ON COMMON INDEX DIVISORS AND MONOGENITY OF CERTAIN NUMBER FIELDS DEFINED BY TRINOMIALS OF TYPE $x^{2r} + ax^m + b$

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Abstract. Let $K = \mathbb{Q}(\theta)$ be a number with $\theta$ a root of an irreducible trinomial of type $F(x) = x^{2r} + ax^m + b \in \mathbb{Z}[x]$. In this paper, based on the $p$-adic Newton polygon techniques applied on decomposition of primes in number fields and the classical index theorem of Ore [17, 32], we study the monogenity of $K$. More precisely, we prove that if $a$ and $1+b$ are both divisible by 32, then $K$ cannot be monogenic. For $m = 1$, we provide explicit conditions on $a$, $b$ and $r$ for which $K$ is not monogenic. We also construct a family of irreducible trinomials which are not monogenic, but their roots generate monogenic number fields. To illustrate our results, we give some computational examples.

1. Introduction and statements of results

Let $K$ be a number field generated by a root $\theta$ of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$ of degree $n$ and $\mathbb{Z}_K$ its ring of integers. The field $K$ is called monogenic if there exists a primitive element $\eta \in \mathbb{Z}_K$ such that $\mathbb{Z}_K = \mathbb{Z}[\eta]$, that is $(1, \eta, \ldots, \eta^{n-1})$ is an integral basis (called a power integral basis) in $K$. The problem of studying the monogenity of number fields and constructing power integral bases is one of the most important problems in algebraic number theory. This problem is intensively studied by several researchers in the last four decades (cf. [2, 7, 8, 10, 13, 14, 19, 20, 27, 28, 33]).

In a series of his papers [19, 20, 22], Győry provided the first general algorithms for deciding whether $K$ is monogenic or not and for determining all power integral bases in $\mathbb{Z}_K$. He also studied in [21] and [23] the question of monogenity in relative extensions. Further, he succeeded to reduce index form equations to system of unit equations (see [24]).

For any element $\eta$ of $\mathbb{Z}_K$, let $\text{ind}(\eta)$ denote the index of the subgroup $\mathbb{Z}[\eta]$ in $\mathbb{Z}_K$. The index of $K$ is defined as follows:

$$i(K) = \gcd \{\text{ind}(\eta) \mid \eta \in \mathbb{Z}_K \text{ and } K = \mathbb{Q}(\eta)\}.$$ 

A prime $p$ dividing $i(K)$ is called a prime common index divisor of $K$. Remark that if $K$ is monogenic, then $i(K) = 1$. Thus, a field possessing a prime common index divisor is not monogenic.

The existence of common index divisor was first established by Dedekind. He show that the cubic number field $K = \mathbb{Q}(\theta)$, where $\theta$ is a root of $x^3 + x^2 - 2x + 8$ cannot be monogenic, since the prime 2 splits completely in $\mathbb{Z}_K$ (see e.g. [30, page 64]). There is an extensive literature on indices of number fields, see e.g [10] by Dedekind, [11] by Engstrom and [31] by Nart.

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In [28], Llorente and Nart studied the index of cubic number fields defined by $x^3+ax+b$. In [10], Davis and Spearman calculated the index of the quartic number field defined by $x^4+ax+b$. In [16], Gaál, Pethő and Pohst gave efficient algorithm for quartic number fields. In [34], Pethő and Ziegler gave an efficient criterion to decide whether the maximal order of a biquadratic field has a unit power integral basis or not. In [33], Pethő and Ziegler studied indices in multiquadratic number fields.

Combining a refined version [24] of the general approach of [19, 20] with an efficient reduction and enumeration algorithms, Gaál and Győry [15], Bilu, Gaál and Győry [7] described algorithms to solve index form equations in quintic resp sextic fields. In [8], Bérczes, Evertse and Győry studied multiply monogenic orders. The books [12, 13] by Bérczes, Evertse and Győry gave detailed surveys on the discriminant form and index form theory. In [26], Jones gave infinite families of non-monogenic trinomials. Also in [27], Jones and Phillips identify classes of monogenic trinomials.

