A SUFFICIENT CONDITION FOR STRONG $F$-REGULARITY

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Abstract. Let $(R, \mathfrak{m}, K)$ be an $F$-finite Noetherian local ring which has a canonical ideal $I \subseteq R$. We prove that if $R$ is $S_2$ and $H^{d-1}_m(R/I)$ is a simple $R(F)$-module, then $R$ is a strongly $F$-regular ring. In particular, under these assumptions, $R$ is a Cohen-Macaulay normal domain.

1. Introduction

Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of positive characteristic and let $x \in \mathfrak{m}$ be a nonzero divisor in $R$. A central question in the study of singularities is whether good properties of the ring $R/xR$ imply good properties of $R$. This is related to whether a type of singularity deforms. It is known that $F$-purity and strong $F$-regularity, two important and well studied types of singularities in positive characteristic, do not deform. This was showed by an example of Fedder [Fed83] for $F$-purity, and by an example of Singh [Sin99], for strong $F$-regularity. However, if $R$ is a Gorenstein ring, both $F$-purity and strong $F$-regularity do deform.

Enescu [Ene03] changed gears by looking at canonical ideals instead of ideals generated by nonzero divisors. Suppose that $R$ has a canonical ideal, i.e. an ideal $I$ such that $I \cong \omega_R$. Note that in a Gorenstein ring, an ideal is generated by a nonzero divisor if and only if it is a canonical ideal. Recently, Ma [Ma12] showed that, under mild assumptions on $R$, if $R/I$ is $F$-pure, then $R$ is also $F$-pure [Ma14, Theorem 3.4]. Inspired by his result, we investigate if $R/I$ being strongly $F$-regular implies that $R$ is strongly $F$-regular or, equivalently, if $R/I$ being $F$-rational implies that $R$ is strongly $F$-regular. Our main result is Theorem 3.8, which is a more general version of the following.

Theorem 1.1. Let $(R, \mathfrak{m}, K)$ be an excellent local ring of dimension $d$ and characteristic $p > 0$. Suppose that $R$ is $S_2$ and it has a canonical ideal $I \cong \omega_R$ such that $R/I$ is $F$-rational. Then $R$ is a strongly $F$-regular ring.

This theorem extends a result of Enescu [Ene03, Corollary 2.9] by dropping the hypotheses of $R$ being a Cohen-Macaulay (normal) domain. We point out that these three conditions are implied by $R$ being strongly $F$-regular. In his work, Enescu uses properties of pseudo-canonical covers, while we focus on an interplay between Frobenius actions and $p^{-1}$-linear maps combined with structural properties of local cohomology. As a consequence of Theorem 1.1, we extend a result of Goto, Hayasaka, and Iai [GHI03, Corollary 2.4] to rings which are not necessarily Cohen-Macaulay. Specifically, we show that, under the assumptions of Theorem 1.1, if $R/I$ is regular, then $R$ is also regular (see Corollary 3.9).

Throughout this article $(R, \mathfrak{m}, K)$ will denote a Noetherian local ring of Krull dimension $d$ and characteristic $p > 0$. $(-)^\vee$ denotes the Matlis dual functor $\text{Hom}_R(-, E_R(K))$. In addition, $\omega_R$ denotes a canonical module for $R$, which is a finitely generated $R$-module satisfying $\omega_R^\vee \cong H^d_\mathfrak{m}(R)$.  


2. Preliminaries

2.1. Canonical modules. In this section we present several facts and properties regarding canonical modules over rings which are not necessarily Cohen-Macaulay. We refer to [Aoy83, HH94b, Ma14] for details.

We recall that not every ring has a canonical module; however, every complete ring has one. In fact, if \( R \) is a homomorphic image of a Gorenstein local ring \((S, n, L)\) of dimension \( n \), then \( \omega_R \cong \text{Ext}^n_S(R,S) \).

**Proposition 2.1** ([Aoy83, Corollary 4.3]). Let \((R, \mathfrak{m}, K)\) be a local ring with canonical module \( \omega_R \). If \( R \) is equidimensional, then for every prime ideal \( P \), \((\omega_R)_P\) is a canonical module for \( R_P \).

