ON LIE ALGEBRA WEIGHT SYSTEMS FOR 3-GRAPHS

Alexander Schrijver

Abstract. A 3-graph is a connected cubic graph such that each vertex is equipped with a cyclic order of the edges incident with it. A weight system is a function $f$ on the collection of 3-graphs which is antisymmetric: $f(H) = -f(G)$ if $H$ arises from $G$ by reversing the orientation at one of its vertices, and satisfies the IHX-equation. Key instances of weight systems are the functions $\varphi_\mathfrak{g}$ obtained from a metric Lie algebra $\mathfrak{g}$ by taking the structure tensor $c$ of $\mathfrak{g}$ with respect to some orthonormal basis, decorating each vertex of the 3-graph by $c$, and contracting along the edges.

We give equations on values of any complex-valued weight system that characterize it as complex Lie algebra weight system. It also follows that if $f = \varphi_\mathfrak{g}$ for some complex metric Lie algebra $\mathfrak{g}$, then $f = \varphi_{\mathfrak{g}'}$ for some unique complex reductive metric Lie algebra $\mathfrak{g}'$. Basic tool throughout is geometric invariant theory.

Keywords: 3-graph, weight system, Lie algebra, Vassiliev knot invariant

Mathematics Subject Classification (2010): 17Bxx, 57M25, 05Cxx

1. Introduction

A 3-graph is a connected, nonempty, cubic graph such that each vertex $v$ is equipped with a cyclic order of the edges incident with $v$. Loops and multiple edges are allowed. Also the ‘vertexless loop’ $\bigcirc$ counts as 3-graph.

3-graphs come up in various branches of mathematics, under several names. They play an important role in studying the Vassiliev knot invariants (see [6]), and in this context the term 3-graph was introduced by Duzhin, Kaishev, and Chmutov [8]. We adopt this name as it is short and settled in [6]. 3-graphs also emerge in the related Chern-Simons topological field theory (Bar-Natan [2], Axelrod and Singer [1]). They are in one-to-one correspondence with cubic graphs that are cellurally embedded on a compact oriented surface, and hence, through graph duality, with triangulations of a compact oriented surface. Moreover, 3-graphs produce a generating set for the algebra of $O(n, \mathbb{C})$-invariant regular functions on the space $\Lambda^3 \mathbb{C}^n$ of alternating 3-tensors, by Weyl’s ‘first fundamental theorem’ of invariant theory [17].

For the Vassiliev knot invariants, ‘weight systems’ are pivotal. For 3-graphs they are defined as follows. Let $\mathcal{G}$ denote the collection of all 3-graphs, and call a function $f : \mathcal{G} \to \mathbb{C}$ a weight system if $f$ satisfies the AS-equation (for antisymmetry): $f(H) = -f(G)$ whenever $H$ arises from $G$ by turning the cyclic order of the edges at one of the vertices of $G$, and the IHX-equation.
Here the cyclic order of edges at any vertex is given by the clockwise order of edges at the vertex. The grey areas in (1) represent the remainder of the 3-graphs, the same in each of these areas.

Any $C_3$-invariant tensor $c = (c_{i,j,k})_{i,j,k=1}^n \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$ gives the partition function $p_c : G \rightarrow \mathbb{C}$, defined by

$$p_c(G) := \sum_{\psi : E(G) \rightarrow [n]} \prod_{v \in V(G)} c_{\psi(e_1),\psi(e_2),\psi(e_3)}$$

for any 3-graph $G$, where, for any $v \in V(G)$, $e_1, e_2, e_3$ denote the edges incident with $v$, in cyclic order. (As usual, $[k] := \{1, \ldots, k\}$ for any $k \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}$. Moreover, if a group $\Gamma$ acts on a set $X$, then $X^\Gamma$ denotes the set of $\Gamma$-invariant elements of $X$.)

Note that (2) is invariant under orthonormal transformations of $c$, and that $p_c(G)$ is the ‘partition function’ of the ‘vertex model’ $c$, in the sense of de la Harpe and Jones [11]. It may also be viewed as ‘edge coloring model’ as in Szegedy [15].

An important class of weight systems is obtained as follows from the structure constants of metric Lie algebras, which roots in papers of Penrose [14] and Murphy [13], and the relevance for knot theory was pioneered by Bar-Natan [2,3] and Kontsevich [12]. (In this paper, all Lie algebras are finite-dimensional and complex.) A metric Lie algebra is a Lie algebra $\mathfrak{g}$ enriched with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ which is $ad$-invariant, that is, $\langle [x,y], z \rangle = \langle x, [y,z] \rangle$ for all $x, y, z \in \mathfrak{g}$. If $b_1, \ldots, b_n$ is an orthonormal basis of $\mathfrak{g}$, the structure constants $(c_{\mathfrak{g}})_{i,j,k}$ of $\mathfrak{g}$ are characterized by $(c_{\mathfrak{g}})_{i,j,k} := \langle [b_i, b_j], b_k \rangle$ for $i, j, k = 1, \ldots, n$. Then $c_{\mathfrak{g}} \in (\mathfrak{g}^{\otimes 3})^{C_3}$, and $\varphi_{\mathfrak{g}} := p_{c_{\mathfrak{g}}}$ is a weight system. Indeed, the set of such structure constants $c_{\mathfrak{g}}$ of metric Lie algebras $\mathfrak{g}$ in $n$ dimensions is equal to the affine variety $V_n$ in $\Lambda^3 \mathbb{C}^n$ determined by the quadratic equations

$$\sum_{a=1}^n (x_{i,j,a}x_{a,k,l} + x_{k,i,a}x_{a,j,l} + x_{j,k,a}x_{a,i,l}) = 0$$

for $i, j, k, l = 1, \ldots, n$.

This directly implies that $p_x$ satisfies the AS- and IHX-equations; that is, $p_x$ is a weight system.

As was shown by Bar-Natan [4], the functions $\varphi_{\mathfrak{sl}(n)}$ connect to basic graph theory properties like edge-colorability and planarity, and the four-color theorem can be expressed as a relation between the zeros and the degree of $\varphi_{\mathfrak{sl}(n)}(G)$ (as polynomial in $n$).

