The solution of unconfined water seepage problem in saturated-unsaturated soil using Bathe algorithm and Signorini condition

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Abstract: Bathe algorithm converts problem of determining inner infiltration face into a material nonlinearity problem, and Signorini condition coverts outflow boundary into conventional water boundary. Because an unsaturated seepage problem is also a nonlinear flow problem, through the joint application of this 2 methods, solving variational inequalities is avoided. In conventional material nonlinearity finite element framework with small modification, using NR algorithm, the solving of unconfined water seepage problems including unsteady and unsaturated water seepage problems is realized. The followings are discussed: (1) the improvement of convergence of Bathe method and its generalized form in a three-dimensional model; (2) use Switching Algorithm of Signorini condition to solve water seepage problems; (3) the under-relaxation methods of improving the convergence and mass conservation when solving unsaturated and unsteady water seepage problems. At last, through classic calculation examples, the calculating accuracy and application fields of the proposed methods are evaluated.

1. Introduction
Currently, unconfined water seepage problem can be solved using either saturated soil water seepage theory or unsaturated soil water seepage theory. Saturated soil water seepage theory is concluded in unsaturated soil water seepage theory, and the governing equations of them are the same[1-2]. However, unsaturated soil water seepage theory thinks that the permeability coefficient of soil depends on not only soil properties, but also pressure head. When unsaturated soil is used to solve water seepage problem, unsaturated soil zone and saturated soil zone are not necessary to be clarified, and this integrates the solving process of saturated and unsaturated water seepage problem, and this method has wide application prospects. But the relation between permeability coefficient and pressure head of unsaturated soil is difficult to acquired, and generally in engineering, the focus is the flow pattern and pressure distribution in saturated soil zone. So in engineering, the relation between permeability coefficient and pressure head is usually simplified into Heaviside Step Function. This simplifying method ensures the solution accuracy of saturated soil zone, at the expense of the solution accuracy of unsaturated soil zone. Bathe method uses this kind of simplification[3].

Bathe method is a kind of fixed-grid method, and the finite-element meshes remains unchanged during solving process. Many other methods of solving unconfined saturated water seepage problem are belong to fixed-grid method, such as Desai’s residual flow method[4], Yutian Zhang’s initial flow
method\textsuperscript{[5]}, as well as famous Lacy method\textsuperscript{[6]}, and so on. Though the derivation basis of these methods is not the same, they eventually need solve a problem caused by Heaviside Function and similar to the rigid-plastic material nonlinearity problem, and uses the method of Newton-Raphson(NR) of classic material nonlinearity finite element framework to solve. Of course, because of the properties of Heaviside Function, it can meet convergence problems when these methods are used to solve unconfined saturated water seepage problem. Yuan Wang\textsuperscript{[7]} introduces element adjustment parameters $\varepsilon_1$ and $\varepsilon_2$ into Heaviside Function to modify it, and improves the convergence of these methods.

Correspond with the fixed-grid method, another kind of method, that can be used to solve unconfined saturated water seepage problem, is mesh(node) adjustment method. This method is always used. But when solving non-homogeneous soil water seepage problem, this method often fail to solve because of difficult to adjust the nodes at boundaries. Besides, because of the change of meshes, it is inconvenience to solve rock-soil coupling problem. Based on this, there is a trend of replacing this method with fixed-grid method.

No matter what the method is, another problem will be inevitably involved: the determination of outflow boundary. When modeling, the exact range of outflow boundary is unknown and need to be determined by iterative calculation. There are many different solutions on this problem. For example, A. Larabi and F. D. E. Smedt\textsuperscript{[8]} set the outflow boundary as constant water boundary, if calculated flow of a node is positive (water soaked into the soil), then set this node as impervious boundary at the beginning of next-step calculation, until convergence. But practices show that the convergence of this method is not good, and oscillation is easy to appear. The method of C. S. Desai and G. C. Li\textsuperscript{[9]} is that a layer of high-permeable elements is attached on the possible outflow boundary. But Hong Zheng\textsuperscript{[10]} points out that possible outflow boundary should satisfy the Signorini condition, and he establishes variational inequality and uses non-fully complementary algorithm to solve. When using Signorini condition to solve unconfined saturated water seepage problem, the outflow boundary and impervious boundary are unified treated, and this can easily solve the problem of determining the outflow boundary. But H. Zheng’s method will eventually solve a set of discrete variational inequalities, and this is quite difficult to solve. Actually, Signorini condition is frequently encountered boundary conditions in contact problems, and there are relatively mature iterative algorithms in conventional finite element solving methods. For example, M. Aitchison and M. W. Poole\textsuperscript{[11]} proposed Switching Algorithm, and this algorithm conveniently realized the solution of Signorini condition, with almost no change in the conventional finite element solution framework.

