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THE NORM OF THE NON-SELF-ADJOINT HARMONIC OSCILLATOR SEMIGROUP

JOE VIOLA

Abstract. We identify the norm of the semigroup generated by the non-self-adjoint harmonic oscillator acting on $L^2(\mathbb{R})$, for all complex times where it is bounded. We relate this problem to embeddings between Gaussian-weighted spaces of holomorphic functions, and we also give a formula for the operator norm of the Weyl quantization of a Gaussian. The method used — identifying the exponents of sharp products of Mehler formulas — is elementary and is inspired by more general works of L. Hörmander, A. Melin, and J. Sjöstrand.

1. Introduction

We consider the non-self-adjoint harmonic oscillator, often called the Davies operator after the contributions of E. Brian Davies,

$$Q_\theta = -e^{-i\theta} \frac{d^2}{dx^2} + e^{i\theta}x^2$$

acting on $L^2(\mathbb{R})$ for $\theta \in (-\pi/2, \pi/2)$. We refer the reader to [3, Section 14.5] for an introduction and a summary of many recent results.

Even though $Q_\theta$ has a compact resolvent and simple real eigenvalues $\{1 + 2k : k \in \mathbb{N}\}$, the operator $e^{-tQ_\theta}$, realized as the graph closure starting on the span of its eigenfunctions, is only bounded on $L^2(\mathbb{R})$ when $t \in \Omega_\theta$ with

$$\Omega_\theta = \{\Re t > 0\} \cap \left\{ t \in \mathbb{C} : |\arg \tanh t| \leq \frac{\pi}{2} - |\theta| \right\}.$$  (1.2)

(See example 2.3 and remark 4.3, which follows [1, Section 1.2.1].) Note that $\Omega_0 = \{\Re t \geq 0\}$, agreeing with the set where $e^{-tQ_0}$ is bounded, and for nonzero $\theta \in (-\pi/2, \pi/2)$ we have $\Omega_\theta = \Omega_0 \cup \frac{i\pi}{2} \mathbb{Z}$.

The purpose of this note is to identify the norm of $e^{-tQ_\theta}$. The problem is trivial for $t = i\pi k \in i\pi \mathbb{Z}$, since then $e^{-i\pi kQ_\theta} = (-1)^k$ is unitary. It is less obvious that $e^{-tQ_\theta}u(x) = -iu(-x)$ and is therefore unitary; see remark 3.1. Otherwise, for all $t \in \mathbb{C} \setminus i\mathbb{R}$ for which $e^{-tQ_\theta}$ is bounded on $L^2(\mathbb{R})$, we have the following formula.

Theorem 1.1. Let $Q_\theta$ be as in (1.1) with $|\theta| < \pi/2$, and let $t \in \Omega_\theta$. Write $\phi = \arg \tanh t \in (-\pi, \pi)$ and

$$A = \frac{1}{2} |\sinh 2t|^2 \left( \cos 2\theta + \cos 2\phi \right).$$

Then, as an operator on $L^2(\mathbb{R})$,

$$\|e^{-tQ_\theta}\| = \left( \sqrt{1 + A} + \sqrt{A} \right)^{-1/2}.$$  (1.4)

Remark 1.2. Note that $\cos 2\phi + \cos 2\theta \geq 0$ if and only if $\cos 2|\phi| \geq \cos 2(\pi/2 - |\theta|)$, which is in turn equivalent to $|\phi| \leq \pi/2 - |\theta|$. In this way, apart from $t \in \frac{i\pi}{2} \mathbb{Z}$ where
\( \phi \) is undefined, we see that \( e^{-tQ_\theta} \) is bounded if and only if the expression in (1.4) is real.

Also note that \( \{ t \in \mathbb{C} : \| e^{-tQ_\theta} \| = 1 \} \) coincides with the boundary \( \partial \Omega_\theta \). This phenomenon does not generalize to higher dimensions; see remark 5.2.

To identify the norm \( \| e^{-tQ_\theta} \| \), and indeed the norm \( \| e^{-tQ_\theta} u \| \) for any \( u \in L^2(\mathbb{R}) \), it suffices to find a unitary equivalence between \( (e^{-tQ_\theta})^* e^{-tQ_\theta} \)1/2 and a well-understood (non-negative self-adjoint) operator.

**Theorem 1.3.** With the assumptions and notations of theorem 1.1, again with \( t \in \Omega_\theta \), let

\[
\delta = \frac{A}{1 + A} \in [0, 1).
\]

Then \( \delta = 0 \) if and only if \( |\phi| + |\theta| = \pi / 2 \), in which case we have a unitary equivalence

\[
((e^{-tQ_\theta})^* e^{-tQ_\theta})^{1/2} \sim e^{-x^2}.
\]

Otherwise, when \( \delta > 0 \), we have a unitary equivalence

\[
((e^{-tQ_\theta})^* e^{-tQ_\theta})^{1/2} \sim e^{-\frac{1}{2} \arctanh(\sqrt{\delta})Q_0}.
\]

A related problem, which is equivalent up to unitary factors described below, concerns embeddings between spaces of holomorphic functions with Gaussian weights. For \( \Phi : \mathbb{C} \to \mathbb{R} \) real-quadratic and verifying \( (\partial_{\mathbb{R}}^2 + \partial_{\mathbb{I}}^2)\Phi > 0 \), let

\[
H_\Phi = \text{Hol}(\mathbb{C}) \cap L^2(\mathbb{C}, e^{-2\Phi(z)} d\mathbb{R}_z d\mathbb{I}_z)
\]

be the space of holomorphic functions with finite weighted-\( L^2 \) norm

\[
\| u \|_\Phi = \left( \int_\mathbb{C} |u(z)|^2 e^{-2\Phi(z)} d\mathbb{R}_z d\mathbb{I}_z \right)^{1/2}.
\]
Given any two such weights $\Phi' \leq \Phi''$, we consider the embedding

$$H_{\Phi'} \ni u \mapsto u \in H_{\Phi''}.$$  

Simple unitary transformations allow us to assume that $\Phi'(z) = \Phi_{1,0}(z) = \frac{1}{2}|z|^2$ and $\Phi''(z) = \Phi_{a,b}(z)$ with

$$(1.6) \quad \Phi_{a,b}(z) = \frac{1}{2}(a|z|^2 - \Re(b|z|^2))$$

for $a > 0$ and $b \geq 0$. The embedding is bounded if and only if $\Phi_{a,b} \succeq \Phi_{1,0}$, that is, if and only if $a - b \geq 1$, and is compact if and only if $a - b > 1$ (see e.g. [1, Proposition 2.4, Corollary 2.6]). We are able to obtain the following result from an elementary optimization among Gaussians.

**Proposition 1.4.** Let $a \geq 1$, let $b \geq 0$, and assume that $a - b > 1$. Define

$$\gamma = \frac{1}{2b} \left(1 - a^2 + b^2 + \sqrt{(a^2 - b^2 - 1)^2 - 4b^2}\right).$$

For $\Phi_{a,b}$ as in (1.6) and $\iota_{a,b}$ the embedding from $H_{\Phi_{1,0}}$ into $H_{\Phi_{a,b}}$ defined in (1.5),

$$||\iota_{a,b}|| = \left(\frac{1 - \gamma^2}{a^2 - (b - \gamma)^2}\right)^{1/4}$$

and the norm is attained for $u_{\gamma}(z) = e^{-\gamma z^2/2}$.

If $a > 1$ and $a - b = 1$, then $||\iota_{a,b}|| = \frac{1}{a}$ and is not attained, but is realized in the limit for $u_{\gamma}(z) = e^{-\gamma z^2/2}$ as $\gamma \to -1^+$.

