The double hypergeometric series for the partition function of the 2D anisotropic Ising model

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Abstract. In 1944 Lars Onsager published the exact partition function of the ferromagnetic Ising model on the infinite square lattice in terms of a definite integral. Only in the literature of the last decade, however, has it been recast in terms of special functions. Until now all known formulas for the partition function in terms of special functions have been restricted to the important special case of the isotropic Ising model with symmetric couplings. Indeed, the anisotropic model is more challenging because there are two couplings and hence two reduced temperatures, one for each of the two axes of the square lattice. Hence, standard special functions of one variable are inadequate to the task. Here, we reformulate the partition function of the anisotropic Ising model in terms of the Kampé de Fériet function, which is a double hypergeometric function in two variables that is more general than the Appell hypergeometric functions. Finally, we present hypergeometric formulas for the generating function of multipolygons of given length on the infinite square lattice, for isotropic as well as anisotropic edge weights. For the isotropic case, the results allow easy calculation, to arbitrary order, of the celebrated series found by Cyril Domb.

Keywords: exact results, solvable lattice models, classical phase transitions

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1. Introduction

Lars Onsager solved the ferromagnetic Ising model with nearest neighbor interactions on the infinite square lattice in 1942 and published the solution in 1944 in a breakthrough paper [1]. Let $J_1$ and $J_1$ denote the couplings, $K_i = J_i/kT$ the reduced temperatures, and $Z(K_1, K_2)$ the partition function per site of the ferromagnetic Ising model on the square lattice. Onsager’s solution is then given by

$$\log(\frac{Z}{2}) = \frac{1}{2\pi^2} \int_0^\pi \log[\cosh 2K_1 \cosh 2K_2 - \sinh 2K_1 \cos \theta_1$$

$$- \sinh 2K_2 \cos \theta_2] d\theta_1 d\theta_2.$$ (1)

Furthermore, let

$$2\kappa_1 = \tanh 2K_1 \sech 2K_2$$

$$2\kappa_2 = \tanh 2K_2 \sech 2K_1.$$ (2)

Then, Onsager showed that

$$\log Z = \frac{1}{2} \log(4 \cosh 2K_1 \cosh 2K_2)$$

$$= \frac{1}{2\pi^2} \int_0^\pi \log[1 - 2\kappa_1 \cos \theta_1 - 2\kappa_2 \cos \theta_2] d\theta_1 d\theta_2$$ (3)

$$= -\frac{1}{2} \sum_{j+k>0}^{\infty} \frac{(2j + 2k - 1)!}{(j!)^2(k!)^2} \kappa_1^{2j} \kappa_2^{2k}. $$ (4)

This last equation (4) is equation (109b) in the 1944 paper [1]. It is easily obtained from equation (3) as follows: (i) first Taylor expanded the logarithm in the Mercator
The double hypergeometric series for the partition function of the 2D anisotropic Ising model series, (ii) then applied the binomial theorem to expand powers of $(2\kappa_1 \cos \theta_1 + 2\kappa_2 \cos \theta_2)$, and (iii) finally performed the double integral of the now-separable integrands as an iterated integral, term by term. For the case of the isotropic Ising model with symmetric couplings $J_1 = J_2 = J$, we can write $K_1 = K_2 = K$ and $\kappa_1 = \kappa_2 = \kappa$ and the above series simplifies considerably (see equation (109c) of [1]).

In the literature of the last decade, the partition function for the isotropic Ising model has been reformulated in terms of special functions (see below). However, the anisotropic model is much more challenging because there are two reduced temperatures—one for each coupling. Here we address this problem by more carefully studying the series (4). We thereby obtain a double hypergeometric reformulation of the partition function for the 2D anisotropic Ising model.

