LOCAL-IN-TIME STRONG SOLUTIONS OF THE COMPRESSIBLE FENE DUMBBELL MODEL OF WARNER TYPE.

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Abstract. We consider a dilute suspension of dumbbells joined by a finitely extendible nonlinear elastic (FENE) connector evolving under the classical Warner potential
\[ U(s) = -\frac{b^2}{2} \log(1 - \frac{2}{s^2}), \quad s \in [0, \frac{b}{2}]. \]
The solvent under consideration is modelled by the compressible Navier–Stokes system defined on the torus \( T^d \) with \( d = 2, 3 \) coupled with the Fokker–Planck equation (Kolmogorov forward equation) for the probability density function of the dumbbell configuration. We prove the existence of a local-in-time unique strong solution to the coupled system. Our result holds true independently of whether or not the centre-of-mass diffusion term is incorporated in the Fokker–Planck equation.

1. Introduction

The interactions of polymer molecules and fluids are of great importance in many areas of applied sciences. Polymeric fluid analysis also has various practical applications including performances in industrial and household items such as paints, lubricants, plastics and in the processing of food stuff, see [11]. A common mathematical model to describe the behaviour of such complex fluids is the FENE dumbbell model introduced by Warner [36], where the polymer molecules are idealized as a bead-spring chain with a finitely extensible nonlinear elastic (FENE) type spring potential. Two beads are connected by a spring which is represented by a vector \( q \in B \), where \( B \subset \mathbb{R}^d, d = 2, 3 \) is a ball with radius \( \sqrt{b} \) around the origin.

On the microscopic level, we describe the evolutionary changes in the distribution of the dumbbell configuration by the Fokker–Planck equation for the polymer density function \( \psi = \psi(t, x, q) \) (depending on time \( t \geq 0 \), spatial position \( x \in \mathbb{R}^d \) and the prolongation vector \( q \in B \) of the spring). On the macroscopic level, we consider a viscous isentropic fluid described by the compressible Navier–Stokes equations for the fluid velocity \( u = u(t, x) \) and density \( \rho = \rho(t, x) \). The beads of the dumbbells, which model the monomers that join to form a polymer chain, unsettle the flow field around the dumbbells once immersed in the fluid. These microscopic effects of the polymer molecules on the fluid motion are described by an elastic stress tensor \( T \). It is meant to describe the random movements of polymer chains/springs and can be modelled using the spring potential \( U \). The potential \( U \) is unbounded on the interval \( [0, b/2) \), \( U(0) = 0 \) and belongs to the class \( U \in C^1([0, b/2); \mathbb{R}_{\geq 0}) \). The elastic spring force \( F : B \subset \mathbb{R}^d \to \mathbb{R}^d \) and associated Maxwellian \( M \) are defined by

\[ F(q) = \nabla_q U \left( \frac{1}{2} |q|^2 \right) = U' \left( \frac{1}{2} |q|^2 \right) q \]

and

\[ M(q) = \frac{e^{-U(\frac{1}{2} |q|^2)}}{\int_B e^{-U(\frac{1}{2} |q|^2)} \, dq} \]

respectively such that \( \int_B M(q) \, dq = 1 \). Several of such models are proposed in the literature, see for example [11] Table 11.5-1]. We will concentrate our attention on the following spring potential introduced

\[ U(s) = -\frac{b^2}{2} \log(1 - \frac{2}{s^2}), \quad s \in [0, \frac{b}{2}]. \]
by Warner \[36\]

\[ U(s) = -\frac{b}{2} \log \left( 1 - \frac{2s}{b} \right), \quad s \in [0,b/2), \quad b > 2, \]

or equivalently, given \eqref{1.1}, the following \textit{elastic spring force}

\[ F(q) = \frac{b q}{b - |q|^2} \]

for \( q \in B \) so that in particular, \( |q|^2 \neq b \). The choice of the above potential or spring force reflects its conformity with physical applications unlike other unrealistic models such as the Hookean dumbbell and Hookean bead-spring models which assume arbitrary large extensions of their polymer chains, see for instance \[13, 25, 33\].

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The coupled system is now described by the compressible Navier–Stokes–Fokker-Planck system. We wish to find the fluid’s density \( \rho : (t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{T}^d \mapsto \rho(t,x) \in \mathbb{R}_{\geq 0} \), the fluid’s velocity field \( u : (t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{T}^d \mapsto u(t,x) \in \mathbb{R}^d \) and the probability density function \( \psi : (t,x,q) \in \mathbb{R}_{\geq 0} \times \mathcal{T}^d \times B \mapsto \psi(t,x,q) \in \mathbb{R}_{\geq 0} \) such that the equations

\[ \partial_t \rho + \text{div}_x (\rho u) = 0, \quad \text{in } \mathcal{T}^d, \]

\[ \partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x \rho \left( \left( \nabla_x u \right) \right) = \text{div}_x \mathcal{S}(\nabla_x u) + \text{div}_x \mathcal{T}(\psi) + \rho f, \]

\[ \partial_t \psi + \text{div}_x (\psi u) = \varepsilon \Delta_x \psi - \text{div}_q (\left( \nabla_x u \right) q \psi) + \frac{A_{11}}{4 \lambda} \text{div}_q \left( M \nabla_q \left( \frac{\psi}{M} \right) \right), \]

are satisfied pointwise a.e. in \( \mathbb{R}_{\geq 0} \times \mathcal{T}^d \times B \) subject to the following initial and boundary conditions

\[ \left( \rho, u \right)_{|t=0} = (\rho_0, u_0) \quad \text{in } \mathcal{T}^d, \]

\[ \psi_{|t=0} = \psi_0 \geq 0 \quad \text{in } \mathcal{T}^d \times B, \]

\[ \psi = 0 \quad \text{on } \mathbb{R}_{\geq 0} \times \mathcal{T}^d \times \partial B. \]

The momentum equation is complemented by Newton’s rheological law

\[ \mathcal{S}(\nabla_x u) = \mu^S \left( \nabla_x u + \nabla_x^T u - \frac{2}{d} \text{div}_x u \mathbb{I} \right) + \mu^B \text{div}_x u \mathbb{I}, \]

with shear viscosity \( \mu^S > 0 \) and bulk viscosity \( \mu^B \geq 0 \); as well as the adiabatic pressure law

\[ p(\rho) = a \rho^\gamma, \quad a > 0, \gamma > 1. \]

The parameter \( A_{11} > 0 \) in \( \mathbb{R}_{\geq 0} \) is the first component of the symmetric positive definite Rouse matrix or connectivity matrix \( A = A_{ij} \) for polymer chains, see \[34\]. The Rouse matrix describes the network
of monomers that combine to form a polymer chain. A detailed analysis of this matrix can be found in [20] [30]. The non-dimensional parameter \( \lambda > 0 \) is the Deborah number \( \text{De} \) and the parameter \( \varepsilon \geq 0 \) is the centre-of-mass diffusion coefficient. The Deborah number, introduced by Reiner [32], measures the time it takes a material to be restored to an equilibrium state following a disturbance and the time it takes to observe the aforementioned material. Most models in the literature usually ignore the centre-of-mass diffusion term \( \varepsilon \Delta_x \psi \) in (1.8). However, an alternative school of thought, see [3, 13, 35], gives justifications for the inclusion of this term. We do not want to enter this discussion. Instead, we provide an approach which is suitable for both cases.

Existence of a solution to the Fokker–Planck equation for a given solenoidal velocity field incorporating the center-of-mass diffusion term has been established by El-Kareh and Leal [18] independently of the Deborah number. The incompressible Navier–Stokes–Fokker–Planck system (when the time evolution of the fluid is described by the incompressible Navier–Stokes equations) for polymeric fluids including centre-of-mass diffusion (the case \( \varepsilon > 0 \)) has been studied considerably. See for example, the works by Barrett, Schwab & Süli [2], Barrett & Süli [3, 4, 5, 6, 7]. All these results derive global-in-time weak solutions for variations of the incompressible Navier–Stokes equation coupled with the Fokker–Planck equation. On the other hand, a unique local-in-time strong solution for the centre-of-mass system was first shown to exist by Renardy [33]. Unfortunately, [33] excludes the physically relevant FENE dumbbell models. The local theory was then revisited by Jourdain, Lelièvre & Le Bris [21] for the stochastic FENE model for the simple Couette flow and by E, Li & Zhang [17] who analysed the incompressible Navier–Stokes equation coupled with a system of SDEs describing the configuration of the spring (rather than the Fokker–Planck equation for their probability distribution). The corresponding deterministic system (the incompressible Navier–Stokes equations coupled with the Fokker–Planck equation) was studied by Li, Zhang & Zhang [24] and Zhang & Zhang [37]. Constantin proved the existence of Lyapunov functionals and smooth solutions in [14] and then derived global-in-time strong solution for the 2-D system in [15] together with Fefferman, Titi & Zarnescu.

If \( \varepsilon = 0 \), the analysis is significantly harder since (1.8) becomes a degenerate parabolic equation which behaves like an hyperbolic equation in the spacetime \((t, x)\)-variable. A global weak solution result to the incompressible Navier–Stokes–Fokker–Planck system for the FENE dumbbell model without centre-of-mass diffusion has recently been achieved in the seminal paper [29] by Masmoudi. The main difficulty is to pass to the limit in the term \( \text{div}_x \left( (\nabla u) q \psi \right) \) on the right-hand side of (1.8) which does not have any obvious compactness properties. Earlier global weak solution results when \( \varepsilon = 0 \) include the work by Lions & Masmoudi [25] for Oldroyd models, Lions & Masmoudi [20] who study the corotational case, and Otto & Tzavaras [31] who study weak solutions for the stationary system. Masmoudi [28] also constructed a local-in-time strong solution to the incompressible Navier–Stokes–Fokker–Planck system for the FENE dumbbell model without centre-of-mass diffusion in [28]. Furthermore, the solution is global near equilibrium, see also Kreml & Pokorný [22]. The corresponding result of [28] in Besov spaces is shown by Luo & Yin [27].

There are a few results in the compressible case. An extensive analysis in the 3D and 2D case has been performed by Barrett & Süli [8, 9] respectively with constant viscosity coefficients and by Feireisl, Lu & Süli [19] with variable viscosity coefficients. However, all of them are concerned with the existence of weak solutions for the problem with centre-of-mass diffusion. Related results include the work by Barrett & Süli [10] for the FENE-P model and by Barrett, Lu & Süli [11] for the Oldroyd-B model. We are not aware of any results on strong solutions. Also, there are no results for the compressible system without centre-of-mass diffusion. We close both gaps in this paper and prove the existence of a unique local-in-time strong solution to (1.6)–(1.8) under the assumption that \( \varepsilon \geq 0 \) (see Theorem 2.3 in the next section for the precise statement).

The strategy of our proof works as follows. Inspired by [12], we rewrite the compressible Navier–Stokes equations - by dividing the momentum equation (1.7) by the density - into a symmetric hyperbolic system perturbed by partial viscosity. For a given elastic potential, we derive higher order energy estimates for the velocity and density (see Section 3 in particular Theorems 3.2 and 3.3). On the other hand, we derive estimates for the polymer density function if the density and viscosity of the fluid are given (see Section 4 in particular Theorems 4.2 and 4.3). This is significantly more complicated than the analysis
of the incompressible Fokker–Planck equation in [28]. In particular, since our fluid is compressible, the solution of the Fokker–Planck equation is no longer transported by the Lagrangian flow as was the case in [28]. As such, it is of little use, if at all, to lift and study the Fokker–Planck equation from the Eulerian description to the Lagrangian description. We therefore solve the Fokker–Planck equation in its entirety in the Eulerian framework. To close the fluid and kinetic equations, we use a fixed-point argument. As is now widely known in contraction arguments, we are faced with the hitherto interesting twist where after showing boundedness of the fixed point map in the natural space, one is unable to show the contraction property in the same space. Indeed, we perform a difference estimate (for two solutions 

\[ u, v \] such that 

\[ \| \partial_x^\alpha (u - v) \|_{L^2} \lesssim_{s, d} \| u \|_{L^\infty_x} \| \nabla_x^s v \|_{L^2_x} + \| v \|_{L^\infty_x} \| \nabla_x v \|_{L^2_x}. \] (2.1) 

For any \( u, v \in W^{s, 2}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d) \), we have that 

\[ \| \partial_x^\alpha (u^2) \|_{L^2_x} \lesssim_{s, d} \| u \|_{L^\infty_x} \| \nabla_x^s u \|_{L^2_x}. \] (2.2) 

We will primarily deal with three independent variables: \( t \geq 0, x \in \mathbb{T}^d \) and \( q \in B \). Here, \( \mathbb{T}^d \) is the flat torus in \( \mathbb{R}^d \) with \( d \in \{2, 3\} \) and \( B := B(0, \sqrt{d}) \subset \mathbb{R}^d \) is a bounded open ball of radius \( \sqrt{d} \in \mathbb{R}_{>0} = (0, \infty) \) centred at \( 0 \in \mathbb{R}^d \). Time and space variables are represented by \( t \) and \( x \) respectively whereas \( q \in B \) is the elongation /conformation vector of a polymer molecule. The spacetime cylinder \( (0, t) \times \mathbb{T}^d \) will sometimes be denoted as \( Q_t \). For functions \( F \) and \( G \) and a variable \( p \), we write \( F \lesssim_p G \) if there exists a generic constant \( c > 0 \) and another such constant \( c(p) > 0 \) which now depends on \( p \) such that \( F \leq cG \) and \( F \leq c(p)G \) respectively.

By \( L^p_x := L^p(\mathbb{T}^d) \) and \( W^{s, p} := W^{s, p}(\mathbb{T}^d) \) for \( 1 \leq p \leq \infty \) and \( s \in \mathbb{N} \), we denote the standard Lebesgue and Sobolev spaces for functions with periodic boundary conditions. For a separable Banach space \( (X, \| \cdot \|_X) \), we denote by \( L^p(0, T; X) \) the space of Bochner-measurable functions \( u : (0, T) \to X \) such that \( \| u \|_X \in L^p(0, T) \). Finally, \( C([0, T]; X) \) is the set of continuous functions \( u : [0, T] \to X \).

2.2. Function spaces. Let us recall some Moser-type inequalities whose proofs can be found in the appendix of [22], let \( \alpha = (\alpha_1, \ldots, \alpha_d) \) be a \( d \)-tuple multi-index of nonnegative integers \( \alpha_i \) such that \( |\alpha| = \alpha_1 + \ldots + \alpha_d \leq s \) for a nonnegative integer \( s \geq 0 \).

- For any \( u, v \in W^{s, 2}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d) \), we have that \( \| \partial_x^\alpha (uv) \|_{L^2_x} \lesssim_{s, d} \| u \|_{L^\infty_x} \| \nabla_x^s v \|_{L^2_x} + \| v \|_{L^\infty_x} \| \nabla_x v \|_{L^2_x}. \) (2.1)

- For any \( u \in W^{s, 2}(\mathbb{T}^d) \), \( \nabla_x u \in L^\infty(\mathbb{T}^d) \) and any \( v \in W^{s-1, 2}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d) \), we have that \( \| \partial_x^\alpha (u^2) - u \partial_x^\alpha u \|_{L^2_x} \lesssim_{s, d} \| \nabla_x u \|_{L^\infty_x} \| \nabla_x^s u \|_{L^2_x} + \| v \|_{L^\infty_x} \| \nabla_x^s u \|_{L^2_x}. \) (2.2)

- If \( 1 \leq |\alpha| \leq 2 \), then for any \( u \in W^{s, 2}(\mathbb{T}^d) \cap C(\mathbb{T}^d) \) and an \( s \)-times continuously differentiable function \( F \) on an open neighbourhood of a compact set \( G = \text{range}[u] \), we also have

\[ \| \partial_x^\alpha F(u) \|_{L^2_x} \lesssim_{s, d} \| \partial_x F \|_{C^{s-1}(G)} \| u \|_{L^\infty_x}^{|\alpha|} \| \partial_x^\alpha u \|_{L^2_x}. \] (2.3)

We now define a couple of weighted spaces for functions depending on the conformation vector. For the real-valued Maxwellian \( M > 0 \), whose precise definition is given by (1.7), and \( p \geq 1 \), we denote by

\[ L^p_M(B) = \{ f \in L^1_{\text{loc}}(B) : \| f \|_{L^p_M(B)}^p < \infty \}, \quad H^1_M(B) = \{ f \in L^1_{\text{loc}}(B) : \| f \|_{H^1_M(B)}^2 < \infty \} \]

the Maxwellian-weighted \( L^p \) and \( H^1 \) spaces over \( B \) with norms

\[ \| f \|_{L^p_M(B)}^p := \int_B M \left| \int_B f \right|^p \text{d}q \quad \text{and} \quad \| f \|_{H^1_M(B)}^2 := \int_B M \left| \nabla_q \int_B f \text{d}q \right|^2 \text{d}q \]

respectively. The following crucial lemma is originally due to [28].
Lemma 2.1. For every $\delta > 0$, there exists $c_\delta > 0$ such that
\[
\left( \int_B \frac{|\psi|}{1+|q|} \, dq \right)^2 \leq \delta \int_B M \left| \nabla_q \frac{\psi}{M} \right|^2 \, dq + c_\delta \int_B M \left| \frac{\psi}{M} \right|^2 \, dq
\]
for all $\psi \in H^1_M(B)$.