It is easy to construct a weight system that is not equal to $\varphi_{\mathfrak{g}}$ for any metric Lie algebra: just take a different Lie algebra for each number of vertices of the 3-graph $G$. More interestingly, Vogel [16] showed that there is a weight system that is no Lie
algebra weight system even when restricted to 3-graphs with 34 vertices.

The AS and IHX conditions are ‘2-term’ and ‘3-term’ relations. We give a characterization of those weight systems that come from an \( n \)-dimensional Lie algebra by adding an ‘\((n+1)!!\)-term’ relation. To illustrate it, for \( n = 2 \) it is the following 6-term relation:

\[
(4) \quad f(\begin{array}{c}
\end{array}) + f(\begin{array}{c}
\end{array}) + f(\begin{array}{c}
\end{array}) = f(\begin{array}{c}
\end{array}) + f(\begin{array}{c}
\end{array}) + f(\begin{array}{c}
\end{array}).
\]

To describe it in general, we need the following notion (cf. [6]). A \( k \)-legged fixed diagram is a graph with trivalent vertices, each equipped with a cyclic order of the edges incident with it, and moreover precisely \( k \) univalent vertices (called legs), labeled \( 1, \ldots, k \). (Connectivity is not required.) Let \( F_k \) denote the collection of all \( k \)-legged fixed diagrams.

So \( F_0 \) is the collection of disjoint unions of 3-graphs. Any function \( f \) on 3-graphs can be extended to \( F_0 \) by the ‘multiplicativity rule’ \( f(G \sqcup H) = f(G)f(H) \), where \( \sqcup \) denotes disjoint union, and setting \( f(\emptyset) := 1 \).

For \( G, H \in F_k \), let \( G \cdot H \) be the graph in \( F_0 \) obtained from the disjoint union of \( G \) and \( H \) by identifying the \( i \)-labeled legs in \( G \) and \( H \) and joining the two incident edges to one edge (thus forgetting these end vertices as vertex), for each \( i = 1, \ldots, k \).

For \( \pi \in S_k \), let \( P_{\pi} \) be the \( 2k \)-legged fixed diagram consisting of \( k \) disjoint edges \( e_1, \ldots, e_k \), where the ends of edge \( e_i \) are labeled \( i \) and \( k + \pi(i) \) (for \( i = 1, \ldots, k \)). Then we call the following \( k! \)-term relation the \( \Delta_k \)-equation:

\[
(5) \quad \sum_{\pi \in S_k} \text{sgn}(\pi)f(P_{\pi} \cdot H) = 0 \quad \text{for each } 2k \text{-legged fixed diagram } H.
\]

**Theorem.** Let \( f : \mathcal{G} \to \mathbb{C} \) be a weight system. Then there exists a complex reductive metric Lie algebra \( \mathfrak{g} \) with \( f = \varphi_{\mathfrak{g}} \) if and only if \( f(\emptyset) \in \mathbb{Z}_+ \) and \( f \) satisfies the \( \Delta_{f(\emptyset)+1} \)-equation. If \( \mathfrak{g} \) exists, it is unique.

Although (5) may look like a linear constraint separately for each fixed number of vertices of 3-graphs (as the AS- and IHX-equations are), it in fact interconnects 3-graphs with different numbers of vertices, since \( P_{\pi} \cdot H \) can be a disjoint union of 3-graphs, taking multiplicativity of \( f \) as above. So (5) is a polynomial relation between \( f \)-values of 3-graphs. It describes \( f \) as common zero of a set of polynomials in the ring \( \mathbb{C}[\mathcal{G}] \) formally generated by the collection \( \mathcal{G} \) of 3-graphs (which are connected by definition), in which the disjoint union \( \sqcup \) is taken as multiplication. We note that, for any \( k \in \mathbb{Z}_+ \), the \( \Delta_k \)-equation implies that \( f(\emptyset) \) is a nonnegative integer strictly less than \( k \) (by taking \( H := P_{\text{id}}, \) where \( \text{id} \) is the identity permutation in \( S_k \)).

The theorem implies that if \( f = \varphi_{\mathfrak{g}} \) for some metric Lie algebra \( \mathfrak{g} \), then \( f = \varphi_{\mathfrak{g}} \) for a unique reductive metric Lie algebra \( \mathfrak{g} \). Indeed, let \( f = \varphi_{\mathfrak{g}} \) for some \( n \)-dimensional Lie algebra \( \mathfrak{g} \). So \( f = p_{\mathfrak{g}} \) and \( f(\emptyset) = n \). We show that \( f \) satisfies the \( \Delta_{n+1} \)-equation.
Take a \(2(n+1)\)-legged fixed diagram \(H\) and consider formulas (2) and (3) for \(f := p_{c_\delta}\). The summations over \(\pi\) and \(\psi\) can be interchanged. For each fixed \(\psi : E(H) \to [n]\), we need to add up \(\text{sgn}(\pi)\) over all those \(\pi \in S_{n+1}\) for which, for each \(i \in [n+1]\), legs \(i\) and \(k + \pi(i)\) of \(H\) have the same \(\psi\)-value. As \(n < n + 1\), there exist two legs \(i, j \in [n + 1]\) of \(H\) with the same \(\psi\)-value. Let \(\sigma \in S_{n+1}\) be the transposition of \(i\) and \(j\). Then we can pair up each \(\pi\) with \(\pi\sigma\), and in the summation they cancel. So for each \(\psi\), the sum is 0, and therefore the \(\Delta_{n+1}\)-equation holds.

These arguments also yield the necessity of the condition in the theorem. We prove sufficiency in Section 2 and uniqueness in Section 3.

By the theorem, if the \(\Delta_{f(\circ)+1}\)-equation holds, then there exist unique (up to permuting indices) simple Lie algebras \(g_1, \ldots, g_t\) and nonzero complex numbers \(\lambda_1, \ldots, \lambda_t\) such that

\[
(6) \quad f(G) = \sum_{i=1}^{t} \lambda_i^{\frac{1}{2}}[\gamma_i(G)] \varphi_{g_i}(G)
\]

for each \(3\)-graph \(G \neq \circ\), taking the Killing forms as metrics. So any linear combination of \(3\)-graphs ‘detected’ (to be nonzero) by a Lie algebra weight system, is detected by a simple Lie algebra weight system.

