LEVEL ALGEBRAS THROUGH BUCHSBAUM* MANIFOLDS

UWE NAGEL*

Abstract. Stanley-Reisner rings of Buchsbaum* complexes are studied by means of their quotients modulo a linear system of parameters. The socle of these quotients is computed. Extending a recent result by Novik and Swartz for orientable homology manifolds without boundary, it is shown that modulo a part of their socle these quotients are level algebras. This provides new restrictions on the face vectors of Buchsbaum* complexes.

1. Introduction

This note is inspired by work by Novik and Swartz [13] and by Athanasiadis and Welker [1]. Its goal is to generalize some results of [13] and to contribute to the fruitful interaction of algebraic, combinatorial, and topological methods in order to study simplicial complexes.

Let $\Delta$ be a finite simplicial complex. Important algebraic properties of it like Cohen-Macaulayness or Buchsbaumness are defined by means of its Stanley-Reisner ring $K[\Delta]$. However, these properties turn out to be topological properties in the sense that they depend only on the homeomorphism type of the geometric realization $|\Delta|$ of $\Delta$. For example, all triangulations of a manifold (with or without boundary) are Buchsbaum, and all triangulations of a sphere or a ball are Cohen-Macaulay over each field.

In [1], Athanasiadis and Welker introduced Buchsbaum* complexes as the $(d-1)$-dimensional Buchsbaum complexes over a field $K$ such that

$$
\dim K \tilde{H}_{d-2}(|\Delta|-p; K) = \dim K \tilde{H}_{d-2}(|\Delta|; K)
$$

for every $p \in |\Delta|$, where $\tilde{H}_j(|\Delta|, K)$ denotes the reduced singular homology of the geometric realization $|\Delta|$. The class of Buchsbaum* complexes includes all doubly Cohen-Macaulay complexes and all triangulations of orientable homology manifolds without boundary (see [11]).

In [12] Novik and Swartz pioneered the study of Buchsbaum complexes by investigating socles of artinian reductions of their Stanley-Reisner rings. Our first main result is an improvement of their description of the socle ([12], Theorem 2.2) for Buchsbaum* complexes. In fact, it provides a numerical characterization of such complexes.

**Theorem 1.1.** Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum simplicial complex. Then $\Delta$ is a Buchsbaum* complex if and only if, for each linear system of parameters $\ell$ of $K[\Delta]$ and each positive integer $j$,

$$
\dim_K [\text{Soc} K[\Delta]/\ell]_j = \binom{d}{j} \beta_{j-1}(\Delta),
$$

where $\beta_j(\Delta) := \dim_K \tilde{H}_j(\Delta; K)$ is the $j$-th reduced Betti number of $\Delta$.

* Part of the work for this paper was done while the author was sponsored by the National Security Agency under Grant Number H98230-09-1-0032.
This extends [13], Theorem 1.3 because each triangulation of an orientable \( K \)-homology manifold without boundary is a Buchsbaum* complex. Similarly, the following result extends [13], Theorem 1.4.

**Theorem 1.2.** Let \( \Delta \) be a \((d - 1)\)-dimensional Buchsbaum* complex, and let \( \ell := \ell_1, \ldots, \ell_d \) be a linear system of parameters for \( K[\Delta] \). Set \( I := \bigoplus_{j=1}^{d-1}[\text{Soc} K[\Delta]/\ell_j] \). Then \((K[\Delta]/\ell)/I\) is a level ring of Cohen-Macaulay type \( \beta_{d-1}(\Delta) \) and socle degree \( d \), that is, the socle of \((K[\Delta]/\ell)/I\) is a \( K \)-vector space of dimension \( \beta_{d-1}(\Delta) \) that is concentrated in degree \( d \).

Being a level ring provides strong restrictions on the Hilbert function of \((K[\Delta]/\ell)/I\). However, the classification of Hilbert functions of level algebras is a wide open problem, and the above result lends further motivation to studying it.

This note is organized as follows. In Section 2 we study the socle of the artinian reduction of a Buchsbaum complex. We observe that these complexes can in fact be characterized by their socle (Corollary 2.2). Moreover, we slightly improve the description of this socle given in [12], Theorem 2.2, by identifying one more piece. Combined with a result in [14], this implies Theorem 1.1.

