On some methods for generating extremely multistable systems

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Abstract. A multistable dynamic system can demonstrate solutions with fundamentally different behavior depending on the choice of their initial conditions, which poses a threat to its use in practical engineering applications. On the other hand, the system’s multistability may be its undisputed advantage when it is used, for example, to hide information in communication systems and audio encryption schemes to improve the performance of secure communications, since in this case a special choice of initial conditions can play the role of a "secret key". In this paper, we propose some approaches to the generation of extremely multistable systems containing an infinite number of attractors using mathematical models of systems in Lurie form.

1. Introduction

The multistability of a real dynamic system can pose a threat in practical engineering applications because the behavior of the system cannot be guaranteed unambiguously. Such a system can demonstrate solutions with fundamentally different behavior depending on the choice of their initial conditions. For this reason, a large number of papers have been recently devoted to the study of multistable systems [1-8]. After finding solutions to the problems of synchronization of oscillations of chaotic systems, many researchers focused on the problems of using chaos in engineering applications. The fact is that the multistability of the system can be its undeniable advantage in the case when it is used, for example, to hide information in communication systems [9] and audio encryption schemes to improve the performance of secure communications [10], because in this case, a special choice of initial conditions can play the role of "secret key". Currently, chaotic signals and systems have been widely used in image encryption [11-13], secure communications [14], weak signal detection [15], and radar systems [16]. Therefore, in recent years there have been many studies devoted to the artificial construction of multistable dynamical systems containing an infinite number of coexisting chaotic attractors. [17-22].

Systems containing an infinite number of coexisting nontrivial attractors are called extremely multistable (infinitely multistable) [17]. The type of extreme multistability demonstrated by the systems constructed in [21,22] is typical only for systems whose order is higher than three. Recently, a class of dynamic systems called offset boostable dynamical systems was considered in the series of studies [17-19]. All mentioned studies have been based on the idea of introducing periodic functions in variable-boostable system proposed in [17]. Systems $\dot{X} = F(X), X = (x_1, x_2, ..., x_n)$ are called an offset boostable system by the variable $x_i$, if the replacement of the variable $x_i$ by $x_i + c$ with a suitable choice of the constant $c$ does not change the dynamics of other variables in the initial phase space. Such systems include cascade systems in particular
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\cdots, \\
\dot{x}_n &= f(\dot{x}_1, \dot{x}_2, \dot{x}_3, \ldots, x).
\end{align*}
\] (1)

Offset boosting of the variables \(x_2, x_3, \ldots, x_n\) in the system (1) can be obtained by introducing extra constants in the first \(n-1\) equations. In fact, by replacing \(x_1 \rightarrow \tilde{x}_1, x_2 \rightarrow \tilde{x}_2 + m, x_3 \rightarrow \tilde{x}_3 + n, \ldots, x_n \rightarrow \tilde{x}_n + p\) we arrive at the system

\[
\begin{align*}
\dot{\tilde{x}}_1 &= \tilde{x}_2 + m, \\
\dot{\tilde{x}}_2 &= \tilde{x}_3 + n, \\
&\cdots, \\
\dot{\tilde{x}}_n &= f(\dot{\tilde{x}}_1, \dot{\tilde{x}}_2, \dot{\tilde{x}}_3, \ldots, \tilde{x}).
\end{align*}
\] (2)

It is easy to see that such a transformation perform offset the variables \(x_2, x_3, \ldots, x_n\), leaving the dynamics of the variable \(\tilde{x}_1 = x_1\) unchanged, since the dynamics of this variable obeys the same equation (1). Obviously, the offset can be performed separately for any of these variables.

The basic idea of designing self-reproducing systems with an infinite \((n-1)\)-D grid of identical attractors, proposed in [17], is as follows: since the system (1) allows the displacement of the phase flow in variables \(x_2, x_3, \ldots, x_n\), we can try to replace these variables with periodic functions. If, after such a replacement, the new system has a chaotic attractor, then it has an infinite \((n-1)\)-D grid of identical attractors. For a system that is not a cascade system, the procedure for introducing periodic functions to generate a system containing a multidimensional grid of attractors is much more complex. The introduction of periodic functions in this case can significantly change the dynamics of the system and lead to the destruction of its attractors.

