THE COTANGENT BUNDLE OF A COMINUSCULE
GRASSMANNIAN

V. LAKSHMIBAI, VIJAY RAVIKUMAR, AND WILLIAM SLOFSTRA

Abstract. A theorem of the first author states that the cotangent bundle of the
type $A$ Grassmannian variety can be embedded as an open subset of a smooth
Schubert variety in a two-step affine partial flag variety. We extend this result to
cotangent bundles of cominuscule generalized Grassmannians of arbitrary Lie type.

1. Introduction

Earlier work of Lusztig and Strickland suggests possible connections between the
conormal varieties to partial flag varieties on the one hand, and affine Schubert
varieties on the other. In particular, Lusztig relates certain orbit closures arising
from the type $A$ cyclic quiver $\hat{A}_h$ to affine Schubert varieties [7]. In the case $h = 2$,
Strickland relates such orbit closures to conormal varieties of determinantal varieties
[9]; furthermore, any determinantal variety can be canonically realized as an open
subset of a Schubert variety in the Grassmannian [6].

Inspired by these results, the first author was interested in finding a relationship
between affine Schubert varieties and conormal varieties to the Grassmannian. As
a first step, she showed that the compactification of the cotangent bundle to the
Grassmannian is canonically isomorphic to a Schubert variety in a two-step affine
partial flag variety [5]. In this paper we extend her result to cominuscule generalized
Grassmannians of arbitrary finite type (such Grassmannians occur in types $A - E$).

1.1. Preliminaries. Let $G_0$ be a simple algebraic group over $\mathbb{C}$ with associated Lie
algebra $\mathfrak{g}_0$ and simple roots $\{\alpha_1, \ldots, \alpha_n\}$. A simple root $\alpha_i$ is cominuscule if the
coefficient of $\alpha_i$ in any positive root of $\mathfrak{g}_0$ (written in the simple root basis) is less
than or equal to 1.

The Weyl group of $G_0$ is generated by simple reflections $S_0 := \{s_1, \ldots, s_n\}$
corresponding to the the simple roots $\{\alpha_1, \ldots, \alpha_n\}$. For any subset $K \subset S_0$, we let
$P_K \subset G_0$ denote the parabolic subgroup whose Weyl group is generated by the
elements of $K$. For $1 \leq i \leq n$, set $S_{0,i} := S_0 \setminus \{s_i\}$, so that $P_{S_{0,i}}$ is a maximal parabolic
subgroup of $G_0$. The manifold $G_0/P_{S_{0,i}}$ is called a generalized Grassmannian of type
$G_0$, and is said to be cominuscule if $\alpha_i$ is cominuscule. For the remainder of the
paper, we fix $m \in [1, n]$ and consider the generalized Grassmannian $X := G_0/P_{S_{0,m}}$
associated to $\alpha_m$. Note that $\alpha_m$ may or may not be cominuscule at this point.

Let $\mathfrak{g}$ denote the affine untwisted Kac-Moody algebra associated to $\mathfrak{g}_0$, and let $\mathcal{G}$
be the corresponding affine Kac-Moody group (see [1] §6). The Dynkin diagram for

\footnote{We use calligraphic font (e.g. $\mathcal{G}$ and $\mathcal{P}_J$) for infinite-dimensional Kac-Moody groups, and non-calligraphic font for finite-dimensional Lie groups (e.g. $G_0$, $P_J$).}
\( \mathfrak{g} \) depends on the Dynkin diagram for \( \mathfrak{g}_0 \), and is shown in Table 1 (see [2] §18.1 or [3] §4.8]). We use the convention that the affine node (sometimes called the special node) is labelled by zero, and similarly let \( \alpha_0 \) and \( s_0 \) be the affine simple root and reflection respectively. The Weyl group \( W \) of \( \mathfrak{g} \) is generated by \( S := \{ s_0, \ldots, s_n \} \), and there is a parahoric subgroup \( \mathcal{P}_K \subset \mathcal{G} \) associated to any subset \( K \subset S \). We let \( \mathcal{X}_K := \mathcal{G}/\mathcal{P}_K \) denote the associated affine flag variety, and \( W_K \subset W \) denote the Weyl group of \( \mathcal{P}_K \), or in other words the subgroup of \( W \) generated by \( K \). For any subsets \( I \subset K \subset S \), let \( W_I^K \subset W_K \) denote the set of minimal length coset representatives of \( W_K/W_I \). In particular, \( W^K := W_S^K \) is the set of minimal length coset representatives of \( W/W_K \), and elements \( w \in W_K \) index Schubert varieties \( \mathcal{X}_K(w) \) of \( \mathcal{X}_K \).

