Summary. - The Lagrangian formulation of classical field theories and in particular general relativity leads to a coordinate-free, fully covariant analysis of these constrained systems. This paper applies multisymplectic techniques to obtain the analysis of Palatini and self-dual gravity theories as constrained systems, which have been studied so far in the Hamiltonian formalism. The constraint equations are derived while paying attention to boundary terms, and the Hamiltonian constraint turns out to be linear in the multimomenta. The equivalence with Ashtekar's formalism is also established. The whole constraint analysis, however, remains covariant in that the multimomentum map is evaluated on any spacelike hypersurface. This study is motivated by the non-perturbative quantization program of general relativity.
1. - Introduction.

When Dirac developed his approach to constrained Hamiltonian systems with the corresponding quantization program, he emphasized that the Hamiltonian formalism is always necessary to quantize a field theory with constraints. Remarkably, he provided a well-defined (though not unique) prescription to define a Hamiltonian function on the whole phase space, a classification of constraints in terms of their Poisson brackets which is immediately relevant for quantization, and an approach to quantum electrodynamics which does not rely on Feynman diagrams and renormalization theory [1-5]. However, any attempt to combine Dirac’s quantization of first-class constrained Hamiltonian systems with the Arnowitt-Deser-Misner geometrodynamical framework for canonical gravity faces very severe technical problems. In other words, the occurrence of the scalar curvature of the spacelike three-surfaces and the product of the three-momenta in the Hamiltonian constraint make it impossible to find exact solutions of the corresponding Wheeler-De Witt equation, as well as interpret the (as yet unknown) physical states of the quantum theory within the geometrodynamical framework [5,6].

More recently, the work by Ashtekar, Rovelli, Smolin and their collaborators on connection dynamics and loop variables has made it possible to cast the constraint equations of general relativity in polynomial form, and then find a large class of solutions to the quantum version of constraints [7-14]. However, the quantum theory via the Rovelli-Smolin transform still suffers from severe mathematical problems in 3+1 space-time dimensions [15], and there appear to be reasons for studying non-perturbative quantum gravity also from a Lagrangian, rather than Hamiltonian, point of view (see below). The aim of this paper is therefore to provide a multisymplectic, Lagrangian framework for general relativity [16-18], to complement the present attempts to quantize general relativity in a non-perturbative way. The motivations of our analysis are as follows.

(i) In the case of field theories, there is not a unique prescription for taking duals, on passing to the Hamiltonian formalism. For example, algebraic and topological duals are different. In turn, this may lead to inequivalent quantum theories.
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(ii) The 3+1 split of the Lorentzian space-time geometry, with the corresponding $\Sigma \times \mathbb{R}$ topology, appears to violate the manifestly covariant nature of general relativity, as well as rely on a very restrictive assumption on the topology [19].

(iii) In the Lagrangian formalism, explicit covariance is instead recovered. The first constraints one actually evaluates correspond to the secondary first-class constraints of the Hamiltonian formalism. At least at a classical level, the Lagrangian theory of constrained systems is by now a rich branch of modern mathematical physics [4,20,21], although the majority of general relativists are more familiar with the Hamiltonian framework.

(iv) In the ADM formalism [5,6], the invariance group is not the whole diffeomorphism group, but a subgroup given by the Cartesian product of diffeomorphisms on the real line with the diffeomorphism group on spacelike three-surfaces. By contrast, in the Lagrangian approach, the invariance group of the theory is the full diffeomorphism group of four-dimensional Lorentzian space-time. This is the unique group responsible for the occurrence of constraints in Einstein’s general relativity, on looking at symmetry properties of the action functional under such a group.

(v) Jet-bundles theory provides a rigorous geometric framework for the hyperbolic problems of classical field theory and, in particular, general relativity. The corresponding evaluation of constraints, on using tetrad formalism, is elegant, very powerful, and well-suited for any attempt to study general relativity as a field theory with constraints.

(vi) Although the elliptic boundary-value problems of Riemannian geometry and quantum gravity via Wick-rotated path integrals enable one to get a better understanding of different approaches to the quantization of gauge fields and gravitation [5], the corresponding perturbative theory is non-renormalizable. Hence the background-field method should be complemented or replaced by a radically different view of space-time theories at the Planck length [22]. The Hamiltonian approach is the first step of the non-perturbative quantization program, but unfortunately it breaks covariance.

We have thus tried to develop a classical multisymplectic analysis of Palatini and self-dual gravity theories which might be applied to a non-perturbative formalism for quantum gravity that preserves covariance. For this purpose, sect. 2 presents a derivation of Einstein’s equations in tetrad form which relies on multisymplectic-geometry techniques (cf.
2. - Multisymplectic form of Einstein’s equations.

Our analysis begins by studying the Palatini action $S_P$ of (Lorentzian) general relativity. This is a real-valued functional of the tetrad $e^a_I$ and the connection one-form $\omega_a^{IJ}$, taking values in the Lorentz Lie-algebra, and given by

$$ S_P \equiv \frac{1}{2} \int_M d^4x \ e^a_I \ e^b_J \ \Omega_{ab}^{IJ} . \quad (2.1) $$

With our notation, $e \equiv \sqrt{-g}$ is the square root of the determinant of the space-time four-metric, and

$$ \Omega_{ab}^{IJ} \equiv \partial_a \omega_b^{IJ} - \partial_b \omega_a^{IJ} + \omega_a^I \omega^K_J - \omega^K_J \omega_a^I $$

is the curvature of the four-dimensional connection one-form $\omega_a^{IJ}$. Moreover, $a, b$ are tangent-space indices, whereas $I, J$ are the usual internal indices [13].

