A Modified Scalar-Tensor-Vector Gravity Theory and the Constraint on its Parameters

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Abstract

A gravity theory called scalar-tensor-vector gravity (STVG) has been recently developed and succeeded in solar system, astrophysical and cosmological scales without dark matter [J. W. Moffat, J. Cosmol. Astropart. Phys. 03, 004 (2006)]. However, two assumptions have been used: (i) \( B(r) = A^{-1}(r) \), where \( B(r) \) and \( A(r) \) are \( g_{00} \) and \( g_{rr} \) in the Schwarzschild coordinates (static and spherically symmetric); (ii) scalar field \( G = \text{Const.} \) in the solar system. These two assumptions actually imply that the standard parametrized post-Newtonian parameter \( \gamma = 1 \). In this paper, we relax these two assumptions and study STVG further by using the post-Newtonian (PN) approximation approach. With abandoning the assumptions, we find \( \gamma \neq 1 \) in general cases of STVG. Then, a version of modified STVG (MSTVG) is proposed through introducing a coupling function of scalar field \( G: \theta(G) \). We have derived the metric and equations of motion (EOM) in 1PN for general matter without specific equation of state and \( N \) point masses firstly. Subsequently, the secular periastron precession \( \dot{\omega} \) of binary pulsars in harmonic coordinates is given. After discussing two PPN parameters (\( \gamma \) and \( \beta \)) and two Yukawa parameters (\( \alpha \) and \( \lambda \)), we use \( \dot{\omega} \) of four binary pulsars data (PSR B1913+16, PSR B1534+12, PSR J0737-3039 and PSR B2127+11C) to constrain the Yukawa parameters for MSTVG: \( \lambda = (3.97 \pm 0.01) \times 10^8 \text{m} \) and \( \alpha = (2.40 \pm 0.02) \times 10^{-8} \) if we fix \( |2\gamma - \beta - 1| = 0 \).

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I. INTRODUCTION

With tremendous advance in the accuracy of observations, Einstein’s general relativity (GR) has passed nearly all the tests in the solar system. However, alternative gravity theories still stand up for explaining some exotic phenomena such as the dark matter and for possible violation of general relativity in the future higher precision experiments. Among them, Moffat [1] proposed a scalar-tensor-vector gravity (STVG) theory in which there are three scalar fields and a vector field besides the metric tensor. Three scalar fields are respectively the scalar field $G$ that origins from the Newtonian gravitational constant, the coupling function of the vector field $\omega$ and the rest mass of the vector field $m$ which controls the coupling range. They all change with space and time. The vector field $\phi_{\mu}$, which is associated with a fifth force charge, corresponds to the exchange of a massive spin 1 boson and couples to the ordinary matter. Through introduced $\phi_{\mu}$, a Yukawa-like force was added to the Newtonian inverse square law. This leads to a satisfied fit to galaxy rotation curves and the Tully-Fisher law [2]. Besides, the theory has been used successfully to explain cosmological observations [3], the motion of galaxy clusters [8], the Bullet Cluster [9], the velocity dispersion profiles of satellite galaxies [10] and globular clusters [11] without exotic dark matter. By studying STVG attentively, we found it uses two assumptions in STVG:

1. The metric components $g_{00}$ and $g_{rr}$ in a spherically symmetric field have the following relationship: $B(r) = A^{-1}(r)$, where $B(r)$ and $A(r)$ are $g_{00}$ and $g_{rr}$ respectively in a Schwarzschild-like coordinate system;

2. For the solar system, the running of gravitational constant $G$ is zero, namely, $G = \text{Const.}$

With the above assumptions, let us analysis what results they lead to. A general form of the metric in the Schwarzschild coordinates reads

$$\text{d}s^2 = -B(r)c^2\text{d}t^2 + A(r)\text{d}r^2 + r^2\text{d}\Omega^2,$$

where $\text{d}\Omega^2 = \text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2$. Comparing Eq. (1) with the following parametrized post-Newtonian metric,

$$\text{d}s^2 = -\left(1 - 2\frac{GM}{c^2 r}\right)c^2\text{d}t^2 + \left(1 + 2\gamma \frac{GM}{c^2 r}\right)\text{d}r^2 + r^2\text{d}\Omega^2,$$

and
within the post-Newtonian precision $O(1/c^2)$, it is clear that $\gamma = 1$ under assumption (i). On the other hand, since only the scalar field $G$ affects $\gamma$ in STVG (for details see subsection D of section II), assumption (ii) gives $\gamma = 1$ directly. In summary, these assumption impose a constraint on the theory: the standard parametrized post-Newtonian (PPN) parameter $\gamma$ equals to 1. Therefore, these two assumptions are reasonable and consistent with the current measurement of $\gamma$.

However, we prefer to a general form in natural science without any imposed limitations. For the first assumption, we have no strong reason to take it. For the second assumption, the search for the Newtonian gravitational constant never stops. For instance, planetary and spacecraft ranging, neutron star binary observations, paleontological and primordial nucleosynthesis data allow one to constrain the variation of $G$ with time \[12\]. This is the reason why we relax these two assumptions and study STVG further. Firstly, Let us investigate the numerical value of $\gamma$ in STVG attentively with abandoning these two assumptions in comparison with Brans-Dicke theory. We expand metric as follows

\[
\begin{align*}
g_{00} &= -1 + \epsilon^2 N + \epsilon^4 L + O(5), \\
g_{0i} &= \epsilon^3 L_i + O(5), \\
g_{ij} &= \delta_{ij} + \epsilon^2 H_{ij} + O(4),
\end{align*}
\]

by using Chandrasekhar’s approach \[13\], we have found $N$ and $H_{ij}$ only depend on the scalar field $G$ and matter in STVG (for details see subsection D of section II) and where $\epsilon = 1/c$ and $O(n)$ means of order $\epsilon^n$. Then, we can compare STVG with Brans-Dicke theory (BD) \[14\]. The action of BD is

\[
S_{BD} = \frac{c^3}{16\pi} \int \left( \phi R - \varsigma_0 g^{\mu\nu} \frac{\phi \Phi_{\mu;\nu}}{\phi} \right) \sqrt{-g} d^4x + S_M,
\]

where $\varsigma_0$ is a coupling constant ($\omega_0$ usually is used in BD, here we use $\varsigma_0$ to avoid confusion with another symbol in the context). For STVG \[1, 3, 4, 15\], the action by only considering the metric and $G$ is

\[
S_{STVG} = \frac{c^3}{16\pi} \int \left( \frac{1}{G} R + \frac{1}{2} g^{\mu\nu} \frac{G_{\mu\nu} G_{,\nu}}{G^3} \right) \sqrt{-g} d^4x + S_M.
\]

Comparing Eq. (6) with Eq. (7), we have $G = 1/\phi$ and $\varsigma_0 = -1/2$. In BD, $\gamma \equiv (\varsigma_0 + 1)/(\varsigma_0 + 2)$ and when $\varsigma_0 \to \infty$, $\gamma = 1$. With the correspondence between BD and STVG, we obtain $\gamma \equiv (\varsigma_0 + 1)/(\varsigma_0 + 2) = 1/3 \neq 1$ in Moffat’s STVG. (But in some Refs. \[3, 5, 7\], the
value of $\zeta_0$ is $1/2$ due to the sign of the scalar field action changed and correspondent $\gamma$ is $3/5$.) In subsection D of section II and Appendix A, we give a strict proof about $\gamma \neq 1$ of STVG in general cases.

Although STVG is a theory that violates EEP, with relaxing the above two assumptions, the departure of $\gamma$ from 1 should not be out of the range restricted by current experiments. For example, the measurement of $\gamma$ in the Cassini experiment gives $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ \cite{16}. The data analysis for this result is conducted under standard PPN framework. One might argue that STVG modifies the Newtonian law by introducing a Yukawa potential, which breaks the PPN framework and therefore the constraint given by Cassini may not be used directly here. Another measurement of $\gamma$ comes from the Lense-Thirring effect (precession of a gyroscope located near a rotating massive body). Lense-Thirring effect will cause advance of the ascending node of the orbit of an earth satellite, which depends only on a force perpendicular to its orbital plane. But the Yukawa force is confined in the orbital plane so that the measurements of Lense-Thirring effect can give a clear and direct constraint on $\gamma$ in STVG. Although the Lense-Thirring effect for LAGEOS and LAGEOSII are at a precision level of 10% \cite{17}, worse than the Cassini experiment, this experiment demands that $\gamma$ can not departs from 1 too much (We will explain this in subsection A of section III in detail). This is the main motivation to propose a modified STVG theory in this paper when we abandon these two assumptions. It is more competitive for present and future experiments.

With discovery of the binary pulsar PSR B1913+16 by Hulse and Taylor \cite{18}, binary pulsars promises an unprecedented opportunity to measure the effects of relativistic gravitation (see \cite{19, 20} for a review). For example, pulsar timing has provided indirect evidence for the existence of gravitational waves \cite{21}, the binary pulsars data can constrain the existence of massive black hole binaries \cite{22}, and the binary pulsars can also test the effects of strong relativistic internal gravitational fields on orbital dynamics \cite{23}. In addition, binary pulsars could help us to test various gravity theories. By fitting the arrival time of pulsars, observational parameters of binary pulsar are obtained in high precision. They are “physical” parameters, “Keplerian” parameters and “post-Keplerian” parameters \cite{24, 25}. A very important class is “post-Keplerian” parameters \cite{26, 27} which include the average rate of periastron shift $\dot{\omega}$, redshift dilation $\gamma$, orbital period derivative $\dot{P}_b$, and two Shapiro-delay parameters $s$ and $r$. It worthy of noted that the periastron advance for binary pulsars
could reach several degrees per year, which is about $10^5$ more than the perihelion advance of Mercury. Hence, the relativistic effects from binary pulsar are more remarkable than other celestial systems. We will take a sample of binary pulsars for testing MSTVG by using their $\dot{\omega}$. In this paper we chose four best studied pulsar binaries. They are respectively PSR B1913+16, PSR B1534+12, PSR J0737-3039, and PSR B2127+11C, which are double neutron star binaries. We mainly focus on $\dot{\omega}$ of these four binaries data to constrain the parameters of MSTVG.

Through introducing a coupling function of the scalar field $G$: $\theta(G)$, we obtain the metric and equations of motion (EOM) for general matter without specific equation of state and $N$ point masses in the first order post-Newtonian (1PN) approximation by Chandrasekhar’s approach. Then, the secular periastron precession of binary pulsars for 1PN in harmonic coordinates is derived. After two PPN parameters ($\gamma$ and $\beta$) and two Yukawa parameters ($\alpha$ and $\lambda$) are discussed and compared, we fix $\gamma = 1$ and $\beta = 1$ and constrain the Yukawa parameters. This constraint coming from binary pulsars systems on $\alpha$ and $\lambda$ are consistent with the results from the solar system such as the earth-satellite measurement of earth gravity, the lunar orbiter measurement of lunar gravity, and lunar laser ranging measurement to constrain the fifth force.

