Noncommutative localization in smooth deformation quantization

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Abstract

In this paper we shall show the equivalence of analytic localization and noncommutative algebraic localization (in the general and in Ore’s sense) for algebras of smooth deformation quantization for several situations. The proofs are based on old work by Whitney, Malgrange and Tougeron on the commutative algebra of smooth function rings from the 60’s and 70’s. We comment on the more general situation of localizations of differential star products of commutative algebras equipped with a multiplicative subset.
Introduction

Localization in commutative algebra means a universal construction where a set of chosen elements in a given commutative ring is made invertible (they will become denominators): the outcome is called a ring of fractions. The classical example is the well-known passage from the integers to the field of rational numbers. It is a very important tool in algebraic and analytical geometry. In differential geometry, however, localization is rather used in the analytic sense, i.e. the passage from globally defined smooth functions to those which are only defined on an open subset. It follows from the classical works by Whitney, Malgrange [23] and Tougeron [38] that these analytical localizations are often isomorphic to certain algebraic localizations in the smooth (or even $C^k$, $k \in \mathbb{N}$) case, see also [11], [28] and the book [24].

For noncommutative algebras the problem of algebraic localization has two solutions: a general construction (see e.g. [20, p.289]), and the better known solution initiated by Ø.Ore [26] in the 1930’s requiring additional conditions on the multiplicative subset, the famous Ore conditions. It turns out that the general construction is rather inexplicit and in some situations not very practical. On the other hand, the more particular Ore localization shares almost all properties of the commutative localization.

In this work we would like to study noncommutative localization of algebras arising in deformation quantization: in this theory –invented by [6] in 1978– formal associative deformations of the algebra of all smooth complex-valued functions on a Poisson manifold, so-called star products, are studied aiming at an interpretation of the noncommutative multiplication of operators used in quantum mechanics. It is well-known that the first order commutator of such a deformation always gives rise to a Poisson bracket, but it is highly non-trivial to show that every Poisson bracket arises as a first order commutator of a deformation: this latter result is the famous Kontsevich formality Theorem, [18]. We consider star products given by formal power series of bidifferential operators (as almost every-one): these multiplications immediately define star products of locally defined functions by suitable ‘restrictions’. We mention that noncommutative localization has been used for the algebras in noncommutative geometry to describe inverses of functions appearing in passing to coordinates, see e.g. the recent work [2], [3], and [4].

In this article we have chosen the algebra of smooth functions on a smooth manifold and not a framework of algebraic or analytical geometry which would have required a sheaf theoretic approach: firstly, historically deformation quantization has been formulated in a differential geometry framework and has farther reaching existence and uniqueness results for smooth functions, secondly, in the smooth world there is no urgent need to pass to sheaves which simplifies the exposition, and thirdly, the commutative algebra of smooth function algebras seems to be of a different, ‘funnier’ kind (not Noetherian) which we liked to rediscover from the classical literature, in particular from the books [38] and [24].

We first show that this analytical localization of star product algebras is isomorphic to the algebraic localization with respect to the set of all those formal power series of smooth functions whose zeroth order term does nowhere vanish on the given open set. As a by-product we have the result that this multiplicative set satisfies the right (and also left) Ore condition.

In a similar way we can show that the set of all germs of a star product algebra at a given point of the manifold –defined in analytic terms– is isomorphic to the noncommutative localization of the complement of the maximal ideal of all those formal power series of functions whose term of order zero vanishes at the point.

We also sketch a more general algebraic framework inspired by the two preceding results: given a commutative associative unital algebra $A$ over a commutative ring $K$, a multiplicative subset $S_0$ of $A$, and a bidifferential star product $\star$ (the bidifferential operators are defined in the well-known
algebraic sense) then the following two constructions can be compared: first, localizing first the
bidifferential operators à la G. Vezzosi [39] there is a star product \( \star_{S_0} \) on \( A_{S_0}[[\lambda]] \), the formal power
series whose terms are in the localized algebra \( A_{S_0} \), and secondly, considering first the natural
‘deformation of \( S_0 \), \( S = S_0 + \lambda A[[\lambda]] \), which is a noncommutative multiplicative subset of the star
product algebra \( (A[[\lambda]], \star) \) there is the noncommutative localization (a priori in the general sense) of
\( A[[\lambda]] \) with respect to the deformation \( S \) of \( S_0 \). Two quite natural questions arise: ‘Does localization
commute with deformation?’ – on which we give a positive answer in case \( S_0 \) has a sort of ‘common
multiple property for sequences’ – and ‘Is \( S \) right or left Ore?’ for which we give an elementary
counterexample in Section 5.

The paper is organised as follows: in the first section we recall some basic concepts of the commutative
algebra of smooth function algebras (where we could not resist the pleasure of the completely
unnecessary study of its prime ideals) following Tougeron’s book [38], and of (non)commutative
localization following Lam’s very nice text-book [20].

In Section 2 we show the first localization result concerning open sets: an important tool is
Tougeron’s fonction aplatisseur [37] which makes a given sequence of locally defined smooth functions
globally defined by multiplication with a single suitable function being nowhere zero on the
given open set. In this proof, we had to make explicit use of the seminorms defining the Fréchet
topology of the smooth function space.

In Section 3 we prove a similar result for germs, heavily relying on the first theorem. Note that
there is slight, but important difference between ‘germs over the formal power series ring \( \mathbb{K}[[\lambda]] \)’
which we describe and the ‘formal power series of germs’ on which we comment in Section 4.

Section 4 is devoted to the above-mentioned discussion of commutative localization of multidiffer-
ential operators (defined in the algebraical way) for any commutative algebra \( A \) (over some unital
commutative ring \( K \)) by general commutative multiplicative subsets \( S_0 \) and its comparison with a
‘natural’ noncommutative localization with respect to \( S = S_0 + \lambda A[[\lambda]] \). The technical tool will be
a localization theorem of algebraic differential operators due to G. Vezzosi, 1997, [39].

In Section 5 we describe a simple example of a multiplicative set of the type \( S = S_0 + \lambda A[[\lambda]] \) in
smooth deformation quantization of the plane which is not Ore.

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1 Review of basic concepts

Let \( K \) be a fixed commutative associative unital ring such that \( 1 = 1_K \neq 0 = 0_K \). All \( K \)-algebras
are supposed to be associative and unital. We shall include unital \( K \)-algebras \( R \) isomorphic to \( \{0\} \)
for which \( 1_R = 0_R \). Note that associative unital rings are always \( \mathbb{Z} \)-algebras in a natural way. In
order to avoid clumsy notation we shall not write \( 1_R, 1_K, 0_R \) or \( 0_K \), but simply 1 and 0 where the
precise interpretation should be clear from the context.
1.1 Review of Commutative Algebra for smooth function algebras

1.1.1 Elementary features of function algebras

For the convenience of the reader (working in differential geometry) we shall give the following elementary survey which can be ignored on first reading. For more information see e.g. [8]. Recall some elementary commutative algebra of unital $K$-algebras of functions on a set: let $X$ be a set, $K$ be a commutative domain, and $R$ be a given unital subalgebra of the $K$-algebra of all the functions $X \to K$. As usual, for any subset $Y \subset X$, let $I(Y) \subset R$ be the set of functions in $R$ vanishing on $Y$ which is always an ideal of $R$ (the *vanishing ideal of $Y$*), and for any subset $J \subset R$ let $Z(J) \subset X$ be the subset of those points of $X$ on which all functions in $J$ vanish. Clearly $Y \subset Z(I(Y))$ and $J \subset Z(I(J))$. Moreover for any two ideals $I_1$ and $I_2$ of $R$ it follows that $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$ (here the fact that $K$ is a domain is used), hence the set of all subsets $Z(I)$, $I$ ideal of $R$, satisfies the axioms of the closed sets of a topology on $X$ called the *Zariski topology on $X$ w.r.t. $R$*. These closed subsets could be called $R$-*algebraic sets*: in the particular case where $K = \mathbb{K}$ is a field, $X = \mathbb{K}^n$, and $R$ the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ these are the *algebraic subsets*, whereas for $R$ being the algebra of analytic functions (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) these sets are called *analytic subsets*. It is well-known that for $X = \mathbb{R}^n$ and $R$ the ring of smooth real-valued functions the Zariski topology of $X = \mathbb{R}^n$ coincides with its usual topology. Returning to the general situation, note that the Zariski closure of a set $Y \subset X$ is equal to $Z(I(Y))$. On the other hand the inclusion $J \subset Z(I(J))$ can sometimes be made more precise by what is called a *Nullstellensatz*: in algebraic geometry over algebraically closed fields $I(Z(J))$ is equal to the set of all polynomials in $R$ such that a certain power is in $J$.

Recall that an ideal $I \subset R$ is called *proper* iff $I \neq R$ iff $1 \notin I$. Moreover, a proper ideal $m$ is called *maximal* iff it is equal to any other proper ideal containing it iff –since $R$ is commutative– the factor algebra $R/m$ is a field. Recall that a *multiplicative set* $S \subset R$ is a subset $S$ of $R$ containing 1 and if $s, s' \in S$ then $ss' \in S$. Moreover a general proper ideal $p$ of $R$ is called a *prime ideal* iff the factor algebra $R/p$ is a domain iff the complementary set $S = R \setminus p$ is a *multiplicative subset*. Recall that *Krull’s Lemma* states that given any multiplicative subset $S \subset R$ and ideal $J$ with $J \cap S = \emptyset$ there is a prime ideal $p \supset J$ with $S \cap p = \emptyset$, see e.g. [23] p.391, Prop. 7.2, Prop. 7.3.

1.1.2 Analytical features of smooth function algebras

Let $X$ be an $N$-dimensional differentiable manifold (whose underlying topological space we shall always assume to be Hausdorff and second countable). Let $\mathbb{K}$ denote either the field of all real numbers, $\mathbb{R}$, or the field of all complex numbers, $\mathbb{C}$. For any real vector bundle $E$ over $X$ we shall denote by the same symbol $E$ its complexification. Consider the $\mathbb{K}$-algebra $A = C^\infty(X, \mathbb{K})$ of all smooth $\mathbb{K}$-valued functions $f$ on $X$. Even in the case where $X$ is an open subset of $\mathbb{R}^n$ the algebraic properties of $A$ are rather different from the function algebras used in algebraic or analytic geometry. There has been much work on that in the past, see e. g. [23], [38], based on the classical works by Whitney. We shall give a short outline of the features we shall need.

The $\mathbb{K}$-vector space $A$ is given a well-known Fréchet topology which can be conveniently defined in the following terms: fix a Riemannian metric $h$ on $X$, and let $\nabla$ denote its Levi-Civita connection. For any nonnegative integer $n$ denote by $S^n T^* X$ the $n$th symmetric power of the cotangent bundle of $X$: its smooth sections can be viewed as smooth functions on the tangent bundle $\tau X : TX \to X$ which are of homogeneous polynomial degree $n$ in the direction of the fibres. For any section $\alpha \in \Gamma^\infty(X, S^n T^* X)$ let $D \alpha \in \Gamma^\infty(X, S^{n+1} T^* X)$ be it is symmetrized covariant derivative w.r.t. $\nabla$ which can be seen as a symmetric version (depending on $\nabla!$) of the exterior derivative. Finally for any smooth function $f : X \to \mathbb{K}$ let $D^n f \in \Gamma^\infty(X, S^n T^* X)$ be the $n$-fold iterated symmetrized
covariant derivative of \( f \), hence \( D^0 f = f, D f = df, D^{n+1} f = D (D^n f) \). For any compact set \( K \subset X \) and any nonnegative integer \( m \) define a system of functions \( p_{K,m} : A \rightarrow \mathbb{R} \) by

\[
p_{K,m}(f) = \max\{|D^n f(v)| \mid n \leq m, \; \tau_X(v) \in K \text{ and } h(v,v) \leq 1\}, \tag{1.1}
\]

which will define an exhaustive system of seminorms, hence a locally convex topological vector space which is known to be metric and sequentially complete, hence Fréchet. It is not hard to see that the choice of another Riemannian metric will give another system of seminorms which is equivalent to the first one. For flat \( \mathbb{R}^n \) equipped with the usual euclidean scalar product these seminorms are easily seen to be equivalent to the usual seminorms used in analysis where the higher partial derivatives are expressed by multi-indices. Pointwise multiplication and evaluation at a point are well-known to be continuous w.r.t. to the Fréchet topology. For later use we shall give the usual definition of multidifferential operators \( D : A \times \cdots \times A \rightarrow A \) of rank \( p \) as a \( p \)-linear map (over the ground field \( \mathbb{K} \)) such that there is a nonnegative integer \( l \) and for each chart \((U,(x^1, \ldots, x^N))\) there are smooth functions \( D^{\alpha_1,\ldots,\alpha_l} : U \rightarrow \mathbb{K} \) indexed by \( p \) multi-indices \( \alpha_1, \ldots, \alpha_p \in \mathbb{N}^N \) such that for each point \( x \in U \), and smooth functions \( f_1, \ldots, f_p \in A \) there is the following local expression

\[
D(f_1, \ldots, f_p)(x) = \sum_{|\alpha_1|,\ldots,|\alpha_l| \leq l} D^{\alpha_1,\ldots,\alpha_l} \frac{\partial^{|\alpha_1|}(f_1|U)}{\partial x^{\alpha_1}}(x) \cdots \frac{\partial^{|\alpha_l|}(f_p|U)}{\partial x^{\alpha_p}}(x) \tag{1.2}
\]

