Majorana Fermions, Exact Mapping between Quantum Impurity Fixed Points with four bulk Fermion species, and Solution of the “Unitarity Puzzle”

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Abstract

Several Quantum Impurity problems with four flavors of bulk fermions have zero temperature fixed points that show non fermi liquid behavior. They include the two channel Kondo effect, the two impurity Kondo model, and the fixed point occurring in the four flavor Callan-Rubakov effect. We provide a unified description which exploits the SO(8) symmetry of the bulk fermions. This leads to a mapping between correlation functions of the different models. Furthermore, we show that the two impurity Kondo fixed point and the Callan-Rubakov fixed point are the same theory. All these models have the puzzling property that the S matrix for scattering of fermions off the impurity seems to be non unitary. We resolve this paradox showing that the fermions scatter into collective excitations which fit into the spinor representation of SO(8). Enlarging the Hilbert space to include those we find simple linear boundary conditions. Using these boundary conditions it is straightforward to recover all partition functions, boundary states and correlation functions of these models.

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Introduction

Quantum Impurity problems with four flavors of bulk fermions such as the 2-channel single impurity and 1 channel two impurity Kondo models have low temperature fixed points that provide some of the simplest non-fermi liquid theories. Exact transport properties and correlation functions have been computed for those fixed points using conformal field theory (CFT).

The CFT treatment provides exact S-matrices for scattering of conduction electrons off the impurity. In all cases mentioned above we find that an incoming fermion that hits the impurity seems to disappear because all matrix elements with outgoing states in the fermion Fock space are zero. We solve this paradox by identifying the states into which the fermions scatter as collective, non local excitations. In order to describe them it is useful to view the four Dirac fermions as eight Majorana fermions, and then notice the presence of an SO(8) symmetry group. The fermions transform in the vector representation of SO(8). We will see that these excitations will transform in a spinorial representation which is also eight dimensional and has the same charge, spin and flavor quantum numbers of the electron. Enlarging the Hilbert space to include these excitations, we show that the boundary condition is linear on the vector and the spinor representations of SO(8). The vector is simply scattered into one of the spinors with unit amplitude. Using this linear boundary condition on spinors and vectors, we recover all CFT results including partition functions, finite size spectra, correlation functions, etc.. In this formalism it is very clear how the original SO(8) symmetry is reduced to the symmetry groups of the various models. Moreover, we show that the 2-impurity single channel non-fermi liquid fixed point and the fixed point occurring in the Callan-Rubakov effect are the same CFT, upon an appropriate identification of the fermions. Our interpretation in terms of SO(8) thus provides a unified picture of all known non-fermi liquid quantum impurity fixed points with four bulk flavors and it leads to an explicit and exact mapping between the theories. As a consequence, we show that there are remarkable and completely unexpected exact mappings between the correlation functions of those different models.

Anisotropic versions of the 2 channel Kondo and 2 impurity Kondo models have also been studied in the Toulouse limit, leading to results for some quantities in agreement with the CFT treatment. The Toulouse limit is fine tuned so that the leading irrelevant operator is absent, giving rise to some ungeneric features. Also, correlation functions and transport properties have not been computed using the Toulouse limit. Our SO(8) Majorana fermion treatment is very much like an exact and isotropic version of the Toulouse limit approximation. It incorporates the leading irrelevant operator, and permits to compute exactly all correlation functions, in agreement with previously obtained CFT results.

In section I we review the various models we will analyze. In section II we describe the fermions in terms of SO(8) representations, we introduce the spinors and we define a new set of fermions which have simple boundary condition at all fixed points. We then analyze the various impurity models using this formalism, showing the connection between them as well as explaining the resolution to the unitarity problem. In section III we calculate correlation functions for the Kondo fixed using the new formalism. We study four point correlators, and the leading irrelevant correction to the two point functions. In section IV we show how the symmetry groups of the various models arise from the original SO(8) symmetry, and we
provide the mapping between correlation functions. In appendix A we analyze more explicitly the relationship between Kondo and Monopole correlators. And in appendix B we compute explicitly the boundary states and partition functions.

I. REVIEW OF THE VARIOUS IMPURITY MODELS

We will be considering in this paper the critical points of various impurity models. We start with a conformal invariant bulk Hamiltonian describing four Dirac (eight Majorana) free fermions. Then, a boundary interaction with a localized degree of freedom is introduced. Conformal invariance is broken, but, in the infrared limit, the theory flows to a boundary fixed point that is conformal invariant. We will study these nontrivial fixed points.

The first model that we will consider is the spin 1/2 two channel Kondo effect, which consists of a spin 1/2 impurity interacting with the spin of two channels (flavors) of electrons. Putting the impurity $\vec{S}$ at the origin of three dimensional space, we see that it interacts only with the s-wave. The problem can be reduced to a 1+1 dimensional theory with a boundary at $x = 0$ where the spin sits. The four Dirac fermions correspond to the spin up and spin down components of the two channels of three dimensional electrons. The Hamiltonian is

$$H = \frac{v_F}{2\pi} \int_0^\infty dr (\psi_{L\alpha j}^*(r)i \frac{d\psi_{L\alpha j}(r)}{dr} - \psi_{R\alpha j}^*(r)i \frac{d\psi_{R\alpha j}(r)}{dr}) + v_F \lambda_K \vec{J}_L(r = 0) \cdot \vec{S}$$

The first term describes the four Dirac fermions $\psi_{\alpha j}$, where $\alpha$ is the spin index and $j$ is the flavor index. The second term describes the interaction of the impurity spin with the total fermion spin current $j^a = \psi_{\alpha j}^*(\sigma^a)_{\alpha}^\beta \psi_{\beta j}$ at the origin. It turns out that under the renormalization group this theory flows in the infrared limit to a new critical point at a finite value of $\lambda_K$. At this critical point the theory is conformal invariant, leading to an infinite number of conservation laws. The charge, spin and flavor symmetries appear in conformal field theory as a $U(1)$ charge $\times$ $SU(2)$ spin $\times$ $SU(2)$ flavor Kac-Moody algebra. The constraints imposed by this algebra lead to a complete classification of the possible theories, one for each value of the spin of the impurity $0 \leq s \leq 1$. In the case $s = 1/2$ we get a nontrivial fixed point that was extensively analyzed using the fusion hypothesis. This procedure consists in imposing an auxiliary boundary condition at $x = l$ and then studying the partition function for the system at finite temperature $T = 1/\beta$. The partition function has the form

$$Z = Tr(e^{-\beta H}) = \sum_i n_i \chi_i(e^{-\beta/l})$$

Where $i \equiv (Q, j_{sp}, j_{fl})$ runs over the primary fields of the theory which are labeled by charge, spin and flavor quantum numbers with $\chi_i$ the corresponding character. $n_i$ indicates the number of times that this primary field appears in the partition function. These numbers, that depends on the boundary conditions at both ends, have to obey certain constraints due to conformal invariance. The fusion procedure provides a way to generate solutions to these constraints. Starting with the primary field content of the theory with free fermion boundary conditions ($n_i^{FF}$), we calculate the primary field content with the formula $n_i^{KF} = N_i^k n_k^{FF}$, where $N_i^k$ are the fusion coefficients for the fields $i$ and $j$. The partition function with a Kondo boundary is obtained fusing with a spin 1/2 operator (i.e. $j = (0, 1/2, 0)$).
An important tool in the analysis of these problems was introduced by Cardy. It is based on the observation that we can view this system as propagating in imaginary time with \(Z = Tr(e^{-\beta H_{\text{kondo}}})\) or as propagating in space between two boundary states so that \(Z = \langle K | e^{-lH_{\text{free}}} | F \rangle\). In this later picture all the information about the boundary interaction is encoded in the Kondo boundary state \(| K \rangle\) and we have just a free Hamiltonian.