In what follows, our conventions and notations generally follow those of [28], the metric signature is (-, +, +, +). Greek indices take the values from 0 to 3, while Latin indices take the values from 1 to 3. A comma denotes a partial derivative, and semicolon denotes a covariant derivative. Bold letters denote spatial vectors. The plot of this paper is as follows. In the next section, the equations of motion and $\dot{\omega}$ for binary of MSTVG in 1PN are given. Subsequently, in the third section, we discuss parameters of MSTVG in detail. We then constrain the parameters of MSTVG by means of fitting $\dot{\omega}$ for four binary pulsars and deal with the details of discussion about results in the forth section. Finally, constraints method and results are outlined in the last section.
II. THE THEORY

A. Action and field equations

Through introducing a coupling function of the scalar field $G$: $\theta(G)$, we adopt the following modified action for STVG

$$S = \int (\mathcal{L}_G + \mathcal{L}_\phi + \mathcal{L}_s)\sqrt{-gd^4x} + S_M(g_{\mu\nu}, \phi_\mu, \Psi),$$

where $\Psi$ denotes all the matter fields. In Eq. (8), the matter fields $\Psi$ interact with both the metric field and the vector field. This means that the trajectory of a free-fall test particle depends not only on the space-time geometry but also on the vector field so that it violates the Einstein equivalence principle (EEP). Although the current experiments verify EEP to a very high accuracy in the Solar System scale, violations of EEP at galactic and cosmological distance scales cannot be ruled out. In Eq. (8), the Lagrangian densities of the gravitational field, vector field and scalar fields are respectively

$$\mathcal{L}_G = \frac{c^3}{16\pi G}(R + 2\Lambda),$$

$$\mathcal{L}_\phi = -\frac{1}{c}\omega \left[ \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}m^2k^2\phi_\mu\phi^\mu - V_\phi(\phi) \right],$$

$$\mathcal{L}_s = -\frac{c^3}{16\pi G} \left[ \frac{1}{2}g^{\mu\nu} \left( \theta(G) \frac{G_\mu G_\nu}{G^2} - \frac{m_\mu m_\nu}{m^2} + \omega_\mu\omega_\nu \right) - \frac{V_G(G)}{G^2} + \frac{V_m(m)}{m^2} - V_\omega(\omega) \right],$$

where $\Lambda$ denotes cosmological constant and $V_X(X)$ denotes the self-interaction potential function of an field. Besides, $k = c^2/h^2$ and $B_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}$. Where $c$ is the ultimate speed of the special theory of relativity and $h$ is the reduced Planck constant. It is noted that in Moffat’s STVG [1, 3, 4, 15], $\theta(G) = -1$ in Eq. (11).

When we omit $\Lambda$ and all the self-interaction potentials $V_X(X)$, equations of gravitational field are obtained by variation of the action (8) with respect to $g^{\mu\nu}$.

$$R_{\mu\nu} = \frac{8\pi G}{c^2} \left[ T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T + \omega \frac{m^2k^2}{c^2}\phi_\mu\phi_\nu + \frac{1}{c^2}\omega \left( B_{\mu\kappa}B_\nu^\kappa - \frac{1}{4}g_{\mu\nu}B_\kappa\lambda B^{\kappa\lambda} \right) \right]$$

$$+ \frac{1}{2} \left( \theta(G) \frac{G_\mu G_\nu}{G^2} + 4G_\mu G_\nu - \frac{m_\mu m_\nu}{m^2} - 2G_{\mu\nu} + \omega_\mu\omega_\nu \right)$$

$$- \frac{1}{2}g_{\mu\nu} \left( \frac{G_{\kappa\lambda}}{G} - 2G_\kappa G^{\kappa\lambda} \right),$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor of matter which is defined by

$$\frac{c}{2}\sqrt{-g}T_{\mu\nu} = -\frac{\delta S_M}{\delta g^{\mu\nu}}.$$
Following [29] and [30, 31], we define mass, current, and stress density as

\[ \sigma \equiv T^{00} + T^{kk}, \]
\[ \sigma_i \equiv cT^{0i}, \]
\[ \sigma_{ij} \equiv c^2 T^{ij}. \]

(14) \hspace{1cm} (15) \hspace{1cm} (16)

It is worth emphasizing that, in these definitions, the material is described by the energy-momentum tensor without specific equation of state.

Variation of the action with respect to \( \phi^\mu \) yields

\[ \omega B^\mu_\nu + B^{\mu \nu} \omega_\nu - \omega m^2 k^2 \phi^\mu = J^\mu, \]

(17)

where \( J^\mu \) is a “fifth force” matter current defined as

\[ \sqrt{-g} J^\nu \equiv - \frac{\delta S_M}{\delta \phi_\nu}. \]

(18)

We further define

\[ J^0 \equiv J^0, \quad J^i \equiv \epsilon J^i. \]

(19)

When we substitute \( k = c^2/\hbar^2 \) and \( B_{\mu \nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu} \) into Eq. (17), we can obtain the Proca equation if \( \omega = \text{Const.} \) and \( J^\mu = 0. \)

Variation with respect to the scalar fields yield respectively

\[ \frac{(\theta(G) + 3)}{G} G^{\kappa \kappa} - 2(\theta(G) + 3) \frac{G_{\kappa \lambda} G^{\kappa \lambda}}{G^2} - \frac{1}{2} \frac{d\theta(G)}{dG} G_{\kappa \lambda} G^{\kappa \lambda} = - \frac{8\pi G}{c^2} \left( T - \omega \frac{m^2 k^2}{c^2} \phi^\mu \phi^\mu \right), \]

(20)

\[ \frac{1}{m} m^\nu_\nu - \frac{1}{m^2} m^{\mu \nu} m_\nu - \frac{1}{Gm} G^{\mu \nu} m_\nu = - \frac{16\pi G}{c^4} \omega m^2 k^2 \phi^\mu \phi^\mu, \]

(21)

\[ \omega^\nu_\nu - \frac{1}{G} G^{\mu \nu} \omega^\mu_\nu = \frac{16\pi G}{c^4} \left( \frac{1}{4} B^{\mu \nu} B_{\mu \nu} + \frac{1}{2} m^2 k^2 \phi^\mu \phi^\mu \right). \]

(22)

B. Perturbation of MSTVG

Following the approach in [13, 32], we deal with MSTVG in the form of a Taylor expansion in the parameter \( \epsilon \equiv 1/c, \) similar to the expansion of the metric tensor in Eqs. (3), (4) and (5). For the expansions of vector field,

\[ \phi_0 = (0) \varphi_0 + \epsilon^2 (2) \varphi_0, \]

(23)

\[ \phi_i = \epsilon (1) \varphi_i. \]

(24)
and for the expansion of scalar field $G$

$$G = G_0(1 + \xi) = G_0 \left( 1 + \epsilon^2 G + \epsilon^4 G \right), \quad (25)$$

$$\theta(G) = \theta_0 + \theta_1 \xi + \frac{1}{2} \theta_2 \xi^2 + \cdots, \quad (26)$$

$$\theta(G) = \theta_0 + \epsilon^2 \theta_1 G + \epsilon^4 \left( \frac{1}{2} \theta_2 (2G)^2 + \theta_1 G \right), \quad (27)$$

$$\frac{d\theta}{dG} = \frac{1}{G_0} \left( \theta_1 + \epsilon^2 \theta_2 G \right), \quad (28)$$

where $G_0$ is the background value of scalar field $G$. The expansions of other scalar fields are

$$m = m_0 \left(1 + \epsilon^2 \frac{m}{m} + \epsilon^4 \frac{m}{m} \right), \quad (29)$$

$$\omega = \omega_0 \left(1 + \epsilon^2 \frac{\omega}{\omega} + \epsilon^4 \frac{\omega}{\omega} \right), \quad (30)$$

where $m_0$ and $\omega_0$ are respectively the background values for the scalar fields $m$ and $\omega$.

C. Gauge condition

We use the gauge condition imposed on the component of the metric tensor proposed by Kopeikin & Vlasov \[33\] as follows:

$$\left( \frac{G_0}{G} \sqrt{-g} g^{\mu\nu} \right)_{,\nu} = 0. \quad (31)$$

Noted that the scalar field $G$ in MSTVG is in the inverse ratio to the scalar field $\phi$ in \[33\]. To 1PN order, this gauge gives

$$\epsilon^2 \left( \frac{1}{2} H_{,i} - \frac{1}{2} N_{,i} - H_{ik,k} - G_{,i} \right) = 0, \quad (32)$$

and

$$\epsilon^3 \left( L_{k,k} - \frac{1}{2} H_{,t} - \frac{1}{2} N_{,t} + G_{,t} \right) = 0. \quad (33)$$

Through covariant divergence of Eq. (17), we derive a useful formula

$$\varphi_{0,t} - \varphi_{k,k} = \frac{1}{m_0^2 \omega_0} \left( J_0^0 + J_1^k \right) + \mathcal{O}(2). \quad (34)$$
D. First order post-Newtonian approximation for MSTVG

Based on fields equations of Eqs. (12), (17), (20), (21) and (22), we obtain the result of MSTVG in 1PN by using the gauge conditions (32), (33) and Eq. (34) as follows.

The equation for $N$ and $G$ are

\[ \Box N = -8\pi G\sigma, \quad (35) \]
\[ \Box G = (1 - \gamma)4\pi G\sigma, \quad (36) \]

where $\Box$ is the D’Alembert operator in the Minkowski spacetime and Newton’s constant $G$ is related to the constant $G_0$ by

\[ G = \frac{4 + \theta_0}{3 + \theta_0}G_0. \quad (37) \]

Metric $H_{ij}$ is

\[ \Box H_{ij} = -8\gamma\pi G\sigma\delta_{ij}, \quad (38) \]

where we can define $H_{ij} \equiv \delta_{ij}V$ and

\[ \gamma \equiv \frac{\theta_0 + 2}{\theta_0 + 4}. \quad (39) \]

Appendix A gives mathematical details of the derivation of this important formula. A rigorous presentation in Appendix B identifies the parameter $\gamma$ defined in Eq. (39) is just the PPN parameter with the same symbol.