where as usual \( |\alpha| = (i_1, \ldots, i_N) = i_1 + \ldots + i_N \) and \( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \) is short for \( \frac{\partial^{i_1+\cdots+i_N}\phi}{\partial (x^1)^{i_1}\cdots \partial (x^N)^{i_N}} \). Note that the value of \( D(f_1, \ldots, f_p) \) at \( x \) only depends on the restriction of the functions \( f_1, \ldots, f_p \) to any open neighbourhood of \( x \): it follows that multidifferential operators can always be localized in the analytical sense that they give rise to unique well-defined multidifferential operators \( D_U \) on \( C^\infty(U, \mathbb{K}) \) for any open subset \( U \subset X \) such that \( D \) and \( D_U \) intertwine the restriction map \( \eta : A \rightarrow C^\infty(U, \mathbb{K}) \) in the obvious way. Multidifferential operators are well-known to be continuous w.r.t. to the Fréchet topology. Furthermore, recall the usual composition rule of multidifferential operators (inherited by the usual rule for multi-linear maps): given two multidifferential operators \( D \) (of rank \( p \)) and \( D' \) (of rank \( q \)) and a positive integer \( 1 \leq i \leq p \) then the map \( D \circ_i D' \) defined by \( (f_1, \ldots, f_{p+q-1}) \mapsto D(f_1, \ldots, f_{i-1}, D'(f_i, \ldots, f_{i+p-1}), f_{i+p}, \ldots, f_{p+q-1}) \) is a multidifferential operator of rank \( p + q - 1 \). This composition rule obviously is compatible with localization in the sense that \( (D \circ_i D')_U = D_U \circ_i D'_U \). Consider for any given point \( x_0 \in X \) the binary relation \( \sim_{x_0} \) on \( A \) defined by \( f \sim_{x_0} g \) iff the two smooth functions \( f \) and \( g \) have the same Taylor series at \( x_0 \) w.r.t. to some chosen chart around \( x_0 \). It is well-known to be an equivalence relation which does not depend on the chosen chart, an equivalence class is called an infinite jet, and the class of \( f \) is called the infinite jet \( j^\infty_{x_0}(f) \) of \( f \), see e.g. [17] p.117 section 12.2. Apart from the trivial case where \( X \) is a point, \( A \) is well-known to have two ‘bad’ features from the point of view of commutative algebra: firstly the product of two non-zero functions with disjoint supports –which exist in \( A– \) clearly vanishes showing that \( A \) has very many nontrivial zero-divisors. Secondly \( A \) is NOT Noetherian: if for each nonnegative integer \( n \) we denote by \( I_n \) the ideal of all smooth \( \mathbb{K} \)-valued functions on \( \mathbb{R} \) vanishing on the closed interval \([\frac{-1}{n+1}, \frac{1}{n+1}]\), then the ascending sequence \( I_n \subseteq I_{n+1} \) never stabilizes after a finite number of steps.

### 1.1.3 Some ideal theory of smooth function algebras

In this paragraph we collect some facts of maximal and prime ideals of \( A = C^\infty(X, \mathbb{K}) \) which are major topics in commutative algebra. The main source will be J.-C. Tougeron’s classic [38].
paragraph can be ignored by the impatient reader.
For a given ideal \( J \) of \( A \) its zero set \( Z(J) \subset X \) is of course a closed subset of \( X \), and the closure \( \overline{J} \) of \( J \) (w.r.t. the Fréchet topology) in \( A \) remains an ideal of \( A \). The vanishing ideal of any set is closed in the Fréchet topology, hence there is the chain of inclusions \( J \subset \overline{J} \subset I(Z(J)) \). Since for any point \( x \) not contained in \( Z(J) \) there is a function \( g \) in the ideal \( J \) not vanishing at \( x \), a simple partition of unity argument shows that any smooth \( K \)-valued function whose support is compact and has empty intersection with \( Z(J) \) must be an element of \( J \). It follows in particular that an ideal \( J \) contains the ideal \( \mathcal{D}(X) \) of all smooth \( K \)-valued functions having compact support iff \( Z(J) = \emptyset \) iff \( J \) is dense in \( A \). In the particular case where \( X \) is compact this means that the only dense ideal is equal to \( A \). Returning to general \( X \) it follows that every proper ideal of \( A \) which is closed w.r.t. the Fréchet topology has a non-empty set of common zeros.
In particular, every closed maximal ideal of \( A \) is equal to the vanishing ideal \( I_{x_0} = I(\{x_0\}) \) of some point \( x_0 \in X \).
In general, for the closure of an ideal there is the very useful Whitney’s Spectral Theorem stating that a function \( g \) belongs to the closure \( \overline{J} \) of an ideal \( J \) of \( A \) iff for each \( x \in X \) there is a function \( h \in J \) (whose choice may depend on \( x \)) whose infinite jet \( j_x^\infty(h) \) is equal to \( j_x^\infty(g) \), see e.g. [38] p. 91, Cor. 1.6., cas \( q = 1 \). Moreover, ideals having finitely many analytic generators are always closed, see e.g. [38] p.119, Cor. 1.6.], but there are also closed ideals having finitely many nonanalytic generators, see e.g. [38] p.104, Rem. 4.7, Exemp. 4.8.].

In the following, given \( x_0 \in X \), denote by \( \mathfrak{J}_{x_0} \) the ideal of \( A \) consisting of all smooth functions vanishing in some neighbourhood of \( x_0 \), and by \( I_{x_0}^\infty \) the ideal of \( A \) consisting of all functions \( f \) such that \( j_{x_0}^\infty(f) = 0 \). Clearly \( \mathfrak{J}_{x_0} \subset I_{x_0}^\infty \subset I_{x_0} \). Consider now a prime ideal \( p \subset A \) of \( A \). We know that it is either dense iff \( Z(p) = \emptyset \) or has a nonempty zero set. For each prime ideal it can be shown that

\[
Z(p) \neq \emptyset \iff \exists \ x_p \in X : Z(p) = \{x_p\} \iff \exists \ x_0 \in X : \mathfrak{J}_{x_0} \subset p \iff \exists \ y_0 \in X : p \subset I_{y_0}. \quad (1.3)
\]

and in case one the four equivalent statements is fulfilled then \( x_0 = y_0 = x_p \), uniquely determined by \( p \).
Indeed, it is obvious that in eqn (1.3) the second statement implies the first which is equivalent to the fourth. Moreover, if \( Z(p) \) contained two distinct points \( x_1, x_2 \in X \) there would be two smooth functions \( \varphi_1, \varphi_2 \in A \) with disjoint supports such that \( \varphi_1(x_1) = 1 = \varphi_2(x_2) \) (whence \( \varphi_1, \varphi_2 \in A \setminus p \)), but \( \varphi_1 \varphi_2 = 0 \) in \( p \) contradicting the fact that \( p \) is prime hence the first, the second, and the last statement of (1.3) are equivalent implying the uniqueness and equality of \( x_p \) and \( y_0 \) in case one of three statements is fulfilled. Moreover supposing that \( Z(p) = \{x_p\} \) then for any \( h \in \mathfrak{J}_{x_p} \) there is \( \varphi \in A \) with \( \varphi(x_p) = 1 \) having its support inside the open neighbourhood of \( x_p \) on which \( h \) vanishes, whence \( \varphi \in A \setminus p \), but \( \varphi h = 0 \in p \) so \( h \in p \) since \( p \) is prime, whence the second statement of (1.3) implies the third. Finally, supposing \( \mathfrak{J}_{x_0} \subset p \) for some \( x_0 \in X \), if there was a \( g \in p \) with \( g(x_0) \neq 0 \) one would find a positive valued function \( h \in \mathfrak{J}_{x_0} \subset p \) and a bump function \( \chi \in A \) such that \( \chi |g|^2 + h \in p \) has only strictly positive values, hence is invertible implying \( p = A \) which contradicts the fact that \( p \) is proper. Hence \( p \subset I_{x_0} \) whence the third statement implies the last in eqn (1.3) and the equality \( x_p = x_0 = y_0 \).
Next we shall look at closed prime ideals of \( A \): fix a point \( x_0 \in X \), then by Borel’s classical Lemma (see e.g. [40] p.332, Satz 5.3.33]) the factor algebra \( A/I_{x_0}^\infty \) is isomorphic to the algebra of formal power series \( K[[x_1, \ldots, x_n]] \) which is a domain whence \( I_{x_0}^\infty \) is a prime ideal which is closed since \( f \mapsto j^k_{x_0}(f) \) is continuous for every nonnegative integer \( k \). Moreover the obvious inclusion \( \mathfrak{J}_{x_0} \subset I_{x_0}^\infty \) implies \( \mathfrak{J}_{x_0} \subset I_{x_0}^\infty \) since \( I_{x_0}^\infty \) is closed. Since for any \( g \in I_{x_0}^\infty \) we have by definition \( j^\infty_{x_0}(g) = 0 = j^\infty_{x_0}(0) \), and for any \( y \in X \setminus \{x_0\} \) there is a smooth function \( \chi \in A \) vanishing in a suitable open neighbourhood of \( x_0 \) and having the constant value 1 in another suitable open neighbourhood of \( y \) it follows that \( \chi g \in \mathfrak{J}_{x_0} \) and \( j_y^\infty(\chi g) = j_y^\infty(g) \) whence \( g \in \mathfrak{J}_{x_0} \) thanks to Whitney’s spectral theorem. This implies
the equality $\overline{J_{x_0}} = I_{x_0}^\infty$. By passing to closures in eqn (1.3) it immediately follows that for any proper closed prime ideal $p$

$$\text{if } p = \overline{p} \neq A \text{ and } Z(p) = \{x_p\} : J_{xp} \subset \overline{J_{x_p}} = I_{x_p}^\infty \subset p \subset J_x.$$ (1.4)

Conversely, another simple application of Whitney’s spectral theorem shows that for any prime ideal $p$ the inclusion $I_{x_0}^\infty \subset p \subset I_{x_0}$ implies that $p$ is proper and closed. Moreover, it follows that for each given $x_0 \in X$ the set of all closed prime ideals $p \subset A$ with $Z(p) = \{x_0\}$ is in bijection with the set of all the prime ideals of the formal power series algebra $\mathbb{K}[[x_1, \ldots, x_n]]$ via the map $p \mapsto p/I_{x_0}^\infty$. These latter prime ideals can be characterized in a purely algebraic way, see e.g. [38, p. 93, Lemme 2.4] (for the particular case where the closed set equals $\{x_0\}$) implying that $I_{x_0}^\infty$ is equal to the intersection of all the powers of $I_{x_0}$.

Note that there are many ‘funny’ non closed prime ideals of $A$ (even in the case where $X$ is compact): by applying Krull’s Lemma to the ideal $J(X)$ and the multiplicative subset generated by an arbitrary fixed function $f$ having non compact support we have the existence of a dense prime ideal which does not contain $f$. Likewise, applying Krull’s Lemma to the ideal $J_{x_0}$ and the multiplicative subset generated by an arbitrary function $g \in I_{x_0}^\infty$ which is not in $J_{x_0}$ we get a proper non closed prime ideal $p$ with $Z(p) = \{x_0\}$, hence containing $J_{x_0}$, but not $I_{x_0}^\infty$.

1.2 (Algebraic) Localization

This section recalls well-known results which we present according to the excellent text-book [20] in a categorically ‘tuned’ version. See also the rather useful review [35] for more aspects.

1.2.1 Commutative Localization

Recall that for any domain $R$ it is always possible to construct a field, called the field of fractions of $R$, by formally inverting all nonzero elements. More generally, recall the localization of a commutative $K$-algebra $R$: let $S \subset R$ be a multiplicative subset (which is characterized by containing the unit and for any two of its elements its product). Then the following binary relation $\sim$ on $R \times S$ defined by

$$(r_1, s_1) \sim (r_2, s_2) \text{ if and only if } \exists s \in S : r_1s_2s = r_2s_1s$$ (1.5)

is an equivalence relation, and the set of all classes (written as (symbolic) fractions $\frac{r}{s}$) forms a commutative $K$-algebra $R_S$ –by means of the usual addition and multiplication rules of fractions– called the quotient ring, and a ring homomorphism (the numerator morphism) $\eta_{(R,S)} = \eta : R \to R_S$ given by $r \mapsto \frac{r}{1}$ which in particular defines the $K$-algebra structure of $R_S$. Let $U(R) \subset R$ denote the multiplicative group of invertible elements of $R$. A morphism of unital $K$-algebras $\Phi : R \to R'$ is called $S$-inverting (for a multiplicative subset $S \subset R$) if for each $s \in S$ the image $\Phi(s)$ is invertible in $R'$, hence $\Phi(S) \subset U(R')$. The following properties of the constructions can be observed:

**Proposition 1.1.** Let $R$ be a commutative $K$-algebra and $S \subset R$ be a multiplicative subset. Then the following is true:

a. $\eta_{(R,S)}(S) \subset U(R_S)$, that is, the homomorphism $\eta_{(R,S)}$ sends elements of $S$ to invertible elements of $R_S$. Moreover, for any commutative unital $K$-algebra $R$ equipped with a multiplicative subset $S \subset R$, the pair $(R_S, \eta_{(R,S)})$ is universal in the sense that any $S$-inverting morphism
of unital $K$-algebras uniquely factorizes, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
R & \xrightarrow{\eta} & RS \\
\downarrow{\alpha} & & \downarrow{f} \\
R' & & R'S'
\end{array}
$$

(1.6)

where $f$ is a morphism of unital $K$-algebras determined by $\alpha$, see e.g. [22, p.55, Ch.III] for definitions of universal objects.

b. Every element of $RS$ can be written as a fraction $\eta(r)\eta(s)^{-1}$, for some $r \in R$ and $s \in S$.

c. $\ker(\eta_{(R,S)}) = \{ r \in R \mid rs = 0 \text{ for some } s \in S \}$.

