The Characterization and the Product of Quasi-Ehresmann Transversals

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Abstract. Wang (Filomat 29(5), 985-1005, 2015) introduced and investigated quasi-Ehresmann transversals of semi-abundant semigroups satisfy conditions (CR) and (CL) as the generalizations of orthodox transversals of regular semigroups in the semi-abundant case. In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal. These results further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case. Moreover, we obtain the main result that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup \(S\) which satisfy the certain conditions is a quasi-ideal quasi-Ehresmann transversal of \(S\).

1. Introduction

The concept of inverse transversals of regular semigroups was introduced by Blyth-McFadden [1]. Since then, inverse transversals have attracted much attention and a series of important results have been obtained and generalized (see [1-5,11,13-21,23-26]). If \(S\) is a regular semigroup, then an inverse transversal of \(S\) is an inverse subsemigroup \(S_0\) which meets \(V(a)\) precisely once for each \(a \in S\) (that is, \(|V(a) \cap S_0| = 1\)), where \(V(a) = \{x \in S | axa = a\text{ and } xax = x\}\) denotes the set of inverses of \(a\). Since orthodox semigroups can be considered as generalizations of inverse semigroups, Chen [2] generalized inverse transversals to orthodox transversals in the class of regular semigroups and gave a construction theorem for regular semigroups with quasi-ideal orthodox transversals. Chen-Guo [4] obtained some important properties associated with orthodox transversals in the general case. Most recently, Kong, Meng, Zhao [13,15,16,17,21] investigated orthodox transversals and obtained some interesting results. Especially, Kong-Meng [17] acquired the characterization for a generalized orthodox transversal to be an orthodox transversal and present a concrete description of the maximum idempotent separating congruence on regular semigroups with orthodox transversals. If the concept of transversals could be introduced in the \(E\)-inversive semigroups, then the congruences [6,7] on them will be characterized more neatly.

The concept of adequate transversals was introduced for abundant semigroups by El-Qallali [5] as an analogue of inverse transversals, and followed by Chen, Guo, Shum, Kong and Wang etc. [3,11,14,18,19]. In [19], the authors shown that the product of any two quasi-ideal adequate transversals of an abundant
lemmas 2.2 and 2.3 in this paper.

In this paper, we continue along the line of [8,17,19,26] by studying quasi-Ehresmann transversals of semi-abundant semigroups which satisfy conditions (CR) and (CL). In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case. The main purpose of this paper is to show that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup satisfies conditions (CR) and (CL) and satisfies the regularity condition is a quasi-ideal quasi-Ehresmann transversal of S.

2. Preliminaries

Let S and S° be semigroups. Throughout this paper, if no confusion, the set of idempotents of S and S° are denoted by E and E°, respectively. For short, the set V(α) ∩ S° is denoted by V_S°(α). If E generates a regular semiband, that is, (E) is a regular subsemigroup of S, then S is said to satisfy the regularity condition. S° is called a quasi-ideal of S, if S°S° ⊆ S°. We list some basic results as follows which are frequently used in this paper.

Definition 2.1[2] Let S be a regular semigroup with an orthodox subsemigroup of S°. Then S° is said to be an orthodox transversal of S, if the following two conditions are satisfied:

1. (∀ a ∈ S) V_S°(a) ≠ ∅;
2. For any a, b ∈ S, if [a, b] ∩ S° ≠ ∅, then V_S°(a)V_S°(b) ⊆ V_S°(ab).

Lemma 2.1[17] Let S be a regular semigroup and S° a subsemigroup of S with V_S°(a) ≠ ∅ for each a ∈ S. Then S° is an orthodox transversal of S if and only if

(∀a, b ∈ S) [V_S°(a) ∩ V_S°(b) ≠ ∅ ⇒ V_S°(a) = V_S°(b)].

The so-called Miller-Clifford theorem will be frequently used in this paper.

Lemma 2.2[12] Let e and f be D-equivalent idempotents of a semigroup S. Then each element a of R_e ∩ L_f has a unique inverse a′ in R_f ∩ L_e, such that aa′ = e and a′a = f;

2. Let a, b ∈ S. Then ab ∈ R_a ∩ L_b if and only if L_a ∩ R_b contains an idempotent.

Let S be a semigroup and a, b ∈ S. By aR°b we mean that xa = ya if and only if xb = yb for all x, y ∈ S. The relation L° can be defined dually. R is a left congruence and L° is a right congruence on S. A semigroup S is called abundant if each L°-class and each R°-class of S contains at least one idempotent. An abundant semigroup S is called quasi-adequate if its idempotents form a band. A band B is called a rectangular band if it satisfies the identity abc = ac for all a, b, c ∈ B. An adequate semigroup is an abundant semigroup in which the idempotents commute.