It can be proved that if \(n := f(\circ) \in \mathbb{Z}_+\), then the \(\Delta_{n+1}\)-equation can be replaced by the equivalent condition that for each \(k \in \mathbb{Z}_+\), the rank of the \(\mathcal{F}_k \times \mathcal{F}_k\) matrix \(C_{f,k} := (f(G \cdot H)_{G,H \in \mathcal{F}_k})\) is at most \(n^k\). A weaker, but also equivalent condition is that there exists an \(m \in \mathbb{Z}_+\) such that the rank of \(C_{f,2(n+1)m}\) is less than the dimension of the space of all \(\text{GL}(d)\)-invariant tensors in \(\mathfrak{gl}(d)^{\otimes m}\), where \(d := n^{n+1} + 1\).

Our proof is based on some basic theorems from invariant theory (Weyl’s first and second fundamental theorem for the orthogonal group, and the unique closed orbit theorem; cf. [5],[10]), and roots in methods used in [7], [9], and [15]. For any \(n \in \mathbb{Z}_+\) and any \(3\)-graph \(G\), let \(p(G)\) be the regular function on \(((\mathbb{C}^n)^{\otimes 3})^{C_3}\) defined by

\[
(7) \quad p(G)(x) := p_x(G) \quad \text{for } x \in (((\mathbb{C}^n)^{\otimes 3})^{C_3})^{C_3}
\]

(cf. 2). Then each \(p(G)\) is \(O(n, \mathbb{C})\)-invariant, and the first fundamental theorem of invariant theory implies that the algebra of \(O(n, \mathbb{C})\)-invariant regular functions on \(((\mathbb{C}^n)^{\otimes 3})^{C_3}\) is generated by \(\{p(G) \mid G \text{ 3-graph}\}\).

Note that \(O(n, \mathbb{C})\) acts naturally on the affine variety \(V_n\) defined by (3), and that for any two metrized Lie algebras \(g\) and \(g'\) one trivially has: \(g = g'\) if and only if \(c_{g}\) and \(c_{g'}\) belong to the same \(O(n, \mathbb{C})\)-orbit on \(V_n\). Moreover, the closed orbit theorem implies that \(\varphi_{g} = \varphi_{g'}\) if and only if the closures of the orbits \(O(n, \mathbb{C}) \cdot c_{g}\) and \(O(n, \mathbb{C}) \cdot c_{g'}\) intersect; that is, if and only if they project to the same point in \(V_n/\!\!/O(n, \mathbb{C})\).

The proof implies that a metric Lie algebra \(g\) is reductive if and only if the orbit \(O(g) \cdot c_{g}\) is closed. Hence, for each \(n\), there is a one-to-one correspondence between the points in the orbit space \(V_n/\!\!/O(n, \mathbb{C})\) and the \(n\)-dimensional complex reductive metric Lie algebras.
2. Proof of the theorem: existence of $g$

Let $f : \mathcal{G} \to \mathbb{C}$ satisfy the $\Delta_{n+1}$-equation (5), where $n := f(\emptyset) \in \mathbb{Z}_+$. As above, we extend $f$ to the collection $\mathcal{F}_0$ of disjoint unions of 3-graphs by the rule that $f(\emptyset) = 1$ and $f(G \sqcup H) = f(G)f(H)$ for all $G, H \in \mathcal{F}_0$ (where $\sqcup$ denotes disjoint union). For any $k$, let $\mathbb{C}\mathcal{F}_k$ be the linear space of formal $\mathbb{C}$-linear combinations of elements of $\mathcal{F}_k$. Any (bi-)linear function on $\mathcal{F}_k$ can be extended (bi-)linearly to $\mathbb{C}\mathcal{F}_k$. Taking $\sqcup$ as product, $\mathbb{C}\mathcal{F}_0$ becomes an algebra (which is equal to $\mathbb{C}[\mathcal{G}]$ described above), and $f$ becomes an algebra homomorphism $\mathbb{C}\mathcal{F}_0 \to \mathbb{C}$. Similarly, $p$ (as defined in (7)) extends to an algebra homomorphism $\mathbb{C}\mathcal{F}_0 \to \mathcal{O}(((\mathbb{C}^n)^{\otimes 3})^{C_3})$. (As usual, $\mathcal{O}(\cdot)$ denotes the algebra of $\mathbb{C}$-valued regular functions on $\cdot$.)

**Proposition 1.** $\text{Ker}(p) \subseteq \text{Ker}(f)$.

**Proof.** Let $\gamma \in \mathbb{C}\mathcal{F}_0$ with $p(\gamma) = 0$. We prove that $f(\gamma) = 0$. As each homogeneous component of $p(\gamma)$ is 0, we can assume that $\gamma$ is a linear combination of graphs in $\mathcal{F}_0$ that all have the same number of vertices, say $k$ (which is necessarily even).

Let $\mathcal{M}$ be the collection of perfect matchings on $[3k]$. We can naturally identify $\mathcal{M}$ with the set of $3k$-legged fixed diagrams with no trivalent vertices and no copies of $\emptyset$.

Let $H$ be the the $3k$-legged fixed diagram with precisely $k$ trivalent vertices $v_1, \ldots, v_k$ and $3k$ legs, where $v_i$ is adjacent to legs $3i - 2, 3i - 1, 3i$, in order. (So $H$ is the disjoint union of $k$ copies of the tri-star $K_{1,3}$.) Then each graph in $\mathcal{F}_0$ with $k$ vertices is equal to $M \cdot H$ for at least one $M \in \mathcal{M}$. Hence we can write

$$\gamma = \sum_{M \in \mathcal{M}} \lambda(M)M \cdot H$$

for some $\lambda : \mathcal{M} \to \mathbb{C}$.

