NEW TYPE SERIES FOR POWERS OF π

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Abstract. Motivated by Ramanujan-type series and Zeilberger-type series, in this paper we investigate two new types of series for powers of π. For example, we prove that

\[
\sum_{k=0}^{\infty} (198k^2 - 425k + 210) \frac{k^3(2k)^3}{4096k} = -\frac{1}{21\pi}
\]

and

\[
\sum_{k=0}^{\infty} \frac{198k^2 - 227k + 47}{(2k)^3} = \frac{3264 - 4\pi^2}{63}.
\]

We also pose many conjectures in this new direction.

1. Introduction

In 1859, G. Bauer got the following identity:

\[
\sum_{k=0}^{\infty} \frac{4k + 1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}.
\]

This began the history of series for π⁻¹. In 1914, S. Ramanujan [16] posed 16 conjectural series of the following form:

\[
\sum_{k=0}^{\infty} \frac{bk + c}{m^k} a(k) = \frac{\lambda \sqrt{d}}{\pi}
\]

where \(b, c, m \in \mathbb{Z}\) with \(bm \neq 0\), \(d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}\) is squarefree, \(\lambda \in \mathbb{Q} \setminus \{0\}\), and \(a(k) \in \mathbb{Z}\) is one of the products

\[
\binom{2k}{k}^3, \binom{2k}{k}^2 \binom{3k}{k}, \binom{2k}{k}^2 \binom{4k}{k}, \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}.
\]

There are totally 36 series of the above type, which are called rational Ramanujan-type series for π⁻¹. For surveys of such series, one may consult [2, 3, 8] and S. Cooper [11, Chapter 14]). In general, a Ramanujan-type series (of type \(R\)) has the form

\[
\sum_{k=0}^{\infty} \frac{P_d(k)}{m^k} a(k) = \frac{\alpha}{\pi^d}, \quad (R)
\]

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where \( P_d(x) \in \mathbb{Z}[x] \) with \( \deg P = d > 0 \), \( m \) is a nonzero integer, \( \alpha \) is an algebraic number, and \( a(k) \in \mathbb{Z} \) for all \( k \in \mathbb{N} = \{0, 1, 2, \ldots\} \). For example, B. Gourevich used the PSLQ algorithm to find the conjectural identity

\[
\sum_{k=0}^{\infty} \frac{168k^3 + 76k^2 + 14k + 1}{2^{20k}} \left( \frac{2k}{k} \right)^7 = \frac{32}{\pi^3}.
\] (1.2)

In this paper we introduce a new type of series different from Ramanujan’s type. A series of type \( S \) has the form

\[
\sum_{k=0}^{\infty} \frac{P(k)Q_{2d}(k)}{m^k} a(k) = \frac{\alpha}{\pi^d},
\] (S)

where \( P(x), Q_{2d}(x) \in \mathbb{Z}[x] \) with \( \deg P > 0 \) and \( \deg Q_{2d} \leq 2d \), \( m \) is a nonzero integer, \( \alpha \) is an algebraic number, and \( a(k) \in \mathbb{Z} \) for all \( k \in \mathbb{N} \). We also investigate what happens if we replace \( \binom{2k}{k} \) in the general summand of the classical Ramanujan-type series by \( \binom{2k}{k+1} = \frac{kC_k}{2} \), where \( C_k \) denotes the Catalan number.

Now we state our first theorem which gives variants of the Ramanujan series involving cubes of central binomial coefficients.

**Theorem 1.1.** We have the following identities:

\[
\sum_{k=0}^{\infty} \frac{(6k^2 - 19k + 6)k^3}{256^k} \left( \frac{2k}{k} \right)^3 = -\frac{1}{12\pi},
\] (S1)

\[
\sum_{k=0}^{\infty} \frac{(13608k^2 + 25050k + 10589)k^3}{256^k} \left( \frac{2k}{k+1} \right)^3 = 27296 - \frac{84604}{\pi},
\] (S1’)

\[
\sum_{k=0}^{\infty} \frac{(18k^2 - 29k + 16)k^3}{(-512)^k} \left( \frac{2k}{k+1} \right)^3 = \sqrt{2} \frac{\sqrt{2}}{24\pi},
\] (S2)

\[
\sum_{k=0}^{\infty} \frac{(83592k^2 + 152922k + 64925)k^3}{(-512)^k} \left( \frac{2k}{k+1} \right)^3 = 625282 \sqrt{2} \frac{\sqrt{2}}{\pi} - 281920,
\] (S2’)

\[
\sum_{k=0}^{\infty} \frac{(198k^2 - 425k + 210)k^3}{4096^k} \left( \frac{2k}{k+1} \right)^3 = -\frac{1}{21\pi},
\] (S3)

\[
\sum_{k=0}^{\infty} \frac{(32473224k^2 + 58012446k + 24235261)k^3}{4096^k} \left( \frac{2k}{k+1} \right)^3
\]
\[
= 667628032 - \frac{2097324016}{\pi}.
\] (S3’)

**Remark 1.1.** Motivated by [27], we may also look at \( p \)-adic congruences corresponding to those identities in Theorem 1.1. For example, inspired by
the identity (S1) we conjecture that for any prime \( p > 3 \) we have

\[
\sum_{k=1}^{(p-1)/2} \frac{k^3 \binom{2k}{k}^3}{256^k} \left(6k^2 - 19k + 6\right) \equiv (-1)^{p+1} p + 2p^3 \frac{48}{48} \pmod{p^4}.
\]

The paper [23] contains many conjectural \( p \)-adic congruences.

For our variants of other classical Ramanujan-type series, the reader may consult Section 4.

In 1993 D. Zeilberger [30] used the WZ method to establish the identity

\[
\sum_{k=1}^{\infty} \frac{21k - 8}{k^3 \binom{2k}{k}^3} = \frac{\pi^2}{6}.
\]

More such series were conjectured by the author [19], and confirmed by J. Guillera and M. Rogers [14]. In 1997, T. Amdeberhan and Zeilberger [1] used the WZ method to obtain the identity

\[
\sum_{k=1}^{\infty} \frac{(-1)^k(205k^2 - 160k + 32)}{k^5 \binom{2k}{k}^5} = -2\zeta(3).
\]

In general, a Zeilberger-type series (of type \( Z \)) has the form

\[
\sum_{k=1}^{\infty} \frac{P_d(k)m^k}{k^{2d+1}a(k)} = cL(d + 1, \chi)
\]

where \( P_d(x) \in \mathbb{Z}[x] \) with \( \deg P_d = d \), \( m \) is a nonzero integer, \( a(k) \) is a product of \( 2d + 1 \) binomial coefficients, \( c \) is a nonzero rational number, and \( L(2d, \chi) \) is the Dirichlet \( L \) function associated with a Dirichlet character \( \chi \).

For example, in 2010 the author conjectured that

\[
\sum_{k=1}^{\infty} \frac{(28k^2 - 18k + 3)(-64)^k}{k^{5} \binom{2k}{k}^4 \binom{3k}{k}^2} = -14\zeta(3)
\]

(cf. [20]) which remains open.

Motivated by the author’s work [25] and the two new identities

\[
\sum_{k=1}^{\infty} \frac{(22k^2 - 7k - 3)64^k}{(2k + 1)(3k + 1)k^3 \binom{2k}{k}^2 \binom{3k}{k}} = 48 - 4\pi^2
\]

and

\[
\sum_{k=1}^{\infty} \frac{(22k^2 + 5k + 1)64^k}{(2k + 1)(3k + 1)k^2 \binom{2k}{k}^2 \binom{3k}{k}} = 4\pi^2 - 16
\]

missing from [25, Theorem 1.2], we introduce series of type \( T \):

\[
\sum_{k=1}^{\infty} \frac{Q_{2d}(k)m^k}{a(k)} = c_0 + c_1 L(d + 1, \chi),
\]

where \( Q_{2d}(x) \in \mathbb{Z}[x] \) with \( \deg Q_{2d} < 2d \), \( m \) is a nonzero integer, \( a(k) \) is a product of \( 2d + 1 \) binomial coefficients, \( c_0 \) and \( c_1 \) are algebraic numbers, and \( \chi \) is a Dirichlet character. We also investigate what happens if we replace
\( \binom{2k}{k} \) in the denominator of a general summand of the Zeilberger-type series \((Z)\) by \( \binom{2k}{k+1} = kC_k \).

Through this paper, we adopt the notation

\[
G = L \left( 2, \left( \frac{-4}{\cdot} \right) \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}
\]

and

\[
K = L \left( 2, \left( \frac{-3}{\cdot} \right) \right) = \sum_{k=1}^{\infty} \frac{(k_3)}{k^2},
\]

where \( (\frac{d}{\cdot}) \) with \( d \equiv 0, 1 \pmod{4} \) is the Kronecker symbol, and \( (\frac{k}{\cdot}) \) is the Legendre symbol. The number \( G \) is usually called the Catalan constant.

Now we state our second theorem.

**Theorem 1.2.** We have the following identities:

\[
\sum_{k=0}^{\infty} \frac{198k^2 - 227k + 47}{\binom{2k}{k}^3} = \frac{3264 - 4\pi^2}{63}, \tag{T1}
\]

\[
\sum_{k=1}^{\infty} \frac{4884741k^2 - 7292783k + 2041168}{\binom{2k}{k+1}^3} = \frac{68102148 - 10093535\pi^2}{126}, \tag{T1'}
\]

\[
\sum_{k=0}^{\infty} \frac{(54k^2 - 33k + 18)(-8)^k}{\binom{2k}{k}^3} = 8 + 2G, \tag{T2}
\]

\[
\sum_{k=1}^{\infty} \frac{(11277k^2 - 13124k + 3212)(-8)^k}{\binom{2k}{k+1}^3} = \frac{38954G - 24679}{3}, \tag{T2'}
\]

\[
\sum_{k=0}^{\infty} \frac{(850k^2 - 1133k + 69)8^k}{\binom{2k}{k}^2 \binom{3k}{k}} = \frac{885 - 12\pi^2}{5}, \tag{T3}
\]

\[
\sum_{k=1}^{\infty} \frac{(4621650k^2 - 7550827k + 1785654)8^k}{\binom{2k}{k+1}^2 \binom{3k}{k+1}} = \frac{6728880 - 1468127\pi^2}{5}, \tag{T3'}
\]

\[
\sum_{k=0}^{\infty} \frac{(18k^2 - 39k - 6)16^k}{\binom{2k}{k}^3} = \frac{16 - \pi^2}{4}, \tag{T4}
\]

\[
\sum_{k=1}^{\infty} \frac{(2778k^2 - 6499k + 1679)16^k}{\binom{2k}{k+1}^3} = \frac{89008 - 20305\pi^2}{12}, \tag{T4'}
\]
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\[
\sum_{k=0}^{\infty} \frac{(315k^2 - 54k + 119)(-27)^k}{(2k)^2 \binom{3k}{k}} = \frac{10 + 216K}{5}, \quad (T5)
\]

\[
\sum_{k=1}^{\infty} \frac{(339115k^2 - 324974k + 646792)(-27)^k}{(2k) \binom{3k}{k+1}} = \frac{2344210 - 5115501K}{30},
\]

\[
\sum_{k=0}^{\infty} \frac{(1298k^2 - 7807k - 3165)64^k}{(2k)^2 \binom{3k}{k+1}} = 195 - 192\pi^2, \quad (T6)
\]

\[
\sum_{k=1}^{\infty} \frac{(347074k^2 - 20527615k + 4631490)64^{k-1}}{(2k)^2 \binom{4k}{2k}} = 387807 - 198173\pi^2,
\]

\[
\sum_{k=0}^{\infty} \frac{(2050k^2 - 5331k - 1357)81^k}{(2k+1)^2 \binom{4k}{2k+1}} = \frac{8800 - 2592\pi^2}{35}, \quad (T7)
\]

\[
\sum_{k=1}^{\infty} \frac{(174397k^2 - 4725049k + 8929776)81^{k-1}}{(2k+1)^2 \binom{4k}{2k+1}} = \frac{43153685 - 44613764\pi^2}{210},
\]

\[
\sum_{k=0}^{\infty} \frac{(50k^2 + 93k + 22)(-144)^k}{(2k)^2 \binom{4k}{2k}} = \frac{243K - 200}{4}, \quad (T8)
\]

\[
\sum_{k=1}^{\infty} \frac{(53725k^2 - 27494k + 3675)(-144)^{k-1}}{(2k+1)^2 \binom{4k}{2k+1}} = \frac{78296 - 122235K}{128}. \quad (T8')
\]

Our third theorem is motivated by the Amdeberhan-Zeilberger identity (1.4).

**Theorem 1.3.** We have

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)^5} (684700k^4 - 1418358k^3 + 1100639k^2 - 365392k + 47327) = \frac{231168 - 64\zeta(3)}{5}.
\]

(T9)

Recall that the Domb numbers are given by

\[
\text{Domb}(n) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{2k}{k} \right)^2 \binom{2(n-k)}{n-k} (n = 0, 1, 2, \ldots).
\]

By H. H. Chan, S. H. Chan and Z. Liu [4] and M. D. Rogers [17],

\[
\sum_{k=0}^{\infty} \frac{5k + 1}{64^k} \text{Domb}(k) = \frac{8}{\sqrt{3} \pi} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{3k + 1}{(-32)^k} \text{Domb}(k) = \frac{2}{\pi}.
\]

(1.6)
Franel numbers of order 4 are those integers
\[ f_n^{(4)} = \sum_{k=0}^{n} \binom{n}{k}^4 \ (n \in \mathbb{N}). \]

In 2005 Yifan Yang discovered the first \( \frac{1}{\pi} \)-series involving Franel numbers of order four:
\[ \sum_{k=0}^{\infty} \frac{4k+1}{36^k} f_k^{(4)} = \frac{18}{\sqrt{15} \pi}. \] (1.7)

For more such series deduced via modular forms, see S. Cooper [9]. Apéry numbers of the second kind are those integers
\[ \beta_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \ (n \in \mathbb{N}). \]

It is known (cf. Table 14.7 of [11, p. 653]) that
\[ \sum_{k=0}^{\infty} \frac{5k+1}{72^k} \binom{2k}{k} \beta_k = \frac{9\sqrt{2}}{2\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{11k+2}{147^k} \binom{2k}{k} \beta_k = \frac{49\sqrt{3}}{10\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{190k+29}{(-828)^k} \binom{2k}{k} \beta_k = \frac{18\sqrt{23}}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{682k+71}{(-15228)^k} \binom{2k}{k} \beta_k = \frac{162\sqrt{17}}{5\pi}. \]

For \( n \in \mathbb{N} \) define the Cooper number
\[ \text{Co}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{k}. \]

S. Cooper [10] deduced the following three identities for \( 1/\pi \):
\[ \sum_{k=0}^{\infty} \frac{39k+10}{(-64)^k} \text{Co}(k) = \frac{64}{\sqrt{7} \pi}, \] (1.8)
\[ \sum_{k=0}^{\infty} \frac{21k+4}{125^k} \text{Co}(k) = \frac{125}{8\pi}, \] (1.9)
\[ \sum_{k=0}^{\infty} \frac{11895k+1286}{(-22)^{3k}} \text{Co}(k) = \frac{22^3}{\sqrt{7} \pi}. \] (1.10)

Now we state our fourth theorem.

