Identities for the Multiple Zeta (Star) Values

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Abstract. In this paper we prove some new identities for multiple zeta values and multiple zeta star values of arbitrary depth by using the methods of integral computations of logarithm function and iterated integral representations of series. By applying the formulas obtained, we prove that the multiple zeta star values whose indices are the sequences $(\bar{1}, \{1\}_m, \bar{1})$ and $(2, \{1\}_m, \bar{1})$ can be expressed polynomially in terms of zeta values, polylogarithms and $\ln(2)$. We also evaluate several restricted sums involving multiple zeta values.

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1. Introduction

Let $s := (s_1, \ldots, s_k)$ be a vector whose components are non-zero integers. We call $l(s) := k$ the depth and $|s| := \sum_{j=1}^{k} |s_j|$ the weight of $s$. If $s_j$ is negative,
we usually denote it by $-s_j$. The multiple zeta value and the multiple zeta star value \([2,4,5,20,25]\) associated to the vector \(s\) are defined respectively by

$$\zeta(s) \equiv \zeta(s_1, \ldots, s_k) := \sum_{n_1 > \cdots > n_k > 0} \prod_{j=1}^k n_j^{-|s_j|} \text{sgn}(s_j)^{n_j}$$ \hspace{1cm} (1.1)

and

$$\zeta^*(s) \equiv \zeta^*(s_1, \ldots, s_k) := \sum_{n_1 \geq \cdots \geq n_k \geq 1} \prod_{j=1}^k n_j^{-|s_j|} \text{sgn}(s_j)^{n_j}.$$ \hspace{1cm} (1.2)

Here for a non-zero integer \(s\),

$$\text{sgn}(s) := \begin{cases} 1, & s > 0, \\ -1, & s < 0. \end{cases}$$

If \(s_1\) is positive, the condition \(s_1 > 1\) is necessary to ensure the convergence of both sums (1.1) and (1.2). For convenience we set \(\zeta(\emptyset) = \zeta^*(\emptyset) = 1\).

The study of the multiple zeta (star) values went back to the correspondence of Euler with Goldbach in 1742–1743 (see [15]). Euler studied double zeta values and established some important formulas. For example, he proved that

$$2\zeta^*(k, 1) = (k + 2) \zeta(k + 1) - \sum_{i=1}^{k-2} \zeta(k - i) \zeta(i + 1), \quad k \geq 2,$$ \hspace{1cm} (1.3)

which, in particular, implies the simplest but nontrivial relation: \(\zeta^*(2, 1) = 2\zeta(3)\) or equivalently \(\zeta(2, 1) = \zeta(3)\). Moreover, he conjectured that the double zeta values would be reducible whenever the weight is odd. The conjecture was first proved by Borweins and Girgensohn [3]. For some interesting results on generalized double zeta values (also called Euler sums), see [1,12].

The systematic study of multiple zeta values began in the early 1990s with the works of Hoffman [13], Zagier [26] and Borwein-Bradley-Broadhurst [2] and has continued with increasing attention in recent years (see \([7,8,10,11]\)). The first systematic study of reductions up to depth 3 was carried out by Borwein and Girgensohn [6], where the authors proved that if \(p + q + r\) is even or less than or equal to 10 or \(p + q + r = 12\), the triple zeta value \(\zeta(q,p,r)\) (or \(\zeta^*(q,p,r)\)) can be expressed as a rational linear combination of products of zeta values and double zeta values. Additionally, it has been discovered in the course of the years that many multiple zeta (star) values admit expressions involving not only zeta values but also the special values of the polylogarithm function

$$\text{Li}_s(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^s} \left(\Re(s) \geq 1, \ x \in [-1,1]\right)$$

at positive integer arguments and \(x = 1/2\). See \([2,4,5,16,17,20–23,25,27]\). For example, Zagier [27] proved that the multiple zeta star values \(\zeta^*(\{2\}^a, 3, \{2\}^b)\)
and the multiple zeta values $\zeta(\{2\}_a, 3, \{2\}_b)$, where $a, b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$, are reducible to polynomials of zeta values, and gave explicit formulae. Hessami Pilehrood et al. [21] and Li [17] provided new proofs of Zagier’s formulas, respectively. Here and below, we denote by $\{s_1, \ldots, s_j\}_d$ the composition formed by repeating $(s_1, \ldots, s_j)$ $d$ times.

Finally, we mention that the set of the multiple zeta values has a rich algebraic structure given by the shuffle and the stuffle relations [13,18,27].

In this paper, we establish some new family of identities for multiple integral representation of series. Then, we apply it to obtain a family of identities relating multiple zeta (star) values to zeta values and polylogarithms. Specially, we present some new recurrence relations for multiple zeta star values whose indices are the sequences $(\bar{1}, \{1\}_m, \bar{1}), (2, \{1\}_m, \bar{1})$, and prove that the multiple zeta values $\zeta(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_k)$ can be expressed as a rational linear combination of zeta values, polylogarithms and $\ln(2)$. Moreover, we also evaluate several restricted sum formulas involving multiple zeta values.

2. Evaluation of $\zeta^*(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta^*(2, \{1\}_m, \bar{1})$

In this section, we will show that the multiple zeta star values $\zeta^*(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta^*(2, \{1\}_m, \bar{1})$ satisfy certain recurrence relations that allow us to write them in terms of zeta values, polylogarithms and $\ln(2)$.

2.1. Four Lemmas and Proofs

Now, we need four lemmas which will be useful in proving Theorems 2.4 and 2.5.

**Lemma 2.1** ([24]). For $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $x \in [-1, 1)$, the following relation holds:

$$
\int_0^x t^{n-1} \ln^m (1-t) dt = \frac{1}{n} (x^n - 1) \ln^m (1-x) + m! \left( \frac{-1}{n} \right)^m \sum_{1 \leq k_m \leq \ldots \leq k_1 \leq n} \frac{1}{k_1 \cdots k_m} - \frac{1}{n} \sum_{i=1}^m (-1)^{i-1} i! \binom{m}{i} \ln^{m-i} (1-x) \sum_{1 \leq k_i \leq \ldots \leq k_1 \leq n} \frac{x^{k_i} - 1}{k_1 \cdots k_1}.
$$

(2.1)

*Proof.* We prove by induction on $m$. Denote the left-hand side of (2.1) by $J(n, m; x)$. For $m = 0$, since

$$
J(n, 0; x) = \int_0^x t^{n-1} dt = \frac{x^n}{n},
$$

we get the result. For $m \geq 1$, using integration by parts, we have the following recurrence relation.
\[J(n, m; x) = \frac{1}{n} (x^n - 1) \ln^m (1 - x) - \frac{m}{n} \sum_{k=1}^{n} J(k, m - 1; x). \quad (2.2)\]