We shall give a more categorical description in the next section.

Remarks (which will only be used in Section 4):

1. Recall the well-known localization of any $R$-module $M$, see e.g. [13, p.397], which is a module $M_S$ with respect to the localized algebra $RS$. It is naturally isomorphic to $RS \otimes_R M$, see e.g. [13, p.398, Prop.7.6].

2. Let $R, R'$ be commutative associative unital $K$-algebras, and $S \subset R, S' \subset R'$ multiplicative subsets, respectively. Then it is straightforward to see that $S \otimes_K S' = \{ s \otimes_K s' \in R \otimes_K R' \mid s \in S; s' \in S' \}$ is a multiplicative subset of the $K$-algebra $R \otimes_K R'$ and that the tensor product of the numerator morphisms $\eta_{(R,S)} \otimes_K \eta_{(R',S')} : R \otimes_K R' \rightarrow RS \otimes_K R'S'$ induces a natural isomorphism of unital $K$-algebras

$$
(R \otimes_K R')_{S \otimes_K S'} \rightarrow RS \otimes_K R'S'.
$$

(1.7)

1.2.2 Noncommutative Localization: General Construction

Let $R$ be an associative unital $K$-algebra which is not necessarily commutative. Again, we call $S \subset R$ a multiplicative subset if for all $s, s' \in S$ we have $ss' \in S$ and $1_R = 1 \in S$. As above, let $U(R) \subset R$ denote the multiplicative subset (which is even a group) of invertible elements of $R$.

Let $K\text{Alg}$ be the category of all associative unital $K$-algebras. Moreover, let $K\text{AlgMS}$ be the category of all pairs $(R, S)$ of associative unital $K$-algebras $R$ with a multiplicative subset $S \subset R$ where the morphisms $(R, S) \rightarrow (R', S')$ are morphisms of unital $K$-algebras $R \rightarrow R'$ mapping $S$ into $S'$. Since any morphism of unital $K$-algebras maps the group of invertible elements in the group of invertible elements there is an obvious functor $U : K\text{Alg} \rightarrow K\text{AlgMS}$ given on objects by $U(R) = (R, U(R))$.

For commutative $K$-algebras, the above localization description in Proposition 1.1.a., gives rise to a functor $L : K\text{AlgMS} \rightarrow K\text{Alg}$ associating to each pair $(R, S)$ the quotient ring $RS$, and it is not hard to see that it is a left adjoint of the functor $U$, see e.g. [22, p.79, Ch.IV] for definitions: the unit of the adjunction gives back the canonical numerator morphism $\eta$, and the counit is an isomorphism since localization w.r.t. the group of all invertible elements is isomorphic to the original algebra.

In the general noncommutative situation such a localization functor $L : K\text{AlgMS} \rightarrow K\text{Alg}$ does also always exist, see e.g. [20, Prop.(9.2), p.289] for a proof. We present it in the following categorical form:
Proposition 1.2. There is an adjunction of functors

\[ \text{KAlgMS} \xleftrightarrow{\mathcal{L}} \mathcal{U} \xrightarrow{\mathcal{O}} \text{KAlg} \]

where \( \mathcal{L} \) is the left adjoint to the above functor \( \mathcal{O} \) such that each component \( \eta_{(R,S)} \) of the unit \( \eta : I_{\text{KAlgMS}} \to \mathcal{U} \mathcal{L} \) of the adjunction satisfies the universal property a. of the previous Proposition [7.4] in the general noncommutative case. We refer to \( \mathcal{L} \) as a localization functor.

For a given \((R, S)\) in \( \text{KAlgMS} \) we denote by \( R_S \) the \( K \)-algebra \( \mathcal{L}(R, S) \) given by the functor \( \mathcal{L} \), and by \( \eta_{(R,S)} : R \to R_S \) the component of the unit of the adjunction. Then \( \eta_{(R,U(R))} : R \to R_{U(R)} \) is an isomorphism, the inverse being the component \( \epsilon_R \) of the counit \( \epsilon : \mathcal{LU} \to I_{\text{KAlg}} \) of the adjunction. Moreover, every element of the \( K \)-algebra \( R_S \) is a finite sum of products of the form \( (\eta = \eta_{(R,S)}) \)

\[ \eta(r_1)(\eta(s_1))^{-1} \cdots \eta(r_N)(\eta(s_N))^{-1} \]

(which may be called ‘multifractions’) with \( r_1, \ldots, r_N \in R \) and \( s_1, \ldots, s_N \in S \) (note that \( r_1 \) or \( s_N \) may be equal to the unit element of \( R \)).

The idea of the proof of [20, Prop.(9.2), p.289] is as follows: (see also the PhD thesis [1, p.144] for details) there is a natural surjective morphism of unita \( \lambda \)-algebras \( \hat{\epsilon}_R \) from the free \( K \)-algebra generated by the \( K \)-module \( R, T_K R, \) to \( R \) which provides us with a natural categorical presentation of \( R \) ‘by generators and relations’: this morphism is given by the \( R \)-component of the counit \( \hat{\epsilon} \) of the well-known adjunction

\[ \text{KMod} \xleftrightarrow{T_K} \mathcal{O} \xrightarrow{\mathcal{K}} \text{KAlg} \]

where \( \mathcal{O} \) is the forgetful functor and \( T_K \) the free algebra functor. Let \( \kappa(R) \subset T_K R \) denote the kernel of \( \hat{\epsilon}_R \). The next step is to add to the generating \( K \)-module \( R \) the free \( K \)-module \( KS \) with basis \( S \), and to consider the two-sided ideal \( \kappa(R, S) \) in the free algebra \( T_K (R \oplus KS) \) generated by \( \kappa(R) \) and by the subsets \( \{(s, 0) \otimes (0, s) - 1_T \mid s \in S \} \) and \( \{(0, s) \otimes (s, 0) - 1_T \mid s \in S \} \) of \( T_K (R \oplus KS) \) where the multiplication \( \otimes \) and the unit \( 1_T \) are taken in the free algebra \( T_K (R \oplus KS) \). The localized algebra \( \mathcal{L}(R, S) = R_S \) is then defined by \( R_S = T_K (R \oplus KS) / \kappa(R, S) \), and the ‘numerator morphism’ \( \eta_{(R,S)} : R \to R_S \) is simply the canonical injection of \( R \) into \( T_K R \subset T_K (R \oplus KS) \) followed by the obvious projection. It follows that for every \( s \in S \) its image \( \eta_{(R,S)}(s) \) has an inverse by construction. The verification that this leads to a well-defined functor \( \mathcal{L} \) which is a left adjoint to the functor \( \mathcal{O} \) is lengthy, but straight-forward.

The preceding construction shows that the functor \( \mathcal{L} \) provides us with an abstract universal numerator map \( \eta_{(R,S)} \) which is \( S \)-inverting in the sense that every \( \eta_{(R,S)}(s), s \in S, \) is invertible in \( R_S \) and a natural isomorphism \( \epsilon_R \) from an algebra to its localization w.r.t. its group of units.

1.2.3 Noncommutative Localization: Ore Localization

Although the preceding general localization construction is always well-defined, it exhibits the following draw-backs which show the need for a more particular localization procedure due to \( \mathcal{O} \). Ore, 1931, [26] which we shall sketch in this Section:

- The construction by generators and relations renders the localized algebra \( R_S \) quite implicit and not always computable.
The next theorem shows that such a right algebra of fractions exists iff

\[ \text{Theorem 1.1.} \]

Let \( R \) be an associative unital \( K \)-algebra, and \( S \subset R \) be a multiplicative subset.

i. A \( \tilde{K} \)-algebra \( \tilde{R}_S \) equipped with a morphism of unital \( \tilde{K} \)-algebras \( \tilde{\eta}_{(R,S)} : R \to \tilde{R}_S \) is said to be a right \( \tilde{K} \)-algebra of fractions of \( (R,S) \) if the following conditions are satisfied:

a. \( \tilde{\eta}_{(R,S)} \) is \( S \)-inverting,

b. Every element of \( \tilde{R}_S \) is of the form \( \tilde{\eta}(r)(\tilde{\eta}(s))^{-1} \) for some \( r \in R \) and \( s \in S \);

c. \( \ker(\tilde{\eta}) = \{ r \in R \mid rs = 0 \} \) for some \( s \in S \) =: \( I_{(R,S)} =: I \).

ii. \( S \) is called a right denominator set if it satisfies the following two properties:

a. For all \( r \in R \) and \( s \in S \) we have \( rS \cap sR \neq \emptyset \) (\( S \) right permutable or right Ore set), i.e. there are \( r' \in R \) and \( s' \in S \) such that \( rs' = sr' \).

b. For all \( r \in R \) and for all \( s' \in S \): if \( s'r = 0 \) then there is \( s \in S \) such that \( rs = 0 \) (\( S \) right reversible).

In case \( R \) is commutative every multiplicative subset is a right denominator set. Moreover the group of all invertible elements \( U(R) \) of any unital \( K \)-algebra is obviously a right denominator set.

The next theorem shows that such a right algebra of fractions exists iff \( S \) is a right denominator set, see also \[20\] Thm (10.6), p.300):

\[ \text{Theorem 1.1.} \]

Let \( R \) be a unital \( K \)-algebra and \( S \subset R \) be a multiplicative subset. Then the following is true:

1. The \( K \)-algebra \( R \) has a right \( K \)-algebra of fractions \( \tilde{R}_S \) with respect to the multiplicative subset \( S \) if and only if \( S \) is a right denominator set.

2. If this is the case each such pair \((\tilde{R}_S, \tilde{\eta})\) is universal in the sense of diagram (1.6) and each \( \tilde{R}_S \) is isomorphic to the canonical localized algebra \( R_S \) of Proposition 1.2.

3. Each \( \tilde{R}_S \) is isomorphic to the quotient set \( RS^{-1} := (R \times S)/\sim \) with respect to the following binary relation \( \sim \) on \( R \times S \)

\[ (r_1, s_1) \sim (r_2, s_2) \iff \exists b_1, b_2 \in R \text{ such that } s_1b_1 = s_2b_2 \in S \text{ and } r_1b_1 = r_2b_2 \in R \quad (1.9) \]
which is an equivalence relation generalizing relation \(1.9\). \(RS^{-1}\) carries a canonical unital \(K\)-algebra structure, i.e., addition and multiplication on equivalence classes \(r_1s_1^{-1}\) and \(r_2s_2^{-1}\) (with \(r_1,r_2 \in R\) and \(s_1,s_2 \in S\)) is given by

\[
   r_1s_1^{-1} + r_2s_2^{-1} = (r_1c_1 + r_2c_2)s^{-1}, \quad \text{and} \quad (r_1s_1^{-1})(r_2s_2^{-1}) = (r_1r')(s_2s')^{-1}
\]

(1.10)

where we have written \(s_1c_1 = s_2c_2 = s \in S\) (with \(c_1 \in S\) and \(c_2 \in R\)) and \(r_2s' = s_1r'\) (with \(s' \in S\) and \(r' \in R\)) using the right Ore property. The numerator morphism \(\eta_I : R \to RS^{-1}\) is given by \(\eta_I(r) = r^{-1}\) for all \(r \in R\).

For a proof, see e.g. [27, p.244, Thm. 25.3] or the PhD thesis \([11\) p.146]).

We shortly describe the idea of the proof: whereas in part 1. the verification of the implication \(\langle i. \rangle \implies \langle ii. \rangle\) in Definition \(1.1\) is straightforward, the converse implication \(\langle ii. \rangle \iff \langle i. \rangle\) of Definition \(1.1\) is much more involved: the traditional ‘steep and thorny way’ (originally set up by Oystein Ore, \([25\]) consists of a concrete construction of the \(K\)-algebra \(RS^{-1}\) upon using the above relation \(1.9\) –which reflects the idea of creating ‘common denominators’– and defining and verifying the canonical \(K\)-algebra structure \(1.10\) on the quotient set \(R \times S/\sim\) by hand which is elementary, but extremely tedious (even the fact that the above relation \(1.9\) is transitive requires some work). We refer to Lam’s book \([20\) p.300-302] for some of the details.

There is a different more elaborate way to prove part 1. and the rest of the theorem (see \([27\) p.244, Thm. 25.3] and \([20\) Remark (10.13), p.302, and footnote 70]): it is instructive to look first at the equivalence relations created by an arbitrary \(S\)-inverting morphism of unital \(K\)-algebras \(\alpha : R \to R'\), the classes being defined by the fibres of the map \(p_\alpha : R \times S \to R'\) given by \(p_\alpha(r,s) = \alpha(r)(\alpha(s))^{-1}\), which is already very close to relation \(1.9\) thanks to the fact that the right fractions \(\alpha(r)(\alpha(s))^{-1}\) form a \(K\)-subalgebra of \(R'\) (here the Ore axiom is needed) it creates an algebra structure on the quotient set isomorphic to the aforementioned subalgebra of \(R'\) whence there is no need of tedious verifications of identities of algebraic structures. The central point then is to construct a unital \(K\)-algebra \(R'\) and an \(S\)-inverting morphism \(\alpha : R \to R'\) whose kernel is minimal, hence equal to \(I_{(R,S)}\) which finally shows that the above algebra \(RS^{-1}\) exists and does everything it should do. For this construction, the following trick is used: after ‘regularizing’ \(R\) by passing to the factor algebra \(\overline{R} = R/I_{(R,S)}\) (where the image multiplicative set \(\overline{S}\) does no longer contain right or left divisors of zero) one looks at the endomorphism algebra of the injective hull \(E\) of the right \(\overline{R}\)-module \(\overline{R}\).

