TWO FAMILY OF NEW IDENTITIES INVOLVING BINOMIAL
COEFFICIENTS AND GENERALIZED HARMONIC NUMBERS

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ABSTRACT. A plenty of summation formulae on harmonic numbers and binomial coefficients have been investigated for a long time. These identities can be applied to analysis of algorithms, number theory and physics. In this text, we provide a family of new summation formulae on generalized harmonic numbers and binomial coefficients.

1. INTRODUCTION

Define generalized harmonic numbers by

\[ H^{(s)}_n = 0 \quad \text{and} \quad H^{(s)}_n = \sum_{j=1}^{n} \frac{1}{j^s}, \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{Z}^{-}, n, s = 1, 2, \ldots \]

When \( s = 1 \), \( H^{(s)}_n \) reduce to the classical harmonic numbers \( H_n \). There are plenty of literature on evaluations of summations involving generalized harmonic numbers, binomial coefficients and special constants. Readers can refer to \[2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22\] on this issue. In \[2, 3\], by employing differential operators to Gauss’s summation formula, Liu etc. obtained a series of identities involving generalized harmonic numbers and binomial coefficients. For example, they produced the following identities.

\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{n})^2}{n!(n+1)!} \cdot O_n = \frac{4}{\pi} \left( 1 - \ln 2 \right), \]

\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{n})^2}{n!(n+1)!} \left( O_n^3 - O_n^{(2)} \right) = \frac{4}{\pi} \left( 2 - 2 \ln 2 + \ln^2 2 \right) - \frac{\pi}{3} \]

\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{n})^2}{n!(n+1)!} \left( O_n^3 - 3O_n O_n^{(2)} + 2O_n^{(3)} \right) = \frac{1}{\pi} \left( 24 - 16 \ln 2 + 4 \ln^2 2 - 4 \ln^3 2 - 4 \pi^2 \ln 2 - 6\zeta(3) \right). \]

In this paper, we extend Gauss’s summation formula to a form with a parameter, and then apply differential operators on it. We establish two family of new identities involving generalized harmonic numbers and binomial coefficients. As applications of our main results, we can obtain companions of many identities appeared in \[2, 3\]. For example, the companions of equations above are

\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{n})^2}{n!(n+1)!} \cdot O_n = \frac{8}{\pi} \left( 3 - 2 \ln 2 \right) - 4, \]
\[
\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)_n^2 \left(O_n^2 - O_n^{(2)}\right) = \frac{4}{\pi} \left(16 - 2\pi - \frac{\pi^2}{3} - 12 \ln 2 + 4 \ln^2 2\right),
\]
\[
\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)_n^2 \left(O_n^3 - 3O_n O_n^{(2)} + 2O_n^{(3)}\right)
= \frac{2}{\pi} \left(120 - 3\pi^2 - 96 \ln 2 + 2\pi^2 \ln 2 + 36 \ln^2 2 - 8 \ln^3 2 - 12\zeta(3)\right) - 24.
\]

This paper is organized as follows. In Section 2, we briefly review some results that we need in our article. In Section 3, we present our main theorems and their proofs. In Section 4, as some applications of main results, we provide a large number of new and closed summation formulae involving generalized harmonic numbers and binomial coefficients by specifying parameters in identities obtained in Section 3.

2. Preliminaries

The well-known Gauss’s summation formula is given by

\[
_{2}F_{1}\left(\begin{array}{c}
 a, b \\
 c
\end{array}\; ; \; 1\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\]

with \(\Re(c-a-b) > 0\). Where the shifted factorial \((x)_n\) is given by

\[(x)_n \equiv 1 \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \ n \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{Z}^-.
\]

And the \(\Gamma\)-function \([1]\) is given by

\[\Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u} \, du \quad \text{for} \ \Re(x) > 0.\]

The digamma function \(\psi(x)\), which is the derivative of the gamma function, is defined by \([14]\)

\[\psi(x) := \frac{d}{dx}\{\ln \Gamma(x)\}.\]

When differentiating the digamma function \(n\) times, one can get the polygamma functions

\[\psi^{(n)}(x) := \frac{d^{n+1}}{dx^{n+1}}\{\ln \Gamma(x)\} = \frac{d^n}{dx^n}\{\psi(x)\}.\]

Another notation that will be used is Bell polynomials \(\Omega_i(t)\), defined by the following generating function

\[\sum_{i \geq 0} \Omega_i(t) \frac{x^i}{i!} = \exp \left\{ \sum_{k \geq 1} \frac{x^k}{k} t_k \right\},\]

where \(t\) is a sequence that is \(t := (t_1, t_2, \ldots)\). We give the first few of Bell polynomials as follow

\[\Omega_0(t) = 1, \quad \Omega_1(t) = t_1, \quad \Omega_2(t) = t_1^2 + t_2, \quad \Omega_3(t) = t_1^3 + 3t_1t_2 + 2t_3.\]

