Difference equations with elliptic coefficients
and quantum affine algebras

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Introduction

The purpose of this paper is to introduce and study a \(q\)-analogue of the holonomic system of differential equations associated to the Belavin’s classical \(r\)-matrix (elliptic \(r\)-matrix equations), or, equivalently, to define an elliptic deformation of the quantum Knizhnik-Zamolodchikov equations invented by Frenkel and Reshetikhin [FR]. In [E], it was shown that solutions of the elliptic \(r\)-matrix equations admit a representation as traces of products of intertwining operators between certain modules over the Lie algebra \(\hat{\mathfrak{sl}}_N\). In this paper, we generalize this construction to the case of the quantum algebra \(U_q(\hat{\mathfrak{sl}}_N)\).

The main object of study in the paper is a family of meromorphic matrix functions of \(n\) complex variables \(z_1,\ldots,z_n\) and three additional parameters \(p, q, s\) – (modified) traces of products of intertwiners between \(U_q(\hat{\mathfrak{sl}}_N)\)-modules. They are a new class of transcendental functions which can be degenerated into many interesting special functions – hypergeometric and \(q\)-hypergeometric functions, elliptic and modular functions, transcendental functions of an elliptic curve, vector-valued modular forms, solutions of the Bethe ansatz equations etc.

The main result of the paper (Theorem 4.3) states that these functions satisfy two holonomic systems of difference equations, (4.18) and (4.19) – the first one has shift parameter \(p\) and elliptic modulus \(s\), and the second one has shift parameter \(s\) and elliptic modulus \(p\).

In Section 1, we recall some properties quantum affine algebras to be used in subsequent sections. Our exposition follows [FR]. At the end of the section, we introduce the quantum analogue of the Sugawara construction for the element \(L_0\) of the Virasoro algebra. A similar construction is described in [ITJMN].

In Section 2, we define intertwiners \(\Phi(z) : M_{\lambda,k} \to M_{\nu,k} \otimes V_z\), where \(M_{\lambda,k}\) is a Verma module and \(V_z\) is an evaluation representation of the quantum affine algebra \((z \in \mathbb{C}^*)\). Following [FR], we classify such intertwiners and prove a difference equation for them (due to Frenkel and Reshetikhin). Our proof uses the quantum Sugawara construction and is less technical than the original proof in [FR], although it relies on essentially the same ideas. At the end of the Section, we reproduce the proof of the quantum Knizhnik-Zamolodchikov equations for correlation functions given in [FR].

In Section 3, we introduce the outer automorphism \(\beta\) of \(U_q(\hat{\mathfrak{sl}}_N)\) corresponding to the rotation of the Dynkin diagram (which is an \(N\)-gon), and define the corresponding operator \(B\) acting on Verma modules. We define a class of meromorphic functions of \(n\) complex variables – traces of products of intertwiners twisted by the operator \(B\). We show that these traces satisfy a holonomic system of difference equations, (4.18) and (4.19), and that these equations can be degenerated into various special functions, as described above.
equations which is a $q$-deformation of the elliptic $r$-matrix system. This system is equivalent to a system of difference equations with elliptic coefficients, and the shift parameter $p$ in this system is independent of the (multiplicative) period $s$ of the elliptic function occurring in its coefficients. Systems of difference equations of this type were considered in [JMN], where they arose as systems satisfied by correlation functions of the 8-vertex model in statistical mechanics. It would be very interesting and useful to clarify the connection between the difference equations of [JMN] and the difference equations of this paper, but at the moment it is not clear how to do it.

In Section 4, we define the fundamental trace – a matrix-valued function whose columns are traces of linearly independent intertwiners. We derive connection relations for the fundamental trace which are the $q$-analogue of the monodromy for the elliptic $r$-matrix equations computed in [E], and the elliptic analogue of the connection matrices for the quantum KZ equations (see [FR]). These connection relations turn out to be equivalent to another holonomic system of difference equations with elliptic coefficients, in which the shift parameter and the elliptic modulus interchange. Thus, we obtain a pair of holonomic systems of difference equations with elliptic coefficients such that each of them is the monodromy (= the set connection relations) for the other system. We call such a pair a double difference system.

In Section 5 we describe some general properties of double difference systems with $N$-dimensional matrix coefficients. We start with conventional holonomic systems and show that a nondegenerate system with rational coefficients is always solvable in meromorphic functions, whereas for systems with elliptic coefficients in more than one variable this is not true. However, if a system with elliptic coefficients is solvable, and one fixes a matrix solution of it, then one can construct a dual system satisfied by this solution, also with elliptic coefficients, which is a system of connection relations for the previous system, and in which the shift parameter and the elliptic modulus interchange. In this way we obtain consistent double difference systems with elliptic coefficients. We give a complete classification of such systems in one variable and one dimension, and compute their solutions. This result can be extended to many variables, but in more than one dimensions the classification is hardly possible. Even the problem of finding a dual system to a given difference system with elliptic coefficients (so that they make a consistent double difference system together) is a difficult transcendental problem (which is no surprise since it is a generalization of the notoriously difficult problems of finding connection matrices for difference systems with rational coefficients, and of computing monodromy of differential equations). In fact, the example given in this paper appears to be the first nontrivial explicit example of a consistent double difference system with elliptic coefficients.

In the appendix, we briefly discuss some limiting cases of the difference equations deduced in the preceding sections and their connections to qKZ equations, elliptic KZ equations, Smirnov’s equations, and Bethe ansatz equations.

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1. Quantum affine algebras.

Let \( g \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) of rank \( r \). Denote by \( <,> \) the standard invariant form on \( g \) with respect to which the longest root has squared length 2.

Let \( h \) denote a Cartan subalgebra of \( g \). The form \( <,> \) defines a natural identification \( h^* \rightarrow h \): \( \lambda \mapsto h_\lambda \) for \( \lambda \in h^* \). We will use the notation \( <,> \) for the inner product in both \( h \) and \( h^* \). For the sake of brevity we will often write \( h_\lambda \) instead of \( h_\lambda \).

Let \( \alpha_i, 1 \leq i \leq r \) be the simple positive roots of \( g \). Let \( \theta \) be the highest root of \( g \) – the positive root such that \( \theta + \alpha_i \) is not a root for any \( i \). Extend the Cartan subalgebra \( h \) by adding a new element \( c \) orthogonal to \( h \). Denote the Cartan subalgebra extended by \( c \) by \( h: \hat{h} = h \oplus \mathbb{C}c \).

Introduce the convenient notation \( \alpha_0 = -\theta, H_i = h_{\alpha_i}, 1 \leq i \leq r, H_0 = -h_\theta + c. \) Let \( a_{ij} = \frac{2<\alpha_i,\alpha_j>}{<\alpha_i,\alpha_i>}, 0 \leq i, j \leq r. \)

Let \( \rho \) be the half sum of positive roots of \( g \), and let \( N = 1+<\rho,\theta> \) be the dual Coxeter number of \( g \).

Let \( t \) be a complex number and \( q = e^t. \) We assume that \( |q| < 1. \) If \( A \) is a number or an operator, by \( q^A \) we will always mean \( e^{tA}. \)

Let \( U_q(\hat{g}) \) be the quantum affine algebra corresponding to \( g \) [Dr1,J]. This algebra with unit is generated by elements \( e_i, f_i, 0 \leq i \leq r, q^{\pm h}, h \in \hat{h} \), satisfying the standard relations

\[
\begin{align*}
q^{h_1}q^{h_2} &= q^{h_1+h_2}, \ 1 \leq h_1, h_2 \leq \hat{h}; \ q^0 = 1; \\
q^h e_i q^{-h} &= q^{\alpha_i(h)} e_i, \ q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \ 0 \leq i \leq r, \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}}, \ 0 \leq i, j \leq r, \\
\sum_{n=1}^{1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n} q_i^n e_i e_j e_i^{1-a_{ij} - n} &= 0, \\
\sum_{n=1}^{1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n} q_i^n f_i f_j f_i^{1-a_{ij} - n} &= 0, \\
q_i &= q^{2/\langle \alpha_i,\alpha_i \rangle}, \ 0 \leq i, j \leq r, i \neq j,
\end{align*}
\]

where \( \binom{m}{n}_q \) is the \( q \)-binomial coefficient: by definition,

\[
\binom{m}{n}_q = \frac{\prod_{k=m-n+1}^{m} (q^k - q^{-k})}{\prod_{k=1}^{m} (q^k - q^{-k})}.
\]

The comultiplication in \( U_q(\hat{g}) \) is defined as follows:

\[
\Delta(q^h) = q^h \otimes q^h, \ h \in \hat{h},
\]
\[
\Delta(e_i) = e_i \otimes q^{H_i} + 1 \otimes e_i,
\]
\[
\Delta(f_i) = f_i \otimes 1 + q^{-H_i} \otimes f_i, \ 0 \leq i \leq r.
\]
The counit is given by
\begin{equation}
\epsilon(e_i) = \epsilon(f_i) = 0, \ 0 \leq i \leq r, \ \epsilon(q^h) = 1, \ h \in \hat{\mathfrak{h}}.
\end{equation}

The antipode is given by
\begin{equation}
S(e_i) = -e_i q^{-H_i}, \ S(f_i) = -q^{H_i} f_i, \ 0 \leq i \leq r, \ S(q^h) = q^{-h}, \ h \in \hat{\mathfrak{h}}.
\end{equation}

Equipped with all these structures, $U_q(\hat{\mathfrak{g}})$ becomes a Hopf algebra.

Let us add a new element $D$ to $U_q(\hat{\mathfrak{g}})$ satisfying the relations
\begin{equation}
[D, e_i] = e_i, \ [D, f_i] = f_i, \ 0 \leq i \leq r, \ [D, h] = 0, h \in \hat{\mathfrak{h}}
\end{equation}
\begin{equation}
\Delta(D) = D \otimes 1 + 1 \otimes D, \ \epsilon(D) = 0, \ S(D) = -D.
\end{equation}

The element $D$ defines a grading on $U_q(\hat{\mathfrak{g}})$: an element $a$ is of degree $n$ if $[D, a] = na$. An element is called homogeneous if it is an eigenvector of the operator of commutation with $D$.

We denote the Cartan subalgebra $\mathfrak{h}$ extended by $\mathfrak{h}$, and the Hopf algebra $U_q(\hat{\mathfrak{g}})$ extended by $D$ by $U_q(\hat{\mathfrak{g}})$.

Let us define two classes of modules over $U_q(\hat{\mathfrak{g}})$: Verma modules and evaluation representations (our definition is slightly different from the standard one; cf. [E]).

The definition of a Verma module of highest weight $\lambda$ and level $k$ is slightly twisted: $M_{\lambda, k}$, ($\lambda \in \mathfrak{h}^*, k \in \mathbb{C}$), is the module freely generated by a vector $v$ satisfying
\begin{equation}
e_i v = 0, \ q^h v = q^{<\lambda + kp, h>} v, \ h \in \mathfrak{h}, \ q^c v = q^{Nk} v, \ Dv = \frac{<\lambda, \lambda>}{2(k + 1)} v.
\end{equation}

Evaluation representations are defined as follows. Let $V$ be a representation of $U_q(\hat{\mathfrak{g}})$ which belongs to category $\mathcal{O}$ when restricted to the subalgebra $U_q(\mathfrak{g})$ generated by $e_i, f_i$, $i \geq 1$, $q^h, h \in \mathfrak{h}$. Consider the representation of $U_q(\hat{\mathfrak{g}})$ in the space of $V$-valued Laurent polynomials $V \otimes \mathbb{C}[z, z^{-1}]$, defined by
\begin{equation}D \circ w(z) = z \frac{dw(z)}{dz}; \ a \circ w(z) = z^n \pi_V(a) w(z), \ a \in U_q(\hat{\mathfrak{g}}), \ \deg(a) = n.
\end{equation}

This representation is reducible. Let $V(z)$ denote the submodule in it spanned by the vectors $v \otimes z^l$, where $v \in V$ runs over all vectors of such weights $\lambda$ that $<\lambda, \rho> - l$ is divisible by $N$. Then $V(z)$ is called the Laurent polynomial representation of the quantum affine algebra associated to $V$.

