FIELDS OF RATIONALITY OF CUSP FORMS

JOHN BINDER

Abstract. In this paper, we prove that for any totally real field $F$, weight $k$, and nebentypus character $\chi$, the proportion of Hilbert cusp forms over $F$ of weight $k$ and character $\chi$ with bounded field of rationality approaches zero as the level grows large. This answers, in the affirmative, a question of Serre. The key intermediate result, which we believe is interesting in its own right, is a Plancherel equidistribution theorem for cusp forms with fixed central character, whose proof builds on earlier work of Shin and Shin-Templier by introducing a bound for certain families of orbital integrals.

1. Introduction

Given a cuspidal Hecke eigenform $f$, define its field of rationality $Q(f)$ to be the number field generated by all its Fourier coefficients $a_n(f)$.

In 1997, Serre proved the following:

Theorem 1.1. [El. Théorème 1] Fix an even weight $k$, a prime $p$, and an integer $A \in \mathbb{Z}_{\geq 1}$. Let $(N_\lambda)$ be a sequence of levels coprime to $p$ with $N_\lambda \to \infty$. As $\lambda \to \infty$, the proportion of cusp forms of level $\Gamma_0(N_\lambda)$ whose field of rationality satisfies $[Q(f) : Q] \leq A$ approaches 0.

The argument was as follows: first, he used a trace formula argument to show that, as $N_\lambda \to \infty$, the eigenvalues of $T_p$ are distributed according to the Plancherel measure on $[-2p^k, 2p^k]$. He then noted that all points have measure zero, and that the set of Weil-$p$-integers of weight $k$ and degree at most $A$ is finite. In particular, the proportion of cusp forms with $[Q(a_p(f)) : Q] \leq A$ must be asymptotically zero.

Serre posited that his theorem could be extended to arbitrary sequences. It is our goal to answer Serre’s question in the affirmative and extend the result in two directions. First, we look at Hilbert cusp forms over an arbitrary totally real field $F$. Second, instead of restricting to cusp forms with trivial character, we allow ourselves to look at forms of an arbitrary (fixed) character.

We’ll fix here some notation that will be in use throughout the paper. Fix a totally real field $F$ with $[F : Q] = n$, a weight $k = (k_1, \ldots, k_n)$, and a level $n \subseteq \mathcal{O}_F$; let $\chi : (\mathcal{O}_F / n)^x \to \mathbb{C}^x$ be a character. Let $B_k(\Gamma_1(n), \chi)$ be a basis of Hecke eigenvalues of weight $k$, level $\Gamma_1(n)$, and character $\chi$. Fix moreover an integer $A \in \mathbb{Z}_{\geq 1}$. We define

$$B_k(\Gamma_1(n), \chi)_{\leq A} = \{ f \in B_k(\Gamma_1(n), \chi) \mid [Q(f) : Q] \leq A \}.$$

In our notation, Serre’s theorem can be rephrased as follows:

Theorem 1.2. Let $F = Q$. Fix an auxiliary prime $p$, an even weight $k$, and an integer $A \geq 1$. Let $n_\lambda \to \infty$ be a sequence of levels with $(n_\lambda, p) = 1$ for all $n_\lambda$. Then

$$\lim_{\lambda \to \infty} \frac{B_k(\Gamma_1(n_\lambda), 1)_{\leq A}}{B_k(\Gamma_1(n_\lambda), 1)} = 0$$

where 1 denotes the trivial character.

Let $\chi : (\mathcal{O}_F / n)^x \to \mathbb{C}^x$ be a character and $k$ be a weight. There is an obstruction to the existence of a cusp form of weight $k$ and character $\chi$. The weight $k$ determines the central character $\chi_\infty$ of the associated automorphic representation at the Archimedean places. As such, if such a cusp form exists, then there must be an automorphic character $Z(F) \backslash Z(k_F) \to \mathbb{C}^\times$ that restricts to $\chi$ on $\mathcal{O}_F^\times$ and $\chi_\infty$ on $Z(F_\infty)$. If such an automorphic character exists, we say $\chi$ occurs in weight $k$. For instance, when $F = Q$, a character $\chi$ occurs in weight $k$ if and only if $\chi(-1) = (-1)^k$. When $F \neq Q$ this requirement is more stringent because $\mathcal{O}_F^\times$ is infinite.
Our Main Theorem is:

**Theorem 1.3.** (Theorem 12.1). Fix a totally real field $F$, a weight $k = (k_1, \ldots, k_n)$, a character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ of conductor $f$ occurring in weight $k$, and an integer $A \geq 1$. Let $(n_\lambda)$ be any sequence of ideals with $f \mid n_\lambda$ and $N(n_\lambda) \to \infty$ as $\lambda \to \infty$. Then

$$\lim_{\lambda \to \infty} \frac{B_k(\Gamma_1(n_\lambda), \chi)_{< A}}{B_k(\Gamma_1(1), \chi)} = 0.$$ 

The key intermediate result in our paper is the Plancherel equidistribution theorem:

**Theorem 1.4.** (Plancherel equidistribution theorem, 11.1). Fix $F$ and let $S$ be a finite set of finite places of $F$. Fix a discrete series representation $\pi_\infty$ of $\text{GL}_2(F_\infty)$ and let $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be an automorphic character of conductor $f$ extending $\chi_{\pi_\infty}$. Let $(n_\lambda)$ be a sequence of levels coprime to $S$, so that $f^S \mid n_\lambda$ and $N(n_\lambda) \to \infty$. As $\lambda \to \infty$, the $S$-components of cuspidal automorphic representations $\pi = \pi_S \otimes \pi_{S,\infty} \otimes \pi_{\infty}$, where $\pi_{S,\infty}$ has conductor dividing $n_\lambda$, and where $\chi_{\pi} = \chi$, are equidistributed according to the Plancherel measure $\hat{\mu}_{S,\pi}^\text{pl}$, when counted with the appropriate multiplicity.

Once this is proved, the Main Theorem follows by relating the field of rationality of certain local representations to their conductors (Proposition 3.6) and from explicit computations with the (fixed central character) Plancherel measure over $\text{GL}_2$ (Remark 8.5). We have stated the Plancherel equidistribution theorem in greater generality than necessary to prove the main theorem. Indeed, our main theorem is (conjecturally) vacuous in certain situations: for instance, if $(\Gamma)$ is a sequence of open-compact subgroups that ‘converge to one’ in the appropriate sense. For instance, if $k = (k_1, \ldots, k_n)$ and there is an $i, j$ with $k_i \neq k_j \mod 2$, then the associated representation is not $C$-algebraic and therefore, at least conjecturally, will not have a finite-degree field of rationality (see [17] section 2) for a discussion of $C$-algebraicity; the failure of $C$-algebraicity for mixed-parity cusp forms is basically [10] Theorem 1.4 (2)). However, because the methods we use to prove the Plancherel equidistribution theorem are representation-theoretic in nature, we can prove it without any algebraicity assumptions.

We will briefly mention three papers that include partial results in this direction, and which are the inspiration for our ideas:

- In [15], Shin proves an equidistribution theorem for Hilbert modular forms of level $\Gamma$, where $(\Gamma)$ is a sequence of open-compact subgroups that ‘converge to one’ in the appropriate sense. For instance, if $(n_\lambda)$ is a nested sequence of ideals of $\mathfrak{o}_F$ whose intersection is the zero ideal, then the sequence $(\Gamma(n_\lambda))$ converges to one, but the sequence $(\Gamma(n_\lambda))$ does not. However, his method is sufficiently general to extend to representations of other algebraic groups.

- In [16], Shin and Templier prove an equidistribution theorem for representations of $G(k_F)$ of increasing level when $G$ is a cuspidal group. In [17] they prove, as a corollary, that if $\mathfrak{p}$ is a sequence of ideals with $\text{ord}_F(n_\lambda) \to \infty$ for some prime $\mathfrak{p}$, then

$$\lim_{\lambda \to \infty} \frac{B_k(\Gamma_1(n_\lambda), 1)_{< A}}{B_k(\Gamma_1(1), 1)} = 0.$$ 

Indeed, the broad ideas for proving our Plancherel equidistribution theorem stem from the proofs of similar theorems in these papers. Like them, we will use the trace formula, Harish-Chandra’s Plancherel theorem, and Sauvageot’s density theorem. However, in our case it is necessary to adapt these existing tools suitably to our situation. We will need versions of the Harish-Chandra Plancherel theorem and the Sauvageot density theorem over local fields to the fixed-central-character setting at least for $\text{GL}_2$.

Moreover, we need a version of the trace formula that both works for the fixed-central-character situation and is sufficiently user-friendly to prove that ‘most’ of the geometric terms vanish asymptotically. It is worth noting here that Arthur has proven a fixed-central-character version of the invariant trace formula from [1] in [3], and used it in [3]. Moreover, in [9], Palm has stated a version of the fixed-central-character trace formula. However, because we will need specific forms of the fixed-central-formula trace formula, we see fit to prove them directly from Arthur’s existing trace formulæ in [1] and [2].
Once these are in place, a key innovation of our paper is a careful asymptotic estimation of the geometric side of the trace formula. Specifically, we will examine the asymptotic behavior of the geometric terms of the trace formula for characteristic functions of $\Gamma_0(\mathfrak{n})$ as $N(\mathfrak{n}) \to \infty$. This builds on the work of Shin and Shin-Templier, who chose sequences of functions whose orbital integrals eventually vanished, and their constant-term computations were simplified because they used characteristic functions of normal subgroups of the maximal compact subgroup $K^\infty$. The function $1_{\Gamma_0(\mathfrak{n})}$ has nonzero orbital integrals for many $\gamma \in \GL_2(F)$, but we will be able to bound these orbital integrals explicitly as $N(\mathfrak{n}) \to \infty$.

The outline of this paper is as follows. In section 2, we discuss the tempered spectrum $\GL_2(L)^{\wedge, t}$ of $\GL_2(L)$ for a $p$-adic field $L$, and recall how it is naturally endowed with the structure of a disjoint union of countably many compact real orbifolds. In section 3, we discuss fields of rationality of these orbital integrals explicitly as $N$ and Sauvageot’s density theorem for fixed central character (at least for $\GL_2$).

In section 6. In section 7, we discuss the Plancherel measure on the tempered spectrum; in section 8, we adapt it to the fixed-central-character setting and prove Harish-Chandra’s Plancherel theorem and Sauvageot’s density theorem for fixed central character (at least for $\GL_2(L)$).

In section 9, we introduce counting measures and construct explicit test functions whose Plancherel transforms count the cusp forms of fixed character, weight, and level. In section 10, we show an asymptotic vanishing result for orbital integrals and constant terms. In section 11, we use the results from section 6-10 to prove the Plancherel equidistribution theorem.

Finally, in section 12, we prove our main theorem. The proof follows from the Plancherel equidistribution theorem and a careful assessment of the explicit (fixed-central-character) Plancherel measure on $\GL_2(F_p)$.

1.1. Notation and Conventions. We will fix here the following conventions:

- $F$ will always refer to a totally real field, and $L, L'$ will always refer to $p$-adic fields. $K$ will be reserved for compact subgroups of $\GL_2(R)$, where $R$ is a local field or an adèle ring $\mathbb{A}_F$.
- The notation $x \mapsto \hat{x}$ takes many uses, so we fix a convention here. We will reserve lower-case Greek letters $\phi, \psi$ for Plancherel transforms of elements of the Hecke algebra of $\GL_2$ (see Definition 5.2). Latin letters $\hat{f}, \hat{h}$ will always denote general functions in $\mathcal{F}_0(\GL_2^2)$ (see Definition 7.4). Both $\hat{f}$ and $\hat{\phi}$ are complex-valued functions on the unitary spectrum of $\GL_2$, but the former is more general. Upper case Greek letters such as $\Phi$ and $\Psi$ are used to denote functions on a subgroup on the center of $\GL_2$. In this case, $\hat{\Phi}$ and $\hat{\Psi}$ will denote their Fourier transforms as functions on a locally compact abelian group.
- Lower-case fraktur letters will refer to integral ideals in $F$ or $L$. $\mathfrak{o}_F, \mathfrak{o}_L$ will always refer to the ring of integers, and $\mathfrak{p}$ will always refer to a prime. $p$ will be reserved for rational primes.
- By a sequence of levels $(\mathfrak{n}_\lambda)$, we mean a sequence $(\mathfrak{n}_\lambda)$ of ideals of $\mathfrak{o}_F$. We always assume $N(\mathfrak{n}_\lambda) \to \infty$.
- Given a representation $\pi$ of $\GL_2(L)$, the conductor $c(\pi)$ will take values $0, 1, 2, \ldots$. For a representation of $\GL_2(\mathbb{A}_F)$, the conductor $\lambda(\pi)$ will always be an ideal in $\mathfrak{o}_F$. As such, if $\pi$ is a representation of $\GL_2(\mathbb{A}_F)$, then $\lambda(\pi) = \prod \mathfrak{p}^{c(\pi_\mathfrak{p})}$.
- All characters $\chi, \chi', \eta, \ldots$ will be unitary characters, and if they are characters on the adèle group $\mathbb{A}_F^\times$, they will be assumed to be trivial on $F^\times$. If $\pi$ is a representation of a $p$-adic or adèle group, its central character will be denoted $\chi_\pi$. A character $\chi_0, \eta_0, \ldots$ will always refer to a character on the elements on absolute 1.
- $\xi$ will always be used to denote a finite-dimensional irreducible representation of $\GL_2(F_\infty)$. If $k = (k_1, \ldots, k_n)$ is a weight, then $\xi_k$ denotes the finite-dimensional complement of the
discrete-series representation associated to any cusp form of weight $k$; in particular, $\xi_k$ will decompose as a tensor product of irreducible representations of the form $\operatorname{Sym}^{k-2}(\mathbb{R}^2) \otimes |\det|^{-\frac{k-2}{2}}$.

1.2. Acknowledgements. I would like to thank Ruthi Hortsch and Nicolas Templier for their helpful suggestions.

I am especially indebted to my adviser, Sug Woo Shin, for his unfailing support and friendship. This paper would have been impossible without him.

2. The Tempered Spectrum of $GL_2(L)$

The goal of this section is to briefly recall some topological properties of the tempered spectrum of $GL_2(L)$ where $L$ is a $p$-adic field. Throughout, $q$ will denote the cardinality of the residue field of $L$. We recall some definitions and preliminary results. Let $GL_2(L)^\wedge$ denote the set of irreducible unitary admissible representations of $GL_2(L)$ (up to isomorphism); in particular, if $\pi \in GL_2(L)^\wedge$, its central character $\chi_{\pi}$ is unitary.

Definition 2.1. Let $G$ be a connected, reductive group over a $p$-adic field $L$ and let $\pi$ be an admissible representation of $G(L)$. We say $(\pi, V_\pi)$ is a discrete series representation if the matrix coefficient $g \mapsto (\pi(g)v, v)$ is in $L^2(G/Z)$ for every $v \in V_\pi$.

We say $\pi$ is tempered if, instead, every matrix coefficient lies in $L^{2+\epsilon}(G/Z)$ for every $\epsilon > 0$.

Throughout this paper, we will denote the set of unitary representations of $G(L)$ as $G(L)^\wedge$, and the set of tempered unitary representations of $G(L)$ as $G(L)^\wedge_t$.

For the rest of the subsection we assume $G = GL_n$. The following results are classical:

Proposition 2.2. A representation $\pi \in GL_n(L)^\wedge$ is a discrete series representation if and only if it is a generalized Steinberg representation $\operatorname{St}(\sigma, m)$ for a unitary supercuspidal representation $\sigma \in GL_m(L)^\wedge$, and $n = md$.

A representation $\pi \in GL_n(L)^\wedge$ is tempered if and only if it is of the form

$$I^G_\mu(\pi_1 \otimes \ldots \otimes \pi_r)$$

where $\pi_i$ is a discrete series representation of $GL_{n_i}(L)$, with $n = n_1 + \ldots + n_r$, and $I^G_\mu$ denotes normalized induction.

Fix a standard parabolic $P$ with Levi subgroup $M$, and let $X_u(M)$ denote the group of unramified unitary characters of $M$. Then $X_u(M)$ acts on the set of discrete series representations $\omega$ of $M$ via $\chi \cdot \omega = \omega \otimes \chi$. Each orbit $O_M$ under the action of $X_u(M)$ naturally acquires the topology of a compact orbifold, and as such the set of discrete series representations of $M$ acquires the topology of a countable union of disjoint compact orbifolds.

Denote by $\Theta$ the set of pairs $(M, O_M)$ where $O_M$ is an orbit of discrete series representations of $M$. Say two pairs $(M, O_M)$ and $(M', O_{M'})$ associated if there is an element $s \in W^G$, the Weyl group of $G$, such that $s \cdot M = M'$ and $s \cdot O_M = O_{M'}$.

The normalized induction functor gives a surjective map

$$\prod_{\Theta/\text{assoc}} (M, O_M) \to GL_n(L)^\wedge_t.$$  

(The fact that $I^G_\mu(\omega)$ is irreducible when $\omega$ is a discrete series representation of $M$ follows from [21, Theorem 4.2]; we note here that this does not hold for general reductive $p$-adic groups).

Moreover, for a given orbit $(M, O_M)$, the stabilizer

$$\operatorname{Stab}(M, O_M) = \{ s \in W^G/W^M : s \cdot M = M, s \cdot O_M = O_M \}$$

acts on $O_M$. The map above descends to a bijection

$$\prod_{\Theta/\text{assoc}} (M, O_M)/\operatorname{Stab}(M, O_M) \sim G^\wedge.$$  

This gives the tempered spectrum of $G$ the structure of a countable disjoint union of compact orbifolds.
Throughout this paper, we will use $O_M$ to refer to an orbit of discrete-series representations of a Levi subgroup $M$ of $G$. We will use $O$ to refer to an orbit in $G^M$; that is, $O$ will refer to the image of an orbit $(M, O_M)$ under the normalized induction functor.

2.1. Tempered Orbits of $\GL_2(L)$. In this subsection, we’ll recall some facts about the tempered orbits of $\GL_2(L)$. We’ll follow the standard practice of writing $\sigma_1 \times \sigma_2$ for $I_{\theta}^L(\sigma_1 \otimes \sigma_2)$ when $\sigma_1 \otimes \sigma_2$ is a discrete series representation of a Levi subgroup $M$. $\pi$ is irreducible since $\sigma_1$ and $\sigma_2$ are unitary.

It is convenient to break partition the set of orbits into four types:

- **Type (1):** $O$ consists of elements $\chi \times \chi'$, where $\chi \chi'^{-1}$ is unramified;
- **Type (2):** $O$ consists of elements $\chi \times \chi'$, where $\chi \chi'^{-1}$ is ramified;
- **Type (3):** $O$ consists of elements $\text{St}(\chi)$ where $\chi$ is a character; and
- **Type (4):** $O$ consists of supercuspidal representations $\pi$.

It is worth recalling the following:

**Definition 2.3.** Let $L'/L$ be a quadratic extension, let $\psi_L$ be an additive character on $L$, and let $\eta$ be a multiplicative character on $L$ that is not $\Gal(L'/L)$-invariant. The *dihedral representation* $\pi_\eta$ of $\GL_2(L)$ is defined as follows. First, let $\omega_{\eta, \psi}$ be the Weil representation of $\text{SL}_2(L)$ on the subspace of functions $f \in C_c^\infty(L')$ satisfying the transformation property

$$f(yv) = \eta(y)^{-1} f(v) \quad \text{for all } v \in E, \ y \in \ker(N_{L'/L}).$$

Upgrade this to a representation $\omega_{\eta, \psi}$ of

$$\GL_2(L)^{L'} \equiv \{ g \in \GL_2(L) : \det(g) \in N_{L'/L}(L'^\times) \}$$

by setting

$$\left(\omega_{\eta, \psi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) (v) = |a|^{1/2} \eta(b) f(bv), \quad a = N_{L'/L}(b).$$

Let $\pi_\eta = \text{Ind}^{\GL_2(L)^{L'}}_{\GL_2(L)}(\omega_{\eta, \psi})$; this is independent of the choice of additive character $\psi$.

We have the following facts:

- (i) The central character of $\pi_\eta$ is $\chi_{L'/L} \cdot \eta|_{F^\times}$. Here $\chi_{L'/L} : L^\times \to C^\times$ is the unique nontrivial character whose kernel is $N_{L'/L}(L'^\times)$.
- (ii) $\pi_\eta \cong \pi_{\eta'}$ if $\eta$ and $\eta'$ are characters on the same quadratic extension $L'$, and $\eta$ and $\eta'$ are $\Gal(L'/L)$-conjugate.
- (iii) If the residue characteristic of $L$ is odd, then all supercuspidal representations of $\GL_2(L)$ are dihedral.
- (iv) If $\chi$ is a character of $L^\times$ then $\pi_\eta \otimes \chi \cong \pi_{\eta \circ (\chi \circ N_{L'/L})}$.

Facts (i) and (iv) are on page 121 of [12], while (iii) is on page 120. Fact (ii) follows by noting that $\pi_\eta$ corresponds to the irreducible Weil representation $I_{\chi_{L'}}^{W(L')}(\eta)$ under the Local Langlands correspondence.

When the residue characteristic of $L$ is odd, we have the following characterization of the orbits:

**Proposition 2.4.** Assume the residue characteristic of $L$ is $p > 2$.

1. The orbits of type (1) are in correspondence with characters $\chi_0 : o_L^\times \to C^\times$.
2. The orbits of type (2) are in correspondence with pairs of characters $\chi_0 \neq \chi'_0 : o_L^\times \to C^\times$.
3. The orbits of type (3) are in correspondence with characters $\chi_0 : o_L^\times \to C^\times$.
4. The orbits of type (4) are in correspondence with pairs $(L', \{\eta_0, \eta'_0\})$ where $L'/L$ is a quadratic extension, and $\{\eta_0, \eta'_0\} : o_L^\times \to C^\times$ is a $\Gal(L'/L)$-conjugate pair of characters with $\eta_0 \neq \eta'_0$.

**Proof.** For the first statement, we note that $\chi_1 \times \chi_2$ and $\chi'_1 \times \chi'_2$ are in the same orbit if $\chi'_1 \chi^{-1}_2$ and $\chi_1 \chi'^{-1}_2$ are unramified. Moreover, $\chi_1 \chi'^{-1}_2$ is unramified, so any two characters differ by an unramified twist. As such, $\chi_1$ and $\chi'_1$ all share the same restriction to $o_L^\times$: this determines a canonical bijection between orbits and characters $\chi_0 : o_L^\times \to C^\times$.

