ON THE COHOMOLOGY OF
THE MODULI SPACE OF σ-STABLE Triples
AND (1,2)-VARIATIONS OF HODGE STRUCTURES

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Abstract

There is an isomorphism between the moduli spaces of holomorphic triples and the Variations of Hodge Structures, VHS. Motivated by special embeddings on the moduli spaces of k-Higgs bundles of rank three, the main objective here is to study the cohomology of the VHS which correspond to critical submanifolds of such moduli spaces, extending those embeddings to moduli spaces of holomorphic triples.

Keywords: Higgs bundles, Hodge bundles, moduli spaces, stable triples, vector bundles.

MSC classes: Primary 14F45; Secondaries 14D07, 14H60.

Introduction

In this work, we study the stabilization of some cohomology groups of the moduli space of holomorphic triples of type (2, 1, \( \tilde{d}_1, \tilde{d}_2 \)) through the comparison with (1, 2)-Variations of Hodge Structures, abreviated as (1, 2)-VHS, which correspond to critical submanifolds of type (1, 2) from the Morse function \( f: \mathcal{M}^k(3, d) \rightarrow \mathbb{R} \) defined by

\[
f(E, \Phi) = \frac{1}{2\pi} \|\Phi\|_{L^2}^2 = \frac{i}{2\pi} \int_X \text{tr}(\Phi \Phi^*)
\]
applied to the moduli spaces of rank three k-Higgs bundles \( \mathcal{M}^k(3, d) \), on a Riemann surface \( X \) of genus \( g \geq 2 \). The co-prime condition \( \text{GCD}(3, d) = 1 \) implies that the moduli space \( \mathcal{M}^k(3, d) \) is smooth.

In this paper, our estimates are based on the embeddings \( \mathcal{M}^k(3, d) \hookrightarrow \mathcal{M}^{k+1}(3, d) \) defined by

\[
i_k: [(E, \Phi^k)] \mapsto [(E, \Phi^k \otimes s_p)]
\]
where \( p \in X \) is an arbitrary fixed point, and \( 0 \neq s_p \in H^0(X, \mathcal{O}_X(p)) \) is a nonzero fixed section of \( \mathcal{O}_X(p) \).

The paper is organized as follows: in section 1 we recall some basic facts about holomorphic triples and Higgs bundles; in section 2, we present the effect of the embeddings on \( \sigma \)-stable triples; in section 3, we discuss the effect of the embeddings at the blow-up level, considering the flip loci, and present an original result, the so-called “Roof Theorem”.

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Theorem. (Theorem 3.1) There exists an embedding at the blow-up level

\[ \tilde{i}_k : \tilde{N}_{\sigma_c(k)} \hookrightarrow \tilde{N}_{\sigma_c(k+1)} \]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{N}_{\sigma_c(k+1)} & \xrightarrow{\tilde{i}_k} & \tilde{N}_{\sigma_c(k)} \\
\downarrow & & \downarrow \\
N_{\sigma_c^+(k+1)} & \xrightarrow{\cong} & N_{\sigma_c^+(k)} \\
\end{array}
\]

where \( \tilde{N}_{\sigma_c(k)} \) is the blow-up of \( N_{\sigma_c^-(k)} = N_{\sigma_c^-(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \) along the flip locus \( S_{\sigma_c^-(k)} \) and, at the same time, represents the blow-up of \( N_{\sigma_c^+(k)} = N_{\sigma_c^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \) along the flip locus \( S_{\sigma_c^+(k)} \).

In section 4 we present the cohomology main results: in subsection 4.1 the stabilization of the cohomology at the blow-up level (Theorem 4.5) for certain indexes:

Theorem. (Theorem 4.5) There is an isomorphism

\[ \tilde{i}_k^* : H^j(\tilde{N}_{\sigma_c(k+1)}, \mathbb{Z}) \xrightarrow{\cong} H^j(\tilde{N}_{\sigma_c(k)}, \mathbb{Z}) \quad \forall j \leq n(k) \]

at the blow-up level, where \( n(k) := \min(\tilde{d}_1 - d_M - \tilde{d}_2 - 1, 2(\tilde{d}_1 - 2\tilde{d}_2 - (2g - 2)) + 1) \).

And hence, the cohomology stabilization of the moduli spaces of triples:

Corollary. (Corollary 4.7) There is an isomorphism

\[ \tilde{i}_k^* : H^j(\tilde{N}_{\sigma_c(k+1)}, \mathbb{Z}) \xrightarrow{\cong} H^j(\tilde{N}_{\sigma_c(k)}, \mathbb{Z}) \quad \forall j \leq \tilde{n}(k) \]

where \( \tilde{n}(k) := \min\left(n(k), 2d_M - \tilde{d}_1 + g - 2\right) \).

Finally, in subsection 4.2 we show the stabilization of the \( (1, 2) \)-VHS cohomology using the isomorphisms between them and the moduli spaces of triples:

Theorem. (Corollary 4.12) There is an isomorphism

\[ H^j(F_{d_1}^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(F_{d_1}^k, \mathbb{Z}) \]

for all \( j \leq \tilde{n}(k) \).
# Preliminary definitions

Let $X$ be a compact Riemann surface of genus $g \geq 2$ and let $K = T^*X$ be the canonical line bundle of $X$. Note that, algebraically, $X$ is also a nonsingular complex projective algebraic curve.

**Definition 1.1.** For a vector bundle $E \to X$, we denote the **rank** of $E$ by $\text{rk}(E) = r$ and the **degree** of $E$ by $\text{deg}(E) = d$. Its **slope** is defined to be

$$\mu(E) := \frac{\text{deg}(E)}{\text{rk}(E)} = \frac{d}{r}. \quad (1.1)$$

A vector bundle $E \to X$ is called **semistable** if $\mu(F) \leq \mu(E)$ for any nonzero $F \subseteq E$. Similarly, a vector bundle $E \to X$ is called **stable** if $\mu(F) < \mu(E)$ for any nonzero $F \not\subseteq E$. Finally, $E$ is called **polystable** if it is the direct sum of stable subbundles, all of the same slope.

## 1.1 Holomorphic Triples

**Definition 1.2.** Holomorphic Triples:

i. A **holomorphic triple** on $X$ is a triple $T = (E_1, E_2, \phi)$ consisting of two holomorphic vector bundles $E_1 \to X$ and $E_2 \to X$ and a homomorphism $\phi : E_2 \to E_1$, i.e., an element $\phi \in \text{Hom}(E_2, E_1)$. Nevertheless, we often abuse of the notation using $\phi \in H^0(\text{Hom}(E_2, E_1))$ for the class of the homomorphism.

ii. A **homomorphism** from a triple $T' = (E'_1, E'_2, \phi')$ to another triple $T = (E_1, E_2, \phi)$ is a commutative diagram of the form:

$$
\begin{array}{ccc}
E'_1 & \xrightarrow{\phi'} & E'_2 \\
\downarrow & & \downarrow \\
E_1 & \xrightarrow{\phi} & E_2
\end{array}
$$

where the vertical arrows represent holomorphic maps.

iii. $T' \subset T$ is a **subtriple** if the sheaf homomorphisms $E'_1 \to E_1$ and $E'_2 \to E_2$ are injective. As usual, a subtriple is called **proper** if $0 \not= T' \subsetneq T$.