If $z_0$ is a nonzero complex number, then the space of all elements of $V(z)$ vanishing at $z_0$ is a $U_q(\hat{\mathfrak{g}})$-submodule of $V(z)$. The corresponding quotient is denoted by $V_{z_0}$. We will say that $V_{z_0}$ is the evaluation representation (at $z_0$) associated to $V$. In particular, if $z_0 = 1$ then $V_{z_0} = V$, so $V$ itself is also an evaluation representation.

The universal quantum $R$-matrix for $U_q(\hat{\mathfrak{g}})$ is defined as follows. Consider the subalgebras $U_q(\mathfrak{g}^+)$ and $U_q(\mathfrak{g}^-)$ in $U_q(\hat{\mathfrak{g}})$ generated by $\{e_i\}$ and $\{f_i\}$, respectively. It is known [Dr1] that there exists a unique pairing between these two algebras, $<\cdot, \cdot>_t$ such that $<e_i, f_j>_t = \frac{1}{t}^{t} \delta_{ij}$, and
\begin{equation}<a, b>_t = \sum_j <a, r_j>_t <b, p_j>_t, \ \Delta(c) = \sum_j p_j \otimes r_j, a, b \in U_q(\mathfrak{g}^+), \ c \in U_q(\mathfrak{g}^-),
\end{equation}
\begin{equation}<c, ab>_t = \sum_j <p_j, a>_t <r_j, b>_t, \ \Delta(c) = \sum_j p_j \otimes r_j, a, b \in U_q(\mathfrak{g}^-), \ c \in U_q(\mathfrak{g}^+).
\end{equation}
Let \( \{a_i\} \) be a basis of \( U_q(n^+) \) consisting of homogeneous vectors, and let \( \{a^i\} \) be the basis of \( U_q(n^-) \) dual to \( \{a_i\} \). Then the universal \( R \)-matrix is given by

\[
(1.10) \quad \tilde{R} = q^{c \otimes d + d \otimes c + \sum_{j=1} x_j \otimes x_j} \sum_i a_i \otimes a^i,
\]

where \( \{x_j\} \) is an orthonormal basis of \( h \), and \( d = \frac{1}{N} (D - \rho) \).

The \( R \)-matrix should be regarded as an infinite expression which makes sense as an operator in the tensor product of Verma modules. Such expressions form an algebra – a completion of the tensor square of the quantum affine algebra, and the \( R \)-matrix is its element.

The \( R \)-matrix satisfies the following quasitriangularity axioms:

\[
(1.11) \quad \tilde{R} \Delta(x) = \Delta^{\text{op}}(x) \tilde{R},
\]

\[
(\Delta \otimes \text{Id})(\tilde{R}) = \tilde{R}_{13} \tilde{R}_{23},
\]

\[
(\text{Id} \otimes \Delta)(\tilde{R}) = \tilde{R}_{13} \tilde{R}_{12},
\]

where \( \Delta^{\text{op}} \) denotes the opposite comultiplication and

\[
\tilde{R} = \sum_i a_i \otimes b_i, \quad \tilde{R}^{\text{op}} = \sum_i b_i \otimes a^i,
\]

\[
\tilde{R}_{12} = \sum_i a_i \otimes b_i \otimes 1 = \tilde{R} \otimes 1,
\]

\[
\tilde{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i,
\]

\[
\tilde{R}_{23} = \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes \tilde{R}.
\]

The matrix \( R \) also satisfies the quantum Yang-Baxter equation:

\[
(1.13) \quad \tilde{R}_{12} \tilde{R}_{13} \tilde{R}_{23} = \tilde{R}_{23} \tilde{R}_{13} \tilde{R}_{12}
\]

We introduce the modified \( R \)-matrices

\[
(1.14) \quad R = q^{-C \otimes D - D \otimes C} \tilde{R}, \quad R(z) = (z^D \otimes 1) R(z^{-D} \otimes 1), \quad C = \frac{c}{N}
\]

Note that \( R(z) \) is a power series in \( z \) which includes only nonnegative powers of \( z \).

The matrix \( R \) satisfies a modified version of the quasitriangularity axioms and quantum Yang-Baxter equations:

\[
R \Delta(x) = \Delta^{\text{op}}_{C}(x) R,
\]

\[
(\Delta \otimes \text{Id})(R(z)) = R_{13}^{C}(z) R_{23}(z),
\]

\[
(\text{Id} \otimes \Delta)(R(z)) = R_{13}^{C}(z) R_{12}(z),
\]

\[
(1.15) R_{12}(z_1/z_2) R_{13}^{C}(z_1/z_3) R_{23}(z_2/z_3) = R_{23}(z_2/z_3) R_{13}^{C}(z_1/z_3) R_{12}(z_1/z_2),
\]

where \( \Delta_{C}(x) = (\otimes \otimes C \otimes \otimes C) \Delta(x)(\otimes \otimes C \otimes \otimes C) \), \( R_{12}^{\pm}(x) = \pm \otimes \otimes C \otimes \otimes $1 R_{12}(x) \otimes \otimes C \otimes \otimes $1 $1 \).
Unlike $\hat{R}$, the element $R$ has a remarkable property: it can be projected to evaluation representations. Namely, for any $z \in \mathbb{C}^*$ and an evaluation representation $V_z$ one can define the elements

$$L_V^+(z) = (\text{Id} \otimes \pi_V)(\hat{R}^{op}) = (\text{Id} \otimes \pi_V)(\hat{R}^{op}(z)) \in U_q(\hat{g}) \hat{\otimes} \text{End}(V),$$

(1.16) $$L_V^-(z) = (\text{Id} \otimes \pi_V)(\hat{R}^{-1}) = (\text{Id} \otimes \pi_V)(\hat{R}^{-1}(z^{-1})) \in U_q(\hat{g}) \hat{\otimes} \text{End}(V),$$

where $\hat{\otimes}$ denotes a completed tensor product. These elements are called quantum currents.

If we pick a basis in $V$ labeled by a set $I$ then the quantum currents can be viewed as matrices, $L_{ij}^\pm$, $i, j \in I$, whose entries are Laurent polynomials of $z$ with values in $U_q(\hat{g})$.

Further, one can define the projection of $R$ into two evaluation representations $V^{1}_{z_1}, V^{2}_{z_2}$:

$$R^{V^{1}V^{2}}(z_1, z_2) = (\pi_{V^{1}_{z_1}} \otimes \pi_{V^{2}_{z_2}})(R).$$

(1.17) This projection turns out to be a power series in $z = z_1/z_2$ with only positive powers of $z$ present. This series converges in a neighborhood of the origin and therefore defines a holomorphic function with values in $\text{End}(V^1 \otimes V^2)$ in the neighborhood of the origin, which we will write as $R^{V^{1}V^{2}}(z)$ or, when no confusion is possible, simply as $R(z)$. One can show that this function extends to a meromorphic function in $\mathbb{C}$ which is a product of a scalar transcendental function $\phi$ and a rational matrix-valued function $\tilde{R}$: $R^{V^{1}V^{2}}(z) = \phi^{V^{1}V^{2}}(z)\tilde{R}^{V^{1}V^{2}}(z)$. The rational function $\tilde{R}$ regarded as a function of $y = \log z$ is a trigonometric solution of the quantum Yang-Baxter equation. It satisfies the unitarity condition $R(z)R_{21}(z^{-1}) = 1$.

Let us define a product operation applicable to quantum currents. Let $a = a_1 \otimes a_2 \in U_q(\hat{g}) \otimes \text{End}(V)$, $b = b_1 \otimes b_2 \in U_q(\hat{g}) \otimes \text{End}(W)$. Define a “product” of $a$ and $b$ by $a \ast b = a_1 b_1 \otimes a_2 b_2$. It is important to distinguish this operation from the usual product.

Let us now write down the commutation relations for currents.

**Proposition 1.1. [FR]** The following relations between power series with values in $\text{End}(M_{\lambda,k} \otimes V^1 \otimes V^2)$ hold true:

$$(1 \otimes R(\frac{z_1}{z_2}))(L^{V^{1}}_{V^1}(z_1) \ast L^{V^{2}}_{V^2}(z_2)) = S_{V^{1}V^{2}}(L^{V^{2}}_{V^2}(z_2) \ast L^{V^{1}}_{V^1}(z_1))(1 \otimes R(\frac{z_1}{z_2})), $$

$$ (1 \otimes R(\frac{z_1}{z_2}))(L^{V^{1}}_{V^1}(z_1) \ast L^{V^{1}}_{V^1}(z_2)) = S_{V^{1}V^{2}}(L^{V^{2}}_{V^2}(z_2) \ast L^{V^{1}}_{V^1}(z_1))(1 \otimes R(\frac{z_1}{z_2})), $$

(1.18)

$$(1 \otimes R(\frac{q^{-k}z_1}{z_2}))(L^{V^{1}}_{V^1}(z_1) \ast L^{V^{2}}_{V^2}(z_2)) = S_{V^{1}V^{2}}(L^{V^{2}}_{V^2}(z_2) \ast L^{V^{1}}_{V^1}(z_1))(1 \otimes R(\frac{q^{-k}z_1}{z_2})),$$

where $S_{V^{1}V^{2}}$ denotes the permutation of the $V^1$ and $V^2$ factors.

To prove this proposition, it is enough to apply the maps $\pi_{V^1} \otimes \pi_{V^2} \otimes \pi_{M_{\lambda,k}}$, $\pi_{M_{\lambda,k}} \otimes \pi_{V^1} \otimes \pi_{V^2}$, and $\pi_{V^1} \otimes \pi_{M_{\lambda,k}} \otimes \pi_{V^2}$ to the quantum Yang-Baxter relation for $\hat{R}$.

Let us now describe a quantum analogue of the Sugawara construction. Let $m : U_q(\hat{g}) \otimes U_q(\hat{g}) \to U_q(\hat{g})$ be the multiplication map. Consider the element

$$u = m((S \otimes \text{Id})(\hat{R}^{op})).$$

Drinfeld showed that this element satisfies the following relations.
Proposition 1.2. [Dr2]

\begin{equation}
\sum S^{-1}(b_i^*) a_i^* = \sum a_i^* \otimes b_i^*; \tag{1.20}
\end{equation}

\begin{equation}
ux^{-1} = S^2(x), \ x \in U_q(\mathfrak{g}); \tag{1.21}
\end{equation}

\begin{equation}
\Delta(u) = (u \otimes u)(\hat{\mathcal{R}}^{\text{op}}\hat{\mathcal{R}})^{-1} = (\hat{\mathcal{R}}^{\text{op}}\hat{\mathcal{R}})^{-1}(u \otimes u). \tag{1.22}
\end{equation}

Thus we have the following proposition.

Proposition 1.3. In the Verma module \(M_{\lambda,k}\)

\begin{equation}
u = q^{2D}. \tag{1.23}\end{equation}

Proof. First of all, we have the equality \(q^{2D} xq^{-2D} = S^2(x)\). This equality can be easily checked: it is enough to check it for the generators \(x = e_i, f_i, q^h\), since both sides of it are automorphisms of the quantum affine algebra. Hence, it follows from the previous proposition that \(uq^{-2D}\) commutes with the quantum affine algebra, so it is a constant. To prove that this constant is 1, it is enough to check that \(uv = q^{2D}v\), where \(v\) is the vacuum vector, which is straightforward.