Let S be an abundant semigroup and U an abundant subsemigroup of S. U is called a **-subsemigroup of S, if for any a ∈ U, there exist idempotents e ∈ L°(S) ∩ U and f ∈ R°(S) ∩ U.

Definition 2.3[5] Let S be an abundant semigroup and S° a **-adequate subsemigroup of S. S° is called an adequate transversal of S, if for each x ∈ S there exist idempotents e, f ∈ S and a unique element x ∈ S° such that x = exf, where eLx° and fKx°.
Let $S$ be a semigroup and $a, b \in S$. That $\tilde{a}R\tilde{b}$ means that $ea = a$ if and only if $eb = b$ for all $e \in E$. The relation $\overline{L}$ can be defined dually. Denote $\overline{H} = \overline{L} \cap R$. In general, $\overline{L}$ is not a right congruence and $\overline{R}$ is not a left congruence. Obviously, $L \subseteq \overline{L}$ and $R \subseteq \overline{R}$. If $a, b \in \text{Reg}S$, the set of regular elements of $S$, then $\tilde{a}R\tilde{b}$ ($a\overline{L} \tilde{b}$) if and only if $\tilde{a} \tilde{R} \tilde{b}$ ($aL \tilde{b}$). On the relation $\overline{R}$ on a semigroup $S$, we have the following useful result.

**Lemma 2.3** Let $S$ be a semigroup and $a \in S, e \in E$. Then the following statements are equivalent:

(1) $eRa$;

(2) $ea = a$ and for all $f \in E, fa = a$ implies $fe = e$.

Now, we state the following fundamental concept of our paper. Semi-abundant semigroups satisfy conditions (CR) and (CL) were introduced by Fountain-Gomes-Gould\[8\].

**Definition 2.3** A semigroup $S$ is called semi-abundant if each $\overline{L}$-class and each $\overline{R}$ -class of $S$ contains idempotents. In particular, if $\overline{L}$ is a right congruence and $\overline{R}$ is a left congruence on a semi-abundant semigroup $S$, then we say that $S$ satisfies conditions (CR) and (CL).

A semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is quasi-Ehresmann if its idempotents form a subsemigroup of $S$. Certainly, regular semigroups are semi-abundant semigroups satisfy conditions (CR) and (CL), and orthodox semigroups are quasi-Ehresmann semigroups. It is easy to see a semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is quasi-Ehresmann if and only if $\text{Reg}S$ is an orthogonal subsemigroup of $S$. Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL). For $K \in \{L, R\}$ and $a \in S$, the $K$-class of $S$ containing $a$ is denoted by $K_a$.

A semi-abundant subsemigroup $U$ of a semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is called a $\sim$-subsemigroup of $S$ if

$$\overline{L}(U) = \overline{L}(S) \cap (U \times U), \overline{R}(U) = \overline{R}(S) \cap (U \times U),$$

and this equivalent to that there exist idempotents $e, f \in U$ such that $e\overline{L}x$ and $f\overline{R}x$ in $S$ for all $x \in U$. Now, let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) and $S^o$ a quasi-Ehresmann $\sim$-subsemigroup of $S$. For any $x \in S$, denote

$$\Omega_f(x) = \{(e, \overline{x}, f) \in E \times S^o \times E : x = e\overline{x}f, e\overline{L}\overline{x}^+; f\overline{R}\overline{x}^+\text{ for some } \overline{x}^+, \overline{x}^+ \in E^o\},$$

and $\Gamma_x = \{\overline{x} : (e, \overline{x}, f) \in \Omega_f(x)\}, I(x) = \{e : (e, \overline{x}, f) \in \Omega_f(x)\}, \Lambda(x) = \{f : (e, \overline{x}, f) \in \Omega_f(x)\}, I = \bigcup_{x \in S} I(x), \Lambda = \bigcup_{x \in S} \Lambda(x)$.

**Lemma 2.4**[26] Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) and $S^o$ a quasi-Ehresmann $\sim$-subsemigroup of $S$. Then $I = \{e \in E : (\exists e^o \in E^o) e\overline{L}e^o\}$ and $\Lambda = \{f \in E : (\exists f^o \in E^o) f\overline{R}f^o\}$.