With $\theta_0 = -1$, we obtain $\gamma = 1/3$ which just reduces to STVG. Then, Eq. (37) becomes

\[ G = \frac{2}{1 + \gamma}G_0. \quad (40) \]

It is evident that $G = G_0$ when $\gamma = 1$. The above shows that only the scalar field $G$ enters the metric $N$ and $H_{ij}$ and we have given a strict proof about $\gamma \neq 1$ in general cases of STVG by Chandrasekhar’s approach. (see Appendix A and B)

Other metric components in 1PN are respectively

\[ \Box L_i = 8(1 + \gamma)\pi G\sigma_i, \quad (41) \]
\[ \Box L = -4\pi G \left[ (3\gamma - 2\beta - 1)N\sigma - 2(1 - \gamma)\sigma_{kk} ight. \\
+ (3\gamma + 1)\varphi_0^{(0)}J^0 + \frac{1}{2}(1 + \gamma)\omega_0\Delta\varphi_0^{(0)}^2 \\
+ 2\gamma\omega_0\varphi_0\Delta\varphi_0^{(0)} \bigg] - \frac{1}{2}\beta\Delta N^2, \quad (42) \]
TABLE I: PPN parameters in MSTVG.

| γ   | β     | ξ | α₁ | α₂ | α₃ | ζ₁ | ζ₂ | ζ₃ | ζ₄ |
|-----|-------|---|----|----|----|----|----|----|----|
| $\frac{\theta_0 + 2}{\theta_0 + 4}$ | $1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2}$ | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |

where $\Delta = \nabla^2$ and

$$\beta \equiv 1 + \frac{\theta_1 (1 - \gamma)^3}{8(1 + \gamma)} = 1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2}. \tag{43}$$

Appendix B identifies $\beta$ in Eq. (43) as the corresponding PPN parameter with the same symbol. In that Appendix all the other PPN parameters except $\gamma$ and $\beta$ are proved all zero for MSVTG. The ten PPN parameters for MSVTG are listed in Table I.

Equations for the vector and other scalar fields are

$$\Box \varphi^{(0)}_0 - m^2_{0}\kappa \varphi^{(0)}_0 = -\frac{J^0}{\omega_0}, \tag{44}$$

$$\Box m^{(2)} = 0, \tag{45}$$

$$\Box \omega^{(2)} = 0. \tag{46}$$

In Eq. (44), if we consider a vacuum case ($J^\mu = 0$), we find that the speed of vector field is not equal to the velocity of light when we assume a plane wave solution which may cause chromatic dispersion in vacuum.

E. Equations of motion for MSTVG in 1PN

Based on Ref. [34], the equations of motion (EOM) derived from $T^\mu_\nu = 0$ is equivalent to $G^\mu_\nu = 0$ if EEP is satisfied. However, STVG violates EEP due to the vector field. So, equations of motion in MSTVG have to be derived from the covariant divergence of the Einstein tensor $G^\mu_\nu \equiv 0$. On the other hand, according to [34], we can infer that the equations of motion in MSTVG must include the contribution of the vector field besides the
matter and metric. The Einstein tensor $G^\mu_\nu$ in MSTVG is
\[
R^\mu_\nu - \frac{1}{2}g^\mu_\nu R = \frac{8\pi G}{c^2} \left[ T^\mu_\nu + \frac{\omega}{c^2} \left( B^\mu_\kappa B^\nu_\kappa - \frac{1}{4} g^\mu_\nu B_\lambda B^\lambda \right) \right] \\
+ \frac{\omega}{c^2} \phi^\mu \phi^\nu - \frac{m^2 k^2}{c^2} \omega g^\mu_\nu \phi_\alpha \phi^\alpha \\
+ \frac{1}{2} \left( \theta(G) G^\mu_\nu + \frac{4}{G^2} G^\mu_\nu - 2 \frac{G^\mu_\nu}{G} \right) \\
- \frac{m^\mu m^\nu}{m^2} + \omega m^\mu \omega^\nu \\
- \frac{1}{4} g^\mu_\nu \left( \theta(G) G^\mu_\kappa G^\nu_\kappa + \frac{8}{G^2} G^\mu_\kappa G^\nu_\kappa - \frac{4}{G} G^\mu_\nu \right) \\
- \frac{m^\mu m^\nu}{m^2} + \omega m^\mu \omega^\nu. \tag{47}
\]

Then, the Bianchi identities $G^\mu_\nu \equiv 0$ yields the momentum equation
\[
\sigma_{i,t} + \sigma_{ik,k} - \frac{1}{2} \sigma_{N,i} + \varphi_{0,i} J^0 \\
+ \epsilon^2 \left( \frac{1}{2} \sigma_{i,t} N_i + \sigma_{i,t} L_i - \frac{1}{2} \sigma_{i,t} L_i \\
+ \frac{5}{2} \sigma_{i,t} V_i - \frac{1}{2} \sigma_{i,t} N_i + \sigma_k L_{i,k} - \sigma_{k,i} L_{k,i} \\
+ \frac{5}{2} \sigma_{i,k} V_{i,k} - \frac{1}{2} \sigma_{i,k} N_{i,k} + \frac{1}{2} \sigma_{k,i} N_{i,k} - \frac{1}{2} \sigma_{k,k} V_{i,k} \\
- \sigma_{i,k} V_{i,k} - \sigma_{i,k} J^0 - \sigma_{i,k} J^0 \\
+ \varphi_{0,i} J^k - \varphi_{k,i} J^k - \varphi_{i,k} (J_{i,t} + J_{k,k}) \right) = 0. \tag{48}
\]

$G^\mu_\nu \equiv 0$ yields the continuity equation
\[
\sigma_{i,t} + \sigma_{k,k} \\
+ \epsilon^2 \left( \frac{3}{2} V_{i,t}\sigma - N_{i,t}\sigma + \frac{3}{2} V_{k,t}\sigma_k - N_{k,t}\sigma_k - \sigma_{k,k,t} + \varphi_{0,k} J^k \\
+ \varphi_{0,i} (J_{i,t} + J_{k,k}) \right) = 0. \tag{49}
\]

F. N-body pointlike for MSTVG in 1PN

Considering an N-body system of nonspinning point masses for simplicity, we follow the notation adopted by [35] and use the matter stress energy tensor as follows
\[
c^2 T^\mu_\nu (x,t) = \sum_a \mu_a (t) \nu^\mu_a \nu^\nu_a \delta (x - y_a (t)), \tag{50}
\]
where \( \delta \) denotes the three-dimensional Dirac distribution, the trajectory of the \( a \)th mass is represented by \( y_a(t) \), the coordinate velocity of the \( a \)th body are \( v_a = dy_a(t)/dt \) and

\[
v_a^\mu \equiv (c, v_a),
\]

and \( \mu_a \) denotes an effective time-dependent mass of the \( a \)th body defined by

\[
\mu_a(t) = \left( \frac{M_a}{\sqrt{g g_{\rho\sigma} v_a^\rho v_a^\sigma}} \right)_a,
\]

(51)

where subscript \( a \) denotes evaluation at the \( a \)th body and \( M_a \) is the constant Schwarzschild mass. Another useful notation is

\[
\tilde{\mu}_a(t) = \mu_a(t) \left[ 1 + \frac{v_a^2}{c^2} \right],
\]

(52)

where \( v_a^2 = v_a^2 \). Both \( \mu_a \) and \( \tilde{\mu}_a \) reduce to the Schwarzschild mass at Newtonian order:

\[
\mu_a = M_a + O(e^2) \quad \text{and} \quad \tilde{\mu}_a = M_a + O(e^2).
\]

Then the mass, current, and stress densities in Eqs. (14), (15) and (16) for the \( N \) point masses read

\[
\sigma = \sum_a \tilde{\mu}_a \delta(x - y_a(t)),
\]

(53)

\[
\sigma_i = \sum_a \mu_a v_i^a \delta(x - y_a(t)),
\]

(54)

\[
\sigma_{ij} = \sum_a \mu_a v_i^a v_j^a \delta(x - y_a(t)).
\]

(55)

We now assume a “fifth force” matter current

\[
cJ^\mu(x, t) = \sum_a Q_a v_a^\mu \delta(x - y_a(t)),
\]

(56)

where the “fifth force charge” is \( Q_a \equiv \kappa \sqrt{G \omega} \mu_a(t) \) which coupled with ordinary mass through coupling function \( \omega \), where \( \kappa \) is a dimensionless constant. This assumption implies that this charge is proportional to ordinary mass as in (15). Substituting Eq. (56) into Eq. (19), we obtain

\[
J^0 = \sum_a \kappa \sqrt{G \omega} \mu_a \delta(x - y_a(t)),
\]

\[
J^i = \sum_a \kappa \sqrt{G \omega} \mu_a v_i^a \delta(x - y_a(t)).
\]

(57)

Then, we obtain the following form in Newtonian approximation,

\[
N = 2 \Delta^{-1} \{-4\pi G \sigma\} = 2 \sum_a \frac{GM_a}{r_a} + O(2),
\]

(58)
and
\[
\Phi_0^{(0)} = \frac{1}{4\pi \omega_0} \int \frac{e^{-m_0 |x-z|}}{|x-z|} J^0(z, t) d^3z
\]
\[
= \frac{\kappa}{4\pi} \sqrt{(1 + \gamma) G} \sum_a \frac{M_a e^{-m_0 k r_a}}{r_a} + \mathcal{O}(2). \tag{59}
\]

Through integration of Eq. (48), we have EOM in Newtonian approximation as follows
\[
\frac{dv^i_a}{dt} = - \sum_{b\neq a} \frac{GM_b}{r_{ab}^3} v^i_{ab} \left\{ 1 - \frac{\kappa^2 \omega_0 (1 + \gamma)}{8\pi} e^{-m_0 k r_{ab}} \left( 1 + m_0 k r_{ab} \right) \right\} + \mathcal{O}(2). \tag{60}
\]

From Eq. (60), the gravitational potential in Newtonian approximation for MSTVG is that
\[
U(r) = - \sum_{b\neq a} \frac{GM_b}{r_{ab}} \left( 1 - \frac{k^2 \omega_0 (1 + \gamma)}{8\pi} e^{-m_0 k r_{ab}} \right). \tag{61}
\]

On the other hand, Fischbach and Talmadge \[36\] provided the following potential
\[
U(r) = - \sum_{b\neq a} \frac{GM_b}{r_{ab}} \left( 1 + \alpha e^{-r_{ab}/\lambda} \right), \tag{62}
\]
which includes a usual Newtonian gravitational potential and a Yukawa-type one. Compared between Eq. (61) and Eq. (62), we define \(m_0 k = 1/\lambda\) and \(\alpha = -\frac{\kappa^2 (1+\gamma)}{8\pi} \omega_0\). Then, Eq. (60) becomes
\[
\frac{dv^i_a}{dt} = - \sum_{b\neq a} \frac{GM_b}{r_{ab}^3} v^i_{ab} \left\{ 1 + \alpha e^{-r_{ab}/\lambda} \left( 1 + \frac{r_{ab}}{\lambda} \right) \right\} + \mathcal{O}(2). \tag{63}
\]

If MSTVG could explain galaxy rotation curves without exotic dark matter, it must have \(\alpha > 0\) from Eq. (63).

