ON $k$-CIRCULANT MATRICES INVOLVING THE FIBONACCI NUMBERS

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Abstract. Let $k$ be a nonzero complex number. In this paper we consider a $k$-circulant matrix whose first row is $(F_1, F_2, \ldots, F_n)$, where $F_n$ is the $n$th Fibonacci number, and investigate the eigenvalues and Euclidean (or Frobenius) norm of that matrix. Also, the upper and lower bounds for the spectral norm of the Hadamard inverse of that matrix are obtained.

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1. INTRODUCTION

In this paper, $k$ is a nonzero complex number and $\mathbb{C}^{n \times n}$ denotes the set of all complex matrices of order $n$. Any $n^{th}$ root of $k$ and any primitive $n^{th}$ root of unity are denoted by $\psi$ and $\omega$, respectively. Symbols $\lambda_j, j = 0, n - 1, |C|, \|C\|_E, \|C\|_2$ and $C^{\circ-1}$ stand for the eigenvalues, the determinant, the Euclidean norm, the spectral norm and the Hadamard inverse of $C \in \mathbb{C}^{n \times n}$, respectively. Namely, for $C = [c_{i,j}] \in \mathbb{C}^{n \times n}$,

$$\|C\|_E = \left( \sum_{i,j=1}^{n} |c_{i,j}|^2 \right)^{\frac{1}{2}}, \|C\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(C^*C)},$$
where $C^*$ is the conjugate transpose of $C$, and $C^{\circ-1} = \left[ c_{i,j}^{-1} \right]$.

Definition 1. A matrix $C$ of order $n$ with the first row $(c_0, c_1, c_2, \ldots, c_{n-1})$ is called a $k$-circulant matrix if $C$ has the following form:

$$C = \begin{bmatrix}
  c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\
  k c_{n-1} & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \\
  k c_{n-2} & k c_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  k c_2 & k c_3 & k c_4 & \cdots & c_0 & c_1 \\
  k c_1 & k c_2 & k c_3 & \cdots & k c_{n-1} & c_0
\end{bmatrix}.$$  (1.1)
We shall write $C = cirn \{ k (c_0, c_1, c_2, \ldots, c_{n-1}) \}$ if a matrix $C$ has the form (1.1). The designation for the order of a matrix can be omitted if the dimension of a matrix is known. Circulant matrices are $k$-circulant matrices for $k = 1$ and skew circulant matrices are $k$-circulant matrices for $k = -1$.

The Fibonacci numbers $\{F_n\}$ satisfy the following recursive relation:

$$F_n = F_{n-2} + F_{n-1}, \quad n \geq 2,$$

(1.2)

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Let $\alpha$ and $\beta$ be the roots of the equation $x^2 - x - 1 = 0$, i.e.

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad \alpha \beta = -1, \quad \alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \quad (1.3)$$

Binet’s formula for the Fibonacci numbers is:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right). \quad (1.4)$$

Let us mention that the Lucas numbers $\{L_n\}$ satisfy the same recursive relation (as the Fibonacci numbers) but with initial conditions $L_0 = 2$ and $L_1 = 1$ and

$$L_n = \alpha^n + \beta^n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \quad (1.5)$$

is Binet’s formula for the Lucas numbers.

The following identities hold for the Fibonacci numbers:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \quad \text{and} \quad \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} = \frac{L_{2n+1} - (-1)^n}{5}. \quad (1.6)$$

More information about these numbers can be found in [2, 3, 8–10, 12, 14, 15].

In [4,13] the authors investigated the determinants and inverses of circulant (of skew circulant) matrices whose first rows are $(F_1, F_2, \ldots, F_n)$ and $(L_1, L_2, \ldots, L_n)$. The paper [5] is devoted to $k$-circulant matrices with the Fibonacci and Lucas numbers, and an upper bound estimation of the spectral norm for such matrices was given in that paper.

