Specialization of integral dependence for modules

TERENCE GAFFNEY\textsuperscript{1} AND STEVEN L. KLEIMAN\textsuperscript{2}

Abstract. We establish the principle of specialization of integral dependence for submodules of finite colength of free modules, as part of the general algebraic-geometric theory of the Buchsbaum–Rim multiplicity. Then we apply the principle to the study of equisingularity of ICIS germs, obtaining results for such equisingularity conditions as Whitney’s Condition A, Thom’s Condition A\(_f\), and the Relative Whitney Condition W\(_f\). Notably, we describe these conditions for analytic families in terms of various numerical invariants, which, for the most part, depend only on the members of a family, not on its total space.

Introduction

Describing the structure of a singular set remains a basic, but elusive, goal of complex-analytic geometry. If the set is a member of an analytic family, then it is often easier to tell when the set’s structure is similar to that of the general member. In this paper, we study analytic families of germs of isolated complete-intersection singularities, or ICIS germs, as a step toward the general study. We develop some algebraic tools and a geometric point of view that enable us to describe many equisingularity conditions for these families in terms of numerical invariants. The invariants, for most conditions, depend only on the individual members of a family, not on its total space.

The basic numerical invariants we use are certain Buchsbaum–Rim multiplicities. They arise from the column space of the Jacobian matrix of a given ICIS germ. This column space is known as the Jacobian module. It is, in a natural way, a submodule of finite colength in a free module over the local ring of the germ, and the associated invariants govern its integral closure. In this context, our main algebraic theorem is a generalization, from ideals to modules, of Teissier’s “principle of specialization of integral dependence,” [34, 3.2, p. 330] and [38, App. I]. We prove the theorem and some related results in Sections 1 and 2. In Section 3, we treat the related notion of strict dependence, which we use to handle those equisingularity conditions that require a little more than integral dependence to ensure that they hold.

In Sections 4, 5, and 6, we apply the results of Sections 1, 2 and 3 to the study of various equisingularity conditions on families of ICIS germs. More specifically, in Section 4, we study Whitney’s Condition A. We prove notably...
that A is satisfied at the origin if every associated multiplicity of the Jacobian module is constant across the family.

In Section 5, we fix a function \( f \) on the total space \( X \), and study Thom’s Condition \( A_f \). Notably, we generalize a celebrated theorem of Lê and Saito [24]: using Parameswaran’s construction in [31], we relate \( A_f \) to information about vanishing cycles as encoded in a sequence of Milnor numbers. In addition, we refine a theorem of Briançon, Maisonobe and Merle’s [1, Thm. 4.2.1, p. 541] in the present setting of a family of ICIS germs: their theorem requires the fulfillment of Condition A and the stratified local topological triviality of every linear projection to the singular locus, whereas ours requires only the constancy of a Buchsbaum–Rim multiplicity, or equivalently, of two Milnor numbers.

Finally, in Section 6, we study \( W_f \), the standard relative form of Whitney’s Condition B. Notably, we establish three necessary and sufficient conditions for \( W_f \) to hold. Two are memberwise conditions: the constancy of two sequences of Milnor numbers, and the constancy of a single Buchsbaum–Rim multiplicity.

The remaining condition is global: denote the locus of central points by \( Y \), and set \( Z := f^{-1}\{0\} \); then both pairs, \((X - Y, Y)\) and \((Z - Y, Y)\) must satisfy Whitney’s Condition B at the origin. The sufficiency of this condition was established by Briançon, Maisonobe and Merle in [1, Thm. 4.3.2, p. 543] in a more general setting, and it is recovered in the present setting via a new proof. The new proof illustrates the use of integral closure methods, and lays the foundation for further progress in the study of \( W_f \) both for nonisolated singularities and for families of isolated singularities.

Some of the present work, together with its extension in [9], is explained and developed by the second author in [19]. See also [21, (1.7)].

Let us now discuss the contents in more detail, stressing the philosophy of our approach. Let \((X, 0) \rightarrow (Y, 0)\) be a map of germs of complex analytic spaces. (As is conventional when dealing with germs, we often let it go without saying that any analytic set, even \( \mathbb{C}^n \) itself, should be replaced by a suitably small neighborhood of the central point, or “origin,” whenever appropriate.) Let \( X(y) \) denote the fiber over the point \( y \in Y \). Assume that the \( X(y) \) are equidimensional of the same dimension \( d \), where \( d \geq 1 \), that \( X \) is equidimensional, and that \( Y \) has dimension at least 1. Let \( \mathcal{E} := \mathcal{O}_X^p \) be a free module of rank \( p \) at least 1. Let \( \mathcal{M} \) be a coherent submodule such that, on each fiber \( X(y) \), the image of the restriction of \( \mathcal{M} \) in that of \( \mathcal{E} \) has finite colength.

Theorem (1.8) is our generalization of Teissier’s principle. Our theorem gives a criterion, valid for any rank \( p \), for a section \( h \) of \( \mathcal{E} \) to be integrally dependent on \( \mathcal{M} \), or equivalently, for the submodule \( \mathcal{H} \) generated by \( h \) and \( \mathcal{M} \) to lie in the integral closure of \( \mathcal{M} \) (in other words, for \( \mathcal{M} \) to be a reduction of \( \mathcal{H} \)). As we show in Sections 5 and 6, Theorem (1.8) is a powerful tool for establishing sufficient conditions for an equisingularity condition to hold.

Since Theorem (1.8) is central to our work, let us outline its proof. Let \( \mathcal{S} \mathcal{E} \) denote the symmetric algebra, and \( \mathcal{R} \mathcal{H} \) and \( \mathcal{R} \mathcal{M} \) the Rees algebras, which are
the subalgebras generated by \( \mathcal{H} \) and \( \mathcal{M} \). Form the analytic homogeneous spectra,

\[
P := \text{Projan}(\mathcal{S}E), \quad Q := \text{Projan}(\mathcal{R}H) \quad \text{and} \quad P' := \text{Projan}(\mathcal{R}M).
\]

Then the inclusion of \( \mathcal{M} \) into \( \mathcal{H} \) induces a well-defined finite map from \( Q \) to \( P' \) if and only if \( \mathcal{M} \) is a reduction of \( \mathcal{H} \); cf. [20, (2.6)].

In Sections 8 and 9 of [20], this setup is used to associate various multiplicities to the pair \((\mathcal{M}, \mathcal{H})\). It is shown there, in Theorem (9.5) and Corollary (9.7), that, if \( \mathcal{M} \) is not a reduction of \( \mathcal{H} \) everywhere, but is so away from the origin, then certain of these multiplicities are positive. In turn, by Corollary (10.2) of [20], this positivity implies, surprisingly enough, that the dimension of the central fiber of \( P'/X \) is of maximal dimension, namely, \( n - 1 \). (The recent paper [21] gives, in (1.4), a substantially shorter, simpler, and more direct proof that this dimension is maximal.) Thus, if we can control \( P' \), then we can force \( \mathcal{H} \) to be in the integral closure of \( \mathcal{M} \).

For each \( y \), let \( e(y) \) denote the \textit{Buchsbaum–Rim multiplicity} of the pair induced on \( X(y) \) by \((\mathcal{E}, \mathcal{M})\); so \( e(y)/(d+p-1)! \) is the coefficient of \( n^{d+p-1} \) in the polynomial whose value for \( n \gg 0 \) is the (vector space) dimension of \( (S_n\mathcal{E}/\mathcal{R}_n\mathcal{M})(y) \).

Just as Teissier did in the case of an ideal, we show that the constancy of \( e(y) \) implies the existence of a reduction of \( \mathcal{M} \) with the minimal number of generators, and therefore yields the desired upper bound on the dimension of the central fiber of \( P'/X \). The proof depends heavily on the upper semicontinuity of \( e(y) \), which we establish in Proposition (1.1). Thus Theorem (1.8) is proved.

We can also study the integral dependence of modules by reducing this study to that of the integral dependence of ideals; compare with [33] and [15, p. 160]. Indeed, in the setup above, let \( \rho(\mathcal{M}) \) be the ideal on \( P \) associated to the homogeneous ideal in the symmetric algebra \( \mathcal{S}E \) generated by \( \mathcal{M} \) viewed in degree 1. By Proposition (3.4), the module \( \mathcal{H} \) is integrally dependent on the module \( \mathcal{M} \) if and only if the ideal \( \rho(\mathcal{H}) \) is integrally dependent on the ideal \( \rho(\mathcal{M}) \).

An important difference between the case of modules and that of ideals shows up when we form the blowup \( B \) of \( P \) with respect to \( \rho(\mathcal{M}) \). Namely, if an ideal \( I \) has finite colength, then the corresponding exceptional divisor is naturally a projective scheme, and its degree is simply the multiplicity of \( I \). On the other hand, even though \( \mathcal{M} \) has finite colength in \( \mathcal{E} \), its associated ideal \( \rho(\mathcal{M}) \) may define a subset of \( P \) of positive dimension, and then the exceptional divisor of \( B \) is naturally a biprojective scheme. So it has a series of bidegrees, its \textit{Segre numbers}, defined by intersecting it with the various powers of the two hyperplane classes. The partial sums of these Segre numbers are the \textit{associated multiplicities} of \( \mathcal{M} \). They can also be computed directly from a set of generators of \( \mathcal{M} \).

In Section 2, we advance the theory of these Segre numbers and associated multiplicities, relating them to the dimension of the exceptional divisor of \( B \). By using these multiplicities instead of the multiplicity of an ideal, we can generalize to ICIS germs many results about families of hypersurface germs. We do so in Section 4 for some results about Whitney’s Condition A.
Section 3 concerns strict dependence. In the case of an ideal $I$, Lejeune and Teissier [27, pp. 46–48] showed that, if $I$ has finite colength, then a germ $h$ is strictly dependent on $I$ if and only if, on every component of the exceptional divisor in the normalized blowup of $I$, the order of vanishing of the pullback of $h$ is strictly greater than the order of vanishing of the pullback of $I$. Proposition (3.5) gives the generalization of this result to the case of a module $M$; instead of blowing up $I$, we blow up $\rho(M)$.

In the case where $I$ defines a family of ideals of finite colength and constant multiplicity, Teissier [38, App. I] used the constancy to force the components of the exceptional divisor to surject onto the parameter space. Then it is easy to show that, if the restriction of a germ $h$ to a general fiber is strictly dependent on the restriction of $I$, then the restriction of $h$ to every fiber is strictly dependent. In the module case, using the associated multiplicities, we make a similar argument and draw a similar conclusion in the course of proving Proposition (4.2).

In Sections 4, 5, and 6, we assume that the fibers $X(y)$ represent ICIS germs, and abusing notation, we also let $Y$ denote the locus of central points. We embed $(X,0)$ in $(\mathbb{C}^n, 0)$ so that $(Y,0)$ is the germ of a linear subspace. In Sections 5, and 6, we also consider a function germ $f: X \to \mathbb{C}$ and its zero set $Z$, and we assume $Z \supset Y$.

Section 4 concerns Whitney’s Condition A. Let $S$ denote the singular locus of $X$, and assume that $S$ is smooth. By definition, Condition A is satisfied by the pair $(X - S, S)$ at $0 \in S$ if the tangent space $T_0S$ lies in every hyperplane obtained as a limit of hyperplanes, each tangent to $X - S$ as the point of contact approaches 0.

Theorem (4.2) asserts that Condition A is satisfied if every associated multiplicity of the Jacobian module of the fiber $X(y)$ is constant in $y$. The theorem is illustrated in Examples (4.3) and (4.4). In the first, we work out, from our viewpoint, Trotman’s example, showing that no fiberwise criterion for Condition A can be necessary as well as sufficient. In the second, we consider a case where $S$ is larger than $Y$.

Section 5 concerns Thom’s Condition $A_f$. Let $\Sigma(f)$ denote the union of the singular points of the fibers of $f$. By definition, $(X - \Sigma(f), Y)$ satisfies $A_f$ at $y \in Y$ if the tangent space to $Y$ at $y$ lies in every limit tangent hyperplane at $y$ to the fibers of $f$. In Lemma (5.1), we connect $A_f$ to the theory of integral closure via the augmented Jacobian module.

Thom introduced $A_f$ as the primary condition guaranteeing the local topological triviality of the family of functions defined by $f$. Condition $A_f$ is also important because of its relationship (which is well understood in only a few cases) to the vanishing cycles. For example, in a recent paper [11], Green and Massey show that certain information about the vanishing cycles implies $A_f$ for families with generalized isolated singularities. In the case where $X = \mathbb{C}^n$ and $\Sigma(f) = Y$, Lê and Saito [24] showed that $A_f$ is implied by the constancy of the Milnor number.

Theorem (5.2) generalizes the Lê–Saito theorem to the case where $X$ is a
complete intersection. This generalization is based on the following construction of Parameswaran’s in [31], which reduces the study of a family of ICIS germs to the study of an isolated singularity defined by the vanishing of a single function inside an ambient space with an isolated singularity. Given an ICIS germ \((X, 0)\) with embedding codimension \(k\), for each \(i\) with \(0 \leq i \leq k\) let \(\mu_i\) be the smallest of all the Milnor numbers of ICIS germs that serve as total spaces of \(i\)-parameter (flat) deformations of \((X, 0)\). Denote this sequence of numbers by \(\mu^*\).

A chain of ICIS germs, each of codimension 1 inside the next, whose sequence of Milnor numbers is \(\mu^*\) is said to be \(\mu^*\)-minimal. Given the germ of a family \(X/Y\) with \(Y\) smooth, Parameswaran constructed a chain of deformations of the family such that the parameter spaces are smooth over \(Y\) and the central chain is \(\mu^*\)-minimal; moreover, if the \(\mu^*\)-sequence is constant in the given family, then the chain of deformations is \(\mu^*\)-minimal.

Given a chain of deformations of \((X, 0)\) that is \(\mu^*\)-minimal, we can look, on the deformation \(X_i\), at the function \(f_{i-1}\) that defines the deformation \(X_{i-1}\), and ask that \(A_{f_{i-1}}\) be satisfied at 0 by the pair \((X_i - Y, Y)\) where \(Y\) is the common singular locus of all the \(X_i\). Theorem (5.2) gives a necessary and sufficient condition for \(A_{f_{i-1}}\) to hold for every \(i\), namely, the constancy of \(\mu^*\).

Indeed, this constancy turns out to be equivalent to the constancy of the multiplicity of the relative augmented Jacobian module of \(X_i\) and \(f_{i-1}\); the principle of specialization of integral dependence then shows that the generators of the relative Jacobian module that come from the partial derivatives along \(Y\) are dependent on the augmented module. The original theorem of Lê and Saito, which is reproved, shows that this dependence is strict at the top of the chain, where the ambient space is just the affine space. It remains to push the strict dependence down the chain. A careful examination of the proof shows a similarity in the role played by the elements in the chain and the associated multiplicities.

We prove the sufficiency as follows. Because the number of generators of the relative augmented Jacobian submodule is always the minimum number needed to generate a module of finite nonzero colength, it follows from Proposition (1.5)(3) that, if \(A_{f_{i-1}}\) holds, then the multiplicity of the augmented Jacobian module of the fibers of \(X_i\) and \(f_{i-1}\) is independent of the parameter value; hence, so is the \(\mu^*\)-sequence.

We also refine, in our case of a family of ICIS germs, a theorem proved by Briançon, Maisonobe and Merle [1, Thm. 4.2.1, p. 541]. Recalled before our Theorem (5.3), their theorem essentially asserts this: \(A_f\) holds along \(Y\) if, for every linear retraction \(r\) to \(Y\), the restriction \(r|(X, Z, Y)\) is stratified locally topologically trivial, and if both pairs \((X - Y, Y)\) and \((Z - Y, Y)\) satisfy Whitney’s Condition A along \(Y\). Our machinery allows us to prove, in Theorem (5.3), that \(A_f\) holds at 0 assuming only, for every \(r\), that the Buchsbaum–Rim multiplicity of the augmented Jacobian module of the fibers of \(X\) and \(f\) is defined and is independent of the parameter value, or equivalently that, the germs of the fibers of the restrictions \(r|X\) and \(r|Z\) have isolated singularities, and their Milnor
numbers are independent of the parameter value.

Our work leads us to conjecture that $A_f$ holds whenever the Buchsbaum–Rim multiplicity is defined and is independent of the parameter value, or the Milnor numbers are independent of the parameter value. After the present work was completed, Massey and the first author ([9], (5.8)) proved this conjecture for families of ICIS germs via a careful study of the conormal variety; see also [19] and [21], (1.7). At the moment, this multiplicity is defined only when $X$ is the total space of a family of ICIS germs. However, we conjecture that, once the theory of multiplicity has been extended to cover modules of infinite colength, then the independence of the multiplicity of the augmented Jacobian module will always be equivalent to $A_f$.

Assume also that $(Z - Y, Y)$ satisfies Whitney’s Condition A at the origin. Then $A_f$ holds if the Milnor numbers of the fibers of the two restrictions $r|_X$ and $r|_Z$ are independent of the parameter value for only a single $r$; see Theorem (5.5).

We prove it via a close analysis of the relative conormal space of $f$ using the principle of specialization of integral dependence to gain control. The hypothesis on $Z$ gives additional information on the relative conormal space, since it contains the conormal space of $Z$.

Section 6 treats the condition $W_f$. By definition, it is satisfied by the pair $(X - \Sigma(f), Y)$ at $y \in Y$ if each tangent plane to the fiber of $f$ at an $x$ in $X - \Sigma(f)$ approaches the tangent plane of $Y$ at $y$ as fast as $x$ approaches $Y$. In Proposition (6.1), we connect $W_f$ to the theory of integral closure via the augmented Jacobian module again; we also recover the integral closure condition of Lê and Teissier [26, Prop. 1.3.8] between ideals on the relative conormal space. In Lemma (6.2) we illustrate the usefulness of our integral closure condition by using it to give a new proof of a basic transversality result of Henry and Merle.

It is natural to ask if there is a numerical invariant of the fibers of $X$ and $f$ over $Y$ whose constancy ensures that $W_f$ is satisfied at 0. An affirmative answer is given in Theorem (6.4): a suitable invariant is the multiplicity $em(y)$ of the product of the maximal ideal of $X(y)$ and its augmented Jacobian ideal. In fact, assuming that each fiber $Z(y)$ has an isolated singularity at 0, we prove that the constancy of $em(y)$ is both necessary and sufficient for $W_f$ to be satisfied. The theorem also proves another necessary and sufficient condition: that $(X - Y, Y)$ and $(Z - Y, Y)$ satisfy both Whitney conditions at 0. The key ingredient in our proof is again Theorem (1.8).

We interpret $em(y)$ topologically in Lemma (6.3); namely, $em(y)$ is equal to a linear combination with certain binomial coefficients of the sum of the Milnor numbers of the plane sections of $X(y)$ and $Z(y)$. Hence the constancy of $em(y)$, and so that of $W_f$, is equivalent to the constancy of the Milnor numbers of these plane sections. This equivalence is also part of Theorem (6.4).

1. Specialization of integral dependence

Let $F: (X, x_0) \to (Y, y_0)$ be a map of germs of complex analytic spaces, which need not be reduced. Assume that the fibers $X(y)$ are equidimensional of the
same dimension $d$ at least 1 and that $Y$ has dimension at least 1. Let $E := \mathcal{O}_X^p$ be a free module of rank $p$ at least 1, and set $r := d + p - 1$. Let $M$ be a coherent submodule of $E$. Set $S := \text{Supp}(E/M)$, and assume that $S$ is finite over $Y$. Finally, for each $y \in Y$, denote the Buchsbaum–Rim multiplicity of the pair that $(E, M)$ induces on $X(y)$ by $e(y)$.

