SIMULTANEOUS UNITARY EQUIVALENCE TO CARLEMAN OPERATORS WITH ARBITRARILY SMOOTH KERNELS

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ABSTRACT. In this paper, we describe families of those bounded linear operators on a separable Hilbert space that are simultaneously unitarily equivalent to integral operators on $L^2(\mathbb{R})$ with bounded and arbitrarily smooth Carleman kernels. The main result is a qualitative sharpening of an earlier result of \cite{7}.

1. INTRODUCTION. MAIN RESULT

Throughout, $\mathcal{H}$ will denote a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the norm $\| \cdot \|_{\mathcal{H}}$, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathbb{C}$, and $\mathbb{N}$, and $\mathbb{Z}$, the complex plane, the set of all positive integers, the set of all integers, respectively. For an operator $A$ in $\mathcal{B}(\mathcal{H})$, $A^*$ will denote the Hilbert space adjoint of $A$ in $\mathcal{B}(\mathcal{H})$.

Throughout, $C(X, B)$, where $B$ is a Banach space (with norm $\| \cdot \|_B$), denote the Banach space (with the norm $\| f \|_{C(X, B)} = \sup_{x \in X} \| f(x) \|_B$) of continuous $B$-valued functions defined on a locally compact space $X$ and vanishing at infinity (that is, given any $f \in C(X, B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\| f(x) \|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$).

Let $\mathbb{R}$ be the real line $(-\infty, +\infty)$ with the Lebesgue measure, and let $L^2 = L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(s) \overline{g(s)} \, ds$$

and the norm $\| f \| = \langle f, f \rangle^{\frac{1}{2}}$.

A linear operator $T : L^2 \to L^2$ is said to be integral if there exists a measurable function $T$ on the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, a kernel, such that, for every $f \in L^2$,

$$(Tf)(s) = \int_{\mathbb{R}} T(s, t) f(t) \, dt$$

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for almost every $s$ in $\mathbb{R}$. A kernel $T$ on $\mathbb{R}^2$ is said to be Carleman if $T(s, \cdot) \in L_2$ for almost every fixed $s$ in $\mathbb{R}$. An integral operator with a kernel $T$ is called Carleman if $T$ is a Carleman kernel. Every Carleman kernel, $T$, induces a Carleman function $t$ from $\mathbb{R}$ to $L_2$ by $t(s) = \overline{T(s, \cdot)}$ for all $s$ in $\mathbb{R}$ for which $T(s, \cdot) \in L_2$.

The integral representability problem for linear operators stems from the work [10] of von Neumann, and is now well enough understood. The problem involves the question: which operators are unitarily equivalent to an integral operator? Now we recall a characterization of Carleman representable operators to within unitary equivalence [5, p. 99], [3, Section 15]:

**Proposition 1.** A necessary and sufficient condition that an operator $S \in \mathcal{A}(\mathcal{H})$ be unitarily equivalent to an integral operator with Carleman kernel is that there exist an orthonormal sequence $\{e_n\}$ such that

$$\|S^*e_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty$$

(or, equivalently, that 0 belong to the right essential spectrum of $S$).

Given any non-negative integer $m$, we impose on a Carleman kernel $K$ the following smoothness conditions:

(i) the function $K$ and all its partial derivatives on $\mathbb{R}^2$ up to order $m$ are in $C(\mathbb{R}^2, \mathbb{C})$,

(ii) the Carleman function $k$, $k(s) = \overline{K(s, \cdot)}$, and all its (strong) derivatives on $\mathbb{R}$ up to order $m$ are in $C(\mathbb{R}, L_2)$.

**Definition 1.** A function $K$ that satisfies Conditions (i), (ii) is called a $SK^m$-kernel [7].

Now we are in a position to formulate our result on simultaneous integral representability of operator families by $SK^m$-kernels.

**Proposition 2** ([7]). If for a countable family $\{B_r \mid r \in \mathbb{N}\} \subset \mathcal{A}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\}$ such that

$$\sup_{r \in \mathbb{N}} \|B_r^*e_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty,$$

then for each fixed non-negative integer $m$ there exists a unitary operator $U_m : \mathcal{H} \to L_2$ such that all the operators $U_mB_rU_m^{-1}$ ($r \in \mathbb{N}$) are bounded Carleman operators having $SK^m$-kernels.

In [7], there is a counterexample which shows that Proposition 2 may fail to be true if the family $\{B_r\}$ is not countable.

