Model independent sum rules
for $B \to \pi K$ decays

JOAQUIM MATIAS

Institut für Theoretische Physik E, RWTH Aachen, D - 52056 Aachen, Germany and
IFAE, Universitat Autònoma de Barcelona, Spain

Abstract

We provide a set of sum rules relating CP-averaged branching ratios and CP-asymmetries of the $B \to \pi K$ modes. They prove to be useful as a mechanism to ‘test’ experimental data given our expectations of the size of isospin breaking. A set of observables emerges providing a simpler interpretation of data in terms of isospin breaking. Moreover, the derivation is done in a completely model independent way, i.e., they can accommodate also New Physics contributions.
B physics offers the possibility to test the mechanism of CP violation in the Standard Model, i.e., a single phase in the quark flavor mixing matrix but also to seek for the first clues of New Physics. Non-leptonic B meson decays, in particular, \( B \rightarrow \pi K \) modes are nicely suited for both purposes. These modes will play a central role in the determination of the weak angle \( \gamma \) of the Unitarity Triangle (UT). There has been an intense activity to improve the theoretical predictions, following different strategies to control their hadronic uncertainties and to extract the relevant information, i.e., \( \gamma \) and the strong phases. One of the main differences between these strategies is the way to deal with strong phases, either using \( SU(3) \) together with other experimental data \([1, 2, 3]\), or predict them from first principles using QCD factorization \([4, 5, 6, 7]\), or using Wick contractions to combine factorization with a parametrization of the penguin amplitude as in \([8]\). In the present work, we will follow the QCD factorization approach also taking into account the impact of annihilation topologies \([6]\). (The potential importance of annihilation was also noted in \([9]\).)

The angle \( \gamma \) of the UT obtained from these modes should be compared with other determinations of \( \gamma \), if a different value is found this would signal New Physics \([11, 12]\). There has been a considerable effort in the last years to find strategies \([1, 2, 10]\) to constrain the angle \( \gamma \).

On the experimental side, B factories have reported recently new data \([13, 14]\) on the set of charged and neutral non-leptonic decay modes: \( B \rightarrow \pi K \). The collected data on the branching ratios of these decay modes are organized in two types of observables. A first type of observables are the CP-averaged branching ratios \([2, 3]\):

\[
R = \frac{\text{BR}(B_d^0 \rightarrow \pi^- K^+) + \text{BR}(B_d^0 \rightarrow \pi^+ K^-)}{\text{BR}(B^+ \rightarrow \pi^+ K^0) + \text{BR}(B^- \rightarrow \pi^- K^0)},
\]

\[
R_c = 2 \frac{\text{BR}(B^+ \rightarrow \pi^0 K^+) + \text{BR}(B^- \rightarrow \pi^0 K^-)}{\text{BR}(B^+ \rightarrow \pi^+ K^0) + \text{BR}(B^- \rightarrow \pi^- K^0)},
\]

\[
R_0 = 2 \frac{\text{BR}(B_d^0 \rightarrow \pi^0 K^0) + \text{BR}(B_d^0 \rightarrow \pi^0 K^0)}{\text{BR}(B^+ \rightarrow \pi^+ K^0) + \text{BR}(B^- \rightarrow \pi^- K^0)}.
\]

We prefer to use these definitions because, in terms of them, the expressions for the sum rules become simpler. Other definitions for the charged and neutral channels that are used in the literature are \( R_* = 1/R_c \) \([3]\) and \( R_n = R/R_0 \) \([2]\). CP-asymmetries are the second type of observables:

\[
A_{CP}^{0+} = \frac{\text{BR}(B^+ \rightarrow \pi^0 K^+) - \text{BR}(B^- \rightarrow \pi^0 K^-)}{\text{BR}(B^+ \rightarrow \pi^0 K^0) + \text{BR}(B^- \rightarrow \pi^0 K^0)},
\]

\[
A_{CP}^{+0} = \frac{\text{BR}(B^+ \rightarrow \pi^+ K^0) - \text{BR}(B^- \rightarrow \pi^- K^0)}{\text{BR}(B^+ \rightarrow \pi^+ K^0) + \text{BR}(B^- \rightarrow \pi^- K^0)},
\]

\[
A_{CP}^{-+} = \frac{\text{BR}(B_d^0 \rightarrow \pi^- K^+) - \text{BR}(B_d^0 \rightarrow \pi^+ K^-)}{\text{BR}(B_d^0 \rightarrow \pi^- K^0) + \text{BR}(B_d^0 \rightarrow \pi^+ K^-)},
\]

\[
A_{CP}^{00} = \frac{\text{BR}(B_d^0 \rightarrow \pi^0 K^0) - \text{BR}(B_d^0 \rightarrow \pi^0 K^0)}{\text{BR}(B_d^0 \rightarrow \pi^0 K^0) + \text{BR}(B_d^0 \rightarrow \pi^0 K^0)}. \tag{2}
\]
Moreover, in the future (possibly at LHCb \cite{15}) we will have at our disposal the time dependent CP-asymmetries of the neutral decay modes \cite{2} that will provide us with additional information. Anyway, measuring all the asymmetries of eq.\(2\) is already a very challenging task and the experimental results should be considered very preliminary.