**Theorem 1.4.** (i) We have
\[ \sum_{k=1}^{\infty} \frac{k(3k+1)^2}{(-32)^k} \text{Domb}(k) = -\frac{2}{\pi}, \] (1.11)
\[ \sum_{k=1}^{\infty} \frac{k^2(5k-3)}{64^k} \text{Domb}(k) = \frac{8\sqrt{3}}{9\pi}. \] (1.12)
(ii) We have the following identities:

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{40k^2 - 42k - 1}{36^k} = \frac{198\sqrt{15}}{25\pi},
\]

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{80k^2 + 16k + 3}{(-64)^k} = -\frac{544\sqrt{15}}{675\pi},
\]

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{33000k^2 - 3410k + 291}{196^k} = \frac{1582\sqrt{7}}{3\pi},
\]

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{2720k^2 + 141k - 7}{(-324)^k} = -\frac{3969\sqrt{5}}{200\pi},
\]

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{156000k^2 - 2205k + 563}{1296^k} = \frac{17901\sqrt{2}}{32\pi},
\]

\[
\sum_{k=1}^{\infty} k f_k^{(4)} \frac{26079360k^2 - 81592k + 33569}{5776^k} = \frac{220628\sqrt{95}}{75\pi},
\]

(iii) We have

\[
\sum_{k=1}^{\infty} k \left(\frac{2k}{72}\right) \beta_k \frac{125k^2 - 300k - 17}{72^k} = \frac{999\sqrt{3}}{8\pi},
\]

\[
\sum_{k=1}^{\infty} k \left(\frac{2k}{147}\right) \beta_k \frac{121k^2 - 78k - 1}{147^k} = \frac{21903\sqrt{3}}{1250\pi},
\]

\[
\sum_{k=1}^{\infty} k \left(\frac{2k}{-828}\right) \beta_k \frac{5234500k^2 + 395850k - 31627}{(-828)^k} = \frac{22302\sqrt{23}}{\pi},
\]

and

\[
\sum_{k=1}^{\infty} k \left(\frac{2k}{-15228}\right) \beta_k \frac{4127975500k^2 - 17838750k + 6386881}{(-15228)^k} = -\frac{3699918\sqrt{7}}{5\pi}.
\]

(iv) We have

\[
\sum_{k=1}^{\infty} k \text{Co}(k) \frac{1365k^2 + 575k + 86}{(-64)^k} = -\frac{7424\sqrt{7}}{147\pi},
\]

\[
\sum_{k=1}^{\infty} k \text{Co}(k) \frac{1029k^2 - 413k + 4}{125^k} = \frac{16375}{96\pi},
\]

and

\[
\sum_{k=1}^{\infty} k \text{Co}(k) \frac{3480060675k^2 + 12712753k - 5211590}{(-22)^{3k}} = \frac{226632032\sqrt{7}}{147\pi}.
\]
Remark 1.2. Actually our method to show Theorem 1.4 allows us to deduce many more similar identities from all the known Ramanujan-type series for $1/\pi$ listed in [11, pp. 647–658].

We will prove Theorem 1.1 in Section 2, and Theorems 1.2-1.3 in Section 3. Section 4 contains our variants of classical Ramanujan-type series not included in Theorem 1.1. Section 5 is devoted to our proof of Theorem 1.4.

Surprisingly, it seems that each non-classical Ramanujan-type series also has related series of type $S$. This enables us to pose many conjectural series of type $S$ in Sections 6-8 motivated by known series of type $R$.

Now we present our general conjecture relating series of type $R$ to series of type $S$.

Conjecture 1.1. Suppose that we have a Ramanujan-type identity $(R)$ with $\limsup_{k \to +\infty} \sqrt[k]{|a(k)|} < |m|$. For any non-constant polynomial $P(x) \in \mathbb{Z}[x]$, there is a polynomial $Q_{2d+1}(x) \in \mathbb{Z}$ of degree at most $2d$ such that

$$\sum_{k=0}^{\infty} \frac{P(k)Q_{2d}(k)}{m^k} a(k) = \frac{\alpha'}{\pi^d}$$

for some $\alpha' \in \mathbb{Q}(\alpha)$.

Now we describe our algorithm to deduce series of type $S$ from a Ramanujan-type series $(R)$ with $a(k)$ a product of $2d + 1$ binomial coefficients.

General Algorithm. Suppose that we have a Ramanujan-type series $(R)$ with $a(k)$ a product of $2d + 1$ binomial coefficients. Given a monic polynomial $P(x) \in \mathbb{Z}[x]$ with $\deg P > 0$, we first write $P(x) = (x - \alpha_1) \cdots (x - \alpha_r)$ with $\alpha_1, \ldots, \alpha_r$ complex numbers. Using the Gosper algorithm (cf. [15]) we find a polynomial $Q_{2d+1}$ of degree at most $2d + 1$ such that $\sum_{k=0}^{\infty} \frac{a(k)}{m^k} Q_{2d+1}(k)$ has a closed form which tends to 0, so that we get

$$\sum_{k=0}^{\infty} \frac{a(k)}{m^k} Q_{2d+1}(k) = 0. \quad (1.26)$$

Note that $(1.26) \times P_d(\alpha_1) - (R) \times Q_{2d+1}(\alpha_1)$ yields an identity of the form

$$\sum_{k=0}^{\infty} \frac{(k - \alpha_1)a(k)}{m^k} R_{2d}(k) = -Q_{2d+1}(\alpha_1) \frac{\alpha}{\pi^d}, \quad (1.27)$$

where $R_{2d}$ is a polynomial of degree at most $2d$. Using the Gosper algorithm, we find

$$\sum_{k=0}^{\infty} \frac{(k - \alpha_1)a(k)}{m^k} \tilde{Q}_{2d+1}(k) = 0, \quad (1.28)$$

where $\tilde{Q}_{2d+1}$ is a polynomial of degree at most $2d + 1$. Then $(1.28) \times R_{2d}(\alpha_2) - \tilde{Q}_{2d+1}(\alpha_2) \times (1.27)$ yields an identity of the form

$$\sum_{k=0}^{\infty} \frac{(k - \alpha_1)(k - \alpha_2)a(k)}{m^k} \check{R}_{2d}(k) = Q_{2d+1}(\alpha_1)\check{Q}_{2d+1}(\alpha_2) \frac{\alpha}{\pi^d}$$
with $\tilde{R}_{2d}$ a polynomial of degree at most $2d$. Continue this process, we finally obtain the exact value of

$$\sum_{k=0}^{\infty} \frac{P(k)\alpha(k)}{m^k}Q_{2d}(k)$$

with $Q_{2d}$ a suitable polynomial of degree at most $2d$.

**Remark 1.3.** In practice this algorithm works well. Actually it could also be used to deduce series of type $T$ from series of type $Z$.

**Example 1.1.** Let’s implement the algorithm to deduce a series of type $S$ with $P(k) = k^2 + 1 = (k + i)(k - i)$ from Ramanujan’s identity

$$\sum_{k=0}^{\infty} \frac{(2k)^3}{256^k}(6k + 1) = \frac{4}{\pi}. \quad (1.29)$$

By Gosper’s algorithm we find that

$$\sum_{k=0}^{n} \frac{(2k)^3}{256^k}(24k^3 - 12k^2 - 6k - 1) = -(2n + 1)^2 \frac{(2n)^3}{256^n}. \quad (1.29)$$

Letting $n \to +\infty$ we get

$$\sum_{k=0}^{\infty} \frac{(2k)^3}{256^k}(24k^3 - 12k^2 - 6k - 1) = 0. \quad (1.30)$$

Note that $6k + 1 = 6(k + i) + 1 - 6i$ and

$$24k^3 - 12k^2 - 6k - 1 = 6(k + i)(4k^2 - (2 + 4i)k + 2i - 5) + 30i + 11.$$ 

Via $(1.29) \times (30i + 11) + (6i - 1) \times (1.30)$ we obtain

$$\sum_{k=0}^{\infty} \frac{(k + i)(2k)^3}{256^k}((k - i)(6i - 1)(2k - 1) + 8) = \frac{11 + 30i}{3\pi}. \quad (1.31)$$

Via the Gosper algorithm, we find that

$$\sum_{k=0}^{\infty} \frac{(k + i)(2k)^3}{256^k}(2(k-i)((132+360i)k^2 - (602+144i)k + 15 - 708i)+1373) = 0. \quad (1.32)$$

Observe that $(1.31) \times 1373 - 8 \times (1.32)$ yields

$$\sum_{k=0}^{\infty} \frac{(k^2 + 1)(2k)^3}{256^k}(192k^2 - 626k - 103) = \frac{1373}{3\pi}. \quad (1.33)$$
We can also deduce other identities similar to (1.33) such as

\[
\sum_{k=0}^{\infty} \frac{(k^2 + 1) \left( \frac{2k}{k} \right)^3}{(-512)^k} (6k + 1) = \frac{11\sqrt{2}}{6\pi},
\]

(1.34)

\[
\sum_{k=0}^{\infty} \frac{(k^2 + 1) \left( \frac{2k}{k} \right)^3}{4096^k} (126504k^2 - 921334k - 109205) = -\frac{1063412}{3\pi}.
\]

(1.35)

2. Proof of Theorem 1.1

Lemma 2.1. Let \( m \) be any nonzero integer, and let \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \).

(i) We have

\[
\sum_{k=0}^{n} \frac{(2k)^3}{m^k} \left( (64 - m)k^3 + 96k^2 + 48k + 8 \right) = 8(2n + 1)^3 \frac{(2n)^3}{m^n},
\]

\[
\sum_{k=0}^{n} \frac{k(2k)^3}{m^k} \left( (64 - m)k^3 + (96 + m)k^2 + 48k + 8 \right) = 8n(2n + 1)^3 \frac{(2n)^3}{m^n},
\]

and

\[
\sum_{k=0}^{n} \frac{k^2(2k)^3}{m^k} \left( (64 - m)k^3 + (96 + 2m)k^2 + (48 - m)k + 8 \right)
= 8n^2(2n + 1)^3 \frac{(2n)^3}{m^n}.
\]

(ii) We have

\[
\sum_{k=0}^{n} \frac{(2k)^3}{(k + 1)^3 m^k} \left( (64 - m)k^3 + (96 - 3m)k^2 + (48 - 3m)k + 8 - m \right)
= -m + 8(2n + 1)^3 \frac{(2n)^3}{(n + 1)^3 m^n},
\]

and

\[
\sum_{k=0}^{n} \frac{k(2k)^3}{(k + 1)^3 m^k} P(k, m) = 8m - 8(2n + 1)^3 (m - 8n + mn) \frac{(2n)^3}{(n + 1)^3 m^n},
\]

where \( P(k, m) \) denotes

\[
(512 - 72m + m^2)k^3 + (768 - 176m + 3m^2)k^2 + (384 - 144m + 3m^2)k + 64 - 40m + m^2.
\]

Also,

\[
\sum_{k=0}^{n} \frac{k^2(2k)^3}{(k + 1)^3 m^k} Q(k, m) + 8m(m + 8)
= 8(2n + 1)^3 (m(m + 8) - 40mn(n + 1) + 2m^2n + 64n^2 + m^2n^2) \frac{(2n)^3}{(n + 1)^3 m^n},
\]
where \( Q(k,m) \) denotes
\[
(4096 - 2624m + 1024m^2 - m^3)k^3 + (6144 - 6464m + 304m^2 - 3m^3)k^2 \\
+ (3072 - 5120m + 296m^2 - 3m^3)k + 512 - 1344m + 96m^2 - m^3.
\]

**Proof.** It is easy to prove all the six identities by induction on \( n \). We find them via the Gosper algorithm (cf. [15]). \( \square \)

**Lemma 2.2** (Sun [25]). We have
\[
\sum_{k=0}^{\infty} \frac{k(6k - 1)(2k^3)}{(2k - 1)^3 256k} = \frac{1}{2\pi}, \quad (2.1)
\]
\[
\sum_{k=0}^{\infty} \frac{(30k^2 + 3k - 2)(2k^3)}{(2k - 1)^3 (-512)k} = \frac{27\sqrt{2}}{8\pi}, \quad (2.2)
\]
\[
\sum_{k=0}^{\infty} \frac{(42k^2 - 3k - 1)(2k^3)}{(2k - 1)^3 4096k} = \frac{27}{8\pi}. \quad (2.3)
\]

**Proof of Theorem 1.1.** By Stirling's formula,
\[
\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}} \text{ as } n \to +\infty.
\]

Thus, for any integer \( m \) with \(|m| > 64\), by Lemma 2.1 we have
\[
\sum_{k=0}^{\infty} \frac{(2k^3)}{m^k} \left( (64 - m)k^3 + 96k^2 + 48k + 8 \right) = 0, \quad (2.4)
\]
\[
\sum_{k=0}^{\infty} \frac{k(2k^3)}{m^k} \left( (64 - m)k^3 + (96 + m)k^2 + 48k + 8 \right) = 0, \quad (2.5)
\]
\[
\sum_{k=0}^{n} \frac{k^2(2k^3)}{m^k} \left( (64 - m)k^3 + (96 + 2m)k^2 + (48 - m)k + 8 \right) = 0; \quad (2.6)
\]
also
\[
\sum_{k=0}^{n} \frac{(2k^3)}{(k + 1)^3 m^k} \left( (64 - m)k^3 + (96 - 3m)k^2 + (48 - 3m)k + 8 \right) = -m, \quad (2.7)
\]
\[
\sum_{k=0}^{n} \frac{k(2k^3)}{(k + 1)^3 m^k} P(k,m) = 8m \quad (2.8)
\]
and
\[
\sum_{k=0}^{n} \frac{k^2(2k^3)}{(k + 1)^3 m^k} Q(k,m) = -8m(m + 8) \quad (2.9)
\]
where \( P(k,m) \) and \( Q(k,m) \) are as in Lemma 2.1(ii).
Now we prove (S3) in details. (The identities (S1) and (S2) can be proved similarly.) Taking \( m = 4096 \) in (2.4)-(2.6), we get the identities

\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{4096^k} \frac{\binom{2k}{k}}{k^3} (504k^3 - 12k^2 - 6k - 1) = 0, \tag{2.10}
\]

\[
\sum_{k=0}^{\infty} \frac{k(2k)^3}{4096^k} \frac{\binom{2k}{k}}{k^3} (504k^3 - 524k^2 - 6k - 1) = 0, \tag{2.11}
\]

\[
\sum_{k=0}^{\infty} \frac{k^2(2k)^3}{4096^k} \frac{\binom{2k}{k}}{k^3} (504k^3 - 1036k^2 + 506k - 1) = 0. \tag{2.12}
\]

Combining the Ramanujan identity

\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4096^k} (42k + 5) = \frac{16}{\pi}
\]

with (2.10), and noting that

\[
12k(210k^2 - 5k + 1) = 60k(42k^2 - k) + 12k = 5\times 12k(42k^2 - k) + 12k,
\]

we obtain

\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{4096^k} 12k(210k^2 - 5k + 1) = \sum_{k=0}^{\infty} \frac{(2k)^3}{4096^k} (5(6k + 1) + 12k) = \frac{16}{\pi}
\]

and hence

\[
\sum_{k=0}^{\infty} \frac{k(2k)^3}{4096^k} (210k^2 - 5k + 1) = \frac{4}{3\pi}.
\]

Adding this and (2.11), we see that

\[
\sum_{k=0}^{\infty} \frac{k^2(2k)^3}{4096^k} (504k^3 - 314k - 11) = \frac{4}{3\pi}.
\]

(2.12) times 11 minus the last formula yields

\[
\sum_{k=0}^{\infty} \frac{k^2(2k)^3}{4096^k} (11 \times 504k^3 - (11 \times 1036 + 504)k^2 + (11 \times 506 + 314)k) = -\frac{4}{3\pi},
\]

which is equivalent to (S3).