Hypergeometric functions belong to the family of special functions [2, 3]. Much insight and understanding can be gained when a definite integral or an infinite series is re-expressed in terms of special functions, because the latter have been widely studied and are often better understood than the former. The analogy with elementary functions can be helpful to make this point clear:

$$\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = -\log(x), \quad |x - 1| < 1$$

$$\int_{a}^{b} \frac{dx}{x} = \log \left( \frac{b}{a} \right), \quad \frac{b}{a} > 0.$$ 

The left-hand sides of the above are precisely equal to the right-hand sides. However, the appearance of the logarithm on the right gives us immediate insight and intuition that may not be so obvious in the expressions to the left. The same kind of insight can be gained from special functions. For further details on the motivation for expressing definite integrals and infinite series in terms of special functions, see [2–8].

In section 2 we review the definition of the prerequisite hypergeometric functions. In section 3 we state our main results, together with the proofs. Section 4 ends with discussion and conclusions.

2. The Kampé de Fériet double hypergeometric function

The generalized hypergeometric function $\, _pF_q$ of a variable $x$ has series expansion $\sum c_n x^n$ such that the ratio $c_{n+1}/c_n$ of successive coefficients is a ratio of polynomials in $n$. In other words, the ratio of coefficients is an a rational function of $n$. The degrees of the polynomials of the numerator and denominator are $p$ and $q+1$ respectively. Let the Pochhammer symbol $(x)_n$ denote the rising factorial for $n = 1, 2, 3 \ldots$ as follows:

$$\begin{align*}
(x)_0 &= 1, \\
(x)_n &= (x)_{n-1}(x + n - 1) \\
&= \frac{\Gamma(x + n)}{\Gamma(x)}.
\end{align*}$$

(5)
The double hypergeometric series for the partition function of the 2D anisotropic Ising model

Then \( pFq \) is defined as follows [2]:

\[
\begin{align*}
pFq[a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x] &= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n x^n}{(b_1)_n(b_2)_n \ldots (b_q)_n n!}. 
\end{align*}
\]  

(6)

For \( p < q + 1 \) the function is entire. In contrast, for \( p > q + 1 \) it has zero radius of convergence. For \( p = q + 1 \) the radius of convergence is 1.

The isotropic Ising model can be expressed in terms of \( pFq \) functions (see below). In contrast, the expression (4) for the anisotropic Ising model contains a power series in two different variables, hence we will need a more general and more powerful hypergeometric function.

The Kampé de Fériet function \( F^{p,q}_{r,s} \) is a double hypergeometric function of higher order in two variables [4, 9] that generalizes the \( pFq \) function. It is defined as

\[
\begin{align*}
F^{p,q}_{r,s}[a_1 \ldots a_p, c_1 \ldots c_r, b_1, b_1'; \ldots; b_q, b_q'; d_1, d_1'; \ldots; d_s, d_s'; x, y] &= \\
&= \sum_{m,n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_{m+n} \prod_{k=1}^{q} [(b_k)_{m}(b_k')_{n}] \prod_{k=1}^{r} (c_k)_{m+n} \prod_{k=1}^{s} [(d_k)_{m}(d_k')_{n}]}{m!n!} x^m y^n.
\end{align*}
\]

(7)

According to convention, if any of the quantities \( p \) or \( q \) in (6) or any of \( p, q, r, s \) in (7) are zero, then a dash is used to indicate the absence of the corresponding parameters, e.g.

\[
\exp(x) = \, \, \, qF0[-; x].
\]

We note in passing that for specific low values of the parameters \( p, q, r, s \), the Kampé de Fériet function can be reduced to the simpler Appell hypergeometric functions, of which there are four. For the particular values of the parameters \( p \) and \( q \) for the anisotropic Ising model (see below), this standard reduction to the Appell functions is not possible, so we do not further discuss this point here.

