LIMITS OF CONJUGACY CLASSES UNDER ITERATES OF HYPERBOLIC ELEMENTS OF Out(\(\mathbb{F}\))

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Dedicated to Lee Mosher on his 60th birthday

Abstract. For a free group \(\mathbb{F}\) of finite rank such that \(\text{rank}(\mathbb{F}) \geq 3\), we prove that the set of weak limits of a conjugacy class in \(\mathbb{F}\) under iterates of some hyperbolic \(\phi \in \text{Out}(\mathbb{F})\) is equal to the collection of generic leaves and lines with endpoints in attracting fixed points of \(\phi\).

As an application we describe the ending lamination set for a hyperbolic extension of \(\mathbb{F}\) by a hyperbolic subgroup of \(\text{Out}(\mathbb{F})\) in a new way and use it to prove results about Cannon-Thurston maps for such extensions. We also use it to derive conditions for quasiconvexity of finitely generated, infinite index subgroups of \(\mathbb{F}\) in the extension group. These results generalize similar results obtained in Mitra [1999], Kapovich and Lustig [2015] and use different techniques.

1. Introduction

Fix a nonabelian free group \(\mathbb{F}\) of finite rank and assume that \(\text{rank}(\mathbb{F}) \geq 3\). The outer automorphism group, \(\text{Out}(\mathbb{F})\), of a free group \(\mathbb{F}\) is defined as \(\text{Aut}(\mathbb{F})\) modulo the group of inner automorphisms of \(\mathbb{F}\). There are many tools in studying the properties of this group. One of them is by using train-track maps introduced in Bestvina and Handel [1992] and later generalized in Bestvina et al. [1997], Bestvina et al. [2000], Feighn and Handel [2011]. The fully irreducible outer automorphisms are elements in \(\text{Out}(\mathbb{F})\) which do not have iterates that leave a conjugacy class of some proper free factor invariant. They are the most well understood elements in \(\text{Out}(\mathbb{F})\). They behave very closely to the pseudo-Anosov homeomorphisms of surfaces with one boundary component, which have been well understood and are a rich source of examples and interesting theorems. Our focus will be on outer automorphisms \(\phi\) which might not be fully irreducible but have a lift \(\Phi \in \text{Aut}(\mathbb{F})\) such that

\[
\exists M > 0, \lambda > 1 \ni \lambda |c| \leq \max \{ |\Phi^M(c)|, |\Phi^{-M}(c)| \} \forall c \in \mathbb{F}
\]

Such classes of outer automorphisms are called hyperbolic outer automorphisms. It was shown by Bestvina and Feighn [1992] and Brinkmann [2000] that an outer automorphism is hyperbolic if and only if it does not have any periodic conjugacy classes.

Our first objective (Section 3) is to classify all weak limits of any conjugacy class \([c]\) under the action of a hyperbolic \(\phi \in \text{Out}(\mathbb{F})\). For this we use the theory of completely split relative train track maps introduced by Feighn-Handel in their

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Recognition Theorem work Feighn and Handel [2011] and later further studied in great details by Handel-Mosher in their body of work about Subgroup Decomposition in Out(\mathbb{F}) Handel and Mosher [2017]. Our main result in this direction is the following result (here \(B_{\text{gen}}(\phi)\) denotes the set of all generic leaves of all attracting laminations for \(\phi\) (See section 2.5) and \(B_{\text{Fix}+}(\phi)\) denotes the set of all lines with endpoints in Fix$_+$(\phi). (See section 2.7)):

**Theorem 3.10** Suppose \(\phi \in \text{Out}(\mathbb{F})\) is a hyperbolic outer automorphism and \([c]\) is any conjugacy class in \(\mathbb{F}\). Then the weak limits of \([c]\) under iterates of \(\phi\) are in

\[
\mathcal{WL}(\phi) := B_{\text{gen}}(\phi) \cup B_{\text{Fix}+}(\phi)
\]

Conversely, any line in \(\mathcal{WL}(\phi)\) is a weak limit of some conjugacy class in \(\mathbb{F}\) under iterates of \(\phi\).

The dynamics of fully irreducible outer automorphisms are very well understood. This theorem, in a way shows that dynamical behavior of hyperbolic outer automorphisms which are not fully irreducible are also well behaved. One of the special properties of a fully irreducible and hyperbolic automorphism \(\phi\) is that the non-attracting subgroup system (see section 2.6) of the unique attracting lamination of \(\phi\) is trivial. This implies that every conjugacy class is attracted to the unique attracting lamination under iterates of \(\phi\). The above theorem goes on to show that a somewhat similar behaviour is observed in the case of hyperbolic automorphisms which are not fully irreducible. More precisely, every conjugacy class is attracted to some attracting lamination under iterates of \(\phi\), which is the content of Lemma 3.1.

The above theorem also implies that \(\phi\) is hyperbolic if and only if every conjugacy class is weakly attracted to some singular line of \(\phi\) under iterates of \(\phi\) (see 3.12).

Theorem 3.10 is used to prove several important results in Section 4. The work of Brinkmann [2000, Theorem 1.2] and Bestvina and Feighn [1992] show that \(\phi\) is hyperbolic if and only if the mapping torus \(G_\phi := \mathbb{F}\langle \phi \rangle \ltimes \mathbb{Z}\) is a Gromov hyperbolic group. We use the above theorem 3.10 to define a set of lines which describe the ending lamination set (\(\Lambda_{\langle \phi \rangle}\)) defined in Mitra [1997] for \(\mathbb{F}\langle \phi \rangle \ltimes \mathbb{Z}\). We prove:

**Lemma 4.4** Let \(\mathcal{H} = \langle \phi \rangle\) for some hyperbolic \(\phi \in \text{Out}(\mathbb{F})\). Then

\[
\Lambda_{\langle \phi \rangle} = \tilde{\mathcal{WL}}(\phi) \cup \tilde{\mathcal{WL}}(\phi^{-1})
\]

This new description of the ending lamination set has very significant implications. First we use this new description of the ending lamination set to prove the following theorem, which partially answers a question of Swarup:

**Theorem 4.8** Consider the exact sequence of hyperbolic groups:

\[
1 \rightarrow \mathbb{F} \rightarrow G_\phi \rightarrow \langle \phi \rangle \rightarrow 1
\]

The Cannon-Thurston map \(\tilde{i} : \partial \mathbb{F} \rightarrow \partial G_\phi\) is a finite-to-one map and cardinality of preimage of each point in \(\partial G_\phi\) is uniformly bounded, the bound depending only on rank of \(\mathbb{F}\).

The proof of this theorem becomes fairly simple once we prove in Proposition 3.6 that every nongeneric generic leaf of an attracting lamination for \(\phi\) is a line in \(B_{\text{Fix}+}(\phi)\). Then we can apply the proposition that the number of lines in \(B_{\text{Fix}+}(\phi)\) for any hyperbolic outer automorphism \(\phi\) is uniformly bounded above by some number depending only on rank of \(\mathbb{F}\) (see Proposition 2.24).
The special case, of the aforementioned theorem, when \( \phi \) is fully irreducible was first proved by [Kapovich and Lustig, 2015, Theorem 5.4]. However, the methods of our proofs are very different; we use the action of \( \phi \) on conjugacy classes of \( \mathbb{F} \) and their proof uses action of \( \phi \) on boundary of outer space.

In Section 4.2 the next application of the new description of the ending lamination set is in understanding quasiconvexity of finitely generated infinite index subgroups of \( \mathbb{F} \) inside \( G_\phi \). Here we are able to give one equivalent condition and also one sufficient condition for quasiconvexity. (For definition of \( \text{Fix}_+ (\phi) \) please see section 2.3).

**Theorem 4.14** Consider the exact sequence

\[
1 \to \mathbb{F} \to G_\phi \to \langle \phi \rangle \to 1
\]

where \( \phi \) is a hyperbolic automorphism of \( \mathbb{F} \). Let \( H \) be a finitely generated infinite index subgroup of \( \mathbb{F} \). Then the following are equivalent:

1. \( H \) is quasiconvex in \( G_\phi \).
2. \( \partial H \cap \{ \text{Fix}_+ (\phi) \cup \text{Fix}_- (\phi) \} = \emptyset \).

Moreover both (1) and (2) are satisfied if every attracting and repelling lamination of \( \phi \) is minimally filling with respect to \( H \).

The proof of this theorem uses a result that was proved in [Handel and Mosher, 2009, 2.23] which states that every attracting lamination \( \Lambda \) of \( \phi \) contains a leaf which is a singular line for \( \phi \) and one of whose ends is dense in \( \Lambda \).

The concept of *minimally filling with respect to* \( H \) is motivated from the idea of *minimally filling*. We make this definition in 4.11 and show that this theorem indeed covers the case for a fully irreducible \( \phi \) in Corollary 4.15.

Dynamics of outer automorphisms have been studied for quite some time using the notion of train track maps which were introduced in the seminal paper of [Bestvina and Handel, 1992]. Then [Bestvina et al., 1997, Proposition 2.4] studied the dynamics of fully irreducible outer automorphisms and proved the existence of an attracting lamination with the property that any finitely generated subgroup of \( \mathbb{F} \) which carries this lamination must be of finite index in \( \mathbb{F} \). Note that this is a much stronger property than just “filling” the free group (not being carried by a proper free factor of \( \mathbb{F} \)) and was called *minimally filling* by Kapovich-Lustig in their work [Kapovich and Lustig, 2015]. We establish a connection between minimally filling and filling (in the sense of free factor support) in Proposition 4.10. The primary obstruction that occurs is due to possible existence filling attracting laminations which are not minimally filling. We then proceed to generalize the notion of minimally filling for the purposes of using it for hyperbolic outer automorphisms which are not fully irreducible (Definition 4.11).

Next we give an algebraic condition on a finitely generated, infinite index subgroup of \( \mathbb{F} \) for it to be quasiconvex in \( G_\phi \) (see Theorem 4.16). It is quite understandable that such a general condition is quite complicated in nature due to complexities that arise in a non-fully irreducible hyperbolic \( \phi \). For example, one might have a filling lamination which may not be minimally filling and whose nonattracting subgroup system is nontrivial implying that not every conjugacy class is attracted to it under iteration of \( \phi \).
Section 4.3 of this paper deals with the Cannon-Thurston maps for a more complicated scenario when we have an exact sequence of hyperbolic groups

$$1 \rightarrow \mathbb{F} \rightarrow G \rightarrow \mathcal{H} \rightarrow 1$$

where $\mathcal{H}$ is non-elementary hyperbolic subgroup of $\text{Out}(\mathbb{F})$. Observe that hyperbolicity of $G$ implies that $\mathcal{H}$ is purely atoroidal, meaning every element of $\mathcal{H}$ is a hyperbolic outer automorphism. We use the Lemma 4.4, where we describe the nature of ending lamination set for a hyperbolic $\phi$ in terms of generic leaves and lines in $B_{\mathbb{F},+}(\phi)$, to derive a list equivalent necessary conditions in Proposition 4.20 that elements of $\mathcal{H}$ must satisfy in order for the extension group to be hyperbolic. In particular it we observe that two hyperbolic outer automorphisms which do not have a common power but have a common attracting lamination cannot both belong to $\mathcal{H}$.

In a very recent work Dowdall et al. [2016] have shown the finiteness of the fibers of Cannon-Thurston map for the special case when $\mathcal{H}$ is convex cocompact. The general case however still remains open. We hope that the results we develop here will be useful for addressing this problem in its full generality and also other interesting questions related to hyperbolicity in $\text{Out}(\mathbb{F})$.

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2. Preliminaries

In this section we give the reader a short review of the definitions and some important results in $\text{Out}(\mathbb{F})$ which are relevant to the theorem that we are trying to prove here. All the results which are stated as lemmas, can be found in full details in [Bestvina et al. 2000, Feighn and Handel 2011, Handel and Mosher 2017a, Handel and Mosher 2017b].

2.1. Weak topology. Given any finite graph $G$, let $\tilde{B}(G)$ denote the compact space of equivalence classes of circuits in $G$ and paths in $G$, whose endpoints (if any) are vertices of $G$. We give this space the weak topology. Namely, for each finite path $\gamma$ in $G$, we have one basis element $\tilde{N}(G, \gamma)$ which contains all paths and circuits in $\tilde{B}(G)$ which have $\gamma$ as its subpath. Let $B(G) \subset \tilde{B}(G)$ be the compact subspace of all lines in $G$ with the induced topology. One can give an equivalent description of $B(G)$ following [Bestvina et al. 2000]. A line is completely determined, upto reversal of direction, by two distinct points in $\partial \mathbb{F}$, since there only one line that joins these two points. We can then induce the weak topology on the set of lines coming from the Cantor set $\partial \mathbb{F}$. More explicitly, let $\tilde{B} = (\partial \mathbb{F} \times \partial \mathbb{F} - \partial \mathbb{F}^2)/Z_2$, where $\partial \mathbb{F}$ is the diagonal and $Z_2$ acts by interchanging factors. We can put the weak topology on $\tilde{B}$, induced by Cantor topology on $\partial \mathbb{F}$. The group $\mathbb{F}$ acts on $\tilde{B}$ with a compact but non-Hausdorff quotient space $\mathcal{B} = \tilde{B}/\mathbb{F}$. The quotient topology is also called the weak topology. Elements of $\mathcal{B}$ are called lines. A lift of a line $\gamma \in \mathcal{B}$ is an
element $\gamma \in \tilde{B}$ that projects to $\gamma$ under the quotient map and the two elements of $\gamma$ are called its endpoints.

One can naturally identify the two spaces $B(G)$ and $B$ by considering a homeomorphism between the two Cantor sets $\partial F$ and set of ends of universal cover of $G$, where $G$ is a marked graph. $Out(F)$ has a natural action on $B$. The action comes from the action of $Aut(F)$ on $\partial F$. Given any two marked graphs $G, G'$ and a homotopy equivalence $f : G \to G'$ between them, the induced map $f_# : \tilde{B}(G) \to \tilde{B}(G')$ is continuous and the restriction $f_# : B(G) \to B(G')$ is a homeomorphism. With respect to the identification $B(G) \approx B \approx B(G')$, if $f$ preserves the marking then $f_# : B(G) \to B(G')$ is equal to the identity map on $B$. When $G = G'$, $f_#$ agree with their homeomorphism $B \to B$ induced by the outer automorphism associated to $f$.