This gives an alternate formula for the result in theorem 1.1, since the question of $||\iota_{a,b}||$ is already understood ([1, Section 1.2.1]; see proposition 4.2) to be equivalent to the question of $||e^{-tQ_{x}}||$: if $t \in \mathbb{T}$, for $a = e^{-4\Re t}$ and $b = ||(e^{-4t} - 1)\sin \theta||$, we have

$$||e^{-tQ_{x}}|| = e^{-\Re t}||\iota_{a,b}||.$$  

In addition, $||\iota_{a,b}||$ represents a type of anisotropic uncertainty principle for holomorphic functions: the farther $||\iota_{a,b}||$ is below 1, the less holomorphic functions are able to concentrate near the origin. (If one could approximate the Dirac delta function $\delta_{0}$ from within $H_{\Phi_{1,0}}$, then $||\iota_{a,b}||$ would obviously be 1.) Indeed, the case $b = 0$ is already known to be unitarily equivalent to the question of the lowest eigenvalue for the harmonic oscillator, as discussed in [1, Section 4.3].

The final result of this note is a formula which gives the norm of the Weyl quantization of a symbol of integrable Gaussian type. (We avoid the limiting case, of a positive semidefinite exponent, to avoid certain difficulties.) With this result one is immediately able to find the operator norm of a broad class of solution operators $e^{-tQ}$ once the Mehler formula (2.3) gives an integrable Gaussian symbol.

**Theorem 1.5.** Let

$$a(x, \xi) = \exp \left(-\frac{1}{2}(x, \xi) \cdot A(x, \xi)\right)$$

for $A$ a symmetric matrix with $\Re A$ positive definite, and let $a^{w}(x, D_{x})$ be the Weyl quantization of $a(x, \xi)$ as in (2.1). With $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_{2n}(\mathbb{C})$, define

$$D = 1 - \frac{1}{4}AJA^*J$$

...
and write

\[ B = A^* + \left(1 + \frac{i}{2} A^* J\right) D^{-1} A \left(1 - \frac{i}{2} J A^*\right). \]

Then, as an operator on \( L^2(\mathbb{R}^n) \),

\[ \|a^w(x,D)\| = (\det D)^{-1/4} \prod_{\mu \in \text{Spec}(-JB) \setminus \{\mu > 0\}} (1 - i\mu)^{-1/2}, \]

with \( \mu \), the eigenvalues of \(-\frac{1}{2}JB\), in the product repeated according to their algebraic multiplicity.

The strategy followed here is straightforward and takes its inspiration from deeper works like [8, 7]. In the case of Theorems 1.1 and 1.3, when \( e^{-tQ}\theta \) is compact, the sharp product of its Mehler formula and that of its adjoint must give, up to symplectic equivalence and a factor depending on \( t \) and \( \theta \), the Mehler formula coming from a harmonic oscillator. While the formulas obtained are explicit, it seems clear that a deeper analysis can be performed to understand results obtained through sometimes opaque direct computations, particularly in higher dimensions.

The plan of the paper is as follows. In the following section, we introduce the standard tools of the Weyl quantization and Mehler formulas and establish some simple results around the harmonic oscillator semigroup and Mehler formulas of real quadratic type. Next, we perform the computations leading to theorems 1.1 and 1.3. We then explain the relationship between the non-self-adjoint harmonic oscillator semigroup and embeddings between Gaussian-weighted spaces by recalling the dimension-one FBI-Bargmann theory and we prove proposition 1.4. Finally, we establish theorem 1.5.

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2. The Weyl quantization, Mehler formulas, and their sharp products

We briefly recall the essential tools of the Weyl quantization, composition of symbols, and the quantization of the linear symplectic group. Being primarily interested in dimension one and symbols which are either polynomials or bounded with all derivatives, we only present a vague sketch of the general theory, which may be found in [9, Sections 18, 21].

When \( a \in C^\infty(\mathbb{R}^{2n}, \mathbb{C}) \) is in an appropriate symbol class (for instance, bounded with all derivatives), the Weyl quantization may be defined weakly for \( u, v \) Schwartz functions as

\[ \langle a^w(x,D)u,v \rangle = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a \left(\frac{x+y}{2},\xi\right) u(y)v(x) \, dy \, d\xi \, dx. \]

The operator \( a^w(x,D) \) then has a continuous extension to a bounded operator on \( L^2(\mathbb{R}^n) \). If \( a(x,\xi) \) is a polynomial in \( (x,\xi) \), the standard computations with the Fourier transform give \( a^w(x,D) \) as a polynomial in the non-commuting operators of multiplication by \( x_j \) and \( D_{x_j} = -i(\partial/\partial x_j) \), for \( j = 1, \ldots, n \).
Example 2.1. If \( q(x, \xi) = ax^2 + b x \xi + c \xi^2 \) for \((x, \xi) \in \mathbb{R}^2\), then

\[
q^w(x, D_x) = ax^2 + \frac{b}{i} \left( x \frac{d}{dx} + \frac{d}{dx} x \right) - c \frac{d^2}{dx^2}.
\]

In particular, the Weyl symbol of \( Q_\theta \) in (1.1) is

\[
q_\theta(x, \xi) = e^{-i \theta} \xi^2 + e^{i \theta} x^2.
\]

The standard symplectic form on \( T^* \mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n_\xi \) is

\[
\sigma((x, \xi), (y, \eta)) = \xi \cdot x - \eta \cdot y.
\]

A (real) linear transformation \( K \) is said to be canonical if, for all \((x, \xi), (y, \eta) \in \mathbb{R}^{2n}\),

\[
\sigma(K(x, \xi), (y, \eta)) = \sigma((x, \xi), (y, \eta)).
\]

In dimension one, this is equivalent to assuming that \( \det K = 1 \).

We say that the symbols \( a(x, \xi) \) and \( b(x, \xi) \) are symplectically equivalent if there exists some real linear canonical transformation \( K \) such that \( a \circ K = b \). It is well known [9, Theorem 18.5.9] that, for any real linear canonical transformation \( K \) on \( \mathbb{R}^{2n} \), there exists a unitary operator \( \mathcal{U} \) on \( L^2(\mathbb{R}^n) \), unique up to a factor of modulus one, which may be written as a composition of changes of variables, multiplication by imaginary Gaussians, and the Fourier transform, such that

\[
\mathcal{U} a^w(x, D_x) \mathcal{U}^* = (a \circ K)^w(x, D_x).
\]

Any positive semidefinite quadratic form on \( \mathbb{R}^2 \) is symplectically equivalent to either \( x^2 \) or \( \mu(x^2 + \xi^2) \) for some \( \mu \geq 0 \); see [9, Theorem 21.5.3].

Any complex-valued quadratic form on \( \mathbb{R}^2 \) for which \( q^{-1}(\{0\}) = \{(0,0)\} \) and \( q(\mathbb{R}^2) \neq \mathbb{C} \), for instance if \( q \) has positive definite real part, is symplectically equivalent to \( \rho q_\theta(x, \xi) \) for \( \theta \in [0, \pi/2) \) and with \( \rho \in \mathbb{C} \setminus \{0\} \); see [10, Section 7.8] and references therein, particularly to [12].