The following three lemmas will be used in proofs of our main results. The first two can be found in the paper \([2]\) and the proof of the third lemma is similar to the proof of Theorem 2.2 in the paper \([2]\).
Lemma 2.1. [2, Theorem 2.1] The ith derivative of the shifted factorial \((a + x)_n\) at \(x = 0\) is
\[
\left. \frac{d^i}{dx^i} (a + x)_n \right|_{x=0} = (-1)^i(a)_n \Omega_i \left( -H_n(a), -H_n^{(2)}(a), \cdots, -H_n^{(i)}(a) \right).
\]

The ith derivative of the reciprocal of the shifted factorial \((a + x)_n^{-1}\) at \(x = 0\) is
\[
\left. \frac{d^i}{dx^i} (a + x)_n^{-1} \right|_{x=0} = (-1)^i(a)_n^{-1} \Omega_i \left( H_n(a), H_n^{(2)}(a), \cdots, H_n^{(i)}(a) \right).
\]

Lemma 2.2. [2, Theorem 2.2] The ith derivatives of the gamma functions \(\Gamma(c + x)\) and \(\Gamma(c - x)\) at \(x = 0\) are
\[
\left. \frac{d^i}{dx^i} \Gamma(c + x) \right|_{x=0} = \Gamma(c) \Omega_i \left( \frac{\psi(c)}{0!}, \frac{\psi'(c)}{1!}, \cdots, \frac{\psi^{(i-1)}(c)}{(i-1)!} \right),
\]
\[
\left. \frac{d^i}{dx^i} \Gamma(c - x) \right|_{x=0} = (-1)^i \Gamma(c) \Omega_i \left( \frac{-\psi(c)}{0!}, \frac{-\psi'(c)}{1!}, \cdots, \frac{-\psi^{(i-1)}(c)}{(i-1)!} \right).
\]

The ith derivatives of the reciprocals of the gamma functions \(\Gamma(c + x)\) and \(\Gamma(c - x)\) at \(x = 0\) are
\[
\left. \frac{d^i}{dx^i} \frac{\Gamma(c + x)}{\Gamma(c + x)} \right|_{x=0} = \Gamma(c) \Omega_i \left( \frac{-\psi(c)}{0!}, \frac{-\psi'(c)}{1!}, \cdots, \frac{-\psi^{(i-1)}(c)}{(i-1)!} \right),
\]
\[
\left. \frac{d^i}{dx^i} \frac{\Gamma(c - x)}{\Gamma(c - x)} \right|_{x=0} = (-1)^i \Gamma(c) \Omega_i \left( \frac{-\psi(c)}{0!}, \frac{-\psi'(c)}{1!}, \cdots, \frac{-\psi^{(i-1)}(c)}{(i-1)!} \right).
\]

Lemma 2.3. The ith derivatives of the function \(\frac{\Gamma(a + x)\Gamma(b + x)}{\Gamma(c + x)\Gamma(d + x)}\) at \(x = 0\) are
\[
\left. \frac{d^i}{dx^i} \frac{\Gamma(a + x)\Gamma(b + x)}{\Gamma(c + x)\Gamma(d + x)} \right|_{x=0} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} \Omega_i (t_1, t_2, \ldots, t_i),
\]
with
\[
t_1 = \frac{\psi(a) + \psi(b) - \psi(c) - \psi(d)}{0!}, t_2 = \frac{\psi'(a) + \psi'(b) - \psi'(c) - \psi'(d)}{1!},
\]
\[
\ldots
\]
\[
t_i = \frac{\psi^{(i-1)}(a) + \psi^{(i-1)}(b) - \psi^{(i-1)}(c) - \psi^{(i-1)}(d)}{(i-1)!}.
\]

3. Main results

We begin with an interesting identity.

Lemma 3.1. For any \(m (m \in \mathbb{N} = \{0, 1, 2, \ldots\})\), we have
\[
\frac{1}{n + m + 1} = \frac{A_{m,1}}{n + 1} + \frac{A_{m,2}}{(n + 1)(n + 2)} + \frac{A_{m,3}}{(n + 1)(n + 2)(n + 3)} + \cdots + \frac{A_{m,m+1}}{(n + 1)(n + 2) \cdots (n + m + 1)},
\]
with
\[
A_{m,j} = \frac{(-1)^j m!}{(m + 1 - j)!} (j = 1, 2, 3, \ldots, m + 1).
\]
Proof. We assume that the equation (3) holds. If we can solve out the value of \( A_{m,1}, A_{m,2}, \ldots, A_{m,m+1} \), then the proof is completed. Multiplying both sides of the equation (3) by \((n+1)(n+2)\cdots(n+m+1)\), we get the following equation

\[
(n+1)(n+2)\cdots(n+m) = A_{m,1}(n+2)(n+3)\cdots(n+m+1) + A_{m,2}(n+3)(n+4)\cdots(n+m+1) + \cdots + A_{m,m}(n+m+1) + A_{m,m+1}.
\]

