A FAMILY OF NON-SCHURIAN $p$-SCHUR RINGS OVER GROUPS OF ORDER $p^3$

KIJUNG KIM

Abstract. Recently, it was proved that every commutative $p$-Schur ring over a group of order $p^3$ is Schurian. In this article, we consider the Schurity problem of non-commutative $p$-Schur rings over groups of order $p^3$. In particular, it is given a family of non-Schurian $p$-Schur rings over groups of order $p^3$.

Key words: $p$-Schur ring; Schurian; Cayley scheme.

1. Introduction

Let $H$ be a finite group. We denote by $QH$ the group algebra of $H$ over the rational number field $Q$. For a nonempty subset $T \subseteq H$, we set $T := \sum_{t \in T} t$ which is treated as an element of $QH$.

A subalgebra $A$ of the group algebra $QH$ is called a Schur ring (briefly S-ring) over $H$ if there exists a partition $Bsets(A) := \{T_0, T_1, \ldots, T_r\}$ of $H$ satisfying the following conditions:

(i) $\{T_i | T_i \in Bsets(A)\}$ is a linear basis of $A$;
(ii) $T_0 := \{1_H\}$;
(iii) $T_i^{-1} := \{t^{-1} | t \in T_i\} \in Bsets(A)$ for all $T_i \in Bsets(A)$.

A S-ring $A$ over a $p$-group $H$ is called a $p$-Schur ring (briefly $p$-S-ring) if the size of every element in $Bsets(A)$ is a power of $p$, where $p$ is a prime.

Let $G$ be a subgroup of $Sym(H)$ containing the right regular representation of $H$. We denote by $T_0 = \{1_H\}, T_1, \ldots, T_r$ the orbits of the stabilizer $G_{1_H}$. The transitivity module $V(H,G_{1_H})$ of $G$ is the vector space spanned by $\{T_i | 0 \leq i \leq r\}$.

It was proved in [13] that $V(H,G_{1_H})$ is a S-ring over $H$. Customarily, an S-ring $A$ over $H$ is called Schurian if $A$ is the transitivity module $V(H,G_{1_H})$ for some group $G$ containing the right regular representation of $H$. A family of S-rings which are not Schurian was given in [13, Theorem 26.4]. It is known that every S-ring over a cyclic $p$-group is Schurian (see [10]). In 1979, M. Klin conjectured that every S-ring over a cyclic group is Schurian. But, it was proved in [4] that there exist non-Schurian S-rings over cyclic groups. Recently, it was proved that every commutative $p$-S-ring over a group of order $p^3$ is Schurian (see [7, 8, 12]). In this article, we consider the Schurity problem of non-commutative $p$-Schur rings over groups of order $p^3$. First of all, we observe non-commutative 7-S-rings over a group of order 7$^3$.

Example 1.1. Let $H_0 = \langle a, b | a^7 = b^7 = 1, ab = ba^8 \rangle$ be a non-abelian group of order 7$^3$. Then the following partitions give two 7-S-rings $\mathcal{A}_i$ over $H_0$ $(i = 1, 2)$:

$Bsets(\mathcal{A}_i) = \{\{l\} | l \in \langle a^7, b \rangle\} \cup \bigcup_{1 \leq j \leq 6} \{a^j(b(a^7)^{z_i(j)})(a^7)^k | 0 \leq k \leq 6\}$.
where \((x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(6)) = (0, 3, 6, 2, 5, 1)\)
and \((x_2(1), x_2(2), x_2(3), x_2(4), x_2(5), x_2(6)) = (0, 4, 2, 5, 6, 1)\).

In Example 1.1, the action of \(\phi\) on \(H_0\) induces \(Bsets(A_1)\), where \(\phi : a \mapsto ab, b \mapsto b\) is an automorphism of \(H_0\). This means that \(A_1\) is Schurian. But, no automorphism of \(H_0\) induces \(Bsets(A_2)\). So, one might be curious what can be said in the case of \(A_2\). An attempt to verify the Schurity of \(A_2\) leads to our results.

In Example 2.1, the group generated by \(\{T^{-1}T \mid T \in Bsets(A_i)\} (i = 1, 2)\) is an elementary abelian group of rank 2. We denote it by \(O^p(A_i)\). Two 7-S-rings have common properties as follows:

(i) For each \(T \in Bsets(A_i), T^{-1}T\) is a proper subgroup of \(O^p(A_i)\);
(ii) For \(T, T' \in Bsets(A_i)\) with \(T O^p(A_i) \neq T' O^p(A_i)\), \(T^{-1}T\) and \(T'^{-1}T'\) are different.

In general, we deal with \(p\)-S-rings satisfying the above properties. Our first step is to determine \(Bsets(A)\). Let \(A\) be a \(p\)-S-ring over a non-abelian group \(H\) of order \(p^3\) satisfying the above properties (see [11] and [12] for details). If \(T_1, \ldots, T_{p-1}\) are elements of \(Bsets(A)\) such that \(T_i^{-1}T_i \neq T_j^{-1}T_j\) for distinct \(1 \leq i, j \leq p - 1\), then we have

\[
Bsets(A) = \{\{l\} \mid l \in O^p(A)\} \cup \bigcup_{1 \leq i \leq p-1} \{T_i z^j \mid 0 \leq j \leq p - 1\},
\]

where \(z\) is a nontrivial element of the center of \(H\). Based on this result, we are able to give a criterion for Schurity of \(A\) and verify the existence of non-Schurian \(p\)-S-rings over groups of order \(p^3\).

This article is organized as follows. In Section 2, we review definitions and known facts about \(S\)-rings. In Section 3, we give a criterion when \(p\)-S-rings over groups of order \(p^3\) with some conditions are Schurian. In Section 4, we introduce a problem related to constructions of \(p\)-S-rings. In Section 5, we construct non-Schurian \(p\)-S-rings over groups of order \(p^3\).

2. Preliminaries

Throughout this article, we use the notations based on [11] [15].

For a group \(H\), we denote by \(R_H\) the set of all binary relations on \(H\) invariant with respect to the right regular representation of \(H\). Then the mapping

\[
2^H \to R_H (T \mapsto R_H(T)),
\]

where \(R_H(T) = \{(h, th) \mid h \in H, t \in T\}\), is a bijection. If \(A\) is a \(p\)-ring over \(H\), then the pair

\[
C(A) = (H, R_H(Bsets(A))),
\]

where \(R_H(Bsets(A)) = \{R_H(T) \mid T \in Bsets(A)\}\), is called a Cayley (association) scheme over \(H\) (see Subsection 2.1 for association schemes).