The purpose of this paper is to study the monogenity of the number field $K = \mathbb{Q}(\theta)$ with $\theta$ a root of an irreducible trinomial of type $F(x) = x^3 + ax^n + b \in \mathbb{Z}[x]$. Recall that in [6], Ben Yakkou and El Fadil studied the non-monogenity of number fields defined by $x^n + ax + b$. More precisely, they gave sufficient conditions on $a, b$ and $m$ for which $K$ admits an odd prime common index divisor. Theses results were generalized in [4] by Ben Yakkou for number fields defined by $x^n + ax^m + b$. Also, in [25], Jakhar, Khanduja and Sangwan studied the problem of the integral closedness of $\mathbb{Z}[\theta]$: they gave necessary and sufficient conditions for a prime $p$ to be a divisor of the index $\text{ind}(\theta)$. However, by Definition of $i(K)$, the divisibility of $\text{ind}(\theta)$ by $p$ is not sufficient to decide if $p$ is a common index divisor of $K$ or not. Therefore, their results does not characterize the prime divisors of indices of these number fields. Therefore, the results obtained in [4, 6, 24] cannot give a complete answer about the monogenity of number fields defined by $x^n + ax^m + b$. For this reason, we have chosen to study this special case separately.

Throughout this paper, for any prime $p$ and any $t \in \mathbb{Z}$, $\nu_p(t)$ stands for the $p$-adic valuation of $t$ and $t_p := \frac{t}{p^{\nu_p(t)}}$. We recall also that the discriminant of the trinomial $F(x) = x^n + ax^m + b$ is

$$\Delta(F) = (-1)^{\frac{n(n-1)}{2}} b^{m-1} (n^{n_1} b^{n_1-m_1} - (-1)^{m_1} m^{m_1} (m-n)^{n_1-m_1} d_0),$$

where $d_0 = \gcd(n, m)$, $n_1 = \frac{n}{d_0}$ and $m_1 = \frac{m}{d_0}$. Recall also that the polynomial $F(x)$ is called monogenic if $\mathbb{Z}_K = \mathbb{Z}[\theta]$. It is important to note that the monogenity of the polynomial $F(x)$ implies the monogenity of the field $K$. But the converse is not true. Let us start with the following theorem, which is in a more general case. It gives infinite parametric families of irreducible non-monogenic trinomials with non-squarefree discriminant, but their roots generate monogenic number fields.
Theorem 1.1.
Let \( F(x) = x^n + ax^m + b \in \mathbb{Z}[x] \) be a monic polynomial with discriminant \( \Delta \). Suppose that there exist a prime \( p \) dividing both \( a \) and \( b \) such that \( \nu_p(b) \geq 2 \), \( \gcd(n, \nu_p(b)) = 1 \), \( \nu_p(a) > (n - m)\nu_p(b) \), and \( \Delta_p \) is square free. Then \( F(x) \) is irreducible over \( \mathbb{Q} \). Let \( K = \mathbb{Q}(\theta) \) be a number field with \( \theta \) a root of \( F(x) \). Then \( F(x) \) is not monogenic \((\mathbb{Z}_K \neq \mathbb{Z}[\theta])\), but \( K \) is monogenic. Moreover, in this case, \( \mathbb{Z}_K = \mathbb{Z}[\alpha] \), with \( \alpha = \frac{a^m}{b^2} \), where \((s, t) \in \mathbb{N}^2\) is the unique positive solution of the Diophantine equation \( \nu_p(b)s - nt = 1 \) with \( 0 \leq s < n \).

Remark 1.2. Theorem 1.1 implies [6, Theorem 2.1], where the special case \( n = p^r \) and \( m = 1 \) is previously considered.

Corollary 1.3.
Let \( K = \mathbb{Q}(\theta) \) be a number field with \( \theta \) a root of a monic irreducible polynomial \( F(x) = x^r + ax^m + b \in \mathbb{Z}[x] \). Assume that there exist a prime \( p \) such that \( \nu_p(a) \geq \nu_p(b) \geq 3 \), \( \nu_p(b) \) is odd, and \( \Delta_p \) is square free. Then \( F(x) \) is irreducible over \( \mathbb{Q} \) and not monogenic. But, \( K \) is monogenic.

Example 1.4.
The polynomial \( F(x) = x^8 + 8x + 8 \in \mathbb{Z}[x] \) has discriminant \( \Delta = 2^{24} \times 1273609 \). Note that 1273609 is a prime. Then \( F(x) \) satisfies the conditions of Corollary 1.3 for \( p = 2 \). Hence, it is irreducible over \( \mathbb{Q} \). Let \( K = \mathbb{Q}(\theta) \) with \( \theta \) is a root of \( F(x) \). Then \( \mathbb{Z}_K \neq \mathbb{Z}[\theta] \). But, \( K \) is monogenic and \( \alpha = \frac{a^m}{2} \) generates a power integral basis of \( \mathbb{Z}_K \).

