L-DUNFORD-PETTIS PROPERTY IN BANACH SPACES

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Abstract. In this paper, we introduce and study the concept of L-Dunford-Pettis sets and L-Dunford-Pettis property in Banach spaces. Next, we give a characterization of the L-Dunford-Pettis property with respect to some well-known geometric properties of Banach spaces. Finally, some complementability of operators on Banach spaces with the L-Dunford-Pettis property are also investigated.

1. Introduction and notation

A norm bounded subset \( A \) of a Banach space \( X \) is called Dunford-Pettis (DP for short) if every weakly null sequence \((f_n)\) in \( X' \) converge uniformly to zero on \( A \), that is,
\[
\lim_{n \to \infty} \sup_{x \in A} |f_n(x)| = 0.
\]

An operator \( T \) between two Banach spaces \( X \) and \( Y \) is completely continuous if \( T \) maps weakly null sequences into norm null ones.

Recall from [11], that an operator \( T : X \to Y \) between two Banach spaces is Dunford-Pettis completely continuous (abb. DPcc) if it carries a weakly null sequence, which is a DP set in \( X \) to norm null ones in \( Y \). It is clear that every completely continuous operator is DPcc. Also every weakly compact operator is DPcc (see Corollary 1.1 of [11]).

A Banach space \( X \) has:

- a relatively compact Dunford-Pettis property (DPcP for short) if every Dunford-Pettis set in \( X \) is relatively compact [5]. For example, every Schur spaces have the DPcP.
- a Grothendieck property (or a Banach space \( X \) is a Grothendieck space) if weak and weak convergence of sequences in \( X' \) coincide. For example, each reflexive space is a Grothendieck space.
- a Dunford-Pettis property (DP property for short) if every weakly compact operator \( T \) from \( X \) into another Banach space \( Y \) is completely continuous, equivalently, if every relatively weakly compact subset of \( X \) is DP.
- a reciprocal Dunford-Pettis property (RDP property for short) if every completely continuous operator on \( X \) is weakly compact.

A subspace \( X_1 \) of a Banach space \( X \) is complemented if there exists a projection \( P \) from \( X \) to \( X_1 \) (see page 9 of [2]).

Recall from [11], that a Banach lattice is a Banach space \((E, \| \cdot \|)\) such that \( E \) is a vector lattice and its norm satisfies the following property: for each \( x, y \in E \) such that \( |x| \leq |y| \), we have \( \|x\| \leq \|y\| \).

We denote by \( c_0, \ell^1, \) and \( \ell^\infty \) the Banach spaces of all sequences converging to zero, all absolutely summable sequences, and all bounded sequences, respectively.

Let us recall that a norm bounded subset \( A \) of a Banach space \( X' \) is called L-set if every weakly null sequence \((x_n)\) in \( X \) converge uniformly to zero on \( A \), that is,
\[ \lim_{n \to \infty} \sup_{f \in A} |f(x_n)| = 0. \] Note also that a Banach space \( X \) has the RDP property if and only if every \( L \)-set in \( X' \) is relatively weakly compact.

In his paper, G. Emmanuelle in [4] used the concept of \( L \)-set to characterize Banach spaces not containing \( \ell^1 \), and gave several consequences concerning Dunford-Pettis sets. Later, the idea of \( L \)-set is also used to establish a dual characterization of the Dunford-Pettis property [6].

The aim of this paper is to introduce and study the notion of \( L \)-Dunford-Pettis set in a Banach space, which is related to the Dunford-Pettis set (Definition 2.1), and note that every \( L \)-set in a topological dual of a Banach space is \( L \)-Dunford-Pettis set (Proposition 2.3). After that, we introduce the \( L \)-Dunford-Pettis property in Banach space which is shared by those Banach spaces whose \( L \)-Dunford-Pettis subsets of his topological dual are relatively weakly compact (Definition 2.6). Next, we obtain some important consequences. More precisely, a characterizations of \( L \)-Dunford-Pettis property in Banach spaces in terms of DPcc and weakly compact operators (Theorem 2.7), the relation between \( L \)-Dunford-Pettis property with DP and Grothendieck properties (Theorem 2.8), a new characterizations of Banach space with DPrcP (resp, reflexive Banach space) (Theorem 2.13 and Corollary 2.14).