It is interesting to analyze the scattering of fermions at the Kondo boundary. We send in a left moving fermion and we try to find out what the outgoing excitations are. This scattering amplitude is related to the correlation function of a left moving with a right moving fermion which is constrained by conformal invariance to be of the form

\[ \langle \psi^{\dagger \alpha j}(z) \psi_{\alpha j}(\bar{z}) \rangle = \frac{S_1}{(z - \bar{z})^2} \]

The scattering amplitude \(S_1\) was shown in to be \(S_1 = 0\). Furthermore, the scattering amplitudes for a fermion to go into any number of fermions is zero so that unitarity seems to be violated. We will address this paradox below.

The second model we will analyze is the four flavor Callan-Rubakov effect which describes the scattering of fermions by a magnetic monopole. This scattering has the surprising feature of catalyzing baryon number violation. In the low energy limit the theory will flow to a conformal field theory having an SU(4) Kac-Moody algebra associated with flavor conservation. The free fermion theory contains also a U(1) current corresponding to baryon number, which is not conserved by the interaction with the monopole. It was found that the low energy theory reduces to a simple change in the boundary condition for this current to

\[ j_U^{(1)}(z) = -j_U^{(1)}(\bar{z})|_{\text{Im}\,z=0} \quad (1.1) \]

This conformal point was described in using a U(1)×SU(4) decomposition of the free fermion theory and then twisting the U(1) boson as implied in. The gauge theory that produces the monopole has a topological \(\theta\) angle which appears as parameter of the low energy theory. It introduces an additional phase in the scattering amplitudes. When we study the scattering of a single fermion by the monopole we also see that unitarity seems to be violated because the correlation functions of a left moving fermion with any number of right moving fermions is zero.

The last model we will discuss is the one channel two impurity spin 1/2 Kondo effect. This model is a step forward in analyzing the effect that interimpurity couplings produce in a realistic system. It consists of two spin 1/2 impurities located at two different points interacting with the spin of one channel (one band) of electrons at those points. It is possible to reduce it to a 1+1 dimensional problem. In the reduction process one needs to introduce two channels of 1+1 dimensional fermions, which together with two spin directions give four bulk fermions. The boundary interaction contains several parameters associated with the interaction of these two channels with both impurities, together with an interimpurity coupling of the form \(K \vec{S}_1 \cdot \vec{S}_2\). For \(K \to \infty\) the impurity forms a singlet and the theory reduces in the infrared to the free fermion theory. If \(K \to -\infty\) the impurity forms a spin one triplet and the fixed point is the same as the one for the spin one two channel Kondo theory, which is a fermi liquid with a phase shift of \(\pi\) at the origin. As the phase shift

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for the fermions can only be zero or \( \pi \) for a particle-hole symmetric Hamiltonian there is a non fermi liquid fixed point for a critical value of the coupling \( K_c \). This intermediate fixed point was analyzed by \(^{11}\). For the critical value of the parameters the charges of the two channels are conserved independently. Associated with each of these U(1) currents there is a boson with a compactification radius such that the U(1) symmetry is enhanced to SU(2). The resulting conformal structure is SU(2)\(_1\)\(\times\)SU(2)\(_1\)\(\times\)SU(2)\(_2\)\(\times\)Ising. The first two SU(2)s correspond to the charges of both channels. The third SU(2) is associated with the spin and it is also necessary to introduce an extra Ising model degree of freedom. It was found in \(^{11}\) that the theory is described just by a change in the boundary condition for the Ising model fermion. The same unitarity paradox we had for the previous cases is present here, namely that a left moving fermion has no overlap with right moving fermions. It was also found that the theory has actually a hidden SO(7) invariance.

II. NEW DESCRIPTION USING SO(8)

A. Free Fermions: \( SO(8) \) and its representations

We will concentrate now on the free fermion theory and consider only the left moving sector. All we will say goes over to the right moving sector in a straightforward way. We have four species of Dirac fermions \( \psi_{\alpha j} \) labeled by the spin \( \alpha = 1, 2 \) (for \( \uparrow, \downarrow \)) and the flavor index \( j = 1, 2 \). We can form eight Majorana fermions \( \chi_a \), \( a = 1, 2, \ldots, 8 \) by taking the real and imaginary parts of the complex dirac fermions. The Hamiltonian is

\[
H^0_L = \frac{1}{2\pi} \int dx \chi_i d \frac{d}{dx} \chi_a
\]

which is invariant under SO(8) rotations of the fermions generated by the currents

\[
j^A(z) = \chi_a(z)(T^A)_{ab}\chi_b(z)
\]

where \( T^A \) are antisymmetric 8\(\times\)8 associated with the generators of SO(8). These currents form a Kac-Moody algebra at level one which has four representations \(^{12}\): a 1 dimensional singlet \((1)\), an 8 dimensional vector \((v)\), and two irreducible spinor representations, \((s)\) and \((c)\), each of dimension 8. The one dimensional singlet corresponds to the identity operator and the vector \((v)\) to the fermions. There are two sectors in the free fermion Hilbert space which arise when we consider different boundary conditions in the space direction

\[
\begin{align*}
\text{NS}_x & : \chi_a(\tau, x + l) = -\chi_a(\tau, x) \quad \text{Neveu-Schwartz} \\
\text{R}_x & : \chi_a(\tau, x + l) = \chi_a(\tau, x) \quad \text{Ramond}
\end{align*}
\]

The spinors appear in the Ramond sector, in fact we can decompose the two sectors in terms of representations as

\[
\text{NS} = (1) + (v), \quad \text{R} = (s) + (c)
\]

We note, for later usage, that if we study the system at finite temperature, considering the system in periodic euclidean time \( t = t + \beta \), we have also two possible boundary conditions
anticommutation relations. These are just constants and we will omit them for the sake of clarity.

\[ \gamma^2 = 1 \]

We have disentangled the two irreducible spinor representations using the chirality operator \( \gamma \) which commutes with SO(8) and satisfies \( \gamma^2 = 1 \). In order to describe the spinors it is useful to bosonize the theory introducing four left moving bosonic fields

\[ : \psi^{(a)} \phi_{\alpha j} : (z) = i \partial_z \phi_{\alpha j} \tag{2.8} \]

These currents are of the form \( \mathcal{J}_a \) and correspond to four commuting elements of the SO(8) algebra, which is the maximum number of commuting generators we can have because the algebra has rank four. They are conventionally called Cartan generators and will be denoted by \( H^1, ..., H^4 \). In terms of these bosons we can write the fermions as

\[ \psi_{\alpha j} = e^{-i \phi_{\alpha j}}, \quad \psi^{(a)} \phi_{\alpha j} = e^{i \phi_{\alpha j}} \tag{2.9} \]

