Exchanging OWL 2 QL Knowledge Bases

Marcelo Arenas  
PUC Chile & Univ. of Oxford, U.K.  
marenas@ing.puc.cl

Elena Botovea  
Free U. of Bolzano, Italy  
botoeva@inf.unibz.it

Diego Calvanese  
Free U. of Bolzano, Italy & TU Vienna, Austria  
calvanese@inf.unibz.it

Vladislav Ryzhikov  
Free U. of Bolzano, Italy  
ryzhikov@inf.unibz.it

Abstract

Knowledge base exchange is an important problem in the area of data exchange and knowledge representation, where one is interested in exchanging information between a source and a target knowledge base connected through a mapping. In this paper, we study this fundamental problem for knowledge bases and mappings expressed in OWL 2 QL, the profile of OWL 2 based on the description logic DL-Lite\(_g\). More specifically, we consider the problem of computing universal solutions, identified as one of the most desirable translations to be materialized, and the problem of computing UCQ-representations, which optimally capture in a target TBox the information that can be extracted from a source TBox and a mapping by means of unions of conjunctive queries. For the former we provide a novel automata-theoretic technique, and complexity results that range from NP to \(\text{NLOGSPACE}\)-completeness.

1 Introduction

Complex forms of information, maintained in different formats and organized according to different structures, often need to be shared between agents. In recent years, both in the data management and in the knowledge representation communities, several settings have been investigated that address this problem from various perspectives: in information integration, uniform access is provided to a collection of data sources by means of an ontology (or global schema) to which the sources are mapped [Lenzerini, 2002]; in peer-to-peer systems, a set of peers declaratively linked to each other collectively provide access to the information assets they maintain [Kementsietsidis et al., 2003; Adjiman et al., 2006; Fuxman et al., 2006]; in ontology matching, the aim is to understand and derive the correspondences between elements in two ontologies [Euzenat and Shvaiko, 2007; Shvaiko and Euzenat, 2013]; finally, in data exchange, the information stored according to a source schema needs to be restructured and translated so as to conform to a target schema [Fagin et al., 2005; Barceló, 2009].

The work we present in this paper is inspired by the latter setting, investigated in databases. We study it, however, under the assumption of incomplete information typical of knowledge representation [Arenas et al., 2011]. Specifically, we investigate the problem of knowledge base exchange, where a source knowledge base (KB) is connected to a target KB by means of a declarative mapping specification, and the aim is to exchange knowledge from the source to the target by exploiting the mapping. We rely on a framework for KB exchange based on lightweight Description Logics (DLs) of the DL-Lite\(_g\) family [Calvanese et al., 2007], recently proposed in [Arenas et al., 2012a; Arenas et al., 2012b]: both source and target are KBs constituted by a DL TBox, representing implicit information, and an ABox, representing explicit information, and mappings are sets of DL concept and role inclusions. Note that in data and knowledge base exchange, differently from ontology matching, mappings are first-class citizens. In fact, it has been recognized that building schema mappings is an important and complex activity, which requires the designer to have a thorough understanding of the source and how the information therein should be related to the target. Thus, several techniques and tools have been developed to support mapping design, e.g., exploiting lexical information [Fagin et al., 2009]. Here, similar to data exchange, we assume that for building mappings the target signature is given, but no further axioms constraining the target knowledge are available. In fact, such axioms are derived from the source KB and the mapping.

We consider two key problems: (i) computing universal solutions, which have been identified as one of the most desirable translations to be materialized; (ii) UCQ-representability of a source TBox by means of a target TBox that captures at best the intensional information that can be extracted from the source according to a mapping using union of conjunctive queries. Determining UCQ-representability is a crucial task, since it allows one to use the obtained target TBox to infer new knowledge in the target, thus reducing the amount of extensional information to be transferred from the source. Moreover, it has been noticed that in many data exchange applications users only extract information from the translated data by using specific queries (usually conjunctive queries), so query-based notions of translation specifically tailored to store enough information to answer such queries have been widely studied in the data exchange area [Madhavan and Halevy, 2003; Fagin et al., 2008; Arenas et al., 2009; Fagin and Kolaitis, 2012].
For these two problems, we investigate both the task of checking membership, where a candidate universal solution (resp., UCQ-representation) is given and one needs to check its correctness, and non-emptiness, where the aim is to determine the existence of a universal solution (resp., UCQ-representation).

We significantly extend previous results in several directions. First of all, we establish results for OWL 2QL [Motik et al., 2012], one of the profiles of the standard Web Ontology Language OWL 2 [Bao et al., 2012], which is based on the DL DL-LiteR. To do so, we have to overcome the difficulty of dealing with null values in the ABox, since these become necessary in the target to represent universal solutions. Also, for the first time, we address disjointness assertions in the TBox, a construct that is part of OWL 2QL. The main contribution of our work is then a detailed analysis of the computational complexity of both membership and non-emptiness for universal solutions and UCQ-representability. For the non-emptiness problem of universal solutions, previous known results covered only the simple case of DL-LiteR, the RDFS fragment of OWL 2QL, in which no new facts can be inferred, and universal solutions always exist and can be computed in polynomial time via a chase procedure (see [Calvanese et al., 2007]).

We show that in our case, instead, the problem is PSPACE-hard, hence significantly more complex, and provide an EXPTime upper bound on a novel approach exploiting two-way alternating automata. We provide also NP upper bounds for the simpler case of ABoxes without null values, and for the case of the membership problem. As for UCQ-representability, we adopt the notion of UCQ-representability introduced in [Arenas et al., 2012a; Arenas et al., 2012b] and extend it to take into account disjointness of OWL2QL. For that case we show NLSPACE-completeness of both non-emptiness and membership, improving on the previously known PTIME upper bounds.

The paper is organized as follows. In Section 2 we give preliminary notions on DLs and queries. In Section 3 we define our framework of KB exchange and discuss the problem of computing solutions. In Section 4 we overview our contributions, and then we provide our results on computing universal solutions in Section 5, and on UCQ-representability in Section 6. Finally, in Section 7 we draw some conclusions and outline issues for future work.

2 Preliminaries

The DLs of the DL-Lite family [Calvanese et al., 2007] of light-weight DLs are characterized by the fact that standard reasoning can be done in polynomial time. We adapt here DL-LiteR, the DL underlying OWL 2QL, and present now its syntax and semantics. Let NC, NR, Nc, Ni be pairwise disjoint sets of concept names, role names, constants, and labeled nulls, respectively. Assume in the following that A ∈ NC and P ∈ NR; in DL-LiteR, B and C are used to denote basic and arbitrary (or complex) concepts, respectively, and R and Q are used to denote basic and arbitrary (or complex) roles, respectively, defined as follows:

\[
R ::= P \mid P^- \\
Q ::= R \mid \neg R \\
B ::= A \mid \exists R \\
C ::= B \mid \neg B
\]

From now on, for a basic role R, we use R to denote P when R = P, and P when R = P-.

A TBox is a finite set of concept inclusions B ⊑ C and role inclusions R ⊑ Q. We call an inclusion of the form B1 ⊑ ¬B2 or R1 ⊑ ¬R2 a disjointness assertion. An ABox is a finite set of membership assertions B(u), R(u, v), where a, b ∈ Na. In this paper, we also consider extended ABoxes, which are obtained by allowing labeled nulls in membership assertions. Formally, an extended ABox is a finite set of membership assertions B(u) and R(u, v), where u, v ∈ (Na ∪ Ni).

Moreover, a(n extended) KB K is a pair ⟨T, A⟩, where T is a TBox and A is an extended ABox.

A signature Σ is a finite set of concept and role names. A KB K is said to be defined over (or simply, over) Σ if all the concept and role names occurring in K belong to Σ (and likewise for TBoxes, ABoxes, concept inclusions, role inclusions and membership assertions). Moreover, an interpretation I of Σ is a pair (ΔI, ℓI), where ΔI is a non-empty domain and ℓI is an interpretation function such that: (1) ΔI ⊆ ΔΣ, for every concept name A ∈ Σ; (2) P ⊆ ΔI × ΔΣ; for every role name R ∈ Σ; and (3) aR ∈ ΔI, for every constant a ∈ Na. Function ℓI is extended to also interpret concept and role constructs:

\[
(\exists R)^I = \{ x ∈ ΔI \mid \exists y ∈ ΔI \text{ such that } (x, y) ∈ R^I \}; \\
(P^-)^I = \{ (y, x) ∈ ΔI × ΔI \mid (x, y) ∈ P^I \}; \\
(\neg B)^I = ΔI \setminus B^I; \\
(\neg R)^I = (ΔI × ΔΣ) \setminus R^I.
\]

Note that, consistently with the semantics of OWL2QL, we do not make the unique name assumption (UNA), i.e., we allow distinct constants a, b ∈ Na to be interpreted as the same object, i.e., aR = bR. Note also that labeled nulls are not interpreted by I.

Let I = (ΔI, ℓI) be an interpretation over a signature Σ. Then I is said to satisfy a concept inclusion B ⊑ C over Σ, denoted by I ⊨ B ⊑ C over Σ, if B ⊆ C; I is said to satisfy a role inclusion R ⊑ Q over Σ, denoted by I ⊨ R ⊑ Q over Σ, if x, y ∈ ΔI, where (x, y) ∈ Q, and I is said to satisfy a TBox T over Σ, denoted by I ⊨ T, if I ⊨ α for every α ∈ T. Moreover, satisfaction of membership assertions over Σ is defined as follows.

A substitution over I is a function h : (Na ∪ Ni) → ΔΣ such that h(a) = aR for every a ∈ Na. Then I is said to satisfy an (extended) ABox A, denoted by I ⊨ A, if there exists a substitution h over I such that:

- for every B(u) ∈ A, it holds that h(u) ∈ B^I; and
- for every R(u, v) ∈ A, it holds that (h(u), h(v)) ∈ R^I.

Finally, I is said to satisfy a(n extended) KB K = ⟨T, A⟩, denoted by I ⊨ K, if I ⊨ T and I ⊨ A. Such I is called a model of K, and we use Mod(K) to denote the set of all models of K. We say that K is consistent if Mod(K) ≠ ∅.

As is customary, given an (extended) KB K over a signature Σ and a membership assertion or an inclusion α over Σ, we use notation K ⊨ α to indicate that for every interpretation I of Σ, if I ⊨ K, then I ⊨ α.

2.1 Queries and certain answers

A k-ary query q over a signature Σ, with k ≥ 0, is a function that maps every interpretation (Δ^k, ℓ^k) of Σ into a k-ary
relation $q^T \subseteq (\Delta^T)^k$. In particular, if $k = 0$, then $q$ is said to be a Boolean query, and $q^T$ is either a relation containing the empty tuple (representing the value true) or the empty relation (representing the value false). Given a KB $K$ over $\Sigma$, the set of certain answers to $q$ over $K$, denoted by $\text{cert}(q, K)$, is defined as:

$$\bigcap_{I \in \text{Mod}(K)} \{ (a_1, \ldots, a_k) \mid \{ a_1, \ldots, a_k \} \subseteq N_a \land (a_1^T, \ldots, a_k^T) \in q^T \}.$$ 

Notice that the certain answer to a query does not contain labeled nulls. Besides, notice that if $q$ is a Boolean query, then $\text{cert}(q, K)$ evaluates to true if $q^T$ evaluates to true for every $I \in \text{Mod}(K)$, and it evaluates to false otherwise.

A conjunctive query (CQ) over a signature $\Sigma$ is a formula of the form $q(x) = \exists \vec{x}. \varphi(\vec{x}, \vec{y})$, where $\vec{x}$, $\vec{y}$ are tuples of variables and $\varphi(\vec{x}, \vec{y})$ is a conjunction of atoms of the form $A(t)$, with $A$ a concept name in $\Sigma$, and $P(t, t')$, with $P$ a role name in $\Sigma$, where each of $t$, $t'$ is either a constant from $N_a$ or a variable from $\vec{x}$ or $\vec{y}$. Given an interpretation $I = (\Delta^I, \cdot^I)$ of $\Sigma$, the answer of $q$ over $I$, denoted by $q^I$, is the set of tuples $\vec{a}$ of elements from $\Delta^I$ for which there exist a tuple $\vec{b}$ of elements from $\Delta^I$ such that $I$ satisfies every conjunct in $\varphi(\vec{a}, \vec{b})$. A union of conjunctive queries (UCQ) over a signature $\Sigma$ is a formula of the form $q(x) = \bigcup_{i=1}^n q_i(x)$, where each $q_i$ ($1 \leq i \leq n$) is a CQ over $\Sigma$, whose semantics is defined as $q^I = \bigcup_{i=1}^n q_i^I$.

### 3 Exchanging OWL 2 QL Knowledge Bases

We generalize now, in Section 3.1, the setting proposed in [Arenas et al., 2011] to OWL 2 QL, and we formalize in Section 3.2 the main problems studied in the rest of the paper.

#### 3.1 A knowledge base exchange framework for OWL 2 QL

Assume that $\Sigma_1$, $\Sigma_2$ are signatures with no concepts or roles in common. An inclusion $E_1 \subseteq E_2$ is said to be from $\Sigma_1$ to $\Sigma_2$, if $E_1$ is a concept or a role over $\Sigma_1$, and $E_2$ is a concept or a role over $\Sigma_2$. A mapping is a tuple $M = (\Sigma_1, \Sigma_2, T_{12})$, where $T_{12}$ is a TBox consisting of inclusions from $\Sigma_1$ to $\Sigma_2$ [Arenas et al., 2012a]. Recall that in this paper, we deal with DL-Lite$\gamma$ TBoxes only, so $T_{12}$ is assumed to be a set of DL-Lite$\gamma$ concept and role inclusions. The semantics of such a mapping is defined in [Arenas et al., 2012a] in terms of a notion of satisfaction for interpretations, which has to be extended in our case to deal with interpretations not satisfying the UNA (and, more generally, the standard name assumption). More specifically, given interpretations $I$, $J$ of $\Sigma_1$ and $\Sigma_2$, respectively, pair $(I, J)$ satisfies TBox $T_{12}$, denoted by $(I, J) \models T_{12}$, if (i) for every $a \in N_a$, it holds that $a^{\Delta_I} = a^{\Delta_J}$, (ii) for every concept inclusion $B \subseteq C$ in $T_{12}$, it holds that $B^{\Delta_J} \subseteq C^{\Delta_J}$, and (iii) for every role inclusion $R \subseteq Q$ in $T_{12}$, it holds that $R^{\Delta_J} \subseteq Q^{\Delta_J}$. Notice that the connection between the information in $I$ and $J$ is established through the constants that move from source to target according to the mapping. For this reason, we require constants to be interpreted in the same way in $I$ and $J$, i.e., they preserve their meaning when they are transferred. Besides, notice that this is the only restriction imposed on the domains of $I$ and $J$ (in particular, we require neither that $\Delta^I = \Delta^J$ nor that $\Delta^I \subseteq \Delta^J$). Finally, $\text{SAT}_M(I)$ is defined as the set of interpretations $\mathcal{J}$ of $\Sigma_2$ such that $(I, J) \models T_{12}$, and given a set $\mathcal{X}$ of interpretations of $\Sigma_1$, $\text{SAT}_M(\mathcal{X})$ is defined as $\bigcup_{I \in \mathcal{X}} \text{SAT}_M(I)$.

The main problem studied in the knowledge exchange area is the problem of translating a KB according to a mapping, which is formalized through several different notions of translation (for a thorough comparison of different notions of solutions see [Arenas et al., 2012a]). The first such notion is the concept of solution, which is formalized as follows. Given a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and KBs $K_1$, $K_2$ over $\Sigma_1$ and $\Sigma_2$, respectively, $K_2$ is a solution for $K_1$ under $M$ if $\text{MOD}(K_2) \subseteq \text{SAT}_M(\text{MOD}(K_1))$. Thus, $K_2$ is a solution for $K_1$ under $M$ if every interpretation of $K_2$ is a valid translation of an interpretation of $K_1$ according to $M$. Although natural, this is a mild restriction, which gives rise to the stronger notion of universal solution. Given $M$, $K_1$ and $K_2$ as before, $K_2$ is a universal solution for $K_1$ under $M$ if $\text{MOD}(K_2) \subseteq \text{SAT}_M(\text{MOD}(K_1))$. Thus, $K_2$ is designed to exactly represent the space of interpretations obtained by translating the interpretations of $K_1$ under $M$ [Arenas et al., 2012a]. Below is a simple example demonstrating the notion of universal solutions. This example also illustrates some issues regarding the absence of the UNA, which has to be given up to comply with the OWL 2 QL standard, and regarding the use of disjointness assertions.

**Example 3.1** Assume $M = \{ (F(\cdot), G(\cdot)), (F'(\cdot), G'(\cdot)), T_{12}) \}$, where $T_{12} = \{ F \subseteq F', G \subseteq G' \}$, and let $\mathbf{k}_1 = \{(T_1, A_1)\}$, where $T_1 = \emptyset$ and $A_1 = \{ F(a), G(b) \}$. Then the ABox $A_2 = \{ F'(a), G'(b) \}$ is a universal solution for $K_1$ under $M$.

Now, if we add a seemingly harmless disjointness assertion $\{ F \subseteq \neg G \}$ to $T_1$, we obtain that $A_2$ is no longer a universal solution (not even a solution) for $K_1$ under $M$. The reason for that is the lack of the UNA on the one hand, and the presence of the disjointness assertion in $T_1$ on the other hand. In fact, the latter forces $a$ and $b$ to be interpreted differently in the source. Thus, for a model $\mathcal{J}$ of $A_2$ such that $a^{\mathcal{J}} = b^\mathcal{J}$ and $F^{\mathcal{J}} = G^{\mathcal{J}} = \{ a^\mathcal{J} \}$, there exists no model $\mathcal{I}$ of $K_1$ such that $(\mathcal{I}, \mathcal{J}) \models T_{12}$ (which would require $a^{\mathcal{I}} = a^{\mathcal{J}}$ and $b^{\mathcal{I}} = b^{\mathcal{J}}$). In general, there exists no universal solution for $K_1$ under $M$, even though $K_1$ and $T_{12}$ are consistent with each other.

A second class of translations is obtained in [Arenas et al., 2012a] by observing that solutions and universal solutions are too restrictive for some applications, in particular when one only needs a translation storing enough information to properly answer some queries. For the particular case of UCQ, this gives rise to the notions of UCQ-solution and universal UCQ-solution. Given a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$, a KB $K_1 = \{(T_1, A_1)\}$ over $\Sigma_1$ and a KB $K_2$ over $\Sigma_2$, $K_2$ is a UCQ-solution for $K_1$ under $M$ if for every query $q \in \text{UCQ}$ over $\Sigma_2$: $\text{cert}(q, (T_1 \cup T_{12}, A_1)) \subseteq \text{cert}(q, K_2)$, while $K_2$ is a universal UCQ-solution for $K_1$ under $M$ if for every query $q \in \text{UCQ}$ over $\Sigma_2$: $\text{cert}(q, (T_1 \cup T_{12}, A_1)) = \text{cert}(q, K_2)$.

Finally, a last class of solutions is obtained in [Arenas et al., 2012a] by considering that users want to trans-
late as much of the knowledge in a TBox as possible, as a lot of effort is put in practice when constructing a TBox. This observation gives rise to the notion of UCQ-representation [Arenas et al., 2012a], which formalizes the idea of translating a source TBox according to a mapping. Next, we present an alternative formalization of this notion, which is appropriate for our setting where disjointness assertions are considered.

Assume that $M = (\Sigma_1, \Sigma_2, T_{12})$ and $T_1, T_2$ are TBoxes over $\Sigma_1$ and $\Sigma_2$, respectively. Then $T_2$ is a UCQ-representation of $T_1$ under $M$ if for every query $q \in \text{UCQ}$ over $\Sigma_2$ and every ABox $A_1$ over $\Sigma_1$ that is consistent with $T_1$:

$$\text{cert}(q, \langle T_1 \cup T_{12}, A_1 \rangle) = \bigcap_{A_2 : A_2 \text{ is an ABox over } \Sigma_2 \text{ that is a UCQ-solution for } A_1 \text{ under } M} \text{cert}(q, \langle T_2, A_2 \rangle).$$

Notice that in the previous definition, $A_2$ is said to be a UCQ-solution for $A_1$ under $M$ if the KB $(\emptyset, A_2)$ is a UCQ-solution for the KB $(\emptyset, A_1)$ under $M$. Let us explain the intuition behind the definition of the UCQ-representation. Assume that $T_1, T_2, M$ satisfy (1). First, $T_2$ captures the information in $T_1$ that is translated by $M$ and that can be extracted by using a UCQ, as for every ABox $A_1$ over $\Sigma_1$ that is consistent with $T_1$ and every UCQ $q$ over $\Sigma_2$, if we choose an arbitrary UCQ-solution $A_2$ for $A_1$ under $M$, then it holds that $\text{cert}(q, \langle T_1 \cup T_{12}, A_1 \rangle) \subseteq \text{cert}(q, \langle T_2, A_2 \rangle)$. Notice that $A_1$ is required to be consistent with $T_1$ in the previous condition, as we are interested in translating data that make sense according to $T_1$. Second, $T_2$ does not include any piece of information that can be extracted by using a UCQ and it is not the result of translating the information in $T_1$ according to $M$. In fact, if $A_1$ is an ABox over $\Sigma_1$ that is consistent with $T_1$ and $q$ is a UCQ over $\Sigma_2$, then it could be the case that $\text{cert}(q, \langle T_1 \cup T_{12}, A_1 \rangle) \subseteq \text{cert}(q, \langle T_2, A_2 \rangle)$ for some UCQ-solution $A_2$ for $A_1$ under $M$. However, the extra tuples extracted by query $q$ are obtained from the extra information in $A_2$, as if we consider a tuple $\vec{a}$ that belong to $\text{cert}(q, \langle T_2, A_2 \rangle)$ for every UCQ-solution $A_2$ for $A_1$ under $M$, then it holds that $\vec{a} \in \text{cert}(q, \langle T_1 \cup T_{12}, A_1 \rangle)$.

**Example 3.2** Assume that $M = (\{F(\cdot), G(\cdot), H(\cdot), D(\cdot)\}, \{F'(\cdot), G'(\cdot), H'(\cdot)\}, T_{12})$, where $T_{12} = \{F \subseteq F', G \subseteq G', H \subseteq H'\}$, and let $T_1 = \{F \subseteq G\}$. As expected, TBox $T_2 = \{F' \subseteq G'\}$ is a UCQ-representation of $T_1$ under $M$. Moreover, we can add the inclusion $D \subseteq H'$ to $T_{12}$, and $T_2$ will still remain a UCQ-representation of $T_1$ under $M$. Notice that in this latter setting, our definition has to deal with some ABoxes $A_1$ that are consistent with $T_1$ but not with $T_1 \cup T_{12}$, for instance $A_1 = \{H(a), D(a)\}$ for some constant $a$. In those cases, Equation (1) is trivially satisfied, since $\text{MOD}(\langle T_1 \cup T_{12}, A_1 \rangle) = \emptyset$ and the set of UCQ-solutions for $A_1$ under $M$ is empty.

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1If disjointness assertions are not allowed, then this new notion can be shown to be equivalent to the original formalization of UCQ-representation proposed in [Arenas et al., 2012a].

### 3.2 On the problem of computing solutions

Arguably, the most important problem in knowledge exchange [Arenas et al., 2011] [Arenas et al., 2012a], as well as in data exchange [Pagin et al., 2005] [Kolaitis, 2005], is the task of computing a translation of a KB according to a mapping. To study the computational complexity of this task for the different notions of solutions presented in the previous section, we introduce the following decision problems. The membership problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and KBs $K_1, K_2$ over $\Sigma_1$ and $\Sigma_2$, respectively. Then the question to answer is whether $K_2$ is a universal solution (resp. universal UCQ-solution) for $K_1$ under $M$. Moreover, the membership problem for UCQ-representations has as input a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and TBoxes $T_1, T_2$ over $\Sigma_1$ and $\Sigma_2$, respectively, and the question to answer is whether there exists a UCQ-representation of $T_1$ under $M$.

In our study, we cannot leave aside the existential versions of the previous problems, which are directly related with the problem of computing translations of a KB according to a mapping. Formally, the non-emptiness problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and a KB $K_1$ over $\Sigma_1$. Then the question to answer is whether there exists a universal solution (resp. universal UCQ-solution) for $K_1$ under $M$. Moreover, the non-emptiness problem for UCQ-representations has as input a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and a TBox $T_1$ over $\Sigma_1$, and the question to answer is whether there exists a UCQ-representation of $T_1$ under $M$.

### 4 Our contributions

In Section3.2 we have introduced the problems that are studied in this paper. It is important to notice that these problems are defined by considering only KBs (as opposed to extended KBs), as they are the formal counterpart of OWL 2 QL. Nevertheless, as shown in Section5 there are natural examples of OWL 2 QL specifications and mappings where null values are needed when constructing solutions. Thus, we also study the problems defined in Section3.2 in the case where translations can be extended KBs. It should be noticed that the notions of solution, universal solution, UCQ-solution, universal UCQ-solution, and UCQ-representation have to be enlarged to consider extended KBs, which is straightforward to do. In particular, given a mapping $M = (\Sigma_1, \Sigma_2, T_{12})$ and TBoxes $T_1, T_2$ over $\Sigma_1$ and $\Sigma_2$, respectively, $T_2$ is said to be a UCQ-representation of $T_1$ under $M$ in this extended setting if in Equation (1) $A_2$ is an extended ABox over $\Sigma_2$ that is a UCQ-solution for $A_1$ under $M$.

The main contribution of this paper is to provide a detailed analysis of the complexity of the membership and non-emptiness problems for the notions of universal solution and UCQ-representation. In Figure 1 we provide a summary of the main results in the paper, which are explained in more detail in Sections 5 and 6. It is important to notice that these results considerably extend the previous known results about these problems [Arenas et al., 2012a] [Arenas et al., 2012b]. In the first place, the problem of computing universal solutions was studied in [Arenas et al., 2012a] for the case of
**Table 1**: Complexity results obtained in the paper about the membership and non-emptiness problems.

| Membership | ABoxes extended ABoxes | Non-emptiness | ABoxes extended ABoxes |
|------------|------------------------|---------------|------------------------|
| Universal solutions | in NP | NP-complete | Universal solutions | in NP | PSPACE-hard, in EXPTIME |
| UCQ-representations | NLOGSPACE-complete | UCQ-representations | NLOGSPACE-complete |

**DL-Lite_{RDF,c}**, a fragment of **DL-Lite_R** that allows neither for inclusions of the form \( B \subseteq \exists R \) nor for disjointness assertions. In that case, it is straightforward to show that every source KB has a universal solution that can be computed by using the chase procedure [Calvanese et al., 2007]. Unfortunately, this result does not provide any information about how to solve the much larger case considered in this paper, where, in particular, the non-emptiness problem is not trivial. In fact, for the case of the notion of universal UCQ-solutions, all the lower and upper bounds provided in Figure 1 are new results, which are not consequences of the results obtained in [Arenas et al., 2012a]. In the second place, a notion of UCQ-representation that is appropriate for the fragment of **DL-Lite_R** not including disjointness assertions was studied in [Arenas et al., 2012a][1][2]. In particular, it was shown that the membership and non-emptiness problems for this notion are solvable in polynomial time. In this paper, we considerably strengthen these results: (i) by generalizing the definition of the notion of UCQ-representation to be able to deal with OWL 2QL, that is, with the entire language **DL-Lite_R** (which includes disjointness assertions); and (ii) by showing that the membership and non-emptiness problems are both NLOGSPACE-complete in this larger scenario.

It turns out that reasoning about universal UCQ-solutions is much more intricate. In fact, as a second contribution of our paper, we provide a PSPACE lower bound for the complexity of the membership problem for the notion of universal UCQ-solution, which is in sharp contrast with the NP and NLOGSPACE upper bounds for this problem for the case of universal solutions and UCQ-representations, respectively (see Figure 1). Although many questions about universal UCQ-solutions remain open, we think that this is an interesting first result, as universal UCQ-solutions have only been investigated before for the very restricted fragment **DL-Lite_{RDF,c}** of **DL-Lite_R** [Arenas et al., 2012a], which is described in the previous paragraph.

### 5 Computing universal solutions

In this section, we study the membership and non-emptiness problems for universal solutions, in the cases where nulls are not allowed (Section 5.1) and are allowed (Section 5.2) in such solutions. But before going into this, we give an example that shows the shape of universal solutions in **DL-Lite_R**.

**Example 5.1** Assume that \( M = \{(F(\cdot), S(\cdot, \cdot)), \{\exists S^-(\cdot, \cdot, \cdot)\}, \{\exists S^+(\cdot, \cdot, \cdot)\}\}\), and let \( K_1 = \langle T_1, A_1 \rangle \), where \( T_1 = \{ F \subseteq \exists S \} \) and \( A_1 = \{ F(a) \} \). Then a natural way to construct a universal solution for \( K_1 \) under \( M \) is to ‘populate’ the target with all implied facts (as it is usually done in data exchange). Thus, the ABox \( A_2 = \{ G'(n) \} \), where \( n \) is a labeled null, is a universal solution for \( K_1 \) under \( M \) if nulls are allowed. Notice that here, a universal solution with non-extended ABoxes does not exist: substituting \( n \) by any constant is too restrictive, ruining universality.

**Example 5.2** Now, assume \( M = \{(F(\cdot), S(\cdot, \cdot), T(\cdot, \cdot)), \{\exists S^-(\cdot, \cdot)\}, \{\exists S^+(\cdot, \cdot)\}, \{S \subseteq S', T \subseteq S'\}, \) and \( K_1 = \langle T_1, A_1 \rangle \), where \( T_1 = \{ F \subseteq \exists S, \exists S^- \subseteq \exists S, \} \) and \( A_1 = \{ F(a), T(a, a) \} \). In this case, we cannot use the same approach as in Example 5.1 to construct a universal solution, as now we would need an infinite number of labeled nulls to construct such a solution. However, as \( S \) and \( T \) are transferred to the same role \( S' \), it is possible to use constant \( a \) to represent all implied facts. In particular, in this case \( A_2 = \{ S'(a, a) \} \) is a universal solution for \( K_1 \) under \( M \).

### 5.1 Universal solutions without null values

We explain here how the NP upper bound for the non-emptiness problem for universal solutions is obtained, when ABoxes are not allowed to contain null values.

Assume given a mapping \( M = (\Sigma_1, \Sigma_2, \tau_{12}) \) and a KB \( K_1 = \langle T_1, A_1 \rangle \) over \( \Sigma_1 \). To check whether \( K_1 \) has a universal solution under \( M \), we use the following nondeterministic polynomial-time algorithm. First, we construct an ABox \( A_2 \) over \( \Sigma_2 \) containing every membership assertion \( \alpha \) such that \( \langle T_1 \cup \tau_{12}, A_1 \rangle \models \alpha \), where \( \alpha \) is of the form either \( B(\cdot) \) or \( R(\cdot, \cdot, \cdot) \), and \( a, b \) are constants mentioned in \( A_1 \). Second, we guess an interpretation \( I \) of \( \Sigma_1 \) such that \( I \models K_1 \) and \( I, \mathcal{U}_{A_2} \models T_{12} \), where \( \mathcal{U}_{A_2} \) is the interpretation of \( \Sigma_2 \) naturally corresponding to \( A_2 \). The correctness of the algorithm is a consequence of the facts that:

a) there exists a universal solution for \( A_1 \) under \( M \) if and only if \( A_2 \) is a solution for \( A_1 \) under \( M \); and
b) \( A_2 \) is a solution for \( A_1 \) under \( M \) if and only if there exists a model \( I \) of \( K_1 \) such that \( (I, \mathcal{U}_{A_2}) \models T_{12} \).

Moreover, the algorithm can be implemented in a non-deterministic polynomial-time Turing machine given that:

(i) \( A_2 \) can be constructed in polynomial time; (ii) if there exists a model \( I \) of \( K_1 \) such that \( (I, \mathcal{U}_{A_2}) \models T_{12} \), then there exists a model of \( K_1 \) of polynomial-size satisfying this condition; and (iii) it can be checked in polynomial time whether \( I \models K_1 \) and \( (I, \mathcal{U}_{A_2}) \models T_{12} \).

In addition, in this case, the membership problem can be reduced to the non-emptiness problem, thus, we have that:

**Theorem 5.3** The non-emptiness and membership problems for universal solutions are in NP.

The exact complexity of these problems remains open. In fact, we conjecture that these problems are in PTIME.\(^3\)

\(^3\)Interpretation \( \mathcal{U}_{A_2} \) can be defined as the Herbrand model of \( A_2 \) extended with fresh domain elements to satisfy assertions of the form \( \exists R(\cdot) \) in \( A_2 \).
We conclude by showing that reasoning about universal UCQ-solutions is harder than reasoning about universal solutions, which can be explained by the fact that TBoxes have bigger impact on the structure of universal UCQ-solutions rather than of universal solutions. In fact, by using a reduction from the validity problem for quantified Boolean formulas, similar to a reduction in \cite{Konev2011}, we are able to prove the following:

**Theorem 5.4** The membership problem for universal UCQ-solutions is PSPACE-hard.

### 5.2 Universal solutions with null values

We start by considering the non-emptiness problem for universal solutions with null values, that is, when extended ABoxes are allowed in universal solutions. As our first result, similar to the reduction above, we show that this problem is PSPACE-hard, and identify the inclusion of inverse roles as one of the main sources of complexity.

To obtain an upper bound for this problem, we use two-way alternating automata on infinite trees (2ATA), which are a generalization of nondeterministic automata on infinite trees \cite{Vardi1998} well suited for handling inverse roles in DL-Lite\(_R\). More precisely, given a KB \( K \), we first show that it is possible to construct the following automata:

- \( \Delta_{K}^{can} \) is a 2ATA that accepts trees corresponding to the canonical model of \( K \) with nodes arbitrary labeled with a special symbol \( G \);
- \( \Delta_{K}^{mod} \) is a 2ATA that accepts a tree if its subtree labeled with \( G \) corresponds to a tree model \( T \) of \( K \) (that is, a model forming a tree on the labeled nodes); and
- \( \Delta_{fin} \) is a (one-way) non-deterministic automaton that accepts a tree if it has a finite prefix where each node is marked with \( G \), and no other node in the tree is marked with \( G \).

