ARITHMETIC OF FUZZY NUMBERS AND INTERVALS -
A NEW PERSPECTIVE WITH EXAMPLES

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Abstract. The article guides the reader through four consecutive definitions of progressing scope. Fuzzy intervals are represented as an ordered pair of functions, and fuzzy numbers are defined as a subcase. The four arithmetic operations are defined utilizing the notion of generalized inverse functions as practiced in statistics.

This new approach has the advantage that in most cases the outcome of an arithmetic operation may be stated explicitly, in closed form, which is not possible using the original definition \cite{Zadeh65} or when working with alpha-cuts.

Although the presented material is original the text is formulated in the style of an introductory textbook and many completely worked examples are provided and graphically illustrated.

0. Summary and Structure

This article is divided into four sections:

Section 1: A first simplified definition of fuzzy numbers, as just members of a set without (arithmetic) structure, resembles the L-R form of Dubois and Prade \cite{DB00}, a form in which any fuzzy number may be represented by suitable change of variables.

What is new in the literature and possibly originative is the method of performing the operations of addition and multiplication by inverting, operating and then re-inverting the characterizing functions.

This novel approach is mathematically equivalent to Lofti A. Zadeh’s original definition \cite{Zadeh65} of binary operations via fuzzy vectors and the extension principle \((\xi \circ \eta)(z) = \sup \\{ \min(\xi(x), \eta(y)) \mid (x, y) \in \mathbb{R}^2 : x \circ z \}\), but far more user friendly.

Section 2: Starting with section 2 we prefer to speak more generally about fuzzy intervals regarding fuzzy numbers a subcase. To make possible the

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progression from Definition 1 (continuous characterizing functions and support on the positive real line) to Definition 2 (semi-continuous characterizing functions, support on the positive reals) the notion of generalized inverse functions as known in statistics is briefly discussed and illustrated. The representation of a fuzzy interval as an ordered pair of functions is adapted.

Section 3 extends the definitions of sections 1 and 2 to the case of characterizing functions of mixed support (that is containing 0).

Section 4: In the concluding section a new, very natural and intuitive definition of fuzzy intervals and arithmetic operations on them is laid out. With the new approach to defining binary operations some new terminology is introduced.
Definition 1. For the purpose of this section: A fuzzy number \( \xi \) is defined by, and identified with a continuous function (called the \textit{characterizing function} of that fuzzy number) \( \xi(\cdot) : \mathbb{R}^+ \rightarrow [0, 1] \) which has the following properties:

1. compact support: the set of points \( x \in \mathbb{R}^+ \), where \( \xi(x) > 0 \) is an open interval \( (l, r) \subset \mathbb{R}^+ \), for some \( l, r \in \mathbb{R}^+ \)
2. there is exactly one point \( x_o \in (l, r) \) such that \( \xi(x_o) = 1 \),
3. \( \xi(\cdot) \) is strictly increasing on the interval \( [l, x_o] \),
4. \( \xi(\cdot) \) is strictly decreasing on the interval \( [x_o, r] \).

Notice that condition (1) of Definition 1 implies that \( \xi_l(l) = 0 \) and \( \xi_r(r) = 0 \).

The collection of fuzzy numbers \( \xi \) satisfying Definition 1 shall be denoted by \( \mathcal{F}_c^0(\mathbb{R}^+) \).

Convention: Throughout the article when writing \( \mathbb{R}^+ \) we mean that \( 0 \in \mathbb{R}^+ \).

For future computational comfort, the increasing (resp. the decreasing) components of \( \xi \) are singled out and treated as separate entities denoted by \( \xi_l \) (resp. \( \xi_r \)). (As in left, right)

For convenience we may now rewrite \( \xi \):

\[
\xi(x) = \begin{cases} 
\xi_l(x) & \text{for } x \in [l, x_o], \\
1 & \text{for } x = x_o, \\
\xi_r(x) & \text{for } x \in [x_o, r]. 
\end{cases}
\]

with \( \xi, \xi_l, \xi_r \) meeting conditions (1)-(4) of Definition 1.

We shall also sometimes write \( \xi_m \) for \( x_o \), where \( \xi(x_o) = 1 \).

(The formally faulty overkilling of \( x_0 \) in the definition of \( \xi \) above, using closed intervals instead of half-closed is intentional and will find its justification at a later point in this article. We want to have \( \xi_l \) and \( \xi_r \) defined on closed intervals to simplify the process of inverting back and forth)

Definition 1" Equivalently one may think of a fuzzy number as the ordered pair \( (\xi_l(\cdot), \xi_r(\cdot)) \), with \( x_0 \) again being implicit from \( \xi_l(x_0) = \xi_r(x_0) = 1 \).

Terminology:
For a given fuzzy number \( \xi \) the functions \( \xi_l : [l, x_o] \rightarrow [0, 1] \) and \( \xi_r : [x_o, r] \rightarrow [0, 1] \) shall be for the purpose of this article called the left and right \textit{fuzzy components} of \( \xi \).

We may colloquially say:"Here we have a real number \( x_0 \) which is fuzzy to the left with \( \xi_l(x) = .... \) and fuzzy to the right with \( \xi_r(x) = .... \)"
Below, in Fig. 1 several examples of “fuzzy 2-s” as per Definition 1 are charted:

\[ \text{Figure 1. Examples of fuzzy numbers: (a) } \xi_l, \xi_r \text{ – linear (b) } \xi_l, \xi_r \text{ – piecewise linear, (c) } \xi \text{ – of sine type} \]

On the other hand: the functions illustrated in Fig. 2 do not characterize fuzzy numbers in the sense of Definition 1. In Fig. 2 (a) the function is not non-negative. In Fig. 2 (b) the set of points, where the function is positive is not a finite interval. Finally, the function in Fig. 2 (c) does not satisfy the monotonicity conditions (3), (4) of Definition 1.

\[ \text{Figure 2. Functions which do not characterize fuzzy numbers: (a) function with negative values, (b) function with unbounded support, (c) monotonicity condition not fulfilled.} \]

**Remark 1.1.** In the scientific community dealing with fuzzy numbers it is customary to denote the values of characterizing functions by \( \alpha \) (not e.g. \( y \)), where \( 0 \leq \alpha \leq 1 \). When analyzing graphs of fuzzy numbers one needs to be ever conscientious about whether and to what degree the ordinate axis is scaled. (We shall later see that \( \text{supp} \xi \circ \eta = \text{supp} \xi \cdot \text{supp} \eta \).) In this presentation we shall look at “small” numbers and as a rule employ non-scaled graphs.

1.1. **Addition and multiplication on** \( F_0^c(\mathbb{R}^+) \).

**Remark 1.2.** Because of the monotonicity conditions (1.3), (1.4) of Definition 1 the fuzzy components \( \xi_l \) and \( \xi_r \) of \( \xi \) have their inverse functions \( \xi_l^{-1} : [0, 1] \to \mathbb{R} \) with \( \xi_l^{-1}(\mathbb{R}) = [l, x_o] \) and \( \xi_r^{-1} : [0, 1] \to \mathbb{R} \) with \( \xi_r^{-1}(\mathbb{R}) = [x_o, r] \), which are again invertible and those new inverses are once more strictly monotone and have values \( \alpha \in [0, 1] \) and so we may say:
1.1.1. Addition. of fuzzy numbers is now defined in the following way:

\[(\xi \oplus \eta)_l(x) = (\xi_l^{-1} + \eta_l^{-1})^{-1}(x),\]
\[(\xi \oplus \eta)_r(x) = (\xi_r^{-1} + \eta_r^{-1})^{-1}(x).\]

This implies that \((\xi \oplus \eta)(x_o) = 1\), where \(x_o = \xi^{-1}(1) + \eta^{-1}(1)\), so using the notation of Definition 1’ we have

\[(\xi \oplus \eta)_m = \xi_m + \eta_m.\]

In interval notation (Definition 1”) we write:

\[(\xi_l, \xi_r) \oplus (\eta_l, \eta_r) = ((\xi_l^{-1} + \eta_l^{-1})^{-1}, (\xi_r^{-1} + \eta_r^{-1})^{-1})\]

Remark 1.3. The operation given in (1.2) is well defined since being the sum of strictly monotone functions \((\xi_l^{-1} + \eta_l^{-1})\) and \((\xi_r^{-1} + \eta_r^{-1})\) are also strictly monotone and therefore are invertible on their natural domain \([0, 1]\).

