Structured Stochastic Linear Bandits

Nicholas Johnson, Vidyashankar Sivakumar, Arindam Banerjee
{njohnson,sivakuma,banerjee@cs.umn.edu}
Department of Computer Science and Engineering
University of Minnesota, Twin Cities

June 21, 2016

Abstract

The stochastic linear bandit problem proceeds in rounds where at each round the algorithm selects a vector from a decision set after which it receives a noisy linear loss parameterized by an unknown vector. The goal in such a problem is to minimize the (pseudo) regret which is the difference between the total expected loss of the algorithm and the total expected loss of the best fixed vector in hindsight. In this paper, we consider settings where the unknown parameter has structure, e.g., sparse, group sparse, low-rank, which can be captured by a norm, e.g., $L_1$, $L_{(1,2)}$, nuclear norm. We focus on constructing confidence ellipsoids which contain the unknown parameter across all rounds with high-probability. We show the radius of such ellipsoids depend on the Gaussian width of sets associated with the norm capturing the structure. Such characterization leads to tighter confidence ellipsoids and, therefore, sharper regret bounds compared to bounds in the existing literature which are based on the ambient dimensionality.

1 Introduction

We consider the stochastic linear bandit problem\cite{16} which proceeds in rounds $t = 1, \ldots, T$ where at each round $t$ the algorithm selects a vector $x_t$ from some decision set $\mathcal{X} \subset \mathbb{R}^p$ and receives a noisy loss defined as $\ell_t(x_t) = \langle x_t, \theta^* \rangle + \eta_t$ where $\theta^*$ is an unknown parameter and $\eta_t$ is martingale noise. The algorithm observes only $\ell_t(x_t)$ at each round $t$ and its goal is to minimize the cumulative loss. We measure its performance by the (pseudo) regret \cite{10} defined as

$$R_T = \sum_{t=1}^T \langle x_t, \theta^* \rangle - \arg\min_{x^* \in \mathcal{X}} \sum_{t=1}^T \langle x^*, \theta^* \rangle . \quad (1)$$

The stochastic linear bandit can be used to model problems in several real-world applications ranging from recommender systems to medical treatments to network security. Frequently, in such applications, one has knowledge of the structure of the unknown parameter $\theta^*$, for example, $\theta^*$ may be sparse, group sparse, or low-rank. Previous works\cite{16} either made no structural assumptions on $\theta^*$ and proved regret bounds\cite{1} of the form $\tilde{O}(p\sqrt{T})$ or assumed $\theta^*$ was $s$-sparse ($s$ non-zero elements) and showed\cite{2} the regret sharpens to $O(\sqrt{spT})$. In this paper, we consider the setting

\footnote{The $\tilde{O}()$ notation selectively hides constants and log terms.}
where $\theta^*$ is any generally structure vector (sparse, group sparse, low-rank, etc.) such that the structure can be captured by some norm ($L_1$, $L_{(1,2)}$, nuclear norm, etc.).

Our approach follows previous works \cite{16, 1, 2} which use the optimism-in-the-face-of-uncertainty principle \cite{10} to design a class of algorithms which construct a confidence ellipsoid $C_t$ such that $\theta^* \in C_t$ across all rounds with high-probability. After which, the algorithm selects a single $x_{t+1}$ by solving a bilinear optimization problem with respect to parameters $x \in \mathcal{X}$ and $\theta \in C_t$.

Our algorithm differs from previous algorithms \cite{16, 1, 2} in two key ways. First, for the initial rounds, we select random samples from the decision set $\mathcal{X}$ to compute an estimate of $\theta^*$ such that the estimate is statistically consistent. The length of the random estimation rounds is dependent on the structure, where, for example, if $\theta^*$ is $s$-sparse scales like $s \log p$ and if $\theta^*$ is unstructured scales like $p$. Second, after the random estimation rounds, we select samples uniformly at random from specific subsets of $\mathcal{X}$. More specifically, previous works selected a sample by solving a bilinear optimization problem however, such an approach only gives one, possibly unique, solution. We build on such works by solving a similar bilinear optimization problem but then center two $L_2$ balls of suitable radii over the parameters $x$ and $\theta$ and select $x_{t+1}$ uniformly at random using such balls.

**Overview of Results.** The main technical challenge in previous works \cite{16, 1, 2} is constructing confidence ellipsoids which contain $\theta^*$ across all rounds with high-probability. The focus of our work is again to construct confidence ellipsoids such that $\theta^* \in C_t$ across all rounds with high-probability but which are general enough to hold for any norm structured $\theta^*$. Moreover, we desire that the ellipsoids are tighter than previous works in order to provide sharper regret bounds. Previous works \cite{16, 1} constructed the confidence ellipsoids by solving a ridge regression problem to compute an estimate $\hat{\theta}_t$ and centered an ellipsoid over the estimate. We generalize such an approach by instead solving a norm regularized regression problem, e.g., Lasso, given the structure of $\theta^*$. We show our construction of $C_t$ contains $\theta^*$ across all rounds with high-probability by extending recent results in structured estimation \cite{11, 13, 21, 5} which rely on i.i.d. samples to active sampling.

The main technical result we show is that the radius of our confidence ellipsoids depend on the Gaussian width\footnote{The Gaussian width is a geometric characterization of the size of a set and the definition is presented in Section A} of sets associated with the structure of $\theta^*$ which leads to tighter confidence ellipsoids than previous works \cite{16, 1} when $\theta^*$ is structured. For example, with an $s$-sparse $\theta^*$ the radius of the confidence ellipsoid scales as $O(\sqrt{s \log p})$ compared to $O(\sqrt{p})$ in the unstructured settings considered in \cite{16, 1}.

The regret bounds for our algorithm follow from the analysis in \cite{16} and depend on the radius of the confidence ellipsoid therefore, our regret bounds scale with the structure of $\theta^*$ as measured by the Gaussian width which leads to sharper regret bounds when $\theta^*$ is generally structured and matches existing bounds when $\theta^*$ is $s$-sparse \cite{2} or unstructured \cite{16}.

\subsection{Previous Works}

Multiarmed bandits have a mature and active literature \cite{10}. One popular algorithm based on the OFU principle and upper confidence bounds is UCB \cite{4} used for the $K$-arm stochastic bandit problem where the algorithm selects a decision $k \in \{1, \ldots, K\}$ and receives a stochastic loss drawn i.i.d. from the $k$th decision’s distribution. The regret was shown to be $O(\Delta \log T)$ where $\Delta$ is the gap in performance between the best and second best decisions.

A similar problem has been considered \cite{3, 19, 15} when a $p$-dimensional feature vector is provided
for each of the $K$ decisions and the expected loss is a linear function of the feature vector and an unknown parameter. For such a problem, a regret bound of $O(\log^{3/2}(K) \sqrt{p} \sqrt{T})$ was shown. However, dependence on the number of decisions $K$ can be problematic when it is large or infinite. For such settings, [16] studied the stochastic linear bandit problem where the decision set is an arbitrary compact set in $\mathbb{R}^p$. They presented the algorithm ConfidenceBall2 based on the OFU principle which computes an estimate $\hat{\theta}_t$ of the unknown parameter $\theta^*$ using ridge regression and constructs an ellipsoidal confidence set around $\hat{\theta}_t$ with radius $\sqrt{\beta_t}$. After which the vectors $x_t$ and $\tilde{\theta}_t$ are selected optimistically from the decision set and confidence ellipsoid respectively, such that $\langle x_t, \tilde{\theta}_t \rangle$ is minimized. They showed how to set $\beta_t$ such that $\theta^*$ stays within the ellipsoid with high-probability for all $t$ and showed a problem independent regret of $\tilde{O}(p \sqrt{T})$ and a problem dependent regret of $\tilde{O}(\sqrt{\beta} \log^3 T)$ where $\Delta > 0$ is the gap between the best and second best extremal points. Note, the regret depends on the ambient dimensionality $p$ and not the number of decisions. Further, [24, 1] showed how to construct tighter confidence ellipsoids, specifically, [1] used a self-normalized tail inequality for vector-valued martingales which decreased the regret by a $\sqrt{\log T}$ multiplicative factor.

Building on such works which had considered the problem without structural assumptions on $\theta^*$, two papers published simultaneously [12, 2] considered the problem where $\theta^*$ is $s$-sparse. [2] followed the same problem setting as [16] and presented a method which can use the predictions of any full information online algorithm with an upper bound on its regret to construct confidence sets. When constructing confidence sets using the algorithm SeqSEW [18] they showed a problem independent regret of $\tilde{O}(\sqrt{s p T})$ and a problem dependent regret of $\tilde{O}(s \Delta \log^2 T)$.

[12] uses initial rounds to estimate the sparse structure similar to our work however, they do not consider the standard stochastic linear bandit problem. Specifically, they define the loss as $\ell_t(x_t) = \langle x_t, \theta^* \rangle + \langle x_t, \eta_t \rangle$ where $\eta_t$ is i.i.d. white noise (not martingale) and they assume the decision set is the unit $L_2$ ball. During the random estimation rounds, they used techniques from compressed sensing to identify the subspace where $\theta^*$ lives then ran ConfidenceBall2 where the decision set is a subset of the subspace and showed a problem independent regret of $\tilde{O}(s \sqrt{T})$.

Such papers show that sharper regret bounds can be obtained when $\theta^*$ is structured however, only for a sparse $\theta^*$. Such results motivate our work to study the regret for any generally norm structured $\theta^*$. We organize the paper as follows. In Section 2 we give background on high-dimensional structured estimation which our analysis builds on. Section 3 we present the problem setting and algorithm. Section 4 we present our main regret bounds from a high-level and provide examples of popular types of structure. Section 5 we present the main technical results of the analysis and point to the detailed proofs in the appendix. Finally, we conclude in Section 6.

2 Background: High-Dimensional Structured Estimation

We rely on recent developments in the analysis of non-asymptotic bounds for structured estimation in high-dimensional statistics. In this section, we will discuss the main results needed for our analysis which can be found in the following papers [11, 7, 13, 22, 27, 8, 5, 6].

In high-dimensional structured estimation, one is concerned with settings in which the dimension $p$ of the parameter $\theta^*$ to be estimated is significantly larger than the sample size $n$, i.e., $p \gg n$. It
is known that for \( n \) i.i.d. Gaussian samples, one can compute an estimate \( \hat{\theta}_n \) using least squares regression which converges to \( \theta^* \) at a rate of \( O\left(\frac{\sqrt{p}}{n}\right) \). The convergence rate can be improved when \( \theta^* \) is structured which is usually characterized as having a small value according to some norm \( R(\cdot) \). For such problems, estimation is performed by solving a norm regularized regression problem

\[
\hat{\theta}_n := \arg\min_{\theta \in \mathbb{R}^p} L(\theta, Z_n) + \lambda_n R(\theta) \tag{2}
\]

where \( L(\cdot, \cdot) \) is a convex loss function, \( Z_n \) is a dataset consisting of i.i.d. pairs \( \{(x_i, y_i)\}_{i=1}^n \) where \( x_i \in \mathbb{R}^p \) is a sample, \( y_i \in \mathbb{R} \) is the response, and \( \lambda_n \) is the regularization parameter.

For such problems, let \( \hat{\theta}_n - \theta^* \) be the estimation error vector, then for a suitably large \( \lambda_n \), [5] showed the error vector deterministically belongs to the restricted error set

\[
E_{r,n} = \left\{ \hat{\theta}_n - \theta^* \in \mathbb{R}^p : R(\hat{\theta}_n) \leq R(\theta^*) + \frac{1}{\rho} R(\hat{\theta}_n - \theta^*) \right\} \tag{3}
\]

where \( \rho > 1 \) is a constant which we fix as \( \rho = 2 \) for ease of exposition. For such a \( \rho \), \( E_{r,n} \) is a restricted set of directions, in particular, the error vector \( \hat{\theta}_n - \theta^* \) cannot be in the direction of \( \theta^* \).

Using the restricted error set, bounds on the estimation error can be established which hold with high-probability under two assumptions. First, the regularization parameter \( \lambda_n \) must satisfy the inequality

\[
\lambda_n \geq 2 R^*(\nabla L(\theta^*, Z_n)) \tag{4}
\]

where \( R^*(\cdot) \) is the dual norm of \( R(\cdot) \). Second, the loss function must satisfy the restricted strong convexity (RSC) condition in the restricted error set \( E_{r,n} \) as illustrated in [22]. Specifically, there exists a \( \kappa > 0 \) such that

\[
L(\hat{\theta}_n) - L(\theta^*) - \langle \nabla L(\theta^*), \hat{\theta}_n - \theta^* \rangle \geq \kappa \|\hat{\theta}_n - \theta^*\|_2^2 \quad \forall \hat{\theta}_n - \theta^* \in E_{r,n}. \tag{5}
\]

For the squared loss, the RSC condition simplifies to the restricted eigenvalue (RE) condition

\[
\frac{1}{n} \|X_n(\hat{\theta}_n - \theta^*)\|_2^2 \geq \kappa \|\hat{\theta}_n - \theta^*\|_2^2 \quad \forall \hat{\theta}_n - \theta^* \in E_{r,n} \tag{6}
\]

where \( X_n \in \mathbb{R}^{n \times p} \) is the design matrix [13]. Under such conditions, the following bound holds with high-probability [22, 5]

\[
\|\hat{\theta}_n - \theta^*\|_2 \leq c \psi(E_{r,n}) \frac{\lambda_n}{\kappa} \tag{7}
\]

where \( \psi(E_{r,n}) = \sup_{u \in E_{r,n}} \frac{R(u)}{\|u\|_2} \) is the norm compatibility constant and \( c > 0 \) is a constant. For an \( s \)-sparse \( \theta^* \), one obtains \( \|\hat{\theta}_n - \theta^*\|_2 \leq O\left(\sqrt{\frac{s \log p}{n}}\right) \) and for a group sparse \( \theta^* \), one obtains \( \|\hat{\theta}_n - \theta^*\|_2 \leq O\left(\sqrt{\frac{s_G (m + \log K)}{n}}\right) \) where \( K \) is the number of groups, \( m \) is the maximum group size, and \( s_G \) is the group sparsity level. Similar bounds can be computed for other types of structure including low-rank.

