Abstract

Stochastic gradient descent ascent (SGDA) and its variants have been the workhorse for solving minimax problems. However, in contrast to the well-studied stochastic gradient descent (SGD) with differential privacy (DP) constraints, there is little work on understanding the generalization (utility) of SGDA with DP constraints. In this paper, we use the algorithmic stability approach to establish the generalization (utility) of DP-SGDA in different settings. In particular, for the convex-concave setting, we prove that the DP-SGDA can achieve an optimal utility rate in terms of the weak primal-dual population risk in both smooth and non-smooth cases. To our best knowledge, this is the first-ever-known result for DP-SGDA in the non-smooth case. We further provide its utility analysis in the nonconvex-strongly-concave setting which is the first-ever-known result in terms of the primal population risk. The convergence and generalization results for this nonconvex setting are new even in the non-private setting. Finally, numerical experiments are conducted to demonstrate the effectiveness of DP-SGDA for both convex and nonconvex cases.

1 Introduction

In recent years, there is a growing interest on studying the minimax problems which involve both mini-
mization over the primal variable $w$ and maximization over the dual variable $v$. Notable examples include

generative adversarial networks (GANs) (Goodfellow et al., 2014), adversarial training (Sinha et al., 2017), algorithmic fairness (Mohri et al., 2019), robust learning (Audibert and Catoni, 2011), and Markov Decision Process (MDP) (Puterman, 2014). Details of these motivating examples are given in Appendix A.

The minimax problem can be formulated as

$$
\min_{w \in W} \max_{v \in V} F(w, v) := \mathbb{E}_{z \sim D}[f(w, v; z)],
$$

where $W \subseteq \mathbb{R}^d_1$ and $V \subseteq \mathbb{R}^d_2$ are two convex domains and $z$ is a random variable from some distribution $D$ taking values in $\mathcal{Z}$. Since the distribution $D$ is usually unknown and one has access only to an i.i.d. training dataset $S = \{z_1, \ldots, z_n\}$, one resorts to solving its empirical minimax problem

$$
\min_{w \in W} \max_{v \in V} F_S(w, v) := \frac{1}{n} \sum_{i=1}^{n} f(w, v; z_i).
$$

One popular optimization algorithm for solving this problem is SGD. Specifically, at iteration $t$, upon receiving a random data point or mini-batch from $S$, it performs gradient descent over $w$ with the stepsize $\eta_{w,t}$ and gradient ascent over $v$ with the stepsize $\eta_{v,t}$. As SGDA is conceptually simple and easy to implement, it is widely deployed in solving minimax problems, e.g., GANs (Goodfellow et al., 2014), adversarial learning (Sinha et al., 2017), and AUC maximization (Ying et al., 2016). Its local convergence analysis for nonconvex-(strongly-)concave problems was established in (Lin et al., 2020). Other variants of SGDA were proposed and studied in (Luo et al., 2020; Nouiehed et al., 2019; Li et al., 2016; Wang et al., 2020a; Martinez et al., 2020; Diana et al., 2021). One other front, collected data often contain sensitive information such as individual records from hospitals, online behavior from social media, and genomic data from cancer diagnosis. Differential privacy (Dwork et al., 2014) has emerged as a well-accepted mathematical definition of privacy which ensures that an attacker

Differentially Private SGDA for Minimax Problems

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gets roughly the same information from the dataset regardless of whether an individual is present or not. Its related technologies have been adopted by Google (Erlingsson et al., 2014), Apple (Ding et al., 2017), and the US Census Bureau (Abowd, 2010). While SGD and SGDA have become the workhorse behind the remarkable progress of machine learning and AI, it is of pivotal importance for developing their counterparts with DP constraints.

Many studies analyze the privacy and utility (generalization) of DP-SGD for the ERM problem that only involves the minimization over \( w \), (e.g., Bassily et al., 2019, 2020; Feldman et al., 2020; Song et al., 2013; Wang et al., 2021, 2020a, 2019b; Wu et al., 2017; Zhou et al., 2020). In contrast, there is little work on analyzing the generalization of minimax optimization algorithms with DP constraints except the recent work of Boob and Guzmán (2021). However, Boob and Guzmán focus on the noisy stochastic extragradient method on convex-concave and smooth settings.

In this paper, we use the algorithmic stability approach to establish the generalization (utility) of SGDA in different forms such as the weak primal-dual population risk and the primal population risk. Our contributions can be summarized as follows.

- We analyze the privacy and utility of DP-SGDA under the convex-concave setting in terms of the weak primal-dual population risk, i.e., \( \max_{v \in V} \mathbb{E}[F(A_w(S), v)] - \min_{w \in W} \mathbb{E}[F(w, A_v(S))] \), where \( (A_w(S), A_v(S)) \) is the output of DP-SGDA. Specifically, we show that it can guarantee \((\epsilon, \delta)\)-DP and achieve the optimal rate \( O\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon} \right) \) for smooth and nonsmooth cases where \( d = \max\{d_1, d_2\} \).

To our best knowledge, this is the first-ever known result for DP-SGDA in the nonsmooth case.

- We further study the utility of DP-SGDA in the nonconvex-strongly-concave case in terms of the primal population risk, i.e., \( R(A_w(S)) = \max_{v \in V} \mathbb{E}[F(A_w(S), v)] \). In particular, under the Polyak-Łojasiewicz (PL) condition of \( F_S \), we prove that the excess primal population risk, i.e., \( R(A_w(S)) - \min_{w \in W} R(w) \), enjoys the rate \( O\left(\frac{1}{n^\gamma} + \frac{d \log(1/\delta)}{n^\gamma \epsilon} \right) \) while guaranteeing \((\epsilon, \delta)\)-DP. The key techniques involve the convergence analysis of \( R_S(A_w(S)) - \min_w R_S(w) \) and the stability analysis for \( A_w(S) \) which are of interest in their own rights. As far as we are aware, these results are the first ones known for DP-SGDA in the nonconvex setting.

- We perform numerical experiments on three benchmark datasets which validate the effectiveness of DP-SGDA for both convex and non-convex cases.

### 1.1 Related Work

Below we briefly discuss some related work.

**Convergence analysis for SGDA.** It is a classical result that SGD can achieve a convergence rate \( O\left(1/\sqrt{T}\right) \) in the convex and concave case (e.g., Nedic and Ozdaglar, 2009; Nemirovski et al., 2009) where \( T \) is the number of iterations. For the nonconvex-(strongly)-concave case, the work of Lin et al. (2020) shows the local convergence of SGDA if the stepsizes \( \eta_w, t \) and \( \eta_v, t \) are chosen to be appropriately different. Other important studies consider variants of SGDA and prove their local convergence for the nonconvex case. Such algorithms include nested algorithms (Rahique et al., 2021) for weakly-convex-weakly-concave problems, multi-step GDA (Nouiehed et al., 2019) under the one-sided PL condition, epoch-wise SGD (Yan et al., 2020), and stochastic recursive SGD (Luo et al., 2020) for nonconvex-strongly-concave problems, to mention but a few.

**Stability and generalization of non-private SGD and SGDA.** The studies of Hardt et al. (2016); Charles and Papailiopoulos (2018); Kuzborskij and Lampert (2018) use uniform stability (Bousquet and Elisseeff, 2002) to derive the generalization of non-private SGD for the convex and smooth case while the convex and nonsmooth case was established by Bassily et al. (2020); Lei and Ying (2020). The nonconvex case under the PL-condition was considered by Charles and Papailiopoulos (2018); Lei and Ying (2021). The stability and generalization of SGDA for minimax problems were studied by Lei et al. (2021) in different forms for convex and nonconvex, smooth, and nonsmooth cases, and by Farnia and Ozdaglar (2021) with focus on the smooth cases.

**DP-SGD and DP-SGDGA.** DP-SGD was shown to attain the optimal excess population risk \( O\left(1/\sqrt{n} + \sqrt{d \log(1/\delta)/n\epsilon} \right) \) in Bassily et al. (2019, 2020); Wang et al. (2021, 2020a) for the convex case. For nonconvex objectives, Wang et al. (2019a) studied the DP Gradient Langevin Dynamics, and Zhang et al. (2021b) studied a multi-stage type of DP-SGD assuming the weakly-quasi-convexity and PL condition. In Xie et al. (2018); Zhang et al. (2018), DP-SGD and its variants together with clipping techniques were employed to train differentially private GANs which showed promising results in applications. However, no utility analysis was given there. Boob and Guzmán (2021) focused on the noisy stochastic extragradient method with DP constraints for minimax problems in the convex-concave and smooth settings and provided its utility analysis using variational inequality (VI) and stability approaches.
2 Problem Formulation

In this section, we introduce necessary notations and the DP-SGDA algorithm.

2.1 Assumptions and Notations

Firstly, we introduce necessary assumptions and notations. A function $h : \mathcal{W} \rightarrow \mathbb{R}$ is said to be convex if, for all $w, w' \in \mathcal{W}$, there holds $h(w') \geq h(w) + \langle \nabla h(w), w - w' \rangle$ where $\nabla$ is the gradient operator and $\langle \cdot, \cdot \rangle$ is the inner product. We say $h$ is $\rho$-strongly-convex if $h - \frac{\rho}{2}\|w\|^2$ is convex, $h$ is concave if $-h$ is convex, and $\rho$-strongly-concave if $-h - \frac{\rho}{2}\|w\|^2$ is convex. Let $[n] := \{1, 2, \ldots, n\}$.

**Definition 1.** Given a function $h : \mathcal{W} \times \mathcal{V} \rightarrow \mathbb{R}$. We say $h$ is convex-concave if for any $v \in \mathcal{V}$, the function $w \mapsto h(w, v)$ is convex and for any $w \in \mathcal{W}$, the function $v \mapsto h(w, v)$ is concave.

**Assumption 1 (A1).** The function $f$ is said to be Lipschitz continuous if there exist $G_w, G_v > 0$ such that, for any $w, w' \in \mathcal{W}, v, v' \in \mathcal{V}$ and $z \in \mathcal{Z}$, $\|f(w, v; z) - f(w', v; z)\|_2 \leq G_w\|w - w'\|_2$, and $\|f(w, v; z) - f(w', v'; z)\|_2 \leq G_v\|v - v'\|_2$.

**Assumption 2 (A2).** For randomly drawn $j \in [n]$, the gradients $\nabla_w f(w, v; z_j)$ and $\nabla_v f(w, v; z_j)$ have bounded variances $B_w$ and $B_v$, respectively.

**Assumption 3 (A3).** The function $f$ is said to be smooth if it is continuously differentiable and there exists a constant $L > 0$ such that for any $w, w' \in \mathcal{W}$, $v, v' \in \mathcal{V}$ and $z \in \mathcal{Z}$,

$$\left\| \nabla_w f(w, v; z) - \nabla_w f(w', v; z) \right\|_2 \leq L \left\| w - w' \right\|_2,$$

We also need the Polyak-Lojasiewicz (PL) condition (Polyak 1964).

**Definition 2.** A function $h : \mathcal{W} \rightarrow \mathbb{R}$ satisfies the PL condition if there exists a constant $\mu > 0$ such that, for any $w \in \mathcal{W}$, $\frac{1}{2}\|\nabla h(w)\|^2 \geq \mu(h(w) - \min_{w' \in \mathcal{W}} h(w'))$.

We refer to Karimi et al. (2016) for a nice discussion of this condition and other general conditions that allow the global convergence of gradient descent.

2.2 DP-SGDA Algorithm

We now move on to the definition of differential privacy and the description of DP-SGDA. Differential privacy was introduced by Dwork et al. (2006, 2014). We say that two datasets $S, S'$ are neighboring datasets if they differ by at most one example.

**Definition 3 (Differential Privacy).** A (randomized) algorithm $A$ is called $(\epsilon, \delta)$-differentially private (DP) if, for all neighboring datasets $S, S'$ and for all events $O$ in the output space of $A$, the following holds

$$\mathbb{P}[A(S) \in O] \leq e^\epsilon \mathbb{P}[A(S') \in O] + \delta.$$

### Algorithm 1 Differentially Private Stochastic Gradient Descent (DP-SGDA) Method

1. **Inputs:** data $S = \{z_i : i \in [n]\}$, privacy budget $\epsilon, \delta$, number of iterations $T$, learning rates $\eta_{w,t}, \eta_{v,t}$, $t = 1, \ldots, T$
2. Compute noise parameters $\sigma_w$ and $\sigma_v$ based on Eq. (3)
3. for $t = 1$ to $T$
4. Sample a mini-batch $I_t = \{i_1, \cdots, i_m \in [n]\}$ uniformly with replacement and compute gradients $g_{w,t} = \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_j})$, $g_{v,t} = \frac{1}{m} \sum_{j=1}^{m} \nabla_v f(w_t, v_t; z_{i_j})$
5. Sample independent noises $\xi_t \sim \mathcal{N}(0, \sigma_w^2 I_{d_t})$ and $\zeta_t \sim \mathcal{N}(0, \sigma_v^2 I_{d_t})$
6. Update

$$w_{t+1} = \Pi_{W}(w_t - \eta_{w,t}(g_{w,t} + \xi_t))$$
$$v_{t+1} = \Pi_{V}(v_t + \eta_{v,t}(g_{v,t} + \zeta_t))$$
7. end for
8. **Outputs:** $(w_T, v_T) = \left(\frac{1}{T} \sum_{t=1}^{T} (w_t, v_t)\right)$ or $(w_T, v_T)$

Specifically, our aim is to design a randomized algorithm satisfying $(\epsilon, \delta)$-DP which solves the empirical minimax problem:

$$\min_{w \in \mathcal{W}} \max_{v \in \mathcal{V}} \left\{ F_S(w, v) = \frac{1}{n} \sum_{i=1}^{n} f(w, v; z_i) \right\}.$$  \hspace{1cm} (2)

Notice that in the standard ERM problem, which involves the minimization only with respect to $w$, DP-SGD (Wu et al., 2017; Song et al., 2013; Bassily et al., 2019; Wang et al., 2020a e.g.) uses the gradient perturbation at each iteration. Specifically, at each iteration of this algorithm, a randomized gradient estimated from a random subset (mini-batch) of $S$ is perturbed by a Gaussian noise and then the model parameter is updated based on this noisy gradient.

Following the same spirit, DP-SGDA (e.g. Xie et al., 2018; Zhang et al., 2018) adds Gaussian noises per iteration to the randomized gradient mapping $(g_{w,t}, g_{v,t}) = (\frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_j}), \frac{1}{m} \sum_{j=1}^{m} \nabla_v f(w_t, v_t; z_{i_j}))$ where the index of example $z_{i_j}$ is from the mini-batch $I_t$. Then, the primal variable $w$ is updated by gradient descent based on the noisy gradient $g_{w,t} + \xi_t$ and the dual variable $v$ is updated by gradient ascent based on the noisy gradient $g_{v,t} + \zeta_t$. The pseudo-code for DP-SGDA is given in Algorithm 1. The noise levels $\sigma_w$, $\sigma_v$ are given by (3) which will be specified soon in Section 3 in order to guarantee $(\epsilon, \delta)$-DP. The notations $\Pi_W(\cdot)$ and $\Pi_V(\cdot)$ denote the projections to $W$ and $V$, respectively. From now on, the notation...
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A denotes the DP-SGDA algorithm and its output is denoted by \( A(S) = (A_w(S), A_v(S)) \).