Let

\begin{equation}\hat{\mathcal{R}} = (q^{-2CD} \otimes 1)\mathcal{R}(q^{2CD} \otimes 1). \tag{1.24}\end{equation}

Proposition 1.4. (Quantum Sugawara construction) The following relation is satisfied in any Verma module:

\begin{equation}q^{2(C+1)D} = m((S \otimes \text{Id})(\hat{\mathcal{R}}^{\text{op}})) \tag{1.25}\end{equation}

Proof. It follows from the definition of \(\hat{\mathcal{R}}\) that the right hand side of (1.54) is equal to \(q^{2CD}u\). But \(u = q^{2D}\), so we get (1.25).

2. Intertwining operators and difference equations

We will be interested in \(U_q(\mathfrak{g})\)-intertwining operators \(\Phi(z) : M_{\lambda,k} \rightarrow M_{\nu,k} \otimes z^{\Delta}V(z)\), where \(\otimes\) denotes the completed tensor product, and \(\Delta\) is a complex number. It turns out that such operators may be nonzero if and only if \(\Delta\) equals \(\frac{<\nu,\nu> - <\lambda,\lambda>}{2(k+1)}\) plus an integer. The shift by an integer is unimportant, so we will assume that \(\Delta = \frac{<\nu,\nu> - <\lambda,\lambda>}{2(k+1)}\).

Proposition 2.1. [FR] Operators \(\Phi\) are in one-to-one correspondence with vectors in \(V\) of weight \(\lambda - \nu\). This correspondence is defined by the action of \(\Phi\) at the vacuum level.

Let \(z_0\) be a nonzero complex number. Evaluation of the operator \(\Phi(z_0)\) at the point \(z_0\) yields an operator \(\Phi(z_0) : M_{\lambda,k} \rightarrow M_{\nu,k} \otimes V\).
Sometimes (when no confusion is possible) we will use the notation $\Phi(z)$ for the operator $\Phi$ evaluated at the point $z \in \mathbb{C}^*$. This will give us an opportunity to regard the operator $\Phi(z)$ as an analytic function of $z$. This analytic function will be multivalued: $\Phi(z) = z^\Delta \Phi^0(z)$, where $\Phi^0$ is a single-valued function on $\mathbb{C}^*$.

Let $W$ be an evaluation representation of $U_q(\hat{g})$. Then the intertwining property for $\Phi(z)$ can be written in the form

$$
(\Phi(z) \otimes \text{Id}) L^+_W(w) = R_W V(q^{-k} w/z) L^+_W(w)(\Phi(z) \otimes \text{Id})
$$

(2.1)

$$
(\Phi(z) \otimes \text{Id}) L^-_W(w) = R_W V(z/w)^{-1} L^-_W(w)(\Phi(z) \otimes \text{Id}).
$$

Relations (2.1) are equalities of maps $M_{\lambda,k} \otimes W \rightarrow M_{\nu,k} \otimes V_z \otimes W$.

To prove these formulas, it is enough to combine the second and third relations of (1.15) with the intertwining relation $\Phi(x) = \Delta(x) \Phi(z)$.

**Remark.** Note that relations (1.18) and (2.1) are a priori satisfied only formally, as equalities between power series. However, since we know that both sides of these equalities extend to meromorphic functions, we can conclude that they are also satisfied analytically for almost all values of the parameters.

Let us now deduce the difference equation for intertwining operators, following the method of Frenkel and Reshetikhin.

Let

$$
U = q^{2C} u.
$$

(2.2)

Introduce the notation $Q = q^{-2(k+1)D}$, $p = q^{-2(1+1)}$. We assume that $|p| < 1$. The quantum Sugawara construction implies that $Q = U^{-1}$ in $M_{\lambda,k}$. Also, the operator $Q$ acts in the Laurent series representation $V(z)$ as follows: $Qv(z) = v(pz)$.

Since $\Phi$ is an intertwiner, we have the following relation between Laurent series in $z$:

$$
\Phi(z) Q^{-1} = \Delta(Q^{-1}) \Phi(z) = (Q^{-1} \otimes Q^{-1}) \Phi(z).
$$

(2.3)

Using the quantum Sugawara formula (1.25) and the identity $(1 \otimes Q^{-1}) \Phi(z) = \Phi(p^{-1}z)$, we obtain

$$
(2.4) \quad \Phi(p^{-1}z) = (Q \otimes 1) \Phi(z) Q^{-1} = (U^{-1} \otimes 1) \Phi(z) U,
$$

where $U$ is defined by (2.2).

Introduce the notation $\Phi(\sum a_i \otimes b_i) = \sum (1 \otimes b_i) \Phi a_i$, $\Phi \in \text{Hom}_C(M_{\lambda,k}, M_{\nu,k} \otimes z^\Delta V(z))$.

**Lemma 2.2.**

$$
(U^{-1} \otimes 1) \Phi(z) U = (L^+_V(q^k z)^{-1} \Phi(z)) \bullet L^-_V(p^{-1} z).
$$

(2.5)

**Proof.** We have $U = u q^{2CD} = \sum_j S(b_j) a_j q^{2CD}$. Therefore, since $\Phi(z)$ is an intertwiner,

$$
\Phi(z) U = \sum \Delta(S(b_j)) \Phi(z) a_j q^{2CD} = \sum (S \otimes S)(\Delta^o p(b_j)) \Phi(z) a_j q^{2CD}.
$$

(2.6)
Following Drinfeld ([Dr2]), introduce the notation \((X \otimes Y \otimes Z) \circ \Phi = (S(Y) \otimes S(Z)) \Phi X\). This defines a right action of the tensor cube of the quantum affine algebra on the space \(\text{Hom}_{\mathbb{C}}(M_{\lambda,k}, M_{\nu,k} \hat{\otimes} Z V(z))\).

Using this notation, we can write (2.6) as follows:

\[
(2.7) \quad \Phi(z) U = (\text{Id} \otimes \Delta^\text{op})(\tilde{\mathcal{R}}) \circ \Phi(z) \cdot q^{2CD}.
\]

Applying (1.11), we get

\[
(2.8) \quad \Phi(z) U = (\mathcal{R}_{12} \mathcal{R}_{13}) \circ \Phi(z) \cdot q^{2CD} = \mathcal{R}_{13} \circ (\mathcal{R}_{12} \circ \Phi(z)) \cdot q^{2CD}.
\]

Let us separately consider the expression \(X = \mathcal{R}_{12} \circ \Phi(z)\) which occurs in (2.8). Using the intertwining property of \(\Phi(z)\) and (1.11), we obtain

\[
(2.9) \quad X = Y \Phi(z), \quad Y = m_{31}((\Delta \otimes S)(\tilde{\mathcal{R}})),
\]

where \(m_{31}(a \otimes b \otimes c) = ca \otimes b\).

Applying (1.11), we find

\[
(2.10) \quad Y = m_{31}((\text{Id} \otimes \text{Id} \otimes S)(\mathcal{R}_{13} \mathcal{R}_{23})) = (S \otimes \text{Id})(\tilde{\mathcal{R}}^{\text{op}}) \cdot (u \otimes 1).
\]

Thus we have

\[
(U^{-1} \otimes 1) \Phi(z) U = (U^{-1} \otimes 1) \left[ (\mathcal{R}_{13} \circ (S \otimes \text{Id})(\tilde{\mathcal{R}}^{\text{op}})(u \otimes 1) \Phi(z) \right] q^{2CD} = \]

\[
(\text{using (2.12)}) \quad \left( q^{-2CD} \otimes 1 \right) \left[ (\mathcal{R}_{13} \circ (S^{-1} \otimes \text{Id})(\tilde{\mathcal{R}}^{\text{op}}) \Phi(z) \right] q^{2CD} =
\]

\[
(\text{using (2.12)}) \quad \left( q^{-2CD} \otimes 1 \right) \left[ (\mathcal{R}_{13} \circ (\tilde{\mathcal{R}}^{\text{op}})^{-1} \Phi(z) \right] q^{2CD} =
\]

\[
(\text{using (2.12)}) \quad \left( q^{-2CD} \otimes 1 \right) \left[ (\tilde{\mathcal{R}}^{\text{op}})^{-1} \Phi(z) \cdot (\text{Id} \otimes S)(\tilde{\mathcal{R}}_{13}) \right] q^{2CD} =
\]

\[
(\text{using (2.12)}) \quad \left( q^{-2CD} \otimes 1 \right) \left[ (\tilde{\mathcal{R}}^{\text{op}})^{-1} \Phi(z) \cdot (S^{-2} \otimes \text{Id})(\tilde{\mathcal{R}}^{-1}_{13}) \right] q^{2CD} =
\]

\[
(\text{using (2.12)}) \quad (\tilde{\mathcal{R}}^{\text{op}})^{-1} \left( 1 \otimes q^{-kD} \right) \left( q^{2kD} \otimes q^{2kD} \right) \Phi(z) \cdot (\mathcal{R}_{13}^{-1} (q^{-2-2k})(1 \otimes q^{-kD})) =
\]

\[
\mathcal{R}_{32}^{-1} (q^k) \Phi(z) \cdot \mathcal{R}_{13}^{-1} (p) =
\]

\[
L^+_V(q^k z)^{-1} \Phi(z) \cdot L^-_V(p^{-1} z).
\]

The lemma together with equation (2.4) implies the following difference equation for \(\Phi(z)\):

**Theorem 2.3.** [FR] The intertwining operator \(\Phi(z)\) satisfies the difference equation

\[
(2.12) \quad \Phi(pz) = L^+_V(pq^k z)(\Phi(z) \cdot L^-_V(z)^{-1}).
\]

Let us now deduce the difference equations for quantum correlation functions. Let \(V^1, \ldots, V^n\) be evaluation representations of \(U_q(\hat{\mathfrak{g}})\), and let \(\Phi^j(z_j) : M_{\lambda_j,k} \rightarrow M_{\lambda_{j-1},k} \hat{\otimes} V_{j}^j\) be \(U_q(\hat{\mathfrak{g}})\)-intertwining operators. Then the product \(\Phi^1(z_1) \ldots \Phi^n(z_n)\) makes sense as an operator \(M_{\lambda_n,0} \rightarrow M_{\lambda_n,0} \hat{\otimes} V^1 \otimes \ldots \otimes V^n\), if \(|z_1| > > |z_n| > > \).


\[
\cdots \implies |z_n|.
\]
Consider the matrix element of this product corresponding to the vacuum vectors in the Verma modules:
\[
(2.13) \quad \Psi(z_1, \ldots, z_N) = \langle v^*_{\lambda_0}, \Phi^1(z_1) \cdots \Phi^n(z_n) v_{\lambda_n} \rangle,
\]
where \( v_{\lambda_n} \) is the highest weight vector of \( M_{\lambda_n} \) and \( v^*_{\lambda_0} \) is the lowest weight vector of \( M_{\lambda_0} \). This function takes values in the tensor product \( V^1 \otimes \cdots \otimes V^n \). It is called the quantum correlation function.

We have
\[
(2.14) \quad \Psi(z_1, \ldots, p z_j, \ldots, z_n) = \langle v^*_{\lambda_0}, \Phi^1(z_1) \cdots \Phi^j(p z_j) \cdots \Phi^n(z_n) v_{\lambda_n} \rangle =
\]
\[
< v^*_{\lambda_0}, \Phi^1(z_1) \cdots \Phi^{j-1}(z_{j-1}) L^+_V(pq^k z_j) \Phi^j(z_j) \cdot L^-_V(z_j)^{-1} \Phi^j+1(z_{j+1}) \cdots \Phi^n(z_n) v_{\lambda_n} > .
\]

Let us now drag \( L^+ \) to the left and \( L^- \) to the right, using commutation relations (2.1). Taking into account the relations
\[
(2.15) \quad (S \otimes \text{Id})(L^+_V(z))(v^*_{\lambda_0} \otimes v) = v^*_{\lambda_0} \otimes q^{\lambda_0} v,
\]
\[
L^-_V(z)^{-1}(v_{\lambda_n} \otimes v) = v_{\lambda_n} \otimes q^{\lambda_n} v,
\]
which follow from the definition of the quantum currents, we will get the following result.