Given a marked graph $G$, a ray in $G$ is an one-sided infinite concatenation of edges $E_0 E_1 E_2 ....$. A ray of $F$ is an element of the orbit set $\partial F / F$. There is a connection between these two objects which can be explained as follows. Two rays in $G$ are asymptotic if they have equal subrays, and this is an equivalence relation on the set of rays in $G$. The set of asymptotic equivalence classes of rays $\rho$ in $G$ is in natural bijection with $\partial F / F$ where $\rho$ in $G$ corresponds to end $\xi \in \partial F / F$ if there is a lift $\tilde{\rho} \subset G$ of $\rho$ and a lift $\tilde{\xi} \in \partial F$ of $\xi$, such that $\tilde{\rho}$ converges to $\tilde{\xi}$ in the Gromov compactification of $G$. A ray $\rho$ is often said to be the realization of $\xi$ if the above conditions are satisfied.

A line(path) $\gamma$ is said to be weakly attracted to a line(path) $\beta$ under the action of $\phi \in Out(F)$, if the $\phi^k(\gamma)$ converges to $\beta$ in the weak topology. This is same as saying, for any given finite subpath of $\beta$, $\phi^k(\gamma)$ contains that subpath for some value of $k$; similarly if we have a homotopy equivalence $f : G \to G$, a line(path) $\gamma$ is said to be weakly attracted to a line(path) $\beta$ under the action of $f_#$ if the $f_#^k(\gamma)$ converges to $\beta$ in the weak topology. The accumulation set of a ray $\gamma$ in $G$ is the set of lines $l \in B(G)$ which are elements of the weak closure of $\gamma$; which is same as saying every finite subpath of $l$ occurs infinitely many times as a subpath $\gamma$. The weak accumulation set of some $\xi \in \partial F$ is the set of lines in the weak closure of any of the asymptotic rays in its equivalence class. We call this the weak closure of $\xi$.

A line $l$ is said to be birecurrent if $l$ if it is in the weak closure of some positive subray of itself and it is in the weak closure of some negative subray of itself. This implies that any finite subpath of $l$ occurs infinitely often in either direction.

### 2.2. Free factor systems and subgroup systems

Given a finite collection $\{K_1, K_2, \ldots, K_s\}$ of subgroups of $F$, we say that this collection determines a free factorization of $F$ if $F$ is the free product of these subgroups, that is, $F = K_1 * K_2 * \ldots * K_s$. The conjugacy class of a subgroup is denoted by $[K_i]$.

A free factor system is a finite collection of conjugacy classes of subgroups of $F$, $\mathcal{K} := \{[K_1],[K_2],\ldots,[K_p]\}$ such that there is a free factorization of $F$ of the form $F = K_1 * K_2 * \ldots * B$, where $B$ is some finite rank subgroup of $F$ (it may be trivial).

There is an action of $Out(F)$ on the set of all conjugacy classes of subgroups of $F$. This action induces an action of $Out(F)$ on the set of all free factor systems. For notation simplicity we will avoid writing $[K]$ all the time and write $K$ instead, when we discuss the action of $Out(F)$ on this conjugacy class of subgroup $K$ or anything regarding the conjugacy class $[K]$. It will be understood that we actually mean $[K]$.
For any marked graph $G$ and any subgraph $H \subset G$, the fundamental groups of the noncontractible components of $H$ form a free factor system. We denote this by $[H]$. A subgraph of $G$ which has no valence 1 vertex is called a core graph. Every subgraph has a unique core graph, which is a deformation retract of the union of the noncontractible components of $H$, implying that free factor system defined by the core of $H$ is equal to the free factor system defined by core of $H$. Conversely, any free factor system can be realized as $[H]$ for some nontrivial core subgraph of some marked graph $G$.

A free factor system $K$ carries a conjugacy class $[c]$ in $F$ if there exists some $[K] \in K$ such that $c \in K$. We say that $K$ carries the line $\gamma \in B$ if for any marked graph $G$ the realization of $\gamma$ in $G$ is the weak limit of a sequence of circuits in $G$ each of which is carried by $K$. An equivalent way of saying this is: for any marked graph $G$ and a subgraph $H \subset G$ with $[H] = K$, the realization of $\gamma$ in $G$ is contained in $H$.

Similarly define a subgroup system $A = \{[H_1], [H_2], \ldots, [H_k]\}$ to be a finite collection of conjugacy classes of finite rank subgroups $H_i < F$.

A subgroup system $A$ carries a conjugacy class $[c] \in F$ if there exists some $[A] \in \mathcal{A}$ such that $c \in A$. Also, we say that $A$ carries a line $\gamma$ if one of the following equivalent conditions hold:

- $\gamma$ is the weak limit of a sequence of conjugacy classes carries by $A$.
- There exists some $[A] \in \mathcal{A}$ and a lift $\tilde{\gamma}$ of $\gamma$ so that the endpoints of $\tilde{\gamma}$ are in $\partial A$.

The following lemma is an important property of lines carried by a subgroup system. The proof is by using the observation that $A < F$ is of finite rank implies that $\partial A$ is a compact subset of $\partial F$.

**Lemma 2.1.** For each subgroup system $\mathcal{A}$ the set of lines carried by $\mathcal{A}$ is a closed subset of $B$.

From [Bestvina et al., 2000] the free factor support of a set of lines $B$ in $B$ is (denoted by $A_{\text{supp}}(B)$) defined as the meet of all free factor systems that carries $B$. We are skipping giving the exact definition of meet here since we have no explicit use for that definition. Roughly speaking, one should think of the free factor support as the smallest (in terms of inclusion of subgroups) free factor system that carries $B$ (for more details please see [Bestvina et al., 2000, Corollary 2.6.5]). If $B$ is a single line then $A_{\text{supp}}(B)$ is single free factor. We say that a set of lines, $B$, is filling if $A_{\text{supp}}(B) = [F]$.

### 2.3. Principal automorphisms and rotationless automorphisms.

Given an outer automorphism $\phi \in \text{Out}(F)$, we can consider a lift $\Phi$ in $\text{Aut}(F)$. We call a lift principal automorphism, if it satisfies certain conditions described below. Roughly speaking, what such lifts guarantees is the the existence of certain lines which are not a part of the attracting lamination but it still fills the free group $F$. Such lines (called singular lines) will be a key tool in describing the set of lines which are not attracted to the attracting lamination of $\phi$.

Consider $\phi \in \text{Out}(F)$ and a lift $\Phi$ in $\text{Aut}(F)$. $\Phi$ has an action on $F$, which has a fixed subgroup denoted by $\text{Fix}(\Phi)$. Consider the boundary of this fixed subgroup $\partial \text{Fix}(\Phi) \subset \partial F$. It is either empty or has exactly two points.

This action action extends to the boundary and is denoted by $\tilde{\Phi} : \partial F \to \partial F$. Let $\text{Fix}(\tilde{\Phi})$ denote the set of fixed points of this action. We call an element $P$ of $\text{Fix}(\tilde{\Phi})$...
attracting fixed point if there exists an open neighborhood $U \subset \partial \mathcal{F}$ of $P$ such that we have $\tilde{\Phi}(U) \subset U$ and for every points $Q \in U$ the sequence $\tilde{\Phi}^n(Q)$ converges to $P$. Let $\text{Fix}_+(\tilde{\Phi})$ denote the set of attracting fixed points of $\text{Fix}(\tilde{\Phi})$. Similarly let $\text{Fix}_-(\tilde{\Phi})$ denote the attracting fixed points of $\text{Fix}(\tilde{\Phi}^{-1})$.

Let $\text{Fix}_N(\tilde{\Phi}) = \text{Fix}(\tilde{\Phi}) - \text{Fix}_-(\tilde{\Phi}) = \partial \text{Fix}(\tilde{\Phi}) \cup \text{Fix}_+(\tilde{\Phi})$. We say that an automorphism $\Phi \in \text{Aut}(\mathcal{F})$ in the outer automorphism class of $\phi$ is a \textit{principal automorphism} if $\text{Fix}_N(\tilde{\Phi})$ has at least 3 points or $\text{Fix}_N(\tilde{\Phi})$ has exactly two points which are neither the endpoints of an axis of a covering translation, nor the endpoints of a generic leaf of an attracting lamination $\Lambda^+_{\phi}$. The set of all principal automorphisms of $\phi$ is denoted by $P(\phi)$.

**Notation:** For sake of simplicity we will abuse the notation in this paper. Since we will be dealing exclusively with hyperbolic outer automorphisms in this paper we know that $\partial \text{Fix}(\Phi)$ is trivial for any automorphism $\Phi$ in the class of $\phi$. Hence $\text{Fix}_N(\tilde{\Phi}) = \text{Fix}_+(\tilde{\Phi})$. We shall abbreviate $\text{Fix}_+(\phi) := \bigcup_{\Phi \in P(\phi)} \text{Fix}_+(\tilde{\Phi})$

Similarly for $\text{Fix}_-(\phi)$.

We have the following lemma from \textit{Levitt and Lustig} \cite{2008} that we shall be using in this paper to show that any conjugacy class limits to either a generic leaf of some attracting lamination of $\phi$ or a singular line of $\phi$ (see proposition 3.6). This result essentially tells us that the attracting fixed points are almost globally attracting, except for the finite number of repelling fixed points.

**Lemma 2.2.** \textit{Levitt and Lustig} \cite{2008}, Theorem I. If $\phi \in \text{Out}(\mathcal{F})$ has no periodic conjugacy classes in $\mathcal{F}$ then there exists an integer $q \geq 1$ such that for each $\Phi \in \text{Aut}(\mathcal{F})$ representing $\phi$ and each $\xi \in \partial \mathcal{F}$, one of the following holds:

1. $\xi \in \text{Fix}_-(\tilde{\Phi}^q)$.
2. The sequence $\tilde{\Phi}^q$ converges to a point in $\text{Fix}_+(\tilde{\Phi}^q)$.

Let $\text{Per}(\tilde{\Phi}) = \bigcup_{k \geq 1} \text{Fix}(\tilde{\Phi}^k)$, $\text{Per}_+(\tilde{\Phi}) = \bigcup_{k \geq 1} \text{Fix}_+(\tilde{\Phi}^k)$ and similarly define $\text{Per}_-(\tilde{\Phi})$ and $\text{Per}_N(\tilde{\Phi})$. We say that $\phi \in \text{Out}(\mathcal{F})$ is rotationless if $\text{Fix}_N(\tilde{\Phi}) = \text{Per}_N(\tilde{\Phi})$ for all $\Phi \in P(\phi)$, and if for each $k \geq 1$ the map $\Phi \rightarrow \tilde{\Phi}^k$ induces a bijection between $P(\phi)$ and $P(\tilde{\phi}^k)$.

The following two important lemmas about rotationless automorphisms are taken from \textit{Feighn and Handel} \cite{2011}.

**Lemma 2.3** \textit{Feighn and Handel} \cite{2011}, Lemma 4.43. \textit{There exists a $K$ depending only upon the rank of the free group $\mathcal{F}$ such that for every $\phi \in \text{Out}(\mathcal{F})$, $\phi^K$ is rotationless.}

The above lemma is heavily used in this paper. Whenever we write “pass to a rotationless power” we intend to use this uniform constant $K$ given by the lemma. Rotationless powers are useful and important since they kill any periodic behaviour (as the following lemma shows) and guarantee the existence of completely split relative train track maps as we shall see later.

**Lemma 2.4** \textit{Feighn and Handel} \cite{2011}. \textit{If $\phi \in \text{Out}(\mathcal{F})$ is rotationless then:}

- Every periodic conjugacy class of $\phi$ is a fixed conjugacy class.
Every free factor system which is periodic under $\phi$ is fixed.

Every periodic direction of $\phi$ is fixed by $\phi$.

**Remark 2.5.** These results show that given any $\phi \in \text{Out}(F)$ one can pass to $\psi = \phi^k$ which is rotationless and hence one can choose $q = 1$ for the rotationless outer automorphism $\psi$ as in Lemma 2.2. This is the form in which we will be using lemma 2.2.

2.4. **Topological representatives and Train track maps.** Given $\phi \in \text{Out}(F)$ a topological representative is a homotopy equivalence $f : G \rightarrow G$ such that $\rho : R_r \rightarrow G$ is a marked graph, $f$ takes vertices to vertices and edges to paths and $\overline{\rho} \circ f \circ \rho : R_r \rightarrow R_r$ represents $R_r$. A nontrivial path $\gamma$ in $G$ is a periodic Nielsen path if there exists a $k$ such that $f_k^\#(\gamma) = \gamma$; the minimal such $k$ is called the period and if $k = 1$, we call such a path Nielsen path. A periodic Nielsen path is indivisible if it cannot be written as a concatenation of two or more nontrivial periodic Nielsen paths.

Given a subgraph $H \subset G$ let $G \setminus H$ denote the union of edges in $G$ that are not in $H$.

Given a marked graph $G$ and a homotopy equivalence $f : G \rightarrow G$ that takes edges to paths, one can define a new map $Tf$ by setting $Tf(E)$ to be the first edge in the edge path associated to $f(E)$; similarly let $Tf(E_i, E_j) = (Tf(E_i), Tf(E_j))$. So $Tf$ is a map that takes turns to turns. We say that a nondegenerate turn is illegal if for some iterate of $Tf$ the turn becomes degenerate; otherwise the turn is legal. A path is said to be legal if it contains only legal turns and it is $r$-legal if it is of height $r$ and all its illegal turns are in $G_{r-1}$.

**Relative train track map.** Given $\phi \in \text{Out}(F)$ and a topological representative $f : G \rightarrow G$ with a filtration $G_0 \subset G_1 \subset \cdots \subset G_k$ which is preserved by $f$, we say that $f$ is a train relative train track map if the following conditions are satisfied:

1. $f$ maps $r$-legal paths to legal $r$-paths.
2. If $\gamma$ is a path in $G$ with its endpoints in $H_r$ then $f_\#(\gamma)$ has its end points in $H_r$.
3. If $E$ is an edge in $H_r$ then $Tf(E)$ is an edge in $H_r$.