### 2.3 Measures of Generalization (Utility)

Since the model \( A(S) \) is only trained based on the training data \( S \), its empirical behavior as measured by \( \mathbb{F}_S \) may not generalize well on test data. Our goal is to investigate the statistical (generalization/utility) behavior of \( A(S) \) on the test data in terms of some population risk. However, unlike the standard statistical learning theory (SLT) setting where there is only a minimization of \( w \), we have different measures of population risk due to the minimax structure (Zhang et al., 2021a; Lei et al., 2021). Let \( \mathbb{E} \) denote the expectation with respect to the randomness of algorithm \( A \) and data \( S \). We are particularly interested in the following two metrics.

**Definition 4 (Weak Primal-Dual (PD) Risk).** The weak Primal-Dual population risk of \( A(S) \), denoted by \( \Delta^w(A_w(S), A_v(S)) \), is defined as

\[
\max_{v \in V} \mathbb{E}[F(A_w(S), v)] - \min_{w \in W} \mathbb{E}[F(w, A_v(S))].
\]

The corresponding weak PD empirical risk, denoted by \( \Delta^w_n(A_w(S), A_v(S)) \), is defined by

\[
\max_{v \in V} \mathbb{E}[\mathbb{F}_S(A_w(S), v)] - \min_{w \in W} \mathbb{E}[\mathbb{F}_S(w, A_v(S))].
\]

**Definition 5 (Primal Risk).** The primal population and empirical risks of \( A(S) \) are respectively defined by

\[
R(A_w(S)) = \max_{v \in V} \mathbb{E}[F(A_w(S), v)],
\]

\[
R_n(A_w(S)) = \max_{v \in V} \mathbb{E}[\mathbb{F}_S(A_w(S), v)].
\]

We refer to \( R(A_w(S)) - \min_{w \in W} R(w) \) as the excess primal population risk.

When \( F \) is strongly-convex-strongly-concave, the point distance from the model \( (A_w(S), A_v(S)) \) to the true saddle point \((w^*, v^*) \) is given by

\[
\mathbb{E}||A_w(S) - w^*||^2 + ||A_v(S) - v^*||^2 \leq \mathcal{O}(\Delta^w(A_w(S), A_v(S)))
\]

where \( || . || \) is the Euclidean norm. The primal risk is more meaningful when one is concerned about the risk with respect to the primal variable.

### 3 Main Results

In this section, we present our main theoretical results for DP-SGDA. For the privacy guarantee, we leverage the moments accountant method (Abadi et al., 2016), which implies tight privacy loss for adaptive Gaussian mechanisms with amplification by subsampling. Below we summarize a specific version of this method that suffices for our purposes.

**Theorem 1.** Let (A1) hold true. Then, there exist constants \( c_1, c_2, c_3 \) so that given the mini-batch size \( m \) and total iterations \( T \), for any \( \epsilon < c_1 m^2 T/n^2 \), Algorithm 1 is \((\epsilon, \delta)\)-differentially private for any \( \delta > 0 \) if we choose

\[
\sigma_w = c_2 G_w \sqrt{T \log\left(\frac{\delta}{\epsilon}\right)} \quad \sigma_v = c_3 G_v \sqrt{T \log\left(\frac{\delta}{\epsilon}\right)}.
\]

The proof of Theorem 1 is given in Appendix [B](#).

**Remark 1.** In practice, given privacy budget \( \epsilon, \delta \) and parameters \( m, T \), the constant \( c_2 \) and hence \( \sigma \) can be found by grid search (Abadi et al., 2016). Here we provide a set of parameters that satisfies the condition in that reference and our Theorem 4. That is, by choosing \( \epsilon \leq 1, \delta \leq 1/n^2 \) and \( m = \max(1, n \sqrt{\epsilon/(4T)}) \), then we have explicit values for the variances as \( \sigma_w = 8 G_w \sqrt{T \log(1/\delta)}, \sigma_v = 8 G_v \sqrt{T \log(1/\delta)} \).

**Remark 2.** Our Algorithm 1 allows the application of independent noises \( \xi_t, \zeta_t \) with different \( \sigma_w, \sigma_v \), respectively. In Boob and Guzmán (2021), a uniform \( \sigma \) is used (Theorem 5.4 there) for both primal and dual variables. In many examples, the primal and dual gradients \( \nabla_w f(w_t, v_t, z_{i_t}), \nabla_v f(w_t, v_t, z_{i_t}) \) enjoy different Lipschitz constants \( (\ell_2, \text{sensitivity}) \). Therefore, our treatment leads to a more delicate way of calibrating the variances of the Gaussian noises. As we shall see in the experiments in Section 4, this treatment enables Algorithm 1 to achieve better performance.

In the subsequent subsections, we present our main contribution of this paper, i.e., the utility bounds of DP-SGDA for the convex-concave and nonconvex-strongly-concave cases, respectively.

#### 3.1 Convex-Concave Case

In this subsection, we present the utility bound of DP-SGDA for the convex-concave case in terms of the weak PD risk of the output \((\bar{w}_T, \bar{v}_T)\) of Algorithm 1.

**Theorem 2.** Assume the function \( f \) is convex-concave, \( \max_{w \in W} ||w||_2 \leq D_w \) and \( \max_{v \in V} ||v||_2 \leq D_v \). Let the stepsizes \( \eta_{w,t} = \eta_{v,t} = \eta \) for all \( t \in [T] \) with some \( \eta > 0 \).

a) Suppose (A1) and (A3) hold true. If we choose \( T \approx n \) and \( \eta \approx 1/\{\max\{\sqrt{n}, \sqrt{d \log(1/\delta)}\}/\epsilon\} \), then Algorithm 1 satisfies

\[
\Delta^w(\bar{w}_T, \bar{v}_T) = \mathcal{O}\{\max\{\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)}, \frac{1}{n \epsilon}\}\},
\]

where \( d = \max\{d_1, d_2\} \).

b) Suppose (A1) holds true. If we choose \( T \approx n^2 \) and \( \eta \approx 1/\{n \max\{\sqrt{n}, \sqrt{d \log(1/\delta)}\}/\epsilon\} \), then Algorithm 1 satisfies

\[
\Delta^w(\bar{w}_T, \bar{v}_T) = \mathcal{O}\{\max\{\frac{1}{\sqrt{n}} \sqrt{d \log(1/\delta)}, \frac{1}{n \epsilon}\}\}.
\]
Its detailed proof can be found in Appendix C. The proof mainly relies on the concept of stability (Bousquet and Elisseeff, 2002; Charles and Papailiopoulos, 2018; Hardt et al., 2016; Kuzborskij and Lampert, 2018). Specifically, the weak PD population risk can be decomposed as follows:

$$\Delta^w(\tilde{w}_T, \tilde{v}_T) = \Delta^w(\tilde{w}_T, \tilde{v}_T) - \Delta^w_\forall(\tilde{w}_T, \tilde{v}_T) + \Delta^w_\forall(\tilde{w}_T, \tilde{v}_T),$$

(4) where the term $\Delta^w(\tilde{w}_T, \tilde{v}_T) - \Delta^w_\forall(\tilde{w}_T, \tilde{v}_T)$ is the generalization error and the term $\Delta^w_\forall(\tilde{w}_T, \tilde{v}_T)$ is the optimization error.

The estimation for the optimization error can be conducted by standard techniques (e.g., Nemirovski et al., 2009). We give a self-contained proof in Appendix C.1. The generalization error is estimated using a concept of weak stability (Lei et al., 2021). Specifically, we say the randomized algorithm $A$ is $\varepsilon$-weakly-stable if, for any neighboring sets $S, S'$ differing at one single datum, there holds

$$\sup_{z \in V} \mathbb{E}_A[f(A_w(S), v; z) - f(A_w(S'), v; z)] + \sup_{w \in W} \mathbb{E}_A[f(w, A_w(S); z) - f(w, A_w(S'); z)] \leq \varepsilon.$$  

We know from (Lei et al., 2021) that $\varepsilon$-weak-stability implies $\Delta^w(A_w(S), A_v(S)) - \Delta^w_\forall(A_w(S), A_v(S)) \leq \varepsilon$.

In Appendix C.2 we prove the weak stability of DP-SGDA (i.e., Algorithm 1) for both smooth and nonsmooth cases. Putting the estimations for the optimization error and generalization error into (4) can yield the bound in Theorem 2. We end this subsection with some remarks.

**Remark 3.** The same optimal utility bound $O(\max\{1, \frac{1}{\sqrt{n}}, \frac{d \log(1/\delta)}{n\varepsilon}\})$ was claimed in Boob and Guzmán (2021) in terms of the VI gap (Theorem 5.4 there). By the monotonicity assumption, their results imply the same bound on the primal-dual risk $\Delta(A_w(S), A_v(S)) = \mathbb{E}\max_v F(A_w(S), v) - \min_w F(w, A_v(S))$ of the minimax problem (1). Such measure of generalization seems to be stronger than ours since $\Delta^w(A_w(S), A_v(S)) \leq \Delta(A_w(S), A_v(S))$. However, the discussion between stability and generalization there is not rigorous. Indeed, borrowing their notations, equation (4.3) from their paper, i.e., $\mathbb{E}_{\beta_j}(F(u; \beta_j), A(S) - u) \leq \mathbb{E}_{\beta_j}(F(u; \beta_j), A(S) - u) + \mathbb{E}(\|A(S) - A(S')\|)$, only holds when $u$ is independent of $\beta_j$. It does not hold when (4.3) is applied in (4.5) with $u_i = \arg\max_{u \in W} \mathbb{E}_{\beta_i}(F(u; \beta_i), A(S) - u)$ since it depends on $\beta_j$.

**Remark 4.** Besides the practical advantage mentioned in Remark 3, our results also possess two theoretical gains compared to Boob and Guzmán (2021). Firstly, when the smoothness assumption holds, Part a) in our Theorem 3 shows the optimal utility with $T = O(n)$ gradient computations, while their single-looped algorithm (Algorithm 1 there) requires $T = O(n^2)$ gradient computations in their Theorem 5.4. Boob and Guzmán (2021) further improved the gradient complexity to $O(n)$ in Theorem 7.4, which, however, requires an extra subroutine algorithm (inner-loop) (Algorithm 2 there). Secondly, we also derive the same optimal bound with only Lipschitz continuous assumption for the nonsmooth case which was not addressed in Boob and Guzmán (2021).

### 3.2 Nonconvex-Strongly-Concave Case

Throughout this subsection, we assume that $f$ is nonconvex-strongly-concave. In this case, we can present utility bounds of DP-SGDA in terms of the primal excess risk, i.e., $R(w_T) - \min_{w \in W} R(w)$, where $w_T$ is the last iterate of Algorithm 1. We also assume that the saddle point of the empirical minimax problem exists, i.e., there exists $(\hat{w}_S, \hat{v}_S)$ such that, for any $w \in W$ and $v \in V$,

$$F_S(\hat{w}_S, v) \leq F_S(\hat{w}_S, \hat{v}_S) \leq F_S(w, \hat{v}_S).$$

Later on, we use the notation $(\hat{w}_S, \hat{v}_S) \in \arg\min_{w \in W} \max_{v \in V} F_S(w, v)$ to indicate that $(\hat{w}_S, \hat{v}_S)$ is a saddle point.

To estimate the primal excess risk, we define $R_S = \min_{w \in W} R_S(w)$, and $R^* = \min_{w \in W} R(w)$. Then, for any $w^* \in \arg\min_w R(w)$ we have the error decomposition:

$$
\mathbb{E}[R(w_T) - R^*] = \mathbb{E}[R(w_T) - R_S(w_T)] + \mathbb{E}[R_S(w_T) - R^*] + \mathbb{E}[R_S(w^*) - R(w^*)] \\
\leq \mathbb{E}[R(w_T) - R_S(w_T)] + \mathbb{E}[R_S(w^*) - R(w^*)] \\
+ \mathbb{E}[R_S(w_T) - R^*],
$$

(5) where the last inequality follows from the fact that $R_S - R_S(w^*) \leq 0$ since $R_S^* = \min_{w \in W} R_S(w)$. The term $\mathbb{E}[R(w_T) - R_S(w_T)] + \mathbb{E}[R_S(w^*) - R(w^*)]$ is called the generalization error which measures the discrepancy between the primal population risk and the empirical one. The term $\mathbb{E}[R_S(w_T) - R^*]$ is the optimization error which characterizes the discrepancy between the primal empirical risk of an output of Algorithm 1 and the least possible one. The estimations for these two errors are described as follows.

**Generalization Error.** We present the bound for the generalization error which is proved again using the stability approach.

We begin with a discussion of the saddle points. While the saddle point $(\hat{w}_S, \hat{v}_S)$ may not be unique,
Assumption 6. the sense of "optimization can help generalization". Lei and Ying, 2021) for the minimization problems in DP-SGDA, assume that \( \Omega_S = \{ \hat{w}_S : (\hat{w}_S, \hat{v}_S) \in \arg\min_{w \in W} \max_{v \in V} F_S(w, v) \} \), where the equivalence between these two sets follows from the uniqueness of \( v_S \) and the fact that \( \hat{w}_S \in \arg\min_{w \in W} F_S(w, \hat{v}_S) \) for any saddle point \( (\hat{w}_S, \hat{v}_S) \).

Recall that \( w_T \) is the iterate of DP-SGDA at time \( T \) based on the training data set \( S \). Likewise, we denote by \( w_T' \) based on the training set \( S' \) which differs from \( S \) at one single datum. Due to the possibly multiple saddle points, we need the following critical assumption for estimating the generalization error.

**Assumption 4 (A4).** For the (randomized) algorithm DP-SGDA, assume that \( \pi_S(\pi_S(w_T)) = \pi_S(w_T') \) for any neighboring sets \( S \) and \( S' \).

Now we can state the results on the generalization error.

**Theorem 3** (Generalization Error). Suppose Assumptions (A1), (A3) (A4) hold true, and assume the function \( f(w, \cdot; z) \) is \( \rho \)-strongly concave and \( F_S(\cdot, v) \) satisfies \( \mu \)-PL condition. Let \( \mu = L/\rho. \) If \( \mathbb{E}[R_S(w_T) - R^*] \leq \varepsilon_T \), then there holds

\[
\mathbb{E}[R(w_T) - R(w^*)] \leq 4G_w^2 \frac{2\mu}{\rho n},
\]

and

\[
\mathbb{E}[R_S(w^*) - R(w^*)] \leq 4G_w^2 \frac{1}{\rho n}.
\]

The proof of Theorem 3 is provided in Appendix D.1.

**Remark 5.** The generalization error bounds given in Theorem 3 indicate that if the optimization error \( \mathbb{E}[R_S(w_T) - R^*] \) is small then the generalization error will be small. This is consistent with the observation in the stability and generalization analysis of SGD (Charles and Papailiopoulos, 2018; Hardt et al., 2016; Lei and Ying, 2021) for the minimization problems in the sense of "optimization can help generalization".