3. Results and proofs

Our first result is the natural continuation of previous results. In the 1970s, Glasser and Onsager obtained the following expression for the partition function of the 2D isotropic Ising model [7, 10]:

\[
\ln Z(K) = \ln(2 \cosh 2K) - \frac{1}{2} + \frac{1}{\pi} E(4\kappa) + \kappa^2 \, _3F_3\left[\begin{array}{c}
\frac{1}{2}, 1, \frac{3}{2}
\end{array}; 16\kappa^2
\right]
\]

where \( E(\cdot) \) denotes the complete elliptic integral of the second kind that can also be written as a \( _2F_1 \) function [2, 7]. In 2011, a simpler hypergeometric formula was discovered

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The double hypergeometric series for the partition function of the 2D anisotropic Ising model by Hucht, Grüneberg and Schmidt using Wolfram Mathematica [6]. In 2015 the identical expression was independently found and a derivation was given [7]:

\[
\ln Z = \ln(2 \cosh 2K) - \kappa_1^2 \frac{\Gamma_3}{2} \left[ \frac{1, 1, 3}{2, 2, 2} ; 4\kappa_2^2 \right].
\]  

(9)

The above two expressions are related via a hypergeometric identity (see equation (33) in [7]). In 2016, the exact low temperature series was expressed as an infinite series in terms of complete and partial Bell polynomials [8]. All of the above results are restricted to the isotropic case and do not apply to the anisotropic Ising model. Here we address this limitation and report an advance for the anisotropic case:

**Theorem 1.** Let \( \kappa_1 \) and \( \kappa_2 \) be given by (2). Then the partition function \( Z(K_1, K_2) \) of the anisotropic Ising model as defined in (1) is given by

\[
\log Z - \frac{1}{2} \log(4 \cosh 2K_1 \cosh 2K_2)
= -3 \kappa_1^2 \kappa_2^2 \left[ \frac{2, 5/2}{2} \right] \left[ \frac{1, 1}{2, 2, 2} ; 4\kappa_1^2, 4\kappa_2^2 \right]
- \frac{1}{2} \kappa_1^2 \kappa_2^2 \left[ \frac{1, 1, 3/2}{2, 2} ; 4\kappa_1^2 \right]
- \frac{1}{2} \kappa_1^2 \kappa_2^2 \left[ \frac{1, 1, 3/2}{2, 2} ; 4\kappa_2^2 \right].
\]  

(10)

**Corollary 1.**

\[
\log Z - \frac{1}{2} \log(4 \cosh 2K_1 \cosh 2K_2)
= -3 \kappa_1^2 \kappa_2^2 \left[ \frac{2, 5/2}{2} \right] \left[ \frac{1, 1}{2, 2, 2} ; 4\kappa_1^2, 4\kappa_2^2 \right]
+ \frac{1}{2} \left[ \log \left( \frac{1}{2} \left( 1 + \sqrt{1 - 4\kappa_1^2} \right) \right) + \log \left( \frac{1}{2} \left( 1 + \sqrt{1 - 4\kappa_2^2} \right) \right) \right].
\]  

(11)

To prove the above, we will use the following supporting lemmas:

**Lemma 1 (Basic identities for rising factorials).** Let \((a)_n\) be defined according to (5). Then,

\[
(n + 1)! = (2)_n, 
\]  

(12)

\[
(2n + 1)! = 2^{2n}(3/2)_n(1)_n, 
\]  

(15)

\[
(2j + 2k + 3)! = 6 \cdot 2^{2j+k}(2j+k)(5/2)_{j+k}. 
\]  

(16)

**Proof.** The first three above follow immediately from the definition of the rising factorial. The claim (15) can be proven by splitting the terms in the factorial into odd and

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even factors and dividing each factor by 2:

\[(2n + 1)! = [1] \times (3)(5)(7) \ldots (2n + 1) \times (2)(4)(6) \ldots (2n) = 2^{2n}(3/2)_n(1)_n. \quad (17)\]