Let \(C = (H, R)\) be a Cayley scheme. For each \(r \in R\), we set

\[
r(1_H) = \{h \in H \mid (1_H, h) \in r\}.
\]

Then the vector space spanned by \(\{r(1_H) \mid r \in R\}\) is a \(p\)-ring over \(H\).

**Theorem 2.1** (See [9]). The correspondence \(A \mapsto C(A), C(A) \mapsto A\) induces a bijection between the \(S\)-rings and Cayley schemes over the group \(H\) that preserves the natural partial orders on these sets.

**Theorem 2.1** means that the \(S\)-rings and Cayley schemes provide two different ways to talk about the same thing. In the rest of this section, we consider definitions in the theory of association schemes and translate them into \(S\)-rings.
2.1. Association schemes. Let $X$ be a nonempty finite set. Let $S$ denote a partition of $X \times X$. Then we say that $(X, S)$ is an association scheme (briefly scheme) if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\} \in S$;
(ii) For each $s \in S$, $s^* := \{(x, y) \mid (y, x) \in s\} \in S$;
(iii) For all $s, t, u \in S$ and $x, y \in X$, $a_{stu} := |xs \cap yt^*|$ is constant whenever $(x, y) \in u$, where $xs := \{y \in X \mid (x, y) \in s\}$.

For each $s \in S$, we abbreviate $a_{s \ast 1_X}$ as $n_s$, which is called the valency of $s$. Put $n_S = \sum_{s \in S} n_s$. We call $n_S$ the order of $S$. A scheme $(X, S)$ is called $p$-valenced if the valency of every element is a power of $p$, where $p$ is a prime. In particular, a $p$-valenced scheme $(X, S)$ is called a $p$-scheme if $n_S$ is also a power of $p$.

Let $P$ and $Q$ be nonempty subsets of $S$. We define $PQ$ to be the set of all elements $s \in P$ such that there exists elements $p \in P$ and $q \in Q$ with $apqs \neq 0$. The set $PQ$ is called the complex product of $P$ and $Q$. If one of the factors in a complex product consists of a single element $s$, then one usually writes $s$ for $\{s\}$.

A nonempty subset $T$ of $S$ is called closed if $TT \subseteq T$. Note that a subset $T$ of $S$ is closed if and only if if $\bigcup_{t \in T} t$ is an equivalence relation on $X$. A closed subset $T$ is called thin if all elements of $T$ have valency 1. The set $\{s \in S \mid n_s = 1\}$ is called the thin radical of $S$ and denoted by $O_p(S)$.

For binary relations $r, s \subseteq X \times X$, we set $r \ast s = \{(\alpha, \gamma) \mid (\alpha, \beta) \in r, (\beta, \gamma) \in s \text{ for some } \beta \in X\}$. We call $r \ast s$ the relational product of $r$ and $s$. Note that $T$ is thin if and only if $T$ is a group under the relational product.

A closed subset $T$ of $S$ is called strongly normal in $S$, denoted by $T \trianglelefteq S$, if $s^*T \subseteq T$ for every $s \in S$. We put $O_S^0(S) := \bigcap_{T \trianglelefteq S} T$ and call it the thin residue of $S$. Note that $O_S^0(S) = \left(\bigcup_{s \in S} s^*s\right)$ (see [14] Theorem 2.3.1)).

Let $(X, S)$ and $(X_1, S_1)$ be association schemes. A bijective map $\phi : X \cup S \to X_1 \cup S_1$ is called an isomorphism if it satisfies the following conditions:

(i) $X^\phi \subseteq X_1$ and $S^\phi \subseteq S_1$;
(ii) For all $x, y \in X$ and $s \in S$ with $(x, y) \in s$, $(x^\phi, y^\phi) \in s^\phi$.

An isomorphism $\phi$ from $X \cup S$ to $X_1 \cup S_1$ is called an automorphism of $(X, S)$ if $s^\phi = s$ for all $s \in S$. We denote by Aut$(X, S)$ the automorphism group of $(X, S)$.

**Lemma 2.2** (See [14]). Let $(X, S)$ be an association scheme. For $u, v, w \in S$, we have the following:

(i) $a_{uvw}n_w = a_{uvw}n_v = a_{vwu}n_u$;
(ii) $n_an_u = \sum_{s \in S} a_{uvs}n_s$.

**Lemma 2.3** (See [11]). Let $(X, S)$ be an association scheme and $r, s \in S \setminus \{1_X\}$. Then $rr^* \cap ss^* = \{1_X\}$ if and only if $a_{r^*s^*} \leq 1$ for all $t \in S$.

For a binary relation $t \subseteq X \times X$ and subsets $E, F \subseteq X$, we define the adjacency matrix $A_t$ of $t \cap (E \times F)$ whose rows and columns are indexed by the elements of $E$ and $F$ as follows:

$$(A_t)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in t; \\ 0 & \text{otherwise.} \end{cases}$$

If both of $E$ and $F$ are $X$, then we usually write $A_t$ without mentioning $t \cap (E \times F)$.

**Lemma 2.4.** Let $(X, S)$ be an association scheme. Fix $x \in X$. For $d, e, f \in S$, each column of the adjacency matrix of $e \cap (xd \times xf)$ has $a_{def}$ 1’s.

**Proof.** Let $A_e$ be the adjacency matrix of $e \cap (xd \times xf)$. Fix $y \in xf$. We count the number of $x'$ such that $(A_e)_{x'y} = 1$. Then it amounts to $|xd \cap ye^*|$. By the definition of association schemes, we have $|xd \cap ye^*| = a_{def}$. \hfill $\square$
Lemma 2.5. Let \((X, S)\) be an association scheme. Fix \(x \in X\). For \(d, f \in S\), we have \(\{s \in S \mid s \cap \{(xd \times xf) \neq \emptyset\}\} = \{s \in S \mid s \in d^* f\}\).

Proof. By Lemma 2.4 we have \(s \cap \{(xd \times xf) \neq \emptyset\} \iff a_{dsf} \neq 0\). By Lemma 2.2 (i), we have \(a_{dsf}n_f = a_{d'sf}n_s\). Then \(a_{dsf} \neq 0 \iff a_{d'sf} \neq 0\). This completes the proof.

Theorem 2.6 (See [5]). Let \((X, S)\) be a non-commutative p-scheme of order \(p^3\) which is not thin. Then \(O^0(S)\) is a thin closed subset of \(S\) isomorphic to \(C_p \times C_p\).