**Remark 6.** Assumption (A4) was introduced in Charles and Papailiopoulos (2018) for studying the stability of SGD in the non-convex case which only involves the minimization over \( w \). (A4) holds true if the saddle point is unique or the two sets of saddle points based on \( S \) and \( S' \), i.e. \( \Omega_S \) and \( \Omega_S' \), do not change too much. Indeed, we can see from the proof given in Appendix D.1 that, if we do not assume (A4), then one restates the estimation in Theorem 3 as

\[
\mathbb{E}[R(w_T) - R_S(w_T)] = O\left( \frac{G_w^2}{\rho n^2} + \frac{G_v^2}{\mu^2} + \frac{1}{\rho n} \right) + \mathbb{E} \left[ \sup_{w_1 \in \Omega_S, w_2 \in \Omega_{S'}} \| w_1 - w_2 \|_2 \right].
\]

We notice the assumption (A4) might be difficult to verify except for some special cases (e.g., \( F_S \) is strongly-convex and strongly-concave). It remains an question to us on how to relax the condition (A4) in the proof.

**Optimization Error.** We now establish the following convergence rate for DP-SGDA.

**Theorem 4** (Optimization Error). Suppose Assumptions (A1) and (A2) hold true, and assume the function \( F_S(w, \cdot) \) is \( \rho \)-strongly concave and \( F_S(\cdot, v) \) satisfies \( \mu \)-PL condition. Let \( \mu = L/\rho. \) If we choose \( \eta_w, \tau = \frac{1}{\mu} \) and \( \eta_v, \tau = \frac{\kappa^2}{\mu^2 T^{2/3}} \), then

\[
\mathbb{E}[R_S(w_{T+1}) - R^*_S] = O\left( \frac{\kappa^3}{\mu^2} \left( \frac{1/m + d \sigma_v^2 + \sigma_w^2}{T^{2/3}} \right) \right).
\]

We provide the proof of Theorem 3 in Appendix D.2.

In the non-private setting, i.e. \( \sigma_w = \sigma_v = 0 \), Theorem 4 implies that the convergence rate in terms of the primal empirical risk is of the order \( \mathcal{O}(\frac{\kappa^3}{\mu^2 T^{2/3}}) \), which is a new result even in the non-private case as far as we are aware of.

In Lin et al. (2020), the local convergence of SGDA in the non-private case was proved in terms of the metric \( \mathbb{E}_T[||\nabla R_S(w_T)||^2] \) where \( \tau \) is chosen uniformly at random from the set \( \{1, 2, \ldots, T\} \). Our analysis is much more involved since it proves the global convergence of the last iterate \( w_T \). Our main idea is to prove the coupled recursive inequalities for two terms, i.e.,\( a_t = R_S(w_t) - R^*_S \) and \( b_t = ||v_t - v_S(w_t)||^2 \) where \( v_S(w_t) = \arg\max_{v \in V} F_S(w_t, v) \), and then carefully derive the the convergence rate for \( a_t + b_t \) by choosing \( \lambda_t \) appropriately. The convergence rate and its proof can be of interest in their own right. One can find more detailed arguments in Appendix D.2.

**Final Utility Bound.** We can derive the following utility bound for DP-SGDA by combining the results in Theorems 3 and 4.

**Theorem 5.** Under the same assumptions of Theorem 3 if we choose \( T \sim n \), \( \eta_w, \tau = \frac{1}{n^{1/3}} \) and \( \eta_v, \tau = \frac{\kappa^2}{n^{1/3} \mu^{2/3}} \), then we have

\[
\mathbb{E}[R(w_{T+1}) - R^*] = O\left( \frac{\kappa^{2.75}}{n^{1/3}} \left( \frac{1}{n^{1/3}} + \sqrt{d \log(1/\delta)} \right) \right).
\]

The proof can be found in Appendix D.3.

**4 Experiments.** In this section, we evaluate the performance of DP-SGDA by taking AUC maximization as an example. Due to space limitation, we present the most significant information and results of our experiments while
more detailed information and additional results are given in Appendix [E] and [F]

4.1 Experimental Settings

Baseline Model. We perform experiments on the problem of AUC maximization with the least square loss to evaluate the DP-SGDA algorithm in linear and non-linear settings (two-layer multilayer perceptron (MLP)). In this case, AUC maximization can be formulated as

$$\min_{\theta \in \Theta} \mathbb{E}_{z,x}[(1 - h(\theta; x) + h(\theta; x'))^2|y = 1, y' = -1],$$

where $h : \Theta \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the scoring function. As shown in [Ying et al., 2016], it is equivalent to a minimax problem:

$$\min_{w=(\theta,a,b)} \max_{v} \mathbb{E}_{z} [f(\theta,a,b,v;z)],$$

where $f = (1 - p)(h(\theta; x) - a)^2[y = 1] + p(h(\theta; x) - b)^2[y = -1] + (1 + v)(ph(\theta; x)1[y = -1] - (1 - p)h(\theta; x)1[y = 1]) - p(1 - pv)^2$ and $p = P[y = 1]$. When $h$ is a linear function, the AUC learning objective above is convex-strongly-concave. On the other hand, when $h$ is a MLP function, it becomes a nonconvex-strongly-concave minimax problem. In addition, following [Liu et al., 2020], we use Leaky ReLu as an activation function for MLP. It was shown in their paper the empirical AUC objective satisfies the PL condition with this choice of $h$. Without a special statement, we set 256 as the number of hidden units in MLP and 64 as the mini-batch size during the training.

Datasets and Evaluation Metrics. Our experiments are based on three popular datasets, namely icjnn1 [Chang and Lin [2011]], MNIST [LeCun et al., 1998], and Fashion-MNIST [Xiao et al., 2017] that have been used in previous studies. For MNIST and Fashion-MNIST, following Gao et al. [2013]; Ying et al. [2016], we transform their classes into binary classes by randomly partitioning the data into two groups, each with an equal number of classes. For icjnn1, we randomly split its original training set into new training (80%) and testing (20%) sets. For MNIST and Fashion-MNIST, we use their original training set and testing set. For each method, the reported performance is obtained by averaging the AUC scores on the test set according to 5 random seeds (for initial $w$ and $v$, sampling and noise generation).

Privacy Budget Settings. In the experiments, we set up five privacy levels: $\epsilon \in \{0.1, 0.5, 1, 5, 10\}$. We also consider three different $\delta$ from $\{1e-4, 1e-5, 1e-6\}$. Due to space limitation, we only report the performance when $\delta = 1e-6$. More results can be found in Appendix [E]. To estimate the Lipschitz constants $G_w$ and $G_v$ (in Theorem 1), we first run the algorithms without adding noise then calculate the maximum gradient norms of AUC loss w.r.t $w$ and $v$ and assign them as $G_w$ and $G_v$, respectively. According to these parameters, we calculate the noise parameter $\sigma$ by applying autodp [1] which is widely used in the existing works (Wang et al., 2019b).

4.2 Results

General AUC Performance vs Privacy. The general performance of all methods under linear and MLP settings of AUC optimization is shown in Table [1] Since the standard deviation of the AUC performance is around [0, 0.1%] and the difference between different methods is very small, we only report the average AUC performance. First, without adding noise into gradients, we can find the NSEG method and our DP-SGDA method have similar performance under the linear case. Furthermore, we can find the performance of the DP-SGDA with MLP model can outperform linear models on all datasets. This is because the non-linear model can learn more information among features than linear models. Second, by adding noise into the gradients, we can find the AUC performance of all models is decreased on all datasets. However, by increasing the privacy budget $\epsilon$, the AUC performance is increased. The reason is that $\epsilon$ and $\sigma$ have opposite trends according to equation (3). The relation between $\epsilon$ and AUC score also verifies our Theorem 2 and 3. Third, to verify our statement in Remark 2 and 3, we compare the $\sigma$ values from NSEG and DP-SGDA on all datasets in Figure [1](a). From the figure, it is clear that the $\sigma$ from NSEG is larger than ours in all $\epsilon$ settings since it is calibrated based on the gradients’ sensitivity from both $w$ and $v$. In fact, the sensitivity w.r.t. $v$ is small as it is a one-dimensional variable for AUC maximization. Therefore, NSEG leads to overestimate on the noise addition towards $v$. From Table [1] we observe our DP-SGDA achieves better AUC score than NSEG under the same privacy budget.

Different Hidden Units. In DP-SGDA under the MLP setting, the hidden unit is one of the most important factors affecting the model performance. Therefore, we compare the AUC performance with respect to the different hidden units in Figure [1](b). If we provide

https://github.com/yuxiangw/autodp
| Datasets  | Methods | ijcnn1 | MLP | MNIST | Fashion-MNIST |
|----------|---------|--------|------|-------|---------------|
|          | Linear  | NSEG   | DP-SGDA | Linear  | NSEG   | DP-SGDA | Linear  | NSEG   | DP-SGDA |
| Original | 92.191  | 92.448 | 96.609 | 93.306 | 93.349 | 99.546 | 96.552  | 96.523 | 98.020  |
| $\epsilon=0.1$ | 90.106 | 91.110 | 92.763 | 91.247 | 91.858 | 97.878 | 95.446 | 95.468 | 95.692 |
| $\epsilon=0.5$ | 90.346 | 91.357 | 95.840 | 91.324 | 92.058 | 98.656 | 95.530 | 95.816 | 96.988 |
| $\epsilon=1$  | 90.355 | 91.371 | 96.167 | 91.330 | 92.070 | 98.705 | 95.534 | 95.834 | 97.102 |
| $\epsilon=5$  | 90.363 | 91.386 | 96.294 | 91.334 | 92.078 | 98.742 | 95.538 | 95.848 | 97.198 |
| $\epsilon=10$ | 90.363 | 91.386 | 96.297 | 91.334 | 92.080 | 98.747 | 95.539 | 95.850 | 97.213 |

Table 1: Comparison of AUC performance in NSEG and DP-SGDA (Linear and MLP settings) on three datasets with different $\epsilon$ and $\delta=1e-6$. The “Original” means no noise ($\epsilon = \infty$) is added in the algorithms.

![Figure 1](image.png)

Figure 1: (a) Comparison of $\sigma$ for NSEG and DP-SGDA (Linear setting) on three datasets with different $\epsilon$ and $\delta=1e-6$. (b) Comparison of AUC performance for SGD and DP-SGDA in MLP settings on three datasets with different hidden units and $\epsilon=1$ and $\delta=1e-6$. (c) Comparison of AUC performance for DP-SGDA (Linear and MLP settings) on three datasets with different batch size and $\epsilon=1$ and $\delta=1e-6$.

a small number of hidden units, the model will suffer from poor generalization capability. Using a large number of hidden units will make the model easier to fit the training set. For SGD (non-private) training, it is often helpful to apply a large number of hidden units, as long as the model does not overfit. In agreement with this intuition, we find the model performance improves with increasing hidden units in Figure 1(b). However, for DP-SGDA training, more hidden units increase the sensitivity of the gradients, which leads to more noise added at each update. Therefore, in contrast to the non-private setting, we find the AUC performance decreases when the number of hidden units increases.

**Different Mini-Batch Size.** From Theorem 1 and Theorem 4, we find mini-batch size can influence the Gaussian noise variances $\sigma^2_w$ and $\sigma^2_v$ as well as the convergence rate. Selecting the mini-batch size must balance two conflicting objectives. On one hand, a small mini-batch size may lead to sub-optimal performance. On the other hand, for large batch sizes, the added noise has a smaller relative effect. Therefore, we show the AUC score for DP-SGDA with different mini-batch sizes in Figure 1(c). The experimental results show that the mini-batch size has a relatively large impact on the AUC performance when the mini-batch size is small.

## 5 Conclusion

In this paper, we have used algorithmic stability to conduct utility analysis of the DP-SGDA algorithm for minimax problems under DP constraints. For the convex-concave setting, we proved that DP-SGDA can attain an optimal rate $O\left(\frac{1}{\sqrt{n\epsilon}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right)$ in terms of the weak primal-dual population risk while providing $(\epsilon, \delta)$-DP for both smooth and nonsmooth cases. For the nonconvex-strongly-concave case, assuming that the empirical risk satisfies the PL condition we proved the excess primal population risk of DP-SGDA can achieve a utility bound $O\left(\frac{1}{n^{1/3}} + \frac{\sqrt{d\log(1/\delta)}}{n^{5/6}\epsilon}\right)$. Experiments on three benchmark datasets illustrate the effectiveness of DP-SGDA.

For future work, it would be interesting to improve the utility bound for the nonconvex-strongly-convex setting. It also remains unclear to us how to establish the utility bound for DP-SGDA when gradient clipping techniques are enforced at each iteration.

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A Motivating Examples

We provide several examples that can be formulated as a stochastic minimax problem. All these examples have corresponding empirical minimax formulations.

**Generative Adversarial Networks (GANs).** GAN is introduced in Goodfellow et al. (2014) which can be regarded as a game between a generator network $G_v$ and a discriminator network $D_w$. The generator network produces synthetic data from random noise $\xi$, while the discriminator network discriminates between the true data and the synthetic data. In particular, a popular variant of GAN named as WGAN (Arjovsky et al., 2017) can be written as a minimax problem

$$\min_w \max_v \mathbb{E}[f(w, v; z, \xi)] := \mathbb{E}_z[D_w(z)] - \mathbb{E}_\xi[D_w(G_v(\xi))].$$

DP-SGDA and its variants were employed to train differential private WGANs by Xie et al. (2018).

**AUC Maximization.** Area Under the ROC Curve (AUC) is a widely used measure for binary classification. Optimizing AUC with square loss can be formulated as

$$\min_{\theta} \mathbb{E}_{z, z'}[(1 - h(\theta; x) + h(\theta; x'))^2 | y = 1, y' = -1]$$

where $h : \Theta \times \mathbb{R}^d \to \mathbb{R}$ is the scoring function for the classifier. It has been shown this problem is equivalent to a minimax problem once auxiliary variables $a, b, v \in \mathbb{R}$ are introduced Ying et al. (2016).

$$\min_{\theta, a, b} \max_v F(\theta, a, b, c) = \mathbb{E}_z[f(\theta, a, b, v; z)]$$

where $f = (1 - p)(h(\theta; x) - a)^2 I[y = 1] + (h(\theta; x) - b)^2 I[y = -1] + 2(1 + v)(pb(\theta; x)I[y = -1] - (1 - p)h(\theta; x)I[y = 1]) - p(1 - p)v^2$ and $p = \mathbb{P}[y = 1]$.