The claim (16) is slightly more complicated but follows from the previous four claims:

\[
(2j + 2k + 3)! = (2j + 2k + 1)! \times (2j + 2k + 2)(2j + 2k + 3)
\]

\[
= 2^{2j+2k}(3/2)_{j+k}(1)_{j+k}(2j + 2k + 2)(2j + 2k + 3)
\]

\[
= 2^{2j+2k+2}(3/2)_{j+k}(1)_{j+k}(j + k + 1)(j + k + 3/2)
\]

\[
= \frac{2^{2j+2k+2}(3/2)_{j+k}(1)_{j+k}(2j + 2k + 3)(5/2)_{j+k}}{(1)_{j+k}(3/2)_{j+k}}
\]

\[
= 6 \cdot 2^{2(j+k)}(2j + 5/2)_{j+k}, \quad (20)
\]

which completes the proof. \(\square\)

Our next lemma deals with the infinite double series in (4). For convenience let us write the partition function in (4) as

\[
\log Z - \frac{1}{2} \log(4 \cosh 2K_1 \cosh 2K_2) = -\frac{1}{2} \Phi(\kappa_1, \kappa_2), \quad (21)
\]

with \(\Phi\) thus defined as

\[
\Phi(x_1, x_2) = \sum_{j+k > 0} \frac{(2j + 2k - 1)!}{(j!)^2(k!)^2} x_1^{2j} x_2^{2k}. \quad (22)
\]

Lemma 2.

\[
\Phi(x_1, x_2) = 6x_1^2 x_2^2 F_{0,2}^{2,1} \left[ \frac{2, 5/2}{2, 2; 2, 2} \right] 4x_1^2 x_2^2
\]

\[
+ x_1^3 F_{2}^{1,1} \left[ \frac{1, 3/2}{2, 2; 4x_1^2} \right] + x_2^3 F_{2}^{1,1} \left[ \frac{1, 3/2}{2, 2; 4x_2^2} \right]. \quad (23)
\]

Proof. If the sum in (22) started from \(j = k = 0\), then the task would be easier, but the sum excludes the point \(j = k = 0\). So we will rewrite \(\Phi\) as a sum of a double series for \(j > 0, k > 0\) and separate single series corresponding to \(j = 0\) and \(k = 0\):

\[
\Phi(x_1, x_2) = \sum_{j,k=1}^{\infty} \frac{(2j + 2k - 1)!}{(j!)^2(k!)^2} x_1^{2j} x_2^{2k} + \sum_{j=1}^{\infty} \frac{(2j - 1)!}{(j!)^2} (x_1^{2j} + x_2^{2j}). \quad (24)
\]
The double hypergeometric series for the partition function of the 2D anisotropic Ising model

We next change the summation indices to start at zero and invoke the identities in lemma 1:

\[
\Phi(x_1, x_2) = \sum_{j,k=0}^{\infty} \frac{(2(j+k)+3)!}{((j+1)!)^2((k+1)!)^2} x_1^{2(j+1)} x_2^{2(k+1)} \\
+ \sum_{j=0}^{\infty} \frac{(2j+1)!}{((j+1)!)^2} (x_1^{2(j+1)} + x_2^{2(j+1)}) \\
= x_1^2 x_2^2 \sum_{j,k=0}^{\infty} \frac{6 \cdot 2^{2(j+k)} (2j+k) (5/2)_{j+k} (2j)_{j+k}}{((2j+1)!)^2 (x_1^{2j} x_2^{2k})} x_1^2 x_2^2 \\
+ x_1^2 \sum_{j=0}^{\infty} \frac{2^{2j} (3/2)_j (1)_{j+k}}{((2j+1)!)^2} x_1^{2j}.
\]

(25)

Finally, from the definitions (6) of the generalized hypergeometric function and (7) of the Kampé de Fériet function, (25) becomes

\[
\Phi(x_1, x_2) = 6x_1^2 x_2^2 \left[ F_{0,2}^{2,1} \left[ \begin{array}{c} 2, 5/2 \\ 1, 1 \\ 2, 2 ; 2, 2 \\ 4x_1^2, 4x_2^2 \end{array} \right] \\
+ x_1^2 \left[ F_{2,3}^{1,3} \left[ \begin{array}{c} 1, 1, 3/2 \\ 2, 2 \\ 1, 3/2, 2 ; 4x_1^2 \end{array} \right] + x_2^2 \left[ F_{3,3}^{1,3} \left[ \begin{array}{c} 1, 1, 3/2 \\ 2, 2 \\ 1, 3/2, 2 ; 4x_2^2 \end{array} \right] \right] \right],
\]

(26)

which completes the proof.