For any topological representative $f : G \rightarrow G$ and exponentially growing stratum $H_r$, let $N(f, r)$ be the number of indivisible Nielsen paths $\rho \subset G$ that intersect the interior of $H_r$. Let $N(f) = \Sigma_r N(f, r)$. Let $N_{min}$ be the minimum value of $N(f)$ that occurs among the topological representatives with $\Gamma = \Gamma_{min}$. We call a relative train track map stable if $\Gamma = \Gamma_{min}$ and $N(f) = N_{min}$. The following result is Theorem 5.12 in [Bestvina and Handel 1992] which assures the existence of a stable relative train track map.

**Lemma 2.6.** Every $\phi \in \text{Out}(F)$ has a stable relative train track representative.

**Splittings, complete splittings and CT’s.** Given relative train track map $f : G \rightarrow G$, splitting of a line, path or a circuit $\gamma$ is a decomposition of $\gamma$ into subpaths $\gamma_0, \gamma_1, \ldots, \gamma_k$, such that for all $i \geq 1$ the path $f_\#(\gamma) = \gamma_0$ $f_\#(\gamma_1)$ $\cdots f_\#(\gamma_k)$. The terms $\gamma_i$ are called the terms of the splitting of $\gamma$.

Given two linear edges $E_1, E_2$ and a root-free closed Nielsen path $\rho$ such that $f_\#(E_i) = E_i, \rho^m$ then we say that $E_1, E_2$ are said to be in the same linear family and any path of the form $E_1 \rho^m E_2$ for some integer $m$ is called an exceptional path.

**Complete splittings:** A splitting of a path or circuit $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_k$ is called complete splitting if each term $\gamma_i$ falls into one of the following categories:
• $\gamma_i$ is an edge in some irreducible stratum.
• $\gamma_i$ is an indivisible Nielsen path.
• $\gamma_i$ is an exceptional path.
• $\gamma_i$ is a maximal subpath of $\gamma$ in a zero stratum $H_r$ and $\gamma_i$ is taken.

**Completely split improved relative train track maps.** A CT or a completely split improved relative train track maps are topological representatives with particularly nice properties. But CTs do not exist for all outer automorphisms. Only the rotationless outer automorphisms are guaranteed to have a CT representative as has been shown in the following Theorem from Feighn and Handel [2011] (Theorem 4.28).

**Lemma 2.7.** For each rotationless $\phi \in \text{Out}(F)$ and each increasing sequence $F$ of $\phi$-invariant free factor systems, there exists a CT $f : G \to G$ that is a topological representative for $\phi$ and $f$ realizes $F$.

The following results are some properties of CT’s defined in Recognition theorem work of Feighn-Handel in Feighn and Handel [2011]. We will state only the ones we need here.

1. **(Rotationless)** Each principal vertex is fixed by $f$ and each periodic direction at a principal vertex is fixed by $Tf$.
2. **(Completely Split)** For each edge $E$ in each irreducible stratum, the path $f(E)$ is completely split.
3. **(vertices)** The endpoints of all indivisible Nielsen paths are vertices. The terminal endpoint of each nonfixed NEG edge is principal.
4. **(Periodic edges)** Each periodic edge is fixed.
5. **(Zero strata)** Each zero strata $H_i$ is contractible and enveloped by a EG strata $H_s, s > i,$ such that every edge of $H_i$ is a taken in $H_s$. Each vertex of $H_i$ is contained in $H_s$ and link of each vertex in $H_i$ is contained in $H_i \cup H_s$.
6. **(Linear Edges)** For each linear edge $E_i$ there exists a root free indivisible Nielsen path $w_i$ such that $f#(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$.
7. **(Nonlinear NEG edges)** Feighn and Handel [2011], Lemma 4.21] Each non-fixed NEG stratum $H_i$ is a single edge with its NEG orientation and has a splitting $f#(E_i) = E_i \cdot u_i$, where $u_i$ is a closed nontrivial completely split circuit and is an indivisible Nielsen path if and only if $H_i$ is linear.

CTs have very nice properties. The reader can look them up Feighn and Handel [2011] for a detailed exposition or Handel and Mosher [2017a] for a quick reference. We list below only a few of them that is needed for us. All three lemmas are present in the aforementioned papers. These properties along with the complete description of the components in a complete splitting of a circuit are the main reasons why will keep working with rotationless powers of a hyperbolic $\phi$ in the next two sections. We can achieve a great deal of control when we iterate random conjugacy classes under $\phi$.

**Lemma 2.8.** Feighn and Handel [2011], Lemma 4.11) A completely split path or circuit has a unique complete splitting.

The following lemma is a crucial component of our proof here. It assures us that as we iterate a circuit $\sigma$ under $f#$ eventually we will achieve a complete splitting at some point and then analyzing the components of such a splitting (as we do in Lemma 3.1) tells us what the possible limits could be.
Lemma 2.9. [Feighn and Handel, 2011, Lemma 4.26] If $\sigma$ is a finite path or a circuit with endpoint in vertices, then $f_{\#}^k(\sigma)$ is completely split for all sufficiently large $k \geq 1$ and $f_{\#}^{k+1}(\sigma)$ has complete splitting that is a refinement of the complete splitting of $f_{\#}^k(\sigma)$.

The following two Lemmas are used in proof of Lemma 3.1, when we give our argument with the zero strata within the proof. When a circuit $\sigma$ is iterated under a hyperbolic $\phi$ and we have a complete splitting for some $f_{\#}^k(\sigma)$, where one of the components of the complete splitting is a path in a zero strata, these lemmas along with Property(5) listed above, tell us that the preceding and the following components (with respect to the component contained in zero strata) in the complete splitting must be an exponentially growing edge.

Lemma 2.10. [Bestvina et al., 2000, Theorem 5.15, eg(i)] Every periodic Nielsen path is fixed. Also, for each EG stratum $H_r$ there exists at most one indivisible Nielsen path of height $r$, upto reversal of orientation, and the initial and terminal edges of this Nielsen path is in $H_r$.

The following lemma is part of [Feighn and Handel, 2011, Lemma 4.24]

Lemma 2.11. If an EG stratum $H_i$ has an indivisible Nielsen path of height $i$ then there is no zero strata enveloped by $H_i$.

2.5. Attracting Laminations and their properties under CTs. For any marked graph $G$, the natural identification $B \approx B(G)$ induces a bijection between the closed subsets of $B$ and the closed subsets of $B(G)$. A closed subset in any of these two cases is called a lamination, denoted by $\Lambda$. Given a lamination $\Lambda \subset B$ we look at the corresponding lamination in $B(G)$ as the realization of $\Lambda$ in $G$. An element $\lambda \in \Lambda$ is called a leaf of the lamination.

A lamination $\Lambda$ is called an attracting lamination for $\phi$ if it is the weak closure of a line $l$ (called the generic leaf of $\lambda$) satisfying the following conditions:

- $l$ is birecurrent leaf of $\Lambda$.
- $l$ has an attracting neighborhood $V$, in the weak topology, with the property that every line in $V$ is weakly attracted to $l$.
- no lift $\bar{l} \in B$ of $l$ is the axis of a generator of a rank 1 free factor of $F$.

We know from [Bestvina et al., 2000] that with each $\phi \in \text{Out}(F)$ we have a finite set of laminations $\mathcal{L}(\phi)$, called the set of attracting laminations of $\phi$, and the set $\mathcal{L}(\phi)$ is invariant under the action of $\phi$. When it is nonempty $\phi$ can permute the elements of $\mathcal{L}(\phi)$ if $\phi$ is not rotationless. For rotationless $\phi$ $\mathcal{L}(\phi)$ is a fixed set. Attracting laminations are directly related to EG stratas. An important result from [Bestvina et al., 2000] section 3 is that there is a unique bijection between exponentially growing strata and attracting laminations, which implies that there are only finitely many elements in $\mathcal{L}(\phi)$.

Dual lamination pairs. We have already seen that the set of lines carried by a free factor system is a closed set and so, together with the lemma that the weak closure of a generic leaf $\lambda$ of an attracting lamination $\Lambda$ is the whole lamination $\Lambda$ tells us that $A_{\text{supp}}(\lambda) = A_{\text{supp}}(\Lambda)$. In particular the free factor support of an attracting lamination $\Lambda$ is a single free factor. Let $\phi \in \text{Out}(F)$ be an outer automorphism and $\Lambda_\phi^+$ be an attracting lamination of $\phi$ and $\Lambda_\phi^-$ be an attracting lamination of $\phi^{-1}$. We say that this lamination pair is a dual lamination pair.
if $A_{\text{supp}}(\Lambda_{\phi}^+) = A_{\text{supp}}(\Lambda_{\phi}^-)$. By Lemma 3.2.4 of Bestvina et al. [2000] there is a bijection between $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ induced by this duality relation.

Tiles: Bestvina-Feighn-Handel introduced the concept of tiles in [Bestvina et al. 2000]. For an edge $E$ in an EG stratum, an unoriented path of the form $f_{\#}^k(E)$ is called a $k$-tile of height $r$. We state below a two important applications of tiles.

Lemma 2.12. (1) [Bestvina et al. 2000, Lemma 3.1.10 item (4)] If $\Lambda_r$ is the unique attracting lamination associated with $H_r$ then every generic leaf can be written as a increasing union of tiles of height $r$.

(2) [Handel and Mosher, 2017b, Lemma 1.57 item (4)] There exists $p$ such that for every $k \geq i \geq 0$, each $k+p$-tile of height $r$ contains every $i$-tile of height $r$.

This lemma will be used by us while analysing the weak limits of a conjugacy class under iterates of $\phi$; when we look at a complete splitting of some $\phi_{\rho_\sigma}^s(\sigma)$ and one of the components in that splitting is an edge in an EG strata, these lemma tell us that such a circuit is weakly attracted to the attracting lamination related to that EG strata.

2.6. Nonattracting subgroup system $A_{na}(\Lambda_{\phi}^+)$. The nonattracting subgroup system of an attracting lamination contains information about lines and circuits which are not attracted to the lamination. The definition of this subgroup system is

Definition 2.13. Suppose $\phi \in \text{Out}(F)$ is rotationless and $f : G \to G$ is a CT representing $\phi$ such that $\Lambda_{\phi}^+$ is an invariant attracting lamination which corresponds to the EG stratum $H_{\phi} \in G$. The nonattracting subgraph $\mathcal{Z}$ of $G$ is defined as a union of irreducible stratas $H_i$ of $G$ such that no edge in $H_i$ is weakly attracted to $\Lambda_{\phi}^+$. This is equivalent to saying that a strata $H_{r} \subset G \setminus \mathcal{Z}$ if and only if there exists $s \geq 0$ some term in the complete splitting of $f_{\#}^s(E_{r})$ is an edge in $H_s$. Define the path $\rho_s$ to be trivial path at any chosen vertex if there does not exist any indivisible Nielsen path of height $s$, otherwise $\rho_s$ is the unique closed indivisible path of height $s$ (from definition of stable train track maps).

The groupoid $\langle \mathcal{Z}, \rho_s \rangle$ - Let $\langle \mathcal{Z}, \rho_s \rangle$ be the set of lines, rays, circuits and finite paths in $G$ which can be written as a concatenation of subpaths, each of which is an edge in $\mathcal{Z}$, the path $\rho_s$ or its inverse. Under the operation of tightened concatenation of paths in $G$, this set forms a groupoid (Lemma 5.6, Handel and Mosher [2017b]).

Define the graph $K$ by setting $K = \mathcal{Z}$ if $\rho_s$ is trivial and let $h : K \to G$ be the inclusion map. Otherwise define an edge $E_p$ representing the domain of the Nielsen path $\rho_s : E_p \to G_s$, and let $K$ be the disjoint union of $\mathcal{Z}$ and $E_p$ with the following identification. Given an endpoint $x \in E_p$, if $\rho_s(x) \in \mathcal{Z}$ then identify $x \sim \rho_s(x)$, otherwise identify $x \sim y$. In this case define $h : K \to G$ to be the inclusion map on $K$ and the map $\rho_s$ on $E_p$. It is not difficult to see that the map $h$ is an immersion. Hence restricting $h$ to each component of $K$, we get an injection at the level of fundamental groups. The nonattracting subgroup system $A_{na}(\Lambda_{\phi}^+)$ is defined to be the subgroup system defined by this immersion.

We will leave it to the reader to look it up in [Handel and Mosher, 2017b] where it is explored in details. We however list some key properties which we will be using and justifies the importance of this subgroup system.
**Lemma 2.14.** \(\text{[Handel and Mosher [2017b]- Lemma 1.5, 1.6]}\)

1. The set of lines carried by \(A_{na}(\Lambda_\phi^+)\) is closed in the weak topology.
2. A conjugacy class \([c]\) is not attracted to \(\Lambda_\phi^+\) if and only if it is carried by \(A_{na}(\Lambda_\phi^-)\).
3. \(A_{na}(\Lambda_\phi^+)\) does not depend on the choice of the CT representing \(\phi\).
4. Given \(\phi, \phi^{-1} \in \text{Out}(F)\) both rotationless elements and a dual lamination pair \(\Lambda_\phi^\pm\) we have \(A_{na}(\Lambda_\phi^+) = A_{na}(\Lambda_\phi^-)\).
5. \(A_{na}(\Lambda_\phi^+)\) is a free factor system if and only if the stratum \(H_r\) is not geometric.
6. \(A_{na}(\Lambda_\phi^+)\) is nonnormal.
7. If \(\{\gamma_n\}_{n \in \mathbb{N}}\) is a sequence of lines such that every weak limit of every subsequence of \(\{\gamma_n\}\) is carried by \(A_{na}(\Lambda_\phi)\) then \(\{\gamma_n\}\) is carried by \(A_{na}(\Lambda_\phi)\) for all sufficiently large \(n\).

2.7. **Singular lines, Extended boundary and Weak attraction theorem.**

In this section we will look at some results from \[\text{Handel and Mosher [2017b]}\] which analyze and identify the set of lines which are not weakly attracted to an attracting lamination \(\Lambda_\phi^+\), given some exponentially growing element in \(\text{Out}(F)\). Most of the results stated here are in terms of rotationless elements as in the original work. However, we note that being weakly attracted to a lamination \(\Lambda_\phi^+\) is not dependent on whether the element is rotationless. All lemmas stated here about rotationless elements also hold for non rotationless elements also, unless otherwise mentioned. This has been pointed out in Remark 5.1 in \[\text{Handel and Mosher [2017b]}\]. The main reason for using rotationless elements is to make use of the train track structure from the CT theory. We will use some of the lemmas to prove lemmas about non rotationless elements which we will need later on.

