INSTANTON CORRECTIONS IN $\mathcal{N} = 2$ SUPERSYMMETRIC THEORIES WITH CLASSICAL GAUGE GROUPS AND FUNDAMENTAL MATTER HYPERMULTIPLETS

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Abstract

We compute instanton corrections to the low energy effective prepotential of $\mathcal{N} = 2$ supersymmetric theories in a variety of cases, including all classical gauge groups and even number of fundamental matter hypermultiplets. To this end, we take profit of a set of first- and second-order equations for the logarithmic derivatives of the prepotential with respect to the dynamical scale expressed in terms of Riemann’s theta-function. These equations emerge in the context of the Whitham hierarchy approach to the low-energy Seiberg–Witten solution of supersymmetric gauge theories. Our procedure is recursive and allows to compute the effective prepotential to arbitrary order in a remarkably straightforward way. General expressions for up to three-instanton corrections are given. We illustrate the method with explicit expressions for several cases.

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1 Introduction

Some five years ago, Seiberg and Witten gave an ansatz for the dominant piece of the effective action governing the light degrees of freedom of $SU(2)\,\mathcal{N} = 2$ super Yang-Mills theory at low energy\cite{1}. It is given in terms of an auxiliary complex algebraic curve $C$ (whose moduli space is identified with the quantum moduli space of the low-energy theory $\mathcal{M}_A$) and a given meromorphic differential, $dS_{SW}$, that induces a special geometry on $\mathcal{M}_A$ (see Refs.\cite{2} for excellent reviews). The solution was soon extended to other gauge groups and matter content by determining both the appropriate complex curve and meromorphic differential \cite{3, 4, 5, 6, 7}, thus leading to a substantial progress in our understanding of $\mathcal{N} = 2$ supersymmetric gauge theories. In particular, the appearance of an auxiliary Riemann surface made it possible to identify remarkable connections with string theory, singularity theory of differentiable maps and integrable systems. The framework that will be used in this paper is strongly inspired by the latter one.

It is well-known that, as long as $\mathcal{N} = 2$ supersymmetry is unbroken, the low energy effective action is given in terms of a holomorphic prepotential $F$. The solution proposed by Seiberg and Witten embodies a prescription to compute this prepotential. However, its explicit evaluation for a given gauge group and matter content is technically involved and it requires to integrate an expression for the B-periods of $dS_{SW}$ as functions of its A-periods. The complexity of this procedure increases rapidly with the rank of the gauge group, even without matter hypermultiplets.

Whereas perturbative contributions to $F$ are exhausted by one-loop diagrams \cite{8}, the non-perturbative part is given by an infinite series of instanton corrections. The importance of instanton calculus lies precisely in the fact that it provides one of the few non-perturbative links between the Seiberg–Witten solution and the microscopic non-abelian field theory that it is supposed to describe effectively at low energies. From the microscopic point of view, the first few instanton contributions to the asymptotic semiclassical expansion of the effective prepotential have been computed for gauge group $SU(N_c)$, and a remarkable agreement with the Seiberg–Witten solution has been found \cite{9}. From the side of the effective theory, several methods for determining the instanton corrections have been developed in the last few years by using the Picard–Fuchs equations \cite{10}, holomorphicity arguments \cite{11}, analytic continuation \cite{12} (also for non-hyperelliptic curves \cite{13}), modular anomaly equations \cite{14}, etc. Among them, we would like to distinguish those methods that lead to recursion relations for the $k$-instanton corrections, as long as they give an implicit expression for the exact solution. For the case of $SU(2)$, recursion relations determining the whole instanton expansion have been obtained both in
the pure gauge theory [15] and when matter is included [10]. These recursion relations were obtained by combining the renormalization group and Picard-Fuchs equations. Also one-instanton corrections for the whole ADE series where obtained along the same lines in [16]. SU($N_c$) with additonal matter in the adjoint representation was considered in [17] from the point of view of the Calogero-Moser model, and in [14] (for ADE groups with dual Coxeter number $k_D \leq 6$) where the modular anomaly equations of softly broken $N = 4$ supersymmetric gauge theories were invoked.

In a recent paper, [18], a new strategy was observed to work very well for the case of pure SU($N_c$). It is based on a set of first- and second-order equations for the logarithmic derivatives of the prepotential with respect to the dynamical scale $\Lambda$, evaluated all over the moduli space

\[
\frac{\partial F}{\partial \log \Lambda} = \frac{\beta}{2\pi i} u_2 ,
\]

\[
\frac{\partial^2 F}{\partial (\log \Lambda)^2} = -\frac{\beta^2}{2\pi i} \frac{\partial u_2}{\partial a^i} \frac{1}{i\pi} \frac{\partial}{\partial u^j} \log \Theta_{E}(0|\tau) ,
\]

where the different quantities entering these expressions will be explained below. Each one of these equations was obtained separately in the last few years by many authors [15, 19, 20, 21, 22]. It was not until very recently that a unifying approach based on the Whitham hierarchy was shown to be useful to obtain both [23]. In Ref. [18], the ansatz for the semiclassical expansion of the prepotential for gauge group SU($N_c$) was inserted in both sides of equations (1.1)–(1.2) with the result of an elegant and systematic procedure that allowed us to compute instanton corrections up to any desired order with relatively little effort. In particular, this method does not require knowledge of the actual solution for the periods $a$ and $a_D$ of $dS_{SW}$, a fact which spares a considerable amount of work.

In this paper we will exhibit the strenght of this method by extending the results of [18] to any classical gauge group with and without matter content. We shall limit ourselves to asymptotically free theories. The obvious question arises, about the validity of equations like (1.2) for all these situations. It turns out that, for our present purposes, the only constraint which seems to be unavoidable within this method is that massive hypermultiplets have to be introduced in pairs, degenerated in mass. This can be understood either from a purely theoretical study, which we shall leave for a separate paper [24], or else, “a posteriori”, for the consistency of the results.

Our aim is that this paper could be useful to anyone interested in finding explicit expressions for the instanton corrections to the effective prepotential of $N = 2$ supersymmetric gauge theories. To this end, general formulas for up to three-instanton corrections will be given and the method to obtain arbitrary higher corrections will be clearly ex-
plained. A number of particular examples will be also worked out for the lower rank groups and small number of flavour-pairs, in order to make them easily available for further comparison even to other methods or, actually, to the results obtained in the microscopic non-Abelian field theory. For the one- and two-instanton corrections, our results coincide with those in the literature while, for most of the three-instanton contributions, our results are new.

2 Recursive Evaluation of the Effective Prepotential

2.1 Review and Notation

For completeness, we shall start by reviewing the rôle of the different ingredients that enter the formulas (1.1) and (1.2). The low-energy dynamics of $\mathcal{N}=2$ supersymmetric theories with classical gauge group $G$ corresponding to $N_c$ colors, and $N_f$ hypermultiplets of mass $m_f = m_{(f+N_f/2)}$ (i.e. degenerated in masses) in the fundamental representation of $G$, can be described in terms of an auxiliary hyperelliptic curve $C$ given by

$$y^2 = (P(\lambda, e_p) + T(\lambda, m_f, \Lambda))^2 - 4\Lambda^\beta F(\lambda, m_f),$$

(2.1)

where $P$ is the characteristic polynomial of $G$, $\Lambda$ is the quantum generated dynamical scale, $\beta$ is the coefficient of the one-loop $\mathcal{N}=2$ beta function and $e_p$ are the eigenvalues of the complex scalar field $\langle \phi \rangle = \sum_p e_p h_p$ (see Ref. [25] for the conventions followed in the notation of Lie group and Lie algebra objects) that belongs to the $\mathcal{N}=2$ vector supermultiplet in the adjoint of $G$. $T$ and $F$ are polynomials that do not depend on the moduli $e_p$ and $T$ is different from zero only when $N_f > N_c$. As pointed out in Ref. [12], when the gauge group is $SU(N_c)$ or $SO(N_c)$ all dependence on $T$ can be absorbed in a redefinition of $e_p$, the effective prepotential remaining untouched. Thus, we can set $T = 0$, and write the hyperelliptic curve $C$ as

$$y^2 = P^2(\lambda, e_p) - 4\Lambda^\beta F(\lambda, m_f).$$

(2.2)

In the case of $Sp(N_c)$, there is a residual value for $T$, $T = \Lambda^{N_c-N_f+2}(\prod_{f=1}^{N_f/2} m_f^2)$. In order to consider all the different cases within a unified framework we will neglect this contribution by setting the mass of one of the degenerated hypermultiplets to zero (say for example $m_{N_f/2} = 0$). Following Ref. [12] this case will be denoted $Sp(N_c)'$, and the corresponding hyperelliptic curve has the form (2.2) with the proviso that, according to our previous remark, the number of hypermultiplets will be at least two, $N_f = N_f' + 2 \geq 2$. 