**Proposition 2.2** ([Ma14, Proposition 2.4]). Let \((R, \mathfrak{m}, K)\) be a local ring with canonical module \( \omega_R \). If \( R \) is equidimensional and unmixed, then the following conditions are equivalent:

1. \( \omega_R \) is isomorphic to an ideal \( I \subseteq R \).
2. \( R \) is generically Gorenstein, i.e. if \( R_p \) is Gorenstein for all \( p \in \text{Min}(R)(= \text{Ass}(R)) \).
3. \( \omega_R \) has rank 1.

Moreover, when any of these equivalent conditions hold, \( I \) is a height one ideal containing a nonzero divisor of \( R \), and \( R/I \) is equidimensional and unmixed [Ma14, Proposition 2.6]. If, in addition, \( R \) is Cohen-Macaulay, then \( R/I \) is Gorenstein.

**Definition 2.3.** Let \( k \) be a positive integer. Recall that a finitely generated \( R \)-module \( M \) is said to satisfy Serre’s condition \( S_k \) (or simply \( M \) is \( S_k \)) if

\[
\text{depth}(M_p) \geq \min\{k; \text{ht}(p)\}
\]

for all \( p \in \text{Spec}(R) \).

\( R \) is \( S_k \) if it satisfies Serre’s condition \( S_k \) as a module over itself. If \( R \) is \( S_2 \), then \( R \) is unmixed. Furthermore, when \((R, \mathfrak{m}, K)\) is local and catenary (e.g. when it is excellent), the \( S_2 \) condition also implies that \( R \) is equidimensional. If \( R \) is excellent and \( S_2 \), then its \( \mathfrak{m} \)-adic completion \( \hat{R} \) is also \( S_2 \), and if \( I \) is a canonical ideal of \( R \), then \( I \hat{R} = \hat{I} \) is a canonical ideal of \( \hat{R} \).

2.2. Methods in positive characteristic. We recall some of the definitions of singularities for rings of positive characteristic. We refer the interested reader to [Hun96, Smi01, ST12, BS13] for surveys and a book on these topics.

Let \((R, \mathfrak{m}, K)\) be a Noetherian local ring of characteristic \( p > 0 \), and let \( F^e : R \to R \) be the \( e \)-th iteration of the Frobenius endomorphism on \( R \), where \( e \) is a positive integer. Let \( M \) be an \( R \)-module. By \( F^e_*(M) \) we denote \( M \) viewed as a module over \( R \) via the action of \( F^e \). Specifically, for any \( F^e_*(m_1), F^e_*(m_2) \in F^e_*M \) and for any \( r \in R \) we have

\[
F^e_*(m_1) + F^e_*(m_2) = F^e_*(m_1 + m_2) \quad \text{and} \quad r \cdot F^e_*(m_1) = F^e_*(r^p m_1).
\]

If \( e = 1 \), we omit \( e \) in the notation. When \( R \) is reduced, the endomorphism \( F^e \) can be identified with the inclusion of \( R \) into \( R^{1/p^e} \), the ring of its \( p^e \)-th roots.

**Definition 2.4.** \( R \) is called \( F \)-finite if \( F_*(R) \) is a finitely generated \( R \)-module.

A local ring \((R, \mathfrak{m}, K)\) is \( F \)-finite if and only if it is excellent and \([K : K^p] < \infty \) [Kun76].
Definition 2.5. $R$ is $F$-pure if $F \otimes 1_M : R \otimes_R M \to R \otimes_R M$ is injective for all $R$-modules $M$. $R$ is $F$-split if the map $R \to F_* R$ splits.

Remark 2.6. If $R$ is an $F$-pure ring, $F$ itself is injective and $R$ must be a reduced ring. We have that $R$ is $F$-split if and only if $R$ is a direct summand of $F_* R$. If $R$ is an $F$-finite ring, $R$ is $F$-pure if and only $R$ is $F$-split [HR74, Lemma 5.1]. As a consequence, if $(R, m, K)$ is $F$-finite, we have that $R$ is $F$-pure if and only if $\hat{R}$ is $F$-pure. If $R$ is $F$-finite, we use the word $F$-pure to refer to both.

Definition 2.7 ([BB11]). We say that an additive map $\phi : M \to M$ is $p^{-e}$-linear if $\phi(p^e v) = r \phi(v)$ for every $r \in R, v \in M$.

There is a bijective correspondence between $p^{-1}$-linear maps on $M$ and $R$-module homomorphisms $F_* M \to M$.

Definition 2.8. We define the ring $R\{F\}$ as $\frac{R(F)}{R(p^e F - F \cdot r \in R)}$, the non-commutative $R$-algebra generated by $F$ with relations $r^p \cdot F = F \cdot r$, for $r \in \hat{R}$.