Section 3 is devoted to the proof of Theorem 1.2. It allows us to establish results on the enumeration of the faces of Buchsbaum* complexes (Theorem 4.4) that improve the corresponding results for arbitrary Buchsbaum complexes in [12]. This is carried out in Section 4.

2. **Socles of artinian reductions**

Throughout this note \( S := K[x_1, \ldots, x_n] \) denotes the polynomial ring in \( n \) variables over a field \( K \).

Let \( M = \bigoplus_{j \in \mathbb{Z}} [M]_j \) be a finitely generated, graded \( S \)-module. Its \( i \)-th local cohomology module with support in the maximal ideal \( m := (x_1, \ldots, x_n) \) is denoted by \( H^i_m(M) \) (see, e.g., [6] and [17]). The **socle** of \( M \) is the submodule

\[ \text{Soc} M := 0 :_M m := \{ y \in M \mid my = 0 \} \]

The module \( M(i) \) is the module with the same structure as \( M \), but with a shifted grading defined by \([M(i)]_j := [M]_{i+j} \). Furthermore we use \( sM \) to denote the direct sum of \( s \) copies of \( M \). If \( M \) has (Krull) dimension \( d \), a sequence \( \ell_1, \ldots, \ell_d \in S \) of linear forms is called a **linear system of parameters** of \( M \) if \( M/\ell := M/(\ell_1, \ldots, \ell_d)M \) has dimension zero. In this case \( M/\ell \) is called an **artinian reduction** of \( M \).

Assume now that \( M \) is a Buchsbaum module. For a comprehensive introduction to the theory of Buchsbaum modules we refer to [19]. Here we only need the following facts about Buchsbaum modules:

- For all \( i \neq \dim M \), \( mH^i_m(M) = 0 \);
- If \( \ell \in S \) is a linear parameter of \( M \), that is \( \dim M/\ell M = \dim M - 1 \), then also \( M/\ell M \) is Buchsbaum and the kernel of the multiplication map \( M(-1) \rightarrow M/\ell M \) has Krull dimension zero.

Thus, the long exact cohomology sequence induced by the multiplication map splits into short exact sequences

\[ 0 \rightarrow H^i_m(M)(-1) \rightarrow H^i_m(M/\ell M) \rightarrow H^{i+1}(M) \rightarrow 0 \quad \text{if} \quad i + 1 < \dim M =: d \]

and ends with

\[ 0 \rightarrow H^{d-1}_m(M)(-1) \rightarrow H^{d-1}_m(M/\ell M) \rightarrow H^d_m(M)(-1) \rightarrow H^d_m(M) \rightarrow 0. \]
Using the first sequence repeatedly one obtains, for every part $\ell_1, \ldots, \ell_j$ of a linear system of parameters of $M$, the isomorphism of graded modules

$$H^i_m(M/(\ell_1, \ldots, \ell_j)M) \cong \bigoplus_{k=0}^{j} \binom{j}{k} H^{i+k}_m(M)(-k)$$ if $i < d - j$.

In the case $i = 0$ and $j = d$, Novik and Swartz [12] established the following result on the socle of the module on the left-hand side.

**Theorem 2.1** ([12], Theorem 2.2). Let $M$ be a finitely generated graded Buchsbaum $S$-module of dimension $d$, and let $\ell := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $M$. Then

$$\text{Soc } M/\ell \cong \left( \bigoplus_{j=0}^{d-1} \binom{d}{j} H^j_m(M)(-d) \right) \oplus Q(-d),$$

where $Q$ is a graded submodule of $\text{Soc } H^d_m(M)$.

In this section we make two comments about this result in the case the module $M$ is a Stanley-Reisner ring.

A simplicial complex $\Delta$ on $n$ vertices is a collection of subsets of $[n] := \{1, \ldots, n\}$ that is closed under inclusion. Its Stanley-Reisner ideal is

$$I_\Delta := (x_{j_1} \cdots x_{j_k} | \{j_1 < \cdots < j_k\} \notin \Delta) \subset S,$$

and its Stanley-Reisner ring is $K[\Delta] := S/I_\Delta$. For each subset $F \subset [n]$, the link of $F$ is the subcomplex

$$\text{lk}_\Delta F := \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Note that $\text{lk}_\Delta \emptyset = \Delta$.