Proof of theorem 1. The claim follows from using lemma 2 in (21).

Proof of corollary 1. To prove the claim (11) starting from theorem 1, it suffices to show that

\[
x_3 F_2 \left[ \begin{array}{c} 1, 1, 3/2 \\ 2, 2 \\ 2, 2 ; x \end{array} \right] = -4 \log \left( \frac{1}{2} (\sqrt{1-x} + 1) \right).
\]

(27)

We begin with the Mercator series and Newton’s generalized binomial theorem:

\[
\log(1 + \sqrt{1-x}) = -\sum_{n=1}^{\infty} \frac{(-1)^n(1-x)^{n/2}}{n} \\
= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{j=0}^{\infty} \binom{n/2}{j} (-x)^j \\
= -\sum_{j=0}^{\infty} (-x)^j \sum_{n=1}^{\infty} \binom{n/2}{j} \frac{(-1)^n}{n}.
\]

(28)

(29)

(30)

Next, we rewrite the binomial coefficient more conveniently as

\[
\binom{n/2}{j} = \frac{(n/2 + 1 - j)_j}{j!}.
\]

(31)
We then get
\[
\log(1 + \sqrt{1 - x}) = -\sum_{j=0}^{\infty} \frac{(-1)^j (n/2 + 1 - j)}{j!} \sum_{n=1}^{\infty} \frac{(-1)^n (n/2 + 1 - j)}{j!} \tag{32}
\]
\[
= -\sum_{j=0}^{\infty} \frac{(-1)^j (n/2 + 1 - j)}{j!} \sum_{n=1}^{\infty} \frac{(-1)^n (n/2 + 1 - j)}{j!} \tag{33}
\]
\[
= -\sum_{j=0}^{\infty} \frac{(-1)^j (n/2 + 1 - j)}{j!} \sum_{n=1}^{\infty} \frac{(-1)^n (n/2 + 1 - j)}{j!} \tag{34}
\]
\[
= -\sum_{j=0}^{\infty} \frac{(-1)^j (n/2 + 1 - j)}{j!} \sum_{n=1}^{\infty} \left[ \frac{(n + 1 - j)}{2n} - \frac{(n + 1/2 - j)}{2n - 1} \right]. \tag{35}
\]

The sum over \( n \) can be done after transforming the rising factorials to gamma functions:
\[
\sum_{n=1}^{\infty} \left[ \frac{(n + 1 - j)}{2n} - \frac{(n + 1/2 - j)}{2n - 1} \right] \tag{36}
\]
\[
= \left[ \frac{\Gamma(-j)}{2 \Gamma(1 - j)^2} - \frac{\sqrt{\pi} \Gamma(-j)}{2 \Gamma(1/2 - j) \Gamma(1 - j)} \right] \tag{37}
\]
\[
= \left[ \frac{\sqrt{\pi} \cos(\pi j) \Gamma(j + 1/2) - \sin(\pi j) \Gamma(j)}{2\pi j} \right] \tag{38}
\]

where in the last step we have used Euler’s reflection formula. The terms for \( j > 0 \) in (35) can be evaluated as usual, but the term for \( j = 0 \) is indeterminate due to the \( j \) in the denominator in (38), so it needs to be evaluated as a limit. We take the limit \( z \to j = 0 \) as follows:
\[
\lim_{z \to 0} \frac{\sqrt{\pi} \cos(\pi z) \Gamma(z + 1/2) - \sin(\pi z) \Gamma(z)}{2\pi z} = -\log 2. \tag{39}
\]