The following result gives infinite parametric families number fields defined by \( x^r + ax^m + b \) their indices are divisible by 2 which garantie the non monogenity of these fields.

Theorem 1.5.
Let \( F(x) = x^r + ax^m + b \) be a monic irreducible trinomial and \( K = \mathbb{Q}(\theta) \) a number fields generated by \( \theta \), a root of \( F(x) \). If \( r \geq 3 \) and \( a \) and \( b + 1 \) are both divisible by 32, then \( 2 \) is a prime common index divisor of \( K \). In particular, \( K \) is not monogenic.

Now, we focus on the case \( m = 1 \). The following Theorem gives sufficient conditions which guarantee the non monogenity of infinite parametric families of number fields defined by trinomials of type \( x^r + ax + b \).

Theorem 1.6.
Let \( K = \mathbb{Q}(\theta) \) be a number field with \( \theta \) root of a monic irreducible trinomial \( F(x) = x^r + ax + b \in \mathbb{Z}[x] \). If one of the following conditions holds

1. \( r \geq 3 \), \( a \equiv 4 \pmod{8} \) and \( b \equiv 3 \pmod{8} \).
2. \( r \geq 4 \), \( a \equiv 8 \pmod{16} \) and \( b \equiv 7 \pmod{16} \).
3. \( r \geq 3 \) and \( (a, b) \in \{(0,31),(16,15)\} \pmod{32} \),

then \( K \) is a prime common index divisor of \( K \). In particular, if one of these conditions holds, then \( K \) is not monogenic.

Remark 1.7. Theorem 1.3(1)(3) implies respectively [3, Theorem 2.3(1)(2)] when the special case \( r = 3 \) is previously studied.

When \( a = 0 \), the following result is an immediate consequence of the above theorem.
Corollary 1.8. If \( b \equiv 1 \pmod{32} \), then the pure number field \( \mathbb{Q}(\sqrt[3]{b}) \) is not monogenic for every natural integer \( r \geq 3 \).

Remark 1.9.
Note that [31, Corollary 2.4] is about trinomials of type \( x^{2r} - 3x + ax^m + b \). It gives sufficient conditions for the prime 3 to be a common index divisor of \( K \). Although formally it includes \( x^2 + ax^m + b \), all statements of Corollary 2.4 of [31] concerns the cases \( k \geq 1 \). Hence do not overlap with Theorems 1.5 and 1.6. Here, we gave sufficient conditions for which the prime 2 is a common index divisor of \( K \).

2. Newton polygons and the Index theorem of Ore

To prove our main results, we need some preliminaries that can be found in details in [3, 4]. For any \( \eta \in \mathbb{Z}_K \), it is well known from [30, Proposition 2.13] that

\[
D(\eta) = (\mathbb{Z}_K : \mathbb{Z}[\eta])^2 \cdot D_K,
\]

(2.1)

where \( D(\eta) \) is the discriminant of the minimal polynomial of \( \eta \) and \( D_K \) is the discriminant of \( K \). Let \( p \) be a prime. In 1878, Dedekind gave the explicit factorization of \( p\mathbb{Z}_K \) when \( p \) does not divide the index \((\mathbb{Z}_K : \mathbb{Z}[\eta]) \) (see [10] and [30, Theorem 4.33]). In 1928, Ø. Ore gave a method for factoring \( F(x) \) in \( \mathbb{Q}_p(x) \), and so, for factoring \( p\mathbb{Z}_K \) when \( F(x) \) is \( p \)-regular (see [32]). This method is based on Newton polygon techniques. His method was developed by Guàrdia, Montes and Nart and [17, 18], see also [29] by Montes and Nart. So, let us recall some fundamental facts on this algorithm.

Let \( \nu_p \) be the discrete valuation of \( \mathbb{Q}_p(x) \) defined on \( \mathbb{Z}_p[x] \) by

\[
\nu_p \left( \sum_{i=0}^{m} a_i x^i \right) = \min \{ \nu_p(a_i), 0 \leq i \leq m \}.
\]