**Markov Decision Process (MDP).** Let $A$ be a finite action space. For any $a \in A$, $P(a) \in [0, 1]^{n \times n}$ is the state-transition probability matrix and $r(a) \in [0, 1]^n$ is the vector of expected state-transition rewards. In the infinite-horizon average-reward Markov decision problem, one aims to find a stationary policy $\pi$ to make an infinite sequence of actions and optimize the average-per-time-step reward $\bar{v}$. By classical theory of dynamics programming Puterman (2014), finding an optimal policy is equivalent as solving the fixed-point Bellman equation

$$\bar{v}^* + h^*_i = \max_{a \in A} \left\{ \sum_{j=1}^n (p_{ij}(a)h^*_j + p_{ij}(a)r_{ij}(a)) \right\}, \quad \forall i$$

where $h \in \mathbb{R}^n$ is the difference-of-value vector. Wang (2017) showed that this problem is equivalent to the minimax problem as follow

$$\min_{h \in \mathcal{H}} \max_{\mu \in \mathcal{U}} \mu^T((P(a) - I)h + r(a))$$

where $\mathcal{H}$ and $\mathcal{U}$ are the feasible regions chosen according to the mixing time and stationary distribution.

**Robust Optimization and Fairness.** The aim is to minimize the worst population risks among multiple scenarios:

$$\min_{w \in W} L(w) = \max_{1 \leq i \leq m} \{ \mathbb{E}_z[f_i(w; z)], \ldots, \mathbb{E}_z[f_m(w; z)] \}$$
This problem can be reformulated as the following stochastic minimax problem

$$\min_{\mathbf{w} \in \mathcal{W}} \max_{\mathbf{v} \in \Delta_m} \sum_{i=1}^{m} \mathbb{E}_{\mathbf{z}}[v_i f_i(\mathbf{w}; \mathbf{z})]$$

where $\Delta_m = \{ \mathbf{v} \in \mathbb{R}^n : v_i \geq 0, \sum_{i=1}^{m} v_i = 1 \}$ denotes the $m$-dimensional simplex. Such robust optimization formulation has been recently proposed to address fairness among subgroups \cite{Mohri2019} and federated learning on heterogeneous populations \cite{Li2019}.

**B Proof of Privacy Guarantee**

Next we provide a lemma on the privacy-amplification by subsampling result, which is a direct application of Theorem 1 in \cite{Abadi2016}.

**Definition 6** \cite{Abadi2016}. Given a function $g : \mathbb{Z}^n \rightarrow \mathbb{R}^d$, we say $g$ has $\Delta(g)$ $\ell_2$-sensitivity if for any neighboring datasets $S, S'$ we have

$$\|g(S) - g(S')\|_2 \leq \Delta(g).$$

**Definition 7** \cite{Abadi2016}. For an (randomized) algorithm $A$, and neighboring datasets $S, S'$ the $\lambda$-th moment is given as

$$\alpha_A(\lambda, S, S') = \log \mathbb{E}_{\mathcal{O} \sim A(S)} \left[ \left( \mathbb{P}[A(S) = O] - \mathbb{P}[A(S') = O] \right)^\lambda \right].$$

The moments accountant is then defined as

$$\alpha_A(\lambda) = \sup_{S, S'} \alpha_A(\lambda, S, S').$$

**Lemma 1** \cite{Abadi2016}. Consider a sequence of mechanisms $\{A_t\}_{t \in [T]}$ and the composite mechanism $A = (A_1, \cdots, A_T)$. Then for any $\lambda$,

$$\alpha_A(\lambda) = \sum_{t=1}^{T} \alpha_{A_t}(\lambda).$$

**Lemma 2** \cite{Abadi2016}. For any $\epsilon$, the mechanism $A$ is $(\epsilon, \delta)$ differentially private for

$$\delta = \min_{\lambda} \alpha_A(\lambda) - \epsilon \lambda.$$

**Lemma 3** \cite{Abadi2016}. Consider a sequence of mechanisms $A_i = g_i(S_i) + \xi_i$ where $\xi_i \sim \mathcal{N}(0, \sigma^2 I)$. Here each function $g_i : \mathbb{Z}^m \rightarrow \mathbb{R}^d$ has $\ell_2$-sensitivity of 1. And each $S_i$ is a subsample of size $m$ obtained by uniform sampling without replacement from $S$, i.e. $S_i \sim (\text{Unif}(S))^m$. There exist constants $c_1$ and $c_2$ so that for any $\epsilon < c_1 m^2 T/n^2$, the composite mechanism $A = (A_1, \cdots, A_T)$ is $(\epsilon, \delta)$-differentially private for any $\delta > 0$ if we choose

$$\sigma \geq \frac{c_2 m \sqrt{T \log(1/\delta)}}{n \epsilon}.$$

**Theorem B.1** (Theorem 1 restated). There exist constants $c_1, c_2$ and $c_3$ so that for any $\epsilon < c_1 T/n^2$, Algorithm 4 is $(\epsilon, \delta)$-differentially private for any $\delta > 0$ if we choose

$$\sigma_\mathbf{w} \geq \frac{c_2 G_\mathbf{w} \sqrt{T \log(1/\delta)}}{n \epsilon} \quad \text{and} \quad \sigma_v \geq \frac{c_3 G_v \sqrt{T \log(1/\delta)}}{n \epsilon}.$$

**Proof.** Let $S = \{z_1, \cdots, z_n\}$ and $S' = \{z'_1, \cdots, z'_n\}$ be two neighboring datasets. At iteration $t$, we first focus on $A^*_t = \frac{1}{m} \sum_{j=1}^{m} \nabla \mathbf{w} f(\mathbf{w}_t, \mathbf{v}_i; \mathbf{z}_{i,j}) + \xi_t$. Since $f(\cdot, \mathbf{v}; \mathbf{z})$ is $G_{\mathbf{w}}$-Lipschitz continuous, it implies for any neighboring datasets $S, S'$,

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \nabla \mathbf{w} f(\mathbf{w}_t, \mathbf{v}_i; \mathbf{z}_{i,j}) - \frac{1}{m} \sum_{j=1}^{m} \nabla \mathbf{w} f(\mathbf{w}_{t}, \mathbf{v}_i; \mathbf{z}'_{i,j}) \right\|_2 \leq \frac{2 G_{\mathbf{w}}}{m}.$$
Therefore we can define \( g_t(S_t) = \frac{1}{2c \sigma_w} \sum_{j=1}^m \nabla_w f(w_t, v_t, z_{i_t}) \) such that \( \Delta(g_t) = 1 \). By Lemma [3], there exist constants \( c_1 \) and \( c_2 \) so that for any \( \epsilon / 2 < c_1 m^2 T / n^2 \), the composite mechanism \( A^w = (A^w_1, \ldots, A^w_T) \) is \((\frac{\epsilon}{2}, \delta)\)-DP for any \( \delta > 0 \) if we choose

\[
\sigma_w = \frac{c_2 \sqrt{T \log(1/\delta)}}{m \epsilon}.
\]

Similarly, since \( A^v = \nabla_w f(w_t, v_t; z_{i_t}) + \zeta_t \) has \( \ell_2\)-sensitivity \( 2G_v / m \), then the choice of \( \sigma_v \) ensures \( A^v = (A^v_1, \ldots, A^v_T) \) is \((\frac{\epsilon}{2}, \delta)\)-DP. According to Lemma [1] and [2] the final output \( A = (A^w_1, A^v_1, \ldots, A^w_T, A^v_T) \) satisfies \((\epsilon, \delta)\)-DP. The proof is complete.

**Proof of Remark 1.** We focus on the case of Lemma [3] as it can easily extend to our Theorem. Following the argument of [Abadi et al. 2016], the algorithm \( A \) is guaranteed to be \((\epsilon, \delta)\)-DP if one can find \( \lambda > 0 \) such that

\[
\frac{\lambda^2 m^2 T}{n^2 \sigma^2} \leq \frac{\lambda \epsilon}{2}, \quad \exp(-\frac{\lambda \epsilon}{2}) \leq \delta, \quad \text{and} \quad \lambda \leq \frac{\sigma^2}{n \sigma^2} = \frac{1}{n}.
\]

Given \( \delta = \frac{1}{n^2} \), the second inequality can be reformulated as \( \lambda \geq \frac{4 \log(n)}{\epsilon} \). Therefore by choosing \( \sigma^2 = \frac{8 m^2 T \log(n)}{n \epsilon^2} \), the first inequality becomes \( \lambda \leq \frac{4 \log(n)}{\epsilon} \), indicating \( \lambda = \frac{4 \log(n)}{\epsilon} \). It suffices to show such choice of \( \lambda \) satisfies the third inequality, which is straightforward by the choice of \( m \) and \( \epsilon \leq 1 \). The proof is complete.

### C Proof for the Convex-Concave Setting in Section 3.1

Recall that the error decomposition (1) given in Section 3.1 that the weak PD risk can be decomposed as follows:

\[
\Delta^w(\bar{w}_T, \bar{v}_T) = \Delta^w(\bar{w}_T, \bar{v}_T) - \Delta^w_S(\bar{w}_T, \bar{v}_T) + \Delta^w_S(\bar{w}_T, \bar{v}_T),
\]

where the term \( \Delta^w(\bar{w}_T, \bar{v}_T) - \Delta^w_S(\bar{w}_T, \bar{v}_T) \) is the generalization error and the term \( \Delta^w_S(\bar{w}_T, \bar{v}_T) \) is the optimization error.

The proof of Theorem 2 involves the estimation of the optimization error which are performed in the subsequent subsection, respectively.

#### C.1 Estimation of Optimization Error

We start by studying the optimization error for Algorithm 1. This is obtained as a direct corollary of [Nemirovski et al. 2009], with the existence of the Gaussian noise’s variance and the mini-batch. Recall that \( d = \max\{d_1, d_2\} \).

**Lemma 4.** Suppose (A1) holds, and \( F_S \) is convex-concave. Let the stepsizes \( \eta_{w, t} = \eta_{v, t} = \eta, \ t \in [T] \) for some \( \eta > 0 \). Then Algorithm 7 satisfies

\[
\mathbb{E}_{\mathcal{A}}[\sup_{v \in \mathcal{V}} F_S(\bar{w}_T, v) - \inf_{w \in \mathcal{W}} F_S(w, \bar{v}_T)] \leq \frac{n G^2_w + G^2_v}{\eta T} + \frac{D^2_w + D^2_v}{\eta T} + \frac{(D_w G_w + D_v G_v)}{\sqrt{m T}} + \eta d (\sigma^2_w + \sigma^2_v).
\]

**Proof.** According to the non-expansiveness of projection and update rule of Algorithm 1 for any \( w \in \mathcal{W} \), we have

\[
\|w_{t+1} - w\|^2 \leq \|w_t - w - \eta \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) \|_2^2
\]

\[
\leq 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) + \xi_t \right) + \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) \|_2^2 + \eta^2 \xi_t \|_2^2
\]

\[
+ 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) \right) + \eta^2 \xi_t \|_2^2
\]

\[
\leq 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + \eta^2 (\sigma^2_w + \sigma^2_v),
\]

\[
\leq 2 \eta^2 (\sigma^2_w + \sigma^2_v)
\]

\[
+ \eta^2 \xi_t \|_2^2 + 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) \right) + 2 \eta^2 (\sigma^2_w + \sigma^2_v),
\]

\[
+ \eta^2 G^2_w + \eta^2 \xi_t \|_2^2 + 2 \eta^2 \left( 1 + \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{i_t}) \right) + 2 \eta (w - w_t, \xi_t).
\]
where in the last inequality we have used \( f(\cdot, v_t, z_{i_t}) \) is \( G_w \)-Lipschitz continuous. According to the convexity of \( F_S(\cdot, v_t) \) we know

\[
2\eta(F_S(w_t, v_t) - F_S(w, v_t)) \leq \|w_t - w\|^2 - \|w_{t+1} - w\|^2 + 2\eta \left( w - w_t, \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + \eta^2 G_w^2 + \eta^2 \|\xi_t\|^2 + 2\eta^2 \left( \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}), \xi_t \right) + 2\eta(w - w_t, \xi_t).
\]

Taking a summation of the above inequality from \( t = 1 \) to \( T \) we derive

\[
2\eta \sum_{t=1}^{T} (F_S(w_t, v_t) - F_S(w, v_T)) \leq 2D_w^2 - 2\eta \sum_{t=1}^{T} \left( w - w_t, \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + T\eta^2 G_w^2 + \eta^2 \sum_{t=1}^{T} \|\xi_t\|^2 + 2\eta^2 \sum_{t=1}^{T} \left( \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}), \xi_t \right) + 2\eta(w - w_t, \xi_t).
\]

It then follows from the concavity of \( F_S(w, \cdot) \) and Schwartz's inequality that

\[
2 \sum_{t=1}^{T} \eta(F_S(w_t, v_t) - F_S(w, v_T)) \leq 2D_w^2 - 2\eta \sum_{t=1}^{T} \left( w - w_t, \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + T\eta^2 G_w^2 + \eta^2 \sum_{t=1}^{T} \|\xi_t\|^2 + 2\eta^2 \sum_{t=1}^{T} \left( \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}), \xi_t \right) + 2\eta(w - w_t, \xi_t).
\]

Since the above inequality holds for all \( w \), we further get

\[
2\eta \sum_{t=1}^{T} (F_S(w_t, v_t) - \min_{w \in \mathcal{W}} F_S(w, v_T)) \leq 2D_w^2 - 2\eta \sum_{t=1}^{T} \left( w - w_t, \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right) + T\eta^2 G_w^2 + \eta^2 \sum_{t=1}^{T} \|\xi_t\|^2 + 2\eta^2 \sum_{t=1}^{T} \left( \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}), \xi_t \right) + 2\eta(w - w_t, \xi_t).
\]

We can take an expectation over both sides of (C.2) and get

\[
2\eta \sum_{t=1}^{T} \mathbb{E}[(F_S(w_t, v_t) - \min_{w \in \mathcal{W}} F_S(w, v_T))] \leq 2D_w^2 + 2D_w \eta \mathbb{E}\left[ \left\| \sum_{t=1}^{T} \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right\|_2 \right] + T\eta^2 G_w^2 + \eta^2 \alpha^2 \sigma_w^2,
\]

where we used that the variance \( \mathbb{E}[\|\xi_t\|^2] = d_t \sigma^2 \), the unbiasedness \( \mathbb{E}[(w_t, \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t))] = 0 \) and the independence \( \mathbb{E}[(\frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}), \xi_t)] = 0, \mathbb{E}[(w - w_t, \xi_t)] = 0. \) According to Jensen’s inequality and \( G_w \)-Lipschitz continuity we further derive

\[
\left( \mathbb{E}\left[ \left\| \sum_{t=1}^{T} \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right\|_2 \right] \right)^2 \leq \mathbb{E}\left[ \left\| \sum_{t=1}^{T} \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{i_t}) - \nabla_w F_S(w_t, v_t) \right\|_2 \right] \leq \frac{T G_w^2}{m}.
\]
Lemma 6. Suppose the function $DP-SGDA$. The lemma is an extension of the uniform argument stability results in Lei et al. (2021) to the case of mini-batch addition does not impact the stability analysis. The stability analysis is given in the following lemma. This

Lemma 5. Without loss of generality, let $z_t = \bar{z}_t$. We say the randomized algorithm $A$ is $\varepsilon$-weakly-stable if, for any neighboring datasets $S, S'$, there holds

$$\sup_z \left( \sup_{v \in V} E_A[f(A_w(S), v; z)] - f(A_w(S'), v; z) \right) \leq \varepsilon.$$  

Lemma 5. [Lei et al. 2021] If $A$ is $\varepsilon$-weakly-stable, then there holds

$$\Delta^w(A_w(S), A_v(S)) - \Delta^w(S, A_v(S)) = \varepsilon.$$  

Since the noise added to the gradient in each iteration is the same for neighboring datasets $S$ and $S'$, the noise addition does not impact the stability analysis. The stability analysis is given in the following lemma. This lemma is an extension of the uniform argument stability results in [Lei et al. 2021] to the case of mini-batch DP-SGDA.

