Some elementary examples of quartics with finite–dimensional motive

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Received: date / Accepted: date

Abstract This small note contains some easy examples of quartic hypersurfaces that have finite–dimensional motive. As an illustration, we verify a conjecture of Voevodsky (concerning smash–equivalence) for some of these special quartics.

Keywords Algebraic cycles · Chow groups · motives · finite–dimensional motives · quartics

Mathematics Subject Classification (2010) 14C15, 14C25, 14C30.

1 Introduction

The notion of finite–dimensional motive, developed independently by Kimura and O’Sullivan [13], [11], [14], [10], [7] has given important new impetus to the study of algebraic cycles. To give but one example: thanks to this notion, we now know the Bloch conjecture is true for surfaces of geometric genus zero that are rationally dominated by a product of curves [13]. It thus seems worthwhile to find concrete examples of varieties that have finite–dimensional motive, this being (at present) one of the sole means of arriving at a satisfactory understanding of Chow groups.

The present note aims to contribute something to the list of examples of varieties with finite–dimensional motive, by considering quartic hypersurfaces. In any dimension, there is one famous quartic known to have finite–dimensional motive: the Fermat quartic

\[(x_0)^4 + (x_1)^4 + \cdots + (x_{n+1})^4 = 0.\]

The Fermat quartic has finite–dimensional motive because it is rationally dominated by a product of curves, and the indeterminacy locus is again of Fermat type [17]. The main result of this note presents, in any dimension, a family of quartics (containing the Fermat quartic) with finite–dimensional motive:

**Theorem (stheorem)** The following quartics have finite–dimensional motive:

(i) a smooth quartic \(X \subset \mathbb{P}^{3k-1}(\mathbb{C})\) given by an equation

\[f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) = 0,\]

where the \(f_i\) define smooth quartic curves;

(ii) a smooth quartic \(X \subset \mathbb{P}^{4k}(\mathbb{C})\) given by an equation

\[f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + (x_{3k})^4 = 0,\]

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where the \( f_i \) define smooth quartic curves;

(iii) a smooth quartic \( X \subset \mathbb{P}^{3k+1}(\mathbb{C}) \) given by an equation

\[
f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + h(x_{3k}, x_{3k+1}) = 0,
\]

where the \( f_i \) define smooth quartic curves;

The proof is an elementary argument, based on the fact that the inductive structure exhibited by Shioda [17], [12] still applies to this kind of hypersurfaces.

To illustrate how nicely finite-dimensionality allows to understand algebraic cycles, we provide an application to a conjecture of Voevodsky’s [21]:

**Proposition (proposition 10)** Let \( X \) be a quartic as in theorem 5 and suppose \( X \) has odd dimension. Then numerical equivalence and smash-equivalence coincide for all algebraic cycles on \( X \).

(For the definition of smash-equivalence, cf. definition 7)

**Conventions** All varieties will be projective irreducible varieties over \( \mathbb{C} \).

For smooth \( X \) of dimension \( n \), we will denote by \( A^j(X) = A_{n-j}(X) \) the Chow group \( CH^j(X) \otimes \mathbb{Q} \) of codimension \( j \) cycles under rational equivalence. The notations \( A^j_{hom}(X) \) and \( A^j_{AJ}(X) \) will denote the subgroup of homologically trivial resp. Abel–Jacobi trivial cycles. The category \( M_{rat} \) will denote the (contravariant) category of Chow motives with \( \mathbb{Q} \)-coefficients over \( \mathbb{C} \). For a smooth projective variety, \( h(X) = (X, \Delta_X, 0) \) will denote its motive in \( M_{rat} \).

The group \( H^*(X) \) will denote singular cohomology with \( \mathbb{Q} \)-coefficients.

## 2 Finite-dimensionality

We refer to [13], [1], [14], [7], [10] for the definition of finite-dimension motive. An essential property of varieties with finite-dimension motive is embodied by the nilpotence theorem:

**Theorem 1 (Kimura [13])** Let \( X \) be a smooth projective variety of dimension \( n \) with finite-dimension motive. Let \( \Gamma \in A^n(X \times X) \) be a correspondence which is numerically trivial. Then there is \( N \in \mathbb{N} \) such that

\[
\Gamma^{\leq N} = 0 \quad \in A^n(X \times X).
\]

Actually, the nilpotence property (for all powers of \( X \)) could serve as an alternative definition of finite-dimension motive, as shown by a result of Jannsen [10, Corollary 3.9].