The notations and terminologies are standard. We use the symbols \( X, Y \) for arbitrary Banach spaces. We denoted the closed unit ball of \( X \) by \( B_X \), the topological dual of \( X \) by \( X' \) and \( T' : Y' \to X' \) refers to the adjoint of a bounded linear operator \( T : X \to Y \).

2. Main results

**Definition 2.1.** Let \( X \) be a Banach space. A norm bounded subset \( A \) of the dual space \( X' \) is called an \( L \)-Dunford-Pettis set, if every weakly null sequence \( (x_n) \), which is a DP set in \( X \) converges uniformly to zero on \( A \), that is, \( \lim_{n \to \infty} \sup_{f \in A} |f(x_n)| = 0 \).

For a proof of the next Proposition, we need the following Lemma which is just Lemma 1.3 of [11].

**Lemma 2.2.** A sequence \( (x_n) \) in \( X \) is DP if and only if \( f_n(x_n) \to 0 \) as \( n \to \infty \) for every weakly null sequence \( (f_n) \) in \( X' \).

The following Proposition gives some additional properties of \( L \)-Dunford-Pettis sets in a topological dual Banach space.

**Proposition 2.3.** Let \( X \) be a Banach space. Then

1. every subset of an \( L \)-Dunford-Pettis set in \( X' \) is \( L \)-Dunford-Pettis,
2. every \( L \)-set in \( X' \) is \( L \)-Dunford-Pettis,
3. relatively weakly compact subset of \( X' \) is \( L \)-Dunford-Pettis,
4. absolutely closed convex hull of an \( L \)-Dunford-Pettis set in \( X' \) is \( L \)-Dunford-Pettis.

**Proof.** (1) and (2) are obvious.

(3) Suppose \( A \subset X' \) is relatively weakly compact but it is not an \( L \)-Dunford-Pettis set. Then, there exists a weakly null sequence \( (x_n) \), which is a DP set in \( X \), a sequence \( (f_n) \) in \( A \) and an \( \epsilon > 0 \) such that \( |f_n(x_n)| > \epsilon \) for all integer \( n \). As \( A \) is relatively weakly compact, there exists a subsequence \( (g_n) \) of \( (f_n) \) that converges weakly to an element \( g \) in \( X' \). But from

\[ |g_n(x_n)| \leq |(g_n - g)(x_n)| + |g(x_n)| \]

\[ \Rightarrow |g_n(x_n)| \to 0 \] as \( n \to \infty \), contradicting the definition of an \( L \)-Dunford-Pettis set.
and Lemma 2.3 we obtain that $|g_n(x_n)| \to 0$ as $n \to \infty$. This is a contradiction.

(4) Let $A$ be a L-Dunford-Pettis set in $X'$, and $(x_n)$ be a weakly null sequence, which is a DP set in $X$. Since

$$\sup_{f \in \alpha(A)} |f(x_n)| = \sup_{f \in A} |f(x_n)|$$

for each $n$, where $\alpha(A) = \{\sum_{i=1}^{\infty} \lambda_i x_i : x_i \in A, \forall i, \sum_{i=1}^{\infty} |\lambda_i| \leq 1\}$ is the absolutely closed convex hull of $A$ (see [1, pp. 148, 151]), then it is clear that $\alpha(A)$ is L-Dunford-Pettis set in $X'$.

We need the following Lemma which is just Lemma 1.2 of [11].

**Lemma 2.4.** A Banach space $X$ has the DPrcP if and only if any weakly null sequence, which is a DP set in $X$ is norm null.

From Lemma 2.4, we obtain the following characterization of DPrcP in a Banach space in terms of an L-Dunford-Pettis set of his topological dual.

**Theorem 2.5.** A Banach space $X$ has the DPrcP if and only if every bounded subset of $X'$ is an L-Dunford-Pettis set.