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Consequently, the case of pure $Sp(N_c)$ will not be attainable from our results though it is quite simple to make the appropriate modifications, as will be explained below. It is convenient, for later use, to list the form of $P$, $F$ and $\beta$ for all classical groups:

| $G$        | $P(\lambda, e_p)$ | $F(\lambda)$ | $\beta/2$ | $l_G$ | $\xi$ | $\xi^+$ | $\xi^-$ | $k_D$ |
|------------|--------------------|---------------|------------|-------|-------|--------|--------|-------|
| $SU(r+1)$  | $\prod_{p=1}^{r+1}(\lambda - e_p)$ | $\prod_{f=1}^{N_f}(\lambda + m_f)$ | $r + 1 - N_f/2$ | $r+1$ | 1     | 1      | 0      | $r+1$ |
| $SO(2r)$   | $\prod_{p=1}^{r}(\lambda^2 - e_p^2)$ | $\lambda^4 \prod_{f=1}^{N_f}(\lambda^2 - m_f^2)$ | $2r - N_f - 2$ | $r$   | 1     | 1      | 1      | $2r-2$|
| $SO(2r+1)$ | $\prod_{p=1}^{r}(\lambda^2 - e_p^2)$ | $\lambda^2 \prod_{f=1}^{N_f}(\lambda^2 - m_f^2)$ | $2r - N_f - 1$ | $r$   | 1     | 1      | 1      | $2r-1$|
| $Sp(2r)^\prime$ | $\prod_{p=1}^{r}(\lambda^2 - e_p^2)$ | $\prod_{f=1}^{N_f}(\lambda^2 - m_f^2)$ | $2r - N_f'$ | $r$   | 2     | 2      | 2      | $r+1$ |

Table 1

The symmetric polynomials $\bar{u}_k(e_p)$ and $t_i(m_f)$ are defined through the expansions

$$P(\lambda, e_p) \equiv \lambda^h - \sum_{k=2}^{h} \lambda^{h-k} \bar{u}_k(e_p) , \quad (2.3)$$

$$F(\lambda, m_f) \equiv \lambda^h + \sum_{k=1}^{h} \lambda^{h-k} t_k(m_f) , \quad (2.4)$$

where $h$ stands, in each case, for the highest power in $P(\lambda, e_p)$ or $F(\lambda, m_f)$. The moduli for each curve (2.2) can be taken to be either the independent roots $e_p$ or the $a^i$ defined as the coefficients of $\langle \phi \rangle = a^i H_i$ in the Chevalley basis, hence linearly related to $e_p$ (see Ref. [25]). Neither of these parameters are invariant under Weyl transformations which, in particular, act by permutation on the $e_p$. On the contrary, the symmetric polynomials (Casimirs) $\bar{u}_k$ provide faithful coordinates for the moduli space of vacua. In particular, $\bar{u}_2 = \frac{1}{2} \tr \phi^2 = (\bar{\xi}^+ + \bar{\xi}^-) \sum_{p=1}^{l_2} e_p^2$, and in general: $\bar{u}_k = \frac{1}{k} \tr \phi^k + \ldots$, the dots standing for homogeneous powers of lower Casimir operators.

The Seiberg–Witten meromorphic differential can be written as

$$dS_{SW}(u_k, m_f) = \left( P' - \frac{PF'}{2F} \right) \frac{\lambda d\lambda}{y} , \quad (2.5)$$

and the quantum relations between the low-energy coordinates of the moduli space $a^i, a_{D,j}$
and the “mean field” order parameters \( u_k(a) = \frac{1}{k} \text{tr}(\phi^k) + \cdots = \bar{u}_k(a') + \mathcal{O}(\Lambda) \) are implicitly given by the period integrals

\[
a^i(u_k, m_f) = \oint_{A^i} dS_{SW}(u_k, m_f) ; \quad a_{Dj}(u_k, m_f) = \oint_{B_j} dS_{SW}(u_k, m_f),
\]

where \( A^i \) and \( B_j \) constitute a symplectic basis of homology cycles with canonical intersections of the hyperelliptic curve (2.2); the effective prepotential \( F \) is implicitly defined by the equation

\[
a_{Di} = \frac{\partial F(a)}{\partial a'^i},
\]

so that its exact determination involves the integration of functions \( a_{Di}(a) \) for which there is not a closed form available. In this context, the existence of an algorithm that let us determine the exact form of \( F \) without going through the actual computation of \( a^i(u_k, m_f) \) and \( a_{Dj}(u_k, m_f) \) is welcome.

As mentioned above, our first ingredient is the set of RG equations (1.1)–(1.2). Strictly speaking, as they stand, they are valid for the pure gauge theory. In presence of matter, the first equation receives an additional term which depends only on the masses and the second one has to be modified for \( \beta = 2 \) where it receives an additional constant contribution [24]. In summary

\[
\frac{\partial F}{\partial \log \Lambda} = \frac{\beta}{2\pi i} u_2 + \varphi(m), \quad (2.8)
\]

\[
\frac{\partial^2 F}{\partial (\log \Lambda)^2} = -\frac{\beta^2}{2\pi i} \frac{\partial u_2}{\partial \omega} \frac{1}{\pi} \partial_{n_j} \log \Theta_E(0|\tau) + \frac{\beta^2}{2\pi i} \Lambda^2 \delta_{\beta,2}. \quad (2.9)
\]

In Eq. (2.9), \( \Theta_E(0|\tau) \) is Riemann’s theta function associated to the hyperelliptic curve \( C \)

\[
\Theta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (\xi|\tau) = \sum_{n_k \in \mathbb{Z}} e^{i\pi \left[ n_i(n_i+\alpha_i)(n_j+\alpha_j)+2(n_i+\alpha_i)(\xi+\beta_i) \right]}, \quad (2.10)
\]

where \( E = \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \) stands for an even half integer characteristic. In almost all the cases this characteristic will be \( E = \left[ \begin{array}{c} 0,...,0 \\ \frac{1}{2},...,\frac{1}{2} \end{array} \right] \), as it is in the case of pure \( SU(N_c) \) [18]. The only exception is \( Sp(2r)'' \) for which the characteristic gets modified to the value \( E = \left[ \begin{array}{c} 0,...,0,0 \\ 0,...,0,\frac{1}{2} \end{array} \right] \).

\(^1\) Although the addition of an \( a'^i \)-independent terms to \( F \) is unphysical from the point of view of the effective theory, the embedding of the Seiberg-Witten solution into the Whitham dynamics fixes them as a function of the bare coupling \( \tau_0 \). A similar behaviour was observed in the study of \( F \) near the strong coupling singularities of \( SU(N_c) N = 2 \) super Yang–Mills theory [20].
The second ingredient is an ansatz for the instanton expansion of the prepotential valid for any classical gauge group $G$ and $N_f$ massive hypermultiplets with paired masses

$$
\mathcal{F}_{G,N_f} = \frac{\tau_0^G}{4\pi i} \sum_{\alpha_+} Z_{\alpha_+}^2 + \frac{i}{4\pi} \xi \sum_{\alpha_+} Z_{\alpha_+}^2 \log \frac{Z_{\alpha_+}^2}{\Lambda^2} - \frac{i}{4\pi} \xi \frac{l_G}{2} \sum_{p=1}^{N_f/2} \sum_{f=1}^{l_G} (e_p + m_f)^2 \log \frac{(e_p + m_f)^2}{\Lambda^2} \\
- \frac{i}{4\pi} \xi \sum_{p=1}^{l_G} \sum_{f=1}^{N_f/2} (e_p - m_f)^2 \log \frac{(e_p - m_f)^2}{\Lambda^2} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k(Z) \Lambda^{k\beta},
$$

(2.11)

where $\alpha_+$ denotes a positive root and $\sum_{\alpha_+}$ is the sum over all positive roots. The set $\{\alpha_i\}_{i=1,...,r}$ stands for the simple roots of the corresponding classical Lie algebra, they generate the root lattice $\Delta = \{\alpha = n^i \alpha_i | n^i \in \mathbb{Z}\}$. The dot product $(\cdot)$ of two simple roots $\alpha_i$ and $\alpha_j$ gives an element of the Cartan matrix, $A_{ij} = \alpha_i \cdot \alpha_j$, and extends bilinearly to arbitrary linear combinations of simple roots. So, for example, for any root $\alpha = n^j \alpha_j \in \Delta$, the quantities $Z_\alpha$ are defined by $Z_\alpha = a \cdot \alpha \equiv a^i A_{ij} n^j$ where $a = a^i \alpha_i$. For non-simply laced Lie algebras this product is not symmetric.