The proofs of (2) and (3) are exactly the same.
For (4), because $p > 2$, every supercuspidal representation of $\pi$ of $GL_2(L)$ is a dihedral representation, so there is a pair $(L', \eta)$ as above such that $\pi = \pi_{\eta}$. The proof will follow once we show that, given characters $\eta, \eta'$ of $L'^\times$, then $\pi_{\eta}$ and $\pi_{\eta'}$ differ by an unramified twist if and only if $\eta$ and $\eta'$ differ by an unramified twist. On the one hand, assume $\eta = \theta \eta'$ for an unramified $\theta : L'^\times \to \mathbb{C}^\times$. Since $\theta$ is unramified we can write $\theta = \chi \circ N_{L'/L}$ for an unramified character $\chi$; then we have

$$\pi_{\eta'} = \pi_{\eta \circ (\chi \circ N_{L'/L})} \cong \pi_{\eta} \otimes \chi.$$ 

On the other hand, if $\chi$ is an unramified character of $L^\times$ and $\pi_{\eta'} = \pi_{\eta} \otimes \chi$, then $\pi_{\eta'} = \pi_{\eta \circ (\chi \circ N_{L'/L})}$, and so $\eta (\chi \circ N_{L'/L}) = \eta'$ or $\eta''$.

Therefore, the supercuspidal orbits are parameterized by pairs of $\{\eta, \pi\}$ up to unramified twist, and giving a character up to unramified twist is the same as giving its restriction to $O_L^\times$, as above, completing the proof. 

It follows immediately from above that if $\pi$ and $\pi'$ are in the same orbit, then $\chi_{\pi} |_{O_L^\times} = \chi_{\pi'} |_{O_L^\times}$. We define $\chi_{\mathcal{O}} = \chi_{\pi} |_{O_L^\times}$ for any $\pi \in \mathcal{O}$.

A list of conductors of tempered representations is given in [12, p. 122]:

- If $\pi = \chi \times \chi'$, then $c(\pi) = c(\chi) + c(\chi')$.
- If $\pi = St(\chi)$, then
  $$c(\pi) = \begin{cases} 1 & \text{if } \chi \text{ is unramified} \\ 2 \cdot c(\chi) & \text{if } \chi \text{ is ramified.} \end{cases}$$

- If $\pi$ is the dihedral representation $\pi_{\eta}$, then
  $$c(\pi) = \begin{cases} 2 \cdot c(\eta) & \text{if } L'/L \text{ is unramified} \\ c(\eta) + 1 & \text{if } L'/L \text{ is ramified.} \end{cases}$$

Since the conductor of a character $\chi$ or $\eta$ depends only on its restriction to $\mathfrak{o}_L^\times$ or $\sigma_L^\times$, we can make the definition:

**Definition 2.5.** Let $\mathcal{O}$ be an orbit in $GL_2(L)^{\wedge t}$. We define its conductor $c(\mathcal{O})$ to be the conductor $c(\pi)$ for any $\pi \in \mathcal{O}$.

3. Preliminaries on Fields of Rationality

Throughout, let $F$ be a totally real field.

**Definition 3.1.** Let $f$ be a Hilbert modular form over $F$ of level $\Gamma_1(n)$, weight $k$, and character $\chi$ that is a Hecke eigenform. Then $\mathbb{Q}(f) \subseteq \mathbb{Q}$ is the field generated by all the Fourier coefficients of $f$.

**Definition 3.2.** Fix a level $n$, a weight $k$, a character $\chi : (\mathfrak{o}_F/n)^\times \to \mathbb{C}^\times$ occurring in weight $k$, and an integer $A \in \mathbb{Z}_{\geq 1}$. We denote by $B_k(\Gamma_1(n), \chi)$ a basis of normalized Hecke eigenforms in $S_k(\Gamma_1(n), \chi)$, and define

$$B_k(\Gamma_1(n), \chi)_{\leq A} = \{ f \in B_k(\Gamma_1(n), \chi) \mid [\mathbb{Q}(f) : \mathbb{Q}] \leq A \}.$$

**Definition 3.3.** Let $G$ be a reductive group over a $p$-adic field $L$ and let $\pi$ be an admissible $G(L)$-representation. The field of rationality $\mathbb{Q}(\pi)$ is the fixed field of the subgroup

$$\{ \sigma \in \text{Aut}(\mathbb{C}) : \sigma \pi \cong \pi \}.$$ 

If $\pi$ is an automorphic representation of $G(A_F)$, then $\pi$ decomposes as $\pi \cong \bigotimes_v \pi_v$, and $\mathbb{Q}(\pi)$ is the compositum of the fields $\mathbb{Q}(\pi_v)$ over the finite places $v$ of $F$.

**Lemma 3.4.** Let $f$ be a Hecke eigenform of weight $k$, level $n$, and character $\chi$, and let $\pi_f$ be the associated $GL_2(A_F)$-representation. Then $\mathbb{Q}(f) = \mathbb{Q}(\pi_f)$.

**Proof.** This is [10, Theorem 1.4 (5)]. We omit the proof. □
3.1. Fields of rationality of tempered orbits of $GL_2(L)$. In this subsection, we switch back to the local theory. Let $p > 2$. We assume $L$ is a $p$-adic field whose residue field has cardinality $q$. Throughout, $\pi$ will denote an irreducible unitary admissible representation of $GL_2(L)$.

**Definition 3.5.** Let $O$ be an orbit in $GL_2(L)^\wedge$. We define $\mathbb{Q}(O)$ to be the intersection of all $\mathbb{Q}(\pi)$ for $\pi \in O$.

The goal of this subsection is to prove the following, in analogy with Corollary 3.12 of [17]:

**Proposition 3.6.** Let the $p$ be the residue characteristic of $L$, and assume $p > 2A + 1$. Let $O$ be a tempered orbit of $GL_2(L)$ of conductor at least 3. Then $|\mathbb{Q}(O) : \mathbb{Q}| > A$.

We begin with three lemmas, which rely on the characterization of orbits given in Proposition 2.4.

**Lemma 3.7.** Let $O$ be the supercuspidal orbit corresponding to $\eta_0 : \mathfrak{o}_L^\times \rightarrow \mathbb{C}^\times$. Let $\eta_0(x) = \zeta$ for some $x \in \mathfrak{o}_L^\times$. Then $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq 2[\mathbb{Q}(O) : \mathbb{Q}]$.

**Proof.** Recall the construction of the dihedral representation in Definition 2.3, and let $\sigma \in \text{Aut}(\mathbb{C})$; it is easy to check if $GL_2(L)^L$ acts on $f$ via $\omega_{\eta,\psi}$, then it acts on $\sigma \circ f$ as $\omega_{\sigma \eta,\sigma \psi}$. This exhibits an isomorphism $\sigma \omega_{\eta,\psi} \cong \omega_{\sigma \eta,\sigma \psi}$.

But the representation $\pi_{\eta}$ is independent of the choice of $\psi$, so upon induction we get $\sigma \pi_{\eta} = \sigma \pi_{\eta} \cong \pi_{\sigma \eta} \cong \pi_{\sigma \eta}$. As such, if $\sigma \pi \cong \pi$ for some $\pi \in O$ then $\sigma$ permutes the character $\eta_0$ with its conjugate $\overline{\eta_0}$ under $\text{Gal}(L/L)$. Therefore, $\sigma$ fixes $(\eta_0 + \overline{\eta_0})(x)$ and $(\eta_0\overline{\eta_0})(x)$ for $x \in \mathfrak{o}_L^\times$, so both these quantities are in $\mathbb{Q}(O)$. As such, $\zeta = \eta_0(x)$ is a root of the polynomial $T^2 - (\eta_0 + \overline{\eta_0})(x)T + (\eta_0\overline{\eta_0})(x) \in \mathbb{Q}(O)[T]$ and so $\zeta$ is of degree at most 2 over $\mathbb{Q}(O)$, completing the proof. □

**Lemma 3.8.** Let $O$ be a Steinberg orbit corresponding to $\chi_0 : \mathfrak{o}_L^\times \rightarrow \mathbb{C}^\times$. Then $\mathbb{Q}(O) \supseteq \mathbb{Q}(\chi_0)$.

**Proof.** Let $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, and let $f : G \rightarrow \mathbb{C}$ satisfy $f(tug) = \chi(t_1t_2)f(g)$ for some $\chi$ with $\chi|_{\mathfrak{o}_L^\times} = \chi_0$. Then $\sigma \circ f$ satisfies $\sigma \circ f(tug) = \sigma(\chi(t_1t_2))$, exhibiting an isomorphism between $\sigma \text{St}(\chi)$ and $\text{St}(\sigma(\chi))$. Now the proof follows exactly as above. □

**Lemma 3.9.** Let $O$ be an orbit consisting of principal series representations corresponding to $\chi_0, \chi_0' : \mathfrak{o}_L^\times \rightarrow \mathbb{C}^\times$, with $\chi_0(x) = \zeta$ for $x \in \mathfrak{o}_L^\times$. Then, $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq 2[\mathbb{Q}(O) : \mathbb{Q}]$.

**Proof.** Assume $\pi = \chi \times \chi'$ and assume $\sigma \pi \cong \pi$. Arguing as above, we have an isomorphism $\sigma \chi \times \chi' \cong (\sigma(\chi)) \times (\sigma(\chi'))$ and therefore $\sigma \chi \times \chi' \cong \chi \times \chi'$ if and only if $\sigma$ permutes $\chi$ and $\chi'$. As such, $\sigma$ fixes both $\chi + \chi'$ and $\chi \chi'$. Therefore, $\chi_0(x) + \chi_0'(x)$ and $\chi_0(x)\chi_0'(x)$ are in $\mathbb{Q}(O)$ for all $x \in \mathfrak{o}_L^\times$.

If $\chi_0(x) = \zeta$ then $\zeta$ is a root of

$$T^2 - (\chi_0(x) + \chi_0'(x))T + \chi_0(x)\chi_0'(x) \in \mathbb{Q}(O)[T].$$

In particular, $\zeta$ is of degree at most 2 over $\mathbb{Q}(O)$, completing the proof. □

With these lemmas in hand, we can prove Proposition 3.6.

**Proof.** From the discussion of conductors before definition 2.5, we see that if $c(O) \geq 3$ then $O$ takes one of the following forms:

- $O$ is a supercuspidal orbit corresponding to $\eta_0 : \mathfrak{o}_L^\times \rightarrow \mathbb{C}^\times$, with $c(\eta_0) \geq 2$.
- $O$ is a Steinberg orbit corresponding to $\chi_0 : \mathfrak{o}_L^\times \rightarrow \mathbb{C}^\times$, with $c(\chi_0) \geq 2$.
- $O$ is a principal-series orbit corresponding to $\chi_0 \times \chi_0'$, where $c(\chi_0) \geq 2$ (up to switching $\chi_0$ and $\chi_0'$).

In the second two cases, $\chi_0$ is nontrivial on $1 + \varpi L_\infty$, a pro-$p$-group and so $\zeta_\varpi \in c(\chi_0)$. In the first case, $\eta_0$ is nontrivial on $1 + \varpi L_\infty$, again a pro-$p$-group, so $\zeta_\varpi \in c(\eta_0)$. Therefore, in all cases, $|\mathbb{Q}(O) : \mathbb{Q}| \geq \frac{1}{2}[\mathbb{Q}(\zeta_\varpi) : \mathbb{Q}] = \frac{q-1}{2} > A$. □
4. **Hecke algebras and Plancherel Transforms**

Throughout this section, $F$ is a totally real field with $[F : \mathbb{Q}] = n$ and $\mathbb{A}$ is the ring of adèles over $F$. $R$ will be used to denote $\mathbb{A}$ or $F_v$ for some place $v$ of $F$.

The definitions and lemmas below will depend upon a choice of Haar measure. Fix once and for all the following Haar measures:

- If $L$ is a $p$-adic field, and $G(L)$ the group of $L$-points of some reductive group, the we choose the Haar measure giving a maximal compact subgroup measure 1.
- We choose the Euler-Poincaré measure on $GL_2(\mathbb{R})$ (see section 5) and the standard Haar measure on $\mathbb{R}^\times$.
- On an adèlic group such as $GL_2(\mathbb{A}_F)$ or $\mathbb{A}_F^\times$, we take the product measure of the local measures just described.

**Definition 4.1.** Let $X$ be a closed subgroup of the center $Z(GL_2(R))$, and let $\chi : X \rightarrow \mathbb{C}^\times$ be a unitary character. The **Hecke algebra** $H(GL_2(R), X, \chi)$ is the convolution algebra of smooth functions $\phi : GL_2(R) \rightarrow \mathbb{C}$ that are compactly-supported modulo $X$ and that satisfy the transformation property

$$\phi(gx) = \phi(g)\chi(x)^{-1} \text{ for all } g \in G, x \in X.$$

**Definition 4.2.** Let $\phi \in H(GL_2(R), X, \chi)$. We define its **Plancherel transform** $\hat{\phi}$ as a complex function on the space of representations $\pi$ with $\chi_\pi|_X = \chi$, by

$$\hat{\phi}(\pi) = tr_X \pi(\phi) = tr \left( v \mapsto \int_{X \backslash GL_2(R)} \phi(g)\pi(g)v \, dg \right).$$

The integrand is well-defined since $\phi(gx)\pi(gx) = \phi(g)\chi^{-1}(x)\chi(x)\pi(g) = \phi(g)\pi(g)$ for all $g \in G, x \in X$.

**Definition 4.3.** Given $(X, \chi), (X', \chi')$ with $X \supseteq X'$ and $\chi|_{X'} = \chi'$, we define the **averaging map**

$$H(GL_2(R), X', \chi') \rightarrow H(GL_2(R), X, \chi)$$

$$\phi \mapsto \overline{\phi}_{X, \chi}$$

where $\overline{\phi}_{X, \chi}$ is defined by

$$\overline{\phi}_{X, \chi}(g) = \int_{X' \backslash X} \phi(gx^{-1})\chi(x)^{-1} \, dx.$$

In this case, we say $\overline{\phi}_{X, \chi}$ is the average of $\phi$ over $X$ with respect to $\chi$.

**Lemma 4.4.** Let $(X, \chi), (X', \chi')$ be as above and let $\phi \in H(GL_2(R), X', \chi')$. Assume the Haar measures on $GL_2(R)/X$, $GL_2(R)/X'$, and $X/X'$ are chosen compatibly. Then for any $\pi$ with $\chi_\pi|_X = \chi$ we have

$$tr_{X'} \pi(\phi) = tr_X \pi(\overline{\phi}_{X, \chi}).$$
Proof. The proof is a quick application of Fubini’s theorem:

\[
\begin{align*}
\text{tr}_{\mathfrak{X}} \pi(\phi_{\mathfrak{X}}, \chi') &= \text{tr} \left( \int_{\mathfrak{X} \setminus \text{GL}_2(R)} \int_{\mathfrak{X} \setminus \mathfrak{X}} \phi(gx) \chi(x) \pi(g) \, dx \, dg \right) \\
&= \text{tr} \left( \int_{\mathfrak{X} \setminus \mathfrak{X}} \chi(x) \int_{\mathfrak{X} \setminus \text{GL}_2(R)} \phi(gx) \pi(g) \, dg \, dx \right) \\
&= \text{tr} \left( \int_{\mathfrak{X} \setminus \mathfrak{X}} \chi(x) \int_{\mathfrak{X} \setminus \text{GL}_2(R)} \phi(g) \pi(gx^{-1}) \, dg \, dx \right) \\
&= \text{tr} \left( \int_{\mathfrak{X} \setminus \mathfrak{X}} \chi(x) \chi(x^{-1}) \int_{\mathfrak{X} \setminus \text{GL}_2(R)} \phi(g) \pi(g) \, dg \, dx \right) \\
&= \text{tr} \left( \int_{\mathfrak{X} \setminus \text{GL}_2(R)} \phi(g) \pi(g) \, dg \right) \\
&= \text{tr}_{\mathfrak{X}} \pi(\phi)
\end{align*}
\]

\[\square\]

**Definition 4.5.** Let \( \phi \in \mathcal{H}(\text{GL}_2(R), \mathfrak{X}', \chi') \), let \( \mathfrak{X} \) be a closed subset of \( Z(\text{GL}_2(R)) \), and let \( \Phi_{\mathfrak{X}} \) be smooth and compactly supported on \( \mathfrak{X} \). We define the **convolution** \( \phi \ast \Phi_{\mathfrak{X}} \) by

\[
(\phi \ast \Phi_{\mathfrak{X}})(g) = \int_{\mathfrak{X}} \phi(gx^{-1})\Phi_{\mathfrak{X}}(x) \, dx.
\]

**Definition 4.6.** Let \( \Phi \) be smooth and compactly supported on \( \mathfrak{X} \), a closed subset of \( Z(\text{GL}_2(R)) \). We define its **Fourier transform** \( \hat{\Phi} : \mathfrak{X} \to \mathbb{C} \) by

\[
\hat{\Phi}(\chi) = \int_{\mathfrak{X}} \Phi(x)\chi(x) \, dx.
\]

Here, we apologize for the re-use of notation. We’ll distinguish the following cases: when we use lower-case Greek letters like \( \phi, \psi \), we mean elements of some Hecke algebra on \( \text{GL}_2(R) \), and \( \hat{\phi}, \hat{\psi} \) will denote their **Plancherel** transforms. Upper-case Greek letters \( \Phi, \Psi \) will always denote functions on some closed subset of the center, and \( \hat{\Phi}, \hat{\Psi} \) will always denote their **Fourier** transforms as functions on a locally compact abelian group.

**Lemma 4.7.** Let \( \Phi_{\mathfrak{X}} \) be as above and let \( \phi \in \mathcal{H}(\text{GL}_2(R), \mathfrak{X}', \chi') \) with \( \mathfrak{X} \supseteq \mathfrak{X}' \). Then

1. \( \phi \ast \Phi_{\mathfrak{X}} \in \mathcal{H}(\text{GL}_2(R), \mathfrak{X}', \chi') \).
2. For any \( \pi \) with \( \chi_{\pi}|_{\mathfrak{X}'} = \chi' \),

\[
\text{tr}_{\mathfrak{X}'} \pi(\phi \ast \Phi_{\mathfrak{X}}) = \hat{\Phi}_{\mathfrak{X}}(\chi_{\pi}|_{\mathfrak{X}}) \cdot \text{tr}_{\mathfrak{X}} \pi(\phi).
\]

**Proof.** (1) is clear, so we will prove (2). We have

\[
\begin{align*}
\text{tr}_{\mathfrak{X}'} \pi(\phi \ast \Phi_{\mathfrak{X}}) &= \text{tr} \left( \int_{\mathfrak{X}' \setminus \text{GL}_2(R)} \int_{\mathfrak{X}} \phi(gx^{-1})\Phi(x)\pi(g) \, dx \, dg \right) \\
&= \text{tr} \left( \int_{\mathfrak{X}} \Phi(x) \int_{\mathfrak{X}' \setminus \text{GL}_2(R)} \phi(gx^{-1})\pi(g) \, dg \, dx \right) \\
&= \text{tr} \left( \int_{\mathfrak{X}} \Phi(x) \int_{\mathfrak{X}' \setminus \text{GL}_2(R)} \phi(g)\pi(gx) \, dg \, dx \right) \\
&= \text{tr} \left( \int_{\mathfrak{X}} \Phi(x) \chi_{\pi}(x) \int_{\mathfrak{X}' \setminus \text{GL}_2(R)} \phi(g)\pi(g) \, dg \, dx \right)
\end{align*}
\]

completing the proof. \[\square\]

We’ll conclude this subsection with a lemma to be used in the upcoming sections:
Lemma 4.8. Let \( L \) be a \( p \)-adic field and let \( \chi: L^\times \to \mathbb{C}^\times \) be a character. Define
\[
\Phi_{M, \chi}(z) = \frac{1}{M^2} \sum_{j=0}^{M-1} \int_{-j \leq \nu_L(z) \leq j} \chi^{-1}(z) \, dz.
\]

Then

1. if \( \chi \) and \( \tau \) differ by a ramified character, then \( \Phi_{M, \chi}(\tau) = 0 \),
2. if \( \tau \neq \chi \) then \( \Phi_{M, \chi}(\tau) \to 0 \) as \( M \to \infty \), and
3. if \( \tau = \chi \) then \( \Phi_{M, \chi}(\tau) = 1 \) for all \( M \).

Proof. We have
\[
\Phi_{M, \chi}(\tau) = \frac{1}{M^2} \sum_{j=0}^{M-1} \int_{q_L^{-1} \leq |z| \leq q_L} (\tau \chi^{-1})(z) \, dz.
\]

If \( \tau \chi^{-1} \) is ramified, then the integral vanishes. If \( \tau \chi^{-1} = 1 \), then the integral is 1 and the sum is \( M^2 \), completing (3). For (2), if \( \tau \chi^{-1} \) is unramified, the sum is \( \frac{1}{M} \int_{-1}^{1} F_m(\tau \chi^{-1}(\pi)) \), where \( F_m \) is the Fejér kernel from real Fourier analysis. For all \( z \neq 1 \) we have \( \frac{1}{M} F_m(z) \to 0 \), finishing the proof. \( \square \)

5. Euler-Poincaré measures and Euler-Poincaré functions

Let \( \xi \) be an irreducible, finite-dimensional representation of \( GL_2(F) \) and let \( \pi_{\xi} \) be its discrete-series complement: that is, for every \( v \mid \infty \), \( \xi_v + \pi_{\xi_v} \) is equivalent to an induced representation in the Grothendieck group. In this section, we will prove the existence of a function \( \phi_{\xi} \in \mathcal{H}(GL_2(F), Z(F), \chi_{\xi}) \) such that for any infinite-dimensional representation \( \pi' \) of \( GL_2(F) \),
\[
\text{tr}_{Z(F)}(\pi')(\phi_{\xi}, \mu_{\text{EP}}) = \begin{cases} (-1)^{|F:q|} & \pi' \cong \pi_{\xi} \\ 0 & \text{otherwise}; \end{cases}
\]

here the trace is taken with respect the Euler-Poincaré measure on \( GL_2(F) \):

\[ \mathcal{H}(GL_2(F), Z(F), \chi_{\xi}) \]

Definition 5.2. Let \( \overline{G} \) be the compact inner form of \( GL_2(\mathbb{R})/Z(\mathbb{R}) \), and let \( \overline{\mathcal{H}} \) be the Haar measure on \( \overline{G} \) of total measure 1. We define the Euler-Poincare measure on \( GL_2(\mathbb{R})/Z(\mathbb{R}) \) as the unique Haar measure such that the induced measure on \( \overline{G} \) is \( \overline{\mathcal{H}} \).