In this paper, we propose methods for constructing extremely multistable systems of order at least 3, possessing a 1-D and \((n-1)\)-D grid of attractors (self-excited or hidden [23]), based on the use of systems in Lurie form. The proposed methods are based on some approaches to the study of multidimensional systems in Lurie form, developed in [24]. In Section 2, a third-order system is constructed that has a 1-D grid of attractors, one third of which are hidden, and two-thirds are self-excited. In Section 4, systems with a 1-D grid of chaotic attractors are constructed. Finally, in Section 3, three-dimensional systems have been constructed with a 2-D grid of chaotic attractors.

2. Generation of Extremely Multistable Systems Using systems in Lurie Form

The system in Lurie form is the system of the following type

\[
\dot{x} = Ax + b\varphi(\sigma), \sigma = c^*x,
\] (3)

where \(A\) is a constant \(n \times n\)-matrix, \(b\) and \(c\) are constant \(n\)-vectors, \(\ast\) is the transposition sign (or Hermitian conjugation in the complex case), and \(\varphi(\sigma)\) is a continuous, piecewise differentiable scalar function. Let \(I\) be the identity matrix. For system (3), we define a fractionally rational function of a complex argument \(p\) as follows: \(\chi(p) = c^*(A - pI)^{-1}b\). Let \(\chi(p) = m(p)[n(p)]^{-1}\), with the polynomial in the denominator of a fraction having degree \(n\) and irreducible with its numerator. In this case, it is said [25] that the transfer function is nondegenerate. Let us put in (3) \(\varphi(\sigma) = \mu\sigma\) i.e. consider the linear system

\[
\dot{x} = (A + \mu bc^*)x
\] (4)
We will suppose that number $\mu$ is chosen so that the system (4) has no periodic solution, i.e. the matrix $A + \mu bc^*$ has no pure imaginary eigenvalues. Since under continuous change of $\mu$ the spectrum of the matrix $A + \mu bc^*$ is continuously changing, then on the $(\sigma, \varphi)$-plane we can construct a sector $S[\nu_1, \nu_2] = \{\sigma, \varphi \colon \nu_1 \leq \frac{\varphi}{\sigma} \leq \nu_2\}$ such that all linear systems with $\varphi = \mu \sigma, \nu_1 \leq \mu \leq \nu_2$ have the same number of eigenvalues in the ring-hand half-plane and do not have them on the imaginary axis. Extending the sector $S[\nu_1, \nu_2]$, we arrive at values $\nu_1 = \mu_j, \nu_2 = \mu_j + 1$, such that

$$1 + \mu_j \lambda(i\omega_j) = 0, 1 + \mu_j+1 \lambda(i\omega_{j+1}) = 0.$$  

(5)

There can obviously be only finitely many numbers $\mu$ for which the condition (5) is fulfilled. We join to them the values $-\infty$ and $\infty$, and, in the case of degenerate matrix, also the number $\mu = 0$, and put

$$\mu_1 = -\infty < \mu_2 < \cdots < \mu_p < \mu_{p+1} = \infty.$$ 

It follows from the definition of the numbers (5) that the matrices of the linear systems for $\mu_k < \mu < \mu_{k+1}$ have the same number $k$ of eigenvalues in the ring-hand half-plane and do not have them on the imaginary axis. Let us call the open sector $S[\mu_k, \mu_{k+1}]$ a sector of linear stability for $k = 0$ or a sector of linear instability of degree $k$ for $k > 0$. Thus, the $(\sigma, \varphi)$-plane is divided by the straight lines $f = \mu_k \sigma$ into $l$ sectors $S[\mu_k, \mu_{k+1}], h = 1, 2, \ldots, l$.