**Definition 1.3.** $\sigma$-Stability, $\sigma$-Semistability and $\sigma$-Polystability:

i. For any $\sigma \in \mathbb{R}$, the **$\sigma$-degree** and the **$\sigma$-slope** of $T = (E_1, E_2, \phi)$ are defined as:

$$\text{deg}_\sigma(T) := \text{deg}(E_1) + \text{deg}(E_2) + \sigma \cdot \text{rk}(E_2),$$

and

$$\mu_\sigma(T) := \frac{\text{deg}_\sigma(T)}{\text{rk}(E_1) + \text{rk}(E_2)} = \mu(E_1 \oplus E_2) + \sigma \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}$$

respectively.

ii. $T$ is then called **$\sigma$-stable** [respectively, **$\sigma$-semistable**] if $\mu_\sigma(T') < \mu_\sigma(T)$ [respectively, $\mu_\sigma(T') \leq \mu_\sigma(T)$] for any proper subtriple $0 \not= T' \subsetneq T$. 


iii. A triple is called \( \sigma \)-polystable if it is the direct sum of \( \sigma \)-stable triples of the same \( \sigma \)-slope.

Now we may use the following notation for moduli spaces of triples:

i. Denote \( r = (r_1, r_2) \) and \( d = (d_1, d_2) \), and then regard
\[
N_\sigma = N_\sigma(r, d) = N_\sigma(r_1, r_2, d_1, d_2)
\]
as the moduli space of \( \sigma \)-polystable triples \( T = (E_1, E_2, \phi) \) such that \( \text{rk}(E_j) = r_j \) and \( \text{deg}(E_j) = d_j \).

ii. Denote by \( N^s_\sigma = N^s_\sigma(r, d) \) the subspace of \( \sigma \)-stable triples.

iii. Call \( (r, d) = (r_1, r_2, d_1, d_2) \) the type of the triple \( T = (E_1, E_2, \phi) \).

The reader may consult the works of Bradlow and García-Prada [4]; and Bradlow, García-Prada and Gothen [5] for the formal construction of the moduli space of triples. Also can see Muñoz, Oliveira and Sánchez [24]; and Muñoz, Ortega and Vázquez-Gallo [25] for other details.

1.2 Higgs Bundles

**Definition 1.4.** A Higgs bundle over \( X \) is a pair \( (E, \Phi) \) where \( E \to X \) is a holomorphic vector bundle and \( \Phi : E \to E \otimes K \) is an endomorphism of \( E \) twisted by \( K \), which is called a Higgs field. Note that \( \Phi \in H^0(X; \text{End}(E) \otimes K) \).

**Definition 1.5.** A subbundle \( F \subset E \) is said to be \( \Phi \)-invariant if \( \Phi(F) \subset F \otimes K \). A Higgs bundle is said to be semistable [respectively, stable] if \( \mu(F) \leq \mu(E) \) [respectively, \( \mu(F) < \mu(E) \)] for any nonzero \( \Phi \)-invariant subbundle \( F \subseteq E \) [respectively, \( F \subset E \)]. Finally, \( (E, \Phi) \) is called polystable if it is the direct sum of stable \( \Phi \)-invariant subbundles, all of the same slope.

Fixing the rank \( \text{rk}(E) = r \) and the degree \( \text{deg}(E) = d \) of a Higgs bundle \( (E, \Phi) \), the isomorphism classes of polystable bundles are parametrized by a quasi-projective variety: the moduli space \( \mathcal{M}(r, d) \). Constructions of this space can be found in the work of Hitchin [18], using gauge theory, or in the work of Nitsure [26], using algebraic geometry methods.

An important feature of \( \mathcal{M}(r, d) \) is that it carries an action of \( \mathbb{C}^* \): \( z \cdot (E, \Phi) = (E, z \cdot \Phi) \). According to Hitchin [18], \( (\mathcal{M}, I, \Omega) \) is a Kähler manifold, where \( I \) is its complex structure and \( \Omega \) its corresponding Kähler form. Furthermore, \( \mathbb{C}^* \) acts on \( \mathcal{M} \) biholomorphically with respect to the complex structure \( I \) by the aforementioned action, where the Kähler form \( \Omega \) is invariant under the induced action \( e^{i\theta} \cdot (E, \Phi) = (E, e^{i\theta} \cdot \Phi) \) of the circle \( \mathbb{S}^1 \subset \mathbb{C}^* \). Besides, this circle action is Hamiltonian, with proper momentum map \( f : \mathcal{M} \to \mathbb{R} \) defined by:
\[
f(E, \Phi) = \frac{1}{2\pi} \|\Phi\|_{L^2}^2 = \frac{i}{2\pi} \int_X \text{tr}(\Phi \Phi^*),
\]
where \( \Phi^* \) is the adjoint of \( \Phi \) with respect to a hermitian metric on \( E \), and \( f \) has finitely many critical values.

There is another important fact mentioned by Hitchin (see the original version in Frankel [7], and its application to Higgs bundles in [18]): the critical points of \( f \) are exactly the fixed points of the circle action on \( \mathcal{M} \).
If \((E, \Phi) = (E, e^{i\theta} \Phi)\) then \(\Phi = 0\) with critical value \(c_0 = 0\). The corresponding critical submanifold is \(F_0 = f^{-1}(c_0) = f^{-1}(0) = N\), the moduli space of stable bundles. On the other hand, when \(\Phi \neq 0\), there is a type of algebraic structure for Higgs bundles introduced by Simpson [28]: a variation of Hodge structure, or simply a \(VHS\), for a Higgs bundle \((E, \Phi)\) is a decomposition:

\[
E = \bigoplus_{j=1}^{n} E_j \quad \text{such that} \quad \Phi : E_j \to E_{j+1} \otimes K \quad \text{for} \ 1 \leq j \leq n - 1.
\]

(1.3)

It has been proved by Simpson [29] that the fixed points of the circle action on \(\mathcal{M}(r, d)\), and so, the critical points of \(f\), are these Variations of the Hodge Structure, \(VHS\), where the critical values \(c_\lambda = f(E, \Phi)\) will depend on the degrees \(d_j\) of the components \(E_j \subset E\). By Morse theory, we can stratify \(\mathcal{M}\) in such a way that there is a nonzero critical submanifold \(F_\lambda := f^{-1}(c_\lambda)\) for each nonzero critical value \(0 \neq c_\lambda = f(E, \Phi)\) where \((E, \Phi)\) represents a fixed point of the circle action, or equivalently, a \(VHS\). We then say that \((E, \Phi)\) is a \((\text{rk}(E_1), \ldots, \text{rk}(E_n))\)-\(VHS\).

