Existence of global solutions to the nonlocal Schrödinger equation on the line

Yi Zhao | Engui Fan

School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai, P. R. China

Correspondence
Engui Fan, School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P. R. China.
Email: faneg@fudan.edu.cn

Funding information
National Science Foundation of China, Grant/Award Numbers: 12271104, 51879045

Abstract
We address the existence of global solutions to the Cauchy problem for the integrable nonlocal nonlinear Schrödinger (nonlocal NLS) equation under the initial data $q_0(x) \in H^{1,1}(\mathbb{R})$ with the $L^1(\mathbb{R})$ small-norm. The nonlocal NLS equation was first introduced by Ablowitz and Musslimani as a new nonlocal reduction of the well-known Ablowitz–Kaup–Newell–Segur system. The main technical difficulty for proving its global well-posedness on the line in $H^1(\mathbb{R})$ is due to the fact that mass and energy conservation laws, being nonlocal, do not preserve any reasonable norm and may be negative. In this paper, we use the inverse scattering transform approach to prove the existence of global solutions in $H^{1,1}(\mathbb{R})$ based on the representation of a Riemann–Hilbert (RH) problem associated with the Cauchy problem of the nonlocal NLS equation. A key of this approach is, by applying the Volterra integral operator and Cauchy integral operator, to establish a Lipschitz bijective map between the solution of the nonlocal NLS equation and reflection coefficients associated with the RH problem. By using the reconstruction formula and estimates on the solution of the time-dependent RH problem, we further affirm the existence of a unique global solution to the Cauchy problem for the nonlocal NLS equation.
1 | INTRODUCTION

This paper is concerned with the existence of global solutions to the Cauchy problem for nonlocal nonlinear Schrödinger (nonlocal NLS) equation

\[ iq_t(t, x) = q_{xx}(t, x) - 2\sigma q^2(t, x)\bar{q}(t, -x), \quad \sigma = \pm 1, \quad x \in \mathbb{R}, \quad t > 0 \]  
\[ q|_{t=0} = q_0 \in H^{1,1}(\mathbb{R}), \]  

where the subscripts denote partial derivatives and \( q(t, x) \) is a complex valued function of the real variables \( x \) and \( t \).

The nonlocal NLS equation (1) was first introduced as an integrable model by Ablowitz and Musslimani in 2013,\(^1\) further, they obtain its Lax pair, an infinite number of conservation laws, and the \( PT \) symmetry. For the initial data with rapidly decaying conditions and nonzero boundary conditions, the soliton solutions to the nonlocal NLS equation (1) were obtained via the inverse scattering transform, respectively.\(^1,2\) In fact, the nonlocal NLS equation was ever derived in a physical application of magnetics.\(^3\) The nonlocal NLS equation (1) also can be regarded as a linear Schrödinger equation

\[ iq_t(t, x) = q_{xx}(t, x) + V[q, t, x]q(t, x) \]  

with a self-induced potential \( V[q, x, t] = -2\sigma q(t, x)\bar{q}(t, -x) \), thus the nonlocal NLS equation (1) is \( PT \) symmetric.\(^4,5\) Since \( PT \) symmetric systems allow for loseless-like propagation due to their balance of gain and loss,\(^6\) they have attracted considerable attention in recent years. The \( PT \) symmetric system is a key model on linear and nonlinear waves\(^7\) and related to the cutting edge research area of modern physics.\(^5,6\) Besides, the nonlocal NLS equation is gauge-equivalent to the unconventional system of coupled Landau–Lifchitz equations and therefore can be useful in the physics of nanomagnetic artificial materials.\(^7\) Possible application of the nonlocal NLS equation is discussed in the context of Alice–Bob systems.\(^8,9\)

Related to the nonlocal NLS equation, Fokas analyzed a (2+1)-dimensional integrable nonlocal NLS equation.\(^10\) Ablowitz introduced new reverse space-time and reverse time nonlocal nonlinear integrable equations and their discrete version.\(^11\) They identified new nonlocal symmetry reductions for the general Ablowitz–Kaup–Newell–Segur (AKNS) system and addressed the scattering problem. Besides, an integrable discrete \( PT \) symmetric “discretization” of the nonlocal NLS equation was obtained from a new nonlocal \( PT \) symmetric reduction of the Ablowitz–Ladik scattering problem.\(^12\) Nonlocal versions of some other integrable equations such as the modified Korteweg–De Vries (KdV) equation and the sine-Gordan equation were investigated.\(^11\)
The long-time behavior of a solution to the nonlocal NLS equation (1) with decaying boundary conditions was investigated via the nonlinear Deift–Zhou steepest-decent method. Recently, the long-time behavior for the nonlocal NLS equation (1) with the step-like initial data was obtained. For the weighted Sobolev initial data \( q_0(x) \in H^{1,1}(\mathbb{R}) \), we obtained the long-time asymptotic behavior for the nonlocal NLS equation (1) in solitonic region. Comparing with the classical NLS equation, the nonlocal NLS equation (1) displays some different characteristics and interesting properties on both exact solutions and long-time asymptotic behavior.

It is well known that the existence of a global solution or the well-posedness of the initial value problem of a partial differential equation is the theoretical guarantee to the long-time asymptotic analysis. Regarding the global existence, Genoud showed that there exists an small soliton profile whose evolution exhibits a blow-up in a finite time. A technical difficulty to prove global well-posedness to the Cauchy problem (1)–(2) comes from the fact that the mass and energy conservation laws to Equation (1) are in the form

\[
I_0 = \int_{\mathbb{R}} q(x,t)\overline{q(-x,t)}dx, \\
I_1 = \int_{\mathbb{R}} [q_x(x,t)\overline{q_x(-x,t)} - \sigma q^2(-x,t)\overline{q(-x,t)}]dx,
\]

which do not preserve any reasonable norm and may be negative. In contrast with this, the mass and energy conservation laws of the classical NLS equation

\[
 iq_t(t,x) = q_{xx}(t,x) - 2\sigma q^2(t,x)\overline{q(t,x)}
\]

allows to obtain a priori estimates for establishing a unique global solution. However, the global existence of the nonlocal NLS equation (1) on the line is still not completely settled to our best knowledge.

In this paper, in order to obtain the existence of global solutions to the nonlocal NLS equation (1), we use an alternative way, namely, the inverse scattering transform approach based on the representation of a Riemann–Hilbert (RH) problem associated with the Cauchy problem (1)–(2). A key of this method is to establish a Lipschitz bijective map between the solution of the nonlocal NLS equation and reflection coefficients associated with the RH problem. In addition, unlike the case of the NLS equation or the derivative NLS equation, the jump matrix of the associated RH problem for the nonlocal NLS equation does not satisfy the Hermitian symmetries and positive definite real part. An \( L^1 \) small-norm condition (4) (see below) on the initial data is imposed to ensure that a solution to the RH problem exists. This condition (4) is also necessary to rule out spectral singularities and eigenvalues of the spectral problem (6) (see below).

To precisely state our main result, we first fix some notations used in this paper.

1.1 Notations

- Let \( I \) be an interval on the real line \( \mathbb{R} \) and \( X \) be a Banach space. \( C(I,X) \) denotes the space of continuous functions on \( I \) taking values in \( X \). It is equipped with the norm

\[
\|q(x)\|_{C(I,X)} = \sup_{x \in I} \|q(x)\|_X.
\]
For the spatial variable $x \in \mathbb{R}$, a weighted space $L^{2,1}(\mathbb{R})$ is specified by

$$L^{2,1}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) \mid \langle \cdot \rangle f \in L^2(\mathbb{R}) \},$$
equipped with the norm $\|f\|_{L^{2,1}(\mathbb{R})} = \|\langle \cdot \rangle f\|_{L^2(\mathbb{R})}$, where $\langle x \rangle = (1 + x^2)^{1/2}$. We further define a weighted Sobolev space by

$$H^{1,1}(\mathbb{R}) := \{ q | q \in L^{2,1}(\mathbb{R}), q_x \in L^{2,1}(\mathbb{R}) \},$$
equipped with the norm

$$\|q\|_{H^{1,1}(\mathbb{R})} = \left( \|q\|_{L^{2,1}(\mathbb{R})}^2 + \|q_x\|_{L^{2,1}(\mathbb{R})}^2 \right)^{1/2}.$$

For the spectral parameter $k \in \mathbb{R}$, define the function space

$$L^{2,1}(\mathbb{R}) := \{ r(k) | kr(k) \in L^2(\mathbb{R}) \},$$
$$\Gamma(\mathbb{R}) := \{ r(k) | r(k) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}), kr(k) \in L^\infty(\mathbb{R}) \},$$
equipped with the norm

$$\|r(k)\|_{\Gamma(\mathbb{R})} = \|r(k)\|_{H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})} + \|kr(k)\|_{L^\infty(\mathbb{R})}.$$

### 1.2 Main results

We now address our main results on the existence of global solutions to the Cauchy problem of the nonlocal NLS equation (1)–(2) as follows.

**Theorem 1.** Let the initial data $q_0 \in H^{1,1}(\mathbb{R})$ with the $L^1(\mathbb{R})$ small-norm condition

$$1 - \|q\|_{L^1(\mathbb{R})} (1 + 2e^{2\|q\|_{L^1(\mathbb{R})}}) > 0. \quad (4)$$

Then, there exists a unique global solution $q \in C([0, \infty), H^{1,1}(\mathbb{R}))$ to the Cauchy problem (1)–(2). Furthermore, the map

$$H^{1,1}(\mathbb{R}) \ni q_0 \mapsto q \in C([0, \infty); H^{1,1}(\mathbb{R})) \quad (5)$$
is Lipschitz continuous.

A key of our approach for proving this theorem is to establish an $L^2$-bijection map between the solution $q$ and the reflection data $r_{1,2}$ by applying the inverse scattering transform theory to the Cauchy problem (1)–(2). The scheme behind the proof can be described as Figure 1. Starting from the given initial data initial-value $q(0, x) \in H^{1,1}(\mathbb{R})$, the direct scattering transform gives rise to the scattering data $r_{1,2}(0; k) \in \Gamma(\mathbb{R})$ and an RH problem associated with the Cauchy problem (1)–(2). Then, the inverse scattering transform goes back to the global solution $q \in C([0, +\infty); H^{1,1}(\mathbb{R})$ to the Cauchy problem (1)–(2) via the RH problem associated with time-dependent scattering data $r_{1,2}(t; k) \in \Gamma$. 
The structure of the paper is as follows. In Section 2, we present the direct scattering transform to the initial value problem (1)–(2) based on its Lax pair. The analyticity, asymptotic, and integrability for the Jost functions and the scattering coefficients are analyzed in details. We establish the Lipschitz continuous mapping from the initial data to the reflection coefficients. In Section 3, we carry out the inverse scattering transform and set up an RH problem associated with the initial value problem (1)–(2), whose existence and uniqueness are further proved via a general vanishing lemma under the $L^1(\mathbb{R})$ small-norm. In Section 4, we reconstruct and estimate the potential associated with the solutions of the RH problem and the reflection coefficients. Further, we establish the Lipschitz continuous mapping from the reflection coefficients to the potential. In Section 5, we perform the time evolution of the reflection coefficients and the RH problem. Furthermore, we prove the existence of a global solution to the initial value problem (1)–(2).

## 2 | DIRECT SCATTERING TRANSFORM

### 2.1 | Jost functions and Lipschitz continuity

The nonlocal NLS equation is integrable and admits the Lax pair

$$
\Phi_x + ik\sigma_3\Phi = U\Phi, \tag{6}
$$

$$
\Phi_t + 2ik^2\sigma_3\Phi = V\Phi, \tag{7}
$$

where the function $\Phi(t, x; k)$ is a matrix-valued function, $k$ is a spectral parameter, and the matrices $U$ and $V$ are given by

$$
U = \begin{pmatrix}
0 & q(t, x) \\
-\sigma \bar{q}(t, -x) & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
A & B \\
C & -A
\end{pmatrix}
$$

with

$$
A = i\sigma q(t, x)\bar{q}(t, -x),
$$

$$
B = 2kq(t, x) + iq_x(t, x),
$$

$$
C = -2k\sigma \bar{q}(t, -x) + i\sigma (\bar{q}(t, -x))_x.
$$
The matrix $\sigma_3$ is the standard Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Denote $\Phi_1(x; k)$ and $\Phi_2(x; k)$ as the first and second column of the matrix $\Phi(x; k)$, respectively, with which we define two normalized Jost functions

$$\varphi_\pm(x; k) = \Phi_1(x; k)e^{ikx}, \quad \phi_\pm(x; k) = \Phi_2(x; k)e^{-ikx}, \quad x \to \pm \infty,$$

then we have

$$\varphi_\pm(x; k) \to e_1 \quad as \quad x \to \pm \infty,$$

$$\phi_\pm(x; k) \to e_2 \quad as \quad x \to \pm \infty. \quad (8)$$

Moreover, the Jost functions $\varphi_\pm(x; k)$ and $\phi_\pm(x; k)$ satisfy the Volterra’s integral equation

$$\varphi_\pm(x; k) = e_1 + \int_{\pm \infty}^x \text{diag}(1, e^{2ik(x-y)})U(q(y))\varphi_\pm(y; k)dy, \quad (9)$$

$$\phi_\pm(x; k) = e_2 + \int_{\pm \infty}^x \text{diag}(e^{-2ik(x-y)}, 1)U(q(y))\phi_\pm(y; k)dy. \quad (10)$$

**Lemma 1.** If $q \in L^1(\mathbb{R})$ and $\|q\|_{L^1(\mathbb{R})} < 1$, then:

- **Boundness:** For every $k \in \mathbb{R}$, there exist unique solutions $\varphi_\pm(\cdot; k) \in L^\infty(\mathbb{R})$ and $\phi_\pm(\cdot; k) \in L^\infty(\mathbb{R})$ of the integral equations (9) and (10), respectively.
- **Analyticity:** For every $x \in \mathbb{R}$, $\varphi_-(x; \cdot)$ and $\phi_+(x; \cdot)$ are continued analytically in $\mathbb{C}^+$, whereas $\varphi_+(x; \cdot)$ and $\phi_-(x; \cdot)$ are continued analytically in $\mathbb{C}^-$. 
- **Symmetry:** The Jost functions $\varphi_\pm(x; k)$ and $\phi_\pm(x; k)$ admit the following symmetries:

$$\varphi_\pm(x; k) = \sigma \Lambda \varphi_\pm(-x; -k), \quad \phi_\pm(x; k) = \Lambda \varphi_\mp(-x; -k), \quad (11)$$

where $\Lambda = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$.