Simple roots can be written in terms of the orthogonal set of unit vectors $\{\epsilon_p\}_{p=1,...,l_G}$. The order parameters $a^i$ and $\epsilon_p$ are related by $\epsilon_p = a \cdot \epsilon_p$. The exact relations and the actual values of $\alpha_i \cdot \epsilon_p$ for each classical gauge group can be found in Ref.[25]. Finally, we also have that $\lambda^k \cdot \alpha_j = \delta^k_j$ define the fundamental weights. In particular, this means that $\alpha_i = \sum_k A_{ik} \lambda^k$. We have also introduced three parameters $\xi$, $\xi^+$ and $\xi^-$, so as to deal with all classical Lie algebras within one single ansatz; $k_D$ denotes the dual Coxeter number. The particular values of these variables for each classical gauge group are shown in Table 1. The coefficient of the one-loop beta function turns out to be given by

$$
\beta/2 = \xi k_D - \frac{N_f}{2} \left( 1 + \frac{\xi^-}{\xi^+} \right).
$$

(2.12)

By expressing the roots $\alpha_+$ in terms of $\epsilon_p$, (i.e., $Z_{\alpha_+}$ in terms of $\epsilon_p$) one can also check that the following relation holds

$$
\xi^+ \sum_{p=1}^{l_G} \epsilon_p^2 = \frac{1}{k_D} \sum_{\alpha_+} Z_{\alpha_+}^2.
$$

(2.13)

### 2.2 The procedure

It is important to notice that we may shift $\tau_0^G$ in (2.11) to any value by appropriately rescaling $\Lambda$, and this will be reflected in our choice for the normalization of the $\mathcal{F}_k(Z)$. We have fixed it in all cases to be $\tau_0^G = 3\beta/2k_D$ so that quadratic terms in $Z_\alpha$ do not contribute to the coupling constant $\tau_{ij}$ (see (2.17) below).
The l.h.s. of Eq.(2.9) can be easily computed from the expansion of the effective prepotential (2.11) to be

\[
\frac{\partial^2 F}{\partial (\log \Lambda)^2} = \frac{1}{2\pi i} \sum_{k=1}^{\infty} (k\beta)^2 F_k(Z) \Lambda^{k\beta},
\]

then comparing (2.14) and (2.9) we get

\[
\sum_{k=1}^{\infty} k^2 F_k \Lambda^{k\beta} = -\frac{\partial u_2}{\partial a^i} \frac{\partial u_2}{\partial a^j} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) + \Lambda^2 \delta_{\beta,2},
\]

such that the instanton correction \( F_k \) can be obtained through a set of recursive relations after expanding the r.h.s. of Eq.(2.15) in powers of \( \Lambda^\beta \).

The expansion of the derivative of the quadratic Casimir in powers of \( \Lambda \) can be obtained from the RG equation (2.8) and (2.11),

\[
\frac{\partial u_2}{\partial a^i} = 2\pi i \frac{\partial^2 F}{\partial a^i \partial \log \Lambda} = \frac{2\xi}{\beta} \sum_{\alpha} Z_{\alpha\beta} \partial_i Z_{\alpha\beta} - \frac{N_f}{\beta} (\xi^+ + \xi^-) \sum_p e_p \partial_i e_p + \sum_{k=1}^{\infty} k F_{k,i} \Lambda^{k\beta}
\]

\[
= \xi^+ \sum_p e_p \partial_i e_p + \sum_{k=0}^{\infty} k F_{k,i} \Lambda^{k\beta} = \sum_{k=0}^{\infty} \mathcal{H}_{2,i}^{(k)} \Lambda^{k\beta},
\]

where \( F_{k,i} = \partial F_k / \partial a^i \) and use has been made of (2.12) and (2.13).

To expand the theta function we need to compute from (2.11) the couplings in the semiclassical region,

\[
\tau_{ij} = \frac{\partial^2 F}{\partial a^i \partial a^j} = \frac{i}{2\pi} \xi \sum_{\alpha} \frac{\partial Z_{\alpha\beta}}{\partial a^i} \frac{\partial Z_{\alpha\beta}}{\partial a^j} \log \left( \frac{Z_{\alpha\beta}}{\Lambda^2} \right) - \frac{i}{2\pi} \xi^+ \sum_{p=1}^{N_f/2} \sum_{f=1}^{N_f/2} \frac{\partial e_p}{\partial a_i} \frac{\partial e_p}{\partial a_j} \log \left( \frac{e_p + m_f}{\Lambda^2} \right) + \frac{1}{2\pi} \sum_{k=1}^{\infty} F_{k,ij} \Lambda^{k\beta},
\]

with \( F_{k,ij} = \frac{\partial^2 F_k}{\partial a^i \partial a^j} \). So the term involving the couplings that appear in the theta function \( \Theta_E \) can be written as

\[
i\pi n^i \tau_{ij} n^j = \sum_{\alpha} \log \left( \frac{Z_{\alpha\beta}}{\Lambda} \right)^{-\xi(\alpha \cdot \alpha)} + \sum_{p,f} \log \left( \frac{e_p + m_f}{\Lambda} \right)^{\xi^+(\alpha \cdot \epsilon_p)^2} +
\]

\[
+ \sum_{p,f} \log \left( \frac{e_p - m_f}{\Lambda} \right)^{\xi^-(\alpha \cdot \epsilon_p)^2} + \frac{1}{2} \sum_{k=1}^{\infty} (\alpha \cdot \mathcal{F}_k^\alpha) \Lambda^{k\beta},
\]

\[7\]
where \( \alpha \mathcal{F}_k'' \alpha \equiv \sum_{i,j} n^i \mathcal{F}_{k,ij} n^j \). Also \( \alpha = n^i \alpha_i \), and we have set
\[
n^i \left( \frac{\partial Z_{\alpha^+}}{\partial a^i} \right) = n^i (\alpha_i \cdot \alpha_+) = \alpha \cdot \alpha_+ ; \quad n^i \left( \frac{\partial \epsilon_p}{\partial a^i} \right) = n^i (\alpha_i \cdot \epsilon_p) = \alpha \cdot \epsilon_p . \tag{2.19}\]
Inserting (2.18) in the Theta function (2.10) with a characteristic \( E = \left[ \begin{array}{c} 0, \ldots, 0 \end{array} \right] \), we obtain
\[
\Theta_E(0|\tau) = \sum_{\alpha \in \Delta} \exp \left[ i \pi n^i \tau_{ij} n^j + i \pi \sum_k n_k \right]
\]
\[
= \sum_{\alpha \in \Delta} (-1)^{\rho \alpha} \prod_{\alpha_+} \left( \frac{Z_{\alpha_+}}{\Lambda} \right)^{-\xi(\alpha \cdot \alpha_+)^2} \prod_{p,f} \left( \frac{e_p + m_f}{\Lambda} \right)^{\xi^+(\alpha \cdot \epsilon_p)^2} \prod_{p,f} \left( \frac{e_p - m_f}{\Lambda} \right)^{\xi^-(\alpha \cdot \epsilon_p)^2}
\]
\[
\times \prod_{k=1}^\infty \exp \left( \frac{1}{2}(\alpha \cdot \mathcal{F}_k'' \alpha) \Lambda^{k\beta} \right)
\]
\[
= \sum_{s=0}^\infty \sum_{\alpha \in \Delta_s} (-1)^{\rho \alpha} \prod_{\alpha_+} Z_{\alpha_+}^{-\xi(\alpha \cdot \alpha_+)^2} \prod_p [R(e_p)]^{(\alpha \cdot \epsilon_p)^2}
\]
\[
\times \prod_{k=1}^\infty \left( \sum_{m=0}^\infty \frac{1}{2^m m!} (\alpha \cdot \mathcal{F}_k'' \alpha)^m \Lambda^{k\beta} \right) \Lambda^{s\beta}
\]
\[
= \sum_{l=0}^\infty \Theta^{(l)} \Lambda^{l\beta} . \tag{2.20}\]
In the previous expression, \( \rho \) is the maximal weight \( \rho = \sum_{i=1}^r \lambda^i \). In the case of \( Sp(N_c)'' \), due to its peculiar characteristic, the dot product \( \rho \cdot \alpha \) needs to be replaced by \( \rho_r \cdot \alpha = n^r \). On the other hand, \( \Delta_s \subset \Delta \) is a subset of the root lattice composed of those lattice vectors \( \alpha \in \Delta_s \) that fulfill the constraint \( \frac{1}{2} \xi^+ \sum_p (\alpha \cdot \epsilon_p)^2 = s \). The Weyl group permutes the \( \epsilon_p \), hence the previous statement is Weyl invariant. In other words, the sets \( \Delta_s \) are unions of Weyl orbits. This fact guarantees that the final result will recombine into Weyl invariant expressions. In (2.20) we have also introduced the polynomial
\[
R(\lambda, m_f) = \prod_{f=1}^{N_f/2} (\lambda + m_f)^{\xi^+} \prod_{f=1}^{N_f/2} (\lambda - m_f)^{\xi^-} \tag{2.21}\]
\[
\equiv \left( \lambda^h + \sum_{i=1}^h q_i(m_f)\lambda^{h-i} \right)^{\xi^+} , \tag{2.22}\]
where \( h \), again, stands for the appropriate highest power. It is at this point where the need for pairwise equal masses enters. Otherwise, we would be dealing with square root factors of the form \( (\lambda \pm m_f)^{1/2} \). Now we can collect the first few terms in the expansion (2.20),
\[
\Theta^{(0)} = 1 , \quad \Theta^{(1)} = \sum_{\alpha \in \Delta_1} (-1)^{\rho \alpha} \prod_{\alpha_+} Z_{\alpha_+}^{-\xi(\alpha \cdot \alpha_+)^2} \prod_p [R(e_p)]^{(\alpha \cdot \epsilon_p)^2} ,
\]
\[ \Theta^{(2)} = \sum_{\alpha \in \Delta_1} (-1)^{p_{\alpha}} \frac{1}{2} \left( \alpha \cdot \mathcal{F}_1'' \alpha \right) \prod_{\alpha^+} Z_{\alpha^+}^{-\xi (\alpha \cdot \alpha^+)^2} \prod_{p} [R(e_p)]^{(\alpha \cdot e_p)^2} \]
\[ + \sum_{\beta \in \Delta_2} (-1)^{p_{\beta}} \prod_{\alpha^+} Z_{\alpha^+}^{-\xi (\beta \cdot \alpha^+)^2} \prod_{p} [R(e_p)]^{(\alpha \cdot e_p)^2} . \]

