Exact $\varphi_{1,3}$ boundary flows in the tricritical Ising model

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Abstract

We consider the tricritical Ising model on a strip or cylinder under the integrable perturbation by the thermal $\varphi_{1,3}$ boundary field. This perturbation induces five distinct renormalization group (RG) flows between Cardy type boundary conditions labelled by the Kac labels $(r,s)$. We study these boundary RG flows in detail for all excitations. Exact Thermodynamic Bethe Ansatz (TBA) equations are derived using the lattice approach by considering the continuum scaling limit of the $A_4$ lattice model with integrable boundary conditions. Fixing the bulk weights to their critical values, the integrable boundary weights admit a thermodynamic boundary field $\xi$ which induces the flow and, in the continuum scaling limit, plays the role of the perturbing boundary field $\varphi_{1,3}$. The excitations are completely classified, in terms of string content, by $(m,n)$ systems and quantum numbers but the string content changes by either two or three well-defined mechanisms along the flow. We identify these mechanisms and obtain the induced maps between the relevant finitized Virasoro characters. We also solve the TBA equations numerically to determine the boundary flows for the leading excitations.

1 Introduction

Quantum Field Theories with a boundary have received a lot of attention recently due to their applications in Condensed Matter, Solid State Physics and String Theory (D-branes). A problem of great interest is the Renormalization Group (RG) flow between different boundary fixed points of a Conformal Field Theory (CFT) that remains conformal in the bulk. Many interesting results have been achieved, and flows have been studied for minimal models and for $c = 1$ CFT (see e.g. \textsuperscript{1} and references therein). Numerical scaling functions for the flow of states interpolating two different boundary conditions can be systematically explored by use of the approximate Truncated Conformal Space Approach (TCSA) \cite{2,3,4}.

A beta function can be defined for the boundary deformations, much the same as for the bulk perturbations of conformal field theories \cite{5}. The conformal boundary conditions play the role of ultraviolet (UV) and infrared (IR) points of the flow. One flows away from the UV fixed point by perturbing with a relevant boundary operator and flows into an IR fixed point attracted by irrelevant boundary operators.

Among the possible boundary perturbations of a CFT there are some integrable perturbations that preserve an infinite number of conservation laws. In this case the flows are amenable to investigation by exact methods such as commuting transfer matrices and Bethe ansatz techniques. One of the most celebrated of these methods is the Thermodynamic Bethe Ansatz (TBA) \cite{6} giving a set of non-linear coupled integral equations governing the scaling functions along the RG flow.

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Boundary TBA equations were first obtained with the usual scattering approach \cite{7, 3} for the groundstate, a few excited states and the boundary entropy in the minimal models. In particular, the groundstate and boundary entopies of the Tricritical Ising Model have been studied \cite{5} within this approach.

In this paper we use a lattice approach to obtain exact TBA equations for all the excitations of the integrable boundary flows of the Tricritical Ising Model (TIM) with central charge $c = \frac{7}{10}$. This model has interesting applications in Solid State Physics and Statistical Physics. Its Kac table of conformal weights $h_{r,s}$ is

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$r$ & 1 & 2 & 3 \\
\hline
s & 1/10 & 3/80 & 3/5 \\
\hline
1 & 0 & 7/16 & 3/2 \\
\hline
\end{tabular}
\end{center}

We have drawn only half of the table, by taking into account the well known $\mathbb{Z}_2$ Kac table symmetry $(r, s) \equiv (4 - r, 5 - s)$. The number of independent chiral primary fields is thus six.

Cardy-type conformal boundary conditions for minimal models with diagonal modular invariant partition functions of type $(A_p, A_{p'})$ are in one-to-one correspondence with the chiral primary fields in the Kac table. So the TIM $(A_3, A_4)$ admits six different conformal boundary conditions which we denote by $B_{(r,s)}$, $r = 1, 2, 3$ and $s = 1, 2$.

Let us consider the TIM defined on a strip of width $L$ in the two-dimensional $(x, t)$ plane so that $0 \leq x \leq L$ and $-\infty < t < +\infty$. Let us impose boundary conditions $B_{(r,s)}$ and $B_{(r',s')}$. In the sequel we are interested in boundaries of type $(r, s)|(1, 1)$ which we denote by $B_{(r,s)} = B_{(r,s)|(1,1)} \equiv B_{(r,1)(1,1)}$. For these boundary conditions the partition function reduces to a single character \cite{9}

\[ Z_{B_{(r,s)}}(q) = Z_{(r,s)|(1,1)}(q) = \chi_{r,s}(q) \] (1.1)

where $\chi_{r,s}(q)$ denotes the character of the irreducible Virasoro representation labelled by $(r, s)$ at central charge $c = \frac{7}{10}$.

For each given boundary condition $B_{(r,s)}$ there is a set of boundary operators $\varphi_{u,v}$ that live on the edge. If we want to keep the boundary condition unchanged all along the edge, these operators must be restricted to the conformal families appearing in the OPE fusion of the Virasoro family $(r, s)$ with itself: $(u, v) \in (r, s) \times (r, s)$. They are distinguished in terms of their conformal dimensions $h_{r,s}$ as relevant $(h_{r,s} < 1)$ and irrelevant $(h_{r,s} > 1)$. Of course only relevant perturbations break scale invariance at the boundary in such a way to get out of the fixed boundary point of a specific boundary condition and flow to another one. So if we want to consider possible relevant boundary perturbations of the TIM, i.e. QFTs described by the action

\[ S = S_{(r,s)} + \lambda \int_{-\infty}^{+\infty} dt \varphi_{u,v}(x = 0, t) \] (1.2)

where $S_{(r,s)}$ denotes the action of TIM with boundary condition $B_{(r,s)}$, we have to restrict to the possibilities as shown in Table \cite{11}

Actually there are two physically different flows for each “pure” perturbation (i.e. containing only one operator $\varphi_{1,2}$ or $\varphi_{1,3}$), flowing to two possibly different IR destinies. This is achieved by taking different signs in the coupling constant in the case of $\varphi_{1,3}$ flows, and real or purely...
Table 1.

| boundary condition | boundary perturbations |
|--------------------|------------------------|
| \( B_{(1,1)} \equiv B_{(3,3)} \) | \(-\) | none |
| \( B_{(2,1)} \equiv B_{(2,4)} \) | 0 | none |
| \( B_{(3,1)} \equiv B_{(1,4)} \) | + | none |
| \( B_{(1,2)} \equiv B_{(3,3)} \) | \(-0\) | \( \varphi_{1,3} \) |
| \( B_{(2,2)} \equiv B_{(2,3)} \) | \( d \) | \( \varphi_{1,2}, \varphi_{1,3} \) |
| \( B_{(3,2)} \equiv B_{(1,3)} \) | \( 0+ \) | \( \varphi_{1,3} \) |

imaginary coupling constant in case of \( \varphi_{1,2} \) ones. The boundary condition \( B_{(2,2)} \) can be perturbed by any linear combination of the fields \( \varphi_{1,2} \) and \( \varphi_{1,3} \). The symbols +, −, 0+, −0, 0, d represent other ways [10, 11] to denote the TIM conformal boundary conditions and we give them only to make contact with the existing literature.

Along the physical flow from UV to IR, the boundary entropy associated to each \( B_{(r,s)} \)

\[
g(r,s) = \left( \frac{2}{5} \right)^{1/4} \frac{\sin \frac{\pi r}{4} \sin \frac{\pi s}{5}}{\sqrt{\sin \frac{\pi}{4} \sin \frac{\pi}{5}}} \]  

(1.3)
decreases [5], so we only expect flows between boundary conditions where the initial boundary entropy is larger than the final boundary entropy. The possible conformal boundary conditions have been studied by Chim [10] and the flows connecting them by Affleck [11]. The picture is summarized in Figure 1 Integrability can be investigated in a manner similar to the bulk perturbations and it turns out that the flows generated by pure \( \varphi_{1,3} \) and \( \varphi_{1,2} \) perturbations are integrable. In contrast, the flow starting at \( B_{(2,2)} \) as a perturbation which is a linear combination of \( \varphi_{1,2} \) and \( \varphi_{1,3} \) is strongly suspected to be non-integrable. Notice the \( Z_2 \) symmetry of Figure 1, which is related to the supersymmetry of the TIM. Investigation of the supersymmetric aspects of the TIM boundary flows is, however, beyond the scope of the present paper.

In this paper we derive exact TBA equations for all excitations of the integrable boundary flows of the TIM from the functional relations for the transfer matrix of the \( A_4 \) lattice RSOS model [12] which is a member of the \( A_L \) Andrews-Baxter-Forrester models [13]. Using general techniques [14, 15] of the lattice approach for turning functional equations into non-linear integral equations, we derive the TBA equations for all excitations and solve them numerically to determine the flows for the leading excitations. Our preliminary results for the flow \( \chi_{1,2} \mapsto \chi_{2,1} \) were announced in [16]. Although we only consider the TIM here, our approach is quite general and should apply, for example, to the integrable boundary flows of all minimal models. A similar lattice approach has been successfully applied [17] to the massive and massless bulk thermal RG flows of the TIM. We stress that we are not addressing here the interesting question of predicting the allowed pattern of flows between fixed points. Rather, given a particular flow, the thrust of the lattice approach is to calculate the flow of all excitations and to predict, excitation by excitation, where these levels flow.

The layout of this paper is as follows. In Section 2, we discuss the TIM as the continuum scaling limit of the \( A_4 \) lattice model. We describe the integrable boundary conditions [15] that give rise, in the continuum scaling limit, to the conformal boundary conditions labelled by the Kac labels \((r, s)\), the associated double row transfer matrices [19] and their functional equations. We summarize the results of O’Brien, Pearce and Warnaar [20] which apply at the conformal fixed point of the TIM before the boundary perturbation \( \xi = \text{Im}(\xi_{\text{latt}}) \) is turned on. In particular, we review the classification of the allowed patterns of zeros of the transfer matrix eigenvalues in terms of \((m, n)\) systems and quantum numbers and summarize the TBA equations and their solutions for
Fig. 1. The boundary flows between TIM conformal boundary conditions. Pure $\varphi_{1,3}$ and $\varphi_{1,2}$ flows are integrable. More information can be found in [11]. The boundary entropy associated with the $+\&−$ boundary condition is linear in the Cardy type boundary entropies: $g_{(+\&−)} = g_{(1,1)} + g_{(3,1)}$.

the finite-size energy spectra in terms of finitized characters. In Section 3, we consider the three boundary flows with constant $r$, namely, $\chi_{1,2} \mapsto \chi_{1,1}$, $\chi_{3,2} \mapsto \chi_{3,1}$ and $\chi_{2,2} \mapsto \chi_{2,1}$. We describe two mechanisms A, B under which the patterns of zeros change during these flows and in each case we show that this is consistent with an explicit map between finitized characters. In each case, we also solve the functional equations to obtain the boundary TBA equations describing the flow. In Section 4, we repeat these considerations for the two boundary flows with variable $r$, namely, $\chi_{1,3} \mapsto \chi_{2,1}$ and $\chi_{1,2} \mapsto \chi_{2,1}$. In these cases we find three mechanisms A, B, C. In Section 5, we discuss the numerical solution of the boundary TBA equations for all five thermal boundary flows. Finally, we finish with some conclusions in Section 6.

2 The tricritical Ising model

The conformal unitary minimal model with central charge $c = 7/10$ describes the critical behaviour of the tricritical Ising model. A lattice realization of this universality class is given by the critical $A_4$ ABF model on the critical line separating regimes III and IV. Since we are interested in working in the presence of a boundary we define, following [20], a double row transfer matrix with $(r_1, a_1)$ and $(r_2, a_2)$ boundary conditions by the following diagrammatic representation

$$D_{r_1 a_1| r_2 a_2}^{N}(u, \xi_1, \xi_2)_{\sigma, \sigma'} = \sum_{\tau_0, \ldots, \tau_N} \lambda - u, \xi_1 \quad \lambda - u, \xi_2 \quad \lambda - u$$

(2.1)
The bulk weights are fixed to their critical values

\[ W(d \ a \ b \ c \ u) = \begin{pmatrix} \sin(\lambda - u) \delta_{a,c} + \sin u \sqrt{S_a S_c} \delta_{b,d} \\ \sin \lambda \end{pmatrix} \] (2.2)

where the heights \( a, b, c, d \in \{1, 2, 3, 4\} \) on any edge must respect the \( A_4 \) adjacency rule. The physical range of the spectral parameter is \( 0 < u < \lambda \) where \( \lambda = \frac{\pi}{\theta} \) is the crossing parameter. The crossing factors are \( S_a = \sin a \lambda / \sin \lambda \).

The most general integrable boundaries for critical bulk weights are given by (3.28) in [18]. They are labelled by a pair \((r, a)\) and depend upon boundary spins \( b, c, d \), the spectral parameter \( u \) and a boundary thermodynamic field \( \xi \) (interaction)

\[ B_{ra}(c \ b \ d \ u, \xi) \] (2.3)

We only need to consider diagonal boundary weights in the sense that \( b = d \). Explicitly, the relevant boundary weights for our purposes are

\[ B_{ra}(c \ b \ a \ d \ u, \xi) = \begin{pmatrix} S_r S_a S_c S_d & \sin(\xi + u) \sin(\lambda r + \xi - u) \\ \sin \lambda S_r \sin(\lambda r + \xi + u) \end{pmatrix} \] (2.4)

where \( r = 1, \ldots, 3 \) and \( a = s = 1, \ldots, 4 \). Here \( a \) and \( b = d = c \pm 1 \) must be adjacent at fusion level \( r \), that is, \( a - b = \pm 1 \) for \( r = 1 \), \( a + b = 4, 6 \) for \( r = 2 \) and \( a + b = 5 \) for \( r = 3 \). The normalization factor chosen here is different from that used in [18]. It does not affect any result but simply keeps the boundary weights finite in the limit \( \text{Im}(\xi) \to \pm \infty \). An obvious and important symmetry is given by \( \xi \to -\xi - r \lambda \)

\[ B_{ra}(c \ b \ a \ d \ u, \xi) = B_{ra}(c \ b \ a \ d \ u, -\xi - r \lambda) \] (2.5)

A special case is obtained setting \( a = 1 \)

\[ B_{r1}(r \pm 1 \ r \ u, \xi) = \begin{pmatrix} S_{r \pm 1} & \sin(\xi + u) \sin(\lambda r + \xi - u) \\ \sin \lambda S_r \sin(\lambda r + \xi + u) \end{pmatrix} = \begin{pmatrix} S_{r \pm 1}^{1/2} & \sin(\xi + u) \sin(\lambda r + \xi + u) \\ \sin \lambda S_r^{1/2} \cosh 2\text{Im}(\xi) \end{pmatrix} \] (2.6)

In this case we can use the equivalent form of (3.32) in [18] arising from its construction

\[ B_{r1}(r \pm 1 \ r \ u, \xi) = -\frac{\sin \lambda}{\cosh 2\text{Im}(\xi)} \prod_{k=1}^{r-2} \frac{\sin^2 \lambda}{\sin(u - (k + 1) \lambda - \xi) \sin(u + k \lambda + \xi)} \]

\[ \times \sum_{\{w\}, \{f_{r1}\}} \left| \begin{array}{ccc} w_{-u-\xi} & w_{-u-\xi} & -u-\xi \\ -u-\xi & -u-\xi & -u-\xi \\ -u-\xi & -u-\xi & -u-\xi \end{array} \right| \begin{array}{c} r \end{array} \begin{array}{c} f_1 \ f_2 \ f_{r-2} \end{array} \begin{array}{c} \xi \ 1 \end{array} \begin{array}{c} 2 \end{array} \] (2.7)
As in [16], we will vary the imaginary part $\text{Im}(\xi)$ of the field $\xi$ to interpolate between the conformal fixed points.

The given bulk and boundary weights satisfy the Yang-Baxter and Boundary Yang-Baxter equations and lead to commuting double row transfer matrices and an integrable model. The double row transfer matrices are also periodic, crossing symmetric and transpose symmetric

$$
D_{r_1 a_1|r_2 a_2}^N(u, \xi_1, \xi_2) = D_{r_1 a_1|r_2 a_2}^N(u + \pi, \xi_1, \xi_2),
$$

$$
D_{r_1 a_1|r_2 a_2}^N(u, \xi_1, \xi_2) = D_{r_1 a_1|r_2 a_2}^N(\lambda - u, \xi_1, \xi_2)
$$

$$
D_{r_1 a_1|r_2 a_2}^N(u, \xi_1, \xi_2) = (D_{r_1 a_1|r_2 a_2}^N(u, \xi_1, \xi_2))^t
$$

for arbitrary complex values of the spectral and boundary parameters.