The symplectic polarization of a quadratic form \( q(x, \xi) : \mathbb{R}^2 \to \mathbb{C} \) is made through the fundamental matrix

\[
F = \frac{1}{2} \begin{pmatrix}
q''_{xx} & q''_{x\xi} \\
-q''_{\xi x} & -q''_{\xi\xi}
\end{pmatrix},
\]

the unique matrix antisymmetric with respect to \( \sigma \) for which

\[
q(x, \xi) = \sigma((x, \xi), F(x, \xi)), \quad \forall (x, \xi) \in \mathbb{R}^{2n}.
\]

Example 2.2. We recall how to use the fundamental matrix to deduce the harmonic oscillator structure of a positive definite quadratic operator in dimension one. Let

\[
q(x, \xi) = ax^2 + 2b x \xi + c \xi^2
\]

be positive in the sense that \( q(x, \xi) > 0 \) for any \((x, \xi) \in \mathbb{R}^2 \setminus \{0\} \). (This implies that the coefficients are real.) We can easily check that

\[
\det(F - \lambda) = \lambda^2 - b^2 + ac = \lambda^2 + \det F.
\]

Note that \( \det F \) must be positive by positivity of \( q \). Writing \( \delta = \det F \) gives \( \text{Spec } F = \pm i \sqrt{\delta} \) and the eigenspaces of \( F \) are

\[
\ker(F - i \sqrt{\delta}) = \{(x, i \gamma x)\}_{x \in \mathbb{C}}, \quad \ker(F + i \sqrt{\delta}) = \{(x, -i \gamma x)\}_{x \in \mathbb{C}}
\]

where \( \gamma = \frac{1}{2}(\sqrt{\delta} + ib) \).
These linear algebra facts are in correspondence with the expression
\[ q(x, \xi) = \sqrt{\frac{\delta}{\Re q}} |\xi - i\gamma x|^2, \]
from which we can check that
\[ q^w(x, D_x) e^{-\gamma x^2/2} = \sqrt{\delta} e^{-\gamma x^2/2} \]
and that \( q(x, \xi) \) is symplectically equivalent to \( \sqrt{\delta}(x^2 + \xi^2) \).

The fundamental matrix allows us to write the Mehler formula [7, Theorem 4.2]
\[
M_q(x, \xi) = \frac{1}{\sqrt{\det \cos F}} \exp(\sigma((x, \xi), (\tan F)(x, \xi)))
\]
for which, at least where \( \Re q \leq 0 \), we have
\[ \exp(q^w(x, D_x)) = M^w_q(x, D_x). \]

**Example 2.3.** For the symbol \( q_\theta(x, \xi) \) in (2.2), we have
\[
F = \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix}.
\]
Since \( F^2 = -I \) for any \( \theta \in \mathbb{R} \), we have the Mehler formula
\[ M_{-tq_\theta}(x, \xi) = \frac{1}{\cosh t} \exp(-\tanh(t)q_\theta(x, \xi)). \]
We note that \( \Omega_\theta \) in (1.2) is the preimage under hyperbolic tangent of the sector
\[ \{ t \in \mathbb{C} \setminus \{0\} : \Re(tq_\theta(x, \xi)) \leq 0 \ \forall (x, \xi) \in \mathbb{R}^2 \} \]
and coincides with the set of \( t \in \mathbb{C} \setminus i\mathbb{R} \) such that \( M_{-tq_\theta}(x, \xi) \) is bounded on \( \mathbb{R}^2 \).

We also recall that, for two symbols \( a(x, \xi) \) and \( b(x, \xi) \) which are bounded with all derivatives, we have
\[
a^w(x, D_x)b^w(x, D_x) = (a\sharp b)^w(x, D_x)
\]
with
\[
(a\sharp b)(x, \xi) = e^{\frac{i}{2}\sigma((D_x, D_\xi), (D_y, D_\eta))} a(x, \xi) b(y, \eta) \bigg|_{(y, \eta) = (x, \xi)}
\]
This is a pseudodifferential operator on \( \mathbb{R}^4 \). The strategy here is simply to perform the eight integrations giving the symbol of \( (e^{-tQ_\theta})^* e^{-tQ_\theta} \), which allows us to show that the symbol is symplectically equivalent to either \( ce^{-sQ_\theta} \) or \( ce^{-x^2} \) for some \( c, s > 0 \).

The sharp product of Gaussian symbols gives a Gaussian symbol. Consider a symbol
\[
a(x, \xi) = e^{q(x, \xi)}, \quad (x, \xi) \in \mathbb{R}^2,
\]
where \( \Re q \leq 0 \). If \( a^w(x, D_x) \) is self-adjoint, then \( q \) is real-valued. In this case, \( q(x, \xi) \) is therefore symplectically equivalent to 0, to \(-x^2\), or to \(-r(x^2 + \xi^2)\) for some \( r > 0 \), according to whether \( a^w(x, D_x) \) is the identity, a bounded but not compact operator, or a compact operator.

Since \( \tanh : [0, \infty) \to [0, 1) \), the Mehler formula for the harmonic oscillator only allows us to analyze \( e^{-r(x^2 + \xi^2)} \) when \( r \in [0, 1) \). When \( r = 1 \), the corresponding operator is (up to a constant) orthogonal projection onto the Gaussian \( e^{-x^2/2} \), reflecting that it is the limit of the Mehler formula for \( e^{-tQ_0} \) as \( t \to \infty \) (see [6,
Section 1.4]). Beyond, when \( r > 1 \), we have by holomorphic continuation a non-positive compact operator.

**Proposition 2.4.** Let \( a_r(x, \xi) = e^{-r(x^2 + \xi^2)} \) for some \( r \in \mathbb{C} \) with \( \Re r > 0 \). Then

\[
a_r^w(x, D_x) = \cosh(\arctanh r)e^{-(\arctanh r)Q_0}
\]

in the sense that, for \( h_k(x) \in \ker(Q_0 - (2k + 1)) \) the Hermite functions,

\[
a_r^w(x, D_x)h_k(x) = \frac{(1 - r)^k}{(1 + r)^{k+1}} h_k(x).
\]

As a consequence, \( \|a_r^w(x, D_x)\| = |1 + r|^{-1} \).

**Proof.** The formula for \( a_r^w(x, D_x)h_k(x) \) is clearly holomorphic in \( r \) for \( \Re r > 0 \), and by the Mehler formula for the harmonic oscillator, for \( r \in (0, 1) \),

\[
a_r^w(x, D_x)h_k(x) = \cosh(\arctanh r)e^{-(2k+1)\arctanh r}h_k(x) = \frac{(1 - r)^k}{(1 + r)^{k+1}} h_k(x)
\]

by usual formulas like \( e^{-\arctanh r} = \sqrt{\frac{1-r}{1+r}} \) when \( r \in (0, 1) \). The formula (2.6) is holomorphic on \( \{\Re r > 0\} \) and therefore coincides with \( a_r^w(x, D_x)h_k(x) \) on all of \( \{\Re r > 0\} \).

The norm of \( a_r^w(x, D_x) \) is then attained on the function \( h_0(x) \), which follows easily from the facts that the \( h_k \) are orthogonal and \( |1-r| < |1+r| \) on \( \{\Re r > 0\} \). \( \square \)

**Corollary 2.5.** Let \( a(x, \xi) = e^{q(x, \xi)} \) with \( q : \mathbb{R}^2 \to \mathbb{C} \) quadratic. If \( a^w(x, D_x) \) is self-adjoint, positive, compact, and is not rank 1, then \( q(x, \xi) \) is symplectically equivalent to \( -r(x^2 + \xi^2) \) for some \( r \in (0, 1) \). If \( a^w(x, D_x) \) is self-adjoint and bounded but not compact, then either \( q = 0 \) or \( q(x, \xi) \) is symplectically equivalent to \( -x^2 \).