Let \( \phi(x) \in \mathbb{Z}[x] \) be a monic polynomial whose reduction modulo \( p \) is irreducible. Upon the Euclidean division by successive powers of \( \phi(x) \), the polynomial \( F(x) \in \mathbb{Z}[x] \) admits a unique \( \phi \)-adic development

\[
F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_n(x)\phi(x)^n,
\]

with \( \deg (a_i(x)) < \deg (\phi(x)) \). For every \( 0 \leq i \leq n \), let \( u_i = \nu_p(a_i(x)) \). The \( \phi \)-Newton polygon of \( F(x) \) with respect to \( \nu_p \) (or to \( p \), briefly) is the lower convex hull of the points \( \{(i, u_i) \mid 0 \leq i \leq n, a_i(x) \neq 0 \} \) in the Euclidean plane, which we denote by \( N_\phi(F) \). The polygon \( N_\phi(F) \) is the union of different adjacent sides \( S_1, S_2, \ldots, S_g \) with increasing slopes \( \lambda_1, \lambda_2, \ldots, \lambda_g \). We shall write \( N_\phi(F) = S_1 + S_2 + \cdots + S_g \). The polygon determined by the sides of negative slopes of \( N_\phi(F) \) is called the \( \phi \)-principal Newton polygon of \( F(x) \) with respect to \( \nu_p \) and will be denoted by \( N_\phi^+(F) \). Note that the length of \( N_\phi^+(F) \) is \( l(N_\phi^+(F)) = \nu_\phi(F(x)) \); the highest power of \( \phi(x) \) dividing \( F(x) \) modulo \( p \).

Let \( S \) be a side of \( N_\phi^+(F) \). Then the length of \( S \), denoted \( l(S) \), is the length of its projection to the horizontal axis and its height, denoted \( h(S) \), is the length of its projection to the vertical axis. Let \( \lambda = -\frac{h(S)}{l(S)} = -\frac{b}{e} \) its slope, where \( e \) and \( h \) are two positive coprime integers. The degree of \( S \) is \( d(S) = \gcd(h(S), l(S)) = \frac{l(S)}{e} \); it is equal to the the number of
segments into which the integral lattice divides $S$. More precisely, if $(s, u_s)$ is the initial point of $S$, then the points with integer coordinates lying in $S$ are exactly
\[(s, u_s), (s + c, u_s - h), \ldots, (s + de, u_s - dh).\]
The natural integer $e = \frac{\ell(S)}{d(S)}$ is called the ramification index of the side $S$ and denoted by $e(S)$.

Let $F_{\phi}$ be the finite field $\mathbb{Z}[x]/(p, \phi(x)) \simeq F_p[x]/(\overline{\phi(x)})$ (note that the ideal $(p, \phi(x))$ is maximal in the ring of polynomials $\mathbb{Z}[x]$). For any abscissa $s \leq i \leq s + de$, we define the following residue coefficient $c_i \in F_{\phi}$:
\[
c_i = \begin{cases} 
0, & \text{if } (i, u_i) \text{ lies strictly above } N_{\phi}^+(F), \\
\frac{a_i(x)}{p^{m_i}} \pmod{(p, \phi(x))}, & \text{if } (i, u_i) \text{ lies on } N_{\phi}^+(F).
\end{cases}
\]
Further, we attach to $S$ the residual polynomial:
\[R_{\lambda}(F)(y) = c_s + c_{s+1}y + \cdots + c_{s+(d-1)c}y^{d-1} + c_{s+de}y^d \in F_{\phi}[y].\]

Now, we recall some related definitions to Ore’s program.

**Definitions 2.1.** Let $F(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial. Let $\overline{F(x)} = \prod_{i=1}^{t} \overline{\phi_i(x)}$ be the factorization of $\overline{F(x)}$ into a product of powers of distinct monic irreducible polynomials in $F_p[x]$. For every $i = 1, \ldots, t$, let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{ir_i}$, and for every $j = 1, \ldots, r_i$, let $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{r_{ijs}} \psi_{ij}^{n_{ij}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $F_{\phi_i}[y]$.

1. For every $i = 1, \ldots, t$, the $\phi_i$-index of $F(x)$, denoted by $\text{ind}_{\phi_i}(F)$, is $\deg(\phi_i)$ multiplied by the number of points with natural integer coordinates that lie below or on the polygon $N_{\phi_i}^+(F)$, strictly above the horizontal axis and strictly beyond the vertical axis.
2. The polynomial $F(x)$ is said to be $\phi_i$-regular with respect to $\nu_p$ if for every $j = 1, \ldots, r_i$, $R_{\lambda_{ij}}(F)(y)$ is separable; $n_{ijs} = 1$.
3. The polynomial $F(x)$ is said to be $p$-regular if it is $\phi_i$-regular for every $1 \leq i \leq t$.