Our aim in this letter is to show how isospin symmetry allows us to obtain relations or sum rules between CP-averaged branching ratios of different modes and also between CP-asymmetries. Some of these relations were known in the limit of no isospin breaking \cite{16}, some in the context of the SM \cite{3, 17} and some of them are new. These sum rules will be derived in a transparent way and they will be valid for any model. They will be useful to understand in a more direct way the implications of the present experimental results in terms of isospin breaking. From our estimate of how reasonably large could be isospin breaking we can try to ‘test’ the experimental results. We will show, in that respect, that present experimental results are quite unexpected. We will first construct these sum rules in a model independent way, then we will analyze the case of the Standard Model in the framework of NLO QCD factorization \cite{6} and, in the last section, we will test the sum rules using present experimental data.

I. Isospin decomposition

In the theoretical description of the decay modes of \(B \to \pi K\), isospin symmetry and its breaking plays a central role. Indeed, the CP-averaged branching ratios \(R\), \(R_c\) and \(R_0\) can be considered as a measure of isospin breaking, i.e., if isospin were an exact symmetry they would be equal to one.

A general amplitude for a hadronic \(B\) decay based on the quark transition level \(\bar{b} \to \bar{s}qq\) is described by an effective lagrangian \cite{18} that includes current-current operators, QCD and electroweak penguins.

Using isospin decomposition one arrives easily to the following relations:

\[
-\sqrt{2}A\left(B^+ \to \pi^0 K^+\right) = A\left(B^+ \to \pi^+ K^0\right) + d_1,
-\left(B^0 \to \pi^- K^+\right) = A\left(B^+ \to \pi^+ K^0\right) + d_2,
\sqrt{2}A\left(B^0 \to \pi^0 K^0\right) = A\left(B^+ \to \pi^+ K^0\right) + d_2 - d_1,
\]

where \(d_1\) and \(d_2\) vanish in absence of isospin breaking. In presence of New Physics the most general expression for \(d_1\) and \(d_2\) is\cite{4}

\[
d_i = |P|\xi_i e^{i\theta_i} \left( e^{i\gamma} - a_i e^{i\phi_{a_i}} - ib_i e^{i\phi_{b_i}} \right),
\]

where \(P\) contain all CP conserving terms of the penguin contribution to \(B^+ \to \pi^+ K^0\). \(\xi_i\) parametrize isospin breaking and they are expected to be small parameters. \(\theta_i, \phi_{a_i}, \phi_{b_i}\) are strong phases and \(\gamma\) and \(ib_i\) parametrize weak phases that change sign under a

\(^2\)Latin subindex \(i\) will always be understood to run from 1 to 2.
CP transformation. We will follow as close as possible the notation of [1 2]. We show explicitly in eq.(4) the dependence on $\gamma$, meaning that $b_1$ and $b_2$ can be non-zero only if there is New Physics. In a similar way, we can parametrize, in general, the amplitude:

$$A \left( B^+ \rightarrow \pi^+ K^0 \right) = |P| e^{i\theta_0} \left( 1 - ib_0 e^{i\phi_{b_0}} \right),$$

with $ib_0$ changing sign under CP and $\theta_0, \phi_{b_0}$ are strong phases.

The Standard Model limit of these parameters can be found in Appendix A.

If we now use eq.(3) to compute eq.(1) we obtain very simple expressions that show in a manifest way the relations between $R$, $R_c$ and $R_0$:

$$R = 1 + u_+,$$

$$R_c = 1 + z_+,$$

$$R_0 = 1 + n_+ = 1 + u_+ - z_+ + k_1.$$  \hspace{1cm} (6)

$$R_0 = 1 + n_+ = 1 + u_+ - z_+ + k_1.$$  \hspace{1cm} (7)

$$R_0 = 1 + n_+ = 1 + u_+ - z_+ + k_1.$$  \hspace{1cm} (8)

In a similar way, after choosing certain combinations of observables, we find the corresponding decomposition for the CP-asymmetries:

$$A_{CP}^{+} R = A_{CP}^{+0} + u_-, \hspace{1cm} (9)$$

$$A_{CP}^{0+} R_c = A_{CP}^{0+} + z_-, \hspace{1cm} (10)$$

$$A_{CP}^{00} R_0 = A_{CP}^{00} + n_- = A_{CP}^{00} + u_- - z_- + k_2.$$ \hspace{1cm} (11)

The exact dependence of $u_{\pm}$, $z_{\pm}$ and $k_1, k_2$ in terms of $d_1, d_2$ can be found in Appendix B. For the following discussion we will only need to notice that $u_{\pm}$ and $z_{\pm}$ contain a piece linear in $\xi_i$ while $k_1$ and $k_2$ are quadratic in $\xi_i$. Consequently, being $\xi_i$ a measure of isospin breaking, one would expect $k_1$ and $k_2$ to be smaller than $u_\pm$ and $z_\pm$.

II. Sum rules

Eqs.(6,7) and eqs.(8,9) provide us with all necessary ingredients to construct a set of ‘sum rules’. Notice that we pretend to find those relations between CP-averaged observables ($R$) and CP-asymmetries ($A_{CP}$) that minimize the impact of isospin breaking, i.e., we should get rid of all terms linear in $\xi_i$.

The first relation is linear and it is obtained by substituting $u_+$ and $z_+$ in eq.(8) by $R$ and $R_c$

$$1) \quad R_0 - R + R_c - 1 = k_1.$$ \hspace{1cm} (12)

Some remarks, concerning eq.(12), are in order here. First, this is an exact relation and it involves terms of order $\xi_i^2$ and higher that measure the amount of isospin breaking. Second, $k_1$ can be interpreted, looking at eq.(35) of Appendix A, as a measure of the misalignment between the isospin breaking contributions to two channels: $\sqrt{2} A (B^+ \rightarrow \pi^0 K^+)$ and $A (B^0 \rightarrow \pi^- K^+)$.