1.1.2. Multiplication. is defined analogically

\[(\xi \odot \eta)_l(x) = (\xi_l^{-1} \cdot \eta_l^{-1})^{-1}(x),\]
\[(\xi \odot \eta)_r(x) = (\xi_r^{-1} \cdot \eta_r^{-1})^{-1}(x).\]

With

\[(\xi \odot \eta)(x_o) = 1\] where \(x_o = \xi^{-1}(1) \cdot \eta^{-1}(1),\)

or

\[(\xi \odot \eta)_m = \xi_m \cdot \eta_m.\]

and again

\[(\xi_l, \xi_r) \odot (\eta_l, \eta_r) = ((\xi_l^{-1} \cdot \eta_l^{-1})^{-1}, (\xi_r^{-1} \cdot \eta_r^{-1})^{-1})\]

The formulae for addition and multiplication of fuzzy numbers given above may be explicated in a convenient albeit quite informal way:

A fuzzy 5 times a fuzzy 3 is a fuzzy 15 and their sum is a fuzzy 8. The tricky part is to compute the left and right fuzzy parts: Here is what you do:

Draw the graphs of the two fuzzy numbers on a piece of paper acting as the the coordinate plane of \(x\) and \(\alpha\). Rotate the piece of paper by 90 degrees to interchange the roles of \(x\) and \(\alpha\). Add or multiply in standard manner as functions the components of the rotated functions. Rotate back the coordinate plane to see the outcome.
1.2. **Algebraic properties of** \( F^0_c(\mathbb{R}^+) \). It follows directly from the definition that the operations of addition “⊕” and multiplication “⊙” are commutative and associative, i.e. for arbitrary \( \xi, \eta, \mu \in F^0_c(\mathbb{R}^+) \),

\[
\xi \oplus \eta = \eta \oplus \xi \quad \text{and} \quad \xi \odot \eta = \eta \odot \xi,
\]

\[
(\xi \oplus \eta) \oplus \mu = \eta \oplus (\xi \oplus \mu) \quad \text{and} \quad (\xi \odot \eta) \odot \mu = \eta \odot (\xi \odot \mu).
\]

Moreover, the standard distributive property

\[
(1.9) \quad \xi \odot (\eta \oplus \mu) = (\xi \odot \eta) \oplus (\xi \odot \mu)
\]

is satisfied as well. (for \(1.9\) to hold, positive support of the characterizing functions is essential. Distributivity will not hold for the extension of section 3).

We shall close this section with several examples illustrating the operations of addition and multiplication.

**Example 1.** The simplest examples of fuzzy numbers, and those that appear in the literature most often, are numbers whose fuzzy components are linear. (Such numbers are commonly referred to as “triangle numbers” and in short denoted by \( tr(l, m, r) \)). So let us consider two such fuzzy numbers \( \xi \) and \( \eta \) defined by

\[
\xi(x) = \begin{cases} 
\xi_l(x) = x - 1 & \text{for } x \in [1, 2], \\
1 & \text{for } x = 2, \\
\xi_r(x) = 3 - x & \text{for } x \in [2, 3],
\end{cases}
\]

and

\[
\eta(x) = \begin{cases} 
\eta_l(x) = \frac{1}{2}x - \frac{5}{2} & \text{for } x \in [5, 7], \\
1 & \text{for } x = 7, \\
\eta_r(x) = -\frac{1}{3}x + \frac{10}{3} & \text{for } x \in [7, 10].
\end{cases}
\]

**Figure 3.** Two fuzzy numbers \( \xi \) and \( \eta \) with linear fuzzy components.

Now whether we want to add or multiply \( \xi \) and \( \eta \) we need to first find the inverse functions \( \xi^{-1}_l, \xi^{-1}_r \) and \( \eta^{-1}_l, \eta^{-1}_r \).
\[\xi_i^{-1}(\alpha) = \alpha + 1, \quad \eta_i^{-1}(\alpha) = 2\alpha + 5,\]
\[\xi_r^{-1}(\alpha) = 3 - \alpha, \quad \eta_r^{-1}(\alpha) = 10 - 3\alpha.\]

To perform the addition we use (1.2). For $\alpha \in [0, 1]$ we obtain
\[\xi_i^{-1}(\alpha) + \eta_i^{-1}(\alpha) = 3\alpha + 6,\]
\[\xi_r^{-1}(\alpha) + \eta_r^{-1}(\alpha) = 13 - 4\alpha.\]

Thus, inverting once again by (1.2) we arrive at
\[\xi \oplus \eta = \begin{cases} 
\left(\xi \oplus \eta\right)_l(x) = \frac{1}{3}x - 2 & \text{for } x \in [6, 9], \\
1 & \text{for } x = 9, \\
\left(\xi \oplus \eta\right)_r(x) = -\frac{1}{4}x + \frac{13}{4} & \text{for } x \in [9, 13].
\end{cases}\]

with $(\xi \oplus \eta)_m = \xi_m + \eta_m = 2 + 7 = 9.$

**Figure 4.** Two fuzzy numbers $\xi, \eta$ and their sum $\xi \oplus \eta$

Fig. 4 shows $\xi, \eta$ and $\xi \oplus \eta$. We note that the fuzzy addition operation “$\oplus$” preserves linearity of components.

Now let us move on to multiplication. Going by (1.5) we get
\[\xi_l^{-1}(\alpha) \cdot \eta_l^{-1}(\alpha) = 2\alpha^2 + 7\alpha + 5,\]
\[\xi_r^{-1}(\alpha) \cdot \eta_r^{-1}(\alpha) = 3\alpha^2 - 19\alpha + 30.\]

for $\alpha \in [0, 1]$. We now need to find the inverse functions of the obtained products. For the function $\xi_l^{-1}(\alpha) \cdot \eta_l^{-1}(\alpha)$ we have $2\alpha^2 + 7\alpha + 5 = x$. Since $\alpha$ changes between 0 and 1, we see that $x$ changes between 5 and 14. Thus we need to solve the quadratic equation $2\alpha^2 + 7\alpha + 5 - x = 0$ for $\alpha \in [0, 1]$ and $x \in [5, 14]$. We calculate the discriminant $\Delta = 9 + 8x$ and see that $\alpha = \frac{-7 + \sqrt{9 + 8x}}{4}$ for $x \in [5, 14]$. The function $\xi_r^{-1}(\alpha) \cdot \eta_r^{-1}(\alpha)$ can be inverted in a similar way (the details are left as an exercise for the reader). Clearly, $(\xi \odot \eta)_m = \xi_m \cdot \eta_m = 2 \cdot 7 = 14$, so finally we obtain
\[(\xi \odot \eta)(x) = \begin{cases} 
\left(\xi \odot \eta\right)_l(x) = \frac{-7 + \sqrt{9 + 8x}}{4} & \text{for } x \in [5, 14], \\
1 & \text{for } x = 14, \\
\left(\xi \odot \eta\right)_r(x) = \frac{19 - \sqrt{112x}}{6} & \text{for } x \in [14, 30].
\end{cases}\]
Remark 1.4. The product $\xi \odot \eta$ of two numbers with linear components has fuzzy components that are not linear, but are rather of a square root type.