The motivation for this paper is the paper [17] in which the authors investigated $k$-circulant matrices with the generalized $r$-Horadam numbers $\{H_{r,n}\}$ which are defined as follows:

$$H_{r,n+2} = f(r)H_{r,n+1} + g(r)H_{r,n}, \quad n \geq 0,$$

where $r \in \mathbb{R}^+$, $H_{r,0} = a$, $H_{r,1} = b$, $a, b \in \mathbb{R}$ and $f^2(r) + 4g(r) > 0$, and presented the upper and lower bounds for the spectral norms of such matrices.
Theorem 1 (Theorem 5. [17]). Let $H = cir_c{k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})}$. 

a) If $|k| \geq 1$, then

$$\left\| H \right\|_2 \leq \sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq \sqrt{\left(a^2(1-|k|^2)+|k|^2\sum_{i=0}^{n-1} H_{r,i}^2 \right)\left(1-a^2+\sum_{i=0}^{n-1} H_{r,i}^2 \right)}$$  \hspace{1cm} (1.7)

b) If $|k| < 1$, then

$$\left| k \right| \sum_{i=0}^{n-1} H_{r,i}^2 \leq \left\| H \right\|_2 \leq \sum_{i=0}^{n-1} n H_{r,i}^2.$$  \hspace{1cm} (1.8)

Also, in [17], the formulae for the eigenvalues and determinant of a $k$-circulant matrix with the generalized $r$-Horadam numbers were derived.

Theorem 2 (Theorem 7. [17]). Let $H = cir_c{k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})}$. Then the eigenvalues of $H$ are:

$$\lambda_j = \frac{kH_{r,n} + (g(r)kH_{r,n-1} - b + af(r))\psi\omega^{-j} - H_{r,0}}{g(r)(\psi\omega^{-j})^2 + f(r)\psi\omega^{-j} - 1}, \quad j = 0, n-1.$$  \hspace{1cm} (1.9)

Theorem 3 (Theorem 8. [17]). Let $H = cir_c{k(H_{r,0}, H_{r,1}, \ldots, H_{r,n-1})}$. Then the determinant of $H$ is:

$$|H| = \frac{(H_{r,0} - kH_{r,n})^n - (g(r)kH_{r,n-1} - b + af(r))^nk}{(1 - k\alpha^n)(1 - k\beta^n)},$$  \hspace{1cm} (1.10)

where $\alpha$ and $\beta$ are the roots of the equation $x^2 - f(r)x - g(r) = 0$.

Let us also mention that, in [16], the authors considered circulant matrices with the generalized $r$-Horadam numbers and obtained the determinants and inverses of such matrices.

In the present paper, we consider the matrix

$$F = cir_c{k(F_1, F_2, \ldots, F_n)}$$  \hspace{1cm} (1.11)

and improve the result in relation to the eigenvalues of (1.11) which can be obtained from (1.9) because the authors did not consider the case when the denominator is equal to zero. Also, we determine the Euclidean norm of (1.11) and derive the upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11). The results are presented in the next section.
2. Results

Let us recall that $\psi$ is any $n^{th}$ root of $k$ and $\omega$ is any primitive $n^{th}$ root of unity. Also, throughout this section, $\alpha$ and $\beta$ are the roots of the equation $x^2 - x - 1 = 0$. In order to obtain the eigenvalues of (1.11), we need the following lemma.

**Lemma 1** (Lemma 4. [1]). The eigenvalues of $C = circ \{c_0, c_1, c_2, \ldots, c_n\}$ are:

$$
\lambda_i = \sum_{j=0}^{n-1} c_i (\psi \omega^{-j})^i, \ j = 0, n - 1.
$$

Moreover, in this case:

$$
c_i = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j (\psi \omega^{-j})^{-i}, \ i = 0, n - 1.
$$

**Theorem 4.** Let $F$ be the matrix as in (1.11). Then the eigenvalues of $F$ are given by the following formulae:

1) If $\psi \omega^{-j} = \frac{1}{\alpha}$, then

$$
\lambda_j = \frac{1}{\sqrt{5}} \left[ n\alpha + \frac{1 - (-1)^n \beta^{2n}}{\sqrt{5}} \right],
$$

2) If $\psi \omega^{-j} = \frac{1}{\beta}$, then

$$
\lambda_j = \frac{1}{\sqrt{5}} \left[ \frac{1 - (-1)^n \alpha^{2n}}{\sqrt{5}} - n\beta \right],
$$

3) If $\psi \omega^{-j} \neq \frac{1}{\alpha}$ and $\psi \omega^{-j} \neq \frac{1}{\beta}$, then

$$
\lambda_j = \frac{k F_{n+1} - 1 + k F_n \psi \omega^{-j}}{(\psi \omega^{-j})^2 + \psi \omega^{-j} - 1}.
$$