Finally, for $s \in \mathbb{N}$ and $p \geq 1$ we define the spaces $W^{s,p}(\mathcal{T}; L^2_M(B))$ and $W^{s,p}(\mathcal{T}; H^1_M(B))$ respectively, as the set of measurable functions $f$ on $\mathcal{T} \times B$ for which the corresponding norm
\[
\|f\|_{W^{s,p}L^2_M} := \sum_{|\alpha| \leq s} \int_{\mathcal{T}} \left( \int_B M \left| \partial_x^{\alpha} \frac{f}{M} \right|^2 \, dq \right)^{\frac{p}{2}} \, dx,
\]
\[
\|f\|_{W^{s,p}H^1_M} := \sum_{|\alpha| \leq s} \left( \int_B M \left| \partial_x^{\alpha} \nabla_q \frac{f}{M} \right|^2 \, dq \right)^{\frac{p}{2}} \, dx
\]
is finite.

2.3. Main result. We start by giving a rigorous definition of solution to the coupled system (1.6)–(1.8).

Definition 2.2 (Strong solution). Let $s \in \mathbb{N}$ and $T > 0$. Assume there is $(\varrho_0, u_0) \in W^{s,2}(\mathcal{T}; \mathbb{R}^d) \times W^{s,2}(\mathcal{T}; L^2_M(B))$, $\psi_0 \in W^{s,2}(\mathcal{T}; L^2_M(B))$ and $f \in C([0,T];W^{s,2}(\mathcal{T}; \mathbb{R}^d))$. We call the triple $(\varrho, u, \psi)$ a strong solution to the system (1.6)–(1.8) with initial condition $(\varrho_0, u_0, \psi_0)$ in the interval $[0,T]$ provided the following holds.

(a) $\varrho$ satisfies
\[
\varrho \in C([0,T]; W^{s,2}(\mathcal{T})), \quad \varrho > 0;
\]
(b) $u$ satisfies
\[
u \in C([0,T]; W^{s,2}(\mathcal{T}; \mathbb{R}^d)) \cap L^2(0,T; W^{s+1,2}(\mathcal{T}; \mathbb{R}^d));
\]
(c) $\psi$ satisfies
\[
\psi \in C([0,T]; W^{s,2}(\mathcal{T}; L^2_M(B))) \cap L^2(0,T; W^{s,2}(\mathcal{T}; H^1_M(B)));
\]
(d) there holds for all $t \in [0,T]$ and all $(x,q) \in \mathcal{T} \times B$
\[
\varrho(t) = \varrho_0 - \int_0^t \text{div}_x(\varrho u) \, d\sigma,
\]
\[
\varrho u(t) = \varrho_0 u_0 - \int_0^t \left[ \text{div}_x (\varrho u \otimes u) + \nabla_x p(\varrho) - \text{div}_x S(\nabla_x u) - \text{div}_x T(\psi) - g f \right] \, d\sigma,
\]
\[
\psi(t) = \psi_0 - \int_0^t \left[ \text{div}_x (\varrho u \psi) - \varepsilon \Delta_x \psi \right] \, d\sigma + \frac{A_{11}}{4\lambda} \int_0^t \text{div}_x \left( M \nabla_q \left( \frac{\psi}{M} \right) \right) \, d\sigma.
\]

We remark that the assumed regularity of the solution $(\varrho, u, \psi)$ as well as the data together with the equations in (d) immediately imply that $\varrho$, $u$ and $\psi$ are differentiable in time so that indeed we obtain (1.6)–(1.8). Similarly, it is easy to see that $\varrho(0) = \varrho_0$, $u(0) = u_0$ and $\psi(0) = \psi_0$. Our main result, stated below in Theorem 2.3, establishes the existence of a unique local-in-time strong solution to the coupled kinetic-fluid system (1.6)–(1.11) satisfying the boundary and initial conditions (1.9)–(1.11).

Theorem 2.3. Let $s \in \mathbb{N}$ satisfy $s > \frac{d}{2} + 2$ and suppose that $\varepsilon \geq 0$. Assume there is $(\varrho_0, u_0) \in W^{s,2}(\mathcal{T}) \times W^{s,2}(\mathcal{T}; \mathbb{R}^d)$, $\varrho_0 > 0$, $\psi_0 \in W^{s,2}(\mathcal{T}; L^2_M(B))$ and $f \in C([0,T]; W^{s,2}(\mathcal{T}; \mathbb{R}^d))$. Then there is $T > 0$ such that there is a unique strong solution $(\varrho, u, \psi)$ to problem (1.6)–(1.8) in the sense of Definition 2.2 on the interval $[0,T]$ with the initial condition $(\varrho_0, u_0, \psi_0)$.

Remark 2.4. One can easily adapt the method used in this paper to solve the equivalent version of Theorem 2.3 on the whole space by imposing the following far field conditions
\[
\varrho \to \overline{\varrho}, \quad u \to 0, \quad \psi \to 0 \quad \text{as} \quad |x| \to \infty,
\]
where $\overline{\varrho} > 0$ is a constant. An extension to bounded domains complemented with the no-slip boundary conditions for the velocity field seems more involved. In particular, a Galerkin approximation cannot be used for the justification of the estimates.
We shall prove Theorem 2.3 in an indirect way by first rewriting (1.6)–(1.7) as a symmetric hyperbolic-parabolic system in terms of \((r, \mathbf{u})\) where \(r = r(\varrho)\) similar to [12]. This reformulation relies on the non-anticipation of possible vacuum region (i.e. \(\varrho \neq 0\)) in the construction of strong solutions to (1.6)–(1.7). Having transformed the original system (1.6)–(1.7) into a symmetric hyperbolic-parabolic one, we derive a priori estimates for \((r, \mathbf{u})\) under the assumption that the given data has enough regularity. To derive this reformulation, we first observe that formerly we have

\[
\partial_t (\varrho \mathbf{u}) = \partial_t \varrho \mathbf{u} + \varrho \partial_t \mathbf{u}, \quad \partial_t \varrho = -\text{div}_x(\varrho \mathbf{u}),
\]

due to (1.6). As such, we can rewrite the momentum balance equation (1.7) as

\[
\varrho \partial_t \mathbf{u} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p(\varrho) = \text{div}_x S(\nabla_x \mathbf{u}) + \text{div}_x T + \varrho \mathbf{f}.
\]

Now since the non appearance of a vacuum state is anticipated for the existence of a strong solution to the compressible system, we can further rewrite the above equation as

\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \frac{1}{\varrho} \nabla_x p(\varrho) = \frac{1}{\varrho} \text{div}_x S(\nabla_x \mathbf{u}) + \frac{1}{\varrho} \text{div}_x T(\psi) + \mathbf{f}.
\]

Finally, if we introduce

\[
r := \sqrt{\frac{2a \gamma}{\gamma - 1}} \varrho^{\frac{\gamma-1}{2}}, \quad D(r) := \frac{1}{\varrho(r)} = \left( \frac{\gamma - 1}{2a \gamma} \right)^{\frac{1}{\gamma-1}} r^{\frac{\gamma}{\gamma-1}},
\]

then the mass-momentum balance equations (1.6)–(1.7) together with the Fokker–Planck equation (1.8) become

\[
\partial_t r + \mathbf{u} \cdot \nabla_x r + \frac{\gamma - 1}{2} r \text{div}_x \mathbf{u} = 0, \tag{2.5}
\]

\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + r \nabla_x r = D(r) \text{div}_x S(\nabla_x \mathbf{u}) + D(r) \text{div}_x T(\psi) + \mathbf{f}, \tag{2.6}
\]

\[
\partial_t \psi + \text{div}_x (\mathbf{u} \psi) = \varepsilon \Delta_x \psi - \text{div}_q((\nabla_x \mathbf{u}) q \psi) + \frac{A_{11}}{4\lambda} \text{div}_q \left( M \nabla_q \left( \frac{\psi}{M} \right) \right), \tag{2.7}
\]

respectively. Setting \(r_0 := \sqrt{\frac{2a \gamma}{\gamma - 1}} \varrho_0^{\frac{\gamma-1}{2}}\) we can endow the above system with the following initial and boundary conditions

\[
(r, \mathbf{u})|_{t=0} = (r_0, \mathbf{u}_0) \quad \text{in } \mathcal{T}^d. \tag{2.8}
\]

\[
\psi|_{t=0} = \psi_0 \geq 0 \quad \text{in } \mathcal{T}^d \times B. \tag{2.9}
\]

\[
\psi = 0 \quad \text{on } \mathbb{R}^{d+1} \times \partial B. \tag{2.10}
\]

Analogous to Definition 2.2 we give in the following the definition of a strong solution to (2.5)–(2.7).

**Definition 2.5** (Strong solution). Let \(s \in \mathbb{N}\) and \(T > 0\). Assume there is \((r_0, \mathbf{u}_0) \in W^{s,2}(\mathcal{T}^d) \times W^{s,2}(\mathcal{T}^d; \mathbb{R}^d), \psi_0 \in W^{s,2}(\mathcal{T}^d; L^2_M(B))\) and \(\mathbf{f} \in C([0, T]; W^{s,2}(\mathcal{T}^d; \mathbb{R}^d))\). We call the triple \((r, \mathbf{u}, \psi)\) a strong solution to the system (2.5)–(2.7) with initial condition \((r_0, \mathbf{u}_0, \psi_0)\) in the interval \([0, T]\) provided the following holds.

(a) \(r\) satisfies

\[r \in C([0, T]; W^{s,2}(\mathcal{T}^d)), \quad r > 0;\]

(b) the velocity \(\mathbf{u}\) satisfies

\[
\mathbf{u} \in C([0, T]; W^{s,2}(\mathcal{T}^d; \mathbb{R}^d)) \cap L^2(0, T; W^{s+1,2}(\mathcal{T}^d; \mathbb{R}^d));
\]

(c) \(\psi\) satisfies

\[
\psi \in C([0, T]; W^{s,2}(\mathcal{T}^d; L^2_M(B))) \cap L^2(0, T; W^{s,2}(\mathcal{T}^d; H^1_M(B)));\]
where the constant \(c\).

**Theorem 3.2.** Let us start with a precise definition of the solution.

We start by solving the fluid system for a given elastic stress tensor and a given external force. That is, for a given \(T\) and \(f\), we want to solve the following system

\[
\begin{align*}
\partial_t r + u \cdot \nabla x r + \frac{\gamma - 1}{2} r \text{div}_x u &= 0, \\
\partial_t u + u \cdot \nabla x u + r \text{div}_x r &= D(r) \text{div}_x \mathbb{S}(\nabla x u) + D(r) \text{div}_x \mathbb{T} + f.
\end{align*}
\]

Let us start with a precise definition of the solution.

**Definition 3.1** (Strong solution). Let \(s \in \mathbb{N}\) and \(T > 0\). Assume there is \((r_0, u_0) \in W^{s,2}(T^{d}) \times W^{s,2}(T^{d}; \mathbb{R}^{d})\), \(T \in C([0, T]; W^{s,2}(T^{d}; \mathbb{R}^{d} x d))\), and \(f \in C([0, T]; W^{s,2}(T^{d}; \mathbb{R}^{d}))\). We call the tuple \((r, u)\) a strong solution to the approximate system \((3.1)-(3.2)\) with initial condition \((r_0, u_0)\) in the interval \([0, T]\) provided the holds.

(a) \(r\) satisfies
\[
r \in C([0, T]; W^{s,2}(T^{d})), \quad r > 0;
\]

(b) \(u\) satisfies
\[
u \in C([0, T]; W^{s,2}(T^{d}; \mathbb{R}^{d})) \cap L^{2}(0, T; W^{s+1,2}(T^{d}; \mathbb{R}^{d}));
\]

(c) there holds for all \(t \in [0, T]\)
\[
r(t) = r_0 - \int_0^t \left[ u \cdot \nabla x r + \frac{\gamma - 1}{2} r \text{div}_x u \right] \text{d}\sigma,
\]
\[
u(t) = u_0 - \int_0^t \left[ u \cdot \nabla x u + r \nabla x r \right] \text{d}\sigma + \int_0^t D(r) \text{div}_x \mathbb{S}(\nabla x u) \text{d}\sigma + \int_0^t D(r) \text{div}_x T \text{d}\sigma + \int_0^t f \text{d}\sigma.
\]

We now formulate our results concerning existence and uniqueness of solutions to \((3.1)-(3.2)\).

**Theorem 3.2.** Let \(s \in \mathbb{N}\) satisfy \(s > \frac{d}{2} + 2\) and \(T_0 > 0\). Assume there is \((r_0, u_0) \in W^{s,2}(T^{d}) \times W^{s,2}(T^{d}; \mathbb{R}^{d})\), \(r_0 > 0\), \(T \in C([0, T_0]; W^{s,2}(T^{d}; \mathbb{R}^{d} x d))\), and \(f \in C([0, T_0]; W^{s,2}(T^{d}; \mathbb{R}^{d}))\). Then there is \(T \in (0, T_0]\) such that there is a strong solution \((r, u)\) to problem \((3.1)-(3.2)\) in the sense of Definition 3.1 with the initial condition \((r_0, u_0)\) in the interval \([0, T]\). Moreover, we have the estimate
\[
\sup_{0 < t \leq T} \left\| (r, u) \right\|_{W^{s,2}}^2 + \int_0^T \left\| u \right\|_{W^{s+1,2}}^2 \text{d}t \leq c \left[ \left\| (r_0, u_0) \right\|_{W^{s,2}}^2 + \int_0^T \left( \left\| f \right\|_{W^{s,2}}^2 + \left\| T \right\|_{W^{s,2}}^2 \right) \text{d}t \right],
\]

where the constant \(c\) depends only on \(\gamma, s\) and \(d\).

The proof of Theorem 3.2 can be found in Section 3.2.
Theorem 3.3. Let the assumptions of Theorem 3.2 be satisfied. Then the solution from Theorem 3.2 is unique. Moreover, for $s' \in \mathbb{N}$ with $s' \leq s - 1$ we have the estimate

$$
\sup_{0 < t < T} \| (r^1(t) - r^2(t), u^1(t) - u^2(t)) \|^2_{W_{x'}^{s'+2,2}} + \int_0^T \| u^1 - u^2 \|^2_{W_{x'}^{s'+1,2}} \, dt
\leq c \exp \left( c \int_0^T \left( \| (r^1, r^2) \|^2_{W_{x'}^{s'+1,2}} + \| (u^1, u^2) \|^2_{W_{x'}^{s'+2,2}} + \| \mathbb{T}^2 \|^2_{W_{x'}^{s'+1,2}} + 1 \right) \, dt \right)
$$

(3.4)

for some $T = T(\sup_t \| u^1 \|_{s,2}, \sup_t \| u^2 \|_{s,2})$. Here $(r^1, u^1)$ and $(r^2, u^2)$ are two strong solutions to (3.1)–(3.9) with data $(r_0, u_0, f, \mathbb{T}^1)$ and $(r_0, u_0, f, \mathbb{T}^2)$ respectively.

The proof of Theorem 3.3 can be found in Section 3.3.