Moreover, for $a_1 = a_2 = 1$, the normalized transfer matrices

$$
t(u) = D_{r_1 1|r_2 1}^N(u, \xi_1, \xi_2) S_{r_1}(u, \xi_1) S_{r_2}(u, \xi_2) S(u) \left[ \frac{\sin(u + 2\lambda) \sin \lambda}{\sin(u + 3\lambda) \sin(u + \lambda)} \right]^{2N}
$$

with

$$
S(u) = \frac{\sin^2(2u - \lambda)}{\sin(2u + \lambda) \sin(2u - 3\lambda)}
$$

$$
S_r(u, \xi) = h_r(u - \xi) h_{-r}(u + \xi) \frac{\cosh 2\text{Im}(\xi)}{\sin \lambda}
$$

$$
h_r(u) = \frac{\sin \lambda \sin(u + (3 - r)\lambda) \sin(u + (1 - r)\lambda)}{\sin u \sin(u - \lambda) \sin(u + 2\lambda)}
$$

satisfy the functional equation

$$
t(u) t(u + \lambda) = 1 + t(u + 3\lambda)
$$

where we have suppressed the $\xi$ dependence. Periodicity, crossing symmetry and transpose symmetry extend to this operator and its factors $S(u), S_r(u, \xi)$. The transfer matrix $D$ is an entire function of $u$ whereas the normalized transfer matrix $t$ is a meromorphic function with poles arising from the normalization factor. We need to determine appropriate choices of the boundaries to interpolate between distinct conformal boundary conditions of types $(r, s)$ by varying the imaginary part of $\xi$. We then need to solve the double row transfer matrix functional equation in the continuum scaling limit for the induced flow of the resulting eigenvalue (energy) spectra.

In the scaling limit, the large $N$ corrections to the eigenvalues\(^1\) of the double row transfer matrix are related to the excitation energies of the associated perturbed conformal field theory by

$$
- \frac{1}{2} \log D_{r_1 a_1|r_2 a_2}^N(u) = N f_b(u) + f_{r_1 a_1|r_2 a_2}(u) + \frac{2\pi \sin 5u}{N} E_{r_1 a_1|r_2 a_2}(\xi) + o\left(\frac{1}{N}\right)
$$

where $f_b$ is the bulk free energy, $f_{r_1 a_1|r_2 a_2}$ is the surface (i.e. boundary dependent) free energy and $E_{r_1 a_1|r_2 a_2}(\xi)$ is a scaling function that, at the boundary critical points, reduces to

$$
E_{r_1 a_1|r_2 a_2}|_{\text{crit}} = \frac{c}{24} + h + n, \quad n \in \mathbb{N}
$$

where $h$ is one of the conformal weights allowed\(^2\) by the boundaries $(r_1 a_1|r_2 a_2)$. Observe that $D_{r_1 a_1|r_2 a_2}^N(u, \xi_1, \xi_2)$ in (2.16) can be a function of up to two boundary parameters but we find that only one boundary parameter $\xi$ needs to flow to reproduce the $\varphi_{1,3}$ thermal boundary flows.

\(^1\)We use $D, t$ to indicate eigenvalues of $D, t$.

\(^2\)If the lattice is wrapped on a cylinder, its conformal partition function is given by a sum of characters determined by a “fusion” of the boundaries $(r_1 a_1|r_2 a_2)$. 
For our computations, we need certain information about the analyticity of the double row transfer matrices encoded in the zeros and poles of their eigenvalues. Because of integrability, \( D_{r_1, a_1}^{N} (u, \xi_1, \xi_2) \) at different spectral parameters \( u, v \) forms a commuting family of operators so that its eigenstates can be chosen independent of the spectral parameter \( u \) and only the eigenvalues depend on \( u \). These eigenvalues are Laurent polynomials in the variable \( z = \exp(iu) \). We have performed numerical diagonalization for small sizes (up to \( N = 16 \) faces in a row). For these sizes, the zeros of the eigenvalues are extracted numerically and we extrapolate their pattern to the limit of large \( N \). We will refer to numerical observations obtained in this way as “numerics on \( D \”).

### 2.1 Critical point: classification of the zeros and TBA

In the scaling limit at the isotropic point \( u = \lambda/2 \), \( B_{r\alpha} \) is critical if the boundary parameter is \( \xi = \lambda/2 \) or \(-\lambda/2\). These two choices yield the fixed and semi-fixed boundaries introduced in \[21\]. Actually, the parameter can be chosen in the regions \( \xi \in [\lambda/2, (5-r-1/2)\lambda] \), \( \xi \in [(1/2-r)\lambda, -\lambda/2] \) respectively. Indeed, these are the maximal intervals where each single boundary contribution remains nonnegative at the isotropic point. The “numerics on \( D \)” shows that the scaling properties (in particular the classification of zeros that will be introduced shortly) do not depend on the actual value of \( \xi \) inside the intervals \[21\], so in this sense Re(\( \xi \)) is an irrelevant variable close to the critical point.

In general, a cylinder partition function is a superposition of characters. However, a single character is obtained if the double row transfer matrix is built \[20\] with one \( r \)-type and one \( s \)-type boundary. The \( r \)-type boundary is obtained from \( B_{r1} \) by choosing \( \xi \in [\lambda/2, (5-r-1/2)\lambda] \) or from \( B_{r+1,1} \) by choosing \( \xi \in [-(1/2+r)\lambda, -\lambda/2] \). At the isotropic point \( u = \lambda/2 \) and with \( \xi = \lambda/2 \) we have explicitly (fixed boundary, “–” case)

\[
B_{r1}^{r-1} \left( \begin{array}{c} r-1 \r \end{array} \begin{array}{c} r \lambda/2 \lambda/2 \end{array} \right) = r-1 \begin{array}{c} r \lambda/2 \lambda/2 \end{array} = 0
\]

\[
B_{r1}^{r+1} \left( \begin{array}{c} r+1 \r \end{array} \begin{array}{c} r \lambda/2 \lambda/2 \end{array} \right) = r+1 \begin{array}{c} r \lambda/2 \lambda/2 \end{array} = \sin \lambda \sqrt{S_{r+1} S_{r}}
\]

Similarly, at \( \xi = -\lambda/2 \) in \( B_{r+1,1} \) we have explicitly (fixed boundary, “+” case)

\[
B_{r+1,1}^{r} \left( \begin{array}{c} r \ r+1 \r+1 \end{array} \begin{array}{c} \lambda/2 \lambda/2 \end{array} \right) = r \begin{array}{c} \lambda/2 \lambda/2 \end{array} = \sin \lambda \sqrt{S_{r+1} S_{r}}
\]

\[
B_{r+1,1}^{r+2} \left( \begin{array}{c} r+2 \ r+1 \r+1 \end{array} \begin{array}{c} \lambda/2 \lambda/2 \end{array} \right) = r+2 \begin{array}{c} \lambda/2 \lambda/2 \end{array} = 0
\]

In both the cases we are left with an alternating chain of \( r, r+1 \) sites on the boundary.

The \( s \)-type boundary weights are obtained as the braid limit of the \( r \)-type boundary weights
with \( r = s \)

\[
\lim_{\text{Im}(\xi) \to -\infty} B_s^s\left( s \pm 1 \begin{array}{c}s \\ s \end{array} | u, \xi \right) = s \pm 1 \begin{array}{c}s \\ s \end{array} = \frac{S_{s \pm 1}}{S_s} e^{i(2\text{Re}(\xi) + s\lambda)} \begin{array}{c}s \\ -2 \sin \lambda \end{array}
\] (2.20)

A similar expression holds for the limit \( \text{Im}(\xi) \to +\infty \). The \( u \) and \( \text{Im}(\xi) \) dependence in the last term cancels out explicitly. The overall factor is immaterial for our purposes because it corresponds to a trivial factor multiplying the transfer matrix. Among the \( r \)- and \( s \)-type boundaries there is the special common boundary condition with \( r = s = 1 \) corresponding to the vacuum.

In the case of one \( r \)-type and one \( s \)-type boundary condition which leads to the single Virasoro character \( \chi_{r,s}(q) \), a detailed analysis of the structure of zeros of the eigenvalues of the double row transfer matrices has been performed in [20]. In the large \( N \) limit, each eigenvalue of the double row transfer matrix is characterized by a specific pattern of zeros, organized as 1-strings or 2-strings in two analyticity strips [20]

Strip 1: \(-\frac{\lambda}{2} < \text{Re}(u) < \frac{3\lambda}{2}\),

Strip 2: \(2\lambda < \text{Re}(u) < 4\lambda\) (2.21)

The 1-strings are single zeros appearing in the center of each strip

\[ \text{Re}(u) = \frac{\lambda}{2} \text{ or } 3\lambda \] (2.22)

while the 2-strings are pairs of zeros \((u, u')\) appearing on the edges of a strip, with the same imaginary part

\[ (\text{Re}(u), \text{Re}(u')) = \begin{cases} (-\frac{\lambda}{2}, \frac{3\lambda}{2}), & \text{strip 1}, \\ (2\lambda, 4\lambda), & \text{strip 2}. \end{cases} \] (2.23)

We use \( m_1, m_2 \) to denote the number of 1-strings in each strip and \( n_1, n_2 \) to denote the number of 2-strings in each strip. For each \((r, s)\) boundary condition these numbers satisfy particular \((m, n)\) systems. The complete classification of the allowed patterns of 1- and 2-strings in terms of \((m, n)\) systems and quantum numbers is summarized in Table 2.

For a generic function \( h(u) \), for example the normalized transfer matrix eigenvalues \( t(u) \), it is natural to introduce two functions \( h_1(x) \), \( h_2(x) \) with reference to the center lines of the two analyticity strips

\[ h_1(x) = h\left(\frac{\lambda}{2} + i\frac{x}{5}\right) \quad \text{for } |\text{Im}(x)| \leq \pi \] (2.24)

\[ h_2(x) = h\left(3\lambda + i\frac{x}{5}\right) \quad \text{for } |\text{Im}(x)| \leq \pi \] (2.25)

At the critical point of the TIM, the TBA equations derived in [20] for the six \((r, s)\) conformal boundary conditions in Table 2 are

\[ \epsilon_1(x) = -\sum_{k=1}^{m_1} \log \tanh\left(\frac{y_k^{(1)} - x}{2}\right) - K \ast \log(1 + s_2 e^{-\epsilon_2(x)}) \] (2.26)

\[ \epsilon_2(x) = 4e^{-x} - \sum_{k=1}^{m_2} \log \tanh\left(\frac{y_k^{(2)} - x}{2}\right) - K \ast \log(1 + s_1 e^{-\epsilon_1(x)}) \] (2.27)
Table 2. Classification, for all \((r, s)\) boundary conditions of the TIM, of the allowed patterns of 1- and 2-strings by \((m, n)\) system and quantum numbers. The parity \(\sigma = \pm 1\) occurs when there are frozen zeros. The parities \(s_1, s_2 = \pm 1\) occur in the TBA equations. The expressions for \(n_1\), are only used on a finite lattice because in the scaling limit \(n_1 \sim N/2 \to \infty\). The number of faces in a row is even or odd according to \(N = (r - s) \mod 2\).

| \(\chi_{r,s}(q)\) | \((m, n)\) system | parities | quantum numbers |
|------------------|------------------|----------|-----------------|
| \(\chi_{1,1}(q)\) | \(m_1, m_2\) even | \(n_1 = (N + m_2)/2 - m_1\) | \(s_1 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2\) |
| \(\chi_{1,2}(q)\) | \(m_1\) odd, \(m_2\) even | \(n_2 = (m_1 - \sigma)/2 - m_2\) | \(s_1 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 - \sigma\) |
| \(\chi_{2,1}(q)\) | \(m_1, m_2\) odd | \(n_2 = (m_1 + 1)/2 - m_2\) | \(s_1 = -1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 + \sigma\) |
| \(\chi_{2,2}(q)\) | \(m_1\) even, \(m_2\) odd | \(n_1 = (N + m_2)/2 - m_1\) | \(s_1 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 - \sigma\) |
| \(\chi_{3,1}(q)\) | \(m_1\) even, \(m_2\) odd | \(n_2 = (m_1 + 1)/2 - m_2\) | \(s_1 = -1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 + \sigma\) |
| \(\chi_{3,2}(q)\) | \(m_1\) even, \(m_2\) odd | \(n_1 = (N + m_2)/2 - m_1\) | \(s_1 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 - \sigma\) |

where \(y_k^{(j)}\) are the (scaled) locations of the 1-strings in the strip \(j = 1, 2\), that is to say, they are zeros of the normalized (and scaled) transfer matrices \(t_i(x) = s_i e^{-\epsilon_i(x)}\). The parities \(s_i\) are given in Table 2. The locations of the zeros are determined by a set of non-degenerate quantum numbers \(n_k^{(i)} \in \mathbb{Z}\) by the quantization conditions

\[
\epsilon_1(y_k^{(2)} - \frac{i\pi}{2}) = i\pi n_k^{(2)}, \quad k = 1, 2, \ldots, m_2 \tag{2.28}
\]

\[
\epsilon_2(y_k^{(1)} - \frac{i\pi}{2}) = i\pi n_k^{(1)}, \quad k = 1, 2, \ldots, m_1 \tag{2.29}
\]

The non-negative quantum numbers \(\{I_k^{(j)}\}\) have the topological meaning that for a given 1-string \(y_k^{(j)}\), \(I_k^{(j)}\) is the number of 2-strings with larger coordinate \(z_k^{(j)} > y_k^{(j)}\). The conventional order of the zeros is \(y_1^{(j)} < y_2^{(j)} < \ldots < y_m^{(j)}\). The notation \((I_1^{(1)}, I_2^{(1)}, I_3^{(1)}, I_4^{(1)}, \ldots, I_m^{(1)})\) uniquely labels states.

The corresponding expression for the scaling energy \(2E_{rs}\) is given by

\[
E_{rs} = \frac{1}{\pi} \sum_{k=1}^{m_1} e^{-y_k^{(1)}} - \int_{-\infty}^{+\infty} dy \frac{e^{-y}}{\pi y^2} \log(1 + s_2 e^{-\epsilon_2(y)}) \tag{2.30}
\]

\[
= \frac{c}{24} + h_{rs} + \frac{(r-s)(s-r-1)}{4} + \frac{m^t C m}{4} - \frac{A \cdot m}{2} + \sum_{j=1}^{m_1} \sum_{k=1}^{m_j} I_k^{(j)}
\]
where $C$ is the $A_2$ Cartan matrix, $A$ is a vector

$$
C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad A = (1, -1) \sigma
$$

and $\sigma = \pm 1$ according to Table 2 if $s = 2, 3$ and $\sigma = 0$ otherwise. The conformal partition function obtained from these energies yields a finitized fermionic character

$$
Z_N(q) = \sum_{\text{all states}} q^{E_{r,s}} = \chi_{\sigma,0}^{(N)}(q) \xrightarrow{N \to \infty} \chi_{\sigma,0}(q)
$$

where the sum is over all states allowed by the finite $(m, n)$ system and

$$
\chi_{\sigma,0}^{(N)}(q) = q^{e_h + h_s + \frac{1}{2}(r-s)(s-r-1)} \sum_{m_1, m_2, \sigma} q^{\frac{1}{2}m' C m - \frac{1}{2}A m} \left[ \begin{array}{c} m_1 + n_1 - \delta_{\sigma,1} \\ m_1 - \delta_{\sigma,1} \end{array} \right] \left[ \begin{array}{c} m_2 + n_2 \\ m_2 \end{array} \right]
$$

with the previous convention on $\sigma$.