The purpose of this paper is to restrict the conclusion of Proposition 2 to arbitrarily smooth Carleman kernels. Now define these kernels.

**Definition 2.** We say that a function $K$ is a $SK^\infty$-kernel ([8], [9]) if it is a $SK^m$-kernel for each non-negative integer $m$. 
Theorem. If for a countable family \( \{ B_r \mid r \in \mathbb{N} \} \subset \mathcal{B}(\mathcal{H}) \) there exists an orthonormal sequence \( \{ v_n \} \) such that
\[
\sup_{r \in \mathbb{N}} \| B_r^* v_n \|_{\mathcal{H}} \to 0 \quad \text{as} \quad n \to \infty,
\]
then there exists a unitary operator \( U_\infty : \mathcal{H} \to L_2 \) such that all the operators \( U_\infty B_r U_\infty^{-1} \) \( (r \in \mathbb{N}) \) are Carleman operators having \( SK^\infty \)-kernels.

This theorem, which is our main result, will be proved in the next section of the present paper. The proof yields an explicit construction of the unitary operator \( U_\infty : \mathcal{H} \to L_2 \). The construction of \( U_\infty \) is independent of those spectral points of \( B_r \) \( (r \in \mathbb{N}) \) that are different from 0, and is defined by \( U_\infty f_n = u_n \) \( (n \in \mathbb{N}) \), where \( \{ f_n \}, \{ u_n \} \) are orthonormal bases in \( \mathcal{H} \) and \( L_2 \), respectively, whose elements can be explicitly described in terms of the operator family.

2. Proof of Theorem

The proof has two steps.

Step 1. Assume that
\[
\sup_{r \in \mathbb{N}} \| B_r \| \leq 1.
\]
This is a harmless assumption, involving no loss of generality; just replace \( B_r \) with \( \frac{B_r}{\| B_r \|} \). Find a subsequence \( \{ e_k \}_{k=1}^\infty \) of the sequence \( \{ v_n \} \) in (1) so that
\[
\sum_k \sup_{r \in \mathbb{N}} \| S_r e_k \|_{\mathcal{H}}^4 \leq \sum_k \sup_{r \in \mathbb{N}} \| r S_r e_k \|_{\mathcal{H}}^4 = M < \infty,
\]
where \( S_r = \frac{1}{r} B_r \) \( (r \in \mathbb{N}) \) (the sum notation \( \sum \) will always be used instead of the more detailed symbol \( \sum_{k=1}^\infty \)). For each \( r \), let
\[
Q_r = (1 - E) S_r, \quad J_r = S_r^* E,
\]
where \( E \) is the orthogonal projection onto the closed linear span \( H \) of the \( e_k \)'s, and observe that
\[
S_r = Q_r + J_r^*.
\]
Assume, with no loss of generality, that \( \dim (1 - E) H = \infty \), and let \( \{ e_k^\perp \}_{k=1}^\infty \) be any orthonormal basis for \( (1 - E) H \). Let \( \{ f_n \}_{n=1}^\infty \) denote any basis in \( \mathcal{H} \) consisting of the elements of the set \( \{ e_k \} \cup \{ e_k^\perp \} \). It follows from (2) that
\[
\sum_n \| J_r f_n \|_{\mathcal{H}} \leq \sum_k \| J_r e_k \|_{\mathcal{H}} \leq \sum_k \sup_{r \in \mathbb{N}} \| S_r e_k \|_{\mathcal{H}} \leq M^4,
\]
and hence that \( J_r \) and \( J_r^* \) are Hilbert–Schmidt operators, for each \( r \).
For each \( h \in \mathcal{H} \), let

(5) \[ d(h) = \sup_{r \in \mathbb{N}} \| J_r h \|_{\mathcal{H}}^{\frac{1}{2}} + \sup_{r \in \mathbb{N}} \| J_r^* h \|_{\mathcal{H}}^{\frac{1}{2}} + \sup_{r \in \mathbb{N}} \| \Gamma_r h \|_{\mathcal{H}}, \]

where, for each \( r \),

(6) \[ \Gamma_r = \Lambda S_r, \quad \text{and} \quad \Lambda = \sum_{k} \frac{1}{k} \langle \cdot, e_k^\perp \rangle_{\mathcal{H}} e_k^\perp. \]

It is clear that \( \Lambda \) and \( \Gamma_r \) (\( r \in \mathbb{N} \)) are Hilbert-Schmidt operators on \( \mathcal{H} \). Prove that

