Linear Strain Tensors and Optimal Exponential of thickness in Korn’s Inequalities for Hyperbolic Shells

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Abstract We perform a detailed analysis of the solvability of linear strain equations on hyperbolic surfaces. Then the rigidity results on the strain tensor of the middle surface is given for non-characteristic regions. Finally, we obtain the optimal constant in the first Korn inequality scales like $h^{4/3}$ for hyperbolic shells, removed the main assumption that the middle surface of the shell is given by a single principal system by Harutyunyan in Arch. Ration. Mech. Anal. 226 (2017), no. 2, 743-766.

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1 Introduction

The goal of the present paper is twofold to study the solvability of linear strain equations and the optimal constant in the first Korn inequality for hyperbolic shells.

The Linear strain equations plays a fundamental role in the theory of thin shells, see [12, 13, 14, 15, 29]. In those works, the linear strain equations have been studied in order to obtain the Γ-limits in a hierarchy of shell(plate) models (introduced in the setting of plates and justified in [4], and in the setting of shells in [14]). In general, the strain equations are translated into a scalar second order partial differential equation,

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which is subject to the geometry of the middle surface. The main observation is that the scalar equation is, respectively, elliptic, hyperbolic or degenerate if the middle surface is of positive, negative or zero Gaussian curvature. The works [15], [12], and [29] are specific to the elliptic, parabolic, and hyperbolic middle surfaces, respectively. A survey on this topic is presented in [13].

Here we present a direct method of solving the linear strain equations for the hyperbolic middle surface, different from [29], which is relatively simple and allows us to obtain a lower regularity on the solution, see Theorem 1.1 later. Fortunately, this regularity implies the rigidity results on the strain tensor of the middle surface which is one of the key ingredients for the optimal constant in the first Korn inequality for hyperbolic shells ([10]).

Originally, Korn’s inequalities were used to prove existence, uniqueness and well-posedness of boundary value problems of linear elasticity (see e.g., [1, 17]). The optimal exponential of thickness in Korn’s inequalities for thin shells represents the relationship between the rigidity and the thickness of a shell when the small deformations take place since Korn’s inequalities are linearized from the geometric rigidity inequalities under the small deformations ([3]). Thus it is the best Korn constant in the Korn inequality that is of central importance (e.g., [2, 16, 18, 19, 21, 22]). Moreover, it is ingenious that the best Korn constant is subject to the Gaussian curvature. The one for the parabolic shell scales like $h^{3/2}$ ([6, 7]), for the hyperbolic shell, $h^{4/3}$ ([10]) and for the elliptic shell, $h$ ([10]). All those results were derived under the main assumption that the middle surface of the shell is given by a single principal coordinate system in order to carry out some necessary computation. This assumption is

\[ S = \{ r(z, \theta) \mid (z, \theta) \in [1, 1 + l] \times [0, \theta_0] \}, \tag{1.1} \]

where the properties

\[ \nabla_{\partial z} \vec{n} = \kappa_z \partial z, \quad \nabla_{\partial \theta} \vec{n} = \kappa_\theta \partial \theta \quad \text{for} \quad p \in S \]

hold.

In the case of the parabolic or hyperbolic shell, a principal coordinate only exists locally ([30]). There is even no such an existence for the elliptic shell. However, the assumption (1.1) in [6, 7, 10] can be removed if the Bochner technique is employed to perform some necessary computation. The Bochner technique provides us the great simplification in computation, for example, see [26] or [28]. It has been done in the cases of the parabolic and elliptic shells in [30]. Here we treat the hyperbolic shell by combining the rigidity lemma of the strain tensor of the middle surface, given in this paper, and the interpolation inequality [11] to obtain that the best constant in Korn’s inequality scales like $h^{4/3}$, removed the assumption (1.1).

Let $M \subset \mathbb{R}^3$ be a $C^3$ surface with the induce metric $g$ and a normal field $\vec{n}$. Let $S \subset M$ be an open bounded set with a regular boundary $\partial S$. Suppose that $S$ is the middle
surface of the shell with thickness \( h > 0 \)
\[
\Omega = \{ x + t\bar{n}(x) \mid x \in S, -h/2 < t < h/2 \}.
\]
A shell \( \Omega \) is said to be hyperbolic if
\[
\kappa(p) < 0 \quad \text{for} \quad p \in \overline{S},
\]
where \( \kappa \) is the Gaussian curvature. Throughout the paper \( \Omega \) is assumed to be hyperbolic.

Let \( y \in H^1(S, \mathbb{R}^3) \) be a displacement of the middle surface \( S \). We decompose \( y \) as
\[
y = W + w\bar{n}, \quad w = \langle y, \bar{n} \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the dot metric of the Euclidean space \( \mathbb{R}^3 \). The (linear) strain tensor of the middle surface (related to the displacement \( y \)) is defined by
\[
\Upsilon(y) = \text{sym} \, DW + w\Pi,
\]
where \( D \) is the Levi-Civita connection of the induced metric \( g \) on \( S \),
\[
\text{sym} \, DW = \frac{1}{2}(DW + DW^T),
\]
and
\[
\Pi(\alpha, \beta) = \langle \nabla_\alpha \bar{n}, \beta \rangle \quad \text{for} \quad \alpha, \beta \in M, \quad x \in S
\]
is the second fundamental form of surface \( M \). \( y \in H^1(S, \mathbb{R}^3) \) is said to be an infinitesimal isometry if
\[
\Upsilon(y) = 0.
\]
For \( U \in L^2(S, T^2) \) given, consider problem
\[
\Upsilon(y) = U \quad \text{for} \quad p \in S. \tag{1.2}
\]

We say that \( S \) is a non-characteristic region if one of the following assumptions (I) and (II) holds.

(I) Let
\[
S = \{ \alpha(t, s) \mid (t, s) \in (0, a) \times (0, b) \},
\]
where \( \alpha : [0, a] \times [0, b] \to M \) is an imbedding map which is a family of regular curves with two parameters \( t, s \) such that
\[
\Pi(\alpha_t(t, s), \alpha_t(t, s)) \neq 0, \quad \text{for all} \quad (t, s) \in [0, a] \times [0, b],
\]
\[
\Pi(\alpha_s(0, s), \alpha_s(0, s)) \neq 0, \quad \Pi(\alpha_s(a, s), \alpha_s(a, s)) \neq 0, \quad \text{for all} \quad s \in [0, b],
\]
\[
\Pi(\alpha_t(0, s), \alpha_s(0, s)) = \Pi(\alpha_t(a, s), \alpha_s(a, s)) = 0, \quad \text{for all} \quad s \in [0, b].
\]
(II) Let

\[ S = \{ \alpha(t, \theta) \mid t \in [0, a], \ \theta \in [0, 2\pi) \}, \quad a > 0, \]

where \( \alpha : [a_0, a_1] \times [0, 2\pi) \to M \) is an imbedding map if \( \alpha(0, \cdot) \) is a closed curve; \( \alpha : (0, a] \times [0, 2\pi) \to M \) is an imbedding map if \( \alpha(0, \cdot) \) is one point, and, for each \( t \in (0, a] \), \( \alpha(t, \cdot) \) is of periodicity \( 2\pi \) in \( \theta \). Moreover, for each \( t \in (0, a] \),

\[ \Pi(\alpha_\theta(t, \theta), \alpha_\theta(t, \theta)) \neq 0 \quad \text{for} \quad \theta \in (0, 2\pi). \]

**Remark 1.1** *If the middle surface is given by one single principal coordinate, that is, the assumption (1.1) holds, then \( S \) is in (II) when for each \( z \in [1, 1+l], r(z, \cdot) \) is a closed curve; otherwise, \( S \) in (I).*

We define \( Q : M_x \to M_x \) by

\[ Q\alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in M_x, \tag{1.3} \]

where \( e_1, e_2 \) is an orthonormal basis of \( M_x \) with the positive orientation. Then the definition of \( Q \) is independent of the choice of a positively orientation orthonormal basis and is the rotation on \( M_x \) by \( \pi/2 \) along the clockwise direction, see [29].

Recall that the shape operator \( \nabla \vec{n} : M_x \to M_x \) is defined by \( \nabla \vec{n}X = \nabla_X \vec{n}(x) \) for \( X \in M_x \). Now we define operators \( \mathcal{T}_i : M_x \to M_x \) by

\[ \mathcal{T}_iX = \frac{1}{2} \left[ X + (-1)^i \varphi(X) Q \nabla \vec{n}X \right] \quad \text{for} \quad X \in M_x, \quad i = 1, 2, \tag{1.4} \]

where

\[ \varphi(X) = \frac{1}{\sqrt{-K}} \text{sign} \Pi(X, X) \tag{1.5} \]

and \( \text{sign} \) is the sign function.

In (I), we shall consider the part boundary data

\[ \langle W, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(0, s) = q_1(s), \quad \langle W, \mathcal{T}_1 \alpha_s \rangle \circ \alpha(a, s) = q_2(s) \quad \text{for} \quad s \in (0, b), \tag{1.6} \]

\[ W \circ \alpha(t, 0) = \phi \quad \text{for} \quad t \in (0, a). \tag{1.7} \]

For convenience, we denote the relations (1.6) and (1.7) by

\[ W\mid_{(I)(a,b)} = (q_1, \phi, q_2). \tag{1.8} \]

In (II) the following boundary data are concerned

\[ W \circ \alpha(a_1, \theta) = \phi \quad \text{for} \quad \theta \in (0, 2\pi), \tag{1.9} \]

which is denoted by \( W\mid_{(II)} = \phi \).