**Definition 1.6.** Fix a point \(p \in X\), and let \(\mathcal{O}_X(p)\) be the associated line bundle to the divisor \(p \in \text{Sym}^1(X) = X\). A \(k\)-Higgs bundle (or Higgs bundle with poles of order \(k\)) is a pair \((E, \Phi^k)\) where:

\[
E \xrightarrow{\Phi^k} E \otimes K(kp)
\]

and where the morphism \(\Phi^k \in H^0(X, \text{End}(E) \otimes K \otimes \mathcal{O}_X(p)^{\otimes k})\) is what we call a Higgs field with poles of order \(k\). The moduli space of \(k\)-Higgs bundles of rank \(r\) and degree \(d\) is denoted by \(\mathcal{M}^k(r, d)\). For simplicity, we will suppose that \(\text{GCD}(r, d) = 1\), and so, \(\mathcal{M}^k(r, d)\) will be smooth.

There is an embedding

\[
i_k : \mathcal{M}^k(r, d) \to \mathcal{M}^{k+1}(r, d) : [(E, \Phi^k)] \mapsto [(E, \Phi^k \otimes s_p)] \quad (1.4)
\]

where \(0 \neq s_p \in H^0(X, \mathcal{O}_X(p))\) is a nonzero fixed section of \(\mathcal{O}_X(p)\).

So far, everything we have said for \(\mathcal{M}(r, d)\) holds also for \(\mathcal{M}^k(r, d)\). Moreover, when the rank is \(r = 3\), the map \(i_k\) induces embeddings of the form

\[
F^k_\lambda \xrightarrow{i_k} F^{k+1}_\lambda \quad \forall \lambda,
\]

and those embeddings induce isomorphisms in cohomology:

\[
H^j(F^{k+1}_\lambda, \mathbb{Z}) \xrightarrow{\cong} H^j(F^k_\lambda, \mathbb{Z})
\]

for certain values of \(j\) and \(k\). To find the range of \(j\) for which the isomorphism holds turns out to be difficult. Hence, it is not obvious how to apply this approach to \(\mathcal{M}^k(3, d)\).

If we restrict the embedding to the critical manifolds of type \((1, 2)\):

\[
F^k_{d_1} \xrightarrow{i_k} F^{k+1}_{d_1}
\]

\[
\left( \begin{array}{cc} E_1 \oplus E_2, & \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \\ \left( \begin{array}{cc} \varphi_k^k & 0 \\ \varphi_{21} & 0 \end{array} \right) \end{array} \right) \mapsto \left( \begin{array}{cc} E_1 \oplus E_2, & \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \\ \left( \begin{array}{cc} \varphi_k^k \otimes s_p & 0 \\ \varphi_{21} & 0 \end{array} \right) \end{array} \right)
\]

(1.5)
for \( \sigma \) where \( \tilde{\sigma} \) is justified since the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
E_1 \oplus E_2, \\
(0, 0, 0)
\end{array}
\end{array}
\xrightarrow{\varphi} (V, V_1, \varphi)
\]

between (1, 2)-VHS and the moduli space of triples, where we denote by \( V_1 = E_2 \otimes K(kp), \) by \( V_2 = E_1, \) by \( \varphi = \varphi^E_2, \) and \( \sigma^E_{hp} = \deg(K(kp)) = 2q - 2 + k, \) induces another embedding:

\[
i_k : \mathcal{N}_{\sigma^E_{h}}(2, 1, \tilde{d}_1, \tilde{d}_2) \rightarrow \mathcal{N}_{\sigma^E_{h} + 1}(2, 1, \tilde{d}_1 + 2, \tilde{d}_2)
\]

\[
(V_1, V_2, \varphi) \mapsto (V_1 \otimes \mathcal{O}_X(p), V_2, \varphi \otimes s_p)
\]

where \( \tilde{d}_1 = \deg(V_1) = d_2 + 2\sigma^E_{hp} \) and \( \tilde{d}_2 = \deg(V_2) = d_1, \) and so, it induces embeddings of the kind:

\[
i_k : \mathcal{N}_{\sigma^c_{h}}(2, 1, \tilde{d}_1, \tilde{d}_2) \rightarrow \mathcal{N}_{\sigma^c_{h} + 1}(2, 1, \tilde{d}_1 + 2, \tilde{d}_2)
\]

and

\[
i_k : \mathcal{N}_{\sigma^c_{h}}(2, 1, \tilde{d}_1, \tilde{d}_2) \rightarrow \mathcal{N}_{\sigma^c_{h} + 1}(2, 1, \tilde{d}_1 + 2, \tilde{d}_2)
\]

for \( \sigma_m < \sigma_c(k) < \sigma_M. \)

\[2\] \( \sigma \)-Stability

For rank \( r = 3, \) the embeddings \( i_k \) preserve \( \sigma \)-stability:

**Lemma 2.1.** A triple \( T \) is \( \sigma \)-stable \( \iff \) \( i_k(T) \) is \( (\sigma + 1) \)-stable.

**Proof.** Recall that \( T = (V_1, V_2, \varphi) \) is \( \sigma \)-stable if and only if \( \mu_\sigma(T') < \mu_\sigma(T) \) for any \( T' \) proper subtriple of \( T. \)

Denote by \( S = i_k(T) = (V_1 \otimes \mathcal{O}_X(p), V_2, \varphi \otimes s_p) \). Is easy to check that \( \mu_{\sigma + 1}(S) = \mu_\sigma(T) + 1. \)

Without lost of generality, we may suppose that any \( S' \) proper subtriple of \( S \) is of the form \( S' = i_k(T') \) for some \( T' \) subtriple of \( T, \) or equivalently:

\[
S' = (V'_1 \otimes \mathcal{O}_X(p), V'_2, \varphi \otimes s_p)
\]

and that there are injective sheaf homomorphisms \( V'_1 \rightarrow V_1 \) and \( V'_2 \rightarrow V_2. \) This statement is justified since the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
S : \\
V_2 \rightarrow V_1 \otimes \mathcal{O}_X(p)
\end{array}
\end{array}
\xrightarrow{\varphi \otimes s_p} \begin{array}{c}
\begin{array}{c}
S' : \\
B \rightarrow A
\end{array}
\end{array}
\xrightarrow{(\varphi \otimes s_p)_{|B}} \begin{array}{c}
\begin{array}{c}
T' : \\
B \otimes s_p^{-1} \rightarrow A \otimes L(-p)
\end{array}
\end{array}
\xrightarrow{(\varphi \otimes s_p)_{|B \otimes s_p^{-1}}} \begin{array}{c}
\begin{array}{c}
T : \\
V_2 \rightarrow V_1 \otimes \mathcal{O}_X(p)
\end{array}
\end{array}
\end{equation}

