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THE ECCENTRIC KOZAI MECHANISM FOR A TEST PARTICLE

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ABSTRACT

We study the dynamical evolution of a test particle that orbits a star in the presence of an exterior massive planet, considering octupole-order secular interactions. In the standard Kozai mechanism (SKM), the planet’s orbit is circular and so the particle conserves vertical angular momentum. As a result, the particle’s orbit oscillates periodically, exchanging eccentricity for inclination. However, when the planet’s orbit is eccentric, the particle’s vertical angular momentum varies and its Kozai oscillations are modulated on longer timescales—we call this the eccentric Kozai mechanism (EKM). The EKM can lead to behavior that is dramatically different from the SKM. In particular, the particle’s orbit can flip from prograde to retrograde and back again, and it can reach arbitrarily high eccentricities given enough time. We map out the conditions under which this dramatic behavior (flipping and extreme eccentricities) occurs and show that when the planet’s eccentricity is sufficiently high, it occurs quite generically. For example, when the planet’s eccentricity exceeds a few percent of the ratio of semimajor axes (outer to inner), around half of randomly oriented test particle orbits will flip and reach extreme eccentricities. The SKM has often been invoked for bringing pairs of astronomical bodies (star–star, planet–star, compact–object pairs) close together. Including the effect of the EKM will enhance the rate at which such matchmaking occurs.

Key words: gravitation – planets and satellites: dynamical evolution and stability

Online-only material: color figures

1. INTRODUCTION

Naoz et al. (2011a) recently showed that in a system with two planets orbiting a star, the inner planet’s orbit can flip from prograde to retrograde and back, and can also reach extremely high eccentricities. Therefore, starting from a system with two prograde planets on distant orbits from their star, the inner planet can both flip its orbital orientation and reach high enough eccentricities to be tidally captured by the star. That might explain the origin of the ∼25% of hot Jupiters whose orbits are retrograde with respect to the spin of their star, as inferred from Rossiter–McLaughlin measurements (e.g., Triaud et al. 2010). However, while the behavior displayed in Naoz et al. (2011a) is intriguing, it has not yet been understood. Furthermore, Naoz et al. (2011a) choose initial eccentricities and inclinations that is intriguing, it has not yet been understood. Moreover, Naoz et al. (2011a) choose initial eccentricities and inclinations that are fairly extreme for planetary systems (e ∼ 0.6, i ∼ 70°), and hence it is not yet clear how generic their results are. The goal of this paper is to understand and map out the mechanism in a simplified system where the inner planet is treated as a massless test particle.

There has been much work in the literature on the orbital dynamics of similar systems. Kozai (1962) considered the evolution of an asteroid perturbed by a circular Jupiter. He focused on secular interactions, meaning that interactions are averaged over the orbital phases of the asteroid and Jupiter. Kozai found a remarkable result: if the asteroid’s orbit is sufficiently inclined (between 39° and 141°), then it cannot remain on a circular orbit. Instead, its eccentricity and inclination oscillate periodically. Furthermore, if the asteroid is highly inclined (∼90°), then its eccentricity will grow over the course of a single Kozai oscillation to ∼1.

The Kozai mechanism has been applied to a variety of astronomical systems. For example, in triple star systems, if the outer star is inclined it can force Kozai oscillations in the inner binary, increasing the inner binary’s eccentricity until tidal dissipation circularizes it into a very tight orbit (e.g., Eggleton & Kiseleva-Eggleton 2001; Fabrycky & Tremaine 2007). Similarly, if a Jupiter-mass planet forms around a star with an inclined companion star, that companion can force Kozai oscillations in the planet until tides damp its orbit, forming it into a hot Jupiter (Wu & Murray 2003; Fabrycky & Tremaine 2007). Analogous scenarios have been proposed that use Kozai oscillations to merge supermassive black holes (Blaes et al. 2002), stellar-mass compact objects (Thompson 2010), and binary minor planets (Perets & Naoz 2009), as well as to produce blue straggler stars (Perets & Fabrycky 2009). See also Naoz et al. (2011b) for a review of other applications of the Kozai mechanism.

In the case considered by Kozai, the asteroid’s vertical angular momentum (= const × √1 − e2 cos i) is conserved because Jupiter’s orbit is circular. Since this is a two-degrees-of-freedom system (eccentricity and inclination) with two conserved quantities (vertical angular momentum and secular energy), it is integrable. But when Jupiter’s orbit is eccentric, the dynamics can no longer be solved analytically. As Kozai noted, “[w]ithout the aid of a high-speed computer, it is rather difficult to estimate the effects of Jupiter’s eccentricity.”