Let $N$ be a coherent submodule of $M$. Let $\mathcal{S}E$ denote the symmetric algebra, and $\mathcal{R}M$ and $\mathcal{R}N$ the subalgebras generated by $M$ and $N$; say

$$\mathcal{S}E = \bigoplus_n S_nE, \; \mathcal{R}M = \bigoplus_n R_nM \text{ and } \mathcal{R}N = \bigoplus_n R_nN$$

are the decompositions into graded pieces. Form the analytic homogeneous spectra,

$$P := \text{Projan}(\mathcal{S}E), \; P' := \text{Projan}(\mathcal{R}M) \text{ and } P'' := \text{Projan}(\mathcal{R}N).$$

Recall that, if $\mathcal{R}M$ is a finitely generated $\mathcal{R}N$-module, then $N$ is called a reduction of $M$, and the sections of $M$ are said to be integrally dependent on $N$.

A different, but equivalent, definition of integral dependence is discussed at the beginning of Section 3 (and a third definition is mentioned there in passing).

If $N$ is a reduction of $M$, then the following three conditions obtain:

(i) *on each fiber $X(y)$ the Buchsbaum–Rim multiplicity arising from $(E,N)$ is defined and equal to the multiplicity $e(y)$ arising from $(E,M)$;*

(ii) $N$ is equal to $E$ at every point $x$ of $X - S$, that is, $\text{Supp}(E/N) \subset S$;

(iii) the inclusion $\mathcal{R}N \hookrightarrow \mathcal{R}M$ induces a finite surjective map $P' \twoheadrightarrow P''$, which is an isomorphism off $S$.

Condition (i) obtains by [20, (6.7a)(iii)(a), p. 204]. Condition (ii) obtains by [20, (2.4), p. 182]. The argument is simple. By Nakayama’s lemma, it suffices to note that the map of fibers $N(x) \to E(x)$ is surjective. However, its image is a vector subspace of $E(x)$; so, if a basis of the former is extended to a basis of the latter, then the image of $\mathcal{R}N(x)$ in the polynomial ring $\mathcal{S}E(x)$ is the subring generated by a subset of variables; yet the larger ring is a finitely generated module over the smaller one. Condition (iii) is clearly not only necessary, but also sufficient, for $N$ to be a reduction of $M$.

The results below are well known in the case $p = 1$; see Lipman’s masterful treatment [28], for example. For arbitrary $p$, Proposition (1.5) relates the existence of a reduction generated by $r$ elements to the constancy of $e(y)$. Its proof involves Proposition (1.1) and Lemmas (1.2) and (1.4). Proposition (1.1) asserts the upper semicontinuity of $e(y)$. Lemma (1.2) asserts that if $N$ is a reduction of $M$ fiberwise, then it is so globally over a dense open subset of $Y$. Example (1.3) shows the necessity of passing to an open subset. Lemma (1.4) gives a geometric criterion for $N$ to be a reduction of $M$. Remark (1.6) suggests that part of Proposition (1.5) should hold in greater generality. Lemma (1.7) gives a necessary and sufficient geometric condition for the existence of a reduction generated by $r$ elements. Finally, Theorem (1.8) rests on all the preceding results; it gives a generalization of Teissier’s principle of the specialization of integral dependence [38, App. 1].
Proposition (1.1) The function \( y \mapsto e(y) \) is Zariski upper semicontinuous.

Proof. For each \( y \), the number \( e(y)/r! \) is the coefficient of \( n^r \) in the polynomial whose value is eventually the (vector space) dimension,

\[
\lambda(n, y) := \dim \left( F_*(S_n E/R_{n, M}) \right)(y).
\]

The polynomial has degree \( r \), or else vanishes. Its value is \( \lambda(n, y) \) for all \( n \) at least \( n_0 \), where \( n_0 \) depends on \( y \). (See [3, bot. p. 213] or [20, (5.10)(i)(ii), pp. 199–200].)

For each \( n \), the \( \mathcal{O}_Y \)-module \( F_*(S_n E/R_{n, M}) \) is coherent; hence, \( y \mapsto \lambda(n, y) \) is upper semicontinuous. Therefore, \( y \mapsto e(y) \) is ‘nondecreasing’; that is, if \( A \) is a Zariski closed irreducible subset of \( Y \) and if \( \eta \) is its generic point, then \( e(\eta) \leq e(y) \) for all \( y \) in \( A \). However, it is less obvious that \( A \) contains a nonempty Zariski relatively open subset \( U \), independent of \( n \), on which \( y \mapsto \lambda(n, y) \) is constant. To prove it, we may replace \( Y \) by \( A \), given its reduced structure, and replace \( X \), \( M \), and so forth by their restrictions.

Form the bigraded \( \mathcal{O}_X \)-algebra \( \mathcal{R}_M \otimes \mathcal{S} E \) and its bigraded module

\[
\mathcal{F} := \bigoplus_{p \geq 0, q \geq 1} \mathcal{F}_{p, q} \text{ where } \mathcal{F}_{p, q} := \mathcal{R}_p \mathcal{M} \mathcal{S}_q \mathcal{E}/\mathcal{R}_{p+1} \mathcal{M} \mathcal{S}_{q-1} \mathcal{E}.
\]

Clearly, \( \mathcal{F} \) is generated by \( \mathcal{F}_{0, 1} \), which is equal to \( \mathcal{S}_1 \mathcal{E}/\mathcal{R}_1 \mathcal{M} \). Therefore, \( \mathcal{F} \) is an \( \mathcal{O}_S \)-module. So \( F_* \mathcal{F} \) is a finitely generated module over \( F_*((\mathcal{R}_M \otimes \mathcal{S} E)|S) \), which is a finitely generated bigraded \( \mathcal{O}_Y \)-algebra. Therefore, by the lemma of generic flatness, there is a nonempty open subset \( U \) of \( Y \) on which \( F_* \mathcal{F} \) is flat. Hence, on \( U \), each \( F_* \mathcal{F}_{p, q} \) is flat.

It follows that, on \( U \), the formation of each \( F_* \mathcal{F}_{p, q} \) commutes with restriction to the fibers. Indeed, since \( F|S \) is finite, \( (F_* \mathcal{F}_{p, q})(y) \) is equal to \( F_*(\mathcal{F}_{p, q}(y)) \) for any \( y \) in \( Y \). Moreover, if \( y \) in \( U \), then the formation of \( \mathcal{F}_{p, q} \) commutes with restriction to \( X(y) \); this claim will hold, clearly, if the natural map,

\[
(\mathcal{R}_p \mathcal{M} \mathcal{S}_q \mathcal{E})(y) \longrightarrow \mathcal{S}_{p+q} \mathcal{E}(y)
\]

is injective. It is trivially injective if \( p = 0 \). Proceeding by induction on \( p \), consider the short exact sequence,

\[
0 \longrightarrow \mathcal{R}_{p+1} \mathcal{M} \mathcal{S}_{q-1} \mathcal{E} \longrightarrow \mathcal{R}_p \mathcal{M} \mathcal{S}_q \mathcal{E} \longrightarrow \mathcal{F}_{p, q} \longrightarrow 0.
\]

Since \( F_* \mathcal{F}_{p, q} \) is flat over \( U \) and since \( F|S \) is finite, also \( \mathcal{F}_{p, q} \) is flat over \( U \). Therefore, the first of these two maps,

\[
(\mathcal{R}_{p+1} \mathcal{M} \mathcal{S}_{q-1} \mathcal{E})(y) \longrightarrow (\mathcal{R}_p \mathcal{M} \mathcal{S}_q \mathcal{E})(y) \longrightarrow \mathcal{S}_{p+q} \mathcal{E}(y),
\]

is injective. The second is injective by induction. So the composition is injective, as required.

Work on \( U \). Since the formation of \( F_* \mathcal{F}_{p, q} \) commutes with restriction, clearly

\[
\sum_{p+q=n} \dim(F_* \mathcal{F}_{p, q})(y) = \lambda(n, y).
\]

Each \( F_* \mathcal{F}_{p, q} \) is flat, so locally free; hence, each function \( y \mapsto \dim F_* \mathcal{F}_{p, q}(y) \) is constant. Therefore, \( y \mapsto \lambda(n, y) \) is constant. Thus the proposition is proved.
Lemma (1.2) Assume that there is a dense Zariski open subset $V$ of $Y$ such that, for each $y$ in $V$, the image in $\mathcal{E}(y)$ of $\mathcal{N}$ is a reduction of the image of $\mathcal{M}$. Then there is a smaller dense Zariski open subset $U$ of $Y$ over which $\mathcal{N}$ is a reduction of $\mathcal{M}$.

Proof. Clearly, it suffices to find a dense Zariski open subset $U$ of $V$ and an integer $k$ such that the inclusion map,

$$\mathcal{N}\mathcal{R}_k\mathcal{M} \to \mathcal{R}_{k+1}\mathcal{M}, \quad (1.2.1)$$

is surjective over $U$. By Nakayama’s lemma, we may assume that $Y$ is reduced. Then there is a dense Zariski open subset $U$ of $Y$ such that the restriction, $(\mathcal{R}_k\mathcal{M})(y) \to S_k\mathcal{E}(y)$, is injective for all $k$ and for all $y$ in $U$; the existence of $U$ was established in the proof of Proposition (1.1). Replace $U$ by $U \cap V$.

By hypothesis, for each $y$ in $U$, there exists a $k$ such that the image of the composition,

$$\mathcal{N}(y)(\mathcal{R}_k\mathcal{M})(y) \to (\mathcal{R}_{k+1}\mathcal{M})(y) \to S_{k+1}\mathcal{E}(y),$$

is equal to the image of the second map. Since the second map is injective, the first map is surjective. Hence, Nakayama’s lemma implies that, at each point of the fiber $X(y)$, the map (1.2.1) is surjective. Therefore, $X(y)$ is contained in the maximal open set on which (1.2.1) is surjective, namely, the complement of the support $S_k$ of the cokernel of (1.2.1).

On the other hand, $S_k \subset S$; in other words, (1.2.1) is surjective at every $x$ off $S := \text{Supp}(\mathcal{E}/\mathcal{M})$, as we’ll now see. Set $y := F(x)$. By hypothesis, the image in $\mathcal{E}(y)$ of $\mathcal{N}$ is a reduction of the image of $\mathcal{M}$. Since $x \notin S$, the image of $\mathcal{M}$ is equal to $\mathcal{M}(y)$ at $x$. Hence, the image of $\mathcal{N}$ is equal to $\mathcal{M}(y)$ at $x$ because of Condition (ii) recalled at the beginning of this section. In other words, the map $\mathcal{N} \to \mathcal{M}(y)$ is surjective at $x$. Therefore, by Nakayama’s lemma, the inclusion $\mathcal{N} \to \mathcal{M}$ is surjective at $x$. So (1.2.1) is surjective at $x$, as claimed.

By hypothesis, $S$ is finite over $Y$. Hence, since $S_k \subset S$, the image $A_k$ of $S_k$ is a closed analytic subset of $Y$. Pick a point $y$ in each component of $U$, and pick a $k$ large enough so that $A_k$ contains none of these $y$. Then $U - A_k$ a dense Zariski open subset $U$ of $V$ on which (1.2.1) is surjective, as required.

Example (1.3) The open subset provided by Lemma (1.2) may have to be strictly smaller than the given open subset (in other words, fiberwise integral dependence does not imply dependence at the level of the total space). For example, let $X$ be the $(s,t)$-plane, $Y$ the $s$-line, and $F: X \to Y$ the projection. Let $\mathcal{E} := \mathcal{O}_X$, let $\mathcal{N} := (st^2 + t^3, st^4)$, and let $\mathcal{M} := (\mathcal{N}, t^3)$. Then, on each fiber of $X/Y$, the ideals induced by $\mathcal{M}$ and $\mathcal{N}$ are equal. However, $\mathcal{N}$ is not a reduction of $\mathcal{M}$ (in other words, $t^3$ is not integrally dependent on $\mathcal{N}$). Indeed, otherwise, under the map from the $u$-line into $X$ given by $u \mapsto (u, -u)$, the ideals $\mathcal{M}$ and $\mathcal{N}$ would induce two ideals, where the second is a reduction of the first; however, $\mathcal{M}$ and $\mathcal{N}$ induce $\langle u^3 \rangle$ and $\langle u^3 \rangle$, and it is easy to see that the latter is not a reduction of the former.
Lemma (1.4) Assume that $X$ is equidimensional and that $\dim P''(x_0) < r$. Set $T := \text{Supp} (\mathcal{E}/\mathcal{N})$ and assume that $T \rightarrow Y$ is finite. Then $\mathcal{N}$ is a reduction of $\mathcal{M}$ if it is so over a dense Zariski open subset of $Y$.

Proof. Apply Corollary (10.7) of [20, p. 225] as follows. Let $A$ be the local ring of $X$ at $x_0$. Then $A$ is Noetherian, universally catenary, and equidimensional. Set $X_0 := \text{Specan} A$. Let $G$, $G'$, and $G''$ be the quasi-coherent sheaves of graded algebras on $X_0$ associated to the stalks of $\mathcal{SE}$, $\mathcal{RM}$, and $\mathcal{RN}$ at $x_0$. Let $P_0$, $P'_0$, and $P''_0$ be their “Projan’s.” Set $M := G$. Then $\text{Supp}(\mathcal{M})$ is equal to $P_0$, so it is equidimensional of dimension $r_0$ with $r_0 := \dim X_0 + p - 1$. Clearly, $\dim X_0 = d + \dim Y$. Let $Y'$ be the closed subset of $X_0$ defined by the stalk of the ideal of $T$. Since $\mathcal{N}$ is equal to $\mathcal{E}$ off $T$, the stalks of $\mathcal{SE}$, $\mathcal{RM}$, and $\mathcal{RN}$ at $x_0$ become equal after localization with respect to any analytic function on $X$ that vanishes along $T$; hence, $G$, $G'$, and $G''$ are equal off $Y'$. Moreover, since $T \rightarrow Y$ is finite, the dimension of $Y'$ is bounded by that of $Y$.

Viewed as a $G''$-module, $M$ gives rise to a quasi-coherent sheaf on $P''_0$. The support $R$ of this sheaf is equal to $P''_0$; indeed, by (6.4)(i) of [20, p. 202], $R$ is equal to the transform of $P_0$, and by (2.6) of [20, p. 183], the latter is equal to $P''_0$. Let $p'': P''_0 \rightarrow X_0$ be the structure map. Then $p''^{-1}x_0$ is a scheme, whose associated analytic space is $P''(x_0)$. Hence

$$\dim (p''^{-1}Y' \cap R) \leq \dim p''^{-1}x_0 + \dim Y' \leq r - 1 + \dim Y = r_0 - 1.$$ 

(The corresponding bound in (10.7) of [20, p. 225] is, unfortunately, incorrectly stated because of a typographer’s error; however, the text suggests that the appropriate inequality is, in fact, not strict.) Moreover, if there is a component of dimension $r_0$ of $p''^{-1}Y' \cap R$ (the $R$ is unnecessary in the present case), then this component maps onto a component $Y'_1$ of $Y'$ such that $\dim Y'_1 = \dim Y$.

By hypothesis, $\mathcal{RM}$ is a finitely generated module over $\mathcal{RN}$ locally over a dense Zariski open subset $U$ of $Y$. Let $Z$ be the preimage in $T$ of $Y - U$, and $Z_0$ the closed subset of $X_0$ defined by the stalk of the ideal of $Z$. Then $\dim Z_0 < \dim Y$, and so $Y'_1$, if it exists, contains a point $\eta$ outside $Z_0$. The stalks $G'_\eta$ and $G''_\eta$ are localizations of the stalks of $\mathcal{RM}$ and $\mathcal{RN}$ at $x_0$ with respect to a certain set of analytic functions on $X$, including some that vanish on $Z$. Hence $G'_\eta$ is a finitely generated module over $G''_\eta$; in the language of [20], $G''_\eta$ is a reduction of $G'_\eta$ for $\mathcal{N}_\eta$. Finally, the preimage of $Y'$ in $P_0$ has no component of dimension $r_0$, because $\dim Y' < \dim X_0$ and $\mathcal{E}$ is free of rank $p$. Therefore, by (10.7) of [20, p. 225], $G'$ is a finitely generated module over $G''_\eta$; in other words, at $x_0$ the stalk of $\mathcal{RM}$ is a finitely generated module over that of $\mathcal{RN}$. Hence (after $X$ and $Y$ are replaced by neighborhoods of $x_0$ and $y_0$ if necessary) $\mathcal{N}$ is a reduction of $\mathcal{M}$, and the proof is complete.

Proposition (1.5) Assume that $X$ is equidimensional.

(1) If $y \mapsto e(y)$ vanishes, then $\mathcal{M} = \mathcal{E}$ and $S = \emptyset$.

(2) If $y \mapsto e(y)$ is constant on $Y$ and nonvanishing, then $S \rightarrow Y$ is surjective, and $\mathcal{M}$ has a reduction generated by $r$ elements.
(3) Assume that $Y$ is smooth, that $X/Y$ is flat with Cohen–Macaulay fibers, and that $S \rightarrow Y$ is surjective. If there exists a reduction of $\mathcal{M}$ generated by $r$ elements, then, conversely, $y \mapsto e(y)$ is constant on $Y$.

**Proof.** Consider (1). Fix $y \in Y$. Since $e(y)$ vanishes, a theorem of Buchsbaum and Rim implies that the image of $\mathcal{M}$ in $\mathcal{E}(y_0)$ is all of $\mathcal{E}(y)$. Since $y$ is arbitrary, $\mathcal{M} = \mathcal{E}$ by Nakayama’s lemma. So $S = \emptyset$ as $S := \text{Supp}(\mathcal{E}/\mathcal{M})$.

Consider (2). Fix $y \in Y$. Since $e(y)$ doesn’t vanish, the image of $\mathcal{M}$ is, obviously, not all of $\mathcal{E}(y)$. Hence $S \rightarrow Y$ is surjective.

After $X$ is replaced by a neighborhood of $x_0$ if necessary, there exist $r$ elements of $\mathcal{M}$ whose images in $\mathcal{E}(y_0)$ generate a reduction of the image of $\mathcal{M}$; see [20, (6.6), p. 203] for example. Let $\mathcal{N}$ be the submodule of $\mathcal{M}$ generated by the elements. Set $T := \text{Supp}(\mathcal{E}/\mathcal{N})$. Since $T(y_0)$ is finite, and since $(Y, y_0)$ is the germ of an analytic space, $T$ is finite over $Y$ after $X$ and $Y$ are replaced by neighborhoods of $x_0$ and $y_0$ if necessary. Hence, for every $y \in Y$, the Buchsbaum–Rim multiplicity $f(y)$ is defined for the pair that $(\mathcal{E}, \mathcal{M})$ induces on $X(y)$. By Proposition (1.1), there is a dense Zariski open subset $U$ of $Y$ on which $y \mapsto f(y)$ is constant and $f(y) \leq f(y_0)$. Then, for all $y$ in $U$,

$$e(y) \leq f(y) \leq f(y_0) = e(y_0) = e(y);$$

the first relation holds because $\mathcal{N} \subseteq \mathcal{M}$, the second because $y \in U$, the third by construction of $\mathcal{N}$, and the last by the hypothesis. Thus $e(y) = f(y)$.

Fix $y \in Y$. For each $x \in X(y)$, let $e(x)$ denote the Buchsbaum–Rim multiplicity at $x$ of the pair that $(\mathcal{E}, \mathcal{M})$ induces on $X(y)$, and let $f(x)$ denote that by $(\mathcal{E}, \mathcal{N})$. Of course, $e(x)$ vanishes if $x \notin S$, and $f(x)$ vanishes if $x \notin T$. Clearly,

$$e(y) = \sum_x e(x) \leq \sum_x f(x) = f(y).$$

Suppose $y \in U$. Then $e(y) = f(y)$. Hence $e(x) = f(x)$ for all $x \in X(y)$. Therefore, for all $x \in X(y)$, the stalk at $x$ of the image of $\mathcal{N}$ in $\mathcal{E}(y)$ is a reduction of that of $\mathcal{M}$ by the generalized theorem of Rees, Corollary (6.8)(a) of [20, p. 207–8] with $M := S\mathcal{E}$. Hence, for each $y$ in $U$, the image in $\mathcal{E}(y)$ of $\mathcal{N}$ is a reduction of that of $\mathcal{M}$. In particular, $T = S$.