Then to verify whether a KB \( K_1 = (T_1, A_1) \) has a universal solution under a mapping \( M = (\Sigma_1, \Sigma_2, T_1) \), we solve the non-emptiness problem for an automaton \( \mathbb{B} \) defined as the product automaton of \( \pi_{\Gamma_{K}}(\Delta_{K}^{can}), \pi_{\Gamma_{K}}(\Delta_{K}^{mod}) \) and \( \Delta_{fin} \), where \( K = (T_1 \cup T_2, A_1, \pi_{\Gamma_{K}}(\Delta_{K}^{can}) \) is the projection of \( \Delta_{K}^{can} \) on a vocabulary \( \Gamma_K \) not mentioning symbols from \( \Sigma_1 \), and likewise for \( \pi_{\Gamma_{K}}(\Delta_{K}^{mod}) \). If the language accepted by \( \mathbb{B} \) is empty, then there is no universal solution for \( K_1 \) under \( M \), whereas a universal solution (possibly of exponential size) exists, and we can compute it by extracting the ABox encoded in some tree accepted by \( \mathbb{B} \). Summing up, we get:

**Theorem 5.5** If extended ABoxes are allowed in universal solutions, then the non-emptiness problem for universal solutions is PSPACE-hard and in EXPTIME.

Interestingly, the membership problem can be solved more efficiently in this scenario, as now the candidate universal solutions are part of the input. In the following theorem, we pinpoint the exact complexity of this problem.

\footnote{If \( K = (T, A) \), then this model essentially corresponds to the chase of \( A \) with \( T \) (see \cite{Konev2011} for a formal definition).}

**Theorem 5.6** If extended ABoxes are allowed in universal solutions, then the membership problem for universal solutions is NP-complete.

### 6 Computing UCQ-representations

In Section 5 we show that the complexity of the membership and non-emptiness problems for universal solutions differ depending on whether ABoxes or extended ABoxes are considered. On the other hand, we show in the following proposition that the use of null values in ABoxes does not make any difference in the case of UCQ-representations. In this proposition, given a mapping \( M \) and TBoxes \( T_1, T_2 \), we say that \( T_2 \) is a UCQ-representation of \( T_1 \) under \( M \) considering extended ABoxes if \( T_1, T_2 \), \( M \) satisfy Equation (\[\]) in Section 5.1 but assuming that \( A_2 \) is an extended ABox over \( \Sigma_2 \) that is a UCQ-solution for \( A_1 \) under \( M \).

**Proposition 6.1** A TBox \( T_2 \) is a UCQ-representation of a TBox \( T_1 \) under a mapping \( M \) if and only if \( T_2 \) is a UCQ-representation of \( T_1 \) under \( M \) considering extended ABoxes.

Thus, from now on we study the membership and non-emptiness problems for UCQ-representations assuming that ABoxes can contain null values.

We start by considering the membership problem for UCQ-representations. In this case, one can immediately notice some similarities between this task and the membership problem for universal UCQ-solutions, which was shown to be PSPACE-hard in Theorem 5.4. However, the universal quantification over ABoxes in the definition of the notion of UCQ-representation makes the latter problem computationally simpler, which is illustrated by the following example.

**Example 6.2** Assume that \( M = (\Sigma_1, \Sigma_2, T_{12}) \), where \( \Sigma_1 = \{ F(\cdot), S_1(\cdot, \cdot), S_2(\cdot, \cdot), T_1(\cdot, \cdot), T_2(\cdot, \cdot) \} \), \( \Sigma_2 = \{ F'(\cdot), S'(\cdot, \cdot), T'(\cdot, \cdot), G'(\cdot) \} \) and \( T_{12} = \{ F \subseteq F', S_1 \subseteq S', S_2 \subseteq S', T_1 \subseteq T', T_2 \subseteq T', \exists T_1' \subseteq \exists G' \} \). Moreover, assume that \( T_1 = \{ F \subseteq \exists S_1, F \subseteq S_2, \exists T_1' \subseteq \exists T_2 \} \) and \( T_2 = \{ F' \subseteq \exists S', \exists S' \subseteq T' \subseteq \exists T_1', \exists T_1' \subseteq \exists G' \} \). If we were to verify whether \( (T_2, \{ F(a) \}) \) is a universal UCQ-solution for \( (T_1, \{ F(a) \}) \) under \( M \) (which it is in this case), then we would first need to construct the path \( \pi = (F'(a), S'(a, n), T'(n, m), G'(n)) \) formed by the inclusions in \( T_2 \) where \( n, m \) are fresh null values, and then we would need to explore the translations according to \( M \) of all paths formed by the inclusions in \( T_1 \) to find one that matches \( \pi \).

On the other hand, to verify whether \( T_2 \) is a UCQ-representation of \( T_1 \) under \( M \), one does not need to execute any “backtracking”, as it is sufficient to consider independently a polynomial number of pieces \( C \) taken from the paths formed by the inclusions in \( T_1 \), each of them of polynomial size, and then checking whether the translation \( C' \) of \( C \) according to \( M \) matches with the paths formed from \( C' \) by the inclusions in \( T_2 \). If any of these pieces does not satisfy this condition, then it can be transformed into a witness that Equation (\[\]) is not satisfied, showing that \( T_2 \) is not a UCQ-representation of \( T_1 \) under \( M \) (as we have a universal quantification over the ABoxes over \( \Sigma_1 \) in the definition of UCQ-representations). In fact, one of the pieces considered in this case is \( C = (T_2(n, m)) \), where \( n, m \) are null values, which
does not satisfy the previous condition as the translation \( C' \) of \( C \) according to \( M \) is \( \langle T'(n, m) \rangle \), and this does not match with the path \( \langle T'(n, m), G'(m) \rangle \) formed from \( C' \) by the inclusions in \( T_2 \). This particular case is transformed into an ABox \( A_1 = \{ T_2(b, c) \} \) and a query \( q = T'(b, c) \land G'(c) \), where \( b, c \) are fresh constants, for which we have that Equation (1) is not satisfied.

Notice that disjointness assertions in the mapping may cause \( \langle T_1 \cup T_1, A_1 \rangle \) to become inconsistent for some source ABoxes \( A_1 \) (which will make all possible tuples to be in the answer to every query), therefore additional conditions have to be imposed on \( T_2 \). To give more intuition about how the membership problem for UCQ-representations is solved, we give an example showing how one can deal with some of these inconsistency issues.

**Example 6.3** Assume that \( M = (\Sigma_1, \Sigma_2, T_{12}) \), where \( \Sigma_1 = \{ F(\cdot), G(\cdot), H(\cdot) \} \), \( \Sigma_2 = \{ F'(\cdot), G'(\cdot), H'(\cdot) \} \) and \( T_{12} = \{ F \sqsubseteq F', G \sqsubseteq G', H \sqsubseteq H' \} \). Moreover, assume that \( T_1 = \{ F \sqsubseteq G \} \) and \( T_2 = \{ F' \sqsubseteq G' \} \). In this case, it is clear that \( T_2 \) is a UCQ-representation of \( T_1 \) under \( M \). However, if we add inclusion \( H \sqsubseteq G^* \) to \( T_{12} \), then \( T_2 \) is no longer a UCQ-representation of \( T_1 \) under \( M \). To see why this is the case, consider an ABox \( A_1 = \{ F(a), H(a) \} \), which is consistent with \( T_1 \), and a query \( q = F'(b) \), where \( b \) is a fresh constant. Then we have that \( \text{cert}(q, \langle T_1 \cup T_1, A_1 \rangle) = \emptyset \) as KB \( \langle T_1 \cup T_1, A_1 \rangle \) is inconsistent, while \( \text{cert}(q, \langle T_2, A_2 \rangle) = \emptyset \) for UCQ-solution \( A_2 = \{ F'(a), H'(a) \} \) for \( A_1 \) under \( M \). Thus, we conclude that Equation (1) is violated in this case.

One can deal with the issue raised in the previous example by checking that on every pair \( (B, B') \) of \( T_1 \)-consistent basic concepts over \( \Sigma_1 \) it holds that: \( (B, B') \) is \( \langle T_1 \cup T_1 \rangle \)-consistent if and only if \( (B, B') \) is \( \langle T_1 \cup T_2 \rangle \)-consistent, and likewise for every pair of basic roles over \( \Sigma_1 \). This condition guarantees that for every ABox \( A_1 \) over \( \Sigma_1 \) that is consistent with \( T_1 \), it holds that: \( \langle T_1 \cup T_2, A_1 \rangle \) is consistent if and only if there exists an extended ABox \( A_2 \) over \( \Sigma_2 \) such that \( A_2 \) is a UCQ-solution for \( A_1 \) under \( M \) and \( \langle T_2, A_2 \rangle \) is consistent. Thus, the previous condition ensures that the sets on the left- and right-hand side of Equation (1) coincide whenever the intersection on either of these sides is taken over an empty set.

The following theorem, which requires of a lengthy and non-trivial proof, shows that there exists an efficient algorithm for the membership problem for UCQ-representations that can deal with all the aforementioned issues.

**Theorem 6.4** The membership problem for UCQ-representations is \textsc{NLogSpace-complete}.

We conclude by pointing out that the non-emptiness problem for UCQ-representations can also be solved efficiently. We give an intuition of how this can be done in the following example, where we say that \( T_1 \) is \textit{UCQ-representable under} \( M \) if there exists a UCQ-representation \( T_2 \) of \( T_1 \) under \( M \).

**Example 6.5** Assume that \( M = (\Sigma_1, \Sigma_2, T_{12}) \), where \( \Sigma_1 = \{ F(\cdot), G(\cdot), H(\cdot) \} \), \( \Sigma_2 = \{ F'(\cdot), G'(\cdot), H'(\cdot) \} \) and \( T_{12} = \{ F \sqsubseteq F', G \sqsubseteq G', H \sqsubseteq H' \} \). Moreover, assume that \( T_1 = \{ F \sqsubseteq G \} \). Then it follows that \( T_1 \cup T_{12} \models F \sqsubseteq G' \), and in order for \( T_1 \) to be UCQ-representable under \( M \), the following condition must be satisfied:

\( (\ast) \) there exists a concept \( B' \) over \( \Sigma_2 \) s.t. \( T_{12} \models F \sqsubseteq B' \), and for each concept \( B \) over \( \Sigma_1 \) with \( T_1 \cup T_{12} \models B \sqsubseteq B' \) it follows that \( T_1 \cup T_{12} \models B \sqsubseteq G' \).

The idea is then to add the inclusion \( B' \sqsubseteq G' \) to a UCQ-representation \( T_2 \) so that \( T_1 \cup T_{12} \models F \sqsubseteq G' \) as well. In our case, concept \( F' \) satisfies the condition \( T_{12} \models F \sqsubseteq F' \), but it does not satisfy the second requirement as \( T_1 \cup T_{12} \models H \sqsubseteq F' \) and \( T_1 \cup T_{12} \models H \sqsubseteq G' \). In fact, \( F' \sqsubseteq G' \) cannot be added to \( T_2 \) as it would result in \( T_{12} \cup T_2 \models H \sqsubseteq G' \), hence in Equation (1), the inclusion from right to left would be violated. There is no way to reflect the inclusion \( F \sqsubseteq G \) in the target, so in this case \( T_1 \) is not UCQ-representable under \( M \).

The proof of the following result requires of some involved extensions of the techniques used to prove Theorem 6.4.

**Theorem 6.6** The non-emptiness problem for UCQ-representations is \textsc{NLogSpace-complete}.

The techniques used to prove Theorem 6.6 which is sketched in the example below.

**Example 6.7** Consider \( M \) and \( T_1 \) from Example 6.3 but assuming that \( T_{12} \) does not contain the inclusion \( H \sqsubseteq F' \). Again, \( T_1 \cup T_{12} \models F \sqsubseteq G' \), but now condition \( (\ast) \) is satisfied. Then, an algorithm for computing a representation essentially needs to take any \( B' \) given by condition \( (\ast) \) and add the inclusion \( B' \sqsubseteq F' \) to \( T_2 \). In this case, \( T_2 = \{ F' \sqsubseteq G' \} \) is a UCQ-representation of \( T_1 \) under \( M \).

7 Conclusions

In this paper, we have studied the problem of KB exchange for OWL 2 QL, improving on previously known results with respect to both the expressiveness of the ontology language and the understanding of the computational properties of the problem. Our investigation leaves open several issues, which we intend to address in the future. First, it would be good to have characterizations of classes of source KBs and mappings for which universal (UCQ-)solutions are guaranteed to exist. As for the computation of universal solutions, while we have pinned-down the complexity of membership for extended ABoxes as \textsc{NP-complete}, an exact bound for the other case is still missing. Moreover, it is easy to see that allowing for inequalities between terms (e.g., \( a \neq b \) in Example 3.1) and for negated atoms in the (target) ABox would allow one to obtain more universal solutions, but a full understanding of this case is still missing. Finally, we intend to investigate the challenging problem of computing universal UCQ-solutions, adopting also here an automata-based approach.

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A Definitions and Preliminary Results

Let $\Sigma$ be a $\text{DL-Lite}_K$ signature; a concept name $A$ (role name $P$) is said to be over $\Sigma$, if $A \in \Sigma$ ($P \in \Sigma$).
A basic role $R$ is said to be over $\Sigma$, if, either it is a role name over $\Sigma$, or $R = P^-$ for a role $P$ over $\Sigma$; a basic concept $B$ is said to be over $\Sigma$, if either it is a concept name, which is over $\Sigma$, or $B = \exists P R$ and $R$ is a basic role over $\Sigma$. We naturally extend these definitions to TBoxes, ABoxes, KBs, and queries; so we can refer to $\Sigma$-TBoxes or TBoxes over $\Sigma$, and analogously for ABoxes, KBs, and queries.

Define relation $\sqsubseteq_T$ to be the reflexive and transitive closure of the following relation on the set of all basic roles over $N_R$:

\[
\{(R_1, R_2) \mid R_1 \sqsubseteq R_2 \in T \lor R_1^- \sqsubseteq R_2^- \in T\},
\]

and let $\sqsubseteq_C$ be the reflexive and transitive closure of the following relation on the set of all basic concepts over $N_C$:

\[
\{(B_1, B_2) \mid B_1 \sqsubseteq B_2 \in T \cup \{(\exists R_1, \exists R_2) \mid R_1 \sqsubseteq_T R_2 \}\}.
\]

Then define the relation $\models$ between $\mathcal{K}$ and the $\text{DL-Lite}_K$ membership assertions over $\Sigma$ as:

\[
\{(\mathcal{K}, B(a)) \mid \text{there exists a basic concept } B' \text{ s.t. } A \models B'(a) \text{ and } B' \sqsubseteq_T B \} \cup
\{(\mathcal{K}, R(a, b)) \mid \text{there exists a basic role } R' \text{ s.t. } A \models R'(a, b) \text{ and } R' \sqsubseteq_T R \}.
\]

Notice that for consistent $\mathcal{K}$, for every membership assertion $\alpha$ it holds that $\mathcal{K} \models \alpha$ if and only if $\mathcal{K} \models \alpha$.

Moreover, for every basic role $R$ over $N_R$, define $[R]$ as \{ $S \mid R \sqsubseteq T S \text{ and } S \sqsubseteq_T R \}$, and then let $\leq_T$ be a partial order on the set $\{[R] \mid R \text{ is a basic role over } N_R\}$ defined as $[R] \leq_T [S]$ if $R \sqsubseteq_T S$.

For each set $[R]$, where $R$ is a basic role, consider an element $w_{[R]}$, witness for $[R]$. Now, define a generating relationship $\sim_K$ between the set $N_a \cup \{w_{[R]} \mid R \text{ is a basic role}\}$, as follows:

- \( a \sim_K w_{[R]}. \) if (1) $\mathcal{K} \models \exists R(a)$; (2) $\mathcal{K} \not\models R(a, b)$ for every $b \in N_a$; (3) $[R'] = [R]$ for every $[R']$ such that $[R'] \leq_T [R]$ and $\mathcal{K} \models \exists R'(a)$.

- $w_{[S]} \sim_K w_{[R]}. \) if (1) $T \models \exists S' \sqsubseteq \exists R$; (2) $[S'] \neq [R]$; (3) $[R'] = [R]$ for every $[R']$ such that $[R'] \leq_T [R]$ and $T \models \exists S' \sqsubseteq R'$.

Denote by path($\mathcal{K}$) the set of all $\mathcal{K}$-paths, where a $\mathcal{K}$-path is a sequence $a\cdot w_{[R_1]}\ldots w_{[R_n]}$ (sometimes we simply write $aw_{[R_1]}\ldots w_{[R_n]}$) such that for every $i \in \{1, \ldots, n\}$. Moreover, for every $\sigma \in \text{path}(\mathcal{K})$, denote by tail($\sigma$) the last element in $\sigma$.

With all the previous notation, we can finally define the canonical model $\mathcal{U}_K$. The domain $\Delta^{\mathcal{U}_K}$ of $\mathcal{U}_K$ is defined as path($\mathcal{K}$), and $a^{\mathcal{U}_K} = a$ for every $a \in N_a$. Moreover, for every concept $A$:

\[
\mathcal{A}^{\mathcal{U}_K} = \{ \sigma \in \text{path}(\mathcal{K}) \mid \mathcal{K} \models A(\text{tail}(\sigma)) \text{ or } \text{tail}(\sigma) = w_{[R]} \text{ and } T \models \exists R^- \sqsubseteq A \},
\]

and for every role $P$, we have that $P^{\mathcal{U}_K}$ is defined as follows:

\[
\{(\sigma_1, \sigma_2) \in \text{path}(\mathcal{K}) \times \text{path}(\mathcal{K}) \mid \mathcal{K} \models P(\text{tail}(\sigma_1), \text{tail}(\sigma_2)); \text{ or } \sigma_2 = \sigma_1 \cdot w_{[R]}, \text{tail}(\sigma_1) \sim_K w_{[R]} \text{ and } [R] \leq_T [P]; \text{ or } \sigma_1 = \sigma_2 \cdot w_{[R]}, \text{tail}(\sigma_2) \sim_K w_{[R]} \text{ and } [R] \leq_T [P^-] \}.
\]

Notice that $\mathcal{U}_K$ defined above can be treated (by ignoring sets $N^{\mathcal{U}_K}$ for some concepts and role names $N$) as a $\Sigma$-interpretation, for any $\Sigma$. Denote also by $\text{Ind}(\mathcal{A})$ the set of constants occurring in $\mathcal{A}$.

Let us point out the similarity of our definition of $\mathcal{U}_K$ with the definition of the canonical model $\mathcal{M}_K$ defined in [Konev et al., 2011]. When $\mathcal{K}$ is consistent, many results proved there for $\mathcal{M}_K$ apply to $\mathcal{U}_K$. In particular, from the proof of Theorem 5 in [Konev et al., 2011] we can immediately conclude:

**Claim A.1** If $\mathcal{K}$ is consistent, $\mathcal{U}_K$ is a model of $\mathcal{K}$.

We are going to introduce the notions of $\Sigma$-types and $\Sigma$-homomorphisms, heavily employed in the proofs.

For an interpretation $\mathcal{I}$ and a signature $\Sigma$, the $\Sigma$-types $t_{\mathcal{I}}^\Sigma(x)$ and $r_{\mathcal{I}}^\Sigma(x, y)$ for $x, y \in \Delta^\mathcal{I}$ are given by:

\[
t_{\mathcal{I}}^\Sigma(x) = \{ B - \text{basic concept over } \Sigma \mid x \in B^\Sigma \},
\]

\[
r_{\mathcal{I}}^\Sigma(x, y) = \{ R - \text{basic role over } \Sigma \mid (x, y) \in R^\Sigma \}.
\]
We also use \( t^a(x) \) and \( r^a(x, y) \) to refer to the types over the signature of all DL-Lite concepts and roles. A \( \Sigma \)-homomorphism from an interpretation \( \mathcal{I} \) to \( \mathcal{I}' \) is a function \( h : \Delta^\mathcal{I} \rightarrow \Delta^\mathcal{I}' \) such that \( h(a^\mathcal{I}) = a^{\mathcal{I}'} \), for all individual names \( a \) interpreted in \( \mathcal{I} \), \( t^a_\mathcal{I}(x) \subseteq t^a_{\mathcal{I}'}(h(x)) \) and \( r^a_\mathcal{I}(x, y) \subseteq r^a_{\mathcal{I}'}(h(x), h(y)) \) for all \( x, y \in \Delta^\mathcal{I} \). We say that \( \mathcal{I} \) is (finitely) \( \Sigma \)-homomorphically embeddable into \( \mathcal{I}' \) if, for every (finite) subinterpretation \( \mathcal{I}_I \) of \( \mathcal{I} \), there exists a \( \Sigma \)-homomorphism from \( \mathcal{I}_I \) to \( \mathcal{I}' \). If \( \Sigma \) is a set of all DL-Lite concepts and roles, we call \( \Sigma \)-homomorphism simply homomorphism.

The claim below from the proof of Theorem 5 in [Konev et al., 2011] establishes the relation between \( \mathcal{U}_K \) and the models of \( \mathcal{K} \).

Claim A.2 For every model \( \mathcal{I} \models \mathcal{K} \), there exists a homomorphism from \( \mathcal{U}_K \) to \( \mathcal{I} \).

Another result follows from Theorem 5 in [Konev et al., 2011]:

Claim A.3 For each consistent KB \( \mathcal{K} \), every UCQ \( q(\bar{x}) \) and tuple \( \bar{a} \subseteq N_a \), it holds \( \mathcal{K} \models q(\bar{a}) \) iff \( \mathcal{U}_K \models q(\bar{a}) \).

It is important to notice that the notion of certain answers can be characterized through the notion of canonical model. Finally, for a signature \( \Sigma \) and two KBs \( \mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle \) and \( \mathcal{K}_2 = \langle \mathcal{T}_2, \mathcal{A}_2 \rangle \), we say that \( \mathcal{K}_1 \)-\( \Sigma \)-query entails \( \mathcal{K}_2 \) if, for all \( \Sigma \)-queries \( q(\bar{x}) \) and all \( \bar{a} \subseteq N_a \), \( \mathcal{K}_1 \models q(\bar{a}) \) implies \( \mathcal{K}_2 \models q(\bar{a}) \). The KBs \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are said to be \( \Sigma \)-query equivalent if \( \mathcal{K}_1 \)-\( \Sigma \)-query entails \( \mathcal{K}_2 \) and vice versa. The following is a consequence of Theorem 7 in [Konev et al., 2011]:

Claim A.4 Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be consistent KBs. Then \( \mathcal{K}_1 \)-\( \Sigma \)-query entails \( \mathcal{K}_2 \) iff \( \mathcal{U}_{\mathcal{K}_2} \) is finitely \( \Sigma \)-homomorphically embeddable into \( \mathcal{U}_{\mathcal{K}_1} \).

B Proofs in Section 5

B.1 Definitions and Preliminary Results: Characterization of Universal Solutions

First, we define the notion of canonical model for extended ABoxes. Let \( \mathcal{A} \) be an extended ABox. Without loss of generality, assume that \( \mathcal{A} \) does not contain assertions of the form \( \exists R(x) \). Then the canonical model of \( \mathcal{A} \), denoted \( \mathcal{V}_A \), is defined as follows: \( \Delta^{\mathcal{V}_A} = \text{Null}(\mathcal{A}) \cup N_a \), where \( \text{Null}(\mathcal{A}) \) is the set of labeled nulls mentioned in \( \mathcal{A} \). For each \( a \in N_a \), \( A^{\mathcal{V}_A} = \{ x \in \Delta^{\mathcal{V}_A} \mid A(x) \in \mathcal{A} \} \) for each atomic concept \( A \), and \( P^{\mathcal{V}_A} = \{ (x, y) \in \Delta^{\mathcal{V}_A} \times \Delta^{\mathcal{V}_A} \mid P(x, y) \in \mathcal{A} \} \) for each atomic role \( P \). Let \( h \) be a function from \( N_a \cup N_I \rightarrow \Delta^{\mathcal{V}_A} \) such that \( h(a) = a^\mathcal{I} \) for every \( a \in N_a \) and \( h(x) = x^\mathcal{I} \) for every \( x \in N_I \). Then \( \mathcal{V}_A \) is a model of \( \mathcal{A} \) with substitution \( h \).

Lemma B.2 For every model \( \mathcal{I} \models \mathcal{A}_2 \), there exists a homomorphism from \( \mathcal{V}_{\mathcal{A}_2} \) to \( \mathcal{I} \).

Proof. Let \( \mathcal{I} \) be a model of \( \mathcal{A}_2 \) with a substitution \( h' \). Then \( h' \) is the desired homomorphism from \( \mathcal{V}_{\mathcal{A}_2} \) to \( \mathcal{I} \).

The following lemma is a characterization of universal solutions in DL-Lite\(_R^\text{pos} \). Recall that in Proposition 4.1 from [Arenas et al., 2012a] we showed that if \( \langle \mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1 \rangle \) is consistent and a KB \( \mathcal{K}_2 \) is a universal solution for \( \langle \mathcal{T}_1, \mathcal{A}_1 \rangle \) under \( \mathcal{M} = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1 \rangle \), then \( \mathcal{T}_2 \) is a trivial TBox, i.e., a TBox that admits the same models as the empty TBox. Therefore, without loss of generality, in the rest of this section when we talk about universal solutions, we mean target ABoxes.

Lemma B.3 Let \( \mathcal{M} = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_12 \rangle \) be a DL-Lite\(_R^\text{pos} \) mapping, \( \mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle \) a DL-Lite\(_R^\text{pos} \) KBs over \( \mathcal{S}_1 \), and \( \mathcal{A}_2 \) an (extended, without inequalities) ABox over \( \mathcal{S}_2 \). Then, \( \mathcal{A}_2 \) is a universal solution (with extended ABoxes) for \( \mathcal{K}_1 \) under \( \mathcal{M} \) if \( \mathcal{V}_{\mathcal{A}_2} \) is (extended) DL-Lite\(_R^\text{pos} \) ABox over \( \mathcal{S}_2 \) without inequalities.

Proof. \((\Rightarrow)\) Let \( \mathcal{A}_2 \) be a universal solution for \( \mathcal{K}_1 \) under \( \mathcal{M} \). Then \( \mathcal{V}_{\mathcal{A}_2} \) is (extended) DL-Lite\(_R^\text{pos} \) ABox over \( \mathcal{S}_2 \). Thus, there exists \( \mathcal{I} \) a model of \( \mathcal{K}_1 \) such that \( \langle \mathcal{I}, \mathcal{V}_{\mathcal{A}_2} \rangle \models \mathcal{M} \). Then \( \mathcal{I} \cup \mathcal{V}_{\mathcal{A}_2} \) is a model of \( \mathcal{K}_1 \). Therefore there is a homomorphism \( h \) from \( \mathcal{U}(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1) \) to \( \mathcal{I} \cup \mathcal{V}_{\mathcal{A}_2} \). As \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are disjoint signatures it follows that \( h \) is a (extended) DL-Lite\(_R^\text{pos} \) homomorphism from \( \mathcal{U}(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1) \) to \( \mathcal{V}_{\mathcal{A}_2} \).
On the other hand, as $A_2$ is a universal solution, $J$, the interpretation of $\Sigma_2$ obtained from $U(T_1 \cup T_{12} - A_1)$ is a model of $A_2$ with a substitution $h'$. This $h'$ is exactly a homomorphism from $V_{A_2}$ to $U(T_1 \cup T_{12} - A_1)$. 

(\Rightarrow) Assume $A_2$ is $\Sigma_2$-homomorphically equivalent to $U(T_1 \cup T_{12} - A_1)$. We show that $A_2$ is a universal solution for $K_1$ under $M$.

First, $A_2$ is a solution for $K_1$ under $M$. Let $J$ be a model of $A_2$, and $h_1$ a homomorphism from $V_{\{a, b\}}$ to $J$. Furthermore, let $h$ be a $\Sigma_2$-homomorphism from $U(T_1 \cup T_{12} - A_1)$ to $V_{A_2}$. Then $h' = h_1 \circ h$ is a $\Sigma_2$-homomorphism from $U(T_1 \cup T_{12} - A_1)$ to $J$. Let $I$ be the interpretation of $\Sigma_1$ defined as the image of $h'$ applied to $K_{\{a\}}$, with $I = h'(h_{\{a\}})$. The $I$ is a model of $K_1$ and $(I, J) \models M$ as $K_1$ and $M$ contain only positive information. Indeed, $A_2$ is a solution for $K_1$ under $M$.

Second, $A_2$ is a universal solution. Let $I$ be a model of $K_1$ and $J$ an interpretation of $\Sigma_2$ such that $(I, J) \models M$. Then, since $U(T_1 \cup T_{12} - A_1)$ is the canonical model of $K_1 \cup T_{12}$, there exists a homomorphism $h$ from $U(T_1 \cup T_{12} - A_1)$ to $I \cup J$ $(I \cup J)$ is a model of $K_1 \cup T_{12}$). In turn, there is a homomorphism $h_1$ from $V_{A_2}$ to $U(T_1 \cup T_{12} - A_1)$, therefore $h' = h \circ h_1$ is a homomorphism from $V_{A_2}$ to $I \cup J$, and a $\Sigma_2$-homomorphism from $V_{A_2}$ to $J$. Hence, $J$ is a model of $A_2$: take $h'$ as the substitution for the labeled nulls. By definition of universal solution, $A_2$ is a universal solution for $K_1$ under $M$. □

The definition below is used in the characterization of universal solutions in the general case. Its purpose is to single out the cases when a universal solution does not exist due to the need to represent in the target a form of negative information (for instance, in the form of inequalities of negated atoms).

**Definition B.4** Let $M = (\Sigma_1, \Sigma_2, T_{12})$ be a DL-Lite$_R$ mapping, and $K_1 = (T_1, A_1)$ a DL-Lite$_R$ KB over $\Sigma_1$. Then, we say that $K_1$ and $M$ are $\Sigma_2$-positive if

(a) for each $b \in B^U(T_1 \cup T_{12} - A_1)$ and $c \in C^U(T_1 \cup T_{12} - A_1)$ with $T_1 \models B \cap C \sqsubseteq \perp$, it is not the case that $b \in \text{InTarget}$ and $c \in \text{InTarget}$,

(b) for each $(b_1, b_2) \in R^U(T_1 \cup T_{12} - A_1)$ and $(c_1, c_2) \in Q^U(T_1 \cup T_{12} - A_1)$ with $T_1 \models R \cap Q \sqsubseteq \perp$ for basic roles $R, Q$, it is not the case that $b_1 \in \text{InTarget}$ and $c_1 \in \text{InTarget}$ for $i = 1, 2$,

(c) for each $(a, b) \in R^U(T_1 \cup T_{12} - A_1)$ and $(a, c) \in Q^U(T_1 \cup T_{12} - A_1)$ with $T_1 \models R \cap Q \sqsubseteq \perp$ for basic roles $R, Q$, it is not the case that $b \in \text{InTarget}$ and $c \in \text{InTarget}$,

where

\[ \text{InTarget} = \{ x \in \Delta^U(T_1 \cup T_{12} - A_1) \mid t^U_{\Sigma_2}(T_1 \cup T_{12} - A_1)(x) \neq \emptyset \} \cup N_a \]

(d) for each $B \sqsubseteq \neg B' \in T_{12}$, $B^U(T_1 \cup T_{12} - A_1) = \emptyset$ and for each $R \sqsubseteq \neg R' \in T_{12}$, $R^U(T_1 \cup T_{12} - A_1) = \emptyset$.

In the following, given a TBox $T$, we denote by $T^{\text{pos}}$ the subset of $T$ without disjointness assertions, and given a KB $K = (T, A)$, we denote by $K^{\text{pos}}$ the KB $(T^{\text{pos}}, A)$ . Moreover, if $M = (\Sigma_1, \Sigma_2, T_{12})$ is a DL-Lite$_R$ mapping, then $M^{\text{pos}}$ denotes the mapping $(\Sigma_1, \Sigma_2, T_{12}^{\text{pos}})$. Finally, we provide a characterization of universal solutions in DL-Lite$_R$.

**Lemma B.5** Let $M = (\Sigma_1, \Sigma_2, T_{12})$ be a DL-Lite$_R$ mapping, $K_1 = (T_1, A_1)$ a DL-Lite$_R$ KBs over $\Sigma_1$, and $A_2$ an (extended, without inequalities, without negation) ABox over $\Sigma_2$. Then, $A_2$ is a universal solution (with extended ABoxes) for $K_1$ under $M$ iff

1. $K_1$ and $M$ are $\Sigma_2$-positive,
2. $A_2$ is a universal solution for $K_1^{\text{pos}}$ under $M^{\text{pos}}$.

**Proof.** ($\Rightarrow$) Let $A_2$ be a universal solution for $K_1$ under $M$. Then $A_2$ is a universal solution for $K_1^{\text{pos}}$ under $M^{\text{pos}}$.

For the sake of contradiction, assume that $K_1$ and $M$ are not $\Sigma_2$-positive, and e.g., (a) does not hold, i.e., there is a disjointness constraint in $T_1$ of the form $B \cap C \sqsubseteq \perp$, such that $b \in B^U(T_1 \cup T_{12} - A_1)$ and $c \in C^U(T_1 \cup T_{12} - A_1)$, and

\[ t^U_{\Sigma_2}(T_1 \cup T_{12} - A_1)(b) \neq \emptyset \quad \text{or} \quad b \in N_a, \]

\[ t^U_{\Sigma_2}(T_1 \cup T_{12} - A_1)(c) \neq \emptyset \quad \text{or} \quad c \in N_a. \]

Let $h$ be a $\Sigma_2$-homomorphism from $U(T_1 \cup T_{12} - A_1)$ to $V_{A_2}$ (it exists by Lemma [B.3]). Then it follows that

\[ t^V_{A_2}(h(b)) \neq \emptyset \quad \text{or} \quad b \in N_a, \]

\[ t^V_{A_2}(h(c)) \neq \emptyset \quad \text{or} \quad c \in N_a. \]
Take a minimal model $\mathcal{J}$ of $A_2$ with a substitution $h'$ such that $h'((b)) = h'((c))$. Assume that both $b$ and $c$ are constants (i.e., $b^J = c^J$). Then, obviously, there exists no model $\mathcal{I}$ of $\Sigma_1$ such that $\mathcal{I} \models K_1$ and $(\mathcal{I}, \mathcal{J}) \models T_{12}$; in every such $\mathcal{I}$, $b^J$ must be equal to $c^J$ which contradicts $B \cap C \subseteq \bot$, and $h' \in B^\mathcal{J}$ and $c^J \notin C^\mathcal{I}$. Now, assume that at least $b$ is not a constant and $t(b) = w_R[b]$ for some role $R$ over $\Sigma_1$ (hence, $b \in \exists R^-(h((\tau_1 \cup \tau_2 \cdot A_1)) \cap T_1 \models \exists R^-(b) \subseteq B$). Let $B' \in \mathcal{I}_{\Sigma_2}^{U(T_1 \cup T_2 \cdot A_1)}(b)$, then by construction of the canonical model, $T_1 \cup T_2 \models \exists R^-(b)'$, by homomorphism, $B'(h((b))) \in A_2$, $h'(h((b))) \in B'^{\mathcal{J}}$, and since $\mathcal{J}$ is a minimal model, $B'^{\mathcal{J}}$ is minimal. As $A_2$ is a universal solution, let $\mathcal{I}$ be a model of $K_1$ such that $(\mathcal{I}, \mathcal{J})$ satisfy $T_{12}$. Then $(\exists R^-(b)')$ is not empty, and by minimality of $B'^{\mathcal{J}}$, it must be the case that $h'(h((b))) \in (\exists R^-(b))$, hence $h'(h((b))) \in B'^{\mathcal{J}}$. By a similar argument, it can be shown that $h'(h((c)))$ must be in $C^\mathcal{J}$. As we took $\mathcal{J}$ such that $h'(h((b))) = h'(h((c)))$, it contradicts that $\mathcal{I}$ is a model of $B \cap C \subseteq \bot$. Contradiction with $A_2$ being a universal solution. Similar to (a) we can derive a contradiction if assume that (b) or (c) does not hold.