Based on Bathe algorithm, this paper proposes an exchange algorithm for water seepage problem, conducts numerical simulation with Signorini condition, and determines water seepage outflow boundary. Through joint application of Bathe algorithm and Signorini condition, the unconfined water seepage problem in saturated soil will be converted into conventional nonlinear constitutive problem. And use conventional NR algorithm to solve unconfined water seepage problem, including unsteady water seepage flow and unsaturated water seepage flow problems.

2. Governing equations of water seepage problem and its finite element discrete form

During the analysis process, assume that the total stress and pore pressure keeps constant, so the governing equations of water seepage problem\textsuperscript{[1]} is:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial h}{\partial z}\right) = \gamma \omega h \frac{\partial h}{\partial t}$$

(1)

where: $h$ is waterhead; $k_x$, $k_y$ and $k_z$ is respectively $x$, $y$ and $z$ direction component of permeability coefficient; $\gamma$ is specific weight of water; $\omega$ is the slope of the curve of volumetric soil water content $\Theta$ along with the change of pore pressure $u$, $\omega = d\Theta/du$; $t$ is time.

After backward difference in time, the finite element discrete form of Equation (1) based on the waterhead (hereinafter referred to h-type) is:
Permeability coefficient matrix of soil; $[B]^i$ is waterhead gradient interpolation matrix of elements; $[N]$ is interpolation function matrix of elements; $[h]^i$, $([h]^i)_{t+\Delta t}$ is respectively column matrix of unknown nodes’ waterhead at the time of $t$ and $\Delta t$; $([Q]^i)_{t+\Delta t}$ is column matrix of nodes’ flow at the time of $t+\Delta t$; $J_{ij}$ is norm of Jacobi matrix; $W_j$ is integral point weights; $\lambda = \gamma_w m_w$; $n_e$ is the total number of elements in model; $n_g$ is the number of integration points of an element; $i$ and $j$ is respectively counter of elements and integration points.

When water is confined, $[T]$, $[C]$ and $\lambda$ are all constants. When a non-saturated water seepage flow, they are not constants, Equation (2a) will be a set of nonlinear equations and need iteration to solve it. For unconfined unsaturated water seepage problem, after step nonlinear permeability coefficient matrix $[K]$ is introduced into Equation (2a), Bathe algorithm is acquired.

3. Bathe algorithm and Signorini condition

3.1. Bathe algorithm

3.1.1 Saturated water seepage flow. A typical unconfined saturated water seepage problem is shown in Fig.1. Heaviside step function $H(h - z)$ is introduced into Bathe algorithm as relative permeability coefficient $k_r$. According to Equation (3a), the saturated permeability coefficient $k_s$ is adjusted and then get permeability coefficient $k$. Solving domain is expanded to the whole domain ABCDA. Permeability coefficient of saturation zone is $k_s$, and permeability coefficient of unsaturated zone is 0. The assumptions of unsaturated zone $\Omega_d$ being anhydrous dry is:

![Fig.1 Sketch of water seepage in earth dam](image)
\[ k = k_s (h - z) k_s = H(h - z) k_s \]  
\[ k_s (h - z) = H(h - z) = \begin{cases} 
1 & (h \geq z) \\
\varepsilon_0 & (h < z) 
\end{cases} \]  

(3a)  

(3b)

where: \( z \) is statical waterhead; \( \varepsilon_0 \) is a small number, theoretically 0, considering the stability of the numerical calculation, take \( \varepsilon_0 \) value as 0.001; \( h \) is the height.

