THE $F$-PURE THRESHOLD OF QUASI-HOMOGENEOUS POLYNOMIALS

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Abstract. Inspired by the work of Bhatt and Singh [BS15] we compute the $F$-pure threshold of quasi-homogeneous polynomials. We first consider the case of a curve given by a quasi-homogeneous polynomial $f$ in three variables $x, y, z$ of degree equal to the degree of $xyz$ and then we proceed with the general case of a Calabi-Yau hypersurface, i.e. a hypersurface given by a quasi-homogeneous polynomial $f$ in $n + 1$ variables $x_0, \ldots, x_n$ of degree equal to the degree of $x_0 \cdots x_n$.

1. Introduction

To any polynomial $f \in K[x_0, \ldots, x_n] = R$, where $K$ is a field of characteristic $p > 0$, one can attach an invariant called the $F$-pure threshold, first defined in [TW04], [MTW05]. This invariant is the characteristic $p$ analogue of the log canonical threshold in characteristic zero. The $F$-pure threshold, which is a rational number, is a quantitative measure of the severity of the singularity of $f$. Smaller values of the $F$-pure threshold correspond to a "worse" singularity.

In this article we focus mainly on the computation of the $F$-pure threshold of quasi-homogeneous polynomials. A polynomial $f$ is called quasi-homogeneous if there exists a $\mathbb{N}$-grading of $K[x_0, \ldots, x_n]$ such that $f$ is homogeneous with respect to this grading.

The first of our two main results is the following:

**Theorem** (see Theorem 4.3). Let $C = \text{Proj}(R/fR)$ be the curve given by a quasi-homogeneous polynomial $f \in K[x, y, z]$ of degree equal to the degree of $xyz$ with an isolated singularity. Then

$$ \text{fpt}(f) = \begin{cases} 
1, & \text{if } C \text{ is ordinary} \\
1 - \frac{1}{p}, & \text{otherwise.}
\end{cases} $$

Here, a curve $C$ is (by definition) ordinary if and only if the map on $H^1(C, \mathcal{O}_C)$ induced by Frobenius is bijective. This theorem is a generalization of the two-dimensional case of the main theorem of Bhatt and Singh [BS15], which says that the $F$-pure threshold of an elliptic curve $E$ given by a homogeneous polynomial $f \in K[x, y, z]$ of degree three is 1 if $E$ is ordinary and $1 - \frac{1}{p}$ otherwise. In contrast to the paper of Bhatt and Singh our proof does not rely on deformation-theoretic arguments and hence gives a more elementary approach to this result.

More generally, we consider the case of a Calabi-Yau hypersurface $X = \text{Proj}(R/fR)$ given by a quasi-homogeneous polynomial $f \in K[x_0, \ldots, x_n] = R$ and relate the $F$-pure threshold of $f$ to a numerical invariant of $X$, namely the order of vanishing of the so-called Hasse invariant. For this, we consider the family $\pi : \mathcal{X} \to \text{Hyp}_w$ of hypersurfaces of degree $w = \deg(x_0 \cdots x_n)$ in the weighted projective space $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$. Our chosen hypersurface $X = \text{Proj}(R/fR)$ gives a point
Let $[X]$ be a point in $\text{Hyp}_w$. The Hasse invariant $H$ of a suitable family $\pi : \mathfrak{X} \to S$ of varieties in characteristic $p$ (for the precise conditions see section 5) is the element in

$$\text{Hom}\left(R^N\pi_*\mathcal{O}_{\mathfrak{X}(1)}, R^N\pi_*\mathcal{O}_\mathfrak{X}\right) \cong \text{Hom}\left((R^N\pi_*\mathcal{O}_\mathfrak{X})^p, R^N\pi_*\mathcal{O}_\mathfrak{X}\right) \cong \text{Hom}\left(\mathcal{O}_S, (R^N\pi_*\mathcal{O}_\mathfrak{X})^{1-p}\right) \cong H^0\left(S, (R^N\pi_*\mathcal{O}_\mathfrak{X})^{1-p}\right)$$

induced by the relative Frobenius $\text{Frob}_{\mathfrak{X}/S}$. Here the relative Frobenius is given by the following diagram

$$\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\pi} & \mathfrak{X}^{(1)} \\
\downarrow{\text{Frob}_\mathfrak{X}} & & \downarrow{\pi^{(1)}} \\
S & \xrightarrow{\text{Frob}_S} & S
\end{array}$$

Now, fix $s \in S$ and an integer $t > 0$ and let $t[s]$ be the order $t$ neighbourhood of $s$. Then, the order of vanishing of the Hasse invariant at the point $s \in S$ is given by $\text{ord}_s(H) = \max\{t \mid i^*H = 0\}$ where $i : t[s] \hookrightarrow S$. The second main result of this paper, generalizing [BS15] to the quasi-homogeneous case, is the following:

**Theorem** (see Theorem 5.1). If $p \geq w(n-2) + 1$, then $\text{fpt}(f) = 1 - \frac{h}{p}$, where $h$ is the order of vanishing of the Hasse invariant at $[X] \in \text{Hyp}_w$ on the deformation space $\mathfrak{X}$ of $X \subset \mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$.

In the homogeneous case this result was proven by Bhatt and Singh in [BS15] by pointing out a connection between the order of vanishing of the Hasse invariant and the injectivity of the map $H^{n-1}(X, \mathcal{O}_X) \twoheadrightarrow H^{n-1}(tX, \mathcal{O}_{tX})$ induced by $\text{Frob}_R$, where $tX$ is the order $t$ neighbourhood of $X$ in $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$. We generalize this statement to the quasi-homogeneous case using local cohomology instead of sheaf cohomology, i.e. we consider the map $a_t$ as a map $H^n_m(R/f)_0 \twoheadrightarrow H^n_m(R/f^t)_0$, which makes our approach rather explicit.

We should also mention the paper [HNBWZ16] of Hernández, Núñez - Betancourt, Witt and Zhang, where the authors compute the possible values of the $F$-pure threshold of a quasi-homogeneous polynomial of arbitrary degree using base $p$ expansions. In particular, as a corollary they get the same list of possible $F$-pure thresholds as we obtain here in the case of a Calabi-Yau hypersurface.

In sections 2 and 3 we set up the notation and extend some results of Bhatt and Singh ([BS15]) to the quasi-homogeneous setting, leading to recovering the results of [HNBWZ16] in the Calabi-Yau case. In section 4 we prove Theorem 4.3 by elementary methods. The final section 5 is dedicated to the proof of Theorem 5.1.

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2. Quasi-homogeneous polynomials with an isolated singularity

In this section we fix some notation and give some basic facts, which we will need later. For further information we refer the reader to [Kun97], for example.

Throughout this article, $K$ will denote a field of characteristic $p > 0$ and

$$R := K[x] := K[x_0, \ldots, x_n]$$

will be the polynomial ring over $K$ in $n + 1$ variables. By

$$m := (x_0, \ldots, x_n)$$

we will denote the maximal ideal of $R$ generated by the variables of $R$.

An $(n + 1)$-tuple $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}_{>0}^{n+1}$ defines a grading on $R = K[x_0, \ldots, x_n]$ by setting

$$\deg(x_i) := \alpha_i$$

and

$$\deg(x_k) := \sum_{i=0}^{n} \alpha_i k_i, \quad k = (k_0, \ldots, k_n)$$

where $k = (k_0, \ldots, k_n)$ is a multi-index and $x^k = x_0^{k_0} \cdots x_n^{k_n}$. It follows that $R = \bigoplus_d R_d$, where

$$R_d := \left\{ \sum_{k=(k_0, \ldots, k_n)} p_k x^k \bigg| \sum_{i=0}^{n} \alpha_i k_i = d, p_k \in K \right\}$$

and the elements of $R_d$ are called quasi-homogeneous polynomials of degree $d$ and type $\alpha$. For an element $f \in R_d$ we have

$$f(\lambda^{\alpha_0} x_0, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_0, \ldots, x_n)$$

for all $\lambda \in K$.

