Abstract. We prove a converse of Yano’s extrapolation theorem for translation invariant operators.

A CONVERSE EXTRAPOLATION THEOREM FOR TRANSFORMATION INVARIANT OPERATORS

TERENCE TAO

1. Introduction

Let $X$ be a compact symmetric space with compact symmetry group $G$, and $r > 0$, $1 < p_0 < \infty$ be numbers; all constants may depend on $r$ and $p_0$. If a linear operator $T$ is bounded on $L^p$, $1 < p < p_0$ with an operator norm of $O((p-1)^{-r})$ as $p \to 1$, then it is a classical extrapolation theorem of Yano [5] that $T$ also maps $L \log^r L(X)$ to $L^1(X)$.

In this paper we show the following converse:

Theorem 1.1. Let $G$, $X$, $p_0$, and $r$ be as above. Suppose $T$ is translation invariant, maps $L \log^r L$ to $L^1$, and is bounded on $L^{p_0}$. Then $T$ is bounded on $L^p$, $1 < p < p_0$ with an operator norm of $O((p-1)^{-r})$.

This theorem is false without the assumption of translation invariance, since $L^p$ is not an interpolation space between $L \log^r L$ and $L^{p_0}$. For a concrete counterexample, take $E$ and $F$ be subsets of $X$ of measure $2^{-N}$ and $N^{-p_0}2^{-N}$ respectively, where $N$ is a large number. Then the operator

$$Tf = 2^N 2^{-r/p_0} \langle f, \chi_E \rangle \chi_F$$

maps $L \log^r L$ to $L^1$ and bounded on $L^{p_0}$, but the $L^p$ operator norm for $1 < p < p_0$ grows polynomially in $N$.

The translation invariance hypothesis is exploited via the following heuristic principle: if $f$ is a function on $X$ supported on a set of measure $O(1/N)$, then there exists $N$ translates of $f$ which are essentially disjoint. This idea is used in factorization theory (see e.g. [1]) and also appears in the abstract theory of covering lemmas (e.g. [2], [3]). The point is that the $(L \log^r L, L^1)$ hypothesis yields more information when applied to the sum of the $N$ translates of $f$ than when applied to just $f$ by itself.

The theorem also holds for $p_0 = \infty$, either by a routine modification of the argument, or by assuming an a priori operator bound on $L^2$ (for instance, applying the theorem with $p_0 = 2$, and re-interpolating the result with $L^\infty$ to obtain a better bound on $L^2$. The theorem also holds of course for $r = 0$ by Riesz convexity.

Although our theorem is phrased for compact spaces, it can be extended to non-compact Lie groups if all operator norms are local. In other words, if $T$ is translation invariant, locally bounded on $L^{p_0}$ and locally maps $L \log^r L$ to $L^1$, then $T$ is also locally bounded on $L^p$, $1 < p < p_0$, with an operator norm of $O((p-1)^{-r})$.

This can be proven either by direct modification of the argument, or by abstract transplantation considerations.
As is well known, the space $L \log L$ is an atomic space generated by the atoms $|E|^{-1} \log(1/|E|) \chi_E$, where $E$ is an arbitrary measure subset of $X$ with $0 < |E| \ll 1$. (For completeness, we provide a proof of this fact in an appendix). As a consequence we have

**Corollary 1.2.** Let $G$, $X$, $p_0$, $r$ be as above, and let $T$ be a translation invariant operator which is bounded on $L^{p_0}(X)$. Then a necessary and sufficient condition for $T$ to be bounded on $L^p$, $1 < p < p_0$, with an operator norm of $O(1/(p-1)^r)$, is that

$$\int |T \chi_E| \lesssim |E| \log(1/|E|)^r$$

for all measurable subsets $E$ of $X$ with $0 < |E| \ll 1$.

In a subsequent paper with Jim Wright [4], we show that certain classes of rough multipliers are bounded from $L \log L$ to $L^1$ for various values of $r$, and apply Theorem 1.1 to deduce sharp bounds for the growth of $L^p$ operator norms.

## 2. The main lemma

We use $A \lesssim B$ to denote the estimate $A \leq CB$ where $C$ is a constant depending on $p_0$, $r$, and the implicit constants in Theorem 1.1, and $A \sim B$ to denote the estimates $B \sim A \lesssim B$.

