ACTION OF THE CREMONA GROUP ON FOLIATIONS ON \( \mathbb{P}^2_C \): SOME CURIOUS FACTS

by

Dominique CERVEAU & Julie DÉSERTI

Abstract. — The Cremona group of birational transformations of \( \mathbb{P}^2_C \) acts on the space \( \mathcal{F}(2) \) of holomorphic foliations on the complex projective plane. Since this action is not compatible with the natural graduation of \( \mathcal{F}(2) \) by the degree, its description is complicated. The fixed points of the action are essentially described by Cantat-Favre in [3]. In that paper we are interested in problems of "aberration of the degree" that is pairs \( (\phi, \mathcal{F}) \in \text{Bir}(\mathbb{P}^2_C) \times \mathcal{F}(2) \) for which \( \deg \phi^* \mathcal{F} < (\deg \mathcal{F} + 1)\deg \phi + \deg \phi - 2 \), the generic degree of such pull-back. We introduce the notion of numerical invariance \( (\deg \phi^* \mathcal{F} = \deg \mathcal{F}) \) and relate it in small degrees to the existence of transversal structure for the considered foliations.

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1. Introduction

Let us consider on the complex projective plane \( \mathbb{P}^2_C \) a foliation \( \mathcal{F} \) of degree \( d \) and a birational map \( \phi \) of degree \( k \). If the pair \( (\mathcal{F}, \phi) \) is generic then \( \deg \phi^* \mathcal{F} = (d + 1)k + k - 2 \). For example if \( \mathcal{F} \) and \( \phi \) are both of degree 2, then \( \phi^* \mathcal{F} \) is of degree 6. Nevertheless one has the following statement which says that "aberration of the degree" is not exceptional:

Theorem A. — For any foliation \( \mathcal{F} \) of degree 2 on \( \mathbb{P}^2_C \) there exists a quadratic birational map \( \psi \) of \( \mathbb{P}^2_C \) such that \( \deg \psi^* \mathcal{F} \leq 3 \).

Holomorphic singular foliations on compact complex projective surfaces have been classified up to birational equivalence by Brunella, McQuillan and Mendes ([11]). Let \( \mathcal{F} \) be a holomorphic singular foliation on a compact complex projective surface \( S \). Let Bir(\( \mathcal{F} \)) (resp. Aut(\( \mathcal{F} \))) denote the group of birational (resp. biholomorphic) maps of \( S \) that send leaf to leaf. If \( \mathcal{F} \) is of general type, then Bir(\( \mathcal{F} \)) = Aut(\( \mathcal{F} \)) is a finite group. In [3] Cantat and Favre classify the pairs \( (\mathcal{S}, \mathcal{F}) \) for which Bir(\( \mathcal{F} \)) (resp. Aut(\( \mathcal{F} \))) is infinite; in the case of \( \mathbb{P}^2_C \) such foliations are given by closed rational 1-forms.

In this article we introduce a weaker notion: the numerical invariance. We consider on \( \mathbb{P}^2_C \) a pair \( (\mathcal{F}, \phi) \) of a foliation \( \mathcal{F} \) of degree \( d \) and a birational map \( \phi \) of degree \( k \geq 2 \). The foliation \( \mathcal{F} \) is numerically invariant under the action of \( \phi \) if \( \deg \phi^* \mathcal{F} = \deg \mathcal{F} \). We characterize such pairs \( (\mathcal{F}, \phi) \) with \( \deg \mathcal{F} = \deg \phi = 2 \) which

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is the first degree with deep (algebraic and dynamical) phenomena, both for foliations and birational maps. We prove that a numerically invariant foliation under the action of a generic quadratic map is special:

**Theorem B.** — Let $\mathcal{F}$ be a foliation of degree 2 on $\mathbb{P}^2_C$ numerically invariant under the action of a generic quadratic birational map of $\mathbb{P}^2_C$. Then $\mathcal{F}$ is transversely projective.

In that statement generic means outside an hypersurface in the space $\text{Bir}_2$ of quadratic birational maps of $\mathbb{P}^2_C$.

For any quadratic birational map $\phi$ of $\mathbb{P}^2_C$ there exists at least one foliation of degree 2 on $\mathbb{P}^2_C$ numerically invariant under the action of $\phi$ and we give "normal forms" for such foliations. We don’t know if the foliations numerically invariant under the action of a non-generic quadratic birational map have a special transversal structure. Problem: for any birational map $\phi$ of degree $d \geq 3$, does there exist a foliation numerically invariant under the action of $\phi$?

A foliation $\mathcal{F}$ on $\mathbb{P}^2_C$ is primitive if $\deg \mathcal{F} \leq \deg \phi^* \mathcal{F}$ for any birational map $\phi$. Foliations of degree 0 and 1 are defined by a rational closed 1-form (it is a well-known fact, see for example [2]). Hence a non-primitive foliation of degree 2 is also defined by a closed 1-form that is a very special case of transversely projective foliations. Generically a foliation of degree 2 is primitive. The following problem seems relevant: classify in any degree the primitive foliations numerically invariant under the action of birational maps of degree $\geq 2$; are such foliations transversely projective or is this situation specific to the degree 2 ? In this vein we get the following statement.

**Theorem C.** — A foliation $\mathcal{F}$ of degree 2 on $\mathbb{P}^2_C$ numerically invariant under the action of a generic cubic birational map of $\mathbb{P}^2_C$ satisfies the following properties:

- $\mathcal{F}$ is given by a closed rational 1-form (Liouvillian integrability);
- $\mathcal{F}$ is non-primitive.

Is it a general fact, i.e. if $\mathcal{F}$ is numerically invariant under the action of $\phi$ and $\deg \phi \gg \deg \mathcal{F}$ is $\mathcal{F}$ Liouvillian integrable ?

The text is organized as follows: we first give some definitions, notations and properties of birational maps of $\mathbb{P}^2_C$ and foliations on $\mathbb{P}^2_C$. In §3 we give a proof of Theorem A, we focus on foliations of degree 2 on $\mathbb{P}^2_C$ that have at least two singular points and then on foliations of degree 2 on $\mathbb{P}^2_C$ with exactly one singular point. The section §4 is devoted to the description of foliations of degree 2 on $\mathbb{P}^2_C$ numerically invariant under the action of any quadratic birational map. This allows us to prove Theorem B. At the end of the paper, §5 we describe the foliations of degree 2 numerically invariant under some cubic birational maps of $\mathbb{P}^2_C$ and establish Theorem C.