### 2.2 Boundary flows

The TIM admits 6 Cardy-type boundary conditions each corresponding to a single Virasoro character. Interpolating between them, as described in [11] and shown in Figure 1, there are 7 integrable renormalization group flows, 5 generated by the perturbing operator $\varphi_{1,3}$ and 2 by $\varphi_{1,2}$. In this paper we consider only those described by the $A_4$ RSOS lattice model of [13], that is to say, the $\varphi_{1,3}$ flows. We expect that the $\varphi_{1,2}$ boundary flows will be given by consideration of the dilute $A$ RSOS lattice realization of the TIM. There is an important reason why $A_4$ cannot describe both the perturbations, namely, the existence of two independent perturbing parameters would suggest the presence of an integrable surface instead of just the integrable line that is known to exist.

From the $A_4$ lattice description of the TIM we observe the presence of similarities between the $\varphi_{1,3}$ flows related by inversion of the spin direction in the Affleck [11] description. Moreover, from the conformal description, we also notice that three flows share the same “active” boundary, namely an $s = 2$ type boundary becomes an $s = 1$ type boundary, while the $r$-type boundary is just a spectator. We therefore divide the flows according to whether $r$ remains constant or not

| Constant $r$ | Variable $r$ |
|--------------|--------------|
| $\chi_{1,2} \equiv (0-) \mapsto \chi_{1,1} \equiv (-)$ | $\chi_{1,2} \equiv (0-) \mapsto \chi_{2,1} \equiv (0)$ |
| $\chi_{3,2} \equiv (0+) \mapsto \chi_{3,1} \equiv (+)$ | $\chi_{3,2} \equiv (0+) \mapsto \chi_{2,1} \equiv (0)$ |
| $\chi_{2,2} \equiv (d) \mapsto \chi_{2,1} \equiv (0)$ | |

We will see that these similarities also arise in our TBA description of the flows.

### 3 Boundary flows with constant $r$

The common feature of this family of flows is the change from an $s = 2$ to an $s = 1$ label in the conformal character $\chi_{r,s}(q)$, keeping $r$ fixed. Clearly, on the lattice this means that the boundary of type $r$ is a spectator and the dynamics is given by the $s$ type boundary.

For convenience, from now on the complex boundary thermodynamic field on the lattice will be denoted $\xi_{\text{latt}}$. From (2.20) we know that $B^{2,1}(u, \xi_{\text{latt}})$ in the limit $\text{Im}(\xi_{\text{latt}}) \to \pm \infty$ (and for all $\text{Re}(\xi_{\text{latt}}) \in \mathbb{R}$) is an $s = 2$ type boundary. We can take advantage of the fact that $\text{Re}(\xi_{\text{latt}})$ is
If we explicitly compute $B^{2,1}$ with an $r$-type boundary on the left having no free boundary parameters, we are left with just one boundary parameter $\xi_{\text{latt}}$.

By (3.2), if $\text{Im}(\xi_{\text{latt}}) = \pm \infty$, on the right we have an $s = 2$-type boundary coupled with an $r$-type on the left, leading to the character $\chi_{r,2}$. If $\text{Im}(\xi_{\text{latt}}) = 0$, the $s = 1$ boundary on the right is coupled with an $r$-type on the left, corresponding to $\chi_{r,1}$. We now introduce a simplified notation $\xi_{\text{latt}} = -\lambda + i\xi/5$ with $\xi$ real and

$$D(u, \xi) \equiv D^{N}_{r,1|2,1}(u, -\lambda + i\xi/5)$$

If we explicitly compute $B^{2,1}(u, -\lambda + i\xi/5)$ we see that it is real for arbitrary $\xi$ and real $u$. The boundary weight on the left and the bulk weights are also real so each single entry of the transfer matrix is real for real $u$. This is nothing but the real analyticity property

$$D^{N}_{r,1|2,1}(u, -\lambda + i\xi/5) = (D^{N}_{r,1|2,1}(u^*, -\lambda + i\xi/5))^*, \quad \text{for all } \xi. \quad (3.4)$$

The same property holds for the normalized transfer matrix. Combining real analyticity, periodicity and crossing symmetry and using the notation introduced in (2.24) and (2.25) we find the following reality condition for $x \in \mathbb{R}$

$$D\left(\frac{\lambda}{2} + i\frac{x}{5}, \xi\right) = (D\left(\frac{\lambda}{2} + i\frac{x}{5}, \xi\right))^*, \quad (3.5)$$

$$D(3\lambda + i\frac{x}{5}, \xi) = (D(3\lambda + i\frac{x}{5}, \xi))^*, \quad (3.6)$$

$$t_j(x, \xi) = t_j(x, \xi)^*. \quad (3.7)$$

so that the transfer matrices are real along the center line of each strip. They also are transpose symmetric (2.10); this implies that they are real symmetric and thus they have real eigenvalues leading to real energies. The previous equations are also true for the eigenvalues. Of course, if $\text{Re}(\xi_{\text{latt}}) \neq -\lambda$, the boundary weights are no longer real, the reality condition is lost, the transfer matrices become complex and so too do the scaling energies. Again, this is confirmed by the “numerics on $D$”.

We now proceed with the flow $\chi_{1,2} \mapsto \chi_{1,1}$ which is the prototype among the flows in this family and so we describe it in detail.

---

3We use $^*$ to indicate complex conjugation.
Fig. 2. The two mechanisms A, B respectively that change string content during the flow $\chi_{1,1} \mapsto \chi_{1,2}$. Note that this is the reverse of the physical flow. These mechanisms are illustrated in the upper-half $u$-plane for the states: A, $(\_\_\_\_) \mapsto (0)_+$; B, $(00) \mapsto (0)_-$. 

3.1 RG flow $\chi_{1,2} \mapsto \chi_{1,1}$

From the previous discussion, the relevant double row transfer matrix is $D(u,\xi) \equiv D_{N_{\chi_{1,1}}2,1}(u, -\lambda + i\xi/5)$ containing one boundary thermodynamic field $\xi$. It is convenient to restrict to $\xi \leq 0$ so that $\xi = -\infty$ corresponds to the ultraviolet (UV) fixed point $(\chi_{1,1},)$ and $\xi = 0$ to the infrared (IR) fixed point $(\chi_{1,1},)$. The adjacency rules force $N$ to be odd. The addition of a column implemented by the boundary interaction (3.1) is consistent with the different parity of the number of faces in $\chi_{1,2}$ and $\chi_{1,1}$, as required by the adjacency rules

$$N_{\text{UV}}, \text{ odd } \mapsto N_{\text{IR}} = N_{\text{UV}} + 1, \text{ even} \quad (3.8)$$

3.1.1 Two mechanisms for changing the string content

As in [16], it is sometimes convenient mathematically to consider flows which are the reverse of the physical flows. In this section the actual flow we consider is $\text{IR} = \chi_{1,1} \mapsto \chi_{1,2} = \text{UV}$.

The pattern of zeros of the eigenvalues $D(u,\xi)$ change along the flow interpolating the extreme configurations described in Table 2. In particular, $m_1$ must change its parity from even to odd. Using (2.2), a simple counting of powers of the variable $\exp(iu)$ shows that $D(u,\xi)$ is a polynomial of order $2(N + 1)$ for $\xi \leq 0$ ($\chi_{1,1}$ is included in this range) and becomes a polynomial of order $2N$ in the limit $\xi \to -\infty$. So, inside the periodicity strip $-\lambda < \text{Re}(u) \leq 4\lambda$, precisely two zeros must “escape to infinity” during the flow, one in the upper- and one in the lower-half $u$-plane. Indeed, from the “numerics on $D$” we find just two mechanisms for changing the string content during the flow. As shown in Figure 2, these involve the objects (1 or 2-string) which are furthest from the real axis in strip 1 at the IR point. Strip 2 is not directly involved.

A. When the furthest object in strip 1 is a 2-string it moves away from the real axis, heading to $+\infty$; then it collapses into a pair of 1-strings; one of them disappears moving to $+\infty$ (it is actually reached only at the UV point), the other remains in the scaling region and becomes the top 1-string in strip 1. Of course, it cannot have a 2-string above it, so it is frozen and in the UV $\sigma = 1$. This pair of 1-strings, called correlated 1-strings, has special features that will be explained in the numerical section. The disappearance of the top 2-string leads to a decrease by one unit in the quantum numbers in strip 1.

B. If the furthest object in strip 1 is a 1-string it moves to $+\infty$. The remaining 1-strings can be excited and are therefore not frozen, so in the UV we have $\sigma = -1$.

This qualitative description of the mechanisms implies the following changes in the values of the
quantum numbers

\[ A: \begin{align*}
I^{(1)}_{m^{\text{IR}}} & \geq 1: \\
& \qquad m^{\text{IR}} \mapsto m^{\text{UV}} = m^{\text{IR}} + 1, \quad \sigma = +1 \\
I^{(1)}_{k^{\text{IR}}} & \mapsto I^{(1)}_{k^{\text{UV}}} = I^{(1)}_{k^{\text{IR}}} - 1, \quad I^{(1)}_{m^{\text{UV}}} = 0
\end{align*} \tag{3.9} \]

\[ B: \begin{align*}
I^{(1)}_{m^{\text{IR}}} & = 0: \\
& \qquad m^{\text{IR}} \mapsto m^{\text{UV}} = m^{\text{IR}} - 1, \quad \sigma = -1
\end{align*} \tag{3.10} \]

Observe that the change of parity of \( m_1 \) is consistent with the known parities at the two endpoints of the flow, as in Table 2. The consistency and completeness of these two mechanisms is demonstrated by the explicit mapping between the finitized characters associated with the UV and IR fixed points, as shown in the next section. Moreover, we observe these same mechanisms when solving the TBA equations.

The application of mechanisms A and B to the first few states is summarized in Table 3.

Table 3. Flow \( \chi_{1,1} \mapsto \chi_{1,2} \) (reverse of the physical flow). We present the explicit mapping of states from IR to UV up to the UV level 6. Here \( n^{\text{IR}}, n^{\text{UV}} \) are the excitation levels above the ground states, respectively \( h = 0 \) and \( h = 1/10 \).

| \( n^{\text{IR}} \) | Mapping of states – mechanism | \( n^{\text{UV}} \) | Mapping of states – mechanism | \( n^{\text{UV}} \) |
|---|---|---|---|---|
| 0 | \(( \) \( \mapsto (0)_+ \) | A | 0 | \((40) \mapsto (4)_- \) | B | 5 |
| 2 | \((00) \mapsto (0)_- \) | B | 1 | \((31) \mapsto (200)_+ \) | A | 5 |
| 3 | \((10) \mapsto (1)_- \) | B | 2 | \((100000) \mapsto (100000)_- \) | B | 5 |
| 4 | \((11) \mapsto (000)_+ \) | A | 3 | \((32) \mapsto (210)_+ \) | A | 6 |
| 4 | \((20) \mapsto (2)_- \) | B | 3 | \((41) \mapsto (300)_+ \) | A | 6 |
| 5 | \((21) \mapsto (100)_+ \) | A | 4 | \((50) \mapsto (5)_- \) | B | 6 |
| 5 | \((30) \mapsto (3)_- \) | B | 4 | \((0000) \mapsto (0000)_- \) | B | 6 |
| 6 | \((000000) \mapsto (000000)_- \) | B | 4 | \((110000) \mapsto (110000)_- \) | B | 6 |
| 6 | \((22) \mapsto (110)_+ \) | A | 5 | \((200000) \mapsto (200000)_- \) | B | 6 |

3.1.2 RG mapping between finitized characters

In this section we show that the two mechanisms A, B are compatible with the counting of states at the two endpoints of the flow, given by the finitized characters.

We observe in (3.9) and (3.10) that, starting in the IR, each state changes unambiguously by precisely one of the two possible mechanisms so that the counting of states is complete. Moreover, the IR finitized character naturally splits into two terms corresponding precisely to the two
mechanisms A, B (see (A.2))

\[
\chi^{(\text{IR})}_{1,1}(q) = q^{-\frac{c}{24}} \sum_{m_1^{\text{IR}},m_2} q^\frac{1}{4} m_1^{\text{IR}} C m_1^{\text{IR}} \left[ \frac{m_1^{\text{IR}} + n_1^{\text{IR}}}{m_1^{\text{IR}}} \right] \left[ \frac{m_2 + n_2}{m_2} \right]
\]

\[
= q^{-\frac{c}{24}} \sum_{A} q^\frac{1}{2} m_1^{\text{IR}} C m_1^{\text{IR}} \left[ \frac{m_1^{\text{IR}} + n_1^{\text{IR}} - 1}{m_1^{\text{IR}}} \right] \left[ \frac{m_2 + n_2}{m_2} \right]
\]

\[
+ q^{-\frac{c}{24}} \sum_{B} q^\frac{1}{2} m_1^{\text{IR}} C m_1^{\text{IR}} \left[ \frac{m_1^{\text{IR}} + n_1^{\text{IR}} - 1}{m_1^{\text{IR}}} \right] \left[ \frac{m_2 + n_2}{m_2} \right]
\]

(3.11)

where we attach the label IR or UV only to the variables that change under the flow. The labels A and B on the sums indicate that the sums on \( m_1^{\text{IR}} \), \( m_2 \) are restricted by the constraints imposed by the two mechanisms (3.9) and (3.10). This can be understood by using (A.3) for the first strip in the second line of (3.11) so that the sum on \( m_1^{\text{IR}} \geq 1 \) required by the mechanism A is manifest. Similarly, the factor related to the first strip in the last line of (3.11) can be rewritten using (A.4) as a sum with the constraint \( I_{m_1} = 0 \) required by mechanism B.

The single IR energy level at the base of a tower of states fixed by the string content \((m_1, m_2)\) maps to a UV energy level according to

A:

\[
q^{-\frac{c}{24}} q^\frac{1}{4} m_1^{\text{IR}} C m_1^{\text{IR}} q^{m_1^{\text{IR}}} \quad \mapsto \quad q^{-\frac{c}{24} + \frac{1}{24}} q^\frac{1}{4} m_1^{\text{UV}} C m_1^{\text{UV}} q^{-\frac{1}{2} (m_1^{\text{UV}} - m_2)}
\]

B:

\[
q^{-\frac{c}{24}} q^\frac{1}{2} m_1^{\text{IR}} C m_1^{\text{IR}} \quad \mapsto \quad q^{-\frac{c}{24} + \frac{1}{24}} q^\frac{1}{4} m_1^{\text{UV}} C m_1^{\text{UV}} q^{-\frac{1}{2} (m_1^{\text{UV}} - m_2)}
\]

(3.12)

(these mappings of energies are fixed by the known expression for the energies at the IR and UV critical points). Using (3.9) and (3.11), the mapping of the q-binomials (counting polynomials) is given by

A:

\[
\left[ \frac{m_1^{\text{IR}} + n_1^{\text{IR}} - 1}{m_1^{\text{IR}}} \right] = \left[ \frac{m_1^{\text{UV}} + n_1^{\text{UV}} - 1}{m_1^{\text{UV}}} \right] = \left[ \frac{m_1^{\text{UV}} + n_1^{\text{UV}} - \delta_{1,\sigma}}{m_1^{\text{UV}}} \right]
\]

B:

\[
\left[ \frac{m_1^{\text{IR}} + n_1^{\text{IR}} - 1}{m_1^{\text{IR}}} \right] = \left[ \frac{m_1^{\text{UV}} + n_1^{\text{UV}}}{m_1^{\text{UV}}} \right] = \left[ \frac{m_1^{\text{UV}} + n_1^{\text{UV}} - \delta_{1,\sigma}}{m_1^{\text{UV}}} \right]
\]

(3.13)

Combining (3.11) to (3.13) we obtain the finitized UV character

\[
\chi^{(\text{UV})}_{1,1}(q) \quad \mapsto \quad q^{-\frac{c}{24} + \frac{1}{24}} \sum_{\sigma,m_1^{\text{UV}},m_2} q^\frac{1}{4} m_1^{\text{UV}} C m_1^{\text{UV}} q^{-\frac{1}{2} (m_1^{\text{UV}} - m_2) \sigma} \left[ \frac{m_1^{\text{UV}} + n_1^{\text{UV}} - \delta_{1,\sigma}}{m_1^{\text{UV}}} \right] \left[ \frac{m_2 + n_2}{m_2} \right]
\]

\[
= \chi^{(\text{UV})}_{1,2}(q)
\]

(3.14)

This shows that the completeness of the IR counting of states, together with the mechanisms A and B, implies the consistency and completeness of the UV counting.