The effect of an eccentric perturber on Kozai oscillations has been considered in a number of example cases. Harrington (1969) integrates the secular octupole equations for four triple star systems and shows that in one case there is a new resonance; in a second case there is chaos; and in the other two cases no new interesting effects arise. Holman et al. (1997) perform a direct numerical integration for a planet forced by a star, but with their parameters (e ∼ 0.008 in our notation, see below), the perturber’s finite eccentricity only leads to a narrow zone of chaos around the Kozai separatrix. Ford et al. (2000) integrate the secular octupole equations for a system of two eccentric planets, where the outer planet has eccentricity = 0.9, and find that the inner planet’s Kozai oscillations are modulated on long timescales, leading to extreme eccentricities. Blaes et al. (2002) examine some examples for triples of supermassive black holes. Naoz et al. (2011a, 2011b) examine some examples for
two planets. Yet a systematic exploration of the effect of an eccentric Kozai perturber is lacking.

In a number of studies the effect of an eccentric perturber on Kozai oscillations has been neglected, even in situations where it should not be (Naoz et al. 2011b). Part of the reason is that treatments of Kozai oscillations often expand the secular Hamiltonian to leading order in the ratio of semimajor axes, i.e., to quadrupole order. At that order, the exterior body’s argument of periapse does not appear, which implies that the particle’s vertical angular momentum is conserved. To see the effect of the eccentricity, one must work to higher order in the ratio of semimajor axes, i.e., at least to octupole order.

In this paper, we extend Kozai’s work to the case of an eccentric planet and map out the resulting behavior, which we call the eccentric Kozai mechanism (EKM). To do so, we use the secular octupole Hamiltonian of Ford et al. (2000), which is also derived in Harrington (1968a), Marchal (1990), Krymolowski & Mazeh (1999), and Yokoyama et al. (2003).

Katz et al. (2011) independently arrived at many of the results presented in this paper; their paper was posted to http://www.arxiv.org at the same time as this one.

This paper is organized as follows. In Section 2, we present the equations of motion, relegating their derivation to Appendix A. In Section 3, we review the standard Kozai mechanism (SKM). In Section 4, the heart of this paper, we map out the EKM. We summarize in Section 5.

2. EQUATIONS OF MOTION

We solve for the orbit of a massless test particle in the presence of an exterior massive planet, including only secular interactions expanded to octupole order. The planet is on a fixed eccentric orbit, and the particle’s orbit is specified by four variables,

\[ \{e, \omega, \theta, \Omega\} \]

which are its eccentricity, argument of periapse, inclination (or its cosine), and longitude of ascending node relative to the planet’s periapse (e.g., Murray & Dermott 2000). In Appendix A, we use the secular octupole Hamiltonian that has been published in the literature to derive the particle’s equations of motion. As we show in Appendix A, although that published Hamiltonian has had its nodes eliminated, one can still use it to derive the full test particle equations of motion—even the equation that requires the nodes.

We summarize the equations here. Defining the particle’s (scaled) total angular momentum and vertical angular momentum as

\[ J = \sqrt{1 - e^2} \]

(2)

\[ J_z = \theta \sqrt{1 - e^2} \]

(3)

the equations of motion may be expressed as partial derivatives of an energy function \( F(e, \omega, \theta, \Omega) \) via

\[ \frac{dJ}{dt} = \frac{\partial F}{\partial \omega} \]

(4)

\[ \frac{dJ_z}{dt} = \frac{\partial F}{\partial \Omega} \]

(5)

\[ \frac{d\omega}{dt} = \frac{\partial F}{\partial e} + \frac{\partial F}{\partial \theta} \frac{J_z}{J} \]

(6)

\[ \frac{d\Omega}{dt} = -\frac{\partial F}{\partial \theta} \frac{J_z}{J} \]

(7)

where \( t \) is proportional to the true time (Equation (A8)). These are Hamilton’s equations for the two pairs of canonically conjugate variables \( \{J, \omega\} \) and \( \{J_z, \Omega\} \), except that we express \( F \) as a function of the non-canonical variables \( e \) and \( \theta \)—for that reason, we call \( F \) the energy function rather than the Hamiltonian. Of course, \( F \) is a constant of the motion. It is given by a quadrupole and an octupole term

\[ F = F_{\text{qu}} + \epsilon F_{\text{oc}} \]

(8)

where the constant

\[ \epsilon = \frac{(a/a_{\text{pl}})^2}{1 - e_{\text{pl}}^2} \]

(9)

here \( a/a_{\text{pl}} \) is the ratio of semimajor axes (inner to outer) and \( e_{\text{pl}} \) is the planet’s eccentricity. The quadrupole piece is

\[ F_{\text{qu}} = -\frac{e^2}{2} + \theta^2 + \frac{5}{2} e^2(1 - \theta^2) \cos(2\omega) \]

(10)

after dropping an irrelevant constant, and the octupole term is

\[ F_{\text{oc}} = \frac{5}{16} \left( e + \frac{3}{4} e^3 \right) [(1 + 11\theta - 5\theta^2 - 15\theta^3) \cos(\omega + \Omega) \]

\[ + (11\theta - 5\theta^2 + 15\theta^3) \cos(\omega - \Omega)] \]

\[ - \frac{175}{64} e^3 [(1 - \theta - \theta^2 + 3\theta^3) \cos(3\omega - \Omega) \]

\[ + (1 + \theta - \theta^2 - 3\theta^3) \cos(3\omega + \Omega)] \]

(11)

A similar form for the energy function is given by Yokoyama et al. (2003).