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2. Some definitions, notations and properties

2.1. About birational maps of $\mathbb{P}^2_C$. — A rational map $\phi$ of $\mathbb{P}^2_C$ is a "map" of the type

$$\phi: \mathbb{P}^2_C \longrightarrow \mathbb{P}^2_C, \hspace{1cm} (x : y : z) \longrightarrow (\phi_0(x,y,z) : \phi_1(x,y,z) : \phi_2(x,y,z))$$

where the $\phi_i$’s are homogeneous polynomials of the same degree and without common factor. The degree of $\phi$ is by definition the degree of the $\phi_i$’s. A birational map $\phi$ of $\mathbb{P}^2_C$ is a rational map having a rational
"inverse" \( \psi \), i.e. \( \phi \circ \psi = \psi \circ \phi = \text{id} \). The first examples are the birational maps of degree 1 which generate the group \( \text{Aut}(\mathbb{P}_2^2) = \text{PGL}(3, \mathbb{C}) \). Let us give some examples of quadratic birational maps:

\[
\begin{align*}
\sigma &: (x : y : z) \rightarrow (yz : xz : xy), \\
\rho &: (x : y : z) \rightarrow (xy : z^2 : yz), \\
\tau &: (x : y : z) \rightarrow (x^2 : xy : y^2 - xz).
\end{align*}
\]

These three maps, which are involutions, play an important role in the description of the set of quadratic birational maps of \( \mathbb{P}_2^2 \).

The birational maps of \( \mathbb{P}_2^2 \) form a group denoted \( \text{Bir}(\mathbb{P}_2^2) \) and called the Cremona group. If \( \phi \) is an element of \( \text{Bir}(\mathbb{P}_2^2) \) then \( \mathcal{O}(\phi) \) is the orbit of \( \phi \) under the action of \( \text{Aut}(\mathbb{P}_2^2) \times \text{Aut}(\mathbb{P}_2^2) \):

\[
\mathcal{O}(\phi) = \{ \ell \phi' | \ell, \ell' \in \text{Aut}(\mathbb{P}_2^2) \}.
\]

A very old theorem, often called Noether Theorem, says that any element of \( \text{Bir}(\mathbb{P}_2^2) \) can be written, up to the action of an automorphism of \( \mathbb{P}_2^2 \), as a composition of quadratic birational maps (\cite{4}). In \cite{5} Chapters 1 & 6 the structure of the set \( \text{Bir}_d \) (resp. \( \text{Bir}_d \)) of birational maps of \( \mathbb{P}_2^2 \) of degree \( \leq d \) (resp. \( d \)) has been studied when \( d = 2 \) and \( d = 3 \).

**Theorem 2.1** (Corollary 1.10, Theorem 1.31, \cite{5}). — One has the following decomposition

\[
\text{Bir}_2 = \mathcal{O}(\sigma) \cup \mathcal{O}(\rho) \cup \mathcal{O}(\tau).
\]

Furthermore

\[
\text{Bir}_2 = \overline{\mathcal{O}(\sigma)}
\]

where \( \mathcal{O}(\sigma) \) denotes the ordinary closure of \( \mathcal{O}(\sigma) \), and

\[
\dim \mathcal{O}(\sigma) = 12, \quad \dim \mathcal{O}(\rho) = 13, \quad \dim \mathcal{O}(\tau) = 14.
\]

Note that there is a more precise description of \( \text{Bir}_3 \) in \cite{5} Chapter 1.

We will further do some computations with birational maps of degree 3. Let us consider the following family of cubic birational maps:

\[
\Phi_{a,b}: (x : y : z) \rightarrow (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)
\]

with \( a, b \in \mathbb{C}, a^2 \neq 4 \) and \( 2b \notin \{a \pm \sqrt{a^2 - 4}\} \). The structure of the set of cubic birational maps is much more complicated (\cite{5} Chapter 6), nevertheless one has the following result.

**Theorem 2.2** (Proposition 6.35, Theorem 6.38, \cite{5}). — The closure of

\[
\mathcal{X} = \{ \mathcal{O}(\Phi_{a,b}) | a, b \in \mathbb{C}, a^2 \neq 4, 2b \notin \{a \pm \sqrt{a^2 - 4}\} \}
\]

in the set of rational maps of degree 3 is an irreducible algebraic variety of dimension 18.

Furthermore the closure of \( \mathcal{X} \) in \( \text{Bir}_3 \) is \( \text{Bir}_3 \).

The "most degenerate model" \cite{1} is up to automorphisms of \( \mathbb{P}_2^2 \)

\[
\Psi: (x : y : z) \rightarrow (xz^2 + y^3 : yz^2 : z^3).
\]

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1. In the following sense: for any \( \phi \) in \( \text{Bir}_3 \) the following inequality holds: \( \dim \mathcal{O}(\phi) \geq \dim \mathcal{O}(\Psi) = 13 \).

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2.2. About foliations.

Definition 2.3. — Let $\mathcal{F}$ be a foliation (maybe singular) on a complex manifold $M$; the foliation $\mathcal{F}$ is a singular transversely projective one if there exists

a) $\pi: P \to M$ a $\mathbb{P}^1$-bundle over $M$,

b) $\mathcal{G}$ a codimension one singular holomorphic foliation on $P$ transversal to the generic fibers of $\pi$,

c) $\varsigma: M \to P$ a meromorphic section generically transverse to $\mathcal{G}$, such that $\mathcal{F} = \varsigma^* \mathcal{G}$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_C$, assume that there exist three rational 1-forms $\theta_0, \theta_1$ and $\theta_2$ on $\mathbb{P}^2_C$ such that

i) $\mathcal{F}$ is described by $\theta_0$, i.e. $\mathcal{F} = \mathcal{F}_{\theta_0}$,

ii) the $\theta_i$’s form a $\mathfrak{sl}(2;\mathbb{C})$-triplet, that is

$$d\theta_0 = \theta_0 \wedge \theta_1, \quad d\theta_1 = \theta_0 \wedge \theta_2, \quad d\theta_2 = \theta_1 \wedge \theta_2.$$  

Then $\mathcal{F}$ is a singular transversely projective foliation. To see it one considers the manifolds $M = \mathbb{P}^2_C, P = \mathbb{P}^2_C \times \mathbb{P}^1_C$, the canonical projection $\pi: P \to M$ and the foliation $\mathcal{G}$ given by

$$\theta = dz + \theta_0 + z\theta_1 + \frac{z^2}{2}\theta_2$$

where $z$ is an affine coordinate of $\mathbb{P}^1_C$, in that case the transverse section is $z = 0$. When one can choose the $\theta_i$’s such that $\theta_1 = \theta_2 = 0$ (resp. $\theta_2 = 0$) the foliation $\mathcal{F}$ is defined by a closed 1-form (resp. is transversely affine).