We set

$$w := \sum_{i=0}^{n} \alpha_i = \deg(x_0 \cdots x_n).$$

A sequence $f_1, \ldots, f_m \in R$ is called a regular sequence if the image of $f_i$ in $R/(f_1, \ldots, f_{i-1})$ is a non-zero-divisor ($1 \leq i \leq m$) and if $(f_1, \ldots, f_m) \neq R$. We say that an element $f \in R_d$ has an isolated singularity if

$$\frac{\partial f}{\partial x_0} \cdots \frac{\partial f}{\partial x_n}$$

is a regular sequence. Furthermore, the Jacobian ideal of $f \in R$ is

$$J(f) := \left( \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

and the Milnor number of $f$ is defined by

$$\mu(f) := \dim_K R/J(f).$$

It is well-known that $f$ has an isolated singularity if and only if $\mu(f) < \infty$. This can be shown with explicit bounds for $\mu(f)$ by using the Poincaré series. The Poincaré series $H_M(T)$ of a finitely generated graded $R$-module $M$ is defined by

$$H_M(T) := \sum_j \dim_K (M_j) T^j,$$

where $M_j$ is the homogeneous part of $M$ of degree $j$. 
First, we want to compute \( H_R(T) \). For this, we use the fact that if \( f \in R \) is a homogeneous element of degree \( d \), which is a non-zero divisor of \( M \), then
\[
H_{M/fM}(T) = (1 - T^d)H_M(T).
\]
(1)

Hence, \( H_{R/x^n}(T) = (1 - T^{\alpha n})H_R(T) \) and inductively we get
\[
1 = H_R/(x_0,\ldots,x_n)(T) = n \prod_{i=0}^n (1 - T^{\alpha_i})H_R(T).
\]

Thus,
\[
H_R(T) = \prod_{i=0}^n \frac{1}{1 - T^{\alpha_i}}.
\]

Now, let \( f_0,\ldots,f_m \in R \) be quasi-homogeneous polynomials of type \( \alpha \) with \( \deg(f_i) = d_i, 0 \leq i \leq m \), such that \( f_0,\ldots,f_m \) is a regular sequence. Then
\[
H_{R/(f_0,\ldots,f_m)}(T) = \prod_{j=0}^m (1 - T^{d_j})H_R(T) = \frac{\prod_{j=0}^m (1 - T^{d_j})}{\prod_{i=0}^n (1 - T^{\alpha_i})}.
\]

It is shown in [Kun97, p. 213] that for \( m = n \) one has
\[
\dim_K(R/(f_0,\ldots,f_n)) = \lim_{T \to 1} \prod_{i=0}^n \frac{(1 - T^{d_i})}{(1 - T^{\alpha_i})} = \prod_{i=0}^n \frac{d_i}{\alpha_i}.
\]

In particular, \( R/(f_0,\ldots,f_n) \) is a finite dimensional \( K \)-algebra.

If we have \( f \in R_d \) with an isolated singularity, then \( \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \) form a regular sequence and we know that \( \deg \left( \frac{\partial f}{\partial x_i} \right) = d - \alpha_i \). Therefore the Milnor number is given by
\[
\mu(f) = \dim_K(R/J(f)) = \prod_{i=0}^n \frac{d - \alpha_i}{\alpha_i}.
\]

### 3. The \( F \)-pure threshold of a quasi-homogeneous polynomial

In [BS15], Bhatt and Singh explain how to compute the \( F \)-pure threshold of a homogeneous polynomial and consider in particular the case of a Calabi-Yau hypersurface. In the following section we extend their results to compute the \( F \)-pure threshold of a quasi-homogeneous polynomial. To a large extent the proofs are analogous to the ones in [BS15].

During this section let \( q = p^e \) be a power of \( p \). Remember that \( R = K[x_0,\ldots,x_n] \) and \( m = (x_0,\ldots,x_n) \). By \( F : R \to R, r \mapsto r^p \) we denote the Frobenius or \( p \)-th power map on \( R \). We denote by
\[
a^{[q]} := (a^q | a \in a)
\]
the Frobenius power of an ideal \( a \subset R \). For \( f \in m \) one defines in [BS15]
\[
\mu_f(q) := \min \left\{ k \in \mathbb{N} | f^k \in m^{[q]} \right\}
\]
and observes that \( \mu_f(1) = 1 \) and that
\[
1 \leq \mu_f(q) \leq q.
\]
Furthermore,
\[ \mu_f(pq) \leq p \mu_f(q), \]
since \( f^{\mu_f(q)} \in m[q] \) implies that \( f^{p \mu_f(q)} \in m[pq] \). Hence, \( \{ \frac{\mu_f(p^e)}{p^e} \}_{e \geq 0} \) is a non-increasing sequence of positive rational numbers and one defines:

**Definition 3.1.** The F-pure threshold of \( f \) is
\[ \text{fpt}(f) := \lim_{e \to \infty} \frac{\mu_f(p^e)}{p^e}. \]

The definition of \( \mu_f(q) \) yields \( f^{\mu_f(q)} - 1 \notin m[q] \) and therefore we get \( f^{p \mu_f(q)} - p \notin m[pq] \). Together with (2) we deduce \( p \mu_f(q) - p + 1 \leq \mu_f(pq) \leq p \mu_f(q) \), which implies
\[ \mu_f(q) = \left\lfloor \frac{\mu_f(pq)}{p} \right\rfloor. \]

Now, let \( f \in R = K[x_0, \ldots, x_n] \) be a quasi-homogeneous polynomial of degree \( d \) and type \( \alpha \) and let \( t \leq q \) be an integer. Then the Frobenius iterate \( F^e : R/fR \to R/fR \) lifts to a map \( R/fR \to R/f^tR \). We compose this map with the canonical surjection \( R/f^qR \to R/f^tR \) and get a map \( \overline{F}_t^e \).

**Lemma 3.2.** Let \( f \in R \) be a quasi-homogeneous polynomial of degree \( d \) and type \( \alpha \) and let \( t \leq q \) be an integer. Then \( \mu_f(q) > q - t \) if and only if \( \overline{F}_t^e : H^\alpha_m(R/fR) \to H^\alpha_m(R/f^tR) \) is injective.

**Proof.** We have a commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & R(-d) & \stackrel{f}{\longrightarrow} & R & \longrightarrow & R/fR & \longrightarrow & 0 \\
\downarrow f^{q-t} F^e & & \downarrow F^e & & \downarrow \overline{F}_t^e & & \\
0 & \longrightarrow & R(-dt) & \stackrel{f^t}{\longrightarrow} & R & \longrightarrow & R/f^tR & \longrightarrow & 0,
\end{array}
\]
which gives an induced diagram of local cohomology modules
\[
\begin{array}{cccccc}
0 & \longrightarrow & H^\alpha_m(R/fR) & \longrightarrow & H^\alpha_{m+1}(R)(-d) & \stackrel{f}{\longrightarrow} & H^\alpha_{m+1}(R) & \longrightarrow & 0 \\
\downarrow \overline{F}_t^e & & \downarrow f^{q-t} F^e & & \downarrow \overline{F}_t^e & & \\
0 & \longrightarrow & H^\alpha_m(R/f^tR) & \longrightarrow & H^\alpha_{m+1}(R)(-dt) & \stackrel{f^t}{\longrightarrow} & H^\alpha_{m+1}(R) & \longrightarrow & 0.
\end{array}
\]

We first show that the map \( F^e \) is injective. For this, it suffices to show that \( F^e \) acts injectively on the socle
\[ \text{Soc} \left( H^\alpha_{m+1}(R) \right) = \left\langle \left[ \frac{1}{x_0 \ldots x_n} \right] \right\rangle, \]
which follows from \( F^e \left( \left[ \frac{1}{x_0 \ldots x_n} \right] \right) = \left[ \frac{1}{f^{-q} x_0 \ldots x_n} \right] \neq 0 \). Now by the five lemma \( \overline{F}_t^e \) is injective if and only if \( f^{q-t} F^e \) is injective. Again by looking at the socle one shows that \( f^{q-t} F^e \) is injective if and only if
\[ f^{q-t} F^e \left( \left[ \frac{1}{x_0 \ldots x_n} \right] \right) = \left[ \frac{f^{q-t}}{x_0 \ldots x_n} \right] \neq 0, \]
which is equivalent to \( f^{q-t} \notin m[q] \). By the definition of \( \mu_f(q) \) this is equivalent to \( \mu_f(q) > q - t \).
\[ \square \]
This cohomological description of $\mu_f(q)$ will be very useful in the following sections. Now we want to compute the $F$-pure threshold of a quasi-homogeneous polynomial. As a first step we give a lower and an upper bound for $\mu_f(p)$ (see Lemma 3.5 and Lemma 3.6) from which one obtains bounds for $\mu_f(p^e)$. We start with some lemmata which are similar to results shown in [HNBWZ16].

**Lemma 3.3.** Let $f \in R = K[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree $d$ and type $\alpha$ with an isolated singularity. Then

$$R_{\geq (n+1)d-2w+1} \subset J(f),$$

where $w = \sum_{i=0}^n \alpha_i$.

**Proof.** Since $f$ has an isolated singularity, there exists a $k \in \mathbb{N}$ such that $m^k \subset J(f)$. This means that $(R/J(f))_i = 0$ for all $i$ greater than some $N \in \mathbb{N}$. Thus, the Poincaré series

$$H_{R/J(f)}(T) = \frac{\prod_{i=0}^n (1 - T^{d-\alpha_i})}{\prod_{i=0}^n (1 - T^{\alpha_i})}$$

must be a polynomial, which means that $\prod_{j=0}^n (1 - T^{d-\alpha_j})$ is divisible by $\prod_{i=0}^n (1 - T^{\alpha_i})$. Therefore

$$\deg(H_{R/J(f)}) = \left( (n+1)d - \sum_{j=0}^n \alpha_j \right) - \sum_{i=0}^n \alpha_i = (n+1)d - 2w.$$

This means that $(R/J(f))_i = 0$ for all $i \geq (n+1)d - 2w + 1$, thus, $R_{\geq (n+1)d-2w+1} \subset J(f)$.