Fix $p$; by Riesz convexity we may assume that $p < 1 + p_0/2$. All of our implicit constants shall be independent of $p$.

The main lemma in the argument is

**Lemma 2.1.** Let $E, F$ be subsets of $X$ with $0 < |E| \leq |F|$. Then we have

$$\int_F |Tf| \lesssim |F|^{1/p'} \left( \frac{1}{p-1} + \log(2 + |F|/|E|)^r \right) \|f\|_p$$

for all $L^p$ functions $f$ supported on $E$.

We remark that without translation invariance, one can only obtain (1) with $\log(2 + |F|/|E|)$ replaced by $\log(2 + 1/|E|)$.

**Proof** Fix $E$, $F$, $f$; we may normalize $\|f\|_p = 1$. Let $h$ denote the function $h = |\chi_F T f|$, and define the quantity $A$ by

$$\|h\|_1 = A|E|^{1/p'};$$

our task is then to show that

$$A \lesssim \left( \frac{1}{p-1} + \log(2 + |F|/|E|)^r \right).$$

Let $N$ be the nearest integer to $\varepsilon/|F|$, where $0 < \varepsilon \ll 1$ is a small constant to be chosen later. The first step in the argument is to construct group elements $\Omega_0, \ldots, \Omega_N \in G$ such that

$$\langle \chi_{\bigcup_{j<J} \Omega_j(F)} h \circ \Omega_j \rangle \leq \frac{1}{2} A|E|^{1/p'}$$
and

\[(5) \quad \langle (\sum_{j<J} |f| \circ \Omega_j)^{p-1}, |f| \circ \Omega_J \rangle \leq 1\]

for all \(0 \leq J \leq N\).

Intuitively, (4) asserts that the \(h \circ \Omega_j\) are essentially disjoint, while (5) asserts that the \(|f| \circ \Omega_j\) are similarly disjoint. For future reference, we note that (5) and the \(L^p\) normalization of \(f\) implies that

\[(6) \quad \int_X (\sum_{j\leq J} |f| \circ \Omega_j)^p - \int_X (\sum_{j<J} |f| \circ \Omega_j)^p \leq C.\]

We now construct the desired group elements. We may let \(\Omega_0\) be arbitrary since (4), (5) are vacuously true for \(J = 0\). Now suppose inductively that \(\Omega_0, \ldots, \Omega_{J-1}\) have already been constructed for some \(0 < J \leq N\) such that (5) (and hence (6)) holds for all previous values of \(J\). We will show that

\[(7) \quad \int_G \langle \chi_{\bigcup_{j<J} \Omega_j(F), h \circ \Omega_J} \rangle \, d\Omega_J \leq \frac{1}{8} |E|^{1/p}.\]

and

\[(8) \quad \int_G \langle (\sum_{j<J} |f| \circ \Omega_j)^{p-1}, |f| \circ \Omega_J \rangle \, d\Omega_J \leq \frac{1}{4}\]

where \(d\Omega_j\) is Haar measure on \(G\). By Markov’s inequality, this implies that a randomly selected \(\Omega_J\) has probability at least \(3/4\) of obeying (4) and probability at least \(3/4\) of obeying (5), and so there exists an \(\Omega_J\) with the desired properties.

From Fubini’s theorem, (2), and the identity

\[\int_G g \circ \Omega(x) \, d\Omega = C \int_X g\]

for all \(x \in X\), the left-hand side of (7) evaluates to

\[C |\bigcup_{j<J} \Omega_j(F)| A|E|^{1/p} \lesssim J|F|A|E|^{1/p} \lesssim \epsilon A|E|^{1/p}.\]

Thus (7) holds if \(\epsilon\) is sufficiently small. The left-hand side of (8) can similarly be evaluated as

\[C(\int_X (\sum_{j<J} |f| \circ \Omega_j)^{p-1})(\int_X |f|).\]

From Hölder we have

\[\int_X |f| \leq |E|^{1/p'} \|f\|_p = |E|^{1/p'},\]

and

\[\int_X (\sum_{j<J} |f| \circ \Omega_j)^{p-1} \leq (J|E|)^{1/p}(\int_X (\sum_{j<J} |f| \circ \Omega_j)^p)^{\frac{p-1}{p}}.\]
On the other hand, from (6) and the induction hypothesis we have
\[ \int_X \left( \sum_{j<J} |f| \circ \Omega_j \right)^p \lesssim J. \]

Combining all these estimates, we see that
\[ \text{LHS of (8)} \lesssim J[E] \lesssim \varepsilon |E|/|F| \lesssim \varepsilon \]

Thus we obtain (8) if \( \varepsilon \) is sufficiently small.