\(\text{i.e. there is a } (1 - 1)\text{-correspondence between the proper subtripes } S' \subset S \text{ and the proper subtripes } T' \subset T. \) Taking \( A = V'_1 \otimes \mathcal{O}_X(p), \) \( B = V'_2 \) and \( T' = (V'_1, V'_2, \varphi), \) we can easily see that \( \mu_{\sigma + 1}(S') = \mu_\sigma(T') + 1 \) and hence:

\[
\mu_{\sigma + 1}(S') < \mu_{\sigma + 1}(S) \iff \mu_\sigma(T') + 1 < \mu_\sigma(T) + 1 \iff \mu_\sigma(T') < \mu_\sigma(T).
\]

Therefore, \( T \) is \( \sigma \)-stable \( \iff \) \( S = i_k(T) \) is \( (\sigma + 1) \)-stable. \( \square \)
Corollary 2.2. The embedding 

\[ i_k : \mathcal{N}_{\sigma(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2) \]

is well defined for any \( \sigma(k) \) such that \( \sigma_m < \sigma(k) < \sigma_M \). In particular, the embedding \( i_k \) restricted to \( F_{d_1}^k \) (see (1.5)) is well defined and we have a commutative diagram of the form:

\[ (E_1 \oplus E_2, \Phi^k) \xymatrix{ \ar[r] & (E_1 \oplus E_2, \Phi^k \otimes s_p) } \]

where \( E_1 = E_2 \otimes K(kp), \ E_2 = E_1, \) and \( \varphi_{21}^k : E_1 \to E_2 \otimes K(kp). \)

Hence, these results allow us to conclude that there is an interesting and important correspondence between the \( \sigma \)-stability values of moduli spaces of holomorphic triples:

\[ \sigma_m(k) \xymatrix{ \ar[r] & \sigma_H(k) \ar[d]_{i_k} \ar[r] & \sigma_M(k) \ar[d]_{i_k} \ar[r] & \sigma'(k) \ar[d]_{i_k} \ar[r] & \sigma_M(k+1) } \]

where \( \sigma_m(k) = \tilde{\mu}_1 - \tilde{\mu}_2, \ \sigma_M(k) = 4(\tilde{\mu}_1 - \tilde{\mu}_2), \ \sigma_H(k) = \text{deg}(K(kp)) = 2g - 2 + k, \) and the correspondence gives us \( \sigma_m(k+1) = \sigma_m(k) + 1, \ \sigma' = \sigma_M(k) + 1, \ \sigma_M(k+1) = \sigma_M(k) + 3, \) and \( \sigma_H(k+1) = \sigma_H(k) + 1. \)

3 Blow-UP and The Roof Theorem

Recall that the blow-up of \( \mathcal{N}_{\sigma_-^c(k)} = \mathcal{N}_{\sigma_-^c(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \) along the flip locus \( S_{\sigma_-^c(k)} \), \( \tilde{\mathcal{N}}_{\sigma_-^c(k)} \) is isomorphic to \( \mathcal{N}_{\sigma_-^c(k)} \), the blow-up of \( \mathcal{N}_{\sigma_-^{c+}(k)} = \mathcal{N}_{\sigma_-^{c+}(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \) along the flip locus \( S_{\sigma_-^{c+}(k)} \). From now on, we will denote just \( \tilde{\mathcal{N}}_{\sigma_c(k)} \) whenever no confusion is likely to arise.

Theorem 3.1. There exists an embedding at the blow-up level

\[ \tilde{i}_k : \tilde{\mathcal{N}}_{\sigma_c(k)} \to \tilde{\mathcal{N}}_{\sigma_c(k+1)} \]
Recall that \( \tilde{\mathcal{L}} \) that deg(\( V \)) note that any triple \( \tilde{\mathcal{L}} \) is a line bundle of degree deg(\( V \)) is a non-trivial extension of a subtriple \( \mathcal{N} \) where \( \tilde{\mathcal{L}} \) is a non-trivial extension of a subtriple \( \mathcal{N} \) where \( \tilde{\mathcal{L}} \) along the flip locus \( S_{\sigma^-}(k) \) and, at the same time, represents the blow-up of \( \tilde{\mathcal{N}}_{\sigma^+(k)} = \mathcal{N}_{\sigma^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \) along the flip locus \( S_{\sigma^+(k)} \). Proof. Recall that \( T \) is \( \sigma \)-stable if and only if \( i_k(T) \) is \((\sigma + 1)\)-stable. Furthermore, by [25], note that any triple \( T \in S_{\sigma^+(k)} \subset \mathcal{N}_{\sigma^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \) is a non-trivial extension of a subtriple \( T' \subset T \) of the form \( T' = (V_1', V_2', \varphi') = (M, 0, \varphi') \) by a quotient triple of the form \( T'' = (V_1'', V_2'', \varphi'') = (L, V_2, \varphi'') \), where \( M \) is a line bundle of degree \( \deg(M) = d_M \) and \( L \) is a line bundle of degree \( \deg(L) = d_L = \tilde{d}_1 - d_M \). Besides, also by [25], the non-trivial critical values \( \sigma_c \neq \sigma_m \) for \( \sigma_m < \sigma < \sigma_M \) are of the form \( \sigma_c = 3d_M - \tilde{d}_1 - \tilde{d}_2 \). Then, we can visualize the embedding \( i_k : T \rightarrow i_k(T) \) as follows:

\[
\begin{array}{cccccccc}
0 & \rightarrow & T' & \longrightarrow & T & \longrightarrow & T'' & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & V_2 & = & V_2 & \rightarrow & 0 \\
0 & \rightarrow & M & \rightarrow & V_1 & \rightarrow & L & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & V_2 & = & V_2 & \rightarrow & 0 \\
0 & \rightarrow & M \otimes O_X(p) & \rightarrow & V_1 \otimes O_X(p) & \rightarrow & L \otimes O_X(p) & \rightarrow & 0
\end{array}
\]

where \( \deg(V_1 \otimes O_X(p)) = \tilde{d}_1 + 2 \) and \( \deg(M \otimes O_X(p)) = d_M + 1 \), and so \( L \otimes O_X(p) \) verifies that \( \deg(L \otimes O_X(p)) = \deg(V_1 \otimes O_X(p)) - \deg(M \otimes O_X(p)) \):

\[
\deg(L \otimes O_X(p)) = d_L + 1 = \tilde{d}_1 - d_M + 1 = \]
\[(\tilde{d}_1 + 2) - (d_M + 1) = \deg(V_1 \otimes O_X(p)) - \deg(M \otimes O_X(p)).\]