### 3.1.3 Solution of the functional equations

To solve the functional equation for the eigenvalues of the double row transfer matrix \( D(u, \xi) \equiv D^{N}_{1,1|2,1}(u, -\lambda + i\xi/5) \), we follow steps similar to those in [20] but taking particular care in managing the terms containing \( \xi \). The functional equation can be rewritten using (2.4) and (2.5) as a coupled system between the two analyticity strips

\[
t_1(x + i \frac{\pi}{2}) t_1(x - i \frac{\pi}{2}) = 1 + t_2(x)
\]

(3.15)

\[
t_2(x + i \frac{\pi}{2}) t_2(x - i \frac{\pi}{2}) = 1 + t_1(x)
\]

(3.16)
These equations can be solved by taking the Fourier transform of the logarithmic derivative of the equations, taking care to remove all of the zeros and poles that can generate a singularity in $\log t_j(x)$. It is convenient to consider separately the order $N$, order 1 and order $1/N$ contributions to $t_j(x)$.

The analyticity of the order $N$ contribution is contained in the last factor of (2.11), leading to a zero of order $2N$ at $u = 3\lambda$ and to two poles of the same order at $u = 2\lambda$, $4\lambda$. These are at the edges of the second strip so they do not effect the RHS terms $1 + t_1$ and $1 + t_2$ within (3.17). The order $N$ contribution $f(u)$ is thus given by the inversion relations

$$f_1(x + i\frac{\pi}{2}) f_1(x - i\frac{\pi}{2}) = 1$$  \hspace{1cm} (3.18)
$$f_2(x + i\frac{\pi}{2}) f_2(x - i\frac{\pi}{2}) = 1$$  \hspace{1cm} (3.19)

The unique solution [14] of these equations with the required analyticity properties is

$$f_1(x) = 1$$  \hspace{1cm} (3.20)
$$f_2(x) = \tanh^{2N}\frac{x}{2}$$  \hspace{1cm} (3.21)

If we divide (3.15), (3.16) by (3.18), (3.19), we obtain a new system of equations containing the function $t_i(x)/f_i(x)$ on the LHS and free of order $N$ zeros and poles in (3.17).

$$\frac{t_1}{f_1}(x + i\frac{\pi}{2}) \frac{t_1}{f_1}(x - i\frac{\pi}{2}) = 1 + t_2(x)$$  \hspace{1cm} (3.22)
$$\frac{t_2}{f_2}(x + i\frac{\pi}{2}) \frac{t_2}{f_2}(x - i\frac{\pi}{2}) = 1 + t_1(x)$$  \hspace{1cm} (3.23)

The analyticity of the order 1 contribution in (3.22), (3.23), represented in Figure 3, is specific to each boundary condition, being contained in the factors $S(u)$, $S_1(u)$ and $S_2(u, \xi_{latt})$ of (2.11). $S(u)$ leads $t_1(x)$ to have a double zero at $x = 0$ and two poles at $x = \pm i\pi$ and leads $t_2(x)$ to have
functions position of the 1-string zeros \( v \) in each strip. It is simple to prove that these functions satisfy
\[
\cosh(2\text{Im}(\xi))
\]
We see that \( S_2(u, -\lambda + i\xi/5) \) has zeros and poles in the second strip only: two single zeros at \( u = 3\lambda \pm i\xi/5 \) \((x = \pm \xi)\) and four single poles at \( u = 2\lambda \pm i\xi/5, u = 4\lambda \pm i\xi/5 \) \((x = \pm i\pi \pm \xi)\). The factor \( \cosh(2\text{Im}(\xi_{\text{latt}})) \) is compensated by the corresponding normalization assumed in \((2.4)\) and always disappears from the equations. Observe that the given poles of order 1 are on the edges of the two analyticity strips so \( 1 + t_2(x) \) and \( 1 + t_1(x) \) in \((3.15)\) and \((3.16)\) are free of poles inside \((3.17)\). They are also non-zero in \((3.17)\). To see this it is enough to observe that all factors \( t_j/f_j(x \pm i\pi/2) \) have no order one zeros in \(|\text{Im}(x)| < \pi/2\). So we conclude that the order 1 contribution satisfies
\[
g_1(x + i\frac{\pi}{2}) g_1(x - i\frac{\pi}{2}) = 1 \tag{3.25}
g_2(x + i\frac{\pi}{2}) g_2(x - i\frac{\pi}{2}) = 1 \tag{3.26}
\]
The unique solution with the analyticity determined by the structure of zeros and poles just discussed is
\[
g_1(x, \xi) = \tanh^2 \frac{x}{2} \tag{3.27}
g_2(x, \xi) = \tanh^2 \frac{x}{2} \tanh \frac{x - \xi}{2} \tanh \frac{x + \xi}{2} \tag{3.28}
\]
Clearly, the functions \( l_j(u) = t_j(u)/(f_j(u)g_j(u, \xi)) \) giving the finite-size corrections satisfy the equations
\[
l_1(x + i\frac{\pi}{2}) l_1(x - i\frac{\pi}{2}) = 1 + t_2(x) \tag{3.29}
l_2(x + i\frac{\pi}{2}) l_2(x - i\frac{\pi}{2}) = 1 + t_1(x) \tag{3.30}
\]
and do not possess the zeros and poles of order \( N \) and 1 that we have already removed. They only have other zeros, given by the 1-string zeros of \( D(u, \xi) \). Observe that these do not effect the functions \( 1 + t_1(x) \) and \( 1 + t_2(x) \), that again are non-zero on the real axis. Assuming \( N \) large, the position of the 1-string zeros \( v^{(j)}_k \) is real\(^4\) and we can remove their contribution by introducing the functions
\[
p_j(x) = \prod_{k=1}^{m_j} \tanh[\frac{1}{2}(5v^{(j)}_k - x)] \tanh[\frac{1}{2}(5v^{(j)}_k + x)], \tag{3.31}
\]
in each strip. It is simple to prove that these functions satisfy
\[
p_j(x + i\frac{\pi}{2}) p_j(x - i\frac{\pi}{2}) = 1. \tag{3.32}
\]
So, as before, we introduce the functions
\[
\tilde{l}_j(x) = \frac{t_j(x)}{f_j(x)g_j(x, \xi)p_j(x)} \tag{3.33}
\]
\(^4\)Observe that the classification of zeros in terms of 1- and 2-strings is only strictly true for \( N \) sufficiently large.
which are completely free of poles and zeros inside (3.17). We can then solve the remaining equations
\[
\tilde{l}_1(x + i\frac{\pi}{2}) \tilde{l}_1(x - i\frac{\pi}{2}) = 1 + t_2(x)
\]
\[
\tilde{l}_2(x + i\frac{\pi}{2}) \tilde{l}_2(x - i\frac{\pi}{2}) = 1 + t_1(x)
\]
by taking the Fourier transform of the logarithmic derivative. The solution is
\[
\log \tilde{l}_1(x) = K^* \log(1 + t_2) + C_1,
\]
\[
\log \tilde{l}_2(x) = K^* \log(1 + t_1) + C_2,
\]
where \(C_j\) are integration constants. The kernel of the integration and the convolution are given by
\[
K(x) = \frac{1}{2\pi \cosh x}, \quad (f * g)(x) = \int_{-\infty}^{+\infty} dy f(x - y)g(y)
\]
Using (3.33) we recombine all the contributions
\[
\log t_1(x) = \log g_1(x, \xi) + \sum_{k=1}^{m_1} \log \left[ \tanh\left(\frac{1}{2} (5v_k^{(1)} - x)\right) \tanh\left(\frac{1}{2} (5v_k^{(1)} + x)\right) \right] + K^* \log(1 + t_2) + C_1,
\]
\[
\log t_2(x) = \log f_2(x) + \log g_2(x, \xi) + \sum_{k=1}^{m_2} \log \left[ \tanh\left(\frac{1}{2} (5v_k^{(2)} - x)\right) \tanh\left(\frac{1}{2} (5v_k^{(2)} + x)\right) \right] + K^* \log(1 + t_1) + C_2
\]
These are the lattice TBA equations. The boundary thermodynamic field \(\xi\) enters the TBA equations only through the boundary factors \(g_1, g_2\).

There are two important simplifications of the previous equations. It will be clear in the sequel that, in all the cases of interest,
\[
\lim_{x \to \pm \infty} g_j(x, \xi) = 1 \text{ or } -1.
\]
This expression and (B.3) from Appendix B can be used in the TBA equations (3.39), (3.40) to compute the limit \(x \to +\infty\). This forces the integration constants to be an integer multiple of \(i\pi\)
\[
C_j \propto i\pi
\]
so that they can be removed from the equations and replaced by a sign, \(s_j = \pm 1\), inside the LHS logarithm: \(\log s_j t_j(x)\). This amounts to fixing the branch of the logarithm. From (3.7) we know that \(t_j(x)\) is real for real \(x\). From the analysis of the analyticity, we also know that \(1 + t_j(x) \neq 0\) for real \(x\) so we find that it never changes its sign. The limit in (3.3) forces it to be positive, \(1 + t_j(x) > 0\), so that the convolution term in the TBA equations is always real.

### 3.1.4 Scaling limit and TBA equations

The order \(1/N\) term in (2.16), giving the scaling behaviour, is obtained in the scaling limit. Guided by the function \(f_2\), that has the well defined behaviour
\[
\lim_{N \to \infty} f_2(x + \log N) = \exp(-4e^{-x}),
\]
we assume the general scaling forms
\[
\hat{h}_j(x) = \lim_{N \to \infty} h_j(x + \log N).
\]
so that (for real $\epsilon_j(x)$) the TBA equations are

$$
\epsilon_1(x) = -\log \hat{g}_1(x, \xi) - \sum_{k=1}^{m_1} \log(\tanh \frac{y^{(1)}_k - x}{2}) - K \log |1 + s_2 e^{-\epsilon_2(x)}|, \\
\epsilon_2(x) = 4e^{-x} - \log \hat{g}_2(x, \xi) - \sum_{k=1}^{m_2} \log(\tanh \frac{y^{(2)}_k - x}{2}) - K \log |1 + s_1 e^{-\epsilon_1(x)}|.
$$

(3.49)

This means that any complex contribution to $\epsilon_j$ comes only from the source terms in the TBA equations. Note that these perturbed TBA equations differ from the critical TBA equations (2.26) and (2.27) only through the appearance of the boundary perturbation terms $\log \hat{g}_1(x, \xi)$ and $\log \hat{g}_2(x, \xi)$.

The boundary thermodynamic field is also scaled so that it survives in the continuum limit

$$
\hat{g}_j(x, \xi) = \lim_{N \to \infty} g_j(x + \log N, \xi - \log N)
$$

(3.50)

In the present case, we find from (3.27), (3.28) that

$$
\hat{g}_1(x, \xi) = 1, \\
\hat{g}_2(x, \xi) = \tanh \frac{x + \xi}{2}
$$

(3.51)

(3.52)

so that only $\hat{g}_2$ remains in the equations. The sign used in the scaling of $g_j$, for the boundary field, $\xi - \log N$, corresponds to the value $\xi = -\infty$ at the UV conformal fixed point ($\chi_{1,2}(q)$) and $\xi = \infty$ at the IR conformal fixed point ($\chi_{1,1}(q)$). To be more general, we will keep track of $\hat{g}_1$ in many of the equations that follow, because in the analysis of other flows this term will be non-trivial.

In the TBA literature, it is standard to define

$$
L_j(x) = \log |1 + \hat{t}_j(x)| = \log |1 + s_j e^{-\epsilon_j(x)}| \in \mathbb{R}.
$$

(3.53)

The zeros of $\hat{t}_j$ are then the zeros of $L_j$ and vice versa, as is seen if we write the TBA equations in the more physical form

$$
L_1(x) = \log \left|1 + s_1 \hat{g}_1(x, \xi) \prod_{k=1}^{m_1} \tanh \frac{y^{(1)}_k - x}{2} e^{K* L_2(x)}\right|, \\
L_2(x) = \log \left|1 + e^{-4e^{-x}} s_2 \hat{g}_2(x, \xi) \prod_{k=1}^{m_2} \tanh \frac{y^{(2)}_k - x}{2} e^{K* L_1(x)}\right|.
$$

(3.54)

(3.55)

Again following [20], we can use (3.36), (3.37) to determine the scaling part of the eigenvalues of $D(u, \xi)$ at the isotropic point $u = \lambda/2$

$$
E(\xi) = \frac{2}{\pi} \sum_{k=1}^{m_1} e^{-y^{(1)}_k} - \int_{-\infty}^{\infty} \frac{dx}{\pi} e^{-x} L_2(x)
$$

(3.56)

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3.1.5 Zeros and quantum numbers

In the scaling limit, the zeros of the double row transfer matrix satisfy

\[ \hat{t}_j(y^{(j)}_k) = 0 \quad \text{for the 1-strings} \] (3.57)
\[ \hat{t}_j(z^{(j)}_k \mp i\pi) = 0 \quad \text{for the 2-strings} \] (3.58)

The normalized transfer matrix itself can have other order 1 zeros from the factors in (2.11). In particular, the point \( x = -\xi \) being a zero for \( g_2(x, \xi) \) is also a zero for \( \hat{t}_2(x, \xi) \), i.e. a zero in the center of the second strip. The string-like zeros can be obtained by a single equation; to see this, we perform the scaling limit and the shift \( x \mapsto x - i\pi/2 \) in (3.15), (3.16)

\[ \hat{t}_1(x) \hat{t}_1(x - i\pi) = 1 + \hat{t}_2(x - i\pi/2) \] (3.59)
\[ \hat{t}_2(x) \hat{t}_2(x - i\pi) = 1 + \hat{t}_1(x - i\pi/2) \] (3.60)

and find that each real solution of

\[ \hat{t}_j(x - i\pi/2) = -1 \] (3.61)

is either a 1-string or a 2-string in the strip \( 3 - j \). The zero \( x = -\xi \) is not contained in the previous equation because it is coupled with a pole, as in Figure 2. The 1-string or 2-string cases are mutually exclusive, as the numerical analysis on small double rows transfer matrices confirm: the zeros generally repel one another and only single zeros are observed. Double zeros can appear, however, at exceptional values of \( \xi \), for example at the point where the 2-string in mechanism A transforms into the correlated 1-strings, as in Figure 2.

The locations of the zeros are fixed by (3.61) so we can define the counting functions

\[ \psi_j(x) = -i \epsilon_j(x - i\pi/2) = -i \log s_j \hat{t}_j(x - i\pi/2) \] (3.62)

that are a multiple of \( \pi \) on a 1 or 2-string and introduce non-degenerate quantum numbers \( n^{(j)}_k \in \mathbb{Z} \) by the quantization conditions

\[ \psi_2(y^{(1)}_k) = -i \epsilon_2(y^{(1)}_k - i\pi/2) = n^{(1)}_k \pi, \quad (n^{(1)}_k + s_2) = 0 \mod 2, \] (3.63)
\[ \psi_1(y^{(2)}_k) = -i \epsilon_1(y^{(2)}_k - i\pi/2) = n^{(2)}_k \pi, \quad (n^{(2)}_k + s_1) = 0 \mod 2. \] (3.64)

Note the inversion of the indices: \( \psi_1 \) is for strip 2 and \( \psi_2 \) for strip 1. The same equations hold for the 2-string locations \( z^{(j)}_l \) so, each time \( \psi_j(x) = n\pi \) is satisfied for an integer \( n \) with the appropriate parity, \( x \) is a 1-string or a 2-string in the strip \( 3 - j \).

The explicit expressions for \( \psi_j \) are given by:

\[ \psi_1(x) = i \log \hat{g}_1(x - i\pi/2, \xi) + i \sum_{k=1}^{m_1} \log \tanh(y^{(1)}_k - x + i\pi/4) \]
\[ - \operatorname{PV} \int_{-\infty}^{+\infty} \frac{dy}{2\pi \sinh(x - y)} L_2(y) \] (3.65)
\[ \psi_2(x) = 4e^{-x} + i \log \hat{g}_2(x - i\pi/2, \xi) + i \sum_{k=1}^{m_2} \log \tanh(y^{(2)}_k - x + i\pi/4) \]
\[ - \operatorname{PV} \int_{-\infty}^{+\infty} \frac{dy}{2\pi \sinh(x - y)} L_1(y) \] (3.66)
where the integral around the singularity \( x = y \) must be understood as a principal value. We take the fundamental branch for each logarithm so that in general \( \log a + \log b \) cannot be identified as \( \log(ab) \). This prescription is convenient in view of the numerical computations and is used also in the TBA equations \( (3.48), (3.49) \).