Finally, assume (d) does not hold, i.e., $B \subseteq \neg B' \in T_{12}$ and $B^U(T_1 \cup T_2 \cdot A_1) = \emptyset$. Note that $A_2$ is an extended ABox, i.e., it contains only assertions of the form $A(u)$, $P((u, v) \in R \cup Q \cup \delta)$. Take a model $\mathcal{J}$ of $A_2$ such that $B'^{\mathcal{J}} = \Delta^{\mathcal{J}}$. Such $\mathcal{J}$ exists as $A_2$ contains only positive facts. Since $A_2$ is a universal solution, there exist a model $\mathcal{I}$ of $K_1$ such that $(\mathcal{I}, \mathcal{J}) \models T_{12}$. Then, $B^\mathcal{J} \neq \emptyset$, and it is easy to see that $(\mathcal{I}, \mathcal{J}) \models B \subseteq \neg B'$ because $B^\mathcal{J} \subseteq \Delta^{\mathcal{J}} \setminus B'^{\mathcal{J}} = \emptyset$. In every case we derive a contradiction, hence $K_1$ and $M$ are $\Sigma_2$-positive.

(⇐) Assume conditions 1-2 are satisfied. We show that $A_2$ is a universal solution for $K_1$ under $M$.

First, $A_2$ is a solution for $K_1$ under $M$. Let $\mathcal{J}$ be a model of $A_2$, then there exists $\mathcal{I}$ a model of $K_1$ such that $(\mathcal{I}, \mathcal{J}) \models T_{12}$.

We show that if $\mathcal{J}$ is a model of $K_1$ and $M$ are $\Sigma_2$-positive. Assume $b$ is a null and $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(b) = \emptyset$. Then by definition of $h'$, $h'(b) = d_b \in \Delta$ (and $d = d_b$). In either case $c$ is a constant, or $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(c) = \emptyset$, we obtain contradiction with $h'(c) = d_b = h'(c)$ (remember, $\Delta$ and $\Delta^\mathcal{J}$ are disjoint). Contradiction rises from the assumption $B \subseteq \neg B$ in $T_{12}$. Next, assume $T_1 \models R \cap Q \subseteq \bot$. Then there exist $d_1, d_2 \in \Delta^\mathcal{I}$ such that $(d_1, d_2) \in R^\mathcal{J}$, and $h'(d_1, d_2) = h'(d_2)$ for $i = 1, 2$. Then it cannot be the case that $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(d_1, d_2) = \emptyset$.

Consider the following cases:

- $b_1$ is a null and $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(b_1) = \emptyset$. Then by definition of $h'$, $h'(b_1) = d_{b_1} \in \Delta$ (and $d_1 = d_{b_1}$).
- $c_1$ is a null and $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(c_1) = \emptyset$, then $h'(c_1) = d_{c_1} = d_1$, hence $c_1 = b_1$ and $(b_1, c_1) \in R^\mathcal{I}$, $(b_1, c_2) \in Q^\mathcal{I}$. By (c) in the definition of $K_1$ and $M$ are $\Sigma_2$-positive, it cannot be the case that $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(b_2) = \emptyset$ or $b_2$ is a constant. Assume $c_2$ is a null and $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(c_2) = \emptyset$. Then $h'(c_2) = d_{c_2} \in \Delta$ and in either case $c_2$ is a constant, or $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(c_2) = \emptyset$, or $\mathcal{I}_{\Sigma_2 \cdot T_1 \cup T_2 \cdot A_1}(c_2) = \emptyset$, we obtain contradiction with $h'(c_2) = d_{c_2} = h'(c_2)$.
- Otherwise we obtain contradiction with $h'(b_1) = d_{b_1} = h'(c_1)$.

The cases $b_2$ or $c_1$ are nulls with the empty $\Sigma_2$-type are covered by swapping $R$ and $Q$ or by taking their inverses. Finally, assume $B \subseteq \neg C \subseteq T_{12}$ and $(T', \mathcal{J}) \not\models T_{12}$, i.e., for some $d \in B^\mathcal{J}$, $d \notin \Delta^\mathcal{J} \setminus C^\mathcal{J}$. Then
there must exist \( b \in B^{|K_1|} \) such that \( h'(b) = d \). Contradiction with (d). Therefore, indeed, \( I \) is a model of \( K_1 \) and \( (I, J) \models T_{12} \). This concludes the proof \( A_2 \) is a solution for \( K_1 \) under \( M \).

Second, \( A_2 \) is a universal solution. Let \( I \) be a model of \( K_1 \) and \( J \) an interpretation of \( \Sigma_2 \) such that \( (I, J) \models T_{12} \). Then, \( I \) is a model of \( K_1^{\text{pos}} \) and \( (I, J) \models T_{12}^{\text{pos}} \), and as \( A_2 \) is a universal solution for \( K_1^{\text{pos}} \) under \( M^{\text{pos}} \), it follows that \( J \) is a model \( A_2 \).

The following lemma shows that \( \Sigma_2 \)-positiveness can be checked in polynomial time.

**Lemma B.6** Let \( M = (\Sigma_1, \Sigma_2, T_{12}) \) be a mapping, and \( K_1 = \langle T_1, A_1 \rangle \) a KB over \( \Sigma_1 \). Then it can be decided in polynomial time whether \( K_1 \) and \( M \) are \( \Sigma_2 \)-positive.

**Proof.** We check (a) as follows:

- for each concept disjointness axiom \( B_1 \cap B_2 \subseteq \bot \in T_{11} \), check for \( i = 1, 2 \) if \( K_1 \models B_i(b_i) \) for some \( b_i \in \text{Ind}(A_1) \) or there exists a \( K_1 \)-path \( x = a \cdot w_{[S_1]} \ldots w_{[S_n]} \) such that \( b_i \in t_{(T_{12}, A_1)}^{U}U_{(T_{12}, A_1)}(x) \) and \( t_{\Sigma_2}^{U}(T_{12}, A_1)(x) \neq \emptyset \). If yes, then (a) does not hold, otherwise it holds.

We check (b) as follows:

- for each role disjointness axiom \( R \cap Q \subseteq \bot \in T_{11} \), check for \( i = 1, 2, 3, 4 \) if \( K_1 \models B_i(b_i) \) for some \( b_i \in \text{Ind}(A_1) \) or there exists a \( K_1 \)-path \( x = a \cdot w_{[S_1]} \ldots w_{[S_n]} \) such that \( b_i \in t_{(T_{12}, A_1)}^{U}U_{(T_{12}, A_1)}(x) \) and \( t_{\Sigma_2}^{U}(T_{12}, A_1)(x) \neq \emptyset \). If yes, then (b) does not hold, otherwise it holds.

We check (c) as follows:

- for each role disjointness axiom \( R \cap R \subseteq \bot \in T_{11} \), check if there exists a \( K_1 \)-path \( x = a \cdot w_{[S_1]} \ldots w_{[S_n]} \) such that \( \exists R_1, \exists R_2 \in \text{Ind}(A_1) \) or there exists a \( K_1 \)-path \( y_i = a' \cdot w_{[Q_1]} \ldots w_{[Q_n]} \) such that \( R_i \in t_{(T_{12}, A_1)}^{U}(T_{12}, A_1)(x, y_i) \) and \( t_{\Sigma_2}^{U}(T_{12}, A_1)(y_i) \neq \emptyset \). If yes, then (c) does not hold, otherwise it holds.

Note that in the previous three checks, it is sufficient to look for paths where \( n \) is bounded by the number of roles in \( K_1 \), moreover in the last check \( |n - n'| = 1 \).

We check (d) as follows:

- for each concept disjointness axiom \( B \subseteq \neg B' \in T_{12} \), check if \( K_1 \) implies that \( B \) is necessarily non-empty. If yes, then (d) does not hold, otherwise

- for each role disjointness axiom \( R \subseteq \neg R' \in T_{12} \), check if \( K_1 \) implies that \( R \) is necessarily non-empty. If yes, then (d) does not hold, otherwise it holds.

It is straightforward to see that each of the checks can be done in polynomial time as the standard reasoning in DL-Lite\(R \) is in NLOGSPACE.

**Lemma B.7** Let \( M = (\Sigma_1, \Sigma_2, T_{12}) \) be a mapping, and \( K_1 = \langle T_1, A_1 \rangle \) a KB over \( \Sigma_1 \) such that \( K_1 \) and \( M \) are \( \Sigma_2 \)-positive. Then, a universal solution (with extended ABoxes) for \( K_1 \) under \( M \) exists iff \( U(T_{11} \cup T_{12}, A_1) \) is \( \Sigma_2 \)-homomorphically embeddable into a finite subset of itself.

**Proof.** (\( \Rightarrow \)) Let \( A_2 \) be an ABox such that \( V_{A_2} \) is a finite subset of \( U(T_{11} \cup T_{12}, A_1) \) and there exists a \( \Sigma_2 \)-homomorphism \( h \) from \( U(T_{11} \cup T_{12}, A_1) \) to \( V_{A_2} \). Then, \( U(\emptyset, A_2) \) is trivially homomorphically embeddable into \( U(T_{11} \cup T_{12}, A_1) \). Hence by Lemma B.5, \( A_2 \) is a universal solution for \( K_1 \) under \( M \).

(\( \Rightarrow \)) Let \( A_2 \) be a universal solution for \( K_1 \) under \( M \). Then \( V_{A_2} \) is \( \Sigma_2 \)-homomorphically equivalent to \( U(T_{11} \cup T_{12}, A_1) \) by Lemma B.5. Let \( h \) be a homomorphism from \( V_{A_2} \) to \( U(T_{11} \cup T_{12}, A_1) \), and \( h(V_{A_2}) \) the image of \( h \). Then, \( h(V_{A_2}) \) is a finite subset of \( U(T_{11} \cup T_{12}, A_1) \), moreover it is homomorphically equivalent to \( V_{A_2} \) and to \( U(T_{11} \cup T_{12}, A_1) \). Therefore, it follows that \( U(T_{11} \cup T_{12}, A_1) \) is \( \Sigma_2 \)-homomorphically embeddable to a finite subset of itself.

**B.2 Definitions and Preliminary Results: The Automata Construction for Theorem S.5**

**Definition of alternating two-way automata**

Infinite trees are represented as prefix closed (infinite) sets of words over \( \mathbb{N} \) (the set of positive natural numbers). Formally, an infinite tree is a set of words \( T \subseteq \mathbb{N}^* \), such that if \( x \cdot c \in T \), where \( x \in \mathbb{N}^* \) and \( c \in \mathbb{N} \), then also \( x \in T \). The elements of \( T \) are called nodes, the empty word \( \epsilon \) is the root of \( T \), and for every \( x \in T \), the nodes \( x \cdot c \), with \( c \in \mathbb{N} \), are the successors of \( x \). By convention we take \( x \cdot 0 = x \), and \( x \cdot i \cdot -1 = x \). The branching degree \( d(x) \) of a node \( x \) denotes the number of successors of \( x \). If
the branching degree of all nodes of a tree is bounded by \( k \), we say that the tree has branching degree \( k \). An infinite path \( P \) of \( T \) is a prefix closed set \( P \subseteq T \) such that for every \( i \geq 0 \) there exists a unique node \( x \in P \) with \( |x| = i \). A labeled tree over an alphabet \( \Sigma \) is a pair \((T, V)\), where \( T \) is a tree and \( V : T \rightarrow \Sigma \) maps each node of \( T \) to an element of \( \Sigma \).

Alternating automata on infinite trees are a generalization of nondeterministic automata on infinite trees, introduced in [9]. They allow for an elegant reduction of decision problems for temporal and program logics [3, 1]. Let \( B(I) \) be the set of positive boolean formulae over \( I \), built inductively by applying \( \land \) and \( \lor \) starting from true, false, and elements of \( I \). For a set \( J \subseteq I \) and a formula \( \phi \in B(I) \), we say that \( J \) satisfies \( \phi \) if and only if, assigning true to the elements of \( J \) and false to those in \( I \setminus J \), makes \( \phi \) true. For a positive integer \( k \), let \( [k] = \{-1, 0, 1, \ldots, k\} \). A two-way alternating tree automaton (2ATA) running over infinite trees with branching degree \( k \), is a tuple \( \mathcal{A} = (\Sigma, Q, \delta, q_0, F) \), where \( \Sigma \) is the input alphabet, \( Q \) is a finite set of states, \( \delta : Q \times \Sigma \times B([k] \times Q) \) is the transition function, \( q_0 \in Q \) is the initial state, and \( F \) specifies the acceptance condition.

The transition function maps a state \( q \in Q \) and an input letter \( \sigma \in \Sigma \) to a positive boolean formula over \( [k] \times Q \). Intuitively, if \( \delta(q, \sigma) = \phi \), then each pair \((c, \phi')\) appearing in \( \phi \) corresponds to a new copy of the automaton going to the direction suggested by \( c \) and starting in state \( \phi' \). For example, if \( k = 2 \) and \( \delta(q_1, \sigma) = ((1, q_2) \land (1, q_3)) \lor ((-1, q_1) \land (0, q_3)), \) when the automaton is in the state \( q_1 \) and is reading the node \( x \) labeled by the letter \( \sigma \), it proceeds either by sending off two copies, in the states \( q_2 \) and \( q_3 \) respectively, to the first successor of \( x \) (i.e., \( x \cdot 1 \)), or by sending off one copy in the state \( q_1 \) to the predecessor of \( x \) (i.e., \( x \cdot -1 \)) and one copy in the state \( q_3 \) to \( x \) itself (i.e., \( x \cdot 0 \)).

A run of a 2ATA \( \mathcal{A} \) over a labeled tree \((T, V)\) is a labeled tree \((T', r)\) in which every node is labeled by an element of \( T \times Q \). A node in \( T_r \) labeled by \((x, q)\) describes a copy of \( A \) that is in the state \( q \) and reads the node \( x \) of \( T \). The labels of adjacent nodes have to satisfy the transition function of \( \mathcal{A} \). Formally, a run \((T_r, r)\) is a \( T \times Q \)-labeled tree satisfying:

- \( \epsilon \in T_r \) and \( r(\epsilon) = (\epsilon, q_0) \).
- Let \( y \in T_r \), with \( r(y) = (x, q) \) and \( \delta(q, V(x)) = \phi \). Then there is a (possibly empty) set \( S = \{(c_1, q_1), \ldots, (c_n, q_n)\} \subseteq [k] \times Q \) such that:
  - \( S \) satisfies \( \phi \) and
  - for all \( 1 \leq i \leq n \), we have that \( y \cdot i \in T_r \), \( x \cdot c_i \) is defined \((x \cdot c_i \in T)\), and \( r(y \cdot i) = (x \cdot c_i, q_i) \).

A run \((T_r, r)\) is accepting if all its infinite paths satisfy the acceptance condition. Given an infinite path \( P \in T_r \), let \( \text{inf}(P) \subseteq Q \) be the set of states that appear infinitely often in \( P \) (as second components of node labels). We consider here Büchi acceptance conditions. A Büchi condition over a state set \( Q \) is a subset \( F \) of \( Q \), and an infinite path \( P \) satisfies \( F \) if \( \text{inf}(P) \cap F \neq \emptyset \).

The non-emptiness problem for 2ATAs consists in determining, for a given 2ATA, whether the set of trees it accepts is nonempty. It is known that this problem can be solved in exponential time in the number of states of the input automaton \( \mathcal{A} \), but in linear time in the size of the alphabet as well as in the size of the transition function of \( \mathcal{A} \).

The automata construction

Now, we are going to construct two 2ATA automata and a one-way non-deterministic automaton to use them as a mechanism to decide the non-emptiness problem for universal solutions. More specifically, let \( \Sigma_1, \Sigma_2 \) be signatures with no concepts or roles in common, and \( K = (\mathcal{T}, A) \) a KB over \( \Sigma_1 \cup \Sigma_2 \), \( N = \{a_1, \ldots, a_n\} \) be the set of individuals in \( A_1 \), \( B \) be the set of basic concepts and \( R \) be the set of basic roles over the signature of \( K \) (that is, over \( \Sigma_1 \cup \Sigma_2 \)). Finally, assume that \( r, G \) are special characters not mentioned in \( N \cup B \cup R \), and let \( P = \{P_{ij} \mid P \text{ is an atomic role over the signature of } K \text{ and } 1 \leq i, j \leq n\} \). Then assuming that \( \Sigma_K = 2^N \cup B \cup R \cup P \cup \{\epsilon\} \) and \( \Gamma_K = \{\sigma \in \Sigma_K \mid r \in \sigma, \sigma \cap N \neq \emptyset, \text{ or every basic concept and every basic role in } \sigma \text{ is over } \Sigma_2\} \), we construct the following automata:

- \( \mathcal{A}_{\text{con}}^\cong_{\Sigma_K} \): The alphabet of this automaton is \( \Sigma_K \), and it accepts trees that are essentially the tree corresponding to the canonical model of \( K \), but with nodes arbitrary labeled with the special character \( G \).
- \( \mathcal{A}_{\text{mod}}^\cong_{\Sigma_K} \): The alphabet of this automaton is \( \Sigma_K \), and it accepts a tree if its subtree labeled with \( G \) corresponds to a tree model \( I \) of \( K \) (tree models are models which from trees on the labeled nulls).
- \( \mathcal{A}_{\text{fin}}^\cong_{\Sigma_K} \): The alphabet of this automaton is \( \Gamma_K \), and it accepts a tree if it has a finite prefix where each node is marked with the special symbol \( G \), and no other node in the tree is marked with \( G \).
Automaton $A_{\text{can}}$ for the canonical model of $K = \langle T, A \rangle$

$A_{\text{can}}$ is a two-way alternating tree automaton (2ATA) that accepts the tree corresponding to the canonical model of the DL-Lite$_R$ KB $K = \langle T, A \rangle$, with nodes arbitrarily labeled with a special character $G$.

Formally, $A_{\text{can}} = \langle \Sigma_K, Q_{\text{can}}, \delta_{\text{can}}, q_0, F_{\text{can}} \rangle$, where

$$Q_{\text{can}} = \{ q_0, q_s, q_{\neg r}, q_d \} \cup \{ q^*_X, q^*_\neg X \mid X \in N \cup B \cup R \cup P \} \cup \{ q_{\exists R}, q_{R} \mid R \in R \},$$

and the transition function $\delta_{\text{can}}$ is defined as follows. Assume without loss of generality that the number of basic roles over the signature of $K$ is equal to $n$ (this can always be done by adding the required assertions to the ABox), and let $f : R \rightarrow \{1, \ldots, n\}$ be a one-to-one function. Then $\delta_{\text{can}} : Q_{\text{can}} \times \Sigma_K \rightarrow B([n] \times Q_{\text{can}})$ is defined as:

1. For each $\sigma \in \Sigma_K$ such that $r \in \sigma$, $\delta_{\text{can}}(q_0, \sigma)$ is defined as:

$$\bigwedge_{i=1}^n \left[ (i, q_s) \land (i, q^*_r) \land (i, q^*_a) \land \left( \bigwedge_{j=1}^n (i, q^*_\neg a_j) \right) \right] \land$$

$$\bigwedge_{j=1}^n \left( \bigwedge_{P \in P : K \models P(a_i, a_j)} (0, q_{P_{ij}}) \land \bigwedge_{P \in P : K \n\models P(a_i, a_j)} (0, q^*_\neg P_{ij}) \right) \land$$

$$\left( \bigwedge_{B \in B : K \models B(a_i)} (i, q^*_B) \right) \land \left( \bigwedge_{B \in B : K \n\models B(a_i)} (i, q^*_\neg B) \right) \land$$

$$\left( \bigwedge_{R \in R : K \models \exists R \n\models \exists R_{\n\models R(a_i)}} (i, q^*_R) \right) \land \bigvee_{R \in R : K \models \exists R \models \exists R_{\models R(a_i)}} \left( i, q^*_\neg R \right) \right]$$

2. For each $\sigma \in \Sigma_K$:

$$\delta_{\text{can}}(q_s, \sigma) = \bigwedge_{i=1}^n \left[ (i, q_s) \land (i, q^*_r) \land \left( \bigwedge_{j=1}^n (i, q^*_\neg a_j) \right) \land \left( i, q_d \right) \lor \bigvee_{R \in R} \left( i, q^*_R \right) \right]$$

3. For each $\sigma \in \Sigma_K$:

$$\delta_{\text{can}}(q_d, \sigma) = \bigwedge_{R \in R} (0, q^*_\neg R) \land \bigwedge_{i=1}^n (i, q_d)$$

4. For each $\sigma \in \Sigma_K$ and each basic role $[R]$ from $R$:

$$\delta_{\text{can}}(q^*_\neg R, \sigma) = \bigwedge_{R' \in R} \left( f(R), q^*_\neg R' \right)$$

5. For each $\sigma \in \Sigma_K$ and each basic role $[R]$ from $R$:

$$\delta_{\text{can}}(q_{\exists R}, \sigma) = (f(R), q_{\exists R})$$

6. For each $\sigma \in \Sigma_K$ such that $\sigma \cap N = \emptyset$ and each basic role $[R]$ from $R$, $\delta_{\text{can}}(q_R, \sigma)$ is defined as

$$\left( \bigwedge_{R' \in R : K \models R \subseteq R'} (0, q^*_R) \right) \land \left( \bigwedge_{R' \in R : K \n\models R \subseteq R'} (0, q^*_\neg R) \right) \land$$

$$\left( \bigwedge_{B \in B : K \models \exists B \subseteq B} (0, q^*_B) \right) \land \left( \bigwedge_{B \in B : K \n\models \exists B \subseteq B} (0, q^*_\neg B) \right) \land$$

$$\left( \bigwedge_{S \in R : K \models \exists S \subseteq S} (0, q^*_S) \right) \land \bigvee_{S \in R : K \n\models \exists S \subseteq S \text{ or } [R] \n\models [S], S \text{ or } S \text{ is not } \tau \text{ minimal}} \left( 0, q^*_\neg S \right) \right]$$

7. For each $\sigma \in \Sigma_K$:

$$\delta_{\text{can}}(q^*_{\neg r}, \sigma) = \begin{cases} \text{true} & \text{if } r \notin \sigma \\ \text{false} & \text{otherwise} \end{cases}$$
where R

Finally, the acceptance condition is \( F_{\text{can}} = Q_{\text{can}} \).

To represent the canonical model \( U_\mathcal{K} \) of \( \mathcal{K} \) as a labeled tree, we label each individual \( x \) with the set of concepts \( B \) such that \( x \in B^{U_\mathcal{K}} \). We also add a basic role \( R \) to the label of \( x \) whenever \((x', x) \in R^{U_\mathcal{K}}\) and \( x \) is not an individual. Moreover, we make sure this tree is an infinite full \( n \)-ary tree, where \( n \) is the number of individuals in \( \text{Ind}(A) \) and basic roles in \( R \). Thus, let \( n^* \) be the set of sequences of numbers from 1 to \( n \) of the form \( n^* = \{ i_1 \cdot i_2 \cdot \ldots \cdot i_m | 1 \leq i_j \leq n, m \geq j \geq 0 \} \), the sequence of length 0 is denoted by \( \epsilon \).

Recall that we have a numbering of individuals \( \{a_1, \ldots, a_n\} = \text{Ind}(A) \), and each role \( R \in R \) can be identified through the number \( f(R) \in \{1, \ldots, n\} \). Therefore, the elements of \( \Delta^U \) can be seen as sequences of natural numbers, namely a sequence \( a_1 \cdot w[R_1] \cdot \ldots \cdot w[R_m] \) corresponds to the numeric sequence \( i \cdot f(R_1) \cdot \ldots \cdot f(R_m) \). However, for better readability we use the original notation as \( a_1 \cdot w[R_1] \cdot \ldots \cdot w[R_m] \).

Note, that \( \Delta^U \subseteq n^* \).

In the following, we assume \( \mathcal{K} \) is fixed and for simplicity we use \( \mathcal{U} \) instead of \( U_\mathcal{K} \).

The tree encoding of the canonical model \( \mathcal{U} \) of \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \) is the \( \Sigma_{\mathcal{K}} \)-labeled tree \( T_\mathcal{U} = (n^*, V^\mathcal{U}) \), such that

- \( V^\mathcal{U}(\epsilon) = \{r\} \cup \{P_{ij} | (a_i, a_j) \in P^\mathcal{U}, P \) is an atomic role\},
- for each \( x \in \Delta^U \):
  \[
  V^\mathcal{U}(x) = \begin{cases}B | x \in B^\mathcal{U} \cup & S | (x', x) \in S^\mathcal{U} \text{ and } x = x' \cdot w[R] \text{ for some role } R \text{ s.t. } [R] \leq \tau [S] \cup \\ {a | a \in \text{Ind}(A) \text{ and } x = a}&&\end{cases}
  \]

Conversely, we can see any \( \Sigma_{\mathcal{K}} \)-labeled tree as a representation of an interpretation of \( \mathcal{K} \), provided that each individual name occurs in the label of only one node, a child of the root. Informally, the domain of this interpretation are the nodes of the tree reachable from the root through a sequence of roles, except the root itself. The extensions of individuals, concepts and roles are determined by the node labels.

Given a \( \Sigma_{\mathcal{K}} \)-labeled tree \((T, V)\), we call a node \( c \) an individual node if \( a \in V(c) \) for some \( a \in \text{Ind}(A) \), and we call \( c \) an a-node if we want to make the precise a explicit. We say that \( T \) is individual unique if for each \( a \in \text{Ind}(A) \) there is exactly one a-node, a child of the root of \( T \).

An individual unique \( \Sigma_{\mathcal{K}} \)-labeled tree \((T, V), \) represents the interpretation \( \mathcal{I}_T \) defined as follows. For each role name \( P \), let:

\[
R_p = \{(x, x \cdot i) | P \in V(x \cdot i) \} \cup \{(x \cdot i, x) | P^\mathcal{U} \in V(x \cdot i) \} \cup
\{(c, c') | a_i \in V(c), a_j \in V(c') \text{ and } P_{ij} \in V(e)\}
\]

and
\[
\Delta^\mathcal{I}_T = \{ (i, x) \in \bigcup_{P \in R} (R_p \cup R_p)^* | i \in \{1, \ldots, n\} \}
\]

where \( R_p \) denotes the inverse of relation \( R_p \). Then the interpretation \( \mathcal{I}_T = (\Delta^\mathcal{I}_T, \mathcal{I}_T) \) is defined as:

\[
a^\mathcal{I}_T = c \text{ such that } a_i \in V(c), \text{ for each } a_i \in \text{Ind}(A)
\]

\[
A^\mathcal{I}_T = \Delta^\mathcal{I}_T \cap \{ A \in V(x) \}, \text{ for each atomic concept } A \in \mathcal{B}
\]

\[
P^\mathcal{I}_T = (\Delta^\mathcal{I}_T \times \Delta^\mathcal{I}_T) \cap R_p, \text{ for each atomic role } P \in \mathcal{R}
\]

**Proposition B.8** The following hold for \( \mathcal{A}_\mathcal{K}^\text{can} \):

- \( T_\mathcal{U} \in L(\mathcal{A}_\mathcal{K}^\text{can}) \).
- for each \((T, V) \in L(\mathcal{A}_\mathcal{K}^\text{can})\), \((T, V)\) is individual unique and \( \mathcal{I}_T \) is isomorphic to \( \mathcal{U} \), the canonical model of \( \mathcal{K} \).

**Proof.** For the first item, assume \( T_\mathcal{U} = (n^*, V^\mathcal{U}) \) is the tree encoding of the universal model \( \mathcal{U} \) of \( \mathcal{K} \). We show that a full run of \( \mathcal{A}_\mathcal{K}^\text{can} \) over \( T_\mathcal{U} \) exists.

The run \((T_r, r)\) is built starting from the root \( \epsilon \), and setting \( r(\epsilon) = (\epsilon, q_0) \). Then, to correctly execute the initial transition, the root has children as follows:
• for each $a_k \in \text{Ind}(A)$
  - a child $k_s$ with $r(k_s) = (a_k, q_s)$,
  - a child $k_{s_r}$ with $r(k_{s_r}) = (a_k, q^*_{s_r})$,
  - a child $k_{s_\alpha_j}$ with $r(k_{s_\alpha_j}) = (a_k, q^*_{s_\alpha_j})$,
  - a child $k^*_R$ for each $R \in \text{B}$ such that $a_k \in B\^R$, with $r(k^*_R) = (a_k, q^*_R)$,
  - a child $k^*_{s_R}$ for each $R \in \text{B}$ such that $a_k \notin B\^R$, with $r(k^*_{s_R}) = (a_k, q^*_{s_R})$,
  - a child $k^*_{s_R}$ for each $R \in \text{B}$ such that $a_k \notin B\^R$, with $r(k^*_{s_R}) = (a_k, q^*_{s_R})$,
  - a child $k^*_{s_R}$ for each $R \in \text{B}$ such that $a_k \notin B\^R$, with $r(k^*_{s_R}) = (a_k, q^*_{s_R})$,
  - a child $k^*_{s_R}$ for each $R \in \text{B}$ such that $a_k \notin B\^R$, with $r(k^*_{s_R}) = (a_k, q^*_{s_R})$.

Note that nodes $y \in T_s$ with $r(y) = (x, q^*_s)$ are leafs of the tree $T_s$, as by the transition function $\delta_{\text{can}}$, all the states of the form $q^*_s$ in $Q_{\text{can}}$ can be satisfied with the empty assignment.

Other nodes, however, can have children. They are defined inductively as follows.

2. Let $y$ be a node in $T_s$ such that $r(y) = (x, q_s)$ for some $x \in n^*$. Moreover, let $i \in \{1, \ldots, n\}$. Then $y$ has
  - a child $y \cdot i_s$ with $r(y \cdot i_s) = (x \cdot i, q_s)$,
  - a child $y \cdot i^*_s$ with $r(y \cdot i^*_s) = (x \cdot i, q^*_s)$,
  - a child $y \cdot i^*_{s_\alpha_j}$ for each $j \in \{1, \ldots, n\}$ with $r(y \cdot i^*_{s_\alpha_j}) = (x \cdot i, q^*_{s_\alpha_j})$,
  - if $x \in \Delta^U$ and for each $R \in \text{R}$ s.t. $f(R) = i, x \cdot w[R] \in \Delta^U$, then
    - a child $y \cdot i^*_R$ with $r(y \cdot i^*_R) = (x \cdot w[R], q^*_R)$,
    - otherwise
      - a child $y \cdot i_d$ with $r(y \cdot i_d) = (x, q_d)$.

3. Let $y$ be a node in $T_s$ such that $r(y) = (x, q_d)$ for some $x \in n^*$. Then $y$ has
  - a child $y \cdot i_d$ for each $i \in \{1, \ldots, n\}$, with $r(y \cdot i_d) = (x \cdot i, q_d)$,
  - a child $y \cdot 0^*_R$ for each $R \in \text{R}$, with $r(y \cdot 0^*_R) = (x, q^*_R)$.

4. Let $y$ be a node in $T_s$ such that $r(y) = (x, q^*_{s_R})$ for some $x \in \Delta^U$ and $R \in \text{R}$. Then $y$ has
  - a child $y \cdot f(R)^*_{s_R}$ for each $R' \in \text{R}$, with $r(y \cdot f(R)^*_{s_R}) = (x \cdot f(R), q^*_R)$.

5. Let $y$ be a node in $T_s$ such that $r(y) = (x, q_{sR})$ for some $x \in \Delta^U$ and $R \in \text{R}$. Then $x \cdot w[R] \in \Delta^U$ and $y$ has
  - a child $y \cdot f(R)^*_{s_R}$ with $r(y \cdot f(R)^*_{s_R}) = (x \cdot w[R], q^*_R)$.

6. Let $y$ be a node in $T_s$ such that $r(y) = (x, q_{sR})$ for some $x \in \Delta^U$ and $R \in \text{R}$. Then $y$ has
  - a child $y \cdot 0^*_{s_R}$ for each $R' \in \text{R}$ s.t. $K \models R \subseteq R'$, with $r(y \cdot 0^*_{s_R}) = (x, q^*_{s_R})$,
  - a child $y \cdot 0^*_{s_{s_R}}$ for each $R' \in \text{R}$ s.t. $K \not\models R \subseteq R'$, with $r(y \cdot 0^*_{s_{s_R}}) = (x, q^*_{s_{s_R}})$,
  - a child $y \cdot 0^*_{s_B}$ for each $B \in \text{B}$ s.t. $K \models \exists R^\-> B$, with $r(y \cdot 0^*_{s_B}) = (x, q^*_{s_B})$,
  - a child $y \cdot 0^*_{s_B}$ for each $B \in \text{B}$ s.t. $K \not\models \exists R^\-> B$, with $r(y \cdot 0^*_{s_B}) = (x, q^*_{s_B})$.

Each node of $T_s$ defined as described above satisfies the transition function $\delta_{\text{can}}$.

It is easy to see that this run is accepting, as for each infinite path $P$ of $\text{T}_s$, either $q_s \in \text{inf}(P)$, or $q_s \notin \text{inf}(P)$ for some $R$. Hence, $T_s \in \mathcal{L}(\text{K}_{\text{can}})$.