By Lemma (1.2), there is a dense Zariski open subset of $Y$ over which $\mathcal{N}$ is a reduction of $\mathcal{M}$. On the other hand, $\dim P''(x_0) < r$ because $\mathcal{N}$ is generated by $r$ elements. Finally, since $S \rightarrow Y$ is surjective, so is $T \rightarrow Y$. Therefore, Lemma (1.4) implies that $\mathcal{N}$ is a reduction of $\mathcal{M}$.

Consider (3). Replacing $\mathcal{M}$ by its reduction, we may assume that $\mathcal{M}$ itself is generated by $r$ elements. Let $\mathcal{J}$ denote the zeroth Fitting ideal of $\mathcal{E}/\mathcal{M}$. Then $\mathcal{O}_X/\mathcal{J}$ is supported by $S$. Since the codimension of $S$ is right, $\mathcal{O}_X/\mathcal{J}$ is Cohen–Macaulay. Since $Y$ is smooth and $S \rightarrow Y$ is finite and surjective, $\mathcal{O}_X/\mathcal{J}$ is therefore flat over $Y$. Hence the function

$$y \mapsto \dim(F_*(\mathcal{O}_X/\mathcal{J})(y))$$
is constant on $Y$. However, since the fibers $X(y)$ are Cohen–Macaulay and since the formation of a Fitting ideal commutes with base change,

$$\dim(F_*(\mathcal{O}_X/J)(y)) = e(y)$$

by some theorems of Buchsbaum and Rim [3, 2.4 p. 207, 4.3 and 4.5 p. 223].

**Remark (1.6)** Considerations involving the expression of $e(y)$ as a sum of intersection numbers suggest that, in Part (3) of Proposition (1.5), the Cohen–Macaulay hypothesis is unnecessary. For example, it is unnecessary when $p = 1$ (so $\mathcal{E} = \mathcal{O}_X$ and $\mathcal{M}$ is an ideal); see Theorem (2.2).

**Lemma (1.7)** The following conditions are equivalent:

(i) There exists a reduction of $\mathcal{M}$ generated by $r$ elements.

(ii) The bound $\dim P'(x_0) < r$ obtains.

**Proof.** Indeed, assume (i) (replacing $X$ and $Y$ if necessary). Say $\mathcal{N}$ is the reduction of $\mathcal{M}$. Then the inclusion $\mathcal{R}\mathcal{N} \hookrightarrow \mathcal{R}\mathcal{M}$ induces a finite surjective map $P' \rightarrow P''$. Hence $\dim P'(x_0) = \dim P''(x_0)$. However, $\dim P''(x_0) < r$ because $\mathcal{N}$ is generated by $r$ elements. Hence (ii) holds.

Conversely, assume (ii). Then (i) follows, for example, from [20, (6.2)(iv), p. 201] applied with $\mathcal{R}(\mathcal{M})$ for $G$ and for $\mathcal{M}$ and with $\mathcal{R}(\mathcal{N})$ for $G'$. The idea is simple. Condition (ii) implies that there are $r$ hyperplanes in $P'$ whose intersection misses the fiber $P'(x_0)$. Let $\mathcal{N}$ be the submodule of $\mathcal{M}$ generated by the $r$ elements corresponding to these hyperplanes, and $Z$ the subspace of $P'$ defined by the vanishing of these $r$ elements. Then the central projection from $P' - Z$ to $P''$ restricts to a finite map over $X - W$ where $W$ is the image of $Z$. Hence, after $X$ and $Y$ are replaced by neighborhoods of $x_0$ and $y_0$ if necessary, $\mathcal{N}$ is a reduction of $\mathcal{M}$, and the proof is complete.)

**Theorem (1.8)** (Specialization of integral dependence) Assume that $X$ is equidimensional, and that $y \mapsto e(y)$ is constant on $Y$. Let $h$ be a section of $\mathcal{E}$ whose image in $\mathcal{E}(y)$ is integrally dependent on the image of $\mathcal{M}$ for all $y$ in a dense Zariski open subset of $Y$. Then $h$ is integrally dependent on $\mathcal{M}$.

**Proof.** If $y \mapsto e(y)$ vanishes, then $\mathcal{M} = \mathcal{E}$ by Part (1) of Proposition (1.5), and so the assertion is trivial. Assume $y \mapsto e(y)$ is nonvanishing. Then, by Part (2) of Proposition (1.5), the map $S \rightarrow Y$ is surjective, and (after $X$ is replaced by a neighborhood of $x_0$ if necessary) there exists a reduction of $\mathcal{M}$ generated by $r$ elements. So Lemma (1.7) implies $\dim P'(x_0) < r$.

Let $\mathcal{H}$ be the submodule of $\mathcal{E}$ generated by $h$ and $\mathcal{M}$. By hypothesis, for all $y$ in a dense Zariski open subset of $Y$, the image in $\mathcal{E}(y)$ of $\mathcal{M}$ is a reduction of the image of $\mathcal{H}$. So, by Lemma (1.2), there is a smaller dense Zariski open subset of $Y$ over which $\mathcal{M}$ is a reduction of $\mathcal{H}$. Therefore, Lemma (1.4) implies that $\mathcal{M}$ is a reduction of $\mathcal{H}$, and the proof is complete.
2. The special fiber of the exceptional divisor

Preserve the setup of Section (1). Let $Z$ denote the analytic subspace of $P$ defined by the sheaf of ideals in $\mathcal{E}$ generated by $\mathcal{M}$. Form the blowup $B$ of $P$ with respect to $Z$, and the exceptional divisor $D$. The main result of this section, Theorem (2.2), relates the condition $\dim D(y_0) < r$ to the constancy on $Y$ of all the associated multiplicities $e^j(y)$ of the pair that $(\mathcal{E}, \mathcal{M})$ induces on the fiber $X(y)$. The definition of the $e^j(y)$ is recalled below. In particular, $e^0(y)$ is equal to $e(y)$, whose constancy was studied in the last section, and part of that study will be needed to prove Theorem (2.2). Not surprisingly, the constancy of $e(y)$ alone does not imply the constancy of all the $e^j(y)$; one instance where it doesn’t is considered in Example (2.3). Half the content of Theorem (2.2) is provided by Lemma (2.1), which gives a geometric description of a dense (Zariski) open subset $U$ of $Y$ on which all the $e^j(y)$ are locally constant. In particular, Lemma (2.1) provides, in a second way, the open set $U$ needed in the proof of Proposition (1.1).

In the case $p = 1$, there is only one possible nonzero associated multiplicity, namely, $e^0(y)$. In this case, $\mathcal{M}$ is the ideal on $X$ of $S$, and $B$ is the blowup of $X$ along $S$. Theorem (2.2) says that, if $X$ is equimultiple along $S$, then the exceptional divisor is equidimensional over $S$ (that is, if every fiber is empty or has the minimal possible dimension $d - 1$), and the converse holds if $Y$ is smooth. A version of the latter was proved in 1969 by Hironaka (according to Remark (2.6) in [28, p. 121]); a few years later, versions of the direct assertion were proved by Teissier (in [34, 3.1, p. 327] and [38, I.1, p. 131, I.3, p. 133]) and by Schikhoff (again according to [28, p. 121]).

The main new technical ingredient in this section is intersection theory. Denote the first Chern classes of the tautological sheaves $\mathcal{O}_P(1)$ and $\mathcal{O}_P(1)$ by $\ell'$ and $\ell$. Denote the blowup of $P(y)$ with respect to $Z(y)$ by $B_y$, and denote the exceptional divisor by $D_y$. Finally, form the Segre numbers $s^i(y)$ of $Z(y)$ in $X(y)$:

$$s^i(y) := \int \ell'^{r-i} \ell^{r-i} [D_y] \text{ for } i = 1, \ldots, r.$$  

Then $e^j(y)$ is defined as the sum of the first $r - j$ of the $s^i(y)$ in [20, (7.1), p. 207]:

$$e^j(y) = \sum_{i=1}^{r-j} s^i(y) \text{ for } j = 0, \ldots, r - 1.$$  

(In fact, in [20, (7.1)], the sum starts with an additional term $s^0(y)$, but that term clearly vanishes here.) With this definition,

$$e(y) = e^0(y)$$

because of [20, (5.1), p. 191] and [20, (5.7), p. 207]. The projection formula with respect to the map $D_y \to Z(y)$ yields

$$s^i(y) = 0 \text{ for } i < d;$$
in particular, $e^{r-d}(y) = s^d(y)$. All the $e^j(y)$ are constant if and only if all the $s^i(y)$ are. However, the $e^j(y)$ are upper semicontinuous, whereas the $s^i(y)$ needn’t be; see Example (2.3).

**Lemma (2.1)** Let $U$ be the open subset of $y$ in $Y$ such that $\dim D(y) < r$ and $Y$ is smooth at $y$. Then on $U$ all the functions $y \mapsto e^j(y)$ and $y \mapsto s^i(y)$ are locally constant.

**Proof.** For each $y \in Y$, the fiber $B(y)$ contains the blowup $B_y$ as a closed subscheme, and the intersection $D \cap B_y$ is equal to the exceptional divisor $D_y$. Hence the intersection product $D \cdot [B_y]$ is equal to the fundamental cycle $[D_y]$. Now, $B(y) - D(y)$ is equal to $B_y - D_y$. So, if $\dim D(y) < r$, then $[B(y)]$ is equal to $[B_y]$, and $D \cdot [B_y]$ is equal to $[D(y)]$; hence

$$s^i(y) = \int \ell^{r-1} \ell^{r-i}[D(y)] \text{ for } i \geq 1.$$ 

If $y \in U$, then the embedding $\iota_y: y \mapsto Y$ is regular; so the operation of pullback along $\iota_y$ commutes with that of pushforward along the proper map $D \to Y$. Hence, $y \mapsto s^i(y)$ is constant on each connected component of $U$ for $i \geq 1$; compare with [4, 10.2, p. 180]. The proof is now complete.

**Theorem (2.2)** Assume that $X$ is equidimensional. If the function $y \mapsto e^j(y)$ is constant on $Y$ for $0 \leq j < p$, then the central fiber $D(x_0)$ of the exceptional divisor $D$ of the blowup $B$ of $P$ is empty or has the minimal possible dimension, $r - 1$. Furthermore, the converse holds if, in addition, $Y$ is smooth at $y_0$.

**Proof.** The converse follows immediately from Lemma (2.1), applied after $Y$ is replaced by $U$. So assume that $e^j(y)$ is constant on $Y$ for all $j$. By way of contradiction, suppose that $D(y_0)$ has a component $D'(y_0)$ of dimension $r$ or more. Replacing $X$ by a neighborhood of $x_0$ if necessary, we may assume that $x_0$ is the unique point of $S(y_0)$. Then $D(y_0) = D(x_0)$. By Proposition (1.5)(2), after $X$ and $Y$ are replaced by neighborhoods of $x_0$ and $y_0$ if necessary, there exists a reduction $\mathcal{N}$ of $\mathcal{M}$ generated by $r$ elements. So Lemma (1.7) yields $\dim P'(x_0) < r$. However, $P'(x_0)$ contains the image of $D'(x_0)$ under the projection of $B$ onto $P'$. Hence the fibers of the map $D'(x_0) \to P'(x_0)$ all have dimension at least 1. However, these fibers are embedded in $P(x_0)$ by the blowup map $B \to P$ for the following reason: by definition of $Z$, its ideal sheaf is a quotient of the pullback of $\mathcal{M}$ to $P$, and so the Rees algebra of the ideal sheaf is a quotient of $\mathcal{R}\mathcal{M}$; correspondingly, $B$ is embedded in $P' \times P$, and the second projection restricts to the blowup map $B \to P$. Now, if $p = 1$, then $P = X$, and so $P(x_0)$ can contain no subspace of dimension at least 1. Thus, if $p = 1$, then the assertion holds.

The proof proceeds by induction on $p$. Suppose $p > 1$. Let $g$ be a general section of $\mathcal{E} := \mathcal{O}_X^p$. Set $\mathcal{E}' := \mathcal{E}/g$ and let $\mathcal{M}'$ denote the image of $\mathcal{M}$. Then $\mathcal{E}'$ is free of rank $p - 1$. Set $Q := \text{Projan}(\mathcal{S}(\mathcal{E}'))$ and $Z' := Z \cap Q$. Obviously $Z'$ is defined by the sheaf of ideals generated by $\mathcal{M}'$. Since $g$ is general, the preimage of $Q$ in $B$ is equal to the blowup $C$ of $Q$ along $Z \cap Q$, and $D \cap C$
is the exceptional divisor $E$, at least after $Y$ is replaced by a neighborhood of $y_0$. Since the fibers of the map $D'(x_0) \rightarrow P'(x_0)$ have dimension at least 1 and are embedded in $P$ by the blowup map, it follows that $Q$ must intersect these fibers. Hence the fiber $E(y_0)$ has dimension $r - 1$ or more. For convenience, denote the $j$th associated multiplicity of the pair that $(E', M')$ induces on $X(y)$ by $e^{ij}(y)$. Since $g$ is general, it follows from [20, (7.1) and (7.2)(iv), pp. 207–8] that $e^{ij}(y)$ is equal to $e^{j+1}(y)$ for $0 \leq j \leq r - 1$ and for all $y$ in a (Zariski) open neighborhood of $y_0$; replace $Y$ by this neighborhood. By hypothesis, $e^{j+1}(y)$ is constant on $Y$. Hence $y \mapsto e^{ij}(y)$ is constant. Thus the induction hypothesis is contradicted, and so $D'(y_0)$ does not exist. The proof is now complete.

**Example (2.3)** Consider the example of Henry and Merle [14, p. 578–9]. In it, the parameter space $Y$ is the affine line $C$, and the total space $X$ is cut out of $C^4 \times Y$ by two equations,

$$X : X_1^2 + X_2^2 + X_3^2 + yX_4 = 0, X_1^4 + X_2^4 + X_3^4 + X_4^2 = 0.$$ 

Set $p := 2$, and let $M$ be the Jacobian module, the column space of the Jacobian matrix with respect to the $X_i$. Henry and Merle proved that $X$ is Whitney equisingular along the $Y$-axis at the origin. Hence, by [8, 1.3, p. 211, 2.6, p. 215], the function $y \mapsto e(y)$ is constant on $Y$. In fact, in the case at hand, it is not hard to see via a direct computation that $e(y) = 36$ for all $y$.

On the other hand, $e^1(y) = 0$ if $y \neq 0$ because, obviously, $X(y)$ has embedding dimension 3 at the origin. However, $e^1(0) \neq 0$ because $e^1(0)$ is the multiplicity of the ideal obtained by taking a generic linear combination of the rows of the Jacobian matrix of $X(0)$; in fact, it is easy to see that $e^1(0) = 4$. Hence, by Theorem (2.2), $D(0)$ must have a “vertical” component. In fact, it is also not hard to see that $D(0)$ consists of two components, one of which maps onto the fiber of $P$ over the origin in $X(0)$.

Finally, $e(y)$ is equal to $e^1(y) + s^3(y)$; so this sum is constant. On the other hand, $e^1(y)$ is upper semicontinuous. Therefore, $s^3(y)$ is lower semicontinuous.

3. **Integral Dependence and Strict Dependence**

In the next sections, we’ll study Whitney’s Condition A, Thom’s Condition $A_f$, and Henry, Merle and Sabbah’s Condition $W_f$, which concern limiting tangent hyperplanes at a singular point of a complex analytic space. To prepare further for this study, in this section and in part of the next one, we’ll recall and develop some material from [6] and [5]. In [34], Teissier made a similar study in the case of families of hypersurfaces with isolated singularities, and his work has been a model for ours.

Let $(X, 0)$ be the germ of a complex analytic space, and $E := O_X^p$ a free module of rank $p$ at least 1. Let $M$ be a coherent submodule of $E$, and $h$ a section of $E$. Given a map of germs $\varphi : (C, 0) \rightarrow (X, 0)$, denote by $h \circ \varphi$ the induced section of the pullback $\varphi^*E$, or $O_X^p$, and by $M \circ \varphi$ the induced submodule. Call $h$ **integrated dependent** (resp., **strictly dependent**) on $M$ at 0 if, for every $\varphi$, the section $h \circ \varphi$
of $\varphi^*\mathcal{E}$ is a section of $\mathcal{M} \circ \varphi$ (resp., of $m_1(\mathcal{M} \circ \varphi)$, where $m_1$ is the maximal ideal of 0 in $\mathcal{C}$). The submodule of $\mathcal{E}$ generated by all such $h$ will be denoted by $\mathcal{M}$, resp., by $\mathcal{M}^\dagger$ (the notation '$\mathcal{M}^\dagger$' is a change from [5]).

To check for integral (resp., strict) dependence, it suffices to use only those $\varphi$ whose image meets any given dense Zariski open subset of $X$. Indeed, if $h \circ \varphi$ is not a section of $\mathcal{M} \circ \varphi$ (resp., of $m_1(\mathcal{M} \circ \varphi)$), then $\varphi$ can be tweaked, preserving this condition, so that the image of $\varphi$ does meet the given open set (see the proof of Prop. 1.7 on p. 304 in [6]).

Let $\mathcal{N}$ be a coherent submodule of $\mathcal{M}$. Then $\mathcal{M} \subset \overline{\mathcal{N}}$ if and only if $\mathcal{N}$ is a reduction of $\mathcal{M}$ in the sense of Section 1 (after $X$ is replaced by a neighborhood of 0 if necessary). Indeed, the present definition of integral closure is taken from [6, 1.3, p. 303]. This definition is shown, on the middle of p. 305 in [6], to be equivalent to Rees’s definition [32, p. 435]. Hence, Theorem 1·5 in [32, p. 437] yields the assertion.

The following result is a simple, but useful, observation.

**Proposition (3.1)** If $\mathcal{N} \subset \mathcal{M} \subset \overline{\mathcal{N}}$, then $\overline{\mathcal{M}} = \overline{\mathcal{N}}$ and $\mathcal{M}^\dagger = \mathcal{N}^\dagger$.

**Proof.** For any map $\varphi: (\mathcal{C}, 0) \rightarrow (X, 0)$, the hypothesis yields

$$N \circ \varphi \subset M \circ \varphi \subset \overline{N} \circ \varphi.$$

By definition, the third term is equal to the first. Hence the first term is equal to the second. Therefore, the definitions yield the assertions.

The next result gives one useful connection between the notions of integral dependence and strict dependence.

**Proposition (3.2)** Fix a set $C$ of generators of $\mathcal{M}$. Then the following conditions on $\mathcal{M}$ and $\mathcal{N}$ are equivalent:

(i) $\mathcal{N} \subset \mathcal{M}^\dagger$;

(ii) $\mathcal{M} \subset \overline{\mathcal{N}'}$ for every coherent submodule $\mathcal{N}'$ of $\mathcal{M}$ such that $\mathcal{N} + \mathcal{N}' = \mathcal{M}$;

(iii) $\mathcal{M} \subset \overline{\mathcal{N}'}$ for every submodule $\mathcal{N}'$ of $\mathcal{M}$ such that $\mathcal{N}'$ is generated by a subset of $C$ and $\mathcal{N} + \mathcal{N}' = \mathcal{M}$.

**Proof.** Assume (i). To prove (ii), take any map $\varphi: (\mathcal{C}, 0) \rightarrow (X, 0)$. Then

$$\mathcal{M} \circ \varphi = N \circ \varphi + N' \circ \varphi \subset m_1(\mathcal{M} \circ \varphi) + N' \circ \varphi.$$

By Nakayama’s lemma, $\mathcal{M} \circ \varphi = N' \circ \varphi$. So (ii) holds. Trivially, (ii) implies (iii).

