ON PAIR CORRELATION AND DISCREPANCY

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Abstract. We say that a sequence \((x_n)_{n \geq 1}\) in \([0, 1)\) has Poissonian pair correlations if

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} = 2s
\]

for all \(s > 0\). In this note we show that if the convergence in the above expression is - in a certain sense - fast, then this implies a small discrepancy for the sequence \((x_n)_{n \geq 1}\). As an easy consequence it follows that every sequence with Poissonian pair correlations is uniformly distributed in \([0, 1)\).

1. Introduction

The concept of Poissonian pair correlations for a sequence \((x_n)_{n \geq 1}\) in \([0, 1)\) was introduced by Rudnick and Sarnak in [5], and has been intensively studied by several authors over the last years (see for instance [2, 3, 6, 7, 8]). Let \(\|\cdot\|\) denote distance to the nearest integer. We say that a sequence \((x_n)_{n \geq 1}\) of real numbers in \([0, 1)\) has Poissonian pair correlations if

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} = 2s
\]

for every \(s > 0\).

In this note we are concerned with the relation between the Poissonian pair correlation property and the notion of uniform distribution. We say that the sequence \((x_n)_{n \geq 1}\) is uniformly distributed, or equidistributed, in \([0, 1)\) if

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : x_n \in [a, b) \} = b - a
\]

for all \(0 \leq a \leq b \leq 1\). It is well-known that uniform distribution does not necessarily imply Poissonian pair correlations. One example confirming this is the Kronecker sequence \(\{n\alpha\}_{n \geq 1}\), which is uniformly distributed for every irrational \(\alpha\), but does not have Poissonian pair correlations.
correlations for any value of $\alpha$. Whether the converse implication holds has until recently remained an open question: Is every sequence in $[0, 1)$ with Poissonian pair correlations uniformly distributed? We answer this question in the affirmative by establishing a quantitative result connecting the speed of convergence in (1.1) to the star-discrepancy $D^*_N$ of the sequence. We recall that the star-discrepancy $D^*_N$ of $(x_n)_{n \geq 1}$ is defined as

$$D^*_N = \sup_{0 \leq a \leq 1} \left| \frac{1}{N} \cdot A_N([0, a)) - a \right|,$$

where $A_N([0, a)) := \#\{1 \leq n \leq N : x_n \in [0, a)\}$, and that $(x_n)_{n \geq 1}$ is uniformly distributed in $[0, 1)$ if and only if $\lim_{N \to \infty} D^*_N = 0$ (see for example [4]).

The main result of this paper is the following.

**Theorem 1.** Let $(x_n)_{n \geq 1}$ be a sequence in $[0, 1)$, and suppose that there exists a function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ which is monotonically increasing in its first argument, and which satisfies

$$\max_{s=1,\ldots,K} \frac{1}{2^s} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \leq F(K, N)$$

for all $N \in \mathbb{N}$ and all $K \leq N/2$. One can then find an integer $N_0 > 0$ such that for $N \in \mathbb{N}$, $N \geq N_0$, and arbitrary $K$ satisfying

$$\min \left( \frac{1}{2} N^{2/5}, \frac{N}{F(K^2, N)} \right) \leq K \leq N^{2/5},$$

we have

$$ND_N^* \leq 5 \cdot \max \left( N^{4/5}, \sqrt{N \cdot F(K^2, N)} \right)$$

where $D^*_N$ is the star-discrepancy of $(x_n)_{n \geq 1}$.

The next result is an easy consequence of Theorem 1.

**Corollary 2.** If the sequence $(x_n)_{n \geq 1}$ in $[0, 1)$ has Poissonian pair correlations, then it is uniformly distributed.

**Proof.** Suppose that $(x_n)_{n \geq 1}$ has Poissonian pair correlations, and fix any $\varepsilon > 0$. We then have

$$\max_{s=1,\ldots,[1/\varepsilon^5]} \frac{1}{2^s} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \leq \varepsilon N,$$

for all sufficiently large $N \geq N(\varepsilon)$. Hence, we may construct a function $F$ satisfying (1.2) where $F(L, N) = \varepsilon N$ for $N \geq N(\varepsilon)$ and $L \leq 1/\varepsilon^5$.

Simultaneously with our proof, another elegant proof of this result was given by Aistleitner et.al. in [1]. However, their approach is less elementary and does not provide the quantitative bound on the star discrepancy given by Theorem 1.
Without loss of generality, we may assume that \( N(\varepsilon) \geq 1/\varepsilon^5 \). If we fix \( K := \lceil 1/\varepsilon^2 \rceil \), then for \( N \geq N(\varepsilon) \) we have
\[
\frac{N}{F(K^2, N)} = \frac{N}{\varepsilon N} = \frac{1}{\varepsilon} \leq K \leq N^{2/5},
\]
and accordingly \( K \) satisfies (1.3). By Theorem 1 it thus follows that
\[
D^*_N \leq \frac{5}{N} \cdot \max \left( N^{4/5}, \varepsilon N \right) = 5\sqrt{\varepsilon}
\]
for \( N \geq N_0 \) (where in particular \( N_0 \geq N(\varepsilon) \)). □

