Aligning Reference Frames Using Quantum States

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We analyze the problem of sending, in a single transmission, the information required to specify an orthogonal trihedron or reference frame through a quantum channel made out of $N$ elementary spins.

We analytically obtain the optimal strategy, i.e., the best encoding state and the best measurement. For large $N$, we show that the average error goes to zero linearly in $1/N$. Finally, we discuss the construction of finite optimal measurements.

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Can a system of $N$ elementary spins be used to communicate in a single transmission the orientation of three mutually orthogonal unit vectors (orthogonal trihedron)? A positive answer would, e.g., enable two distant parties (Alice and Bob) to establish a common reference frame using just a quantum channel. This question was addressed twenty years ago by Holevo [1] who concluded that if such a quantum system has a well defined total spin $J$ the best the sender (Alice) can attempt to achieve is to transmit the orientation of at most one of the three vectors. There has recently been renewed interest in this simpler, more manageable, problem of sending a single direction, and reformulations and extensions of the original question abound in the literature [2, 3, 4, 5, 6, 7, 8, 9, 10] (related issues can also be found in [11]). In all the cases, optimal communication involves collective (entangled) measurements and an accurate choice of the messenger quantum states.

In this letter, we will be concerned with the more complex problem of sending the information that specifies an orthogonal trihedron (OT). We will demonstrate that by encoding the relevant geometrical information in a particular class of states one overcomes the limitations foreseen by Holevo and a good transmission is possible. These states can be written as a simple superposition of states belonging to each of the irreducible representations of SU(2) that appear in the Hilbert space of the $N$ spins. They have maximal third component of the total spin or $J_z$. The quality of the optimal communication strategy is shown to increase with $N$ and in the limit $N \to \infty$ the average error, $\langle h \rangle$, goes to zero. For large $N$ we obtain an analytical estimate of this error, $\langle h \rangle \gtrsim 8/N$. We would like to emphasize that despite the apparent difficulty of the problem [12], an analytical treatment is possible, which provides us with a physical insight of the underlying quantum aspects involved in the communication process.

Let us suppose Alice has a system of $N$ spins which she wishes to use to tell Bob an OT, $n = \{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. By performing quantum measurements, Bob will be able to reconstruct this OT with some accuracy and will make the guess $n' = \{\vec{n}'_1, \vec{n}'_2, \vec{n}'_3\}$. The obvious parametrization of the different OTs is provided by the Euler angles $\alpha, \beta, \gamma$, of the rotations that map $n_0 = \{\vec{x}, \vec{y}, \vec{z}\}$ into $n, n'$. We will use $g$ as a shorthand for the three Euler angles, i.e., $g = (\alpha, \beta, \gamma)$. Following Holevo [1], we may quantify the quality of the communication strategy by evaluating the mean value of the error (or average error) defined for each individual measurement by

$$h(g, g') = \sum_{a=1}^{3} |\vec{n}_a - \vec{n}'_a|^2 = \sum_{a=1}^{3} |\vec{n}_a(g) - \vec{n}_a(g')|^2.$$  \hspace{1cm} (1)

Assuming the OT are chosen from an isotropic distribution, and denoting by $p_{g'}(g)$ the conditional probability of Bob guessing $n(g')$ if Alice’s OT is $n(g)$, one has

$$\langle h \rangle = \int dg \int dg' h(g, g') p_{g'}(g),$$ \hspace{1cm} (2)

where $dg$ is the Haar measure of the rotation group, SU(2), which in terms of the Euler angles reads $dg = \sin \beta d\beta d\alpha d\gamma / 8\pi^2$. Covariance implies that (3) can be written as

$$\langle h \rangle = \int dg h(g, 0) p_{0}(g),$$ \hspace{1cm} (3)

where $0$ stands for $(\alpha, \beta, \gamma) = (0, 0, 0)$. One can easily check that

$$h(g, 0) = 6 - 2tr U(1)(g),$$ \hspace{1cm} (4)