Finally, assume (i) fails. Then there exists a $\varphi$ such that $N \circ \varphi$ is not contained in $m_1(\mathcal{M} \circ \varphi)$. Let $h$ be a section of $N$ such that $h \circ \varphi$ is not contained in $m_1(\mathcal{M} \circ \varphi)$. Supplement $h$ by elements of $C$ to obtain a basis of the vector space $\mathcal{M} \circ \varphi / m_1(\mathcal{M} \circ \varphi)$, or what is the same, a basis of $\mathcal{M}/m\mathcal{M}$ where $m$ is the maximal ideal of 0 in $\mathcal{O}_X$. Let $\mathcal{N}'$ be the submodule of $\mathcal{M}$ generated by these elements of $C$. Then, by Nakayama’s lemma, $\mathcal{N} + \mathcal{N}' = \mathcal{M}$. Moreover, by construction, $h \circ \varphi$ is not contained in $\mathcal{N}' \circ \varphi$. Thus (iii) fails, and the proof is complete.
The following lemma is a useful generalization of Proposition 1.6 in [5], and
the following proof is a little different.

Lemma (3.3) For a section \( h \) of \( \mathcal{E} := \mathcal{O}_X^p \) to be integrally dependent (resp.,
strictly dependent) on \( \mathcal{M} \) at 0, it is necessary that, for all maps \( \varphi : (\mathbf{C}, 0) \rightarrow (X, 0) \) and \( \psi : (\mathbf{C}, 0) \rightarrow (\text{Hom}(\mathbf{C}^p, \mathbf{C}), \lambda) \) with \( \lambda \neq 0 \), the function \( \psi(h \circ \varphi) \) on \( \mathbf{C} \)
belong to the ideal \( \psi(\mathcal{M} \circ \varphi) \) (resp., to \( m_1 \psi(\mathcal{M} \circ \varphi) \)).

Conversely, it is sufficient that this condition obtain for every \( \varphi \) whose image
meets any given dense Zariski open subset of \( X \). Furthermore, if 0 lies in the
cosupport \( \text{Supp}(\mathcal{E}/\mathcal{M}) \) of \( \mathcal{M} \), then it is sufficient that the condition obtain for
every such \( \varphi \) and for every \( \psi \) that carries \( \mathcal{M} \circ \varphi \) into \( m_1 \).

Proof. The first assertion follows directly from the definitions.

Conversely, given any \( \varphi \), (after \( X \) is replaced by a neighborhood of 0 if neces-
sary) there exists a basis \( e_1, \ldots, e_p \) for \( \varphi^* \mathcal{E} \) such that \( \mathcal{M} \circ \varphi \) is equal to the
submodule generated by \( t^{n_1}e_1, \ldots, t^{n_r}e_r \) for suitable integers \( n_i \) and \( r \), where \( t \)
is the coordinate function on \( \mathbf{C} \). Say \( h \circ \varphi \) expands as \( a_1 e_1 + \cdots + a_p e_p \).

Then clearly \( h \circ \varphi \) is a section of \( \mathcal{M} \circ \varphi \) (resp., of \( m_1 (\mathcal{M} \circ \varphi) \)) if and only if \( a_i = b_i t^{n_i} \)
(resp., \( a_i = b_i t^{n_i+1} \)) for a suitable \( b_i \) for \( 1 \leq i \leq r \) and \( a_i = 0 \) for \( r < i \leq p \).

Form the dual basis \( e'_1, \ldots, e'_p \). Then \( a_i = e'_i (h \circ \varphi) \) for all \( i \).
Moreover, \( e'_i (\mathcal{M} \circ \varphi) \) is equal to the ideal generated by \( t^{n_i} \) for \( 1 \leq i \leq r \), and to 0 for \( r < i \leq p \).

Hence \( h \circ \varphi \) lies in \( \mathcal{M} \circ \varphi \) (resp., in \( m_1 (\mathcal{M} \circ \varphi) \)) if (and only if) the condition
obtains for the \( p \) maps \( \psi \) corresponding to \( e'_1, \ldots, e'_p \), and for each of these \( \psi \),

Thus the second assertion holds.

Suppose \( 0 \in \text{Supp}(\mathcal{E}/\mathcal{M}) \). Then either \( r = p \) and \( n_j > 0 \) for some \( j \), or \( r < p \).
Suppose first \( r < p \). Fix \( i \) with \( r < i \leq p \), and let \( \psi \) correspond to \( e'_i \).
Then the condition implies that \( a_i = 0 \). Now, fix \( i \) with \( 1 \leq i \leq r \), and let \( \psi \) correspond
to \( t e'_i + e'_{i+1} \). Then \( \lambda \neq 0 \), and \( \psi \) carries \( \mathcal{M} \circ \varphi \) into \( m_1 \). The condition implies that \( t a_i = b_i t^{n_i+1} \)
(resp., \( t a_i = b_i t^{n_i+2} \)) for some \( b_i \). Thus the third assertion
hods when \( r < p \).

Suppose \( r = p \). Reorder the \( e_i \) so that \( n_i \leq n_{i+1} \) for each \( i \). Say \( n_j = 0 \), but
\( n_{j+1} > 0 \). Fix \( i \) with \( j < i \leq p \), and let \( \psi \) correspond to \( e'_i \).
Then \( \lambda \neq 0 \), and \( \psi \) carries \( \mathcal{M} \circ \varphi \) into \( m_1 \). The condition implies that \( a_i = b_i t^{n_i} \)
(resp., \( a_i = b_i t^{n_i+1} \)) for some \( b_i \). Now, fix \( i \) with \( 1 \leq i \leq j \). Then \( n_i = 0 \), so \( a_i = a_i t^{n_i} \). Thus \( h \) is
integrally dependent on \( \mathcal{M} \) at 0.

To handle strict dependence, let \( \psi \) correspond to \( t e'_i + e'_{i+1} \). Then \( \lambda \neq 0 \), and
\( \psi \) carries \( \mathcal{M} \circ \varphi \) into \( m_1 \). The condition implies that \( t a_i + a_{i+1} = b_i(t + t^{n_{j+1}}) \)
for some \( b_i \). Now, \( n_{j+1} > 0 \) and \( a_{i+1} = b_{j+1} t^{n_{j+1}+1} \). Hence, \( a_i = b'_i t^{n_{i+1}} \) for a
suitable \( b'_i \). Thus the third assertion holds, and the proof is complete.

It is often convenient to work on the space \( P \) of Section 1; obviously, \( P = X \times \mathbf{P}^{p-1} \) since \( \mathcal{E} := \mathcal{O}_X^p \).

The section \( h \) of \( \mathcal{E} \) and the submodule \( \mathcal{M} \) of \( \mathcal{E} \)
generate ideals on \( P \); denote them by \( \rho(h) \) and \( \rho(\mathcal{M}) \). Note that \( \rho(h) \) is locally
principal. The next result gives a translation of the two notions of dependence
into this context, thereby reducing the study of dependence on the module \( \mathcal{M} \)
on the germ \( (X, 0) \) to that of the ideal \( \rho(\mathcal{M}) \) on the more global space \( P \).
**Proposition (3.4)** A necessary and sufficient condition for a section $h$ of $\mathcal{E}$ to be integrally dependent (resp., strictly dependent) on $\mathcal{M}$ at $0$ is that, at each point of $V(\rho(\mathcal{M}))$ lying over $0 \in X$, a generator of $\rho(h)$ be integrally dependent (resp., strictly dependent) on $\rho(\mathcal{M})$.

**Proof.** To give a map $\phi: (C, 0) \to (P, (0, l))$ is the same as to give a pair of maps $\varphi: (C, 0) \to (X, 0)$ and $\psi: (C, 0) \to (\text{Hom}(C^p, C), \lambda)$ where $\lambda$ corresponds to $l$ (although $\psi$ is determined only up to multiplication by a function that doesn’t vanish at $0 \in C$). Hence, the assertion follows from Lemma (3.3).

It is also convenient to work with the normalized blowup, with its structure map,

$$\pi: NB_{\rho(\mathcal{M})}(P) \to P,$$

and with its exceptional divisor $E$. (After replacing $X$ by a neighborhood of $0$ if necessary, we may assume that each component of $E$ meets the fiber over $0$.) The next result relates the two notions of dependence to vanishing of the ideal $\rho(h) \circ \pi$ on the components of $E$.

**Proposition (3.5)** Let $h$ be a section of $\mathcal{E}$, and $Y$ a closed analytic subset of the image of $E$ in $X$.

(1) A necessary and sufficient condition for $h$ to be integrally dependent on $\mathcal{M}$ at $0$ is that, along each component of $E$, the ideal $\rho(h) \circ \pi$ vanish to order at least the order of vanishing of $\rho(\mathcal{M}) \circ \pi$.

(2) A necessary and sufficient condition for $h$ to be strictly dependent on $\mathcal{M}$ at every $y \in Y$ is that, along each component $V$ of $E$, the ideal $\rho(h) \circ \pi$ lie in the product $I(Y, V)\rho(\mathcal{M})\circ \pi$, where $I(Y, V)$ denotes the ideal of the reduced preimage of $Y$ in $V$; in particular, if $V$ projects into $Y$, then this condition simply requires the ideal $\rho(h) \circ \pi$ to vanish to order strictly greater than the order of vanishing of $\rho(\mathcal{M}) \circ \pi$.

**Proof.** Consider (1). Proposition (3.4) reduces the assertion to the case of an ideal, and this case is treated in [40,p.330, 1.4 Prop. 2].

Consider (2). At each $b \in V$, the ideal $\rho(\mathcal{M}) \circ \pi$ is generated by a single section $g \circ \pi$ where $g$ is a suitable section of $\mathcal{M}$, and the ideal $\rho(h) \circ \pi$ is generated by a multiple $k(\rho(g) \circ \pi)$ where $k$ is a meromorphic function. In these terms, the condition in (2) says that $k$ is holomorphic and vanishes at $b$ if $b$ projects into $Y$.

Hence the condition in (2) holds if and only if, for every map

$$\beta: (C, 0) \to (NB_{\rho(\mathcal{M})}(P), b)$$

such that $\phi := \pi \circ \beta$ is not constant and such that the image of $\beta$ meets the complement of $V(\rho(\mathcal{M}))$, the function $k \circ \beta$ vanishes at $0 \in C$. Now, $\rho(h) \circ \phi$ is generated at $0$ by $(k \circ \beta)(g \circ \phi)$ if $k$ is holomorphic; moreover, $k$ is holomorphic, if $h$ is integrally dependent on $\mathcal{M}$ at image of $b$ by (1). Furthermore, we can factor any map $\phi: (C, 0) \to (P, (y, l))$ through $NB_{\rho(\mathcal{M})}(P)$. Therefore, the assertion follows from Proposition (3.4).
4. Whitney’s Condition A

In this section, we use the theory developed in the preceding sections to study Whitney’s Condition A. After introducing the setup, we prove a lemma, which relates limit tangent hyperplanes with the notions of strict dependence and integral dependence; the statement and proof are, more or less, found in Section 2 of [5]. Then we prove the main result of the section, Theorem (4.2), which asserts that Whitney’s Condition A holds under the constancy of certain Buchsbaum–Rim multiplicities on the fibers. Finally, we illustrate the theorem with two examples.

Let \((X, 0)\) be a complex analytic subgerm of \((\mathbb{C}^n, 0)\) defined by the vanishing of a map of germs \(F: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)\). Call the \(\mathcal{O}_X\)-submodule of the normal module to \(X\) in \(\mathbb{C}^n\) generated by all the partial derivatives of \(F\) the \(\text{(absolute)}\) Jacobian module of \(F\), and denote it by \(JM(F)\); more precisely, \(JM(F)\) is the image of the canonical map,

\[
\text{Hom}_X(\Omega^1_{\mathbb{C}^n}|X, \mathcal{O}_X) \to \text{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X),
\]

where \(\mathcal{I}\) is the ideal of \(X\) in \(\mathbb{C}^n\). Since \(\mathcal{I}\) is generated by the \(p\) coordinate functions of \(F\), the displayed map is given by the Jacobian matrix \(DF\), and \(JM(F)\) is simply the submodule of the free module \(\mathcal{O}_X^p\) generated by the columns of \(DF\). Note in passing that this module \(\mathcal{O}_X^p\) contains the target \(\text{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)\), which is an abstract \(\mathcal{O}_X\)-module and is known as the \(\text{normal module}\) of \(X\) in \(\mathbb{C}^n\), but the embedding depends on the choice of the \(p\) generators of \(\mathcal{I}\); moreover, this embedding is an isomorphism if \(X\) is a complete intersection of codimension \(p\), but not in general.

Given an analytic map germ \(g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0)\), let \(JM(F)_g\) denote the submodule of \(JM(F)\) generated by the “partials” \(\partial F/\partial v\) for all vector fields \(v\) on \(\mathbb{C}^n\) tangent to the fibers of \(g\), that is, for all \(v\) that map to the 0-field on \(\mathbb{C}^l\); call \(JM(F)_g\) the \(\text{relative Jacobian module}\) with respect to \(g\). For example, if \(g\) is the projection onto the space of the last \(l\) variables of \(\mathbb{C}^n\), then \(JM(F)_g\) is simply the submodule generated by all the partial derivatives of \(F\) with respect to the first \(n - l\) variables.

Call a hyperplane in \(\mathbb{C}^n\) through \(0\) a \(\text{limit tangent hyperplane}\) of \((X, 0)\) if it is the limit of hyperplanes tangent to \(X\) at nonsingular points approaching \(0\) along an analytic arc. Now, let \((S, 0)\) be a smooth subgerm of \((\mathbb{C}^n, 0)\) defined by the vanishing of a map of germs \(g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0)\) with \(l = n - m\) where \(m := \dim S\), and let \(T_0 S\) denote its tangent space at \(0\). Finally, denote the singular locus of \(X\) by \(\Sigma\).

The following lemma describes the limit tangent hyperplanes in general and those that contain \(T_0 S\) in particular in terms of Jacobian modules. The lemma and its proof are, more or less, the statement and proof of Theorem 2.4 of [5].

**Lemma (4.1)** \(\text{Preserve the conditions above.}\)
(1) A hyperplane \( H \), defined by the vanishing of a linear function \( h: \mathbb{C}^n \rightarrow \mathbb{C} \), is a limit tangent hyperplane of \((X,0)\) if and only if \( JM(F)_h \) is not a reduction of \( JM(F) \).

(2) No hyperplane containing \( T_0 S \) is a limit tangent hyperplane of \((X,0)\) if \( JM(F)_g \) is a reduction of \( JM(F) \).

(3) Every limit tangent hyperplane of \((X,0)\) contains \( T_0 S \) — in other words, the pair \((X - \Sigma, S)\) satisfies Whitney’s Condition A at the origin — if and only if \( JM(F)_g \) is contained in \( JM(F)^\dagger \).

**Proof.** A hyperplane \( M \) is a limit tangent hyperplane of \((X,0)\) if and only if there exist maps \( \varphi(t): (\mathbb{C},0) \rightarrow (X,0) \) and \( \psi(t): (\mathbb{C},0) \rightarrow (\text{Hom}(\mathbb{C}^n, \mathbb{C}), \lambda) \) with \( \lambda \neq 0 \) such that \( \varphi(t) \) is a nonsingular point of \( X \) for \( t \neq 0 \) and such that, for a suitable \( k \), the limit,

\[
\lim_{t \to 0}(1/t^k)(\psi(t)DF(\varphi(t))),
\]

exists and is a conormal vector to \( M \). This condition means that, given a vector field \( v \) on \( \mathbb{C}^n \), the vector \( v(0) \) lies in \( M \) if and only if

\[
\psi(t)DF(\varphi(t))v(\varphi(t)) \in m_1(\psi(t)JM(F) \circ \varphi(t)),
\]

where \( m_1 \) is the maximal ideal of \( 0 \) in \( \mathbb{C} \). Obviously, \( DF(\varphi(t))v(\varphi(t)) \) is equal to \( \partial F/\partial v \circ \varphi(t) \). Hence, \( v(0) \in M \) if and only if

\[
\psi(t)(\partial F/\partial v) \circ \varphi(t) \in m_1(\psi(t)JM(F) \circ \varphi(t)). \tag{4.1.1}
\]

Consider (1). If the vector field \( v \) is tangent to the fibers of \( h \), then \( v(0) \in H \), and every vector in \( H \) is a \( v(0) \) for some such \( v \). Hence, if \( H = M \), then (4.1.1) holds for every \( v \) tangent to the fibers of \( h \); whence, \( JM(F)_h \) is not a reduction of \( JM(F) \), thanks to Lemma (3.3) applied with \( \partial F/\partial v \) for \( h \) and with \( JM(F) \) for \( M \). Conversely, if \( JM(F)_h \) is not a reduction of \( JM(F) \), then, by Lemma (3.3), there exists a pair of maps \( \varphi(t) \) and \( \psi(t) \) such that (4.1.1) holds for every \( v \) tangent to the fibers of \( h \); whence, the corresponding \( M \) contains every vector in \( H \), and so \( M = H \). Thus (1) holds.

Consider (2) and (3). Since \( S \) is smooth and \( l = n - m \), the germ \( g \) is a submersion. Hence, if the vector field \( v \) is tangent to the fibers of \( g \), then \( v(0) \in T_0 S \), and every vector in \( T_0 S \) is a \( v(0) \) for some such \( v \). If \( JM(F)_g \) is a reduction of \( JM(F) \), then (4.1.1) fails for some such \( v \) by Lemma (3.3), and therefore \( v(0) \notin M \); thus (2) holds. Finally, (4.1.1) implies that \( v(0) \) lies in every limit tangent \( M \) if and only \( \partial F/\partial v \) is contained in \( JM(F)^\dagger \), thanks to Lemma (3.3); thus (3) holds. The proof of the lemma is now complete.

The following theorem gives a sufficient fiberwise numerical criterion for the condition to hold. The proof involves a delicate interplay among the absolute and two relative Jacobian modules of \( F \).
Theorem (4.2) Let \( Y := \mathbb{C}^m \) be the space of the first \( m \) coordinates in \( \mathbb{C}^n \) where \( 1 \leq m < n \), and set \( l := n - m \). Assume that, under the projection \( r: \mathbb{C}^n \to Y \), the subspace \( X \) of \( \mathbb{C}^n \) becomes the total space of a family of complete intersections \( X(y) \) of codimension \( p \) defined by the maps \( F(y): \mathbb{C}^l \to \mathbb{C}^p \) given by \( F(y)(z) := F(y, z) \). Assume that the \( X(y) \) have isolated singularities, which trace out the smooth subgerm \( (S, 0) \) of \( (X, 0) \). Finally, let \( e^j(y) \) be the \( j \)th associated Buchsbaum–Rim multiplicity of the Jacobian module \( JM(F(y)) \) in \( O^p_{X(y)} \), and assume that the function \( y \mapsto e^j(y) \) is constant on \( (Y, 0) \) for \( 0 \leq j < p \). Then \( (X - S, S) \) satisfies Whitney’s Condition A along \( S \).

Proof. Form the ideal \( \rho(JM(F)_r) \) on \( X \times \mathbb{P}^{p-1} \), form the corresponding normalized blowup, and form its exceptional divisor \( E \). Then every component of \( E \) projects onto \( Y \); indeed, this conclusion follows from Theorem (2.2) because, by hypothesis, the functions \( y \mapsto e^j(y) \) are constant.

