TWO POINT EXTREMAL GROMOV-WITTEN INVARIANTS OF HILBERT SCHEMES OF POINTS ON SURFACES

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ABSTRACT. Given an algebraic surface $X$, the Hilbert scheme $X^{[n]}$ of $n$-points on $X$ admits a contraction morphism to the $n$-fold symmetric product $X^{(n)}$ with the extremal ray generated by a class $\beta_n$ of a rational curve. We determine the two point extremal GW-invariants of $X^{[n]}$ with respect to the class $d\beta_n$ for a simply-connected projective surface $X$ and the quantum first Chern class operator of the tautological bundle on $X^{[n]}$. The methods used are vertex algebraic description of $H^\bullet(X^{[n]})$, the localization technique applied to $X=P^2$, and a generalization of the reduction theorem of Kiem-J. Li to the case of meromorphic 2-forms.

1. INTRODUCTION

The Hilbert scheme $X^{[n]}$ of $n$-points of an algebraic surface $X$ is a crepant resolution of the $n$-fold symmetric product $X^{(n)}$. The extremal ray of the contraction map $\pi: X^{[n]} \to X^{(n)}$ is generated by the class of a rational curve $\beta_n$ in $X^{[n]}$. The $k$-point extremal Gromov-Witten invariants on $X^{[n]}$ is defined by

$$\langle A^1, \ldots, A^k \rangle_{0,k,d} = \int_{\overline{\mathcal{M}}_{0,k}(X^{[n]}, d)} \ev^* (A^1 \otimes \cdots \otimes A^k), \ A^i \in H^\bullet(X^{[n]}, \mathbb{C}).$$

When $X$ is a simply-connected projective surface, the 1-point extremal Gromov-Witten invariants of $X^{[n]}$ are computed in [L-Q]. The main goal of this paper is to compute the 2-point extremal Gromov-Witten invariants of $X^{[n]}$.

Besides its own interest, this research is motivated by the following two reasons. The first comes from a conjecture of Y. Ruan. Since $X^{(n)} = X^n/S_n$ is an orbifold, one may ask a McKay correspondence type question relating the cohomology ring of $X^{[n]}$ with the orbifold cohomology ring of $X^{[n]}$. The orbifold cohomology ring $H^\bullet_{CR}(X^n/S_n)$ was defined by Chen and Ruan in [C-R] (see also [AGV]). Using the extremal Gromov-Witten invariants, Ruan defined a “restricted” quantum cohomology ring $H^\bullet_{\pi}(X^{[n]})$ of the Hilbert scheme $X^{[n]}$ and conjectured in [Ruan] that $H^\bullet_{CR}(X^n/S_n)$ is isomorphic to $H^\bullet_{\pi}(X^{[n]})$ as rings (see also [B-G]). The orbifold cohomology ring $H^\bullet_{CR}(X^n/S_n)$ was computed by Fantechi-Göttsche and Uribe independently in [F-G, Ur]. Thus the complete determination of the cohomology ring of $X^{[n]}$ depends upon the computation of the extremal Gromov-Witten invariants on $X^{[n]}$.

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The other motivation is the vertex algebraic nature of the cohomology ring of $X^{[n]}$. Grojnowski [Gro] and Nakajima [Na1] discovered that the direct sum of the cohomology groups of $X^{[n]}$ over all $n$ is a highest weight irreducible representation of a Heisenberg algebra. From the work of Lehn [Lehn], Lehn-Sorger [L-S], Li-Qin-Wang [LQW], and Costello-Grojnowski [C-G] on the cohomology ring of $X^{[n]}$, the ring structure of $X^{[n]}$ can be understood via Chern characters of the tautological bundle on $X^{[n]}$ twisted by cohomology classes of $X$. In particular, the first Chern class of the tautological bundle plays a fundamental role. The right viewpoint is quantum cohomology ring $H$ expressed in terms of Heisenberg operators. In the same spirit, to understand the bundle on $X$ that the first Chern class should be regarded as an operator on the cohomology class operator, which comes from the first Chern class acting on the cohomology ring structure of $X^{[n]}$, is determined in [O-P]. The method used there is via localization, and the key is to compute the quantum first Chern class operator. In this paper, we shall determine the quantum first Chern class operator when $X$ is projective and simply-connected. As a consequence, the two point extremal Gromov-Witten invariants are determined.

The main technique used in this paper is a generalized version of the reduction theorem of the virtual cycle of the moduli space $\mathfrak{M}_{g,k}(X^{[n]}, d)$ of stable maps to the Hilbert scheme $X^{[n]}$. The reduction theorem was first observed by Lee and Parker in symplectic geometry. The algebro-geometric treatment of it was done by Kiem and the first author in [K-L]. The original theorem deals with projective surfaces $X$ with a holomorphic two-form. In order to cover general surfaces, we extend the reduction theorem to cover the case where only meromorphic sections of $\Omega_X^2$ are used. We now briefly outline the argument used in this paper.

The Hilbert scheme $X^{[n]}$ contains a rational curve
$$\{ \xi_{x_0} + x_1 + \ldots + x_{n-2} \in X^{[n]} \mid \text{Supp}(\xi_{x_0}) = x_0 \},$$
where $x_0, x_1, \ldots, x_{n-2}$ are fixed points on $X$. This curve is a generator of the extremal ray of the contraction map
$$\pi: X^{[n]} \to X^{(n)}.$$
Let $\beta_n$ represent the class of this curve in $H_2(X^{[n]}, \mathbb{Z})$.

A stable map $\varphi \in \mathfrak{M}_{g,k}(X^{[n]}, d)$ can be factorized through the product of punctual Hilbert schemes
$$\varphi = (\varphi_1, \ldots, \varphi_l): C \to \prod X_{p_i}^{[n_i]} \subset X^{[n]},$$
where $\varphi_*(C) = d\beta_n$ and $\varphi_i$ is a morphism from $C$ to the punctual Hilbert scheme $X_{p_i}^{[n_i]}$ consisting of closed subschemes of $X$ of length $n_i$ supported at a fixed point $p_i$.

Suppose $X$ admits a holomorphic 2-form $\theta$. The reduction theorem basically says that the virtual cycle $[\mathfrak{M}_{g,k}(X^{[n]}, d)]^{vir}$ is a homology class of a much smaller space consisting of stable maps $\varphi$ satisfying that for each $i$, either $\varphi_i$ is a constant map or its support $p_i$ lies in the vanishing locus of $\theta$. To extend the reduction
To an arbitrary projective surface, we have to consider meromorphic two-forms on $X$. The main part of the section §4 is to prove the reduction theorem for a general surface. The conclusion is similar except we replace the vanishing locus of a holomorphic two-form by the zero locus and pole locus of a meromorphic two-form.

Using the reduction theorem, we conclude that the two-point extremal Gromov-Witten invariants satisfy a universal formula with universal coefficients to be determined. To get the explicit expressions for the universal coefficients, we study the equivariant two-point extremal Gromov-Witten invariants on the projective plane $\mathbb{P}^2$ equipped with a torus action. Using the localization, the computation is reduced to that on the complex plane $\mathbb{C}^2$ which was done by Okounkov and Pandharipande in [O-P]. By the divisor axiom in the Gromov-Witten theory, we also get the formula for the quantum first Chern class operator expressed in terms of Heisenberg operators.

In this paper, all the homology and cohomology classes on $X^{[n]}$ are expressed in terms of Nakajima’s basis. Let $a_i(\alpha)$ be the Heisenberg operators on the direct sum of the cohomology of Hilbert schemes $X^{[n]}$ over all $n$, where $\alpha$ is a cohomology class on $X$. Let $M(q)$ be the quantum first Chern class operator $c_1(O^{[n]} \cup \pi)$ where $O^{[n]}$ is the tautological bundle on $X^{[n]}$ and $\cup \pi$ is the extremal quantum cup product defined via extremal Gromov-Witten invariants. We have the following theorem.

**Theorem 1.1.** As an operator, the quantum first Chern class operator $M(q)$ can be expressed in terms of Heisenberg operators as

$$M(q) = \sum_{k>0} \left( \frac{k}{2} (-q)^{\frac{k}{2}} + \frac{1}{2} \left( \frac{1}{-q} + 1 \right) a_{-k} a_k (\tau_2^* [K_X]) \right)$$

$$- \frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} + a_{-k} a_{k+\ell} a_\ell \right) (\tau_3^* [X]).$$

Again by the divisor axiom in the Gromov-Witten theory, the two-point extremal Gromov-Witt invariants of $X^{[n]}$ is thus completely determined. As a corollary, Ruan’s conjecture that

$$H^\bullet_{CR}(X^n/S_n) \cong H^\bullet_\tau(X^{[n]})$$

is also verified for two-point case.

The paper is organized as follows. In Section two, we review some basic facts and notations about cohomology $H^*(X^{[n]})$ of the Hilbert scheme $X^{[n]}$, especially Nakajima’s treatment of $H^*(X^{[n]})$. We review the extremal Gromov-Witten invariants of Hilbert schemes. In Section three, we generalize the reduction technique via holomorphic two-forms of [K-L] to the case of meromorphic sections. In Section four, we prove the universality of two-point extremal Gromov-Witten invariants of $X^{[n]}$ based on the reduction theorem. As the result, the mentioned Gromov-invariants only depend on certain universal coefficients. In the next two Sections, we study the operator $M(q)$ and determine these universal coefficients by working on the projective plane with a torus action. We achieve this by using the localization technique to reduce to the computation of equivariant Gromov-Witten invariants on the complex plane, which was already done in [O-P]. In Section seven, we determine the explicit formula for the quantum first Chern class operator and the two-point extremal Gromov-Witten invariants of the Hilbert scheme. We also verify Ruan’s conjecture for two-point case.
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2. Preliminary on Hilbert schemes

2.1. The cohomology groups $H^*(X^{[n]}, \mathbb{C})$ of the Hilbert scheme. For a smooth, simply-connected projective surface $X$, we let $X^{[n]}$ be the Hilbert scheme of length-$n$ zero dimensional subschemes of $X$. The Hilbert scheme $X^{[n]}$ is smooth; its Hilbert-Chow morphism $\pi: X^{[n]} \to X^{(n)}$ is a crepant resolution of the symmetric product $X^{(n)}$.

It is known that the cohomology groups of $X^{[n]}$ form an infinite dimensional vector space

$$H^*_{X^{[n]}} \cong \bigoplus_{n=0}^{\infty} \bigoplus_{k=0}^{4n} H^k(X^{[n]}, \mathbb{C})$$

that is the highest weight irreducible representation of a Heisenberg algebra

$$\{a_i(\alpha)\}_{i \in \mathbb{Z}, \alpha \in H^*(X)}$$

of which the operators $a_i(\alpha)$ satisfy the Heisenberg commutation relation

$$[a_i(\alpha), a_j(\beta)] = -i\delta_{i+j,0} \int_X \alpha \cup \beta, \quad (2.1)$$

and the highest weight vector

$$|0\rangle = 1 \in H^0(\text{pt}, \mathbb{C}).$$

We also know the generators of the vector space $H^*(X^{[n]}, \mathbb{C})$: it is generated by elements of the forms

$$A^\nu = A^\nu(\alpha_1, \ldots, \alpha_r) = a_{-\nu_1}(\alpha_1) \cdots a_{-\nu_r}(\alpha_r)|0\rangle$$

where $\nu: \nu_1 \geq \nu_2 \geq \ldots \geq \nu_r$ is a partition of $n$ and $\alpha_1, \ldots, \alpha_r$ are cohomology classes of $X$.