However, in the logarithmic derivative, the Theta function appears in the denominator so we shall need the expansion of \( \Theta(0|\tau)^{-1} \) in terms of \( \Lambda \). We can write this expansion as
\[ \Theta(0|\tau)^{-1} = \sum_{l=0}^{\infty} \Xi^{(l)}(\Theta) \Lambda^{2Nl} . \] (2.23)

Here \( \Xi^{(0)}(\Theta) = 1 \) and for \( \Xi^{(l)}(\Theta) \) we can write in general
\[ \Xi^{(l)}(\Theta) = \sum_{(p_1, \ldots, p_k) \in \mathbb{N}^k} \chi(p_1, \ldots, p_k) \prod_{i=1}^{l} (\Theta^{(i)})^{p_i} , \] (2.24)
where the coefficients \( \chi \) are parametrized by the partition elements \((p_1, \ldots, p_k)\). The first few values for these parameters are, for example,
\[ \chi(1) = -1 , \quad \chi(2,0) = 1 , \quad \chi(0,1) = -1 , \quad \chi(3,0,0) = -1 , \quad \chi(1,1,0) = 2 , \quad \chi(0,0,1) = -1 , \]
and using these values we can immediately obtain the lower \( \Xi^{(l)}(\Theta) \).

Next, we compute the derivative of the theta function with respect to the period matrix
\[ \frac{1}{i\pi} \partial_{\tau_{ij}} \Theta_{E}(0, \tau) = \sum_{\alpha \in \Delta_1} \sum_{k} n_i^j \exp \left[ i\pi n_k^k \tau_{kl} + i\pi \sum_{k} n_k \right] \]
\[ = \sum_{s=1}^{\infty} \sum_{\alpha \in \Delta_1} \sum_{k} (-1)^{p_{\alpha}} \left( \lambda^i \cdot \alpha \right) \left( \lambda^j \cdot \alpha \right) \prod_{\alpha^+} Z_{\alpha^+}^{-\xi (\alpha \cdot \alpha^+)^2} \prod_{p} [R(e_p)]^{(\alpha \cdot e_p)^2} \]
\[ \times \prod_{k=1}^{\infty} \exp \left( \frac{1}{2} \left( \alpha \cdot \mathcal{F}_k'' \alpha \right) \Lambda^{k\beta} \right) \Lambda^{s\beta} \]
\[ \equiv \sum_{p=1}^{\infty} \Theta^{(p)}_{ij} \Lambda^{p\beta} . \] (2.25)

Now, collecting all the pieces and inserting them back into (2.9), we find for \( \mathcal{F}_k(Z) \) the following expression:
\[ \mathcal{F}_k(Z) = -k^{-2} \sum_{p+q+l=k-1} \sum_{i,j} \mathcal{H}^{(p)}_{2,i} \mathcal{H}^{(q)}_{2,j} \Theta^{(k-p-q-l)}_{ij} \Xi^{(l)} , \] (2.26)
in terms of previously defined coefficients. If we look at the factors on the r.h.s. of Eq. (2.26), it is easy to see that they involve \( F_1, F_2, \ldots \) up to \( F_{k-1} \). In fact, although both \( \mathcal{H}^{(p)}_2 \) and \( \Theta^{(p)} \) depend on \( F_1, \ldots, F_p \), the indices within parenthesis reach at most the value \( k-1 \) as \( \Theta_{ij}^{(0)} = 0 \). Moreover \( \Theta_{ij}^{(k)} \) depends on \( F_1, \ldots, F_{k-1} \) since the vector \( \alpha = 0 \) is missing from the lattice sum. This “lucky accident” has its origin in the particular form of the characteristic in the semiclassical (duality) frame, and seems to be an essential feature in order to build up a recursive procedure to compute all the instanton coefficients by starting just from the perturbative contribution to \( \mathcal{F}(a) [26] \). For the first few cases we may develop (2.26) to find

\[
\begin{align*}
F_1 &= -\sum_{ij} \mathcal{H}^{(0)}_{2,i} \mathcal{H}^{(0)}_{2,j} \Theta_{ij}^{(1)} \\
F_2 &= -\frac{1}{4} \sum_{ij} \left( \Theta_{ij}^{(2)} \mathcal{H}^{(0)}_{2,i} \mathcal{H}^{(0)}_{2,j} + \Theta_{ij}^{(1)} (2 \mathcal{H}^{(1)}_{2,i} \mathcal{H}^{(0)}_{2,j} - \mathcal{H}^{(0)}_{2,i} \mathcal{H}^{(0)}_{2,j} \Theta_{ij}^{(1)}) \right)
\end{align*}
\]

After some algebraic manipulations (2.26) admits the following general form

\[
F_k = -\frac{1}{k^2} \sum_{s=1}^{k} \sum_{\alpha \in \Delta_s} (-1)^{\rho \cdot \alpha} \prod_{\alpha_+} Z_{\alpha+}^{-\xi(\alpha-\alpha_+)^2} \prod_p [R(e_p)]^{(\alpha-e_p)^2} \Phi_{k+1-s}(\alpha) , \tag{2.27}
\]

where the functions \( \Phi_k(\alpha) \) depend on \( \mathcal{F}_1, \ldots, \mathcal{F}_{k-1} \) and have to be evaluated case by case. For the first few we have

\[
\begin{align*}
\Phi_1(\alpha) &= Z_{\alpha(G)}^2 , \\
\Phi_2(\mathcal{F}_1, \alpha) &= \mathcal{F}_1 + 2(\alpha \cdot \mathcal{F}_1) Z_{\alpha(G)} + \frac{1}{2} (\alpha \cdot \mathcal{F}_1') Z_{\alpha(G)}^2 , \\
\Phi_3(\mathcal{F}_1, \mathcal{F}_2, \alpha) &= 4 \mathcal{F}_2 + 4(\alpha \cdot \mathcal{F}_2) Z_{\alpha(G)} + (\alpha \cdot \mathcal{F}_1')^2 + \frac{1}{2} (\alpha \cdot \mathcal{F}_1') (\mathcal{F}_1 + 2(\alpha \cdot \mathcal{F}_1) Z_{\alpha(G)}) \\
&\quad + \frac{1}{8} (\alpha \cdot \mathcal{F}_1')^2 Z_{\alpha(G)}^2 + \frac{1}{2} (\alpha \cdot \mathcal{F}_2') \alpha Z_{\alpha(G)}^2 , 
\end{align*}
\]

where \( \alpha \cdot \mathcal{F}_k = n^k \mathcal{F}_{k,i} \). Expressions (2.27)-(2.30) make patent the iterative character of the procedure. \( Z_{\alpha(G)} \) stands for \( n^i \mathcal{H}^{(0)}_{2,i} \), and for simply laced groups, \( Z_{\alpha(G)} = Z_{\alpha} \) while for non-simply laced, the exact form will be given below. In the case \( \beta = 2 \), as we can see from Eq. (2.13) the first instanton correction acquires a shift \( \tilde{\mathcal{F}}_1 = \mathcal{F}_1 + 1 \). It soon becomes clear that, except for the simplest cases, the concrete evaluation of the \( \mathcal{F}_k \) has to be carried out by symbolic computation. In the next sections we illustrate our procedure with explicit examples in several cases for the lower rank groups.
A last word concerning the possibility to split the masses is in order. Generically, all resulting expressions involve powers of the degenerated masses \( \{ m_f, f = 1, ..., N_f/2 \} \). They must be recovered from the exact result for arbitrary masses in the coincidence limit \( m_f = m_{f+N_f} \). The possibility to go back unambiguously only happens for low powers of \( m_f \). As a thumb rule, we have checked in several cases that the following prescription does the job: for \( SU(N_c) \) and when powers of \( m_f \) are not higher than 2, \( m_f \rightarrow \frac{1}{2}(m_f + m_{f+N_f}) \) and \( m_f^2 \rightarrow m_f m_{f+N_f} \); while for the rest of the groups and for (even) powers of \( m_f \) not higher than 4, \( m_f^2 \rightarrow \frac{1}{2}(m_f^2 + m_{f+N_f}^2) \) and \( m_f^4 \rightarrow m_f^2 m_{f+N_f}^2 \). Typically these cases only occur in \( F_1 \). Another way to see this is to observe that if we write down \( F_1 \) in terms of the \( q_i(m_f), f = 1, ..., N_f/2 \) as given in (2.22), these factors always appear precisely in those combinations that build up the \( t_k = t_k(q_i) \) as given in (2.4), which are valid for arbitrary masses.