To show the second item, let $(T, V) \in \mathcal{L}(\text{K}_{\text{can}})$ and $(T_r, r)$ an accepting run of $(T, V)$. First, assume $T$ is not individual unique, that is,
• there exists an o-node \( x \) in \( T \), such that \( x \) is not a child of the root, or
• there exist two nodes \( i \) and \( j \) in \( T \) such that \( a \in V(i) \) and \( a \in V(j) \).

In the former case, let \( x' \) be the parent of \( x, x' \neq e \), then there exists a node \( y' \in T_r \) with \( r(y') = (x', q_0) \) and a node \( y \in T_r \) with \( r(y) = (x, q'_0) \), which contradicts that \( (T_r, r) \) is an accepting run of \( (T, V) \) as \( a \in V(x) \). In the latter case, assume \( a \) is equal to \( a_1 \). Then we get contradiction with \( \delta_{can}(q_0, \sigma) \).

Hence, \( T \) is individual unique. Let \( I_T \) be the interpretation represented by \( T \). We show that \( I_T \) is isomorphic to \( U \) by constructing a function \( h \) from \( D^{x_{2}} \) to \( D^{y_{2}} \) and showing that it is a one-to-one and onto homomorphism. We construct \( h \) by induction on the length of the sequence \( x \in D^{x_{2}} \).

Initially, as \( T \) is individual unique, we set for each \( i \in \{1, \ldots, n\} \), \( h(i) = a_i \), where \( a_i \in V(i) \). Note that by definition of \( U \), \( a_i \in D^{y_{2}} \) and by definition of \( I_T \), \( i \in D^{x_{2}} \). Then the following holds for \( i, j \in \{1, \ldots, n\} \).

1. for an atomic role \( P \), \((i, j) \in D^{x_{2}} \) iff \((a_i, a_j) \in D^{y_{2}} \); let \((i, j) \in D^{x_{2}} \), by definition of \( I_T \) it follows that \( P_{ij} \in V(x) \). Assume \( K \not\models P(a_i, a_j) \), then \((0, q_{\alpha_{ij}}) \in \delta_{can}(q_0, V(\epsilon)) \) and in \( T_r \) there exists a node \( y, r(y) = (\epsilon, q_{\alpha_{ij}}) \), hence \( y \) does not satisfy the condition on a run. Contradiction with \((T_r, r)\) being accepting. Therefore, indeed \( K \models P(a_i, a_j) \) and \((a_i, a_j) \in D^{y_{2}} \). Similarly for the other direction.

2. for a basic concept \( B \), \( i \in D^{x_{2}} \) iff \( a_i \in D^{y_{2}} \); let \( i \in D^{x_{2}} \), by definition of \( I_T \) it follows that \( B \in V(i) \). Assume \( K \not\models B(a_i) \), then \((i, q_{\alpha_{ij}}) \in \delta_{can}(q_0, V(\epsilon)) \) and there exists \( y \in T_r \) with \( r(y) = (i, q_{\alpha_{ij}}) \). We get contradiction as \( y \) does not satisfy the condition on a run. Therefore, indeed \( K \models B(a_i) \) and \( a_i \in D^{y_{2}} \). Similarly for the other direction.

For the inductive step we prove two auxiliary claims.

Claim B.9 (1) \( \text{Let } i \cdot f(R) \in D^{x_{2}} \text{ for some } i \in \{1, \ldots, n\}. \) Then \( K \models \exists R(a_i), K \not\models R(a_i, a_j) \text{ for each } j \in \{1, \ldots, n\} \) and \( R \) is a \( \leq_{\tau} \)-minimal such role.

\textbf{Proof}. Assume \( K \not\models \exists R(a_i) \), or \( K \models R(a_i, a_j) \) for some \( j \in \{1, \ldots, n\} \), or \( R \) is not a \( \leq_{\tau} \)-minimal such role.

Then by definition of \( \delta_{can}(q_0, V(\epsilon)) \) and of a run, there exists a node \( y = \epsilon \cdot t_{\alpha_{ij}}^{a_i} \) in \( T_r \) such that \( r(y) = (i, q_{\alpha_{ij}}) \) and by \( \delta_{can}(q_{\alpha_{ij}}, V(\epsilon)) \) it is required that \( R' \not\in V(x \cdot f(R)) \) for each \( R' \in R \). It means that \( i \cdot f(R) \) is not connected to \( i \) through any role. Contradiction with \( i \cdot f(R) \) being in \( D^{x_{2}} \).

Claim B.10 (2) \( \text{Let } x \cdot f(R) \in D^{x_{2}} \text{, } \text{len}(x) \geq 2 \text{ and there exists } y \in T_r \text{ with } r(y) = (x, q_{S}) \text{. Then } K \models \exists S^{-} \subseteq \exists R, [S^{-}] \not\models [R] \text{ and } R \text{ is a } \leq_{\tau} \text{-minimal such role.} \)

\textbf{Proof}. For the sake of contradiction assume \( K \not\models \exists S^{-} \subseteq \exists R \). Then by definition of \( \delta_{can}(q_{S}, V(x)) \) and of a run, there exists a node \( y' = y \cdot t_{\alpha_{ij}}^{a_i} \) in \( T_r \) such that \( r(y') = (x, q_{\alpha_{ij}}) \) and by \( \delta_{can}(q_{\alpha_{ij}}, V(x)) \) it is required that \( R' \not\in V(x \cdot f(R)) \) for each \( R' \in R \). It means that \( x \cdot f(R) \) is not connected to \( x \) through any role. Contradiction with \( x \cdot f(R) \) being in \( D^{x_{2}} \).

By the same argument it can be shown that \([S^{-}] \not\models [R] \) and \( R \) is \( \leq_{\tau} \)-minimal.
Let \((x, x \cdot f(R)) \in R^{T_r}\) for some role \(R\). By contradiction assume \((h(x), h(x) \cdot w[R]) \notin R^{A^l}\), this implies that \(\mathcal{K} \not\models R \subseteq R'\). Hence, \((0, q^*_A) \in \delta_{can}(q_R, V(x \cdot f(R)))\), and in \(T_r\) there is a node \(y'' = y''_0 \cdot 0^*_A\) with \(r(y''_0) = (x \cdot f(R), q^*_A)\). We get a contradiction with \(T_r\) being a run as by definition of \(T_r, R' \subseteq V(i \cdot f(R))\). Similarly for the other direction.

Finally, let \(x \cdot f(R) \in A^{T_r}\) for some concept \(A\), and assume \(h(x) \cdot w[R] \notin A^{A^l}\). The latter implies that \(\mathcal{K} \not\models \exists R \subseteq A\). Hence, \((0, q^*_A) \in \delta_{can}(q_R, V(x \cdot f(R)))\), and in \(T_r\) there is a node \(y'' = y''_0 \cdot 0^*_A\) with \(r(y''_0) = (x \cdot f(R), q^*_A)\). We get a contradiction with \(T_r\) being a run as by definition of \(T_r, A \subseteq V(x \cdot f(R))\). Similarly for the other direction.

\[\square\]

**Automaton \(A^\mathbb{K}_{mod}\) for a model of \(\mathcal{K} = (T, A)\).**

\(A^\mathbb{K}_{mod}\) is a 2ATA on infinite trees that accepts a tree if its subtree labeled with \(G\) corresponds to a tree model \(\mathcal{I}\) of \(\mathcal{K}\). Formally, \(A^\mathbb{K}_{mod}\) is defined as the tuple \(\langle \Sigma, Q_{mod}, \delta_{mod}, q_0, F_{mod} \rangle\), where

\[Q_{mod} = \{q_0\} \cup \{q_X | X \in \mathbb{N} \cup \mathbb{B} \cup \mathbb{R} \cup P\}\]

\[F_{mod} = Q_{mod} \times \Sigma \rightarrow \mathbb{B}([n] \times Q_{mod})\] defined as follows:

1. For each \(\sigma \in \Sigma\) such that \(\{r, G\} \subseteq \sigma\), \(\delta_{mod}(q_0, \sigma)\) is defined as:

\[\bigwedge_{i=1}^n \left[ (i, q_{a_i}) \land \left( \bigwedge_{A \in B: \mathbb{K} = A(a_{a_i})} (i, q_A) \right) \land \left( \bigwedge_{P \in \mathbb{R}: \mathbb{K} = P(a_i, a_j)} (0, q_{P_{a_j}}) \right) \right] \]

2. For each \(\sigma \in \Sigma\) such that \(\{r, G\} \subseteq \sigma\) and each \(P_{ij} \in P\):

\[\delta_{mod}(q_{P_{ij}}, \sigma) = (i, q_{\exists P}) \land (j, q_{\exists P})\]

3. For each \(\sigma \in \Sigma\) such that \(\sigma \cap \mathbb{N} = \{a_i\}\) and each atomic role \(P\) in the signature of \(\mathcal{K}\):

\[\delta_{mod}(q_{\exists P}, \sigma) = \left( \bigvee_{j=1}^n (j, q_P) \right) \lor \left( \bigvee_{j=1}^n (-1, q_{P_{ij}}) \right)\]

\[\delta_{mod}(q_{\exists P}, \sigma) = \left( \bigvee_{j=1}^n (j, q_{P_{ij}}) \right) \lor \left( \bigvee_{j=1}^n (-1, q_{P_{ij}}) \right)\]

4. For each \(\sigma \in \Sigma\) such that \(\sigma \cap \mathbb{N} = \emptyset\) and each basic role \(R \in \mathbb{R}\),

\[\delta_{mod}(q_{\exists R}, \sigma) = (0, q_{\exists R}) \lor \left( \bigvee_{i=1}^n (i, q_{R_i}) \right)\]

5. For each \(\sigma \in \Sigma\) such that \(\sigma \cap \mathbb{N} = \emptyset\) and each basic role \(R \in \mathbb{R}\):

\[\delta_{mod}(q_{R}, \sigma) = \left( \bigwedge_{R' \in \mathbb{R}: \mathbb{K} = R \subseteq R'} (0, q_{R'}) \right) \land (0, q_{\exists R}) \land (-1, q_{\exists R})\]

6. For each \(\sigma \in \Sigma\) and each \(B \in \mathbb{B}\):

\[\delta_{mod}(q_{B}, \sigma) = \bigwedge_{B' \in \mathbb{B}: \mathbb{K} = B \subseteq B'} (0, q_{B'})\]

7. For each \(\sigma \in \Sigma\) and each \(X \in \mathbb{B} \cup \mathbb{R} \cup \mathbb{N} \cup P\):

\[\delta_{mod}(q_{X}, \sigma) = \begin{cases} 
\text{true} & \text{if } G \in \sigma \text{ and } X \in \sigma \\
\text{false} & \text{otherwise}
\end{cases}\]

If there are several entries of \(\delta_{mod}\) for the same \(q \in Q_{mod}\) and \(\sigma \in \Sigma_{mod}\), \(\delta_{mod}(q, \sigma) = \phi_1, \ldots, \phi_m\), then we assume that \(\delta_{mod}(q, \sigma) = \bigwedge_{i=1}^m \phi_i\).

Given a model \(\mathcal{I}\), a path \(\pi\) from \(x\) to \(x'\), \(x, x' \in \Delta^{T_r}\), is a sequence of the form \(x = x_1, x_2, \ldots, x_m, x_{m+1} = x'\), \(m \geq 0\), such that \(x_i \in \Delta^T\) and \((x_i, x_{i+1}) \in R^{T_r}_{i}\) for some \(R_i\), and \(m\) is the length of \(\pi\). A model \(\mathcal{I}\) of \(\mathcal{K} = (T, A)\) is said to be a tree model if for each \(x \in \Delta^{T} \setminus \text{Ind}(A)\) there exists a unique shortest path from \(x\) to \(\text{Ind}(A)\). The depth of an object \(x\) in a tree model \(\mathcal{I}\), denoted \(\text{dep}(x)\), is the length of the shortest path from \(x\) to \(\text{Ind}(A)\). It is said that \(x'\) is a successor of \(x\), \(x' \in \text{succ}(x)\) if \(x\) belongs to the path from \(x'\) to \(\text{Ind}(A)\) and \(\text{dep}(x') = \text{dep}(x) + 1\).

Note that given a tree-model \(\mathcal{I}\) of \(\mathcal{K}\) with branching degree \(n\), each domain element of \(\mathcal{I}\) can be seen as an element of \(n^*\). For \(x' \in \Delta^{T}\) with \(\text{dep}(x') = m \geq 0\), we assume a one-to-one numbering \(g_{m,x'}(x)\) of each \(x \in \text{succ}(x')\), such that \(1 \leq g_{m,x'}(x) \leq n\). Then \(x \in \Delta^{T}\) corresponds to
Then the successor relationship in a child transition, the root has children as follows:

**Proof.**

Given a labeled tree \( T, V \), the restriction of \( T \) on \( G \) is a set \( T_G \) such that \( T_G \subseteq T \) and for each \( x \in T \):

\[ x \in T_G \text{ iff } G \subseteq V(x). \]

Given a labeled tree \( (T, V) \) and a run \((T_r, r)\), the interpretation represented by \( T \) and \( T_r \), denoted, \( \mathcal{I}_{T,T_r} \), is defined similarly to \( \mathcal{I}_T \):

\[
\begin{align*}
\Delta_{T,T_r} & = \Delta_T, \\
a_{T,T_r} & = a_T, \\
A_{T,T_r} & = \Delta_T \cap \{ x \mid A \in V(x) \text{ and there exists } y \in T_r \text{ with } r(y) = (x, q_A) \}, \\
P_{T,T_r} & = (\Delta_T \times \Delta_T) \cap \{ (x, x') \in R_P \mid \text{there exists } y \in T_r \text{ s.t. } r(y) = (x', q_P) \text{ or } r(y) = (x, q_{P^-}) \},
\end{align*}
\]

**Proposition B.11** The following hold for \( \mathcal{K}^{\text{mod}}_K \):

- Let \( I \) be a tree model of \( K \) with branching degree \( n \). Then \( T_{I,G} \in L(\mathcal{K}^{\text{mod}}_K) \).

- for each \( (T, V) \in L(\mathcal{K}^{\text{mod}}_K) \), if \( T_G \) is an individual unique tree and \((T_r, r)\) is a corresponding run, then \( \mathcal{I}_{T,G,T_r} \) is a model of \( K \).

**Proof.** For the first item, assume \( T_{I,G} = (n^*, V^{I,G}) \) is the tree encoding of a model \( I \) of \( K \). We show that a full run of \( \mathcal{K}^{\text{mod}}_K \) over \( T_{I,G} \) exists.

The run \((T_r, r)\) is built starting from the root \( \epsilon \), and setting \( r(\epsilon) = (\epsilon, q_0) \). Then, to correctly execute the initial transition, the root has children as follows:

- for each \( a_k \in \text{ind}(A) \)
  - a child \( k_{a_k} \) with \( r(k_{a_k}) = (a_k, q_{a_k}) \),
  - a child \( k_A \) for each \( A \in B \) such that \( a_k \in A \), with \( r(k_A) = (a_k, q_A) \),
- a child \( k_{P,a_k,a_j} \) for each \( a_k, a_j \in \text{ind}(A) \) and each atomic role \( P \) such that \( (a_k, a_j) \in P^* \), with \( r(k_{P,a_k,a_j}) = (\epsilon, q_{P,a_k,a_j}) \).

Then the successor relationship in \( T_r \) is defined inductively as follows.

2. Let \( y \) be a node in \( T_r \) such that \( r(y) = (x, q_{P_i}) \) for \( x = \epsilon \) and \( P_i \in R \). Then \( y \) has

- a child \( y \cdot i \) with \( r(y \cdot i) = (x, i, q_{\exists P_i}) \),
- a child \( y \cdot j \) with \( r(y \cdot j) = (x, j, q_{\exists P^-}) \).

3. Let \( y \) be a node in \( T_r \) such that \( r(y) = (x, q_{P_i}) \) for some \( x \in \Delta^*, V^{I,G}(x) \cap N = \{ a_i \}, R \subseteq R \), and \( R_{ij} \) denotes \( P_{ij} \) if \( R = P \) and \( P_{ji} \) if \( R = P^- \) for some atomic role \( P \). Then \( y \) has

- if \( R \in V^{I,G}(x, j) \) for some \( j \)
  - a child \( y \cdot j \) with \( r(y \cdot j) = (x, j, q_R) \),
- if \( R_{ij} \) is\( P_{ij} \) if \( R = P \) and \( P_{ji} \) if \( R = P^- \) for some atomic role \( R \). Then \( y \) has

- if \( R \in V^{I,G}(y \cdot i) \) for some \( i \)
that there is such that where each node is marked with the special symbol \((x, q_R)\).

- a child \((y \cdot i_R)\) with \(r(y \cdot i_R) = (x \cdot i, q_R)\).
- if \(R^- \in V^{L,G}(x)\)
  - a child \((0_R^-)\) with \(r(0_R^-) = (x, q_R)\).

5. Let \(y\) be a node in \(T_r\) such that \(r(y) = (x, q_R)\) for some \(x \in \Delta^I\) and \(R \in \mathbb{R}\). Then \(y\) has
  - a child \((y \cdot 0_R^-)\) for each \(R^- \in \mathbb{R}\) s.t. \(r(y \cdot 0_R^-) = (x, q_R^\prime)\),
  - a child \((y \cdot 0_{3R^-})\) with \(r(y \cdot 0_{3R^-}) = (x, q_{3R^-})\),
  - a child \((y \cdot -1_{3R})\) with \(r(y \cdot -1_{3R}) = (x, q_{3R})\).

6. Let \(y\) be a node in \(T_r\) such that \(r(y) = (x, q_B)\) for some \(x \in \Delta^I\) and \(B \in \mathbb{B}\). Then \(y\) has
  - a child \((y \cdot 0_B)\) for each \(B^- \in \mathbb{B}\) s.t. \(r(y \cdot 0_B) = (x, q_{B^\prime})\).

Since \(I\) is a model of \(K, T_r\) satisfies the transition function \(\delta_{mod}\). In particular, in the rules 3 and 4 in the inductive definition of \(T_r\), there will exist a node \(x' \in \Delta^I\) such that \((x, x') \in R^I\), hence at least one of the conditions will be satisfied.

It is easy to see that this run is accepting, as for each infinite path \(P\) of \(T_r, q_R \in inf(P)\) for some \(R\). Hence, \(L_{T_G} \in L(A_K^{mod})\).

To show the second item, let \((T, V) \in L(A_K^{can})\) and \((T_r, r)\) an accepting run of \((T, V)\). Moreover, let \(T_G\) be a tree (i.e., prefix closed) and individual unique. Then \(L_{T_G, T_r}\) is defined and it can be shown that \(L_{T_G, T_r}\) a model of \(K\):

1. for each \(i \in \{1, \ldots, n\}, K \models B(a_i) \implies a_i \in B^{T_G, T_r}\),
2. for each \(i, j \in \{1, \ldots, n\}, K \models P(a_i, a_j) \implies (a_i, a_j) \in P^{T_G, T_r}\),
3. if \(x \in B^{T_G, T_r}\), then \(x \in B^{T_G, T_r}\) for each \(B^- \in \mathbb{B}\) s.t. \(K \models B \subseteq B^\prime\),
4. if \((x, x') \in R^{T_G, T_r}\), then \((x, x') \in R^{T_G, T_r}\) for each \(R^- \in \mathbb{R}\) s.t. \(K \models R \subseteq R^\prime\),
5. if \(x \in B^{T_G, T_r}\) and \(K \models B \subseteq B^-\), then there exists \(x' \in T_G\) such that \((x, x') \in R^{T_G, T_r}\).
6. if \((x, x') \in S^{T_G, T_r}\) and \(K \models S^- \subseteq S\), then there exists \(x'' \in T_G\) such that \((x', x'') \in R^{T_G, T_r}\).

We show items 5 and 6 hold, the rest can be shown by analogy.

Assume \(x \in B^{T_G, T_r}\) and \(K \models B \subseteq B^-\) for some concept \(B\) and \(R\). Then by definition of \(L_{T_G, T_r}\), we have that \(B, G \in V(x)\) and there exist a node \(y \in T_r\) with \(r(y) = (x, q_B)\). Since \(T_r\) is a run and by definition of \(\delta_{mod}\), there exists a node \(y' = y \cdot 0_{3R^-} \in T_r\) such that \(r(y') = (x, q_{3B^-})\). Then, if \(V(x) \cap N = \emptyset\), there exists a node \(y'' = y' \cdot z \in T_r\) such that \(r(y'') = (x \cdot i, q_R) \lor r(y'' = (x, q_{3R^-})\). If \(V(x) \cap N = a_i\), there exists a node \(y'' = y' \cdot z \in T_r\) such that \(r(y'') = (x \cdot j, q_R) \lor r(y'' = (x, q_{3R^-})\). In any case, it is easy to see that there is \(x' \in T\) with \(G \in V(x')\) (i.e., \(x' \in T_G\)) such that \((x, x') \in R^{T_G, T_r}\).

Assume that \((x, x') \in S^{T_G, T_r}\) and \(x'\) is a successor of \(x\). Then \(S, G \in V(x')\) and there exists \(y \in T_r\) such that \(r(y) = (x', q_{3S^-})\). Since \(T_r\) is a run, there exists a node \(y' = y \cdot 0_{3S^-} \subseteq S\) such that \(r(y') = (x', q_{3S^-})\). Further, as \(K \models S^- \subseteq S\), there exists \(y'' = y' \cdot 0_{3R^-} \subseteq S\) with \(r(y'') = (x', q_{3R^-})\) and as above we obtain that there is \(x'' \in T\) with \(G \in V(x'')\) (i.e., \(x'' \in T_G\)) such that \((x', x'') \in R^{T_G, T_r}\).

Assume now that \((x, x') \in S^{T_G, T_r}\) and \(x\) is a successor of \(x'\). Then \(S^- \subseteq G \in V(x)\) and there exists \(y \in T_r\) such that \(r(y) = (x, q_{3S^-})\). Since \(T_r\) is a run, there exists a node \(y' = y \cdot -1_{3S^-} \subseteq S\) such that \(r(y') = (x', q_{3S^-})\) (recall that \(x' = x \cdot -1\)). Further, as \(K \models S^- \subseteq S\), there exists \(y'' = y' \cdot 0_{3S^-} \subseteq S\) with \(r(y'') = (x', q_{3S^-})\) and as above we obtain that there is \(x'' \in T\) with \(G \in V(x'')\) (i.e., \(x'' \in T_G\)) such that \((x', x'') \in R^{T_G, T_r}\).

Thus, \(L_{T_G, T_r}\) a model of \(K\). □

**Automaton \(A_{fin}\)**

\(A_{fin}\) is a one-way non-deterministic automaton on infinite trees that accepts a tree if it has a finite prefix where each node is marked with the special symbol \(G\), and no other node in the tree is marked with \(G\).

Formally, \(A_{fin} = (K, Q_{fin}, \delta_{fin}, q_0, F_{fin})\), where \(Q_{fin} = \{q_0, q_1\}\), \(F_{fin} = \{q_1\}\) and transition function \(\delta_{fin} : Q_{fin} \times K \rightarrow B([n] \times Q_{fin})\) is defined as follows:

1. For each \(\sigma \in \Gamma_K\):

   \[
   \delta(q_0, \sigma) = \bigg\{ \begin{array}{ll}
   \bigwedge_{i=1}^{n} (i, q_0), & \text{if } G \in \sigma \\
   \bigwedge_{i=1}^{n} (i, q_1), & \text{if } G \notin \sigma
   \end{array}
   \bigg\}
   \]
2. For each $\sigma \in \Gamma_K$:

$$\delta(q_1, \sigma) = \begin{cases} 
\bigwedge_{i=1}^{n}(i, q_1), & \text{if } G \notin \sigma \\
\text{false} & \text{if } G \in \sigma
\end{cases}$$

### B.3 Proof of Theorem 5.3

**Proof.** We prove that the non-emptiness problem for universal solutions is in NP. Assume we are given a mapping $M = (S_1, S_2, T_{12})$ and a source KB $K_1 = (T_1, A_1)$, and we want to decide whether there exists a universal solution for $K_1$ under $M$ (all ABoxes are considered to be OWL 2 QL ABoxes without inequalities).

First, we check whether $K_1$ and $M$ are $S_2$-positive. This check can be done in polynomial time, and if it was successful, then by Lemma B.5, it remains to verify whether there exists a universal solution for $K_1$ under $M^{pos}$.

Second, we construct the maximal target OWL 2 QL ABox, a candidate for universal solution. Let $A_2$ be the ABox over $S_2$ containing every membership assertion $\alpha$ of the form $B(a)$ or $R(a, b)$ such that $(T_1^{pos} \cup T_{12}^{pos}, A_1) \models \alpha$, $a, b \in \text{Ind}(A_1)$, $B$ is a basic concept and $R$ is a basic role. Then $A_2$ is of polynomial size, and

**Lemma B.12** A universal solution for $K_1^{pos}$ under $M^{pos}$ exists iff $A_2$ is a solution for $K_1^{pos}$ under $M^{pos}$.

**Proof.** $(\Rightarrow)$ Assume a universal solution for $K_1^{pos}$ under $M^{pos}$ exists. As it follows from Lemma B.7, there exists a universal solution $A_3$ such that $U_{A_3} \subseteq U_{(T_1, \cup T_{12}^{pos}, A_1)}$, hence $A_3 \subseteq A_2$. As $A_3$ is a solution, there exists $\mathcal{I}$ such that $\mathcal{I} \models K_1^{pos}$ and $(\mathcal{I}, U_{A_3}) \models T_{12}^{pos}$. It follows that for each model $\mathcal{J}$ of $A_2$, $\mathcal{J} \supseteq U_{A_2} \supseteq U_{A_3}$, and therefore $(\mathcal{I}, \mathcal{J}) \models T_{12}^{pos}$. By definition of solution, $A_2$ is a solution.

$(\Leftarrow)$ Assume $A_2$ is a solution for $K_1^{pos}$ under $M^{pos}$. Then $A_2$ is a universal solution follows from the proof of Lemma B.3. Since $A_2$ is an OWL 2 QL ABox, we conclude that a universal solution for $K_1^{pos}$ under $M^{pos}$ exists. 

Thus, it remains only to check whether $A_2$ is a solution. We need the following result to perform this check in NP.

**Lemma B.13** Let $A_2$ be an (extended) ABox over $S_2$ such that it is a solution for $K_1^{pos}$ under $M^{pos}$. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I}$ is of polynomial size, $\mathcal{I}$ is a model of $K_1^{pos}$ and $(\mathcal{I}, \mathcal{V}_{A_2}) \models T_{12}^{pos}$.

**Proof.** Assume $A_2$ is a solution for $K_1^{pos}$ under $M^{pos}$, then for each model of $A_2$, in particular for $\mathcal{V}_{A_2}$, there exists $\mathcal{I}'$ such that $\mathcal{I}'$ is a model of $K_1^{pos}$ and $(\mathcal{I}', \mathcal{V}_{A_2}) \models T_{12}^{pos}$. Suppose $\mathcal{I}'$ is more than polynomial, then since $(\mathcal{I}', \mathcal{V}_{A_2}) \models T_{12}^{pos}$ it follows $B_{\mathcal{I}'} \subseteq \Delta_{A_2}$ and $R_{\mathcal{I}'} \subseteq \Delta_{A_2} \times \Delta_{A_2}$ for each basic concept $B$ and role $R$ that appear on the left hand side of some inclusion in $T_{12}^{pos}$. Therefore, we construct an interpretation $\mathcal{I}$ of polynomial size as follows:

- $\Delta_{\mathcal{I}} = \Delta_{A_2} \cup N_a \cup \{d\}$, for a fresh domain element $d$,
- $\alpha_{\mathcal{I}} = \alpha$ for $\alpha \in N_a$,
- $A_{\mathcal{I}} = (A_{\mathcal{I}'} \cap \Delta_{A_2}) \cup \{d \mid \text{if } A_{\mathcal{I}'} \backslash \Delta_{A_2} \neq \emptyset\}$ for each atomic concept $A$,
- $R_{\mathcal{I}} = (R_{\mathcal{I}'} \cap (\Delta_{A_2} \times \Delta_{A_2})) \cup \\
\{ (a, d) \mid (a, b) \in R_{\mathcal{I}'} \backslash (\Delta_{A_2} \times \Delta_{A_2}), a \in (\exists R)_{\mathcal{I}'} \cap \Delta_{A_2}\} \cup \\
\{ (d, a) \mid (b, a) \in R_{\mathcal{I}'} \backslash (\Delta_{A_2} \times \Delta_{A_2}), a \in (\exists R^-)_{\mathcal{I}'} \cap \Delta_{A_2}\} \cup \\
\{ (d, d) \mid (a, b) \in R_{\mathcal{I}'} \backslash (\Delta_{A_2} \times \Delta_{A_2}), a \notin (\exists R)_{\mathcal{I}'} \cap \Delta_{A_2}, b \notin (\exists R^-)_{\mathcal{I}'} \cap \Delta_{A_2}\}$

for each atomic role $R$.

Note that $\mathcal{V}_{A_2}$ interprets all constants as themselves, and $\mathcal{I}'$ agrees on interpretation of constants with $\mathcal{V}_{A_2}$, for this reason $\Delta_{\mathcal{I}} \supseteq N_a$.

It is straightforward to verify that $\mathcal{I}$ is a model of $K_1^{pos}$; clearly, $\mathcal{I}$ is a model of $A_1$, we show $\mathcal{I} \models T_{12}^{pos}$.

Assume, $T_{12}^{pos} \models B \subseteq C$ for basic concepts $B, C$, and $b \in B^{\mathcal{I}}$. If $b \in \Delta_{\mathcal{I}} \cap \Delta_{A_2}$, then since $\mathcal{I}' \models B \subseteq C$, we have that $b \in C_{\mathcal{I}'}$, which implies $b \in C^{\mathcal{I}}$. Otherwise, $b = d$ and for some $c \in \Delta_{\mathcal{I}} \cap \Delta_{A_2}$, $c \in B^{\mathcal{I}}$, therefore $c \in C^{\mathcal{I}}$, and thus by definition of $\mathcal{I}$, $d \in C^{\mathcal{I}}$. Role inclusions are handled similarly. Moreover, as $\mathcal{I}$ and $\mathcal{I}'$ agree on all concepts and roles that appear on the left hand side of $T_{12}^{pos}$, it follows that $(\mathcal{I}, \mathcal{V}_{A_2}) \models T_{12}^{pos}$. Hence, $\mathcal{I}$ is the interpretation of polynomial size we were looking for.

Finally, the NP algorithm for deciding the non-emptiness problem for universal solutions is as follows:
1. verify whether $\mathcal{K}_1$ and $\mathcal{M}$ are $\Sigma_2$-positive, if yes, 
2. compute $\mathcal{A}_2$, the $\Sigma_2$-closure of $\mathcal{A}_1$ with respect to $T^{\text{pos}}_1 \cup T^{\text{pos}}_{12}$. 
3. guess a source interpretation $\mathcal{I}$ of polynomial size.
4. If $\mathcal{I} \models \mathcal{K}_1^{\text{pos}}$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models T^{\text{pos}}_{12}$, then a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists, and $\mathcal{A}_2$ is a universal solution, otherwise a universal solution does not exist.

Note that steps 1,2 and 4 can be done in polynomial time, hence this algorithm is in fact an NP algorithm. Below we prove the correctness of the algorithm.

Assume $\mathcal{I} \models \mathcal{K}_1^{\text{pos}}$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models T^{\text{pos}}_{12}$. Then $\mathcal{A}_2$ is a solution: for each model $\mathcal{J}$ of $\mathcal{A}_2$, it holds $\mathcal{U}_{\mathcal{A}_2} \subseteq \mathcal{J}$, therefore $(\mathcal{I}, \mathcal{J}) \models T_{12}$. By Lemma B.12 we obtain that a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists, and from its proof it follows that $\mathcal{A}_2$ is a universal solution. Thus, the algorithm is sound.

We show the algorithm is complete. Assume $\mathcal{I} \not\models \mathcal{K}_1^{\text{pos}}$ or $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \not\models T^{\text{pos}}_{12}$, and to the contrary, $\mathcal{A}_2$ is a solution. The by Lemma B.13 there exists a model $\mathcal{I}'$ of $\mathcal{K}_1^{\text{pos}}$ of polynomial size such that $(\mathcal{I}', \mathcal{U}_{\mathcal{A}_2}) \models T^{\text{pos}}_{12}$. Contradiction with the guessing step. Therefore, $\mathcal{A}_2$ is not a solution and there exists no universal solution. Thus, the algorithm is complete.

As a corollary we obtain an upper bound for the membership problem.

**Theorem B.14** The membership problem for universal solutions is in $\text{NP}$. 

\[ \Box \]

**B.4 Proof of Theorem 5.5**

**Proof.** First we provide the PSPACE lower bound, and then present the EXPSPACE automata-based algorithm for deciding the non-emptiness problem for universal solutions with extended ABoxes.

**Lemma B.15** The non-emptiness problem for universal solutions with extended ABoxes in DL-Lite$^\mathcal{R}$ is PSPACE-hard.

**Proof.** The proof is by reduction of the satisfiability problem for quantified Boolean formulas, known to be PSPACE-complete. Suppose we are given a QBF

$$\phi = Q_1X_1 \ldots X_n \bigwedge_{j=1}^m C_j$$

where $Q_i \in \{\forall, \exists\}$ and $C_j, 1 \leq j \leq m$, are clauses over the variables $X_i, 1 \leq i \leq n$.

Let $\Sigma_1 = \{A, Y^k_1, X^k_1, S_i, T_i, Q^k_1, P^k_i, R^k_j, R^l_j \mid 1 \leq j \leq m, 1 \leq i \leq n, 0 \leq l \leq n, k \in \{0,1\}\}$ where $A, Y^k_1, X^k_1$ are concept names and the rest are role names. Let $T_1$ be the following TBox over $\Sigma_1$ for $1 \leq j \leq m, 1 \leq i \leq n$ and $k \in \{0,1\}$:

$$A \sqsubseteq \exists S_0^-, \exists S^+_1, \exists S^+_i, \exists Q^k_i, \exists P^k_i, \exists R^k_j, \exists R^l_j$$
$$\exists(Q^k_1)^- \sqsubseteq Y^k_1$$
$$\exists(P^k_i)^- \sqsubseteq X^k_1$$
$$A \sqsubseteq \exists T^{-}_0, \exists T^+_1, \exists T^+_i, \exists P^k_i, \exists R^k_j, \exists R^l_j$$
$$\exists(X^k_1)^- \sqsubseteq \exists R^{i-1}_j$$

and $\mathcal{A}_1 = \{A(a)\}$. Let $\Sigma_2 = \{A', Z^0_1, Z^1_1, S', R'_j\}$ where $A', Z^0_1, Z^1_1$ are concept names and $S', R'_j$ are role names, $\mathcal{M} = (\Sigma_1, \Sigma_2, T_{12})$, and $T_{12}$ the following set of inclusions:

$$A \sqsubseteq A', S_i \sqsubseteq S', T_i \sqsubseteq T'_j$$
$$Y^k_1 \sqsubseteq Z^k_1$$
$$X^k_1 \sqsubseteq Z^k_1$$

We verify that $\models \phi$ if and only if $\mathcal{U}(T_1 \cup T_{12}, \mathcal{A}_1)$ is $\Sigma_2$-homomorphically embeddable into a finite subset of itself. The latter, in turn, is equivalent to the existence of a universal solution for $\mathcal{K}_1 = (T_1, \mathcal{A}_1)$ under $\mathcal{M}$, which is shown in Lemma B.7.