**Proof.** We only give the proof for the Jost function $\varphi_-(x; k)$, others can be given by a similar procedure.

For a vector function $f(x; k) = (f_1(x; k), f_2(x; k))^T$, define a operator $K$ by

$$(Kf)(x; k) = \int_{-\infty}^x \text{diag}(1, e^{2ik(x-y)})U(q(y))f(y; k)dy. \quad (12)$$

Then, the integral equation (9) for $\varphi_-(x; k)$ can be rewritten as

$$\varphi_-(x; k) = e_1 + K\varphi_-(x; k). \quad (13)$$

Since $q \in L^1(\mathbb{R})$, for every $k \in \mathbb{C}^+$, we have

$$\|Kf(\cdot; k)\|_{L^\infty(\mathbb{R})} \leq \|q\|_{L^1(\mathbb{R})}\|f(\cdot; k)\|_{L^\infty(\mathbb{R})}, \quad (14)$$
where the norm is defined by
\[ \|f(\cdot;k)\|_{L^\infty(\mathbb{R})} = \|f_1(\cdot;k)\|_{L^\infty(\mathbb{R})} + f_2(\cdot;k)\|_{L^\infty(\mathbb{R})}. \]

Due to the small-norm assumption \( \|q\|_{L^1(\mathbb{R})} < 1 \), the operator \( K \) is a contraction from \( L^\infty(\mathbb{R}) \) to \( L^\infty(\mathbb{R}) \). By the Banach fixed point theorem, there exists a unique solution \( \varphi_-(\cdot;k) \in L^\infty(\mathbb{R}) \) of the operator equation (13) for every \( k \in \mathbb{C}^+ \).

To prove the analyticity of \( \varphi_-(x;\cdot) \) in \( \mathbb{C}^+ \), we define the Neumann series
\[ w(x;k) = \sum_{n=0}^{\infty} w_n(x;k) \]
with
\[ w_0 = e_1, \quad w_{n+1}(x;k) = \int_{-\infty}^{x} \text{diag}(1,e^{2ik(x-y)})U(q(y))w_n(y;k)dy. \]

We see that
\[ \|w_{n+1}(x;k)\|_{L^\infty} \leq \frac{1}{n!}\|U(q)\|_{L^1}^n. \]

Consequently, the Neumann series \( w(x;k) \) converges absolutely and uniformly for every \( x \in \mathbb{R} \) and every \( k \in \mathbb{C}^+ \). As a result, \( \varphi_-(x;\cdot) \) is analytic in \( \mathbb{C}^+ \) for every \( x \in \mathbb{R} \).

Lemma 2. Let \( q \in L^1(\mathbb{R}) \) and \( \|q\|_{L^1(\mathbb{R})} < 1 \). For every \( x \in \mathbb{R} \), the Jost functions \( \varphi_\pm(x;k) \) and \( \phi_\pm(x;k) \) satisfy the following limits as \( |k| \) approaches to infinity in their analytic domains such that \( |\text{Im}k| \to \infty \)
\[ \lim_{|k| \to \infty} \varphi_\pm(x;k) = e_1, \quad (15) \]
\[ \lim_{|k| \to \infty} \phi_\pm(x;k) = e_2. \quad (16) \]

If in addition, \( q \in C(\mathbb{R}) \), then
\[ \lim_{|k| \to \infty} 2i\sigma k[\varphi_\pm(x;k) - e_1] = s_\pm(x)e_1 + \bar{q}(-x)e_2, \quad (17) \]
\[ \lim_{|k| \to \infty} 2i\sigma k[\phi_\pm(x;k) - e_2] = \sigma q(x)e_1 - s_\pm(x)e_2, \quad (18) \]
where
\[ s_\pm(x) = \int_{-\infty}^{x} q(y)\bar{q}(-y)dy. \quad (19) \]

Proof. Again, we only give the proof for the Jost function \( \varphi_- \). Let \( \varphi_- = [\varphi_-^{(1)}, \varphi_-^{(2)}]^t \). It follows from (9) that
\[ \varphi_-^{(1)}(x;k) = 1 + \int_{-\infty}^{x} q(y)\varphi_-^{(2)}(y;k)dy, \quad (20) \]
\[
\phi^-(2)(x; k) = -\sigma \int_{-\infty}^{x} e^{2ik(x-y)} \tilde{q}(-y) \phi^-(1)(y; k) dy. \tag{21}
\]

We had the result \(\phi_-(x; k) \in L^\infty_x(\mathbb{R})\) thanks to Lemma 1. Employing \(q \in L^1(\mathbb{R})\), we have

\[
\lim_{|k| \to \infty} \phi^-(2)(x; k) = 0 \tag{22}
\]

by Lebesgue’s dominated convergence theorem. Similarly, we arrive at

\[
\lim_{|k| \to \infty} \phi^-(1)(x; k) = 1
\]

in view of \(q \in L^1(\mathbb{R})\) and \(\phi_-(x; k) \in L^\infty_x(\mathbb{R})\). This completes the proof of (15) for \(\phi_-(x; k)\).

If in addition \(q \in C(\mathbb{R})\), then for small \(\delta > 0\), we rewrite (21) as

\[
\phi^-(2)(x; k) = \int_{-\infty}^{x-\delta} e^{2ik(x-y)} \eta(y; k) dy + \eta(x; k) \int_{x-\delta}^{x} e^{2ik(x-y)} dy \tag{23}
\]

\[
+ \int_{x-\delta}^{x} e^{2ik(x-y)} [\eta(y; k) - \eta(x; k)] dy = I_1 + I_2 + I_3,
\]

where \(\eta(x; k) = -\sigma \tilde{q}(-x) \phi^-(1)(x; k)\).

Since \(\eta(\cdot; k) \in L^1(\mathbb{R})\), we have

\[
|I_1| \leq e^{-2\delta \Im k} \|\eta(\cdot; k)\|_{L^1(\mathbb{R})}.
\]

Since \(\eta(\cdot; k) \in C(\mathbb{R})\), we derive

\[
|I_3| \leq \|\eta(\cdot; k) - \eta(x; k)\|_{L^\infty(x-\delta, x)} \left| \int_{x-\delta}^{x} e^{2ik(x-y)} dy \right|
\]

\[
\leq \frac{1}{2\Im k} \|\eta(\cdot; k) - \eta(x; k)\|_{L^\infty(x-\delta, x)}.
\]

Direct integration yields

\[
I_2 = \frac{1}{2ik} (e^{2ik\delta} - 1) \eta(x; k).
\]

Choose \(\delta = (\Im k)^{-\frac{1}{2}}\) such that \(\delta \to 0\) as \(|k| \to \infty\). Then, \(kI_1\) and \(kI_2\) approach to zero as \(|k| \to \infty\). Therefore, we obtain

\[
\lim_{|k| \to \infty} k \phi^-(2)(x; k) = \lim_{|k| \to \infty} kI_2 = \frac{\sigma}{2i} \tilde{q}(-x). \tag{24}
\]

Multiplying (20) with \(k\) and substituting (24) leads to

\[
\lim_{|k| \to \infty} k(\phi^-(1) - 1) = \frac{\sigma}{2i} \int_{-\infty}^{x} q(y) \tilde{q}(-y) dy. \tag{25}
\]
This completes the proof of (17) for \( \varphi_- \). □

**Proposition 1** (see Ref. [27]). If \( w \in L^2(\mathbb{R}) \), then

\[
\sup_{x \in \mathbb{R}} \left\| \int_{-\infty}^{x} e^{2ik(x-y)} w(y) dy \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \| w \|_{L^2(\mathbb{R})}.
\] (26)

And if \( w \in H^1(\mathbb{R}) \), then

\[
\sup_{x \in \mathbb{R}} \left\| 2ik \int_{-\infty}^{x} e^{2ik(x-y)} w(y) dy + w(x) \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \| \partial_x w \|_{L^2(\mathbb{R})}.
\] (27)

Moreover, if \( w \in L^{2,1}(\mathbb{R}) \), then for every \( x_0 \in \mathbb{R}^- \), we have

\[
\sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \int_{-\infty}^{x} e^{2ik(x-y)} w(y) dy \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \| w \|_{L^{2,1}(-\infty, x_0)}.
\] (28)

In addition, if \( w \in H^{1,1}(\mathbb{R}) \), then for every \( x_0 \in \mathbb{R}^- \), we have

\[
\sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \left( 2ik \int_{-\infty}^{x} e^{2ik(x-y)} w(y) dy + w(x) \right) \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \| \partial_x w \|_{L^{2,1}(-\infty, x_0)},
\] (29)

where \( \langle x \rangle = (1 + x^2)^{\frac{1}{2}} \).

**Proof.** The bounds (26)–(28) were given in Ref. [27]. It remains to prove the estimate (29). For every \( x \in \mathbb{R} \) and \( k \in \mathbb{R} \), define

\[
f(x; k) = \int_{-\infty}^{x} e^{2ik(x-y)} w(y) dy = \int_{-\infty}^{0} e^{-2iky} w(x + y) dy.
\]

Using the Plancherel's theorem, we have

\[
\| f(x; k) \|_{L^2_k}^2 = \pi \int_{-\infty}^{0} |w(x + y)|^2 dy = \pi \int_{-\infty}^{x} |w(y)|^2 dy.
\] (30)

Integrating by parts, then we derive

\[
2ik f(x; k) + w(x) = \int_{-\infty}^{x} e^{2ik(x-y)} \partial_y w(y) dy,
\]
thus we obtain that for $y \leq x \leq 0$,

$$\left\| 2ik \int_{-\infty}^{x} e^{2ik(x-y)}w(y)dy + w(x) \right\|_{L_k^2(\mathbb{R})}^2$$

$$= \pi \int_{-\infty}^{x} |\partial_y w(y)|^2 dy$$

$$\leq \frac{\pi}{1 + x^2} \int_{-\infty}^{x} (1 + y^2)|\partial_y w(y)|^2 dy,$$

which implies the estimate (29) for every $x_0 \in \mathbb{R}^-$. \hfill \Box

**Lemma 3.** If $q \in L^{2,1}(\mathbb{R})$ and $\|q\|_{L^1(\mathbb{R})} < 1$, then for every $x \in \mathbb{R}^\pm$, we have

$$\varphi_\pm(x; \cdot) - e_1 \in H^1(\mathbb{R}), \quad \phi_\pm(x; \cdot) - e_2 \in H^1(\mathbb{R}).$$

Moreover, if $q \in H^{1,1}(\mathbb{R})$, then for every $x \in \mathbb{R}^\pm$, we have

$$2i\sigma k(\varphi_\pm(x; k) - e_1) - (s_\pm(x)e_1 + \bar{q}(-x)e_2) \in H^1_k(\mathbb{R}),$$

$$2i\sigma k(\phi_\pm(x; k) - e_2) - (\sigma q(x)e_1 - s_\pm(x)e_2) \in H^1_k(\mathbb{R}).$$

**Proof.** Again, we only give the proof for $\varphi_-$. Recall $\varphi_- = e_1 + K\varphi_-$ where the operator $K$ is defined in Lemma 1. Subtracting $(I - K)e_1$ from $(I - K)\varphi_- = e_1$ leads to

$$(I - K)(\varphi_- - e_1) = e_1 - (I - K)e_1 = he_2,$$

where $h(x; k) = -\sigma \int_{-\infty}^{x} e^{2ik(x-y)}\bar{q}(-y)dy$.

It is sufficient to prove (32) for $\varphi_-$ that the operator $I - K$ is invertible and bounded, $h(x; k) \in L^{\infty}_x(\mathbb{R}; L^2_k(\mathbb{R}))$, and $\partial_k \varphi_-(x; k) \in L^{\infty}_x(\mathbb{R}; L^2_k(\mathbb{R}))$.

Recall the definition of the operator $K$ in Lemma 1, we have

$$\|K^n f(x; k)\|_{L^{\infty}_x L^2_k} \leq \frac{1}{n!} \|U(q)\|_{L^1}^n \|f(x; k)\|_{L^{\infty}_x L^2_k},$$

where $\|f\|_{L^{\infty}_x L^2_k} = \sup_{x \in \mathbb{R}} \|f\|_{L^2_k(\mathbb{R})}$ and $\|U(q)\|_{L^1} = 2\|q\|_{L^1}$, which means the operator $I - K$ is invertible and bounded. Besides, we obtain

$$\|(I - K)^{-1}\| \leq \sum_{n=0}^{\infty} \frac{1}{n!}(\|U(q)\|_{L^1})^n = e^{2\|q\|_{L^1}}.$$

Since $q \in L^{2,1}(\mathbb{R})$, it follows from (26) that

$$\sup_{x \in \mathbb{R}} \|h(x; k)\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \|q\|_{L^2(\mathbb{R})},$$
which implies $h(x; k) \in L^\infty_x(\mathbb{R}; L^2_k(\mathbb{R}))$. Employing (28), we get that for every $x_0 \in \mathbb{R}^-$,

$$\sup_{x \in (-\infty, x_0)} \| (x) h(x; k) \|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \| q \|_{L^{2,1}(-\infty, x_0)}.$$  \hfill (39)

Combining (37) with (39), we acquire

$$\sup_{x \in (-\infty, x_0)} \| (x)(\varphi_-(x; k) - e_1) \|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} e^2 \| q \|_{L^1(\mathbb{R})} \| q \|_{L^{2,1}(\mathbb{R})}. \hfill (40)$$

Let

$$v(x; k) = \begin{pmatrix} \partial_k \varphi_-(x; k) \\ \partial_k \varphi_-(x; k) - 2ix \varphi_-(x; k) \end{pmatrix}.$$  \hfill (41)

Direct calculation yields

$$(I - K)v = \begin{pmatrix} \partial_k \varphi_-(x; k) - \int_{-\infty}^{x} q(y)(\partial_k \varphi_-(y; k) - 2iy \varphi_-(y; k))dy \\ \partial_k \varphi_-(x; k) - 2ix \varphi_-(x; k) + \int_{-\infty}^{x} \sigma e^{2i(k(x-y))} \varphi_-(y; k) \partial_k \varphi_-(y; k)dy + \int_{-\infty}^{x} \sigma e^{2i(k(x-y))} \partial_k \varphi_-(y; k) \partial_k \varphi_-(y; k)dy \end{pmatrix}.$$  \hfill (42)

Taking derivative with $k$ for the integral equation (9) gives

$$\partial_k \varphi_-(x; k) = \int_{-\infty}^{x} q(y) \partial_k \varphi_-(y; k)dy,$$

$$\partial_k \varphi_-(x; k) = - \sigma \int_{-\infty}^{x} e^{2i(k(x-y))} \varphi_-(y; k) \partial_k \varphi_-(y; k)dy - 2i \sigma \int_{-\infty}^{x} (x - y) e^{2i(k(x-y))} \varphi_-(y; k) \partial_k \varphi_-(y; k)dy,$$

and utilizing the integral equation (9) for $\varphi_-(x; k)$, we derive that

$$(I - K)v = h_1 e_1 + (h_2 + h_3)e_2$$  \hfill (43)

with

$$h_1(x; k) = 2i \int_{-\infty}^{x} yq(y) \varphi_-(y; k)dy,$$

$$h_2(x; k) = 2i \sigma \int_{-\infty}^{x} ye^{2i(k(x-y))} \varphi_-(y; k) \varphi_-(y; k) - 1)dy,$$  \hfill (44)

$$h_3(x; k) = 2i \sigma \int_{-\infty}^{x} ye^{2i(k(x-y))} \varphi_-(y; k)dy.$$

Using the Minkowski’s inequality of the integral form, we find that for every $x_0 \in \mathbb{R}^-$,

$$\sup_{x \in (-\infty, x_0)} \| h_1(x; k) \|_{L^2_k(\mathbb{R})} \leq 2 \| q \|_{L^1(\mathbb{R})} \sup_{x \in (-\infty, x_0)} \| (x) \varphi_-(x; k) \|_{L^2_k(\mathbb{R})}.$$
Due to the imbedding of $L^{2,1}(\mathbb{R})$ into $L^1(\mathbb{R})$ and the bound (40), we infer that $\sup_{x \in (-\infty, x_0)} \|h_1(x; k)\|_{L^2_k(\mathbb{R})}$ is finite.