Observe that \( S_0 = S \setminus \{ s_0 \} \). Let \( S_m := S \setminus \{ s_m \} \) and \( J := S_{0,m} = S \setminus \{ s_0, s_m \} \). Let \( w_i \) be the maximal element of \( W^I_{S_i} \), where \( i \in \{ 0, m \} \). It is a standard fact that \( \mathcal{X}_{S_m}(w_0) \cong \mathcal{X}_{J}(w_0) \cong X \) (see Lemma 2.3). The basis of this note is the following elementary but crucial observation:

**Lemma 1.1.** If \( \alpha_m \) is cominuscule in \( \mathfrak{g}_0 \) then \( \mathcal{X}_J(w_0) \) and \( \mathcal{X}_J(w_m) \) are isomorphic.

**Proof.** The list of cominuscule simple roots in each type is well known. We indicate the cominuscule simple roots for each Dynkin diagram (up to diagram automorphism) in the left column of Table 1, and the corresponding untwisted affine Dynkin diagram in the right column. In each case the Dynkin diagram of \( W_{S_0} \) is isomorphic to the Dynkin diagram of \( W_{S_m} \), and this isomorphism identifies \( \alpha_m \) with the affine root \( \alpha_0 \). Consequently \( \mathcal{X}_J(w_0) \) and \( \mathcal{X}_J(w_m) \) are isomorphic. \( \square \)

1.2. **Results for cominuscule varieties.** Consider the Schubert variety \( Y := \mathcal{X}_J(w_0w_m) \) in \( \mathcal{X}_J \). The Kac-Moody group \( \mathcal{G} \) acts on \( \mathcal{X}_J \) by left multiplication, and since \( G_0 \) is the Levi subgroup of \( \mathcal{P}_{S_0} \subset \mathcal{G} \), we can regard \( Y \) as a \( G_0 \)-variety.

In fact \( Y \) can naturally be considered as a \( G_0 \)-homogeneous fibre bundle over \( X \). More precisely:

**Theorem 1.2.** The affine Schubert variety \( Y = \mathcal{X}_J(w_0w_m) \) is stable under the left action of \( G_0 \subset \mathcal{G} \), and the natural projection \( Y \to \mathcal{X}_{S_0}(w_0) \cong X \) is a \( G_0 \)-homogeneous fibre bundle map with fibre \( \mathcal{X}_J(w_m) \). In particular \( Y \) is smooth.

Our main result is that if \( X \) is cominuscule then \( Y \) is a natural compactification of the cotangent bundle \( T^*X \):

**Theorem 1.3.** If \( X \) is cominuscule, then the fibre \( \mathcal{X}_J(w_m) \) is isomorphic to \( X \), and there is a \( G_0 \)-equivariant map \( \tilde{\mu} : T^*X \to Y \) of fibre bundles over \( X \), under which \( T^*X \) is isomorphic to a dense open subset of \( Y \).

We prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3. In order to prove Theorem 1.3 we explicitly construct the \( G_0 \)-equivariant embedding \( \tilde{\mu} : T^*X \to Y \), which maps the base \( X \) isomorphically onto the Schubert variety \( \mathcal{X}_J(w_0) \), and maps the fibre over the identity to a dense open subset of the Schubert variety \( \mathcal{X}_J(w_m) \).