Because Equation (3a) and (3b) contains Heaviside step function \( H(h - z) \), this will result to oscillation during solving process and difficult to converge. In order to improve its convergence and accuracy of numerical integration, Yuan Wang\(^7\) introduces element adjustment parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) into Heaviside Function to modify it, and convert Equation (3), which is similar to rigid-plastic problem, to the problem which is similar to ideal elastoplastic problem (as shown in Fig.2).

![Fig.2 Relation between relative permeability and pressure waterhead](image)

If pressure waterhead of a integration point of a element is less than \( \varepsilon_1 \), then this integration point make a minimum contribution to the flow of the node ( \( k_r = \varepsilon_0 \) ). If pressure waterhead of a integration point of a element is greater than \( \varepsilon_2 \), then this integration point make a maximum contribution to the flow of the node ( \( k_r = 1 \) ). And if pressure waterhead is between \( \varepsilon_1 \) and \( \varepsilon_2 \), then linear interpolation. After adjustment according to this method, Equation (3b) becomes:

\[ k_r = \begin{cases} 
1 & (h - z \geq \varepsilon_2) \\
\frac{h - z - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} & (\varepsilon_1 < h - z < \varepsilon_2) \\
\varepsilon_0 & (h - z \leq \varepsilon_1) 
\end{cases} \]  

(4)

The key of Equation (4) is the value of \( \varepsilon_1 \) and \( \varepsilon_2 \). S. J. Lacy and H. Prevost\(^6\) provides the method of taking the value of \( \varepsilon_1 \) for rectangular meshes of plane problem, that is, the value of \( \varepsilon_1 \) is the the maximum value of pressure waterhead difference between the lowest node and the lowest integration point of a rectangular element. The method of taking the value of \( \varepsilon_2 \) is similar to \( \varepsilon_1 \)\(^7\), that is, the value of \( \varepsilon_2 \) is the the maximum value of pressure waterhead difference between the highest node and the highest integration point of a rectangular element. Extend this method to other elements, including 3D elements, and the results are shown in Fig.3. Noting that: the value of \( \varepsilon_1 \) and \( \varepsilon_2 \) is proportional to \( l \) (when 2D problems, \( l \) is the position along the another direction; when 3D
problems, $l$ is vector sum of two position vectors along the two other directions), so for 3D elements.

\[
\begin{align*}
\begin{array}{c}
\text{(a)}\; \varepsilon_1 = -\max(f(\alpha)) = -0.5h(\sqrt{1+1/h} + 1) \\
\text{(b)}\; \varepsilon_2 = -\max(f(\alpha)) = -0.5h(\sqrt{1+1/h} + 1)
\end{array}
\end{align*}
\]

Fig.3 Values of correction coefficients $\varepsilon_1$ and $\varepsilon_2$.

3.1.2 Unsaturated water seepage flow. Permeability coefficient of unsaturated water seepage flow depends on effective soil water saturation, and is a highly nonlinear problem. Frequently-used models of permeability coefficient for unsaturated soil are Van Genuchten model$^{[12]}$, Fredlund model$^{[13]}$, Gardner model$^{[14]}$, and so on. Though Gardner model is not necessarily suitable for the actual soil, it makes Equation (1) easy to linearize and easy to calculate analytic solutions.

Because Bathe algorithm is realize through nonlinear constitutive of permeability coefficient (Equation (3) or (4)), converting Equation (3) or (4) into Van Genuchten model, Fredlund model or other nonlinear constitutive model of permeability coefficient can conveniently realize the simulation of unsaturated water seepage flow. Furthermore, because the nonlinearity of constitutive model of unsaturated water seepage flow is lower than Equation (3), its convergence is generally better than Equation (3) or (4).

3.2. Signorini condition of outflow boundary and its realization
Bathe algorithm effectively converts problem of determining inner infiltration face into material nonlinearity problem. However, there is a problem to be solved, that is, the determination of outflow point G (as shown in Fig.1). GF is outflow boundary, and GCBE is impervious boundary.