Definition 2.9. We say that an $R$-module $M$ has a Frobenius action, if there is an additive map $F : M \to M$ such that $F(ru) = r^p F(u)$ for $u \in M$ and $r \in \hat{R}$.

There is a natural equivalence between $R\{F\}$-modules and $R$-modules with a Frobenius action. In addition, every Frobenius action of $M$ corresponds to an $R$-module homomorphism $M \to F_* M$. If $R$ is complete and $F$-finite, then $(F_* M)^\vee \cong F_* (M^\vee)$ [BB11, Lemma 5.1]. Then, there is an induced map $F_* (M^\vee) \cong (F_* M)^\vee \to (M)^\vee$, which gives a correspondence between Frobenius actions on $M$ and $p^{-1}$-linear maps on $M^\vee$.

In this case, we have that the Frobenius map $F : R \to R$ induces a Frobenius action on $H^d_m(R)$. Suppose $R$ has a canonical module $\omega_R$. Let $\Phi : \omega_R \to \omega_R$ be the $p^{-1}$-linear map corresponding to the Matlis dual of $F : H^d_m(R) \to H^d_m(R)$. We will refer to $\Phi$ as the trace map of $\omega_R$.

Definition 2.10. A local ring $(R, m, K)$ of dimension $d$ is called $F$-rational if it is Cohen-Macaulay and $H^d_m(R)$ is simple as a $R\{F\}$-module.

We point out that this is not the original definition introduced by Hochster and Huneke [HH90], which is in terms of tight closure: $R$ is $F$-rational if the ideals generated by parameters are tightly closed. However, both definitions are equivalent due to Smith [Smi97, Theorem 2.6]. $F$-rational local rings have nice singularities; for example, they are normal domains.

Definition 2.11 ([HH90]). An $F$-finite ring $R$ is strongly $F$-regular if for all nonzero elements $c \in R$, the $R$-linear homomorphism $\varphi : R \to F^e_* (R)$ defined by $\varphi(1) = F^e_* (c)$ splits for $e \gg 0$.

Theorem 2.12 ([HH89, Theorem 3.1 c]]). Every $F$-finite regular ring is strongly $F$-regular.

It is a well-known fact that strong $F$-regularity implies $F$-rationality. This could be seen from the relation that these notions have with tight closure (see [HH90]).

Definition 2.13 ([Sch10]). Suppose that $R$ is an integral domain. The test ideal $\tau(R) \subseteq R$ is defined as the smallest non-zero compatible ideal of $R$. 

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τ(R) is the big test ideal originally defined by Hochster and Huneke [HH90] in terms of tight closure. Schwede [Sch10, Theorem 6.3] proved that the definition above is equivalent. We have that R is strongly F-regular if and only if τ(R) = R. Furthermore, τ(R) = \text{ann}_{E_R(K)}0_{E_R(K)}^e, where 0_{E_R(K)}^e denotes the tight closure of 0 in the injective hull E_R(K) of K. We have that R is strongly F-regular if and only if 0_{E_R(K)}^e = 0.

Remark 2.14. One can define strongly F-regular rings for rings that are not F-finite by requiring that for all nonzero elements c ∈ R, the R-linear homomorphism \varphi : R → F_e^c(R) defined by \varphi(1) = F_e^c(c) is pure for \( e \gg 0 \). If R is F-finite, this definition is equivalent to Definition 2.11. Furthermore, R is strongly F-regular if and only if τ(R) = \text{ann}_{E_R(K)}0_{E_R(K)}^e = R, as in the F-finite case [Smi93, Theorem 7.1.2]. Regular rings are strongly F-regular also for non F-finite rings. This can be proven using the Gamma construction [HH94a, Discussion 6.11 & Lemma 6.13] and Aberbach and Enescu’s results on base change for test ideals [AE03, Corollary 3.8].

Remark 2.15. Let \( \phi : R → R \) be a \( p^{-e} \)-linear map. For every ideal \( J ⊆ R \) we have \( \phi(J^{[p^r]} ) ⊆ J \). If \( \phi \) is surjective then equality holds. Furthermore, if \( \phi \) is surjective then R is F-pure. In fact, there exists an element \( r ∈ R \) such that \( \phi(r) = 1 \), and then the R-linear homomorphism \( \varphi : F_e^r R → R \) defined by \( \varphi(F_e^r x) = \phi(rx) \) gives the desired splitting.