Classical examples of singular transversely projective foliations are given by Riccati foliations.

Definition 2.4. — A Riccati equation is a differential equation of the following type

$$\mathcal{E}_R: y' = a(x)y^2 + b(x)y + c(x)$$

where $a$, $b$ and $c$ are meromorphic functions on an open subset $\mathcal{U}$ of $\mathbb{C}$. To the equation $\mathcal{E}_R$ one associates the meromorphic differential form

$$\omega_{x_R} = dy - \left( a(x)y^2 + b(x)y + c(x) \right) dx$$

defined on $\mathcal{U} \times \mathbb{C}$. In fact $\omega_{x_R}$ induces a foliation $\mathcal{F}_{\omega_{x_R}}$ on $\mathcal{U} \times \mathbb{P}^1_C$ that is transverse to the generic fiber of the projection $\mathcal{U} \times \mathbb{P}^1_C \to \mathcal{U}$. One can check that

$$\theta_0 = \omega_{x_R}, \quad \theta_1 = - (2a(x)y + b(x)) dx, \quad \theta_2 = - 2a(x) dx$$

is a $\mathfrak{sl}(2;\mathbb{C})$-triplet associated to the foliation $\mathcal{F}_{\omega_{x_R}}$.

We say that $\omega_{x_R}$ is a Riccati 1-form and $\mathcal{F}_{\omega_{x_R}}$ is a Riccati foliation.

Let $S$ be a ruled surface, that is a surface $S$ endowed with $f: S \to C$, where $C$ denotes a curve and $f^{-1}(c) \simeq \mathbb{P}^1_C$. Let us consider a singular foliation $\mathcal{F}$ on $S$ transverse to the generic fibers of $f$. The foliation $\mathcal{F}$ is transversely projective.

Recall that a foliation $\mathcal{F}$ is radial at a point $m$ of the surface $M$ if in local coordinates $(x,y)$ around $m$ the foliation $\mathcal{F}$ is given by a holomorphic 1-form of the following type

$$\omega = x dy - y dx + \text{h.o.t.}$$

Let us denote by $\mathbb{F}(n;d)$ the set of foliations of degree $d$ on $\mathbb{P}^n_C$ (see [2]). The following statement gives a criterion which asserts that an element of $\mathbb{F}(2;2)$ is transversely projective.
Proposition 2.5. — Let $\mathcal{F} \in \mathbb{F}(2; 2)$ be a foliation of degree 2 on $\mathbb{P}_C^2$. If a singular point of $\mathcal{F}$ is radial, then $\mathcal{F}$ is transversely projective.

Proof. — Assume that the singular point is the origin 0 in the affine chart $z = 1$, the foliation $\mathcal{F}$ is thus defined by a 1-form of the following type

$$\omega = x dy - y dx + q_1 dx + q_2 dy + q_3 (x dy - y dx)$$

where the $q_i$’s denote quadratic forms. Let us consider the complex projective plane $\mathbb{P}_C^2$ blown up at the origin; this space is denoted by $\text{Bl}(\mathbb{P}_C^2, 0)$. Let $\pi: \text{Bl}(\mathbb{P}_C^2, 0) \to \mathbb{P}_C^2$ be the canonical projection. Then $\pi^* \mathcal{F}$ is transverse to the generic fibers of $\pi$, and in fact transverse to all the fibers excepted the strict transforms of the lines $xq_1 + yq_2 = 0$. Hence the foliation $\pi^* \mathcal{F}$ is transversely projective; since this notion is invariant under the action of a birational map, $\mathcal{F}$ is transversely projective.

Remark 2.6. — The same argument can be involved for foliations of degree 2 on $\mathbb{P}_C^2$ having a singular point with zero 1-jet.

Remark 2.7. — The closure of the set $\Delta_R$ of foliations in $\mathbb{F}(2; 2)$ having a radial singular point is irreducible, of codimension 2 in $\mathbb{F}(2; 2)$.

3. Proof of Theorem A

We establish Theorem A in two steps: we first look at foliations that have at least two singular points and then at foliations with exactly one singular point.

3.1. Foliations of degree 2 on $\mathbb{P}_C^2$ with at least two singularities. —

Proposition 3.1. — For any $\mathcal{F} \in \mathbb{F}(2; 2)$ with at least two distinct singularities there exists a quadratic birational map $\psi \in \mathcal{O}(\rho)$ such that $\deg \psi^* \mathcal{F} \leq 3$.

Proof. — In homogeneous coordinates $\mathcal{F}$ is described by a 1-form

$$\omega = q_1 y z \left( \frac{dy}{y} - \frac{dz}{z} \right) + q_2 x z \left( \frac{dz}{z} - \frac{dx}{x} \right) + q_3 x y \left( \frac{dx}{x} - \frac{dy}{y} \right)$$

where

$$q_1 = a_0 x^2 + a_1 y^2 + a_2 z^2 + a_3 x y + a_4 x z + a_5 y z, \quad q_2 = b_0 x^2 + b_1 y^2 + b_2 z^2 + b_3 x y + b_4 x z + b_5 y z,$$

$$q_3 = c_0 x^2 + c_1 y^2 + c_2 z^2 + c_3 x y + c_4 x z + c_5 y z.$$

Up to an automorphism of $\mathbb{P}_C^2$ one can suppose that $(1 : 0 : 0)$ and $(0 : 1 : 0)$ are singular points of $\mathcal{F}$, that is $a_1 = b_0 = c_0 = c_1 = 0$. If $c_3 \neq 0$, resp. $c_3 = 0$ and $b_4 \neq 0$, resp. $c_3 = b_4 = 0$, then let us consider the quadratic birational map $\psi$ of $\mathcal{O}(\rho)$ defined as follows

$$\psi: (x : y : z) \mapsto \left( xy : z^2 + \frac{b_3 - c_4 + \sqrt{(b_3 - c_4)^2 + 4b_4 c_3}}{2c_3} y z : y z \right),$$

resp.

$$\psi: (x : y : z) \mapsto \left( xy : z^2 + y z : - \frac{b_3 - c_4}{b_4} y z \right),$$

resp. $\psi = \rho$. A direct computation shows that $\psi^* \omega = \chi\omega'$ where $\omega'$ denotes a homogeneous 1-form of degree 4. The foliation $\mathcal{F}'$ defined by $\omega'$ has degree at most 3.