The only adjustable parameter in the equations of motion other than the initial conditions is the constant \( \epsilon \). That constant encodes the properties of the planet. In the SKM, the planet’s orbit is circular (\( \epsilon = 0 \)). Hence \( F \) is independent of \( \Omega \), and thus \( J_z \) is a constant of the motion. But if the planet’s eccentricity is not zero (\( \epsilon > 0 \)), then the octupole term allows \( J_z \) to change. If the planet is either nearly circular (\( e_{\text{pl}} \ll 1 \)) or distant (\( a/a_{\text{pl}} \ll 1 \)), then \( \epsilon \) is very small and the evolution is typically similar to the SKM (except at extremely high inclinations). But as we shall show, if \( \epsilon \gtrsim 0.01 \), then the octupole term can lead to qualitatively new behavior that is not found in the \( \epsilon = 0 \) limit.
The three left panels of Figure 1 show sample trajectories, with the same, but with $J_z = 0.6$. For circulating trajectories, the minimum $e$ and $\theta$ occur at $\omega = 0$. The dashed horizontal lines show the cosine of the critical Kozai inclination, $\sqrt{3/5}$. The separatrix—which separates circulating from librating trajectories—always has $e_0 = 0$ and $|\theta_0| = \sqrt{3/5}$. The colored curves have values of $F$ as labeled; the black curves have the following values of $F$: $[−1.44, −.64]$ (left panels, librating); $[1.44]$ (left panels, circulating); $0.25$ (right panels, librating); and $.64, 1$ (right panels, circulating). (A color version of this figure is available in the online journal.)

3. THE STANDARD KOZAI MECHANISM ($\epsilon = 0$)

We review the SKM to set the stage for the EKM. For the SKM, the planet’s orbit is circular ($\epsilon = 0$), i.e., $F = F_{qu}$. Hence there are two constants of the motion, $F$ and $J_z$, and the motion is regular. Each trajectory may be labeled by the values of $F$ and $J_z$. Inserting Equation (3) into Equation (10) immediately determines $e$ as a function of $\omega$ as well as $\theta$ as a function of $\omega$.

The three left panels of Figure 1 show sample trajectories, with fixed $J_z = 0.2$ and various values of $F$; the right panels show the same, but with $J_z = 0.4$. For our purposes, the most important properties of the SKMs are as follows.

1. Because $J_z$ is constant, each trajectory traces out a curve in the $e$-$\theta$ plane, which we call a “Kozai curve.”

2. There are two classes of trajectories, librating and circulating. On circulating trajectories, $e$ and $|\theta|$ are smallest at $\omega = 0$, and they are largest at $\omega = \pm \pi/2$. The separatrix has $e_0 \equiv e_{|\omega=0} = 0^2$ and $|\theta_{0,\pi/2}| = \sqrt{3/5}$.

3. On a trajectory that has $e_0 \ll 1$, the largest $e$ is

$$e^2_{\pm\pi/2} \approx 1 - \frac{5}{3} \theta_0^2 \approx 1 - \frac{5}{3} J_z^2$$

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4. Given $F$ and $J_z$, the minimal $e$ on a circulating trajectory satisfies

$$e_0^2 = \frac{1}{2} (F - J_z^2).$$

Figure 2, left panel, shows a sequence of Kozai curves that have the same value of $F = 0.16$ and differing $J_z$. The left boundaries of the sequence (i.e., the values $\{e_0, \theta_0\}$ for each Kozai curve) trace out a curve in the $e$-$\theta$ plane, which we call an “energy curve.” That curve is plotted in green in the right panel of Figure 2. An energy curve is a curve of constant $F_{qu}|_{\omega=0}$. Its form is given by Equation (13) with $J_z^2 = e_0^2 (1 - \theta_0^2)$. The right panel of Figure 2 also shows other energy curves with different energies $F_{qu}$.