\footnote{We drop the second argument when it is clear from the context.}
3 Structured Bandits: Problem and Algorithm

Here, we will formally define the problem, mention the assumptions under which our analysis works, and present our algorithm. The results and analysis are presented in subsequent sections.

3.1 Problem Setting

We consider the stochastic linear bandit problem [16, 1] where in each round \( t = 1, \ldots, T \) the algorithm selects a \( p \)-dimensional vector \( x_t \) from the decision set \( \mathcal{X} \) and receives a loss of \( \ell_t(x_t) = \langle x_t, \theta^* \rangle + \eta_t \). Our focus is on settings where the unknown parameter \( \theta^* \in \mathbb{R}^p \) is structured which we characterize as having a small value according to some norm \( \hat{R}(\cdot) \).

The goal of the algorithm is to minimize its cumulative loss \( \sum_t \ell_t(x_t) \) and we measure the performance of the algorithm in terms of the fixed cumulative (pseudo) regret defined as

\[
R_T = \sum_{t=1}^T \langle x_t, \theta^* \rangle - \min_{x^* \in \mathcal{X}} \sum_{t=1}^T \langle x^*, \theta^* \rangle .
\] (8)

We require that the algorithm’s regret grows sub-linearly in \( T \), i.e., \( R_T = o(T) \), and desire it grows with the structure of \( \theta^* \) rather than the ambient dimensionality \( p \) with high-probability. The following assumptions under which our analysis holds are standard in the literature [16, 1, 2].

3.2 Assumptions and Definitions

**Assumption 1** The decision set \( \mathcal{X} \subset \mathbb{R}^p \) is a compact (closed and bounded) convex set with non-empty interior. For ease of exposition, we assume \( \mathcal{X} \subset B_2^p \), the (closed) unit \( L_2 \) ball defined as \( B_2^p = \{ x \in \mathbb{R}^p : \|x\|_2 \leq 1 \} \), to avoid scaling factors.

**Assumption 2** The noise is a bounded martingale difference sequence (MDS), i.e., \( |\eta_t| \leq B, \mathbb{E}[\eta_t] < \infty, \mathbb{E}[\eta_t | F_{t-1}] = 0 \forall t \) where \( F_t = \{ x_1, \ldots, x_{t+1}, \eta_1, \ldots, \eta_t \} \) is a filtration (sequence of \( \sigma \)-algebras).

We assume bounded noise for simplicity however, the results hold for any sub-Gaussian noise (refer to Section A for definitions of sub-Gaussian and related quantities).

**Assumption 3** We assume the unknown parameter \( \theta^* \) is fixed for all rounds, the structure is known, for example, for an \( s \)-sparse \( \theta^* \) the value of \( s \) is known, and \( \|\theta^*\|_2 = 1 \).

**Assumption 4** The number of rounds \( T \) is known a priori.

**Definition 1** The Gaussian width [13] of a set \( A \) is defined as \( w(A) = \mathbb{E} [\sup_{u,v \in A} \langle g, u \rangle] \) where the expectation is over \( g \) which is a zero mean, unit variance Gaussian random variable.

**Definition 2** For a set \( A, \phi(A) = \sup_{u,v \in A} \|u-v\|_2 = \sup_{u \in A} \|u\|_2 \) measures the diameter of \( A \).

**Definition 3** The restricted error set is defined as \( E_{r,t} := \left\{ \hat{\theta}_t - \theta^* \in \mathbb{R}^p : R(\hat{\theta}_t) \leq R(\theta^*) + \frac{1}{2}R(\hat{\theta}_t - \theta^*) \right\} \) and the largest such error set is \( E_{r,max} = \arg\max_{E_r \in \{E_{r,1}, \ldots, E_{r,T}\}} w(E_r) \).

**Definition 4** The set \( A_t \) is a spherical cap constructed as \( A_t := \text{cone}(E_{r,t}) \cap S^{p-1} \) where \( S^{p-1} \) is the unit sphere in \( p \)-dimensions and \( A_{max} := \text{cone}(E_{r,max}) \cap S^{p-1} \) is the largest such cap.

**Definition 5** Each \( x_t \) has sub-Gaussian norm (refer to Section A for the definition of sub-Gaussian norm) satisfying \( \|x_t\|_{\psi_2} \leq K \) for some absolute constant \( K \). This follows from Assumption [11]

**Definition 6** The unit norm \( R(\cdot) \) ball is \( \Omega_R := \{ u \in \mathbb{R}^p : R(u) \leq 1 \} \). The norm compatibility constant with respect to vectors in the restricted error set at round \( t \) is \( \psi(E_{r,t}) = \sup_{u \in E_{r,t}} \frac{R(u)}{\|u\|_2} \).

5
Algorithm 1 Structured Stochastic Linear Bandit

1: Input: \( p, \mathcal{X}, R(\cdot), T, E_{r, \max}, A_{\max}, \Omega_R, \gamma, \epsilon, c_0, c', C \)
2: Set \( \beta = C\psi(E_{r, \max})(w(\Omega_R) + \sqrt{\gamma^2 + \log T \phi(\Omega_R)}/2) \)
3: Play \( n = c'w^2(A_{\max})(\epsilon^2 + \log T) \) uniform i.i.d. random vectors \( x_{1:n} \in \mathcal{X} \) and receive losses \( \ell_{1:n} \)
4: For \( t = n, \ldots, T \)
5: Compute \( X_t = [x_1 \ldots x_t]^T, y_t = [\ell_1(x_1) \ldots \ell_t(x_t)]^T \), and \( D_t = X_t^T X_t \)
6: Set \( \lambda_t = c_0(w(\Omega_R) + \sqrt{\gamma^2 + \log T})/\sqrt{t} \)
7: Compute \( \hat{\theta}_t = \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{t} \| y_t - X_t \theta \|^2 + \lambda_t R(\theta) \)
8: Construct \( C_t := \{ \theta : \| \theta - \hat{\theta}_t \|_{2, D_t} \leq \beta \} \)
9: Compute \((x'_{t+1}, \theta'_{t+1}) := \arg\min_{x \in \mathcal{X}, \theta \in C_t \cap S^{p-1}} \langle x, \theta \rangle \)
10: Play \( x_{t+1} \sim \text{Uniform}(\mathcal{X} \cap \tilde{B}_2^p(x'_{t+1}, \|x'_{t+1}\|_2/2)) \) and receive loss \( \ell_{t+1}(x_{t+1}) \)
11: End For

3.3 Algorithm

For the initial \( t = 1, \ldots, n = c'w^2(A_{\max})(\epsilon^2 + \log T) \) rounds where \( c' > 0 \) is a constant, our algorithm selects vectors \( x_{1:n} := \{x_1, \ldots, x_n\} \) uniformly at random from \( \mathcal{X} \) and receives the corresponding losses \( \ell_{1:n} := \{\ell_1(x_1), \ldots, \ell_n(x_n)\} \). The length of such random estimation rounds depends on the Gaussian width of the largest spherical cap induced by the structure of \( \theta^* \) and a parameter \( \epsilon > 0 \) which controls the success probability and will become clear in the analysis in Section 5. The random estimation rounds can be considered the “burn-in” period similar to the use of a barycentric spanner or identity matrix as in [16, 1].

After the loss \( \ell_n(x_n) \) is received in round \( n \), the algorithm constructs an \((n \times p)\)-dimensional design matrix \( X_n = [x_1 \ldots x_n]^T \), a sample covariance matrix \( D_n = X_n^T X_n \), and an \( n \)-dimensional response vector \( y_n = [\ell_1(x_1) \ldots \ell_n(x_n)]^T \). The algorithm then computes an estimate \( \hat{\theta}_n \) by solving a norm regularized regression problem, constructs a confidence ellipsoid using the Mahalanobis distance defined as \( \| \theta - \hat{\theta}_n \|_{2, D_n} = \sqrt{\langle \theta - \hat{\theta}_n, D_n(\theta - \hat{\theta}_n) \rangle} \), then selects a sample to play. Specifically, the algorithm performs the following four main steps sequentially in each round thereafter.

For each \( t = n, \ldots, T \):

1. Compute an estimate:
   \[
   \hat{\theta}_t := \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{t} \| y_t - X_t \theta \|^2 + \lambda_t R(\theta)
   \]
2. Construct a confidence ellipsoid:
   \[
   C_t := \{ \theta \in \mathbb{R}^p : \| \theta - \hat{\theta}_t \|_{2, D_t} \leq \beta \}
   \]
3. Compute an optimal solution:
   \[
   (x'_{t+1}, \theta'_{t+1}) := \arg\min_{x \in \mathcal{X}, \theta \in C_t \cap S^{p-1}} \langle x, \theta \rangle
   \]
4. Play vector \( x_{t+1} \sim \text{Uniform}(\mathcal{X} \cap \tilde{B}_2^p(x'_{t+1}, \|x'_{t+1}\|_2/2)) \) and receive loss \( \ell_{t+1}(x_{t+1}) \)

where \( \tilde{B}_2^p(x'_{t+1}, \|x'_{t+1}\|_2/2) \) is a closed \( L_2 \) ball centered at \( x'_{t+1} \) with radius \( \|x'_{t+1}\|_2/2 \). After receiving the loss \( \ell_{t+1}(x_{t+1}) \), the design matrix \( X_{t+1} \) and response vector \( y_{t+1} \) are updated with \( x_{t+1} \) and \( \ell_{t+1}(x_{t+1}) \) respectively. Then, the sample covariance matrix \( D_{t+1} = X_{t+1}^T X_{t+1} \) is recomputed and the regularization parameter \( \lambda_{t+1} \) is updated.
3.3.1 Discussion

Step 1. An estimate is computed by solving a norm regularized regression problem following existing results discussed in Section 2. This generalizes previous works [16, 11] which only consider computing an estimate by solving the ridge regression problem.

Step 2. A confidence ellipsoid is constructed in order to allow the algorithm to explore in certain directions. Since the confidence ellipsoid is defined as \( C_t = \{ \theta \in \mathbb{R}^p : \| \theta - \hat{\theta}_t \|_{2,D_t} \leq \beta \} \), we focus on bounds for \( \| \hat{\theta}_t - \theta^* \|_{2,D_t} \). Extending the results in Section 2 we will show in Section 5 high-probability bounds on the estimation error of the form \( \| \hat{\theta}_t - \theta^* \|_{2,D_t} \leq \psi(E_{r,t}) \frac{\lambda_t}{\kappa} \sqrt{t} \). Therefore, setting \( \beta \) to the right hand side will give bounds such that \( \theta^* \in C_t \) with high-probability. The value of \( \beta \) then depends on two key terms: the regularization parameter \( \lambda_t \) and the restricted eigenvalue (RE) constant \( \kappa \) detailed in (6). The value of \( \lambda_t \) is set by the user and we will provide an explicit characterization of its value in Section 5. Moreover, the estimation error bound will only hold when the RE constant \( \kappa \) is positive and we will show in Section 5 that after a suitable number of random estimation rounds and by selecting samples via Step 3, it will be positive for all rounds.

Steps 3 and 4. These steps are motivated from the regret analysis established in [16] and the need to satisfy the RE condition. Let the instantaneous regret at round \( t+1 \) be defined as \( r_{t+1} = \langle x_{t+1}, \theta^* \rangle - \langle x^*, \theta^* \rangle \) where \( x^* = \arg\min_{x \in \mathcal{X}} \langle x, \theta^* \rangle \). As shown in [16], by selecting an \( x_{t+1} \) and \( \hat{\theta}_{t+1} \) via

\[
(x_{t+1}, \hat{\theta}_{t+1}) := \arg\min_{x, \theta \in \mathcal{X}} \langle x, \theta \rangle
\]  

the instantaneous regret can be upper bounded as \( r_{t+1} = \langle x_{t+1}, \theta^* \rangle - \langle x^*, \theta^* \rangle \leq \langle x_{t+1}, \hat{\theta}_{t+1} \rangle - \langle x_{t+1}, \hat{\theta}_{t+1} \rangle \) because we optimize over both \( x \) and \( \theta \). Therefore, one obtains the following inequality \( \langle x_{t+1}, \hat{\theta}_{t+1} \rangle \leq \langle x^*, \theta^* \rangle \) on which the entire regret analysis relies. We will use the regret analysis from [16] therefore, we need to select an \( x_{t+1} \) and \( \hat{\theta}_{t+1} \) such that the above inequality holds.