Then the inductive hypothesis implies that the right-hand side of (2.2) is equal to

\[
\begin{align*}
&\frac{1}{n} (x^n - 1) \ln^m (1 - x) + m! \left( \frac{(-1)^m}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{1 \leq k_{m-1} \leq \ldots \leq k_1 \leq j} \frac{1}{k_1 \cdots k_{m-1}} \right) \ln^{m-1}(1-x) \sum_{j=1}^{n} \frac{1}{j} \sum_{1 \leq k_i \leq \ldots \leq k_1 \leq j} \frac{x^{k_i} - 1}{k_1 \cdots k_i} \\
&- \frac{m}{n} \ln^{m-1}(1-x) \sum_{j=1}^{n} \frac{x^j - 1}{j},
\end{align*}
\]

(2.3)

from which we deduce the desired result. This completes the proof of Lemma 2.1. \(□\)

**Lemma 2.2.** For integers \(m \geq 0, n \geq 0\) and \(x \in (0, 1)\), the following integral identity holds:

\[
\int_{0}^{x} t^n (\ln t)^m dt = \sum_{l=0}^{m} \frac{m!}{l!} \left( \frac{(-1)^l}{n+1} \right)^{l+1} x^{n+1} \ln^{m-l}(x). \quad (2.4)
\]

**Proof.** Using the formula

\[
\int_{0}^{x} t^n \ln^m (t) dt = \frac{x^{n+1}}{n+1} \ln^m (x) - \frac{m}{n+1} \int_{0}^{x} t^n \ln^{m-1}(t) dt,
\]

one may get the result by induction on \(m\). \(□\)

**Lemma 2.3** ([23]). For a positive integer \(m\), the following identity holds:

\[
\int_{0}^{1} \frac{\ln^m (1+t) \ln (1-t)}{1+t} dt = \frac{1}{m+1} \ln^{m+2}(2) - \zeta(2) \ln^m(2) \\
- \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \left\{ \sum_{l=1}^{k-1} \frac{l!}{l} \left( \frac{1}{k} \right)^{l} \left( \ln^{m-l}(2) \right) \text{Li}_{l+2} \left( \frac{1}{2} \right) \right\}.
\]

(2.5)
Proof. Using the variable substitution $1 + t = 2x$, we have
\[
\int_0^1 \ln^m (1 + t) \ln (1 - t) \frac{dt}{1 + t} = \int_{1/2}^1 \ln^m (2x) \ln (2 - 2x) \frac{dx}{x}
\]
\[
= \sum_{k=0}^m \binom{m}{k} \ln^{m-k+1} (2) \int_{1/2}^1 \ln^k (x) \frac{dx}{x}
\]
\[
+ \sum_{k=0}^m \binom{m}{k} \ln^{m-k} (2) \int_{1/2}^1 \ln^k (x) \ln (1 - x) \frac{dx}{x}.
\]
Then by the power series expansion of the logarithm function
\[
\ln (1 - x) = -\sum_{n=1}^\infty \frac{x^n}{n}, \quad x \in [-1, 1)
\]
and (2.4), we easily obtain (2.5).
\[
\square
\]
Lemma 2.4 ([23]). For any integer $m \geq 1$ and any real $x \in [0, 1]$, the following identity holds:
\[
\int_0^x \ln^m (1 + t) \frac{dt}{t} = \frac{1}{m+1} \ln^{m+1} (1 + x) + m! \left( \zeta (m + 1) - \operatorname{Li}_{m+1} \left( \frac{1}{1+x} \right) \right)
\]
\[
- m! \sum_{j=1}^m \frac{\ln^{m-j+1} (1 + x)}{(m-j+1)!} \operatorname{Li}_j \left( \frac{1}{1+x} \right).
\] (2.6)

Proof. Applying the change of variable $t \to w - 1$, we have
\[
\int_0^x \ln^m (1 + t) \frac{dt}{t} \underset{w = 1 + t}{=} \int_0^{1+x} \ln^m (w) \frac{dw}{w} \underset{u = w^{-1}}{=} \int_1^{(1+x)^{-1}} \ln^m (u) \frac{du}{u} \left( 1 + x \right)^{-1}
\]
\[
= (-1)^{m+1} \left\{ \int_1^{(1+x)^{-1}} \frac{\ln^m (u)}{u} \frac{du}{1-u} + \int_1^{(1+x)^{-1}} \frac{\ln^m (u)}{1-u} \frac{du}{u} \right\}
\]
\[
= \frac{1}{m+1} \ln^{m+1} (1 + x) + (-1)^{m+1} \int_1^{(1+x)^{-1}} \frac{\ln^m (u)}{1-u} \frac{du}{u}.
\]
Then with the help of (2.4) we deduce (2.6).
\[
\square
2.2. Two Theorems and Proofs

Now, we state our main results.

**Theorem 2.5.** For any positive integer \( m \), we have the recurrence relation

\[
\zeta^* (\bar{1}, \{1\}_m, \bar{1}) = \frac{(-1)^m}{m!} \zeta(2) \ln^m(2)
\]

\[
- \frac{(-1)^m}{(m+1)!} \sum_{i=1}^{m} (-1)^{i+1} \binom{m+1}{i} (\ln 2)^{m+1-i} \left\{ \zeta^* (\bar{1}, \{1\}_{i-1}, \bar{1}) - \zeta^* (\bar{1}, \{1\}) \right\}
\]

\[
+ \frac{(-1)^m}{m!} \sum_{k=1}^{m} \frac{m}{k} (-1)^{k+1} \left\{ \sum_{l=1}^{k} \binom{k}{l} (\ln 2)^{m-l} \text{Li}_{l+2} \left( \frac{1}{2} \right) \right\}
\]

\[
- k! (\ln 2)^{m-k} \zeta (k+2)
\]

(2.7)

where

\[
\zeta^* (\bar{1}, \bar{1}) = \frac{\zeta(2) + \ln^2(2)}{2}.
\]

**Proof.** Multiplying (2.1) by \((-1)^{n-1}\) and summing with respect to \( n \), we get

\[
\int_0^x \frac{\ln^m (1-t)}{1+t} dt = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^x t^{n-1} \ln^m (1-t) dt
\]

\[
= \ln^m (1-x) \sum_{n=1}^{\infty} \frac{x^n - 1}{n} (-1)^{n-1}
\]

\[
+ m! (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \frac{x^{k_m}}{k_1 \cdots k_m}
\]

\[
- \sum_{i=1}^{m-1} (-1)^{i-1} \binom{m}{i} \ln^{m-i} (1-x)
\]

\[
+ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{k_1} - 1}{k_1 \cdots k_i}.
\]