Even in presence of isospin breaking if the new contributions to these channels are equal, i.e., $d_1 = d_2$ then $k_1$ would be exactly zero. On the contrary, if the isospin contribution to these channels have opposite
sign then $k_1$ would be maximal. We should look at data to discern which of the two scenarios is closer to the one realized in nature.

Eq. (12) was first obtained in [13], but with $k_1$ equal to zero and in the SM case in [3]. Here, we provide a model independent expression for $k_1$:

$$k_1 = \frac{-1}{1 + b_0^2} \left\{ \xi_1 \xi_2 \text{Re} \left[ e^{i(\theta_1 - \theta_2)} \left( e^{i\gamma} - a_1 e^{i\phi_{a_1}} - ib_1 e^{i\phi_{b_1}} \right) \left( e^{-i\gamma} - a_2 e^{-i\phi_{a_2}} + ib_2 e^{-i\phi_{b_2}} \right) \right] ight\} - \xi_1^2 \left[ (1 + a_1^2 + b_1^2) - 2a_1 \cos \phi_{a_1} \cos \gamma - 2b_1 \cos \phi_{b_1} \sin \gamma \right] + \left\{ \begin{array}{c} \gamma \to -\gamma \\ b_{1,2} \to -b_{1,2} \end{array} \right\} \quad (13)$$

We have checked, explicitly, that taking the SM limit (Appendix A) of eq. (13) and the strong phase $\omega \to 0$, eq. (13) reduces to the expression found in [3].

Following exactly the same steps, substituting $u_-$ and $z_-$ in eq. (11), we obtain the second sum rule in a quite transparent way:

$$\text{II) } \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{+0} R + \mathcal{A}_{\text{CP}}^{0+} R_0 - \mathcal{A}_{\text{CP}}^{+0} = k_2. \quad (14)$$

This is also an exact relation and it was found in [3] in the SM case. We can also interpret $k_2$ from eq. (14) as a measure of the misalignment between $d_1$ and $d_2$, then the same conditions that force $k_1$ to vanish also apply to $k_2$. But, more specifically, $k_2$ measures the importance of weak phase differences between $d_1$ and its CP conjugate and between $d_2$ and its CP conjugate, i.e., $k_2 = 0$ if $d_1 = \overline{d}_1$ and $d_2 = \overline{d}_2$. The model independent expression for $k_2$ is:

$$k_2 = \frac{-1}{1 + b_0^2} \left\{ \xi_1 \xi_2 \text{Re} \left[ e^{i(\theta_1 - \theta_2)} \left( e^{i\gamma} - a_1 e^{i\phi_{a_1}} - ib_1 e^{i\phi_{b_1}} \right) \left( e^{-i\gamma} - a_2 e^{-i\phi_{a_2}} + ib_2 e^{-i\phi_{b_2}} \right) \right] ight\} + 2 \xi_1^2 \left[ a_1 \sin \phi_{a_1} \sin \gamma - b_1 \sin \phi_{b_1} \cos \gamma - a_1 b_1 \sin(\phi_{a_1} - \phi_{b_1}) \right] + \left\{ \begin{array}{c} \gamma \to -\gamma \\ b_{1,2} \to -b_{1,2} \end{array} \right\} \quad (15)$$

We have also checked that its SM limit agrees with [3]. In Fig. 4 and 2 we have plotted sum rule I and II versus $\gamma$ using the predictions for the Standard Model in the framework of NLO QCD factorization [3]. We have restricted ourselves to the first quadrant for $\gamma$ since it is the region where the standard Unitarity Triangle Analysis (UTA) expects $\gamma$ to lie. Indeed the precise value is $[8, 19]$:

$$\gamma = (54.8 \pm 6.2)^0.$$

We have plotted the result taking into account two cases for the estimate of the uncertainty coming from the annihilation topologies (see [3]): low uncertainty (a) and extremely large uncertainty (b). The low uncertainty case corresponds to the more realistic situation of taking the parameter $\varrho_A$ (defined in [3]) equal to one and the large uncertainty case corresponds to the extreme case (very conservative) of $\varrho_A$ equal to two. The gradient from light to dark it is used for further reference in Figs. 3-9 to indicate the different values of $\gamma$. Notice that around the UTA value of $\gamma$, the values of $k_1$ and $k_2$ are extremely small: $-0.01 \lesssim k_1 \lesssim 0.05$ and $-0.030 \lesssim k_2 \lesssim 0.005$. 

The following set of sum rules are obtained in a completely different way. The aim is to find the simplest combinations of CP-averaged branching ratios \( R \) that would be strongly correlated if isospin were an exact symmetry. They are obtained combining eq. (6-8) to construct a quantity of order \( \xi_i^2 \), dividing by one of the \( R \)'s \( (R, R_c \text{ or } R_0) \) and reinserting sum rule I to simplify the expression. The result is the following three sum rules:

III) \( R = R_0 R_c + k_3, \) (16)
with \( k_3 = z_+ (z_+ - u_+) - k_1 - k_1 z_+ . \)

IV) \( R_c = - \frac{R_0}{R} + 2 + k_4, \) (17)
with \( k_4 = (u_+ z_+ + k_1)/(1 + u_+). \) And, finally,
Table 1: Strongly correlated observables associated to sum rules III-V

| III  | $\mathcal{O}_1^{III} = R$ | $\mathcal{O}_2^{III} = R_0 R_c$ |
|------|----------------|-------------------|
| IV   | $\mathcal{O}_1^{IV} = R_c$ | $\mathcal{O}_2^{IV} = -R_0 / R + 2$ |
| V    | $\mathcal{O}_1^V = R_0$ | $\mathcal{O}_2^V = -R_c / R + 2$ |

$V) \quad R_0 = -\frac{R_c}{R} + 2 + k_5. \hspace{1cm} (18)$

with $k_5 = k_1 + u_+(u_+ - z_+)/ (1 + u_+)$. This sum rule can be related with one proposed in $^3$ (but with the inverse $R/R_c$) for the SM case and in an approximate form, i.e., keeping only the term $\xi_i^2$.