Now, we state Ore’s theorem which will be often used in the proof of our results (see [18 Theorem 1.19], [29 and 32]):

**Theorem 2.2.** (Ore’s Theorem)
Let $K$ be a number field generated by $\theta$, a root of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$. Under the above notations, we have:

1. $\nu_p(\mathbb{Z}_K : \mathbb{Z}[\theta]) \geq \sum_{i=1}^{t} \text{ind}_{\phi_i}(F)$.

Moreover, the equality holds if $F(x)$ is $p$-regular
2. If $F(x)$ is $p$-regular, then
\[p\mathbb{Z}_K = \prod_{i=1}^{t} \prod_{j=1}^{r_i} \prod_{s=1}^{n_{ij}} \mathbb{Z}_{p_i}^{e_{ij}},\]
where $e_{ij}$ is the ramification index of the side $S_{ij}$ and $f_{ijs} = \deg(\phi_i) \times \deg(\psi_{ijs})$ is the residue degree of $p_{ijs}$ over $p$.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1.
Reducing modulo $p$, one has $F(x) \equiv \phi^n \pmod{p}$, where $\phi(x) = x$, because $p$ divides both $a$ and $b$. Further, we have $N_{\phi}^+(F) = S$ has only one side of degree 1 (because $\gcd(n, \nu_p(b)) = 1$) with slope $\lambda = -\frac{\nu_p(b)}{n}$. Its attached residual polynomial $R_{\lambda}(F)(y)$ is irreducible over $\mathbb{F}_p$. By Theorem 1.19, $F(x)$ is irreducible over $\mathbb{Q}_p$. Hence, it is irreducible over $\mathbb{Q}$. On the other hand, by using Theorem 2.2, we see that

$$\nu_p(\mathbb{Z}_K : \mathbb{Z}[\theta]) \geq \text{ind}_\phi(F) = \frac{(n-1)(\nu_p(b)-1)}{2} \geq 1.$$ 

Thus, $p$ divides $(\mathbb{Z}_K : \mathbb{Z}[\theta])$. So, the polynomial $F(x)$ is not monogenic.

Let $L = \mathbb{Q}_p(\theta)$ and $\omega$ the unique valuation of $L$ extending $\nu_p$ (note that $\mathbb{Q}_p$ is a Henselian field). Let $(s, t) \in \mathbb{Z}^2$ be the unique solution of the Diophantine equation $\nu_p(x)s - nt = 1$ with $0 \leq s < n$ and $\alpha = \frac{\theta^s}{p^t}$. Note that $\alpha \in \mathbb{Z}_K$ if and only if $\omega(\alpha) \geq 0$. Since $N_{\phi}^+(F) = S$ has a single side of slope $\lambda = -\frac{\nu_p(b)}{n}$, we conclude that $\omega(\theta) = \frac{\nu_p(b)}{n}$. So,

$$\omega(\alpha) = \omega\left(\frac{\theta^s}{p^t}\right) = s\omega(\theta) - t = \frac{s\nu_p(b) - tn}{n} = \frac{1}{n}.$$ 

Since $s$ and $n$ are coprime, $K = \mathbb{Q}(\alpha)$. Let $H(x)$ be the minimal polynomial of $\alpha$ over $\mathbb{Q}$. By the formula relating roots and coefficients of a monic polynomial, we conclude that

$$H(x) = x^n + \sum_{i=1}^{n}(-1)^i s_i x^{n-i},$$

where $s_i = \sum_{k_1 < \cdots < k_i} \alpha_{k_1} \cdots \alpha_{k_i}$, and $\alpha_1, \ldots, \alpha_n$ are the $\mathbb{Q}_p$-conjugates of $\alpha$. Since there is a unique valuation extending $\nu_p$ to any algebraic extension of $\mathbb{Q}_p$, we conclude that $\omega(\alpha_i) = 1/n$ for every $i = 1, \ldots, n$ (recall that the valuation $\omega$ is invariant under the $K$-embedding actions). Thus, $\nu_p(s_n) = \omega(\alpha_1 \cdots \alpha_n) = 1$ and $\nu_p(s_i) \geq i/n$ for every $i = 1, \ldots, n-1$. That means that $H(x)$ is a $p$-Eisenstein polynomial. Hence, $p$ does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. On the other hand, every prime $q \neq p$ does not divide $(\mathbb{Z}_K : \mathbb{Z}[\theta])$, because $\Delta_p$ is square free (see [30 Proposition 2.13]). By definition of $\alpha$, $p$ is the unique positive prime integer candidate to divide $(\mathbb{Z}[\alpha] : \mathbb{Z}[\theta])$. Consequently, $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.