**Lemma 3.4.** We have

$$\left( m^{[q]} :_R R_{\geq (n+1)d-2w+1} \right) \setminus m^{[q]} \subset R_{\geq (q+1)w-(n+1)d}.$$

**Proof.** Suppose the statement is false, then there exists a monomial

$$\lambda := x_0^{q-1-b_0} \cdots x_n^{q-1-b_n} \subset \left( m^{[q]} :_R R_{\geq (n+1)d-2w+1} \right) \setminus m^{[q]}$$

of degree

$$\deg(\lambda) = (q-1)w - \sum_{i=0}^n b_i \alpha_i < (q+1)w - (n+1)d.$$

Equivalently,

$$\sum_{i=0}^n b_i \alpha_i > (q-1)w - (q+1)w + (n+1)d = (n+1)d - 2w,$$

thus the monomial $\eta := x_0^{b_0} \cdots x_n^{b_n}$ is an element of $R_{\geq (n+1)d-2w+1}$. Therefore, $x_0^{q-1} \cdots x_n^{q-1} = \lambda \cdot \eta \in m^{[q]}$, which is a contradiction.

This yields a lower and an upper bound for $\mu_f(q)$.

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This page contains a detailed explanation of cohomological descriptions in the context of $F$-pure thresholds for quasi-homogeneous polynomials. It introduces lemmata that are similar to results in [HNBWZ16], providing lower and upper bounds for $\mu_f(p)$ and $\mu_f(p^e)$. The proofs involve analyzing the Poincaré series of the quotient ring and bounding the degree of these series to derive the desired inclusions.
Lemma 3.5. Let \( f \in R \) be a quasi-homogeneous polynomial of degree \( d \) and type \( \alpha \) with an isolated singularity. If \( p \nmid \mu_f(q) \), then
\[
\mu_f(q) \geq \frac{w(q+1) - nd}{d}.
\]

Proof. With \( k := \mu_f(q) \) we have \( f^k \in m^{[q]} \). The partial derivatives \( \frac{\partial}{\partial x_i} \) map \( m^{[q]} \) to \( m^{[q]} \) and therefore
\[
k f^{k-1} \frac{\partial f}{\partial x_i} \in m^{[q]}
\]
for all \( i \). Since \( k \) is non-zero in \( K \), it follows
\[
f^{k-1} J(f) \subset m^{[q]}.
\]
By the definition of \( k \) we know that \( f^{k-1} \notin m^{[q]} \) so that \( f^{k-1} \in (m^{[q]} :_R J(f)) \setminus m^{[q]} \). By Lemma 3.3 and Lemma 3.4 it follows that
\[
\left( m^{[q]} :_R J(f) \right) \setminus m^{[q]} \subset \left( m^{[q]} :_R R_{\geq (n+1)d - 2w + 1} \right) \setminus m^{[q]} \subset R_{\geq (q+1)w - (n+1)d}.
\]
This means that \( f^{k-1} \in R_{\geq (q+1)w - (n+1)d} \). Hence,
\[
d(k - 1) = \deg(f^{k-1}) \geq w(q + 1) - (n + 1)d
\]
and therefore \( k \geq \frac{w(q+1)-nd}{d} \). \( \square \)

Lemma 3.6. Let \( f \in R \) be a quasi-homogeneous polynomial of degree \( d \) and type \( \alpha \). Then
\[
\mu_f(q) \leq \left\lfloor \frac{wq - w + 1}{d} \right\rfloor
\]
for all \( q = p^r \).

Proof. For \( k \in \mathbb{N} \) one has \( f^k \in m^{[q]} \) if \( dk \geq w(q - 1) + 1 \). Thus,
\[
\mu_f(q) = \min \left\{ k | f^k \in m^{[q]} \right\} \leq \left\lfloor \frac{wq - w + 1}{d} \right\rfloor.
\]
\( \square \)

The following lemma explains how to compute \( \mu_f(pq) \) if \( \mu_f(q) \) is given, when certain conditions are fulfilled.

Lemma 3.7. Let \( f \in R \) be a quasi-homogeneous polynomial of degree \( d \) and type \( \alpha \) with an isolated singularity.

1. If \( \frac{\mu_f(q)-1}{q-1} = \frac{w}{d} \) for some \( q = p^s \), then \( \frac{\mu_f(pq)-1}{pq-1} = \frac{w}{d} \).
2. Suppose \( p \geq nd - d - w + 1 \). If \( \frac{\mu_f(q)}{q} < \frac{w}{d} \) for some \( q = p^s \), then \( \mu_f(pq) = p \mu_f(q) \).

In particular,
\[
\frac{\mu_f(pq)}{pq} = \frac{\mu_f(q)}{q},
\]
thus the sequence \( \left\{ \frac{\mu_f(p^r)}{p^r} \right\}_{p^r \geq q} \) is constant.

Proof. To prove the first statement, assume that \( \frac{\mu_f(q)-1}{q-1} = \frac{w}{d} \) for some \( q = p^s \). Then \( f^{\mu_f(q)-1} \) has degree
\[
d(\mu_f(q) - 1) = d \cdot \frac{w}{d} \cdot (q - 1) = w(q - 1).
\]
Since we also know that $f^{(q) - 1}$ generates $\text{Soc}(R/m^{[q]})$, it follows that $f^{(q) - 1}$ generates $\text{Soc}(R/m)$. Therefore, $(f^{(q) - 1})_{q=1}^{\infty}$ generates $\text{Soc}(R/m^{[q]})$, so
\[
(m_f(q) - 1) \frac{pq - 1}{q - 1} = m_f(pq) - 1.
\]
Rearranging the terms one obtains
\[
\frac{m_f(pq)-1}{q-1} = \frac{m_f(q)-1}{q-1} = \frac{w}{q}.
\]
In order to prove the second statement, note that by equation (2) we only need to show that $m_f(pq) < pm_f(q)$ cannot occur. Therefore, suppose $m_f(pq) < pm_f(q)$. First we will show that $m_f(pq)$ cannot be a multiple of $p$. Suppose $m_f(pq) = pl$ for some $l$. By equation (3) it follows that $pm_f(q) = pl = m_f(pq)$, which is a contradiction. We can now use Lemma 3.5 and get
\[
m_f(pq) \geq \frac{w(pq + 1) - nd}{d}.
\]
Using $m_f(pq) < pm_f(q)$ and the second assumption, namely $dm_f(q) < wq$, we get
\[
w(pq + 1) - nd \leq dm_f(pq) \leq dm_f(q) \leq wq - p - d.
\]
This gives $p \leq nd - d - w$, which contradicts our assumption on $p$. □

To see how one can calculate the $F$-pure threshold of a quasi-homogeneous polynomial using Lemma 3.5, 3.6 and 3.7 let us consider an example.

**Example 3.8.** Let $f = xy^2 + x^3$, then $f$ is quasi-homogeneous of degree 8 and type $\alpha = (2,3)$. Let $p \neq 2$. We claim that
\[
fpt(f) = \begin{cases} 
\frac{5}{8}, & \text{if } p \equiv 1 \pmod{8} \\
\frac{5}{8} - \frac{1}{8}, & \text{if } p \equiv 5 \pmod{8} \\
\frac{5}{8} - \frac{3}{5}, & \text{if } p \equiv 7 \pmod{8}.
\end{cases}
\]

By Lemma 3.6 we get
\[
m_f(p) \leq \left\lceil \frac{5p - 4}{8} \right\rceil = \left\lceil \frac{p - 3p + 4}{8} \right\rceil \leq p - 1.
\]
With Lemma 3.3 it follows
\[
m_f(p) \geq \left\lfloor \frac{5p - 3}{8} \right\rfloor.
\]
Since $\left\lfloor \frac{5p - 4}{8} \right\rfloor \leq \left\lceil \frac{5p - 3}{8} \right\rceil$, we conclude that $m_f(p) = \left\lfloor \frac{5p - 3}{8} \right\rfloor$.

First, let $p \equiv 1 \pmod{8}$, then $m_f(p) = \frac{5}{8}p + \frac{3}{8}$. With the first part of Lemma 3.7 it follows that $m_f(q) = \frac{5}{8}q + \frac{3}{8}$. Hence, $fpt(f) = \lim_{e \to \infty} \frac{m_f(p^e)}{p^e} = \frac{5}{8}$.

Secondly, let $p \equiv 3 \pmod{8}$, then $m_f(p) = \frac{5}{8}p + \frac{1}{8}$. Since we cannot use Lemma 3.7 we compute $m_f(p^2)$ in the same way we computed $m_f(p)$. We get $m_f(p^2) = \frac{5}{8}p^2 + \frac{3}{8}$. With the first part of Lemma 3.7 it follows $m_f(q) = \frac{5}{8}q + \frac{3}{8}$ for all $q = p^e \geq p^2$. Hence, $fpt(f) = \lim_{e \to \infty} \frac{m_f(p^e)}{p^e} = \frac{5}{8}$.