In first-order theory, tetrad and connection are regarded as independent variables. Since we are aiming to use a Lagrangian version of first-order theory in terms of one-jet bundles (see appendix), it is useful to bear in mind that in general relativity one takes a fibre bundle whose base is space-time, and whose fibres are isomorphic to the Cartesian product of the space of Lorentzian four-metrics with the space of linear connections. In the language of tetrads and connection one-forms used in (2.1)-(2.2), one takes a fibre bundle $Y$ which, in local coordinates, reads

$$ \left( x^a, e^a_I, \omega_a^{IJ} \right) . $$
To obtain the corresponding one-jet bundle $J^1(Y)$ (see appendix), one is thus led to consider the multivelocity $V_{ab}^I$ corresponding to the tetrad, and the multivelocity $W_{ab}^{IJ}$ corresponding to the connection one-form. In local coordinates, our one-jet bundle $J^1(Y)$ is therefore represented by

$$
\left( x^a, e^a_I, \omega^I_a, V_{ab}^I, W_{ab}^{IJ} \right).
$$

This leads to the Lagrangian

$$
L \equiv \frac{e}{2} e^a_I e^b_J \left( W_{ab}^{IJ} - W_{ba}^{IJ} + [\omega_a, \omega_b]^{IJ} \right), \tag{2.3}
$$

and hence to the Cartan four-form on $J^1(Y)$ as

$$
\Theta_L = \left( L - \frac{\partial L}{\partial W_{ab}^{IJ}} W_{ab}^{IJ} \right) d^4x + \frac{\partial L}{\partial W_{ab}^{IJ}} d\omega_a^{IJ} \wedge d^3x_b
$$

$$
= \frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) \left[ d\omega_a^{IJ} \wedge d^3x_b + \omega_a^I \wedge \omega_b^K \wedge d^4x \right]. \tag{2.4}
$$

Note that $V_{ab}^I$ does not contribute, since derivatives of the tetrad do not occur in this first-order theory for vanishing torsion. The corresponding multisymplectic five-form $\Omega_L$ is obtained by exterior differentiation of $\Theta_L$ as $\Omega_L \equiv d\Theta_L$. To write down field equations, one now considers a vector field $U$ tangent to $J^1(Y)$. It has the form

$$
U = U^d \frac{\partial}{\partial x^d} + U^I_1 \frac{\partial}{\partial e^I_1} + U^d_{ab} \frac{\partial}{\partial \omega_a^{IJ}} + U^d_{bl} \frac{\partial}{\partial V_{bl}^I} + U^d_{ab} \frac{\partial}{\partial W_{ab}^{IJ}}.
$$

One then takes the contraction of $\Omega_L$ with $U$. The pull-back of the resulting geometric object by means of the tangent lift $j^{(1)}(\varphi)$ of sections of the fibre bundle $Y$, leads to the Euler-Lagrange field equations. In our case one obtains

$$
\text{i}_U \Omega_L = -\frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) e^L_h \left. de^b_L \wedge d\omega_a^{IJ} \wedge d^2x_{bm} U^m \right.
$$

$$
+ \frac{e}{2} \left[ \left( de^a_I \right) e^b_J + e^a_I \left( de^b_J \right) - \left( de^b_I \right) e^a_J - e^b_I \left( de^a_J \right) \right] \wedge d\omega_a^{IJ} \wedge d^2x_{bl} U^l
$$

$$
+ \frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) e^L_h \left( de^h_L \wedge d^3x_m U^m \right) \omega_a^I \wedge \omega_b^K J.
$$
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\[-\frac{e}{2} \left[ (de^a_I)e^b_J + e^a_I(de^b_J) - (de^b_I)e^a_J - e^b_I(de^a_J) \right] \omega_a^I \omega_b^K \wedge d^3 x_l U^l \]
\[-\frac{e}{2} \left( e^a_J e^b_J - e^b_J e^a_J \right) \left[ (d\omega_a^I)_K \omega_b^K + \omega_a^I (d\omega_b^J) \right] \wedge d^3 x_l U^l \]
\[+ \frac{e}{2} \left[ U^a_I e^b_J + U^b_I e^a_J - U^b_I e^a_J - U^a_I e^b_J \right] \left[ d\omega_a^{IJ} \wedge d^3 x_b + \omega_a^I \omega_b^J d^4 x \right] \]
\[-\frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) e^L_h U^h_L \left[ d\omega_a^{IJ} \wedge d^3 x_b + \omega_a^I \omega_b^J d^4 x \right] \]
\[+ \frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) U^a_{IJ} e^L_h \left[ d\omega_a^{IJ} \wedge d^3 x_b \right] \]
\[-\frac{e}{2} U^a_{IJ} \left[ (de^a_I)e^b_J + e^a_I(de^b_J) - (de^b_I)e^a_J - e^b_I(de^a_J) \right] \wedge d^3 x_b \]
\[+ \frac{e}{2} \left( e^a_I e^b_J - e^b_I e^a_J \right) \left( U^a_I \omega_b^K + \omega_a^I \omega_b^J \right) d^4 x \,. \tag{2.6} \]

Such a lengthy equation is indeed necessary, since it involves many contributions which cannot be obvious to non-expert readers, and their interpretation needs a careful thinking.