Since \( S \) is smooth of dimension \( m \), there is a map germ \( g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0) \) such that \( (S, 0) = (g^{-1}0, 0) \). Moreover, by the generic Whitney lemma, Whitney’s Condition A holds on a dense Zariski open subset \( U \) of \( S \). Hence, for \( s \in U \),

\[
JM(F)_g \subset JM(F)^\dagger \quad \text{at } s \tag{4.2.1}
\]

by Proposition (4.1). Replacing \( U \) by a smaller subset, we may assume that the map \( S \to Y \) is unramified at \( s \). Then, at \( s \), the sum \( JM(F)_g + JM(F)_r \) is all of \( JM(F) \). Hence, at \( s \),

\[
JM(F) \subset \overline{JM(F)_r}
\]

by (i)⇒(ii) of Proposition (3.2).

Hence, this inclusion holds everywhere on \( X \) by Proposition (3.5)(1) since every component of \( E \) projects onto \( Y \); apply the proposition twice, first the necessity assertion with \( U \) for \( Y \), and then the sufficiency assertion. (In fact, here we could appeal to Theorem (1.8) instead, and thus use only the constancy of \( e^0(y) \); however, the constancy of all the \( e^j(y) \) is used in an essential way in the next paragraph.) Therefore, by Proposition (3.1),

\[
JM(F)^\dagger = JM(F)^\dagger_r \tag{4.2.2}
\]

everywhere on \( X \).

Again, since every component of \( E \) projects onto \( Y \), Proposition (3.5)(2) implies that the inclusion,

\[
JM(F)_g \subset JM(F)^\dagger_r
\]

holds everywhere on \( S \) because it holds on \( U \) by virtue of (4.2.1) and (4.2.2). Therefore, again by virtue of (4.2.2), the inclusion (4.2.1) holds along \( S \). Consequently, Proposition (4.1) implies that the pair \( (X - S, S) \) satisfies Whitney’s Condition A along \( S \), and the proof is complete.
**Example (4.3)** The fiberwise numerical criterion of Theorem (4.2), although sufficient, is not necessary. In fact, it is impossible to have a necessary and sufficient numerical criterion for Whitney’s Condition A that depends only on the members of the family. This observation was made by Trotman [41, Prop. 5.1, p. 147] on the basis of the following example of his:

\[ X : w^a = y^b v^c + v^d \]

and \( S \) is the \( y \)-axis. Here, different values of \( b \) can give essentially different parameterizations of the same collection of plane curves. Trotman determined when Whitney’s Condition A is satisfied, and when it isn’t.

Let’s look, from our point of view, at a special case of Trotman’s example, the “Whitney umbrella of type \( b \),”

\[ X : w^2 - v^3 + v^2 y^b = 0. \]

Since \( X \) is the total space of a one-parameter family of plane curves, \( m = 1 \) and \( l = 2 \) and \( p = 1 \); furthermore, \( e^0(y) \) is simply the (ordinary) multiplicity of the Jacobian ideal. Here, \( X(y) \) is a nodal cubic for \( y \neq 0 \), and \( X(0) \) is a cuspidal cubic. So \( e^0(0) = 3 \) and \( e^0(y) = 2 \) for \( y \neq 0 \), as is easy to check. Finally, it is well known that Whitney’s Condition A is satisfied if \( b \geq 2 \), but not if \( b = 1 \). This fact will now be checked as an illustration of the use of Lemma (4.1)(3).

Set \( F := w^2 - v^3 + v^2 y^b \). Let \( g: (C^3, 0) \to (C^2, 0) \) be the projection onto the \((w, v)\)-plane. Then \( JM(F)_g \) is generated by the partial derivative \( \partial F/\partial y \), and so we have to show that \( \partial F/\partial y \in JM(F)^\dagger \) holds if \( b \geq 2 \), but fails if \( b = 1 \). So consider a map \( \varphi(t): (C, 0) \to (X, 0) \), say with coordinate functions,

\[ w(t) = \alpha t^i + \cdots, \quad v(t) = \beta t^j + \cdots, \quad y(t) = \gamma t^k + \cdots, \]

with \( i, j, k > 0 \). Since \( DF = (2w, v(3v + 2y^b), bv^2y^{b-1}) \), we’re asking about the condition,

\[ 2j + (b - 1)k > \min(i, 2j + j'), \quad (4.2.3) \]

where \( j' \geq 0 \) and \( j' = 0 \) unless \( j = bk \) and \( 3\beta + 2\gamma^b = 0 \). We have to establish this condition if \( b \geq 2 \), and show that it fails if \( b = 1 \) for a suitable choice of \( \varphi(t) \). Now,

\[ \alpha^2 t^{2i} + \cdots = (\beta^2 t^2 j + \cdots)(\beta t^i - \gamma t^{bk} + \cdots). \quad (4.2.4) \]

So there are three cases to consider. First, suppose that \( j > bk \). Then (4.2.4) implies that \( 2i = 2j + bk < 3j \). So \( i < 3j/2 \). So (4.2.3) holds for any \( b \geq 0 \). Second, suppose either that \( j < bk \) or that \( j = bk \) and \( \beta \neq \gamma^b \). Then (4.2.4) implies that \( 2i = 3j \). So \( i = 3j/2 \). So again (4.2.3) holds for any \( b \geq 0 \). Finally, suppose that \( j = bk \) and \( \beta = \gamma^b \). If \( b \geq 2 \), then (4.2.3) holds because \( k > 0 \) and \( j' = 0 \). However, if \( b = 1 \), then (4.2.3) need not hold. For instance, (4.2.3) does not hold if

\[ w(t) = t^2, \quad v(t) = t, \quad y(t) = t - t^2, \]

although (4.2.4) does hold. Thus Whitney’s Condition A is satisfied if \( b \geq 2 \), but not if \( b = 1 \).
Example (4.4) In Theorem (4.2), the smooth subgerm \((S,0)\) of \((X,0)\) is customarily taken to be the plane of the first \(m\) variables, but it needn’t be. In fact, the projection of \((S,0)\) onto \((Y,0)\) may be allowed to ramify at 0. For example, suppose that \(X\) is defined as follows:

\[ X : (w^2 - y)^2 - x^2 = 0. \]

Then \(X(y)\) is a binodal quartic for \(y \neq 0\) and \(X(0)\) is a tacnodal quartic. Moreover, \(S\) is the parabola,

\[ S : w^2 = y, \quad x = 0. \]

It is easy to check that \(e(y) = 4\) for all \(y\). Hence, by Theorem (4.2), Whitney’s Condition A is satisfied. In fact, it is obvious geometrically that the condition is satisfied. Indeed, \(X = X_+ \cup X_-\) where

\[ X_+ : (w^2 - y) + x = 0 \quad \text{and} \quad X_- : (w^2 - y) - x = 0. \]

The two components \(X_+\) and \(X_-\) are smooth, and they meet transversally along their intersection, which is \(S\). Therefore, each limit tangent hyperplane at 0 is either the tangent plane to \(X_+\) or that to \(X_-\), so contains the tangent line to \(S\).

5. Thom’s Condition \(A_f\)

In this section, we use the theory developed in Sections 2 to 4 to study Thom’s Condition \(A_f\). After introducing the setup, we prove a lemma, which is similar to Lemma (4.1), and relates limit tangent hyperplanes to level hypersurfaces with the notions of strict dependence and integral dependence. Then we prove a generalization of the Lê–Saito theorem. Finally, we prove some variations of a special case of a recent result of Briançon, Maisonobe and Merle’s.

Let \((X,0)\) be a complex analytic germ defined by the vanishing of a map of germs \(F : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)\) with \(p \geq 0\); if \(p = 0\), then \(F = \emptyset\) and \(X = \mathbb{C}^n\). Let \(f : (\mathbb{C}^n,0) \to (\mathbb{C},0)\) be the germ of a complex analytic function. Form the \(p + 1\) by \(n\) matrix \(D(F;f)\) by augmenting the Jacobian matrix \(DF\) at the bottom with the gradient \(df\). Call the submodule of the free module \(\mathcal{O}_{\mathbb{C}^p+1}^\times\), generated by the columns of \(D(F;f)\), the augmented Jacobian module and denote it by \(JM(F;f)\).

More intrinsically, \(JM(F;f)\) may be viewed as follows. Identify \((X,0)\) with the graph of \(f|(X,0)\), which is a germ in \((\mathbb{C}^{n+1},0)\). This germ is defined by the vanishing of the map \(G : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{p+1},0)\) whose components are \(F\) and \(f - z\), where \(z\) is the last coordinate function on \((\mathbb{C}^{n+1},0)\). Then \(JM(F;f) = JM(G)_z\); that is, they are the same submodule of \(\mathcal{O}_{\mathbb{C}^p+1}^\times\). Now, \(JM(G)_z\) depends only on the (abstract) normal module of the graph of \(f|(X,0)\), not on the choice of generators of this module (nor on the choice of coordinates on \(\mathbb{C}^n\)); see the beginning of Section 4. Thus \(JM(F;f)\) depends only on the restriction \(f|(X,0)\); in other words, a second function germ on \((\mathbb{C}^n,0)\) with the same restriction as \(f\) gives rise to the same augmented Jacobian module, viewed as a submodule of the normal module of the graph. Moreover, given \(F\) and \(f\), this normal module may
be viewed as a submodule of the free module $O_X^{b+1}$, and the latter two modules are equal if $X$ is a complete intersection of codimension $p$.

Given an analytic map germ $g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0)$, let $JM(F; f)_g$ denote the submodule of $JM(F; f)$ generated by the columns of “partial derivatives” with respect to the vector fields on $\mathbb{C}^n$ tangent to the fibers of $g$; in other words,

$$JM(F; f)_g := JM(G)_{(g, z)},$$

where $(g, z): (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{l+1}, 0)$ has components $g$, $z$. Call $JM(F; f)_g$ the relative augmented Jacobian module with respect to $g$.

If $p = 0$, or $F = \emptyset$, then write $JM(\cdot; f)$ and $JM(\cdot; f)_g$ for $JM(F; f)$ and $JM(F; f)_g$. Then $JM(\cdot; f)$ is simply the ideal on $\mathbb{C}^n$ generated by all the partial derivatives of $f$, and $JM(\cdot; f)_g$ is the subideal generated by the “partials” with respect to the vector fields on $\mathbb{C}^n$ tangent to the fibers of $g$.

Call a hyperplane in $\mathbb{C}^n$ through 0 a limit tangent hyperplane of the fibers (or level hypersurfaces) of $f|X$ if it is the limit of hyperplanes tangent to the fibers of $f|X$ at points where $f|X$ is a submersion and that approach 0 along an analytic arc. Now, let $(S, 0)$ be a smooth subgerm of $(\mathbb{C}^n, 0)$ defined by the vanishing of a map of germs $g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0)$ with $l = n - m$ where $m := \dim S$, and let $T_0S$ denote its tangent space at 0. Assume that $f|X$ is a submersion on the smooth locus of $X - S$. The following lemma describes the limit tangent hyperplanes of the fibers of $f|X$ in general and those that contain $T_0S$ in particular in terms of augmented Jacobian modules.

**Lemma (5.1)** Preserve the conditions above.

1. A hyperplane $H$, defined by the vanishing of a linear function $h: \mathbb{C}^n \to \mathbb{C}$, is a limit tangent hyperplane of the fibers of $f|X$ if and only if $JM(F; f)_h$ is not a reduction of $JM(F; f)$.

2. No hyperplane containing $T_0S$ is a limit tangent hyperplane of the fibers of $f|X$ if $JM(F; f)_g$ is a reduction of $JM(F; f)$.

3. Every limit tangent hyperplane of the fibers of $f|X$ contains $T_0S$ — in other words, the pair $(X - S, S)$ satisfies Thom’s Condition $A_f$ at the origin — if and only if $JM(F; f)_g$ is contained in $JM(F; f)^\dagger$.

**Proof.** With $(F, f)$ in place of $F$, the proof is essentially the same as that of Proposition (4.1), because of the following observation: at a nonsingular point $x$ of $X$, the Jacobian matrix of $(F, f)$ has maximal rank because $x$ is not a critical point of $f|X$; moreover, the row space of the matrix is the conormal module in $\mathbb{C}^n$ to the fiber through $x$ of $f|X$.

Parameswaran [31] generalized the Lê–Ramanujam theorem [23, Thm 2.1, p. 69] from a family of hypersurfaces to a family of germs with isolated complete-intersection singularities (ICIS germs) as follows. To an ICIS germ $(X, 0)$, he associated [31, Def. 1, p. 324] the sequence of numbers,

$$\mu_* := \mu_0, \mu_1, \ldots, \mu_k,$$
where \( k \) is the embedding codimension and where \( \mu_i \) is the smallest Milnor number of any ICIS germ that serves as the total space of a flat deformation of \((X, 0)\) with a smooth parameter space of dimension \( i \). Parameswaran noted [31, Rmk., p. 324] that \( \mu_k = 0 \) and that \( \mu_i > 0 \) for \( i < k \). Given a chain (or nested sequence) of deformations \((X_i, 0) \to (Y_i, 0)\) for \( 1 \leq i \leq k \) such that each \( Y_i \) is of dimension \( i \), call the chain \( \mu_\ast\)-minimal if the Milnor number of \((X_i, 0)\) is equal to \( \mu_i \).

Parameswaran proved [31, Lem. 3, p. 325] that there exists a \( \mu_\ast\)-minimal chain where each \( Y_i \) is smooth. He said [31, Def. 6, p. 331] that two ICISs have the same topological type if each has a \( \mu_\ast\)-minimal chain such that the two chains are embedded homeomorphic, and he proved that this notion does not depend on the choice of chains. Finally, Parameswaran proved [31, Thm. 2, p. 332] that, in a family of ICISs of dimension other than 2, the topological types of the members are the same if their \( \mu_\ast\)-sequences are the same; this is his generalization of the Lé–Ramanujam theorem.

Our next result stands to the Lé–Saito theorem [24, Thm., p. 793] as Parameswaran’s result stands to the Lé–Ramanujam theorem; in fact, our result also asserts the converse to the Lé–Saito theorem, and generalizes it. Consider a family of ICIS germs, \((X, 0) \to (Y, 0)\) with section \( \sigma(Y, 0) \to (X, 0) \) where \( Y \) is smooth, and let \( k \) be the embedding codimension of \((X(0), 0)\). Parameswaran constructed a chain of deformations \((X_i, 0) \to (Y_i, 0)\) over \( 0 \in Y \) form a \( \mu_\ast\)-minimal chain (see the beginning of the proof of [31, Thm. 2, p. 332]). Call such a chain a Parameswaran chain if in addition, for each \( y \in Y \) in a neighborhood of 0, the fibers \((X_i(y), 0) \to (Y_i(y), 0)\) over \( y \in Y \) form a \( \mu_\ast\)-minimal chain. Parameswaran also noted that the latter condition holds if the \( \mu_\ast\)-sequence of \((X(y))\) is constant in \( y \) on a neighborhood of 0 in \( Y \).

Fix a Parameswaran chain. Then \( \mu_k = 0 \); so \((X_k, 0)\) may be identified with \((\mathbb{C}^n, 0)\) where \( n := \dim X_k \). For convenience, set \( X_0 := X \) and \( Y_0 := Y \). For \( 0 \leq i < k \), let \( f_i : (X_k, 0) \to (\mathbb{C}^1, 0) \) be a function that cuts \( X_i \) out of \( X_{i+1} \). Finally, let \( S \) denote the singular locus of \( X \). Then, for \( i < k \), the singular locus of \( X_i \) is also \( S \), and \( f_i | X_{i+1} \) is a submersion off \( S \). Moreover, the image of \( Y \) under \( \sigma \) lies in \( S \), and \( S \) is finite over \( Y \).

**Theorem (5.2)** In the above setup, the pair \((X_{i+1} - \sigma(Y), \sigma(Y))\) satisfies Thom’s Condition \( A_{f_i} \) at the origin for \( 0 \leq i < k \) if and only if the \( \mu_\ast\)-sequence of \((X(y), 0)\) is constant in \( y \) on a neighborhood of 0 in \( Y \).

**Proof.** Let \( h : X_k \to Y \) be the projection. For \( 0 \leq i < k \), set

\[
F_i := (f_{k-1}, \ldots, f_{i+1}) \quad \text{and} \quad \mathcal{M}_i := JM(F_i; f_i)_h \subset \mathcal{O}_{X_{i+1}}^{k-i},
\]

and let \( J_i \) be the zeroth Fitting ideal of \( \mathcal{O}_{X_{i+1}}^{k-i} / \mathcal{M}_i \). Then, by the theorem of Lé [22, Thm. 3.7.1, p. 130] and Greuel [12, Kor. 5.5, p. 263], the colength of the induced ideal \( J_i \mathcal{O}_{X_{i+1}}(y, 0) \) is equal to the sum of the Milnor numbers of the germs.
\((X_{i+1}(y), 0)\) and \((X_i(y), 0)\). Hence the colength is independent of \(y \in Y\) near 0 for all \(i\) if and only if the Milnor numbers are so, since the Milnor numbers are upper semicontinuous by \(29\), bot. p. 126.

Now, \(X_{i+1}(y)\) has dimension \(d + i + 1\) where \(d\) is the dimension of \(X(0)\), and \(\mathcal{M}_i\) is generated by \(d + k\) sections; hence, by virtue of some theorems of Buchsbaum and Rim [3, 2.4, 4.3, 4.5], the colength of the ideal \(J_0\mathcal{O}_{X_{i+1}(y),0}\) is equal to the Buchsbaum–Rim multiplicity, \(e(i, y)\) say, of the image of \(\mathcal{M}_i\) in \(\mathcal{O}_{X_{i+1}(y),0}\). Therefore, since the chain is Parameswaran, the \(\mu_+\)-sequence of \((X(y), 0)\) is independent of \(y\) near 0 if and only if, for each \(i\), the multiplicity \(e(i, y)\) is independent of \(y\) near 0.

The next part of the proof has some similarities with the beginning of the proof of Theorem (4.2). Assume for the moment that \(S = \sigma(Y)\). Since \(S\) is smooth, there exists a map germ \(g: (X_k, 0) \to (C^l, 0)\), where \(l := \text{cod}(S, X_k)\), such that \((S, 0) = (g^{-1}(0), 0)\). Moreover, by [17, Thm. 1, p. 242], Thom’s Condition \(A_{f_i}\) is satisfied by \((X_{i+1}, S)\) at \(s\) for all \(s\) in some dense Zariski open subset \(U_i\) of \(S\). Hence, by Proposition (5.1), given \(s \in U_i\),

\[
JM(F_i; f_i)g \subset JM(F_i; f_i)\uparrow \text{ at } s, \tag{5.2.1}
\]

and this relation holds at \(s = 0\) if and only if \((X_{i+1}, S)\) satisfies \(A_{f_i}\) at 0. Since the projection \(S \to Y\) is an isomorphism, the sum \(JM(F_i; f_i)g + \mathcal{M}_i\) is all of \(JM(F_i; f_i)\). Hence, by Part (i) \(\Rightarrow\) (ii) of Proposition (3.2) and Proposition (3.1),

\[
JM(F_i; f_i) = \overline{\mathcal{M}_i} \text{ at } s \tag{5.2.2}
\]

for \(s \in U_i\), and this relation holds at \(s = 0\) if \((X_{i+1}, S)\) satisfies \(A_{f_i}\) at 0.

Now, assume that the pair \((X_{i+1} - \sigma(Y), \sigma(Y))\) satisfies \(A_{f_i}\) at 0 for each \(i\); in particular, this assumption means, by convention, that \(\sigma(Y)\) is smooth, so \(S = \sigma(Y)\). Then, (5.2.2) holds at \(s = 0\); so \(\mathcal{M}_i\) is a reduction of \(JM(F_i; f_i)\) over a neighborhood of 0 in \(Y\). Hence the cosupport of \(\mathcal{M}_i\) is just \(\sigma(Y)\). Since \(\mathcal{M}_i\) is generated by the right number of sections, Proposition (1.5)(3) says that \(e(i, y)\) is independent of \(y\) near 0, hence constant along \(\sigma(Y)\). Therefore, by the first paragraph, the \(\mu_+\)-sequence of \((X(y), 0)\) is independent of \(y\) near 0.