We finish with a straightforward observation regarding our ability to pick out functions of harmonic oscillators via their norms on Gaussians

\[
u_\gamma(x) = \left(\frac{\pi}{\Re \gamma}\right)^{-1/4} e^{-\frac{2}{\Re \gamma} x^2}
\]

for \( \gamma \in \mathbb{C}_+ \),

\[
\mathbb{C}_+ = \{ \gamma \in \mathbb{C} : \Re \gamma > 0 \}.
\]

This observation is motivated by \( f(t) = e^{-t} \), for use in section 4.2.

**Proposition 2.6.** Let \( Q = q^w(x, D_x) \) for \( q : \mathbb{R}^2 \to [0, \infty) \) quadratic, real-valued, and positive semidefinite. Let \( f : [0, \infty) \to (0, \infty) \) be a strictly decreasing Borel function, so that \( f(Q) \) may be defined by the functional calculus. Then

\[
\|f(Q)\| = \sup_{\gamma \in \mathbb{C}_+} \|f(Q)u_\gamma\|_{L^2(\mathbb{R})}.
\]

Furthermore, we may identify the form of \( Q \) by examining where this norm is attained in the following way.

(i) We have \( \|f(Q)\| < f(0) \) if and only if \( Q \) is unitarily equivalent to \( \lambda Q_0 \) for \( \lambda = f^{-1}(f(0)) \). In this case, there exists a unique \( \gamma_0 \in \mathbb{C}_+ \) for which \( \|f(Q)\| = \|f(Q)u_{\gamma_0}\| \). Consequently, \( q(x, \xi) = \frac{\lambda}{\Re \gamma_0} |\xi - i\gamma_0 x|^2 \).

(ii) If \( \|f(Q)\| = f(0) \), then \( \|f(Q)u_\gamma\| = f(0) \) for some \( \gamma \in \mathbb{C}_+ \) if and only if \( \|f(Q)u_\gamma\| = f(0) \) for all \( \gamma \in \mathbb{C}_+ \) if and only if \( Q = 0 \).
(iii) If \( \|f(Q)\| = f(0) \) but \( \|f(Q)u_\gamma\| \neq f(0) \) for some (all) \( \gamma \in \mathbb{C}_+ \), then there exists a unique \( \gamma_0 \in \partial \mathbb{C}_+ \cup \{+\infty\} \) such that
\[
\lim_{\gamma \to \gamma_0} \|f(Q)u_\gamma\|,
\]
and in this case there is some \( \lambda > 0 \) such that \( q(x, \xi) = \lambda x^2 \) if \( \gamma_0 = +\infty \) or \( q(x, \xi) = \lambda(\xi - i\gamma_0x)^2 \) if \( \gamma_0 \in \partial \mathbb{C}_+ = i\mathbb{R} \).

**Proof.** We know that \( q \) is symplectically equivalent to either \( 0, x^2 \), or \( \lambda(x^2 + \xi^2) \) for some \( \lambda > 0 \); by conjugating with some unitary operator, we begin by assuming that \( Q \) is in one of these forms. In this case, (i) is obvious since \( \text{Spec} \lambda Q_0 = \lambda(1 + 2\mathbb{N}) \) and \( \lambda Q_0 u_1 = \lambda u_1 \). It is also easy to see (iii) because
\[
\sup_{\|u\|=1} \|f(x^2)u\| = f(0) = \lim_{\gamma \to +\infty} \|f(x^2)u_\gamma\|.
\]
The only case remaining is \( Q = 0 \), where the characterization is obvious.

The result follows by undoing the unitary transformation corresponding to the symplectic transformation \( q \) to a normal form. \( \square \)

## 3. The non-self-adjoint harmonic oscillator

We now compute the norm of the non-self-adjoint harmonic oscillator via the sharp product of its Mehler formula with its adjoint. In the boundary case \( t \in \partial \Omega_0 \setminus i\mathbb{R} \), the sharp product is not of symbols bounded with all derivatives (since for instance \( |\partial^k_x e^{ix^2}| \) grows like \( |x|^k \), but following [1, Theorem 2.9] the family is strongly continuous in \( t \) and it suffices to check the formula on the eigenfunctions of \( Q_0 \) which decay superexponentially along with their Fourier transforms.

Write \( T = \tanh t \). The symbol of \( e^{-tQ_0} \) is \( M_{-tq_0} (x, \xi) \) from example 2.3, and the symbol of \( (e^{-tQ_0})^* \) is easily seen to be \( M_{-tq_0} = M_{-tq_0}^* \). We compute the symbol of \( (e^{-tQ_0})^* e^{-tQ_0} \) via the sharp product (2.4) and the Fourier transform of a Gaussian; note that \( t \in \Omega_0 \) implies that \( T e^{i\theta} \neq 0 \) and that \( \Re(T e^{i\theta}), \Re(T e^{i\theta}) \geq 0 \). Writing \( Z = (x, \xi, y, \eta) \) and similarly for \( Z^* \) and \( Z_* \), we compute
\[
e^{-i\sigma((D_x, D_\xi), (D_y, D_\eta))} e^{-i\tau_q(x, \xi)} e^{-tq_0(y, \eta)}
\]
\[
= (2\pi)^{-4} \int \frac{e^{iZ^* (Z - Z_*)}}{|T|^4} \int e^{i\tau_q(x, \xi)} e^{i\tau_q(y, \eta)} dZ_* dZ^*
\]
\[
= (2\pi)^{-4} \sqrt{\frac{16T^2 \pi^6}{|T|^4}} \int e^{i(\pi x^2 + \pi \xi^2) - \frac{i}{2} q_0(x, \xi) - \frac{i}{2} q_0(y, \eta)} d\xi^* d\eta^*
\]
\[
= \frac{1}{1 + e^{2i\theta |T|^2}} e^{-T \phi(y, \eta) + \frac{1}{T - e^{i\theta} + e^{-i\theta}} (ix + T e^{-i\theta} \eta)^2 + \frac{1}{T - e^{i\theta} + e^{i\theta}} (ix + T e^{-i\theta} \eta)^2}.
\]

Writing the exponent \( \phi \) in a manageable fashion does not seem obvious. We take inspiration from the importance of
\[
f(z) = z + z^{-1}
\]
in simplifying Mehler formulas for the harmonic oscillator and define
\[
\phi = \arg T, \quad A_\theta = |T| e^{i\theta}.
\]
This is so we can write

\[
\left( \frac{1}{T} e^{-i\theta} + e^{-i\theta} T \right)^{-1} = \frac{e^{-i\phi}}{f(A_{-\theta})}.
\]

In addition, \( A_{\theta} = A_{-\theta} \) and \( f(z) \) is holomorphic. This is useful in isolating complex conjugates because we know in advance that the exponent will be real-valued, and we also know from the form of \( Q_\theta \) that exchanging \( x \) and \( \xi \) should exchange \( \theta \) and \( -\theta \).