When \( X \) is minuscule rather than cominuscule, it is natural to replace \( \mathcal{G} \) with a twisted affine Kac-Moody group. Theorem 1.2 still holds in this case, but as we show in Section 4, Theorem 1.3 does not hold. In this case the variety \( Y \) is not the compactification of the cotangent bundle \( T^*X \), but of a different bundle over \( X \).
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2. **The fibre bundle structure on Y**

Given $I \subset K \subset S$, we can write any $w \in W^I$ uniquely as $w = vu$, where $v \in W^K$ and $u \in W^I_K$. In this case the projection $G/P_I \to G/P_K$ induces a projection $\mathcal{X}_I(w) \to \mathcal{X}_K(v)$, and the generic fibre of this projection is $\mathcal{X}_I(u)$. We say $w = vu$ is a **parabolic decomposition** with respect to $K$.

For any $v \in W$, we define $\text{Supp}(v) := \{s \in S \mid s \leq v\}$ to be the set of simple reflections contained in a reduced expression for $v$. For any $u \in W$, let $D^I(u) := \{s \in S \mid su \leq_I u\}$, where $\leq_I$ is the Bruhat order on $W/W_I$. We have the following proposition from [8] Theorem 2.3 and Proposition 3.2:

**Proposition 2.1.** The projection $\mathcal{X}_I(w) \to \mathcal{X}_K(v)$ is a fibre bundle with fibre $\mathcal{X}_I(u)$ if and only if $\text{Supp}(v) \cap K \subset D^I(u)$.

When the condition $\text{Supp}(v) \cap K \subset D^I(u)$ is satisfied, we say that $w = vu$ is a **Billey-Postnikov decomposition** with respect to $K$.

Recall that for any $s \in S$, we have $sw \leq_I w$ if and only if $\mathcal{X}_I(w)$ is stable under left multiplication by the rank 1 parahoric subgroup $P_{\{s\}}$. It follows that if $L = D^I(w)$, then $\mathcal{X}_I(w)$ is stable under the action of the parahoric subgroup $P_L$ ([1], see also [8] Lemma 3.9).

**Lemma 2.2.** Let $y = w_0w_m$, so $Y = \mathcal{X}_I(y)$.

(a) $y = w_0w_m$ is a Billey-Postnikov decomposition with respect to $S_m$.

(b) $D^I(y) = S_0$.

**Proof.** Since $w_i$ is maximal in $W^I_{S_i}$, we know that $D^I(w_i) = S_i$, for $i \in \{0, m\}$. It is clear that $w_0w_m$ is a parabolic decomposition with respect to $S_m$, and $\text{Supp}(w_0) \cap S_m = S_0 \cap S_m = J \subset D^I(w_m)$, proving part (a).

For part (b), if $z \in W_{S_0}$ then $zw_0 \leq_I w_0$, and hence $zw_0 = v_0z'$, where $v_0 \in W^I_{S_0}$ and $z' \in W_J$. Similarly $z'w_m = v_mz''$, where $v_m \in W^I_{S_m}$ and $z'' \in W_J$. So $zy = v_0v_mz'' \leq_I y$, and hence $S_0 \subset D^I(y)$. But $D^I(y)$ must be a proper subset of $S$, so $D^I(y) = S_0$.

Given $K \subseteq S$, the Levi subgroup $G_K$ of $P_K$ is a Kac-Moody group with Weyl group $W_K$. Since $G$ is affine, if $K$ is a strict subset of $S$ then $G_K$ is finite-dimensional, and similarly $W_K$ is finite. In order to prove Theorem 1.2 we need the following standard lemma:

**Lemma 2.3.** If $K, I \subsetneq S$ and $w \in W^{K \cap I}_K$, then $P_{K,I} := G_K \cap P_I$ is the parabolic subgroup of $G_K$ corresponding to the subgroup $W_{K \cap I} \subseteq W_K$, and $\mathcal{X}_I(w)$ is isomorphic to a Schubert variety in the flag variety $G_K/P_{K,I}$. In particular, if $w$ is the maximal element of $W^{K \cap I}_K$ then $\mathcal{X}_I(w)$ is isomorphic to $G_K/P_{K,I}$.
Now the proof of Theorem 1.2 follows immediately from Lemma 2.2.