3.1. The Galerkin approximation. We start with a regularized system which contains cut-offs in the nonlinear term to render them globally Lipschitz. It can be solved by a standard Galerkin approximation. To be precise, we consider for $R \gg 1$ the system

$$
\partial_t r + \varphi_R(\| u \|_{2,\infty}) \left[ u \cdot \nabla_x r + \frac{2}{R^2} r \div_x u \right] = 0,
$$

(3.5)

$$
\partial_t u + \varphi_R(\| u \|_{2,\infty}) \left[ u \cdot \nabla_x u + r \nabla_x r \right] = \varphi_R(\| u \|_{2,\infty}) \div_x S(\nabla_x u)
+ \varphi_R(\| u \|_{2,\infty}) \div_x \mathbb{T} + f,
$$

(3.6)

$$
r(0) = r_0, \quad u(0) = u_0,
$$

(3.7)

where $\varphi_R : [0, \infty) \to [0, 1]$ are smooth cut-off functions satisfying

$$
\varphi_R(y) = \begin{cases} 
1, & 0 \leq y \leq R, \\
0, & R + 1 \leq y.
\end{cases}
$$

To begin with, observe that for any $u \in C([0, T]; W^{2,\infty}(T^d))$, the transport equation (3.5) admits a classical solution $r = r(u)$, uniquely determined by the initial datum $r_0$. In addition, for a certain universal constant $c$, we have the estimates

$$
\frac{1}{R} \exp (-cRt) \leq \exp (-cRt) \inf_{t \in T^d} r_0 \leq r(\cdot, t) \leq \exp (cRt) \sup_{t \in T^d} r_0 \leq R \exp (cRt),
$$

(3.8)

$$
|\nabla_x r(\cdot, t)| \leq \exp (cRt) |\nabla_x r_0| \leq R \exp (cRt), \quad t \in [0, T].
$$

Next, we consider the orthonormal basis $\{\psi_m\}_{m=1}^\infty$ of the space $L^2(T^d)$ formed by trigonometric functions and set

$$
X_n = \text{span} \{\psi_1, \ldots, \psi_n\}, \quad \text{with the associated projection} \ P_n : L^2 \to X_n.
$$

We look for approximate solutions $u^n$ of (3.6) belonging to $C([0, T]; X_n)$, satisfying

$$
\partial_t \int_{T^d} u^n \cdot \psi_i \, dx + \varphi_R(\| u^n \|_{2,\infty}) \int_{T^d} \left[ u^n \cdot \nabla_x u^n + r^n \nabla_x r^n \right] \cdot \psi_i \, dx
= \varphi_R(\| u^n \|_{2,\infty}) \int_{T^d} D(r^n) \div_x S(\nabla_x u^n) \cdot \psi_i \, dx
+ \varphi_R(\| u^n \|_{2,\infty}) \int_{T^d} D(r^n) \div_x \mathbb{T} \cdot \psi_i \, dx + \int_{T^d} f \cdot \psi \, dx, \quad i = 1, \ldots, n.
$$

(3.9)

$$
u^n(0) = u_0^n := P_n u_0.
$$

Here $r^n = r(u^n)$ is the solution to (3.6) with $u = u^n$. As all norms on $X_n$ are equivalent, solutions of (3.9) can be obtained in a standard way by means of the Banach fixed point argument. Specifically, we have to show that the mapping $u \mapsto \mathcal{T} u : X_n \to X_n$,
\[
\int_{T^d} \mathcal{F} u \cdot \psi_i \, dx = \int_{T^d} u_0 \cdot \psi_i \, dx - \int_0^t \varphi_R(\|u\|_{2,\infty}) \int_{T^d} \left[ u \cdot \nabla_x u + r[u] \nabla_x \cdot u \right] \cdot \psi_i \, dx \, dt
\]
\[
+ \int_0^t \varphi_R(\|u\|_{2,\infty}) \int_{T^d} D(r[u]) \nabla_x S(\nabla_x u) \cdot \psi_i \, dx \, dt
\]
\[
+ \int_0^t \varphi_R(\|u\|_{2,\infty}) \int_{T^d} D(r[u]) \nabla_x \cdot \psi_i \, dx \, dt + \int_0^t \int_{T^d} f \cdot \psi \, dx \, dt, \quad i = 1, \ldots, n.
\]

is a contraction on \(B = C([0, T^*]; X_n)\) for \(T^*\) sufficiently small. For \(r_1 = r[v_1], r_2 = r[v_2]\), we get
\[
d(r_1 - r_2) + v_1 \cdot \nabla_x (r_1 - r_2) \, dt - \frac{\gamma - 1}{2} \nabla_x v_1 (r_1 - r_2) \, dt
\]
\[
= -\nabla_x r_2 \cdot (v_1 - v_2) - \frac{\gamma - 1}{2} r_2 \nabla_x (v_1 - v_2) \, dt,
\]
where we have set
\[
v_1 = \varphi_R(\|u_1\|_{2,\infty})u_1, \quad v_2 = \varphi_R(\|u_2\|_{2,\infty})u_2.
\]

Consequently, we easily deduce that
\[
\sup_{0 \leq t \leq T^*} \left\| r[u_1] - r[u_2] \right\|^2_{L^2} \leq T^* c(n, R, T) \sup_{0 \leq t \leq T^*} \left\| u_1 - u_2 \right\|^2_{X_n}
\]
noting that \(r_1, r_2\) coincide at \(t = 0\) and that \(r_j, \nabla_x r_j\) are bounded by a constant depending on \(R\), recall (3.8). As a consequence of (3.8), (3.11) and the equivalence of norms on \(X_n\) we can show that the mapping \(\mathcal{F}\) satisfies the estimate
\[
\|\mathcal{F}u_1 - \mathcal{F}u_2\|^2_\mathcal{B} \leq T^* c(n, R, T) \|u_1 - u_2\|^2_\mathcal{B}.
\]

The inequality (3.12) shows that \(\mathcal{F}\) is a contraction provided we choose \(T^* > 0\) small enough. A solution \((r^n, u^n)\) to (3.5), (3.9) on the whole interval \([0, T]\) can be obtained by decomposing it into small subintervals and gluing the corresponding solutions together.

### 3.2. A priori estimates.

Consider the solution \((r^n, u^n)\) to (3.5), (3.9) constructed in the previous section. From the maximum principle, we gain the estimate
\[
r^n(t, x) \leq \sup_{x \in T^d} r_0(x) \exp\left( \int_0^t \varphi_R(\|u^n\|_{2,\infty}) \left| \nabla_x u^n \right|_{L^\infty} \, dt \right) \leq \sup_{x \in T^d} r_0(x) \exp(cR t) \leq K_R
\]
as well as
\[
r^n(t, x) \geq \inf_{x \in T^d} r_0(x) \exp\left( -\int_0^t \varphi_R(\|u^n\|_{2,\infty}) \left| \nabla_x u^n \right|_{L^\infty} \, dt \right) \geq \inf_{x \in T^d} r_0(x) \exp(-cR t) \geq \frac{1}{K_R}
\]
hold for all \((t, x) \in [0, T] \times T^d\) with a constant \(K_R\). Similarly, we have
\[
|\nabla_x r^n(t, x)| \leq \sup_{x \in T^d} |\nabla_x r_0(x)| \exp\left( \int_0^t \varphi_R(\|u^n\|_{2,\infty}) \|u^n\|_{W^{1,\infty}} \, dt \right)
\]
\[
\leq \sup_{x \in T^d} r_0(x) \exp(cR t) \leq K_R.
\]

Furthermore, per the definition (2.4) and the pressure law (1.13), it follows from (3.13), (3.15) that
\[
\|D(r^n)\|_{W^{1,\infty}} + \|D(r^n)^{-1}\|_{L^\infty} \leq cK_R.
\]

Additionally, it follows from (2.3) and (3.13) that
\[
\|D(r^n)\|_{W^{2,\infty}} \leq cK_R \|r^n\|_{W^{2,\infty}}.
\]

With the above preparation, we now derive uniform a priori bounds for the solution \((r^n, u^n)\) of the coupled mass and momentum balance equations (3.5), (3.9) given \((r_0, u_0, \mathcal{T}, f)\) as assumed in Theorem 3.2. In particular, we suppose \(s > 2 + \frac{d}{2}\). In what follows, the constants hidden in \(\lesssim\) depend on \(R\) only.
through the constant $K_R$ from (3.13)–(3.17) but are otherwise independent of $R$. We proceed by applying $\partial_x^n$ to (3.18). This yield
\[
\partial_t \partial_x^n u^n + \varphi_R(\|u^n\|_{2,\infty}) \left[ u^n \cdot \nabla_x \partial_x^n r^n + \frac{\gamma - 1}{2} r^n \text{div}_x \partial_x^n u^n \right] = J^n_1 + J^n_2
\]
(3.18)
in $Q_T$, where
\[
J^n_1 = \varphi_R(\|u^n\|_{2,\infty}) \left[ u^n \cdot \partial_x^n \nabla_x r^n - \partial_x^n (u^n \cdot \nabla_x r^n) \right],
\]
\[
J^n_2 = \varphi_R(\|u^n\|_{2,\infty}) \left[ \frac{\gamma - 1}{2} (r^n \partial_x^n \text{div}_x u^n - \partial_x^n (r^n \text{div}_x u^n)) \right],
\]
and
\[
\partial_t \partial_x^n u^n + \varphi_R(\|u^n\|_{2,\infty}) \left[ u^n \cdot \nabla_x \partial_x^n u^n + r^n \nabla_x \partial_x^n r^n - D(r^n) \text{div}_x \partial_x^n (\nabla_x \partial_x^n u^n) - D(r^n) \text{div}_x \partial_x^n \nabla_x \partial_x^n u^n \right]
= \partial_x^n f^n + I^n_1 + I^n_2 + I^n_3 + I^n_4
\]
in $X_n'$, where
\[
I^n_1 = \varphi_R(\|u^n\|_{2,\infty}) \left[ u^n \cdot \partial_x^n \nabla_x u^n - \partial_x^n (u^n \cdot \nabla_x u^n) \right],
\]
\[
I^n_2 = \varphi_R(\|u^n\|_{2,\infty}) \left[ r^n \partial_x^n \nabla_x r^n - \partial_x^n (r^n \nabla_x r^n) \right],
\]
\[
I^n_3 = \varphi_R(\|u^n\|_{2,\infty}) \left[ -D(r^n) \partial_x^n \text{div}_x \nabla_x u^n + \partial_x^n (D(r^n) \text{div}_x \nabla_x u^n) \right],
\]
\[
I^n_4 = \varphi_R(\|u^n\|_{2,\infty}) \left[ -D(r^n) \partial_x^n \text{div}_x \nabla_x u^n + \partial_x^n (D(r^n) \text{div}_x \nabla_x u^n) \right],
\]
respectively. We obtain from (3.22) the estimates
\[
\|J^n_1\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \left( \|
abla_x u^n\|_{L^\infty_T} \|\nabla_x r^n\|_{L^2_T} + \|
abla_x u^n\|_{L^2_T} \right),
\]
(3.20)
and
\[
\|J^n_2\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \left( \|
abla_x r^n\|_{L^\infty_T} \|\nabla_x u^n\|_{L^2_T} + \|\text{div}_x u^n\|_{L^2_T} \right),
\]
(3.21)
for the right-hand side of (3.18). We also obtain the estimates
\[
\|I^n_1\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \|\nabla_x u^n\|_{L^\infty_T} \|\nabla_x u^n\|_{L^2_T},
\]
\[
\|I^n_2\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \|\nabla_x r^n\|_{L^\infty_T} \|\nabla_x r^n\|_{L^2_T},
\]
\[
\|I^n_3\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \left( \|\nabla_x D(r^n)\|_{L^\infty_T} \|\nabla_x \nabla_x u^n\|_{L^2_T} + \|\text{div}_x \nabla_x u^n\|_{L^\infty_T} \|\nabla_x D(r^n)\|_{L^2_T} \right),
\]
(3.22)
for the indicated terms on the right-hand side of (3.19). For the last term in (3.19) we have
\[
\|I^n_4\|_{L^2_T} \lesssim \varphi_R(\|u^n\|_{2,\infty}) \left( \|\nabla_x D(r^n)\|_{L^\infty_T} \|\nabla_x u^n\|_{L^2_T} + \|\text{div}_x u^n\|_{L^\infty_T} \|\nabla_x D(r^n)\|_{L^2_T} \right).
\]
(3.23)
Now if we multiply (3.18) by $\partial_x^n r^n$ and integrate by parts when necessary, we gain
\[
\frac{d}{dt} \int_{T^d} |\partial_x^n r^n|^2 \, dx + (\gamma - 1) \varphi_R(\|u^n\|_{2,\infty}) \int_{T^d} r^n \text{div}_x \partial_x^n u^n \partial_x^n r^n \, dx
= \varphi_R(\|u^n\|_{2,\infty}) \int_{T^d} |\partial_x^n r^n|^2 \, dx + 2 \int_{T^d} (J^n_1 + J^n_2) \partial_x^n r^n \, dx.
\]
(3.24)
It follows from (3.20), (3.21) and (3.15) that,
\[
\varphi_R(\|u^n\|_{2,\infty}) \int_{T^d} |\partial_x^n r^n|^2 \, dx + 2 \int_{T^d} (J^n_1 + J^n_2) \partial_x^n r^n \, dx
= \int_{T^d} \partial_x^n r^n \left( \varphi_R(\|u^n\|_{2,\infty}) \partial_x^n r^n \text{div}_x u^n + 2(J^n_1 + J^n_2) \right) \, dx
\leq \varphi_R(\|u^n\|_{2,\infty}) \|\partial_x^n r^n\|_{L^2_T} \left( \|u^n\|_{W^{1,\infty}_x} \|r^n\|_{W^{\infty}_x} + \|r^n\|_{W^{\infty}_x} \|u^n\|_{W^{\infty}_x} \right)
\lesssim R \|(r^n, u^n)\|_{W^{\infty}_x}^2
\]
with a constant depending only on \( s, d, T \) and \( K_R \). By substituting (3.25) into (3.24), integrating in time and summing over \( \alpha \) such that \( |\alpha| \leq s \), we obtain for any \( t > 0 \),

\[
\|r^n(t)\|_{W^{s,2}_{x}}^2 + (\gamma - 1)\varphi_R(\|u^n\|_{2,\infty}) \sum_{|\alpha| \leq s} \int_{Q_t} r^n \text{div}_x \partial^n_x u^n \partial^n_x r^n \, dx \, ds \\
\lesssim \|r_0\|_{W^{s,2}_{x}}^2 + R \int_0^t \|(r^n, u^n)\|_{W^{s,2}_{x}}^2 \, ds.
\]  

(3.26)

Now if we test (3.17) by \( \partial^n_x u^n \in X_n \), integrate by parts where necessary and then integrate in time, we gain

\[
\|\partial^n_x u^n(t)\|_{L^2_x}^2 + 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} r^n \text{div}_x \partial^n_x u^n \partial^n_x r^n \, dx \, ds \\
+ 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} D(r^n) \mathcal{S}(\nabla_x \partial^n_x u^n) : \nabla_x \partial^n_x u^n \, dx \, ds \\
= \|\partial^n_x u^n\|_{L^2_x}^2 + 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} \|\partial^n_x u^n\|_{2,\infty}^2 \text{div}_x u^n \, dx \, ds \\
- 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} \nabla_x D(r^n) \partial^n_x \Sigma : \partial^n_x u^n \, dx \, ds \\
- 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} \nabla_x D(r^n) \mathcal{S}(\nabla_x \partial^n_x u^n) \cdot \partial^n_x u^n \, dx \, ds \\
+ 2\varphi_R(\|u^n\|_{2,\infty}) \int_{Q_t} \nabla_x r^n \cdot \partial^n_x r^n \partial^n_x u^n \, dx \, ds \\
+ 2\int_{Q_t} (\partial^n_x f + I^n_1 + I^n_2 + I^n_3 + I^n_4) \cdot \partial^n_x u^n \, dx \, ds \\
=: K^n_1 + \ldots + K^n_p.
\]

(3.27)

We can estimate the \( K^n_p \)'s as follows. Firstly, we have by (3.16)

\[
|K^n_2| \lesssim R \int_0^t \|\partial^n_x u^n\|_{L^2_x}^2 \, ds.
\]  

(3.28)

Furthermore, we can estimate \( K^n_3 \) using (3.13) by

\[
|K^n_3| \lesssim \int_0^t \|\nabla_x \partial^n_x u^n\|_{W^{s,2}_{x}} \, ds \lesssim \int_0^t \|\nabla_x \partial^n_x u^n\|_{W^{s,2}_{x}}^2 \, ds + \int_0^t \|(r^n, u^n)\|_{W^{s,2}_{x}}^2 \, ds.
\]  

(3.29)

Also, by (3.15) we have that

\[
|K^n_4| \leq cK_R \int_{Q_t} |\nabla_x \partial^n_x u^n| \, dx \, ds \leq c\delta K_R^2 \int_0^t \|\nabla_x \partial^n_x u^n\|_{W^{s,2}_{x}}^2 \, ds + \delta \int_0^t \|u^n\|_{W^{s+1,2}_{x}}^2 \, ds.
\]  