Now let $p \equiv 5 \pmod{8}$, then $m_f(p) = \frac{5}{8}p - \frac{1}{8}$. With the second part of Lemma 3.7 it follows that $\left\{ \frac{m_f(q)}{q} \right\}$ is a constant sequence. Hence, $fpt(f) = \frac{5}{8} - \frac{1}{8}$. Note that $\frac{m_f(q)}{q} = \frac{5}{8} - \frac{1}{8}$.

The last case is $p \equiv 7 \pmod{8}$. Here $m_f(p) = \frac{5}{8}p - \frac{3}{8}$. With the second part of Lemma 3.7 it follows that $\left\{ \frac{m_f(q)}{q} \right\}$ is a constant sequence. Hence, $fpt(f) = \frac{5}{8} - \frac{3}{8}$. Now let $p \equiv 5 \pmod{8}$. Here $m_f(p) = \frac{5}{8}p - \frac{3}{8}$. With the second part of Lemma 3.7 it follows that $\left\{ \frac{m_f(q)}{q} \right\}$ is a constant sequence. Hence, $fpt(f) = \frac{5}{8} - \frac{3}{8}$.
As a consequence of the above lemmata we obtain the following theorem, which is very similar to the one given in [BS15] for homogeneous polynomials.

**Theorem 3.9.** Let \( f \in K[x_0, \ldots, x_n] \) be a quasi-homogeneous polynomial of degree \( w = \sum_{i=0}^{n} \alpha_i \) with an isolated singularity. Then

1. \( \mu_f(p) = p - h \), where \( 0 \leq h \leq n - 1 \) is an integer.
2. \( \mu_f(pq) = \mu_f(q)p \) for all \( q \) with \( q \geq n - 1 \).
3. If \( p \geq n - 1 \), then \( \mu_f(fpt) = 1 - \frac{h}{p} \), where \( 0 \leq h \leq n - 1 \).

**Proof.** First, suppose \( \mu_f(p) = p \). Then by Lemma 3.7 (1) we get \( \mu_f(q) = q \) for all \( q \), so the first two assertions follow.

Now, suppose \( \mu_f(p) < p \). Then by Lemma 3.5 it follows that \( \mu_f(p) \geq \frac{w(p + 1) - nw}{w} = p + 1 - n \), which gives \( \mu_f(p) = p - h \) with \( 0 < h \leq n - 1 \). To prove the second assertion, suppose \( \mu_f(pq) < p\mu_f(q) \) (see equation (2)). Then \( p \not| \mu_f(pq) \), since otherwise \( \mu_f(pq) = p\mu_f(q) \) by equation (3). Thus Lemma 3.5 yields

\[
\mu_f(pq) \geq pq + 1 - n.
\]

Since \( \mu_f(p) \leq p - 1 \), it follows with equation (2) that \( \mu_f(q) \leq q - \frac{4}{p} \) and therefore

\[
\mu_f(pq) \leq p\mu_f(q) - 1 \leq pq - q - 1.
\]

Combining these two results we get

\[
pq + 1 - n \leq \mu_f(pq) \leq pq - q - 1,
\]

which is equivalent to \( q \leq n - 2 \). The third assertion easily follows from (1) and (2). \( \square \)

In the homogeneous case, Bhatt and Singh [BS15] relate the integer \( h \) that appears in Theorem 3.9 to the so-called Hasse invariant. The aim of the next two sections is to answer the following question: What is \( h \) in the quasi-homogeneous case? For this, we first consider the case of a curve and then we pass on to the case \( n > 2 \).

### 4. The case of a curve

In this section let \( R = K[x, y, z] \), where \( K \) is perfect and let \( \mathfrak{m} = (x, y, z) \) be the maximal ideal of \( R \). Let

\[
\deg(x) = \alpha_x, \deg(y) = \alpha_y \text{ and } \deg(z) = \alpha_z
\]

and let \( f \in R \) be a quasi-homogeneous polynomial of degree \( w = \alpha_x + \alpha_y + \alpha_z \) and type \( \alpha = (\alpha_x, \alpha_y, \alpha_z) \) with an isolated singularity. By

\[
C = \text{Proj}(R/fR)
\]

we denote the curve given by \( f \). Then by [Rei] Proposition 3.3 and 3.4] the curve \( C \) is in fact projective, thus by [CL98, Exposé III, 4.4.] \( C \) is ordinary if and only if the map

\[
F : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)
\]

induced by Frobenius is bijective. Since \( K \) is perfect, \( F \) is bijective if and only if \( F \) is injective.
In order to prove the main theorem of this section, we need the following two lemmata.

**Lemma 4.1.** If the coefficient of \((xyz)^{p-1}\) in \(f^{p-1}\) is nonzero, then \(\text{fpt}(f) = 1\). Otherwise \(\text{fpt}(f) = 1 - 1/p\).

**Proof.** First, suppose that the coefficient of \((xyz)^{p-1}\) in \(f^{p-1}\) is nonzero. This means that the coefficient of \(1/(xyz)^{p-1}\) is nonzero. By Lemma 3.2 this is equivalent to \(\mu_f(p) > p - 1\). Since \(\mu_f(p) \leq p\), we get that \(\mu_f(p) = p\). Therefore,

\[
\mu_f(p) = q
\]

by Lemma 3.7 (1) and it follows \(\text{fpt}(f) = 1\).

Now, let the coefficient of \(\frac{1}{xyz}\) in \(\frac{f^{p-1}}{(xyz)^p}\) be zero, which implies that the coefficient of \(\frac{1}{xyz}\) in \(\frac{f^{p-1}}{(xyz)^p}\) is zero. Again, by Lemma 3.2 this is equivalent to \(\mu_f(p) \leq p - 1\), thus

\[
\frac{\mu_f(p)}{p} \leq 1 - \frac{1}{p} < 1.
\]

By Lemma 3.7 (2) it follows that the sequence \(\left\{ \frac{\mu_f(q)}{q} \right\}_q\) is constant. Since \(\text{fpt}(f) \in \left\{ 1, 1 - \frac{1}{p} \right\}\) by Theorem 3.9 (3), this gives \(\text{fpt}(f) = \frac{\mu_f(p)}{p} = 1 - \frac{1}{p}\). \(\square\)

If \(f \in K[x_0, \ldots, x_n]\) is quasi-homogeneous of degree \(w = \sum_{i=0}^n \alpha_i\) (where \(\alpha_i = \deg (x_i)\)), then a similar argument as above shows: If \(p \geq w(n-2) + 1\), then \(\text{fpt}(f) = 1\) if and only if the coefficient of \((x_0 \cdots x_n)^{p-1}\) in \(f^{p-1}\) is nonzero.

**Lemma 4.2.** Let \(C = \Proj(R/fR)\) be the curve given by the quasi-homogeneous polynomial \(f \in K[x, y, z]\) of degree \(w\) and type \(\alpha\) with an isolated singularity. Then \(H^1(C, \mathcal{O}_C) = \text{Soc}\left( H^2_m(R/fR) \right)\).

**Proof.** Using local cohomology [11L, Theorem 13.21], we get

\[
H^1(C, \mathcal{O}_C) = H^2_m(R/fR)_0
\]

thus it is enough to show that \(H^2_m(R/fR)_0 = \text{Soc}\left( H^2_m(R/fR) \right)\). By Lemma 3.2 we know that \(H^2_m(R/fR)\) is a submodule of \(H^3_m(R)(-w)\) and we know that

\[
\text{Soc}\left( H^3_m(R)(-w) \right) = \left\langle \left[ \frac{1}{xyz} \right] \right\rangle,
\]

which is the degree zero part of \(H^3_m(R)(-w)\). Since

\[
m \cdot H^2_m(R/fR)_0 \subset H^2_m(R/fR)_{>0} = 0,
\]

it follows that \(H^2_m(R/fR)_0 \subseteq \text{Soc}\left( H^2_m(R/fR) \right)\). Furthermore, since \(H^2_m(R/fR) \neq 0\), it follows

\[
\text{Soc}\left( H^2_m(R/fR) \right) = H^2_m(R/fR) \cap \text{Soc}\left( H^3_m(R)(-w) \right) \neq 0
\]

(see Lemma 3.2). Therefore, \(H^2_m(R/fR)_0 = \text{Soc}\left( H^2_m(R/fR) \right)\). \(\square\)

Using these two lemmata, we deduce the main theorem of this section, which generalizes the two-dimensional case of the main theorem of [BS15] to the quasi-homogeneous case.
Theorem 4.3. Let $C = \text{Proj}(R/fR)$ be the curve given by the quasi-homogeneous polynomial $f \in K[x, y, z]$ of degree $w$ and type $\alpha$ with an isolated singularity. Then
\[
\text{fpt}(f) = \begin{cases} 
1, & \text{if } C \text{ is ordinary} \\
1 - \frac{1}{p}, & \text{otherwise}.
\end{cases}
\]

Proof. The curve $C$ is ordinary if and only if $F : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$ is injective. Using Lemma 4.2 we have shown that $C$ is ordinary if and only if
\[
F : \text{Soc} \left( H^2_m(R/fR) \right) \to \text{Soc} \left( H^2_m(R/fR) \right)
\]
is injective. But this is equivalent to the fact that
\[
\tilde{F}^1 : H^2_m(R/fR) \to H^2_m(R/fR)
\]
is injective (see Lemma 3.2). Equivalently, the coefficient of $\frac{1}{xyz}$ in $\frac{f^p-1}{(xyz)^p}$ (which is the coefficient of $(xyz)^{p-1}$ in $f^{p-1}$) is nonzero. By Lemma 4.1 the result follows. □

We conclude this section with some examples.