Hence, \(\sigma_c(k + 1)\) verifies that \(\sigma_c(k + 1) = \sigma_c(k) + 1:\)

\[
\sigma_c(k + 1) = \begin{eqnarray*}
3 \deg(M \otimes O_X(p)) - \deg(V_1 \otimes O_X(p)) - \deg(V_2) = \\
3d_M + 3 - \tilde{d}_1 - 2 - \tilde{d}_2 = (3d_M - \tilde{d}_1 - \tilde{d}_2) + 1 = \sigma_c(k) + 1
\end{eqnarray*}
\]

and where \(i_k(T') = (M \otimes O_X(p), 0, \varphi' \otimes s_p)\) is the maximal \(\sigma_c^+(k+1)\)-destabilizing subtriple of \(i_k(T)\).

Similarly, also by [25], any triple \(T \in S_{\sigma_c^{-}}(k) \subset N_{\sigma_c^{-}}(k)\) is a non-trivial extension of a subtriple \(T' \subset T\) of the form \(T' = (V_1', V_2', \varphi') = (L, V_2, \varphi')\) by a quotient triple of the form \(T'' = (V_1'', V_2'', \varphi'') = (M, 0, \varphi'')\), where \(M\) is a line bundle of degree \(\deg(M) = d_M\) and \(L\) is a line bundle of degree \(\deg(L) = d_L = \tilde{d}_1 - d_M\). Then, the embedding \(i_k : T \mapsto i_k(T)\) looks like:

\[
\begin{array}{cccccc}
0 & \rightarrow & T' & \rightarrow & T & \rightarrow & T'' & \rightarrow & 0 \\
0 & \rightarrow & V_2 & \rightarrow & V_2 & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & L & \rightarrow & V_1 & \rightarrow & M & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & V_2 & \rightarrow & V_2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & L \otimes O_X(p) & \rightarrow & V_1 \otimes O_X(p) & \rightarrow & M \otimes O_X(p) & \rightarrow & 0 \\
\end{array}
\]

where \(i_k(T') = (L, V_2, \varphi')\) is the maximal \(\sigma_c^+(k + 1)\)-destabilizing subtriple of \(i_k(T)\).

Hence, \(i_k\) restricts to the flip loci \(S_{\sigma_c^+}(k)\) and \(S_{\sigma_c^-}(k)\). Recall that, by definition, the blow-up of \(N_{\sigma_c^+}(k)\) along the flip locus \(S_{\sigma_c^+}(k)\) is the space \(\tilde{N}_{\sigma_c}(k)\) together with the projection

\[
\pi : \tilde{N}_{\sigma_c}(k) \rightarrow N_{\sigma_c^+}(k)
\]

where \(\pi\) restricted to \(N_{\sigma_c^+}(k) - S_{\sigma_c^+}(k)\) is an isomorphism and the \textbf{exceptional divisor} \(E^+ = \pi^{-1}(S_{\sigma_c^+}(k)) \subset \tilde{N}_{\sigma_c}(k)\) is a fiber bundle over \(S_{\sigma_c^+}(k)\) with fiber \(\mathbb{P}^{n-k-1}\), where \(n = \dim(N_{\sigma_c^+}(k))\) and \(k = \dim(S_{\sigma_c^+}(k))\). So, the embedding can be extended to \(E^+\) in a natural way. Same argument remains valid when we consider \(\tilde{N}_{\sigma_c}(k)\) as the blow-up of \(N_{\sigma_c^-}(k)\) along the flip locus \(S_{\sigma_c^-}(k)\) with exceptional divisor \(E^- = \pi^{-1}(S_{\sigma_c^-}(k)) \subset \tilde{N}_{\sigma_c}(k)\). Therefore, the embedding can be extended to the whole \(\tilde{N}_{\sigma_c}(k)\).

\[\square\]

\textbf{Remark 3.2.} The construction of the blow-up may be found in the book of Griffiths and Harris [11].
4 Cohomology

We want to prove that the embedding $i_k : F_{d_1}^k \hookrightarrow F_{d_1}^{k+1}$ induces an isomorphism in cohomology:

$$H^j(F_{d_1}^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(F_{d_1}^k, \mathbb{Z})$$

for certain $j$, or equivalently:

$$H^j(N_{r+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(N_r, \mathbb{Z}),$$

where we denote $N_{r+1} = N_{r+1}(2, 1, \tilde{a}_1, \tilde{a}_2)$.

We do that in two steps. First, in subsection 4.1, we get that

$$H^j(N_{r+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(N_r, \mathbb{Z})$$

for all critical $\sigma_c = \sigma_c(k)$ such that $\sigma_m(k) < \sigma_c(k) < \sigma_M(k)$, and for all $j \leq \tilde{n}(k)$, where the bound $\tilde{n}(k)$ is known. We first analyze the embedding restricted to the flip loci, $i_k : S^k_{\sigma_c(k)} \hookrightarrow S^k_{\sigma_c(k+1)}$ and $i_k : S^k_{\sigma_c(k)} \hookrightarrow S^k_{\sigma_c(k+1)}$. For simplicity, we will denote from now on $S^k = S^k_{\sigma_c(k)}$ and $S^k_{+} = S^k_{\sigma_c(k+1)}$ whenever no confusion is likely to arise about the critical value.

Finally, in subsection 4.2, we stabilize the cohomology of the $(1,2)$-VHS, using useful results from the work of Bradlow, Garcia-Prada, Gothen [5].

4.1 Cohomology of $N_{\sigma_c}(2, 1, \tilde{a}_1, \tilde{a}_2)$

**Theorem 4.1.** There is an isomorphism

$$i_k^* : H^j(S^k_{\sigma_c+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(S^k_{\sigma_c}, \mathbb{Z})$$

for all $j \leq \tilde{a}_1 - d_M - \tilde{a}_2 - 1 = d_2 - d_1 + 2\sigma_H(k) - d_M$, where $d_j = \deg(E_j)$, $\tilde{d}_j = \deg(\tilde{E}_j)$, $d_M = \deg(M)$, and $\sigma_H(k) = \deg(K(kp)) = 2g - 2 + k$.