Additionally, recall the RE condition in (6)

\[
\frac{1}{t} \| X_t (\hat{\theta}_t - \theta^*) \|_2^2 \geq \kappa \| \hat{\theta}_t - \theta^* \|_2^2 \quad \forall \hat{\theta}_t - \theta^* \in E_{r,t}.
\]

We must have \( \kappa > 0 \) for the estimation error bound used to compute \( \beta \) to hold. Therefore, in order to show such a \( \kappa \) exists, we need samples which are not too correlated otherwise the design matrix will be ill-conditioned.

To use the regret analysis and satisfy the RE condition, we cannot exactly follow existing work [16, 11, 2] and select an \( x_{t+1} \) by solving (12) since we may obtain a single unique solution and the rows of the design matrix will be too correlated. Instead, we select samples uniformly at random from specific subsets of \( \mathcal{X} \) which spreads the samples out enough to show the RE condition holds. Moreover, as we will show in Section 5 for any random sample \( x_{t+1} \) we select, we can deterministically compute a \( \hat{\theta}_{t+1} \in C_t \) such that the inequality \( \langle x_{t+1}, \hat{\theta}_{t+1} \rangle \leq \langle x^*, \theta^* \rangle \) holds.

Steps 1, 2, and 4 can be performed efficiently, in particular, there are several efficient methods for computing the estimate in (9) for common regularizers, e.g., \( L_1 \), \( L_{(1,2)} \), nuclear norm, etc. \[17, 23, 3\]. Step 3 is computationally difficult in general (similar to all previous work) however, for simple decision sets such as the unit the \( L_2 \) ball, a solution can be computed efficiently by solving the corresponding quadratically constrained quadratic program. Our algorithm for structured stochastic linear bandits is presented in Algorithm 4.
4 Regret Bound for Structured Bandits

Here, we present the main result which is a high-probability bound on the regret of Algorithm 1 and show examples for popular types of structure. The analysis of the bound is presented in Section 3.1. First, we review some of the assumptions from Section 3.2. We assume $\mathcal{X}$ is a compact convex set with non-empty interior. Further, we assume $\mathcal{X} \subseteq B_2^p$, the unit $L_2$ ball, for ease of exposition. Examples of such decision sets include: $L_p$ balls for $1 \leq p < \infty$, ellipsoids, polytopes, norm cones, and hypercubes. We assume the noise $\eta_t$ is a bounded MDS where $|\eta_t| \leq B, \forall t$ and the number of rounds $T$ is known a priori.

Further, we recall a few definitions introduced in Section 3.2 which will help interpret the main result. Under such assumptions, we present the main result in a high-level form, which hides the exact nature of the constants involved. A more explicit form of the constants is presented in the appendix.

The main result consists of two theorems for the problem independent and problem dependent nature of the constants involved. A more explicit form of the constants is presented in the appendix.

The main result consists of two theorems for the problem independent and problem dependent settings [11]. Let $\mathcal{E}$ be the set of all extremal points. The problem independent setting occurs when the difference between the expected loss of the best extremal point $x^*$ and the expected loss of the second best extremal point is zero, i.e., $\Delta = \inf_{x \in \mathcal{E}} \langle x, \theta^* \rangle - \langle x^*, \theta^* \rangle = 0$. Such a setting occurs, for example, when the decision set is the unit $L_2$ ball. The problem dependent setting occurs when $\Delta > 0$, for example, when the decision set is a polytope.

**Theorem 1 (Problem Independent Regret Bound)** For any $\epsilon, \gamma > 0$, choose the radius of the ellipsoid in Algorithm 1 as

$$\beta = c_0 \psi(E_{r,\text{max}}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T \frac{\phi(\Omega_R)}{2}} \right).$$  \hspace{1cm} (13)

Then, for any $T > c' w^2(A_{\text{max}}) (\epsilon^2 + \log T)$, with probability at least $1 - \frac{1}{4} \exp(-w^2(A_{\text{max}})\epsilon^2) - c_2 \exp(-\gamma^2)$, the fixed cumulative regret of Algorithm 1 is at most

$$R_T \leq O \left( \psi(E_{r,\text{max}}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T} \right) \sqrt{\log T} \sqrt{T \log T} \right),$$  \hspace{1cm} (14)

where $c', c_0, c_1, c_2 > 0$ are constants.

**Theorem 2 (Problem Dependent Regret Bound)** For any $\epsilon, \gamma > 0$, choose the radius of the ellipsoid in Algorithm 1 as

$$\beta = c_0 \psi(E_{r,\text{max}}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T \frac{\phi(\Omega_R)}{2}} \right).$$  \hspace{1cm} (15)

Then, for any $T > c' w^2(A_{\text{max}}) (\epsilon^2 + \log T)$, with probability at least $1 - \frac{1}{4} \exp(-w^2(A_{\text{max}})\epsilon^2) - c_2 \exp(-\gamma^2)$, the fixed cumulative regret of Algorithm 1 with a decision set which has non-zero gap $\Delta > 0$ is at most

$$R_T \leq O \left( \psi^2(E_{r,\text{max}}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T} \right)^2 \log T / \Delta \right),$$  \hspace{1cm} (16)

where $c', c_0, c_1, c_2 > 0$ are constants.
4.1 Examples

We present the problem independent regret of popular types of structured $\theta^*$ using Theorem 1 and the values of $\psi(E_{r,\text{max}})$ and $w(\Omega_R)$ from [12, 13, 14]. The problem dependent regret can be similarly computed. Only unstructured and sparse structures have been considered [16, 1, 2, 12]. No previous works have considered any other types of structure including group sparse and low-rank.

**Example 1 (Unstructured)** For problems where $\theta^*$ is not structured, we simply use $R(\theta) = \|\theta\|_2^2$ and solve the ridge regression problem

$$\hat{\theta}_t = \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{t} \| y_t - X_t \theta \|_2^2 + \lambda_t \|\theta\|_2^2 = \left( X_t^\top X_t + \lambda_t I_{p \times p} \right)^{-1} X_t^\top y_t. \quad (17)$$

We compute the regret by plugging in the values $\psi(E_{r,\text{max}}) = O(1)$ and $w(\Omega_R) = O(\sqrt{p})$ to obtain a regret of $O(p\sqrt{T})$. Such a regret matches [16, 1] up to log and constant factors.

**Example 2 (Sparse)** For problems where $\theta^*$ is $s$-sparse ($s$ non-zeros), one common regularizer to induce sparse solutions is $R(\theta) = \|\theta\|_1$. With such a regularizer, we solve the Lasso problem

$$\hat{\theta}_t = \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{t} \| y_t - X_t \theta \|_2^2 + \lambda_t \|\theta\|_1. \quad (18)$$

We compute the regret by plugging in the values $\psi(E_{r,\text{max}}) = O(\sqrt{s})$ and $w(\Omega_R) = O(\sqrt{\log p})$ to obtain a regret of $O(\sqrt{s\log p} / \sqrt{T})$ which matches [2] up to log and constant factors. Note, it is worse than the regret from [12] which is $O(s\sqrt{T})$ however, they consider a different noise model in the loss function.

**Example 3 (Group Sparse)** Let $\{1, \ldots, p\}$ be an index set of $\theta^*$, $G = \{G_1, \ldots, G_K\}$ be a known set of $K$ groups which define a disjoint partitioning of the index set. For group sparse problems, one common regularizer is $R(\theta) = \sum_{i=1}^{K} ||\theta_{G_i}||_2$ where $\theta_{G_i}$ is a vector with elements equal to $\theta$ for indices in $G_i$ and 0 otherwise. With such a regularizer, we solve the group lasso problem

$$\hat{\theta}_t = \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{t} \| y_t - X_t \theta \|_2^2 + \lambda_t \sum_{i=1}^{K} ||\theta_{G_i}||_2. \quad (19)$$

With maximum group size $m = \max_i |G_i|$ and subset $S_G \subset \{1, \ldots, K\}$ of the groups with cardinality $s_G$ which denotes the number of active groups, we compute the regret by plugging in the values $\psi(E_{r,\text{max}}) = O(\sqrt{s_G})$ and $w(\Omega_R) = O(\sqrt{m + \log K})$ to obtain a regret of $O(\sqrt{s_G(m + \log K)} \sqrt{T})$.

**Example 4 (Low-Rank)** Let $\Theta^* \in \mathbb{R}^{d \times p}$ be a matrix with rank $r$ and we select the matrix $X_t \in \mathbb{R}^{d \times p}$ at each round. Define the loss we receive as $\ell_t(X_t) = \text{trace}(X_t^\top \Theta^*) + \eta_t$. For problems where the rank of $\Theta^*$ is small, for example, $r \leq \min(d, p)$, one common regularizer to use is the nuclear norm $R(\Theta) = \|\Theta\|_* = \sum_{j=1}^{\min(d, p)} \sigma_j(\Theta)$ where $\sigma_j(\Theta)$ are the singular values of the $\Theta$. With such a regularizer, we solve the trace-norm regularized least squares problem

$$\hat{\Theta}_t = \arg\min_{\Theta \in \mathbb{R}^{d \times p}} \frac{1}{t} \sum_{i=1}^{t} \left( y_i - \text{trace}(X_i^\top \Theta) \right)^2 + \lambda_t \|\Theta\|_* . \quad (20)$$

We compute the regret by plugging in the values $\psi(E_{r,\text{max}}) = O(\sqrt{T})$ and $w(\Omega_R) = O(\sqrt{d + p})$ from [21] to obtain a regret of $O(\sqrt{r(d + p)} \sqrt{T})$.  

9
5 Overview of the Analysis

The analysis starts from a regret result established in [16]. Let \( r_t = \langle x_t, \theta^* \rangle - \langle x^*, \theta^* \rangle \) denote the instantaneous regret acquired by the algorithm on round \( t \) where \( x^* = \arg\min_{x \in X} \langle x, \theta^* \rangle \) is the optimal vector. Then for Algorithm 1, as long as we have \( \theta^* \in C_t \) over all rounds \( t \), [16, Theorem 6] shows that \( \sum_{t=1}^{T} r_t^2 \leq 8 \beta^2 p \log T \). Then, to establish a problem independent regret bound we directly apply the Cauchy-Schwarz inequality to get

\[
R_T = \sum_{t=1}^{T} r_t \leq \left( \sum_{t=1}^{T} r_t^2 \right)^{1/2} \leq \beta \sqrt{8p T \log T} , \tag{21}
\]

which holds conditioned on \( \theta^* \in C_t \) over all rounds \( t \). Moreover, for a problem dependent regret bound, we follow the proof of [16, Theorem 1] which shows

\[
R_T = \sum_{t=1}^{T} r_t \leq \frac{T}{\Delta} \sum_{t=1}^{T} r_t^2 \Delta \leq \frac{8p \beta^2 \log T}{\Delta} , \tag{22}
\]

which holds conditioned on \( \theta^* \in C_t \) over all rounds \( t \).

The focus of our analysis is then to choose a \( \beta \) such that the condition holds with high-probability uniformly over all rounds. From Algorithm 1, since \( C_t := \{ \theta : \| \theta - \hat{\theta}_t \|_{2, D_t} \leq \beta \} \) and we want to have \( \theta^* \in C_t \), we focus on bounds for \( \| \hat{\theta}_t - \theta^* \|_{2, D_t} \), the instantaneous estimation error. Building on ideas for high-dimensional structured estimation as discussed in Section 2, deterministic bounds on the instantaneous estimation error can be obtained under two assumptions. First, we need to choose the regularization parameter \( \lambda_t \) such that

\[
\lambda_t \geq 2 R^* \left( \frac{1}{T} X_t^\top (y_t - X_t \theta^*) \right) . \tag{23}
\]

Second, for all \( \hat{\theta}_t - \theta^* \in E_{r,t} \), we need to have the restricted eigenvalue (RE) condition for constant \( \kappa > 0 \)

\[
\inf_{\theta_t - \theta^* \in E_{r,t}} \frac{1}{t} \| X_t (\hat{\theta}_t - \theta^*) \|_2^2 \geq \kappa \| \hat{\theta}_t - \theta^* \|_2^2 . \tag{24}
\]

Under these two assumptions, following existing analysis for high-dimensional estimation, we have the following theorem (refer to Section 3 for the proof).

**Theorem 3** Assume that the RE condition is satisfied in the set \( E_{r,t} \) with parameter \( \kappa \) and \( \lambda_t \) is suitably large. Then for any norm \( R(\cdot) \), we have for constant \( c > 0 \)

\[
\| \hat{\theta}_t - \theta^* \|_{2, D_t} \leq c \psi(E_{r,t}) \frac{\lambda_t}{\kappa} \sqrt{t} . \tag{25}
\]

In the Sections 4 and 5 we show that the two assumptions in fact hold with high-probability. In particular, for the assumption in (23), we show the following result.