The bosons satisfy the periodicity condition

\[ \phi_{\alpha j}(z) = \phi_{\alpha j}(z) + 2\pi \tag{2.10} \]

which in SO(8) language it just means that a rotation by 2\( \pi \) leaves the system invariant. The spinors however are expected to acquire a minus sign under a 2\( \pi \) rotation. We also know that the spinors have conformal dimension 1/2. These requirements fix the spinors to be

\[
\begin{align*}
\begin{pmatrix} c_\mu (z) \\ s_\mu (z) \end{pmatrix} & \equiv 
\begin{pmatrix}
e^{\frac{i}{2}(\phi_{1,1}+\phi_{2,1}+\phi_{1,2}+\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}-\phi_{2,1}+\phi_{1,2}-\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}+\phi_{2,1}-\phi_{1,2}+\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}-\phi_{2,1}-\phi_{1,2}-\phi_{2,2})}
\end{pmatrix} & \equiv 
\begin{pmatrix}
e^{\frac{i}{2}(\phi_{1,1}+\phi_{2,1}+\phi_{1,2}-\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}-\phi_{2,1}+\phi_{1,2}+\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}+\phi_{2,1}-\phi_{1,2}+\phi_{2,2})} \\
e^{\frac{i}{2}(\phi_{1,1}-\phi_{2,1}-\phi_{1,2}-\phi_{2,2})}
\end{pmatrix} 
\end{align*}
\]

We have disentangled the two irreducible spinor representations using the chirality operator \( \gamma \) which commutes with SO(8) and satisfies \( \gamma^2 = 1 \). It is represented in our case by \( (-1)^{2Q} \) where \( Q \) is the total charge corresponding to the operator

\[ j^{\text{ch}}(z) \equiv \frac{1}{2} \sum_{\alpha,j=1,2} : \psi^{(a)} \phi_{\alpha j} : (z) = i \partial_z \phi^{\text{ch}}(z) \tag{2.12} \]

where \( \phi^{\text{ch}}(z) \equiv \frac{1}{2} \sum_{\alpha,j=1,2} \phi_{\alpha j}(z) \tag{2.13} \)

\* Actually, we have to insert additional operators in front of the exponentials to ensure proper anticommutation relations. These are just constants and we will omit them for the sake of clarity.
In order to simplify later formulas we have adopted a slightly unconventional normalization for the charge operator that assigns charges ±1/2 for the fermions \(2.9\). We introduce also bosonic fields associated to the third component of the spin and the third component of flavor

\[
\begin{align*}
    j^{sp}(z) &= \frac{1}{2} : \psi^{\dagger \alpha j}(\sigma^z)^{\alpha}_j \psi_{\beta j} : (z) = i \partial_z \phi^{sp}(z) = \frac{1}{2} \sum_{\alpha, j=1,2} (\sigma^z)^{\alpha}_j i \partial_z \phi_{\alpha j}(z) \\
    j^{fl}(z) &= \frac{1}{2} : \psi^{\dagger \alpha i}(\tau^z)^{\alpha}_i \psi_{\beta j} : (z) = i \partial_z \phi^{fl}(z) = \frac{1}{2} \sum_{\alpha, j=1,2} (\tau^z)^{\alpha}_j i \partial_z \phi_{\alpha j}(z)
\end{align*}
\]

which together with \(2.13\) form a set of three commuting generators. By orthogonality we find a fourth one

\[
    j^X(z) = \frac{1}{2} : \psi^{\dagger \alpha i}(\sigma^z)^{\alpha}_j (\tau^z)^{\alpha}_i \psi_{\beta j} : (z) = i \partial_z \phi^X(z) = \frac{1}{2} \sum_{\alpha, j=1,2} (\sigma^z)^{\alpha}_j (\tau^z)^{\alpha}_j i \partial_z \phi_{\alpha j}(z) \quad (2.15)
\]

These four bosons constitute a new set of Cartan generators. They are related to the old ones by

\[
\begin{align*}
    J_0^h &= \tilde{H}^1 = \frac{1}{2}(H^1 + H^2 + H^3 + H^4) \\
    J_0^{sp} &= \tilde{H}^2 = \frac{1}{2}(H^1 - H^2 + H^3 - H^4) \\
    J_0^{fl} &= \tilde{H}^3 = \frac{1}{2}(H^1 + H^2 - H^3 - H^4) \\
    J_0^X &= \tilde{H}^4 = \frac{1}{2}(H^1 - H^2 - H^3 + H^4) \quad (2.16)
\end{align*}
\]

which is the relation between the old bosons \(2.8\) and the new bosons \(\phi^h, \phi^{sp}, \phi^{fl}, \phi^X\). We rewrite now the vector and spinor representations as

\[
c_\mu(z) \equiv \begin{pmatrix} e^{i\phi^h} \\ e^{i\phi^{sp}} \\ e^{i\phi^{fl}} \\ e^{i\phi^X} \\ e^{-i\phi^h} \\ e^{-i\phi^{sp}} \\ e^{-i\phi^{fl}} \\ e^{-i\phi^X} \end{pmatrix}, \quad s_\mu(z) = \begin{pmatrix} e^{\frac{i}{2}(\phi^h + \phi^{sp} + \phi^{fl} + \phi^X)} \\ e^{\frac{i}{2}(\phi^h - \phi^{sp} + \phi^{fl} - \phi^X)} \\ e^{\frac{i}{2}(\phi^h + \phi^{sp} - \phi^{fl} - \phi^X)} \\ e^{\frac{i}{2}(\phi^h - \phi^{sp} - \phi^{fl} + \phi^X)} \\ e^{\frac{i}{2}(\phi^h + \phi^{sp} + \phi^{fl} - \phi^X)} \\ e^{\frac{i}{2}(\phi^h - \phi^{sp} - \phi^{fl} + \phi^X)} \\ e^{\frac{i}{2}(\phi^h + \phi^{sp} - \phi^{fl} - \phi^X)} \\ e^{\frac{i}{2}(\phi^h - \phi^{sp} + \phi^{fl} + \phi^X)} \end{pmatrix}
\]

We see that, in terms of the new bosons, the vector \(v\) has taken the form of the spinor and the spinor \(c\) has taken the form of the vector. This surprising property of SO(8), namely that we can interchange the 8 dimensional representations by choosing a different basis of Cartan generators, is called triality and leads to a very simple and unified picture for the impurity problems. Clearly we cannot have changed the physics when we chose this
new basis of Cartan generators so the correlation functions are the same, using either basis. An important point in understanding this triality operation is that in the original basis the eigenvalues of the Cartan generators $H^1, \cdots, H^4$ for the vector representation are integers (such that the sum is an odd number). By means of (2.10) we see that the new generators have half integer eigenvalues (such that the sum is an even number).

Finally we like to note that the last two rep’s in (2.17) differ only in the sign of the boson $\phi^X$. This will become a crucial observation later on.

We introduce now a new set of Dirac fermions
\[
\psi_{ch} = e^{-i\phi^{ch}}, \quad \psi_{sp} = e^{-i\phi^{sp}}, \quad \psi_{fl} = e^{-i\phi^{fl}}, \quad \psi_X = e^{-i\phi^X}
\] (2.18)
which form the components of the $(c)$ spinor (2.17). The currents (2.13) have the following form
\[
j^{ch} =: \psi^{\dagger ch} \psi_{ch} : \quad j^{sp3} =: \psi^{\dagger sp} \psi_{sp} : \quad j^{fl3} =: \psi^{\dagger fl} \psi_{fl} :
\] (2.19)
in terms of these fermions. We introduce eight Majorana fermions by taking the real and imaginary parts of the Dirac fermions (2.18)
\[
\chi_A^1 = \frac{\psi^{\dagger A} + \psi_A}{2} = \cos \phi^A
\]
\[
\chi_A^2 = \frac{\psi^{\dagger A} - \psi_A}{2i} = \sin \phi^A
\] (2.20)
where the label $A$ runs over the four fermions (2.18).