The symmetric group $S_{3k}$ acts naturally on $\mathcal{F}_{3k}$ (by permuting leg-labels). Let $Q$ be the group of permutations $\sigma \in S_{3k}$ with $H^\sigma = H$. (So $Q$ is the wreath product of the cyclic group $C_3$ with $S_k$. It stabilizes the partition $\{\{3i - 2, 3i - 1, 3i\} \mid i = 1, \ldots, k\}$ of $[3k]$ and permutes each class in this partition cyclically.) Since $M^\sigma \cdot H = M \cdot H^\sigma^{-1} = M \cdot H$ for each $M \in \mathcal{M}$ and $\sigma \in Q$, we can assume that $\lambda$ is invariant under the action of $Q$ on $\mathcal{M}$.

Define linear functions $F_M$ (for $M \in \mathcal{M}$) and $F$ on $(\mathbb{C}^n)^{\otimes 3k}$ by

$$F_M(a_1 \otimes \cdots \otimes a_{3k}) := \prod_{ij \in M} a_i^T a_j \quad \text{and} \quad F := \sum_{M \in \mathcal{M}} \lambda(M)F_M,$$

for $a_1, \ldots, a_{3k} \in \mathbb{C}^n$. ($ij$ stands for the unordered pair $\{i, j\}$; so $ij = ji$.) Note that $F_M(x^{\otimes k}) = p(M \cdot H)(x)$ for any $x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$. Hence $F(x^{\otimes k}) = p(\gamma)(x) = 0$. We show that this implies that $F = 0$.

Indeed, suppose $F(u_1 \otimes \cdots \otimes u_k) \neq 0$ for some $u_1, \ldots, u_k \in (\mathbb{C}^n)^{\otimes 3}$. Since $F$ is $Q$-invariant (as $\lambda$ is $Q$-invariant), we can assume that each $u_i$ is $C_3$-invariant.
For \( y \in \mathbb{C}^k \), define the \( C_3 \)-invariant tensor \( b_y := y_1 u_1 + \cdots + y_k u_k \). As \( F \) is \( Q \)-invariant, the coefficient of the monomial \( y_1 \cdots y_k \) in the polynomial \( F(b_y^\otimes k) \) is equal to \( k! \cdot F(u_1 \cdots \otimes u_k) \neq 0 \). So the polynomial is nonzero, hence \( F(b_y^\otimes k) \neq 0 \) for some \( y \in \mathbb{C}^k \), a contradiction. Therefore, \( F = 0 \).

Define for each multiset \( N \) of singletons and unordered pairs from \([3k]\) the monomial \( q_N \) on \( S^2 \mathbb{C}^{3k} \) (= the set of symmetric matrices in \( \mathbb{C}^{3k \times 3k} \)), and define moreover the polynomial \( q \) on \( S^2 \mathbb{C}^{3k} \) by:

\[
q_N(X) := \prod_{i,j \in N} X_{i,j} \quad \text{and} \quad q := \sum_{M \in M} \lambda(M)q_M,
\]

for \( X = (X_{i,j}) \in S^2 \mathbb{C}^{3k} \). Note that for each monomial \( \mu \) on \( S^2 \mathbb{C}^{3k} \) there is a unique multiset \( N \) of singletons and unordered pairs from \([3k]\) with \( \mu = q_N \). Now \( F = 0 \) implies

\[
q(X) = 0 \quad \text{if rank}(X) \leq n.
\]

Indeed, if \( \text{rank}(X) \leq n \), then there exist \( a_1, \ldots, a_{3k} \in \mathbb{C}^n \) such that \( X_{i,j} = a_i^T a_j \) for all \( i, j = 1, \ldots, 3k \). By [9] and (10), \( q(X) = F(a_1 \otimes \cdots \otimes a_{3k}) = 0 \), proving (11).

By the second fundamental theorem of invariant theory (cf. [10] Theorem 12.2.12), (11) implies that \( q \) belongs to the ideal in \( \mathcal{O}(S^2 \mathbb{C}^{3k}) \) generated by the \( (n+1) \times (n+1) \) minors of \( X \in S^2 \mathbb{C}^{3k} \). That is, \( q \) is a linear combination of polynomials \( \det(X_{I,J})q_N(X) \), where \( I, J \subseteq [3k] \) with \( |I| = |J| = n+1 \) and where \( N \) is a multiset of singletons and unordered pairs from \([3k]\). Here \( X_{I,J} \) denotes the \( I \times J \) submatrix of \( X \).

Now such triples \( I, J, N \) occur in two kinds: (1) those with \( I \cap J = \emptyset \) and \( N \) a perfect matching on \([3k]\) \( \setminus (I \cup J) \), in which case all monomials occurring in \( \det(X_{I,J})q_N(X) \) are equal to \( q_M \) for some \( M \in \mathcal{M} \); and (2) all other triples \( I, J, N \), in which case none of the monomials occurring in \( \det(X_{I,J})q_N(X) \) is equal to \( q_M \) for some \( M \in \mathcal{M} \). Since \( q(X) \) consists completely of monomials \( q_M \) with \( M \in \mathcal{M} \), we can ignore all triples of kind (2), and conclude that \( q(X) \) is a linear combination of \( \det(X_{I,J})q_N(X) \) with \( I, J, N \) of kind (1).

For any \( M \in \mathcal{M} \), define \( \Gamma(q_M) := M \cdot H \), and extend \( \Gamma \) linearly to linear combinations of the \( q_M \) for \( M \in \mathcal{M} \). Then \( \Gamma(q) = \gamma \). Moreover, for each \( I, J, N \) of kind (1), by the \( \Delta_{n+1} \)-equation for \( f \), \( f(\Gamma(\det(X_{I,J})q_N(X))) = 0 \). As \( \gamma \) is a linear combination of elements \( \Gamma(\det(X_{I,J})q_N(X)) \), we have \( f(\gamma) = 0 \), as required.