Denote the set of lines not attracted to \(\Lambda_\phi^+\) by \(B_{na}(\Lambda_\phi^+)\). The non-attracting subgroup system carries partial information about such lines as we can see in Lemma 2.14. Other obvious lines which are not attracted are the generic leaves of \(\Lambda_\phi^-\). There is another class of lines, called singular lines, which we define below, which are not weakly attracted to \(\Lambda_\phi^+\).

**Definition 2.15.** Define a singular line for \(\phi\) to be a line \(\gamma \in B\) if there exists a principal lift \(\Phi\) of some rotationless iterate of \(\phi\) and a lift \(\tilde{\gamma}\) of \(\gamma\) such that the endpoints of \(\tilde{\gamma}\) are contained in \(\text{Fix}_{\phi}((\Phi) \subset \partial F\).

The set of all singular lines of \(\phi\) is denoted by \(B_{\text{sing}}(\phi)\).

The lemma [Lemma 2.1, \text{Handel and Mosher [2017b]}] below summarizes this discussion.

**Lemma 2.16.** Given a rotationless \(\phi \in \text{Out}(F)\) and an attracting lamination \(\Lambda_\phi^+\), any line \(\gamma\) that satisfies one of the following three conditions is in \(B_{na}(\Lambda_\phi^+)\).

1. \(\gamma\) is carried by \(A_{na}(\Lambda_\phi^+)\).
2. \(\gamma\) is a generic leaf of some attracting lamination for \(\phi^{-1}\).
3. \(\gamma\) is in \(B_{\text{sing}}(\phi^{-1})\).

But these are not all lines that constitute \(B_{na}(\Lambda_\phi^+)\). A important theorem in \[\text{Theorem 2.6, Handel and Mosher [2017b]}\], stated below, tells us that there is way to concatenate lines from the three classes we mentioned in the above lemma which will...
also result in lines that are not weakly attracted to \( \Lambda^\pm_{\phi} \). These are all possible types of lines in \( B_{na}(\Lambda^\pm_{\phi}) \). A simple explanation of why the concatenation is necessary is, one can construct a line by connecting the base points of two rays, one of which is asymptotic to a singular ray in the forward direction of \( \phi \) and the other is asymptotic to a singular ray in the backward direction of \( \phi \). This line does not fall into any of the three categories we see in the lemma above. The concatenation process described in [Handel and Mosher 2017b] takes care of such lines. We will not describe the concatenation here, but the reader can look up section 2.2 in [Handel and Mosher 2017b]. The following definition is by Handel and Mosher:

**Definition 2.17.** Let \( A \in A_{na}\Lambda^\pm_{\phi} \) and \( \Phi \in P(\phi) \), we say that \( \Phi \) is \( A \)-related if 
\[ \text{Fix}_N(\hat{\Phi}) \cap \partial A \neq \emptyset. \]
Define the extended boundary of \( A \) to be
\[ \partial_{ext}(A,\phi) = \partial A \cup \left( \bigcup_{\Phi} \text{Fix}_N(\hat{\Phi}) \right) \]
where the union is taken over all \( A \)-related \( \Phi \in P(\phi) \).

Let \( B_{ext}(A,\phi) \) denote the set of lines which have end points in \( \partial_{ext}(A,\phi) \); this set is independent of the choice of \( A \) in its conjugacy class. Define
\[ B_{ext}(\Lambda^\pm_{\phi}) = \bigcup_{A \in A_{na}\Lambda^\pm_{\phi}} B_{ext}(A,\phi) \]

It is worth noting that the sets of lines mentioned in Lemma 2.18 are not necessarily pairwise disjoint. But if we have a line \( \sigma \in B_{na}(\Lambda^\pm_{\phi}) \) that is birecurrent then the situation is much simpler. In that case \( \sigma \) is either a generic leaf of some attracting lamination for \( \phi^{-1} \) or \( \sigma \) is carried by \( A_{na}\Lambda^\pm_{\phi} \). This takes us to the following result due to Handel and Mosher that we need to prove an important result about asymptotic behaviour of leaves of attracting laminations in Proposition 4.7 where we use the conclusion from the “moreover” part of the theorem to conclude that if two leaves of attracting laminations of \( \phi \) can be asymptotic then they are both singular lines of \( \phi \).

**Lemma 2.18.** [Handel and Mosher, 2017b, Theorem G] If \( \phi,\psi = \psi^{-1} \in \text{Out}(\mathbb{F}) \) are rotationless and \( \Lambda^\pm_{\phi} \) is a dual lamination pair then
\[ B_{na}(\Lambda^-_{\phi},\psi) = B_{ext}(\Lambda^\pm_{\phi},\phi) \cup B_{gen}(\phi) \cup B_{sing}(\phi) \]

Moreover the set of lines in \( B_{na}(\Lambda^-_{\phi},\psi) \) are closed under concatenation. More precisely if \( l',l'' \) are two lines in \( B_{na}(\Lambda^-_{\phi},\psi) \) with one asymptotic end \( P \) say, and two other distinct endpoints \( Q',Q'' \) for \( l',l'' \) respectively, then either there exists some \([A] \in A_{na}(\Lambda^\pm_{\phi}) \) so that \( P,Q',Q'' \in \partial_{ext}(A,\phi) \) or there is a principal lift \( \Phi \) such that all three points \( P,Q',Q'' \) are in \( \text{Fix}_N(\hat{\Phi}) \).

This result also plays a key role in the proof of Proposition 3.10 where we show that weak limit of any conjugacy class under iterates of a hyperbolic outer automorphism is either a generic leaf or a singular line and uses the understanding of \( B_{ext}(\Lambda^\pm_{\phi},\phi) \).

**Remark 2.19.** For our purposes, where \( \phi \) is hyperbolic, \( \text{Fix}_N(\hat{\Phi}) = \text{Fix}_+(\hat{\Phi}) \).
From the work of Feighn and Handel in Recognition theorem we extract the following lemma by assuming $\phi$ is hyperbolic. The original statement is much more general.

**Lemma 2.20.** [Feighn and Handel, 2011, Lemma 4.36] Let CT $f : G \rightarrow G$ represent a rotationless $\phi \in \text{Out}(F)$, and $\hat{f} : \hat{G} \rightarrow \hat{G}$ is a lift. If a vertex $v$ is fixed by $f$ and $E$ is a non-fixed edge in some EG strata or superlinear NEG strata originating at $v$ with a fixed initial direction, such that $f_E^k(E) = E \cdot u$, then there exists a splitting $f^k(E) = E \cdot u$ such that: $\hat{R} = \hat{E} \cdot \hat{u} \cdot f^k_\#(\hat{u}) \cdot f^2_\#(\hat{u}) \cdot \ldots \cdot f^k_\#(\hat{u}) \ldots$ is a ray and its endpoint is $\xi \in \text{Fix}_+(\hat{\Phi})$ for some principal lift $\hat{\Phi}$. Moreover, if $E$ is an edge in some EG strata then the weak accumulation set of $\xi$ is the unique attracting lamination associated to EG strata which contains $E$.

Conversely, every point $\xi \in \text{Fix}_+(\hat{\Phi})$ is obtained by iteration of a some nonfixed edge with a fixed initial vertex.

Note that in this case $\hat{\Phi}$ may not be a principal lift and the ray may not be a singular ray as we have described it here. What we are really interested in is the conclusion in the last sentence about accumulation set about the endpoint. The fact that the weak closure of any attracting fixed point contains an attracting lamination is used several times in this paper. In particular note that if we have a conjugacy class that gets attracted to a singular line under iterates of $\phi$, then it is also weakly attracted to the attracting laminations which appear in the closure of the endpoints of this singular line.

The following lemmas are applications of the above result which, put together, essentially tell that iteration of any edge whose initial vertex is principal will give us a ray whose endpoint is an attracting fixed point.

**Lemma 2.21.** Let $v$ be a principal vertex in a CT $f : G \rightarrow G$ representing a rotationless $\phi \in \text{Out}(F)$. Then for every direction $d$ originating from $v$, $f_E^N(d)$ is a fixed direction for some $N > 0$.

*Proof.* The number of directions originating at any given vertex is finite. If a direction is not periodic then upon iteration of $f_E^k$ it must repeat some direction after $N'$ iterations. But since $f$ is rotationless every periodic direction has period one. So, $N'$ is bounded by valence of $v$. Repeat this for every direction and taking the least common multiple we get $N$ such that $f_E^N(d)$ has a fixed direction for any direction $d$. \qed

**Lemma 2.22.** Let $v$ be a principal vertex in a CT $f : G \rightarrow G$ representing a rotationless $\phi \in \text{Out}(F)$ and $E'$ be a non-fixed EG edge originating at $v$. Then $f^k_\#(E')$ weakly accumulates on a singular ray as $k \rightarrow \infty$.

*Proof.* By using the previous lemma we know that the initial direction of $f^k_\#(E')$ is fixed for all $k \geq N$, where $N$ is as in Lemma 2.22. Hence we can write $f^k_\#(E') = E \cdot u$. where $E$ is a principal edge. Since the initial direction of $E$ is a fixed direction, we have $f_\#(E) = E \cdot u'$. So, $f_\#(Eu) = E \cdot u' f_\#(u)$. Because, if $f_\#(u)$ cancels out all of $u'$, then we can write $f_\#(u) = u'' u''$ and then in $u''$ having an $E$ in the beginning must be followed by $E$, since the initial direction of $f^k_\#(E')$ is principal. This is not possible by definition of $f_\#$. Thus, the splitting is indeed there. So, $f^{k+1}_\#(E') = E \cdot u' f_\#(u)$. Upon iteration we see that this ray $R'$ accumulates on a singular ray $R$ with initial direction $E$. 


Next we note that since every finite subpath of $R'$ occurs as a subpath of $R$. This is because each tile of $E'$ is contained in a tile of $E$. Hence the weak closure of $R'$ is contained in the weak closure of $R$. But the weak closure of $R$ is just $\Lambda^+$. Hence weak closure of $R'$ is the same set and the two rays are asymptotic.

□

Structure of Singular lines: We now state a lemma which is a collection of lemmas from the subgroup decomposition work of Handel and Mosher that tells us the structure of singular lines and guarantees that one of the leaves of any attracting lamination is a singular line.

Lemma 2.23. [Handel and Mosher, 2009, Lemma 3.5, Lemma 3.6], [Handel and Mosher, 2017a, Lemma 1.63]
Let $\phi \in \text{Out}(F)$ be rotationless and hyperbolic and $l \in B_{\text{sing}}(\phi)$ then the following are true:

1. $l = R\alpha R'$ where $R$ for some singular rays $R \neq R'$ and some path $\alpha$ which is either trivial or a Nielsen path. Conversely, any such line is a singular line.

2. If $\Lambda \in \mathcal{L}(\phi)$ then there exists a leaf of $\Lambda$ which is a singular line and one of its ends is dense in $\Lambda$.

We include a short sketch of the proof here for sake of completeness, since this result is fundamental to the work done in this paper here. For more technical details please refer to the original proof.

Proof. Let $f : G \to G$ be a CT representing $\phi$.

Sketch of proof for (1): Suppose $l$ is a singular line, then there exists a principal lift $\Phi$ of $\phi$ which fixes the endpoints of this line, which are attracting fixed points. Use Lemma 2.20 to conclude that there are singular rays $R, R'$ which converge to these endpoints. Join the initial endpoints of $R, R'$ by a path $\tilde{\alpha}$. Then the endpoints of $\tilde{\alpha}$ are fixed and the projection to $G$, say $\alpha$, is a Nielsen path. However, the line $R\alpha R'$ may not be locally injective at endpoints of $\alpha$. But using the lemma that $\alpha$ has a unique decomposition into concatenation of indivisible Nielsen paths and fixed edges, one can choose $R, R'$ such that it is locally injective as is done in the original proof in [Handel and Mosher, 2009, Lemma 3.5]. The converse part follows straight from definition of a singular line. This completes the sketch of proof for (1).

Sketch of proof for (2) [Handel and Mosher, 2009, Lemma 3.6]: Let $H_r$ be the unique EG strata corresponding to $\Lambda$. Using the lemma that every generic leaf has a complete splitting (since it can be written as a increasing union of tiles) one concludes that a generic leaf $l$ has a splitting of the form $R \cdot E \cdot R'$, where $E$ is an edge in $H_r$ whose initial vertex is principal and initial direction is fixed (existence of $E$ is due to [Feighn and Handel, 2011, Lemma 3.19]). Consider a further splitting of $l = R \cdot E \cdot L \cdot R''$, where $L$ is the longest initial segment of $R'$ which can be expressed as a concatenation of indivisible Nielsen paths and fixed edges. Recall that there is an upper bound on the number of components in the complete splitting of any path which occurs as a concatenation of fixed edges and indivisible Nielsen paths. Using this, choose $l$ to maximize the number of components of $\alpha$. Now iterate $l$ by $f_\#$ to get a leaf (since $\Lambda$ is invariant under $f_\#$) and this leaf will be a singular line due to choice of the splitting above.
Finiteness of singular lines: Before we end this overview about singular lines we prove a lemma that shows that $B_{\text{sing}}(\phi)$ is a finite set. We will need this result to prove the uniform finiteness of fibers of Cannon-Thurston maps.

Recall that the normal subgroup Inn($\mathbb{F}$) acts on $\text{Aut}(\mathbb{F})$ by conjugation and the orbits of this action define an equivalence relation on $\text{Aut}(\mathbb{F})$ called isogredience. The set of principal automorphisms representing some $\phi \in \text{Out}(\mathbb{F})$ is invariant under isogredience. Also any two elements of $\text{Aut}(\mathbb{F})$ which are in the same equivalence class have the same attracting fixed points at the boundary. [Feighn and Handel, 2011, Remark 3.9] shows that the number of isogredience classes of principal automorphisms are finite.

Let $B_{\text{Fix}+}(\phi)$ denote the set of lines with endpoints in $\text{Fix}+_{\hat{\phi}}$. Proposition 2.24.

For any hyperbolic $\phi \in \text{Out}(\mathbb{F})$ the set of lines $B_{\text{Fix}+}(\phi)$ is a finite set and its cardinality is bounded above by a number depending only on $\text{rank}(\mathbb{F})$.

The author thanks Lee Mosher for help with the proof.