4. THE ECCENTRIC KOZAI MECHANISM ($\epsilon > 0$)

When the planet is eccentric ($\epsilon > 0$) there is only a single conserved quantity, the secular energy $F$. Therefore the particle’s trajectories are more complicated and can even be chaotic. Figure 3 shows two sample trajectories in the $e$-$\theta$ plane for $\epsilon = 0.01$. Both trajectories have the same energy $F = 0.16$, but different initial conditions consistent with that energy. Rather than being confined to a single Kozai curve, the particle evolves from one Kozai curve to another. As long as $\epsilon \ll 1$, all of these Kozai curves have nearly the same $F_{qu}$, and therefore the trajectories in Figure 3 follow along the tracks displayed in the left panel of Figure 2.

Trajectory ‘a’ in Figure 3 evolves through $\theta = 0$. Its orbit flips from prograde ($\theta > 0$) to retrograde ($\theta < 0$) and back again. Furthermore, its eccentricity approaches unity. In fact, the flipping of an orbit is closely tied to its eccentricity reaching unity. This can be seen from Equation (12), which implies that a Kozai curve that has $\theta_0 \approx 0$ reaches $e \approx 1$. (It never reaches exactly $e = 1$; see below.) Trajectory ‘b’ never flips, and its eccentricity does not approach unity.

The left panels of Figure 4 show the temporal evolution of $\theta$ and $e$ for trajectory ‘a.’ The bottom left panel shows two angles that appear as arguments of cosine terms in $F_{oc}$. When the particle’s orbit is prograde, the angle $\omega + \Omega$ librates and $\omega - \Omega$ circulates; and when it is retrograde, those two angles switch

(i.e., inclination close to 90°), the largest eccentricity is nearly unity.
The initial conditions for trajectory 'a' are those variables are initialized to time (Equation (A8)). Only points up to time \( t = \theta \) are plotted, longer runs yield identical plots, albeit with more densely packed points. In particular, orbit 'b' never flips. The initial conditions for trajectory 'a' are \( [e, \omega, \theta, \Omega] = [0.192, 0, 0.3, \pi] \) and for trajectory 'b' those variables are initialized to \( [0.0913, 0, 0.38, 0] \). (A color version of this figure is available in the online journal.)

![Figure 3](image1.png)

Figure 3. EKM: two sample trajectories ('a' and 'b') with \( F = 0.16, \epsilon = 0.01 \), and different initial conditions. When the planet is eccentric, the test particle no longer traces out a single Kozai curve, but evolves from one Kozai curve to another, approximately tracing out a sequence of Kozai curves that have a fixed quadrupole energy (Figure 2, left panel). Orbit 'a' flips repeatedly (i.e., crosses \( \theta = 0 \)) and reaches extreme eccentricities, and orbit 'b' does not. Although only points up to time \( t = 190 \) are plotted, longer runs yield identical plots, albeit with more densely packed points. In particular, orbit 'b' never flips. The initial conditions for trajectory 'a' are \( [e, \omega, \theta, \Omega] = [0.192, 0, 0.3, \pi] \) and for trajectory 'b' those variables are initialized to \( [0.0913, 0, 0.38, 0] \). (A color version of this figure is available in the online journal.)

![Figure 4](image2.png)

Figure 4. EKM: temporal evolution of the two trajectories depicted in Figure 3. Trajectory 'a' reaches extreme eccentricities when it flips. Its evolution is regular, with \( \omega+\Omega \) librating when it is prograde, and \( \omega-\Omega \) librating when it is retrograde. Trajectory 'b' is chaotic and never flips. The time \( t \) is proportional to the true time (Equation (A8)).

![Figure 5](image3.png)

Figure 5. EKM: zoomed-in evolution of trajectories 'a' and 'b' at early times (Figure 3). In the top panels, the red lines show that \( J_z \) controls the envelope of both \( \theta \) and \( e \) (via Equation (12)). In the bottom panels, the angles are in units of radians/\( \pi \) (though unlike in Figure 4, the angles are first reset with modulo \( 2\pi \)). The time \( t \) is proportional to the true time (Equation (A8)).

from \( -\pi \) to 0. Hence the orbit always circulates (see Figure 1). The top left panel also shows \( J_z \). Whereas in the SKM \( J_z \) = const, here \( J_z \) changes in a nearly step-wise fashion. There are sharp jumps in \( J_z \) when \( e \) and \( \theta \) change rapidly; these are forced by the octupolar contribution \( F_{\infty} \). The long-term evolution of \( J_z \) controls the envelopes of both \( \theta \) (Figure 5, top left panel) and \( e \) (Figure 5, middle left panel; see Equation (12)). Successive maxima of \( e \) occur in discrete steps. Therefore even when \( J_z \) crosses through zero, the maximum \( e \) is never precisely unity. Nonetheless, as time evolves, the maximum \( e \) reaches approaches closer and closer to unity.