With this in mind, we can easily compute the limits of \( \psi_j \) for \( x \to \pm \infty \). For \( x \to -\infty \), the dominant term in the expression of \( L_2 \) and \( \psi_2 \) is the exponential, so we have

\[
L_2(x) \sim e^{-4e^{-x}} \to 0, \quad \text{as } x \to -\infty,
\]

\[
\psi_2(x) \sim 4e^{-x} \to \infty, \quad \text{as } x \to -\infty
\]

These reflect the emergence of an infinite number of 2-strings in strip 1 as the size of the double row transfer matrix goes to infinity, as in Section 3.1.4. This dense filling of 2-strings forces the 1-strings in the first strip only to be far apart from the \(-\infty\) row, so that the convolution disappears due to the principal value. We take

\[
\lim_{x \to -\infty} \psi_1(x) = i \log \hat{g}_1(-\infty - i \frac{\pi}{2}, \xi).
\]

In the present case, we have

\[
\lim_{x \to -\infty} \psi_1(x) = 0, \quad \text{for } \xi > -\infty.
\]

The limit \( x \to +\infty \) requires some care. We assume that \( L_j(+\infty) \) is finite, as is confirmed by numerical computations, so that the contribution from the convolution disappears due to the principal value. In general, there are two possibilities

\[
\psi_j(+\infty) = \begin{cases} 
\log(1 + \hat{s}_1 \hat{g}_1(-\infty, \xi)) = \begin{cases} 
\log 2 \\
- \infty 
\end{cases} 
& \text{if } y_m^{(j)} < +\infty, \\
\log(1 + \hat{s}_1 \hat{g}_1(+\infty - i \frac{\pi}{2}, \xi)) - \pi m_j & \text{if } y_m^{(j)} = +\infty.
\end{cases}
\]

In the present case, from the TBA computations at the IR critical point \((\chi_{1,1}(q))\), we know that \( \hat{g}_m^{(j)} < +\infty \) in both strips so, away from the UV point, the previous equations reduce to

\[
\psi_j(+\infty) = -\pi m_j, \quad \xi > -\infty.
\]

Our main conclusion is that generally the functions \( \psi_j \) are globally decreasing, with \( m_1, m_2 \) nonnegative and possibly large and again this is confirmed by numerics. There are a few exceptions that give rise to interesting behaviours that will be discussed later. The monotonicity is the ingredient we need to relate the two families \( \{n_k^{(j)}\} \) and \( \{I_k^{(j)}\} \) of non-degenerate and nonnegative quantum numbers. As in the critical case in Section 2.4.1 and in [20], we start from the limit \( \psi_j(+\infty) \). The 1 and 2-strings occur at multiples of \( \pi \) with parity dictated by \( (3.63) \) and \( (3.64) \), so decreasing from one object to the next, the function \( \psi_j \) increases by \( 2\pi \). The ordering of the zeros is \( y_1^{(j)} < y_2^{(j)} < \ldots < y_{m_j}^{(j)} \) so above the zero \( y_k^{(j)} \) there are \( I_k^{(j)} \) 2-strings (by definition) and \( m_j - k \) 1-strings.

An expression for the parities \( s_j \) is still missing. If we assume the IR critical \((\chi_{1,1}(q))\) TBA equations can be perturbed by the “small” function \( \log \hat{g}_2 = \log \tanh((x + \xi)/2) \) with large \( \xi \), we are led to maintain the even parity of \( m_1, m_2 \) and the parities \( s_1 = s_2 = 1 \), as in Table 2. This forces

\[
n_k^{(1)}, n_k^{(2)} \text{ odd}
\]
We observed that \( \psi_j(+\infty) \) cannot be a zero. At least close to the IR region, we have \( y^{(j)}_k < +\infty \) in both strips, consistent with the limits \((3.73)\) where \( m_j \) are both even. The first available position for a zero is \( \psi_j(+\infty) + \pi = \pi(1 - m_j) \), yielding the following expressions for the quantum numbers

\[
\begin{align*}
 n_k^{(1)} &= 2(I_k^{(1)} + m_1 - k) + 1 - m_2, \\
 n_k^{(2)} &= 2(I_k^{(2)} + m_2 - k) + 1 - m_1.
\end{align*}
\]

Observe that \((3.72)\) is not predictive because we need to know in advance if a zero is at \( +\infty \) to get the proper solution.

A final comment on the zeros is useful. On the lattice, the 1- and 2-strings are zeros of the transfer matrix \( D(u, \xi) \). In addition to that, the normalized transfer matrix \( t(u, \xi) \) can have other zeros and poles. After the scaling limit, only the functions \( \hat{t}_j(x, \xi) \) remain and, from the previous derivation, we see that \( \hat{t}_2(x, \xi) \) has a zero at \( x = -\xi \). Restricting \( x \) to the real axis, the zeros of \( \hat{t}_j(x, \xi) \) and of \( L_j(x) \) coincide \((3.53)\) and can be either 1-strings or the boundary dependent zero \( x = -\xi \). In terms of the TBA functions \( L_j \) (or \( e_j \)) and \( \psi_j \), the characterization of the lattice quantities is: the 1-strings in strip \( j \) are the real zeros in common between \( L_j(x) \) and \( \psi_{3-j}(x) - 2\pi n \); the 2-strings in strip \( j \) are the real zeros of \( \psi_{3-j}(x) - 2\pi n \) and not of \( L_j(x) \); the boundary zeros are the real zeros of \( L_2(x) \) and not of \( \psi_1 - 2\pi n \).

As a complement to the description of the mechanisms and to the TBA equations, we summarize the main results concerning this flow starting from the IR point:

**mechanism A before the collapse and mechanism B:**

\[
m_1, m_2 \text{ even, } \quad n_2 = \frac{m_2}{2} - m_2 \geq 0, \quad s_1 = s_2 = 1,
\]

\[
 n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2,
\]

\[
 n_k^{(2)} = 2(I_k^{(2)} + m_2 - k) + 1 - m_1;
\]

**mechanism A after the collapse point:**

\[
m_1^4 = m_1 + 2, \quad n_{m_1^4-1}^{(1)} = n_{m_2}^{(1)} = 1 - m_2,
\]

the values of \( m_2, n_2, n_k^{(2)}, n_k^{(1)} \) with \( k < m_1^4 - 1, s_j \), remain unchanged.

### 3.2 RG flow \( \chi_{3,2} \mapsto \chi_{3,1} \)

As pointed out in Section 2.2, this flow is dual to \( \chi_{1,2} \mapsto \chi_{1,1} \) and many of the considerations of Section 3.1 apply to it. For this flow we consider the double row transfer matrix \( D(u, \xi) \equiv D_{3,1}^N(u, \xi_{\text{latt}}) \) with an odd number of faces \( N \) in accord with the adjacency rules. We find the same change of parity in the number of columns \((3.8)\) and the same two mechanisms A and B \((3.9), (3.10)\) changing the string content during the flow. The actual parities of \( N, m_i \) are different but the changes are still fixed by the mechanisms A and B. The application of mechanisms A and B to the first few states is given in Table 4. Again, we find a mapping between finitized characters

\[
\chi_{3,1}^{(\text{IR})}(q) \mapsto \chi_{3,2}^{(\text{UV})}(q)
\]

\[
N_{\text{IR}} \mapsto N_{\text{UV}} = N_{\text{IR}} - 1
\]

showing the consistency of the mechanisms with the IR and UV counting as in Section 3.1.2.

In solving the functional equations, as noticed in Section 3.1.3, the order 1 behaviour is specific to each boundary condition. Here we need to consider \( S(u), S_3(u, \lambda/2), S_2(u, \xi_{\text{latt}}) \) from \((3.11)\) to
Table 4. Flow $\chi_{3,1} \rightarrow \chi_{3,2}$ (reverse of the physical flow). We present the explicit mapping of states from IR to UV up to the UV level 6. Here $n^{\text{IR}}$, $n^{\text{UV}}$ are the excitation levels above the ground states, respectively $h = \frac{3}{2}$ and $h = \frac{3}{5}$.

| $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ | $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ |
|-----------------|---------------------------------|----------------|-----------------|---------------------------------|----------------|
| 0               | $(0|0) \rightarrow (0|0)$       | B              | 0               | $(4|1) \rightarrow (3|0|0)$       | A              |
| 1               | $(1|0) \rightarrow (1|0)$       | B              | 1               | $(3|2) \rightarrow (2|1|0)$       | A              |
| 2               | $(2|0) \rightarrow (2|0)$       | B              | 2               | $(0|0|0|1) \rightarrow (0|0|0|1)$  | B              |
| 2               | $(1|1) \rightarrow (0|0|0|0)$   | A              | 2               | $(1|0|0|0) \rightarrow (1|0|0|0)$  | B              |
| 3               | $(3|0) \rightarrow (3|0)$       | B              | 3               | $(6|0) \rightarrow (6|0)$        | B              |
| 3               | $(2|1) \rightarrow (1|0|0|0)$   | A              | 3               | $(5|1) \rightarrow (4|0|0)$       | B              |
| 4               | $(4|0) \rightarrow (4|0)$       | B              | 4               | $(4|2) \rightarrow (3|1|0)$       | A              |
| 4               | $(3|1) \rightarrow (2|0|0|0)$   | A              | 4               | $(3|3) \rightarrow (2|2|0)$       | A              |
| 5               | $(0|0|0|0) \rightarrow (0|0|0|0)$| B              | 4               | $(1|0|0|1) \rightarrow (1|0|0|1)$  | B              |
| 5               | $(5|0) \rightarrow (5|0)$       | B              | 5               | $(1|1|0|0) \rightarrow (1|1|0|0)$  | B              |

2.**R**. The only difference with Section 3.1.3 is the term $S_3(u, \lambda/2)$ that replaces $S_1(u)$

$$S_3(u, \lambda/2) = -\frac{\sin^2 \lambda}{\sin(u + \frac{\lambda}{2}) \sin(u - \frac{\lambda}{2})}. \quad (3.78)$$

It simply adds to Figure 3 poles at $u = -\lambda/2$, $u = 3\lambda/2$, that is, $x = \mp i\pi$. Being outside (3.14) they are not relevant, so we obtain the same order 1 system (3.25), (3.26), the same solution (3.27), (3.28) and the same scaling terms (3.51), (3.52):

$$\hat{g}_1(x) = 1 \quad (3.79)$$

$$\hat{g}_2(x) = \tanh \frac{x + \xi}{2} \quad (3.80)$$

The order $1/N$ analyticity is also specific to each boundary condition in the sense that the parity of $m_j$ in (3.31) is boundary dependent, but the way to proceed is exactly the same as before. In conclusion, the equations obtained in Section 3.1.4 hold true also for this case.

We summarize the description of this flow starting from the IR point:

**mechanism A before the collapse and mechanism B:**

$$m_1 \geq 2 \text{ even, } m_2 \text{ odd, } n_2 = \frac{m_1}{2} - m_2 \geq 0, \quad s_1 = 1, \ s_2 = -1,$$

$$n_k^{(1)} = 2(l_k^{(1)} + m_1 - k) + 1 - m_2,$$

$$n_k^{(2)} = 2(l_k^{(2)} + m_2 - k) + 1 - m_1.$$
Table 5. Flow \( \chi_{2,1} \mapsto \chi_{2,2} \) (reverse of the physical flow). We present the explicit mapping of states from IR to UV up to the UV level 6. Here \( n_{IR} \), \( n_{UV} \) are the excitation levels above the ground states, respectively \( h = 7/16 \) and \( h = 3/80 \).

| \( n_{IR} \) | Mapping of states – mechanism | \( n_{UV} \) | Mapping of states – mechanism | \( n_{UV} \) |
|-------------|--------------------------------|-------------|--------------------------------|-------------|
| 0           | \((0|0) \mapsto (0|0)_-\)       | B           | 0                             | \((11|0) \mapsto (11|0)_-\)       | B           |
| 1           | \((1|0) \mapsto (00|0)_+\)      | A           | 1                             | \((10|1) \mapsto (10|1)_-\)      | B           |
| 2           | \((2|0) \mapsto (10|0)_+\)      | A           | 2                             | \((5|0) \mapsto (40|0)_+\)      | A           |
| 3           | \((000|0) \mapsto (000|0)_-\)   | B           | 2                             | \((300|0) \mapsto (300|0)_-\)   | B           |
| 4           | \((3|0) \mapsto (20|0)_+\)      | A           | 3                             | \((210|0) \mapsto (21|0)_-\)    | B           |
| 4           | \((100|0) \mapsto (10|0)_-\)    | B           | 3                             | \((200|1) \mapsto (20|1)_-\)    | B           |
| 5           | \((000|1) \mapsto (00|1)_-\)    | B           | 3                             | \((110|1) \mapsto (11|1)_-\)    | B           |
| 4           | \((4|0) \mapsto (30|0)_+\)      | A           | 4                             | \((111|0) \mapsto (000|00|0)_+\)| A           |
| 5           | \((200|0) \mapsto (20|0)_-\)    | B           | 4                             | \((6|0) \mapsto (50|0)_+\)      | A           |

### 3.3 RG flow \( \chi_{2,2} \mapsto \chi_{2,1} \)

The treatment of the flow \( \chi_{2,2} \mapsto \chi_{2,1} \) is very similar to the previous flows. The relevant transfer matrix is now \( D(u, \xi) = D_{2,1}|2,1(u, \xi_{latt}) \) which has an even number of faces \( N \). Again, we find the same change of parity in the number of columns \( \xi \) and the same two mechanisms A and B \( \chi \), \( \xi \), as illustrated in Table 4. We also find a mapping between finitized characters

\[
\chi_{2,1}^{(N_{IR})}(q) \mapsto \chi_{2,2}^{(N_{UV})}(q) \quad (3.81)
\]

\[
N_{IR} \mapsto N_{UV} = N_{IR} - 1
\]

showing the consistency of the mechanisms with the IR and UV counting as in Section 3.1.2.

In solving the functional equations, we need to take care of the order 1 analyticity. With reference to Section 3.1.3 we need to consider the term

\[
S_{2}(u, \frac{\lambda}{2}) = \frac{\sin^{2}\lambda \sin(u + \frac{3}{2}\lambda) \sin(u + \frac{7}{2}\lambda)}{\sin^{2}(u - \frac{1}{2}\lambda) \sin(u + \frac{3}{2}\lambda) \sin(u + \frac{7}{2}\lambda)}.
\quad (3.82)
\]

All of its zeros and poles are independent of \( \xi \) and disappear in the scaling limit. Many of them disappear also at the lattice level: the factors \( \sin(u + \frac{3}{2}\lambda) \sin(u + \frac{7}{2}\lambda) \) in the denominator are cancelled by equal terms in \( D(u, \xi) \). Indeed, \( \sin(u + \frac{3}{2}\lambda) \) explicitly appears in \( B_{2,1}^{(2)}(u, \lambda/2) \); \( \sin(u + \frac{7}{2}\lambda) \) does not explicitly appear but we can prove its existence by crossing symmetry: \( D(u, \xi) = D(\lambda - u, \xi) \) implies that if \( u = -3\lambda/2 \) is a zero, then \( \lambda - u = 5\lambda/2 \equiv -5\lambda/2 \) must also be a zero (the last equality is true because of periodicity). The remaining factors contain zeros at \( u = -\lambda/2, u = 3\lambda/2, \) which are irrelevant because they are outside of \( \chi_{1,1} \), and a double pole at \( u = \lambda/2 \) that compensates a double zero from \( S(u) \). So, already at the lattice level, \( g_{1}(x) = 1 \). For \( g_{2} \), all the considerations in Section 3.1.3 can be repeated here.

We conclude that the same TBA equations, quantization conditions and energy expression as in Section 3.1.4 hold for this flow.