(7) \[ d(e_k) \to 0 \quad \text{as} \quad k \to \infty. \]

Using known facts about Hilbert–Schmidt operators (see [2, Chapter III]), write the following relations

(8) \[
\sum_{r \in \mathbb{N}} \sup_{k} \| J_r^* e_k \|_{\mathcal{H}}^2 \leq \sum_{r \in \mathbb{N}} \sum_{k} \| J_r^* e_k \|_{\mathcal{H}}^2 \leq \sum_{r \in \mathbb{N}} \| J_r^* \|_{\mathcal{H}}^2 = \sum_{r \in \mathbb{N}} \| J_r \|_{\mathcal{H}}^2
\]

\[
= \sum_{r} \sum_{k} \| J_r e_k \|_{\mathcal{H}}^2 = \sum_{r} \sum_{k} \| S_r^* e_k \|_{\mathcal{H}}^2
\]

\[
\leq \sum_{r} \frac{1}{r^2} \sum_{k} \sup_{r \in \mathbb{N}} \| r S_r^* e_k \|_{\mathcal{H}}^2 \leq \frac{M^8 \pi^2}{6},
\]

where \( \| \cdot \|_2 \) is the Hilbert–Schmidt norm. Observe also that

(9) \[
\sum_{k} \sup_{r \in \mathbb{N}} \| \Gamma_r e_k \|_{\mathcal{H}}^2 \leq \sum_{r} \sum_{k} \| \Gamma_r e_k \|_{\mathcal{H}}^2
\]

\[
\leq \sum_{r} \| \Gamma_r \|_{\mathcal{H}}^2 = \sum_{r} \| r \|_{\mathcal{H}}^2 = \sum_{r} \| S_r^* \Lambda f_n \|_{\mathcal{H}}^2
\]

\[
\leq \sum_{r} \frac{1}{r^2} \sum_{k} \| \Lambda e_k^\perp \|_{\mathcal{H}}^2 = \sum_{r} \frac{1}{r^2} \sum_{k} \frac{1}{k^2} = \frac{\pi^4}{36}.
\]

Then (7) follows immediately from (8), (9), (2), and (3).

**Notation.** If an equivalence class \( f \in L_2 \) contains a function belonging to \( C(\mathbb{R}, \mathbb{C}) \), then we shall use \([f]\) to denote that function.

Take any orthonormal basis \( \{u_n\} \) for \( L_2 \) which satisfies conditions:

(a) the terms of the derivative sequence \( \{[u_n]^{(i)}\} \) are in \( C(\mathbb{R}, \mathbb{C}) \), for each \( i \) (here and throughout, the letter \( i \) is reserved for all non-negative integers),

(b) \( \{u_n\} = \{g_k\}_{k=1}^\infty \cup \{h_k\}_{k=1}^\infty \), where \( \{g_k\}_{k=1}^\infty \cap \{h_k\}_{k=1}^\infty = \emptyset \), and, for each \( i \),

(10) \[ \sum_{k} H_{k,i} < \infty \quad \text{with} \quad H_{k,i} = \| [h_k]^{(i)} \|_{C(\mathbb{R}, \mathbb{C})} \quad (k \in \mathbb{N}), \]
there exist a subsequence \( \{x_k\}_{k=1}^\infty \subset \{e_k\} \) and a strictly increasing sequence \( \{n(k)\}_{k=1}^\infty \) of positive integers such that, for each \( i \),

\[
\sum_k d(x_k) (G_{k,i} + 1) < \infty \quad \text{with} \quad G_{k,i} = \|g_k^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} \quad (k \in \mathbb{N}),
\]

\[
\sum_k kH_{n(k),i} < \infty.
\]

**Remark.** Let \( \{u_n\} \) be an orthonormal basis for \( L_2 \) such that, for each \( i \),

\[
[u_n]^{(i)} \in C(\mathbb{R}, \mathbb{C}) \quad (n \in \mathbb{N}),
\]

\[
\|u_n\|^{(i)}_{C(\mathbb{R}, \mathbb{C})} \leq D_n A_i \quad (n \in \mathbb{N}),
\]

\[
\sum_k D_{n_k} < \infty,
\]

where \( \{D_n\}_{n=1}^\infty \) and \( \{A_i\}_{i=0}^\infty \) are sequences of positive numbers, and \( \{n_k\}_{k=1}^\infty \) is a subsequence of \( \mathbb{N} \) such that \( \mathbb{N} \setminus \{n_k\}_{k=1}^\infty \) is a countable set. By (7), the basis \( \{u_n\} \) satisfies Conditions (a)-(c) with \( h_k = u_{n_k} \ (k \in \mathbb{N}) \) and \( \{g_k\}_{k=1}^\infty = \{u_n\} \setminus \{h_k\}_{k=1}^\infty \).