**Proof.** Based on Lemma 1 and (1.4), it follows:

1) Suppose that $\psi \omega^{-j} = \frac{1}{\alpha}$. Then,

$$
\lambda_j = \sum_{i=0}^{n-1} F_{i+1} (\psi \omega^{-j})^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{n-1} \left[ \alpha^{i+1} - \beta^{i+1} \right] \left( \frac{1}{\alpha} \right)^i
$$

$$
= \frac{1}{\sqrt{5}} \left[ \alpha \sum_{i=0}^{n-1} 1 - \beta \sum_{i=0}^{n-1} \left( \frac{\beta}{\alpha} \right)^i \right] = \frac{1}{\sqrt{5}} \left[ n\alpha - \beta \frac{1 - (\frac{\beta}{\alpha})^n}{1 - \frac{\beta}{\alpha}} \right]
$$

$$
= \frac{1}{\sqrt{5}} \left[ n\alpha + \frac{1 - (-1)^n \beta^{2n}}{\sqrt{5}} \right].
$$
2) Suppose that \( \psi \omega^{-j} = \frac{1}{q} \). Then,
\[
\lambda_j = \sum_{i=0}^{n-1} F_{i+1}(\psi \omega^{-j})^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{n-1} \left[ \alpha^i + \beta^{i+1} \right] \left( \frac{1}{\beta} \right)^i
\]
\[
= \frac{1}{\sqrt{5}} \left[ \alpha \sum_{i=0}^{n-1} \left( \frac{\alpha}{\beta} \right)^i - \beta \sum_{i=0}^{n-1} \right] = \frac{1}{\sqrt{5}} \left[ \alpha - \frac{(q/\beta)^n}{1-q} - n\beta \right]
\]
\[
= \frac{1}{\sqrt{5}} \left[ 1 - (-1)^n \alpha^{2n} \right] - n\beta.
\]
3) Suppose that \( \psi \omega^{-j} \neq \frac{1}{q} \) and \( \psi \omega^{-j} \neq \frac{1}{p} \). Then, \( \lambda_j \) follows from (1.9).

The previously obtained result will be illustrated by the following example.

**Example 1.** Let
\[
F = circ\{9-4\sqrt{5}(1,1,2,3,5,8)\}
\]
i.e.
\[
F = \begin{bmatrix}
1 & 1 & 2 & 3 & 5 & 8 \\
8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 & 5 \\
5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 \\
3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 \\
2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 \\
9-4\sqrt{5} & 2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1
\end{bmatrix}
\]
Since \( n = 6 \) and \( k = 9-4\sqrt{5} \) i.e. \( \psi = -\beta \) and \( \omega = \frac{1}{2} + i \frac{\sqrt{3}}{2} \), based on Theorem 4, it follows that
\[\star: \psi \omega^0 = \frac{1}{q}, \text{ so } \lambda_0 \text{ is obtained based on 1) of Theorem 4: } \lambda_0 = -29 + 15\sqrt{5};\]
\[\star: \psi \omega^{-j} \neq \frac{1}{q} \text{ and } \psi \omega^{-j} \neq \frac{1}{p}, \text{ for } j = 1,2, \text{ so } \lambda_j, \text{ for } j = 1,2, \text{ are obtained based on 3) of Theorem 4: } \lambda_{1,5} = -\frac{1}{2} \left[ 51 - 23\sqrt{5} \pm i \sqrt{3}(29 - 13\sqrt{5}) \right], \lambda_{2,4} = 7 - 3\sqrt{5} \mp i \sqrt{3}(29 - 13\sqrt{5}), \lambda_3 = 72 - 32\sqrt{5};\]

Bearing in mind that \( |F| = \prod_{j=0}^{n-1} \lambda_j \), it follows that
\[|F| = -238300041216 - 106571018240\sqrt{5}.\]
Let us remark, in relation to the previous example, that the determinant of \( F = \text{circ}\{9, 4, 7, 1, 2, 3, 5, 8\} \) is not possible to obtain using the result of Theorem 3.

The next theorem is devoted to determining the Euclidean norm of (1.11). The following formula will be needed.

For all \( x \),

\[
\sum_{i=1}^{n-1} i x^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2}.
\]  

(2.6)

**Theorem 5.** Let \( F \) be the matrix as in (1.11). Then the Euclidean norm of \( F \) is:

\[
\|F\|_E = \frac{1}{5} \left\{ n \left[ L_{2n+1} - (-1)^n \right] + (|k|^2 - 1) \left[ -L_{2n} + (n-1)L_{2n+1} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n \right] \right\}.
\]  

(2.7)

