The partial $r$-Bell polynomials

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Abstract. In this paper, we show that the $r$-Stirling numbers of both kinds, the $r$-Whitney numbers of both kinds, the $r$-Lah numbers and the $r$-Whitney-Lah numbers form particular cases of family of polynomials forming a generalization of the partial Bell polynomials. We deduce the generating functions of several restrictions of these numbers. In addition, a new combinatorial interpretations is presented for the $r$-Whitney numbers and the $r$-Whitney-Lah numbers.

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1 Introduction

The exponential partial Bell polynomials $B_{n,k}(x_1, x_2, \ldots) := B_{n,k}(x_j)$ in an infinite number of variables $x_j$, $(j \geq 1)$, introduced by Bell [1], as a mathematical tool for representing the $n$-th derivative of composite function. These polynomials are often used in combinatorics, statistics and also mathematical applications. They are defined by their generating function

$$\sum_{n \geq k} B_{n,k}(x_j) \frac{t^n}{n!} = \left( \sum_{m \geq 1} x_m \frac{t^m}{m!} \right)^k,$$

and are given explicitly by the formula

$$B_{n,k}(a_1, a_2, \ldots) = \sum_{\pi(n,k)} \frac{n!}{k_1! \cdots k_n!} \left( \frac{a_1}{1!} \right)^{k_1} \left( \frac{a_2}{2!} \right)^{k_2} \cdots \left( \frac{a_n}{n!} \right)^{k_n}, \quad (1)$$

where

$$\pi(n,k) = \{k = (k_1, \ldots, k_n) \in \mathbb{N}^n : k_1 + k_2 + \cdots + k_n = k, \quad k_1 + 2k_2 + \cdots + nk_n = n \}.$$

It is well-known that for appropriate choices of the variables $x_j$, the exponential partial Bell polynomials reduce to some special combinatorial sequences. We mention the following special cases:

$$\begin{align*}
\begin{bmatrix} n \\ k \end{bmatrix} &= B_{n,k}(0!, 1!, 2!, \cdots), \text{ unsigned Stirling numbers of the first kind,} \\
n \begin{bmatrix} k \\ n \end{bmatrix} &= B_{n,k}(1, 1, 1, \ldots), \text{ Stirling numbers of the second kind,} \\
n \begin{bmatrix} n \\ k \end{bmatrix} &= B_{n,k}(1!, 2!, 3!, \cdots), \text{ Lah numbers,} \\
(\begin{bmatrix} n \\ k \end{bmatrix})^{kn-k} &= B_{n,k}(1, 2, 3, \cdots), \text{ idempotent numbers.}
\end{align*}$$
For more details on these numbers, one can see [1, 4, 7, 8, 10].

In 1984, Broder [2] generalized the Stirling numbers of both kinds to the so-called \( r \)-Stirling numbers. In this paper, after recalling the partition polynomials, we give a unified method for obtaining a class of special combinatorial sequences, called the exponential partial \( r \)-Bell polynomials for which the \( r \)-Stirling numbers and other known numbers appear as special cases. In addition, these polynomials generalize the exponential partial Bell polynomials and possess some combinatorial interpretations in terms of set partitions.

2 The partial \( r \)-Bell polynomials

First of all, to introduce the partial \( r \)-Bell polynomials, we may give some combinatorial interpretations of the partial Bell polynomials. Below, for \( B_{n,k} (a_1, a_2, a_3, . . . ) \), we use \( B_{n,k} (a_l) \) and sometimes we use \( B_{n,k} (a_1, a_2, a_3, . . . ) \) and for \( B_{n,k}^{(r)} (a_1, a_2, . . . ; b_1, b_2, . . . ) \), we use \( B_{n,k}^{(r)} (a_l; b_l) \) and sometimes we use \( B_{n,k}^{(r)} (a_1, a_2, . . . ; b_1, b_2, . . . ) \).

**Theorem 1** Let \( (a_n; n \geq 1) \) be a sequence of nonnegative integers. Then, we have

- the number \( B_{n,k} (a_l) \) counts the number of partitions of a \( n \)-set into \( k \) blocks such that the blocks of the same cardinality \( i \) can be colored with \( a_i \) colors,
- the number \( B_{n,k} ((l-1)!a_l) \) counts the number of permutations of a \( n \)-set into \( k \) cycles such that any cycle of length \( i \) can be colored with \( a_i \) colors, and,
- the number \( B_{n,k} (!a_l) \) counts the number of partitions of a \( n \)-set into \( k \) ordered blocks such that the blocks of cardinality \( i \) can be colored with \( a_i \) colors.

**Proof.** For a partition of a finite \( n \)-set that is decomposed into \( k \) blocks, let \( k_i \) be the number of blocks of the same cardinality \( i, i = 1, . . . , n \). Then, the number to choice such partition is

\[
\frac{n!}{k_1! (1)^{k_1} k_2! (2)^{k_2} \cdots k_n! (n)^{k_n}}, \quad k = (k_1, . . . , k_n) \in \pi (n, k),
\]

and, the number to choice such partition for which the blocks of the same cardinality \( i \) can be colored with \( a_i \) colors is

\[
\frac{n!}{k_1! (1)^{k_1} k_2! (2)^{k_2} \cdots k_n! (n)^{k_n} (a_1)^{k_1} (a_2)^{k_2} \cdots (a_n)^{k_n}}, \quad k = (k_1, . . . , k_n) \in \pi (n, k),
\]

Then, the number of partitions of a \( n \)-set into \( k \) blocks of cardinalities \( k_1, k_2, . . . , k_n \) such that the blocks of the same length \( i \) can be colored with \( a_i \) colors is

\[
\sum_{k \in \pi(n, k)} \frac{n!}{k_1! (1)^{k_1} \cdots k_n! (n)^{k_n} (a_1)^{k_1} (a_2)^{k_2} \cdots (a_n)^{k_n}} = B_{n,k} (a_l).
\]

For the combinatorial interpretations of \( B_{n,k} ((l-1)!a_l) \) and \( B_{n,k} (!a_l) \), we can proceed similarly as above. \( \square \)

**Definition 2** Let \( (a_n; n \geq 1) \) and \( (b_n; n \geq 1) \) be two sequences of nonnegative integers. The number \( B_{n+r,k+r}^{(r)} (a_l; b_l) \) counts the number of partitions of a \( (n+r) \)-set into \( (k+r) \) blocks such that:
the $r$ first elements are in different blocks,

- any block of the length $i$ with no elements of the $r$ first elements, can be colored with $a_i$ colors,
- any block of the length $i$ with one element of the $r$ first elements, can be colored with $b_i$ colors.