2. Proof of Theorem 1

For a fixed pair of integers \((N, K)\), where \( K \) satisfies (1.3), we introduce the notation
\[
H(N, K) := \frac{5}{N} \cdot \max \left( N^{4/5}, \sqrt{N \cdot F(K^2, N)} \right).
\]
Aiming for a proof by contradiction, we assume that \( ND^*_N > H(N, K) \) for infinitely many pairs \((N, K)\). That is, there exist integers \( 1 < N_1 < N_2 < \ldots \) and corresponding integers \( K_1, K_2, \ldots \) satisfying (1.3), as well as real numbers \( B_1, B_2, \ldots \in (0, 1) \), such that either
\[
\# \{ 1 \leq n \leq N_j : x_n \in [0, B_j) \} - N_j B_j > H(N_j, K_j) \quad (2.1)
\]
for every \( j \), or
\[
\# \{ 1 \leq n \leq N_j : x_n \in [0, B_j) \} - N_j B_j < -H(N_j, K_j) \quad (2.2)
\]
for every \( j \). We assume in what follows that (2.1) holds (the case when (2.2) holds is treated analogously). Note that (2.1) implies
\[
N_j - N_j B_j - H(N_j, K_j) > 0. \quad (2.3)
\]
Let \( N := N_j, K := K_j, B := B_j \) and \( H := H(N_j, K_j) \) for some fixed \( j \). We now consider the distribution of the points \( x_n \) into subintervals of \([0, 1)\) of length \( K/N \). Let
\[
A_i := \# \left\{ 1 \leq n \leq N : x_n \in \left[ i \cdot \frac{K}{N}, (i + 1) \cdot \frac{K}{N} \right) \right\}
\]
for \( i = 0, 1, \ldots, \lceil N/K \rceil - 1 \), and let
\[
A_{\lfloor N/K \rfloor} := \# \left\{ 1 \leq n \leq N : x_n \in \left[ \lfloor N/K \rfloor \cdot \frac{K}{N}, 1 \right) \right\}.
\]
Moreover, for arbitrary positive integers \( l \), let
\[A_l := A_{l \mod (\lfloor N/K \rfloor + 1)}.
\]
If we introduce the notation
\[
\mathcal{H}_L := \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{LK}{N} \right\}
\]
for \( L = 1, 2, \ldots, K \), then
\[
\left| \frac{1}{2LK} \mathcal{H}_L - N \right| \leq F(K^2, N). \tag{2.4}
\]

We have that
\[
\mathcal{H}_L \geq \sum_{i=0}^{\lfloor N/K \rfloor} (A_i(A_i - 1) + 2A_i(A_{i+1} + \cdots + A_{i+L-1}))
\]
\[
= \sum_{i=0}^{\lfloor N/K \rfloor} ((A_i + \cdots + A_{i+L-1})^2 - (A_{i+1} + \cdots + A_{i+L-1})^2) - N
\]
\[
=: 2LN \cdot \gamma_L - N,
\]
where
\[
\gamma_L = \frac{1}{2LN} \sum_{i=0}^{\lfloor N/K \rfloor} ((A_i + \cdots + A_{i+L-1})^2 - (A_{i+1} + \cdots + A_{i+L-1})^2).
\]

Thus, we get
\[
\frac{1}{2LN} \cdot \mathcal{H}_L \geq \gamma_L - \frac{1}{2LK}. \tag{2.5}
\]