When one evaluates the pull-back of \( i_U \Omega_L \) by means of the tangent lift

\[
\left( j^{(1)}(\varphi) \right) \equiv \left( x^a, e^a_I(x), \omega_a^{IJ}(x), \frac{\partial e^a_I}{\partial x^b}(x), \frac{\partial \omega_a^{IJ}}{\partial x^b}(x) \right) ,
\]

the terms \( de^h_L \) and \( d\omega_a^{IJ} \) occurring in (2.6) take the forms

\[
de^h_L = e^h_{L,f} dx^f , \tag{2.7}
\]
\[
d\omega_a^{IJ} = \omega_a^{IJ,l} dx^l . \tag{2.8}
\]

Thus, by using the identities

\[
dx^f \wedge d^3 x_l = \delta^f_l d^4 x , \tag{2.9}
\]
\[
dx^f \wedge dx^h \wedge d^2 x_{bc} = \left( \delta^f_b \delta^h_c - \delta^f_c \delta^h_b \right) d^4 x , \tag{2.10}
\]

and defining

\[
\delta e^h_L \equiv U^m e^h_{L,m} - U^h_L , \tag{2.11}
\]
\[
\delta \omega_a^{IJ} \equiv U^m \omega_a^{IJ,m} - U_a^{IJ} , \tag{2.12}
\]
The field equations
\[ \left( j^{(1)}(\varphi) \right)^* \left( i_U \Omega_L \right) = 0 \] (2.14)
are found to take the form
\[ e \left( \delta e^h_L \right) G^L_h + \left( \delta \omega_{a^{IJ}} \right) F^a_{IJ} = 0 , \] (2.15)
where
\[ G^L_h \equiv 2e^b_I \left[ \Omega_{bh}^{IL} - \frac{1}{2} e^d_J e^L_h \Omega_{bd}^{IJ} \right] , \] (2.16)
\[ F^a_{IJ} \equiv D_b \left[ e \left( e^b_I e^a_J - e^a_I e^b_J \right) \right] . \] (2.17)
Since \( e^h_L \) and \( \omega_{a^{IJ}} \) are independent, Eq. (2.15) implies
\[ e^b_I \left[ \Omega_{bh}^{IL} - \frac{1}{2} e^d_J e^L_h \Omega_{bd}^{IJ} \right] = 0 , \] (2.18)
\[ D_b \left[ e \left( e^b_I e^a_J - e^a_I e^b_J \right) \right] = 0 . \] (2.19)
Eqs. (2.18) are the Einstein equations, while eqs. (2.19) express a property of a connection which is completely determined by the tetrad [13].

Note that in (2.11)-(2.12) the terms involving partial derivatives of the field variables are the horizontal part of the variation, and remaining terms represent the vertical part of the variation. Hence (2.11)-(2.12) have a very clear geometric meaning, and they lead to a considerable simplification of a lengthy calculation. Moreover, (2.13) defines the covariant derivatives of the tetrad. The connection \( D \) is a Lorentz connection which annihilates the Minkowskian metric \( \eta_{IJ} \) on the internal space. The definition (2.13) is an additional condition we are imposing, since the action of \( D \) on space-time indices is not defined \textit{a priori} [13].
3. - Multimomenta.

The analysis of constraint equations in sect. 4 makes it necessary to describe some properties of the multimomenta corresponding to the multivelocities defined in sect. 2. The multimomenta of a field theory are defined as the derivatives of the Lagrangian with respect to the multivelocities. In general relativity, the multimomenta resulting from the Lagrangian (2.3) are defined as the densities

\[ \tilde{p}_{IJ}^{ab} \equiv 2 \frac{\partial L}{\partial W_{ab}^{IJ}} = e p_{IJ}^{ab}, \]  

(3.1)

\[ \tilde{\pi}^{bI}_a \equiv 2 \frac{\partial L}{\partial V^{a}_b I} = 0, \]  

(3.2)

where the \( p_{IJ}^{ab} \) are bivectors defined as

\[ p_{IJ}^{ab} \equiv e_a^I e_b^J - e_b^I e_a^J. \]  

(3.3)

Of course, the multiplicative factor in (3.1)-(3.2) is unessential, and is introduced for convenience. Note that the multimomenta \( \tilde{\pi}^{bI}_a \) vanish, and this reflects that torsion vanishes in general relativity.

From the definition (3.3), we note that the bivectors \( p_{IJ}^{ab} \) satisfy the following commutation relations:

\[ \left[ p^{ab}_{IJ}, p^{cd}_{IJ} \right] = p^{ac}_{IJ} g_{bd} + p^{bd}_{IJ} g_{ac} - p^{ad}_{IJ} g_{bc} - p^{bc}_{IJ} g_{ad}. \]  

(3.4)

In particular, on a spacelike hypersurface, Eq. (3.4) may be used to derive the identity

\[ \left[ p^{0i}_{IJ}, p^{j0}_{IJ} \right] = p^{ij}_{IJ} g^{00} - p^{i0}_{IJ} g^{0j} - p^{0j}_{IJ} g^{i0}, \]  

(3.5)

where the index 0 refers to the time coordinate and the indices \( i, j \) refer to the spatial coordinates. This property will be useful in sect. 5. The multimomenta correspond to 36
variables (6 for each bivector density), therefore equation (3.5) expresses the eighteen variables associated to the $\tilde{p}^{ij}_{IJ}$ as quadratic functions of the $\tilde{p}^{0i}_{IJ}$ bivector densities. In the following sections the constraint equations are all expressed in terms of the multimomenta.