In Bathe algorithm, outflow point G can be obtained using extrapolation method, but this is quite difficult$^{[10]}$, especially on 3D problems. Taking advantage of Signorini condition, outflow boundary GF and impervious boundary GCBE are treated together in this paper. And it isn’t difficult to find that, boundary FGCBE satisfies the following Signorini boundary boundary:

\[
\begin{align*}
\begin{array}{c}
h - y \leq 0, q_n \leq 0 \\
(h - y)q_n = 0
\end{array}
\end{align*}
\]

3.3. Switching Algorithm of Signorini condition
In this paper, Switching Algorithm (SA) proposed by J. M. Aitchison and M. W. Poole$^{[11]}$. The advantage of SA is that Signorini boundary boundary can be used to numerical simulation at the expense of just small changes, and conventional finite element solution framework does not have to change. The details of SA are the followings:

1) Initialize possible outflow boundary as impervious boundary, and start iteration calculation.

2) During iteration calculation, according to the results of the previous iteration, possible outflow
boundary can be determined in the next iteration as followings:

(1) If the boundary of a node, which is on the possible outflow boundary, is impervious boundary in previous iteration, but in current iteration, the calculated pressure waterhead is greater than 0 (its initial value is 0.001), then set this boundary as constant water boundary, and the value of waterhead is statical head.

(2) If the boundary of a node, which is on the possible outflow boundary, is set as constant water boundary in previous iteration, but calculated water seepage flow is greater than 0 in current iteration, then set this boundary as outflow boundary.

It can be found in examples behind that the convergence rate of this algorithm is very fast, and this algorithm has little impact on the overall solution efficiency.

4. Finite element implementation

4.1. NR iteration

Substituting Equation (3a) and (4) into (2a), and using Switching Algorithm (Equation (5)), a FEM program is compiled. NR iteration method is used to solve, and within a time step $\Delta t$, iteration equations are shown as followings:

$$
\begin{align*}
[A]^{-1} \{ \Delta h \}^k &= \{ R \}^k \\
[A]^{-1} &= [T]^{-1} + \frac{[C]}{\Delta t} \\
\{ R \}^k &= \{ Q \}_b - [F]^{k-1-1} \\
\{ F \}^{k-1} &= [T]^{-1}[h]^{k-1} + \frac{[C]}{\Delta t} \left[ [h]^{k-1} - \{ R \}^0 \right] \\
\{ h \}^k &= \{ h \}^{k-1} + s \{ \Delta h \}^k
\end{align*}
$$

(6)

Convergence condition of this iteration is:

$$
\| [R]^{k} \| / \| [F]^{k-1} \| \leq TOL
$$

(7)

When using modified NR method to solve, $[A]^{-1}$ just need to be changed into $[A]^0$. Thus, $[A]^{-1}$ needn’t to be decomposed at each time step if Signorini condition is satisfied. Generally, the efficiency of modified NR is higher than NR.

In Equation (6) and (7), $k$ is the current iteration step, $TOL$ is convergence tolerance and its value is generally set 0.01. $s$ is relaxation factor, and according to convergence, the range of its value is $0 < s \leq 1.0$. When $s$ take a small value, oscillation of iteration results can be suppressed. But if the value of $s$ is quite small, convergence rate will be slowed. So the value of $s$ need to be adjusted according to specific circumstances.

4.2. Under-relaxation (UR) process on $k_r$ and $m_w$

During using Equation (6) to solve with iteration and updating $[T]^{-1}$ and $[C]^{-1}$, under-relaxation process on $k_r$ and $m_w$ can effectively suppress the oscillation of iteration results[15]. The the results of numerical experiments in this paper show that, using UR2 under-relaxation method[15] on $k_r$, that is $k_r$ takes the average value of previous two iteration results, and this can significantly suppress the oscillation of Equation (6) and improve the convergence of NR algorithm. Use UR1 under-relaxation method[15] on $m_w$, that is:
\[ m^k_w = \frac{\theta^{k-1} - \theta^0}{u^{k-1} - u^0} \]  

Equation (8) can significantly improve the mass conservation of discrete-formed Equation (6).