We summarize the description of this case starting from the IR point:
mechanism A before the collapse and mechanism B:

\[ \begin{align*}
  m_1, m_2 & \text{ odd, } \\
  n_2 &= \frac{m_1+1}{2} - m_2 \geq 0, \quad s_1 = s_2 = -1, \\
  n_k^{(1)} &= 2(i_k^{(1)} + m_1 - k) + 1 - m_2, \\
  n_k^{(2)} &= 2(i_k^{(2)} + m_2 - k) + 1 - m_1;
\end{align*} \]

mechanism A after the collapse:

\[ \begin{align*}
  m_1^A &= m_1 + 2, \\
  n_{m_1^A - 1}^{(1)} &= n_{m_1^A}^{(1)} = 1 - m_2, \\
  \text{the values of } m_2, n_2, n_k^{(2)}, n_k^{(1)} \text{ with } k < m_1^A - 1, s_j, \text{ remain unchanged.}
\end{align*} \]

4 Boundary flows with variable $r$

4.1 RG flow $\chi_{3,2} \mapsto \chi_{2,1}$

For the flow $\chi_{3,2} \equiv \chi_{1,3} \mapsto \chi_{2,1}$ we consider the double row transfer matrix $D_N^{1,1|3,1}(u, \xi_{\text{latt}})$ with the trivial $B^{1,1}$ boundary on the left, with no free parameter, and $B^{3,1}(u, \xi_{\text{latt}})$ on the right with the parameter $\xi_{\text{latt}} \in \mathbb{C}$. The number of faces $N$ is even. The limit $\Im(\xi_{\text{latt}}) \to \pm \infty$ on $B^{3,1}(u, \xi_{\text{latt}})$ reproduces the $s = 3$ boundary mechanism $A$ and, for $\Im(\xi_{\text{latt}}) = 0$, $B^{3,1}(u, \Re(\xi_{\text{latt}}))$ yields an $r = 2$ type boundary (2.18) if $\Re(\xi_{\text{latt}}) \in [-5/2\lambda, -\lambda/2]$. If we choose $\xi_{\text{latt}} = -3\lambda/2 + i\xi/5$ with $\xi$ real,

\[
B^{3,1}
\left[
\begin{array}{ccc}
3 & 1 & 3 \\
3 & 3 & 3
\end{array}
\right]
, u - \frac{3}{2} \lambda + i\frac{\xi}{5}
= \sqrt{\frac{S_{5+1}}{S_3}}
\frac{1 - \sin(2\lambda + i\frac{\xi}{5})}{\sin(2\lambda + i\frac{\xi}{5})}
\sin \lambda \cosh 2\Im(\xi)
\tag{4.1}
\]

is real analytic in $u$. Notice that any choice other than $\Re(\xi_{\text{latt}}) \neq -3\lambda/2$ would lead to the loss of real analyticity, that is a key property to get real flows. From the previous equation, we also see that each single entry of the transfer matrix is real for real $u$. This leads to the real analyticity for the whole matrix

\[
D(u, \xi) \equiv D_N^{1,1|3,1}(u, -\frac{3}{2} \lambda + i\frac{\xi}{5}) = (D_N^{1,1|3,1}(u^*, -\frac{3}{2} \lambda + i\frac{\xi}{5}))^*, \quad \xi \in \mathbb{R}.
\tag{4.2}
\]

The same property holds for the normalized transfer matrix. Combining real analyticity, periodicity and crossing symmetry and using the notation introduced in (2.24 2.25) we obtain the reality conditions for $x \in \mathbb{R}$

\[
\begin{align*}
D\left(\frac{\lambda}{2} + i\frac{x}{5}, \xi\right) &= (D\left(\frac{\lambda}{2} + i\frac{x}{5}, \xi\right))^*, \\
D(3\lambda + i\frac{x}{5}, \xi) &= (D(3\lambda + i\frac{x}{5}, \xi))^*, \\
t_j(x, \xi) &= t_j(x, \xi)^*, \quad j = 1, 2.
\end{align*}
\tag{4.3}
\]

The transfer matrix is transpose symmetric (2.10); with the previous equations, this implies that it is real symmetric so that its eigenvalues and the scaling energies are also real. The previous equations are also true for the corresponding eigenvalues. The integrable flow

\[
D_N^{1,1|3,1}(u, -\frac{3}{2} \lambda + i\frac{\xi}{5}) \to \begin{cases} 
\chi_{1,3} & \text{if } \xi \to \pm\infty \\
\chi_{2,1} & \text{if } \xi = 0
\end{cases}
\tag{4.4}
\]

thus provides a lattice description of the renormalization group flow $\chi_{1,3} \to \chi_{2,1}$. The two ranges $\xi \geq 0$ and $\xi \leq 0$ actually describe the same flow. For later convenience, we choose $\xi \leq 0$. 

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Table 6. Classification, for alternative realizations of (1, 3) and (2, 1) boundary conditions of the TIM not shown in Table 2 of the allowed patterns of 1- and 2-strings by \((m, n)\) system and quantum numbers. The parity \(\sigma = \pm 1\) occurs when there are frozen zeros. The parities \(s_1, s_2 = \pm 1\) occur in the TBA equations. The expressions for \(n_1\), only used on a finite lattice because in the scaling limit \(n_1 \sim N/2 \to \infty\). The number of faces in a row is even or odd according to \(N\) mod 2. The energy expression (2.34) still holds in these cases.

| \(\chi_{r,s}(q)\) | \((m, n)\) system | parities | quantum numbers |
|--------------------|------------------|----------|----------------|
| \(\chi_{1,3}^{(N)}(q)\) \(N \in \text{even}\) | \(m_1, m_2\) odd \(n_1 = (N + m_2 + \sigma)/2 - m_1\) \(n_2 = (m_1 - \sigma)/2 - m_2\) | \(s_1 = 1\) \(s_2 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2 - \sigma\) \(n_k^{(2)} = 2(I_k^{(2)} + m_2 - k) + 1 - m_1 + \sigma\) |
| \(\chi_{2,1}^{(N)}(q)\) \(N \in \text{even}\) | \(m_1\) odd, \(m_2\) even \(n_2 = (m_1 + 1)/2 - m_2\) \(n_1 = (N + m_2)/2 - m_1\) | \(s_1 = -1\) \(s_2 = 1\) | \(n_k^{(1)} = 2(I_k^{(1)} + m_1 - k) + 1 - m_2\) \(n_k^{(2)} = 2(I_k^{(2)} + m_2 - k) + 1 - m_1\) |

If we compare the transfer matrix under consideration with those of Sections 3.1 and 3.3, we see that we are using inequivalent lattice realizations of the same conformal characters. Indeed, the parities and boundary spins of the corresponding chains are different

\[
\chi_{1,3} \equiv \chi_{3,2} : \quad D_{1,1,3,1}^N(u, \pm i\infty) \neq D_{3,1,2,1}^{N'}(u, \pm i\infty)
\]

\[
\chi_{2,1} : \quad D_{1,1,3,1}^N(u, -\frac{3}{2}\lambda) \neq D_{2,1,1,1}^{N'}(u, \frac{1}{2})
\]  

(4.5)

\(N\) being even, \(N'\) odd, the left transfer matrices are used in the present section and the right ones were used in the constant \(r\) flows. This means that we cannot use the rules given in Table 2.

We used “numerics on \(D^r\)” to confirm the scaling behaviour in (4.4) and to classify the pattern of zeros appropriate for these lattice realizations. The results are shown in Table 6. We notice that for \(\chi_{1,3}(q)\) we obtain the same \((m, n)\) system as described in Table 2 after a shift of the parity of the chain \(N_{1,3} = N_{3,2} + 1\). So, we have two different transfer matrices that give rise to the same finitization of the conformal character. In contrast, for \(\chi_{2,1}(q)\) we obtain a new finitized character, built from a different \((m, n)\) system. We argue that this must be the case from TBA considerations. Indeed, we expect two flows related by duality to enter \(\chi_{2,1}\). Because of duality, we expect the same boundary terms in the TBA equations, so the only way to generate different flows is to start from different and inequivalent patterns of zeros.

4.1.1 Three mechanisms for changing the string content

It is again convenient to consider the reversed (unphysical) flow, from \(\text{IR} = \chi_{2,1}\) to \(\text{UV} = \chi_{1,3}\).

The pattern of zeros of \(D(u, \xi)\) changes along the flow: \(m_1\) remains odd while \(m_2\) changes from even to odd, according to Table 6. \(D(u, \xi)\) is a polynomial of order \(2N + 2\) for \(\xi > -\infty\) in the variable \(\exp(iu)\) inside the periodicity strip \(-\lambda < \text{Re}(u) \leq 4\lambda\) and becomes of order \(2N\) at the UV point itself. We therefore expect precisely two zeros to “escape to infinity” along the flow, one for each complex half plane \((\text{Im}(u) < 0\text{ or } \text{Im}(u) > 0)\). The “numerics on \(D^r\)” shows three active mechanisms that, for the most part, only involve the furthest zeros from the real axis in the second strip. They are distinct from the mechanisms described for the constant \(r\) flows but show many similarities. They were described in [16] for the flow \(\chi_{1,2}\) to \(\chi_{2,1}\), dual to the present case. More detailed “numerics on \(D^r\)” allows us to be more precise in describing the 2-string movements, as given in Figure 3 and Table 7.
Fig. 4. The three mechanisms A, B, C respectively that change string content during the flow $\chi_{2,1} \rightarrow \chi_{1,3}$ which is the reverse of the physical flow. These mechanisms are illustrated for the states: A, $(000|00) \rightarrow (000|0)_+$; B, $(111|00) \rightarrow (000|1)_-$; C, $(0) \rightarrow (0|0)_-$.

A. If the top object in both strips is a 1-string, the top 1-string in strip 2 moves to $+\infty$, decoupling from the system exactly at the UV point where it produces a frozen state with $\sigma = 1$.

B. If there is a top 2-string in strip 1 and a top 1-string in strip 2, they move away from the real axis towards $+\infty$. A 2-string comes in strip 2 becoming the top 2-string. Consequently, the quantum numbers in strip 1 decrease by 1 and those in strip 2 increase by 1.

C. If there is a top 2-string in strip 2, it moves away from the real axis and collapses into a pair of correlated 1-strings. The furthest of these 1-strings from the real axis moves to $+\infty$, reaching it precisely at the UV point. The 1-string closest to the real axis remains in the scaling region and becomes the top 1-string at the UV point. The quantum numbers in strip 2 thus decrease by 1.

These mechanisms are initially observed by the “numerics on $D$” then verified solving the TBA equations. For the mechanism B, the description of the movement of the 2-strings is still incomplete.

We can imagine a first scenario where the relevant 2-strings remain exactly located on the border of the corresponding strip so that they need to reach $+\infty$ to be able to move from one strip to the other. If this is the case, a quite unusual exchange between the 1 and 2 strings in the second strip must occur. It can be avoided in a second scenario where the 1-string only reaches $+\infty$; the zeros forming the 2-string leave the first strip by the two sides pointing toward the second strip where they become the top 2-string. If this is the case, to keep track of the movement of the 2-string we need to solve a much more complicated system of functional equations involving

$$
\hat{t}_1(x) \hat{t}_2(x + i\frac{3}{2}\pi) = 1 + \hat{t}_2(x - i\frac{\pi}{2}), \quad -\pi < \text{Im}(x) < 0.
$$

A 2-string located between the first and second strip can be fixed by $\hat{t}_2(x + i\frac{3}{2}\pi) = 0$ leading to

$$
1 + \hat{t}_2(x - i\frac{\pi}{2}) = 0, \quad -\frac{\pi}{2} < \text{Im}(x) < 0
$$

where both the real and imaginary parts of $x$ must be determined. This is equivalent to analytically continue $\hat{t}_1$, $\hat{t}_2$ out of the fundamental interval $[3.17]$. Actually, the TBA equations are completely independent of the behaviour of the 2-strings so we can still compute scaling functions even if we miss the complete description of the movement of the 2-strings.

The description of the mechanisms A,B,C forces the following changes for the various parameters (if no label IR or UV is used the corresponding parameter does not change during the flow):

A. $I_{m_1}^{(1)} = I_{m_2}^{(2)} = 0$:

$$
m_2^{\text{IR}} \rightarrow m_2^{\text{UV}} = m_2^{\text{IR}} - 1, \quad \sigma = 1.
$$
B. $I_{m_1}^{(1)\text{IR}} > 0$, $I_{m_2}^{(2)\text{IR}} = 0$:

\[
m_2^{\text{IR}} \mapsto m_2^{\text{UV}} = m_2^{\text{IR}} - 1, \quad \sigma = -1,
\]
\[
I_k^{(1)\text{IR}} \mapsto I_k^{(1)\text{UV}} = I_k^{(1)\text{IR}} - 1, \quad k = 1, \ldots, m_1,
\]
\[
I_k^{(2)\text{IR}} \mapsto I_k^{(2)\text{UV}} = I_k^{(2)\text{IR}} + 1, \quad k = 1, \ldots, m_2^{\text{IR}} - 1.
\]

(4.9)

C. $I_{m_2}^{(2)\text{IR}} > 0$:

\[
m_2^{\text{IR}} \mapsto m_2^{\text{UV}} = m_2^{\text{IR}} + 1, \quad \sigma = -1, \quad I_{m_2}^{(2)\text{UV}} = 0,
\]
\[
I_k^{(2)\text{IR}} \mapsto I_k^{(2)\text{UV}} = I_k^{(2)\text{IR}} - 1, \quad k = 1, \ldots, m_2^{\text{IR}}.
\]

(4.10)

The correlated 1-strings introduced here are in strip 2 while those introduced in Section 3.1.1 were in strip 1 (and connected to the frozen case).

Table 7. Flow $\chi_2,1 \mapsto \chi_1,3$ (reverse of the physical flow). We present the explicit mapping of states from IR to UV up to the UV level 6. Here $n^{\text{IR}}$, $n^{\text{UV}}$ are the excitation levels above the ground states, respectively $h = 7/16$ and $h = 3/5$.

| $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ | $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ |
|-----------------|--------------------------------|----------------|-----------------|--------------------------------|----------------|
| 0               | (0) $\mapsto$ (0|0|0)$_-$       | C              | 0               | (110|00) $\mapsto$ (110|0|0)$_+$  | A              | 4               |
| 1               | (1) $\mapsto$ (1|0|0)$_-$       | C              | 1               | (100) $\mapsto$ (100|0|0)$_-$    | C              | 5               |
| 2               | (2) $\mapsto$ (2|0|0)$_-$       | C              | 2               | (5) $\mapsto$ (5|0|0)$_-$         | C              | 5               |
| 3               | (000|00) $\mapsto$ (000|0|0)$_+$ | A              | 2               | (111|00) $\mapsto$ (000|1|0)$_-$ | B              | 5               |
| 3               | (3) $\mapsto$ (3|0|0)$_-$       | C              | 3               | (300|00) $\mapsto$ (300|0|0)$_+$ | A              | 5               |
| 4               | (100|00) $\mapsto$ (100|0|0)$_+$ | A              | 3               | (210|00) $\mapsto$ (210|0|0)$_+$ | A              | 5               |
| 4               | (000) $\mapsto$ (000|0|0)$_-$   | C              | 4               | (6) $\mapsto$ (6|0|0)$_-$         | C              | 6               |
| 4               | (4) $\mapsto$ (4|0|0)$_-$       | C              | 4               | (200) $\mapsto$ (200|0|0)$_-$    | C              | 6               |
| 5               | (200|00) $\mapsto$ (200|0|0)$_+$ | A              | 4               | (110) $\mapsto$ (110|0|0)$_-$    | C              | 6               |

4.1.2 RG mapping between finitized characters

In this section we show that the three mechanisms A,B,C are compatible with the counting of states, given by the finitized characters [22], at the two conformal endpoints of the flow.

From the definitions, we observe that for each IR state there is precisely one applicable mechanism so that the counting of states is complete. Moreover, using [A2], the IR finitized character...
naturally splits into three terms precisely associated with the three mechanisms

\[
\chi^{(N)}_{2,1}(q) = q^{\Delta_{2,1}-\frac{1}{2}} \sum_{m_1, m_2} q^{m_1} \sum_{A,B,C} q^{m_1} \sum_{m_1} \left[ m_{1+n_1}^{IR} \right] \left[ m_{2+n_2}^{IR} \right]
\]

Here and below we attach the labels IR, UV to the variables that change under the flow. The labels A,B,C on the sums indicate that the sums on \( m_1, m_2 \) are restricted by the constraints imposed by the corresponding mechanism. The sum constrained by A can be understood using (A.4) for both the \( q \)-binomial factors in (4.11) and corresponds to summing under the constraint \( I^{(1)}_{m_1} = I^{(2)}_{m_2} \). Similarly, the sum on B becomes apparent using (A.3) for the first strip \( q \)-binomial and (A.4) for the second strip \( q \)-binomial and likewise the sum on C uses (A.3) for the second strip only. The relations in Table 4 were used to rewrite the \( q \)-binomials as functions of \( m_1, m_2, N \) only.