**Proof.** Recall that, according to [25, Theorem 4.8.], $S^k_{\sigma_c} = \mathbb{P}(V)$ is the projectivization of a bundle $V \to N^L_{\sigma_c} \times N^R_{\sigma_c}$ of rank $\mathrm{rk}(V) = -\chi(T'', T')$, where $N^L_{\sigma_c} = N^L_{\sigma_c}(1, 1, \tilde{d}_1 - d_M, \tilde{a}_2) \cong J^{d_2} \times \mathrm{Sym}^{d_1 - d_M - d_2}(X)$ and $N^R_{\sigma_c} = N^R_{\sigma_c}(1, 0, d_M, 0) \cong J^{d_M}(X)$, and where any triple $T = (V_1, V_2, \varphi) \in S^k_{\sigma_c} \subset N^L_{\sigma_c}(2, 1, \tilde{d}_1, \tilde{a}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (L, V_2, \varphi')$ by a quotient of the form $T'' = (V_1'', V_2'', \varphi'') = (M, 0, \varphi'')$, where $M$ is a line bundle of degree $\deg(M) = d_M$ and $L$ is a line bundle of degree $\deg(L) = d_L = \tilde{d}_1 - d_M$. Then, the embedding $i_k : T \to i_k(T)$ restricts to:

$$\xymatrix{(V_1', V_2', \varphi') \ar@{^{(}->}[r] & (V_1' \otimes \mathcal{O}_X(p), V_2', \varphi' \otimes s_p)}$$

$$\xymatrix{N'_{\sigma_c} \ar@{^{(}->}[r]^{i_k} & \ar[d]_\cong N'_{\sigma_c+1}}$$

$$\xymatrix{J^{d_2} \times \mathrm{Sym}^{d_1 - d_M - d_2}(X) \ar[d]_\cong_{i_k} \ar[r] & J^{d_2} \times \mathrm{Sym}^{d_1 - d_M - d_2 + 1}(X) \ar@{^{(}->}[r] & ([V_2], \mathrm{div}(\varphi')) \ar[r] & ([V_2], \mathrm{div}(\varphi' \otimes s_p))}$$
because $\sigma_c(k + 1) = \sigma_c(k) + 1$, and $d_M(k + 1) = d_M(k) + 1$, and because, by the proof of the Roof Theorem 3.1, $i_k$ restricts to the flip locus $S^k_\sigma$.

Similarly, $i_k$ restricts to:

\[
(V''_1, 0, 0) \xrightarrow{i_k} (V''_1 \otimes O_X(p), 0, 0)
\]

\[
\begin{array}{c}
\mathcal{N}'_{\sigma_c} \approx \mathcal{N}'_{\sigma_c + 1} \\
\mathcal{J}^d M \approx \mathcal{J}^d M
\end{array}
\]

So, by Macdonald [21, (12.2)],

\[
i^*_k : H^j(\mathcal{N}'_{\sigma_c + 1}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}'_{\sigma_c}, \mathbb{Z}) \forall j \leq d_1 - d_M - \tilde{d}_2 - 1,
\]

and hence

\[
i^*_k : H^j(S^k_{\sigma_c + 1}, \mathbb{Z}) \xrightarrow{\cong} H^j(S^k_{\sigma_c}, \mathbb{Z}) \forall j \leq d_1 - d_M - \tilde{d}_2 - 1. \quad \square
\]

Similarly, for the flip locus $S^k_\sigma = S_{\sigma_c(k)}$ we have:

**Theorem 4.2.** There is an isomorphism

\[
i^*_k : H^j(S^k_{\sigma_c + 1}, \mathbb{Z}) \xrightarrow{\cong} H^j(S^k_{\sigma_c}, \mathbb{Z})
\]

for all $j \leq 2d_M - \tilde{d}_1 + g - 2 = 2d_M - \left(\tilde{d}_2 + 2\sigma_H(k)\right) + g - 2$, where $d_j = \deg(E_j)$, $d_{\tilde{d}} = \deg(E_{\tilde{d}})$, $d_M = \deg(M)$, and $\sigma_H(k) = \deg(K(kp)) = 2g - 2 + k$.

**Proof.** Quite similar argument to the one presented above, except for the detail that this time is the other way around: according also to [25, Theorem 4.8.], $S^k_\sigma = \mathbb{P}(V)$ is the projectivization of a bundle $V \to N^c_\sigma \times N^c_\sigma$ of rank $\text{rk}(V) = -\chi(T^c, T)$, but this time $N^c_\sigma = N_c(1, 0, d_M, 0) \cong \mathcal{J}^M(X)$, and $N^c_\sigma = N_c(1, 1, -d_1 - d_M, \tilde{d}_2) \cong \mathcal{J}^M \times \text{Sym}^d - d - 2d_2 - 2(X)$ and where any triple $T = (V_1, V_2, \varphi) \in S^k_\sigma \subset N^c_\sigma(2, 1, \tilde{d}_1, \tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V'_1, V'_2, \varphi') = (M, 0, \varphi')$ by a quotient triple of the form $T'' = (V'_1, V''_2, \varphi'') = (L, V_2, \varphi'')$, where $M$ is a line bundle of degree $\deg(M) = d_M$, and $L$ is a line bundle of degree $\deg(L) = d_L = \tilde{d}_1 - d_M$. \quad \square

**Theorem 4.3.** There is an isomorphism

\[
i^*_k : H^j(N^c_\sigma(\sigma_c + 1), \mathbb{Z}) \xrightarrow{\cong} H^j(N^c_\sigma(\sigma_c(k)), \mathbb{Z}) \quad \forall j \leq 2\left(\tilde{d}_1 - 2\tilde{d}_2 - (2g - 2)\right) + 1.
\]

Since the behavior of $N^c_\sigma$, where $\sigma_c = \sigma_c - \varepsilon$, is the same that the one of $N^c_\sigma$, where $\sigma_m = \sigma_m + \varepsilon$, is enough to prove the following lemma:

**Lemma 4.4.** The relative homology groups

\[
H^j(N^c_\sigma(\sigma_m + k), N^c_\sigma(\sigma_m(k)); \mathbb{Z}) = 0
\]

are trivial for all $j \leq 2\left(\tilde{d}_1 - 2\tilde{d}_2 - (2g - 2)\right)$.\]
Proof. Note that $\mathcal{N}_{\sigma_2(k)} = \emptyset$, hence $\mathcal{N}_{\sigma_2(k)} = S_+^k$, and according to [25, Theorem 4.10.], any triple $T = (V_1, V_2, \varphi) \in S_+^k = \mathcal{N}_{\sigma_2(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) = \mathcal{N}_{\sigma_2(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (V_1, 0, 0)$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (0, V_2, 0)$. Hence, there is a map

$$
\pi : \mathcal{N}_{\sigma_m} \to \mathcal{N}(2, \tilde{d}_1) \times \mathcal{J}^{\tilde{d}_2}(X)
$$

where the inverse image $\pi^{-1}(\mathcal{N}(2, \tilde{d}_1) \times \mathcal{J}^{\tilde{d}_2}(X)) = \mathbb{P}^N$ has rank $N = -\chi(T'', T') = \tilde{d}_1 - 2\tilde{d}_2 - (2g - 2)$, and the proof follows.