2.2. S-rings. Let \(A\) be a S-ring over \(H\). We say that a subgroup \(K\) of \(H\) is an \(A\)-subgroup if \(K \in A\). For each \(A\)-subgroup \(E\) of \(H\), one can define a subring \(A_E\) by setting

\[A_E = A \cap QE.\]

It is easy to see that \(A_E\) is a S-ring over \(E\) and

\[\text{Bssets}(A_E) = \{T \mid T \in \text{Bssets}(A), T \subseteq E\}.\]

An automorphism of \(C(A)\) taking \(1_H\) to \(1_H\) is called the automorphism of \(A\), which is denoted by \(\text{Aut}(A)\). One can see that

\[\text{Aut}(A) = \text{Aut}(H, R_H(\text{Bssets}(A)))_{1_H},\]

\[\text{Aut}(H, R_H(\text{Bssets}(A))) = \text{Aut}(A)H_R,\]

where \(H_R\) is the right regular representation of \(H\).

Lemma 2.7. Let \(A\) be a S-ring over \(H\). Let \(T_1, T_2\) be elements of \(\text{Bssets}(A)\). Then the following are equivalent.

1. \(T_1 \cdot T_2 = \sum_{T \in \text{Bssets}(A)} a_{R_H(T_1)R_H(T_2)R_H(T)} T_1 T_2\)
2. \(R_H(T_1)R_H(T_2) = \sum_{T \in \text{Bssets}(A)} a_{R_H(T_1)R_H(T_2)R_H(T)} R_H(T_1)T_2\)

Based on Theorem 2.1, we state the following results in [14].

Proposition 2.8. Let \(A\) be a S-ring over \(H\) and \(m\) an element of \(H\). If \(T, \{m\} \in \text{Bssets}(A)\), then \(Tm = \{tm \mid t \in T\}\) lies in \(\text{Bssets}(A)\).

Proposition 2.9. Let \(A\) be a S-ring over \(H\). If \(T \in \text{Bssets}(A)\), then the stabilizer \(\text{St}_R(T) := \{h \in H \mid Th = T\}\) is an \(A\)-subgroup of \(H\).

Proposition 2.10. Let \(A\) be a p-S-ring over a group \(H\) of order \(p^m\). Then

1. the group \(O_p(\mathcal{O}) := \{h \in H \mid \{h\} \in \text{Bssets}(A)\}\) is a nontrivial \(A\)-subgroup,
2. the group \(O^0(\mathcal{O}) := \{\{T^{-1}T \mid T \in \text{Bssets}(A)\}\}\) is a proper \(A\)-subgroup,
3. there exists a series \(H_0 = \{1_H\} < H_1 < \cdots < H_m = H\) of \(A\)-subgroups such that \([H_{i+1} : Hi] = p\).

Remark 2.1. \(O_p(\mathcal{O})\) and \(O^0(\mathcal{O})\) correspond to the thin radical and thin residue in the theory of association schemes, respectively.

3. Schurity of p-S-rings

It is known that every 2-S-ring of a group of order 8 is commutative and Schurian (see [2]). From now on, we assume that \(p\) is an odd prime. It is well known that there are two non-abelian groups of order \(p^3\) up to isomorphism, namely

\[H_1 = \langle a, b \mid a^p = b^p = 1, ab = ba^{p+1}\rangle,\]
\[H_2 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.\]

Throughout this section, we assume that \(A\) is a non-commutative p-S-ring over \(H_i (i = 1, 2)\) such that

\[(1) \quad |T| = p\] for each \(T \in \text{Bssets}(A) \setminus \text{Bssets}(A_{O^p(A)})\).
(2) \(|\{\text{St}_R(T) \mid T \in \text{Bssets}(A) \setminus \text{Bssets}(A_{O^\theta(A)})\}| = p - 1.\)

By Theorem 2.6 we have

\[ O^\theta(A) = O_\theta(A) \cong C_p \times C_p. \]

Without loss of generality, we can assume \(O^\theta(A) = \langle a^p, b \rangle\) over \(H_1\) and \(O^\theta(A) = \langle b, c \rangle\) over \(H_2\). For convenience, we omit the subindex \(i\) of \(H_i\) and denote \(O^\theta(A)\) by \(L\).

**Lemma 3.1.** For each \(T \in \text{Bssets}(A) \setminus \text{Bssets}(A_L)\), \(\text{St}_R(T)\) is not the center \(Z(H)\) of \(H\).

**Proof.** Suppose that \(\text{St}_R(T) = Z(H)\) for some \(T \in \text{Bssets}(A) \setminus \text{Bssets}(A_L)\). Then \(T = hZ(H)\) for some \(h \in H \setminus L\). Note that \(h = a^i b^j\) for some \(1 \leq i \leq p - 1\) and \(0 \leq j \leq p - 1\). Since \(T \cdot T = p \cdot h^2 Z(H)\) and \(|h^2 Z(H)| = p\), we have \(h^2 Z(H) \in \text{Bssets}(A) \setminus \text{Bssets}(A_L)\) by (11). Since \(b^j \in \text{Bssets}(A)\) for each \(1 \leq j \leq p - 1\), it follows from Proposition 2.10(ii) that \(b^jT \in \text{Bssets}(A) \setminus \text{Bssets}(A_L)\). These imply that every element of \(\text{Bssets}(A) \setminus \text{Bssets}(A_L)\) has the form \(a^i b^j Z(H)\) for some \(1 \leq i \leq p - 1\), \(0 \leq j \leq p - 1\). By Proposition 2.10(ii), we have \(L = Z(H)\), which contradicts \(L \cong C_p \times C_p\). \(\square\)

**Lemma 3.2** (See [3]). If \(A\) is Schurian, then the order of \(\text{Aut}(A)\) is \(p\).

**Proof.** Let \((X, S) = (H, R_H(\text{Bssets}(A)))\). For convenience, we denote \(\text{Aut}(X, S)\) by \(G\). By the orbit-stabilizer property, we have \(|G| = |G_\alpha||X|\) for a fixed \(\alpha \in X\). Since \(|X| = r_1 p^3\), it suffices to verify \(|G_\alpha| = p\).

Let \(r_1\) be an element of \(S\) \setminus \(O_\theta(S)\). We pick an element \(\beta\) of \(X\) such that \(\beta \in \alpha r_1\).

Under the assumption that \((X, S)\) is Schurian, we have

\[ |G_\alpha| = |G_{\alpha, \beta}||\alpha r_1| = |G_{\alpha, \beta}|p \]

by the orbit-stabilizer property. We shall show \(|G_{\alpha, \beta}| = 1\).