We set \((y,\eta) = (x,\xi)\) in the above computation and simplify the exponent and the factor according to the notations just introduced. With \( M_{-t_q\theta} \) as in example 2.3 and with the sharp product defined by (2.4), we have that the Weyl symbol of \( e^{-iQ_\theta} e^{-itQ_\theta} \) is

\[
(M_{-t_q\theta} M_{-t_q\theta})(x,\xi) = \frac{2}{|f(A_\theta)| \sinh 2t} e^{p(x,\xi)}
\]

with

\[
p(x,\xi) = -e^{i\phi}(A_\theta x^2 + A_{-\theta} \xi^2) + \frac{e^{-i\phi}}{f(A_{-\theta})} (ix + e^{i\phi} A_\theta \xi)^2 + \frac{e^{-i\phi}}{f(A_\theta)} (i\xi - e^{i\phi} A_{-\theta} \xi)^2.
\]

The coefficient of \( x^2 \) is

\[
\frac{1}{2} \frac{\partial^2_p p(x,\xi)}{\partial x^2} = -e^{i\phi} A_\theta - \frac{e^{-i\phi}}{f(A_{-\theta})} + \frac{e^{-i\phi}}{f(A_\theta)} (e^{i\phi} A_\theta)^2 = -2\Re \left( \frac{e^{i\phi}}{f(A_\theta)} \right),
\]

using that \( e^{i\phi} A_\theta f(A_\theta) = e^{i\phi} (1 + A_\theta^2) \) to combine the first and third terms. Continuing with similar computations, we arrive at

\[
p(x,\xi) = -2\Re \left( \frac{e^{i\phi}}{f(A_\theta)} \right) x^2 + 4\Im \left( \frac{A_\theta}{f(A_\theta)} \right) x \xi - 2\Re \left( \frac{e^{i\phi}}{f(A_{-\theta})} \right) \xi^2.
\]

We identify the harmonic oscillator equivalent to \( p''(x, D_x) \) following example 2.2. The fundamental matrix of \( p \) is

\[
F = 2 \begin{pmatrix}
3 \left( \frac{A_\theta}{f(A_\theta)} \right) & -\Re \left( \frac{e^{i\phi}}{f(A_\theta)} \right) \\
\Re \left( \frac{e^{i\phi}}{f(A_\theta)} \right) & -3 \left( \frac{A_\theta}{f(A_\theta)} \right)
\end{pmatrix}.
\]

Expanding real and imaginary parts using complex conjugates,

\[
\det F = 4 \left( -3 \left( \frac{A_\theta}{f(A_\theta)} \right)^2 + \Re \left( \frac{e^{i\phi}}{f(A_{-\theta})} \right) \Re \left( \frac{e^{i\phi}}{f(A_\theta)} \right) \right)
\]

\[
= \left( \frac{A_\theta}{f(A_\theta)} - \frac{A_{-\theta}}{f(A_{-\theta})} \right)^2 + \left( \frac{e^{i\phi}}{f(A_{-\theta})} + \frac{e^{-i\phi}}{f(A_\theta)} \right) \left( \frac{e^{i\phi}}{f(A_\theta)} + \frac{e^{-i\phi}}{f(A_{-\theta})} \right)
\]

\[
= 1 + A_\theta^2 f(A_{-\theta}) + 1 + A_{-\theta}^2 f(A_\theta) + 2 \cos 2\phi - 2A_\theta^2 f(A_{-\theta}) f(A_\theta)
\]

\[
= \frac{1}{f(A_\theta) f(A_{-\theta})} \left( A_{\theta} f(A_{-\theta}) + A_{-\theta} f(A_\theta) + 2 \cos 2\phi - 2A_\theta^2 \right)
\]

\[
= \frac{2 \cos 2\theta + 2 \cos 2\phi}{|f(A_\theta)|^2}.
\]
We compute furthermore that
\[ |f(A_\theta)|^2 = |(|T| + |T|^{-1}) \cos \theta + i(|T| - |T|^{-1}) \sin \theta|^2 \]
\[ = |T|^2 + |T|^{-2} + 2 \cos 2 \theta \]
\[ = |T|^2 + |T|^{-2} - 2 \cos 2 \phi + 2 \cos 2 \theta + 2 \cos 2 \phi \]
\[ = \left| |T|e^{i\phi} - \frac{1}{|T|e^{i\phi}} \right|^2 + 2 \cos \theta + 2 \cos 2 \phi. \]

Noting that \( T = |T|e^{i\phi} \), that \( T + T^{-1} = \frac{2}{\sinh t} \) and writing
\[ A = \frac{1}{2} \sinh t |\cos 2 \theta + \cos 2 \phi|^2 = \left| \frac{1}{2} f(A_\theta) \sinh 2t \right|^2 - 1 \]
as in \( (1.3) \),
\[ \det F = \frac{A}{1 + A}. \]

Note that \( t \in \Omega_{\theta} \) if and only if both \( \Re t > 0 \) and \( |\phi| \leq \pi/2 - \theta \). We see that \( \cos 2 \theta + \cos 2 \phi = 0 \) if and only if \( |\phi| = \pi/2 - \theta \). In this case, \( p(x, \xi) \) is symplectically equivalent to \(-x^2\) and the coefficient \( 2 |f(A_\theta) \sinh(2t)|^{-1} \) is 1, so the conclusions of theorems 1.1 and 1.3 for \( t \in \partial \Omega_{\theta} \cap \{ \Re t > 0 \} \) follow.

Otherwise, by corollary 2.5, we know that \( \det F \in (0, 1) \). Since \( p(x, \xi) \) must be negative definite when \( t \in \text{int} \, \Omega_{\theta} \), we have that the symbol of \( (e^{-tQ_\theta})^*e^{-tQ_\theta} \) is symplectically equivalent to
\[ (1 + A)^{-1/2}e^{-\sqrt{\det F}(x^2 + \xi^2)}. \]

From proposition 2.4, we obtain the norm \( (1.4) \) of \( \|e^{-tQ_\theta}\| \) in \( (1.4) \) as
\[ \|e^{-tQ_\theta}e^{-tQ_\theta}\|^{1/2} = \left( \sqrt{1 + A} (1 + \sqrt{\det F}) \right)^{-1/2} = \left( \sqrt{1 + A} + \sqrt{A} \right)^{-1/2}. \]

The unitary reduction of \( (e^{-tQ_\theta})^*e^{-tQ_\theta} \) follows similarly: since
\[ \cosh \text{arctanh} \sqrt{\det F} (1 - \det F)^{-1/2} = \sqrt{1 + A}, \]
we see that \( (1.1) \) is already the Mehler formula for \( e^{-\text{arctanh} (\sqrt{\det F})Q_\theta} \), so taking the square root amounts to dividing the exponent by 2. This completes the proof of theorems 1.1 and 1.3.

Remark 3.1. The case \( t \in \frac{i\pi}{2} \mathbb{Z} \) should be treated separately since \( \phi = \text{arctanh} t \) is not well-defined. We recall that the eigenfunctions of \( Q_\theta \) may be realized as
\[ u_k(x) = h_k(e^{i\theta/2}x) \in \ker(Q_\theta - (1 + 2k)), \]
for \( h_k \) the Hermite functions. In particular, the \( u_k \) inherit the parity of the Hermite functions, \( u_k(-x) = (-1)^k u_k(x) \), and we recall that the span of the \( u_k \) is dense in \( L^2(\mathbb{R}) \). In this way, if \( t = i\pi j \in i\pi \mathbb{Z} \) then \( e^{-tQ_\theta} u_k(x) = e^{i\pi j} e^{2\pi i j k} u_k(x) = (-1)^j u_k(x) \) which clearly extends by continuity to all of \( L^2(\mathbb{R}) \).

To complete the analysis of all \( t \in \frac{i\pi}{2} \mathbb{Z} \) it suffices to treat the case where \( t = i\pi/2 \), where \( e^{-iQ_\theta} u_k(x) = -i(-1)^k u_k(x) \). By parity it is straightforward to see that \( V_j = \text{Span}\{ u_k \}_{k-j \in \mathbb{Z}} \) for \( j = 0, 1 \) are orthogonal. Relying on the completeness of
Finally, as is known from return to equilibrium (for instance, [11, Theorem 2.2] or [1, Theorem 4.2]),
\[
\lim_{\Re t \to \infty} e^{\Re t} \|e^{-tQ_0}\| = \frac{1}{\sqrt{\cos \theta}},
\]
which is the norm of the first spectral projection for \(e^{-tQ_0}\) (see [4]).