Setting \( n = -1, -2, -3, \ldots, -m, -m - 1 \) in the formula above respectively, the following matrix equation can be established

\[
MX = B
\]

with

\[
M = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
1 & 2 & 2 \times 1 & \ldots & 0 \\
1 & 3 & 3 \times 2 \times 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & m-1 & (m-1)(m-2) & \ldots & 0 \\
1 & m & m(m-1) & \ldots & m(m-1)\cdots2\cdots1
\end{pmatrix},
\]

\[
X = \begin{pmatrix}
A_{m,m+1} \\
A_{m,m} \\
\vdots \\
A_{m,2} \\
A_{m,1}
\end{pmatrix}, 
\]

\[
B = \begin{pmatrix}
(-1)^m m! \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}.
\]

This is an upper triangular matrix, and it is easily can be found that \( A_{m,m+1} = (-1)^m m!, A_{m,m} = (-1)^{m-1} m!, \ldots, A_{m,2} = (-1)^4 m!, A_{m,1} = (-1)^0 m! \). The Lemma 3.1 is proven.

Lemma 3.2. For \( i \in \mathbb{N} (i \neq a, b, c) \), any \( a, b, c \in \mathbb{R}_+ \) and \( \Re(c-a-b) > 0 \), we have

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{(n+i)!} = \frac{(c-i)_i}{(a-i)i(b-i)_i} \binom{2}{1} \binom{c-i}{a-i,b-i} - \sum_{k=0}^{i-1} \frac{(c-i+k)_{i-k}}{(a-i+k)(b-i+k)_{i-k}} \frac{1}{k!}
\]

Proof. According to the definition of shifted factorials,

\[
(a)_n = \frac{(a-i)_{n+i}}{(a-i)_i}, \quad (b)_n = \frac{(b-i)_{n+i}}{(b-i)_i}, \quad (c)_n = \frac{(c-i)_{n+i}}{(c-i)_i}.
\]

Therefore,

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{(n+i)!} = \frac{(c-i)_i}{(a-i)i(b-i)_i} \binom{2}{1} \binom{c-i}{a-i,b-i} - \sum_{k=0}^{i-1} \frac{(c-i+k)_{i-k}}{(a-i+k)(b-i+k)_{i-k}} \frac{1}{k!}
\]
\[
\begin{align*}
&= \frac{(c - i)_i}{(a - i)_i (b - i)_i} \sum_{n=0}^{\infty} \frac{(a - i)_{n+i} (b - i)_{n+i}}{(c - i)_{n+i}} \frac{1}{(n+i)!} \\
&= \frac{(c - i)_i}{(a - i)_i (b - i)_i} \left[ 2F_1 \left( \frac{a - i, b - i}{c - i}; 1 \right) - \sum_{k=0}^{i-1} \frac{(a - i)_k (b - i)_k}{(c - i)_k} \frac{1}{k!} \right] \\
&= \frac{(c - i)_i}{(a - i)_i (b - i)_i} 2F_1 \left( \frac{a - i, b - i}{c - i}; 1 \right) - \sum_{k=0}^{i-1} \frac{(c - i + k)_{i-k}}{(a - i + k)_{i-k} (b - i + k)_{i-k}} \frac{1}{k!}.
\end{align*}
\]

Corollary 3.3. Let \( m \in \mathbb{N} \) and \( m + 1 < c \), then the following identity holds:
\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} = \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j}{(a - j)_j (b - j)_j} 2F_1 \left( \frac{a - j, b - j}{c - j}; 1 \right) \\
- \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{i-1} \frac{(c - i + k)_{i-k}}{(a - i + k)_{i-k} (b - i + k)_{i-k}} \frac{1}{k!}.
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.

Proof. According to the Lemma 3.1 we have
\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} = A_{m,1} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{(n+1)!} + A_{m,2} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{(n+2)!} + \cdots \\
+ A_{m,m+1} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{(n + m + 1)!}.
\]
Taking \( i = 1, 2, \ldots, m+1 \) in the Lemma 3.2, the RHS of the equation above can be transformed into
\[
\sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j}{(a - j)_j (b - j)_j} 2F_1 \left( \frac{a - j, b - j}{c - j}; 1 \right) - \sum_{j=1}^{m+1} \sum_{k=0}^{i-1} \frac{A_{m,j} (c - i + k)_{i-k}}{(a - i + k)_{i-k} (b - i + k)_{i-k} k!}.
\]

For convenience, we let
\[
\begin{align*}
\psi^{(i)}(c - a - b + j) - \psi^{(i)}(c - a) &:= P_i(a, b, c, j), \\
\psi^{(i)}(c - j) + \psi^{(i)}(c - a - b + j) - \psi^{(i)}(c - a) - \psi^{(i)}(c - b) &:= S_i(a, b, c, j), \\
\Omega_i \left( H_{j}(a - j), H^{(2)}_{j}(a - j), \ldots, H^{(i)}_{j}(a - j) \right) &:= \alpha_i(a, j), \\
\Omega_i \left( P_{0}(a, b, c, j), P_{1}(a, b, c, j), \ldots, P_{i-1}(a, b, c, j) \right) &:= \beta_i(a, b, c, j), \\
\Omega_i \left( H_{j-k}(a - j + k), H^{(2)}_{j-k}(a - j + k), \ldots, H^{(i)}_{j-k}(a - j + k) \right) &:= \tau_i(a, j, k), \\
\Omega_i \left( -H_{j}(c - j), -H^{(2)}_{j}(c - j), \ldots, -H^{(i)}_{j}(c - j) \right) &:= \eta_i(c, j)
\end{align*}
\]
Differentiating the equation (6) at $x = (5)$.