The complex $\Delta$ is a Cohen-Macaulay or a Buchsbaum complex over $K$ if $K[\Delta]$ has the corresponding property. Alternatively, a combinatorial-topological characterization of Cohen-Macaulay complexes is due to Reisner [14]. Schenzel extended his result in [15].

A simplicial complex $\Delta$ is Buchsbaum (over $K$) if and only if $\Delta$ is pure and the link of each vertex of $\Delta$ is Cohen-Macaulay (over $K$).

Our first observation about Theorem 2.1 states that its description of the socle in fact characterizes Buchsbaum simplicial complexes.

**Corollary 2.2.** Let $\Delta$ be a simplicial complex of dimension $d - 1$, and let $\ell := \ell_1, \ldots, \ell_d$ be a linear system of parameters of its Stanley-Reisner ring $K[\Delta]$. Then $\Delta$ is Buchsbaum if and only if

$$\text{Soc } K[\Delta]/\ell \cong \left( \bigoplus_{j=1}^{d-1} \binom{d}{j} H^j_m(K[\Delta])(-j) \right) \oplus Q(-d),$$

where $Q$ is a graded submodule of $\text{Soc } H^d_m(K[\Delta])$.

**Proof.** By Theorem 2.1 it is enough to show sufficiency. The formula for the socle implies that, for each $j \in \{1, \ldots, d-1\}$, the module $H^j_m(K[\Delta])$ is annihilated by $m$ and finitely generated, hence it is a finite-dimensional $K$-vector space. Therefore $\Delta$ must be a Buchsbaum complex (see, e.g., [15] or [10]).

Our second observation identifies a piece of the module $Q$ occurring in Theorem 2.1.

In its proof we use

$$e(P) := \sup \{j \in \mathbb{Z} | [P]_j \neq 0\}$$

to denote the end of a graded module $P$. Note that $e(N) = -\infty$ if $N$ is trivial.
Proposition 2.3. Let $\Delta$ be a Buchsbaum simplicial complex of dimension $d - 1$, and let $\ell := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $K[\Delta]$. Then

$$\text{Soc } K[\Delta]/\ell \cong \left( \bigoplus_{j=1}^{d-1} \left( \bigoplus_{j=1}^d \right. \left. H^j_m(K[\Delta])(-j) \right) \right) \oplus ([H^d_m(K[\Delta])]/0_S)(-d) \oplus Q'(-d),$$

where $[H^d_m(K[\Delta])]/0_S$ is the submodule of $H^d_m(K[\Delta])$ generated by its elements of degree zero and where $Q'$ is a graded submodule of $\text{Soc } H^d_m(K[\Delta])$ that vanishes in all non-negative degrees.

Proof. Hochster’s formula (see [17], Theorem 4.1) provides that all intermediate cohomology modules $H^0_m(K[\Delta]), \ldots, H^{d-1}_m(K[\Delta])$ are concentrated in degree zero and that $H^d_m(K[\Delta])$ vanishes in all positive degrees.

Set $\ell_j := \ell_1, \ldots, \ell_j$. Since $\ell_d = \ell$ and $H^d_m(K[\Delta]/\ell_d) \cong K[\Delta]/\ell_d$, our claim follows, once we have shown that, for all $j \in \{0, \ldots, d\}$,

$$e(H^{d-j}_m(K[\Delta]/\ell_j)) \leq j \quad \text{and} \quad \dim_K H^{d-j}_m(K[\Delta]/\ell_j) = \dim_K H^d_m(K[\Delta]).$$

Indeed, if $j = 0$, this is true. Let $j < d$. Multiplication by $l_{j+1}$ on $K[\Delta]/\ell_j$ induces the long exact cohomology sequence

$$0 \to H^{d-j-1}_m(K[\Delta]/\ell_j) \to H^{d-j-1}_m(K[\Delta]/\ell_{j+1}) \to H^{d-j}_m(K[\Delta]/\ell_j)(-1) \to H^{d-j}_m(K[\Delta]/\ell_j).$$