**Theorem 2.4.** [FR] The quantum correlation functions satisfy the following system of linear difference equations.
\[
(2.16) \quad \Psi(z_1, \ldots, p z_j, \ldots, z_n) = R_{j-1}^{V_j V_{j-1}} \left( \frac{p z_j}{z_{j-1}} \right) \cdots R_j^{V_j V_1} \left( \frac{p z_j}{z_1} \right) (q^{\lambda_0+\lambda_n}) |V_j \times
\]
\[
R_n^{V_n V_j} \left( \frac{z_n}{z_j} \right)^{-1} \cdots R_{j+1}^{V_{j+1} V_j} \left( \frac{z_{j+1}}{z_j} \right)^{-1} \Psi(z_1, \ldots, z_j, \ldots, z_n).
\]

**3. Difference equations for traces of intertwiners.**

From now on the letter \( \mathfrak{g} \) will denote the Lie algebra \( \mathfrak{sl}_N(\mathbb{C}) \) of traceless \( N \times N \) matrices with complex entries. The dual Coxeter number of this algebra is \( N \), and the rank is \( N-1 \). The Cartan subalgebra \( \mathfrak{h} \) is the subalgebra of diagonal matrices.

Let \( B \) be the \( N \times N \) matrix of zeros and ones corresponding to the cyclic permutation (12...\( N \)):
\[
B = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

Define an inner automorphism \( \beta \) of \( \mathfrak{h} \): \( \beta(a) = B a B^{-1}, a \in \mathfrak{h} \). This automorphism has order \( N \). We can extend it to \( \tilde{\mathfrak{h}} \) by making it act trivially on \( c \) and \( D \).

The automorphism \( \beta \) can be extended to an outer automorphism of \( U_q(\tilde{\mathfrak{g}}) \) defined by the relations
\[
(3.1) \quad \beta(c) = c, \quad \beta(f_j) = f_j, \quad \beta(h) = \alpha^\beta(h), \quad h \in \tilde{\mathfrak{h}}.
\]
where the subscripts are regarded modulo \( N \). This outer automorphism also has order \( N \) and can be interpreted as the rotation of the Dynkin diagram of \( \hat{\mathfrak{g}} \) (which is a regular \( N \)-gon) through the angle \( 2\pi/N \).

The action of \( \beta \) in \( U_q(\hat{\mathfrak{g}}) \) preserves degree, hence, it preserves the polarization. Therefore, it transforms Verma modules into Verma modules. In other words, we can regard \( \beta \) as an operator \( B : M_{\lambda,k} \to M_{\beta(\lambda),k} \), where by convention \( \beta(\lambda)(h) = \lambda(\beta^{-1}(h)) \). This operator intertwines the usual action of \( U_q(\hat{\mathfrak{g}}) \) and the action twisted by \( \beta: \beta(a)Bw = Bab, a \in U_q(\hat{\mathfrak{g}}), w \in M_{\lambda,k} \).

Let \( \nu = \beta^{-1}(\lambda) \), and let \( \Phi^j(z_j) \) be as above (cf. section 2).

We assume that the representations \( V^j \) are finite-dimensional and irreducible when restricted to \( U_q(\hat{\mathfrak{g}}) \). It is easy to show that for any such representation \( V \) there exists a unique, up to a constant, operator \( B : V \to V \) such that \( Bab = \beta(a)Bv \) for \( a \in U_q(\hat{\mathfrak{g}}), v \in V \) – it follows from the fact that \( V \) twisted by \( \beta \) is isomorphic to \( V \). The operator \( B \) gives rise to a well defined automorphism \( \beta \) of \( \text{End}(V) \):

\[
\beta(E) = BEB^{-1}.
\]

Let \( s \) be a complex number, \( 0 < |s| < 1 \). Following the idea of Bernard [Ber], Frenkel, Reshetikhin ([FR], Remark 2.3) (see also [E],[EK]), introduce the following formal power series in \( z_1/z_2, z_2/z_3, \ldots, sz_n/z_1 \):

\[
(3.2) \quad F(z_1, \ldots, z_n|s) = \text{Tr} \left|_{M_{\lambda,k}} \left( \Phi^1(z_1) \ldots \Phi^n(z_n)B \right) \right|_{B^{-D}}.
\]

It is not difficult to prove that this series defines an analytic function when \( |z_1| >> |z_2| >> \cdots >> |z_n| >> |sz_1| \). This function takes values in \( V^1 \otimes \cdots \otimes V^n \). From now on it will be the main object of our study.

It turns out that the \( n \)-point trace \( F(z_1, \ldots, z_n|s) \) defined by (3.2) satisfies a remarkable system of difference equations involving elliptic solutions of the quantum Yang-Baxter equation for \( \mathfrak{sl}_N \). Let us deduce these equations.

According to (2.14), we have

\[
F(z_1, \ldots, pz_j, \ldots, z_n|s) = \text{Tr} \left|_{M_{\lambda,k}} \left( \Phi^1(z_1) \ldots \Phi^j(pz_j) \ldots \Phi^n(z_n)B \right) \right|_{B^{-D}} = \\
(3.3) \quad \text{Tr} \left|_{M_{\lambda,k}} \left( \Phi^1(z_1) \ldots \Phi^{j-1}(z_{j-1})L^+_{V,j}(pq^kz_j) \Phi^j(z_j) \Phi^j+1(z_{j+1}) \ldots \Phi^n(z_n)B \right) \right|_{B^{-D}}.
\]

We would like to deduce a difference equation for \( F \).

First of all, we need to describe some properties of the automorphism \( \beta \). Since \( \beta \) is a degree preserving automorphism, it preserves the universal \( R \)-matrix:

\[
(3.4) \quad (\beta \otimes \beta)(\hat{\mathcal{R}}) = \hat{\mathcal{R}}, \quad (\beta \otimes \beta)(\hat{\mathcal{R}}) = \hat{\mathcal{R}}.
\]

Let \( \hat{\mathcal{R}}^M = (\beta^M \otimes 1)(\mathcal{R}) \). It is obvious that \( (\beta^I \otimes \beta^J)(\mathcal{R}) = \mathcal{R}^{I-J} \).

Introduce the following notation:

\[
L^+_{V}(z)^{M} = (\text{Id} \otimes \pi_{V_z})(\mathcal{R}^{M})^{\text{op}},
\]

\[
L^-_{V}(z)^{M} = (\text{Id} \otimes \pi_{V_z})(\mathcal{R}^{M})^{-1},
\]

\[
R^{1}V^{2}z^{I}M = (\pi_{V^{1}}(z) \otimes \pi_{V^{2}})(\mathcal{R}^{M}).
\]

Let \( W \) be a finite dimensional representation of \( U_q(\hat{\mathfrak{g}}) \). Let \( \Phi : M_{\beta(\lambda),k} \to M_{\lambda,k} \otimes W \) be an intertwining operator. Introduce a new function \( \tilde{F}^{IJ}(x, y|s) \) with values in \( W \otimes \text{End}V \otimes \text{End}V \) defined by

\[
\tilde{F}^{IJ}(x, y|s) = \text{Tr} \left|_{\hat{M}_{\lambda,k} \otimes W} \left( I_{+}(pq^kz_{i})L_{+}(I_{-}(y)) \right) \Phi \cdot (I_{-}(y))^{-1}B \right|_{B^{-D}}.
\]
Our plan is to show that \( \tilde{F} \) satisfies a system of difference equations, by dragging \( L^+ \) around the circle from right to left and \( L^- \) from left to right using the commutation relations for quantum currents and the property of trace: \( \text{Tr}(ab) = \text{Tr}(ba) \). Before we do so, we need a few identities.

First, we have the following generalization of the third equation in (1.18):

\[
(3.7) \quad L_{V_1}^+(z_1)^I_3 R_{32}(q^k z_1/z_2)_{J+1} (L_{V_2}^-(z_2)^I_2)^{-1} = (L_{V_2}^-(z_2)^I_2)^{-1} R_{32}(q^{-k} z_1/z_2)_{J+1} L_{V_1}^+(z_1)^I_3,
\]

which implies that

\[
(3.8) \quad (L_{V_2}^-(z_2)^I_2)^{-1} * L_{V_1}^+(z_1)^I_3 = R_{32}^r(q^k z_1/z_2)_{J+1} R_{32}^{l r}(q^{-k} z_1/z_2)_{J+1} L_{V_1}^+(z_1)^I_3 * (L_{V_2}^-(z_2)^I_2)^{-1},
\]

where \( R_{32}^r \) implies that the first component of \( R \) is applied from the right and the second one from the left, and \( R_{32}^{l r} \) implies that the first component of \( R \) is applied from the left, and the second one from the right. These relations are equalities between elements of the product \( U_q(\mathfrak{g}) \otimes \text{End} V \otimes \text{End} V \). The notation \( L(z)^I_2 \) and \( L(z)^I_3 \) implies that the second component of \( L(z) \) operates in the second and third factor of this tensor product, respectively. To prove these relations, it is enough to apply the automorphism \( 1 \otimes \beta^{-I} \otimes \beta^{-J} \) to the quantum Yang-Baxter relation for the \( R \)-matrix, and then project the obtained relation to the corresponding product of representations of the quantum affine algebra.

Next, we have the identities

\[
(3.9) \quad L_{V}^-(z)^{-1} s^{-D} = s^{-D} L_{V}^-(s^{-1} z)^{-1}, \quad s^{-D} L_{V}^+(z) = L_{V}^+(sz)s^{-D},
\]

and

\[
(3.10) \quad L_{V}^-(z)^{-1} B = B \beta^{-1}(L_{V}^-(z))^{-1}, \quad B L_{V}^+(z) = \beta(L_{V}^+(z)) B.
\]

Finally, we have the following \( \beta \)-twisted versions of relations (2.1):

\[
(3.11) \quad \Phi_1 L_{V}(x)^I_3 = R_{31}(q^{-k} x)_I L_{V}(x)^I_3 \Phi_1,
\]

\[
(L_{V}(y)^I_2)^{-1} R_{12}(y^{-1})_I \Phi_1 = \Phi_1 (L_{V}(y)^I_2)^{-1},
\]

These relations are equalities of elements of \( \text{Hom}(M_{\lambda,k}, M_{\nu,k} \otimes W) \otimes \text{End} V \otimes \text{End} V \), and the meaning of the subscripts 1,2, and 3 is as above.