(3.30)

where \( \delta > 0 \) is arbitrary. By using (3.10),

\[
|K^n_5| \leq cK_R \int_{Q_t} |\nabla_x \partial^n_x u^n| \, dx \, ds \\
\leq \delta \int_0^t \|u^n\|_{W^{s+1,2}_{x}}^2 \, ds + c\delta K_R^2 \int_0^t \|u^n\|_{W^{s,2}_{x}}^2 \, ds,
\]  

(3.31)

whereas by (3.15)

\[
|K^n_6| \lesssim \int_0^t \left( \|\partial^n_x r^n\|_{L^2_x}^2 + \|\partial^n_x u^n\|_{L^2_x}^2 \right) \|\nabla_x r^n\|_{L^\infty_x} \, ds \lesssim \int_0^t \|(r^n, u^n)\|_{W^{s,2}_{x}}^2 \, ds.
\]  

(3.32)
Finally, we use (3.22)–(3.24) to obtain the following estimate

\[
|K_p^n| \lesssim \int_0^t \varphi_R(\|u^n\|_{L^2}) \|u^n\|_{W^{2,2}} \left( \|f\|_{W^{2,2}} + \|I_1^n\|_{L^2} + \|I_2^n\|_{L^2} + \|I_3^n\|_{L^2} + \|I_4^n\|_{L^2} \right) \, d\sigma
\ lesssim \int_0^t \varphi_R(\|u^n\|_{L^2}) \|u^n\|_{W^{1,2}} \left( \|f\|_{W^{2,2}} + \|u^n\|_{W^{1,2}} + \|u^n\|_{W^{1,2}} + \|\nabla D(r^n)\|_{L^2} \right) \, d\sigma
\]

(3.33)

It therefore follows from (3.13)–(3.17) that for any $\delta > 0$, we can find $c_\delta = c_\delta(s, d, K_R) > 0$ such that

\[
|K_p^n| \leq c_\delta \int_0^t \|f\|_{W^{2,2}}^2 \, d\sigma + c_\delta R^2 \int_0^t \|r^n\|_{W^{2,2}}^2 \, d\sigma + c_\delta \int_0^t \|T\|_{W^{2,2}}^2 \, d\sigma + \delta \int_0^t \|u^n\|_{W^{1,2}}^2 \, d\sigma.
\]

(3.34)

If we now substitute (3.28)–(3.34) into (3.27) with a small enough choice of $\delta$, sum over $\alpha$ such that $|\alpha| \leq s$ and use again (3.13), we obtain

\[
\|u^n(t)\|_{W^{2,2}}^2 - 2\varphi_R(\|u^n\|_{L^2}) \sum_{|\alpha| \leq s} \int_0^t r^n \nabla \cdot \partial_x^\alpha u^n \partial^2_x r^n \, dx \, d\sigma + \int_0^t \|u^n\|_{W^{1,2}}^2 \, d\sigma
\]

(3.35)

\[
\lesssim \|u_0^n\|_{W^{2,2}}^2 + \int_0^t \left( \|f\|_{W^{2,2}}^2 + \|T\|_{W^{2,2}}^2 \right) \, d\sigma + R^2 \int_0^t \|r^n, u^n\|_{W^{1,2}}^2 \, d\sigma
\]

Now if we multiply (3.35) by $\frac{3\alpha}{2}$ (which is always positive) and sum the resulting inequality with (3.20), we obtain

\[
\|r^n(t), u^n(t)\|_{W^{2,2}}^2 + \int_0^t \|u^n\|_{W^{1,2}}^2 \, d\sigma
\]

(3.36)

\[
\lesssim \|r^n_0, u^n_0\|_{W^{2,2}}^2 + \int_0^t \left( \|f\|_{W^{2,2}}^2 + \|T\|_{W^{2,2}}^2 \right) \, d\sigma + R^2 \int_0^t \|r^n, u^n\|_{W^{1,2}}^2 \, d\sigma
\]

with a constant $c$ depending only on $\gamma, \delta, s, d$ and $K_R$. Now since

\[
g_1(t) := c \left[ \|r^n_0, u^n_0\|_{W^{2,2}}^2 + \int_0^t \left( \|f\|_{W^{2,2}}^2 + \|T\|_{W^{2,2}}^2 \right) \, d\sigma \right]
\]

is non-decreasing in $t$, it follows from Gronwall’s lemma that

\[
\sup_{t \in (0, T)} \|r^n, u^n\|_{W^{2,2}}^2 + \int_0^T \|u^n\|_{W^{1,2}}^2 \, dt \leq g_1(T) \exp(cR^2T).
\]

(3.37)

for any given $T > 0$. Recall that the constant $c$ in $g_1$ only depends on $R$ via the constant $K_R$ from (3.13)–(3.17). It is now standard to pass to the limit in (3.5), (3.9) in order to obtain a global-in-time solution $\left( r_R, u_R \right)$ to (3.1), (3.9) with

\[
r_R \in C(0, T; W^{s/2}(\mathbb{R}^d)), \quad r > 0, \quad u_R \in C(0, T; W^{s/2}(\mathbb{R}^d)) \cap L^2(0, T; W^{s+1/2}(\mathbb{R}^d)).
\]

For any fixed $R \gg 1$ we can find $T_0 \ll 1$ such that

\[
\|u_R(t)\|_{W^{2,2}} \leq R
\]

(3.38)

for all $t \leq T_0$. Consequently, $\left( r_R, u_R \right)$ is a solution to (3.1)–(3.2) as the cut-offs are not seen. Moreover, we can assume (by further decreasing $T_0$ if necessary) that the constant $K_R$ in the limit version of (3.13)–(3.17) does not depend on $R$. This implies the required a priori estimate and finishes the proof of Theorem 3.2 (with a constant independent of $R$).
3.3. Difference estimate for fluid system. The purpose of this subsection is to show uniqueness of solutions to (3.1)–(3.2) and hence prove Theorem 3.3. Based on the estimates from the previous subsection (that is Theorem 3.2) the constructed solutions possess enough regularity provided the existence interval is chosen small enough. Let \((r^i, u^i), i = 1, 2\) be two solutions of (3.1)–(3.2) with data \((r_0, u_0, f, T)\) and \((r_0, u_0, f, \mathbb{T}^2)\) respectively defined in an interval \([0, T_0]\). We set

\[
\mathcal{R} := \max \left\{ \sup_{0 < t < T_0} \|u^1\|_{s, 2}, \sup_{0 < t < T_0} \|u^2\|_{s, 2} \right\}.
\]  

We obtain versions of (3.13)–(3.17) for \(r^1\) and \(r^2\) with a constant \(K_{\mathcal{R}}\). In particular, we have for \(r = r^1, r^2\)

\[
\frac{1}{K_{\mathcal{R}}} \leq \inf_{x \in \mathbb{T}^d} r_0(x) \exp(-c_{\mathcal{R}} t) \leq r(t, x) \leq \sup_{x \in \mathbb{T}^d} r_0(x) \exp(c_{\mathcal{R}} t) \leq K_{\mathcal{R}},
\]  

\[
|\nabla_x r(t, x)| \leq \sup_{x \in \mathbb{T}^d} r_0(x) \exp(c_{\mathcal{R}} t) \leq K_{\mathcal{R}},
\]

\[
\|D(r)\|_{W^{1, \infty}_{x, t}} + \|D(r)^{-1}\|_{L^\infty_{x, t}} \leq c K_{\mathcal{R}}, \quad \|D(r)\|_{W^{s, 2}_{x, t}} \leq c K_{\mathcal{R}} \|r\|_{W^{s, 2}_{x, t}}.
\]

In the following we derive estimates for the difference of \(r^1\) and \(r^2\). As in Section the constants \(K_{\mathcal{R}}\) only depend on \(\mathcal{R}\) via \(K_{\mathcal{R}}\) but are independent of \(r^1\).

Set \(r^{12} = r^1 - r^2, u^{12} = u^1 - u^2, \mathbb{T}^{12} = \mathbb{T}^1 - \mathbb{T}^2\) so that \((r^{12}, u^{12})\) satisfies

\[
\partial_t r^{12} + u^{12} \cdot \nabla_x r^{12} + u^{12} \cdot \nabla_x u^{12} + \frac{\gamma - 1}{2} r^{12} \text{div}_x u^{12} + \frac{\gamma - 1}{2} r^{12} \text{div}_x u^{2} = 0,
\]

\[
\partial_t u^{12} + u^1 \cdot \nabla_x u^{12} + u^{12} \cdot \nabla_x u^{2} + r^1 \nabla_x r^{12} + r^{12} \nabla_x r^2 = D(r) \text{div}_x S(\nabla_x u^{12}) + (D(r^1) - D(r^2)) \text{div}_x S(\nabla_x u^2) + D(r^1) \text{div}_x T^{12} + (D(r^1) - D(r^2)) \text{div}_x \mathbb{T}^2,
\]

subject to the following initial condition

\[
(r^{12}, u^{12})|_{t=0} = (r_0^{12}, u_0^{12}) = (0, 0) \quad \text{in } \mathbb{T}^d.
\]

Let the multi-index \(\alpha\) satisfy

\[
|\alpha| \leq s' \leq s - 1
\]

with \(s > \frac{d}{2} + 2\). By applying \(\partial_x^\alpha\) to (3.42), we obtain

\[
\partial_t \partial_x^\alpha r^{12} + u^{12} \cdot \nabla_x \partial_x^\alpha r^{12} + u^{12} \cdot \nabla_x \partial_x^\alpha u^{12} + \frac{\gamma - 1}{2} r^{12} \text{div}_x \partial_x^\alpha u^{12} + \frac{\gamma - 1}{2} r^{12} \text{div}_x \partial_x^\alpha u^2 = J_1^\alpha + J_2^\alpha + J_3^\alpha + J_4^\alpha
\]

such that

\[
\|J_1^\alpha\|_{L^2_T} \lesssim \|\nabla_x u^{12}\|_{L^\infty_T} \|\nabla_x r^{12}\|_{L^2_T} + \|\nabla_x r^{12}\|_{L^2_T} \|\nabla_x u^{12}\|_{L^2_T},
\]

\[
\|J_2^\alpha\|_{L^2_T} \lesssim \|\nabla_x u^{12}\|_{L^\infty_T} \|\nabla_x r^{12}\|_{L^2_T} + \|\nabla_x u^{12}\|_{L^2_T} \|\nabla_x u^{2}\|_{L^2_T},
\]

\[
\|J_3^\alpha\|_{L^2_T} \lesssim \|\nabla_x r^{12}\|_{L^2_T} \|\nabla_x u^{12}\|_{L^2_T} + \|\text{div}_x u^{12}\|_{L^2_T} \|\nabla_x r^{12}\|_{L^2_T},
\]

\[
\|J_4^\alpha\|_{L^2_T} \lesssim \|\nabla_x r^{12}\|_{L^2_T} \|\nabla_x u^{12}\|_{L^2_T} + \|\text{div}_x u^{12}\|_{L^2_T} \|\nabla_x r^{12}\|_{L^2_T}.
\]

On the other hand, the application of \(\partial_x^\alpha\) to (3.44) yields

\[
\partial_t \partial_x^\alpha u^{12} + u^1 \cdot \nabla_x \partial_x^\alpha u^{12} + u^{12} \cdot \nabla_x \partial_x^\alpha u^2 + r^1 \nabla_x \partial_x^\alpha r^{12} + r^{12} \nabla_x \partial_x^\alpha r^2 = D(r^1) \text{div}_x S(\nabla_x \partial_x^\alpha u^{12}) + (D(r^1) - D(r^2)) \text{div}_x S(\nabla_x \partial_x^\alpha u^2) + D(r^1) \text{div}_x \partial_x^\alpha T^{12} + (D(r^1) - D(r^2)) \text{div}_x \partial_x^\alpha \mathbb{T}^2 + I_1^\alpha + I_2^\alpha + I_3^\alpha + I_4^\alpha + I_5^\alpha + I_6^\alpha
\]
where
\begin{align}
\|I_0^n\|_{L^2_x} & \lesssim \|\nabla_x u^1\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2_x} + \|\nabla_x u^{12}\|_{L^2} \|\nabla^s_x u^1\|_{L^2}, \quad (3.52) \\
\|I_0^n\|_{L^2_x} & \lesssim \|\nabla_x u^{12}\|_{L^2} \|\nabla^s_x u^1\|_{L^2} + \|\nabla_x u^1\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2}, \quad (3.53) \\
\|I_0^n\|_{L^2_x} & \lesssim \|\nabla_x u^{12}\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2} + \|\nabla_x u^1\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2}, \quad (3.54) \\
\|I_0^n\|_{L^2_x} & \lesssim \|\nabla^s_x u^{12}\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2} + \|\nabla_x u^{12}\|_{L^2} \|\nabla^s_x u^1\|_{L^2}, \quad (3.55)
\end{align}

and
\begin{align}
\|I_3^n\|_{L^2_x} & \lesssim \|\nabla_x (D(r^1))\|_{L^2} \|\nabla^s_x \mathsf{S}(\nabla_x u^{12})\|_{L^2} + \|\text{div}_x \mathsf{S}(\nabla_x u^{12})\|_{L^2} \|\nabla^s_x D(r^1)\|_{L^2}, \quad (3.56) \\
\|I_4^n\|_{L^2_x} & \lesssim \|\nabla_x (D(r^1) - D(r^2))\|_{L^2} \|\nabla^s_x \mathsf{S}(\nabla_x u^2)\|_{L^2} + \|\text{div}_x \mathsf{S}(\nabla_x u^2)\|_{L^2} \|\nabla^s_x (D(r^1) - D(r^2))\|_{L^2}, \quad (3.57) \\
\|I_5^n\|_{L^2_x} & \lesssim \|\nabla_x (D(r^1))\|_{L^2} \|\nabla^s_x \mathsf{S}(\nabla_x u^{12})\|_{L^2} + \|\text{div}_x \mathsf{S}(\nabla_x u^{12})\|_{L^2} \|\nabla^s_x (D(r^1) - D(r^2))\|_{L^2}, \quad (3.58) \\
\|I_6^n\|_{L^2_x} & \lesssim \|\nabla_x (D(r^1) - D(r^2))\|_{L^2} \|\nabla^s_x \mathsf{S}(\nabla_x u^2)\|_{L^2} + \|\text{div}_x \mathsf{S}(\nabla_x u^2)\|_{L^2} \|\nabla^s_x (D(r^1) - D(r^2))\|_{L^2}. \quad (3.59)
\end{align}

Testing (3.47) with \(\partial_x^r r^{12}\) yields
\begin{align}
\frac{d}{dt} \int_{\Omega^1} |\partial_x^r r^{12}|^2 \, dx + (\gamma - 1) \int_{\Omega^1} r^1 \text{div}_x \partial_x^r u^{12} \partial_x^r r^{12} \, dx = \int_{\Omega^1} |\partial_x^r r^{12}|^2 \text{div}_x u^2 \, dx \\
- (\gamma - 1) \int_{\Omega^1} r^{12} \text{div}_x \partial_x^r u^2 \partial_x^r r^{12} \, dx - 2 \int_{\Omega^1} \partial_x^r r^{12} \partial_x^r u^1 \cdot u^{12} \, dx + 2 \int_{\Omega^1} (J_1^0 + J_2^0 + J_3^0) \cdot \partial_x^r r^{12} \, dx. \quad (3.60)
\end{align}

Now note that from Hölder’s inequality, (3.48), Young’s inequality and Sobolev embedding \(W_x^{s',2} \hookrightarrow W_x^{1,\infty}\), we can obtain the following estimate
\begin{align}
\int_{\Omega^1} J_1^0 \cdot \partial_x^r r^{12} \, dx \, d\sigma \leq \int_0^t \|J_1^0\|_{L^2_x} \|\partial_x^r r^{12}\|_{L^2_x} \, d\sigma \lesssim \int_0^t \left( \|\nabla_x u^{12}\|_{L^2} \|\nabla^s_x u^{12}\|_{L^2_x} \|\partial_x^r r^{12}\|_{L^2_x} \right) \left( \|\nabla x u^{12}\|_{L^2_x} \|\nabla^s_x u^{12}\|_{L^2_x} \|\partial_x^r r^{12}\|_{L^2_x} \right) \, d\sigma \lesssim \int_0^t \|(r^{12}, u^{12})\|^2_{W_x^{s',2}} \|r^1\|_{W_x^{s'+1,2}} \|u^2\|_{W_x^{s'+1,2}} \, d\sigma. \quad (3.61)
\end{align}