Since $H^{d-j-1}_m(K[\Delta]/\ell_j) \cong \bigoplus_{i=0}^j \left( H^{d-j-1+i}_m(K[\Delta])(-i) \right)$ vanishes in each degree $k > j$, we get $e(H^{d-j}_m(K[\Delta]/\ell_{j+1})) \leq j + 1$ and

$$\dim_K H^{d-j}_m(K[\Delta]/\ell_{j+1}) = \dim_K H^{d-j}_m(K[\Delta]/\ell_j),$$

as required. \hfill \Box

Remark 2.4. The above argument shows that for an arbitrary, not necessarily Buchsbaum, simplicial complex $\Delta$, the module $[H^d_m(K[\Delta])]/0_S$ contributes to the socle of $K[\Delta]/\ell$. This also follows from Lemma 2.3 in [2].

We now identify an instance where the module $Q'$ appearing in the previous result vanishes.

Corollary 2.5. Let $\Delta$ be a Buchsbaum simplicial complex of dimension $d - 1$, and let $\ell := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $K[\Delta]$. Then

$$\text{Soc } K[\Delta]/\ell \cong \left( \bigoplus_{j=1}^{d-1} \left( \bigoplus_{j=1}^d \right. \left. H^j_m(K[\Delta])(-j) \right) \right) \oplus ([H^d_m(K[\Delta])]/0_S)(-d).$$

Proof. According to [1], Proposition 2.8, the socle of $H^d_m(K[\Delta])$ is concentrated in degree zero. Hence Proposition 2.3 gives the claim. \hfill \Box

The last result proves Theorem 1.2 because $H^d_m(K[\Delta]) = [H^d_m(K[\Delta])]/0$ if $j \neq d$ and $\dim_K H^d_m(K[\Delta])/0 = \beta_{j-1}(\Delta)$ for all $j$ by Hochster’s formula.

3. Level quotients

The goal of this section is to establish Theorem 1.2. Recall that an artinian graded $K$-algebra $A$ is a level ring if its socle is concentrated in one degree, that is, $[\text{Soc } A]_j = 0$ if $j \neq e(A)$.

Let $\Delta$ be a simplicial complex on $[n]$. Recall that each subset $F \subseteq [n]$ induces the following simplicial subcomplexes of $\Delta$: the star

$$\text{st}_\Delta F := \{ G \in \Delta : F \cup G \in \Delta \},$$
and the deletion
\[ \Delta_{-F} := \{ G \in \Delta : F \cap G = \emptyset \}. \]

If \( F \notin \Delta \), then \( \text{lk}_\Delta F = \text{st}_\Delta F = \emptyset \).

Consider any vertex \( k \in [n] \). Then \( \text{st}_\Delta k \) is the cone over the link \( \text{lk}_\Delta k \) with apex \( k \). Hence its Stanley-Reisner ideal is \( I_{\text{st}_\Delta k} = I_\Delta : x_k \). Furthermore, the Stanley-Reisner ideal of the deletion \( \Delta_{-k} \) considered as a complex on \( [n] \) is \( (x_k, J_{\Delta_{-k}}) = (x_k, I_\Delta) \), where \( J_{\Delta_{-k}} \subset S \) is the extension ideal of the Stanley-Reisner ideal of \( \Delta_{-k} \) considered as a complex on \( [n] \setminus \{k\} \). Thus, we get the short exact sequence.

\[
0 \to (K[\text{st}_\Delta k]/\ell)(-1) \to K[\Delta]/\ell \to K[\Delta_{-k}]/\ell \to 0,
\]

As preparation, we need the following technical result.