To generate an extremely multistable system of the form (3), we use one result obtained in [24]. We present the formulation of the theorem established in [24] in a form convenient for us.

**Theorem.** Suppose that in the system (3), the matrix $A$ has one zero eigenvalue, $n - 1$ eigenvalues with negative real parts, the transfer function $\chi(p)$ does not degenerate, $\lim_{p \to \infty} p \chi(p) = -e^*b > 0$ and the following conditions are satisfied:

1) There are numbers $\mu_1 < 0, \mu_2 > 0, \lambda_1 > 0$ such that for all $\omega \geq 0$ the following relation is true

$$Re[1 + \mu_1 \lambda(i\omega - \lambda_1)]^*[1 + \mu_2 \lambda(i\omega - \lambda_1)] > 0.$$  

(6)

Wherein, for some $\tilde{\mu} \in (\mu_1, \mu_2)$, the matrix $A + \tilde{\mu} bc^*$ has exactly two eigenvalues with positive real parts and does not have them in the strip $-\lambda_1 \leq Rep \leq 0$.

2) There exists a number $\lambda_2 > 0$ such that the matrix $A + \lambda_2 I$ has one positive eigenvalue, and for all $\omega \geq 0$ the following inequality holds

$$Re\chi(i\omega - \lambda_2) + \nu|\chi(i\omega - \lambda_2)|^2 \leq 0.$$  

(7)

for some $\nu > 0$.

Then we can specify such a $\Delta$-periodic continuous function $\varphi(\sigma)$ having exactly two zeros on the period that the system (3) will have in each strip $X_k = \{x : (k - 1)\Delta \leq e^*x \leq k\Delta\} (k = 0, \pm 1, \pm 2, \ldots)$ any given number of orbitally asymptotically stable cycles.

The algorithm for constructing a function $\varphi(\sigma)$ with the necessary properties is described in [24]. The main idea of this algorithm, speaking somewhat inaccurately, is to construct this function so that its graph stays alternately long enough in the sectors of linear stability and linear instability of degree 2 of system (3). It is clear that if system (3) with a periodic nonlinearity and two equilibrium states on a period has at least three cycles on each period, then it has an infinite number of hidden cycles. The latter means that the basin of attraction of these cycles will not intersect with a small neighborhood of any of the equilibrium states of the system.

**Example 1.** Let us consider a system of the form (3) with a transfer function $\chi(p) = (p^3 + 3p^2 + 1.92p)^{-1}$ For such a transfer function, condition (6) of the theorem stated above is satisfied for $\mu_1 = -0.5, \mu_2 = 10, \lambda_1 = 1.6$, and also condition (7) of this theorem for $\lambda_2 = 0.4, \nu = 0.352$. Sector $(-\infty; 0)$ is the sector of instability of degree 1, sector
(0, 5.76) is the sector of linear stability, sector (5.76, ∞) is the sector of instability of degree 2. Following the procedure described in [24], we construct an odd periodic function \(\varphi(\sigma)\) of period \(\Delta = 0.12(e^{12} + 2)\) so that its graph alternately “long enough” remains in each of sectors (5.76, ∞) and (0, 5.76), and the system has three orbital asymptotically stable cycles in each period. One of these cycles will be hidden, and the regions of attraction of the other two will intersect with a small neighborhood of the equilibrium states of the system. By virtue of the periodicity of the nonlinearity built, the system will have an infinite number of cycles, one third of which will be hidden and two thirds will be self-excited.