We now use the properties of the three mechanisms to map this expression for the finitized IR character (4.11) into the finitized UV character. An IR energy level at the base of a tower of states with string content fixed by \((m_1, m_2)\) maps to a UV energy level according to the energy expression (2.38) that holds at the two conformal endpoints of the flow

A: \( q^{\Delta_{2,1}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{IR}Cm^{IR}} \quad \mapsto \quad q^{\Delta_{1,3}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{UV}Cm^{UV}} \quad q^{-\frac{1}{2}(m_1-m_2^{UV})} \)
B: \( q^{\Delta_{2,1}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{IR}Cm^{IR}} \quad q^{m_1} \quad \mapsto \quad q^{\Delta_{1,3}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{UV}Cm^{UV}} \quad q^{\frac{1}{2}(m_1-m_2^{UV})} m_2^{UV} \)
C: \( q^{\Delta_{2,1}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{IR}Cm^{IR}} \quad q^{m_2^{IR}} \quad \mapsto \quad q^{\Delta_{1,3}-\frac{1}{2}} q^{\frac{1}{2}m_{1}^{UV}Cm^{UV}} \quad q^{\frac{1}{2}(m_1-m_2^{UV})} \)

Using (4.8) to (4.10) we rewrite the \( q \)-binomials appearing in the last three lines of (4.11). Also
taking into account the mapping of the energies (4.12) we obtain

\[
\chi_{2,1}^{(N)}(q) \xrightarrow{\Delta_{1,3} - \frac{\lambda}{2}} \sum_A q^{+\Delta_{1,3} - \frac{\lambda}{2}} \left( \sum_B q^{+\Delta_{1,3} - \frac{\lambda}{2}} \frac{m UV C m UV}{q^2 (m_1 - m_2 UV)} + \frac{(N+1 UV - 1)/2}{m_1} - \frac{(m_1 - 1)/2}{m_2 UV} \right) + \sum_C q^{+\Delta_{1,3} - \frac{\lambda}{2}} \left( \sum_B q^{+\Delta_{1,3} - \frac{\lambda}{2}} \frac{m UV C m UV}{q^2 (m_1 - m_2 UV)} + \frac{(N+1 UV - 1)/2}{m_1} - \frac{(m_1 - 1)/2}{m_2 UV - 1} \right) \]

\[
= q^{+\Delta_{1,3} - \frac{\lambda}{2}} \sum_A q^{+\Delta_{1,3} - \frac{\lambda}{2}} \left( \sum_B q^{+\Delta_{1,3} - \frac{\lambda}{2}} \frac{m UV C m UV}{q^2 (m_1 - m_2 UV)} + \frac{(N+1 UV - 1)/2}{m_1} - \frac{(m_1 - 1)/2}{m_2 UV - 1} \right) \]

\[
= \chi_{1,3}^{(N)}(q) \tag{4.13}
\]

The last equality in (4.13) follows from the \(q\)-binomial identities in Appendix A. Again this argument shows the consistency of the IR and UV counting of states with the three mechanisms A,B,C.

### 4.1.3 Order 1 analyticity and TBA equations

The steps required to solve the functional equation for the scaling functions \(t_j(x, \xi)\) described in Sections 3.1.3 to 3.1.5 also apply for the flow \(\chi_{1,3} \to \chi_{2,1}\), the only differences being related to the boundary terms \(g_j\) and the order 1 analyticity contained in the factors \(S_1(u), S(u), Z(u, \xi)\) of (2.11). \(S_1(u)\) cancels with terms from \(B^{1,1}\) and disappears. Because of the factor \(S(u), t_1(x)\) and \(t_2(x)\) both have a double zero at \(x = 0\) and two simple poles at \(x = \pm i\pi\), independent of \(\xi\). We now have

\[
S_3(u, \xi) = \frac{\sin \lambda \sin(u + \lambda + \xi) \sin(u + 3\lambda - \xi) \cosh 2i\xi}{\sin(u + \xi) \sin(u + 2\lambda - \xi) \sin(u - \lambda - \xi) \sin(u + 2\lambda + \xi)} \tag{4.14}
\]

where \(\xi = -3\lambda/2 + i\xi/5\), so we find simple zeros at \(u = \lambda/2 + i\xi/5\), that are in the first strip at \(x = \pm \xi\), and simple poles at \(u = -\lambda/2 + i\xi/5\), \(3\lambda/2 \pm i\xi/5\) that are in the first strip at \(x = \pm i\pi + \xi\) (all four combinations of signs). Remarkably, the second strip is free of zeros and poles dependent on \(\xi\). This leads to the same plot as in Figure 3 with the exchange of the two strips. Observe that the indicated poles are on the edges of the strips and the zeros in the middle so, using (4.14), (4.16) we conclude that \(1 + t_1(x)\) and \(1 + t_2(x)\) are free of poles and are nonzero inside \(|\text{Im}(x)| < \pi/2\) (4.17). This implies that the order 1 factors can be removed from the functional system (3.15), (3.16) using

\[
g_1(x + i\pi/2) g_1(x - i\pi/2) = 1 \tag{4.15}
\]

\[
g_2(x + i\pi/2) g_2(x - i\pi/2) = 1 \tag{4.16}
\]

with the indicated analytic properties. This system has the same form as for the constant \(r\) flows (3.25), (3.26), the difference being in the analytic content. The solution is the previous one (3.27),
but with the exchange of the two strips
\[
g_1(x, \xi) = \tanh^2 \frac{x}{2} \tanh \frac{x - \xi}{2} \tanh \frac{x + \xi}{2}
\]
\[
g_2(x, \xi) = \tanh^2 \frac{x}{2}
\]
so the scaling limit yields
\[
\hat{g}_1(x, \xi) = \tanh \frac{x + \xi}{2}
\]
\[
\hat{g}_2(x, \xi) = 1.
\]
We are led to the same TBA equations (3.48), (3.49), energy expression (3.56) and quantization conditions (3.63) to (3.66) as in the constant \( r \) flows. Similarly, the expressions for the quantum numbers (3.75), (3.76) are not modified from the IR expressions so that
\[
m_1 \text{ odd}, \quad m_2 \text{ even}, \quad s_1 = -1, \quad s_2 = 1.
\]

As a complement to the given description of the mechanisms and to the TBA equations, we summarize the main results concerning this flow starting from the IR point:

**mechanism A and B, mechanism C before the collapse:**
\[
m_1 \text{ odd}, \quad m_2 \text{ even}, \quad n_2 = \frac{m_1+1}{2} - m_2 \geq 0, \quad s_1 = -1, \quad s_2 = 1,
\]
\[
n_k^{(1)} = 2(i_k^{(1)} + m_1 - k) + 1 - m_2,
\]
\[
n_k^{(2)} = 2(i_k^{(2)} + m_2 - k) + 1 - m_1;
\]

**mechanism C after the collapse:**
\[
m_2^A = m_2 + 2, \quad n_{m_2^A}^{(2)} = m_2^{(2)} = 1 - m_1,
\]
the values of \( m_1, n_1, n_k^{(1)} \), \( n_k^{(2)} \) with \( k < m_2^A - 1, s_j \), remain unchanged.

### 4.2 RG flow \( \chi_{1,2} \mapsto \chi_{2,1} \)

The preliminary treatment of this case was presented in the letter [16] but we now have a more complete understanding of the mechanisms of movement of the zeros and of the role played by \( \text{Re}(\xi_{\text{latt}}) \) along the flow so we need to correct and clarify some statements.

The analysis done in the previous section for the flow \( \chi_{1,3} \mapsto \chi_{2,1} \) can be repeated for this flow with \( D(u, \xi) \equiv D_{1,1,2,1}^N(u, \xi_{\text{latt}}) \) and \( \xi_{\text{latt}} = \frac{3}{2} \lambda + i\xi/5 \). According to (2.18) and (2.20), this choice of the boundary field gives an \( r = 2 \) type boundary for \( \xi = 0 \) and an \( s = 2 \) type boundary for \( \xi \to \pm \infty \). Moreover, this is the only value of \( \text{Re}(\xi_{\text{latt}}) \) that leads to real analytic transfer matrices and, consequently, to real positive energies. We need to analyse the order 1 analyticity. The relevant term is
\[
S_2(u, \xi_{\text{latt}}) = \frac{\sin \lambda \sin(\xi_{\text{latt}} - u - \lambda) \sin(\xi_{\text{latt}} + u + 3\lambda) \cosh 2\text{Im}(\xi_{\text{latt}})}{\sin(\xi_{\text{latt}} - u) \sin(\xi_{\text{latt}} - u - 2\lambda) \sin(\xi_{\text{latt}} + u - \lambda) \sin(\xi_{\text{latt}} + u + 2\lambda)}
\]
that has exactly the same zeros and poles obtained for the previous flow (4.14): in the first strip there are two single zeros at \( u = \lambda/2 \pm i\xi/5 \) (\( x = \pm \xi \)) and four single poles at \( u = -\lambda/2 \pm i\xi/5, u = 3\lambda/2 \pm i\xi/5 \) (\( x = \pm i\pi \pm \xi \)), with no zeros or poles in the second strip. This leads to the same TBA equations and the same boundary term
\[
\hat{g}_1(x, \xi) = \tanh \frac{x + \xi}{2}
\]
\[
\hat{g}_2(x, \xi) = 1
\]
Table 8. Flow $\chi_{2,1} \rightarrow \chi_{1,2}$ (reverse of the physical flow). We present the explicit mapping of states from IR to UV up to the UV level 6. Here $n^{\text{IR}}, n^{\text{UV}}$ are the excitation levels above the ground states, respectively $h = 7/16$ and $h = 1/10$.

| $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ | $n^{\text{IR}}$ | Mapping of states – mechanism | $n^{\text{UV}}$ |
|-----------------|-------------------------------|-----------------|-----------------|-------------------------------|-----------------|
| 0               | $ (0|0) \mapsto (0)_+$         | A               | 0               | $ (5|0) \mapsto (4)_-$         | B               |
| 1               | $ (1|0) \mapsto (0)_-$         | B               | 1               | $ (11|0) \mapsto (11)_+$      | A               |
| 2               | $ (2|0) \mapsto (1)_-$         | B               | 2               | $ (20|0) \mapsto (20)_+$      | A               |
| 3               | $ (3|0) \mapsto (2)_-$         | B               | 3               | $ (11|0) \mapsto (11|0)_-$    | C               |
| 4               | $ (00|0) \mapsto (00)_+$       | A               | 3               | $ (20|1) \mapsto (20|0)_-$    | C               |
| 4               | $ (00|1) \mapsto (00|0)_-$     | C               | 4               | $ (6|0) \mapsto (5)_-$        | B               |
| 4               | $ (4|0) \mapsto (3)_-$         | B               | 4               | $ (11|0) \mapsto (00)_-$      | B               |
| 4               | $ (10|0) \mapsto (10)_+$       | A               | 4               | $ (21|0) \mapsto (21)_+$      | A               |
| 5               | $ (10|1) \mapsto (10|0)_-$     | C               | 5               | $ (30|0) \mapsto (30)_+$      | A               |

obtained in Section 4.1, the only difference being in the following parameters that we take from [16]: $m_1, m_2$ are both odd, $s_1 = s_2 = -1$ as at the IR starting point.

The mechanisms were explained in [16] and the mapping between characters (see also Table 8)

$$\chi_{2,1}^{(N)}(q) \mapsto \chi_{1,2}^{(N)}(q)$$

(4.25)

was explicitly computed showing the consistency of the mechanisms with the IR and UV counting.

Notice that $\hat{t}_j(x)$ actually has no trace of the “pole” discussed erroneously there, instead the zero of $\hat{g}_1$ at $x = -\xi$ gives a zero to $\hat{t}_1(-\xi)$ and, correspondingly, a zero to $L_1(-\xi)$.

We summarize the main results concerning this flow starting from the IR point:

**mechanism A and B, mechanism C before the collapse:**

$$m_1, m_2 \text{ odd}, \quad n_2 = \frac{m_1 + 1}{2} - m_2 \geq 0, \quad s_1 = s_2 = -1,$$

$$n^{(1)}_k = 2(t^{(1)}_k + m_1 - k) + 1 - m_2,$$

$$n^{(2)}_k = 2(t^{(2)}_k + m_2 - k) + 1 - m_1;$$

**mechanism C after the collapse:**

$$m_A^2 = m_2 + 2, \quad n^{(2)}_{m_A^2 - 1} = n^{(2)}_{m_2} = 1 - m_1,$$

the values of $m_1, n_1, n^{(1)}_k, n^{(2)}_k$ with $k < m_A^2 - 1, s_j$, remain unchanged.

### 5 Numerical solution of TBA equations

The TBA equations (3.48), (3.49) can be solved analytically at the two conformal fixed points, which occur at the endpoints $\xi = \pm \infty$ of the flow, leading to the results in [20]. In particular, the energy expression (2.30) and the finitized characters can be obtained analytically. It appears that the TBA equations cannot be solved analytically along the flow, however, with numerical computations we can obtain in detail the behaviour of the flow for intermediate values of $\xi$ to interpolate between the known conformal fixed points.
Our numerical algorithm to solve the TBA equations and auxiliary equations for the zeros is iterative. We make suitable initial guesses for the pseudo-energies and the locations of the zeros close to one of the conformal fixed points. We then iteratively update in turn the pseudo-energies and locations of the zeros to find new values until the iteration scheme converges to the required accuracy. We then increment or decrement the value of the field $\xi$ and repeat the process until we scan the full range in $\xi$. This procedure is natural for the pseudo-energies in the TBA equations but iteration of the quantization conditions for the location of the zeros requires inversion of some complicated expressions associated with phases that wind along the flow. From the expression for $\psi_2$, we can extract the zero $y^{(1)}_\ell$ by one of the following:

**first strip algorithms:**
- **exp**: inversion of the exponential term $e^{-x}$;
- **g_2**: inversion of the boundary term, for the constant $r$ flows only;
- **$\psi_2$**: scan the domain of the function $\psi_2(x) - \pi n^{(1)}_\ell$ to find its zero;
- **phase**: inversion of one of the phases $i \log \tanh(y^{(2)}_\ell - y^{(1)}_\ell + \frac{i \pi}{4})$ (not used here).

Similarly, from the expression for $\psi_1$, we can extract the zero $y^{(2)}_\ell$ by one of the following:

**second strip algorithms:**
- **g_1**: inversion of the boundary term, for the variable $r$ flows only;
- **$\psi_1$**: scan the domain of the function $\psi_1(x) - \pi n^{(2)}_\ell$ to find its zero;
- **phase**: inversion of one of the phases $i \log \tanh(y^{(1)}_\ell - y^{(2)}_\ell + \frac{i \pi}{4})$.

We start with the simplest choice, algorithm **exp, phase** in strips 1, 2 respectively. The computations at the critical points in $\mathbb{R}$ can all be done with this choice and do not require the introduction of other algorithms. We emphasize that these different algorithms are not equivalent under iteration and it can happen that some schemes fail to converge in some intervals. In every case, however, the validity of our use of different algorithms in different intervals is confirmed by the fact that the plots of the scaling energies, and the locations of all of their associated zeros, join smoothly in some typically small overlapping interval of convergence.

The **contraction mapping theorem**, says that if a mapping $f : M \rightarrow M$ on a complete metric space $M$ is a contraction then there exists a unique fixed point $x_0 = f(x_0)$ and all the sequences obtained under iteration starting from an arbitrary initial point $x \in M$ converge to the fixed point. For our purposes, we take $M \subseteq \mathbb{R}$ to be a closed interval and a contraction means that the first derivative is $|f'(x)| \leq k < 1$ inside $M$, for a fixed $k$. In particular, $|f'(x_0)|$ is a measure of the rate of convergence. We will see that, in some cases, $|f'(x_0)| \gg 1$ and although the fixed point exists it cannot be reached by iterating the mapping $f$.