To show the existence of a basis \( \{u_n\} \) satisfying (13)-(15), consider a Lemarié-Meyer wavelet,

\[
u(s) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\xi(\frac{1}{2}+s)} \text{sign} \xi b(|\xi|) \, d\xi \quad (s \in \mathbb{R}),
\]

with the bell function \( b \) belonging to \( C^\infty(\mathbb{R}) \) (for construction of the Lemarié-Meyer wavelets we refer to [6], [1], § 4], [4] Example D, p. 62). In this case, \( u \) belongs to the Schwartz class \( \mathcal{S}(\mathbb{R}) \), and hence all the derivatives \( [u]^{(i)} \) are in \( C(\mathbb{R}, \mathbb{C}) \). The “mother function” \( u \) generates an orthonormal basis for \( L_2 \) by

\[
u_{j,k}(s) = 2^{\frac{j}{2}} u(2^j s - k) \quad (j, k \in \mathbb{Z}).
\]

Rearrange, in a completely arbitrary manner, the orthonormal set \( \{u_{j,k}\}_{j,k \in \mathbb{Z}} \) into a simple sequence, so that it becomes \( \{u_n\}_{n \in \mathbb{N}} \). Since, in view of this rearrangement, to each \( n \in \mathbb{N} \) there corresponds a unique pair of integers \( j_n, k_n \), and conversely, we can write, for each \( i \),

\[
\|u_n\|^{(i)}_{C(\mathbb{R}, \mathbb{C})} = \|u_{j_n,k_n}\|^{(i)}_{C(\mathbb{R}, \mathbb{C})} \leq D_n A_i,
\]

where

\[
D_n = \begin{cases} 2^{j_n^2} & \text{if } j_n > 0, \\ \left( \frac{1}{\sqrt{2}} \right)^{|j_n|} & \text{if } j_n \leq 0, \end{cases} \quad A_i = 2^{(i+\frac{1}{2})^2} \|u\|^{(i)}_{C(\mathbb{R}, \mathbb{C})}.
\]

Whence it follows that if \( \{n_k\}_{k=1}^\infty \subset \mathbb{N} \) is a subsequence such that \( j_{n_k} \to -\infty \) as \( k \to \infty \), then

\[
\sum_k D_{n_k} < \infty.
\]

Thus, the basis \( \{u_n\} \) satisfies Conditions (13)-(15).
Let us return to the proof. Let \( \{ x_k^\perp \}_{k=1}^\infty = \{ e_k^\perp \}_{k=1}^\infty \cup \{ x_k \}_{k=1}^\infty \setminus \{ e_k \}_{k=1}^\infty \), and observe that \( \{ f_n \}_{n=1}^\infty = \{ x_k \}_{k=1}^\infty \cup \{ x_k^\perp \}_{k=1}^\infty \).

Now construct a candidate for the desired unitary operator in the theorem. Define a unitary operator \( U_\infty : \mathcal{H} \to L_2 \) on the basis vectors by setting
\[
U_\infty x_k^\perp = h_k, \quad U_\infty x_k = g_k \quad \text{for all } k \in \mathbb{N},
\]
in the harmless assumption that, for each \( k \in \mathbb{N} \),
\[
U_\infty f_k = u_k, \quad U_\infty e_k^\perp = h_{n(k)},
\]
where \( \{ n(k) \} \) is just that sequence which occurs in Condition (c).

**Step 2.** The verification that \( U_\infty \) in (16) has the desired properties is straightforward. Fix an arbitrary \( r \in \mathbb{N} \) and put \( T = U_\infty S_r U_\infty^{-1} \). Once this is done, the index \( r \) may be omitted for \( S_r, J_r, Q_r, \Gamma_r \).

Write the Schmidt decomposition
\[
J = \sum_n s_n \langle \cdot, p_n \rangle_\mathcal{H} q_n,
\]
where the \( s_n \) are the singular values of \( J \) (eigenvalues of \( (J^*J)^{1/2} \)), \( \{ p_n \} \), \( \{ q_n \} \) are orthonormal sets (the \( p_n \) are eigenvectors for \( J^*J \) and the \( q_n \) are eigenvectors for \( JJ^* \)).