The same results hold for the homology groups of $X^{[n]}$. The infinite dimensional vector space

$$H_\ast_{X^{[n]}} \cong \bigoplus_{n=0}^{\infty} \bigoplus_{k=0}^{4n} H_k(X^{[n]}, \mathbb{C})$$

is the highest weight irreducible representation of a Heisenberg algebra

$$\{a_i(e)\}_{i \in \mathbb{Z}, e \in H_*(X)}$$

where the operators $a_i(e)$ satisfy the Heisenberg commutation relation

$$[a_i(e), a_j(e')] = -i\delta_{i+j,0} \int_X PD^{-1}e \cup PD^{-1}e', \quad (2.2)$$

and the highest weight vector

$$|0\rangle = 1 \in H^0(\text{pt}, \mathbb{C}).$$

The homology group $H_*(X^{[n]}, \mathbb{C})$ is also generated by classes

$$A_\nu = A_\nu(e_1, \ldots, e_r) = a_{-\nu_1}(e_1) \cdots a_{-\nu_r}(e_r)|0\rangle.$$
It is known that $PD^{-1}A_\nu = A^\nu$ if we take $\alpha_i = PD^{-1}e_i$.

The homology class $A_\nu$ has a geometric description. If we represent $e_i$ by geometric representatives $Z_i \subset X$, then $A_\nu(e_1, \ldots, e_r)$ is a multiple of the closure of the set

$$\{\xi_1 + \ldots + \xi_r \in X^{[n]} \mid \text{Supp}(\xi_i) = x_i, x_i \neq x_j \text{ for } i \neq j; \ell(\xi_i) = \nu_i\}$$

One more useful remark: under the pairing $\langle A^\lambda, A^\mu \rangle = \int_{X^{[n]}} A^\lambda \cup A^\mu$, the operator $a_i(\alpha)$ is the adjoint operator of $(-1)^i a_{-i}(\alpha)$.

A convention used through out the paper: for a subvariety $Y$ of a variety $X$, we use $[Y]$ to represent the cohomology class in $H^*(X)$ dual to $Y$.

For the details of the results quoted here, the readers can consult [Na2].

### 2.2. Extremal Gromov-Witten invariants of Hilbert schemes

It is known that the Hilbert-Chow morphism $\pi: X^{[n]} \to X^{(n)}$ contracts the curve class $\beta_n$ mentioned in the introduction; the class $\beta_n$ also spans the extremal ray of the morphism $\pi$ (see [LQZ, V]).

In this paper, we shall investigate the moduli space of genus zero stable morphisms to $X^{[n]}$ with a $d$-multiple of the extremal curve $\beta_n$ as the fundamental class; we denote this moduli space by $\mathfrak{M}_{0,k}(X^{[n]}, d)$, where $k$ stands for the number of marked points on the domains of the morphisms. The $k$-point extremal Gromov-Witten invariants are defined via

$$\langle A^1, \ldots, A^k \rangle_{0,k,d} = \int_{[\mathfrak{M}_{0,k}(X^{[n]}, d)]^{vir}} ev^*(A^1 \otimes \ldots \otimes A^k), \quad A^i \in H^*(X^{[n]}, \mathbb{C}), \quad (2.3)$$

where $ev: \mathfrak{M}_{0,k}(X^{[n]}, d) \to X^{[n]} \times \ldots \times X^{[n]}$ is the evaluation map at $k$ marked points.

Using the extremal Gromov-Witten invariants (2.3), Ruan in [Ruan] defined an extremal quantum cup-product structure on $H^*(X^{[n]})$ as follows: by denoting

$$\langle A^1, A^2, A^3 \rangle_{qc}(q) = \sum_{d>0} \langle A^1, A^2, A^3 \rangle_{0,3,d} \cdot q^d$$

and denoting

$$\langle A^1, A^2, A^3 \rangle_{qc} = \langle A^1, A^2, A^3 \rangle_{qc}(q)|_{q=-1},$$

he defined the quantum corrected cup product $A^1 \cup_{qc} A^2$ by

$$\langle A^1 \cup_{qc} A^2, A^3 \rangle = \langle A^1 \cup A^2, A^3 \rangle + \langle A^1, A^2, A^3 \rangle_{qc}.$$

The cohomology group $H^*(X^{[n]})$ with so defined quantum product $\cup_{qc}$ is denoted by $H^*_{qc}(X^{[n]})$.

Chen and Ruan defined a ring structure on the orbifold cohomology group $H^*_c(M/G)$ of the quotient of a manifold $M$ by a finite group $G$. We denote this cohomology ring by $H^*_cR(M/G)$. Applying this to the quotient $X^{(n)} = X^n/S_n$, we obtain the Chen-Ruan orbifold cohomology ring $H^*_cR(X^{(n)})$. The cohomological crepant resolution conjecture formulated by Ruan that relates the Chen-Ruan cohomology ring of an orbifold with the quantum corrected cohomology ring of its crepant resolution, in the case of Hilbert schemes, is of the following form:

**Conjecture 2.1** (Ruan). $H^*_cR(X^{(n)}) \cong H^*_{qc}(X^{[n]})$ as rings.
Since the ring $H^*_{CR}(X^{(n)})$ was computed in [F-G, Ur], to verify this conjecture, we need to derive an explicit form of the quantum corrected cohomology ring of $H^*_{\pi}(X^{[n]})$, which will be the task of the most of the remaining sections.

3. The Reduction Lemma

Our approach relies on the reduction theorem of the virtual cycle of the moduli space $M_{g,k}(X^{[n]}, d)$ observed by Lee-Parker in symplectic geometry. The algebro-geometric treatment is the localization technique worked out by Kiem and the first author in [K-L]. To begin with, we suppose the surface $X$ admits a non-trivial holomorphic differential two-form $\theta \in \Gamma(\Omega^2_X)$. Then following Beauville, $\theta$ induces a holomorphic two form $\theta^{[n]}$ of the Hilbert scheme $X^{[n]}$, and by the result of [K-L], it defines a regular cosection of the obstruction sheaf of $M = M_g(X^{[n]}, d)$:

$$\eta: Ob_M \rightarrow O_M.$$ (3.1)

Here $Ob_M$ is the obstruction sheaf and $O_M$ is the structure sheaf of $M$. We remark that marked points don’t play any role here. For simplicity, we only consider the case without marked points.

This cosection reduces the virtual cycle of $M$ to a smaller subset of it.

Lemma 3.1 ([K-L]). Let $\Lambda \subset M$ be the loci of points over which $\eta$ fails to be surjective. Then the virtual cycle $[M]^{vir} \in H_*(\Lambda)$.

To identify the vanishing loci of $\eta$, we recall the vanishing criterion stated in [K-L]: $\eta$ vanishes at $\varphi \in M$ if the image of $\varphi_* : T_{C_{reg}} \rightarrow TX^{[n]}$ lies in the null space of

$$\vartheta^{[n]} : TX^{[n]} \rightarrow T^\vee X^{[n]}.$$ (3.2)

To pinpoint such $\varphi$, we notice that because of our choice of the fundamental class of stable morphisms under investigation, the composite of any $\varphi \in M$ with the Hilbert-Chow morphism

$$\pi: X^{[n]} \rightarrow X^{(n)}$$ (3.3)

is a constant map. Let

$$Spt: M \rightarrow X^{(n)}$$

be the induced map. Then in case $Spt(\varphi) = \sum n_ip_i$, the morphism $\varphi$ factors through the product of punctual Hilbert schemes:

$$\varphi = (\varphi_1, \cdots, \varphi_l): C \rightarrow \prod X^{[n]}_{p_i} \subset X^{[n]}$$ (3.4)

in which each $\varphi_i$ is a morphism from $C$ to the punctual Hilbert scheme $X^{[n]}_{p_i}$. (Here, $X^{[m]}_p$ is the preimage $\pi^{-1}(mp)$ of $mp \in X^{(m)}$.) In the following, we call the collection $\varphi = (\varphi_i)$ the standard decomposition of $\varphi$ and call $p_i$ the support of $\varphi_i$. Note that the collection $\{\varphi_i\}$ is canonical except the ordering; the ordering depends on the ordering of points in $Spt(\varphi)$.

Lemma 3.2. Let $\Lambda_\theta \subset M$ be the set of those $\varphi \in M$ whose decompositions $\varphi = (\varphi_i)$ satisfying that for each $i$ either $\varphi_i$ is a constant or its support $p_i = Spt(\varphi_i)$ lies in the vanishing locus of $\theta$. Then the locus where $\eta$ fails to be surjective is contained in $\Lambda_\theta$. 
Proof. Suppose \( \varphi \in \mathcal{M} \) lies in the vanishing loci of \( \eta \) and suppose \( \varphi = (\varphi_i) \) is its decomposition. By the criterion stated in [K-L], \( \eta(\varphi) = 0 \) if and only if \( \varphi_*(\text{TC}_{\text{reg}}) \) lies in the null space of \( \theta^{[n]} \). Because at a zero-dim subscheme \( \xi \) that is a union of \( l \) mutually disjoint zero-dim subscheme \( \xi_i \in X^{[n_i]} \),

\[
T_\xi X^{[n]} = \bigoplus_{i=1}^l T_{\xi_i} X^{[n_i]}
\]

and the form \( \theta^{[n]} \) is a direct sum of \( \theta^{[n_i]} \), the image space \( \varphi_*(\text{TC}_{\text{reg}}) \) lies in the null space of \( \theta^{[n]} \) if and only if \( \varphi_i*(\text{TC}_{\text{reg}}) \) lies in the null space of \( \theta^{[n_i]} \) for all \( i \). Now suppose \( \xi_i \) is supported at a single point \( p_i \). By the work of Beauville, the form \( \theta^{[n_i]} \) is non-degenerate along \( X^{[n_i]} \), if \( p_i \notin \theta^{-1}(0) \). Applying this to the support of \( \varphi_i \), we obtain the desired inclusion, thus proving the Lemma.

This reduction Lemma is sufficient for our application in case we have a regular section \( \theta \in H^0(X, K_X) \). For general surfaces, we might not have such sections. Instead, we will work with meromorphic sections of \( K_X \) and show that such sections will provide us the reduction lemma we need.

To this end, we let

\[
f : \mathcal{C} \longrightarrow X^{[n]} \quad \text{and} \quad \pi : \mathcal{C} \rightarrow \mathcal{M}
\]

be the universal family of \( \mathcal{M} \). As shown in [L-T], the obstruction sheaf \( \mathcal{O}_{\mathcal{M}} \) of \( \mathcal{M} \) is a quotient sheaf of \( R^1\pi_* f^* T_X^{[n]} \). We then pick a locally free sheaf \( E \) that surjects onto \( R^1\pi_* f^* T_X^{[n]} \), of which the later surjects onto the obstruction sheaf \( \mathcal{O}_{\mathcal{M}} \). We let \( E \) be the vector bundle on \( \mathcal{M} \) whose sheaf of sections is \( E \). Then the construction of virtual cycle provides us a cone cycle \( V \in C_* E \) whose intersection with the zero section of \( E \) gives rise to the virtual cycle \( [\mathcal{M}]^{\text{vir}} \).