To prove Theorems 1.5 and 1.6, we will use the following lemma which gives a sufficient condition for a prime $p$ to be a prime common index divisor of a given field $K$ (see [10] and [30] Theorems 4.33 and 4.34).

Lemma 3.1.
Let $p$ be a prime and $K$ a number field. For every positive integer $f$, let $L_p(f)$ be the
number of distinct prime ideals of \( \mathbb{Z}_K \) lying above \( p \) with residue degree \( f \) and \( N_p(f) \) be the number of monic irreducible polynomials of \( \mathbb{F}_p[x] \) of degree \( f \). If \( L_p(f) > N_p(f) \) for some positive integer \( f \), then \( p \) is a common index divisor of \( K \).

Remark 3.2.
Note that the condition \( i(K) = 1 \) is not sufficient for the monogenity of \( K \). The index of the pure cubic number field \( K = \mathbb{Q}(\sqrt[3]{175}) \) equals 1, but \( K \) is not monogenic as its index form equation is \( 5x^3 - 7y^3 = \pm 1 \) and has no integral solutions.

**Proof of Theorem 1.5.** By hypothesis, 2 divides \( a \) and does not divide \( b \). Thus, \( F(x) = \phi_1(x)^{2^r} \) in \( \mathbb{F}_2[x] \), where \( \phi_1(x) = x - 1 \). Write
\[
F(x) = (x - 1 + 1)^{2^r} + ax^m + b = \sum_{j=1}^{2^r} \binom{2^r}{j} \phi_1(x)^j + ax^m + 1 + b. \tag{3.1}
\]
Since \( a \) and \( b + 1 \) are both divisible by 32, \( \nu_2(a) \) and \( \nu_2(1 + b) \geq 5 \). So,
\[
\nu_2(ax^m + 1 + b) \geq 5.
\]
Let \( ax^m + 1 + b = \sum_{j=1}^{m} b_j \phi_1(x)^j \) be the \( \phi \)-adic development of \( ax^m + 1 + b \) where \( b_j \in \mathbb{Z} \).

Note that \( \nu_2(b_j) \geq 5 \) for all \( j = 1, \ldots, m \). It follows that \( F(x) = \sum_{j=0}^{2^r} a_j \phi_1(x)^j \), where \( a_0 = b_0 \) and
\[
a_j = \begin{cases} b_j + \binom{2^r}{j}, & \text{if } 1 \leq j \leq m, \\ \binom{2^r}{j}, & \text{if } m < j \leq 2^r. \end{cases}
\]
Recall that \( \nu_2 \left( \binom{2^r}{j} \right) = r - \nu_2(j) \) for every \( j = 1, \ldots, 2^r - 1 \) (see \cite{4} Lemma 3.4]). It follows that \( \mu_0 = \nu_2(\alpha_0) \geq 5 \) and
\[
\mu_j = \nu_2(\alpha_j) \geq \min\{5, r - \nu_2(j)\}, \quad \text{if } 1 \leq j \leq m,
\]
\[
\mu_j = r - \nu_2(j), \quad \text{if } m < j \leq 2^r.
\]

Since \( r \geq 3 \), \( N_{\phi_1}^+ = S_{1,1} + \cdots + S_{1,t-2} + S_{1,t-1} + S_{1,t} \) has \( t \) sides with \( t \geq 4 \) joining the points \( \{(j, \mu_j), \ j = 0, \ldots, 2^r\} \) in the euclidean plane. The last three sides have degree 1 each and ramification index \( 2^{r-1} \) each. More precisely, the part \( S_{1,t-2} + S_{1,t-1} + S_{1,t} \) is the polygon joining the points \( (2^{r-3}, 3), (2^{r-2}, 2), (2^{r-1}, 1) \) and \( (2^r, 0) \) (see FIGURE 1). Thus the residual polynomials \( R_{t-k}(F)(y) \) are irreducible in \( \mathbb{F}_{\phi_1}[y] \) as they are of degree 1 each for \( k = 0, 1, 2 \). Applying Theorem 2.2, one has:
\[
2\mathbb{Z}_K = p_{1,t-2,1}^{2^{r-1}} \cdot p_{1,t-1,1}^{2^{r-1}} \cdot p_{1,t,1}^{2^{r-1}} a,
\]
where \( p_{1,t-2,1}, p_{1,t-1,1} \) and \( p_{1,t,1} \) are three prime ideals of \( \mathbb{Z}_K \) of residue degree 1 each, and \( a \) is a non-zero ideal of \( \mathbb{Z}_K \) provided by the other segments of \( N_{\phi_1}^+(F) \). So, the monic irreducible factor \( \phi_1(x) \) of \( F(x) \) modulo 2 provides at least three prime ideals of residue degree 1 each lying above 2 in \( \mathbb{Z}_K \). So, \( L_2(1) \geq 3 > 2 = N_2(1) \). By Lemma 3.1, 2 divides \( i(K) \). Hence, \( K \) is not monogenic. \( \square \)
Proof of Theorem 1.6.