**Definition 2.4**[26] Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) and $S^o$ a quasi-Ehresmann $\sim$-subsemigroup of $S$. Then $S^o$ is called a quasi-Ehresmann transversal of $S$ if the following three conditions hold:

1. $\Gamma_x \neq \emptyset$ for all $x \in S$;
2. is $i \in I$ and $si \in \text{Reg}S$ implies $si \in E$ for all $i \in I$ and $s \in E^o$;
3. $s\lambda \in \Lambda$ and $s\lambda \in \text{Reg}S$ implies $s\lambda \in E$ for all $\lambda \in \Lambda$ and $s \in E^o$.

**3. Two characterizations of quasi-Ehresmann transversals**

Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with the set of idempotents $E$ and $S^o$ a quasi-Ehresmann $\sim$-subsemigroup of $S$ with the set of idempotents $E^o$. $S^o$ is called a generalized quasi-Ehresmann transversal of $S$ if $\Gamma_x \neq \emptyset$ for all $x \in S$.

In the following, we shall give two characterizations for a generalized quasi-Ehresmann transversal to
be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case.

**Theorem 3.1** Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal $S'$ Then $S'$ is a quasi-Ehresmann transversal of $S$ if and only if

$$(\forall a, b \in \text{Reg}(S), \ [V_S(a) \cap V_S(b) \neq \emptyset \Rightarrow V_{S'}(a) = V_{S'}(b)].$$

**Proof.** (Sufficiency) Let $f \in E^s, e \in 1$ with $eLe' \in E^s$. By means of $S'$ is quasi-Ehresmann and $efLef' \in E^s$ we have

$$fe' \cdot ef \cdot fe' = f \cdot e'e \cdot ff \cdot e' = fe' = fe'$$

$$ef \cdot fe' \cdot ef = e \cdot ff \cdot e'e \cdot f = ef \cdot fe' = ef.$$  

Thus $fe' \in V_S(fe') \cap V_S(ef)$. Then, by the condition, we obtain $V_S(fe') = V_S(ef)$. From $S'$ is quasi-Ehresmann, we deduce that $E^s$ is a band and so is the semilattice $Y$ of rectangular bands $E_\alpha(\alpha \in Y)$. Since $e'f$ and $fe'$ are in the same rectangular band, and so are inverses of each other. Hence $e'f \in V_S(ef)$ and so

$$ef = (ef)(e'f)(ef) = (ef)^2$$

That is $ef$ is idempotent and we have in fact proved $IE^s \subseteq E$.

If $fe'$ is regular, take $x \in V_S(fe')$ and $x' \in V_S(fe')$. Then $exf$ is idempotent and $exf \in V(fe')$ with $eLe'xf \in S'$. Let $(e'xf)' \in E^s$ with $(e'xf)' \subseteq eLe'xf$ since $L$ is a left congruence. Then $exfL(e'xf)' \in E^s$ and so $(e'xf)' \in V_S(exf) \cap V_S(e'xf)'$. From the assumption and $S'$ is quasi-Ehresmann, we have $V_S(exf) = V_S((e'xf)')$ and hence $V_S(exf) \subseteq E^s$. Meanwhile we deduce that the regular elements of $S'$ form an orthodox subsemigroup of $S'$, and so $fx'e' \in V_S(e'xf)$ since $e', f \in E^s$. Hence

$$fx'e' \cdot exf \cdot fx'e' = fx'e' \cdot e'xf \cdot fx'e' = fx'e'$$

and

$$exf \cdot fx'e' \cdot exf = e \cdot e'xf \cdot fx'e' \cdot e'xf = e \cdot e'xf = exf$$

since $e'Le$ with $e', e'$ are idempotent. So, $fx'e' \in V_S(exf)$. Similarly, one can prove that $e'xf \in V_S(fe') \cap V_S(xfe')$. Thus $fx'e' \in E^s$ and $V_S(xfe') = V_S(xfe') \subseteq E^s$ and consequently $x \in E^s$. Therefore $e'xf \in E^s$ and

$$fe = fe \cdot e'xf \cdot fe = fe \cdot exf \cdot e'xf \cdot fe = fex \cdot ex \cdot xfe.$$  

Premultiplying and postmultiplying by $x$, we obtain

$$x = xfxex = xfxex \cdot fe' \cdot xfxex = xfe'x.$$  

Thus $fx'e'x$ with $fe'x \in E^s$, from which we deduce that $fexxf = fe'x \cdot xfxex$. By means of $fexxf, xfe \in E^s$, we have $fx'e'x \in V_S(xfe)$. It is obvious that $xf \in V(fe')$ and $xf \in E^s$ implies that $V_S(fe') = V_S(xfe)$ and so $fe'x \in V_S(fe')$. Therefore $fe = fe \cdot fe'xf \cdot fe = fef(ef)efxe'xf \cdot fe = fexfe'x \cdot fe = fef xfe$ since ef is idempotent, and so $fe$ is idempotent. Up to now, we have in fact proved if $fe$ is regular, then it is idempotent. Dually, we can proved that $E^s \Lambda \subseteq E$ and if for all $\Lambda \in \Lambda, f \in E^s$, if $\Lambda f$ is regular, then it is idempotent.