Next we prove (S3)’ in details. (The identities (S1’) and (S2’) can be proved similarly.) For any positive integer \( k \), clearly

\[
\frac{(2k)}{2k - 1} = 2 \left( \frac{2(k - 1)}{k - 1} \right).
\]
Thus, by (2.3) we have
\[
\frac{27}{8\pi} - 1 = \sum_{k=1}^{\infty} \frac{42k^2 - 3k - 1}{4096^k} \left( \frac{2}{k} \binom{2(k-1)}{k-1} \right)^3
\]
\[
= \frac{8}{4096} \sum_{k=0}^{\infty} \frac{42(k+1)^2 - 3(k+1) - 1}{4096^k} \cdot \frac{(2k)^3}{(k+1)^3}
\]
and hence
\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{(k+1)^3 4096^k} (42k^2 + 81k + 38) = \frac{1728}{\pi} - 512. \tag{2.13}
\]
On the other hand, applying (2.7) with \(m = 4096\), we get
\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{(k+1)^3 4096^k} (504k^3 + 1524k^2 + 1530k + 511) = 512. \tag{2.14}
\]
Note that \((2.14) \times 38 - (2.13) \times 511\) yields
\[
\sum_{k=0}^{\infty} \frac{k(2k)^3}{(k+1)^3 4096^k} (2128k^2 + 4050k + 1861) = 31232 - \frac{98112}{\pi}. \tag{2.15}
\]
Taking \(m = 4096\) in (2.8), we obtain
\[
\sum_{k=0}^{\infty} \frac{k(2k)^3}{(k+1)^3 4096^k} (257544k^3 + 775180k^2 + 777222k + 259585) = 512. \tag{2.16}
\]
Observe that \((2.16) \times 1861 - (2.15) \times 259585\) gives
\[
\sum_{k=0}^{\infty} \frac{k^2(2k)^3}{(k+1)^3 4096^k} (78162k^2 + 145175k + 64431) = \frac{4153360}{\pi} - 1321984. \tag{2.17}
\]
Putting \(m = 4096\) in (2.9), we get
\[
\sum_{k=0}^{\infty} \frac{k^2(2k)^3}{(k+1)^3 4096^k} (130830840k^3 + 392743412k^2 + 392994810k + 131082751)
\]
\[
= 667628032 - \frac{2097324016}{\pi}, \tag{2.18}
\]
which is equivalent to the desired (S3') since \(\binom{2k}{k+1} = \frac{k}{k+1} \binom{2k}{k}\). \(\square\)
3. PROOFS OF THEOREMS 1.2 AND 1.3

Lemma 3.1. Let \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \).

(i) We have the following identities:

\[
\sum_{k=1}^{n} \frac{m^k}{k^3 \binom{2k}{k}} ((m - 64)k^3 + (2m + 96)k^2 - 48k + 8) = -m + \frac{(n + 1)^2 m^{n+1}}{(2n)^3},
\]

\[
\sum_{k=1}^{n} \frac{m^k}{k^2 \binom{2k}{k}} ((m - 64)k^3 + (m + 96)k^2 - 48k + 8) = -m + \frac{(n + 1)m^{n+1}}{(2n)^3},
\]

\[
\sum_{k=1}^{n} \frac{m^k}{k^3 \binom{2k}{k}} ((m - 64)k^3 + 96k^2 - 48k + 8) = -m + \frac{m^{n+1}}{(2n)^3},
\]

(ii) We have

\[
\sum_{k=1}^{n} \frac{(k + 1)m^k}{k^3 \binom{2k}{k}} P_1(k, m) = m(m - 432) + (216(n + 2) - m(n + 1)) \frac{m^{n+1}}{(2n)^3},
\]

\[
\sum_{k=1}^{n} \frac{(k + 1)^2 m^k}{k^3 \binom{2k}{k}} P_2(k, m) = m(m^2 - 216m + 186624)
\]

\[
= - \frac{m^{n+1}}{(2n)^3}
\times (186624(n + 1) - 216m(n^2 + 1) + m^2(2n + 1) - 648mn + m^2n^2 + 46656n^2),
\]

where

\[
P_1(k, m) = (280m - m^2 - 13824)k^3 + (56m + 20736)k^2
\]

\[
- (8m + 10368)k + 1728
\]

and

\[
P_2(k, m) = (2985984 - 60480m + 280m^2 - m^3)k^3
\]

\[
- (4478976 + 58752m + 8m^2)k^2 + (2239488 + 15552m)k
\]

\[
- (1728m + 373248).
\]

Proof. It is easy to prove the five identities by induction on \( n \). We find them via the Gosper algorithm (cf. [15]). \( \square \)
Proof of (T1). Lemma 3.1(i) with \( m = 1 \) gives the identities

\[
\sum_{k=1}^{n} \frac{63k^3 - 98k^2 + 47k - 8}{k(\binom{2k}{k})^3} = 1 - \frac{(n + 1)^2}{(\binom{2n}{n})^3},
\]

\[
\sum_{k=1}^{n} \frac{63k^3 - 97k^2 + 48k - 8}{k^2(\binom{2k}{k})^3} = 1 - \frac{n + 1}{(\binom{2n}{n})^3},
\]

\[
\sum_{k=1}^{n} \frac{63k^3 - 96k^2 + 48k - 8}{k^3(\binom{2k}{k})^3} = 1 - \frac{1}{(\binom{2n}{n})^3}.
\]

Letting \( n \to +\infty \) we get

\[
\sum_{k=1}^{\infty} \frac{63k^3 - 98k^2 + 47k - 8}{k(\binom{2k}{k})^3} = 1, \tag{3.1}
\]

\[
\sum_{k=1}^{\infty} \frac{63k^3 - 97k^2 + 48k - 8}{k^2(\binom{2k}{k})^3} = 1, \tag{3.2}
\]

\[
\sum_{k=1}^{\infty} \frac{63k^3 - 96k^2 + 48k - 8}{k^3(\binom{2k}{k})^3} = 1. \tag{3.3}
\]

Note that (3.3) minus (3.1) yields the identity

\[
\sum_{k=1}^{\infty} \frac{21k^2 - 32k + 9}{k^2(\binom{2k}{k})^3} = \frac{1}{3} - \frac{\pi^2}{18}. \tag{3.4}
\]

Via (3.2)×9 + 8×(3.4), we obtain

\[
\sum_{k=1}^{\infty} \frac{567k^2 - 705k + 176}{k(\binom{2k}{k})^3} = \frac{35}{3} - \frac{4}{9} \pi^2. \tag{3.5}
\]

Observe that (3.1)×22+(3.5) yields the identity

\[
\sum_{k=1}^{\infty} \frac{198k^2 - 227k + 47}{(\binom{2k}{k})^3} = \frac{303 - 4\pi^2}{63},
\]

which is equivalent to (T1) since \( 303 + 47 \times 63 = 3264 \). \( \square \)

Proof of (T1'). Lemma 3.1(ii) with \( m = 1 \) gives the identities

\[
\sum_{k=1}^{n} \frac{k + 1}{k^3(\binom{2k}{k})^3} \left( 13545k^3 - 20792k^2 + 10376k - 1728 \right)
= 431 - \frac{215n + 431}{(\binom{2n}{n})^3}.
\]
and
\[
\sum_{k=1}^{n} \frac{(k + 1)^2}{k^3 \binom{2k}{k}^3} \left(2925783k^2 - 4537736k^2 + 2255040k - 374976\right) = 186409 - \frac{46441n^2 + 185978n + 186409}{(2n)^3}.
\]

Letting \(n \to +\infty\) we get
\[
\sum_{k=1}^{n} \frac{(k + 1)^2}{k^3 \binom{2k}{k}^3} \left(2925783k^2 - 4537736k^2 + 2255040k - 374976\right) = 186409.
\]

They can be rewritten as follows:
\[
\sum_{k=1}^{n} \frac{k + 1}{k^3 \binom{2k}{k}^3} ((k + 1)(13545k^2 - 34337k + 44713) - 46441) = 431,
\]
\[
\sum_{k=1}^{n} \frac{(k + 1)^2}{k^3 \binom{2k}{k}^3} ((k + 1)(2925783k^2 - 7463519k + 9718559) - 10093535) = 186409.
\]

We may rewrite (3.3) in the form
\[
\sum_{k=1}^{\infty} \frac{(k + 1)(63k^2 - 159k + 207) - 215}{k^3 \binom{2k}{k}^3} = 1,
\]
(3.8)
and rewrite (1.3) in the form
\[
\sum_{k=1}^{\infty} \frac{21(k + 1) - 29}{k^3 \binom{2k}{k}^3} = \frac{\pi^2}{6}.
\]
(3.9)
Note that (3.8) \(\times 29 - 215 \times (3.9)\) yields
\[
\sum_{k=1}^{\infty} \frac{k + 1}{k^3 \binom{2k}{k}^3} ((609k^2 - 1537k + 496) = \frac{29}{3} - \frac{215}{18}\pi^2,
\]
which has the equivalent form
\[
\sum_{k=1}^{\infty} \frac{k + 1}{k^3 \binom{2k}{k}^3} ((k + 1)(609k - 2146) + 2642) = \frac{29}{3} - \frac{215}{18}\pi^2.
\]
(3.10)
Via (3.6) \(\times 2642 + 46441 \times (3.10)\) we obtain
\[
\sum_{k=1}^{\infty} \frac{(k + 1)^2}{k^3 \binom{2k}{k}^3} (166446k^2 - 290399k + 85904) = \frac{22153}{3} - \frac{46441}{18}\pi^2,
\]
which has the equivalent form
\[
\sum_{k=1}^{\infty} \frac{(k+1)^2}{k^3(\frac{2k}{k})^3}(k+1)(166446k - 456845) + 542749) = \frac{22153}{3} - \frac{46441}{18}\pi^2.
\]  
(3.11)

Observe that \((3.7) \times 542749 + 10093535 \times (3.11)\) yields
\[
\sum_{k=1}^{\infty} \frac{(k+1)^3}{k^3(\frac{2k}{k})^3}(2041168 - 7292783k + 4884741k^2) = \frac{68102148 - 10093535\pi^2}{126},
\]
which is equivalent to the desired \((T1')\) since \((\frac{2k}{k+1}) = \frac{k}{k+1}(\frac{2k}{k})\). This concludes the proof. □

**Remark 3.1.** In the spirit of our proofs of \((T1)\) and \((T1')\), we can prove \((T2),(T2'),(T4),(T4')\) similarly by using Lemma 3.1 and the known identities
\[
\sum_{k=1}^{\infty} \frac{3k-1}{k^3(\frac{2k}{k})^3}(-8k)^k = -2G \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3(\frac{2k}{k})^3} = \pi^2
\]
(cf. [14]).

**Lemma 3.2.** Let \(m \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\).
(i) We have the following identities:
\[
\sum_{k=1}^{n} \frac{m^k}{k^3(\frac{2k}{k})^3(\frac{3k}{k})}((m-108)k^3 + (2m+162)k^2 + (m-78)k + 12)
\]
\[
= -m + \frac{(n+1)^2m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}},
\]
\[
\sum_{k=1}^{n} \frac{m^k}{k^2(\frac{2k}{k})^3(\frac{3k}{k})}((m-108)k^3 + (m+162)k^2 - 78k + 12)
\]
\[
= -m + \frac{(n+1)m^{n+1}}{\binom{2n}{n}\binom{3n}{n}},
\]
\[
\sum_{k=1}^{n} \frac{m^k}{k^3(\frac{2k}{k})^3(\frac{3k}{k})}((m-108)k^3 + 162k^2 - 78k + 12)
\]
\[
= -m + \frac{m^{n+1}}{\binom{2n}{n}\binom{3n}{n}}.
\]
(ii) We have
\[
\sum_{k=1}^{n} \frac{(k+1)m^k}{k^3(\frac{2k}{k})^3(\frac{3k}{k})}P_1(k, m) = m(m - 720) + (360(n+2) - m(n+1))\frac{m^{n+1}}{\binom{2n}{n}\binom{3n}{n}}.
\]
and

\[
\sum_{k=1}^{n} \frac{(k + 1)^2 m^k}{k^3(\frac{2k}{k})^2(\frac{4k}{2k})^2} P_2(k, m) - m(m^2 - 372m + 518400)
\]
\[
= - m^{n+1} \left( \frac{n+1}{\left(\frac{2n}{n}\right)^2 \left(\frac{4n}{2n}\right)} \right)
\times (m^2(n + 1)^2 - 366mn(n + 3) + 129600n^2 + 518400(n + 1) - 372m),
\]

where

\[
P_1(k, m) = (468m - m^2 - 38880)k^3 + (90m + 58320)k^2
- (12m + 28080)k + 4320
\]

and

\[
P_2(k, m) = (13996800 - 169128m + 474m^2 - m^3)k^3
- (20995200 + 160380m + 12m^2)k^2
+ (10108800 + 41112m)k - 1555200
\]

Lemma 3.3. Let \(m \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\).