Lemma 6. Suppose the function $F_S$ is convex-concave. Let the stepsizes $\eta_t, \eta_{\epsilon} = \eta$ for some $\eta > 0$.

a) \textbf{Assume (A1) and (A3) hold}, then Algorithm \[ satisfies

$$\Delta^w(w_T, v_T) - \Delta^w(S, v_T) \leq \frac{4\sqrt{2}}{\sqrt{n}} \left( G_w + G_v \right)^2 \eta \exp(2K^2\eta^2/2),$$

b) \textbf{Assume (A1) holds}, then Algorithm \[ satisfies

$$\Delta^w(w_T, v_T) - \Delta^w(S, v_T) \leq 4\sqrt{2} \eta \left( G_w + G_v \right)^2 \left( T + \frac{T}{n} \right).$$

\textbf{Proof.} Without loss of generality, let $S = \{z_1, \ldots, z_n\}, S' = \{z'_1, \ldots, z'_n\}$ be neighboring datasets differing by the last element, i.e. $z_n \neq z'_n$. Let $\{w_t, v_t\}, \{w'_t, v'_t\}$ be the sequence produced by Algorithm \[ w.r.t. $S$ and $S'$, respectively. We first prove Part a). Analogous to the discussions in [Lei et al. 2021], in the case $n \notin I_t$ we have

$$\left\| \left( \begin{array}{c} w_{t+1} - w'_{t+1} \\ v_{t+1} - v'_{t+1} \end{array} \right) \right\|^2 \leq \left( 1 + L^2 \eta^2 \right) \left( \left\| \left( \begin{array}{c} w_t - w'_t \\ v_t - v'_t \end{array} \right) \right\|^2ight).$$
If \( n \in I_t \), then it follows that

\[
\left\| \frac{w_{t+1} - w'_{t+1}}{v_{t+1} - v'_{t+1}} \right\|_2^2 \leq \left( \left\| \frac{w_t - w'_t}{v_t - v'_t} \right\|_2^2 + \frac{m(1 + L^2\eta^2)}{n} \left( \frac{m - 1}{m} \right) \right) \left( 1 + \frac{1 + p}{n} \right) \eta^2 (G_w^2 + G_v^2) \exp \left( \sum_{j=1}^{t} \eta^2 \right) \left( \sum_{k=1}^{t} \exp \left( \frac{L^2 \eta^2 + p/n}{2} \right) \right). \tag{C.6}
\]

where in the last inequality we used the elementary inequality \((a + b)^2 \leq (1 + p)a^2 + (1 + p)b^2\) \((p > 0)\). Since \( I_t \) are drawn uniformly at random with replacement, the event \( n \notin I_t \) happens with probability \( 1 - m/n \) and the event \( n \in I_t \) happens with probability \( m/n \). Therefore, we know

\[
\mathbb{E}_{t} \left[ \left\| \frac{w_{t+1} - w'_{t+1}}{v_{t+1} - v'_{t+1}} \right\|_2^2 \right] \leq \frac{(n - m)(1 + L^2\eta^2)}{n} \left( \frac{m}{n} \left( \frac{m - 1}{m} \right) \right) \left( 1 + \frac{1 + p}{n} \right) \eta^2 (G_w^2 + G_v^2) \exp \left( \sum_{j=1}^{t} \eta^2 \right) \left( \sum_{k=1}^{t} \exp \left( \frac{L^2 \eta^2 + p/n}{2} \right) \right). \tag{C.6}
\]

Applying this inequality recursively, we derive

\[
\mathbb{E}_A \left[ \left\| \frac{w_{t+1} - w'_{t+1}}{v_{t+1} - v'_{t+1}} \right\|_2^2 \right] \leq \frac{4(1 + p)}{n} \eta^2 (G_w^2 + G_v^2) \exp \left( \sum_{j=1}^{t} \eta^2 \right) \left( \sum_{k=1}^{t} \exp \left( \frac{L^2 \eta^2 + p/n}{2} \right) \right). \tag{C.6}
\]

By the elementary inequality \( 1 + a \leq \exp(a) \), we further derive

\[
\mathbb{E}_A \left[ \left\| \frac{w_{t+1} - w'_{t+1}}{v_{t+1} - v'_{t+1}} \right\|_2^2 \right] \leq \frac{4(1 + p)}{n} \eta^2 (G_w^2 + G_v^2) \exp \left( \sum_{j=1}^{t} \eta^2 \right) \left( \sum_{k=1}^{t} \exp \left( \frac{L^2 \eta^2 + p/n}{2} \right) \right). \tag{C.6}
\]

By taking \( p = n/t \) we get

\[
\mathbb{E}_A \left[ \left\| \frac{w_{t+1} - w'_{t+1}}{v_{t+1} - v'_{t+1}} \right\|_2^2 \right] \leq \frac{4e(G_w^2 + G_v^2)(1 + t/n)}{n} \exp \left( \sum_{j=1}^{t} \eta^2 \right) \left( \sum_{k=1}^{t} \eta^2 \right). \tag{C.6}
\]

Now by the Lipschitz continuity and Jensen’s inequality we ave

\[
\sup_{z} \left( \sup_{w \in V} E_A[f(A_w(S), v; z) - f(A_w(S'), v; z)] + \sup_{w \in V} E_A[f(w, A_v(S); z) - f(w, A_v(S'); z)] \right) \leq G_w E_A[\|w_T - w'_T\|_2] + G_v E_A[\|v_T - v'_T\|_2] \leq \frac{4\sqrt{T + T^2/n}}{\sqrt{n}}(G_w + G_v)^2 \eta \exp(L^2 T \eta^2/2).
\]
According to Lemma [5] we know
\[ \Delta_w^w(\bar{w}_T, \bar{v}_T) - \Delta_S^w(\bar{w}_T, \bar{v}_T) \leq \frac{4 \sqrt{e(T + T^2/n)} (G_w + G_v)^2 \eta \exp(L^2 T^2 / 2)}{\sqrt{n}}. \]

Next we focus on Part b). We consider two cases at the \( t \)-th iteration. If \( n \notin I_t \), then analogous to the discussions in [Lei et al. 2021] we can show
\[
\left\| \left( \begin{array}{c} w_{t+1} - w'_{t+1} \\ v_{t+1} - v'_{t+1} \end{array} \right) \right\|_2^2 \leq \left\| \left( \begin{array}{c} w_t - \frac{1}{m} \sum_{j=1}^{m} \nabla_w f(w_t, v_t; z_{ij}) - w'_t \\ v_t - \frac{1}{m} \sum_{j=1}^{m} \nabla_v f(w_t, v_t; z_{ij}) - v'_t \end{array} \right) \right\|_2^2 \\
\leq \left\| \left( \begin{array}{c} w_t - w'_t \\ v_t - v'_t \end{array} \right) \right\|_2^2 + 4(G_w^2 + G_v^2) \eta^2.
\]

(C.7)

Combining the preceding inequality with (C.6) and using the probability of \( n \notin I_t \), we derive
\[
\mathbb{E}_{n_t} \left[ \left\| \left( \begin{array}{c} w_{t+1} - w'_{t+1} \\ v_{t+1} - v'_{t+1} \end{array} \right) \right\|_2^2 \right] \leq \frac{n - 1}{n} \left( \left\| \left( \begin{array}{c} w_t - w'_t \\ v_t - v'_t \end{array} \right) \right\|_2^2 + 4(G_w^2 + G_v^2) \eta^2 \right) \\
+ \frac{1 + p}{n} \left\| \left( \begin{array}{c} w_t - w'_t \\ v_t - v'_t \end{array} \right) \right\|_2^2 + \frac{4(1 + 1/p)}{n} (G_w^2 + G_v^2) \eta^2 \\
= (1 + p/n) \left\| \left( \begin{array}{c} w_t - w'_t \\ v_t - v'_t \end{array} \right) \right\|_2^2 + 4(G_w^2 + G_v^2) \eta^2(1 + 1/(np)).
\]

Applying this inequality recursively implies that
\[
\mathbb{E}_A \left[ \left\| \left( \begin{array}{c} w_{t+1} - w'_{t+1} \\ v_{t+1} - v'_{t+1} \end{array} \right) \right\|_2^2 \right] \leq 4(G_w^2 + G_v^2) \eta^2 (1 + 1/(np)) \sum_{k=1}^{t} \left( 1 + \frac{p}{n} \right)^{t-k} \\
= 4(G_w^2 + G_v^2) \eta^2 \left( 1 + \frac{1}{np} \right) \left( \frac{1}{n} \right)^t - 1 \right) = 4(G_w^2 + G_v^2) \eta^2 \left( \frac{n}{p} + 1 \right) \left( \frac{n}{p} \right)^t - 1. 
\]

By taking \( p = n/t \) in the above inequality and using \((1 + 1/t)^t \leq e\), we get
\[
\mathbb{E}_A \left[ \left\| \left( \begin{array}{c} w_{t+1} - w'_{t+1} \\ v_{t+1} - v'_{t+1} \end{array} \right) \right\|_2^2 \right] \leq 16(G_w^2 + G_v^2) \eta^2 \left( t + \frac{t^2}{n} \right).
\]

Now by the Lipschitz continuity and Jensen’s inequality we ave
\[
\sup_{z} \left( \sup_{v \in V} \mathbb{E}_A[f(A_w(S), v; z) - f(A_w(S'), v; z)] + \sup_{w \in W} \mathbb{E}_A[f(w, A_v(S); z) - f(w, A_v(S'); z)] \right) \\
\leq G_w \mathbb{E}_A[\|\bar{w}_T - w'\|_2] + G_v \mathbb{E}_A[\|\bar{v}_T - v'\|_2] \leq 4\sqrt{2}(G_w + G_v)^2 \eta^2 \left( \sqrt{T} + \frac{T}{n} \right).
\]

According to Lemma [5] we know
\[
\Delta_w^w(\bar{w}_T, \bar{v}_T) - \Delta_S^w(\bar{w}_T, \bar{v}_T) \leq 32(G_w + G_v)^2 \eta^2 \left( \sqrt{T} + \frac{T}{n} \right).
\]

C.3 Proof of Theorem [2]

Finally we are ready to present the proof of Theorem [2]

**Theorem C.1** (Theorem [2] restated). Suppose the function \( F_S \) is convex-concave. Let the stepsizes \( \eta_w, t = \eta_v, t = \eta, t = [T] \) for some \( \eta > 0 \).
Proof of Theorem 2. We first focus on Part a). According to Part a) of Lemma 6 we know
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) = O\left( \max\{G^2_w + G^2_v, (G_w + G_v)^2, D^2_w + D^2_v, D_w G_w + D_v G_v\} \right) \]
and by Lemma 6, we know
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) \leq \frac{4\sqrt{c(T + T^2/n)}(G_w + G_v)^2\eta \exp(L^2T\eta^2/2)}{\sqrt{n}} \]
and
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) \leq \frac{\eta(G^2_v + G^2_v)}{2} + \frac{D^2_w + D^2_v}{2\eta T} + \frac{D_w G_w + D_v G_v}{\sqrt{mT}} + \eta d(\sigma^2_w + \sigma^2_v). \]
Combining the above two quantities we have
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) \leq \frac{4\sqrt{c(T + T^2/n)}(G_w + G_v)^2\eta \exp(L^2T\eta^2/2)}{\sqrt{n}} + \frac{\eta(G^2_v + G^2_v)}{2} + \frac{D^2_w + D^2_v}{2\eta T} + \frac{D_w G_w + D_v G_v}{\sqrt{mT}} + \eta d(\sigma^2_w + \sigma^2_v). \]
Furthermore, by Theorem 1 we know
\[ \sigma^2_w = O\left( \frac{G^2_w T \log(1/\delta)}{n^2 c^2} \right), \quad \sigma^2_v = O\left( \frac{G^2_v T \log(1/\delta)}{n^2 c^2} \right). \]
Plugging it back into (C.8) we have
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) = O\left( \sqrt{\frac{T + T^2/n}{n}}(G_w + G_v)^2\eta \exp(L^2T\eta^2/2) \right) \]
\[ \quad + \frac{\eta(G^2_v + G^2_v)}{2} + \frac{D^2_w + D^2_v}{2\eta T} + \frac{D_w G_w + D_v G_v}{\sqrt{mT}} + \eta d(\sigma^2_w + \sigma^2_v). \]
By picking \( T \approx n \) and \( \eta \approx 1/(L \max\{\sqrt{n}, \sqrt{d \log(1/\delta)} / \epsilon\}) \) we have \( \exp(L^2T\eta^2/2) = O\left( \min\{1, \frac{\eta^2}{\log(1/\delta)} \} \right) = O(1) \) and
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) = O\left( \max\{G^2_w + G^2_v, (G_w + G_v)^2, D^2_w + D^2_v, D_w G_w + D_v G_v\} \right) \]
We now turn to Part b). According to Lemma 6 Part b) we know
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) - \Delta^w(\tilde{w}_T, \tilde{v}_T) \leq 4\sqrt{2\eta(G_w + G_v)^2\left( \sqrt{T} + \frac{T}{n} \right)}. \]
Similar to Part a) we have
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) = O\left( \eta(G_w + G_v)^2 \left( \sqrt{T} + \frac{T}{n} \right) + \frac{\eta(G^2_w + G^2_v)}{2} + \frac{D^2_w + D^2_v}{2\eta T} + \frac{D_w G_w + D_v G_v}{\sqrt{mT}} + \eta d(\sigma^2_w + \sigma^2_v) \right). \]
By picking \( T \approx n^2 \) and \( \eta \approx 1/(n \max\{\sqrt{n}, \sqrt{d \log(1/\delta)} / \epsilon\}) \) we have
\[ \Delta^w(\tilde{w}_T, \tilde{v}_T) = O\left( \max\{G^2_w + G^2_v, (G_w + G_v)^2, D^2_w + D^2_v, D_w G_w + D_v G_v\} \right) \]
The proof is complete.
D Proofs for the Nonconvex-Strongly-Concave Setting in Section 3.2

In this section, we will provide the proofs for the theorems in Section 3.2.

D.1 Proof of Theorem 3 (Generalization Error)

We first focus on the generalization error $\mathbb{E}[R(w_T) - R_S(w_T)]$. Firstly, we introduce a lemma that bridges the generalization and the uniform argument stability. We modify the lemma so that it satisfies our needs.