(2.8)

On the other hand, by integration by parts, we obtain

\[
\lim_{x \to -1} \left\{ \int_0^x \frac{\ln^m (1-t)}{1+t} dt - \ln^m (1-x) \ln (1+x) \right\}
\]

\[
= \lim_{x \to -1} \left\{ m \int_0^x \frac{\ln^{m-1} (1-t) \ln (1+t)}{1-t} dt \right\}
\]
\[= m \int_{0}^{1} \frac{\ln^{m-1}(1-t) \ln(1+t)}{1-t} \, dt \]
\[= -m \int_{0}^{1} \frac{\ln^{m-1}(1+t) \ln(1-t)}{1+t} \, dt.\]

Hence, replacing \(m - 1\) by \(m\), letting \(x\) approach to \(-1\) in (2.8), and combining it with (2.5), we deduce (2.7).

Noting that, taking \(x \to 1\) in (2.8), we have
\[\int_{0}^{1} \frac{\ln^{m}(1-t)}{1+t} \, dt = m!(-1)^{m-1} \zeta^{*}(\bar{1}, \{1\}_m).\]

On the other hand, applying the change of variable \(t \to 1-x\) to the integral above and using (2.4), we obtain
\[\int_{0}^{1} \frac{\ln^{m}(1-t)}{1+t} \, dt = (-1)^{m}m!\text{Li}_{m+1}\left(\frac{1}{2}\right).\]

Therefore, we conclude that
\[\zeta^{*}(\bar{1}, \{1\}_m) = -\text{Li}_{m+1}\left(\frac{1}{2}\right), \quad m \in \mathbb{N}_0. \quad (2.9)\]

**Theorem 2.6.** For any positive integer \(m\), we have the recurrence relation
\[\zeta^{*}(2, \{1\}_m, \bar{1}) = \frac{m+2}{(m+3)!} (-1)^{m}\ln^{m+3}(2) + (m+2)(-1)^{m} \left( \zeta(m+3) - \text{Li}_{m+3}\left(\frac{1}{2}\right) \right) \]
\[+ (m+2)(-1)^{m} \sum_{j=1}^{m+2} \ln^{m+3-j}(2) (m+3-j)! \text{Li}_{j}\left(\frac{1}{2}\right) - \frac{3}{2} \frac{(-1)^{m}}{m+1)!}\zeta(2)\ln^{m+1}(2) \]
\[+ \frac{(-1)^{m}}{(m+1)!}\sum_{i=1}^{m} (-1)^{i-1} i! \left(\frac{m+1}{i}\right) (\ln^{m+1-i}(2)) \]
\[\times \left\{ \zeta^{*}(2, \{1\}_{i-1}, \bar{1}) - \zeta^{*}(2, \{1\}_i) \right\}, \quad (2.10)\]

where
\[\zeta^{*}(2, \bar{1}) = \frac{1}{4}\zeta(3) - \frac{3}{2} \zeta(2) \ln(2).\]

**Proof.** Similarly as in the proof of Theorem 2.5, we consider the integral
\[\int_{0}^{1} \frac{\ln^{m+1}(1-t)}{t} \, dt = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{1} t^{n-1}\ln^{m}(1-t) \, dt. \quad (2.11)\]
Putting $x \rightarrow -1$ in (2.1), we deduce that

$$
\int_{0}^{1} t^{n-1} \ln^{m} (1 - t) dt
= \frac{1}{n} (\ln^{m}(2)) ((-1)^{n} - 1) + m! \left( \frac{-1}{n} \right)^{m} \zeta_{n}^{*} (\{1\}_{m-1}, \emptyset)
- \frac{1}{n} \sum_{i=1}^{m-1} (-1)^{i-1} i! \left( \frac{m}{i} \right) (\ln^{m-i}(2)) \{ \zeta_{n}^{*} (\{1\}_{i-1}, \emptyset) - \zeta_{n}^{*} (\{1\}_{i}) \} \right). (2.12)
$$

Setting $x \rightarrow 1$ in (2.6) and combining (2.11) with (2.12), we obtain the desired result (2.10). □

By considering the case of $m = 2$ in (2.7) and (2.10) we get

$$
\zeta^{*} (1, 1, \emptyset) = \frac{1}{8} (3) + \frac{1}{2} (2) \ln(2) - \frac{1}{6} \ln^{3}(2),
$$

$$
\zeta^{*} (2, 1, \emptyset) = \frac{1}{8} \ln^{2}(2) + 3 \text{Li}_{4} \left( \frac{1}{2} \right) - 3 (4) - \frac{3}{2} \zeta (2) \ln^{2}(2) + \frac{7}{8} \zeta (3) \ln(2).
$$

From [24] we have

$$
\zeta^{*} (2, \{1\}_{m}) = (m + 1) \zeta (m + 2), \quad m \in \mathbb{N}_{0}.
$$

Therefore, from Theorems 2.5 and 2.6, we know that the multiple zeta star values $\zeta^{*} (\emptyset, \{1\}_{m}, \emptyset)$ and $\zeta^{*} (2, \{1\}_{m}, \emptyset)$ can be expressed as a rational linear combination of zeta values, polylogarithms and $\ln(2)$.

### 3. Some Results on Multiple Zeta Values

In this section, we use certain multiple integral representations to evaluate several multiple zeta values.

We first introduce some notations. For positive integers $n, k$ and non-zero integers $s_{1}, \ldots, s_{k}$, the multiple harmonic number $\zeta_{n} (s_{1}, \ldots, s_{k})$ and the multiple harmonic star number $\zeta_{n}^{*} (s_{1}, \ldots, s_{k})$ are defined by

$$
\zeta_{n} (s_{1}, s_{2}, \ldots, s_{k}) := \sum_{n \geq n_{1} > n_{2} > \cdots > n_{k} \geq 1} \prod_{j=1}^{k} n_{j}^{-|s_{j}|} \text{sgn}(s_{j})^{n_{j}}, (3.1)
$$

and

$$
\zeta_{n}^{*} (s_{1}, s_{2}, \ldots, s_{k}) := \sum_{n \geq n_{1} \geq n_{2} \geq \cdots \geq n_{k} \geq 1} \prod_{j=1}^{k} n_{j}^{-|s_{j}|} \text{sgn}(s_{j})^{n_{j}}, (3.2)
$$

respectively. We also set $\zeta_{n} (\emptyset) = \zeta_{n}^{*} (\emptyset) = 1$.