4. - Constraints.

When one studies classical mechanics and classical field theory one learns that, if the Lagrangian is invariant under the action of a group, then by virtue of Noether’s theorem there exist functions which are constant along solutions of the equations of motion. In the case of the invariance under a gauge group or the diffeomorphism group of general relativity, such first integrals always vanish along solutions of the equations of motion. Hence the first-class constraints of a field theory result from Noether’s theorem through the action of the gauge group or the group of space-time diffeomorphisms [23,24]. The multimomentum map is the mathematical tool which enables one to describe these properties of classical fields. With the notation of Eqs. (A.1)-(A.2), the multimomentum map on a section of our jet-bundle $J^1(Y)$ is defined by the expression [23,24]

$$J(\xi) = \left[ e a^I \left( \xi_a^I - e_a^I \delta^b \right) + e \frac{\partial L}{\partial \omega_{IJ}} \left( \xi_{I}^{IJ} - \omega_{IJ} \delta^b \right) + \frac{e}{2} \Omega_{IJ} \xi^c \right] d^3x_c. \quad (4.2)$$

In the Hamiltonian framework, setting to zero the integral of the multimomentum map $J_\xi$ on a spacelike hypersurface $\Sigma$ leads to the first-class constraints of the theory [23,24]. Moreover, in the case of null hypersurfaces, second-class constraints also occur [25], and hence they cannot be derived from the multimomentum map. On passing from spacelike
to null hypersurfaces, discontinuities occur in the normal to the hypersurface and in the induced three-metric [25]. Hence there might be a covariant constraint analysis for any spacelike hypersurface, and an independent, covariant constraint analysis for any null hypersurface. [We are grateful to Luca Lusanna for bringing this open problem to our attention]

To express our \( J(\xi) \), from

\[
\frac{\partial L}{\partial e^a_{I,c}} = 0 ,
\]

and

\[
\xi^I_a = -\xi^b_{,a} \omega^I_b + (D_a \lambda)^I_J ,
\]

it follows (on using (3.3))

\[
I_\Sigma[\xi] \equiv \int_\Sigma J(\xi) = \frac{1}{2} \int_\Sigma \left[ e p^{ac}_{IJ} \left( \xi^b_{,a} \omega^I_b - (D_a \lambda)^I_J + \omega^I_J a, b \xi^b \right) \right. \\
+ \left. \frac{1}{2} e p^{ab}_{IJ} \Omega^{IJ}_{ab} \xi^c \right] d^3 x_c ,
\]

where one has defined the covariant derivative with respect to the tetrad indices (cf. (2.13))

\[
(D_a \lambda)^I_J \equiv (\partial_a \lambda)^I_J + [\lambda, \omega_a]^I_J .
\]

Thus, taking the integral of the multimomentum map on the spacelike hypersurface \( \Sigma \), integrating by parts for the term in \( \lambda^I_J \), and defining

\[
\sigma^{ac} \equiv p^{ac}_{IJ} \lambda^I_J ,
\]

one obtains

\[
I_\Sigma[\xi] = \frac{1}{2} \int_\Sigma \lambda^I_J (D_a \tilde{p}^{ac}_{IJ}) d^3 x_c + \frac{1}{2} \int_\Sigma \left[ (\tilde{p}^{ac}_{IJ})(L_\xi \omega)_a^I_J + \frac{1}{2} \tilde{p}^{ab}_{IJ} \Omega^{IJ}_{ab} \xi^c \right] d^3 x_c \\
- \frac{1}{2} \int_\Sigma \partial_a \sigma^{ac} d^3 x_c .
\]

Again from integration by parts in the second integral, and defining

\[
\rho^{ac} \equiv \tilde{p}^{ac}_{IJ} \omega^I_b \xi^b ,
\]
one finds
\[ I_\Sigma[\xi] = \frac{1}{2} \int_\Sigma \left[ -\xi^b (\tilde{p}^{ac} L J) (\omega^I a, J - \omega^I b, J) - \xi^b \tilde{p}^{ac} a, J \omega^I a, b + \frac{1}{2} \tilde{p}^{ab} L J \Omega^I c \frac{1}{2} \int_\Sigma \partial_a \sigma^{ac} d^3 x_c \right. \]
\[ + \frac{1}{2} \int_\Sigma \partial_a \rho^{ac} d^3 x_c + \frac{1}{2} \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c - \frac{1}{2} \int_\Sigma \partial_a \sigma^{ac} d^3 x_c \] \quad \text{.} \tag{4.10}