We have the following.
Theorem 1.1 (i) Let $S$ be given in (I). Then there is a constant $C > 0$ such that, for any $q_1, q_2 \in L^2(0, b)$, and $\phi \in L^2((0, a), \mathcal{X})$ and any $U \in L^2(S, T^2)$, there exists a unique solution $y$ to problem (1.2) with the data (1.6) and (1.7) satisfying
\[
\|W\|_{L^2(S, \mathcal{X})} \leq C(\|U\|_{L^2(S, T^2)}^2 + \|\phi\|_{L^2((0, a), \mathcal{X})}^2 + \|q_1\|_{L^2(0, b)}^2 + \|q_2\|_{L^2(0, b)}^2),
\]
where $y = W + w\bar{n}$.

(ii) Let $S$ be given in (II). Then there is a constant $C > 0$ such that, for any $\phi \in L^2((0, 2\pi), \mathcal{X})$ and any $U \in L^2(S, T^2)$, there exists a unique solution $y$ to problem (1.2) with the data (1.9) satisfying
\[
\|W\|_{L^2(S, \mathcal{X})} \leq C(\|U\|_{L^2(S, T^2)}^2 + \|\phi\|_{L^2((0, 2\pi), \mathcal{X})}^2),
\]
where $y = W + w\bar{n}$.

Remark 1.2 A roundabout solvability to problem (1.2) has been given in [29] where the matching property is proven and some $\Gamma$-limits are obtained for the hyperbolic shell.

It follows immediately from Theorem 1.1 the corollary below.

Corollary 1.1 Let $S$ be in (I). Then there is a constant $C > 0$ such that, for any $y = W + w\bar{n} \in H^1(S, \mathbb{R}^3)$ there exists an infinitesimal $y^0 \in H^1(S, \mathbb{R}^3)$ with the boundary data
\[
W^0|_{(I)(a, b)} = \left(\langle W, \mathcal{T}_2\alpha_s \rangle \circ \alpha(0, \cdot), W \circ \alpha(-\cdot, 0), \langle W, \mathcal{T}_1\alpha_s \rangle \circ \alpha(0, \cdot)\right)
\]
satisfying
\[
\|W - W^0\|_{L^2(S, \mathcal{X})} \leq C\|Y(y)\|_{L^2(S, T^2)},
\]
where $y^0 = W^0 + w^0\bar{n}$.

Let $S$ be given in (II). Then there is a constant $C > 0$ such that, for any $y = W + w\bar{n} \in H^1(S, \mathbb{R}^3)$ there exists an infinitesimal $y^0 \in H^1(S, \mathbb{R}^3)$ with the boundary data
\[
W^0|_{(II)} = W \circ \alpha(a_1, \cdot)
\]
satisfying the estimate (1.12).

We have a rigidity lemma on the strain tensor of the middle surface.

Theorem 1.2 Let $S$ be a non-characteristic region. Then there is a constant $C > 0$ such that
\[
\|W\|_{L^2(S)} \leq C\|Y(y)\|_{L^2(S)},
\]
\[
\|w\|_{L^2(S)}^2 \leq C(\|Dw\|_{L^2(S)}\|Y(y)\|_{L^2(S)} + \|Y(y)\|_{L^2(S)}^2),
\]
for all $y = W + w\bar{n} \in H^1(S, \mathbb{R}^3)$ with $w|_{\partial S \times (-h/2, h/2)} = 0$ and $W|_{(I)(a, b)} = 0$, or $W|_{(II)} = 0$. 

\[\]
We combine [11, Theorem 3.1] with Theorem 1.2 by an argument as in [10] to obtain

\textbf{Theorem 1.3} Let \( S \) be a non-characteristic region. Then there are \( C > 0, h_0 > 0 \), independent of \( h > 0 \), such that

\[
\|\nabla y\|_{L^2(\Omega)}^2 \leq \frac{C}{h^{4/3}} \|\text{sym} \nabla y\|_{L^2(\Omega)}^2
\]

(1.15)

for all \( h \in (0,h_0) \) and \( y = W + w \bar{n} \in H^1(\Omega, \mathbb{R}^3) \) with \( w|_{\partial S \times (-h/2,h/2)} = 0 \) and \( W|_{\partial (a,b)} = 0 \), or \( W|_{\partial I} = 0 \). Moreover, the exponential of the thickness \( h \) in (1.15) is optimal.

\textbf{Remark 1.3} The results in Theorems 1.2 and 1.3 are given in [10] when the middle surface is given by one single principal coordinate.

\section{Linear Strain Equations}

On \( \mathbb{R}^2 \) we consider the solvability of problem

\[
\begin{aligned}
&f_{1x1}(x) = a_{11}(x)f_1(x) + a_{12}(x)f_2(x) + p_1(x), &\text{for } x = (x_1,x_2) \in \mathbb{R}^2, \\
f_{2x2}(x) = a_{21}(x)f_1(x) + a_{22}(x)f_2(x) + p_2(x),
\end{aligned}
\]

(2.1)

where \((p_1,p_2)\) is given and \((f_1,f_2)\) is the unknown and \(a_{ij} \in L^\infty\).

We shall work out some basic regions in which problem (2.1) is uniquely solvable when \((p_1,p_2)\) and some data on part of boundary are given.

A curve \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) : [a,b] \rightarrow \mathbb{R}^2 \) is said to be noncharacteristic if

\[
\gamma_1'(t)\gamma_2'(t) \neq 0 \quad \text{for} \quad t \in [a,b].
\]

Let \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) : [0,t_0] \rightarrow \mathbb{R}^2 \) be a noncharacteristic curve such that

\[
\gamma_1'(t) > 0, \quad \gamma_2'(t) < 0 \quad \text{for} \quad t \in (0,t_0).
\]

(2.2)

Set

\[
E(\gamma) = \{(x_1,x_2) \in \mathbb{R}^2 \mid \gamma_1 \circ \gamma_2^{-1}(x_2) < x_1 < \gamma_1(t_0), \gamma_2(t_0) < x_2 < \gamma_2(0)\}.
\]

(2.3)

Consider the boundary data

\[
f \circ \gamma(t) = q(t) \quad \text{for} \quad t \in (0,t_0).
\]

(2.4)

\textbf{Proposition 2.1} For any \( q \in L^2((0,t_0), \mathbb{R}^2) \) and \( p \in L^2(E(\gamma), \mathbb{R}^2) \) given, there exists a unique solution \( f = (f_1,f_2) \in L^2(E(\gamma), \mathbb{R}^2) \) to problem (2.1) with the data (2.4) satisfying

\[
\|f\|_{L^2(E(\gamma), \mathbb{R}^2)}^2 \leq C(\|q\|_{L^2(0,t_0), \mathbb{R}^2}^2 + \|p\|_{L^2(E(\gamma), \mathbb{R}^2)}^2),
\]

(2.5)

\[
\|f_1(\gamma_1(t_0)), f_2(\gamma_2(t_0))\|_{L^2(\gamma_2(t_0), \gamma_2(0))}^2 \leq C(\|q\|_{L^2(0,t_0), \mathbb{R}^2}^2 + \|p\|_{L^2(E(\gamma), \mathbb{R}^2)}^2),
\]

(2.6)

\[
\|f_2(t, \gamma_2(0))\|_{L^2(\gamma_2(t), \gamma_2(t_0))}^2 \leq C(\|q\|_{L^2(0,t_0), \mathbb{R}^2}^2 + \|p\|_{L^2(E(\gamma), \mathbb{R}^2)}^2).
\]

(2.7)
Let $\beta = (\beta_1, \beta_2) : [0, t_1] \to \mathbb{R}^2$ be a noncharacteristic curve such that
\[
\beta'(t) > 0, \quad \beta''(t) > 0 \quad \text{for} \quad t \in [0, t_1]. \tag{2.8}
\]

Set
\[
P(\beta) = \{ (x_1, x_2) \mid \beta_1(0) < x_1 < \beta_1 \circ \beta_2^{-1}(x_2), \quad \beta_2(0) < x_2 < \beta_2(t_1) \},
\]
and consider the boundary data
\[
f_1(\beta_1(0), x_2) = q_1(x_2), \quad x_2 \in (\beta_2(0), \beta_2(t_1)); \quad f_2 \circ \beta(t) = q_2(t), \quad t \in [0, t_1]. \tag{2.9}
\]

**Proposition 2.2** For any $q_1 \in L^2(\beta_2(0), \beta_2(t_1)), q_2 \in L^2(0, t_1),$ and $p \in L^2(E(\gamma), \mathbb{R}^2)$ given, there exists a unique solution $f = (f_1, f_2) \in L^2(P(\beta), \mathbb{R}^2)$ to problem (2.1) with the data (2.9) satisfying
\[
\|f\|_{L^2(P(\beta), \mathbb{R}^2)} \leq C(\|q_1\|^2_{L^2(\beta_2(0), \beta_2(t_1))} + \|q_2\|^2_{L^2(0, t_1)} + \|p\|^2_{L^2(P(\beta), \mathbb{R}^2)}). \tag{2.10}
\]

Let $\gamma = (\gamma_1, \gamma_2) : (0, 0) \to \mathbb{R}^2$ and $\beta = (\beta_1, \beta_2) : (0, t_1) \to \mathbb{R}^2$ be two noncharacteristic curves with $\gamma(0) = \beta(0)$ such that (2.2) and (2.8) hold, respectively. We further assume that
\[
\sup_{t \in [0, t_0]} [\gamma_1(t) + \gamma_2(t)] < \beta_1(t_1) + \beta_2(t_1), \quad \beta_1(t_1) + \beta_2(t_1) - \beta_2(0) \leq \gamma_1(t_0). \tag{2.11}
\]