(i) We have the following identities:

\[
\sum_{k=1}^{n} \frac{m^k}{k(\frac{2k}{k})^2(\frac{4k}{2k})} (m - 256)k^3 + (2m + 384)k^2 + (m - 176)k + 24
\]
\[
= - m + \frac{(n + 1)^2 m^{n+1}}{\left(\frac{2n}{n}\right)^2 \left(\frac{4n}{2n}\right)},
\]

\[
\sum_{k=1}^{n} \frac{m^k}{k^2(\frac{2k}{k})^2(\frac{4k}{2k})} ((m - 256)k^3 + (2m + 384)k^2 - 176k + 24)
\]
\[
= - m + \frac{(n + 1)m^{n+1}}{\left(\frac{2n}{n}\right)^2 \left(\frac{4n}{2n}\right)},
\]

\[
\sum_{k=1}^{n} \frac{m^k}{k^3(\frac{2k}{k})^2(\frac{4k}{2k})} ((m - 256)k^3 + 384k^2 - 176k + 24)
\]
\[
= - m + \frac{m^{n+1}}{\left(\frac{2n}{n}\right)^2 \left(\frac{4n}{2n}\right)}.\]

(ii) We have

\[
\sum_{k=1}^{n} \frac{(k + 1)m^k}{k^3(\frac{2k}{k})^2(\frac{4k}{2k})} P_1(k, m) = m(m - 1680) + (840(n + 2) - m(n + 1)) \frac{m^{n+1}}{\left(\frac{2n}{n}\right)^2 \left(\frac{4n}{2n}\right)}
\]
and
\[\sum_{k=1}^{n} \frac{(k + 1)^2 m^k}{k^3 C_{2k}^k C_{2k}^k} P_2(k, m) = m(m^2 - 904m + 2822400)\]
\[= - \frac{m^{n+1}}{(2n)^2 (2n)} \times (m^2(n + 1)^2 - 872mn(n + 3) + 705600n(n + 4) - 904m + 2822400),\]
where
\[P_1(k, m) = (1096m - m^2 - 215040)k^3 + (200m + 322560)k^2 - (24m + 147840)k + 20160\]
and
\[P_2(k, m) = (180633600 - 928832m + 1128m^2 - m^3)k^3 - (270950400 + 853120m + 24m^2)k^2 + (124185600 + 209088m)k - (16934400 + 20160m).\]

Using Lemmas 3.2 and 3.3 as well as some results in [25], we can prove all the remaining formulas in Theorem 1.2.

**Proof of Theorem 1.3.** By Gosper’s algorithm, we find the identity
\[\sum_{k=1}^{n} \frac{(-1)^k}{k^5 C_{2k}^k} (1025k^5 - 2560k^4 + 2560k^3 - 1280k^2 + 320k - 32) = -1 + \frac{(-1)^n}{(2n)^{n}}\]
which can be easily proved by induction on \(n\). Letting \(n \to +\infty\) we get
\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5 C_{2k}^k} (1025k^5 - 2560k^4 + 2560k^3 - 1280k^2 + 320k - 32) = -1.\]
Adding this to (1.4) we get
\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4 C_{2k}^k} (205k^4 - 512k^3 + 512k^2 - 215k + 32) = -\frac{1 + 2\zeta(3)}{5}. \quad (3.12)\]

By Gosper’s algorithm, we find that
\[\sum_{k=1}^{n} \frac{(-1)^k}{k^4 C_{2k}^k} (1025k^5 - 2559k^4 + 2560k^3 - 1280k^2 + 320k - 32) = -1 + \frac{(-1)^n(n + 1)}{(2n)}\]
and hence
\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4 C_{2k}^k} (1025k^5 - 2559k^4 + 2560k^3 - 1280k^2 + 320k - 32) = -1. \quad (3.13)\]
Adding (3.12) and (3.13), we obtain
\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 C_{2k}^k} (1025k^4 - 2354k^3 + 2048k^2 - 768k + 105) = -\frac{2}{5}(3 + \zeta(3)). \quad (3.14)\]
By Gosper’s algorithm, we find that
\[
\sum_{k=1}^{n} \frac{(-1)^k}{k^3 \binom{2k}{k}^5} (1025k^5 - 2558k^4 + 2561k^3 - 1280k^2 + 320k - 32) = -1 + \frac{(-1)^n (n + 1)^2}{\binom{2n}{n}}
\]
and hence
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}^5} (1025k^5 - 2558k^4 + 2561k^3 - 1280k^2 + 320k - 32) = -1. \quad (3.15)
\]
Note that \((3.14) \times 32 + 105 \times (3.15)\) yields
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 \binom{2k}{k}^5} (1025k^5 - 2557k^4 + 193577k^2 - 68864k + 9024)
= -\frac{64}{5} (3 + \zeta(3)) - 105. \quad (3.16)
\]
By Gosper’s algorithm, we find that
\[
\sum_{k=1}^{n} \frac{(-1)^k}{k^2 \binom{2k}{k}^5} (1025k^5 - 2557k^4 + 2563k^3 - 1279k^2 + 320k - 32) = -1 + \frac{(-1)^n (n + 1)^3}{\binom{2n}{n}}
\]
and hence
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 \binom{2k}{k}^5} (1025k^5 - 2557k^4 + 2563k^3 - 1279k^2 + 320k - 32) = -1. \quad (3.17)
\]
Via \((3.16) + 282 \times (3.17)\) we get
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k \binom{2k}{k}^5} (289050k^4 - 613449k^3 + 486976k^2 - 167101k + 21376)
= -\frac{64}{5} (3 + \zeta(3)) - 105 - 282. \quad (3.18)
\]
By Gosper’s algorithm, we find that
\[
\sum_{k=1}^{n} \frac{(-1)^k}{k \binom{2k}{k}^5} (1025k^5 - 2556k^4 + 2566k^3 - 1276k^2 + 321k - 32) = -1 + \frac{(-1)^n (n + 1)^4}{\binom{2n}{n}}
\]
and hence
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k \binom{2k}{k}^5} (1025k^5 - 2556k^4 + 2566k^3 - 1276k^2 + 321k - 32) = -1. \quad (3.19)
\]
Observe that \((3.18) + 668 \times (3.19)\) gives
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{\binom{2k}{k}^5} (684700k^4 - 1418358k^3 + 1100639k^2 - 365392k + 47327)
= -\frac{5467 + 64\zeta(3)}{5},
\]
which is equivalent to the desired (T9) since

\[
47327 - \frac{5467}{5} = \frac{231168}{5}.
\]

This concludes our proof of Theorem 1.3. □

4. Other variants of Ramanujan-type series for \( \frac{1}{\pi} \)

Lemma 4.1. Let \( m \in \mathbb{Z} \setminus \{0\} \) and \( n \in \mathbb{N} \).

(i) We have

\[
\sum_{k=0}^{n} \frac{(2k)^2 (3k)}{m^k} \left( (108 - m)k^3 + 162k^2 + 78k + 12 \right)
= 6(2n + 1)(3n + 1)(3n + 2) \frac{(2n)^2 (3n)}{m^n},
\]

and

\[
\sum_{k=0}^{n} \frac{k(2k)^2 (3k)}{m^k} \left( (108 - m)k^3 + (162 + m)k^2 + 78k + 12 \right)
= 6n(2n + 1)(3n + 1)(3n + 2) \frac{(2n)^2 (3n)}{m^n},
\]

(ii) We have

\[
\sum_{k=0}^{n} \frac{(2k)^2 (3k)}{(k + 1)^2 m^k} \left( (108 - m)k^3 + (162 - 2m)k^2 + (78 - m)k + 12 \right)
= 6(2n + 1)(3n + 1)(3n + 2) \frac{(2n)^2 (3n)}{(n + 1)^2 m^n},
\]

and

\[
\sum_{k=0}^{n} \frac{k(2k)^2 (3k)}{(k + 1)^2 m^k} \left( (108 - m)k^3 + (162 - m)k^2 + (78 + m)k + m + 12 \right)
= 6n(2n + 1)(3n + 1)(3n + 2) \frac{(2n)^2 (3n)}{(n + 1)^2 m^n}.
\]

Proof. It is easy to prove the five identities by induction on \( n \). We find them by the Gosper algorithm. □
Theorem 4.1. We have the following identities:

\[ \sum_{k=0}^{\infty} \frac{(150k^2 - 51k + 89)k^3{\binom{2k}{k}}^2{\binom{3k}{k}}}{(-192)^k} = \frac{32\sqrt{3}}{15\pi}, \quad (S4) \]

\[ \sum_{k=0}^{\infty} \frac{(1550k^2 + 3029k + 1481){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{(-192)^k} = -\frac{80\sqrt{3}}{\pi}, \quad (S4') \]

\[ \sum_{k=0}^{\infty} \frac{(738k^2 - 4023k + 745)k^3{\binom{2k}{k}}^2{\binom{3k}{k}}}{216^k} = -\frac{24\sqrt{3}}{\pi}, \quad (S5) \]

\[ \sum_{k=0}^{\infty} \frac{(702k^2 + 1491k + 787){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{216^k} = \frac{228\sqrt{3}}{\pi}, \quad (S5') \]

\[ \sum_{k=0}^{\infty} \frac{(159426k^2 - 292761k + 153995)k^3{\binom{2k}{k}}^2{\binom{3k}{k}}}{(-12)^{3k}} = \frac{96\sqrt{3}}{\pi}, \quad (S6) \]

\[ \sum_{k=0}^{\infty} \frac{(604962k^2 + 1206195k + 601183){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{(-12)^{3k}} = -\frac{6864\sqrt{3}}{\pi}, \quad (S6') \]

\[ \sum_{k=0}^{\infty} \frac{(900k^2 - 2097k + 929)k^3}{1458^k} = -\frac{27}{20\pi}, \quad (S7) \]

\[ \sum_{k=0}^{\infty} \frac{(56025k^2 + 112584k + 56551){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{1458^k} = \frac{6615}{4\pi}, \quad (S7') \]

\[ \sum_{k=0}^{\infty} \frac{(173502k^2 - 354087k + 181205)k^3}{(-8640)^k} = \frac{32\sqrt{15}}{5\pi}, \quad (S8) \]

\[ \sum_{k=0}^{\infty} \frac{(2086398k^2 + 4169997k + 2083585){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{(-8640)^k} = -\frac{11504\sqrt{15}}{5\pi}, \quad (S8') \]

\[ \sum_{k=0}^{\infty} \frac{(216711k^2 - 473742k + 226495)k^3}{15^{3k}} = -\frac{60\sqrt{3}}{\pi}, \quad (S9) \]

\[ \sum_{k=0}^{\infty} \frac{(620730k^2 + 1243839k + 623087){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{15^{3k}} = \frac{16935\sqrt{3}}{4\pi}, \quad (S9') \]

\[ \sum_{k=0}^{\infty} \frac{(36512550k^2 - 76893003k + 39703217)k^3}{(-48)^{3k}} = \frac{768\sqrt{3}}{5\pi}, \quad (S10) \]

\[ \sum_{k=0}^{\infty} \frac{Q_{-48}(k){\binom{2k}{k+1}}^2{\binom{3k}{k}}}{(-48)^{3k}} = -3538560\sqrt{3}, \quad (S10') \]
where
\[ Q_{-48}(k) := 18009838350k^2 + 36017730237k + 18007890593. \]

Also,
\[ \sum_{k=0}^{\infty} \frac{(30437550k^2 - 64346463k + 33372157)k^3 \binom{2k}{k}^2 \binom{3k}{k}}{(-326592)^k} = \frac{864\sqrt{7}}{35\pi}, \tag{S11} \]
and
\[ \sum_{k=0}^{\infty} Q(k) \binom{2k}{k+1}^2 \binom{3k}{k} \frac{1}{(-326592)^k} = -11756880 \frac{\sqrt{7}}{7\pi}. \tag{S11'} \]

where
\[ Q(k) := 38518093350k^2 + 77034773577k + 38516679853. \]

Moreover,
\[ \sum_{k=0}^{\infty} \frac{(1227699410778k - 2613341265669k + 1370770039375)k^3 \binom{2k}{k}^2 \binom{3k}{k}}{(-300)^{3k}} = 12000 \frac{\sqrt{3}}{\pi}, \tag{S12} \]
and
\[ \sum_{k=0}^{\infty} Q_{-300}(k) \binom{2k}{k+1}^2 \binom{3k}{k} \frac{1}{(-300)^{3k}} = -13499994000 \frac{\sqrt{3}}{\pi}, \tag{S12'} \]
where
\[ Q_{-300}(k) = 16746723121124538k^2 + 33493438799463639k + 16746715678300691. \]

Proof. By Lemma 4.1(i), for any \( m \in \mathbb{Z} \) with \( |m| > 108 \), we have
\[ \sum_{k=0}^{\infty} \frac{(2k)^2 \binom{3k}{k}}{m^k} \left((108 - m)k^3 + 162k^2 + 78k + 12\right) = 0, \tag{4.1} \]
\[ \sum_{k=0}^{\infty} \frac{k(2k)^2 \binom{3k}{k}}{m^k} \left((108 - m)k^3 + (162 + m)k^2 + 78k + 12\right) = 0, \tag{4.2} \]
and
\[ \sum_{k=0}^{n} \frac{k^2(2k)^2 \binom{3k}{k}}{m^k} \left((108 - m)k^3 + (162 + 2m)k^2 + (78 - m)k + 12\right) = 0. \tag{4.3} \]
We now prove (S4) in details. Taking $m = -192$ in the last three identities, we get

\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)}{(-192)^k} (50k^3 + 27k^2 + 13k + 2) = 0, \quad (4.4)
\]

\[
\sum_{k=0}^{\infty} k \frac{(2k)^2 (3k)}{(-192)^k} (50k^3 - 5k^2 + 13k + 2) = 0, \quad (4.5)
\]

\[
\sum_{k=0}^{\infty} k^2 \frac{(2k)^2 (3k)}{(-192)^k} (50k^3 - 37k^2 + 45k + 2) = 0. \quad (4.6)
\]

Combining the known identity

\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)}{(-192)^k} (5k + 1) = \frac{4}{\sqrt{3}\pi} \quad (4.7)
\]

with (4.4), we get

\[
\sum_{k=0}^{\infty} k \frac{(2k)^2 (3k)}{(-192)^k} (50k^2 + 27k + 3) = -\frac{8}{\sqrt{3}\pi}.
\]

Combining this with (4.5), we see that

\[
\sum_{k=0}^{\infty} k^2 \frac{(2k)^2 (3k)}{(-192)^k} (150k^3 - (3 \times 5 + 2 \times 50)k^2 + (3 \times 13 - 2 \times 27)k) = \frac{16\sqrt{3}}{\pi},
\]

i.e.,

\[
\sum_{k=0}^{\infty} k^2 \frac{(2k)^2 (3k)}{(-192)^k} (150k^3 + 91k^2 + 45k + 2) = \frac{16\sqrt{3}}{5\pi}. \quad (4.8)
\]

(4.6)$\times 3$ plus (4.8)$\times 2$ yields

\[
\sum_{k=0}^{\infty} k^2 \frac{(2k)^2 (3k)}{(-192)^k} (150k^3 + (60 - 111)k^2 + (135 - 46)k) = \frac{32}{5\sqrt{3}\pi},
\]

which is equivalent to (S4).