By the Lemma 2.3, we have

Theorem 3.4. For any positive real numbers $a, b$ and $c - a - b > 0$, with $l \in \mathbb{N}, m \in \mathbb{N}$ and $m + 1 < c$, we have

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} \Omega_i \left( -H_n(a), -H_n^{(2)}(a), \ldots, -H_n^{(l)}(a) \right)$$

$$= \sum_{j=1}^{m+1} A_{m,j} (c - j) \Gamma(c - j) \Gamma(c - a - b + j) \Gamma(c - a) R_i(a, b, c; j) - \sum_{j=1}^{m+1} A_{m,j} T_i(a, b, c; j),$$

where

$$R_i(a, b, c; j) = \sum_{h=0}^{l} \left( \frac{l}{h} \right) a_{l-h}(a, j) \beta_h(a, b, c, j),$$

$$T_i(a, b, c; j) = \sum_{k=0}^{j-1} \frac{(c - j + k)_{j-k}}{(a - j + k)_{j-k} (b - j + k)_{j-k}} \tau_i(a, j, k) \frac{1}{k!},$$

and $A_{m,j}$ is the same as in Lemma 3.1.

Proof. Setting $a + x \to a$ in formula (3), it follows that

$$\sum_{n=0}^{\infty} \frac{(a + x)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1}$$

$$= \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_{j}(a + x - j)_{j}(b - j)_{j}}{(a - j + x)_{j}(b - j)_{j}} {2F_1} \left( \frac{a + x - j, b - j}{c - j}; 1 \right) - \sum_{j=1}^{m+1} A_{m,j} Q(a + x, b, c; j).$$

Differentiating the equation (6) at $x = 0$, one has

$$\frac{d^l}{dx^l} \left( \sum_{n=0}^{\infty} \frac{(a + x)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} \right) \bigg|_{x=0} = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} \frac{d^l(a + x)_n}{dx^l} \bigg|_{x=0}.$$
\[
\frac{d^l}{dx^l} \left( \frac{\Gamma(c - j) \Gamma(c - a - b + j - x)}{\Gamma(c - a - x) \Gamma(c - b)} \right) \bigg|_{x=0} = \frac{\Gamma(c - j) \Gamma(c - a - b + j)}{\Gamma(c - b) \Gamma(c - a)} (-1)^j \beta_j(a, b, c, j).
\]

Hence, it follows that
\[
\frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0} = \sum_{h=0}^{l} \binom{l}{h} (-1)^{l-h} \frac{1}{(a - j)_j} \alpha_{l-h}(a, j) \Gamma(c - j) \frac{\Gamma(c - a - b + j)}{\Gamma(c - a)} \beta_h(a, b, c, j)
\]
\[
= (-1)^j \sum_{h=0}^{l} \binom{l}{h} \frac{1}{(a - j)_j} \Gamma(c - a - b + j) \alpha_{l-h}(a, j) \beta_h(a, b, c, j)
\]
\[
= (-1)^j \frac{1}{(a - j)_j} \frac{\Gamma(c - a - b + j)}{\Gamma(c - a)} \sum_{h=0}^{l} \binom{l}{h} \alpha_{l-h}(a, j) \beta_h(a, b, c, j).
\]

Therefore,
\[
\frac{d^l}{dx^l} \left[ \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j}{(a + x - j)_j(b - j)_j} {}_2F_1 \left( \frac{a + x - j, b - j}{c - j}; 1 \right) \right] \bigg|_{x=0} = \sum_{j=1}^{m+1} A_{m,j} (c - j)_j \frac{\Gamma(c - j) \Gamma(c - a - b + j)}{\Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]
\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j \Gamma(c - j) \Gamma(c - a - b + j)}{(b - j)_j \Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]
\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j \Gamma(c - j) \Gamma(c - a - b + j)}{(b - j)_j \Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]
\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j \Gamma(c - j) \Gamma(c - a - b + j)}{(b - j)_j \Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]
\[
= \sum_{j=1}^{m+1} A_{m,j} (c - j)_j \frac{\Gamma(c - j) \Gamma(c - a - b + j)}{\Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]
\[
= \sum_{j=1}^{m+1} A_{m,j} (c - j)_j \frac{\Gamma(c - j) \Gamma(c - a - b + j)}{\Gamma(c - b) \Gamma(c - a)} \frac{d^l}{dx^l} \left( \frac{1}{(a + x - j)_j} \frac{\Gamma(c - a - b + j - x)}{\Gamma(c - a - x)} \right) \bigg|_{x=0}
\]