where $U(1)$ is the SU(2) irreducible representation of spin $j$, whose elements we write as $\Sigma_{mm'}(j)(g) = \langle j, m | U(1)(g) | j, m' \rangle$. One also has $t \equiv tr U(1)(g) = \sum_m \Sigma_{mm}(j)(g) = \cos \beta (1 + \cos \beta) \cos (\alpha + \gamma)$. We see that the values of $t$ lay in the real interval $[-1, 3]$. The value $t = 3$ corresponds to perfect determination of Alice’s OT and implies that $\langle h \rangle = 0$. Note also that $\langle h \rangle = 6 - 2t$. Random guessing implies $t = 0$ ($\langle h \rangle = 6$), while perfect determination of one axis and random guessing of the remaining two imply $t = 1$ ($\langle h \rangle = 4$).
The most general quantum state Alice can use has the form $|A(g)\rangle = U(g)|A\rangle$. Here $U(g) = \bigotimes_j U_j^{(j)}$ and

$$|A\rangle = \sum_j |A_j\rangle = \sum_{j,m} A_{m}^{j}|j, m\rangle; \quad \sum_{j,m} |A_{m}^{j}|^2 = 1, \quad (5)$$

where $j$ runs from 0 to $N/2$ (for simplicity we will only consider $N$ even) and $m$ runs from $-j$ to $j$. $|A\rangle$ is a fixed reference state associated with the OT $n_0$.

Likewise, we may write a reference state $|B\rangle$ from which we can construct Bob’s projectors of his Positive Operator Valued Measurement (POVM) $\{p_j\}$. The general form of the state is

$$|B\rangle = \sum_j \sqrt{2j+1}|B_j\rangle; \quad |B_j\rangle = \sum_m B_{m}^{j}|j, m\rangle, \quad (6)$$

where the square root is introduced for later convenience, and the projectors are

$$O(g) = U(g)|B\rangle\langle B|U\dagger(g). \quad (7)$$

We will first consider continuum POVMs for simplicity but finite ones can also be constructed, as will be explained below. The condition $\mathbb{I} = \int dg\ O(g)$ requires that

$$\sum_{m=-j}^{j} |B_{m}^{j}|^2 = 1, \quad \forall j, \quad (8)$$

as can be easily shown with the help of the orthogonality relations

$$\int dg\ D_{MM'}^{(j)}(g)D_{M'M'}^{(j')}\dagger(g) = \frac{\delta_{jj'}\delta_{MM'}\delta_{mm'}}{2j+1}. \quad (9)$$

Quantum Mechanics tells us that $p_0(g) = \langle\langle B|U(g)|A\rangle\rangle^2$, hence we have

$$\langle t\rangle = \int dg\ |\langle B|U(g)|A\rangle|^2 \text{tr} U(1). \quad (10)$$

In terms of the components of $|A\rangle$ and $|B\rangle$ the last expression reads

$$\langle t\rangle = \sum_{ijj'} \sum_{mnn'} A_{n}^{i} A_{n'}^{i} B_{m}^{j} B_{m'}^{j'} M_{mm'n'n'}^{ijj'}, \quad (11)$$

where

$$M_{mm'n'n'}^{ijj'} = \sqrt{(2l+1)(2j+1)} \times \int dg\ \text{tr} U(1)\ D_{Mm}^{(i)}(g)D_{M'm'}^{(j')}\dagger(g)$$

$$= \sqrt{(2l+1)(2j+1)} \times \sum_{m} \langle 1Mjm|m\rangle \langle 1Mjm'|m'\rangle, \quad (12)$$

and the last terms in brackets are the usual Clebsch-Gordan coefficients.

The optimal strategy is the one that maximizes $\langle t\rangle$. It is tempting to introduce Lagrange multipliers $\lambda$ and $\mu^j$ for the normalization constraints $[3]$ and $[8]$ respectively and follow the standard maximization procedure. Analytical results along this line seem hard to obtain $[12]$. We will, thus, try to develop a more physical picture of Eqs. $[9]$, $[12]$ which will lead us to a stunning simplification of the problem.