5. Applications

According to above algorithm, four calculation samples are conducted and analyzed. First two of them are saturated and steady water seepage problem, and last two of them are unsaturated and unsteady water seepage problem. Equation (4) is used as seepage model to solve saturated and steady water seepage problem. However, unsaturated water seepage model must to be used to solve unsaturated and unsteady water seepage problem, one is Gardner model, and the other is Van Genuchten model. Parameters for solving unsaturated water seepage model in calculation samples are shown in Table 1.

| Name of model | Type of soil | Saturation parameters | Non-saturation parameters |
|---------------|--------------|----------------------|---------------------------|
|               |              | \(k_s/(\text{m.s}^{-1})\) | \(m_w/(\text{kPa})\) | \(\alpha\) | \(n\) | \(\theta_s\) | \(\theta_r\) |
| Gardner       | Silt         | 1.2\times10^{-5}     | 0.5\times10^{-6}         | 2.00   | -    | 0.40   | 0.09   |
| Van Genuchten | Sandy soil   | 1.4\times10^{-3}     | 1.0\times10^{-3}         | 5.02   | 2.36 | 0.35   | 0.05   |
|               | Silt         | 5.5\times10^{-5}     | 1.0\times10^{-5}         | 0.68   | 1.96 | 0.40   | 0.05   |

Note: \(\alpha\) and \(n\) are fitting parameters.

For each calculation sample, both 2D and 3D model are calculated, and compare the calculated results with the published results or analytical solutions of some literature. During calculating, six-nodes quadratic triangular element and ten-nodes quadratic tetrahedral element are respectively used in 2D and 3D models.

5.1. Saturated and steady water seepage problem

Two calculation samples for saturated and steady water seepage problem are shown in Fig.4, and these two samples are widely used to Verify other methods\(^6,16,18\). The first calculation sample is the typical Muskat problem. AD and BE are water boundaries, the heights of water are respectively 10m and 2m; EC is possible outflow boundary, and other boundaries are impervious boundary. The second calculation sample is similar to the first one, the difference is that soil is not a single. The permeability coefficient of the zone at the right side of line \(mm\) is 10 times of left. Fig.4 shows the calculated infiltration face using modified NR algorithm. It can be found that: (1) The calculated results of this paper are consistent with published research results of other researchers, and in the first calculation sample, the calculated location of outflow point is in good agreement with the analytical solution in literature \(^19\); The calculated results of 2D is consistent with the results of 3D, and this indicates that the value taking method for \(\varepsilon_1\) and \(\varepsilon_2\) (as shown in Fig.3) is feasible; (3) Both the calculated results of 2D and 3D indicates that the calculated results are not sensitive to the size of element; (4) Equation (4) can be effective to be used to solve saturated water seepage flow problems.

Comparison of calculating efficiency of NR and modified NR algorithm is shown in Table 2. From Table 2, it can be found that modified NR algorithm is more steady and high-efficiency than NR algorithm to solve saturated water seepage flow problems. On the flow of seepage, all calculated results are consistent with analytic solutions. It can also be found from Table 2 that the efficiency of determining outflow boundary using Signorini condition is very high, and number of iterations is small relative to overall calculating number, especially in the example #2. In the example #2, because the soil is not homogeneous, a large number of calculations are used to determine the inner infiltration face.
Fig. 4 Verification for saturated steady water seepage problems