Example 4.4. Up to permutation there are three solutions $(a, b, c) \in \mathbb{N}^3$ of the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, namely
\[
(3, 3, 3), (2, 4, 4) \text{ and } (2, 3, 6).
\]
We will consider the corresponding elliptic singularities that occur in the classification of Arnold directly after the ADE-singularities (see for example [AGZV85]):
\[
\tilde{E}_6 = P_8 : \quad x^3 + y^3 + z^3 + \lambda xyz = 0, \\
\tilde{E}_7 = X_9 : \quad x^2 + y^4 + z^4 + \lambda xyz = 0, \\
\tilde{E}_8 = J_{10} : \quad x^2 + y^3 + z^6 + \lambda xyz = 0.
\]
The aim is to compute the $F$-pure threshold of these three polynomials. In order to do this by Lemma 4.1 it is enough to compute the coefficient of $(xyz)^{p-1}$ in the $(p-1)$-th power of the respective polynomial.

Let us start with $f_3 = x^3 + y^3 + z^3 + \lambda xyz$, which is (quasi)-homogeneous of degree 3 and type $\alpha = (1, 1, 1)$. First, we compute $f_3^{p-1}$:
\[
(x^3 + y^3 + z^3 + \lambda xyz)^{p-1} = \sum_{n=0}^{p-1} \binom{p-1}{n} \lambda^{p-1-n}(xyz)^{p-1-n}(x^3 + y^3 + z^3)^n
\]
\[
= \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n-k}{n-k} \lambda^{p-1-n} x^{3k+p-1-n} y^{3l+p-1-n} z^{3(n-k-l)+p-1-n}.
\]
Thus, in order to compute the coefficient of $(xyz)^{p-1}$, we need to solve the following three equations:
\[
3k = n, 3l = n \text{ and } 3(n-k-l) = n.
\]
Using this, the coefficient of $(xyz)^{p-1}$ in $f_3^{p-1}$ is
\[
\varphi(\lambda) = \sum_{s=0}^{\lfloor p-1 \rfloor} \binom{p-1}{3s} \lambda^{p-1-3s} = \sum_{s=0}^{\lfloor p-1 \rfloor} \frac{(3s)!}{s!} (-1)^{3s} \lambda^{p-1-3s},
\]
since \((p - 1)^{3s} \equiv (-1)^{3s} \mod p\). Therefore,
\[
fpt(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}
\]

Now, let us consider \(f_\lambda = x^2 + y^4 + z^4 + \lambda xyz\), which is quasi-homogeneous of degree 4 and type \(\alpha = (2, 1, 1)\). Similar to the above we compute that the coefficient of \((xyz)^{p - 1}\) in \(f_\lambda^{p - 1}\) is given by
\[
\varphi(\lambda) = \sum_{s=0}^{\frac{p-1}{4}} \frac{(4s)!}{(2s)!((s)!)^2} \lambda^{p - 1 - 4s}.
\]
Therefore,
\[
fpt(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}
\]

Lastly, we consider \(f_\lambda = x^2 + y^3 + z^6 + \lambda xyz\), which is quasi-homogeneous of degree 6 and type \(\alpha = (3, 2, 1)\). The coefficient of \((xyz)^{p - 1}\) in \(f_\lambda^{p - 1}\) is given by
\[
\varphi(\lambda) = \sum_{s=0}^{\frac{p-1}{6}} \frac{(6s)!}{(3s)!((2s)!)((s)!)} \lambda^{p - 1 - 6s}.
\]
Therefore,
\[
fpt(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}
\]

**Remark 4.5.** Consider the period
\[
\psi(\lambda) := \frac{1}{(2\pi)^3} \oint \frac{\lambda xyz \, dx \, dy \, dz}{f_\lambda} \quad (x, y, z)
\]
of \(f_\lambda = x^a + y^b + z^c + \lambda xyz\), where \((a, b, c)\) is one of the three triples of Example 4.4 and \(\lambda \neq 0\). One can compute that
\[
\psi(\lambda) = \sum_{n=0}^{\infty} \left( \frac{-1}{\lambda} \right)^n \left[ \left( \frac{x^a + y^b + z^c}{xyz} \right)_0^n \right] - 0,
\]
where \([-]_0\) denotes the degree zero part. One can show that the corresponding polynomial
\[
\psi(\lambda) = \sum_{n=0}^{\frac{p-1}{w}} \left( \frac{-1}{\lambda} \right)^n \left[ \left( \frac{x^a + y^b + z^c}{xyz} \right)_0^n \right] - 0
\]
is equal to the polynomial \(\varphi(\lambda)\) computed in Example 4.4.

**Example 4.6.** Next, we want to consider the \(T_{a,b,c}\) - singularities given by
\[
f_\lambda = x^a + y^b + z^c + \lambda xyz \quad \text{with } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1 \text{ and } \lambda \neq 0.
\]
Since \(f_\lambda\) is not quasi-homogeneous, we can not use Lemma 4.1. Instead, we remember that the \(F\)-pure threshold of \(f_\lambda\) was defined by \(fpt(f_\lambda) = \lim_{e \to \infty} \frac{\mu_{f_\lambda}(p^e)}{p^e}\) with \(\mu_{f_\lambda}(p^e) = \min \{ k \in \mathbb{N} | f_\lambda^{p^k} \in \mathfrak{m}[p^e] \}\). In the following we will show that the coefficient
of \((xyz)^{q-1}\) in \(f^q_\lambda\) is 1, where \(q = p^e\). This means that \(f^q_\lambda \notin \mathfrak{m}^{[q]}\), but obviously \(f^q_\lambda \in \mathfrak{m}^{[q]}\). Thus, \(\mu_{f_\lambda}(p^e) = p^e\) and therefore \(\text{fpt}(f_\lambda) = 1\) for all \(\lambda\).

Now, it remains to compute the coefficient of \((xyz)^{q-1}\) in \(f^q_\lambda\):

\[
\left(x^a + y^b + z^c + \lambda xyz\right)^{q-1} = \sum_{n=0}^{q-1} \binom{q-1}{n} \lambda^{q-1-n}(xyz)^{q-1-n}(x^a + y^b + z^c)^n
\]

\[
= \sum_{n=0}^{q-1} \sum_{k=0}^{n} \binom{n-k}{k} \lambda^{q-1-n} x^{ak+q-1-n} y^{bl+q-1-n} c(n-k-l)+q-1-n.
\]

We have to solve the equations

\[
ak = n, bl = n \text{ and } c(n - k - l) = n
\]

but since \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1\), the third equation is never satisfied except for \(n = k = l = 0\).

Therefore, the coefficient of \((xyz)^{q-1}\) in \(f^q_\lambda\) is \(\varphi(\lambda) = \lambda^{q-1} \equiv 1\).

5. The case \(n > 2\)

Now, let us come back to the situation of Theorem 3.9. Remember that we consider \(R = K[x_0, \ldots, x_n]\) with maximal ideal \(\mathfrak{m} = (x_0, \ldots, x_n)\) and let \(f \in R\) be a quasi-homogeneous polynomial of degree \(w = \sum_{i=0}^{n} \alpha_i\) with an isolated singularity, where \(\alpha_i = \deg(x_i)\).

Similar to the homogeneous case (BS15) we want to relate the integer \(h\) that appears in Theorem 3.9 to the order of vanishing of the Hasse invariant on some deformation space of \(X = \text{Proj}(R/fR)\). For this, let us first fix some more notation.

