Basic Fourier series: convergence on and outside the $q$-linear grid

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Abstract. A $q$-type Hölder condition on a function $f$ is given in order to establish (uniform) convergence of the corresponding basic Fourier series $S_q[f]$ to the function itself, on the set of points of the $q$-linear grid.

Furthermore, by adding others conditions, one guaranties the (uniform) convergence of $S_q[f]$ to $f$ on and "outside" the set points of the $q$-linear grid.

Key words and phrases: $q$-trigonometric functions, $q$-Fourier series, Basic Fourier expansions, uniform convergence, $q$-linear grid.

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1. Introduction

Basic Fourier expansions on $q$-quadratic and on $q$-linear grids were first considered in [8] and in [7], respectively. Recently, in [10], sufficient conditions for (uniform) convergence of the $q$-Fourier series in terms of basic trigonometric functions $S_q$ and $C_q$, on a $q$-linear grid, were given. In [19] it was established an "addition" theorem for the corresponding basic exponential function, being these functions equivalent to the ones introduced by H. Exton in [12]. Following the unified approach of M. Rahman in [18], these functions can be seen as analytic linearly independent solutions of the initial value problem

$$\frac{\delta f(x)}{\delta x} = \lambda f(x), \quad f(0) = 1,$$

where $\delta$ is the symmetric $q$-difference operator acting on a function $f$ by

$$(1.1) \quad \delta f(x) = f(q^{1/2}x) - f(q^{-1/2}x),$$

with $0 < q < 1$. Then, from (1.1),

$$(1.2) \quad \frac{\delta f(x)}{\delta x} = \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{x(q^{1/2} - q^{-1/2})}.$$

There exists an important relation between this difference operator and the $q$-integral. The $q$-integral is defined by

$$\int_0^a f(x) d_q x = a(1 - q) \sum_{n=0}^{\infty} f(aq^n) q^n$$
and

\[(1.3) \quad \int_{a}^{b} f(x) dq(x) = \int_{0}^{b} f(x) dq(x) - \int_{0}^{a} f(x) dq(x).\]

From (1.2) and (1.3) it follows

\[(1.4) \quad \int_{-1}^{1} \delta f(x) dq(x) = q^{\frac{1}{2}} \left\{ \left[ f\left(q^{\frac{1}{2}}\right) - f\left(-q^{\frac{1}{2}}\right) \right] - \left[ f(0^{+}) - f(0^{-}) \right] \right\},\]

hence, one have the following formula \[10\] for \(q\)-integration by parts:

\[(1.5) \quad \int_{-1}^{1} g\left(q^{\frac{1}{2}}x\right) \delta q(x) dq(x) = -\int_{-1}^{0} f\left(q^{\frac{1}{2}}x\right) \delta q(x) dq(x) + q^{\frac{1}{2}} \left\{ \left[ (fg)\left(q^{\frac{1}{2}}\right) - (fg)\left(-q^{\frac{1}{2}}\right) \right] - \left[ (fg)(0^{+}) - (fg)(0^{-}) \right] \right\} .\]

These functions satisfy an orthogonality relation \[7, 12\] where the corresponding inner product is defined in terms of the \(q\)-integral (1.4). In \[7\], it was proved that they form a complete system and analytic bounds on their roots were derived.

As we will refer in section 2, the above \(q\)-trigonometric functions can be written using the Third Jackson \(q\)-Bessel function (or the Hahn-Exton \(q\)-Bessel function). In \[5\], analytic bounds were derived for the zeros of this function—which includes, as particular cases, the corresponding results established in \[7\]—and recently, in \[4\], it was shown that they define a complete system.

Throughout this paper we will follow the notation used in \[13\] which is now standard.

The publications \[7, 8, 9, 10, 20, 21\] are the most affiliated with this work. For other type of expansions (sampling theory) or related topics see \[1, 2, 3, 5, 6\].

2. The \(q\)-Linear Sine and Cosine. Properties.

The initial value problem

\[\frac{\delta f(x)}{\delta x} = \lambda f(x), \quad f(0) = 1,\]

has the analytic solution \[7\]

\[(2.1) \quad \exp_{q}[\lambda(1 - q)x] = \sum_{n=0}^{\infty} \frac{[\lambda(1 - q)x]^{n}q^{(n^{2} - n)/4}}{(q; q)_{n}},\]

which is a standard \(q\)-analog of the classical exponential function \[13, 18\]. The \(q\)-linear sine and cosine, \(S_{q}(z)\) and \(C_{q}(z)\), are then defined by

\[\exp_{q}iz := C_{q}(z) + iS_{q}(z).\]

From \[2.1\] we get

\[C_{q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{n(n^{2} - (1/2))z^{2n}}}{(q^{2}; q^{2})_{n}} = \frac{1}{1-q} \phi_{1}\left(0; q^{2}, q^{1/2}z^{2}\right)\]

\[S_{q}(z) = \frac{z}{1-q} \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{n(n+1/2)z^{2n}}}{(q^{2}; q^{3}; q^{2})_{n}} = \frac{z}{1-q} \phi_{1}\left(0; q^{3}, q^{3/2}z^{2}\right),\]
which can be written in terms of the third Jackson $q$-Bessel function (or, Hahn-Exton $q$-Bessel function) \cite{15, 17, 22}

$$J_{\nu}^{(3)}(z; q) := z^{\nu} \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \phi_1 \left( \begin{array}{c} 0 \\ q^{\nu+1} ; q, qz^2 \end{array} \right)$$

as

$$C_q(z) = q^{-3/8} \left( \frac{q^2; q^2}{(q; q^2)_{\infty}} \right)^{1/2} z^{1/2} J_{-3/4}^{(3)} \left( q^{-3/4} z^2 \right),$$

$$S_q(z) = q^{1/8} \left( \frac{q^2; q^2}{(q; q^2)_{\infty}} \right)^{1/2} z^{1/2} J_{1/4}^{(3)} \left( q^{-1/4} z^2 \right).$$

They satisfy \cite{7}

\begin{align}
\frac{\delta C_q(\omega z)}{\delta z} &= -\frac{\omega}{1-q} S_q(\omega z), \\
\frac{\delta S_q(\omega z)}{\delta z} &= \frac{\omega}{1-q} C_q(\omega z),
\end{align}

and, when $\omega$ is such that $S_q(\omega) = 0$,

\begin{equation}
[C_q(\omega)]^{-1} = C_q(\omega^{-1/2}) = C_q(q^{1/2} \omega). \tag{2.4}
\end{equation}

It is known \cite{7} that the roots of $C_q(z)$ and $S_q(z)$ are real, simple and countable. Further, because $C_q(z)$ and $S_q(z)$ are respectively even and odd functions, the roots of $C_q(z)$ and $S_q(z)$ are symmetric and we will denote the positive zeros of the function $S_q(z)$ by $\omega_k$, $k = 1, 2, \ldots$, with $\omega_1 < \omega_2 < \omega_3 < \ldots$.