Example 2. Let us consider the fuzzy number $\xi(x) = \chi_{[0,\pi]}(x) \cdot \sin x$ and decompose and represent it as in Definition 1':

$$
\xi(x) = \begin{cases} 
\sin x & \text{for } x \in [0, \frac{\pi}{2}], \\
1 & \text{for } x = \frac{\pi}{2} \\
\sin x & \text{for } x \in [\frac{\pi}{2}, \pi]. 
\end{cases}
$$

We would like to calculate $\xi \odot \xi$, that is, $\sin^{\odot 2}$. We have $\xi_l(x) = \sin x$ for $x \in [0, \frac{\pi}{2}]$ and $\xi_r(x) = \sin x$ on $[\frac{\pi}{2}, \pi]$. Thus $\xi_l^{-1}(\alpha) = \arcsin \alpha$ for $\alpha \in [0, 1]$, and $\xi_r^{-1}(\alpha) = \pi - \arcsin(\alpha)$ for $\alpha \in [0, 1]$. Both these functions are shown in Fig. 5.

In order to find $(\xi \odot \xi)_l$ we need to invert the function $x = \arcsin^2 \alpha$. Similarly, for $(\xi \odot \xi)_r$ we need to invert $x = (\pi - \arcsin(\alpha))^2$. In this way we obtain

$$
\xi \odot \xi = \begin{cases} 
(\xi \odot \xi)_l(x) = \sin(\sqrt{x}) & \text{for } x \in [0, \frac{\pi^2}{4}], \\
1 & \text{for } x = \frac{\pi^2}{4}, \\
(\xi \odot \xi)_r(x) = \sin(\pi - \sqrt{x}) = \sin(\sqrt{x}) & \text{for } x \in [\frac{\pi^2}{4}, \pi^2]. 
\end{cases}
$$

Remark 1.5. We see that in the realm (domain) of fuzzy numbers we have an identity $\sin^{\odot 2}(x) = \sin(\sqrt{x})$ with supports $[0, \pi]$ and $[0, \pi^2]$ respectively. This is not a coincidence but an example indicating of how functions of fuzzy quantities work.
Fig. 7. \((\xi_l^{-1})^2, (\xi_r^{-1})^2\) and \(\sin^2\). For comparison (dotted blue line) \(\sin^2\).

Example 3. Let \(\xi\) be as in the previous example, that is, \(\xi(x) = \chi_{[0,\pi]}(x) \sin x\) and \(\eta(x) = \chi_{[\pi/2, 3\pi/2]} \cos x\). For \(\eta\) we have \(\eta_l^{-1}(\alpha) = \arcsin(\alpha) + \frac{3\pi}{2}\) and \(\eta_r^{-1}(\alpha) = \arcsin(-\alpha) + 5\pi/2\) for \(\alpha \in [0, 1]\). Thus, we invert the functions \(x = \arcsin(\alpha)(\arcsin \alpha + 3\pi/2)\) and \(x = (\arcsin(-\alpha) + \pi)(\arcsin(-\alpha) + 5\pi/2)\) to arrive at (see Fig. 8)

\[
\xi \odot \eta = \begin{cases} 
(\xi \odot \eta)_l = \sin \left(-\frac{3\pi}{4} + \frac{1}{2} \sqrt{\frac{9}{4}\pi^2 + 4x}\right) & \text{for } x \in [0, \pi^2], \\
1 & \text{for } x = \pi^2, \\
(\xi \odot \eta)_r = \sin \left(\frac{7\pi}{4} - \frac{1}{2} \sqrt{\frac{9}{4}\pi^2 + 4x}\right) & \text{for } x \in [\pi^2, 5\pi^2/2]. 
\end{cases}
\]

or in one piece

\[
\xi \odot \eta = \chi_{[0,5\pi/2]}(x) \cdot -\sin \left(\frac{1}{4} \pi + \frac{1}{4} \sqrt{9\pi^2 + 16x}\right)
\]

Fig. 8. \(\xi(x) = \chi_{[0,\pi]}(x) \sin x, \eta(x) = \chi_{[3\pi/2,5\pi/2]} \cos x\), and the product \(\xi \odot \eta\).
2. $F_c(\mathbb{R}^+)$

In section 1 we defined a class of fuzzy numbers $F^0_c(\mathbb{R}^+)$, where the subscript “c” stands for “compact” to indicate that the characterizing functions defining the numbers have compact support, and the superscript “0” indicates that these functions are also continuous.

Notably the real numbers $\mathbb{R}$ are beyond the scope of Definition 1. In this section a second, wider definition is brought in to include the standard real numbers and also the interval numbers (that is, the characteristic functions of intervals). Thus, we will allow a characterizing function $\xi(\cdot)$ to be equal to 1 on a closed interval $[m, m]$ (in the class $F^0_c(\mathbb{R}^+)$ we had $m = m$). Our findings and statements will henceforth formulated in terms “fuzzy intervals” and we will refer to fuzzy numbers as to the subcase that $m = m = x_0$.

Moreover, we shall not longer require that the fuzzy components $\xi_l$ and $\xi_r$ be strictly monotone and continuous, and instead require only simple monotonicity and semi-continuity.

This new class shall be denoted by $F_c(\mathbb{R}^+)$.  

**Definition 2.** A fuzzy interval $\xi$ from the class $F_c(\mathbb{R}^+)$ is characterized by a function $\xi(\cdot) : \mathbb{R}^+ \to [0, 1]$ with the following properties:

1. compact support: the closure of the points $x \in \mathbb{R}^+$, where $\xi(x) > 0$ is a closed interval $[l, r] \subset \mathbb{R}^+$,

2. The set of points $x \in \mathbb{R}^+$ where the characterizing function $\xi(\cdot)$ attains value 1 is a closed interval $[m, m] \subset [l, r]$, in other words $\{x \in \mathbb{R} : \xi(x) = 1\} = [m, m] \subseteq [l, r]$.

3. the function $\xi_l(x) := \xi(x)$ for $x \in [l, m]$ is non-decreasing and right-continuous,

4. the function $\xi_r(x) := \xi(x)$ for $x \in [m, r]$ is non-increasing and left-continuous.

**Remark 2.1.** Note that conditions (1) – (4) in Definition 2 imply that $\xi(\cdot)$ is upper semi-continuous, that is

$$\limsup_{x \to x_0} \xi(x) \leq \xi(x_0), \quad \forall x_0 \in \mathbb{R}^+.$$  

As in section 1 and Definition 1’ the characterizing function of a fuzzy interval $\xi \in F_c(\mathbb{R}^+)$ can be rewritten as

**Definition 2’.**

$$\xi(x) = \begin{cases} 
\xi_l(x) & \text{for } x \in [l, m], \\
1 & \text{for } x \in [m, m], \\
\xi_r(x) & \text{for } x \in [m, r].
\end{cases}$$

(Again $\xi_l(m) = \xi_r(m) = 1$ is an overkill, but we need this for inversion)
As in the preceding section a fuzzy interval (number) is uniquely determined by its fuzzy components $\xi_l$ and $\xi_r$. The interval $[\underline{m}, \overline{m}]$ need not be stated explicitly, but usually will be for sake of clarity.

**Definition 2”** Again a representation by and ordered pair $(\xi_l, \xi_r)$ where $\xi_l$ and $\xi_r$ meet the conditions of Definition 2 above seems intuitive and highly desirable, especially when speaking of fuzzy intervals (not fuzzy numbers).