**Proof of Theorem 1.2.** By part (b) of Lemma 2.2, the variety \( Y = \mathcal{X}_f(w_0w_m) \) is stable under the left action of \( G_0 \). The base \( \mathcal{X}_{S_{m}}(w_0) \) is clearly \( G_0 \)-stable as well, and the natural projection \( Y \to \mathcal{X}_{S_{m}}(w_0) \) is \( G_0 \)-equivariant. By part (a) of Lemma 2.2 and Proposition 2.1, the projection \( Y \to \mathcal{X}_{S_{m}}(w_0) \) is a \( G_0 \)-homogeneous fibre bundle with fibre \( \mathcal{X}_f(w_m) \).

Now the Levi subgroup \( G_{S_0} \) of \( \mathcal{P}_{S_0} \) is simply \( G_0 \). Since \( S_m \cap S_0 = J \) and \( w_0 \) is the maximal element of \( W_{S_0}^J \), we can alternately set \( I = S_m \) and \( J = P_0 \) in Lemma 2.3 to get \( \mathcal{X}_{S_{m}}(w_0) \cong \mathcal{X}_f(w_0) \cong X = G_0/P_J \). Similarly \( \mathcal{X}_f(w_m) \) is isomorphic to the flag variety \( G_{S_m}/P_{S_m,J} \). Since \( Y \) is a fibre bundle with smooth fibre and base, it follows that \( Y \) is smooth. \( \square \)

### 3. The cotangent bundle

If \( X \) is cusinimcuse then \( \mathcal{X}_f(w_m) \) is isomorphic to \( X \) by Lemma 1.1. To prove Theorem 1.3, we explicitly construct the map \( T^*X \hookrightarrow Y \). Let \( B \) be the Borel subgroup of the Kac-Moody group \( G \) (in the literature \( B \) is also known as the Iwahori subgroup of \( G \)). For convenience, we write \( G_i \) for the Levi subgroup \( G_{S_i} \) of the parahoric subgroup \( \mathcal{P}_{S_i} \subset G \), where \( i \in \{0, m\} \) (in particular \( G_0 \) is the same as before). We let \( B_i := G_i \cap B \) be the induced Borel of \( G_i \), and \( P_i := B_iW_iB_i = G_i \cap \mathcal{P}_{J_i} \). Finally, let \( U_i \subset P_i \) be the unipotent radical of \( P_i \). As in the previous section, \( P_0 = P_J \), \( X = G_0/P_0 \), and moreover \( \mathcal{X}_f(w_i) \cong G_i/P_i \) for \( i \in \{0, m\} \).

We will also need to use the underlying Lie algebras. We assume the standard construction of \( g \), in which

\[
g \cong g_0 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d
\]

as a vector space (see [2 §18.1] or [3 §7.2]). Let \( h \) be the Cartan subalgebra of the Kac-Moody algebra \( g \). Let \( g_i \subset g \) be the (finite-dimensional) Lie algebra of \( G_i \), where \( i \in \{0, m\} \). Let \( u_i \subset g_i \) be the (nilpotent) Lie algebra of \( U_i \). Finally, let \( u_m \) be the opposite nilpotent radical to \( u_m \) inside \( g_m \). We consider the linear map

\[
\phi : u_0 \to g \text{ defined by } x \mapsto x \otimes z^{-1}.
\]

In order to prove Theorem 1.3 we will need the following lemma.