Proof. Let $\text{rank}(\mathbb{F}) = N$. There are only finitely many isogredience classes of principal automorphisms in $P(\phi)$. Pick representatives $\Phi_1, \Phi_2, \ldots, \Phi_K$ as principal automorphisms that represent these isogredience classes. Since $\phi$ is hyperbolic $\text{Fix}_i(\Phi_i) = \text{Fix}_i(\hat{\Phi}_i)$ for all $i = 1, 2, \ldots, K$. If $a_i = |\text{Fix}_i(\hat{\Phi}_i)|$ for $i = 1, 2, \ldots, K$ then the main inequality of counting indexes in [Gaboriau et al., 1998, Theorem 1'] gives us

\[ a_1 + a_2 + \ldots + a_K \leq 2N \]

Since any line in $B_{\text{sing}}(\phi)$ is obtained by joining two points in $\text{Fix}_i(\hat{\Phi}_i)$ for some principal lift $\hat{\Phi}_i$, the above inequality shows that maximum number of possible lines in $B_{\text{sing}}(\phi)$ is uniformly bounded above by some function of $N$.

Now, coming to $B_{\text{Fix}+}(\phi)$, by using Lemma 2.20 we can deduce that the endpoints of these lines are ones that are defined by singular rays. So, $B_{\text{Fix}+}(\phi)$ is obtained by joining endpoints of lines in $B_{\text{sing}}(\phi)$, hence the cardinality of $B_{\text{Fix}+}(\phi)$ is uniformly also bounded above by some number depending on $\text{rank}(\mathbb{F})$. □

2.8. Weak attraction theorem. The following lemma is very important for our purposes. The form in which we will be using it is: given a line $l$ such that one endpoint of $l$, $\xi$ say, is carried by some $A_{\text{na}}(\Lambda_+^\phi)$ (i.e. there exists some $[A] \in A_{\text{na}}(\Lambda_+^\phi)$ such that $\xi \in \partial A$) but the other point of $l$ is not in $\partial A$ then the lemma tells us that the line gets attracted to either the attracting lamination or the repelling lamination (or both). We call it the Weak attraction theorem.

Lemma 2.25 (Handel and Mosher [2017b], Corollary 2.17, Theorem H). Let $\phi \in \text{Out}(\mathbb{F})$ be a rotationless and exponentially growing. Let $\Lambda_+^\phi$ be a dual lamination pair for $\phi$. Then for any line $\gamma \in \mathcal{B}$ not carried by $A_{\text{na}}(\Lambda_+^\phi)$ at least one of the following hold:

1. $\gamma$ is attracted to $\Lambda_+^\phi$ under iterations of $\phi$.
2. $\gamma$ is attracted to $\Lambda_-^\phi$ under iterations of $\phi^{-1}$. 

Moreover, if \( V_\phi^+ \) and \( V_\phi^- \) are attracting neighborhoods for the laminations \( \Lambda_\phi^+ \) and \( \Lambda_\phi^- \) respectively, there exists an integer \( l \geq 0 \) such that at least one of the following holds:

- \( \gamma \in V_\phi^- \).
- \( \phi^l(\gamma) \in V_\phi^+ \).
- \( \gamma \) is carried by \( A_{na}(\Lambda_\phi^\pm) \).

**Lemma 2.26.** Suppose \( \phi, \psi \in \text{Out}(F) \) are two exponentially growing automorphisms with attracting laminations \( \Lambda_\phi^+ \) and \( \Lambda_\psi^+ \) respectively. If a generic leaf \( \lambda \in \Lambda_\phi^+ \) is in \( B_{na}(\Lambda_\psi^+) \) then the whole lamination \( \Lambda_\phi^+ \subset B_{na}(\Lambda_\psi^+) \).

**Proof.** Recall that a generic leaf is birecurrent. Hence, \( \lambda \in B_{na}(\Lambda_\psi^+) \) implies that \( \lambda \) is either carried by \( A_{na}(\Lambda_\psi^+) \) or it is a generic leaf of some element of \( \mathcal{L}(\psi^{-1}) \). First assume that \( \lambda \) is carried by \( A_{na}(\Lambda_\psi^+) \). Then using Lemma 2.14 item 1, we can conclude that \( \Lambda_\phi^+ \) is carried by \( A_{na}(\Lambda_\psi^+) \).

Alternatively, if \( \lambda \) is a generic leaf of some element \( \Lambda_\psi^- \in \mathcal{L}(\psi^{-1}) \), then the weak closure \( \overline{\lambda} = \Lambda_\phi^+ = \Lambda_\psi^- \) and we know \( \Lambda_\psi^- \) does not get attracted to \( \Lambda_\psi^+ \). Hence, \( \Lambda_\phi^+ \subset B_{na}(\Lambda_\psi^+) \). \( \square \)

**Lemma 2.27.** Let \( \phi \in \text{Out}(F) \) be rotationless and \([c]\) be some conjugacy class in \( F \). If \( \Lambda^- \) is a repelling lamination for \( \phi \) and \( V^- \) is an attracting neighborhood for \( \Lambda^- \), then there are only finitely many values of \( k > 0 \) such that \( \phi^k([c]) \notin V^- \).

**Proof.** Since \( \phi \) is rotationless, \( \Lambda^- \) is \( \phi \) invariant and so \( \phi^{-1}(V^-) \subset V^- \). Since any weak neighborhood can be defined by some finite subpath of a generic leaf ([Bestvina et al., 2000, Corollary 3.1.11]), we see that under iterates of \( \phi^{-1} \) this subpath grows exponentially and since by [Bestvina et al., 2000, Lemma 3.1.16] a circuit cannot be a generic leaf, we have that there exists some \( K \) such that \( [c] \notin \phi^{-K}(V^-) \). Hence for every \( k \geq K \), \( \phi^k([c]) \notin V^- \).

\( \square \)

### 3. Main Theorem

In this section the goal is to list all the possible weak limits of \( \phi^k([c]) \) as \( k \to \infty \), where \( \phi \) is a hyperbolic outer automorphism of \( F \) and \([c]\) is any conjugacy class in \( F \). Let us denote this collection of weak limits by \( \mathcal{WLC}(\phi) \). We shall see from Theorem 3.10 that

\[ \mathcal{WLC}(\phi) = B_{gen}(\phi) \cup B_{sing}(\phi) \]

The set on the right hand side of the equality is a very well understood collection of lines in the theory of \( \text{Out}(F) \). Recall that it appears in the description of the set of nonattracted lines in 2.13 and this shows that the dynamics of conjugacy classes under iterates of a hyperbolic outer automorphism is very well controlled. There is an immediate observation one can make here; if a line \( \gamma \) is a weak limit of some conjugacy class under iterates of a hyperbolic \( \phi \), then \( \gamma \) is not weakly attracted to any repelling lamination of \( \phi \).

We begin this section with a lemma that, in some sense, is the heart of the proof here. It does a detailed analysis of the components of any complete splitting of a circuit under iterates of a hyperbolic outer automorphism. A clear understanding
of the proof of this lemma gives a very clear indication of why the above equality is what one should expect.

**Notation:** EG strata is an abbreviation for “exponentially growing strata”, NEG strata for “non-exponentially growing strata”, CT stands for “completely split relative train track maps”, \( B_{\text{gen}}(\phi) \) denotes the collection of all generic leaves of all attracting laminations for \( \phi \) and \( B_{\text{sing}}(\phi) \) denotes the collection of all singular lines of \( \phi \).

**Lemma 3.1.** Let \( \phi \in \text{Out}(F) \) be a rotationless and hyperbolic and \( f : G \longrightarrow G \) be any CT representing \( \phi \). If \( \sigma \) is a circuit in \( G \) then the terms that appear in a complete splitting of \( \sigma \) must contain an EG edge and hence is exponentially growing. Moreover, \( \sigma \) is attracted to some element of \( L(\phi) \).

**Proof.** By using Lemma 2.9 we know that \( f^k(\sigma) \) is completely split for all sufficiently large \( k \).

Existence of linear-NEG strata and geometric strata in \( G \) automatically guarantees the existence of a conjugacy class that is fixed by \( \phi \) which is not possible since \( \phi \) is hyperbolic. Hence no term in any complete splitting of \( f^k(\sigma) \) can be a linear edge or exceptional path or edge in a geometric EG strata.

If \( G \) has any superlinear-NEG strata, then consider \( E' = H_r \) to be the non-fixed NEG stratum of lowest height. By (NEG edges, 2.4) there exists a circuit \( u_r \subset G_{r-1} \) such that \( f^k(E') = E' \cdot u_r \) is a splitting and \( u_r \) is completely split. The terms of its complete splitting are edges in irreducible stratum, indivisible Nielsen paths, fixed edges or maximal subpaths in zero strata which are taken. We claim that some term of such a splitting must be an EG edge.

No term in splitting of \( u_r \) can be a linear NEG edge since \( \phi \) is hyperbolic. If we have taken, maximal subpaths in zero strata then we are done because one needs to pass through an EG stratum to enter and exit a zero strata (Envelope, 2.4) and the only way to do this would be if we had EG edges appearing as terms of complete splitting of \( u_r \) (on either side of the splitting component contained in the zero strata) since all zero strata are contractible and EG strata that contain indivisible Nielsen path do not envelope zero strata (see Lemma 2.11) and indivisible Nielsen paths of higher EG strata have both their endpoints in the same strata by Lemma 2.10.

So if we do not have paths contained in some zero strata then the only remaining possibilities for the components of complete splitting of \( u_r \) remaining are that of EG edges, fixed edges and indivisible Nielsen paths (since we have assumed that \( E_r \) has lowest height among NEG edges). If we have only fixed edges and indivisible Nielsen paths appearing as terms of complete splitting of \( u_r \) then it would imply \( u_r \) is a closed Nielsen path, which is not true since \( E' \) is superlinear NEG strata. Hence some term of complete splitting of \( u_r \) must be an EG edge. Now for superlinear NEG edges of height \( s > r \), one can similarly show by an induction argument that some term in the complete splitting of \( f^k(u_s) \) must be an EG edge, for some \( k > 0 \).

Since we have ruled out the possibility of any linear NEG edge and geometric strata in \( G \), the only choices we have in the complete splitting of \( f^k(\sigma) \) are EG edges, superlinear NEG edge, fixed edges, indivisible Nielsen paths and maximal connecting subpaths in a zero strata which are taken. Presence of only fixed edges and indivisible Nielsen paths would imply existence of a periodic circuit and violate the hyperbolicity assumption on \( \phi \). So we must have at least an EG edge, superlinear NEG edge or a taken connecting path in a zero stratum. If the last
case happens then, by the same argument we gave for superlinear-NEG edges, on either side of this subpath of zero strata the terms of complete splitting must be EG edges. Also, if we have a superlinear NEG edge, then by same argument above, an EG edge appears as a term of complete splitting in \(f^t_\#(\sigma)\), for some \(t > 0\). Hence some term in the complete splitting of \(f^t_\#(\sigma)\) must be an EG edge.

Thus \(\sigma\) is exponentially growing and is weakly attracted to the attracting laminations which are associated to the EG strata whose edges appear in some complete splitting of \(f^k_\#(\sigma)\) for some \(k > 0\).

□

Notice that the above lemma implies that given any conjugacy class \([c]\) in \(F\), there exists at least one one nonattracting subgroup system \(A_{na}(\Lambda_+^{\phi})\) which does not carry \([c]\). We shall see shortly in Lemma 3.5 that this behaviour is also observed in any line that occurs as a weak limit of \([c]\) under iterates of \(\phi\).

**Lemma 3.2.** Let \(\phi \in \text{Out}(F)\) be rotationless and hyperbolic. Then the weak closure of every point in \(\xi \in \text{Fix}_+(\phi)\) contains an element of \(\Lambda \in \mathcal{L}(\phi)\). Moreover, if the principal edge whose iteration generates the singular ray with endpoint \(\xi\) is an EG edge then \(\Lambda\) is the unique attracting lamination associated to the EG strata that contains the principal edge.

**Proof.** Let \(f : G \longrightarrow G\) be CT representing \(\phi\). Given any point \(\xi \in \text{Fix}_+(\phi)\), it is obtained by iteration of an edge \(E\) whose initial vertex is principal and initial direction is fixed (such an edge is called principal edge) (use Lemma 2.9). If \(E\) is an EG edge, and \(k \to \infty\), \(f^k_\#(E)\) generates a singular ray with endpoint in \(\xi\) then the closure of \(\xi\) equals the unique attracting lamination associated to the EG strata that contains \(E\).

On the other hand if \(E\) is superlinear NEG edge then \(f_\#(E) = E \cdot u\) is a splitting and the circuit \(u\) is completely split. Arguing just as we did in the previous lemma, some term in the complete splitting of \(f^k_\#(u)\) will be an EG edge and hence the ray generated by \(f^k_\#(E)\), as \(k \to \infty\), is a singular ray which weakly accumulates on some attracting lamination \(\Lambda \in \mathcal{L}(\phi)\) (where \(\Lambda\) is the unique attracting lamination associated to the EG strata whose edge appears in some complete splitting of \(f^k_\#(u)\)). This implies that the weak closure of \(\xi\) contains \(\Lambda\).

This shows that the weak closure of \(\xi\) is exactly the unique attracting lamination associated to the strata that contains the edge \(E\) by using Lemma 2.20. □

Our next goal is to inspect the dynamical nature of a line \(\gamma\) which appears as a weak limit of some conjugacy class under iterates of some hyperbolic outer automorphism \(\phi\). We shall prove in Lemma 3.3 and Lemma 3.5 that any such line gets attracted to some attracting lamination of \(\phi\) (under iterates of \(\phi\)) but is never attracted to any repelling lamination of \(\phi\) (under iterates of \(\phi^{-1}\)). Notice that this implies \(\gamma \in \mathcal{B}_{na}(\Lambda_\phi^+)\) for every repelling lamination of \(\phi\) and \(\gamma\) is not carried by \(A_{na}(\Lambda_\phi^+)\) for at least one attracting lamination of \(\phi\).