As we mentioned before, the class $F_c(\mathbb{R}^+)$ includes the standard positive numbers $\mathbb{R}^+$ and the interval numbers. Indeed, every $\lambda \in \mathbb{R}^+$ can be identified with the delta function $\delta_\lambda : \mathbb{R}^+ \to \{0, 1\}$ such that $\delta_\lambda(x) = 1$ iff $x = \lambda$ and 0 elsewhere. *(Convention: We use $\chi_{\{\lambda\}}$ and $\delta_\lambda(x)$ interchangeably).*

Clearly, $\delta_\lambda \in F_c(\mathbb{R}^+)$, which can be seen by taking $l = \underline{m} = \overline{m} = r = \lambda$ and $\xi_l = \xi_r = \chi_{\{\lambda\}}$ in Definition 2’. For an interval number interpreted as its own characteristic function $\chi_{[a,b]}$ of the interval $[a, b] \subset \mathbb{R}^+$ the understanding is the same. We have $\chi_{[a,b]} \in F_c(\mathbb{R}^+)$, which can be seen by taking $l = \underline{m} = a$ and $b = \overline{m} = r$ and $\xi_l = \chi_{\{a\}}, \xi_r = \chi_{\{b\}}$ in Definition 2’.

We shall see later that the arithmetic which will be defined below for $F_c(\mathbb{R}^+)$ is consistent with the standard arithmetic of $\mathbb{R}^+$, as well as the set arithmetic for intervals.

**2.1. Generalized Inverse Function.** The (not strictly) monotone functions $\xi_l, \xi_r$ of Definition 2’ are not necessarily invertible. In order to be able to employ, as before in section 1 formulae (1.2) and (1.5) to define the operations “$\oplus$“, “$\odot$” we shall presently introduce and then apply the notion (appearing in statistics in the form of generalized inverses of CDFs, or quantile functions) of a *generalized inverse function*:

The price to pay for relaxing the conditions of continuity and strict monotonicity and instead assume one-sided continuity and non-strict monotonicity is that we encounter two types of problems in the process of finding the inverse function:

Take a function $h : I \mapsto [0, 1]$ defined on a closed interval $I$, that is one-sided continuous and non-strictly monotone. The following two types of behavior are an obstacle for finding a useful inverse function:

1. There is a point $x_0$, where the function $h$ has a jump discontinuity. In this case the image $h(I) \neq [0, 1]$. So in principle the function is invertible, but the domain of the inverse function will not be the full interval $[0, 1]$. Therefore, (1.2), and (1.5) cannot be applied because addition or multiplying such an inverse by another function’s inverse whose domain is not the full interval $[0, 1]$ either, is simply not well defined.

2. The function $h$ is constant on some interval $[x_0, x_0]$. 
In this case the function is not injective, and therefore, not invertible at all in the classical sense.

In order to overcome these problems in extending formulae (1.2) and (1.5) we shall, for our present purposes, single out two types of generalized inverse functions. For a non-decreasing right-continuous function $f : I \to J$ defined on a closed interval $I$ and taking values in a closed interval $J$ (resp. non-increasing left-continuous function $g : I \to J$) we define two types of generalized inverse functions by

\begin{align*}
  f^{-1}_{\inf}(\alpha) &= \inf \{ x : f(x) \geq \alpha \} \\
  g^{-1}_{\sup}(\alpha) &= \sup \{ x : g(x) \leq \alpha \}
\end{align*}

for $\alpha \in J$.

To see how the generalized inverse procedure works in practice let us consider the case of a non-decreasing right-continuous function $f$ (which may serve as the left fuzzy component $\xi_l$ of a fuzzy interval $\xi$ as in Definition 2).

(1) Let $f$ be a non-decreasing right-continuous function with a jump discontinuity at the point $x_0$. Let $f(x_0) = \alpha_0$. Let us call $\underline{x}_0 = \alpha_0$ and $\overline{x}_0 = \sup \{ f(x) : x < x_0 \}$. Then for all $\alpha \in [\underline{x}_0, \overline{x}_0]$ we have that

$$f^{-1}_{\inf}(\alpha) = \inf \{ x : f(x) \geq \alpha \} = x_0,$$

that is, the generalized inverse function is constant on the closed interval $[\underline{x}_0, \overline{x}_0]$.

(2) Let $f$ be constant on the interval $[x_0, \overline{x}_0]$ with $f(x) = \alpha_0$ for $x$ in the interval. In this case clearly,

$$f^{-1}_{\inf}(\alpha_0) = \inf \{ x : f(x) \geq \alpha_0 \} = \overline{x}_0,$$

while

$$\inf \{ f^{-1}_{\inf}(\alpha) : \alpha > \alpha_0 \} = \overline{x}_0.$$

Thus, the generalized inverse has a jump discontinuity at $\alpha_0$.

For a (not strictly) decreasing function $g$, which in what is coming may serve as the right fuzzy component of a fuzzy interval $\xi$ the reasoning is analogous, except that “\( \inf \)” on each occurrence must be replaced by “\( \sup \)”, and inequality signs reversed.

In the light of the above we see that the procedure of finding the generalized inverse works in a very civilized way: Jump discontinuities of the input convert to intervals where the output inverted function is constant. And the intervals where the input is constant convert to jump discontinuities of the output. These two observations are illustrated below for fragments of two possible left components:
Again for a non-increasing left-continuous function $g$ serving as the right fuzzy part $\xi_r$ of a fuzzy interval $\xi$ given as in Definition 2 the procedure of finding the general inverse works analogously substituting “sup” for “inf” and reversing inequality signs on each occurrence.

It is absolutely fundamental that with the assumptions we have made on $f$ and $g$ the following identities hold:

**Corollary 2.2.**

\[(f^{-1})_{\text{inf}}^{-1} = f\]
\[(g^{-1})_{\text{sup}}^{-1} = g\]

**Remark 2.3.** We also explicitly note, that for a left fuzzy component $\xi_l(\cdot)$ which is by definition increasing and right-continuous its generalized inverse $\xi_l^{-1} : [0, 1] \to \mathbb{R}^+$ is still increasing (only fuzzy convexity transforms into concavity), but left-continuous.

Observe also that $\xi_l^{-1}(0) = l, \xi_l^{-1}(1) = m$.

Likewise $\xi_r^{-1} : [0, 1] \to \mathbb{R}^+$ is a decreasing, right-continuous function and $\xi_r^{-1}(1) = m$ and $\xi_r^{-1}(r) = 0$

2.1.1. *Graphs of inverted fuzzy components.* In what follows some example graphs are displayed:
The infimum and supremum inverse of the delta function $\delta_\lambda$ is the constant function $\alpha = \lambda$ (with domain $[0, 1]$):

\[ \text{Figure 9. inverse (infimum and supremum) of a real number} \]

Below the infimum and supremum generalized inverses of a step function are confronted for comparison:

(a) infimum inverse of a step function  
(b) supremum inverse of the same step function
Intended as an exercise to help get accustomed to the procedure of generalized inversion here is now a freely patched together "fantasy" function which includes both jumps and constant parts and its generalized inverse:

\[
(fanta)_{l}(x) = \begin{cases}
\frac{1}{4}x & x \in \left[\frac{1}{4}, \frac{2}{3}\right) \\
\sin(3x - \frac{1}{2}) & x \in \left[\frac{2}{3}, \frac{6}{7}\right) \\
\frac{5}{8} & x \in \left[\frac{6}{7}, \frac{9}{8}\right) \\
\frac{1}{6}x & x \in \left[\frac{9}{8}, \frac{1}{2}\right]
\end{cases}
\]

\[
(fanta)_{l}^{-1}(\alpha) = \begin{cases}
\frac{1}{4} & \alpha \in \left[0, \frac{1}{16}\right] \\
4\alpha & \alpha \in \left(\frac{1}{16}, \frac{1}{6}\right] \\
\frac{2}{3} & \alpha \in \left(\frac{1}{6}, \sin(1) - \frac{1}{2}\right] \\
\frac{1}{3} \arcsin(\alpha + \frac{1}{2}) + \frac{1}{3} & \alpha \in \left(\sin(1) - \frac{1}{2}, \sin(\frac{11}{7}) - \frac{1}{2}\right] \\
\frac{1}{6} & \alpha \in \left(\sin(\frac{11}{7}) - \frac{1}{2}, \frac{5}{16}\right] \\
\frac{9}{8} & \alpha \in \left(\frac{5}{8}, \frac{15}{16}\right] \\
\frac{4}{5} & \alpha \in \left(\frac{15}{16}, 1\right]
\end{cases}
\]