Notice that the expressions for $k_3$, $k_4$, and $k_5$ are model independent and their general expressions are obtained using the expressions of $u_+$, $z_+$ in Appendix B and $k_1$ of eq.(13).

Sum rules III to V will allow us to define a set of observables. These observables are chosen in such a way to be strongly correlated by isospin, i.e., they should have the same value except for corrections of order $\xi_i^2$. They are given in Table I. A nice graphical interpretation of these sum rules can be obtained by plotting $\mathcal{O}_1^{\alpha}$ versus $\mathcal{O}_2^{\alpha}$ ($\alpha = \text{III, IV, V}$) for different values of $\gamma$. In Figs.3-5, we shown the prediction for these observables for the SM using NLO QCD factorization $^3$ taking into account the uncertainty coming from annihilation topologies (low (a) and high (b)). The region presented in Figs.3-5 corresponds to varying all the parameters (amplitudes and strong phases) within the predictions of NLO QCD factorization for the SM. All figures are restricted to values of $\gamma$ ($0 \leq \gamma \leq \pi/2$) inside the first quadrant, according to the SM fit from other measurements. Shading should be understood in the following way: lighter region corresponds exclusively to low values of $\gamma$, following the pattern of Fig.1-2, while the dark region can correspond to large or small values of $\gamma$ inside the first quadrant, since they cannot be distinguished in the plots.

In absence of isospin breaking both observables should fall in the diagonal of Figs.3-5, with $\mathcal{O}_1^{\alpha} = 1$. If isospin breaking is small $O_1^{\alpha}$ and $O_2^{\alpha}$ should stay near the diagonal. The deviation from one along the diagonal gives an idea of the isospin breaking terms of order $\xi_i$ (remember that $R$, $R_c$ and $R_0$ measure isospin breaking of this size). This is useful to have an idea of the maximal size of this breaking. Notice that it also implies that each pair of observables ($O_1^{\alpha}$, $O_2^{\alpha}$) is chosen in such a way to present the same deviation of order $\xi_i$, independently of the model.

More interestingly, deviations from the diagonal would measure isospin breaking contributions of order $\xi_i^2$. It implies that if isospin is not badly broken, we can estimate that the deviations from the diagonal will be smaller than the square of the maximal deviation from one along the diagonal. For instance, in Fig.3b, the maximal...
Figure 3: Correlation between $O_{1}^{III}$ and $O_{2}^{III}$ evaluated in the SM using NLO QCD factorization, for low (a) and large (b) uncertainty coming from annihilation topologies. The lighter region corresponds to the lowest values of $\gamma$ inside the first quadrant.

Figure 4: Correlation between $O_{1}^{IV}$ and $O_{2}^{IV}$ of sum rule IV, conventions as in Fig.3.

Figure 5: Correlation between $O_{1}^{V}$ and $O_{2}^{V}$ of sum rule V, conventions as in Fig.3.
deviation from one along the diagonal is approximately 0.5, then the maximal expected deviation from the diagonal would be 0.25 and, indeed, this is the case. This rule applies to all figures evaluated using NLO QCD factorization.

An experimental measurement of these observables very far away from the diagonal and the SM allowed region would require the contribution of isospin breaking New Physics. Present data favors this situation as we will see in the next section.

Finally, following the same strategy as in sum rule III-V, we can construct a set of three sum rules involving the CP-asymmetries using the translation table

\[
\begin{align*}
\text{VI)} & \quad \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00} R_0 = 1 + (1 + \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right) + k_6. \\
\text{VII)} & \quad \mathcal{A}_{\text{CP}}^{0+} R_c = \mathcal{A}_{\text{CP}}^{0+} + \frac{\mathcal{A}_{\text{CP}}^{+} R - \mathcal{A}_{\text{CP}}^{00} R_0}{1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}} + k_7.
\end{align*}
\]

Here \(k_6\) is given by \(k_6 = z_- (z_- - u_-) - k_2 - k_2 z_-\).

\[
\begin{align*}
\text{VIII)} & \quad \mathcal{A}_{\text{CP}}^{00} R_0 = \mathcal{A}_{\text{CP}}^{00} + \frac{\mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00} R_c}{1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}} + k_8.
\end{align*}
\]

Here \(k_8\) is given by, \(k_8 = k_2 + u_- (u_- - z_-) / (1 + u_-)\).

