Tomograms and the quest for single particle nonlocality

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Abstract. By using a tomographic approach to quantum states, we rise the problem of nonlocality within a single particle (single degree of freedom). We propose a possible way to look for such effects on a qubit. Although a conclusive answer is far from being reached, we provide some reflections on the foundational ground.

1. Introduction
Quantum entanglement is associated with the specific nonlocal correlations among the parts of a quantum system that has no classical analog [1]. This assumes that the entangled system should consist of two or more parts. Although recently much interest has been dedicated to single particle entanglement, it relies to different degrees of freedom, hence to different parts of the system (subsystems).

Typical Bell-type experiments [2] involve, beside entangled (singlet) states, noncommuting observables (on each subsystem). Thus, the nonlocal character might not solely be ascribed to the property of states (entanglement), but also to uncertainty principle (e.g. correlations that arise due to the noncommuting character of observables). As such it could somehow emerge even in a single system (single degree of freedom). Here, we address this possibility by resorting to quantum tomography in order to fix the meaning of nonlocality in this context.

Results along this direction might shed light on the basic principles of quantum mechanics, like the uncertainty principle, perhaps pointing out some form of self entanglement.

2. Qubit tomograms
The tomographic description can be applied to the systems with both continuous and discrete variables. Here we are interested in case of discrete variables, because we are going to deal with the “smallest” system-a qubit. Thus we briefly recall the spin tomography [4]. For a system with defined angular momentum $j$ we employ throughout the paper the vectors $\{ |m\rangle, m = -j, \ldots, j \}$ of $j_z$ eigenstates basis. Then, any state-density operator can be represented as

$$\rho = \int d \vec{n} \sum_{m=-j}^{j} w(m; \vec{n}) K(m, \vec{n}), \tag{1}$$
where \( \vec{n} \) is a unit vector determining a point on Bloch sphere characterized by the Euler angles \((\varepsilon, \vartheta, \varphi)\). Furthermore, \( K(m, \Omega) \) is the operator whose matrix elements are given by

\[
\langle m' | K(m, \Omega) | m'' \rangle = \frac{(-1)^{m+m''}}{8\pi^2} \sum_{k=0}^{2j} (2k+1)^2 \\
\times \sum_{l=-j}^{j} \langle l | D(\vec{n}) | 0 \rangle W^{j,j-l}_{m',-m-l} W^{j,l}_{m,-m,l},
\]

with

\[
D(\vec{n}) \equiv \exp \left[ -i (\varepsilon j_x + \vartheta j_y + \varphi j_z) \right]
\]

the rotation operator and

\[
W^{a,b,c}_{\alpha,\beta,\gamma} \equiv (-1)^{a-b-c} \sqrt{\frac{(a + b - c)!(a - b + c)!(a + b + c)!}{(a + b + c + 1)!}} \\
\times \sum_{t} (-1)^{t} \sqrt{\frac{(a + \alpha)!(a - \alpha)!(b + \beta)!(b - \beta)!(c + \gamma)!(c - \gamma)!}{t!(c - b + t + \alpha)!(c - a + t - \beta)!(a + b - c - t)!(a - t - \alpha)!(b - t + \beta)!}},
\]

the Wigner \(3j\)-symbol [3] (where the sum is taken over all integers \(t\) for which the factorials all have nonnegative arguments).