For positive integers $k, s_{1}, \ldots, s_{k}$, we define the multiple polylogarithm function by
\[ \text{Li}_{s_1, s_2, \ldots, s_k} (x) := \sum_{n_1 > \cdots > n_k > 0} \frac{x^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}, \quad (3.3) \]

where \( x \) is a real number satisfying \( 0 \leq x < 1 \). If \( s_1 > 1 \), \( x \) can be 1.

For integers \( n \) and \( k \), let \( s(n, k) \) denote the (unsigned) Stirling number of the first kind (see [9]). It is known that the Stirling numbers of the first kind satisfy a recurrence relation in the form

\[ s(n, k) = s(n - 1, k - 1) + (n - 1) s(n - 1, k), \quad n, k \in \mathbb{N}, \]

with \( s(n, k) = 0, n < k, s(n, 0) = s(0, k) = 0, s(0, 0) = 1 \).

We need the following lemma.

**Lemma 3.1** ([23, 24]). For any integer \( k > 0 \) and any real \( x \in [-1, 1) \), we have

\[ \ln^k (1 - x) = (-1)^k k! \sum_{n=1}^{\infty} \frac{x^n}{n^n} \zeta_{n-1} (\{1\}_{k-1}), \quad (3.4) \]

and

\[ s(n, k) = (n - 1)! \zeta_{n-1} (\{1\}_{k-1}). \quad (3.5) \]

**Proof.** The proof is based on the two identities

\[ \ln^{k+1} (1 - x) = - (k + 1) \int_0^x \frac{\ln^k (1 - t)}{1 - t} \, dt, \quad k \in \mathbb{N}_0 \quad (3.6) \]

and

\[ \ln^k (1 - x) = (-1)^k k! \sum_{n=k}^{\infty} \frac{s(n, k)}{n!} x^n, \quad -1 \leq x < 1. \quad (3.7) \]

By induction and the Cauchy product formula, we have

\[ \ln^{k+1} (1 - x) = - (k + 1) \int_0^x \frac{\ln^k (1 - t)}{1 - t} \, dt \]

\[ = (-1)^{k+1} (k + 1)! \sum_{n=1}^{\infty} \frac{1}{n + 1} \sum_{i=1}^{n} \frac{\zeta_{i-1} (\{1\}_{k-1})}{i} x^{n+1} \]

\[ = (-1)^{k+1} (k + 1)! \sum_{n=1}^{\infty} \frac{\zeta_n (\{1\}_k)}{n + 1} x^{n+1}. \]

Since \( \zeta_n (\{1\}_k) = 0 \) for \( n < k \), we deduce (3.4). Then, comparing the coefficients of \( x^n \) in (3.4) and (3.7), we obtain (3.5).

It is clear that from (3.4), we deduce

\[ \frac{\ln^k (1 + x)}{1 - x} = (-1)^k k! \sum_{n=1}^{\infty} \zeta_{n-1} (\{1\}_{k-1}) x^{n-1}, \quad x \in (-1, 1), \quad (3.8) \]
and
\[
\frac{\ln^k (1 - x)}{1 + x} = (-1)^k k! \sum_{n=1}^{\infty} (-1)^{n-1} \zeta_{n-1} (\bar{1}, \{1\}_{k-1}) \frac{x^{-1}}{1 + x}, \quad x \in (-1, 1).
\]  

(3.9)

Then the main results are the following theorems and corollary.

**Theorem 3.2.** For integers \( m, k \in \mathbb{N}_0 \), we have
\[
\zeta (\bar{1}, \{1\}_m, \bar{1}, \{1\}_k) = (-1)^{m+1} \operatorname{Li}_{k+2,\{1\}_m} \left( \frac{1}{2} \right).
\]  

(3.10)

**Proof.** To prove (3.10), we consider the following multiple integral
\[
M_m (k) := \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \int_0^{t_m} \frac{\ln^{k+1} (1 - t_{m+1})}{1 + t_{m+1}} dt_{m+1}.
\]  

(3.11)

Using (3.9) and the power series expansion
\[
\frac{1}{1 + x} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}, \quad x \in (-1, 1)
\]
we find that
\[
\frac{\ln^{k+1} (1 - t_{m+1})}{(1 + t_1) \cdots (1 + t_m) (1 + t_{m+1})} = (-1)^{k+m} (k + 1)!
\sum_{n_1, n_2, \ldots, n_{m+1} = 1}^{\infty} (-1)^{n_1+n_2+\cdots+n_{m+1}} \zeta_{n_1-1} (\bar{1}, \{1\}_k) t_1^{n_{m+1}-1} t_2^{n_m-1} \cdots t_{m+1}^{n_1-1}.
\]  

(3.12)

Applying \( \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \frac{\ln^{k+1} (1 - t_{m+1})}{1 + t_{m+1}} dt_{m+1} \) to (3.12), we obtain
\[
M_m (k) = (-1)^{k+m} (k + 1)! \sum_{n_1, n_2, \ldots, n_{m+1} = 1}^{\infty} (-1)^{n_1+n_2+\cdots+n_{m+1}} \zeta_{n_1-1} (\bar{1}, \{1\}_k) \frac{t_1^{n_{m+1}-1} t_2^{n_m-1} \cdots t_{m+1}^{n_1-1}}{n_1}
\times \int_0^1 t_1^{n_{m+1}-1} dt_1 \cdots \int_0^{t_{m-1}} t_1^{n_2-1} dt_2
\]
\[
= (-1)^{k+m} (k + 1)!
\sum_{n_1, n_2, \ldots, n_{m+1} = 1}^{\infty} (-1)^{n_1+n_2+\cdots+n_{m+1}} \frac{\zeta_{n_1-1} (\bar{1}, \{1\}_k)}{n_1 (n_1 + n_2) \cdots (n_1 + \cdots + n_{m+1})}
\]
\[
(-1)^{k+m} (k+1)! \sum_{n_1 > \cdots > n_{m+1} \geq 1} \zeta_{m+1-1}(\{\bar{1}\}, \{1\}_k) (-1)^{n_1} \n_{n_2 \cdots n_{m+1}} (k+1) \\
= (-1)^{k+m} (k+1)! \zeta(\{\bar{1}\}, \{1\}_m, \{1\}_k). 
\]

(3.13)