By this proposition, there exists a linear function \( \Phi : p(\mathbb{C}^n) \to \mathbb{C} \) such that \( \Phi \circ p = f \). Then \( \Phi \) is an algebra homomorphism, since for \( G, H \in \mathcal{F}_0 \) one has \( \Phi(p(G)p(H)) = \Phi(p(G \sqcup H)) = f(G \sqcup H) = f(G)f(H) = \Phi(p(G))\Phi(p(H)) \).

By the first fundamental theorem of invariant theory,

\[
\mathcal{O}(\langle \mathbb{C}^n \rangle^{\otimes 3} C_3)^{O(n)} = p(\mathbb{C}^n)
\]

(setting \( O(n) := O(n, \mathbb{C}) \)). So \( \Phi \) is an algebra homomorphism \( \mathcal{O}(\langle \mathbb{C}^n \rangle^{\otimes 3} C_3)^{O(n)} \to \mathbb{C} \).
C. Hence the affine $O(n)$-variety

$$V := \{ x \in ((\mathbb{C}^n)^{\otimes 3})^G_3 \mid q(x) = \Phi(q) \text{ for each } q \in \mathcal{O}((\mathbb{C}^n)^{\otimes 3})^{O(n)} \}$$

is nonempty (as $O(n)$ is reductive). By (12) and by substituting $q = p(G)$ in (13),

$$V := \{ x \in ((\mathbb{C}^n)^{\otimes 3})^G_3 \mid p_x = f \}.$$

Hence as $V \neq \emptyset$ there exists $c \in ((\mathbb{C}^n)^{\otimes 3})^G_3$ with $p_c = f$. We choose $c$ such that the orbit $O(n) \cdot c$ is a closed. This is possible by the unique closed orbit theorem (cf. Brion [5]), which also implies that $c$ is contained in each nonempty $O(n)$-invariant closed subset of $V$.

Then $c$ gives the required Lie algebra:

**Proposition 2.** $c = c_\mathfrak{g}$ for some complex reductive metric Lie algebra $\mathfrak{g}$.

**Proof.** We extend $p(G)$ to a function $\hat{p}$ on fixed diagrams as follows. For each $k$ and $G \in F_k$, let $\hat{p}(G) : ((\mathbb{C}^n)^{\otimes 3})^{G_3} \rightarrow (\mathbb{C}^n)^{\otimes k}$ be defined by

$$\hat{p}(G)(x) := \sum_{\psi : E(G) \rightarrow [n]} \prod_{v \in V_3(G)} x_{\psi(e_1),\psi(e_2),\psi(e_3)} \otimes_{j=1}^k b_{\psi(e_j)}$$

for $x \in ((\mathbb{C}^n)^{\otimes 3})^{G_3}$, where $V_3(G)$ is the set of trivalent vertices of $G$, $e_1, e_2, e_3$ are the edges incident with $v$, in order, and $\varepsilon_j$ is the edge incident with leg labeled $j$ (for $j = 1, \ldots, k$). Moreover, $b_1, \ldots, b_n$ is the standard basis of $\mathbb{C}^n$.

Then for all $G, H \in F_k$ and $x \in ((\mathbb{C}^n)^{\otimes 3})^{G_3}$,

$$\hat{p}(G)(x) \cdot \hat{p}(H)(x) = p(G \cdot H)(x),$$

where $\cdot$ denotes the standard inner product on $(\mathbb{C}^n)^{\otimes k}$.

**Claim.** For each $k$ and $\tau \in \mathbb{C}F_k$, if $f(\tau \cdot H) = 0$ for each $H \in F_k$, then $\hat{p}(\tau)(c) = 0$.

**Proof.** As $\hat{p}(\tau)$ is $O(n)$-equivariant, it suffices to show that $\hat{p}(\tau)$ has a zero $x$ in $V$, since then the $O(n)$-stable closed set $\{ x \in V \mid \hat{p}(\tau)(x) = 0 \}$ is nonempty, and hence must contain $c$.

Suppose that no such zero exists. Then the functions $\hat{p}(\tau)$ and $p(G) - f(G)$ (for $G \in \mathcal{G}$) have no common zero. Hence, by the Nullstellensatz, there exist regular functions $s : ((\mathbb{C}^n)^{\otimes 3})^{G_3} \rightarrow (\mathbb{C}^n)^{\otimes k}$ and $g_1, \ldots, g_t : ((\mathbb{C}^n)^{\otimes 3})^{G_3} \rightarrow \mathbb{C}$, and $G_1, \ldots, G_t \in \mathcal{G}$ such that

$$\hat{p}(\tau)(x) \cdot s(x) + \sum_{i=1}^t (p(G_i)(x) - f(G_i))g_i(x) = 1$$

for all $x \in ((\mathbb{C}^n)^{\otimes 3})^{G_3}$. Now $\hat{p}(\tau)$ and $p(G_1), \ldots, p(G_t)$ are $O(n)$-equivariant. Hence, by applying the Reynolds operator, we can assume that also $s$ and $g_1, \ldots, g_k$ are $O(n)$-
equivariant. Then by the first fundamental theorem of invariant theory, $s = \hat{p}(\beta)$ for some $\beta \in \mathbb{C}\mathcal{F}_k$, and $g_i = p(\gamma_i)$ for some $\gamma_i \in \mathbb{C}\mathcal{F}_0$, for $i = 1, \ldots, t$. This gives, with (16),

\begin{equation}
1 = \hat{p}(\tau)(c) \cdot \hat{p}(\beta)(c) + \sum_{i=1}^t (p(G_i)(c) - f(G_i))p(\gamma_i)(c) = p(\tau \cdot \beta)(c) + \sum_{i=1}^t (p(G_i) - f(G_i))p(\gamma_i)(c) = f(\tau \cdot \beta) + \sum_{i=1}^t (f(G_i) - f(G_i))f(\gamma_i) = 0,
\end{equation}

a contradiction, proving the Claim.