The next step is to work out \(N\) and \(V\) easily in 1PN approximation as
\[
N = 2 \sum_a \frac{GM_a}{r_a} + \epsilon^2 \left\{ \sum_a \frac{GM_a}{r_a} \left[ 4v^2_a - (n_a v_a)^2 \right] + 2(2 - 3\gamma) \sum_a \sum_{b\neq a} \frac{G^2 M_a M_b}{r_a r_{ab}} \right\} + \mathcal{O}(3), \tag{64}
\]
\[
V = 2\gamma \sum_a \frac{GM_a}{r_a} + \mathcal{O}(2), \tag{65}
\]
by the relation of \(\tilde{\mu}_a\) that
\[
\tilde{\mu}_a = M_a \left\{ 1 + \epsilon^2 \left[ \left( N - \frac{3}{2} V \right)_a + \frac{3}{2} v^2_a \right] + \mathcal{O}(4) \right\}
\]
\[
= M_a \left\{ 1 + \epsilon^2 \left( 2 - 3\gamma \right) \sum_{b\neq a} \frac{GM_b}{r_{ab}} + \frac{3}{2} v^2_a \right\} + \mathcal{O}(4). \tag{66}
\]
where \( r_a = |\mathbf{x} - \mathbf{y}_a| \) and \( r_{ab} = |\mathbf{y}_a - \mathbf{y}_b| \). For \( L_i \),

\[
L_i = -2(1 + \gamma) \sum_a \frac{G M_a}{r_a} v_a^i.
\] (67)

Because \( \alpha \) in the Newtonian order is very tiny (see Table III), we omit any coupling terms in the magnitude of \( \alpha \epsilon^2 \). Thus, it yields

\[
L = 2(3\gamma - 2\beta - 1) \sum_a \sum_{b \neq a} \frac{G^2 M_a M_b}{r_a r_{ab}} - 2(1 - \gamma) \sum_a \frac{G M_a}{r_a} v_a^2
\]

\[
-2\beta \sum_a \sum_b \frac{G^2 M_a M_b}{r_a r_{ab}} + \mathcal{O}(1, \alpha).
\] (68)

At last, by integration of Eq. (48), we obtain EOM for MSTVG in 1PN:

\[
\frac{dv_a^i}{dt} = - \sum_{b \neq a} \frac{G M_b}{r_{ab}^2} n_{ab}^i - \alpha \sum_{b \neq a} \frac{G M_b}{r_{ab}^2} n_{ab}^i \left( 1 + \frac{r_{ab}}{\lambda} \right) e^{-r_{ab}/\lambda}
\]

\[
+ \epsilon^2 \left\{ 2(\gamma + \beta) \sum_{b \neq a} \frac{G^2 M_b^2}{r_{ab}^3} n_{ab}^i + (2\gamma + 2\beta + 1) \sum_{b \neq a} \frac{G^2 M_a M_b}{r_{ab}^3} n_{ab}^i \right. 
\]

\[
+ \sum_{b \neq a} \frac{G M_b}{r_{ab}^2} \left[ - \gamma v_a^2 - (1 + \gamma) v_b^2 + \frac{3}{2} (n_{ab} v_a)^2 + 2(1 + \gamma) (v_a v_b) \right] n_{ab}^i 
\]

\[
+ \sum_{b \neq a} \frac{G M_b}{r_{ab}^2} \left[ 2(1 + \gamma) (n_{ab} v_a) - (2\gamma + 1) (n_{ab} v_b) \right] v_{ab}^i 
\]

\[
+ \sum_{b \neq a} \sum_{c \neq a, b} \frac{G^2 M_b M_c}{r_{ab}^2 r_{bc}^2} \left[ (2\beta - 1) \frac{1}{r_{bc}} + 2(\beta + \gamma) \frac{1}{r_{ac}} - \frac{r_{ab}}{2r_{bc}^2} (n_{bc} n_{ab}) \right] n_{ab}^i 
\]

\[
- \frac{1}{2} (4\gamma + 3) \sum_{b \neq a} \sum_{c \neq a, b} \frac{G^2 M_b M_c}{r_{ab}^2 r_{bc}^2} n_{bc}^i \right\} + \mathcal{O}(\epsilon^4, \epsilon^2 \alpha),
\] (69)

where \( n_{ab}^i = (y_a^i - y_b^i)/r_{ab} \), \( \mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b \), and scalar products are denoted with parentheses, e. g., \( (n_{ab} v_{ab}) = \mathbf{n}_{ab} \cdot \mathbf{v}_{ab} \). Eq. (69) will return to Einstein-Infeld-Hoffmann (EIH) equations of motion in the PPN formalism \([24, 37]\) when we eliminate the contribution of the vector field in MSTVG.

G. Secular periastron precession of binary pulsar for MSTVG in 1PN

When we consider only two body (\( M_1 \) and \( M_2 \)) in Eq. (69), EOM of a point-mass binary for MSTVG in 1PN yields

\[
\ddot{\mathbf{r}} = \mathbf{a}_N + \mathbf{a}_{1PN},
\] (70)
TABLE II: Summary of the PPN parameters $\gamma$ and $\beta$ in several gravity theories.

| Parameter | GR | ST | MST | STVG | MSTVG | $\lambda E$ | TeVeS |
|-----------|----|----|-----|------|--------|-------------|-------|
| $\gamma$ | $1$ | $\frac{2n+1}{2n+2}$ | $1 - \frac{2n^2}{1+\omega_0^2}$ | $\frac{1}{3}$ | $\frac{\theta_\alpha}{\theta_\beta}$ | $1$ | $1$ |
| $\beta$  | $1$ | $1 + \frac{\omega_0}{(2n_0+3)(2n_0+4)^2}$ | $1 + \frac{\beta_0n_0^2}{2(1+\alpha_0^2)^2}$ | $1$ | $1 + \frac{\theta_\beta}{2(\theta_0+3)(\theta_0+4)^2}$ | $1$ | $1$ |

Violation of EEP?

where

\[ a_N = -\frac{GM}{r^2} \left[ 1 + \alpha \left( 1 + \frac{r}{\lambda} \right) \exp \left( -\frac{r}{\lambda} \right) \right] n, \] (71)

\[ a_{1 PN} = -\frac{GM}{c^2 r^2} \left\{ \left[ (\gamma + 3\nu)v^2 - 2(\gamma + \beta + \nu) \frac{GM}{r} - \frac{3}{2} \nu r^2 \right] n - 2 \left( 1 + \gamma - \nu \right) \dot{r} v \right\}, \] (72)

where $M = M_1 + M_2$, $\dot{r} = (n_1 \dot{v}_1)$, $v = v_1 - v_2$, $\nu = M_1 M_2 / M^2$, $r = r_12$, $n = n_12$ and $a = dv_12/dt$. Compared with [38], additional term comes from the Yukawa force, namely, the contribution of the vector field in MSTVG. By the aid of the averaging method [39], the secular periastron advance for a binary pulsar in 1PN is

\[ \frac{d\omega}{dt} = \frac{1}{2} \frac{n ap^2}{\lambda^2} e^{-p/\lambda} + (2 + 2\gamma - \beta) \frac{GMn}{p} \epsilon^2, \] (73)

where $n^2a^3 = GM$, $p = a(1 - e^2)$, $a$ is the semi-major axis and $e$ is the eccentricity of the binary. Here the periastron shift of Yukawa force is

\[ \frac{d\omega}{dt}_{Yukawa} = \frac{1}{2} \frac{n ap^2}{\lambda^2} e^{-p/\lambda}, \] (74)

which is different from the result given by [40]. In this paper, we keep the angular momentum in the process of using the averaging method instead of approximate treatments such as $\exp(-r/\lambda) \approx 1 - r/\lambda$ [40] to calculate the $\dot{\omega}$.

III. SUMMARY OF PARAMETERS FOR MSTVG

There are four parameters in MSTVG. They are two PPN parameters ($\gamma$, $\beta$) and two Yukawa parameters ($\alpha$, $\lambda$). In this section, we mainly discuss these four parameters.
A. Review and discussion on PPN parameters

For a slow motion and weak field, PPN formalism introduces 10 parameters in a post-Newtonian metric to include various gravity theories \[41, 42\]. PPN formalism can be traced back to Eddington-Robertson-Schiff formalism \[43, 44, 45\], which introduce two PPN parameters $\gamma$ and $\beta$ by only considering one point mass. At the same time, these two parameters were endowed with some kind of meaning. For example, $\gamma$ denotes the level of space curvature and $\beta$ can be treated as described the nonlinearity in the superposition law \[24\]. Table II lists the results of $\gamma$ and $\beta$ in: (1) General relativity (GR); (2) Scalar-Tensor theory \[46, 47\] (ST, where $\omega_0$ and $\omega_1$ are coefficients of the coupling function of scalar field $\phi$: $\theta(\phi) = \omega_0 + \omega_1 \xi + \mathcal{O}(\xi^2)$ if $\phi = \phi_0(1 + \xi)$); (3) Multi-Scalar-Tensor theory \[30\] (MST, where $\alpha_0 = \partial \ln A(\phi_0)/\partial \phi_0$, $\beta_0 = \partial \alpha(\phi_0)/\partial \phi_0$); (4) Scalar-Tensor-Vector theory \[1\] (STVG); (5) Modified Scalar-Tensor-Vector theory in this paper (MSTVG); (6) Einstein-aether theory \[48\] ($\mathcal{E}$); (7) The tensor-vector-scalar theory provided by \[49\] (TeVeS, the results $\gamma$ and $\beta$ given by \[50\]).

For different theories listed in Table III EEP is satisfied except MST, STVG and MSTVG. PPN formalism only works in validity of EEP so that comparisons with PPN parameters of all kinds of gravity theories, especially MST, STVG and MSTVG, is only phenomenological. We are going to argue that even though the PPN formalism is only valid under EEP, $\gamma$ should not depart from 1 too much in STVG. Generally, the deflection of light can be used to test the parameter $\gamma$. Equations of motion for photons in 1PN contain only two metric coefficients: $N$ and $V$. For MSTVG, equations of motion for photons in 1PN still contain the contribution of vector field besides the two metric coefficients. When considering only one point mass, equations of motion for photons in 1PN for MSTVG is as follows

\[
\frac{d^2 x^i}{dt^2} = -\frac{GM}{r^3} r^i - \alpha \frac{GM}{r^3} r^i \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda} + \epsilon^2 \left[ -\gamma \frac{GM}{r^3} \frac{dx^i}{dt} \right]^2 + 2(1 + \gamma) \frac{GM}{r^3} \left( r \cdot \frac{dx}{dt} \right) \frac{dx^i}{dt}, \tag{75}
\]

\[
0 = g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}. \tag{76}
\]

Let us assume the Newtonian order solution of these equation is

\[
x^i_N \equiv c \hat{n}^i(t - t_0), \tag{77}
\]
where $|\hat{n}| = 1$ and light in the Newtonian order travels in a straight line at constant speed $c$. Furthermore, we assume the first order solution of these equation is

$$x^i \equiv c\hat{n}^i(t - t_0) + x^i_p,$$

where $\hat{n}^i$ is the initial emitting direction of a light signal. Then, substituting Eq. (78) into Eqs. (75)-(76), we obtain

$$\frac{d^2x_p}{dt^2} = -\frac{GM}{r^3}\left[1 + \gamma + \alpha \left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}\right]\mathbf{r} + 2(1 + \gamma)\frac{GM}{r^3}\hat{n}(\hat{n} \cdot \mathbf{r}),$$

$$\hat{n} \cdot \frac{dx_p}{dt} = -\epsilon(1 + \gamma)\frac{GM}{r}.$$ (79)