**Remark 3.3.** In the final remark of [7], Hörmander observes that \(e^{-ax^2+ibx^2}\), for \(a, b \in \mathbb{R}\) small and \(a > 0\), is not realized as the Weyl symbol of \(e^{-\bar{q}^\omega(x, \xi)}\) for \(\bar{q} : \mathbb{R}^2 \to \mathbb{C}\) quadratic, small, and verifying \(\Re q \geq 0\). We note that this is not a local phenomenon and that it does not require that the real part of the exponent is only positive semidefinite.

Let \(q : \mathbb{R}^2 \to \mathbb{C}\) be a quadratic form obeying \(\Re q \leq 0\) and \(q\) and which is elliptic in the sense that \(q^{-1}\{0\} = \{(0, 0)\}\). There is therefore some \(\theta \in [0, \pi/2)\) and \(\rho\) with \(|\arg(-\rho)| \leq \theta\) for which we have a symplectic equivalence \(q \sim \rho q_0\) where \(q_0\) is as in (2.2). One method for identifying \(\rho\) and \(\theta\) is detailed in [10, Proposition 8]; we note that \(\rho^2 = \det F\) for \(F\) the fundamental matrix. So long as \(\rho \neq -1\), we may realize \(\rho q_0\) as the exponent of the Mehler formula as in (2.3),
\[
M_{\arctanh(\rho)q_0} = \frac{1}{\cosh(\arctanh(\rho))} e^{\rho q_0}.
\]
This representation is unique up to the choice of \(\arctanh(\rho)\). Furthermore, the exponent \(\rho q_0\) cannot be realized as the Mehler formula for any other quadratic form, elliptic or not, which can be seen by a quick check of the remaining elliptic model \(\alpha(\xi - \lambda x)(\xi - \mu x)\) with \(3\lambda \lambda \mu > 0\) (see [13, Section 3.1]) and the non-elliptic model \(ax^2 + 2bx\xi\) (since, up to symplectic equivalence, we may assume that \(q(0, 1) = 0\).

In conclusion, an elliptic quadratic form \(q\) with negative semidefinite real part may be realized as the exponent in a Mehler formula if and only if \(\rho = -\sqrt{-\det F} \neq -1\), but the Mehler formula only corresponds to a negative semidefinite quadratic form if \(\arctanh(-\rho)\) may be chosen so that \(|\arg \arctanh(-\rho)| \leq \pi/2 - \theta\) for \(\theta \in [0, \pi/2)\) depending on which \(\rho q_0\) is symplectically equivalent to \(q\). Otherwise, the Mehler formula can be understood as corresponding to a holomorphic continuation (or some other extension) of the semigroup.
4. Embeddings between Fock spaces

4.1. Setting and equivalence with the non-self-adjoint harmonic oscillator semigroup. We recall that we are interested in embeddings between spaces defined in (1.5),

\[ t_{a,b} : H_{\Phi_{a,b}} \ni u \mapsto u \in H_{\Phi_{a,b}}, \]

for weights \( \Phi_{a,b} \) as in (1.6) and parameters \( a, b > 0 \) with \( a - b \geq 1 \). We begin by recalling the relationship between \( t_{a,b} \) and \( e^{-tQ_\theta} \) following [1]. This will be accomplished through a simplified one-dimensional version of the FBI (Fourier-Bros-Iagolnitzer) transform theory; see, for instance, [15, Chapter 13] for an in-depth discussion.

Let \( \varphi(z, x) = \frac{1}{2} \alpha z^2 + \beta zx + \frac{1}{2} \gamma x^2 \)

with \( \beta \neq 0 \) and \( \Im \gamma > 0 \). Then

\[ T_\varphi f(z) = c_\varphi \int_{\mathbb{R}} e^{i\varphi(z,x)} f(x) \, dx, \]

with

\[ c_\varphi = \frac{\beta}{2^{1/2} \pi^{3/4} (\Im \gamma)^{1/4}}, \]

is unitary from \( L^2(\mathbb{R}) \) to \( H_{\Phi}(\mathbb{C}) \) when

\[ \Phi(z) = \sup_{x \in \mathbb{R}} (\Im \varphi(z, x)) \]

\[ = \frac{1}{4 \Im \gamma} |\beta z|^2 - \frac{1}{4 \Im \gamma} \Re((\beta z)^2) - \frac{1}{2} \Im(\alpha z^2). \]

Conjugation with \( T_\varphi \) allows us to compose symbols with the linear canonical transformation, which is now allowed to be complex,

\[ K = -\frac{1}{\beta} \begin{pmatrix} \alpha & -1 \\ \beta^2 - \alpha \gamma & \gamma \end{pmatrix}. \]

That is,

\[ T_\varphi a^w(x, D_x) T_\varphi^* = (a \circ K)^w(x, D_x), \]

where the Weyl quantization on \( H_{\Phi} \) is realized along a certain contour. To avoid complications, we will only apply this rule to polynomial (quadratic) symbols where the usual formulas apply.

Example 4.1. (See also [14, Example 2.6].) For \( \mu \in \mathbb{C} \setminus \{0\} \) to be determined and \( |\theta| < \pi/2 \), let

\[ \varphi(z, x) = i e^{i \theta} \left( \frac{1}{2} (x^2 + (\mu z)^2) - \sqrt{2} \mu zx \right). \]

This is chosen so that

\[ K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i e^{i \theta} \\ i e^{-i \theta} & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \]

which gives

\[ (q_\theta \circ K)(x, \xi) = 2ix\xi. \]

Therefore, for \( \mathcal{T} \) as in (4.1) with \( \varphi \) above, we have

\[ \mathcal{T} Q_\theta \mathcal{T}^* = 2z \frac{d}{dz} + 1. \]
The corresponding weight is
\[ \Phi(z) = \frac{1}{2\cos \theta} |\mu z|^2 + \frac{1}{2\cos \theta} \Re(ie^{i\theta} \sin \theta(\mu z)^2). \]

To set the coefficient of $|z|^2$ to $\frac{1}{2}$ and to modify the argument of the coefficient of $z^2$, we choose \( \mu = i\sqrt{-ie^{i\theta}\cos \theta} \) and obtain
\[ \Phi(z) = \frac{1}{2} \left( |z|^2 - (\sin \theta) \Re \right). \]

(4.2)

Since example 4.1 above gives a reduction from $Q_\theta$ to $2z + \frac{2}{z} + 1$, we have [1, Proposition 2.1, Theorems 2.9 and 2.12] a unitary equivalence between $e^{-tQ_\theta}$ acting on $L^2(\mathbb{R})$ and the map
\[ e^{-t(2z + \frac{2}{z} + 1)}u(z) = e^{-t}u(e^{2t}z) \]
acting on $H_\Phi$ with $\Phi$ as in (4.2), which is in turn only a unitary factor, the change of variables
\[ Cu(z) = e^{2t}u(e^{2t}z), \]
away from the embedding $H_\Phi \ni u \mapsto e^t u \in H_{\Phi(e^{2t})}.$

As a final reduction, write
\[ w(z) = Wu(z) = u(e^{i\phi}e^{\sin \theta(e^{i\theta}z)^2/2} \]
for $\phi \in \mathbb{R}$ to be determined. We compute that
\[ \|w\|_{\Phi,1,0}^2 = \int |w(z)|^2 e^{-|z|^2} dRz d\bar{z} z \]
\[ = \int |w(e^{-i\phi}z)|^2 e^{-|z|^2} dRz d\bar{z} z \]
\[ = \int |u(z)|^2 e^{-|z|^2 + \sin \theta Rz^2} dRz d\bar{z} z \]
\[ = \|u\|_{\Phi}^2. \]

We also obtain that
\[ \|u(z)\|_{\Phi(e^{2t})}^2 = \int |u(z)|^2 e^{-|z|^2 + \sin \theta R(e^{2t}z)^2} dRz d\bar{z} z \]
\[ = \int |w(z)|^2 e^{-e^{4t}R|z|^2 + \sin \theta R(e^{2t}z)^2} dRz d\bar{z} z. \]

Choosing $\phi = \frac{1}{2} \arg(e^{4t}-1)$ puts the exponent in the form of $-2\Phi_{a,b}$, so $\|w\|_{\Phi,1,0} = \|w\|_{\Phi_{a,b}}$ with $a = e^{4Rt}$ and $b = |\sin \theta (e^{4t} - 1)|$.