Rearranging the terms above, the equation \([5]\) is proven. \(\Box\)
Theorem 3.5. For any positive real numbers $a, b, c$ and $c - a - b > 0$, with $l(l \in \mathbb{N}), m \in \mathbb{N}$ and $m + 1 < c$, we have

\begin{equation}
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n! \, n + m + 1} \Omega_l \left( H_n(c), H_n^{(2)}(c), \ldots, H_n^{(l)}(c) \right) = \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j}{(a - j)_j(b - j)_j} \frac{\Gamma(c - j) \Gamma(c - a + b + j) \Gamma(c - a) \Gamma(c - b)}{\Gamma(c - b + x) \Gamma(c - a + x)} \xi_l(a, b, c, j) - \sum_{j=1}^{m+1} A_{m,j} \tilde{T}_l(a, b, c, j),
\end{equation}

where

\begin{align*}
\tilde{T}_l(a, b, c, j) &= \sum_{k=0}^{j-1} \frac{(c - j + k)_j}{(b - j + k)_j} \frac{\eta_k(a, b, c, j)}{k!}, \\
\tilde{\eta}_l(a, b, c, j) &= \sum_{h=0}^{l} \binom{l}{h} (-1)^h \eta(-h(a, c, j))
\end{align*}

and $A_{m,j}$ is the same as in Lemma 2.1.

Proof. Setting $c + x \rightarrow x$ in identity (4), one has

\begin{equation}
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n! \, n + m + 1} \Gamma(c - j) \Gamma(c - a + b + j) = \sum_{j=1}^{m+1} A_{m,j} \frac{(c - j)_j}{(a - j)_j(b - j)_j} 2F_1 \left( \begin{array}{c} a - j, b - j \\ c + x - j \end{array} ; 1 \right) - \sum_{j=1}^{m+1} A_{m,j} Q(a, b, c + x; j).
\end{equation}

Firstly, we differentiate the LHS of equation (8) $l$ times at $x = 0$, it follows that

\begin{align*}
\frac{d^l}{dx^l} \left( \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n! \, n + m + 1} \right) \bigg|_{x=0} &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n! \, n + m + 1} (-1)^l \Omega_l \left( H_n(c), H_n^{(2)}(c), \ldots, H_n^{(l)}(c) \right) \\
\frac{d^l}{dx^l} \frac{(c - j)_j}{(a - j)_j(b - j)_j} 2F_1 \left( \begin{array}{c} a - j, b - j \\ c + x - j \end{array} ; 1 \right) \bigg|_{x=0} &= \frac{\Gamma(c - j) \Gamma(c - a + b + j) \Gamma(c - a) \Gamma(c - b)}{\Gamma(c - b + x) \Gamma(c - a + x)} \xi_l(a, b, c, j),
\end{align*}

\begin{align*}
\frac{d^l}{dx^l} (c + x - j)_j \bigg|_{x=0} &= (-1)^l (c - j)_j \eta_l(a, b, c, j).
\end{align*}

So

\begin{align*}
\frac{d^l}{dx^l} \left( \sum_{j=1}^{m+1} A_{m,j} \frac{(c + x - j)_j}{(a - j)_j(b - j)_j} 2F_1 \left( \begin{array}{c} a - j, b - j \\ c + x - j \end{array} ; 1 \right) \right) \bigg|_{x=0} &= \sum_{j=1}^{m+1} A_{m,j} \frac{1}{(a - j)_j(b - j)_j} \frac{d^l}{dx^l} \left( \frac{\Gamma(c - j + x) \Gamma(c - a + b + j + x) \Gamma(c - a) \Gamma(c - b)}{\Gamma(c - b + x) \Gamma(c - a + x)} \right) \bigg|_{x=0}
\end{align*}
Corollary 3.6. Setting \( l = 1 \) in Theorem 3.4, there holds the following identity:

\[
\begin{align*}
= \sum_{j=1}^{m+1} & A_{m,j} \frac{1}{(a-j)_{j}(b-j)_{j}} \sum_{h=0}^{l} \left( \frac{l}{h} \right) (-1)^{l-h}(c-j)_{j} \eta_{h-k}(c,j) \\
\cdot & \frac{\Gamma(c-j)\Gamma(c-a-b+j)}{\Gamma(c-a)\Gamma(c-b)} \xi_{h}(a,b,c,j)
\end{align*}
\]

\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c-j)_{j}}{(a-j)_{j}(b-j)_{j}} \frac{\Gamma(c-j)\Gamma(c-a-b+j)}{\Gamma(c-a)\Gamma(c-b)} (-1)^{l} \\
\cdot \sum_{h=0}^{l} \left( \frac{l}{h} \right) (-1)^{h} \eta_{h-k}(c,j) \xi_{h}(a,b,c,j).
\]