**Lemma 3.3.** Let \(\phi\) be a rotationless hyperbolic outer automorphism and \(f : G \longrightarrow G\) be a CT representing \(\phi\). Suppose \(\sigma\) is a circuit in \(G\). If \(\gamma\) is a weak limit of \(\sigma\) under action of \(f_\#\), then \(\gamma\) is not attracted to any element of \(\mathcal{L}(\phi^{-1})\).
Proof. Suppose on the contrary $\gamma$ gets attracted to some $\Lambda^-$. If $V^-$ is an attracting neighborhood of $\Lambda^-$ then $\phi^t_\#(\gamma) \in V^-$ for some $t \geq 1$. This implies that $\gamma \in \phi^t_\#(V^-)$. Since $\phi^t_\#(\Lambda^-) = \Lambda^-$ for all $t \geq 1$ we can conclude that $\gamma$ contains some subpath $\alpha$ of a generic leaf of $\Lambda^-$. So $\alpha$ is contained in $f^k_\#(\sigma)$ as a subpath for all sufficiently large $k$.

But this would mean that $f^k_\#(\sigma) \in V^-$ for infinitely many value of $k$, which contradicts the lemma that $\sigma$ cannot be attracted to $\Lambda^-$ by iterates of $f^k_\#$ (Lemma 2.27).

\[ \square \]

**Corollary 3.4.** Let $\phi \in \text{Out}(F)$ be rotationless and hyperbolic. If $\gamma$ is line which is a weak limit of some conjugacy class in $F$ then no endpoint of any lift of $\gamma$ is in $\text{Fix}_-(\phi)$.

Proof. Suppose $\gamma$ had an endpoint $\xi \in \text{Fix}_-(\phi)$. Since $\phi$ is hyperbolic, so is $\phi^{-1}$ and using this we derive a contradiction to Lemma 3.3.

Choose a CT $f' : G' \to G'$ representing some rotationless power of $\phi^{-1}$. Using Lemma 2.20 every point in $\text{Fix}_-(\phi)$ is obtained by iteration of an edge $E'$ whose initial vertex is principal and initial direction is fixed. By using Lemma 3.2 we get that the weak closure of $\xi$, and hence $\gamma$, will contain an attracting lamination $\Lambda^-$ associated to the EG strata $E'$. This means $\gamma$ is attracted to $\Lambda^-$. Which contradicts Lemma 3.3.

\[ \square \]

**Lemma 3.5.** Let $\phi$ be a hyperbolic automorphism and suppose $\phi^k(c)$ converges weakly to some line $\gamma$ as $k \to \infty$. Then $\gamma$ is weakly attracted to at least one element of $\mathcal{L}(\phi)$.

Proof. Replacing $\phi$ by some $\phi^N$ if necessary, we may assume that $\phi$ is rotationless. Let $f : G \to G$ be a CT representing $\phi$ and $\sigma$ be the realization of the conjugacy class $c$ in $G$. Note that if $\gamma \in B_{\text{gen}}(\phi) \cup B_{\text{sing}}(\phi)$ then weak closure of $\gamma$ contains at least one attracting lamination of $\phi$ and hence by definition, $\gamma$ is attracted to that attracting lamination. Hence we are left with the case when $\gamma \notin B_{\text{gen}}(\phi) \cup B_{\text{sing}}(\phi)$.

By using Corollary 3.4 we know that no endpoint of any lift of $\gamma$ is in $\text{Fix}_-(\phi)$. Also Lemma 3.3 tells us that $\gamma$ is not a generic leaf of some repelling lamination of $\phi$. Hence using Lemma 2.18 we can conclude that if $\gamma$ is not attracted to some attracting lamination $\Lambda^+$ then it must be carried by $\mathcal{A}_{\text{na}}(\Lambda^+)$. This implies that there is some lift $\tilde{\gamma}$ of $\gamma$ such that both endpoints of $\tilde{\gamma}$ are carried by $\partial A$ for some $[A] \in \mathcal{A}_{\text{na}}(\Lambda^+)$. Therefore we can conclude that if $\gamma$ is not attracted to any element of $\mathcal{L}(\phi)$ then is carried by $\mathcal{A}_{\text{na}}(\Lambda^+_i)$ for every element $\Lambda^+_i \in \mathcal{L}(\phi)$ where $i = 1, 2, ..., K$ for some $K < \infty$. So there are $[A_i] \in \mathcal{A}_{\text{na}}(\Lambda^+_i)$ such that $\partial A_i$ carries both endpoints of some lift of $\gamma$. Hence $\cap \partial A_i \neq \emptyset$. Thus $\cap A_i \neq \emptyset$ and so there exists a conjugacy class $[c]$ which is carried by $\mathcal{A}_{\text{na}}(\Lambda^+_i)$ for every $i$. But this contradicts Lemma 3.1. Therefore $\gamma$ must be attracted to some element of $\mathcal{L}(\phi)$.

\[ \square \]

The following proposition gives us one side of the inclusion of Theorem 3.10.

**Proposition 3.6.** Let $f : G \to G$ be a CT representing a rotationless hyperbolic element of $\text{Out}(F)$ and $\sigma$ is a circuit in $G$. If $l$ is a weak limit of $\sigma$ under the action...
of $f_\#$ which is not a singular line for $\phi$, then either $l$ must be a generic leaf of some element of $\mathcal{L}(\phi)$ or both endpoints of $l$ must be in $\text{Fix}_+(\phi)$.

**Proof.** Suppose on the contrary that $l \notin \mathcal{B}_{\text{gen}}(\phi)$. Given that $l$ is a generic leaf, and since we know that $l$ is not weakly attracted to any element of $\mathcal{L}(\phi^{-1})$ we can apply Lemma 2.18 and deduce that $l \in \mathcal{B}_{\text{ext}}(\Lambda^\pm, \phi)$ for every dual lamination pair of $\phi$. But there exists at least one element $\Lambda^+ \in \mathcal{L}(\phi)$, to which $l$ is attracted (by Lemma 3.5), which implies that $l$ is not carried by $\mathcal{A}_{na}(\Lambda^\pm)$. If we denote the two distinct endpoints of $l$ by $P$ and $Q$, then at least one endpoint, $P$ say, is in $\text{Fix}^+(\phi)$. We now proceed to give an argument by contradiction.

Suppose that $P \in \text{Fix}_+(\phi)$ but $Q \notin \text{Fix}_+(\phi)$. In this case notice that $Q$ must be in $\text{Fix}_-(\phi)$, since that would violate corollary 3.4. Since $Q \notin \text{Fix}_+(\phi)$, iteration of $Q$ by $\phi^{-t}$ converges to a point $Q^-$ in $\text{Fix}_-(\phi^{-t})$ (for some $t \geq 1$) by Lemma 2.2. But this implies $l$ is attracted to some $\Lambda^+_Q$ (since $\text{Fix}_+(\phi)$ and $\text{Fix}_-(\phi)$ are disjoint and hence $P \neq Q$), where $\Lambda^-_Q$ is in the weak closure of $Q^-$ (by using Lemma 3.2). But this contradicts Lemma 3.4. Hence this case is not possible.

Therefore $P \in \text{Fix}_+(\phi)$ and $Q \in \text{Fix}_+(\phi)$ and we get the desired conclusion.

Note that since we have assumed that $l$ is not a singular line there does not exist a principal lift that fixes both $P$ and $Q$, hence this case covers more than singular lines.

□

We now state an important lemma from the Subgroup Decomposition work of Handel and Mosher. This lemma can be used to conclude that every line in $\mathcal{B}_{\text{sing}}(\phi)$ and every leaf in any attracting lamination for $\phi$ occurs as a weak limit of some conjugacy class under action of $\phi$.

**Lemma 3.7.** [Handel and Mosher, 2017a, Lemma 1.52] For each $P \in \text{Fix}_+(\phi)$ there is a conjugacy class $[a]$ which is weakly attracted to every line in the weak accumulation set of $P$.

**Corollary 3.8.** Let $\phi$ be a rotationless hyperbolic outer automorphism and $\Lambda \in \mathcal{L}(\phi)$ be an attracting lamination. Then every nongeneric leaf (if it exists) of $\Lambda$ must be a line with both endpoints in $\text{Fix}_+(\phi)$.

**Proof.** Recall that Lemma 2.23 says that every attracting lamination $\Lambda \in \mathcal{L}(\phi)$ contains a singular line as a leaf, one of whose ends is dense in $\Lambda$. Lemma 3.7 tells us that there exists a conjugacy class $[a]$ which is weakly attracted to every line in $\Lambda$.

The proof now follows directly from Proposition 3.6. □

**Remark 3.9.** It is worth noting that this corollary is very special to hyperbolic outer automorphisms and will fail if $\phi$ is not hyperbolic. Once we have linear NEG edges or geometric strata the extended boundary takes a much more complex structure and one can easily construct examples where this lemma will fail.

We finally state and prove the main theorem of this section which classifies all weak limits of conjugacy classes under iterations of a hyperbolic outer automorphism.

**Notation:** Let $\mathcal{B}_{\text{Fix}_+}(\phi)$ denote the set of all lines with endpoints in $\text{Fix}_+(\phi)$. Note that $\mathcal{B}_{\text{sing}}(\phi) \subset \mathcal{B}_{\text{Fix}_+}(\phi)$. 

Theorem 3.10. Suppose \( \phi \in \text{Out}(F) \) is a hyperbolic outer automorphism and \([c]\) is any conjugacy class in \(F\). Then the weak limits of \([c]\) under iterates of \(\phi\) are in \(\mathcal{WL}(\phi) := B_{\text{gen}}(\phi) \cup B_{\text{Fix}+}(\phi)\).

Conversely, any line in \(\mathcal{WL}(\phi)\) is a weak limit of some conjugacy class in \(F\) under action of \(\phi\).

Proof. We may assume without loss that \(\phi\) is rotationless, since the work of Feighn and Handel in [Feighn and Handel 2011] show that there exists some number \(K\) such that \(\phi^K\) is rotationless for any \(\phi \in \text{Out}(F)\) and it is obvious that the set of weak limits of conjugacy classes is invariant under passing to a finite power.

If \(\gamma\) occurs as a weak limit of some conjugacy class under action of \(\phi\), Proposition 3.6 guarantees that it is in \(B_{\text{gen}}(\phi) \cup B_{\text{Fix}+}(\phi)\).

The proof of the converse part directly follows by using Lemma 3.2 and Lemma 3.7, since every attracting lamination \(\Lambda\) of \(\phi\) has a leaf which is a singular line and one of whose ends is dense in \(\Lambda\) by using Lemma 2.23.

\(\square\)

Remark 3.11. Note that the situation is much simpler in case of fully irreducible and hyperbolic \(\phi\) since the unique attracting lamination of such an element does not contain any nongeneric leaves. Hence the equality just reduces to \(\mathcal{WL}(\phi) := B_{\text{gen}}(\phi) \cup B_{\text{sing}}(\phi)\).

We end this section with a corollary which characterizes hyperbolicity in terms of weak limits of conjugacy classes.

Corollary 3.12. Let \(\phi \in \text{Out}(F)\) be rotationless. Then \(\phi\) is hyperbolic if and only if every conjugacy class \([c]\) is weakly attracted to some line in \(B_{\text{sing}}(\phi)\).

Proof. Follows directly from Theorem 3.10 and Lemma 3.2. \(\square\)

4. Applications

In this section we will look at some applications of the results we proved in the previous section. The first half of this section deals with Cannon-Thurston maps for a hyperbolic \(\phi \in \text{Out}(F)\) and quasiconvexity of infinite index, finitely generated subgroups of \(F\) in the extension group \(G = F \rtimes \phi \mathbb{Z}\). In this half of the section we carefully develop the ideas and comment on possible motivations which leads to the Theorems 4.8, 4.14, 4.16.

The next half of the section generalizes the results we prove in the first half to the case when we replace \(\phi\) by a Gromov-hyperbolic and purely atoroidal group (recall that in such a group every element is hyperbolic). Except for description of the ending lamination set we will generally be brief about the proofs since they are almost identical to the ones in the first half of this section.

Remark 4.1. There are a couple of important points that we would like to clarify before we proceed with the applications. This is for benefit of readers who are not familiar with standard terminologies in the weak attraction language.

1. A “line” \(l \in \overset{\sim}{B}\) is not just a geodesic in \(F\) joining two points in \(\partial F \times \partial F\). Recall that \(\overset{\sim}{B} = \{\partial F \times \partial F - \Delta\}/(\mathbb{Z}_2)\) (see Preliminaries 2.1). The action of \(\mathbb{Z}_2\) on \(\partial F \times \partial F\) is by interchanging endpoints. So a line of \(\overset{\sim}{B}\) is unoriented and flip-invariant. \(\overset{\sim}{B}\) carries the weak topology induced from Cantor topology.
on \(\partial F\). Elements of \(\mathcal{B} = \tilde{B}/F\), are projections of line in \(\tilde{B}\) in the quotient space which is compact but not Hausdorff. Elements of \(\mathcal{B}\) are also referred to as lines.

(2) When we say \(\phi^k([c])\) converges to a line \(l\) or \(l\) is a weak limit of \(\phi^k([c])\) as \(k \to \infty\) it is equivalent to saying every subpath \(\alpha\) of \(l\) occurs as subpath of \(\phi^k([c])\) for all sufficiently large \(s\).

Lifting to \(\tilde{B}\), if \(\tilde{\alpha}\) is a subpath of \(\tilde{l}\) then either \(\tilde{\alpha}\) or \(\tilde{\alpha}^{-1}\) occurs as a subword of some cyclic permutation of a word \(w\) representing \(\phi^s([c])\). To summarize, \(\phi^k([c])\) converges to a line \(l\) or \(l\) is a weak limit of \(\phi^k([c])\) as \(k \to \infty\) is equivalent to saying that for every lift \(\tilde{l} \in \tilde{B}\) and every subword \(\tilde{\alpha}\) of \(\tilde{l}\), either \(\tilde{\alpha}\) or \(\tilde{\alpha}^{-1}\) occurs as a subword of some cyclic permutation of a word \(w\) representing \(\phi^s([c])\) for all sufficiently large \(s\).