**Lemma 7 (Lei et al. (2021)).** Let $A$ be a randomized algorithm and $\epsilon > 0$. If for all neighboring datasets $S, S'$, there holds

$$\mathbb{E}_A[\|A_w(S) - A_w(S')\|_2] \leq \epsilon.$$

Furthermore, if the function $F(w, \cdot)$ is $\rho$-strongly-concave and Assumptions (A3) hold, then the primal generalization error satisfies

$$\mathbb{E}_{S,A}[R(A_w(S)) - R_S(A_w(S))] \leq (1 + L/\rho)G_w\epsilon.$$

The next proposition states the set of saddle points is unique with respect to the variable $v$ when $F_S(w, \cdot)$ is strongly concave

**Proposition 1.** Assume $F_S(w, \cdot)$ is $\rho$-strongly concave with $\rho > 0$. Let $(\hat{w}_S, \hat{v}_S)$ and $(\hat{w}_S', \hat{v}_S')$ be two saddle points of $F_S$. Then we have $\hat{v}_S = \hat{v}_S'$.

**Proof.** Given $\hat{w}_S$, by the strong concavity, we have

$$F_S(\hat{w}_S, \hat{v}_S') \geq F_S(\hat{w}_S, \hat{v}_S) + \langle \nabla \hat{v} F_S(\hat{w}_S, \hat{v}_S), \hat{v}_S - \hat{v}_S' \rangle + \frac{\rho}{2} \|\hat{v}_S - \hat{v}_S'\|^2_2.$$

Since $(\hat{w}_S, \hat{v}_S)$ is a saddle point of $F_S$, it implies $\hat{v}_S$ attains maximum of $F_S(\hat{w}_S, \cdot)$. By the first order optimality we know $\langle \nabla \hat{v} F_S(\hat{w}_S, \hat{v}_S), \hat{v}_S - \hat{v}_S' \rangle \geq 0$ and therefore

$$F_S(\hat{w}_S, \hat{v}_S') \geq F_S(\hat{w}_S, \hat{v}_S) + \frac{\rho}{2} \|\hat{v}_S - \hat{v}_S'\|^2_2 \geq F_S(\hat{w}_S', \hat{v}_S') + \frac{\rho}{2} \|\hat{v}_S - \hat{v}_S'\|^2_2,$$

where in the second inequality we used $(\hat{w}_S', \hat{v}_S')$ is also a saddle point of $F_S$. Similarly, given $\hat{w}_S'$ we can show

$$F_S(\hat{w}_S', \hat{v}_S') \geq F_S(\hat{w}_S', \hat{v}_S) + \frac{\rho}{2} \|\hat{v}_S - \hat{v}_S'\|^2_2.$$

Adding (D.1) and (D.2) together implies that $\rho \|\hat{v}_S - \hat{v}_S'\|^2_2 \leq 0$. This implies $\hat{v}_S = \hat{v}_S'$ which completes the proof.

Recall that $\pi_S : \mathcal{W} \to \mathcal{W}$ is the projection onto the set of saddle points $\Omega_S = \{\hat{w}_S : (\hat{w}_S, \hat{v}_S) \in \text{arg min max } F_S(w, v)\}$, i.e., $\pi_S(w) = \text{arg min}_{\hat{w}_S \in \Omega_S} \frac{1}{2} \|w - \hat{w}_S\|^2_2$. Proposition 1 makes sure the projection is well-defined. The next lemma shows that PL condition implies QG condition. The proof follows straightforward from [Karimi et al. (2016)] and we omit it for brevity.

**Lemma 8.** Suppose the function $F_S(\cdot, v)$ satisfies $\mu$-PL condition. Then $F_S$ satisfies the QG condition with respect to $w$ with constant $4\mu$, i.e.,

$$F_S(w, v) - F_S(\pi_S(w), v) \geq 2\mu\|w - \pi_S(w)\|^2_2, \ \forall v \in V.$$

With the help of Assumption 1 and the preceding lemmas, we can derive the uniform argument stability.

**Lemma 9.** Assume (A1) and (A3) hold. Assume $F_S(\cdot, v)$ satisfies PL condition with constant $\mu$ and $F_S(w, \cdot)$ is $\rho$-strongly concave. Let $A$ be a randomized algorithm. If for any $S$, $\mathbb{E}[\|A_w(S) - \pi_S(A_w(S))\|_2] = O(\epsilon_A)$, then we have

$$\mathbb{E}[\|A_w(S) - A_w(S')\|_2] \leq O(\epsilon_A) + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_w^2}{\rho\mu}}.$$
Proof. Let \((\pi_S(A_w(S)), \hat{v}_S) \in \arg \min w \max_v F_S(w, v)\) and \((\pi_{S'}(A_w(S')'), \hat{v}_{S'})\) defined in the similar way. By triangle inequality we have
\[
E[|A_w(S) - A_w(S')|_2^2] \leq E[|A_w(S) - \pi_S(A_w(S))|_2^2 + |\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))|_2^2 + E[|A_w(S') - \pi_{S'}(A_w(S'))|_2^2]
= ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 + O(\varepsilon_A).
\]
Since \(\pi_S(A_w(S)) \in \arg \min_{w \in Y} F_S(w, \hat{v}_S)\) and by Assumption 4 we know that \(\pi_S(A_w(S))\) is the closest optimal point of \(F_S\) to \(\pi_{S'}(A_w(S'))\). And since \(\hat{v}_S\) is fixed, by Lemma 5 we have
\[
2\mu ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 \leq F_S(\pi_{S'}(A_w(S'))', \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S).
\]
Similarly, we have
\[
2\mu ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 \leq F_S(\pi_{S'}(A_w(S'))', \hat{v}_S) - F_S(\pi_{S'}(A_w(S')'), \hat{v}_S).
\]
Summing up the above two inequalities we have
\[
4\mu ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 \leq F_S(\pi_{S'}(A_w(S'))', \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S) + F_S(\pi_S(A_w(S)), \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S).
\]
On the other hand, by the \(\rho\)-strong convexity of \(F_S(\cdot, v)\) and \(\hat{v}_S = \arg \max_v F_S(\pi_S(A_w(S)), v)\), we have
\[
\frac{\rho}{2} ||\hat{v}_S - \hat{v}_S'||_2^2 \leq F_S(\pi_S(A_w(S)), \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S).
\]
Similarly, we have
\[
\frac{\rho}{2} ||\hat{v}_S - \hat{v}_S'||_2^2 \leq F_S(\pi_{S'}(A_w(S'))', \hat{v}_S) - F_S(\pi_{S'}(A_w(S')'), \hat{v}_S).
\]
Summing up the above two inequalities we have
\[
\rho ||\hat{v}_S - \hat{v}_S'||_2^2 \leq F_S(\pi_S(A_w(S)), \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S) + F_S(\pi_{S'}(A_w(S')), \hat{v}_S) - F_S(\pi_{S'}(A_w(S')), \hat{v}_S).
\]
Summing up (D.3) and (D.4) rearranging terms, we have
\[
4\mu ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 + \rho ||\hat{v}_S - \hat{v}_S'||_2^2 \leq F_S(\pi_{S'}(A_w(S'))', \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S) + F_S(\pi_S(A_w(S)), \hat{v}_S) - F_S(\pi_S(A_w(S)), \hat{v}_S).
\]
where the second inequality is due to Lipschitz continuity of \(f\), the third inequality is due to Cauchy-Schwarz inequality. Therefore
\[
2\sqrt{\frac{\mu}{\rho}} ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2 \leq \sqrt{4\mu ||\pi_S(A_w(S)) - \pi_{S'}(A_w(S'))||_2^2 + \rho ||\hat{v}_S - \hat{v}_S'||_2^2} \leq \frac{1}{n} \sqrt{\frac{G_w^2}{\mu} + \frac{4G_v^2}{\rho}}.
\]
The proof is complete. \(\square\)

We are now ready to present the generalization error of Algorithm 1 in terms of \(w_T\).

**Theorem D.1.** Assume (A1) and (A3) hold. Assume \(F_S(\cdot, v)\) satisfies PL condition with constant \(\mu\) and \(f(w, \cdot; z)\) is \(\rho\)-strongly concave. For Algorithm 1, the iterates \(\{w_t, v_t\}\) satisfies the following inequality
\[
E[R(w_T) - R_S(w_T)] \leq (1 + \frac{L}{\rho})G\left(\sqrt{\frac{\varepsilon_T}{2\mu} + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_v^2}{\rho\mu}}}ight).
\]
Proof. Since $R_S$ satisfies $\mu$-PL, by Lemma 8 and Theorem 4, we have

$$\mathbb{E}[\|w_T - \pi(w_T)\|_2] \leq \sqrt{\mathbb{E}[\|w_T - \pi(w_T)\|_2^2]} \leq \sqrt{\mathbb{E}[\frac{1}{2\mu}(R_S(w_T) - R_S^*)]} \leq \sqrt{\frac{\varepsilon_T}{2\mu}}.$$ 

By Lemma 9, we have

$$\mathbb{E}[\|w_T - w'_T\|_2] \leq \frac{1}{\sqrt{2\mu}} + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_w^2}{\rho\mu}}.$$ 

By Part b) of Lemma 7, we have

$$\mathbb{E}[R(w_T) - R_S(w_T)] \leq (1 + \frac{L}{\rho})G_w\left(\frac{\varepsilon_T}{2\mu} + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_w^2}{\rho\mu}}\right).$$

The proof is complete. \qed

The next theorem establishes the generalization bound for the empirical maximizer of a strongly concave objective, i.e. $\mathbb{E}[R_S(w^*) - R(w^*)]$. The proof follows from Shalev-Shwartz et al. [2009].

**Theorem D.2.** Assume (A1) holds. Assume $F_S(w, \cdot)$ is $\rho$-strongly concave. Assume that for any $w$ and $S$, the function $v \mapsto F_S(w, v)$ is $\rho$-strongly-concave. Then

$$\mathbb{E}[R_S(w^*) - R(w^*)] \leq \frac{4G_w^2}{\rho n}.$$ 

**Proof.** We decompose the term $\mathbb{E}[R_S(w^*) - R(w^*)]$ as

$$\mathbb{E}[R_S(w^*) - R(w^*)] = \mathbb{E}[F_S(w^*, \hat{v}_S^*) - F(w^*, v^*)] = \mathbb{E}[F_S(w^*, \hat{v}_S^*) - F(w^*, \bar{v}_S^*)] + \mathbb{E}[F(w^*, \bar{v}_S^*) - F(w^*, v^*)],$$

where $\bar{v}_S^* = \arg\max_v F_S(w^*, v)$. The second term $\mathbb{E}[F(w^*, \bar{v}_S^*) - F(w^*, v^*)] \leq 0$ since $(w^*, v^*)$ is a saddle point of $F$. Hence it suffices to bound $\mathbb{E}[F_S(w^*, \hat{v}_S^*) - F(w^*, \bar{v}_S^*)]$. Let $S' = \{z_1', \ldots, z_n'\}$ be drawn independently from $\rho$. For any $i \in [n]$, define $S(i) = \{z_1, \ldots, z_i-1, z_i', z_{i+1}, \ldots, z_n\}$. Denote $\hat{v}_{S(i)}^* = \arg\max_{v \in V} F_{S(i)}(w^*, v)$. Then

$$F_S(w^*, \hat{v}_{S(i)}^*) - F_S(w^*, \hat{v}_{S(i)}^*) = \frac{1}{n} \sum_{j \neq i} \left(f(w^*, \hat{v}_{S(i)}^*; z_j) - f(w^*, \hat{v}_{S(i)}^*; z_i') + 1/n \left(f(w^*, \hat{v}_{S(i)}^*; z_i) - f(w^*, \hat{v}_{S(i)}^*; z_i')\right)\right)$$

$$= \frac{1}{n} \left(f(w^*, \hat{v}_{S(i)}^*; z_i') - f(w^*, \hat{v}_{S(i)}^*; z_i') + 1/n \left(f(w^*, \hat{v}_{S(i)}^*; z_i) - f(w^*, \hat{v}_{S(i)}^*; z_i')\right)\right)$$

$$+ F_{S(i)}(w^*, \hat{v}_{S(i)}^*) - F_{S(i)}(w^*, \hat{v}_{S(i)}^*)$$

$$\leq \frac{1}{n} \left|f(w^*, \hat{v}_{S(i)}^*; z_i) - f(w^*, \hat{v}_{S(i)}^*; z_i')\right| + 1/n \left|f(w^*, \hat{v}_{S(i)}^*; z_i) - f(w^*, \hat{v}_{S(i)}^*; z_i')\right|$$

$$\leq \frac{2G_w}{n} \left\|\hat{v}_{S(i)}^* - \hat{v}_{S(i)}^*\right\|_2,$$ 

(D.5)

where the first inequality follows from the fact that $\hat{v}_{S(i)}^*$ is the maximizer of $F_{S(i)}(w^*, \cdot)$ and the second inequality follows the Lipschitz continuity. Since $F_S$ is strongly-concave and $\hat{v}_S^*$ maximizes $F_S(w^*, \cdot)$, we know

$$\frac{\rho}{2} \left\|\hat{v}_S^* - \hat{v}_{S(i)}^*\right\|_2^2 \leq F_S(w^*, \hat{v}_S^*) - F_S(w^*, \hat{v}_{S(i)}^*).$$

Combining it with (D.5), we get $\left\|\hat{v}_S^* - \hat{v}_{S(i)}^*\right\|_2 \leq 4G_w/(\rho n)$. By Lipschitz continuity, the following inequality holds for any $z$

$$\left|f(w^*, \hat{v}_S^*; z) - f(w^*, \hat{v}_{S(i)}^*; z)\right| \leq \frac{4G_w^2}{\rho n}.$$ 

Since $z_i$ and $z_i'$ are i.i.d., we have

$$\mathbb{E}[F(w^*, \hat{v}_S^*)] = \mathbb{E}[F(w^*, \hat{v}_{S(i)}^*)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[f(w^*, \hat{v}_{S(i)}^*; z_i)\right],$$
Lemma 12. Assume

Proof. Because $R_S$ is $L + L^2/\rho$-smooth by Lemma 10, we have

$$R_S(w_{t+1}) - R_S^* \leq R_S(w_t) - R_S^* + \langle \nabla R_S(w_t), w_{t+1} - w_t \rangle + \frac{L + L^2/\rho}{2} \|w_{t+1} - w_t\|^2$$

$$= R_S(w_t) - \eta_{w,t} \langle \nabla R_S(w_t), w_t - w_t \rangle - \eta_{w,t} \sum_{j=1}^m \nabla_w f(w_t, v_t; z_{ij}^*) + \xi_t$$

$$+ \frac{(L + L^2/\rho)\eta_{w,t}^2}{2} \frac{1}{m} \| \nabla f(w_t, v_t; z_{ij}^*) + \xi_t \|^2.$$
Lemma 13. \( R_s \) satisfies PL condition with constant \( \mu \). Taking this conditional expectation of both sides, we get

\[
\mathbb{E}_t[R_S(w_{t+1}) - R_S^*] = R_S(w_t) - R_S^* - \eta_{w,t}(\nabla R_S(w_t), \nabla w F_S(w_t, v_t)) + \frac{(L + L^2/\rho)\eta^2_{w,t}}{2} \frac{1}{m} \sum_{j=1}^m \nabla w f(w_t, v_t; z_{t,j}) - \nabla w F_S(w_t, v_t) + \nabla w F_S(w_t, v_t) - \xi_t\] 

\[
\leq R_S(w_t) - R_S^* - \eta_{w,t}(\nabla R_S(w_t), \nabla w F_S(w_t, v_t)) + \frac{(L + L^2/\rho)\eta^2_{w,t}}{2} \frac{1}{m} \sum_{j=1}^m \mathbb{E}_t[\nabla w f(w_t, v_t; z_{t,j}) - \nabla w F_S(w_t, v_t)]^2 + \frac{(L + L^2/\rho)\eta^2_{w,t}}{2} \frac{B_w^2}{m} + d\sigma^2_w
\]

where in first inequality since \( \mathbb{E}_t[\frac{1}{m} \sum_{j=1}^m \nabla w f(w_t, v_t; z_{t,j}) - \nabla w F_S(w_t, v_t)]^2 \leq \frac{B_w^2}{m} \) and \( \mathbb{E}_t[\xi_t^2] = d\sigma^2_w \), and the last inequality we use \( \eta_t \leq 1/(L + L^2/\rho) \). Because \( R_S \) satisfies PL condition with \( \mu \) by Lemma 11, we have

\[
\mathbb{E}_t[R_S(w_{t+1}) - R_S^*] \leq (1 - \mu\eta_{w,t})(R_S(w_t) - R_S^*) + \frac{\eta_{w,t}}{2} \||\nabla R_S(w_t) - \nabla w F_S(w_t, v_t)|^2 \] 

\[
+ \frac{(L + L^2/\rho)\eta^2_{w,t}}{2} \frac{B_w^2}{m} + d\sigma^2_w \]

where the second we use \( F_S \) is \( L \)-smooth. Now taking expectation of both sides yields the claimed bound. The proof is complete.