**Theorem 4** For any \( \gamma > 0 \) and for absolute constant \( L > 0 \), with probability at least \( 1 - L \exp(-\gamma^2) \), the following bound holds uniformly for all rounds \( t = 1, \ldots, T \):

\[
R^* \left( \frac{1}{T} X_t^\top (y_t - X_t \theta^*) \right) \leq 2LKB \left( w(O_R) + \sqrt{\gamma^2 + \log \frac{\phi(O_R)}{2}} \right) \sqrt{t} . \tag{26}
\]
Then, from (23), for $c_0 = 4LKB$ we set $\lambda_t$ as

$$
\lambda_t \geq c_0 \frac{\left( w(\Omega_R) + \gamma^2 + \log T \psi(\Omega_R) / 2 \right)}{\sqrt{t}}.
$$

(27)

As for the assumption in (24), we show the following result.

**Theorem 5** For constants $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ and any $\epsilon > 0$, with probability at least $1 - c_0 \exp(-w^2(A_{\max})\epsilon^2)$ the following holds uniformly for all rounds $t = 1, \ldots, T$:

$$
\inf_{\hat{\theta}_t - \theta^* \in E_{r,t}} \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|^2 \geq c_1 \left( 1 - c_2 \frac{w(A_{\max}) \sqrt{c_3 \epsilon^2 + c_4 \log T}}{\sqrt{t}} \right) - c_5 \frac{w(A_{\max}) \left( 2 + \sqrt{c_6 \epsilon^2 + c_7 \log T} \right)}{\sqrt{t}}.
$$

After $t \geq c' w^2(A_{\max}) (\epsilon^2 + \log T)$, the quantity will be positive for some constant $c'$.

Theorem 5 shows that after round $t$ crosses a suitably scaled version of $w^2(A_{\max})$ then there exists a constant $\kappa$ such that the RE assumption holds with high-probability. Note that this requirement implies a phase shift at which point the estimator starts to work and forms the basis of sampling the arms i.i.d. for the initial set of rounds in Algorithm 1.

For a bound on the instantaneous ellipsoidal estimation error, we plug in the value of $\lambda_t$ from (27) into (25) and use the norm compatibility constant of the largest restricted error set to obtain

$$
\|\hat{\theta}_t - \theta^*\|_{2,D_t} \leq C\psi(E_{r,\max}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T \phi(\Omega_R) / 2} \right)
$$

where $C = c_0 c / \kappa$ is a constant which holds with high-probability across all rounds $t = 1, \ldots, T$. Therefore, if we set

$$
\beta = C\psi(E_{r,\max}) \left( w(\Omega_R) + \sqrt{\gamma^2 + \log T \phi(\Omega_R) / 2} \right)
$$

(28)

the confidence ellipsoid $C_t$ will contain $\theta^*$ across all rounds with high-probability. Substituting our $\beta$ into the regret bounds in (21) and (22) gives our main result in Theorem 1 and Theorem 2.

### 6 Conclusions

We studied the stochastic linear bandit problem under structural assumptions on $\theta^*$ and focused on constructing confidence ellipsoids which contain the unknown parameter $\theta^*$ across all rounds with high-probability. We showed how to construct such confidence ellipsoids which are general enough to hold for any norm structured $\theta^*$ and are tighter than previous works leading to sharper regret bounds which, for the problem independent regret scales as $\tilde{O}(\psi(E_{r,\max}) w(\Omega_R) \sqrt{T})$ and for the problem dependent regret scales as $\tilde{O}(\psi^2(E_{r,\max}) w^2(\Omega_R) T \log T / \Delta)$. For unstructured and $s$-sparse $\theta^*$, such regret bounds match existing results on the standard stochastic linear bandit problem. For all other types of structured $\theta^*$ including group sparse and low-rank, the bounds are sharper.

**Acknowledgements:** The research was supported by NSF grants IIS-1447566, IIS-1422557, CCF-1451986, CNS-1314560, IIS-0953274, IIS-1029711, and by NASA grant NNX12AQ39A. The authors also acknowledge support from Adobe, IBM, and Yahoo.
Appendix

A Definitions and Background

The following definitions and lemmas can be found in [5, 6, 27].

Definition 7 A random variable \( x \) is sub-Gaussian if the moments satisfies
\[
[\mathbb{E}|x|^p]^{\frac{1}{p}} \leq K \sqrt{p}
\]  
for any \( p \geq 1 \) with constant \( K \). The minimum value of \( K \) is called the sub-Gaussian norm of \( x \) and denoted by \( \|x\|_{\psi_2} \).

Additionally, every sub-Gaussian random variable satisfies
\[
P(|x| > t) \leq \exp \left( 1 - \frac{t^2}{\|x\|_{\psi_2}^2} \right)
\]  
for all \( t \geq 0 \).

Definition 8 A random vector \( X \in \mathbb{R}^p \) is sub-Gaussian if the one-dimensional marginals \( \langle X, x \rangle \) are sub-Gaussian random variables for all \( x \in \mathbb{R}^p \). The sub-Gaussian norm of \( X \) is defined as
\[
\|X\|_{\psi_2} = \sup_{x \in S^{p-1}} \|\langle X, x \rangle\|_{\psi_2}
\]  

Definition 9 For any set \( A \in \mathbb{R}^p \), the Gaussian width of the set \( A \) is defined as
\[
w(A) = \mathbb{E} \left[ \sup_{u \in A} \langle g, u \rangle \right]
\]  
where the expectation is over \( g \sim \mathcal{N}(0, I_{p \times p}) \) which is a vector of independent zero-mean unit-variance Gaussian random variables.

Lemma 1 For any bounded random variable \( |X| \leq B \), then \( X \) is a sub-Gaussian random variable with \( \|X\|_{\psi_2} \leq B \).

Lemma 2 Consider a sub-Gaussian random vector \( X \) with sub-Gaussian norm \( K = \max_i \|X_i\|_{\psi_2} \), then, for vector \( a \), \( Z = \langle X, a \rangle \) is a sub-Gaussian random variable with sub-Gaussian norm \( \|Z\|_{\psi_2} \leq CK\|a\|_2 \) for absolute constant \( C \).

B Ellipsoid Bound

Theorem 3 Assume that the RE condition is satisfied in the set \( E_{r,t} \) with parameter \( \kappa \) and \( \lambda_t \) is suitably large. Then for any norm \( R(\cdot) \) we have for constant \( c > 0 \)
\[
\|\hat{\theta}_t - \theta^*\|_{2,D_t} \leq c\psi(E_{r,t}) \frac{\lambda_t}{\kappa} \sqrt{t}.
\]  
(33)
Proof: Proof of Theorem 3
For any $\hat{\theta}_t - \theta^* \in E_{r,t}$ and by the definition of a convex function
\[ \mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*) \geq \langle \nabla \mathcal{L}(\theta^*), \hat{\theta}_t - \theta^* \rangle. \]
Moreover, by the definition of a dual norm we have
\[ |\langle \nabla \mathcal{L}(\theta^*), \hat{\theta}_t - \theta^* \rangle| \leq R^*(\nabla \mathcal{L}(\theta^*)) R(\hat{\theta}_t - \theta^*). \]
By construction following (4) from Section 2, for any $\rho > 0$ (not just $\rho = 2$) we get
\[ R^*(\nabla \mathcal{L}(\theta^*)) \leq \lambda_t \rho \]
which implies
\[ |\langle \nabla \mathcal{L}(\theta^*), \hat{\theta}_t - \theta^* \rangle| \leq \frac{\lambda_t \rho}{\rho} R(\hat{\theta}_t - \theta^*) \]
\[ \Rightarrow \langle \nabla \mathcal{L}(\theta^*), \hat{\theta}_t - \theta^* \rangle \geq -\frac{\lambda_t \rho}{\rho} R(\hat{\theta}_t - \theta^*). \]
Therefore,
\[ \mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*) \geq -\frac{\lambda_t \rho}{\rho} R(\hat{\theta}_t - \theta^*) \]
\[ \Rightarrow |\mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*)| \leq \frac{\lambda_t \rho}{\rho} R(\hat{\theta}_t - \theta^*). \]
By the definition of the norm compatibility constant $\psi(E_{r,t}) = \sup_{u \in E_{r,t}} \frac{R(u)}{\|u\|_2}$ we have $R(\hat{\theta}_t - \theta^*) \leq \|\hat{\theta}_t - \theta^*\|_2 \psi(E_{r,t})$ which implies
\[ |\mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*)| \leq \frac{\lambda_t \rho}{\rho} \|\hat{\theta}_t - \theta^*\|_2 \psi(E_{r,t}). \]
Therefore, for the squared loss, since $\mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*) = \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|_2^2$ we get
\[ |\mathcal{L}(\hat{\theta}_t) - \mathcal{L}(\theta^*)| = \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|_2^2 = \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|_2^2. \]
Therefore,
\[ \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|_2^2 \leq \frac{\lambda_t \rho}{\rho} \|\hat{\theta}_t - \theta^*\|_2 \psi(E_{r,t}). \]
Using the bound in (7) from Section 2 for $\|\hat{\theta}_t - \theta^*\|_2$ we obtain
\[ \frac{1}{t} \|X_t(\hat{\theta}_t - \theta^*)\|_2^2 \leq \frac{\lambda_t \rho}{\rho} \psi(E_{r,t}) \frac{\lambda_t}{\kappa} \psi(E_{r,t}). \]
Finally, noting $\|X_t(\hat{\theta}_t - \theta^*)\|_2^2 = \|\hat{\theta}_t - \theta^*\|_{2, D_t}^2$, multiplying each side by $t$, and taking the square root of both sides we get the final bound
\[ \|\hat{\theta}_t - \theta^*\|_{2, D_t} \leq c \psi(E_{r,t}) \frac{\lambda_t}{\kappa} \sqrt{t} \]
for constant $c > 0$ which ends the proof.
C Algorithm

In this section, we will show that selecting an $x_{t+1}$ following Algorithm 1 that we can compute a $\tilde{\theta}_{t+1}$ such that the inequality $\langle x_{t+1}, \tilde{\theta}_{t+1} \rangle \leq \langle x^*, \theta^* \rangle$ holds. In Algorithm 1 we select $x_{t+1}$ by computing $\langle x'_{t+1}, \theta'_{t+1} \rangle = \arg\min_{x \in X, \theta \in C_t \cap S^p-1} \langle x, \theta \rangle$ and sampling uniformly at random from a closed $L_2$ ball centered at $x'_{t+1}$ with radius $\|x'_{t+1}\|_2/2$. In the following lemma, we show a specific way of computing such a sample which shows why the radius is set to such a value. Moreover, we will prove that we can deterministically compute a $\tilde{\theta}_{t+1}$ such that the inequality above holds.

Lemma 3 For a decision set $X$ and a confidence ellipsoid $C_t$, if we compute
\[
\langle x'_{t+1}, \theta'_{t+1} \rangle = \arg\min_{x \in X} \langle x, \theta \rangle \quad \theta \in C_t \cap S^p-1
\]
and set $x_{t+1}$ and $\tilde{\theta}_{t+1}$ as
\[
x_{t+1} = x'_{t+1} + \xi_x v, \quad \tilde{\theta}_{t+1} = \theta'_{t+1} + \xi_\theta u
\]
where $v$ is a random vector such that $\|v\|_2 \leq 1$ and $x'_{t+1} + \xi_x v \in X$, $u = -x'_{t+1}/\|x'_{t+1}\|_2$, $\xi_x = \|x'_{t+1}\|_2/2$, and $\xi_\theta = 1$ then the inequality $\langle x_{t+1}, \tilde{\theta}_{t+1} \rangle \leq \langle x^*, \theta^* \rangle$ holds.

Proof: First, observe
\[
\langle x_{t+1}, \tilde{\theta}_{t+1} \rangle = \left( x'_{t+1} + \frac{\|x'_{t+1}\|_2^2}{2} v, \frac{x'_{t+1}}{\|x'_{t+1}\|_2} - \frac{x'_{t+1}}{\|x'_{t+1}\|_2} \right)
\]
\[
= \langle x'_{t+1}, \theta'_{t+1} \rangle - \frac{\|x'_{t+1}\|_2^2}{2} \langle v, \theta'_{t+1} \rangle - \frac{1}{2} \langle v, x'_{t+1} \rangle.
\]
We need to show that
\[
\langle x_{t+1}, \tilde{\theta}_{t+1} \rangle - \|x_{t+1}\|_2 + \frac{\|x'_{t+1}\|_2^2}{2} \langle v, \theta'_{t+1} \rangle - \frac{1}{2} \langle v, x'_{t+1} \rangle \leq \langle x'_{t+1}, \theta'_{t+1} \rangle
\]
\[
\Rightarrow \langle v, \theta'_{t+1} \rangle \leq \|x_{t+1}\|_2 + \frac{1}{2} \langle v, x_{t+1} \rangle
\]
\[
\Rightarrow -\frac{\|x'_{t+1}\|_2^2}{2} \leq \frac{1}{2} \langle v, x'_{t+1} \rangle \quad \text{(since } |\langle v, \theta'_{t+1} \rangle| \leq 1)\]
\[
\Rightarrow 0 \leq \langle v, x'_{t+1} \rangle + \|x'_{t+1}\|_2.
\]
(34)
From the Cauchy-Schwarz inequality we have
\[ |\langle v, x'_{t+1} \rangle| \leq \|v\|_2 \|x'_{t+1}\|_2 \]
\[ \Rightarrow \langle v, x'_{t+1} \rangle \geq -\|v\|_2 \|x'_{t+1}\|_2 \quad \text{(since } v \text{ is a unit vector)} \]

Plugging this in (34) completes the proof.