For \( j \neq 0 \) the term with \( \sin(\pi j) \) vanishes. We can then simplify the expression using Legendre’s duplication formula for the gamma function and applying lemma 1:
\[
\log(1 + \sqrt{1 - x}) = \log 2 - \sum_{j=1}^{\infty} \frac{1}{j!} \frac{\sqrt{\pi} \cos(\pi j) \Gamma\left(j + \frac{1}{2}\right)}{2\pi j} (-x)^j \tag{40}
\]
\[
\log\left(\frac{1}{2}(1 + \sqrt{1 - x})\right) = -\sum_{j=1}^{\infty} \frac{1}{j!} \frac{\sqrt{\pi} \Gamma\left(j + \frac{1}{2}\right)}{2\pi j} x^j \tag{41}
\]
\[
= -\sum_{j=1}^{\infty} \frac{2^{1-2j} \Gamma(2j)}{2j \Gamma(j) j!} x^j \tag{42}
\]
\[
= -x \sum_{j=0}^{\infty} \frac{2^{-2j-2} \Gamma(2j + 2)}{(j + 1)!^2} x^j \tag{43}
\]
Theorem 1 allows the generating function of multipolygons on the infinite square lattice to be reformulated in terms of hypergeometric functions. By multipolygon on the infinite square lattice is meant a connected or disconnected simple graph all of whose nodes have even degrees. Let $t_1$ be the weight of edges in the first lattice axis direction and $t_2$ be the edge weight for the other direction. For convenience, we will say that $t_1$ is the weight of edges of the first type (say, horizontal edges) and $t_2$ the weight of edges of the second type (say, vertical edges). Let $\Lambda_N(t_1,t_2)$ be the generating function for the number of multipolygons on a finite square lattice with $N = M^2$ sites, such that the coefficient of the term of degree $t_1^m t_2^n$ gives the number of multipolygons with $m$ edges of the first type and $n$ edges of the second type. The case $t_1 = t_2$, which we will call the isotropic case, has been widely studied [11]. The anisotropic case $t_1 \neq t_2$ has received somewhat less attention [12].

We define the generating function $\Lambda(t_1,t_2)$ for anisotropic multipolygons on the infinite square lattice in terms of a limit when the number of sites goes to infinity:

$$\Lambda(t_1,t_2) = \lim_{N \to \infty} \frac{[\Lambda_N(t_1,t_2)]^1}{N}. \tag{47}$$

The connection between the function $\Lambda(t_1,t_2) = 1 + t_1^2 t_2^2 + t_1^2 t_2^4 + t_1^4 t_2^4 \ldots$ and the low-temperature series for the partition function of the 2D Ising model is well known. Moreover, the self-dual property of the Ising model on the square lattice leads to a similar relation for the high-temperature series. Let

$$u_i = \tanh(K_i), \tag{48}$$
$$v_i = \exp(-2K_i), \tag{49}$$

for $i = 1, 2$ be high- and low-temperature variables respectively. Then the following are well known [11, 12]:

$$Z = 2 \cosh K_1 \cosh K_2 \Lambda(u_1, u_2) \tag{50}$$
$$Z = \exp(K_1) \exp(K_2) \Lambda(v_1, v_2). \tag{51}$$

From either of these two expressions, we can obtain the double hypergeometric formulation of $\Lambda(t_1,t_2)$:

**Theorem 2.** Let $\Lambda(t_1,t_2)$ be the generating function for multipolygons with anisotropic weights $t_1$ and $t_2$ for the two directions on the infinite square lattice, defined according
The double hypergeometric series for the partition function of the 2D anisotropic Ising model to (47). Let \( \Phi \) be given by (26). Then,

\[
\log \Lambda(t_1, t_2) = \frac{1}{2} \log \left[ \left( t_1^2 + 1 \right) \left( t_2^2 + 1 \right) \right] - \frac{1}{2} \Phi \left[ \frac{t_1 (1 - t_2^2)}{\left( t_1^2 + 1 \right) \left( t_2^2 + 1 \right)}, \frac{t_2 (1 - t_1^2)}{\left( t_1^2 + 1 \right) \left( t_2^2 + 1 \right)} \right].
\]