In order to clarify further the connection with the original SU(2)×SU(2)×U(1) description we build the SU(2) groups from these fermions. The SU(2) algebra can be realized with three Majorana fermions. So $\chi_{sp}^1, \chi_{sp}^2$ together with $\chi^X_2$ gives rise to the SU(2)$^{spin}$ group. In a similar fashion $\chi_X^1$ combines with the other two flavor fermions to give SU(2)$^{flavor}$. So that the other components of the SU(2) currents read
\[
j^{sp+} = \psi^{\dagger sp} \chi_2^X = e^{i\phi^{sp}} \sin \phi^X, \quad j^{sp-} = \psi_{sp} \chi_2^X = e^{-i\phi^{sp}} \sin \phi^X
\] (2.21)
\[
j^{fl+} = \psi^{\dagger fl} \chi_1^X = e^{i\phi^{fl}} \cos \phi^X, \quad j^{fl-} = \psi_{fl} \chi_1^X = e^{-i\phi^{fl}} \cos \phi^X
\] (2.22)

At these point we have two sets of fermions (and the corresponding bosons) related to each other in a nonlocal way that correspond to two ways of choosing Cartan generators for the SO(8) symmetry algebra. The importance of this construction is that while the boundary condition is nonlinear in terms of the original fermions $\psi_{\alpha j}$ it is indeed linear in the new fermions (2.18) as we will see below.

**B. Spin 1/2 two channel Kondo fixed point**

The fixed point is described by a boundary conformal field theory. The SU(2)×SU(2)×U(1) symmetry of the Kondo model, together with conformal invariance is preserved at the boundary. This means that the currents must satisfy the boundary condition
The operators in 2.13 2.14 are just some of these currents, this implies that the left and right components of the bosons \( \phi^{ch}, \phi^{fl}, \phi^{sp} \) become equal at the boundary, up to a possible constant that will be determined later. We will determine the boundary condition of the boson \( \phi^X \) in two ways. First let us consider scattering off the boundary. It was found in \( [\text{8}] \) that a left moving fermion \( \psi_{L\alpha j}(z) \) has zero probability to scatter into a right moving fermion. The fermion is a primary field under the Kac Moody symmetry and there are only a finite number of primary fields into which the fermions can scatter. The only one that has the same \( SU(2) \times SU(2) \times U(1) \) quantum numbers is the \( (s) \) spinor. A glance at 2.17 convinces us that the boundary condition is

\[
\phi^X_L(z) = -\phi^X_R(\bar{z}) \mid_{\text{Im} z = 0}
\] (2.24)

We see therefore that this twist of the \( x \)-boson is equivalent to the the fact that the left moving fermions have zero overlap with right moving fermions. The spinors appear in the fermionic theory as collective excitations. The appearance of solitons upon scattering from a boundary was also seen in the simpler case of a scalar field in \( [\text{8}] \).

The second way of deducing this boundary condition is based in the observation that \( j^X = i\partial \phi^X \) is the \( (3,3) \) component of a flavor-spin one primary field \( ((Q, j^{sp}, j^{fl}) = (0, 1, 1)) \). The correlation functions of left and right moving fields of this type, calculated in \( [\text{8}] \) show the presence of a minus sign upon reflection at the boundary. This leads to the boundary condition 2.24.

Demanding that the other two components of the flavor and spin currents 2.22 are conserved we find

\[
\phi^{fl}_L(z) = \phi^{fl}_R(\bar{z}) \mid_{\text{Im} z = 0} \quad \phi^{sp}_L(z) = \phi^{sp}_R(\bar{z}) + \pi \mid_{\text{Im} z = 0}
\] (2.25)

so we fixed some of the constants we mentioned above. The charge boson will, in general, involve an arbitrary constant that will appear as an extra phase in the scattering amplitudes. We will set it to zero putting

\[
\phi^{ch}_L(z) = \phi^{ch}_R(\bar{z}) \mid_{\text{Im} z = 0}
\] (2.26)

This corresponds to the particle hole symmetric case. We will show that this model has enhanced symmetry.

In terms of the fermions 2.18 the boundary conditions are linear

\[
\psi_{L,ch} = \psi_{R,ch}, \quad \psi_{L,fl} = \psi_{R,fl} \quad \psi_{L,sp} = -\psi_{R,sp}, \quad \psi_{L,x} = \psi_{R,x}^\dagger
\] (2.27)

while in terms of the original fermions they read

\[
\psi_{Loa}(\bar{z}) = e^{i\pi j_\alpha} S_{Roa}(\bar{z}) \quad S_{Loa}(\bar{z}) = e^{i\pi j_\alpha} \psi_{Roai}(\bar{z})
\] (2.28)

where \( j_\alpha = \pm \frac{1}{2} \) is the spin quantum number of the operator.

This is a remarkably simple picture of the boundary condition: the three Majorana fermions associated with the \( SU(2)^{spin} \) group acquire a minus sign. In this picture it is
clear that the original SO(8) symmetry is broken down to SO(3)×SO(5). The group SO(5) contains the expected SU(2)\textsuperscript{flavor}×U\textsuperscript{charge}(1) plus the following extra generators

\[ \cos \phi^X e^{\pm i\phi^ch} e^{\pm i\phi^{sp}\pm i\phi^{fl}} \quad (2.29) \]

which transform particles to holes. Therefore this symmetry will only be present in the particle hole symmetric case. Indeed, it is easy to see that it is broken if we include a constant in the boundary condition for \( \phi^ch \) \( (\phi^ch_L = \phi^ch_R + \delta|_{Imz=0} ) \).

Using the bosonic representation \( 2.17 \) and the boundary conditions \( 2.24 \) the problem of calculating correlation functions in this theory is reduced to a simple free field theory exercise. Using this bosons we can also write explicitly the Kondo boundary state and calculate the partition function of the theory. We leave these calculations for Appendix \( B \), but let us anticipate that they will agree with the results found using the fusion method.

C. Monopole fixed point

The Monopole theory is described in terms of four fermion flavors \( \Psi_1, \ldots, \Psi_4 \), the natural U(1)×SU(4) group is broken to SU(4) due to a change in the boundary condition for the U(1) baryon number current to

\[ j^U_{L}(z) = -j^U_R(\bar{z})|_{Imz=0} \quad (2.30) \]