Next we prove (S4'). Applying 4.1(ii) with $m = -192$, we have

\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)}{(k+1)^2(-192)^k} (50k^3 + 91k^2 + 45k + 2) = 0, \quad (4.9)
\]

\[
\sum_{k=0}^{\infty} \frac{k(2k)^2 (3k)}{(k+1)^2(-192)^k} (50k^3 + 59k^2 - 19k - 30) = 0. \quad (4.10)
\]

As

\[
50k^3 + 91k^2 = 45k + 2 - 10(k+1)(k+1)^2 = -(19k^2 + 25k + 8),
\]
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from (4.9) and (4.7) we obtain

$$\sum_{k=0}^{\infty} \frac{(2k)^2}{(k+1)^2(-192)^k} (19k^2 + 25k + 8) = \frac{40\sqrt{3}}{3\pi}.$$ (4.11)

Note that (4.9) $\times 4$ (4.11) yields

$$\sum_{k=0}^{\infty} \frac{k^2 (2k)^2}{(k+1)^2(-192)^k} (40k^2 + 69k + 31) = -\frac{8\sqrt{3}}{3\pi}.$$ (4.12)

Via (4.10) $\times 31 + 30 \times (4.12)$ we obtain

$$\sum_{k=0}^{\infty} \frac{k^2 (2k)^2}{(k+1)^2(-192)^k} (1550k^2 + 3029k + 1481) = -\frac{80\sqrt{3}}{\pi},$$

which is equivalent to the desired (S4′) since $(\begin{array}{c} 2k \\ k+1 \end{array}) = \frac{k}{k+1} (\begin{array}{c} 2k \\ k \end{array})$.

All the remaining formulas in Theorem 4.1 can be proved similarly. We omit the details. □ □

Lemma 4.2. Let $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$.

(i) We have

$$\sum_{k=0}^{n} \frac{(2k)}{m^k} \frac{(2k)^2}{(4k)} ((256 - m)k^3 + 384k^2 + 176k + 24) = 8(2n+1)(4n+1)(4n+3) \frac{(2n)^2}{m^n},$$

and

$$\sum_{k=0}^{n} \frac{k^2 (2k)}{m^k} \frac{(2k)^2}{(4k)} ((256 - m)k^3 + (384 + 2m)k^2 + 176k + 24) = 8n(2n+1)(4n+1)(4n+3) \frac{(2n)^2}{m^n},$$

and

$$\sum_{k=0}^{n} \frac{k^2 (2k)}{m^k} \frac{(2k)^2}{(4k)} ((256 - m)k^3 + (384 + 2m)k^2 + (176 - m)k + 24) = 8n^2(2n+1)(4n+1)(4n+3) \frac{(2n)^2}{m^n}.$$ (2n)

(ii) We have

$$\sum_{k=0}^{n} \frac{(2k)}{(k+1)^2 m^k} ((256 - m)k^3 + (384 - 2m)k^2 + (176 - m)k + 24) = 8(2n+1)(4n+1)(4n+3) \frac{(2n)^2}{(n+1)^2 m^n}.$$
and
\[ \sum_{k=0}^{\infty} \frac{k(2k)^2(4k)}{(k+1)^2m^k} \left( (256 - m)k^3 + (384 - m)k^2 + (176 + m)k + m + 24 \right) \]
\[ = 8n(2n + 1)(4n + 1)(4n + 3) \frac{(2n)^2(4n)}{(n + 1)^2m^n}. \]

**Proof.** It is easy to prove the five identities by induction on \( n \). We find them by the Gosper algorithm. \( \square \)

Using Lemma 4.2 and the method we prove Theorem 4.1, we obtain the following theorem.

**Theorem 4.2.** We have the following identities:

\[ \sum_{k=0}^{\infty} \frac{(854k^2 - 3639k + 910)k^3(2k)^2(4k)}{648k} = -\frac{243}{14\pi}, \quad (S13) \]
\[ \sum_{k=0}^{\infty} \frac{(2107k^2 + 4359k + 2249)(2k)^2(4k)}{648k} = \frac{567}{\pi}, \quad (S13') \]
\[ \sum_{k=0}^{\infty} \frac{(400k^2 - 496k + 327)k^3(2k)^2(4k)}{(-1024)^k} = \frac{9}{5\pi}, \quad (S14) \]
\[ \sum_{k=0}^{\infty} \frac{(19840k^2 + 39324k + 19481)(2k)^2(4k)}{(-1024)^k} = -\frac{1000}{\pi}, \quad (S14') \]
\[ \sum_{k=0}^{\infty} \frac{(6016k^2 - 14856k + 6215)k^3(2k)^2(4k)}{48^{2k}} = -\frac{27\sqrt{3}}{4\pi}, \quad (S15) \]
\[ \sum_{k=0}^{\infty} \frac{(75776k^2 + 152520k + 76729)(2k)^2(4k)}{48^{2k}} = \frac{1746\sqrt{3}}{\pi}, \quad (S15') \]
\[ \sum_{k=0}^{\infty} \frac{(1326650k^2 - 2420121k + 1281559)(2k)^2(4k)}{(-63)^k} = 1944\frac{\sqrt{7}}{5\pi}, \quad (S16) \]

and
\[ \sum_{k=0}^{\infty} \frac{(8242975k^2 + 16441878k + 8198387)(2k)^2(4k)}{(-63)^k} = -\frac{106515\sqrt{7}}{2\pi}. \quad (S16') \]

We also have
\[ \sum_{k=0}^{\infty} \frac{(320k^2 - 678k + 337)k^3(2k)^2(4k)}{12^{4k}} = -\frac{243\sqrt{2}}{10240\pi}. \quad (S17) \]
\[
\sum_{k=0}^{\infty} \frac{(6671360k^2 + 13350714k + 6679321)(2k)\binom{2k}{k+1}^2}{12^{4k}} = \frac{70065\sqrt{2}}{4\pi}, \quad (S17')
\]

\[
\sum_{k=0}^{\infty} \frac{(2576k^2 - 5136k + 2635)k^3}{(-3 \times 2^{12})^k} (2k)\binom{2k}{k+1}^2 = \frac{2\sqrt{3}}{7\pi}, \quad (S18)
\]

\[
\sum_{k=0}^{\infty} \frac{(796544k^2 + 1591612k + 795059)}{(-3 \times 2^{12})^k} (2k)\binom{2k}{k+1}^2 = \frac{-8176\sqrt{3}}{3\pi}, \quad (S18')
\]

\[
\sum_{k=0}^{\infty} \frac{(2428400k^2 - 5044368k + 2635)}{(-2^{10}3^4)^k} (2k)\binom{2k}{k+1}^2 = \frac{243}{5\pi}, \quad (S19)
\]

\[
\sum_{k=0}^{\infty} \frac{(2475740800k^2 + 4950772932k + 2475031103)}{(-2^{10}3^4)^k} (2k)\binom{2k}{k+1}^2 = \frac{-2238840}{\pi}, \quad (S19')
\]

\[
\sum_{k=0}^{\infty} \frac{(38400k^2 - 80696k + 41609)}{284k} (2k)\binom{2k}{k+1}^2 = \frac{-49\sqrt{3}}{1080\pi}, \quad (S20)
\]

\[
\sum_{k=0}^{\infty} \frac{(1967513600k^2 + 3935104168k + 1967590547)}{284k} (2k)\binom{2k}{k+1}^2 = \frac{3764915\sqrt{3}}{27\pi}, \quad (S20')
\]

\[
\sum_{k=0}^{\infty} \frac{(423089968k^2 - 891891888k + 463383905)}{(-2^{12}4^{15})^k} (2k)\binom{2k}{k+1}^2 = \frac{972\sqrt{5}}{35\pi}, \quad (S21)
\]

\[
\sum_{k=0}^{\infty} q(k)\binom{2k}{k+1}^2 = -716633568\sqrt{5} \frac{\sqrt{5}}{5\pi}, \quad (S21')
\]

where

\[
q(k) := 28206829076608k^2 + 56413556154372k + 28206727074305.
\]

We also have

\[
\sum_{k=0}^{\infty} \frac{(643835623600k^2 - 1361740501968k + 711617288021)}{(-2^{10}21^4)^{2k}} = \frac{11907}{5\pi}, \quad (S22)
\]

and

\[
\sum_{k=0}^{\infty} \frac{q_k(2k)^2}{(-2^{10}21^4)^{2k}} = \frac{-263473491960}{\pi}, \quad (S22')
\]
where \( q_k \) denotes

\[
6959113049907200 k^2 + 1391822523134211732k + 695911219634175403
\]

Moreover,

\[
\sum_{k=0}^{\infty} \frac{(13462400k^2 - 28347528k + 14689591)k^3}{1584^{2k}} = \frac{243\sqrt{11}}{140\pi}, \quad (S23)
\]

and

\[
\sum_{k=0}^{\infty} \frac{u_k k^2 (4k)^2}{1584^{2k}} = 16936290\sqrt{\frac{11}{\pi}}, \quad (S23')
\]

where

\[
u_k := 1868912998400k^2 + 3737843883096k + 1868930883259.
\]

Also,

\[
\sum_{k=0}^{\infty} \frac{(234440315200k^2 - 497134511862k + 260991361673)k^3}{396^{4k}} = -\frac{264627\sqrt{2}}{71680\pi}, \quad (S24)
\]

and

\[
\sum_{k=0}^{\infty} \frac{v_k k^2 (4k)^2}{396^{4k}} = 90382094430705\sqrt{\frac{2}{4\pi}}, \quad (S24')
\]

where

\[
v_k := 1042214305665206809600k^2 + 20844286081501973765862k + 1042214304583676781623.
\]

**Lemma 4.3.** Let \( m \in \mathbb{Z} \setminus \{0\} \) and \( n \in \mathbb{N} \).

(i) We have

\[
\sum_{k=0}^{n} \frac{k(2k)k(3k)k(6k)}{m^k} \left( (1728 - m)k^3 + 2592k^2 + 1104k + 120 \right) = 24(2n + 1)(6n + 1)(6n + 5) \frac{(2n)(3n)(6n)}{m^n},
\]

\[
\sum_{k=0}^{n} \frac{k(2k)k(3k)k(6k)}{m^k} \left( (1728 - m)k^3 + (2592 + m)k^2 + 1104k + 120 \right) = 24n(2n + 1)(6n + 1)(6n + 5) \frac{(2n)(3n)(6n)}{m^n},
\]
and
\[
\sum_{k=0}^{n} k^2 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \frac{k^3 + (2592 + 2m)k^2 + (1104 - m)k + 120}{m^k} = 24n^2(2n + 1)(6n + 1)(6n + 5) \frac{(2n) \binom{3n}{n} \binom{6n}{3n}}{m^n}.
\]

(ii) We have
\[
\sum_{k=0}^{n} \frac{(2k) \binom{3k}{k} \binom{6k}{3k}}{(k + 1)^2 m^k} \left((1728 - m)k^3 + (2592 - 2m)k^2 + (1104 - m)k + 120\right)
= 24(2n + 1)(6n + 1)(6n + 5) \frac{(2n) \binom{3n}{n} \binom{6n}{3n}}{(n + 1)^2 m^n}
\]
and
\[
\sum_{k=0}^{n} \frac{k(2k) \binom{3k}{k} \binom{6k}{3k}}{(k + 1)^2 m^k} \left((1728 - m)k^3 + (2592 - m)k^2 + (1104 + m)k + m + 120\right)
= 24n(2n + 1)(6n + 1)(6n + 5) \frac{(2n)^2 \binom{4n}{2n}}{(n + 1)^2 m^n}.
\]

Proof. It is easy to prove the five identities by induction on \(n\). We find them by the Gosper algorithm. \(\square\)

For any \(k \in \mathbb{N}\), we clearly have
\[
\binom{3k}{k + 1} = \frac{2k}{k + 1} \binom{3k}{k},
\]
which is similar to \(\binom{2k}{k+1} = \frac{k}{k+1} \binom{2k}{k}\). In view of this and Lemma 4.3, by our method to prove Theorem 4.1 we can establish the following theorem.

**Theorem 4.3.** (i) We have the following identities:
\[
\sum_{k=0}^{\infty} \frac{(23968k^2 - 71188k + 25539)k^3 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{20^{3k}} = -\frac{1875\sqrt{5}}{56\pi},
\]
\[
\sum_{k=0}^{\infty} \frac{(269696k^2 + 545164k + 275383) \binom{2k}{k+1} \binom{3k}{k+1} \binom{6k}{3k}}{20^{3k}} = \frac{50750\sqrt{5}}{3\pi},
\]
\[
\sum_{k=0}^{\infty} \frac{(90706k^2 - 168589k + 88872)k^3 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-2^{15})^k} = \frac{150\sqrt{2}}{7\pi},
\]
and
\[
\sum_{k=0}^{\infty} \frac{(5821200k^2 + 11623266k + 5801911) \binom{2k}{k+1} \binom{3k}{k+1} \binom{6k}{3k}}{(-2^{15})^k} = -\frac{261184\sqrt{2}}{3\pi}.
\]
(ii) We have
\[
\sum_{k=0}^{\infty} \frac{(457326k^2 - 308241k + 342923)k^3 \binom{2k}{k} \binom{3k}{3k}^6}{{(-15)}^3k} = \frac{5000\sqrt{15}}{7\pi}, \quad (S27)
\]
and
\[
\sum_{k=0}^{\infty} \frac{(933849k^2 + 1843866k + 910277)k^3 \binom{2k}{k+1} \binom{3k}{3k}^6}{{(-15)}^3k} = -\frac{27125\sqrt{15}}{\pi}, \quad (S27')
\]
Moreover,
\[
\sum_{k=0}^{\infty} \frac{(12826k^2 - 27741k + 13298)k^3 \binom{2k}{k} \binom{3k}{3k}^6}{(2 \times 30^3)k} = -\frac{625\sqrt{15}}{972\pi}, \quad (S28)
\]
\[
\sum_{k=0}^{\infty} \frac{(4941882k^2 + 9895613k + 4953661)k^3 \binom{2k}{k+1} \binom{3k}{3k}^6}{(2 \times 30^3)k} = \frac{56375\sqrt{15}}{3\pi}. \quad (S28')
\]
(iii) We have
\[
\sum_{k=0}^{\infty} \frac{(18126342k^2 - 37421775k + 19111480)k^3 \binom{2k}{k} \binom{3k}{3k}^6}{(-96)^3k} = \frac{50\sqrt{6}}{\pi}, \quad (S29)
\]
and
\[
\sum_{k=0}^{\infty} \frac{(680205988k^2 + 13603203918k + 6801143345)k^3 \binom{2k}{k+1} \binom{3k}{3k}^6}{(-96)^3k} = -\frac{2358976\sqrt{6}}{\pi}, \quad (S29')
\]
(iv) We have
\[
\sum_{k=0}^{\infty} \frac{(2248722k^2 - 4689621k + 2357878)k^3 \binom{2k}{k} \binom{3k}{3k}^6}{66^3k} = -\frac{275\sqrt{33}}{28\pi}, \quad (S30)
\]
\[
\sum_{k=0}^{\infty} \frac{(7147721k^2 + 142985853k + 71508077)k^3 \binom{2k}{k+1} \binom{3k}{3k}^6}{66^3k} = \frac{32956\sqrt{33}}{\pi}, \quad (S30')
\]
\[
\sum_{k=0}^{\infty} \frac{(2161071858k^2 - 449774503k + 2312761384)k^3 \binom{2k}{k} \binom{3k}{3k}^6}{(-3 \times 160^3)k} = \frac{1250\sqrt{30}}{9\pi}, \quad (S30')
\]
and
\[
\sum_{k=0}^{\infty} \frac{w_k \binom{2k}{k} \binom{3k}{3k}^6}{(-3 \times 160^3)k} = -\frac{273064000\sqrt{30}}{3\pi}, \quad (S31)
\]
where

\[ w_k := 8126882714192k^2 + 16253686120778k + 8126803400291 \]