Set
\[
\Xi(\beta, \gamma) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid \beta_1(0) < x_1 < \beta_1(t_1), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < \beta_2 \circ \gamma_1^{-1}(x_1) \}
\]
\[
\cup\{ (x_1, x_2) \in \mathbb{R}^2 \mid \beta_1(t_1) \leq x_1 < \gamma_1(t_1), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < -x_1 + \beta_1(t_1) + \beta_2(t_1) \}. \tag{2.12}
\]

Consider the boundary data
\[
f_1 \circ \beta(t) = q_1(t) \quad \text{for} \quad t \in (0, t_1), \tag{2.13}
\]
\[
f \circ \gamma(t) = \dot{q}(t) \quad \text{for} \quad t \in (0, t_0). \tag{2.14}
\]

**Proposition 2.3** For any $q_1 \in L^2(0, t_1), \dot{q} \in L^2((0, t_0), \mathbb{R}^2),$ and $p \in L^2(\Xi(\beta, \gamma), \mathbb{R}^2)$ given, there is a unique solution $f = (f_1, f_2) \in L^2(\Xi(\beta, \gamma), \mathbb{R}^2)$ to problem (2.1) with the data (2.13) and (2.14) satisfying
\[
\|f\|^2_{L^2(\Xi(\beta, \gamma))} \leq C(\|q_1\|^2_{L^2(0, t_1)} + \|\dot{q}\|^2_{L^2((0, t_0), \mathbb{R}^2)} + \|p\|^2_{L^2(\Xi(\beta, \gamma))}). \tag{2.15}
\]

Let $\beta$ and $\gamma$ be noncharacteristic curves with $\beta(0) = \gamma(0)$ such that (2.2) and (2.8) hold, respectively. Let $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2) : (0, \hat{t}_1) \to \mathbb{R}^2$ be noncharacteristic such that
\[
\beta_1(t_1) + \beta_2(t_1) = \hat{\beta}_1(\hat{t}_1) + \hat{\beta}_2(\hat{t}_1),
\]
Consider the boundary data

\[ \beta_1(t_1) + \beta_2(t_1) \leq \gamma_1(t_0) + \gamma_2(0), \quad \gamma(t_0) = \hat{\beta}(0), \quad \hat{\beta}_1'(t) > 0, \quad \hat{\beta}_2'(t) > 0 \quad \text{for} \quad t \in (0, \hat{t}_1). \]

Set

\[ \Phi(\hat{\beta}, \hat{\beta}) = \{ (x_1, x_2) | \beta_1(0) < x_1 < \beta_1(t_1), \gamma_2 \circ \gamma_1^{-1}(x_1) \leq x_2 < \beta_2 \circ \gamma_1^{-1}(x_1) \} \]

\[ \cup \{ (x_1, x_2) | \beta_1(t_1) \leq x_1 < \gamma_1(0), \gamma_2 \circ \gamma_1^{-1}(x_1) \leq x_2 < -x_1 + \beta_1(t_1) + \beta_2(t_1) \} \]

\[ \cup \{ (x_1, x_2) | \gamma_1(t_0) \leq x_1 < \hat{\beta}_1(\hat{t}_1), \hat{\beta}_2 \circ \hat{\beta}_1^{-1}(x_1) < x_2 < -x_1 + \beta_1(t_1) + \beta_2(t_1) \}. \]

Consider the boundary data

\[ f_1 \circ \beta(t) = q_1(t), \quad t \in (0, t_1); \quad f_2 \circ \hat{\beta}(t) = q_2(t), \quad t \in (0, \hat{t}_1), \quad (2.16) \]

\[ f \circ \gamma(t) = q(t) \quad \text{for} \quad t \in (0, t_0). \quad (2.17) \]

**Proposition 2.4** Let \( q_1, q_2, \) and \( q \) be given \( L^2 \) functions. Then problem (2.1) admits a unique solution \( f \in L^2(\Phi(\beta, \hat{\beta}), \mathbb{R}^2) \) with the data (2.16) and (2.17). Moreover, the following estimates hold

\[ \|f\|_{L^2(\Phi(\beta, \hat{\beta}))}^2 \leq C(\|q_1\|_{L^2(0, t_0)}^2 + \|q_2\|_{L^2(0, \hat{t}_1)}^2 + \|p\|_{L^2(\Phi(\beta, \hat{\beta}))}^2). \]

The proofs of Propositions 2.1 and 2.4 will be given by the following lemmas.

**Lemma 2.1** Let \( T > 0 \) be given. There is a \( \varepsilon_T > 0 \) such that if \( |\gamma(0)| \leq T \) and \( \max \{ \gamma_1(t_0) - \gamma(0), \gamma_2(0) - \gamma_2(t_0) \} < \varepsilon_T \), then the results in Proposition 2.1 hold.

**Proof** The proof is broken into several steps as follows.

**Step 1.** Let \( x = (x_1, x_2) \in E(\gamma) \) be given. We integrate the first equation in (2.1) with respect to the variable \( x_1 \) over \((\gamma \circ \gamma_1^{-1}(x_2), x_1)\) where \( x_2 \in (\gamma_2 \circ \gamma_1^{-1}(x_1), x_2) \) is fixed to have

\[ f_1(x_1, x_2) = q_1 \circ \gamma \circ \gamma_1^{-1}(x_2) + \int_{\gamma_1 \circ \gamma_1^{-1}(x_2)}^{x_1} (a_{11}f_1 + a_{12}f_2 + p_1)(\zeta_1, x_2)d\zeta_1. \quad (2.18) \]

Then integrating the second equation in (2.1) over \((\gamma_2 \circ \gamma_1^{-1}(x_1), x_2)\) with respect to the variable \( x_2 \) yields

\[ f_2(x_1, x_2) = q_2 \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} (a_{12}f_1 + a_{22}f_2 + p_2)(\zeta_2, x_1)d\zeta_2. \quad (2.19) \]

**Step 2.** Let \( p \) and \( q \) be given. We define an operator \( B : L^2(E(\gamma), \mathbb{R}^2) \rightarrow L^2(E(\gamma), \mathbb{R}^2) \) by

\[ Bf = \left( q_1 \circ \gamma \circ \gamma_2^{-1}(x_2), q_2 \circ \gamma \circ \gamma_1^{-1}(x_1) \right) + \left( \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)}^{x_1} (a_{11}f_1 + a_{12}f_2 + p_1)(\zeta_1, x_2)d\zeta_1, \right. \]

\[ \left. \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} (a_{12}f_1 + a_{22}f_2 + p_2)(\zeta_2, x_1)d\zeta_2, \right. \quad (2.20) \]

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for any \( f = (f_1, f_2) \in L^2(E(\gamma), \mathbb{R}^2) \). It is easy to check that \( f \in L^2(E(\gamma), \mathbb{R}^2) \) solves (2.1) with the data (2.4) if and only if \( B f = f \).

Next, we shall prove that there is a \( 0 < \varepsilon_T \leq 1 \) such that when \( |\gamma(0)| \leq T \) and \( 0 < \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T \), the map \( B : L^2(E(\gamma), \mathbb{R}^2) \to L^2(E(\gamma), \mathbb{R}^2) \) is contractible. Thus the existence and uniqueness of solutions follow from Banach’s fixed point theorem.

In fact, for \( f = (f_1, f_2), \hat{f} = (\hat{f}_1, \hat{f}_2) \in L^2(E(\gamma), \mathbb{R}^2) \), it follows from (2.20) that

\[
B f - B \hat{f} = \left( \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)} [a_{11}(f_1 - \hat{f}_1) + a_{12}(f_2 - \hat{f}_2)](\zeta_1, x_2)d\zeta_1, \right.
\]

\[
\left. \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)} [a_{12}(f_1 - \hat{f}_1) + a_{22}(f_2 - \hat{f}_2)](x_1, \zeta_2)d\zeta_2 \right),
\]

which yields

\[
|B f - B \hat{f}|^2 \leq C \varepsilon \left[ \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)} |f - \hat{f}|^2(\zeta_1, x_2)d\zeta_1 + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)} |f - \hat{f}|^2(x_1, \zeta_2)d\zeta_2 \right],
\]

for \( x = (x_1, x_2) \in E(\gamma) \), where \( \varepsilon = \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} \). Thus we obtain

\[
\|B f - B \hat{f}\|^2_{L^2} \leq C T \varepsilon^2 \|f - \hat{f}\|^2_{L^2},
\]

(2.21)
i.e., the map \( B : L^2(E(\gamma), \mathbb{R}^2) \to L^2(E(\gamma), \mathbb{R}^2) \) is contractible if \( \varepsilon \) is small.

**Step 3.** Let map \( B : L^2(E(\gamma), \mathbb{R}^2) \to L^2(E(\gamma), \mathbb{R}^2) \) be defined in Step 2 and let \( f \in L^2(E(\gamma), \mathbb{R}^2) \) be the solution to problem (2.1) with the data (2.4). It follows from (2.20) and (2.21) that

\[
\|f\|_{L^2(E(\gamma))} = \|B f\|_{L^2(E(\gamma))} \leq \|B (0)\|_{L^2(E(\gamma))} + \|B f - B (0)\|_{L^2(E(\gamma))} \leq C T (\|q\|_{L^2(0,t_0)} + \|p\|_{L^2(E(\gamma))}) + C T \varepsilon \|f\|_{L^2(E(\gamma))}.
\]

Thus, the estimate (2.5) follows if \( \varepsilon \) is small. Moreover, it follows from (2.18) that

\[
|f_1(\gamma_1(t_0), x_2)|^2 \leq C T |q_1 \circ \gamma \circ \gamma_2^{-1}(x_2)|^2 + C T \varepsilon \int_{\gamma_2 \circ \gamma_1^{-1}(x_2)} (|f|^2 + |p_1|^2)(\zeta_1, x_2)d\zeta_1,
\]

for \( x_2 \in (\gamma_2(t_0), \gamma_2(0)) \), which yield the estimate (2.6) by (2.5). A similar argument gives (2.7).