Finally, \( w(m; \vec{n}) \) represents the tomograms, describing the probability distribution to measure the corresponding value of spin projection \(m\) along the rotated direction \(\vec{n}\):

\[
w(m; \vec{n}) = \langle m | D(\vec{n}) \rho D^\dagger(\vec{n}) | m \rangle.
\]

They satisfy the normalization condition \( \sum_{m=-j}^{j} \omega(m, \vec{n}) = 1 \). Now, for a qubit one has to simply consider \(j = 1/2\) and \(j_x = \sigma_x/2, j_y = \sigma_y/2, j_z = \sigma_z/2\) with \(\sigma s\) the usual Pauli operators. Then, it is well known that the density operator for qubit state can be expressed in terms of only three tomograms corresponding e.g. to the \(x, y\) and \(z\) directions [4], however this is not relevant for our purposes. Using the tomographic relation (1) between a quantum state and a set of probability distributions we can translate the quantum measurement in a classical language. First we note, by Eq.(3), that the set of tomograms corresponds to the probability distributions of the same observable, \(j_z\), measured in different (rotated) reference frames. Then the problem of wave function collapse reduces to the problem of a reduction of the probability distribution which occurs as soon as we “pick” a classical value of the classical random observable in the classical framework. Nevertheless, measurement on a reference frame instantaneously affects the distributions on the others (due to the underlying uncertainty principle).

In this sense nonlocality seems intrinsically present in a single system and should emerge as correlations among distributions on different reference frames (i.e. correlations of noncommuting observables measurement results). It immediately follows the question of whether such correlations can be reproduced by any hidden variable theory.

3. Qubit nonlocality

To address the above question, we consider the simultaneous measurement of spin projection along two directions specified by vectors \(\vec{a}, \vec{b} \in \mathbb{R}^3\). From Ref. [5] we can get the POVM elements for such a joint measurements as

\[
\Pi_{r,s}(\vec{a}, \vec{b}) = \left( \frac{1}{4} + rs \vec{a} \cdot \vec{b} \right) 1 + \left( r\vec{a} + s\vec{b} \right) \cdot \vec{\sigma}/2,
\]
where \( r, s = \pm 1/2 \) are the possible measurement results and \( \vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z) \) represents the vector of Pauli operators.

Due to the unsharpness of the measurements, the vectors \( \vec{a}, \vec{b} \) are constrained by

\[
\|\vec{a} + \vec{b}\| + \|\vec{a} - \vec{b}\| \leq 2. \tag{5}
\]

If we consider a qubit state \( \rho \), the probability of outcomes \( r, s \) along \( \vec{a}, \vec{b} \) reads

\[
P_{r,s}(\vec{a}, \vec{b}) = \text{Tr} \left[ \rho \Pi_{r,s}(\vec{a}, \vec{b}) \right]. \tag{6}
\]

Then, we can write the correlation of measurement results. In doing so we suppose to have outcomes of the type \( \pm 1 \) (rather than \( \pm 1/2 \)), thus obtaining

\[
E(\vec{a}, \vec{b}) = \sum_{r, s = \pm 1/2} 4 rs P_{r,s}(\vec{a}, \vec{b}). \tag{7}
\]

Given the measurement correlations (7) one can test the nonlocal character of the quantum state through some Bell like inequality.

### 3.1. Our attempt

Let us consider the CHSH inequality [6]

\[
|E(\vec{a}, \vec{b}) + E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}) - E(\vec{a}', \vec{b}')| \leq 2. \tag{8}
\]

We restrict our attention to the \( x - z \) plane and consider

\[
\vec{a} \propto (0, 0, 1), \quad \vec{a}' \propto (\sin \phi, 0, \cos \phi),
\]

\[
\vec{b} \propto (\sin(2\phi), 0, \cos(2\phi)). \tag{9}
\]

with \( 0 \leq \phi \leq \pi/2 \). Moreover, we take \( \rho \equiv |\psi\rangle\langle\psi| \) with

\[
|\psi\rangle = \cos \left( \frac{\theta}{2} \right) |+1/2\rangle + \sin \left( \frac{\theta}{2} \right) |-1/2\rangle, \quad 0 \leq \theta < 2\pi. \tag{12}
\]

We are now going to distinguish the four possible correlations (7). In each case we assume the condition (5) satisfied with equality and the two vectors having the same norm.