The right panels of Figure 5 show the corresponding zoom-in for trajectory 'b'. The bottom right panel shows that \( \omega \) switches from circulation to libration and back again. That explains why the 'b' trajectory is chaotic. Similar behavior is shown in Holman et al. (1997).

Figure 6 shows a zoom-in of the values of \( 1 - e \) over the course of the first 10 flips for trajectory 'a'. Near the time that the orbit flips, \( e \) reaches nearly unity and \( 1 - e \) becomes small. The typical minimum \( 1 - e \) near a flip is \( \sim 5 \times 10^{-6} \). For real astrophysical problems, if the inner body's \( e \) becomes too large it can penetrate the star or feel other effects such as tides or general relativistic precession. Whether that happens before the first flip depends on the parameters of the system of interest. We are currently investigating the properties of the minima of \( 1 - e \), including their distribution and how the properties depend on \( e \) (J. Teyssandier et al. 2011, in preparation).

A global view of the dynamics is provided by surfaces of section. Figure 7 maps out the behavior when \( e = 0.01 \). We plot a point whenever \( \omega = 2n\pi \) for integer \( n \). This may be interpreted as follows. Consider trajectory ‘a’ of Figure 3. When

\[ \omega + \Omega \]
Figure 6. EKM, $\epsilon = 0.01$. Zoom-in of $1 - \epsilon$ for trajectory 'a' of Figures 3–5 up to $t = 750$. The points are separated in time by $\Delta t = 10^{-4}$. Near the time that the inclination flips from retrograde to prograde or vice versa, the eccentricity reaches nearly unity and appears as a spike in this figure. The 10 spikes here show the eccentricity during the first 10 flips. For this trajectory, the typical maximum $\epsilon$ near a flip is $\epsilon \sim 1 - (5 \times 10^{-6})$; in the third flip the eccentricity reaches $1 - (1.2 \times 10^{-7})$.

(A color version of this figure is available in the online journal.)

Figure 7. EKM, $\epsilon = 0.01$. Surfaces of section with various values of $F$; points are plotted whenever $\omega = 2n\pi$ for integer $n$. For each $F$, if the values of $\theta_0$ were plotted against the value of $e_0$, they would lie along an energy curve (Figure 2, right panel). The colors in that figure correspond to the colors in this one. Consider for example the $F = 0.16$ panels (green points). If one imagines extending the green $F = 0.16$ energy curve in Figure 2 out of the plane of the paper into a half-cylinder, with the third dimension being the value of $\Omega$ then the green points shown in this figure would cover the surface of that cylinder.

(A color version of this figure is available in the online journal.)

$\omega = 2n\pi$ that trajectory hits the green energy curve depicted in the right panel of Figure 2. At those times, we plot in Figure 7 the value of $\epsilon = e_0$ versus $\Omega$ and $\theta = \theta_0$ versus $\Omega$ (curves labeled ‘a,’ middle panels). Also shown in those green middle panels are other trajectories with the same energy $F = 0.16$, including trajectory ‘b.’ If all of the green points were plotted against each other in the $e_0 - \theta_0$ plane, they would trace out the energy curve labeled 0.16 in Figure 2. Equivalently, one can imagine extending that energy curve out of the plane of the paper into a half-cylinder, with the third dimension being the value of $\Omega$. The $F = 0.16$ surfaces of section of Figure 7 cover the surface of that half-cylinder. Of course, for a given energy, the maximum $|\theta_0|$ is $\sqrt{F}$ and the maximum $e_0$ is $\sqrt{F/2}$. Beyond those values, the energy curve does not exist (see the caption of Figure 2).

The other panels in Figure 7 may be interpreted similarly with each pair of panels at fixed $F$ lying along a single energy curve of Figure 2 (with corresponding colors). We therefore now have a virtually complete view of the $\epsilon = 0.01$ dynamics.

One new behavior caused by the octupole term is the appearance of chaos. Regular orbits appear as curves in the surfaces of section, while chaotic orbits appear as a smattering of points. For example, it is apparent from Figure 7 that trajectory ‘a’ is regular and ‘b’ is chaotic. Chaos always occurs near $e_0 = 0$. That is because the Kozai separatrix always has $e_0 = 0$ (Section 3), and chaos is caused by the crossing of the Kozai separatrix, when $\omega$ transitions from librating to circulating and back to librating (e.g., Figure 5).