Fix \( \varepsilon \); all constants may now implicitly depend on \( \varepsilon \). By telescoping (6) we have
\[ \int_X \left( \sum_{j \leq N} |f| \circ \Omega_j \right)^p \lesssim N \lesssim |F|^{-1}. \quad (9) \]

Let \( \epsilon_j = \pm 1 \) be an arbitrary assignment of signs. Then the function
\[ \sum_{j \leq N} \epsilon_j f \circ \Omega_j \]
has a \( \mathcal{L}^p \) norm of \( O(|F|^{-1/p}) \) and is supported on a set of measure \( O(N |E|) = O(|E|/|F|) \). We now apply

**Lemma 2.2.** Let \( g \) be a function supported on a set \( E \subset E \). Then
\[ \|g\|_{L^{\log r}} \lesssim \left( \frac{1}{p-1} + \log(2 + \frac{1}{|E|}) \right)^r |E|^{1/p'} \|g\|_p. \]

**Proof** We divide into two cases, \( |E| \geq 2^{-2r/(p-1)} \) and \( |E| \leq 2^{-2r/(p-1)} \). We normalize
\[ \|g\|_p = (p-1)^r \]
in the first case and
\[ \|g\|_p = (\log \frac{1}{|E|})^{-r} |E|^{-1/p'} \]
in the second; in either case our task reduces to showing that
\[ \int_E |g| \log(2 + |g|)^r \lesssim 1. \]

We may restrict ourselves to the set
\[ E' = \{ x \in E : |g(x)| \geq 2 + |E|^{-1} \log^{-r}(2 + \frac{1}{|E|}) \}, \]
since the contribution outside of \( E' \) is clearly acceptable. In this set \( \log(2 + |g|) \) may of course be replaced by \( \log |g| \).

The function \( \frac{\log r}{|g|^{p-1}} \) is increasing for \( 1 \leq t < e^{r/(p-1)} \) and decreasing for \( t > e^{r/(p-1)} \), with a global maximum of \( \frac{(r/e)^r}{(p-1)} \). We thus have
\[ \frac{\log(|g|)^r}{|g|^{p-1}} \lesssim \frac{1}{(p-1)^r} \]
in the first case and
\[ \frac{\log(|g|)^r}{|g|^{p-1}} \lesssim |E|^{p-1} \log^{pr} \frac{1}{|E|} \]
if the second case. In either case the claim follows by multiplying this estimate by \( |g|^p \) and integrating, using the \( L^p \) normalization of \( g \). \( \blacksquare \)
From this lemma we obtain
\[ \| \sum_{j \leq N} \epsilon_j f \circ \Omega_j \|_{L^\log} \leq |E|^{1/p'} |F|^{-1} \left( \frac{1}{p-1} + \log(2 + \frac{|F|}{|E|}) \right)^r. \]

Since \( T \) is translation invariant and maps \( L^\log \) to \( L^1 \), we thus have
\[ \| \sum_{j \leq N} \epsilon_j T f \circ \Omega_j \|_1 \leq |E|^{1/p'} |F|^{-1} \left( \frac{1}{p-1} + \log(2 + \frac{|F|}{|E|}) \right)^r. \]

Randomizing the signs \( \epsilon_j \) and taking expectations using Khinchin’s inequality, we obtain
\[ \left\| \left( \sum_{j \leq N} |T f \circ \Omega_j|^2 \right)^{1/2} \right\|_1 \leq |E|^{1/p'} |F|^{-1} \left( \frac{1}{p-1} + \log(2 + \frac{|F|}{|E|}) \right)^r. \]

In particular, we have
\[ (10) \quad \left\| \left( \sum_{j \leq N} (h \circ \Omega_j)^2 \right)^{1/2} \right\|_1 \leq |E|^{1/p'} |F|^{-1} \left( \frac{1}{p-1} + \log(2 + \frac{|F|}{|E|}) \right)^r. \]