On the half-period \([0, \frac{\Delta}{2}]\), we define the function \(\varphi(\sigma)\) as follows:

\[
\varphi(\sigma) = \begin{cases} 
6\sigma, & 0 \leq \sigma \leq 0.01, \\
0.06, & 0.01 \leq \sigma \leq 0.02, \\
0.06e^{200\sigma - 4}, & 0.02 \leq \sigma \leq 0.038, \\
0.06e^{11.2 - 200\sigma}, & 0.038 \leq \sigma \leq 0.05, \\
0.06e^{12}, & 0.05 \leq \sigma \leq 0.08, \\
0.06e^{200\sigma - 14.8}, & 0.08 \leq \sigma \leq 0.1, \\
0.06e^{25.2 - 200\sigma}, & 0.1 \leq \sigma \leq 0.12, \\
0.06e^{12} - \sigma + 0.12, & 0.12 \leq \sigma \leq 0.06e^{12} + 0.12.
\end{cases}
\]

Figure 1. The graph of function \(\varphi(\sigma)\) on a half-period.  

Figure 2. Projection on the plane \((x_1, x_2)\) of the fragment 1-D grid of attractors.

Figure 1 shows the graph of function \(\varphi(\sigma)\) on a half-period. Figure 2 shows the projections of the trajectories of the system (3) on the plane \((x_1, x_2)\) with an offset of one period. From each triple of cycles, the “middle” cycle is hidden.

3. System Containing a 1-D Grid of Hidden Chaotic Attractors

Suppose the existence of a number \(k\) such that the polynomial \(n(p) + km(p)\) has a zero root. Then the system (3) can be written as

\[
\dot{x} = A_1x + bg(\sigma), \quad \sigma = c^*x,
\]

where \(g(\sigma) = \varphi(\sigma) - k\sigma, A_1 = A + kbc^*\), and the matrix \(A_1\) has zero eigenvalue. Let \(A_1d = 0\). From the assumption that the function \(\chi(p)\) is nondegenerate, it follows [25] \(c^*d = \mu \neq 0\). Let \(g(\sigma_0) = 0\). Then \(x_0 = \sigma_0\mu^{-1}d\) is the equilibrium state of system (8).
Let the function $g(\sigma)$ in system (8) be odd, $g(\pm \sigma) = 0$, and $g(\sigma_0) \neq 0$ when $\sigma \neq \pm \sigma_0, \sigma \neq 0$. Then there are exactly three equilibrium states of system (8): $x = 0, x = \pm \sigma_0 \mu^{-1} d$. Suppose that the system under consideration has a chaotic attractor $\Omega$, located entirely in the strip $\Pi = \{x : -\sigma_1 < c^* x < \sigma_1\}$, where $\sigma_1 \geq \sigma_0$. That is, for any $x_0 \in \Omega$, the relation $c^* x(t, x_0) \subset \Pi$ holds at $t \geq 0$. Replacing in (8) the function $g(\sigma)$ by the $2\sigma_1$-periodic function $\psi(\sigma)$, which coincides with $g(\sigma)$ on the segment $[-\sigma_1, \sigma_1]$, we obtain a system with an infinite number of equilibrium states containing 1-D grid of attractors obtained by shifting the attractor of system (8) in the direction of the vector $d$.

**Example 2.** Consider the system (8) with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} -19 \\ -3.5 \\ -3.2 \end{pmatrix}.$$ 

For this system, sector $(-\infty, 0)$ is the sector of instability of degree 1, sector $(0, 0.08842)$ is the sector of linear stability, sector $(0.08843, 1.00979)$ is the sector of instability of degree 2, and finally sector $(1.0098, +\infty)$ is the sector of linear stability. As nonlinearity, we choose the function $g(\sigma) = 1.5 \text{atan}(\sigma) - 0.28 \sigma$. It turns out that in the case under consideration, system (8) has a pair of hidden chaotic twin attractors, which were discovered using the search procedure for hidden attractors proposed in [26] and are presented in Figure 3. These attractors can be detected by numerical integration of the system with initial conditions $(\pm 0.177, \pm 0.512, \mp 0.445)$. Now we replace in system (8) the nonlinearity $g(\sigma)$ with a periodic function of period $\Delta = 15.4522$. Figure 4 shows the projection onto the plane $(x_1, x_2)$ of a fragment of 1-D grid of hidden chaotic attractors of new system.