The Euler-Poincare measure on \( GL_2(F) / Z(F) \) is given by the product measure under the identification
\[
\frac{GL_2(F)}{Z(F)} \cong \prod_{v \mid \infty} \frac{GL_2(F_v)}{Z(F_v)} \cong \prod_{v \mid \infty} \frac{GL_2(\mathbb{R})}{Z(\mathbb{R})}.
\]

To construct \( \phi_{\xi} \), it’s enough to have local functions \( \phi_{\xi_v} \) and let \( \phi_{\xi} = \prod_v \phi_{\xi_v} \).

Let \( K'_v = F_{v,0} \cdot O(2)_v \subseteq GL_2(F_v) \). For an irreducible finite-dimensional representation \( \xi_v \) of \( GL_2(F_v) \) and an admissible representation \( \pi_v \) such that \( \xi_v \) and \( \pi_v \) have the same central character on \( F_{v,0} \), we define the Euler-Poincaré characteristic:
\[
\chi_{\text{EP}}(\pi_v \otimes \xi'_v) = \sum_{i \geq 0} (-1)^i \dim H^i(\text{Lie } GL_2(F_v), K'_v, \pi_v \otimes \xi'_v); 
\]

(here the cohomology is \( (\mathfrak{g}, K) \)-Lie algebra cohomology).

Clozel and Delorme [6 Théorème 3] have constructed a function \( \phi_{\xi_v} \in \mathcal{H}(GL_2(F_v), F_{v,0}, \chi_{\xi_v}) \) such that
\[
\text{tr}_{F_{v,0}}(\pi_v(\phi_{\xi_v}, \mu_{\text{EP}}) = \chi_{\text{EP}}(\pi_v \otimes \xi'_v).
\]

It is moreover well-known that \( \chi_{\text{EP}}(\pi_v \otimes \xi'_v) = 0 \) unless \( \pi_v \) has the same infinitesimal character as \( \xi_v \) (see the bottom of page 43 of [10]). Since \( \pi_v \) and \( \xi_v \) also have the same central character (restricted to \( F_{v,0} \)), then the Langlands classification for admissible representations of \( GL_2(\mathbb{R}) \) tells us that if \( \text{tr} \pi_v(\phi_{\xi_v}) \neq 0 \), then \( \pi_v \) must be of one of the following three forms:
• \( \pi_v = \xi_v \)
• \( \pi_v \) is the discrete series complement of \( \xi_v \); i.e., there is an exact sequence
  
  \[ 0 \to \xi_v \to \mu_1 \times \mu_2 \to \pi_v \to 0 \]

  where \( \mu_1 \times \mu_2 \) is the representation induced from the character \( \mu_1 \otimes \mu_2 \) on the Borel subgroup.
• If \( \mu_1, \mu_2 \) is as above, then \( \pi_v = (\mu_1 \cdot \text{sgn}) \times \mu_2. \)

However, in the third case, \( \pi_v \) is in the continuous series, and since \( \text{tr} \pi'_v(\phi_{\xi_v}) = 0 \) for all other continuous-series representations \( \pi'_v \), then we must have \( \text{tr} \pi_v(\phi_{\xi_v}) = 0 \). We have therefore proved:

**Proposition 5.3.** Assume \( \pi'_v \) is infinite-dimensional, that \( \chi_{\text{EP}}(\pi'_v \otimes \xi'_v) \neq 0 \), and that \( \chi_{\pi'_v} \) and \( \chi_{\xi_v} \) agree on \( A_{G,\infty} \). Then \( \pi'_v \) is the discrete-series complement of \( \xi'_v \).

If \( \pi_{\xi_v} \) is the discrete-series complement of \( \xi_v \), then \( \text{tr}_{F_{v,>0}} \pi_{\xi_v}(\phi_{\xi_v}) = -1 \) (see the fact at the top of page 44 of [16]). By replacing \( \phi_{\xi_v} \) with \( g \to \frac{1}{2}(\phi_{\xi_v}(g) + \chi_{\xi_v}(-1)\phi_{\xi_v}(-g)) \), we may assume \( \phi_{\xi_v} \in \mathcal{H}(GL_2(F_v), Z(F_v), \chi_{\xi_v}) \). In this case we have

\[ \text{tr}_{F_{v,>0}} \pi_{\xi_v}(\phi_{\xi_v}) = -1. \]

Here we are making a choice of Haar measure that will be in effect for the rest of the paper: the Haar measure on \( GL_2(F_v)/Z(F_v) \) is chosen so that the finite group \( Z(F_v)/F_{v,>0} \cong \{ \pm 1 \} \) gets total measure 1, the measure on \( GL_2(F_v)/F_{v,>0} \) is the Euler-Poincaré measure, and the measures are compatible under

\[ 1 \to \frac{Z(F_v)}{F_{v,>0}} \to \frac{GL_2(F_v)}{F_{v,>0}} \to \frac{GL_2(F_v)}{Z(F_v)} \to 1. \]

We will need later that \( \phi_{\xi_v}(1) = -\dim(\xi_v) \). This basically follows from the Plancherel theorem for real groups, and is proven at the bottom of p. 276 in [2].

Let \( \xi = \bigotimes_v \xi_v \), and let \( \phi_{\xi} = \prod_v \phi_{\xi_v} \). Its discrete series complement is \( \bigotimes_v \pi_{\xi_v} \). We have proven the following:

**Corollary 5.4.** Let \( F \) be a totally real field and let \( \xi \) be an irreducible finite-dimensional representation of \( GL_2(F_\infty) \), whose complementary discrete series representation is \( \pi_\xi \). Then there is a function \( \phi_{\xi} \in \mathcal{H}(GL_2(F_\infty), Z(F_\infty), \chi_\xi) \) such that

• for any generic representation \( \pi \) of \( GL_2(F_\infty) \),
  \[ \text{tr}_{Z(F_\infty)} \pi(\phi_\xi) = \begin{cases} (-1)^{|F:Q|} & \text{if } \pi = \pi_\xi \\ 0 & \text{otherwise} \end{cases} \]

• \( \phi_{\xi}(1) = (-1)^{|F:Q|} \dim \xi \).

**Proof.** The only point that needs to be made is that \( \xi \) is generic (i.e. has a Whittaker model) if and only if it is infinite-dimensional at every place. \( \square \)

### 6. The Trace Formula

The goal of this section is to recall the versions of Arthur’s trace formulae for \( GL_2 \) that will be of use to us, and to specialize them to the fixed-central-character setting. It is worth noting here that an invariant version of the fixed-central-character trace formula has been proven in [3]. Palm ([9]) has stated a different version of the fixed-central-character trace formula following Jacquet-Langlads. However, because we want to use specific versions of the fixed-central-character trace formula, we see fit to prove the forms we need.

Throughout, we will use Arthur’s notation:

• \( GL_2(\mathbb{A})^1 \) is the set of \( g \in GL_2(\mathbb{A}) \) with \( |\det g| = 1 \).
• \( A_{G,\infty} \) is the image of the diagonal embedding \( \mathbb{R}_{>0} \hookrightarrow GL_2(F_\infty) \).

Then \( GL_2(\mathbb{A}) = GL_2(\mathbb{A})^1 \times A_{G,\infty} \), and \( GL_2(F) \subseteq GL_2(\mathbb{A})^1 \).

We begin with a definition.

**Definition 6.1.** Let \( \phi : GL_2(\mathbb{A}) \to \mathbb{C} \) be smooth and compactly-supported modulo center.
• Let $\gamma \in \text{GL}_2(\mathbb{A})$ and let $G_\gamma(\mathbb{A})$ be its centralizer in $\text{GL}_2(\mathbb{A})$. We define the orbital integral

$$O_\gamma(\phi) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} \phi(g^{-1} \gamma g) \, dg.$$ 

• Let $\gamma \in T(\mathbb{A}^\infty)$. We define the constant term

$$Q_\gamma(\phi) = \int_{K^\infty} \int_{A^\infty} \phi \left( k^{-1} \gamma \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k \right) \, da \, dk.$$ 

It is worth noting that if $\phi$ is a product of local functions, then the constant terms and orbital integrals decompose as a product of local constant terms and local orbital integrals.

**Definition 6.2.** Let $\gamma_v \in \text{GL}_2(F_v)$. We say $\gamma_v$ is elliptic if it is semisimple and the split component of the center of the centralizer $G_{\gamma_v}$ is $A_G(F_v)$. Equivalently, $\gamma_v$ is either central, or it is semisimple but not diagonalizable in $\text{GL}_2(F_v)$.

Let $\phi = \prod_v \phi_v \in C_c^\infty(\text{GL}_2(\mathbb{A}))$. We say $\phi$ is cuspidal at a place $v$ if for every element $\gamma_v \in \text{GL}_2(F_v)$ that is not central or elliptic, the orbital integral $O_{\gamma_v}(\phi_v)$ vanishes.

Here we note that the Euler-Poincare functions $\phi_\xi$ at $\infty$ from the previous section are cuspidal (see, for instance, page 267 of [2]). This will allow us to use simpler forms of the trace formula.

When $F = \mathbb{Q}$, the arguments on 267-268 and Theorem 6.1 of [2] show:

**Theorem 6.3.** Let $\phi = \phi_\infty \phi_\xi \in \mathcal{H}(\text{GL}_2(\mathbb{A}), A_{G,\infty}, \chi_{A_{G,\infty}})$, where $\phi_\xi$ is as in section Corollary 5.4. Then

$$\sum_\pi \text{tr}_{A_{G,\infty}}(\phi) = \sum_{\gamma \in Z(\mathbb{Q})} (-1)^{\eta(\gamma)} \text{vol}(G(\mathbb{Q})A_{G,\infty} \backslash G(\mathbb{A}))(\phi(\gamma))$$

$$+ \sum_{\gamma \in G(\mathbb{Q}) \backslash Z(\mathbb{Q})} C(G, \gamma) O_\gamma(\phi)$$

$$+ \sum_{\gamma \in T(\mathbb{Q})} C(T, \gamma) Q_\gamma(\phi).$$

On the left-hand side, $\pi$ is taken over discrete automorphic representations on which $A_{G,\infty}$ acts trivially. On the right-hand side, $C(G, \gamma)$, $C(T, \gamma)$ are constants that depend only on $\gamma$ and not on $\phi$.

The explicit values of the constants $C(G, \gamma)$ and $C(T, \gamma)$ are unnecessary for our purposes; the interested reader can see Theorem 6.1, and the subsequent remark, in [2] or (4.2), (4.3), and (4.4) of [13].

**Theorem 6.4.** [1 Corollary 7.5]. Let $F$ an arbitrary totally real field, let $\phi' \in C_c^\infty(\text{GL}_2(\mathbb{A}))$ be cuspidal at two distinct (possibly infinite) places, and let $\phi = \overline{\phi'} \in \mathcal{H}(\text{GL}_2(\mathbb{A}), A_{G,\infty}, \chi_{A_{G,\infty}})$ be its average over $A_{G,\infty}$ (see Definition 4.3). Then

$$\sum_\pi \text{tr}_{A_{G,\infty}}(\phi) = \sum_{\gamma \in Z(F)} \text{vol}(G(F)A_{G,\infty} \backslash G(\mathbb{A}))(\phi(\gamma))$$

$$+ \sum_{\gamma \in G(F) \backslash Z(F)} C(G, \gamma) O_\gamma(\phi).$$

Here the left-hand side is as in the previous theorem, and $C(G, \gamma) = \text{vol}(G_\gamma(F)A_{G,\infty} \backslash G_\gamma(\mathbb{A}))$.

**Proof.** Examining (3.3) and (4.4) in [1], Arthur proves this for when $\phi$ is the restriction of some $\phi'$ to $G(\mathbb{A})^1$, where $\phi'$ is cuspidal at 2 places. However, because $\text{GL}_2(\mathbb{A}) = A_{G,\infty} \times \text{GL}_2(\mathbb{A})^1$, there is no harm done by instead averaging over $A_{G,\infty}$. □
6.1. Trace formulae for fixed central character. The goal of the rest of the section is to extend these trace formulae to a trace formula with fixed central character, at least in a form that we will need.

We begin with a definition.

Definition 6.5. Let \( \mathfrak{x} \) be a closed subgroup of \( Z(\mathbb{A}) \) containing \( A_{G,\infty} \) such that \( Z(F)\cap \mathfrak{x} \) is discrete and cocompact in \( A_{G,\infty}\setminus \mathfrak{x} \). Let \( \phi \in \mathcal{H}(\text{GL}_2(\mathbb{A}), \mathfrak{x}, \chi) \). Define

\[
I_{\text{spec}}(\mathfrak{x}, \chi, \phi) = \sum_{\chi \ast |x = \chi} \text{tr}_x(\phi)
\]

(where the sum runs over discrete automorphic representations \( \pi \)), and

\[
I_{\text{geom}}(\mathfrak{x}, \chi, \phi) = \sum_{\gamma \in Z(F)\setminus \mathfrak{x}} \text{vol}(G(F)\setminus \text{GL}_2(\mathbb{A}))\phi(\gamma) + \sum_{\gamma \in (G(F)\setminus Z(F))\setminus \mathfrak{x}} C(G, \gamma) \text{vol}(Z(F)A_{G,\infty}\setminus \mathfrak{x})^{-1}O(\phi)
\]

\[
+ \sum_{\gamma \in T(F)\setminus \mathfrak{x}} C(T, \gamma) \text{vol}(Z(F)A_{G,\infty}\setminus \mathfrak{x})^{-1}Q(\phi)
\]

On both sides, the Haar measure on \( G \) is a product of the Euler-Poincare measure at \( \infty \) with the canonical measure at all finite places. We note that on the geometric side, each term is independent of the representative of \( G(F)\setminus \mathfrak{x} \) chosen since \( \chi \) is trivial on \( Z(F) \) and \( \phi \in \mathcal{H}(\text{GL}_2(\mathbb{A}), \mathfrak{x}, \chi) \).

We wish to prove the following:

Theorem 6.6. (Trace formula for \( \text{GL}_2 \) and fixed central character). Let \( F \) be a totally real field and let \( \phi \in \mathcal{H}(\text{GL}_2(\mathbb{A}), Z(\mathbb{A}), \chi) \). Assume either that \( F = \mathbb{Q} \), or that \( \phi \) is cuspidal at two places. Then

\[
I_{\text{spec}}(Z(\mathbb{A}), \chi, \phi) = I_{\text{geom}}(Z(\mathbb{A}), \chi, \phi).
\]

We’ll need two lemmas:

Lemma 6.7. (Poisson summation for ad\`ele groups). Let \( G = (\mathbb{A}^\times)^1 \) and let \( G^\vee \) be its dual. We say \( f : G \to \mathbb{C} \) is nice if it is Schwartz at every infinite place and is compactly-supported and locally constant at each finite place.

Let \( \Lambda \) be a discrete, cocompact subgroup of \( G \) and let \( \Lambda^\vee \) be the dual subgroup in \( G^\vee \). Then \( \Lambda^\vee \) is discrete, and for any nice function \( f \), we have

\[
\sum_{\lambda \in \Lambda} f(\lambda) = \text{vol}(G/\Lambda)^{-1} \sum_{\lambda^\vee \in \Lambda^\vee} \hat{f}(\lambda^\vee)
\]

Proof. To see that \( \Lambda^\vee \) is discrete, we note that \( \Lambda^\vee \) is the dual of \( G/\Lambda \), a compact group.

Under this duality, the dual measure on \( \Lambda^\vee \) gives each point the measure \( \text{vol}(G/\Lambda)^{-1} \). Let \( f'(g) = \sum_{\lambda \in \Lambda} f(g\lambda) \). This is \( L^1 \) and smooth on \( G/\Lambda \) since \( f \) is nice. We therefore have

\[
\sum_{\lambda \in \Lambda} f(\lambda) = f'(1_G) = \text{vol}(G/\Lambda)^{-1} \sum_{\lambda^\vee \in \Lambda^\vee} \hat{f}(\lambda^\vee) = \text{vol}(G/\Lambda)^{-1} \sum_{\lambda^\vee \in \Lambda^\vee} \hat{f}(\lambda^\vee)
\]

where the middle equality holds by Fourier inversion.

Lemma 6.8. Let \( \phi = \prod_v \phi_v \in \mathcal{H}(\text{GL}_2(\mathbb{A}), Z(\mathbb{A}), \chi) \) and let \( \widehat{\mathfrak{g}}_F \) be the maximal compact subgroup of \( (\mathbb{A}^\times)^\times \). Then there is a function \( \phi_0 = \prod_v \phi_{0,v} \in \mathcal{H}(\text{GL}_2(\mathbb{A}), \widehat{\mathfrak{g}}_F, \chi) \) such that

\[
\phi(g) = \int_{Z(\mathbb{A})} \phi_0(gz^{-1})\chi^{-1}(z) \, dz
\]

and such that, given \( \gamma_v \in \text{GL}_2(F_v) \), if \( O_{\gamma_v}(\phi_v) = 0 \), then \( O_{\gamma_v}(\phi_{0,v}) = 0 \).
Proof. If \( v \) is finite, then
\[
\phi_{0,v}(g_v) = \begin{cases} \text{vol}(O_{F,v})\phi_v(g_v) & |\det g_v|_v = 1 \text{ or } q_v \\ 0 & \text{otherwise} \end{cases}
\]
suffices. It is easy to check that \( \phi_{0,v} \) averages to \( \phi_v \), and it is compactly supported modulo the center. Finally, the orbital integrals vanish automatically unless \( |\det\tau_v| = 1 \) or \( q_v \), and in this case we have \( O_{\gamma}(\phi_{0,v}) = O_{\gamma}(\phi_v) \).

If \( v \) is a real place, we may assume by twisting that \( \chi_v = 1 \). Let \( \Phi_\infty : \mathbb{R}^\infty \to \mathbb{C} \) be smooth and compactly supported, with
\[
\int_{\mathbb{R}^\infty} \Phi_\infty(x) \, dx = 2.
\]
Let \( \phi_{0,v}(g_v) = \Phi_\infty(\det(g_v))\phi_v(g_v) \); this is compactly supported since \( e \) is compactly supported and because \( \phi_v \) is compactly supported modulo center. Given \( g \in \text{GL}_2(F_v) \) we have
\[
\int_{\mathbb{R}} \phi_{0,v}(gz) \, dz = \int_{\mathbb{R}^\infty} \phi_v(gz) \Phi_\infty(z^2 \det(g)) \, dz = \phi_v(g) \int_{\mathbb{R}^\infty} \Phi(z^2 \det(g)) \, dz = \phi_v(g)
\]
where the last equality holds because \( \Phi\) integrates to 2 over \( \mathbb{R}^\infty \). Finally, the orbital integral hypothesis holds because \( O_{\gamma}(\phi_{0,v}) = \Phi_\infty(\det(\tau_v))O_{\gamma}(\phi_v) \).

We’ll now prove Theorem 6.6. Before proving it in full detail, we’ll briefly outline the plan. By Lemma 6.8, we can pick a function \( \phi_0 \) that is cuspidal at two places and so that \( \overline{\phi_0} \) \( Z(\mathbb{A}) \cdot \chi = \phi \).

We’ll then produce nice functions \( \Psi_M : Z(\mathbb{A}) \to \mathbb{C} \) so that, as \( M \to \infty \), we have \( \Psi_M(\chi) = 1 \), and \( \Psi_M(\chi') \to 0 \) for any \( \chi' \neq \chi \). Letting \( \phi_M = \phi_0 \Psi_M \), we will use the Fourier-theoretic lemmas from section 4 to show that the spectral side of the trace formula for \( \phi_M \) approaches the spectral side of the trace formula for \( \phi \). On the geometric side, we’ll use Poisson summation to show that
\[
I_{\text{geom}}(Z(\mathbb{A}), \chi, \phi) = \text{vol}(Z(F)A_{G,\infty} | Z(\mathbb{A}))^{-1} \lim_{M \to \infty} I_{\text{geom}}(A_{G,\infty}, \chi_{A_{G,\infty}}, \phi_M),
\]
completing the proof.

Proof. Let \( \phi \in \mathcal{H}(\text{GL}_2(\mathbb{A}), Z(\mathbb{A}), \chi) \). We will write
\[
Z(\mathbb{A}) \cong F^\infty_v \prod_{v | \infty} F^\infty_v \cong A_{G,\infty} \{ \pm 1 \}^n \mathbb{R}^{n-1} \prod_{v | \infty} F^\infty_v
\]
where \( \prod_{v | \infty} \) denotes the restricted product with respect to \( \sigma^\infty_v \). We may assume that \( \phi \) takes the form
\[
\phi = \phi_{\pm} \prod_{v} \phi_v
\]
where \( \phi_{\pm} \) is a function on \( \{ \pm 1 \}^n \), \( \phi_{\mathbb{R}} \) is a function on \( \mathbb{R}^{n-1} \), and \( \phi_v \) is a function on \( F_v \) for all finite places \( v \).

Pick
\[
\phi'_v = \prod_{v} \phi'_{0,v} \in \mathcal{H}(\text{GL}_2(\mathbb{A}), \sigma^\infty_v \cdot \{ \pm 1 \}^n, \chi)
\]
so that
\[
\phi(g) = \int_{Z(\mathbb{A})} \phi'_v(gz^{-1}) \chi^{-1}(z) \, dz.
\]
and so that \( \phi'_v \) is cuspidal at all places where \( \phi \) is cuspidal. Let \( \phi_0 = \sigma^\infty_0 \), (where Haar measures on \( A_{G,\infty}, Z(\mathbb{A}) \), and \( Z(\mathbb{A})/A_{G,\infty} \) are chosen compatibly).

Recall from Lemma 4.7 that
\[
\Phi_{M,\chi,p}(z_p) = \frac{1}{M^2} \sum_{j=0}^{M-1} \int_{\mathbb{A}^{n-1}} \chi^{-1}(z) \, dz
\]
and let
\[
\Phi_{M,\chi,\mathbb{R}}(z_\mathbb{R}) = \chi(2\mathbb{R})^{-1} \frac{c_F}{M^{n-2}} e^{-\frac{z_\mathbb{R}^2}{2|z_\mathbb{R}|^2}}
\]
where \( c_F \) is chosen so that
\[
\int_{\mathbb{R}^{n-1}} \Phi_{M,\chi}(z) \chi(z) \, dz = 1.
\]
for all \( M \), under the Haar measure on \( \mathbb{R}^{n-1} \) induced by the Euler-Poincaré measure on \( Z(F_\infty) \setminus A_{G,\infty} \).

Let \( \Psi_M : Z(\mathbb{A}) \to \mathbb{C} \) be the function with
\[
\Psi_{M,\pm} = \chi_{\pm}^{-1}, \\
\Psi_{M}(z) = \Phi_{M,\chi}(z), \\
\Psi_{M,p}(z) = \Phi_{M,p}(z),
\]
where \( M \geq N(p) \) and \( M < N(p) \).

