AN ISOSPECTRAL DEFORMATION ON AN ORBIFOLD QUOTIENT OF A NILMANIFOLD

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Abstract. We construct a Laplace isospectral deformation of metrics on an orbifold quotient of a nilmanifold. Each orbifold in the deformation contains singular points with order two isotropy. Isospectrality is obtained by modifying a generalization of Sunada's Theorem due to DeTurck and Gordon.

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1. Introduction

A Riemannian orbifold (see [S], [Sc]) is a mildly singular generalization of a Riemannian manifold. For example the quotient space of a Riemannian manifold under an isometric, properly discontinuous group action is a Riemannian orbifold [T]. First defined in 1956 by I. Satake, orbifolds have proven useful in many settings including the theory of 3-manifolds, symplectic geometry, and string theory.

The local structure of a Riemannian orbifold is given by the orbit space of a Riemannian manifold under the isometric action of a finite group. If a point, $p$, in the manifold is fixed under the group action, the corresponding element of the orbit space, $\bar{p}$, is called a singular point of the orbifold. The isotropy type of a point $\bar{p}$ in the orbit space is the isomorphism class of the isotropy group of a point $p$ in the manifold that projects to $\bar{p}$ under the quotient. The singular set of an orbifold is the set of all singular points of the orbifold.

The tools of spectral geometry can be transferred to the setting of Riemannian orbifolds by exploiting the well-behaved local structure of these spaces (see [C], [St]). As in the manifold setting, the spectrum of the Laplace operator of a compact Riemannian orbifold is a sequence $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow +\infty$ where each eigenvalue has finite multiplicity. We say that two orbifolds are isospectral if their Laplace spectra agree.

In this note we show that the formulation of Sunada’s Theorem found in [DG] can be used to obtain isospectral deformations on Riemannian orbifolds with non-trivial singular sets. We prove this fact in Section 2 by observing that the proof of Theorem 2.7 in [DG] does not require that the action of the discrete subgroup $\Gamma$ be free. In Section 3 we display an example of an isospectral deformation of metrics on an orbifold quotient of a nilmanifold.

The only other known examples of non-manifold isospectral deformations on orbifolds were recently obtained by Sutton using a blend of the torus action method and the Sunada technique [Su]. Other examples of non-manifold isospectral orbifolds include pairs with boundary in [BCDS], and in [BW]: isospectral flat 2-orbifolds that

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are not conjugate (in terms of lengths of closed geodesics) [DR]; a \((2m)\)-manifold isospectral to a \((2m)\)-orbifold on \(m\)-forms [GR]; pairs of isospectral orbifolds for which the maximal isotropy groups have different orders [R] and [Sch]; and arbitrarily large finite families of isospectral orbifolds in [SSW].

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2. Isospectral deformations on orbifolds

In this section we observe that the generalization of Sunada’s method found in [DG] can be further generalized to include isospectral deformations of metrics on orbifolds. In what follows we will assume that \(G\) is a Lie group with simply connected identity component \(G_0\). We let \(\Gamma\) be a discrete subgroup of \(G\) such that \(G = \Gamma G_0\) and \((G_0 \cap \Gamma) \backslash G_0\) is compact.

Given an automorphism \(\Phi : G \to G\), we say that \(\Phi\) is an \(\textit{almost-inner automorphism}\) if for each \(x \in G\) there exists an element \(a \in G\) such that \(\Phi(x) = axa^{-1}\). More generally, if \(\Phi : G \to G\) is an automorphism such that for each \(\gamma \in \Gamma\) there exists \(a \in G\) satisfying \(\Phi(\gamma) = a\gamma a^{-1}\), we say that \(\Phi\) is an \(\textit{almost-inner automorphism of } G\) \(\textit{relative to } \Gamma\). We denote the set of all almost-inner automorphisms of \(G\) (resp. almost-inner automorphisms of \(G\) with respect to \(\Gamma\)) by \(\text{AIA}(G)\) (resp. \(\text{AIA}(G; \Gamma)\)).

We have the following theorem.

**Theorem 2.1.** [DG] Let \(G, G_0, \text{ and } \Gamma\) be as above with \(G_0\) nilpotent and let \(\Phi \in \text{AIA}(G; \Gamma)\). Suppose that \(G\) acts effectively and properly on the left by isometries on a Riemannian manifold \((M, g)\) and that \(\Gamma\) acts freely on \(M\) with \(\Gamma \backslash M\) compact. Then, letting \(g\) denote the submersion metric, \((\Phi(\Gamma) \backslash M, g)\) is isospectral to \((\Gamma \backslash M, g)\).

The proof of Theorem 2.1 is based on work by Donnelly in [D] concerning the existence of a heat kernel on a manifold \(M\) which admits a properly discontinuous (but not necessarily free) action by a group \(\Gamma\). Donnelly shows that if \(\Gamma \backslash M\) is compact, then there exists a unique heat kernel on \(M\). Furthermore, Donnelly gives the following relationship between the heat kernels on \(M\) and on \(\Gamma \backslash M\).