Using equation (2.19), and denoting by \( I^{\text{ess}}_\Sigma[\xi] \) the part of \( I_\Sigma[\xi] \) not involving total divergences, one can show that
\[ I^{\text{ess}}_\Sigma[\xi] = \frac{1}{2} \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c + \frac{1}{2} \int_\Sigma \left[ - (\tilde{p}^{ac} L J) \Omega^I ad^{IJ} + \frac{1}{2} \tilde{p}^{ab} L J \Omega^I ab \delta^d c \right] \xi^d d^3 x_c \]
\[ = \frac{1}{2} \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c - \frac{1}{2} \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c - \frac{1}{2} \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c \] \quad \text{,} \tag{4.11}

where \( \lambda^{IJ} \) and \( \xi^d \) are independent and arbitrary quantities. Then from imposing \( I_\Sigma[\xi] = 0 \),
\[ \int_\Sigma \partial_a \sigma^{ac} d^3 x_c = \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c = 0 \] \quad \text{,} \tag{4.12}

and
\[ \int_\Sigma \partial_a \rho^{ac} d^3 x_c = \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c = 0 \] \quad \text{.} \tag{4.13}

Note that the total divergences appearing in (4.12)-(4.13) lead to boundary terms evaluated on the two-surface \( \partial \Sigma \). Hence sufficient conditions for the vanishing of such boundary terms are as follows: (i) \( \Sigma \) has no boundary; (ii) \( \tilde{p}^{ab} L J \) vanishes at \( \partial \Sigma \); (iii) \( \lambda^{IJ} \) vanishes at \( \partial \Sigma \) (in (4.12)); (iv) \( \xi^b \) or \( \omega^I a, J \) vanishes at \( \partial \Sigma \) (in (4.13)). Although these conditions are (rather) restrictive, from now on we will always assume that (i) or (ii) or both (iii) and (iv) hold. Hence (4.12)-(4.13) reduce to
\[ \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c = 0 \] \quad \text{,} \tag{4.14}

and
\[ \int_\Sigma \lambda^{IJ} (D_a \tilde{p}^{ac} L J) d^3 x_c = 0 \] \quad \text{.} \tag{4.15}
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5. - Reduction of constraints to Ashtekar’s form.

In this section we show that the previous equations reproduce Ashtekar’s results for a
Palatini Lagrangian [13]. For this purpose, we choose the adapted local coordinates defined
by the condition $x^0 = \text{constant}$, while the remaining three coordinates are denoted by the
spatial indices $i, j, k$. In view of the covariance of the Lagrangian formalism, it is enough
to prove the equivalence with Ashtekar’s constraint analysis on spacelike hypersurfaces in
this particular coordinate system. The constraints (4.14)-(4.15) are then found to take the
form (from the arbitrariness of $\lambda^{IJ}$ and $\xi^d$)

$$(D_a p^{a0})_{IJ} = 0 , \quad (5.1)$$

which corresponds to the Gauss constraint [9,13], and

$$\text{Tr} \left[ \bar{\rho}^{a0} \Omega_{ad} - \frac{1}{2} \bar{\rho}^{ab} \Omega_{ab} \delta_d^0 \right] = 0 . \quad (5.2)$$

Note that (5.1) reflects the invariance of general relativity under local Lorentz transforma-
tions, and leads to six independent constraint equations. They may be further split into
internal rotations and boosts [9,13]. The last four constraints (5.2) can be split into two
parts, for $d \neq 0$ one has

$$\text{Tr} \left[ \bar{\rho}^{a0} \Omega_{aj} \right] = \text{Tr} \left[ \bar{\rho}^{i0} \Omega_{ij} \right] = 0 , \quad (5.3)$$

which can be identified with the vector constraint; and for $d = 0$

$$\text{Tr} \left[ \bar{\rho}^{a0} \Omega_{a0} - \frac{1}{2} \bar{\rho}^{ab} \Omega_{ab} \right] = \text{Tr} \left[ \bar{\rho}^{i0} \Omega_{i0} - \frac{1}{2} \bar{\rho}^{i0} \Omega_{i0} - \frac{1}{2} \bar{\rho}^{0i} \Omega_{0i} - \frac{1}{2} \bar{\rho}^{ij} \Omega_{ij} \right]$$

$$= -\frac{1}{2} \text{Tr} \left[ \bar{\rho}^{ij} \Omega_{ij} \right] = 0 , \quad (5.4)$$

which is the Hamiltonian constraint. To see that (5.4) implies the Hamiltonian constraint
in its usual form, we point out that from (3.5)

$$\text{Tr} \left[ \bar{\rho}^{ij} \Omega_{ij} \right] = 2 e^{-1} (g^{00})^{-1} \text{Tr} \left[ \bar{\rho}^{i0} \bar{\rho}^{j0} \Omega_{ij} \right] + 2 (g^{00})^{-1} g^{ij} \text{Tr} \left[ \bar{\rho}^{i0} \Omega_{ij} \right] . \quad (5.5)$$

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Then from (5.3)-(5.5) it follows that

\[ \text{Tr} \left[ \tilde{p}^{0i} \tilde{p}^{0j} \Omega_{ij} \right] = 0 , \]  

(5.6)

which is the familiar constraint equation quadratic in the momenta [13].

6. - Preservation of constraints.