**Theorem 4.5.** There is an isomorphism

$$i_k^* : H^j(\mathcal{N}_{\sigma_2(k + 1)}, \mathbb{Z}) \cong H^j(\mathcal{N}_{\sigma_2(k)}, \mathbb{Z}) \quad \forall j \leq n(k)$$

at the blow-up level, where $n(k) := \min(\tilde{d}_1 - d_M - \tilde{d}_2 - 1, 2(\tilde{d}_1 - 2\tilde{d}_2 - (2g - 2)) + 1)$.

Proof. By the Roof Theorem 3.1, $i_k$ lifts to the blow-up level. We will denote $\mathcal{N}^k_+ = \mathcal{N}_{\sigma_2(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ and $\mathcal{N}^k = \mathcal{N}_{\sigma_2(k)}$ its blow-up along the flip locus $S_+^k = S_{\sigma_2}(k)$. Recall that, from the construction of the blow-up, there is a map $\pi_+ : \mathcal{N}^k \to S_+^k$ such that

$$0 \to \pi_+^*(H^j(\mathcal{N}^k)) \to H^j(\mathcal{N}^k) \to H^j(\mathcal{E}^k) / \pi_+^*(H^j(S_+^k)) \to 0$$

splits where $\mathcal{E}^k = \pi_+^{-1}(S_+^k)$ is the so-called exceptional divisor. Hence, the following diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \pi_+^*(H^j(\mathcal{N}^k)) \\
\cong & \uparrow & \cong \\
0 & \rightarrow & \pi_+^*(H^j(\mathcal{N}^{k+1})) \\
\end{array}
$$

commutes for all $j \leq n(k)$, and the theorem follows.

**Corollary 4.6.** There is an isomorphism

$$i_k^* : H^j(\mathcal{N}_{\sigma_2(k + 1)}, \mathbb{Z}) \cong H^j(\mathcal{N}_{\sigma_2(k)}, \mathbb{Z}) \quad \forall j \leq \tilde{n}(k)$$

where $\tilde{n}(k) := \min(n(k), 2d_M - \tilde{d}_1 + g - 2)$.

Proof. Recall that $\mathcal{N}^k = \mathcal{N}_{\sigma_2(k)}$ is also the blow-up of $\mathcal{N}^k_+ = \mathcal{N}_{\sigma_2(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ along the flip locus $S_+^k = S_{\sigma_2}(k)$, so there is a map $\pi_+ : \mathcal{N}^k \to S_+^k$ such that

$$0 \to \pi_+^*(H^j(\mathcal{N}^k)) \to H^j(\mathcal{N}^k) \to H^j(\mathcal{E}^k) / \pi_+^*(H^j(S_+^k)) \to 0$$

splits:

$$H^j(\mathcal{N}^k) = \pi_+^*(H^j(\mathcal{N}^k)) \oplus H^j(\mathcal{E}^k) / \pi_+^*(H^j(S_+^k)),$$

and by Theorem 4.2 and Theorem 4.5, the result follows.

**Corollary 4.7.** There is an isomorphism

$$i_k^* : H^j(\mathcal{N}_{\sigma_2(k + 1)}, \mathbb{Z}) \cong H^j(\mathcal{N}_{\sigma_2(k)}, \mathbb{Z}) \quad \forall j \leq \tilde{n}(k).$$
4.2 Cohomology of the \((1, 2)\)-VHS

So far, we stabilize the cohomology of \(\mathcal{N}_{\sigma, (k)}\) for any critical \(\sigma, (k)\). The following results will allow us to generalize the stabilization for all \(\sigma \in I = [\sigma_m (k), \sigma_M (k)]\):

**Theorem 4.8** ([5, Th. 7.7]). Assume that \(r_1 > r_2\) and \(\frac{d_1}{r_1} > \frac{d_2}{r_2}\). Then the moduli space \(\mathcal{N}^s_L = \mathcal{N}^s_L (r_1, r_2, d_1, d_2)\) is smooth of dimension 

\[
(g - 1)(r_1^2 + r_2^2 - r_1 r_2) - r_1 d_2 + r_2 d_1 + 1,
\]

and is birationally equivalent to a \(\mathbb{P}^N\)-fibration over \(M^s (r_1 - r_2, d_1 - d_2) \times M^s (r_2, d_2)\), where \(M^s (r, d)\) is the moduli space of stable bundles of degree \(r\) and degree \(d\), and 

\[
N = r_2 d_1 - r_1 d_2 + r_1 (r_1 - r_2) (g - 1) - 1.
\]

In particular, \(\mathcal{N}^s_L (r_1, r_2, d_1, d_2)\) is non-empty and irreducible.

If \(\text{GCD} (r_1 - r_2, d_1 - d_2) = 1\) and \(\text{GCD} (r_2, d_2) = 1\), the birational equivalence is an isomorphism.

Moreover, in all cases, \(\mathcal{N}_L = \mathcal{N}_L (r_1, r_2, d_1, d_2)\) is irreducible and hence, birationally equivalent to \(\mathcal{N}^s_L\).

Here and after, \(\sigma_L\) represents the largest critical value in the open interval \([\sigma_m, \sigma_M]\), and \(\mathcal{N}_L\) (respectively \(\mathcal{N}^s_L\)) denotes the moduli space of \(\sigma\)-polystable (respectively \(\sigma\)-stable) triples for \(\sigma_L < \sigma < \sigma_M\). \(\mathcal{N}_L\) is so-called the ‘large \(\sigma\)’ moduli space (see [5]).

**Theorem 4.9** ([5, Th. 7.9]). Let \(\sigma\) be any value in the range \(\sigma_m < 2g - 2 \leq \sigma < \sigma_M\), then \(\mathcal{N}^s_L\) is birationally equivalent to \(\mathcal{N}^s_L\). In particular it is non-empty and irreducible.

**Corollary 4.10** ([5, Cor. 7.10]). Let \((r, d) = (r_1, r_2, d_1, d_2)\) be such that 

\[
\text{GCD} (r_2, r_1 + r_2, d_1 + d_2) = 1.
\]

If \(\sigma\) is a generic value satisfying \(\sigma_m < 2g - 2 \leq \sigma < \sigma_M\), then \(\mathcal{N}_\sigma\) is birationally equivalent to \(\mathcal{N}_L\), and in particular it is irreducible.

**Proof.** \(\mathcal{N}_\sigma = \mathcal{N}^s_L\) if \(\text{GCD} (r_2, r_1 + r_2, d_1 + d_2) = 1\) and \(\sigma\) is generic. In particular, \(\mathcal{N}_L = \mathcal{N}^s_L\), and the result follows from the last theorem. The reader may see the full details in [5].