**Theorem 2.2.** [D] Let \(\Gamma\) act properly discontinuously on \(M\) with compact quotient \(\overline{M} = \Gamma \backslash M\). Suppose that \(F\) is a fundamental domain for \(\Gamma \backslash M\). If \(\bar{x}, \bar{y} \in \overline{M}\) then set

\[
E(t, \bar{x}, \bar{y}) = \sum_{\gamma \in \Gamma} E(t, x, \gamma \cdot y)
\]

where \(x, y \in F\), \(\bar{x} = \pi(x)\), and \(\bar{y} = \pi(y)\). If \(E\) is the heat kernel of \(M\), the sum on the right converges uniformly on \([t_1, t_2] \times F \times F\), \(0 < t_1 \leq t_2\), to the heat kernel on \(\overline{M}\).

Notice that since the action of \(\Gamma\) need not be free, the quotient space \(\overline{M}\) may not be a manifold.

Theorem 2.1 relies on the fact that two manifolds \((M_1, g_1)\) and \((M_2, g_2)\) are isospectral if and only if they have the same heat trace, i.e. \(\int_{M_1} E_1(t, x, x) \, dx = \int_{M_2} E_2(t, x, x) \, dx\), where \(E_i\) denotes the heat kernel on \(M_i\). In particular, the proof uses Theorem 2.2 to pull the heat trace back from the quotient \(\Gamma \backslash M\) to the cover.
$M$ in order to use combinatorial arguments to reexpress the heat trace on $\Gamma \backslash M$. The new expression of the heat trace makes it evident that, when comparing the heat trace of $(\Gamma \backslash M, g)$ with the heat trace of $(\Phi(\Gamma) \backslash M, g)$, if certain volumes (which depend only on $\Gamma$ and $\Phi(\Gamma)$) are equal then the respective heat traces are equal. DeTurck and Gordon show that when $\Phi$ is an almost-inner automorphism, these volumes are in fact equal, and hence $(\Gamma \backslash M, g)$ and $(\Phi(\Gamma) \backslash M, g)$ are isospectral.

We note that, as with Theorem 2.2, the proof of Theorem 2.1 does not rely on the freeness of the action of $\Gamma$ on $M$. Therefore we make the following generalization of Sunada’s theorem.

**Theorem 2.3.** Suppose that $G$, $G_0$, and $\Gamma$ are as above and $G_0$ is nilpotent. Suppose that $G$ acts effectively and properly discontinuously on the left by isometries on $(M, g)$ with $\Gamma \backslash M$ compact. Let $\Phi \in \text{Aut}(G; \Gamma)$. Then, letting $g$ denote the submersion metric, the quotient orbifolds $(\Gamma \backslash M, g)$ and $(\Phi(\Gamma) \backslash M, g)$ are isospectral.

### 3. Examples

Now we apply Theorem 2.3 to give an example of a nontrivial isospectral deformation on an orbifold. We first note the following.

**Lemma 3.1.** If $\Phi$ is an isomorphism of $G$ and $\Gamma$ acts on $M$ in such a way that there exists a diffeomorphism $\Psi$ of $M$ satisfying $\Psi(g \cdot x) = \Phi(g) \cdot \Psi(x)$ for all $g \in G$ and $x \in M$, then $(\Phi(\Gamma) \backslash M, g)$ is isometric to $(\Gamma \backslash M, \Psi^* g)$.

Applying Theorem 2.3 in conjunction with Lemma 3.1 will allow us to produce an isospectral deformation on a fixed orbifold $\Gamma \backslash M$.

In Appendix B to [DG], K. B. Lee translates Theorem 2.1 to the setting of infranilmanifolds. For a group $G$ we have that $\text{Aut}(G) \ltimes G$ acts on $G$ by $(\phi, g) \cdot h = g\phi(h)$. Consider the case when $G$ is a simply connected nilpotent Lie group and $\Gamma$ is a uniform discrete subgroup of $G$. Take $\Pi$ to be a finite extension of $\Gamma$ in $\text{Aut}(G) \ltimes G$. If the action of $\Pi$ on $G$ is free, then $\Pi \backslash G$ is an infranilmanifold. Lee observes that by setting $\Gamma$, $G_0$, and $G$ from Theorem 2.1 equal to $\Pi$, $G$ and $\Pi G$, and assuming that the action of $\Pi$ on $G$ is free, we can find isospectral deformations on infranilmanifolds. We note that a priori, the action of $\Pi G$ on $G$ need not be free. Thus by working in this setting we introduce the possibility of finding isospectral orbifold quotients of $G$.

Lee gives a specific example to illustrate his case. His example is based on a similar example found in [GW2].