As we mentioned before, the zeros of the function $S_q(z)$ form a discrete set of symmetric points in the real line. In \cite{7} page 145, it was shown that the set of positive zeros $\omega_k$, $k = 1, 2, \ldots$ of the function $S_q(z)$, verify the following analytic bounds:

If $0 < q < \beta_0$, where $\beta_0$ is the root of $(1-q^2)^2 - q^3$, $0 < q < 1$, then

$$q^{-k+\alpha_k+1/4} < \omega_k < q^{-k+1/4}, \quad k = 1, 2, \ldots,$$

where

$$\alpha_k \equiv \alpha_k(q) = \frac{\log \left[ \frac{1 - q^{2k+1}}{1-q^2} \right]}{2 \log q}, \quad k = 1, 2, \ldots.$$

According to Remark 1 in \cite{7} page 145, the previous result can be restated in the following form:

**Theorem A** For every $q$, $0 < q < 1$, $K$ exists such that if $k \geq K$ then

$$\omega_k = q^{-k+\epsilon_k+1/4}, \quad 0 < \epsilon_k < \alpha_k(q).$$

By using Taylor expansion one finds out that

\begin{equation}
\alpha_k(q) = \mathcal{O}(q^{2k}) \quad \text{as} \quad k \to \infty. \tag{2.5}
\end{equation}

Theorem 4.1 of \cite{7} page 139] settle the orthogonality relations:
Theorem B Considering \( \mu_k = (1 - q)C_q(q^{1/2}\omega_k)S'_q(\omega_k) \) we have
\[
\int_{-1}^{1} C_q(q^{1/2}\omega_k x)C_q(q^{1/2}\omega_m x)dx = \begin{cases} 
0 & \text{if } k \neq m \\
2 & \text{if } k = 0 = m \\
\mu_k & \text{if } k = m \neq 0 
\end{cases}
\]
\[
\int_{-1}^{1} S_q(q\omega_k x)S_q(q\omega_m x)dx = \begin{cases} 
0 & \text{if } k \neq m \vee k = 0 = m \\
q^{-1/2}\mu_k & \text{if } k = m \neq 0 
\end{cases}
\]

The Completeness Theorem [7] page 153, where a misprint is corrected, states the following:

**Theorem C** Let \( f(\omega_k z) = C_q(q^{1/2}\omega_k z) + i S_q(q\omega_k z) \) where the \( \omega_k, \omega_0 = 0 < \omega_1 < \omega_2 < \ldots \) are the non-negative roots of \( S_q(z) \). Suppose that
\[
\int_{-1}^{1} g(z)f(\omega_k z)dz = 0 \quad , \quad k = 0, 1, 2, \ldots
\]
where \( g(z) \) is bounded on \( z = \pm q^j \), \( j = 0, 1, 2, \ldots \). Then, \( g(z) \equiv 0 \), i.e., \( g(\pm q^j) = 0 \) for all \( j = 0, 1, 2, \ldots \).

To end this section we write down the Theorem 6.2 of [7] page 150:

**Theorem D** If \( S_q(\omega_k) = 0 \) then, for \( n = 0, 1, 2, \ldots \),
\[
S_q(q^{1+n}\omega_k) = S_q(q\omega_k) \sum_{j=0}^{n} (-1)^j q^{j(j+1)} \frac{(q^{1+n-j}; q)_{2j+1}}{(q; q)_{2j+1}} (\omega_k^2)^j ,
\]
\[
C_q(q^{1+n}\omega_k) = C_q(q^2\omega_k) \sum_{j=0}^{n} (-1)^j q^{j(j-1)} \frac{(q^{1+n-j}; q)_{2j}}{(q; q)_{2j}} (\omega_k^2)^j .
\]

3. The Fourier Coefficients

As a consequence of the orthogonality relations of Theorem B, we may consider formal Fourier expansions of the form
\[
(3.1) \quad f(x) \sim S_q[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[ a_k C_q(q^{1/2}\omega_k x) + b_k S_q(q\omega_k x) \right] ,
\]
with \( a_0 = \int_{-1}^{1} f(t)dt \) and, for \( k = 1, 2, 3, \ldots \),
\[
(3.2) \quad a_k = \frac{1}{\mu_k} \int_{-1}^{1} f(t)C_q(q^{1/2}\omega_k t)dt
\]
\[
(3.3) \quad b_k = \frac{q^{1/2}}{\mu_k} \int_{-1}^{1} f(t)S_q(q\omega_k t)dt ,
\]
where
\[
(3.4) \quad \mu_k = (1 - q)C_q(q^{1/2}\omega_k)S'_q(\omega_k) .
\]
In order to study the convergence of the series \( (3.1)-(3.4) \), it becomes clear that we need to know the behavior of the factor \( \mu_k \) of the denominator as \( k \to \infty \), which is equivalent to control the behavior of \( S'_q(\omega_k) \) and \( C_q(q^{1/2}\omega_k) \) as \( k \to \infty \).

Theorem 3.2 from [10] asserts that

**Theorem E** At least for \( 0 < q \leq (1/51)^{1/50} \),

\[
S'_q(\omega_k) = \frac{2}{1-q}q^{-(k-\frac{1}{2}-\epsilon_k)^2 S_k},
\]

where \( S_k \) satisfies \( \liminf_{k \to \infty} |S_k| > 0 \).