For $\phi = \exists X_1 \forall X_2 \exists X_3 (X_1 \land (X_2 \lor \neg X_3))$, $\Sigma_2$-reduct of $\mathcal{U}(T_1 \cup T_{12}, \mathcal{A}_1)$ can be depicted as follows:
where each edge $\longrightarrow$ is labeled with $S'$, each edge $\longrightarrow$ is labeled with $S'_j R'_i$ for $1 \leq j \leq m$, and the labels of edges $\longrightarrow$ are shown to the left of each infinite and finite path. The labels of the nodes (if any) are shown next to each node.

Let $C_{\text{inf}}$ and $C_{\text{fin}}$ be the parts of $\mathcal{U}_1$ generated using the first 9 axioms and the last 9 axioms of $T_1$ respectively. Note that $C_{\text{inf}}$ is infinite, while $C_{\text{fin}}$ is finite. One can show that $C_{\text{inf}}$ is $\Sigma_2$-homomorphically embeddable into $C_{\text{fin}}$ (which is equivalent to $\mathcal{U}(T_{11} \cup T_{12}, A_1)$ is $\Sigma_2$-homomorphically embeddable into $C_{\text{fin}}$) iff $\phi$ is satisfiable.

The rest of the proof follows the line of the proof of Theorem 11 in [Konev et al., 2011].

$(\Rightarrow)$ Suppose $\models \phi$. We show that the canonical model $\mathcal{U}(T_{11} \cup T_{12}, A_1)$ is $\Sigma_2$-homomorphically embeddable into a finite subset of itself. More precisely, let us denote with $C_{\text{inf}}$ the subset of $T_1$ consisting of the first 9 axioms, and $C_{\text{fin}}$ the subset of $T_1$ consisting of the last 9 axioms. Then $\mathcal{U}(T_{11} \cup T_{12}, A_1) = \mathcal{U}(T_{11} \cup T_{12}, A_1) \cup \mathcal{U}(T_{12} \cup T_{12}, A_1)$, and we construct a $\Sigma_2$-homomorphism $h : \Delta \mathcal{U}(T_{11} \cup T_{12}, A_1) \rightarrow \Delta \mathcal{U}(T_{12} \cup T_{12}, A_1)$. In the following we use $U_{\text{inf}}$ to denote $\mathcal{U}(T_{11} \cup T_{12}, A_1)$, and $U_{\text{fin}}$ to denote $\mathcal{U}(T_{12} \cup T_{12}, A_1)$.

We begin by setting $h(t_{\text{inf}}) = t_{\text{fin}}$. Then we define $h$ in such a way that, for each path $\pi$ in $U_{\text{inf}}$ of length $i + 1 \leq n$, $h(\pi)$ is a path $t_{\text{fin}} w_1 \ldots w_i$ of length $i + 1$ in $U_{\text{fin}}$ and it defines an assignment $a_{h(\pi)}$ to the variables $X_1, \ldots, X_i$ by taking, for all $1 \leq i' \leq i$,

$$a_{h(\pi)}(X_{i'}) = \top \iff t_{\text{fin}} w_1 \ldots w_{i'} \in (X_{i'})_{t_{\text{fin}}}$$

$$a_{h(\pi)}(X_{i'}) = \bot \iff t_{\text{fin}} w_1 \ldots w_{i'} \notin (X_{i'})_{t_{\text{fin}}}$$

Such assignments $a_{h(\pi)}$ will satisfy the following:

(a) the QBF obtained from $\phi$ by removing $Q_1 X_1 \ldots Q_i X_i$ from its prefix is true under $a_{h(\pi)}$.

For the paths of length 0 the $\Sigma_2$-homomorphism $h$ has been defined and (a) trivially holds. Suppose that we have defined $h$ for all paths in $U_{\text{inf}}$ of length $i + 1 \leq n$. We extend $h$ to all paths of length $i + 2$ in $U_{\text{inf}}$ such that (a) holds. Let $\pi$ be a path of length $i + 1$. In $U_{\text{fin}}$ we have

$$\text{tail}(h(\pi)) \leadsto (T_{11} \cup T_{12} \cup T_{12}) \ w_{t_{\text{fin}}}^{U_{\text{fin}}}, \text{ and } \ h(\pi) \cdot w_{t_{\text{fin}}}^{U_{\text{fin}}} \in (X_{i+1})_{t_{\text{fin}}}, \text{ for } k = 0, 1.$$
We know that \( \models \phi \) and so, by (a), the QBF obtained from \( \pi \) by removing \( Q_1, X_1 \ldots Q_i, X_i \) is true under either \( a_{\mathcal{K}}(\pi) \cup \{ X_i = T \} \) or \( a_{\mathcal{K}}(\pi) \cup \{ X_i = \bot \} \). We set \( h(\pi \cdot w_{[S_i]}^{\mathcal{U}_{\inf}}) = h(\pi) \cdot w_{[S_i]}^{\mathcal{U}_{\fin}} \) with \( k = 1 \) in the former case, and \( k = 0 \) in the latter case. Either way, (a) holds.

Consider now in \( \mathcal{U}_{\inf} \) a path \( \pi \) of length \( n + 1 \) from \( \alpha^{\mathcal{U}_{\inf}} \) to \( w_{[R_i]}^{\mathcal{U}_{\inf}} \). By construction, we have
\[
h(\pi) = \alpha^{\mathcal{U}_{\fin}} \cdot w_{[R_i]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_k]}^{\mathcal{U}_{\fin}}.
\]
Next, on the one hand, the path \( \pi \) in \( \mathcal{U}_{\inf} \) has \( m \) infinite extensions of the form \( \pi \cdot w_{[R_i]}^{\mathcal{U}_{\inf}} \cdot w_{[R_j]}^{\mathcal{U}_{\inf}} \cdot \ldots \), for \( 1 \leq j \leq m \). On the other hand, as \( \models \phi \), by (a), for each clause \( C_j \), there is some \( 1 \leq i' \leq n \) such that \( h(\pi) \) contains \( w_{[R_{i'}]}^{\mathcal{U}_{\inf}} \) if \( X_{i'} \in C_j \), or \( w_{[R_{i'}]}^{\mathcal{U}_{\inf}} \) if \( \neg X_{i'} \in C_j \). We set for each \( 1 \leq l \leq n - i' \)
\[
h(\pi \cdot w_{[R_i]}^{\mathcal{U}_{\inf}} \cdot \ldots \cdot w_{[R_j]}^{\mathcal{U}_{\inf}}) = \alpha^{\mathcal{U}_{\fin}} \cdot w_{[R_i]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{n-i'-1}]}^{\mathcal{U}_{\fin}},
\]
for each \( n + 1 \geq l > n - i' \)
\[
h(\pi \cdot w_{[R_i]}^{\mathcal{U}_{\inf}} \cdot \ldots \cdot w_{[R_j]}^{\mathcal{U}_{\inf}}) = \alpha^{\mathcal{U}_{\fin}} \cdot w_{[R_i]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_j]}^{\mathcal{U}_{\fin}} \cdot w_{[R_{j+i-1}]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{j+i-1}]}^{\mathcal{U}_{\fin}},
\]
and for each \( l > n + 1 \)
\[
h(\pi \cdot w_{[R_i]}^{\mathcal{U}_{\inf}} \cdot \ldots \cdot w_{[R_j]}^{\mathcal{U}_{\inf}}) = \alpha^{\mathcal{U}_{\fin}} \cdot w_{[R_i]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{j+i-1}]}^{\mathcal{U}_{\fin}} \cdot w_{[R_{j+i-1}]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{j+i-1}]}^{\mathcal{U}_{\fin}},
\]
where \( i' = (n - l + 1) \mod 2 \). It is immediate to verify that \( h \) is a \( \Sigma_2 \)-homomorphism from \( \mathcal{U}_{\inf} \) to \( \mathcal{U}_{\fin} \).

\((\Leftarrow)\) Let \( h \) be a \( \Sigma_2 \)-homomorphism from \( \mathcal{U}_{\inf} \) to \( \mathcal{U}_{\fin} \). We show that \( \models \phi \).

Let \( \pi \) be a path of length \( n+1 \), \( \pi = (\alpha^{\mathcal{U}_{\inf}}, w_1 \ldots w_n, \mathcal{U}_{\inf}) \). Then \( (\alpha^{\mathcal{U}_{\inf}}, \pi_1), (\pi_1, \pi_{i+1}) \in S^{\mathcal{U}_{\inf}} \), where \( \pi_i = \alpha^{\mathcal{U}_{\inf}} \cdot w_1 \ldots w_i \), for \( 1 \leq i \leq n - 1 \). Furthermore, let \( Z_k^1, Z_k^2, \ldots, Z_k^n \) be the concepts containing subpaths of \( h(\pi) \). We show that for every \( 1 \leq j \leq m \), the clause \( C_j \) contains at least one of the literals \( \{ X_i \mid k_i = 1, 1 \leq i \leq n \} \cup \{ \neg X_i \mid k_i = 0, 1 \leq i \leq n \} \).

Validity of \( \phi \) will follow.

Consider a path of the form \( \pi \cdot w_{[R_i]}^{\mathcal{U}_{\inf}} \cdot \ldots \cdot w_{[R_j]}^{\mathcal{U}_{\inf}} \) in \( \mathcal{U}_{\inf} \). Then its \( h \)-image in \( \mathcal{U}_{\fin} \) must be of the form
\[
\alpha^{\mathcal{U}_{\fin}} \cdot w_{[R_i]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{n+i-1}]}^{\mathcal{U}_{\fin}} \cdot w_{[R_{n+i-1}]}^{\mathcal{U}_{\fin}} \cdot \ldots \cdot w_{[R_{n+i-1}]}^{\mathcal{U}_{\fin}},
\]
for some \( 1 \leq i \leq n \), \( i' = 0 \) or \( i' = 1 \), and \( k_i = 0 \) or \( k_i = 1 \). If \( k_i = 0 \), then \( C_j \) must contain \( \neg X_i \), otherwise \( X_i \).

\begin{lemma}
Lemma B.16 The non-emptiness problem for universal solutions is in EXPTIME. For a given DL-Lite\textsubscript{R} mapping \( \mathcal{M} \) and a given DL-Lite\textsubscript{R} KB \( \mathcal{K}_1 \), if a universal solution \( \mathcal{A}_2 \) (an extended ABox without inequalities) exists, then it is at most exponentially large in the size of \( \mathcal{K}_1 \cup \mathcal{M} \).
\end{lemma}

\textbf{Proof}. First, we provide an algorithm for checking existence of a universal solution with extended ABoxes in DL-Lite\textsubscript{K\textsuperscript{pos}}. Given a DL-Lite\textsubscript{K\textsuperscript{pos}} mapping \( \mathcal{M} = (\Sigma_1, \Sigma_2, T_{12}) \), to verify that a universal solution for \( (T_{12}, A_1) \) under \( \mathcal{M} \) exists, we first check for non-emptiness of the automaton \( \mathcal{B} \) defining the intersection of the automata \( \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{can}}), \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{mod}}) \), and \( \mathcal{A}_{\mathcal{fin}} \), where \( \mathcal{K} = (T_1 \cup T_{12}, A_1), \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{can}}) \) is the projection of \( A_{\mathcal{K}}^{\text{can}} \) on the vocabulary \( \Gamma_{\mathcal{K}} \), and likewise for \( \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{mod}}) \). If the language accepted by \( \mathcal{B} \) is empty, then there is no universal solution, otherwise a universal solution exists and it is exactly the tree accepted by \( \mathcal{B} \).

\begin{proposition}
Proposition B.17 Let \( \mathcal{M} = (\Sigma_1, \Sigma_2, T_{12}) \) be a DL-Lite\textsubscript{K\textsuperscript{pos}} mapping, and \( \mathcal{K}_1 = (T_{12}, A_1) \) a DL-Lite\textsubscript{R} KB over \( \Sigma_1 \). Then, a universal solution with extended ABoxes for \( \mathcal{K}_1 \) under \( \mathcal{M} \) exists iff the language of the automata \( \mathcal{B} = \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{can}}) \cap \mathcal{A}_{\mathcal{fin}} \cap \pi_{\mathcal{K}}(A_{\mathcal{K}}^{\text{mod}}) \), where \( \mathcal{K} = (T_1 \cup T_{12}, A_1), \) is non-empty.
\end{proposition}

\textbf{Proof}. \((\Leftarrow)\) Assume that \( L(\mathcal{B}) \neq \emptyset \) and \( T \in L(\mathcal{B}) \). Let \( T_G \) be the subtree of \( T \) defined by the \( G \) labels, and \( L_{T,G} \) the interpretation represented by \( T_G \). Then from the definition of \( \mathcal{B} \) it follows that
1. $\mathcal{I}_{T,G}$ is a finite interpretation of $\Sigma_2$ and $\mathcal{I}_{T,G} \subseteq \mathcal{U}_{(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1)}$.

2. there exists an interpretation $\mathcal{I}$ of $\Sigma_1$ such that $\mathcal{I} \cup \mathcal{I}_{T,G}$ is a model of $\langle \mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1 \rangle$.

Since $\mathcal{I}_{T,G}$ is finite, let $\mathcal{A}_{T,G}$ be the ABox over $\Sigma_2$ such that $\mathcal{U}_{A_{T,G}} = \mathcal{I}_{T,G}$. Then, $\mathcal{A}_{T,G}$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$ (by the second item). We show it is a universal solution. Let $\mathcal{J}$ be an interpretation of $\Sigma_2$ such that for some model $\mathcal{I}$ of $\mathcal{K}_1$, $(\mathcal{I}, \mathcal{J}) \models \mathcal{M}$. Then, since $\mathcal{U}_{(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1)}$ is the canonical model of $\langle \mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1 \rangle$, there exists a homomorphism from $\mathcal{U}_{(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1)}$ to $\mathcal{I} \cup \mathcal{J}$ ($\mathcal{I} \cup \mathcal{J}$ is a model of $\langle \mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1 \rangle$). In particular, there is a homomorphism from $\mathcal{I}_{T,G}$ to $\mathcal{I} \cup \mathcal{J}$, and as $\mathcal{I}_{T,G}$ and $\mathcal{I}$ are interpretations of disjoint signatures, there is a homomorphism $h$ from $\mathcal{I}_{T,G}$ to $\mathcal{J}$. Hence, $\mathcal{J}$ is a model of $\mathcal{A}_{T,G}$; take $h$ as the substitution for the labeled nulls. By definition of universal solution, $\mathcal{A}_{T,G}$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$.

$(\Rightarrow)$ Assume a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists. Then by Lemma B.7 there exists a universal solution $\mathcal{A}_2$ such that $\mathcal{V}_{\mathcal{A}_2} \subseteq \mathcal{U}_{(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}_1)}$. Therefore, the language of $\mathcal{B}$ is not empty.

As a corollary of Lemma B.5, Lemma B.6, Lemma B.7, and Proposition B.17, we obtain the exponential time upper bound of the non-emptiness problem for universal solutions with extended ABoxes in DL-Lite$\mathcal{R}$.

B.5 Proof of Theorem 5.6

Proof. We show that the membership problem for universal solutions with extended ABoxes is NP-complete by first proving the lower bound, and then the upper bound.

**Lemma B.18** The membership problem for universal solutions with extended ABoxes is NP-hard.

**Proof.** The proof is by reduction of 3-colorability of undirected graphs known to be NP-hard. Suppose we are given an undirected graph $G = (V, E)$. Let $\Sigma_1 = \{\text{Edge}\}$ and $\Sigma_2 = \{\text{Edge}'\}$. Let $r, g, b \in N_a$, $V \subseteq N_i$ and

$$
\begin{align*}
\mathcal{A}_1 &= \{ \text{Edge}(r, g), \text{Edge}(g, r), \text{Edge}(r, b), \text{Edge}(b, r), \text{Edge}(g, b), \text{Edge}(b, g) \},
\mathcal{T}_1 &= \{ \},
\mathcal{T}_{12} &= \{ \text{Edge} \subseteq \text{Edge}' \},
\mathcal{A}_2 &= \{ \text{Edge}'(r, g), \text{Edge}'(g, r), \text{Edge}'(r, b), \text{Edge}'(b, r), \text{Edge}'(g, b), \text{Edge}'(b, g) \} \cup \{ \text{Edge}'(x, y), \text{Edge}'(y, x) \mid (x, y) \in E \}.
\end{align*}
$$

Note that the nodes in $G$ become labeled nulls in $\mathcal{A}_2$.

We show that $G$ is 3-colorable if and only if $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$.

$(\Rightarrow)$ Suppose $G$ is 3-colorable. Then it follows that there exists a function $h$ that assigns to each vertex from $V$ one of the colors $\{r, g, b\}$ such that if $(x, y) \in E$, then $h(x) \neq h(y)$, hence $h$ is a homomorphism from $G$ to the undirected graph $\langle \{r, g, b\}, \{(r, g), (g, b), (b, r)\} \rangle$.

We prove that $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Obviously, $\mathcal{K}_1$ and $\mathcal{M}$ are $\Sigma_2$-positive. Thus, it remains to verify that $\mathcal{V}_{\mathcal{A}_2}$ is $\Sigma_2$-homomorphically equivalent to $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$. First, it is easy to see that $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$ is $\Sigma_2$-homomorphically embeddable into $\mathcal{V}_{\mathcal{A}_2}$. Second, $\mathcal{S}$ is also a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$, thus $\mathcal{V}_{\mathcal{A}_2}$ is homomorphically embeddable into $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$. $(\Leftarrow)$ Suppose now $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Then by Lemma B.3, it follows that $\mathcal{V}_{\mathcal{A}_2}$ is $\Sigma_2$-homomorphically equivalent to $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$. Let $h$ be a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$. Then $h$ assigns to each labeled null $x \in \Delta_{\mathcal{A}_2}$ some constant $a \in \Delta_{\mathcal{A}_1}$, and it is easy to see that $h$ is an assignment for the vertices in $V$ that is a 3-coloring of $G$.

**Lemma B.19** The membership problem for universal solutions with extended ABoxes is in NP.

**Proof.** Assume we are given a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, a source KB $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$, and a target ABox $\mathcal{A}_2$. We want to decide whether $\mathcal{A}_2$ is a universal solution with extended ABoxes for $\mathcal{K}_1$ under $\mathcal{M}$ (ABoxes without inequalities).

We need the following proposition that provides an upper bound for checking existence of homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$.

**Proposition B.20** Deciding whether $\mathcal{V}_{\mathcal{A}_2}$ is homomorphically embeddable into $\mathcal{U}_{(\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{A}_1)}$ can be done in NP in the size of $\mathcal{K}_1$, $\mathcal{M}$ and $\mathcal{A}_2$. 


Proof. First, if there exists a homomorphism $h$ from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$, then there exists a polynomial size witness $A_3$ such that $V_{A_2} \subseteq U_{(T_1 \cup T_2, A_1)}$ and $h$ is a homomorphism from $V_{A_2}$ to $V_{A_3}$. Then $|A_3| \leq |A_2|$. Therefore, to verify that such $h$ exists, it is sufficient to compute $A_3$ and then to check whether $V_{A_3}$ can be homomorphically mapped into $V_{A_3}$.

Second, there exists a witness $A_3$ such that $V_{A_3} \subseteq U_{(T_1 \cup T_2, A_1)}$ and every $x \in \Delta A_3$ is a path of polynomial length in the size of $T_1 \cup T_2$ and $A_2$ (more precisely, of length smaller or equal $2m$, where $m$ is the size of $T_1 \cup T_2 \cup A_2$). Proof: let $h$ be a homomorphism from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$ and $A_3$ an ABox such that $V_{A_3} = h(V_{A_3})$. Assume that $x \in \Delta A_3$ and the length of $x$ is more than $2m$. Then $x$ is not connected to $\text{Ind}(A_1)$ in $A_3$, i.e., there exists no path $R_1(x_1, x_2), \ldots , R_n(x_n, x_{n+1})$ with $x_1 = x$, $x_{n+1} = a \in \text{Ind}(A_1), R_i(x_i, x_{i+1}) \in A_3$ (otherwise it contradicts $V_{A_3} = h(V_{A_2})$). Let $C$ be the maximal connected subset of $A_3$ with $x \in \Delta C$, i.e., $\Delta C \cap \Delta A_3 \subseteq C$ and for each $C' \subseteq C$, $\Delta C' \cap \Delta C \subseteq \Delta C' \neq \emptyset$, moreover $\Delta C \cap \text{Ind}(A_1) = \emptyset$. Let $y$ be the path (in the sense of path($I(T_1 \cup T_2, A_1)$)) of minimal length in $C$, it exists and is unique since $V_{A_3} \subseteq U_{(T_1 \cup T_2, A_1)}$ and there are no constants in $C$, and for each $x \in C, x = y \cdot w[R_1] \ldots w[R_n]$ for some $n$. Further assume $\text{tail}(y) = w[R_i]$, then let $y'$ be a path of the minimal length in $\Delta U_{(T_1 \cup T_2, A_1)}$ with $\text{tail}(y') = w[R_i]$. Then the length of $y'$ is bounded by the size of $T_1 \cup T_2$ and the length of each $y' \cdot w[R_1] \ldots w[R_{n'}]$ where $y' \cdot w[R_1] \ldots w[R_{n'}] \in C$, is bounded by the size of $T_1 \cup T_2 \cup A_2$. Now, define a new function $h': \Delta V_{A_2} \rightarrow \Delta U_{(T_1 \cup T_2, A_1)}$ such that $h'(x) = h(x)$ if $h(x) \in C, h'(x) = y' \cdot w[R_1] \ldots w[R_{n'}]$. It is easy to see that $h'$ is a homomorphism from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$. We can continue this iteratively until we get that for every $x \in \Delta A_3, x$ is a path of length bounded by $2m$, where $A_3$ is an ABox such that $V_{A_3} = h'(V_{A_3})$.

Finally, our algorithm for checking existence of a homomorphism from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$ is as follows:

1. compute (guess) $A_3$ (in NP):
   - for each $x \in \Delta A_2$ we guess $y \in \Delta U_{(T_1 \cup T_2, A_1)}$ such that there exists a path from $\text{Ind}(A_1)$ to $y$ of polynomial length,
   - Let $W$ be the set of all $y$ guessed above, then
     $$A_3 = \{ A(x) \ | \ x \in W, \text{tail}(x) = w[R_i], T_1 \cup T_2 \models \exists R^a \subseteq A, A \in \Sigma_2 \} \cup \{ S(x', x) \ | \ x, x' \in W, x = x' \cdot w[R_i], T_1 \cup T_2 \models R \subseteq S, S \in \Sigma_2 \},$$
     $$\forall A_3 \subseteq U_{(T_1 \cup T_2, A_1)}, \Delta A_3 = W \text{ and } A_3 \text{ is of polynomial size.}$$

2. check whether there exists a homomorphism from $V_{A_2}$ to $V_{A_3}$ (in NP).

   We prove that the above described procedure is correct.

   Assume, we computed $A_3$ and there exists a homomorphism $h$ from $V_{A_2}$ to $V_{A_3}$. Then since $V_{A_3} \subseteq U_{(T_1 \cup T_2, A_1)}$, it follows that $h$ is a homomorphism from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$.

   Now, assume that there exists no homomorphism from $V_{A_2}$ to $V_{A_3}$, and by contradiction there exists a homomorphism from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$. Then, we showed that there exists a homomorphism $h'$ from $V_{A_2}$ to $U_{(T_1 \cup T_2, A_1)}$ and an ABox $A_3$ such that $V_{A_3} = h'(V_{A_2})$ and the length of every $x \Delta A_3$ is bounded by $2m$, where $m$ is the size of $T_1 \cup T_2 \cup A_2$. Contradiction with step 1.

Then the membership check for universal solutions with extended ABoxes can be done as follows:

1. verify whether $K_1$ and $M$ are $\Sigma_2$-positive, if yes
2. check whether $T_2$ is equivalent to the empty TBox, if yes
3. check whether $A_2$ is a solution with extended ABoxes for $K_{\text{pos}}^\Sigma$ under $M_{\text{pos}}$, if yes
4. check whether $A_2$ is homomorphically embeddable into $U_{T_1 \cup T_2, A_1}$. If yes, then $K_2$ is a universal solution for $K_1$ under $M$, otherwise it is not.

Steps 1 and 2 can be done in polynomial time. Step 3 can be done in NP similarly to Theorem 5.3.1 guess an interpretation $I$ of $\Sigma_1$ of polynomial size, check whether $I$ is a model of $K_{\text{pos}}^\Sigma$ and $(\Sigma, V_{A_2}) \models T_{\text{pos}}^\Sigma$.

If yes, then $A_2$ is a solution: let $J$ be a model of $A_2$ and $h$ a homomorphism from $V_{A_2}$ to $J$. Then, let $T_{J}$ be the image of $h$ applied to $I$, $T_{J} = h(I)$. Then $T_{J}$ is a model of $K_{\text{pos}}^\Sigma$ and $(\Sigma, J) \models T_{\text{pos}}^\Sigma$, hence indeed, $A_2$ is a solution. Step 4 is feasible in NP, therefore in overall the membership check can be done.
B.6 Proof of Theorem 5.4

Proof. The proof is by reduction of the satisfiability problem for quantified Boolean formulas, known to be PSPACE-complete. Suppose we are given a QBF

$$\phi = Q_1 X_1 \ldots Q_m X_n \bigwedge_{j=1}^m C_j$$

where $Q_i \in \{\forall, \exists\}$ and $C_j, 1 \leq j \leq m$, are clauses over the variables $X_i, 1 \leq i \leq n$.

Let $\Sigma_1 = \{A, Y_i^k, X_i^k, S_l, T_i, Q_i^k, P_i^k, R_j^l, R_j^i | 1 \leq j \leq m, 1 \leq i \leq n, 0 \leq l \leq n, k \in \{0, 1\}\}$ where $A, Y_i^k, X_i^k$ are concept names and the rest are role names. Let $T_1$ be the following TBox over $\Sigma_1$ for $1 \leq j \leq m, 1 \leq i \leq n$ and $k \in \{0, 1\}$:

$$A \sqsubseteq \exists S_0^i \exists S_{i-1}^i \exists Q_i^k \exists R_j^l \exists R_j^i$$

and $A_1 = \{A(a)\}$.

Further, let $\Sigma_2 = \{A', Z_i^0, Z_i^1, S', R_j^0, R_j^i, T_i', T_i^l, T_i^j, R_j^i, R_j^j | 1 \leq j \leq m, 1 \leq i \leq n, k \in \{0, 1\}\}$ where $A', Z_i^0, Z_i^1$ are concept names and $S', R_j^0, R_j^i, T_i', T_i^l, T_i^j, R_j^i, R_j^j$ are role names, $\mathcal{M} = (\Sigma_1, \Sigma_2, T_{12})$.

Finally, let $A_2 = \{A'(a)\}$, and $T_2$ the following target TBox for $1 \leq j \leq m, 1 \leq i \leq n$ and $k \in \{0, 1\}$:

$$P_i^k \sqsubseteq P_i^k$$

Finally, let $A_2 = \{A'(a)\}$, and $T_2$ the following target TBox for $1 \leq j \leq m, 1 \leq i \leq n$ and $k \in \{0, 1\}$:

We verify that $\models \phi$ if and only if $\langle T_2, A_2 \rangle$ is a universal UCQ-solution for $\mathcal{K}_1 = \langle T_1, A_1 \rangle$ under $\mathcal{M}$. From Claim [A.4] it follows that $\langle T_2, A_2 \rangle$ is a universal UCQ-solution for $\mathcal{K}_1 = \langle T_1, A_1 \rangle$ under $\mathcal{M}$ if $\mathcal{U}(T_1 \cup T_{12}, A_1)$ is finitely $\Sigma_2$-homomorphically equivalent to $\mathcal{U}(T_{12}, A_2)$. Therefore, we are going to show that $\models \phi$ if and only if $\mathcal{U}(T_1 \cup T_{12}, A_1)$ is finitely $\Sigma_2$-homomorphically equivalent to $\mathcal{U}(T_{12}, A_2)$.

The rest of the proof is similar to Lemma [B.15]
C.1 Basic Preliminary Results

Lemma C.1 Let $\mathcal{K} = \langle T, A \rangle$ be a KB, $a, b \in N_a$, $\sigma \in \Delta^{\mathcal{K}}$, and tail$(\sigma) \rightsquigarrow_{\mathcal{K}} w_{[R]}$. Then,

(i) $B \in t^{\mathcal{K}}(a)$ iff $A \models B'(a)$ and $T \models B' \subseteq B$;
(ii) $R \in r^{\mathcal{K}}(a, b)$ iff $A \models R'(a, b)$ and $T \models R' \subseteq R$;
(iii) $B \in t^{\mathcal{K}}(\sigma w_{[R]})$ iff $T \models \exists R' \subseteq B$;
(iv) $R \in r^{\mathcal{K}}(\sigma, \sigma w_{[R]})$ iff $T \models R' \subseteq R$.

Proof. For (i), assume, first, $B$ is a concept name, then the proof straightforwardly follows from the definition of $U_{\mathcal{K}}$. Let $B = \exists R$ for a role $R$, we show the “only if” direction. By the definition of $U_{\mathcal{K}}$ it follows either $a \rightsquigarrow_{\mathcal{K}} w_{[R]}$ for some role $R'$ such that $T \models R' \subseteq R$ or $K \models R(a, b)$ for some $b \in N_a$. In the first case $K \models \exists R'(a)$ and $T \models \exists R' \subseteq B$ by the definition of $\rightsquigarrow$. It is then immediate that $A \models B'(a)$ and $T \models B' \subseteq B$ for some concept $B'$. In the second case, there is a role $R''$ such that $A \models R''(a)$ and $T \models R'' \subseteq B$, so the result follows with $B' = \exists R''$. The “if” direction is similar using the definition of $U_{\mathcal{K}}$ and $\rightsquigarrow$, which concludes the proof of (i). The proof for (ii) is analogous.

For (iii), assume, first, $B$ is a concept name, then the proof straightforwardly follows from the definition of $U_{\mathcal{K}}$. Let $B = \exists S$ for a role $S$, we, first, show the “only if” direction. It follows there exists $\sigma' \in \Delta^{\mathcal{K}}$ such that $(\sigma w_{[R]}, \sigma') \in S^{\mathcal{K}}$. From the definition of $U_{\mathcal{K}}$ it should be clear that either $\delta' = \delta$ and $T \models R \subseteq S'$, or $\delta' = \sigma w_{[R]}(\sigma w_{[R]})$ for a role $R'$ such that $w_{[R]} \rightsquigarrow_{\mathcal{K}} w_{[R']}$ and $T \models R' \subseteq S$. Then, from $w_{[R]} \rightsquigarrow_{\mathcal{K}} w_{[R']} we can also conclude $T \models \exists R \subseteq \exists R'$. One can see that in both cases above it follows $T \models \exists R \subseteq \exists S$, which concludes the proof of the “only if” direction. The “if” direction is similar using the definition of $U_{\mathcal{K}}$ and $\rightsquigarrow$.

Lemma C.2 Let $\langle T, A \rangle$ and $\langle T', A' \rangle$ be the KBs, such that:

(i) $T \subseteq T'$,
(ii) $A \models B(a)$ implies $A' \models B(a)$ and $A \models R(a, b)$ implies $A' \models R(a, b)$, for all $a, b \in N_a$, concepts $B$ and roles $R$.

Then, for each $\sigma \in \Delta^{\mathcal{T}(T, A)}$ there exists $\delta \in \Delta^{\mathcal{T}(T', A')}$ such that

(iii) $t^{\mathcal{T}(T, A)}(\sigma) \subseteq t^{\mathcal{T}(T', A')}(\delta)$,
(iv) $r^{\mathcal{T}(T, A)}(a, \sigma) \subseteq r^{\mathcal{T}(T', A')}(a, \delta)$ for all $a \in N_a$.

Proof. Consider, first, the case $\sigma = b \in N_a$, then set $\delta = b$ and we show (iii). Consider $B \in t^{\mathcal{T}(T, A)}(\sigma)$, it follows by Lemma C.1, $A \models B'(b)$ and $T \models B' \subseteq B$, for some concept $B'$. Then, by (i) it follows $T' \models B' \subseteq B$ and by (ii) it follows $A' \models B'(b)$, therefore, by Lemma C.1 we obtain $B \in t^{\mathcal{T}(T', A')}(\delta)$. The proof for (iv) is analogous.

Now, assume the lemma holds for $\sigma' \in \Delta^{\mathcal{T}(T, A)}$; we show it also holds for $\sigma = \sigma' w_{[R]} \in \Delta^{\mathcal{T}(T, A)}$ for a role $R$. By the definition of $U_{\mathcal{T}(T, A)}$ it follows tail$(\sigma') \rightsquigarrow_{\mathcal{T}(T, A)} w_{[R]}$ and so $\exists R \in t^{\mathcal{T}(T, A)}(\sigma')$. By Lemma C.1 it follows

$$\begin{align*}
T \models \exists R' \subseteq B & \text{ for each } B \in t^{\mathcal{T}(T, A)}(\sigma) \\
T \models R \subseteq \exists Q & \text{ for each } Q \in t^{\mathcal{T}(T, A)}(a, \sigma)
\end{align*}$$

(1) (2)

On the other hand, observe by our induction hypothesis that there exists $\delta' \in \Delta^{\mathcal{T}(T', A')}$ such that $t^{\mathcal{T}(T', A')}(\sigma') \subseteq t^{\mathcal{T}(T', A')}(\delta')$; therefore, $\exists R \in t^{\mathcal{T}(T', A')}(\delta')$. It follows there exists $\delta'' \in \Delta^{\mathcal{T}(T, T', A)}$ such that $(\delta', \delta'') \in R^{\mathcal{T}(T, A')}. We select $\delta$ (for $\sigma$) equal to $\delta''$; using (1), (2) and Lemma C.1 one can easily show (iii) and using (2) and (ii) one can show (iv).