This reminds us of the condition \( 2.24 \) for \( \phi^X \). So it is natural to make a correspondence between the fermions \( \Psi_1, \ldots, \Psi_4 \) of the monopole theory with the ones of the Kondo theory in such a way that the baryon number becomes the U(1)\textsuperscript{X} charge of the Kondo theory

\[
\begin{pmatrix}
\Psi_1^\dagger, 1 \\
\Psi_2 \\
\Psi_3 \\
\Psi_4^\dagger, 4 \\
\Psi_1^\dagger, 2 \\
\Psi_2 \\
\Psi_3 \\
\Psi_4^\dagger, 3 \\
\Psi_1^\dagger, 3 \\
\Psi_4^\dagger, 4 \\
\end{pmatrix}
= 
\begin{pmatrix}
\psi_{1,1}^\dagger \\
\psi_{1,1}^\dagger \\
\psi_{2,1}^\dagger \\
\psi_{2,2}^\dagger \\
\psi_{1,1} \\
\psi_{2,1} \\
\psi_{2,2} \\
\psi_{1,2} \\
\psi_{2,2} \\
\end{pmatrix}
= 
\begin{pmatrix}
e^{\frac{i}{2}(\phi^{ch}+\phi^{sp}+\phi^{fl}+\phi^X)} \\
e^{\frac{i}{2}(\phi^{ch}-\phi^{sp}+\phi^{fl}-\phi^X)} \\
e^{\frac{i}{2}(\phi^{ch}+\phi^{sp}-\phi^{fl}-\phi^X)} \\
e^{\frac{i}{2}(\phi^{ch}-\phi^{sp}-\phi^{fl}+\phi^X)} \\
e^{\frac{i}{2}(\phi^{ch}+\phi^{sp}+\phi^{fl}+\phi^X)} \\
e^{-\frac{i}{2}(\phi^{ch}-\phi^{sp}+\phi^{fl}-\phi^X)} \\
e^{-\frac{i}{2}(\phi^{ch}+\phi^{sp}-\phi^{fl}+\phi^X)} \\
e^{-\frac{i}{2}(\phi^{ch}-\phi^{sp}+\phi^{fl}-\phi^X)} \\
e^{\frac{i}{2}(\phi^{ch}+\phi^{sp}+\phi^{fl}+\phi^X)} \\
\end{pmatrix} \quad (2.31)
\]

The bosons \( \phi^{ch}, \phi^{sp}, \phi^{fl} \) give rise to the SU(4) group. The fifteen generators of SU(4) can be written as

\[ i\partial \phi^{ch} \quad i\partial \phi^{fl} \quad i\partial \phi^{sp} \quad e^{\pm i\phi^{ch} \pm i\phi^{fl}} \quad e^{\pm i\phi^{sp} \pm i\phi^{fl}} \quad (2.32) \]

All these generators should be conserved at the boundary so

\[ \phi^ch_L(z) = \phi^ch_R(\bar{z})|_{Imz=0} \quad \phi^{sp}_L(z) = \phi^{sp}_R(\bar{z})|_{Imz=0} \quad \phi^{fl}_L(z) = \phi^{fl}_R(\bar{z})|_{Imz=0} \quad (2.33) \]

In terms of the fermions \( 2.18 \) these conditions can be rewritten, together with the \( \phi^X \) condition \( 2.24 \) as
\[ \psi_{L,ch} = \psi_{R,ch} \quad \psi_{L,fl} = \psi_{R,fl} \quad \psi_{L,sp} = \psi_{R,sp} \quad \psi_{L,x} = \psi_{R}^{\dagger}x \] (2.34)

As the sign of just one Majorana fermion is changed, we discover a hidden SO(7) symmetry in this theory. This symmetry is present only in the case where the \( \theta \) angle is zero. A non-zero \( \theta \) angle changes the boundary condition for \( \phi^{X} \) to

\[ \phi_{L}^{X}(z) = -\phi_{R}^{X}(\bar{z}) + \theta|_{\text{Im}z=0} \] (2.35)

which breaks SO(7) down to SO(6)~SU(4) again.

The boundary state and the partition function are described in Appendix B, they agree with the ones found in\(^4\). We see that the only difference between the Kondo boundary condition 2.27 and the Monopole 2.34 is the minus sign for the fermion \( \psi_{fl} \). This implies a relationship between the correlation functions of the two theories which is described in Appendix A.

D. Two impurity Kondo problem

We will first establish the connection between the description used in ref.\(^{11}\) in terms of SU(2)\(_{\text{charge}}^{1}\)×SU(2)\(_{\text{charge}}^{2}\)×SU(2)\(_{\text{spin}}^{2}\)×Ising and our four fermions 2.18. The two SU(2)\(_{\text{charge}}^{1}\) groups can be represented in terms of two bosons \( \phi^{1}, \phi^{2} \) at the SU(2) compactification radius. They can be constructed as the following linear combinations

\[ \phi^{1} = \frac{\phi^{ch} + \phi^{fl}}{\sqrt{2}} \quad \phi^{2} = \frac{\phi^{ch} - \phi^{fl}}{\sqrt{2}} \]

The Ising fermion can be identified with the Majorana fermion \( \chi_{X}^{1} \) from 2.20. The SU(2)\(_{\text{spin}}^{2}\) has the same representation as before 2.21. The boundary condition in this model corresponds to flipping the sign of the Ising fermion 11

\[ \chi_{L1}^{X} = -\chi_{R1}^{X} \]

We see that this is the same boundary condition as in 2.34 except for an interchange \( \chi_{1}^{X} \leftrightarrow \chi_{2}^{X} \) so that the two are the same conformal field theory. For example, the partition functions are the same. They were calculated independently in\(^4\) and in\(^{11}\). We can also see from this boundary condition the SO(7) symmetry of the theory, which is indeed the same group as in the Monopole fixed point.

### III. CORRELATION FUNCTIONS FOR THE TWO CHANNEL KONDO THEORY

A. Correlation function for the two channel Kondo effect at the fixed point

Using the bosonized picture 2.17 together with the boundary conditions for the bosons 2.24, 2.25 and 2.26 we can calculate all correlation functions of the theory in a simple fashion. As an example we will calculate the following correlation function...
\begin{align}
\langle \psi_{L11}(z_1)\psi_{L12}^\dagger(z_2)\psi_{R11}^\dagger(z_3)\psi_{R22}(z_4) \rangle &=
\langle e^{-\frac{i}{\hbar}(\phi_R^{\dagger\pi}+\phi_R^{\pi}+\phi_R^{\dagger}+\phi_R)(z_1)}e^{\frac{i}{\hbar}(\phi_R^{\dagger\pi}-\phi_R^{\pi}+\phi_R^{\dagger}-\phi_R)(z_2)}e^{\frac{i}{\hbar}(\phi_R^{\dagger\pi}+\phi_R^{\pi}+\pi-\phi_R^{\dagger}-\phi_R)(z_3)} \times
e^{-\frac{i}{2}(\phi_R^{\dagger\pi}-\phi_R^{\pi}+\phi_R^{\dagger}-\phi_R)(z_4)} \rangle 
= \frac{-1}{||z_1-z_3||(z_2-z_3)(z_1-z_4)(z_2-z_4)||^2}.
\end{align}

(3.1)

In a similar fashion we can calculate all others. Strictly speaking we should include
cocycle operators that ensure proper fermionic commutation relations for the exponentials.
However, it can be shown that they are not necessary if we put the fermions first in some
standard order and then replace them by the exponentials.

We will calculate the four point functions of fermions and spin operators for the two
channel Kondo problem. We will show that all four point correlation functions can be related
to the one having four fermions. This is so because, due to the Kondo boundary condition,
two of these fermions turn into spin operators. So no new calculations are necessary.

We are interested in the case were we have two right moving operators and two left
moving operators. We will use the boundary condition \ref{bc2} to express the right movers in
terms of left movers analytically continued past the impurity. The correlation function of
four fermions was analyzed in \ref{3.2}. They calculated

\begin{align}
\langle \psi_{La1}(z_1)\psi_{L1}^{\dagger\beta j}(z_2)\psi_{R1j}(z_3)\psi_{R1}\dagger(\bar{z}_4) \rangle_K &= e^{-i\pi(j_\beta+j_\alpha)}\langle \psi_{La1}(z_1)\psi_{L1}^{\dagger\beta j}(z_2)S_{L\beta j}(z_3)S_{R}^{\dagger\alpha i}(\bar{z}_4) \rangle\text{F} \quad (3.2)
\end{align}
in formula (4.11) of that paper.