Notice that Eqs. $[9]$, $[12]$ can also be written in a compact form as

$$\langle t\rangle = \sum_{ij} \sum_{l} \frac{\sqrt{(2l+1)(2j+1)}}{3} \langle B_{l}^{j}\tilde{B}_{l}^{j'}|P_{1}|A_{l}^{j'}\tilde{A}_{l}^{j}\rangle, \quad (13)$$

where $|A_{l}^{j'}\tilde{A}_{l}^{j}\rangle = |A_{l}^{j}\otimes|\tilde{A}_{l}^{j}\rangle$, the state $|\tilde{A}_{l}\rangle$ is the time reversed of $|A_{l}\rangle$, i.e., $A_{m}^{l} = (-1)^{m} A_{m}^{l}$ (and similarly for $|B_{l}^{j}\tilde{B}_{l}^{j}\rangle$ and $P_{1}$) and $P_{1}$ is the projector over the Hilbert space of the representation of total spin $J = 1$. Our aim is to compute

$$\langle t\rangle_{\text{max}} = \max_{AB} \langle t\rangle, \quad (14)$$

where the maximization is over all $A_{m}^{l}$ and $B_{m}^{l}$ subject to the normalization conditions in $[3]$ and $[8]$. The Schwarz inequality implies

$$\langle B_{l}^{j}\tilde{B}_{l}^{j'}|P_{1}|A_{l}^{j'}\tilde{A}_{l}^{j}\rangle \leq ||P_{1}|A_{l}^{j'}\tilde{A}_{l}^{j}\rangle|| ||P_{1}|B_{l}^{j}\tilde{B}_{l}^{j}\rangle||, \quad (15)$$

where the equality holds iff

$$P_{1}|A_{l}^{j'}\tilde{A}_{l}^{j}\rangle = \mu^{j} P_{1}|B_{l}^{j}\tilde{B}_{l}^{j}\rangle \quad \forall j, l. \quad (16)$$

Hence, to compute $\langle t\rangle_{\text{max}}$, we can restrict ourselves to a smaller parameter space, where $|A_{l}\rangle$ and $|B_{l}\rangle$ are constrained through $[9]$. This is equivalent to consider the states $|A\rangle$ such that

$$A_{m}^{l} = C_{m}^{l} B_{m}^{l}, \quad \text{with} \quad \sum_{j} |C_{j}|^2 = 1, \quad (17)$$

i.e., we only need to consider the set of parameters $\{C_{j}, B_{j}\}$. This we can prove, e.g., by induction on $j$ using $[10]$ with $l = j + 1$ and starting with the trivial case $j = 0$ $[13]$. Eq. $[17]$ is easy to understand from the physical point of view. It just tells us that, for an optimal communication, the messenger states $|A(g)\rangle$ must be as similar as possible to the states $|B(g)\rangle$ on which the measuring device projects $[6]$. We next substitute back in $[13]$ to obtain

$$\langle t\rangle_{\text{max}} = \max_{BC} \sum_{ij} C_{i} M_{B}^{ijj'} C_{j'}, \quad (18)$$

where

$$M_{B}^{jj'} = \frac{\sqrt{(2j+1)(2j'+1)}}{3} \langle B_{j}^{j'}\tilde{B}_{j}^{j'}|P_{1}|B_{j}^{j'}\tilde{B}_{j}^{j'}\rangle \quad (19)$$
and the maximization is over all $B_i^j$ and $C^j$ subject to the normalizations \( [8] \) and \( [17] \).

Let us now discuss some properties of the matrix $M_B$ defined by \( [19] \). We first note that $M_B$ is tridiagonal, i.e., $M_B^{jj'} = 0$ if $|j - j'| > 1$, and symmetric. It is manifestly non-negative, i.e., $M_B^{jj'} \geq 0$ for all $j, j'$ and, most importantly, it is rotationally invariant: any reference state of the form $|B'\rangle = U(g)|B\rangle$ is equally as good as $|B\rangle$.

We next compute bounds for the diagonal ($M_B^{jj}$) and off diagonal ($M_B^{j+1,j}$) entries of $M_B$. We have

\[
0 \leq M_B^{jj} = \frac{2j+1}{3} \langle (B^j B^j)|10\rangle^2 \leq \frac{2j+1}{3} \\
\times \left[ \sum_{m'} |B_{m'}^j|^2 \max_m \langle jm|j-10\rangle \right] = \frac{j}{j+1},
\]

where we have used rotational invariance to orient the (real) vector $P_j |B^j B^j\rangle$ along the $z$ ($m = 0$) axes. As for the off diagonal entries, the Schwarz inequality leads to

\[
0 \leq M_B^{j+1,j} = \frac{\sqrt{(2j+1)}[2j+3]}{3} \sum_{m''} |B_{m''}^j|^2 \\
\times \max_{M} \left( \sum_{M} (jM - j + 1 m|M) \right) = \sqrt{\frac{2j+1}{2j+3}},
\]