We assume that any block with 0 color does not appear in partitions.

On using this definition, the following theorem gives an interesting relation which help us to find a family of polynomials generalize the above numbers.

On using combinatorial arguments, the partial $r$-Bell polynomials admit the following expression.

**Theorem 3** For $n \geq k \geq r \geq 1$, the partial $r$-Bell polynomials can be written as

$$B_{n,k}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots) = \frac{(n-r)!}{(k-r)!} \sum_{n_1 + \cdots + n_k = n - r - k} b_{n_1+1} \cdots b_{n_r+1} \frac{a_{n_r+1} \cdots a_{n_k+1}}{n_1! \cdots n_r! (n_r+1)! \cdots (n_k+1)!}.$$ 

**Proof.** Consider the $(n + r)$-set as union of two sets $R$ which contains the $r$ first elements and $N$ which contains the $n$ last elements. To partition a $(n + r)$-set into $k + r$ blocks $B_1, \ldots, B_{k+r}$ given as in Definition 2 let the elements of $R$ be in different $r$ blocks $B_1, \ldots, B_r$.

There is $\frac{1}{k!} \binom{n}{(n_1, \ldots, n_{k+r})} b_{n_1+1} \cdots b_{n_{r+1}+1} a_{n_{r+1}+1} \cdots a_{n_{k+1}}$ ways to choose $n_1, \ldots, n_{k+r}$ in $N$ on using colors, such that

- $n_1 \geq 0, \ldots, n_r \geq 0 : n_1, \ldots, n_r$ to be, respectively, in $B_1, \ldots, B_r$ with $b_{n_1+1} \cdots b_{n_r+1}$ ways to color these blocks,
- $n_{r+1} \geq 1, \ldots, n_{k+r} \geq 1 : n_{r+1}, \ldots, n_{k+r}$ to be, respectively, in $B_{r+1}, \ldots, B_{k+r}$ with $\frac{1}{k!} a_{n_{r+1}+1} \cdots a_{n_k+1}$ ways to color these blocks.

Then, the total number of colored partitions is

$$B_{n+r,k+r}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots) = \frac{1}{k!} \sum_{n_1 + \cdots + n_k = n - r - k} \binom{n}{(n_1, \ldots, n_{k+r})} b_{n_1+1} \cdots b_{n_{r+1}+1} a_{n_{r+1}+1} \cdots a_{n_{k+1}},$$

where $M_{n,k} = \{(n_1, \ldots, n_k) : n_1 + \cdots + n_k = n, (n_1, \ldots, n_r, n_{r+1}-1, \ldots, n_k - 1) \in \mathbb{N}^k\}$. 

On using Theorem 3 we may state that:

**Corollary 4** We have

$$\sum_{n \geq k} B_{n+r,k+r}^{(r)}(a_1; b_1) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{j \geq 1} \frac{a_j}{j!} \right)^k \left( \sum_{j \geq 0} \frac{b_j}{j!} \right)^r.$$ 

**Proof.** From Theorem 3 we get

$$\sum_{n \geq k} B_{n+r,k+r}^{(r)}(a_1; b_1) \frac{t^n}{n!} = \sum_{n \geq k} \left( \frac{1}{k!} \sum_{n_1 + \cdots + n_k = n - r - k} b_{n_1+1} \cdots b_{n_r+1} a_{n_{r+1}+1} \cdots a_{n_{k+1}} \frac{n_1! \cdots n_r! (n_r+1)! \cdots (n_k+1)!}{n_1! \cdots n_r! (n_{r+1}+1)! \cdots (n_{k+1}+1)!} \right) \frac{t^n}{n!}$$

$$= \frac{1}{k!} \sum_{n_1 \geq 0, \ldots, n_r \geq 0} b_{n_{r+1}+1} \cdots b_{n_{k+1}+1} a_{n_{r+1}+1} \cdots a_{n_{k+1}} \frac{n_{r+1}! \cdots n_{k+1}!}{n_1! \cdots n_r! (n_{r+1}+1)! \cdots (n_{k+1}+1)!} \frac{t^n}{n!}$$

$$= \frac{1}{k!} \left( \sum_{j \geq 1} \frac{a_j}{j!} \right)^k \left( \sum_{j \geq 0} \frac{b_j}{j!} \right)^r.$$
To give an explicit expression of the number $B^{(r)}_{n+r,k+r}(a; b)$ generalizing the formula (1), we use the Touchard polynomials defined in [3] as follows. Let $(x_i; i \geq 1)$ and $(y_i; i \geq 1)$ be two sequences of indeterminates, the Touchard polynomials

$$T_{n,k}(x_j, y_j) \equiv T_{n,k}(x_1, \ldots, x_n; y_1, \ldots, y_n), \quad n = k, k + 1, \ldots,$$

are defined by $T_{0,0} = 1$ and the sum

$$T_{n,k}(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_{\Lambda(n,k)} \left[ \frac{n!}{k_1!k_2! \cdots} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} \cdots \right] \left[ \frac{1}{r_1!r_2! \cdots} \left( \frac{y_1}{1!} \right)^{r_1} \left( \frac{y_2}{2!} \right)^{r_2} \cdots \right],$$

where

$$\Lambda(n,k) = \left\{ (k_1, k_2, \ldots) : k_i \in \mathbb{N}, \sum_{i \geq 1} k_i = k, \sum_{i \geq 1} i(k_i + r_i) = n \right\},$$

and admits a vertical generating function given by

$$\sum_{n=k}^{\infty} T_{n,k}(x_1, x_2, \ldots; y_1, y_2, \ldots) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{i \geq 1} \frac{t^i}{i!} \right)^k \exp \left( \sum_{i \geq 1} \frac{y_i}{i!} t^i \right), \quad k = 0, 1, \ldots \quad (3)$$