With respect to \( S_k \) from the previous theorem we have the following lemma:

**Lemma 3.1.** There exists a constant \( B \), independent of \( k \), such that \( |S_k| \leq B \), \( k = 1, 2, 3, \ldots \).

**Proof.** The expression of \( S_k \) is given [7, page 147] by

\[
S_k = \sum_{n=0}^{\infty} \frac{(-1)^n q^{m-n-k+1/2+\epsilon_k}}{(q^2, q^3, q^2)_n} = (-1)^k \sum_{m=-k}^{\infty} \frac{(-1)^m q^{m-n-k+1/2+\epsilon_k}}{(q^2, q^3, q^2)_m + k}.
\]

For \( k \) large enough, by Theorem A and (2.5), \( 1/2 + \epsilon_k > 0 \) hence

\[
|S_k| \leq \sum_{m=-k}^{\infty} \frac{|mq^{m-n-k+1/2+\epsilon_k}|}{(q^2, q^3, q^2)_m + k} \leq \frac{2}{(q^2; q)\infty} \sum_{m=1}^{\infty} mq^{m-1} \leq B
\]

which completes the proof since the infinite series on the right member is convergent.

We observe that the constant \( B \), as well as \( S_k \), depend on the parameter \( q \).

The behavior of \( C_q(q^{1/2}\omega_k) \) as \( k \to \infty \) will be known by the corresponding behavior of \( C_q(\omega_k) \) and by (2.4). Theorem 3.3 of [10] establishes

**Theorem F** At least for \( 0 < q \leq (1/50)^{1/49} \),

\[
C_q(\omega_k) = q^{-(k-\epsilon_k)^2} R_k,
\]

where \( |R_k| < \frac{2}{(1-q)(q; q)\infty} \) and \( \liminf_{k \to \infty} |R_k| > 0 \).

To end this section, we collect the Theorems 4.1, 4.2 and 4.3 of [10]:

**Theorem G** If \( c \in \mathbb{R} \) exists such that, as \( k \to \infty \),

\[
\int_{-1}^{1} f(t)C_q(q^{1/2}\omega_k t) \, dt = O(q^{ck}) \quad \text{and} \quad \int_{-1}^{1} f(t)S_q(q\omega_k t) \, dt = O(q^{ck})
\]

then, at least for \( 0 < q \leq (1/51)^{1/50} \), the \( q \)-Fourier series \( (3.1) \) is pointwise convergent at each fixed point \( x \in V_q = \{ \pm q^n-1 : n \in \mathbb{N} \} \).

**Theorem H** If \( c > 1 \) exists such that, as \( k \to \infty \),

\[
\int_{-1}^{1} f(t)C_q(q^{1/2}\omega_k t) \, dt = O(q^{ck}) \quad \text{and} \quad \int_{-1}^{1} f(t)S_q(q\omega_k t) \, dt = O(q^{ck})
\]
then, the \( q \)-Fourier series \( S_q \), at least for \( 0 < q \leq (1/51)^{1/50} \), converges uniformly on \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

**Theorem 1** If \( f \) is a bounded function on the set \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \), and the \( q \)-Fourier series \( S_q[f](x) \) converges uniformly on \( V_q \), then its sum is \( f(x) \) whenever \( x \in V_q \).

4. Convergence condition on the function

Denoting the \( q \)-Fourier coefficients of a function \( f \) by \( a_k(f(x)) \) and \( b_k(f(x)) \), \( k = 1, 2, 3, \ldots \), using \( \|C_q\|_{\mathfrak{M}_2} \), \( \|C_q\|_{\mathfrak{M}_3} \), \( \|C_q\|_{\mathfrak{M}_4} \) one have, by \( \|C_q\|_{\mathfrak{M}_5} \),

\[
(4.1) \quad a_k(f(x)) = \frac{1 - q}{q^{1/2} \omega_k \mu_k} \int_{-1}^{1} S_q(q \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} d_q t - \frac{1 - q}{q \omega_k} b_k \left( \frac{\delta f(q^{1/2} x)}{\delta x} \right)
\]

and

\[
(4.2) \quad b_k(f(x)) = \frac{q - 1}{q^2 \omega_k \mu_k} \left\{ q^{1/2} \left[ f(q^{-1}) - f(-q^{-1}) \right] C_q \left( q^{1/2} \omega_k \right) - q^{1/2} \left[ f(0^+) - f(0^-) \right] - \int_{-1}^{1} C_q \left( q^{1/2} \omega_k t \right) \frac{\delta f(q^{1/2} t)}{\delta t} d_q t \right\}
= \frac{1 - q}{q^{1/2} \omega_k} \left\{ a_k \left( \frac{\delta f(q^{-1/2} x)}{\delta x} \right) + q^{1/2} \left[ \frac{f(0^+) - f(0^-)}{\mu_k} - \frac{f(q^{-1}) - f(-q^{-1})}{(1 - q) S_q'(\omega_k)} \right] \right\}.
\]

The conjugation of this last two identities with Theorem H enables us to deduce conditions on the function \( f \) in order to guarantee uniform convergence of the corresponding Fourier series \( S_q[f] \). In its statement, we will consider the notation

\[
L_q^\infty[-1, 1] = \{ f : \sup \{ |f(\pm q^{n-1})| : n \in \mathbb{N} \} < \infty \}
\]

and the following definition:

**Definition 4.1** If two constants \( M \) and \( \lambda \) exist such that

\[
(4.3) \quad \left| f(\pm q^{n-1}) - f(\pm q^n) \right| \leq M q^{\lambda n}, \quad n = 0, 1, 2, \ldots ,
\]

then the function \( f \) is said to be \( q \)-linear Hölder of order \( \lambda \).