Let $\text{AS} \in \mathbb{C}\mathcal{F}_3$ and $\text{IHX} \in \mathbb{C}\mathcal{F}_4$ be extracted from the the $\text{AS}$- and $\text{IHX}$-equations; that is,

\begin{equation}
\text{AS} := \bigwedge_3 + \bigwedge_3, \quad \text{IHX} := \bigwedge_3 - \bigwedge_3 + \bigwedge_3.
\end{equation}

As $f$ is a weight system, $f(\text{AS} \cdot H) = 0$ for each $H \in \mathcal{F}_3$ and $f(\text{IHX} \cdot H) = 0$ for each $H \in \mathcal{F}_4$. Hence the Claim implies that $\hat{p}(\text{AS})(c) = 0$ and $\hat{p}(\text{IHX})(c) = 0$. Therefore, $c = c_\mathfrak{g}$ for some metric Lie algebra $\mathfrak{g}$ (cf. (3)).

We show that $\mathfrak{g}$ is reductive. For this it suffices to show that the orthogonal complement $Z(\mathfrak{g})^\perp$ of the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is semisimple (as then $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 0$, so $Z(\mathfrak{g})$ is nondegenerate).

Suppose to the contrary that $Z(\mathfrak{g})^\perp$ contains a nonzero abelian ideal $I$. We can assume that $I$ is a minimal nonzero ideal. Then $I \subseteq I^\perp$, since, by the minimality of $I$, either $[\mathfrak{g}, I] = 0$, hence $I \subseteq Z(\mathfrak{g}) \subseteq I^\perp$, or $[\mathfrak{g}, I] = I$, hence $\langle I, I \rangle = \langle [\mathfrak{g}, I], I \rangle = \langle \mathfrak{g}, [I, I] \rangle = 0$.

So $I + Z(\mathfrak{g}) \subseteq I^\perp$. This implies that we can choose a subspace $A$ of $I^\perp$ with $I \cap A = 0$ and $I + A = I^\perp$ such that

\begin{equation}
Z(\mathfrak{g}) = (I \cap Z(\mathfrak{g})) + (A \cap Z(\mathfrak{g})).
\end{equation}

Then $A$ is nondegenerate, since $A \cap A^\perp = A \cap I^\perp \cap A^\perp = A \cap (I + A)^\perp = A \cap I = 0$.

So also $A^\perp$ is nondegenerate. As $I \subseteq A^\perp$ and $\dim(A^\perp) = 2 \dim(I)$, there exists a self-orthogonal subspace $C$ of $A^\perp$ with $I \cap C = 0$ and $I + C = A^\perp$. Then $\dim(C) = \dim(I)$ (as $\dim(A) = n - 2 \dim(I)$).

Now define, for any nonzero $\alpha \in \mathbb{C}$, $\varphi_\alpha : \mathfrak{g} \to \mathfrak{g}$ by

\begin{equation}
\varphi_\alpha(x) = \begin{cases} 
\alpha^{-1}x & \text{if } x \in I, \\
\alpha x & \text{if } x \in C, \\
x & \text{if } x \in A.
\end{cases}
\end{equation}

So $\varphi_\alpha \in O(n)$.

Let $\pi_A$ denote the orthogonal projection $\mathfrak{g} \to A$. Then
forms as metrics. If \( \phi \) as \( \mathcal{A} \)

**Proposition 3.** Let

We first show uniqueness if the metrics are the Killing forms.

3. Proof of the theorem: uniqueness of \( g \)

We first show uniqueness if the metrics are the Killing forms.

\[
(22) \quad \lim_{\alpha \to 0} c_\alpha \cdot \varphi^{\otimes 3}_\alpha = c_\alpha \cdot \pi^{\otimes 3}_A,
\]

where \( \cdot \) denotes the standard inner product on \((\mathbb{C}^n)^{\otimes 3}\). To prove \((22)\), choose \( x, y, z \in I \cup C \cup A \). If \( x, y, z \in A \), then for each nonzero \( \alpha \in \mathbb{C} \):

\[
(23) \quad c_\alpha \cdot \varphi^{\otimes 3}_\alpha (x \otimes y \otimes z) = c_\alpha \cdot (x \otimes y \otimes z) = c_\alpha \cdot \pi^{\otimes 3}_A(x \otimes y \otimes z).
\]

If not all of \( x, y, z \) belong to \( A \), let \( k \) be the number of \( x, y, z \) belonging to \( I \) minus the number of \( x, y, z \) belonging to \( C \). Then

\[
(24) \quad \lim_{\alpha \to 0} c_\alpha \cdot \varphi^{\otimes 3}_\alpha (x \otimes y \otimes z) = \lim_{\alpha \to 0} c_\alpha \cdot (x \otimes y \otimes z) = 0.
\]

The last equality follows from the fact that if \( k \geq 0 \), then we may assume (by symmetry) that \( x \in I \) and \( z \notin C \), so \( z \in I + A = A^\perp \). Hence \( c_\alpha \cdot (x \otimes y \otimes z) = \langle [x, y], z \rangle = 0 \), as \( [x, y] \in I \).

This proves \((22)\). Hence, as \( O(n) \cdot c_\alpha \) is closed, there exists \( \varphi \in O(n) \) such that

\[
(25) \quad c_\alpha \cdot \pi^{\otimes 3}_A = c_\alpha \cdot \varphi^{\otimes 3}.
\]

This implies

\[
(26) \quad \varphi(I + C + Z(g)) \subseteq Z(g).
\]

To see this, by \((20)\), \( I + C + Z(g) = I + C + (A \cap Z(g)) \). Now choose \( x \in I \cup C \cup (A \cap Z(g)) \). Then for all \( y, z \in g \), using \((25)\):

\[
(27) \quad \langle [\varphi(x), \varphi(y)], \varphi(z) \rangle = \langle [\pi_A(x), \pi_A(y)], \pi_A(z) \rangle = 0.
\]

Indeed, if \( x \in I + C = A^\perp \) then \( \pi_A(x) = 0 \). If \( x \in A \cap Z(g) \), then \( \pi_A(x) = x \), and \((27)\) follows as \( x \in Z(g) \). As \((27)\) holds for all \( y, z \in g \), \( \varphi(x) \) belongs to \( Z(g) \).