Now let us consider correlation functions with spin operators. We will show that those
cases can be put in a form similar to the one above. The case of two left moving fermions
and two right moving spin operators reduces simply to a trivial free fermion four point
correlation function.

The nontrivial cases are when we have a left moving fermion and spin operator and
similarly for the right movers. Let us start with

\begin{align}
\langle \psi_{La1}(z_1)S_{L\beta j}(z_2)\psi_{L}^{\dagger\alpha i}(z_3)S_{R}^{\dagger\beta j}(\bar{z}_4) \rangle_K &= e^{-i\pi(j_\beta+j_\alpha)}\langle \psi_{La1}(z_1)S_{L\beta j}(z_2)S_{L}^{\dagger\alpha i}(z_3)\psi_{L}^{\dagger\beta j}(\bar{z}_4) \rangle\text{F}
\end{align}

We see that we can read the result for this correlation function from \ref{3.2} making in that
formula the replacements $z_2 \rightarrow \bar{z}_4$, $z_3 \rightarrow z_2$, $\bar{z}_4 \rightarrow z_3$. The formula has ambiguities of factors
of $-1$ but those are present because there is a square root branch cut in the correlation
function of the fermion in the presence of a spin operator. Another correlation function is

\begin{align}
\langle \psi_{La1}(z_1)S_{L}^{\dagger\alpha i}(z_2)\psi_{R}^{\dagger\beta j}(z_3)S_{R\beta j}(\bar{z}_4) \rangle_K &= e^{-i\pi(j_\beta+j_\alpha)}\langle \psi_{La1}(z_1)S_{L}^{\dagger\alpha i}(z_2)S_{R\beta j}(z_3)\psi_{L}^{\dagger\beta j}(\bar{z}_4) \rangle\text{F}
\end{align}
The result can be obtained from \ref{3.2} by making the replacements $z_2 \rightarrow \bar{z}_3$, $z_3 \rightarrow \bar{z}_4$, $\bar{z}_4 \rightarrow \bar{z}_2$. So that we get

\begin{align}
\langle \psi_{La1}(z_1)S_{L}^{\dagger\alpha i}(z_2)\psi_{R}^{\dagger\beta j}(z_3)S_{R\beta j}(\bar{z}_4) \rangle_K &= \frac{e^{i\pi(j_\alpha-j_\beta)}(\delta_\alpha^{\dagger}\delta_\beta\delta_\beta^{\dagger}\delta_\alpha^{\dagger} \delta_\alpha^{\dagger} \delta_\beta^{\dagger})}{||z_1-z_3||(z_2-z_3)(\bar{z}_4-z_3)(\bar{z}_4-z_2)||^{1/2}}
\end{align}

In this way we see that these correlation functions have the same form that the ones
for four fermions. Let us remark that with this bosonized formalism it is also straightforward
to calculate higher point correlation functions.
B. Correlation functions away from the fixed point

This formalism enables us to calculate low temperature corrections to the scaling behavior. They are obtained adding to the action the leading irrelevant operator, which is the boundary operator with lowest possible dimension that respects all the symmetries of the problem. For our case this operator is \( \phi_0 = J_{\text{spin}} \phi_{\text{spin}}^{1} \) where \( \phi_{\text{spin}}^{1} \) is a charge zero, flavor-spin zero and spin one primary field (the quantum numbers are \((Q, j_{\text{spin}}, j_{\text{flavor}}) = (0, 1, 0))

The correction term is

\[
S_{\text{irr}} = \lambda_0 \int dt i \chi_1 \chi_2 \chi_3
\]

where we have expressed \( \phi_0 \) in terms of the three Majorana fermions associated with the \( \text{SU(2)}_{\text{spin}} \) group. The coupling \( \lambda_0 \) has dimension 1/2 and is proportional to \( T^{1/2} \). In terms of the bosons 2.18 the correction to the action becomes

\[
S_{\text{irr}} = -\lambda_0 \sqrt{2} \int dt \partial \phi_{\text{sp}} \sin \phi_X
\]

We saw that the left-right fermion green function vanishes at the fixed point. So it is specially interesting to calculate the leading irrelevant correction since it is the dominant contribution.

The first order correction is given by

\[
\langle \psi_{R++}^{1}(z')(-S_{\text{int}})\psi_{L++}(z) \rangle = (-i)\lambda_0 \sqrt{2} \int dt e^{\frac{i}{2}(-\phi_{ch} - \phi_{sp} - \phi_{fl} + \phi_X)} \partial \phi_{sp} \sin \phi_X e^{\frac{i}{2}(\phi_{ch} + \phi_{sp} + \phi_{fl} + \phi_X)} = -i\lambda_0 2\sqrt{2}(z' - z)^{-3/2}
\]

IV. SYMMETRY GROUPS AND BOUNDARY CONDITIONS

The bulk Hamiltonian for eight Majorana fermions has an O(8) symmetry. The fermions transform in the vector representation and the two SO(8) spinors form a single 16 dimensional O(8) spinor. We saw above that by using the triality transformation we generate an SO(8) group under which the fermions are in a spinor representation. If we enlarge this group to O(8) the original fermions together with the \((s)\) spinor 2.17 form a single 16 dimensional representation which will be denoted by \( \eta = (\psi_{\alpha j}, s_{\mu}) \). The extra group elements in O(8) that do not belong to SO(8) interchange the original fermions with the \((s)\) spinor. Note that these elements act locally on the fermions 2.18. The bulk free fermion theory is invariant under the transformations

\[
\eta_L = G_L \eta_L \quad \quad \quad \eta_R = G_R \eta_R \quad \quad \quad (4.1)
\]

where \( G_L \) and \( G_R \) are two independent group elements of O(8). If we have a boundary condition of the form

\[
\eta_L = R \eta_R |_{Imz=0} \quad \quad \quad (4.2)
\]

with \( R \in \text{O(8)} \) then the transformations in 4.1 have to satisfy
Notice that this equation implies that if \( G_L \in \text{SO}(8) \) then also \( G_R \in \text{SO}(8) \) for any \( R \) in \( \text{O}(8) \). This implies that if we have a theory with only the fermions (no \((s)\) spinor) then these transformations leave them as fermions. In the problems we treated in this paper the boundary condition can be put in this form with \( R \notin \text{SO}(8) \) which means that the boundary condition \( 4.2 \) interchanges the fermions with the \((s)\) spinor.

For the free fermion theory \( R=1 \) so that \( G_L = G_R \) but for a general group element \( R \) we see from \( 4.3 \) that \( G_L = G_R \) implies that \( G_L \) commutes with \( R \). We will call the set of such transformations the “proper” symmetry group of the theory. If we allow different transformation laws for left and right movers then the symmetry group of all these theories is always the same. Instead, “proper” symmetries correspond to charges that are conserved at the boundary. We can determine \( R \) for the different theories by its action on the \((c)\) spinor which is in the vector representation of \( \text{O}(8) \). In the Monopole theory we see from \( 2.34 \) that \( R_M \) can be represented by a diagonal matrix that differs from the identity matrix only by the sign of one entry. This implies that the “proper” symmetry group in this case is \( Z_2 \times \text{O}(7) \) which reduces to the \( \text{SO}(7) \) we had before if we consider only the elements continuously connected with the identity. The same situation is true for the two impurity Kondo effect since it is the same conformal field theory as the one describing the Monopole.