By integrating (3.60) in time, we can treat the corresponding terms on the right side of the equation as in (3.61). Indeed, some are easier to tackle. Subsequently, it follows from (3.48–3.51) and the continuous embedding \(W_x^{s'+1,2} \hookrightarrow W_x^{s',2}\) that
\begin{align}
\|r^{12}(t)\|^2_{W_x^{s',2}} + (\gamma - 1) \sum_{|s| \leq s'} \int_{\Omega^1} r^1 \text{div}_x \partial_x^r u^{12} \partial_x^r r^{12} \, dx \, d\sigma \\
\lesssim \int_0^t \|(r^{12}, u^{12})\|^2_{W_x^{s',2}} \left( \|r^1\|_{W_x^{s'+1,2}} + \|u^2\|_{W_x^{s'+1,2}} \right) \, d\sigma. \quad (3.62)
\end{align}

Note that we need \(W_x^{s'+1,2}\)-regularity of \(r^1\) and \(u^2\) in the above because of the second and third terms on the right-hand side of (3.60). Also note that by (3.46), \(s' + 1 \leq s\) and as such, \(W_x^{s,2}\) is contained in
Similarly, by Young’s inequality
\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega_t} |\partial_\alpha^\beta u^{12}|^2 \, dx & - 2 \int_{\Omega_t} r^1 \nabla_x \partial_\alpha^\beta u^{12} \partial_\alpha^\beta r^{12} \, dx + 2 \int_{\Omega_t} D(r^1) \mathcal{S}(\nabla_x \partial_\alpha^\beta u^{12}) : \nabla_x \partial_\alpha^\beta u^{12} \, dx \\
& = \int_{\Omega_t} |\partial_\alpha^\beta u^{12}|^2 \, div_x u^1 \, dx - 2 \int_{\Omega_t} u^{12} \nabla_x \partial_\alpha^\beta u^2 \cdot \partial_\alpha^\beta u^{12} \, dx \\
& - 2 \int_{\Omega_t} r^{12} \nabla_x \partial_\alpha^\beta r^{2} \cdot \partial_\alpha^\beta u^{12} \, dx - 2 \int_{\Omega_t} D(r^1) \partial_\alpha^\beta \nabla^T \cdot \partial_\alpha^\beta u^{12} \, dx \\
& - 2 \int_{\Omega_t} \nabla_x D(r^1) \left[ \mathcal{S}(\nabla_x \partial_\alpha^\beta u^{12}) + \partial_\alpha^\beta \nabla^T \right] : \partial_\alpha^\beta u^{12} \, dx + 2 \int_{\Omega_t} \nabla_x r^{1} \cdot \partial_\alpha^\beta r^{12} \partial_\alpha^\beta u^{12} \, dx \\
& - 2 \int_{\Omega_t} (D(r^1) - D(r^2)) \div_x \mathcal{S}(\nabla_x \partial_\alpha^\beta u^2) \cdot \partial_\alpha^\beta u^{12} \, dx \\
& - 2 \int_{\Omega_t} (D(r^1) - D(r^2)) \div_x \partial_\alpha^\beta \nabla^T \cdot \partial_\alpha^\beta u^{12} \, dx \\
& + 2 \int_{\Omega_t} (I_1^r + \ldots + I_k^r) \partial_\alpha^\beta u^{12} \, dx.
\end{aligned}
\]  

We can now integrate (3.63) in time and estimate the right-hand terms. First of all, we can use Sobolev’s inequality to obtain,
\[
\int_{\Omega_t} |\partial_\alpha^\beta u^{12}|^2 \, div_x u^1 \, dx \, d\sigma \lesssim \int_0^t \|u^{12}\|^2_{W^{1,2}_x} \|u^1\|_{W^{3,2}_x} \, d\sigma
\]
whereas in combination with Hölder’s inequality,
\[
\int_{\Omega_t} u^{12} \nabla_x \partial_\alpha^\beta u^2 \cdot \partial_\alpha^\beta u^{12} \, dx \, d\sigma \lesssim \int_0^t \|u^{12}\|^2_{W^{1,2}_x} \|u^2\|_{W^{3,2}_x} \, d\sigma.
\]  

Similarly, by Young’s inequality
\[
\begin{aligned}
2 \int_{\Omega_t} r^{12} \nabla_x \partial_\alpha^\beta r^{2} \cdot \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \lesssim \int_0^t \|r^{12}\|^2_{W^{3,2}_x} \|r^2\|_{W^{3,2}_x} \, d\sigma + \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \|r^2\|_{W^{3,2}_x} \, d\sigma
\end{aligned}
\]  

Using (3.40), Sobolev’s embedding and (3.42) yields
\[
\begin{aligned}
2 \int_{\Omega_t} D(r^1) \partial_\alpha^\beta \nabla^T : \nabla_x \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \leq \delta \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \, d\sigma + c_\delta \int_0^t \|\nabla^T\|^2_{W^{3,2}_x} \, d\sigma
\end{aligned}
\]  

for any \( \delta > 0 \). Similarly, we obtain
\[
\begin{aligned}
2 \int_{\Omega_t} \nabla_x D(r^1) \mathcal{S}(\nabla_x \partial_\alpha^\beta u^{12}) : \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \leq \delta \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \, d\sigma + c_\delta \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \, d\sigma
\end{aligned}
\]  

for any \( \delta > 0 \). Also, (3.41) yields
\[
\begin{aligned}
2 \int_{\Omega_t} \nabla_x D(r^1) \partial_\alpha^\beta \nabla^T : \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \lesssim \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \, d\sigma + \int_0^t \|\nabla^T\|^2_{W^{3,2}_x} \, d\sigma
\end{aligned}
\]  

and
\[
\begin{aligned}
2 \int_{\Omega_t} \nabla_x r^{1} \cdot \partial_\alpha^\beta r^{12} \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \lesssim \int_0^t \|r^{12}\|^2_{W^{3,2}_x} \, d\sigma + \int_0^t \|u^{12}\|^2_{W^{3,2}_x} \, d\sigma.
\end{aligned}
\]  

As consequence of (3.42) we obtain
\[
\begin{aligned}
2 \int_{\Omega_t} (D(r^1) - D(r^2)) \div_x \mathcal{S}(\nabla_x \partial_\alpha^\beta u^2) \cdot \partial_\alpha^\beta u^{12} \, dx \, d\sigma & \lesssim \int_0^t \|u^{12}\|_{W^{3,2}_x} \|r^{12}\|_{W^{3,2}_x} \|u^2\|_{W^{3,2}_x} \, d\sigma.
\end{aligned}
\]
In conclusion

\[-2 \int_{Q_t} (D(r^1) - D(r^2)) \text{div}_x \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds = 2 \int_{Q_t} \nabla_x (D(r^1) - D(r^2)) \cdot \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds \]
\[+ 2 \int_{Q_t} (D(r^1) - D(r^2)) \partial_x^2 T^2 : \partial_x^2 \nabla_x u^{12} \, dx \, ds \]
\[\quad = 2 \int_{Q_t} D'(r^1) (\nabla_x r^1 - \nabla_x r^2) \cdot \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds \]
\[+ 2 \int_{Q_t} (D'(r^1) - D'(r^2)) \nabla_x r^2 : \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds \]
\[+ 2 \int_{Q_t} (D(r^1) - D(r^2)) \partial_x^2 T^2 : \partial_x^2 \nabla_x u^{12} \, dx \, ds \]

Using (3.40), (3.41) and Sobolev’s embedding we can estimate these terms as follows

\[
2 \int_{Q_t} D'(r^1) (\nabla_x r^1 - \nabla_x r^2) \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| u^{12} \|_{W^{2,2}} r^{12} \| r^{12} \|_{W^{2,2}} \| T^2 \|_{W^{2,2}} \, ds,
\]

\[
2 \int_{Q_t} (D'(r^1) - D'(r^2)) \nabla_x r^2 : \partial_x^2 T^2 : \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| u^{12} \|_{W^{2,2}} r^{12} \| r^{12} \|_{W^{2,2}} \| T^2 \|_{W^{2,2}} \, ds,
\]

\[
2 \int_{Q_t} (D(r^1) - D(r^2)) \partial_x^2 T^2 : \partial_x^2 \nabla_x u^{12} \, dx \, ds \leq \delta \int_0^t \| u^{12} \|_{W^{2,2}}^2 \| T^2 \|_{W^{2,2}} \, ds + c \int_0^t \| r^{12} \|_{W^{2,2}}^2 \| T^2 \|_{W^{2,2}} \, ds.
\]

As a result of (3.40) – (3.55) and the continuous embedding $W^{x'+2} \hookrightarrow W_1^{2,\infty}$, we also have

\[
2 \int_{Q_t} (I^a + \ldots + I^b) \cdot \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| u^{12} \|_{W^{2,2}}^2 \| (r^{12}, u^{12}) \|_{W^{x'+2}} (\| (r^{12}, r^{12}) \|_{W^{x'+2}} + \| (u^{12}, u^{12}) \|_{W^{x'+2}}) \, ds.
\]

Furthermore, due to (3.32)

\[
2 \int_{Q_t} I^a \cdot \partial_x^2 u^{12} \, dx \, ds \leq \delta \int_0^t \| u^{12} \|_{W^{2,2}}^2 \| r^{12} \|_{W^{x'+2}} \| u^{2} \|_{W^{x'+2}} \, ds (3.71)
\]

as well as

\[
2 \int_{Q_t} I^b \cdot \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| u^{12} \|_{W^{2,2}} \| r^{12} \|_{W^{x'+2}} \| u^{2} \|_{W^{x'+2}} \, ds (3.72)
\]

because of $W^{x'+2} \hookrightarrow W_2^{2,\infty}$. Finally,

\[
2 \int_{Q_t} I^a \cdot \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| u^{12} \|_{W^{2,2}} r^{12} \| T^2 \|_{W^{2,2}} \, ds + \int_0^t \| u^{12} \|_{W^{2,2}} r^{12} \| T^2 \|_{W^{2,2}} \, ds,
\]

\[
2 \int_{Q_t} I^a \cdot \partial_x^2 u^{12} \, dx \, ds \lesssim \int_0^t \| r^{12} \|_{W^{2,2}} \| u^{12} \|_{W^{x'+2}} \| T^2 \|_{W^{2,2}} \, ds (3.73)
\]

In conclusion

\[
\| u^{12}(t) \|_{W^{2,2}}^2 - 2 \sum_{|\alpha| \leq s'} \int_{Q_t} r^1 \text{div}_x \partial_x^2 \partial_x^2 u^{12} \partial_x^2 u^{12} \, dx \, ds + \int_0^t \| u^{12}(\sigma) \|_{W^{x'+2}}^2 \, d\sigma
\]
\[\lesssim \int_0^t \| u^{12} \|_{W^{2,2}}^2 \, ds (3.74)
\]
Summing \((3.62)\) with the product of \(\frac{\lambda}{2}\) and \((3.74)\) then yield
\[
\|(r^{12}(t), u^{12}(t))\|_{W_{s+2}^{2}}^2 + \int_0^t \|u^{12}(\sigma)\|_{W_{s+1}^{2}}^2 \, d\sigma \\
\lesssim \int_0^t \|(r^{12}, u^{12})\|_{W_{s+2}^{2}}^2 \left(\|(r^1, r^2)\|_{W_{s+2}^{2}}^2 + \|(u^1, u^2)\|_{W_{s+2}^{2}}^2 + \|T^2\|_{W_{s+2}^{2}}^2 + 1\right) \, d\sigma
\]
\[
\lesssim \int_0^t \|T^{12}\|_{W_{s+2}^{2}}^2 \, d\sigma\ |
\|W_{s+1}^{2}\|ight)\]

Applying Gronwall’s lemma finishes the proof of Theorem \(5.3\) with a constant depending on \(K_{\mathcal{R}}\). This implies uniqueness. As in Section \(3.2\) we can now decrease the time \(T\) (depending on \(\sup_{0 < t < T_0} \|u^1\|_{s,2}\) and \(\sup_{0 < t < T_0} \|u^2\|_{s,2}\), recall the definition of \(\mathcal{R}\) in \((8.34)\)) such that the constant is independent of \(\mathcal{R}\) which completes the proof of Theorem \(5.3\).

**Remark 3.4.** We remind the reader that the condition \((8.30)\) means that in particular, \(s' + 2 \leq s + 1\) which is the threshold differentiability exponent of the solutions \(u^i, i = 1, 2\) to the momentum equation \((2.6)\).

## 4. Solving the Fokker–Planck Equation

The aim of this section is to solve the Fokker–Planck equation
\[
\partial_t \psi + \text{div}_x (u \psi) = \varepsilon \Delta_x \psi - \text{div}_q (\nabla_x u \psi) + \frac{A_{11}}{4 \lambda} \text{div}_q \left( M \nabla_q \left( \frac{\psi}{M} \right) \right)
\]
with \(\varepsilon \geq 0\) and \(A_{11}, \lambda > 0\). Here, \(u\) is a given smooth function and we recall that the Maxwellian is given by
\[
M(q) = \int_B e^{-U\left(\frac{1}{2}q^2\right)} \, dq, \quad U(s) = -\frac{b}{2} \log \left(1 - \frac{2s}{b}\right), \quad s \in [0, b/2)
\]
with \(b > 2\). Let us start with a precise definition of the solution.

**Definition 4.1** (Strong solution). Let \(s \in \mathbb{N}\) and \(T > 0\). Assume there is \(\psi_0 \in W^{s,2}(T^d; L_M^2(B))\) and \(u \in C([0, T]; W^{s,2}(T^d; \mathbb{R}^d)) \cap L^2(0, T; W^{s+1,2}(T^d; \mathbb{R}^d))\). We call \(\psi\) a strong solution to the system \((1.1)\) with initial condition \(\psi_0\) in the interval \([0, T]\) provided the following holds.

(a) \(\psi\) satisfies
\[
C([0, T]; W^{s,2}(T^d; L_M^2(B))) \cap L^2(0, T; W^{s,2}(T^d; H_M^2(B)));
\]
(b) there holds for all \(t \in [0, T]\)
\[
\psi(t) = \psi_0 - \int_0^t \left[ \text{div}_x (u \psi) - \varepsilon \Delta_x \psi + \text{div}_q (\nabla_x u \psi) \right] \, d\sigma + \frac{A_{11}}{4 \lambda} \int_0^t \text{div}_q \left( M \nabla_q \left( \frac{\psi}{M} \right) \right) \, d\sigma.
\]

As in the case of Definition \(2.2\) differentiability in time as well as the correct initial datum follows from the equation in (b). We now formulate our results concerning well-posedness and uniqueness for \((1.1)\).

**Theorem 4.2.** Let \(s \in \mathbb{N}\) satisfy \(s > \frac{d}{2} + 2\), \(T > 0\) and suppose that \(\varepsilon \geq 0\). Assume there is \(\psi_0 \in W^{s,2}(T^d; L_M^2(B))\) and \(u \in C([0, T_0]; W^{s,2}(T^d; \mathbb{R}^d)) \cap L^2(0, T_0; W^{s+1,2}(T^d; \mathbb{R}^d))\). Then there is a strong solution \(\psi\) to problem \((1.1)\) in the sense of Definition \(4.1\) defined on the interval \([0, T]\) with the initial condition \(\psi_0\). Moreover, we have the estimate
\[
\sup_{0 < t < T} \|\psi\|_{W^{s,2}_x L^2_M}^2 + 2\varepsilon \int_0^T \|u\|_{W^{s+1,2}_x L^2_M}^2 \, dt + \frac{A_{11}}{2 \lambda} - 4 \delta \int_0^T \|\psi\|_{W^{s,2}_x L^2_M}^2 \, dt \\
\leq c \exp \left( c \int_0^T \|u\|_{W^{s+1,2}_x L^2_M}^2 \, dt \right) \|\psi_0\|_{W^{s,2}_x L^2_M}^2.
\]
with a constant depending only on \(s, b, d, \delta\) and \(T\).
The proof of Theorem 4.2 can be found in Section 4.1.

**Theorem 4.3.** Let the assumptions of Theorem 4.2 be satisfied. Then the solution from Theorem 4.2 is unique. Moreover, for $s' \in \mathbb{N}$ with $s' \leq s - 1$ we have the estimate

\[
\sup_{0 < t < T} \left\| \psi^1 - \psi^2 \right\|_{H_{x,t}^{s',2}L_M^2}^2 + 2\varepsilon \int_0^T \left\| \psi^1 - \psi^2 \right\|_{H_{x,t}^{s',2}L_M^2}^2 dt + \left( \frac{A_{11}}{2\lambda} - 8\delta \right) \int_0^T \left\| \psi^1 - \psi^2 \right\|_{H_{x,t}^{s',2}L_M^2}^2 dt \leq c \exp \left( c \int_0^T \left( \left\| u^1 \right\|_{H_{x,t}^{s',2}L_M^2}^2 + 1 \right) \right) \int_0^t \left\| u^1 - u^2 \right\|_{H_{x,t}^{s',2}L_M^2}^2 \left\| \psi^1, \psi^2 \right\|_{H_{x,t}^{s',2}L_M^2}^2 dt,
\]

where $\psi^1$ and $\psi^2$ are two strong solutions to (4.4) with data $(\psi_0, u^1)$ and $(\psi_0, u^2)$ respectively.