By the similar method, we get

\[
\begin{align*}
\frac{d^{l}}{dx^{l}} \left( \sum_{j=1}^{m+1} A_{m,j} Q(a,b,c+x;j) \right) \bigg|_{x=0}
\end{align*}
\]

\[
= \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{1}{(a-j+k)_{j-k}(b-j+k)_{j-k} k!} \frac{1}{x^{k}} \left( (c+x-j+k)_{j-k} \right) \bigg|_{x=0}
\]

\[
= \sum_{j=1}^{m+1} A_{m,j} (-1)^{l} \sum_{k=0}^{j-1} \frac{(c-j+k)_{j-k}}{(a-j+k)_{j-k}(b-j+k)_{j-k} k!} \frac{1}{x^{k}} \delta(c,j,k).
\]

Rearranging the formulae above, the equation (7) is proven. \( \square \)

Corollary 3.6. Setting \( l = 1 \) in Theorem 3.4, there holds the following identity:

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n} \ n! \ n+m+1} H_{n}(a)
\end{align*}
\]

\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c-j)_{j}}{(a-j)_{j}(b-j)_{j}} \frac{\Gamma(c-j)\Gamma(c-a-b+j)}{\Gamma(c-a)\Gamma(c-b)} (-P_{0}(a,b,c,j) - H_{j}(a-j))
\]

\[
+ \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{(c-j+k)_{j-k}}{(a-j+k)_{j-k}(b-j+k)_{j-k} k!} H_{j-k}(a-j+k).
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.

Corollary 3.7. Setting \( l = 2 \) in Theorem 3.4, there holds the following identity:

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n} \ n! \ n+m+1} H_{n}^{2}(a)
\end{align*}
\]

\[
= \sum_{j=1}^{m+1} A_{m,j} \frac{(c-j)_{j}\Gamma(c-j)\Gamma(c-a-b+j)}{(a-j)_{j}(b-j)_{j}\Gamma(c-a)\Gamma(c-b)} \left\{ \begin{array}{l}
H_{j}^{2}(a-j) + H_{j}^{(2)}(a-j) \\
+2H_{j}(a-j)P_{0}(a,b,c,j) \\
+P_{0}(a,b,c,j)^{2} + P_{j}(a,b,c,j)
\end{array} \right\}
\]

\[
- \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{(c-j+k)_{j-k}}{(a-j+k)_{j-k}(b-j+k)_{j-k} k!} H_{j-k}^{2}(a-j+k) + H_{j-k}^{(2)}(a-j+k)
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.
Corollary 3.8. Setting \( l = 3 \) in Theorem 3.5, there holds the following identity:

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{H_n^3(a) - 3H_n(a)H_n^{(2)}(a) + 2H_n^{(3)}(a)}{n!} n + m + 1
\]

\[= (-1)^m \sum_{j=0}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{(c - j + k)_{j-k}}{(a + j + k)_{j-k}} \frac{1}{k!} \]

\[
\cdot \left\{ H_{j-k}^3(a - j + k) + 3H_{j-k}(a - j + k)H_{j-k}^{(2)}(a - j + k) + 2H_{j-k}^{(3)}(a - j + k) \right\},
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.

Corollary 3.9. Setting \( l = 1 \) in Theorem 3.5, there holds the following identity:

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} H_n(c)
\]

\[= \sum_{j=1}^{m+1} \frac{(c - j)_j}{(a - j)_j} \frac{1}{(b - j)_j} \frac{\Gamma(c - j)\Gamma(c - a + b + j)}{\Gamma(c - a)\Gamma(c - b)} \left( -S_0(a, b, c, j) - H_j(c - j) \right)
\]

\[+ \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{(c - j + k)_{j-k}}{(a + j + k)_{j-k}} \frac{H_{j-k}(c - j + k)}{k!},
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.

Corollary 3.10. Setting \( l = 2 \) in Theorem 3.5, there holds the following identity:

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} H_n^2(c) + H_n^{(2)}(c)
\]

\[= \sum_{j=1}^{m+1} \frac{(c - j)_j}{(a - j)_j} \frac{1}{(b - j)_j} \frac{\Gamma(c - j)\Gamma(c - a + b + j)}{\Gamma(c - a)\Gamma(c - b)} \left( H_j^2(c - j) - H_j^{(2)}(c - j) \right)
\]

\[+ 2H_j(c - j)S_0(a, b, c, j) + S_0(a, b, c, j)^2 + S_1(a, b, c, j)
\]

\[- \sum_{j=1}^{m+1} A_{m,j} \sum_{k=0}^{j-1} \frac{(c - j + k)_{j-k}}{(a + j + k)_{j-k}} \frac{H_{j-k}^2(c - j + k) - H_{j-k}^{(2)}(c - j + k)}{k!},
\]

where \( A_{m,j} \) is the same as in Lemma 3.1.