By a similar argument as in the proof of Lemma 2.1, we obtain Lemma 2.2 below.

For \( z = (z_1, z_2) \in \mathbb{R}^2, a_1 > 0, \) and \( a_2 > 0 \) given, let

\[
R(z, a) = (z_1, z_1 + a_1) \times (z_2, z_2 + a_2), \quad a = (a_1, a_2).
\]

(2.22)

Consider the boundary data

\[
f_1(x_1, z_2) = q_1(x_1), \quad f_2(z_1, x_2) = q_2(x_2)
\]

(2.23)

for \( x_1 \in [z_1, z_1 + a_1] \) and \( x_2 \in [z_2, z_2 + a_2] \), respectively.
Lemma 2.2 Let $T > 0$ be given. There is a $\varepsilon_T > 0$ such that if $|z| \leq T$ and $0 < a_i < \varepsilon_T$, then for $q_i \in L^2(z_i, z_i + a_i)$ and $p \in L^2(R(z,a), \mathcal{R}^2)$ given, there exists a unique solution $f = (f_1, f_2) \in L^2(R(z,a), \mathcal{R}^2)$ to problem (2.1) with the data (2.23) satisfying estimate

$$\|f\|^2_{L^2(R(z,a))} \leq C_T(\|q_1\|^2_{L^2(z_1,z_1+a_1)} + \|q_2\|^2_{L^2(z_2,z_2+a_2)} + \|p\|^2_{L^2(R(z,a))}).$$

(2.24)

Proof of Proposition 2.1. Let $T > 0$ be given such that

$$E(\gamma) \subset \{ x \in \mathcal{R}^2 \mid |x| < T \}.$$

Let $\varepsilon_T > 0$ be given small such that Lemmas 2.1-2.2 hold. We divide the curve $\gamma$ into $m$ parts with the points $\tau_0 = 0$, $\tau_0 < \tau_1 < \cdots < \tau_m = t_0$ such that

$$|\gamma(\tau_{i+1}) - \gamma(\tau_i)| = \frac{\varepsilon_T}{2}, \quad 0 \leq i \leq m - 2, \quad |\gamma(t_1) - \gamma(\tau_{m-1})| \leq \frac{\varepsilon_T}{2}.$$

For simplicity, we assume that $m = 3$. The other cases can be treated by a similar argument.

In the case of $m = 3$, we have

$$E(\gamma) = (\cup_{i=0}^2 E_i) \cup (\cup_{i=1}^3 R_i) \quad (2.25)$$

where

$$E_i = \{ x \in E(\gamma) \mid \gamma_1(\tau_i) < x_1 < \gamma_1(\tau_{i+1}), \gamma_2(\tau_{i+1}) < x_2 < \gamma_2(\tau_i) \}, \quad i = 0, 1, 2,$$

$$R_1 = [\gamma_1(\tau_1), \gamma_1(\tau_2)] \times [\gamma_2(\tau_1), \gamma_2(0)], \quad R_2 = [\gamma_1(\tau_2), \gamma_1(t_1)] \times [\gamma_2(\tau_2), \gamma_2(\tau_1)],$$

$$R_3 = [\gamma_1(\tau_2), \gamma_1(t_0)] \times [\gamma_2(\tau_1), \gamma_2(0)].$$

From Lemma 2.1, problem (2.1) admits a unique solution $f^i \in L^2(E_i, \mathcal{R}^2)$ for each $i = 0, 1, 2, 3$, respectively, with the corresponding data and the corresponding estimates. We define $f \in L^2(\cup_{i=0}^2 E_i, \mathcal{R}^2)$ by

$$f(x) = f^i(x) = (f_1^i(x), f_2^i(x)) \quad \text{for} \quad x \in E_i \quad \text{for} \quad i = 0, 1, 2.$$

We extend the domain of $f$ from $\cup_{i=0}^2 E_i$ to $E(\gamma)$ by the following way. By Lemma 2.2, we define $f \in L^2(R_i, \mathcal{R}^2)$ to be the solution $u^i \in L^2(R_i, \mathcal{R}^2)$ to problem (1.2) with the data

$$u_1^i(\gamma_1(\tau_i), x_2) = f_1^{i-1}(\gamma_1(\tau_i), x_2) \quad \text{for} \quad x_2 \in [\gamma_2(\tau_i), \gamma_2(\tau_{i-1})],$$

$$u_2^i(x_1, \gamma_2(\tau_i)) = f_2^i(x_1, \gamma_2(\tau_i)) \quad \text{for} \quad x_1 \in [\gamma_1(\tau_i), \gamma_1(\tau_{i+1})],$$

for $i = 1, 2$, respectively. Then we extend $f$ on $R_3$ to be the solution $u^3$ of (1.2) with the data

$$u_1^3(\gamma_1(\tau_2), x_2) = u_1^1(\gamma_1(\tau_2), x_2) \quad \text{for} \quad x_2 \in [\gamma_2(\tau_1), \gamma_2(0)],$$

$$u_2^3(x_1, \gamma_2(\tau_1)) = f_2^3(x_1, \gamma_2(\tau_1)) \quad \text{for} \quad x_1 \in [\gamma_1(\tau_2), \gamma_1(\tau_{i+1})].$$
\[ u_2^3(x_1, \gamma_2(\tau_2)) = u_2^2(x_1, \gamma_2(\tau_2)) \quad \text{for} \quad x_1 \in [\gamma_1(\tau_2), \gamma_1(t_0)]. \]

To complete the proof, we have to show \( f \in L^2(E(\gamma), \mathbb{R}^2) \). Consider the subregion
\[ \tilde{E} = E_0 \cup E_1 \cup R_1. \]

Since \( |\gamma(\tau_2) - \gamma(0)| \leq \varepsilon_T \), Lemma 2.1 insures that problem (1.2) admits a unique solution \( \tilde{f} \in L^2(\tilde{E}, \mathbb{R}^2) \) with the corresponding data. Then the uniqueness implies that \( f(x) = \tilde{f}(x) \) for \( x \in \tilde{E} \). By a similar argument, we show that \( f \in L^2(E(\gamma), \mathbb{R}^2) \) is a solution to problem (1.2) with the data (2.4).

The estimates (2.5)-(2.7) follow from the ones in Lemmas 2.1 and 2.2. \( \square \)

**Proofs of Propositions 2.2-2.4.** By similar arguments as in the proof of Proposition 2.1, we complete the proofs. The details are omitted. \( \square \)

We shall solve (1.2) locally in asymptotic coordinate systems and then paste the local solutions together. A chart \( \psi(p) = (x_1, x_2) \) on \( M \) is said to be an *asymptotic coordinate system* if
\[ \Pi(\partial x_1, \partial x_1) = \Pi(\partial x_2, \partial x_2) = 0. \] (2.26)

If \( M \) is hyperbolic, an asymptotic coordinate system exists locally([24]).

Let \( p \in M \) be given. Then there is an asymptotic coordinate system \( \psi : \mathcal{N} \rightarrow \mathbb{R}^2 \) with \( \psi(q) = (x_1, x_2) \) such that (2.26) hold for \( q \in \mathcal{N} \), where \( \mathcal{N} \) is a neighbourhood of \( p \). Let
\[ G = \begin{pmatrix} g_{ij}(q) \end{pmatrix}, \quad g_{ij} = \langle \partial x_i, \partial x_j \rangle. \]

Then
\[ \Pi^2(\partial x_1, \partial x_2) = -\kappa \det G. \]

We may assume that \( \Pi(\partial x_1, \partial x_2) > 0 \) and set
\[ \omega = \Pi(\partial x_1, \partial x_2) = \sqrt{-\kappa \det G} \quad \text{for} \quad q \in \mathcal{N}. \] (2.27)

In an asymptotic coordinate system, equation (1.2) takes the form (2.28) below.

**Proposition 2.5** Let \( M \) be a hyperbolic orientated surface and let \( \psi(p) = (x_1, x_2) : \mathcal{N}(\subset M) \rightarrow \mathbb{R}^2 \) be an asymptotic coordinate system on \( M \) with the positive orientation. Then equation (1.2) is equivalent to problem
\[
\begin{cases}
W_{1x_1} = \Gamma_{11}^1 W_1 + \Gamma_{11}^2 W_2 + U_{11}, \\
W_{2x_1} = \Gamma_{22}^1 W_1 + \Gamma_{22}^2 W_2 + U_{22},
\end{cases}
\] (2.28)

where
\[ W_i = \langle W, \partial x_i \rangle, \quad U_{ij} = U(\partial x_i, \partial x_j), \]
and $\Gamma_{ij}^k$ are the Christoffel symbols for $1 \leq i, j, k \leq 2$. Moreover, if $(W_1, W_2)$ solves problem (2.28), then

$$w = \frac{1}{\omega}[U_{12} - \frac{1}{2}(W_{1x_2} + W_{2x_1}) + \Gamma_{12}^1 W_1 + \Gamma_{12}^2 W_2]. \quad (2.29)$$

**Proof** Problem (1.2) is equivalent to

$$\Upsilon(y)(\partial x_i, \partial x_j) = U(\partial x_i, \partial x_j) \quad \text{for} \quad 1 \leq i, j \leq 2.$$ 

Then the equations, $\Upsilon(y)(\partial x_i, \partial x_i) = U(\partial x_i, \partial x_i)$ for $i = 1, 2$, yield problem (2.28) since $\Pi(\partial x_i, \partial x_i) = 0$. In addition, (2.29) follows from the equation $\Upsilon(y)(\partial x_1, \partial x_2) = U(\partial x_1, \partial x_2)$. \hfill $\square$

We also need the following lemmas 2.3-2.5, whose proofs are given in [29].