D Bound on Regularization Parameter \( \lambda_t \)

We will prove the following main theorem.

**Theorem 4** For any \( \gamma > 0 \) and for absolute constant \( L > 0 \), with probability at least \( 1 - L \exp(-\gamma^2) \), the following bound holds uniformly for all \( t = 1, \ldots, T \):

\[
R^* \left( \frac{1}{t} X_t^\top (y_t - X_t \theta^*) \right) \leq 2LKB \left( \frac{w(\Omega_R) + \sqrt{\gamma^2 + \log T \phi(\Omega_R)}}{\sqrt{t}} \right).
\]

**Proof:** Proof of Theorem 4

Recall the regularization parameter \( \lambda_t \) needs to satisfy the inequality

\[
\lambda_t \geq \rho R^* (\nabla L(\theta^*, Z_t)) = \rho R^* \left( \frac{1}{t} X_t^\top (y_t - X_t \theta^*) \right)
\]

for \( \rho > 1 \). Two issues of the right hand side are (1) the expression depends on the unknown parameter \( \theta^* \) and (2) the expression is a random variable since it depends on \( n \) vectors selected uniformly at random from the decision set \( X \) and a sequence of random noise terms \( \eta_1, \ldots, \eta_t \). We can remove the dependence on \( \theta^* \) by observing that \( y_t - X_t \theta^* \) is precisely the \( t \)-dimensional noise vector \( \omega_t = [\eta_1 \ldots \eta_t]^\top \). Therefore,

\[
R^* \left( \frac{1}{t} X_t^\top \omega_t \right) = R^* \left( \frac{1}{t} X_t^\top \omega_t \right).
\]

By the definition of the dual norm \( R^* \left( \frac{1}{t} X_t^\top \omega_t \right) = \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \). The proof involves showing that \( \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \) is a martingale difference sequence (MDS) which concentrates as a sub-Gaussian random variable. Then, using a generic chaining argument, we show the supremum of such a quantity also concentrates as a sub-Gaussian random variable.

We begin by observing that

\[
\frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle = \frac{1}{\sqrt{t}} \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle.
\]

We will save one of the \( \frac{1}{\sqrt{t}} \) terms for later and now proceed to show how \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) concentrates.
D.1 \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) Concentrates as a Sub-Gaussian

First, let
\[
\frac{1}{\sqrt{t}} \left\langle X_t^\top \omega_t, u \right\rangle = \|u\|_2 \frac{1}{\sqrt{t}} \left\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \right\rangle = \|u\|_2 \frac{1}{\sqrt{t}} \left\langle X_t^\top \omega_t, q \right\rangle
\]
where \( q = \frac{u}{\|u\|_2} \). We focus on the term \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, q \rangle \). We can construct a martingale difference sequence (MDS) by observing that
\[
\left\langle X_t^\top \omega_t, q \right\rangle = \langle \omega_t, X_t q \rangle = \sum_{\tau=1}^{t} \eta_{\tau} \langle x_{\tau}, q \rangle = \sum_{\tau=1}^{t} z_{\tau}
\]
for \( z_{\tau} = \eta_{\tau} \langle x_{\tau}, q \rangle \). Recall from Assumption 2 the filtration is defined as \( F_t = \{x_1, \ldots, x_{t+1}, \eta_1, \ldots, \eta_t\} \).
Each \( z_{\tau} \) can be seen as a MDS since
\[
\mathbb{E}[z_{\tau} | F_{\tau-1}] = \mathbb{E}[\eta_{\tau} \langle x_{\tau}, q \rangle | F_{\tau-1}] = \langle x_{\tau}, q \rangle \cdot \mathbb{E}[\eta_{\tau} | F_{\tau-1}] = 0
\]
because \( x_{\tau} \) is \( F_{\tau-1} \) measurable and \( \eta_{\tau} \) is \( F_{\tau} \) measurable. Additionally, each \( z_{\tau} \) follows a sub-Gaussian distribution with parameter \( KB \) because \( \|\eta_{\tau} \langle x_{\tau}, q \rangle\|_{\psi_2} \leq KB \) (Assumption 2 and Definition 5). Since each \( z_{\tau} \) is a bounded MDS, we can use the Azuma-Hoeffding inequality to show that the sum \( \sum_{\tau=1}^{t} z_{\tau} \) concentrates as a sub-Gaussian with parameter \( KB \). For all \( \gamma \geq 0 \)
\[
P\left( \left| \sum_{\tau=1}^{t} z_{\tau} \right| \geq \gamma \right) = P\left( \left| \left\langle X_t^\top \omega_t, q \right\rangle \right| \geq \gamma \right)
\]
\[
= P\left( \left| \left\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \right\rangle \right| \geq \gamma \right) \leq 2 \exp\left( \frac{-\gamma^2}{2tK^2B^2} \right)
\]
\[
= P\left( \frac{1}{\sqrt{t}} \left| \left\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \right\rangle \right| \geq \zeta \right) \leq 2 \exp\left( \frac{-\zeta^2}{2K^2B^2} \right)
\]
where \( \zeta = \gamma / \sqrt{t} \) which implies \( \gamma = \sqrt{t} \zeta \). From [42] and [30] in Definition 7 (Section A) we can see that the term \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \rangle \) concentrations as a sub-Gaussian with \( \|\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \rangle\|_{\psi_2} \leq KB \).
Next, we show that the term \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) also concentrations as a sub-Gaussian with \( \|\langle X_t^\top \omega_t, u \rangle\|_{\psi_2} \leq \|u\|_2KB \) using [42] as
\[
P\left( \frac{1}{\sqrt{t}} \left| \left\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \right\rangle \right| \geq \zeta \right)
\]
\[
= P\left( \|u\|_2 \frac{1}{\sqrt{t}} \left| \left\langle X_t^\top \omega_t, \frac{u}{\|u\|_2} \right\rangle \right| \geq \|u\|_2 \zeta \right)
\]
\[
= P\left( \frac{1}{\sqrt{t}} \left| \left\langle X_t^\top \omega_t, u \right\rangle \right| \geq \epsilon \right) \leq 2 \exp\left( \frac{-\epsilon^2}{2\|u\|_2^2K^2B^2} \right)
\]
where \( \epsilon = \|u\|_2 \zeta \) which implies \( \zeta = \epsilon / \|u\|_2 \). The reason we went through showing the above is because the generic chaining argument we will invoke to bound \( \sup_{R(u) \leq 1} \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) requires that \( \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) is a sub-Gaussian random variable.
D.2 Bound on \( \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \) via Generic Chaining

We obtain a high-probability bound on \( \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \) using a generic chaining argument from [25, 26]. This involves (1) showing that the absolute difference of two sub-Gaussian processes concentrates as a sub-Gaussian, (2) showing the expectation over the supremum of the absolute difference of two sub-Gaussian processes is upper bounded by the sub-Gaussian width of a set from which the processes are indexed from, and (3) showing the supremum of a sub-Gaussian process is concentrated around its expectation and therefore, around the sub-Gaussian width with high-probability.

(1) Sub-Gaussian Process Concentration

First, we show that the absolute difference of two sub-Gaussian processes concentrates as a sub-Gaussian. Let \( Y_u = \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, u \rangle \) indexed by \( u \in \Omega_R \) and \( Y_v = \frac{1}{\sqrt{t}} \langle X_t^\top \omega_t, v \rangle \) indexed by \( v \in \Omega_R \) be two zero-mean (since they are both a MDS sum), random symmetric processes (since \( E^v \) on \( v \)). Bound on \( \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \) we obtain the following bound on the absolute difference of two sub-Gaussian random processes \( Y_u \) and \( Y_v \) as

\[
P \left( \frac{1}{\sqrt{t}} \left| \langle X_t^\top \omega_t, u - v \rangle \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{\epsilon^2}{2\|u - v\|_2^2 K^2 B^2} \right) \tag{44}
\]

which shows \( |Y_u - Y_v| \) concentrates as a sub-Gaussian random variable with \( \|Y_u - Y_v\|_{\psi_2} = \|u - v\|_{2 K B} \).

(2) Bound on \( \mathbb{E} \left[ \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \right] \)

In order to establish a high-probability bound on \( \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \) we need to prove a bound on \( \mathbb{E} \left[ \sup_{R(u) \leq 1} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \right] \). To prove such a bound, we will apply a generic chaining argument for upper bounds on such sub-Gaussian processes. For the generic chaining argument, we will need the result in \( [42] \) and the following lemma.

Lemma 4 ([25], Theorem 2.1.5) Consider two processes \((Y_u)_{u \in \Omega_R}\) and \((X_u)_{u \in \Omega_R}\) indexed by the same set. Assume that the process \((X_u)_{u \in \Omega_R}\) is Gaussian and that the process \((Y_u)_{u \in \Omega_R}\) satisfies the condition

\[
\forall \epsilon > 0, \forall u, v \in \Omega_R, P(|Y_u - Y_v| \geq \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2}{d(u, v)^2} \right) \tag{45}
\]

where \( d(u, v) \) is a distance function which we assume is \( d(u, v) = \|u - v\|_2 \) for the set \( \Omega_R \). Then we have

\[
\mathbb{E} \left[ \sup_{u, v \in \Omega_R} \frac{1}{t} \langle X_t^\top \omega_t, u \rangle \right] \leq L \mathbb{E} \left[ \sup_{u \in \Omega_R} X_u \right] \tag{46}
\]

where \( L \) is an absolute constant.
First, notice that $E \left[ \sup_{u \in \Omega_R} X_u \right]$ is exactly the Gaussian width $w(\Omega_R)$ of the set $\Omega_R$ as seen by the Definition 9 (Section A). For our purposes, we make one modification to the above lemma similar to [6, Theorem 8]. In (41), we see that $|Y_u - Y_v|$ concentrates as a sub-Gaussian with parameter $\|u - v\|_2KB$. To bound the expectation of two sub-Gaussian processes, we scale the Gaussian width by the additional term $KB$ to get

$$E \left[ \sup_{u,v \in \Omega_R} |Y_u - Y_v| \right] \leq LKB \left( \sup_{u \in \Omega_R} X_u \right) = LKBw(\Omega_R).$$

(47)

This shows for two sub-Gaussian processes $Y_u$ and $Y_v$, the expectation of the supremum of their absolute difference is upper bounded by the Gaussian width scaled by the sub-Gaussian norm, i.e., the sub-Gaussian width.

The second result we need is the following lemma.

**Lemma 5 ([25], Lemma 1.2.8)** If the process $(Y_u)_{u \in \Omega_R}$ is symmetric then

$$E \left[ \sup_{u,v \in \Omega_R} |Y_u - Y_v| \right] = 2E \left[ \sup_{u \in \Omega_R} Y_u \right].$$

(48)

We know from above that our processes $Y_u = \frac{1}{\sqrt{t}} \langle X_\top t \omega_t, u \rangle$ and $Y_v = \frac{1}{\sqrt{t}} \langle X_\top t \omega_t, v \rangle$ are symmetric. As such we get the following lemma.

**Lemma 6** From (44) we can see that the condition of Lemma 4 is satisfied in the sub-Gaussian case so using Lemma 4 and Lemma 5 for some absolute constant $L$ we obtain

$$E \left[ \sup_{u,v \in \Omega_R} |Y_u - Y_v| \right] = 2E \left[ \sup_{u \in \Omega_R} |Y_u| \right] \leq 2LKBw(\Omega_R).$$

(49)

(3) Concentration of $\sup_{R(u) \leq 1} \frac{1}{\sqrt{t}} \langle X_\top \omega_t, u \rangle$

To complete the argument, we need the following lemma.