**Table 2** Efficiency comparison of NR and modified NR methods

| SN of samples | Model | Iterations | Solving Time/s | Flow/(m³·d⁻¹) | Analytical Solution + |
|---------------|-------|------------|----------------|----------------|----------------------|
|               | Dimen| MS/m       | mNR | NR   | mNR | NR   | mNR | NR |               |
|               | sions|            |     |      |     |      |     |    |               |
| 1             | 2D   | 1.00       | 12(5)| 8(4) | 0.01| 0.01| 9.55| 9.56| 9.60          |
|               |      | 0.25       | 16(9)| 12(9)| 0.04| 0.06| 9.59| 9.59|                |
|               | 3D   | 1.00       | 12(6)| 8(6) | 0.04| 0.06| 9.55| 9.55|                |
|               |      | 0.25       | 14(13)| 12(11)| 8.50| 9.22| 9.59| 9.59|                |
| 2             | 2D   | 1.00       | 95(3)| 505(45)| 0.33| 2.36| 17.42| 17.44| 17.45          |
|               |      | 0.25       | 1208(13)| NC | 5.79| NC | 17.45| NC |                |
|               | 3D   | 1.00       | 117(4)| 303(63)| 0.53| 3.16| 17.50| 17.50| 17.46          |
|               |      | 0.25       | 1097(46)| 785(168)| 152.23| 602.13| 17.46| 17.46|                |

Note: (1) MS means mesh’s size; NR means Newton-Raphson algorithm; mNR means modified Newton-Raphson algorithm; NC means Not Converge. (2) The numbers inside of brackets mean the number of iteration for determining the outflow boundary using Signorini condition and switching algorithm. (3) “+” means that the solutions are calculated based on Dupuit assumption (according to the research results of O. D. L. Strack[20], the calculated solutions based on Dupuit assumption are exact solutions.

Besides, if Equation (3) is chosen as seepage constitutive model, only sample #1 with big-size element can converge well, others can’t reach the convergence criteria of Equation (7), neither NR algorithm nor modified NR algorithm, and neither 2D models nor 3D models. This indicates that Equation (4), which is modified with \( \varepsilon_1 \) and \( \varepsilon_2 \), has a better convergence.

5.2. Unsaturated and unsteady water seepage problem
5.2.1. Water infiltration problem into soil pillar. This problem is one-dimensional unsaturated and unsteady infiltration into homogeneous pillar problem, as shown in Fig.5(a). Gardner model is used as infiltration model, and its parameters is shown in Table 1. The boundary conditions are: (1) when \( t \geq 0 \), \( B \) is constant water-head boundary, and \( H_b = 0.0m \); (2) when \( t > 0 \), \( A \) is inlet boundary, and its seepage strength \( q = 0.1m/(d.m^2) \).

The mesh sizes of both 2D and 3D model are all 0.05m. The total length of time is 5 d, and there are 154 time steps. The length of a step in first 100 time steps is 0.005d, the length of a step in next 50 time steps is 0.01d, and the length of a step in the last 4 time steps is 1d. The calculated results are shown in Fig.5(b). From Fig.5(b), it can be found that the calculated results are consistent with analytic solutions in literature [21].

(a) Model of numerical calculation time  
(b) Pressure waterhead distribution at different Fig.5 Unsaturated soil column infiltration problem

In terms of mass conservation, \( Q_b \) is defined as the water flow of entering the model from boundary in each time step, \( Q_b = \left( \sum P_{k-1}^{n} \right) \Delta t \). \( Q_s \) is defined as the amount of water stored in the model because of the changes of volumetric water content of saturated soil, \( Q_s = \int (\theta - \theta_0) J dV = \sum_{i=0}^{n} \sum_{j=1}^{n} (\theta - \theta_0) J_j |w_j| \).