Now we are in a position to compute \( \tilde{F} \). Combining relations (3.6)-(3.8) with (2.1), we get

\[
\tilde{F}^{I+1,J}(sx, y | s) = R_{32}^r(q^{-2} sx/y)_{J+1} R_{32}^{l r}(psx/y)_{J+1} R_{31}(psx)_{J+1} \tilde{F}^{I,J}(x, y | s),
\]

(3.12)

\[
\tilde{F}^{I,J-1}(x, s^{-1} y | s) = R_{32}^r(q^{-2} sx/y)_{J+1} R_{32}^{l r}(psx/y)_{J+1} R_{12}(s)_{J-1} \tilde{F}^{I,J}(x, y | s).
\]

Let \( G^{I,J} = B_2 I B_3^{-1} \tilde{F}^{I,J} B_2^{-J} B_3^I \). Then we have

\[
G^{I+1,J}(sx, y | s) = R_{32}^r(q^{-2} sx/y) R_{32}^{l r}(psx/y)_{J+1} R_{31}(psx)_{J+1} G^{I,J}(x, y | s) B_3^{-1} B_2,
\]

(3.13)

\[
G^{I,J-1}(x, s^{-1} y | s) = R_{32}^r(q^{-2} sx/y) R_{32}^{l r}(psx/y)_{J+1} R_{12}(s)_{J-1} G^{I,J}(x, y | s) B_2.
\]
Let
\[ T_0(x, y) = B_3^l R_{31} (p s x)^{-1} R_{32}^{lr} \left( \frac{p s x}{y} \right) R_{32}^l \left( \frac{q^{-2} s x}{y} \right)^{-1} (B_3^r)^{-1}, \]
and let
\[ T_1(x, y) = B_2^l R_{12}^r \left( \frac{s}{y} \right) R_{32}^{lr} \left( \frac{p s x}{y} \right) R_{32}^l \left( \frac{q^{-2} s x}{y} \right)^{-1} (B_2^r)^{-1}, \]

and let
\[ P_0(x, y) = T_0(x, y) T_0(s x, y) \ldots T_0(s^{-N} x, y), \]
\[ P_1(x, y) = T_1(x, y) T_1(x, s^{-1} y) \ldots T_1(x, s^{-N+1} y), \]

where \( T_0, T_1, P_0, P_1 \in \text{End}(W \otimes \text{End}V \otimes \text{End}V). \) Let \( \mathcal{G}^{00} \) be denoted by \( \mathcal{G} \). Then we have the following equation:
\[ \mathcal{G}(x, y|s) = \prod_{M=0}^{\infty} P_0(x, s^{MN} y) \prod_{M=0}^{\infty} P_1(0, s^{-MN} y) \mathcal{G}(0, \infty|s) \]

Let us now consider the expression \( \mathcal{G}(0, \infty|s) \). We have
\[ \mathcal{G}(0, \infty|s) = \]
\[ \text{Tr} |_{M, k} ( (q^{\sum x_i \otimes x_i - C \otimes \rho})^3 \Phi_1 \bullet (q^{\sum x_i \otimes x_i + C \otimes \rho})_2 B s^{-D}) = \]
\[ (q^{k(\rho \otimes 1 - 1 \otimes \rho)})_{23} \text{Tr} |_{M, k} ( (q^{\sum x_i \otimes x_i})^3 (q^{\sum (x_i \otimes 1 + 1 \otimes x_i) \otimes x_i})_{12} \Phi_1 B s^{-D}) \]

Apart from that, we have the identity
\[ \text{Tr} |_{M, k} ( h \Phi_1 B s^{-D} ) + h \text{Tr} |_{M, k} ( \Phi_1 B s^{-D}) = \]
\[ \text{Tr} |_{M, k} ( \Phi_1 h B s^{-D} ) = \text{Tr} |_{M, k} ( \beta^{-1}(h) \Phi_1 B s^{-D}) , \ h \in \mathfrak{h}, \]

which implies that
\[ \text{Tr} |_{M, k} ( h \Phi_1 B s^{-D} ) = (\beta^{-1} - 1)^{-1}(h) \text{Tr} |_{M, k} ( \Phi_1 B s^{-D}). \]

Hence, if \( \phi \) is any analytic function then
\[ \text{Tr} |_{M, k} ( \phi(h) \Phi_1 B s^{-D} ) = \phi((\beta^{-1} - 1)^{-1}(h)) \text{Tr} |_{M, k} ( \Phi_1 B s^{-D}). \]

For brevity introduce the notation \( \chi = (\beta^{-1} - 1)^{-1}. \) Then expression (3.17) can be rewritten in the form
\[ \mathcal{G}(0, \infty|s) = \]
\[ q^{1 \otimes (k \rho \otimes 1 - 1 \otimes k \rho) + \sum_i (\chi(x_i) \otimes 1 \otimes x_i - (1 + \chi)(x_i) \otimes x_i \otimes 1)} \left[ \text{Tr} |_{M, k} ( \Phi_1 B s^{-D} ) \otimes \text{Id} \otimes \text{Id} \right]. \]
Let

\[ P(x, y) = \]
\[ \prod_{M=0}^{\infty} P_0(x, s^{MN} y) \prod_{M=0}^{\infty} P_1(0, s^{-MN} y) q^{1 \otimes (k \rho \otimes 1 - 1 \otimes k \rho)} + \sum_i \chi(x_i) \otimes 1 \otimes x_i - (1 + \chi)(x_i) \otimes x_i \otimes 1) \]  

In our situation \( W = V_1 \otimes V_2 \otimes \cdots \otimes V_n, V = V_i \), and \( \text{End} V_i \) naturally acts on \( W \). Therefore, it makes sense to consider the \( \text{End} W \)-valued function

\[ X_i(z_1, \ldots, z_n | s) = m_{321}(P^{(i)}(z, z)), \]
where \( P^{(i)}(x, y) \in \text{End} W \otimes \text{End} V_i \otimes \text{End} V_i \) is defined by (3.22) with \( V = V_i \), and by definition \( m_{321}(a \otimes b \otimes c) = cba \).

Now it remains to observe that equations (2.12), (3.16), (3.21), and (3.22) imply the following difference equation for the function \( F \):

**Theorem 3.1.** The function \( F(z_1, \ldots, z_n | s) \) satisfies the difference equations

\[ F(z_1, \ldots, p z_i, \ldots, z_n | s) = X_i(z_1, \ldots, z_n | s) F(z_1, \ldots, z_i, \ldots, z_n | s) \]

Note that the coefficients of these difference equations are meromorphic functions which we have explicitly represented as (contracted) infinite products of trigonometric \( R \)-matrices. In the next section we will show that they can be expressed in terms of elliptic functions.

### 4. Monodromy equations

The quasi-classical limit (i.e. the limit \( q \to 1 \)) of equations (3.24) is the system of elliptic KZ equations described in [E]. For this system one can define the notion of monodromy which turns out to be expressed by products of \( R \)-matrices. It is a natural question what is the quantum analogue of monodromy. In this section we will give an answer to this question. The answer is that the role of monodromy is played by another system of difference equations with elliptic coefficients which are products of \( R \)-matrices depending on spectral parameters.

Let us first describe how to interchange the order of intertwining operators.

Let \( \Phi^{w, \lambda, \nu}(z) : M_{\lambda, k} \to M_{\nu, k} \) be the intertwining operator such that \( \langle v^*_w, \Phi^{w, \lambda, \nu}(z) v_{\lambda} \rangle = w, w \in V^{\lambda - \nu} \). Suppose that \( z_1, z_2 \) are nonzero complex numbers, and we have a product \( \Phi^{w_1, \lambda_1, \lambda_0}(z_1) \Phi^{w_2, \lambda_2, \lambda_1}(z_2) : M_{\lambda_2, k} \to M_{\lambda_0, k} \) on \( V^1 \otimes V^2 \) where \( V^1 \) and \( V^2 \) are finite dimensional representations of \( U_q(\hat{g}) \). The question is: can this product be expressed in terms of products of the form \( \Phi(z_2) \Phi(z_1) \)? Of course, we can only talk about such an expression after analytic continuation, since the former is defined for \( |z_1| >> |z_2| \), and the latter for \( |z_1| << |z_2| \). However, if we apply analytic continuation, the answer to the question is positive, and given by the following theorem.

**Theorem 4.1.** (see [FR]) Let \( x_{iv} \) be a basis of \((V^1)^{\nu - \lambda_0}\), and let \( y_{iv} \) be a basis of \((V^2)^{\lambda_2 - \nu}\). Then

\[ R^{V^1 V^2} \left( \frac{z_{21}}{z_{12}} \right) - 1 \Phi^{x_{iv}, \lambda_1, \lambda_0}(z_1) \Phi^{y_{iv}, \lambda_1, \lambda_2, \lambda_1}(z_2) = \]
\[ A \sum E^{\lambda_2, \lambda_0}_{ijrs} \frac{z_{12}}{z_{21}} V^1 V^2 P(x_{iv}, \nu, \lambda_0(z_2) \Phi^{x_{ju}, \lambda_2, \nu}(z_1), \]

\[ (4.1) \]
where $A$ is the analytic continuation, $E^{λ,μ}$ is a matrix called the matrix of exchange coefficients, and $P$ is the permutation: $V^1 \otimes V^2 \rightarrow V^2 \otimes V^1$.

Clearly, the matrix $E^{λ,μ}(z)$ (we drop the superscripts $V^1, V^2$ when no confusion is possible) represents a linear operator $(V^1 \otimes V^2)^{λ-μ} \rightarrow (V^2 \otimes V^1)^{λ-μ}$. Therefore, if we define

$$E^{λ}(z) = \oplus_μ E^{λ,μ}(z),$$

then $E^{λ}(z)$ will correspond to an operator: $V^1 \otimes V^2 \rightarrow V^2 \otimes V^1$.

Let us introduce some convenient notation. Define the operators

$$E_j(z)_{V^1,\ldots,V^n} : V^1 \otimes \cdots \otimes V^j \otimes V^{j+1} \otimes \cdots \otimes V^n \rightarrow V^1 \otimes \cdots \otimes V^{j+1} \otimes V^j \otimes \cdots \otimes V^n$$

as follows: if $v_i \in V^i, 1 \leq i \leq n$, and $hv_i = \chi_i(h)v_i, \chi_i \in \mathfrak{h}, h \in \mathfrak{h}$, then

$$(4.3) E_j(z)_{V^1,\ldots,V^n}(v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes E^{λ,j+1}(z)^{V^jV^{j+1}}(v_j \otimes v_{j+1}) \otimes \cdots \otimes v_n,$$

where $λ_j$ are defined by

$$(4.4) λ_j = (β - 1)^{-1} \left( \sum_{i=1}^{n} \chi_i \right) + \sum_{i=1}^{j} \chi_i, \ 0 \leq j \leq n.$$

Let $t_j$ denote the elementary transposition $(jj+1)$ in the symmetric group $S_n$.

For every $σ \in S_n$ set $E_j(z)^σ = E_j(z)_{V^{σ(1)},\ldots,V^{σ(n)}}$. This operator maps $V^{σ(1)} \otimes \cdots \otimes V^{σ(n)}$ to $V^{t_jσ(1)} \otimes \cdots \otimes V^{t_jσ(n)}$ (we make a convention that for two permutations $σ_1, σ_2$ $σ_1 σ_2(j) = σ_1(σ_2(j)), 1 \leq j \leq n$, i.e. the factors in a product of permutations are applied from right to left).

Let us describe some properties of the matrices of exchange coefficients. First of all, $E_j(z)^σ$ is not a single-valued function of $z$, so one should consider the function $B_j(ζ)^σ = E_j(e^ζ)^σ$.

**Proposition.** [FR] The functions $B_j(ζ)$ have the following properties:

(i) double periodicity:

$$(ia): B_j(ζ + \log p)^σ = B_j(ζ)^σ;$$

$$(ib): B_j(ζ + 2π\sqrt{-1})^σ = LB_j(ζ)^σ L', \ (log p = -2t(k + 1))$$

where $L, L'$ are constant matrices;

(ii) the quantum braid (Yang-Baxter) relation:

$$(4.6) B_j(ζ_1-ζ_2)^{ld}B_j(ζ_1-ζ_3)^{t_j}B_j(ζ_2-ζ_3)^{t_{j+1}} = B_j(ζ_2-ζ_3)^{ld}B_j(ζ_1-ζ_3)^{t_j}B_j(ζ_1-ζ_2)^{t_{j+1}};$$

(iii) unitarity:

$$(4.7) B_j(ζ)^{ld}B_j(-ζ)^{t_j} = 1.$$

**Remarks.** 1. Properties (ib), (ii) and (iii) follow directly from the definition of $B_j$. Property (ia) follows from the fact that matrix elements of products of intertwiners satisfy the quantum KZ equations.

2. Statement (i) implies that the matrix elements of $B_j$ must express in terms of elliptic functions. Frenkel and Reshetikhin showed that in some special cases these functions are the elliptic solutions of the quantum Yang-Baxter equation which occur in statistical mechanics.

From now on we will be assuming that the matrices of exchange coefficients are known.