On the other hand, applying the change of variables 

\[
t_i \mapsto 1 - t_{m+2-i}, \quad i = 1, 2, \ldots, m+1
\]

to \(M_m(k)\), and noting the fact that

\[
\frac{1}{(2-t_1) \cdots (2-t_m)(2-t_{m+1})} = \sum_{n_1, n_2, \ldots, n_{m+1} = 1}^{\infty} \frac{t_1^{n_{m+1}-1} t_2^{n_m-1} \cdots t_{m+1}^{n_1-1}}{2^{n_1+n_2+\cdots+n_{m+1}}},
\]

we have

\[
M_m(k) = \int_0^1 \ln^{k+1}(t_1) \frac{dt_1}{2-t_1} \int_0^{t_1} \frac{dt_2}{2-t_2} \cdots \int_0^{t_{m-1}} \frac{dt_m}{2-t_m} \int_0^{t_{m-1}} \frac{dt_{m+1}}{2-t_{m+1}} \\
= \sum_{n_1, \ldots, n_{m+1} = 1}^{\infty} \frac{1}{2^{n_1+\cdots+n_{m+1}}} \int_0^{t_{m+1}} \frac{dt_{m+1}}{2^{n_1+\cdots+n_{m+1}}} \\
= \sum_{n_1, \ldots, n_{m+1} = 1}^{\infty} \frac{1}{2^{n_1+\cdots+n_{m+1}}} \\
= \sum_{n_1, \ldots, n_{m+1} = 1}^{\infty} \frac{1}{2^{n_1+\cdots+n_{m+1}}} \\
= \left( k + 1 \right)! \left( -1 \right)^{k+1} \sum_{n_1, \ldots, n_{m+1} = 1}^{\infty} \frac{1}{2^{n_1+\cdots+n_{m+1}}} \\
= \left( k + 1 \right)! \left( -1 \right)^{k+1} \sum_{1 \leq n_{m+1} < \cdots < n_1}^{\infty} \frac{1}{n_{m+1} \cdots n_1} \frac{k+2}{2^{n_1}} \\
= \left( k + 1 \right)! \left( -1 \right)^{k+1} \text{Li}_{k+2, \{1\}_m} \left( \frac{1}{2} \right). 
\]

(3.14)

Therefore, combining (3.13) and (3.14) we deduce the desired result. \(\square\)
Theorem 3.3. For integers $m, k \in \mathbb{N}_0$, we have
\[
\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_k) = \frac{(-1)^{m+k+1}}{k!} \sum_{j=0}^{k} (-1)^j (\ln 2)^{k-j} \binom{k}{j} \times \left\{ \zeta(m+2, \{1\}_j) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m+1-l)!} \text{Li}_{l+1,\{1\}_j} \left( \frac{1}{2} \right) \right\}. \tag{3.15}
\]

Proof. Using (3.4), we find that
\[
\int_{0}^{1} \frac{(\ln x)^k \ln^{m+1} \left( 1 - \frac{x}{2} \right) dx}{x} = (-1)^{m+1} (m+1)! \sum_{n=1}^{\infty} \zeta_{n-1} \frac{\{1\}_m}{n^{2n}} \int_{0}^{1} x^{n-1} \ln^k(x) dx
\]
\[
= (-1)^{m+k+1} (m+1)!k! \sum_{n=1}^{\infty} \zeta_{n-1} \frac{\{1\}_m}{n^{k+2} 2^n}
\]
\[
= (-1)^{m+k+1} (m+1)!k! \text{Li}_{k+2,\{1\}_m} \left( \frac{1}{2} \right). \tag{3.16}
\]
Then it is readily that
\[
\text{Li}_{k+2,\{1\}_m} \left( \frac{1}{2} \right) = (-1)^{m+k+1} (m+1)! \frac{1}{k!} \int_{0}^{1} \frac{(\ln x)^k \ln^{m+1} \left( 1 - \frac{x}{2} \right) dx}{x}, \tag{3.17}
\]
\[
\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_k) = (-1)^k \frac{1}{(m+1)!k!} \int_{0}^{1} \frac{(\ln x)^k \ln^{m+1} \left( 1 - \frac{x}{2} \right) dx}{x}. \tag{3.18}
\]
On the other hand, setting $x = 2(1 - u)$ in the above integral, we conclude that
\[
\int_{0}^{1} \frac{(\ln x)^k \ln^{m+1} \left( 1 - \frac{x}{2} \right) dx}{x}
\]
\[
= \int_{\frac{1}{2}}^{1} \frac{(\ln 2 + \ln (1 - u))^k \ln^{m+1}(u) du}{1 - u}
\]
\[
= \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{2} \right)^j \ln^j (1 - u) \ln^{m+1}(u) du
\]
\[
= \sum_{j=0}^{k} (-1)^{j} (\ln 2)^{k-j} j! \left( \sum_{n=1}^{\infty} \zeta_{n-1} \{1\} \right) \frac{1}{n} \int_{1/2}^{1} u^{n-1} \ln^{m+1}(u) du
\]
\[
= (m + 1)!(-1)^{m+1} \sum_{j=0}^{k} (-1)^{j} (\ln 2)^{k-j} j! \binom{k}{j}
\times \left\{ \zeta(m + 2, \{1\}) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m + 1 - l)!} \text{Li}_{l+1,\{1}\} \left( \frac{1}{2} \right) \right\}.
\] (3.19)

Thus, substituting (3.19) into (3.18), we obtain the desired result. □

From Theorems 3.2 and 3.3, we immediately derive the following special case of the multiple zeta values.

**Corollary 3.4** (Conjectured in [2]). For any integer \( m \in \mathbb{N}_{0} \), the following identities hold:

\[
\text{Li}_{2,\{1\}}(\frac{1}{2}) = \zeta(m + 2) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m + 1 - l)!} \text{Li}_{l+1,\{1\}} \left( \frac{1}{2} \right)
\] (3.20)

\[
\zeta(\overline{1}, \{1\}_m, \overline{1}) = (-1)^{m+1} \left\{ \zeta(m + 2) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m + 1 - l)!} \text{Li}_{l+1,\{1\}} \left( \frac{1}{2} \right) \right\}.
\] (3.21)

**Proof.** Setting \( k = 0 \) in (3.10) and (3.15) we easily deduce the results. □

Note that Corollary 3.4 is an immediate corollary of Zlobin’s result (see [28, Theorem 9]).