So we have \((26)\), which implies \( \dim(I + C + Z(g)) \leq \dim(Z(g)) \), so \( C \subseteq Z(g) \), hence \( C = 0 \), as \( C \cap Z(g) \subseteq C \cap (I + A) = 0 \). Therefore, \( \dim(I) = \dim(C) = 0 \), contradicting \( I \neq 0 \). Concluding, \( g \) is reductive.

### 3. Proof of the theorem: uniqueness of \( g \)

We first show uniqueness if the metrics are the Killing forms.

**Proposition 3.** Let \( g \) and \( g' \) be complex semisimple Lie algebras with their Killing forms as metrics. If \( \varphi_g = \varphi_{g'} \) then \( g = g' \).

**Proof.** As \( \varphi_g = \varphi_{g'} \), we know \( \dim(g) = \varphi_g(\bigcirc) = \varphi_{g'}(\bigcirc) = \dim(g') \). Let \( h \) and \( h' \) be real compact forms in \( g \) and \( g' \) respectively. Since the Killing forms are negative definite on \( h \) and \( h' \), we can assume that the inner product spaces underlying \( h \) and \( h' \) both are \( \mathbb{R}^n \) with standard negative definite inner product, and that \( c_g \) and \( c_{g'} \) belong to \(((\mathbb{R}^n)^{\otimes 3})_{C^\circ} \).
Suppose \( \mathfrak{g} \neq \mathfrak{g}' \). Hence the orbits \( O(n, \mathbb{R}) \cdot \mathfrak{c}_\mathfrak{g} \) and \( O(n, \mathbb{R}) \cdot \mathfrak{c}_{\mathfrak{g}'} \) are disjoint compact subsets of \( (\mathbb{R}^n)^{\otimes 3}C_3 \). By the Stone-Weierstrass theorem, there exists a real-valued polynomial \( q \) on \( (\mathbb{R}^n)^{\otimes 3}C_3 \) such that \( q(x) \leq 0 \) for each \( x \in O(n, \mathbb{R}) \cdot \mathfrak{c}_\mathfrak{g} \) and \( q(x) \geq 1 \) for each \( x \in O(n, \mathbb{R}) \cdot \mathfrak{c}_{\mathfrak{g}'} \). Applying the Reynolds operator, we may assume that \( q \) is \( O(n, \mathbb{R}) \)-invariant. By the first fundamental theorem of invariant theory, \( q \) belongs to the algebra generated by \( \{ p(G) \mid G \text{ 3-graph} \} \). However, \( p(G)(c_\mathfrak{g}) = \varphi_\mathfrak{g}(G) = \varphi_{\mathfrak{g}'}(G) = p(G)(c_{\mathfrak{g}'}) \) for each 3-graph \( G \). So \( q(c_\mathfrak{g}) = q(c_{\mathfrak{g}'}) \), contradicting \( q(c_\mathfrak{g}) \leq 0 \) and \( q(c_{\mathfrak{g}'}) \geq 1 \). \( \square \)

For each complex metric Lie algebra \( \mathfrak{g} \) of positive dimension, define
\[
(28) \quad \varphi'_\mathfrak{g} := \frac{1}{\dim(\mathfrak{g})} \varphi_\mathfrak{g}.
\]
From Proposition \( 3 \) we derive the next proposition.

**Proposition 4.** Let \( \mathfrak{g} \) and \( \mathfrak{g}' \) be complex simple metric Lie algebras. If \( \varphi'_\mathfrak{g} = \varphi'_\mathfrak{g}' \), then \( \mathfrak{g} = \mathfrak{g}' \).

**Proof.** Let \( B \) and \( B' \) denote the bilinear forms associated with \( \mathfrak{g} \) and \( \mathfrak{g}' \), respectively, and let \( K \) and \( K' \) be the Killing forms of \( \mathfrak{g} \) and \( \mathfrak{g}' \), respectively. Since \( \mathfrak{g} \) and \( \mathfrak{g}' \) are simple, there are nonzero \( \alpha, \alpha' \in \mathbb{C} \) such that \( B = \alpha K \) and \( B' = \alpha' K' \). Then
\[
(29) \quad \varphi_{\mathfrak{g}, B}(\Theta) = \alpha^{-1} \varphi_{\mathfrak{g}, K}(\Theta) = -\alpha^{-1} K^{\otimes 3}(c_{\mathfrak{g}, K}, c_{\mathfrak{g}, K}) = \alpha^{-1} \dim(\mathfrak{g}),
\]
and similarly \( \varphi_{\mathfrak{g}'}, B'(\Theta) = \alpha'^{-1} \dim(\mathfrak{g}') \). Since \( \varphi_{\mathfrak{g}, B}(\Theta) = \varphi_{\mathfrak{g}', B'}(\Theta) \), this implies \( \alpha = \alpha' \). So \( \varphi_{\mathfrak{g}, K} = \varphi_{\mathfrak{g}', K'} \), hence we can assume that \( \alpha = 1 \), so \( B = K \) and \( B' = K' \).

Now let \( \tilde{\mathfrak{g}} \) be the direct sum of \( \dim(\mathfrak{g}') \) copies of \( \mathfrak{g} \). Similarly, let \( \tilde{\mathfrak{g}'} \) be the direct sum of \( \dim(\mathfrak{g}) \) copies of \( \mathfrak{g}' \). So \( \dim(\tilde{\mathfrak{g}}) = \dim(\tilde{\mathfrak{g}'}) \), and for each 3-graph \( G \), as \( \varphi'_\mathfrak{g} = \varphi'_\mathfrak{g}' \) and as \( G \) is connected:
\[
(30) \quad \varphi_{\tilde{\mathfrak{g}}}(G) = \dim(\mathfrak{g}') \varphi_{\mathfrak{g}}(G) = \dim(\mathfrak{g}) \varphi_{\mathfrak{g}'}(G) = \varphi_{\mathfrak{g}'}(G).
\]
Hence by Proposition \( 3 \) \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}'} \), and so \( \mathfrak{g} = \mathfrak{g}' \). \( \square \)

We note that also if \( \mathfrak{g} \) is a complex 1-dimensional metric Lie algebra and \( \mathfrak{g}' \) is a complex simple metric Lie algebra, then \( \varphi'_\mathfrak{g} \neq \varphi'_\mathfrak{g}' \), since \( \varphi'_\mathfrak{g}(\Theta) = 0 \) while \( \varphi'_\mathfrak{g}'(\Theta) \neq 0 \). This and Proposition \( 4 \) is used to prove the last proposition, which settles the theorem.