(v) We have

\[
\sum_{k=0}^{\infty} \frac{(273732850062k^2 - 572136425667k + 296241014776)k^3 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-960)^3k} = \frac{5000\sqrt{15}}{21\pi},
\]

and

\[
\sum_{k=0}^{\infty} \frac{x_k \binom{2k}{k+1} \binom{3k}{k+1} \binom{6k}{3k}}{(-960)^3k} = -235929568000\frac{\sqrt{15}}{3\pi}, \tag{S32'}
\]

where

\[ x_k := 357405080886027216k^2 + 741810113296569594k + 357405032410460843. \]

Also,

\[
\sum_{k=0}^{\infty} \frac{(6491502k^2 - 13521457k + 6955771)k^3 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{255^3k} = -42500\sqrt{255} \frac{1}{413343\pi}, \tag{S33}
\]

and

\[
\sum_{k=0}^{\infty} \frac{y_k \binom{2k}{k+1} \binom{3k}{k+1} \binom{6k}{3k}}{255^3k} = 2349045125\frac{\sqrt{255}}{54\pi}. \tag{S33'}
\]

where

\[ y_k := 15274005325299k^2 + 30548121249166k + 15274115917127. \]

(vi) We have

\[
\sum_{k=0}^{\infty} \frac{p(k)k^3 \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-5280)^3k} = \frac{13750\sqrt{330}}{7\pi}, \tag{S34}
\]

and

\[
\sum_{k=0}^{\infty} \frac{q(k)\binom{2k}{k+1} \binom{3k}{k+1} \binom{6k}{3k}}{(-5280)^3k} = -107945164712000\frac{\sqrt{330}}{\pi}, \tag{S34'}
\]

where

\[ p(k) = 2417841826472898k^2 - 5065781116806693k + 2634006768739304 \]

and

\[ q(k) = 382825470402996808454064k^2 + 765650940493903329223926k + 38282547090906516313597. \]
Also,
\[
\sum_{k=0}^{\infty} \frac{P(k)k^3 \binom{2k}{k} \binom{3k}{3k} \binom{6k}{3k}}{(-640320)^{3k}} = -\frac{83\sqrt{10005}}{\pi} \tag{S35}
\]
and
\[
\sum_{k=0}^{\infty} \frac{Q(k)k^{2k} \binom{2k}{k+1} \binom{3k}{3k} \binom{6k}{3k}}{(-640320)^{3k}} = -46696654461704580256000 \sqrt{10005} \pi , \tag{S35'}
\]
where
\[
P(k) := 3726784819871553194540063287782k^2 \\
- 7783860761103294083667021327391k \\
+ 4057075941237594195269253626425
\]
and
\[
Q(k) := 16263888939999998834192505020394081197549.
\]

5. Proof of Theorem 1.4

Proof of Theorem 1.4(i). Let \( u_n = \text{Domb}(n)/64^n \) and \( v_n = \text{Domb}(n)/(-32)^n \) for \( n \in \mathbb{N} \). By the Zeilberger algorithm, we find the recurrence

\[
(n + 1)^3u_n - 2(2n + 3)(5n^2 + 15n + 12)u_{n+1} + 64(n + 2)^3u_{n+2} = 0
\]

and

\[
(n + 1)^3v_n + (2n + 3)(5n^2 + 15n + 12)v_{n+1} + 16(n + 2)^3v_{n+2} = 0.
\]

Thus,

\[
0 = \sum_{n=0}^{\infty} (n + 1)^3u_n - 2\sum_{n=0}^{\infty} (2(n + 1) + 1)(5(n + 1)^2 + 5(n + 1) + 2)u_{n+1}
\]

\[
+ 64\sum_{n=0}^{\infty} (n + 2)^3u_{n+2}
\]

\[
= \sum_{k=0}^{\infty} (k + 1)^3u_k - 2\sum_{k=1}^{\infty} (2k + 1)(5k^2 + 5k + 2)u_k + 64\sum_{k=2}^{\infty} k^3u_k
\]

\[
= \sum_{k=0}^{\infty} (\left( (k + 1)^3 - 2(2k + 1)(5k^2 + 5k + 2) + 64k^3 \right) u_k
\]

\[
+ 2(2 \times 0 + 1)(5 \times 0^2 + 5 \times 0 + 2)u_0 - 64(0^3u_0 + 1^3u_1)
\]

\[
= 3\sum_{k=0}^{\infty} (15k^3 - 9k^2 - 5k - 1)u_k
\]
and hence
\[ \sum_{k=0}^{\infty} (15k^3 - 9k^2 - 5k - 1)u_k = 0. \] (5.1)

Similarly,
\[ 0 = \sum_{k=0}^{\infty} \left((k + 1)^3 + (2k + 1)(5k^2 + 5k + 2) + 16k^3\right)v_k \]
\[ = 3 \sum_{k=0}^{\infty} (9k^3 + 6k^2 + 4k + 1)v_k \]
and hence
\[ \sum_{k=0}^{\infty} (9k^3 + 6k^2 + 4k + 1)v_k = 0. \] (5.2)

Combining (5.1) and (5.2) with the two known identities in (1.6), we immediately get the desired (1.12) and (1.11). \[ \square \]

Proof of Theorem 1.4(ii). The second part of Theorem (1.4) can be proved by the same method we prove Theorem (1.4)(i). Here we just provide a proof of (1.13) in details. Let \( w_n = \frac{f_n^{(4)}}{36^n} \) for \( n \in \mathbb{N} \). By the Zeilberger algorithm, we find the recurrence
\[ 324(n+2)^3w_{n+2} - 18(2n+3)(3n^2+9n+7)w_{n+1} - (n+1)(4n+3)(4n+5)w_n = 0. \]

Thus,
\[ 0 = 324 \sum_{n=0}^{\infty} (n + 2)^3w_{n+2} - 18 \sum_{n=0}^{\infty} (2(n + 1) + 1)(3(n + 1)^2 + 3(n + 1) + 1)w_{n+1} \]
\[ - \sum_{n=0}^{\infty} (n + 1)(4n + 3)(4n + 5)w_n \]
\[ = 324 \sum_{k=2}^{\infty} k^3w_k - 18 \sum_{k=1}^{\infty} (2k + 1)(3k^2 + 3k + 1)w_k \]
\[ - \sum_{k=0}^{\infty} (k + 1)(4k + 3)(4k + 5)w_k \]
\[ = \sum_{k=0}^{\infty} \left(324k^3 - 18(2k + 1)(3k^2 + 3k + 1) - (k + 1)(4k + 3)(4k + 5)\right)w_k \]
\[ - 324 \times 1^3w_1 + 18(2 \times 0 + 1)(3 \times 0^2 + 3 \times 0 + 1) \]
and hence
\[ \sum_{k=0}^{\infty} (200k^3 - 210k^2 - 137k - 33)w_k = 0. \] (5.3)
By (1.7),
\[ \sum_{k=0}^{\infty} 33(4k + 1)w_k = \frac{33 \times 18}{\sqrt{15\pi}}. \] (5.4)

Adding (5.3) and (5.4), we immediately obtain (1.13). □

Our method for proving parts (i)-(ii) of Theorem 1.4 also works for parts (iii)-(iv) of Theorem 1.4. We omit the proof details.

6. CONJECTURES FOR SERIES ONLY INVOLVING BINOMIAL COEFFICIENTS

Motivated by the author’s conjectural identity
\[ \sum_{k=1}^{\infty} \frac{48^k}{k(2k-1)\binom{2k}{k}\binom{4k}{2k}} = \frac{15}{2}K \]
(cf. [22, 24]), we make the following conjecture.

**Conjecture 6.1.** We have
\[ \sum_{k=0}^{\infty} \frac{(4k^2 - 30k + 17)48^k}{k(2k-1)\binom{2k}{k}\binom{4k}{2k}} = \frac{45K - 44}{2}. \]

**Remark 6.1.** By induction, for each \( n = 1, 2, 3, \ldots \) we have
\[ \sum_{k=1}^{n} \frac{(4k^2 - 28k + 3)48^k}{k\binom{2k}{k}\binom{4k}{2k}} = 12 - \frac{12(n + 1)48^n}{\binom{2n}{n}\binom{4n}{2n}}. \]
and
\[ \sum_{k=1}^{n} \frac{(4k^2 - 40k - 9)48^k}{\binom{2k}{k}\binom{4k}{2k}} = -12 + \frac{12(n + 1)^248^n}{\binom{2n}{n}\binom{4n}{2n}}. \]

In view of Theorem 1.3 and the author’s conjectural identity (1.5), we pose the following conjecture obtained via the PSLQ algorithm.

**Conjecture 6.2.** (i) We have
\[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k+1)/5}P(k) = \frac{8103654862170335619 + 4368545100830839178\zeta(3)}{5}, \]
where
\[ P(k) = 54430524632163842275k^4 - 132483674356197881281k^3 + 121816306858962351437k^2 - 47590274284953796032k + 6700636215039814272. \]

(ii) We have
\[ \sum_{k=0}^{\infty} \frac{(-64)^k}{\binom{2k}{k}\binom{4k}{3k}}(676704k^4 - 1205388k^3 + 1140374k^2 - 152237k + 78797) = 35733 - 1344\zeta(3), \]
NEW TYPE SERIES FOR POWERS OF $\pi$

$$\sum_{k=1}^{\infty} \frac{(-64)^{k-1}}{\binom{2k}{k+1}^4 \binom{3k}{k}} Q(k) = 7(14488697756 + 4718909979\zeta(3))$$

and

$$\sum_{k=1}^{\infty} \frac{(-64)^{k} R(k)}{\binom{2k}{k+1}^4 \binom{3k}{k+1}} = -86166921288937568 - 74477398755902744\zeta(3),$$

where

$$Q(k) = 9152858507744k^4 - 18103487906940k^3 + 16104889340010k^2 - 5519172201903k + 668801335410$$

and

$$R(k) = 152571345867547488k^4 - 32544501335126032k^3 + 295511129648313866k^2 - 10644946934061699k + 13378286508841890$$

**Remark 6.2.** In the spirit of our method to prove Theorems 1.1-1.3 as well as the general algorithm to deduce series of type $S$, parts (i) and (ii) of Conjecture 6.2 look equivalent to (1.4) and (1.5) respectively but we have not done this in details.

Motivated by B. Gourevich’s conjectural identity (1.2) and Guillera’s conjectural identity

$$\sum_{k=1}^{\infty} \frac{256^k}{k^7 \binom{2k}{k}^7} (21k^3 - 22k^2 + 8k - 1) = \frac{\pi^4}{8},$$

(cf. [13, Section 4]), we pose the following conjecture via the PSLQ algorithm.

**Conjecture 6.3.** (i) We have

$$\sum_{k=1}^{\infty} \frac{S(k)}{2^{20k}} k^7 \binom{2k}{k}^7 = -\frac{1}{24\pi^3}$$

and

$$\frac{1}{32} \sum_{k=1}^{\infty} \frac{T(k)}{2^{20k}} \binom{2k}{k+1}^7 = \frac{5570231394467396545747200}{234314585133056805845704703} \pi^3$$

where

$$S(k) := 56448k^6 - 347200k^5 + 854280k^4 - 1145956k^3 + 851214k^2 - 339967k + 56160$$
and

\[
T(k) = 362334901725047543340617856k^6 + 1557477795272579082461315904k^5 \\
+ 238729637785482351932510816k^4 + 1269650426797215833274563064k^3 \\
- 372612359665735835469802516k^2 - 62039596962808879309367722k \\
- 170581863683533832164571735
\]

(ii) We have

\[
\sum_{k=0}^{\infty} \frac{U(k)256^k}{(2^k)^7} = \frac{9984 - \pi^4}{48}
\]

and

\[
\sum_{k=1}^{\infty} \frac{V(k)256^k}{(2^k_{k+1})^7} = \frac{103034584550200896294336 - 47118227214655104697325\pi^4}{48},
\]

where

\[
U(k) := 14112k^6 - 44464k^5 + 49490k^4 - 41069k^3 + 8155k^2 - 4749k - 210.
\]

and

\[
V(k) := 105195631551721406964324k^6 - 375522390327670972174376k^5 \\
+ 50801003024769047270138k^4 - 37698957318614973646723k^3 \\
+ 156139327775481582503524k^2 - 33942569977747809706722k \\
+ 3027100288061502033993.
\]

7. NEW TYPE SERIES INVOLVING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

For \(b, c \in \mathbb{Z}\) and \(n \in \mathbb{N}\), the generalized central trinomial coefficient \(T_n(b, c)\) denotes the coefficient of \(x^n\) in the expansion of \((x^2 + bx + c)^n\). The author \([21, 24, 25, 26]\) posed totally 9 types of conjectural series for \(1/\pi\) involving generalized central trinomial coefficients. Here we consider their variants of type \(S\).