Perhaps the most dramatic new behavior caused by the octupole term is that orbits can flip orientation (i.e., cross $\theta = 0$), and, as a consequence, reach arbitrarily high values of eccentricity. From the top panels of Figure 7, it is clear which orbits exhibit this behavior, and which do not, at $\epsilon = 0.01$. In particular, any curve that crosses $\theta_0 = 0$ is a regular orbit that flips, and any smattering of points that straddle the $\theta_0 = 0$ line is a chaotic orbit that flips. If one makes the correspondence between the orbits in the top panels and those in the bottom, one infers that orbits with $e_0 = 0$ will always flip provided that $|\theta_0| \lesssim 0.2$ (i.e., $i_0 \gtrsim \cos^{-1}0.2 \sim 80^\circ$ for prograde orbits). For $e_0 \neq 0$, one infers from the blue ($F = 0.25$) panels that even orbits with $\theta_0$ as large as 0.4 ($i_0$ as small as 66°) can flip.

Figure 8 is the main result of this paper. It summarizes where flipping orbits occur for various values of $\epsilon$. Toward the top of the plot (large $\theta_0$ and small $i_0$), there are no flipping orbits.
But as $\theta_0$ decreases, an increasing number of orbits can flip. Each curve marks, for a given $\epsilon$, the location at which the first flipping orbit occurs, if $\Omega$ is judiciously chosen. Above the curve there are no flipping orbits and below there are some—typically around half of the orbits below the curve can flip. We made this plot by numerically integrating Equations (4)–(7) with a variety of initial conditions. Each integration was initialized with $\theta = 0$ and $\omega = 0$, and we scanned through values of initial $\epsilon$ and $\Omega$ to determine which flipping orbit reached the largest values of $\theta_0$ for a given initial $\epsilon$. We then plotted the extreme values of $\theta_0$ and $\theta$ for that orbit. (Note that all such trajectories initialized with the same initial $\epsilon = \epsilon_0$ share the same energy curve.)

A striking feature of Figure 8 is the fairly sudden transition when $\epsilon$ exceeds a few percent. For example, focusing on orbits with $\epsilon_0 = 0$, when $\epsilon = 0.01$ flipping orbits can occur only if $|\theta_0| \lesssim 0.2$, i.e., $i_0 \gtrsim 80^\circ$ (for prograde), as stated above. But for $\epsilon = 0.03$ flips can occur for $|\theta_0| \sim 0.5$, i.e., $i_0 \sim 60^\circ$ (for prograde). Note that if test particle orbits are randomly oriented, they will be uniformly distributed in $\theta$, and therefore half of the orbits will have $\theta < 0.5$. Of course, this is only a rough estimate for the fraction of flipping orbits, since it depends also on the values of the other variables ($\epsilon$, $\omega$, $\Omega$).

The curves in Figure 8 rise as $\epsilon_0$ increases from 0 and then suddenly turn over. The reason for the sudden turnover is apparent from a glance at the surfaces of section (Figure 7). In the blue ($F = 0.25$) section, there are flipping orbits centered both on $\Omega = 0$ and $\Omega = \pi$, with the latter ones reaching to a larger $|\theta_0|$. But in the magenta ($F = 0.36$) section, there are no more flipping orbits centered at $\Omega = \pi$. It is that discontinuity that gives rise to the sharp break in Figure 8.

Our $\epsilon = 0.01$ curve in Figure 8 agrees with the numerically calculated boundary shown in Figure 5 of Katz et al. (2011). Katz et al. (2011) also derive a theoretical expression for this curve in terms of integrals of elliptic functions that agrees quite well with the results of numerical simulations.

5. SUMMARY

In this paper, we map out the behavior of a test particle that is forced by a massive eccentric planet, working under the secular octupole approximation.

In Section 2, we give the evolutionary equations. These are derived in Appendix A, where we show how one may use the published Hamiltonian—which has had its nodes eliminated—to derive the full equations of motion. There is a single parameter, $\epsilon$, that characterizes deviations from the SKM (Equation (9)). The SKM is recovered when $\epsilon \ll 1$, i.e., when the planet is either nearly circular or very distant.

In Section 3, we review the SKM. In the $\epsilon$–$\theta$ plane (where $\theta \equiv \cos i$), the particle traces out a Kozai curve because its $J_2$ is fixed. A sequence of Kozai curves with the same energy $F$ outline an energy curve, which is a curve of constant secular energy, evaluated at $\omega = 0$.

In Section 4, we map out what happens when the planet’s orbit is eccentric. Our principal results are as follows.

1. In the $\epsilon$–$\theta$ plane, a single trajectory evolves from one Kozai curve to another, all of which abut the same energy curve (Figure 3).

2. For trajectories where the orbit flips, the eccentricity becomes extremely high, and as time evolves, the maximum eccentricity reached approaches closer and closer to unity. That is because every time $\theta$ crosses through zero, $|\theta_0|$ becomes very small. And given enough zero-crossings, the smallest $|\theta_0|$ reached becomes arbitrarily small, and hence the maximum eccentricity becomes arbitrarily close to unity (as determined by Equation (12)).