Introduce an auxiliary operator \( A \) by
\[
A = \sum_n \frac{s_n}{2} \langle \cdot, p_n \rangle_\mathcal{H} q_n,
\]
and observe that, by the Schwarz inequality,
\[
\| Af \|_\mathcal{H} = \left\| (J^*J)^{1/2} f \right\|_\mathcal{H} \leq \| J f \|_\mathcal{H}^{1/2},
\]
\[
\| A^* f \|_\mathcal{H} = \left\| (JJ^*)^{1/2} f \right\|_\mathcal{H} \leq \| J^* f \|_\mathcal{H}^{1/2}
\]
if \( \| f \| = 1 \).

Since \( \{ e_k^\perp \}_{k=1}^\infty \) is an orthonormal basis for \( (1 - E)H \), (3) implies that
\[
Q = \sum_k \langle \cdot, S^* e_k^\perp \rangle_\mathcal{H} e_k^\perp.
\]
Whence, using (17), one can write
\[
P f = \sum_k \left\langle f, T^* h_{n(k)} \right\rangle h_{n(k)} \quad (f \in L_2)
\]
where \( P = U_\infty Q U_\infty^{-1} \). By (5),
\[
T^* h_{n(k)} = \sum_n \langle S^* e_k^\perp, f_n \rangle_\mathcal{H} u_n = k \sum_n \langle e_k^\perp, \Gamma f_n \rangle_\mathcal{H} u_n \quad (k \in \mathbb{N}).
\]
Prove that, for any fixed \( i \), the series
\[
\sum_n \langle e_k^\perp, \Gamma f_n \rangle_\mathcal{H} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})
\]
converge in the norm of $C(\mathbb{R}, \mathbb{C})$. Indeed, all these series are pointwise dominated on $\mathbb{R}$ by one series
\[ \sum_n \| f_n \|_{\mathcal{H}} \| u_n \|^{(i)}(s), \]
which converges uniformly in $\mathbb{R}$ because its component subseries
\[ \sum_k \| f_x \|_{\mathcal{H}} \| g_k \|^{(i)}(s), \sum_k \| f_x \|_{\mathcal{H}} \| h_k \|^{(i)}(s) \]
are in turn dominated by the convergent series
\[ \sum_k d(x_k) G_{k,i}, \sum_k \| f \| \| H_{k,i} \|, \]
respectively (see (16), (5), (11), (10)). Whence it follows via (21) that, for each $k \in \mathbb{N}$,
\[ \left\| \left[ T^* h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \leq C_i k, \]
with a constant $C_i$ independent of $k$. Consider functions $P : \mathbb{R}^2 \to \mathbb{C}$, $p : \mathbb{R} \to L_2$, defined, for all $s, t \in \mathbb{R}$, by
\[ P(s, t) = \sum_k \left[ h_{n(k)} \right](s) \overline{\left[ T^* h_{n(k)} \right](t)}, \]
\[ p(s) = \overline{P(s, \cdot)} = \sum_k \left[ h_{n(k)} \right](s) T^* h_{n(k)}. \]
The termwise differentiation theorem implies that, for each $i$ and each integer $j \in [0, +\infty)$,
\[ \frac{\partial^{i+j} P}{\partial s^i \partial t^j}(s, t) = \sum_k \left[ h_{n(k)} \right]^{(i)}(s) \overline{\left[ T^* h_{n(k)} \right]^{(j)}(t)}, \]
\[ \frac{d^j p}{ds^j}(s) = \sum_k \left[ h_{n(k)} \right]^{(i)}(s) T^* h_{n(k)}, \]
since, by (22) and (12), the series displayed converge (absolutely) in $C(\mathbb{R}^2, \mathbb{C})$, $C(\mathbb{R}, L_2)$, respectively. Thus, $\frac{\partial^{i+j} P}{\partial s^i \partial t^j} \in C(\mathbb{R}^2, \mathbb{C})$, and $\frac{d^j p}{ds^j} \in C(\mathbb{R}, L_2)$. Observe also that, by (12) and (23), the series (20) (viewed, of course, as one with terms belonging to $C(\mathbb{R}, \mathbb{C})$) converges (absolutely) in $C(\mathbb{R}, \mathbb{C})$-norm to the function
\[ [P f](s) \equiv \langle f, p(s) \rangle \equiv \int_{\mathbb{R}} P(s, t) f(t) \, dt. \]
Thus, $P$ is an integral operator, and $P$ is its $SK^\infty$-kernel.