**Conjecture 7.1.** We have

\[
\sum_{k=1}^{\infty} \frac{3054600k^2 - 16826114k + 11236485}{(-256)^k} k^3 \binom{2k}{k}^2 T_k(1, 16) = -\frac{1952307}{5\pi}, \quad (I1S)
\]

\[
\sum_{k=1}^{\infty} \frac{357600k^2 - 239434k + 401075}{(-1024)^k} k^3 \binom{2k}{k}^2 T_k(34, 1) = \frac{25983}{10\pi}, \quad (I2S)
\]

\[
\sum_{k=1}^{\infty} \frac{28823880k^2 - 740215234k + 3516311133}{4096^k} k^3 \binom{2k}{k}^2 T_k(194, 1) \]
\[
= -\frac{152854918}{3\pi}, \quad (I3S)
\]
\[ \sum_{k=1}^{\infty} \frac{1336776k^2 - 5896258k + 4457117}{4096k} k^3 \binom{2k}{k}^2 T_k(62, 1) = -\frac{188698\sqrt{3}}{7\pi}. \]

Also,
\[ \sum_{k=1}^{\infty} \frac{12k^2 - 28k - 9}{256k} \binom{2k}{k}^2 T_k(8, -2) = -\frac{4(\sqrt{8} + 6\sqrt{2} + 3\sqrt{2})}{3\pi}, \]
\[ \sum_{k=1}^{\infty} \frac{162k^2 - 969k + 872}{256k} \binom{2k}{k}^2 T_k(8, -2) = \frac{708\sqrt{2} - 1951\sqrt{8} + 6\sqrt{2}}{12\pi}, \]
and
\[ \sum_{k=1}^{\infty} \frac{31392k^2 - 277274k + 594637}{256k} k^3 \binom{2k}{k}^2 T_k(8, -2) = \frac{90563\sqrt{8} + 6\sqrt{2} - 221844\sqrt{2}}{12\pi}. \]

**Conjecture 7.2.** We have the following identities:

\[ \sum_{k=1}^{\infty} k^3 \left(3401775k^2 - 37884933k + 65097406\right) \binom{2k}{k} \binom{3k}{k} T_k(18, 6) \]
\[ = \frac{158875\sqrt{3}}{\pi}, \]
\[ \sum_{k=1}^{\infty} k^3 a_k \binom{2k}{k} \binom{3k}{k} T_k(10, 1) = -\frac{3441159450\sqrt{3}}{7\pi}, \]
where
\[ a_k = 34073404820k^2 - 166944861551k + 136909066683, \]
\[ \sum_{k=1}^{\infty} k^3 a'_k \binom{2k}{k} \binom{3k}{k} T_k(198, 1) = -\frac{1285805325750\sqrt{3}}{\pi}, \]
where
\[ a'_k = 138317121900k^2 - 11341624063599k + 169057291391203, \]
\[ \sum_{k=1}^{\infty} \frac{515565k^2 - 1452888k + 614707}{243k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(26, 729) \]
\[ = -\frac{64}{98415\pi} (195423\sqrt{3} + 532925\sqrt{15}), \]
\[ \sum_{k=1}^{\infty} \frac{1326294k^2 - 4598217k + 2285731}{(-5400)^k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(70, 3645) \]
\[ = -\frac{2}{2187\pi} (78460655\sqrt{3} + 13402757\sqrt{15}), \]
\[ \sum_{k=1}^{\infty} \frac{1\cdot 5}{(-5400)^k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(70, 3645) \]
\[ = -\frac{2}{2187\pi} (78460655\sqrt{3} + 13402757\sqrt{15}), \]
\[
\sum_{k=1}^{\infty} \frac{4076131815k^2 - 7828831071k + 7124292568}{(-13500)^k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(40, 1548) \\
= \frac{25}{91854\pi} \left(32473732248\sqrt{3} - 33657641611\sqrt{6}\right)
\]

(II12S)

\[
\sum_{k=1}^{\infty} \frac{17276571k^2 - 9174528k + 19362029}{(-675)^k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(15, -5) \\
= \frac{159680563\sqrt{5}}{3456\pi}
\]

(II13S)

\[
\sum_{k=1}^{\infty} \frac{5837598k^2 - 6981399k + 6325061}{(-1944)^k} k^3 \binom{2k}{k} \binom{3k}{k} T_k(18, -3) \\
= \frac{11336\sqrt{3}}{\pi}
\]

(II14S)

**Conjecture 7.3.** We have the following identities:

\[
\sum_{k=1}^{\infty} \frac{k(133120k^2 - 28704k + 2669)}{(-168^2)^k} \binom{2k}{k} \binom{4k}{2k} T_k(7, 4096) \\
= \frac{21(14481\sqrt{42} + 5650\sqrt{210})}{256\pi}
\]

\[
\sum_{k=1}^{\infty} \frac{k^2(2733056k^2 - 1413552k + 1735253)}{(-168^2)^k} \binom{2k}{k} \binom{4k}{2k} T_k(7, 4096) \\
= \frac{63(26395881\sqrt{42} + 13904950\sqrt{210})}{40960\pi}
\]

and

\[
\sum_{k=1}^{\infty} \frac{k^3 b_k}{(-168^2)^k} \binom{2k}{k} \binom{4k}{2k} T_k(7, 4096) \\
= \frac{189}{1310720\pi} \left(181371717913\sqrt{42} - 200347079650\sqrt{210}\right)
\]

(III15S)

where

\[
b_k = 17768990720k^2 - 81509977344k + 73509891901.
\]

**Remark 7.1.** We omit many other conjectural identities of type S arising from the author’s series for 1/π of type III (cf. [21]).

**Conjecture 7.4.** We have the following identities:

\[
\sum_{k=1}^{\infty} \frac{k^3 c_k}{(-48^2)^k} \binom{2k}{k}^2 T_{2k}(7, 1) = \frac{7110645399}{5\pi}
\]

(IV1S)
and
\[ \sum_{k=1}^{\infty} k^3 d_k \left( \frac{2k}{k} \right)^2 T_{2k}(10, -2) = -\frac{20525967\sqrt{6}}{4\pi}, \quad \text{(IV19S)} \]

where
\[ c_k = 253607171350k^2 - 231835223289k + 29591225514 \]
and
\[ d_k = 220693694k^2 - 1203431385k + 1673732470. \]

Remark 7.2. We omit many other conjectural identities of type S arising from the author’s series for $1/\pi$ of type IV (cf. [21]).

Conjecture 7.5. We have
\[ \sum_{k=1}^{\infty} k e^{(1)}_k \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = -\frac{318713456\sqrt{105}}{49\pi} \]
where\[ e^{(1)}_k := 24501204k^2 + 511623432k + 17450401, \]
\[ \sum_{k=1}^{\infty} k^2 e^{(2)}_k \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = -\frac{11867637235544\sqrt{105}}{343\pi} \]
where\[ e^{(2)}_k := 1543522869252k^2 + 354135873240k + 326925084163, \]
and
\[ \sum_{k=1}^{\infty} k^3 e^{(3)}_k \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = -\frac{127447736489940892\sqrt{105}}{2401\pi} \quad \text{(V1S)} \]
where\[ e^{(3)}_k := 41310253635000948k^2 - 37929668348178936k + 56105265685716343. \]

Also,
\[ \sum_{k=1}^{\infty} \left( \frac{k^2 + 1}{-240} \right)^3 \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = -\frac{429939571410644\sqrt{105}}{343\pi}, \]
where\[ e_k := 263893057256556k^2 + 46715038856064k + 6234077522125. \]

Remark 7.3. This is motivated by the author’s conjectural identity
\[ \sum_{k=0}^{\infty} \frac{1638k + 277}{(-240)^{3k}} \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = \frac{44\sqrt{105}}{\pi} \quad \text{(V1)} \]
(cf. [21, 24]) confirmed in [28]. We also conjecture that
\[ \sum_{k=0}^{\infty} \frac{47675628k^3 + 995541624k^2 + 437210809k + 79678364}{(-240)^{3k}} \left( \frac{2k}{k} \right) \left( \frac{3k}{k} \right) T_{3k}(62, 1) = 0. \]
Conjecture 7.6. (i) We have
\[
\sum_{k=1}^{\infty} \frac{5939142726k^2 - 50217677843k + 47679239989}{(2113^3)^k}k^3T_k(10,11^2)^3 = 125366398162515\sqrt{2} \over 644204\pi,
\] (VI1S)

(ii) We have
\[
\sum_{k=1}^{\infty} \frac{k^3f(k)}{(-80)^3 k}T_k(22,21^2)^3 = -\frac{317993969514116005\sqrt{5}}{64827\pi},
\] (VI2S)
where
\[
f(k) := 182075646906594k^2 - 658193121766971k + 498776294291290.
\]

(iii) We have
\[
\sum_{k=1}^{\infty} \frac{k^3g(k)}{(-288)^3 k}T_k(62,95^2)^3 = -\frac{9}{30008125\pi}(a\sqrt{2} + b\sqrt{14}),
\] (VI3S)
where
\[
a = 2778571005723952224834213458, b = 2335639592156477133727790625,
\]
and
\[
g(k) := 29814661986490996566930k^2 - 138270524135932079678425k + 91756685418870140080439.
\]

Conjecture 7.7. (i) We have
\[
\sum_{k=1}^{\infty} \frac{k^3h_k}{450^k} \binom{2k}{k}^2T_k(6,2)^2 = -\frac{1525516918799600750400}{7\pi},
\] (VII1S)
where
\[
h_k := 1717562453635471595698k^2 - 20741022469043431508721k + 29930651775020896516895.
\]

Also,
\[
\sum_{k=1}^{\infty} \frac{761948702208k^2 - 4717770389584k + 2473620838841}{282^k}k^3\binom{2k}{k}T_k(4,9)^2 = -\frac{49}{46656\pi}(1744223791168\sqrt{3} + 307854304805\sqrt{6}),
\] (VII2S)

and
\[
\sum_{k=1}^{\infty} \frac{k^3\ell_k}{22^{2k}} \binom{2k}{k}T_k(5,1)^2 = -\frac{861143519145937597955\sqrt{7}}{1536\pi},
\] (VII3S)
where
\[ \ell_k := 25818737554793894400k^2 - 148156855009332208624k + 136699546718553502681. \]

(ii) We have
\[ \sum_{k=1}^{\infty} \frac{k^3 m_k}{46^{2k}} \binom{2k}{k} T_k(7, 1)^2 = -\frac{675829671880947233751995193\sqrt{7}}{2458624\pi}, \]
and
\[ \sum_{k=1}^{\infty} \frac{k^3 n_k}{5177196} \binom{2k}{k} T_k(171, -171)^2 = \frac{350688948772953536866818998433157048375\sqrt{7}}{1024\pi}, \]
where
\[ m_k := 60405940502779625588736k^2 - 194226790034061510982928k + 113445302068806184048447. \]
Also,
\[ \sum_{k=1}^{\infty} \frac{23575340288k^2 - 30724817208k + 70092391411}{(-108)^k} k^3 \binom{2k}{k} T_k(3, -3)^2 = \frac{83673484461\sqrt{7}}{512\pi}, \]
\[ = \frac{83673484461\sqrt{7}}{512\pi}, \] (VII5S)
and
\[ \sum_{k=1}^{\infty} \frac{k^3 n_k}{1024\pi} \binom{2k}{k} T_k(171, -171)^2 = \frac{350688948772953536866818998433157048375\sqrt{7}}{1024\pi}, \]
where
\[ n_k := 10084570997509032944421348793965204251648k^2 - 19349061584246199246547660635419016030648k + 10272620865354618637287798800250376143247. \]
Moreover,
\[ \sum_{k=1}^{\infty} \frac{k^3 p(k)}{434^{2k}} \binom{2k}{k} T_k(73, 576)^2 = -\frac{1941824970252133568742562638158781654482433600879227\sqrt{6}}{28311552\pi}, \]
where
\[ p(k) := 8486970607342072180410168982128577900986728448k^2 - 33680299486772296440807451280403962121626343552k + 17329177465492357868335964242267514081171615873. \]
**Conjecture 7.8.** We have
\[
\sum_{k=1}^{\infty} \frac{4158282880k^2 - 22813966696k + 40778481375}{(-50)^k} \frac{k^3T_k(4,1)T_k(1,-1)^2}{24416\pi} = -\frac{29231055627965}{124416\pi},
\] (VIII1S)
\[
\sum_{k=1}^{\infty} \frac{q(k)}{3240} k^3T_k(7,1)T_k(10,10)^2 = -\frac{260284882234938220643277936\sqrt{5}}{7\pi},
\] (VIII2S)
where
\[
q(k) := 498872838526220266559278065k^2 - 7594176317285717732725105481k + 15117694789961557184766131894.
\]
Also,
\[
\sum_{k=1}^{\infty} \frac{r_k}{(-2430)^k} k^3T_k(8,1)T_k(5,-5)^2 = \frac{4114638030580119213489\sqrt{15}}{7168\pi},
\] (VIII3S)
and
\[
\sum_{k=1}^{\infty} \frac{s_k}{(-29700)^k} k^3T_k(14,1)T_k(11,-11)^2 = \frac{23782586468921446583421682328565\sqrt{5}}{3584\pi},
\] (VIII4S)
where
\[
r_k := 1921491176698673825280k^2 - 2877654926621770908536k + 2447787507626240957299
\]
and
\[
s_k := 46821333085900052992621725937920k^2 - 71580243731698230860035196766952k + 47098473251985583734378482612735.
\]
**Conjecture 7.9.** We have
\[
\sum_{k=1}^{\infty} \frac{P_1(k)}{3136k^3} \binom{2k}{k} T_k(14,1)T_k(17,16) = -\frac{70662375766828100018060759}{24\pi},
\] (IX1S)
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and
\[ \sum_{k=1}^{\infty} \frac{P_2(k)}{3136k^3} \binom{2k}{k} T_k(2,81)T_k(14,81) = -\frac{49}{2916\pi}(16031198128567258016 + 6632521068118111355\sqrt{5}) \] (IX2S)

where

\[ P_1(k) := 8566416619450156417918110k^2 - 73211973506397665606012003k \]
\[ + 109289868688706912564582106 \]

and

\[ P_2(k) := 650373941133830880k^2 - 11795256486234065124k \]
\[ + 5657835264086004473 \]

Those $T_n = T_n(1,1)$ ($n \in \mathbb{N}$) are central trinomial coefficients. Motivated by the author’s conjectural identity
\[ \sum_{k=1}^{\infty} \frac{(105k - 44)T_{k-1}}{k^2 \binom{2k}{k}^2 3^{k-1}} = 6 \log 3 + \frac{5\pi}{\sqrt{3}} \]
(cf. [25, (10.1)]), we pose the following conjecture.