Followed the approach of Ref. [24], by using Eqs. (79)-(80), the light deflection angle up to 1PN approximation for MSTVG is

$$\Delta \phi = 2\epsilon^2\frac{GM}{d}\left(1 + \gamma + \alpha e^{-d/\lambda}\right),$$

where $d$ represents the coordinate radius at the point of closest approach of the ray. Then the deviation from GR is

$$\delta \phi \equiv \frac{\Delta \phi_{MSTVG} - \Delta \phi_{GR}}{\Delta \phi_{GR}} = \frac{1}{2}\left(\gamma - 1 + \alpha e^{-d/\lambda}\right),$$

which comes from two parts: PPN parameter $\gamma$ and the vector field. If $\gamma = 1/3$, the case of Eq. (81) and (82) will return to STVG [1]. Here, one would claim that the part from the vector field contributes to remaining 2/3 which could make up $\gamma \neq 1$ in STVG. However, this is not the case. When considering the Lense-Thirring drag on the orbit of an earth-satellite, the longitude of the ascending node $\Omega$ and the argument of pericenter $\omega$ vary with time by using the averaging method [39]:

$$\frac{d\Omega}{dt} = \frac{(1 + \gamma)Ge^2J_{\text{earth}}}{a^3(1 - e^2)^{3/2}},$$

$$\frac{d\omega}{dt} = \frac{2(1 + \gamma)Ge^2J_{\text{earth}}}{a^3(1 - e^2)^{3/2}}\cos i,$$

where $J_{\text{earth}}$ is the angular momentum of the earth and $i$ is the inclination. Eqs. (83) and (84) return to GR when $\gamma = 1$. In Lense-Thirring effect, it worthy of note that $d\Omega/dt$, namely Eq. (83), only depends on the force which is perpendicular to the orbital plane. In
STVG, however, the Yukawa force can not affect \(d\Omega/dt\) because it is a radial force in the orbital plane. For this reason, \(d\Omega/dt\) can limit \(\gamma\) very closely even though STVG violate the EEP. This is the reason why we have to modified STVG.

Now, we mainly focus on the three gravity theories that include a vector field. They are respectively \(\mathcal{A}\) \[48\], TeVeS \[49\] and STVG in this paper. \(\mathcal{A}\) is a tensor-vector theory in which the vector field is massless, unitary and time like. This theory investigates preferred frames and violation of Lorentz invariance. There are five gravitational and aether wave modes for \(\mathcal{A}\)-theory \[51\]. Recently, Xie and Huang \[52\] presented a 2PN approximation of \(\mathcal{A}\)-theory and found that the linearized waves with the spin-0 and spin-1 modes in \(\mathcal{A}\) will propagate with infinite velocities if the 2PN light deflection angle in \(\mathcal{A}\)-theory is identical with that of GR. MOdified Newtonian Dynamics (MOND) \[53, 54, 55\] has gained recognition as a successful scheme for explaining galaxy rotation curves without invoking dark matter. TeVeS is a scenario of relativistic MOND, which includes a unit massless timelike vector and a scalar field. In TeVeS, modified Newtonian acceleration is from two positive dimensionless parameters \(K\) and \(k\) but not scalar or vector field. TeVeS passes the usual solar system tests and provides a specific formalism for constructing cosmological models. For STVG, there are three scalar fields, one vector field of rest mass \(m\) besides the metric field. Both STVG and TeVeS try to explain galaxy rotation curves without dark matter, but the modified Newtonian acceleration in STVG is from the vector field.

For MSTVG in 1PN, other eight PPN parameters except \(\gamma\) and \(\beta\) are all zero. Measurement of the deflection of light passing Jupiter by Very Long Baseline Interferometer (VLBI) gives \(\gamma - 1 = (-1.7 \pm 4.5) \times 10^{-4}\) \[56\]. The most precise measurement of \(\gamma\) comes from the Cassini experiment by Doppler tracking, which gives \(\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}\) \[16\]. By comparing the masses of 15 elliptical lensing galaxies from Sloan Lens Advanced Camera for Surveys (SLACS) on kiloparsec scales (about \(10^{19}\)m), Bolton et al. \[57\] constrain \(\gamma = 0.98 \pm 0.07\) in 1\(\sigma\) confidence level (CL). For \(\beta\), although its accuracy level is lower than that of \(\gamma\), some experiments give the limits of \(\beta\). For example, \(\beta - 1 = 3 \times 10^{-3}\) comes from the perihelion shift of Mercury \[58\] and \(\beta - 1 = 2.3 \times 10^{-4}\) comes from the Nordtvedt effect \[59, 60\]. Tests of \(\gamma\) and \(\beta\) are basically going on the solar system scale. However, according to present-day experiment data, the values of these two parameters (\(\gamma\) and \(\beta\)) are independent of the scale of the testing systems. In order to obtain \(\gamma = 1\) and \(\beta = 1\), the parameter \(\theta_0\) in MSTVG must go to infinity with \(\theta_1\) growing slower than \(\theta_0^3\).
### TABLE III: Yukawa parameters \( \alpha \) and \( \lambda \) in macroscopical range.

|                      | \( |\alpha| \)       | \( \lambda \)          | Ref.   |
|----------------------|----------------------|-------------------------|--------|
| SDSS                 | 0.35 \( \pm \) 0.9   | 10Mpc/h < \( \lambda \) < 100Mpc/h | [63]   |
| 2dFGRS               | 0.025 \( \pm \) 1.7  | 10Mpc/h < \( \lambda \) < 100Mpc/h | [63]   |
| Pioneer anomaly 10/11, |                      |                         |        |
| Galileo, and Ulysses data | 10\(^{-3}\)      | 200AU                    | [69]   |
| Long range limit     | > 10\(^{-4}\)       | > 70AU                   | [70]   |
| Pioneer anomaly (STVG) | 10\(^{-3}\)      | 47 \pm 1AU               | [71]   |
| Constraint from Sun (STVG) | 10\(^{-8}\)  | 10\(^{15}\) m           | [15]   |
| Constraint from Earth (STVG) | 10\(^{-13}\)  | 10\(^{13}\) m           | [15]   |
| Planetary data (EPM2004) | 10\(^{-12}\) \( \sim \) 10\(^{-13}\) | 1AU                      | [65]   |
| Constraint on solar system | (0.3 \( \pm \) 2.7) \( \times \) 10\(^{-11}\) | 0.2 \( \pm \) 0.4AU       | [40]   |
| Planetary motions     | < 10\(^{-9}\)       | < 0.18AU                 | [66]   |
| LLR                   | > 10\(^{-10}\)      | 10\(^{7}\) m < \( \lambda \) < 10\(^{13}\) m | [36, 62] |
| Planetary data        | > 10\(^{-9}\)       | > 10\(^{9}\) m           | [36]   |
| Constraint on intermediate-range gravity | \( \approx \) 10\(^{-8}\) | \( 1.2 \times 10^{7}\) m < \( \lambda \) < 3.8 \( \times \) 10\(^{8}\) m | [64]   |
| LAGEOS-lunar          | 10\(^{-8}\) \( < |\alpha| < \) 10\(^{-5}\) | 10\(^{5}\) m < \( \lambda \) < 10\(^{9}\) m | [36]   |
| Earth-LAGEOS         | 10\(^{-6}\) \( < |\alpha| < \) 10\(^{-3}\) | 10\(^{4}\) m < \( \lambda \) < 10\(^{6}\) m | [36]   |

### B. Review and discussion on Yukawa parameters

Since Fischbach et al. [61] suggest a possible gravity-like “fifth” fundamental force in macroscopic scale, it evokes some interest in many theories which intend to unify gravity with other known forces. The presence of the fifth force could be detected by searching for apparent deviations from Newtonian gravity. For instance, the fifth force would arise from the exchange of a new ultra-light boson which coupled to ordinary matter with a strength comparable to gravity. Typically, through added a hypothetical Yukawa force to the Newtonian potential, this modified potential per mass takes the form:

\[
U(r) = -\frac{GM}{r} \left( 1 + \alpha e^{-r/\lambda} \right),
\]

where \( \alpha \) represents the strength of the Yukawa coupling, and \( \lambda \) represents its length scale.
In Table III, we list some results about $\alpha$ and $\lambda$ in macroscopical range from $100\text{Mpc}$ to $10^4\text{m}$. Fischbach and Talmadge [36] consider the model of Eq. (85) in astronomical tests which provide tight constraints on Yukawa parameters ($\alpha$ and $\lambda$). Typically, these results are based on testing $G(r)M_\odot$ values deduced for different planets through the following equation

$$a = -\frac{G(r)M}{r^3}r,$$  \hspace{1cm} (86)

where

$$G(r) = G\left[1 + \alpha \left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}\right].$$  \hspace{1cm} (87)

Planetary data gives $|\alpha| > 10^{-9}$ and $\lambda > 10^9\text{m}$ (see Table III). The constraints at larger ranges from laboratory, geophysical, and astronomical data are essentially unchanged (for detail, see Figure 2.13 of [36]). As to the constraint from LLR, Adelberger et al. [62] updated the result to include recent LLR data and gave $|\alpha| > 10^{-10}$ and $10^7 < \lambda < 10^{13}\text{m}$.

By using two large-scale structure surveys: the Sloan Digital Sky Survey (SDSS) and the two-degree Field Galaxy Redshift Surveys (2dFGRS) on megaparsec scales, Sealfon et al. [63] considered two models about Poisson equation which deviated from gravitational inverse-square law to constraint $\alpha$ with marginalized over $\lambda$ from $10\text{Mpc}/h$ to $100\text{Mpc}/h$. Where $h$ is the value of Hubble constant in units of $100\text{km/s/Mpc}$. The potential of the first model is

$$\Phi(r) = -G \int d^3r' \frac{\rho(r')}{|r - r'|} \left[1 + \alpha \left(1 - e^{-|r-r'|/\lambda}\right)\right].$$  \hspace{1cm} (88)

After integration of Eq. (88), it has the same result as the Eq. (85). On this large-scale structure scales, $\alpha = 0.025\pm1.7$ for 2dFGRS and $\alpha = -0.35\pm0.9$ for SDSS at $68\%$ confidence level (CL) through fitting the power spectrum measurements from SDSS and 2dFGRS.

Li and Zhao [64] consider Eqs. (86)-(87) and constrain the Yukawa parameters. Using the earth-satellite measurement of earth gravity, the lunar orbiter measurement of lunar gravity, and lunar laser ranging measurement ([64]), the result from constraint on the two Yukawa parameters are $\alpha = 10^{-8} - 5 \times 10^{-8}$ and $\lambda = 1.2 \times 10^7\text{m}-3.8 \times 10^8\text{m}$.