3 Results for Simply-Laced Lie Algebras

We start by giving the concrete expression for \( Z_{\alpha(G)} = n^i \mathcal{H}_{2,i}^{(0)} \). In the case of \( A_r \) and \( D_r \) Lie algebras, we have that \( \sum_p e_p^2 = a^ia^jA_{ij} \) and also that \( \xi^+ = 1 \). Hence

\[
Z_{\alpha(G)} = n^i \mathcal{H}_{2,i}^{(0)} = n^i \sum_p e_p \partial_i e_p = n^i a^j A_{ij} = a \cdot \alpha = Z_{\alpha}
\]

We shall define \( \Delta_0 = \prod_{a_+} Z_{a_+}^2 \). To avoid confusion we must mention that, although we have expressed all results in terms of Weyl invariant polynomial combinations \( \bar{u}_k(a^i) \), for notational clearness we shall drop the bar.

3.1 \( SU(r + 1) \) with \( N_f \) hypermultiplets

The only asymptotically free theories that we can consider within our approach, for these groups, are \( N_f = 2, ..., 2r \). Let us list some of the results that we have obtained by using our formulas (we omit the case of \( SU(N_c) \) without matter which can be found in [R]):

3.1.1 \( SU(2) \)

\( N_f = 2 \) Using \( u_2 = a^2 \), we found the following corrections:

\[
\mathcal{F}_1 = \frac{u_2 + m^2}{2u_2}, \tag{3.1}
\]
\begin{align*}
\mathcal{F}_2 &= \frac{u_2^2 - 6u_2m^2 + 5m^4}{64u_2^3}, \\
\mathcal{F}_3 &= \frac{5u_2^2m^2 - 14u_2m^4 + 9m^6}{192u_2^5}.
\end{align*}
(3.2)

The one- and two-instanton contributions coincide with those computed in [12] (after adjusting $\Lambda$ to $\Lambda/2$). In the one-instanton correction, as discussed before, it is possible to split the masses, $m^2 \to m_1m_2$, so that the result for non-degenerated (ND) matter hypermultiplets is

$$
\mathcal{F}_{1,ND} = \frac{u_2 + m_1m_2}{2u_2},
$$
(3.4)
in agreement with the result in Ref.[12]. Also, the case of one hypermultiplet can then be considered by letting $m_2 \to \infty$ while keeping $\Lambda m_2$ finite an equal to the square of the dynamical scale $\Lambda^2$ that corresponds to $N_f = 1$.

3.1.2 SU(3)

To express our results, we introduce Weyl invariant combinations in terms of the $a_i$-variables, $u_2 = a_1^2 + a_2^2 - a_1a_2$ and $u_3 = a_1a_2(a_1 - a_2)$. For the case $N_f = 4$, we will denote $q_1 = m_1 + m_2$ and $q_2 = m_1m_2$. We obtained:

**$N_f = 2$**

\begin{align*}
\mathcal{F}_1 &= \frac{(2u_2^2 + 6m^2u_2 - 18mu_3)/\Delta_0}, \\
\mathcal{F}_2 &= \frac{(5u_2^6 + 153m^4u_2^4 + 162m^2u_2^5 - 1998m^2u_3u_2^3 - 414mu_3u_2^4 + 1701m^4u_3^2u_2 + 4374m^2u_3^2u_2^2 + 162u_3^2u_2^2 + 729u_3^4 - 2916m^3u_3^3 - 2673mu_3^4u_2^2)/\Delta_0^3}, \\
\mathcal{F}_3 &= \frac{(48u_2^{10} + 12320m^6u_2^7 + 31792m^4u_2^8 + 4992m^2u_2^9 - 366624m^5u_3u_2^6 - 12032mu_3u_2^8 - 253088m^3u_3u_2^7 + 478116m^6u_3^2u_2^4 + 2276856m^4u_3^2u_2^5 + 529236m^2u_3^2u_2^6 + 5600u_3^2u_2^5 - 3684852m^5u_3^3u_2^4 - 4654800m^3u_3^4u_2^4 - 394524mu_3^3u_2^6 + 994356m^6u_3^4u_2^2 + 7097544m^4u_3^4u_2^2 + 3969648m^2u_3^4u_2^3 + 105192u_3^4u_2^4 - 1469664m^5u_3^5u_2^2 - 4878468m^3u_3^5u_2^2 - 1571724mu_3^5u_2^6 + 1364688m^2u_3^6 + 215784u_3^6u_2^2)/\Delta_0^5}.
\end{align*}
(3.5)–(3.7)
Again, the splitting of the one-instanton correction can be done by letting $m^2 \to m_1 m_2$ and $m \to (m_1 + m_2)/2$,

$$\mathcal{F}_{1,ND} = (2u_2^2 + 6m_1 m_2 u_2 - 9(m_1 + m_2)u_3)/\Delta_0 , \quad (3.8)$$

and the reduction of hypermultiplets mentioned above is immediate. Our results (3.5), (3.6) and (3.8) agree with those obtained in [12].

$N_f = 4$

$$\mathcal{F}_1 = (2u_2^3 - 9u_3^2 - 6q_1 u_2 u_3 + 2(q_1^2 + 2q_2)u_2^2 + 6q_2^2 u_2 - 18q_1 q_2 u_3)/\Delta_0 , \quad (3.9)$$

and the length of the expressions grows rapidly. Again, having in mind that $\mathcal{F}_{1,ND}$ should be linear in the polynomials of the masses it is possible to carefully split the masses, $m_1^2 \to m_1 m_3$, $m_2^2 \to m_2 m_4$, $m_1 \to (m_1 + m_3)/2$ and $m_2 \to (m_2 + m_4)/2$ (or, in terms of the polynomials of the masses, $2q_1 \to t_1$, $q_1^2 + 2q_2 \to t_2$, $2q_1 q_2 \to t_3$ and $q_2^2 \to t_4$) so that

$$\mathcal{F}_{1,ND} = (2u_2^3 - 9u_3^2 - 3t_1 u_2 u_3 + 2t_2 u_2^2 - 9t_3 u_3 + 6t_4 u_2)/\Delta_0 , \quad (3.10)$$

and then one can reduce it to an odd number of matter hypermultiplets in the way mentioned before [12].

3.1.3 $SU(4)$

As the expressions become too long, we will only display the one-instanton correction. To express our results, we introduce the classical values of the Weyl invariant Casimirs $u_2 = a_1^2 + a_2^2 + a_3^2 - a_1 a_2 - a_2 a_3$, $u_3 = a_1 a_2 (a_1 - a_2) + a_2 a_3 (a_2 - a_3)$ and $u_4 = a_1^2 a_2 a_3 - a_1 a_2^2 a_3 - a_1^2 a_3^2 + a_1 a_2 a_3^2$. For the case $N_f = 4$, we will denote $q_1 = m_1 + m_2$ and $q_2 = m_1 m_2$.

$N_f = 2$

$$\mathcal{F}_1 = (8m^2 u_2^3 + 6u_2^2 u_2 - 8mu_3 u_2^2 - 36u_3^2 m^2 + 32m^2 u_2 u_4 - 16u_2^2 u_4 + 96mu_3 u_4 - 64u_4^2)/\Delta_0 . \quad (3.11)$$

$N_f = 4$

$$\mathcal{F}_1 = (2u_2^3 u_2^2 - 8u_4 u_2^3 + 12u_4^2 u_2 - 32u_4^2 u_2 - 18q_1 u_3^3 + 64q_1 u_4 u_3 u_2 + 6(q_1^2 + 2q_2)u_3^2 u_2 - 16(q_1^2 + 2q_2)u_4 u_2^2 - 64(q_1^2 + 2q_2)u_4^2 - 8q_1 q_2 u_3 u_2^2 + 96q_1 q_2 u_4 u_3 + 8q_2 u_2^3 - 36q_2 u_3^2 + 32q_2^2 u_4 u_2)/\Delta_0 . \quad (3.12)$$
3.2 $SO(2r)$ with $N_f$ hypermultiplets

The only asymptotically free theories that we can consider within our approach, for these groups, are $N_f = 0, 2, \ldots, 2(r-2)$. Notice that the case $N_f = 0$ corresponds to vanishing values of $\xi^+$ and $\xi^-$ in (2.11), which in the formulas of the instanton corrections implies $R = 1$. Let us list some of the result one can easily obtain by using our formulas:

3.2.1 $SO(4)$

For this group the classical values of the Casimir operators in terms of the $a_i$ are given by $u_2 = 2a_1^2 + 2a_2^2$, $u_4 = -(a_1^4 + a_2^4 - 2a_1^2a_2^2)$, and we can only consider the pure case ($N_f = 0$).