In all cases, we prove that $K$ is not monogenic by showing that that 2 divides $i(K)$. Since in all cases, 2 divides $a$ and does not divide $b$, $F(x) = (x - 1)^{2^r}$ in $\mathbb{F}_2[x]$. Write

$$F(x) = (x - 1 + 1)^{2^r} + a(x - 1 + 1) + b$$

$$= (x - 1)^{2^r} + 2^{r-1} \sum_{j=2}^{2^r} \left(2^r\right)^j (x - 1)^j + (2^r + a)(x - 1) + 1 + a + b. \quad (3.2)$$

Let $\phi_1(x) = x - 1$, $\mu = \nu_2(2^r + a)$ and $\nu = \nu_2(1 + a + b)$. It follows that by the above $\phi$-development (3.2) of $F(x)$ that the principal Newton polygon $N_{\phi_1}^+(F)$ with respect to $\nu_2$ is the Newton polygon joining the points $\{(0, \nu), (1, \mu)\} \cup \{(2^j, r - \nu_2(j)), j = 1, \ldots, r\}$.

(1) If $r \geq 3$, $a \equiv 4 \pmod{8}$ and $b \equiv 3 \pmod{8}$, then $\mu = 2$ and $\nu \geq 3$. It follows that $N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}$ has three sides of degree 1 each joining the points $(0, \nu), (1, 2), (2^r-1, 1)$ and $(2^r, 0)$ in the euclidean plane (see FIGURE 2 for $r = 5$). Thus, $R_{\lambda_k}(F)(y) = 1 + y$ is irreducible in $\mathbb{F}_{\phi_1}[y] \simeq \mathbb{F}_2[y]$ as it is of degree 1 for $i = 1, 2, 3$. So, $F(x)$ is 2-regular. Applying Theorem 2.2 one gets:

$$2\mathbb{Z}_K = p_{111} \cdot p_{121}^{2^{r-1}-1} \cdot p_{131}^{2^{r-1}}$$

where $p_{1ki}$ is a prime ideal of $\mathbb{Z}_K$ of residue degree $f(p_{1ki}/2) = 1$ for $k = 1, 2, 3$. So, there are three prime ideals of $\mathbb{Z}_K$ of residue degree 1 each lying above 2. Applying Lemma 3.1 for $p = 2$ and $f = 1$, we see that 2 divides $i(K)$. Consequently, $K$ cannot be monogenic.
(2) If \( r \geq 4 \), \( a \equiv 8 \pmod{16} \) and \( b \equiv 7 \pmod{16} \), then \( \mu = 3 \) and \( \nu \geq 4 \). It follows that \( N_{\phi_1^+}(F) = S_{11} + S_{12} + S_{13} + S_{14} \) has 4 sides of degree 1 each joining the points \((0, \nu), (1, 3), (2^{r-2}, 2), (2^{r-1}, 1) \) and \((2^r, 0)\) in the Euclidean plane with respective slopes \( \lambda_{11} \leq -1, \lambda_{12} = -\frac{1}{2^{r-1}}, \lambda_{13} = -\frac{1}{2^2} \) and \( \lambda_{14} = -\frac{1}{2^3} \) (see Figure 3 for \( r = 6 \)). Thus, \( R_{\lambda_{14}}(F)(y) \) are irreducible over \( \mathbb{F}_{\phi_1} \simeq \mathbb{F}_2 \) as they are of degree 1. It follows that \( F(x) \) is 2-regular. By using Theorem 2.2, we see that
\[
2\mathbb{Z}_K = p_{111} \cdot p_{121}^{2^{r-2}-1} \cdot p_{131}^{2^{r-2}} \cdot p_{141}^{2^{r-1}},
\]
where \( p_{1k1} \) are prime ideals of \( \mathbb{Z}_K \) of residue degree \( f(p_{1k1}/2) = 1 \) for \( k = 1, 2, 3, 4 \). Thus, for \( p = 2 \), we have \( 4 = L_2(1) > N_2(1) = 2 \). By using Lemma 3.1, \( 2 \) divides \( i(K) \). Hence, \( K \) cannot be monogenic.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{\( N_{\phi_1^+}(F) \) with respect to \( \nu_2 \) where \( r = 6, \ a \equiv 8 \pmod{16} \) and \( b \equiv 7 \pmod{16} \).}
\end{figure}