**Lemma 3.1.** Let \( \Delta \) be a \( (d-1) \)-dimensional Buchsbaum complex, and let \( k \) be a vertex of \( \Delta \) such that \( \text{lk}_\Delta k \) is 2-CM and \( \Delta_k \) is Buchsbaum of dimension \( d-1 \). Then, for every linear system of parameters \( \ell := \ell_1, \ldots, \ell_d \) of \( K[\Delta] \), there is an exact sequence of graded modules

\[
0 \to (K[\text{st}_\Delta k]/\ell_j)(-1) \to K[\Delta]/\ell_j \to K[\Delta_{-k}]/\ell_j \to 0.
\]

**Proof.** For \( j \in [d] \), set \( \ell_j := \ell_1, \ldots, \ell_j, \ell_{j+1} \). We will show that there is a short exact sequence

\[
0 \to (K[\text{st}_\Delta k]/\ell_j)(-1) \to K[\Delta]/\ell_j \to K[\Delta_{-k}]/\ell_j \to 0.
\]

Since \( \Delta \) is Buchsbaum, the long exact cohomology sequence induced by Sequence (1) provides the exact sequence

\[
0 \to H^d_m(K[\Delta]) \xrightarrow{\varphi} H^d_m(K[\Delta_{-k}]) \to H^d_m(K[\text{st}_\Delta k])(-1)
\]

Using that the modules \( H^d_m(K[\Delta]) \) and \( H^d_m(K[\Delta_{-k}]) \) are concentrated in degree zero, we see that every non-trivial element in \( \text{coker} \varphi \) gives a socle element of \( H^d_m(K[\text{st}_\Delta k]) \). However, since \( \text{lk}_\Delta k \) is 2-CM, the socle of \( H^d_m(K[\text{st}_\Delta k])(-1) \) is concentrated in degree \( d > 0 \). We conclude that \( \varphi \) is an isomorphism.

Since \( \text{st}_\Delta k \) is Cohen-Macaulay and has dimension \( d-1 \), it follows that

\[
H^i_m(K[\Delta]) \cong H^i_m(K[\Delta_{-k}]) \quad \text{for all } i \neq d.
\]

Using that \( \Delta \) and \( \Delta_{-k} \) are Buchsbaum this implies, for \( j = 0, \ldots, d-1 \),

\[
H^0_m(K[\Delta]/\ell_j) \cong H^0_m(K[\Delta_{-k}]/\ell_j).
\]

We now show exactness of Sequence (2) by induction. Let \( j \in \{0, \ldots, d-1\} \). Using the induction hypothesis, multiplication by \( \ell_{j+1} \) induces the following commutative diagram
Since \( K[\Delta] / \ell_j \) and \( K[\Delta_{-k}] / \ell_j \) are Buchsbaum rings, we conclude that
\[
0 :_{K[\Delta] / \ell_j} \ell_{j+1} \cong H^0_m(K[\Delta] / \ell_j) \cong H^0_m(K[\Delta_{-k}] / \ell_j) \cong 0 :_{K[\Delta_{-k}] / \ell_j} \ell_{j+1}.
\]
Hence, the Snake lemma provides the exact sequence
\[
0 \to (K[\text{st}_\Delta k] / \ell_{j+1})(-1) \to K[\Delta] / \ell_{j+1} \to K[\Delta_{-k}] / \ell_{j+1} \to 0,
\]
as desired. \( \Box \)

Note that the above result remains true if one uses a system of parameters of arbitrary (positive) degrees. In the special case when \( \Delta \) is an orientable homology manifold, Lemma \( \ref{lemma:face-lem} \) essentially reduces to \( \ref{lemma:face-lem-dim} \), Proposition 4.24.

The following result is more general than Theorem \( \ref{thm:face-enum} \). However, to take full advantage of this generality one needs an analogue of Theorem \( \ref{thm:face-enum} \) for 2-Buchsbaum complexes.

**Theorem 3.2.** Let \( \Delta \) be a \((d - 1)\)-dimensional 2-Buchsbaum simplicial complex such that \( \beta_{d-1}(\Delta) \neq 0 \), and let \( \ell := \ell_1, \ldots, \ell_d \) be a linear system of parameters of \( K[\Delta] \).

Set \( I := \bigoplus_{j=1}^d [\text{Soc} K[\Delta] / \ell_j] \). Then \( (K[\Delta] / \ell)/I \) is a level ring of Cohen-Macaulay type \( \beta_{d-1}(\Delta) \).