**Theorem 4.1.** If \( f \in L_q^\infty[-1, 1] \) is a \( q \)-linear Hölder function of order \( \lambda > \frac{1}{2} \) and satisfies \( f(0^+) = f(0^-) \), then, at least for \( 0 < q \leq (1/50)^{1/49} \), the corresponding \( q \)-Fourier series \( S_q[f] \) converges uniformly to \( f \) on the set of points \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

**Proof.** From \( \|C_q\|_{\mathfrak{M}_2} \) and \( \|C_q\|_{\mathfrak{M}_3} \) one have

\[
(4.4) \quad \int_{-1}^{1} f(t) C_q \left( q^{1/2} \omega_k t \right) d_q t = \mu_k a_k(f) = -\frac{1 - q}{q^{1/2} \omega_k} \int_{-1}^{1} S_q(q \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} d_q t.
\]
Similarly, from (4.3) and (4.2),

\[
\int_{-1}^{1} f(t) S_q(q \omega_k t) \, dq_t = q^{-1/2} \mu_k b_k(f) = q^{-1/2} \mu_k \frac{\delta}{\delta t} f(q^{1/2} t),
\]

\[
\frac{q^{-1}}{q \omega_k} \left\{ q^{1/2} \left[ f(q^{-1}) - f(-q^{-1}) \right] C_q \left( q^{1/2} \omega_k \right) - \int_{-1}^{1} C_q \left( q^{1/2} \omega_k t \right) \frac{\delta f(q^{1/2} t)}{\delta t} \, dq_t \right\}.
\]

By Cauchy-Schwarz inequality we have

\[
\left| \int_{-1}^{1} S_q(q \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} \, dq_t \right| \leq \left( \int_{-1}^{1} S_q^2(q \omega_k t) \, dq_t \right)^{1/2} \left( \int_{-1}^{1} \left( \frac{\delta f(q^{1/2} t)}{\delta t} \right)^2 \, dq_t \right)^{1/2}
\]

and

\[
\left| \int_{-1}^{1} C_q(q^{1/2} \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} \, dq_t \right| \leq \left( \int_{-1}^{1} C_q^2(q^{1/2} \omega_k t) \, dq_t \right)^{1/2} \left( \int_{-1}^{1} \left( \frac{\delta f(q^{1/2} t)}{\delta t} \right)^2 \, dq_t \right)^{1/2}
\]

Using the orthogonality relations of Theorem B we may write

\[
q^{1/2} \int_{-1}^{1} S_q^2(q \omega_k t) \, dq_t = \int_{-1}^{1} C_q^2(q^{1/2} \omega_k t) \, dq_t = \mu_k = (1-q)C_q(q^{1/2} \omega_k) S_q'(\omega_k),
\]

thus (4.6) and (4.7) become, respectively,

\[
\left| \int_{-1}^{1} S_q(q \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} \, dq_t \right| \leq q^{-1/2} (1-q)^{1/2} \left( C_q(q^{1/2} \omega_k) S_q'(\omega_k) \right)^{1/2} \left( \int_{-1}^{1} \left( \frac{\delta f(q^{1/2} t)}{\delta t} \right)^2 \, dq_t \right)^{1/2}
\]

and

\[
\left| \int_{-1}^{1} C_q(q^{1/2} \omega_k t) \frac{\delta f(q^{1/2} t)}{\delta t} \, dq_t \right| \leq (1-q)^{1/2} \left( C_q(q^{1/2} \omega_k) S_q'(\omega_k) \right)^{1/2} \left( \int_{-1}^{1} \left( \frac{\delta f(q^{1/2} t)}{\delta t} \right)^2 \, dq_t \right)^{1/2}.
\]

Now, using the corresponding definitions of the q-integral and of the operator \( \delta \), one finds that

\[
\int_{-1}^{1} \left( \frac{\delta f(q^{1/2} t)}{\delta t} \right)^2 \, dq_t = (1-q) \sum_{n=0}^{\infty} \left\{ \left[ f(q^n) - f(q^{n+1}) \right]^2 + \left[ f(-q^n) - f(-q^{n+1}) \right]^2 \right\} q^{-n}
\]
hence, since \( f \) is \( q \)-linear Hölder of order \( \lambda > \frac{1}{2} \), by (4.3),
\[
\int_{-1}^{1} \left( \frac{\delta f(q^t\omega t)}{\delta t} \right)^2 \, dq_t \leq 2M^2(1-q) \sum_{n=0}^{\infty} q^{(2\lambda-1)n} = \frac{2(1-q)M^2}{1-q^{2\lambda-1}}.
\]
In a similar way we obtain
\[
\int_{-1}^{1} \left( \frac{\delta f(q^{-\frac{1}{2}}t)}{\delta t} \right)^2 \, dq_t \leq \frac{2(1-q)M^2}{1-q^{2\lambda-1}}.
\]
Thus, (4.8) and (4.9) become, respectively,
\[
\left| \int_{-1}^{1} S_q(q\omega_k t) \frac{\delta f(q^t\omega t)}{\delta t} \, dq_t \right| \leq \frac{\sqrt{2}q^{-\frac{1}{2}}(1-q)M}{\sqrt{1-q^{2\lambda-1}}} \left( C_q \left( q^{\frac{1}{2}\omega_k} S_q'(\omega_k) \right) \right)^\frac{1}{2},
\]
and
\[
\left| \int_{-1}^{1} C_q \left( q^{\frac{1}{2}\omega_k} \right) \frac{\delta f(q^{-\frac{1}{2}}t)}{\delta t} \, dq_t \right| \leq \frac{\sqrt{2}(1-q)M}{\sqrt{1-q^{2\lambda-1}}} \left( C_q \left( q^{\frac{1}{2}\omega_k} S_q'(\omega_k) \right) \right)^\frac{1}{2}.
\]
Finally, using (4.12) and (4.13) in (4.4) and (4.5), respectively, by Theorems A, E, F and identity (2.4), as well as Lemma 3.1, one concludes that the conditions of Theorem H are fulfilled with, for instance, \( E, F \) and identity (2.4), as well as Lemma 3.1, one concludes that the conditions of Theorem H are fulfilled with, for instance, \( c = 3/2 \), thus the \( q \)-Fourier series (4.14), at least for \( 0 < q \leq (1/50)^{1/49} \), converges uniformly on the set \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \), hence, by Theorem I, under the same restriction on \( q \),
\[
S_q[f](x) = f(x), \quad \forall x \in V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \}.
\]
\[ \square \]

A simple analysis of the previous theorem shows immediately that the behavior of the function \( f \) at the origin is crucial to study the convergence of the \( q \)-Fourier series \( S_q[f] \). Consider, then, the following concept:

**Definition 4.2** A function \( f \) is said to be almost \( q \)-linear Hölder of order \( \lambda \) if two constants \( M, \lambda \) and a positive integer \( n_0 \) exist such that
\[
\left| f(\pm q^{-1}) - f(\pm q^n) \right| \leq M q^{\lambda n}
\]
holds for every \( n \geq n_0 \).

