New Suns in the Cosmos. IV. The Multifractal Nature of Stellar Magnetic Activity in Kepler Cool Stars

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Abstract

In the present study, we investigate the multifractal nature of a long-cadence time series observed by the Kepler mission for a sample of 34 M dwarf stars and the Sun in its active phase. Using the Multifractal Detrending Moving Average algorithm, which enables the detection of multifractality in nonstationary time series, we define a set of multifractal indices based on the multifractal spectrum profile as a measure of the level of stellar magnetic activity. This set of indices is given by the (\(A, \Delta \alpha, C, H\))-quartet, where \(A\), \(\Delta \alpha\), and \(C\) are related to geometric features from the multifractal spectrum and the global Hurst exponent \(H\) describes the global structure and memorability of time series dynamics. As a test, we measure these indices and compare them with a magnetic index defined as \(S_{PE}\) and verify the degree of correlation among them. First, we apply the Poincaré plot method and find a strong correlation between the \(S_{PE}\) index and one of the descriptors that emerges from this method. As a result, we find that this index is strongly correlated with long-term features of the signal. From the multifractal perspective, the \(S_{PE}\) index is also strongly linked to the geometric properties of the multifractal spectrum except for the \(H\) index. Furthermore, our results emphasize that the rotation period of stars is scaled by the \(H\) index, which is consistent with Skumanich’s relationship. Finally, our approach suggests that the \(H\) index may be related to the evolution of stellar angular momentum and a star’s magnetic properties.

Key words: methods: data analysis – stars: activity – stars: solar-type – Sun: rotation

1. Introduction

Stellar activity is strongly related to extreme and transient phenomena caused by the dissipation of magnetic energy when magnetic fields evolve from complex to simple topologies according to the rate of stellar rotation (de Freitas & De Medeiros 2013; Oswalt & Barstow 2013). Indeed, as cited by Saar & Brandenburg (2002), the magnetic cycle period and rotation period are correlated in such a manner that slow rotators have larger magnetic cycles. Furthermore, the global level of magnetic activity changes over time following cycles, and the local levels change on different timescales that are associated with high and low intermittent fluctuations, which change on timescales ranging from a few seconds to several hours (flares) and from days to weeks (active regions), respectively. However, this level of activity depends on the rotation of stars. As mentioned by Mathur et al. (2014a), the most rapidly rotating stars show the shortest activity cycles, while more slowly rotating stars generally present cycle periods similar to the Sun or longer. Depending on the star mass, the variability in stellar behavior at small and large scales is a fundamental factor that can be used to better understand stellar magnetism phenomena (de Freitas et al. 2013; Oswalt & Barstow 2013). Because the magnetic activity that emerges from the stellar surface is a physical mechanism produced by the stellar dynamo, the long-term variations due to the rotational period play an important role in understanding the level of activity on the surface of the star. Specifically, for low-mass stars, the observed magnetism is associated with the external convective envelope, where strong mass motions of conductive plasma induce a magnetic field through a cyclic dynamo process (Brun et al. 2015). On the other hand, fossil magnetism is not the result of these same mechanisms. This kind of magnetism arises from the trapping of the magnetic flux after stellar formation or an initial dynamo phase and may therefore be the mechanism for long-timescale stability without the dynamo effect (Duez et al. 2010; Duez & Mathis 2010). However, it is necessary to investigate the effects of magnetic activity as a source of noise responsible for stellar microvariability when modeling different features in the frequency spectrum (Karoff et al. 2013a).

In particular, fully convective stars show granulation signatures on their surfaces (Karoff et al. 2013a). This type of target is particularly attractive because of different physical characteristics, particularly the transition from partially convective Sun-like stars to fully convective stars, likely around the spectral type M3/M4 (Reiners 2012). As Houdek et al. (1999) indicated, inside these stars, acoustic oscillations are excited by the outer convection zone, where their amplitudes at the surfaces can be observed. Thus, the signatures of both granulation and oscillations are included in the noise background of the time series at high frequencies. The signatures at low frequencies can be caused by the rotational modulation of long-lived sunspots (Lanza et al. 2004).