Every left multiplication with elements of \(\overline{R}\) can nonuniquely be extended to \(E\), and the extensions of left multiplications with elements of \(\overline{S}\) turn out to be invertible (here the Ore axiom is needed). \(R'\) will then be given by the subalgebra generated by all extensions of left multiplications and the inverses of left multiplications with elements of \(\overline{S}\) modulo the two-sided ideal of all \(\overline{R}\)-linear maps \(E \to E\) vanishing on \(\overline{R}\): this will resolve the ambiguity of extension, and \(\overline{R}\) injects in \(R'\), the injection being \(\overline{S}\)-inverting.

Moreover, in any noncommutative domain (no nontrivial zero divisors) which is right Noetherian (i.e., where every ascending chain of right ideals stabilizes) the subset of nonzero elements is always a right denominator set (see \([20\) p.304, Cor. (10.23)] or \([7\) p.14, Beisp. 2.3 b\])). In particular, this applies to every universal enveloping algebra over a finite-dimensional Lie algebra (over a field \(\mathbb{K}\) of characteristic zero) and for the Weyl-algebra generated by \(\mathbb{K}^2\). On the other hand, for the free algebra \(R = T_{\mathbb{K}}V\) generated by a vector space \(V\) of dimension \(\geq 2\) over a field \(\mathbb{K}\) of characteristic zero (which is well-known to be isomorphic to the universal enveloping algebra of the free Lie algebra generated by \(V\)) the multiplicative subset of all nonzero elements is neither a right nor a left denominator set: for two linearly independent elements \(v\) and \(w\) in \(V\) we clearly have

Footnote 1: We are indebted to A. Eduque for having pointed out this reference to us.
$vR \cap wR = \{0\}$. Hence the above statement about universal enveloping algebras does no longer apply to infinite-dimensional Lie algebras like the free Lie algebra generated by $V$. Moreover inverse images of right denominator subsets are in general no right denominator subsets as the example of the natural homomorphism $T_K V \to S_K V$ of the free to the free commutative algebra generated by $V$ shows: as $S_K V$ is a commutative domain, the subset $S = S_K V \setminus \{0\}$ is a right denominator set whereas its inverse image $T_K V \setminus \{0\}$ is not. On the other hand every homomorphic image of a right (or left) Ore set clearly is again a right (or left) Ore set. However, there may be subsets of right (or left) denominator sets which are no longer right (or left) denominator sets, as we shall see later in Section 5.

1.3 Star products

We want to recall some basic definitions and facts about the deformation quantization of smooth manifolds and star products, see [6], [40] for more information.

Given a $\mathbb{K}$-vector space $V$ we denote by $V[[\lambda]]$ the $\mathbb{K}[[\lambda]]$-module of formal power series. An element of $v \in V[[\lambda]]$ can be written uniquely as $v = \sum_{i=0}^{\infty} v_i \lambda^i$ with $v_i \in V$, and for a given $v \in V[[\lambda]]$ and $i \in \mathbb{N}$ we shall always write $v_i \in V$ for the $i$th component of $v$ as a formal power series. We also note that for two $\mathbb{K}$-vector spaces $V,W$ we have $\text{Hom}_{\mathbb{K}[[\lambda]]}(V[[\lambda]],W[[\lambda]]) \cong \text{Hom}_\mathbb{K}(V,W)[[\lambda]]$.

In the following considerations of differential geometry we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and for any smooth differentiable manifold $X$ we write $C^\infty(X) = C^\infty(X,\mathbb{K})$.

**Definition 1.2** (Star product). A (formal) star product $\star$ on a manifold $X$ is a $\mathbb{K}[[\lambda]]$-bilinear operation $C^\infty(X)[[\lambda]] \times C^\infty(X)[[\lambda]] \to C^\infty(X)[[\lambda]]$ —which can always be written as a formal series $f \star g = \sum_{k=0}^{\infty} \lambda^k C_k(f,g)$ for all $f,g \in C^\infty(X)$— satisfying the following properties for all $f,g \in C^\infty(X)$ (see section 1.1.2 for definitions and notations):

- $\sum_{i=0}^{k} (C_i \circ_1 C_{k-i} - C_i \circ_2 C_{k-i}) = 0$, $\forall k \geq 0$,
- $C_0(f,g) = fg$,
- $1 \star f = f \star 1 = f$,

with $\mathbb{K}$-bilinear operators $C_k : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ which we always assume to be bidifferential operators.

**Remark 1.1.** It follows from the first equation of Definition 1.2 that $\star$ is associative.

Note that every star product $\star$ can be analytically localized to an associative star product $\star_U$ defined on $C^\infty(U)[[\lambda]]$ by the localization of all the bidifferential operators $C_k$ to $C_k U$ (see section 1.1.2 for more details).

The following well-known explicit star product $\star_s$ on $\mathbb{R}^2$ with coordinates $(x,p)$ will be used in the sequel:

$$f \star_s g = \sum_{k=0}^{\infty} \frac{\lambda^k \partial f \partial^k g}{k! p^k \partial x^k}$$

(1.11)

for any two functions $f,g \in C^\infty(\mathbb{R}^2)$. In the physics literature $\lambda$ corresponds to $(-i\hbar)$. Moreover, for functions polynomial in the ‘momenta’ $p$ it is obvious that the above series converges, and for $\lambda = 1$ one obtains the usual formula for the symbol calculus of multiplication of differential operators on the real line (where partial derivatives are always brought to the right and replaced by the new variable $p$).

The star commutator for $a,b \in C^\infty(X)[[\lambda]]$ is defined by $[a,b]_\star = a \star b - b \star a$. As usual, the star commutator satisfies the Leibniz-identity, i.e. $[a,b \star c]_\star = [a,b]_\star \star c + b \star [a,c]_\star$, and the Jacobi-identity
and thus defines the structure of a non-commutative Poisson algebra. Also the adjoint action is a derivation of $C^\infty(X)[[\lambda]]$ for all $a \in C^\infty(X)[[\lambda]]$.

From this it can easily be deduced that the first order term of a star product defines a Poisson bracket as follows

$$\{f, g\} = \frac{1}{2}(C_1(f, g) - C_1(g, f)) = \frac{1}{2\lambda}[f, g]|_{\lambda=0} \text{ for } f, g \in C^\infty(X).$$ (1.12)

For $C^\infty(X)$ it is well-known that every Poisson bracket comes from a unique Poisson structure $\pi$ which is a smooth bivector field $\pi$, i.e. a smooth section in $\Lambda^2TX$ satisfying the identity $[\pi, \pi]|_S = 0$ where $[,]_S$ denotes the Schouten bracket, see e.g. [10], p.84-87: the relation is $\{f, g\} = \pi(df, dg)$.

The very difficult converse problem whether the Poisson bracket associated to any given Poisson structure $\pi$ arises as the first order commutator of a star product had been solved by M. Kontsevich, see [18].

The following considerations will only be used in Section 5: two star products $\star, \star'$ are called equivalent if there exists a formal power series of differential operators $T = \text{id} + \sum_{k=1}^\infty \lambda^k T_k$, with $T(1) = 1$ such that $T(f) \star T(g) = T(f \star' g)$ for all $f, g \in C^\infty(X)[[\lambda]]$. The operator $T$ in the above definition is always invertible and indeed, given a star product $\star, f \star' g := T^{-1}(T(f) \star T(g))$ always gives a new equivalent star product. Two equivalent star products clearly give rise to the same Poisson bracket.

For the star product (1.11) there is the following well-known transformation $T = e^{-\lambda \Delta}$ with $\Delta(f) = \partial^2 f / \partial x \partial p$: together with the $\mathbb{K}$-linear (and not $\mathbb{K}[[\lambda]]$-linear) involution $L : A[[\lambda]] \to A[[\lambda]]$ given by $L(\sum_{r=0}^\infty \lambda^r f_r) = \sum_{r=0}^\infty (-\lambda)^r f_r$ we get –setting $V = L \circ T$

$$\left(V(f)\right) \star_s \left(V(g)\right) = V\left(g \star_s f\right)$$ (1.13)

which can easily be checked on exponential functions $(x, p) \mapsto e^{ax+by}$ with $a, b \in \mathbb{K}$.

2 Noncommutative localization of smooth star products on open subsets

Let $(X, \pi)$ be a Poisson manifold, let $\star = \sum_{k=0}^\infty \lambda^k C_k$ be a star product on $(X, \pi)$, and let $\Omega \subset X$ be a fixed open set. We set $K = \mathbb{K}[[\lambda]]$, and consider the $K$-algebra $(R = C^\infty(X)[[\lambda]], \star)$. Moreover, since the star product $\star$ only involves bidifferential operators, it restricts to a star product $\star_\Omega$ on formal power-series $\phi \in R_\Omega := C^\infty(\Omega, \mathbb{K})[[\lambda]]$ such that $(R_\Omega, \star_\Omega)$ is also a $K$-algebra. It follows that the restriction map $\eta_\Omega = \eta : R \to R_\Omega : f \mapsto f|_\Omega$ is a morphism of unital $K$-algebras. We define the following subsets $S_\Omega \subset C^\infty(X, \mathbb{K})$ and $S \subset R$:

$$S_\Omega := \{g_0 \in C^\infty(X, \mathbb{K}) \mid \forall x \in \Omega : g_0(x) \neq 0\} \quad \text{and} \quad S := S_\Omega + \lambda R.$$ (2.1)

Clearly, $S_\Omega$ is a commutative multiplicative subset of $C^\infty(X, \mathbb{K})$. Since the constant function 1 is in $S$, and for any $g, h \in S$ we have $(g \star h)_0(x) = g_0(x)h_0(x) \neq 0$ (for all $x \in \Omega$) it follows that $S$ is a multiplicative subset of the unital $K$-algebra $R$.

We can now consider the noncommutative localization of $R$ with respect to $S$ and compare it with the unital $K$-algebra $R_\Omega$:

**Theorem 2.1.** Using the previously fixed notations we get for any open set $\Omega \subset X$:

1. $(R_\Omega, \star_\Omega)$ equipped with the restriction morphism $\eta$ constitutes a right $K$-algebra of fractions for $(R, S)$.
2. As an immediate consequence we have that $S$ is a right denominator set.

3. This implies in particular that the algebraic localization $RS^{-1}$ of $R$ with respect to $S$ is isomorphic to the concrete localization $R_{\Omega}$ as unital $K$-algebras.

Proof. 1. We have to check properties (i.a.), (i.b.), and (i.c.) of Definition 1.1.

- “$\eta$ is $S$-inverting” (property (i.a.)): indeed, this is a classical reasoning from deformation quantization which we shall repeat for the convenience of the reader. Let $g \in S$ and $\gamma = \eta(g)$ its restriction to $\Omega$. Take $\psi \in R_{\Omega}$ and try to solve the equation $\gamma * \Omega \psi = 1$. At order $k = 0$ we get the condition $\gamma_0 \psi_0 = 1$, but since $\gamma_0(x) \neq 0$ for all $x \in \Omega$ the function $x \mapsto \psi_0(x) := \gamma_0(x)^{-1}$ is well-defined and smooth in $C^\infty(\Omega, K)$. Suppose by induction that the functions $\psi_0, \ldots, \psi_k \in C^\infty(\Omega, K)$ have already been found in order to satisfy equation $\gamma * \Omega \psi = 1$ up to order $k$. At order $k + 1 \geq 1$ the condition reads

$$0 = (\gamma * \Omega \psi)_{k+1} = \sum_{l, p, q = 0}^{k+1} C_l(\gamma_p, \psi_q) = \gamma_0 \psi_{k+1} + F_{k+1}(\psi_0, \ldots, \psi_k, \gamma_0, \ldots, \gamma_{k+1})$$

where the term starting with $F_{k+1}$ denotes the difference $(\gamma * \Omega \psi)_{k+1} - \gamma_0 \psi_{k+1}$ which obviously does not contain $\psi_{k+1}$. Again, since $\gamma_0$ is nowhere zero on $\Omega$ the function $\psi_{k+1}$ can be computed from this equation by multiplying with $x \mapsto \gamma_0(x)^{-1}$. Hence there is a solution $\psi \in R_{\Omega}$ of equation $\gamma * \Omega \psi = 1$. In a completely analogous way there is a solution $\psi' \in R_{\Omega}$ of the equation $\psi' * \Omega \gamma = 1$.