The proof of Theorem 4.3 can be found in Section 4.2.

### 4.1. A priori estimates

In order to justify the following calculations, we need to work with an approximate system. Following [28], we consider an orthonormal basis $\{\phi_n, \ldots, \phi_n\}$ of eigenfunction of the operator $L\psi = -\nabla q(M\nabla q(\frac{\psi}{M}))$ on $L_M^2$ with the domain

\[
D(L) = \left\{ \psi \in L_M^2 : \psi \in H_{x,t}^1, \quad \nabla q \left( M\nabla q \left( \frac{\psi}{M} \right) \right) \in L_M^2, \quad \nabla q \left( \frac{\psi}{M} \right) \rvert_{\partial B} = 0 \right\},
\]

where the boundary condition is interpreted in the weak sense. We denote $Y_n = \text{span}\{\phi_1, \ldots, \phi_n\}$. Furthermore, we introduce a cut-off in the last term in order to avoid the blow-up of $M$ for $q$ close to $\partial B$. So let $\chi_n = \chi_n(q) \in C^1(\bar{B})$, $B := B(0, \sqrt{\delta})$ where $\chi_n = 1$ in $B(0, \sqrt{\delta} - \frac{2}{n})$ and $\chi_n = 0$ when $|q| \geq \sqrt{\delta} - \frac{2}{n}$. It will be needed to ensure that certain terms belong to $L^2(B)$. Similar to Section 3.1, we consider a smooth orthonormal basis $\{\omega_m\}_{m=1}^\infty$ of the space $L^2(\mathcal{T}^d)$ with $X_n = \text{span}\{\omega_1, \ldots, \omega_n\}$. We denote by $\Pi_x : L^2(\mathcal{T}^d) \to X_n$ and $\Pi_q : L^2(M^d(B)) \to Y_n$ the corresponding orthogonal projections. We aim to solve for $n \in \mathbb{N}$,

\[
\partial_t \int_{\mathcal{T}^d \times B} \psi^n \phi_i \omega_j \, dq \, dx + \int_{\mathcal{T}^d \times B} \nabla_x \left( M u^q \Pi_{q \psi^n} \phi_i \omega_j \right) \, dq \, dx = \int_{\mathcal{T}^d \times B} \varepsilon \Delta_x \psi^n \phi_i \omega_j \, dq \, dx - \int_{\mathcal{T}^d \times B} \nabla_q \left( \nabla_x u \right) q \chi_n \psi^n \phi_i \omega_j \, dq \, dx + \int_{\mathcal{T}^d \times B} \frac{A_{11}}{4\lambda} \nabla_q \left( \frac{\psi^n}{M} \right) \phi_i \omega_j \, dq \, dx, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.
\]

Equation (4.4) is an ODE which can be solved locally in time. We start by showing an estimate for $\psi^n$ in $L^2_{x,t}L^2_M$ which will imply global solvability of (1.1). We can take $\Pi_{q \frac{\psi^n}{M}}$ as test function in (4.4) and integrate over $Q_t \times B$ for which we obtain

\[
\frac{1}{2} \int_{\mathcal{T}^d \times B} M \left| \psi^n(\sigma) \right|_{M}^2 \, dq \, d\sigma \bigg|_{\sigma = 1} + \varepsilon \int_{Q_t \times B} M \left| \nabla_x \psi^n \right|_{M}^2 \, dq \, d\sigma + \frac{A_{11}}{4\lambda} \int_{Q_t \times B} M \left| \nabla_q \psi^n \right|_{M}^2 \, dq \, d\sigma - \int_{Q_t \times B} \chi_n M(\nabla_x u) q \frac{\psi^n}{M} \cdot \nabla_q \Pi_{q \frac{\psi^n}{M}} \, dq \, d\sigma
\]

\[
+ \int_{Q_t \times B} \left( \frac{1}{2} \chi_n M(\nabla_x u) \Pi_{q \frac{\psi^n}{M}} \right) \Pi_{q \frac{\psi^n}{M}} \, dq \, d\sigma - \int_{Q_t \times B} M \chi_n \div_x u \left| \Pi_{q \frac{\psi^n}{M}} \right|_{M}^2 \, dq \, d\sigma := (I)^n + (II)^n + (III)^n.
\]
As a consequence of Young’s inequality, Hölder’s inequality and $|q|^2 < b$, we obtain the estimates

\[ (I)^n \leq \frac{1}{2} \int_{Q_t \times B} M \left| \nabla_q \psi^n_M \right|^2 \, dq \, dx \, d\sigma + c_b(b) \int_{Q_t} \left| \nabla_x u \right|^2 \int_B M \left| \psi^n_M \right|^2 \, dq \, dx \, d\sigma, \]

\[ (II)^n \leq \frac{1}{2} \int_0^t \| \text{div}_x u \|_{L^\infty_x} \int_{T^4 \times B} M \left| \psi^n_M \right|^2 \, dq \, dx \, d\sigma, \]

\[ (III)^n \leq \frac{1}{2} \int_{Q_t \times B} M \left| \psi^n_M \right|^2 \, dq \, dx \, d\sigma + \frac{1}{2} \int_{Q_t} \| \text{div}_x u \|_{L^2_x} \int_B M \left| \psi^n_M \right|^2 \, dq \, dx \, d\sigma, \]

for any $\delta > 0$. Note that we also took into account continuity of $\Pi^n_q$ on $L^2_M$ and $H^1_M$. All three terms together can be bounded by

\[ \delta \int_0^t \| \psi^n \|_{L^2_x H^1_t M}^2 \, d\sigma + c \int_0^t \left( 1 + \| u \|^2_{W^{1,2}_x} \right) \| \psi^n \|_{L^2_x L^2_t M}^2 \, d\sigma \]

using Sobolev’s embedding. Plugging this into (4.15) and using Gronwall’s lemma yields

\[ \sup_{0 < t < T} \| \psi^n \|_{L^2_x L^2_t M}^2 \leq c \exp \left( c \int_0^T \| u \|^2_{W^{1,2}_x} \, d\sigma \right) \| \psi_0 \|_{L^2_x L^2_t M}^2 \]

(4.6)

since $\varepsilon \geq 0$ and $\frac{4\varepsilon}{\chi} - \delta > 0$ for $\delta > 0$ small. Note that (4.6) implies there is a global solution to (4.4).

Next apply $\partial^2 \psi$ to (4.4) to obtain

\[ \partial_t \partial_q \psi^n + M \chi_n \Pi^n_q \frac{\psi^n}{M} \text{div}_x \partial^2 \psi + M \chi_n u \cdot \nabla_x \partial^2 \psi + M \chi_n \Pi^n_q \frac{\psi^n}{M} \]

\[ = \varepsilon \Delta_x \partial^2 \psi^n + \text{div}_q \left( \chi_n (\nabla_x \partial^2 \psi) u \right) \psi^n + A_{11} \text{div}_q \left( M \nabla_q \partial^2 \psi \left( \frac{\psi^n}{M} \right) \right) + K_1^n + K_2^n + \text{div}_q K_3^n \]

(4.7)

in $(X_n \otimes Y_n)'$, where

\[ K_1^n = M \chi_n \Pi^n_q \frac{\psi^n}{M} \partial^2 \psi + \text{div}_x \left[ M \chi_n \Pi^n_q \frac{\psi^n}{M} \text{div}_x u \right], \]

\[ K_2^n = u \cdot \nabla_x \Pi^n_q \frac{\psi^n}{M} - \partial^2 \psi \left[ u \cdot \nabla_x \Pi^n_q \frac{\psi^n}{M} \right], \]

\[ K_3^n = \left( \partial^2 \psi \nabla_x u \right) \chi_n \psi^n - \partial^2 \psi \left( \nabla_x u \right) \chi_n \psi^n. \]

As a consequence of (2.21) and $|q| < \sqrt{b}$, it follows that

\[ \| K_1^n \|_{L^2_x M} \lesssim_b \| \nabla_x \Pi^n_q \frac{\psi^n}{M} \|_{L^2_x M} \| \nabla_x u \|_{L^2_x M} + M \| \text{div}_x u \|_{L^2_x M} \]

\[ \| K_2^n \|_{L^2_x M} \lesssim_b \| \nabla_x u \|_{L^2_x M} \| \nabla^2 \Pi^n_q \frac{\psi^n}{M} \|_{L^2_x M} + M \| \nabla^2 \Pi^n_q \frac{\psi^n}{M} \|_{L^2_x M} \| \nabla_x u \|_{L^2_x M}, \]

\[ \| K_3^n \|_{L^2_x M} \lesssim_b \| \nabla_x u \|_{L^2_x M} \| \nabla^2 \psi^n \|_{L^2_x M} \| \nabla_x u \|_{L^2_x M} + \| \nabla_x u \|_{L^2_x M} \| \nabla^2 \psi^n \|_{L^2_x M}. \]

(4.8)

(4.9)

(4.10)

We multiply (4.4) by $\partial^2 \psi$ and integrate over $Q_t \times B$ to gain

\[ \frac{1}{2} \int_{T^4 \times B} M \left| \partial^2 \frac{\psi^n}{M} \right|^2 \, dq \, dx \, d\sigma \]

\[ + \frac{A_{11}}{4\varepsilon} \int_{Q_t \times B} M \left| \nabla_q \partial^2 \psi \right|^2 \, dq \, dx \, d\sigma = \int_{Q_t \times B} M \left( \nabla_q \partial^2 \psi \right) u \left( \nabla_q \partial^2 \psi \right) \frac{\psi^n}{M} \, dq \, dx \, d\sigma \]

\[ + \int_{Q_t \times B} \left( K_1^n + K_2^n \right) \cdot \partial^2 \psi \frac{\psi^n}{M} \, dq \, dx \, d\sigma - \int_{Q_t \times B} K_3^n \cdot \nabla_q \partial^2 \psi \frac{\psi^n}{M} \, dq \, dx \, d\sigma. \]
In order to estimate the terms on the right-hand side, we repeatedly use the embedding $W_{s}^{2} \hookrightarrow W_{s}^{1,\infty}$ which follows from $s \geq 1 + \frac{d}{2}$. Furthermore, we use the continuity of $\Pi_{q}^{n}$ on $L_{s}^{2}$ and $H_{s}^{2}$. By Young’s and Hölder’s inequalities (note that $|q|^{2} < b$), we obtain the estimate

$$
\left| \int_{Q_{t} \times B} M(\nabla_{x} \partial_{x}^{2} u)_{q}^{\chi_{n_{q}}} Q_{n_{q}}^{\psi_{n}} M_{q} \dd q \dd x \dd \sigma \right|
\leq \delta \int_{Q_{t} \times B} M \left| \nabla_{x} \partial_{x}^{2} u_{q}^{\chi_{n_{q}}} Q_{n_{q}}^{\psi_{n}} M_{q} \right|^{2} \dd q \dd x \dd \sigma + c_{3} \int_{Q_{t} \times B} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma
$$

(4.12)

for any $\delta > 0$ as well as

$$
\left| \int_{Q_{t} \times B} M_{q} \frac{1}{2} M(\nabla_{x} \partial_{x}^{2} u)_{q}^{\chi_{n_{q}}} Q_{n_{q}}^{\psi_{n}} M_{q} \dd q \dd x \dd \sigma \right|
\leq \frac{1}{2} \int_{Q_{t} \times B} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma
$$

(4.13)

Furthermore, we have

$$
\left| \int_{Q_{t} \times B} M_{q} \frac{1}{2} M(\nabla_{x} \partial_{x}^{2} u)_{q}^{\chi_{n_{q}}} Q_{n_{q}}^{\psi_{n}} M_{q} \dd q \dd x \dd \sigma \right|
\leq \frac{1}{2} \int_{Q_{t} \times B} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma + \frac{1}{2} \int_{Q_{t} \times B} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma
$$

(4.14)

and for any $\delta > 0$,

$$
\left| \int_{Q_{t} \times B} M_{q} \frac{1}{2} M(\nabla_{x} \partial_{x}^{2} u)_{q}^{\chi_{n_{q}}} Q_{n_{q}}^{\psi_{n}} M_{q} \dd q \dd x \dd \sigma \right|
\leq \frac{1}{2} \int_{Q_{t} \times B} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma + \int_{0}^{t} \left| \nabla_{x} \partial_{x}^{2} u \right|^{2} \dd q \dd x \dd \sigma
$$

(4.15)

Analogously, we use Young’s inequality to estimate $K_{1}^{n}$ and $K_{2}^{n}$. Substituting (4.12)–(4.15) into (4.11) and choosing $\delta$ small enough yields

$$
\|\psi^{n}(\sigma)\|_{W_{s}^{2} L_{s}^{2}}^{2} \leq \int_{0}^{t} \left( \frac{A_{11}}{2M} - \frac{4\delta}{M} \right) \dd t + \int_{0}^{t} \left( \|\nabla u\|_{W_{s}^{2} L_{s}^{2}} + \|\nabla u\|_{W_{s}^{2} L_{s}^{2}}^{2} \right) \dd t
$$

(4.16)

On the other hand, by (4.8)–(4.10) we have

$$
\sum_{i=1}^{3} \int_{Q_{t} \times B} \frac{1}{M} |K_{i}^{n}|^{2} \dd q \dd x \dd \sigma
\leq \int_{0}^{t} \left( \sup_{x \in T_{q}} \left| \nabla_{x} u \right|_{L_{s}^{2}}^{2} \right) \dd t + \int_{0}^{t} \left( \|\nabla u\|_{W_{s}^{2} L_{s}^{2}} + \|\nabla u\|_{W_{s}^{2} L_{s}^{2}}^{2} \right) \dd t
$$

(4.17)
Substituting (4.17) into (4.16) yields

\[
\|\psi^n(\sigma)\|^2_{W^{2,1}_M} \big|_{\sigma=0} + 2\varepsilon \int_0^\sigma \|\psi^n\|^2_{W^{2,1}_M} d\sigma + \left( \frac{A_{11}}{2\lambda} - 4\delta \right) \int_0^\sigma \|\psi^n\|^2_{W^{2,1}_M} d\sigma \\
\lesssim \int_0^\sigma (1 + \|u\|^2_{W^{2,1}_M}) \|\psi^n\|^2_{W^{2,1}_M} d\sigma.
\]

(4.18)

By Gronwall’s lemma we obtain

\[
\sup_{0 \leq t \leq T} \|\psi^n(t)\|^2_{W^{2,1}_M} + 2\varepsilon \int_0^T \|\psi^n\|^2_{W^{2,1}_M} dt + \left( \frac{A_{11}}{2\lambda} - 4\delta \right) \int_0^T \|\psi^n\|^2_{W^{2,1}_M} dt \\
\leq c \exp \left( c \int_0^T \|u\|^2_{W^{2,1}_M} dt \right) \|\psi_0\|^2_{W^{2,1}_M}
\]

(4.19)

uniformly in \( n \). It is now standard to pass to the limit \( n \to \infty \) in (4.14) and to show that (4.19) also holds in the limit. The proof of Theorem 4.2 is hereby complete.