Obviously that every \( q \)-linear Hölder function of order \( \lambda \) is almost \( q \)-linear Hölder function of order \( \lambda \).

**Corollary 4.2.** If a function \( f \in L^\infty([-1,1]) \) is almost \( q \)-linear Hölder of order \( \lambda > \frac{1}{2} \) and satisfies \( f(0^+) = f(0^-) \) then, at least for \( 0 < q \leq (1/50)^{1/49} \), the corresponding \( q \)-Fourier series \( S_q[f] \) converges uniformly to \( f \) on the set of points \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

**Proof.** By hypothesis, \( f \) is almost \( q \)-linear Hölder of order \( \lambda > 1/2 \), i.e., it satisfies (4.14). Then the relations (4.10) and (4.11) now become
\[
\int_{-1}^{1} \left( \frac{\delta f(q^t\omega t)}{\delta t} \right)^2 \, dq_t \leq \frac{2(1-q)M^2 q^{n_0}}{1-q^{2\lambda-1}}.
\]
and
\[
\int_{-1}^{1} \left( \frac{\delta f(q^{\frac{1}{2}}t)}{\delta t} \right)^2 dtq_t \leq \frac{2(1 - q)M_2^2 q^{r_0}}{1 - q^{2\lambda - 1}},
\]
respectively, where \( M_1 \) and \( M_2 \) are constants. Therefore, using the above inequalities in formulas \( 1.8 \) and \( 1.9 \) we get two new inequalities that differ from \( 1.12 \) and \( 1.13 \) only by a constant in the corresponding right hand side. Hence, the conclusion on the uniform convergence follows.

\[ \square \]

Corollary 4.3. If \( f \in L^\infty_q[-1,1] \) satisfies \( f(0^+) = f(0^-) \) and there exists a neighborhood of the origin where the function \( f \) is continuous and piecewise smooth then, at least for \( 0 < q \leq (1/50)^{1/49} \), the corresponding \( q \)-Fourier series \( S_q[f] \) converges uniformly to \( f \) on the set of points \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

Proof. It's just a consequence of the fact that a function \( f \) that is continuous and piecewise smooth at any neighborhood of the origin satisfies a Lipschitz condition \( 1.6 \), page 204. Thus, it satisfies a Hölder condition of order 1 on that neighborhood and so, by Corollary 4.2, the uniform convergence follows.

\[ \square \]

5. Convergence on and outside the \( q \)-linear grid

The convergence of the basic Fourier series \( 3.1 \) always refer to the discrete set of the points of the \( q \)-linear grid \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

Two important questions arise at this moment:

- The above mentioned \( q \)-Fourier series also converges outside the points of the \( q \)-linear grid?
- In that case, to what function it converges?

Next theorem will give a positive answer to both questions.

Theorem 5.1. Let \( f \in L^\infty_q[-1,1] \) and suppose that \( c \in \mathbb{R}^+ \) exists such that, as \( k \to \infty \),
\[
\int_{-1}^{1} f(t)C_q(q^{\frac{1}{2}}t) dt = O\left(q^{(k+c)^2}\right), \quad \int_{-1}^{1} f(t)S_q(q^t) dt = O\left(q^{(k+c-\frac{1}{2})^2}\right).
\]

If \( f \) is analytic inside \( C_\delta = \{ z \in \mathbb{C} : |z| < \delta \} \), where \( \delta \) is a positive quantity such that \( 0 < \delta \leq q^{-\sigma} \) with \( 0 < \sigma < c \), then, at least for \( 0 < q \leq (50/151)^{\frac{1}{50}} \),
\[
f(z) = S_q[f](z) \quad \text{in} \quad C_\delta = \{ z \in \mathbb{C} : |z| < \delta \}.
\]