**Proof.** Since we know that \( \dim_K(K[\Delta] / \ell)_d = \beta_{d-1}(\Delta) \) is it enough to show that the socle of \( (K[\Delta] / \ell)/I \) vanishes in all degree \( j < d \). By definition of \( I \), this is true if \( j = d - 1 \).

Let \( z \in K[\Delta] / \ell \) be an element of degree \( j \leq d - 2 \) such that \( m \cdot z \subseteq I \subseteq \text{Soc} K[\Delta] / \ell \). We have to show that \( z \) is already in \( \text{Soc} K[\Delta] / \ell \).

To this end let \( k \in [n] \) be any vertex of \( \Delta \). Since \( \Delta \) is 2-Buchsbaum, \( \text{lk}_\Delta k \) is 2-CM by Miyazaki (\( \ref{lemma:face-lem-dim} \), Lemma 4.2). Thus, we may apply Lemma \( \ref{lemma:face-lem-dim} \). We rewrite the exact sequence therein as
\[
0 \to (K[\text{st}_\Delta k] / \ell)(-1) \overset{\varphi}{\to} K[\Delta] / \ell \to K[\Delta] / (\ell, x_k) \to 0.
\]
This sequence implies that there is some \( y \in K[\text{st}_\Delta k] / \ell \) of degree \( j \) such that \( \varphi(y) = x_k \cdot z \).
Since \( x_k \cdot z \) is in \( \text{Soc} K[\Delta] / \ell \), we conclude that \( y \) is in the socle of \( K[\text{st}_\Delta k] / \ell \). However, since \( \text{lk}_\Delta k \) is 2-CM of dimension \( d - 2 \) and \( \text{st}_\Delta k \) is the cone over \( \text{lk}_\Delta k \), it follows that the socle of \( K[\text{st}_\Delta k] / \ell \) is concentrated in degree \( d - 1 > j = \deg y \). This shows that \( y = 0 \), thus \( x_k \cdot z = 0 \). Since this is true for every vertex \( k \), we have shown \( m \cdot z = 0 \), as required. \( \Box \)

4. **Face enumeration**

Now we discuss how Theorems \( \ref{thm:face-enum} \) and \( \ref{thm:face-enum-dim} \) imply upper and lower bounds on the face vector of a Buchsbaum* complex \( \Delta \).

The face or \( f \)-vector of a \((d - 1)\)-dimensional simplicial complex \( \Delta \) is the sequence
\[
f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)),
\]
where \( f_j(\Delta) \) is the number of \( j \)-dimensional faces of \( \Delta \). The same information is encoded in the \( h \)-vector \( h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta)) \), which is defined by
\[
\frac{h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d}{(1-t)^d} := \sum_{j \geq 0} \dim_K(K[\Delta])_j t^j = \sum_{j=-1}^{d-1} \frac{f_j(\Delta) t^{j+1}}{(1-t)^{j+1}}.
\]
More explicitly, this gives
\[
h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}(\Delta) \quad \text{and} \quad f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i(\Delta).
\]
The $h'$-vector $h'(\Delta) := (h'_0(\Delta), \ldots, h'_d(\Delta))$ is defined by

$$h'_j(\Delta) := h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta).$$

If $\Delta$ is Buchsbaum, then its $h'$-vector is again a Hilbert function because, according to [15],

$$h'_j = \dim_K [K[\Delta]/\ell]_j$$

if $\ell := \ell_1, \ldots, \ell_d$ is a linear system of parameters of $K[\Delta]$.

Following [12], we define the $h''$-vector $h''(\Delta) := (h''_0(\Delta), \ldots, h''_d(\Delta))$ of $\Delta$ by

$$h''_j(\Delta) := h'_j(\Delta) - \binom{d}{j} \beta_{j-1}(\Delta) = h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j} (-1)^{j-i-1} \beta_{i-1}(\Delta) \quad \text{if } i < d$$

and

$$h''_d(\Delta) := \beta_{d-1}(\Delta).$$

The key for our purposes is that in case $\Delta$ is Buchsbaum* $h''(\Delta)$ also is a Hilbert function.