We consider the family \(\pi : \mathcal{X} \to \text{Hyp}_w\) of hypersurfaces of degree \(w\) in the weighted projective space \(\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)\). Our chosen hypersurface \(X = \text{Proj}(R/fR)\) gives a point \([X]\) in \(\text{Hyp}_w\). Set

\[
G := \sum_{i=1}^{m} \tilde{s}_i g_i \in K[x_0, \ldots, x_n, \tilde{s}_1, \ldots, \tilde{s}_m],
\]

where \(\{g_1, \ldots, g_m\} \subset K[x_0, \ldots, x_n]\) is the set of monomials of degree \(w\) and we set \(\deg(\tilde{s}_i) := 0\) for \(1 \leq i \leq m\) (such that \(G\) is quasi-homogeneous of degree \(w\)). The family \(\pi\) of hypersurfaces of degree \(w\) in \(\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)\) is given by

\[
\mathcal{X} = \text{Proj}_{\mathbb{P}^m-1}(\mathcal{O}_{\mathbb{P}^m-1}[x_0, \ldots, x_n] / G) \xrightarrow{i} \text{Proj}_{\mathbb{P}^m-1}(\mathcal{O}_{\mathbb{P}^m-1}[x_0, \ldots, x_n]) \xrightarrow{\pi} \mathbb{P}^m-1
\]

where \(i\) is a closed immersion. If \(f = \sum_{i=1}^{m} f_i g_i, f_i \in K\), is the defining equation of \(X\) in the weighted projective space \(\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)\), then \(X\) is the fiber over \((f_1, \ldots, f_m)\), i.e. \([X] \in \text{Hyp}_w\) is equal to \(V(\tilde{s}_i - f_i | i \in \{1, \ldots, m\})\). The aim of this section is to prove the following theorem:
**Theorem 5.1.** If $p \geq w(n-2)+1$, then the integer $h$ in Theorem 3.9 is the order of vanishing of the Hasse invariant at $[X] \in \text{Hyp}_w$ on the deformation space $\mathfrak{X}$ of $X \subset \mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$ described above.

First, let us recall the definition of the Hasse invariant of a family of varieties in characteristic $p$ (see for example [BS15]). Fix a proper flat morphism $\pi : \mathfrak{X} \to S$ of relative dimension $N$ between noetherian $\mathbb{F}_p$-schemes and assume that the formation of $R^N\pi_*\mathcal{O}_X$ is compatible with base change, i.e. if we have the following pullback diagram

$$
\begin{array}{ccc}
\mathfrak{X} \times S & \xrightarrow{i(1)} & \mathfrak{X} \\
\downarrow{\pi(1)} & & \downarrow{\pi} \\
T & \xrightarrow{i} & S,
\end{array}
$$

then $i^*R^N\pi_*\mathcal{O}_X \cong R^N\pi_*{(i(1))^*}\mathcal{O}_X$. Furthermore, assume that $R^N\pi_*\mathcal{O}_X$ is a line bundle.

Consider the Frobenius twist $\mathfrak{X}^{(1)} = \mathfrak{X} \times_{\text{Frob}_S} S$ of $\mathfrak{X}$, which gives the following diagram

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\text{Frob}_X} & \mathfrak{X} \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathfrak{X}^{(1)} & \xrightarrow{\pi^{(1)}} & \mathfrak{X} \\
\downarrow{\pi} & & \downarrow{\pi} \\
S & \xrightarrow{\text{Frob}_S} & S.
\end{array}
$$

By the base change assumption we have

$$R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}} \cong \text{Frob}_S^* R^N\pi_*\mathcal{O}_X \cong (R^N\pi_*\mathcal{O}_X)^p.$$

The relative Frobenius $\text{Frob}_X/S$ induces a map $\mathcal{O}_{\mathfrak{X}^{(1)}} \to (\text{Frob}_X/S)_*\mathcal{O}_X$ and this induces a map

$$H : R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}} \to R^N\pi_*^{(1)}(\text{Frob}_X/S)_*\mathcal{O}_X \in \text{Hom}\left(R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}}, R^N\pi_*\mathcal{O}_X\right),$$

which is called the Hasse invariant of the family $\pi$ (here we used that $\pi = \pi^{(1)} \circ \text{Frob}_X/S$). Since

$$\text{Hom}\left(R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}}, R^N\pi_*\mathcal{O}_X\right) \cong \text{Hom}\left((R^N\pi_*\mathcal{O}_X)^p, R^N\pi_*\mathcal{O}_X\right)$$

$$\cong \text{Hom}\left(\mathcal{O}_S, (R^N\pi_*\mathcal{O}_X)^{1-p}\right)$$

$$\cong H^0\left(S, (R^N\pi_*\mathcal{O}_X)^{1-p}\right),$$

the Hasse invariant $H$ is a section of a line bundle.

Next, we want to give the definition of the order of vanishing of the Hasse invariant. For this, fix $s \in S$ and an integer $t \geq 0$. Let $t[s]$ be the order $t$ neighbourhood of $s$, i.e. it is defined by the $t$-th power of the ideal defining $s$, and let $t\mathfrak{X}_s \subset \mathfrak{X}$ respectively $t\mathfrak{X}_s^{(1)} \subset \mathfrak{X}^{(1)}$ be the corresponding neighbourhoods of the fibers of $\pi$ respectively of
follows from [Mum66, p. 51] or EGA III ([Gro63, 7.7]), since
[H]
Thus, the order of vanishing of the Hasse invariant at
(see for example [Ogu01, p. 35]). The compatibility of
[R]
shows that
[π]
relative dimension
[P]
since
[t]
some
Lemma 5.3.
and similarly
Lemma 5.2. The order of vanishing of the Hasse invariant \( H \) at the point \( s \in S \) is
\( \text{ord}_s(H) = \max \{ t|φ_t = 0 \} \).
Later, we will need the following reformulation of the Hasse invariant (for a proof see [BS15]):
Lemma 5.3. If \( ψ_t : H^N(X_s, O_{X_s}) \to H^N(tX_s, O_{tX_s}) \) induced by \( \text{Frob}_X \) is nonzero for some \( t \leq p \), then \( \text{ord}_s(H) + 1 = \min \{ t|ψ_t \neq 0 \} \).
Now, let us come back to our family \( π \) of hypersurfaces of degree \( w \) in \( \mathbb{P}^n(α_0, \ldots, α_n) \). Using diagram (1), one can check that \( π \) is a proper morphism of relative dimension \( n - 1 \) between noetherian \( \mathbb{F}_p \)-schemes and Lemma 9.3.4 of [FGI+05] shows that \( π \) is also flat. Furthermore, one can prove that \( R^{n-1}π_∗O_X \) is a line bundle (see for example [Ogu01] p. 35). The compatibility of \( R^{n-1}π_∗O_X \) with base change follows from [Mum66, p. 51] or EGA III ([Gro63, 7.7]), since \( H^n(X_s, O_{X_s}) = 0 \).
Thus, the order of vanishing of the Hasse invariant at \( [X] \in \text{Hyp}_p \) on the space of hypersurfaces \( X \) is defined.
In order to start with the proof of Theorem 5.1 let us first remark that it suffices to consider the affine situation, i.e. we work on the left side of the following diagram
\[
\begin{align*}
\text{Proj}_{\mathbb{A}^m \setminus 0}(O_{\mathbb{A}^m \setminus 0}[x_0, \ldots, x_n]/G) & \longrightarrow \text{Proj}_{\mathbb{P}^m-1}(O_{\mathbb{P}^m-1}[x_0, \ldots, x_n]/G) \\
(\mathbb{A}^m \setminus 0) \times \mathbb{P}^n(α_0, \ldots, α_n) & \longrightarrow \text{Proj}_{\mathbb{P}^m-1}(O_{\mathbb{P}^m-1}[x_0, \ldots, x_n]) \\
\mathbb{A}^m \setminus 0 & \longrightarrow \mathbb{P}^{m-1}.
\end{align*}
\]
Remember that \( f = \sum_{i=1}^{m} f_i g_i \) is the defining equation of \( X \) and \( G = \sum_{i=1}^{m} s_i g_i \). Changing coordinates via \( s_i = \hat{s}_i - f_i \), one obtains \( X' := \pi^{-1}(\{X\}) = \text{Proj} \left( \hat{R}/(F, s)\hat{R} \right) \), where \( \hat{R} = K[x_0, \ldots, x_n, s_1, \ldots, s_m] \), \( F = f + \sum_{i=0}^{m} s_i g_i \) and \( s = (s_1, \ldots, s_m) \). Furthermore, let \( tX \) respectively \( tX' \) be the order \( t \) neighbourhoods of \( X \) in \( \mathbb{P}^n(\alpha_0, \ldots, \alpha_n) \) respectively \( X' \) in \( \mathcal{X} \), i.e. \( \begin{align*}
tX &= \text{Proj} \left( R/f^t R \right) \quad \text{and} \quad tX' = \text{Proj}_{K[s]/s^t} \left( \hat{R}/(F, s^t)\hat{R} \right).
\end{align*} \)