\qed
3.2. Foliations of degree 2 on \( \mathbb{P}^2 \) with exactly one singularity. — Such foliations have been classified:

**Theorem 3.2** ([6]). — Up to automorphisms of \( \mathbb{P}^2 \) there are four foliations of degree 2 on \( \mathbb{P}^2 \) having exactly one singularity. They are described in affine chart by the following 1-forms:

- \( \Omega_1 = x^2 \, dx + y^2 \, (x \, dy - y \, dx) \),
- \( \Omega_2 = x^2 \, dx + (x + y^2)(x \, dy - y \, dx) \),
- \( \Omega_3 = xy \, dx + (x + y^2)(x \, dy - y \, dx) \),
- \( \Omega_4 = (x + y^2 - x^2y) \, dy + x(x + y^2) \, dx \).

**Proposition 3.3.** — There exists a quadratic birational map \( \psi_1 \in \mathcal{O}(\mathfrak{p}) \) such that \( \deg \psi_1^* \mathcal{F}_{\Omega_1} = 2 \); furthermore \( \mathcal{F}_{\Omega_1} \) has a rational first integral and is non-primitive.

For \( k = 2, 3 \), there is no birational map \( \phi_k \) such that \( \deg \phi_k^* \mathcal{F}_{\Omega_k} = 0 \) but there is a \( \psi_k \in \mathcal{O}(\mathfrak{r}) \) such that \( \deg \psi_k^* \mathcal{F}_{\Omega_k} = 1 \). In particular \( \mathcal{F}_{\Omega_2} \) and \( \mathcal{F}_{\Omega_3} \) are non-primitive.

There is a quadratic birational map \( \psi_4 \in \mathcal{O}(\mathfrak{t}) \) such that \( \deg \psi_4^* \mathcal{F}_{\Omega_4} = 3 \) and \( \mathcal{F}_{\Omega_4} \) is primitive.

**Remark 3.4.** — If \( \phi = (x^2 : xy : xz + y^2) \), then \( \deg \phi^* \mathcal{F}_{\Omega_2} = \deg \phi^* \mathcal{F}_{\Omega_3} = 2 \). A contrario we will see later there is no quadratic birational map \( \phi \) such that \( \deg \phi^* \mathcal{F}_{\Omega_4} = 2 \) (see Corollary 4.15).

**Corollary 3.5.** — For any element \( \mathcal{F} \) of \( \mathcal{F}(2;2) \) with exactly one singularity there exists a quadratic birational map \( \psi \) such that \( \deg \psi^* \mathcal{F} \leq 3 \).

**Proof of Proposition 3.3** — The foliation \( \mathcal{F}_{\Omega_1} \) is given in homogeneous coordinates by

\[
\Omega_1' = (x^2z - y^3) \, dx + xy^2 \, dy - x^3 \, dz;
\]

if \( \psi_1 : (x : y : z) \rightarrow (x^2 : xy : yz) \) then

\[
\psi_1^* \Omega_1' \wedge (y(2xz - y^2) \, dx + x(y^2 - xz) \, dy - x^2 \, ydz) = 0.
\]

The foliation \( \mathcal{F}_{\Omega_1} \) has a rational first integral and is non-primitive, it is the image of a foliation of degree 0 by a cubic birational map:

\[
(x^3 : x^2y : x^2z + y^3/3) \wedge (z \, dx - x \, dz) = 0.
\]

The foliation \( \mathcal{F}_{\Omega_2} \) is described in homogeneous coordinates by

\[
\Omega_2' = (x^2z - xy - y^3) \, dx + x(xz + y^2) \, dy - x^3 \, dz;
\]

let us consider the birational map \( \psi_2 : (x : y : z) \rightarrow (x^2 : xy : xz - 2x^2 - 2xy - y^2) \) then

\[
\psi_2^* \Omega_2' \wedge ((xz - yz) \, dx + xz \, dy - x^2 \, dz) = 0.
\]

One can verify that

\[
(2 + \frac{1}{x} + \frac{2y}{x} + \frac{y^2}{x^2}) \exp \left( -\frac{y}{x} \right)
\]

is a first integral of \( \mathcal{F}_{\Omega_2} \); it is easy to see that \( \mathcal{F}_{\Omega_2} \) has no rational first integral so there is no birational map \( \phi_2 \) such that \( \deg \phi_2^* \mathcal{F}_{\Omega_2} = 0 \).

The foliation \( \mathcal{F}_{\Omega_3} \) is given in homogeneous coordinates by the 1-form

\[
\Omega_3 = y(xz - x^2 - y^2) \, dx + x(x^2 + y^2) \, dy - x^2 \, ydz;
\]

if \( \psi_3 : (x : y : z) \rightarrow (x^2 : xy : xz + y^2/2) \) then

\[
\psi_3^* \Omega_3 \wedge (y(z - x) \, dx + x^2 \, dy - xy \, dz) = 0.
\]
The function
\[ \left( \frac{y}{x} \right) \exp \left( \frac{1}{2} \frac{y^2}{x^2} - \frac{1}{x} \right) \]
is a first integral of \( \mathcal{F}_{\Omega_4} \) and \( \mathcal{F}_{\Omega_3} \) has no rational first integral so there is no birational map \( \phi_3 \) such that \( \deg \phi_3 \mathcal{F}_{\Omega_3} = 0 \).

Let us consider the birational map of \( \mathbb{P}^2 \) given by
\[ \psi_4: (x:y:z) \rightarrow (-x^2:xy:y^2-xz) \]
In homogeneous coordinates \( \Omega'_4 = x(xz+y^2) dx + (xz^2+y^2z-x^2y)dy + (xyz-y^3-x^3)dz \); a direct computation shows that
\[ \psi_4 \mathcal{F}_{\Omega'_4} \cap (x^3y^2-x^2y^2+y^3z-2x^2yz) dx + (x^3y-4y^4-x^2z^2+3xy^2z)dy + x(2y^3-x^3-xyz)dz = 0. \]
The foliation \( \mathcal{F}_{\Omega_4} \) has no invariant algebraic curve so \( \mathcal{F}_{\Omega_4} \) is not transversely projective ([6, Proposition 1.3]). In fact a foliation of degree 2 without invariant algebraic curve is primitive; as a consequence \( \mathcal{F}_{\Omega_4} \) is a primitive foliation. \( \square \)

4. Numerical invariance

In the sequel num. inv. means numerically invariant.