- “Every $\phi \in R_{\Omega}$ is equal to $\eta(f) * \Omega \eta(g)^{n-1}$ for some $f \in R$ and $g \in S$” (property (i.b.)): the main idea is to transfer the proof of Lemme 6.1 of Jean-Claude Tougeron’s book [38, p.113] to the non-commutative situation. Let $\phi = \sum_{i=0}^{\infty} \lambda^i \phi_i \in R_{\Omega}$. We then fix the following data which we get thanks to the fact that $X$ and therefore each open set $\Omega$ is a second countable locally compact topological space: there is a sequence of compact sets $(K_n)_{n \in \mathbb{N}}$ of $X$, a sequence of open sets $(W_n)_{n \in \mathbb{N}}$, and a sequence of smooth functions $(g_n)_{n \in \mathbb{N}} : X \to \mathbb{R}$ such that

$$\bigcup_{n \in \mathbb{N}} K_n = \Omega,$$

and

$$\forall n \in \mathbb{N} : K_n \subset W_n \subset \overline{W_n} \subset K_{n+1} \text{ and } g_n(x) = \begin{cases} 1 & \text{if } x \in W_n, \\ 0 & \text{if } x \not\in K_{n+1}, \\ y \in [0,1] & \text{else}. \end{cases}$$

We denote by $\gamma_j$ the restriction $\eta(g_j)$ of $g_j$ to $\Omega$ for each nonnegative integer $j$. The idea is to define the denominator function $g$ as a (non formal!) converging sum $g = \sum_{j=0}^{\infty} \varepsilon_j g_j$. Choose a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ of strictly positive real numbers such that

$$\forall j \in \mathbb{N} : \varepsilon_j p_{K_{j+1}, j}(g_j) < \frac{1}{2^j} \text{ and } \forall i \leq j \in \mathbb{N} : \varepsilon_j \sum_{l=0}^{j} p_{K_{l+1}, j}(C_l(\phi_{i-l}, g_j)) < \frac{1}{2^j}$$

(see eqn (1.1) for the definition of the seminorms $p_{K_{m,n}}$) which is possible since for each nonnegative integer $j$ there are only finitely many seminorms and functions involved. For all nonnegative integers $i, j, N$ we define the functions $g_{(N)} \in C^\infty(X, K)$, and $\psi_{ij}, \psi_{(i,N)} \in C^\infty(\Omega, K)$:

$$g_{(N)} := \sum_{j=0}^{N} \varepsilon_j g_j, \quad \psi_{ij} := \sum_{l=0}^{i} C_l(\phi_{i-l}, \gamma_j), \quad \psi_{(i,N)} := \sum_{j=0}^{N} \varepsilon_j \psi_{ij} = \sum_{l=0}^{i} C_l(\phi_{i-l}, \gamma(N)),$$
and since \( \text{supp}(g_j) \subset K_{j+1} \subset \Omega \), hence \( \text{supp}(g(N)) \subset K_{N+1} \subset \Omega \), there are unique functions \( f_{ij} \in C^\infty(X, \mathbb{K}) \) such that

\[
f_{ij}(x) := \begin{cases} 
\psi_{ij}(x) & \text{if } x \in \Omega, \\
0 & \text{if } x \notin \Omega.
\end{cases}
\]

hence \( \eta(f_{ij}) = \psi_{ij} \) and \( \text{supp}(f_{ij}) \subset K_{j+1} \).

For each nonnegative integer \( N \) we set \( f_{(i,N)} := \sum_{j=0}^{N} \varepsilon_j f_{ij} \in C^\infty(X, \mathbb{K}) \) with \( \text{supp}(f_{(i,N)}) \subset K_{N+1} \).

Clearly, \( \eta(f_{(i,N)}) = \phi_{(i,N)} \).

We shall now prove that both sequences \( (g(N))_{N \in \mathbb{N}} \), and for each nonnegative integer \( i \), \( (f_{(i,N)})_{N \in \mathbb{N}} \) are Cauchy sequences in the complete metric space \( C^\infty(X, \mathbb{K}) \). First, it is obvious that for any two compact subsets \( K, K' \) and nonnegative integers \( N, N' \) we always have for all \( f \in C^\infty(\mathbb{R}^n, \mathbb{K}) \)

\[
\text{if } K \subset K' \text{ and } m \leq m' \text{ then } p_{K,m}(f) \leq p_{K',m'}(f).
\]

(2.2)

Fix a nonnegative integer \( i \). Let \( \epsilon \in \mathbb{R}, \epsilon > 0, K \subset X \) a compact subset, and \( m \in \mathbb{N} \). Then there is a nonnegative integer \( N_0 \) such that

\[
\frac{1}{2^{N_0}} < \epsilon, \quad m \leq N_0, \quad \text{and } i \leq N_0.
\]

Then for all nonnegative integers \( N, p \) with \( N \geq N_0 \) we get (since for all \( j \in \mathbb{N} \) such that \( N + 1 < j \) we have \( m \leq N_0 \leq N \leq j \) and \( i \leq N \), and \( \text{supp}(f_{ij}) \subset K_{j+1} \subset K_{j+1} \))

\[
p_{K,m}(f_{(i,N+p)} - f_{(i,N)}) = p_{K,m} \left( \sum_{j=N+1}^{N+p} \epsilon_j f_{ij} \right) \leq \sum_{j=N+1}^{N+p} \epsilon_j p_{K,m}(f_{ij}) = \sum_{j=N+1}^{N+p} \epsilon_j p_{K,m}(f_{ij})
\]

\[
\leq \sum_{j=N+1}^{N+p} \epsilon_j p_{K_{j+1},m}(\sum_{l=0}^{i} C_l(\psi_{i-l}, g_j)) \leq \sum_{j=N+1}^{N+p} \epsilon_j p_{K_{j+1},m}(C_l(\psi_{i-l}, g_j))
\]

\[
< \sum_{j=N+1}^{N+p} \frac{1}{2^j} = \frac{1}{2^N} \left( 1 - \frac{1}{2^p} \right) < \frac{1}{2^{N_0}} < \epsilon.
\]

It follows that for each \( i \in \mathbb{N} \) the sequence \( (f_{(i,N)})_{N \in \mathbb{N}} \) is a Cauchy sequence in the locally convex vector space \( C^\infty(X, \mathbb{K}) \) hence converges to a smooth function \( f_i = \sum_{j=0}^{\infty} \epsilon_j f_{ij} \). Replacing in the above reasoning the function \( \phi_0 \) by the constant function \( 1 \) on \( \Omega \) it follows that the sequence \( (g(N))_{N \in \mathbb{N}} \) converges to a smooth function \( g : X \rightarrow \mathbb{R} \). Now let \( x \in \Omega \). Then there is a nonnegative integer \( J_0 \) such that \( x \in K_{J_0} \). It follows from the nonnegativity and the definition of all the \( g_j \) and from the strict positivity of \( \epsilon_j \) that

\[
g(x) = \sum_{j=0}^{\infty} \epsilon_j g_j(x) \geq \epsilon_{J_0} g_{J_0}(x) = \epsilon_{J_0} > 0
\]

(2.3)

showing that \( g \) takes strictly positive values on \( \Omega \) whence \( g \in S \).

Now let \( x \notin \Omega \). Then for any \( v \in T_x X \) with \( h(v, v) \leq 1 \) we have that

\[
\forall m \in \mathbb{N} : \quad (D^m g(N))(v) = \sum_{j=0}^{N} \epsilon_j (D^m g_j)(v) = 0
\]

because each \( g_j \) has compact support in \( K_{j+1} \subset \Omega \). Since \( g(N) \rightarrow g \) for \( N \rightarrow \infty \) it follows by the continuity of differential operators and evaluation functionals that \( D^m g(N)(v) \rightarrow D^m g(v) \), and hence

\[
\forall x \in X \setminus \Omega, \forall m \in \mathbb{N}, \forall v \in T_x X, \quad h(v, v) \leq 1 : \quad (D^m g)(v) = 0,
\]

(2.4)
and in a completely analogous manner
\[ \forall x \in X \setminus \Omega, \forall m \in \mathbb{N}, \forall v \in T_x X, h(v, v) \leq 1 : (D^m f_i)(v) = 0. \]

Hence the infinite jets of all the functions \( g \) and \( f_i, i \in \mathbb{N} \), vanish outside the open subset \( \Omega \). J.-C. Tougeron calls the function \( g \) fonction aplatisseur for the family \( (\phi_i)_{i \in \mathbb{N}} \) in case \( C_l = 0 \) for \( l \geq 1 \). Now we get
\[ (\phi \ast_U \eta(g_N))_i = \sum_{l=0}^{i} C_l(\phi_{i-l}, \eta(g_N)) = \psi_{(i,N)} = \eta(f_{(i,N)}). \]

Since the restriction map \( \eta : C^\infty(X, \mathbb{K}) \to C^\infty(\Omega, \mathbb{K}) \) is continuous (where the Fréchet topology on \( C^\infty(\Omega, \mathbb{K}) \) is induced by those seminorms \( p_{K,m} \) where \( K \subset \Omega \) as are the bidifferential operators \( C_l \) we can pass to the limit \( N \to \infty \) in the above equation and get
\[ \phi \ast \Omega \eta(g) = \sum_{i=0}^{\infty} \lambda^i(\phi \ast \Omega \eta(g))_i = \sum_{i=0}^{\infty} \lambda^i \eta(f_i) =: \eta(f). \]

Since \( g \in S \) it follows that \( \eta(g) \) is invertible in \( R_\Omega \) by property (i.a) of Definition \[ \ref{definition} \] and the preceding equation implies \( \phi = \eta(f) \ast \Omega \eta(g)^{*-1} \) thus proving property (i.b) of Definition \[ \ref{definition} \].

- The kernel of \( \eta \) is equal to the space of functions \( f \in R \) such that there is \( g \in S \) with \( f \ast g = 0 \) (property (i.c) of Definition \[ \ref{definition} \]. Clearly if there is \( f \in R \) and \( g \in S \) such that \( f \ast g = 0 \) then \( \eta(f) \ast \Omega \eta(g) = 0 \), and since \( \eta(g) \) is invertible in \( R_\Omega \) we have \( \eta(f) = 0 \).

Conversely, if \( f \in R \) such that \( \eta(f) = 0 \), then for all integers \( i \in \mathbb{N} \) and for all \( x \in \Omega \) we have \( f_i(x) = 0 \). Hence the infinite jet of each \( f_i \) vanishes at each point \( x \in \Omega \) since \( \Omega \) is open. Take the fonction aplatisseur \( g \in S \) constructed in the preceding part of the proof for \( \phi_0 = 1, \phi_i = 0 \) for all \( i \geq 1 \). Then we get
\[ \forall x \in X : (f \ast g)_i(x) = \sum_{l=0}^{i} C_l(f_{i-l}, g)(x) = \begin{cases} 0 & \text{if } x \in \Omega \text{ since every jet of each } f_i \text{ vanishes in } \Omega, \\ 0 & \text{if } x \notin \Omega \text{ since every jet of } g \text{ vanishes outside of } \Omega, \end{cases} \]

where we have used eqn \[ \ref{equation} \] for the second alternative of the above statement. This proves part 1. of the theorem.

Statements 2. and 3. are immediate consequences of 1. and Theorem \[ \ref{theorem} \].

**Remarks:** For zero Poisson structure and trivial deformation \( C_l = 0 \) for all \( l \geq 1 \) the above result specializes upon restricting to terms of order 0 to the classical result that algebraic and analytic localization with respect to an open subset \( \Omega \subset X \) are isomorphic for the commutative \( \mathbb{K} \)-algebra \( C^\infty(X, \mathbb{K}) \).

Moreover, since for any closed set \( F \subset X \) Tougeron’s above construction gives us a smooth function \( g : X \to \mathbb{R} \) which is nowhere zero on the open set \( \Omega = X \setminus F \) and zero outside \( \Omega \), hence on \( F \), one gets the well-known result that the Zariski topology on \( X \) induced by the commutative \( \mathbb{K} \)-algebra \( C^\infty(X, \mathbb{K}) \) coincides with the usual manifold topology because each set \( Z(I) \) is closed by continuity of all the functions in the ideal \( I \), and conversely every closed set \( F \) is the zero set \( Z(gA) \) of the ideal \( gA \) (where \( A = C^\infty(X, \mathbb{K}) \)).

Finally, note that the numerator morphism \( \eta \) is injective iff the open set \( \Omega \) is dense which is quite easy to see.
3 Noncommutative germs for smooth star products

Let \((X, \pi)\) again be a Poisson manifold, and let \(\star = \sum_{i=0}^{\infty} \lambda^i C_i\) be a bidifferential star product. Let \(K = \mathbb{K}[\![\lambda]\!]\), and we denote the unital \(K\)-algebra \((\mathcal{C}^\infty(X, \mathbb{K})[\![\lambda]\!], \star)\) by \(R\). For any open set \(U \subset X\) let \(R_U\) denote the unital \(K\)-algebra \((\mathcal{C}^\infty(U, \mathbb{K})[\![\lambda]\!], \star_U)\), where \(\star_U\) denotes the obvious action of the bidifferential operators in \(\star\) to the local functions in \(\mathcal{C}^\infty(U, \mathbb{K})\). We write \(R_X = R\). For any two open sets with \(U \supset V\), denote by \(\eta^V_U : R_U \rightarrow R_V\) be the restriction morphism where we write \(\eta^V_U\) for \(\eta^V_U\). Clearly, for \(U \supset V \supset W\) one has the categorical identities \(\eta^V_W \circ \eta^U_V = \eta^U_W\) and \(\eta^V_U = \text{id}_U\). Denoting by \(\mathcal{X}\) the topology of \(X\) it is readily checked that the family \((R_U)_{U \subset X}\) with the restriction morphisms \(\eta^U_V\) defines a sheaf of \(K\)-algebras over \(X\), see e.g. the book [15] for definitions.