![Figure 3. Hidden chaotic twin attractors of the system (8).](image1)

![Figure 4. The projection onto the plane $(x_1, x_2)$ of a fragment of 1-D grid of hidden chaotic twin attractors.](image2)

The points belonging to the hidden attractors of the 1-D grid presented in Figure 4 were searched as follows. Let $x_0$ be any point on the attractor of system (8), and $d$ be the eigenvector of the matrix $A_1$ corresponding to its zero eigenvalue. Set $s = \Delta d(c^* d)^{-1}$. Then the points $x_j = x_0 + js, j = 0, \pm 1, \pm 2, \ldots$ belong to the attractors of the 1-D grid. All attractors of 1-D grid have the same Lyapunov exponents $\Lambda_1 = 0.094, \Lambda_2 = 0, \Lambda_3 = -1.015$ and the Kaplan-Yorke dimension $D_{KY} = 2.093$. 

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Example 3. Consider the classic Chua system
\[
\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y, \\
f(x) &= rx + 0.5(m - r)(|x + 1| - |x - 1|).
\end{align*}
\]

Let us set the following parameter values: \(\alpha = 9.8, \beta = 13.37, m = -1.099, r = -0.7143\). Putting \(g(x) = x + f(x)\), we obtain a system of the form (8), in which
\[
A_1 = \begin{pmatrix}
0 & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix},
\]
\[
b = \begin{pmatrix}
-\alpha \\
0 \\
0
\end{pmatrix},
\]
\[
c = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}.
\] (9)

The system (8) - (9) has three equilibrium states: \((\mp r - m + 1, 0, \pm r - m + 1)\) and \((0, 0, 0)\). In this case, a double scroll chaotic attractor, presented in Figure 5, is excited from a neighborhood of symmetric equilibrium states.

![Figure 5. Double scroll chaotic attractor of system (8)-(9).](image)

![Figure 6. Graphs of functions \(g(\sigma)\) (solid line) and \(\psi(\sigma)\) (dashed line).](image)

We set \(\sigma_0 = (r - m)(r + 1)^{-1}\) and replace the nonlinearity in system (8) - (9) with a 4.4\(\sigma_0\)-periodic function \(\psi(\sigma)\) which coincides with \(g(\sigma)\) on \([-1.6\sigma_0, 1.6\sigma_0]\). Graphs of functions \(g(\sigma)\) (solid line) and \(\psi(\sigma)\) (dashed line) are shown in Figure 6. The system with a new nonlinearity has a 1-D grid of identical double scroll attractors, a fragment of which is shown in Figure 7. Lyapunov exponents of these attractors \(\Lambda_1 = 0.3479, \Lambda_2 = 0, \Lambda_4 = -3.0063\) and their Kaplan-Yorke dimension \(D_{KY} = 2.113\).

It is clear that the above method does not allow generating systems that contain grids of attractors of dimensions greater than 1. In Section 4, we show that multidimensional system (3) can always be reduced using a non-singular linear transformation to a cascade type system. The latter circumstance will allow us, in particular, to use system (3) to construct a system with a 2-D grid of identical attractors.

4. Generation of Systems with a Multidimensional Grid of Chaotic Attractors

The following statement is well known[25]: "Two systems of the form (3) with the same transfer function \(\chi(p)\) are equivalent up to a non-singular linear transformation of their coordinates". If
\[
\chi(p) = \frac{c_0 + c_1 p + \ldots + c_{n-1} p^{n-1}}{a_0 + a_1 p + \ldots + a_{n-1} p^{n-1} + p^n},
\]
then, as shown in [25], system (3) with a non-singular linear transformation \( x = My \) can be reduced to the form

\[
\dot{y} = A_2 y + b_2 \varphi(\sigma), \quad \sigma = c_2^* y
\] (10)

\[
A_2 = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0 & -a_1 & -a_2 & \ldots & -a_{n-1}
\end{pmatrix},
\] (11)

\[
b_2 = \text{col}(0, 0, \ldots, 0, 1), \\
c_2 = \text{col}(-c_0, -c_1, \ldots, -c_{n-1}).
\]

In this case, the transformation matrix \( M \) can be found as a solution to the system of equations

\[ AM = A_2 M, \quad b = M b_2, \quad c = M^* c. \] (12)

System (10) - (11) is a cascade type system. If system (3) had an attractor, then system (10) - (11) obviously also has an attractor that you can try to “clone” by introducing periodic functions into the system. Thus, one can try to construct a multistable system containing a \((n-1)\)-D grid of attractors.