Let us define the fundamental trace $E(ζ, ζ^{-1})$. 

Theorem 4.2. The fundamental trace is a function \( F \) of \( z_1, \ldots, z_n, p, s, q \) with values in the space \( \text{End}(V^1 \otimes \cdots \otimes V^n) \) defined by the property: if \( v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \) is a vector in \( V^1 \otimes \cdots \otimes V^n \), \( v_i \in V^i \), \( 1 \leq i \leq n \), and \( h v_i = \chi_i(h) v_i \), \( \chi_i \in \mathfrak{h}^* \), \( h \in \mathfrak{h} \), then

\[
F = \text{Tr} |_{\text{End}(V^1 \otimes \cdots \otimes V^n)} \left( \Phi_{v_1}^{z_1}, \ldots, \Phi_{v_n}^{z_n} B s^{-D} \right),
\]

where \( \lambda_j \) are defined by \((4.4)\).

It follows from Section 3 that the fundamental trace satisfies \( p \)-difference equations \((3.24)\). Below we will show that it also satisfies another system of \( s \)-difference equations that this solution exists and is unique. It can be represented as an infinite ordered product of \( R \)-matrices.

Indeed, let us carry the operator \( \Phi(z_i) \) from left to right. We have to interchange it with \( \Phi(z_{i+1}), \ldots, \Phi(z_n) \), then with \( B \) and \( s^{-D} \) (this will change \( \Phi(z_i) \) to \((1 \otimes B) \Phi(s^{-1} z_i)) \), then to shift it to the left side of the product, using the property \( \text{Tr}(ab) = \text{Tr}(ba) \), and then interchange it with \( \Phi(z_1), \ldots, \Phi(z_{i-1}) \). This procedure allows us to deduce a difference equation for \( F \). Before we write it down, let us introduce some notation.

Let \( \sigma_{jm} = t_{m-1} \cdots t_{j+1} t_j, \ j < m \leq n, \sigma_{jm} = t_m \cdots t_{j-2} t_{j-1}, \ 1 \leq m < j \).

Theorem 4.2. The fundamental trace \( F \) satisfies the relations:

\[
R_{j+1j}^{V+1V} \left( \frac{z_j+1}{z_j} \right) \cdots F(z_1, \ldots, z_j, z_{j+1}, \ldots, z_n|s) = F(z_1, \ldots, z_{j+1}, z_j, \ldots, z_n|s) E_j \left( \frac{z_j}{z_{j+1}} \right) \text{Id};
\]

\[
R_{j+1j}^{V+1V} \left( \frac{z_j+1}{z_j} \right) \cdots R_{j+1j}^{V+1V} \left( \frac{z_j+1}{z_j} \right) \cdots B_j^{-1} R_{nj}^{V V} \left( \frac{z_n}{z_j} \right) \cdots E_j \left( \frac{z_j}{z_{j+1}} \right) \text{Id},
\]

where \( B_j \) denotes the action of \( B \) in \( V^j \).

Now consider the system of difference equations

\[
G(z_1, \ldots, s^{-1} z_i, \ldots, z_n) = M_j(z_1, \ldots, z_n) G(z_1, \ldots, z_i, \ldots, z_n),
\]

\[
M_j(z_1, \ldots, z_n) = R_{j+1j}^{V+1V} \left( \frac{z_j+1}{z_j} \right) \cdots R_{j+1j}^{V+1V} \left( \frac{z_j+1}{z_j} \right) \cdots B_j^{-1} R_{nj}^{V V} \left( \frac{z_n}{z_j} \right) \cdots E_j \left( \frac{z_j}{z_{j+1}} \right) \text{Id}.
\]

This is one of the versions of the quantum KZ equations. Let \( G_0(z_1, \ldots, z_n|s) \) denote the fundamental solution of this system of equations – the solution with values in \( \text{End}(V^1 \otimes \cdots \otimes V^n) \) such that \( G_0 \sim \prod_{i=1}^N \frac{\log M_i}{z_i / z_{i+1}} \) as \( z_i / z_{i+1} \to \infty \), where \( M_i \) is the limit of \( M_i(z_1, \ldots, z_n) \) when \( z_i / z_{i+1} \to \infty \). It follows from the theory of difference equations that this solution exists and is unique. It can be represented as an infinite ordered product of \( R \)-matrices.

Introduce the matrix-valued function

\[
K(z_1, \ldots, z_n|s) = G_0^{-1}(z_1, \ldots, z_n|s) F(z_1, \ldots, z_n|s). 
\]
Theorem 4.2 is equivalent to the statement that this function satisfies the difference equation

\[ K(z_1, \ldots, z_j, \ldots, z_n|s) = \]

\[ K(z_1, \ldots, s^{-1}z_j, \ldots, z_n|s)E_{j-1} \left( \frac{s^{-1}z_j}{z_j-1} \right) \ldots E_1 \left( \frac{s^{-1}z_j}{z_1} \right) B_j^{-1} \frac{z_j}{z_n} \ldots E_j \left( \frac{z_j}{z_{j+1}} \right) \text{Id}. \]

(4.12)

On the other hand, formula (3.24) implies that the function \( K \) satisfies another difference equation:

\[ K(z_1, \ldots, pz_i, \ldots, z_n|s) = \]

\[ G_0^{-1}(z_1, \ldots, pz_i, \ldots, z_n)X_i(z_1, \ldots, z_n|s)G_0(z_1, \ldots, z_n|s)K(z_1, \ldots, z_i, \ldots, z_n|s). \]

(4.13)

Define the operators \( D_j \) acting on \( V^1 \otimes \cdots \otimes V^n \) as follows:

\[ D_j(v_1 \otimes \cdots \otimes v_n) = \frac{\lambda_j \lambda_j - \lambda_{j-1} \lambda_{j-1}}{2(k+1)}(v_1 \otimes \cdots \otimes v_n), \]

where \( \lambda_j \) are defined by (4.4). From the definition of the fundamental trace it follows that \( \mathcal{F}(z_1, \ldots, z_n|s) = \mathcal{F}_0(z_1, \ldots, z_n|s)Z_1^{D_1} \ldots Z_n^{D_n} \), where \( \mathcal{F}_0 \to 1 \) as \( s \to 0 \) and \( z_i/z_{i+1} \to \infty \).

Let

\[ \Theta(z|p) = \prod_{m \geq 0} (1 - p^m z)(1 - p^{m+1} z^{-1})(1 - p^{m+1}). \]

Then the function

\[ \varepsilon(z, \alpha|p) = z^\alpha \frac{\Theta(zp^\alpha|p)}{\Theta(z|p)} \]

is \( p \)-periodic: \( \varepsilon(pz, \alpha|p) = \varepsilon(z, \alpha|p) \). Therefore, the function

\[ K_0(z_1, \ldots, z_n|s) = K(z_1, \ldots, z_n|s) \prod_i \varepsilon(z, -D_i|p) \]

is a single-valued meromorphic function in the region \( z_i \neq 0 \) which still satisfies equation (4.13).

Thus, we have proved the following theorem.

**Theorem 4.3.** The function \( K_0 \) satisfies a pair of matrix difference equations – a \( p \)-difference equation from the left:

\[ K_0(z_1, \ldots, pz_j, \ldots, z_n|s) = Y_j(z_1, \ldots, z_n|s)K_0(z_1, \ldots z_j, \ldots, z_n|s), \]

and an \( s \)-difference equation from the right:

\[ K_0(z_1, \ldots, s^{-1}z_j, \ldots, z_n|s) = K_0(z_1, \ldots, z_n|s)U_j(z_1, \ldots, z_n|s). \]
where

\[ Y_j(z_1, ..., z_n|s) = G_0^{-1}(z_1, ..., z_n|s)X_j(z_1, ..., z_n|s)G_0(z_1, ..., z_n|s), \]

\[
U_j(z_1, ..., z_n|s) = \\
\left( \prod_i \varepsilon(z, D_i|p) \right) E_{j-1} \left( \frac{z_j}{z_{j-1}} \right)^{\sigma_{j-1}} \cdots E_1 \left( \frac{z_j}{z_1} \right)^{\sigma_1} B_j^{-1} E_{n-1} \left( \frac{s z_j}{z_n} \right)^{\sigma_{n-1}} \cdots E_j \left( \frac{s z_j}{z_{j+1}} \right)^{\text{Id}} \times \\
\left( \prod_i \varepsilon(z, -D_i|p) \right).
\]

We know that the function \( U_j \) is a product of \( p \)-periodic functions, so it is \( p \)-periodic (elliptic): \( U_j(z_1, ..., p z_i, ..., z_n) = U_j(z_1, ..., z_i, ..., z_n) \). Therefore, we have

**Corollary.** The function \( Y_j \) is \( s \)-periodic for all \( j \):

\[ Y_j(z_1, ..., s z_i, ..., z_n) = Y_j(z_1, ..., z_i, ..., z_n). \]

**Proof.** Since \( U_j \) is \( p \)-periodic, the function \( K_0(z_1, ..., p z_i, ..., z_n) \) satisfies (4.19) as long as \( K_0(z_1, ..., z_i, ..., z_n) \) does. Thus, the function \( K_0(z_1, ..., p z_i, ..., z_n)K_0(z_1, ..., z_i, ..., z_n)^{-1} \) has to be \( s \)-periodic. But this function is exactly \( Y_i \), Q.E.D.

5. Difference equations with elliptic coefficients and double difference systems

In this section we survey the theory of systems of difference equations, in particular those with elliptic coefficients, and introduce the notion of a double difference system which naturally arises from this theory.

Let \( p \) and \( s \) denote two nonzero complex numbers such that \( 0 < |p|, |s| < 1 \). If \( f(z) = f(z_1, ..., z_n) \) is a matrix-valued function of \( n \) complex variables then let

\[
P_i f(z) = f(z_1, ..., p z_i, ..., z_n), \\
S_i f(z) = f(z_1, ..., s z_i, ..., z_n).
\]

We will say that a meromorphic function \( f \) in \( \mathbb{C}^* \) is \( p \)-elliptic (respectively, \( s \)-elliptic) if \( P_i f = f \) (respectively \( S_i f = f \)) for all \( i \).

Let \( a_i(z) \), \( 1 \leq i \leq n \) be arbitrary meromorphic functions in \( \mathbb{C}^* \) with values in \( N \times N \) matrices. Consider the system of difference equations

\[ P_i f(z) = a_i(z) f(z), \]

where \( f \) is an \( N \times N \)-matrix valued function.

**Definition.** A system of \( p \)-difference equations is called consistent if there exists a meromorphic solution \( f \) to this system whose determinant is not identically zero.

The following obvious proposition gives necessary conditions for consistency of equations (5.2).
Proposition 5.1. If system (5.2) has a nonzero meromorphic solution $f$ such that $\det(f)$ is not identically zero, then

$$P_i a_j(z) \cdot a_i(z) = P_j a_i(z) \cdot a_j(z).$$

Definition. If a system of difference equations satisfies (5.3), it is called holonomic.

Thus, any consistent system must be holonomic.

Remark. It is obvious that if system (5.2) admits a meromorphic solution $f$ whose determinant is not identically zero then all meromorphic solutions of (5.2) form an $N^2$-dimensional vector space over the field of $p$-elliptic functions: all of them have the form $f g$, where $g$ is a matrix-valued $p$-elliptic function.

In the special case when the coefficients $a_i$ are rational functions the above necessary conditions are also sufficient:

Proposition 5.2. Any holonomic system of $p$-difference equations with rational coefficients is consistent.

Proof. We start with proving a technical lemma.

Lemma. If $a_i$ are rational for all $i$ then there exist positive real numbers $r_j$, $1 \leq j \leq n$, such that the multiannulus $A = \{(z_1, ..., z_n) | r_j |p| \leq |z_j| \leq r_j, 1 \leq j \leq n\}$ does not contain a singular point of $a_i^{\pm 1}$ for any $1 \leq i \leq n$.