Let $G$ be the Lie group

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) | x_i, y_i, z_i \in \mathbb{R}\},$$

where group multiplication is defined by

$$(x_1, \ldots, z_2)(x_1', \ldots z_2') = (x_1 + x_1', \ldots, y_1 + y_2', z_1 + z_1' + x_1 y_1' + x_2 y_2', z_2 + z_2' + x_1 y_2').$$

Suppose that $\Gamma$ is the integer lattice in $G$ and define $\Phi_1 : G \to G$ by

$$\Phi_1(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, y_1, y_2, z_1, z_2 + ty_2),$$

where $t \in [0, 1)$. In the original example Gordon and Wilson show that each $\Phi_1$ is an almost-inner automorphism so, applying Lemma 3.1, $(\Phi_1(\Gamma) \backslash G, g)$ is an isospectral deformation on $\Gamma \backslash G$. They also show that the deformation is nontrivial.
In his example, Lee defines $\alpha \in \text{Aut}(G) \ltimes G$ by

$$\alpha(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, -y_1, -y_2, -z_1, -z_2 + \frac{1}{2})$$

and lets $\Pi = \Gamma \cup \alpha \Gamma$. Since $\alpha$ commutes with $\Phi_t$ for all $t$, we can extend each $\Phi_t$ to an element $\Phi_t$ of $\text{AIA}(\Pi \Gamma; \Pi \Gamma)$. If $g$ is a $\Pi G$-invariant metric on $G$, then for each $t$, $(\Phi_t(\Pi \backslash G, g))$ is isospectral to $(\Pi \backslash G, g)$.

Lee implicitly assumed that the action of $\Pi$ on $G$ is free. However, we can see by closer inspection that the action of $\Pi$ on $G$ is not free. For example, any point of the form $(x_1, x_2, 0, 0, 0, \frac{1}{2})$ is fixed by $\alpha \in \Pi G$. In fact the set of all fixed points of the action of $\Pi$ on $G$ is:

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{R}^6 : x_1, x_2 \in \mathbb{R}, y_1, y_2, z_1 \in \frac{1}{2}\mathbb{Z}, z_2 = \frac{n}{2} + \frac{1}{4}\}$$

where $n$ is any integer. The isotropy group of a point in this set has the form,

$$\{1, (\Phi, (0, 0, 2y_1, 2y_2, 2z_1, 2z_2))\}.$$ 

So we see that $\Pi \backslash G$ is an orbifold containing singular points with $\mathbb{Z}_2$ isotropy type. Thus Lee’s example is an illustration of Theorem 2.3: after applying Lemma 3.1 with $\Psi = \Phi_t$ and $\Phi = \tilde{\Phi}_t$, we have an isospectral deformation of metrics on the orbifold $\Pi \backslash G$.

This example is a nontrivial deformation. Indeed, suppose that $\tau : (\Pi \backslash G, g) \rightarrow (\Pi G, \Phi_t^* g)$ is an isometry. Then because $G$ is simply connected and $\Pi$ is discrete, $G$ is the simply connected cover of $\Pi \backslash G$. Thus $\tau$ lifts to an isometry, also called $\tau$ from $(G, g)$ to $(G, \Phi_t^* g)$. Since $G$ is a nilpotent Lie group $\tau$ must be an element of $\text{AIA}(G) \ltimes G$ (see [GW1]). Furthermore, because $\tau$ is a lift we have that $\tau \circ \Pi \circ \tau^{-1} = \Pi$ within the transformation group $\text{Aut}(G) \ltimes G$. On the other hand, $G$ is normal in $\text{Aut}(G) \ltimes G$ so conjugation by $\tau$ maps $G$ to itself. Therefore, conjugation by $\tau$ leaves $\Gamma$ invariant. This implies that $\tau$ must descend to an isometry $\tau : (\Gamma \backslash G, g) \rightarrow (\Gamma \backslash G, \Phi_t^* g)$. However, from [GW2] we know that no such isometry can exist. Thus $(\Pi \backslash G, g)$ cannot be isometric to $(\Pi G, \Phi_t^* g)$.

Note that Lee’s example can be modified to produce an isospectral deformation on a manifold. For example, suppose that we define $\beta : G \rightarrow G$ by

$$\beta(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, y_1, y_2, -z_1, z_2 + \frac{1}{2}).$$

Letting $\Pi = \Gamma \cup \beta \Gamma$ we see that since $\beta^2$ is simply translation by $(0, 0, 0, 0, 0, 1)$, $\Pi$ is a finite extension of $\Gamma$. Since $\beta$ commutes with the maps $\Phi_t$ defined above, we can extend each $\Phi_t$ to an element $\tilde{\Phi}_t$ of $\text{AIA}(\Pi \Gamma; \Pi \Gamma)$. Finally by direct computation we can see that the action of $\Pi$ on $G$ has no fixed points. Thus we have an isospectral deformation of metrics on a manifold. The proof that the deformation is nontrivial is identical to the one given above.

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