**Notation:** By \(\mathcal{WL}(\phi)\) we denote all the lifts of lines in \(\mathcal{WL}(\phi)\) to \(\mathcal{B}\).

4.1. **Canon-Thurston maps for hyperbolic \(\phi\).** Let \(\Gamma\) be a word-hyperbolic group and \(H < \Gamma\) is a word-hyperbolic subgroup. If the inclusion map \(i : H \to \Gamma\) extends to a continuous map of the boundaries \(\hat{i} : \partial H \to \partial \Gamma\) then \(\hat{i}\) is called a Cannon-Thurston map. When it does exist, it is an interesting question to know what its properties are. Its precise behavior is captured by the notion of **Ending laminations**. The original definition was given in Mitra [1997] and for Free groups it was later modified and used in Kapovich and Lustig [2015] by Kapovich and Lustig. Let

\[
1 \to F \to G \to \mathcal{H} \to 1
\]

be an exact sequence of hyperbolic groups where \(\mathcal{H} < \text{Out}(F)\). Then Mitra defined:

**Definition 4.2.** Let \(z \in \partial \mathcal{H}\) and \(\{\phi_n\}\) be a sequence of vertices on a geodesic joining 1 to \(z\) Cayley graph of \(\mathcal{H}\). Define

\[
\Lambda_{z,[c]} = \{l \in \partial F \times \partial F | w \text{ subword of } l \Rightarrow \exists n \ni w \text{ or } w^{-1} \text{ subword of } \phi_n([c])\}
\]

\[
\Lambda_z = \bigcup_{c \in F \setminus \{1\}} \Lambda_{z,[c]}
\]

\[
\Lambda_H = \bigcup_{z \in \partial \mathcal{H}} \Lambda_z
\]

In Mitra [1997, Lemma 3.3] shows that \(\Lambda_z\) is independent of the choice of the sequence \(\phi_n\).

Strictly speaking Mitra’s definition is much more general, but we have adapted the definition as the special case for \(\text{Out}(F)\).

The Cannon-Thurston map, when it exists, identifies the endpoints of the certain leaves of ending lamination. Our goal here is to understand the class of leaves that get identified by this map by using the theory of attracting laminations and singular lines.

Let \(\Gamma_\phi\) denote the mapping torus for a hyperbolic \(\phi \in \text{Out}(F)\). The precise statement for the behaviour of Cannon-Thurston map is given by:

**Theorem 4.3.** Mitra, 1998. **Theorem 4.11** If \(\phi \in \text{Out}(F)\) is a hyperbolic outer automorphism then the Cannon-Thurston map \(\hat{i} : \partial F \to \partial \Gamma_\phi\) exists. Moreover, \(\hat{i}(X) = \hat{i}(Y)\) if and only if the line \(l \in \partial F \times \partial F\) joining \(X\) to \(Y\) is in \(\Lambda_z\) for some \(z \in \partial \mathcal{H}\).
We now proceed to give the description of $\Lambda_z$ using our work in the earlier section.

If $\phi \in \text{Out}(F)$ is hyperbolic then by Brinkmann’s work [Brinkmann, 2000] we have an exact sequence
\[ 1 \to F \to G \to \langle \phi \rangle \to 1 \]
of hyperbolic groups where $G = F \rtimes \varphi \mathbb{Z}$ is the mapping torus of $\phi$.

**Lemma 4.4.** Let $\mathcal{H} = \langle \phi \rangle$ for some hyperbolic $\phi \in \text{Out}(F)$. Then
\[ \Lambda_{\langle \phi \rangle} = \tilde{WL}(\phi) \cup \tilde{WL}(\phi^{-1}) \]

**Proof.** In Definition 4.2 if we let $\mathcal{H} = \langle \phi \rangle$ we see that $\partial \mathcal{H}$ has exactly two points, $z_1$ and $z_2$ say, and the sequences that converge to these points are $\phi^n$ and $\phi^{-n}$ respectively. If we use our observations in (2) of Remark 4.1, we see that the set of lines in $\Lambda_{z_1}$, $\mathcal{L}_c$ are exactly the lines $l \in \tilde{B}$ such that $[c]$ weakly converges to the projection of $l$ in $B$ under iteration of $\phi#$. So, $\Lambda_{z_1} = \tilde{WL}(\phi)$. Similarly $\Lambda_{z_2} = \tilde{WL}(\phi^{-1})$. Thus we have $\Lambda_{\langle \phi \rangle} = \tilde{WL}(\phi) \cup \tilde{WL}(\phi^{-1})$. \qed

Kapovich and Lustig in [Kapovich and Lustig, 2015, Theorem 5.4] showed the following:

**Theorem 4.5.** For a hyperbolic and fully irreducible $\phi \in \text{Out}(F)$, the Cannon-Thurston map $\hat{\iota} : \partial F \to \partial \Gamma_\phi$ is a finite-to-one map and the cardinality of the preimage set of each point in $\partial \Gamma_\phi$ is bounded by $2N$, where $N = \text{rank}(F)$.

We shall improve this theorem by removing the fully irreducible assumption. However the uniform bound that we provide is not a sharp bound like the one obtained above (see [Kapovich and Lustig, 2015, Remark 5.9].

We use the following lemma by [Handel and Mosher, 2017b, Lemma 2.15] to prove proposition 4.7 which is the main technical result in this section that tells us exactly which elements of $WL(\phi)$ are identified. A baby version of the lemma first appeared in [Handel and Mosher, 2011, Lemma 3.3] for the fully irreducible case.

**Lemma 4.6.** For any rotationless $\phi$ and generic leaves $l', l''$ of $\phi$, if some end of $l'$ is asymptotic to some end of $l''$ then both $l', l'' \in B_{\text{Fix}+}(\phi)$.

Notice that in the following proposition, by extending our set from $B_{\text{sing}}(\phi)$ to $B_{\text{Fix}+}(\phi)$, we can generalize from generic leaves to include nongeneric leaves also.

**Proposition 4.7.** Let $\phi$ be rotationless and hyperbolic and suppose $l', l''$ are two leaves in $\bigcup \Lambda_i$, where $\Lambda_i$’s are attracting laminations of $\phi$. If some end of $l'$ is asymptotic to some end of $l''$ then both $l', l'' \in B_{\text{Fix}+}(\phi)$.

To summarize, only lines in $B_{\text{Fix}+}(\phi)$ can be asymptotic.

**Proof.** Let $f : G \to G$ be a CT representing $\phi$. Denote the endpoints of $l'$ by $Q', P$ and the endpoints of $l''$ by $Q'', P$, where $P$ is the common endpoint.

If both the leaves are generic then we are done by Lemma 4.6. If both $l', l''$ are nongeneric then they are already in $B_{\text{Fix}+}(\phi)$ by Lemma 4.8. It remains to consider the case when $l'$ is nongeneric leaf but $l''$ is generic. The following claim completes the proof.

**Claim:** If $l'$ is a nongeneric leaf and $l''$ is a generic leaf which are asymptotic then $l'' \in B_{\text{Fix}+}(\phi)$. 

Proof of claim: Denote the endpoints of \( l', l'' \) by \( P, Q' \) respectively, where \( P \) is the common endpoint. Let \( \Lambda_+^j \) denote the attracting lamination that contains \( l'' \). Then both \( l', l'' \) are in \( B_{na}(\Lambda_+^j) \). Recall that the set of lines in \( B_{na}(\Lambda_+^j) \) are closed under concatenation (by Lemma 2.18). So consider the line \( l \) obtained by concatenation of \( l' \) and \( l'' \), which has endpoints at \( Q', Q'' \). Then by using the work of Handel and Mosher 2.18, there exists a principal lift \( \Phi \) of \( \phi \) such that either all three of \( P, Q', Q'' \) are in \( \text{Fix}_+(\hat{\Phi}) \) or all three points are in \( \partial_{\text{ext}}(A, \phi) \) for some \([A] \in A_{na}(\Lambda_+^j)\).

If the first conclusion is true then \( l'' \in B_{\text{sing}}(\phi) \) by definition. If the second conclusion is true, then \( l'' \) cannot have an endpoint carried by \( A_{na}(\Lambda_+^j) \) because it is generic leaf of \( \Lambda_+^j \) and so the only remaining possibility is that \( P, Q'' \) are both in \( \text{Fix}_+(\phi) \). This gives us that \( l'' \in B_{\text{Fix}_+}(\phi) \).

\[ \square \]

We now use this Proposition to prove our main result about Cannon-Thurston maps for hyperbolic \( \phi \).

**Theorem 4.8.** Consider the exact sequence of hyperbolic groups:

\[ 1 \to \mathbb{F} \to G \to \langle \phi \rangle \to 1 \]

The Cannon-Thurston map \( \hat{i} : \partial \mathbb{F} \to \partial G \) is a finite-to-one map and cardinality of preimage of each point in \( \partial G \) is bounded above by a number depending only on \( \text{rank}(\mathbb{F}) \).

**Proof.** The map \( \hat{i} \) identifies two points in \( \partial \mathbb{F} \) if and only if there is a line in \( \hat{WL}(\phi) \cup \hat{WL}(\phi^{-1}) \) which connects the points. If three or more points are identified then we get asymptotic lines in either \( \hat{WL}(\phi) \) or \( \hat{WL}(\phi^{-1}) \) (not both).

The above proposition 4.7 tells us that only the lines in \( B_{\text{Fix}_+}(\phi) \) or \( B_{\text{Fix}_-}(\phi) \) can be asymptotic. But for any given \( \phi \), both \( B_{\text{Fix}_+}(\phi) \) and \( B_{\text{Fix}_-}(\phi) \) are finite sets with cardinality uniformly bounded above by Proposition 2.24.

\[ \square \]

### 4.2. Quasiconvexity in extension of \( \mathbb{F} \) by \( \phi \)

Next we proceed to show another application of our work. It is related to quasiconvexity of subgroups of \( \mathbb{F} \) in the extension group \( G \). First we quote a result due to Mitra:

**Lemma 4.9.** [Mitra, 1999, Lemma 2.1] [Mj and Rafi, 2017, Lemma 2.4] Consider the exact sequence

\[ 1 \to \mathbb{F} \to G \to \mathcal{H} \to 1 \]

of hyperbolic groups. If \( H \) is a finitely generated infinite index subgroup of \( \mathbb{F} \), then \( H \) is quasiconvex in \( G \) if and only if it does not carry a leaf of the ending lamination.

We use our description of the ending lamination of hyperbolic \( \phi \) to prove an interesting result about quasiconvexity of finitely generated subgroups of \( \mathbb{F} \). The first step is to connect the concept of filling in the sense of free factor supports with the concept of “minimally filling”. (see section 2.2)

Kapovich and Lustig in their work of ending laminations [Kapovich and Lustig, 2015, Proposition A.2] showed that for a hyperbolic fully irreducible outer automorphism, its ending lamination set is minimally filling. A set of lines \( S \) is said to be minimally filling if there does not exist any finitely generated infinite index subgroup of \( \mathbb{F} \) that carries a line of \( S \). Note that minimally filling implies filling in
Proposition 4.10. Suppose $S$ is a set of lines in $\mathbb{F}$ that is filling in the sense of free factor supports and there does not exist any free factor of any finite index subgroup of $F$ that carries a line in $S$ then $S$ is minimally filling in $F$. Converse also holds.

Proof. Using Corollary 1 from the work of Burns [1999], we know that any finitely generated subgroup of $F$ can be realized as a free factor of some finite index subgroup of $F$. This together with the additional hypothesis in the lemma implies $S$ is minimally filling.

The converse part follows directly from definitions and the observation that any free factor of any finite index subgroup of $F$ is finitely generated and infinite index in $F$. □

The following definition generalizes the “minimally filling” so that we can use it for hyperbolic outer automorphisms which are not fully irreducible. The idea behind this definition is the property that the free factor support of lamination is either all of $[\mathbb{F}]$ or every line is carried by some proper free factor system $[F^i]$ where $F^i$ is a proper free factor of $F$.

Definition 4.11. Consider a finitely generated subgroup of infinite index $H < \mathbb{F}$.

A set of lines $S$ in $\mathbb{F}$ said to be minimally filling with respect to $H$ if for every finitely generated subgroup $H' < \mathbb{F}$ containing $H$ as a proper free factor, no finitely generated infinite index subgroup in $H'$ carries a line of $S$.

Similarly we say a set of lines $\Lambda$ in $B$ is minimally filling with respect to $H$ if every lift of $\Lambda$ in $\tilde{B}$ is minimally filling with respect to $H$.

It is easy to see that $S$ is minimally filling with respect to $H$ if and only if no finitely generated subgroup of $\mathbb{F}$ containing $H$ as a proper free factor, carries a line in $S$. However we will use the aforementioned definition since it is easier to relate with the standard definition of minimally filling. The following lemma establishes the connection.

Lemma 4.12. If $S$ is minimally filling in $\mathbb{F}$ if and only if $S$ is minimally filling with respect to every finitely generated, infinite index subgroup of $\mathbb{F}$.

Proof. To see the forward direction, let $\mathbb{F} \geq H' \geq H$ be finitely generated subgroups such that $H$ is a proper free factor of $H'$. If $H'$ carries a line in $S$ then there exists a proper free factor $K < H'$ of $H'$ which carries this line. But then we know that $K$ must have infinite index in $H'$, hence in $\mathbb{F}$, which violates that $S$ is minimally filling.

Conversely, suppose $S$ is minimally filling with respect to every finitely generated, infinite index subgroup of $\mathbb{F}$. If there exists some finitely generated infinite index subgroup $H < \mathbb{F}$ which carries a line in $S$, then by using the result of Burns [1999], we know that there exists a finite index (hence finitely generated) subgroup $H' < \mathbb{F}$ which contains $H$ as a free factor (if $H$ is a free factor of $\mathbb{F}$ then take $H' = \mathbb{F}$). But $H$ has infinite index in $H'$ since it is a proper free factor of $H'$ and $H'$ has finite index in $\mathbb{F}$, hence $S$ is not minimally filling with respect to $H$. Which is a contradiction. □

Let us elaborate on the purpose of giving this definition.
Proposition 4.13. Consider the exact sequence
\[ 1 \rightarrow \mathbb{F} \rightarrow G \rightarrow \langle \phi \rangle \rightarrow 1 \]
where \( \phi \in \text{Out}(\mathbb{F}) \) hyperbolic.

Let \( H \) be a finitely generated, infinite index subgroup of \( \mathbb{F} \). If every attracting and every repelling lamination of \( \phi \) is minimally filling with respect to \( H \) then \( H \) is quasiconvex in \( G \).

Proof. Suppose \( H \) is not quasiconvex. Then by Lemma 4.9 it carries a leaf of the ending lamination set. Without loss we may assume that it carries a line in \( \tilde{WL}(\phi) \). Since the closure of projection of any line in \( \tilde{WL}(\phi) \) contains an attracting lamination and \( \partial H \) is compact, \( H \) must carry lift of a generic leaf of some \( \Lambda_i \in \mathcal{L}(\phi) \).