Let \(\gamma\) be an element of \(\alpha r_1\) such that \(\gamma \in \beta t\) for some \(t \in \{s \in S \mid r_1 s = r_1\} \setminus \{Z_X\}\). \(G_{\alpha, \beta}\) fixes \(\gamma\) since \(a_{r_1 t} r_1 = 1\). Thus, \(G_{\alpha, \beta}\) fixes \(\gamma\) for \(\alpha r_1\).

We consider an arbitrary element \(r_2 \in S\setminus O_\theta(S)\) such that \(r_1 \neq r_2\). For \(\delta \in \alpha r_2\), we assume that there exists \(g \in G_{\alpha, \beta}\) such that \(\delta^g \neq \delta\). Then there exist \(s_1, s_2 \in S\) such that \((\beta, \delta), (\beta, \delta^g) \in s_1\) and \((\gamma, \delta), (\gamma, \delta^g) \in s_2\). This means \(a_{s_1 s_2}^r 1 \geq 2\) and \(a_{s_1 s_2}^r 1 \geq 2\).

We divide our consideration into the following cases.

(i) If \(r_1 O^\theta(S) = r_2 O^\theta(S)\), then \(s_1, s_2 \in O_\theta(S)\). This contradicts \(a_{s_1 s_2}^r 1 = 1\). Thus, \(G_{\alpha, \beta}\) fixes each element of \(\alpha r_2\). This implies that \(G_{\alpha, \beta}\) fixes each element of \(\alpha s\) with \(r_1 O^\theta(S) = s O^\theta(S)\).

(ii) If \(r_1 O^\theta(S) \neq r_2 O^\theta(S)\), then \(s_1, s_2 \in S\setminus O_\theta(S)\). This contradicts \(a_{r_2 s_1 r_1} 1 = 1\).

Whichever the case may be, \(G_{\alpha, \beta}\) fixes each element of \(\alpha s\) for all \(s \in S\). Hence, we have \(|G_{\alpha, \beta}| = 1\). \(\square\)

In the rest of this section, we consider the association scheme \((H, R_H(\text{Bssets}(A))\)) corresponding to the S-ring \(A\). We fix an element \(z\) of \(Z(H)\).

**Lemma 3.3.** Let \(T_1, \ldots, T_{p-1}\) be elements of \(\text{Bssets}(A) \setminus \text{Bssets}(A_L)\) such that \(\text{St}_R(T_i) \neq \text{St}_R(T_j)\) for distinct \(1 \leq i, j \leq p - 1\). Then we have

\[ \text{Bssets}(A) = \{|l| \mid l \in L\} \cup \bigcup_{1 \leq i \leq p - 1} \{T_i z^j \mid 0 \leq j \leq p - 1\}. \]

**Proof.** By Lemma 3.1 we have \(T_i \neq T_i z^j\) for each \(z^j \in Z(H)\). It follows from Proposition 2.8 that \(T_i z \in \text{Bssets}(A)\). This completes the proof. \(\square\)
From now on, we assume that for each \( R_H(T) \in R_H(\text{Bsets}(A)) \), the rows and columns of the adjacency matrix of \( R_H(T) \) are indexed by the following:

(i) Assume that each \( T_i \) has a fixed ordering of elements in itself;
(ii) Assume that \( \text{Bsets}(A_L) \) has a fixed ordering of elements in itself;
(iii) A fixed ordering of \( L \) and \( T_i \)'s is

\[
(L, T_1, T_1z, \ldots, T_1z^{p-1}, T_2, T_2z, \ldots, T_2z^{p-1}, \ldots, T_{p-1}, T_{p-1}z, \ldots, T_{p-1}z^{p-1})
\]

where \( L \) and \( T_i z^j = (1 \leq i \leq p, 0 \leq j \leq p - 1) \) are given in \( [3] \).

**Lemma 3.4.** Let \( T, T' \) and \( T'' \) be elements of \( \text{Bsets}(A) \setminus \text{Bsets}(A_L) \) such that

\[
R_H(T) \cap (T' \times T'') \neq \emptyset.
\]

Then we have the following:

(i) The adjacency matrix of \( R_H(T) \cap (T' \times T'') \) is a permutation matrix;
(ii) \( R_H(Tz^k) \cap (T' \times T'') \neq \emptyset \) for each \( 1 \leq k \leq p - 1 \).

**Proof.** It follows from Lemma 2.5 that \( R_H(T) \in R_H(T') \ast R_H(T'') \). Put \( \mathbf{O}^\theta(R_H) := \mathbf{O}^\theta(R_H(\text{Bsets}(A))) \). Since \( R_H(T) \cap (T' \times T'') \neq \emptyset \), we have \( R_H(T') \mathbf{O}^\theta(R_H) \neq R_H(T'') \mathbf{O}^\theta(R_H) \). This means \( \text{St}_R(T') \neq \text{St}_R(T'') \). This implies \( A_{R_H(T')} A_{R_H(T'')} = \sum_{h \in R_H(T')} \mathbf{O}^\theta(R_H)_h A_k \). It follows from Lemmas 2.4 that the adjacency matrix of \( R_H(T) \cap (T' \times T'') \) is a permutation matrix. It follows from Lemma 2.5 that \( R_H(Tz^k) \cap (T' \times T'') \neq \emptyset \) for each \( 1 \leq k \leq p - 1 \).

From now on, for \( E, F, G \subseteq H \), we denote \( \{(e, f, g) \mid e \in E, f \in F, g \in G \} \) by \((E \times F \times G)^G \).

We introduce a notion of compatibility given in \( [3, 4] \). Denote by \( r(\beta, \gamma) \) the element of \( R_H(\text{Bsets}(A)) \) containing \((\beta, \gamma)\). We define the subset \( \Gamma_1 \) of the symmetric group on \( H \) with respect to \( \text{Bsets}(A) \) such that

(i) for \( \sigma \in \Gamma_1 \), \( L \) is the set of fixed points of \( \sigma \);
(ii) for \( \sigma \in \Gamma_1 \) and \( T \in \text{Bsets}(A) \) with size \( p \), \( \sigma \) is a \( p \)-cycle on \( T \) such that

\[
r(\beta, \gamma) = r(\beta^\sigma, \gamma^\sigma) \quad \text{for each} \quad (\beta, \gamma) \in T \times T.
\]

For \( T, T' \in \text{Bsets}(A) \) and \( \sigma \in \Gamma_1 \), we say that \( T \) and \( T' \) are compatible with respect to \( \sigma \)

\[
r(\beta, \gamma) = r(\beta^\sigma, \gamma^\sigma) \quad \text{for each} \quad (\beta, \gamma) \in T \times T'.
\]

We shall write \( T \sim_\sigma T' \) if \( T \) and \( T' \) are compatible with respect to \( \sigma \), otherwise \( T \sim_\sigma T' \).