(Necessity) By [26, Theorem 3.6 (4)], the condition is necessary. □

**Theorem 3.2** Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal $S'$. Then $S'$ is a quasi-Ehresmann transversal if and only if for any regular elements $a \in S$, $b \in S'$, if $aba$ is regular, then $V_S(a)V_S(b) \subseteq V_S(ab)$; and if $ab$ is regular, then $V_S(b)V_S(a) \subseteq V_S(ab)$. 

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Proof. (Necessity) For any regular elements $a \in S$, $b \in S'$, take $a^r \in V_S(a)$, $b^r \in V_S(b)$, if $S'$ is a quasi-Ehresmann transversal, then by the definition, $aa^r b \in E^s \subseteq E$. If $ba$ is regular, take $(ba)^r \in V_S(ba)$, then

$$
(b^r b a^r)(a ba)^r (b^r b a^r) = b^r (ba a)(ba)(b^r b a) a^r = b^r (ba) a^r = b^r (ba) a = b^r b a^r.
$$

Thus $b^r b a^r$ is regular and so $b^r b a^r \in E^{t E} \subseteq E$. Therefore

$$
a^r b^r \cdot ba \cdot a^r b^r = a^r (aa^r b) b (aa^r b) b^r = a^r \cdot a a^r b \cdot b^r = a^r b^r
$$

and so $V_S(a) V_S(b) \subseteq V_S(ba)$. Similarly, if $ab$ is regular, then $V_S(b) V_S(a) \subseteq V_S(ab)$.

(Sufficiency) For any regular elements $t_1, t_2 \in S'$, if $V(t_1) \cap V(t_2) \neq \emptyset$, take $t \in V(t_1) \cap V(t_2)$ and $t_1^r \in V_S(t_1)$. From $t_2 L t_1 t_1^r L t_2^r$, by Lemma 2.2, $t_2 R t_1 t_1^r L t_2^r$ and $(t_2 t_1 t_2^r t_1^r)^2 = t_2 (t_1 t_1^r t_2 t_1^r) = t_2 t_1 t_2^r$ since by the assumption

$$
t_1^r t_2 \in V_S(t_1).
$$

Similarly, $t_2 L t_1 t_2^r L t_1^r$ with $t_1^r t_2 \in E$. Thus

$$
t_1^r t_2^r t_1^r t_2 = (t_1^r t_2^r t_1^r t_2 t_2^r) t_1^r t_2 = (t_2 t_1^r t_2^r t_1^r t_2) t_1^r = t_2 t_1^r t_1^r t_2 = t_2 t_1^r = t_2.
$$

Hence $t_1^r \in V_S(t_2)$, that is, $V_S(t_1) \cap V_S(t_2) \neq \emptyset$. Therefore $V_S(t_1) = V_S(t_2)$ since the regular elements of $S'$ form an orthodox subsemigroup of $S$.

For any $e \in S$, if $V_S(e) \cap E^s \neq \emptyset$, take $f \in V_S(e) \cap E^s$. Then for any $e' \in V_S(e)$, we have $e \in V(f) \cap V(e')$ and so by the above result, $V_S(e) = V_S(e')$. Consequently, $e'$ is an inverse of $f$ in $S'$ and $e' \in E^s$ since $S'$ is quasi-Ehresmann. That is, if $V_S(e) \cap E^s \neq \emptyset$, then $V_S(e) \subseteq E^s$.

Let $e, f \in I$ with $e L f$. Take $h \in E^s$ such that $h L e L f$, then $h \in V_S(e) \cap V_S(f)$. For any $g \in V_S(e)$, by the above result we have $g \in E^s$. It is easy to see that $ghg \in V_S(g e g)$ and $gh g \in V_S(geg) = V_S(g)$. Then $g f g$ and $g$ have a common inverse $gh g$. Consequently $g h g \cdot g f g \cdot gh g = gh g$ and thus $g f g = g$. Since $g e L e L f$, by Lemma 2.2, $f g R f$ and so $g f g = f$. Thus $g \in V_S(f)$ and so $V_S(e) \subseteq V_S(f)$. Similarly, we have the reverse inclusion and hence $V_S(e) = V_S(f)$. Dually, if $e, f \in I$ with $e R f$, then $V_S(e) = V_S(f)$.