Definition 2.16 ([Sch10]). Let \( \phi : R → R \) be a \( p^{-e} \)-linear map and let \( J ⊆ R \) be an ideal. J is \( \phi \)-compatible if \( \phi(J) ⊆ J \). An ideal J is said to be compatible if it is \( \phi \)-compatible for all \( p^{-e} \)-linear maps \( \phi : R → R \) and all \( e ∈ \mathbb{N} \).

Compatible ideals in the definition above were used previously in different contexts [MR85, Smi97, LS01, HT04].

Remark 2.17. If R is a Gorenstein F-finite ring, we have that \( \text{Hom}_R(F_e^r R, R) \cong F_e^r R \) as \( F_e^r R \)-modules, and the isomorphism is given by precomposition with multiplication by elements in \( F_e^r R \). Let \( \Phi \) be the \( p^{-1} \)-map corresponding to a generator of \( \text{Hom}_R(F_e^r R, R) \cong F_e^r R \) as a \( F_e^r R \)-module. If an ideal J is \( \Phi \)-compatible, then it is compatible [ST12, Theorem 3.7].

3. Canonical ideals and strong F-regularity

We start by proving preparation lemmas that imply that under the hypotheses of Theorem 1.1, R is F-pure.

Lemma 3.1. Let \((R, m, K)\) be a Noetherian local ring of dimension \( d \). Suppose that R has a canonical ideal \( I ⊆ R \) such that \( \text{H}_m^{d-1}(R/I) \) is a simple \( R\{F\} \)-module. Then the Frobenius map \( F : \text{H}_m^{d-1}(R/I) → \text{H}_m^{d-1}(R/I) \) is injective.

Proof. Since the Frobenius action on \( \text{Ker}(F) ⊆ \text{H}_m^{d-1}(R/I) \) is trivial, we have that \( \text{Ker}(F) \) is an \( R\{F\} \)-submodule. Then either \( \text{Ker}(F) = 0 \) or \( \text{Ker}(F) = \text{H}_m^{d-1}(R/I) \), because \( \text{H}_m^{d-1}(R/I) \) is simple. If \( \text{Ker} F = \text{H}_m^{d-1}(R/I) \), then every \( R \)-submodule of \( \text{Ker} F \) is an \( R\{F\} \)-module. Since \( \text{H}_m^{d-1}(R/I) \) is a simple \( R\{F\} \)-module, \( \text{Ker} F = \text{H}_m^{d-1}(R/I) \) must be a simple \( R \)-module, that is \( \text{H}_m^{d-1}(R/I) \cong R/m = k. \) Since \( \text{dim}(R/I) = d - 1 \), this is only possible if \( d = 1 \) and \( R/I \) is zero-dimensional. However, if \( \text{dim}(R/I) = 0 \), we have \( \text{H}_m^{d-1}(R/I) = R/I = R/m \) and \( \text{Ker}(F) = 0 \) in this case. ∎
Remark 3.2. Suppose that $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module. From the short exact sequence $0 \to I \to R \to R/I \to 0$, we obtain an exact sequence of $R\{F\}$-modules

$$H_m^{d-1}(R/I) \longrightarrow H_m^d(I) \longrightarrow H_m^d(R) \longrightarrow 0.$$ 

The map $H_m^d(I) \to H_m^d(R)$ is not injective by [Ma14, Lemma 3.3], and its kernel is a non-zero $R\{F\}$-submodule of $H_m^d(I)$. Since $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module, the first map in the sequence above must be injective. Hence, when $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module, we have a short exact sequence of $R\{F\}$-modules

$$0 \longrightarrow H_m^{d-1}(R/I) \longrightarrow H_m^d(I) \longrightarrow H_m^d(R) \longrightarrow 0.$$ 

Remark 3.3. Note that $H_m^{d-1}(R/I)$ and $H_m^d(R)$ are simple $R\{F\}$-modules if and only if $H_m^{d-1}(\hat{R}/\hat{I})$ and $H_m^d(\hat{R})$ are simple $\hat{R}\{F\}$-modules.

Lemma 3.4. Let $(R, \mathfrak{m}, K)$ be an excellent local ring of dimension $d$. Suppose that $R$ is equidimensional and unmixed, and it has a canonical ideal $I \subsetneq R$. If $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module, then $H_m^{d-1}(R/I) \cong (R/I)^\vee$.