The corresponding set of correlated observables associated to these sum rules is shown in Table 2 and their SM predictions using NLO QCD factorization are presented in Figs. 6-8. It is remarkable the extreme correlation of some observables like the pair \(O_{\text{VI}}^1 - O_{\text{VI}}^2\) or \(O_{\text{VII}}^1 - O_{\text{VII}}^2\). Notice that sum rules VI to VIII are chosen

| Table 2: Strongly correlated observables associated to sum rules VI-VIII |
|-----------------|-----------------|
| VI \(O_{\text{VI}}^1 = \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00} - 1 + (1 + \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right)\) | VI \(O_{\text{VI}}^2 = \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00} + \left(1 + \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}\right) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right)\) |
| VII \(O_{\text{VII}}^1 = \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00} R_0 = 1 + (1 + \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00} R_c - \mathcal{A}_{\text{CP}}^{00}\right) + k_6.\) | VII \(O_{\text{VII}}^2 = \mathcal{A}_{\text{CP}}^{0+} + \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}\right) + k_7.\) |
| VIII \(O_{\text{VIII}}^1 = \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00} R_0 = 1 + (1 + \mathcal{A}_{\text{CP}}^{00} R_0 - \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right) + k_8.\) | VIII \(O_{\text{VIII}}^2 = \mathcal{A}_{\text{CP}}^{0+} + \left(1 + \mathcal{A}_{\text{CP}}^{0+} R - \mathcal{A}_{\text{CP}}^{00}\right) \left(1 + \mathcal{A}_{\text{CP}}^{0+} R_c - \mathcal{A}_{\text{CP}}^{00}\right) + k_8.\) |
Figure 6: Correlation between $O_{1}^{\text{VI}}$ and $O_{2}^{\text{VI}}$, conventions as in Fig.3.

Figure 7: Correlation between $O_{1}^{\text{VII}}$ and $O_{2}^{\text{VII}}$, conventions as in Fig.3.

Figure 8: Correlation between $O_{1}^{\text{VIII}}$ and $O_{2}^{\text{VIII}}$ conventions as in Fig.3.
in such a way that can be combined easily with sum rules III to V to eliminate $R$, $R_c$ or $R_0$ keeping the dependence on $k_1$ and $k_2$. For example, combining sum rule III with sum rule VI, we can get rid of $R$ or sum rule IV with VII eliminates $R_c$ and V with VIII cancels $R_0$.

### III. Numerical results and tests of the sum rules

In this section we will evaluate the sum rules presented in the previous section using available experimental data [13, 14] on CP-averaged branching ratios and CP-asymmetries (see Table 3). We will start obtaining the basic block elements of the sum rules, i.e, the set of CP-averaged branching ratios:

$$
R = 1.00 \pm 0.18, \\
R_c = 1.41 \pm 0.29, \\
R_0 = 1.21 \pm 0.35.
$$

(23)

They imply the following values for the deviations from unity,

$$
u_+ = 0.00 \pm 0.18, \\
z_+ = 0.41 \pm 0.29, \\
n_+ = 0.21 \pm 0.35.
$$

(24)

Following the construction of sum rule I eq.(12) from eqs.(6-8), and taking the experimental values for $u_+$, $z_+$ and $n_+$ eq.(24) we find a first surprising result. While $u_+$, $z_+$ and $n_+$ are quantities of order $\xi_i$, the experimental value of $k_1$, a quantity of order $\xi_i^2$, obtained from eq.(12) is

$$
k_1 = 0.62 \pm 0.45,
$$

(25)

with an unexpectedly large central value, although with a large error. The reason is that $z_+$ goes in the wrong direction and requires $k_1$ to be very large in order to compensate it and to reproduce the experimental result for $R_0$. Moreover, if we compare this experimental result with the SM prediction using NLO QCD factorization we find that the central value of the experimental result is one order of magnitude larger than the prediction. However, given the large experimental error we are still only one standard deviation away from the the SM prediction, so it is crucial that experimentalist try to reduce this error.

If the error is reduced and the central value does not change drastically, one should conclude that $\xi_i$ is not small enough to be considered as a good expansion parameter and that there is some new mechanism that generates a very large isospin breaking contribution. On the contrary, if future data prefer a central value smaller by an order of magnitude with reduced errors then the small isospin breaking scenario of the SM will be again in good shape and sum rule I will be fulfilled. It is interesting to notice that $z_+$ seems to be quite large as compared to $u_+$ and both $z_+$ and $k_1$ depend on $d_1$, so the problem seems to affect more to $d_1$ while $d_2$ is inside the expectations. $d_1$ is related to the contributions of the charged channel: $A(B^+ \rightarrow \pi^0 K^+)$. If these experimental results are confirmed it seems that one should look for a type of isospin breaking New Physics affecting more the charged than the neutral channel.

---

Notice that in eq.(11) we are neglecting the small phase space difference between $B^\pm \rightarrow \pi^\pm K^0$ and $B^0_d \rightarrow \pi^\pm K^\mp, \pi^0 K^0$. 

---

10
Table 3: Measured branching ratios and CP-asymmetries for $B \to \pi K$ modes. The branching ratios correspond to the average of the three experiments BELLE, Babar and CLEO and are taken from [8]. CP-asymmetries correspond to CLEO data [14].

| BR($B_0^d \to \pi^\pm K^\mp$) | 17.2 $\pm$ 1.6 |
|-------------------------------|-------------------|
| BR($B^\pm \to \pi^0 K^\mp$)  | 12.1 $\pm$ 1.7 |
| BR($B^\pm \to \pi^\pm K^0$)  | 17.2 $\pm$ 2.6 |
| BR($B_0^0 \to \pi^0 K^0$)    | 10.4 $\pm$ 2.6 |
| $A_{CP}^{\pm 0}$              | $-0.18 \pm 0.24$ |
| $A_{CP}^{0+}$                 | $0.29 \pm 0.23$  |
| $A_{CP}^{-+}$                 | $0.04 \pm 0.16$  |