**Theorem 3.5.** Let \( T_1, T_2, \ldots, T_{p-1} \) be elements of \( \text{Bsets}(A) \setminus \text{Bsets}(A_L) \) such that

\[
\text{St}_R(T_i) \neq \text{St}_R(T_j)
\]

for distinct \( 1 \leq i, j \leq p - 1 \). Then \( A \) is Schurian if and only if there exists a nontrivial element \( \sigma \in \Gamma_1 \) such that \( T_i \sim_\sigma T_j \) for all \( 1 \leq i, j \leq p - 1 \).

**Proof.** We consider \((H, R_H(\text{Bsets}(A)))\) corresponding to \( A \). By Lemma 3.2, \( A \) is Schurian if and only if there exists a nontrivial element \( \sigma \in \Gamma_1 \) such that \( T \sim_\sigma T' \) for all \( T, T' \in \text{Bsets}(A) \). It suffices to prove the sufficiency. Note that \( T \sim_\sigma T' \) for all \( \sigma \in \Gamma_1 \), \( T \in \text{Bsets}(A_L) \) and \( T' \in \text{Bsets}(A) \setminus \text{Bsets}(A_L) \).

From \( \sigma \) given in the assumption, we find a nontrivial element \( \sigma' \) of \( \Gamma_1 \) such that \( T \sim_\sigma T' \) for all \( T, T' \in \text{Bsets}(A) \).

Define \( \sigma' \in \text{Sym}(H) \) such that

(i) \( \sigma'|_{T_i} = \sigma|_{T_i} \) for all \( 1 \leq i \leq p - 1 \),
(ii) \( h^\sigma z^l = (hz^l)^\sigma \) for all \( h \in T_i, (1 \leq i \leq p - 1), 1 \leq l \leq p - 1 \).
First of all, we verify \( \sigma' \in \Gamma_1 \). For a given \( T_z \), we show \( r(\beta, \gamma) = r(\beta', \gamma') \) for each \((\beta, \gamma) \in T_z \times T_z \). There exists \( T \in Bsets(\mathcal{A})_1 \) such that \((\beta, \gamma) \in R_H(T) \cap (T_z \times T_z)\). Since \((R_{I_{T}}(T) \cap (T \times T))^{\ast} = R_H(T) \cap (T_z \times T_z)\), there exists \((\beta_0, \gamma_0) \in R_{I_{T}}(T) \cap (T \times T)\) such that \((\beta_0, \gamma_0)^{\ast} = (\beta, \gamma)\). Thus, it follows from the definition of \( \sigma' \) that \((\beta', \gamma') = ((\beta_0, \gamma_0^{\ast})^{\ast}, (\gamma_0 z_0^{\ast})^{\ast}) = (\beta_0^{\ast}, \gamma_0^{\ast}) = (\beta_0^{\ast}, \gamma_0^{\ast}) = (\beta_0^{\ast}, \gamma_0^{\ast}) \in R_H(T)^{\ast} = R_H(T)\).

Next, we verify \( T_i \sim_{\sigma'} T_i z_l \) for all \( 1 \leq l \leq p - 1 \). Let \((\beta, \gamma) \in T_i \times T_i z_l \). Then there exists \( T \in Bsets(\mathcal{A})_1 \) such that \((\beta, \gamma) \in R_H(T) \cap (T_i \times T_i z_l)\), and we have \( \gamma = \beta z^m \) for some \( t \in S_{R}(T) \) and \( m. \) Since \( \beta t = q \beta \) for some \( q \in H \), we have \((\beta, \gamma) = (\beta, \beta z^m) = (\beta, q z^m) \in r(1, q z^m)\). It follows from the definition of \( \sigma' \) that \((\beta', \gamma') = (\beta, \beta z^m) \in r(\beta, (\beta)') r((\beta t)^{\ast}, (\beta t)^{\ast} z^m) = r(\beta, \beta t) r((\beta t)^{\ast}, (\beta t)^{\ast} z^m) = r(\beta, \beta t) r(1, z^m) = r(1, q) r(q, q z^m) = r(1, q z^m)\). Thus, \( T_i \sim_{\sigma'} T_i z_l \) for all \( 1 \leq l \leq p - 1 \).

By the same argument, for each \( T_i z_l \) we have \( T_i z_l \sim_{\sigma'} T_i z_l z_m \) for all \( 1 \leq m \leq p - l - 1 \). This implies that \( T_i z_l \sim_{\sigma'} T_i z_n \) for all \( 0 \leq l, n \leq p - 1 \).

Finally, we verify \( T_i z_l \sim_{\sigma'} T_i z_l z_m \) for distinct \( 1 \leq i, j \leq p - 1 \) and all \( 1 \leq l, n \leq p - 1 \). Now we say that \((T_i \times T_j) \) is equivalent to \((T_i z_l \times T_j z_m)\) if the adjacency matrix of \( R_H(T) \cap (T_i \times T_j) \) coincides with that of \( R_H(T') \cap (T_i z_l \times T_j z_m) \) for some \( T, T' \in Bsets(\mathcal{A}) \). For convenience, we denote it by \((T_i \times T_j) \simeq (T_i z_l \times T_j z_m)\). We show that \((T_i \times T_j) \) is equivalent to \((T_i z_l \times T_j z_m)\) for all \( 0 \leq l, n \leq p - 1 \).

**Claim 1:** \((T_i \times T_j) \simeq (T_i z_l \times T_j z_m)\). There exists \( T \in Bsets(\mathcal{A}) \) such that \((\beta, \gamma) \in R_H(T) \cap (T_i \times T_j) \) and \((T_i z_l \times T_j z_m) \) is equivalent to \((T_i z_l \times T_j z_m)\) for all \( 1 \leq l, n \leq p - 1 \).

By the definition of \( R_H(T) \), we have \((\beta z_l, \gamma z_l) \in R_{H}(T)\). Since the orderings of \( T_i z_l \) and \( T_j z_l \) are depending on \( T_i \) and \( T_j \) respectively, it is easy to see that the adjacency matrix of \( R_{H}(T) \cap (T_i \times T_j) \) coincides with that of \( R_{H}(T) \cap (T_i z_l \times T_j z_l)\). This completes the proof of Claim 1.