Performing a similar analysis yields

$$\sup_{x \in (-\infty, x_0)} \|h_2(x; k)\|_{L^2_k(\mathbb{R})} \leq 2\|q\|_{L^1(\mathbb{R})} \sup_{x \in (-\infty, x_0)} \|(x)(\varphi_{-}^{(1)}(x; k) - 1)\|_{L^2_k(\mathbb{R})} < +\infty,$$

and utilizing the estimate (26), we find that

$$\sup_{x \in \mathbb{R}} \|h_3(x; k)\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \|q\|_{L^{2,1}(\mathbb{R})} < +\infty. \quad (43)$$

As a result, we conclude that $v(x; k) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$.

Since $x \varphi_{-}^{(1)}(x; k) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$ given the estimate (40), we obtain $\partial_k \varphi_{-}(x; k) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$. This completes the proof of (32) for $\varphi_{-}$.

Moreover, if $q \in H^{1,1}(\mathbb{R})$, using the operator $I - K$ again leads to

$$(I - K)(k(\varphi_{-} - e_1) - \frac{\sigma}{2i}(s_{-}(x)e_1 + \bar{q}(-x)e_2)) = khe_2 - \frac{\sigma}{2i}(I - K)(s_{-}(x)e_1 + \bar{q}(-x)e_2)$$

Let

\[ h(x; k) = -\frac{\sigma}{2i} \begin{pmatrix} 2i k \int_{-\infty}^{x} e^{2ik(x-y)} \bar{q}(-y)dy - \frac{\sigma}{2i} \bar{q}(-x) - \frac{1}{2i} \int_{-\infty}^{x} e^{2ik(x-y)} \bar{q}(-y)s_{-}(y)dy \end{pmatrix}. \]

we have

$$\begin{aligned} (I - K)(k(\varphi_{-} - e_1) - \frac{\sigma}{2i}(s_{-}(x)e_1 + \bar{q}(-x)e_2)) = h e_2. \quad (44) \end{aligned}$$

On account of the estimate (27), we infer that

$$\sup_{x \in \mathbb{R}} \left\| \int_{-\infty}^{x} e^{2ik(x-y)} \bar{q}(-y)s_{-}(y)dy \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \left\| \partial_x \bar{q} \right\|_{L^2(\mathbb{R})} < +\infty. \quad (45)$$

As $q \in L^2(\mathbb{R})$, we have $\bar{q}(-x)s_{-}(x) = \bar{q}(-x) \int_{-\infty}^{x} q(y)\bar{q}(-y)dy \in L^2(\mathbb{R})$. Then, employing the estimate (26), we have

$$\sup_{x \in \mathbb{R}} \left\| \int_{-\infty}^{x} e^{2ik(x-y)} \bar{q}(-y)s_{-}(y)dy \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} \left\| \bar{q}(-x)s_{-}(x) \right\|_{L^2(\mathbb{R})} < +\infty. \quad (46)$$

Subsequently, it follows from (45) and (46) that $h(x; k) \in L^\infty_x(\mathbb{R}; L^2_k(\mathbb{R}))$. Consequently, We conclude from (44) that $k(\varphi_{-}(x; k) - e_1) - \frac{\sigma}{2i}(s^{(1)}(x)e_1 + \bar{q}(x)e_2)$ belongs to $L^\infty_x(\mathbb{R}; L^2_k(\mathbb{R}))$ for every $x \in \mathbb{R}$ because the operator $I - K$ is invertible and bounded.
Besides, it follows from (28), (29), (37), and (44) that

$$\sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \left( k(\phi_-(x; k) - e_1) - \frac{\sigma}{2i} (s_-(x)e_1 + \bar{q}(-x)e_2) \right) \right\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} e^{2\|q\|_{L^1(\mathbb{R})}} (\|\bar{q}(-x) s_-(x)\|_{L^{2,1}(\mathbb{R})} + \|\partial_x q\|_{L^{2,1}(\mathbb{R})}) < +\infty. \tag{47}$$

It remains to prove that for every $x \in \mathbb{R} \pm$, we have $k \partial_k \varphi_\pm(x; k) \in L^2_k(\mathbb{R})$. Let

$$\bar{v}(x; k) = \begin{pmatrix}
 k \partial_k \varphi_-(1)(x; k) \\
 k \partial_k \varphi_-(2)(x; k) - 2ix \left( k \varphi-(2)(x; k) - \frac{\sigma}{2i} \bar{q}(-x) \right)
\end{pmatrix}.$$  

Long but direct calculation yields

$$(I - K) \bar{v} = \bar{h}_1 e_1 + (\bar{h}_2 + \bar{h}_3 + \bar{h}_4) e_2, \tag{48}$$

where

$$\bar{h}_1(x; k) = 2i \int_{-\infty}^x yq(y) \left( k\varphi-(2)(y; k) - \frac{\sigma}{2i} \bar{q}(-y) \right) dy,$$
$$\bar{h}_2(x; k) = 2i \sigma k \int_{-\infty}^x e^{2ik(x-y)} y\bar{q}(-y) dy + \sigma x \bar{q}(-x),$$
$$\bar{h}_3(x; k) = 2i \sigma \int_{-\infty}^x e^{2ik(x-y)} y\bar{q}(-y) \left( k(\varphi-(1)(y; k) - 1) - \frac{\sigma}{2i} s_-(y) \right) dy,$$
$$\bar{h}_4(x; k) = \int_{-\infty}^x e^{2ik(x-y)} y\bar{q}(-y)s_-(y) dy. \tag{49}$$

Utilizing the Fubini’s theorem, we have

$$\sup_{x \in (-\infty, x_0)} \|\bar{h}_1(x; k)\|_{L^2_k(\mathbb{R})} \leq 2\|q\|_{L^1(\mathbb{R})} \sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \left( k\varphi-(2)(x; k) - \frac{\sigma}{2i} \bar{q}(-x) \right) \right\|_{L^2_k(\mathbb{R})},$$
$$\sup_{x \in (-\infty, x_0)} \|\bar{h}_3(x; k)\|_{L^2_k(\mathbb{R})} \leq 2\|q\|_{L^1(\mathbb{R})} \sup_{x \in (-\infty, x_0)} \left\| \langle x \rangle \left( k(\varphi-(1)(x; k) - 1) - \frac{\sigma}{2i} s_-(x) \right) \right\|_{L^2_k(\mathbb{R})}, \tag{50}$$

which are finite due to (47). Using (27), we have

$$\sup_{x \in \mathbb{R}} \|\bar{h}_2(x; k)\|_{L^2_k(\mathbb{R})} \leq \sqrt{\pi} (\|\bar{q}\|_{L^2(\mathbb{R})} + \|x \partial_x q\|_{L^2(\mathbb{R})}) < +\infty. \tag{51}$$

Since $q \in L^{2,1}(\mathbb{R})$, we have $x \bar{q}(-x)s_-(x) = x \bar{q}(-x) \int_{-\infty}^x q(y)\bar{q}(-y) dy \in L^2(\mathbb{R})$. Then, employing the estimate (26), we have

$$\sup_{x \in \mathbb{R}} \|\bar{h}_4(x; k)\|_{L^2_k(\mathbb{R})} \leq c \|\bar{q}(-x)s_-(x)\|_{L^2(\mathbb{R})} < +\infty. \tag{52}$$

Using the above results (50)–(52), together with the boundness and invertibility of the operator $I - K$, we obtain $\bar{v}(x; k) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$. Since $x(k\varphi-(2)(x; k) - \frac{\sigma}{2i} \bar{q}(-x)) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$ by (47), we finally conclude that $k \partial_k \varphi_-(x; k) \in L^\infty_x((-\infty, x_0); L^2_k(\mathbb{R}))$. \[\square\]
Lemma 4. Let $q \in H^{1,1}(\mathbb{R})$ and $\|q\|_{L^1(\mathbb{R})} < 1$. The mappings
\begin{equation}
L^{1,1}(\mathbb{R}) \ni q \mapsto [\varphi_\pm(x; k) - e_1, \phi_\pm(x; k) - e_2] \in L^\infty_x (\mathbb{R}^\pm; H^1_k(\mathbb{R})),
\end{equation}
\begin{equation}
H^{1,1}(\mathbb{R}) \ni q \mapsto [\Phi_\pm, \Phi_\pm] \in L^\infty_x (\mathbb{R}^\pm; H^1_k(\mathbb{R}))
\end{equation}
are Lipschitz continuous, here
\begin{align*}
\dot{\varphi}_\pm &= 2i\sigma k (\varphi_\pm(x; k) - e_1) - (s_\pm(x)e_1 + \bar{q}(-x)e_2), \\
\dot{\phi}_\pm &= 2i\sigma k (\phi_\pm(x; k) - e_2) - (\sigma q(x)e_1 - s_\pm(x)e_2).
\end{align*}

Proof. Let $q, \tilde{q} \in L^{1,1}(\mathbb{R})$. Let the functions $[\varphi_\pm, \phi_\pm]$ and $[\tilde{\varphi}_\pm, \tilde{\phi}_\pm]$ denote the corresponding Jost functions, respectively. Again, we only prove the statement for $\varphi_-$. It is enough to prove that there is a constant $c$ such that for every $x \in \mathbb{R}^\pm$,
\begin{equation}
\|\varphi_-(x; \cdot) - \tilde{\varphi}_-(x; \cdot)\|_{H^1(\mathbb{R})} \leq c \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})}.
\end{equation}
Using (35), we have
\begin{align}
\varphi_- - \tilde{\varphi} &= (\varphi_- - e_1) - (\tilde{\varphi}_- - e_1) \\
&= (I - K)^{-1} h e_2 - (I - K)^{-1} \tilde{h} e_2 \\
&= (I - K)^{-1} (h - \tilde{h}) e_2 + (I - K)^{-1} (K - \tilde{K})(I - \tilde{K})^{-1} \tilde{h} e_2,
\end{align}
where
\begin{equation}
h(x; k) - \tilde{h}(x; k) = -\sigma \int_{-\infty}^{x} e^{2ik(x-y)} (\bar{q}(y) - \tilde{q}(y)) dy.
\end{equation}
For the first term of (56), it follows from (26) that
\begin{equation}
\sup_{x \in \mathbb{R}} \|h(x; \cdot) - \tilde{h}(x; \cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{\pi} \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})} \leq \sqrt{\pi} \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})}.
\end{equation}
For the second term of (56), recall the boundness (36) of the operator $K$, we have
\begin{equation}
\|(K - \tilde{K}) f\|_{L^\infty_x L^2_k} \leq 2 \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})},
\end{equation}
where we have used the embedding of $L^{2,1}(\mathbb{R})$ into $L^1(\mathbb{R})$.
Note that $\tilde{h} \in L^\infty_x (\mathbb{R}; L^2_k(\mathbb{R}))$, we obtain
\begin{equation}
\|\varphi_-(x; \cdot) - \tilde{\varphi}_-(x; \cdot)\|_{L^2(\mathbb{R})} \leq c \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})}.
\end{equation}
Repeating an analogous analysis for (41) for $v = (\partial_k \varphi_-^{(1)}, \partial_k \varphi_-^{(2)} - 2ix \varphi_-^{(2)})f$, we find
\begin{equation}
\|\varphi_-(x; \cdot) - \tilde{\varphi}_-(x; \cdot)\|_{H^1(\mathbb{R})} \leq c \|q - \tilde{q}\|_{L^{2,1}(\mathbb{R})}.
\end{equation}
Performing a similar manipulation leads to other statements. □
2.2 Scattering data and Lipschitz continuity

Both the Jost functions $\varphi_\pm(x; k)e^{-ikx}$ and $\phi_\pm(x; k)e^{ikx}$ satisfy the first-order linear equation (6), so there exists a linear dependence

\[
(\varphi_-(x; k), \phi_-(x; k)) = (\varphi_+(x; k), \phi_+(x; k)) \begin{pmatrix} a(k) & c(k)e^{-2ikx} \\ b(k)e^{2ikx} & d(k) \end{pmatrix},
\]

for every $x \in \mathbb{R}$ and every $k \in \mathbb{R}$.