\( \Psi_M \) is compactly supported on \( Z(\mathbb{A}^\infty) \) and Schwartz on \( Z(F_\infty) \). We note that the lemmas above, and Arthur’s trace formulae, apply equally well if we take our functions to be Schwartz at \( \infty \).

Let \( \phi_M = \phi_0 \ast \Psi_M \). If we are in the situation where \( \phi_0 \) is cuspidal at two places, we must check that \( \phi_M \) is also cuspidal at two places. If the places are finite this is elementary. If the places are infinite, we need one more step. We extend \( \Psi_{M,R} \) to \( \Psi'_{M,R} \) on all of \( Z(F_\infty) \) by taking
\[
\Psi'_{M,R}(y) = c' \prod_{v \mid \infty} e^{- \frac{1}{2\pi} |y_v|^2}
\]
where \( c' \) is chosen so that \( \Psi'_{M,\infty} \) averages to \( \Psi_{M,R} \) over \( A_{G,\infty} \). Then we note that \( \phi_0' \ast \Psi'_{M,R} \) is cuspidal wherever \( \phi_0' \) is.

We now will show that
\[
I_{\text{spec}}(Z(\mathbb{A}), \chi, \phi) = \lim_{M \to \infty} I_{\text{spec}}(A_{G,\infty}, \chi_{A_{G,\infty}}, \phi_M)
\]
\[
= \lim_{M \to \infty} I_{\text{geom}}(A_{G,\infty}, \chi_{A_{G,\infty}}, \phi_M)
\]
\[
= I_{\text{geom}}(Z(\mathbb{A}), \chi, \phi).
\]
The middle equality follows from the existing automorphic trace formulae (Theorems 6.3 and 6.4); we need to show the first and third equalities.

For the first equality, for any automorphic representation \( \pi \),
\[
\text{tr} \pi(\phi_M) = \hat{\Psi}_M(\chi_{\pi}) \cdot \text{tr} \pi(\phi_0) \quad \text{by Lemma 4.7 (ii)}.
\]
At each finite place \( v \), we have already shown that if \( \chi_{\pi,v} \) and \( \chi_v \) differ by a ramified character, then \( \pi_v(\phi_{0,v}) = 0 \). If \( \chi_{\pi,v} \) and \( \chi_v \) differ by an unramified character, then for \( M \geq q_v \) we have
\[
\text{tr} \pi_v(\phi_{M,v}) = \frac{1}{M} F_M((\chi_{\pi,v} \chi_v^{-1})(\omega)) \text{tr} \pi_v(\phi_{0,v})
\]
which is equal to \( \text{tr} \pi_v(\phi_{0,v}) \) if \( \chi_{\pi,v}(\omega) = \chi_v(\omega) \), and approaches 0 otherwise.

At \( \infty \) we do a similar computation. First, if \( \chi_{\pi,\infty} \neq \chi_{\infty} \), then \( \pi_{\infty}(\phi_{0,\infty}) \) vanishes, and therefore so does \( \pi_{\infty}(\phi_{M,\infty}) \). Otherwise, we have
\[
\text{tr} \pi_{\infty}(\phi_{M,\infty}) = \hat{\Psi}_{M,R}(\chi_{\pi,\infty}) \text{tr} \pi_{\infty}(\phi_{0,\infty})
\]
This is equal to 1 for all \( M \) if \( \chi_{\pi,\infty} = \chi_{\infty} \), and approaches 0 otherwise.

As such, for any \( \pi \) with \( \chi_{\pi} \neq \chi \), we have \( \text{tr} \pi(\phi_M) \to 0 \) and \( |\text{tr} \pi(\phi_M)| \leq |\text{tr} \pi(\phi_0)| \). Then the Lebesgue dominated convergence theorem gives
\[
\lim_{M \to \infty} I_{\text{spec}}(A_{G,\infty}, \chi_{A_{G,\infty}}, \phi_M) = \sum_{\chi_{\pi} = \chi} \text{tr} \pi(\phi_0)
\]
\[
= \sum_{\chi_{\pi} = \chi} \text{tr} Z(\mathbb{A}) \pi(\phi) \quad \text{by Lemma 4.4}
\]
\[
= I_{\text{spec}}(Z(\mathbb{A}), \chi, \phi)
\]
For the second equality, we will show that
\[
\lim_{M \to \infty} \sum_{\gamma \in Z(F)} \phi_M(\gamma) = \text{vol}(Z(F) \setminus A_{G,\infty})^{-1} \phi(1).
\]
Let $\Theta_M = \phi_M|_{Z(\hat{A})}$ (so that $\hat{\Theta}_M$ is its Fourier transform). By Poisson summation we have

$$\lim_{M \to \infty} \sum_{\gamma \in Z(F)} \phi_M(\gamma) = \lim_{M \to \infty} \sum_{\gamma \in Z(F)} \Theta_M(\gamma)$$

$$= \text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1} \lim_{M \to \infty} \sum_{\tau:Z(\hat{A}) \to \mathbb{C}^\times \atop \tau(Z(F))=1} \hat{\Theta}_M(\tau)$$

$$= \text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1} \lim_{M \to \infty} \sum_{\tau:Z(\hat{A}) \to \mathbb{C}^\times \atop \tau(Z(F))=1} \hat{\Theta}_0(\tau)\hat{\Psi}_M(\tau)$$

We have shown above that $\hat{\Psi}_M(\chi) = 1$ and that $\hat{\Psi}_M(\tau) \to 0$ if $\tau \neq \chi$. Since $\Theta_0$ is smooth and compactly supported, $\sum_{\tau} \hat{\Theta}_0(\tau)$ is absolutely summable. Therefore, by dominated convergence, the final limit is simply $\text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1}\hat{\Theta}_0(\chi)$. But

$$\hat{\Theta}_0(\chi) = \int_{Z(\hat{A})} \phi_0(z)\chi(z)\,dz = \phi(1)$$

by construction, completing the claim.

We can similarly show that

$$O_\gamma(\phi) = \text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1} \sum_{\gamma^{-1} \in Z(F)} O_\gamma(\phi);$$

this follows from putting $\Theta_M(z) = O_{\gamma z}(\phi_M)$ and once again applying Fourier summation. The analogous result for constant terms follows similarly. We therefore have

$$\lim_{M \to \infty} I_{\text{geom}}(A_{G,\infty}, \chi, \phi_M) = \lim_{M \to \infty} \sum_{\gamma \in Z(F)} \text{vol}(G(F)A_{G,\infty}\setminus \text{GL}_2(\hat{A}))\phi_M(\gamma)$$

$$+ \lim_{M \to \infty} \sum_{\gamma \in G(F)\setminus Z(F) \atop \gamma \text{ semisimple}} C(G, \gamma)O_\gamma(\phi_M)$$

$$+ \lim_{M \to \infty} \sum_{\gamma \in T(F)\setminus Z(F) \atop \gamma \text{ elliptic}} C(T, \gamma)Q_\gamma(\phi)$$

$$= \frac{\text{vol}(G(F)A_{G,\infty}\setminus \text{GL}_2(\hat{A}))}{\text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))} \cdot \phi(1)$$

$$+ \sum_{\gamma \in G(F)\setminus Z(F) \atop \gamma \text{ semisimple} \atop \gamma \in \text{elliptic}} C(G, \gamma)\text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1}O_\gamma(\phi)$$

$$+ \sum_{\gamma \in T(F)\setminus Z(F) \atop \gamma \in \text{elliptic}} C(T, \gamma)\text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))^{-1}Q_\gamma(\phi)$$

$$= I_{\text{geom}}(Z(\hat{A}), \chi, \phi)$$

where the last equality holds because we have $\text{vol}(\text{GL}_2(F)Z(\hat{A})\setminus \text{GL}_2(\hat{A})) = \frac{\text{vol}(\text{GL}_2(F)A_{G,\infty}\setminus \text{GL}_2(\hat{A}))}{\text{vol}(Z(F)A_{G,\infty}\setminus Z(\hat{A}))}$. This completes the proof.

7. The Plancherel Measure on the Tempered Spectrum

We begin by discussing the canonical measure on the tempered spectrum.

**Definition 7.1.** Let $i : X \to Y$ be a surjective, finite map of orbifolds equipped with measures $\mu_X, \mu_Y$. We say $i$ **locally preserves measures** if there is an open $X' \subseteq X$ so that $\mu_X(X - X') = 0$, $\mu_Y(i(X - X')) = 0$, and so that there is an open cover $\{U_\alpha\}$ of $X'$ such that, for each $U \subseteq U_\alpha$, $\mu_X(U) = \mu_Y(i(U))$. 

This definition may be ugly, but has the following useful property: if \( i : X \to Y \) locally preserves measures and \( E \subseteq X \) is an open fundamental domain for the map (so that \( E \to Y \) is injective and covers \( Y \) up to a set of measure 0), then for any function \( h : Y \to \mathbb{C} \) we have

\[
\int_Y f \, d\mu_Y = \int_E (f \circ i) \, d\mu_X.
\]

We’ll use this to define the canonical measure on a tempered orbit \( \mathcal{O} \). To this end, let \( \mathcal{O} \) be such an orbit and pick a discrete series representation \( \omega_0 \) of a Levi subgroup \( M \) so that \( I_M^G(\omega_0) \in \mathcal{O} \). This gives a surjective map \( X_u(M) \to \mathcal{O} \) via \( \chi_M \mapsto I_M^G(\omega_0 \otimes \chi_M) \).

**Definition 7.2.** Let \( M \subseteq \text{GL}_2(L) \) be a Levi subgroup with center \( Z(M) \). Let \( \mathcal{O} \) be an orbit in \( \text{GL}_2(L)^{\text{r,t}} \) induced from \( M \). Consider the surjective, finite maps

\[
X_u(Z(M)) \xrightarrow{i} X_u(M) \xrightarrow{j} \mathcal{O}.
\]

We give \( X_u(Z(M)) \) the Haar measure with total measure 1. If measures \( d\chi_M \) on \( X_u(M) \) and \( d\pi \) on \( \mathcal{O} \) are chosen so that \( i, j \) locally preserve measures, then we call \( d\pi \) the canonical measure on \( \mathcal{O} \).

**Remark 7.3.** Note that this definition differs from the definition in Aubert-Plymen’s paper [5]: they say \( i : X \to Y \) locally preserves measures if \( \mu_Y(V) = \mu_X(i^{-1}V) \) for measurable subsets \( V \) of \( Y \). They define the canonical measure on \( \mathcal{O} \) so that

\[
X_u(A_M) \to X_u(M) \to \mathcal{O}_M
\]

preserves measures (under their definition): then \( \mathcal{O}_M \to \mathcal{O} \) locally preserves measure under our definition. We believe this to be incorrect, since it would give the Steinberg orbit a canonical measure of 1. However (in view of their later computations, which we believe to be both elegant and correct), Harish-Chandra’s Plancherel theorem will later determine that a Steinberg orbit must have canonical measure 2.

We compute the canonical measures of the four types of orbits.

1. If \( \mathcal{O} \) is a principal series orbit corresponding to \( \chi_0 \times \chi_0 \), then \( M = T \) and so the map \( X_u(T) \to X_u(Z(T)) \) is an isomorphism. The map \( X_u(T) \to \mathcal{O} \) is a double cover outside a set of measure zero, so \( \omega(\mathcal{O}) = 1/2 \).
2. If \( \mathcal{O} \) is a principal series orbit corresponding to \( \chi_0 \times \chi'_0 \), then \( M = T \) and both maps are isomorphisms, so the canonical measure is 1.
3. If \( \mathcal{O} \) is a Steinberg orbit, then \( M = G \) and \( X_u(G) \to X_u(Z(G)) \) is a double cover, so the canonical measure of \( X_u(G) \) is 2. The map \( X_u(G) \to \mathcal{O} \) is an isomorphism so we have \( \omega(\mathcal{O}) = 2 \).
4. If \( \mathcal{O} \) is a supercuspidal orbit, let \( r(\pi) \) be the order of the group \( \{ \chi \in X_u(G) : \pi \otimes \chi \cong \pi \} \) (note that this is independent of \( \pi \in \mathcal{O} \)). As above, the canonical measure of \( X_u(G) \) is 2, and \( X_u(G) \to \mathcal{O} \) is a cover of degree \( r(\pi) \), so \( \omega(\mathcal{O}) = \frac{2}{r(\pi)} \).

**Definition 7.4.** Let \( \mathcal{F}_0(\text{GL}_2(L)^\wedge) \) denote the set of bounded functions \( \hat{f} : \text{GL}_2(L)^\wedge \to \mathbb{C} \) that is supported on a finite number of Bernstein components and whose restriction to the tempered spectrum is continuous outside a set of canonical measure 0.

**Remark 7.5.** Throughout this paper, we will use \( \hat{f}, \hat{h} \) to denote arbitrary functions in \( \mathcal{F}_0(\text{GL}_2(L)) \) or \( \mathcal{F}_0(\text{GL}_2(F_S)) \). We will refer to \( \hat{\phi}, \hat{\psi} \) only when referring to a Plancherel transform of a function living in some Hecke algebra (see Definition 4.1). This convention is in line with [15] [16].

Note that, when restricted to the tempered spectrum, \( \hat{\phi} \) is supported on finitely many orbits and is smooth with respect to the structure of \( \mathcal{O} \) as a real manifold.

We recall Harish-Chandra’s Plancherel theorem, following Waldspurger [19]:

**Theorem 7.6.** (Harish-Chandra’s Plancherel Theorem). There is a continuous function \( \nu^{\text{pl}} : \text{GL}_2(L)^{\text{r,t}} \to \mathbb{C} \), called the Plancherel density function, such that, for any \( \phi \in C_c^\infty(\text{GL}_2(L)) \) we have

\[
\phi(1) = \int_{\text{GL}_2(L)^{\text{r,t}}} \nu^{\text{pl}}(\pi) \hat{\phi}(\pi) \, d\pi
\]
where $d\pi$ is the canonical measure.

For any $f \in \mathcal{F}_0(\text{GL}_2(L)^\omega)$, we define

$$\hat{\mu}^\text{pl}(f) = \int_{\text{GL}_2(L)^\omega} \nu^\text{pl}(\pi) \hat{f}(\pi) \, d\pi.$$ 

Remark 7.7. The Plancherel density function is described explicitly in [19] as

$$\nu^\text{pl}(I_0^F \omega) = \gamma(G|M)^{-1} j(\omega)^{-1} d(\omega).$$

The $\gamma$ factor is as described and computed on p. 241. The term $d(\omega)$ is the formal degree of $\omega$; this is defined by the condition that

$$\int_{A.M \setminus M} \langle \omega(m)v_1, \tilde{v}_1 \rangle \langle v_2, \tilde{\omega}(m)v_2 \rangle \, dm = d(\omega)^{-1} \langle v_1, \tilde{v}_2 \rangle \langle v_2, \tilde{v}_1 \rangle$$

for $v_1, v_2 \in V_\omega$ and $\tilde{v}_1, \tilde{v}_2 \in V_{\tilde{\omega}}$, where $\tilde{\omega}$ is the contragredient representation.

Finally, the $j(\omega)$ is the scalar given by an intertwining operator $I_0^F \omega \to I_0^F \omega$; these intertwining operators are defined in ch. I.

Note the dependence on Haar measures: $\hat{\phi}(\pi)$ depends on a Haar measure on $G$, whereas $j(\omega)^{-1}$ and $d(\omega)$ depend inversely on Haar measures on $N, M$ respectively (where $N$ is the unipotent radical of $P = MN$); we choose Haar measures $dg, dm, dn, dk$ so that $dk$ is the restriction of $dg$ to the maximal compact subgroup $K$ and so that

$$\int_G \phi(g) \, dg = \int_M \int_N \int_K \phi(mnk) \, dk \, dn \, dm$$

for any $\phi \in C_c^\infty(G)$.

Remark 7.8. The Plancherel theory carries over to a finite product of $p$-adic fields. In our situation, if $F$ is a number field and $S$ a finite set of finite places of $F$ we may define

$$\text{GL}_2(F_S) = \prod_{v \in S} \text{GL}_2(F_v)$$

so that

$$\text{GL}_2(F_S)^\omega = \prod_{v \in S} \text{GL}_2(F_v)^\omega.$$ 

If we define $\hat{\mu}^\text{pl}_S$ as the product of the Plancherel measures on $\text{GL}_2(F_v)^\omega$ then Harish-Chandra’s Plancherel density theorem go through exactly as desired. In particular, $\phi_S(1) = \hat{\mu}^\text{pl}_S(\hat{\phi}_S)$.

Remark 7.9. We will need explicit descriptions of the Plancherel density on orbits $O \subseteq \text{GL}_2(L)^\omega$.

Parts (1), (2), (3), and part of (4) have been computed by [15], and we recall the results here, with appropriate citations in [5], and give an explicit value of the formal degree for supercuspidal representations, as computed in [7]. Note that Aubert-Plymen’s function $\mu_{G|M}(\omega)$ satisfies

$$\mu_{G|M}(\omega)c(G|M)^{-2} \gamma(G|M)^{-1} = \gamma(G|M) j(\omega)^{-1}.$$ 

Here $c(G|M)$ is defined as in Waldspurger directly following the definition of $\gamma(G|M)$; for $G = \text{GL}_2(L)$, we have $c(G|M) = 1$ for all Levi subgroups $M$.

(1) If $O$ corresponds to $\chi_0 \times \chi_0$, then $\omega(O) = \frac{1}{2}$. We have $M = T$ so $\gamma(G|M) = \frac{q+1}{q}$, and $d(\omega) = 1$. [5 Thm. 4.4] then gives

$$\mu_{G|M}(\chi \otimes \chi') = \frac{(q+1)^2}{q^2} \left| \frac{1 - (\chi' \chi^{-1})(\omega)}{1 - q^{-1}(\chi' \chi^{-1})(\omega)} \right|^2$$

so that

$$\nu^\text{pl}(\chi \otimes \chi') = \frac{q + 1}{q} \left| \frac{1 - (\chi' \chi^{-1})(\omega)}{1 - q^{-1}(\chi' \chi^{-1})(\omega)} \right|^2.$$ 

Integrating this function on $S^1 \times S^1$ yields 2, so that $\hat{\mu}^\text{pl}(O) = 1$.

We remark that the density function is independent of choice of uniformizer since $\chi_2$ and $\chi_1$ differ by an unramified character.
(2) If $\mathcal{O}$ is a principal series orbit corresponding to $\chi_0 \times \chi'_0$, then the canonical measure is $1$. [5, Thm 4.3] says that $\nu^{pl}$ is constant on such orbits and equal to
\[
\gamma(G|M)q^{c(\chi_0^{-1})} = \frac{q + 1}{q}q^{c(\chi_0^{-1})}
\]
so that $\hat{\mu}^{pl}(\mathcal{O}) = \frac{q + 1}{q}q^{c(\chi_0^{-1})}$.

(3) If $\mathcal{O}$ is a Steinberg orbit, then $M = G$ and thus the canonical measure is $2$. The $\gamma$ and $j$-terms are uniformly $1$, so we simply need to find the formal degree. The formal degree of a Steinberg representation of $\text{GL}_2(L)$ is $\frac{q-1}{2}$, so $\nu^{pl} = \frac{q-1}{2}$ and $\hat{\mu}^{pl}(\mathcal{O}) = q - 1$; see [5] (17) or [7] (2.2.2) and note that the formal degree of a Steinberg is invariant under twisting by any unitary character.

(4) If $\mathcal{O}$ is a supercuspidal orbit, then the same logic as above says that $\nu^{pl}(\pi) = d(\pi)$ and that this is constant on $\mathcal{O}$, so that $\hat{\mu}^{pl}(\mathcal{O}) = \frac{2d(\pi)}{r(\pi)}$.

Let $\pi = \pi_\eta$ for $\eta : L^\times \to \mathbb{C}^\times$, with conductor $c(\eta)$. By fact (iv) after Definition 2.3, if $L'/L$ is unramified then $r(\pi) = 2$, and if $L'/L$ is ramified then $r(\pi) = 1$. Moreover, the formal degrees are computed in [7] Theorem 2.2.8. From the remark between (2.1.2) and (2.1.3) of that article, we deduce that $\alpha(\eta)$ is the minimal conductor of all characters of the form $\eta \cdot (\chi \circ N_{L'/L})$, as $\chi$ ranges over all characters of $L^\times$.

Then Theorem 2.2.8 proves that if $L'/L$ is unramified, then $d(\pi) = (q-1)q^{\alpha(\eta) - 1}$ (note that the quantity given in 2.2.8 must be multiplied by $\frac{2}{q-1}$ because they choose their Haar measure so that $\text{vol}(KZ/Z) = d(\text{St}) = \frac{q-1}{2}$, whereas we choose it to be $1$, and the formal degree depends inversely on the choice of Haar measure). Similar logic says that if $L'/L$ is ramified, then $d(\pi) = \frac{1}{2}(q^2 - 1)q^{\frac{\alpha(\eta)}{2} - 1}$. We note that $\alpha(\eta)$ is even since if $L'/L$ is ramified and $\eta$ is trivial on $1 + p^nL^\times_\mathbb{R}$ then we can pick $\chi$ so that $\eta \cdot (\chi \circ N_{L'/L})$ is trivial on $1 + p^nL^\times_\mathbb{R}$.

We conclude this chapter by stating Sauvageot’s density theorem [11] Thm. 7.3:

**Theorem 7.10.** (Sauvageot’s Density Theorem). Let $\hat{f}_s \in \mathcal{F}_0(\text{GL}_2(F_s)\chi)$ and fix $\epsilon > 0$. Then there are functions $\phi_s, \psi_s \in C_c(\text{GL}_2(F_s))$ such that

(i) $|\hat{f}_s(\pi) - \hat{\phi}_s(\pi)| \leq \psi_s(\pi)$ for all $\pi \in \text{GL}_2(F_s)\chi$, and
(ii) $\hat{\mu}^{pl}(\psi_s) < \epsilon$.

**Remark 7.11.** It is worth noting that there is a larger class of functions $\mathcal{F}(\text{GL}_2(F_s)\chi) \supseteq \mathcal{F}_0(\text{GL}_2(F_s)\chi)$ for which Sauvageot’s density theorem holds; in fact, in Sauvageot’s paper he proves that the space $\mathcal{F}$ is precisely the set of functions that can be approximated. We omit the definition since it will be unnecessary for our purposes.

**Remark 7.12.** It is worth comparing the tempered orbits of our situation to Weinstein’s inertial types at finite places [20]. Using our characterisation of orbits $\mathcal{O}$, we see that if $\pi$ and $\pi'$ are tempered representations in the same orbit, then their associated Weil-Deligne representations $\rho(\pi)$, $\rho'(\pi)$ have the same restriction to the inertia subgroup $I_L$ and the same monodromy operator. As such, two tempered representations are in the same orbit if and only if they have the same inertial type.