In this section we determine the foliations \( \mathcal{F} \) of \( \mathbb{P}(2;2) \) num. inv. under the action of \( \sigma \) (resp. \( \rho \), resp. \( \tau \)). Note that if \( \phi \) is a birational map of \( \mathbb{P}^2_\mathbb{C} \) and \( \ell \) an element of \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \) then \( \deg(\phi \ell)^* \mathcal{F} = \deg \phi^* \mathcal{F} \); hence following Theorem 2.1 we get the description of foliations num. inv. under the action of a quadratic birational map of \( \mathbb{P}^2_\mathbb{C} \).

Lemma 4.1. — An element \( \mathcal{F} \) of \( \mathbb{P}(2;2) \) is num. inv. under the action of \( \sigma \) if and only if it is given up to permutations of coordinates and standard affine charts by 1-forms of the following type
\[ \begin{align*}
\text{either} & \quad \omega_1 = y(x+ey)dx + (bx+\delta y+ax^2+\gamma y)dy, \\
\text{or} & \quad \omega_2 = (\delta+\beta y+\kappa y^2)dx + (\alpha+\varepsilon x+\chi x^2)dy,
\end{align*} \]
where \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \) (resp. \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \)) are complex numbers such that \( \deg \mathcal{F}_{\omega_1} = 2 \) (resp. \( \deg \mathcal{F}_{\omega_2} = 2 \)).

Proof. — The foliation \( \mathcal{F} \) is defined by a homogeneous 1-form \( \omega \) of degree 3. The map \( \sigma \) is an automorphism of \( \mathbb{P}^2_\mathbb{C} \setminus \{xyz=0\} \) so if \( \sigma^* \omega = P \omega' \), with \( \omega' \) a 1-form of degree 3 and \( P \) a homogeneous polynomial then \( P = x^iy^jz^k \) for some integers \( i, j, k \) such that \( i+j+k = 4 \). Up to permutation of coordinates it is sufficient to look at the four following cases: \( P = x^4, P = x^3y, P = x^2y^2 \) and \( P = x^2yz \). Let us write \( \omega \) as follows
\[ \omega = q_1yz \left( \frac{dy}{y} - \frac{dz}{z} \right) + q_2xz \left( \frac{dx}{x} - \frac{dz}{z} \right) + q_3xy \left( \frac{dx}{x} - \frac{dy}{y} \right) \]
where
\[ \begin{align*}
q_1 &= a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz, \\
q_2 &= b_0x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4xz + b_5yz, \\
q_3 &= c_0x^2 + c_1y^2 + c_2z^2 + c_3xy + c_4xz + c_5yz.
\end{align*} \]
Computations show that \( x^4 \) (resp. \( x^3y \)) cannot divide \( \sigma^* \omega \). If \( P = x^2yz \) then \( \sigma^* \omega = P \omega' \) if and only if
\[ \begin{align*}
c_0 &= 0, & b_0 &= 0, & a_2 &= 0, & b_2 &= 0, & a_1 &= 0, & c_1 &= 0, & b_4 &= 0, & c_3 &= 0, & b_3 &= c_4,
\end{align*} \]
that gives \( \omega_1 \). Finally one has \( \sigma^* \omega = x^2y^2\omega' \) if and only if
\[ \begin{align*}
c_1 &= 0, & c_0 &= 0, & b_0 &= 0, & a_1 &= 0, & b_4 &= 0, & c_3 &= 0, & a_5 &= 0, & b_3 &= c_4, & c_5 &= a_3; \]
in that case we obtain $\omega_2$.
\[\square\]

**Proposition 4.2.** — A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathcal{O}(\sigma)$ is Aut($\mathbb{P}^2_C$)-conjugate either to a foliation of type $\mathcal{F}_{\omega_1}$, or to a foliation of type $\mathcal{F}_{\omega_2}$; in particular it is transversely projective.

**Proof.** — Let $\phi$ be an element of $\mathcal{O}(\sigma)$ such that $\deg \phi^* \mathcal{F} = 2$; the map $\phi$ can be written $\ell_1 \sigma \ell_2$ where $\ell_1$ and $\ell_2$ denote automorphisms of $\mathbb{P}^2_C$. By assumption the degree of $(\ell_1 \sigma \ell_2)^* \mathcal{F} = \ell_2^* (\sigma^*(\ell_1^* \mathcal{F}))$ is 2. Hence $\deg \sigma^*(\ell_1^* \mathcal{F}) = 2$ and the foliation $\ell_1^* \mathcal{F}$ is num. inv. under the action of $\sigma$. Since $\ell_1^* \mathcal{F}$ and $\mathcal{F}$ are conjugate and since the notion of transversal projectivity is invariant by conjugacy it is sufficient to establish the statement for $\sigma = \sigma'$. The proposition thus follows from the fact that 1-forms of Lemma 4.1 are Riccati ones (up to multiplication).

\[\square\]

**Remark 4.3.** — For generic values of parameters $\alpha$, $\beta$, $\gamma$, $\epsilon$, $\kappa$ a foliation of type $\mathcal{F}_{\omega_1}$ given by the corresponding form $\omega_1$ is not given by a closed meromorphic 1-form. This can be seen by studying the holonomy group of $\mathcal{F}_{\omega_1}$, that can be identified with a subgroup of PGL($2;\mathbb{C}$) generated by two elements $f$ and $g$. For generic values of the parameters $f$ and $g$ are also generic, in particular the group $\langle f, g \rangle$ is free. When $\mathcal{F}_{\omega_1}$ is given by a closed 1-form, then the holonomy group is an abelian one.

Remark that a contrario the foliations given by 1-forms of type $\mathcal{F}_{\omega_2}$ are conjugate to a foliation of type $\mathcal{F}_{\omega_2}$; in particular it is transversely projective.

**Remark 4.4.** — Let $\Delta_1$ denote the closure of the set of elements of $\mathbb{F}(2;2)$ conjugate to a foliation of type $\mathcal{F}_{\omega_1}$. The following inclusion holds: $\Delta_2 \subset \Delta_1$.

Note also that $\Delta_1$ is contained in $\Delta_R$ (see Remark 2.7).