Corollary 3.11. Setting \( l = 3 \) in Theorem 3.5, there holds the following identity:

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{n!} \frac{1}{n + m + 1} \left[ H_n^3(c) + 3H_n(c)H_n^{(2)}(c) + 2H_n^{(3)}(c) \right]
\]
In this section, we give some applications of our main results, and we provide many closed summation expressions involving binomial coefficients and generalized harmonic numbers.

4. Applications

4.1. l=1.

Identity 4.1 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{((n+1)!)^2} O_n = \frac{8}{\pi} (3 - 2 \ln 2) - 4.
\]

Identity 4.2 \((a = \frac{1}{4}, b = \frac{3}{4}, c = 2, m = 0)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{3}{4})_n}{((n+1)!)^2} H_n \left(\frac{1}{4}\right) = \frac{81 \sqrt{3}}{16 \pi} (4 - 3 \ln 3) - \frac{27}{16}.
\]

Identity 4.3 \((a = \frac{1}{4}, b = \frac{3}{4}, c = 2, m = 0)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{3}{4})_n}{((n+1)!)^2} H_n \left(\frac{1}{4}\right) = \frac{128 \sqrt{2}}{9 \pi} \left(\frac{\pi}{2} - 3 \ln 2 + \frac{5}{3}\right) - \frac{64}{9}.
\]

Identity 4.4 \((a = \frac{1}{4}, b = \frac{6}{7}, c = 2, m = 0)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{7})_n (\frac{6}{7})_n}{((n+1)!)^2} H_n \left(\frac{1}{7}\right) = \frac{324}{25 \pi} \left(\sqrt{3 \pi} - 4 \ln 2 - 3 \ln 3 + \frac{14}{5}\right) - \frac{216}{25}.
\]

Identity 4.5 \((a = \frac{1}{4}, b = \frac{1}{3}, c = 3, m = 1)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{7})_n^2}{n!(n+2)!(n+2)} O_n = \frac{64}{27} - \frac{32}{3 \pi} + \frac{128 \ln 2}{27 \pi}.
\]

Identity 4.6 \((a = \frac{1}{3}, b = \frac{2}{3}, c = 3, m = 1)\).

\[
\sum_{n=0}^{\infty} \frac{(\frac{2}{3})_n (\frac{2}{3})_n}{n!(n+2)!(n+2)} H_n \left(\frac{2}{3}\right) = \frac{1701}{640} - \frac{2187 \sqrt{3}}{320 \pi} \left(\frac{73}{9} - 7 \ln 3\right).
\]
Identity 4.7 \((a = \frac{1}{4}, b = \frac{3}{4}, c = 3, m = 1 \text{ in } \text{[19]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!(n+2)!(n+2)} H_n \left(\frac{1}{4}\right) = \frac{2048}{441} - \frac{2048\sqrt{2}}{147\pi} \left(\frac{13\pi}{75} + \frac{1429}{1575} - \frac{26\ln 2}{25}\right).
\]

Identity 4.8 \((a = \frac{1}{6}, b = \frac{5}{6}, c = 3, m = 1 \text{ in } \text{[19]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!(n+2)!(n+2)} H_n \left(\frac{1}{6}\right) = \frac{124416}{21175} - \frac{23328}{148225\pi} \left(31\sqrt{3}\pi + \frac{7449}{55} - 124\ln 2 - 93\ln 3\right).
\]

Identity 4.9 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{((n+1)!)^2} H_{n+1} = \frac{16}{\pi} (3 - 4\ln 2).
\]

Identity 4.10 \((a = \frac{1}{3}, b = \frac{2}{3}, c = 2, m = 0 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{((n+1)!)^2} H_{n+1} = \frac{81\sqrt{3}}{16\pi} \left(7 - 6\ln 3\right).
\]

Identity 4.11 \((a = \frac{1}{4}, b = \frac{3}{4}, c = 2, m = 0 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{((n+1)!)^2} H_{n+1} = \frac{128\sqrt{2}}{27\pi} (13 - 18\ln 2).
\]

Identity 4.12 \((a = \frac{1}{6}, b = \frac{5}{6}, c = 2, m = 0 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{((n+1)!)^2} H_{n+1} = \frac{648}{25\pi} \left(\frac{31}{5} - 4\ln 2 - 3\ln 3\right).
\]

Identity 4.13 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 3, m = 1 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(n+2)!(n+2)} H_{n+2} = \frac{64}{27\pi} (-5 + 8\ln 2).
\]

Identity 4.14 \((a = \frac{1}{3}, b = \frac{2}{3}, c = 3, m = 1 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{n!(n+2)!(n+2)} H_{n+2} = \frac{2187\sqrt{3}}{32000\pi} (140\ln 3 - 143).
\]

Identity 4.15 \((a = \frac{1}{4}, b = \frac{3}{4}, c = 3, m = 1 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!(n+2)!(n+2)} H_{n+2} = \frac{2048\sqrt{2}}{1157625\pi} (16380\ln 2 - 10849).
\]

Identity 4.16 \((a = \frac{1}{6}, b = \frac{5}{6}, c = 3, m = 1 \text{ in } \text{[12]})\).