**i)**

\[
\vec{a} \equiv \frac{1}{\sqrt{1 + \sin \phi}} (0, 0, 1) \quad \vec{b} \equiv \frac{1}{\sqrt{1 + \sin \phi}} (\sin \phi, 0, \cos \phi) \quad \Rightarrow E(\vec{a}, \vec{b}) = \frac{\cos \phi}{1 + \sin \phi}. \tag{13}
\]

**ii)**

\[
\vec{a} \equiv \frac{1}{\sqrt{1 + \sin(2\phi)}} (0, 0, 1) \quad \vec{b}' \equiv \frac{1}{\sqrt{1 + \sin(2\phi)}} (\sin(2\phi), 0, \cos(2\phi)) \quad \Rightarrow E(\vec{a}, \vec{b}') = \frac{\cos(2\phi)}{1 + \sin(2\phi)}. \tag{14}
\]
iii)\[
\begin{align*}
\vec{a'} \equiv (\sin \phi, 0, \cos \phi) \\
\vec{b} \equiv (\sin \phi, 0, \cos \phi)
\end{align*}
\Rightarrow E(\vec{a'}, \vec{b}) = 1.
\]

(15)

iv)\[
\begin{align*}
\vec{a'} \equiv \frac{1}{\sqrt{1+\sin \phi}} (\sin \phi, 0, \cos \phi) \\
\vec{b'} \equiv \frac{1}{\sqrt{1+\sin \phi}} (\sin(2\phi), 0, \cos(2\phi))
\end{align*}
\Rightarrow E(\vec{a'}, \vec{b'}) = \frac{\cos \phi}{1 + \sin \phi}.
\]

(16)

Putting together Eqs.(13), (14), (15), (16) into Eq.(8), it is easy to see that the inequality is always verified (for any pure state of the qubit). We remark that the Bell inequality (8) has been derived by focusing on one and the same physical system and analyzing correlations between measurement outcomes of different (non compatible) observables. As such, it shares some analogies with the arguments of temporal Bell inequality initiated by Leggett and Garg [7] where, instead of considering correlations between observables of spatially separated systems, it is considered one and the same physical system and correlations between measurement outcomes at different times.

4. Some Foundational Reflections

Although we have not found violations of Bell inequality (8), we cannot draw firm conclusions about the rised problem. In fact many other Bell-type inequalities could be considered, and moreover the effect could be sought in systems living in larger Hilbert spaces, even in continuous variable systems (which is an ongoing work).

However, we can provide some reflections on the foundational ground. We can conceptually analyze the two possible scenarios:

- (Case A) impossibility to violate any Bell inequality;
- (Case B) possibility to violate some Bell inequality.

These scenarios bring us to the following reflections:

**Case A: Entanglement as basic level.** The Case A would be favorable to the assumption that the basic level of physical world could be the entanglement. This simple position may have important epistemological implications, like the rejection of individual object, and the rejection of individual intrinsic properties. As consequence, it is not possible to give a definition of the individual object in a spatio-temporal location and it is not possible to characterize the properties of the objects, in order to distinguish it from other ones. In other words, if we adopt the entanglement as basic level, we accept the philosophy of the relations and we renounce at the possible existence of intrinsic properties while we accept relational properties.

We remember, for instance, that a mathematical model based on the relationist principle accept that the position of an object can only be defined respect to other matter. We do not venture in the philosophical implications of the relationalism, as the monism which affirm that there are not distinction a priori between physical entities. An important advantage of these approach is the possibility to eliminate the privileged role of the observer. This is Rovelli’s approach to quantum mechanics [8] where the founding postulate is the impossibility to talk about properties of systems in the abstract, but only of properties of systems relative to one system (we can never juxtapose properties relative to different systems). Relational quantum mechanics is not
the claim that reality is described by the collection of all properties relatives to all systems, rather, reality admits one description per each (observing) system, and any such description is internally consistent. As Einstein’s original motivation with EPR was not to question locality, but rather to question the completeness of quantum mechanics, so the relation interpretation can be interpreted as the discovery of the incompleteness of the description of reality that any single observer can give. In this particular sense, relational quantum mechanics can be said to show the “incompleteness” of single-observer Copenhagen interpretation.