**Figure 10.** the infimum generalized inverse (blue) of a "fantasy" function (green)
2.2. **Addition and multiplication on** \( \mathcal{F}_c(\mathbb{R}^+) \). Addition and multiplication in \( \mathcal{F}_c(\mathbb{R}^+) \) are defined in the same way as in the class \( \mathcal{F}_0(\mathbb{R}^+) \), namely we extend formulae (1.2) and (1.5) by application of the generalized inverse functions as defined in (2.2) and (2.3):

We set

\[
(\xi \oplus \eta)_l = ((\xi_l)^{-1}_{\inf} + (\eta_l)^{-1}_{\inf})^{-1}_{\sup} \\
(\xi \oplus \eta)_r = ((\xi_r)^{-1}_{\sup} + (\eta_r)^{-1}_{\sup})^{-1}_{\inf}
\]

and

\[
(\xi \odot \eta)_l = ((\xi_l)^{-1}_{\inf} \cdot (\eta_l)^{-1}_{\inf})^{-1}_{\sup} \\
(\xi \odot \eta)_r = ((\xi_r)^{-1}_{\sup} \cdot (\eta_r)^{-1}_{\sup})^{-1}_{\inf}
\]

At a later stage, when there is no ambiguity one might want to opt to drop the “inf” and “sup” subscripts and simply write (1.2) and (1.5) as in section 1.

**Remark 2.4.** Clearly, the sum and the product of left-continuous (resp. right-continuous) functions are left-continuous (resp. right-continuous), thus the above operations are well defined.

**Example 4 (Multiplication of interval numbers).** The basic example of a fuzzy interval is the abstraction of a real interval \( A = [a, \overline{a}] \) by its characterising function \( \xi_A = \chi_{[a, \overline{a}]} \).

Let us multiply two interval numbers \( \xi_A = \chi_{[2, 3]} \) and \( \xi_B = \chi_{[5, 6]} \). The result will be, as expected, \( \xi_A \odot \xi_B = \chi_{[10, 18]} \), as shown in Fig. 11.

Going by the definiton, one piece at a time, step by step:

\[
(\xi_A)_l = \chi_{\{2\}} \text{ for } x \in \mathbb{R} \\
(\xi_B)_l = \chi_{\{5\}} \text{ for } x \in \mathbb{R} \\
(\xi_A)_l^{-1}(\alpha) = 2 \text{ for } \alpha \in [0, 1] \\
(\xi_B)_l^{-1}(\alpha) = 5 \text{ for } \alpha \in [0, 1] \\
(\xi_A)_l^{-1}(\alpha) \cdot (\xi_B)_l^{-1}(\alpha) = 2 \cdot 5 = 10 \text{ for } \alpha \in [0, 1] \\
(\xi_A \odot \xi_B)_l = ((\xi_A)_l^{-1} \cdot (\xi_B)_l^{-1})^{-1}_{\sup} = (\alpha = 10)^{-1}_{\sup} = \chi_{\{10\}} \text{ for } x \in \mathbb{R}
\]

The computation of the righthand side \( (\xi_A \odot \xi_B)_r \) is analogous.
In general, for $0 \leq a \leq \overline{a}$ and $0 \leq b \leq \overline{b}$ we may always use
\begin{equation}
\chi_{[a,\overline{a}]} \odot \chi_{[b,\overline{b}]} = \chi_{[ab,\overline{ab}]};
\chi_{[a,\overline{a}]} \oplus \chi_{[b,\overline{b}]} = \chi_{[a+b,\overline{a+b}]}.
\end{equation}
which is of course perfectly consistent with interval arithmetic for the non-negative reals.

**Notation:** We write $\delta_{\lambda^{-1}} = \delta_{\lambda_{\sup}^{-1}} = \lambda$ for $\alpha \in [0, 1]$ or simply $\alpha = \lambda$

**Notation:** For a given $\lambda \in \mathbb{R}^+$ the corresponding function $\delta_{\lambda}(x)$ shall be, in short, denoted by $\lambda$, that is, $\lambda \odot \xi$ means $\delta_{\lambda} \odot \xi$.

**Example 5** (Multiplication of a fuzzy interval by a real number). We calculate
\[(\lambda \odot \xi)_{\ell}(x) = (\delta_{\lambda_{\inf}^{-1}} \cdot \delta_{\xi_{\inf}^{-1}})_{\sup}^{-1}(x) = (\lambda \cdot \xi_{\inf}^{-1})_{\sup}^{-1}(x)
= \sup\{s : \lambda \cdot \xi_{\inf}(s) \geq x\}
= \sup\{s : \xi_{\inf}(s) \geq x/\lambda\}
= (\xi_{\inf})_{\sup}^{-1}(x/\lambda)
= \xi_{\ell}(x/\lambda).
\]
Repeating the calculation for $\xi_{r}$, we have established the following property:
\begin{equation}
(\lambda \odot \xi)(x) = \xi(x/\lambda)
\end{equation}
Below $2 \odot \chi_{[3,4]}(x) \sqrt{4-x} = \chi_{[6,8]}(x) \sqrt{4-x/2}$ is shown as an example:

![Figure 12. Multiplication of a fuzzy interval by a real number](image)
Property (2.10) provides for the following formula which is crucial for statistical analysis of fuzzy data:

\[
\frac{1}{n} \odot (\xi \oplus \xi \oplus \cdots \oplus \xi) = \frac{1}{n} \odot \bigoplus_{i=1}^{n} \xi = \xi,
\]

Indeed,

\[
\frac{1}{n} \odot \bigoplus_{i=1}^{n} \xi_r(x) = \frac{1}{n} \odot \left( n \odot \xi_{r, \text{sup}}^{-1}(x) \right)_{\text{inf}}^{-1} = \frac{1}{n} \odot \xi_r(x/n) = \xi_r(x),
\]

and a similar check can be done for \( \xi_l(x) \).

**Example 6** (Adding a real number to a fuzzy interval). Multiplying a fuzzy interval by a real number yields an inverse dilation of the fuzzy interval. Similarly, it can be easily shown that adding a real number to a fuzzy interval corresponds to a negative translation:

\[
(\lambda \oplus \xi)(x) = \xi(x - \lambda)
\]

**Example 7.** In the last example of this subsection we shall provide calculations and graphs of addition and multiplication of a fuzzy interval \( \xi \) and a fuzzy number \( \eta \).

For \( \eta \) we take the already considered function \( \chi_{[0, \pi]} \cdot \sin \):

\[
\eta = \begin{cases} 
\eta_l = \sin(x) & x \in [0, \frac{\pi}{2}] \\
1 & x = \frac{\pi}{2} \\
\eta_r = \sin(x) & x \in [\frac{\pi}{2}, \pi]
\end{cases}
\]

This example’s \( \xi \) has a linear part \( \xi_l \) and a step function as \( \xi_r \):

\[
\xi = \begin{cases} 
\xi_l = 8x - 1 & x \in [\frac{1}{4}, \frac{1}{4}] \\
1 & x \in [\frac{1}{4}, 1] \\
\xi_r = \begin{cases} 
1 & x = 1 \\
0.3 & x \in (1, 2] \\
0.1 & x \in (2, 4]
\end{cases}
\end{cases}
\]

Here the function \( \xi_r \) is not invertible in the classical sense.