It is easy to see that if $a \in Reg S$, then for any $a^r \in V_S(a)$, we have $V_S(a^r) = V_S(a) a^r V_S(a^r)$. For $a, b \in Reg S$, if $V_S(a) \cap V_S(b) \neq \emptyset$, take $e^c \in V_S(a) \cap V_S(b)$. Then $V_S(a) = V_S(c^e a) a^e V_S(ac^e)$ and $V_S(b) = V_S(c^e b) b^e V_S(bc^e)$. It follows from $ac^e \subseteq I$ and $ac^e L e^c b$ that $V_S(ac^e) = V_S(bc^e)$. Similarly, $V_S(c^e a) = V_S(c^e b)$. Therefore $V_S(a) = V_S(b)$ and so by Theorem 3.1 $S'$ is a quasi-Ehresmann transversal. \qed

Obviously, a regular semigroup with an orthodox transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal. Comparing Lemma 2.1 with Theorem 3.1, and Definition 2.1 with Theorem 3.2, it is illustrated by these two points of view that the transversal is a quasi-Ehresmann transversal. Thus, quasi-Ehresmann transversals are the generalization of orthodox transversals in the semi-abundant case.

By means of the properties of adequate transversal[3,Theorem3.3], one can easily observe that an abundant semigroup with an adequate transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a quasi-Ehresmann transversal.

In the following, we will investigate when a quasi-Ehresmann transversal is an orthodox transversal and when a quasi-Ehresmann transversal is an adequate transversal, respectively. We have the following results.

**Theorem 3.3** Let $S'$ be a quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). Then

(i) $S'$ is an orthodox transversal of $S$ if and only if $S$ is a regular semigroup.

(ii) if $S$ and $S'$ are abundant, then $S'$ is an adequate transversal of $S$ if and only if $S'$ is an adequate semigroup.
Lemma 4.1
Let $S$ be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and $H$ a subset of $S$. Then

1. $HS^o = HS$ and $S^oH = S^oH$;
2. $HS^o$ and $S^oH$ are both subsemigroups and quasi-ideals of $S$;
3. For any $x \in RegS$, if $|V(x) \cap H| \geq 1$, then $|V(x) \cap HS^o| \geq 1$ and $|V(x) \cap S^oH| \geq 1$.

Proof. (1) Let $h \in H, x \in S$ and $s \in S^o$. Then $h = chtf_k, f_k \in H, x = hxs, hxs \in HS^oSS^o \subseteq HS^o$. It is obvious that $hs = hfs \in HS^o$ and $HS^o = HS^o$. Similarly, $S^oH = S^oH$.

(2) It is easy to see $HS^o \subseteq H \subseteq S^oSS^o \subseteq HS^o, HS^o$ is a subsemigroup of $S$. Similarly, $HS^o \subseteq S^oSS^o \subseteq HS^o$ and $HS^o$ is a quasi-ideal of $S$. There is a dual result for $S^oH$.

(3) For any regular element $x \in S$, take $x' \in V(x) \cap H$, then for any $x' \in V(x) \cap H$, $x'x't \in V(x) \cap HS^o = V(x) \cap HS^o$, that is $|V(x) \cap HS^o| \geq 1$. Similarly, $|V(x) \cap S^oH| \geq 1$.

Lemma 4.2 Let $S^o, S^o$ be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). For every $a \in RegS$, we have $V_{S^o}(a) = V_{S^o}(a) \cdot a \cdot V_{S^o}(a)$.

Proof. Let $a^o \in V_{S^o}(a), a^o \in V_{S^o}(a)$. Then $a^oaa^o \in S^oSS^o = S^oS^o$ and $a^oaa^o \in V(a)$, and so $V_{S^o}(a) \cdot a \cdot V_{S^o}(a) \subseteq$
Let $S$. For every $x y^a \in V_{S \cdot S}(a)$, we have

$$a = ax^a y^a, \quad x y^a = x y^a \cdot a \cdot x y^a.$$

Hence

$$x y^a = x y^a \cdot a x y^a = x y^a \cdot a \cdot a x y^a.$$

and

$$x y^a a a^0 \in S^0 S^0 \subseteq S^0, \quad a x y^a \in S^0 S^0 \subseteq S^0,$$

On the other hand,

$$a x y^a a a^0 = a x y^a a x y^a a a^0 = x y^a a x y^a a a^0.$$

Thus $x y^a a a^0 \in V_{S}(a)$ and dually, $a x y^a \in V_{S}(a)$. Therefore $V_{S \cdot S}(a) \subseteq V_S(a) \cdot a \cdot V_S(a)$.