Lemma C.3 Let $\mathcal{K} = \langle T, A \rangle$ and assume $a \rightsquigarrow_{\mathcal{K}} w_{[R]}$ for some basic role $R$. Then there exists a basic concept $B$, such that $A \models B(a)$, and:

(i) $a \rightsquigarrow_{\mathcal{T}(T, B(a))} w_{[R]}$;
(ii) $t^{\mathcal{T}_{\mathcal{K}}}(aw_{[R]}) \subseteq t^{\mathcal{T}(T, B(a))}(ow_{[R]})$;
(iii) $r^{\mathcal{T}_{\mathcal{K}}}(a, aw_{[R]}) \subseteq r^{\mathcal{T}(T, B(a))}(o, ow_{[R]})$.

Proof. Consequence of Lemma C.1.
Lemma C.4 Let $A$ be an ABox, $B$ a set of basic concepts, and $T, T'$ TBoxes. Let $\mathcal{B} = \langle T, \{B(b) \mid B \in B\}\rangle$, and assume $y \in \Delta^{U_b}$. If $\sigma \in \Delta^{U(T \cup T', A)}$ and $B \subseteq t^{U(T \cup T', A)}(\sigma)$, then there exists $\delta \in \Delta^{U(T \cup T', A)}$ such that

(i) $t^{U_b}(y) \subseteq t^{U(T \cup T', A)}(\delta)$

(ii) $t^{U_b}(a, y) \subseteq t^{U(T \cup T', A)}(\sigma, \delta)$

Proof. Straightforward consequence of Lemma C.2

Lemma C.5 For each $\sigma \in \Delta^{U(T_1 \cup T_2, A)}$ and $B' \in t^{U(T_1 \cup T_2, A)}(\sigma)$ one of the following holds:

(i) there exists a concept $B$ over $\Sigma$ such that $B \in t^{U(T_1 \cup T_2, A)}(\sigma)$ and $T_{12} \vdash B \subseteq B'$;

(ii) $t^{U(T_1 \cup T_2, A)}(\sigma) = \{B'\}$.

Proof. Using Lemma C.1 and considering the structure of $T_1 \cup T_2$ with $\Sigma \cap \Xi = \emptyset$. The case (ii) occurs, when $\text{tail}(\sigma) = \emptyset$ for a role $Q$ is over $\Xi$.

Lemma C.6 A DL-Lite KB $\langle T, A \rangle$ is consistent iff

(i) $B, B'$ is $T$-consistent for each pair of basic concepts $B, B'$ and each $a \in \text{Ind}(A)$ such that $A \models B(a)$ and $A \models B'(a)$;

(ii) $R, R'$ is $T$-consistent for each pair of roles $R, R'$ and each $a, b \in \text{Ind}(A)$ such that $A \models R(a, b)$ and $A \models R'(a, b)$;

Proof. ($\Rightarrow$) Assume (ii) is violated, so there exist $B_1, B_2$ and $a \in \text{Ind}(A)$ such that $A \models B_1(a), A \models B_2(a)$, and $\langle T, \{B_1(a), B_2(a)\}\rangle$ is inconsistent. It follows that $U_\mathcal{T}(\{B_1(a), B_2(a)\})$ is not a model of $\langle T, \{B_1(a), B_2(a)\}\rangle$, so there is $\sigma \in \Delta^{U(T.(B_1(a), B_2(a)))}$ and a disjointness assertion $B \cap C \subseteq \perp \in T$ (note that inclusion assertions $B \supseteq C \in T$ cannot cause inconsistency) such that $B, C \in t^{U(T.(B_1(a), B_2(a)))}(\delta)$. Obviously, $\langle T, \{B_1(a), B_2(a)\}\rangle \subseteq t^{U(T.(A))}(\sigma)$, then by Lemma C.2 we obtain $\delta \in \Delta^{U(T, A)}$ such that $B, C \in t^{U(T, A)}(\delta)$. Hence, $U_{T, A}$ is not a model of $\langle T, A \rangle$, which contradicts Claim A.1 since $\langle T, A \rangle$ is consistent.

(\Leftarrow) The proof is analogous to the modification of the proof above.

Lemma C.7 If a KB $\langle T, A \rangle$ is consistent, then for all $\delta, \sigma \in \Delta^{U(T, A)}$,

(i) $B \in T$-consistent for each $B \in t^{U(T, A)}(\delta)$;

(ii) $R$ is $T$-consistent for each $R \in t^{U(T, A)}(\delta, \sigma)$.

Proof. Similar to Lemma C.6

C.2 Homomorphism Lemmas

Here we present a series of important lemmas used in the proof the main results in the following sections.

Lemma C.8 Assume a mapping $M = \langle \Sigma, \Xi, T_1, T_2 \rangle$, ABoxes $A$ and $A'$ over, respectively, $\Sigma$ and $\Xi$, and a $\Xi$-TBox over $T_2$. If $U_{T_2, A}$ is $\Xi$ homomorphically embeddable into $U_{A'}$, then $U_{T_2, A}$ is homomorphically embeddable into $U_{T_2, A'}$.

Proof. Consider the $\Xi$ homomorphism $h : \Delta^{U(T_1, A)} \to \Delta^{U(A')}$ from $U_{T_2, A}$ to $U_{A'}$. we are going to construct the $\Sigma$ homomorphism $h' : U_{\Sigma(T_2, A)} \to U_{\Sigma(T_2, A')}$ from $U_{\Sigma(T_2, A)}$ to $U_{\Sigma(T_2, A')}$. Initially, we define $h'(a) = a$, let us immediately verify that $t^{U_{\Sigma(T_2, A')}_{\Sigma}(a)} \subseteq t^{U_{\Sigma(T_2, A')}_{\Sigma}(h'(a))}$. Notice that by the definition of $h$ we have:

\[ t^{U_{\Sigma(T_2, A')}_{\Sigma}(a)}(a) \subseteq t^{U_{\Sigma(A')}_{\Sigma}(h(a))}, \]

\[ h(a) = h'(a). \]

Let $C \in t^{U_{\Sigma(T_2, A')}_{\Sigma}(a)}$, it follows by Lemma C.1 (i) there exists $B$ over $\Sigma$, such that $A \models B(a)$ and $\Sigma \cup T_1 \vdash B \subseteq C$. Taking into account the shape of $T_2$ and $T_1$, it follows also there exists $D$ over $\Xi$ such that $T_{12} \vdash B \subseteq D$ and $T_2 \vdash D \subseteq C$. Observe that $B \in t^{U_{\Sigma(T_1, A')}_{\Sigma}(a)}$, then by Lemma C.1 (i) and (ii)
it follows $D \in \mathfrak{t}_{\Xi}^{U(T_{12} \cup A)}(a)$ and taking into account $\mathfrak{t}_{\Xi}^{U(T_{12} \cup A)}(a, \sigma)$ we conclude $D \in \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(h'(a))$. Finally, using again Lemma C.1(i) and (iii) we obtain $C \in \mathfrak{r}_{\Xi}^{U(T_{12} \cup A)}(h'(a))$. The proof that $\mathfrak{r}_{\Xi}^{U(T_{12} \cup A)}(a, b) \subseteq \mathfrak{r}_{\Xi}^{U(T_{12} \cup A)}(h'(a), h'(b))$ for all constants $a$ and $b$ is analogous.

Now we show how to define $h'$ for $\sigma = aw[\gamma] \in \text{path}([T_{2} \cup T_{12}, A])$. It follows a $^\gamma \rightarrow [T_{2} \cup T_{12}, A]$ $w[\gamma]$, then two cases are possible:

1. $R$ is over $\Sigma$;
2. $R$ is over $\Xi$.

In case (I) it follows $a \rightarrow [T_{12}, A]$ $w[\gamma]$ and by the condition of the current lemma it follows there is $\delta \in \Delta^{U(\theta^\prime)}$ such that:

\[ \mathfrak{t}_{\Xi}^{U(T_{12} \cup A)}(aw[\gamma]) \subseteq \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\delta), \quad (5) \]

\[ \mathfrak{r}_{\Xi}^{U(T_{12} \cup A)}(a, aw[\gamma]) \subseteq \mathfrak{r}_{\Xi}^{U(\theta^\prime)}(a, \delta), \quad (6) \]

Then, using Lemma C.2 with $A' = A$, $T = \emptyset$, $T' = T_{2}$ we obtain $\gamma \in \Delta^{U(\theta^\prime)}$ such that

\[ \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\delta) \subseteq \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\gamma), \quad (7) \]

\[ \mathfrak{r}_{\Xi}^{U(\theta^\prime)}(a, \delta) \subseteq \mathfrak{r}_{\Xi}^{U(\theta^\prime)}(a, \gamma), \quad (8) \]

Now define $h'(\sigma) = \gamma$; we need to show

\[ \mathfrak{t}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(\sigma) \subseteq \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\gamma), \quad (9) \]

\[ \mathfrak{r}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(a, \sigma) \subseteq \mathfrak{r}_{\Xi}^{U(\theta^\prime)}(a, \gamma). \quad (10) \]

For (I) consider the set $\bar{S} = \{ B \in \Xi | T_{12} \vdash \exists \bar{R} \subseteq B \}$ and observe that $\bar{B} \subseteq \mathfrak{t}_{\Xi}^{U(T_{12} \cup A)}(aw[\gamma])$ and also by Lemma C.1 and the structure of $T_{2} \cup T_{12}$, for each $B' \in \mathfrak{t}_{\Xi}^{U(T_{12} \cup A)}(aw[\gamma])$ there exists $B \in \bar{B}$ such that $T_{2} \vdash B \subseteq B'$. By (5) and (7) we obtain $\bar{B} \subseteq \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\gamma)$; then using Lemma C.1 it can be easily verified $B' \in \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\gamma)$ for all $B'$ as above, which concludes the proof of (9). The (10) is analogous using (6), (8), and the set $\bar{S} = \{ S \in \Xi | T_{12} \vdash \bar{R} \subseteq S \}.

Consider the case (II) using Lemma C.1 and the structure of $T_{2} \cup T_{12}$ and $A$, one can show:

\[ \exists \bar{R} \in \mathfrak{t}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(a), \quad (11) \]

\[ T_{2} \vdash \bar{R} \subseteq B \text{ for all } B \in \mathfrak{t}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(\sigma), \quad (12) \]

\[ T_{2} \vdash \bar{R} \subseteq S \text{ for all } S \in \mathfrak{r}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(a, \sigma). \quad (13) \]

Provided that the homomorphism $h'$ is defined for $a$, it follows $\exists \bar{R} \in \mathfrak{t}_{\Xi}^{U(T_{2} \cup A)}(h'(a))$, therefore, there exists $\gamma \in \Delta^{U(\theta^\prime)}$ such that $R \in \mathfrak{r}_{\Xi}^{U(T_{2} \cup A)}(h'(a), \gamma)$. Now define $h'(\sigma) = \gamma$; we need to show

\[ \mathfrak{t}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(\sigma) \subseteq \mathfrak{t}_{\Xi}^{U(\theta^\prime)}(\gamma), \quad (14) \]

\[ \mathfrak{r}_{\Xi}^{U(T_{2} \cup T_{12} \cup A)}(a, \sigma) \subseteq \mathfrak{r}_{\Xi}^{U(\theta^\prime)}(a, \gamma). \quad (15) \]

For (II) consider (12) and Lemma C.1 similarly, for (II) consider (13).

Assume now $\sigma = \sigma[aw[\gamma]]$ and the homomorphism from $U(T_{2} \cup T_{12} \cup A)$ to $U(T_{2} \cup A)$ is defined for $\sigma'$. The proof is done in the same way as for the case (II) all the statements are valid if one substitutes $a$ by $\sigma'$. □

Let $\mathcal{M} = (\Sigma, \Xi, T_{12})$ be a mapping, and, $T_{1}$ and $T_{2}$, respectively, $\Sigma$- and $\Xi$-TBoxes. Define KBs $\mathcal{S}_{\bar{B}} = (\bar{T}_{1} \cup T_{12}, \{ B(a) \})$ and $\mathcal{X}_{\bar{B}} = \{ T_{2} \cup T_{12}, \{ B(o) \} \}$ for a basic concept $B$ over $\Sigma$. We slightly abuse the notation and write $\mathcal{S}_{A}$ to denote the KB $(T_{1} \cup T_{12}, A)$ for a given ABox $A$, analogously we use $\mathcal{X}_{A}$ to denote $(T_{2} \cup T_{12}, A)$. We show

**Lemma C.9** Let $\mathcal{A}$ be an ABox over $\Sigma$ and assume for each concept $B$, role $R$, and all $\sigma, \delta \in \Delta^{U_{\mathcal{S}_{A}}}$ such that

(i) $B \in \mathfrak{t}_{\Sigma}^{\mathcal{S}_{A}}(\sigma)$,

(ii) $R \in \mathfrak{r}_{\Sigma}^{\mathcal{S}_{A}}(\sigma, \delta)$,
the following conditions hold

(iii) \( t_{\Sigma}^{X_B}(o) \subseteq t_{\Sigma}^{S_B}(o) \);
(iv) \( T_2 \cup T_1 \vdash R \subseteq R' \) implies \( T_2 \cup T_1 \vdash R \subseteq R' \) for all roles \( R' \) over \( \Sigma \);
(v) for each role \( R \) such that \( o \rightarrow_{X_B} w_{[R]} \) there exists \( y \in \Delta t_{\Sigma}^{S_B} \) such that

(a) \( t_{\Sigma}^{L_B}(aw_{[R]}) \subseteq t_{\Sigma}^{J_B}(y) \),
(b) \( r_{\Sigma}^{L_B}(a, aw_{[R]}) \subseteq r_{\Sigma}^{J_B}(a, y) \).

Then \( U_{\Sigma}^{X_A} \) is finitely homomorphically embeddable into \( U_{\Sigma}^{S_A} \).

**Proof.** Let \( A \) as above and assume the condition of the lemma are satisfied. We build a mapping \( h \) from \( \text{path}(X_A) \) to \( \text{path}(S_A) \) such that for any finite subinterpretation of \( U_{\Sigma}^{X_A} \) the restriction of \( h \) to it is a homomorphism to \( U_{\Sigma}^{S_A} \). Initially, we define \( h(a) = a \), let us immediately verify that \( t_{\Sigma}^{L_B}(a) \subseteq t_{\Sigma}^{S_B}(a) \). Let \( C \in t_{\Sigma}^{L_B}(a) \), it follows by Lemma C.1(i) there exists \( B \) over \( \Sigma \) such that \( A \models B(a) \) and \( T_2 \cup T_1 \vdash B \subseteq C \). Observe that \( B \in t_{\Sigma}^{S_B}(a) \); now if \( C \) is over \( \Sigma \) it follows \( C = B \), so \( C \in t_{\Sigma}^{S_B}(a) \) and the proof is done. Otherwise, \( C \in s_{\Sigma}^{S_B}(a) \), then by C.1(i) obtain \( B \in t_{\Sigma}^{S_B}(a) \). The proof of \( r_{\Sigma}^{L_B}(a, b) \subseteq r_{\Sigma}^{S_B}(a, b) \) is analogous using Lemma C.1(iii) and current(iv).

Now we show how to define \( h \) for \( \sigma = aw_{[R]} \in \text{path}(X_A) \). It follows \( o \rightarrow_{X_B} w_{[R]} \), then by Lemma C.5 with \( K = X_A \) there exists \( B \) over \( \Sigma \) such that \( A \models B(a) \), \( o \rightarrow_{X_B} w_{[R]} \), and

\[
\begin{align*}
  t_{\Sigma}^{L_B}(aw_{[R]}) &\subseteq t_{\Sigma}^{S_B}(w_{[R]}) & (16) \\
  r_{\Sigma}^{L_B}(a, aw_{[R]}) &\subseteq r_{\Sigma}^{S_B}(a, w_{[R]}). & (17)
\end{align*}
\]

We are going to show now there exists \( y \in \Delta t_{\Sigma}^{S_B} \) such that

\[
\begin{align*}
  t_{\Sigma}^{S_B}(w_{[R]}) &\subseteq t_{\Sigma}^{J_B}(y) & (18) \\
  r_{\Sigma}^{S_B}(a, w_{[R]}) &\subseteq r_{\Sigma}^{J_B}(a, y). & (19)
\end{align*}
\]

Assume, first, \( t_{\Sigma}^{L_B}(aw_{[R]}) = \emptyset \), then also \( r_{\Sigma}^{L_B}(a, aw_{[R]}) = \emptyset \); it remains to observe that from \( A \models B(a) \) it follows \( h(a) = a \), then by \( \vdash_\Sigma \) we obtain \( y \) satisfying \( (18) \) and \( (19) \).

Assume now \( t_{\Sigma}^{L_B}(aw_{[R]}) \neq \emptyset \), it follows \( B = \exists R, t_{\Sigma}^{L_B}(aw_{[R]}) = \{ \exists R \} \), and \( r_{\Sigma}^{L_B}(a, aw_{[R]}) = r_{\Sigma}^{S_B}(a, w_{[R]}) \). Since \( B = \exists R \), there must exist a role \( Q \) such that \( o \rightarrow_{S_B} w_{[Q]} \) and \( T_1 \cup T_2 \vdash Q \subseteq R \), we choose \( w_{[Q]} \) to be the required \( y \); it is immediate to see \( t_{\Sigma}^{L_B}(aw_{[R]}) \subseteq t_{\Sigma}^{S_B}(w_{[Q]}) \), and \( r_{\Sigma}^{L_B}(a, aw_{[R]}) \subseteq r_{\Sigma}^{S_B}(a, w_{[R]} \subseteq r_{\Sigma}^{S_B}(a, y) \). To prove also \( t_{\Sigma}^{S_B}(aw_{[R]}) \subseteq t_{\Sigma}^{J_B}(y) \) and \( r_{\Sigma}^{S_B}(a, w_{[R]}) \subseteq r_{\Sigma}^{J_B}(a, y) \) we are going to use \( \vdash_\Sigma \) and \( \vdash_\Sigma \) but we need \( \exists R' \in t_{\Sigma}^{L_B}(\sigma) \) and \( R \in t_{\Sigma}^{L_B}(\sigma, \delta) \) for some \( \sigma', \delta \in \Delta t_{\Sigma}^{L_B} \). To get the latter two facts it is sufficient to notice \( R \in t_{\Sigma}^{L_B}(a) \) (since \( o \rightarrow_{X_B} w_{[R]} \) and \( t_{\Sigma}^{L_B}(a) \subseteq t_{\Sigma}^{S_B}(a) \) proven above.

The proof of \( t_{\Sigma}^{L_B}(aw_{[R]}) \subseteq t_{\Sigma}^{S_B}(y) \) is as follows: assume \( B' \in t_{\Sigma}^{L_B}(aw_{[R]}) \), then since \( R \) is over \( \Sigma \) it follows \( B' \subseteq t_{\Sigma}^{L_B}(aw_{[Q]}) \). By \( \Sigma R' \in t_{\Sigma}^{L_B}(\sigma) \) and \( \exists R' \in t_{\Sigma}^{L_B}(\sigma) \), then since \( T_1 \cup T_2 \vdash Q \subseteq R \) it follows \( t_{\Sigma}^{L_B}(aw_{[Q]}) \subseteq t_{\Sigma}^{S_B}(w_{[Q]}) \), and we obtain \( B' \in t_{\Sigma}^{S_B}(y) \). The proof of \( r_{\Sigma}^{L_B}(a, aw_{[R]}) \subseteq r_{\Sigma}^{S_B}(a, y) \) is analogous using \( R \in t_{\Sigma}^{S_B}(\sigma, \delta) \) and \( T_1 \cup T_2 \vdash Q \subseteq R \). We finished showing there exists \( y \in \Delta t_{\Sigma}^{S_B} \) such that \( (18) \) and \( (19) \).

To continue the proof consider \( \{ B \} \subseteq t_{\Sigma}^{L_B}(a) \) and Lemma C.4 with \( T = T_1 \cup T_2 \) and \( T' = \emptyset \); there exists \( \delta \in \Delta t_{\Sigma}^{L_B} \) such that \( t_{\Sigma}^{S_B}(y) \subseteq t_{\Sigma}^{S_B}(\delta) \) and \( r_{\Sigma}^{S_B}(a, y) \subseteq r_{\Sigma}^{S_B}(a, \delta) \). It follows now using \( (16) \) and \( (18) \) that \( t_{\Sigma}^{L_B}(aw_{[R]}) \subseteq t_{\Sigma}^{L_B}(\delta) \). Analogously using \( (17) \) and \( (19) \) one obtains \( r_{\Sigma}^{L_B}(a, aw_{[R]}) \subseteq r_{\Sigma}^{S_B}(a, \delta) \).

We show how to define the homomorphism for \( \sigma w_{[R]} \in \text{path}(X_A) \) with \( \tau(\sigma) = w_{[R]} \); given that the homomorphism for \( h(\sigma) \) is defined. It follows \( w_{[R]} \rightarrow_{X_A} w_{[R]} \) and by definition of \( \rightarrow_\Sigma \) and the structure of \( T_2 \cup T_1 \) we obtain \( T_2 \cup T_1 \vdash \exists R' \subseteq \exists R \) and \( R \) is a \( \Sigma \) role different from \( R' \). By Lemma C.1 it also follows \( \{ \exists R' \} \subseteq t_{\Sigma}^{S_B}(\delta) \). Since \( h \) is a homomorphism, \( \{ \exists R', \exists R \} \subseteq t_{\Sigma}^{S_B}(\delta) \) for \( \delta = h(\sigma) \in \Delta t_{\Sigma}^{S_B} \). We use Lemma C.5 to obtain \( B \) over \( \Sigma \) such that \( B \in t_{\Sigma}^{S_B}(\delta) \) and \( T_2 \cup T_1 \vdash B \subseteq \exists R \). Notice that such \( B \) exists: since \( \exists R' \neq \exists R \) are different concepts, \( \vdash_\Sigma \) of Lemma C.5 is excluded, so \( \vdash_\Sigma \) holds.
Then in $X_B$ we have that $o \rightarrow_X B$ $w_{[R]}$ for a $\Xi$ role $R$, and the proof continues analogously to the proof for the case $\sigma = aw_{[R]}$ above using the conditions (ii), (iii) and Lemmas C.3 to obtain $\delta'$ in $\Delta^{\Xi_B}$ such that $t^{\Xi_B}(\sigma \omega_{[R]}) \subseteq t^{\Xi_B}(\delta')$ and $r^{\Xi_B}(\sigma, \omega_{[R]}) \subseteq r^{\Xi_B}(\delta, \delta')$. We assign $h(\sigma \omega_{[R]}) = \delta'$.

Thus, we defined the mapping $h$ that is clearly a $\Xi$-homomorphism from each finite subinterpretation of $U(T_1 \cup T_1, \mathcal{A})$ into $U(T_2 \cup T_2, \mathcal{A})$.

Lemma C.10 Let $\mathcal{A}$ be an ABox over $\Sigma$ and assume for each concept $B$, role $R$, and all $\sigma, \delta \in \Delta^{\Xi_B}$ such that

(i) $B \in t_{\Xi_B}^B(\sigma)$,

(ii) $R \in r_{\Xi_B}^B(\sigma, \delta)$

the following conditions hold

(iii) $t_{\Xi_B}^B(o) \subseteq t_{\Xi_B}(o)$;

(iv) $T_1 \cup T_2 \vdash R \subseteq R'$ implies $T_2 \cup T_2 \vdash R \subseteq R'$ for all roles $R'$ over $\Sigma$;

(v) for each role $R$ such that $o \rightarrow_{\Xi_B} w_{[R]}$ there exists $y \in \Delta^{\Xi_B}$ such that

(a) $t_{\Xi}^{\Xi_B}(aw_{[R]}) \subseteq t_{\Xi_B}(y)$,

(b) $r_{\Xi}^{\Xi_B}(o, aw_{[R]}) \subseteq r_{\Xi_B}(o, y)$.

Then $U_{\Xi_B}$ is finitely $\Xi$-homomorphically embeddable into $U_{\Xi_A}$.

Proof. Assume the condition of the lemma is satisfied, and let $\mathcal{A}$ be an ABox over $\Sigma$. We build a mapping $h$ from $\text{path}(S_A)$ to $\text{path}(X_A)$ such that for any finite subinterpretation of $U_{\Xi_B}$ the restriction of $h$ to it is a $\Xi$-homomorphism to $U_{\Xi_A}$. Initially, we define $h(a) = a$, let us immediately verify that $t_{\Xi}^{\Xi_B}(a) \subseteq t_{\Xi_B}(a)$. Let $B' \in t_{\Xi_B}(a)$, if follows by Lemma C.3 there exists $B$ over $\Sigma$ such that $A \models B(a)$ and $T_1 \cup T_2 \vdash B' \subseteq B$. Observe that $B \in t_{\Xi_B}(a)$, then by (iii) $B' \in t_{\Xi_B}(o)$, so $T_2 \cup T_2 \vdash B \subseteq B'$. Finally, using Lemma C.3 obtain $B' \in t_{\Xi_B}(a)$. The proof of $r_{\Xi_B}^{\Xi_B}(a, b) \subseteq r_{\Xi_B}(a, b)$ is analogous using Lemma C.3 and current (iv).

Now we show how to define $h$ for $\sigma = aw_{[R]} \in \text{path}(S_A)$. It follows $a \rightarrow_{\Xi_A} w_{[R]}$ and by Lemma C.3 (with $K = S_A$) we obtain $B$ over $\Sigma$ such that $A \models B(a)$, $o \rightarrow_{\Xi_B} w_{[R]}$, and

$$t_{\Xi_B}^{\Xi_B}(aw_{[R]}) \subseteq t_{\Xi_B}(y) \quad (20)$$
$$r_{\Xi_B}^{\Xi_B}(a, aw_{[R]}) \subseteq r_{\Xi_B}(o, aw_{[R]}). \quad (21)$$

Notice that $B \in s_{\Xi_B}^B(a)$ (that is, (ii)), then by (iv) there exists $y \in \Delta^{\Xi_B}$ such that

$$t_{\Xi_B}^{\Xi_B}(w_{[R]}) \subseteq t_{\Xi_B}(y) \quad (22)$$
$$r_{\Xi_B}^{\Xi_B}(o, w_{[R]}) \subseteq r_{\Xi_B}(o, y). \quad (23)$$

Since $\{B\} \subseteq t_{\Xi_B}(a)$, by Lemma C.3 (with $T = T_2 \cup T_1$, $T' = \emptyset$) there exists $\delta \in \Delta^{\Xi_B}$ such that $t_{\Xi_B}(y) \subseteq t_{\Xi_B}(\delta)$ and $r_{\Xi_B}(o, y) \subseteq r_{\Xi_B}(a, \delta)$. It follows now using (20) and (22) that $t_{\Xi_B}^{\Xi_B}(aw_{[R]}) \subseteq t_{\Xi_B}(\delta)$. Analogously using (21) and (23) one obtains $r_{\Xi_B}^{\Xi_B}(a, aw_{[R]}) \subseteq r_{\Xi_B}(a, \delta)$. We assign $h(\sigma) = \delta$.

We show how to define the homomorphism for $aw_{[R]} \in \text{path}(S_A)$ with $\sigma = \sigma'w_{[R]}$ given that the homomorphism $h(\sigma)$ and $h(\sigma')$ is defined. It follows $w_{[R]} \rightarrow_{\Xi_B} w_{[R]}$ and it that case $R'$ is over $\Sigma$ by the structure of $T_1 \cup T_2$. Analogously to the proof of Lemma C.3 it can be verified $o \rightarrow_{\Xi_B}(aw_{[R]}) \subseteq w_{[R]}$ and

$$t_{\Xi_B}^{\Xi_B}(aw_{[R]}) \subseteq t_{\Xi_B}(o) \quad (24)$$
$$r_{\Xi_B}^{\Xi_B}(a, aw_{[R]}) \subseteq r_{\Xi_B}(o, aw_{[R]}). \quad (25)$$

Observe that $\exists R' \in t_{\Xi_B}^{\Xi_B}(\sigma)$ (that is, (ii)), then by (v) there is $y \in \Delta^{\Xi_B}$ satisfying (a) and (b). Given the structure of $T_2 \cup T_2$ two cases are possible:
(III) \( y \in \Delta^{\mathcal{T}_2, (B(o) \in B)} \) for the set \( B \) of all concepts \( B \) over \( \Xi \) such that \( \mathcal{T}_2 \vdash \exists R' \sqsubseteq B \),

\begin{align*}
\mathfrak{t}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, w | R') & \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(y), \quad \text{(26)} \\
\mathfrak{r}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, w | R') & \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, y). \quad \text{(27)}
\end{align*}

(IV) \( o \rightsquigarrow x \in \mathcal{R}_{\Xi}^{R''} \) \( w | R'' \) for the set \( B \) of all concepts \( B \) over \( \Xi \) such that \( \mathcal{T}_2 \vdash \exists R' \sqsubseteq B \),

\begin{align*}
\mathfrak{t}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, w | R') & \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(y), \quad \text{(28)} \\
\mathfrak{r}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, w | R') & \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, y). \quad \text{(29)}
\end{align*}

Consider (III) then, \( B \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (h(\sigma))} \), since obviously \( B \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma)} \) and \( h \) is a homomorphism on \( \sigma \). By Lemma C.4 (with \( T = \mathcal{T}_2 \) and \( T' = \mathcal{T}_1 \)) we obtain \( \delta \in \Delta^{\mathcal{T}_2, (\sigma)} \) such that \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(y) \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\delta)} \) and \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (B(o) \in B)}(o, y) \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\delta)} \). Note that using (24) and (26) we obtain \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\delta)} \); also using (25) and (27) we obtain \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\delta)} \). We assign \( h(\sigma w | R') = \delta \) which concludes the proof.

Consider (IV) at this point we need

\begin{align*}
B & \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma')} \quad \text{(31)} \\
R & \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma')} \quad \text{(32)}
\end{align*}

for \( R = \{ R'' \mid \mathcal{T}_2 \vdash R'' \sqsubseteq R'' \} \). Indeed, (31) follows since \( \exists R' \in \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma')} \), by the definition of \( B \), and Lemma C.1(i) and (iii). For (32) let \( R'' \in R \) it follows \( [R''] \leq [R''] \leq \mathcal{T}_1 \cup \mathcal{T}_2 \ [R''] \). Then by the definition of \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma')} \) obtain \( R'' \in \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma')} \), so obviously \( R'' \in \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma')} \).

Observe that, since \( h \) is a \( \Xi \)-homomorphism on \( \sigma' \) and (31), it follows

\begin{align*}
B & \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma')} \quad \text{(33)}
\end{align*}

and distinguish two subcases:

(V) \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma w | R')} = \emptyset \);

(VI) \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma w | R')} \neq \emptyset \).

In case (V) consider (28), (33) and Lemma C.4 to obtain \( \delta \in \Delta^{\mathcal{T}_2, (\sigma w | R')} \) such that \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')}(y) \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')}(\delta) \). Then using (24) and (29) one obtains \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\delta)} \). Taking \( \delta = h(\sigma w | R') \) completes the proof of the first subcase.

In the alternative case (VI) it follows by (25) that \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \neq \emptyset \) therefore \( \emptyset = o \) (c.f. (28)). We assign \( h(\sigma w | R') = h(\sigma') \) and we prove \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \subseteq \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (h(\sigma'))} \), and \( \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma, \sigma w | R')} \subseteq \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (h(\sigma'), h(\sigma'))} \).

Indeed, let \( B \in \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \), by (24) \( B \in \mathfrak{s}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \), then by (29) there exists \( B' \in B \) such that \( \mathcal{T}_2 \vdash B' \sqsubseteq B \). Using (33) and Lemma C.1(iii) obtain \( B \in \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (h(\sigma'))} \).

Let now \( Q \in \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \), by (25) it follows \( Q \in \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (\sigma w | R')} \), then by (30) there exists \( R'' \in R \) such that \( \mathcal{T}_2 \vdash R'' \sqsubseteq Q \). Since \( h \) is a homomorphism on \( \sigma' \) and (32) obtain \( R'' \in \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (h(\sigma'), h(\sigma'))} \). By the definition of \( \mathfrak{t}_{\Xi}^{\mathcal{T}_2, (h(\sigma), h(\sigma'))} \) we conclude also \( Q \in \mathfrak{r}_{\Xi}^{\mathcal{T}_2, (h(\sigma), h(\sigma'))} \). This concludes the proof of the second subcase and the whole case (IV). We have shown how to define \( h \) for \( \sigma w | R' \in \text{path}(\mathcal{S}_A) \) so that \( h \) is a \( \Xi \)-homomorphism.