Conversely, assume that the \(\mu_+\)-sequence of \((X(y), 0)\) is independent of \(y\) near 0. Then so is the multiplicity \(e(i, y)\) for each \(i\) by the conclusion of the first paragraph. Consider the multiplicity of the image of \(\mathcal{M}_i\) in \(\mathcal{O}_{X_{i+1}(y),i}^{k-i}\); it is the sum of the multiplicities at each point of the fiber \(X_{i+1}(y)\). So it is at least \(e(i, y)\), and the two are equal at \(y = 0\). However, the former is upper semicontinuous in \(y\) by Proposition (1.1). Hence, the two are equal for all \(y\) near 0. Therefore, the cosupport of \(\mathcal{M}_i\) is equal to \(\sigma(Y)\) over a Zariski open subset of \(Y\), which we may assume is all of \(Y\). Hence \(S = \sigma(Y)\). Hence \(\mathcal{M}_i\) is a reduction of \(JM(F_i; f_i)\) over \(h(U_i)\) because (5.2.2) holds for \(s \in U_i\). Therefore, Theorem (1.8) implies that \(\mathcal{M}_i\) is a reduction of \(JM(F_i; f_i)\).

Form the ideal \(\rho(\mathcal{M}_i)\) on \(X_{i+1} \times \mathbb{P}^{k-i-1}\). Form the corresponding normalized blowup, its structure map,

\[
\pi: NB_{\rho(\mathcal{M}_i)}(X_{i+1} \times \mathbb{P}^{k-i-1}) \to X_{i+1} \times \mathbb{P}^{k-i-1},
\]
and its exceptional divisor $E$. Since $\mathcal{M}_i$ is a reduction of $JM(F_i; f_i)$, Proposition (3.5)(1) yields the inclusion,

$$\rho(JM(F_i; f_i)_g) \circ \pi \subset \rho(\mathcal{M}_i) \circ \pi.$$  (5.2.3)

Moreover, this inclusion is strict along each component of $E$ that projects onto $S$ because (5.2.1) holds for $s \in U_i$. Finally, to complete the proof, it suffices, by Proposition (5.1), to prove that the inclusion is strict along each remaining component $E_1$ of $E$.

Let $D$ be the exceptional divisor of the blowup itself of $X_{i+1} \times \mathbf{P}^{k-i-1}$ along $\rho(\mathcal{M}_i)$. Then $E$ is the preimage of $D$. Since $\mathcal{M}_i$ has $d + k$ generators, $D$ lies in $\mathbf{P}^{d+k-1} \times S \times \mathbf{P}^{k-i-1}$. Now,

$$\dim E_1 = \dim(X_{i+1} \times \mathbf{P}^{k-i-1}) - 1 = d + k - 1 + \dim Y,$$

and $\dim S = \dim Y$. Also, $E_1$ does not project onto $S$. Hence $E_1$ cannot project onto a point of $\mathbf{P}^{k-i-1}$. Therefore, if $k = 1$, then no such $E_1$ can exist, and the proof is complete in this case.

The proof proceeds by induction on $k$. Let $L$ be a general hyperplane in $\mathbf{P}^{k-i-1}$. Then the intersection

$$\mathbf{P}^{d+k-1} \times X_{i+1} \times L \bigcap B_{\rho(\mathcal{M}_i)}(X_{i+1} \times \mathbf{P}^{k-i-1})$$

is equal to the blowup $B$ of $X_{i+1} \times L$ along the ideal $\rho$ induced by $\rho(\mathcal{M}_i)$. Then $B$ contains the image $b$ of a general point of $E_1$ because $L$ does. Choose a map $\beta: (C, 0) \to (B, b)$ whose image does not lie entirely in the exceptional divisor, and let

$$\varphi: (C, 0) \to (X_{i+1}, 0) \text{ and } \psi: (C, 0) \to (\text{Hom}(C^{k-i}, C), \lambda)$$

be the maps arising from the composition of $\beta$ and the structure map $B \to X_{i+1} \times \mathbf{P}^{k-i-1}$. If (5.2.3) is not strict along $E_1$, then

$$\psi(\rho(JM(F_i; f_i)_g) \circ \varphi) = \psi(\mathcal{M}_i \circ \varphi).$$  (5.2.4)

Since $L$ is general, it is spanned by $k - i - 1$ general points. These points correspond to $k - i - 1$ general linear combinations $g_{k-2}, \ldots, g_i$ of the $k - i$ functions $f_{k-1}, \ldots, f_i$, in fact, to combinations of the functions cutting the $Y_i$ out of the $Y_{i+1}$. These functions define a chain of deformations $X'_j/Y'_j$ for $0 \leq j \leq k - 1$ where $X'_0 = X_0$ and $Y'_0 = Y_0$. For $j < k - 1$, the singular locus of $X_j$ is $S$, and $f_j|X_{j+1}$ is a submersion off $S$. Moreover, by the upper semicontinuity of Mihlom numbers, the chain is Paranasaran, and the $\mu$-sequence of $(X'_j(y), 0)$ is independent of $y$ in a neighborhood of $0$ in $Y$. Thus the induction hypothesis applies. Set $G_{i+1} := (g_{k-1}, \ldots, g_{i+2})$. Then (5.2.4) becomes

$$\psi(\rho(JM(G_{i+1}; g_{i+1})_g) \circ \varphi) = \psi(\rho(JM(G_{i+1}; g_{i+1})_h) \circ \varphi).$$

However, this equation contradicts the induction hypothesis; indeed, thanks to Lemma (3.3) and Proposition (5.1), the equation implies that $A_g_{i+1}$ is not satisfied by the pair $(X_{i+1} - S, S)$ at $0$. The proof is now complete.
Briançon, Maisonobe and Merle found a relation between Whitney’s Condition A and Thom’s Condition A, while working at the level of the total space [1, Thm. 4.2.1, p. 541]. In essence, they proved this. Consider a pair \((X,Y)\) consisting of an analytic subspace \(X\) of \(\mathbb{C}^n\), and a linear subspace \(Y\) contained in \(X\). Consider a function germ \(f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\). Set \(Z := f^{-1}(0) \cap X\), and assume that \(Z\) contains \(Y\). If both \(X - Y\) and \(Z - Y\) are smooth, if both pairs, \((X - Y, Y)\) and \((Z - Y, Y)\), satisfy Whitney’s Condition A along \(Y\), if \(f\) is submersive on \(X - Y\), and if, given the germ of any linear retraction, the restriction \(r|((X,Z,Y)\) is stratified locally topologically trivial, then the pair \((X - Y, Y)\) satisfies Thom’s Condition A along \(Y\).

Our next result, Theorem (5.3) shows that, when \(X\) is a complete intersection and \(Y\) is its singular locus, then the condition of stratified triviality can be replaced by a numerical condition, which is not only sufficient, but also necessary. The numerical condition is this: for every \(r\) and for all \(y \in Y\), the Buchsbaum–Rim multiplicity,
\[
e(r, y) := e(JM(F; f)_{(r,y)}),
\]
is defined and constant in \(y\), where \(F: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)\) defines \(X\) as a complete intersection and where \(JM(F; f)_{(r,y)}\) stands for the image of \(JM(F; f)_r\) in \(\mathcal{O}_{(r^{-1}(y) \cap X), 0}^{p+1}\).

In Theorem (5.3), it is unnecessary to assume that Whitney’s Condition A is satisfied. Moreover, in Corollary (5.4), we recover the original theorem of Briançon, Maisonobe and Merle for ICIS germs in a refined form: Condition A need be satisfied simply at 0, and the topological trivializations of \(r|((X,Y)\) and \(r|((Z,Y)\) need not be compatible.

Furthermore, when Condition A is satisfied simply by \((Z - Y, Y)\) at 0, then a much weaker condition will do. It requires the constancy of \(e(r, y)\) only for a single \(r\). In other words, the condition depends only on the individual fibers of \(r|X\) and not on how they fit together to form \(X\). See Theorem (5.5). In fact, Condition A is unnecessary here too, as Massey and the first author proved in [9, (5.8)] after the present work was completed; see also [19] and [21, (1.7)].

The Buchsbaum–Rim multiplicity \(e(r, y)\) is defined if and only if the ideal \(JM(F; f)_{(r,y)}\) of \(\mathcal{O}_{X,y}\) has finite colength; hence, since \(X\) is a complete intersection, if and only if the germs of the fibers of the restriction \(r|Z\) have isolated singularities. If so, then the germs of the fibers of the restriction \(r|X\) have isolated singularities too, and the following Milnor numbers are defined:
\[
\mu((r^{-1}(y) \cap X), 0) \text{ and } \mu((r^{-1}(y) \cap Z), 0).
\]
These two Milnor numbers sum to \(e(r, y)\) thanks to the theorem of Lê and Greuel and some theorems of Buchsbaum and Rim; see the first paragraph of the
proof of Theorem (5.2). Since the two Milnor numbers are upper semicontinuous by [29, bot. p. 126], they are independent of \( y \) if and only if \( e(r, y) \) is so. Hence we may reformulate the three results below that involve the independence of \( e(r, y) \) by replacing this condition with the independence of the two Milnor numbers.

**Theorem (5.3)** In the setup of Briançon, Maisonobe and Merle described above, assume that \( X \) is a complete intersection, and that both \( X - Y \) and \( Z - Y \) are smooth. Then the critical set \( \Sigma(f) \) represents the same germ as \( Y \), and the pair \((X - Y, Y)\) satisfies \( A_f \) at 0 if and only if, for every linear retraction \( r: (\mathbb{C}^n, 0) \to (Y, 0) \), the Buchsbaum–Rim multiplicity \( e(r, y) \) is defined and is independent of \( y \) for all \( y \in Y \) near 0.

**Proof.** To a certain degree, the proof is similar to those of Theorems (4.2) and (5.2). Let \( g: (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0) \) be a map germ such that \((Y, 0) = (g^{-1}0, 0)\) and \( l := \text{cod}(Y, \mathbb{C}^n) \). Then, for all \( y \) in some dense Zariski open subset \( U \) of \( Y \), the pair \((X - Y, Y)\) satisfies \( A_f \) at \( y \), and so

\[
JM(F; f)_g \subset JM(F; f)^\dagger \text{ at } y \in U
\]

by Proposition (5.1)(3). Hence, for every retraction \( r \), Proposition (3.2) implies that \( JM(F; f)_r \) is a reduction of \( JM(F; f) \) at \( y \in U \).

Fix \( r \), and assume that \( e(r, y) \) is defined and independent of \( y \in Y \). Consider the image of \( JM(F; f)_r \) in \( \mathcal{O}^{p+1}_{r^{-1}(y) \cap X} \). It has finite colength for \( y = 0 \), so for all \( y \) in a neighborhood of 0, which we may assume is all of \( Y \). Since \( JM(F; f)_r \) is generated by the right number of sections, Proposition (1.5)(3) yields the constancy of its multiplicity. This multiplicity is the sum of the multiplicities at each point of the fiber \( r^{-1}(y) \cap X \). So it is at least \( e(r, y) \), and the two are equal at \( y = 0 \). Hence, the two are equal for all \( y \) near 0. Therefore, the cospupport of \( JM(F; f)_r \) is equal to \( Y \). Since this cospupport, \( \Sigma(f) \), and \( Y \) are always nested, the three are equal. In particular, \( \Sigma(f) \) represents the same germ as \( Y \).

Therefore, Theorem (1.8) implies that \( JM(F; f)_r \) is a reduction of \( JM(F; f) \) everywhere. Hence Lemma (5.1)(1) implies that no hyperplane containing \( Ker r \) is a limit tangent hyperplane of the fibers of \( f|X \). Now, given a hyperplane \( H \) that does not contain \( Y \), there exists a retraction \( r: \mathbb{C}^n \to Y \) such that \( H \) contains \( Ker r \). Therefore, \((X - Y, Y)\) satisfies Thom’s Condition \( A_f \) at the origin.

Conversely, assume that \( \Sigma(f) \) represents the same germ as \( Y \), and that the pair \((X - Y, Y)\) satisfies \( A_f \) at 0. Then \( JM(F; f)_g \) is contained in \( JM(F; f)^\dagger \) by Proposition (5.1)(3). Hence, for every retraction \( r \), Proposition (3.2) implies that \( JM(F; f)_r \) is a reduction of \( JM(F; f) \) at 0. Since \( JM(F; f)_r \) is generated by the right number of sections, Proposition (1.5)(3) yields the constancy of \( e(r, y) \) for all \( y \in Y \) near 0. The proof is now complete.

**Corollary (5.4)** In the setup of Briançon, Maisonobe and Merle described above, assume that \( X \) is a complete intersection, that both \( X - Y \) and \( Z - Y \) are smooth, and that both pairs \((X - Y, Y)\) and \((Z - Y, Y)\) satisfy Whitney’s
Condition $A$ at 0. Assume that, for every retraction $r$, the restrictions $r|(X,Y)$ and $r|(Z,Y)$ are topologically trivial. Then $(X - Y,Y)$ satisfies $A_f$ at 0.

**Proof.** We are about to prove that, after $X$ is replaced by a smaller representative, the Milnor numbers, $\mu((r^{-1}(y) \cap X), 0)$ and $\mu((r^{-1}(y) \cap Z), 0)$, are independent of $y$. Hence, by the discussion just before Theorem (5.3), the Buchsbaum–Rim multiplicity $e(r,y)$ is independent too. Hence the assertion follows from Theorem (5.3).

Since $(X - Y,Y)$ satisfies $A$ at 0, it follows that, over a sufficiently small neighborhood of 0 in $Y$, the fibers of the restriction of $r|X$ are smooth except at points of $Y$. Indeed, reasoning as in the last paragraph of the proof of Theorem (5.3), but using Lemma (4.1)(3) in place of Proposition (3.2), we find that, on a sufficiently small neighborhood, $JM(F)_r$ is a reduction of $JM(F)$. Hence, the cosupports of both these modules represent the same germ; otherwise, there would be a map of germs $\varphi: (C,0) \to (X,0)$ whose image lies in the former cosupport, but not in the latter, and then the pullbacks, $JM(F)_r \circ \varphi$ and $JM(F) \circ \varphi$ would not be equal. However, the former cosupport is the singular locus $Y$ of $X$, and the latter cosupport is the union of the singular loci of the fibers $r^{-1}(y) \cap X$ of $r|X$.

Let $\Phi_y$ be the Milnor fiber of $r^{-1}(y) \cap X$, at its only singular point $y$. If $y$ is close enough to 0, then there is a short exact sequence of reduced integral homology groups,

$$0 \to \widetilde{H}_{n-p}(\Phi_y) \to \widetilde{H}_{n-p}(\Phi_0) \to \widetilde{H}_{n-p}(r^{-1}(y) \cap X) \to 0;$$

it is obtained from a versal deformation of $(X,0)$, see the top of p. 121 in [29]. Hence, if the Milnor number of $r^{-1}(0) \cap X$ is strictly greater than that of $r^{-1}(y) \cap X$, then the first map cannot be surjective, and so $r^{-1}(y) \cap X$ is not contractible.

Replacing $X$ with a smaller representative, we may assume that $r^{-1}(0) \cap X$ is contractible by [29, (2.4)]. Then $r^{-1}(y) \cap X$ is contractible too, since the restriction $r|(X,Y)$ is topologically trivial. Therefore, the Milnor number of $r^{-1}(y) \cap X$ is independent of $y$. Similarly, the Milnor number of $r^{-1}(y) \cap Z$ is independent of $y$ for $y$ near 0, and the proof is complete.

**Theorem (5.5)** In the setup of Briançon, Maisonobe and Merle described above, above, assume that $X$ is a complete intersection, that both $X - Y$ and $Z - Y$ are smooth, and that the $(Z - Y,Y)$ satisfies Whitney’s Condition $A$ at 0. Then the following conditions are equivalent:

(i) the critical set $\Sigma(f)$ represents the same germ as $Y$, and the pair $(X - Y,Y)$ satisfies $A_f$ at 0;

(ii) for every linear retraction $r: (C^n, 0) \to (Y,0)$, the Buchsbaum–Rim multiplicity $e(r,y)$ is independent of $y$ for all $y \in Y$ near 0;

(ii') for some linear retraction $r: (C^n, 0) \to (Y,0)$, the Buchsbaum–Rim multiplicity $e(r,y)$ is independent of $y$ for all $y \in Y$ near 0.

**Proof.** Condition (i) implies (ii) by Theorem (5.3), and trivially (ii) implies (ii'). So assume (ii'), and let’s prove (i). Note that the proof of Theorem (5.3)
yields this: $\Sigma(f)$ represents the same germ as $Y$, and no hyperplane containing $\text{Ker}(r)$ is a limit tangent hyperplane of the fibers of $f|X$. Now, in $\mathbf{P}^{n-1}$, the subspace of hyperplanes containing $\text{Ker}(r)$ has dimension $k-1$ where $k := \dim Y$. Since this subspace doesn’t meet the space of limit tangent hyperplanes of the fibers of $f|X$, the latter space must have dimension at most $n - k - 1$.

We now use an observation due to D. Massey and M. Green (pers. com.). Form the relative conormal variety $C(X, f)$: by definition, it is the closure in $\mathbf{C}^n \times \mathbf{P}^{n-1}$ of the locus of the pairs $(x, H)$ where $x$ is a simple point of the level hypersurface surface $(f^{-1}fx) \cap X$ and $H$ is a tangent hyperplane at $x$. Intersect $C(X, f)$ with the hypersurface $(f^{-1}0) \times \mathbf{P}^{n-1}$. Each component must have dimension at least $n - 1$ because $C(X, f)$ has dimension $n$. Hence no component can project onto a proper subset of $Y$; otherwise, the fiber of $C(X, f)$ over 0 would have dimension at least $n - k$, but this fiber is simply the space of limit tangent hyperplanes of the fibers of $f|X$, and so it has dimension at most $n - k - 1$ by the paragraph above. Moreover, if $(x, H)$ is a point of the intersection with $x \in Z - Y$, then $H$ must be tangent to $Z$ because $f$ is a submersion off $Y$ by hypothesis. Thus, each component of the intersection either surjects onto $Y$ or lies in the conormal variety $C(Z)$; the latter is, by definition, the closure of the locus of the pairs $(x, H)$ where $x$ is a simple point of $Z$ and $H$ is a tangent hyperplane at $x$.

By hypothesis, $(Z, Y)$ satisfies Whitney’s Condition A at 0; in other words, the preimage of $Y$ in $C(Z)$ lies in $C(Y)$, the space of hyperplanes containing $Y$. Moreover, for all $y$ in a Zariski open subset of $Y$, the pair $(X - Y, Y)$ satisfies $A_f$ at $y$; in other words, the fiber $C(X, f)(y)$ lies in $C(Y)$. Hence, any irreducible subset of $C(X, f)$ that projects onto $Y$ must lie in $C(Y)$. Therefore, each component of the intersection above lies in $C(Y)$. So $C(X, f)(0)$ lies in $C(Y)$; in other words, $(X - Y, Y)$ satisfies $A_f$. The proof is now complete.

If $Y$ has dimension 1, then we have the following version of Theorem (5.5). It is a numerical criterion for Thom’s Condition $A_f$, which involves only a single retraction $r$ and not Whitney’s Condition $A$.

**Corollary (5.6)** In the setup of Briançon, Maisonobe and Merle described above, assume that $X$ is a complete intersection, that both $X - Y$ and $Z - Y$ are smooth, and that $Y$ has dimension 1. Assume that, for each hyperplane $H$ transverse to $Y$ at the origin, the Milnor numbers of $H \cap Z$ and the Milnor number of a general hyperplane slice are independent of $H$. Then the following conditions are equivalent:

(i) the critical set $\Sigma(f)$ represents the same germ as $Y$, and the pair $(X - Y, Y)$ satisfies $A_f$ at 0;
(ii) for some linear retraction $r: (\mathbf{C}^n, 0) \rightarrow (Y, 0)$, the Buchsbaum–Rim multiplicity $e(r, y)$ is independent of $y$ for all $y \in Y$ near 0;
(ii') for every linear retraction $r: (\mathbf{C}^n, 0) \rightarrow (Y, 0)$, the Buchsbaum–Rim multiplicity $e(r, y)$ is independent of $y$ for all $y \in Y$ near 0.