**Theorem 5** We have

$$B^{(r)}_{n+r,k+r}(a; b) = \sum_{\Lambda(n,k,r)} \left[ \frac{n!}{k_1!k_2! \cdots} \left( \frac{a_1}{1!} \right)^{k_1} \left( \frac{a_2}{2!} \right)^{k_2} \cdots \right] \left[ \frac{r!}{r_0!r_1! \cdots} \left( \frac{b_1}{0!} \right)^{r_0} \left( \frac{b_2}{1!} \right)^{r_1} \cdots \right],$$

where

$$\Lambda(n,k,r) = \left\{ (k, r) : (k_i : i \geq 1); (r_i : i \geq 0) : \quad (k_i \in \mathbb{N}, r_i \in \mathbb{N}, \sum_{i \geq 1} k_i = k, \sum_{i \geq 1} r_i = r, \sum_{i \geq 1} i(k_i + r_i) = n \right\}.$$

**Proof.** Setting

$$\pi(n, k, j) = \left\{ k = (k_1, \ldots, k_n; r_1, \ldots, r_n) : \sum_{i=1}^{n} k_i = k, \sum_{i=1}^{n} r_i = j, \sum_{i \geq 1} i(k_i + r_i) = n \right\},$$

$$\Pi(n, k, r) = \left\{ k = (k_1, \ldots, k_n; r_0, \ldots r_n) : \sum_{i=1}^{n} k_i = k, \sum_{i=0}^{n} r_i = r, \sum_{i \geq 1} i(k_i + r_i) = n \right\},$$

$$T_{n,k,s}(a; b_{i+1}) = \sum_{\pi(n,k,s)} \frac{n!}{k_1! \cdots k_n!r_1! \cdots r_n!} \left( \frac{a_1}{1!} \right)^{k_1} \cdots \left( \frac{a_n}{n!} \right)^{k_n} \left( \frac{b_1}{0!} \right)^{r_0} \left( \frac{b_2}{1!} \right)^{r_1} \cdots \left( \frac{b_{n+1}}{n!} \right)^{r_n}.$$

On using Corollary 3 we obtain

$$\sum_{u \geq k} \left( \exp \left( -b_1 u \right) \sum_{r \geq 0} B^{(r)}_{n+r,k+r}(a; b) \frac{u^r}{r!} \right) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{j \geq 1} \frac{a_j t^j}{j!} \right)^k \exp \left( u \sum_{j \geq 1} \frac{b_{j+1} t^j}{j!} \right).$$
Upon using (3), the last expression shows that
\[
\exp\left(-b_1 u\right) \sum_{r \geq 0} B_{n+r,k+r}^{(r)}(a_i; b_j) \frac{u^r}{r!} = T_{n,k}(a_1, \ldots, a_n; u b_2, \ldots, u b_{n+1})
\]
\[
= \sum_{\pi(n,k)} \frac{n!}{k_1! \cdots k_n! r_1! \cdots r_n!} \frac{a_1}{1!} \cdots \frac{a_n}{n!} \frac{b_1}{0!} \cdots \frac{b_{n+1}}{n!} \frac{r_1}{1!} \cdots \frac{r_n}{n!} u^{r_1 + \cdots + r_n}
\]
\[
= \sum_{s \geq 0} u^s \sum_{\pi(n,k,s)} \frac{n! s!}{k_1! \cdots k_n! r_1! \cdots r_n!} \frac{a_1}{1!} \cdots \frac{a_n}{n!} \frac{b_1}{0!} \cdots \frac{b_{n+1}}{n!} \frac{r_1}{1!} \cdots \frac{r_n}{n!} u^{r_1 + \cdots + r_n}
\]
\[
= \sum_{s \geq 0} s! T_{n,k,s}(a_i; b_{j+1}) \frac{u^s}{s!}.
\]

So, we obtain
\[
\sum_{r \geq 0} B_{n+r,k+r}^{(r)}(a_i; b_j) \frac{u^r}{r!} = \exp\left(b_1 u\right) \sum_{r \geq 0} s! T_{n,k,s}(a_i; b_{j+1}) \frac{u^s}{s!}
\]
\[
= \sum_{r \geq 0} \frac{u^r}{r!} \sum_{j=0}^{r} \binom{r}{j} j! b_1^{r-j} T_{n,k,j}(a_i; b_{j+1}).
\]

Then
\[
B_{n+r,k+r}^{(r)}(a_i; b_j)
\]
\[
= \sum_{j=0}^{r} \binom{r}{j} b_1^{r-j} j! T_{n,k,j}(a_i; b_{j+1})
\]
\[
= \sum_{r_0=0}^{r} \frac{b_1^{r_0}}{r_0!} \sum_{\pi(n,k,r_0-j)} \frac{n! r_1!}{k_1! \cdots k_n! r_1! \cdots r_n!} \frac{a_1}{1!} \cdots \frac{a_n}{n!} \frac{b_1}{0!} \cdots \frac{b_{n+1}}{n!} \frac{r_1}{1!} \cdots \frac{r_n}{n!}
\]
\[
= \sum_{\Pi(n,k,r)} \frac{n! r_1!}{k_1! \cdots k_n! r_0! r_1! \cdots r_n!} \frac{a_1}{1!} \cdots \frac{a_n}{n!} \frac{b_1}{0!} \cdots \frac{b_{n+1}}{n!} \frac{r_1}{1!} \cdots \frac{r_n}{n!}.
\]

The elements of \(\Lambda(n, k, r)\) can be reduced to those of \(\Pi(n, k, r)\) because we get necessarily \(k_j = r_{j+1} = 0\) for \(j \geq n + 1\). Thus, the expression of \(B_{n+r,k+r}^{(r)}(a_i; b_j)\) results. \(\square\)

### 3 Some properties of the partial \(r\)-Bell polynomials

Other combinatorial processes give the following identity.