3. The vertical angular momentum $J_z$ varies gradually in the EKM, unlike in the SKM where it is constant. Its evolution controls the evolution of both $\theta$ and $\epsilon$ (Figure 4). Even when the value of $\epsilon$ is small, and hence the temporal evolution of $J_z$ is slow, if $J_z$ crosses through zero, the dramatic flipping and extreme eccentricity behavior occurs (Figure 3, left panel).

4. Chaotic behavior occurs when the Kozai separatrix is crossed. In that case, $\omega$ transitions from librating to circulating and back. See also Holman et al. (1997).

5. Surfaces of section, taken whenever $\omega = 2\pi \epsilon$, provide an graphical global map of the dynamics. Each energy curve gives rise to two panels in Figure 7 (top and bottom). By stepping through the energy curves, one acquires a complete map of the dynamics for a given value of $\epsilon$. From the surfaces of section, one can immediately see which orbits exhibit flipping and extreme eccentricity, and which do not. One can also see which orbits are chaotic and which are not. All four combinations occur: flipping and non-chaotic (‘a’), flipping and chaotic, non-flipping and non-chaotic, non-flipping and chaotic (‘b’).

6. Figure 8 is the main result of this paper. For each $\epsilon$, it gives the curve in the $\epsilon_0$–$\theta_0$ plane at which the first flipping orbit occurs (along with the accompanying extreme eccentricity behavior). Above the curve no flipping orbits occur and below it an increasing number occur. There is a fairly sharp transition in the smallest inclination at which orbits can flip. For $\epsilon \sim 0.01$, the inclination must exceed $\sim 80^\circ$ to flip (for prograde orbits with $\epsilon_0 = 0$); but when $\epsilon \sim 0.03$, it need only exceed $\sim 60^\circ$, implying that around half of randomly oriented test particle orbits will flip in that case.

We have made a number of approximations in order to simplify the analysis. First, we used the secular octupole approximation, which is poor when $\epsilon$ is too large because in that case the particle comes close to the planet. We have made a few comparisons with $N$-body integrations. One example is shown in Appendix B, which displays the result of an $N$-body integration of trajectory ‘a’ of Figures 3–6. Other examples are shown in Naoz et al. (2011b). Thus far the agreement is quite good, but more should be done to determine the domain of validity of the secular octupole approximation. Second, we took the inner orbiting body to be massless. While that is appropriate for a planet being forced by a companion star, it is not for two planets, which is the main motivation for this paper (Naoz et al. 2011a). We are currently extending the work presented here to two massive planets; in that case, there is a generalized $\epsilon$ that includes the masses (Naoz et al. 2011b). And third, we ignored all physical effects beyond Newtonian gravity. In the absence of such effects, if the particle’s orbit flips then its eccentricity will reach arbitrarily close to unity given enough time. But in reality, if the particle reaches too high an eccentricity, it will either plunge into the star or its eccentricity growth will be halted by other precessional forces, such as those induced by tidal distortions, rotational distortions, or general relativity (e.g., Sterne 1939; Wu & Goldreich 2002; Wu & Lithwick 2011). We leave a study of the competition between these various effects to future work.7 For the orbit to flip at all, the typical maximum eccentricity reaches, becomes very small. And given enough zero-crossings, the smallest $|\theta_0|$ reached becomes arbitrarily small, and hence the maximum eccentricity becomes arbitrarily close to unity (as determined by Equation (12)).

7 Katz et al. (2011) consider the effect of general relativistic precession.
eccentricity during a flip (as displayed, e.g., in Figure 6) must not lead to a periapse which is too small. That can always be accomplished by increasing the semimajor axes of both the particle and the perturbing planet. However, the semimajor axes cannot be arbitrarily large or the time between flips will become longer than the age of the system.

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APPENDIX A

OCTUPOLE SECULAR EQUATIONS OF MOTION FOR A TEST PARTICLE

The secular interaction energy between an interior and exterior planet (unprimed and primed, respectively) can be written as

$$E_\text{con} = -G \frac{mnm'}{a^2} f(\alpha, e', e, g', g, \theta), \quad (A1)$$