**Lemma 2.3** ([29]) There is a $\sigma_0 > 0$ such that, for all $p \in S$, there exist asymptotic coordinate systems $\psi : B(p, \sigma_0) \to \mathbb{R}^2$ with $\psi(p) = (0, 0)$, where $B(p, \sigma_0)$ is the geodesic plate in $M$ centered at $p$ with radius $\sigma_0$.

**Lemma 2.4** ([29]) Let $\gamma : [0, a] \to M$ be a regular curve without self intersection points. Then there is a $\sigma_0 > 0$ such that, for all $p \in \{ \gamma(t) \mid t \in (0, a) \}$, $S(p, \sigma_0)$ has at most two intersection points with $\{ \gamma(t) \mid t \in [0, a] \}$, where $S(p, \sigma_0)$ is the geodesic circle centered at $p$ with radius $\sigma_0$. If $p = \gamma(0)$, or $\gamma(a)$, then $S(p, \sigma_0)$ has at most one intersection point with $\{ \gamma(t) \mid t \in [0, a] \}$.

**Lemma 2.5** ([29]) Let $p_0 \in M$ and let $B(p_0, \sigma)$ be the geodesic ball centered at $p_0$ with radius $\sigma > 0$. Let $\gamma : [-a, a] \to B(p_0, \sigma)$ and $\beta : [-b, b] \to B(p_0, \sigma)$ be two noncharacteristic curves of class $C^1$, respectively, with

$$\gamma(0) = \beta(0) = p_0, \quad \Pi(\dot{\gamma}(0), \dot{\beta}(0)) = 0.$$ 

Let $\tilde{\psi} : B(p_0, \sigma) \to \mathbb{R}^2$ be an asymptotic coordinate system. Then there exists an asymptotic coordinate system $\psi : B(p_0, \sigma) \to \mathbb{R}^2$ with $\psi(p_0) = (0, 0)$ such that

$$\psi(\gamma(t)) = (t, -t) \quad \text{for} \quad t \in [-a, a], \quad (2.30)$$

$$\beta_1'(s) > 0, \quad \beta_2'(s) > 0 \quad \text{for} \quad s \in [-b, b], \quad (2.31)$$

where $\psi(\beta(s)) = (\beta_1(s), \beta_2(s))$. Moreover, for $X = X_1 \partial x_1 + X_2 \partial x_2$ with $\Pi(X, X) \neq 0$, we have

$$g(X)Q\nabla \vec{n} X = \begin{cases} X_1 \partial x_1 - X_2 \partial x_2, & X_1 X_2 > 0, \\ -X_1 \partial x_1 + X_2 \partial x_2, & X_1 X_2 < 0, \end{cases} \quad (2.32)$$

where $g(X)$ and $Q$ are given in (1.5) and (1.3), respectively.
Denote
\[ S(0, s_0) = \{ \alpha(t, s) | t \in (0, a), \ s \in (0, s_0) \} \quad \text{for} \quad s_0 \in [0, b]. \quad (2.33) \]
Then \( S = S(0, b). \)

**Lemma 2.6** Let \( S \) be given in (I). There is a \( 0 < \eta \leq b \) such that problem (1.2) admits a unique solution \( y = W + w \) with the data
\[ W|_{(a, \eta)} = (q_1, \phi, q_2) \]
to satisfy
\[ \|W\|_{L^2(S(0,\eta),\mathcal{X})} \leq C(\|U\|_{L^2(S,T^2)} + \|q_1\|_{L^2(0,b)} + \|\phi\|_{L^2(0,a),\mathcal{X}} + \|q_2\|_{L^2(0,b)}). \quad (2.34) \]

**Proof** Let \( \sigma_0 > 0 \) be given small such that the claims in Lemmas 2.3 and 2.4 hold, where \( \gamma(t) = \alpha(t,0) \) in Lemma 2.4. We divide the curve \( \alpha(t,0) \) into \( m \) parts with the points \( \lambda_i = \alpha(t_i,0) \) such that
\[ \lambda_0 = \alpha(0,0), \quad \lambda_m = \alpha(a,0), \quad d(\lambda_i, \lambda_{i+1}) = \frac{\sigma_0}{3}, \quad 0 \leq i \leq m - 2, \quad d(\lambda_{m-1}, \lambda_m) \leq \frac{\sigma_0}{3}, \]
where \( t_0 = 0, \ t_1 > 0, \ t_2 > t_1, \cdots, \) and \( t_m = a > t_{m-1} \). For simplicity, we assume that \( m = 3 \). The other cases can be treated by a similar argument.

We shall construct a local solution in a neighborhood of the curve \( \alpha(\cdot,0) \) by the following steps.

**Step 1.** Let \( \hat{s}_0 > 0 \) be small such that
\[ \alpha(0, s) \in B(\lambda_0, \sigma_0) \quad \text{for} \quad s \in [0, \hat{s}_0]. \]
From Lemma 2.5, there is asymptotic coordinate system \( \psi_0(p) = x : B(\lambda_0, \sigma_0) \rightarrow \mathbb{R}^2 \) with \( \psi_0(\lambda_0) = (0, 0) \) such that
\[ \psi_0(\alpha(t,0)) = (t, -t) \quad \text{for} \quad t \in [0, t_2], \quad (2.35) \]
\[ \beta'_{01}(s) > 0, \quad \beta'_{02}(s) > 0 \quad \text{for all} \quad s \in [0, \hat{s}_0], \quad (2.36) \]
where \( \psi_0(\beta_0(s)) = (\beta_{01}(s), \beta_{02}(s)) \) and \( \beta_0(s) = \alpha(0, s) \). Let \( 0 < s_0 \leq \hat{s}_0 \) be given such that \( \beta_{01}(s_0) + \beta_{02}(s_0) \leq t_2 \). Let \( \gamma_0(t) = \psi_0(\alpha(t,0)) = (t, -t) \). Set
\[ \Xi(\beta_0, \gamma_0) = \{ x \in \mathbb{R}^2 | 0 < x_1 \leq \beta_{01}(s_0), \ -x_1 \leq x_2 < \beta_{02} \circ \beta_{01}^{-1}(x_1) \} \]
\[ \cup \{ x \in \mathbb{R}^2 | \beta_{01}(s_0) < x_1 < t_2, \ -x_1 < x_2 < -x_1 + \beta_{01}(s_0) + \beta_{02}(s_0) \}. \quad (2.37) \]
Next, since \( \alpha_{s}(s, 0) = \beta'_{01}(s) \partial x_1 + \beta'_{02}(s) \partial x_2 \), from (2.32) and (2.36), we have
\[ \varrho(\alpha_{s}) Q \nabla \bar{u} \alpha_{s}(s, 0) = \beta'_{01}(s) \partial x_1 - \beta'_{02}(s) \partial x_2, \]
that is, by (1.4),
\[ \mathcal{T}_2 \alpha_s(s, 0) = \beta_{01}'(s) \partial x_1. \]  
(2.38)

Set
\[ W_i = \langle W, \partial x_i \rangle, \quad U_{ij} = U(\partial x_i, \partial x_j) \quad \text{for} \quad 1 \leq i, j \leq 2, \]
\[ \phi(t) = \phi_1(t) \partial x_1 + \phi_2(t) \partial x_2. \]

From (2.38) the boundary data \( \langle W, \mathcal{T}_2 \alpha_s \rangle = q_1(s) \) and \( W \circ \alpha(t, 0) = \phi(t) \) are equivalent to
\[ W_1 \circ \beta_0(s) = \frac{q_1(s)}{\beta_{01}'(s)}, \quad (W_1, W_2) \circ \gamma_0(t) = \left( \phi_1(t)g_{11} + \phi_2(t)g_{12}, \phi_1(t)g_{12} + \phi_2(t)g_{22} \right). \]  
(2.39)

From Proposition 2.5, solvability of problem (1.2) with \( W|_I = (q_1, q_2) \) on \( S \cap \psi_0^{-1}(\Xi(\beta_0, \gamma_0)) \) is equivalent to that of problem (2.28) over the region \( \Xi_1(\beta_0, \gamma_0) \) with the boundary data (2.39).

By Proposition 2.3, problem (2.28) admits a unique solution \( (W_1, W_2) \in L^2(\Xi(\beta_0, \gamma_0)) \) with the corresponding boundary data (2.39). Thus, we have obtained a solution, denoted by \( y_0 = W^0 + w^0 \tilde{n} \) to problem (1.2) on \( S_0 = S \cap \psi_0^{-1}(\Xi(\beta_0, \gamma_0)) \) with the data
\[ \langle W^0, \mathcal{T}_2 \alpha_s \rangle = q_1(s) \quad \text{for} \quad s \in [0, \hat{s}_0]; \quad W^0 \circ \alpha(t, 0) = \phi \quad \text{for} \quad t \in [0, t_2], \]
where
\[ W^0 = (g^{11}W_1 + g^{12}W_2) \partial x_1 + (g^{12}W_1 + g^{22}W_2) \partial x_2, \quad \left( g^{ij} \right) = G^{-1}, \]
and \( w^0 \) is given by the formula (2.29).