Next, a meromorphic section \( \theta \) of \( K_X \), viewed as a meromorphic section of \( \Omega_X^2 \), induces a meromorphic section \( \theta^{[n]} \) of \( \Omega_X^{2[n]} \), and hence a meromorphic homomorphism

\[
\eta : E \longrightarrow \mathcal{O}_{\mathcal{M}}.
\]

We let \( D_0 \) (resp. \( D_\infty \)) be the vanishing (resp. pole) divisor of \( \theta \).

Adopting the proof of the previous lemma, we immediately see that the degeneracy loci of \( \eta \):

\[
\text{Deg}(\eta) = \{ \varphi \in \mathcal{M} \mid \text{either } \eta \text{ is undefined or fails to be surjective at } \varphi \}
\]

is contained in the set of all \( \varphi = (\varphi_i) \) such that either for some \( i \) the support of \( \varphi_i \) is contained in \( D_0 \cup D_\infty \), or for each \( i \) the map \( \varphi_i \) is a constant.

It is the purpose of the remaining section to prove the reduction Lemma, which says that the virtual cycle \( [\mathcal{M}]^{\text{vir}} \) lies in a much smaller set than \( \text{Deg}(\eta) \).

**Proposition 3.3.** Let \( \Lambda_0 \subset \mathcal{M} \) be the subset that consists of those \( \varphi \in \mathcal{M} \) whose decompositions \( \varphi = (\varphi_i) \) have the property that for each \( i \) either \( \varphi_i \) is constant or the support of \( \varphi_i \) lies in the union \( D_0 \cup D_\infty \). Then the virtual cycle \( [\mathcal{M}]^{\text{vir}} \) is supported in \( \Lambda_0 \).

We first investigate the behavior of \( \eta \) over where it is undefined. For this, we introduce a partition of the moduli stack \( \mathcal{M} \) based on the standard stratification of \( X^{(n)} \). Recall that the standard stratification of \( X^{(n)} \) is indexed by the set of all partitions of \( n \), and that to each partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_l) \) the stratum \( X^\lambda \) is

\[
X^\lambda = \{ z = \sum_{i=1}^l \lambda_i x_i \in X^{(n)} \mid x_1, \cdots, x_l \in X \text{ are distinct} \}.
\]
Using preimages of the support map $\text{Spt} : \mathcal{M} \to X^{(n)}$ mentioned in (3.3), we obtain a partition of $\mathcal{M}$ indexed by $\lambda$: $\mathcal{M}_\lambda = \text{Spt}^{-1}(X^\lambda)$, each endowed with the reduced stack structure.

To proceed, we shall split the maps in $\mathcal{M}_\lambda$ into $l$ individual maps. To achieve this, we need an ordering of the points occurring in the support of elements in $X^\lambda$. We let

$$\psi_\lambda : X^{(\lambda_1)} \times \cdots \times X^{(\lambda_l)} \to X^{(n)}$$

be the map that sends $(\xi_1, \ldots, \xi_l)$ to $\sum \xi_i$. Within the domain of $\psi_\lambda$, we let $B_\lambda$ be the open subset of all $(\xi_1, \ldots, \xi_l)$ such that the support of $\xi_i$ are mutually disjoint. Clearly, $\psi_\lambda(B_\lambda) = X^\lambda$, and $\psi_\lambda : B_\lambda \to X^\lambda$ is étale.

Using $B_\lambda$, we form $\mathcal{U}_\lambda$ and the projection

$$j_\lambda : \mathcal{U}_\lambda = \mathcal{M} \times_{X^{(n)}} B_\lambda \to \mathcal{M};$$

we let

$$f_\lambda : C_\lambda \to X^{[n]}, \quad \pi_\lambda : C_\lambda \to \mathcal{U}_\lambda$$

be the pull back to $\mathcal{U}_\lambda$ of the universal family $f$ via $j_\lambda$. Because elements in $\mathcal{U}_\lambda$ are

$$(\varphi, (\xi_1, \ldots, \xi_l)) \in \mathcal{M} \times_{X^{(n)}} B_\lambda$$

with support $\text{Spt}(\varphi) = \sum \xi_i$, as in (3.4) $\varphi$ canonically splits into $l$ maps $(\varphi_1, \cdots, \varphi_l)$ so that the support of $\varphi_i$ is $\xi_i$. Obviously, this construction can be carried over to the family $f_\lambda$. In this way, we obtain $l$ morphisms

$$f_{\lambda,i} : C_\lambda \to X^{[\lambda_i]}$$

such that over each closed point $(\varphi, (\xi_i)) \in \mathcal{U}_\lambda$ the morphism $f_{\lambda,i}$ is the $\varphi_i$ alluded before.

Next we look at the obstruction sheaf $\mathcal{O}_{\mathcal{M}}$. Recall that in constructing the virtual cycle we have picked a locally free sheaf $\mathcal{E}$ surjects onto the sheaf $R^1 \pi_* f^* \mathcal{T}_X^{[n]}$. As shown in [L-T], we can pick $\mathcal{E}$ so that over each $\mathcal{U}_\lambda$, we have direct sum decomposition

$$j_\lambda^* \mathcal{E} = \mathcal{E}_{\lambda,1} \oplus \cdots \oplus \mathcal{E}_{\lambda,l}$$

and surjective homomorphisms

$$\mathcal{E}_{\lambda,i} \twoheadrightarrow R^1 \pi_{\lambda,i} f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]}$$

that fits into the following commutative diagram

$$\begin{array}{ccc}
\mathcal{E} & \to & R^1 \pi_* f^* \mathcal{T}_X^{[n]} \\
\mathcal{E}_{\lambda,1} \oplus \cdots \oplus \mathcal{E}_{\lambda,l} & \to & R^1 \pi_{\lambda,i} f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]} \oplus \cdots \oplus R^1 \pi_{\lambda,i} f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]} \\
\mathcal{E}_{\lambda,i} & \to & R^1 \pi_{\lambda,i} f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]} \\
\end{array}$$

We next look at the meromorphic homomorphism $\eta$. Following its construction, $\eta$ is the composite

$$\mathcal{E} \to R^1 \pi_* f^* \mathcal{T}_X^{[n]} \to R^1 \pi_* f^* \Omega_X^{[n]} \to R^1 \pi_* \omega_{\mathcal{E}|f^* \mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}$$

in which the second arrow is induced by applying $\theta^{[n]}$, the third arrow by $f^*$ and the last isomorphism by Serre’s duality. Similarly, replacing $\mathcal{E}$ by $\mathcal{E}_{\lambda,i}$ and replacing $f^* \mathcal{T}_X^{[n]}$ by $f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]}$, we obtain a homomorphism

$$\eta_{\lambda,i} : j_\lambda^* \mathcal{E} \to R^1 \pi_{\lambda,i} f^*_{\lambda,i} \mathcal{T}_X^{[\lambda_i]} \to \mathcal{O}_{\mathcal{U}_\lambda}. $$
These individual (meromorphic) homomorphisms fit into the identity
\[ j_\lambda^* \eta = \eta_{\lambda,1} + \cdots + \eta_{\lambda,t}, \]
over where all make sense.

We now let \( \Lambda_{\lambda,i} \subset \mathcal{U}_\lambda \) be those \( (\varphi_1, (\xi_i)) \) such that either \( \varphi_i \) are constant or the supports \( \text{Spt}(\varphi_i) \) of \( \varphi_i \) satisfy
\[ \text{Spt}(\varphi_i) \cap (D_0 \cup D_\infty) \neq \emptyset. \]
Mimicking the proof in [K-L], we immediately see that each \( \eta_{\lambda,i} \) is surjective away from \( \Lambda_{\lambda,i} \). Then applying results in [K-L], we obtain

**Lemma 3.4.** Away from \( \Lambda_{\lambda,i} \), the pull back cone \( j_\lambda^* \mathcal{N} \subset j_\lambda^* \mathcal{E} \) is contained in the kernel of \( \eta_{\lambda,i} \). Namely,
\[ j_\lambda^* \mathcal{N}|_{\mathcal{U}_\lambda - \Lambda_{\lambda,i}} \subset \ker\{\eta_{\lambda,i} : j_\lambda^* \mathcal{E} \to \mathcal{O}_{\mathcal{U}}\}. \]

**Proof of Proposition 3.3.** First we shall transform the problem from stacks to schemes. Let \( N \subset E \) be the cone over \( \mathcal{M} \) whose intersection with the zero section gives the virtual cycle \( [\mathcal{M}]^{\text{vir}} \). We let \( N_\alpha \) be the irreducible components of \( N \) with \( c_0 \) its multiplicities. For each \( \alpha \), we let \( T_\alpha \subset \mathcal{M} \) be the image stack of the projection \( N_\alpha \to \mathcal{M} \); we pick a proper variety \( T_\alpha \) and a morphism
\[ \phi_\alpha : T_\alpha \to \mathcal{M} \]
so that \( \phi_\alpha \) is generically finite. Since \( \mathcal{M} \) has a projective coarse moduli space, such \( T_\alpha \) does exist.

We then let \( E_\alpha \) be the pull back vector bundle \( \phi_\alpha^* \mathcal{E} \), and let \( N_\alpha \subset E_\alpha \) be the subvariety so that under the projection \( E_\alpha \to E|_{\mathcal{Y}_\alpha} \) the variety \( N_\alpha \) maps generically finitely onto \( N_\alpha \). Since both \( N_\alpha \) and \( T_\alpha \) are irreducible, such \( N_\alpha \) is unique. Finally, we let \( s_\alpha \) be the zero section of \( E_\alpha \); let \( d_\alpha \) be the degree of the map \( \phi_\alpha \), which is identical to the degree of \( N_\alpha \to N_\alpha \). Then
\[ [\mathcal{M}]^{\text{vir}} = \sum_\alpha c_0 d_\alpha^{-1} \cdot \phi_\alpha^* s_\alpha^*[N_\alpha]. \]

The reduction Proposition will follow from the following reduction statement for each of the class \( s_\alpha^*[N_\alpha] \):
\[ s_\alpha^*[N_\alpha] \in H_*(\Lambda_\alpha, \mathbb{Q}), \quad \text{where } \Lambda_\alpha = \phi_\alpha^{-1}(\Lambda_\theta). \quad (3.8) \]

We now prove this statement for any fixed \( \alpha \). We first cover \( T_\alpha \) by open subsets \( V_\alpha \) so that each of its image \( \phi_\alpha(V_\alpha) \) is contained in the image of some \( \mathcal{U}_\lambda \to \mathcal{M} \). By choosing \( V_\alpha \) small enough, we can assume that \( V_\alpha \to \mathcal{M} \) lifts to \( \phi_\alpha : V_\alpha \to \mathcal{U}_\lambda \). Using this, we can pull back the direct sum decomposition (3.6) and the meromorphic homomorphisms \( \eta_{\lambda,i} \):
\[ E_\alpha|_{V_\alpha} \cong \phi_\alpha^* \mathcal{E}_{\lambda,1} \oplus \cdots \oplus \phi_\alpha^* \mathcal{E}_{\lambda,t} \xrightarrow{\oplus \phi_\alpha^* \eta_{\lambda,i}} \mathcal{O}_{V_\alpha} \oplus \cdots \oplus \mathcal{O}_{V_\alpha}. \]
We denote \( E_{\alpha,i} = \phi_\alpha^* \mathcal{E}_{\lambda,i} \) and \( \eta_{\alpha,i} = \phi_\alpha^* \eta_{\lambda,i} \).