Case B: Uncertainty principle as basic level. The Case B would show a sort of self-entanglement and would be favorable to the assumption that the basic level of physical world could be the uncertainty principle. As we know, Heisenberg’s relation express ontological restrictions on the experiments that we can perform on quantum systems. The relation introduce a subject-object separation metaphorically called “the Heisenberg cut”. For these reasons, there are many interpretations of the uncertainty principle. First, we note that the usual formalism of quantum theory does not incorporate notion such a “simultaneous observations”, and thus no statement about them can be deduced from the same formalism. The question if the theoretical structure or the quantitative laws of quantum theory can be indeed derived on the basis of the uncertainty principle, as the same Heisenberg wished, is open. Recently, a proposal to construct quantum mechanics as a theory of “principle” was provided by Bub [9]; but this proposal does not use the uncertainty principle as one of its fundamental principles. Heisenberg’s relation cut acts as a boundary between potentiality and actuality, a definite boundary between a quantum system and a classical apparatus. According to this position, in the world of potentiality should be possible to have precise value of measurable quantities: we see an evident contradiction with the assumption that physical quantities do not exist before a measurement process. In the perspective of the above relational approach to quantum mechanics, Dickson [10] proposes an original interpretation of uncertainty principle based on a refreshing reminder on the foundations of dynamics. According Dickson, the formulation of dynamical laws requires the notion of inertial frames. The tomographic approach seems in line with this idea.

We retain that the basic problem is how uncertainty principle consider the fundamental concept of “individuality” of a quantum event. First, we need to understand the definition of a quantum process, and not only to focus our attention on the unavoidable “disturbance” or “physical influence” of the observer on the observed. However, the new concept of nonlocality would change our vision of physical reality; probably we cannot anymore speak about simple individuality. The concept of individuality should be revisited. For instance, a forced equivalence between information and individuality (underlying a physical reality) is claimed by Zeilinger [11], putting forward an idea which connects the concept of information with the notion of elementary systems.

5. Conclusions
We may conclude that both possible scenarios, entanglement or uncertainty principle as basic level, renounce in different ways to establish every ontological statute of the physical reality. The “entanglement” world renounces to intrinsic properties for relational properties with the advantage of eliminating the observer (the observer becomes relational). The “uncertainty principle” world seems to accept the challenge of primary properties (however, it does not go to the heart of the matter), but introduce the “observer” with the function to “evidence” these properties. The subjectivism of the observer, drives away any ontological statute of the physical reality.

References
[1] J. Bell, *Speakable and Unspeakable in Quantum Mechanics*, (Cambridge University Press, Cambridge, 1987).
[2] A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. 49, 1804 (1982); A. Aspect, Ph. Grangier and G. Roger, Phys. Rev. Lett. 49, 91 (1982).

[3] See e.g., A. Messiah, Quantum Mechanics, Vol. 2, North-Holland, Amsterdam (1962).

[4] O. V. Man’ko and V. I. Man’ko, JETP 85, 430 (1997); V. V. Dodonov and V. I. Ma’ko, Phys. Lett. A 229, 335 (1997).

[5] P. Busch and P. J. Lahti, Nuovo Cimento 18, 4 (1995).

[6] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).

[7] A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985).

[8] C. Rovelli, Intl. J. theor. Phys. 35, 1637-1678 (1996).

[9] J. Bub, Studies in History and Philosophy of Modern Physics 31, 75 (2000).

[10] M. Dickson, Studies in History and Philosophy of moder Physics 35, 195 (2004).

[11] J. Kofler and A. Zeilinger, Sciences et Avenir Hors-Serie No. 148, October/November 2006.