C.3 Proof of Proposition 6.1

This proof can be obtained as an easy consequence of the following

Lemma C.11 Let \( \mathcal{M} = (\Sigma, \Xi, \mathcal{T}_2) \) be a mapping, and \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), respectively, \( \Sigma \)- and \( \Xi \)-TBoxes, \( q(\bar{x}) \) a \( \Xi \)-query, and \( A \) a \( \Sigma \)-ABox. Then

\[ \bigcap_{A' \text{ is a } \Sigma \text{-ABox, s.t. it is UCQ solution for } A \text{ under } \mathcal{T}_1} \text{cert}(q, \langle \mathcal{T}_2, A' \rangle) \subseteq \bigcap_{A' \text{ extended } \Sigma \text{-ABox, s.t. it is UCQ solution for } A \text{ under } \mathcal{T}_1} \text{cert}(q, \langle \mathcal{T}_2, A' \rangle). \]
Proof. Consider a tuple of constants \( \bar{a} \) such that \( \langle T_2, A' \rangle \models q[\bar{a}] \) for all \( \Xi\)-ABoxes \( A' \), such that \( A' \) is a UCQ-solution for \( A \) under \( T_2 \). Assume an extended ABox \( A' \), such that it is a UCQ-solution for \( A \); we are going to show \( \langle T_2, A' \rangle \models q[\bar{a}] \). If \( \langle T_2, A' \rangle \) is inconsistent, the proof is done; otherwise, take an interpretation \( I \models \langle T_2, A' \rangle \). It follows there exists a substitution over \( I \), such that \( h(u) \in B^I \) for every \( B(u) \in A \) and \( (h(u), h(v)) \in R^I \) for all \( R(u, v) \in A \). We associate with every null \( n \) in \( A' \) a fresh (w.r.t. constants in \( A', \bar{a} \), and \( q(\bar{x}) \) constant \( a_n \) in \( N_n \); then take \( A' \) the result of the substitution of each \( n \) by \( a_n \) in \( A' \). Consider an interpretation \( I' \), such that it is equal to \( I \), except for \( a_n \), such that \( n \) is a null in \( A' \), we set \( a_n^{T'} = h(n) \). It should be clear that \( I' \models \langle T_2, A' \rangle \), then we obtain \( I' \models q[\bar{a}] \). It remains to show \( I \models q[\bar{a}] \); for that assume \( \bar{x} = (x_1, \ldots, x_n) \), \( \bar{a} = (a_1, \ldots, a_n) \), and

\[
q(\bar{x}) = \exists y_1, \ldots, y_m \varphi(\bar{x}, y_1, \ldots, y_m, b_1, \ldots, b_k),
\]

where \( b_i \) are constants and \( \varphi \) a quantifier-free formula. It follows, there exist \( d_1, \ldots, d_n, e_1, \ldots, e_m, f_1, \ldots, f_k \in \Delta^{T'} \), such that \( d_i = a_i^{T'} \), \( f_i = b_i^{T'} \), and

\[
I' \models \varphi(d_1, \ldots, d_n, e_1, \ldots, e_m, f_1, \ldots, f_k).
\]

It remains to observe that all of \( d_i, e_i, f_i \) belong to the interpretation of the same concepts/roles in \( I \) as in \( I' \), and \( a_i^{T'} = d_i, b_i^{T'} = f_i \). Therefore, \( I \models \varphi(d_1, \ldots, d_n, e_1, \ldots, e_m, f_1, \ldots, f_k) \), and, finally, \( I \models q[\bar{a}] \). □

C.4 Proof of Proposition 6.2

The result is proved in Theorem C.16 which is based on a series of lemmas.

Lemma C.12 Let \( M = (\Sigma, \Xi, T_2) \) be a mapping, and \( T_1 \) and \( T_2 \), respectively, \( \Sigma \)- and \( \Xi \)-TBoxes. Then \( T_2 \) is a UCQ-representation of \( T_1 \) under \( \Sigma \) if and only if \( \langle T_1 \cup T_2, A \rangle \) is \( \Xi \)-query equivalent to \( \langle T_2 \cup T_2, A \rangle \) for every ABox \( A \) over \( \Sigma \) such that \( \langle T_1, A \rangle \) is consistent.

Proof. We first prove the following:

Proposition C.13 Let \( M = (\Sigma, \Xi, T_2) \) be a mapping, and \( T_1 \) and \( T_2 \), respectively, \( \Sigma \)- and \( \Xi \)-TBoxes, \( A \) a \( \Sigma \)-ABox, such that \( \langle T_1, A \rangle \) is consistent, \( q(\bar{x}) \) a \( \Xi \) query and \( \bar{a} \) a tuple of constants. Then \( \langle T_2 \cup T_2, A \rangle \models q[\bar{a}] \) iff \( \langle T_2, A' \rangle \models q[\bar{a}] \) for all \( \Xi \)-ABoxes \( A' \) such that \( A' \) is a UCQ-solution for \( A \) under \( M \).

Proof. \((\Rightarrow)\) Let \( A \) and \( A' \) as above; we show \( \langle T_2, A' \rangle \) \( \Xi \) query entails \( \langle T_2 \cup T_2, A \rangle \). Notice that since \( A' \) is a UCQ-solution, it follows \( \langle T_2, A \rangle \) \( \Xi \) query entails \( A' \); and since \( A \) is consistent, \( \langle T_2, A \rangle \) is consistent as well. Using Claim A.4, we obtain that \( U_{\langle T_2, A \rangle} \) is \( \Xi \) homomorphically embeddable into \( U_{A'} \). By Lemma C.3 it follows \( U_{\langle T_2 \cup T_2, A \rangle} \) is \( \Xi \) homomorphically embeddable into \( U_{\langle T_2, A' \rangle} \). Now, if \( \langle T_2 \cup T_2, A \rangle \) is inconsistent, it can be shown in the way similar to the proof of Lemma C.4 that \( \langle T_2, A' \rangle \) is inconsistent, then the proof is done. Otherwise, we use Claim A.4 to conclude \( \langle T_2, A' \rangle \) \( \Xi \) query entails \( \langle T_2 \cup T_2, A \rangle \).

\((\Leftarrow)\) Let \( A \), \( q(\bar{x}) \), and \( \bar{a} \) as above; assume \( \langle T_2, A' \rangle \models q[\bar{a}] \) for all solutions \( A' \) for \( A \) under \( T_2 \). We are going to show \( \langle T_2 \cup T_2, A \rangle \models q[\bar{a}] \). If \( \langle T_2 \cup T_2, A \rangle \) is inconsistent, the proof is done; assume the opposite, then we will show \( U_{\langle T_2 \cup T_2, A \rangle} \models q[\bar{a}] \), using Theorem A.3 the proof will be done. Consider \( U_{\langle T_2, A \rangle} \) and define the set

\[
\Theta_{T_2, A} = \text{Ind}(A) \cup \{au|I| \in \Delta^{U_{\langle T_2, A \rangle}} | a \in \text{Ind}(A) \text{ and } R \text{ over } \Sigma\}.
\]

For each \( \sigma \in \Theta_{T_2, A} \) define \( t_\sigma = a \) if \( \sigma = a \in \text{Ind}(A) \), and \( t_\sigma = a_\sigma \) for a fresh w.r.t. \( A \), \( \bar{a} \), and \( q(\bar{x}) \) constant \( a_\sigma \), otherwise. Now, define

\[
A' = \{B(t_\sigma) | B \text{ basic conc. over } \Xi, \sigma \in B^{U_{\langle T_2, A \rangle}} \cap \Theta_{T_2, A}\} \cup
\]

\[
\{P(t_\sigma, t_\sigma) | P \text{ role name over } \Xi, (\sigma, \sigma') \in P^{U_{\langle T_2, A \rangle}} \cap (\Theta_{T_2, A} \times \Theta_{T_2, A})\}.
\]

It is straightforward to build a \( \Xi \)-homomorphism from \( \langle T_2, A \rangle \) to \( A' \) and use Claim A.4 to show \( A' \) is a UCQ-solution for \( A \) under \( T_2 \). Consider now \( U_{\langle T_2, A' \rangle} \) and a mapping \( g: \Delta^{U_{\langle T_2, A' \rangle}} \rightarrow \Delta^{U_{\langle T_2 \cup T_2, A \rangle}} \) defined in the following way:

\[
g(au|I_1| \ldots w|I_n|) = \begin{cases} \sigma u|I_1| \ldots w|I_n|, & \text{if } a = t_\sigma \text{ and } \sigma \in \Theta_{T_2, A}, \\ a, & \text{otherwise,} \end{cases}
\]

where \( n \geq 0 \). Notice that \( g \) is not a homomorphism, however, using the definitions of \( U_{\langle T_2, A' \rangle} \) and \( U_{\langle T_2 \cup T_2, A \rangle} \) one can straightforwardly verify

\[
\begin{align*}
\textbf{t}(T_2, A') \subseteq & \textbf{t}(T_2 \cup T_2, A) (g(\delta)), \\
\textbf{r}(T_2, A') \subseteq & \textbf{r}(T_2 \cup T_2, A) (g(\delta), g(\delta')),
\end{align*}
\]

(34)
for all $\delta, \delta' \in \Delta^U(T_2, A')$. This is sufficient to prove in the way analogous to the proof of LemmaC.6 that $U_{(T_2, A')}$ is consistent. Using ClaimA.3 one can obtain $U_{(T_2, A') \models \varphi[i]}$. Finally, observe
\[ g(a) = a, \]
for all $a$ in $\text{Ind}(A), \bar{a},$ or $q(\bar{x})$; then using (34), (35), (36) in the same way as the proof of ClaimA.4 one can show $U_{(T_2 \cup T_{12}, A) \models q[\bar{d}]}$, which concludes the proof.

Now, given a $\Sigma$ ABox $A$ such that $\langle T_1, A \rangle$ is consistent, we show that $\langle T_1 \cup T_{12}, A \rangle$ is $\Xi$-query equivalent to $\langle T_2 \cup T_{12}, A \rangle$ if and only if for every $\Xi$ query $q(\bar{x})$ it holds
\[ \text{cert}(q, \langle T_1 \cup T_{12}, A \rangle) = \bigcap_{A' \text{ consistent for } A \text{ under } T_{12}} \text{cert}(q, \langle T_2, A' \rangle). \]
(\Rightarrow) Let $q(\bar{x})$ be a $\Xi$ query, it follows $\text{cert}(q, \langle T_1 \cup T_{12}, A \rangle) = \text{cert}(q, \langle T_2 \cup T_{12}, A \rangle)$, and we easily obtain (37) using PropositionC.13 (\Leftarrow) Let $q(\bar{x})$ be a $\Xi$ query, we need to show $\text{cert}(q, \langle T_1 \cup T_{12}, A \rangle) = \text{cert}(q, \langle T_2 \cup T_{12}, A \rangle)$, which is easily concluded using PropositionC.13 and (37).

Lemma C.14 The $\Xi$-TBox $T_2$ is a UCQ-representation of $\Sigma$-TBox $T_1$ under the mapping $M = \langle \Sigma, \Xi, T_{12} \rangle$ if and only if the following conditions hold:

(i) for each pair of $T_1$-consistent concepts $A, B'$ over $\Sigma, B, B'$ is $T_1 \cup T_{12}$-consistent iff $B, B'$ is $T_2 \cup T_{12}$-consistent;

(ii) for each $T_1$-consistent role $R, R'$ over $\Sigma$, $R, R'$ is $T_1 \cup T_{12}$-consistent iff $R, R'$ is $T_2 \cup T_{12}$-consistent;

(iii) for each $T_1 \cup T_{12}$-consistent concept $B$ over $\Sigma$ and each $B'$ over $\Sigma_2$, $T_1 \cup T_{12} \models B \iff B' \models T_2 \cup T_{12} \models B$.

(iv) for each $T_1 \cup T_{12}$-consistent role $R$ over $\Sigma$ and each $R'$ over $\Sigma_2$, $T_1 \cup T_{12} \models R \iff R' \models T_2 \cup T_{12} \models R'$.

(v) for each $B \in \text{cons}_{\Sigma}(T_1 \cup T_{12})$ over $\Sigma$ and each role $R$ such that $\varphi \models_{\Sigma, B} w[R]$, there exists $y \in \Delta^\Sigma_{B}$ such that
\[ \begin{align*}
(a) & \quad t^\Sigma_{B} (\text{owl}[R]) \subseteq t^\Sigma_{B} (y), \\
(b) & \quad r^\Sigma_{B} (\text{owl}[R]) \subseteq r^\Sigma_{B} (y).
\end{align*} \]

(vi) for each $B \in \text{cons}_{\Sigma}(T_1 \cup T_{12})$ over $\Sigma$ and each role $R$ such that $\varphi \models_{\Sigma, B} w[R]$, there exists $y \in \Delta^\Sigma_{B}$ such that
\[ \begin{align*}
(a) & \quad t^\Sigma_{B} (\text{owl}[R]) \subseteq t^\Sigma_{B} (y), \\
(b) & \quad r^\Sigma_{B} (\text{owl}[R]) \subseteq r^\Sigma_{B} (y).
\end{align*} \]

Proof. (\Leftarrow) Let the conditions above hold for $T_1, T_2$ and $T_{12}$. Let $A$ be an ABox over $\Sigma$ such that $\langle T_1, A \rangle$ is consistent, we show $\mathcal{S}_{\bar{A}}$ is $\Xi$-query equivalent to $\mathcal{X}_{\bar{A}}$. Observe that $\mathcal{S}_{\bar{A}}$ is consistent iff $\mathcal{X}_{\bar{A}}$ is consistent. Indeed, if $\mathcal{S}_{\bar{A}}$ is inconsistent then by LemmaC.6 one of the following holds:

(VII) $B_1, B_2$ is $T_1 \cup T_{12}$-inconsistent for some basic concepts $B_1, B_2$ and $a \in \text{Ind}(A)$ such that $\vdash A \models B_1(a), A \models B_2(a)$;

(VIII) $R_1, R_2$ is $T_1 \cup T_{12}$-inconsistent for some roles $R_1, R_2$ and $a, b \in \text{Ind}(A)$ such that $\vdash A \models R_1(a, b), A \models R_2(a, b)$.

Consider (VII) and observe that by LemmaC.6, $B_1, B_2$ are $T_1$ consistent. Then by (i) $B_1, B_2$ are $T_2 \cup T_{12}$-inconsistent and again by LemmaC.6 $\mathcal{X}_{\bar{A}}$ is inconsistent. The proof for the case of (VIII) is similar using (ii).

First, assume $\mathcal{S}_{\bar{A}}$ is inconsistent, it follows $\mathcal{S}_{\bar{A}} \models q[\bar{a}]$ for all $\bar{a} \subseteq \text{Ind}(A)$ and $\Xi$-queries $q$. By the paragraph above, $\mathcal{X}_{\bar{A}}$ is inconsistent, so $\mathcal{X}_{\bar{A}} \models q[\bar{a}]$ for all $\bar{a} \subseteq \text{Ind}(A)$ and $\Xi$-queries $q$, and so $\mathcal{S}_{\bar{A}}$ is $\Xi$-query equivalent to $\mathcal{X}_{\bar{A}}$.

Now assume $\mathcal{S}_{\bar{A}}$ is consistent, by LemmaC.7 each $B$ is $T_1 \cup T_{12}$-consistent for all $\delta, \sigma \in \Delta^\Sigma_{\bar{A}}$, each $B$ such that $\in \Delta^\Sigma_{\bar{A}}(\delta)$, and each $R$ such that $R \in \Delta^\Sigma_{\bar{A}}(\delta, \sigma)$. It follows from (iii) and (vi) that all the conditions of LemmaC.10 are satisfied, therefore we conclude $\mathcal{U}_{\bar{A}}$ is finitely $\Xi$-homomorphically embeddable into $\mathcal{U}_{\mathcal{X}_{\bar{A}}}$. Since $\mathcal{X}_{\bar{A}}$ is consistent, then we can apply TheoremA.4 to obtain $\mathcal{X}_{\bar{A}}$ $\Xi$-query
entails $S_A$. On the other hand, conditions (iii) and (iv) imply that all the conditions of Lemma C.9 are satisfied, therefore we conclude $U_{X_A}$ is finitely $\Xi$-homomorphically embeddable into $U_{S_A}$ and $S_A \subseteq \Xi$-query entails $X_A$ by Theorem A.3. We again obtain $S_A \subseteq \Xi$-query equivalent to $X_A$.

$\Rightarrow$ Assume, by contraction, one of the conditions (i) or (vi) is not satisfied. We produce a $T_1$-consistent ABox $A$ over $\Sigma$ and an instance $\Xi$-query $q[s]$ such that it is not the case that $S_A \models q \text{ if } X_A \models q$.

Assume, first, the condition (i) is violated. Then we take $A = \{B_1(o), B_2(o)\}$ violating it and $q = B_1(o)$ for some constant $a \neq o$. If $B_1, B_2$ are $T_1 \cup T_1$-consistent, but $T_2 \cup T_1$ inconsistent, it follows $S_A \models q$ and $X_A \models q$, and the opposite holds if $B_1, B_2$ are $T_2 \cup T_1$-consistent, but $T_1 \cup T_1$-inconsistent. If (iii) is violated, the proof is analogous.

Let now the condition (iii) be violated for $B \in \text{cons}_C(T_1 \cup T_1)$ over $\Sigma$. Assume, first, there is $B' \in U_{X_B} \setminus U_{S_B}$, then we take $q = B'(o)$. By definition of $U_{S_B}, U_{X_B}$ and Lemma C.11 it follows $U_{S_B} \not\models q$ and $U_{X_B} \models q$. Then by Claim A.3 it follows $S_B \models q$ and $X_B \models q$. The opposite follows if there exists $B' \in t_{U_{X_B}}(o) \setminus t_{U_{S_B}}(o)$, which completes the proof for this case. If (iv) is violated, the proof is analogous.

To prove the case when (vi) is violated, we need an additional lemma below. Before we present it, notice that, w.l.o.g., one can consider UCQ’s with atoms over basic concepts $B(t)$; one can convert such a UCQ into the one over the standard syntax by using fresh existentially quantified variables.

**Lemma C.15** Let $T$ TBox, $B$ a concept, $\bar B$ and $\bar R$ the sets of concepts and roles, respectively, and the instance query

$$q_{\bar B, \bar R} = \exists x ( \bigwedge_{B' \in \bar B} B'(x) \land \bigwedge_{R' \in \bar R} R'(o, x)).$$

Then $U_{(\cdot \cdot \cdot)} \models q_{\bar B, \bar R}$ iff there exists $y \in \Delta U_{(\cdot \cdot \cdot)}$, such that

(i) $\bar B \subseteq t_{U_{(\cdot \cdot \cdot)}}(y)$,
(ii) $\bar R \subseteq r_{U_{(\cdot \cdot \cdot)}}(o, y)$.

**Proof.** Straightforward using Lemma C.4 and the definition of $U_{(\cdot \cdot \cdot)}$. \hfill $\square$

Now, assume (vi) is violated, so there exists $B \in \text{cons}_C(T_1 \cup T_1)$ over $\Sigma$ and a role $R$ such that $o \not\vdash_{S_B} w_{[R]}$ and for all $y \in \Delta U_{S_B}$ either $t_{U_{S_B}}(o_{[R]}) \not\subseteq t_{U_{S_B}}(y)$ or $t_{U_{S_B}}(o, w_{[R]}) \not\subseteq r_{U_{S_B}}(o, y)$. Then, by Lemma C.15 with $\bar B = t_{U_{S_B}}(o_{[R]})$, $\bar R = r_{U_{S_B}}(o, w_{[R]})$ and $T = T_1 \cup T_1$ it follows $U_{S_B} \not\models q_{\bar B, \bar R}$. On the other hand, by Lemma C.15 with $T = T_2 \cup T_1$ it follows $U_{X_B} \not\models q_{\bar B, \bar R}$. Using Claim A.3 we then obtain $S_B \models q_{\bar B, \bar R}$ and $X_B \not\models q_{\bar B, \bar R}$.

The case when (vi) is violated is analogous to the case above. The proof is complete. \hfill $\square$

**Theorem C.16** The membership problem for UCQ-representability is NLOGSPACE-complete.

**Proof.** The lower bound can be obtained by the reduction from the directed graph reachability problem, which is known to be NLogSpace-hard: given a graph $G = (V, E)$ and a pair of vertices $v_k, v_m \in V$, decide if there is a directed path from $v_k$ to $v_m$. To encode the problem, we need a set of $\Sigma$ concept names \{V_i \mid v_i \in V\} and a set of $\Xi$ concept names \{V_i' \mid v_i \in V\}. Consider $T_1 = \{V_k \subseteq V_m\} \cup \{V_i \subseteq V_j \mid (v_i, v_j) \in E\}$, $T_2 = \{V_i \subseteq V_j' \mid (v_i, v_j) \in E\}$ and $T_3 = \{V_i' \subseteq V_j' \mid (v_i, v_j) \in E\}$. One can easily verify that the condition (iii) of Lemma C.14 is satisfied iff there is a directed path from $v_k$ to $v_m$ in $G$, whereas the other conditions of Lemma C.14 are satisfied trivially. Therefore,

**Proposition C.17** There is a directed path from $v_k$ to $v_m$ in $G$ iff $T_2$ is a representation for $T_1$ under $\mathcal{M} = \{\Sigma, \Xi, T_1\}$.

This concludes the proof of the lower bound. For the upper bound, we show that the conditions (i) through (vi) of Lemma C.14 can be verified in NLOGSPACE. It is well known (see, e.g., [Artale et al., 2009]), that given a pair of DL-Lite_RR concepts $B, B'$, and a TBox $T$, it can be verified in NLOGSPACE, if $B, B'$ is $T$ consistent (using an algorithm, based on directed graph reachability solving procedure); the same holds for a pair of DL-Lite_R roles $R, R'$. The same algorithm can be straightforwardly adopted to check, if $T \vdash B \subseteq B'$ or $T \vdash R \subseteq R'$. Therefore, clearly, the conditions (i) through (vi) can be verified in NLOGSPACE.

The conditions (vi) and (vi) are slightly more involved; first of all, observe that, given a concept $B$ and a role $R$, it can be checked in NLOGSPACE, whether $o \not\vdash_{(\cdot \cdot \cdot)} w_{[R]}$, using an algorithm based on the directed graph reachability solving procedure. At the same time, given $z \in \{o\} \cup \{w_{[R]} \mid R - \text{ role}\}$,
we can verify, if there exists \( y \in \Delta^T(o) \) with \( z = \text{tail}(y) \): we “follow” the sequence of roles \( R_1, \ldots, R_n = R \) (with \( n \geq 0 \)) in the way that when we “guess” \( R_{i+1} \), we check \( w_{[R_i]} \) (by the algorithm, similar to the one for checking \( o \supseteq \{B(o)\} \) \( w_{[R]} \)), and “forget” \( R_i \).

Furthermore, in a similar way, as testing \( T \vdash B \subseteq B' \), one can, check for a concept \( B' \), if \( B' \in t_{\Xi}(o) \) in \( \text{NLOGSPACE} \); the same holds for checking if a role \( R' \in r_{\Xi}(o) \), and, then, for checking \( B' \in t_{\Xi}(y) \) for \( y \in \Xi \). By combining the algorithms outlined above, one can produce a procedure that checks the conditions \( \textbf{[v]} \) and \( \textbf{[vi]} \) in \( \text{NLOGSPACE} \).

\[ \square \]

**D Non-emptiness Problem for UCQ-representability**

The definitions that follow are needed for the non-emptiness problem of UCQ-representability. Let the mapping \( M = (\Sigma, \Xi, T_1) \), \( T_1 \) and \( T_2 \) TBoxes over, respectively, \( \Sigma \) and \( \Xi \). For a pair of concepts \( B', C' \) be over \( \Sigma \), we say that \( T_1 \cup T_12 \) is closed under inclusion between \( B' \) and \( C' \) if the following is satisfied for each \( T_1 \)-consistent concept \( B \) over \( \Sigma \):

\[ \textbf{(IX)} \quad T_1 \cup T_12 \vdash B \subseteq B' \implies T_1 \cup T_12 \vdash B \subseteq C'; \]

\[ \textbf{(X)} \quad \text{if } B' = \exists Q', \text{ then } \exists Q'^{-} \in t_{\Xi}^{T_12}(o) \implies o \supseteq \Sigma_B w_{[Q]} \text{ for some role } Q \text{ such that } Q'^{-} \in r_{\Xi}^{T_12}(o, w_{[Q]}), \]

Then, for a pair \( R', Q' \) of roles over \( \Xi \), we say \( T_1 \cup T_12 \) is closed under inclusion between \( R' \) and \( Q' \) if the following is satisfied:

\[ \textbf{(XI)} \quad T_1 \cup T_12 \vdash R \subseteq R' \implies T_1 \cup T_12 \vdash R \subseteq Q' \text{ for each } T_1 \text{-consistent role } R \text{ over } \Sigma; \]

\[ \textbf{(XII)} \quad T_1 \cup T_12 \text{ is closed under inclusion between } \exists R' \text{ and } \exists Q'; \]

\[ \textbf{(XIII)} \quad T_1 \cup T_12 \text{ is closed under inclusion between } \exists R'^{-} \text{ and } \exists Q'^{-}. \]

Next, we say \( T_1 \cup T_12 \) is closed under disjointness between \( B' \) and \( C' \) if the following is satisfied:

\[ \textbf{(XIV)} \quad \text{for each } T_1 \cup T_12 \text{-consistent pair of concepts } B, C \text{ over } \Sigma \text{ it is not the case } T_1 \cup T_12 \vdash B \subseteq B' \text{ and } T_1 \cup T_12 \vdash C \subseteq C'; \]

\[ \textbf{(XV)} \quad \text{for each } T_1 \cup T_12 \text{-consistent concept } B \text{ over } \Sigma \text{ and each role } R \text{ such that } o \supseteq \Sigma_B \text{ it is not the case } B', C' \in t_{\Xi}^{T_12}(o). \]

Then, \( T_1 \cup T_12 \) is closed under disjointness between \( R' \) and \( Q' \) if the following is satisfied:

\[ \textbf{(XVI)} \quad \text{for each } T_1 \cup T_12 \text{-consistent pair of roles } R, Q \text{ over } \Sigma \text{ it is not the case } T_1 \cup T_12 \vdash R \subseteq R' \text{ and } T_1 \cup T_12 \vdash Q \subseteq Q'; \]

\[ \textbf{(XVII)} \quad \text{for each } T_1 \cup T_12 \text{-consistent concept } B \text{ over } \Sigma \text{ and each role } R \text{ such that } o \supseteq \Sigma_B \text{ it is neither the case } R', Q' \in r_{\Xi}^{T_12}(o) \text{ nor } R'^{-}, Q'^{-} \in r_{\Xi}^{T_12}(o, w_{[Q]}). \]

Define a generating pass for a concept \( B \) over \( \Sigma \) as a pair \( \pi = ((C_0, C_1, \ldots, C_n), L) \), where \( (C_0, C_1, \ldots, C_n) \) a is tuple of concepts of the length greater or equal 1, \( C_0 = B \), and for each \( 1 \leq i \leq n \) it holds \( C_i = \exists Q_i \) for some role \( Q_i \); then \( L \) is a labeling function

\[ L : C_i \cup C_i \times C_j \mapsto 2^\Xi \text{-concepts } \cup 2^\Xi \text{-roles} \]

such that \( L(C_i, C_j) = \emptyset \) for \( j \neq i + 1 \). It is said that a generating pass \( \pi \) for \( B \) is \( \text{conform with } T_1 \cup T_12 \) if the following is satisfied:

\[ \textbf{(XVIII)} \quad \exists Q \in L(C_i) \text{ or } \exists Q = C_i \text{ for all } 0 \leq i < n \text{ and roles } Q \text{ such that } C_{i+1} = \exists Q^{-}; \]

\[ \textbf{(XIX)} \quad \text{for each } 0 \leq i \leq n \text{ and } B' \in L(C_i) \text{ there exists } C' \text{ over } \Xi \text{ such that } T_1 \cup T_12 \vdash C_i \subseteq C' \text{ and } T_1 \cup T_12 \text{ is closed under inclusion between } C' \text{ and } B'; \]

\[ \textbf{(XX)} \quad \text{for each } 0 \leq i < n \text{, role } Q \text{ such that } C_{i+1} = \exists Q^{-} \text{ and } R' \in L(C_i, C_{i+1}) \text{ there exists } Q' \text{ over } \Xi \text{ such that } T_1 \cup T_12 \vdash Q \subseteq Q' \text{ and } T_1 \cup T_12 \text{ is closed under inclusion between } Q' \text{ and } R'. \]

**D.1 Basic Preliminary Results**

**Lemma D.1** Let \( M = (\Sigma, \Xi, T_1) \) be a mapping, and a \( \Sigma \)-TBox \( T_2 \) be a representation for a \( \Xi \)-TBox \( T_1 \) under \( T_12 \). Then \( T_1 \cup T_12 \) is closed under:

\[ \text{(i) inclusion between concepts } B' \text{ and } C' \text{ (roles } R' \text{ and } Q') \text{ for all } B', C' \text{ over } \Xi \text{ (} R', Q' \text{ over } \Xi \text{) such that } T_2 \vdash B' \subseteq C' \text{ (} T_2 \vdash R' \subseteq Q'. \]
(ii) disjointness between concepts $B'$ and $C'$ (roles $R'$ and $Q'$) for all $B'$, $C'$ over $\Sigma$ (and $R'$, $Q'$ over $\Sigma$) such that $T_2 \vdash B' \subseteq D'$, $T_2 \vdash C' \subseteq E'$, and $(D' \cap E' \subseteq \bot) \in T_2$ for some concepts $D'$, $E'$ over $\Sigma$.

(iii) disjointness between $B'$ and $B'' (R'$ and $R'')$ for all $T_2$ inconsistent concepts $B'$ (roles $R'$).

**Proof.** We assume that $T_2$ is a representation, but (ii) or (iii) is violated, and derive a contradiction. Let, first, (ii) be violated for concepts, i.e., there are $B', C'$ over $\Sigma$ such that $T_2 \vdash B' \subseteq C'$ and $T_1 \cup T_12$ is not closed under inclusion between $B'$ and $C'$. Then, (iii) or (iv) must be violated for some $B \in consc(T_1 \cup T_12)$ over $\Sigma$. Assume (iii) is violated, i.e., $B' \in t_{\Xi}^{Ucb}(o)$ and $C' \notin t_{\Xi}^{Ucb}(o)$. By Lemma C.14(iii) we get the contradiction. If (iv) is violated, then $B' \in t_{\Xi}^{Ucb}(o)$ and for all roles $Q$ such that $o \rightarrow_{SB} w[q]$ and $Q' \notin r_{\Xi}^{Ucb}(o, ow[q])$ it holds $C' \notin t_{\Xi}^{Ucb}(o)$. By Lemma C.14(iii) obtain $E'$ over $\Sigma$ such that $T_2 \vdash B' \subseteq C'$ it follows there exists a role $Q$ such that $o \rightarrow_{SB} w[q]$, $Q' \notin r_{\Xi}^{Ucb}(o, ow[q])$, and $C' \in t_{\Xi}^{Ucb}(ow[q])$. Using (iv) we obtain a contradiction.

Finally, assume (iii) is violated for concepts, i.e., there are $B', C', D', E'$ over $\Sigma$ such that $T_2 \vdash B' \subseteq D'$, $T_2 \vdash C' \subseteq E'$, and $(D' \cap E' \subseteq \bot) \in T_2$ and $T_1 \cup T_12$ is not closed under disjointness between $B'$ and $C'$. Then, (iv) or (v) must be violated. If it is (iv) there is a $T_1 \cup T_12$-consistent pair of concepts $B, C$ such that $B' \in t_{\Xi}^{Ucb}(o)$ and $C' \in t_{\Xi}^{Ucb}(o)$. By Lemma C.14(iii) obtain $B' \in t_{\Xi}^{Ucb}(o)$ and $C' \in t_{\Xi}^{Ucb}(o)$, then by $T_2 \vdash B' \subseteq D'$, $T_2 \vdash C' \subseteq E'$ and $(D' \cap E' \subseteq \bot) \in T_2$ it follows the pair $B, C$ is $T_2 \cup T_12$-consistent. We obtained a contradiction. If (v) is violated there is $B \in consc(T_1 \cup T_12)$ over $\Sigma$ and a role $R$ such that $o \rightarrow_{SB} w[r]$ and $B', C' \in t_{\Xi}^{Ucb}(ow[r])$. By Lemma C.14(iv) obtain $y \in \Delta_{\Xi}^{Ucb} such that $B', C' \in t_{\Xi}^{Ucb}(y)$. Using Lemmas C.4 and C.7 we obtain the contradiction. We obtained a contradiction. If (v) is violated there is $B \in consc(T_1 \cup T_12)$ over $\Sigma$ and a role $R$ such that $o \rightarrow_{SB} w[r]$ and $B', C' \in t_{\Xi}^{Ucb}(ow[r])$. By Lemma C.14(iv) obtain $y \in \Delta_{\Xi}^{Ucb}$ such that $B' \in t_{\Xi}^{Ucb}(y)$. Using Lemmas C.4 and C.7 we obtain the contradiction. We obtained a contradiction. If (v) is violated there is $B \in consc(T_1 \cup T_12)$ over $\Sigma$ and a role $R$ such that $o \rightarrow_{SB} w[r]$ and $B', C' \in t_{\Xi}^{Ucb}(ow[r])$. By Lemma C.14(iv) obtain $y \in \Delta_{\Xi}^{Ucb}$ such that $B' \in t_{\Xi}^{Ucb}(y)$. Using Lemmas C.4 and C.7 we obtain the contradiction. We obtained a contradiction. If (v) is violated there is $B \in consc(T_1 \cup T_12)$ over $\Sigma$ and a role $R$ such that $o \rightarrow_{SB} w[r]$ and $B', C' \in t_{\Xi}^{Ucb}(ow[r])$.
(iii) For each $T_1 \cup T_{12}$-consistent concept $B$ over $\Sigma$ and each role $R$ such that $\alpha \models_{\mathcal{S}_B} w_{[R]}$ there exists a generating pass $\pi = \{(C_0, \ldots, C_n), L\}$ for $B$ conform with $T_1 \cup T_{12}$, such that:

(a) $\mathcal{E}_n^B(\alpha, \omega_{[R]} \in L(C_n))$,

(b) $\mathcal{E}_n^B(\alpha, \omega_{[R]} \in L(C_0, C_n))$.

(iv) For each $T_{12}$-consistent pair of concepts $B_1, B_2$ over $\Sigma$, such that $B_1, B_2$ is $T_1 \cup T_{12}$-inconsistent, there are concepts $B, C$ such that one of the following holds:

(a) $B, C \in \{B_1, B_2\}$ and one of the following holds:

1. $T_{12} \models B \subseteq B', T_{12} \models C \subseteq C'$, and $T_1 \cup T_{12}$ is closed under disjointness between $B'$ and $C'$.

2. $T_{12} \models B \subseteq B'$, $T_{12} \models C \subseteq \bot$, and $T_1 \cup T_{12}$ is closed under inclusion between $B'$ and $C'$.

(b) $\exists R \in \{B_1, B_2\}$ and one of the following holds:

1. $T_{12} \models \exists R^{-} \subseteq B'$, $T_{12} \models \exists R^{-} \subseteq C'$, and $T_1 \cup T_{12}$ is closed under disjointness between $B'$ and $C'$.

2. $T_{12} \models \exists R^{-} \subseteq B'$, $T_{12} \models \exists (\exists R^{-} \cap C' \subseteq \bot)$, and $T_1 \cup T_{12}$ is closed under inclusion between $B'$ and $C'$.

(v) For all $T_1$-consistent pairs of roles $R_1, R_2$, such that $R_1, R_2$ is $T_1 \cup T_{12}$-inconsistent one of the following holds:

(a) there are roles $R, Q \in \{R_1, R_2\}$ and $R', Q'$ over $\Sigma$ such that one of the following holds:

1. $T_{12} \models R \subseteq R'$, $T_{12} \models Q \subseteq Q'$, and $T_1 \cup T_{12}$ is closed under disjointness between $R'$ and $Q'$.