**Proposition 6** We have
\[
B_{n+r,k+r}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots) = \sum_{i=0}^{r} \binom{r}{i} \binom{n}{i} b_1^{i} a_1^{r-i} B_{n-j+r-i,k-j+r-i}^{(r-i)}(0, a_2, a_3, \ldots; 0, b_2, b_3, \ldots).
\]  \(\text{(4)}\)

**Proof.** Consider the \((n+r)\)-set as union of two sets \(R\) which contains the \(r\) first elements and \(N\) which contains the \(n\) last elements. Choice \(i\) elements in \(R\) and \(j\) elements in \(N\) to form \(i+j\) singletons.
Because each singleton can be colored with \( b_1 \) colors if it is in \( \mathbf{R} \) and \( a_1 \) colors if it is in \( \mathbf{N} \), then, the number of the colored singletons is \( \binom{\ell}{r} \binom{\ell}{j} b_1^j a_1^r \). The elements not really used is of number \( r - i + n - j \) which can be partitioned into \( r - i + k - j \) colored partitions with non singletons (such that the \( r - i \) first elements are in different blocks) in \( B_{n-j+r-i,k-j+r-i}(0, a_2, a_3, \ldots ; 0, b_2, b_3, \ldots) \) ways. Then, for a fixed \( i \) and a fixed \( j \), there are \( \binom{\ell}{r} \binom{\ell}{j} b_1^j a_1^r B_{n-j+r-i,k-j+r-i}(0, a_2, a_3, \ldots ; 0, b_2, b_3, \ldots) \) colored partitions. So, the number of all colored partitions is

\[
\sum_{i=0}^{r} \sum_{j=0}^{k} \binom{r}{i} \binom{n}{j} b_1^j a_1^r B_{n-j+r-i,k-j+r-i}(0, a_2, a_3, \ldots ; 0, b_2, b_3, \ldots) = B_{n+r,k+r}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots).
\]

On using Corollary 4 or Theorem 5, we can verify that

**Proposition 7** We have

\[
B_{n+r,k+r}^{(r)}(xa_i; yb_i) = x^k y^r B_{n+r,k+r}^{(r)}(a_i; b_i),
\]

\[
B_{n+r,k+r}^{(r)}(x^i a_i; x^j b_i) = x^{n+r} B_{n+r,k+r}^{(r)}(a_i; b_i),
\]

\[
B_{n+r,k+r}^{(r)}(x^{i-1} a_i; x^{j-1} b_i) = x^{n-k} B_{n+r,k+r}^{(r)}(a_i; b_i).
\]

The relations of the following proposition generalize some of the known relations on partial Bell polynomials.

**Proposition 8** We have

\[
\sum_{j=1}^{n} \binom{n}{j} a_j B_{n+r-j,k+r-1}^{(r)}(a_i; b_i) = kB_{n+r,k+r}^{(r)}(a_i; b_i),
\]

\[
\sum_{j=1}^{n} \binom{n}{j} b_j B_{n+r-j,k+r-1}^{(r-1)}(a_i; b_i) = rB_{n+r,k+r}^{(r)}(a_i; b_i)
\]

and

\[
\sum_{j=1}^{n} ja_j \binom{n}{j} B_{n+r-j,k+r-1}^{(r)}(a_i; b_i) + \sum_{j=1}^{n} jb_j \binom{n}{j} B_{n+r-j,k+r-1}^{(r-1)}(a_i; b_i) = (n + r) B_{n+r,k+r}^{(r)}(a_i; b_i).
\]

**Proof.** On using Corollary 4 we deduce that

\[
\frac{\partial}{\partial a_j} B_{n+r,k+r}^{(r)}(a_i; b_i) = \binom{n}{j} B_{n-j+r,k+1+r}^{(r)}(a_i; b_i),
\]

\[
\frac{\partial}{\partial b_j} B_{n+r,k+r}^{(r)}(a_i; b_i) = \binom{n}{j} B_{n-j+r-1,k+r-1}^{(r-1)}(a_i; b_i).
\]

Then, by derivation the two sides of (5) in first time respect to \( x \) and in second time respect to \( y \), we obtain

\[
\sum_{j=1}^{n} \binom{n}{j} a_j B_{n+r-j,k+r-1}^{(r)}(a_i; yb_i) = kx^{k-1} y^r B_{n+r,k+r}^{(r)}(a_i; b_i),
\]

\[
\sum_{j=1}^{n} \binom{n}{j} b_j B_{n+r-j,k+r-1}^{(r-1)}(a_i; yb_i) = rx^{k} y^{r-1} B_{n+r,k+r}^{(r)}(a_i; b_i),
\]
and by derivation the two sides of (6) respect to $x$, we obtain

$$\sum_{j=1}^{n} j x^{j-1} a_j \binom{n}{j} B_{n+r-j,k+r-1}^{(r)} (a_l x^j; b_l y^j) + r \sum_{j=1}^{n} j x^{j-1} b_j \binom{n}{j-1} B_{n-j+r-1,k+r-1}^{(r-1)} (a_l x^j; b_l y^j)$$

$$= (n + r) x^{n+r-1} B_{n+r,k+r}^{(r)} (a_l; b_l).$$

The three relations of the proposition follow by taking $x = y = 1$. □

The partial $r$-Bell polynomials can be expressed by the partial bell polynomials as follows.