**Conjecture 2 (Kimura [13])** All smooth projective varieties have finite-dimension motive.

We are still far from knowing this, but at least there are quite a few non-trivial examples:

**Remark 3** The following varieties have finite-dimension motive: abelian varieties, varieties dominated by products of curves [13], \( K \)3 surfaces with Picard number 19 or 20 [15], surfaces not of general type with \( p_g = 0 \) [5, Theorem 2.11], certain surfaces of general type with \( p_g = 0 \) [6, 16, 22], Hilbert schemes of surfaces to have finite-dimension motive [5], generalized Kummer varieties [24, Remark 2.9(ii)], threefolds with nef tangent bundle [8] (an alternative proof is given in [19, Example 3.16]), fourfolds with nef tangent bundle [9], log–homogeneous varieties in the sense of [2] (this follows from [5, Theorem 4.4]), certain threefolds of general type [20, Section 8], varieties of dimension \( \leq 3 \) rationally dominated by products of curves [19, Example 3.15], varieties \( X \) with \( A^i_{AJ}(X) = 0 \) for all \( i \) [18, Theorem 4], products of varieties with finite-dimension motive [13].

**Remark 4** It is an embarassing fact that up till now, all examples of finite-dimension motives happen to lie in the tensor subcategory generated by Chow motives of curves, i.e. they are “motives of abelian type” in the sense of [12]. On the other hand, there exist many motives that lie outside this subcategory, e.g. the motive of a very general quintic hypersurface in \( \mathbb{P}^3 \) [4, 7.6].
3 Main

Theorem 5 The following quartics have finite–dimensional motive:

(i) a smooth quartic $X \subset \mathbb{P}^{3k-1}(\mathbb{C})$ given by an equation

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) = 0,$$

where the $f_i$ define smooth quartic curves;

(ii) a smooth quartic $X \subset \mathbb{P}^{k}(\mathbb{C})$ given by an equation

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + (x_3)^4 = 0,$$

where the $f_i$ define smooth quartic curves;

(iii) a smooth quartic $X \subset \mathbb{P}^{3k+1}(\mathbb{C})$ given by an equation

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + h(x_{3k}, x_{3k+1}) = 0,$$

where the $f_i$ define smooth quartic curves;

Proof The proof is by induction, using Shioda’s trick in the guise of the following proposition (this is [12, Remark 1.10]):

Proposition 6 (Katsura–Shioda [12]) Let $Z \subset \mathbb{P}^{m_1+m_2}$ be a smooth hypersurface of degree $d$ defined by an equation

$$g_1(x_0, \ldots, x_{m_1}) + g_2(x_{m_1+1}, \ldots, x_{m_1+m_2}) = 0.$$

Let $Z_1$ resp. $Z_2$ be the smooth hypersurfaces of dimension $m_1$ resp. $m_2 - 1$, defined as

$$g_1(x_0, \ldots, x_{m_1}) + y^d = 0,$$

resp.

$$g_2(x_{m_1+1}, \ldots, x_{m_1+m_2}) + z^d = 0.$$

Then there exists a dominant rational map

$$\phi: Z_1 \times Z_2 \dashrightarrow Z,$$

and the indeterminacy of $\phi$ is resolved by blowing up the locus

$$\left(Z_1 \cap (y = 0)\right) \times \left(Z_2 \cap (z = 0)\right) \subset Z_1 \times Z_2.$$

The induction base is $k = 1$. In this case, the quartic defined in (i) has finite–dimensional motive because it is a curve; the quartic surface defined in (ii) has finite–dimensional motive because it is dominated by a product of curves [5, Example 11.3]; case (iii) is OK because any quartic threefold $X$ has

$$A^*_X(X) = 0$$

and as such has finite–dimensional motive [18, Theorem 4].

Next, we suppose the theorem is true for $k - 1$.