2. $T_{12} \models R \subseteq R'$, $T_{12} \models Q \subseteq Q'$, and $T_1 \cup T_{12}$ is closed under inclusion between $R'$ and $Q'$.

(b) there exist $B, C \in \{\exists R_1, \exists R_2\}$ or $B, C \in \{\exists R^{-}_1, \exists R^{-}_2\}$ such that one of the following holds:

1. $T_{12} \models B \subseteq B'$, $T_{12} \models C \subseteq C'$, and $T_1 \cup T_{12}$ is closed under disjointness between $B'$ and $C'$.

2. $T_{12} \models B \subseteq B'$, $T_{12} \models C \subseteq \bot$, and $T_1 \cup T_{12}$ is closed under inclusion between $B'$ and $C'$.

Proof. ($\leftrightarrow$) Assume the conditions (i)-(v) are satisfied, we construct a TBox $T_2$ and prove it is a UCQ-representation for $T_1$ under $\mathcal{M}$. The required $T_2$ will be given as the union of the five sets of axioms presented below. First, take $B \in \text{cons}_{\Sigma}(T_1 \cup T_{12})$ over $\Sigma$, $B' \in \mathcal{E}_n^B(o)$, then let $ax_1(B, B') = \{C' \subseteq B'\}$ for $C'$ given by the condition (ii). For $R \in \text{cons}_{\Sigma}(T_1 \cup T_{12})$ over $\Sigma$ and $R'$ over $\Sigma$, such that $T_1 \cup T_{12} \models R \subseteq R'$, define $ax_3(R, R') = \{Q' \subseteq R'\}$ for $Q'$ by the condition (iii). For each $B \in \text{cons}_{\Sigma}(T_1 \cup T_{12})$ over $\Sigma$ and each role $R$ such that $\alpha \models_{\mathcal{S}_B} w_{[R]}$, define the set $ax_3(B, R)$ from the generating pass $(C_0, \ldots, C_n)$ for $B$ conform with $T_1 \cup T_{12}$ that satisfies (iii). Take $ax_3(B, R)$ equal to the set of all axioms $C' \subseteq B' \models$ (XIX) and all axioms $Q' \subseteq R'$ satisfying (XX). Now let $B, B_2$ be a $T_1$-consistent and $T_1 \cup T_{12}$-inconsistent pair of $\Sigma$ concepts, then define a set $ax_4(B_1, B_2)$ to be equal to $\{B' \cap C' \subseteq \bot\}$ for the corresponding $B'$ and $C'$, if (iv)(a) or (iv)(b) is satisfied; and $\{R' \cap Q' \subseteq \bot\}$ for the corresponding $R'$ and $Q'$, if (iv)(b) is satisfied. On the other hand, define $ax_4(B_1, B_2)$ to be equal to $\{B' \cap C' \subseteq \bot\}$ for the corresponding $B'$ and $C'$, if (iv)(a) or (iv)(b) is satisfied; and $\{R' \cap Q' \subseteq \bot\}$ for the corresponding $R'$ and $Q'$, if (iv)(b) is satisfied. Finally, we define $ax_5(R_1, R_2)$ for $T_1$-consistent and $T_1 \cup T_{12}$-inconsistent pair of $\Sigma$ roles $R_1, R_2$ analogously to $ax_4(B_1, B_2)$ using the conditions (v)(b) and (v)(b). Finally we have:

$$
T_2 = \bigcup_{B \in \text{cons}_{\Sigma}(T_1 \cup T_{12}) \text{ over } \Sigma, B' \in \mathcal{E}_n^B(o)} ax_3(B, B') \cup \bigcup_{R \in \text{cons}_{\Sigma}(T_1 \cup T_{12}) \text{ over } \Sigma, R' \text{ over } \Sigma, T_1 \cup T_{12} \models R \subseteq R'} ax_4(R, R') \cup \bigcup_{B \in \text{cons}_{\Sigma}(T_1 \cup T_{12}) \text{ over } \Sigma, \alpha \models_{\mathcal{S}_B} w_{[R]} \text{ over } \Sigma, B, B_1 \text{ conc. over } \Sigma, T_1 \not\models \text{consist. and } T_1 \cup T_{12} \not\models \text{consist.}} ax_5(B, R) \cup \bigcup_{R_0, R_1 \text{ roles over } \Sigma, R_0 \models \text{consist. and } T_1 \cup T_{12} \not\models \text{consist.}} ax_2(R_0, R_1)
$$

We need the following intermediate result:
Lemma D.3 For all concepts $B', C' \in \Xi$ (roles $R', Q'$ over $\Xi$), if $T_2 \vdash B' \subseteq C'$ ($T_2 \vdash R' \subseteq Q'$) then $T_1 \cup T_2$ is closed under inclusion between $B'$ and $C'$ ($R'$ and $Q'$).

Proof. Notice that for all concepts $B'$ and $C'$ (roles $R'$ and $Q'$) such that $(B' \subseteq C') \in T_2$ (($R' \subseteq Q'$) $\in T_2$) it holds $T_1 \cup T_2$ is closed under inclusion between $B'$ and $C'$ ($R'$ and $Q'$). First we prove the statement of the lemma for roles, if $T_2 \vdash R' \subseteq Q'$ there is a sequence of roles $Q_1, \ldots, Q_n$ such that $Q_1 = R'$, $Q_n = Q'$, and for each $1 \leq i < n$ one of the following holds:

$$(\text{XXII}) \quad (Q_i \subseteq Q_{i+1}) \in T_2$$

$$(\text{XXII}) \quad (Q_i^\neg \subseteq Q_{i+1}^\neg) \in T_2$$

We show $T_1 \cup T_2$ is closed under inclusion between $R'$ and $Q_i$ by induction on $i$. For $i = 1$ the proof is trivial, assume $T_1 \cup T_2$ is closed under inclusion between $R'$ and $Q_i$, we show now its closure under inclusion between $R'$ and $Q_{i+1}$. Let, first, (\text{XXII}) we show (\text{XI}) assume $T_1 \cup T_2 \vdash R \subseteq R'$ for some $R \in \mathbb{cons}_\Xi(T_1 \cup T_2)$ over $\Sigma$, since $T_1 \cup T_2$ is closed under inclusion between $R'$ and $Q_i$, it follows by (\text{XI}) $T_1 \cup T_2 \vdash R \subseteq Q_i$, then, again by closure under inclusion between $Q_i$ and $Q_{i+1}$ obtain $T_1 \cup T_2 \vdash R \subseteq Q_{i+1}$.

To show (\text{XI}) we need to prove (\text{IX}) and (\text{X}) for $B' = \exists R'$ and $C' = \exists Q_{i+1}$. For (\text{IX}) assume $\exists R' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ for some $B \in \mathbb{cons}_\Xi(T_1 \cup T_2)$ over $\Sigma$; since $T_1 \cup T_2$ is closed under inclusion between $R'$ and $Q_i$, it follows by (\text{XI}) that $\exists Q_i \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$, and again by closure under inclusion between $Q_i$ and $Q_{i+1}$ obtain $\exists Q_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$. For (\text{X}) assume $\exists R' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ for some $B \in \mathbb{cons}_\Xi(T_1 \cup T_2)$ over $\Sigma$ and consider two cases: $R' = Q_i$ and $R' \neq Q_i$. In the first case $\alpha \rightarrow_{\mathbb{S}_B} w_{[Q]}$ for some role $R'$ such that $R' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ and $\exists Q_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ immediately follows, since $T_1 \cup T_2$ is closed under inclusion between $Q_i$ and $Q_{i+1}$.

Assume $R' \neq Q_i$, since $T_1 \cup T_2$ is closed under inclusion between $R'$ and $Q_i$, it follows by (\text{XII}) and the structure of $S_B$ that $\alpha \rightarrow_{\mathbb{S}_B} w_{[Q]}$ for some role $R'$ over $\Sigma$ such that $R' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ and $\exists Q_i \in t_{\Xi}^{U_{\mathbb{S}_B}}(ow_{[Q]})$. Since $S_B$ is consistent, it can be easily shown that $\exists Q_i \in \mathbb{cons}_\Xi(T_1 \cup T_2)$. Observe now that $\exists Q_i \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$; then since $T_1 \cup T_2$ is closed under inclusion between $Q_i$ and $Q_{i+1}$ and obtain $\exists Q_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(ow_{[Q]})$. Finally, it follows $\exists Q_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(ow_{[Q]})$, which completes the proof for the case (\text{XI}). The proof for the case (\text{XII}) is analogous.

To prove the lemma for concepts we exploit that $T_2 \vdash B' \subseteq C'$ implies there exists a sequence of $\Xi$ concepts $B_1, \ldots, B_n$ such that $B_1 = B'$, $B_n = C'$, and for each $1 \leq i < n$ one of the following holds:

$$(\text{XXIII}) \quad (B_i \subseteq B_{i+1}) \in T_2$$

$$(\text{XXIV}) \quad B_i = \exists R', B_{i+1} = \exists Q', (R' \subseteq Q') \in T_2$$

$$(\text{XXV}) \quad B_i = \exists R', B_{i+1} = \exists Q', (R' \subseteq Q') \in T_2$$

We show $T_1 \cup T_2$ is closed under inclusion between $B'$ and $B_i$ by induction on $i$. For $i = 1$ the proof is trivial, assume $T_1 \cup T_2$ is closed under inclusion between $B'$ and $B_i$, we show now its closure under inclusion between $B'$ and $B_{i+1}$. First we consider the case of $B_i$ and $B_{i+1}$ are as in (\text{XXIII}). To show (\text{IX}) assume $B \in \mathbb{cons}_\Xi(T_1 \cup T_2)$ over $\Sigma$ and $B' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$; by closure under inclusion between $B'$ and $B_i$ and (\text{IX}) it follows $B_i \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$, then by closure under inclusion between $B_i$ and $B_{i+1}$ and (\text{IX}) obtain $B_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$.

To show (\text{X}) assume $B' = \exists Q'$ and $\exists Q' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$. If $B' = B_i$, then $\alpha \rightarrow_{\mathbb{S}_B} w_{[Q]}$ for some role $Q$ such that $Q' \in r_{\Xi}^{U_{\mathbb{S}_B}}(o, ow_{[Q]})$ and $B_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(ow_{[Q]})$ by closure under inclusion between $B_i$ and $B_{i+1}$, and (\text{IX}) $\alpha \rightarrow_{\mathbb{S}_B} w_{[Q]}$ for a role $Q$ over $\Sigma$ such that $Q' \in r_{\Xi}^{U_{\mathbb{S}_B}}(o, ow_{[Q]})$ and $B_i \in t_{\Xi}^{S_B}(ow_{[Q]})$. Since $S_B$ is consistent, it follows $\exists R' \in \mathbb{cons}_\Xi(T_1 \cup T_2)$, then by closure under inclusion between $B_i$ and $B_{i+1}$ and (\text{IX}) we conclude $B_{i+1} \in t_{\Xi}^{U_{\mathbb{S}_B}}(ow_{[H]})$, which concludes the proof. The proof for the cases (\text{XXIII}) and (\text{XXIV}) is analogous.

We return to the proof of $(\Leftarrow)$ of Lemma D.2 we prove $T_2$ above is a representation of $T_1$ under $T_2$ by showing the conditions (\text{I}) - (\text{VI}) of Lemma D.14 are satisfied. We start from (\text{III}) consistency conditions will be shown in the end.) Let $B \in \mathbb{cons}_\Xi(T_1 \cup T_2)$ over $\Sigma$, $B' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$, then $B' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$ follows straightforwardly from current (\text{I}). Assume now some $B' \in t_{\Xi}^{U_{\mathbb{S}_B}}(o)$, it follows $T_1 \cup T_2 \vdash B \subseteq C'$.
and $T_2 \vdash C' \subseteq B'$ for some concept $C'$ over $\Sigma$; by Lemma $D.3$, if $C' \in t_{\Sigma}^{U_{S_{B}}}(o)$ implies $B' \in t_{\Sigma}^{U_{S_{B}}}(o)$, and since $T_2 \vdash B \subseteq C'$, conclude $B' \in t_{\Sigma}^{U_{S_{B}}}(o)$. The proof that $\text{(iv)}$ of Lemma $D.2$ is satisfied is analogous to the proof that $\text{(iii)}$ is satisfied above, using current $\text{(iii)}$ and Lemma $D.3$.

The $\text{(v)}$ of Lemma $D.2$ follows straightforwardly from current $\text{(iii)}$ the definition of a $\Xi$ pass conform with $T_1 \cup T_2$, and the structure of $T_2$. To show $\text{(vi)}$ of Lemma $D.2$, assume $B \in cons_{C}(T_1 \cup T_2)$ over $\Sigma$ and a role $Q$ such that $o \leadsto_{X_{B}} w_{[Q]}$. We first consider the case $Q$ over $\Sigma$, by the structure of $T_2$ (see the proof that $\text{(iii)}$ of Lemma $C.14$ is satisfied), then it follows $\exists Q \in t_{\Sigma}^{U_{S_{B}}}(o)$. If $t_{\Sigma}^{U_{S_{B}}}(w_{[Q]}) = \{ \Xi Q \}$, the proof is done; otherwise, $T_2 \vdash \exists Q' \subseteq C'$ for some $C' \neq \Xi Q'$, then by Lemma $D.3$ and $\text{(X)}$ it follows there exists $R$ such that $o \leadsto_{S_{B}} w_{[R]}, Q \in t_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$ and $C' \in t_{\Sigma}^{U_{S_{B}}}(o)$; also by $C' \neq \Xi Q'$ and the structure of $S_{B}$ it follows $R$ is over $\Sigma$. Notice that $\exists R' \subseteq cons_{C}(T_1 \cup T_{12})$ since $S_{B}$ is consistent. For each (other) $C' \in t_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]})$ we show $C' \in t_{\Sigma}^{U_{S_{B}}}(w_{[Q]})$. Indeed, it follows $T_2 \vdash \exists Q' \subseteq C'$, then using Lemma $D.3$, $\exists Q' \in t_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$, we can conclude $C' \in t_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$, and so $C' \in t_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]})$. To show $r_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]}) \subseteq r_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$, consider that $R \in cons_{C}(T_1 \cup T_{12})$ and assume $Q' \in r_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]}); it follows $T_2 \vdash Q \subseteq Q'$, then by Lemma $D.3$ and $\text{(XI)}$ and $T_1 \cup T_{12} \vdash R \subseteq Q$ if $Q' \in r_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]})$, which concludes the proof.

Consider now the case $Q$ over $\Sigma$, then, clearly, $o \leadsto_{S_{B}} w_{[R]}$ and $Q \in r_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$ for some role $R$ over $\Sigma$. We show now $t_{\Sigma}^{U_{S_{B}}}(ow_{[R]}) \subseteq t_{\Sigma}^{U_{S_{B}}}(ow_{[Q]}), \text{let} C' \in t_{\Sigma}^{U_{S_{B}}}(ow_{[Q]}), \text{then} T_2 \vdash \exists Q' \subseteq B'$ and $T_2 \vdash B' \subseteq C'$ for some $B'$ over $\Sigma$. It follows $B' \in t_{\Sigma}^{U_{S_{B}}}(ow_{[R]})$, then by Lemma $D.3$ and $\text{(IX)}$ obtain $C' \subseteq t_{\Sigma}^{U_{S_{B}}}(ow_{[R]})$. The proof of $r_{\Sigma}^{U_{S_{B}}}(o, ow_{[Q]}) \subseteq r_{\Sigma}^{U_{S_{B}}}(o, ow_{[R]})$ is analogous.

Now we show that the consistency conditions of Lemma $C.14$ are satisfied. For $\text{(ii)}$ assume a pair $B_1, B_2$ of $T_1$ consistent and $T_1 \cup T_{12}$-inconsistent concepts; then $B_1, B_2$ is $T_2 \cup T_{12}$-inconsistent follows easily from current $\text{(iv)}$ and definition of $T_2$. Assume $B_1, B_2$ are $T_1$ consistent and $T_2 \cup T_{12}$-inconsistent; it follows there exists $\delta, \sigma \in \Delta_{X_{B_{1}(o), B_{2}(o)}}$ such that one of the following holds:

\[ \text{(XXVI)} \] There are concepts $C, C' \in t_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ such that $(C \cap C' \subseteq \bot) \subseteq T_2 \cup T_{12};$

\[ \text{(XXVII)} \] There are roles $Q, Q' \in r_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ such that $(Q \cap Q' \subseteq \bot) \subseteq T_2 \cup T_{12};$

Assume the sake of contradiction that $B_1, B_2$ is $T_1 \cup T_{12}$ is consistent. By Lemma $C.7$, it follows for each $\delta, \sigma \in \Delta_{U_{S_{B}}(o, B_{1}(o), B_{2}(o))}$, every $B \in t_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ and $R \in r_{\Sigma}^{U_{S_{B}}(o, B_{1}(o), B_{2}(o))}(\delta, \sigma)$ are $T_1 \cup T_{12}$-consistent. By the structure of $T_2$ for all such $B$ and $R$ the conditions $\text{(iii)}$ and $\text{(v)}$ of Lemma $C.10$ are satisfied (see the proof that $\text{(iii)}$ and $\text{(iv)}$ of Lemma $C.14$ are satisfied above). Then, by Lemma $C.10$ we have that there exist $\delta, \sigma \in \Delta_{X_{B_{1}(o), B_{2}(o)}}$ such that one of the following holds:

\[ \text{(XXVIII)} \] There are concepts $C, C' \in t_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ such that $(C \cap C' \subseteq \bot) \subseteq T_2 \cup T_{12};$

\[ \text{(XXIX)} \] There are roles $Q, Q' \in r_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ such that $(Q \cap Q' \subseteq \bot) \subseteq T_2 \cup T_{12};$

Assume $\text{(XXVIII)}$ and observe that w.l.o.g. $C'$ is over $\Sigma$, whereas $C \in \Sigma \subseteq \Xi \Sigma$. If $C \in \Sigma$ it follows $(C \cap C') \in T_{12}$ and we immediately have the contradiction to the fact that $B_1, B_2$ is $T_1 \cup T_{12}$ consistent. So let $C$ be over $\Sigma$, it follows $(C \cap C') \in T_2$, and $T_1 \cup T_{12}$ is closed under disjointness between $C$ and $C'$ by the definition of $T_2$. Consider, first, the case $\delta \neq \alpha$: by Lemma $C.1, C, C' \in S_{B} \subseteq U_{S_{B}}(o)$ for the role $Q$ such that $\text{tail}(\delta) = w_{[Q]}$. If $Q$ is over $\Sigma$ we derive the contradiction because $\exists Q' \subseteq T_1 \cup T_{12}$-consistent and $\text{[XIV]}$. On the other hand, if $Q$ is over $\Xi$, it can be seen by the structure of $U_{S_{B}}(o, B_{1}(o), B_{2}(o))$ that $o \leadsto_{S_{B}} w_{[Q]}$ for some $B \in cons_{C}(T_1 \cup T_{12})$ over $\Sigma$; then $C, C' \subseteq t_{\Sigma}^{U_{S_{B}}}(ow_{[Q]})$ and we derive the contradiction because of $\text{[XV]}$. Finally, consider the case $\delta = \alpha$, then by the structure of $U_{S_{B}}(o, B_{1}(o), B_{2}(o))$ there are concepts $B, D \subseteq \{ B_1, B_2 \}$ such that $C \subseteq t_{\Sigma}^{U_{S_{B}}}(o)$ and $C' \subseteq t_{\Sigma}^{U_{S_{B}}}(o)$. By $\text{[XIV]}$ we again have a contradiction.

Assume $\text{(XXIX)}$, then again, assuming $Q$ is over $\Sigma$ produces an immediate contradiction; if, however, $Q$ is over $\Sigma$, we obtain by the definition of $T_2$, that $T_1 \cup T_{12}$ is closed under disjointness between $Q$ and $Q'$. By the structure of $U_{S_{B}}(o, B_{1}(o), B_{2}(o))$ we need to consider two cases: $\sigma = w_{[R]}, Q, Q' \in r_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\delta, \sigma)$ and $\delta = \sigma w_{[R]}, Q, Q' \in r_{\Sigma}^{U_{X_{B_{1}(o), B_{2}(o)}}}(\sigma, \delta)$. In the first case, $o \leadsto_{S_{B}} w_{[R]}$ for some $B \in cons_{C}(T_1 \cup T_{12})$ over $\Sigma$ and $Q, Q' \in r_{\Sigma}^{U_{S_{B}}}(o)$; using $\text{[XXVII]}$ we derive the contradiction. The second case is proved analogously using $\text{[XVII]}$. 

Thus, assuming the pair $B_1, B_2$ is $T_1 \cup T_{12}$-consistent produces a contradiction, therefore $B_1, B_2$ is $T_1 \cup T_{12}$ inconsistent. This concludes the proof that (ii) of Lemma C.2 is satisfied. Analogously, using (v) Lemma C.10 (XIV) [XV] [XVI] [XVII] it can be shown that (ii) of Lemma D.2 is satisfied, which concludes the proof (\(\Rightarrow\)) of Lemma D.2.

(\(\Rightarrow\)) Assume $T_2$ is a representation for $T_1$ under $T_{12}$, we show that (iv) - (iii) are satisfied. For (iv) assume a $T_1$-consistent pair of concepts $B_1, B_2$, such that $B_1, B_2$ is $T_1 \cup T_{12}$ inconsistent; it follows by Lemma C.14 (iv) that $X_{(B_1(o), B_2(o))}$ is inconsistent. Then, one of the following holds:

(XXX) There are concepts $C, C' \in \text{t}_{12}(B_1(o), B_2(o)) (\delta)$ such that $(C \cap C' \subseteq \bot) \in T_2 \cup T_{12}$.

(XXXI) There are roles $Q, Q' \in \text{t}_{12}(B_1(o), B_2(o)) (\delta, \sigma)$ such that $(Q \cap Q' \subseteq \bot) \in T_2 \cup T_{12}$.

Assume (XXX) is the case and notice that w.l.o.g. we can assume $C'$ is over $\Sigma$ and $C$ is over $\Sigma \cup \Xi$. Let, first, $\delta = o$, by the structure of $X_{(B_1(o), B_2(o))}$ it follows there are $B \in \{B_1, B_2\}$ and $B'$ is over $\Xi$ such that $T_{12} \models B \subseteq B'$ and $T_{12} \models B' \subseteq C'$. Suppose $C$ is over $\Xi$, then it follows $(C \cap C' \subseteq \bot) \in T_2$ and, again, there are $D \in \{B_1, B_2\}$ and $D'$ is over $\Xi$ such that $T_{12} \models D \subseteq D'$ and $T_{12} \models D' \subseteq C'$. By Lemma D.1 (iii) it follows $T_1 \cup T_{12}$ is closed under disjointness between $B'$ and $D'$, so (iv) - (ii) is satisfied. Suppose $C$ is over $\Sigma$, then $(C \cap C' \subseteq \bot) \in T_2$, and by the structure of $X_{(B_1(o), B_2(o))}$ it follows $C \subseteq \{B_1, B_2\}$. By Lemma D.1 (iii) it follows $T_1 \cup T_{12}$ is closed under inclusion between $B'$ and $C'$, so (iv) - (ii) is satisfied. Consider now the case $\delta = w_{[R]}$ with $R'$ over $\Xi$. By Lemma C.4 it is the case $T_{12} \models R' \subseteq C$, $T_{12} \models R' \subseteq C'$. If $o \not\sim X_{(B_1(o), B_2(o))} \{w[R']\}$, then by the structure of $X_{(B_1(o), B_2(o))}$ it follows $T_{12} \models B \subseteq B'$ and $o \not\sim (T_{2}, (B'(o))) \{w[R']\}$ for some $B \in \{B_1, B_2\}$, $B'$ over $\Xi$; also by Lemmas C.4 and C.7 it follows the concept $B'$ is $T_2$ inconsistent. Since by Lemma D.1 (iii) $T_1 \cup T_{12}$ is closed under the disjointness between $B'$ and $B''$, it follows (iv) - (ii) is satisfied. If it is not the case $o \not\sim X_{(B_1(o), B_2(o))} \{w[R']\}$, it follows $\exists R \in \{B_1, B_2\}$ and $T_{12} \models \exists R \subseteq C$, $T_{12} \models \exists R \subseteq C'$. Now we can consider the argument above with $B = D = \exists R'$ to conclude either that either (iv) or (ii) is satisfied.

Finally, consider the case $\delta = w_{[R]}$ with $R'$ over $\Sigma$. By Lemma C.4 it is the case $T_{12} \models R' \subseteq C$, $T_{12} \models R' \subseteq C'$. If $o \not\sim X_{(B_1(o), B_2(o))} \{w[R']\}$, then by the structure of $X_{(B_1(o), B_2(o))}$ it follows $T_{12} \models B \subseteq B'$ and $o \not\sim (T_{2}, (B'(o))) \{w[R']\}$ for some $B \in \{B_1, B_2\}$, $B'$ over $\Sigma$; also by Lemmas C.4 and C.7 it follows the concept $B'$ is $T_2$ inconsistent. Since by Lemma D.1 (iii) $T_1 \cup T_{12}$ is closed under the disjointness between $B'$ and $B''$, it follows (iv) - (ii) is satisfied. If it is not the case $o \not\sim X_{(B_1(o), B_2(o))} \{w[R']\}$, it follows $\exists R \in \{B_1, B_2\}$ and $T_{12} \models \exists R \subseteq C$, $T_{12} \models \exists R \subseteq C'$. Now we can consider the argument above with $B = D = \exists R'$ to conclude either that either (iv) or (ii) is satisfied.

Assume (XXX) is the case and notice that w.l.o.g. we can assume $Q'$ is over $\Xi$ and $Q$ is over $\Sigma \cup \Xi$. By the structure of $U_{X}(B_1(o), B_2(o))$ we need to consider two cases: $\sigma = \delta w_{[R]}, Q, Q' \in \text{t}_{12}(B_1(o), B_2(o)) (\delta, \sigma)$ and $\sigma = w_{[R]}, Q, Q' \in \text{t}_{12}(B_1(o), B_2(o)) (\delta, \sigma)$. We show only the first case, the second case is analogous. Assume $\sigma = o, o \not\sim X_{(B_1(o), B_2(o))} \{w[R]\}$ for $R$ over $\Sigma$, and $Q$ over $\Xi$. It follows $T_{12} \models R \subseteq R'$ and $T_{12} \models R' \subseteq Q'$, and also $T_{12} \models R \subseteq S$ and $T_{12} \models S \subseteq Q'$ for some $R', S$ over $\Sigma$. Since $(Q \cap Q' \subseteq \bot) \in T_2$ by Lemma D.1 (iii) we get $T_1 \cup T_{12}$ is closed under inclusion between $R'$ and $S$, so (iv) - (ii) is satisfied. Let $Q \in \Sigma$, it follows $o \not\sim X_{(B_1(o), B_2(o))} \{w[R]\}$ and $R = Q$. It follows also $T_{12} \models Q \subseteq R'$ and $T_{12} \models R' \subseteq Q'$, then by Lemma D.1 (iii) we get $T_1 \cup T_{12}$ is closed under inclusion between $R'$ and $Q'$, so, since $(Q \cap Q' \subseteq \bot) \in T_2$, we conclude (iv) - (ii) is satisfied. Consider now the case $o \not\sim X_{(B_1(o), B_2(o))} \{w[R]\}$ for $R$ over $\Sigma$, which implies $Q$ is over $\Sigma$ and $(Q \cap Q' \subseteq \bot) \in T_2$; then $T_1 \cup T_{12}$ is closed under disjointness between $B'$ and $B''$, so (iv) - (ii) is satisfied. This concludes the proof for the case $\sigma = o$.

Assume $\sigma = w_{[R']}$ for $R'$ over $\Sigma$, this implies $o \not\sim X_{B}(B_1(o), B_2(o)) \{w[R']\}$ and we lead the proof analogously to the case above to show there is a $T_2$ inconsistent $B'$ such that $T_{12} \models R' \subseteq B'$ and (iv) - (ii) is satisfied. If $\sigma = w_{[R']}$ for $R'$ over $\Xi$ it can be easily verified (iv) - (ii) or (iv) - (ii) is satisfied. This concludes the proof that (v) is satisfied; then (v) can be shown analogously.

To show (i) is satisfied assume $B \in \text{cons}_{C}(T_1 \cup T_{12})$ over $\Sigma$ and $B' \in \text{t}_{12}(B, o)$. By Lemma C.14 (iii) it follows $B' \in \text{t}_{12}(B, o)$, so there exists $C'$ over $\Xi$ such that $T_{12} \models B \subseteq C'$ and $T_{12} \models C' \subseteq B'$. By Lemma D.1 (i) it follows $T_1 \cup T_{12}$ is closed under inclusion between $C'$ and $B''$; then (iii) can be shown analogously.

Finally, we show (iii) is satisfied; assume $B \in \text{cons}_{C}(T_1 \cup T_{12})$ over $\Sigma$ and $o \not\sim S_{B} \{w[R]\}$ for some role $R$, by Lemma C.14 (v) it follows there exists $y \in \Delta_{X_{B}}$ such that $t_{12}(B, o[w[R]]) \subseteq t_{12}(y)$, and $t_{12}(o, ow_{[R]}) \subseteq \text{t}_{12}(o, y)$. By the structure of $X_{B}$ it follows there exists a sequence of concepts $(C_0, \ldots, C_n) = (B, \exists Q_{1}, \ldots, \exists Q_{n})$ such that $T_{2} \cup T_{12} \models C_i \subseteq \exists Q$ for all $0 \leq i < n$ and roles $Q$ such
that $C_{i+1} = \exists Q^-$, $T_2 \cup T_{12} \vdash C_n \subseteq B'$ for all $B' \in \mathbf{t}^{U_{S_B}}(ow|R)$, and $r^{U_{S_B}}(o, ow|R) \neq \emptyset$ implies $n = 1$ and $T_2 \cup T_{12} \vdash Q \subseteq R'$ for all $R' \in r^{U_{S_B}}(o, ow|R)$ and $Q$ such that $C_1 = \exists Q^-$. We define a generating pass for $B$ conform with $T_1 \cup T_{12}$ as follows: $L(C_n) = s^{S_B}_{n}(w|R)$, $L(C_1, C_n) = q^{S_B}_{n}(o, w|R)$, $L(C_i) = \{\exists Q \mid C_{i+1} = \exists Q^-, B \subseteq \exists Q\}$ for all $0 \leq i < n$, and $L(C_i, C_j) = \emptyset$ for $j \neq i + 1$. It can be straightforwardly verified that (XVIII) holds, then also (XIX) and (XX) follow using Lemma D.1. We have shown (iii) is satisfied, which concludes the proof of Lemma D.2. □

**Theorem D.4** The non-emptyness problem for UCQ-representability is $\text{NLOGSPACE}$-complete.

**Proof.** As in the case of Theorem C.16 the lower bound is shown by the reduction from the directed graph reachability problem, however, we need a slightly more involved encoding.

**Lemma D.5** The non-emptyness problem for UCQ-representability is $\text{NLOGSPACE}$-hard.

**Proof.** To encode the graph $G = (V, E)$, we need a set of $\Sigma$-concept names $\{V_i \mid v_i \in V\} \cup \{S, F, X, Y\}$ and a set of $\Xi$-concept names $\{V'_i \mid v_i \in V\} \cup \{S', X', Y'\}$. Consider the TBox

$$T_1 = \{V_i \subseteq V_j \mid (v_i, v_j) \in E\} \cup \{S \subseteq V_k, V_m \subseteq F, X \subseteq Y\},$$

where $v_k$ and $v_m$ are, respectively, the initial and final vertices. Then, let

$$T_{12} = \{V_i \subseteq V'_i \mid v_i \in V\} \cup \{S \subseteq S', S \subseteq X', F \subseteq Y', X \subseteq X', Y \subseteq Y\};$$

we will show:

**Proposition D.6** There is a directed path from $v_k$ to $v_m$ in $G$ iff there exists a representation for $T_1$ under $\mathcal{M} = (\Sigma, \Xi, T_{12})$.

Indeed, using Lemma D.2 there exists a representation iff the condition (i) is satisfied. By the structure of $T_1 \cup T_{12}$ one can see that it is the case iff $T_1 \cup T_{12}$ is closed under the inclusion between $X'$ and $Y'$. The latter is the case iff $T_1 \cup T_{12} \vdash S \subseteq X'$ implies $T_1 \cup T_{12} \vdash S \subseteq Y'$, and that holds iff $T_1 \vdash S \subseteq F$, which is the case iff there exists a path from $v_k$ to $v_m$ in $G$. This completes the proof of Lemma D.2. □

To show the upper bound, we prove that the conditions (ii)(v) of Lemma D.2 can be checked in $\text{NLOGSPACE}$. In fact, these conditions can be checked using the algorithm, based on directed graph reachability solving procedure, similar to the proof of Theorem C.16. The only new case is the condition (iii) to verify that there exists a generating pass $\pi = ((C_0, \ldots, C_n), L)$ for a concept $B$ conform with $T_1 \cup T_{12}$, we can use the following procedure, running in $\text{NLOGSPACE}$. First, we take $C_0 = B$ and decide, if the pass ends here (i.e., $n = 1$). If we decided so, it only remains to take $L(C_0) = \mathbf{t}^{U_{S_B}}(ow|R)$, for $S_B$ and $R$ as in the condition (iii) and verify (XIX). This verification can be performed in $\text{NLOGSPACE}$, similarly to the method described in the proof of Theorem C.16. If, on the other hand, we decide, that the pass continues, we “guess” $C_1 = \exists Q^-$ for some role $Q$, and verify that for some $L(C_0) \subseteq \{\exists Q\}$ the (XVIII) and (XIX) are satisfied. Now, if we decide that the pass stops, it remains to take $L(C_1) = \mathbf{t}^{U_{S_B}}(ow|R)$ and $L(C_0, C_1) = r^{U_{S_B}}(o, ow|R)$, for $S_B$ and $R$ as in the condition (iii) and verify (XIX) and (XX). If, on the contrary, we decide that the pass continues, we can “forget” $C_0$, “guess” $C_2$, and proceed with it in the same way, as we did with $C_1$. Finally, when we reach the concept $C_n$, such that the algorithm decides to stop, it remains to verify (XIX) for $L(C_n) = \mathbf{t}^{U_{S_B}}(ow|R)$. It should be clear that whenever the generating pass $\pi = ((C_0, \ldots, C_n), L)$ for a concept $B$ conform with $T_1 \cup T_{12}$ exists, we can find it by the above non-deterministic procedure. □