Proof. We first notice that
\[
C_q(q^{\frac{1}{2}}w_kz) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(q^2, q^2)^{n}} q^{\frac{1}{2}n} w_k^{2n} z^{2n}
\]
and
\[
S_q(q^w_kz) = \frac{q^w_kz}{1-q} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(q^2, q^2)^{n}} z^{2n} w_k^{2n} z^{2n}
\]
hence, for sufficiently large values of $k$, by Theorem A, whenever $|z| \leq q^{-\sigma}$,
\[
| C_q \left( q^{k} \omega_k z \right) | \leq \sum_{n=0}^{\infty} q^n \frac{q^{n(n-1)}}{(q^2, q^3 ; q^2)^n} q^{2n(1-k+\epsilon_k)} (q^{-\sigma})^{2n}
\]
\[
\leq \frac{q^n k^{-\frac{1}{2} + \sigma - \epsilon_k}^2}{(q ; q)_\infty} \sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2
\]
(5.3)
\[
\text{and}
\]
\[
| S_q \left( q^{\omega_k z} \right) | \leq \frac{q^{n(k-k+\frac{1}{2} + \epsilon_k - \sigma)}^2}{(q ; q)_\infty} \sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2
\]
\[
\leq \frac{q^{n(k-k+\frac{1}{2} + \epsilon_k - (k-\frac{1}{2} + \sigma - \epsilon_k))^2}}{q^{n-k+\frac{1}{2} - \sigma + \epsilon_k)^2}} \sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2
\]
(5.4)
An easy calculation shows that
\[
\sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} + \epsilon_k - \sigma)^2 = \sum_{n=0}^{k-1} q^n (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2 + \sum_{n=k}^{\infty} q^n (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2
\]
\[= \sum_{m=0}^{k-1} q^m (n-k+\frac{1}{2} + \sigma - \epsilon_k)^2 + \sum_{m=k}^{\infty} q^m (n-k+\frac{1}{2} - \sigma + \epsilon_k)^2.
\]
thus, if
\[
| \sigma | < \frac{1}{2},
\]
for sufficiently large values of $k$,
\[
\sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} + \epsilon_k - \sigma)^2 < \sum_{m=0}^{k-1} q^m + \sum_{m=k}^{\infty} q^m < 2 \sum_{m=0}^{\infty} q^m = \frac{2}{1-q}.
\]
In a similar way, for a given $p \in \mathbb{N}_0$, if
\[
| \sigma | < \frac{1}{2} + p
\]
(5.5)
then, for sufficiently large values of $k$,
\[
\sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} + \epsilon_k - \sigma)^2 < 2p + \frac{2}{1-q}
\]
(5.6)
With the same reasoning we get, again for sufficiently large values of $k$,
\[
\sum_{n=0}^{\infty} q^n (n-k+\frac{1}{2} + \epsilon_k - \sigma)^2 < 2p + \frac{2}{1-q}
\]
(5.7)
Hence, by (5.3), (5.4) and (5.6), (5.7), we may write, respectively, for $k$ large enough,
\[
\left| C_q \left( q^{k} \omega_k z \right) \right| \leq \frac{2p(1-q) + 2}{(q ; q)_\infty} q^{-(k-\frac{1}{2} + \sigma - \epsilon_k)^2}
\]
(5.8)
and
\[
\left| S_q \left( q^{\omega_k z} \right) \right| \leq \frac{2p(1-q) + 2}{(q ; q)_\infty} q^{2k+\epsilon_k -(k-k+\frac{1}{2} + \sigma - \epsilon_k)^2}
\]
(5.9)
This way, for $k$ large enough, using (3.2) and (3.3), Theorems E and F, relation (2.4) and inequality (5.9), at least for $0 < q \leq \sqrt[51]{1}$,
\[
\left| a_k C_q \left( q^z \omega_k z \right) \right| \leq 2p(1-q) + \frac{2}{(1-q)^2(q; q)_\infty^2} \left| \int_{-1}^{1} f(t)C_q \left( q^z \omega_k t \right) d_q t \right| q^{-k(k-\frac{1}{2}+\sigma)\frac{2}{2}-2k+2+2\epsilon_k} \frac{1}{|S_k|}.
\]
By hypothesis (5.1), we may suppose that $c_1 \in \mathbb{R}^+$ and $M_1 > 0$ exist such that, for $k$ large enough,
\[
\left| \int_{-1}^{1} f(t)C_q \left( q^z \omega_k t \right) d_q t \right| \leq M_1 q^{(k+c_1)^2}.
\]
In that case we have
\[
\left| a_k C_q \left( q^z \omega_k z \right) \right| \leq 2M_1 p(1-q) + \frac{2}{(1-q)^2(q; q)_\infty^2} q^{(k-\frac{1}{2}+\sigma)\frac{2}{2}-2k+2+2\epsilon_k} \frac{1}{|S_k|}
\]
hence, if $1 + 2(c_1 - \sigma) > 1$, i.e., if $\sigma < c_1$ then, taking into account Theorem A and (2.4), and the Theorems E and F, at least for $0 < q \leq \sqrt[51]{1}$,
\[
\left| a_k C_q \left( q^z \omega_k z \right) \right| \leq A_1 q^{\theta_1 k},
\]
where $A_1$ and $\theta_1$ are positive constants.

Analogously, for $k$ large enough, (3.3) and (3.4), Theorems E and F, relation (2.4) and inequality (5.9),
\[
\left| b_k S_q \left( q^z \omega_k z \right) \right| \leq 2M_2 p(1-q) + \frac{2}{(1-q)^2(q; q)_\infty^2} \left| \int_{-1}^{1} f(t)S_q \left( q^z \omega_k t \right) d_q t \right| q^{-k(k-\frac{1}{2}+\sigma)\frac{2}{2}-2k+2+2\epsilon_k} \frac{1}{|S_k|}
\]
so, again by hypothesis (5.1), if we admit that $c_2 \in \mathbb{R}^+$ and $M_2 > 0$ exist such that
\[
\left| \int_{-1}^{1} f(t)S_q \left( q^z \omega_k t \right) d_q t \right| \leq M_2 q^{(k+c_2-\frac{1}{2})^2},
\]
then,
\[
\left| b_k S_q \left( q^z \omega_k z \right) \right| \leq 2M_2 p(1-q) + \frac{2}{(1-q)^2(q; q)_\infty^2} q^{(k-\frac{1}{2}+\sigma)\frac{2}{2}-2k+2+2\epsilon_k} \frac{1}{|S_k|}.
\]
Similarly, if $2 + 2(c_2 - \sigma) > 1$, i.e., if $\sigma < c_2$ then, at least for $q$ such that $0 < q \leq \sqrt[51]{1}$,
\[
\left| b_k S_q \left( q^z \omega_k z \right) \right| \leq A_2 q^{\theta_2 k},
\]
being $A_2$ and $\theta_2$ positive constants.

We remark that in (5.1) we may choose $p$ sufficiently large in order that one has
\[
\frac{1}{2} - p < 0 < \sigma < \min \{c_1, c_2\} \leq \frac{1}{2} + p,
\]
thus, replacing $c_1$ and $c_2$ from (5.10) and (5.12) by $c = \min \{c_1, c_2\}$, respectively, we conclude, through (5.11) and (5.13), that the conditions (5.1) guaranty the uniform convergence of the $q$-Fourier series (5.1) in $C_{q^{-\sigma}} = \{ z \in \mathbb{C} : |z| < q^{-\sigma} \}$ if $\sigma$ satisfies (5.11). This way, under this condition on $\sigma$, we have, by Theorem H,
\[
f(x) = S_q[f](x) \quad \text{whenever} \quad x \in V_q,
\]
since \( V_q \subset C_{q^{-\sigma}} \), where \( V_q = \{ q^{n-1} : n \in \mathbb{N} \} \) is the corresponding set of Theorem I and \( C_{q^{-\sigma}} \) is the interior of the circle of the complex plane with center at the origin and radius \( q^{-\sigma} \).