Proof of Theorem \[\text{[5.7]}\] For \( 1 \leq t \leq p \) we consider the following commutative diagram

\[
\begin{array}{cccc}
\hat{R}/(F, s) & \xrightarrow{\text{Frob}_R} & \hat{R}/(F^p, s^{[p]}) & \xrightarrow{a} \hat{R}/(F, s^p) & \xrightarrow{\bar{pr}_1} \hat{R}/(F, s^t) & \xrightarrow{\bar{pr}_2} \hat{R}/(F, s) \\
R/f & \xrightarrow{\text{Frob}_R} & R/f^{[p]} & \xrightarrow{h_1} & R/f^p & \xrightarrow{pr_1} R/f^t & \xrightarrow{pr_2} R/f,
\end{array}
\]

where the maps \( \bar{pr}_1, \bar{pr}_2, pr_1 \) and \( pr_2 \) are the evident projections and the map \( a \) is given by the inclusion \( (F^p, s^{[p]}) \subset (F, s^p) \), since \( (s_1^{[p]}, \ldots, s_m^{[p]}) \subset (s_1, \ldots, s_m)^p \). The maps \( h_1, h_2 \) and \( h_3 \) are defined as follows: Obviously, we have a map \( \varphi_1 : R \hookrightarrow \hat{R} \twoheadrightarrow \hat{R}/F^p \to \hat{R}/(F^p, s^{[p]}) \), which induces the map \( h_1 \) if and only if \( f^p \in \ker(\varphi_1) = (F^p, s^{[p]}) \), which follows from

\[
f^p = \left( F - \sum_{i=1}^{m} s_i g_i \right)^p = F^p + \left( -\sum_{i=1}^{m} s_i g_i \right)^p \in (F^p, s^{[p]}).
\]

Similarly, we have a map \( \varphi_2 : R \hookrightarrow \hat{R} \twoheadrightarrow \hat{R}/F \to \hat{R}/(F, s^t) \), which induces the map \( h_3 \) if and only if \( f^t \in \ker(\varphi_2) = (F, s^t) \). But this is true, since

\[
f^t = \left( F - \sum_{i=1}^{m} s_i g_i \right)^t = h \cdot F + g \cdot \left( \sum_{i=1}^{m} s_i g_i \right)^t \in (F, s^t)
\]

for some \( h, g \in \hat{R} \). The same argument for \( t = p \) gives \( h_2 \).

Passing to cohomology and taking the degree zero parts yields the following commutative diagram

\[
\begin{array}{ccc}
H^n_m(\hat{R}/(F, s)) & \xrightarrow{b_0} & H^n_m(\hat{R}/(F, s^t)) \\
H^n_m(R/f) & \xrightarrow{a_0} & H^n_m(R/f^t),
\end{array}
\]

(5)

In order to prove Theorem \[\text{[5.1]}\] we now show the following four equalities

\[
\text{ord}_s(H) + 1 = \min \left\{ t \mid H^{n-1}(X', \mathcal{O}_{X'}) \xrightarrow{b_1} H^{n-1}(tX', \mathcal{O}_{tX'}) \text{ injective} \right\}
\]

\[
= \min \left\{ t \mid H^{n-1}(X, \mathcal{O}_X) \xrightarrow{a_1} H^{n-1}(tX, \mathcal{O}_{tX}) \text{ injective} \right\}
\]

\[
= \min \left\{ t \mid H^n_m(R/f^t R) \xrightarrow{F_1} H^n_m(R/f R) \text{ injective} \right\}
\]

\[
= h + 1.
\]

(6)
and then clearly we get $h = \text{ord}_s(H)$ (here we used [ILL+07] Theorem 13.21] to replace local cohomology by sheaf cohomology).

The first equality follows by Lemma 5.3. For the proof of the third equation we need the following lemma about the injectivity of Frobenius on the negatively graded part of local cohomology modules:

**Lemma 5.4.** Let $f \in R$ be a quasi-homogeneous polynomial of degree $d$ and type $\alpha$ with an isolated singularity. Let $t \leq p$. Then for $p \geq nd - w - td + 1$ the Frobenius action

$$
\widetilde{F}^{-1}_t : [H^0_m(R/fR)]_{<0} \to [H^0_m(R/f^tR)]_{<0}
$$

is injective.

**Proof.** As in the proof of Lemma 3.2 for $e = 1$ we have the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & [H^0_m(R/fR)]_{\leq-1} & \longrightarrow & [H^0_m(R)]_{\leq-d-1} & \longrightarrow & \cdots \\
\downarrow F^{-1}_t & & \downarrow f^{p-t}F & & & \\
0 & \longrightarrow & [H^0_m(R/f^tR)]_{\leq-p} & \longrightarrow & [H^0_m(R)]_{\leq-d-t-p} & \longrightarrow & \cdots 
\end{array}
$$

and again it suffices to prove the injectivity of $f^{p-t}F$.

For this, let $\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right]$ be an element of $[H^0_m(R)]_{\leq-d-1}$, where $g$ is quasi-homogeneous of degree $\text{deg}(g) \leq w \frac{q}{p} - d - 1$ for some $q$. Suppose that $\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right] \in \text{Ker} \left(f^{p-t}F\right)$, which means that

$$
0 = f^{p-t}F \left(\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right]\right) = \left[\frac{f^{p-t}g}{(x_0 \cdots x_n)^q}\right].
$$

Therefore, $f^{p-t}g^{p} \in \mathfrak{m}[q]$. Let

$$
k := \min \left\{ l | f^l g^{p} \in \mathfrak{m}[q] \right\},
$$

then $0 \leq k \leq p - t$. We want to show that $k = 0$, so suppose $k \neq 0$. Arguing as in the proof of Lemma 3.3, we get $f^{k-1}g^{p}J(f) \subset \mathfrak{m}[q]$. By Lemma 3.3 and Lemma 3.4, it follows that

$$
f^{k-1}g^{p} \in \left(\mathfrak{m}[q] :_R J(f)\right) \setminus \mathfrak{m}[q] \subset \left(\mathfrak{m}[q] :_R R_{\geq(n+1)d-2w+1}\right) \setminus \mathfrak{m}[q] \subset R_{(q+1)w - (n+1)d},
$$

which implies that $(k - 1)d + p \text{deg}(g) = \text{deg}(f^{k-1}g^{p}) \geq (q + 1)w - (n + 1)d$. Since $k \leq p - t$ and $\text{deg}(g) \leq w \frac{q}{p} - d - 1$, a short computation yields $p \leq nd - w - td$, which is a contradiction. Therefore, $k = 0$, which means that $g^{p} \in \mathfrak{m}[q]$. Thus, $\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right] = 0$ and therefore, $f^{p-t}F$ is injective.

Using this, we can now prove the third equality of (3). Since $p \geq w(n - 2) + 1 \geq w(n - 1) + 1$ for $1 \leq t \leq p$ and using Lemma 5.4, we know that $\widetilde{F}^{-1}_t$ is injective in negative degrees. Then, since $a_t = \left(\widetilde{F}^{-1}_t\right)^0$, the asserted equality follows.

Next, we prove the fourth equation. By the first part of Theorem 3.9 we know that $\mu_f(p) = p - h$ and this gives the two inequalities $\mu_f(p) > p - h - 1 = p - (h + 1)$ and
Now let us consider the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & f_1^{-1} / f_1 & \rightarrow & R / f_1 & \rightarrow & R / f_1^{t-1} & \rightarrow & 0 \\
& & \downarrow h_4 & & \downarrow h_1 & & \downarrow h_1 & & \\
0 & \rightarrow & (F, s^{-1}) / (F, s^t) & \rightarrow & \tilde{R} / (F, s^t) & \rightarrow & \tilde{R} / (F, s^{-1}) & \rightarrow & 0.
\end{array}
\]

Since \( f_1^{-1} \in (F, s^{-1}) \), the map \( \varphi_3 : f_1^{-1} \rightarrow (F, s^{-1}) \rightarrow (F, s^{-1}) / (F, s^t) \) induces \( h_4 \). Passing to local cohomology we obtain the following diagram

\[
\begin{array}{ccccccccc}
H_{m-1}^{-1}(R / f_1^{t-1})_0 & \rightarrow & H_{m}^{-1}(f_1^{-1} / f_1)_0 & \rightarrow & H_{m}^{-1}(R / f_1)_0 & \rightarrow & H_{m}^{-1}(R / f_1^{t-1})_0 & \rightarrow & 0 \\
& \downarrow \varphi_t & & \downarrow \varphi_t & & \downarrow \varphi_t & & \\
H_{m-1}^{-1}(\tilde{R} / (F, s^{-1}))_0 & \rightarrow & H_{m}^{-1}((F, s^{-1}) / (F, s^t))_0 & \rightarrow & H_{m}^{-1}(\tilde{R} / (F, s^t))_0 & \rightarrow & H_{m}^{-1}(\tilde{R} / (F, s^{-1}))_0.
\end{array}
\]

The rest of the proof mainly consists of the following three steps: We first show that \( b \) is injective and then conclude that it suffices to prove the injectivity of \( \phi_t \) for all \( t \) in order to show that \( c_t \) is injective for all \( t \). As a last step, we prove the injectivity of \( \phi_t \) for all \( t \).