**Proof.** We can arrive at the claim through either the low- or the high-temperature formulations, due to the self-dual property. For completeness, we show both approaches. First note that

\[
\sqrt{4 \cosh 2 K_1 \cosh 2 K_2} = \sqrt{(u_1^2 + 1)(u_2^2 + 1)}.
\]

Similarly,

\[
\sqrt{4 \cosh 2 K_1 \cosh 2 K_2} \exp(2 K_1) \exp(2 K_2) = \sqrt{(v_1^2 + 1)(v_2^2 + 1)}.
\]

Next observe that \( \kappa_1 \) and \( \kappa_2 \) can be expressed in terms of the high- and low-temperature variables as

\[
\kappa_1 = \frac{u_1 (1 - u_2^2)}{(u_1^2 + 1)(u_2^2 + 1)} = \frac{v_2 (v_1^2 - 1)}{(v_1^2 + 1)(v_2^2 + 1)},
\]

\[
\kappa_2 = \frac{u_2 (1 - u_1^2)}{(u_2^2 + 1)(u_1^2 + 1)} = \frac{v_1 (v_2^2 - 1)}{(v_2^2 + 1)(v_1^2 + 1)}.
\]

The claim (52) follows from substituting these expressions for \( \kappa_i \) into (21) and then using either (50) and (53) for the high-temperature variable or else (51) with (54) for the low-temperature variable.

The above result simplifies considerably for the isotropic case \( t_1 = t_2 = t \). The next result is known to experts in the field, but for some reason never seems to have been published as such:

**Corollary 2.** Let \( \Lambda(t) \) denote the generating function \( \Lambda(t, t) \) of multipolygons on the infinite square lattice with isotropic weights \( t \). Then,

\[
\log \Lambda(t) = \log (t^2 + 1) - \frac{t^2(t^2 - 1)^2}{(t^2 + 1)^2} 4F3 \left[ \begin{array}{c} 1, 1, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2 \end{array} ; \frac{16t^2(t^2 - 1)^2}{(t^2 + 1)^4} \right].
\]

**Proof.** The claim follows from (9), (21) and theorem 2 with \( t_1 = t_2 = t \).

**4. Discussion and conclusion**

We first comment on a possible avenue for further research. There is a deep connection between the isotropic Ising model on the one hand and spanning trees on the infinite square lattice on the other hand. Guttmann and Rogers have defined a generating

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function that generalizes the spanning tree constant \[13\]. Subsequently, it was shown that this spanning tree generating function is related to the partition function of the isotropic Ising model in a precise way \[14\]. The connection is due to the fact that the same Mahler measure appears in both, which in turn is due to the random walk structure function for the square lattice. A natural question now arises. Let us assume that one allows the spanning tree generating function to have anisotropic weights in some suitably defined manner. Then, does the known connection between spanning trees and the isotropic Ising model generalize to the anisotropic model? We tentatively believe that the answer is yes, and hope that others will take interest in finding a conclusive answer.

We also comment on how the results were originally intuited prior to the discovery of the actual proofs. The results above were not first proven or even intuited using computer algebra systems, theorem-proving software, artificial intelligence, machine learning, and so forth. Wolfram Mathematica software, for example, cannot evaluate the integral in (3) and is unable to recognize the series for the definition \( \Phi \) in (22). In fact, the Kampé de Fériet function (7) is not part of the repertoire of special functions included in Mathematica (at the time of this writing). Instead, theorem 1 was the result of traditional mathematical detective work, starting from (4). After re-expressing the infinite double series in the form (22) of \( \Phi \), we calculated the ratio of successive terms of the series, in the two variables separately and also together. We thus saw that these ratios are rational functions of the degrees of the two variables—precisely the mathematical signature of hypergeometric series. It was then just a matter of finding the suitable hypergeometric function. We first looked at the Appell functions but they are not of high enough order. The Kampé de Fériet function was the next natural candidate. What made the task slightly more difficult is that both the Appell and Kampé de Fériet functions are double hypergeometric and therefore not found in the textbooks. For example, the Special Functions by Andrews et al. \[2\] and Special Functions: A Graduate Text by Beals and Wong \[3\] are both considered to be authoritative texts, but neither book deals with the Appell or Kampé de Fériet functions. Once the relevant definitions were found, however, the proofs came quite naturally.