Let \(x_0\) a fixed point in \(X\), and let \(\mathcal{X}_{x_0} \subset \mathcal{X}\) the set of all open sets containing \(x_0\). We recall the definition of the stalk at \(x_0\), \(R_{x_0}\) of the sheaf \((R_U)_{U \subset X}\) whose elements are called germs at \(x_0\): it is defined as the inductive limit (or colimit, see [22]) \(\lim_{U \subset X \ni x_0} R_U\). In order to perform computations we recall the more down-to-earth definition: let \(\tilde{R}_{x_0}\) be the disjoint union of all the \(R_U\), i.e. the set of all pairs \((U, f)\) where \(U\) is an open set containing \(x_0\) and \(f \in \mathcal{C}^\infty(U, \mathbb{K})[\![\lambda]\!]\). Define an addition \(+\) and a multiplication \(\times\) on these pairs by

\[
(U, f) + (V, g) := (U \cap V, \eta^U_{U \cap V}(f) + \eta^V_{U \cap V}(g)) \quad \text{and} \quad (U, f) \times (V, g) := (U \cap V, \eta^U_{U \cap V}(f) \star_{U \cap V} \eta^V_{U \cap V}(g)),
\]

and it is easily checked that the addition is associative and commutative, that the multiplication is associative, and that there is the distributive law. Furthermore, the sum of \((U, f)\) and \((V, 0)\) equals \((U \cap V, \eta^U_{U \cap V}(f))\) which is equal to \((U, f) \times (V, 1) = (V, 1) \times (U, f)\). Next the binary relation \(\sim_{x_0}\) defined by

\[
(U, f) \sim_{x_0} (V, g) \iff \exists W \in \mathcal{X}_{x_0} \text{ with } W \subset U \cap V : \eta^U_V(f) = \eta^V_W(g)
\]

turns out to be an equivalence relation. Denoting by \(R_{x_0}/\sim_{x_0}\) the quotient set \(\tilde{R}_{x_0}/\sim_{x_0}\) and by \(\eta^U_{x_0} : R_U \rightarrow R_{x_0}\) the restriction of the canonical projection \(\tilde{R}_{x_0} \rightarrow R_{x_0}\) to \(R_U \subset \tilde{R}_{x_0}\) (where \(\eta^U_{x_0}\) will be shortened by \(\eta_{x_0} : R \rightarrow R_{x_0}\)) it is easy to see that the above addition and multiplication passes to the quotient, that all the zero elements \((U, 0)\) are equivalent as are all the unit elements \((U, 1)\), and that this defines the structure of a unital associative \(K\)-algebra denoted by \((R_{x_0}, \star_{x_0})\) on the quotient set such that all maps \(\eta^U_{x_0} : (R_U, \star_U) \rightarrow (R_{x_0}, \star_{x_0})\) are morphisms of unital \(K\)-algebras.

Note the following equations for all open sets \(U \supset V\):

\[
\eta^V_{x_0} \circ \eta^U_{x_0} = \eta^U_{x_0}.
\]

Define the following subsets \(S = S(x_0)\) and \(J = J_{x_0}\) of \(R\):

\[
S = S(x_0) = \{ g \in R \mid g_0(x_0) \neq 0 \} \quad \text{and} \quad J = J_{x_0} = \{ g \in R \mid g_0(x_0) = 0 \}.
\]

It is easy to see that \(S = R \setminus J\), that \(S\) is a multiplicative subset of \(R\), and that \(J_{x_0}\) is a maximal ideal of \(R\) (the quotient \(R/J\) is isomorphic to the quotient \(K/(\lambda K) \cong \mathbb{K}\) which is a field).

We now have the following analog of Theorem 2.3:

**Theorem 3.1.** Using the previously fixed notations we get for any point \(x_0 \in X\):

1. \((R_{x_0}, \star_{x_0})\) together with the morphism \(\eta_{x_0} : R \rightarrow R_{x_0}\) constitutes a right \(K\)-algebra of fractions for \((R, S(x_0))\).

2. As an immediate consequence we have that \(S(x_0)\) is a right denominator set.
3. This implies in particular that the algebraic localization $RS^{-1}$ of $R$ with respect to $S = S(x_0)$ is isomorphic to the concrete stalk $R_{x_0}$ as unital $K$-algebras.

Proof. 1. Once again, we have to check properties (i.a.), (i.b.), and (i.c) of Definition 1.1.

• “$η_{x_0}$ is $S$-inverting” (property (i.a.)): indeed, let $g ∈ S(x_0)$. Since $g_0(x_0) ≠ 0$ there is an open neighbourhood $U$ of $x_0$ such that $g_0(y) ≠ 0$ for all $y ∈ U$. Hence the restriction $η_U(g)$ is invertible in $(R_U, ⋆_U)$ by Theorem 2.1. Using eqn (3.1) we see that $η_{x_0}(g) = η_{x_0}(η_U(g))$, and the r.h.s. is invertible in $R_{x_0}$ as the image of an invertible element $η_U(g)$ in $R_U$ with respect to the morphism of unital $K$-algebras $η_{x_0}.

• “Every $φ ∈ R_{x_0}$ is equal to $η_{x_0}(f) ⋆_{x_0} η_{x_0}(g)^{-1}$ for some $f ∈ R$ and $g ∈ S(x_0)$” (property (i.b.)): indeed, let $φ ∈ R_{x_0}$. By definition of $R_{x_0}$ as a quotient set there is an open neighbourhood $U$ of $x_0$ and an element $ψ ∈ R_U$ with $η^U_{x_0}(U, ψ) = φ$. According to the preceding Theorem 2.1 there are elements $f, g ∈ R$ with $g_0(y) ≠ 0$ for all $y ∈ U$ such that $η_U(f) = ψ ⋆_U η_U(g)$. In particular, $g_0(x_0) ≠ 0$, hence $g ∈ S(x_0)$. Applying $η^U_{x_0}$ to the preceding equation we get (upon using eqn 3.1)

$$η_{x_0}(f) = η^U_{x_0}(η_U(f)) = (η^U_{x_0}(ψ)) ⋆_{x_0} (η^U_{x_0}(η_U(g))) = φ ⋆_{x_0}(η_{x_0}(g))$$

proving the result since $g ∈ S(x_0)$ and $η_{x_0}(g)$ is invertible in the unital $K$-algebra $(R_{x_0}, ⋆_{x_0})$.

• The kernel of $η_{x_0}$ is equal to the space of functions $f ∈ R$ such that $g ∈ S(x_0)$ with $f ⋆ g = 0$ (property (i.c)). Indeed, given $f ∈ R$ with $η_{x_0}(f) = 0$ then there is an open neighbourhood $W$ of $x_0$ such that $η_W(f) = η_W(0) = 0$. By the preceding Theorem 2.1 there is an element $g ∈ S_W ⊂ S(x_0)$ (which can be chosen to be a fonction aplatissaire) such that $f ⋆ g = 0$. This proves 1. of the theorem.

2. and 3. are immediate consequences of part 1. and Theorem 1.1. □

Warning: The stalk $R_{x_0}$ is taken in the sense of sheaves of $K[[\lambda]]$-algebras. Another interpretation would be two consider the sheaf $(C^∞(U, K))_{U ∈ ∑_{i_0}}$ of commutative $K$-algebras and the classical stalk $C^∞(X, K)_{x_0}$: in a completely analogous fashion it can be shown that it is isomorphic to the algebraic localization with respect to the multiplicative set of functions which do not vanish at $x_0$. However the $K[[\lambda]]$-module $C^∞(X, K)_{x_0}[[\lambda]]$ is NOT in general isomorphic to the above $R_{x_0}$: if $f = ∑ l=0 ^∞ λ^lf_l$ is a series of smooth functions such that $f_l$ vanishes on an closed ball of radius $ε_l > 0$ around $x_0$ where $ε_l → 0$ (for $l → ∞$) and is non-zero outside, then the germ of each $f_l$ vanishes, but there is no open neighbourhood of $x_0$ such that $f$ restricted to that neighbourhood vanishes which would imply that the ‘$K[[\lambda]]$-germ of $f$’ vanishes. We shall come back to this problem in Section 4.

4 Commutatively localized star products

In this section we shall describe a more algebraic framework to generalize the two preceding sections. Let in the following $K$ be a fixed unital associative commutative ring. Unadorned tensor products ⊗ are always with respect to $K$, hence ⊗ = ⊗.$K$. 4.1 Algebraic (multi)differentail operators and their localization

We shall first recall the well-known theory of algebraic (multi)differentail operators and their localization, see e.g. [19], [23], [21], [10] p.566-578, and [39]: let $A$ be commutative associative unital $K$-algebra. We shall need the theory only for $A$ and its tensor products over $K$, but as usual–indulging in some more generality has the benefit of being more economic for the computations: let $M$ and $N$ be left $A$-modules. For each $a ∈ A$ fix the following $K$-linear maps $L_a$, $R_a$, and $ad_a$
from the $K$-module $\text{Hom}_K(M,N)$ to itself defined in the following way for all $\phi \in \text{Hom}_K(M,N)$ and $m \in M$:

\[
(L_a(\phi))(m) = a(\phi(m)), \quad (R_a(\phi))(m) = \phi(am), \quad \text{ad}_a(\phi) = L_a(\phi) - R_a(\phi)
\]  

(4.1)

which obviously all commute. Then a $K$-linear map $\phi : M \to N$ is called a differential operator of order $k \in \mathbb{N}$ with respect to the $K$-algebra $A$ iff for all $a_1, \ldots, a_{k+1} \in A$ we have $(\text{ad}_{a_1} \circ \cdots \circ \text{ad}_{a_{k+1}})(\phi) = 0$. It is well-known that the set of all differential operators of order $k$ forms an $A$-$A$-bimodule (w.r.t. $L_a$ and $R_a$), and that these bimodules form an increasing filtration (indexed by the order) of the $A$-$A$-bimodule $\text{Hom}_K(M,N)$ whose union in $\text{Hom}_K(M,N)$ is called the $A$-$A$-bimodule of all differential operators. The $A$-$A$-bimodule of all differential operators of order $0$ is clearly identical to the set of all $A$-$A$-linear maps. Moreover, the composition $\psi \circ \phi$ of a differential operator $\phi : M \to N$ of order $k_1$ and a differential operator $\psi : M \to P$ (where $P$ is another $A$-module) of order $k_2$ is a differential operator $M \to P$ of order $k_1 + k_2$. Therefore there is a category $A$-$\text{Moddiff}$ whose objects are $A$-modules and morphisms differential operators. Let $A'$ be another unital associative commutative $K$-algebra, and $M'$, $N'$ be $A'$-modules. If $\phi : M \to N$ and $\phi' : M' \to N'$ are differential operators of order $k$ and $k'$, respectively, with respect to $A$ and $A'$, respectively, then

\[
\phi \otimes \phi' : M \otimes M' \to N \otimes N'
\]

is a differential operator of order $k + k'$ with respect to $A \otimes A'$, (4.2)

which follows from the obvious equation $\text{ad}_{a \otimes a'}(\phi \otimes \phi') = (\text{ad}_a(\phi)) \otimes (R_{a'}(\phi')) + (L_a(\phi)) \otimes (\text{ad}_{a'}(\phi'))$ for all $a, a' \in A$, and its iterations. Moreover, if $\chi : A \to A'$ is a $K$-algebra morphism and $\phi' : M' \to N'$ a differential operator of order $k'$ with respect to $A'$ it is obvious that $\phi'$ is also a differential operator of the same order $k'$ with respect to $A$ whence there is an obvious restriction functor from $A'$-$\text{Moddiff}$ to $A$-$\text{Moddiff}$. In the particular case of $A' = A_S$, the algebra of quotients of $A$ with respect to a fixed multiplicative subset $S \subset A$, and $\chi = \eta$, the numerator morphism, this restriction functor has a left adjoint which amounts to the localization of differential operators as has been shown by G. Vezzosi in his PhD-thesis, see [39, Prop. 3.3]:

**Theorem 4.1** (G. Vezzosi, 1997). Given the $K$-algebra $A$ and the multiplicative subset $S$ there is a covariant functor $(\cdot)_S : A$-$\text{Moddiff} \to A_S$-$\text{Moddiff}$ which is left adjoint to the above restriction functor $A$-$\text{Moddiff} \leftarrow A_S$-$\text{Moddiff}$ induced by the numerator morphism $A \to A_S$; on objects it is given by the localization of modules $M \to M_S$, and each differential operator $D : M \to N$ of order $k$ w.r.t. $A$ is mapped to the following differential operator $D_S : M_S \to N_S$ of the same order $k$ w.r.t. $A_S$ defined as follows for all $m \in M$ and $s \in S$

\[
D_S \left(\frac{m}{s}\right) = \sum_{r=1}^{k+1} \binom{k+1}{r} (-1)^{r+1} D_S(\frac{s^{-r-1}m}{s^r}). 
\]

(4.3)

In particular, $D_S$ is uniquely determined by its values $D_S \left(\frac{m}{s}\right) = \frac{D(m)}{s}$ for all $m \in M$, and it follows that $(D \circ D')_S = D_S \circ D'_S$ whenever the composition $D \circ D'$ makes sense.

The proof is quite technical: eqn (4.3) is motivated by the fact that if $D_S : M_S \to N_S$ is a differential operator of order $k$ satisfying $D_S(m/1) = D(m)/1$ then -by definition- it satisfies $0 = -(1/s)^{k+1}(\text{ad}_{s^{-1}/s}(D_S))(m/s)$ for all $m \in M$ and $s \in S$ which gives eqn (4.3). The right hand side of eqn (4.3) can be defined for any $K$-linear map $M \to N$ as a set-theoretic map $M \times S \to N_S$, and the fact that it only depends (first in a set-theoretical way) on the fraction $\frac{m}{s}$ is shown by induction over the order of the differential operator $D$. Note also that it can be shown a posteriori
that the integer $k$ in eqn (4.3) can be replaced by any integer $k' \geq k$ without changing the left hand side.