Like what we have done in [K-L], we shall pick (smooth) almost splittings of the above homomorphisms. To control the behavior of these splittings near \( \phi_\alpha^{-1}(\Lambda_{\lambda,i}) \), we pick a small (analytic) neighborhood \( \Lambda_{\lambda,i} \), of \( \Lambda_{\lambda,i} \subset \mathcal{U}_\lambda \) so that it deformation retracts to \( \Lambda_{\lambda,i} \).

Over \( V_\alpha \), we then pick a smooth section \( \delta_{\alpha,i} \in C^\infty(V_\alpha, E_{\alpha,i}) \) so that
\[ \delta_{\alpha,i}^{-1}(0) \subset \phi_\alpha^{-1}(\Lambda_{\lambda,i}); \]
(2) the support of $\delta_{a,i}$, which is the (analytic) closure of $\{\delta_{a,i} \neq 0\}$, is disjoint from $\phi^{-1}_{a,i}(\Lambda_{\lambda,i})$;

(3) for $w \in V_a$ with $\delta_{a,i}(w) \neq 0$, $\eta_{a,i}(\delta_{a,i}(w))$ is a positive real number.

By first picking a smooth function $h$ of $E_{a,i}$ over $V_a - \phi^{-1}_{a,i}(\Lambda_{\lambda,i})$ so that $\eta_{a,i} \circ h = 1$ and then multiplying it by a cut off function, we obtain the desired smooth section $\delta_{a,i}$.

Because of our choice of $\delta_{a,i}$, the section

$$\delta_a = \sum_{i=1}^t \delta_{a,i} \in C^\infty(V_a, E_a)$$

has the property that

(1) $\delta_a(w) = 0$ if and only if $\delta_{a,i}(w) = 0$ for all $i$;

(2) in case $\delta_a(w) \neq 0$, then the fiber of $N_\alpha$ over $w$, $N_\alpha|_w$, lies in the kernel of $\eta_{a,i}(w): E_a|_w \to \mathbb{C}$ for all $i$ of which $\delta_{a,i}(w) \neq 0$.

Therefore, away from $\delta_a^{-1}(0)$ the cone $N_\alpha|_{V_a}$ lies in the kernel of $\delta_a: E_a|_{V_a} \to \mathbb{C}_{V_a}$.

Our last step is to pick a partition of unity $g_a: V_a \to \mathbb{R}_{\geq 0}$ of the covering $\{V_a\}$; namely $\{g_a > 0\} \subseteq V_a$ (its closure in $V_a$ is compact) and $\sum_a g_a \equiv 1$ on $T_\alpha$. The sum

$$\delta_\alpha = \sum_a g_a \cdot \delta_a$$

is then a smooth section of $E_\alpha$. It is direct to check that away from $\delta_\alpha = 0$, the cone $N_\alpha$ is disjoint from $\delta_\alpha$. This proves that

$$s^*_\alpha[N_\alpha] \in H_2(\delta_a^{-1}(0)).$$

It remains to pinpoint the set $\delta_a^{-1}(0)$. This time, because $\eta_{a,i}(\delta_{a,i}(w)) \geq 0$ whenever it makes sense, $\delta_a(w) \neq 0$ if and only if for some $a$: $g_a(w) > 0$. Then $\delta_a(w) = 0$ implies $\delta_{a,i}(w) = 0$ for all $i$, which imply that $w \in \cap_i \phi^{-1}_{a,i}(\Lambda_{\lambda,i})$. Therefore, $\delta_a^{-1}(0) \subseteq \cup_a \phi^{-1}_{a,i}(\Lambda_{\lambda})$. Finally, because of our choice of $\Lambda_{\lambda,i}$, we can retract $\cup_a \phi^{-1}_{a,i}(\Lambda_{\lambda})$ to $\Lambda_\alpha = \phi^{-1}_{a,i}(\Lambda_a)$ in $T_\alpha$. This proves that $s^*_\alpha[N_\alpha] \in H_2(\Lambda_a)$. This proves the reduction Proposition. }

\section{Universality of two point Gromov-Witten invariants}

In this section, we will use the technique developed in the previous section to prove a structure result on the two-point extremal Gromov-Witten invariants of a general algebraic surface.

We consider the moduli space $\mathcal{M}_{0,2}(X^{[n]}, d)$ of two point genus zero stable morphisms to $X^{[n]}$ of the $d$-multiple of the extremal curve class $\beta_n$ as the fundamental class. To determine the two point Gromov-Witten invariants of this moduli space, we need to investigate all possible

$$\langle A_1, A_2 \rangle_{0,2,d}^{X^{[n]}} \text{ for } A_1, A_2 \in H^*(X^{[n]}).$$

Using the reduction Proposition of the previous section, we shall prove in this section that, with $A_1$ chosen among the Nakajima basis, all but a few such numbers vanish; and, for those that don’t, their values only depend on the intersection of $K_X$ with the relevant curve classes involved.
To this end, we shall first recall the Nakajima basis of $H^*(X^{[n]})$ and their geometric representatives. We let $\mu^1$, $\mu^2$ and $\mu^3$ be three partitions of lengths $\ell(\mu^1) = r$, $\ell(\mu^2) = s$ and $\ell(\mu^3) = t$; we write

$$\mu^1: \mu^2 \geq \mu^3 \geq \ldots \geq \mu^r,$$

and write $\mu^2$ and $\mu^3$ accordingly. For the point class $q \in H^4(X)$, curve classes $c_1, \ldots, c_s \in H^2(X)$ and the fundamental class $[X] \in H^0(X)$, the triple $\mu = (\mu^1, \mu^2, \mu^3)$ gives a cohomology class

$$A^\mu_c = a_{-\mu^1}(q) \ldots a_{-\mu^r}(q)a_{-\mu^2}(c_1) \ldots a_{-\mu^3}(c_s)a_{-\mu^3}([X]) \ldots a_{-\mu^3}([X])|0|; \quad (4.1)$$

it is a class in $H^*(X^{[n]})$ if $|\mu^1| + |\mu^2| + |\mu^3| = n$. By going through all possible $\mu^i$ and classes $c_i$, the above form a basis of the cohomology groups of $X^{[n]}$. To proceed, we keep one such homology class $A^\mu_c$ as in (4.1) and pick three more partitions $\lambda^1$, $\lambda^2$ and $\lambda^3$ of lengths $a$, $b$ and $c$; pick curve classes $e_1, \ldots, e_b \in H^2(X)$ and form

$$A^\lambda_a = a_{-\lambda^1}(q) \ldots a_{-\lambda^r}(q)a_{-\lambda^2}(e_1) \ldots a_{-\lambda^2}(e_0)a_{-\lambda^3}([X]) \ldots a_{-\lambda^3}([X])|0|. \quad (4.2)$$

Again, it is a cohomology class of $X^{[n]}$ if $|\lambda^1| + |\lambda^2| + |\lambda^3| = n$. Our immediate task is to investigate the possibility of the vanishing of $\langle A^\lambda_a, A^\mu_c \rangle_{0,2,d}$. For this, we need the geometric representatives of the Poincaré dual of $A^\mu_c$ and $A^\lambda_a$. We pick points $q_1, \ldots, q_r, p_1, \ldots, p_a$ in $X$; pick Riemann surfaces $C_i$ and $E_j$ that represent the Poincaré dual of the classes $c_i$ and $e_j$, respectively. Without lose of generality, we can pick these points and Riemann surfaces in general position that any subcollection of them intersects transversally. Then the Poincaré dual of $A^\mu_c$ is represented by

$$A^\mu_c = a_{-\mu^1}(q_1) \ldots a_{-\mu^r}(q_r)a_{-\mu^2}(C_1) \ldots a_{-\mu^3}(C_a)a_{-\mu^3}(X) \ldots a_{-\mu^3}(X)|0|. \quad A^\mu_c$$

is a multiple of the closure of the following subset of $X^{[n]}$:

$$\left\{ \xi^1_1 + \ldots + \xi^1_1 + \xi^2_1 + \ldots + \xi^2_1 + \xi^3_1 + \ldots + \xi^3_1 \in X^{[n]}, \text{Supp}((\xi^3_j) = y_j \in X \mid \text{Supp}(\xi^3_j) = x_j \in C_j, \text{Supp}(\xi^3_j) = q_j) \right\}. \quad$$

Because the expected dimension of the moduli space $\mathcal{M}_{0,2}(X^{[n]}, d)$ is $2n - 1$, $\langle A^\lambda_a, A^\mu_c \rangle_{0,2,d} \neq 0$ is possible only if

$$\text{deg} A^\lambda_a + \text{deg} A^\mu_c = \exp. \dim \mathcal{M}_{0,2}(X^{[n]}, d) = 2n - 1.$$

Then because the operators $a_{-k}(q)$, $a_{-k}(c_i)$ and $a_{-k}([X])$ increase cohomology degrees by $2k + 2$, $2k$ and $2k - 2$ respectively, the cohomology degree of $A^\mu_c$ is $2(n + \ell(\mu^1) - \ell(\mu^3))$. Therefore, the above identity forces

$$\left(\ell(\lambda^3) - \ell(\mu^1)\right) + \left(\ell(\mu^3) - \ell(\lambda^1)\right) = 1. \quad (4.3)$$

Furthermore, by the reduction Proposition of the previous section, $\langle A^\lambda_a, A^\mu_c \rangle_{0,2,d} \neq 0$ only if for the set $\Lambda_0$ defined there, with $\pi: X^{[n]} \rightarrow X^{(n)}$,

$$\Lambda_0 \cap \pi^{-1}(A^\lambda_a) \cap \pi^{-1}(A^\mu_c) \neq \emptyset, \quad (4.4)$$

Proposition 4.1. Suppose $d > 0$ and $\langle A^\lambda_a, A^\mu_c \rangle_{0,2,d} \neq 0$, then

$$\ell(\lambda^3) = \ell(\mu^1) + \delta \quad \text{and} \quad \ell(\mu^3) = \ell(\lambda^1) + 1 - \delta, \quad \text{for either} \ \delta = 0 \text{ or } 1.$$
In case $\delta = 0$ holds, then $\lambda^3 = \mu^3$ as partitions; and there exists an integer $\ell = \mu^3 = \lambda^3$ for some integers $i$ and $j$ such that the partition $\lambda^3$ is obtained from $\mu^3$ with $\ell$ deleted, and the partition $\mu^2$ is obtained from $\lambda^2$ with $\ell$ deleted.