**Remark 4.5.** — The notion of num. inv. is not related to the dynamic of the map (see 3 for example); the foliations num. inv. by the involution $\sigma$ ("without dynamic") are conjugate to the foliations num. inv. by $A \sigma$, $A \in$ Aut($\mathbb{P}^2_C$), which has a rich dynamic for generic $A$.

The foliations of $\mathbb{F}(2;2)$ invariant by $\sigma$ are particular cases of num. inv. foliations:

**Proposition 4.6.** — An element of $\mathbb{F}(2;2)$ invariant by $\sigma$ is given up to permutations of coordinates and affine charts

- either by $y(1+y) \, dx + (\beta x + \alpha y + \alpha x^2 + \beta xy) \, dy$,
- or by $y(1-y) \, dx + (\beta x - \alpha y + \alpha x^2 - \beta xy) \, dy$,
- or by $y \, dx + (\alpha + \epsilon x + A \alpha x^2) \, dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

**Proof.** — With the notations of Lemma 4.1 one has

$$\sigma^* \omega_1 = -y(\epsilon + \kappa y) \, dx - (\gamma x + \alpha y + \delta x^2 + \beta xy) \, dy;$$

thus $\sigma^* \omega_1 \wedge \omega_1 = 0$ if and only if either $\gamma = \beta$, $\delta = \alpha$, $\epsilon = \kappa$, or $\gamma = -\beta$, $\delta = -\alpha$, $\epsilon = -\kappa$.

One has $\sigma^* \omega_2 = -(\kappa + \beta y + \delta y^2) \, dx - (\gamma + \epsilon x + A \alpha x^2) \, dy$ and $\omega_2 \wedge \sigma^* \omega_2 = 0$ if and only if $\gamma = \alpha$, $\delta = 0$ and $\kappa = 0$.

\[\square\]

**Remark 4.7.** — The foliations associated to the two first 1-forms with parameters $\alpha$, $\beta$ of Proposition 4.6 are conjugate by the automorphism $(x,y) \mapsto (x,-y)$.

**Lemma 4.8.** — A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ is num. inv. under the action of $\rho$ if and only if $\mathcal{F}$ is given in affine chart
• either by $\omega_3 = y(\kappa + \epsilon y + \lambda y^2)\,dx + (\beta + \kappa x + \delta y + \gamma \epsilon y + \alpha y^2 - \lambda \epsilon x^2)\,dy$,
• or by $\omega_4 = y(\mu + \delta x + \gamma y + \epsilon xy)\,dx + (\alpha + \beta x + \lambda y + \delta x^2 + \kappa \epsilon xy - \epsilon \delta x^2)\,dy$,
• or by $\omega_5 = (\lambda + \gamma y + \kappa xy + \epsilon y^2)\,dx + (\beta + \delta x + \alpha x^2)\,dy$,

where the parameters are such that the degree of the corresponding foliations is 2.

**Proof.** — Let us take the notations of the proof of Lemma 4.1. The map $\rho$ is an automorphism of $\mathbb{P}_\mathbb{C}^2 \setminus \{yz = 0\}$ so if $\rho^*\omega = P\omega'$ with $\omega'$ a 1-form of degree 3 and $P$ a homogeneous polynomial then $P = y^j z^k$ for some integer $j, k$ such that $j + k = 4$. We have to look at the four following cases: $P = z^4, P = y z^3, P = y^2 z^2, P = y^3 z$ and $P = y^4$. Computations show that $y^4$ (resp. $y^3 z$) cannot divide $\rho^*\omega$. If $P = z^4$ then $\rho^*\omega = P\omega'$ if and only if

\[
c_0 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad b_4 = 0, \quad b_2 = 0, \quad a_0 = c_4, \quad b_3 = c_4, \quad a_4 = 2c_2 - b_5;
\]

this gives the first case $\omega_3$. The equality $\rho^*\omega = y z^3 \omega'$ holds if and only if

\[
b_0 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_1 = 0, \quad a_1 = 0, \quad b_2 = 0, \quad a_0 = 2c_4 - b_3
\]

and we obtain $\omega_4$. Finally one has $\rho^*\omega = y^2 z^2 \omega'$ if and only if

\[
c_1 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad a_5 = 0, \quad a_1 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_5 = a_3
\]

which corresponds to $\omega_5$. \hfill $\square$

**Proposition 4.9.** — The foliations of type $\mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_4}$ are transversely projective. In fact the $\mathcal{F}_{\omega_5}$ are transversely affine and the $\mathcal{F}_{\omega_6}$ are Riccati ones.

**Proof.** — A foliation of type $\mathcal{F}_{\omega_6}$ is described by the 1-form

\[
\theta_0 = dx - \frac{(\beta + \delta y + \alpha x^2) + (\kappa + \gamma y - \lambda y^2) x}{y(\kappa + \epsilon y + \lambda y^2)}\,dy
\]

and it is transversely affine; to see it consider the $sl(2; \mathbb{C})$-triplet

\[
\theta_0, \quad \theta_1 = \frac{\kappa + \gamma y - \lambda y^2}{y(\kappa + \epsilon y + \lambda y^2)}\,dy, \quad \theta_2 = 0.
\]

A foliation of type $\mathcal{F}_{\omega_5}$ is given by

\[
dy + \frac{\lambda + (\gamma + \kappa) y + \epsilon y^2}{\beta + \delta x + \alpha x^2}\,dx
\]

and thus is a Riccati foliation. In fact the fibration $x/z = \text{constant}$ is transverse to $\mathcal{F}_{\omega_5}$ that generically has three invariant lines. \hfill $\square$

We don’t know if the $\mathcal{F}_{\omega_7}$ are transversely projective. For generic values of the parameters a foliation of type $\mathcal{F}_{\omega_7}$ hasn’t meromorphic uniform first integral in the affine chart $z = 1$. Thus if $\mathcal{F}_{\omega_7}$ is transversely projective then it must have an invariant algebraic curve different from $z = 0$ (see [7]). We don’t know if it is the case. A foliation of degree 2 is conjugate to a generic $\mathcal{F}_{\omega_7}$ (by an automorphism of $\mathbb{P}_\mathbb{C}^2$) if and only to has an invariant line (say $y = 0$) with a singular point (say 0) and local model $2x\,dy - y\,dx$. The closure of the set of such foliations has codimension 2. Note that the three families $\mathcal{F}_{\omega_7}, \mathcal{F}_{\omega_8}$ and $\mathcal{F}_{\omega_9}$ have non trivial intersection. The set $\{\mathcal{F}_{\omega_7}\}$ contains many interesting elements such that the famous Euler foliation given by $y^2\,dx + (y - x)\,dy$; this foliation is transversely affine but is not given by a closed rational 1-form.