We have now to prove explicitly that the constraints (4.14)-(4.15) are preserved in our multisymplectic approach. For this purpose, we begin by requiring that the following Lie derivative:

\[ I \equiv \int_{\Sigma} L_\eta \left[ \text{Tr} \left( \tilde{p}^{ac} \Omega_{ad} - \frac{1}{2} \tilde{p}^{ab} \Omega_{ab} \delta^c_d \right) \right] d^3 x_c , \]  

(6.1)

should vanish, where \( L_\eta \) denotes the Lie derivative along a smooth vector field \( \eta \). The integral (6.1) results from taking differences of the constraints evaluated at two generic spacelike hypersurfaces \( \Sigma \) and \( \Sigma' \), related by a one-parameter flow generated by \( \eta \). Thus, since the Lie derivative along \( \eta \) of a weight-\( w \) vector density \( \tilde{V} \) reads [26]

\[ \left( L_\eta \tilde{V} \right)^a = \partial_b \left( \tilde{V}^a \eta^b - \tilde{V}^b \eta^a \right) + \eta^a \partial_b \tilde{V}^b + (w - 1) \tilde{V}^a \partial_b \eta^b , \]  

(6.2)

one finds that (6.1) takes the form

\[ I = 2 \int_{\partial \Sigma} \tilde{G}^c_{ab} \xi^d \eta^b d^2 x_{bc} + \int_{\Sigma} \eta^a \tilde{G}^b_{ac} \nabla_b \xi^c d^3 x_a + \int_{\Sigma} \eta^a \xi^c \nabla_b \tilde{G}^b_{ac} d^3 x_a , \]  

(6.3)

where \( \tilde{G}^a_{ab} \equiv e^L \tilde{c} e^a_L \) (see (2.16)), and \( \nabla \) is the (torsion-free) connection which annihilates the tetrad. Hence \( \nabla_a \tilde{V}^a = \partial_a \tilde{V}^a \) [26]. The first term in (6.3) vanishes by virtue of the boundary conditions imposed in sect. 4, and the second integral vanishes along solutions of the field equations. Last, the third integral vanishes by virtue of the contracted Bianchi identities with respect to the space-time connection \( \nabla \).
Similarly, one finds that also the Gauss-law constraint (4.14) is preserved, since
\[
\int_{\Sigma} \sum_{\lambda_i} \left( \eta^I \left( D_a \tilde{p}^{abc} \right)_{IJ} \right) d^3 x_c = 2 \int_{\partial \Sigma} \eta^b \lambda^I \left( D_a \tilde{p}^{abc} \right)_{IJ} d^2 x_{bc} 
\]
\[
+ \int_{\Sigma} \eta^c \left( \partial_b \lambda^I \right) \left( D_a \tilde{p}^{abc} \right)_{IJ} d^3 x_c + \int_{\Sigma} \eta^c \lambda^I \partial_b \left( D_a \tilde{p}^{abc} \right)_{IJ} d^3 x_c .
\] (6.4)

Again, the boundary term vanishes by virtue of the boundary conditions of sect. 4, and the second term vanishes on imposing the field equations. Moreover, the third term on the right-hand side of (6.4) vanishes by virtue of the identity
\[
\partial_b \left( D_a \tilde{p}^{abc} \right)_{IJ} = \frac{1}{2} \left[ \Omega_{ba} \tilde{p}^{ab} \right]_{IJ} = 0 .
\] (6.5)

Another way to derive these results is to consider the four-dimensional volume integral of \( \nabla_c (\tilde{G}^c \xi^d) \), taken over a region \( V \) whose boundary is the disjoint union of two hypersurfaces \( \Sigma \) and \( \Sigma' \). On applying the Leibniz rule and imposing the field equations, one finds the equation
\[
\int_{\Sigma'} \tilde{G}^c \xi^d d^3 x_c - \int_{\Sigma} \tilde{G}^c \xi^d d^3 x_c = \int_{V} \xi^d \nabla_c \tilde{G}^c d^4 x .
\] (6.6)

Again, the preservation of constraints is achieved by virtue of the contracted Bianchi identities. A similar argument can be applied to prove the preservation of (4.14) (cf. (6.4)).

In a Palatini formalism, one has also to consider the extra constraints corresponding to second-class constraints [13]. In the Hamiltonian formulation, second-class constraints arise since one starts from tetrads and connection one-forms, while the constraint analysis makes it convenient to replace the tetrads by the momenta. To prove equivalence between these two formulations it is then necessary to restrict the momenta so as to recover general relativity. One then finds six primary second-class constraints and six secondary second-class constraints [13]. This leads to the two degrees of freedom of real general relativity (page 59 of ref. [13]). A similar problem occurs in our analysis, if the action (2.1) is re-expressed in terms of the multimomenta. One has then to consider additional conditions obeyed by the multimomenta.
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In the light of (3.1) and (3.3), the second-class constraint equations found in ref. [13] take the form

\[ n_c n_d \epsilon^{IJKL} \tilde{P}^{ac}_{IJ} \tilde{P}^{bd}_{KL} = 0, \]  

\[ n_d n_f \epsilon^{IJKL} \tilde{P}^{cd}_I \tilde{P}^{af}_{MJ} D_c \left( n_h \tilde{P}^{bh} \right)_{KL} + \tilde{P}^{bf}_{MJ} D_c \left( n_h \tilde{P}^{ah} \right)_{KL} = 0, \]  

where \( n^a \) is the unit timelike normal to the spacelike hypersurface \( \Sigma \). Note that our constraint equations (4.14)-(4.15) and (6.7)-(6.8) are all expressed in terms of the multimomenta.