**Corollary 4.1.** Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum* complex. Using the notation of Theorem 1.2, set $K(\Delta) := (K[\Delta]/\ell)/I$. Then $K(\Delta)$ is a level algebra whose Hilbert function is given by

$$\dim_K [K(\Delta)]_j = h''_j(\Delta).$$

**Proof.** This follows by combining Theorem 1.1 and Theorem 1.2. □

In order to use this information, we recall a result about Hilbert functions.

**Notation 4.2.** (i) We always use the following convention for binomial coefficients: If $a \in \mathbb{R}$ and $j \in \mathbb{Z}$ then

$$\binom{a}{j} := \begin{cases} \frac{a(a-1)\cdots(a-j+1)}{j!} & \text{if } j > 0 \\ 1 & \text{if } j = 0 \\ 0 & \text{if } j < 0. \end{cases}$$

(ii) Let $b > 0$ and $d \geq 0$ be integers. Then there are uniquely determined integers $m_s, \ldots, m_{d+1}$ such that $m_{d+1} \geq 0$, $n + d - 2 \geq m_d > m_{d-1} > \ldots > m_s \geq s \geq 1$, and

$$b = m_{d+1}\binom{n-1+d}{d} + \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \ldots + \binom{m_s}{s}.$$

This is called the $d$-binomial expansion of $b$. For any integer $j$ we set

$$b^{(d)} := m_{d+1}\binom{n+d}{d+1} + \binom{m_d+1}{d+1} + \binom{m_{d-1}+1}{d} + \ldots + \binom{m_s+1}{s+1}.$$

(iii) If $b = 0$, then we put $b^{(d)} := 0$ for all $d \in \mathbb{Z}$.

Assume $b > 0$. Note that then $s = d+1$ if and only if $\binom{n-1+d}{d}$ divides $b$. Furthermore, $(m_d) + (m_{d-1}) + \ldots + (m_s)$ is the standard $d$-binomial representation of $b - m_{d+1}\binom{n-1+d}{d}$.

We are ready to state a generalization of Macaulay’s characterization of Hilbert functions of algebras to modules, which has been proven by Hulett [9] in characteristic zero and by Blancfort and Elias [3] in arbitrary characteristic.

**Theorem 4.3.** For a numerical function $h : \mathbb{N}_0 \to \mathbb{N}_0$, the following conditions are equivalent:
(a) In non-negative degrees $h$ is the Hilbert function of a module over $S = K[x_1, \ldots, x_n]$ that is generated in degree zero, i.e., there is a graded $S$-module $M$ whose minimal generators have degree zero such that $h(j) = \dim_K[M]_j$ whenever $j \geq 0$.

(b) For all integers $j \geq 0$,

$$h(j + 1) \leq h(j)^{(j)}.$$

**Proof.** This follows from [3], Theorem 3.2.

The following result provides restrictions on the face vectors of Buchsbaum* complexes with given Betti numbers. The upper bound strengthens [12], Theorem 4.3, for Buchsbaum complexes in the case of Buchsbaum* complexes.

**Theorem 4.4.** Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum* complex on $n$ vertices. Then its $h'$-vector $(h'_0, \ldots, h'_d)$ and $h''$-vector $(h''_0, \ldots, h''_d)$ satisfy $h' = h'_0 = 1$, $h'_1 = h''_1 = n - d$, $h''_d = \beta_{d-1}(\Delta)$, and

(a) $h'_{j+1} \leq \min \left\{ (h''_j)^{(j)}, (h''_{j+2})^{(d-j-2)} + \beta_j(\Delta) \binom{d}{j+1} \right\}$ if $1 \leq j \leq d - 2$;

(b) $h''_{d-j} \geq \frac{h''_j}{\beta_{d-1}(\Delta)}$ if $1 \leq j \leq d - 1$.

**Proof.** (a) We use the notation of Corollary 4.1. The inequality $h'_{j+1} \leq (h''_j)^{(j)}$ follows as in [12], Theorem 4.3, by applying Theorem 4.3 to the algebra $(\overline{K[\Delta]}/I)/I$, $S$.