**Proof.** Since $\langle A^3_{c_1}, A^3_{c_2}, \ldots, A^3_{c_d} \rangle \neq \emptyset$, there is an $(f, \Sigma)$ in the intersection (4.4). We let its support be the zero-cycle

$$\text{Spt}(f) = m_1 x_1 + \cdots + m_k x_k, \quad x_1, \ldots, x_k \text{ distinct}.$$ Since $f \in \pi^{-1}(A^3_{c_1})$, we have maps $u_1: [a] \rightarrow [k]$, $u_2: [b] \rightarrow [k]$ and $u_3: [c] \rightarrow [k]$, where $[k]$ is the set of integers $\{1, \ldots, k\}$, such that $x_{u_1(i)} = p_i$, $x_{u_2(i)} \in E_i$, that the coproduct $u_1 \sqcup u_2 \sqcup u_3 : [a] \coprod [b] \coprod [c] \rightarrow [k]$ is surjective, and that

$$m_1 = \sum_{i \in u_1^{-1}(l)} \lambda^3_i + \sum_{i \in u_2^{-1}(l)} \lambda^3_i + \sum_{i \in u_3^{-1}(l)} \lambda^3_i. \quad (4.5)$$

For the same reason, since $f \in \pi^{-1}(A^3_{c_2})$, we have maps $v_1$, $v_2$ and $v_3$ from $[s]$, $[t]$ and $[r]$ to $[k]$, respectively, such that $x_{v_1(i)} = q_i$, $x_{v_2(i)} \in C_i$, that the coproduct $v_1 \sqcup v_2 \sqcup v_3$ is surjective, and that

$$m_1 = \sum_{i \in v_1^{-1}(l)} \mu^1_i + \sum_{i \in v_2^{-1}(l)} \mu^2_i + \sum_{i \in v_3^{-1}(l)} \mu^3_i. \quad (4.6)$$

We first show that $\ell(\lambda^3) \geq \ell(\mu^1)$. Indeed, since $q_1, \ldots, q_r$ are distinct, $v_1 : [r] \rightarrow \text{Im}(v_1)$ is an isomorphism. But then because of our general position requirement on the points $p_i$ and $q_j$'s and of the Riemann surfaces $C_i$ and $E_i$'s, $q_1, \ldots, q_r$ do not lie on $E_1, \ldots, E_r$, $\text{Im}(v_1) \cap (\text{Im}(u_1) \cup \text{Im}(u_2)) = \emptyset$. Hence $\text{Im}(v_1) \subset \text{Im}(u_3)$ since $u_1 \sqcup u_2 \sqcup u_3$ is surjective. This proves

$$\ell(\mu^1) = \# \text{Im}(v_1) \leq \# \text{Im}(u_3) \leq \ell(\lambda^3).$$

For the same reason, we have $\ell(\mu^3) \geq \ell(\lambda^1)$. Combined with (4.3), we obtain the first conclusion of the Proposition.

Now suppose $\delta = 0$; namely, $\ell(\mu^1) = \ell(\lambda^3)$, then all the identities above hold. In particular, $u_3$ is an isomorphism $[c] \cong \text{Im}(u_3) \cong \text{Im}(v_1)$.

We next show that $\text{Im}(v_3) = \text{Im}(u_1) \cup \{k_0\}$ for an integer $k_0 \in [k] - \text{Im}(u_1)$. First, following the same reason as before, we have $\text{Im}(u_1) \subset \text{Im}(v_3)$. Because $\# \text{Im}(u_1) = \ell(\lambda^1)$ and because $\ell(\lambda^1) + 1 = \ell(\mu^3)$, either $\text{Im}(v_3) = \text{Im}(u_1)$ or $\text{Im}(v_3) = \text{Im}(u_1) \cup \{k_0\}$ for an $k_0 \in [k] - \text{Im}(u_1)$. We will show that only the later can happen.

For this, we decompose $f$ into $k$ individual morphisms $f_i: \Sigma \rightarrow X^{[m_i]}_{x_i}$, where $X^{[m]}_{x}$ is the preimage of $m x \in X^{(m)}$ under the Hilbert-Chow morphism $X^{(m)} \rightarrow X^{(m)}$. Because $\pi(f(\Sigma)) = \sum m_i x_i$, such decomposition is possible. Since $d > 0$, there is at least one $k_0$ so that $f_{k_0}$ is non-constant. By the characterization of $\Lambda_0$, this is possible only if $x_{k_0} \in D_0 \cup D_\infty$. Because of this, $x_{k_0} \neq p_i$'s, and thus $k_0 \not\in \text{Im}(u_1)$; also $k_0 \not\in \text{Im}(u_3)$ because $\text{Im}(u_3) = \text{Im}(v_1)$. Thus there is an $i_0 \in [k] - \{k_0\}$ such that $u_2(i_0) = k_0$. Thus $x_{k_0} \in E_{i_0} \cap (D_0 \cup D_\infty)$, which then exclude the possibility that $x_{k_0} \in C_i$'s. Hence $k_0 \in \text{Im}(v_3)$; and $\text{Im}(v_3) = \text{Im}(u_1) \cup \{k_0\}$. Note that this also proves that $f_{k_0}$ is the only non-constant $f_j$'s.

Now let $i \in [k] - \{i_0\}$ and consider $x_{u_2(i)}$. Because

$$\{x_l \mid l \in \text{Im}(v_1) \cup \text{Im}(v_3)\} \subset \{q_1, \ldots, q_r\} \cup \{p_1, \ldots, p_a\} \cup (E_{i_0} \cap (D_0 \cup D_\infty)),$$
For some $j \in [s]$ so that $u_2(i) = v_2(j)$, and consequently $x_{u_2(i)} \in E_i \cap C_j$. Because $C_i$'s and $E_j$'s are in general positions, once $x_{u_2(i)} \in E_i \cap C_j$, it does not lie in any other $E_i$'s and $C_j$'s. In particular, $u_2^{-1} \circ v_2$ defines an isomorphism $[s] \to [b] - \{i_0\}$.

Combined, we see that the map $v_1 \sqcup u_2 \sqcup v_3$ is injective and thus is an isomorphism. Similarly, $v_1 \sqcup v_2 \sqcup v_3$ is also an isomorphism. Therefore, by (4.6) and (4.7), we have

$$
\lambda_1^{u_1^{-1}(i)} = \mu_1^{v_1^{-1}(i)}, \quad \lambda_2^{u_2^{-1}(i)} = \mu_2^{v_2^{-1}(i)} \quad \text{and} \quad \lambda_3^{u_3^{-1}(i)} = \mu_3^{v_3^{-1}(i)}
$$

when $i \in \text{Im}(u_1)$, $i \in \text{Im}(u_2) - \{i_0\}$, and $i \in \text{Im}(u_3)$, respectively.

Putting them together, we have proved, in case $\delta = 0$, that $\lambda^3 = \mu^3$ as partitions, that there is an integer $\ell$ so that $\lambda^3$ is $\mu^3$ with $\ell$ aded, and that $\lambda^3$ is $\mu^3$ with $\ell$ deleted. Furthermore, the decomposition of $f$ has all but one component constant; the non-constant component is the one associated to the part $\ell$. \hfill $\square$

Let $A_e^{\lambda - \lambda^2}$ be the cohomology class on $X^{[n]}$ obtained from $A_e^{\lambda}$ in (4.2) with $a_{-\lambda^3_1}(e_j)$ deleted. Similarly, we can define $A_e^{\lambda - \lambda^2_1}, A_e^{\lambda - \lambda^2_1 - \lambda^2_2}$, etc. For example,

$$
A_e^{\lambda - \lambda^2} = a_{-\lambda^3_1}(q) \ldots a_{-\lambda^3_1}(e_2) \ldots a_{-\lambda^3_2}(e_3) a_{-\lambda^3_1}([X]) \ldots a_{-\lambda^3_2}([X])(0).
$$

**Corollary 4.2.** Suppose $\lambda$ and $\mu$ fit into the case $\delta = 0$ in Proposition 4.1, then

$$
\langle A_e^{\lambda}, A_e^{\mu} \rangle_{0,2,d} = \sum_{\mu_1^i = \lambda^2_i} \langle A_e^{\lambda - \lambda^2_1}, A_e^{\mu - \mu^3_1} \rangle \cdot \langle A_{-\lambda^3_1}(e_j)(0), a_{-\lambda^3_1}([X])(0) \rangle_{0,2,d}.
$$

**Proof.** The proof is obvious and is omitted. \hfill $\square$

Thus we only need to determine $\langle a_{-\ell}(e)(0), a_{-\ell}([X])(0) \rangle_{0,2,d}^{X^{[\ell]}}$. For this we have

**Lemma 4.3.** There exists a universal function $c_\ell$, such that for any positive integers $\ell$ and $d$, and homology class $e \in H^2(X)$,

$$
\langle a_{-\ell}(e)(0), a_{-\ell}([X])(0) \rangle_{0,2,d} = c_{\ell,d} \cdot (e \cdot c_1(K_X)).
$$

**Proof.** We first introduce the universal constant $c_{\ell,d}$. We let $U$ be a smooth analytic surface, let $\theta_+ \in H^0(U, K_U)$ be an analytic section vanishing along a smooth curve $C \subset U$, and let $E \subset U$ be another smooth curve that intersects transversally with $C$ at a single point $p \in U$. We can form the Hilbert scheme $U^{[\ell]}$ as an analytic space and form the moduli of stable morphisms $\mathcal{M}_{0,2}(U^{[\ell]}, d)$. By choosing $U$ as an analytic open subset of a smooth algebraic surface, both $U^{[\ell]}$ and the moduli space are analytic spaces of a projective scheme and of a Deligne-Mumford stack; thus their existence are well established.

We now apply the localization by holomorphic two-form to this moduli space. First, the form $\theta_+$ allows us to represent the virtual cycle $\delta$ of $\mathcal{M}_{0,2}(U^{[\ell]}, d)$ as a homology class in the Borel-Moore homology group $H_{BM}^*(\Lambda_{\theta_+})$, of the set $\Lambda_{\theta_+}$ that is defined in (3.2). By the explicit construction of $\Lambda_{\theta_+}$, the Cartesian product

$$
\Lambda_{\theta_+} \times U^{[\ell]} \langle a_{-\ell}(E)(0) \rangle \longrightarrow \Lambda_{\theta_+}
$$

$$
\downarrow \quad \quad \downarrow_{ev_1}
$$

$$
a_{-\ell}(E)(0) \longrightarrow U^{[\ell]}
$$
is compact. Here \( \ell \) is the tautological embedding and \( e_{v_1} \) is the morphism defined by evaluating on the first marked point. Hence because \( U^{[\ell]} \) is smooth, the Gysin map

\[
\iota^* : H_*^{BM}(\Lambda_{\theta_+}) \rightarrow H_*(\Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0]))
\]
sends a Borel-Moore homology class to ordinary homology class.