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!(n+2)!(n+2)} H_{n+2} = \frac{23328}{57066625\pi} (95480\ln 2 + 71610\ln 3 - 141819).
\]
4.2. \( l=2 \).

Identity 4.17 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{((n+1)!)^2} \left( H_{n+1}^2 + H_{n+1}(2) \right) = \frac{32}{3\pi} \left( 24 - \pi^2 - 36 \ln 2 + 24 \ln^2 2 \right).
\]

Identity 4.18 \((a = \frac{1}{2}, b = \frac{3}{2}, c = 2, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2 (\frac{3}{2})_n^2}{((n+1)!)^2} \left( H_{n+1}^2 + H_{n+1}(2) \right) = \frac{81\sqrt{3}}{16\pi} \left( 45 - 2\pi^2 - 42 \ln 3 + 18 \ln^2 3 \right).
\]

Identity 4.19 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{((n+1)!)^2} \left( H_{n+1}^2 + H_{n+1}(2) \right) = \frac{128\sqrt{2}}{81\pi} \left( 320 - 15\pi^2 - 468 \ln 2 + 324 \ln^2 2 \right).
\]

Identity 4.20 \((a = \frac{1}{2}, b = \frac{3}{2}, c = 2, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{((n+1)!)^2} \left( H_{n+1}^2 + H_{n+1}(2) \right) = \frac{648}{25\pi} \left( \frac{1872}{25} - \frac{11}{3} \pi^2 - \frac{62}{5} \ln 432 + \ln^2 432 \right).
\]

Identity 4.21 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 3, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n+2)!} \left( H_{n+2}^2 + H_{n+2}(2) \right) = \frac{128}{27\pi} \left( 67 - 2\pi^2 - 92 \ln 2 + 48 \ln^2 2 \right) - 8.
\]

Identity 4.22 \((a = \frac{1}{2}, b = \frac{2}{3}, c = 3, m = 0 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{2}{3})_n}{(n+1)!(n+2)!} \left( H_{n+2}^2 + H_{n+2}(2) \right) = \frac{729\sqrt{3}}{1600\pi} \left( 5969 - 200\pi^2 - 5220 \ln 3 + 1800 \ln^2 3 \right) - 9.
\]

Identity 4.23 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 3, m = 1 \text{ in } (13))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!(n+2)!(n+2)!} \left( H_{n+2}^2 + H_{n+2}(2) \right) = \frac{128}{243\pi} \left( -109 + 6\pi^2 + 180 \ln 2 - 144 \ln^2 2 \right).
\]

4.3. \( l=3 \).

Identity 4.24 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0 \text{ in } (11))\).
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{((n+1)!)^2} \left( O_n^3 - 3O_nO_n^{(2)} + 2O_n^{(3)} \right) = \frac{2}{\pi} \left( 120 - 3\pi^2 - 96 \ln 2 + 2\pi^2 \ln 2 + 36 \ln^2 2 - 8 \ln^3 2 - 12\zeta(3) \right) - 24.
\]

where \( \zeta(3) \) is the Apéry constant.
Identity 4.25 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 3, m = 1\) in (11)).
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2}}{n!(n+2)!(n+2)} \left(O_{n}^{3} - 3O_{n}O_{n}^{(2)} + 2O_{n}^{(3)}\right) \\
= \frac{16}{81\pi} \left(80\pi + \frac{27}{2}\pi^{3} + 494\ln 2 - 6\pi^{2}\ln2 - 168\ln^{2}2 + 24\ln^{3}2 + 36\zeta(3) - \frac{1978}{3}\right).
\]

Identity 4.26 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 2, m = 0\) in (14)).
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2}}{((n + 1)!)^{2}} \left(H_{n+1}^{3} + 3H_{n+1}H_{n+1}^{(2)} + 2H_{n+1}^{(3)}\right) \\
= \frac{32}{\pi} \left(60 - 3\pi^{2} - 96\ln 2 + 4\pi^{2}\ln2 + 72\ln^{2}2 - 32\ln^{3}2 - 12\zeta(3)\right).
\]

Identity 4.27 \((a = \frac{1}{2}, b = \frac{1}{2}, c = 3, m = 1\) in (14)).
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2}}{n!(n+2)!(n+2)} \left(H_{n+2}^{3} + 3H_{n+2}H_{n+2}^{(2)} + 2H_{n+2}^{(3)}\right) \\
= \frac{128}{27\pi} \left(5\pi^{2} + \frac{436}{3}\ln 2 - 8\pi^{2}\ln2 - 120\ln^{2}2 + 64\ln^{3}2 + 24\zeta(3) - \frac{772}{9}\right).
\]

Concluding remarks The main results in this text allow us to derive more closed expressions involving binomial coefficients and harmonic numbers. We believe most of our identities are new in the literature and are not amenable to a mathematical computer package.

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