**Example 4.** Let us return to the system considered in Example 2. This is a cascade type system, so you can try to replace two variables in this system with periodic functions so that the new system has a 2-D grid of hidden chaotic attractors. Making the replacement \( x_2 \rightarrow \sin(0.5x_2), x_3 \rightarrow \sin x_3 \), we get the system

\[
\begin{align*}
\dot{x}_1 &= \sin 0.5x_2, \\
\dot{x}_2 &= \sin x_3, \\
\dot{x}_3 &= -\sin 0.5x_2 - \sin x_3 + \varphi(-19x_1 - 3.5 \sin 0.5x_2 - 3.2 \sin x_3).
\end{align*}
\] (13)

The system (13) has an infinite number of stable equilibrium states of the form \((0, 4\pi k, 2\pi m), k \in \mathbb{Z}, m \in \mathbb{Z}\) with eigenvalues of the Jacobi matrix \((-0.0254 \pm 1.5452i, -4.8531),\)
as well as an infinite number of equilibrium states of the form \((±x_1, 4\pi k, 2\pi m), k \in \mathbb{Z}, m \in \mathbb{Z}\) with eigenvalues of the Jacobi matrix \((1.2726, -0.7278 ± 1.173i)\). Here \(x_1\) is the solution of the equation \(1.5atan(19x_1) = 5.32x_1\). The system (13) also has a 2-D grid of hidden chaotic attractors with the same Lyapunov exponents \(\Lambda_1 = 0.082, \Lambda_2 = 0, \Lambda_3 = 0.888\) and the Kaplan-Yorke dimension \(D_{KY} = 2.092\). A fragment of this grid (projection on the plane \((x_2, x_3)\)) is shown in Figure 8.

**Figure 8.** Fragment of a 2-D grid of attractors systems (13).

**Figure 9.** Double scroll chaotic attractor of system (15).

**Example 5.** Let us now return to the Chua system discussed in Example 3. For the selected parameter values, its transfer function is

\[
\chi(p) = \frac{9.8p^2 + 9.8p + 131.026}{p^3 + p^2 + 3.57p}
\]

(14)

According to the transfer function (14), we write out the cascade system of the form (10) - (11)

\[
\dot{x} = y, \\
\dot{y} = z, \\
\dot{z} = -3.57y - z + \varphi(\sigma), \\
\sigma = 131.026x - 9.8y - 9.8z.
\]

(15)

From the neighborhood of the equilibrium states \((±\sigma_0(131.026)^{-1}, 0, 0)\) of system (15), a double scroll chaotic attractor is excited, shown in Figure 9. The Lyapunov exponents of this attractor are \(\Lambda_1 = 0.3479, \Lambda_2 = 0, \Lambda_3 = -3.0063\) and its Kaplan-Yorke dimension is \(D_{KY} = 2.113\).

Replace in system (15) the variables \(y\) and \(z\) with periodic functions: \(y \rightarrow 0.06525 \tan(16y), z \rightarrow 0.125 \tan(8z)\). The new system

\[
\dot{x} = 0.06525 \tan(16y), \\
\dot{y} = 0.125 \tan(8z), \\
\dot{z} = -0.223125 \tan(16y) - 0.125 \tan(8z) + \\
\phantom{=} + \varphi(-131.026x - 0.6125 \tan(16y) - 1.225 \tan(8z)).
\]
has an infinite number of equilibrium states of the form \((0, \frac{\pi k}{16}, \frac{\pi m}{8})\) and \((\pm \sigma_0, \frac{\pi k}{16}, \frac{\pi m}{8})\) and a 2-D grid of chaotic attractors-clone excited from neighborhoods of equilibrium states \((0, \frac{\pi k}{16}, \frac{\pi m}{8})\). These attractors are shown in Figure 10 (projection onto the plane \((y, z)\)). All attractors have the same Lyapunov exponents \(\Lambda_1 = 0.3098, \Lambda_2 = 0, \Lambda_3 = -3.3132\) and the Kaplan-Yorke dimension \(D_{KY} = 2.093\).