**Proof.** ($\Leftarrow$) Let $(x_n)$ be a weakly null sequence, which is a DP set in $X$. As

$$\|x_n\| = \sup_{f \in B_{X'}} |f(x_n)|$$

for each $n$, and by our hypothesis, we see that $\|x_n\| \to 0$ as $n \to \infty$. By Lemma 2.3 we deduce that $X$ has the DPrcP.

($\Rightarrow$) Assume by way of contradiction that there exist a bounded subset $A$, which is not an L-Dunford-Pettis set of $X'$. Then, there exists a weakly null sequence $(x_n)$, which is a Dunford-Pettis set of $X$ such that $\sup_{f \in A} |f(x_n)| > \epsilon > 0$ for some $\epsilon > 0$ and each $n$. Hence, for every $n$ there exists some $f_n$ in $A$ such that $|f_n(x_n)| > \epsilon$.

On the other hand, since $(f_n) \subset A$, there exist some $M > 0$ such that $\|f_n\|_{X'} \leq M$ for all $n$. Thus,

$$|f_n(x_n)| \leq M \|x_n\|$$

for each $n$, then by our hypothesis and Lemma 2.4, we have $|f_n(x_n)| \to 0$ as $n \to \infty$, which is impossible. This completes the proof.

**Remark 1.** Note by Proposition 2.3 assertion (3) that every relatively weakly compact subset of a topological dual Banach space is L-Dunford-Pettis. The converse is not true in general. In fact, the closed unit ball $B_{\ell_\infty}$ of $\ell_\infty$ is L-Dunford-Pettis set (see Theorem 2.5), but it is not relatively weakly compact.

We make the following definition.

**Definition 2.6.** A Banach space $X$ has the L-Dunford-Pettis property, if every L-Dunford-Pettis set in $X'$ is relatively weakly compact.

As is known a DPcc operator is not weakly compact in general. For example, the identity operator $Id_{\ell^1} : \ell^1 \to \ell^1$ is DPcc, but it is not weakly compact.

In the following Theorem, we give a characterizations of L-Dunford-Pettis property of Banach space in terms of DPcc and weakly compact operators.

**Theorem 2.7.** Let $X$ be a Banach space, then the following assertions are equivalent:

1. $X$ has the L-Dunford-Pettis property,
2. for each Banach space $Y$, every DPcc operator from $X$ into $Y$ is weakly compact,
3. every DPcc operator from $X$ into $\ell_\infty$ is weakly compact.
Proof. (1) ⇒ (2) Suppose that $X$ has the L-Dunford-Pettis property and $T : X \to Y$ is DPcc operator. Thus $T'(B_Y')$ is an L-Dunford-Pettis set in $X'$. So by hypothesis, it is relatively weakly compact and $T$ is a weakly compact operator.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) If $X$ does not have the L-Dunford-Pettis property, there exists an L-Dunford-Pettis subset $A$ of $X'$ that is not relatively weakly compact. So there is a sequence $(f_n) \subseteq A$ with no weakly convergent subsequence. Now, we show that the operator $T : X \to \ell^\infty$ defined by $T(x) = (f_n(x))$ for all $x \in X$ is DPcc but it is not weakly compact. As $(f_n) \subseteq A$ is L-Dunford-Pettis set, for every weakly null sequence $(x_m)$, which is a DP set in $X$ we have

$$\|T(x_m)\| = \sup_n |f_n(x_m)| \to 0, \text{ as } m \to \infty,$$

so $T$ is a Dunford-Pettis completely continuous operator. We have $T'(\{\lambda_n\}_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$ for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)'$. If $e'_n$ is the usual basis element in $\ell^1$, then $T'(e'_n) = f_n$, for all $n \in N$. Thus, $T'$ is not a weakly compact operator and neither is $T$. This finishes the proof.

Theorem 2.8. Let $E$ be a Banach lattice.
If $E$ has both properties of DP and Grothendieck, then it has the L-Dunford-Pettis property.