In the Kondo problem we see from \( 2.27 \) that \( R_K \) differs from the identity by the sign of three entries so that the proper group is \( \text{O}(3) \times \text{O}(5) \). This again reduces to \( \text{SO}(3) \times \text{SO}(5) \) if we consider the part connected with the identity.

The correlation functions of any theory with a boundary condition of the form \( 4.2 \) can be obtained from the bulk free theory by simply replacing \( \eta^a_L(z) \to R^a_b \eta^a_L(z) \) so that we have

\[
\langle \eta^a_L(z_1) \ldots \eta^a_L(z_n) \eta^b_R(\bar{w}_1) \ldots \eta^c_R(\bar{w}_m) \rangle_{\text{boundary}} = R^{a_1}_{b_1} \ldots R^{a_n}_{b_n} \langle \eta^b_R(z_1) \ldots \eta^b_R(z_n) \eta^c_R(\bar{w}_1) \ldots \eta^c_R(\bar{w}_m) \rangle_{\text{bulk}}
\]

Notice that if we want to calculate a correlation function for left and right moving fermions in the Kondo or Monopole model we have to know the correlation function in the bulk for fermions and spinors, which is a simple problem in free field theory once we use the bosonized forms \( 2.17 \).

This implies also a linear relationship between the correlation functions of the Monopole and Kondo theories. These models are specified by the group elements \( R_M \) and \( R_K \) appearing in the boundary condition. Defining \( S = R_K R_M^{-1} \) we have the following relation

\[
\langle \psi_L(z_1) \ldots \psi_L(z_n) \psi_R(\bar{w}_1) \ldots \psi_R(\bar{w}_m) \rangle_{\text{Kondo}} = \langle S \psi_R(z_1) \ldots S \psi_R(z_n) \psi_R(\bar{w}_1) \ldots \psi_R(\bar{w}_m) \rangle_{\text{Monopole}}
\]

In appendix A we work out this correspondence more explicitly.

**V. CONCLUSIONS**

We gave in this paper a unified description of several impurity problems. The description was based in using the symmetry group of the free theory \( \text{SO}(8) \). We used the triality
transformation and we enlarged the set of states to include the \((s)\) spinor. In this language
the boundary condition is linear and corresponds to an element of \(O(8)\). We have solved
the unitarity paradox by realizing that the fermions are scattered to the \((s)\) spinor. We
found the relation between the correlation functions of the two channel Kondo effect and
the four flavor Monopole theory. We also established that the two impurity Kondo theory
is the same as the four flavor Monopole theory. We have reduced the problem of calculating
correlation functions to a free field theory exercise.

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APPENDIX A: MAPPING BETWEEN THE KONDO AND MONOPOLE
CORRELATORS

We remarked above that the Monopole and the Kondo correlation functions are linearly
related. To make this more explicit we define

\[
\psi^K_\mu = \begin{pmatrix}
\psi^{\dagger,1,1} \\
\psi^{\dagger,2,1} \\
\psi^{\dagger,1,2} \\
\psi^{\dagger,2,2} \\
\psi_{1,1} \\
\psi_{2,1} \\
\psi_{1,2} \\
\psi_{2,2}
\end{pmatrix} \quad \text{and} \quad \psi^M_\mu = \begin{pmatrix}
\psi^{\dagger,1} \\
\psi_2 \\
\psi_3 \\
\psi^{\dagger,3} \\
\psi^{\dagger,2} \\
\psi^{\dagger,4} \\
\psi_1 \\
\psi_4
\end{pmatrix}
\] (A1)

and we set them equal for right movers \(\psi^K_R = \psi^M_R\). We know that the difference between the
Kondo and the Monopole boundary conditions is just the phase associated with the third
component of the spin \(2.25\). So for the left movers we have \(\psi^M_L = S\psi^K_L\) where \(S = e^{-i\pi J_{sp}^0}\)
is simply a phase, different for different components of \(\psi_\mu\). We have set the \(\theta\) angle to zero
for simplicity. Using this we can obtain the Monopole correlation functions in terms of the
Kondo correlators. For example

\[
\langle \Psi_1(z_1)\Psi_2(z_2)\Psi_3(\bar{z}_3)\Psi_4(\bar{z}_4) \rangle = -\langle \psi^L_{11}(z_1)\psi^{\dagger,21}_L(z_2)\psi^{\dagger,12}_R(\bar{z}_3)\psi^R_{22}(\bar{z}_4) \rangle
\] (A2)

where the factor \(-1\) came from the phase in the boundary condition. All correlation functions
of the Monopole theory can be mapped on a correlation function of the Kondo theory.
We have checked explicitly that this is indeed true for all four point functions which were
calculated independently and using different methods in \cite{5} for the Kondo model and in \cite{4}
for the Monopole theory.

APPENDIX B: BOUNDARY STATES AND PARTITION FUNCTIONS

We start by constructing the free fermion boundary state. It is convenient to work in
terms of the four bosons \(\phi^c, \phi^p, \phi^h, \phi^X\). We denote their four \(U(1)\) charges collectively


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by \( \vec{J} = (J_{0}^{ch}, J_{0}^{sp}, J_{0}^{fl}, J_{0}^{X}) \) and their eigenvalues by the four component vector \( \vec{k} \). The components of \( \vec{k} \) are either all integer or all half integer. In the Neveu-Schwartz sector \( \sum_{i=1}^{4} k_{i} = \text{even} \), and in the Ramond sector this sum is odd.

The free fermion boundary state is

\[
|F\rangle = \exp \left( \sum_{i=1}^{4} \frac{\alpha_{L}^{i} \alpha_{R}^{i}}{n} \right) \sum_{\vec{k} \in \Lambda} |\vec{k} \rangle_{L} - |\vec{k} \rangle_{R}
\]

(B1)

where \( \alpha_{L}^{i}, \alpha_{R}^{i} \) are the left and right moving oscillatory modes of the four bosons and \( \vec{k} \) are their momenta defined on a lattice \( \Lambda \) which depends on the sector we are considering.

From the boundary conditions on the bosons 2.24 2.25 2.26 we conclude that the Kondo boundary state is

\[
|K\rangle = e^{i \pi J_{R}^{sp}} R_{L} |F\rangle = \exp \left( \sum_{i=1}^{3} \frac{\alpha_{L}^{i} \alpha_{R}^{i}}{n} \right) \sum_{\vec{k} \in \Lambda} e^{i \pi k_{2}} (k_{1}, k_{2}, k_{3}, -k_{4})_{L} - (k_{1}, k_{2}, k_{3}, k_{4})_{R}
\]

(B2)

where \( R_{L} \) reverses the left part of the boson \( \phi^{X} \).