The next lemma characterizes the descent behavior of \( v_t \).

Lemma 13. Assume (A2) and (A3) hold. Assume \( F_S(\cdot, v) \) satisfies PL condition with constant \( \mu \) and \( F_S(w, \cdot) \) is \( \rho \)-strongly convex. Let \( \hat{v}_S(w) = \arg \max_{v \in \mathcal{V}} F_S(w, v) \). For Algorithm 3 and any \( \epsilon > 0 \), the iterates \( \{w_t, v_t\} \) satisfies the following inequality

\[
\mathbb{E}[\|v_{t+1} - \hat{v}_S(w_{t+1})\|^2] \leq ((1 + \frac{1}{\epsilon})2L^4/\rho \eta^2_{w,t} + (1 + \epsilon)(1 - \rho \eta_{w,t}))\mathbb{E}[\|v_t - \hat{v}_S(w_t)\|^2] + (1 + \frac{1}{\epsilon})\eta^2_{w,t}2L^2/\rho^3 \left( \frac{B_w^2}{m} + d\sigma^2_w \right)
\]

\[
+ (1 + \frac{1}{\epsilon})4L^2/\rho^2(L + L^2/\rho)\eta^2_{w,t}\mathbb{E}[R_S(w_t) - R_S^*] + (1 + \epsilon)\eta^2_{w,t}\left( \frac{B_w^2}{m} + d\sigma^2_w \right).
\]

Proof. By Young’s inequality, we have

\[
\|v_{t+1} - \hat{v}_S(w_{t+1})\|^2 \leq (1 + \epsilon)\|v_{t+1} - \hat{v}_S(w_t)\|^2 + (1 + \frac{1}{\epsilon})\|v_t - \hat{v}_S(w_t)\|^2.
\]

For the term \( \|\hat{v}_S(w_t) - \hat{v}_S(w_{t+1})\|^2 \), since \( \hat{v}_S(\cdot) \) is \( L/\rho \)-Lipschitz by Lemma 10, taking conditional expectation, we have

\[
\mathbb{E}_t[\|\hat{v}_S(w_{t+1}) - \hat{v}_S(w_t)\|^2] \leq L^2/\rho^2 E_t[\|w_{t+1} - w_t\|^2] = L^2/\rho^2 \eta^2_{w,t} E_t[\|\frac{1}{m} \sum_{j=1}^m \nabla w f(w_t, v_t; z_{t,j}) + \xi_t\|^2]
\]

\[
\leq L^2/\rho^2 \eta^2_{w,t} \|\nabla w F_S(w_t, v_t)\|^2 + L^2/\rho^2 \eta^2_{w,t} \left( \frac{B_w^2}{m} + d\sigma^2_w \right)
\]

\[
\leq 2L^2/\rho^2 \eta^2_{w,t} \|\nabla w F_S(w_t, v_t)\|^2 + 2L^2/\rho^2 \eta^2_{w,t} \|\nabla R_S(w_t)\|^2 + L^2/\rho^2 \eta^2_{w,t} \left( \frac{B_w^2}{m} + d\sigma^2_w \right)
\]

\[
\leq 2L^4/\rho^2 \eta^2_{w,t} \|\hat{v}_S(w_t) - v_t\|^2 + 2L^2/\rho^2 \eta^2_{w,t} \|\nabla R_S(w_t)\|^2 + L^2/\rho^2 \eta^2_{w,t} \left( \frac{B_w^2}{m} + d\sigma^2_w \right),
\]

\[
\leq 2L^4/\rho^2 \eta^2_{w,t} \|\hat{v}_S(w_t) - v_t\|^2 + 2L^2/\rho^2 \eta^2_{w,t} \|\nabla R_S(w_t)\|^2 + L^2/\rho^2 \eta^2_{w,t} \left( \frac{B_w^2}{m} + d\sigma^2_w \right),
\]

\[
\leq 2L^4/\rho^2 \eta^2_{w,t} \|\hat{v}_S(w_t) - v_t\|^2 + 2L^2/\rho^2 \eta^2_{w,t} \|\nabla R_S(w_t)\|^2 + L^2/\rho^2 \eta^2_{w,t} \left( \frac{B_w^2}{m} + d\sigma^2_w \right),
\]
Lemma 14. Assume (A2) and (A3) hold. Assume $F_S(\cdot, \nu)$ satisfies PL condition with constant $\mu$ and $F_S(\nu, \cdot)$ is $\rho$-strongly concave. Define $a_t = \mathbb{E}[R_S(w_t) - R_S(w^*)]$ and $b_t = \mathbb{E}[\|\dot{v}_S(w_t) - v_t\|^2]$. For Algorithm 4 if $\eta_{w,t} \leq 1/(L + L^2/\rho)$ and $\eta_{\nu,t} \leq 1/L$, then for any non-increasing sequence $\{\lambda_t > 0\}$ and $\epsilon > 0$, the iterates $\{w_t, v_t\} \in \mathcal{T}$ satisfy the following inequality

$$a_{t+1} + \lambda_{t+1}b_{t+1} \leq k_{1,t}a_t + k_{2,t}\lambda_tb_t + \frac{(L + L^2/\rho)\eta_{w,t}^2}{2}\left(\frac{B_w^2}{m} + \delta\sigma_w^2\right) + 2(1 + \frac{1}{\epsilon})\lambda_tL^2/\rho^2\eta_{w,t}^2\left(\frac{B_w^2}{m} + \delta\sigma_w^2\right) + \lambda_t(1 + \epsilon)\eta_{\nu,t}^2\left(\frac{B_w^2}{m} + \delta\sigma_w^2\right),$$

where

$$k_{1,t} = (1 - \mu\eta_{w,t}) + \lambda_t(1 + \frac{1}{\epsilon})4L^2/\rho^2(L + L^2/\rho)\eta_{w,t}^2,$$

$$k_{2,t} = \frac{L^2\eta_{w,t}^2}{2}\lambda_t + (1 + \epsilon)(1 - \rho\eta_{\nu,t}) + (1 + \frac{1}{\epsilon})2L^4/\rho^2\eta_{\nu,t}^2.$$
Therefore Lemma 14 can be simplified as

\[ a_{t+1} + \lambda_{t+1} b_{t+1} \leq ((1 - \mu) \eta_{w,t} + \lambda_{t+1} (1 + \frac{1}{\epsilon}) 4 L^2 / \rho^2 (L + L^2 / \rho) \eta_{w,t}^2) a_t + \frac{L^2 \eta_{w,t}^2}{2} + \lambda_{t+1} (1 + (1 - \rho \eta_{v,t}) + \lambda_{t+1} (1 + \frac{1}{\epsilon}) 2 L^4 / \rho^2 \eta_{w,t}^2) b_t + \frac{(L + L^2 / \rho) \eta_{w,t}^2}{2} (B_w^2 / m + d \sigma_w^2) + 2 (1 + \frac{1}{\epsilon}) \lambda_{t+1} L^2 / \rho^2 \eta_{w,t}^2 (B_w^2 / m + d \sigma_w^2) + \lambda_{t+1} (1 + (1 + \frac{1}{\epsilon}) 2 L^4 / \rho^2 \eta_{w,t}^2) b_t + \frac{(L + L^2 / \rho) \eta_{w,t}^2}{2} (B_w^2 / m + d \sigma_w^2) + 2 \lambda_{t+1} L^2 / \rho^2 \eta_{w,t}^2 (B_w^2 / m + d \sigma_w^2) + \lambda_{t+1} (1 + (1 + \frac{1}{\epsilon}) 2 L^4 / \rho^2 \eta_{w,t}^2) b_t + \frac{(L + L^2 / \rho) \eta_{w,t}^2}{2} (B_w^2 / m + d \sigma_w^2) + 2 (1 + \frac{1}{\epsilon}) \lambda_{t+1} L^2 / \rho^2 \eta_{w,t}^2 (B_w^2 / m + d \sigma_w^2) + \lambda_{t+1} (1 + (1 + \frac{1}{\epsilon}) 2 L^4 / \rho^2 \eta_{w,t}^2) b_t\]

where the first inequality we used \( \lambda_{t+1} \leq \lambda_t \). The proof is completed.

\[ \square \]

We are now ready to state the convergence theorem of Algorithm 1.

**Theorem D.4** (Theorem 4 restated). Assume (A2) and (A3) hold. Assume \( F_S(\cdot, \nu) \) satisfies PL condition with constant \( \mu \) and \( F_S(w, \cdot) \) is \( \rho \)-strongly concave. Assume \( \mu \leq 2L^2 \) and \( \kappa = \frac{L}{\rho} \). For Algorithm 1, if \( \eta_{w,t} = O(\frac{1}{m}) \) and \( \eta_{v,t} = O(\frac{\epsilon^2 \max(1, \sqrt{\kappa} / \mu)}{m^{2/3}}) \), then the iterates \( \{w_t, v_t\} \in T \) satisfy the following inequality

\[ \mathbb{E}[R_S(w_{T+1}) - R_*] = O(\min \left\{ \frac{1}{L} \frac{1}{\mu} \right\} \frac{B_w^2 / m + d \sigma_w^2}{T^{2/3}}) + \max \left\{ 1, \left( \frac{L \kappa}{\mu} \right)^3 \frac{B_w^2 / m + d \sigma_w^2}{T^{2/3}} \right\}. \]  

(D.8)

Furthermore, if \( \sigma_w, \sigma_v \) are given by (3), we have

\[ \mathbb{E}[R_S(w_{T+1}) - R_*] = O(\min \left\{ \frac{1}{L} \frac{1}{\mu} \right\} \frac{B_w^2 / m + d \sigma_w^2}{T^{2/3}} + \frac{G^2 d T^{1/3} \log(1/\delta)}{n^2 \epsilon^2} ) + \max \left\{ 1, \left( \frac{L \kappa}{\mu} \right)^3 \frac{B_w^2 / m + d \sigma_w^2}{T^{2/3}} + \frac{G^2 d T^{1/3} \log(1/\delta)}{n^2 \epsilon^2} \right\}. \]  

(D.9)

**Proof.** Since \( \eta_{v,t} \leq 1/L \), we can pick \( \epsilon = \frac{\rho \eta_{v,t}}{2(1 - \rho \eta_{v,t})} \). Then we have \( (1 + \epsilon)(1 - \rho \eta_{v,t}) = 1 - \frac{\rho \eta_{v,t}}{2} \) and \( 1 + \frac{1}{\epsilon} \leq 2 \frac{\rho \eta_{v,t}}{2} \). Therefore Lemma 14 can be simplified as

\[ k_{1,t} \leq (1 - \mu \eta_{w,t}) + \lambda_t \frac{8 L^2 / \rho^2 (L + L^2 / \rho) \eta_{w,t}^2}{\rho \eta_{v,t}}, \]

\[ k_{2,t} \leq \frac{L^2 \eta_{w,t}}{2 \lambda_t} + 1 - \frac{\rho \eta_{v,t}}{2} + \frac{4 L^4 / \rho^2 \eta_{w,t}^2}{\rho \eta_{v,t}}. \]

If we choose \( \lambda_t = \frac{4 L^2 \eta_{w,t}}{m \eta_{v,t}} \) and \( \eta_{w,t} \leq \min \left\{ \frac{\sqrt{\tau}}{8 \epsilon^2 \sqrt{L + L^2 / \rho}}, \frac{1}{4 \sqrt{2 \delta}} \right\} \eta_{v,t} \), then further we have \( k_{1,t} \leq 1 - \frac{\mu \eta_{w,t}}{2} \) and...
where we used $\mu \leq 2L^2$. Taking $\eta_{w,t} = \frac{2}{\mu}$ and $\eta_{v,t} = \max \{8\kappa^2/(L^2/\rho), 4\sqrt{2}\kappa^2\}$, and multiplying the preceding inequality with $t$ on both sides, there holds

$$t(a_{t+1} + \lambda_{t+1} b_{t+1}) \leq (t-1)(a_t + \lambda_t b_t) + \frac{2(L + L^2/\rho)}{\mu} \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) + \frac{32L^4/\rho^3}{\mu t^{2/3}} \min \left\{ \frac{8\kappa^2}{6\sqrt{L^2 + L_\rho^2}}, \frac{1}{4\sqrt{2\kappa^2}} \right\}^2 \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) + \frac{16L^2 \max \{8\kappa^2/(L^2/\rho)/\mu, 4\sqrt{2}\kappa^2\}}{2\mu^2 \rho t^{2/3}} \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) \right).$$

Applying the preceding inequality inductively from $t = 1$ to $T$, we have

$$T(a_{T+1} + \lambda_{T+1} b_{T+1}) \leq \frac{2(L + L^2/\rho)}{\mu} \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) \log(T) + \frac{32L^4/\rho^3}{\mu d\sigma_w^2} \min \left\{ \frac{8\kappa^2}{6\sqrt{L^2 + L_\rho^2}}, \frac{1}{4\sqrt{2\kappa^2}} \right\}^2 \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) T^{1/3} + \frac{16L^2 \max \{8\kappa^2/(L^2/\rho)/\mu, 4\sqrt{2}\kappa^2\}}{2\mu^2 \rho} \left( \frac{B_w^2}{m} + d\sigma_w^2 \right) T^{1/3}.$$

Consequently,

$$E[R_S(w_{T+1}) - R^*_S] \leq a_{T+1} + \lambda_{T+1} b_{T+1} \leq \frac{2(L + L^2/\rho)(B_w^2/m + d\sigma_w^2) \log(T)}{\mu^2} + \frac{32L^4/\rho^3}{\mu d\sigma_w^2} \min \left\{ \frac{8\kappa^2}{6\sqrt{L^2 + L_\rho^2}}, \frac{1}{4\sqrt{2\kappa^2}} \right\}^2 \frac{1}{T^{2/3}} + \frac{16(B_w^2/m + d\sigma_w^2) L^2 \max \{8\kappa^2/(L^2/\rho)/\mu, 4\sqrt{2}\kappa^2\}}{2\mu^2 \rho} \frac{1}{T^{2/3}}.$$

Therefore, the estimation (D.8) follows from the fact that $\kappa = L/\rho$. The result in Theorem 4 follows by observing $\min \left\{ 1, \sqrt{L_\rho/\mu} \right\} \frac{L_\rho^3}{\mu^2} \geq \min \left\{ \frac{1}{L_\rho}, \frac{1}{\mu} \right\}$. Substituting the values of $\sigma_w, \sigma_v$, i.e., $\sigma_w = \frac{c_3 G_w \sqrt{T \log(\frac{1}{\vartheta})}}{\mu}$ and $\sigma_v = \frac{c_3 G_v \sqrt{T \log(\frac{1}{\vartheta})}}{\mu}$, into (D.8) yields the desired estimation (D.9).