Finally, we note that theorem 2 and corollary 2 allow very easy explicit evaluation of the multipolygon generating function to arbitrary order. For the anisotropic case we get from (52)

\[
\Lambda(t_1, t_2) = 1 + t_1^2 t_2^2 + t_1^4 t_2^4 + t_1^2 t_2^6 + 3 t_1^4 t_2^4 + t_1^2 t_2^8 + 6 t_1^4 t_2^6 + 6 t_1^6 t_2^8 + 6 t_1^8 t_2^{10} + 10 t_1^4 t_2^8 + 22 t_1^6 t_2^{10} + 10 t_1^8 t_2^{12} + t_1^{10} t_2^{14} + \cdots
\]

(58)

For the isotropic case, the formula (57) generates, to arbitrary order, the celebrated series for the generating function of multipolygons found by Domb \[15\] in 1949 and whose coefficients have been incorporated into the On-Line Encyclopedia of Integer Sequences (OEIS) \[16\]:

\[
\Lambda(t) = 1 + t^4 + 2t^6 + 5t^8 + 14t^{10} + 44t^{12} + 152t^{14} + 566t^{16} + 2234t^{18} + 9228t^{20} + 39520t^{22} + 174271t^{24} + 787246t^{26} + 3628992t^{28} + 17019374t^{30} + O(t^{32}).
\]

(59)

https://doi.org/10.1088/1742-5468/ac0f71
The double hypergeometric series for the partition function of the 2D anisotropic Ising model

In fact, equation (57) leads to very efficient computation of the series. At the time of this writing, OEIS lists the following Wolfram Mathematica code for generating 25 terms of the sequence A002890 [16]:

```mathematica
(*For 25 terms, a PC computation lasts less than half an hour*)
m = 48 (*max y exponent*);
coes = CoefficientList[Series[
  Log[(1 + y^2)^2 - 2*y*(1 - y^2)*
   (Cos[2*Pi*u] + Cos[2*Pi*v])], {y, 0, m}], y] // Rest;

nint[f_, {n_}] :=
  If[n == 2 || OddQ[n], 0, Print[n];
  Integrate[Integrate[f, {u, 0, 1}], {v, 0, 1}] ];
fy = MapIndexed[nint, coes].Table[y^k, {k, 1, m}];
CoefficientList[Series[Exp[fy/2], {y, 0, m}], y^2]
(*Jean-François Alcover, Mar 19 2013*)
```

The above calculation takes several minutes on a PC computer [16], as mentioned in the commented text on the first line of code. The reason that it takes so much time to calculate 25 terms is that the above code depends on explicit integration of Onsager’s formula to obtain the sequence. In contrast, equation (57) is ‘already integrated’, hence it should allow very much faster computation of the same sequence. In the following code below, we have used (57) and the Mathematica implementation of the \( pFq \) hypergeometric function to generate the identical coefficients to the code above:

```mathematica
CoefficientList[Series [E^(-((- t^-2 (-1 + t^-2)^2
  HypergeometricPFQ[{1, 1, 3/2, 3/2}, {2, 2, 2},
   (16 t^-2 (-1 + t^-2)^2)/(1 + t^-2)^4)/(1 + t^-2)^4) (1 + t^-2),
   {t, 0, 48}], t^-2]
(* GM Viswanathan 2021 *)
```

The latter code takes less than 0.1 s to generate 25 terms of the series, whereas the former code takes several minutes. The speedup is by a factor larger than \( 10^3 \).

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The double hypergeometric series for the partition function of the 2D anisotropic Ising model

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