$N_f = 0$

\[ \mathcal{F}_1 = 2^2u_2/\Delta_0, \quad (3.13) \]

\[ \mathcal{F}_2 = 2(5u_2^3 - 60u_2u_4)/\Delta_0^3, \quad (3.14) \]

\[ \mathcal{F}_3 = 2^5(3u_2^5 - 120u_2^3u_4 + 240u_2^2u_3)/\Delta_0^5. \quad (3.15) \]

3.2.2 $SO(6)$

For this group we have $u_2 = 2a_1^2 + 2a_2^2 + 2a_3^2 - 2a_1a_2 - 2a_1a_3 - 2a_1a_3 - 2a_1^2a_2 - 2a_1^2a_3 + 2a_1^2a_3 - 2a_1^2a_3 - 2a_1^2a_3 - 2a_1^2a_3 + 2a_1^2a_3 + 2a_1^2a_3 + 2a_1^2a_3$ $u_6 = a_1^4a_2^2 + a_1^2a_2^2 - 2a_1^4a_2a_3 + 2a_1^2a_2a_3 + a_1^4a_2^2 - 2a_1^2a_2a_3 + 2a_1^2a_3^2 - 2a_1^2a_3 - 2a_1^2a_3$, and we can consider the cases $N_f = 0$ and $N_f = 2$.

$N_f = 0$

\[ \mathcal{F}_1 = 2^2(-u_2u_4 - 9u_6)/\Delta_0. \quad (3.16) \]

\[ \mathcal{F}_2 = 2(-5u_2^5u_4 - 43u_2^3u_4^3 + 172u_2u_4^5 - 60u_2^3u_4u_6 - 647u_2^4u_4^2u_6 - 1701u_2^2u_4^3u_6 + 36u_2^5u_6) 
- 1827u_2^3u_4u_6^2 - 3276u_2^4u_6^2 + 1323u_2^2u_4^2u_6^2 - 14337u_2^2u_6^3 - 21627u_4u_6^3)/\Delta_0^3. \quad (3.17) \]

$N_f = 2$

\[ \mathcal{F}_1 = 2^2(-m^4u_2u_4 - 9m^4u_6 - 4m^2u_4^2 + u_4^2u_2 + 12m^2u_2u_6 - 4u_6u_2^2 - 3u_4u_6)/\Delta_0. \quad (3.18) \]

Again, the splitting of the masses is possible for the one-instanton correction. This happens for all the classical groups.
3.2.3 $SO(8)$

In this case the expressions of the Casimir operators in terms of the $a_i$ gets too long so we are not going to list them here. We have found the following expressions:

$N_f = 0$

$$F_1 = 2^2(9u_2^3u_8 - u_2^2u_4u_6 + 3u_2u_6^2 + 32u_2u_4u_8 + 48u_6u_8 - 4u_4^2u_6)/\Delta_0 .$$  (3.19)

$N_f = 2$

$$F_1 = 2^2(9m^4u_2^3u_8 - m^4u_2^2u_4u_6 + 3m^4u_2u_6^2 + 32m^4u_2u_4u_8 + 48m^4u_6u_8 - 4m^4u_4^2u_6$$
$$-4m^2u_6^2u_2^2 - 12m^2u_4u_6^2 - u_2u_4u_6^2 - 9u_6^3 + 12m^2u_4u_8u_2^2 + 32m^2u_8u_4^2 + 4u_8u_4^2u_2$$
$$+16m^2u_8u_6u_2 - 3u_8u_6u_2^2 + 32u_8u_6u_4 + 128m^2u_8^2 - 48u_8^2u_2)/\Delta_0 .$$  (3.20)

The one-instanton corrections agree with those computed in Ref.[12].

4 Results for non Simply-Laced Lie Algebras

4.1 $SO(2r + 1)$ with $N_f$ hypermultiplets

In this case the form of $Z_{\alpha(G)}$ is different from the simply-laced cases. Indeed, using the fact that

$$\alpha_i \cdot \epsilon_q = \delta_{i,q} - \delta_{i+1,q}, \quad \alpha_r \cdot \epsilon_q = 2\delta_{r,q} ,$$

and setting $\xi^+ = 1$, we have

$$Z_{\alpha(G)} = n^iH_{2,i}^{(0)} = n^i \sum_p e_p\partial_i e_p$$
$$= n^i \sum_p e_p(\alpha_i \cdot \epsilon_p) = \sum_{i=1}^{r-1} (e_i - e_{i+1}) n^i + 2e_r n^r$$
$$= Z_\alpha + Z_{\alpha_r} n^r .$$  (4.1)

The only asymptotically free theories that we can consider within our approach, for $SO(2r + 1)$, are $N_f = 0, 2, \ldots, 2r - 2$. Notice that the case $N_f = 0$ corresponds, as in $SO(2r)$, to take in (2.11) $\xi^+ = \xi^- = 0$ which in the formulas of the instanton corrections means to set $R = 1$. Let us list some of the results that we have obtained:
4.1.1 $SO(5)$

We can consider within our approach the cases $N_f = 0$ and $N_f = 2$. For this group we have $u_2 = 2a_1^2 + 4a_2^2 - 4a_1a_2$, $u_4 = -(a_1^4 - 4a_1^3a_2 + 4a_1^2a_2^2)$. We found:

$\underline{N_f = 0}$

\[
F_1 = -2^3 u_4 / \Delta_0 ,
\]
\[
F_2 = 2(u_2^3 u_4^2 - 76u_2 u_4^3) / \Delta_0^3 ,
\]
\[
F_3 = 2^7 (3u_2^2 u_4^3 - 232u_2 u_4^5 + 176u_4^6) / 3 \Delta_0^5 .
\]

$\underline{N_f = 2}$

\[
F_1 = 2^2 (-2m^4 u_4 + 2m^2 u_2 u_4 - u_2^3 u_4 - 2u_4^3) / \Delta_0 ,
\]
\[
F_2 = 2(m^8 u_2^3 u_4^4 - 76m^8 u_2 u_4^5 + 152m^6 u_2^2 u_4^3 - 32m^6 u_4^5 - 78m^4 u_2^3 u_4 + 168m^4 u_2 u_4^5 + 12m^2 u_2 u_4^3 - 88m^2 u_2^2 u_4^4 + 96m^2 u_4^5 - u_2^5 u_4^3 - 60u_2 u_4^5 + u_4^3) / \Delta_0^3 ,
\]
\[
F_3 = 2^6 (6m^{12} u_4^5 - 464m^{12} u_2^5 u_4^5 + 352m^{12} u_4^6 - 9m^{10} u_2^6 u_4^4 + 1256m^{10} u_2^3 u_4^5
\]
\[-1744m^{10} u_2^5 u_4^5 + 3504m^{12} u_2^6 u_4^4 - 3m^{12} u_2 u_4^5 - 1212m^8 u_2^5 u_4^5 - 960m^8 u_4^7
\]
\[+2976m^6 u_2 u_4^7 - 3024m^6 u_2^3 u_4^6 + 498m^6 u_2 u_4^7 - 2864m^4 u_2^3 u_4^7 + 1054m^4 u_4^9
\]
\[-86m^4 u_2^5 u_4^7 + 736m^4 u_4^8 - 1104m^2 u_2 u_4^8 + 824m^2 u_3 u_4^7 - 137m^2 u_2^5 u_4^6
\]
\[+5m^2 u_2^7 u_4^5 + 240u_2^5 u_4^7 - 128u_4^9 - 56u_2 u_4^7 + 5u_4^6 / 3 \Delta_0^5 .
\]

4.1.2 $SO(7)$

Here we have $u_2 = 2a_1^2 + 2a_2^2 + 4a_2^2 - 2a_1a_2 - 4a_2 a_3$, $u_4 = -(a_1^4 + a_2^4 - 2a_1^3 a_2 - 2a_1a_2^3 - 4a_2^3 a_3 + 3a_1^2 a_2^2 + 8a_1 a_3^2 + 4a_2^2 a_3^2 + 8a_1^2 a_2 a_3 + 8a_1 a_2^2 a_3 - 8a_1 a_2 a_3^2)$, $u_6 = a_1^4 a_2^2 + a_1^2 a_2^4 - 2a_1 a_2^3 - 4a_2 a_2 a_3 + 8a_1^2 a_2 a_3 - 4a_2^2 a_3^2 + 4a_1^2 a_3^2 - 8a_1^2 a_2 a_3^2 + 4a_1^2 a_2^3)$ and for $N_f = 4$ we also denote $q_2 = m_1^2 + m_2^2$ and $q_4 = m_1^4 m_2^2$. We can consider the cases $N_f = 0$, $N_f = 2$ and $N_f = 4$.