Now consider
\[
\Gamma_K := \min_{x_1, \ldots, x_N} \max_{L=1, 2, \ldots, K} \gamma_L,
\]
where by \( \min_{x_1, \ldots, x_N} \) we mean the minimum over all configurations of the points \( x_1, \ldots, x_N \) satisfying (2.1). If we define
\[
Z_L := \frac{1}{2LN} \sum_{i=0}^{\lfloor N/K \rfloor} (A_i + A_{i+1} + \cdots + A_{i+L-1})^2,
\]
then
\[
\gamma_L = Z_L - \frac{L-1}{L} \cdot Z_{L-1},
\]
and thus
\[
\Gamma_K = \min_{x_1, \ldots, x_N} \max \left( Z_1, Z_2 - \frac{1}{2}Z_1, \ldots, Z_K - \frac{K-1}{K}Z_{K-1} \right),
\]
We have
\[
\max \left( Z_1, Z_2 - \frac{1}{2}Z_1, \ldots, Z_K - \frac{K-1}{K}Z_{K-1} \right) \geq \frac{2}{K+1}Z_K.
\]
To see this, assume to the contrary that \( Z_1 \) and \( Z_L - (L-1)Z_{L-1}/L \) are all less than \( 2Z_K/(K+1) \). Then by successive insertions we get the contradiction \( Z_K < Z_K \). Hence, we have
\[
\Gamma_K \geq \min_{x_1, \ldots, x_N} \frac{2}{K+1} \cdot Z_K. \tag{2.7}
\]
Let us now estimate
\[
\min_{x_1, \ldots, x_N} Z_K = \frac{1}{2K^2N} \min_{A_0, A_1, \ldots, A_{[N/K]}} \sum_{i=0}^{[N/K]} \left( A_i + A_{i+1} + \cdots + A_{i+K-1} \right)^2,
\]
where the minimum on the right hand side is taken over all possible values of \(A_0, A_1, \ldots, A_{[N/K]}\) provided that the points \(x_1, \ldots, x_N\) satisfy (2.1). By definition, we have \(A_0 + \cdots + A_{[N/K]} = N\). Introducing the notation \(G_i = A_i + A_{i+1} + \cdots + A_{i+K-1}\), we thus get
\[
\sum_{i=0}^{[N/K]} G_i = K \cdot \sum_{i=0}^{[N/K]} A_i = KN. \tag{2.8}
\]
Moreover, by invoking condition (2.1) on the distribution of \(x_1, \ldots, x_N\), we have
\[
\sum_{i=-K+1}^{[NB/K]} G_i \geq K \sum_{i=0}^{[NB/K]} A_i \geq K(NB + H), \tag{2.9}
\]
and consequently
\[
\sum_{i=[NB/K]+1}^{[N/K]-K} G_i \leq K(N(1-B) - H). \tag{2.10}
\]
We get
\[
\min_{x_1, \ldots, x_N} Z_K \geq \frac{1}{2K^2N} \min_{G_0, G_1, \ldots, G_{[N/K]}} \sum_{i=0}^{[N/K]} G_i^2, \tag{2.11}
\]
where the minimum on the right hand side is taken over all positive reals \(G_0, G_1, \ldots, G_{[N/K]}\) satisfying (2.8) – (2.10). It is an easy exercise to verify that this minimum is attained when
\[
G_i = \frac{K(NB + H)}{K + \lceil NB/K \rceil} \quad \text{for} \quad i = -K + 1, \ldots, \left\lfloor \frac{NB}{K} \right\rfloor,
\]
and
\[
G_i = \frac{K(N(1-B) - H)}{[N/K] - K - \lceil NB/K \rceil} \quad \text{for} \quad i = \left\lceil \frac{NB}{K} \right\rceil + 1, \ldots, \left\lfloor \frac{N}{K} \right\rfloor - K.
\]
Note that since \(K \leq N^{2/5}\) and \(H \geq 5N^{4/5}\), we have \(K^2 \leq H/5\), and hence by (2.3) both the numerator and the denominator of these \(G_i\) are positive. Thus, we get
\[
\frac{1}{2K^2N} \min_{G_0, G_1, \ldots, G_{[N/K]}} \sum_{i=0}^{[N/K]} G_i^2 \geq \frac{1}{2K^2N} \left( \frac{K^2(NB + H)^2}{K + \lceil NB/K \rceil} + \frac{K^2(N(1-B) - H)^2}{[N/K] - K - \lceil NB/K \rceil} \right) \tag{2.12}
\]
\[
\geq \frac{K}{2} \left( 1 + \frac{H^2}{2N^2} \right)
\]
for all \( N > N_0 \). For the final inequality in (2.12), we have again used that \( H \geq 5N^{5/4} \) and \( K^2 \leq H/5 \).

Finally, by combining (2.12), (2.11) and (2.7), we find the lower bound
\[
\Gamma_K \geq \frac{K}{K+1} \left( 1 + \frac{H^2}{2N^2} \right).
\]

From the definition (2.6) of \( \Gamma_K \) and (2.5), it follows that
\[
\max_{L=1,\ldots,K} \frac{1}{2LKN} \mathcal{H}_L > \Gamma_K - \frac{1}{2K} \geq 1 + \frac{H^2}{4N^2} - \frac{2}{K},
\]
and recalling (2.4), we get
\[
\frac{1}{N} F(K^2, N) + 1 \geq \max_{L=1,\ldots,K} \frac{1}{2LKN} \mathcal{H}_L > 1 + \frac{H^2}{4N^2} - \frac{2}{K}.
\]
This implies that
\[
H^2 < \frac{8N^2}{K} + 4NF(K^2, N)
\]
\[
\leq 12 \max \left( \frac{N^2}{K}, NF(K^2, N) \right)
\]
\[
< 25 \max \left( N^{8/5}, NF(K^2, N) \right) = H^2,
\]
which is a contradiction. Thus, our assumption (2.1) must be incorrect, and the proof of Theorem 1 is complete. (Note that the last inequality above is trivially true if \( N^2/K \leq NF(K^2, N) \); In the opposite case we have \( K < N/F(K^2, N) \), and by the condition (1.3) imposed on \( K \) we then get \( K \geq N^{2/5}/2 \), and consequently \( N^2/K \leq 2N^{8/5} \).)

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