Next, let $p$ be a positive integer, let $M_1, \ldots, M_p$, $N$ be $A$-modules, and $k = (k_1, \ldots, k_p) \in \mathbb{N}^p$ a multi-index. Recall that a $K$-linear map $C : M := M_1 \otimes \cdots \otimes M_p \to N$ is called a multidifferential operator of rank $p$ of order $k$ with respect to $A$—which is sometimes also called a polydifferential operator—iff for each integer $1 \leq i \leq p$ and for all $m_1 \in M_1, \ldots, m_{i-1} \in M_{i-1}$, $m_{i+1} \in M_{i+1}, \ldots, m_p \in M_p$ the $K$-linear map $M_i \to N$ given by $m_i \mapsto C(m_1 \otimes \cdots \otimes m_p)$ is a differential operator of order $k_i$. For the particular case $A = \mathcal{C}^\infty(X)$ for a smooth manifold $X$ this algebraic definition is well-known to coincide with the analytic definition, see e.g. [10] p. 575, Satz A.5.2.] which means that in local charts an (algebraically defined) multidifferential operator looks as in eqn (1.2).

For our purposes it is more convenient to use the following formulation: note that the $K$-module $M_1 \otimes \cdots \otimes M_p$ is a module with respect to the unital commutative associative $K$-algebra $A^\otimes p = A \otimes \cdots \otimes A$ ($p$ tensor factors) in a natural way, and that $N$ also can be viewed as a $A^\otimes p$-module by means of $(a_1 \otimes \cdots \otimes a_p)n = a_1 \cdots a_p n$ for all $a_1, \ldots, a_p \in A$ and $n \in N$. Let $\Phi : M_1 \otimes \cdots \otimes M_p \to N$ be a $K$-linear map. If it is a differential operator of order $k$ with respect to $A^\otimes p$ it is easy to see by restricting to $1 \otimes \cdots 1 \otimes a_i 1 \otimes \cdots 1 \in A^\otimes p$, $1 \leq i \leq p$, $a_i \in A$, that $\Phi$ is a multidifferential operator of rank $p$ and order $(k_1, \ldots, k_p)$ with respect to $A$. Conversely, for any $a \in A$ and any integer $1 \leq r \leq p$ writing $L_a, R^r_a, \text{ad}^r_a$ for the following $K$-linear maps from $\text{Hom}_K(M_1 \otimes \cdots \otimes M, N)$ to itself given by (for all $m \in M$) $(L_a(C))(m) = aC(m)$, $(R^r_a(C))(m) = C(a^{(r)}m)$ (where $a^{(r)} = 1^{(r-1)} \otimes a \otimes 1^{(p-r)}$), and $\text{ad}_a^{(r)} = L_a - R^r_a$, there is the easy identity for all $a_1, \ldots, a_p \in A$

$$\text{ad}_{a_1} \otimes \cdots \otimes \text{ad}_{a_p} = \sum_{r=1}^p L_{a_1} \circ \cdots \circ L_{a_{r-1}} \circ \text{ad}^{(r)}_{a_r} \circ R^r_{a_{r+1}} \circ \cdots \circ R^p_{a_p}.$$ 

By iteration this shows that if $C$ is a multidifferential operator of rank $p$ and order $k_1 + \cdots + k_p$ w.r.t. $A$ then $C$ is a differential operator of order $k_1 + \cdots + k_p$ w.r.t. $A^\otimes p$. Hence

$$\{\text{multidifferential operators of rank } p \text{ w.r.t. } A\} = \{\text{differential operators w.r.t. } A^\otimes p\}. \quad (4.4)$$

With this identification, given a multiplicative subset $S \subset A$ it is now straight-forward to localize multidifferential operators by localizing them as differential operators w.r.t. $A^\otimes p$ taking the multiplicative subset $S^\otimes p \subset A^\otimes p$ (which is the obvious iteration of Remark 2 before eqn (1.7) upon using Vezzosi’s Theorem 4.1) Note that it is easy to see that the localization of the $A$-module $N$ w.r.t. the multiplicative subset $S$ is naturally isomorphic to the localization of $N$ seen as a $A^\otimes p$-module w.r.t. the multiplicative subset $S^\otimes p$. We are interested in the particular case where all the $A$ modules $M_1, \ldots, M_p, N$ are equal to $A$ for which we state the preceding considerations in the following

**Proposition 4.1.** Let $S_0 \subset A$ be a multiplicative subset, let $A_{S_0}$ be the ordinary commutative localization of $A$ w.r.t. $S_0$, and let $\eta(A_{S_0}) = \eta : A \to A_{S_0}$ be the numerator morphism. Let $C$ be a multidifferential operator of rank $p$ from $A^\otimes p$ to $A$.

Then there exists a unique multidifferential operator of rank $p$, $C_{S_0}$, from $(A_{S_0})^\otimes p$ to $A_{S_0}$ such that $\eta \circ C = C_{S_0} \circ \eta_{S_0}$.

Furthermore, given another multidifferential operator $C'$ of rank $p'$ we have $(C \circ_i C')_{S_0} = C_{S_0} \circ_i C'_{S_0}$ for each integer $1 \leq i \leq p$.

**Proof.** The first part follows from the above considerations. The second part follows from the equation $C \circ_i C' = C \circ (\text{id}^\otimes(i-1) \otimes C' \otimes \text{id}^\otimes(p-i))$ seen as composition of differential operators w.r.t. the $K$-algebra $A^\otimes(p+p'-1)$ and multiplicative subset $S_0^\otimes(p+p'-1)$ using eqn (4.2).
4.2 Commutatively localized algebraic star products

Observe now that the Definition 1.2 of star products can be generalized to any commutative associative unital $K$-algebra $A$ whence the significance ‘bidifferential’ for the $K$-bilinear maps $C_k : A \times A \to A$ is now given by the algebraic definition outlined in the preceding Section 1.1.

We have

Proposition 4.2. Let $A$ be a commutative unital $K$-algebra and a differential star product $\star = \sum_{i=0}^{\infty} \lambda^i C_i$ on $R := A[[\lambda]]$ where the $C_i$ are bidifferential operators on $A$. For any multiplicative subset $S_0 \subset A$ there exists a unique star product $\star_{S_0}$ on $A_{S_0}[[\lambda]]$ such that the numerator map $\eta$ canonically extended as a $K[[\lambda]]$-linear map (also denoted $\eta$) $A[[\lambda]] \to A_{S_0}[[\lambda]]$ is a morphism of unital $K[[\lambda]]$-algebras.

Proof. This follows from the previous Proposition 1.2 by considering the localization of the bidifferential operators $C_i$. It remains associative since the localization is compatible with the compositions $\circ_1$ and $\circ_2$.

With the above structures $A, S_0, \star$ we set $R = A[[\lambda]]$ and consider the following rather natural subset

$$S := S_0 + \lambda A[[\lambda]] \subset R = A[[\lambda]].$$

which can be called the canonical deformation of the multiplicative subset $S_0$. Then we have the

Proposition 4.3. The subset $S = S_0 + \lambda R$ is a multiplicative subset of the algebra $(R, \star)$, and its image under $\eta$ consists of invertible elements of the $K[[\lambda]]$-algebra $(A_{S_0}[[\lambda]], \star_{S_0})$.

It follows that there is a canonical morphism of unital algebras over $K[[\lambda]]$

$$\Phi : \left( (A[[\lambda]])_S, \star_S \right) \to (A_{S_0}[[\lambda]], \star_{S_0}).$$

where the localization $(A[[\lambda]])_S$ is the general construction, see Proposition 1.2.

Indeed, since the deformation terms of $\star$ come in higher orders of $\lambda$ it is clear that $S$ is multiplicative. Since $\eta(S_0)$ is invertible in $A_{S_0}$ this also holds for the image under $\eta$ of the canonical deformation $S$ of $S_0$, see the reasoning in the beginning of the proof of Theorem 2.1 which is completely algebraic. The existence of the algebra morphisms $\Phi$ is then clear from the universal property of the localized algebra, see Proposition 1.2.

Here we come to two general problems:

1. Does localization commute with deformation?

Meaning: is the above morphism $\Phi$ (4.6) an isomorphism?

2. Is $S$ a right (or left) denominator set?

Note that even in the commutative case, i.e. the localization of an algebra $R[[\lambda]]$ where $R$ is commutative, the map $\Phi$ is not always an isomorphism. This has already been noted in [3].

For localization with respect to open sets (see section 2 ($S_0 = S_\Omega$) the morphism $\Phi$ is an isomorphism, and $S = S_\Omega + \lambda R$ is a left and right denominator set. However, $\Phi$ is not injective for the germs $(S_0 = A \setminus J_{x_0})$ in section 3 as the warning at the end of the section indicates although $S = S_0 + \lambda R$ is a left and right denominator set.

One reason why $\Phi$ is in general not an isomorphism is that $(A[[\lambda]])_S$ is in general no longer a topologically free $K[[\lambda]]$-module, see e.g. [16] p.388-391 for all the details. Given a $K[[\lambda]]$-module $M$ there is a natural topology with basis induced by the (descending) filtration $\{\lambda^k M\}_{k \in \mathbb{N}}$. The
space $M$ is complete if for every sequence $(m_i) \subset M$ the series $\sum_{i=0}^{\infty} m_i \lambda^i$ convergences in $M$. It is Hausdorff iff $\bigcap_{i=0}^{\infty} \lambda^i M = \{0\}$ iff $\{0\}$ is closed in the $\lambda$-adic topology. A $K[[\lambda]]$-linear map between two $K[[\lambda]]$-modules is always continuous.

Next, a $K[[\lambda]]$-module is called (topologically) free if it is isomorphic to a $K[[\lambda]]$-module of the form $V[[\lambda]]$ for some $K$-module $V$. We have $V[[\lambda]] = V \hat{\otimes}_K K[[\lambda]]$. Note that here the tensor product is not the algebraic tensor product, but its completion in the $\lambda$-adic topology, see e.g. [16] p. 390-391. Moreover recall that the $\lambda$-torsion of a $K[[\lambda]]$ module $M$ is the set of elements $m \in M$ for which $\lambda m = 0$. There is the following well-known characterization, see e.g. [16] p.390, Prop. XVI.2.4.:

**Proposition 4.4.** A $K[[\lambda]]$-module $M$ is topologically free if and only if it is complete and Hausdorff in the $\lambda$-adic topology and $\lambda$-torsion free. In this case $M \cong (M/\lambda M)[[\lambda]]$.

Since completeness and Hausdorffness are preserved by isomorphism, $(A[[\lambda]])_S$ needs to be complete and Hausdorff for $\Phi$ to be an isomorphism. In fact, it needs to be topologically free.

If $(A[[\lambda]])_S$ is not Hausdorff, $\Phi$ is not injective, since then $\Phi^{-1}(0) \neq \{0\}$, since it is a closed subset in the $\lambda$-adic topology.

The example of germs (Section 3) is an example of this: Consider the example at the end of Section 3. Then for any $k \in \mathbb{N}$, we have $\sum_{i=0}^{k} \lambda^i f_i = 0 \in R_{x_0}$ since it vanishes on the ball of radius $\epsilon_k$ around $x_0$ (if we choose the sequence $(\epsilon_i)$ monotone). This means $f = \lambda^k \sum_{i=0}^{\infty} \lambda^i f_{i-k}$ so $f \in \lambda^k R_{x_0}$ for all $k$ but as stated before $f \neq 0$.

**Proposition 4.5.** Consider the situation of Proposition 4.3. If $(A[[\lambda]])_S$ is complete then the map $\Phi$ is surjective.

**Proof.** Let $a_0 \in A$ and $s_0 \in S_0$. We have $\Phi(a_0 \ast_S (s_0)^{-1}) = \Phi(a) \ast_{S_0} \Phi(s^{-1}) = \frac{a}{s_0} + \lambda r$ with $r = \frac{a}{s_1} \in (A)s_0[[\lambda]]$. Recursively one can find $a_i \in A, s_i \in S_0$, such that $\Phi(a_0 \ast_S (s_0)^{-1} - \sum_{i=1}^{\infty} \lambda^i a_i \ast_S (s_i)^{-1} = \frac{a}{s_0}$. The series on the left hand side converges since we assume $A[[\lambda]]_S$ to be complete.

More generally, it is always possible to extend the map $\Phi$ to the completion of $A[[\lambda]]_S$ due to the completeness of $(A_0)s_0[[\lambda]]$ and continuity of $\Phi$. The previous proposition implies that this extension is surjective.

It may be interesting to develop a noncommutative localization along the lines of Section 1.2 in particular in the spirit of Proposition 1.2 and/or Theorem 1.1 for complete unital associative $K[[\lambda]]$-algebras whose multiplicative subsets have some additional properties.