**Claim 2:** \((T_i \times T_j) \simeq (T_i z_l \times T_j z_l)\). There exists \( T \in Bsets(\mathcal{A}) \) such that \((\beta, \gamma) \in R_{H}(T) \cap (T_i \times T_j) \) for all \( 1 \leq l, n \leq p - 1 \). For \( T, T z_m \in Bsets(\mathcal{A}) \), we have \( R_{H}(T) \cap (T_i \times T_j) = (\{1_H\} \times T z_m) \cap (T_i \times T_j) \) and \( R_{H}(T z_m) \cap (T_i \times T_j z_m) = (\{1_H\} \times T z_m) \cap (T_i \times T_j z_m)\). This implies that the adjacency matrix of \( R_{H}(T) \cap (T_i \times T_j) \) coincides with that of \( R_{H}(T z_m) \cap (T_i \times T_j z_m)\). This completes the proof of Claim 2.

By Claims 1 and 2, we have \((T_i \times T_j) \simeq (T_i z_l \times T_j z_l) \) for all \( 1 \leq l, n \leq p - 1 \). This means that \((T_i \times T_j) \simeq (T_i z_l \times T_j z_l) \) for all \( 1 \leq l, n \leq p - 1 \). So, it is easy to see that \( T_i z_l \sim_{\sigma'} T_i z_m \).

Therefore, we have \( T \sim_{\sigma'} T' \) for all \( T, T' \in Bsets(\mathcal{A}) \). \( \square \)

4. Construction of Suitable Sequences

In this section, we introduce a notion related to the construction of \( p \)-S-rings.

Let \( \mathbb{Z}_p = \{0, 1, \ldots, p - 1\} \). A sequence \((x_1, x_2, \ldots, x_{p-1})\) is called suitable if \( \{x_1, x_2, \ldots, x_{p-1}\} = \mathbb{Z}_p \setminus \{l\} \) for some \( l \in \mathbb{Z}_p \) such that \( x_1 = 0, \ x_i + i \equiv x_{p-i} \pmod{p} \) for each \( 1 \leq i \leq \frac{p-1}{2} \).

**Example 4.1.** In \( \mathbb{Z}_p \), \( x_1 \mid 1 \leq i \leq p - 1 \) is suitable, where \( x_i \equiv \frac{(p-1)}{2}(i-1) \pmod{p} \).

**Example 4.2.** In \( \mathbb{Z}_7 \), \((0, 4, 2, 5, 6, 1)\) and \((0, 2, 3, 6, 4, 1)\) are suitable.
Example 4.3. In \( \mathbb{Z}_{11} \), \((0, 6, 10, 3, 4, 9, 7, 2, 8, 1)\) and \((0, 4, 10, 5, 3, 8, 9, 2, 6, 1)\) are suitable.

Problem 4.1. Find all of suitable sequences of \( \mathbb{Z}_p \), where \( p \) is an odd prime.

In the case of \( p \equiv 3 \) (mod 4), we construct a suitable sequence. The following proposition is a generalization of first cases in Examples 4.2 and 4.3, which is different from Example 4.1.

Proposition 4.1. For each odd prime \( p = 4k + 3 \), there exists a suitable sequence with \( x_2 = \frac{p-1}{2} \).

Proof. Let \( (x_i) \) be the suitable sequence given in Example 4.1. First of all, we observe the following fact.

Claim 1: \((x_2, x_4, \ldots, x_{2k+2}) = (2k + 1, 2k, \ldots, k + 1)\) and \((x_{p-2k-2}, \ldots, x_{p-2}, x_{p-2}) = (3k + 3, \ldots, 2k + 4, 2k + 3)\).

Proof. We denote \( \mathcal{C} \cup \{ \frac{p+1}{2} \} = \{ x \mid k + 1 \leq x \leq 3k + 3 \} \) and \( \{ x_{p-i} - x_1 \mid x, x_{p-i} \in \mathcal{C}, 2 \leq i \leq 2k \} \cup \{ x_{p-2k-2} - x_{2k+2} \} = \{ 2, 4, \ldots, 2k, 2k + 1 \} \).

Rearranging \( 2k + 2 \) elements in \( \mathcal{C} \cup \{ \frac{p+1}{2} \} \), we construct a new suitable sequence.

Claim 2: There exists a sequence \((y_2, y_4, \ldots, y_{2k}, y_{p-2k-2}, y_{2k+2}, y_{p-2k}, \ldots, y_{p-4}, y_{p-2})\) such that

\[
\begin{align*}
y_{p-2l} - y_{2l} &\equiv 2l \pmod{p} \quad \text{for} \quad 1 \leq l \leq k, \\
y_{2k+2} - y_{p-2k-2} &\equiv 2k + 1 \pmod{p}
\end{align*}
\]

First of all, we consider the case that \( k \) is even. Put 

\[
(y_{2k+2}, y_{p-2k-2}) = (x_{p-2k-4}, x_{2k}) \quad \text{and} \quad (y_{p-2}, y_2) = (x_{p-4}, \frac{p+1}{2}).
\]

Then we have \( y_{2k+2} - y_{p-2k-2} \equiv 2k + 1 \pmod{p} \) and \( y_{p-2} - y_2 \equiv 2 \pmod{p} \).

For each \( 2 \leq l \leq k \), put 

\[
\begin{align*}
(y_{p-2l}, y_{2l}) &= (x_{p-2l+1}, x_{2l+1}) \quad \text{if} \quad l \text{ is even}, \\
(y_{p-2l}, y_{2l}) &= (x_{p-2l+1}, x_{2l+1}) \quad \text{if} \quad l \text{ is odd}.
\end{align*}
\]

Computing \( y_{p-2l} - y_{2l} \) based on Claim 1, we have \( y_{p-2l} - y_{2l} \equiv 2l \pmod{p} \).

Next, we consider the case that \( k \) is odd. Put 

\[
(y_{2k+2}, y_{p-2k-2}) = (x_{p-2k}, x_{2k+2}) \quad \text{and} \quad (y_{p-2}, y_2) = (x_{p-4}, \frac{p+1}{2}).
\]

For each \( 2 \leq l \leq k \), put

\[
\begin{align*}
(y_{p-2l}, y_{2l}) &= (x_{p-2l+1}, x_{2l+1}) \quad \text{if} \quad l \text{ is even}, \\
(y_{p-2l}, y_{2l}) &= (x_{p-2l+1}, x_{2l+1}) \quad \text{if} \quad l \text{ is odd}.
\end{align*}
\]

Then we have \( y_{2k+2} - y_{p-2k-2} \equiv 2k + 1 \pmod{p} \), \( y_{p-2} - y_2 \equiv 2 \pmod{p} \) and \( y_{p-2l} - y_{2l} \equiv 2l \pmod{p} \) for \( 1 \leq l \leq k \). This completes the proof of Claim 2.