**Step 1:** The map \( b \) is injective: To prove this, we show that \( H_{m-1}^{-1}(\tilde{R} / (F, s^{-1})) = 0 \). For \( t = 1 \) this is clear. For \( t = 2 \) we get

\[
H_{m-1}^{-1}(\tilde{R} / (F, s^{-1})) = H_{m-1}^{-1}(\tilde{R} / (F, s)) = H_{m-1}^{-1}(R / f) = 0.
\]

Now let \( t \geq 3 \) and consider the exact sequence

\[
0 \rightarrow (F, s^{-1}) / (F, s^t) \rightarrow \tilde{R} / (F, s^t) \rightarrow \tilde{R} / (F, s^{-1}) \rightarrow 0.
\]

This gives the long exact sequence

\[
H_{m-1}^{-1}((F, s^{-1}) / (F, s^t)) \rightarrow H_{m-1}^{-1}(\tilde{R} / (F, s^t)) \rightarrow H_{m-1}^{-1}(\tilde{R} / (F, s^{-1})).
\]

By induction, we know that \( H_{m-1}^{-1}(\tilde{R} / (F, s^{-1})) = 0 \) and in the following we will show that \( H_{m-1}^{-1}((F, s^{-1}) / (F, s^t)) = 0 \). Using this and the exact sequence above, it follows that \( H_{m-1}^{-1}(\tilde{R} / (F, s^t)) = 0 \). Therefore, it remains to show that
$H_{m}^{n-1} \left( \left( F, s^{t-1} \right) / (F, s^t) \right) = 0$. For this, we compute

$$(F, s^{t-1}) / (F, s^t) = (F + s^t + s^{t-1}) / (F + s^t)$$

$$\cong (s^{t-1}) / ((s^{t-1} \cap (F + s^t)))$$

$$\cong (s^{t-1} / s^t) / F(s^{t-1} / s^t)$$

$$\cong (s^{t-1} / s^t) / f (s^{t-1} / s^t)$$

$$\cong R/f \otimes_K (s^{t-1} / s^t).$$

Hence, using \[\text{ILL}^{07}, \text{Proposition 7.15}\], we get

$$H_{m}^{n-1} \left( \left( F, s^{t-1} \right) / (F, s^t) \right) \cong H_{m}^{n-1}(R/f \otimes_K (s^{t-1} / s^t))$$

$$\cong H_{m}^{n-1}(R/f) \otimes_K (s^{t-1} / s^t) = 0.$$

**Step 2:** The second step of the proof is to show that it suffices to prove the injectivity of $\phi_t$ for all $t$, in order to prove the injectivity of $c_t$ for all $t$. Again, we do this by induction on $t$. It is easy to see that $c_1$ is injective if and only if $\phi_1$ is injective. Now, suppose $c_1, \ldots, c_{t-1}$ are injective, then by a diagram chase and using that $b$ and $\phi_t$ are injective, one can prove that $c_t$ is also injective.

**Step 3:** Now, as a last step, we prove the injectivity of

$$\phi_t : H_{m}^{n} (f^{t-1} / f^t)_{0} \to H_{m}^{n} \left( \left( F, s^{t-1} \right) / (F, s^t) \right)_{0}.$$

For this, consider the projective resolution

$$0 \longrightarrow f^t \longrightarrow f^{t-1} \longrightarrow f^{t-1} / f^t \longrightarrow 0$$

of $f^{t-1} / f^t$ (remark that $(f^t)$ can be identified with $R(-tw)$). Tensoring the sequence

$$0 \longrightarrow R(-w) \xrightarrow{f} R \longrightarrow R/f \longrightarrow 0$$

with $s^{t-1} / s^t$ over $K$ yields the projective resolution

$$0 \longrightarrow R(-w) \otimes_K (s^{t-1} / s^t) \xrightarrow{f} R \otimes_K (s^{t-1} / s^t) \longrightarrow R/f \otimes_K (s^{t-1} / s^t) \longrightarrow 0$$

of $R/f \otimes_K (s^{t-1} / s^t) \cong (F, s^{t-1}) / (F, s^t)$ (see \[\text{II}\]). Altogether, we have the following situation

$$0 \longrightarrow (f^{t-1}) \cdot f \longrightarrow f^{t-1} \longrightarrow f^{t-1} / f^t \longrightarrow 0$$

$$0 \longrightarrow R(-w) \otimes_K (s^{t-1} / s^t) \xrightarrow{f} R \otimes_K (s^{t-1} / s^t) \longrightarrow R/f \otimes_K (s^{t-1} / s^t) \longrightarrow 0,$$

where the map $\theta$ is induced by $h_4$ and sends $f^{t-1}$ to $\left( \sum_{j=1}^{m} s_j g_j \right)^{t-1}$ considered as an element of $R/f \otimes_K (s^{t-1} / s^t) \cong (F, s^{t-1}) / (F, s^t)$, i.e. if we define $\{f_j\}$ to be a basis of the polynomials of degree $t-1$ in the variables $y_1, \ldots, y_m$ then

$$\theta (f^{t-1}) = \sum_j c_j f_j (g_1, \ldots, g_m) \otimes f_j (s_1, \ldots, s_m)$$
for some coefficients \( c_j \in K \). If we apply the functor \( \text{Hom}_R (-, R(-w))_0 \) to the above diagram, we get a map
\[
\psi_t : \text{Hom}_R (R(-w) \otimes_K (s^{t-1}/s^t), R(-w))_0 \to \text{Hom} (f^t, R(-w))_0,
\]
which sends a map \( \varphi \) to \( \varphi \circ \theta \) and induces a map
\[
\psi_t : \text{Ext}^1_R \left( (F, s^{t-1}) / (F, s^t), R(-w) \right)_0 \to \text{Ext}^1_R \left( f^{t-1}/f^t, R(-w) \right)_0.
\]

Here,
\[
\text{Ext}^1_R \left( f^{t-1}/f^t, R(-w) \right) = \text{Hom}_R \left( f^t, R(-w) \right) / f \cdot \text{Hom}_R \left( f^{t-1}, R(-w) \right)
\]
and
\[
\text{Ext}^1_R \left( (F, s^{t-1}) / (F, s^t), R(-w) \right) = \text{Hom}_R \left( R(-w) \otimes_K (s^{t-1}/s^t), R(-w) \right) / f \cdot \text{Hom}_R \left( R \otimes_K (s^{t-1}/s^t), R(-w) \right).
\]

By the functoriality of the local duality (see [BH93, Theorem 3.6.19]) the map \( \psi_t \) is equal to the map \( \phi_t^\vee \), since \( \phi_t \) is induced by the map \( \theta \) by passing to local cohomology and then taking the degree zero parts.

Now, the idea is to prove the surjectivity of \( \phi_t^\vee \) instead of proving the injectivity of \( \phi_t \) by using the above description as Ext-modules. Therefore, we want to examine \( \psi_t \) more closely:
\[
\text{Hom}_R \left( R(-w) \otimes_K (s^{t-1}/s^t), R(-w) \right)_0 \xrightarrow{\simeq} \left( R(-w) \otimes_K (s^{t-1}/s^t) \right)_0
\]
\[
\text{Hom} (f^t, R(-w))_0 \xrightarrow{\simeq} \text{Hom} (R(-tw), R(-w))_0 \xrightarrow{\simeq} R((t-1)w)_0 = R_{(t-1)w},
\]
where the first vertical isomorphism is given by the identification of \( f^t \) with \( R(-tw) \) and the second vertical isomorphism is given by \( \varphi \mapsto \varphi(1) \).

Using the above, we get
\[
\theta (rf^t) = \theta (r \cdot f \cdot f^{t-1}) = rf \theta (f^{t-1}) = rf \sum_j c_j f_j (g_1, \ldots, g_m) \otimes f_j (s_1, \ldots, s_m)
\]
for some coefficients \( c_j \in K \). For an element \( 1 \otimes \delta_i \in \left( R(-w) \otimes_K \left( s^{t-1}/s^t \right) \right)_0 \), where \( \{ \delta_i \} \) is a dual basis of \( s^{t-1}/s^t \), we have
\[
\psi_t (1 \otimes \delta_i) = \left[ rf^t \mapsto (1 \otimes \delta_i) \left( \theta (rf^t) \right) \right]
\]
\[
= \left[ rf^t \mapsto (1 \otimes \delta_i) \left( rf \sum_j c_j f_j (g_1, \ldots, g_m) \otimes f_j (s_1, \ldots, s_m) \right) \right]
\]
\[
= \left[ rf^t \mapsto rf c_i f_i (g_1, \ldots, g_m) \right].
\]
As an element of $\text{Hom}(R(-tw), R(-w))_0$ this map is given by $r \mapsto rc_i f_i (g_1, \ldots, g_m)$ and via the last vertical isomorphism this map is sent to $c_i f_i (g_1, \ldots, g_m) \in R(t-1)w$. With these observations the surjectivity of $\psi_t$ becomes clear, since the set $\{f_i\}$ forms a basis of the space of polynomials of degree $t - 1$.

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