**Conjecture 7.10.** We have
\[ \sum_{k=1}^{\infty} \frac{(33k^2 + 32k + 8)T_{k-1}(8,-2)}{k(2k+1)^2 \binom{2k}{k}^2 18^{k-1}} = 3 \log 2 \]

and
\[ \sum_{k=1}^{\infty} \frac{(15k^3 + 8k^2 - 4k - 2)T_{k-1}(2,-1)}{k^2(2k+1)^2 \binom{2k}{k}^2} = \frac{\pi}{4\sqrt{2}}. \]

Also,
\[ \sum_{k=0}^{\infty} \frac{(32340k^2 - 28975k - 63)T_k}{\binom{2k}{k}^2 3^k} = 1024 + 216 \log 3 - \frac{388\sqrt{3}}{9} \pi, \]
\[ \sum_{k=0}^{\infty} \frac{(625k^3 - 2125k^2 + 1735k - 249)T_k(2,-1)}{\binom{2k}{k}^2} = \frac{9\pi}{\sqrt{2}} - 384, \]
\[ \sum_{k=0}^{\infty} \frac{(11979k^3 - 50303k^2 + 43769k - 9725)T_k(8,-2)}{\binom{2k}{k}^2 18^k} = 36(\log 2 - 288). \]

8. Other series of type S

The author [25, (8.1)] observed the identity
\[ \sum_{k=0}^{\infty} \frac{145k + 9}{900k} \beta_k T_k(52,1) = \frac{285}{\pi}. \]

Motivated by this we formulate the following conjecture.
Conjecture 8.1. (i) We have
\[ \sum_{k=0}^{\infty} \frac{k t_k^{(1)}}{900^k} \beta_k T_k(52, 1) = -\frac{498507105}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{k^2 t_k^{(2)}}{900^k} \beta_k T_k(52, 1) = -\frac{64232846618769}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{k^3 t_k^{(3)}}{900^k} \beta_k T_k(52, 1) = -\frac{1002142163415074991}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{(k^2 + 1) t_k^+}{900^k} \beta_k T_k(52, 1) = -\frac{2512943751517782}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{(k^2 - 1) t_k^-}{900^k} \beta_k T_k(52, 1) = -\frac{213755454707472}{\pi}, \]
where
\[ t_k^{(1)} = 31642625k^2 - 170395425k - 234814102, \]
\[ t_k^{(2)} = 4936966494550k^2 - 65077136905125k + 134224328497289, \]
\[ t_k^{(3)} = 2822066506655501225k^2 - 4544582337949545635k + 13491619388632261894, \]
\[ t_k^+ = 71236503338400k^2 - 613367241395375k + 111921391484847, \]
\[ t_k^- = 40085552483775k^2 - 2952180399125k - 125724302470098. \]

(ii) We have
\[ \sum_{k=0}^{\infty} \frac{k^j \beta_k T_k(52, 1)}{900^k} P_j(k) = 0 \]
for all \( j = 0, 1, 2 \), where
\[ P_0(k) = 601209875k^3 - 3237513075k^2 + 357434077k + 299104263, \]
\[ P_1(k) = 3493703960875k^3 - 48937877104875k^2 + 11052256574445k + 21410948872923, \]
\[ P_2(k) = 450165200053206075k^3 - 726162978455240695k^2 + 21683271651287795913k - 334047387805024997. \]

Motivated by Theorem 1.4, we pose the following two conjectures.
Conjecture 8.2. We have

\[
\sum_{k=1}^{\infty} \frac{(k-1)(9k+1)}{(-32)^k} k^2 \text{Domb}(k) = \frac{4}{3\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{27k^2 - 12k + 17}{(-32)^k} k^3 \text{Domb}(k) = \frac{2}{\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{51k^2 - 60k + 89}{(-32)^k} k^4 \text{Domb}(k) = \frac{-2}{3\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{801k^2 - 796k + 1371}{(-32)^k} k^5 \text{Domb}(k) = \frac{-248}{3\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{12339k^2 - 19308k + 46729}{(-32)^k} k^6 \text{Domb}(k) = \frac{4886}{3\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{15k^2 - 51k + 22}{64^k} k^3 \text{Domb}(k) = \frac{-8\sqrt{3}}{27\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{110k^2 - 453k + 229}{64^k} k^4 \text{Domb}(k) = \frac{-8\sqrt{3}}{45\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{1145k^2 - 7119k + 8368}{64^k} k^5 \text{Domb}(k) = \frac{7832\sqrt{3}}{135\pi},
\]
\[
\sum_{k=1}^{\infty} \frac{376560k^2 - 3010921k + 4944543}{64^k} k^6 \text{Domb}(k) = \frac{3050552\sqrt{3}}{405\pi}.
\]

Also,

\[
\sum_{k=0}^{\infty} \frac{k \text{Domb}(k)}{(-32)^k} (27k^3 - 6k^2 + 9k + 2) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^2 \text{Domb}(k)}{(-32)^k} (54k^3 - 51k^2 + 58k + 3) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^3 \text{Domb}(k)}{(-32)^k} (9k^3 - 9k^2 + 15k + 1) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^4 \text{Domb}(k)}{(-32)^k} (9k^3 - 80k^2 + 99k - 124) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^5 \text{Domb}(k)}{(-32)^k} (1116k^3 - 319k^2 + 2808k + 2443) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^6 \text{Domb}(k)}{(-32)^k} (21987k^3 - 214467k^2 + 316525k - 651453) = 0,
\]
and
\[
\sum_{k=0}^{\infty} \frac{k \text{Domb}(k)}{64^k} (45k^3 - 75k^2 - 21k - 5) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^2 \text{Domb}(k)}{64^k} (45k^3 - 153k^2 + 71k - 3) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^3 \text{Domb}(k)}{64^k} (25k^3 - 105k^2 + 59k - 3) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^4 \text{Domb}(k)}{64^k} (15k^3 + 377k^2 - 1827k + 979) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^5 \text{Domb}(k)}{64^k} (132165k^3 - 1108949k^2 + 2059839k - 381319) = 0,
\]
\[
\sum_{k=0}^{\infty} \frac{k^6 \text{Domb}(k)}{64^k} (113957k^3 - 6418845k^2 - 10015633k + 43834497) = 0.
\]

**Remark 8.1.** Though we have proved (1.11), we are unable to prove the first identity in Conjecture 8.2. This is because we could not prove the auxiliary identity
\[
\sum_{k=0}^{\infty} (27k^3 - 6k^2 + 9k + 2)v'_k = 0
\]
with \(v'_k = \frac{k \text{Domb}(k)}{(-32)^k}\). By the Zeilberger algorithm, we find the recurrence
\[(n + 1)^4 v'_n + n(2n + 3)(5n^2 + 15n + 12)v'_{n+1} + 16n(n + 1)(n + 2)^2 v'_{n+2} = 0\]
from which we have
\[
\sum_{k=0}^{\infty} (27k^4 - 39k^3 + 32k^2 - 3k - 1)v'_k = 0.
\]
Note that this is different from the desired (8.5).

**Conjecture 8.3.** We have the following identities:
\[
\sum_{k=1}^{\infty} \frac{1048k^2 - 5526k + 2651}{36^k} k^3 f_k^{(4)} = -\frac{3618\sqrt{15}}{625\pi}, \quad \text{(Y1S)}
\]
\[
\sum_{k=1}^{\infty} \frac{304k^2 - 576k + 521}{(-64)^k} k^3 f_k^{(4)} = -\frac{25696\sqrt{15}}{151875\pi}, \quad \text{(Y2S)}
\]
\[
\sum_{k=1}^{\infty} \frac{157260600k^2 - 373237478k + 150818747}{196^k} k^3 f_k^{(4)} = -\frac{952574\sqrt{7}}{5\pi}, \quad \text{(Y3S)}
\]
\[
\sum_{k=1}^{\infty} \frac{1691296k^2 - 3241287k + 1756217}{(-324)^k} k^3 f_k^{(4)} = \frac{20434599\sqrt{5}}{20000\pi}, \quad (Y4S)
\]

\[
\sum_{k=1}^{\infty} \frac{833112800k^2 - 1793574801k + 878937679}{1296^k} k^3 f_k^{(4)} = -\frac{149900521\sqrt{2}}{10240\pi}, \quad (Y5S)
\]

\[
\sum_{k=1}^{\infty} \frac{47808294003072k^2 - 102482715691400k + 52422407372915}{5776^k} k^3 f_k^{(4)} = -\frac{122626206796\sqrt{95}}{625\pi}. \quad (Y6S)
\]

**Conjecture 8.4.** (i) We have
\[
\sum_{k=1}^{\infty} \frac{6700k^2 - 25077k + 6239}{96^k} k^3 \binom{2k}{k} = -\frac{5787\sqrt{2}}{40\pi},
\]

\[
\sum_{k=1}^{\infty} \frac{100563606k^2 - 179847747k + 97593215}{(-400)^k} k^3 \binom{2k}{k} = \frac{642350}{3\pi},
\]

where \( f_k \) denotes the Franel number \( \sum_{j=0}^{k} (\binom{k}{j})^3 \).

(ii) We have
\[
\sum_{k=1}^{\infty} \frac{21k^2 - 109k - 26}{32^k} k^2 T_k Z_k = \frac{8(250 + 259\sqrt{5})}{375\pi},
\]

\[
\sum_{k=1}^{\infty} \frac{1170k^2 - 14621k + 19673}{32^k} k^2 T_k Z_k = -\frac{88(51250 + 24079\sqrt{5})}{375\pi},
\]

\[
\sum_{k=1}^{\infty} \frac{885285k^2 - 14121391k + 29366404}{32^k} k^3 T_k Z_k = \frac{8(21456450 + 3244519\sqrt{5})}{125\pi},
\]

where
\[
Z_k = \sum_{j=0}^{k} \left( \binom{k}{j} \binom{2j}{j} \binom{2(k-j)}{k-j} \right)
\]

as introduced by D. Zagier [29].

(iii) We have
\[
\sum_{k=1}^{\infty} \frac{148778208k^2 - 813461721k + 717359335}{(-100)^k} k^3 S_k(1, 25) = -\frac{19433301932825}{995328\pi},
\]

where
\[
S_k(b, c) = \sum_{j=0}^{k} \left( \binom{k}{j}^2 T_j(b, c) T_{k-j}(b, c) \right).
\]
Remark 8.2. For known \( \frac{1}{\pi} \)-series involving Franel numbers, see [5, 6]. Part (ii) of Conjecture 8.4 is motivated by the author’s discovery (cf. [25, (5.1)])

\[
\sum_{k=0}^{\infty} \frac{5k + 1}{32^k} T_k Z_k = \frac{8}{3\pi} (2 + \sqrt{5}),
\]

while part (iii) is inspired by the identity

\[
\sum_{k=0}^{\infty} (3k + 1) \frac{S_k(1, 25)}{(-100)^k} = \frac{25}{8\pi}
\]

obtained by the author (cf. [25, (1.91)]).

Conjecture 8.5. (i) For \( n \in \mathbb{N} \) let

\[
a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n + 2k}{2k} \binom{2k}{k} (-324)^{n-k}.
\]

Then

\[
\sum_{k=0}^{\infty} \frac{k(2k) a_k}{2160^k} \left(3750642k^2 - 27417879 - 200413\right) = \frac{6028830}{\pi},
\]

\[
\sum_{k=0}^{\infty} \frac{k^2(2k) a_k}{2160^k} \left(7297838982k^2 + 44777877957k - 249182264449\right) = \frac{56371645440}{\pi},
\]

and

\[
\sum_{k=0}^{\infty} \frac{k^3(2k) a_k}{2160^k} \left(1296246139663698k^2 - 604571242044753k + 1636493480946725\right)
\]

\[
= -\frac{5602326343530}{7\pi}.
\]

Also,

\[
\sum_{k=0}^{\infty} \frac{(2k) a_k}{2160^k} (36414k^3 - 266193k^2 - 234124k - 66987) = 0,
\]

\[
\sum_{k=0}^{\infty} \frac{k(2k) a_k}{2160^k} (30114378k^3 + 40060179k^2 + 29646217k + 7732736) = 0,
\]

and

\[
\sum_{k=0}^{\infty} \frac{k^2(2k) a_k}{2160^k} P(k) = 0
\]

where

\[
P(k) = 281579848704k^3 - 131104215018k^2 + 356871786301k - 7684943257.
\]

(ii) For \( n \in \mathbb{N} \) let

\[
b_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n + 2k}{2k} \binom{2k}{k} \left(-\frac{2}{3}\right)^{3k}.
\]
Then
\[
\sum_{k=0}^{\infty} \frac{k(2k)_{\{k\}} b_k}{(-20)^k} (1440k^2 - 10980k - 559) = \frac{3}{8\pi} (4255\sqrt{6} + 5904\sqrt{15}),
\]
\[
\sum_{k=0}^{\infty} \frac{k^2(2k)_{\{k\}} b_k}{(-20)^k} (6192k^2 - 99894k + 37303) = -\frac{3}{64\pi} (168935\sqrt{6} + 171648\sqrt{15}),
\]
and
\[
\sum_{k=0}^{\infty} \frac{k^3(2k)_{\{k\}} b_k}{(-20)^k} (93108288k^2 - 1995628400k + 241469595) = \frac{3}{128\pi} (3562267392\sqrt{15} - 403810135\sqrt{6}).
\]

**Remark 8.3.** Conjecture 8.5 is motivated by the author’s conjectural series
\[
\sum_{k=0}^{\infty} \frac{357k + 103}{2160^k} \binom{2k}{k} a_k = \frac{90}{\pi} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{24n + 5}{(-20)^k} \binom{2k}{k} b_k = \frac{3}{2\pi} (5\sqrt{6} + 4\sqrt{15})
\]
(cf. [22, (4.34)] and [26, (5.7)]).

**Conjecture 8.6.** For \(n \in \mathbb{N}\) let
\[
c_n = \sum_{k=0}^{n} \frac{5}{k} \binom{2k}{k} \left(\frac{(2n-k)}{n-k}\right)^2.
\]
Then
\[
\sum_{k=0}^{\infty} \frac{k(2k)_{\{k\}} c_k}{576^k} (68992k^2 - 129336k - 9691) = \frac{27}{2\pi} (2092 + 1577\sqrt{2}),
\]
\[
\sum_{k=0}^{\infty} \frac{k^2(2k)_{\{k\}} c_k}{576^k} (60781952k^2 - 396958056k + 220310929) = -\frac{81}{4\pi} (3688232 + 2390989\sqrt{2}),
\]
\[
\sum_{k=0}^{\infty} \frac{(k^2 + 1)(2k)_{\{k\}} c_k}{576^k} (25180969856k^2 - 118523107336k - 20012768875) = -\frac{9}{4\pi} (37415939872 + 18625995101\sqrt{2}),
\]
\[
\sum_{k=0}^{\infty} \frac{k^3(2k)_{\{k\}} c_k}{576^k} (197398592384k^2 - 1384797901272k + 855551417353) = \frac{243}{56\pi} (4385109711\sqrt{2} - 1128081200),
\]
and
\[
\sum_{k=0}^{\infty} \frac{k^4(2k)_{\{k\}} c_k \cdot Q(k)}{576^k} \cdot (204062865832352 + 52494895975809\sqrt{2}),
\]
where
\[ Q(k) = 5366018489638016k^2 - 50377737067775448k + 54028245883253707. \]

**Remark 8.4.** Conjecture 8.6 is motivated by the author’s conjectural series
\[ \sum_{k=0}^{\infty} \frac{28k + 5}{576^k} \binom{2k}{k} c_k = \frac{9}{\pi}(2 + \sqrt{2}) \]
(cf. [21, (8)]).

As there are many Ramanujan-type series, we cannot list all of their variants of type \( S \) here. We now conclude our paper and hope that our conjectures will stimulate further research.

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