$\underline{N_f = 0}$

\[
F_1 = 2^3 (u_2 u_6 + 3u_4 u_6) / \Delta_0 ,
\]
\[ \mathcal{F}_2 = 2u_6^2(-u_2^6u_4^3 - 12u_2^4u_4^4 - 48u_2^2u_4^5 - 64u_4^6 - 76u_2u_4u_6) \\
-839u_2^5u_4^2u_6 - 3057u_2^3u_4^3u_6 - 3588u_2u_4^4u_6 - 44u_2^6u_6^2 - 1695u_2^4u_4u_6^2 \\
-10827u_2^2u_4^2u_6^2 - 16308u_2u_4u_6^3 + 1863u_2u_4u_6^3 - 567u_2u_4u_6^3 - 18225u_6^4)/\Delta_0^3. \] (4.9)

\[ N_f = 2 \]
\[ \mathcal{F}_1 = 2^3u_6(m^4u_2^2 + 3m^4u_4 + m^2u_2u_4 + 9m^2u_6 - 3u_2u_6)/\Delta_0. \] (4.10)

\[ N_f = 4 \]
\[ \mathcal{F}_1 = 2^3u_6(q_4^2u_2^2 + 3q_4^2u_4 - 3(q_4^2 + 2q_4)u_2u_6 + (q_4^2 + 2q_4)u_4^2 + q_2q_4u_2u_4 + 9q_2q_4u_6 + 3q_2u_4u_6 + 4q_2u_2u_4 - 45u_6^2 - 32u_2u_4u_6 - 8u_2^3u_6 + 7u_4^3 + 2u_2^2u_4^2)/\Delta_0. \] (4.11)

Again, the one-instanton corrections agree with previous results [12].

### 4.2 Sp(2r) with \( N_f \) hypermultiplets

In this subsection we are going to consider the case of \( Sp(2r)^n \), \textit{i.e.}, the case of \( Sp(2r) \) with two massless hypermultiplets and \( N_f' = N_f - 2 \) matter hypermultiplets. In this case the form of \( Z_{\alpha(G)} \) is, as in the case of \( SO(2r + 1) \), different from the simply-laced cases. Now, using

\[ \alpha_i \cdot \epsilon_q = \delta_{i,q} - \delta_{i+1,q} \quad \alpha_r \cdot \epsilon_q = \delta_{r,q}, \]

we see that

\[ Z_{\alpha(G)} = n^i H_{2,i}^{(0)} \]
\[ \equiv 2n^i \sum_p e_p \partial_i e_p = 2n^i \sum_p e_p (\alpha_i \cdot e_p) = 2 \left[ \sum_{i=1}^{r-1} (e_i - e_{i+1}) n^i + e_r n^r \right] \]
\[ = 2Z_{\alpha} - Z_{\alpha,n^r}. \] (4.12)

The only asymptotically free theories that we can consider within our approach, for \( Sp(2r)^n \), are \( N_f' = 0, 2, \ldots, 2r - 2 \). Notice that the case \( N_f' = 0 \) now means to put \( R(e_p) = e_p^4 \) cause we are considering two massless hypermultiplets. Let us list some of the results that we have obtained:

### 4.2.1 Sp(4)

For this group we can consider the cases \( N_f' = 0 \) and \( N_f' = 2 \). We have \( u_2 = 2a_1^2 + a_2^2 - 2a_1a_2, u_4 = -(a_1^4 - 2a_1^2a_2 + a_1^2a_2^2) \).
\[ N_f' = 0 \]

\[ F_1 = \frac{u_2}{2\Delta_0}, \quad (4.13) \]

\[ F_2 = \frac{(5u_2^5 + 43u_2^3u_4 + 172u_2u_4^2)}{4\Delta_0^3}, \quad (4.14) \]

\[ F_3 = 2^2(9u_2^9 + 143u_2^7u_4 + 927u_2^5u_4^2 + 2840u_2^3u_4^3 + 5680u_2u_4^4)/3\Delta_0^5. \quad (4.15) \]

\[ N_f' = 2 \]

\[ F_1 = \frac{(m^4u_2 + 4m^2u_4 - u_2u_4)}{2\Delta_0}, \quad (4.16) \]

\[ F_2 = \frac{(5m^8u_2^5 + 43m^6u_2^3u_4 + 172m^8u_2u_4^2 + 12m^6u_2^2u_4^2 - 24m^6u_2u_4^3 + 532m^6u_4^4 + 54m^4u_2^3u_4^2 - 264m^4u_2u_4^3 + 152m^2u_2u_4^3 - 32m^2u_4^4 - 5u_2u_4^3 + 60u_2u_4^4)}{4\Delta_0^3}, \quad (4.17) \]

\[ F_3 = 2^2(9m^{12}u_2^9 + 143m^{12}u_2^7u_4 + 927m^{12}u_2^5u_4^2 + 2840m^{12}u_2^3u_4^3 + 5680m^{12}u_2u_4^4 + 28m^{10}u_2^8u_4 + 304m^{10}u_2^6u_4^2 + 1536m^{10}u_2u_4^4 + 1408m^{10}u_4^4 + 1972m^8u_2^6u_4^2 + 6600m^8u_2^4u_4^4 + 1280m^8u_2u_4^6 + 752m^2u_4^6 + 7760m^6u_2u_4^4 + 1792m^6u_4^6 + 7760m^4u_2u_4^4 + 321m^4u_2u_4^2 + 5416m^4u_2^3u_4^2 - 3584m^2u_2^3u_4^4 + 1280m^2u_4^6 + 752m^2u_2u_4^4 - 9u_2u_4^3 + 360u_2u_4^3 - 720u_2u_4^3)/3\Delta_0^5. \quad (4.18) \]

4.2.2 \( Sp(6) \)

For this group we can consider the cases \( N_f' = 0, N_f' = 2 \) and \( N_f' = 4 \). We let \( u_2 = 2a_1^2 + 2b_2^2 + a_3^2 - 2a_1a_2 - 2a_2a_3, u_4 = -(a_1^4 + a_2^4 - 2a_1^2a_2 - 2a_2^2a_3 - 2a_1a_2^2 + 3a_1a_2^2 + a_2a_3^2 - a_3a_2^2 + a_1a_2a_3 + 4a_1a_2a_3 - 2a_1a_2a_3), u_6 = a_1a_2^2 - a_2a_3^2 + a_1a_2^2 - 2a_1a_2a_3 + 4a_1a_2a_3 - 2a_1a_2a_3 - a_1a_2a_3 - 2a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3 \). For \( N_f' = 4 \), and we also have \( q_2 = m_1^2 + m_2^2 \) and \( q_4 = m_1m_2 \).

\[ N_f' = 0 \]

\[ F_1 = 2(3u_2u_6 - u_2u_4 - 4u_4)/\Delta_0, \quad (4.19) \]

\[ F_2 = 2^4(-5u_2^6u_4^6 - 60u_2^4u_4^6 - 240u_2^2u_4^4 - 320u_4^8 + 43u_2^7u_4^3u_6 + 546u_2^5u_4^3u_6 + 2304u_2^3u_4^5u_6) \]
+3232u_2^6u_6 - 172u_2^8u_4u_6 - 2255u_2^6u_4^2 - 9715u_2^4u_4^3u_6 - 13272u_2^2u_4^4u_6 + 2064u_4^5u_6 \\
+180u_2^3u_6^2 - 1107u_2^5u_4u_6^3 - 18975u_2^3u_4^2u_6^3 - 46908u_2u_4u_6^3 + 2439u_4^4u_6^2 \\
-3240u_2^2u_4u_6^4 - 57672u_4^2u_6^4 + 19197u_2u_6^5)/\Delta_0^3 .