Proof of the Lemma. Since $a_i$ are all rational, the singular set for the collection of functions $\{a_i^{\pm 1}, 1 \leq i \leq n\}$ is an affine algebraic variety in $\mathbb{C}^n$. By our agreement it is not the entire $\mathbb{C}^n$, therefore it is a subset in some algebraic hypersurface prescribed by the equation $P(z_1, ..., z_n) = 0$, where $P$ is a polynomial: $P = \sum a_{m_1}...a_{m_n} z_1^{m_1} ... z_n^{m_n}$.

Let $r_i$ be defined according to the rule: $r_n = r$, $r_{j-1} = e^{r_j}$. Denote the corresponding annulus $A$ defined in the statement of the lemma by $A_r$. Let $T$ be the lexicographically highest nonzero term in the polynomial $P$ (i.e. it has the highest possible degree of $z_1$, and among the terms with that degree of $z_1$ it has the highest degree of $z_2$, and so on). Then it is easy to see that $\lim_{r \to \infty} \frac{P}{T} |_{A_r} = 1$ (i.e. the highest term dominates all the others). Therefore, if $r$ is sufficiently large, the polynomial $P$ cannot vanish on $A_r$ (because $T$ does not). Thus, all the functions $a_i^{\pm 1}$ are regular in $A_r$, Q.E.D.

Now let us prove our proposition. Pick an annulus $A$ satisfying the condition of the Lemma. Now make this annulus into an $n$-dimensional complex torus (abelian variety) as follows: identify two points $(z_1, ..., z_i, ..., z_n)$ and $(z_1, ..., p z_i, ..., z_n)$ of $A$ whenever $|z_i| = r_i$. Denote the resulting quotient space by $E^*_p$ (it is nothing else but the $n$-th Cartesian power of the elliptic curve $E_p = \mathbb{C}^*/\{z \sim p z\}$). System of difference equations (5.2) can now be interpreted as a gluing condition for a rank $N$ holomorphic vector bundle over $E^*_p$, and meromorphic $\mathbb{C}^N$-valued solutions of (5.2) can be viewed as meromorphic sections of this bundle. But it is known from elementary algebraic geometry that any holomorphic vector bundle over a smooth projective variety has nonzero meromorphic sections, and the dimension of the space of meromorphic sections over the field of rational functions is equal to the rank of the bundle. In our case it implies that (5.2) has $N \mathbb{C}^N$-valued solutions which are linearly independent at a generic point. These solutions can be combined into an $N \times N$-matrix solution which will have a nonzero determinant at a generic point. Q.E.D.
However, in general a holonomic system of $p$-difference equations with meromorphic coefficients does not have to be consistent. A counterexample already exists for systems with elliptic coefficients.

**Example.** Consider the system of difference equations:

\begin{align}
 f(pz, w) &= f(z, w), \quad f(z, pw) = a(z) f(w), 
\end{align}

where $a$ is a non-constant $p$-elliptic function in $\mathbb{C}^*$. It is clear that this system satisfies identities (5.3). Still, let us show that it does not have a nonzero meromorphic solution.

Suppose that $f$ is a nonzero meromorphic solution of (5.4). Then for all $w$ but a countable set of them $f(z, w)$ is a meromorphic function in $z$. Also, the order of $f(z, w)$ as a function of $z$ at any fixed point $z_0$ is an essentially constant function of $w$ – it has the same value $d(z_0)$ at all points $w$ except for a countable set of them. However, the second equation in (5.4) shows that if $z_0$ is a pole of $a$ of order $k$ then

\begin{align}
 \text{ord}_{z_0} f(z, p^m w) = \text{ord}_{z_0} f(z, w) - km,
\end{align}

so for sufficiently large $m$ the order of $f(z, p^m w)$ at $z_0$ will become less than $d(z_0)$ – contradiction.

**Remark.** Of course, one can find plenty of nonzero solutions of (5.4) with essential singularities, e.g.

\begin{align}
 f(z, w) &= \frac{\Theta(w|p)}{\Theta(wa(z)|p)},
\end{align}

where $\Theta$ is defined by (4.15).

In spite of this example, it is very easy to show the existence of meromorphic solutions for a single difference equation.

**Proposition 5.3.** Let $a(z)$ be a meromorphic $N \times N$-matrix function in $\mathbb{C}^*$ such that $\det(a)$ is not identically zero. Then the difference equation

\begin{align}
 f(pz) &= a(z) f(z)
\end{align}

has a meromorphic solution whose determinant is not identically zero.

**Proof.** Let $r$ be a positive real number with the property: the function $a(z)$ is defined on the circles $|z| = r$ and $|z| = |p|r$, and its determinant does not vanish anywhere on these circles. Let $A = \{z \in \mathbb{C}^* | r \geq |z| \geq |p|r\}$ be the annulus squeezed between these circles, and let $E_p = A/(z \sim pz$ if $|z| = r)$ be the elliptic curve obtained by gluing the boundaries of the annulus $A$ to each other. Difference equation (5.6) can now be interpreted as a gluing condition for a rank $N$ holomorphic vector bundle over $E_p$. The rest of the argument is as in the proof of Proposition 5.2.

Now consider system (5.2) in which the coefficients $a_i$ are $s$-elliptic. Then the system has a new property: if $f(z)$ is a solution then $S_i f(z)$ is a solution as well. If $\det(f)$ is not identically zero, this property implies that there exist $p$-elliptic matrix-valued functions $b_i(z)$, $1 \leq i \leq n$, such that

\begin{align}
 S_i f(z) = f(z) b_i(z).
\end{align}

We see that there is another system of difference equations with elliptic coefficients (now $s$-elliptic) satisfied by $f$. Thus, we are naturally lead to introduce a new notion of a double difference system.
Definition. A double difference system is a system of difference equations of the form

\[ P_i f(z) = a_i(z) f(z), \]
\[ S_i f(z) = f(z) b_i(z), \quad i = 1, \ldots, n \]

where \( f \) is an \( N \times N \)-matrix valued function, and \( a_i, b_i \) are meromorphic \( N \times N \)-matrix valued functions whose determinants are not identically zero.

Let us now study double difference systems.

Suppose that \( f \) is a solution of (5.8) whose determinant is not identically equal to zero. Then one must have a consistency condition

\[ a_i(z) f(z) P_i b_j(z) = S_j a_i(z) f(z) b_j(z), \]

since both sides of (5.9) are equal to \( P_i S_j f(z) \), according to (5.8). The simplest way to satisfy this condition automatically is to set

\[ S_j a_i = a_i, \quad P_i b_j = b_j, \]

which is the same as to say that \( a_i \) are \( s \)-elliptic and \( b_j \) are \( p \)-elliptic. Besides this, we have the usual consistency conditions

\[ P_i a_j(z) \cdot a_i(z) = P_j a_i(z) \cdot a_j(z), \]
\[ S_i b_j(z) \cdot b_i(z) = S_j b_i(z) \cdot b_j(z). \]

Definition. A system of the form (5.8) satisfying consistency conditions (5.10) and (5.11) is called an elliptic double difference system.

Elliptic double difference systems are exactly those arising from systems (5.2) with \( s \)-elliptic coefficients. Our discussion shows that they are the most natural examples of double difference systems.

Note that in an elliptic double difference system, the coefficients of the \( s \)-difference equations play the role of monodromy (or connection) matrices for the \( p \)-difference equations, and vice versa.

Of course, (5.10) and (5.11) are only necessary and by no means sufficient conditions of consistency of system (5.8). This is demonstrated by the following proposition giving a necessary and sufficient condition of existence of a nondegenerate solution for an elliptic double difference system with constant coefficients.

Definition. We say that the numbers \( p, s \) are generic if for \( m, k \in \mathbb{Z} \) \( p^m = s^k \) if and only if \( m = k = 0 \). We say that \( p, s \) are strictly generic if they generate a dense subgroup in \( \mathbb{C}^* \).

Proposition 5.4. Let \( p, s \) be strictly generic, and let \( a_i \) and \( b_i \) be constant matrices for all \( i \). Then system (5.8) has a meromorphic solution \( f \) with \( \text{det}(f) \) not identically equal to 0 if and only if there exist invertible \( N \times N \) matrices \( R, L \) and diagonal matrices \( M_i, 1 \leq i \leq n \), with integer entries, such that

\[ a_i = L p^M_i L^{-1}, \quad b_i = R^{-1} s^M_i R, \quad 1 \leq i \leq n. \]
Proof.

Sufficiency. The function

\[ f(z) = L \prod_{i=1}^{n} z_i^{M_i} R \]

is a solution of (5.8) whose determinant is not identically equal to zero.

Necessity. Assume that (5.8) has a meromorphic solution \( f \) whose determinant is not identically zero. Then this solution has to be holomorphic and nondegenerate everywhere. Indeed, suppose that \( z_0 \) is a singularity of \( f \). Then (5.8) implies that for all \( k_i, m_i \in \mathbb{Z} \prod_i (P_i^k, S_i^m)z_0 \) is a singularity of \( f \) as well. But since \( p, s \) are strictly generic, the set of these points is dense in \( \mathbb{C}^n \) — a contradiction which implies the holomorphicity of \( f \). Applying the same argument to \( f^{-1} \), we get the nondegeneracy.

Now we need to use a simple lemma from the classical theory of difference equations:

**Lemma.** The difference equation \( f(pz) = af(z) \) has a holomorphic nondegenerate solution if and only if the matrix \( a \) is diagonalizable, and its entries are integer powers of \( p \). This solution, if it exists, has the form \( Lz^M R \), where \( M \) is a diagonal matrix with integer entries, \( L, R \) are invertible matrices, and \( a = Lp^M L^{-1} \).

This lemma together with the commutativity condition \([a_i, a_j] = 0\) which follows from (5.11), implies that the matrices \( a_i \) simultaneously diagonalize in a certain basis, and their eigenvalues are integer powers of \( p \). In other words, there exists an invertible matrix \( L \) and diagonal matrices with integer entries \( M_1, ..., M_n \) such that \( a_i = Lp^{M_i} L^{-1} \), and \( f = Lz_1^{M_1} \ldots z_n^{M_n} R \), from which we get (5.12).

**Proposition 5.5.** If \( p, s \) are strictly generic then the dimension of the space of solutions of any double difference system (5.8) (over \( \mathbb{C} \)) is less than or equal to \( N^2 \).

**Proof.** Let \( z_0 \) be a regular point of the coefficients of (5.8) and all their \( p, s \)-translates. Such a point obviously exists since the singular set has codimension 1. Then \( f \) must be regular at this point, and its value there determines its value anywhere else, since the \( p, s \)-translates of \( z_0 \) form a dense set in \( \mathbb{C}^n \).

**Remarks.**

1. It is easy to show that Propositions 5.3 and 5.4 are true for generic \( p, s \) which are not necessarily strictly generic.
2. The dimension of the space of solutions in Proposition 5.5 can be exactly \( N^2 \): this happens when \( a_i \) and \( b_i \) are scalar matrices.
3. It follows from Proposition 5.5 that in the case \( N = 1 \) and generic \( p, s \), if a double difference system has a nonzero solution, it is unique up to a constant factor.

One of the first major problems in the theory of elliptic double difference systems is the consistency problem:

**Consistency problem I.** Classify all sets \( \{a_i, b_i, 1 \leq i \leq n\} \) for which (5.8) has a solution whose determinant is not identically zero.

Another formulation of this problem is:

**Consistency problem II.** For a given set of coefficients \( \{a_i, 1 \leq i \leq n\} \) find all possible sets \( \{b_i, 1 \leq i \leq n\} \) for which (5.8) has a solution whose determinant is not identically zero.
In general, this problem is very difficult, and it is not clear how to approach it, even in the one variable case. However, in the case \( N = 1 \) (scalar-valued functions) it can be solved completely as described below.