Given the linear dependence, we have the following properties of the scattering coefficients for $k \in \mathbb{R}$,

- The scattering coefficients have the Wronskian’s expressions
  \[
  a(k) = W(\varphi_-(0; k), \phi_+(0; k)), \\
  b(k) = W(\varphi_+(0; k), \varphi_-(0; k)), \\
  d(k) = W(\varphi_+(0; k), \phi_-(0; k)).
  \]

- The scattering coefficients satisfy the symmetry
  \[
  \overline{a(-\bar{k})} = a(k), \quad c(k) = -\sigma \overline{b(-\bar{k})}, \quad \overline{d(-\bar{k})} = d(k).
  \]

- The determinant of the scattering matrix is
  \[
  a(k)d(k) + \sigma b(k)b(-\bar{k}) = 1.
  \]

- The scattering coefficients satisfy the asymptotics
  \[
  a(k) \rightarrow 1 \quad \text{as} \quad |k| \rightarrow \infty, \\
  d(k) \rightarrow 1 \quad \text{as} \quad |k| \rightarrow \infty, \\
  b(k) \rightarrow 0 \quad \text{as} \quad |k| \rightarrow \infty.
  \]

- The scattering coefficients admit the integral expression:
  \[
  a(k) = 1 + \int_{-\infty}^{+\infty} q(y)\varphi^{(2)}_-(y; k)dy, \\
  b(k) = -\sigma \int_{-\infty}^{+\infty} e^{-2i\bar{k}y} \tilde{q}(-y)\varphi^{(1)}_+(y; k)dy, \\
  d(k) = 1 - \sigma \int_{-\infty}^{+\infty} \tilde{q}(-y)\phi^{(1)}_-(y; k)dy.
  \]
Define the reflection coefficients

\[ r_1(k) = \frac{b(k)}{a(k)}, \quad r_2(k) = \frac{b(-k)}{d(k)}, \quad k \in \mathbb{R}. \]  

(67)

\textbf{Lemma 5.} If \( q \in L^{2,1}(\mathbb{R}) \) and \( \|q\|_{L^1(\mathbb{R})} < 1 \), then the function \( a(k) \) is continued analytically in \( \mathbb{C}^+ \), whereas the function \( d(k) \) is continued analytically in \( \mathbb{C}^- \), in addition,

\[ a(k) - 1, \ b(k), \ d(k) - 1 \in H^1(\mathbb{R}). \]

Moreover, if \( q \in H^{1,1}(\mathbb{R}) \), then

\[ b(k) \in L^{2,1}(\mathbb{R}), \ kb(k) \in L^\infty(\mathbb{R}). \]

\textit{Proof.} By Lemma 1, we obtained that the Jost functions \( \varphi_-(0; k) \) and \( \phi_+(0; k) \) are continued analytically in \( \mathbb{C}^+ \), whereas \( \varphi_+(0; k) \) and \( \phi_-(0; k) \) are continued analytically in \( \mathbb{C}^- \). Thus, we can derive that \( a(k) \) and \( d(k) \) are continued analytically in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \), respectively, by using the Wronskian’s expressions (62).

The scattering coefficients \( a(k) \) can be rewritten as

\[ a(k) - 1 = \varphi_-^{(1)}(0; k)\phi_+^{(2)}(0; k) - \varphi_+^{(2)}(0; k)\phi_-^{(1)}(0; k) - 1 \]

\[ = (\varphi_-^{(1)}(0; k) - 1)(\phi_+^{(2)}(0; k) - 1) + (\phi_+^{(2)}(0; k) - 1) \]

\[ + (\varphi_-^{(1)}(0; k) - 1) - \varphi_+^{(2)}(0; k)\phi_-^{(1)}(0; k). \]  

(68)

Since \( \varphi_-(0; k) - e_1 \in H^1_k(\mathbb{R}), \phi_+(0; k) - e_2 \in H^1_k(\mathbb{R}) \), and the space \( H^1_k(\mathbb{R}) \) is a Banach algebra, we conclude that \( a(k) - 1 \in H^1_k(\mathbb{R}) \).

Utilizing an analogous method, we have

\[ d(k) - 1 = \phi_+^{(1)}(0; k)\phi_-^{(2)}(0; k) - \varphi_+^{(2)}(0; k)\varphi_-^{(1)}(0; k) - 1 \]

\[ = (\varphi_+^{(1)}(0; k) - 1)(\phi_-^{(2)}(0; k) - 1) + (\phi_-^{(2)}(0; k) - 1) \]

\[ + (\varphi_+^{(1)}(0; k) - 1) - \varphi_+^{(2)}(0; k)\varphi_-^{(1)}(0; k), \]  

(69)

then we obtain \( d(k) - 1 \in H^1_k(\mathbb{R}) \).

For the scattering coefficients \( b(k) \), we find

\[ b(k) = \varphi_-^{(1)}(0; k)\varphi_+^{(2)}(0; k) - \varphi_+^{(2)}(0; k)\varphi_-^{(1)}(0; k) \]

\[ = (\varphi_+^{(1)}(0; k) - 1)\varphi_-^{(2)}(0; k) + \varphi_-^{(2)}(0; k) \]

\[ - \varphi_+^{(2)}(0; k)(\varphi_-^{(1)}(0; k) - 1) - \varphi_+^{(2)}(0; k). \]  

(70)

Again, using the Banach algebra property, we get \( b(k) \in H^1_k(\mathbb{R}) \).
Moreover, if \( q \in H^{1,1}(\mathbb{R}) \), we rewrite \( k b(k) \) as
\[
k b(k) = k \varphi_+^{(1)}(0; k) \varphi_-^{(2)}(0; k) - k \varphi_+^{(2)}(0; k) \varphi_-^{(1)}(0; k)
\]
\[:= \varphi_+^{(1)}(0; k) (k \varphi_-^{(2)}(0; k) - \frac{\sigma}{2i} \tilde{q}(0)) - \varphi_+^{(2)}(0; k) (k \varphi_-^{(1)}(0; k) - \frac{\sigma}{2i} \tilde{q}(0))
\]
\[+ \frac{\sigma}{2i} \tilde{q}(0) (\varphi_+^{(1)}(0; k) - 1) - \frac{\sigma}{2i} \tilde{q}(0) (\varphi_+^{(1)}(0; k) - 1). \tag{71}\]

Recall (32) and (33), we find all the terms in (71) are in \( L^2_k(\mathbb{R}) \). Therefore, we arrive at \( b(k) \in L^{2,1}_k(\mathbb{R}) \).

As well, due to (32), (33), and the imbedding of \( H^1(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \), we find all the terms in (71) are in \( L^\infty_k(\mathbb{R}) \). Therefore, we obtain \( k b(k) \in L^\infty(\mathbb{R}) \).

\[ \square \]

**Lemma 6.** Let \( q \in H^{1,1}(\mathbb{R}) \) and \( \|q\|_{L^1(\mathbb{R})} < 1 \). The mappings
\[
L^{2,1}_k(\mathbb{R}) \ni q \mapsto a(k) - 1, \ b(k), \ d(k) - 1 \in H^{1}_k(\mathbb{R}), \tag{72}
\]
\[
H^{1,1}(\mathbb{R}) \ni q \mapsto b(k) \in \Gamma(\mathbb{R}) \tag{73}
\]
are Lipschitz continuous.

**Proof.** From the representations (68), (71), and the Lipschitz continuity of the Jost function \( \varphi_\pm \) and \( \phi_\pm \), we can obtain the Lipschitz continuity of the scattering coefficients. \( \square \)

**Lemma 7.** If \( q \in L^{2,1}(\mathbb{R}) \) with \( L^1 \) small-norm such that \( \|q\|_{L^1(\mathbb{R})} e^2 \|q\|_{L^1(\mathbb{R})} < 1 \), then the spectral problem (6) admits no eigenvalues or resonances, that is, the scattering coefficients \( a(k) \) and \( d(k) \) admit no zeros in \( \mathbb{C}^+ \cup \mathbb{R} \) and \( \mathbb{C}^- \cup \mathbb{R} \), respectively.

**Proof.** The small-norm condition implies that \( \|q\|_{L^1(\mathbb{R})} < 1 \). Recall that \( \varphi_- = e_1 + K \varphi_- \) in Lemma 1 and the operator \( I - K \) is invertible and bounded from (37). Using (14), we reach that for every \( k \in \mathbb{C}^+ \),
\[
\|\varphi_-(\cdot; k) - e_1\|_{L^\infty} = \|(I - K)^{-1}\| \|K e_1(\cdot; k)\|_{L^\infty} \leq e^2 \|q\|_{L^1}. \tag{74}
\]
Employing (66), we derive for every \( k \in \mathbb{C}^+ \),
\[
|a(k)| \geq 1 - \left| \int_{\mathbb{R}} q(y) \varphi_-^{(2)}(y; k) dy \right| > 1 - \|q\|_{L^1} e^2 \|q\|_{L^1}. \tag{75}
\]
Due to the continuity of \( a(k) \), we obtain \( |a(k)| \geq 1 - \|q\|_{L^1} e^2 \|q\|_{L^1} \) for \( k \in \mathbb{R} \). Therefore, we may take \( \|q\|_{L^1(\mathbb{R})} \) small enough such that
\[
|a(k)| \geq 1 - \|q\|_{L^1(\mathbb{R})} e^2 \|q\|_{L^1(\mathbb{R})} > 0, \ k \in \mathbb{R}, \tag{75}
\]
then \( a(k) \) admits no zeros in \( \mathbb{C}^+ \cup \mathbb{R} \). Carrying out a similar manipulation for \( d(k) \), we see that \( d(k) \) admits no zeros in \( \mathbb{C}^- \cup \mathbb{R} \). \( \square \)
Lemma 8. If \( q \in L^{2,1}(\mathbb{R}) \) with \( L^1(\mathbb{R}) \) small-norm condition (4), then for every \( k \in \mathbb{R} \), we have \( |r_{1,2}(k)| < 1 \).

Proof. Rewrite (66) for \( b(k) \) as

\[
  b(k) = -\sigma \int_{\mathbb{R}} e^{-2i ky} \bar{q}(-y) (\varphi^{(1)}(y; k) - 1) dy - \sigma \int_{\mathbb{R}} e^{-2i ky} \bar{q}(-y) dy.
\]

Note that the condition (75) is naturally satisfied under the condition (4). For every \( k \in \mathbb{R} \), applying (74), (75), and (4), we obtain

\[
  |b(k)| \leq \|\varphi^{(1)}(\cdot; k) - 1\|_{L^\infty} \|q\|_{L^1} + \|q\|_{L^1} \leq \|q\|_{L^1(\mathbb{R})} + \|q\|_{L^1(\mathbb{R})} e^2 \|q\|_{L^1(\mathbb{R})}
\]

\[
  < 1 - \|q\|_{L^1(\mathbb{R})} e^2 \|q\|_{L^1(\mathbb{R})} \leq a(k),
\]

which yields

\[
  |r_1(k)| = \frac{|b(k)|}{|a(k)|} < 1.
\]

Similarly, we get \( |r_2(k)| < 1 \). \( \square \)

For the reflection coefficients \( r_{1,2}(k) \), we have the following results:

Lemma 9. If \( q \in L^{2,1}(\mathbb{R}) \) such that the condition (4) is satisfied, then we have

\[
  r_{1,2}(k) \in H^1(\mathbb{R}).
\]

Moreover, if \( q \in H^{1,1}(\mathbb{R}) \), then we have

\[
  r_{1,2}(k) \in \Gamma(\mathbb{R}).
\]

As well, the mapping

\[
  H^{1,1}(\mathbb{R}) \ni q \mapsto (r_1, r_2) \in \Gamma(\mathbb{R})
\]

is Lipschitz continuous.

Proof. Due to the bound (75) and a similar bound of \( d(k) \), the property of \( r_{1,2}(k) \) follows from the Proposition 5.

Let \((r_1, r_2)\) and \((\tilde{r}_1, \tilde{r}_2)\) denote the reflection coefficients corresponding to \( q \) and \( \tilde{q} \), respectively. Owing to

\[
  r_1 - \tilde{r}_1 = \frac{b - \tilde{b}}{a} + \frac{b((\tilde{a} - 1) - (a - 1))}{a \tilde{a}},
\]

the Lipschitz continuity of \( r_1 \) follows from the Lipschitz continuity of \( a - 1 \) and \( b \). \( \square \)
3 | INVERSE SCATTERING TRANSFORM

In this section, we will set up an RH problem and show the existence and uniqueness of the solution to the RH problem for the given data $r_{1,2}(k) \in \Gamma(\mathbb{R})$ satisfying $|r_{1,2}(k)| < 1$.

3.1 | Setup of an RH problem

Define the matrix-valued functions

\[
M(x;k) = \begin{cases}
\left[ \frac{\varphi-(x;k)}{a(k)}, \phi+(x;k) \right], & k \in \mathbb{C}^+ , \\
\left[ \varphi+(x;k), \frac{\phi-(x;k)}{d(k)} \right], & k \in \mathbb{C}^- .
\end{cases}
\] (78)

It follows from the linear dependence (61) and (63) that for $k \in \mathbb{R}$,

\[
\varphi- = a(k)\varphi+ + b(k)e^{2ikx}\phi+ ,
\] (79)

\[
\phi- = -\sigma b(-k)e^{-2ikx}\varphi+ + d(k)\phi+ .
\] (80)

Rewriting (80) as

\[
\varphi+ = \sigma b(-k)e^{-2ikx} \frac{d(k)}{d(k)} \phi+ + \frac{\phi-}{d(k)} .
\] (81)

And utilizing (79) and (81), we have

\[
\frac{\varphi-}{a(k)} = \varphi+ + \frac{b(k)e^{2ikx}}{a(k)}\phi+ = \left( 1 + \frac{b(k)\sigma b(-k)}{a(k)d(k)} \right)\varphi+ + \frac{b(k)e^{2ikx}}{a(k)} \frac{\phi-}{d(k)} .
\] (82)

Therefore, we obtain

\[
M+(x;k) - M-(x;k) = M-(x;k)S(x;k) , \quad k \in \mathbb{R} ,
\] (83)

where

\[
S(x;k) = \begin{pmatrix}
\sigma r_1(k)r_2(k) & \sigma r_2(k)e^{-2ikx} \\
r_1(k)e^{2ikx} & 0
\end{pmatrix} .
\]

The RH problem can be described as follows:

**RH problem 1.** Find a $2 \times 2$ matrix-valued function $M(x;k)$ with the following properties:

- $M(x;\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2\times 2}$ is analytic.
The limits of $M(x; k)$ as $k$ approaches $\mathbb{R}$ from the upper and lower half-plane exist and are continuous on $\mathbb{R}$, moreover, they satisfy

$$M_+(x; k) - M_-(x; k) = M_-(x; k)S(x; k), \quad k \in \mathbb{R}.$$  

(84)

$M(x; k)$ satisfies the asymptotic

$$M(x; k) \to I, \quad |k| \to \infty, k \in \mathbb{C} \setminus \mathbb{R}.$$

3.2 Solvability of the RH problem

Before looking for the solution to the RH problem, we show some preliminary knowledge that will be used in the subsequent section, which has been given in the previous work.  