Let $X \subset \mathbb{P}^{3k-1}$ be a quartic as in (i). According to proposition [5], $X$ is rationally dominated by the product $X_1 \times S$, where $S$ is a $K3$ surface as in [5, Example 11.3], and $X_1$ is of type

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + y^4 = 0.$$

We have seen that $S$ has finite–dimensional motive, and $X_1$ has finite–dimensional motive by induction. $X$ is dominated by the blow–up of $X_1 \times S$ with center $Z_1 \times C$, where $C \subset S$ is a curve, and $Z_1 \subset X_1$ is of type

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_{k-1}(x_{3k-6}, x_{3k-5}, x_{3k-4}) = 0.$$

The product $Z_1 \times C$ has finite–dimensional motive (by induction), hence so has the blow–up and so has $X$. 


Let \( X \subset \mathbb{P}^{3k} \) be a quartic as in (ii). Now \( X \) is rationally dominated by \( X_1 \times X_2 \), where \( X_1 \) is of type
\[
 f(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_{k-1}(x_{3k-6}, x_{3k-5}, x_{3k-4}) + y^4 = 0,
\]
and \( X_2 \) is a fourfold of type
\[
 f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + (x_{3k})^4 + z^4 = 0.
\]
Both have finite–dimensional motive by induction. To resolve the indeterminacy, we need to blow up \( X_1 \times X_2 \) in the center of type \( Z_1 \times Z_2 \), where \( Z_1 \subset X_1 \) is of type
\[
 f(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_{k-1}(x_{3k-6}, x_{3k-5}, x_{3k-4}) = 0.
\]
This \( Z_1 \) has finite–dimensional motive (by induction), and so does \( Z_2 \) (any smooth quartic threefold has finite–dimensional motive). It follows that \( X \) has finite–dimensional motive.

Lastly, let \( X \subset \mathbb{P}^{3k+1} \) be a quartic as in (iii). Then \( X \) is rationally dominated by \( X_1 \times C \), where \( X_1 \) is of type
\[
 f(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) + y^4 = 0,
\]
and \( C \) is a curve. By induction, we may suppose \( X_1 \) has finite–dimensional motive (because we have already treated case (ii)). The indeterminacy locus is of the form \( Z_1 \times \{ \text{points} \} \), where \( Z_1 \) is of type
\[
 f(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) + \cdots + f_k(x_{3k-3}, x_{3k-2}, x_{3k-1}) = 0.
\]
This \( Z_1 \) also has finite–dimensional motive (because we have already treated case (i)), hence so does the blow–up and so does \( X \).

4 Application: Voevodsky’s conjecture

**Definition 7** (Voevodsky [21]) Let \( X \) be a smooth projective variety. A cycle \( a \in A^r(X) \) is called smash–nilpotent if there exists \( m \in \mathbb{N} \) such that
\[
a^m := a \times \cdots \times a = 0 \quad \text{(m times)} \quad \text{in} \quad A^{mr}(X \times \cdots \times X).
\]

Two cycles \( a, a' \) are called smash–equivalent if their difference \( a - a' \) is smash–nilpotent. We will write \( A^r_{\otimes}(X) \subset A^r(X) \) for the subgroup of smash–nilpotent cycles.

**Conjecture 8** (Voevodsky [22]) Let \( X \) be a smooth projective variety. Then
\[
 A^r_{\text{num}}(X) \subset A^r_{\otimes}(X) \quad \text{for all } r.
\]

**Remark 9** It is known [11 Théorème 3.33] that conjecture [8] implies (and is strictly stronger than) conjecture [2].

**Proposition 10** Let \( X \) be a smooth quartic of the type as in theorem [5] Suppose the dimension \( n \) of \( X \) is odd. Then
\[
 A^r_{\text{num}}(X) \subset A^r_{\otimes}(X) \quad \text{for all } r.
\]

**Proof** As \( X \) is a hypersurface, the Künneth components are algebraic and the Chow motive of \( X \) decomposes
\[
h(X) = h_0(X) \oplus \bigoplus_j \mathbb{L}(n_j) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]

Since \( A^r_{\text{num}}(L(n_j)) = 0 \), we have
\[
 A^r_{\text{num}}(X) = A^r_{\text{num}} \left( h_0(X) \right).
\]
The motive \( h_0(X) \) is oddly finite–dimensional. (Indeed, as \( n \) is odd we have that the motive \( \text{Sym}^m h_0(X) \in \mathcal{M}_{\text{hom}} \) is 0 for some \( m \gg 0 \). By finite–dimensionality, the same then holds in \( \mathcal{M}_{\text{rat}} \).) The proposition now follows from the following result (which is [13 Proposition 6.1], and which is also applied in [11] where I learned this):
Proposition 11 (Kimura [13]) Let $M \in \mathcal{M}_{\text{rat}}$ be oddly finite–dimensional. Then
\[ A^r(M) \subset A^r_0(M) \text{ for all } r. \]

Acknowledgements This note is a belated echo of the Strasbourg 2014—2015 groupe de travail based on the monograph [23]. Thanks to all the participants for the pleasant and stimulating atmosphere. Many thanks to Yasuyo, Kai and Len for lots of enjoyable post–work aperitifs.