On the other side, again by the uniform convergence of the \( q \)-Fourier series \( S_q[f](x) \) on \( C_{q^{-\sigma}} \), since the terms of the mentioned \( q \)-Fourier series are entire functions we then have that the \( q \)-series is analytic inside \( C_{q^{-\sigma}} \). From the continuity of both members of the above equality it results \( f(0) = S_q[f](0) \). Thus, if \( f \) is analytic inside \( C_{\delta} = \{ z \in \mathbb{C} : |z| < \delta \} \), where \( 0 < \delta \leq q^{-\sigma} \), then \( f(z) \) and \( S_q[f](z) \) are analytic inside \( C_{\delta} \) and coincide in a set with a limit point in the interior of such circle; by the principle of analytic continuation [11] Corollary 4.4.1, the above mentioned functions must coincide in the whole set \( C_{\delta} \), which proves [5.2].

\[ \square \]

### 6. Examples

In this section we will present four examples of \( q \)-Fourier series and study the corresponding questions about convergence.

**Example 1:** \( g(x) = |x| \)

The basic Fourier series of the absolute value function is given [10] by

\[
S_q[g](x) = \frac{1}{1 + q} - 2q^{-\frac{1}{2}}(1 - q) \sum_{k=1}^{\infty} \frac{1 - C_q \left( q^{\frac{1}{2}} \omega_k \right)}{\omega_k^2 C_q \left( q^{\frac{1}{2}} \omega_k \right) S'_q(\omega_k)} C_q \left( q^{\frac{1}{2}} \omega_k x \right).
\]

Conditions of Theorem H are fulfilled [10] with, for instance, \( c = 2 \). Thus, at least for \( 0 < q \leq (1/50)^{1/49} \), the \( q \)-Fourier series of the function \( f(x) = |x| \) converges uniformly on the set \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \) so, under the same restrictions on \( q \), by Theorem I,

\[
|x| = \frac{1}{1 + q} - 2q^{-\frac{1}{2}}(1 - q) \sum_{k=1}^{\infty} \frac{1 - C_q \left( q^{\frac{1}{2}} \omega_k \right)}{\omega_k^2 C_q \left( q^{\frac{1}{2}} \omega_k \right) S'_q(\omega_k)} C_q \left( q^{\frac{1}{2}} \omega_k x \right)
\]

for all \( x \in V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \).

Now, we may obtain the same conclusion in a easier way through Theorem [11] by simple arguing that the absolute value function

- is bounded on \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \),
- is continuous at the origin,
- and satisfies the \( q \)-linear Hölder condition of order 1 since

\[
\left| \pm q^{n-1} - \pm q^n \right| \leq (1-q)q^{n-1}.
\]

Thus, by Theorem [11] the same conclusion over the uniform convergence follows. Notice that Corollaries [12] or [13] also apply.

Given a function \( f \), it is important to point out that Theorem [11] or its Corollaries [12] and [13] enable one to decide over the uniform convergence of the \( q \)-Fourier series \( S_q[f] \) without the need to compute the corresponding coefficients: only requires a short study of the function itself.
Example 2: \( h(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \)

In this example, the conditions of Theorem H were not satisfied \( \text{(10), Remark 3} \). It was shown, using Theorem G, that the \( q \)-Fourier series

\[
S_q[h](x) = 2 \sum_{k=1}^{\infty} \frac{1 - C_q \left( q^{\frac{x}{2}} \omega_k \right)}{\omega_k C_q \left( q^{\frac{x}{2}} \omega_k \right) S_q'(\omega_k)} S_q(q\omega_k x)
\]

is (pointwise) convergent at each (fixed) point \( x \in V_q \). Theorem 4.1 doesn’t apply too (neither its corollaries) since \( h(0^+) \neq h(0^-) \).

Example 3: \( H^{(a)}(x) = \begin{cases} -1 & \text{se } x \leq a \\ 1 & \text{se } x > a \end{cases} \) \( (a > 0) \)

Once \( 0 < q < 1 \) is fixed, denote by \( n_a \) the least positive integer \( j \) such that \( q^j < a \), i.e., \( n_a = \lfloor \log_q a \rfloor + 1 \). Then

\[
a_0 = -2q^{n_a}
\]

and, for \( k = 1, 2, 3, \ldots \),

\[
a_k = \frac{2(1 - q)}{q^{-\frac{1}{2} + n_a} \omega_k^{\mu_k}} \left[ C_q \left( q^{\frac{1}{2} + n_a} \omega_k \right) - C_q \left( q^{-\frac{1}{2} + n_a} \omega_k \right) \right].
\]

By Theorem D,

\[
C_q \left( q^{\frac{1}{2} + n_a} \omega_k \right) - C_q \left( q^{-\frac{1}{2} + n_a} \omega_k \right) = q^{-\frac{1}{2} + n_a} \omega_k S_q(q^{n_a} \omega_k),
\]

thus

\[
a_k = -\frac{2(1 - q)S_q(q^{n_a} \omega_k)}{\omega_k \mu_k} = \frac{2}{\omega_k C_q \left( q^{\frac{1}{2}} \omega_k \right) S_q'(\omega_k)} S_q(q^{n_a} \omega_k).
\]

For \( k = 1, 2, 3, \ldots \) we have

\[
b_k = -\frac{2(1 - q)}{\omega_k^{\mu_k}} \left[ S_q \left( q^{1 + n_a} \omega_k \right) - S_q(q^{n_a} \omega_k) \right] - S_q(q^{n_a} \omega_k).
\]

By Theorem D,

\[
S_q \left( q^{1 + n_a} \omega_k \right) - S_q(q^{n_a} \omega_k) = -q^{n_a} \omega_k C_q \left( q^{\frac{1}{2} + n_a} \omega_k \right),
\]

so, by \( \text{(2.4)} \),

\[
b_k = \frac{2(1 - q)}{\omega_k^{\mu_k}} \left[ C_q \left( q^{\frac{1}{2} + n_a} \omega_k \right) - C_q \left( q^{\frac{1}{2}} \omega_k \right) \right] + \frac{2}{\omega_k C_q \left( q^{\frac{1}{2}} \omega_k \right) S_q'(\omega_k)} S_q(q^{n_a} \omega_k).
\]
hence, substituting (6.1), (6.2) and (6.3) into (6.4) it becomes

\[ S_q[H^{(a)}](x) = -q^{n_a} - \]

\[ 2 \sum_{k=1}^{\infty} S_q(q^{n_a} \omega_k) C_q(q^{\frac{a}{2}} \omega_k x) + \left[ C_q(q^{\frac{a}{2}} \omega_k) - C_q(q^{\frac{a}{2}+n_a} \omega_k) \right] S_q(q \omega_k x) \]