For main-sequence stars, the study of stellar magnetic activity focuses on the analysis of signatures of fluctuations in short- and long-lived variations because of different physical processes, which manifest in characteristic timescales from oscillation (>2000 \(\mu\)Hz) to rotation (<1 \(\mu\)Hz) (Karoff et al. 2013a). These processes are generated by complex temporal dynamics in the stellar photosphere and atmosphere. In general,
the mechanisms responsible for the magnetic activity, such as interactions among rotation, convection, and the magnetic field, which generate the differential rotation, are not completely understood (Brun et al. 2011; Das Chagas et al. 2016). In this case, it is necessary to investigate the statistical relationships among longer-term variations because of the stellar magnetism cycle and among shorter-term variations because of the correlated noises that differ from Gaussian noise (Carter & Winn 2009). It is important to highlight that although the short-term variations are dominated by granulation and oscillations, while the long-term variations are related to rotation and magnetism, the spectra of M dwarfs are dominated by photon noise at approximately 200 μHz and above, depending on the magnitude of the star. For these targets, both the granulation noise and oscillations are at much higher frequencies than the Nyquist limit. Hence, the power spectra of these targets at low frequencies are dominated by the rotation period and its harmonics and a power slope related to magnetic effects. As our proposal is to study M dwarfs using long-cadence Kepler data, this brief review is very relevant and crucial in the context of this paper.

Since the pioneering results of Wilson & Vainu Bappu (1957), a.k.a. the WB effect, several efforts have been made to measure the level of magnetic activity in photometric and spectroscopic observations. In general, this effect relates the absolute magnitude \( M_V \) and separation between the outer edges of the CaII K emission line \( \Delta \log[W_r] \) in \( \text{km s}^{-1} \); see, e.g., Oswalt & Barstow (2013). From the empirical correlation \( \log[W_r] - M_V \), a linear slope is extracted as a measure of the level of magnetic activity. Another magnetic activity proxy is the average Mount Wilson S index proposed by Middelkoop (1982). The physical index \( R_{\text{HK}} \) is also used to measure the magnetic index and is related to the S index by \( R_{\text{HK}} = C_{B-V}S \), where \( C_{B-V} \) denotes a color-dependent correction factor (see Cincunegui et al. 2007). Noyes et al. (1984) investigated the chromospheric magnetic activity index \( R_{\text{HK}} \) as a function of both rotation period and spectral type through the Rossby number \( R_n = P_{\text{rot}}/\tau_r \), where \( \tau_r \) is the convective turnover time and \( P_{\text{rot}} \) denotes the rotational period. Recently, Mathur et al. (2014b) suggested a new magnetic index based on the global \( S_{\text{ph}} \) index, which García et al. (2010) define as the standard deviation of the entire time series. Considering the magnetic activity fluctuations with time, Mathur et al. (2014b) proposed the mean value of \( S_{\text{ph}} \) as a global magnetic activity index, which was divided into independent subsamples of length \( k \times P_{\text{rot}} \), where \( k \) is an integer number set to 5 and \( P_{\text{rot}} \) is the rotational period of the star. This procedure has the great advantage of accounting for the effects of rotation of the star, allowing for the measurement of the magnetic index during the minimum (or maximum) regime of the magnetic cycle. The next step taken by the authors was to correct the value of the \( S_{\text{ph}} \) index for photon noise. To this end, they made use of the magnitude correction from Jenkins et al. (2010) to estimate the corrections to apply to the Kepler stars. Hence, they ensure that \( S_{\text{ph}} \) is dominated by the magnetism of each star, eliminating any effects due to granulation. Furthermore, these authors assume this new index as a counterpart of the spectroscopic Mount Wilson S index.