Proof. By Remark 3.3 we can assume that $R$ is complete. By Remark 3.2 we have a short exact sequence of $R\{F\}$-modules $0 \to H_m^{d-1}(R/I) \to H_m^d(I) \to H_m^d(R) \to 0$. Taking the Matlis dual we get an exact sequence of $R$-modules:

$$0 \longrightarrow H_m^d(R)^\vee \cong J \longrightarrow H_m^d(I)^\vee \cong R \longrightarrow H_m^{d-1}(R/I)^\vee \longrightarrow 0,$$

where $J \cong \omega_R$ is potentially another canonical ideal for $R$. We then get that $H_m^{d-1}(R/I)^\vee \cong R/J =: \omega_{R/I}$ is a canonical module for $R/I$, and we want to show that $J = I$. We have a homomorphism of $R/I$-modules:

$$R/I \longrightarrow \text{Hom}_{R/I}(\omega_{R/I}, \omega_{R/I}) \cong \text{Hom}_{R/I}(R/J, R/J) \cong R/J,$$

which is just the map induced by the inclusion $I \subseteq J$. Since $R$ is equidimensional and unmixed, so is $R/I$, and the kernel of the above map is trivial [HH94b]. In addition, this kernel is $J/I$. Therefore, $J = I$ and $H_m^{d-1}(R/I) \cong (R/I)^\vee$. \hfill \square

Proposition 3.5. Let $(R, \mathfrak{m}, K)$ be an $F$-finite Noetherian local ring of dimension $d$. Suppose that $R$ is $S_2$ and it has a canonical ideal $I \subsetneq R$ such that $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module. Then, $R/I$ is an $F$-pure ring. As a consequence, $R$ is an $F$-pure ring.

Proof. By Remarks 3.3 and 2.6, we may assume that $R$ is a complete ring. We have that the Frobenius action on $H_m^{d-1}(R/I)$ is injective by Lemma 3.1. This induces a surjective $p^{-1}$-linear map on $R/I = (H_m^{d-1}(R/I))^\vee$. Then, $R/I$ is $F$-pure by Remark 2.15. Therefore, $R$ is also $F$-pure [Ma14, Theorem 3.4]. \hfill \square

The simplicity of $H_m^{d-1}(R/I)$ forces $R/I$ to have mild singularities, as we show in the following result. This result will be needed in the proof of Theorem 3.8.

Theorem 3.6. Let $(R, \mathfrak{m}, K)$ be a Noetherian local $F$-finite ring of dimension $d$. Suppose that $R$ is equidimensional and unmixed, and that it has a canonical ideal $I \subsetneq R$ such that $H_m^{d-1}(R/I)$ is a simple $R\{F\}$-module. Then $R/I$ is a strongly $F$-regular Gorenstein ring.

\[ \text{□} \]
Proof. We note that $R$ is strongly $F$-regular and Gorenstein if and only if its completion is also strongly $F$-regular and Gorenstein. We can assume that $R$ is complete by Remark 3.3. By Lemma 3.4, it follows that $H_{m}^{d-1}(R/I) \cong \omega_{R/I} \cong R/I$. To prove that $R/I$ is Gorenstein, it remains to show that it is Cohen-Macaulay. Then, it suffices to show that $R/I$ is strongly $F$-regular. Let $\Phi : R/I \to R/I$ be the $p^{-1}$-linear map which is dual to the Frobenius action on $H_{m}^{d-1}(R/I)$. We note that $\Phi$ is surjective by Lemma 3.1. Let $c \in R/I$ be a nonzero element and set $J := \bigcup_{n \in \mathbb{N}} \Phi^{n}(c(R/I))$. We have that $J$ is an ideal of $R/I$ that contains $c$. In addition, $\Phi(J) = J$ by Remark 2.15 and Lemma 3.1. Since $c \neq 0$, $J$ is a nonzero ideal compatible with $\Phi$, and thus $J^{\vee}$ corresponds to a nonzero $R\{F\}$-submodule of $H_{m}^{d-1}(R/I)$. But the latter is a simple $R\{F\}$-module, therefore $J^{\vee} = H_{m}^{d-1}(R/I)$, and hence $J = R/I$. In particular, there exists an element $r \in R/I$ and an integer $N$ such that $\Phi^{N}(rc) = 1$. Let $\varphi : F_{*}^{N}(R/I) \to R/I$ be the $R/I$-linear map defined by $\varphi(F_{*}^{N}x) = \Phi^{N}(rx)$ for all $x \in R/I$. We have that $\varphi(F_{*}^{c}c) = 1$. Since $0 \neq c \in R/I$ was chosen arbitrarily, we conclude that $R/I$ is strongly $F$-regular. \hfill $\Box$

We recall a result of Goto, Hayasaka, and Iai [GHI03] that is needed to prove Theorem 1.1.