We let \( j^* \) be the Gysin map associated to the tautological inclusion and the second evaluation map:

\[
\Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0]) \xrightarrow{\text{ev}_2} \Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0]) \xrightarrow{\iota^*} U^{[\ell]},
\]

Note that \( (a_{-\ell}(E)[0]) \) is a closed subset of \( (a_{-\ell}(X)[0]) \), and thus

\[
\Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0]) \times_{U^{[\ell]}} (a_{-\ell}(X)[0]) = \Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0]).
\]

Then an easy dimension count gives us

\[
j^* \iota^* \delta \in H_0(\Lambda_{\theta_+} \times_{U^{[\ell]}} (a_{-\ell}(E)[0])).
\]

We let \( c_{\ell,d} \) be the degree of this cycle.

We remark that by the construction of the localized virtual cycle \( \delta \) and by the property of the Gysin maps, the so defined number is universal in the sense that it does not depend on the choice of the surface \( U \), the form \( \theta_+ \) and the curve \( C \), so long as \( \theta_+^{-1}(0) \) intersects \( E \) transversally at a single point.

We define another universal constant \( c^-_{\ell,d} \) similarly. We keep the surface \( U \), the curve \( C \), but replace the holomorphic two form \( \theta_+ \) by a meromorphic two-form \( \theta_- \) that has no vanishing divisor and has a smooth pole divisor \( E \) that intersects \( C \) transversally at a single point. Then we take localized virtual cycle \( \delta_- \in H_*^{BM}(\Lambda_{\theta_-}) \) of \( \mathcal{M}_{0,2}(U^{[\ell]}, d) \), and define \( c^-_{\ell,d} \) to be the degree of the class

\[
j^* \iota^* \delta_- \in H_0(\Lambda_{\theta_-} \times_{U^{[\ell]}} (a_{-\ell}(E)[0])).
\]

To proceed, we represent the Poincaré dual of \( e \) as

\[
P.D^{-1}(e) = \alpha_1[E_1] - \alpha_2[E_2] + \alpha_0[E_0],
\]

where \( E_1 \) and \( E_2 \) are two smooth very ample divisors, \( E_0 \subset X \) is a Riemann surface disjoint from \( D_0 \cup D_\infty \), and \( \alpha_1, \alpha_2 \) are non-negative rational numbers. Then because

\[
\langle (a_{-\ell}([E_0])[0], (a_{-\ell}([X])[0]) \rangle_{0,2,d} = 0,
\]

by the linearity of GW-invariants, we have

\[
\langle (a_{-\ell}(e))[0], (a_{-\ell}([X])[0]) \rangle_{0,2,d} = \sum_{i=1}^{2} (-1)^{i-1} \alpha_i \langle (a_{-\ell}([E_i])[0], (a_{-\ell}([X])[0]) \rangle_{0,2,d}.
\]

Then since we can arrange \( E_i \) to intersects \( D_0 \) and \( D_\infty \) transversally, the above sum is

\[
((\alpha_1[E_1] - \alpha_2[E_2]) \cdot [D_0]) c_{\ell,d} + ((\alpha_1[E_1] - \alpha_2[E_2]) \cdot [D_\infty]) c^-_{\ell,d}.
\]

In case \( c^-_{\ell,d} = -c_{\ell,d} \), then it becomes

\[
((\alpha_1[E_1] - \alpha_2[E_2]) \cdot ([D_0] - [D_\infty]) c_{\ell,d} = (e \cdot c_1(K_X)) c_{\ell,d},
\]

as desired.
The identity $c_{\ell,d}^- = -c_{\ell,d}$ is easy to see. We let $X$ be a smooth K3 surface. Since its Hilbert scheme is holomorphic symplectic, all its GW-invariants vanish. On the other hand, since $K_X$ is trivial, we can find a meromorphic two form $\theta$ so that $D_0$ and $D_\infty$ are non-empty. Following what we just proved, say take $E_2 = \emptyset$, we have

$$0 = [E_1] \cdot [D_0] c_{\ell,d} + [E_1] \cdot [D_\infty] c_{\ell,d}$$

for all smooth divisor $E_1$. This proves $c_{\ell,d}^- = c_{\ell,d}$, and thus the Lemma. 

From the proof of the previous Proposition and the Lemma above, we can get the following result.

**Corollary 4.4.** Let $\lambda$ and $\mu$ be as in Corollary 4.2, then

$$\sum_{d \geq 0} \langle A^n, c_1(\mathcal{O}[n]), A'^n \rangle_{0,3,d} q^d = A^n \cup c_1(\mathcal{O}[n]) \cup A'^n + \sum_{\mu^k = \lambda^j = \ell} \langle A^n, A'^n \rangle \sum_{d > 0} d c_{\ell,d} \langle a_j, K_X \rangle q^d.$$

**5. The quantum first Chern class operator**

The Hilbert scheme $X^{[n]}$ admits a universal subscheme $Z_n \subset X^{[n]} \times X$, $Z_n = \{ (\xi, x) | x \in \text{Supp}(\xi) \}$. The sheaf $\pi_1(\mathcal{O}_{Z_n})$ on $X^{[n]}$, where $\pi_1$ is the first factor projection, is a locally free sheaf of rank $n$. We use $\mathcal{O}^{[n]}$ to denote this sheaf. The first Chern class $c_1(\mathcal{O}^{[n]})$, treated as an operator on the cohomology ring $H^*(X^{[n]})$ via the cup product, plays the fundamental role in determining the ring structure of $H^*(X^{[n]})$ (see [Lehn, LQW, C-G]). Naturally, the action of $c_1(\mathcal{O}^{[n]})$ on $H^*(X^{[n]})$ via the quantum cup product should play the equally important role in the quantum cohomology ring $H^*_q(X^{[n]})$. We call $c_1(\mathcal{O}^{[n]}) \cup_\pi$ the quantum first Chern class operator on $H^*_q(X^{[n]})$ when it acts via the quantum product defined in subsection 2.2.

When $X = \mathbb{C}^2$, the equivariant quantum first Chern class is determined in [O-P] via localization technique. The main task of the rest of the paper is to determine the quantum first Chern class operator for simply-connected surfaces.

Consider the operator

$$M(q) = \sum_{k > 0} \left( k \frac{(-q)^k}{(-q)^k - 1} - \frac{q}{1 + q} \right) a_{-k} a_\tau \langle \tau_2, [K_X] \rangle$$

$$- \sum_{k > 0} \frac{k - 1}{2} a_{-k} a_\tau \langle \tau_2, [K_X] \rangle - \frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k + \ell} + a_{-k - \ell} a_{k} \right) \langle \tau_3, [X] \rangle.$$

Here for $k \geq 1$, $\tau_k : H^*(X) \to H^*(X^k)$ is the linear map induced by the diagonal embedding $\tau_k : X \to X^k$, and $a_{m_1} \ldots a_{m_k} (\tau_k, (\alpha))$ denotes $\sum_j a_{m_1}(\alpha_{j,1}) \ldots a_{m_k}(\alpha_{j,k})$.

From Lehn’s result [Lehn] (see [Q-W] also), $M(0)$ is the first Chern class operator,

$$M(0)(A'^n) = c_1(\mathcal{O}^{[n]}_X) \cup A'^n.$$
To calculate $M(q)(A^\mu_c)$, it suffices to carry out the following computations:

\[ a_{-k}a_k(\tau_{2s}[K_X])(A^\mu_c), \quad a_{-k-a_{-\ell}}a_{k+\ell}(\tau_{3s}[X])(A^\mu_c), \quad a_{-k-a_{-\ell}}a_k(\tau_{3s}[X])(A^\mu_c). \]  

(5.1)

For the first term in (5.1), we have

\[
a_{-k}a_k(\tau_{2s}[K_X])A^\mu_c = \sum_{i=1}^{s} (-\mu_i^2) a_{-k}([K_X \cdot C_i])A^{\mu - \mu_i^2}_c + \sum_{i=1}^{t} (-\mu_i^3) a_{-k}(\tau_{2s}[C_i])A^{\mu - \mu_i^3}_c.
\]

(5.2)

For the second term in (5.1), we have

\[
a_{-k-a_{-\ell}}a_k(\tau_{3s}[X])A^\mu_c = \sum_{i=1}^{r} (-\mu_i^1) a_{-k-a_{-\ell}}(\tau_{2s}[g_i])A^{\mu - \mu_i^1}_c + \sum_{i=1}^{s} (-\mu_i^2) a_{-k-a_{-\ell}}a_k(\tau_{2s}[C_i])A^{\mu - \mu_i^2}_c + \sum_{i=1}^{t} (-\mu_i^3) a_{-k-a_{-\ell}}a_k(\tau_{2s}[X])A^{\mu - \mu_i^3}_c.
\]

(5.3)

For the third term in (5.1), we have

\[
a_{-k-a_{-\ell}}a_k(\tau_{3s}[X])A^\mu_c = \sum_{i=1}^{r} \sum_{j=1}^{t} k\ell a_{-\ell-a_{-\ell}}([g_i])A^{\mu - \mu_i^1 - \mu_j^1}_c + \sum_{i=1}^{s} \sum_{j=1}^{t} k\ell a_{-\ell-a_{-\ell}}([C_i \cdot C_j])A^{\mu - \mu_i^2 - \mu_j^2}_c + \sum_{i=1}^{s} \sum_{j=1}^{t} k\ell a_{-\ell-a_{-\ell}}([C_i])A^{\mu - \mu_i^2 - \mu_j^3}_c + \sum_{i=1}^{t} \sum_{j=1}^{s} k\ell a_{-\ell-a_{-\ell}}([X])A^{\mu - \mu_i^1 - \mu_j^3}_c + \sum_{i=1}^{t} \sum_{j=1}^{s} k\ell a_{-\ell-a_{-\ell}}([C_j])A^{\mu - \mu_i^3 - \mu_j^3}_c + \sum_{i=1}^{t} \sum_{j=1}^{s} k\ell a_{-\ell-a_{-\ell}}([X])A^{\mu - \mu_i^3 - \mu_j^3}_c.
\]

(5.1)

The following result is the analogue of the Proposition 4.1.

**Proposition 5.1.** If $\langle A^\mu_c, M(q)(A^\mu_c) \rangle$ is not a constant function of $q$, then either $\ell(\mu^3) = \ell(\lambda^1) + 1$ and $\ell(\lambda^3) = \ell(\mu^1)$, or $\ell(\mu^3) = \ell(\lambda^1)$ and $\ell(\lambda^3) = \ell(\mu^1) + 1$. 
In addition, assume $\ell(q^3) = \ell(q^1) + 1$ and $\ell(q^3) = \ell(q^1)$. Then $q^3 = q^1$ as partitions, and there exists an integer $\ell = \ell_3 = \lambda_2^3$ for some integers $i$ and $j$ such that the partition $\lambda^1$ is obtained from $\mu^3$ with $\ell$ deleted, and the partition $\mu^2$ is obtained from $\lambda^2$ with $\ell$ deleted.

**Proof.** One can check easily, as cohomology class, $M(q)(A^\ell_q)$ is of cohomology degree $\deg A^\ell_q + 2$. Now take $A^\lambda_q$ with $\deg A^\lambda_q = 4n - 2 - \deg A^\ell_q$. Therefore

$$\ell(\lambda^1) - \ell(q^3) + \ell(q^1) - \ell(q^3) + 1 = 0, \quad \text{i.e.} \quad (\ell(q^3) - \ell(q^1)) + (\ell(q^3) - \ell(q^1)) = 1.
$$

Since the second term and the third term in (5.1) only contribute to the constant term of $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle$, we only need to consider the first term in (5.1).