**Theorem 3.5.** For integers \( m, k \in \mathbb{N}_{0} \), we have

\[
\zeta(\overline{1}, \{1\}_m, \overline{1}, \{1\}_k) = \frac{(-1)^{k}}{(m + 1)!(k + 1)!} \left\{ (k + 1)(\ln 2)^{m+1} I(k)
- (m + k + 2) I(m + k + 1)
- \sum_{i=1}^{m} \frac{\ln 2}{i!} \zeta(\overline{1}, \{1\}_{m-i}, \overline{1}, \{1\}_k) \right\},
\] (3.22)

where \( I(k) \) denotes the left hand side of (2.5), namely

\[
I(k) := \int_{0}^{1} \ln^{k} (1 + x) \ln (1 - x) \frac{dx}{1 + x} \quad (k \in \mathbb{N}),
\]

with

\[
I(0) = \frac{\ln^{2}(2) - \zeta(2)}{2}.
\]
Proof. By a similar argument as in the proof of Theorem 3.2, we consider the following multiple integral

\[
J_m(k) := \int_0^1 \frac{1}{1 + t_1} dt_1 \cdots \int_0^{t_m} \frac{1}{1 + t_{m-1}} dt_{m-1} \\
\int_0^{t_m} \frac{1}{1 + t_m} dt_m \int_0^{t_m} \frac{\ln^k (1 + t_{m+1})}{1 - t_{m+1}} dt_{m+1}.
\]  
(3.23)

Using (3.8), we deduce that

\[
\int_0^x \frac{1}{1 + t} dt \int_0^t \frac{\ln (1 + u)}{1 - u} du = (-1)^{k-1} k! \sum_{n=1}^\infty \frac{\zeta_{n-1} (\{1\}, \{1\})}{n} (-1)^n x^n.
\]  
(3.24)

Hence, combining (3.23) and (3.24), we easily get

\[
J_m(k) = (-1)^{m+k} k! \zeta (\{1\}, \{1\}) = \frac{\ln 2}{J_{m-1}(k)} - \sum_{i=1}^{m-1} (-1)^i - \frac{\ln 2}{i!} J_{m-i}(k) + \frac{(-1)^{m-1}}{(m-1)!} \int_0^{t_m} \frac{\ln^{m-1} (1 + t_1)}{1 - t_1} dt_1 \int_0^{t_1} \frac{\ln (1 + t_2)}{1 - t_2} dt_2.
\]  
(3.25)
Therefore, substituting (3.25) into (3.26), we have

\[
J_m(k) = (-1)^{m+k-1-k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, \bar{1}, \bar{1}, \{1\}_{k-1})} 
+ \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{\ln^{m-1}(1+t_1)}{1+t_1} \left( \int_0^{t_1} \frac{\ln (1+t_2)}{1-t_2} dt_2 \right) dt_1.
\]

Moreover, we note that the integral in the right-hand side of (3.27) can be written as

\[
\int_0^1 \frac{\ln^{m-1}(1+t_1)}{1+t_1} \left( \int_0^{t_1} \frac{\ln (1+t_2)}{1-t_2} dt_2 \right) dt_1 = \lim_{x \to 1} \left\{ \int_0^x \frac{\ln^{m-1}(1+t_1)}{1+t_1} \left( \int_0^{t_1} \frac{\ln (1+t_2)}{1-t_2} dt_2 \right) dt_1 \right\} 
= \frac{1}{m} \lim_{x \to 1} \left\{ \int_0^x \frac{\ln^m(1+x) \ln (1+t)}{1-t} dt \right\} 
= \frac{1}{m} \left\{ k(\ln 2)^m \int_0^1 \frac{\ln^{k-1}(1+t) \ln (1-t)}{1+t} dt 
- (m+k) \int_0^1 \frac{\ln^{m+k-1}(1+t) \ln (1-t)}{1+t} dt \right\}.
\]

Therefore, (3.25), (3.27) and (3.28) yield the desired result. \qed

From Lemma 2.3 and Theorem 3.5, we have the conclusion: if \( m, k \in \mathbb{N}_0 \), then the multiple zeta values \( \zeta(\bar{1}, \{1\}_m, \bar{1}, \bar{1}, \{1\}_k) \) can be expressed as a rational linear combination of zeta values, polylogarithms and \( \ln(2) \).

We close this section with several examples.

**Example 3.1.** By (3.21) and (3.22), we have

\[
\zeta(\bar{1}, 1, \bar{1}) = \frac{1}{8} \zeta(3) - \frac{1}{6} \ln^3(2),
\]

\[
\zeta(\bar{1}, \bar{1}, 1) = -\frac{1}{4} \zeta(3) + \frac{1}{2} \zeta(2) \ln(2) - \frac{1}{6} \ln^3(2),
\]

\[
\zeta(\bar{1}, 1, 1, \bar{1}) = \text{Li}_4 \left( \frac{1}{2} \right) + \frac{1}{12} \ln^4(2) + \frac{7}{8} \zeta(3) \ln(2) - \frac{1}{2} \zeta(2) \ln^2(2) - \zeta(4),
\]
\[ \zeta(1, 1, 1, 1) = 3 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{1}{6} \ln^4(2) + \frac{23}{8} \zeta(3) \ln(2) - \zeta(2) \ln^2(2) - 3 \zeta(4), \]
\[ \zeta(1, 1, 1, 1) = -3 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{1}{12} \ln^4(2) - \frac{11}{4} \zeta(3) \ln(2) - \frac{3}{4} \zeta(2) \ln^2(2) + 3 \zeta(4). \]

4. Some Evaluation of Restricted Sum Involving Multiple Zeta Values

In [23], we considered the following restricted sum involving multiple zeta values

\[ (-1)^{m+1} \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta(1, \{1\}_{m-i}, 3, \{1\}_k) \]
\[ + (-1)^{k+1} \sum_{i=0}^{k} \frac{(\ln 2)^i}{i!} \zeta(1, \{1\}_{k-i}, 3, \{1\}_m), \]

and gave explicit expression in terms of multiple zeta values of depth less than \( \max\{k + 3, m + 3\} \), where \( m, k \in \mathbb{N}_0 \). Moreover, we also proved that the restricted sum

\[ \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta(1, \{1\}_{m-i}, 2, \{1\}_k) \]

can be expressed by zeta values and polylogarithms. Therefore, for any \( m, k \in \mathbb{N}_0 \), the multiple zeta values \( \zeta(1, \{1\}_m, 2, \{1\}_k) \) can be represented as a polynomial of zeta values and polylogarithms with rational coefficients. As an example, we have the following explicit formula for weight 4

\[ \zeta(1, 2) = 3 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{1}{8} \ln^4(2) + \frac{23}{8} \zeta(3) \ln(2) - \zeta(2) \ln^2(2) - 3 \zeta(4). \]

In this section, we consider the general restricted sum

\[ (-1)^{m+1} \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta(1, \{1\}_{m-i}, p + 3, \{1\}_k) \]
\[ + (-1)^{p+k+1} \sum_{i=0}^{k} \frac{(\ln 2)^i}{i!} \zeta(1, \{1\}_{k-i}, p + 3, \{1\}_m), \]

where \( m, k, p \in \mathbb{N}_0 \).