Moffat and Toth [15] express the acceleration law in STVG as

$$a = -\frac{G_{\text{eff}}M}{r^3}r,$$  \hspace{1cm} (89)

where

$$G_{\text{eff}} = G\left[1 + \alpha - \alpha \left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}\right].$$  \hspace{1cm} (90)
With observations or experiments performed within the solar system or in Earthbound laboratories, the authors estimate $\lambda = 5 \times 10^{15} \text{m}$, $|\alpha| = 3 \times 10^{-8}$ for the Sun, and $\lambda = 8.7 \times 10^{12} \text{m}$, $|\alpha| = 9 \times 10^{-14}$ for the Earth.

By analysis of EPM2004 ephemerides, Iorio [65] constrains on the strength and range of a Yukawa-like fifth force through considering potential Eq. (85) and gives $|\alpha| = 10^{-12} - 10^{-13}$, $\lambda \approx 1 \text{AU}$. By using the same potential Eq. (85), Iorio [66] constrains on the range and the strength of a Yukawa-like fifth force with planetary perihelia by the EPM2004 ephemerides: $\lambda < 0.18 \text{AU}$ and $|\alpha| < 10^{-9}$. Through considering Eqs. (89)-(90), Iorio [40] obtains $\lambda = 0.2 \pm 0.4 \text{AU}$ and $|\alpha| = (0.3 \pm 2.7) \times 10^{-11}$ by constraint on perihelion shift of Mercury.

With Pioneer 10/11 launched, their radiometric tracking data have consistently indicated the presence of a small anomaly which is $a_p = (8.74 \pm 1.33) \times 10^{-10} \text{m/s}^2$, directed toward the Sun [67]. This apparent anomalous constant acceleration has been similarly shown in Galileo and Ulysses data [68]. In order to explain this anomaly, some of recent efforts are looking for new physics. Based on this idea, many modified Newton inverse square laws were provided. John et al. [69] ruled out a lot of potential causes and considered a gravitational potential with introducing an additional Yukawa force but this modified potential is a little different from Eq. (85) and has the following form

$$V(r) = -\frac{GM}{(1 + \alpha)r} \left(1 + \alpha e^{-r/\lambda}\right).$$ (91)

Through identifying the last term of Eq. (91) as the Pioneer 10/11, Galileo and Ulysses acceleration, John et al. [69] obtain $\alpha = -1 \times 10^{-3}$ for $\lambda = 200 \text{AU}$.

Reynaud and Jaekel [70] also discussed the relation between long range tests of the Newton law and the anomaly recorded on Pioneer 10/11 probes through considering the potential of Eq. (85). With a power expansion of Eq. (85) in terms of $r/\lambda$, Reynaud and Jaekel [70] obtain a constant anomalous acceleration on the range of distances probed by Pioneer 10/11 only if $\lambda > 10 \text{AU}$ ($\alpha = -\lambda^2/\Lambda^2$ and $\Lambda = 6300 \text{AU}$). It yields $\alpha < -10^{-4}$ or $|\alpha| > 10^{-4}$. In STVG, Brownstein and Moffat [71] considered the anomaly recorded on Pioneer 10/11 probes through Eqs. (89)-(90) and obtained $\alpha = (1.00 \pm 0.02) \times 10^{-3}$, $\lambda = 47 \pm 1 \text{AU}$. For these results, it can be seen that a long-range Yukawa deviation from the Newton potential can be treated as a constant acceleration.

From above analysis, the Yukawa parameters depend on the scale of testing system. In galactic scale, Yukawa parameters would play a big role to explain flat rotation curve [2].
TABLE IV: Parameters in binary pulsars.

| Pulsars            | $e$    | $M$ ($M_\odot$) | $P$ (day) | $<\dot{\omega}>$ ($^\circ$yr$^{-1}$) | Ref. |
|--------------------|--------|-----------------|-----------|--------------------------------------|------|
| PSR B1913+16       | 0.6171338(4) | 2.8281(2) | 0.322997462727(5) | 4.226595(5) | [76] |
| PSR B1534+12       | 0.2736767(1) | 2.679(2) | 0.420737299153(4) | 1.755805(3) | [72] |
| PSR J0737-3039     | 0.0877775(9) | 2.58708(16) | 0.10225156248(5) | 16.89947(68) | [75] |
| PSR B2127+11C      | 0.681395(2) | 2.71279(13) | 0.33528204828(5) | 4.4644(1) | [74] |

However, in the outer solar system, Yukawa parameters could provide to a certain extent the Pioneer anomalous constant acceleration. Then, in the inner solar system, the strength coupled with gravitation about the Yukawa force, $\alpha$, is very small and the range, $\lambda$, is large. These results lead to the fifth force negligible in inner solar system. If future experiments confirm nonentity of dark matter in the universe, it will provide a possible existence of a fifth force. Until then, the fifth force must work in galactic scale even cosmological scale. From this point of view, constraint on the Yukawa parameters ($\alpha$ and $\lambda$) in MSTVG has somewhat practical significance.

In view of these discussion about the four parameters in MSTVG: PPN parameters ($\gamma$, $\beta$) and the Yukawa parameters ($\alpha$, $\lambda$), the two PPN parameters are independence of the system scale in the current experiments. However, the case for the two Yukawa parameters are not. In the next section, the orbital data of pulsar binaries are used to fit the Yukawa parameters, in which the PPN parameters will be fixed as $\gamma = \beta = 1$.

IV. CONSTRAINTS ON THE YUKAWA PARAMETERS BY DOUBLE NEUTRON STAR BINARIES IN MSTVG

Double neutron star binaries provide us more impressive tests of general relativity than other systems. For example the fraction which will merge due to gravitational wave emission is larger. Besides, a neutron star is rather compact and its companion hardly effect its shape. These systems are highly valuable for measuring the effects of gravity and testing gravitational theories. Four samples used in this paper are PSR B1913+16, PSR B1534+12, PSR J0737-3039 and PSR B2127+11C which are listed in Table IV. The numbers inside a pair of parentheses are the 1$\sigma$ error of its corresponding quantity.

PSR B1913+16 is discovered in 1974 by using the Arecibo 305m antenna [18]. The orbit
FIG. 1: The plane of $\lambda$-$\alpha$ constrained by $\dot{\omega}$ of four double neutron star binaries, where the real line (green) denotes PSR B1913+16, the thinnest dashed line (red) denotes PSR B1534+12, the thickest dashed line (blue) denotes PSR J0737-3039, and the dotted line (magenta) denotes PSR B2127+11C. It can be seen that the four curves are nearly cross at one point.

FIG. 2: The enlarged diagram of the region around the cross point in Figure 1. The legend is the same as in Figure 1.
FIG. 3: Each curve in Fig. 2 is separated into two curves to show the bounds due to the observational 1σ error. Please read the context for the details. Figure legend is the same as in Figure 1.

FIG. 4: Confidence level contours of 68.3% (red), 95.4% (green) and 99.73% (blue) in the plane of $\lambda$-$\alpha$ constrained by four double neutron star binaries. The plus sign denotes the 68.3% values for the Yukawa parameters of the MSTVG scenario which are: $\lambda = (3.97 \pm 0.01) \times 10^8$ m and $\alpha = (2.40 \pm 0.02) \times 10^{-8}$, the minimum value of $\chi^2$ is 21.47.
has evolved since the binary system was initially discovered, in precise agreement with the loss of energy due to gravitational wave emission predicted by Einstein’s General Theory of Relativity. PSR B2127+11C is located in the globular cluster Messier 15. This system appears to be a clone of PSR B1913+16 (see Table IV). For PSR B1534+12, its pulses are significantly stronger and narrower than those of PSR B1913+16. PSR J0737-3039 was discovered during a pulsar search carried out using a multibeam receiver with 64m radio telescope in 2003. And we can see this system has shorter orbital period, smaller eccentricity and larger periastron advance. When we fix $\gamma = 1$ and $\beta = 1$, each $\dot{\omega}$ in Table IV will confine the values of $\alpha$ and $\lambda$ in a curve through Eq.(73) (see Fig. 1 and Fig. 2).

In Fig. 1 the trajectories of four double neutron star binaries in the plane of the Yukawa parameters have been shown. From Fig. 1 we can see that the evolution of $\alpha - \lambda$ by PSR B1913+16 and PSR B2127+11C is very similar. For PSR B1534+12, the evolution of $\alpha - \lambda$ is mostly flat. Besides, the evolution of $\alpha - \lambda$ for PSR J0737-3039 is very steep. In Fig. 1 we can also see that the four trajectories almost meet together at $\lambda/10^8 \approx 4$. Fig. 2 is the enlarged drawing of Fig. 1 at a smaller scale. On the other hand, Damour & Esposito-Farèse constraint two 2PN parameters of MST by using four different binary pulsars data in 1σ confidence level. Based on the same method of 30, we constrain the Yukawa parameters of MSTVG in 1σ confidence level. Firstly, we plot an 1σ constraint imposed by double neutron star binaries. Each set of binary data leads to a reduced $\chi^2$: $\chi^2_{\text{binary}}(\alpha, \lambda) = \frac{(\dot{\omega}_{\text{theory}} - \dot{\omega}_{\text{observation}})^2}{\sigma_{\text{observation}}^2}$, equivalent to the 1σ constraint $-\sigma_{\text{observation}} < \dot{\omega}_{\text{theory}} - \dot{\omega}_{\text{observation}} < \sigma_{\text{observation}}$. The bounds by four double neutron star binaries allowed regions of the $\lambda$-$\alpha$ plane are displayed in Fig. 3. Clearly four binaries data favor only a small region of the Yukawa parameters. To combine the constraints on $\alpha$ and $\lambda$ coming from different double neutron star binaries experiments, we have added their individual $\chi^2$ as if they were part of a total experiment with uncorrelated Gaussian errors like the analysis method of 30:

$$\chi^2_{\text{total}}(\alpha, \lambda) = \chi^2_{1913+16}(\alpha, \lambda) + \chi^2_{1534+12}(\alpha, \lambda) + \chi^2_{0737-3039}(\alpha, \lambda) + \chi^2_{2127+11C}(\alpha, \lambda).$$

Therefore, we plot the contour level $\Delta \chi^2_{\text{total}}(\alpha, \lambda) = 2.3$, $\Delta \chi^2_{\text{total}}(\alpha, \lambda) = 6.17$, and $\Delta \chi^2_{\text{total}}(\alpha, \lambda) = 11.8$, where $\Delta \chi^2_{\text{total}}(\alpha, \lambda) = \chi^2_{\text{total}}(\alpha, \lambda) - \langle \chi^2_{\text{total}}(\alpha, \lambda) \rangle_{\text{min}}$, defines respectively for two degrees of freedom the 68.3%, 95.4% and 99.73% confidence levels represented in Fig. 4. The 1σ fit values for the Yukawa parameters are $\lambda = (3.97 \pm 0.01) \times 10^8$ m and $\alpha = (2.40 \pm 0.02) \times 10^{-8}$ with $\chi^2_{\text{min}} = 21.47$. With these results, we go back to discuss the light deflection for Eq. (82). If we consider that a light ray which passes the Sun at the solar radius, it yields
\[ \delta \phi = 3.33 \times 10^{-9}. \] This means the deviation of MSTVG from GR for defection of light may should be tested in the future accuracy level.