**Proposition 9** We have

$$B_{n+r,k+r}^{(r)} (a_l; b_l) = \left( \begin{array}{c} n + r \\ r \end{array} \right) \sum_{j=1}^{n} \left( \begin{array}{c} n + r \\ j \end{array} \right) B_{j,k} (a_l) B_{n-j,r} (b_l).$$

**Proof.** This proposition follows from the expansion

$$t^r \sum_{n \geq k} B_{n+k+r}^{(r)} (a_l; b_l) \frac{t^n}{n!} = \frac{t^r}{k!} \left( \sum_{j \geq 1} a_j t^j \frac{j!}{j!} \right)^k \left( \sum_{j \geq 1} b_j t^j \frac{j!}{j!} \right)^r$$

$$= \frac{1}{k!} \left( \sum_{j \geq 1} a_j t^j \frac{j!}{j!} \right)^k \left( \sum_{j \geq 1} b_j t^j \frac{j!}{j!} \right)^r$$

$$= r! \left( \sum_{i \geq k} B_{i,k} (a_l) \frac{t^i}{i!} \right) \left( \sum_{j \geq k} B_{j,r} (b_l) \frac{t^j}{j!} \right).$$

□

**Proposition 10** We have

$$\left( \begin{array}{c} n \\ r \end{array} \right) B_{n+k,k+r}^{(r)} (a_l; b_l) = \left( \begin{array}{c} n \\ k \end{array} \right) B_{n-k+r,k+r}^{(k)} (b_l; a_l), \quad n \geq 2 \max (k, r).$$

**Proof.** From Corollary 4 we get

$$\frac{t^r}{r!} \sum_{n \geq k} B_{n+k+r}^{(r)} (a_l; b_l) \frac{t^n}{n!} = \frac{1}{k! r!} \left( \sum_{j \geq 1} j a_j t^j \frac{j!}{j!} \right)^k \left( \sum_{j \geq 1} j b_j t^j \frac{j!}{j!} \right)^r$$

and by the symmetry respect to $(k, (a_j))$ and $(r, (b_j))$ in the last expression we get

$$\frac{t^r}{r!} \sum_{n \geq k} B_{n+k,r}^{(r)} (a_l; b_l) \frac{t^n}{n!} = \frac{1}{k!} \sum_{n \geq r} B_{n+r,k+r}^{(k)} (b_l; a_l) \frac{t^n}{n!}.$$

So, we obtain the desired identity. □
4 New combinatorial interpretations of the $r$-Whitney numbers

The $r$-Whitney numbers of both kinds $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ are introduced by Mező [6] and the $r$-Whitney-Lah numbers $L_{m,r}(n,k)$ are introduced by Cheon and Jung [5, 9]. Some of the properties of these numbers are given in [5]. In this paragraph we use the combinatorial interpretation of the partial $r$-Bell polynomials given above to deduce a new combinatorial interpretations for the numbers $|w_{m,r}(n,k)|$, $W_{m,r}(n,k)$ and $L_{m,r}(n,k)$.

The $r$-Whitney numbers of the first kind $w_{m,r}(n,k)$ are given by their generating function

$$
\sum_{n\geq k} w_{m,r}(n,k) \frac{t^n}{n!} = \frac{1}{k!} (\ln (1 + mt))^k \left((1 + mt)^{-\frac{1}{m}}\right)^r.
$$

So that

$$
w_{m,r}(n,k) = (-1)^{n-k+r} B_{n+r,k+r}^{(r)} \left((l-1)!m^{l-1}; (m+1)(2m+1) \cdots ((l-1)(m+1))\right).
$$

This means that the absolute $r$-Whitney number of the first kind $|w_{m,r}(n,k)|$ counts the number of partitions of a $n$-set into $k$ blocks such that
- the $r$ first elements are in different blocks,
- any block of cardinality $i$ and no contain an element of the $r$ first elements can be colored with $(i - 1)!m^{i-1}$ colors, and,
- any block of cardinality $i$ and contain one element of the $r$ first elements can be colored with $(m+1)(2m+1) \cdots ((i-1)(m+1))$ colors.

The $r$-Whitney numbers of the second kind $W_{m,r}(n,k)$ are given by their generating function

$$
\sum_{n\geq k} W_{m,r}(n,k) \frac{t^n}{n!} = \frac{1}{k!} \left(\frac{\exp(mt) - 1}{m}\right)^k \exp(rt).
$$

So that

$$
W_{m,r}(n,k) = B_{n+r,k+r}^{(r)} \left(m^{l-1}; 1\right).
$$

This means that the $r$-Whitney number of the second kind $W_{m,r}(n,k)$ counts the number of partitions of a $n$-set into $k$ blocks such that
- the $r$ first elements are in different blocks,
- any block of cardinality $i$ and no contain any element of the $r$ first elements can be colored with $m^{i-1}$ colors, and,
- any block of cardinality $i$ and contain one element of the $r$ first elements can be colored with one color.

The $r$-Whitney-Lah numbers $L_{m,r}(n,k)$ are given by their generating function

$$
\sum_{n\geq k} L_{m,r}(n,k) \frac{t^n}{n!} = \frac{1}{k!} \left(t(1 - mt)^{-1}\right)^k \left((1 - mt)^{-\frac{1}{m}}\right)^r.
$$

So that

$$
L_{m,r}(n,k) = B_{n+r,k+r}^{(r)} \left(l!m^{l-1}; 2(m+2) \cdots ((l-1)(m+2))\right).
$$

This means that the $r$-Whitney number of the second kind $L_{m,r}(n,k)$ counts the number of partitions of a $n$-set into $k$ blocks such that
- the $r$ first elements are in different blocks,
- any block no contain an element of the $r$ first elements and is of length $i$ can be colored with $i!m^{i-1}$ colors, and,
- any block of cardinality $i$ and contain one element of the $r$ first elements can be colored with $2(m+2) \cdots ((i-1)(m+2))$ colors.
5 Application to the $r$-Stirling numbers of the second kind

From Definition 2 we may state that the number $B_{n,k}^{(r)}(a_l) := B_{n,k}^{(r)}(a_l, a_l)$ counts the number of partitions of $n$-set into $k$ blocks such that the blocks of the same cardinality $i$ can be colored with $a_i$ colors (such cycles with $a_i = 0$ does not exist) and the $r$ first elements are in different blocks.