Using Burns [1999] one has a finite index subgroup \( H' \subset \mathbb{F} \) such that \( H \) is a proper free factor of \( H' \) (since \( H \) is infinite index in \( \mathbb{F} \)), and hence infinite index in \( H' \) and \( H \) carries lift of a leaf of \( \Lambda_i \). This violates that \( \Lambda_i \) is minimally filling with respect to \( H \). Hence \( H \) must be quasiconvex in \( G \).

\[ \square \]

The following result establishes certain dynamical conditions for a finitely generated, infinite index \( H < \mathbb{F} \) to be quasiconvex in the extension group \( G \). We give one equivalent condition and one sufficient condition for \( H \) to be quasiconvex.

Theorem 4.14. Consider the exact sequence
\[ 1 \rightarrow \mathbb{F} \rightarrow G \rightarrow \langle \phi \rangle \rightarrow 1 \]
where \( \phi \) is a hyperbolic automorphism of \( \mathbb{F} \). Let \( H \) be a finitely generated infinite index subgroup of \( \mathbb{F} \). Then the following are equivalent:

1. \( H \) is quasiconvex in \( G \)
2. \( \partial H \cap \{ \text{Fix}_+(\phi) \cup \text{Fix}_-(\phi) \} = \emptyset \).

Moreover both (1) and (2) are satisfied if every attracting and repelling lamination of \( \phi \) is minimally filling with respect to \( H \).

Proof. The conclusion in the “moreover” part is discussed in Proposition 4.13. We proceed to prove the equivalence of (1) and (2).

Suppose \( H \) is quasiconvex in \( G \). Since \( H \) is finite rank subgroup of \( \mathbb{F} \), \( \partial H \) is compact in \( \partial \mathbb{F} \) the set of lines carried by \( \partial H \) is closed in the weak topology. If \( P \in \partial H \cap \{ \text{Fix}_+(\phi) \cup \text{Fix}_-(\phi) \} \neq \emptyset \) then weak closure of \( P \) must contain either an attracting lamination or a repelling lamination for \( \phi \). Hence \( H \) contains leaves of the ending lamination set and this contradicts that \( H \) quasiconvex by using Lemma 4.9.

Conversely, if \( \partial H \cap \{ \text{Fix}_+(\phi) \cup \text{Fix}_-(\phi) \} = \emptyset \), then \( H \) does not carry a leaf of any ending lamination set, hence \( H \) is quasiconvex. To see this use Theorem 3.10 and Lemma 2.23 which states that every attracting (repelling) lamination carries a singular line which has endpoints in \( \text{Fix}_+(\phi) \cup \text{Fix}_-(\phi) \).

\[ \square \]

We make the following observation regarding the special case when \( \phi \) is fully irreducible. This case is well known and was done in Mitra [1999] and Kapovich and Lustig [2015].

Corollary 4.15. If \( \phi \) is fully irreducible, then any finitely generated infinite index subgroup is quasiconvex in the extension group \( G \).
Proof. This follows from the work of [Bestvina et al., 1997, Proposition 2.4] and [Kapovich and Lustig, 2013, Proposition A.2] where both papers prove that $\Lambda$ is minimally filling in $F$ and hence minimally filling with respect to any finitely generated infinite index subgroup of $F$, where $\Lambda$ is the unique attracting lamination of $\phi$. Similarly for the unique repelling lamination. □

Now we proceed to give an algebraic condition for quasiconvexity:

**Theorem 4.16.** Consider the exact sequence

$$1 \to F \to G \to \langle \phi \rangle \to 1$$

where $\phi$ is a hyperbolic automorphism of $F$.

If $H$ is a finitely generated, infinite index subgroup of $F$ such that:

1. $H$ is contained in a proper free factor of $F$.
2. $H$ does not contain any subgroup which is also a subgroup of some $\phi$ periodic free factor.

Then $H$ is quasiconvex in $G$. Moreover if $\phi$ does not have an attracting lamination that fills in the sense of free factor support, condition (2) is a sufficient condition for quasiconvexity of $H$ in $G$.

Proof. Suppose $H$ is not quasiconvex. Then by Lemma 4.9, $H$ carries a leaf of the ending lamination set. Since the attracting and repelling laminations are paired by free factor supports, (see 2.5) without loss assume that $H$ carries lift of a line in $WL(\phi)$. Since $H$ is of finite rank, $\partial H$ is compact and set of lines carried by $H$ is closed in the weak topology. This implies that $H$ carries lift of an attracting lamination $\Lambda_i \in L(\phi)$. Let $[F_i]$ be the (necessarily $\phi$-invariant, see proof of Lemma 3.2.4, Bestvina et al. 2000) free factor support of $\Lambda_i$.

**Case 1: $F_i$ is not proper:** In this case we use condition (1) to get a contradiction. Since $H$ is contained in a proper free factor $H'$ say, the free factor support of lines carried by $H$ is contained in $H'$. This implies $F_i < H''$ for some conjugate of $H'$, but since $F_i = F$ we get a contradiction.

**Case 2: $F_i$ is a proper free factor:** In this case we conclude that $H \cap gF_i g^{-1}$ is nonempty for some $g \in F$ and hence a subgroup of a $\phi$ invariant free factor. This violates condition (2).

The “moreover” part is a consequence of the proof of Case 2. □

4.3. Ending Laminations for purely atoroidal $\mathcal{H}$. We now proceed to define the notion of weak limit set of a group $\mathcal{H} \in \text{Out}(F)$, which is purely atoroidal i.e. every element of $\mathcal{H}$ is a hyperbolic outer automorphism. For this we will use the ideas developed so far to prove the special case in 3.10 and use the notion of ending laminations as a guide. It is to be noted that extending the definition from $WL(\phi)$ to $WL(\mathcal{H})$ is not entirely obvious at a first glance, however familiarity with description of the ending lamination set (see remark 4.1) will be of great help. So for the rest of the section let us fix the following notations and assumptions:

**Assumptions:**

1. $1 \to F \to G \to \mathcal{H} \to 1$ is an exact sequence of hyperbolic groups where $\mathcal{H}$ is non-elementary and purely atoroidal.
2. $z \in \partial \mathcal{H}$ is point in the Gromov boundary of $\mathcal{H}$.
(3) \( \phi_n \to z \) means \( \phi_n \) is a sequence of vertices in the Cayley graph of \( \mathcal{H} \) so that they lie on a geodesic joining 1 to \( z \) in the compactified space \( \mathcal{H} \cup \partial \mathcal{H} \).

So whenever we use the notation \( \phi_n \) we mean a term in the sequence as described here.

(4) If \( S \) is a set of lines in \( \mathcal{B} \), we denote its lift to \( \tilde{\mathcal{B}} \) by \( \tilde{S} \).

**Definition 4.17.** Let \( z \in \partial \mathcal{H} \) and \( \phi_n \) be a sequence in \( \mathcal{H} \) converging to \( z \). Define the sets

\[
\text{WL}(\phi_n, [c]) = \{ l \in \mathcal{B} | l \text{ is a weak limit of } \phi_{n \#}^{k}(c) \text{ as } k \to \infty \}
\]

\[
\tilde{\text{WL}}(z, [c]) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \tilde{\text{WL}}(\phi_j, [c])
\]

So a line \( \tilde{l} \) is in \( \tilde{\text{WL}}(z, [c]) \) if and only if given any finite subpath \( \alpha \) of \( \tilde{l} \) and \( M \geq 1 \) there exists \( j > M \) such that \( \alpha \) is a subpath of \( \phi_j^{k}(c) \) for some \( k > 0 \). This is equivalent to saying that every line in \( \tilde{\text{WL}}(z, [c]) \) is a weak limit of lines \( \tilde{l_j} \in \tilde{\text{WL}}(\phi_j, [c]) \).

This implies that \( \alpha \) is a subpath of \( \phi_j^{k}(c') \) for some \( c' \in \mathbb{F} - \{1\} \). Hence we deduce that for any subpath \( \alpha \) of \( \tilde{l} \) and any \( M > 0 \), there exists \( j > M \) and a conjugacy class \( [c'] \) such that \( \alpha \) is subpath of \( \phi_j^{k}(c') \). This implies \( \tilde{\text{WL}}(z, [c]) \subset \overline{\Lambda}_z = \Lambda_z \).

Hence we have the following corollary:

**Corollary 4.18.** If for \( z \in \partial \mathcal{H} \), \( \Lambda_z \) is the ending lamination set defined by Mitra, then:

\[
\tilde{\text{WL}}(z, [c]) \subset \Lambda_z
\]

Note that \( \tilde{\text{WL}}(z, [c]) \) is a closed set by construction.

As we can see that the ending lamination set is far more complicated when \( \mathcal{H} \) is not a cyclic group. One of the big problems one faces is if a set of lines that occurs as a attracting or repelling lamination for every element of \( \mathcal{H} \). Recall that stabilizers of attracting (repelling) laminations of a fully irreducible \( \phi \in \text{Out}(\mathbb{F}) \) are virtually cyclic [Bestvina et al., 1997, Theorem 2.14]. But this may not be true if \( \phi \) is hyperbolic but reducible even if the lamination fills (in the sense of free factor supports).

Infact something worse can happen. Since \( \mathcal{H} \) is hyperbolic the stabilizer subgroup of lamination inside \( \mathcal{H} \) if not virtually cyclic will necessarily contain a nonabelian free group. This is due to the lemma that Bestvina, Feighn and Handel showed Tits alternative holds for \( \text{Out}(\mathbb{F}) \) [Bestvina et al., 2000] and Feighn, Handel proved every abelian subgroup of \( \text{Out}(\mathbb{F}) \) is virtually \( \mathbb{Z}^n \) for some \( n \) [Feighn and Handel, 2000]. In order to make a meaningful conclusion about Cannon-Thurston maps for \( G \) we need to avoid these situations. The following results guarantee that such things do not occur.

**Lemma 4.19.** Suppose \( \phi, \psi \) are two hyperbolic elements of \( \text{Out}(\mathbb{F}) \) contained in \( \mathcal{H} \). Denote the ending laminations \( \Lambda_z, \Lambda_{z'} \), where \( z = \phi^\infty, z' = \psi^\infty \) are in \( \partial \mathcal{H} \). Let \( l, l' \in \Lambda_z \cup \Lambda_{z'} \). Also suppose that \( \phi \) and \( \psi \) do not have a common attracting fixed point. Under these assumptions, if \( l, l' \) have an asymptotic end, then
(1) \( \phi \) and \( \psi \) have a common attracting lamination.

(2) Both \( l, l' \) are lifts of lines in either \( \mathcal{B}_{\text{Fix}_+}(\phi) \) or \( \mathcal{B}_{\text{Fix}_+}(\psi) \).

**Proof.** Due to the conclusion from Proposition 4.7 we may assume that \( l \in \Lambda_z, l' \in \Lambda_{z'} \). We split the proof into cases:

**Case 1:** If \( l, l' \) are both generic leaves of \( \phi \) and \( \phi \) and share an endpoint \( P \), then the weak closure of \( P, \overline{P} = \Lambda_\phi^+ = \Lambda_\phi^+ \), where \( \Lambda_\phi^+ \) and \( \Lambda_\psi^+ \) are attracting laminations for \( \phi \) and \( \psi \) respectively. So \( l, l' \) can be treated as lifts of generic leaves of \( \Lambda_\phi^+ \) and (since both are birecurrent leaves) the conclusion (2) now follows directly from Proposition 4.7.

**Case 2:** Suppose \( l \) is lift of some generic leaf of \( \phi \), \( l' \) is lift of a line \( \mathcal{B}_{\text{Fix}_+}(\psi) \) and they are asymptotic with common endpoint \( P \). Thus \( \overline{P} = \Lambda_\phi^+ \supseteq \Lambda_\psi^+ \). If the inclusion is proper then the leaves of \( \Lambda_\psi^+ \) are nongeneric leaves of \( \Lambda_\psi^+ \). But by using Corollary 3.8 we know that these nongeneric leaves are lines in \( \mathcal{B}_{\text{Fix}_+}(\phi) \), which is a finite set by using Proposition 2.24. But since \( \Lambda_\psi^+ \) is an uncountable set we must have \( \Lambda_\phi^+ = \Lambda_\psi^+ \). Now apply Proposition 4.7 (on \( \psi \)) to get the second conclusion.

\( \square \)

The proposition below gives some necessary conditions that a pair of elements of \( \mathcal{H} \) must satisfy in order for the extension group to be hyperbolic. Recall that Dowdall-Taylor proved that for the extension group to be hyperbolic, every element of \( \mathcal{H} \) must be hyperbolic.

**Proposition 4.20.** If \( \phi, \psi \in \mathcal{H} \) are two points such that \( \phi, \psi \) do not have a common power, then the following equivalent conditions are satisfied:

1. \( \phi^\infty, \psi^\infty \) are distinct points in \( \partial \mathcal{H} \).
2. They do not have a common attracting lamination.
3. They do not have common attracting fixed point.
4. No leaf of an attracting lamination or singular line of \( \phi \) is asymptotic to a leaf of an attracting lamination or singular line for \( \psi \).

**Proof.** Using Mitra [1997, Proposition 5.1] we know that \( \phi^\infty \) and \( \psi^\infty \) are distinct points if and only if no line in \( \Lambda_{\phi^\infty} \) has any endpoint common with a line in \( \Lambda_{\psi^\infty} \). Hence they do not have a common attracting lamination by using Proposition 4.4. The reverse direction is true by Theorem 3.10. This establishes the equivalence between (1) and (2). The equivalence between (1) and the others follow similarly by using 3.10 and 4.4.

\( \square \)

**Remark 4.21.** Note that the above proposition combined with Mitra’s result [Mitra 1997, Proposition 5.1] and Theorem 3.10 shows that for any \( z \in \partial \mathcal{H} \) which is not the endpoint of an axis, its corresponding ending lamination set \( \Lambda_z \) does not contain any line which is equal or asymptotic to a leaf or a singular line of any element of \( \mathcal{H} \). This shows that the techniques we used for showing finiteness of fibers of Cannon-Thurston maps cannot be directly applied in the more general setting when \( \mathcal{H} \) is not (virtually) cyclic.

We end this paper with a question: Is the converse to Corollary 4.18 true?

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LIMITS OF CONJUGACY CLASSES UNDER ITERATES OF HYPERBOLIC ELEMENTS OF Out(F)

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