**Theorem 4.11.** There is an isomorphism 

\[
i^j_\ast : H^j (\mathcal{N}^{k+1}_{\sigma_H}, \mathbb{Z}) \rightarrow H^j (\mathcal{N}^k_{\sigma_H}, \mathbb{Z}) \quad \forall j \leq \tilde{n} (k)
\]

where \(\mathcal{N}^k_{\sigma_H} = \mathcal{N}_{\sigma_H (k)} (2, 1, \tilde{d}_1, \tilde{d}_2)\), \(\sigma_H = \sigma_H (k) = 2g - 2 + k\), and 

\[
\tilde{n} (k) = \min (\tilde{d}_1 - d_M - \tilde{d}_2 - 1, 2 (\tilde{d}_1 - 2 \tilde{d}_2 - (2g - 2)) + 1, 2d_M - \tilde{d}_1 + g - 2)
\]

as above mentioned.

**Proof.** In this case \(\text{GCD} (1, 3, \tilde{d}_1 + \tilde{d}_2) = 1\) trivially, and \(\sigma_H = \sigma_H (k)\) satisfies 

\[
\sigma_m < 2g - 2 \leq \sigma_H (k) < \sigma_M.
\]

Hence, by Theorem 4.9 and Corollary 4.10 \(\mathcal{N}^{k+1}_{\sigma_H} = \mathcal{N}_{\sigma_H (k)} (2, 1, \tilde{d}_1, \tilde{d}_2)\) is birationally equivalent to \(\mathcal{N}_{L (k)} = \mathcal{N}_{L (k)} (2, 1, \tilde{d}_1, \tilde{d}_2)\), which is equal to \(\mathcal{N}^s_{L (k)} = \mathcal{N}^s_{L (k)} (2, 1, \tilde{d}_1, \tilde{d}_2)\) also by Theorem 4.9, where \(L (k)\) is the maximal critical value, depending on \(k\) in this case. The isomorphism then follows by Corollary 4.7.

**Corollary 4.12.** There is an isomorphism 

\[
i_\ast : H^j (F_{d_1}^{k+1}, \mathbb{Z}) \rightarrow H^j (F_{d_1}^k, \mathbb{Z})
\]

for all \(j \leq \tilde{n} (k)\) induced by the embedding 1.5.
M.S. of Triples and (1,2)-VHS

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References

[1] M. F. Atiyah, *K-Theory*, W. A. Benjamin, New York–Amsterdam, 1967.

[2] M. F. Atiyah and R. Bott, “Yang–Mills equations over Riemann surfaces”, *Phil. Trans. Roy. Soc. London A* 308 (1982), 523–615.

[3] Bento, S., “Topologia do Espaço Moduli de Fibrados de Higgs Torcidos”, Tese de Doutoramento, Universidade do Porto, Porto, Portugal, 2010.

[4] S. B. Bradlow and O. García-Prada, “Stable triples, equivariant bundles and dimensional reduction”, *Math. Ann.* 304 (1996), 225–252.

[5] S. B. Bradlow, O. García-Prada and P. B. Gothen, “Moduli spaces of holomorphic triples over compact Riemann surfaces”, *Math. Ann.* 328 (2004), 299–351.

[6] S. B. Bradlow, O. García-Prada and P. B. Gothen, “Homotopy groups of moduli spaces of representations”, *Topology* 47 (2008), 203–224.

[7] T. Frankel, “Fixed points and torsion on Kähler manifolds”, *Ann. Math.* 70 (1959), 1–8.

[8] W. Fulton, *Algebraic Topology, A First Course*, Springer, New York, 1995.

[9] P. B. Gothen, “The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface”, *Int. J. Math.* 5 (1994), 861–875.

[10] P. B. Gothen and R. A. Zúñiga-Rojas, “Stratifications on the moduli space of Higgs bundles”, *Portugaliae Mathematica*, EMS, 74 (2017), 127-148.

[11] P. Griffiths, and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[12] G. Harder and M. S. Narasimhan, “On the cohomology groups of moduli spaces of vector bundles on curves”, *Math. Ann.* 212 (1975), 215–248.

[13] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[14] A. Hatcher, *Vector Bundles and K-Theory*, unpublished. (http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html)

[15] T. Hausel, “Geometry of Higgs bundles”, Ph.D. thesis, Cambridge, 1998.
[16] T. Hausel and M. Thaddeus, “Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles”, *Proc. London Math. Soc.* **88** (2004), 632–658.

[17] T. Hausel and M. Thaddeus, “Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles”, *J. Amer. Math. Soc.* **16** (2003), 303–329.

[18] N. J. Hitchin, “The self-duality equations on a Riemann surface”, *Proc. London Math. Soc.* **55** (1987), 59–126.

[19] D. Husemoller, *Fibre Bundles*, third edition, Graduate Texts in Mathematics **20**, Springer, New York, 1994.

[20] I. M. James, ed., *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995.

[21] I. G. Macdonald, “Symmetric products of an algebraic curve”, *Topology* **1** (1962), 319–343.

[22] E. Markman, “Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces”, *J. reine angew. Math.* **544** (2002), 61–82.

[23] E. Markman, “Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces”, *Adv. Math.* **208** (2007), 622–646.

[24] V. Muñoz, A. Oliveira and J. Sánchez, “Motives and the Hodge Conjecture for the Moduli Spaces of Pairs”, *Asian Journal of Mathematics*, Vol. **19** (2015), 281–306.

[25] V. Muñoz, D. Ortega and M. J. Vázquez-Gallo, “Hodge polynomials of the moduli spaces of pairs”, *Int. J. Math.* **18** (2007), 695–721.

[26] N. Nitsure, “Moduli space of semistable pairs on a curve”, *Proc. London Math. Soc.* **62** (1991), 275–300.

[27] S. S. Shatz, “The decomposition and specialization of algebraic families of vector bundles”, *Compos. Math.* **35** (1977), 163–187.

[28] C. T. Simpson, “Constructing variations of Hodge structures using Yang–Mills theory and applications to uniformization”, *J. Amer. Math. Soc.* **1** (1988), 867–918.

[29] C. T. Simpson, “Higgs bundles and local systems”, *Publ. Math. IHÉS* ?, (1992), 5–95.

[30] R.A. Zúñiga-Rojas, “Stabilization of the Homotopy Groups of The Moduli Space of k-Higgs Bundles”, to appear in *Revista Colombiana de Matemáticas*. *arXiv:1702.07774v3[mathAT]*

[31] R. A. Zúñiga-Rojas, “Homotopy groups of the moduli space of Higgs bundles”, Ph. D. thesis, Porto, 2015.