We claim that if an inertial type $\tau_{\infty} = (\tau_p)_{p | \infty}$ is unramified outside the finite set $S$ of finite places, and $\tau_p$ corresponds to the orbit $\mathcal{O}_p$, then
\[
d(\tau_{\infty}) = \hat{\mu}^{pl}_{S}(\prod_{p \not\in S} \mathcal{O}_p).
\]

When $\mathcal{O}$ is non-supercuspidal, this follows simply by comparing $d(\tau_p)$ in [20] pp. 1390, 1393] to $\hat{\mu}^{pl}(\mathcal{O}_p)$ as given in Remark 7.9.

There is a discrepancy when $\mathcal{O}$ is a supercuspidal orbit corresponding to a character $\eta$ on a ramified extension. We believe this to be a minor miscomputation. The value of $\dim(\tau(\pi))$ is given on page 1394 and should be equal to $|\text{GL}_2(O_F) : J^0|$, where $J^0$ is given as in (3) on page 1398. An explicit computation of $J^0$ shows that the index is $(q^2 - 1)q^{\frac{\epsilon(\eta)}{2}}$, not $(q^2 - 1)q^{c(\eta) - 2}$ as on page 1394. This matches up with the Plancherel measure of the supercuspidal orbit as given in Remark 7.9 (4), following [7].
8. Plancherel measures for fixed central character

The goal of this section is to extend the results of the previous section to the fixed-central-character situation.

Let \( \chi : L^\times \to \mathbb{C}^\times \) be a unitary character; we let \( \text{GL}_2(L)^{\wedge, \chi} \) be the collection of tempered representations of central character \( \chi \). Given an orbit \( O \), let \( O_\chi \) be the set of \( \pi \in O \) with \( \chi\pi = \chi \); then \( O_\chi \) is either empty or an orbifold of codimension 1 in \( O \). Assume henceforth that \( O_\chi \) is nonempty, and take \( \pi_0 \in O \) with \( \chi\pi_0 = \chi \). If \( \pi \) is induced from a discrete series representation \( \omega_0 \) of \( M \), then the surjection

\[
X_u(M) \to O \quad \tau \mapsto I^G_\omega(\omega_0 \otimes \tau)
\]

restricts to a surjection

\[
X_u(M)_0 \to O_\chi
\]

where \( X_u(M)_0 \) is the kernel of the restriction map \( X_u(M) \to X_u(Z(G)) \).

Henceforth, if \( Y \to Z \) is a finite surjective map of orbifolds, say \( E \subseteq Y \) is a nice fundamental domain if \( E \to Z \) is injective and surjective, and there is an open set \( U \subseteq Y \) with \( U \subseteq E \subseteq \overline{U} \).

**Lemma 8.1.** Let \( E \) be a nice fundamental domain of \( X_u(M)_0 \to O \), let \( D_0 \) be a nice fundamental domain of \( X_u(G) \to X_u(Z(G)) \), and let \( D \) be the image of \( D_0 \) under \( X_u(G) \to X_u(M) \). Then \( D \times E \to X_u(M) \) is injective, and its image is a nice fundamental domain for \( X_u(M) \to O \).

**Proof.** First, for any \( \chi \in X_u(M) \) and \( d \in D \), if \( \chi \mapsto \pi \), then \( d\chi \mapsto \pi \otimes d \). This follows because the map \( X_u(M) \to O \) is decomposes as

\[
X_u(M) \to O_M \to O,
\]

where the first map is \( \chi \mapsto \omega_0 \otimes \chi \) for a chosen basepoint \( \omega_0 \) and the second map is just normalized induction, and because if \( \chi_G \) is a character of \( G \) with restriction \( \chi_M \) on \( M \), then

\[
I^G_\omega(\omega \otimes \chi_M) = I^G_\omega(\omega) \otimes \chi_G.
\]

It’s clear that no two distinct pairs \( de, d'e' \) can map to the same element in \( O \). If this were true, then because multiplying by \( e \) or \( e' \) does not change the central character, and \( d, d' \) are chosen in a fundamental domain for \( X_u(G) \to X_u(Z(G)) \), then \( d' = d \), and then twisting by \( d^{-1} \) proves \( e' = e \), since \( E \) is a nice fundamental domain for \( X_u(M)_0 \to O_\chi \). Surjectivity is basically the same; fix \( \pi \in O \), and then twist by an element of \( D \) so that \( \chi_{\pi \otimes d} = \chi_{\pi_0} \); then there is an \( e \in E \) with \( e \mapsto \pi \otimes d \).

To see that the image of \( D \times E \) is a nice fundamental domain, pick open subsets \( E' \subseteq E \) in \( X_u(M)_0 \) and \( D'_0 \subseteq D_0 \) in \( X_u(G) \), with image \( D' \) in \( X_u(M) \), then clearly \( D'E' \) is open in \( X_u(M) \) (since \( X_u(G)X_u(M)_0 = X_u(M) \)), and its closure contains \( DE \), completing the proof. \( \Box \)

**Definition 8.2.** Define the canonical measure \( d\pi_\chi \) on \( O_\chi \) as follows; equip \( X_u(M)_0 \) with the Haar measure whose total volume is the canonical measure of \( X_u(M) \), and choose the canonical measure on \( O_\chi \) so that the map \( X_u(M)_0 \to O_\chi \) locally preserves measures.

**Proposition 8.3.** Let \( dy \) be the Haar measure on \( \hat{L}^\times \) giving each unramified orbit total measure 1. Then for any \( f \in L^1(\text{GL}_2(L)^{\wedge, \chi}) \), we have

\[
\int_{\text{GL}_2(L)^{\wedge, \chi}} \hat{f} \ d\pi = \int_{\hat{L}^\times} \int_{\text{GL}_2(L)^{\wedge, \chi}} \hat{f} \ d\pi_\chi \ d\chi.
\]

**Proof.** By summing over all orbits, we just need to prove this for \( \hat{f} \) supported on a single orbit \( O \). Consider the nice fundamental domain \( DE \subseteq X_u(M) \) constructed in Lemma 8.1; if we fix \( d \), we that \( dE \) maps surjectively and injectively onto \( O_{\chi \otimes dZ(G)} \), where \( dZ(G) \) is the restriction of \( d \) to \( Z(G) \).
Therefore, if we pull back \( \hat{f} \) to \( \hat{f}_{DE} \) on \( DE \), we have
\[
\int_{DE} \hat{f} \, d\pi = \int_{DE} \hat{f}_{DE}(de) \, dd \, de \\
= \int_D \int_E \hat{f}_{DE}(de) \, de \, dd \\
= \int_{O_\gamma \neq \emptyset} \int_{O_\gamma} \hat{f}(\pi) \, d\mu_\chi \, d\chi
\]
where the last equality holds because fixing \( dE \) maps surjectively and injectively onto \( O_\chi \otimes_{Z(G)} \), and because the total measure on \( DE \) is the same as the total measure on \( O \).

\[\text{□}\]

**Corollary 8.4.** Let \( dx \) be as above and let \( \nu_{pl}^0 \) be the Plancherel density function from the previous section, and let \( \hat{\mu}_\chi^1 = \nu_{pl}^1 \chi \) on \( GL_2(L)^{\wedge,t,x} \). For any \( \hat{f} \in \mathcal{F}_0(GL_2(L)^x) \), we have
\[
\hat{\mu}_\chi^1(\hat{f}) = \int_{L^x} \hat{\mu}_\chi^1(\hat{f}) \, d\chi.
\]

**Proof.** Apply the above proposition to the function \( \pi \mapsto \nu_{pl}^1(\pi) \hat{f}(\pi) \). \( \text{□} \)

With this in hand, we will give an explicit discussion of the fixed-central-character Plancherel measure on \( GL_2(L)^{\wedge,t,x} \).

(1) If \( O_\chi \) is a principal series orbit corresponding to \( \chi_0 \times \chi_0 \) (where we must have \( \chi_0^2 = \chi_0 = \chi_0^2 \)), assume the basepoint is \( \psi = \chi' \otimes \chi' \) (with \( \chi'^2 = \chi \)). Then \( O_\chi \) is topologically isomorphic to \( S^1 / \Sigma_2 \), where \( \Sigma_2 \) acts via \( z \mapsto \bar{z} \). The canonical measure is \( 1/2 \), and the Plancherel density function is
\[
\nu_{pl}^1(\chi_1 \times \chi_2) = q^{+1} \left| \frac{1 - \chi_2 \chi_1^{-1} \omega}{1 - q^{-1} \chi_2 \chi_1^{-1} \omega} \right|^2
\]
so the total Plancherel measure of this orbit is 1.

(2) If \( O_\chi \) is a principal series orbit corresponding to \( \chi_0 \times \chi_0' \), where \( \chi_0 \neq \chi_0' \), then \( O_\chi \) is topologically isomorphic to \( S^1 \) and has canonical measure 1. The Plancherel density function is uniformly equal to \( q^{\nu(\chi_0' \chi_0^{-1})} \frac{1}{\omega} \), so the total Plancherel measure is \( q^{\nu(\chi_0' \chi_0^{-1})} \frac{\omega - 1}{\omega} \).

(3) If \( O_\chi \) is a Steinberg orbit, then \( O_\chi \) consists of two points. The total canonical measure is 2, so \( d\pi_\chi \) is the counting measure. Since the formal degree \( d(\pi) \) is \( \frac{1}{2} \), then the Plancherel measure in this case is \( \frac{1}{2} \) times the counting measure.

(4) If \( O_\chi \) is a supercuspidal orbit, let \( \pi_\chi = \pi \). If \( r(\pi) = 1 \), then \( O_\chi \) consists of two points, and the total canonical measure is 2, so the Plancherel measure is the counting measure. If \( r(\pi) = 2 \), then \( O_\chi \) consists of one point, but the canonical measure is 1, so once again we get the counting measure. Therefore, the Plancherel measure of a point is simply \( d(\pi) \), and \( d(\pi) \) is as computed in [Theorem 2.2.8] and described in Remark 7.9, (4) above.

**Proposition 8.5.** (Harish-Chandra’s Plancherel theorem for fixed central character). Let \( \phi \in \mathcal{H}(GL_2(L), Z(L), \chi) \). Then
\[
\phi(1) = \hat{\mu}_\chi^1(\phi).
\]

**Proof.** We will use the Fourier-theoretic ideas from section 4. Let
\[
\phi_0(g) = \begin{cases} \phi(g) & |\det(g)|_L = 1 \text{ or } qL \\ 0 & \text{otherwise} \end{cases}
\]
so that \( \overline{\phi}_0 \chi = \phi \), and therefore if \( \chi_\pi = \chi \) we have \( \text{tr}_{Z(L)}(\pi(\phi)) = \text{tr} \pi(\phi_0) \) by Lemma 4.4.

Let \( A_j \) be the annulus \( \{ z \in L^x : -j \leq v_L(z) \leq j \} \), and let
\[
\Psi_{M,\chi}(z) = M \Phi_{M,\chi} = \frac{1}{M} \sum_{j=0}^{M-1} 1_{A_j}(z) \chi^{-1}(z)
\]
where $\Phi_{M, \chi}$ is as in Lemma 4.8. Define
\[
\phi_M(g) = (\phi_0 \ast \Psi_{M, \chi})(g)
\]
\[
= \frac{1}{M} \sum_{j=0}^{M-1} \int_{A_j} \phi_0(gz)\chi(z)\,dz
\]
so that
\[
\phi(1) = \lim_{M \to \infty} \phi_M(1) = \lim_{M \to \infty} \hat{\mu}_M(\hat{\phi}_M).
\]
Moreover, we have
\[
\lim_{M \to \infty} \hat{\mu}_M(\hat{\phi}_M) = \lim_{M \to \infty} \int_{L^g} \text{tr} \pi(\phi_M) \, d\hat{\mu}_M(\pi)
\]
\[
= \lim_{M \to \infty} \int_{L^g} \text{tr} \pi(\phi_M) \, d\hat{\mu}_M(\pi) \, d\chi'
\]
\[
= \lim_{M \to \infty} \int_{L^g} \hat{\Psi}_{M, \chi}(\chi') \cdot \text{tr} \pi(\phi_0) \, d\hat{\mu}_M(\pi) \, d\chi'
\]
by Lemma 4.7
\[
= \lim_{M \to \infty} \int_{L^g} \hat{\Psi}_{M, \chi}(\chi') \left( \int_{\chi = \chi'} \text{tr} \pi(\phi_0) \, d\hat{\mu}_M(\pi) \right) \, d\chi'
\]
By Lemma 4.8, $\hat{\Psi}_{M, \chi}(\chi') = 0$ if $\chi'\chi^{-1}$ is ramified. If $\chi'\chi^{-1}$ is ramified, then $\hat{\Psi}_{M, \chi}(\chi') = F_M(\chi'\chi^{-1}(\pi))$, where $F_M : S^1 \to \mathbb{C}$ is the Fejér kernel. Since
\[
\int_{S^1} F_M(s) \, ds = 1
\]
and
\[
\lim_{M \to \infty} \int_{S^1 - U} F_M(s) \, ds = 0
\]
for any open neighborhood of 1 in $S^1$, then for any continuous function $h : S^1 \to \mathbb{C}$ we have
\[
\lim_{M \to \infty} \int_{S^1} F_M(s)h(s) \, ds = h(1).
\]
As such, we have
\[
\lim_{M \to \infty} \int_{L^g} \hat{\Psi}_{M, \chi}(\chi') \left( \int_{\chi = \chi'} \text{tr} \pi(\phi_0) \, d\hat{\mu}_M(\pi) \right) \, d\chi' = \int_{\chi = \chi} \text{tr} \pi(\phi_0) \, d\hat{\mu}_M(\pi)
\]
\[
= \int_{\chi = \chi} \text{tr} \pi(\phi) \, d\hat{\mu}_M(\pi) \quad \text{by Lemma 4.4}
\]
\[
= \hat{\mu}_M(\hat{\phi})
\]
completing the proof. \qed

To complete this section, we prove Sauvageot’s density theorem for fixed central character. This will follow from

**Proposition 8.6.** (Sauvageot density for fixed central character). Let $f_\chi : \text{GL}_2(L) \to \mathbb{C}$ be bounded, supported on a finite number of orbits, and continuous outside a set of measure 0. Then for any $\epsilon > 0$, there are functions $\phi_\chi, \psi_\chi \in \mathcal{H}(\text{GL}_2(L), Z(L), \chi)$ with

- $|\hat{\phi}_\chi - \hat{f}_\chi| \leq \psi_\chi$, and
- $\hat{\mu}_M(\psi_\chi) < \epsilon$.

**Proof.** Let $D$ be a nice fundamental domain for $X_u(G) \to X_u(Z(G))$, and define $\hat{f}$ on $\text{GL}_2(L)^g$ by $\hat{f}(\pi) = \hat{f}_\chi(\pi \odot d)$, where $d \in D$ is chosen uniquely so that $\chi \pi \odot d_{Z(G)} = \chi$. Note that $\hat{f}$ remains bounded and supported on a finite number of orbits. If $C_\chi$ is the set where $\hat{f}_\chi$ is continuous, then $\hat{f}$ is continuous on $C_\chi D$, which has measure equal to the measure of $O$. 

Therefore, we can choose functions $\phi, \psi \in C_c^\infty(\text{GL}_2(L))$ with $|\hat{\phi} - \hat{f}| \leq \psi$ and $\hat{\mu}_\chi^\text{pl}(\hat{\psi}) \leq \epsilon$. Let $U_\chi$ be the unramified orbit of $\chi$ in $\hat{L}$. Then we have
$$\int_{U_\chi} \hat{\mu}_\chi^\text{pl}(\hat{\psi}) d\chi' \leq \int_{\hat{L}} \hat{\mu}_\chi^\text{pl}(\hat{\psi}) d\chi' < \epsilon$$
and since $U_\chi$ has measure 1, there is a $\chi' \in U_\chi$ with $\hat{\mu}_\chi^\text{pl}(\hat{\psi}) \leq \psi$. Let $\chi' = \chi \cdot dZ(G)$ for some $d \in D$.

Let $\phi \otimes d$ be defined as $(\phi \otimes d)(g) = \phi(g)d(g)$, so that $\hat{\phi} \otimes d(\pi) = \hat{\phi}(\pi \otimes d)$. As such, we have
$$\hat{\mu}_\chi^\text{pl}(\hat{\psi} \otimes d) = \hat{\mu}_\chi^\text{pl}(\hat{\psi}) \leq \epsilon,$$
since the Plancherel density function is unchanged by twisting by an unramified character.

Letting $\psi_\chi$ be the average of $\psi \otimes d$ with respect to $\chi$, and defining $\phi_\chi$ similarly, we see that $\phi_\chi$ and $\psi_\chi$ satisfy (1) and (2) in the statement of the proposition, completing the proof. \qed

**Remark 9.7.** If $F$ is a global field and $S$ a finite set of finite places, we can replace $L$ by $F_S$ above. The proof is exactly the same, except that we would need to twist our functions at each place $v \in S$.

## 9. Counting Measures and Test Functions

In this section, we switch back to the global setting. We’ll adapt the counting measure of (9.4) of [16] to our setting. Throughout this section we fix

- a totally real field $F$,
- an automorphic character $\chi : \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ extending $\chi_\xi$,
- a finite set $S$ of finite places. We set $F_S = \prod_{v \in S} F_v$, so that $\text{GL}_2(F_S) = \prod_{v \in S} \text{GL}_2(F_v)$.

We also fix the following notation:

- $K_v = \text{GL}_2(\mathfrak{o}_{F,v})$ for any finite place $v$,
- $\xi$ is an irreducible finite-dimensional representation of $\text{GL}_2(F_\infty)$, with discrete-series complement $\pi_\xi$ and Clozel-Delorme function $\phi_\xi$ (see Corollary 5.4),
- $\phi_S \in \mathcal{H}(\text{GL}_2(F_S), Z(F_S), \chi_S)$ with Plancherel transform $\hat{\phi}_S$ on $\text{GL}_2(F_S)^{\wedge, \chi_S}$,
- $\hat{f}_S, \hat{h}_S$ denote elements of $\mathcal{F}_0(\text{GL}_2(F_S))^\wedge$ (or $\mathcal{F}_0(\text{GL}_2(F_S)^{\wedge, \chi_s})$), and
- $\phi^{S, \infty}$ is a product of smooth functions $\phi_v \in \mathcal{H}(\text{GL}_2(F_v), F_v^{\times}, \chi_v)$ for finite places $v \not\in S$. We will assume that $\phi_v = 1_{F_v^{\times} K_v}$ at all but finitely many places, and such that $\phi_v$ is supported on $F_v^{\times} K_v$ everywhere.

We will begin with a definition:

**Definition 9.1.** Fix $S$ and $\chi$ as above. Given a tuple $(\hat{f}_S, \hat{\phi}^{S, \infty}, \xi)$, we define the multiset
$$\mathcal{F} = \mathcal{F}_{\text{disc}, \chi}(\hat{f}_S, \hat{\phi}^{S, \infty}, \xi),$$
where for an admissible automorphic representation $\pi = \pi_S \otimes \pi^{S, \infty} \otimes \pi_\infty$ with $\chi = \pi$, $\pi$ occurs in $\mathcal{F}$ with multiplicity
$$a_{\mathcal{F}}(\pi) = (-1)^{[F:Q]} m_{\text{disc}}(\pi) \cdot \hat{f}_S(\pi_S) \cdot \hat{\phi}^{S, \infty}(\pi^{S, \infty}) \cdot \text{tr}_{Z(F_{\infty})} \pi(\phi_\xi).$$
Here $m_{\text{disc}}(\pi)$ is the multiplicity of $\pi$ in the discrete spectrum of $\text{GL}_2(\mathbb{A})$.

Define $\mathcal{F}_{\text{cusp}, \chi}$ similarly, but with $m_{\text{disc}}$ replaced by $m_{\text{cusp}}$.

It follows from Harish-Chandra’s finiteness theorem that, for $\mathcal{F}$ as defined above, $a_{\mathcal{F}}(\pi) = 0$ for all but finitely many $\pi$. Moreover, $m_{\text{disc}}(\pi) = 0$ or 1 by strong multiplicity one.

**Definition 9.2.** Given a multiset $\mathcal{F}$, say $\pi \in \mathcal{F}$ if $a_{\mathcal{F}}(\pi) \neq 0$. If $\mathcal{F}$ is finite, we define
$$|\mathcal{F}| = \sum_{\pi} a_{\mathcal{F}}(\pi).$$

**Remark 9.3.** We have borrowed the multiset notation from [15] and [16], but we have both simplified and generalized to match our needs. For instance, we have eliminated their set $S_1$ (or rather, assumed $S_1$ is empty) and let $S = S_0$. On the other hand, we have generalized their insistence that $\phi^{S, \infty}$ be an idempotent element corresponding to an open-compact subgroup; this will slightly simplify our proof, and will be strictly necessary when we show a partial extension of our result to newforms. We have also restricted to an arbitrary fixed central character.
Definition 9.4. Fix an irreducible finite dimensional representation $\xi$ of $GL_2(F_\infty)$, an automorphic character $\chi$ extending $\chi_\xi$, and $\phi^{S,\infty} \in H(GL_2(\mathbb{A}_{\infty}), Z(\mathbb{A}_{\infty}), \chi^{S,\infty})$. We define the counting measures $\hat{\mu}_{\phi^{S,\infty}, \chi}$ and $\hat{\mu}_{\phi^{S,\infty}, \chi}$ as linear functionals on $\mathcal{F}_0(GL_2(F_\infty))$ by

$$ \hat{\mu}_{\phi^{S,\infty}, \chi}(\hat{f}_S) = \frac{|\mathcal{F}_{\text{cusp}, \chi}(\hat{f}_S, \phi^{S,\infty}, \xi)|}{\tau_Z(G) \cdot \phi^{S,\infty}(1) \cdot \dim \xi} $$

and

$$ \hat{\mu}_{\phi^{S,\infty}, \chi}(\hat{f}_S) = \frac{|\mathcal{F}_{\text{disc}, \chi}(\hat{f}_S, \phi^{S,\infty}, \xi)|}{\tau_Z(G) \cdot \phi^{S,\infty}(1) \cdot \dim \xi} $$

Here $\tau_Z(G)$ is the measure of $GL_2(F)\mathbb{Z} \backslash GL_2(\mathbb{A})$, computed using the Euler-Poincaré measure at infinity and the canonical measure at all finite places.

The residual spectrum of $GL_2(\mathbb{A})$ consists of one-dimensional representations, so if $\dim \xi > 1$ then $\mathcal{F}_{\text{cusp}} = \mathcal{F}_{\text{disc}}$ as a multitset.