The idea is to explicitly present a nonzero solution to the \( p \)-part of the elliptic double difference system, and then find the corresponding functions \( b_i \). Then the general form of the functions \( b_i \) is \( b_i^* = b_i \phi(z) \), where \( h \) is a \( p \)-elliptic function.

For simplicity we assume that \( n = 1 \); the case \( n > 1 \) is analogous but somewhat more technical. Our purpose now is to find a nonzero solution of the equation

\[
f(pz) = a(z)f(z),
\]

where \( a \) is an \( s \)-elliptic function.

It is known that any scalar-valued elliptic function can be written as a constant times a ratio of products of theta functions. More precisely, for every \( s \)-elliptic function \( a(z) \) of one variable there exist a unique constant \( C \in \mathbb{C} \), \( k \in \mathbb{Z} \), and a unique finite set \( A \subset \{ z \in \mathbb{C}^* | |s| \leq |z| < 1 \} \) with a function \( \nu : A \to \mathbb{Z} \setminus \{0\} \), such that

\[
a(z) = Cz^k \prod_{x \in A} \Theta(z/x|s)^{\nu(x)};
\]

and conversely, the function \( a \) given by (5.14) is \( s \)-elliptic iff \( \sum_{x \in A} \nu(x) = 0 \), and

\[
\prod_{x \in A} x^{\nu(x)} = s^{-k}.
\]

Therefore, it is enough to be able to find meromorphic solutions to the equations

\[
(5.15) \quad f(pz) = Cf(z), \quad f(pz) = z^k f(z), \quad f(pz) = \Theta(z/x|s)f(z)
\]

− then a solution of \( f(pz) = a(z)f(z) \) can be obtained as a product of such solutions.

A meromorphic solution of \( f(pz) = Cf(z) \) is given by

\[
(5.16) \quad f(z) = \frac{\Theta(z|p)}{\Theta(Cz|p)},
\]

and

\[
(5.17) \quad f(sz)/f(z) = \frac{\Theta(sz|p)\Theta(Cz|p)}{\Theta(Cs|p)\Theta(z|p)}.
\]

A meromorphic solution of \( f(pz) = z^k f(z) \) is given by

\[
(5.18) \quad f(z) = (-\Theta(z|p))^{-k},
\]

and

\[
(5.19) \quad f(sz)/f(z) = \left( \frac{\Theta(z|p)}{\Theta(sz|p)} \right)^k.
\]

Finally, a meromorphic solution of \( f(pz) = \Theta(z/x|s)f(z) \) is given by

\[
(5.20) \quad f(z) = \prod_{i,j=0}^{\infty} (1-p^i q^j z/x)^{-1} \prod_{i,j=1}^{\infty} (1-p^i q^j x/z)\Theta(z|p)\Theta(\phi(s)z|p)^{-1},
\]
where $\phi(s) = \prod_{m \geq 1}(1 - s^m)$ is the Euler product, and

\begin{equation}
(5.21) \quad f(sz)/f(z) = \frac{\Theta(z/x|p) \Theta(sz|p) \Theta(\phi(s)z|p)}{\phi(p) \Theta(z|p) \Theta(\phi(s)z|p)}.
\end{equation}

The above formulas allow us to construct a solution of $f(pz) = a(z)f(z)$ for any elliptic function $a$ as a ratio of products of expressions (5.16),(5.18),(5.20), which helps us solve the consistency problem.

Let $K(p)$ be the field of $p$-elliptic functions, and let $K(p)^*$ be its multiplicative group. Also, let $K(p)'$ be the multiplicative group generated by constants and the functions $z$, $\Theta(z/x|p)$. Let $L(p,s)'$ be the subgroup of $K(p)'$ consisting of functions of the form $f(z)/f(sz)$, $f \in K(p)'$. Let $K(p,s)' = K(p)'/L(p,s)'$, and let $K(p,s)^* = K(p)^*/(L(p,s)' \cap K(p)^*)$.

Let $F : K(s)' \to K(p,s)'$ be the group homomorphism defined by:

\begin{equation}
(5.22) \quad C \mapsto (5.17), \ z^k \mapsto (5.19), \ \Theta(z/x|s) \mapsto (5.21), \ |s| \leq |x| < 1,
\end{equation}

It is easy to check that $F$ restricts to a homomorphism $F : K(s)^* \to K(p,s)^*$. Moreover, it is clear that the kernel of $F$ is $L(s,p)'$, so $F$ gives rise to a map $F_0 : K(s,p)^* \to K(p,s)^*$. The map $F_0$ is obviously a group isomorphism.

Now we can formulate the necessary and sufficient condition for consistency of system (5.8). If $a \in K(p)^*$, let $\phi_{ps}(a)$ be the image of $a$ in $K(p,s)^*$.

**Proposition 5.6.** The system of difference equations

\begin{equation}
(5.23) \quad f(pz) = a(z)f(z), \ f(sz) = f(z)b(z)
\end{equation}

in which $a$ is $s$-elliptic and $b$ is $p$-elliptic, is consistent if and only if $F_0(\phi_{ps}(a)) = \phi_{sp}(b)$.

Finally let us describe an explicit formula for solutions of double difference systems, which applies in the case when $p, s$ are strictly generic. Consider the system

\begin{equation}
(5.24) \quad f(pz) = a(z)f(z), \ f(sz) = f(z)b(z)
\end{equation}

where $a, b, f$ are $N \times N$-matrix valued functions, and $a$ is $s$-elliptic, $b$ is $p$-elliptic. Assume that $f$ is a meromorphic solution to (5.24), and let $z_0$ be a point at which all matrices $a(p^mz)$, $b(s^nz)$, $m, n \in \mathbb{Z}$, are regular and nondegenerate. Let $z \in \mathbb{C}^*$. Let $\{m_j\}$, $\{n_j\}$ be sequences of integers such that $\lim_{j \to \infty} p^{m_j} s^{n_j} z_0 = z$. Such sequences can be constructed easily with the help of continuous fraction expansions. Then the following formula is valid:

**Proposition 5.7.**

\begin{equation}
(5.25) \quad f(z) = \lim_{j \to \infty} \prod_{j=m_k-1}^{0} a(p^j z_0) f(z_0) \prod_{j=0}^{n_k-1} b(s^j z_0).
\end{equation}

Existence of this limit and the fact that it equals $f(z)$ follows directly from (5.24).

Formula (5.25) determines $f(z)$ up to a finite number of unknown parameters – entries of $f(z_0)$. It can be generalized to the case of several variables.
Let us now construct another consistent elliptic double difference system related to system (4.18), (4.19).

Consider an elliptic double difference system of the form

\begin{equation}
P_i f(z, p, s) = a_i(z, p, s) f(z, p, s), \quad S_i f(z, p, s) = f(z, p, s) b_i(z, p, s),
\end{equation}

where \( a_i, b_i, f \) are \( N \times N \) matrix valued functions whose determinant is not identically zero, and let us assume that \( b_i(z, p, ps) = b_i(z, p, s) \). It is obvious that this condition is satisfied for system (4.18), (4.19). Assume that \( f \) is a solution to the system, and let

\begin{equation}
c(z, p, s) = f(z, p, ps) f(z, p, s)^{-1}.
\end{equation}

Then we can write two equations for \( c \). First of all,

\begin{equation}
P_i S_i f(z, p, ps) = f(z, p, ps) b_i(z, p, ps) = c(z, p, s) f(z, p, s) b_i(z, p, s);
\end{equation}

\( P_i S_i f(z, p, ps) = P_i S_i c(z, p, s) P_i S_i f(z, p, s) = P_i S_i c(z, p, s) a_i(z, p, s) f(z, p, s) b_i(z, p, s), \)

which implies:

\begin{equation}
P_i S_i c(z, p, s) = c(z, p, s) a_i(z, p, s)^{-1}.
\end{equation}

On the other hand,

\begin{equation}
P_i f(z, p, ps) = a_i(z, p, ps) c(z, p, s) f(z, p, s) = P_i c(z, p, s) a_i(z, p, s) f(z, p, s),
\end{equation}

from which we have

\begin{equation}
a_i(z, p, ps) c(z, p, s) = P_i c(z, p, s) a_i(z, p, s)
\end{equation}

Equations (5.29) and (5.31) together imply:

\begin{equation}
S_i c(z, p, s) = S_i a_i(z, p, ps) c(z, p, s).
\end{equation}

Thus, we have shown that the function \( c(z, p, s) \) satisfies a new elliptic double difference system (5.29), (5.32), which involves the functions \( a_i \) and does not involve \( b_i \).

Now assume that as \( s \to 0 \), \( f(z, p, s) \) has a finite limit \( f(z, p, 0) \) which is known and generically nondegenerate. (This property holds for system (4.18), (4.19): the fundamental trace converges to the highest matrix element of the intertwiner as \( s \to 0 \)). Then one can write the following formula for \( f \):

\begin{equation}
f(z, p, s) = \prod_{j=0}^{\infty} c(z, p, p^j s)^{-1} f(z, p, 0).
\end{equation}

Thus, if we knew an explicit expression for \( c \), we could get a more structured expression for \( f \) than that coming from (5.25), which would also be free from unknown parameters.

Unfortunately, for system (4.18), (4.19) it is not clear how to compute \( c \) explicitly; however, the quasiclassical limit of \( c \) can be understood. This quasiclassical limit, i.e. the first term of the Taylor expansion of \( c \) near \( q = 1 \) (where \( f = K_0 \)) is equal to the right hand side coefficient of the “moduli equation” for the fundamental trace of the classical affine Lie algebra ([E], Eqn. (3.24)) – the equation characterizing the derivative of the fundamental trace with respect to the modular parameter of the corresponding elliptic curve. This fact is reassuring, since it gives rise to a hope that the method of [E] can somehow be generalized to the quantum case, which would allow one to produce some kind of explicit expression for the function \( c \), and hence for the fundamental trace. \( F \).
Appendix: limiting cases

Let us briefly describe some interesting limiting cases. These cases correspond to some special (limiting) values of the parameters $p, q, s$.

Case 1. The quantum KZ limit: $s = 0$. In this case, the fundamental trace transforms into the highest matrix element of a product of intertwiners, system (3.24) becomes the quantum KZ system of Frenkel and Reshetikhin, and relations (4.9) become the connection relations for the quantum KZ system. In this limit, one gets $q$-hypergeometric functions and their generalizations.

Case 2. The elliptic KZ limit: $q \to 1$, $p = q^{-2(k+1)}$, $k$ is fixed. In this case, the fundamental trace becomes the fundamental trace for the classical affine Lie algebra, system (3.24) degenerates to the elliptic $r$-matrix system involving Belavin’s elliptic $r$-matrix, and relations (4.9) degenerate to the monodromy relations for the elliptic $R$-matrix equations (see [E]). In this limit, one gets transcendental functions of an elliptic curve, vector-valued modular forms etc.

Case 3. The Yangian limit: $q = e^\varepsilon$, $p = q^{-2(k+1)}$, $k$ is fixed, $z_i = e^{\varepsilon x_i}$, $\varepsilon \to 0$. In this case, the degeneration of the fundamental trace should be something like the fundamental trace for the (doubled) Yangian of $\mathfrak{sl}_N$, and the equations (3.24) should converge to a trigonometric deformation of the Smirnov’s equations [cf [LS]].

This limit is still unexplored.

Case 4. The critical limit: $p \to 1$. In this case, equations (3.24) transform into an elliptic analogue of the Bethe ansatz equations. The Bethe ansatz equations are obtained if one combines this limit with the limit $s \to 0$ (cf. [TV]).

Besides these, there are many other unexplored limiting cases which are easier to study than the general case. It is expected that studying these limiting cases, one should be able to get interesting information about various classes of special functions arising in representation theory of Lie algebras and quantum groups.

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