At a Lagrangian level, however, there are no primary constraints [4,20], since one deals with the pull-back on a manifold corresponding to the primary-constraint submanifold of the Hamiltonian formalism. The problem remains to derive the constraints (if any) corresponding to the secondary second-class constraints of the Hamiltonian formalism (of course, constraints cannot be divided into first- and second-class in a Lagrangian framework, since no Poisson brackets exist). They cannot be derived from the multimomentum map, which only applies to the analysis of constraints which are a counterpart of first-class constraints. In ref. [20], we were able to use the Gotay-Nester Lagrangian analysis in a finite-dimensional model to derive the Lagrangian counterpart of secondary second-class constraints. However, we do not yet know whether that technique can be extended to our formulation of general relativity.

For this purpose, we are currently investigating a more general set of equations obeyed by the multimomenta, i.e. (cf. [26,27])

\[ \epsilon_{abcd} \tilde{P}^{ab}_{IJ} \tilde{P}^{cd}_{KL} = \tilde{T}_{[IJKL]}, \]  

where \( \tilde{T} \) is a tensor density proportional to \( \epsilon_{IJKL} \). Eq. (6.9) admits, as a particular case, the condition of simplicity of the multimomenta (e.g. set \( I = K, J = L \) in (6.9)) in the abstract indices \( a, b \), as well as the analogous of (6.7) where the roles of space-time and internal indices are interchanged, i.e. \( n^J n^L \tilde{T}_{[IJKL]} = 0 \), where \( n^J \equiv e_a^J n^a \).
7. - Self-dual gravity.

The Hamiltonian formulation of self-dual gravity has received careful consideration in the current literature [13]. The corresponding formalism, however, is not manifestly covariant. To overcome this problem, Samuel [28], and, independently, Jacobson and Smolin [29,30], proposed a Lagrangian approach based on a self-dual action. The key idea is to take complex (co)tetrads on a real Lorentzian four-manifold [31], and then express the action functional in terms of the tetrad and of the self-dual part of the connection:

\[ +\omega \equiv \frac{1}{2} \left( \omega - i^*\omega \right). \]  \hspace{1cm} (7.1)

Remarkably, the complex self-dual action

\[ S_{SD} \equiv \frac{1}{2} \int_M d^4x \ e^a e^I e^b e^J \Omega_{abIJ}(+\omega) \]  \hspace{1cm} (7.2)

is related to the real Palatini action (2.1) by [32]

\[ S_{SD}[e, +\omega(e)] = \frac{1}{2} S_P[e, \omega(e)] - i \frac{1}{8} \int_M d^4x \ e^a e^I \epsilon^b e^{JKL} \Omega_a e^{JKL}(\omega(e)), \]  \hspace{1cm} (7.3)

where the second term in (7.3) vanishes for vanishing torsion, since then \( \Omega_a e^{JKL} = 0 \). Hence the resulting field equations for real general relativity are equivalent, and the corresponding constraints are first-class only (before imposing reality conditions) and polynomial (page 64 of ref. [13]).

The self-dual equations are hence obtained by replacing the full connection with the self-dual connection in secs. 2, 4, 5 and 6. Thus, on defining

\[ +\sigma^{ac} \equiv +\hat{\sigma}^{ac}_{IJ} \lambda^{IJ}, \]  \hspace{1cm} (7.4)

\[ +\rho^{ac} \equiv +\hat{\rho}^{ac}_{IJ} +\omega^{IJ}_b \xi^b, \]  \hspace{1cm} (7.5)
the constraint equations become (cf. (4.12)-(4.13))

\[
\int_{\Sigma} \partial_a \sigma^{ac} \, d^3 x_c - \int_{\Sigma} \lambda^IJ (D_a + \tilde{p}^{ac})_{IJ} \, d^3 x_c = 0
\]

(7.6)

\[
\int_{\Sigma} \partial_a \rho^{ac} \, d^3 x_c - \int_{\Sigma} \text{Tr} \left[ \tilde{p}^{ac} \Omega_{ad} (\omega) - \frac{1}{2} \tilde{p}^{ab} \Omega_{ab} (\omega) \delta^c_d \right] \xi^d \, d^3 x_c = 0
\]

(7.7)

In other words, in the most recent presentations [13,31], the equivalence between general relativity with a Palatini action and its self-dual version is proved. In this paper we follow the same argument and we find entirely analogous results, while the multisymplectic framework enables one to preserve covariance. Moreover, no extra constraints are necessary to recover the original content of the theory, when the action (7.2) is re-expressed in terms of the multimomenta (cf. [13]). As a last step, one has to impose suitable reality conditions to recover real general relativity [13,31]. Their formulation in the multisymplectic framework is not studied in our paper, and is a subject for further research.

We think one should emphasize again that, in the canonical-gravity approach to self-duality, one takes complex (co)tetrads on real space-time four-manifolds. By contrast, in other branches of modern relativity [33], one is interested in four-complex-dimensional complex-Riemannian manifolds. In such a case, no complex conjugation can be defined, since this map is not invariant under holomorphic coordinate transformations, and no four-real-dimensional sub-manifold can in general be singled out [33]. Hence the problem of reality conditions to recover real general relativity cannot even be addressed in the complex-Riemannian framework. The corresponding theory of self-duality involves the Weyl spinors, and is not equivalent to the model outlined in this section [33].