(4.20)

\[ N_f' = 2 \]

\[ \mathcal{F}_1 = 2(3m^4u_2u_6 - m^4u_2^2u_4 - 4m^4u_4^2 - 4m^2u_2^2u_6 - 12m^2u_4u_6 - u_2u_4u_6 - 9u_6^2)/\Delta_0 \] (4.21)

\[ N_f' = 4 \]

\[ \mathcal{F}_1 = 2(3q_1^2u_2u_6 - q_1^2u_2^2u_4 - 4q_1^2u_4^2 - 4q_1q_2u_2^2u_6 - 12q_1q_2u_4u_6 - (q_2^2 + 2q_4)u_2u_4u_6 \\
-9(q_2^2 + 2q_4)u_6^2 + 12q_2u_2u_6^2 - 4q_2u_4u_6^2 - 3u_4u_6^2 - 4u_2^2u_6^2 + u_2u_4u_6)/\Delta_0 . \] (4.22)

### 4.3 The case of pure \( Sp(2r) \)

As we discussed above, the case of \( Sp(2r) \) without matter hypermultiplets cannot be obtained from our previous formulas, as long as we are considering at least two massless hypermultiplets. Nevertheless, one can treat this case separately in an analogous way. In fact, we can fix our ansatz for the effective prepotential (2.11) to the one first considered by Ito and Sasakura \[27\] by setting

\[ \xi = 1 \quad \xi^+ = \xi^- = 0 \quad \text{and} \quad \tau_0 = 3 . \] (4.23)

Now, we can introduce the effective prepotential into Eq.(2.9) and the same kind of formulas for the instanton correction would be obtained, provided we have for this case a characteristic \( E = \begin{bmatrix} 0 & \cdots & 0 \\
\frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix} \). Note that, being \( N_f = 0 \), we must set \( R = 1 \) in our formulas.

We also need the value of \( Z_{\alpha(G)} \) which turns out to be the same as that in \( Sp(2r)^n \), \( i.e. \ Z_{\alpha(G)} = 2Z_\alpha - n_rZ_{\alpha_r} \).

#### 4.3.1 \( Sp(4) \)

For this group we have, as we saw before, \( u_2 = 2a_1^2 + a_2^2 - 2a_1a_2, u_4 = -(a_1^4 - 2a_1^3a_2 + a_1^2a_2^2) \). In terms of them the first instanton corrections are

\[ \mathcal{F}_1 = 2^2(u_2^2 + 4u_4)/\Delta_0 , \] (4.24)

\[ \mathcal{F}_2 = 2^6(5u_2^7 + 59u_2^5u_4 + 232u_2^3u_4^2 + 304u_2u_4^3)/\Delta_0^3 , \] (4.25)
\[ \mathcal{F}_3 = 2^{14}(9u_2^{12} + 184u_2^{10}u_4 + 1526u_2^8u_4^2 + 6496u_2^6u_4^3 + 14656u_2^4u_4^4 \\
+ 15872u_2^3u_4^5 + 5632u_4^6)/3\Delta_0^5. \] (4.26)

### 4.3.2 \textit{Sp}(6)

For this group we have \( u_2 = 2a_1^2 + 2a_2^2 + a_3^2 - 2a_1a_2 - 2a_2a_3 \), \( u_4 = -(a_1^4 + a_1^2 - a_1^3a_2 - 2a_1^3a_3 - 2a_1a_2a_3 + 3a_1^2a_2^2 + 2a_1^2a_3^2 + a_2^2a_3^2 - 4a_1^2a_2a_3 + 4a_1a_2^3a_3 - 2a_1a_2a_3^2) \), \( u_6 = a_1^4a_2^2 - 2a_1^3a_3^2 + a_1^2a_2^4 - 2a_1a_2a_3 + 4a_1^3a_2^3a_3 - 2a_1^2a_2^3a_3 + a_1^4a_3^2 - 2a_1^3a_2^3a_3 + a_1^2a_2^3a_3^2 \), and the first instanton corrections are

\[ \mathcal{F}_1 = 2^5(u_2^2u_4^2 + 4u_2^3 - 4u_2u_6 - 18u_2u_4u_6 - 27u_6^2)/\Delta_0, \] (4.27)

\[ \mathcal{F}_2 = 2^{12}(-5u_2^6u_4 - 60u_2^4u_6 - 240u_2^2u_4^3 - 320u_4^4 + 59u_2^7u_4^5u_6 + 738u_2^5u_4^6u_6 \\
+ 3072u_2^3u_4^7u_6 + 4256u_2^5u_4^6u_6 - 232u_2^8u_4^5u_6 - 3021u_2^6u_4^4u_6^2 - 12699u_2^4u_4^3u_6^2 \\
- 15736u_2^3u_4^2u_6^3 + 6288u_2^7u_4u_6^3 + 304u_2^9u_4^3u_6^3 + 4120u_2^4u_4^2u_6^3 + 15518u_2^3u_4^3u_6^3 + 1716u_2^3u_4u_6^3 \\
- 55728u_2^4u_4^2u_6^3 - 16u_2^6u_4^4 + 5928u_2^4u_4^4u_6^3 + 54486u_2^3u_4^3u_6^4 + 113373u_2^3u_4^3u_6^4 - 216u_2^5u_4^5 \\
- 41148u_2^4u_4^4u_6^4 + 39447u_2^3u_4^4u_6^4 + 182250u_2^3u_4^3u_6^5 - 729u_2^5u_4^5u_6^5 + 89667u_4^6u_6^6)/\Delta_0^5. \] (4.28)

The one-instanton corrections agree with those computed in Ref.\textsuperscript{27}.

## 5 Concluding Remarks

In the present paper, we have shown how instanton corrections to the effective prepotential of \( \mathcal{N} = 2 \) supersymmetric theories can be computed in a variety of cases including all classical gauge groups and even number of degenerated fundamental matter hypermultiplets, up to arbitrary order. As compared to other approaches developed in the literature, we should stress that the one presented in this paper has an important feature in that it does not require an explicit knowledge of the BPS spectrum as a function of the moduli, at the same time that it allows to consider a huge variety of cases within a unified framework. Also, being recursive, it admits an easy implementation on a computer. We have illustrated the remarkable simplicity of our procedure by displaying many explicit expressions which should be quite useful for further comparison with the results obtained by other means.

Conversely, our results admit a second reading: They could be thought of as a highly non-trivial test of the connection between the Seiberg–Witten solution to the low energy
dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories, and the theory of Whitham (adiabatic) deformations of a given integrable system [18, 24]. In this sense, it is important to remind that the new equation (2.9), which is a key ingredient of our procedure, is originated in the latter framework, as it is shown in detail in Ref. [24].

Aside from being an interesting mathematical problem by itself, the embedding of the Seiberg-Witten solution within a Whitham hierarchy seems to be the appropriate framework for the study of many physical phenomena. For example, the so-called slow Whitham times can be consistently thought of as spurion vector supermultiplets that can be used to break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 0$ with non-quadratic Casimir operators [18]. In this way, the Whitham hierarchy can be interpreted as a family of supersymmetry breaking deformations of the original theory associated with the higher Casimir operators of the gauge group. This issue generalizes to the $\mathcal{N} = 0$ case the family of $\mathcal{N} = 1$ supersymmetry breaking terms considered, for instance, in Ref. [28].

The key feature of the Whitham formalism lies on the fact that, as the dependence on the slow times is encoded in the prepotential, it is possible to obtain the exact effective potential of the theory, in the spirit of [18, 29]. This allows to perform a detailed study, both qualitative and quantitative, of the vacuum state of the theory once supersymmetry is broken, as well as of the appearance of monopole condensates, mass gaps, etc. Preliminary results on this program were published in Ref. [18]. The formalism is even useful near the Argyres–Douglas singularities, where non-local degrees of freedom become simultaneously massless, provided one approaches them along any of the submanifolds where a unique monopole gets massless [18].

There is another place where deformations of the prepotential by means of the Whitham times are relevant. It is the study of contact terms in the twisted version of $\mathcal{N} = 2$ gauge theories, where these new variables play the rôle of sources for insertions of certain class of operators in the generating functional [19, 20, 30, 31].

Another interesting point is given by the uses of our starting equations (2.8)–(2.9) to study the strong coupling expansion of the prepotential near the singularities of the quantum moduli space, as it was done in Ref. [26] for the case of pure $SU(N_c)$. In particular, these equations provide us with a set of non-trivial constraints that facilitate the study of the couplings between different magnetic photons, originally found in Ref. [32], that take place at such points. The expansion of the prepotential near the maximal points by other methods, as the deformations of the auxiliary singular Riemann manifold [33], is not sensitive to such kind of terms.

Several interesting questions remain open aside from the ones just mentioned. For example, the case of arbitrary masses cannot be treated within our approach, except
for the one-instanton correction (which, on the other hand, is enough for leading order comparison purposes). In fact, from the Whitham hierarchy side, one can show that indeed the formulas used in this paper are insufficient to tackle the generic scenario, though it seems to be possible to refine the formalism in order to extend its applicability to some cases of unpaired masses [24]. The additional corrections that appear in the generic case are, nevertheless, quite difficult to manage with. Finally, another avenue for future research is, certainly, the connection of this formalism with the string theory and D-brane approach to supersymmetric gauge theories, where some steps has already been given in the last few years [34].

We believe that these matters deserve further study.

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