Now we are ready to state and prove our main results. Note that our proof of Theorem 4.1 is based on Lemma 3.1.
Theorem 4.1. For integers $m, k, p \in \mathbb{N}_0$, we have

$$(-1)^{m+1} \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta \left( \bar{1}, \{1\}_{m-i}, p + 3, \{1\}_k \right)$$

$$+ (-1)^{p+k+1} \sum_{i=0}^{k} \frac{(\ln 2)^i}{i!} \zeta \left( \bar{1}, \{1\}_{k-i}, p + 3, \{1\}_m \right)$$

$$= \frac{(-1)^m}{(m+1)!} (\ln 2)^{m+1} \zeta \left( \bar{1}, \{1\}_m, p + 3, \{1\}_k \right)$$

$$+ \frac{(-1)^{p+k}}{(k+1)!} (\ln 2)^{k+1} \zeta \left( \bar{1}, \{1\}_k, p + 3, \{1\}_m \right)$$

$$+ \sum_{i=0}^{p} (-1)^i \zeta \left( 2 + i, \{1\}_m \right) \zeta \left( p + 2 - i, \{1\}_k \right).$$

(4.1)

Proof. Similarly as in the proof of Theorem 3.5, we consider the multiple integral

$$R_{m,k}(p) := \int_{0<t_{m+p+2}<\cdots<t_1<1} \frac{\ln^k \left( 1 + t_{m+p+2} \right)}{(1 + t_1) \cdots (1 + t_m) t_{m+1} \cdots t_{m+p+1} t_{m+p+2}} dt_1 \cdots dt_{m+p+2}.$$

Then with the help of (3.4), we easily deduce that

$$R_{m,k}(p) = (-1)^{m+k} k! \zeta \left( \bar{1}, \{1\}_m, p + 3, \{1\}_{k-1} \right).$$

(4.2)

On the other hand, by a similar argument as in the proof of (3.26) (using integration by parts), we get that

$$(-1)^{m+k} k! \zeta \left( \bar{1}, \{1\}_m, p + 3, \{1\}_{k-1} \right)$$

$$= (-1)^{m+k+1} k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta \left( \bar{1}, \{1\}_{m-1-i}, p + 3, \{1\}_{k-1} \right)$$

$$+ \frac{(-1)^{m-1}}{(m-1)!} \int_{0}^{1} \frac{\ln^{m-1} \left( 1 + t_1 \right)}{1 + t_1} dt_1 \int_{0}^{t_1} \frac{1}{t_2} dt_2$$

$$\cdots \int_{0}^{t_{p+1}} \frac{1}{t_{p+2}} \int_{0}^{t_{p+2}} \frac{\ln^k \left( 1 + t_{p+3} \right)}{t_{p+3}} dt_{p+3}$$

$$= (-1)^{m+k+1} k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta \left( \bar{1}, \{1\}_{m-1-i}, p + 3, \{1\}_{k-1} \right)$$
\[ + \frac{(-1)^{m+k-1}}{m!} k! (\ln 2)^m \zeta(p+3, \{1\}_{k-1}) \]
\[ + \frac{(-1)^m}{m!} \int_0^1 \frac{\ln^m (1+t_1)}{t_1} dt_1 \int_0^1 \frac{dt_2}{t_2} \cdots \int_0^1 \frac{dt_{p+1}}{t_{p+1}} \int_0^{t_{p+1}} \frac{\ln^k (1+t_{p+2})}{t_{p+2}} dt_{p+2} \]

\[ (4.3) \]

Hence, we have
\[ \int_0^1 \frac{\ln^m (1+t_1)}{t_1} dt_1 \int_0^1 \frac{dt_2}{t_2} \cdots \int_0^1 \frac{dt_{p+1}}{t_{p+1}} \int_0^{t_{p+1}} \frac{\ln^k (1+t_{p+2})}{t_{p+2}} dt_{p+2} \]
\[ = (-1)^k m! k! \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\overline{1}, \{1\}_{m-i}, p+3, \{1\}_{k-1}) \]
\[ + (-1)^k k! (\ln 2)^m \zeta(p+3, \{1\}_{k-1}) \]  

\[ (4.4) \]

Moreover, applying the same argument as above, we deduce that
\[ \int_0^1 \frac{\ln^m (1+t_1)}{t_1} dt_1 \int_0^1 \frac{dt_2}{t_2} \cdots \int_0^1 \frac{dt_{p+1}}{t_{p+1}} \int_0^{t_{p+1}} \frac{\ln^k (1+t_{p+2})}{t_{p+2}} dt_{p+2} \]
\[ + (-1)^p \int_0^1 \frac{\ln^k (1+t_1)}{t_1} dt_1 \int_0^1 \frac{dt_2}{t_2} \cdots \int_0^1 \frac{dt_{p+1}}{t_{p+1}} \int_0^{t_{p+1}} \frac{\ln^m (1+t_{p+2})}{t_{p+2}} dt_{p+2} \]
\[ = (-1)^{m+k} k! m! \sum_{i=0}^{p} (-1)^i \zeta(\overline{2}+i, \{1\}_{m-i}) \zeta(p+2-i, \{1\}_{k-1}) \]  

\[ (4.5) \]

Thus, combining (4.4) and (4.5), we obtain the desired result. \[ \square \]

Taking \( p = 0 \) in Theorem 4.1, we get the following corollary which was first proved in [23].

**Corollary 4.2.** For integers \( m, k \in \mathbb{N}_0 \), we have
\[ (-1)^{m+1} \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta(\overline{1}, \{1\}_{m-i}, 3, \{1\}_{k}) \]
\[ + (-1)^{k+1} \sum_{i=0}^{k} \frac{(\ln 2)^i}{i!} \zeta(\overline{1}, \{1\}_{k-i}, 3, \{1\}_{m}) \]
\[ = \frac{(-1)^m}{(m+1)!} (\ln 2)^{m+1} \zeta(3, \{1\}_{k}) + \frac{(-1)^k}{(k+1)!} (\ln 2)^{k+1} \zeta(3, \{1\}_{m}) \]
\[ + \zeta(\overline{2}, \{1\}_{m}) \zeta(\overline{2}, \{1\}_{k}) \]  

\[ (4.6) \]
Theorem 4.3. For integers $m, k, p \in \mathbb{N}_0$, the following identity holds:

$$
(-1)^{k+p}(m+1)!(k+1)! \sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta \left(1, \{1\}_{m-i}, 2, \{1\}_p, 2, \{1\}_k \right) \\
+ (-1)^{m+1}(k+1)!(m+1)! \sum_{i=0}^{k} \frac{(\ln 2)^i}{i!} \zeta \left(1, \{1\}_{k-i}, 2, \{1\}_p, 2, \{1\}_m \right) \\
+ (-1)^{k+p}(k+1)!(\ln 2)^{m+1} \zeta \left(2, \{1\}_p, 2, \{1\}_k \right) \\
+ (-1)^{m+1}(m+1)!(\ln 2)^{k+1} \zeta \left(2, \{1\}_p, 2, \{1\}_m \right) \\
= (-1)^{m+k+p+1}(m+1)!(k+1)! \zeta \left(2, \{1\}_m \right) \zeta \left(1, \{1\}_p, 2, \{1\}_k \right) \\
+ (-1)^{m+k}(k+1)!(m+1)! \zeta \left(2, \{1\}_k \right) \zeta \left(1, \{1\}_p, 2, \{1\}_m \right) \\
+ (-1)^{m+k+p+1}(k+1)!(m+1)! \\
\sum_{i=1}^{p} (-1)^i \zeta \left(1, \{1\}_{i-1}, 2, \{1\}_m \right) \zeta \left(1, \{1\}_{p-i}, 2, \{1\}_k \right). \quad (4.7)
$$

Proof. The proof of Theorem 4.3 is similar as the proof of Theorem 4.1. We consider the iterated integral

$$
Q_{m, k}(p) := \int_{0 < t_{m+p+2} < \cdots < t_1 < 1} \frac{\ln^{k} (1 + t_{m+p+2})}{(1 + t_1) \cdots (1 + t_m) (1 + t_{m+2}) \cdots (1 + t_{m+p+1})} dt_1 \cdots dt_{m+p+2}.
$$

By a similar argument as in the proof of formula (4.1), we obtain the following identities

$$
Q_{m, k}(p) = (-1)^{k+m+p+k!} \zeta \left(1, \{1\}_{m-1}, 2, \{1\}_{p-1}, 2, \{1\}_{k-1} \right), \quad (4.8)
$$

$$
\int_{0}^{1} \frac{\ln^{m} (1 + t_1)}{t_1} dt_1 \int_{0}^{1} \frac{1}{1 + t_2} dt_2 \cdots \int_{0}^{1} \frac{1}{1 + t_{p+1}} dt_{p+1} \int_{0}^{1} \frac{\ln^{k} (1 + t_{p+2})}{t_{p+2}} dt_{p+2} \\
= (-1)^{k+p+m!} \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta \left(1, \{1\}_{m-i-1}, 2, \{1\}_{p-1}, 2, \{1\}_{k-1} \right) \\
+ (-1)^{p+k} k! (\ln 2)^{m} \zeta \left(2, \{1\}_{p-1}, 2, \{1\}_{k-1} \right), \quad (4.9)
$$
and

\[ \int_0^1 \frac{\ln^m (1 + t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{1 + t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{1 + t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k (1 + t_{p+2})}{t_{p+2}} dt_{p+2} \]

\[ + (-1)^{p} \int_0^1 \frac{\ln^k (1 + t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{1 + t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{1 + t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^m (1 + t_{p+2})}{t_{p+2}} dt_{p+2} \]

\[ = (-1)^{m+k+p} m! k! \zeta \left( \bar{2}, \{1\}_{m-1} \right) \zeta \left( \bar{1}, \{1\}_{p-1}, 2, \{1\}_{k-1} \right) + (-1)^{m+k} k! m! \zeta \left( \bar{2}, \{1\}_{k-1} \right) \zeta \left( \bar{1}, \{1\}_{p-1}, 2, \{1\}_{m-1} \right) \]

\[ + (-1)^{m+k+p} k! m! \sum_{i=1}^{p-1} (-1)^i \zeta \left( \bar{1}, \{1\}_{i-1}, 2, \{1\}_{m-1} \right) \zeta \left( \bar{1}, \{1\}_{p-i-1}, 2, \{1\}_{k-1} \right). \]

(4.10)

Hence, combining (4.8)–(4.10), we easily deduce the desired result. \( \square \)

Setting \( p = 1, k = m = 0 \) in Theorem 4.3, we get

\[ \zeta \left( \bar{1}, 2, 1, 2 \right) + \zeta \left( \bar{2}, 1, 2 \right) \ln(2) + \zeta \left( \bar{2} \right) \zeta \left( \bar{1}, 1, 2 \right) = \frac{1}{2} \zeta^2 \left( \bar{1}, 2 \right). \]

We finally remark that proceeding in a similar method, it is possible to evaluate other multiple zeta values. For example, using our method, we obtain the following explicit integral representations and closed form representations of multiple zeta values:

\[ \zeta \left( \{ \bar{1} \}_{2p+2}, \{1\}_k \right) = (-1)^{p+1} \text{Li}_{k+2}, \{2\}_p \left( \frac{1}{2} \right) \]

(4.11)

\[ = \frac{(-1)^{k+p}}{(k+1)!} \int_{0 < t_{2p+1} < \cdots < t_1 < 1} \ln^{k+1} (1 - t_{2p+1}) \prod_{j=1}^{p} \{(1 + t_{2j-1})(1 - t_{2j})\}(1 + t_{2p+1}) dt_1 \cdots dt_{2p+1}, \]

\[ \zeta \left( \{ \bar{1} \}_{2p+1}, \{1\}_k \right) = \frac{(-1)^{k+p+1}}{(k+1)!} \int_{0 < t_{2p} < \cdots < t_1 < 1} \ln^{k+1} (1 + t_{2p}) \prod_{j=1}^{p} \{(1 + t_{2j-1})(1 - t_{2j})\} dt_1 \cdots dt_{2p}, \]

(4.12)
\[
\sum_{i=0}^{m} \frac{(\ln 2)^i}{i!} \zeta \left( \overline{1}, \{1\}_{m-i}, \{1\}_{2p}, \{1\}_k \right) \\
= \frac{(-1)^{k+p+1}}{(k+1)m!} \int_{0 < t_2p < \cdots < t_1 < 0} \ln^m (1 + t_1) \ln^{k+1} (1 + t_{2p}) \prod_{j=1}^{p} \left( (1 + t_{2j-1}) (1 - t_j) \right) dt_1 \cdots dt_{2p}, \quad (4.13)
\]

\[
\zeta \left( \overline{1}, \{1\}_m, \{1\}_{2p+1}, \{1\}_k \right) = (-1)^{m+p+1} \text{Li}_{k+2} \{2\}_{2p}, \{1\}_m \left( \frac{1}{2} \right). \quad (4.14)
\]

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