**Lemma 7 ([26], Theorem 2.2.27)** If the process $(Y_u)$ satisfies (45) or similarly (44) for the sub-Gaussian case then for $\epsilon > 0$ one has

$$P \left( \sup_{u,v \in \Omega_R} |Y_u - Y_v| \geq L\left( \gamma_2(\Omega_R, d(u,v)) + \epsilon \Delta(\Omega_R) \right) \right) \leq L \exp(-\epsilon^2).$$

(50)

Note, the function $\Delta(\Omega_R) = \sup_{u,v \in \Omega_R} d(u,v)$ is the diameter of the set $\Omega_R$. For our setting, $d(u,v) = \|u - v\|_2$ so we replace $\Delta(\Omega_R)$ with $\phi(\Omega_R)$ as detailed in Definition 2 in Section 3.2. The specifics of the $\gamma_2(\cdot, \cdot)$ function are not necessary for this work since we can bound it and simplify Lemma 7 by using the following lemma.
Lemma 8 ([26], Theorem 2.4.1) For some universal constant \( L \) we have
\[
\frac{1}{L} \g_2(\Omega_R, d(u, v)) \leq \mathbb{E} \left[ \sup_{u \in \Omega_R} Y_u \right] \leq L \g_2(\Omega_R, d(u, v)).
\] (51)

Combining Lemma 7 with Lemma 8, using Lemma 6, and our definitions of \( Y_u \) and \( Y_v \) for any \( \epsilon > 0 \) we get

Lemma 9
\[
P \left( \sup_{R(u) \leq 1} \frac{1}{\sqrt{t}} \left| \left< X^T t, \omega_t, u \right> \right| \geq 2LKBw(\Omega_R) + \epsilon \right) \leq L \exp \left( -\left( \frac{\epsilon}{LKB\phi(\Omega_R)} \right)^2 \right). \] (52)

Proof: Proof of Lemma 9
\[
P \left( \sup_{u, v \in \Omega_R} |Y_u - Y_v| \geq L(\gamma_2(\Omega_R, d(u, v)) + \zeta \Delta(\Omega_R)) \right)
= P \left( \sup_{u, v \in \Omega_R} |Y_u - Y_v| \geq L \gamma_2(\Omega_R, d(u, v)) + \epsilon \right)
\leq P \left( \sup_{u, v \in \Omega_R} |Y_u - Y_v| \geq \mathbb{E} \left[ \sup_{u, v \in \Omega_R} |Y_u - Y_v| \right] + \epsilon \right)
= P \left( \sup_{u \in \Omega_R} |Y_u| \geq 2 \mathbb{E} \left[ \sup_{u \in \Omega_R} |Y_u| \right] + \epsilon \right)
= P \left( \sup_{R(u) \leq 1} \frac{1}{\sqrt{t}} \left| \left< X^T t, \omega_t, u \right> \right| \geq 2LKBw(\Omega_R) + \epsilon \right) \leq L \exp \left( -\left( \frac{\epsilon}{LKB\phi(\Omega_R)} \right)^2 \right).
\]

where the first line comes from the left-hand side of Lemma 7, the second line comes from the fact that \( \Delta(\Omega_R) \leq \gamma_2(\Omega_R, d(u, v)) \) from [26] Definition 2.2.19, the third line comes from Lemma 8, the fourth line comes from Lemma 5, the fifth line comes from Lemma 6 and the last line follows from our construction of the process \( Y_u \) and the right-hand side of Lemma 7.

Dividing the other \( \sqrt{t} \) through and setting \( \epsilon/\sqrt{t} = \alpha 2LKBw(\Omega_R)/\sqrt{t} \) we get

Lemma 10
\[
P \left( R^\star \left( \frac{1}{\sqrt{t}} X^T t, \omega_t \right) \right) \geq 2LKB(1 + \alpha) \frac{w(\Omega_R)}{\sqrt{t}} \right) \leq L \exp \left( -\left( \frac{2\alpha w(\Omega_R)}{\phi(\Omega_R)} \right)^2 \right). \] (53)

19
Proof: Proof of Lemma 10

\[ P \left( R^* \left( \frac{1}{\sqrt{t}} X_t^\top \omega_t \right) \geq 2LKB \left( \frac{w(\Omega_R)}{\sqrt{t}} + \epsilon \right) \right) \leq L \exp \left( - \frac{\epsilon^2}{LKB \phi(\Omega_R)} \right) \]

\[ P \left( R^* \left( \frac{1}{\sqrt{t}} X_t^\top \omega_t \right) \geq \frac{1}{\sqrt{t}} \right) \leq L \exp \left( - \frac{\sqrt{t} \gamma}{LKB \phi(\Omega_R)} \right) \]

\[ P \left( R^* \left( \frac{1}{\sqrt{t}} X_t^\top \omega_t \right) \geq \frac{2LKB w(\Omega_R)}{\sqrt{t}} + \gamma \right) \leq L \exp \left( - \frac{\sqrt{t} \alpha^2 LKB w(\Omega_R)}{\sqrt{t}} \right) \]

\[ P \left( R^* \left( \frac{1}{\sqrt{t}} X_t^\top \omega_t \right) \geq 2LKB (1 + \alpha) \frac{w(\Omega_R)}{\sqrt{t}} \right) \leq L \exp \left( - \frac{\alpha^2 w(\Omega_R)}{\phi(\Omega_R)} \right) \]

where the first inequality is from Lemma 9, the second inequality is from multiplying both sides by \( \frac{1}{\sqrt{t}} \) and setting \( \gamma = \frac{\epsilon}{\sqrt{t}} \), and the third inequality is from setting \( \gamma = \alpha 2LKB \frac{w(\Omega_R)}{\sqrt{t}} \).

Lemma 11 gives a high-probability bound on the value of \( R^* \left( X_t^\top \omega_t \right) \) for round \( t \) but to complete the proof of Theorem 4 we need a bound which holds simultaneously for all rounds \( T \) with high-probability. To obtain such a bound, we can set \( \alpha^2 = \left( \sqrt{\gamma^2 + \log T} \right) \frac{w(\Omega_R)}{2w(\Omega_R)} \) and apply a union bound for all \( t \)

\[ \bigcup_{t=1}^{T} P \left( R^* \left( \frac{1}{\sqrt{t}} X_t^\top \omega_t \right) \geq 2LKB \left( 1 + \sqrt{\gamma^2 + \log T} \right) \frac{w(\Omega_R)}{\sqrt{t}} \right) \]

\[ \leq \sum_{t=1}^{T} L \exp \left( - \left( \gamma^2 + \log T \right) \frac{w(\Omega_R)}{2w(\Omega_R)} \right) \]

\[ = L \sum_{t=1}^{T} \exp \left( - \gamma^2 - \log T \right) \]

\[ = L \sum_{t=1}^{T} \exp \left( - \gamma^2 \right) \times \frac{1}{T} \]

\[ = L \exp \left( - \gamma^2 \right) \]

Rearranging the terms ends the proof of Theorem 4.

E  Restricted Eigenvalue (RE) Condition

We will prove the following theorem.

Theorem 5 For constants \( c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0 \) and any \( \epsilon > 0 \), with probability at least \( 1 - c_0 \exp(-w^2(A_{\text{max}})^2) \) the following will hold uniformly for all rounds \( t = 1, \ldots, T \):

\[ \inf_{u \in E_{r,t}} \frac{1}{t} \| X_t u \|^2 \geq c_1 \left( 1 - c_2 \frac{w(A_{\text{max}}) \sqrt{c_3 \epsilon^2 + c_4 \log T}}{\sqrt{t}} \right) - c_5 \frac{w(A_{\text{max}}) \left( 2 + \sqrt{c_0 \epsilon^2 + c_7 \log T} \right)}{\sqrt{t}} \]
After \( t \geq c'w^2(A_{\text{max}})(\epsilon^2 + \log T) \), the quantity will be positive for some constant \( c' \).

**Proof**:

Proof of Theorem 5

For a design matrix \( X_t \) with \( t \) rows, a response vector \( y_t \), and parameter \( \kappa \) the RE condition is

\[
\frac{1}{t} \| y_t - X_t\hat{\theta}_t \|^2 - \frac{1}{t} \| y_t - X_t\theta^* \|^2 - \frac{1}{t} \langle X_t^\top (y_t - X_t\theta^*), \hat{\theta}_t - \theta^* \rangle \geq \kappa \| \hat{\theta}_t - \theta^* \|^2.
\]

\[
\Rightarrow \frac{1}{t} \| X_t(\hat{\theta}_t - \theta^*) \|^2 \geq \kappa \| \hat{\theta}_t - \theta^* \|^2.
\] (54)

We need the above equation to be satisfied \( \forall \hat{\theta}_t - \theta^* \in E_{r,t} \) for Theorem 3 to hold. Note, the restricted error set has dependence on \( t \) because at each round we compute a new estimate \( \hat{\theta}_t \). Refer to (3) in Section 2 to review the definition and see why we need to make this distinction. To that end, we consider the following problem

\[
\inf_{\hat{\theta}_t - \theta^* \in \text{cone}(E_{r,t})} \frac{1}{t} \| X_t(\hat{\theta}_t - \theta^*) \|^2 \geq \kappa \| \hat{\theta}_t - \theta^* \|^2.
\] (55)

Clearly if (55) is true then it is true for all \( \hat{\theta}_t - \theta^* \in E_{r,t} \) since \( E_{r,t} \subseteq \text{cone}(E_{r,t}) \). Additionally, since only the direction matters and not the magnitude we consider just the vectors on the spherical cap \( A_t := \text{cone}(E_{r,t}) \cap S^{p-1} \)

\[
\inf_{u \in A_t} \frac{1}{t} \| X_t u \|^2 \geq \kappa \| u \|^2
\] (56)

where \( S^{p-1} \) is the unit sphere in \( \mathbb{R}^p \). Since \( \| u \|^2 = 1 \) for all \( u \in A_t \) we simply focus on

\[
\inf_{u \in A_t} \frac{1}{t} \| X_t u \|^2 \geq \kappa
\] (57)

which suffices in proving the RE condition for the restricted error set.

Now, to show a bound, we perform the following decomposition. Let \( X_t = [x_1, \ldots, x_t]^\top \), then

\[
\frac{1}{t} \| X_t u \|^2 = \frac{1}{t} \sum_{i=1}^t \langle x_i, u \rangle^2 = \frac{1}{t} \sum_{i=1}^t (x_i - \mu_i, u)^2 - \frac{1}{t} \sum_{i=1}^t \langle \mu_i, u \rangle^2 + \frac{2}{t} \sum_{i=1}^t \langle x_i, u \rangle \langle \mu_i, u \rangle
\]

\[
= \frac{1}{t} \sum_{i=1}^t (x_i - \mu_i, u)^2 + \frac{2}{t} \sum_{i=1}^t \langle \mu_i, u \rangle \langle x_i - \mu_i, u \rangle + \frac{1}{t} \sum_{i=1}^t \langle \mu_i, u \rangle^2
\]

where \( \mu_i = \mathbb{E}[x_i|F_{i-1}] \) and we define the filtration to be \( F_{i-1} = \{x_1, \ldots, x_{i-1}, \eta_1, \ldots, \eta_{i-1}\} \). Taking the infimum

\[
\inf_{u \in A_t} \frac{1}{t} \| X_t u \|^2 = \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^t (x_i - \mu_i, u)^2 + \inf_{u \in A_t} \frac{2}{t} \sum_{i=1}^t \langle \mu_i, u \rangle \langle x_i - \mu_i, u \rangle + \frac{1}{t} \sum_{i=1}^t \langle \mu_i, u \rangle^2
\]

\[
\geq \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^t (x_i - \mu_i, u)^2 - \sup_{u \in A_t} \frac{2}{t} \sum_{i=1}^t \langle x_i - \mu_i, u \rangle
\] (58)

where the inequality follows from \( |\langle \mu_i, u \rangle| \leq 1 \) due to our assumption that \( \mathcal{X} \subseteq B_0^n \) to avoid scaling factors. Suitable scaling modifications can be made to remove the assumption. To obtain the bounds we have to bound the quantities \( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^t (x_i - \mu_i, u)^2 \) and \( \sup_{u \in A_t} \frac{2}{t} \sum_{i=1}^t \langle x_i - \mu_i, u \rangle \).
1. **Bound for** \( \sup_{u \in A_t} \frac{2}{T} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \)

Observe, for all \( i \) that \( x_i - \mu_i \) is a bounded vector-valued MDS such that \( \| x_i - \mu_i \|_{\psi_2} \leq K \) (see Definition 5). Therefore, by the Azuma-Hoeffding inequality we obtain

\[
P \left( \frac{1}{\sqrt{t}} \left| \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \right| \geq \gamma \right) \leq 2 \exp \left( -\frac{\gamma^2}{2 \| u \|_2^2 K^2} \right). \tag{59}
\]

Therefore, for \( u, v \in A_t \)

\[
P \left( \frac{1}{\sqrt{t}} \left| \sum_{i=1}^{t} \langle x_i - \mu_i, u - v \rangle \right| \geq \gamma \right) \leq 2 \exp \left( -\frac{\gamma^2}{2 \| u - v \|_2^2 K^2} \right). \tag{60}
\]

From (59) and using the generic chaining argument [26] similar to our \( \lambda_t \) analysis (Section D) it follows that for an absolute constant \( L > 0 \),

\[
2E \left[ \sup_{u \in A_t} \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \right] \leq 2LKw(A_t). \tag{61}
\]

Therefore,

\[
P \left( \sup_{u \in A_t} \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \geq 2LKw(A_t) + \alpha \right) \leq L \exp \left( -\left( \frac{\alpha}{LK \phi(A_t)} \right)^2 \right). \tag{62}
\]

Setting \( \alpha = LKw(A_t)\zeta \) gives

\[
P \left( \sup_{u \in A_t} \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \geq LKw(A_t)(2 + \zeta) \right) \leq L \exp \left( -\left( \frac{\zeta w(A_t)}{\phi(A_t)} \right)^2 \right). \tag{63}
\]

Now, since the set \( A_t \) changes each round and the bound must hold across all rounds, we put the bound in terms of the largest spherical cap \( A_{\text{max}} \) which is defined in Definition 4. Then, setting \( \zeta^2 = \left( \epsilon^2 \phi^2(A_{\text{max}}) + \log T \left( \frac{\phi^2(A_{\text{max}})}{w^2(A_{\text{max}})} \right) \right) \), and taking a union bound such that across all rounds we have

\[
\bigcup_{t=1}^{T} \left\{ \sup_{u \in A_t} \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \geq LKw(A_{\text{max}}) \right\} \leq L \sum_{t=1}^{T} \exp \left( -\epsilon^2 \phi^2(A_{\text{max}}) \left( \frac{w^2(A_{\text{max}})}{\phi^2(A_{\text{max}})} \right) \right) - \log T \left( \frac{\phi^2(A_{\text{max}})}{w^2(A_{\text{max}})} \left( \frac{w^2(A_{\text{max}})}{\phi^2(A_{\text{max}})} \right) \right)
\]

\[
= L \sum_{t=1}^{T} \exp \left( -\epsilon^2 w^2(A_{\text{max}}) - \log T \right)
\]

\[
= L \sum_{t=1}^{T} \exp \left( -\epsilon^2 w^2(A_{\text{max}}) \right) \times \frac{1}{T}
\]

\[
= L \exp \left( -\epsilon^2 w^2(A_{\text{max}}) \right).
\]
Dividing the other $\sqrt{t}$ through and multiplying by 2 we obtain
\begin{equation}
\sup_{u \in A_t} \frac{2}{t} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle \leq \frac{2LK w(A_{\max}) \left( 2 + \sqrt{\epsilon^2 \phi^2 (A_{\max})} + \log T \left( \frac{\phi^2 (A_{\max})}{w^2 (A_{\max})} \right) \right)}{\sqrt{t}}
\end{equation}
which holds uniformly across all rounds with probability at least $1 - L \exp \left( -\epsilon^2 w^2 (A_{\max}) \right)$.