**Lemma 4.3** Let $S^0$ be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). For any $x, y \in S$, there exist $x, y \in \Gamma_S, \not\Gamma_S$ such that $x = x_0 x_1 e, e \subseteq \Gamma_S, f \subseteq \Gamma_S$ for some $x^0, x^1 \in E^0$ and $y = y_0 y_1 e, e \subseteq \Gamma_S, f \subseteq \Gamma_S$ for $y^0, y^1 \in E^0$. Then

1. $xy = e \not\Gamma_S e y_0 y_1 e, e \subseteq \Gamma_S, f \subseteq \Gamma_S$ and $x y^a \in S^0$ since $S^0$ is a quasi-ideal. Since $\not\Gamma_S, \Gamma_S$ are left congruences and $\not\Gamma, \Gamma$ are right congruences, we have

$$e_0 (\not\Gamma_S e y_0 y_1 e) \subseteq \Gamma \subseteq E^0 \subseteq E$$

where $e_0 (\not\Gamma_S e y_0 y_1 e)$ is an $E^0$-family of $\not\Gamma_S e y_0 y_1 e$ for $\not\Gamma_S e y_0 y_1 e \subseteq S^0$.

Therefore the above properties valid.

In what follows $S^0$ and $S^0$ will denote a pair of quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and $E^0$ and $E^0$ will denote the idempotents of them respectively to avoid confusion. For the sake of simplicity, in $S^0$, we still denote the typical idempotent that $a \subseteq \Gamma_S, a^\ast$ respectively. For any $x, y \in S$, we write $x = x_0 x_1 e$ and $x = x_0 x_1 e$ as the decompositions of $x$ in $S^0$ and $S^0$ respectively. Then $x \in S^0$ has the same meaning as in Definition 2.4.

More precisely, $i_0, \lambda_\ast \in E$ and $x^0, x^1 \in E^0$ with $x^0 \subseteq \Gamma \subseteq E^0$ and $x^1 \subseteq \Gamma \subseteq E^0$, and so $i_0 \subseteq \Gamma \subseteq E^0$. Let $S^0$ and $S^0$ be quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). Denote

$$I(S^0, S^0) = \{ a a^0 : a \in \text{Reg} S \cap S^0, a^0 \in V_{S}(a) \},$$

$$\Lambda(S^0, S^0) = \{ a^\ast a : a \in \text{Reg} S \cap S^0, a^0 \in V_{S}(a) \}.$$

**Theorem 4.4** Let $S^0$ and $S^0$ be a pair of quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then

$$I(S^0, S^0) = \Lambda(S^0, S^0) = I_0 \cap \Lambda_0.$$
Proof. For any $aa^o \in I(S^o, S^o)$, where $a \in \text{Reg}S \cap S^o$, $a^o \in V_{S^o}(a)$, certainly, $a \in V_{S^o}(a^o)$ and so $aa^o = a^oa^o \in \Lambda(S^o, S^o)$. Thus $I(S^o, S^o) \subseteq \Lambda(S^o, S^o)$ and dually $\Lambda(S^o, S^o) \subseteq I(S^o, S^o)$. Consequently, $I(S^o, S^o) = \Lambda(S^o, S^o)$ and we denote it by $W$. From the above definitions, it is clear that $W \subseteq I_o \cap \Lambda_o$.

Conversely, suppose that $x \in I_o \cap \Lambda_o$. Since $x \in \Lambda_o$, we have $x = x^o x$ for some $x^o \in V_{S^o}(x)$ with $x^o \in E_{S^o}$ and so $x^o = xx^o$. Similarly, $x \in I_o$, implied that $x = xx^o$ for some $c^o \in V_{S^o}(x)$ with $c^o \in E_{S^o}$, and so $x^o = xx^o$. Let $x^{oo} \in V_{S^o}(x^o)$. From $c^o \in \mathcal{L}x^o \mathcal{R}x^o \mathcal{X}x^o \mathcal{L}x^{oo} \mathcal{L}x^o \mathcal{X}x^o \mathcal{L}x^{oo}$ with $x^o x^o x^{oo} \in E^o I_o \subseteq E^o$ since $S^o$ is a quasi-ideal and $S$ satisfies the regularity condition. Thus $x^o \in \mathcal{L}x^{oo} \mathcal{R}x^o$.