The approach to generating extremely multistable systems considered in this section allows using numerous well-known results concerning the existence of chaotic attractors of systems in Lurie form for constructing dynamic systems with a multidimensional grid of chaotic attractors. Now we will illustrate the implementation of the idea described above with the example of the system considered in [18], for which the fact of the existence of a chaotic attractor is known.

**Example 6.** Let us consider a system of the form (3) with

\[
A = \begin{pmatrix} 0 & 1 & -0.26 \\ 1 & 0 & -1 \\ 0 & 1.64 & -0.7264 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

In [18], it is indicated that the system under consideration has a chaotic attractor that can be visualized numerically at the start of a computational procedure from point \(q = \text{col}(-0.1, 0, -0.25)\).

Here \(\chi(p) = (1 - p^2)(p^3 - 0.7264p^2 + 0.64p - 0.3)^{-1}\). Therefore, by a non-degenerate linear transformation, this system can be transformed to the form

\[
\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = 0.3x - 0.64y - 0.7264z + (z - x)^2.
\]

Finding the matrix \(M\) from relations (12) and multiplying the matrix \(M^{-1}\) by the vector \(q\), we find the point \(q_1 = \text{col}(0.10725011, -0.02788503, 0.14274989)\). Starting from this point, the computational procedure reaches the chaotic attractor of system (17), the projection of which on the \((y, z)\) plane is shown in Figure 11.

Now let us replace in system (17) the variables \(y\) and \(z\) with periodic functions: \(y \rightarrow 0.125 \sin(8y), z \rightarrow 0.125 \sin(8z)\). If the system thus obtained has a chaotic attractor, then,
as mentioned above, it has an infinite 2-D grid of identical attractors-clone. The numerical experiment shows that the described scenario is implemented in this case. Herewith, the attractors of the new system are excited from an arbitrarily small neighborhood of its equilibrium states \((0, \frac{\pi k}{4}, \frac{\pi m}{4})\), \(k \in Z, m \in Z\). The projection of a fragment of the grid of self-excited identical attractors of the new system onto the plane \((y, z)\) is shown in Figure 12.

![Figure 11. Hidden attractor of Sprott system.](image)

**Figure 11.** Hidden attractor of Sprott system.

![Figure 12. 2-D grid of chaotic attractors of system (17) after the introduction of periodic functions.](image)

**Figure 12.** 2-D grid of chaotic attractors of system (17) after the introduction of periodic functions.

5. Conclusions

The method for constructing multidimensional grids of strange attractors has potential application in chaos-based engineering such as secure communication and weak signal detection, where initial conditions are important for determining the dynamics of the systems. In chaos-based secure communication, the unpredictability of the initial condition can additionally enhance the security of communication [29]. Chaos also has potential application in weak signal detection because chaotic systems are sensitive to certain signals and immune to noise at the same time, while the multistability with infinitely many attractors provides the possibility of intermittent transitions between order and chaos which is helpful for signal detection [30].

In the present paper, we discuss various approaches to the generation of extremely multistable systems containing an infinite grid of hidden attractors. Special attention is paid to multidimensional systems in Lurie form. Methods of transformation of such systems have been discussed, which allow to bring them to cascade type systems by choosing a special basis. Thus, it is possible, using numerous examples of the existence of hidden attractors in systems in Lurie form, to generate systems with an infinite grid of hidden attractors, without resorting to exhaustive computer search.

6. References

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