Proof. Suppose that $T : E \to Y$ is DPcc operator. As $E$ has the DP property, it follows from Theorem 1.5 [11] that $T$ is completely continuous.

On the other hand, $\ell^1$ is not a Grothendieck space and Grothendieck property is carried by complemented subspaces. Hence the Grothendieck space $E$ does not have any complemented copy of $\ell^1$. By [10], $E$ has the RDP property and so the completely continuous operator $T$ is weakly compact. From Theorem 2.7 we deduce that $E$ has the L-Dunford-Pettis property.

Remark 2. Since $\ell^\infty$ has the Grothendieck and DP properties, it has the L-Dunford-Pettis property.

Let us recall that $K$ is an infinite compact Hausdorff space if it is a compact Hausdorff space, which contains infinitely many points.

For an infinite compact Hausdorff space $K$, we have the following result for the Banach space $C(K)$ of all continuous functions on $K$ with supremum norm.

Corollary 2.9. If $C(K)$ contains no complemented copy of $c_0$, then it has L-Dunford-Pettis property.

Proof. Since $C(K)$ contains no complemented copy of $c_0$, it is a Grothendieck space [3]. On the other hand, $C(K)$ be a Banach lattice with the DP property, and by Theorem 2.8 we deduce that $C(K)$ has L-Dunford-Pettis property.

Corollary 2.10. A DPrc space has the L-Dunford-Pettis property if and only if it is reflexive.

Proof. $(\Rightarrow)$ If a Banach space $X$ has the DPrcP, then by Theorem 1.3 of [11], the identity operator $1d_X$ on $X$ is DPcc. As $X$ has the L-Dunford-Pettis property, it follows from Theorem 2.7 that $1d_X$ is weakly compact, and hence $X$ is reflexive.

$(\Leftarrow)$ Obvious.

Remark 3. Note that the Banach space $\ell^1$ is not reflexive and has the DPrcP, then from Corollary 2.10 we conclude that $\ell^1$ does not have the L-Dunford-Pettis property.

Theorem 2.11. If a Banach space $X$ has the L-Dunford-Pettis property, then every complemented subspace of $X$ has the L-Dunford-Pettis property.
Proof. Consider a complemented subspace $X_1$ of $X$ and a projection map $P : X \to X_1$. Suppose $T : X_1 \to \ell^\infty$ is DPcc operator, then $TP : X \to \ell^\infty$ is also DPcc. Since $X$ has L-Dunford-Pettis, by Theorem 2.7, $TP$ is weakly compact. Hence $T$ is weakly compact, also from Theorem 2.7 we conclude that $X_1$ has L-Dunford-Pettis, and this completes the proof.

Let $X$ be a Banach space. We denote by $L(X, \ell^\infty)$ the class of all bounded linear operators from $X$ into $\ell^\infty$, by $W(X, \ell^\infty)$ the class of all weakly compact operators from $X$ into $\ell^\infty$, and by $DPcc(X, \ell^\infty)$ the class of all Dunford-Pettis completely continuous operators from $X$ into $\ell^\infty$.

Recall that Bahreini in [2] investigated the complementability of $W(X, \ell^\infty)$ in $L(X, \ell^\infty)$, and she proved that if $X$ is not a reflexive Banach space, then $W(X, \ell^\infty)$ is not complemented in $L(X, \ell^\infty)$. In the next theorem, we establish the complementability of $W(X, \ell^\infty)$ in $DPcc(X, \ell^\infty)$.

We need the following lemma of [7].

Lemma 2.12. Let $X$ be a separable Banach space, and $\phi : \ell^\infty \to L(X, \ell^\infty)$ is a bounded linear operator with $\phi(e_n) = 0$ for all $n$, where $e_n$ is the usual basis element in $c_0$. Then there is an infinite subset $M$ of $N$ such that for each $\alpha \in \ell^\infty(M)$, $\phi(\alpha) = 0$, where $\ell^\infty(M)$ is the set of all $\alpha = (\alpha_n) \in \ell^\infty$ with $\alpha_n = 0$ for each $n \notin M$.