Using these boundary states it is easy to calculate the different partition functions of the theory. Actually without much more effort we can insert in the partition function the a similar set of angles where \( \vec{J} \) by \( \vec{J} \) by

\[
\text{the projection factor } (1 + (\frac{1}{i} k_{\alpha} = \text{even}) \text{ and in the Ramond sector this sum is odd.}
\]

The twist in the \( \phi^{X} \) boson leaves only the states with \( k_{4} = 0 \) and will produce a factor of \( 1/ \Pi(1 + q^{2n}) \) from the oscillator modes. The result is
\[ Z_{KF}^{NS} = \frac{1}{2} \prod_{i=1}^{3} (\sum (q^2)^{n_2/2} e^{i\alpha_i}) + \prod_{i=1}^{3} (\sum (q^2)^{n_2/2} e^{i(\alpha_i + \pi)}) \]

Setting \( \alpha = 0 \) and putting the result in terms of \( w = e^{-\pi \beta / l} \) we obtain

\[ Z_{KF}^{NS} = \frac{1}{2} \sum w^{\pm 2} \sum w^{\pm (n+\frac{1}{2})^2} \sum w^{\pm i} \]

Which agrees with the result obtained in [1] using the fusion hypothesis. In the case where we have the Kondo boundary condition on both ends we could compute \( \langle K | q^{L_0 + L_0} e^{i(\bar{\varphi}_L J_L - \bar{\varphi}_R J_R)} | K \rangle \). This gives the same result as the free fermion partition function but with \( \alpha_1 \to \bar{\alpha}_4 = \bar{\varphi}_{4L} + \varphi_{4R} \) and \( \alpha_i = \alpha_i \) for \( i = 1, 2, 3 \). This reflects the change in the \( U(1)^X \) boson boundary condition. Note that if we set all angles to zero we obtain exactly the same result, the angles enable us to distinguish between the two situations. This partition function however does not agree with the one calculated using fusion because we have here states propagating that are in the (R,NS) sector. If we restrict just to states in the (NS,NS) sector we obtain

\[ Z_{KK}^{NS} = \frac{1}{2} \prod_{i=1}^{4} (\sum (q^2)^{n_2/2} e^{i\alpha_i}) + \prod_{i=1}^{4} (\sum (q^2)^{n_2/2} e^{i(\alpha_i + \pi)}) \]

which is indeed the result that the fusion hypothesis produces.

We can also repeat the same calculations including, in the free fermion boundary state, also states in the R sector. In the open string picture this would correspond to the NS sector with only states of even fermion number. The calculations are very similar to the ones outlined here, in this case we do not project on states for which the sum of \( k_i \) is even. The results agree with the ones that one could calculate using fusion. And, again, when both boundaries are Kondo we have to project on the sectors (NS,NS) and (R,R).

Now we consider the Monopole partition functions. The Monopole boundary state is just

\[ |M \theta \rangle = e^{i0\cdot J_L} R | F \rangle \]

where we have included also the effect of a non vanishing \( \theta \) angle. We can write the partition function immediately from the expressions we had for the Kondo case. We want however to make connection with the description using SU(4) symmetry so let us analyze the SU(4) characters first. The SU(4) group has three Cartan generators that we can associate to the bosons (\( \partial \phi_1, \partial \phi_2, \partial \phi_3 \)) while the other twelve generators are given by \( e^{i(\pm \phi_i \pm \phi_j)} \) (with \( i \neq j \)). The Kac-Moody algebra SU(4) has four representations: 1, 4, 4, 6, (we labeled them according to their dimensions) of weights: 0, \( \frac{1}{8}, \frac{3}{8}, \frac{3}{2} \) respectively. The states have charges characterized by \( e^{i(k_1 \phi_1 + k_2 \phi_2 + k_3 \phi_3)} \) with \( k_i \) all integer or all half integer. The representation to which this state belongs depends on the sum of \( k_i \)

\[
k_1 + k_2 + k_3 = \begin{cases} 
\text{even} & (1) \\
\text{odd} & (6) \\
3/2 + \text{even} & (4) \\
3/2 + \text{odd} & (4)
\end{cases}
\]
The SU(4) non specialized characters are obtained inserting an exponential of the form $e^{i\sum_{j}a_{j}}$ where $j_i$ are the Cartan generators of SU(4). With the conventions $\vec{n} = (n_1, n_2, n_3)$, $\vec{\gamma} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\vec{a} = (\alpha_1, \alpha_2, \alpha_3)$ they read

$$\chi_{SU(4)}(q, \vec{\alpha}) = \frac{1}{2} \sum_{\vec{n}} (q)^{\vec{n}^2 + 2 \vec{n} \cdot \vec{\alpha}} (1 + (-1)^{n_1 + n_2 + n_3}) \frac{q^{3/4}f(q)^3}{f(q)} \tag{B8}$$

$$\chi_{SU(4)}(q, \vec{\alpha}) = \frac{1}{2} \sum_{\vec{n}} (q)^{\vec{n}^2 + 2 \vec{n} \cdot \vec{\alpha}} (1 - (-1)^{n_1 + n_2 + n_3}) \frac{q^{3/4}f(q)^3}{f(q)} \tag{B9}$$

$$\chi_{SU(4)}(q, \vec{\alpha}) = \frac{1}{2} \sum_{\vec{n}} (q)^{\vec{n}^2 + 2 \vec{n} \cdot \vec{\alpha}} (1 + (-1)^{n_1 + n_2 + n_3}) \frac{q^{3/4}f(q)^3}{f(q)} \tag{B10}$$

$$\chi_{SU(4)}(q, \vec{\alpha}) = \frac{1}{2} \sum_{\vec{n}} (q)^{\vec{n}^2 + 2 \vec{n} \cdot \vec{\alpha}} (1 - (-1)^{n_1 + n_2 + n_3}) \frac{q^{3/4}f(q)^3}{f(q)} \tag{B11}$$

Now we analyze the U(1) characters. The non specialized characters for the different values of $Q$ ($Q = 0, \pm 1/2, 1$) are

$$\chi_{SU(4)}(q, \alpha) = \sum_{\vec{n}} q^{(Q + 2\vec{n})^2 + 2 \vec{n} \cdot \vec{\alpha}} \frac{q^{1/4}f(q)}{f(q)} \tag{B12}$$

We find that the partition function [4] can be rewritten as

$$Z_{FF}^{NS}(q^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \chi_0^{SU(4)}(q^2, \alpha_4) \chi_1^{SU(4)}(q^2, \vec{\alpha}) + \chi_{1/2}^{SU(4)}(q^2, \alpha_4) \chi_{4}^{SU(4)}(q^2, \vec{\alpha})$$

$$+ \chi_{-1/2}^{SU(4)}(q^2, \alpha_4) \chi_{4}^{SU(4)}(q^2, \vec{\alpha}) + \chi_{1}^{SU(4)}(q^2, \alpha_4) \chi_{6}^{SU(4)}(q^2, \vec{\alpha}) \tag{B13}$$

The partition function with the Monopole boundary conditions on one side and free fermions on the other is

$$Z_{M,\theta,F}^{NS} = \frac{1}{(q^2)^{1/24} \prod_{i=1}^{3}(1 + q^{2n})} \chi_1^{SU(4)}(q^2, \vec{\alpha})$$

Notice that the relationship with the Kondo partition function is very simple

$$Z_{K,F}^{NS}(q^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = Z_{M,F}^{NS}(q^2, \alpha_1, \alpha_2, \alpha_4 - \pi, \alpha_3, \alpha_4)$$

The partition function with Monopole boundary conditions on both sides and different theta angles is

$$Z_{M,\theta,M'\theta'}^{NS} = \chi_0^{SU(4)}(q^2, \alpha_4) \chi_1^{SU(4)}(q^2, \vec{\alpha}) + \chi_1^{SU(4)}(q^2, \alpha_4) \chi_6^{SU(4)}(q^2, \vec{\alpha})$$

where $\vec{\alpha} = \vec{\alpha}_4 + \varphi_4L + \varphi_4R - \theta + \theta'$ and again we have projected on to (NS,NS) states. When we set all $\varphi = 0$ these expressions agree with the ones in [4].
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