For $a_n = 1$, $n \geq 1$, we get the know $r$-Stirling numbers of the second kind

$$\begin{bmatrix} n \cr k \end{bmatrix}_r = B_{n,k}^{(r)}(1, 1, \cdots)$$

which counts the number of partitions of a $n$-set into $k$ blocks such that the $r$ first elements are in different blocks.

For $a_n = 1$, $n \geq m$ and $a_n = 0$, $n \leq m - 1$, we get the $m$-associated $r$-Stirling numbers of the second kind

$$\begin{bmatrix} n \cr k \end{bmatrix}_{m}^\uparrow_r = B_{n,k}^{(r)}(m, 1, 1, \cdots)$$

which counts the number of partitions of a $n$-set into $k$ blocks such that the cardinality of any block is at least $m$ elements and the $r$ first elements are in different blocks.

For $a_n = 0$, $n \geq m + 1$ and $a_n = 1$, $n \leq m$, we get the $m$-truncated $r$-Stirling numbers of the second kind

$$\begin{bmatrix} n \cr k \end{bmatrix}_{m}^\downarrow_r = B_{n,k}^{(r)}(m, 1, 0, 0, \cdots)$$

which counts the number of partitions of a $n$-set into $k$ blocks such that the cardinality of any block is $\leq m$ elements and the $r$ first elements are in different blocks.

For $a_{2n - 1} = 0$ and $a_{2n} = 1$, $n \geq 1$, we get the $r$-Stirling numbers of the second kind in even parts

$$\begin{bmatrix} n \cr k \end{bmatrix}_{\text{even}}^r = B_{n,k}^{(r)}(0, 1, 0, 1, 0, \cdots)$$

which counts the number of partitions of a $n$-set into $k$ blocks such that the cardinality of any block is even and the $r$ first elements are in different blocks.

For $a_{2n - 1} = 1$ and $a_{2n} = 0$, $n \geq 1$, we get the $r$-Stirling numbers of the second kind in odd parts

$$\begin{bmatrix} n \cr k \end{bmatrix}_{\text{odd}}^r = B_{n,k}^{(r)}(1, 0, 1, 0, \cdots)$$

which counts the number of partitions of a $n$-set into $k$ blocks such that the cardinality of any block is odd and the $r$ first elements are in different blocks.

6 Application to the $r$-Stirling numbers of the first kind

We start this application by giving a second combinatorial interpretation of the partial $r$-Bell polynomials.

Proposition 11 The number $B_{n,k}^{(r)}((l - 1)!a_l) := B_{n,k}^{(r)}((l - 1)!a_l, (l - 1)!a_l)$ counts the number of permutations of a $n$-set into $k$ cycles such that the cycles of the same length $i$ can be colored with $a_i$ colors (such cycles with $a_i = 0$ does not exist) and the $r$ first elements are in different cycles.
Proof. Let $\Pi_{r,k}$ be the set of partitions $\pi$ of the set $n$-set into $k$ blocks such that the blocks of the same cardinality $i$ posses $(i - 1)!a_i$ colors and the $r$ first elements are in different blocks, and, $P_{r,k}$ be the set of permutations $P$ of the elements of the set $n$-set into $k$ cycles such that the cycles of the same length $i$ can be colored with $a_i$ colors the $r$ first elements are in different cycles. The application $\varphi : \Pi_{r,k} \to P_{r,k}$ which associate any (colored) partition $\pi$ of $\Pi_{r,k}$, $\pi = S_1 \cup \cdots \cup S_k$, $1 \leq k \leq n$, a (colored) permutation $P$ of $P_{r,k}$, $P = C_1 \cup \cdots \cup C_k$, such that the elements of $C_i$ are exactly those of $S_i$. It is obvious that the application $\varphi$ is bijective.

For $a_n = (n - 1)!$, $n \geq 1$, we get the known $r$-Stirling numbers of the first kind

$$\left[ \begin{array}{c} n \\ k \end{array} \right] _ r = B_{n,k}^{(r)} (0!, 1!, 2!, \cdots )$$

which counts the number of permutations of a $n$-set into $k$ cycles such that the $r$ first elements are in different cycles.

For $a_n = (n - 1)!$, $n \geq m$ and $a_n = 0$, $n \leq m - 1$, we get the $m$-associated $r$-Stirling numbers of the first kind

$$\left[ \begin{array}{c} n \\ k \end{array} \right] _ {m\uparrow} = B_{n,k}^{(r)} \left( 0, \cdots , 0, (m - 1)!, m!, \cdots \right)$$

which counts the number of permutations of a $n$-set into $k$ cycles such that the length of any cycle is equal at least $m$ and the $r$ first elements are in different cycles.

For $a_n = (n - 1)!$, $n \leq m$ and $a_n = 0$, $n \geq m + 1$, we get the $m$-truncated $r$-Stirling numbers of the first kind

$$\left[ \begin{array}{c} n \\ k \end{array} \right] _ {m\downarrow} = B_{n,k}^{(r)} \left( 0!, \cdots , (m - 1)!, 0, 0, \cdots \right)$$

which counts the number of permutations of a $n$-set into $k$ cycles such that the length of any cycle is $\leq m$ and the $r$ first elements are in different cycles.

For $a_{2n-1} = 0$ and $a_{2n} = (2n - 1)!$, $n \geq 1$, we get the $r$-Stirling numbers of the first kind in cycles of even lengths

$$\left[ \begin{array}{c} n \\ k \end{array} \right] _ {even} = B_{n,k}^{(r)} (0, 1!, 0, 3!, 0, 5!, 0, \cdots )$$

which counts the number of permutations of a $n$-set into $k$ cycles such that the length of any cycle is even and the $r$ first elements are in different cycles.

For $a_{2n-1} = (2n - 2)!$ and $a_{2n} = 0$, $n \geq 1$, we get the $r$-Stirling numbers of the first kind in cycles of odd lengths

$$\left[ \begin{array}{c} n \\ k \end{array} \right] _ {odd} = B_{n,k}^{(r)} (0!, 0, 2!, 0, 4!, 0, \cdots )$$

which counts the number of permutations of a $n$-set into $k$ cycles such that the length of any cycle is odd and the $r$ first elements are in different cycles.