Theorem 2.13. If $X$ does not have the L-Dunford-Pettis property, then $W(X, \ell^\infty)$ is not complemented in $DPcc(X, \ell^\infty)$.

Proof. Consider a subset $A$ of $X'$ that is L-Dunford-Pettis but it is not relatively weakly compact. So there is a sequence $(f_n)$ in $A$ such that has no weakly convergent subsequence. Hence $S : X \to \ell^\infty$ defined by $S(x) = (f_n(x))$ is an DPcc operator but it is not weakly compact. Choose a bounded sequence $(x_n)$ in $B_X$ such that $(S(x_n))$ has no weakly convergent subsequence. Let $X_1 = \langle x_n \rangle$, the closed linear span of the sequence $(x_n)$ in $X$. It follows that $X_1$ is a separable subspace of $X$ such that $S/X_1$ is not a weakly compact operator. If $g_n = f_n/X_1$, we have $(g_n) \subseteq X_1'$ is bounded and has no weakly convergent subsequence.

Now define the operator $T : \ell^\infty \to DPcc(X, \ell^\infty)$ by $T(\alpha)(x) = (\alpha_n f_n(x))$, where $x \in X$ and $\alpha = (\alpha_n) \in \ell^\infty$. Then

\[ ||T(\alpha)(x)|| = \sup_n |\alpha_n f_n(x)| \leq ||\alpha|| \cdot ||f_n|| \cdot ||x|| < \infty. \]

We claim that $T(\alpha) \in DPcc(X, \ell^\infty)$ for each $\alpha = (\alpha_n) \in \ell^\infty$.

Let $\alpha = (\alpha_n) \in \ell^\infty$ and let $(x_m)$ be a weakly null sequence, which is a DP set in $X$. As $(f_n)$ is L-Dunford-Pettis set $\sup_n |f_n(x_m)| \to 0$ as $m \to \infty$. So we have

\[ ||T(\alpha)(x_m)|| = \sup_n |\alpha_n f_n(x_m)| \leq ||\alpha|| \cdot \sup_n |f_n(x_m)| \to 0, \]

as $m \to \infty$. Then this finishes the proof that $T$ is a well-defined operator from $\ell^\infty$ into $DPcc(X, \ell^\infty)$.

Let $R : DPcc(X, \ell^\infty) \to DPcc(X_1, \ell^\infty)$ be the restriction map and define

\[ \phi : \ell^\infty \to DPcc(X_1, \ell^\infty) \quad \text{by} \quad \phi = RT. \]

Now suppose that $W(X, \ell^\infty)$ is complemented in $DPcc(X, \ell^\infty)$ and $P : DPcc(X, \ell^\infty) \to W(X, \ell^\infty)$ is a projection. Define $\psi : \ell^\infty \to W(X_1, \ell^\infty)$ by $\psi = RPT$. Note that as $T(e_n)$ is a one rank operator, we have $T(e_n) \in W(X, \ell^\infty)$. Hence

\[ \psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n) \]
for all $n \in N$. From Lemma 2.12, there is an infinite set $M \subseteq N$ such that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in \ell^\infty(M)$. Thus $\phi(\chi_M)$ is a weakly compact operator. On the other hand, if $e'_n$ is the usual basis element of $\ell^1$, for each $x \in X_1$ and each $n \in M$, we have

$$(\phi(\chi_M))'(e'_n)(x) = f_n(x).$$

Therefore $(\phi(\chi_M))'(e'_n) = f_n/X_1 = g_n$ for all $n \in M$. Thus $(\phi(\chi_M))'$ is not a weakly compact operator and neither is $\phi(\chi_M)$. This contradiction ends the proof. 

As a consequence of Theorem 2.7 and Theorem 2.13, we obtain the following result.

**Corollary 2.14.** Let $X$ be a Banach space. Then the following assertions are equivalent:

1. $X$ has the L-Dunford-Pettis property,
2. $W(X, \ell^\infty) = DPcc(X, \ell^\infty)$,
3. $W(X, \ell^\infty)$ is complemented in $DPcc(X, \ell^\infty)$.

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