**Lemma 3.1.** The map \( \phi : u_0 \to u_m \) is a \( P_0 \)-equivariant isomorphism of vector spaces.

**Proof.** Let \( R \) denote the set of roots of \( g \), with simple roots \( \Delta := \{\alpha_0, \ldots, \alpha_n\} \). The simple roots of \( g_0 \) and \( g_m \) are the subsets of \( \Delta \) obtained by omitting \( \alpha_0 \) and \( \alpha_m \) respectively. For any subalgebra \( a \subset g \), we let \( R(a) \) denote the set of \( h \)-weights of \( a \), and let \( R^+(a) \) and \( R^-(a) \) denote the subsets of positive and negative roots respectively. Let \( \theta \) be the highest root of \( g_0 \), and let \( \delta = \alpha_0 + \theta \) be the basic imaginary root of \( g \) ([2 §17.1] or [3 §5.6]).

We can describe the set of roots of \( g \) by

\[
R(g) = \{\alpha + k\delta \mid \alpha \in R(g_0), k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}.
\]

The set of positive roots of \( g \) is given by

\[
R^+(g) = R^+(g_0) \cup \{\alpha + k\delta \in R \mid \alpha \in R(g_0), k \in \mathbb{Z}_{>0}\} \cup \{k\delta \mid k \in \mathbb{Z}_{>0}\}.
\]
Note that $R(u_0) \subset R^+(g_0)$ and $R(u_m) \subset R^+(g_m)$. Using the simple roots of $g_0$ and $g_m$, the roots of $u_0$ and $u_m$ can then be written

$$R(u_0) = \left\{ \sum_{i=1}^{n} a_i \alpha_i \in R^+(g) \mid a_m = 1 \right\},$$

and

$$R(u_m) = \left\{ a_0 \alpha_0 + \sum_{i \in [1, n] \setminus \{m\}} a_i \alpha_i \in R^+(g) \mid a_0 = 1 \right\},$$

where the requirement that $a_m = 1$ (resp. $a_0 = 1$) follows from the fact that $\alpha_m$ is cominuscule in $g_0$ (resp. $\alpha_0$ is cominuscule in $g_n$).

Every root of $g_0$ can be written uniquely as $\theta - \sum_{i=1}^{n} a_i \alpha_i$ where $a_i \geq 0$ for all $1 \leq i \leq n$. Since $\alpha_m$ is cominuscule, the coefficient of $\alpha_m$ in $\theta$ is equal to 1. Using the previous description of $R(u_0)$, it follows that $\alpha \in R(g_0)$ is an element of $R(u_0)$ if and only if

$$\alpha = \theta - \sum_{i \in [1, n] \setminus \{m\}} a_i \alpha_i$$

for some coefficients $a_i \geq 0$ (in particular, note that any $\alpha$ of this form cannot belong to $R^-(g_0)$, since it will have positive $\alpha_m$-coefficient).

Note that for any $\alpha \in R(u_0)$, the homomorphism $\phi$ maps $g_0$ isomorphically onto $g_{\alpha - \delta}$. Thus the $h$-weights of $\phi(u_0)$ are precisely

$$\left\{ \alpha - \delta \in R \mid \alpha = \theta - \sum_{i \in [1, n] \setminus \{m\}} a_i \alpha_i, \text{ where } a_i \geq 0 \text{ for } i \in [1, n] \setminus \{m\} \right\}$$

$$= \left\{ -(\theta + \delta) - \sum_{i \in [1, n] \setminus \{m\}} a_i \alpha_i \in R^- \mid a_i \geq 0 \text{ for all } i \in [1, n] \setminus \{m\} \right\}.$$

This latter set is exactly the negative of the $h$-weights of $u_m$, since $(-\theta + \delta) = \alpha_0$, and since $\alpha_0$ is cominuscule in $g_m$ as in Lemma 3.2. We conclude that $\phi(u_0) = u_m^{-}$. Since $\phi$ is a clearly bijective, it is a vector space isomorphism.