Both characterizing functions are pictured below in one graph. The middle parts are marked in dotted grey and the left and right fuzzy components are respectively green and blue.
We have
\[ \xi^{-1}_l(\alpha) = \frac{1 - \alpha}{8} \quad \text{for } \alpha \in [0, 1] \]
and
\[ \xi^{-1}_{r,\text{sup}}(\alpha) = \begin{cases} 
4 & \text{for } \alpha \in [0, 0.1) \\
2 & \text{for } \alpha \in [0.1, 0.3) \\
1 & \text{for } \alpha \in [0.3, 1] 
\end{cases} \]
Let us write out \( \eta^{-1}(\alpha) \) as in Example 2
\[ \eta^{-1}_l(\alpha) = \arcsin(\alpha) \quad \text{for } \alpha \in [0, 1], \]
\[ \eta^{-1}_r(\alpha) = \pi - \arcsin(\alpha) \quad \text{for } \alpha \in [0, 1]. \]
We receive
\[ \xi^{-1}_l + \eta^{-1}_l = \frac{1 - \alpha}{8} + \arcsin(\alpha) \quad \text{for } \alpha \in [0, 1] \]
and
\[ \xi^{-1}_r + \eta^{-1}_r = \begin{cases} 
4 + \pi - \arcsin(\alpha) & \text{for } x \in [0, 0.1) \\
2 + \pi - \arcsin(\alpha) & \text{for } x \in [0.1, 0.3) \\
1 + \pi - \arcsin(\alpha) & \text{for } x \in [0.3, 1] 
\end{cases} \]
\[ \xi^{-1}_l \cdot \eta^{-1}_l = \frac{1 - x}{8} \cdot \arcsin(\alpha) \quad \text{for } \alpha \in [0, 1] \]
and
\[ \xi^{-1}_r \cdot \eta^{-1}_r = \begin{cases} 
4 \cdot (\pi - \arcsin(\alpha)) & \text{for } \alpha \in [0, 0.1) \\
2 \cdot (\pi - \arcsin(\alpha)) & \text{for } \alpha \in [0.1, 0.3) \\
1 \cdot (\pi - \arcsin(\alpha)) & \text{for } \alpha \in [0.3, 1] 
\end{cases} \]
After inverting back once more we obtain

\[
\xi \odot \eta = \begin{cases} 
\arcsin(\alpha) \cdot \left(\frac{\alpha - 1}{8}\right) = x, & x \in [0, \frac{\pi}{8}], \; \alpha \in [0, 1] \\
\left[\frac{\pi}{8}, \frac{\pi}{2}\right] \\
\phi(x) & 
\end{cases}
\]

where

\[
\phi(x) = \begin{cases} 
sin(1^{-1}x) & x \in [(\pi - \arcsin(1)) \cdot 1, (\pi - \arcsin(0.3)) \cdot 1] \\
0.3 & x \in [(\pi - \arcsin(0.3)) \cdot 1, (\pi - \arcsin(0.3)) \cdot 2] \\
sin(2^{-1}x) & x \in [(\pi - \arcsin(0.1)) \cdot 2, (\pi - \arcsin(0.1)) \cdot 2] \\
0.1 & x \in [(\pi - \arcsin(0.1)) \cdot 2, (\pi - \arcsin(0.1)) \cdot 4] \\
sin(4^{-1}x) & x \in [(\pi - \arcsin(0.1)) \cdot 4, (\pi - \arcsin(0)) \cdot 4]
\end{cases}
\]

Similarly
\[
\begin{align*}
\xi \oplus \eta &= \begin{cases} 
(\xi \oplus \eta)_l &= \{\arcsin(\alpha) + (\frac{\alpha + 1}{8}) = x\}, \quad x \in [0, \frac{\pi}{8}, \frac{\pi}{8} + \frac{1}{8}]; \\
(\xi \oplus \eta)_m &= \left[\frac{\pi}{8} + \frac{1}{8}, \frac{\pi}{8} + 1\right] \\
(\xi \oplus \eta)_r &= \phi(x)
\end{cases}
\end{align*}
\]

with
\[
\phi(x) = \begin{cases} 
\sin(x - 1) & x \in [(\arcsin(1) + 1, (\pi - \arcsin(0.3)) + 1] \\
0.3 & x \in ((\pi - \arcsin(0.3)) + 1, (\pi - \arcsin(0.3)) + 2] \\
\sin(x - 2) & x \in ((\pi - \arcsin(0.3)) + 2, (\pi - \arcsin(0.1)) + 2] \\
0.1 & x \in ((\pi - \arcsin(0.3)) + 2, (\pi - \arcsin(0.1)) + 4] \\
\sin(x - 4) & x \in ((\pi - \arcsin(0.1)) + 2, (\pi - \arcsin(0)) + 4]
\end{cases}
\]

Concluding the calculations of the particular components of the outcome let us notice that:

- The components \((\xi \odot \eta)_l\) and \((\xi \oplus \eta)_l\) are in a form often encountered in practice, where the function is given in implicit terms because providing a straightforward closed form representation is impossible.
- Functions \((\xi \odot \eta)_r\) and \((\xi \oplus \eta)_r\) are given in explicit terms where one could have applied (1.2) directly or simpler (2.10) and (2.12) as always taking extreme care to ascertain the proper endpoints of intervals.
- The (meaningless) middle part can be always easily written out using (2.9).

Here are the graphs of \(\xi\) and \(\eta\) again:

Before conducting operations

\[\text{Graph of } \xi, \eta\]

after addition

\[\text{Graph of } \xi \odot \eta, \xi \oplus \eta\]
and after multiplication

\[ \xi \odot \eta \]

2.3. \textbf{Algebraic properties of the class} \( \mathcal{F}_c(\mathbb{R}^+) \). From the general properties of functions and their inverses we see that the operations “\( \oplus \)”, “\( \odot \)” defined on \( \mathcal{F}_c(\mathbb{R}^+) \) are commutative and associative. We also have the zero element \( \delta_0 \) and the multiplicative identity \( \delta_1 \), so \( \mathcal{F}_c(\mathbb{R}^+) \cup \{ \delta_0 \} \) is a commutative semiring.

In general, the elements of \( \mathcal{F}_c(\mathbb{R}^+) \) do not have inverse elements, which can be easily seen in the case of interval numbers:

Indeed, for addition, using (2.9) we would have to have

\[ [a, b] \oplus [x, y] = [0, 0], \]

so \( x = -a, y = -b \). But \( [-a, -b] \) with \( a < b \) does not have the right orientation meaning that the called for interval \( [x, y] \) does not exist (even when allowing for negatives, as we will in section 3).

For multiplication, using (2.9) we would have to have

\[ [a, b] \odot [x, y] = [1, 1], \]

but \( x = \frac{1}{a} \geq y = \frac{1}{b} \), meaning again there is no inverse in the strict, non-fuzzy sense. Note however that \( [\frac{1}{b}, \frac{1}{a}] \) will pass for a fuzzy 1.
Throughout sections 1 and 2 we have been assuming the support of the characterizing functions to be a finite closed interval lying within $\mathbb{R}^+$. This assumption greatly facilitates understanding the essence of how fuzzy arithmetic works. In Definition 3 below this assumption is dropped. Negative and mixed (i.e. containing zero) supports are also considered. Definition 3 is what one would normally encounter in a textbook and this particular version is taken from [Viertl96]. Subsequently the definition is translated into the language used in the preceding sections.

Here is the compact form:

**Definition 3.** A fuzzy interval $\xi$ is determined by its characterizing function $\xi(\cdot)$ which is a real function of one real variable $x$ obeying the following

1. $\xi(\cdot) : \mathbb{R} \mapsto [0, 1]$.
2. $\exists x_0 \in \mathbb{R}$ such that $\xi(x_0) = 1$.
3. $\xi(\lambda x_1 + (1 - \lambda)x_2) \geq \min (\xi(x_1), \xi(x_2))$ (fuzzy-convexity).
4. $\xi(\cdot)$ is upper semi-continuous. ($\lim_{x_n \to x_0} \xi(x_n) \leq \xi(x_0)$)
5. $\xi(\cdot)$ has compact support.