Proposition 3.7 ([GHI03, Corollary 2.4]). Let $(S, m, K)$ be a Cohen-Macaulay local ring which has a canonical ideal $I \subset S$ such that $S/I$ is a regular local ring. Then $R$ is regular.

Now, we are ready to prove our main theorem.

Theorem 3.8. Let $(R, m, K)$ be a Noetherian local $F$-finite ring of dimension $d$ and characteristic $p > 0$. Suppose that $R$ is $S_{2}$ and it has a canonical ideal $I \subset R$ such that $H_{m}^{d-1}(R/I)$ is a simple $R\{F\}$-module. Then $R$ is a strongly $F$-regular ring.

Proof. Under our assumptions on $R$, we have that $\tau(\hat{R}) \cap R = \tau(R)$ [LS01, Theorem 2.3] and $IR$ is a canonical ideal for $\hat{R}$. Thus, it suffices to prove our claim assuming that $R$ is complete by Remark 3.3. By Theorem 3.6, $I$ is a height one prime ideal. Since $R$ is equidimensional, we have that $(\omega_{R})_{I}$ is a canonical module for $R_{I}$, and in particular $IR_{I}$ is a canonical ideal for $R_{I}$. Since $R_{I}$ is a one-dimensional Cohen-Macaulay local ring and $R_{I}/IR_{I}$ is a field, we have that $R_{I}$ must be regular by Proposition 3.7.

We finish proving the theorem by means of contradiction. Assume that $R$ is not strongly $F$-regular. Let $\tau(R)$ be the test ideal of $R$. We claim that $\tau(R) \subset I$. Let $N = \text{ann}_{E_{R}(K)}(\tau(R))$, which is a submodule of $E_{R}(K) \cong H_{m}^{d}(I)$ compatible with every Frobenius action on $E_{R}(K)$ (see [LS01]). In particular, $N$ is an $R\{F\}$-submodule of $H_{m}^{d}(I)$. As $H_{m}^{d-1}(R/I)$ is a simple $R\{F\}$-submodule of $H_{m}^{d}(I)$, it must be contained in $N$. In fact, they cannot be disjoint because they both contain the socle of $H_{m}^{d}(I)$. Taking annihilators in $R$ and applying Matlis duality, we get

$$\tau(R) = \text{ann}_{R}(N) \subset \text{ann}_{R}(H_{m}^{d-1}(R/I)) = I.$$ 

Since the test ideal defines the non-strongly $F$-regular locus [LS01, Theorem 7.1], $R_{I}$ is not strongly $F$-regular. This is a contradiction because every regular ring is strongly $F$-regular by Theorem 2.12. \hfill $\Box$

We are now ready to prove Theorem 1.1. We proceed by reducing to the $F$-finite case via gamma construction. This method is well-known to the experts. We refer to [HH94a, Discussion 6.11 & Lemma 6.13] for definitions and properties.
Proof of Theorem 1.1. Completion does not change the assumption that $R/I$ is $F$-rational by Remark 3.3. In addition, if $\hat{R}$ is strongly $F$-regular, then so is $R$, because $\tau(R)\hat{R} = \tau(\hat{R})$ [LS01, Theorem 2.3]. We now consider a $p$-base for $K^{1/p}$, $\Lambda$, and a cofinite set $\Gamma \subseteq \Lambda$. Consider the faithfully flat extension $R \to R^\Gamma$ given by the gamma construction. We have that $R^\Gamma$ is a complete $F$-finite local ring with maximal ideal $m^\Gamma := mR^\Gamma$ and residue field $K^\Gamma \cong K \otimes_R R^\Gamma$. Since $R$ is complete, there exists a Gorenstein local ring $(S, n, K)$, with $\dim(R) = \dim(S)$, such that $R$ is a homomorphic image of $S$. By functoriality of the Gamma construction, we have that $S^\Gamma$ maps homomorphically onto $R^\Gamma$. Furthermore, the map $S \to S^\Gamma$ is local and flat, with $K^\Gamma$ as closed fiber. Since $S$ is a Gorenstein ring, so is $S^\Gamma$. Because $I$ is a canonical module of $R$, we have that $I \cong \Hom_S(R, S)$. Since $S \to S^\Gamma$ is faithfully flat, we have

