y_0: y_1 : y_2] \mapsto \left( [1 : 1 : 1], [y_0: y_1 : y_2] \right),
$$

which agrees with $\psi \circ \tilde{\phi}_d$.

3.7. Instead of the transpose $\Gamma_p^\top$ of the graph of $F^e: X_m \to X_m$, one can also look at the negative curves $\Gamma_p \subseteq X_m \times X_m$, which are given by pulling back the diagonal along the relative Frobenius of $\rho_2: X_m \times X_m \to X_m$. Their images under the rational map $\rho$ of Corollary 3.4 are given by the parametrised rational curves

$$
C_1 \to R_m, \\
[y_0: y_1 : y_2] \mapsto \left( [y_1^d y_2^d : y_0^d y_1^d : y_0^d y_2^d], [y_0^p: y_1^p : y_2^p] \right),
$$

where $dm = p^e - 1$ as usual. These are obtained from the curve $\tilde{C}_0$ of Corollary 3.6 by pulling back the strict transform of $\tilde{C}_0$ under $\psi^{-1}: R_m^e \sim \to R_m$ along the relative Frobenius $F^e_{\rho_2^{-1}/C_1}$. Note that the images of these curves in $\mathbb{P}^2$ differ from the curves $C_d$ by the Cremona transformation

$$
[x_0: x_1 : x_2] \mapsto [x_0^{-1}: x_1^{-1}: x_2^{-1}].
$$

Finally, we relate the multiplicity of $C_d$ at $[1 : 1 : 1]$ to point counts on the Fermat curve $X_m$ if $dm = p^e - 1$ for some positive integer $e$.

3.8. Lemma. — If $dm = p^e - 1$, then a point $x \in C_d$ maps to $[1 : 1 : 1]$ in $C_d$ if and only if there exists $y \in X_m(F_p^e)$ with nonzero coordinates mapping to $x$ under the $m$-th power map $X_m \to C_1$. In particular,

$$
\text{mult}_{[1:1:1]} C_d = \frac{|X_m(F_p^e)| - 3m}{m^2}.
$$

Proof. The first statement follows since $X_m \to C_1$ is surjective and a point $y = [y_0: y_1 : y_2]$ on $X_m$ with nonzero coordinates maps to $[1 : 1 : 1]$ under the $(p^e - 1)$-st power map $X_m \to C_d$ if and only if $y \in X_m(F_p^e)$. 

For the second statement, note that \( \text{mult}_{[1:1:1]} \mathcal{C}_d \) equals the number of preimages of [1:1:1] in \( C_1 \), since \( \phi_d : C_1 \to \tilde{C}_d \) is an isomorphism by Corollary 1.3. The result now follows since \( X_m \to C_1 \) is finite étale of degree \( m^2 \) away from the coordinate axes, so each point \( x \in C_1 \setminus V(x_0x_1x_2) \) has exactly \( m^2 \) preimages in \( X_m \).

For example, if \( p^v \equiv -1 \pmod{m} \) for some positive integer \( v \), then
\[
\vert X_m(F_{p^v}) \vert = 1 - \frac{(m-1)(m-2)}{2} p^{e/2} + p^e
\]
whenever \( p^v \equiv 1 \pmod{m} \) [SK79, Lem. 3.3].

### 4. Remarks towards characteristic 0

#### 4.1. Although the Bounded Negativity Conjecture is currently still open in characteristic 0, the Weak Bounded Negativity Conjecture is known [Hao19]: for any smooth projective complex surface \( X \) and any \( g \in \mathbb{N} \), there exists a constant \( b(X, g) \) such that \( C^2 \geq -b(X, g) \) for every reduced curve \( C = \sum_i C_i \) whose components \( C_i \) have geometric genus at most \( g \).

Our examples in the Main Theorem certainly violate this, and, as we now verify, arise from the failure of the logarithmic Bogomolov–Miyaoka–Yau inequality for the pair \((R_m, \tilde{C}_d)\) when \( d \) is large with respect to \( m \). In the next three paragraphs, assume \( \text{char} \, k = p > 0 \) and \( dm = p^e - 1 \). To ease notation, write \((R, \tilde{C})\) for \((R_m, \tilde{C}_d)\). We will use logarithmic sheaves of differentials; see for example [EV92, §2].