4.2. Difference estimate for the Fokker-Planck equation. Given the estimates from the previous section (see, in particular, Theorem 4.2) the solution possesses enough regularity to justify the following calculations. Let \( \psi^1 \) and \( \psi^2 \) be two solutions of (4.1) with data \( (u^1, \psi_0) \) and \( (u^2, \psi_0) \) respectively. We set \( \psi^{12} = \psi^1 - \psi^2 \) and \( u^{12} = u^1 - u^2 \) so that \( \psi^{12} \) solves

\[
\partial_t \psi^{12} + \nabla_x (u^{12} \psi^{12}) + \nabla_x (u^{12} \psi^1) = \varepsilon \Delta_x \psi^{12} - \nabla_q (\nabla_x u^1 \psi^{12}) \\
- \nabla_q (\nabla_x u^2 \psi^2) + \frac{A_{11}}{4\lambda} \nabla_q \left( M \nabla_q \left( \frac{\psi^{12}}{M} \right) \right)
\]

(4.20)

subject to the following initial and boundary conditions

\[
u^{12}|_{t=0} = u^{12}_0 = 0 \quad \text{in } \mathcal{T}^d, \\
\psi^{12}|_{t=0} = \psi^{12}_0 = 0 \quad \text{in } \mathcal{T}^d \times B, \\
\psi^{12} = 0 \quad \text{on } [0, T] \times \mathcal{T}^d \times \partial B.
\]

(4.21-4.23)

We now wish to establish a priori estimates in the spirit of Section 4.1 for the difference \( \psi^{12} \). We will repeatedly use the embedding \( W^{s,2}_x \hookrightarrow W^{3,\infty}_x \). We start by proving a counterpart of (4.6). Testing (4.20) with \( \frac{\psi^{12}}{M} \) and integrating over \( Q_t \times B \) yield

\[
\frac{1}{2} \int_{T^d \times B} M \frac{\psi^{12}(t)}{M}^2 \, dq \, dx \, ds + \int_{Q_t \times B} \nabla_q \frac{\psi^{12}}{M} \, dq \, dx \, ds \\
+ \frac{A_{11}}{4\lambda} \int_{Q_t \times B} \nabla_q \frac{\psi^{12}}{M} \, dq \, dx \, ds \\
+ \int_{Q_t \times B} M (\nabla_x u^{12}) \frac{\psi^{12}}{M} \cdot \nabla_q \frac{\psi^{12}}{M} \, dq \, dx \, ds \\
+ \int_{Q_t \times B} M (\nabla_x u^{12}) \frac{\psi^{12}}{M} \cdot \nabla_x \frac{\psi^{12}}{M} \, dq \, dx \, ds \\
- \int_{Q_t \times B} u^{12} \cdot \nabla_x \frac{\psi^{12}}{M} \, dq \, dx \, ds \\
- \int_{Q_t \times B} \psi^{12} \, dq \, dx \, ds =: J_1 + \ldots + J_6
\]

(4.24)
The terms on the right-hand side can be estimated as follows:

\[
J_1 \lesssim \int_0^t \| \nabla_x u^1 \|_{L^\infty} \| \psi^{12} \|_{L^2 L^2_{s,s}} \| \psi^{12} \|_{L^2 H^1_{s,s}} \, d\sigma
\]
\[
\leq \delta \int_0^t \| \psi^{12} \|_{L^2 H^1_{s,s}}^2 \, d\sigma + c_\delta \int_0^t \| u^1 \|_{W^{1,2}}^2 \| \psi^{12} \|_{L^2 L^2_{s,s}}^2 \, d\sigma,
\]
\[
J_2 \lesssim \int_0^t \| \nabla_x u^{12} \|_{L^\infty} \| \psi^1 \|_{L^2 L^2_{s,s}} \| \psi^{12} \|_{L^2 H^1_{s,s}} \, d\sigma
\]
\[
\leq \delta \int_0^t \| \psi^{12} \|_{L^2 H^1_{s,s}}^2 \, d\sigma + c_\delta \int_0^t \| u^{12} \|_{W^{1,2}}^2 \| \psi^1 \|_{L^2 L^2_{s,s}}^2 \, d\sigma,
\]
\[
J_3, J_5 \lesssim \int_0^t \| u^2 \|_{W^{1,2}} \| \psi^{12} \|_{L^2 L^2_{s,s}}^2 \, d\sigma
\]
\[
J_4, J_6 \lesssim \int_0^t \| \nabla_x u^{12} \|_{L^\infty} \| \psi^1 \|_{W^{1,2} L^2_{s,s}} \| \psi^{12} \|_{L^2 L^2_{s,s}} \, d\sigma
\]
\[
\lesssim \int_0^t \| \psi^{12} \|_{L^2 L^2_{s,s}}^2 \, d\sigma + \int_0^t \| u^{12} \|_{W^{1,2}}^2 \| \psi^1 \|_{W^{1,2} L^2_{s,s}}^2 \, d\sigma.
\]

where \( \delta > 0 \) is arbitrary. Inserting the above into (4.24) yields

\[
\| \psi^{12}(t) \|_{L^2 L^2_{s,s}}^2 + 2\varepsilon \int_0^t \| \psi^{12} \|_{W^{1,2} L^2_{s,s}}^2 \, d\sigma + \left( \frac{A_{11}}{2\lambda} - 4\delta \right) \int_0^t \| \psi^{12} \|_{L^2 H^1_{s,s}}^2 \, d\sigma
\]
\[
\lesssim \int_0^t \| u^{12} \|_{W^{1,2}}^2 \| (\psi^1, \psi^2) \|_{W^{1,2} L^2_{s,s}}^2 \, d\sigma + \int_0^t \left( \| (u^1, u^2) \|_{W^{1,2}}^2 + 1 \right) \| \psi^{12} \|_{L^2 L^2_{s,s}}^2 \, d\sigma.
\]

Finally, we obtain from Gronwall’s lemma

\[
\| \psi^{12}(t) \|_{L^2 L^2_{s,s}}^2 + 2\varepsilon \int_0^t \| \psi^{12} \|_{W^{1,2} L^2_{s,s}}^2 \, d\sigma + \left( \frac{A_{11}}{2\lambda} - 4\delta \right) \int_0^t \| \psi^{12} \|_{L^2 H^1_{s,s}}^2 \, d\sigma \leq c \exp \left( c \int_0^t \left( \| (u^1, u^2) \|_{W^{1,2}}^2 \right) \, d\sigma \right) \int_0^t \| u^{12} \|_{W^{1,2}}^2 \| (\psi^1, \psi^2) \|_{W^{1,2} L^2_{s,s}}^2 \, d\sigma.
\] (4.25)

Now we turn to higher order estimates. Let the multi-index \( \alpha \) satisfy

\[
|\alpha| \leq s' \leq s - 1
\] (4.26)

and apply \( \partial_x^{\alpha} \) to (4.20) to obtain

\[
\partial_t \partial_x^{\alpha} \psi^{12} + \psi^{12} \nabla_x \partial_x^{\alpha} u^2 + \psi^1 \nabla_x \partial_x^{\alpha} u^{12} + u^2 \cdot \nabla_x \partial_x^{\alpha} \psi^{12} + u^{12} \cdot \nabla_x \partial_x^{\alpha} \psi^1
\]
\[
= \varepsilon \Delta_x \partial_x^{\alpha} \psi^{12} - \nabla_q ( (\nabla_x \partial_x^{\alpha} u^1) q \psi^{12} ) - \nabla_q ( (\nabla_x \partial_x^{\alpha} u^{12}) q \psi^2)
\]
\[
+ \frac{A_{11}}{4\lambda} \nabla_q \left( M \nabla_q \partial_x^{\alpha} \left( \frac{\psi^{12}}{M} \right) \right) + K_1^a + K_1^b + K_2^a + K_2^b + \text{div}_q K_3^a + \text{div}_q K_3^b
\] (4.27)

where

\[
K_1^a = \psi^{12} \partial_x^{\alpha} \nabla_x u^2 - \partial_x^{\alpha} [\psi^{12} \nabla_x u^2], \quad K_1^b = \psi^1 \partial_x^{\alpha} \nabla_x u^{12} - \partial_x^{\alpha} [\psi^1 \nabla_x u^{12}],
\]
\[
K_2^a = u^2 \cdot \partial_x^{\alpha} \nabla_x \psi^{12} - \partial_x^{\alpha} [u^2 \cdot \nabla_x \psi^{12}], \quad K_2^b = u^{12} \cdot \partial_x^{\alpha} \nabla_x \psi^1 - \partial_x^{\alpha} [u^{12} \cdot \nabla_x \psi^1],
\]
\[
K_3^a = (\partial_x^{\alpha} \nabla_x u^1) q \psi^{12} - \partial_x^{\alpha} [ (\nabla_x u^1) q \psi^{12}], \quad K_3^b = (\partial_x^{\alpha} \nabla_x u^{12}) q \psi^2 - \partial_x^{\alpha} [ (\nabla_x u^{12}) q \psi^2].
\]
whereas the rest of the right-hand terms of (4.28) can be estimated exactly as in (4.12)–(4.15). In analogy to (4.22) recall (4.32). Now note that

\[
\begin{aligned}
\left| \int_{Q_t \times B} M \mathbf{u}^{12} \cdot \nabla x \frac{\partial_y u}{M} - \partial_y v \frac{\partial_y u}{M} \right| dx \, ds \\
\lesssim \int_{Q_t \times B} M \left\| \frac{\partial_y u}{M} \right\|_{L_2}^2 \, dx \, ds &+ \int_0^t \left\| \mathbf{u}^{12} \right\|_{W^{s',2,2}} \, dx \, ds \\
&+ \int_0^t \left( \left\| (\mathbf{u}, \mathbf{u}) \right\|_{W^{s+1,2}} + 1 \right) \left\| \psi^{12} \right\|_{W^{s',2,2}} \, dx \, ds &+ \int_0^t \left\| \mathbf{u}^{12} \right\|_{W^{s',2,2}} \left\| (\psi^1, \psi^2) \right\|_{W^{s',2,2}} \, dx \, ds \\
&\quad + \int_{Q_t \times B} \frac{1}{M} \left( |K^a_1|^2 + |K^b_1|^2 + |K^a_2|^2 + |K^b_2|^2 + |K^a_3|^2 + |K^b_3|^2 \right) \, dx \, ds,
\end{aligned}
\]

where the rest of the right-hand terms of (4.28) can be estimated exactly as in (4.12)–(4.15). In analogy with (4.16), we obtain

\[
\begin{aligned}
\left\| \psi^{12} (t) \right\|_{W^{s',2,2}} + 2 \varepsilon \int_0^t \left\| \psi^{12} \right\|_{W^{s'+1,2} H^2} \, ds &+ \left( \frac{A_{11}}{2\lambda} - 8\delta \right) \int_0^t \left\| \psi^{12} \right\|_{W^{s'+1,2} H^2} \, ds \\
\lesssim \int_0^t \left( \left\| (\mathbf{u}, \mathbf{u}) \right\|_{W^{s+1,2}} + 1 \right) \left\| \psi^{12} \right\|_{W^{s',2,2}} \, dx \, ds &+ \int_0^t \left\| \mathbf{u}^{12} \right\|_{W^{s',2,2}} \left\| (\psi^1, \psi^2) \right\|_{W^{s',2,2}} \, dx \, ds \\
&\quad + \int_{Q_t \times B} \frac{1}{M} \left( |K^a_1|^2 + |K^b_1|^2 + |K^a_2|^2 + |K^b_2|^2 + |K^a_3|^2 + |K^b_3|^2 \right) \, dx \, ds,
\end{aligned}
\]

where \( c = c(b, d) \).

The commutator-terms can be estimated analogously to (4.17) such that

\[
\begin{aligned}
&\int_{Q_t \times B} \frac{1}{M} \left( |K^a_1|^2 + |K^b_1|^2 + |K^a_2|^2 + |K^b_2|^2 + |K^a_3|^2 + |K^b_3|^2 \right) \, dx \, ds \\
\lesssim \int_0^t \left( \left\| \nabla x \mathbf{u} \right\|^2_{L_\infty} + |\nabla x \mathbf{u}| \right\|^2_{L_2} + \left\| \nabla x \mathbf{u} \right\|^2_{L_2} + |\nabla x \mathbf{u}| \right\|^2_{L_2} \, dx \, ds \\
&+ \int_0^t \left( \left\| \nabla x \mathbf{u} \right\|^2_{L_2} + \left\| \nabla x \mathbf{u} \right\|^2_{L_2} \right) \int_{Q_t \times B} \left( \left\| \nabla x \frac{\partial_y u}{M} \right\|_{L_2}^2 + M \left\| \nabla x \frac{\partial_y u}{M} \right\|_{L_2}^2 \right) \, dx \, ds \\
\lesssim \int_0^t \left( \left\| (\mathbf{u}, \mathbf{u}) \right\|_{W^{s'+1,2}} \right\|^2_{W^{s',2,2}} \, dx \, ds &+ \int_0^t \left\| \mathbf{u}^{12} \right\|_{W^{s',2,2}} \left\| (\psi^1, \psi^2) \right\|_{W^{s',2,2}} \, dx \, ds,
\end{aligned}
\]

where the hidden constant is \( c = c(s, b, d) \). Substituting (4.31) into (4.30) yields

\[
\begin{aligned}
\left\| \psi^{12} (t) \right\|_{W^{s',2,2}} + 2 \varepsilon \int_0^t \left\| \psi^{12} \right\|_{W^{s'+1,2} H^2} \, ds &+ \left( \frac{A_{11}}{2\lambda} - 8\delta \right) \int_0^t \left\| \psi^{12} \right\|_{W^{s'+1,2} H^2} \, ds \\
\lesssim \int_0^t \left( \left\| (\mathbf{u}, \mathbf{u}) \right\|_{W^{s'+1,2}} + 1 \right) \left\| \psi^{12} \right\|_{W^{s',2,2}} \, dx \, ds &+ \int_0^t \left\| \mathbf{u}^{12} \right\|_{W^{s',2,2}} \left\| (\psi^1, \psi^2) \right\|_{W^{s',2,2}} \, dx \, ds.
\end{aligned}
\]
As a consequence of Gronwall’s lemma, we obtain

\[
\sup_{0 < t < T} \left\| \psi^1 - \psi^2 \right\|_{W_{x,2}^rL_{M}^2}^2 + 2\varepsilon \int_0^T \left\| \psi^1 - \psi^2 \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt + \left( \frac{A_1}{2\lambda} - 8\delta \right) \int_0^T \left\| \psi^1 - \psi^2 \right\|_{W_{x,2}^rH_{M}^1}^2 \, dt \\
\leq c \exp \left( c \int_0^T \left\| (u^1, u^2) \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt \right) \int_0^T \left\| u^1 - u^2 \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt
\]

(4.33)

which completes the proof of Theorem 4.3.

5. The coupled system

After solving the fluid system and the Fokker-Planck equation both independently from each other in the two previous sections, we are now in the position to solve the coupled system. This shall be done by a fixed point argument which finally leads to the proof of the main result from Theorem 2.6. Set

\[
X^* = L^\infty(0, T; W^{r,2}(\mathbb{R}^d; L_M^2(B))) \cap L^2(0, T; W^{r+1,2}(\mathbb{R}^d; H_M^1(B)))
\]

which is a Banach space equipped with the norm

\[
\left\| \cdot \right\|_{X^*} = \sup_{0 < t < T} \left\| \cdot \right\|_{W_{x}^{r,2}L_{M}^2}^2 + \int_0^T \left\| \cdot \right\|_{W_{x,2}^rH_{M}^1}^2 \, dt.
\]

For \( \tilde{\psi} \in X^* \), let \((r, \mathbf{u})\) be the unique solution to (3.1 - 3.2) with data \((r_0, \mathbf{u}_0, \mathbf{f}, T(\tilde{\psi}))\). The existence of such a solution of class

\[ r \in C([0, T]; W^{r,2}(\mathbb{R}^d)), \quad r > 0, \quad \mathbf{u} \in C([0, T]; W^{r,2}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^2(0, T; W^{r+1,2}(\mathbb{R}^d; \mathbb{R}^d)), \]

is guaranteed by Theorems 3.2 and 3.3 noticing that \( \tilde{\psi} \in X^* \) implies \( T(\tilde{\psi}) \in C([0, T]; W^{r,2}(\mathbb{R}^d; \mathbb{R}^d)) \) by Lemma 2.1. Now, given \((r, \mathbf{u})\) with the regularity above, we can solve equation (4.1) using Theorems 1.2 and 1.3. Hence we obtain a unique \( \psi \in X^* \). We denote the mapping \( \tilde{\psi} \mapsto \psi \) by \( \mathfrak{T} \). We start with the following lemma.

Lemma 5.1. Let \( s > \frac{d}{2} + 2 \). There is \( T > 0 \) and \( \mathfrak{R} > 0 \) such that \( \mathfrak{T} : \mathfrak{B}^s_{\mathfrak{R}} \to \mathfrak{B}^s_{\mathfrak{R}} \), where \( \mathfrak{B}^s_{\mathfrak{R}} \) is the closed ball in \( X^* \) with radius \( \mathfrak{R} \).