5.1 Flow $\chi_{1,2} \mapsto \chi_{1,1}$

The scaling energies for the states in Table 3 are presented in Figure 5. The second strip is a spectator and is not affected by mechanisms A and B in (3.9), (3.10), so it always has $m_2$ even and is not relevant in the following discussion. As usual, for numerical convenience, we describe the reverse of the physical flow.

From Table 3 we begin by following the simpler **mechanism B** that first occurs in the excitation $(00) \mapsto (0)_-$ and has only one strip of zeros. We start decreasing $\xi$ from the IR point $\xi \rightarrow +\infty$ with algorithm **exp** and make the following observations:

- the numerical iteration converges in the interval $\xi \gtrsim -2$ but fails for smaller values;
Fig. 5. Scaling energies for the flow $\chi_{1,2} \mapsto \chi_{1,1}$. The list of states is given in Table 3. The intermediate region of the mechanism A levels (shown dashed) are schematic and have not been obtained from the solution of the TBA equations.
the zero \( y_2^{(1)} \) moves towards \(+\infty\), consistent with the lattice description of mechanism B while \( y_1^{(1)} \) moves slowly in the finite region, as shown in Figure 6.

If we denote by \( f(x) \) the mapping obtained by inverting the exponential term \( e^{-x} \) in (3.66) to solve for \( x \) and follow the value of \( f'(y_2^{(1)}) \) as we vary \( \xi \) we see that \( f'(y_2^{(1)}) \lesssim 0.1 \) for large \( \xi \gg 0 \) but \( f'(y_2^{(1)}) \to -\infty \) for \( \xi \to -\infty \) while it remains bounded for the other zero, \( |f'(y_1^{(1)})| \lesssim 0.1 \), for all values of \( \xi \). This is a clear indication that the zero \( y_2^{(1)} = y_{m_1}^{(1)} \) cannot be reached by iteration. We certainly know that it exists because in the whole interval \( \xi \gtrsim -2 \) we can track it and at \( \xi = -2 \) it is at a finite position, \( y_2^{(1)} \approx 1.49 \). An analytic estimation of \( f'(y_2^{(1)}) \) shows that the dominant behaviour is \( f'(y_2^{(1)}) \sim -e^{-y_2^{(1)}} \) if \( y_2^{(1)} \) is sufficiently large, as we expect in the region \( \xi \to -\infty \), see also Figure 6. These considerations show that we need a different algorithm to determine the location of this zero.

If we try with algorithm \( g_2 \) we observe the opposite behaviour for the mapping \( f(x) \) obtained by inverting the boundary term \( \log g_2 \) in (3.66): \( f'(y_2^{(1)}) \to -\infty \) for large \( \xi \gg 0 \) and \( f'(y_2^{(1)}) \) is small for \( \xi \lesssim -1.5 \) so we can start from the UV point and combine the results from this algorithm with the previous ones, thanks to the existence of a small overlapping region of applicability.

Actually, we find that the algorithm \( \psi_2 \) is more general in the sense that it can be applied everywhere, being intrinsically non-iterative. We always use this choice for the largest first strip zero \( y_{m_1}^{(1)} \) (the other first strip zeros can be fixed with the slightly faster algorithm \( \text{exp} \)). A similar problem shows up even in the second strip for higher excited states so we use the non-iterative algorithm \( \psi_1 \) for all the second strip zeros \( y_k^{(2)}, k = 1, \ldots, m_2 \).

Once the appropriate algorithms are applied, the numerical analysis of the states proceeds smoothly from IR to UV and the predictions of mechanism B are completely confirmed. In particular, the IR description of zeros and quantum numbers can be used all along the flow and \( y_{m_1}^{(1)} \) only reaches \(+\infty\) at the UV critical point.

Fig. 6. Movement of the zeros versus \( \xi \) for the state \((000|00)_- \mapsto (0000|00)\), described by mechanism B, in the flow \( \chi_{1,2} \mapsto \chi_{1,1} \). Solid lines from bottom to top are \( y_1^{(1)}, \ldots, y_4^{(1)} \), dashed lines from bottom to top are \( y_1^{(2)}, y_2^{(2)} \). This picture is consistent with the lattice predictions in part B of Fig. 2.
Fig. 7. Scaling energy and movement of the zeros for the state \((0)_+ \mapsto ()\), involving mechanism A, in the flow \(\chi_{1,2} \mapsto \chi_{1,1}\). In the lower plot, the solid line corresponds to the position of the correlated 1-strings and the dashed one to the top 2-string in strip 1.

The mechanism A requires some care. We examine the ground state ( ), as in Table 3 starting in the IR where there are no zeros and decrease \(\xi\). This procedure must fail somewhere because we expect the top 2-string in strip 1 to transform into a pair of correlated 1-strings, from Figure 2. Keeping track of the top 2-string (with algorithm \(\text{exp}\)) we see that the TBA equations reproduce exactly the movement toward infinity, as shown in Figure 7 up to a value \(\xi \gtrsim -6.5\). We next start from the other end of the flow, the UV point, now increasing \(\xi\) from \(-\infty\). There and only there, one of the correlated 1-strings sits exactly at \(+\infty\) and is essentially decoupled from the system, restoring the correct odd parity of \(m_1\). Away from the UV point we need to consider both of the correlated 1-strings, so now \(m_1 = 2\). To get the code working, we need to use the algorithm \(g_2\) for the largest zero \(y_{2}^{(1)}\) (\(y_{m_1}^{(1)}\) in general) and use the usual algorithm \(\text{exp}\) for the other zero \((k = 1, \ldots, m_1 - 1\), in general). We reproduce the expected behaviour, as shown in Figure 7 for \(\xi \lesssim -9.5\).

In the interval \(\xi \in (-9.5, -6.5)\) containing the collapse region the TBA equations are difficult to solve numerically. Indeed, the indicated zeros are not in the standard positions; in terms of \(u\) their real part is not fixed and must be determined as well as the imaginary part. In solving (3.63) to obtain the position of the two 1-strings for \(\xi \lesssim -9.5\) we see they have the same quantum number \(n_{1}^{(1)} = n_{2}^{(1)} = 1\) as they had when they were a 2-string (\(\xi \gtrsim -6.5\)). This explains the name correlated 1-strings because they are on a different footing from the ordinary 1 and 2-strings. At
criticality, the furthest among them is at \(+\infty\) and the closest is the so called frozen zero already observed in \([20]\). The monotonic decreasing behaviour of the function \(\psi_2(x)\) assumed to introduce the quantum numbers is observed to fail in the present case, in a neighbourhood of the furthest zero among the correlated 1-strings. In this way the space to allocate two degenerate quantum numbers is created. We see the same behaviour in all the mechanism A states.

We have a consistency check. At the critical point \(\chi_{1,2}\), using the data in Table \([2]\) we have \(m_1\) odd, \(m_2\) even, \(n_k^{(1)}\) even and \(n_k^{(2)}\) odd. Comparing with the the limits \((3.72)\), we have agreement only if \(\psi_j(+\infty) = -\pi m_j\) and \(\psi_j(j) < +\infty\). For the first strip, \(\psi_2(+\infty) = -2\pi\) is an even multiple of \(\pi\) and \(n_k^{(1)}\) is even so, in addition to the known zeros labelled 1, \ldots, \(m_1\), we have room for another zero located at \(+\infty\). This zero was not counted in \([20]\) because it is “almost” decoupled from the system but now we know exactly its origin and its role from the off critical behaviour. In the mechanism A case, that is the frozen case \((\sigma = +1)\), this zero is the partner of the frozen one in the correlated 1-strings; in the mechanism B case, corresponding to the unfrozen case \((\sigma = -1)\), it is the zero that left the finite region; in this case it is a normal 1-string.

Repeating the previous argument for the second strip, it seems that we have room for an additional zero. Actually, it is only there at the conformal point \(\chi_{1,2}\); during the flow it cannot exist either at \(+\infty\) or at a finite location so we don’t need to consider its presence at all.

5.2 Flows \(\chi_{3,2} \leftrightarrow \chi_{3,1}\) and \(\chi_{2,2} \leftrightarrow \chi_{2,1}\)

These flows behave similarly to \(\chi_{1,2} \leftrightarrow \chi_{1,1}\) described in the previous section and the algorithms to solve the TBA system are exactly the same. In Figures \([8]\) and \([9]\) we plot the scaling energies for the states in Tables \([4]\) and \([5]\) respectively. They are complete up to the top level in each plot.

A comment for the second of these flows is useful. At the IR point and all along the flow except at the UV point, the value of \(n_k^{(2)}\) is even and, in addition, \(\psi_1(\infty) = 0\) is even \((3.71)\) so that at \(x = -\infty\) there can be a zero \(y_1^{(2)}\). Actually, a 1-string occurs whenever its quantum number equals the number of 2-strings in the same strip, \(I_1^{(2)} = n_2\). In this case, the zero sticks to its position for all values of \(\xi\) and the TBA equations can be slightly simplified taking the appropriate limit.

5.3 Flows \(\chi_{1,3} \leftrightarrow \chi_{2,1}\) and \(\chi_{1,2} \leftrightarrow \chi_{2,1}\)

We consider first \(\chi_{1,3} \leftrightarrow \chi_{2,1}\), displayed in Figure \([10]\). It doesn’t present any new difficulties with respect to the description in Section \([5.2]\). In this case, we have three algorithms for each strip to get the position of the zeros and we need to choose those that converge in the appropriate regimes and depending on the mechanism.

In general the first strip zeros labelled by 1, \ldots, \(m_1 - 1\) do not require special care and the simplest algorithm \(exp\) can be applied to them. So, in the following discussion, we refer just to the remaining first strip zero \(m_1\) and the second strip zeros. In general, they can be determined by the algorithms \(\psi_j, j = 1, 2\) except for a few exceptions.

Considering mechanism A, we just have a zero escaping to \(+\infty\) which causes no problems for the numerical solution. The flow of these states can be followed without interruptions from IR to UV.

Considering mechanism B, we start from the IR and observe that the movement of the 2-strings depicted in Figure \([4]\) doesn’t influence the TBA equations, as expected because they are just spectators; in fact the scaling energies are determined by the 1-string dynamics only. The zero moving toward infinity, \(y_1^{(2)}\), is most conveniently determined by inverting the boundary term \(\hat{g}_1\).

Considering mechanism C, we start from the IR and we expect the iteration to fail because new objects must appear in the region where the transformation of the zeros depicted in Figure \([11]\) occurs, namely \(-9.5 \leq \xi \leq -6.5\). Data for smaller values \(\xi \lesssim -9.5\) can be obtained by starting
Fig. 8. Scaling energies for the flow $\chi_{3,2} \mapsto \chi_{3,1}$. The list of states is given in Table 4. The intermediate region of the mechanism A levels (shown dashed) are schematic and have not been obtained from the solution of the TBA equations.
Fig. 9. Scaling energies for the flow $\chi_{2,2} \mapsto \chi_{2,1}$. The list of states is given in Table 5. The intermediate region of the mechanism A levels (shown dashed) are schematic and have not been obtained from the solution of the TBA equations.
Fig. 10. Scaling energies for the flow $\chi_{1,3} \rightarrow \chi_{2,1}$. The list of states is given in Table 7. The intermediate region of the mechanism $C$ levels (shown dashed) are schematic and have not been obtained from the solution of the TBA equations.
Fig. 11. Scaling energies for the flow $\chi_{1,2} \mapsto \chi_{2,1}$. The list of states is given in Table 8. The intermediate region of the mechanism C levels (shown dashed) are schematic and have not been obtained from the solution of the TBA equations.
from the UV point and including the zero at infinity, \( y_{m^{2}R+2}^{(2)} = +\infty \). Again, this particular zero is most conveniently determined by inverting the boundary term \( \hat{g}_1 \).

A preliminary discussion of the flow \( \chi_{1,2} \leftrightarrow \chi_{2,1} \) was given in [16]. It is very similar to the flow \( \chi_{1,3} \leftrightarrow \chi_{2,1} \). For completeness, we present our current, more accurate numerical results in Fig. 11.

### 6 Conclusions

In this paper we have used a lattice approach to derive exact TBA equations for all excitations of the 5 \( \varphi_{1,3} \) integrable boundary flows of the tricritical Ising model. We have shown that, along these boundary flows, the patterns of zeros classifying the states can change by one of 2 or 3 mechanisms which have been explicitly identified. These mechanisms produce precise mappings between the relevant finitized characters describing the finitized energy spectra at the conformal UV and IR fixed points. The TBA equations were also solved numerically to determine the interpolating boundary flows for the leading excitations.

Even for the TIM there remain other questions of interest. First, it is desirable to have a comparison of our results with the results of the TCSA. The methods used here could usefully be generalized to study other flows, such as \( \chi_{2,2}(q) \leftrightarrow \chi_{1,1}(q) + \chi_{3,1}(q) \), which involve linear combinations of Virasoro characters. It would also be of interest to study the missing 2 \( \varphi_{1,2} \) integrable boundary flows, but presumably, this would involve a study of the relevant dilute A lattice model. Likewise, our approach should be applied to the boundary flows of the TIM from the superconformal perspective, especially since the integrable boundary conditions on the lattice that correspond to the superconformal boundary conditions in the continuum scaling limit are already known [23]. Similarly, it remains to incorporate the flow of boundary entropies into our approach. Ultimately, of course, the considerations of this paper should be generalized to all the minimal models.

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### Appendix A. Gaussian polynomials

The expressions for the finitized characters are based on the following relations for the \( q \)-binomials:

\[
\begin{align*}
\binom{m+n}{m} &= \sum_{I_1=0}^{n} \sum_{I_2=0}^{I_1} \cdots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\cdots+I_m} \\
\binom{n}{m} &= \binom{n-1}{m-1} + q^m \binom{n-1}{m}
\end{align*}
\]  

(A.1)

(A.2)

The following expressions are useful to understand the contribution of each mechanism to the full character. Indeed, the restriction \( I_m > 0 \) leads to

\[
\sum_{I_1=1}^{n} \sum_{I_2=1}^{I_1} \cdots \sum_{I_m=1}^{I_{m-1}} q^{I_1+\cdots+I_m} = q^n \sum_{I_1'=0}^{n-1} \sum_{I_2'=0}^{I_1'} \cdots \sum_{I_m'=0}^{I_{m-1}'} q^{I_1'+\cdots+I_m'} = q^n \binom{m+n-1}{m}
\]  

(A.3)
whereas the restriction $I_m = 0$ leads to
\[
\sum_{I_1=0}^{n} \sum_{I_2=0}^{I_1} \ldots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\ldots+I_{m-1}} = \left[ \frac{m-1+n}{m-1} \right].
\] (A.4)

Appendix B. Braid limit

In this section we consider the braid limit of the normalized transfer matrix and its eigenvalues. By direct computation, we obtain the following limits:

\[
\lim_{\text{Im}(u) \to \pm \infty} e^{\pm 4iu} \sum_g \begin{bmatrix} r, r, \ldots, r \end{bmatrix} = \lim_{\text{Im}(u) \to \pm \infty} e^{\pm 2iu} e^{\pm i\lambda} \frac{\delta_{bd}}{4\sin^2 \lambda},
\]
(B.1)

\[
\lim_{\text{Im}(u) \to \pm \infty} e^{\pm 4iu} \sum_g \begin{bmatrix} r, r, r, \ldots, r \end{bmatrix} = \frac{e^{\pm 2i\lambda} \cos \lambda}{8\sin^2 \lambda \sin 2\xi_1 \sin 2\xi_2}.
\]
(B.2)

We can recursively apply the first result to the normalized transfer matrix to remove all the columns then use the second expression to take care of the boundary terms. This leads to the conclusion that $t(u, \xi_1, \xi_2)$ is diagonal and the actual value of the limit is the same for all eigenvalues and all integrable boundary conditions, that is to say that it is proportional to the identity. For a generic eigenvalue we thus have

\[
\lim_{\text{Im}(u) \to \pm \infty} t(u, \xi_1, \xi_2) = 2 \cos \lambda = \frac{1 + \sqrt{5}}{2}.
\]
(B.3)

In [20] the same expression was boundary dependent because of different normalizations in the right boundary term.

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