For the same model of the Newtonian potential modified by the fifth force (Eq. (85)), we obtain the same results as in LLR \[86, 62\], constraint on intermediate-range gravity \[64\] and LAGEOS-lunar \[36\] listed in Table III. It tells us that the limits of binary pulsar systems on the Yukawa parameters for MSTVG are basically consistent with the solar system. From the other view, it tells us that \(|2\gamma - \beta - 1| \) almost equals to 0. When considering three parameters (\(\alpha, \lambda\) and \(|2\gamma - \beta - 1|\)) based on Eq. (13) in 1\(\sigma\) level, we do not obtain any reasonable results and need more precision binary pulsars data.

V. CONCLUSIONS AND PROSPECTS

With relaxed the two assumptions taken in STVG, which actually hold when the scalar field \(G\) is a constant field, it is shown that the post-Newtonian parameter \(\gamma \neq 1\) by using Chandrasekhar’s approach and a modified scheme of scalar-tensor-vector gravity theory (MSTVG) is then proposed by introducing a coupling function of the scalar field \(G\): \(\theta(G)\). Started with the action in MSTVG, the equations of motion of MSTVG in the first post-Newtonian order (1PN) for general matter without specific equation of state and N point masses are obtained. The secular periastron shift for binary pulsar in 1PN is derived. From the results of MSTVG in 1PN, there are four parameters: two PPN parameters \(\gamma\) and \(\beta\) and two Yukawa parameters \(\alpha\) and \(\lambda\). After pointed out their independence of system scale for \(\gamma\) and \(\beta\), discussion about the dependence of system scale for the Yukawa parameters is touched. Furthermore, with the precondition of \(|2\gamma - \beta - 1| = 0\), \(\alpha\) and \(\lambda\) in MSTVG with applied 4 double neutron star binaries data (PSR B1913+16, PSR B1534+12, PSR J0737-3039, PSR B2127+11C) are constrained: \(\lambda = (3.97 \pm 0.01) \times 10^8\) m and \(\alpha = (2.40 \pm 0.02) \times 10^{-8}\). It has been shown that the limits of binary pulsars systems on MSTVG are basically consistent with the results from the solar system such as the earth-satellite measurement of earth gravity, the lunar orbiter measurement of lunar gravity, and lunar laser ranging measurement to constrain the fifth force. For future applications in binary pulsars systems, it can help us to distinguish between different gravity theories and MSTVG. If MSTVG is proved to correct, besides success in solar system, astrophysical and cosmological scales without dark matter, there must exist a wave of vector field which does not equal to
the light speed. Besides, there also exist violations of the Einstein equivalence principle in large scale in future experiments.

In Appendix B, we calculate the standard PPN parameters of the MSTVG, ignoring the vector field due to the coupling between it and the matter fields. When the vector field is included, some new super potentials might be introduced that would cause the appearance of new parameters and the numerical values of some PPN parameters might be affected. This is a subject of future research.

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APPENDIX A: CALCULATION OF $\gamma$

In this appendix, we will give a detailed calculation of $\gamma$ in MSTVG by using Chandrasekhar’s approach. The contravariant components of the metric tensor are

\begin{align*}
  g^{00} &= -1 - \varepsilon^2 N - \varepsilon^4 \left( N^2 + L \right), \\
  g^{0i} &= \varepsilon^3 L_i, \\
  g^{ij} &= \left( \delta_{ij} - \varepsilon^2 H_{ij} \right).
\end{align*}

(A1) \hspace{2cm} (A2) \hspace{2cm} (A3)

Accordingly,

\[ \sqrt{-g} = \left[ 1 + \frac{1}{2} \varepsilon^2 (H - N) \right]. \]

(A4)

We can now evaluate the Christoffel symbols with the aid of the metric coefficients given
in Eqs. (3)-(5) and (A1)-(A3):

\[ \Gamma_{00}^0 = -\epsilon^3 \frac{1}{2} N_{,t} - \epsilon^5 \frac{1}{2} \left( NN_{,t} + L_{,t} + L_k N_{,k} \right), \]  
(A5)

\[ \Gamma_{0i}^0 = -\epsilon^2 \frac{1}{2} N_{,i} - \epsilon^4 \frac{1}{2} \left( L_{,i} + NN_{,i} \right), \]  
(A6)

\[ \Gamma_{ij}^0 = \epsilon^3 \left( \frac{1}{2} H_{ij,t} - \frac{1}{2} L_{i,j} - \frac{1}{2} L_{j,i} \right), \]  
(A7)

\[ \Gamma_{00}^i = -\epsilon^2 \frac{1}{2} N_{,i} + \epsilon^4 \left( L_{,i} - \frac{1}{2} L_{,i} + \frac{1}{2} N_{,k} H_{ik} \right), \]  
(A8)

\[ \Gamma_{0j}^i = \epsilon^3 \left( \frac{1}{2} L_{i,j} - \frac{1}{2} L_{j,i} + \frac{1}{2} H_{ij,t} \right), \]  
(A9)

\[ \Gamma_{jk}^i = \epsilon^2 \frac{1}{2} \left( H_{ij,k} + H_{ik,j} - H_{jk,i} \right). \]  
(A10)

Then, the general expression for the Ricci tensor is

\[ R_{ij} = \epsilon^2 \left( \frac{1}{2} N_{,ij} - \frac{1}{2} H_{ij,kk} + \frac{1}{2} H_{ik,kj} + \frac{1}{2} H_{jk,ki} - \frac{1}{2} H_{ij} \right), \]  
(A11)

\[ R_{0i} = \epsilon^3 \left( -\frac{1}{2} H_{,it} + \frac{1}{2} H_{ik,ki} - \frac{1}{2} L_{i,tt} + \frac{1}{2} L_{,kk} \right), \]  
(A12)

\[ R_{00} = -\epsilon^2 \frac{1}{2} N_{,kk} \]

\[ \quad + \epsilon^4 \left( -\frac{1}{4} N_{,k} N_{,k} - \frac{1}{2} H_{,tt} + \frac{1}{2} N_{,t} H_{tt} + \frac{1}{2} N_{,k} H_{kl} + \frac{1}{2} N_{,k} H_{kk,tt} \right), \]  
(A13)

Turning to the components of the energy-momentum tensor, we find

\[ T_{00} = \sigma - \epsilon^2 (2 N \sigma + \sigma_{kk}), \]  
(A14)

\[ T_{0i} = -\epsilon \sigma_i, \]  
(A15)

\[ T_{ij} = \epsilon^2 \sigma_{ij}, \]  
(A16)

and

\[ T = -\sigma + \epsilon^2 \left( N \sigma + 2 \sigma_{kk} \right). \]  
(A17)

The (0, 0) component for $N$ from Eq. (12):

\[ \epsilon^2 \left[ -\frac{1}{2} \Box N \right] = \epsilon^2 \left[ 4\pi G_0 \sigma + \frac{1}{2} \delta_{ij} \Box \sigma \right]. \]  
(A18)

The $(i, j)$ component for $H_{ij}$ from Eq. (12):

\[ \epsilon^2 \left[ \frac{1}{2} N_{,ij} - \frac{1}{2} H_{ij,tt} + \frac{1}{2} H_{ik,ki} + \frac{1}{2} H_{jk,ki} - \frac{1}{2} H_{ij} \right] = \epsilon^2 \left[ 4\pi G_0 \delta_{ij} \sigma - G_{,ij} + \frac{1}{2} \delta_{ij} \Box \sigma \right]. \]  
(A19)
Using the gauge equation \( (32) \), we obtain

\[
\epsilon^2 \left[ -\frac{1}{2} H_{ij,kk} \right] = \epsilon^2 \left[ 4\pi G_0 \delta_{ij} \sigma - \frac{1}{2} \delta_{ij} \square^{(2)} G \right].
\]

For \( G \) from Eq. \( (20) \)

\[
\epsilon^2 \left[ (\theta_0 + 3) \square^{(2)} G \right] = \epsilon^2 8\pi G_0 \sigma
\]

Based on Eqs. \( (A18) \) and \( (A21) \), we obtain

\[
\square N = -\frac{\theta_0 + 4}{\theta_0 + 3} 8\pi G_0.
\]

Based on Eqs. \( (A20) \) and \( (A21) \), we obtain

\[
\square H_{ij} = -\frac{\theta_0 + 2}{\theta_0 + 3} 8\pi G_0 \delta_{ij}.
\]

If we define \( H_{ij} = \delta_{ij} V \), then we obtain

\[
\square V = -\frac{\theta_0 + 2}{\theta_0 + 3} 8\pi G_0,
\]

and

\[
\gamma = \frac{V}{N} = \frac{\theta_0 + 2}{\theta_0 + 4}.
\]

When \( \theta_0 = -1 \), it returns to the case of STVG and gives \( \gamma = 1/3 \). It is also explicit that \( \gamma = 1 \) when the scalar field \( G \) is a constant field. In this special case the terms with \( G \) in Eqs. \( (A18) \) and \( (A20) \) disappear and \( \gamma = 1 \) holds for both STVG and MSTVG. One might wonder at Eq. \( (A21) \), which does not allow \( G \) to be zero. But Eq. \( (A21) \) comes from Eq. \( (20) \), the field equation for the scaler field \( G \): Eq. \( (20) \) does not exist when \( G \) is not a variable field.

The next step is to identify the parameter \( \gamma \) defined in Eq. \( (A25) \) as the corresponding PPN parameter. This is done in Appendix B.

**APPENDIX B: DERIVATION OF THE PPN PARAMETERS FOR MSTVG**

In this appendix, we mainly derive the PPN parameters of MSTVG. For this purpose, we must transfer our coordinate into the standard PPN coordinate and then derive PPN parameters of MSTVG in comparison with PPN formalism \[24\]. But, it is shown that MSTVG violates EEP based on Eq. \( (8) \) and can not be brought into the standard PPN
formalism. To compare with the PPN metric, however, we provisionally ignore the vector field of MSTVG, which causes the violation of EEP.

We consider the material composing the various bodies of the system to behave like an ideal fluid. Following [77], the energy-momentum tensor can be written in the following form in the ideal fluid case

\[ c^2 T^{\mu \nu} = \rho (c^2 + \Pi) u^\mu u^\nu + (g^{\mu \nu} + u^\mu u^\nu) p, \]

(B1)

where \( \rho \) denotes the rest-mass density, \( p \) is the pressure, \( \Pi \) is the specific internal energy, and \( u^\mu = dx^\mu / c d\tau \) is the dimensionless 4-velocity and we obtain in 1PN

\[ T^{00} = \rho \left[ 1 + \frac{1}{c^2} (\Pi + v^2 + N) \right] + \mathcal{O}(c^{-4}), \]

(B2)

\[ T^{0i} = \rho \frac{v^i}{c} + \mathcal{O}(c^{-3}), \]

(B3)

\[ T^{ij} = \frac{1}{c^2} (\rho v^i v^j + p \delta_{ij}) + \mathcal{O}(c^{-4}), \]

(B4)

where \( v^i \) is the coordinate velocity of the corresponding material element.