Therefore, it follows from Claims 1 and 2 that \((x_1, x_2, \ldots, x_{2k-1}, y_2k, y_{2k+1}, y_{2k+2}, y_{2k+3}, x_{2k+4}, \ldots, y_{p-2}, x_{p-1})\) is suitable. \(\square\)
Proposition 4.2. Let $A$ be the vector space spanned by $\{T \mid T \in \{0\} \cup \{T_i \mathbf{a}_j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}\}$ such that $L = \langle \mathbf{a}_i, \mathbf{a}_j \rangle, T_i = a^i \langle \mathbf{a}_j \rangle, x_i \in L$ is suitable. Then $A$ is a $p$-S-ring over $H_1$.

Proof. Put $Bsets(A) = \{\{0\} \cup \{T_i \mathbf{a}_j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}\}$. Since $x_i + i \equiv x_{p-i} \pmod{p}$, we have

$$T_i^{-1} = \langle \mathbf{a}_i \rangle a_i^{-1} = \langle \mathbf{a}_i \rangle a_i^{p^2 - p} = a^{p-1} \langle \mathbf{a}_i \rangle a_i^{p(p-1)} = \langle \mathbf{a}_i \rangle a_i^{p(p-1)} = a^{p-1} \langle \mathbf{a}_i \rangle a_i^{p(p-1)} = \langle \mathbf{a}_i \rangle a_i^{p(p-1)} = T_{p-i} \in \text{Bsets}(A).$$

For $T_i, T_j \in \text{Bsets}(A)$,

$$T_i T_j = a^i \langle \mathbf{a}_i \rangle a^j \langle \mathbf{a}_j \rangle = \cdots = a^{i+j} \langle \mathbf{a}_{i+j} \rangle \langle \mathbf{a}_{i+j} \rangle = \left\{ \begin{array}{ll} a^p \langle \mathbf{a}_i \rangle & \text{if } j = p - i; \\ a^{i+j} L & \text{otherwise.} \end{array} \right.$$ 

This means that $T_i \cdot T_j$ is written as a linear combination of the elements in $\{T \mid T \in \text{Bsets}(A)\}$. We leave to the reader to check the remaining part of proof. □

Similarly, we have the following.

Proposition 4.3. Let $A$ be the vector space spanned by $\{T \mid T \in \{0\} \cup \{T_i \mathbf{a}_j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}\}$ such that $L = \langle \mathbf{a}_i, \mathbf{a}_j \rangle, T_i = a^i \langle \mathbf{a}_j \rangle, x_i \in L$ is suitable. Then $A$ is a $p$-S-ring over $H_2$.

5. Non-Schurian $p$-S-rings arisen from suitable sequences

In this section, we show that a $p$-S-ring over $H_1$ arisen from a suitable sequence is non-Schurian. For convenience, we omit the subindex $1$ of $H_1$ By Proposition 4.2 there exists a suitable sequence such that $x_i = 0, x_2 = \frac{p+1}{2}, x_3 = p - 1, \ldots, x_{p-1} = 1$.

Using Proposition 4.2 and this sequence, one can get a $p$-S-ring $A$ such that

$$\text{Bsets}(A) = \{\{0\} \cup \{T_i \mathbf{a}_j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}\}.$$

By Theorem 5.5 if $A$ is Schurian, then there exists an element $\sigma \in \Gamma_1$ such that $T_1 \sim_\sigma T_2 \sim_\sigma T_3 \sim_\sigma T_1$. In the rest of this section, we prove that any $\sigma \in \Gamma_1$ with $T_1 \sim_\sigma T_2 \sim_\sigma T_3$ does not satisfy $T_1 \sim_\sigma T_3$.

From now on, we assume that for all $1 \leq i \leq p-1$,

$$(a^i, a^i \mathbf{a}_i, a^i \langle \mathbf{a}_i \rangle, a^i \langle \mathbf{a}_i \rangle^2, \ldots, a^i \langle \mathbf{a}_i \rangle^{p-1})$$

is a fixed ordering of $T_i$.

Lemma 5.1. Let $T_i = a^i \langle \mathbf{a}_i \rangle, T_j = a^j \langle \mathbf{a}_j \rangle (i \leq j)$ and $T_k = a^k \langle \mathbf{a}_k \rangle$ be elements of $\{T \mid T \in \{0\} \cup \{T_i \mathbf{a}_j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}\}$ such that $R_H(T_k) \cap (T_i \times T_j) = \emptyset$. Then we have

$$R_H(T_k) \cap (T_i \times T_j) = \{(\{i\} \times a^k \langle \mathbf{a}_k \rangle) \times (a^i \langle \mathbf{a}_i \rangle) \cap (a^i \langle \mathbf{a}_i \rangle) \times a^j \langle \mathbf{a}_j \rangle)\}.$$
Note that $k + i = j$. For each $0 \leq l \leq p - 1$, there exist some $m, n$ such that 
\[(ba^{p^2}a^{p(p-1)i})^m (ba^{p^2}a^{p(p-1)i})^n = (ba^{p^2}a^{p(p-1)i})^l.\]

The above equation implies
\[(i) \ (x_k + (p - 1)i)m + x, n \equiv x, l \pmod{p},\]
\[(ii) \ m + n \equiv l \pmod{p}.\]

It follows from the system of congruences that
\[(x_i - x_k - (p - 1)i)n \equiv (x_j - x_k - (p - 1)i)l \pmod{p}.\]

\[\square\]

**Remark 5.1.** By Lemma 3.3, the adjacency matrix of $R_H(T_k) \cap (T_i \times T_j) \neq \emptyset$ is a permutation matrix. In (6), there exists a unique $n$ for a given $l$. This means that for each element of $T_j$, it is possible to determine a unique element of $T_i$ with respect to $R_H(T_k)$.

Using Lemma 5.1, we determine the adjacency matrices of $R_H(T_1) \cap (T_1 \times T_2)$, $R_H(T_1) \cap (T_2 \times T_3)$ and $R_H(T_2) \cap (T_1 \times T_3)$, where $T_1, T_2, T_3 \in \text{Bsets}(A) \setminus \text{Bsets}(A_L)$.

**Case 1:** $R_H(T_1) \cap (T_1 \times T_2)$.

In (5), if $i = k = 1$ and $j = 2$, then we have 
\[-(p - 1)n \equiv \left(\frac{p + 1}{2} - (p - 1)\right)l \pmod{p}.