The canonical module $\omega_{\overline{K(\Delta)}}$ of $\overline{K(\Delta)} = (K[\Delta]/[\ell])/I$ is generated in degree $-d$ because $\overline{K(\Delta)}$ is level. Notice that, for all integers $j$,

$$\dim_K[\omega_{\overline{K(\Delta)}}]_{-j} = \dim_K[\overline{K(\Delta)}]_j = h''_j.$$  

Hence Theorem 4.3 provides

$$h'_{j+1} - \beta_j(\Delta) \binom{d}{j+1} = h''_{j+1} \leq (h''_{j+2})^{(d-j-2)},$$

which completes the proof of (a).

(b) is a consequence of Theorem 2 in [18].

**Remark 4.5.** (i) It is a wide open problem to characterize the Hilbert functions of artinian level algebras. A systematic study of them was begun in [8].

(ii) The bound in Part (b) is never attained. This will be shown in the forthcoming paper [3].

(iii) According to Söderberg ([16], Theorem 4.7), the $h''$-vector also has to satisfy the determinantal condition

$$\begin{vmatrix}
  h''_{j-1} & h''_j & h''_{j+1} \\
  r_{j-1} & r_j & r_{j+1} \\
  r_{d-j+1} & r_{d-j} & r_{d-j-1}
\end{vmatrix} \geq 0$$

for all integers $j$, where $r_j := \binom{n-1+j}{j}$. Söderberg’s result depends on a joint conjecture with Boij in [3] that has been proven by Eisenbud and Schreyer [7].

Notice that Söderberg’s condition characterizes the $h$-vectors of artinian level algebras up to multiplication by a rational number.

**Acknowledgement.** The author would like to thank Isabella Novik, Christos Athanasiadis, and an anonymous referee for helpful comments.
References

[1] C. A. Athanasiadis, V. Welker, *Buchsbaum* complexes, Preprint, 2009; also available at arXiv:0909.1931

[2] E. Babson, I. Novik, *Face numbers and nongeneric initial ideals*, Electron. J. Comb. 11 (2006), Special volume in honor of R. Stanley, #R25, 23 pp.

[3] C. Blancafort, J. Elias, *On the growth of the Hilbert function of a module*, Math. Z. 234 (2000), 507–517.

[4] M. Boij, J. Söderberg, *Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture*, J. London Math. Soc. 78 (2008), 78–101.

[5] M. Boij, J. Migliore, R. Miró-Roig, U. Nagel, F. Zanello, in preparation.

[6] W. Bruns, J. Herzog, *Cohen-Macaulay rings. Rev. ed.*. Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, 1998.

[7] D. Eisenbud, F. Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. 22 (2009), 859–888.

[8] A.V. Geramita, T. Harima, J. Migliore, and Y.S. Shin, *The Hilbert function of a level algebra*, Mem. Amer. Math. Soc. 186 (2007), No. 872.

[9] H. Hulett, *A generalization of Macaulay’s theorem*, Comm. Algebra 23 (1995), 1249–1263.

[10] E. Miller, I. Novik, E. Swartz, *Face rings of simplicial complexes with singularities*, Preprint, 2010; also available at arXiv:1001.2812.

[11] M. Miyazaki, *On 2-Buchsbaum complexes*, J. Math. Kyoto Univ. 30 (1990), 367–392.

[12] I. Novik, E. Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math. 222 (2009), 2059–2084.

[13] I. Novik, E. Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. 145 (2009), 993–1000.

[14] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. 21 (1976), 30–49.

[15] P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. 178 (1981), 125–142.

[16] J. Söderberg, *Graded Betti numbers and h-vectors of level modules*, Preprint, 2006; also available at arXiv:0803.1645

[17] R. P. Stanley, *Combinatorics and commutative algebra. Second edition*. Progress in Mathematics 41, Birkhäuser Boston 1996.

[18] R. P. Stanley, *Cohen-Macaulay Complexes*, Higher Combinatorics, M. Aigner Ed., Reidel, Dordrecht and Boston (1977), 51–62.

[19] J. Stückrad, W. Vogel: *Buchsbaum rings and applications. An interaction between algebra, geometry and topology*, Springer-Verlag, Berlin, 1986.

[20] E. Swartz, *Face enumeration: From spheres to manifolds*, J. Eur. Math. Soc. 11 (2009), 449–485.

Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027, USA

E-mail address: uwenagel@ms.uky.edu