There is an equivalent (for a proof of this see [Viertl96]) and very convenient definition formulated in the widespread language of so-called $\alpha$-cuts which a physicist would rather refer to as level-sets or isolines:

**Definition 3’.** A fuzzy interval $\xi$ is determined by its characterizing function $\xi(\cdot)$ which is a real function of one real variable $x$ obeying the following:

1. $\xi : \mathbb{R} \mapsto [0, 1]$.
2. $\forall \alpha \in (0, 1]$ the so-called $\alpha$-cut $C_\alpha(\xi) = \{x \in \mathbb{R} : \xi(x) \geq \alpha\}$ is a compact interval.
3. The support of $\xi(\cdot)$, $\text{supp}[\xi(\cdot)] := \{x \in \mathbb{R} : \xi(x) > 0\}$ is bounded.

**Remark 3.1.** Note that $C_\alpha(\xi) = \{x \in \mathbb{R} : \xi(x) \geq \alpha\} = [\xi_{\text{inf}}^{-1}(\alpha), \xi_{\text{sup}}^{-1}(\alpha)]$, and this is the bridge to how we have been operating.

**Definition 3”** As in sections 1 and 2 we may equivalently define a fuzzy interval (number) $\xi$ to be defined by an ordered pair $(\xi_l(\cdot), \xi_r(\cdot))$ of two real functions $\xi_l(\cdot) : \mathbb{R} \to [0, 1]$ and $\xi_r(\cdot) : \mathbb{R} \to [0, 1]$ of one real variable $x$ such that:

1. $\xi_l(\cdot)$ is increasing, right-continuous and has support on some closed interval $[l, m]$
2. $\xi_r(\cdot)$ is decreasing, left-continuous and has support on some closed interval $[m, r]$ such that $m \leq m$
3. $\xi_l(m) = \xi_r(m) = 1$
(2') To differentiate fuzzy numbers from fuzzy intervals we additionally request: \( \xi^{-1}_l(1) = \{\underline{m}\} \), and \( \xi^{-1}_r = \{\overline{m}\} \).

Before we go any further let us introduce some shorthand:

**Notation:** We set:

\[
\xi^{-1}_{\text{inf}}(\alpha) := \xi_d(\alpha) \quad \text{as in downwards fuzzy}
\]

and

\[
\xi^{-1}_{\text{sup}}(\alpha) := \xi_u(\alpha) \quad \text{as in upwards fuzzy}
\]

Just like in normal crisp (not fuzzy) interval arithmetic we define for all four arithmetic operations:

\[
(\xi \circ \eta)_d := \min(\xi_d \circ \eta_d, \xi_d \circ \eta_u, \xi_u \circ \eta_d, \xi_u \circ \eta_u)
\]

and

\[
(\xi \circ \eta)_u := \max(\xi_d \circ \eta_d, \xi_d \circ \eta_u, \xi_u \circ \eta_d, \xi_u \circ \eta_u),
\]

with “\( \circ \)” standing for any of the four operations: “\( \oplus \)”, “\( \ominus \)”, “\( \otimes \)”, “\( \oslash \)”. Then

\[
(\xi \circ \eta)_l := ((\xi \circ \eta)_d)^{-1}_{\text{sup}}
\]

and

\[
(\xi \circ \eta)_r := ((\xi \circ \eta)_u)^{-1}_{\text{inf}}
\]

And similar to (2.1) before

\[
\xi \circ \eta = \begin{cases} 
((\xi \circ \eta)_d)^{-1}_{\text{sup}} \\
1 \\
((\xi \circ \eta)_u)^{-1}_{\text{inf}} 
\end{cases}
\]

Or more compact, in interval notation:

\[
\xi \circ \eta = ((\xi \circ \eta)_l, (\xi \circ \eta)_r) = (((\xi \circ \eta)_d)^{-1}_{\text{sup}}, ((\xi \circ \eta)_u)^{-1}_{\text{inf}})
\]

We now separately turn our attention to each of the four arithmetic operations:
3.1. The four operations.

3.1.1. Addition. Because real addition is monotone in both variables we find that \((\xi \oplus \eta)_d = \xi_d + \eta_d\) and \((\xi \oplus \eta)_u = \xi_u + \eta_u\) and so (4.1) reduces to (2.6):

\[ (3.9) \quad (\xi \oplus \eta)_l = ((\xi \oplus \eta)_d, (\xi \oplus \eta)_u) = ((\xi_d + \eta_d)^{-1}_{\text{sup}}, (\xi_u + \eta_u)^{-1}_{\text{inf}}). \]

3.1.2. Subtraction. Define

\[ \xi \ominus \eta := \xi \oplus -\eta \]

where \(-\eta = -(\eta_l, \eta_r) = (-\eta_r, -\eta_l)\)

Then fuzzy subtraction is defined as addition above.

Remark 3.2. Note that \(\xi \ominus \xi = (\xi_l - \xi_r, \xi_r - \xi_l)\) is a symmetrical fuzzy zero but certainly not \(\delta_0\). (Its support is \(\text{supp}[\xi \ominus \xi] = [l - r, r - l]\)).

3.1.3. Multiplication.

\[ (3.10) \quad (\xi \odot \eta)_l = ((\min(\xi_d \cdot \eta_d, \xi_d \cdot \eta_d, \xi_u \cdot \eta_d, \xi_u \cdot \eta_u))^{-1}_{\text{sup}}, \max(\xi_d \cdot \eta_d, \xi_d \cdot \eta_u, \xi_u \cdot \eta_d, \xi_u \cdot \eta_u))^{-1}_{\text{inf}}. \]

Unfortunately, because unlike in the case of addition the real multiplication operator is not monotone the formula (4.1) does not reduce automatically to (2.7) and calculations can become very laborious and error prone when done by hand. At the very end of section 4 some examples of multiplied triangle numbers of mixed support are graphically illustrated.

Remark 3.3. If \(\xi, \eta \in \mathcal{F}_c(\mathbb{R}^+)\) (2.7) applies as demonstrated in section 2.

3.1.4. Division. Division is defined as the fuzzy inverse of multiplication by setting

\[ (3.11) \quad [(\xi \oslash \eta)_d, (\xi \oslash \eta)_u] = [\min(\frac{\xi_d}{\eta_d}, \frac{\xi_d}{\eta_u}, \frac{\xi_u}{\eta_d}, \frac{\xi_u}{\eta_u}), \max(\frac{\xi_d}{\eta_d}, \frac{\xi_d}{\eta_u}, \frac{\xi_u}{\eta_d}, \frac{\xi_u}{\eta_u})]. \]

For the above to be well defined we must assume 0 \(\notin\) \(\text{supp} \eta_l \cup \text{supp} \eta_r\)

Remark 3.4. For \(\eta_d\) left-continuous \(1/\eta_d\) is right-continuous and likewise for \(\eta_u\) right-continuous we have \(1/\eta_u\) left continuous, so the quotients in (3.11) are all semi-continuous “in the right way”.