For any function $h \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, the Cauchy operator is defined as

$$C(h)(k) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(s)}{s - k} \, ds, \quad k \in \mathbb{C} \setminus \mathbb{R},$$  

(85)

and the Plemelj projection is given by

$$P^\pm(h)(k) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(s)}{s - (k \pm \varepsilon i)} \, ds, \quad k \in \mathbb{R}.$$  

(86)

**Proposition 2** (see Ref. [27]). For every $h \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, the Cauchy operator $C(h)$ is analytic off the real line, decays to zero as $|k| \to \infty$, and approaches to $P^\pm(h)$ almost everywhere when a point $k \in \mathbb{C}^\pm$ approaches to a point on the real axis by any nontangential contour from $\mathbb{C}^\pm$. If $1 < p < \infty$, then there exists a positive constant $C_p(C_{p=2} = 1)$ such that

$$\|P^\pm(h)\|_{L^p} \leq C_p \|h\|_{L^p}.$$  

(87)

If $h \in L^1(\mathbb{R})$, then the Cauchy operator admits the following asymptotic limit in either $\mathbb{C}^+$ or $\mathbb{C}^-$:

$$\lim_{|k| \to \infty} kC(h)(k) = -\frac{1}{2\pi i} \int_{\mathbb{R}} h(s) \, ds.$$  

(88)

**Lemma 10.** Let $r_{1,2}(k) \in H^1(\mathbb{R})$ satisfy $|r_{1,2}(k)| < 1$, then there exist positive constants $c_-$ and $c_+$ such that for every $x \in \mathbb{R}$ and every column-vector $g \in \mathbb{C}^2$, we have

$$\text{Re} \, g^*(I + S(x; k))g \geq c_-g^*g, \quad k \in \mathbb{R}$$  

(89)

and

$$\|(I + S(x; k))g\| \leq c_+\|g\|, \quad k \in \mathbb{R},$$  

(90)

where the asterisk denotes the Hermite conjugate.

**Proof.** The original scattering matrix $S(x; k)$ is not Hermitian due to the fact that there is no relationship between $a(k)$ and $d(k)$. Hence it is difficult to use the theory of Zhou [29] to obtain a
unique solution to the RH problem. Therefore, we define Hermitian part of $S(x; k)$ by

$$S_H(x; k) = \frac{1}{2}(S(x; k) + S^*(x; k))$$

$$= \begin{pmatrix}
\sigma \text{Re}(r_1 r_2) & \frac{1}{2}(\bar{r}_1 + \sigma r_2)e^{-2ikx} \\
\frac{1}{2}(r_1 + \sigma \bar{r}_2)e^{2ikx} & 0
\end{pmatrix}$$

(91)

Since $|r_{1,2}(k)| < 1$, the 2-order principle minor of the matrix $I + S_H$

$$1 + \sigma \text{Re}(r_1 r_2) - \frac{1}{4}|r_1 + \sigma \bar{r}_2|^2 = 1 - \frac{1}{4}|r_1 - \sigma r_2|^2 > 0,$$

which indicates that the 1-order principle minor $1 + \sigma \text{Re}(r_1 r_2) > 0$. Thus, the matrix $I + S_H$ is positive definite.

In view of the algebra theory, for a Hermitian matrix, there exists an unitary matrix $A$ such that

$$A^*(I + S_H)A = \text{diag}(\mu_+, \mu_-),$$

(92)

where $\mu_{\pm}$ are the eigenvalues of the matrix $I + S_H$

$$\mu_{\pm}(k) = \frac{2 + \sigma \text{Re}(r_1 r_2) \pm \sqrt{\text{Re}^2(r_1 r_2) + |r_1 + \sigma \bar{r}_2|^2}}{2}.$$  

Note $\mu_+(k) > \mu_-(k) > 0$ as $I + S_H$ is positive definite. And it follows from $r_{1,2}(k) \to 0$ that $\mu_-(k) \to 1$ as $|k| \to \infty, k \in \mathbb{R}$. Together with $r_{1,2}(k) \in H^1(\mathbb{R})$, there exists a positive constant $c_-$ such that $\mu_- > c_-$. Consequently, for every $g \in \mathbb{C}^2$, utilizing (92), we have

$$c_- g^*g < \mu_- g^*g \leq \text{Reg}^*(I + S(x; k))g = g^*(I + S_H)g,$$

this completes the proof of the bound (89).

Calculating $(I + S(x; k))g$ componentwise and utilizing $|r_{1,2}(k)| < 1$ gives that

$$\| (I + S(x; k))g \|^2 \leq 2(1 + |r_1| + |r_2|)^2\|g\|^2$$

$$+ 2\text{Re}\left\{((\sigma + r_1 r_2)\bar{r}_2 + r_1)e^{2ikx}g^{(1)}g^{(2)}\right\},$$

(93)

$$\leq \left((|r_1| + 1)^2 + (|r_2| + 1)^2 + (|r_1| + |r_2|)^2\right)\|g\|^2,$$

here the norm for a 2-component vector $f$ is $\|f\|^2 = |f^{(1)}|^2 + |f^{(2)}|^2$. Therefore, we take

$$c_+ = \sup_{k \in \mathbb{R}} \sqrt{(|r_1| + 1)^2 + (|r_2| + 1)^2 + (|r_1| + |r_2|)^2} < +\infty,$$

then one obtain the bound (90). □

Introduce a transformation

$$\Psi_{\pm}(x; k) = M_{\pm}(x; k) - I,$$
then we obtain a new RH problem for $\Psi(x; k)$, the jump condition and asymptotic behavior are

$$
\Psi_+(x; k) - \Psi_-(x; k) = \Psi_-(x; k) S(x; k) + S(x; k), \quad k \in \mathbb{R},
$$

$$
\Psi_\pm(x; k) \to 0, \quad |k| \to \infty, \quad k \in \mathbb{C} \setminus \mathbb{R}.
$$

**Lemma 11.** Let $r_{1,2}(k) \in H^1(\mathbb{R})$ satisfy $|r_{1,2}(k)| < 1$, then for every $F(k) \in L_k^2(\mathbb{R})$, there exists a unique solution $\Psi(k) \in L_k^2(\mathbb{R})$ of the equation

$$
(I - P_S^-)\Psi_-(k) = F(k), \quad k \in \mathbb{R},
$$

where $P_S^-\Psi_- = P^-(\Psi_- S)$.

**Proof.** Since $I - P_S^-$ is a Fredholm operator of the index zero, by Fredholm’s alternative theorem, there exists a unique solution of the equation $(I - P_S^-)\Psi_-(k) = F(k)$ if and only if the zero solution of the equation $(I - P_S^-)g = 0$ is unique in $L_k^2(\mathbb{R})$.

Assume that there exists a function $g(k) \in L_k^2(\mathbb{R})$ and $g(k) \neq 0$ such that $(I - P_S^-)g = 0$. Define two analytic functions in $\mathbb{C} \setminus \mathbb{R}$ as

$$
g_1(k) = C(gS)(k), \quad g_2(k) = C(gS)^*(k).
$$

The functions $g_1(k)$ and $g_2(k)$ are well defined due to $S(k) \in L_k^2(\mathbb{R}) \cap L_\infty^\alpha(\mathbb{R})$.

We integrate the function $g_1(k)g_2(k)$ along the semicircle of radius $R$ centered at zero in $\mathbb{C}^+$. It follows from Cauchy theorem that

$$
\int g_1(k)g_2(k)dk = 0.
$$

Since $g(k)S(k) \in L_k^1(\mathbb{R})$, using (88) yields $g_{1,2}(k) = \mathcal{O}(k^{-1})$, $|k| \to \infty$. Hence, the integral on the arc approaches to zero as the radius approaches to infinity. Therefore, we obtain

$$
0 = \int_{\mathbb{R}} g_1(k)g_2(k)dk = \int_{\mathbb{R}} P^+(gS)[P^-(gS)]^* dk
$$

$$
= \int_{\mathbb{R}} [P^-(gS) + gS][P^-(gS)]^* dk.
$$

Utilizing the assumption $P^-(gS) = g$, we have

$$
\int_{\mathbb{R}} g(I + S)g^* dk = 0.
$$

By Lemma 10, we get $\text{Re } g(I + S)g^* > c_- g^* g$ with $c_-$ is the unique solution to the equation $(I - P_S^-)g = 0$ in $L_k^2(\mathbb{R})$. Finally there exists a unique solution to the equation $(I - P_S^-)\Psi_-(k) = F(k)$. □

**Lemma 12.** Let $r_{1,2}(k) \in H^1(\mathbb{R})$ satisfy $|r_{1,2}(k)| < 1$, then for every $x \in \mathbb{R}$, there exist unique solutions $\Psi_\pm(x; k) \in L_k^2(\mathbb{R})$ of the equation

$$
\Psi_+(x; k) - \Psi_-(x; k) = \Psi_- S(x; k) + S(x; k), \quad k \in \mathbb{R}.
$$

Moreover, $\Psi_\pm(x; k)$ are analytic for $k \in \mathbb{C}^\pm$. 
Proof. Owing to $\mathcal{S}(x;k) \in L^2_k(\mathbb{R})$, we have $P^-(S)(k) \in L^2_k(\mathbb{R})$ by (87). Then, we infer from Lemma 11 that for every $x \in \mathbb{R}$, there exists a unique solution $\Psi_-(x;k) \in L^2_k(\mathbb{R})$ to the equation

$$
\Psi_-(x;k) = P^-(\Psi_-(x;k)S(x;k) + S(x;k)), \quad k \in \mathbb{R}.
$$

(102)

Based on the existence of $\Psi_-(x;k)$, we define a function $\Psi_+(x;k)$ as

$$
\Psi_+(x;k) = P^+(\Psi_-(x;k)S(x;k) + S(x;k)), \quad k \in \mathbb{R}.
$$

(103)

Besides, analytic extensions of $\Psi_\pm(x;k)$ to $k \in \mathbb{C}^\pm$ are defined by the Cauchy operator

$$
\Psi_\pm(x;k) = C(\Psi_-(x;k)S(x;k) + S(x;k)), \quad k \in \mathbb{C}^\pm.
$$

(104)

Finally we obtain the solution $\Psi_\pm(x;k) \in L^2_k(\mathbb{R})$ of Equation (101). Moreover, given the property of the Cauchy operator and the Plemelj projection operator, the solutions $\Psi_\pm(x;k)$ are analytic functions for $k \in \mathbb{C}^\pm$. □

Lemma 13. Let $r_{1,2}(k) \in H^1(\mathbb{R})$ satisfy $|r_{1,2}(k)| < 1$, then the operator $(I - P^-_S)^{-1}$ is bounded from $L^2_k(\mathbb{R})$ to $L^2_k(\mathbb{R})$, and there exists a constant $c$ that only depends on $\|r_{1,2}(k)\|_{L^\infty_k}$ such that

$$
\|(I - P^-_S)^{-1}f\|_{L^2_k} \leq c\|f\|_{L^2_k}.
$$

(105)

Proof. For every $f(k) \in L^2_k(\mathbb{R})$, it follows from Lemma 11 that there exists a solution $\Psi(k) \in L^2_k(\mathbb{R})$ to $(I - P^-_S)\Psi(k) = f(k)$. Note that $P^+ - P^- = I$, we decompose the function $\Psi(k)$ into $\Psi = \Psi_+ - \Psi_-$ with

$$
\Psi_- - P^-(\Psi_-S) = P^-(f), \quad \Psi_+ - P^-(\Psi_+S) = P^+(f).
$$

(106)

Since $P^\pm(f) \in L^2_k(\mathbb{R})$, by Lemma 11, there exist unique solutions $\Psi_\pm(k) \in L^2_k(\mathbb{R})$ of Equation (106), which implies the decomposition is unique. Therefore, we only need the estimates of $\Psi_\pm$ in $L^2_k(\mathbb{R})$.

To deal with $\Psi_-$, define two analytic functions in $\mathbb{C}\setminus\mathbb{R}$

$$
g_1(k) = C(\Psi_-S)(k), \quad g_2(k) = C(\Psi_-S + f)^*(k),
$$

Analogous manipulation as the proof of Lemma 11, we integrate on the semicircle in the upper half-plane and have

$$
\oint g_1(k)g_2(k)dk = 0.
$$

(107)

Since $g_1(k) = O(k^{-1})$ and $g_2(k) \to 0$ as $|k| \to \infty$, we have

$$
0 = \int_{\mathbb{R}} P^+(\Psi_-S)[P^-(\Psi_-S + f)]^*dk

= \int_{\mathbb{R}} (P^-(\Psi_-S) + \Psi_-S)[P^-(\Psi_-S + f)]^*dk

= \int_{\mathbb{R}} (\Psi_- - P^-(f) + \Psi_-S)\Psi_-^*dk.
$$

(108)
Using the bound (89) and the Hölder inequality, there exists a positive constant $c_-$ such that
\[
c_- \|\Psi_-\|_{L^2}^2 \leq \text{Re} \int_{\mathbb{R}} \Psi_- (I + S) \Psi^*_- \, dk = \text{Re} \int_{\mathbb{R}} P^-(f) \Psi^*_- \, dk \leq \|f\|_{L^2} \|\Psi_-\|_{L^2},
\] (109)
this completes the estimates of $\Psi_-:
\[
\|\left(I - \mathcal{P}_S^+\right)^{-1} P^- f\|_{L^2_k} \leq c_-^{-1} \|f\|_{L^2_k}.
\] (110)

To deal with $\Psi_+$, define two functions in $\mathbb{C} \setminus \mathbb{R}$
\[
g_1(k) = C(\Psi_+ S)(k), \quad g_2(k) = C(\Psi_+ S + f)^*(k).
\]
Performing the similar procedure leads to
\[
0 = \oint g_1(k) g_2(k) \, dk
= \int_{\mathbb{R}} P^-(\Psi_+ S)[P^+(\Psi_+ S + f)]^* \, dk
= \int_{\mathbb{R}} [\Psi_+ - P^+(f)][\Psi_+ (I + S)]^* \, dk,
\] (111)
where we have used (106). Using the bounds (89) and (90), there are positive constants $c_-$ and $c_+$ such that
\[
c_- \|\Psi_+\|_{L^2}^2 \leq \text{Re} \int_{\mathbb{R}} \Psi_+ (I + S)^* \Psi^*_+ \, dk = \text{Re} \int_{\mathbb{R}} P^+(f)(I + S)^* \Psi^*_+ \, dk \leq c_+ \|f\|_{L^2} \|\Psi_+\|_{L^2},
\]
which means
\[
\|\left(I - \mathcal{P}_S^+\right)^{-1} P^+ f\|_{L^2_k} \leq c_+^{-1} c_+ \|f\|_{L^2_k}.
\] (112)
Combining (110) and (112), we obtain
\[
\|\left(I - \mathcal{P}_S^+\right)^{-1} f\|_{L^2_k} \leq c \|f\|_{L^2_k},
\]
where $c$ is a constant that only depends on $\|r_{1,2}(k)\|_{L^\infty_k}$.