Let’s continue the analysis of sum rules involving only CP-averaged branching ratios, i.e, III to V. In this case we will evaluate their associated observables and the $\xi_i^2$ isospin breaking given by $k_3$, $k_4$ and $k_5$. The results are the following:

|   | $O_1^{III}$ | $O_2^{III}$ | $k_3$   |
|---|-------------|-------------|---------|
| III) | 1.00 $\pm$ 0.18 | 1.70 $\pm$ 0.71 | $-0.70 \pm 0.62$ |
| IV)  | 1.41 $\pm$ 0.29 | 0.79 $\pm$ 0.32 | 0.62 $\pm$ 0.43 |
| V)   | 1.21 $\pm$ 0.35 | 0.59 $\pm$ 0.23 | 0.62 $\pm$ 0.43 |

These results imply that, with present experimental data, none of the sum rules III to V seems to follow the expected behavior of a small $\xi_i^2$ term, i.e, to fall in or near the diagonal of Figs. 3 to 5. According to the discussion of the previous section, while eq.(24) corresponds to the deviation from one along the diagonal and measures the isospin breaking of order $\xi_i$, $k_j$ with $j = 3, 4, 5$ of eq.(26) gives an idea of the deviation from the diagonal (more precisely $k_j/\sqrt{2}$). We observe that, similarly to what happens with sum rule I, data force quantities that are formally of order $\xi_i^2$ like $k_j$ to be of the same size or larger than quantities of order $\xi_i$ like $u_+$ or $z_+$ of eq.(24).

Moreover, this experimental result is in conflict, at least for the central values, with the predictions for the SM using NLO QCD factorization shown in Figs. 3 to 5.

The set of sum rules II and VI to VIII involves the still non measured $A_{CP}^{00}$, so it is not possible to compare data with the SM predictions. However, under certain assumptions on isospin and future data we can explore what type of information they can give us on this CP-asymmetry.

From Table 3 it is possible to evaluate $u_-$ and $z_-$, eq.(3) and eq.(10):

\[ u_- = 0.22 \pm 0.29, \quad z_- = 0.59 \pm 0.41. \]  (27)
Figure 9: Sum rule I: (a) evaluated for the SM but keeping free the strong phases, (b) example of a model with $\xi_i = \xi_i^{SM}$, free strong phases but $a_i^{SM}/2 < a_i < 2a_i^{SM}$ with no new weak phases ($b_i = 0$).

We can also evaluate some of the observables of sum rule VI to VIII:

VI) $O_{VI}^1 = 0.04 \pm 0.16$.

VII) $O_{VII}^1 = 0.41 \pm 0.33$.

VIII) $O_{VIII}^2 = -0.48 \pm 0.37$. (28)

If we assume that future experimental results change and indicate that isospin breaking is indeed small, then, we can get an estimate of $A_{00}^{00CP}R_0 (O_{VIII}^1)$ from sum rule VIII since

$$O_{VIII}^1 \sim O_{VIII}^2.$$

We can do a similar exercise with sum rules VI and VII. Also using sum rule II we can get an estimate

$$A_{00}^{00CP}R_0 \sim O_{VI}^1 - O_{VII}^1 + A_{CP}^{\pm0},$$

that obviously would be in agreement with the estimate of eq.(29).

To end with the tests of the sum rules let’s see what happens if one would like to use specific models of New Physics. In this case it is necessary to evaluate for each model the new contributions to $\xi_{1,2}$, $a_{1,2}$, $b_{0,1,2}$ and to use the more general expressions for $k_1$ given in eq.(13), $k_2$ in eq.(15) and $u_\pm$ and $z_\pm$ as given in Appendix B. The expected main effect of New Physics will be new contributions to the electroweak penguin parameters $a_1$, $a_2$ $\xi_2$ ($q$ and $dC$, respectively, in the Standard Model, see Appendix A) and possible new weak phases contributing to $b_0$, $b_1$ and $b_2$.

We show in Fig.9a an example of the difference between the prediction for the SM of sum rule I, using the general expression for $k_1$ eq.(13), but this time keeping strong phases free (opposite to Fig.8 where the strong phases are predicted from NLO QCD factorization) versus the prediction in Fig.9b of a generic model of New Physics that induces important contributions to the electroweak penguins. This is useful to give us an idea of how important are the hadronic uncertainties coming from the strong
phases. We see comparing Fig.9a with Fig.1a or Fig.1b that having a prediction for the strong phases changes dramatically the situation, however still there is an important region of Fig.9b non overlapping with Fig.9a. This is an example of a region that could only be explained by New Physics. This is useful to establish the line between new isospin violating physics and possible hadronic uncertainties coming from our model dependence in predicting strong phases. In both cases, SM with free strong phases and New Physics, we find a much better agreement with present experimental data for $k_1$ eq.(25) than in Fig.1a-b. The reason of the decreasing behavior of $k_1$ with $\gamma$ can be understood analytically taking eq.(13) in the SM limit (Appendix A) and observing that the maximal value of $k_1$ for $\gamma = \pi/2$ is approximately $(1 + q)$ times smaller than the maximal $k_1$ for $\gamma = 0$. (Notice that since in this approach the strong phase $\omega$ can take any value, the maximal value of $k_1$ corresponds precisely to $\omega \sim \pi$ opposite to the NLO QCD factorization prediction of $\omega \sim 0$).

In supersymmetry, for instance, there are certain contributions that could be sizeable, in particular, those involving gluino-quark-squark where a contribution of order $\alpha_s/m_{\text{susy}}^2$ can compete with Standard Model contributions of order $\alpha/M_W^2$ and can be as large as twice the SM predictions [12]. Moreover, if New Physics contains new weak phases, they could contribute to $b_0$, $b_1$ and $b_2$. A global analysis of the contributions from different models can be found in [12].