**Proof.** The hypothesis on the Milnor numbers is exactly what’s needed to
conclude by Corollary 3.9 of [7] that Whitney’s Condition A holds for the pair $(Z - Y, Y)$ at 0. The result now follows from Theorem (5.5).

Here is the idea behind the proof of Corollary 3.9 of [7]. Because $Y$ has dimension 1, a hyperplane $H : h = 0$ that does not contain $Y$ can intersect $Y$ only at 0. Now, [7] makes a study of the hyperplane sections of $Z$ at 0 by hyperplanes transverse to $Y$. On the basis the principle of specialization of integral dependence, Theorem (1.8) above, it is shown in Proposition 2.7 and Theorem 3.3 of [7] that the hypothesis on the Milnor numbers implies that the submodule $JM(F, f)_h$ is a reduction of $JM(F, f)$ in $O_{Z,0}^{p+1}$. Hence Lemma (4.1)(3) implies that $H$ is not a limiting tangent hyperplane to $Z$ at 0.

6. The relative condition $W_f$

In this section, given a map germ $f$ on $(X, 0)$, we study the condition $W_f$. It is a standard relative form of Whitney’s Condition B, and reduces to B, in the form of Verdier’s Condition $W$ [42, Sect. 1], when $f$ is constant. In our first result, $X$ and $f$ are arbitrary, but then we begin specializing as more hypotheses are needed. In fact, we proceed to observe that $W_f$ is a rather strong condition unless $f$ is a function germ, and from then on, we assume that $f$ is a function. Finally, in our last two result, we assume that $X$ is the total space of a family of ICIS germs.

Condition $W_f$ generalizes Teissier’s condition of ‘c-equisingularity’ (see [26, top, p. 550]). It strengthens Thom’s Condition $A_f$, and so is sometimes called the strict Thom condition. Although $W_f$ is defined using Euclidean distances, we prove in Proposition (6.1) that $W_f$ is equivalent to a condition of integral dependence on a modified Jacobian module, obtained by “vertical” differentiation. Thus this module becomes the natural source for numerical invariants that depend only on the members of a family $X/Y$, rather than on the total space $X$.

The Thom–Mather second isotopy lemma readily implies that, if $f$ is a nonconstant function and if $W_f$ is satisfied, then the pair $X, f$ is topologically right trivial over $Y$. Indeed, Thom, Mather, Teissier, Verdier, and others introduced and developed methods of integrating vector fields that yield this triviality. Namely, we can lift to $X$ a constant vector field tangent to $Y$ so that the lift is corrugated (Fr. rugueux), hence integrable, and is tangent to the fibers of $f$ on $X/Y$, so that the integral gives a continuous flow on $X$. If we choose the field carefully, we can show that, after $X$ is replaced by a neighborhood of 0, there is a homeomorphism $h: X(0) \times Y \to X$ such that $fh = (f|X(0)) \times 1_Y$, as required. Similarly, it is possible to generalize the statements and proofs of Corollary 3.6 and Theorem 3.8 of [6].

Our main result, Theorem (6.4), characterizes $W_f$, when $X$ is a family of ICIS germs and $f$ is a nonconstant function, in three ways: (1) by the constancy of the Buchsbaum–Rim multiplicity of a modified Jacobian module, (2) by the constancy of two sequences of Milnor numbers, and (3) by the fulfillment by
two pairs, of the absolute Whitney conditions. The necessity of (3) is trivial; its sufficiency is not new, but was established by Briançon, Maisonobe and Merle in [1, Thm. 4.3.2, p. 543] in a more general setting using a different approach.

We prove the theorem using Proposition (6.1) and Lemma (6.3). The latter expresses the Buchsbaum–Rim multiplicity in (1) as the weighted sum of the Milnor numbers. This lemma is proved using the polar multiplicity formula and the relative polar transversality result of Henry and Merle. The latter is given a new proof in Lemma (6.2), and this proof illustrates, for a second time, the usefulness of Proposition (6.1) and of the methods of integral dependency.

A lemma similar to Lemma (6.3) is involved implicitly in the proof of one of the main results, Theorem 1, in [8]. That proof does not rely on the principle of specialization of integral dependence, our Theorem (1.8); indeed, the principle had not yet been established. However, the principle yields a new proof of the implication (iv) ⇒ (iii) of [8, Thm. 2], and this proof is in the spirit of Teissier’s original proof [34] for the case of a hypersurface. In fact, the new proof is simply a special case of the first part of the proof of the implication (iv) ⇒ (i) of our Theorem (6.4); it is the case where \( f \) is constant (so vanishes). On the other hand, (as the referee pointed out), it is possible to prove this implication in the spirit of [8], using ordinary multiplicities of polar varieties.

To begin the formal discussion, fix a pair \((X,Y)\) consisting of a reduced equidimensional analytic subspace \(X\) of \(\mathbb{C}^n\) and a linear subspace \(Y\) of \(\mathbb{C}^n\) contained in \(X\). Assume \(l > p + q\). Fix a map germ \(f: (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)\) whose restriction \(f|_Y(0)\) is a submersion onto a smooth closed analytic subgerm of \((\mathbb{C}^q, 0)\), and assume that there is a smooth, dense, and open analytic subset \(X_0\) of \(X\) such that \(f|_{X_0}(0)\) is a submersion onto its image and has equidimensional fibers.

Recall from Definition 1.3.7 on p. 550 in [26] (compare [16, pp. 228–9]) that \((X_0, Y)\) satisfies the condition \(W(f)\) at 0 if there exist a (Euclidean) neighborhood \(U\) of 0 in \(X\) and a constant \(C > 0\) such that, for all \(y\) in \(U \cap Y\) and all \(x\) in \(U \cap X_0\), we have

\[
\text{dist}(T_yY(f(y)), T_xX(f(x))) \leq C \text{ dist}(x, Y)
\]

where \(T_yY(f(y))\) and \(T_xX(f(x))\) are the tangent spaces to the indicated fibers of the restrictions \(f|_Y\) and \(f|_X\). This condition depends only on the restrictions \(F|X\) and \(f|X\), and not on the embeddings of \(X\) into \(\mathbb{C}^n\) and of \(f(X)\) into \(\mathbb{C}^q\).

Conditions like \(W(f)\), which are defined by analytic inequalities, often can be re-expressed algebraically in terms of integral dependence. For \(W(f)\) itself, this job was done by Navarro in a 1980 unpublished manuscript according to Remarque 1.2(c) on p. 229 of [16]. Later the job was done in print by Lê and Teissier. In Proposition 1.3.8 on p. 550 of [26], they translated \(W(f)\) into a condition of integral dependence between ideals on the relative conormal variety \(C(X, f)\), whose definition was recalled in the proof of Theorem (5.5). We recover their result below in Proposition (6.1).

Proposition (6.1) also gives another condition of integral dependence equivalent to \(W(f)\), and this is the condition of importance to us here. It is the condition
mentioned above, requiring that one modified Jacobian module be dependent on another. Before we can state and prove the proposition formally, we must define these modules precisely.

Say that \((X, 0)\) is defined by the vanishing of \(F: (C^n, 0) \to (C^p, 0)\). Generalizing the constructions in Section 5, form the corresponding augmented Jacobian module \(JM(F; f)\); namely, first form the \(p+q\) by \(n\) matrix \(D(F; f)\) by augmenting the Jacobian of \(F\) at the bottom with the Jacobian of \(f\); then \(JM(F; f)\) is the \(O_X\)-submodule of the free module \(O_X^{p+q}\), generated by the columns of \(D(F; f)\).

Say \(C^n = C^l \times Y\), and form the corresponding projections,

\[ r: C^n \to Y \text{ and } g: C^n \to C^l. \]

Form the corresponding relative augmented Jacobian modules,

\[ JM(F; f)_r \text{ and } JM(F; f)_g; \]

by definition, these are the submodules of \(JM(F; f)\) generated by the partial derivatives with respect to the first \(l\) variables on \(C^n\) and with respect to the remaining \(n-l\) variables. Finally, let \(m_Y\) be the ideal of \(Y\) in \(C^n\).

The abstract module \(JM(F; f)\) is determined as a quotient of \(O_X^n\), but not as a submodule of \(O_X^{p+q}\), by the (germ of the) embedding of \(X\) in \(C^n\) and by the restriction \(X \to S\), where \(S\) is the image \(f(X)\) viewed as an abstract space. Namely, \(JM(F; f)\) is the unique torsion free quotient that restricts to the normal sheaf on \(X_0\). So the submodules \(JM(F; f)_g\) and \(JM(F; f)_r\) too are determined abstractly by \(X \to S\), given the splitting \(C^n = C^l \times Y\). Finally, it is clear from its definition that the relative conormal variety \(C(Y, f)\) is determined by the embedding of \(X\) in \(C^n\) and by the restriction \(X \to S\) of \(f\).

**Proposition (6.1)** In the setup above, the following conditions are equivalent:

(i) the pair \((X_0, Y)\) satisfies \(W_f\) at 0;

(ii) the module \(JM(F; f)_g\) is integrally dependent on \(m_Y JM(F; f)_r\);

(iii) the module \(JM(F; f)_r\) is integrally dependent on \(m_Y JM(F; f)_g\);

(iii) along the preimage in \((C(X, f)\) of 0, the ideal of \((C(Y, f) \cap C(X, f)\) is integrally dependent on the ideal of the preimage of \(Y\).

**Proof.** We’ll prove that (ii′) is equivalent to each of the other conditions. First consider the notion of integral dependency involved in (ii) and (ii′); it is defined abstractly in Section 1, but it can be treated as discussed in Section 3, using the embedding of \(JM(F; f)\) in \(O_X^{p+q}\).

That (ii) implies (ii′) is trivial. Conversely, assume (ii′). Then \(JM(F; f)_g\) is contained in the strict closure \(JM(F; f)\). Hence \(JM(F; f)\) is integrally dependent on \(JM(F; f)_r\), by Prop. (3.2) because \(JM(F; f)\) is the sum of \(JM(F; f)_r\) and \(JM(F; f)_g\). So \(m_Y JM(F; f)\) is integrally dependent on \(m_Y JM(F; f)_r\). Thus (ii′) implies (ii).

To prove the equivalence of (i) and (ii′), let \(e_1, \ldots, e_n\) be a vector space basis of \(C^n\), and \(f_1, \ldots, f_{n-1}\) one of \(Y\). Then the matrix products \(D(F, f) \cdot e_i\) generate \(JM(F; f)\), and the products \(D(F, f)_g \cdot f_j\) generate \(JM(F; f)_g\). Let \(y_1, \ldots, y_l\) be
a set of coordinate functions on $C^i$. Then the products $y_kD(F, f) \cdot e_i$ generate $m_Y JM(F; f)$. So Proposition 1.11 on p. 306 of [6] says that (ii’) holds if and only if the following condition holds: there exist a neighborhood $U'$ of 0 in $X$ and a constant $C' > 0$ such that, for any $\psi: U' \to \text{Hom}(C^{p+q}, C)$ and any $x \in U'$, we have

$$\sup_j |\psi(x) \cdot D(F, f)_g(x) \cdot f_j| \leq C' \sup_{i,k} |y_k(x)\psi(x) \cdot D(F, f)(x) \cdot e_i|.$$ 

The sup on the right is equal to

$$\sup_k |y_k(x)| \sup_i |\psi(x) \cdot D(F, f)(x) \cdot e_i|.$$ 

Adjusting the constant $C'$, we may replace the inequality above by

$$\|\psi(x) \cdot D(F, f)_g(x)\| \leq C' \text{dist}(x, Y)\|\psi(x) \cdot D(F, f)(x)\|.$$ 

Set $u := \psi(x) \cdot D(F, f)(x)$. Then this inequality holds if and only if, for every unit vector $v$ in $T_yY(f(y))$, the following inequality holds:

$$|(u, v)| \leq C' \text{dist}(x, Y)\|u\|. \quad (6.1.1)$$ 

Here, we may replace $u$ by its complex conjugate.

If (i) holds, then the preceding inequality (6.1.1) holds with $U' := U$ and $C' := C$, at least for an $x$ in $U \cap X_0$, because, by definition,

$$\text{dist}(A, B) := \sup_{u \in B^\perp - \{0\}} \sup_{v \in A - \{0\}} \frac{|(u, v)|}{\|u\|\|v\|}.$$ 

By continuity, the inequality (6.1.1) also holds for an $x$ in $X - X_0$, because the latter set is nowhere dense in $X$. Thus (i) implies (ii’).

Conversely, (ii’) implies (i). Indeed, given any $x \in U \cap X_0$ and $u \in B^\perp - \{0\}$, where $B := T_xX(f(x))$, there is a $\psi: U' \to \text{Hom}(C^{p+q}, C)$ such that $\psi(x)$ is equal to the conjugate of $u$; so we may take $U := U'$ and $C := C'$.

Finally, the equivalence of (iii) and (ii’) follows from the version of Proposition (3.4) for integral dependence given in Remark (10.8)(ii) on p. 229 of [20]. In the latter, $E$ is not necessarily locally free, but we take $P := \text{Projan}(\mathcal{RE})$. (The proof is entirely different, and does not involve any form of the valuative criterion.) In the case at hand, take $E$ to be $JM(F; f)$. Then $P$ is just $C(X, f)$.

The ideal of the preimage in $P$ of $Y$ is just $\rho(m_Y JM(F; f))$ because $m_Y$ is the ideal of $Y$. Furthermore, $\rho(JM(F; f)_g)$ is the ideal of $C(Y, f) \cap C(X, f)$, because a hyperplane $\{w = 0\}$ of $C^n$ contains $Y$ if and only if the coefficients of the last $n - l$ coordinate functions of $w$ are zero, and because the functions corresponding to these coefficients are given on $C(X, f)$ by the columns of $D(F, f)_g$. The proof is now complete.
Previous work on $W_f$ has involved the central fiber of the exceptional divisor of the blowup of the relative conormal variety $C(X, f)$ along the preimage of $Y$. For example, Henry, Merle and Sabbah proved in Conséquence 2 on p. 234 of [16] that, if this fiber is of minimal dimension and if $W_f$ holds generically on $Y$, then it also holds at 0. This conclusion also follows from Proposition (6.1); indeed, if the fiber is of minimal dimension and if (iii) holds generically, then it also holds at 0 by Böger’s celebrated criterion of integral dependence of ideals (see [20, (10.9)] and [21, (1.4)]) for the generalization of this criterion to modules. Conversely, assume that $f$ is a function or assume the more general condition of the “absence of blowup in codimension 0” of Section 4 of [16]. Then the fiber is of minimal dimension if $W_f$ holds everywhere on $Y$. This converse follows from Théorème 6.1 on p. 262 and Proposition 3.3.1 on p. 239 of [16]. Recently, the first author found a new proof using generic plane sections and methods of integral dependence; the details will appear elsewhere.

Unless $f$ is a function (as it will be in our remaining three results), $W_f$ is a very strong condition. It implies that $f$ is analytically right trivial already in this case: $X$ is $\mathbb{C}^n$, the critical set $\Sigma(f)$ of $f$ is reduced and is defined by the maximal minors of the Jacobian matrix $D(f)$, and, for all $y \in Y$, the restriction of the map germ $f(y) : (\mathbb{C}^l, 0) \to (\mathbb{C}^q, 0)$ to its critical set is a finite map onto its discriminant. Indeed, say $f$ is nontrivial. Then both these latter sets have dimension $q - 1$. Moreover, as $y$ varies, the union $\Sigma_Y(f)$ of these critical sets is equal to $\Sigma(f)$, because $\Sigma_Y(f)$ is the cosupport of $JM(\,; f)_r$ and $\Sigma(f)$ is the cosupport of $JM(\,; f)$; furthermore, the second module is integrally dependent on the first by Proposition (6.1). (In fact, (6.1) is stronger than necessary, and (5.1) will do after it is generalized from a function to a map, a straightforward job; thus, already $A_f$ implies the analytic triviality of $f$.)

Consider the map germ $F := (f, r)$, with target $(\mathbb{C}^q \times Y^k, 0)$. The critical set of $F$ is just $\Sigma_Y(f)$, and we just proved that the latter is equal to $\Sigma(f)$. Since $F$ is finite on $\Sigma(f)$, its discriminant, $\Delta(F)$, is a set of codimension 1. Now, $\Delta(F)$ projects onto the discriminant $\Delta(f)$, which is a proper subset of $(\mathbb{C}^q, 0)$. Hence $\Delta(F)$ is equal to $\Delta(f) \times Y$. Now, $\Sigma(F)$ is smooth of dimension $k + q - 1$ on a dense Zariski open subset $U$. Shrinking $U$ if necessary, on $U$, the rank of $D(F)$ is $k + q - 1$, again because $F$ is finite on its critical set. Hence, $\text{Ker} D(F)$ is transverse to $\Sigma(f)$ on $U$. Therefore, because maximal minors define $\Sigma(f)$ with reduced structure, $F$ is the unfolding of a Morse function at points of $U$. Hence, by Theorem 1 on p. 726 of [2], $F$ is analytically right trivial.

Next we turn to the transversality result. Let $P$ be a linear space through 0 in $\mathbb{C}^n$ of codimension $i$ say, with $i \leq \dim X$, and let $\Pi$ be the relative polar variety of $f|_X$ with $P$ as pole. By definition, $\Pi$ is the closure in $X$ of the locus of simple points $x$ of the level hypersurface surface $X(f(x))$ such that there exists a hyperplane that is tangent to $X(f(x))$ at $x$ and that contains $P$. In other words, $\Pi$ is the projection to $X$ of the preimage $\nu^{-1}P^*$ where $\nu : C(X, f) \to \mathbb{P}^{n-1}$ is the projection and $P^*$ is the set of hyperplanes containing $P$. So, if $P$ is general,
Let \( \pi: \mathbb{C}^n \to \mathbb{C}^i \) be a linear map with kernel \( P \). Assume \( f \) is a nontrivial function, and let \( \Sigma(f) \) be the critical locus of \( f \) (which includes the singular locus of \( X \)). Then \( \Pi \cup \Sigma(f) \) is cut out of \( X \) by the maximal minors of the Jacobian matrix of the map \( \mathbb{C}^n \to \mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^i \) with components \( F, f, \) and \( \pi \). Hence, if \( P \) is general and if \( \dim \Sigma(f) < i \), then \( \Pi \) is Cohen–Macaulay if \( X \) is.

Remarkably, although \( \Pi \) is defined using \( P \), nevertheless the two spaces are transverse at 0 if \( P \) is general and \( f \) is a nonconstant function. This is an important result. Related results were proved in the absolute case (the case where \( f \) is constant) by Teissier in [34, 2.7–2.9], in [35, Thm. 7, p. 623] and in [37, Thm. 1, p. 269] for a hypersurface \( X \), and by Lê and Teissier in [25, (4.1.8), p. 569] for an arbitrary \( X \). Teissier proved the relative result (where \( f \) is a nonconstant function) for an arbitrary \( X \) in [39, pp. 40–41], deriving it from his general idealistic Bertini theorem. This transversality result was also proved, at about the same time, by Henry and Merle [13, Cor. 2, p. 195].

The general relative polar transversality result is reproved next in a new way, using the theory of the \( W_f \) condition, especially Proposition (6.1) and the relative generic Whitney lemma. The latter was proved by Navarro, according to Henry, Merle, and Sabbah in Remarque 5.1.1 on p. 255 of [16], and they generalized it (using the normalized blowup of \( C(X, f) \) along the preimage of \( Y \) in the spirit of Hironaka and of Teissier) in their Théorème 5.1 on the same page.