2. **Bound for** \( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle^2 \)

To prove a bound on \( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \langle x_i - \mu_i, u \rangle^2 \) we use the following result from \( \cite{[6]} \) which we extend to martingales in Section \( \cite{E.1} \).

**Lemma 11 (\cite{[6]}, Theorem 12)** Let \( Z_t \in \mathbb{R}^{t \times p} \) be a design matrix with bounded martingale difference sequence, anisotropic sub-Gaussian rows, i.e., \( \mathbb{E}[z_i] < \infty, \mathbb{E}[z_i | F_{i-1}] = 0 \) where \( F_{i-1} \) is a filtration, \( \mathbb{E}[z_i z_i^\top] = \Sigma \), and \( \| z_i \Sigma^{-1/2} \|_2 \leq K \forall i \). Then, for absolute constants \( c_0, c > 0 \), and any \( \gamma > 0 \) with probability at least \( 1 - 2 \exp( -c_0 w^2 (A_t) \gamma^2) \), we have
\begin{equation}
\inf_{u \in A_t} \frac{1}{t} \| Z_t u \|_2^2 \geq \lambda_{\min}(\Sigma | A_t) \left( 1 - c \frac{w(A_t) \gamma}{\sqrt{t}} \right)
\end{equation}
where \( \lambda_{\min}(\Sigma | A_t) = \inf_{u \in A_t} u^\top \Sigma u \) is the restricted minimum eigenvalue of \( \Sigma \) restricted to \( A_t \subseteq S^{p-1} \).

Note, we have made a slight change in the theorem as stated in \( \cite{[6]} \) to put the probability in terms of a parameter \( \gamma \). We did this by setting \( \theta = c_1 c_4 K^2 \frac{w(A_t) \gamma}{\sqrt{t}} \) at the bottom of page 31 and the rest follows through with some algebra. See \( \cite{[6]} \) for more details.

Given Lemma 11 let \( z_i = x_i - \mu_i \) which is an MDS for all \( i \) and then the design matrix \( Z_t = [z_1, \ldots, z_t]^\top \). Each row \( z_i \in Z_t \) will have a covariance matrix \( \mathbb{E}[z_i z_i^\top] = \Sigma_i \) such that \( \| z_i \Sigma_i^{-1/2} \|_2 \leq \| z_i \Sigma_i^{-1/2} \|_2 \leq K \) for some constant \( c_1 \).

We can immediately apply Lemma 11 to get a result in terms of \( \lambda_{\min}(\Sigma_i | A_t) \) however, we must be careful to ensure that \( \lambda_{\min}(\Sigma_i | A_t) \neq 0 \). Next, we will argue that for all rows of \( Z_t \) that \( \lambda_{\min}(\Sigma_i | A_t) > 0 \). The following two paragraphs can be skipped if it is clear that the minimum eigenvalue of such a centered convex subset with non-empty interior of an \( L_2 \) ball is positive.

Under the assumption that the decision set \( \mathcal{X} \) is compact convex with non-empty interior (Assumption 11), at each round \( t = 1, 2, \ldots, T \) a single solution \( x_t' \) is computed via (11) in Section 3 over which a closed \( L_2 \) ball with radius \( \xi_x > 0 \) is centered \( B_2^p (x_t', \xi_x) \). A single solution \( x_t \) is drawn uniformly at random from the set \( \mathcal{X} \cap B_2^p (x_t', \xi_x) \) which is a subset of an \( L_2 \) ball. Now, if we define the set \( Z_t = \{ x - \mu_t : x \in \mathcal{X} \cap B_2^p (x_t', \xi_x) \} \) then \( Z_t \) is a subset of an \( L_2 \) ball centered at the origin.

Given this, we will use a proof by contradiction. If we do not restrict ourselves to the set \( A_t \), clearly \( \lambda_{\min}(\Sigma_t | A_t) \geq \lambda_{\min}(\Sigma_t) \), then we desire a bound of the form for some \( \nu \)
\begin{equation}
\lambda_{\min}(\Sigma_t) = \inf_{u \in S^{p-1}} u^\top \Sigma_t u \geq \nu > 0.
\end{equation}

Assume for a moment that \( \lambda_{\min}(\Sigma_t) = 0 \). Then, compute the eigenvalue decomposition of \( \Sigma_t \) as \( \Sigma_t = VAV^\top \) where \( V = [v_1, \ldots, v_p] \) are the eigenvectors of \( \Sigma_t \). If we can believe our assumption that
\[ \lambda_{\min}(\Sigma_t) = 0 \text{ if this implies that } \mathbb{E}_z \mathbb{I}_t((z, v_p)^2) = 0. \] If we define the set \( Z_{v_p} = \{ z \in Z_t : \langle z, v_p \rangle = 0 \} \) then for our assumption to be true it must be true that \( P(z \in Z_{v_p}) = 1 \) a.s. However, since there is zero probability density outside of \( Z_t \) this implies the density is concentrated on a subspace. Such an implication cannot be true because the span of \( Z_t \) is \( \mathbb{R}^p \), i.e., the set contains all directions. Therefore, our assumption is false and \( \lambda_{\min}(\Sigma_t) \neq 0 \) which implies there exists some constant \( \nu \) such that \( \lambda_{\min}(\Sigma_t|A_t) \geq \lambda_{\min}(\Sigma_t) \geq \nu > 0 \) for all \( t \).

Now, given the argument that \( \lambda_{\min}(\Sigma_i) > 0 \forall i \), we define \( \lambda_{\min}(\Sigma_{1:t}) = \min \{ \lambda_{\min}(\Sigma_1), \ldots, \lambda_{\min}(\Sigma_t) \} \). Then, we can use Lemma 11 with the largest spherical cap \( A_{\max} \) to obtain the following bound which holds for any of the \( t \) rounds and any \( \zeta > 0 \).

\[
P \left( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \langle x_i - \mu_t, u \rangle^2 \leq \lambda_{\min}(\Sigma_{1:t}) \left( 1 - c \frac{w(A_{\max})}{\sqrt{t}} \right) \right) \leq 2 \exp\left(-c_0 w^2(A_{\max}) \zeta^2 \right). \tag{66}
\]

Setting \( \zeta^2 = \frac{c_0^2}{c_0 w^2(A_{\max})} + \frac{\log T}{c_0 w^2(A_{\max})} \) and applying a union bound we get the following bound which holds simultaneously for all rounds \( t = 1, \ldots, T \) and any \( \epsilon > 0 \).

\[
\bigcup_{t=1}^{T} P \left( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \langle x_i - \mu_t, u \rangle^2 \leq \lambda_{\min}(\Sigma_{1:t}) \left( 1 - c \frac{w(A_{\max})}{\sqrt{t}} \right) \right) \leq \sum_{t=1}^{T} 2 \exp \left( -c_0 w^2(A_{\max}) \left( \frac{c_0^2}{c_0 w^2(A_{\max})} + \frac{\log T}{c_0 w^2(A_{\max})} \right) \right) \tag{67}
\]

\[
= \sum_{t=1}^{T} 2 \exp \left( -w^2(A_{\max}) \epsilon^2 - c_0 w^2(A_{\max}) \left( \frac{\log T}{c_0 w^2(A_{\max})} \right) \right) \]

\[
= \sum_{t=1}^{T} 2 \exp \left( -w^2(A_{\max}) \epsilon^2 \right) \times \frac{1}{T} \]

\[
= 2 \exp \left( -w^2(A_{\max}) \epsilon^2 \right). \]

Combining (68), (69), and (67) with probability at least \( 1 - 2L \exp(-w^2(A_{\max})\epsilon^2) \) we obtain

\[
\inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \| X_t u \|_2^2 \geq \lambda_{\min}(\Sigma_{1:t}) \left( 1 - c \frac{w(A_{\max})}{\sqrt{t}} \right) - 2L K \frac{w(A_{\max}) \left( 2 + \sqrt{2 \phi^2(A_{\max}) + \log T} \frac{\phi^2(A_{\max})}{w^2(A_{\max})} \right)}{\sqrt{t}}. \tag{68}
\]

For some constant \( C > 0 \) it is true that \( C - \frac{C}{2} - \frac{C}{2} > 0 \) (where we have chosen to divide \( C \) by 2 and 4 somewhat arbitrarily) therefore, setting \( C = \lambda_{\min}(\Sigma_{1:t}) \), if we can show when

\[
c \lambda_{\min}(\Sigma_{1:t}) \frac{w(A_{\max})}{\sqrt{t}} \geq \frac{C}{2}. \tag{69}
\]
and when
\[ w(A_{\text{max}}) \left( 2 + \sqrt{2 \epsilon^2 (A_{\text{max}})^2} + \log T \frac{\epsilon^2 (A_{\text{max}})^2}{w^2(A_{\text{max}})} \right) \leq \frac{C}{4} \] (70)
then \( \inf_{u \in A_t} \frac{1}{t} \sum_{i=1}^{t} \|X_t u\|_2^2 > 0 \) will be satisfied.

With some algebraic manipulations, we can see that (69) is satisfied when
\[ t \geq 4 c^2 \lambda_{\text{min}}^2 (\Sigma_{1:t}) w^2(A_{\text{max}}) \left( \frac{c^2}{c_0} + \frac{\log T}{c_0 w^2(A_{\text{max}})} \right) / C^2 \]
and (70) is satisfied when
\[ t \geq 64 L^2 K^2 w^2(A_{\text{max}}) \left( 2 + \sqrt{\epsilon^2 (A_{\text{max}})^2} + \log T \frac{\epsilon^2 (A_{\text{max}})^2}{w^2(A_{\text{max}})} \right)^2 / C^2 . \]

Therefore, the RE condition will be satisfied when \( t \geq c' w^2(A_{\text{max}})(\epsilon^2 + \log T) \) for some constant \( c' > 0 \) which completes the proof.

\[ \text{E.1 Anisotropic Sub-Gaussian Design Extension to Martingales} \]

We extend [6, Theorem 12] to martingale difference samples which is an application of Theorem D from [20]. Theorem D relies on Lemma 1.2 [20] which shows concentrations for i.i.d random samples and is the only part of the proof which needs to be modified for martingales. As such, we present an extension of Lemma 1.2 to martingales which generalizes Theorem D to martingales which can be applied to prove Theorem 11. Note, the following result can be considered independent from the rest of the paper and, as such, the notation is not inherited but will be re-defined here. Theorem D is as follows.

**Lemma 12 (Mendelson et al. Theorem D)** There exists absolute constants \( c_1, c_2 \) for which the following holds. Let \((\Omega, \mu)\) be a probability space, set \( F \) be a subset of the unit sphere of \( L_2(\mu) \), i.e., \( F \subseteq S_{L_2} = \{ f : \|f\|_{L_2} = 1 \} \), and assume that \( \text{diam}(F, \|\cdot\|_{\psi_2}) = \alpha \). Then, for any \( \theta > 0 \) and \( n \geq 1 \) satisfying
\[ c_1 \alpha \gamma_2(F, \|\cdot\|_{\psi_2}) \leq \theta \sqrt{n} \] (71)
with probability at least \( 1 - \exp \left( -c_2 \frac{\alpha^2 n}{\theta^2} \right) \),
\[ \sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} f^2(x_i) - \mathbb{E}[f^2] \right| \leq \theta . \] (72)

In [20], the proof assumes the samples \( x_1, \ldots, x_n \in \mathbb{R}^p \) are i.i.d. isotropic sub-Gaussian random vectors. For the problem we are considering, the samples \( x_1, \ldots, x_n \in \mathbb{R}^p \) are a bounded martingale difference sequence (MDS), i.e., \( \|x_i\|_2 \leq A, \mathbb{E}[x_i] \leq \infty, \mathbb{E}[x_i | F_{i-1}] = 0 \) where \( F_i = \{ x_1, \ldots, x_i \} \) is a filtration. Also we consider the following class of functions:
\[ F := \left\{ f_u : f_u(\cdot) = \frac{1}{\sqrt{\mathbb{E}[(x_i, u)^2 | F_{i-1}]}} \langle \cdot, u \rangle = \frac{1}{\sqrt{u^T \Sigma_i u}} \langle \cdot, u \rangle, u \in A \subseteq S^{p-1} \right\} . \] (73)
Lemma 13: For a bounded martingale difference sequence \( x_1, \ldots, x_n \in \mathbb{R}^p \) where each \( x_i \) is bounded as \( \|x_i\|_2 \leq A \), there exists absolute constants \( c_1, c_2 > 0 \) for which the following holds. Let \( F \) be the set of linear functionals over the unit sphere \( F := \left\{ \frac{1}{\sqrt{u^T \Sigma_i u}}(\cdot, u) : u \in A \subseteq S^{p-1} \right\} \) and assume that \( \text{diam}(F, \cdot, \psi_2) = \alpha \). Then, for any \( \theta > 0 \) and \( n \geq 1 \) satisfying

\[
c_1 \alpha^2 (F, \cdot, \psi_2) \leq \theta \sqrt{n}
\]

with probability at least \( 1 - \exp\left( -c_2 \frac{\alpha^2}{n} \right) \),

\[
\sup_{u \in S^{p-1}} \left| \frac{1}{n} \sum_{i=1}^{n} \frac{\langle x_i, u \rangle^2}{u^T \Sigma_i u} - 1 \right| \leq \theta
\]

where \( X = [x_1, \ldots, x_n]^T \in \mathbb{R}^{n \times p} \) is the design matrix and \( \Sigma_i = \mathbb{E}[x_i x_i^T] \in \mathbb{R}^{p \times p} \) is the population covariance matrix for sample \( x_i \).