**Proposition 4.10.** — A foliation $\mathcal{F} \in \mathbb{P}(2; 2)$ num. inv. under the action of an element of $\Theta^1(\rho)$ is conjugate to a foliation either of type $\mathcal{F}_{\omega_7}$, or of type $\mathcal{F}_{\omega_8}$, or of type $\mathcal{F}_{\omega_9}$. 
Let us look at special num. inv. foliations, those invariant by $\rho$.

**Proposition 4.11.** — An element of $\mathcal{F}(2;2)$ invariant by $\rho$ is given by a 1-form of one of the following type

- $y(1-y)\,dx + (\beta + x)dy$,
- $y^2\,dx + (-1+y)dy$,
- $y(1-y)(\gamma + \delta x)\,dx + (1+y)(\alpha + \beta x + \delta^2)dy$,
- $y(1+y)(\gamma + \delta x)\,dx + (1-y)(\alpha + \beta x + \delta^2)dy$,
- $(1-y^2)\,dx + (\beta + \delta x + \alpha^2)dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

**Corollary 4.12.** — An element of $\mathcal{F}(2;2)$ invariant by $\rho$ is defined by a closed 1-form.

**Remark 4.13.** — The third and fourth cases with parameters $\alpha, \beta, \gamma, \delta$ are conjugate by the automorphism $(x,y) \mapsto (x,-y)$.

From Lemmas 4.1 and 4.8 one gets the following statement.

**Proposition 4.14.** — A foliation num. inv. by an element of $\mathcal{O}(\phi)$, with $\phi = \sigma, \rho$, preserves an algebraic curve.

**Corollary 4.15.** — There is no quadratic birational map $\phi$ of $\mathbb{P}^2_1$ such that $\deg \phi^* \mathcal{F}_\Omega = 2$.

**Proof.** — The foliation $\mathcal{F}_\Omega$ has no invariant algebraic curve ([16 Proposition 1.3]); according to Proposition 4.14 it is thus sufficient to show that there is no birational map $\phi \in \mathcal{O}(\tau)$ such that $\deg \phi^* \mathcal{F}_\Omega = 2$ that can be established with a direct and tedious computation.

**Remark 4.16.** — The map $\rho$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with

$$
\ell_1 = (z-y:y-x:y), \quad \ell_2 = (y+z:z:x), \quad \ell_3 = (x+z:y-z:z).
$$

We are interested by the "intermediate" degrees of a numerically invariant foliation $\mathcal{F}$, that is the sequence $\deg \mathcal{F}$, $\deg (\ell_1 \sigma)^* \mathcal{F}$, $\deg (\ell_1 \sigma^2 \sigma \ell_3)^* \mathcal{F} = \deg \mathcal{F}$. A tedious computation shows that for generic values of the parameters the sequence is 2, 5, 2. We schematize this fact by the diagram

```
      5
     /|
    2| 2
   /|
  2| 2
```

A similar argument to Lemma 4.1 yields to the following result.

**Lemma 4.17.** — An element $\mathcal{F}$ of $\mathcal{F}(2;2)$ is num. inv. under the action of $\tau$ if and only if $\mathcal{F}$ is given in affine chart by a 1-form of type

$$
\omega_6 = (-\delta x + \alpha y - \varepsilon x^2 + \theta xy + \beta y^2 + \kappa x^2 y + \mu xy^2 + \lambda y^3)\,dx + (-3\alpha x + \xi x^2 + 2(\delta - \beta)xy + \alpha y^2 - \kappa x^3 - \mu x^2 y - \lambda xy^2)\,dy
$$

where the parameters are such that $\deg \mathcal{F}_\omega = 2$.

We don’t know the qualitative description of foliations of type $\mathcal{F}_{\omega_6}$. For example we don’t know if the $\mathcal{F}_{\omega_6}$ are transversely projective. If it is the case, this implies the existence of invariant algebraic curves, and that fact is unknown.
Remark 4.20. — A foliation $\mathcal{F} \in \mathbb{F}(2; 2)$ num. inv. under the action of an element of $\Theta(\tau)$ is conjugate to $\mathcal{F}_\omega$, for suitable values of the parameters.

Let us describe some special num. inv. foliations under the action of $\tau$, those invariant by $\tau$.

**Proposition 4.19.** — An element of $\mathbb{F}(2; 2)$ invariant by $\tau$ is given

- either by $(-\varepsilon x^2 + \theta xy + \beta y^2 + \varepsilon xy^2 - (\xi + \theta)y^3) \, dx + x(\xi x - 2\beta y + \varepsilon xy + (\xi + \theta)y^2) \, dy$,
- or by $(-\delta x + \alpha y + \frac{1}{2}\delta y^2 + \kappa y^3 + \mu x^2 y + \lambda y^3) \, dx - (3\alpha x - \delta xy - \alpha y^2 + \kappa x^3 + \mu x^2 y + \lambda xy^2) \, dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

The foliations associated to the first 1-form are transversely affine.

**Proof.** — The 1-jet at the origin of the 1-form

$$\omega = (-\varepsilon x^2 + \theta xy + \beta y^2 + \varepsilon xy^2 - (\xi + \theta)y^3) \, dx + x(\xi x - 2\beta y + \varepsilon xy + (\xi + \theta)y^2) \, dy$$

is zero so after one blow-up $\mathcal{F}_\omega$ is transverse to the generic fiber of the Hopf fibration; furthermore as the exceptional divisor is invariant, $\mathcal{F}_\omega$ is transversely affine. \(\square\)

**Remark 4.20.** — The map $\tau$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_2 \sigma \ell_4$ with

$$\ell_1 = (x : y : x - 2y : -x + y : z), \quad \ell_2 = (x + z : x : y), \quad \ell_3 = (-y : x - 3y + z : x), \quad \ell_4 = (y - x : z - 2x : 2x - y).$$

Let us consider a foliation $\mathcal{F}$ num. inv. under the action of $\tau$; set $\mathcal{F}' = \ell_1^* \mathcal{F}$. We compute the intermediate degrees:

$$\deg^* \mathcal{F}' = 5, \quad \deg(\sigma \ell_2 \sigma)^* \mathcal{F}' = 4, \quad \deg(\sigma \ell_3 \sigma \ell_2 \sigma)^* \mathcal{F}' = 5.$$

To summarize:

```
  \[\begin{array}{ccc}
  5 & 5 \\
  4 & & \\
  2 & 2 &
  \end{array}\]
```

**5. Higher degree**

We will now focus on similar questions but with cubic birational maps of $\mathbb{P}_C^2$ and elements of $\mathbb{F}(2; 2)$. The generic model of such birational maps is:

$$\Phi_{a, b}: (x : y : z) \rightarrow (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}, \ a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$.