We express the relation between $e^{-tQ_\theta}$ and the embedding from $H_{\Phi_{1,0}}$ to $H_{\Phi_{a,b}}$ in the following diagram, where $\mathcal{T}$ and $\Phi$ are the FBI-Bargmann transform and weight from example 4.1, the map $\mathcal{C}$ is the change of variables in (4.3), and $W$ is in (4.4).
We also sum up this discussion in the following proposition, essentially due to [1]. Note that this proposition is valid for all $t \in \mathbb{C}$, where $e^{-tQ}$ is the graph closure starting from the span of the eigenfunctions of $Q_\theta$.

**Proposition 4.2.** Let $\theta \in (-\pi/2, \pi/2)$ and let $Q_\theta$ be as in (1.1). For $t \in \mathbb{C}$, write $a = e^{4\Re t}$ and $b = |(e^{4t} - 1)\sin \theta|$. There exist unitary maps $\mathcal{U} : L^2(\mathbb{R}) \to H_{\Phi_{1,0}}$ and $\mathcal{V} : L^2(\mathbb{R}) \to H_{\Phi_{a,b}}$ such that

$$ \mathcal{V} e^{-tQ} \mathcal{U}^* = e^{t_{a,b}} $$

for $t_{a,b} : H_{\Phi_{1,0}} \ni u \mapsto u \in H_{\Phi_{a,b}}$.

**Remark 4.3.** We check that the conditions $a - b \geq 1$ from proposition 4.2 and $t \in \overline{\Omega}_\theta$ with the closure of $\Omega_\theta$ from (1.2) are equivalent, supposing as usual that $|\theta| < \pi/2$. When $\theta = 0$, the condition $a - b \geq 1$ reduces to $e^{-4\Re t} \geq 1$, which agrees with $\Omega_0 = \{\Re t \geq 0\}$. When $\theta \neq 0$ and $\Re t = 0$, it is easy to see that $a - b \geq 1$ if and only if $t \in \frac{i\pi}{2} \mathbb{Z}$. We will therefore suppose that $\theta \neq 0$ and $\Re t > 0$.

So long as $\Re t > 0$, the condition $a - b \geq 1$ with $a$, $b$ as in proposition 4.2 is equivalent to

$$ e^{4\Re t} - 1 \geq \sinh 2\Re t \geq |\sin \theta|. $$

(4.5)

Taking the square of the reciprocal (since $\theta \neq 0$ and $\Re t > 0$), we see that $a - b \geq 1$ is equivalent to

$$ \left|\frac{e^{4t} - 1}{(e^{4\Re t} - 1)^2} - 1\right| \leq \cot^2 |\theta|. $$

(4.6)

On the other hand, note that so long as $\Re t \neq 0$,

$$ \arg \tanh t = \arg \left((e^t - e^{-t})(e^{it} + e^{-it})\right) = \arctan \left(\frac{\sin 2\Im t}{\sinh 2\Re t}\right). $$

So for $\theta \neq 0$ and $|\theta| < \pi/2$, we have that $t \in \Omega_\theta$ defined in (1.2) if and only if $\Re t > 0$ and, squaring both sides,

$$ \frac{(\sin 2\Im t)^2}{(\sinh 2\Re t)^2} \leq \cot^2 |\theta|. $$

(4.7)

Equivalence of (4.5) and (4.7) follows from the computation

$$ \frac{|e^{4t} - 1|^2}{(e^{4\Re t} - 1)^2} - 1 = \frac{|e^{2t}| \sin 2t|^2}{e^{4\Re t} \sinh 2\Re t} - 1 = \frac{\sinh 2t|^2 - \sinh 2\Re t|^2}{(\sinh 2\Re t)^2} = \frac{(\sin 2\Im t)^2}{(\sinh 2\Re t)^2}, $$

which in turn follows from

$$ |\sin 2t|^2 = \frac{1}{4} (e^{2it} - e^{-2it})(e^{2t} - e^{-2t}) $$

$$ = \frac{1}{4} (2 \cosh 4\Re t - 1 + 2 \cos 4\Im t) = \frac{(\sinh 2\Re t)^2 + (\sin 2\Im t)^2}{(\sinh 2\Re t)^2}. $$

### 4.2. Computation of the norm.

By proposition 4.2 and theorem 1.3, we know in advance that the operator $((t_{a,b})^* t_{a,b})^{1/2}$ is of harmonic oscillator or heat semigroup type, meaning that it is unitarily equivalent — but with an operator corresponding complex canonical transformation — to a semigroup generated by a harmonic oscillator or, in the borderline case, a heat equation. Following proposition 2.6, we may obtain the norm as the supremum over Gaussians $u_\gamma$ defined in (2.7). While
the FBI-Bargmann transforms preserve Gaussians, in the sense of kernels of linear forms in \((x, D_x)\) described in [7, Section 5], the set of integrable Gaussians is transformed from \(\{u_\gamma(x) : \gamma \in C_+ \} \subset L^2(\mathbb{R})\) to \(\{u_\gamma(z) : |\gamma| < 1\} \in H_{\Phi_{1,0}}\), and the boundary \(\{u_\gamma(x) : \gamma \in i\mathbb{R} \cup \{\pm \infty\}\},\) with convention \(u_+ = \delta_0\), is replaced by \(\{u_\gamma(z) : |\gamma| = 1\}\). This reasoning allows us to conclude that, instead of finding \(\|\epsilon_{a,b}\|\) by maximizing over all \(u \in H_{\Phi_{1,0}}\), it suffices to maximize over the set of Gaussians in \(H_{\Phi_{1,0}}\):

\[\|\epsilon_{a,b}\| = \sup_{|\gamma| < 1} \frac{\|u_\gamma\|_{\Phi_{a,b}}}{\|u_\gamma\|_{\Phi_{1,0}}}\]

Maximizing this quantity is an elementary exercise which we detail below.

In view of theorem 1.5, this strategy could also be used to obtain the norm of any operator \(e^{-tQ}\), when this operator is compact, with \(Q\) quadratic and supersymmetric as in [1].

In the case \(a - b \geq 1\), the supremum is attained in the limit. Whenever \(|\gamma| < 1\), we find, by changing variables to replace \(\Re(\gamma z^2)\) by \(|\gamma|\Re(z^2)\), that

\[\|u_\gamma\|_{\Phi_{1,0}}^2 = \frac{\pi}{\sqrt{1 - |\gamma|^2}}\]

A similar computation gives that, when \(|\gamma| < 1\) and \(a - b \geq 1\), then

\[\frac{\|u_\gamma\|_{\Phi_{a,b}}}{\|u_\gamma\|_{\Phi_{1,0}}} = \left(\frac{1 - |\gamma|^2}{a^2 - |b - \gamma|^2}\right)^{1/4}\]

Naturally, we maximize the fourth power of this quantity.