The proof follows using analogous arguments from [20]. For \( f_u \in F \), we define the random variables \( Z_{f_u} \) and \( W_{f_u} \) as

\[
Z_{f_u} = \frac{1}{n} \|u\|^2 \sum_{i=1}^{n} f^2(x_i) - \mathbb{E}[f^2] = \frac{1}{n} \|u\|^2 \sum_{i=1}^{n} \left( \frac{\langle x_i, u \rangle^2}{u^T \Sigma_i u} - 1 \right),
\]

\[
W_{f_u} = \left( \frac{1}{n} \|u\|^2 \sum_{i=1}^{N} f^2(x_i) \right)^{1/2} = \left( \frac{1}{n} \|u\|^2 \sum_{i=1}^{n} \frac{\langle x_i, u \rangle^2}{u^T \Sigma_i u} \right)^{1/2}.
\]

We prove the following lemma, which is an analogous result to Lemma 1.2 in [20].

Lemma 14: There exists an absolute constant \( c_1 > 0 \) for which the following holds. For every \( f_u, f_v \in F \) and every \( \epsilon > 2 \) we have,

\[
P(W_{f_u - f_v} \geq \epsilon \|f_u - f_v\|_2) \leq 2 \exp(-c_1 n \epsilon^4).
\]

Also, for every \( u > 0 \),

\[
P\left( |Z_{f_u}| \geq \epsilon u^2 \right) \leq 2 \exp(-c_1 n \epsilon^2),
\]

\[
P(|Z_{f_u} - Z_{f_v}| \geq \epsilon \|f_u - f_v\|_2) \leq 2 \exp(-c_1 n \epsilon^2).
\]

Proof: First, we prove [18]. The process \( W_{f_u - f_v} \) is defined as follows:

\[
W_{f_u - f_v} = \left( \frac{1}{n} \|u - v\|^2 \sum_{i=1}^{n} \frac{\langle x_i, u - v \rangle^2}{(u - v)^T \Sigma_i (u - v)} \right)^{1/2}.
\]
We define the random variable $z_i = \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} - E \left[ \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} | F_{i-1} \right]$ which implies $z_i$ is a bounded MDS as $E[z_i | F_{i-1}] = 0$ and $|z_i| \leq cA^2 = B$ for some constant $c$. Therefore, by the Azuma-Hoeffding inequality,

$$P\left( \frac{1}{n} \sum_{i=1}^{n} z_i \geq t \right) \leq 2 \exp \left( \frac{nt^2}{2B^2} \right)$$

$$= P\left( \frac{1}{n} \sum_{i=1}^{n} \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} - E \left[ \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} | F_{i-1} \right] \geq t \right)$$

$$= P\left( \frac{1}{n} \sum_{i=1}^{n} \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} - 1 \geq t \right)$$

$$= P\left( \frac{1}{n} \left\| u-v \right\|^2 \sum_{i=1}^{n} \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} - 1 \geq t \left\| u-v \right\|^2 \right)$$

$$= P \left( W_{f_a-f_v}^2 - \left\| u-v \right\|^2 \geq t \left\| u-v \right\|^2 \right)$$

$$= P \left( W_{f_a-f_v}^2 \geq c_2 B^2 \epsilon^2 \left\| u-v \right\|^2 \right) \text{ (Setting } t = c_2 B^2 \epsilon^2 - 1)$$

$$= P \left( W_{f_a-f_v}^2 \geq c_2 \left\| f_u - f_v \right\|_{\psi_2}^2 \right)$$

$$= P \left( W_{f_a-f_v} \geq c \left\| f_u - f_v \right\|_{\psi_2} \right) \leq 2 \exp \left( -c_1 n \epsilon^4 \right).$$

The first equality follows by replacing the value of $z_i$, the second by noting that $E \left[ \frac{(x_i,u-v)^2}{(u-v)^T \Sigma_i (u-v)} | F_{i-1} \right] = 1$, the third by multiplying both sides by $\left\| u-v \right\|^2$, the fifth by setting $t = c_2 \epsilon^2 B^2 - 1$ for some constant $c_2 > 0$ and noting that $\left\| f_u - f_v \right\|_{\psi_2} = \left\| u-v \right\|^2 \sup_{\left\| u-v \right\|^2 \leq \epsilon} \left\langle x_i, \frac{u-v}{\left\| u-v \right\|^2} \right\rangle^2 = c_2 B^2 \left\| u-v \right\|^2$, and then taking the square root.

We use similar arguments to prove (79). Let $z_i = \frac{(x_i,u)^2}{u^T \Sigma_i u} - E \left[ \frac{(x_i,u)^2}{u^T \Sigma_i u} | F_{i-1} \right]$. By the argument given earlier, $z_i$ is a bounded MDS and $|z_i| \leq cB$ for some constant $c$. Using the Azuma-Hoeffding inequality we obtain

$$P\left( \frac{1}{n} \left\| u \right\|^2 \sum_{i=1}^{n} z_i \geq t \right)$$

$$= P\left( \frac{1}{n} \left\| u \right\|^2 \sum_{i=1}^{n} \frac{(x_i,u)^2}{u^T \Sigma_i u} - E \left[ \frac{(x_i,u)^2}{u^T \Sigma_i u} | F_{i-1} \right] \geq t \right)$$

$$= P\left( \frac{1}{n} \left\| u \right\|^2 \sum_{i=1}^{n} \frac{(x_i,u)^2}{u^T \Sigma_i u} - 1 \geq t \right)$$

$$= P\left( \left\| Z_{f_a} \right\| \geq t \right) \leq 2 \exp \left( -\frac{nt^2}{2c^2 B^2} \right).$$

Let $t = \epsilon cB$ and noting that $\alpha^2 = cB$ we get the following bound for some constant $c_1 > 0$

$$P \left( \left\| Z_{f_a} \right\| \geq \epsilon \alpha^2 \right) \leq 2 \exp(-c_1 n \epsilon^2). \quad (82)$$
For the proof of (80) note that

\[ Z_{fu} - Z_{fv} = \frac{1}{n} \sum_{i=1}^{n} \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v}. \]  

(83)

Let \( z_i = \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} - \mathbb{E} \left[ \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} \bigg| F_{i-1} \right]. \) Again \( z_i \) is a bounded MDS with \( |z_i| \leq \alpha \|f_u - f_v\|_{\psi^2}. \)

\[
P\left( \frac{1}{n} \left| \sum_{i=1}^{n} z_i \right| \geq t \right) 
= P\left( \frac{1}{n} \left( \sum_{i=1}^{n} \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} - \mathbb{E} \left[ \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} \bigg| F_{i-1} \right] \right) \geq t \right) 
= P\left( \frac{1}{n} \left( \sum_{i=1}^{n} \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} \right) \geq t \right) 
= P\left( |Z_{fu} - Z_{fv}| \geq \epsilon \alpha \|f_u - f_v\|_{\psi^2} \right) \leq 2 \exp(-c_1 n \epsilon^2). \]

The second inequality follows since \( \mathbb{E} \left[ \frac{(x_i, u)^2}{u^T \Sigma_i u} - \frac{(x_i, v)^2}{v^T \Sigma_i v} \bigg| F_{i-1} \right] = 0 \) and the last equality follows by setting \( t = c_2 B \alpha \|f_u - f_v\|_{\psi^2} \). This concludes the proof of Lemma 14 and following the same proof of Theorem D using Lemma 14 instead of Lemma 1.2 \[20\] we prove Lemma 13.

Finally, for the proof of Lemma 11, we follow the same application as in \[6\] Theorem 12. This concludes the proof of Lemma 11.

References

[1] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In *Advances in Neural Information Processing Systems (NIPS)*, 2011.

[2] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Online-to-Confidence-Set Conversions and Application to Sparse Stochastic Bandits. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2012.

[3] Peter Auer. Using confidence bounds for exploitation-exploration trade-offs. *Journal of Machine Learning Research*, 3(3):397–422, 2003.

[4] Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47(2):235–256, 2002.

[5] Arindam Banerjee, Sheng Chen, Farideh Fazayeli, and Vidyashankar Sivakumar. Estimation with Norm Regularization. In *Neural Information Processing Systems (NIPS)*, 2014.

[6] Arindam Banerjee, Sheng Chen, Farideh Fazayeli, and Vidyashankar Sivakumar. Estimation with Norm Regularization. *arXiv:1505.02294v3*, November 2015.
[7] Peter Bickel, Yaacov Ritov, and Alexandre Tsybakov. Simultaneous analysis of Lasso and Dantzig selector. *The Annals of Statistics*, 37(4):1705–1732, 08 2009.

[8] Stephane Boucheron, Gabor Lugosi, and Pascal Massart. Concentration Inequalities: A Nonasymptotic Theory of Independence. Oxford University Press, 2013.

[9] Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2010.

[10] Sébastien Bubeck and Nicolò Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Foundations and Trends in Machine Learning*, 5(1):1–122, 2012.

[11] Emmanuel Candes and Terence Tao. The Dantzig selector: statistical estimation when p is much larger than n. *The Annals of Statistics*, 35(6):2313–2351, 2007.

[12] Alexandra Carpentier and Remi Munos. Bandit Theory Meets Compressed Sensing for High-dimensional Stochastic Linear Bandit. *International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2012.

[13] Venkat Chandrasekaran, Benjamin Recht, Pablo Parrilo, and Alan Willsky. The Convex Geometry of Linear Inverse Problems. *Foundations of Computational Mathematics*, 12(6):805–849, 2012.

[14] Sheng Chen and Arindam Banerjee. Structured estimation with atomic norms: General bounds and applications. In *Advances in Neural Information Processing Systems (NIPS)*, 2015.

[15] Wei Chu, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandits with linear payoff functions. In *Artificial Intelligence and Statistics Conference (AISTATS)*, 2011.

[16] Varsha Dani, Thomas Hayes, and Sham Kakade. Stochastic Linear Optimization Under Bandit Feedback. In *Conference on Learning Theory (COLT)*, 2008.

[17] Ingrid Daubechies, Michel Defrise, and Christine De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on Pure and Applied Mathematics*, 57(11):1413–1457, 2004.

[18] Sebastien Gerchinovitz. Sparsity regret bounds for individual sequences in online linear regression. In *Conference on Learning Theory (COLT)*, 2011.

[19] Lihong Li, Wei Chu, John Langford, and Robert Schapire. A Contextual-Bandit Approach to Personalized News Article Recommendation. *International World Wide Web Conference (WWW)*, 2010.

[20] Shahar Mendelson, Alain Pajor, and Nicole Tomczak-Jaegermann. Reconstruction and subgaussian operators in asymptotic geometric analysis. *Geometric and Functional Analysis*, 17 (4):1248–1282, 2007.

[21] Sahand Negahban and Martin J. Wainwright. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *The Annals of Statistics*, 39(2):1069–1097, 2011.
[22] Sahand Negahban, Pradeep Ravikumar, Martin Wainwright, and Bin Yu. A Unified Framework for High-Dimensional Analysis of M-Estimators with Decomposable Regularizers. *Statistical Science*, 27(4):538–557, 2012.

[23] Neal Parikh and Stephen Boyd. Proximal Algorithms. *Foundations and Trends in Optimization*, 1(3):123–231, 2014.

[24] Paat Rusmevichientong and John Tsitsiklis. Linearly parameterized bandits. *Mathematics of Operations Research*, 35(2):395–411, 2010.

[25] Michel Talagrand. *The Generic Chaining*. Springer Monographs in Mathematics. Springer-Verlag, 2005.

[26] Michel Talagrand. *Upper and Lower Bounds for Stochastic Processes*. Springer-Verlag, 2014.

[27] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. In *Compressed Sensing*, pages 210–268. Cambridge University Press, 2012.