We define a curve on \( S_0 \) by
\[ \beta_1(s) = \psi_0^{-1} \circ \gamma_{t_1}(s) \quad \text{for} \quad s \in [0, s_{t_1}], \]  
(2.40)
where
\[ \gamma_{t_1}(s) = (s + t_1, s - t_1), \quad s_{t_1} = \begin{cases} t_1 & \text{if} \quad t_1 \in (0, \frac{t_2}{2}], \\ t_2 - t_1 & \text{if} \quad t_1 \in (\frac{t_2}{2}, t_2). \end{cases} \]

Then \( \beta_1(s) \) is noncharacteristic and
\[ \Pi(\beta_1(0), \alpha(t_1, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0. \]  
(2.41)

Using the identity (2.18) where \( f = (W_1, W_2), \ p = (U_{11}, U_{22}) \) and \( q = \phi \), we obtain
\[ \| \langle W^0, \mathcal{T}_2 \beta_1' \circ \beta_1 \|_{L^2(0, s_{t_1})} \|_2 \leq C(\| U \|_{L^2(S, T^2)}^2 + \| \phi \|_{L^2(0, a, X)}^2), \]  
(2.42)

Let
\[ \hat{S}_0 = S \cap \psi_0^{-1}(\Xi_1(\beta_0, \gamma_0)) \cap \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \leq 2t_1 \}. \]
It follows from Proposition 2.2 that
\[
\|W^0\|_{L^2(\tilde{s}_0, x)}^2 \leq C(\|U\|_{L^2(S, T^2)}^2 + \|q_1\|_{L^2(0, b)}^2 + \|\phi\|_{L^2(0, a, x)}^2 + \|q_2\|_{L^2(0, b)}^2).
\] (2.43)

**Step 2.** Let the curve $\beta_1$ be given in (2.40). Let $\delta_1 > 0$ be small such that
\[
\beta_1(s) \in B(\lambda_1, \sigma_0) \quad \text{for} \quad s \in [0, \delta_1].
\]
From the noncharacteristicness of $\beta_1(s)$ and the relation (2.41) and Lemma 2.5 again, there exists an asymptotic coordinate system $\psi_1(p) = x : B(\lambda_1, \sigma_0) \to \mathbb{R}^2$ with $\psi_1(\lambda_1) = (0, 0)$ and
\[
\psi_1(\alpha(t + t_1, 0)) = (t, -t) \quad \text{for} \quad t \in [0, t_3 - t_1],
\]
\[
\beta_1(s) > 0, \quad \beta_1''(s) > 0 \quad \text{for} \quad s \in [0, s_1],
\]
where $\psi_1(\beta_1(s)) = (\beta_1(s), \beta_1''(s))$. Since $\psi_1(\beta_1(0)) = (0, 0)$, we take $0 < s_1 \leq \delta_1$ such that
\[
\beta_1(s_1) + \beta_1''(s_1) \leq t_3 - t_1.
\]

By a similar argument in Step 1, we obtain a unique solution $y_1 = W^1 + w^1\tilde{n}$ to problem (1.2) on $S_1$ with the data
\[
\langle W^1, T_2\beta_1 \rangle \circ \beta_1(s) = \langle W^0, T_2\beta_1 \rangle \circ \beta_1(s) \quad \text{for} \quad s \in [0, s_1],
\]
\[
W \circ \alpha(t, 0) = \phi \circ \alpha(t, 0) \quad \text{for} \quad t \in [t_1, t_3],
\]
where
\[
S_1 = S \cap \psi_1^{-1}[\Xi(\beta_1, \gamma_1)], \quad \gamma_1(t) = \psi_1(\alpha(t + t_1, 0)),
\]
y_0 = W^0 + w^0\tilde{n}$ is the solution of (1.2) on $S_0$, given in Step 1, and
\[
\Xi(\beta_1, \gamma_1) = \{ x \in \mathbb{R}^2 | 0 < x_1 < \beta_1(1)(s_1), -x_1 \leq x_2 < \beta_1(1)(1) - x_1 \} \cup \{ x \in \mathbb{R}^2 | \beta_1(1)(s_1) < x_1 < t_3 - t_1, -x_1 < x_2 < -x_1 + \beta_1(1)(s_1) + \beta_1''(s_1) \}.
\]

As in Step 1, we define a curve on $\Omega_1$ by
\[
\beta_2(s) = \psi_1^{-1}(s + t_2 - t_1, s + t_1 - t_2) \quad \text{for} \quad s \in [0, s_{t_2}],
\]
where
\[
s_{t_2} = t_2 - t_1 \quad \text{if} \quad t_2 - t_1 \leq \frac{t_3 - t_1}{2}; \quad s_{t_2} = t_3 - t_2 \quad \text{if} \quad t_2 - t_1 > \frac{t_3 - t_1}{2}.
\]
Then $\beta_2(s)$ is noncharacteristic and
\[
\Pi(\beta_2(0), \alpha_1(t_2, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0,
\] (2.44)
and the following estimate holds
\[
\|W^1, T_2 \beta_2' \circ \beta_2 \|^2_{L^2(0, s_{t_2})} \leq C(||U||^2_{L^2(\mathbb{R}^2)} + ||\phi||^2_{L^2((0,a),x)}).
\]

Let
\[
\tilde{S}_1 = S \cap \psi_1^{-1}[\Xi_1(\beta_1, \gamma_1))] \cap \{x \in \mathbb{R}^2 \mid x_1 - x_2 \leq 2(t_2 - t_1)\}.
\]
As in Step 1, using the estimates in Proposition 2.3, we have
\[
\|W^1\|_{L^2(\tilde{S}_1)}^2 \leq C(||U||_{L^2(\mathbb{R}^2)}^2 + ||q_1||_{L^2(0,0)}^2 + ||\phi||_{L^2(0,a),x}^2 + ||q_2||_{L^2(0,a)}^2). \tag{2.45}
\]

**Step 3.** Let \( \hat{s}_2 > 0 \) be small such that
\[
\beta_2(s), \quad \alpha(a, s) \in B(\lambda_2, \sigma_0) \quad \text{for} \quad s \in [0, \hat{s}_2].
\]

Let \( \psi_2(p) = x : B(\lambda_2, \sigma_0) \to \mathbb{R}^2 \) be an asymptotic coordinate system with \( \psi_2(\lambda_2) = (0,0) \),
\[
\psi_2(\alpha(t + t_2, 0)) = (t, -t) \quad \text{for} \quad t \in [0, a - t_2], \tag{2.46}
\]
and
\[
\beta_{21}(s) > 0, \quad \beta_{22}(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2],
\]
where \( \psi_2(\beta_2(s)) = (\beta_{21}(s), \beta_{22}(s)) \). Next, we prove that
\[
\beta_{31}(s) > 0, \quad \beta_{32}(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2], \tag{2.47}
\]
where \( \psi_2(\beta_3(s)) = (\beta_{31}(s), \beta_{32}(s)) \) and \( \beta_3(s) = \alpha(a, s) \), by contradiction. Since \( \beta_3(s) \) is noncharacteristic, using (2.46) and the assumption \( \Pi(\alpha_1(a, 0), \beta_3'(0)) = 0 \), we have
\[
\beta_{31}'(0) = \beta_{32}'(0); \quad \text{thus} \quad \beta_{31}'(s)\beta_{32}'(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2].
\]

Let
\[
z(t, s) = \alpha_1(t, s) + \alpha_2(t, s), \quad \psi_2(\alpha(t + t_2, s)) = (\alpha_1(t, s), \alpha_2(t, s)).
\]
Let (2.47) be not true, that is, \( \beta_{31}'(s) < 0, \beta_{32}'(s) < 0 \) for \( s \in [0, \hat{s}_2] \). Thus
\[
z(0, s) = \beta_{21}(s) + \beta_{22}(s) > \beta_{21}(0) + \beta_{22}(0) = 0 \quad \text{for} \quad s \in (0, \hat{s}_2),
\]
\[
z(a, s) = \beta_{31}(s) + \beta_{32}(s) < \beta_{31}(0) + \beta_{32}(0) = 0 \quad \text{for} \quad s \in (0, \hat{s}_2).
\]
Let \( t(s) \in (0, a - t_2) \) be such that
\[
\alpha_1(t(s), s) + \alpha_2(t(s), s) = 0 \quad \text{for} \quad s \in (0, \hat{s}). \tag{2.48}
\]
Since \( \alpha_1t(0, 0) = 1 \) and \( \alpha(t + t_2, s) \) are noncharacteristic for all \( s \in [0, \hat{s}] \), we have \( \alpha_{1t}(t, s) > 0 \)
and
\[
0 < \alpha_1(0, s) < \alpha_1(t(s), s) < \alpha_1(a - t_2, s) = \beta_{31}(s) < \beta_{31}(0) = a - t_2.
\]

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Thus, equality (2.48) means that \( \alpha(\alpha_1(t(s),s),0) = \alpha(t(s),s) \), which is a contradiction since \( \alpha : [0,a] \times [a,b] \to M \) is an imbedding map.