Since $\| S^* e_k \|_{\mathcal{H}} = \| J e_k \|_{\mathcal{H}}$ for all $k$ (see (3)), from (2) it follows via (19) that the operator $A$ defined in (18) is nuclear, and hence
\[ \sum_n s_n \| a \| < \infty, \]
Then, according to (18), a kernel which induces the nuclear operator \( F = U_\infty J^* U_\infty^{-1} \) can be represented by the series

\[
(25) \quad \sum_n s_n^t U_\infty A^* q_n(s) U_\infty A p_n(t)
\]

convergent almost everywhere in \( \mathbb{R}^2 \). The functions used in this bilinear expansion can be written as the series convergent in \( L_2 \):

\[
U_\infty A p_k = \sum_n \langle p_k, A^* f_n \rangle_\mathcal{H} u_n, \quad U_\infty A^* q_k = \sum_n \langle q_k, A f_n \rangle_\mathcal{H} u_n \quad (k \in \mathbb{N}).
\]

Show that, for any fixed \( i \), the functions \([U_\infty A p_k]^{(i)}\), \([U_\infty A^* q_k]^{(i)}\) (\( k \in \mathbb{N} \)) make sense, are all in \( C(\mathbb{R}, \mathbb{C}) \), and their \( C(\mathbb{R}, \mathbb{C}) \)-norms are bounded independent of \( k \). Indeed, all the series

\[
\sum_n \langle p_k, A^* f_n \rangle_\mathcal{H} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, A f_n \rangle_\mathcal{H} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})
\]

are dominated by one series

\[
\sum_n (\|A^* f_n\| + \|A f_n\|) |[u_n]^{(i)}(s)|.
\]

This series converges uniformly in \( \mathbb{R} \), since it consists of two uniformly convergent in \( \mathbb{R} \) subseries

\[
\sum_k (\|A^* x_k\| + \|A x_k\|) |[g_k]^{(i)}(s)|,
\]

\[
\sum_k (\|A^* x_k^\perp\| + \|A x_k^\perp\|) |[h_k]^{(i)}(s)|,
\]

which are dominated by the following convergent series

\[
\sum_k d(x_k) G_{k,i}, \quad \sum_k 2\|A\| H_{k,i},
\]

respectively (see (5), (19), (11), (10)). Thus, for functions \( F : \mathbb{R}^2 \to \mathbb{C}, f : \mathbb{R} \to L_2 \), defined by

\[
F(s, t) = \sum_n s_n^t [U_\infty A^* q_n] (s) U_\infty A p_n (t),
\]

\[
f(s) = F(s, \cdot) = \sum_n s_n^t [U_\infty A^* q_n] (s) U_\infty A p_n,
\]

one can write, for all non-negative integers \( i, j \) and all \( s, t \in \mathbb{R} \),

\[
\frac{\partial^{i+j} F}{\partial s^i \partial t^j} (s, t) = \sum_n s_n^t [U_\infty A^* q_n]^{(i)} (s) U_\infty A p_n^{(j)} (t),
\]

\[
\frac{d^i f}{d s^i} (s) = \sum_n s_n^t [U_\infty A^* q_n]^{(i)} (s) U_\infty A p_n,
\]

where the series converge in \( C(\mathbb{R}^2, \mathbb{C}), C(\mathbb{R}, L_2) \), respectively, because of (24). This implies that \( F \) is a \( SK^\infty \)-kernel of \( F \).
In accordance with (4), we have, for each \( f \in L^2 \),
\[
(Tf)(s) = \int_{\mathbb{R}} P(s,t)f(t)\,dt + \int_{\mathbb{R}} F(s,t)f(t)\,dt
\]
\[
= \int_{\mathbb{R}} (P(s,t) + F(s,t))f(t)\,dt
\]
for almost every \( s \in \mathbb{R} \). Therefore \( T \) is a Carleman operator, and that kernel \( K \) of \( T \), which is defined by \( K(s,t) = P(s,t) + F(s,t) \) \((s, t \in \mathbb{R})\), inherits the \( SK^\infty \)-kernel properties from its terms. Consequently, \( K \) is a \( SK^\infty \)-kernel of \( T \).

Since scalar factors do not alter the relevant smoothness conditions, the Carleman operators \( U_\infty B_r U_\infty^{-1} = rU_\infty S_r U_\infty^{-1} \) \((r \in \mathbb{N})\) have \( SK^\infty \)-kernels as well. The proof of the theorem is complete.

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