If we integrate the trivial pointwise estimate
\[ \left( \sum_{j \leq J} (h \circ \Omega_j)^2 \right)^{1/2} \geq \left( \sum_{j<J} (h \circ \Omega_j)^2 \right)^{1/2} + h \circ \Omega_J \left( 1 - \chi_{\bigcup_{j<J} \Omega_j(F)} \right) \]
using (2) and (4), we obtain
\[ \left\| \left( \sum_{j \leq J} (h \circ \Omega_j)^2 \right)^{1/2} \right\|_1 \geq \left\| \left( \sum_{j<J} (h \circ \Omega_j)^2 \right)^{1/2} \right\|_1 + \frac{1}{2} A |E|^{1/p'}. \]

Telescoping this for all \( 1 \leq J \leq N \), we obtain
\[ \left\| \left( \sum_{j \leq N} (h \circ \Omega_j)^2 \right)^{1/2} \right\|_1 \geq \frac{1}{2} N A |E|^{1/p'} \sim A |E|^{1/p'} |F|^{-1}. \]

Comparing this with (10) we obtain (3) as desired.

3. Conclusion of the argument

We are now ready to prove Theorem 1.1. By duality, it suffices to prove the bilinear form estimate
\[ (11) \quad |\langle T f, g \rangle| \lesssim \frac{1}{(p-1)^r}. \]

for all \( f, g \) such that \( \|f\|_p = 1 \), \( \|g\|_{p'} = 1 \).

Fix \( f, g \); we may assume that \( f, g \) are non-negative. Let \( f^* : \mathbb{R}^+ \to \mathbb{R}^+ \) be the non-increasing left-continuous re-arrangement of \( f \), so that \( \|f^*\|_p = 1 \) and
\[ (12) \quad |\{ x : f(x) > f^*(\alpha) \}| \leq \alpha. \]

Similarly define \( g^* \).
For any integers $q \geq 1$ and $k < C$, define $f_{k,q}$ to be the restriction of $f$ to the set \( \{ x : f^* (2^{qk}) < f(x) \leq f^* (2^{qk}) \} \). Since $X$ has finite measure, we thus have
\[
f = \sum_k f_{k,q}.
\]
Similarly define $g_{k,q}$.

Usually one takes $q = 1$, but because of our desire for sharp bounds as $p \to 1$ it shall be more appropriate to choose $q$ so that $q \sim \frac{1}{p - 1}$.

By the triangle inequality, (11) will now follow from the estimates
\[
\sum_{k,l : k \geq l + 1} |\langle T f_{k,q}, g_{l,q} \rangle| \lesssim 1
\]
and
\[
\sum_{k,l : k \leq l} |\langle T f_{k,q}, g_{l,q} \rangle| \lesssim q^r.
\]

Let us first prove (13). By splitting
\[
f_{k,q} = \sum_{k' = qk}^{qk + q - 1} f_{k',1}, \quad g_{l,q} = \sum_{l' = ql}^{ql + q - 1} g_{l',1}
\]
we see that the left-hand side of (13) is majorized by
\[
\sum_{k', l' : k' > l'} |\langle T f_{k',1}, g_{l',1} \rangle|.
\]
Since $T$ is bounded on $L^p$, we may use Hölder’s inequality to majorize this by
\[
\sum_{k', l' : k' > l'} \|f_{k',1}\|_p \|g_{l',1}\|_{p_0}.
\]
From (12) and the definition of $f_{k,q}$ we have
\[
\|f_{k',1}\|_p \lesssim 2^{k'/p_0} f^*(2^{k'})
\]
and similarly
\[
\|g_{l',1}\|_p \lesssim 2^{l'/p_0} g^*(2^{l'}).\]
Thus the left-hand side of (13) is majorized by
\[
\sum_{k', l' : k' > l'} (2^{k'/p} f^*(2^{k'})) (2^{l'/p_0} g^*(2^{l'})) 2^{-(k' - l')(1 - \frac{1}{p_0})}.
\]
The estimate (13) then follows from Young’s inequality for bilinear forms. Indeed, the first expression in parentheses has an $l^p$ norm comparable to $\|f^*\|_p = 1$, the second expression in parentheses has an $l^{p_0}$ norm comparable to $\|g^*\|_{p'} = 1$, and convolution kernel is summable with $l^1$ norm of $O(1)$ since we are assuming $p < (1 + p_0)/2$.