Proof. Choosing \( T \) sufficiently small we obtain from Theorem 3.2 that

\[
\sup_{0 < t < T} \left\| (r, \mathbf{u}) \right\|_{W_{x}^{r,2}L_{M}^2}^2 + \int_0^T \left\| \mathbf{u} \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt \leq c(r_0, \mathbf{u}_0, \mathbf{f}) + c \int_0^T \left\| T(\tilde{\psi}) \right\|_{W_{x}^{r,2}L_{M}^2}^2 \, dt.
\]

We infer further for arbitrary \( \delta > 0 \)

\[
\sup_{0 < t < T} \left\| (r, \mathbf{u}) \right\|_{W_{x}^{r,2}L_{M}^2}^2 + \int_0^T \left\| \mathbf{u} \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt \leq c + Tc_3 \sup_{0 < t < T} \left\| \tilde{\psi} \right\|_{W_{x}^{r+2,2}L_{M}^2}^2 + \delta \int_0^T \left\| \tilde{\psi} \right\|_{W_{x}^{r+2,2}H_{M}^1}^2 \, dt
\]

\[
\leq c + (Tc_3 + \delta)\mathfrak{R}^2.
\]

using Lemma 2.1. Now, we first choose \( \delta = \mathfrak{R}^{-2} \) and then \( T \) so small such that \( Tc_3 \leq 1 \). Consequently, the right-hand side is bounded by a constant only depending on the given data. On the other hand, we obtain from Theorem 4.4

\[
\sup_{0 < t < T} \left\| \tilde{\psi} \right\|_{W_{x}^{r+2,2}L_{M}^2}^2 + \int_0^T \left\| \tilde{\psi} \right\|_{W_{x}^{r+2,2}H_{M}^1}^2 \, dt \leq c \exp \left( c \int_0^T \left\| \mathbf{u} \right\|_{W_{x}^{r+1,2}L_{M}^2}^2 \, dt \right) \left\| \psi_0 \right\|_{W_{x}^{r+2,2}L_{M}^2}^2 \leq c,
\]

where the constant only depends on the data (but we have to choose \( T \) small enough). It is bounded by \( \mathfrak{R}^2 \) provided the latter one was chosen large enough compared to the data \((r_0, \mathbf{u}_0, f, \psi_0)\). \( \square \)

As a by-product of the proof of Lemma 5.1, we obtain the following corollary.
Corollary 5.2. Let \((r, u)\) be the unique solution to (3.1)-(3.2) with data \((r_0, u_0, f, T(\bar{\psi}))\). Under the assumptions of Lemma 5.1 we have

\[
\sup_{0 < t < T} \left\| (r, u) \right\|^2_{W^2 \times 2} + \int_0^T \left\| u \right\|^2_{W^{s+1, 2}} dt \leq c,
\]

where \(c\) only depends on the data \((r_0, u_0, f)\).

In the next step we have to show that \(\mathcal{F}\) is a contraction. Unfortunately, we are unable to show this on \(X^s\). However, we can work in the set

\[
Y^s_{s'} := \{ \bar{\psi} \in X^{s'} : s' \leq s - 1, \quad \bar{\psi} \in 2_{\mathcal{R}}\}.
\]

We have the following result.

Lemma 5.3. Let \(s > \frac{d}{2} + 2\). There is \(T > 0\) and \(\mathcal{R} > 0\) such that \(\mathcal{F} : Y^s_{s'} \to Y^s_{s'}\) is a contraction.

Proof. Let \((r^1, u^1)\) and \((r^2, u^2)\) be two strong solutions to (3.1)-(3.2) with data \((r_0, u_0, f, T(\bar{\psi}^1))\) and \((r_0, u_0, f, T(\bar{\psi}^2))\) respectively. We obtain from Theorem 3.3

\[
\int_0^T \left\| u^1 - u^2 \right\|^2_{W^{s+1, 2}} dt \leq c \exp \left( c \int_0^T \left( \left\| (r^1, r^2) \right\|^2_{W_x^{s+1, 2}} + \left\| (u^1, u^2) \right\|^2_{W_x^{s+1, 2}} + \left\| T(\bar{\psi}^2) \right\|^2_{W_x^{s+1, 2}} + 1 \right) dt \right)
\times \int_0^T \left| T(\bar{\psi}^1) - T(\bar{\psi}^2) \right| \left\| r^1 \right\|_{W_x^{s+1, 2}} \left\| r^2 \right\|_{W_x^{s+1, 2}} dt
\leq c(\mathcal{R}) \int_0^T \left| T(\bar{\psi}^1) - T(\bar{\psi}^2) \right| \left\| r^1 \right\|_{W_x^{s+1, 2}} \left\| r^2 \right\|_{W_x^{s+1, 2}} dt
\]

using also Lemma 5.1 (together with Lemma 2.1) and Corollary 5.2. On the other hand, let \(\psi^1\) and \(\psi^2\) be two strong solutions to (4.1) with data \((\psi_0, u^1)\) and \((\psi_0, u^2)\) respectively. Then Theorem 4.3 tells us that

\[
\sup_{0 < t < T} \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, L^2}} + \int_0^T \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, H^1}} dt
\leq c \exp \left( c \int_0^T \left\| (u^1, u^2) \right\|^2_{W_x^{s+1, 2}} \int_0^T \left\| u^1 - u^2 \right\|^2_{W_x^{s+1, 2}} \left\| (\psi^1, \psi^2) \right\|^2_{W_x^{s+1, 2}} \right) \int_0^T \left\| u^1 - u^2 \right\|^2_{W_x^{s+1, 2}} dt
\leq c(\mathcal{R}) \int_0^T \left\| u^1 - u^2 \right\|^2_{W_x^{s+1, 2}} dt
\]

using again Lemma 5.1 and Corollary 5.2. Combining both shows

\[
\sup_{0 < t < T} \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, L^2}} + \int_0^T \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, H^1}} dt \leq c(\mathcal{R}) \int_0^T \left| T(\bar{\psi}^1) - T(\bar{\psi}^2) \right| \left\| r^1 \right\|_{W_x^{s+2, 2}} \left\| r^2 \right\|_{W_x^{s+2, 2}} dt.
\]

Finally, we infer from Lemma 2.1 (with a suitable choice of \(\delta\))

\[
\sup_{0 < t < T} \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, L^2}} + \int_0^T \left\| \psi^1 - \psi^2 \right\|^2_{W_x^{s+2, H^1}} dt
\leq c(\mathcal{R}) T \sup_{0 < t < T} \left\| \bar{\psi}^1 - \bar{\psi}^2 \right\|^2_{W_x^{s+2, L^2}} + \frac{1}{2} \int_0^T \left\| \bar{\psi}^1 - \bar{\psi}^2 \right\|^2_{W_x^{s+2, H^1}} dt.
\]

The claim follows provided we choose \(T\) small enough to guarantee \(c(\mathcal{R}) T \leq \frac{1}{2}\).

Proof of Theorem 2.6 The claim of Theorem 2.6 follows from Lemma 5.3 due to Banach’s fixed point theorem.

Proof of Theorem 2.8 We have proved Theorem 2.6. Assume that the data \((\bar{q}_0, u_0, \psi_0, f)\) satisfy the hypothesis of Theorem 2.6. Setting \(r_0 := \sqrt{\frac{2 c^2}{\gamma - 1} \bar{q}_0^{-2}}\) we see that the initial data \((r_0, u_0, \psi_0, f)\) satisfy the
assumptions of Theorem 2.6 (in particular, \( r_0 \) is strictly positive). We obtain a unique strong solution \((r, u, \psi)\) to (2.6)–(2.7) with positive \( r \). Using the transformation
\[
\varrho := \left( \frac{\gamma - 1}{2a\gamma} \right)^{\frac{1}{\gamma - 1}} r^{\frac{2}{\gamma - 1}}
\]
it is now straightforward to see that \((\varrho, u, \psi)\) is the unique strong solution to \((1.6)–(1.8)\) defined in the same existence interval.

A consequence of the proof of Theorem 5.2 is the result below. It gives a blowup criterion for the Cauchy problem of the Navier–Stokes–Fokker–Planck system.

**Corollary 5.4.** Let \((\varrho, u, \psi)\) be a strong solution to problem (1.6)–(1.8) in the sense of Definition 2.2 on the interval \([0, T_{\max})\) with the data \((\varrho_0, u_0, \psi_0, f)\). If \(T_{\max} < \infty\) is the maximal existence time, then
\[
\limsup_{T \to T_{\max}} \left[ \|u\|_{W^{2,\infty}} + \|\text{div}_x \mathbf{T}(\psi)\|_{L^\infty_x} \right] = \infty.
\]

**Proof.** Assume that the preamble and assumption of Corollary 5.4 holds true but that the conclusion is false. Then
\[
\limsup_{T \to T_{\max}} \left[ \|u\|_{W^{2,\infty}} + \|\text{div}_x \mathbf{T}(\psi)\|_{L^\infty_x} \right] \lesssim R.
\]
for some \( R < \infty \) which in turn yield
\[
\varrho(t, x) \leq c(R) \quad \text{for all} \quad (t, x) \in [0, T_{\max}] \times \mathcal{T}^d
\]
by virtue of the maximum principle for (1.6). Per the transformation (2.4), it follows that (3.10) still hold and thus, validating the a priori estimates established in Sections 3.2 and 4.1 on \([0, T_{\max})\). The equivalence of (1.6)–(1.7) and (2.5)–(2.6) and the fact that \((\varrho, u, \psi)\) satisfies the a priori estimates means that
\[
\varrho \in C([0, T_{\max}]; W^{1,a}(\mathcal{T}^d)),
\]
\[
u \in C([0, T_{\max}]; W^{s,2}(\mathcal{T}^d) \cap L^2(0, T_{\max}; W^{s+1,2}(\mathcal{T}^d))),
\]
\[
\psi \in C([0, T_{\max}]; W^{s,2}(\mathcal{T}^d; L^2_M(B))) \cap L^2(0, T_{\max}; W^{s,2}(\mathcal{T}^d; H^1_M(B))).
\]
Now note that the density remains positive for a positive initial density. It means that we can take \((\varrho, u, \psi, f)\) as a new data so that Theorem 2.3 establishes the existence of a time \(T_{\max} > T_{\max}^*\) on which a strong solution exist. This contradicts the fact that \(T_{\max}\) is maximal.

**REFERENCES**

[1] Barrett, J.W., Lu, Y., Süli, E.: Existence of large-data finite-energy global weak solutions to a compressible Oldroyd-B model. *Commun. Math. Sci.*, 15(5), 1265–1323 (2017).

[2] Barrett, J.W., Schwab, C., Süli, E.: Existence of global weak solutions for some polymeric flow models. *Math. Models Methods Appl. Sci.*, 15(6), 939–983 (2005).

[3] Barrett, J.W., Süli, E.: Existence of global weak solutions to some regularized kinetic models for dilute polymers. *Multiscale Model. Simul.*, 6(2), 506–546 (2007).

[4] Barrett, J.W., Süli, E.: Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off. *Math. Models Methods Appl. Sci.*, 18(6), 935–971 (2008).

[5] Barrett, J.W., Süli, E.: Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains. *Math. Models Methods Appl. Sci.*, 21(6), 1211–1289 (2011).

[6] Barrett, J.W., Süli, E.: Existence and equilibration of global weak solutions to kinetic models for dilute polymers II: Hookean-type models. *Math. Models Methods Appl. Sci.*, 22(5), 1150,024, 84 (2012).

[7] Barrett, J.W., Süli, E.: Existence of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers with variable density and viscosity. *J. Differential Equations*, 253(12), 3610–3677 (2012).

[8] Barrett, J.W., Süli, E.: Existence of global weak solutions to compressible isentropic finitely extensible bead-spring chain models for dilute polymers. *Mathematical Models and Methods in Applied Sciences*, 26(03), 469–568 (2016).

[9] Barrett, J.W., Süli, E.: Existence of global weak solutions to compressible isentropic finitely extensible nonlinear bead-spring chain models for dilute polymers: the two-dimensional case. *J. Differential Equations* 261(1), 592–626 (2016).

[10] Barrett, J.W., Süli, E.: Existence of large-data global-in-time finite-energy weak solutions to a compressible FENE-P model. *Math. Models Methods Appl. Sci.*, 28(10), 1929–2000 (2018).
[11] Bird, R.B., Curtiss, C.F., Armstrong, R.C., Hassager, O.: *Dynamics of polymeric liquids. vol. 2: Kinetic theory*. Wiley, New York (1987).

[12] Breit, D., Feireisl, E., Hofmanová, M.: Local strong solutions to the stochastic compressible Navier-Stokes system. *Comm. Partial Differential Equations*, 43(2), 313–345 (2018).

[13] Chemin, J.Y., Masmoudi, N.: About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.* 33(1), 84–112 (2001).

[14] Constantin, P.: Nonlinear Fokker-Planck Navier-Stokes systems. *Commun. Math. Sci.* 3(4), 531–544 (2005).

[15] Constantin, P., Fefferman, C., Titi, E.S., Zarnescu, A.: Regularity of coupled two-dimensional nonlinear Fokker-Planck and Navier-Stokes systems. *Comm. Math. Phys.* 270(3), 789–811 (2007).

[16] Degond, P., Liu, H.: Kinetic models for polymers with inertial effects. *Netw. Heterog. Media*, 4(4), 625–647 (2009).

[17] E., W., Li, T., Zhang, P.: Well-posedness for the dumbbell model of polymeric fluids. *Comm. Math. Phys.* 248(2), 409–427 (2004).

[18] El-Kareh, A.W., Leal, L.G.: Existence of solutions for all Deborah numbers for a non-Newtonian model modified to include diffusion. *Journal of Non-Newtonian Fluid Mechanics*, 33(3), 257–287 (1989).

[19] Feireisl, E., Lu, Y., Suli, E.: Dissipative weak solutions to compressible Navier-Stokes-Fokker-Planck systems with variable viscosity coefficients. *J. Math. Anal. Appl.* 443(1), 322–351 (2016).

[20] Gordon, M.: From Riemann’s metric to the graph metric, or applying Occam’s razor to entanglements. *Polymer*, 20(11), 1349-1356 (1979).

[21] Jourdain, B., Lelièvre, T., Le Bris, C.: Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.* 209(1), 162–193 (2004).

[22] Klainerman, S., Majda, A.: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.* 34(4), 481–524 (1981).

[23] Kreml, O., Pokorný, M.: On the local strong solutions for the FENE dumbbell model. *Discrete Contin. Dyn. Syst. Ser. S*, 3(2), 311–324 (2010).

[24] Li, T., Zhang, H., Zhang, P.: Local existence for the dumbbell model of polymeric fluids. *Comm. Partial Differential Equations* 29(5-6), 903–923 (2004).

[25] Lions, P.L., Masmoudi, N.: Global existence of weak solutions to some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2), 131–146 (2000).

[26] Lions, P.L., Masmoudi, N.: Global existence of weak solutions to some micro-macro models. *C. R. Math. Acad. Sci. Paris*, 345(1), 15–20 (2007).

[27] Luo, W., Yin, Z.: Global existence and well-posedness for the FENE dumbbell model of polymeric fluids. *Nonlinear Anal. Real World Appl.*, 37, 457–488 (2017).

[28] Masmoudi, N.: Well-posedness for the FENE dumbbell model of polymeric fluids. *Comm. Pure Appl. Math.*, 61(12), 1685–1714 (2008).

[29] Masmoudi, N.: Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Invent. Math.* 191(2), 427–500 (2013).

[30] Nitta, K.h.: A graph-theoretical approach to statistics and dynamics of tree-like molecules. *J. Math. Chem.*, 25(2-3), 133–143 (1999).

[31] Otto, F., Tzavaras, A.E.: Continuity of velocity gradients in suspensions of rod-like molecules. *Comm. Math. Phys.* 277(3), 729–758 (2008).

[32] Reiner, M.: The deborah number. *Physics today* 17(1), 62 (1964).

[33] Renardy, M.: An existence theorem for model equations resulting from kinetic theories of polymer solutions. *SIAM J. Math. Anal.*, 22(2), 313–327 (1991).

[34] Rouse Jr, P.E.: A theory of the linear viscoelastic properties of dilute solutions of coiling polymers. *The Journal of Chemical Physics* 21(7), 1272–1280 (1953).

[35] Schieber, J.D.: Generalized brownian configuration fields for fokker–planck equations including center-of-mass diffusion. *Journal of non-newtonian fluid mechanics*, 135(2-3), 179–181 (2006).

[36] Warner, H.R.: Kinetic theory and rheology of dilute suspensions of finitely extendible dumbbells. *Industrial & Engineering Chemistry Fundamentals*, 11(3), 379–387 (1972).

[37] Zhang, H., Zhang, P.: Local existence for the FENE-dumbbell model of polymeric fluids. *Arch. Ration. Mech. Anal.* 181(2), 373–400 (2006).