So, $n \equiv \frac{p + 3}{2}l \pmod{p}$. The adjacency matrix of $R_H(T_1) \cap (T_1 \times T_2)$ is associated with
\[(6) \ 2n \equiv 3l \pmod{p}.

**Case 2:** $R_H(T_1) \cap (T_2 \times T_3)$.

In (5), if $i = 2, j = 3$ and $k = 1$, then we have 
\[-\left(\frac{p + 1}{2} - (p - 1)\right)2n \equiv ((p - 1) - (p - 1)2)l \pmod{p}.

So, $-\frac{p + 3}{2}n \equiv l \pmod{p}$. The adjacency matrix of $R_H(T_1) \cap (T_2 \times T_3)$ is associated with
\[(7) \ 5n \equiv 2l \pmod{p}.

**Case 3:** $R_H(T_2) \cap (T_1 \times T_3)$.

In (5), if $i = 1, j = 3$ and $k = 2$, then we have 
\[-\left(\frac{p + 1}{2} - (p - 1)\right)n \equiv ((p - 1) - \frac{p + 1}{2} - (p - 1))l \pmod{p}.

So, $(p - 1)n \equiv l \pmod{p}$. The adjacency matrix of $R_H(T_2) \cap (T_1 \times T_3)$ is associated with
\[(8) \ (p - 1)n \equiv l \pmod{p}.

In (5), (7) and (8), if $l = 0$, then $n = 0$. So, we have 
\[(a, a^2) \in R_H(T_1), (a^2, a^3) \in R_H(T_1) \text{ and } (a, a^3) \in R_H(T_2)\]

for $(a, a^2) \in T_1 \times T_2$, $(a^2, a^3) \in T_2 \times T_3$ and $(a, a^3) \in T_1 \times T_3$.

For each $l_i$ ($0 \leq i \leq p - 1$), there exists a unique $n_{l_i}$ with respect to (5).

So, it is possible to define a permutation $\sigma_1 \in \Gamma_1$ such that $T_1 \sim_{\sigma_1} T_2$ and $\sigma_1|_{T_1} = (l_0, l_1, \ldots, l_{p-1})$, $\sigma_1|_{T_2} = (n_{l_0}, n_{l_1}, \ldots, n_{l_{p-1}})$. For each $n_{l_i}$, applying the same argument for (7), it is possible to determine a permutation $\sigma \in \Gamma_1$ such that
By Lemma 3.2 and Theorem 3.5, it suffices to check either $T_1 \sim_\sigma T_3$ or $T_1 \not\sim_\sigma T_3$. For the given $\sigma \in \Gamma_1$ with $T_1 \sim\sigma T_2 \sim\sigma T_3$, it induces the orbit of $(a, a^3)$ in $T_1 \times T_3$. In order to describe it, we connect $2m \equiv 3l \pmod{p}$ and $5n \equiv 2m \pmod{p}$ given in (6) and (7). So, we have

$$5n \equiv 3l \pmod{p}. \quad (9)$$

Note that $T_1 \sim_\sigma T_3$ if and only if (8) and (9) give the same set of orbits in $T_1 \times T_3$.

Suppose $T_1 \sim_\sigma T_3$. Then $(n, l) = (p - 1, 1)$ satisfies (8). Substituting $(p - 1, 1)$ for $(n, l)$ in (9), we obtain $5p \equiv 8 \pmod{p}$. In order to hold $5p \equiv 8 \pmod{p}$, we shall show that $p$ must be even. Suppose to the contrary that $p = 2k + 1$. Then

$$5(2k + 1) \equiv 8 \pmod{2k + 1} \quad \Leftrightarrow \quad 10k \equiv 3 \pmod{2k + 1} \quad \Leftrightarrow \quad 0 \equiv 8 \pmod{2k + 1}.$$

This is a contradiction. But, $p = 2k$ contradicts the assumption that $p$ is an odd prime.

Therefore, it is impossible to exist $\sigma \in \Gamma_1$ such that $T_1 \sim_\sigma T_2 \sim_\sigma T_3 \sim_\sigma T_1$. We conclude that $\mathcal{A}$ is non-Schurian.

Remark 5.2. By a parallel argument, it is possible to give a non-Schurian $p$-S-ring over $H_2$.

Acknowledgement

The author would like to thank anonymous referees for their valuable comments.

References

[1] S. Evdokimov, I. Ponomarenko, On a family of Schur rings over a finite cyclic group, Algebra Anal. 13 (2001) 139–154.
[2] A. Hanaki, I. Miyamoto, Classification of association schemes of small order, Online catalogue. http://kissme.shinshu-u.ac.jp/as
[3] M. Hirasaka, K. Kim, Association schemes in which the thin residue is an elementary abelian $p$-group of rank 2, J. Algebra 450 (2016) 298–315.
[4] M. Hirasaka, K.-T. Kim, J.R. Park, Every 3-equivalenced association scheme is Frobenius. J. Algebraic Combin. 41 (2015), 217-228.
[5] K. Kim, Characterization of $p$-schemes of prime cube order, J. Algebra 331 (2011) 1–10.
[6] , Characterization of $p$-valenced schemes of order $p^2q$, J. Algebra 379 (2013) 230–240.
[7] , On $p$-Schur rings over abelian groups of order $p^3$, Comm. Algebra 42 (2014) 4456–4463.
[8] , Commutative $p$-Schur rings over non-abelian groups of order $p^3$, Bull. Korean Math. Soc. (2014) 1689–1696.
[9] M.Kh. Klin, The axiomatics of cellular rings, Investigations in the algebraic theory of combinatorial objects, Vissoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1985, pp.6–32 (in Russian).
[10] M. Muzychuk, I. Ponomarenko, Schur rings, Europ. J. Combin. 30 (2009) 1526–1539.
[11] , On pseudocyclic association schemes, ARS Math. Contemp. 5 (2012) 1–25.
[12] P. Spiga, G. Wang, An answer to Hirasaka and Muzychuk: Every $p$-Schur ring over $C_p^3$ is Schurian, Discrete Math. 308 (2008) 1760–1763.
[13] H. Wielandt, Finite Permutation Groups, Academic Press, Berlin, 1964.
[14] P.-H. Zieschang, An Algebraic Approach to Association Schemes, Lecture Notes in Mathematics 1628, Springer, Berlin, 1996.
[15] , Theory of Association Schemes, Springer Monographs in Mathematics, Springer, Berlin, 2005.

Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

E-mail address: knukkj@pusan.ac.kr