If $\ell(q^3) - \ell(q^1) < 0$, then $\ell(q^3) \geq \ell(q^1) + 2$. Then, for the terms in the summation of (5.2) and (5.3), the number of terms $a_{-k}([X])$ is more than $\ell(q^1)$, thus $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle = 0$.

Similarly, by symmetry, if $\ell(q^3) - \ell(q^1) < 0$, we also have $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle = 0$.

Therefore if $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle$ is not a constant function of $q$, we must have $\ell(q^3) - \ell(q^1) \geq 0$ and $\ell(q^3) - \ell(q^1) \geq 0$. Thus either $\ell(q^3) = \ell(q^1) + 1$ and $\ell(q^3) = \ell(q^1)$, or $\ell(q^3) = \ell(q^1)$ and $\ell(q^3) = \ell(q^1) + 1$.

Assume $\ell(q^3) - \ell(q^1) = 1$ and $\ell(q^3) = \ell(q^1)$. If $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle$ is not a constant function of $q$, then $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle$ cannot be a constant function of $q$, where

$$M(q) = M(q) - M(0).$$

The contribution from the terms in the summation of (5.2) must be zero since the number of terms $a_{-k}([pt])$ there is $\ell(q^1) + 1 > \ell(q^3)$.

Each term in the summation of (5.3) has one more $a_{-k}([\text{curve}])$ term and one less $a_{-\ell}([X])$ term than $A^\ell_q$, thus $\langle A^\lambda_q, M(q)(A^\ell_q) \rangle$. By the Heisenberg commutation relation, we prove the Proposition. □

**Corollary 5.2.**

$$\langle A^\lambda_q, M(q)(A^\ell_q) \rangle - \langle A^\lambda_q, M(0)(A^\ell_q) \rangle = \sum_{1 \leq i \leq \ell, \mu(q^1), 1 \leq j \leq \ell - \mu(q^1)} \langle A^\lambda_q, A^\ell_q \rangle \langle E_j, K_X \rangle (-1)^{\ell} \ell \ell'^{-1} - \frac{q}{1 + q}. $$

### 6. The projective plane

In Proposition 4.1 in §4, we see that the two point extremal Gromov-Witten invariants can be reduced to the computation of the case $\langle a_{-n}([C])0, a_{-n}([X])0 \rangle_{0,2,d}$ where $C$ is a curve. By Lemma 4.3, it suffices to carry out the computation for a particular surface. We choose the projective plane. Since it is a toric surface, we can use computations in [O-P] for the affine plane.

The main purpose of this section is to prove the following result.

**Proposition 6.1.** Let $L$ be a line on the projective plane $X$. We have

$$\sum_{d=0}^\infty \langle a_{-n}([L])0, c_1(O^{[n]}), a_{-n}([X])0 \rangle_{0,3,d} q^d = \langle a_{-n}([L])0, M(q)(a_{-n}([X])0) \rangle. \quad (6.1)$$


Since $X$ is a toric surface, we can use the localization technique to compute the corresponding equivariant Gromov-Witten invariants. Since both sides of (6.1) are non-equivariant, we get the conclusion of the Proposition 6.1. The equivariant set-up is as follows.

Let $[z_1, z_2, z_3]$ represent a point in the projective space $X = \mathbb{P}^2$, and $T = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ act on $X$ by

$$(s_1, s_2, s_3) \cdot [z_1, z_2, z_3] = [s_1z_1, s_2z_2, s_3z_3], \quad \text{for } (s_1, s_2, s_3) \in T.$$

We have $H^*_T(pt) = \mathbb{C}[t_1, t_2, t_3]$.

Let $Y$ be a $T$-stable subvariety of $X$. We use $[Y]$ to represent the equivariant class $Y \times_T ET$. By the abuse of notation, we also use $[Y]$ to represent the corresponding dual cohomology class $D^{-1}[Y]$ where $D: H^k_T(X) \to H^k_T(X)$ is the Poincaré duality morphism. Note the unusual convention on the degree of the equivariant homology: if $Y$ has real codimension $k$ in $X$, then $[Y]$ is a class in $H^k_T(X)$ (see [Vas]).

There are three $T$-fixed points:

$q_1 = [1, 0, 0], \quad q_2 = [0, 1, 0], \quad q_3 = [0, 0, 1].$

At $q_1$, under the identification $[z_1, z_2, z_3] = [1, z_2/z_1, z_3/z_1]$, the group $T$ acts as

$$(s_1, s_2, s_3) \cdot (z_2/z_1, z_3/z_1) = (s_2s_1^{-1}z_2/z_1, s_3s_1^{-1}z_3/z_1) \quad \text{for } (s_1, s_2, s_3) \in T.$$

The normal bundle $N_1$ of $q_1$ in $X$, as a $T$-module, is isomorphic to $T_1^{-1}T_2 \oplus T_1^{-1}T_3$, where $T_i$ is the one-dimensional representation given by $(s_1, s_2, s_3) \to s_i$. Similarly, we also have

$$N_2 = T_2^{-1}T_1 \oplus T_2^{-1}T_3, \quad N_3 = T_3^{-1}T_1 \oplus T_3^{-1}T_2$$

as $T$-modules, where $N_i$ is the normal bundle of $q_i$ in $X$ regarded as a $T$-module.

Let $L_1$ be the line in $X$ passing through $q_2$ and $q_3$, $L_2$ be the line passing through $q_1$ and $q_3$, and $L_3$ be the line passing through $q_1$ and $q_2$. Near $q_1$, $L_3$ is given by the equation $z_3/z_1 = 0$ and near $q_2$, $L_3$ is given by the equation $z_3/z_2 = 0$. Thus by the localization, in the localized equivariant cohomology

$$H^*_T(X) = H^*_T(X) \otimes_{\mathbb{C}[t_1, t_2, t_3]} \mathbb{C}(t_1, t_2, t_3),$$

we have

$$[L_3] = \frac{(t_2-t_1)q_1}{(t_2-t_1)(t_3-t_1)} + \frac{(t_1-t_2)q_2}{(t_1-t_2)(t_3-t_2)} = \frac{[q_1]}{(t_3-t_1)} + \frac{[q_2]}{(t_3-t_2)}. \quad (6.2)$$

Similarly we have

$$[L_2] = \frac{[q_1]}{t_2-t_1} + \frac{[q_3]}{t_2-t_3},$$

$$[L_1] = \frac{[q_2]}{t_1-t_2} + \frac{[q_3]}{t_1-t_3},$$

$$[X] = \frac{[q_1]}{(t_3-t_1)(t_2-t_1)} + \frac{[q_2]}{(t_3-t_2)(t_1-t_2)} + \frac{[q_3]}{(t_1-t_3)(t_2-t_3)}. \quad (6.3)$$
The anti-canonical class $-K_X$ can be written as, via localization,

$$[-K_X] = [L_1] + [L_2] + [L_3]$$

$$= \frac{[q_2]}{t_1 - t_2} + \frac{[q_3]}{t_1 - t_3} + \frac{[q_1]}{t_2 - t_1} + \frac{[q_1]}{t_2 - t_3} + \frac{[q_1]}{(t_3 - t_1)} + \frac{[q_2]}{(t_3 - t_2)}$$

$$= \frac{(t_2 + t_3 - 2t_1)[q_1]}{(t_2 - t_1)(t_3 - t_1)} + \frac{(t_1 + t_3 - 2t_2)[q_2]}{(t_1 - t_2)(t_3 - t_2)} + \frac{(t_1 + t_2 - 2t_3)[q_3]}{(t_1 - t_3)(t_2 - t_3)}. \quad (6.4)$$

We also have, by the excess intersection formula

$$[q_1] \cdot [q_1] = (i_{1*}[q_1]) \cdot (i_{1*}[q_1]) = i_{1*}([q_1] \cdot i_{1*}[q_1]) = e^q(N_1)[q_1] = (t_3 - t_1)(t_2 - t_1)[q_1].$$

Here $i_k : q_k \to X$ is the embedding. We don’t distinguish between $[q_1]$ with $i_{1*}[q_1]$ unless it is necessary.

Let’s introduce a convention. Let $Y_i$ be $T$-stable subvarieties of $X$, $1 \leq i \leq k$. We write

$$a_{n_1} \ldots a_{n_k} ([Y_1 \times \ldots \times Y_k]) = a_{n_1} ([Y_1]) \ldots a_{n_k} ([Y_k]).$$

Recall the operator

$$M(q) = \sum_{k>0} \left( \frac{k}{2} \left( \frac{(-q)^k + 1}{(-q)^k - 1} - \frac{1}{2} \left( \frac{(-q)^k + 1}{(-q)^k - 1} \right) a_{-k} a_k \tau_2 \cdot [K_X] \right) \right)$$

$$- \frac{1}{2} \sum_{k, \ell > 0} (a_{-k} a_{-\ell} a_{k+\ell} + a_{-k} a_k a_{\ell}) \tau_3 \cdot [X]. \quad (6.5)$$

With all the notations ready, let’s prove the Proposition 6.1.

**Proof of Proposition 6.1.** Take $L = L_3$. Given $d > 0$, we have

$$\langle a_{-n}([L_3])0, c_1(\mathcal{O}^{|n|}), a_{-n}([X])0 \rangle_{0,3,d}$$

$$= d \langle a_{-n}([L_3])0, a_{-n}([X])0 \rangle_{0,2,d}$$

$$= d \langle a_{-n} \left( \frac{[q_1]}{(t_3 - t_1)} + \frac{[q_2]}{(t_3 - t_2)} \right) 0, \right.$$  

$$\left. a_{-n} \left( \frac{[q_1]}{(t_3 - t_1)(t_2 - t_1)} + \frac{[q_2]}{(t_3 - t_2)(t_1 - t_2)} + \frac{[q_1]}{(t_1 - t_3)(t_2 - t_3)} \right) 0 \rangle_{0,2,d}$$

$$= \frac{1}{(t_3 - t_1)^2(t_2 - t_1)} \langle a_{-n}([q_1])0, a_{-n}([q_1])0 \rangle_{0,2,d}$$

$$+ \frac{1}{(t_3 - t_2)^2(t_1 - t_2)} \langle a_{-n}([q_2])0, a_{-n}([q_2])0 \rangle_{0,2,d}. \quad (6.6)$$
Let’s introduction following notations for convenience.

\[ M_1'(q) = \frac{t_2 + t_3 - 2t_1}{(t_2 - t_1)(t_3 - t_1)} \sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) a_{-k}(q) a_k(q), \]

\[ M_2'(q) = \frac{t_3 + t_3 - 2t_2}{(t_1 - t_2)(t_3 - t_2)} \sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) a_{-k}(q) a_k(q), \]

\[ M_3'(q) = \frac{t_1 + t_3 - 2t_3}{(t_1 - t_3)(t_2 - t_3)} \sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) a_{-k}(q) a_k(q). \]

\[ M_1'' = \frac{1}{2} \sum_{k>0} \left( \frac{t_3 - t_1}{(t_2 - t_1)} \right) a_{-k}(q) a_{k+\ell}(q) - a_{-\ell}(q) a_k(q), \]

\[ M_2'' = \frac{1}{2} \sum_{k>0} \left( \frac{t_2 - t_2}{(t_1 - t_2)} \right) a_{-k}(q) a_{k+\ell}(q) - a_{-\ell}(q) a_k(q), \]

\[ M_3'' = \frac{1}{2} \sum_{k>0} \left( \frac{t_2 - t_3}{(t_1 - t_3)} \right) a_{-k}(q) a_{k+\ell}(q) - a_{-\ell}(q) a_k(q). \]