#### 4.2. Lemma. — The Chern numbers of the pair \((R, \tilde{C})\) are
\[
c_1^2(R, \tilde{C}) := c_1^2(\Omega_R^1(\log \tilde{C})) = d(m-3) - m^2 + 6,
\]
\[
c_2(R, \tilde{C}) := c_2(\Omega_R^1(\log \tilde{C})) = m^2 + 1.
\]
In particular, the Chern slopes \( c_1^2(R, \tilde{C})/c_2(R, \tilde{C}) \) are unbounded for fixed \( m \) and growing \( d \).

**Proof.** The logarithmic sheaf of differentials fit into a short exact sequence
\[
0 \to \Omega^1_R \to \Omega^1_R(\log \tilde{C}) \to \mathcal{O}_{\tilde{C}} \to 0,
\]
so \( c_1^2(R, \tilde{C}) = (K_R + \tilde{C})^2 \) and \( c_2(R, \tilde{C}) = c_2(\Omega_R^1) + \tilde{C}(K_R + \tilde{C}) \). Since \( R \) is the blowup of \( \mathbb{P}^2 \) in \( m^2 \) points, we get \( K_R^2 = 9 - m^2 \) and \( c_2(\Omega_R^1) = 3 + m^2 \), so the result follows from the computations of the intersection numbers in Corollary 1.3.

#### 4.3. Lemma. — If \( m > 3 \) and \( d \) is such that
\[
\chi(2(K_R + \tilde{C})) = d(m-3) - m^2 + 5 > 0
\]
then \( H^0(R, 2(K_R + \tilde{C})) \neq 0 \). In particular, \( K_R + \tilde{C} \) is pseudoeffective.

**Proof.** The Euler characteristic statement follows from Riemann–Roch, so it remains to show that \( H^0(R, 2(K_R + \tilde{C})) \neq 0 \) once \( \chi(2(K_R + \tilde{C})) > 0 \). But \( H^2(R, 2(K_R + \tilde{C})) = H^0(R, -K_R - 2\tilde{C})^\vee \), and the latter vanishes since \( \tilde{C} \) is effective and \( -K_R = \mathcal{O}_R(3-m, 1) \) by Lemma 1.1.
For $d$ large with respect to $m$, this shows that $(R, \tilde{C})$ falls into the final case considered in [Hao19, §1.2, Case 2], and that the failure of Weak Bounded Negativity stems from the failure of the logarithmic Bogomolov–Miyaoka–Yau inequality:

**4.4. Corollary.** — If $m > 3$ and $d > \frac{5m^2 - 2}{m - 1}$, then $K_R + \tilde{C}$ is pseudoeffective and $c_1^2(R, \tilde{C})/c_2(R, \tilde{C}) > 4$.

Moreover, the pair $(R, \tilde{C})$ does not lift to the second Witt vectors $W_2(k)$.

*Proof.* The first part follows from **Lemma 4.2** and **Lemma 4.3**. The final statement follows from [Lan16, Proposition 4.3], since $(R, \tilde{C})$ violates the logarithmic Bogomolov–Miyaoka–Yau inequality. 

**4.5.** On the other hand, the surface $R_m$ itself does lift to characteristic 0. This gives new examples of surfaces $X \to \text{Spec} \mathbb{Z}$ such that almost all special fibres $X_{\overline{F}_p}$ (namely those with $p \nmid m$) violate bounded negativity. The same property holds for the square $C \times C$ of a curve $C \to \text{Spec} \mathbb{Z}$ of genus $\geq 2$, which is the classical counterexample to bounded negativity in positive characteristic. However, the rational surface $\hat{X} = R_m$ has the additional property that the specialisation maps $\text{NS}(X_{\overline{Q}}) \to \text{NS}(X_{\overline{F}_p})$ are isomorphisms for every prime $p \nmid m$.

**4.6. Question.** — Is it possible to determine the effective cone of $R_m$ for some $m \geq 4$? How does it depend on the characteristic of $k$?