**Proof.** From the definition of the Euclidean norm, using (1.4), (1.5), (1.6) and (2.6), we obtain:

\[
\begin{align*}
\|F\|_E^2 &= \sum_{i,j=1}^{n} |f_{i,j}|^2 \\
&= n F_1^2 + [ (n-1) + |k|^2 ] F_2^2 + [(n-2) + 2|k|^2] F_3^2 + \cdots + [1 + (n-1)|k|^2] F_n^2 \\
&= \sum_{i=0}^{n-1} (n-i) F_{i+1}^2 + |k|^2 \sum_{i=1}^{n-1} i F_{i+1}^2 \\
&= n \sum_{i=0}^{n-1} F_{i+1}^2 + (|k|^2 - 1) \sum_{i=1}^{n-1} i F_{i+1}^2 \\
&= \frac{1}{5} \left\{ n \left[ L_{2n+1} - (-1)^n \right] + (|k|^2 - 1) \sum_{i=1}^{n-1} i \left[ \frac{\alpha^{2i+2} - 2(\alpha \beta)^i + 1 + \beta^{2i+2}}{\alpha^2} \right] \right\} \\
&= \frac{n}{5} \left[ L_{2n+1} - (-1)^n \right] + \frac{|k|^2 - 1}{5} \left[ -n \alpha^{2n} + (n-1) \alpha^{2n+2} + \frac{5}{4} \frac{n}{2} + \frac{1-2n}{2}(-1)^n \right] \\
&= \frac{n}{5} \left[ L_{2n+1} - (-1)^n \right] + \frac{|k|^2 - 1}{5} \left[ -n L_{2n} + (n-1)L_{2n+2} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n \right]
\end{align*}
\]
\[ \frac{n}{5} \left[ L_{2n+1} - (-1)^n \right] + \frac{|k|^2 - 1}{5} \left[ -L_{2n} + (n - 1)L_{2n+1} + \frac{5}{2} + \frac{1 - 2n}{2}(-1)^n \right]. \]

Therefore,
\[ \| F \|_K = \sqrt{\frac{1}{5} \left\{ n \left[ L_{2n+1} - (-1)^n \right] + (|k|^2 - 1) \left[ -L_{2n} + (n - 1)L_{2n+1} + \frac{5}{2} + \frac{1 - 2n}{2}(-1)^n \right] \right\}}. \]

The upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11) will be given by the following theorem. We use the well-known inequalities
\[ \frac{\| C \|_E}{\sqrt{n}} \leq \| C \|_2 \leq \| C \|_E, \]
which hold for any matrix \( C \) of order \( n \), and the following lemma.

Lemma 2 ([7]). Let \( M = \left[ m_{i,j} \right] \) and \( N = \left[ n_{i,j} \right] \) be matrices of order \( m \times n \). Then
\[ \| M \circ N \|_2 \leq r_1(M) \circ c_1(N), \]
where \( M \circ N = \left[ m_{i,j}n_{i,j} \right] \) is the Hadamard product (or Schur product),
\[ r_1(M) = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |m_{i,j}|^2 \quad \text{and} \quad c_1(N) = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |n_{i,j}|^2. \]

More information about the Hadamard product can be found in [6, 11].

Theorem 6. Let \( F \) be the matrix as in (1.11).
1) If \( |k| \geq 1 \), then
\[ \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} \leq \| F^{\circ -1} \|_2 \leq \sqrt{n(1 + (n - 1)|k|^2)}. \]
2) If \( |k| < 1 \), then
\[ |k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} \leq \| F^{\circ -1} \|_2 \leq n. \]