Consider the left adjoint action of $p_0 := \text{Lie}(P_0)$ on $g$. Under this action, each element of the weight space $g_\beta \subset p_0$ maps $g_\alpha$ into $g_{\alpha + \beta}$ whenever $\alpha + \beta \in R(g)$, and annihilates $g_\alpha$ otherwise. Recall that $R(p_0) = R^+(g_0) \cup \{ \sum_{i=1}^{n} a_i \alpha_i \in R^{-}(g_0) \mid a_m = 0 \}$, and observe that both $u_0$ and $u_m^{-}$ are stable under the left adjoint action of $p_0$, and moreover that $\phi$ is $p_0$-equivariant. It follows that $\phi$ is $P_0$-equivariant. \qed

Using the map $\phi : u_0 \to u_m^{-}$, we construct a map

$$\Phi : u_0 \to \mathcal{X}_J = \mathcal{G}/\mathcal{P}_J : x \mapsto [\exp(\phi(x)) \cdot \mathcal{P}_J].$$

**Lemma 3.2.** $\Phi$ is a $P_0$-equivariant algebraic isomorphism from $u_0$ to an open dense subset of $\mathcal{X}_J(w_m)$.

**Proof.** The exponential map $u_m^{-} \to \exp(u_m^{-}) =: U_m^{-}$ is an algebraic isomorphism, and $U_m^{-} \cong U_m^{-} \cdot \{ e \mathcal{P}_J \}$ is an open dense subset of $\mathcal{X}_J(w_m) = G_m/P_m$, where $e \in \mathcal{G}$ is the identity. Since $\phi$ is a $P_0$-equivariant bijection and $P_0 \subset \mathcal{P}_J$, the result follows. \qed
We can now finish the proof of the main theorem.

Proof of Theorem 1.3. As in Section 2 we let $Y = \mathcal{X}_f(y)$, where $y = w_0w_m$. The cotangent bundle of $X$ is

$$T^*X = G_0 \times_{P_0} u_0,$$

the quotient of $G_0 \times u_0$ by the $P_0$-action $p \cdot (g, x) = (gp, p^{-1}x)$. We can define a map

$$\mu : G_0 \times u_0 \to \mathcal{X}_f(y) : (g, x) \mapsto g \cdot \Phi(x),$$

where we use the fact that $\Phi(x) \in \mathcal{X}_f(w_m) \subset \mathcal{X}_f(y)$, which is stable under the left action of $G_0$ by Theorem 1.2. But $\mu$ is $P_0$-equivariant, so we get an induced map

$$\tilde{\mu} : G_0 \times_{P_0} u_0 \to \mathcal{X}_f(y).$$

The cotangent bundle map $T^*X \to X$ sends $(g, x) \mapsto [gP_0]$. Since the projection $G \to G/P_{S_m}$ sends $g \cdot \Phi(x) \mapsto [gP_{S_m}]$, we conclude that the diagram

$$\begin{array}{ccc}
G_0 \times_{P_0} u_0 & \longrightarrow & \mathcal{X}_f(y) \\
\downarrow & & \downarrow \\
G_0/P_0 & \cong & \mathcal{X}_{S_m}(w_0)
\end{array}$$

commutes, and thus $\tilde{\mu}$ is a morphism of $G_0$-homogeneous fibre bundles. Over $[eP_0]$, this map restricts to $\Phi : u_0 \to \mathcal{X}_f(w_m)$, which is injective and has open dense image in the fibre $\mathcal{X}_f(w_m)$. We conclude that the total map $G_0 \times_{P_0} u_0 \to \mathcal{X}_f(y)$ is injective and has open image. $\square$

4. Minuscule Grassmannians

A Grassmannian $X = G_0/P_{S_{0,n}}$ is minuscule if $\alpha_m^\vee$ is cominuscule in the dual root system. The minuscule and cominuscule Grassmannians coincide in types $A$, $D$, and $E$, but are disjoint in the other types. There are just two families of Grassmannians which are minuscule but not cominuscule: $\text{SO}(2n + 1)/P_{S_{0,n}}$, the Grassmannian corresponding to the root $\alpha_n$ in type $B_n$, and $\text{Sp}(2n)/P_{S_{0,1}}$, the Grassmannian corresponding to the root $\alpha_1$ in $C_n$. The corresponding Dynkin diagrams are listed in Table 2. As algebraic varieties, $\text{Sp}(2n)/P_{S_{0,1}}$ is isomorphic to $\mathbb{P}^{2n-1}$ and $\text{SO}(2n + 1)/P_{S_{0,n}}$ is isomorphic to $\text{SO}(2n + 2)/P_n \cong \text{SO}(2n + 2)/P_{n+1}$, so each minuscule Grassmannian is isomorphic to a cominuscule Grassmannian. However, the minuscule Grassmannians are distinct as homogeneous spaces, and their cotangent bundles are distinct as homogeneous bundles.