7 Application to the $r$-Lah numbers

Then, similarly to the partial Bell polynomials, we establish the following (third) combinatorial interpretation of the partial $r$-Bell polynomials.
Proposition 12 The number $B_{n,k}^{(r)}(q_1, \ldots, q_k) := B_{n,k}^{(r)}(q_1, \ldots, q_k)$ counts the number of partitions of a $n$-set into $k$ ordered blocks such that the blocks of the same cardinality $i$ can be colored with $q_i$ colors (such block with $a_i = 0$ does not exist) and the $r$ first elements are in different blocks.

Proof. Let $\Pi_{r,k}$ be the set of partitions $\pi$ of the $n$-set into $k$ blocks such that the blocks of the same cardinality $i$ can be colored with $i$ colors and the $r$ first elements are in different blocks, and, $\Pi'^{ord}_{r,k}$ be the set of partitions $\pi^{ord}$ of the $n$-set into $k$ ordered blocks such that the blocks of the same length $i$ can be colored with $a_i$ the $r$ first elements are in different blocks. The application $\varphi : \Pi'_{r,k} \to \Pi'^{ord}_{r,k}$ which associate a (colored) partition $\pi$ of $\Pi'_{r,k}$, $\pi = S_1 \cup \cdots \cup S_k$, $1 \leq k \leq n$, a (colored) partition of ordered blocks $\pi^{ord}$ of $\Pi'^{ord}_{r,k}$, $\pi^{ord} = P_1 \cup \cdots \cup P_k$, such that the elements of $P_i$ are exactly those of $S_i$.

It is obvious that the application $\varphi$ is bijective. \qed

For $a_n = n!$, $n \geq 1$, we get the $r$-Lah numbers

$$\binom{n}{k}_r = B_{n,k}^{(r)}(1!, 2!, 3!, \ldots).$$

The $r$-Lah number $\binom{n}{k}_r$ counts the number of partitions of a $n$-set into $k$ ordered blocks such that the $r$ first elements are in different blocks.

For $a_n = 0$, $n \leq m - 1$, and $a_n = n!$, $n \geq m$, we get the $m$-degenerate $r$-Lah numbers

$$\binom{n}{k}^{m\uparrow}_r = B_{n,k}^{(r)}(0, \cdots, 0, m!, (m + 1)!),$$

which counts the number of partitions of a $n$-set into $k$ ordered blocks such that the cardinality of any block is $\geq m$ and the $r$ first elements are in different blocks.

For $a_n = n!$, $n \leq m$, and $a_n = 0$, $n \geq m + 1$, we get the $m$-truncated $r$-Lah numbers

$$\binom{n}{k}^{m\downarrow}_r = B_{n,k}^{(r)}(1!, \cdots, m!, 0, 0, \cdots),$$

which counts the number of partitions of a $n$-set into $k$ ordered blocks such that the cardinality of any block is $\leq m$ and the $r$ first elements are in different blocks.

For $a_{2n-1} = 0$ and $a_{2n} = (2n)!$, $n \geq 1$, we get the $r$-Lah numbers in blocks of even cardinalities

$$\binom{n}{k}^{even}_r = B_{n,k}^{(r)}(0, 2!, 0, 4!, 0, 6!, 0, \cdots),$$

which represents the number of partitions of a $n$-set into $k$ ordered blocks such that the cardinality of any block is even and the $r$ first elements are in different blocks.

For $a_{2n-1} = (2n-1)!$ and $a_{2n} = 0$, $n \geq 1$, we get the $r$-Lah numbers in blocks of even cardinalities

$$\binom{n}{k}^{odd}_r = B_{n,k}^{(r)}(1!, 0, 3!, 0, 5!, 0, \cdots)$$

which represents the number of partitions of a $n$-set into $k$ ordered blocks such that the cardinality of any block is odd and the $r$ first elements are in different blocks.
8 Application to sum of independent random variables

It is known that for a sequence of independent random variables \( \{X_n\} \) with all its moments exist and are the same, \( \mu_n = E(X^n) \) we have
\[
E(S_n^p) = (\frac{n + p}{p})^{-1} B_{n+p,p}(l\mu_{l-1}).
\]
The following theorem generalizes this result.

**Theorem 13** Let \( \{X_n\} \) and \( \{Y_n\} \) be two independent sequences of independent random variables with all their moments exist and are the same, \( \mu_n = E(X^n) \), \( \nu_n = E(Y^n) \) and let
\[
S_{p,q} = X_1 + \cdots + X_p + Y_1 + \cdots + Y_q.
\]
Then we have
\[
E(S_{p,q}^n) = \left(\frac{n + p}{p}\right)^{-1} B_{n+p+p+q}^{(q)}(l\mu_{l-1}, \nu_{l-1}).
\]

**Proof.** Let \( \varphi_X(t) \) be the common generating function of moments for \( X_n, n \geq 1 \), \( \varphi_Y(t) \) be the common generating function of moments for \( Y_n, n \geq 1 \), and, \( \varphi_{S_{p,q}}(t) \) be the generating function of moments of \( S_{p,q} \). Then, in first part, we get
\[
t^p \varphi_{S_{p,q}}(t) = E(t^p \exp(tS_{p,q})) = (E(t \exp(tX_1)))^p (E(\exp(tY_1)))^q
\]
\[
= \left( \sum_{j \geq 1} j\mu_{j-1} \frac{t^j}{j!} \right)^p \left( \sum_{j \geq 0} \nu_{j} \frac{t^j}{j!} \right)^q
\]
\[
= \sum_{n \geq p} B_{n+p+q}^{(q)}(l\mu_{l-1}, \nu_{l-1}) \frac{t^n}{n!},
\]
and, in second part, we have
\[
t^p \varphi_{S_{p,q}}(t) = \sum_{j \geq 0} E(S_{p,q}^j) \frac{t^{j+p}}{j!^2} = \sum_{n \geq p} \frac{n!}{(n-p)!} E(S_{p,q}^{n-p}) \frac{t^n}{n!}.
\]

\[\square\]