### 4.3 A particular result generalizing the restriction to open sets, Section 2

Let $A$ be a $K$-algebra. Suppose that the multiplicative set $S_0 \subset A$ has the following property

$$\forall \text{ sequence } (s_n)_{n \in \mathbb{N}} \in S_0 \ni \text{ sequence } (b_n)_{n \in \mathbb{N}} \in A \text{ and } s \in S_0 : \text{ s.t. } \forall n \in \mathbb{N} : s_n b_n = s. \quad (4.7)$$

Note that for a sequence having only a finite number of pairwise different terms this is always trivially satisfied by choosing for $s$ the common multiple of all the members in the associated finite set of the sequence. Moreover in the uninteresting case where $S_0$ contains 0 the above property (4.7) is trivially satisfied by choosing the constant 0-sequence for $(b_n)_{n \in \mathbb{N}}$. Returning to the general case, we shall refer to property (4.7) as $\sigma CM$ (something like ‘countable common multiple’). A similar property has been considered in [12] [33]. However, there the common multiple $s$ is only considered to be different from 0. Note however they consider domains, so $s$ is no zero divisor. This implies that one can consider the multiplicative set $S'$ generated by $S$ and $s$. Further the localization with respect to $S$ embeds injectively into the localization with respect to $S'$.
Proposition 4.6. For any open set $\Omega \subset X$ of a smooth manifold the multiplicative subset $S_\Omega = \{ g \in A \mid \forall x \in \Omega : g(x) \neq 0 \}$ appearing in Section 2 has the $\sigma CM$-property.

Indeed this follows from the proof of Theorem 2.1 in the trivial case where all the bidifferential operators of strictly positive order vanish, and where we set $\phi(x) = \sum_{n=0}^{\infty} \lambda^n (1/s_n(x))$ for all $x \in \Omega$, and the function appâtisseur $g$ (see eqn (2.3)) will be the desired element $s \in S_\Omega$. This construction is due to J.-C. Tougeron [37].

The main result of this subsection is the following

Proposition 4.7. Suppose that the multiplicative subset $S_0 \subset A$ satisfies the $\sigma CM$ property. Then the morphism $\Phi_s$, see eqn (4.7), is an isomorphism, and the deformed multiplicative subset $S = S_0 + AA[[\lambda]]$ is right and left denominator subset of the algebra $R$.

Proof. We first note the following easy, but important property of general differential operators $D : M \to N$ of order $k$ where $M$ and $N$ are arbitrary $A$-modules: for any $a \in A$ and $n \in \mathbb{N}$ with $n \geq k$ there are differential operators $\tilde{D}_[a], \tilde{D}[a] : M \to N$ of order $k$ such that for all $m \in M$

$$D(a^n m) = a^{n-k} \tilde{D}_[a](m) \quad \text{and} \quad a^n D(m) = \tilde{D}_[a](a^{n-k} m).$$

Indeed write $R^n_a = (L_a - \text{ad}_a)^n$ for the term on the left of the first equation, and $L^n_a = (\text{ad}_a + R_a)^n$ for the term on the left of the second equation, apply the binomial theorem, and use that all maps $\text{ad}_a^l(D)$ are differential operators of order $k - l \leq k$ and $\text{ad}_a^k(D) = 0$.

Next, let $* = \sum_{n=0}^{\infty} \lambda^n C_n$ be the star product with algebraic bidifferential operators $C_n$, $n \in \mathbb{N}$. We can assume that each $C_n$ has a ‘bi-order’ $(k_n, k_n)$ with $k_n \in \mathbb{N}$ for each $n \in \mathbb{N}$ (of course $k_0 = 0$), and for each $n \in \mathbb{N}$ we define the nonnegative integer $\kappa_n := \max\{0, k_0, k_1, \ldots, k_n\}$.

We shall show that $(A[A[[\lambda]], \ast_{S_0})$ is a right algebra of fractions of $(A[[\lambda]], \ast, S)$ along the (algebra-sized) lines of the proof of Thm 2.1.

- It follows from the previous section (and from the beginning of the proof of Theorem 2.1) that the numerator morphism $\eta : A[[\lambda]] \to A[S_0[[\lambda]]]$ is $S$-inverting.
- “Every $\phi = \sum_{n=0}^{\infty} \lambda^n \frac{a_n}{s_n} \in A[S_0[[\lambda]]]$ is equal to $\eta(f) \ast_{S_0} \eta(g)^{s_{0^{-1}}}$ for some $f = \sum_{n=0}^{\infty} \lambda^n a_n \in A[[\lambda]]$ and $g \in S^n$; here of course $a_0, a_1, \ldots \in A$, $a_0, a_1, \ldots \in A$, and $s_0, s_1, \ldots \in S$. We make the ansatz $g = s \in S_0$ of a ‘fonction appâtisseur’, and consider

$$\phi \ast_{S_0} \frac{S}{1} = \sum_{u=0}^{n} C_u s_0 \left( \frac{a_{n-u}}{s_{n-u}} \right) = \sum_{u=0}^{n-1} \sum_{v=1}^{\kappa_n+1} \left( \frac{\kappa_n + 1}{v} \right) (-1)^{v+1} C_u \left( \frac{s_{n-u}^{-1} a_{n-u}}{s_{n-u}} \right)$$

We have to choose $s \in S_0$ in such a way as to ‘kill the denominators occurring on the right hand side of the preceding equation’: thanks to the $\sigma CM$ property, for the sequence $((s_0 s_1 \cdots s_n)^{2k_n+1})_{n \in \mathbb{N}}$ which is in $S_0$ there is a sequence $(b_n)_{n \in \mathbb{N}}$ and $s \in S_0$ such that for each $n \in \mathbb{N}$ we have

$$\forall n \in \mathbb{N} : (s_0 s_1 \cdots s_n)^{2k_n+1} b_n = s.$$ (4.10)

Clearly, in each of the numerators of the fractions on the right hand side of eqn (4.9) the above $s$ can be written as a product of $s_{n-u}^{2k_n+1} c_{n,u}$ with $c_{n,u}$ is a product of $b_n$ and some factors of the above sequence. By the first equation of (4.8) we can pull $s_{n-u}^{\kappa_n+1}$ out of the second argument of the bidifferential operator in the numerator, and this factor in the numerator cancels each denominator. This shows that there is $f \in A[[\lambda]]$ such that $\phi \ast_{S_0} \eta(s) = \eta(f)$ and since $\eta(s)$ is $S_{S_0}$-invertible, the statement is proved.

- The kernel of $\eta$ is equal to the space of elements $f \in R$ such that there is $g \in S$ with $f \ast g = 0$: indeed, the statement $f = \sum_{n=0}^{\infty} \lambda^n f_n \in A[[\lambda]]$ is such that $f = \eta(f) = 0$ is equivalent to the
statement for each $n \in \mathbb{N}$ there is $s_n \in S_0$ such that $f_n s_n = 0$. In order to get an idea of $g \in S_0$ we again make the ansatz $g = s \in S_0$ and we compute for each $n \in \mathbb{N}$

$$
(f \star s)_n = \sum_{u=0}^{n} C_u(f_{n-u}, s). 
$$

(4.11)

We now take the same element $s$ constructed in the preceding part of the proof satisfying eqn (4.10) with respect to the above $s_0, s_1, \ldots \in S_0$ each killing $f_0, f_1, \ldots$. As in the preceding part, we can pull a factor $s_{n-u}^{n+1}$ out of the second argument of the bidifferential operator $C_u$ (upon using the first equation of eqn (4.8)), and we put it then into the first argument of the resulting bidifferential operator where a factor of $s_{n-u}$ remains in front of $f_{n-u}$ which gives zero (upon using the second equation of (4.8)). It follows that this choice of $s$ makes all the terms in eqn (4.11) vanish which shows the kernel of $\eta$ is contained in the subset of all $f$ killed by right multiplication of some $g \in S$. The other inclusion is trivial since $\eta$ is an $S$-inverting morphism of algebras, and $f \star g = 0$ for some $g \in S$ implies $\eta(f) \star s_0 \eta(g) = 0$ implying $\eta(f) = 0$ since $\eta(g)$ is invertible in $A_{S_0}[[\lambda]]$. It is obvious that the preceding constructions can be done for left fractions etc. by interchanging the arguments in the bidifferential operators. This proves the Proposition since $(A_{S_0}[[\lambda]], \star, S_0)$ is a right (and left) algebra of fractions of $(A[[\lambda]], \star, S)$ in the sense of Definition 1.1.

Note that the property $\sigma CM$ is NOT satisfied for any ‘interesting’ multiplicative subset $S_0$ of a Noetherian domain $A$ where we suppose that $S_0$ does not contain 0: we assume that there is a noninvertible element $s_0$ in $S_0$ because otherwise both localizations are isomorphic to the original algebra $(A[[\lambda]], \star)$. Then the sequence of principal ideals $(s_0^n)_{n \in \mathbb{N}}$ clearly equals the sequence of powers $(I^n)_{n \in \mathbb{N}}$ with $I = s_0 A$, and Krull’s Intersection Theorem (see e.g. [8, p.200, Ch.III 3.2, Corollary]) states that $\cap_{n \in \mathbb{N}} s_0^n A = \{0\}$ whence for the sequence $(s_0^n)_{n \in \mathbb{N}}$ no sequence $(a_n)_{n \in \mathbb{N}}$ can be found to satisfy property $\sigma CM$.

5 Non Ore multiplicative subsets in deformation quantization

The following example provides a multiplicative subset $S$ of a deformed algebra $(R = A[[\lambda]], \star)$ which is of the deformation type $S_0 + \lambda R$ (where $S_0$ is a multiplicative subset of $A$) which fails to satisfy the Ore condition, but is a subset of a large right denominator subset of $R$. This shows that the second problem we raised in the previous Section [4] does not seem to be immediately trivial.

Consider $A = C^\infty(\mathbb{R}^2, \mathbb{R})$ with the standard star product $\star$ given by formula (1.11). Let $R = A[[\lambda]]$, and let $\Omega \subset \mathbb{R}^2$ be the dense open set of all $(x, p) \in \mathbb{R}^2$ where $x \neq 0$. Set

$$
S_0 := \{1, x, x^2, \ldots \} \subset A \text{ and } S := S_0 + \lambda R \subset R.
$$

(5.1)

Recalling the multiplicative subset $S_\Omega = \{g \in A \mid \forall x \in \Omega : g(x) \neq 0\}$ we have the

**Proposition 5.1.** The subset $S \subset R$ is a multiplicative subset of $(R, \star)$ which is contained in the right denominator subset $S_\Omega + \lambda R \subset R$ (see section 2), but which is neither right nor left Ore.

**Proof.** Since $x^m \star x^n = x^{m+n}$ it is clear that $S$ is a multiplicative subset of $R$ which clearly is a subset of $S_\Omega$. Next pick a smooth real-valued function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties

$$
\forall p \in \mathbb{R} : 0 \leq \chi(p) \leq 1, \text{ supp}(\chi) \subset \left[ -\frac{1}{3}, \frac{1}{3} \right], \text{ and } \forall p \in \left[ -\frac{1}{6}, \frac{1}{6} \right] : \chi(p) = 1,
$$

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which is well-known to exist, and define the smooth functions $r \in A \subset \mathbb{R}$ and $s \in S_0 \subset S$ by

$$r(x,p) := \sum_{n=0}^{\infty} \chi(p-n) \frac{(p-n)^n}{n!} \quad \text{and} \quad s(x,p) = x$$

where $r$ is well-defined as a locally finite sum whose terms have mutually disjoint supports. We shall only need the following property of $r$ which is easy to see:

$$\forall n,k \in \mathbb{N} : \quad \frac{\partial^k r}{\partial p^k}(0,n) = \begin{cases} 0 & \text{if } 0 \leq k \leq n - 1, \\ 1 & \text{if } k = n. \end{cases} \quad (5.2)$$

We remark that there are also real analytic functions $r : \mathbb{R}^2 \to \mathbb{R}$ having the preceding property (5.2): it suffices to take the real part of the holomorphic function constructed by Weierstrass’s elementary factors, see e.g. [30, p. 303, Thm. 15.9].

Next note that an element $s' \in R$ is contained in $S$ iff there is a unique nonnegative integer $m$ and a unique smooth function $g \in \lambda A[[\lambda]]$ (i.e. $g_0 = 0$) such that $s'(x,p) = x^m + g$. For any such $s' \in S$ and $r' \in R$ we set

$$\mathcal{R}(r', s') := \sum_{k=0}^{\infty} \lambda^k \mathcal{R}_k(r', s') := r \star s' - x \star r'$$

which is a kind of deviation from the right Ore property for general $s' \in S$ and $r' \in R$. It is easy to compute that

$$\forall 0 \leq k \leq m : \quad \mathcal{R}_k(r', s')(x,p) = \binom{m}{k} \frac{\partial^k r}{\partial p^k}(x,p)x^{m-k} + \sum_{l=0}^{k-1} \frac{1}{l!} \frac{\partial^l r}{\partial p^l}(x,p) \frac{\partial^l g_{k-l}}{\partial x^l}(x,p) - xr'(x,p)$$

where the empty sum (occurring for $k = 0$) is defined to be 0. Using property (5.2) it is immediate that

$$\forall m \in \mathbb{N}, \forall g \in \lambda R, \forall r' \in R : \quad \mathcal{R}_m(r', s')(0,m) = 1 \neq 0, \quad \text{hence} \quad \mathcal{R}_m(r', s') \neq 0$$

showing that for the given $r \in R$, $s \in S$ there are no $r' \in R$ and $s' \in S$ satifying the right Ore condition. An easy application of eqn (1.13) using the fact that $S$ is obviously stable by the bijection $V$ shows that it also fails to satisfy the left Ore condition. 

This example shows the difference between the general noncommutative localization according to Proposition 1.2 and the localization with respect to multiplicative subsets satisfying the Ore conditions, see Theorem 1.1. The localization of $R$ w.r.t $S$ exists, and its elements are multifractions, see eqn (1.8): but mapping it into the localization with respect the the bigger Ore subset $S_\Omega + \lambda R$ helps to transform all the multifractions into simple right (or left) fractions.

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