(3) The case \( r = 3 \) is previously studied in \([3, \text{Theorem } 2.3(2)]\); we have shown that \( 2 \) divides \( i(K) \). Also, by Theorem 1.5, we have \( r \geq 3, a \equiv 0 \pmod{32} \) and \( b \equiv 31 \pmod{32} \), then \( K \) is not monogenic. Assume now that \( r \geq 4, a \equiv 16 \pmod{32} \) and \( b \equiv 15 \pmod{32} \), then \( \mu \geq 4 \) and \( \nu \geq 5 \). Thus,
\[
N_{\phi_1^+}(F) = S_{11} + \ldots + S_{1 \cdot t-2} + S_{1 \cdot t-1} + S_{1t}
\]
has \( t \) sides with \( t \geq 4 \). The last three sides have degree 1 each. More precisely, for \( k = 0, 1, 2, S_{1 \cdot t-k} \) is the side joining the points \((2^{r-k-1}, k+1) \) and \((2^{r-k}, k)\). Their respective slopes are: \( \lambda_{1 \cdot t-2} = -1, \lambda_{1 \cdot t-1} = -1, \lambda_{1t} = -1 \). It follows that \( R_{\lambda_{1 \cdot t-k}}(F)(y) \) is irreducible in \( \mathbb{F}_{\phi_1}[y] \) as it is of degree 1 for \( k = 0, 1, 2 \). Hence, \( F(x) \) is 2-regular. By applying Theorem 2.2, one has:
\[
2\mathbb{Z}_K = p_{1 \cdot t-2, 1}^{2^{r-1}} \cdot p_{1 \cdot t-1, 1}^{2^{r-2}} \cdot p_{1, t, 1}^{2^{r-3}} \cdot \mathfrak{a},
\]
where \( \mathfrak{a} \) is a proper ideal of \( \mathbb{Z}_K \) provided by the other sides of \( N_{\phi_1^+}(F) \), and \( p_{1, t-k, 1} \) is a prime ideal of \( \mathbb{Z}_K \) of residue degree \( f(p_{1, t-k, 1}/2) = 1 \) for \( k = 1, 2, 3 \). So, there are at least three prime ideals of \( \mathbb{Z}_K \) of residue degree 1 each lying above 2. By Lemma 3.1, \( 2 \) divides \( i(K) \). Consequently, \( K \) cannot be monogenic. \( \square \)
To illustrate our results, we give some numerical examples.

**Examples 3.3.**

Let $K = \mathbb{Q}(\theta)$ be a number field generated by a root of a monic irreducible polynomial $F(x) = x^2 + ax + b$.

1. If $F(x) = x^8 + 12x + 3$, then $F(x)$ is irreducible as it is a 3-Eisenstein polynomial. By Theorem 1.6(1), the field $K$ cannot be monogenic. Further, since the polynomial $F(x)$ is 2-regular, by [18], a 2-integral basis of $K$ is given by $(1, \theta, \theta^2, \frac{25+\theta^2}{2}, 21+15\theta+3\theta^2, 7+5\theta^2+3\theta^3+\theta^4, \frac{13+\theta^2+\theta^3+\theta^4+\theta^5+\theta^6}{4})$.

Let $I_2(x_2, x_3, \ldots, x_8) = \pm 1$ be the 2-index form equation associated to the above 2-integral basis of $K$ which is a Diophantine equation of degree 28, where the coefficients are in $\mathbb{Z}_2$, the localization of $\mathbb{Z}$ at $p = 2$ (see [1]). This equation has no solution due to $K$ being non-monogenic.

2. If $F(x) = x^{16} + 24x^{15} + 8$, then $\Delta(F) = 2^{90} \times 7 \times 43 \times 297780173783117618557237901061955691$.

By Corollary 1.3, $F(x)$ is not monogenic, but $K$ is monogenic and $\alpha = \theta^{11}$ generates a power integral basis of $\mathbb{Z}_K$.

3. The pure number field $\mathbb{Q}(\sqrt[6]{65})$ is not monogenic.

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