### 3.3 Estimate on solutions to the RH problem

Next, we come back to the RH problem for $M(x; k)$. Denote the functions $M_\pm$ column-wise
\[
M_\pm(x; k) = [\mu_\pm(x; k), \nu_\pm(x; k)].
\]
Then, the functions $\Psi_\pm$ can be written column-wise as
\[
\Psi_\pm(x; k) = [\mu_\pm(x; k) - e_1, \nu_\pm(x; k) - e_2].
\]
According to (102) and (103), we have
\[
\mu_\pm(x; k) - e_1 = P_\pm(\psi_\pm S)^{(1)}(x; k) = P_\pm(M_-S)^{(1)}(x; k), \quad k \in \mathbb{R}
\] (113)

and
\[
\nu_\pm(x; k) - e_2 = P_\pm(\psi_\pm S)^{(2)}(x; k) = P_\pm(M_-S)^{(2)}(x; k), \quad k \in \mathbb{R}.
\] (114)

Combining (113) with (114), we obtain
\[
M_\pm(x; k) = I + P_\pm(M_-S)(x; \cdot)(k), \quad k \in \mathbb{R}.
\] (115)

By Lemma 12, there exist unique solutions \(M_\pm(x; k)\) to Equation (115). Further, analytic extensions of \(M_\pm(x; k)\) to \(k \in \mathbb{C}^\pm\) are
\[
M_\pm(x; k) = I + C(M_-S)(x; \cdot)(k), \quad k \in \mathbb{C}^\pm.
\] (116)

**Lemma 14.** Let \(r_{1,2}(k) \in H^1(\mathbb{R})\) satisfy \(|r_{1,2}(k)| < 1\), then there exists a constant \(c\) only depending on \(\|r_{1,2}\|_{L^\infty}\) such that for every \(x \in \mathbb{R}\), the functions \(M_\pm(x; k)\) satisfy
\[
\|M_\pm(x; \cdot) - I\|_{L^2_k} \leq c(\|r_1\|_{L^2_k} + \|r_2\|_{L^2_k}).
\] (117)

**Proof.** Due to \(r_{1,2}(k) \in H^1(\mathbb{R}) \cap L^2(\mathbb{R})\), we get \(r_{1,2}(k) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\) and \(S(x; k) \in L^2_k(\mathbb{R})\). Moreover, there exists a constant \(c\) only depending on \(\|r_{1,2}\|_{L^\infty}\) such that
\[
\|S(x; k)\|_{L^2_k} \leq c(\|r_1\|_{L^2_k} + \|r_2\|_{L^2_k}).
\] (118)

Employing the bound (13) and (87), we obtain
\[
\|M_\pm - I\|_{L^2_k} = \|\psi_\pm\|_{L^2_k} \leq c(\|r_1\|_{L^2_k} + \|r_2\|_{L^2_k}),
\] (119)

where we have used the equation \((I - P_{S}^-)\Psi_\pm = P^+_S\), and \(c\) is another constant only depending on \(\|r_{1,2}\|_{L^\infty}\).

The estimates on the reflection coefficients \(r_{1,2}(k)\) is given as follows with Fourier theory and the residue theorem.

**Proposition 3** (see Ref. [27]). For every \(x_0 \in \mathbb{R}^+\) and every \(r_{1,2} \in H^1(\mathbb{R})\), we have
\[
\sup_{x \in (x_0, +\infty)} \|\langle x \rangle P^+(r_2(k)e^{-2ikx})\|_{L^2_k} \leq \|r_2\|_{H^1},
\] (120)

\[
\sup_{x \in (x_0, +\infty)} \|\langle x \rangle P^-(r_1(k)e^{2ikx})\|_{L^2_k} \leq \|r_1\|_{H^1},
\] (121)

where \(\langle x \rangle = (1 + x^2)^{1/2}\). Moreover, we have
\[
\sup_{x \in \mathbb{R}} \|P^+(r_2(k)e^{-2ikx})\|_{L^\infty_k} \leq \frac{1}{\sqrt{2}} \|r_2\|_{H^1},
\] (122)

\[
\sup_{x \in \mathbb{R}} \|P^-(r_1(k)e^{2ikx})\|_{L^\infty_k} \leq \frac{1}{\sqrt{2}} \|r_1\|_{H^1}.
\] (123)
Furthermore, if \( r_{1,2} \in L^{2,1}(\mathbb{R}) \), then we have

\[
\sup_{x \in \mathbb{R}} \| \mathcal{P}^+(k r_2(k) e^{-2ikx}) \|_{L_k^2} \leq \| kr_2(k) \|_{L_k^2},
\]

(124)

\[
\sup_{x \in \mathbb{R}} \| \mathcal{P}^-(k r_1(k) e^{2ikx}) \|_{L_k^2} \leq \| kr_1(k) \|_{L_k^2}.
\]

(125)

In order to obtain the estimates on the vector columns \( \mu_-(x; k) - e_1 \) and \( \nu_+(x; k) - e_2 \) that will be needed in the subsequent section, we rewrite functions \( \mu_-(x; k) - e_1 \) and \( \nu_+(x; k) - e_2 \) by (115) as

\[
\mu_-(x; k) - e_1 = \mathcal{P}^- (M_- S(x; \cdot))^{(1)}(k) = \mathcal{P}^- (r_1(k) e^{2ikx} \nu_+(x; k))(k), k \in \mathbb{R}
\]

(126)

and

\[
\nu_+(x; k) - e_2 = \mathcal{P}^+(M_- S(x; \cdot))^{(2)}(k) = \sigma \mathcal{P}^+(r_2(k) e^{-2ikx} \mu_-(x; k))(k), k \in \mathbb{R},
\]

(127)

where we have used the fact

\[
M_- S = [\mu_-, \nu_-] \begin{pmatrix}
\sigma r_1 r_2 & \sigma r_2 e^{-2ikx} \\
r_1 e^{2ikx} & 0
\end{pmatrix}
\]

(128)

\[
= [\sigma r_1 r_2 \mu_- + r_1 e^{2ikx} \nu_-, \sigma r_2 e^{-2ikx} \mu_-]
\]

and the identity

\[
\nu_+ = \sigma r_2 e^{-2ikx} \mu_- + \nu_-
\]

(129)

follows from (84)

\[
[\mu_+, \nu_+] = [\mu_-, \nu_-] \begin{pmatrix}
1 + \sigma r_1 r_2 & \sigma r_2 e^{-2ikx} \\
r_1 e^{2ikx} & 1
\end{pmatrix}.
\]

(130)

Then introduce a function

\[
M(x; k) = [\mu_-(x; k) - e_1, \nu_+(x; k) - e_2],
\]

(131)

which satisfies

\[
M - \mathcal{P}^+(MS_+) - \mathcal{P}^-(MS_-) = F
\]

(132)

with

\[
F(x; k) = [\mathcal{P}^-(r_1 e^{2ikx}) e_2, \mathcal{P}^+(\sigma r_2 e^{-2ikx}) e_1],
\]

\[
S_+(x; k) = \begin{pmatrix} 0 & \sigma r_2 e^{-2ikx} \\ 0 & 0 \end{pmatrix}, \quad S_-(x; k) = \begin{pmatrix} 0 & 0 \\ r_1 e^{2ikx} & 0 \end{pmatrix}.
\]
Lemma 15. Let \( r_{1,2}(k) \in H^1(\mathbb{R}) \) satisfy \(|r_{1,2}(k)| < 1\), then for every \( x_0 \in \mathbb{R}^+ \), the solution of (126) and (127) satisfies

\[
\sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle \mu_{-}^{(2)}(x; k) \right\|_{L_k^2} \leq c \| r_1 \|_{H^1},
\]

(133)

\[
\sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle \nu_{+}^{(1)}(x; k) \right\|_{L_k^2} \leq c \| r_2 \|_{H^1},
\]

(134)

where \( c \) is a constant that only depends on \( \| r_{1,2} \|_{L^\infty} \). In addition, if \( r_{1,2} \in \Gamma(\mathbb{R}) \), then we have

\[
\sup_{x \in (x_0, +\infty)} \left\| \partial_x \mu_{-}^{(2)}(x; k) \right\|_{L_k^2} \leq c(\| r_1 \|_{\Gamma} + \| r_2 \|_{\Gamma}),
\]

(135)

\[
\sup_{x \in (x_0, +\infty)} \left\| \partial_x \nu_{+}^{(1)}(x; k) \right\|_{L_k^2} \leq c(\| r_1 \|_{\Gamma} + \| r_2 \|_{\Gamma}),
\]

(136)

where \( c \) is another constant that depends on \( \| r_{1,2} \|_{L^\infty} \) and \( \| kr_{1,2}(k) \|_{L^\infty} \).

Proof. Note \( P^+ - P^- = I \) and \( S_+ + S_- = (I - S_+)S \), Equation (132) can be rewritten as

\[
G - P^-(GS) = F
\]

(137)

with \( G = M(I - S_+) \). And the matrix \( G(x; k) \) is written component-wise as

\[
G(x; k) = \begin{pmatrix}
\mu_{-}^{(1)}(x; k) - 1 & \nu_{+}^{(1)} - \sigma r_2 e^{-2ikx} (\mu_{-}^{(1)}(x; k) - 1) \\
\mu_{-}^{(2)}(x; k) & \nu_{+}^{(2)} - 1 - \sigma r_2 e^{-2ikx} \mu_{-}^{(2)}(x; k) 
\end{pmatrix}.
\]

(138)

Comparing the second row of \( F(x; k) \) with \( G(x; k) \) and utilizing the bound (105), we have

\[
\sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle \mu_{-}^{(2)}(x; k) \right\|_{L_k^2} \leq c \sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle P^- (r_1 e^{2ikx}) \right\|_{L_k^2},
\]

(139)

\[
\left\| \nu_{+}^{(2)} - 1 - \sigma r_2 e^{2ikx} \mu_{-}^{(2)}(x; k) \right\|_{L_k^2} \leq c \left\| P^- (r_1 e^{2ikx}) \right\|_{L_k^2},
\]

where \( c \) is a constant that depends on \( \| r_{1,2} \|_{L^\infty} \). Substituting the bound (121) into (139), we obtain the estimate (133).

Similarly, comparing the first row of \( F(x; k) \) and \( G(x; k) \) yields

\[
\| \mu_{-}^{(1)}(x; k) - 1 \|_{L_k^2} \leq c \| P^+ (r_2 e^{-2ikx}) \|_{L_k^2},
\]

(140)

\[
\| \nu_{+}^{(1)}(x; k) - \sigma r_2 e^{-2ikx} (\mu_{-}^{(1)}(x; k) - 1) \|_{L_k^2} \leq c \| P^+ (r_2 e^{-2ikx}) \|_{L_k^2}.
\]

(141)

Employing \( r_2 \in L^\infty(\mathbb{R}) \) and the triangle inequality, we have

\[
\sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle \nu_{+}^{(1)}(x; k) \right\|_{L_k^2} \leq c \sup_{x \in (x_0, +\infty)} \left\| \langle x \rangle P^+ (r_2 e^{-2ikx}) \right\|.
\]

(141)

Substituting the bound (120) into (141), we obtain the estimate (134).

Taking derivative in \( x \) of (132), we obtain

\[
\partial_x M - P^+ (\partial_x M) S_+ - P^- (\partial_x M) S_- = \bar{F}
\]

(142)
with
\[
F = \partial_x F + P^+ M \partial_x S_+ + P^- M \partial_x S_- \\
= 2i \left[ e_2 P^- (kr_1(k)e^{2ikx}), \sigma e_1 P^+ (-kr_2(k)e^{-2ikx}) \right] \\
+ 2i \left( P^-(kr_1(k)e^{2ikx}v_+(x;k)) \quad \sigma P^+ (-kr_2(k)e^{-2ikx}\mu_+(x;k) - 1) \right).
\]

According to the estimates (139) and (140), together with the triangle equality, we obtain
\[\mu_-(x;k) - e_1 \in L_\infty^x((-\infty,x_0);L^2_k(\mathbb{R}))\]
and
\[\nu_+(x;k) - e_2 \in L_\infty^x((-\infty,x_0);L^2_k(\mathbb{R}))\]. On account of the bound (124), (125), and \(kr_{1,2}(k) \in L^\infty(\mathbb{R})\), we conclude that \(\tilde{F}\) belongs to \(L^\infty_x((-\infty,x_0);L^2_k(\mathbb{R}))\).

Repeating the analysis for (137), we derive the estimates (135) and (136).

4 | RECONSTRUCTION AND ESTIMATES OF THE POTENTIAL

Comparing the 2-element of the limits (17) lead to a reconstruction formula
\[
\bar{q}(-x) = 2i\sigma \lim_{\text{Im}k>0, k \to \infty} k\varphi^{(2)}_-(x;k).
\] (143)

As well, comparing the 2-element of the limits (18) leads to another reconstruction formula
\[
q(x) = 2i \lim_{\text{Im}k>0, k \to \infty} k\varphi^{(1)}_+(x;k).
\] (144)

Remark 1. From the symmetry (11), we reduce that
\[
\sigma \varphi^{(2)}_-(x;-\bar{k}) = \varphi^{(1)}_+(x;k),
\]
which implies that two reconstruction formulae (143) and (144) are compatible.