where $\theta \equiv \cos I$ and the other quantities are the mutual inclination $I$, argument of periapse $g$, eccentricity $e$, semimajor axis $a$, ratio of semimajor axes (inner to outer) $\alpha$, mass $m$, and Newton’s constant $G_N$. (The notation here differs slightly from the body of the paper in order to connect to previous treatments of the nodes, i.e., the longitudes of ascending nodes ($\lambda$) have been eliminated (Naoz et al. 2011b). We shall use the same initial conditions as displayed in the caption of Figure 3 for trajectory ‘a.’ The perturbing “planet” was chosen to have mass $m' = 0.03M_\odot$, eccentricity $e' = 0.3$, and semimajor axis $a' = 100$ AU. The test particle’s initial semimajor axis is $a = 3.03$ AU, as implied by the value of $e = 0.01$. Note that the true time $T$ is related to the scaled time $\tau$ displayed on the $x$-axis via $T = \tau \times 2.32$ Myr (Equation (A8)). We integrated the test particle’s equations with the symplectic DH algorithm contained in the SWIFTER package (http://www.boulder.swri.edu/swifter/). We additionally modified the SWIFTER code to include an adaptive time step, where the time step relaxes to 2% of the inverse of the angular velocity at periapse; the relaxation time at which this happens is chosen to be five test particle orbital periods. After the flip occurs, the value of $1 - e$ becomes extremely small $\sim 10^{-5}$. As a result, the periapse becomes extremely small and it becomes costly to integrate further. (The integration shown in this figure took over a month to perform.) (A color version of this figure is available in the online journal.)

$$G'$$ satisfies Hamilton’s equations, i.e., $dG'/dt = -\partial E_\text{con}/\partial g$ and $dG'/dt = -\partial E_\text{con}/\partial g'$, we find that to leading order in $m$, $dG'/dt = -(d/dt)G\theta$, i.e., $dH/dt = (d/dt)E_\text{con}$ which proves that if we replace $g' \rightarrow \pi - h$ in $E_\text{con}$ we will end up with the correct Hamilton equation for $H$.

$A.2. \text{Equations of Motion}$

We may rescale all momenta by an arbitrary constant without changing the equations of motion as long as we rescale the Hamiltonian by the same constant. Therefore defining

$$J \equiv \frac{G}{m\sqrt{G_N M_\star a}} = \sqrt{1 - e^2}, \quad (A3)$$

$$J_z \equiv J \cos I \equiv \theta \sqrt{1 - e^2}. \quad (A4)$$

the rescaled Hamiltonian is

$$\mathcal{H}(J, g; J_z, h) \equiv \frac{E_\text{con}}{m\sqrt{G_N M_\star a}} \quad (A5)$$

$$= -\frac{m'}{M_\star} \Omega_\star a^2 f(\alpha, e', e, g', g, \theta), \quad (A6)$$

$G'$ satisfies Hamilton’s equations, i.e., $dG'/dt = -\partial E_\text{con}/\partial g$ and $dG'/dt = -\partial E_\text{con}/\partial g'$, we find that to leading order in $m$, $dG'/dt = -(d/dt)G\theta$, i.e., $dH/dt = (d/dt)E_\text{con}$ which proves that if we replace $g' \rightarrow \pi - h$ in $E_\text{con}$, we will end up with the correct Hamilton equation for $H$.

$A.1. \text{Proof}$

Our proof is based on the triangle equality $G^2 + G'^2 + 2GG'\theta = (H + H')^2$, which expresses the square of the vertical angular momentum in two equivalent ways. Taking the time derivative of the triangle equality, and noting that both $G$ and $G'$ satisfy Hamilton’s equations, i.e., $dG'/dt = -\partial E_\text{con}/\partial g$ and $dG'/dt = -\partial E_\text{con}/\partial g'$, we find that to leading order in $m$, $dG'/dt = -(d/dt)G\theta$, i.e., $dH/dt = (d/dt)E_\text{con}$ which proves that if we replace $g' \rightarrow \pi - h$ in $E_\text{con}$, we will end up with the correct Hamilton equation for $H$. 

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where \( \Omega_* \) is the orbital angular speed of the test particle around the star.

The expression for \( f \) (defined via Equation (A1)) is derived, e.g., in Ford et al. (2000), to octupole order. We re-express their Equation (22) in the limit \( m, m' \ll M_* \) as

\[
f = \frac{3}{8} \left( \frac{1}{1-e'^2} \right)^{3/2} \left( F_{qu} + \frac{a e'}{1-e'^2} F_{oc} \right),
\]

(A7)

where \( F_{qu} \) and \( F_{oc} \) are displayed above (Equations (10) and (11), with \( g \equiv \omega \) and \( h \equiv \Omega \)) after correcting the sign of the octupole term (Ford et al. 2004) and replacing \( g' \to \pi - h \).

The equations of motion are Hamilton’s equations for Hamiltonian (A6). These are displayed explicitly in the body of the paper (Equations (4)–(7)) after defining the rescaled time

\[
t \equiv T \times \frac{m'}{M_*} \alpha^3 \frac{1}{8 (1-e'^2)^{3/2}},
\]

(A8)

where \( T \) is the true time.

APPENDIX B

\textbf{N-BODY SIMULATION OF TRAJECTORY ‘A’}

Figure 9 shows the result of an \( N \)-body simulation of trajectory ‘a’ of Figures 3–6. See the figure’s caption for detail.

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