8. - Results and open problems.

This paper has studied Palatini and self-dual gravity theories by using tetrad formalism and multisymplectic techniques in Lorentzian four-manifolds (cf. [34]). Our results are as follows.
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(i) The first-class constraint equations of Palatini theory are given by (4.12)-(4.13). Interestingly, boundary terms occur, and they vanish under the sufficient conditions listed at the end of sect. 4.

(ii) The Hamiltonian constraint of general relativity is linear in the multimomenta, as shown in (5.3)-(5.5). Indeed, on studying Palatini formalism, Eq. (3) of chapter 4 of ref. [13] implicitly expresses this property. Our analysis, however, makes it more evident. The multisymplectic framework of sect. 2, and the multimomenta formalism of the following sections, seem to add evidence in favour of the Lagrangian point of view being able to supplement the Hamiltonian formalism to get a better understanding of the gravitational field. Moreover, the linearity in the multimomenta of all constraint equations may have far reaching consequences for the quantization program.

It should be emphasized that our analysis has not improved the understanding of the initial-value problem in general relativity. However, our equations are covariant in that they do not depend on a particular spacelike or null hypersurface, and they naturally lead to study the space of multimomenta. Hence they are a step towards a covariant formulation of relativistic theories of gravitation regarded as constrained systems. Within this framework, it may be interesting to analyze 2+1 gravity and theories with non-vanishing torsion.

A naturally occurring question is whether our classical analysis can be used to formulate an approach to non-perturbative quantum gravity which improves the results obtained within the more familiar Hamiltonian framework [13]. For this purpose, it appears necessary to get a better understanding of covariant Poisson brackets [35], and possibly of the geometric quantization program [36,37]. Hence we do not yet know the key features of the resulting quantum theory. However, the elegance of the mathematical formalism, and its wide range of applications at the classical level, make us feel that new perspectives in non-perturbative quantum gravity are in sight.
Since our paper is primarily addressed to physicists interested in general relativity, we limit ourselves to a very brief outline of some geometric ideas used in our investigation. An extended treatment may be found in ref. [23].

In our paper, the notation \( J^1(Y) \) means what follows [24]. Let \( X \) be a manifold and let \( Y \) be a fibre bundle having \( X \) as its base space, with projection map \( \pi_{XY} \). A fibre is then given by \( \pi_{XY}^{-1}(x) \), where \( x \in X \). Moreover, let \( \gamma : T_xX \to T_yY \) be a linear map between the tangent space to \( X \) at \( x \) and the tangent space to \( Y \) at \( y \in \pi_{XY}^{-1}(x) \). Now, given a point \( y \) belonging to the fibre \( Y_x \) through \( x \in X \), we consider all \( \gamma \) maps relative to \( y \in Y_x \). This leads to a fibre bundle \( J^1(Y) \) having the fibre bundle \( Y \) as its base space and fibres given by the \( \gamma \) maps. Such a \( J^1(Y) \) is called the one-jet bundle on \( Y \).

If \( \varphi^{(A)}(x^\mu) \) is a section of \( Y \), the tangent lift of \( \varphi^{(A)} \) to a section of \( J^1(Y) \) is denoted by \( j^{(1)}(\varphi) \). It is given by the map

\[
j^{(1)} : \varphi \rightarrow \left( x^\mu, \varphi^{(A)}(x^\mu), \frac{\partial \varphi^{(A)}(x^\mu)}{\partial x^\nu} \right). \tag{A.1}\]

The Legendre map, whose pull-back appears in Eq. (4.1), is a function

\[
\mathcal{F}\mathcal{L} : J^1(Y) \rightarrow \left[ J^1(Y) \right]^*, \tag{A.2}
\]

where \( \left[ J^1(Y) \right]^* \) is a fibre bundle having \( Y \) as its base space, and whose fibre through \( y \in Y \) is given by the affine maps on the elements of the fibre of \( J^1(Y) \), with coefficients in the bundle of \( n + 1 \) forms on \( X \) at \( x \). Hence \( \left[ J^1(Y) \right]^* \) is called the dual of the one-jet bundle \( J^1(Y) \). If, in local coordinates, \( J^1(Y) \) is described by \( \left( x^\mu, y^{(A)}, v^{(A)}_{(A)} \right) \), the expression of its dual in local coordinates is given by \( \left( x^\mu, y^{(A)}, p, p^{(A)}_{(A)} \right) \), where the definitions of momenta and of Legendre map yield

\[
p^{(A)}_{(A)} \equiv \frac{\partial L}{\partial v^{(A)}_{(A)}}, \tag{A.3}\]
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\[ p \equiv L - p_{(A)}^{\mu} v^{(A)}_{\mu}. \quad (A.4) \]

* * *

The authors are much indebted to Giuseppe Marmo for encouraging their work and teaching them all what they know about Lagrangian field theory. Our research was supported in part by the European Union under the Human Capital and Mobility Program. Gabriele Gionti is grateful to the military authorities of the Military District of Trieste, and in particular to the Head of the Military District, Col. Luciano Monaco, for making it possible for him to conclude this work during his military service.

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