We notice that Example 2 follows from Example 4 by computing the limit \( n_a \to \infty \), i.e., when \( a \to 0 \). Again by Theorem D,

\[ S_q(q^{n_a} \omega_k) = S_q(q \omega_k) \sum_{j=0}^{n_a-1} (-1)^j q^{j(j+\frac{1}{2})} \frac{q^{n_a-j}; q}{q; q_{2j+1}} \omega_k^{2j} \]

and

\[ C_q(q^{\frac{a}{2}+n_a} \omega_k) = C_q(q^{\frac{a}{2} \omega_k}) \sum_{j=0}^{n_a-1} (-1)^j q^{j(j+\frac{1}{2})} \frac{q^{n_a-j}; q}{q; q_{2j+1}} \omega_k^{2j} \]

thus, since \( S_q(q \omega_k) = -\omega_k C_q(q^{1/2} \omega_k) \) for \( k = 1, 2, 3, \ldots \),

\[ \int_{-1}^{1} H^{(a)}(x) C_q(q^{\frac{a}{2}} \omega_k x) d_q t = 2(1-q) C_q(q^{\frac{a}{2} \omega_k}) \sum_{j=0}^{n_a-1} (-1)^j q^{j(j+\frac{1}{2})} \frac{q^{n_a-j}; q}{q; q_{2j+1}} \omega_k^{2j} \]

and

\[ \int_{-1}^{1} H^{(a)}(x) S_q(q \omega_k x) d_q t = 2q^{-\frac{a}{2}} (1-q) C_q(q^{\frac{a}{2} \omega_k}) \times \]

\[ \left[ \sum_{j=0}^{n_a-1} (-1)^j q^{j(j+\frac{1}{2})} \frac{q^{n_a-j}; q}{q; q_{2j+1}} \omega_k^{2j} - 1 \right] \]

For each fixed \( a > 0 \), at least for \( 0 < q \leq (1/50)^{1/49} \), the \( q \)-Fourier series (6.4) converges uniformly on the set \( V_q = \{ \pm q^{n_a-1} : n \in \mathbb{N} \} \); in fact, after some computations, one verifies that the conditions of Theorem H are satisfied with, for instance, \( c = 2 \), hence, whenever \( x \in V_q \) and under the above restriction on \( q \), we may write by Theorem I,

\[ H^{(a)}(x) \equiv -q^{q^n} - \]

\[ 2 \sum_{k=1}^{\infty} S_q(q^{n_a} \omega_k) C_q(q^{\frac{a}{2}} \omega_k x) + \left[ C_q(q^{\frac{a}{2}} \omega_k) - C_q(q^{\frac{a}{2}+n_a} \omega_k) \right] S_q(q \omega_k x) \]

Another approach is the following: one easily check that \( H^{(a)} \in L_q^\infty[-1, 1] \), \( H^{(a)}(0^+) = 0 = H^{(a)}(0^-) \) and \( H^{(a)} \) is almost \( q \)-linear Hölder of order bigger then \( \frac{1}{2} \) since

\[ |H^{(a)}(q^{n_a-1}) - H^{(a)}(q^n)| = 0, \quad n \geq n_a + 1 = \lfloor \log_q a \rfloor + 2. \]

By Corollary 4.2 the \( q \)-Fourier series \( S_q[H^{(a)}] \) converges uniformly on the set \( V_q \), thus (6.5) follows.
Example 4: \( f(x) = x^m \)

In [10] Proposition 6.1] it was presented the Fourier expansion of the function \( f(x) = x^m \), \( m = 0, 1, 2, \ldots, \) in terms of the functions \( C_q \) and \( S_q \):

\[
S_q[x^m](x) = \sum_{k=1}^{\infty} \frac{(-1)^m}{S_q^2(\omega_k)} \sum_{i=0}^{[\frac{m-1}{2}]} \frac{(-1)^i q^{(i+1)(i-m+1)}}{\omega_k^{2i+2}(q; q)_{m-2i}} C_q(q^2 \omega_k x) + q^\frac{1}{2} (-1)^m \frac{1}{S_q^2(\omega_k)} \sum_{i=0}^{[\frac{m-3}{2}]} \frac{(-1)^i q^{(i+1)(i-m-3)}}{\omega_k^{2i+2}(q; q)_{m-2i}} S_q(q \omega_k x),
\]

where \([x]\) denotes the greatest integer which does not exceed \( x \) and we will take as zero a sum where the superior index is less than the inferior one.

Furthermore, it was proved that the conditions of Theorem H are fulfilled with \( c = 2 \). Thus, at least for \( 0 < q \leq (1/50)^{1/49} \), the \( q \)-Fourier series of the function \( f(x) = x^m \) converges uniformly on the set \( V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \} \), so, by Theorem I,

\[
x^m = S_q[x^m](x) \quad \text{whenever} \quad x \in V_q = \{ \pm q^{n-1} : n \in \mathbb{N} \}.
\]

We notice that the conditions of Theorem 4.1 are trivial checked when \( f(x) = x^m \).

Now, since \( f \) satisfies the conditions of Theorem 5.1 with, for instance, \( c = 1 \) and \( f \) is an entire function then, by Theorem 5.14,

\[
S_q[x^m](x) = x^m, \quad \forall x \in C_\delta = \{ z \in \mathbb{C} : |z| < \delta \}
\]

where \( 0 < \delta < q^{-\sigma} \) and \( 0 < \sigma < 1 \).

Concluding remarks. We notice that Theorem 4.1 or Corollaries 4.2 and 4.3 are \( q \)-analogs of the corresponding classical theorems on uniform convergence for trigonometric Fourier series. See, for instance, Theorem 1 of [16] page 204 or Theorem 55 of [14] page 41.

Mathematica\textregistered suggests that Theorems 4.1 and 5.1 remain valid for \( 0 < q < 1 \). It’s an open question and to prove it a different technic is required.

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