For example, the curves in $\mathbb{P}^2$ cut out by the polynomials $g_{m-1}$ of **Lemma 2.6** are smooth of genus $\frac{(m-2)(m-3)}{2}$ in characteristic 0 [RVVZ01, Thm. 1], and the equation $g_{m-1}h_2 = h_{m+1}$ shows that $V(g_{m-1}) \cup V(x_0 - x_1) \cup V(x_1 - x_2) \cup V(x_2 - x_0)$ contains

$$Z_m' := V \left( \begin{array}{c} x_0x_1^{m+1} - x_1^{m+1}x_0, \\ x_1x_2^{m+1} - x_2^{m+1}x_2, \\ x_2x_0^{m+1} - x_0^{m+1}x_0 \end{array} \right) = Z_m \cup \left\{ [s : t : 0], [s : 0 : t], [0 : s : t] \mid s^m = t^m \right\} .$$

Since $V(g_{m-1})$ has self-intersection $(m - 1)^2$ and passes through the $m^2 - 3m + 2$ points of $Z_m$, whose coordinates are pairwise distinct, its strict transform on $R_m$ has self-intersection $m - 1$. On the further blowup $R_m'$ of $\mathbb{P}^2$ in $Z_m'$, the strict transform has self-intersection $-2m + 2$, but unlike the situation described in 2.8, there does not appear to be an obvious way to produce infinitely many negative curves on a single rational surface this way.

When $m = p^e$ for some prime $p$, the specialisation to characteristic $p$ collapses $Z_m$ onto the point $[1 : 1 : 1]$, and the smooth curve $V(g_{m-1})$ becomes a rational curve that is highly singular at $[1 : 1 : 1]$. Even though these curves are not negative yet (see 2.8), taking different values of $e$ does give infinitely many curves on the same rational surface.

**4.7.** As far as we are aware, all known counterexamples to bounded negativity on a smooth projective surface $X$ over an algebraically closed field $k$ of characteristic $p > 0$ consist of a family $C_i$ of curves on $X$ for which there exist constants $a, b$ such that $C_i^2 = ap^i + b$ for all $i \in \mathbb{N}$.

**4.8. Question.** — If $X$ is a surface over an algebraically closed field $k$ of characteristic $p > 0$, is there a finite set $\{(a_i, b_i) \in \mathbb{Q}^2 \mid i \in I\}$ such that all integral curves $C \subseteq X$ with $C^2 < 0$ satisfy

$$C^2 = a_ip^e + b_i$$
for some positive integer $e$ and some $i \in I$? If not, is there some other way in which the self-intersections of negative curves on $X$ are “not too scattered”?

We can also consider the following uniform version:

4.9. Question. — If $X \to S$ is a smooth projective surface over a finitely generated integral base scheme $S$, does there exist a finite set $\{(a_i, b_i) \in \mathbb{Q}^2 \mid i \in I\}$ such that every geometrically integral curve $C \subseteq X_s$ of negative self-intersection in a fibre $X_s$ with $\text{char} \kappa(s) > 0$ satisfies

$$C^2 = a_i p^e + b_i$$

for some positive integer $e$ and some $i \in I$, where $p = \text{char} \kappa(s)$?

For example, for the surfaces $R_m \to \text{Spec} \mathbb{Z}[1/m]$ and the curves $\tilde{C}_d$ of the Main Theorem, we may take $a = \frac{3-m}{m}$ and $b = \frac{-3}{m}$, which do not depend on the characteristic of $\kappa(s)$.

4.10. Despite the failure of bounded negativity in positive characteristic, a positive answer to Question 4.9 still implies bounded negativity in characteristic 0 via reduction modulo primes. Indeed, the minimum $b_{\min} = \min\{b_i \mid i \in I\}$ is a lower bound for the self-intersection $C^2$ of a geometrically integral curve $C$ in the generic fibre, since the specialisations $C_s$ of $C$ satisfy $C^2_s = C^2$ for all $s \in S$ and remain geometrically integral for $s$ in a dense open set $U \subseteq S$, and

$$\bigcap_{p \in P} \left\{ a_i p^e + b_i \mid i \in I, e \in \mathbb{Z}_{>0} \right\} \subseteq \left[ b_{\min}, \infty \right)$$

for any infinite set of primes $P$. Thus, Question 4.9 is a natural analogue of the Bounded Negativity Conjecture in positive characteristic.

Acknowledgements

We thank Johan de Jong, Joaquín Moraga, Takumi Murayama, Will Sawin, and John Sheridan for helpful discussions. RvDdB was partly supported by the Oswald Veblen Fund at the Institute for Advanced Study.

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