9.1. Test Functions for Counting Cusp Forms. We begin by defining the test functions we'll use to count cusp forms:

Definition 9.5. Let $\chi$ be an automorphic character with conductor $f(\chi)$ and let $\mathfrak{n}$ be a nonzero ideal in $\mathfrak{o}_v$ with $f(\chi) \mid \mathfrak{n}$. We define $\phi_{n,\chi} \in H(GL_2(\mathbb{A}_{\infty}), Z(\mathbb{A}_{\infty}), \chi^{\infty})$ as a product of local factors, as follows

- At all places $p$ not dividing $\mathfrak{n}$, $\phi_{n,\chi,p}$ is supported on $F_{p}^\times K_p$, with $\phi(z \cdot K_p) = \chi_p^{-1}(z)$.
- Otherwise, if $\text{ord}_p(\mathfrak{n}) = r$, then $\phi_{n,\chi,p}$ is supported on $F_{p}^\times \Gamma_0(p^r)$, and

$$ \phi_{n,\chi,p}(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \text{vol}(\Gamma_0(p^r))^{-1} \chi_p^{-1}(a). $$

Lemma 9.6. Let $\pi_p$ have central character $\chi_p$ and let $\text{ord}_p(\mathfrak{n}) = r$. Then $\text{tr} \pi(\phi_{n,\chi,p})$ is the dimension of the space of vectors $v \in V_p$ such that $\pi(\gamma)v = \chi(a)v$ for any $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(p^r)$.

Proof. Let

$$ \phi_0(g) = \begin{cases} \phi_{n,\chi,p}(g) & |\det(g)| = 1 \\ 0 & \text{otherwise} \end{cases} $$

so that $\phi_{n,\chi,p}$ is the average of $\phi_0$ with respect to $\chi_p$. As such, for any $\pi_p$ with $\chi_{\pi_p} = \chi_p$, we have $\text{tr} \pi(\phi_{n,\chi,p}) = \text{tr} \pi(\phi_0)$.

On the other hand, it is elementary to check that $\pi(\phi_0)$ is a projection from $V_p$ onto the space of vectors $v$ so that $\pi(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})v = \chi_p(a) \cdot v$ for $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in \Gamma_0(p^r)$. This completes the proof.

Proposition 9.7. Fix the following data:

- A finite set $S$ of finite places;
- an irreducible finite-dimensional representation $\xi$ of $GL_2(F_\infty)$, with complementary discrete series representation $\pi_\xi$;
- an automorphic character $\chi$ of conductor $f$ extending $\chi_\xi$;
- a nonzero ideal $\mathfrak{n}$ of $\mathfrak{o}_v$ with $f \mid \mathfrak{n}$. Write $\mathfrak{n} = n_Sn^S$, where $n_S$ is divisible only by primes in $S$ and $n^S$ is coprime to $S$; and
- a function $\hat{h}_S \in \mathcal{F}_0(GL_2(F_\infty))$.

Let \( \mathcal{F} = \mathcal{F}_{\text{cusp}, \chi}(\hat{h}_S \cdot \hat{\phi}_{n,\chi}, \hat{\phi}_{n^S,\chi}, \xi) \). Then \( \mathcal{F} \) counts the cuspidal $GL_2(\mathbb{A})$-representations with $\chi_\pi = \chi$, $\pi_\infty \cong \pi_\xi$, and conductor $\mathfrak{d}$ dividing $\mathfrak{n}$; such a representation $\pi$ is counted with multiplicity $\hat{h}_S(\pi_\mathfrak{d})d(n/\mathfrak{d})$.

Proof. Let $\pi$ be an irreducible cuspidal automorphic representation with central character $\chi$ and conductor $\mathfrak{d}$. Then it has a Whittaker model, so each of its archimedean components has a Whittaker model. If $\pi_\mathfrak{d}$ is generic and $\text{tr} \pi_\mathfrak{d}(\phi_{\xi,\mathfrak{d}}) \neq 0$, then $\pi_\mathfrak{d} = \pi_\xi$, and $\text{tr} \pi_\mathfrak{d}(\phi_{\xi,\mathfrak{d}}) = -1$ by Corollary 5.4.

By Lemma 9.6 and the classical result of Borel-Casselman (see Theorem 4.24 and the discussion before Remark 4.25 of [S]), we have that $\text{tr}^{S,\infty}(\phi_{n^S,\chi}) = d(n^S/\mathfrak{d}^S)$. Similarly, $\hat{h}_S(\pi_\mathfrak{d})\hat{\phi}_{n_S,\chi}(\pi_\mathfrak{d}) = \hat{h}_S(\pi_\mathfrak{d})d(n_S/\mathfrak{d}_S)$, completing the proof.
Lemma 9.11. In Theorem 9.10, the second equality implies the first. The second statement follows from the first as in the proof of Corollary 9.8.

Theorem 9.10. Following:

By writing the trace as a product of local traces, it's enough to show that if \( \operatorname{ord}_p(n) = 0 \), then \( \hat{\pi}(\phi_{n,\chi}) = \hat{\pi}_S(\xi, \chi) \).

Proof. This follows directly from the previous proposition and the correspondence between cusp forms and cuspidal representations, once we note the following two facts:

- If \( f \) is a cusp form of weight \( k \) and \( \xi \) is as above, then \( \pi_{f,\infty} = \pi_{\xi} \) \[10 \] Theorem 1.4]; and
- If \( f \) is an \emph{newform} of level \( \mathfrak{d} \) and character \( \chi \), then the multiplicity of \( f \) in \( S_k(\Gamma_0(n), \chi) \) is \( d(n/\mathfrak{d}) \).

\[ \blacksquare \]

Corollary 9.9. Let \( \xi, \chi, f, n \) be as above. Define \( \phi_{n,\chi}^{\text{new}} \in \mathcal{H}(\GL_2(A^\infty), Z(A^\infty), \chi^\infty) \) by

\[
\phi_{n,\chi}^{\text{new}} = \begin{cases} 
\phi_{n,\chi} & \text{ord}_p(n/\mathfrak{d}) = 0 \\
\phi_{n,\chi} - 2 \cdot \phi_{n/\mathfrak{d},\chi} & \text{ord}_p(n/\mathfrak{d}) = 1 \\
\phi_{n,\chi} - 2 \cdot \phi_{n/\mathfrak{d},\chi} + \phi_{n/\mathfrak{d}^2,\chi} & \text{ord}_p(n/\mathfrak{d}) \geq 2
\end{cases}
\]

Assume \( \xi \) is a discrete-series representation with \( \chi_\xi = \chi_\infty \). If

\[
\mathcal{F} = \mathcal{F}_{\text{cusp},\chi}(\hat{h}_S(\phi_{n,\chi}^{\text{new}}, \phi_{n^2,\chi}^{\text{new}}, \xi))
\]

then \( |\mathcal{F}| \) counts the number of automorphic representations \( \pi \) of exact conductor \( n \), \( \chi_\pi = \chi_\xi \), and \( \pi_\infty = \pi_\xi \), with multiplicity \( a_F(\pi) = \hat{h}_S(\pi_S) \).

When \( \xi = \xi_\xi \) as in Corollary 9.8, \( |\mathcal{F}| \) counts the newforms of weight \( k \), level \( n \), and conductor \( \chi \) with multiplicity \( a_F(f) = \hat{h}_S(\pi_{f,S}) \).

Proof. The second statement follows from the first as in the proof of Corollary 9.8.

To prove the first statement, we just need to prove that \( \phi_{n,\chi}^{\text{new}}(\pi) \) is 1 if \( f(\pi) = n \), and zero otherwise. By writing the trace as a product of local traces, it’s enough to show that if \( \text{ord}_p(n) = r \), then we need to show that \( \hat{\phi}_{n,\chi}^{\text{new}}(\pi_p) = 1 \) if \( c(\pi_p) = r \) and zero otherwise.

If \( c(\pi_p) = r \), then \( \phi_{n,\chi}^{\text{new}}(\pi_p) = 1 \) and \( \phi_{n/\mathfrak{d},\chi}^{\text{new}}(\pi_p) = \hat{\phi}_{n/\mathfrak{d}^2,\chi}^{\text{new}}(\pi_p) = 0 \).

If \( c(\pi_p) > r \) then evidently \( \phi_{n,\chi}^{\text{new}}(\pi_p) = 0 \) if \( c(\pi_p) = r' \leq r - 1 \), then we have

\[
\phi_{n,\chi}^{\text{new}}(\pi_p) = \phi_{n,\chi}^{\text{new}}(\pi_p) - 2 \cdot \phi_{n/\mathfrak{d},\chi}^{\text{new}}(\pi_p) + \phi_{n/\mathfrak{d}^2,\chi}^{\text{new}}(\pi_p)
= (r + 1 - r') - 2 \cdot (r - r') + (r - 1 - r')
= 0
\]

\[ \blacksquare \]

Recall the counting measures defined in (9.4). The goal of the next sections is to prove the following:

Theorem 9.10. (Plancherel equidistribution theorem). Let \( S, \hat{h}_S \) be a finite set of places, \( \xi \) an irreducible finite-dimensional \( \GL_2(F_\infty) \)-representation, \( \chi \) an automorphic character with \( \chi_\infty = \chi_\xi \), and \( (n_\lambda) \) a sequence of levels divisible by \( \mathfrak{d}^\infty \) and coprime to \( S \), such that \( N(n_\lambda) \to \infty \). For simplicity let \( \hat{\mu}_{S,\lambda} = \hat{\mu}_{S,\phi_n,\xi} \), with superscript cusp or disc. Then

\[
\lim_{\lambda \to \infty} \hat{\mu}_{S,\lambda}(\hat{f}_S) = \lim_{\lambda \to \infty} \hat{\mu}_{S,\lambda}(\hat{f}_S) = \hat{\mu}_{S,\xi}^{\text{pl}}(\hat{f}_S).
\]

We conclude this section with a lemma, which will start the proof of the theorem.

Lemma 9.11. In Theorem 9.10, the second equality implies the first.
Proof. If \( \dim \xi > 1 \) then any discrete automorphic representation \( \pi \) with \( \pi_\infty = \pi_\xi \) is cuspidal, and so we are done in this case.

Otherwise, assume \( \dim \xi = 1 \). Fix \( \tilde{f}_S \) and let \( \tilde{\iota} \) be the characteristic function of \( \GL_2(F_S)^{\text{disc}} \) in \( \GL_2(F_S)^\wedge \). The Plancherel measure is supported on the tempered spectrum, so \( \tilde{\mu}^{\text{pl}}(\tilde{f}_S) = \tilde{\mu}^{\text{pl}}(\tilde{f}_S \cdot \tilde{\iota}) \).

For a positive function \( \tilde{h}_S \), we have

\[
0 \leq \tilde{\mu}_\lambda^{\text{disc}}(\tilde{h}_S) - \tilde{\mu}_\lambda^{\text{cusp}}(\tilde{h}_S) \leq \tilde{\mu}_\lambda^{\text{disc}}(\tilde{h}_S) - \tilde{\mu}_\lambda^{\text{disc}}(\tilde{h}_S \cdot \tilde{\iota}) ::
\]

this follows because \( \tilde{h}_S \) is positive and because every discrete representation that is tempered at \( S \) is cuspidal, since the residual spectrum of \( \GL_2(K) \) consists of one-dimensional representations. (It is conjectured that every cuspidal representation is, in fact, tempered everywhere; this is the generalization of Ramanujan-Petersson conjecture). We therefore have

\[
|\tilde{\mu}_\lambda^{\text{disc}}(\tilde{f}_S) - \tilde{\mu}_\lambda^{\text{cusp}}(\tilde{f}_S)| \leq (\tilde{\mu}_\lambda^{\text{disc}} - \tilde{\mu}_\lambda^{\text{cusp}})(|\tilde{f}_S|)
\]

\[
\leq \tilde{\mu}_\lambda^{\text{disc}}(|\tilde{f}_S| - \tilde{\iota} \cdot |\tilde{f}_S|)
\]

As \( \lambda \to \infty \), the final term approaches \( \tilde{\mu}^{\text{pl}}(|\tilde{f}_S| - \tilde{\iota} \cdot |\tilde{f}_S|) \). But the Plancherel measure is supported on the tempered spectrum, so this is zero, finishing the proof. \( \square \)

10. Asymptotic Bounds on Constant Terms and Orbital Integrals

The goal of this section is to bound the constant terms \( Q_\gamma(1_{\mathbb{Z}(\mathbb{A}^\infty \Gamma_0(n))}) \) and orbital integrals \( O_\gamma(1_{\mathbb{Z}(\mathbb{A}^\infty \Gamma_0(n))}) \). We will begin by computing local orbital integrals and constant terms and then summarize the global consequences in subsection 10.3.

Throughout, we will choose measures on \( G = \GL_2(F_p) \), \( T \) the diagonal torus, \( N \) the subgroup of upper-triangular unipotent matrices, and \( K_p = \GL_2(o_{F_p}) \) so that maximal compact subgroups are given measure 1; this also ensures that \( dq = dt \, dn \, dk \) under the Iwasawa decomposition \( G = TNK \).

The key tool will be an analysis of the Bruhat-Tits tree for \( SL_2 \). We recall a definition:

Definition 10.1. Consider the \( p \)-adic field \( F_p \). The Bruhat-Tits tree \( X \) of \( SL_2(F_p) \) is a graph consisting of the following data:

- The set of vertices is the set of equivalence classes of rank-two lattices \( \Lambda \subseteq F_p^2 \), with \( \Lambda \sim \Lambda' \) if they differ only by a scalar multiple.

- Two equivalence classes \([\Lambda], [\Lambda']\) are adjacent if and only if there are lattices \( \Lambda \in [\Lambda], \Lambda' \in [\Lambda'] \) such that \( \Lambda \supsetneq \Lambda' \supseteq \Xi \cdot \Lambda \).

We briefly recall some facts:

1. The degree of every vertex \( v \in X \) is \( q + 1 \). To see this, fix a lattice \( \Lambda \). If \( \Lambda' \subset \Lambda \) with index \( q \), then \( \Xi \Lambda \subset \Lambda' \subset \Lambda \), and so \( \Lambda' \) corresponds uniquely to a one-dimensional subspace in \( \Lambda/\Xi \Lambda \cong F_q^2 \). On the other hand, if \( \Lambda' \supset \Lambda \) with index \( q \), then \( \Lambda' \) is equivalent to \( \Xi \Lambda \), which is a sublattice of \( \Lambda \) of index \( q \). Moreover, if \( \Lambda_a \sim \Lambda \) then all index-q sublattices of \( \Lambda_a \) are equivalent to an index-q sublattice of \( \Lambda \).

2. \( X \) is a tree \([14, \text{Theorem 1}] \).

Let \( \{e_1, e_2\} \) be the standard basis of \( F_p^2 \). \( X \) has a distinguished line \( A_0 \) whose vertices correspond to the lattices with bases \( \{e_1, \Xi e_2\} \); this is known as the standard apartment. For fixed \( g \in \GL_2(F_p) \), \( A = g \cdot A_0 \) is called an apartment. Given a vertex \( w \) and an apartment \( A \), let \( d(w, A) \) be the distance from \( w \) to \( A \). Because \( X \) is a tree, there is a unique vertex \( w' \in A \) such that \( d(w, A) = d(w, w') \); we define \( \delta_A(w) = \Xi e_1 \).

By the Iwasawa decomposition, every vertex has an associated lattice \( \Lambda \) with basis \( \{e_1, a e_1 + \Xi e_2\} \), where \( s \in \mathbb{Z} \) and \( a \in F_p \). We denote this vertex by \( \omega_{s,a} \). Note that \( \omega_{s,a} = \omega_{s', a'} \) if and only if \( s = s' \) and \( a - a' \in o_{F_p} \), and that \( \omega_{s,a} \in A_0 \) if and only if \( a \in o_{F_p} \). It is elementary to check by induction that if \( a \not\in o_{F_p} \), then \( \delta(\omega_{s,a}, A_0) = -v_p(a) \) and that \( \delta(A_0, \omega_{s,a}) = \delta(\omega_{s,a}, A_0) \); this follows because \( \omega_{s,a} \) is adjacent to \( \omega_{s+a,1} \).

We say a set of vertices \( \{w_0, \ldots, w_r\} \) is a segment (of length \( r \)) if \( d(w_i, w_j) = i - j \) for all \( 0 \leq i, j \leq r \).

The action of \( \GL_2(F_p) \) on the set of lattices in \( F_p^2 \) descends to an action on \( X \) by graph automorphisms. We have the following:
Proposition 10.4. Let \( t = (t_1 0 0 ) \in \mathbb{R}_p \). Then \( d(w, A_0) \leq v_p(t_1 - t_2) \).

Proof. Write \( w = w_{a,s} \), and note that \( t \) fixes \( w_{a,s} \) if and only if

\[
\begin{pmatrix} t_1 & (t_1 - t_2)a \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & x^s \end{pmatrix}^{-1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in K \cdot Z
\]

which occurs if and only if \( (t_1 - t_2)a \in \mathcal{O}_{F,p} \).

Since \( d(w_{a,s}, A_0) = -v_p(a) \), this completes the proof. \( \square \)

10.1. Computation of Constant Terms. Let \( t = (t_1 0 0 ) \in \mathbb{R}_p \) and let \( w \in X \) be a vertex. Then \( t \) fixes \( w \) if and only if \( d(w, A_0) \leq v_p(t_1 - t_2) \).

Lemma 10.3. Let \( t = (t_1 0 0 ) \in \mathbb{R}_p \) and let \( w \in X \) be a vertex. Then \( t \) fixes \( w \) if and only if \( d(w, A_0) \leq v_p(t_1 - t_2) \).

Proof. Write \( w = w_{a,s} \), and note that \( t \) fixes \( w_{a,s} \) if and only if

\[
\begin{pmatrix} t_1 & (t_1 - t_2)a \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & x^s \end{pmatrix}^{-1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in K \cdot Z
\]

which occurs if and only if \( (t_1 - t_2)a \in \mathcal{O}_{F,p} \).

Since \( d(w_{a,s}, A_0) = -v_p(a) \), this completes the proof. \( \square \)

Proposition 10.4. Let \( t_1 \neq t_2 \in \mathcal{O}^X_{F,p} \). Then

\[
Q_z(1_{\mathbb{R}_p}) \leq \begin{cases} 
1 & r \leq v_p(t_1 - t_2) \\
2q^{v_p(t_1 - t_2)} \text{vol}(\mathbb{R}_p) & r > v_p(t_1 - t_2)
\end{cases}
\]

Proof. Fix a strictly upper-triangular matrix \( n \) and note that for any \( k \in K \) we can only have \( k^{-1}tnk \in K \) if \( n \in K \). Since \( t_1 \neq t_2 \), there is a \( g \) so that \( g^{-1}tnk = g \) and therefore the set \( X^{tn} \) of vectors fixed by \( tn \) is of the form \( g \cdot X^t \). If \( A = g \cdot A_0 \), then \( w \in X^{tn} \) if and only if \( d(w, A) \leq v_p(t_1 - t_2) \).

Fix \( n \in N \cap K \); we have \( k^{-1}tnk \in Z \cdot \Gamma_0(\mathbb{R}_p) \) if and only if the segment \( k \cdot S_r \subset X^{tn} \). We note that the initial vertex of \( k \cdot S_r \) is \( w_{0,0} \); we will show that there number of such segments contained in \( X^{tn} \) is at most \( [K : \Gamma_0(\mathbb{R}_p)] \) if \( r \leq v(t_1 - t_2) \), and is at most \( 2q^{v_p(t_1 - t_2)} \) otherwise. The first statement is obvious simply by counting the total number of segments of length \( r \) with a given initial point.

For the second case, we note the following: since \( X \) is a tree, if \( S = \{w_0, \ldots, w_l\} \) is a segment with \( d(w_1, A) > d(w_0, A) \) then \( d(w_{i+1}, A) > d(w_i, A) \) for all \( i \). As such, if \( k \cdot S_r \) is a segment contained in \( X^{tn} \), then for all \( 1 \leq i \leq r - v_p(t_1 - t_2) \), we have \( d(w_i, A) \leq d(w_i, A) \). As such, we claim that there are at most \( 2q^{v_p(t_1 - t_2)} \) segments of the form \( k \cdot S_r \) contained in \( X^{tn} \). Because \( k \in K \), we have \( w_0' = w_0 \). For each \( 1 \leq i \leq r - v_p(t_1 - t_2) \), if \( w_{i-1} \notin A \), then \( w_i \) is the unique neighbor of \( w_{i-1} \) with \( d(w_i, A) < d(w_{i-1}, A) \). If \( w_{i-1} \in A \) and \( w_{i-2} \notin A \), then \( w_i \) must be one of the two neighbors of \( w_{i-1} \) in \( A \). Finally, if \( w_{i-1}, w_{i-2} \in A \), then \( w_i \) must be the other neighbor of \( w_{i-1} \) in \( A \). Finally, if \( i > r - v_p(t_1 - t_2) \), then \( w_i \) can be any of the \( q \) neighbors of \( w_{i-1} \) which are not equal to \( w_{i-2} \). This completes the proof of the claim.

Therefore, for any \( n \in N \cap K \) we have

\[
\int_{K} 1_{\mathbb{R}_p}(k^{-1}tnk) \, dk \leq \begin{cases} 
1 & r \leq v_p(t_1 - t_2) \\
2q^{v_p(t_1 - t_2)} \text{vol}(\mathbb{R}_p) & r > v_p(t_1 - t_2)
\end{cases}
\]

and so integrating over \( n \in N \cap K \) completes the proof. \( \square \)

We will also need to compute the constant term \( Q_z(1_{\mathbb{R}_p}) \) for a central element \( z \).

Proposition 10.5. Let \( z \in Z(F_p) \). Then

\[
Q_z(1_{\mathbb{R}_p}) = \begin{cases} 
2^{-r} & r = 2k + 1 \\
q^{-r} & r = 2k0
\end{cases}
\]

In particular, \( Q_z(1_{\mathbb{R}_p}) \leq q^{-r/2} \).
Proof. We can assume that \( z = 1 \) and once again find the fixed subspace \( X^n \) for \( n = \binom{b}{0} \in K \). Let \( w_{a,s} \) be as in the beginning of the section. Since

\[
\begin{pmatrix}
1 & a \\
0 & \omega^s
\end{pmatrix}^{-1} \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & a \\
0 & \omega^s
\end{pmatrix} = \begin{pmatrix}
1 & b \omega^s \\
0 & 1
\end{pmatrix}
\]

we see that \( w_{a,s} \in X^n \) if and only if \( s \geq -v(b) \). In particular, if \( b_{A_0}(w, w_{0,s}) = d(w, w_{0,s}) \leq s + v(b) \). Alternatively, \( X^n \) is the union of balls of radius \( s + v(b) \) around \( w_{0,s} \in A_0 \), for \( s \geq -v(b) \).