It remains to prove (14). From Lemma 2.1 and (12) we have
\[
\int_{g > g^*(2^{qk+q})} |T f_{k,q}| \lesssim (2^{qk+q})^{1/p'} \left( \frac{1}{p - 1} + \log(2 + \frac{2^{qk+q}}{2^{qk+q}}) \right) \|f_{k,q}\|_{p'}.
\]
From the definition of $q$ and the assumptions on $k$, $l$, this simplifies to
\[ \int_{g > g^*(2^q)} |T f_{k,q}| \lesssim 2^{qk'/p'} q^r (1 + l - k)^r \|f_{k,q}\|_p. \]

From Hölder’s inequality we thus have
\[ |\langle T f_{k,q}, g_{k,q} \rangle| \lesssim g^*(2^q) 2^{qk'/p'} q^r (1 + l - k)^r \|f_{k,q}\|_p. \]

Thus the left-hand side of (14) is majorized by
\[ q^r \sum_{k,l,k \leq l} \|f_{k,q}\|_p (2^{qk'/p'} g^*(2^q)) 2^{-q(l-k)/p'} (1 + l - k)^r. \]

The claim then follows again from Young’s inequality and the choice of $q$, since the sequence $\|f_{k,q}\|_p$ is in $l^p$, the sequence $2^{-q(l-k)/p'} g^*(2^q)$ has an $l^{p'}$ norm comparable to $\|g^*\|_p = 1$, and the convolution kernel is integrable uniformly in $p$.

4. Appendix: Atomic decomposition of Orlicz spaces

In this section we show that every $L \log L(X)$ function $f$ can be decomposed into a convex linear combination of atoms $|E|^{-1} \log(1/|E|)^r \chi_E$ with $0 < |E| \ll 1$.

We first observe that any function $f$ supported on a set of measure $2^{-k}$ and having a sup norm of $k^{-r} 2^{-k}$ can easily be decomposed in this manner, since bounded functions can be written as convex linear combinations of characteristic functions.

Now let $f$ be a general $L \log L(X)$ function; we may normalize so that
\[ \int_X |f| \log(2 + |f|) = 1. \]

We may also assume without loss of generality that $f$ is non-negative and is supported on a set of measure $\ll 1$.

Let $f^*$ and $f_{k,q}$ be as before. For each integer $k < -C$, we define
\[ c_k = |k|^{-1} 2^k f^*(2^k) \]
and
\[ a_k(x) = f_{k,1}/c_k. \]

Clearly $f = \sum_{k < -C} c_k a_k$. From (12) and the previous discussion, the $a_k$ are convex linear combinations of atoms uniformly in $k$, so it suffices to show that the $c_k$ are summable, i.e. that
\[ \sum_{k < -C} |k|^{-1} 2^k f^*(2^k) \lesssim 1. \]

Since $f$ is non-decreasing, we may bound this expression by
\[ C \int_{0 \leq t \leq 1} f^*(t) \log(1/t)^{-r} dt. \]

The portion of the integral where $f^*(t) \leq t^{-1/2}$ is clearly acceptable, so we may assume that $f^*(t) \geq t^{-1/2}$. But then we may estimate the above by
\[ C \int_{0 \leq t \leq 1} f^*(t) \log(2 + f^*(t))^{-r} dt = C \int f \log^r (2 + f) = C \]
as desired.

References

1. J. Bourgain, Besicovitch-type maximal operators and applications to Fourier analysis, Geom. and Funct. Anal. 22 (1991), 147–187.
2. A. Córdoba, Maximal functions, covering lemmas and Fourier multipliers, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, pp. 29–50, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979.
3. M. de Guzmán, Real variable methods in Fourier analysis, North-Holland Mathematics Studies, 46. Notas de Matemática [Mathematical Notes], 75. North-Holland Publishing Co., Amsterdam-New York, 1981.
4. T. Tao, J. Wright, Endpoint multiplier theorems of Marcinkiewicz type, submitted.
5. S. Yano, Notes on Fourier analysis. XXIX. An extrapolation theorem. J. Math. Soc. Japan 3, (1951). 296–305.

Department of Mathematics, UCLA, Los Angeles, CA 90024
E-mail address: tao@math.ucla.edu