**Lemma 5.1.** — An element $\mathcal{F}$ of $\mathbb{F}(2; 2)$ is num. inv. under the action of $\Phi_{a, b}$ if and only if $\mathcal{F}$ is given in affine chart

- either by $\omega_1 = y(\alpha + \gamma) \, dx - x(\alpha + \kappa) \, dy$,
- or by $\omega_2 = b(b^2 - ab + 1 + (a - 2b)y + y^2) \, dx + (b^2 - ab + 1 + (ab - 2)x + x^2) \, dy$,

where the parameters are such that $\deg \mathcal{F}_\omega = \deg \mathcal{F}_\omega = 2$. 

Remark 5.2. — Remark that the foliations $F_{o_7}$ do not depend on the parameters of $\Phi_{a,b}$, that is, the $F_{o_7}$ are num. inv. by all $\Phi_{a,b}$, whereas the $F_{o_8}$ only depend on $a$ and $b$.

Furthermore $F_{o_7}$ is num. inv. by $\sigma$ and $\rho$.

Proposition 5.3. — Any $F \in F(2; 2)$ num. inv. under the action of $\Phi_{a,b}$, and more generally any $F \in F(2; 2)$ num. inv. under the action of a generic cubic birational map of $\mathbb{P}_C^2$, satisfies the following properties:

- $F$ is given by a rational closed 1-form;
- $F$ is non-primitive.

Proof. — Let us establish those properties for $F_{o_7}$.

For generic values of $\alpha$, $\gamma$ and $\kappa$ one can assume up to linear conjugacy that $F_{o_7}$ is given by

$$\eta' = y(1+y)dx - x(1+x)dy$$

that gives up to multiplication

$$\frac{dx}{x(1+x)} \to \frac{dy}{y(1+y)}$$

which is closed. A foliation of type $F_{o_7}$ is also described in homogeneous coordinates by the 1-form

$$\eta = yz(y+z)dx - xz(x+z)dy + xy(x-y)dz.$$

One has

$$\sigma^*\eta = xyz(- (y+z)dx + (x+z)dy + (x-y)dz)$$

so $F_{o_7}$ is non-primitive.

The idea and result are the same for the foliations $F_{o_8}$ (except that it gives a birational map $\phi$ such that $\deg \phi^* F_{o_8} = 1$).

Let us consider an element $F$ of $F(2; 2)$ num. inv. under the action of a birational map of degree $\geq 3$; is $F$ defined by a closed 1-form ?

Remark 5.4. — The foliations $F_{o_7}$ are contained in the orbit of the foliation $F_{\eta'}$.

Remark 5.5. — Any map $\Phi_{a,b}$ can be written $\ell_1\sigma\ell_2\sigma\ell_3$ with $\ell_2 = (x^*y + z^*z : x^*y + *z : x^*x + *y + *z)$ (see [5, Proposition 6.36]). Let us consider the birational map $\xi = \alpha\ell_2\sigma$ with

$$\ell_2 = (ay + bz : cy + ez : fx + gy + hz) \in \text{Aut}(\mathbb{P}_C^2).$$

As in Lemma 5.1 there are two families of foliations $\mathcal{F}_1$, $\mathcal{F}_2$ of degree 2, one that does not depend on the parameters of $\xi$ and the other one depending only on the parameters of $\xi$, such that $\xi^*\mathcal{F}_1$ and $\xi^*\mathcal{F}_2$ are of degree 2. One question is the following: what is the intermediate degree ? A computation shows that for generic parameters $\deg \sigma^*\mathcal{F}_1 = 4$ and that $\deg \sigma^*\mathcal{F}_2 = 2$. This implies in particular that $F_{o_8}$ is num. inv. under the action of $\sigma$. For $\mathcal{F}_1$ and $F_{o_7}$ one has

```
4
\downarrow
2
\quad \downarrow
2
```

and for $\mathcal{F}_2$ and $F_{o_8}$

```
2 \\
\quad \longrightarrow \quad 2 \\
\quad \longrightarrow \quad 2
```
Let us now consider the "most degenerate" cubic birational map 
\[ \Psi : (x : y : z) \mapsto (x^2 + y^3 : yz^2 : z^3). \]

Lemma 5.6. — An element \( \mathcal{F} \) of \( \mathbb{F}(2;2) \) is num. inv. under the action of \( \Psi \) if and only if \( \mathcal{F} \) is given in affine chart by 
\[ \omega_0 = (-\alpha + \beta y + \gamma z) \, dx + (\varepsilon - 3\beta x + \kappa y - 3\gamma x y + \lambda y^2) \, dy \]
where the parameters are such that \( \deg F_{\omega_0} = 2 \). In particular \( \mathcal{F} \) is transversely affine.

Remark 5.7. — The map \( \psi \) can be written \( \ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_2 \sigma \ell_7 \) with
\[
\begin{align*}
\ell_1 &= (z - y : y : y - x), & \ell_2 &= (y + z : z : x), & \ell_3 &= (-z : -y : x - y), \\
\ell_4 &= (x + z : x : y), & \ell_5 &= (-y : x - 3y + z : x), & \ell_6 &= (-x : -y - z : x + y), \\
\ell_7 &= (x + y : z - y : y).
\end{align*}
\]
As previously we consider the problem of the intermediate degrees; if \( \mathcal{F}' = \ell_1' \mathcal{F} \), a computation shows that for generic parameters
\[
\begin{align*}
\deg \sigma^* \mathcal{F}' &= 4, & \deg(\sigma \ell_2 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma)^* \mathcal{F}' &= 5, \\
\deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma)^* \mathcal{F}' &= 5, \\
\deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* \mathcal{F}' &= 4,
\end{align*}
\]
that is

![Diagram](attachment:diagram.png)

We have not studied the quadratic foliations numerically invariant by (any) cubic birational transformation. It is reasonable to think that such foliations are transversally projective.

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