For fixed \( n \), the volume of the set

\[
\{ k \in K : k^{-1} nk \in Z \cdot \Gamma_0(p^n) \}
\]

is the product of \( \text{vol}(\Gamma_0(p^n)) \) with the number of segments \( \{w_0, \ldots, w_r\} \) whose basepoint is \( w_0 = w_{0,0} \) and which are contained in \( X^n \). Let \( n = \binom{b}{0} \). If \( v_p(b) \geq r \) then all length-\( r \) segments with basepoint \( w_0 \) are contained in \( X^n \), so the total volume is \( 1 \). If \( r > v_p(b) \), then for any \( i \leq \left\lfloor \frac{r - v(b)}{2} \right\rfloor \) we must have \( w_i = w_{0,i} \); for each subsequent step there are \( q \) choices, so the total number of segments contained in \( X^n \) is \( q^{\frac{r(v(b))}{2}} \).

As such, we compute

\[
\int_N \int_K 1_{Z: \Gamma_0(p^n)}(k^{-1} nk) \, dk \, dn = q^{-r} + \frac{1}{q^{r-1}(q+1)} \sum_{j=0}^{r-1} (q-1)^{-1} q^{j/2}.
\]

An elementary computation using induction shows that this is equal to the quantity stated. \( \square \)

10.2. Computation of Orbital Integrals. The goal of this section is to prove:

**Proposition 10.6.** Let \( \gamma \) be a non central, semisimple element of \( \text{GL}_2(F_p) \). Then

\[
O_\gamma(1_{Z: \Gamma_0(p^n)}) \leq 2 \cdot \text{vol}(\Gamma_0(p^n)) \cdot O_\gamma(1_{Z: K})^2.
\]

We’ll break this into two cases: the case where \( \gamma \) is elliptic, and the case where \( \gamma \) is non-elliptic.

**Lemma 10.7.** If \( \gamma \) is elliptic then the set \( X^\gamma \) is finite.

**Proof.** We can compute the fixed set directly, assuming \( \gamma \in K \) by conjugating and multiplying by an element of the center. If \( \gamma \) is elliptic then it is conjugate to a matrix of the form

\[
\begin{pmatrix}
x & y \\
\alpha y & x
\end{pmatrix}
\]

where \( \alpha \) is either a unit that is not a square, or \( \alpha \) is a uniformizer.

If \( \alpha \) is a unit, then the \( X^\gamma \) is the single point \( \{w_{0,0}\} \). If \( \alpha \) is a uniformizer, then \( X^\gamma \) consists of those vertices \( w \) with \( d(w, S_1) \leq v(y) \), where \( S_1 \) is the length-one segment \( \{w_{0,0}, w_{0,1}\} \). In either case, \( X^\gamma \) is finite. \( \square \)

We will now prove Proposition 10.6.

**Proof.** (Of Proposition 10.6). Assume first that \( \gamma \) is elliptic, and by conjugating assume \( \gamma \in \Gamma_0(p^n) \). Then \( O_\gamma(1_{Z: \Gamma_n}) \) is the cardinality of \( X^\gamma \). As such, for a given length \( r \), there are at most \( O_\gamma(1_{Z: \Gamma_n})^2 \) segments of length \( r \) contained in \( X^\gamma \) since each segment is determined uniquely by its two endpoints. For a given segment \( S_1' \), the volume of the set \( \{ g \in G_{\gamma} \setminus \text{GL}_2(F_p) : g \cdot S_r = S'_r \} \) is \( \text{vol}(\Gamma_0(p^n)) \). This finishes the proof when \( \gamma \) is elliptic.

If \( \gamma \) is diagonalizable, we can assume \( \gamma = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in K \). In this case, [18 Lemma 9] tells us that

\[
O_\gamma(1_{Z: \Gamma_0(p^n)}) = |D_M^G(\gamma)|_{p}^{-1/2} Q_\gamma(1_{Z: \Gamma_0(p^n)})
\]

where \( D_M^G(\gamma) \) is the determinant of \( 1 - \text{Ad}(\gamma) \) acting on \( \text{Lie}(G)/\text{Lie}(T) \). In our situation we have

\[
|D_M^G(\gamma)| = \left| \left( 1 - \frac{t_1}{t_2} \right) \left( 1 - \frac{t_2}{t_1} \right) \right| = |t_1 - t_2|^2.
\]
First, this lemma and Proposition 10.4 prove that $O_\gamma(1_{Z \cdot K_p}) = |t_1 - t_2|^{-1} = q^{e(t_1 - t_2)}$. Applying these results to $1_{Z \cdot \Gamma(p^r)}$ gives

$$O_\gamma(1_{Z \cdot \Gamma_0(p^r)}) \leq 2 \cdot O_\gamma(1_{Z \cdot K})^2 \cdot \text{vol}(\Gamma_0(p^r))$$

completing the proof. \qed

10.3. Summary of global consequences. We summarize the global consequences for use in subsequent sections below:

**Proposition 10.8.** Let $\gamma \in \text{GL}_2(F)$ be semisimple and let $\mathfrak{n} \subseteq \mathfrak{o}_F$ be an ideal. Then

1. If $\gamma \in Z(F)$, then
   $$Q_\gamma(1_{Z(\hat{\mathfrak{n}}) \Gamma_0}(\mathfrak{n})) \leq N(\mathfrak{n})^{-1/2}$$

2. If $\gamma = (t_1 \ 0 \ \ 0 \ t_2) \in T(F) - Z(F)$, then
   $$Q_\gamma(1_{Z(\hat{\mathfrak{n}}) \Gamma_0}(\mathfrak{n})) \leq |N_F/Q(t_1 - t_2)| \cdot 2^{P(n)} \cdot N(\mathfrak{n})^{-1}$$

   where $P(n)$ is the number of primes dividing $\mathfrak{n}$.

3. If $\gamma \in \text{GL}_2(F) - Z(F)$ is semisimple, then
   $$O_\gamma(1_{Z(\hat{\mathfrak{n}}) \Gamma_0}(\mathfrak{n})) \leq O_\gamma(\hat{K} \cdot \hat{\mathfrak{n}}) \cdot 2^{P(n)} \cdot N(\mathfrak{n})^{-1}.$$

**Proof.** This follows from Propositions 10.4, 10.5, and 10.6 upon decomposing the orbital integrals and constant terms as a product of local orbital integrals and constant terms. \qed

Because $2^{P(n)} \cdot N(\mathfrak{n})^{-1}$ decreases as $o(N(\mathfrak{n})^{-1+\epsilon})$ for every $\epsilon > 0$, we have the following

**Corollary 10.9.** For every semisimple, noncentral $\gamma \in \text{GL}_2(F)$ and every $\epsilon > 0$, there is a $C_{\epsilon, \gamma} > 0$ such that

$$Q_\gamma(1_{Z(\hat{\mathfrak{n}}) \Gamma_0}(\mathfrak{n})) \leq C_{\epsilon, \gamma} \cdot N(\mathfrak{n})^{-1+\epsilon}$$

for all ideals $\mathfrak{n} \subseteq \mathfrak{o}_{F, p}$.

11. The Plancherel Equidistribution Theorem

In this section, we use the results of the previous section to prove our key intermediate result:

**Theorem 11.1.** (Plancherel equidistribution theorem). Fix a finite set of places $S$. Let $\xi$ be a finite-dimensional $\text{GL}_2(F_S)$-representation, let $\chi$ be a character of conductor $\mathfrak{f}$ with $\chi_\infty = \chi_\xi$, and let $\hat{f}_\mathfrak{f} \in \mathcal{F}_0(\text{GL}_2(F_S)^{\xi, \chi})$. Let $(\mathfrak{n}_\lambda) \to \infty$ be a sequence of levels coprime to $S$ with $\mathfrak{f}^S | \mathfrak{n}_\lambda$ and $N(\mathfrak{n}_\lambda) \to \infty$. Then

$$\lim_{\lambda \to \infty} \mu^\text{cusp}_{\phi_{\mathfrak{n}, \lambda, \xi}}(\hat{f}_\mathfrak{f}) = \lim_{\lambda \to \infty} \mu^\text{disc}_{\phi_{\mathfrak{n}, \lambda, \xi}}(\hat{f}_\mathfrak{f}) = \lim_{\lambda \to \infty} \hat{\mu}^\text{pl}_{\chi}(\hat{f}_\mathfrak{f}).$$

Before the proof, we’ll need a lemma:

**Lemma 11.2.** Fix a compact set $C_S$ of $\text{GL}_2(F_S)/Z(F_S)$. Then there are only finitely semisimple conjugacy classes $\{\gamma\} \in \text{GL}_2(F)/Z(F)$ such that $\{\gamma^\infty\}$ intersects $C_S K_S^{\infty, \infty}$, and such that $\gamma$ is elliptic at all infinite places.

**Proof.** First, because $|\det \gamma|_p = 1$ for all $p \notin S$, and $|\det \gamma|_S$ can be chosen to lie in the finite set $I_S/P_S^2$ (where $I_S$ is the group of ideals divisible only by primes in $S$, and $P_S$ is the subgroup of principal ideals), then $|\det \gamma|_S$ can be chosen in a finite set. Because $\mathfrak{a}_S^{\infty}/(\mathfrak{a}_S^\infty)^2$ is finite, we can actually assume that $\det \gamma$ lies in a finite set by shifting by an element of $Z(F)$.

Because $\{\gamma\}$ intersects $C_S K_S^{\infty, \infty}$, its trace lies in some fractional ideal $\mathfrak{a}$ in $F$. Let $\mathfrak{a}_\infty$ be image of $\mathfrak{a}$ under $F \hookrightarrow \mathbb{R}^n$. Fix a determinant $D \in F^\times$. If $\gamma$ is elliptic at each infinite place we must have $\text{tr}(\gamma)^2 \leq 4D_{\mathfrak{a}}$ for each infinite place, so $\text{tr}(\gamma)$ lies in some compact set. Since $\mathfrak{a}_\infty$ is a lattice, then there are at most finitely many traces $\gamma$ can take for each determinant. A semisimple conjugacy class is determined by its trace and determinant, completing the proof. \qed

We now prove the theorem.
Proof. For simplicity we write $\phi_\lambda = \phi_{\sigma_\lambda, \chi} \in \mathcal{H}(\text{GL}_2(A^{\infty}), Z(A^{\infty}), \chi^{\infty})$ and $\hat{\mu}_{\lambda}^{\text{disc}} = \hat{\mu}_{\phi_{\lambda}, \xi, \chi}^{\text{disc}}$.

Let’s first assume that $\hat{f}_S = \phi_S$ for some $\phi_S \in \mathcal{H}(\text{GL}_2(F_S), Z(F_S), \chi_S)$. In this case, we have

$$I_{\text{spec}}(Z(A), \chi, \phi_S \phi_\lambda \phi_\xi) = \sum_\pi (\text{tr } \pi_S(\phi_S)) \cdot (\text{tr } \pi^{S, \infty}(\phi_\lambda)) \cdot (\text{tr } \pi^{\infty}(\phi_\xi))$$

$$= (-1)^{|F:q|} |F| |\mathcal{F}_{\text{disc}, \chi}(\phi_S, \phi_\lambda, \xi)|$$

where in each sum, $\pi$ runs over the discrete automorphic representations of $\text{GL}_2(A)$ with central character $\chi$.

As such, we have

$$\hat{\mu}_{\lambda}^{\text{disc}}(\phi_S) = (-1)^{|F:q|} |F| \frac{I_{\text{spec}}(Z(A), \chi, \phi_S \cdot \phi_\lambda \cdot \phi_\xi)}{\tau_Z(G) \cdot \phi_\lambda(1) \cdot \dim(\xi)}$$

$$= (-1)^{|F:q|} |F| \frac{I_{\text{geom}}(Z(A), \chi, \phi_S \cdot \phi_\lambda \cdot \phi_\xi)}{\tau_Z(G) \cdot \phi_\lambda(1) \cdot \dim(\xi)}$$

Recall from Definition 6.5 that $I_{\text{geom}}$ consists of three terms: a central term, an sum of orbital integrals of elliptic elements, and a sum of constant terms of diagonal elements. By Lemma 11.2, there are only finitely many nonvanishing orbital integrals, and there are only finitely many constant terms because in a given compact subset of $T(A^{\infty})$ there are only finitely many cosets $T(F)/Z(F)$. The central term of $(-1)^{|F:q|} I_{\text{geom}, \lambda}(Z(A), \chi, \phi_S \cdot \phi_\lambda \cdot \phi_\xi)$ is simply

$$(-1)^{|F:q|} \tau_Z(G) \cdot \phi_S(1) \cdot \phi_\lambda(1) \cdot \phi_\xi(1) = \tau_Z(G) \cdot \phi_S(1) \cdot \phi_\lambda(1) \cdot \dim(\xi)$$

so upon dividing by $\tau_Z(G) \cdot \phi_\lambda(1) \cdot \dim(\xi)$ we are left with $\phi_S(1) = \hat{\mu}_{S, \lambda}^{\text{pl}}(\phi_S)$.

Each orbital integral term in $I_{\text{geom}, \lambda}$ is of the form

$$D(\gamma) \cdot O_{\gamma_2}(\phi_S) \cdot O_{\gamma, \infty}(\phi_\lambda) \cdot O_{\gamma_\infty}(\phi_\xi).$$

As we let $\lambda \to \infty$, the only nonconstant term is $O_{\gamma, \infty}(\phi_\lambda)$. Upon dividing by $\phi_\lambda(1)$ and taking absolute values, this is bounded by

$$O_{\gamma, \infty}(1) = O_{\gamma_2(\phi_S)} Z(A^{\infty}).$$

This goes to zero by Corollary 10.9. The same argument shows that each constant term vanishes asymptotically.

As such, we have

$$\lim_{\lambda \to \infty} \hat{\mu}_{\lambda}^{\text{disc}}(\phi_S) = \lim_{\lambda \to \infty} (-1)^{|F:q|} \frac{I_{\text{geom}}(Z(A), \chi, \phi_S \cdot \phi_\lambda \cdot \phi_\xi)}{\tau_Z(G) \cdot \phi_\lambda(1) \cdot \dim(\xi)}$$

$$= \phi_S(1)$$

$$= \hat{\mu}_{S, \lambda}^{\text{pl}}(\phi_S)$$

This completes the proof of the equidistribution theorem for Plancherel transforms $\hat{\phi}_S$ of functions $\phi_S \in \mathcal{H}(\text{GL}_2(F_S), Z(F_S), \chi_S)$. When $\hat{f}_S \in \mathcal{F}_0(\text{GL}_2(F_S)^\wedge)$ is arbitrary, we use Sauvageot’s density theorem for fixed central character. (This is exactly as in Shin and Templier’s proof of Corollary 9.22 in [16], except that we use Sauvageot’s density theorem for fixed central character. We repeat the proof here for completeness).

Fix $\epsilon > 0$ and pick $\phi_S, \psi_S \in \mathcal{H}(\text{GL}_2(F_S), Z(F_S), \chi)$ such that $|\hat{f}_S - \phi_S| \leq \psi_S$ on $\text{GL}_2(F_S)^\wedge, \chi$ and so that $\hat{\mu}_{\lambda}^{\text{pl}}(\phi_S) < \epsilon/3$. Then we have

$$|\hat{\mu}_{\lambda}^{\text{pl}}(\hat{f}_S) - \hat{\mu}_{\lambda}^{\text{disc}}(\hat{f}_S)| \leq |\hat{\mu}_{\lambda}^{\text{pl}}(\hat{f}_S - \phi_S)| + |\hat{\mu}_{\lambda}^{\text{pl}}(\phi_S) - \hat{\mu}_{\lambda}^{\text{disc}}(\phi_S)| + |\hat{\mu}_{\lambda}^{\text{disc}}(\phi_S - \hat{f}_S)|$$

$$\leq |\hat{\mu}_{\lambda}^{\text{pl}}(\psi_S)| + |\hat{\mu}_{\lambda}^{\text{pl}}(\phi_S) - \hat{\mu}_{\lambda}^{\text{disc}}(\phi_S)| + |\hat{\mu}_{\lambda}^{\text{disc}}(\psi_S)|$$

The first term is at most $\epsilon/3$. The second term approaches 0 as $\lambda \to \infty$, so it is eventually at most $\epsilon/3$. The third term approaches $|\hat{\mu}_{\lambda}^{\text{pl}}(\psi_S)| < \epsilon/3$ as $\lambda \to \infty$, so for large $\lambda$ it is eventually at most $\epsilon/3$. Therefore, for large $\lambda$ we have $|\hat{\mu}_{\lambda}^{\text{pl}}(\hat{f}_S) - \hat{\mu}_{\lambda}^{\text{disc}}(\hat{f}_S)| < \epsilon$, finishing the proof. □
Corollary 11.3. Fix a weight $k$ and let $\chi$ of conductor $f$ occurring in weight $k$. Let $(n_\lambda)$ be a sequence of levels divisible by $f$. Then
\[
\dim S_k(\Gamma_1(n_\lambda), \chi) = \tau_k(G) \cdot [\text{GL}_2(\mathfrak{o}_F) : \Gamma_0(n)] \cdot \dim(\xi_k) + o(N(n)^{1/2})
\]
as $\lambda \to \infty$.

If $F \neq \mathbb{Q}$ then the error term is $o(N(n)^\epsilon)$.

Proof. Apply the above to $S = \emptyset$ with $\phi_S = 1$, use the bounds on the constant terms and orbital integrals in Proposition 10.8 and Corollary 10.9, and note that when $F \neq \mathbb{Q}$, there are no constant terms on the geometric side of the trace formula.

Remark 11.4. It is worth here comparing our results to those of [20]. First, Shin has computed
\[
\tau_k(\text{GL}_2(F)) = (-1)^{|F:\mathbb{Q}|} \zeta_F(-1) 2^{1-|F:\mathbb{Q}|}
\]
(see (iii) in the proof of Lemma 6.2 in [15]), whereas Weinstein’s main term counting the number of cusp forms of fixed inertial type is
\[
(-1)^{|F:\mathbb{Q}|} \zeta_F(-1) 2^{1-|F:\mathbb{Q}|} \cdot h_F \cdot \dim(\xi) \cdot [\text{GL}_2(\mathfrak{o}_F) : \Gamma_0(n)].
\]
The discrepancy occurs because he fixes an inertial type, which only determines the central character on $Z(F) \cdot \hat{\mathcal{A}}^* : F^\infty \subseteq Z(\hat{A})$.

This subgroup has index $h_F$. As such, given an inertial type $\tau$, and $\pi$ of inertial type $\pi, \chi_\pi$ may be one of $h_F$ different characters.

We also have a Plancherel equidistribution theorem for newforms. Since the proof is the same in spirit as the Plancherel equidistribution theorem, we give a sketch:

Corollary 11.5. Let $\chi$ be a character with conductor $f$ and let $\hat{f}_S \in \mathcal{F}_0(\text{GL}_2(F_S)^{\wedge \chi})$. Let $(n_\lambda) \to \infty$ be a sequence of levels coprime to $S$ with $f^2 | n_\lambda$ and $N(n_\lambda) \to \infty$. Then
\[
\lim_{\lambda \to \infty} \hat{\mu}^\sup_{\phi_{n_\lambda}^{\text{new}}} \xi, \chi(\hat{f}_S) = \lim_{\lambda \to \infty} \hat{\mu}^\text{disc}_{\phi_{n_\lambda}^{\text{new}}} \xi, \chi(\hat{f}_S) = \hat{\mu}^\text{pl}_{\chi}(\hat{f}_S).
\]

Proof. We can assume the conductor $f$ is not divisible by any primes of norm 2. In this case, a quick computation shows
\[
\frac{\phi_{n_\lambda}^{\text{new}}}{\phi_{n_\lambda}^{\text{new}}(1)} = c_0 \frac{\phi_{n_\lambda}^{\text{new}}/p, \chi, p}{\phi_{n_\lambda}^{\text{new}}/p, \chi, p(1)} + 2c_1 N(p)^{-1} \frac{\phi_{n_\lambda}^{\text{new}}/p, \chi, p}{\phi_{n_\lambda}^{\text{new}}/p, \chi, p(1)} + c_2 N(p)^2 \frac{\phi_{n_\lambda}^{\text{new}}/p^2, \chi, p}{\phi_{n_\lambda}^{\text{new}}/p^2, \chi, p(1)}
\]
where $c_0, c_1, c_2$ are real constants of absolute value at most $2$.

Therefore, if $\text{ord}_p(n) = r$, Proposition 10.8 tell us that the orbital integrals
\[
O_p(\phi_{n_\lambda}^{\text{new}})
\]
are bounded in absolute value by $16O_p(1_{Z(F_S)K^\infty_p})^2 \cdot N(p^r)$. As such, we get a bound on the global orbital integral:
\[
|O_\gamma(\phi_S \phi_{n_\lambda}^{\text{new}} \phi_{\infty})| \leq C' 16P(n)O_\gamma(1_{Z(\hat{\mathcal{A}}^\infty)K^\infty \hat{A}})^2 N(n)^{-1},
\]
for some constant $C'$ depending only on $\phi_{\infty}$ and $\phi_S$; this goes to zero as $o(N(n)^{-1+\epsilon})$.

The analogous proof works for constant terms, and the corollary follows exactly as in Theorem 11.1; we apply the trace formula and then use Sauvageot density to adapt to the case when $\hat{f}_S \in \mathcal{F}_0(\text{GL}_2(F_S)^{\wedge \chi})$ is arbitrary. \qed

12. Proof of the Main Theorem

The goal of this section is to prove our main theorem:

Theorem 12.1. Let $F$ be a totally real field, $k$ a weight with $k_1 \equiv \ldots \equiv k_n \mod 2$, $\chi$ an automorphic character occurring in weight $k$, and $(n_\lambda)$ a sequence of levels. Let $B_k(\Gamma_1(n_\lambda), \chi)$ denote the standard basis of Hecke eigenforms of weight $k$, level $n$, and character $\chi$. For any $A \in \mathbb{Z}_{>1}$, let $B_k(\Gamma_1(n_\lambda), \chi)_{\leq A}$ be the subset consisting of those forms with $[Q(f) : Q] \leq A$. Then
\[
\lim_{\lambda \to \infty} \frac{|B_k(\Gamma_1(n_\lambda), \chi)_{\leq A}|}{|B_k(\Gamma_1(n_\lambda), \chi)|} = 0.
\]

Proof. We’ll begin the proof with three lemmas:
Lemma 12.2. Fix \( \epsilon > 0 \). Then there is a \( P_0 \in \mathbb{Z} \) so that, for all rational primes \( p > P_0 \), all places \( \mathfrak{p} | p \), and all levels \( n \) with \( \text{ord}_p(n) \geq 3 \), we have

\[
\frac{\#B_k(\Gamma_1(n), \chi) \leq_A}{\#B_k(\Gamma_1(n), \chi)} < \epsilon.
\]

Proof. By Proposition 3.6, for \( p > 2A + 1 \), if \( \mathfrak{p} | p \), \( \text{ord}_p(n) \geq 3 \), and \( f \) is a newform of level \( n \), then \( |Q(f) : \mathbb{Q}| > A \). We will henceforth assume \( p > 2A + 1 \).