Proof. From the definition of the Euclidean norm, it follows that
\[ \| F^{\circ -1} \|_2^2 = \sum_{i=0}^{n-1} \left( n - i \right) \frac{1}{F_{i+1}^2} + |k|^2 \sum_{i=1}^{n-1} \frac{1}{F_{i+1}^2}. \]
1) If $|k| \geq 1$, then
\[
\| F^{-1} \|_E^2 \geq \sum_{i=0}^{n-1} (n-i) \frac{1}{F_{i+1}^2} + \sum_{i=1}^{n-1} \frac{1}{F_{i+1}^2} = n \sum_{i=0}^{n-1} \frac{1}{F_{i+1}^2} = n n \frac{1}{F_{n+1}^2} 
\]
\[
\geq \left( \sum_{i=1}^{n} \frac{1}{F_i^2} \right) \geq n \sum_{i=1}^{n} \frac{1}{F_i^2} = (n F_n)^2 \geq \frac{n^2}{F_n F_{n+1}} 
\]
\[
= \frac{5n^2}{L_{2n+1} - (-1)^n} 
\]
Therefore,
\[
\| F^{-1} \|_E \geq \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} 
\]
We conclude from (2.8) that
\[
\| F^{-1} \|_2 \geq \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} 
\]
Now, we shall obtain the upper bound for the spectral norm of $F^{-1}$. Let $R$ and $S$ be the following matrices:
\[
R = \begin{bmatrix}
\frac{1}{F_1} & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n} \\
k & \frac{1}{F_1} & \frac{1}{F_2} & \cdots & \frac{1}{F_{n-1}} \\
k & k & \frac{1}{F_1} & \cdots & \frac{1}{F_{n-2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k & k & k & \cdots & \frac{1}{F_1} 
\end{bmatrix}
\]
\[
S = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\frac{1}{F_n} & 1 & 1 & \cdots & 1 \\
\frac{1}{F_{n-1}} & \frac{1}{F_n} & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots & 1 
\end{bmatrix} 
\]
Then,
\[
r_1(R) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |r_{i,j}|^2} = \sqrt{1 + (n-1)|k|^2} 
\]
and
\[
c_1(S) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |s_{i,j}|^2} = \sqrt{n} 
\]
Since $F^{-1} = R \circ S$, based on Lemma 2, we can write
\[
\| F^{-1} \|_2 \leq r_1(R) \circ c_1(S) = \sqrt{n(1 + (n-1)|k|^2)} 
\]
2) If $|k| < 1$, then

$$
\| F^o_1 \|_E^2 \geq \sum_{i=0}^{n-1} (n-i) |k|^2 \frac{1}{F_{i+1}^2} + \sum_{i=1}^{n-1} i |k|^2 \frac{1}{F_{i+1}^2} = n|k|^2 \sum_{i=0}^{n-1} \frac{1}{F_{i+1}^2}
$$

$$
= n|k|^2 \sum_{i=1}^{n} \frac{1}{F_i^2} \geq n|k|^2 \sum_{i=1}^{n} \frac{1}{F_n^2} = |k|^2 \left( \frac{n}{F_n} \right)^2 \geq |k|^2 \frac{n^2}{F_n F_{n+1}}
$$

$$
= |k|^2 \frac{5n^2}{L_{2n+1} - (-1)^n}.
$$

Therefore,

$$
\| F^o_1 \|_E \geq |k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.
$$

We conclude from (2.8) that

$$
\| F^o_1 \|_2 \geq |k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.
$$

Now, we shall obtain the upper bound for the spectral norm of $F^o_1$. Let $Q$ and $W$ be the following matrices:

$$
Q = \begin{bmatrix}
\frac{1}{F_1} & 1 & 1 & \cdots & 1 \\
\frac{k}{F_n} & \frac{1}{F_1} & 1 & \cdots & 1 \\
\frac{k}{F_{n-1}} & \frac{k}{F_n} & \frac{1}{F_1} & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{k}{F_2} & \frac{k}{F_3} & \frac{k}{F_4} & \cdots & \frac{1}{F_1}
\end{bmatrix}
\quad \text{and} \quad
W = \begin{bmatrix}
1 & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n} \\
1 & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_{n-1}} \\
1 & 1 & 1 & \cdots & \frac{1}{F_{n-2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{bmatrix}
$$

Then,

$$
r_1(Q) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |q_{i,j}|^2} = \sqrt{n}
$$

and

$$
c_1(W) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |w_{i,j}|^2} = \sqrt{n}.
$$

Since $F^o_1 = Q \circ W$, based on Lemma 2, we can write

$$
\| F^o_1 \|_2 \leq r_1(Q) \circ c_1(W) = n.
$$

□
3. CONCLUSION

In this paper, we investigated the eigenvalues, the Euclidean norm and the upper and lower bounds for the spectral norm of the Hadamard inverse of

\[ F = \text{circ}_k(F_1, F_2, \ldots, F_n), \]

where \( F_n \) is the \( n^{th} \) Fibonacci number and \( k \) is a nonzero complex number.

From the fact that the eigenvalues of an upper triangular matrix are the diagonal entries, the eigenvalues of a semicirculant matrix (i.e. a \( k \)-circulant matrix for \( k = 0 \)) with the first row \((F_1, F_2, \ldots, F_n)\) are: \( \lambda_j = 1, (j = 0, n-1) \). The Euclidean norm of such (semicirculant) matrix can be obtained from (2.7) i.e. in (2.7) \( k \) can be equal to 0. Semicirculant matrices are not Hadamard invertible.

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