Let \( 0 < s_2 \leq \hat{s}_2 \) be such that

\[
\beta_{21}(s_2) + \beta_{22}(s_2) \leq a - t_2.
\]

Then take \( 0 < s_3 < b \) such that

\[
\beta_{21}(s_2) + \beta_{22}(s_2) = \beta_{31}(s_3) + \beta_{32}(s_3).
\]

Set

\[
\Phi(\beta_2, \beta_3) = \{ x \in \mathbb{R}^2 | 0 < x_1 < \beta_{21}(s_2), -x_1 < x_2 < \beta_{22} \circ \beta_{21}^{-1}(x_1) \}
\]

\[
\cup \{ x \in \mathbb{R}^2 | \beta_{21}(s_2) \leq x_1 < a - t_2, -x_1 \leq x_2 < -x_1 + \beta_{21}(s_2) + \beta_{22}(s_2) \}
\]

\[
\cup \{ x \in \mathbb{R}^2 | a - t_2 < x_1 < \beta_{31}(s_3), \beta_{32} \circ \beta_{31}^{-1}(x_1) < x_2 < -x_1 + \beta_{31}(s_3) + \beta_{32}(s_3) \}.
\]

We again let

\[
W_i = (W, \partial x_i), \quad U_{ij} = U(\partial x_i, \partial x_j),
\]

where \( x = \psi_2 \). We consider the boundary data of \((W_1, W_2)\) on \( \Phi(\beta_2, \beta_3) \). We set

\[
W_1 \circ \beta_2(s) = \frac{1}{\beta_{21}(s)} (W_1, T_2 \beta'_2) \circ \beta_2(s), \quad (W_1, W_2)(t, -t) = (\phi_1(t) g_{11} + \phi_2(t) g_{12}, \phi_1(t) g_{12} + \phi_2(t) g_{22}),
\]

where \( \phi = \phi_1 \partial x_1 + \phi_2 \partial x_2 \). Moreover, it follows from (2.47) and (2.32) that the boundary data \( (W, T_1 \alpha_s) \circ \alpha(a, s) = q_2(s) \) yields

\[
W_2 \circ \beta_3(s) = \frac{q_2(s)}{\beta_{32}^2(s)}.
\]

We apply Proposition 2.4 with \( f = (W_1, W_2) \) to obtain a solution \( y^2 = W^2 + w^2 \tilde{n} \) to problem (1.2) with the corresponding boundary data which satisfies

\[
\|W^2\|^2_{L^2(\hat{S}_2, \mathcal{X})} \leq C(\|U\|^2_{L^2(S, T^2)} + \|q_1\|^2_{L^2(0, \tilde{b})} + \|\phi\|^2_{L^2(0, \tilde{b}), \mathcal{X}} + \|q_2\|^2_{L^2(0, \tilde{b})}). \tag{2.49}
\]

**Step 4.** We define

\[
W = W^i, \quad w = w^i \quad \text{for} \quad p \in \hat{S}_i \quad \text{for} \quad i = 0, 1, 2.
\]

Let \( \eta > 0 \) be small such that

\[
\alpha(t, s) \in \hat{S}_0 \cup \hat{S}_1 \cup \hat{S}_2 \quad \text{for} \quad (t, s) \in (0, a) \times (0, \eta).
\]

Then \( y = W + w \tilde{n} \) on \( S(0, \eta) \) will be a solution to (1.2) with the corresponding boundary data if we show that

\[
W^0(p) = W^1(p) \quad \text{for} \quad p \in S_0 \cap S_1; \quad W^1(p) = W^2(p) \quad \text{for} \quad p \in S_1 \cap S_2. \tag{2.50}
\]
Since 
\[ \langle W^1, T_2 \beta'_1 \rangle \circ \beta_1(s) = \langle W^0, T_2 \beta'_1 \rangle \circ \beta_1(s) \quad \text{for} \quad s \in [0, s_1], \]
\[ W^1 \circ \alpha(0, t) = \phi = W^0 \circ \alpha(0, t) \quad \text{for} \quad t \in [t_1, t_2], \]
from the uniqueness in Proposition 2.3, we have 
\[ W^0 \circ \psi_0^{-1}(x) = W^1 \circ \psi_1^{-1}(x) \quad \text{for} \quad x \in \Xi(\beta_0, \gamma_0) \cap \Xi(\beta_1, \gamma_1), \]
which yields the first identity in (2.50). A similar argument shows that the second identity in (2.50) is true.

Finally, the estimate (2.34) follows from (2.43), (2.45), and (2.49). \(\square\)

Proof of Theorem 1.1
(a) Let \( S \) be given in (I).

Let \( \mathcal{N} \) be the set of all \( 0 < \eta \leq b \) such that the claims in Lemma 2.6 hold. We shall prove
\[ b \in \mathcal{N}. \]

Let \( \eta_0 = \sup_{\eta \in \mathcal{N}} \eta \). Then \( 0 < \eta_0 \leq b \). Thus there is a unique solution \( y = W + w \) on \( S(0, \eta_0) \) to (1.2) with the data \( W(t(a, \eta_0)) = (q_1, \phi, q_2) \).

Step 1. We claim \( \eta_0 = b \). By contradiction. We assume that \( \eta_0 < b \).

Let \( \sigma_0 > 0 \) be given small such that the claims in Lemmas 2.3 and 2.4 hold, where \( \gamma(t) = \alpha(t, \eta_0) \) in Lemma 2.4. As in the proof of Lemma 2.6, we divide the curve \( \alpha(t, \eta_0) \) into \( m \) parts with the points \( \lambda_i = \alpha(t_i, \eta_0) \) such that
\[ \lambda_0 = \alpha(0, \eta_0), \quad \lambda_m = \alpha(a, \eta_0), \quad d(\lambda_i, \lambda_{i+1}) = \frac{\sigma_0}{3}, \quad 0 \leq i \leq m - 2, \quad d(\lambda_{m-1}, \lambda_m) \leq \frac{\sigma_0}{3}, \]
where \( t_0 = 0, \ t_1 > 0, \ t_2 > t_1, \ldots, \ t_m = a > t_{m-1} \). Again, we shall treat the case of \( m = 3 \) for simplicity.

Let \( \psi_0(p) = x : B(\lambda_0, \sigma_0) \to \mathbb{R}^2 \) be an asymptotic coordinate system with \( \psi_0(\lambda_0) = (0, 0) \) such that
\[ \psi_0(\alpha(t, \eta_0)) = (t, -t) \quad \text{for} \quad t \in [0, t_2], \]
\[ \zeta_1'(s) > 0, \quad \zeta_2'(s) > 0 \quad \text{for} \quad s \in [\eta_0 - \varepsilon_0, \eta_0 + \varepsilon_0], \]
where
\[ \psi_0(\alpha(0, s)) = (\zeta_1(s), \zeta_2(s)) \quad \text{for} \quad s \in [\eta_0 - \varepsilon_0, \eta_0 + \varepsilon_0]. \]

For \( \varepsilon_0 > 0 \), let
\[ \beta_0(s) = \psi_0(\alpha(0, s + \eta_0 - \varepsilon_0)) = (\beta_{01}(s), \beta_{02}(s)) \quad \text{for} \quad s \in [0, 2\varepsilon_0], \]
\[ \gamma_0(t) = \psi(\alpha(t, \eta_0 - \varepsilon_0)) = (\gamma_{01}(t), \gamma_{02}(t)) \quad \text{for} \quad t \in [0, t_2], \]
where $\beta_{0i}(s) = \xi(s + \eta_0 - \varepsilon_0)$ for $i = 1, 2$. Let $\varepsilon_0 > 0$ be given small such that there is $t_2^0 > 0$ satisfying

$$\gamma_{01}(t_2^0) = t_2, \quad \beta_{02}(0) + t_2 \geq 0.$$ 

Set

$$\Xi(\beta_0, \gamma_0) = \{ x \mid \beta_{01}(0) < x_1 \leq 0, \gamma_{02} \circ \gamma_{01}^{-1}(x_1) < x_2 < \beta_{02} \circ \beta_{01}^{-1}(x_1) \}$$

$$\cup \{ x \mid 0 < x_1 < t_2, \gamma_{02} \circ \gamma_{01}^{-1}(x_1) < x_2 < -x_1 \},$$

$$S_0 = S \cap \psi_0^{-1}(\Xi(\beta_0, \gamma_0)) \cap \{ x \mid x_1 - x_2 \leq 2t_1 \}.$$ 

From Proposition 2.3, the solution $y = W + w\tilde{n}$ can be extended on to $S_0$ and the following estimates

$$\|W\|_{L^2(S_0, X)}^2 \leq C(\|U\|_{L^2(S, T^2)}^2 + \|q_1\|_{L^2(0, b)}^2 + \|\phi\|_{L^2(0, a), X}^2 + \|q_2\|_{L^2(0, b)}^2)$$

hold.

By a similar argument as in Steps 2 and 3 in the proof of Lemma 2.6, we proceed to obtain $0 < \varepsilon_2 \leq \varepsilon_1 \leq \varepsilon_0$ such that the solution $y = W + w\tilde{n}$ is extended to the domain $S(0, \eta_0 + \varepsilon_2)$ and the estimates

$$\|W\|_{L^2(S(0, \eta_0 - \varepsilon_2, \eta_0 + \varepsilon_2), X)}^2 \leq C(\|U\|_{L^2(S, T^2)}^2 + \|q_1\|_{L^2(0, b)}^2 + \|\phi\|_{L^2(0, a), X}^2 + \|q_2\|_{L^2(0, b)}^2)$$

holds, that is, $\eta_0 + \varepsilon_2 \in \mathcal{N}$, contradicting with the definition of $\eta_0$.

**Step 2.** Let $\varepsilon > 0$ be given small. We extend $q \in L^2(0, b)$ to $\hat{q} \in L^2(0, b + \varepsilon)$ such that

$$\|\hat{q}\|_{L^2(0, b + \varepsilon)} \leq C\|q\|_{L^2(0, b)} \quad \text{for} \quad q \in L^2(0, b).$$

By a similar argument as in Step 1, we show that $b \in \mathcal{N}$.