For the choice \( q_0 = 1 \) and \( q_j = 0 \) if \( j \geq 1 \) in the last theorem, we may state that:

**Corollary 14** Let \( \{X_n\} \) be a sequence of independent random variables with all their moments exist and are the same, \( \mu_n = E(X^n) \) and
\[
S_{p,q} = X_1 + \cdots + X_p + q.
\]
Then we have
\[
E(S_{p,q}^n) = \left(\frac{n + p}{p}\right)^{-1} B_{n+p+p+q}^{(q)}(l\mu_{l-1}, 1).
\]

**Example 1** Let \( \{X_n\} \) be a sequence of independent random variables with the same law of probability \( U(0,1) \).
\[
S_{p,q} = X_1 + \cdots + X_p + r.
\]
Then we have
\[
\left\{\binom{n+p+r}{p+r}\right\}_r = \binom{n+p}{p} E(S_{p,r}^n).
\]
Theorem 17 Let \( X \) be a sequence of independent discrete random variables with the same law of probability \( p_j := P(X_1 = j) \), \( j \geq 0 \) we have \( P(S_p = n) = \frac{p!}{(n+p)!} B_{n+p,p}^{(q)} \). The following theorem generalizes this result.

Theorem 15 Let \( \{X_n\} \) and \( \{Y_n\} \) be two independent sequences of independent random variables with \( p_j := P(X_n = j) \), \( q_j := P(Y_n = j) \) and let

\[
S_{p,q} = X_1 + \cdots + X_p + Y_1 + \cdots + Y_q.
\]

Then we have

\[
P(S_{p,q} = n) = \frac{p!}{(n+p)!} B_{n+p,q,p+q}^{(q)} \cdot l! p_{l-1}, (l-1)! q_{l-1}.
\]

Proof. It suffices to take in the last theorem \( q_0 = 1 \) and \( q_j = 0 \) if \( j \geq 1 \).

\[
\sum_{n \geq p} P(S_{p,q} = n - p) t^n = t^p \sum_{s \geq 0} P(S_{p,q} = s) t^s
\]
\[
= t^p E(t^{S_{p,q}})
\]
\[
= \left( \sum_{j \geq 1} P_{j-1} t^j \right)^p \left( \sum_{j \geq 0} q_j t^j \right)^q
\]
\[
= p! \sum_{n \geq p} B_{n+p,q,p+q}^{(q)} \cdot l! p_{l-1}, (l-1)! q_{l-1} \frac{t^n}{n!}.
\]

This gives \( P(S_{p,q} = n) = \frac{p!}{(n+p)!} B_{n+p,q,p+q}^{(q)} \cdot l! p_{l-1}, (l-1)! q_{l-1} \). \( \square \)

For the choice \( q_0 = 1 \) and \( q_j = 0 \) if \( j \geq 1 \) in the last theorem, we may state that:

Corollary 16 Let \( \{X_n\} \) be a sequence of independent discrete random variables with the same law of probability \( p_j := P(X_1 = j) \) and

\[
S_{p,q} = X_1 + \cdots + X_p + q.
\]

Then we have

\[
P(S_{p,q} = n) = \frac{p!}{(n+p)!} B_{n+p,q,p+q}^{(q)} \cdot l! p_{l-1}, (l-1)! q_{l-1}.
\]

9 Application on the successive derivatives of a function

Let \( F(x) = \sum_{n \geq 0} f_n x^n \in C^\infty(0) \) and \( G(x) = \sum_{n \geq 1} g_n \frac{(x-a)^n}{n!} \in C^\infty(a) \). It is shown in [4] that

\[
\frac{d^n}{dx^n} (F(G(x))) \bigg|_{x=a} = \sum_{k=0}^{n} f_k B_{n,k} (g_j).
\]

The following theorem gives a similar result on using the partial \( r \)-Bell polynomials.

Theorem 17 Let \( F, G \) be as above and \( H(x) = \sum_{n \geq 1} h_n \frac{(x-a)^n}{n!} \in C^\infty(a) \). Then, we have

\[
\frac{d^n}{dx^n} \left( \frac{d}{dx} H(x) \right)^r F(G(x)) \bigg|_{x=a} = \sum_{k=0}^{n} f_k B_{n+r,k+r} (g_j, h_j).
\]
Proof. This follows from
\[\sum_{n \geq 0} \left( \sum_{k=0}^{n} f_k B_{n+r,k+r}^{(r)}(g_j, h_j) \right) \frac{(x - a)^n}{n!} = \sum_{k \geq 0} f_k \sum_{n \geq k} B_{n+r,k+r}^{(r)}(g_j, h_j) \frac{(x - a)^n}{n!} = \left( \sum_{j \geq 0} h_{j+1} \frac{(x - a)^j}{j!} \right)^r \sum_{k \geq 0} f_k \left( \sum_{j \geq 1} g_j \frac{(x - a)^j}{j!} \right)^k = \left( \frac{d}{dx} H(x) \right)^r F(G(x)).\]

For the choice \( F(x) = \exp(x) \), we obtain:

**Corollary 18** For \( G, H \in C^\infty(a) \) with \( G(0) = 0 \), we have
\[\frac{d^n}{da^n} \left( \left( \frac{d}{da} H(a) \right)^r \exp(G(a)) \right) = \exp(G(a)) \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} \left( \frac{d^j}{da^j} G(a), \frac{d^j}{da^j} H(a) \right).\]

**Example 2** Let \( G(a) = \frac{\exp(ma) - 1}{m} \) and \( H(a) = \exp(a) \). On using Corollary 18 and the generating function of the numbers \( W_{m,r}^{(n,k)} \) given above, we get
\[\frac{d^n}{da^n} \left( \exp \left( \frac{\exp(ma)}{m} + ra \right) \right) = \exp \left( \frac{\exp(ma)}{m} + ra \right) \sum_{k=0}^{n} W_{m,r}^{(n,k)} \exp(mak).\]

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