References

1. Y. André, Motifs de dimension finie (d’après S.-I. Kimura, P. O’Sullivan,..), Séminaire Bourbaki 2003/2004, Astérisque 299 Exp. No. 929, viii, 115—145,
2. M. Brion, Log homogeneous varieties, in: Actas del XVI Coloquio Latinoamericano de Algebra, Revista Matemática Iberoamericana, Madrid 2007, arXiv: [math/0609669]
3. M. de Cataldo and L. Migliorni, The Chow groups and the motive of the Hilbert scheme of points on a surface, Journal of Algebra 251 no. 2 (2002), 824—848,
4. P Deligne, La conjecture de Weil pour les surfaces K3, Invent. Math. 15 (1972), 206—226,
5. B. van Geemen, Kuga–Satake varieties and the Hodge conjecture, in: The Arithmetic and Geometry of Algebraic Cycles, Banff 1998 (B. Gordon et alii, eds.), Kluwer Dordrecht 2000,
6. V. Guletski˘ı and C. Pedrini, The Chow motive of the Godeaux surface, in: Algebraic Geometry, a volume in memory of Paolo Francia (M.C. Beltrametti et alii, editors), Walter de Gruyter, Berlin New York, 2002,
7. F. Ivorra, Finite dimensional motives and applications (following S.-I. Kimura, P. O’Sullivan and others), in: Autour des motifs, Asian-French summer school on algebraic geometry and number theory, Volume III, Panoramas et synthèses, Société mathématique de France 2011,
8. J. Iyer, Murre’s conjectures and explicit Chow–Künneth projectors for varieties with a nef tangent bundle, Transactions of the Amer. Math. Soc. 361 (2008), 1667—1681,
9. J. Iyer, Absolute Chow–Künneth decomposition for rational homogeneous bundles and for log homogeneous varieties, Michigan Math. Journal Vol.60, 1 (2011), 79—91,
10. U. Jannsen, On finite–dimensional motives and Murre’s conjecture, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,
11. B. Kahn and R. Sebastian, Smash-nilpotent cycles on abelian 3–folds, Math. Res. Letters 16 (2009), 1007—1010,
12. T. Katsura and T. Shioda, On Fermat varieties, Tohoku Math. J. Vol. 31 No. 1 (1979), 97—115,
13. S. Kimura, Chow groups are finite dimensional, in some sense, Math. Ann. 331 (2005), 173—201,
14. J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013,
15. C. Pedrini, On the finite dimensionality of a K3 surface, Manuscripta Mathematica 138 (2012), 59—72,
16. C. Pedrini and C. Weibel, Some surfaces of general type for which Bloch’s conjecture holds, to appear in: Period Domains, Algebraic Cycles, and Arithmetic, Cambridge Univ. Press, 2015,
17. T. Shioda, The Hodge conjecture for Fermat varieties, Math. Ann. 245 (1979), 175—184,
18. C. Vial, Projectors on the intermediate algebraic Jacobians, New York J. Math. 19 (2013), 793—822,
19. C. Vial, Remarks on motives of abelian type, to appear in Tohoku Math. J.,
20. C. Vial, Chow–Künneth decomposition for 3– and 4–folds fibred by varieties with trivial Chow group of zero–cycles, J. Alg. Geom. 24 (2015), 51—80,
21. V. Voevodsky, A nilpotence theorem for cycles algebraically equivalent to zero, Internat. Math. Research Notices 4 (1995), 187—198,
22. C. Voisin, Bloch’s conjecture for Catanese and Barlow surfaces, J. Differential Geometry 97 (2014), 149—175,
23. C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014,
24. Z. Xu, Algebraic cycles on a generalized Kummer variety, [arXiv:1506.04297v1].