From Eqs. (14)-(15) and (B2)-(B4), we obtain

\[ \sigma = \rho \left[ 1 + \frac{1}{c^2} (\Pi + 2v^2 + N) \right] + 3 \frac{p}{c^2} + \mathcal{O}(c^{-4}), \]

(B5)

\[ \sigma_i = \rho v^i + \mathcal{O}(c^{-2}), \]

(B6)

\[ \sigma_{kk} = \rho v^2 + 3p + \mathcal{O}(c^{-2}). \]

(B7)

Then, we rewrite Eqs. (35) and (42) with abandoning the vector field

\[ \Delta \left[ N + \epsilon^2 L \right] = -8\pi G \rho \]

\[ \epsilon^2 \left\{ -8\pi G \rho \left[ N + 2v^2 + \Pi + 3\frac{p}{\rho} \right] \right\} \]

\[ -4\pi G \rho \left[ \frac{3\theta_0 + 2}{\theta_0 + 4} N - 2 \left( 1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2} \right) N - N \right. \]

\[ -2 \left( 1 - \frac{\theta_0 + 2}{\theta_0 + 4} \right) (v^2 + 3\frac{p}{\rho}) \] \[ - \left. \frac{1}{2} \left( 1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2} \right) \Delta N^2 \right\} + N_{\mu \nu} \]

(B8)
Then, we obtain

\[
\Delta N = -8\pi G \rho,
\]

\[
\Delta L = -4\pi G \rho \left[\frac{3}{\theta_0 + 4} N + N - 2 \left(1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2}\right) N + \left(\frac{2\theta_0 + 2}{\theta_0 + 4} + 2\right) v^2 + 2\Pi \right] + 6\theta_0 + 2\theta_0 + 4 \rho \left[\frac{3}{\theta_0 + 4} \theta_0 + 2 \theta_0 + 4 \right] - \frac{1}{2} \left(1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2}\right) \Delta N^2 + N_{\text{tt}},
\]

and Eqs. (38), (41) are rewritten as

\[
\Delta H_{ij} = -8\frac{\theta_0 + 2}{\theta_0 + 4} \pi G \rho \delta_{ij},
\]

\[
\Delta L_i = 8 \left(1 + \frac{\theta_0 + 2}{\theta_0 + 4}\right) \pi G \rho v^i.
\]

After reference [24], we define the following superpotentials

\[
U(x, t) \equiv G \int \frac{\rho(x', t)}{|x - x'|} d^3x',
\]

\[
\chi(x, t) \equiv -G \int \rho(x', t)|x - x'| d^3x',
\]

\[
\Phi_1(x, t) \equiv \int \frac{\rho(x', t)v^2}{|x - x'|} d^3x',
\]

\[
\Phi_2(x, t) \equiv \int \frac{\rho(x', t)U'}{|x - x'|} d^3x',
\]

\[
\Phi_3(x, t) \equiv \int \frac{\rho(x', t)\Pi'}{|x - x'|} d^3x',
\]

\[
\Phi_4(x, t) \equiv \int \frac{\rho(x', t)}{|x - x'|} d^3x',
\]

\[
V_i(x, t) \equiv \int \frac{\rho(x', t)v_i}{|x - x'|} d^3x',
\]

\[
W_i(x, t) \equiv \int \frac{\rho(x', t)[v' \cdot (x - x')] (x^i - x'^i)}{|x - x'|^3} d^3x',
\]

\[
\Phi_w(x, t) \equiv \int \rho(x', t) \rho(x'', t) \frac{x - x'}{|x - x'|^3} \left(\frac{x' - x''}{|x' - x''|} - \frac{x - x''}{|x - x''|}\right) d^3x' d^3x'',
\]

\[
\Phi_i(x, t) \equiv \int \frac{\rho(x', t)[v' \cdot (x - x')]^2}{|x - x'|^3} d^3x'.
\]

From the above, we have the following relation

\[
\Delta \chi = -2U,
\]

\[
\chi_{\text{tt}} = V_i - W_i.
\]
With the above definition of the gravitational potentials, it yields solution of the metric for MSTVG without the vector field in the following forms

\[ N = 2U, \]  
\[ L = -2 \left( 1 + \frac{1}{2(\theta_0 + 3)(\theta_0 + 4)^2} \right) U^2 + \left( 2\frac{\theta_0 + 2}{\theta_0 + 4} + 2 \right) \Phi_1 \]
\[ + 2 \left[ 3\frac{\theta_0 + 2}{\theta_0 + 4} - 2 \left( 1 + \frac{1}{2(\theta_0 + 3)(\theta_0 + 4)^2} \right) \right] \Phi_2 + 2\Phi_3 \]
\[ + 6\frac{\theta_0 + 2}{\theta_0 + 4} \Phi_4 - \chi_{tt}, \]  
\[ L_i = -2 \left( 1 + \frac{\theta_0 + 2}{\theta_0 + 4} \right) V_i, \]  
\[ H_{ij} = 2\frac{\theta_0 + 2}{\theta_0 + 4} \delta_{ij} U, \]  
\[ (B25) \]
\[ (B26) \]
\[ (B27) \]
\[ (B28) \]

In order to obtain PPN parameters, we must transfer our coordinate system into the standard PPN one. And then we could compare the metric of MSTVG with the PPN metric in the standard PPN coordinate system and finally derive the PPN parameters of MSTVG. When the above is compared with the standard PPN metric (see Eqs. (B43)-(B45), the only superpotential which does not appear in the PPN metric is \( \chi_{tt} \) in Eq. (B26). Therefore, it is necessary to transform the coordinates to gauge off this term. Based on the Eq. (4.38) in Ref [24], an infinitesimal coordinate transformation is introduced between the standard PPN coordinate system and ours:

\[ x_{\mu,PN} = x^\mu + \epsilon^2 \xi^\mu (x^\alpha), \]  
\[ (B29) \]

where

\[ \xi_0 = \lambda_1 \chi_{0}, \quad \xi_i = \lambda_2 \chi_{i}. \]  
\[ (B30) \]

The relation between the metrics before and after the gauge transformation, \( g_{\mu \nu} \) and \( \tilde{g}_{\mu \nu} \) respectively, are shown by Eq. (4.46) in Ref [24]

\[ \tilde{g}_{ij} = g_{ij} - \epsilon^2 \lambda_2 \chi_{ij}, \]  
\[ (B31) \]
\[ \tilde{g}_{0i} = g_{0i} - \epsilon^3 (\lambda_1 + \lambda_2) \chi_{t i}, \]  
\[ (B32) \]
\[ \tilde{g}_{00} = g_{00} - \epsilon^4 2\lambda_1 \chi_{tt} - \epsilon^4 2\lambda_2 (U^2 + \Phi_w - \Phi_2). \]  
\[ (B33) \]

Due to the spatial part of the PPN metric and our metric are all diagonal and isotropic, we thus choose \( \lambda_2 = 0 \) through Eqs. (B31) and substitute Eq. (B28) into Eqs. (B31):

\[ \tilde{g}_{ij} = \left( 1 + \epsilon^2 2\frac{\theta_0 + 2}{\theta_0 + 4} U \right) \delta_{ij} \]  
\[ (B34) \]
For Eqs. (B32) and (B33), we have

\[ \bar{g}_{0i} = -\epsilon^3 \frac{1}{2} \left( 1 + \frac{\theta_0 + 2}{2\theta_0 + 4} \right) V_i - \epsilon^3 \lambda_1 \chi_{ti}, \]

\[ \bar{g}_{00} = g_{00} - \epsilon^4 \frac{1}{2} \lambda_1 \chi_{tt}, \]

\[ \bar{g}_{0i} = -\epsilon^3 \left( -\frac{1}{2} \left( \frac{\theta_0 + 2}{2\theta_0 + 4} + 2 + \lambda_1 \right) V_i + \lambda_1 W_i \right), \]

\[ \bar{g}_{00} = g_{00} - \epsilon^4 \left( -\frac{1}{2} \left( \frac{\theta_0 + 2}{2\theta_0 + 4} + 2 \right) \Phi_1 + 2 \Phi_3 + 6 \frac{\theta_0 + 2}{\theta_0 + 4} \Phi_4 \right), \]

by using Eqs. (B25), (B26) and (B27). It is noted that there is no existence of superpotential \( \chi_{tt} \) in the standard PPN framework, we then obtain the following value through Eq. (B36)

\[ \lambda_1 = -\frac{1}{2}, \]

When we substitute \( \lambda_2 = 0 \) and \( \lambda_1 = -\frac{1}{2} \) into Eq. (B29), the transformation between our reference system and the standard PPN reference system is shown as

\[ t_{PN} = t + \epsilon^4 \frac{1}{2} \chi_t + O(5), \]

\[ x^i_{PN} = x^i. \]

Through this transformation, our metric in the PPN coordinate system become

\[ \bar{g}_{ij} = \left( 1 + \epsilon^2 \frac{\theta_0 + 2}{\theta_0 + 4} U \right) \delta_{ij}, \]

\[ \bar{g}_{0i} = -\epsilon^3 \frac{1}{2} \left( \frac{\theta_0 + 2}{\theta_0 + 4} + 3 \right) V_i - \epsilon^3 \frac{1}{2} W_j, \]

\[ \bar{g}_{00} = -1 + \epsilon^2 U + \epsilon^4 \left\{ -2 \left( 1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2} \right) U^2 + \left( \frac{2 \theta_0 + 2}{\theta_0 + 4} + 2 \right) \Phi_1 + 2 \Phi_3 + 6 \frac{\theta_0 + 2}{\theta_0 + 4} \Phi_4 \right\}, \]

by using Eqs. (B34)-(B36).

On the other hand, with the above definition of the gravitational potentials, the standard
PPN metric \[24\] reads

\[
\bar{g}_{ij} = (1 + \epsilon^2 2\gamma U)\delta_{ij}, \tag{B43}
\]

\[
\bar{g}_{0i} = \epsilon^3 \left[ -\frac{1}{2} (4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi) V_i - \frac{1}{2} (1 + \alpha_2 - \zeta_1 + 2\xi) W_i \right], \tag{B44}
\]

\[
\bar{g}_{00} = -1 + \epsilon^2 2U + \epsilon^4 \left\{ -2\beta U^2 - 2\xi \Phi_w + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi) \Phi_1 + 2(1 + \alpha_3) \Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi) \Phi_4 \\
- (\zeta_1 - 2\xi) \Phi_0 \right\} \tag{B45}
\]

Thus, the PPN parameters of MSTVG without the vector field have the following forms by comparison between Eqs. (B40)-(B42) and (B43)-(B45)

\[
\gamma = \frac{\theta_0 + 2}{\theta_0 + 4}, \tag{B46}
\]

\[
\beta = 1 + \frac{\theta_1}{2(\theta_0 + 3)(\theta_0 + 4)^2}, \tag{B47}
\]

\[
\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0. \tag{B48}
\]
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