### D.3 Proof of Theorem 5

Theorem D.5 (Theorem 5 restated). Assume (A1) and (A3) hold. Assume $F_S(\cdot, v)$ satisfies PL condition with constant $\mu$ and $f(w, \cdot, z)$ is $\rho$-strongly concave. For SGDA, if $E[R_S(w_T) - R^*_S] = O(\varepsilon_T)$, then iterates $\{w_t, v_t\}$ satisfies the following inequality

$$E[R(w_T) - R^*] = O(\varepsilon_T + (1 + \frac{L}{\rho}) G_w \left( \frac{\varepsilon_T}{2\mu} + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_v^2}{\rho \mu}} \right) + \frac{4G^2}{\rho \mu}).$$

Furthermore, if we choose $T = O(n), \eta_{w,t} = O(\frac{1}{\mu \delta})$ and $\eta_{v,t} = O(\frac{\kappa^2 \max(1, \sqrt{\kappa/\mu})}{\mu^{1/3}})$, then

$$E[R(w_T) - R^*] = O(\frac{\kappa^{2.75}}{\mu^{1.75}} \left( \frac{1}{n^{1/3}} + \frac{\sqrt{d \log(1/\delta)}}{n^{1/6}} \frac{1}{\epsilon} \right)).$$
Proof. For any $w^* \in \arg \min_w R(w)$, recall that we have the error decomposition (5), which is

$$
E[R(w_T) - R^*] = E[R(w_T) - R_S(w_T)] + E[R_S(w_T) - R_S^*] + E[R_S^* - R^*] + E[R^* - R(w^*)],
$$

where the inequality is by $R_S^* - R_S(w^*) \leq 0$. By Theorem D.1 we have

$$
E[R(w_T) - R_S(w_T)] \leq (1 + \frac{L}{\rho}) G_w \left( \sqrt{\epsilon_T^2 + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_v^2}{\rho \mu}}} \right).
$$

And by Theorem D.2 we have

$$
E[R_S^* - R(w^*)] \leq \frac{4G_v^2}{\rho n}.
$$

We can plug the above two inequalities into (5), and get

$$
E[R(w_T) - R^*] = O(\epsilon_T + (1 + \frac{L}{\rho}) G_w \left( \sqrt{\epsilon_T^2 + \frac{1}{n} \sqrt{\frac{G_w^2}{4\mu^2} + \frac{G_v^2}{\rho \mu}}} \right) + \frac{4G_v^2}{\rho n}).
$$

Now by the choice of $\eta_{w,t}, \eta_{v,t}$, and Theorem 4, we have $\epsilon_T = O(\frac{\epsilon_T^3}{n^{1/3}} \frac{1/m + d(\sigma_w^2 + \sigma_v^2)}{T^{2/3}})$. Assume $m$ is a constant. Plugging $\epsilon_T$ into the preceding inequality and letting $T = O(n)$ yields the second statement.

E Additional Experimental Details

E.1 Source Code

For the purpose of review, the source code is accessible in the supplementary file.

E.2 Computing Infrastructure Description

All algorithms are implemented in Python 3.6 and trained and tested on an Intel(R) Xeon(R) CPU W5590 @3.33GHz with 48GB of RAM and an NVIDIA Quadro RTX 6000 GPU with 24GB memory. The PyTorch version is 1.6.0.

E.3 Description of Datasets

In experiments, we use three benchmark datasets. Specifically, ijcnn1 dataset from LIBSVM repository, MNIST dataset and Fashion-MNIST dataset are from LeCun et al. (1998), and Xiao et al. (2017). The details of these datasets are shown in Table E.1. For the ijcnn1 dataset, we normalize the features into $[0,1]$. For MNIST and Fashion-MNIST datasets, we first normalize the features of them into $[0,1]$ then normalize them according to the mean and standard deviation.

| Datasets      | #Classes | #Training Samples | #Testing Samples | #Features |
|---------------|----------|-------------------|------------------|-----------|
| ijcnn1        | 2        | 39,992            | 9,998            | 22        |
| MNIST         | 10       | 60,000            | 10,000           | 784       |
| Fashion-MNIST | 10       | 60,000            | 10,000           | 784       |

Table E.1: Statistical information of each dataset for AUC optimization.

E.4 Training Settings

The training settings for NSEG and DP-SGDA on all datasets are shown in Table E.2.
Datasets | Learning Rate | Epochs | Projection Size
--- | --- | --- | ---
| | Ori | DP | Ori | DP | Ori | DP |
NSEG | 64 | 300 | 300 | 350 | 350 | 1000 | 15 | 100 | 100 |
MNIST | 64 | 11 | 11 | 5 | 5 | 100 | 15 | 2 | 2 |
Fashion-MNIST | 64 | 11 | 11 | 5 | 5 | 100 | 15 | 3 | 3 |
DP-SGDA (Linear) | 64 | 300 | 300 | 350 | 350 | 100 | 15 | 10 | 10 |
MNIST | 64 | 11 | 11 | 5 | 5 | 100 | 15 | 2 | 2 |
Fashion-MNIST | 64 | 11 | 11 | 5 | 5 | 100 | 15 | 3 | 3 |
DP-SGDA (MLP) | 64 | 3000 | 3001 | 500 | 501 | 100 | 10 | 100 | 100 |
MNIST | 64 | 900 | 1000 | 100 | 210 | 10 | 10 | 2 | 2 |
Fashion-MNIST | 64 | 900 | 1000 | 100 | 210 | 10 | 10 | 2 | 2 |

Table E.2: Training settings for each model and each dataset.

Algorithm 2 DP-SGDA for AUC Maximization

1: **Inputs:** Private dataset $S = \{z_i : i \in [n]\}$, privacy budget $\epsilon, \delta$, number of iterations $T$, learning rates $\{\gamma_t, \lambda_t\}_{t=1}^T$, initial points $(\theta_0, a_0, b_0, v_0)$
2: Compute $n_+ = \sum_{i=1}^n \mathbb{I}[y_i = 1]$ and $n_- = \sum_{i=1}^n \mathbb{I}[y_i = -1]$
3: Compute noise parameters $\sigma_1$ and $\sigma_2$ based on Eq. (3)
4: for $t = 1$ to $T$ do
5: Randomly select a batch $S_t$
6: For each $j \in I_t$, compute gradient $\nabla_\theta f(\theta_t, a_t, b_t, v_t; z_j), \nabla_a f(\theta_t, a_t, b_t, v_t; z_j), \nabla_b f(\theta_t, a_t, b_t, v_t; z_j)$ and $\nabla_v f(\theta_t, a_t, b_t, v_t; z_j)$ based on Eq. (E.1)
7: Sample independent noises $\xi_t \sim \mathcal{N}(0, \sigma_1^2 I_{d+2})$ and $\zeta_t \sim \mathcal{N}(0, \sigma_2^2 I_{d+2})$
8: Update
9: end for
10: **Outputs:** $(\theta_T, a_T, b_T, v_T)$ or $(\bar{\theta}_T, \bar{a}_T, \bar{b}_T, \bar{v}_T)$

E.5 DP-SGDA for AUC Maximization

AUC maximization with square loss can be reformulated as

$$F(\theta, a, b, v) = \mathbb{E}_x[(1-p)(h(\theta; x) - a)^2\mathbb{I}[y = 1] + p(h(\theta; x) - b)^2\mathbb{1}[y = -1] + 2(1 + v)(ph(\theta; x)\mathbb{1}[y = -1] - (1 - p)h(\theta; x)\mathbb{1}[y = 1])] - p(1 - p)v^2]$$

where $z = (x, y)$ and $p = \mathbb{P}[y = 1]$. The empirical risk formulation is given as

$$F_S(\theta, a, b, v) = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{n_+} (h(\theta; x_i) - a)^2\mathbb{1}[y_i = 1] + \frac{1}{n_-} (h(\theta; x_i) - b)^2\mathbb{1}[y_i = -1] + \frac{1}{n} h(\theta; x_i)\mathbb{1}[y_i = 1] - \frac{1}{n} h(\theta; x_i)\mathbb{1}[y_i = -1] - \frac{1}{n} v^2 \right\}$$
Differentially Private SGDA for Minimax Problems

For any subset \( S_t \) of size \( m \), let \( I_t \) denote the set of indices in \( S_t \), the gradients of any \( j \in I_t \) are given by

\[
\nabla_\theta f(\theta, a, b, \nu; z_j) = \frac{2}{n_+} (h(\theta; x_j) - a) \nabla h(\theta; x_j) I[y_j = 1] + \frac{2}{n_-} (h(\theta; x_j) - b) \nabla h(\theta; x_j) I[y_j = -1]
\]

\[
+ 2(1 + \nu) \left( \frac{1}{n_+} \nabla h(\theta; x_j) I[y_j = 1] - \frac{1}{n_-} \nabla h(\theta; x_j) I[y_j = -1] \right)
\]

\[
\nabla_a f(\theta, a, b, \nu; z_j) = \frac{2}{n_+} (a - h(\theta; x_j)) I[y_j = 1], \quad \nabla_b f(\theta, a, b, \nu; z_j) = \frac{2}{n_-} (b - h(\theta; x_j)) I[y_j = -1]
\]

\[
\nabla_\nu f(\theta, a, b, \nu; z_j) = 2 \left( \frac{1}{n_+} h(\theta; x_j) I[y_j = 1] - \frac{1}{n_-} h(\theta; x_j) I[y_j = -1] \right) - \frac{2}{n} \nu
\]

(E.1)

The pseudo-code can be found in Algorithm 2.

F Additional Experimental Results

We show the details of NSEG and DP-SGDA (Linear and MLP settings) performance with using five different \( \epsilon \in \{0.1, 0.5, 1.5, 5, 10\} \) and three different \( \delta \in \{1e-4, 1e-5, 1e-6\} \) in Table F.1. From Table F.1, we can find that the performance will be decreased when decrease the value of \( \delta \) in the same \( \epsilon \) settings. The reason is that the small \( \delta \) is corresponding to a large value of \( \sigma \) based on Theorem 1. A large \( \sigma \) means a large noise will be added to the gradients during the training updates. Therefore, the AUC performance will be decreased as \( \delta \) decreasing. On the other hand, we can find that our DP-SGDA(Linear) outperforms NSEG under the same settings. This is because the NSEG method will add a larger noise than DP-SGDA into the gradients in the training and we have discussed this detail in the Section 4.2.

We also compare the \( \sigma \) values from NSEG and DP-SGDA methods on all datasets in Figure F.1 (a) with setting \( \delta=1e-5 \) and (b) \( \delta=1e-4 \). From the figure, it is clear that the \( \sigma \) from NSEG is larger than ours in all \( \epsilon \) settings. This implies the noise generated from NSEG is also larger than ours.

| Datasets | ijcm1 | MNIST | Fusion-MNIST |
|----------|------|-------|--------------|
|          | Linear | MLP   | Linear | MLP | Linear | MLP |
| Original | NSEG | DP-SGDA | DP-SGDA | NSEG | DP-SGDA | DP-SGDA |
| \( \epsilon=0.1 \) | 92.191 | 92.448 | 96.609 | 93.306 | 93.349 | 99.546 |
| \( \epsilon=0.5 \) | 90.231 | 91.229 | 94.020 | 91.285 | 91.962 | 98.300 |
| \( \epsilon=1 \) | 90.352 | 91.366 | 96.108 | 91.328 | 92.067 | 98.703 |
| \( \epsilon=5 \) | 90.358 | 91.376 | 96.316 | 91.331 | 92.073 | 98.722 |
| \( \epsilon=10 \) | 90.363 | 91.385 | 96.326 | 91.334 | 92.079 | 98.746 |

| \( \delta=1e-5 \) |
| \( \epsilon=0.1 \) | 90.168 | 91.169 | 93.274 | 91.266 | 91.910 | 98.092 |
| \( \epsilon=0.5 \) | 90.349 | 91.362 | 96.029 | 91.326 | 92.063 | 98.675 |
| \( \epsilon=1 \) | 90.357 | 91.373 | 96.209 | 91.330 | 92.071 | 98.714 |
| \( \epsilon=5 \) | 90.363 | 91.384 | 96.300 | 91.334 | 92.079 | 98.743 |
| \( \epsilon=10 \) | 90.363 | 91.386 | 96.301 | 91.334 | 92.080 | 98.747 |

| \( \delta=1e-6 \) |
| \( \epsilon=0.1 \) | 90.106 | 91.110 | 92.763 | 91.247 | 91.858 | 97.878 |
| \( \epsilon=0.5 \) | 90.346 | 91.357 | 95.840 | 91.324 | 92.058 | 98.650 |
| \( \epsilon=1 \) | 90.355 | 91.371 | 96.167 | 91.330 | 92.070 | 98.705 |
| \( \epsilon=5 \) | 90.363 | 91.383 | 96.294 | 91.334 | 92.078 | 98.742 |
| \( \epsilon=10 \) | 90.363 | 91.386 | 96.297 | 91.334 | 92.080 | 98.747 |

Table F.1: Comparison of \( \text{AUC} \) performance in NSEG and DP-SGDA (Linear and MLP settings) on three datasets with different \( \epsilon \) and different \( \delta \). The “Original” means no noise (\( \epsilon = \infty \)) is added in the algorithms.
Figure F.1: Comparison of $\sigma$ in NSEG and DP-SGDA (with Linear setting) on three datasets with different $\epsilon$ and (a) $\delta=1e^{-5}$ and (b) $\delta=1e^{-4}$. 