**Lemma (6.2)** (Relative polar transversality) In the setup above, the relative polar variety \( \Pi \) and its pole \( P \) are transverse at 0 if \( P \) is general and \( f \) is a nonconstant function.

**Proof.** First note that the result is obvious if \( i = \dim X \), as then \( \Pi = X \). Now, consider the “Grassmann modification” \( \tilde{X} \), which is formed as follows. Let \( G \) be the Grassmann variety of all linear spaces of codimension \( i \) through 0 in \( \mathbb{C}^n \), let \( \tilde{C} \) be the tautological subbundle in \( \mathbb{C}^n \times \tilde{G} \), and let \( \alpha: \tilde{C} \to \mathbb{C}^n \) and \( \beta: \tilde{C} \to G \) be the projections. Set \( \tilde{X} := \alpha^{-1}X \) and \( \tilde{X}_0 := \alpha^{-1}X_0 \). Then \( \tilde{X}_0 \) is smooth since \( X_0 \) and \( \alpha \) are smooth. Moreover, since \( i < \dim X \), the 0-section \( 0 \times G \) of \( \tilde{C} \) lies in \( \tilde{X} \).

Set \( \tilde{f} := f \circ \alpha \). Then \( \tilde{f}|_{X_0} \) is a submersion onto its image and has equidimensional fibers because \( \alpha \) is smooth with equidimensional fibers. Hence, since \( \tilde{f} \) is a function, by the relative generic Whitney lemma, \( W_{\tilde{f}} \) is satisfied by \( (\tilde{X}_0, 0 \times G) \) at \( (0 \times P) \) if \( P \) lies in an appropriate dense Zariski open subset of \( G \). Assume \( P \) does so.

Proceeding via contradiction, assume that \( P \) and \( \Pi \) are not transverse at 0. Then there is a curve \( \phi: (\mathbb{C}, 0) \to (\Pi, 0) \) tangent to \( P \); moreover, in view of the definition of \( \Pi \), we may assume that, for \( u \neq 0 \), we have \( \phi(u) \in X_0 \). Since the projection \( C(X, f) \to X \) is proper, \( \phi \) lifts to a map germ \( \phi' \) from \( (\mathbb{C}, 0) \) to \( C(X, f) \) whose image lies in \( \nu^{-1}P^* \). So \( \nu \phi'(u) \) represents a hyperplane \( H_u \) containing \( P \) and, if \( u \neq 0 \), tangent to the fiber of \( f|X \) through \( \phi(u) \). Let \( h: \mathbb{C}^n \to \mathbb{C} \) be a linear functional whose kernel is equal to \( H_0 \). Then \( JM(F; f)_h \).
is not a reduction of $JM(F; f)$ by Lemma (5.1)(i); in fact, the proof shows that $JM(F; f)_h \circ \phi$ is not equal to $JM(F; f) \circ \phi$. We’ll now prove that they are equal since $W_f$ is satisfied; then we’ll have a contradiction.

In $\mathbb{C}^n$, choose an $i$-dimensional linear space $T$ through $0$ and transverse to $P$. Then the various $i$-codimensional spaces transverse to $T$ form a Zariski neighborhood of $P \in G$, which we may identify with $\text{Hom}(\mathbb{C}^i, \mathbb{C}^{n-i})$. In coordinates, a matrix $(a_{\mu,\nu})$ corresponds to the $i$-codimensional space with equations $t_\mu = \sum a_{\mu,\nu} z_\nu$ where $t_1, \ldots, t_i; z_1, \ldots, z_{n-i}$ are coordinates on $\mathbb{C}^n$ split as $T \times P$. Over $\text{Hom}(\mathbb{C}^i, \mathbb{C}^{n-i})$, the bundle $\tilde{C}$ is trivial, so equal to $P \times \text{Hom}(\mathbb{C}^i, \mathbb{C}^{n-i})$.

In coordinates, $\alpha: \tilde{C} \to \mathbb{C}^n$ becomes

$$
\alpha(z_1, \ldots, z_{n-i}; a_1,1, \ldots, a_{k,n-i}) = (z_1, \ldots, z_{n-i}; \sum a_{1,\nu} z_\nu, \ldots, \sum a_{k,\nu} z_\nu).
$$

Given a function germ $\gamma$ on $(\mathbb{C}^n, 0)$, note that we have

$$
\begin{align*}
\partial \gamma \circ \alpha/\partial a_{\mu,\nu} &= z_\nu \partial \gamma/\partial t_\mu \circ \alpha, \\
\partial \gamma \circ \alpha/\partial z_\nu &= \partial \gamma/\partial z_\nu \circ \alpha + \sum \mu a_{\mu,\nu} \partial \gamma/\partial t_\mu \circ \alpha.
\end{align*}
$$

(6.2.1) (6.2.2)

Set $\tilde{F} := F \circ \alpha$. Let $\delta$ denote the projection of $P \times \text{Hom}(\mathbb{C}^i, \mathbb{C}^{n-i})$ onto $P$, and let $m_G$ denote the ideal on $\tilde{X}$ generated by the $z_\nu$. Since $W_f$ is satisfied by $(\tilde{X}_0, 0 \times G)$ at $(0 \times P)$, by Proposition (6.1) the module $JM(\tilde{F}; \tilde{f})_\beta$ is integrally dependent on the product $m_G JM(\tilde{F}; \tilde{f})_\beta$. Now, Equation (6.2.1) implies that

$$
JM(\tilde{F}; \tilde{f})_\delta = m_G (JM(F; f)_\tau \circ \alpha)
$$

where $\tau: \mathbb{C}^n \to P$ denotes the projection with kernel $T$.

Since the curve $\phi$ is tangent to $P$ at $0$, there is a lift $\tilde{\phi}: (\mathbb{C}, 0) \to (\tilde{X}, 0 \times O)$. Hence, by the preceding paragraph,

$$
(m_G (JM(F; f)_\tau \circ \alpha)) \circ \tilde{\phi} \subseteq (m_G JM(\tilde{F}; \tilde{f})_\beta) \circ \tilde{\phi}.
$$

The term on the left is equal to $(m_G \circ \tilde{\phi})(JM(F; f)_\tau \circ \phi)$, and that on the right, to $(m_G \circ \phi)(JM(\tilde{F}; \tilde{f})_\beta \circ \tilde{\phi})$. So the inclusion above is equivalent to this one:

$$
JM(F; f)_\tau \circ \phi \subseteq JM(\tilde{F}; \tilde{f})_\beta \circ \tilde{\phi}.
$$

Combined with Equation (6.2.1), this inclusion implies the following equation:

$$
JM(F; f) \circ \phi = JM(\tilde{F}; \tilde{f})_\beta \circ \tilde{\phi}.
$$

Now, Equation (6.2.2) yields the inclusion,

$$
JM(\tilde{F}; \tilde{f})_\beta \circ \tilde{\phi} \subseteq JM(F; f)_\zeta \circ \phi + m(JM(F; f)_\tau \circ \phi),
$$

where $\zeta: \mathbb{C}^n \to T$ is the projection with kernel $P$ and where $m$ is the maximal ideal of $(\mathbb{C}, 0)$. Hence Nakayama’s lemma yields the equation,

$$
JM(F; f)_\zeta \circ \phi = JM(\tilde{F}; \tilde{f})_\beta \circ \tilde{\phi}.$$
The term on the right is equal to $JM(F; f) \circ \phi$ by the preceding equation, and the term on the left lies in $JM(F; f)_h \circ \phi$ since $P$ lies in $H$. Therefore,

$$ JM(F; f)_h \circ \phi = JM(F; f) \circ \phi. $$

Thus we have obtained the desired contradiction, and the proof is complete.

In our final two results, a key role is played by the level hypersurface,

$$ Z := f^{-1}(0) \cap X. $$

Assume $Z \supset Y$, and assume $f|X$ is a nonconstant function. Given $y \in Y$, set

$$ X(y) := r^{-1}(y) \cap X \text{ and } Z(y) := r^{-1}(y) \cap Z. $$

Finally, assume that $(X(y), 0)$ and $(Z(y), 0)$ are ICIS germs, and that $(X, 0)$ is given by the vanishing of $F: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ as a complete intersection of codimension $p$.

Form the Buchsbaum–Rim multiplicity,

$$ eM(y) := e(m_y JM(F; f)_r(y)), $$

where $JM(F; f)_r(y)$ stands for the image of $JM(F; f)_r$ in $\mathcal{O}^{p+1}_{X(y), 0}$. Since $m_y$ induces the maximal ideal $m_y$ and $JM(F; f)_r(y)$ is equal to the augmented Jacobian module $JM(F|X(y); f|X(y))$ of the restricted functions, we have

$$ eM(y) = e(m_y JM(F|X(y); f|X(y))), $$

which is an invariant of the fiber over $y \in Y$.

**Lemma (6.3)** In the setup above, fix $y$, and for $i = 0, \ldots, l-p$, let $\mu_i(X(y), 0)$ and $\mu_i(Z(y), 0)$ denote the Milnor numbers of the sections by a general linear space $P_i$ of codimension $i$ in $\mathbb{C}^l$. Then

$$ eM(y) = \sum_{i=0}^{l-p} \binom{l-1}{i} (\mu_i(X(y), 0) + \mu_i(Z(y), 0)). $$

**Proof.** Let $\Pi^i_y$ denote the $i$-dimensional relative polar subscheme of $f|X(y)$ with $P_i$ as pole. It follows from the polar multiplicity formula [20, Thm. (9.8)(i)] (compare with [15, 4.2.7] and [8, S3]) that

$$ eM(y) = \sum_{i=0}^{l-p} \binom{l-1}{i} m(\Pi^i_y, 0) $$

where $m(\Pi^i_y, 0)$ is the ordinary multiplicity at 0 of $\Pi^i_y$. In particular, $m(\Pi^{l-p}_y, 0)$ is simply the multiplicity of $X(y)$ at 0; so it is equal to $\mu_{l-p}(X(y), 0) + 1$. By convention, $\mu_{l-p}(X_y, 0)$ and $\mu_{l-p-1}(Z_y, 0)$ and $\mu_{l-p-1}(Z_y, 0)$ are the ordinary multiplicities at 0 diminished by 1, and $\mu_{l-p}(Z_y, 0) = 1$.

In general, since $\Pi^i_y$ is Cohen–Macaulay,

$$ m(\Pi^i_y, 0) = \dim \mathcal{O}_{\Pi^i_y, 0}/\mathcal{I}(L) \quad (6.2.1) $$

25 April 1988

Specialization of integral dependence for modules
where $\mathcal{I}(L)$ is the ideal of any linear space $L$ of codimension $i$ in $C^l$ that is transverse to $\Pi^i_y$. By the preceding lemma, we may take $P_i$ for $L$. Then, for $i < l - p$, the right side is equal to $\mu_i(X(y),0) + \mu_i(Z(y),0)$ by the theorem of Lê and Greuel. The asserted formula follows immediately, and the proof is complete.

**Theorem (6.4)** In the setup of Lemma (6.3), let $\Sigma(f)$ denote the critical set of $f$, and $\Sigma_Y(f)$ the union of the critical sets of the restrictions $f|X(y)$. Then the following four conditions are equivalent:

(i) the germs of $\Sigma(f)$ and $Y$ are equal, and the pair $(X - Y, Y)$ satisfies $W_f$ at 0;

(ii) the germs of $\Sigma_Y(f)$ and $Y$ are equal, and both pairs $(X - Y, Y)$ and $(Z - Y, Y)$ satisfy the absolute Whitney conditions at 0;

(iii) the Milnor numbers of the sections, $\mu_i(X(y),0)$ and $\mu_i(Z(y),0)$, are constant in $y \in Y$ near 0;

(iv) the multiplicity $e_m(y)$ is constant in $y \in Y$ near 0.

**Proof.** First of all, (i) implies (ii). Indeed, $\Sigma_Y(f)$ and $\Sigma(f)$ represent the same germ by the argument given in the third paragraph before Lemma (6.2). Furthermore, $T_xX(f(x)) \subset T_xX$ and, if $x \in Z$, then $T_xX(f(x)) = T_xZ$. Hence, the analytic inequalities required by (ii) are automatically satisfied when (i) holds. Second, (ii) implies (iii); indeed, this implication is virtually the assertion of Théorème (10.1) on p. 223 of [30]. Third, (iii) implies (iv) by Lemma (6.3).

Lastly, assume (iv). Then the germs of $\Sigma(f)$ and $Y$ are equal by the upper semicontinuity argument in the proof of Theorem (5.3), this time applied to $m_YJM(F; f)_r$. Now, since $f$ is a function, by the relative generic Whitney lemma, $W_f$ holds at a general point of $Y$. Hence, generically $JM(F; f)_g$ is integrally dependent on $m_YJM(F; f)_r$ by Proposition (6.1). Therefore, since (iv) holds, this dependency holds at 0 by the principle of specialization of integral dependence, Theorem (1.8). So Proposition (6.1) implies that $(X - \Sigma(f), Y)$ satisfies $W_f$ at 0. Thus (i) holds and the proof is complete.

**References**

[1] J. Briançon, P. Maisonobe and M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. 117 (1994), 531–50.

[2] J. W. Bruce, A. A. Du Plessis, and L. C. Wilson, *Discriminants and liftable vector fields*, J. Alg. Geom. 3 (1994), 725–53.

[3] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. II. Depth and multiplicity*, Trans. Amer. Math. Soc. 111 (1963), 197–224.

[4] W. Fulton, “Intersection Theory,” Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge · Band 2, Springer–Verlag, Berlin, 1984.

[5] T. Gaffney, *Aureoles and integral closure of modules*, in “Stratifications, Singularities and Differential II. Travaux en Cours, 55,” Herman, Paris, (1997), 55–62.

[6] T. Gaffney, *Integral closure of modules and Whitney equisingularity*, Invent. Math. 107 (1992), 301–22.
[7] T. Gaffney, *Equisingularity of plane sections, $t_1$ condition, and the integral closure of modules*, in “Real and Complex Singularities” Proceedings of the Third International Workshop on Real and Complex Singularities at Sao Carlos, Brasil 1994, W. L. Marar (ed.) Pitman Research Notes in Mathematics 333 (1995) 95-111.

[8] T. Gaffney, *Multiplicities and equisingularity of ICIS germs*, Invent. Math. 123 (1996), 209–20.

[9] T. Gaffney and D. Massey, *Trends in equisingularity theory*, to appear in the proceedings of a 1996 symposium in honor of CTC Wall, singularities volume, W. Bruce and D. Mond (eds.), Cambr. U. Press, to appear..

[10] M. D. Green, *Dissertation, Northeastern University*, 1997.

[11] M. D. Green and D. B. Massey, *Vanishing cycles and Thom’s $a_f$ conditions*, Preprint 1996.

[12] G. M. Greuel, “Der Gauss–Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten,” Dissertation, Göttingen (1973), Math. Ann. 214 (1975), 235–66.

[13] J.P.G. Henry and M. Merle, *Limites d’espaces tangents et transversalité de variétés polaires*, in “Proc. La Rábida, 1981.” J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle (eds.) Springer Lecture Notes in Math. 961 (1982), 189–99.

[14] J.P.G. Henry and M. Merle, *Limites de normales, conditions de Whitney et éclatement d’Hironaka, “Singularities,”* Proc. Symposia pure math. vol. 40, part 1, Amer. Math. Soc. (1983), 575–84.

[15] J.P.G. Henry and M. Merle, *Conormal Space and Jacobian module. A short dictionary*, in “Proceedings of the Lille Congress of Singularities,” J.-P. Brassaelet (ed.), London Math. Soc. Lecture Notes 201 (1994), 147–74.

[16] J.P.G. Henry, M. Merle, and C. Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann. Scient. Éc. Norm. Sup. 17 (1984), 227–68.

[17] H. Hironaka, *Stratification and flatness*, in “Real and complex singularities, Nordic Summer School, Oslo, 1976,” Sijthoff & Noordhoff, (1977), 199–265.

[18] D. Kirby and D. Rees, *Multiplicities in graded rings I: The general theory*, in “Commutative algebra: syzygies, multiplicities, and birational algebra” W. J. Heinzer, C. L. Huneke, J. D. Sally (eds.), Contemp. Math. 159 (1994), 209–67.

[19] S. L. Kleiman *Equisingularity, multiplicity, and dependence*, to appear in the proceedings of a conference in honor of M. Fiorentini, Marcel Dekker, 1998.

[20] S. Kleiman and A. Thorup, *A geometric theory of the Buchsbaum–Rim multiplicity*, J. Algebra 167 (1994), 168–231.

[21] S. Kleiman and A. Thorup, *Conormal geometry of maximal minors*, alg-geom/970818.

[22] D. T. Lê, *Calculation of Milnor number of isolated singularity of complete intersection*, Funct. Anal. Appl. 8 (1974), 127–31.

[23] D. T. Lê and C. P. Ramanujam, *The invariance of Milnor’s number implies the invariance of the topological type*, Amer. J. Math. 98 (1976), 67–78.

[24] D. T. Lê and K. Saito, *La constance du nombre de Milnor donne des bonnes stratifications*, C. R. Acad. Sci. Paris 277 (1973), 793–95.

[25] D. T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières* Annals Math. 114 (1981), 457–91.

[26] D. T. Lê and B. Teissier, *Limites de’espaces tangent en géométrie analytique*, Comment. Math. Helvetici 63 (1988), 540–78.

[27] M. Lejeune-Jalabert and B. Teissier, *Clôture integrale des ideaux et equisingularité, chapitre 1* Publ. Inst. Fourier. (1974).

[28] J. Lipman, *Equimultiplicity, reduction and blowing-up*, in “Commutative algebra: analytic methods,” Dekker Lecture Notes in Pure and Applied Math. 68 (1982), 111–48.

[29] E.J.N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society lecture note series 77, Cambridge University Press, 1984..
[30] V. Navarro, *Conditions de Whitney et sections planes*, Invent. Math. 61 (1980), 199–226.

[31] A. J. Parameswaran, *Topological equisingularity for isolated complete intersection singularities*, Compositio Math. 80 (1991), 323–36.

[32] D. Rees, “Reduction of modules,” Math. Proc. Camb. Phil. Soc. 101 (1987), 431–49.

[33] D. Rees, *Gaffney’s problem*, manuscript dated Feb. 22 1989.

[34] B. Teissier, *Cycles évanescent, sections planes et conditions de Whitney*, in “Singularités à Cargèse,” Astérisque 7–8 (1973), 285–362.

[35] B. Teissier, *Introduction to equisingularity problems* Proc. Symposia pure math. vol. 29 Amer. Math. Soc. (1975), 575–84.

[36] B. Teissier, *The hunting of invariants in the geometry of the discriminant*, in “Real and complex singularities, Oslo 1976,” P. Holm (ed.), Sijthoff & Noordhoff (1977), 565–678.

[37] B. Teissier, *Variétés polaires. I* Invent. Math. 40 (1977), 267–92.

[38] B. Teissier, *Résolution simultanée et cycles évanescent*, in “Sém. sur les singularités des surfaces.” Proc. 1976–77. M. Demazure, H. Pinkham and B. Teissier (eds.) Springer Lecture Notes in Math. 777 (1980), 82–146.

[39] B. Teissier, *Variétés polaires locales: quelques résultats*, in “Journées complexes,” Institut Elie Cartan, Nancy, March 1981.

[40] B. Teissier, *Multiplicités polaires, sections planes, et conditions de Whitney*, in “Proc. La Rábida, 1981.” J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle (eds.) Springer Lecture Notes in Math. 961 (1982), 314–491.

[41] D. Trotman, *On the canonical Whitney stratification of algebraic hypersurfaces*, Sem. sur la géométrie algébrique réelle, dirigé par J.-J. Risler, Publ. math. de l’université Paris VII, Tome I (1986), 123–52.

[42] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini–Sard*, Invent. Math. 36 (1976), 295–312.