Certainly, $x^o \mathcal{R}x^o \mathcal{X}x^o \mathcal{L}x^o$ and so by Lemma 2.2, $x^{oo} \mathcal{R}x^o \mathcal{X}x^o \mathcal{L}x^o \mathcal{X}x^o$ and $x^o x^o x^o \mathcal{H}x^o x^o \in I_o \cap \Lambda_o$. Consequently, $x^o x^o x^o = x$ since $x \in E$ and $x^o x^o \cdot x^2 \in I_o E^o \subseteq E$. Also $(x^o x^o x^o)^2 = x^o x^o (x^o x^o x^o) x^o = x^o x^o x^o = x^o x^o x^o$ and $x^o x^o x^o \in E$. Therefore

$$x^o \cdot x^o x^o \cdot x^o = xx^o = x^o$$ and $$x^{oo} x^o \cdot x^o \cdot x^{oo} x^o = x^o x^o x^o = x^o x^o$$

and so $x^{oo} x^o \in V_{S^o}(x)$. Hence $x = x^o \cdot x^o x^o \in I(S^o, S^o) = W$. \hfill \Box

Theorem 4.5 Let $S^o$ and $S^o$ be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then $S^o S^o$ is a quasi-ideal quasi-Ehresmann transversal of $S$.

Proof. It is evident that $S^o S^o$ is a subsemigroup and a quasi-ideal of $S$. For any $x \in S^o S^o$, there exist $s^o \in S^o, t^o \in S^o$ such that $x = s^o t^o$. It follows from $S^o$ is a quasi-ideal of $S$ and Lemma 4.3 that $e_o(s^o f_o e_o)^+ \in I_o E^o = I_o$ and we denote it by $e_o$. It is obvious that $I_o \in E_{S^o}$ since $s^o \in S^o$ and so from $e_o \mathcal{R}I_o \in E_{S^o}$ we deduce that $e_o \in I_o \cap \Lambda_o$. Thus by Theorem 4.4 there exists $a \in \text{Reg}(S^o)$ such that $e_o = a^o a^o$ and so

$$e_o = e_o(s^o f_o e_o)^+ = a^o a^o(s^o f_o e_o)^+ \in S^o S^o.$$  

Similarly, $x^o \in S^o S^o$. Thus $e_o, \lambda_x \in E_{S^o S^o}$, and so from $e_o \mathcal{R}x \mathcal{L} \lambda_x$ we deduce that $S^o S^o$ is semi-abundant. It is a routine matter to show that $e_o \mathcal{R}(S^o S^o) x$, $\lambda_x$, thus $S^o S^o$ is a ~semi-abundant subsemigroup of $S$.

Let $e$ be an idempotent of $S^o S^o$. Then $e = a^o a^o$ for some $a \in S^o$, $s \in S^o$. Since $(asa)(asa)(asa)(asa) = asa(asa)(asa)(asa) = asa(asa)$, we have $s \in V_{S^o}(asa)$, so that $e = asa(asa)(asa)(asa)$, and $s \in S^o$, each idempotent of $S^o S^o$ is of the form $b^o b^o$ for some regular element $b \in S^o$. Let $e$ and $f$ be idempotents of $S^o S^o$. Then $e = b^o b^o$ and $f = c^o c^o$ for some regular elements $b, c \in S^o$ with $b^o \in V_{S^o}(b)$ and $c^o \in V_{S^o}(c)$. For any $l \in E^o$, by the regularity condition, $l c^o c^o$ is regular and so $l c^o c^o \in E^o$. Since $S^o$ is a quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). Thus $l c^o c^o \in E \cap S^o = E^o \subseteq E$ and $S^o S^o$ is a quasi-Ehresmann semigroup.

For any $x \in S^o$, there exist $a, b \in \text{Reg}S$ such that $e_o = a a^o a^o, \lambda_x = b b^o$, where $a^o \in V_{S^o}(a), b^o \in V_{S^o}(b)$. Thus

$$x = e_o x \lambda_x = a a^o x b b^o = a a^o x a^o x b b^o b b^o,$$

where $a^o \in V_{S^o}(a^o), b^o \in V_{S^o}(b^o)$, and consequently

$$e_o = a a^o \mathcal{L} a^o a^o d^o \in E_{S^o S^o}, \lambda_x = b b^o \mathcal{R} b b^o b b^o \in E_{S^o S^o}.$$  