In this 2D sample (3D models are similar to 2D models), the curves of cumulative mass ratio \( \sum Q_i / \sum Q_b \) as time is shown in Fig.6. In Fig.6, \( STS \) means small time step, and under this condition, the total calculating time is 0.05d/step×100steps + 0.01d/step×50steps + 1d/step×4steps; \( LTS \) means small time step, and under this condition, the total calculating time is 0.5d/step×10steps + 0.1d/step×5steps + 1d/step×4steps; \( FM \) means fine mesh, and its size is 0.05m and equals to the size of mesh in Fig.5(b); \( CM \) means coarse mesh, and its size is 0.2m. Fig.6 shows the impact of six kinds of conditions, such as different mass matrix updating method, the size of mesh, the length of time step, and so on, on mass conservation. It can be found from Fig.6 that, UR1 method is good in terms of mass conservation, and the length of time step and the size of mesh has little impacts on it. Under the three conditions (UR1+SLS+FM, UR1+LLS+FM, as well as UR1+SLS+FM, as shown in Fig.6), cumulative mass ratios of each time step are all 1, and the phenomenon of non-conservation of mass, pointed out by M. A. Celia [22] when using h-type discrete form (Equation (2)), does not appear. The reason is that, when solving mass matrix \([C]\), the calculation method of \( m_p \) is different. M. A. Celia [22] used UR0 method, and as a contrast, both UR0 and UR1 method are used in this paper. From
Fig. 6, it can be found that, when using UR0 method, the calculated results of cumulative mass ratios are relatively poor, especially in the beginning of infiltration, the conditions of permeability coefficient and water content change dramatically at this time. This indicates that UR0 method is not suitable for h-type discrete form.

Meanwhile, the numerical calculation results show that, the efficiency of NR algorithm is better than modified NR algorithm if the demand of accuracy is the same. When the boundary conditions are the same, if NR algorithm is used, the calculating time of 2D and 3D models is respectively 1.31s and 31.9s, and if modified NR algorithm is used, the calculating time of 2D and 3D models is respectively 36.0s and 322.2s. So, mNR algorithm is much slower than NR algorithm.

Besides, if UR2 method is not chosen and UR0 is chosen to process $k_r$, NR algorithm will not converge, and convergence of the modified NR algorithm also deteriorates.

5.2.2. Multi-layers soil slope infiltration problem. The second unsaturated and unsteady water seepage problem is 3-layers soil slope infiltration problem. The calculating model is shown in Fig. 7, and in this model, the soil in middle layer is silt, and sandy soil is in the top and bottom layer. Van Genchten model is chosen as infiltration model, and the parameters are shown in Table 1. The boundary conditions are: (1) When $t \geq 0$, $AFE$ is water boundary, $h_A = 0.0m$, $h_E = 0.2m$, $h_E = 0.3m$, $ED$ is outflow boundary, $CBA$ is impervious boundary; (2) when $t > 0$, $DC$ is infiltration boundary, and infiltration strength is $q = 2.1 \times 10^{-4} m/(s.m^2)$. And the initial condition is the steady-state waterhead under the boundary condition (1).

The characteristics of this problem is multiple outflow boundaries. J. J. Rulon[23] studied this problem using sand tank experiments. The element’s sizes of 2D and 3D models used to simulate this problem are all 0.05m, and the time step is 25s. Fig. 7 shows the changes of infiltration face with time. From Fig. 7, it can be found that, the calculated results of this paper are consistent with the results of professional software GeoStudio and experimental results. This indicates that the method proposed in this paper can be used to calculate multiple outflow boundaries problem.
6. Conclusions
Through joint application of Bathe algorithm and Signorini condition, the unconfined water seepage problem in saturated and unsaturated soil is converted into conventional nonlinear constitutive problem, and solving variational inequalities is avoided. In conventional material nonlinearity finite element framework with small modification, using NR algorithm, the solving of unconfined water seepage problems including unstable and unsaturated water seepage problems is realized. The following conclusions are obtained:

(1) The Switching Algorithm of Signorini condition can be effective to realize the simulation of outflow boundary, including multiple outflow boundaries.

(2) In terms of unconfined saturated water seepage, Equation (4), which is modified with $\varepsilon_1$ and $\varepsilon_2$, has a better convergence, whether 2D or 3D models.

(3) In terms of elastic-perfectly plastic constitutive model (Equation (4)) of unconfined saturated water seepage, modified NR algorithm is steady and high efficiency. But in terms of unsaturated and unsteady water seepage, NR algorithm is more efficient.

(4) For h-type FEM discrete form of unsaturated and unsteady water seepage (Equation (2a)), UR1 method can ensure mass conservation of calculated results, but UR0 method is difficult to reach that.

(5) In terms of relative permeability coefficient $k_r$, UR2 under-relaxation method can significantly restrain the oscillation during solving and improve the convergence of NR algorithm.

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