The terms \( a_{-n}(q)|0\rangle, a_{-n}(q)|0\rangle \) in (6.6) was calculated in [O-P]. In fact

\[
\sum_{d=1}^{\infty} d \langle a_{-n}(q)|0\rangle, a_{-n}(q)|0\rangle_{0,2,d} q^d = \sum_{d=1}^{\infty} \langle a_{-n}(q)|0\rangle, c_1(O^n), a_{-n}(q)|0\rangle_{0,3,d} q^d = \langle a_{-n}(q)|0\rangle, (M_1'(q) + M_2'(q) - M_3'(q)) a_{-n}(q)|0\rangle. \quad (6.7)
\]

Now we have

\[
\sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) a_{-k} a_k(q) = \sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) a_{-k} a_k(q) \left( \frac{q_1}{q_1\times q_1} \right) + \frac{t_1 + t_2 - 2t_3}{(t_1 - t_2)(t_3 - t_2)} \left[ q_2 \times q_2 \right] + \frac{t_1 + t_2 - 2t_3}{(t_1 - t_3)(t_2 - t_3)} \left[ q_3 \times q_3 \right]
\]

\[
= \sum_{k>0} \left( \frac{k}{2} - \frac{1}{2} \right) \left( \frac{q_1}{q_1\times q_1} \right) a_{-k} a_k(q) \left( \frac{t_2 + t_3 - 2t_1}{(t_2 - t_1)(t_3 - t_1)} \right) + \frac{t_1 + t_2 - 2t_3}{(t_1 - t_2)(t_3 - t_2)} \left[ q_2 \times q_2 \right] + \frac{t_1 + t_2 - 2t_3}{(t_1 - t_3)(t_2 - t_3)} \left[ q_3 \times q_3 \right]
\]

\[
= M_1'(q) + M_2'(q) + M_3'(q), \quad (6.8)
\]
Similarly, we also have
\[
\frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} - a_{-k-\ell} a_{k} a_{\ell} \right) (\tau_{3+}[X])
\]
\[
= \frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} - a_{-k-\ell} a_{k} a_{\ell} \right) \left( \frac{q_1 \times q_1 \times q_1}{(t_3 - t_1)(t_2 - t_1)} \right)
\]
\[
+ \frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} - a_{-k-\ell} a_{k} a_{\ell} \right) \left( \frac{q_2 \times q_2 \times q_2}{(t_3 - t_2)(t_1 - t_2)} \right)
\]
\[
+ \frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} - a_{-k-\ell} a_{k} a_{\ell} \right) \left( \frac{q_3 \times q_3 \times q_3}{(t_2 - t_3)(t_1 - t_3)} \right)
\]
\[
= M''_1 + M''_2 + M''_3
\]
(6.9)

The first equality comes from \( \tau_{3+}(q_i) = q_i \times q_i \times q_i \) and the localization formula for \( X \) expressed in terms of fixed points \( q_i \).

Combination of the formulae (6.6), (6.7), (6.8) and (6.9) gives the conclusion of the Proposition. \( \square \)

7. General surfaces and applications

7.1. General surfaces. One-point extremal Gromov-Witten invariants on the Hilbert scheme \( X^{[n]} \) for a simply-connected projective surface \( X \) are computed in [L-Q]. As a consequence, the 3-point extremal Gromov-Witten invariants are computed for \( X^{[2]} \) and the Ruan’s Cohomological Crepant Resolution Conjecture holds in this case.

In this section, we will determine two-point extremal Gromov-Witten invariants of \( X^{[n]} \).

**Theorem 7.1.** Let \( X \) be a simply connected projective surface. Then
\[
\sum_{d \geq 0} \langle A^\lambda_c, c_1(\mathcal{O}^{[n]}) \rangle A^\mu_c = \langle A^\lambda_c, M(q) A^\mu_c \rangle.
\]
(7.1)

**Proof.** Let \( L(q) \) denote the left hand side of (7.1) and \( R(q) \) denote the right hand side of (7.1).

If \( \ell(\lambda^3) - \ell(\mu^3) + \ell(\mu^3) - \ell(\lambda^1) \neq 1 \), both \( L(q) \) and \( R(q) \) equal to zero for the cohomological degree reason.

If \( \ell(\lambda^3) - \ell(\mu^3) + \ell(\mu^3) - \ell(\lambda^1) = 1 \), but \( \ell(\lambda^3) \neq \ell(\mu^1) \) and \( \ell(\mu^3) \neq \ell(\lambda^1) \), by Proposition 4.1 and Proposition 5.1, we have

\[
L(q) = A^\lambda_c \cup c_1(\mathcal{O}^{[n]}) \cup A^\mu_c = L(0), \quad R(q) = \langle A^\lambda_c, M(0) \cup A^\mu_c \rangle = R(0).
\]

Now \( L(0) = R(0) \) follows from Lehn’s result in [Lehn] (see [Q-W] as well).

Next let’s assume without loss of generality that \( \ell(\mu^3) = \ell(\lambda^1) + 1 \) and \( \ell(\lambda^3) = \ell(\mu^1) \).

If \( \lambda^3 \neq \mu^1 \), by Proposition 4.1 and Proposition 5.1, both \( L(q) \) and \( R(q) \) are constant functions of \( q \). Therefore \( L(q) = L(0) = R(0) = R(q) \).

If \( \lambda^3 = \mu^1 \), from Corollary 4.4 and Lemma 4.3, we get the formula
\[
L(q) = L(0) + \sum_{\substack{1 \leq i \leq t, \ 1 \leq j \leq b, \\ \mu^1 = \lambda^3}} \langle A^{\lambda^3 - \lambda^2}_c, A^{\mu^3 - \mu^2}_c \rangle \langle E_j, K_X \rangle \left( \sum_{d > 0} d c_d q^d \right).
\]
From the discussion in §5, we get
\[
R(q) = \langle A_e^\lambda, M(0) A_e^\mu \rangle + \langle A_e^\lambda, (M(q) - M(0)) A_e^\mu \rangle
\]
\[
= \langle A_e^\lambda, M(0) A_e^\mu \rangle + \sum_{1 \leq i \leq t, 1 \leq j \leq b, \nu_i = \lambda_j} \langle A_e^{\lambda_i - \lambda_j}, A_e^{\mu_i - \nu_j} \rangle \langle E_j, X \rangle (-1)^{\ell,i} q^{(\ell - q) \ell \ell(\ell - q) \ell - 1} - \frac{q}{1 + q} \right).}
\]

Now we need to determine \( c_{\ell,d} \) explicitly. Since \( c_{\ell,d} \) is independent of the surface \( X \), it suffices to consider the case \( X = \mathbb{P}^2, A_e^\lambda = a_\ell(L) \) and \( A_e^\mu = a_\ell(X) \) where \( L \) is a line in \( X \). By Proposition 6.1, we have
\[
-3 \sum_{d > 0} dc_{\ell,d} q^d = \langle a_{-\ell}(L)(0), (M(q) - M(0)) a_{-\ell}([X])(0) \rangle = -3(-1)^{\ell,i} q^{(\ell - q) \ell \ell(\ell - q) \ell - 1} - \frac{q}{1 + q} \right).
\]

Combination of all the formulae above gives us \( L(q) = R(q) \). □

Thus we see that the quantum first Chern class has an explicit formula, i.e., it is the operator \( M(q) \).

### 7.2. Application to Ruan's conjecture

In [Q-W], a vertex algebraic study of the cohomology \( H_{CR}(X^{(n)}) \) was carried out. There is an irreducible Heisenberg action \( \{ p_i(\alpha) \}_{i \in \mathbb{Z}, \alpha \in H^*(X)} \) on \( H_{CR}(X^{(n)}) \) with a highest weight vector \( |0\rangle \). Therefore there is a natural isomorphism
\[
\Phi: H^*(X^{[n]}) \rightarrow H_{CR}^*(X^{(n)})
\]
as vector spaces. There is a counterpart of the first Chern class of the tautological bundle on \( H_{CR}^*(X^{(n)}) \) defined in [Q-W]. This class \( O^1(1_X, n) \in H_{CR}^*(X^{(n)}) \) defines an operator \( b \) on \( H_{CR}^*(X^{(n)}) \) via the Chen-Ruan product, which plays the similar role as \( c_1(O^{(n)}) \). It has the expression
\[
b = -\frac{1}{2} \sum_{k, \ell > 0} \left( p_{-k} p_{-\ell} p_{k+\ell} + p_{-k-\ell} p_k p_\ell \right)(\tau_{3*}[X]).
\]

Therefore, a consequence of the Conjecture 2.1 for the operator \( b \) on \( H_{CR}^*(X^{(n)}) \) and the quantum first Chern class operator on \( H_{CR}^*(X^{[n]}) \) is the following equation
\[
\langle A_e^\lambda, c_1(O^{[n]}) \cup_{\tau} A_e^\mu \rangle = \langle \Phi(A_e^\lambda), O^1(1_X, n) \cup_{CR} \Phi(A_e^\mu) \rangle.
\]  
(7.2)

Recall that \( c_1(O^{[n]}) \cup_{\tau} \) is the operator \( M(-1) \). Therefore the way to prove the formula (7.2), under the identification of Heisenberg operators \( a_\ell(\alpha) \rightarrow p_\ell(\alpha) \), is to prove the operator
\[
M(-1) = -\frac{1}{2} \sum_{k, \ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} + a_{-k-\ell} a_k a_\ell \right)(\tau_{3*}[X]).
\]

Note that
\[
\frac{k}{2} (-q)^k + 1 - \frac{1}{2} (-q) + 1
\]
is well defined at \( q = -1 \) by using L'Hospital's rule. In fact,
\[
\left( \frac{k}{2} (-q)^k + \frac{1}{2} (-q) + 1 \right) \bigg|_{q=-1} = 0,
\]
and therefore
\[
M(-1) = -\frac{1}{2} \sum_{k,\ell > 0} \left( a_{-k} a_{-\ell} a_{k+\ell} + a_{-k-\ell} a_k a_{\ell} \right) (\tau_3 [X]).
\]
Thus we proved the formula (7.2).

Remark 7.2. When the surface \( X \) is \( K3 \), the combination of results in [L-S] and [F-G, Ur] verifies Ruan’s conjecture. When \( X = \mathbb{P}^2 \), it is is shown in [ELQ] that the three-point extremal GW-invariants for \( X^{[3]} \) can be reduced to two-point extremal GW-invariants. By the result in this paper, the three-point extremal GW-invariants for \( X^{[3]} \) are determined.

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