Let \( \mathfrak{p} | p \), let \( f \in B_k(\Gamma_1(n), \chi) \) and assume \( f \) satisfies \( |Q(f) : \mathbb{Q}| \leq A \). As such, \( f \) must come from a newform of level \( \mathfrak{d} \) with \( \text{ord}_p(\mathfrak{d}) \leq 2 \).

Write \( n = p^B \mathfrak{t} \). We can write

\[
S_k(\Gamma_1(n), \chi) = \bigoplus_{f(\chi)|n} S^\text{new}_k(\Gamma_1(\mathfrak{d}), \chi)^{d(n/\mathfrak{d})}
\]

and let

\[
S_k(\Gamma_1(n), \chi)^{\leq 2} = \bigoplus_{f(\chi)|n, \text{ord}_p(\mathfrak{d}) \leq 2} S^\text{new}_k(\Gamma_1(\mathfrak{d}), \chi)^{d(n/\mathfrak{d})}.
\]

We claim

\[
\dim S_k(\Gamma_1(n), \chi)^{\leq 2} \leq (B - 1) \dim S_k(\Gamma_1(p^2\mathfrak{t}), \chi).
\]

To prove this, it’s enough to show that for any \( \mathfrak{d} \) dividing \( p^2\mathfrak{t} \), the multiplicity of \( S^\text{new}_k(\Gamma_1(\mathfrak{d}), \chi) \) in \( S_k(\Gamma_1(n), \chi) \) is bounded above by \( (B - 1) \) times its multiplicity in \( S_k(\Gamma_1(p^2\mathfrak{t}), \chi) \). We note that the multiplicity of \( S^\text{new}_k(\Gamma_1(p^2\mathfrak{a}), \chi) \) in \( S_k(\Gamma_1(n), \chi) \) is

\[
(B - b + 1)d(t/\mathfrak{a}),
\]

and the multiplicity of \( S^\text{new}_k(\Gamma_1(p^2\mathfrak{a}), \chi) \) in \( S_k(\Gamma_1(p^2\mathfrak{t}), \chi) \) is \( (3 - b)d(t/\mathfrak{a}) \). Since \( x + y + 1 \leq (x + 1)(y + 1) \) for nonnegative \( x, y \), we have

\[
(B - b + 1) \leq (B - 1)(3 - b)
\]

proving the claim.

By Corollary 11.3, for large \( N(n) \), we have

\[
\alpha[\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(n)] \leq \dim S_k(\Gamma_1(n), \chi) \leq \beta[\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(n)]
\]

for some constants \( \alpha, \beta \). In particular, we’ll assume \( p \) is large enough that this holds whenever \( N(n) \geq p^2 \).

Therefore, assuming \( p \) is large enough, we have

\[
\frac{\dim S_k(\Gamma_1(n), \chi)^{\leq 2}}{\dim S_k(\Gamma_1(n), \chi)} \leq (B - 1) \frac{\dim S_k(\Gamma_1(p^2\mathfrak{t}), \chi)}{\dim S_k(\Gamma_1(n), \chi)}
\]

\[
\leq (B - 1) \frac{\beta \cdot [\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(p^2)][\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(t)]}{\alpha \cdot [\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(p^2)][\text{GL}_2(\mathfrak{O}_F) : \Gamma_0(t)]}
\]

\[
\leq \frac{\beta}{(B - 1)N(p)^2 - B}.
\]

Note that \( (B - 1)N(p)^2 - B \) is decreasing on \( B \geq 3 \) assuming \( N(p) \geq 2 \), so we just need to pick \( p \) large enough so that \( \frac{p - 1}{2} > A \) and \( \frac{2p}{\mathfrak{p} \mathfrak{p}^2} < \epsilon \), finishing the proof.

Recall from section 3 that, given an orbit \( \mathcal{O} \), \( \mathbb{Q}(\mathcal{O}) \) is the intersection of \( \mathbb{Q}(\pi) \) for \( \pi \in \mathcal{O} \).

Lemma 12.3. Fix a character \( \chi \) and \( \epsilon > 0 \). Let \( B_p \subseteq \text{GL}_2(F_p)^{\chi} \) be the union of those orbits \( \mathcal{O}_\chi \) with \( e(\mathcal{O}_\chi) \leq 2 \) and \( [\mathbb{Q}(\mathcal{O}_\chi) : \mathbb{Q}] > A \). Then if \( \mathfrak{p} \) is coprime to \( f(\chi) \) and \( N(p) \) is sufficiently large, we have

\[
\hat{\mu}_{x_p}^\text{pl}(B_p) > 1 - \epsilon
\]

where \( \phi_{p^2, \chi, \mathfrak{p}} \) is the \( \mathfrak{p} \)-component of the function \( \phi_{p^2, \chi} \) defined in 9.5.
Proof. Given \( A \), let \( f(A) \) be large enough that if \( \zeta \) is any root of unity with \( [\mathbb{Q}(\zeta) : \mathbb{Q}] \leq 2A \), then \( \zeta \in \mathbb{Q}(\zeta f(A)) \). Because \( (\mathfrak{p}_F/p)^k \) is cyclic of order \( q - 1 \), there are at least \( q - 1 - f(A) \) characters \( \chi_0 : \mathfrak{p}_F^* \to \mathbb{C}^* \) of conductor 1 with \( [\mathbb{Q}(\chi(x)) : \mathbb{Q}] > 2A \) for some \( x \). As such, there are at least \( q - 1 - f(A) \) conjugate pairs of such characters. If \( \mathcal{O} \) is the principal-series orbit corresponding to such a pair, then \( [\mathbb{Q}(\mathcal{O}) : \mathbb{Q}] > A \) by Proposition 3.6. By our computations after Corollary 8.3, each such orbit has Plancherel measure \( q + 1 \), so the total Plancherel measure of these orbits is at least \( \frac{1}{2} \).

We must also count supercuspidal orbits. We note that if \( L'/L \) is ramified, then the map \( (\mathfrak{o}_L/\mathfrak{w}_L)^k \to (\mathfrak{o}_{L'/\mathfrak{w}_L})^k \) is an isomorphism, so there are no characters \( \eta_0 : \mathfrak{g}_{L'}^* \to \mathbb{C}^* \) of conductor 1 such that \( \eta_0 \neq \eta_0' \). Therefore, all such supercuspidal representations of conductor 2 come from characters \( \eta : L^\times \to \mathbb{C}^* \) where \( L'/L \) are unramified, and \( \eta \) has conductor 1. The central character of \( \pi_\eta \) is the map \( x \to (-1)^{v(x)} \cdot \eta(x) \).

As such, if \( \eta \) has conductor 1 and \( \pi_\eta \) has unramified central character, then \( \eta_0 \) factors through \( \mathfrak{o}_{L'/\mathfrak{w}_L}^* \to \mathbb{C}^* \); this is cyclic of order \( q + 1 \). Let \( \eta_0 \) send the generator to a root of unity \( \zeta \). Then \( \zeta^q + 1 = 1 \), and if \( \eta = \eta_0 \) we have \( \zeta^q = \zeta \); thus \( \eta_0 \neq \eta_0 \) if \( \eta_0 \) does not take values outside of \( \pm 1 \). As such, there are \( \frac{q + 1 - f(A)}{2} \) conjugate pairs \( \{ \eta_0, \eta_0' \} \) with \( \eta_0(x) = \zeta^r \) for \( r > f(A) \).

Therefore, there are at least \( \frac{q + 1 - f(A)}{2} \) supercuspidal representations \( \pi \) of conductor 2 and fixed unramified central character with \( [\mathbb{Q}(\pi) : \mathbb{Q}] > A \). Each has formal degree \( q - 1 \) by Remark (??) and 7.9 (4), and so the total measure of these is \( (q - 1)^\frac{q + 1 - f(A)}{2} \). Therefore, the total Plancherel measure of all the orbits of conductor 2 with large enough field of rationality is at least \( q^2 - 1 - f(A) \).

On the other hand,

\[
\widehat{\mu}_x^{\text{pl}}(\phi_{\mathfrak{p}^2,\chi,\mathfrak{p}}) = \phi_{\mathfrak{p}^2,\chi,\mathfrak{p}}(1) = q(q + 1)
\]

and

\[
\lim_{q \to \infty} \frac{q^2 - 1 - f(A)}{q(q + 1)} = 1
\]

completing the proof.

Lemma 12.4. Fix a character \( \chi \) and let \( \mathfrak{p} \) be any prime that does not divide the conductor \( \mathfrak{f} \) of \( \chi \).
Let \( (n_\lambda) \) be a sequence of levels that are divisible by \( \mathfrak{f} \) and prime to \( \mathfrak{p} \). Then

\[
\lim_{\lambda \to \infty} \frac{\#B_k(\Gamma_1(n_\lambda), \chi) \leq A}{\#B_k(\Gamma_1(n_\lambda), \chi)} = 0.
\]

Proof. This argument is due to Serre in the case \( F = \mathbb{Q} \) (see [13]) and all changes are simply cosmetic. We repeat it here for reference and completeness.

Let \( f \) be a modular form of level \( n_\lambda \) with \( [\mathbb{Q}(f) : \mathbb{Q}] \leq A \). Then \( [\mathbb{Q}(a_\mathfrak{p}(f)) : \mathbb{Q}] \leq A \). Because \( n_\lambda \) is prime to \( \mathfrak{p} \), then \( a_\mathfrak{p} \) is the sum of two \( \text{Weil-q} \)-integers \( \alpha, \beta \) of weight \( \max\{k_1\} - 1 \) [10] Thm 1.4 (1). Since \( [\mathbb{Q}(a_\mathfrak{p}) : \mathbb{Q}] \leq A \), then \( Q(\alpha), Q(\beta) \) are of degree at most \( 2A \) over \( \mathbb{Q} \). Thus \( \alpha, \beta \) must satisfy monic polynomials in \( \mathbb{Z}[x] \) of degree at most \( 2A \) and whose coefficients are bounded for fixed \( q \) and max \( \{k_1\} \); thus \( \alpha, \beta \) take only finitely many values.

In this case, \( \pi_{f,\mathfrak{p}} \) is the unramified representation \( \chi_1 \times \chi_2 \), where \( \chi_1, \chi_2 \) take a uniformizer to \( \alpha, \beta \). As such, \( \pi_{f,\mathfrak{p}} \) takes only finitely many possible values in the unramified orbit \( \mathcal{O}^X \). Let \( \mathfrak{I}_{0,\leq A} \) be the characteristic function of this finite set of points. Because the Plancherel measure on the unramified orbit has no germs, we have \( \widehat{\mu}_x^{\text{pl}}(\mathfrak{I}_{0,\leq A}) = 0 \), and because it is supported on finitely many points, it lives in \( \mathcal{F}_0(\text{GL}_2(F_\mathfrak{p}))^\times \). As such, the Plancherel equidistribution theorem completes the lemma.

With this in hand, we complete the proof of the Main Theorem.

Fix \( A \in \mathbb{Z}_{\geq 0} \) and \( \epsilon > 0 \). Fix \( P > 2A + 1 \) so that for any rational prime \( p > P \), and any prime \( \mathfrak{p} \) of \( F \) lying above \( p \), we have

\[
\frac{\widehat{\mu}_x^{\text{pl}}(B_\mathfrak{p})}{\widehat{\mu}_x^{\text{pl}}(\phi_{\mathfrak{p}^2,\chi,\mathfrak{p}})} > 1 - \epsilon
\]
(where $B_p$ is as in Lemma 12.3), and such that for any $n$ with $\text{ord}_p(n) \geq 3$, we have

$$\frac{\#B_k(\Gamma_1(n), \chi_{\leq A})}{\#B_k(\Gamma_1(n), \chi)} < \epsilon$$

Fix, once and for all, primes $p_1, \ldots, p_r$ coprime to the conductor $f$, with $p_i | p_i > P$, and so that

$$\prod_{i=1}^r \frac{q_i - 1}{q_i + 1} < \epsilon.$$ 

This is possible because the Dedekind zeta function $\zeta_F$ has a pole at 1, and because $\frac{q_i - 1}{q_i + 1} \leq 1 - \frac{1}{q_i}$.

With this in hand, let $t = (t_1, \ldots, t_r)$ be a tuple, where $t_i$ is either 0, 1, 2, or $' \geq 3'$. For such a tuple, define the subsequence $(n_{\lambda, t})$, where $n_\lambda \in \{n_{\lambda, t}\}$ if $\text{ord}_{p_i}(n_\lambda) = t_i$ for $i = 1, \ldots, r$ (and where if $t_i = ' \geq 3'$ then $\text{ord}_{p_i}(n_{\lambda, t}) \geq 3$). This breaks the sequence $(n_\lambda)$ into finitely many subsequences, each of which can be assumed to be either empty or infinite.

We’ll show that for every $t$, if $\lambda$ is sufficiently high we have

$$\frac{\#B_k(\Gamma_1(n_{\lambda, t}), \chi_{\leq A})}{\#B_k(\Gamma_1(n_{\lambda, t}), \chi)} < \epsilon.$$ 

If $t_i$ is $' \geq 3'$ for some $i$, we are done because each $p_i$ is chosen to be sufficiently large. If $t_i = 0$ for some $i$, we are done by Lemma 12.4.

If $t_i = 2$ for some $i$, let $S = \{p_i\}$, and let $n'_{\lambda, t} = n_{\lambda, t}/p_i^2$. Then we have

$$\frac{\#B_k(\Gamma_1(n_{\lambda, t}), \chi_{\leq A})}{\#B_k(\Gamma_1(n_{\lambda, t}), \chi)} \leq 1 - \frac{|F_{\text{cusp}}(\hat{1}_{B_{p_i}}, \hat{\phi}_{n'_{\lambda, t}, \chi}, \xi_k)|}{|F_{\text{cusp}}(\hat{1}_{B_{p_i}^2}, \hat{\phi}_{n'_{\lambda, t}, \chi}, \xi_k)|}
\rightarrow 1 - \frac{\hat{\mu}_{p_i, \chi}(B_{p_i})}{\hat{\mu}_{p_i, \chi}(B_{p_i}^2)} < \epsilon$$

(where the third line follows by the Plancherel Equidistribution Theorem) so that eventually we have

$$\frac{\#B_k(\Gamma_1(n_{\lambda, t}), \chi_{\leq A})}{\#B_k(\Gamma_1(n_{\lambda, t}), \chi)} < \epsilon.$$ 

The only remaining case is $t_1 = \ldots = t_r = 1$. Let $C_p$ be the finite set consisting of the two Steinberg points of conductor 1 and central character $\chi_p$ along with the finite set of points $\{\pi_p\}$ in the unramified orbit where the Frobenius eigenvalues are Weil-$q$-integers of small enough degree. As in Lemma 12.4, the characteristic function $\hat{1}_{C_p}$ lives in $\mathcal{F}_0(\text{GL}_2(F_p)^\wedge)$ (the set of discontinuities is a finite set in the unramified orbit), and moreover we have

$$\hat{\mu}_{\chi_p, p}^\text{pl}(\hat{1}_{C_p}) = q - 1,$$

because there are two Steinberg representations with central character $\chi_p$ (corresponding to its two square roots) and because each has formal degree $\frac{q-1}{2}$.

As such, if we take $S = \{p_1, \ldots, p_r\}$ and

$$\hat{f}_S = \prod_{i=1}^r \hat{1}_{C_p},$$
then the same logic as in the $t_i = 2$ case tells us:

$$\limsup_{\lambda \to \infty} \frac{\# B_k(\Gamma_1(n_{\lambda,t}), \chi) \leq A}{\# B_k(\Gamma_1(n_{\lambda,t}), \chi)} \leq \frac{\hat{\rho}_{S,\chi}^I(f_S)}{\hat{\rho}_{S,\chi}^I(f_{1,p_1 \ldots p_r,\chi})} = \prod_{i=1}^r \frac{\hat{\rho}_{S,\chi}^I(C_p)}{\phi_{p_i,\chi}(1)} = \prod_{i=1}^r \frac{q_i - 1}{q_i + 1} \leq \epsilon$$

so in particular, we eventually have

$$\frac{\# B_k(\Gamma_1(n_{\lambda,t}), \chi) \leq A}{\# B_k(\Gamma_1(n_{\lambda,t}), \chi)} < \epsilon$$

for this subsequence. This completes the proof. □

We also have a partial result for fields of rationality of newforms, which we briefly discuss. We would like to say that, as the norm of our ideals increases, that a smaller percentage of the set of newforms has bounded field of rationality. But there is an obstruction to our methods. Consider, for example, the case where $F = \mathbb{Q}$, $\chi$ is the trivial character, and $(n_{\lambda})$ is the sequence

$$(2, 2 \cdot 3, 2 \cdot 3 \cdot 5, \ldots).$$

In this case, for any prime $p$, our sequence will eventually satisfy $\text{ord}_p(n_{\lambda}) = 1$, so in particular, the associated representation will always be Steinberg. Indeed, if we pick any finite set of primes $S$ and look at the representations $\pi_{f, S}$, the field of rationality at these places will eventually be $\mathbb{Q}$. However, the corollary below shows that this is, in effect, the only obstruction.

**Corollary 12.5.** Let $B_{k}^{\text{new}}(\Gamma_1(n), \chi)$ be the canonical basis of newforms of weight $k$, character $\chi$ occurring in weight $k$, and level $n$, and define $B_{k}^{\text{new}}(\Gamma_1(n), \chi) \leq A$ as above. Let $(n_{\lambda})$ be a sequence of levels satisfying the following condition:

For any $P \in \mathbb{Z}$, there is a finite set $p_1, \ldots, p_r$ with $p_i \mid p_i > P$,

and for all sufficiently high $\lambda$, there is an $i$ so that $\text{ord}_{p_i}(n_{\lambda}) \neq 1$.

Then

$$\lim_{\lambda \to \infty} \frac{B_{k}^{\text{new}}(\Gamma_1(n_{\lambda}), \chi) \leq A}{B_{k}^{\text{new}}(\Gamma_1(n_{\lambda}), \chi)} = 0.$$  

**Proof.** Fix $\epsilon > 0$ and let $P$ be large enough that the results of Lemmas 12.2 and 12.3 hold. By hypothesis we can find primes $p_1, \ldots, p_r$ such that, for $\lambda$ sufficiently high we have $\text{ord}_{p_i}(n_{\lambda}) \neq 1$ for some $i$. Break the sequence into subsequences $(n_{\lambda,i,x})$ where $i = 1, \ldots, r$ and $x = 0, 2$ or $\geq 3$; say $n_{\lambda} \in \{n_{\lambda,i,x}\}$ if $\text{ord}_{p_i}(n_{\lambda}) = x$. Each subsequence eventually satisfies:

$$\frac{B_{k}^{\text{new}}(\Gamma_1(n_{\lambda,i,x}), \chi) \leq A}{B_{k}^{\text{new}}(\Gamma_1(n_{\lambda,i,x}), \chi)} < \epsilon$$

by the same logic as in the proof of the main theorem. □

**References**

[1] J. Arthur, *The invariant trace formula II. Global Theory*, J. Amer. Math. Soc. 1 (1988), 501-554

[2] J. Arthur, *The L2-Lefschetz numbers of Hecke operators*, Invent. Math. 97 (1989), 257-290.

[3] J. Arthur, *A Stable trace formula. I. General expansions*, J. Inst. Math. Jussieu 1 (2002), 175-277.

[4] J. Arthur, *The endoscopic classification of representations; orthogonal and symplectic groups*, American Mathematical Society, Providence, RI, 2013.

[5] A.-M. Aubert and R. Plymen, *Plancherel measure for GL(n, F) and GL(m, D): explicit formulas and Bernstein decomposition*, J. Number Theory 112 (2005), 26-66.

[6] L. Clozel and P. Delorme, *Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs II*, Ann. Sci. Éc. Norm. Supér. 23 (1990), 193-228.
[7] L. Corwin, A. Moy, and P. Sally, Degrees and formal degrees for division algebras and GL_n over a p-adic field, Pac. J. Math. 141 (1990), no. 1, 21-45.

[8] S. Gelbart, Automorphic forms on adele groups, Princeton University Press, Princeton, (1975).

[9] M. Palm, Explicit GL(2) trace formulas and mixed Weyl laws. arXiv:1212.4282v1 [math.NT].

[10] A. Raghuram and N. Tanabe, Notes on the arithmetic of Hilbert modular forms. J. Ramanujan Math. Soc. 26 (2011), no. 3, pp. 261-319.

[11] F. Sauvageot, Principe de densité pour les groupes réductifs, Compos. Math. 108 (1997), 151-184.

[12] R. Schmidt, Some remarks on local newforms for GL(2), J. Ramanujan Math. Soc. 17 (2002), 115-147.

[13] J.-P. Serre, Répartition asymptotique des valeurs propres de l’opérateur de Hecke T_p, J. Amer. Math. Soc. 10 (1997), no. 1, 75-102.

[14] J.-P. Serre, Trees, Springer-Verlag, Berlin-Heidelberg (1980).

[15] S.W. Shin, Automorphic Plancherel density theorem. Israel J. Math. 192 (2012), 83-120.

[16] S.W. Shin and N. Templier, Sato-Tate theorem for families and low-lying zeros of automorphic L-functions, with appendices by Robert Kottwitz and Raf Cluckers, Julia Gordon, and Immanuel Halupczok, to appear in Invent. Math.

[17] S.W. Shin and N. Templier, On Fields of rationality for automorphic representation, Compos. Math. 150 (2014), no. 12, 2003-2053.

[18] G. van Dijk, Computation of certain induced characters of p-adic groups, Math. Ann. 199 (1972) 229-240.

[19] J.-L. Waldspurger, La formule de Plancherel d’après Harish-Chandra. J. Inst. Math. Jussieu 2, (2003), 235-333.

[20] J. Weinstein, Hilbert modular forms with prescribed ramification, International Mathematics Research Notices 2009 (2009), no. 8, 1388-1420.

[21] A.V. Zelevinsky, Induced representations of reductive p-adic groups. II. On irreducibly representations of GL(n), Ann. Sci. Éc. Norm. Supér. 13 (1980), no. 2, 165-210.