$$I \otimes_R R^\Gamma \cong \Hom_S(R, S) \otimes_S S^\Gamma \cong \Hom_{S^\Gamma}(R \otimes_S S^\Gamma, S^\Gamma) \cong \Hom_{S^\Gamma}(R^\Gamma, S^\Gamma).$$

Therefore, $I \otimes_R R^\Gamma \cong \omega_{R^\Gamma}$ is a canonical module for $R^\Gamma$. In addition, $I \otimes_R R^\Gamma \cong IR^\Gamma$; therefore, $I^\Gamma := IR^\Gamma \subseteq R^\Gamma$ is a canonical ideal of $R^\Gamma$. Now consider the flat local homomorphism $R/I \to R^\Gamma/I^\Gamma$ induced by the gamma construction above. We have that $R/I$ is excellent because it is complete. In addition, $R/I$ is a Cohen-Macaulay domain because it is $F$-rational. Furthermore, the closed fiber is $K^\Gamma$, and $R/I$ is Gorenstein by Lemma 3.4. We have that $R^\Gamma/I^\Gamma$ is $F$-rational because parameter ideals are tightly closed in $R^\Gamma/I^\Gamma$ [Abe01, Proposition 3.2]. Since $R^\Gamma$ is $F$-finite, we conclude that $R^\Gamma$ is strongly $F$-regular by Theorem 3.8. In this case, $\tau(R^\Gamma) = R^\Gamma$, so that $0^*_F(k_V) = 0$. Now, we apply [AE03, Corollary 3.8] to get that $\tau(R)R^\Gamma = \tau(R^\Gamma) = R^\Gamma$. Therefore, $\tau(R) = R$, and so, $R$ is strongly $F$-regular. \qed

As a corollary of this result, we weaken the Cohen-Macaulay assumption in Proposition 3.7 to $S_2$ for excellent rings of positive characteristic.

Corollary 3.9. Let $(R, m, K)$ be an excellent local ring of positive characteristic. Suppose that $R$ is $S_2$, and it has a canonical ideal $I \subseteq R$ such that $R/I$ is regular. Then $R$ is regular.

Proof. As $R/I$ is regular, it is an $F$-rational ring. By Theorem 1.1 $R$ is strongly $F$-regular; therefore, $R$ is Cohen-Macaulay. We can now apply Proposition 3.7 to get the desired result. \qed

Finally, we give an example which shows that the sufficient condition for strong $F$-regularity in Theorem 3.8 is not necessary. That is, if $R$ is strongly $F$-regular, we cannot always find a canonical ideal $I \subseteq R$ such that $H^d_{m^{-1}}(R/I)$ is a simple $R$-$F$-module, or equivalently such that $R/I$ is strongly $F$-regular.

Example 3.10. Let $K$ be a field with $\text{char}(K) \neq 3$, and consider the two dimensional domain

$$R = K[[s^3, s^2t, st^2, t^3]] \cong K[[x, y, z, w]]/J,$$

where $J = (z^2 - yw, yz - wx, y^2 - xz)$. Then $R$ has type two, and it is a direct summand of $K[[s, t]]$; therefore, it is strongly $F$-regular [HH89, Theorem 3.1 (e)]. Any canonical ideal of $R$ is two-generated, say $I = (f, g)$. Denote by $m$ the maximal ideal of $R$. If $R/I$ was strongly $F$-regular, then it would be regular because it is a one-dimensional ring. This would be equivalent to $\dim_K(m/(m^2 + I + J)) = 1$. Since $J \subseteq m^2$, we have

$$\dim_K\left(\frac{m}{m^2 + I + J}\right) = \dim_K\left(\frac{m}{m^2 + (f, g)}\right) \geq 2.$$
Remark 3.11. If $R$ is an $F$-finite normal domain, one can use a more geometric argument to show that if $R/I$ is strongly $F$-regular, then so is $R$. This is done by choosing a canonical divisor on $\text{Spec}(R)$ corresponding to the ideal $I \cong \omega_R$, and by using $F$-adjunction [Sch09].

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