In conclusion, we have presented a set of sum rules that allow for an easy test of experimental data concerning the size of isospin breaking. They are derived in a model independent way but applied explicitly to the case of the Standard Model in the framework of NLO QCD factorization. Out of these sum rules a set of observables are proposed that permits a simple interpretation of data in terms of isospin breaking as a function of the position of the experimental point in their combined graphs. The predicted results of these sum rules for the SM in QCD factorization are compared with present data, showing in most cases unexpectedly large central values although with still too large experimental errors. It is of crucial interest to reduce these experimental errors to confirm or falsify the strong deviations from the SM predictions and to discern if experimental data fall in a non-overlapping region as in Fig.9a-b.

Acknowledgments

It is a real pleasure to thank the theory group of Aachen: M. Beneke, W. Bernreuther, L. Sehgal, T. Feldmann, K. Stergios, T. Teubner, N. Düchtung and Sonya, for the warm hospitality and nice atmosphere during my stay there. Special thanks to Martin Beneke for very useful discussions and comments and also for understanding my personal decision, also to Werner Bernreuther for his generous support and help during my stay. I acknowledge financial support by BMBF during my stay in Germany and by Ministerio de Ciencia y Tecnologia in Spain.
References

[1] R. Fleischer, *Phys. Lett.* B365 (1996) 399; *Phys. Lett.* B459 (1999) 306; M. Neubert and J.R. Rosner, *Phys. Rev. Lett.* 81 (1998) 5076; *Phys. Lett.* B441 (1998) 403.

[2] A.J. Buras and R. Fleischer, *Eur. Phys. J.* C11 (1999) 93; *Eur. Phys. J.* C16 (2000) 97.

[3] M. Neubert, JHEP 02 (1999) 014.

[4] M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, *Phys. Rev. Lett.* 83, (1999) 1914; *Nucl. Phys.* B591 (2000) 313. See also: M. Neubert, hep-ph/0012204.

[5] M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, hep-ph/007256.

[6] M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, hep-ph/0104110.

[7] T. Muta, A. Sugamoto, M.Z. Yang and Y.D. Yang, *Phys. Rev.* D62 (2000) 094020; M.Z. Yang and Y.D. Yang, *Phys. Rev.* D62 (2000) 114019; M.Z. Yang and Y.D. Yang, hep-ph/0012208; X.G. He, J.P. Ma and C.Y. Wu, *Phys. Rev.* D63 (2001) 094004.

[8] M. Ciuchini, E. Franco, G. Martinelli, M. Pierini, L. Silvestrini, hep-ph/0104018.

[9] Y.Y. Keum, H.-n Li and A.I. Sanda, hep-ph/0004004; *Phys. Rev.* D63 (2001) 054008; Y.Y. Keum and H.-n. Li, *Phys. Rev.* D63 (2001) 074006.

[10] R. Fleischer, *Phys. Lett.* B435 (1998) 221; M. Neubert, JHEP 9902 (1999) 014; hep-ph/9910530, hep-ph/9904231; *Nucl.Phys.Proc.Suppl.* 86 (2000) 477; M. Gronau, *Phys. Rev.* D62 (2000) 014031; X.-G. He, C.-L. Hsueh, J.-Q. Shi, *Phys. Rev. Lett.* 84 (2000) 18; Zhi-zhong Xing, *Phys.Lett.* B488 (2000) 162; X.G. He, Y.K. Hsiao, J.Q. Shi, Y.L. Wu, Y.F. Zhou, *Phys.Rev.* D62 (2000) 036007; hep-ph/0011337; A. F. Falk, A. A. Petrov, *Phys.Rev.Lett.* 85 (2000) 252; R.E. Blanco, C. Gobel, R. Mendez-Galain, hep-ph/0007105; C.S. Kim, S. Oh, hep-ph/0009082, hep-ph/0103319.

[11] R. Fleischer and J. Matias, *Phys. Rev.* D61 (2000) 074004.

[12] Y. Grossman, M. Neubert, A.L. Kagan, JHEP 9910 (1999) 029.

[13] D.Cronin-Hennessy et al. [CLEO Collaboration], hep-ex/0001010; D.M. Asner et al. [CLEO Collaboration], hep-ex/0103040; G. Cavoto, [Babar Collaboration], talk given at the XXXVI Rencontres de Moriond QCD; B. Casey [Belle Collaboration], talk given at the XXXVI Rencontres de Moriond QCD.

[14] S. Chen et al. [CLEO Collaboration], *Phys. Rev. Lett.* 85 (2000) 525.
[15] See the report B Decays at LHC, P. Ball et al., hep-ph/0003238.

[16] H.J. Lipkin, hep-ph/9809347.

[17] M. Gronau and J.L. Rosner, Phys. Rev. D59 (1999) 113002.

[18] G. Buchalla, A.J. Buras and M.E. Lautenbacher, Rev. Mod. Phys. 68 (1996) 1125.