Remark 3.5. For \(\xi, \eta \in \mathcal{F}_c(\mathbb{R}^+)\) (3.11) becomes just

\[ [(\xi \oslash \eta)_d, (\xi \oslash \eta)_u] = \left[ \frac{\xi_d}{\eta_u}, \frac{\xi_u}{\eta_d} \right]. \]

Let us now return to our very first worked through example 1 from section 1, \(\xi\) as in (1.10), \(\eta\) as in (1.11):
Example 8. (fuzzy division) Going by (3.11) and Remark 3.5:

\[
\frac{\xi_d}{\eta_u} = \frac{\xi_l^{-1}}{\eta_r^{-1}} = \frac{\alpha + 1}{10 - 3\alpha} \quad \text{for } \alpha \in [0, 1]
\]

and hence

\[
(\xi \odot \eta)_l = \frac{10x - 1}{1 + 3x} \quad \text{for } x \in [1, 2] \odot [7, 10] = \left[\frac{1}{10}, \frac{2}{7}\right]
\]

Likewise

\[
\frac{\xi_u}{\eta_d} = \frac{\xi_r^{-1}}{\eta_l^{-1}} = \frac{3 - \alpha}{2\alpha + 5} \quad \text{for } \alpha \in [0, 1]
\]

and hence

\[
(\xi \odot \eta)_r = \frac{3 - 5x}{2x + 1} \quad \text{for } x \in [2, 3] \odot [5, 7] = \left[\frac{2}{7}, \frac{3}{5}\right]
\]

Figure 15. Fuzzy division of two triangle numbers

4. $\mathcal{F}_c(\mathbb{R})^{-1}$ - REVERSING THE POINT OF VIEW

Especially in the case when fuzzy intervals are actively designed to model an economic situation to begin with as in management science, as opposed to being an expression of uncertainty in measurement of the outcome of an experiment in a natural science, it is probably more natural and certainly more straightforward for computation to start with the right-hand sides of equations (3.1) and (3.2) as primary from the very outset. Therefore we propose to adopt the following definition, and associated terminology:

**Definition 4.** We understand a fuzzy (generalized) interval

\[A^* \in [0, 1]^\mathbb{R} \times [0, 1]^\mathbb{R}\]

with **fuzzy endpoints** $a_d(\cdot), a_u(\cdot)$ to be an ordered pair of functions in shorthand denoted by $A^* := [a_d(\cdot), a_u(\cdot)]$ such that:

1. $a_d(\cdot) : [0, 1] \mapsto \mathbb{R}$ is increasing and left-continuous
Furthermore we demand in consistence with Definition 2(2) and Remark 2.3:

\[ (1') \lim_{\alpha \to 0^+} a_d(\alpha) = a_d(0) \]
\[ (2') \lim_{\alpha \to 1^-} a_u(\alpha) = a_u(1) \]

Terminology:
- We suggest to call the (inverted) characterizing functions \( a_d(\cdot), a_u(\cdot) \) the fuzzy endpoints of the fuzzy interval \([a_d, a_u]\).
- We propose to denote \( a_d(1) = a_d \) and \( a_u(1) = a_u \) and refer to the real interval \( A = [\underline{a}, \overline{a}] \) as the (real) core (that is the real interval we are generalizing) of \( A^* \).
- We now define the support of \( A^* \) to be \( \text{supp} A^* = [d, u] \) with \( d = \alpha_d(0) \) and \( u = \alpha_u(0) \).

4.1. Arithmetic operations on \( \mathcal{F}_c(\mathbb{R})^{-1} \). For two fuzzy intervals \( A^* = [a_d, a_u] \) and \( B^* = [b_d, b_u] \) all four arithmetic operations on \( \mathcal{F}_c(\mathbb{R})^{-1} \) are then defined simply by:

\[ A^* \circ B^* = [\min(a_d \circ b_d, a_d \circ b_u, a_u \circ b_d, a_u \circ b_u), \max(a_d \circ b_d, a_d \circ b_u, a_u \circ b_d, a_u \circ b_u)] \]

as in crisp interval arithmetic with “\( \circ \)” denoting any of the four operations: “\( \oplus, \ominus, \odot, \oslash \)”.

All comments and simplifications of section 3 apply with the appropriate modifications.

Example 9. Fuzzy numbers are then fuzzy intervals in the sense above for the case the interval \([\underline{a}, \overline{a}]\) is degenerated, that is \( a_d = a_u = x_0 \) for some \( x_0 \in \mathbb{R} \).

Example 10. Real intervals \([\underline{a}, \overline{a}]\) correspond to fuzzy endpoints
- \( a_d(\alpha) = a \) for \( \alpha \in [0, 1] \)
- \( a_u(\alpha) = \overline{a} \) for \( \alpha \in [0, 1] \)

Example 11. Triangle numbers \( tr_{(l,m,r)} \) were characterized in section 1 by

\[ tr(x) = \begin{cases} 
 tr_l(x) = \frac{x-l}{m-l} & \text{for } x \in [l, m] \\
 1 & \text{for } x = m \\
 tr_r(x) = \frac{r-x}{r-m} & \text{for } x \in [m, r]
\end{cases} \]

We now write simply

\[ tr^*_{{(d,x_0,u)}}(\alpha) = [\alpha(x_0 - d) + d, u - \alpha(u - x_0)], \quad \alpha \in [0, 1] \]

Returning to Examples 1 and 8 in which we considered the two triangle numbers \( tr_1 = tr(1, 2, 3) \) and \( tr_2 = tr(5, 7, 10) \) we change the point of view...
and write:
\[
tr_1^*(\alpha) = [\alpha(2 - 1) + 1, 3 - \alpha(3 - 2)] = [\alpha + 1, 3 - \alpha]
\]
\[
tr_2^*(\alpha) = [\alpha(7 - 5) + 5, 10 - \alpha(10 - 7)] = [2\alpha + 5, 10 - 3\alpha]
\]
Then by (4.1), Remark 3.3 and Remark 3.5
\[
tr_1^* \oplus tr_2^* = [(\alpha + 1) + (10 - 3\alpha), (3 - \alpha) + (2\alpha + 5)] = [10 - 2\alpha, 8 - 3\alpha]
\]
\[
tr_1^* \odot tr_2^* = [(\alpha + 1) - (2\alpha + 5), (3 - \alpha) - (10 - 3\alpha)] = [-\alpha - 4, -7 + 2\alpha]
\]
\[
tr_1^* \otimes tr_2^* = \frac{\alpha + 1}{10 - 3\alpha}, \frac{3 - \alpha}{2\alpha + 5}
\]
very simply and straightforward, without having to go through the hassle of inverting and re-inverting.

To recapitulate:
The main motivation and advantage of this section’s approach is: By changing the point of view right from the start, when we set up a model and in doing so define a set of fuzzy intervals, the cumbersome procedure of (twice!) inverting normally un-invertible functions is avoided.

This is especially true when characterizing functions of mixed support are involved, and we close this section with some random impressions of various multiplied triangle numbers of mixed support, \((tr_1^*: \text{green}, \ tr_2^*: \text{blue}, \ tr_1^* \odot tr_2^*: \text{red})\) that have been rewritten as in Example 11:
4.2. **Algebraic properties of the class** $\mathcal{F}_c(\mathbb{R})$ (**equivalently** $\mathcal{F}_c(\mathbb{R})^{-1}$).

As was the case for $\mathcal{F}_c^0(\mathbb{R}^+)$, $\mathcal{F}_c(\mathbb{R}^+)$ we have associativity and commutativity.

Compared to sections 1 and 2 we gain subtraction and division (Division could have been introduced from the outset in section 1 by 3.11). We real-
ize however that neither really constitutes an inverse operation in the group
sense to the basic operations of addition and multiplication.

The distributive property of $\mathcal{F}_c^0(\mathbb{R}^+)$, $\mathcal{F}_c(\mathbb{R}^+)$ is lost. We only retain sub-
distributivity in the sense that for every $\alpha \in [0,1]$ (that is for every $\alpha$—cut,
at every $\alpha$—level) for three fuzzy intervals $[a_d(\alpha), a_u(\alpha)]$, $[b_d(\alpha), b_u(\alpha)]$ and $[c_d(\alpha), c_u(\alpha)]$,

\[(4.4) \quad [a_d, a_u] \odot ([b_d, b_u] \oplus [c_d, c_u]) \subseteq ([a_d, a_u] \odot [b_d, b_u]) \oplus ([a_d, a_u] \odot [c_d, c_u])\]

holds as sets.

Graphically the above means that the graph of the left hand side of (4.4) lies entirely within the graph of the right hand side.

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