**(b) Let $S$ be given in (II).**

If $\alpha(0, \cdot)$ is a closed curve, by similar arguments as in the proof of (a), we solve problem (1.2) locally and paste the local solutions together to obtain the desired solution. The details are omitted.

Next, let us assume that $\alpha(0, \cdot) = p_0 \in M$ is one point. Let $\sigma_0 > 0$ be given small such that $B(p_0, \sigma_0) \subset S$ and there is an asymptotic coordinate system $\psi = x : B(p_0, \sigma_0) \to \mathbb{R}^2$. Set

$$\beta(t, s) = \psi^{-1}\left(\varepsilon(1 - t)(1 - s) + ts\varepsilon, (1 - t)s\varepsilon + t\varepsilon(1 - s)\right) \quad \text{for} \quad (t, s) \in (0, 1) \times (0, 1).$$

It is easy to check that for $\varepsilon > 0$ small,

$$\mathcal{N}_\varepsilon = \{ \alpha(t, s) \mid (t, s) \in (0, 1) \times (0, 1) \} \subset B(p_0, \sigma_0)$$

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is a neighbourhood of \( p_0 \), which is a non-characteristic region.

We fixed \( t_0 > 0 \) small such that

\[
\alpha(t, \theta) \in \mathcal{N}_\varepsilon \text{ for } (t, \theta) \in [0, t_0] \times (0, 2\pi].
\]

We first solve problem (1.2) on the region

\[
S_0 = \{ \alpha(t, \theta) \mid (t, \theta) \in [t_0, a] \times (0, 2\pi] \}
\]

with the boundary data \( W |_{(II)} = \phi \) to have the solution \( y = W + w\bar{n} \). Then we solve problem (1.2) on the region \( \mathcal{N}_\varepsilon \) with the corresponding the boundary data which is from \( W \) to have the solution \( y^1 \) on \( \mathcal{N}_\varepsilon \). Finally, we paste \( y \) and \( y^1 \) together to obtain the desired solution.

\[
\Box
\]

**Proof of Theorem 1.2** The estimate (1.13) follows from Corollary 1.1 immediately.

Next, we prove the estimate (1.14). It follows from the identity (2.52) below that

\[
\int_S w^2 |\Pi|^2 dg = \int_S w\langle \Pi, w\Pi \rangle dg = \int_S [w\langle \Pi, \Upsilon(y) \rangle - w\langle \Pi, DW \rangle] dg
\]

\[
= -\int_{\partial S} w\Pi(W, \nu) d\partial S + \int_S [w\langle \Pi, \Upsilon(y) \rangle + \Pi(W, Dw) + w\text{tr}_g \iota(W) D\Pi] dg
\]

\[
\leq C [\|w\|_{L^2(S)} (\|\Upsilon(y)\|_{L^2(S)} + \|W\|_{L^2(S)}) + \|Dw\|_{L^2(S)} \|W\|_{L^2(S)}].
\]

Thus the estimate (1.14) follows from (2.51) and (1.13).

\[
\Box
\]

**Lemma 2.7** For \((W, w) \in H^1(S, \mathcal{X}) \times H^1(S)\), we have

\[
\text{div}_g [w \iota(W)\Pi] = \Pi(W, Dw) + w\text{tr}_g \iota(W)D\Pi + w\langle \Pi, Dw \rangle \quad \text{for } \quad p \in S. \tag{2.52}
\]

**Proof** Let \( p \) be given. Let \( E_1, E_2 \) be a frame field normal at \( p \) such that

\[
\nabla_{E_i(p)} \bar{n} = \lambda_i E_i(p), \quad D_{E_i(p)} E_j = 0 \quad \text{for } \quad 1 \leq i, j \leq 2.
\]

Thus we have at \( p \)

\[
\text{div}_g [w \iota(W)\Pi] = E_1[w\Pi(W, E_1)] + E_2[w\Pi(W, E_2)]
\]

\[
= E_1(w)\Pi(W, E_1) + E_2(w)\Pi(W, E_2) + wD\Pi(W, E_1, E_1) + wD\Pi(W, E_2, E_2)
\]

\[
+ w\Pi(D_{E_1} W, E_1) + w\Pi(D_{E_2} W, E_2)
\]

\[
= \Pi(W, Dw) + w\text{tr}_g \iota(W)D\Pi + w\lambda_1 \langle D_{E_1} W, E_1 \rangle + w\lambda_2 \langle D_{E_2} W, E_2 \rangle
\]

\[
= \Pi(W, Dw) + w\text{tr}_g \iota(W)D\Pi + w\langle \Pi, Dw \rangle.
\]

\[
\Box
\]
3 Optimal Exponential

We need an interpolation inequality from [11]. This result is also established in [30] where the existence of a local principal coordinate is not assumed but the Dirichlet boundary conditions are needed to hold on the thin faces of the shell.

**Theorem 3.1** ([11]) Suppose that for each \( p \in \overline{S} \) there exists locally a principal coordinate at \( p \). Then there are \( C > 0, h_0 > 0 \), independent of \( h > 0 \), such that

\[
\|\nabla y\|^2 \leq C\left(\frac{\|\langle y, \overline{n}\rangle\|\|\text{sym} \nabla y\|}{h} + \|y\|^2 + \|\text{sym} \nabla y\|^2\right)
\]

(3.1)

for all \( h \in (0, h_0) \) and \( y \in H^1(\Omega, \mathbb{R}^3) \).

From [30, Proposition 2.1], if \( \kappa(p) < 0 \), a local principal coordinate exists at \( p \). Thus, the estimates (3.1) hold when \( S \) is a non-characteristic region.

By defining \( \nabla \overline{n}\overline{n} = 0 \), we introduce an 2-order tensor \( p(y) \) on \( \mathbb{R}^3 \) by

\[
p(y)(\overline{\alpha}, \overline{\beta}) = \langle \nabla \nabla \overline{n}\overline{\alpha} y, \overline{\beta} \rangle \quad \text{for} \quad \overline{\alpha}, \overline{\beta} \in \mathbb{R}^3.
\]

(3.2)

Moreover, we need the following lemma from [30].

**Lemma 3.1** ([30]) Let \( y = W + w\overline{n} \in H^2(\Omega, \mathbb{R}^3) \) be given. Then

\[
|\nabla y + tp(y)|^2 = |DW + w\Pi|^2 + |Dw - i(W)\Pi|^2 + |W_t|^2 + w_t^2,
\]

(3.3)

\[
|\text{sym} \nabla y + t \text{sym} p(y)|^2 = |\Upsilon(y)|^2 + \frac{1}{4}|X(y)|^2 + w_t^2 \quad \text{for} \quad (p, t) \in S \times (-h/2, h/2),
\]

(3.4)

where

\[
\Upsilon(y) = \text{sym} DW + w\Pi, \quad X(y) = Dw - i(W)\Pi + W_t.
\]

**Proof of Theorem 1.3** It follows from (3.2)-(3.4) that

\[
(1 - Ch)^2|\nabla y|^2 \leq |\nabla y + tp(y)|^2 \leq (1 + Ch)^2|\nabla y|^2,
\]

\[
(1 - Ch)^2|\text{sym} \nabla y|^2 \leq |\text{sym} \nabla y + t \text{sym} p(y)|^2 \leq (1 + Ch)^2|\text{sym} \nabla y|^2.
\]

From Theorem 1.2, we thus have

\[
\|w\|^2_{L^2(S)} \leq C[(|DW - i(W)\Pi|_{L^2(S)} + |W|_{L^2(S)})\|\Upsilon(y)\|_{L^2(S)} + \|\Upsilon(y)\|^2_{L^2(S)}]
\]

\[
\leq C(\|\nabla y\|_{L^2(S)}\|\Upsilon(y)\|_{L^2(S)} + \|\Upsilon(y)\|^2_{L^2(S)}).
\]

(3.5)

We integrate the above inequality in \( t \in (-h/2, h/2) \) to have

\[
\|w\| \leq C(\sqrt{\|\nabla y\|\|\text{sym} \nabla y\| + \|\text{sym} \nabla y\|}).
\]
Thus, by Holder’s inequality, we obtain

\[ \frac{1}{h} \|w\| \|\text{sym} \nabla y\| \leq C \left( \varepsilon \|\nabla y\|^2 \right)^{1/4} \left( \frac{\|\text{sym} \nabla y\|^2}{\varepsilon^{1/3} h^{4/3}} \right)^{3/4} + C \left( \frac{\|\text{sym} \nabla y\|^2}{h} \right) \]

\[ \leq C \left( \frac{\varepsilon \|\nabla y\|^2}{4} + \frac{3}{4} \frac{\|\text{sym} \nabla y\|^2}{\varepsilon^{1/3} h^{4/3}} \right) + C \left( \frac{\|\text{sym} \nabla y\|^2}{h} \right) \]

\[ \leq C \varepsilon \|\nabla y\|^2 + C \varepsilon \frac{\|\text{sym} \nabla y\|^2}{h^{4/3}}, \quad (3.6) \]

for \( \varepsilon > 0 \) small.

In addition, from (3.5) and (1.13), we have

\[ \|y\|^2 \leq C \varepsilon \|\nabla y\|^2 + C \varepsilon \|\text{sym} \nabla y\|^2, \quad (3.7) \]

for \( \varepsilon > 0 \) small.

Inserting (3.6) and (3.7) into (3.1), we obtain (1.15).

To complete the proof, we need to construct an Ansatz. From [30, Proposition 2.1], there is a local principal coordinate on \( S \). In such a local principal coordinate, the Ansatz has been given in [10]. □

Compliance with Ethical Standards
Conflict of Interest: The author declares that there is no conflict of interest.
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