[19] M. Ciuchini et. al, hep-ph/0012308.
Appendix A: Standard Model limit

In order to make contact with the SM, we write down the parameters (amplitudes and phases) for the specific case of the SM, using the notation of \[5\]

\[\xi_1 e^{i\theta_1} \rightarrow \frac{\epsilon_3/2 e^{i(\phi_P + \phi + \pi)}}{\sqrt{1 + \epsilon_a^2 \cos^2 \gamma - 2 \epsilon_a \cos \phi \cos \eta}}, \quad a_1 e^{i\phi_1} \rightarrow q e^{i\omega}, \quad b_1 e^{i\phi_1} \rightarrow 0,\]

\[\xi_2 e^{i\theta_2} \rightarrow \frac{\epsilon_T e^{i(\phi_P + \phi_T + \pi)}}{\sqrt{1 + \epsilon_a^2 \cos^2 \gamma - 2 \epsilon_a \cos \phi \cos \eta}}, \quad a_2 e^{i\phi_2} \rightarrow q e^{i\omega C}, \quad b_2 e^{i\phi_2} \rightarrow 0,\]

\[b_0 e^{i\phi_0} \rightarrow \frac{\epsilon_a \sin \gamma e^{i(\eta - \delta)}}{\sqrt{1 + \epsilon_a^2 \cos^2 \gamma - 2 \epsilon_a \cos \phi \cos \eta}}, \quad \theta_0 \rightarrow \phi_P + \delta, \quad (31)\]

where

\[\cos \delta = \frac{1 - \epsilon_a \cos \phi \cos \eta}{\sqrt{1 + \epsilon_a^2 \cos^2 \gamma - 2 \epsilon_a \cos \phi \cos \eta}}, \quad \sin \delta = \frac{-\epsilon_a \cos \phi \sin \eta}{\sqrt{1 + \epsilon_a^2 \cos^2 \gamma - 2 \epsilon_a \cos \phi \cos \eta}}. \quad (32)\]

Appendix B: \(u_\pm, z_\pm\) and \(k_{1,2}\)

The dependence of \(u_\pm = u \pm \pi\) and \(z_\pm = z \pm \pi\) on \(d_{1,2}\) is given by

\[u = \frac{2}{x} \text{Re} \left[ A \left( B^+ \rightarrow \pi^+ K^0 \right) d_2^* \right] + \frac{1}{x} |d_2|^2,\]

\[z = \frac{2}{x} \text{Re} \left[ A \left( B^+ \rightarrow \pi^+ K^0 \right) d_1^* \right] + \frac{1}{x} |d_1|^2, \quad (33)\]

and the corresponding CP conjugates

\[\bar{u} = \frac{2}{x} \text{Re} \left[ A \left( B^- \rightarrow \pi^- \bar{K}^0 \right) \bar{d}_2^* \right] + \frac{1}{x} |\bar{d}_2|^2,\]

\[\bar{z} = \frac{2}{x} \text{Re} \left[ A \left( B^- \rightarrow \pi^- \bar{K}^0 \right) \bar{d}_1^* \right] + \frac{1}{x} |\bar{d}_1|^2. \quad (34)\]

The parameters \(k_1\) and \(k_2\) are

\[k_1 = \frac{2}{x} \left( |d_1|^2 + |\bar{d}_1|^2 - \text{Re}[d_1 d_2^*] - \text{Re}[\bar{d}_1 \bar{d}_2^*] \right), \quad (35)\]

\[k_2 = \frac{2}{x} \left( |d_1|^2 - |\bar{d}_1|^2 - \text{Re}[d_1 d_2^*] + \text{Re}[\bar{d}_1 \bar{d}_2^*] \right), \quad (36)\]

\(^4\)We understand \(P\) in \[3\] as \(P = |P| e^{i\phi_P}\). If one would like to make contact with the notation of \[3\] it is necessary to identify our \(\phi_T\) with \(\phi\) of \[3\] and our \(\phi_T + \phi_P\) with \(\phi_T\) of \[3\].
with \( x = 2(1 + b_0^2)|P|^2 \). Notice that the dependence on \(|P|\) cancels in \( u, z, k_1 \) and \( k_2 \).

From eqs. (33)-(34) and eq. (4) we can derive a set of model independent expressions for \( u, \pi, z \) and \( \pi \):

\[
\begin{align*}
  u &= \frac{\xi_2}{1 + b_0^2} \left\{ \text{Re} \left[ e^{i(\theta_0 - \theta_2)} \left( 1 - i b_0 e^{i \phi_0} \right) \left( e^{-i \gamma} - a_2 e^{-i \phi_{a_2}} + i b_2 e^{-i \phi_{b_2}} \right) \right] \\
  &\quad + \frac{\xi_2}{2} \left[ (1 + a_2^2 + b_2^2) - 2 a_2 \cos(\phi_{a_2} - \gamma) + 2 b_2 \sin(\phi_{a_2} + \gamma) - 2 a_2 b_2 \sin(\phi_{a_2} - \phi_{b_2}) \right] \right\}.
\end{align*}
\]

Its CP conjugated \( \pi \) can be obtained from eq. (37) by changing the sign of the weak phases:

\[
\gamma \rightarrow -\gamma, \quad b_{0,2} \rightarrow -b_{0,2}.
\]

\( z \) is also obtained from eq. (37) substituting amplitudes and phases of \( d_2 \) by those of \( d_1 \):

\[
\xi_2 \rightarrow \xi_1, \quad a_2 \rightarrow a_1, \quad b_2 \rightarrow b_1, \quad \theta_2 \rightarrow \theta_1, \quad \phi_{a_2} \rightarrow \phi_{a_1}, \quad \phi_{b_2} \rightarrow \phi_{b_1}.
\]

A similar substitution to eq. (38) will allow us to obtain the CP conjugate \( \pi \) from \( z \):

\[
\gamma \rightarrow -\gamma, \quad b_{0,1} \rightarrow -b_{0,1}.
\]