Since $a a^o a^o x b b^o \lambda_x = a a^o a^o x \lambda_x = a a^o a^o x$, we have $a a^o a^o x b b^o \mathcal{R}a a^o a^o x$. From $x \mathcal{R} x \mathcal{R} e_o$ and $\mathcal{R}$ is a left congruence we deduce that

$$a a^o a^o x \mathcal{R} a a^o a^o e_o = d^o d^o \in E_{S^o S^o}.$$  

Similarly,

$$a a^o a^o x b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \mathcal{L} b b^o \in E_{S^o S^o}.$$  

Consequently, $x = e_o(a a^o a^o x b b^o \lambda_x)$ with $e_o, \lambda_x \in E, e_o \mathcal{L}(a a^o a^o x b b^o \lambda_x)^+ = a a^o a^o \in E_{S^o S^o}$ and $\lambda_x \mathcal{R}(a a^o a^o x b b^o \lambda_x)^+ = b b^o \in E_{S^o S^o}$. Therefore, $S^o S^o$ is a generalized quasi-Ehresmann transversal of $S$. 

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For regular elements \(c \in S, d \in S^eS^0\), take \(c' \in V_{S^eS^0}(c), d' \in V_{S^eS^0}(d)\), then by Lemma 4.2, there exist \(c' \in V_{S^e}(c), c'' \in V_{S^e}(c), d'' \in V_{S^e}(d), d'' \in V_{S^e}(d)\), such that \(c' = c''c'\), \(d' = d''d'\). Since \(d \in S^eS^0, d'' \in V_{S^e}(d)\), we have \(d \in V_{S^e}(d'')\). By Lemma 4.2, there exist \((d'')^c \in V_{S^e}(d''), (d'')^c \in V_{S^e}(d'')\), such that \(d = (d'')^c d''(d'')^c\). So
\[
c'c''d''c' = c'c''c''d'' = c'c'(d'')^c d''(d'')^c = c'c'(d'')^c c = \lambda_o \subseteq \lambda_o,
\]
and \(c'c''d''\) is idempotent. On the other hand,
\[
d''d'c = d d''d'' c c = d d''c = (d'')^c d''(d'')^c c = (d'')^c c = \lambda_o \subseteq \lambda_o,
\]
and \(d''d'c \in E\). Thus
\[
c''d''c = c \cdot c''d'' \cdot c''d' \cdot d = c c''d''d' = cd,
\]
d''d''c' = d''d''c = d''d''c = d''d''c = d''d''c = d'd'c,
and so \(V_{S^eS^o}(d)V_{S^eS^o}(c) \subseteq V_{S^eS^o}(cd)\). Similarly, \(V_{S^eS^o}(c)V_{S^eS^o}(d) \subseteq V_{S^eS^o}(dc)\).

It follows from Theorem 3.2 that \(SS^o\) is a quasi-Ehresmann transversal. Since \(SS^o\) is a quasi-ideal, therefore \(SS^o\) is a quasi-ideal quasi-Ehresmann transversal of \(S\).

**Theorem 4.6** Let \(S\) be a semi-abundant semigroup satisfying conditions (CR) and (CL) and the the regularity condition. If \(S\) has a quasi-ideal quasi-Ehresmann transversal, then all quasi-ideal quasi-Ehresmann transversals of \(S\) form a rectangular band.

**Proof.** If \(SS^o\) is a quasi-ideal quasi-Ehresmann transversal of \(S\), then \(SS^o = S^o\). To see this, for \(s' \in S^o, s'' = s'(s')^c \in SS^o\), hence \(SS^o \subseteq SS^0\) and the reverse inclusion is obvious. By Theorem 4.5, all quasi-ideal quasi-Ehresmann transversals of \(S\) form a semigroup and so form a band.

Let \(SS^o, SS^0, SS^1\) be arbitrary three quasi-ideal quasi-Ehresmann transversals of \(S\). For any \(a \in SS^o, x \in S, b \in SS^0\), we have
\[
a''x b = a''x e(b^o)^c b \in SS^o SS^0 \subseteq SS^1, a''b = a''(a'')^c b \in SS^0 SS^o \subseteq SS^1 SS^0,
\]
where \(b^o \in E\) and \(e(b^o)^c \in E^0\). Thus \(SS^o SS^0 \subseteq SS^1 SS^0 = SS^1 SS^0\). For every \(a' \in SS^o, b' \in SS^0\), then
\[
a''b' = a''f(x')^c f(x')^c b' \in SS^0 SS^1 SS^0 = SS^1 SS^0,
\]
with \((f(x'))^c\) is an inverse in \(SS^0\) of \(f(x')^c\). Thus \(SS^0 SS^1 = SS^1 SS^0\) and therefore all quasi-ideal quasi-Ehresmann transversals of \(S\) form a rectangular band.

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