**Proposition 5.** Let \( \mathfrak{g} \) and \( \mathfrak{g}' \) be complex reductive metric Lie algebras. If \( \varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'} \) then \( \mathfrak{g} = \mathfrak{g}' \).

**Proof.** As \( \mathfrak{g} \) and \( \mathfrak{g}' \) are reductive, we can write
\[
(31) \quad \mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \text{ and } \mathfrak{g}' = \bigoplus_{j=1}^{m'} \mathfrak{g}'_j.
\]
where each $g_i$ and $g_j'$ is either simple or 1-dimensional. Then, since 3-graphs are connected,

\begin{equation}
\sum_{i=1}^{m} \varphi_{g_i} = \varphi_g = \sum_{j=1}^{m'} \varphi_{g_j'}.
\end{equation}

So we can assume that $g_i \neq g_j'$ for all $i \in [m]$ and $j \in [m']$. Hence, by Proposition 4 and the remark thereafter, there exist finitely many 3-graphs $G_1, \ldots, G_k$ such that for all $i \in [m]$ and $j \in [m']$ there exists $t \in [k]$ with $\varphi_{g_i}'(G_t) \neq \varphi_{g_j}'(G_t)$. That is, for each $i \in [m]$ and $j \in [m']$, the following vectors $y_i, z_j \in \mathbb{C}^k$:

\begin{equation}
y_i := (\varphi_{g_i}'(G_1), \ldots, \varphi_{g_i}'(G_k)) \quad \text{and} \quad z_j := (\varphi_{g_j}'(G_1), \ldots, \varphi_{g_j}'(G_k))
\end{equation}

are distinct. So there exists a polynomial $q \in \mathbb{C}[x_1, \ldots, x_k]$ such that $q(y_i) = 0$ for each $i = 1, \ldots, m$ and $q(z_j) = 1$ for each $j = 1, \ldots, m'$. Now set $\gamma := q(G_1, \ldots, G_k)$, taking formal linear sums of 3-graphs and applying the following composition of 3-graphs $G$ and $H$ as product ([8]): take the disjoint union of $G$ and $H$, choose an edge $uv$ of $G$ and an edge $u'v'$ of $H$, and replace them by $uu'$ and $vv'$. Let $F$ be the 3-graph thus arising. Then for any complex simple or 1-dimensional metric Lie algebra $g$: $\varphi_{g_i}'(F) = \varphi_{g_i}'(G) \varphi_{g_i}'(H)$, independently of the choice of $uv$ and $u'v'$ (see Proposition 7.18 in [6]).

We extend each $\varphi_{g_i}'$ and $\varphi_{g_j}'$ linearly to $\gamma$. Then $\varphi_{g_i}'(\gamma) = q(y_i) = 0$ for each $i = 1, \ldots, m$ while $\varphi_{g_j}'(\gamma) = q(z_j) = 1$ for each $j = 1, \ldots, m'$. Hence $\varphi_{g_i}(\gamma) = 0$ for each $i = 1, \ldots, m$ and $\varphi_{g_j}'(\gamma) = \dim(g_j')$ for each $j = 1, \ldots, m'$. Therefore, by (32), $m' = 0$. Similarly, $m = 0$.

References

[1] S. Axelrod, I.M. Singer, Chern-Simons perturbation theory II, Journal of Differential Geometry 39 (1994), 173–213.

[2] D. Bar-Natan, Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory, Ph.D. Thesis, Harvard University, Cambridge, Mass., 1991.

[3] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995) 423–472.

[4] D. Bar-Natan, Lie algebras and the four color theorem, Combinatorica 17 (1997) 43–52.

[5] M. Brion, Introduction to actions of algebraic groups, Les cours du C.I.R.M. 1 (2010) 1–22.

[6] S. Chmutov, S. Duzhin, J. Mostovoy, Introduction to Vassiliev Knot Invariants, Cambridge University Press, Cambridge, 2012.
[7] J. Draisma, D. Gijswijt, L.Lovász, G. Regts, A. Schrijver, Characterizing partition functions of the vertex model, *Journal of Algebra* 350 (2012) 197–206.

[8] S.V. Duzhin, A.I. Kaishev, S.V. Chmutov, The algebra of 3-graphs, *Proceedings of the Steklov Institute of Mathematics* 221 (1998) 157–186.

[9] M.H. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, *Journal of the American Mathematical Society* 20 (2007) 37–51.

[10] R. Goodman, N.R. Wallach, *Symmetry, Representations, and Invariants*, Springer, Dordrecht, 2009.

[11] P. de la Harpe, V.F.R. Jones, Graph invariants related to statistical mechanical models: examples and problems, *Journal of Combinatorial Theory, Series B* 57 (1993) 207–227.

[12] M. Kontsevich, Feynman diagrams and low-dimensional topology, in: *First European Congress of Mathematics Volume II*, Birkhäuser, Basel, 1994, pp. 97–121.

[13] T. Murphy, On the tensor system of a semisimple Lie algebra, *Proceedings of the Cambridge Philosophical Society (Mathematical and Physical Sciences)* 71 (1972) 211–226.

[14] R. Penrose, Applications of negative dimensional tensors, in: *Combinatorial Mathematics and Its Applications* (D.J.A. Welsh, ed.), Academic Press, London, 1971, pp. 221–244.

[15] B. Szegedy, Edge coloring models and reflection positivity, *Journal of the American Mathematical Society* 20 (2007) 969–988.

[16] P. Vogel, Algebraic structures on modules of diagrams, *Journal of Pure and Applied Algebra* 215 (2011) 1292–1339.

[17] H. Weyl, *The Classical Groups — Their Invariants and Representations*, Princeton University Press, Princeton, New Jersey, 1946.