It follows from (143), (116), and (78) that
\[
\bar{q}(-x) = 2i\sigma \lim_{|k| \to \infty} kC((M_- S)_1).
\] (145)

Since \(r_{1,2} \in \Gamma(\mathbb{R})\), we have \(S(x;\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\). Besides, the estimate (117) implies that \(M_- S = (M_- - I)S + S \in L^1(\mathbb{R})\). Therefore, we arrive at \(M_- = (M_- - I)S + S \in L^1(\mathbb{R})\). Subsequently, applying (88) to (145), we obtain
\[
\bar{q}(-x) = -\frac{\sigma}{\pi} \int_\mathbb{R} r_1(k)e^{2ikx} \left( \nu^{(2)}_-(x;k) + \sigma r_2(k)e^{-2ikx}\mu^{(2)}_-(x;k) \right) dk \\
= -\frac{\sigma}{\pi} \int_\mathbb{R} r_1(k)e^{2ikx}v^{(2)}_+(x;k) dk,
\] (146)

where we have used the identity \(v^{(2)}_+ = \sigma r_2(k)e^{-2ikx}\mu^{(2)}_- + \nu^{(2)}_-\) due to (129).
Performing the same manipulation for (144) yields
\[ q(x) = -\frac{\sigma}{\pi} \int_{\mathbb{R}} r_2(k) e^{-2ikx} \mu_+^{(1)}(x; k) dk. \] (147)

**Lemma 16.** Let \( r_{1,2}(k) \in \Gamma(\mathbb{R}) \) satisfy \( |r_{1,2}(k)| < 1 \), then \( q \in H^{1,1}(\mathbb{R}^+) \). Moreover, we have
\[ \| q \|_{H^{1,1}(\mathbb{R}^+)} \leq c(\| r_1 \|_{\Gamma(\mathbb{R})} + \| r_2 \|_{\Gamma(\mathbb{R})}), \] (148)
where \( c \) is a constant that depends on \( \| r_{1,2} \|_{L^\infty} \) and \( \| kr_{1,2} \|_{L^\infty} \).

**Proof.** We rewrite (147) for \( q(x) \) as
\[ q(x) = -\frac{\sigma}{\pi} \int_{\mathbb{R}} r_2(k) e^{-2ikx} dk - \frac{\sigma}{\pi} \int_{\mathbb{R}} r_2(k) e^{-2ikx}(\mu_+^{(1)}(x; k) - 1) dk. \] (149)
Recall the results from the Fourier theory. For a function \( r(k) \in L^2(\mathbb{R}) \), by Parseval’s equation, we have
\[ \| r \|_{L^2} = \| \hat{r} \|_{L^2}, \] (150)
where the function \( \hat{r} \) denotes the Fourier transform with the definition
\[ \hat{r}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} r(k) e^{-ikx} dk. \]
Since \( r_2 \in H^1(\mathbb{R}) \), the first term of (149) belongs to \( L^{2,1}(\mathbb{R}) \) due to the property \( \hat{\partial_k r}(k) = x\hat{r}(x) \).
Let
\[ I(x) = \int_{\mathbb{R}} r_2(k) e^{-2ikx}(\mu_+^{(1)}(x; k) - 1) dk. \] (151)
Substituting (127) into (151) and applying the Fubini’s theorem yields
\[ I(x) = \sigma \int_{\mathbb{R}} r_2(k) e^{-2ikx} \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{r_1(s) e^{2isx} \nu_+^{(1)}(s)}{s - (k - i\varepsilon)} ds dk \]
\[ = -\sigma \int_{\mathbb{R}} r_1(s) e^{2isx} \nu_+^{(1)}(s) \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{r_2(k) e^{-2ikx}}{k - (s + i\varepsilon)} dk ds \]
\[ = -\sigma \int_{\mathbb{R}} r_1(k) e^{2ikx} \nu_+^{(1)}(k) P^+(r_2(k) e^{-2ikx})(k) dk. \] (152)
Therefore, for every \( x_0 \in \mathbb{R}^+ \), utilizing the Hölder’s inequality and the estimates (120) and (134), we find
\[ \sup_{x \in (x_0, +\infty)} \| \langle x \rangle^2 I(x) \|_{L^\infty} \leq \sup_{x \in (x_0, +\infty)} \| \langle x \rangle \nu_+^{(1)}(x; k) \|_{L^2_k} \]
\[ \times \sup_{x \in (x_0, +\infty)} \| \langle x \rangle P^+(r_2(k) e^{-2ikx}) \|_{L^2_k} \leq c \| r_2 \|_{H^1}^2, \] (153)
where $c$ is a constant and only depends on $\|r_{1,2}\|_L^\infty$, further, we obtain

$$
\|\langle x \rangle I(x)\|_{L^2(\mathbb{R}^+)} \leq c \|r_2\|_{H^1}^2,
$$

(154)

where $c$ is another constant only depends on $\|r_{1,2}\|_L^\infty$. Combining the results of the two terms of (149) leads to

$$
\|q(x)\|_{L^{2,1}(\mathbb{R}^+)} \leq c(1 + \|r_2\|_{H^1}) \|r_2\|_{H^1}.
$$

(155)

This completes the proof of $q \in L^{2,1}(\mathbb{R}^+)$. By the Fourier theory, the derivative of the first term of (149) belongs to $L^2(\mathbb{R})$. For the second term $I(x)$, we differentiate $I(x)$ in $x$ and obtain

$$
I'(x) = \frac{\partial}{\partial x} \int_{\mathbb{R}} r_2(k)e^{-2ikx}\left(\mu^{(1)}(x; k) - 1\right) dk
$$

$$
= -2i \int_{\mathbb{R}} kr_2(k)e^{-2ikx}\left(\mu^{(1)}(x; k) - 1\right) dk
$$

$$
+ \int_{\mathbb{R}} r_2(k)e^{-2ikx} \frac{\partial \mu^{(1)}(x; k)}{\partial x} dk
$$

(156)

$$
= 2i \int_{\mathbb{R}} r_1(k)e^{2ikx}\nu^{(1)}_+(x; k)P^+(kr_2(k)e^{-2ikx})(k) dk
$$

$$
- 2i \int_{\mathbb{R}} kr_1(k)e^{2ikx}\nu^{(1)}_+(x; k)P^+(r_2(k)e^{-2ikx})(k) dk
$$

$$
- \int_{\mathbb{R}} r_1(k)e^{2ikx} \partial_x \nu^{(1)}_+(x; k)P^+(r_2(k)e^{-2ikx})(k) dk,
$$

where we have used Equation (126) and the Fubini’s theorem.

Utilizing the estimates (120), (122), (124), (134), and (136), we find that for every $x_0 \in \mathbb{R}^+$,

$$
\sup_{x \in (x_0, +\infty)} |\langle x \rangle I'(x)| \leq 2\|r_1\|_{L^\infty} \sup ||\langle x \rangle \nu^{(1)}_+(x; k)||_{L^2_k} \sup ||P^+(kr_2(k)e^{-2ikx})||_{L^2_k}
$$

$$
+ 2||kr_1||_{L^2} \sup ||\langle x \rangle \nu^{(1)}_+(x; k)||_{L^2_k} \sup ||P^+(r_2(k)e^{-2ikx})||_{L^\infty_k}
$$

$$
+ ||r_1||_{L^\infty} \sup ||\partial_x \nu^{(1)}_+(x; k)||_{L^2_k} \sup ||\langle x \rangle P^+(r_2(k)e^{-2ikx})||_{L^2_k}
$$

$$
\leq c(||r_1||_F ||r_2||_F (||r_1||_F + ||r_2||_F),
$$

which implies that

$$
\|\langle x \rangle I'(x)\|_{L^2(\mathbb{R}^+)} \leq c||r_1||_F ||r_2||_F (||r_1||_F + ||r_2||_F),
$$

(157)

with $c$ is another constant that depends on $\|r_{1,2}\|_{L^\infty}$ and $\|kr_{1,2}\|_{L^\infty}$. Subsequently, we obtain $I'(x) \in L^{2,1}(\mathbb{R}^+)$. Finally we conclude that $q \in H^{1,1}(\mathbb{R}^+)$. The estimate (167) can be obtained from (155) and (157) with another constant $c$ depending on $\|r_{1,2}\|_{L^\infty}$ and $\|kr_{1,2}\|_{L^\infty}$. 

□
Lemma 17. Let \( r_{1,2}(k) \in \Gamma(\mathbb{R}) \) satisfy \(|r_{1,2}(k)| < 1\), then the mapping
\[
\Gamma(\mathbb{R}) \ni (r_1, r_2) \mapsto q \in H^{1,1}(\mathbb{R}^+)
\] (158)
is Lipschitz continuous.

Proof. Let \((r_1, r_2), (\tilde{r}_1, \tilde{r}_2) \in \Gamma(\mathbb{R})\). Let the functions \(q\) and \(\tilde{q}\) be the corresponding potentials, respectively. We will show that there exists a constant \(c\) that depends on \(\|r_{1,2}\|_{L_\infty}\) and \(\|kr_{1,2}\|_{L_\infty}\) such that
\[
\|q - \tilde{q}\|_{H^{1,1}(\mathbb{R}^+)} \leq c(\|r_1 - \tilde{r}_1\|_{\Gamma(\mathbb{R})} + \|r_2 - \tilde{r}_2\|_{\Gamma(\mathbb{R})}).
\] (159)

From (147) and (149), we have
\[
q - \tilde{q} = -\frac{\sigma}{\pi} \int_\mathbb{R} (r_2 - \tilde{r}_2)e^{-2ikx} dk - \frac{\sigma}{\pi} \int_\mathbb{R} (r_2 - \tilde{r}_2)e^{-2ikx}(\mu^{(1)}_-(x; k) - 1)dk
\] - \frac{\sigma}{\pi} \int_\mathbb{R} \tilde{r}_2(k)e^{-2ikx}(\mu^{(1)}_-(x; k) - \tilde{\mu}^{(1)}_-(x; k))dk.
\] (160)

Repeating the analysis in the proof of Lemma 16, we obtain the Lipschitz continuity of \(q\). \(\square\)

Lemma 18. Let \( r_{1,2}(k) \in \Gamma(\mathbb{R}) \) satisfy \(|r_{1,2}(k)| < 1\), then \(q \in H^{1,1}(\mathbb{R}^-)\). Moreover, we have
\[
\|q\|_{H^{1,1}(\mathbb{R}^-)} \leq c(\|r_1\|_{\Gamma(\mathbb{R})} + \|r_2\|_{\Gamma(\mathbb{R})}),
\] (161)
where \(c\) is a constant that depends on \(\|r_{1,2}\|_{L_\infty}\) and \(\|kr_{1,2}\|_{L_\infty}\).

Proof. We rewrite (146) as
\[
\tilde{q}(-x) = -\frac{\sigma}{\pi} \int_\mathbb{R} r_1(k)e^{2ikx} dk - \frac{\sigma}{\pi} \int_\mathbb{R} r_1(k)e^{2ikx}(\nu^{(2)}_+(x; k) - 1)dk.
\] (162)

Let
\[
\hat{r}_1(-x) = \int_\mathbb{R} r_1(k)e^{-2ik(-x)} dk.
\] (163)

According to the Fourier theory, we have \(-x\hat{r}_1(-x) = \hat{\delta}_k r_1(k)(x)\) and \(\|x\hat{r}_1(-x)\|_{L^2(\mathbb{R})} = \|\hat{\delta}_k r_1(k)\|_{L^2(\mathbb{R})}\). Due to \(r_1 \in H^1(\mathbb{R})\) and the Parseval’s equation, we obtain
\[
\|\hat{r}_1(-x)\|_{L^{2,1}(\mathbb{R})} \leq \|\hat{r}_1(-x)\|_{L^2(\mathbb{R})} + \|x\hat{r}_1(-x)\|_{L^2(\mathbb{R})} = \|r_1\|_{H^1(\mathbb{R})}.
\] (164)

Let
\[
I_2(x) = -\frac{\sigma}{\pi} \int_\mathbb{R} r_1(k)e^{2ikx}(\nu^{(2)}_+(x; k) - 1)dk.
\]

Repeating the analysis in the proof of Lemma 16, we obtain \(q \in H^{1,1}(\mathbb{R}^-)\) and
\[
\|q\|_{H^{1,1}(\mathbb{R}^-)} \leq c(\|r_1\|_{\Gamma(\mathbb{R})} + \|r_2\|_{\Gamma(\mathbb{R})}),
\] (165)
where \(c\) is a constant that depends on \(\|r_{1,2}\|_{L_\infty}\) and \(\|kr_{1,2}\|_{L_\infty}\). \(\square\)
By a similar procedure as Lemma 17, we have the following results:

**Lemma 19.** Let \( r_{1,2}(k) \in \Gamma(\mathbb{R}) \) satisfy \(|r_{1,2}(k)| < 1\), then the mapping
\[
\Gamma(\mathbb{R}) \ni (r_1, r_2) \mapsto q \in H^{1,1}(\mathbb{R}^-)
\] (166)
is Lipschitz continuous.

Summarizing the results from Lemma 16 to Lemma 19, we have the following proposition:

**Proposition 4.** Let \( r_{1,2}(k) \in \Gamma(\mathbb{R}) \) satisfy \(|r_{1,2}(k)| < 1\), then we have \( q(x) \in H^{1,1}(\mathbb{R}) \) and

\[
\|q\|_{H^{1,1}(\mathbb{R})} \leq c(\|r_1\|_{\Gamma(\mathbb{R})} + \|r_2\|_{\Gamma(\mathbb{R})}).
\] (167)

Moreover, the mapping
\[
\Gamma(\mathbb{R}) \ni (r_1, r_2) \mapsto q \in H^{1,1}(\mathbb{R})
\] (168)
is Lipschitz continuous.

## 5 | EXISTENCE OF GLOBAL SOLUTIONS

### 5.1 | Time evolution of scattering data

From Section 2 to Section 4, for the initial data \( q(0, x) \in H^{1,1}(\mathbb{R}) \), we only consider the spatial spectral problem (6) and obtain its unique normalized solution
\[
\varphi_{\pm}(0, x; k) \rightarrow e_1, \quad \phi_{\pm}(0, x; k) \rightarrow e_2, \quad x \rightarrow \pm \infty,
\] (169)
which cannot satisfy the time spectral problem (7) since they are short of a function of the time \( t \).