Suppose $\alpha_m$ is minuscule but not cominuscule, and let $g$ and $G$ be the affine twisted Kac-Moody algebra and group associated to $\mathfrak{g}_0$ (see [2], §18.4] or [3], and [4], §6]). The proof of Theorem 1.2 still works in this setting, and consequently the affine Schubert variety $Y = \mathcal{X}_f(w_0w_m) \subset \mathcal{X}_f : = G/J$ is a fibre bundle over $X$ with fibre $\mathcal{X}_f(w_m)$. Furthermore, following Lemma 1.1 we have $\mathcal{X}_f(w_m) \cong \mathcal{X}_f(w_0) \cong X$ (see Table 2 for the proof).

With all these pieces in place, we might expect that $Y$ is a compactification of $T^*X$ as in the cominuscule case. However, the argument from the cominuscule setting breaks down at this point. Specifically, the argument from Section 3 shows
that $Y$ is a compactification of the homogeneous vector bundle $T := G_0 \times_{P_0} u_m^-$ on $X$. However, $T$ is not the cotangent bundle of $X$. Indeed, by the following Lemma, $T$ splits as the direct sum of two $G_0$-homogeneous vector bundles on $X$, whereas $T^*X$ does not.

**Lemma 4.1.** As $P_0$-modules, $u_m^-$ splits as the direct sum of two submodules, while $u_0$ does not.

**Proof.** Let $\delta = \alpha_0 + \theta_0$ be the basic imaginary root of $\mathfrak{g}$, where $\theta_0$ is the highest short root of $\mathfrak{g}_0$ ([2, §17.1] or [3, §8.3]). For any subalgebra $\mathfrak{a} \subset \mathfrak{g}$, let $R_s(\mathfrak{a})$ (resp. $R_l(\mathfrak{a})$) denote the set of real short (resp. long) $\mathfrak{h}$-weights of $\mathfrak{a}$ (see [2, §17.2] or [3, §5.1]). The set of roots of $\mathfrak{g}$ is given by

$$\{\alpha + k\delta : \alpha \in R_s(\mathfrak{g}_0), k \in \mathbb{Z}\} \cup \{\alpha + 2k\delta : \alpha \in R_l(\mathfrak{g}_0), k \in \mathbb{Z}\} \cup \{k\delta : k \in \mathbb{Z} \neq 0\}.$$ 

Moreover, the $\mathfrak{h}$-weights of $u_0$ and $u_m^-$ are given by

$$R(u_0) = \left\{ \sum_{i \in [1,n]} a_i \alpha_i \in R^+(\mathfrak{g}) : a_m \in \{1, 2\} \right\},$$

$$R(u_m^-) = \left\{ a_0 \alpha_0 + \sum_{i \in [1,n] \setminus \{m\}} a_i \alpha_i \in R^-(\mathfrak{g}) : a_0 \in \{-1, -2\} \right\}.$$ 

Write $u_m^- = u_{m,s}^- \oplus u_{m,l}^-$, where $u_{m,s}^- := \oplus_{\alpha \in R_s(u_m^-)} \mathfrak{g}_\alpha$ and $u_{m,l}^- := \oplus_{\alpha \in R_l(u_m^-)} \mathfrak{g}_\alpha$. The short (resp. long) $\mathfrak{h}$-weights of $u_m^-$ are precisely those with $\alpha_0$ coefficient $a_0 = -1$ (resp. $a_0 = -2$) in the simple root basis. It follows that the left adjoint action of $P_0$ preserves the long and short roots of $u_m^-$, and hence $T = G_0 \times_{P_0} u_{m,s}^- \oplus G_0 \times_{P_0} u_{m,l}^-$ is a direct sum of two homogeneous vector bundles. On the other hand $u_0$ does not split as a $P_0$-module, since $\mathfrak{p}_0$ of $P_0$ can take short roots of $u_0$ (which have $\alpha_m$ coefficient 1) to long roots (which have $\alpha_m$ coefficient 2).