where, actually, the sum over $M$ in the second line is independent of $m$. It is straightforward to verify that the particular choice

\[
|B_{\text{op}}\rangle = \sum_j \sqrt{2j+1} |j,j\rangle \iff B_{\text{op}m}^j = \delta_{m}^j
\]

saturates the two upper bounds \( [20] \) and \( [21] \) simultaneously. Hence

\[
|M_B^{jj'}| \leq M_B^{jj'} = M_B^{jj'},
\]

for all $j, j'$ and $|B\rangle$. The matrix $M_B^{jj'}$ is

\[
M_{\text{op}} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \sqrt{\frac{2j-1}{2j+1}} & \cdots & 0 \\
\sqrt{\frac{2j-1}{2j+1}} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \frac{1}{\sqrt{3}} & \sqrt{\frac{2j-1}{2j+1}} \\
0 & \cdots & \frac{1}{\sqrt{3}} & \sqrt{\frac{2j-1}{2j+1}} \\
\end{pmatrix},
\]

\( J = N/2 \) is the maximum spin of the system.

We now go back to \( [13] \) and compute $\langle t \rangle_{\text{max}}$. We first note that, $\langle t \rangle_{\text{max}} = \max_B \lambda(B)$, where $\lambda(B)$ is the maximal eigenvalue of the matrix $M_B$. Since it is non-negative, Eq. \( [23] \) implies \( [4] \)

\[
\langle t \rangle_{\text{max}} = \max_B \lambda(B) = \lambda(B_{\text{op}}) \equiv \lambda_{\text{op}}.
\]

We thus have simplified the problem to that of computing $\lambda_{\text{op}}$, the maximal eigenvalues of $M_{\text{op}}$ in \( [24] \). This can be done proceeding along the same lines as in \( [1] \).

We would like to emphasize that the calculation relies on the fact that the maximal value of each entry of $M_B$ is reached simultaneously, e.g. for the single state $|B_{\text{op}}\rangle$. This is, a priori, a rather unexpected property which, however, provides a remarkable simplification of the calculation.

The result obtained and the form of the optimal state, $|B_{\text{op}}\rangle$, agree with our physical intuition as we now briefly discuss. If Alice’s state has a well defined total spin (i.e. it is an eigenstate of $J^2$), $M_B$ becomes diagonal and $\langle t \rangle_{\text{max}} = J/(J+1) = N/(N+2)$. In terms of the average error, $\langle h \rangle = 4(N+3)/(N+2)$, thus, at most $(N \to \infty)$ $\langle h \rangle = 4$. In average, Bob cannot determine more than just one axes of Alice’s trihedron. The structure of the state $|B_{\text{op}}\rangle$ is such that, within each irreducible representation, the determination of a single axes is optimal \( [3] \) (this is the best Alice could do if she only was allowed to use a single irreducible representation). At the same time, $|B_{\text{op}}\rangle$ is as different of an eigenstate of $J_z$ as it can possibly be (if $J_{\text{op}}|B_{\text{op}}\rangle \propto |B_{\text{op}}\rangle$, Alice would be able to communicate only a single axes).

For small $N$, one can easily obtain analytic expressions for $\langle t \rangle_{\text{max}}$ (see table). For large $N$ it suffices to give simple lower and upper bounds for $\langle t \rangle_{\text{max}}$. A useful upper bound is provided by the condition $\langle t \rangle_{\text{max}} \leq \max_j \sum_{j'} M_{\text{op}}^{jj'}$. A lower bound is obtained computing $\Delta = \sum_{jj'} C^j M_{\text{op}}^{jj'} C^{j'}$ for any normalized vector with components $C^j$.

A judicious choice is $C^j \propto \sqrt{2j-1}(N/2-j)^j$. The maximum of $\Delta$ occurs at $p \approx \sqrt{3}/4$. We obtain

\[
3 - \frac{4}{N} + O(N^{-4/3}) \lesssim \langle t \rangle_{\text{max}} \lesssim 3 - \frac{4}{N} + O(N^{-2}).
\]

It is now clear that perfect determination of the trihedron, $\langle t \rangle_{\text{max}} = 3$, is reached in the asymptotic limit, and $\langle t \rangle$ approaches three at most linearly in $1/N$. Finally, we have performed a linear fit obtaining

\[
\langle t \rangle_{\text{max}} \sim 3 - \frac{4}{N} - \frac{9.4}{N^{4/3}} + \ldots,
\]

which is completely consistent with \( [26] \).