The partial derivative with respect to \(\Im\gamma\) is

\[\frac{\partial}{\partial \Im\gamma} \left(\frac{1 - |\gamma|^2}{a^2 - |b - \gamma|^2}\right) = \frac{2(\Im\gamma)(1 - a^2 - 2b\Re\gamma + b^2)}{(a^2 - |b - \gamma|^2)^2}\]

and using \(a - b \geq 1\) and \(|\gamma| < 1\),

\[1 - a^2 - 2b\Re\gamma + b^2 = 1 - (a - b)(a + b) - 2b\Re\gamma \leq 1 - (a + b) - 2b\Re\gamma \leq 1 - a - b - 2b = 1 - a + b \leq 0\]

Furthermore, apart from the trivial case \((a, b) = (1, 0)\), one of the two inequalities must be strict. The derivative therefore has the same sign as \(-\Im\gamma\), giving a global maximum when \(\Im\gamma = 0\).

Assuming, as we therefore may, that \(\gamma \in \mathbb{R}\),

\[\frac{d}{d\gamma} \left(\frac{1 - \gamma^2}{a^2 - (b - \gamma)^2}\right) = -2\frac{b\gamma^2 + (a^2 - b^2 - 1)\gamma + b}{(a^2 - (\gamma - b)^2)^2}\]

The maximum is therefore at the positive zero of the numerator

\[\gamma = \frac{1}{2b} \left(1 - a^2 + b^2 + \sqrt{(a^2 - b^2 - 1)^2 - 4b^2}\right)\]

as in proposition 1.4. (The condition \(a - b > 1\) implies that this is the only root satisfying \(|\gamma| \leq 1\).) When \(a - b = 1\), we obtain the same result in the limit \(\gamma \rightarrow -1^+\). This proves proposition 1.4.
5. Formula in arbitrary dimension

Here, we compute the sharp product of two Gaussian symbols; this result may also be found in [5, Theorem (5.6)]. We state the result for Gaussian symbols which tend to zero as \((x, \xi) \to \infty\) in order to avoid the significant and interesting complications discussed in [7, pp. 427-436].

**Proposition 5.1.** Let \(A_1\) and \(A_2\) be symmetric matrices with positive definite real parts, and for \(j = 1, 2\), let

\[
a_j(x, \xi) = \exp \left( -\frac{1}{2} (x, \xi) \cdot A_j(x, \xi) \right).
\]

With \(J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C})\), define

\[
D = 1 - \frac{1}{4} A_2 J A_1 J, \\
B = A_1 + \left( 1 + \frac{i}{2} A_1 J \right) D^{-1} A_2 \left( 1 - \frac{i}{2} J A_1 \right).
\]

Then

\[
a_1 \ast a_2(x, \xi) = \frac{1}{\sqrt{\det D}} \exp \left( -\frac{1}{2} (x, \xi) \cdot B(x, \xi) \right).
\]

**Proof.** Taking the Fourier transform in \(Z = (X, Y) = (x, \xi, y, \eta)\) gives that

\[
e^{\frac{i}{2} \sigma \left( (D_x, D_\xi), (D_y, D_\eta) \right)} a_1(x, \xi) a_2(y, \eta) = \frac{1}{\sqrt{\det(A_1 A_2)}} \int e^{iZ \cdot \tilde{A}^{-1} Z} dZ^*.
\]

The exponent is \(i Z^* - \frac{1}{2} Z^* \cdot \tilde{A} Z^*\) where

\[
\tilde{A} = \begin{pmatrix} A_1^{-1} & -\frac{i}{2} J \\ \frac{i}{2} J & A_2^{-1} \end{pmatrix}.
\]

From the formula for the Fourier transform of a Gaussian, we obtain that

\[
e^{\frac{i}{2} \sigma \left( (D_x, D_\xi), (D_y, D_\eta) \right)} a_1(x, \xi) a_2(y, \eta) = \frac{1}{\sqrt{\det A_1 A_2}} \exp \left( -\frac{1}{2} Z^* \cdot \tilde{A}^{-1} Z \right).
\]

We note that \(D\) defined in (5.1) is invertible because

\[
\langle A_2^{-1} X, DX \rangle = \langle X, A_2^{-1} X \rangle + \frac{1}{4} (J X, A_1 J X),
\]

which has positive real part whenever \(X \neq 0\). We perform row reduction to find \(\tilde{A}^{-1}\) as follows:

\[
\begin{pmatrix} 1 & \frac{i}{2} A_1 J \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D^{-1} A_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{i}{2} J & 1 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{A} = 1.
\]

Observing that \(\det \tilde{A}^{-1} = \det A_1 D A_2\) and expanding \((X, X) \cdot \tilde{A}^{-1}(X, X)\) gives the proposition. \(\square\)

We finish with the proof of theorem 1.5.
Proof. With $B$ and $D$ as in the statement of theorem 1.5, we have from proposition 5.1 that the symbol of $(a^w(x, D_x))^*a^w(x, D_x)$ is

$$a(x, \xi)^*a(x, \xi) = \frac{1}{\sqrt{\det D}} \exp \left( -\frac{1}{2} (x, \xi) \cdot B(x, \xi) \right).$$

Since $(a^w)^*a^w$ is a bounded self-adjoint compact operator on $L^2(\mathbb{R}^n)$, we have that $B$ is real symmetric and positive definite. Therefore the exponent is symplectically equivalent to some harmonic oscillator symbol, and since the eigenvalues of the fundamental matrix are symplectic invariants, we have that

$$-\frac{1}{2} (x, \xi) \cdot B(x, \xi) \sim -\sum_{j=1}^n \mu_j (x^2 + \xi^2)$$

for $\mu_j$ the eigenvalues of $-\frac{1}{2}JB$ for which $\mu_j/i > 0$, repeated for multiplicity.

Therefore, up to unitary equivalence, $(a^w)^*a^w$ can be written as a tensor product of harmonic oscillator semigroups. By proposition 2.4 and positivity of $(a^w)^*a^w$, we have that $\mu_j \in (0, 1)$ for all $\mu_j$ and the conclusion of theorem 1.5. \qed

Remark 5.2. The formula obtained from theorem 1.5 does not immediately give a simple expression for the norm: for instance, theorem 1.1 does not immediately reveal certain simple facts like that $\|e^{-tQ_0}\| = e^{-\Re t}$ whenever $\Re t \geq 0$.

For another example, we consider the class of operators considered in [2], with symbols

$$q(x, \xi) = \frac{1}{2} M(\xi + ix) \cdot (\xi - ix), \quad (x, \xi) \in \mathbb{R}^{2n},$$

for $M$ an $n$-by-$n$ matrix. Writing $D_x = -i\nabla_x$, the corresponding Weyl quantization is

$$Q(x, D_x) = \frac{1}{2} M(D_x + ix) \cdot (D_x - ix) + \frac{1}{2} \text{Tr} M.$$

The operator $e^{-tQ}$ is bounded if and only if $\|e^{-tM}\| \leq 1$, in which case

$$\|e^{-tQ}\| = e^{-\frac{1}{2} \Re(t \text{Tr} M)}.$$

(This is the case $N = 0$ of [2, Corollary 2.9].) It seems possible that one can deduce this result via theorem 1.5, but it certainly does not seem easy.

Nearly any such example shows that the boundary of the set where $e^{-tQ}$ is bounded does not coincide with those $t \in \mathbb{C}$ for which $\|e^{-tQ}\| = 1$; the author is partial to

$$M = \begin{pmatrix} 0 & -a \\ a & 1 \end{pmatrix}, \quad a \in \mathbb{R}\{0\}.$$  

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