Let $\mathfrak{h}_0$ and $H_0$ denote the Cartan subalgebra and subgroup of $\mathfrak{g}_0$ and $G_0$ respectively. An $H_0$-module $M$ is *attractive* if there is some $\omega$ in $\mathfrak{h}_0$ such that $\alpha(\omega) > 0$ for all $\mathfrak{h}_0$-weights $\alpha$ of $M$. The fact that $Y$ cannot be the compactification of $T^*X$ follows from the following more general result.

**Lemma 4.2.** Given $P_0$-modules $U$ and $V$, suppose there exists an element $\omega \in \mathfrak{h}_0$ with the property that $\alpha(\omega) > 0$ for any $\mathfrak{h}_0$-weight $\alpha$ of $U$ or $V$. Furthermore, if both $G_0 \times_{P_0} U$ and $G_0 \times_{P_0} V$ embed as open dense homogeneous $G_0$-bundles in a homogeneous $G_0$-fibre bundle $Y$, then $U$ and $V$ are isomorphic as $P_0$-modules.

**Proof.** We can think of $U$ and $V$ as open dense subsets of the fibre over the identity in $Y$. As such, the intersection of $U$ and $V$ is non-empty. Let $y$ be a point of the intersection. Since $\alpha(\omega) > 0$ for all $\mathfrak{h}_0$-weights $\alpha$ of $U$ or $V$, the limit

$$\lim_{n \to \infty} \exp(-n\omega).y$$

exists and is equal to both $0_U$ and $0_V$, the zero elements of $U$ and $V$, which in particular must be equal. The sets $U$ and $V$ are both open, and $0 := 0_U = 0_V$ is a $P_0$-fixed point in both $U$ and $V$, so $U \cong T_0U = T_0V \cong V$ as $P_0$-modules.
Corollary 4.3. There is no open embedding of $T^*X$ into $Y$ as $G_0$-homogeneous fibre bundles over $X$.

Proof. Suppose on the contrary that such an embedding exists. Note that $\alpha(\omega_m) > 0$ for all $\alpha \in R(u_0)$, where $\omega_m \in h_0$ is the fundamental weight dual to the simple root $\alpha_m$. Moreover, the roots of $u_m^-$ are all of the form $\alpha - \delta$ or $\alpha - 2\delta$ with $\alpha \in R(u_0)$.

Since $\delta(\omega_m) = 0$, it follows that $\beta(\omega_m) > 0$ for all $\beta \in R(u_m^-)$ (indeed, $u_0$ and $u_m^-$ have the same $h_0$-weights since $\delta(\omega) = 0$ for any $\omega \in h_0$). By Lemma 4.2, it follows that $u_0$ and $u_m^-$ are isomorphic as $P_0$-modules, contradicting Lemma 4.1.

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E-mail address: lakshmibai@neu.edu
E-mail address: vijayr@cmi.ac.in
E-mail address: wslofstra@math.ucdavis.edu
Table 1. Finite type Dynkin diagrams with cominuscule simple root marked in black (left column), and the corresponding affine Dynkin diagrams with both the cominuscule and the additional affine root marked in black (right column).

Table 2. Finite type Dynkin diagrams with minuscule simple root marked in black (left column), and the corresponding twisted affine Dynkin diagrams with both the minuscule and the additional affine root marked in black (right column). We use Kac’s notation for the twisted affine Dynkin diagrams. Note that in Dynkin’s notation, $A_{2n-1}^{(2)}$ is denoted $\tilde{B}_n$, and $D_{2n+1}^{(2)}$ is denoted $\tilde{C}_n$. 