We now turn our attention to the construction of POVM’s with a finite number of outcomes, as they are the only ones that can be physically realized. The main idea is stated in \( [11] \). There, we introduced the concept of set of directions isotropically distributed. In the context of the present letter the term directions has to be generalized to elements of the group. We say that a finite set
Notice that the set of points \((g_r), r = 1, \cdots, N(J)\), of elements of SU(2) is isotropically distributed up to spin \(J\), if there exist positive weights \(\{c_r\}\) such that the following orthogonality relation holds for any \(j, j' \leq J\):

\[
\sum_{r=1}^{N(J)} c_r D_m^{(j)}(g_r) D_{m'}^{(j')}(g_r) = \frac{C_j}{2j + 1} \delta_m^m' \delta_n^n' \delta_j^{j'},
\]

where \(C_j = \sum_{r=1}^{N(J)} c_r\). This discrete version of (23) is only valid up to a certain value \(J\), the larger \(J\) is, the larger \(N(J)\) must be chosen. Working along the same lines as in [10] one can show that the angular dependence on \(\alpha\) and \(\gamma\) can be trivially satisfied choosing \(N + 1\) equidistant angles for each variable. The only non-trivial conditions concern the set \(\{\beta_r\}\), which is required to satisfy \(\sum_r c_r P_L(\cos \beta_r) = 0\) (\(1 \leq L \leq 2J\)), where \(P_L\) is the Legendre polynomial of degree \(L\). The procedure to solve this equation is described in [10] (see also [12]). This recipe yields a finite optimal POVM for any value of \(N\). In general, however, one can find equally optimal POVM’s with a smaller number of outcomes. Ideally one would be interested in finding the minimal ones, however, as far as we are aware, the solution is not known for arbitrary \(J\) and general groups [13].

Nevertheless, the minimal POVM for the first non-trivial case of two spins is not difficult to find. Consider the simplest normalized reference state that leads to an optimal POVM:

\[
|B\rangle = \frac{\sqrt{3}}{2}|1, 1\rangle + \frac{1}{2}|0, 0\rangle,
\]

It is easy to verify that the four projectors \(O_r = U(g_r)|B\rangle\langle B|U^\dagger(g_r)\), with

\[
\alpha_r = (r - 1) \frac{\pi}{2}, \quad \gamma_r = \pi - \alpha_r, \quad \cos \theta_r = \frac{1}{r}, \quad r \leq 3
\]

\[
\alpha_4 = 0, \quad \gamma_4 = 0, \quad \cos \theta_4 = 1, \quad (30)
\]

satisfy the POVM condition \(\sum_{r=1}^{4} O_r = I\). Since the Hilbert space has dimension four, the minimal number of outcomes for any measurement is also four. This measurement is therefore finite, minimal, and optimal. In fact, it is a von Neumann measurement as \(O_i O_i = \delta_{ii} O_i\).

Notice that the set of points \((g_r)\) do not satisfy the orthogonality conditions (28) for all the \(m, m'\) values, but it does for the relevant ones. It is the particular structure of the state (29) what enables us to construct a POVM with only four outcomes.

We conclude that it is feasible to use quantum systems to encode the orientation of a reference frame. The optimal strategy involves the use of encoding states which are remarkably simple and have a clear physical interpretation. The average error of the transmission is seen to approach zero linearly in \(1/N\). Finally, we give a recipe for constructing finite optimal POVMs and present an example of a minimal one for the simple case \(N = 2\).

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\[\begin{array}{ccccccc}
N & 2 & 3 & 5 & 10 & 50 & 100 \\
(t)_{\max} & \frac{2\pi}{12} & \frac{2\pi}{14} & \frac{2\pi}{16} & \frac{2\pi}{30} & 1.6708 & 2.6202 & 2.9362 & 2.9707
\end{array}\]

**TABLE I:** Maximal value of \(t\) vs. the number of spins \(\{g_r\}, r = 1, \cdots, N(J)\), of elements of SU(2) is isotropically distributed up to spin \(J\), if there exist positive weights \(\{c_r\}\) such that the following orthogonality relation holds for any \(j, j' \leq J\):