For every \( t \in [0, T] \), we define the normalized Jost functions of the Lax pair (6) and (7)
\[
\varphi_{\pm}(t, x; k) = \varphi_{\pm}(0, x; k)e^{-2ik^2t},
\] (170)
\[
\phi_{\pm}(t, x; k) = \phi_{\pm}(0, x; k)e^{2ik^2t},
\] (171)
with the potential \( q(0, x) \in H^{1,1}(\mathbb{R}) \). It follows that for every \( t \in [0, T] \), we have
\[
\varphi_{\pm}(t, x; k) \rightarrow e^{-2ik^2t}e_1 \quad x \rightarrow \pm \infty,
\]
\[
\phi_{\pm}(t, x; k) \rightarrow e^{2ik^2t}e_2 \quad x \rightarrow \pm \infty.
\]
Repeating the analysis as the proof of Lemma 1, we prove that there exist unique solutions of the Volterra’s integral equations for Jost functions \( \varphi_{\pm}(t, x; k) \) and \( \phi_{\pm}(t, x; k) \), which admits the same analytic property as the functions \( \varphi_{\pm}(0, x; k) \) and \( \phi_{\pm}(0, x; k) \).
As well, for every \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\) and every \(k \in \mathbb{R}\), the Jost functions \(\varphi_\pm (t, x; k)\) and \(\phi_\pm (t, x; k)\) are supposed to satisfy the scattering relation

\[
\begin{align*}
\varphi_- (t, x; k) &= a(t; k) \varphi_+(t, x; k) + b(t; k) e^{2ikx} \phi_+(t, x; k), \\
\phi_- (t, x; k) &= c(t; k) e^{-2ikx} \varphi_+(t, x; k) + d(t; k) \phi_+(t, x; k).
\end{align*}
\]

By the Crammer’s law and the evolution relation (170)–(171), we obtain the evolution of the scattering coefficients

\[
\begin{align*}
a(t; k) &= W(\varphi_-(0, 0; k) e^{-2ik^2t}, \phi_+(0, 0; k) e^{2ik^2t}) = a(0; k), \\
b(t; k) &= W(\varphi_+(0, 0; k) e^{-2ik^2t}, \varphi_-(0, 0; k) e^{-2ik^2t}) = b(0; k) e^{-4ik^2t}, \\
d(t; k) &= W(\varphi_+(0, 0; k) e^{2ik^2t}, \phi_-(0, 0; k) e^{-2ik^2t}) = d(0; k).
\end{align*}
\]

Direct calculation shows that the reflection coefficients are given by

\[
\begin{align*}
r_1(t; k) &= \frac{b(t; k)}{a(t; k)} = \frac{b(0; k)}{a(0; k)} e^{-4ik^2t} = r_1(0; k) e^{-4ik^2t}, \\
r_2(t; k) &= \frac{\overline{b(t; -k)}}{\overline{d(t; k)}} = \frac{\overline{b(0; k)}}{\overline{d(0; k)}} e^{4ik^2t} = r_2(0; k) e^{4ik^2t},
\end{align*}
\]

where \(r_{1,2}(0; k)\) are the initial reflection data founded from the initial data \(q(0, x)\).

**Proposition 5.** If \(r_{1,2}(0; k) \in \Gamma(\mathbb{R})\), then for every \(T > 0\), and all \(t \in [0, T]\), we have \(r_{1,2}(t; k) \in \Gamma(\mathbb{R})\).

**Proof.** By (172), we obtain

\[
\|r_{1,2}(t; \cdot)\|_{L^{2,1}(\mathbb{R})} = \|r_{1,2}(0; \cdot)\|_{L^{2,1}(\mathbb{R})}.
\]

Using Lemma 9, we have

\[
\|\bar{k}r_{1,2}(t; \cdot)\|_{L^\infty(\mathbb{R})} = \|kr_{1,2}(0; \cdot)\|_{L^\infty(\mathbb{R})}.
\]

For every \(t \in [0, T]\), we have

\[
\begin{align*}
\|\partial_k r_1(t; \cdot)\|_{L^2(\mathbb{R})} &= \|\partial_k r_1(0; \cdot) - 8iktr_1(t; k)\|_{L^2(\mathbb{R})} \\
&\leq \|\partial_k r_1(0; \cdot)\|_{L^2(\mathbb{R})} + 8T\|r_1(0; \cdot)\|_{L^{2,1}}, \\
\|\partial_k r_2(t; \cdot)\|_{L^2(\mathbb{R})} &= \|\partial_k r_2(0; \cdot) + 8iktr_2(t; k)\|_{L^2(\mathbb{R})} \\
&\leq \|\partial_k r_2(0; \cdot)\|_{L^2(\mathbb{R})} + 8T\|r_2(0; \cdot)\|_{L^{2,1}}.
\end{align*}
\]

Therefore, we infer that \(r_{1,2}(t; \cdot) \in \Gamma(\mathbb{R})\) for every \(t \in [0, T]\) since \(r_{1,2}(0; \cdot) \in \Gamma(\mathbb{R})\).

**5.2 | Proof of Theorem 1**

Finally we give the proof of Theorem 1 as follows.
As well, the constraint $|r_{1,2}(t;k)| < 1$ remains valid for every $t \in [0,T]$. Performing a similar analysis as Lemmas 16–19, we can establish a time-dependent RH problem for $r_{1,2}(t;k)$ for every $t \in [0,T]$:

**RH problem 2.** Find a $2 \times 2$ matrix-valued function $M(x,t;k)$ with the following properties:

- $M(x,t;\cdot): \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times2}$ is analytic.
- The limits of $M(x,t;k)$ as $k$ approaches $\mathbb{R}$ from the upper and lower half-plane exist and are continuous on $\mathbb{R}$, moreover, they satisfy
  \[ M_+(x,t;k) - M_-(x,t;k) = M_-(x,t;k)S(x,t;k), \quad k \in \mathbb{R}. \]  \hspace{1cm} (178)

- $M(x,t;k)$ satisfies the asymptotic
  \[ M(x,t;k) \to I, \quad |k| \to \infty, k \in \mathbb{C}\setminus\mathbb{R}. \]  \hspace{1cm} (181)

And we can also address the existence and uniqueness of the solution to the RH problem. Further, the potential $q(t,x)$ can be recovered from the reflection coefficients $r_{1,2}(t;k)$ as

\[ q(t,x) = 2i \lim_{k \to \infty} kM_{12}(x,t;k). \]  \hspace{1cm} (179)

Next we will prove that the potential $q(t,x)$ obtained by (179) is the solution of Equation (1). Define two operators for the solution $M(x,t;k)$ of the RH problem 2

\[ \mathcal{L}M = M_x + ik[\sigma_3,M] - UM, \]  \hspace{1cm} (180)

\[ \mathcal{N}M = M_t + 2ik^2[\sigma_3,M] - VM, \]  \hspace{1cm} (181)

where $U = i[\sigma_3,M_I]$ and $V = 2kU - i\sigma_3U^2 + i\sigma_3U_x$ with $M = I + \frac{M_1}{k} + \frac{M_2}{k^2} + \cdots$. Then, the function $\mathcal{L}M$ and $\mathcal{N}M$ satisfy the RH problem as follows:

\[ M_+(x,t;k) - M_-(x,t;k) = M_-(x,t;k)S(x,t;k), \]  \hspace{1cm} (182)

\[ M(x,t;k) = O(k^{-1}), \quad |k| \to \infty, k \in \mathbb{C}\setminus\mathbb{R}. \]  \hspace{1cm} (183)

Due to the uniqueness of the solution of the RH problem, we have

\[ \mathcal{L}M = 0, \quad \mathcal{N}M = 0. \]  \hspace{1cm} (184)

From the compatibility of (184), the potential $q(t,x)$ recovered from (179) is the solution of Equation (1).

Moreover, the potential $q(t,x)$ belongs to $H^{1,1}(\mathbb{R})$ for every $t \in [0,T]$ and is Lipschitz continuous of $r_{1,2}(t;k)$. Thus, we have

\[ \|q(t;\cdot)\|_{H^{1,1}} \leq c_1(\|r_1(t;\cdot)\|_{\Gamma} + \|r_2(t;\cdot)\|_{\Gamma}) \]

\[ \leq c_2(r_1(0;\cdot)\|_{\Gamma} + \|r_2(0;\cdot)\|_{\Gamma}) \leq c_3\|q_0\|_{H^{1,1}}, \]  \hspace{1cm} (185)
where the positive constants $c_1$, $c_2$, and $c_3$ depend on $(r_{1,2}\|L^\infty, \|kr_{1,2}\|L^\infty), (T, \|r_{1,2}\|L^\infty, \|kr_{1,2}\|L^\infty)$, and $(T, \|q_0\|_{H^{1,1}})$, respectively.

Next, we show the solution $q(t, x)$ is continuous with respect to $t \in [0, T]$ under the $H^{1,1}(\mathbb{R})$ norm. Let $t \in [0, T]$ and $|\Delta t| < 1$ such that $t + \Delta t \in [0, T]$, then with the Lipschitz continuity from $q(t, x)$ to $r_{1,2}(t; k)$ in Proposition 4, we have

$$
\|q(t + \Delta t, x) - q(t, x)\|_{H^{1,1}(\mathbb{R})} \\
\leq c(\|r_1(t + \Delta t; k) - r_1(t; k)\|_{\Gamma(\mathbb{R})} + \|r_2(t + \Delta t; k) - r_2(t; k)\|_{\Gamma(\mathbb{R})}) \\
\leq c|\Delta t|(\|r_1(0; k)\|_{\Gamma} + \|r_2(0; k)\|_{\Gamma}) \leq c|\Delta t| \to 0, \Delta t \to 0.
$$

Assume $q \in C([0, T], H^{1,1}(\mathbb{R}))$ blows up in a finite time, that is, there exists a constant $T_{\text{max}}$ such that

$$
\lim_{t \to T_{\text{max}}} \|q(t; \cdot)\|_{H^{1,1}(\mathbb{R})} = \infty. \quad (186)
$$

According to the estimates (176), (177), and (185), we find that $c_3$ in (185) remains finite for every $T > 0$. This contradicts on the assumption (186). Therefore, we can extend the local solution $q \in C([0, T], H^{1,1}(\mathbb{R}))$ to the whole time line $q \in C([0, \infty), H^{1,1}(\mathbb{R}))$.

ACKNOWLEDGMENTS
This work is supported by the National Science Foundation of China (Grant No. 12271104,51879045).

CONFLICT OF INTEREST STATEMENT
The authors have no conflicts to disclose.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available within the paper.

ORCID
Engui Fan https://orcid.org/0000-0002-8816-8398

REFERENCES
1. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear Schrödinger equation. Phys Rev Lett. 2013;110:064105.
2. Ablowitz MJ, Luo XD, Musslimani ZH. Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions. J Math Phys. 2018;59:011501.
3. Makris KG, El-Ganainy R, Christodoulides DN, Musslimani ZH. Beam dynamics in $PT$ symmetric optical lattices. Phys Rev Lett. 2008;100:103904.
4. Musslimani ZH, Makris KG, El-Ganainy R, Christodoulides DN. Optical solitons in $PT$ periodic potentials. Phys Rev Lett. 2008;100:030402.
5. Konotop VV, Yang J, Zezyulin DA. Nonlinear waves in $PT$-symmetric systems. Rev Modern Phys. 2016;88:035002.
6. Bender CM, Boettcher S. Real spectra in non-Hermitian Hamiltonians having $PT$ symmetry. Phys Rev Lett. 1998;80:5243.
7. Gadzhimuradov TA, Agalarov AM. Towards a gauge-equivalent magnetic structure of the nonlocal nonlinear Schrodinger equation. Phys Rev A. 2016;93:062124.
8. Lou SY. Alice-Bob systems, $P_{s^{-1}-T_d-C}$ symmetry invariant and symmetry breaking soliton solutions. *J Math Phys*. 2018;59:083507.

9. Lou SY, Huang F. Alice-Bob physics: coherent solutions of nonlocal KdV systems. *Sci Rep*. 2017;7:1-11.

10. Fokas AS. Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. *Nonlinearity*. 2016;29:319.

11. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear equations. *Stud Appl Math*. 2017;139:7-59.

12. Ablowitz MJ, Haberman R. Resonantly coupled nonlinear evolution equations. *J Math Phys*. 1975;16:2301-2305.

13. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. *J Math Phys*. 2019;60:031504.

14. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal focusing nonlinear Schrödinger equation for a family of step-like initial data. *Commun Math Phys*. 2021;382:87-121.

15. Rybalko Y, Shepelsky D. Long-time asymptotics for the nonlocal nonlinear Schrödinger equation with step-like initial data. *J Differ Equ*. 2021;270:694-724.

16. Li GZ, Yang YL, Fan EG. Long time asymptotic behavior for the nonlocal nonlinear Schrödinger equation in solitonic region, *Sci. China, Ser A*, in press.

17. Deift P, Zhou X. *Long-time behavior of the non-focusing nonlinear Schrödinger equation-a case study*. Lectures in Mathematical Sciences, Graduate School of Mathematical Sciences, University of Tokyo; 1994.

18. Dieng M, McLaughlin KDTR. *Dispersive asymptotics for linear and integrable equations by the Dbar steepest descent method*. In: PD Miller, PA Perry, JS Sulem eds. Nonlinear dispersive partial differential equations and inverse scattering. Fields Inst. Comm., Springer; 2019:253-291.

19. Borghese M, Jenkins R, McLaughlin KDTR, Miller P. Long-time asymptotic behavior of the focusing nonlinear Schrödinger equation. *Ann I H Poincaré Anal*. 2018;35:897-920.

20. Boutet de Monvel A, Kotlyrov VP, Shepelsky D. Foucusing NLS equation: long-time dynamics of step-like initial data. *Int Math Res Notices*. 2011;7:1613-1653.

21. Boutet de Monvel A, Lenells J, Shepelsky D. The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics. *Commun Math Phys*. 2021;383:893-952.

22. Boutet de Monvel A, Lenells J, Shepelsky D. The focusing NLS equation with step-like oscillating background: the genus 3 sector. *Commun Math Phys*. 2022;390:1081-1148.

23. Fromm S, Lenells J, Quirchmayr R. The defocusing nonlinear Schrödinger equation with step-like oscillatory initial data, arXiv: 2104.03714v1, 2021.

24. Jenkins R. Regularization of a sharp shock by the defocusing nonlinear Schrödinger equation. *Nonlinearity*. 2015;28:2131-2180.

25. Genoud F. Instability of an integrable nonlocal NLS. *C R Math Acad Sci Paris*. 2017;355:299-303.

26. Zhou X. $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms. *Commun Pure Appl Math*. 1998;51:0697-0731.

27. Pelinovsky DE, Shimabukuro Y. Existence of global solutions to the derivative NLS equation with the inverse scattering transform method. *Int Math Res Notices*. 2018;18:5663-5728.

28. Deift P, Zhou X. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Commun Pure Appl Math*. 2003;56:1029-1077.

29. Zhou X. The Riemann-Hilbert problem and inverse scattering. *SIAM J Math Anal*. 1989;20:966-986.

30. Beals R, Coifman RR. Scattering and inverse scattering for first order systems. *Commun Pure Appl Math*. 1984;37:39-90.

31. Beals R, Coifman RR. Inverse scattering and evolution equations. Commun. *Pure Appl Math*. 1985;38:29-42.

**How to cite this article:** Zhao Y, Fan E. Existence of global solutions to the nonlocal Schrödinger equation on the line. *Stud Appl Math*. 2024;152:111–146. https://doi.org/10.1111/sapm.12636