Recently, de Freitas et al. (2013) showed that the rotational periods of a sample of solar-type stars from the CoRoT database are linked to the Hurst index \( H \) by a simple logarithmic relationship (see De Medeiros et al. 2013). More recently, de Freitas et al. (2016) have shown that the level of complexity in stellar activity is associated with the degree of multifractality and the asymmetry of the multifractal spectrum. More specifically, the authors note that short time series are characterized by strong long-range correlations because of rotational modulation, where the flicker noise is used as a good candidate to explain the changes in the multifractal index. As mentioned by de Freitas et al. (2016), the flicker noise (Bastien et al. 2013) is the stellar granulation measured in the time domain that appears at lower frequencies and on timescales shorter than 8 hr. The granulation signal is widely used as a proxy for the stellar surface gravity (Mathur et al. 2011; Bastien et al. 2016; Kallinger et al. 2016). Generally, this multifractal approach suggests that the growth of the rotation rate destroys the “multifractal diversity,” which is given by the index \( D_\alpha \) and indicates which different rotation regimes affect local structures (Aschwanden & Parnell 2002; Aschwanden 2011). In other words, this approach implies that a high long-term persistence (the longest period) has a low level of complexity, i.e., it has a strong system memorability (Tang et al. 2015).

In Section 2, we first describe a geometric technique, a.k.a. Poincaré plot, that relates different levels of variability in short- to long-term periods. We also define a set of the magnetic activity indices from multifractal formalism based on the geometric features of the generalized fractal dimension spectrum. In this context, we use the statistical procedure suggested by de Freitas et al. (2016). In Section 3, we compare these indices with the average \( S_{\text{ph}} \) index developed by Mathur et al. (2014b) using a sample of 34 M notably low-mass dwarfs, which were observed by the Kepler mission and initially treated by these authors. In this section, we also discuss the physical implications of the \( S_{\text{ph}} \) index and its relationship with the geometric properties of the multifractal spectrum. In the final section, we present our final comments and conclusions.

2. Statistical Background

Several methods are used for analyzing the behavior of the standard deviation (or variance) of a time series, among which we can point to a self-similar statistical method known as the Poincaré plot (Khandoker et al. 2013), as well as other methods that assume the time series as a self-affine fractal, known as scaled windowed variance (SWV) analysis (Seu mont 2010) and rescaled range \( (R/S) \) analysis (Hurst 1951; Mandelbrot & Wallis 1969; de Freitas et al. 2013). The degree of the robustness of the methods follows from the Poincaré plot to the \( R/S \) analysis.

In this context, the method applied by Mathur et al. (2014b) is, for purposes of fast inspection, more similar to the Poincaré plot. This conclusion is due to the simple fact that the Poincaré plot analyzes only the global aspect of the fluctuations, while the SWV and \( R/S \) methods were developed for assessing the behavior of the fluctuations at different timescales. The intention of the authors was to find a metric adapted to each star to compare their magnetism instead of fixing the timescale and comparing their variability. In this way, the \( S_{\text{ph}} \) index is a metric linked to spots and thus to the surface magnetism. It is noteworthy that the Poincaré plot is widely used in cases where the signal is stationary.
2.1. Poincaré Plot

A time series \( x(t) \) comprises a deterministic function \( p(t, \bar{P}) \), which represents a type of global trend, and a set of short-term variations, which are denoted by random walk \( r(t) \) and stationary noise \( \eta(t) \) components; thus, \( x(t) = p(t, \bar{P}) + r(t) + \eta(t) \) (de Freitas et al. 2016). A global trend can be modeled by nonlinear functions or smoothers, whereas a short-term structure can be treated using a differencing filter given by \( x_{n+1} - x_n \), where \( n \) denotes the number of points. According to Feigelson & Jogesh Babu (2012), the combination of these procedures constructs a return map, which simultaneously fits long-term deterministic trends and periodicities and short-term autocorrelated fluctuations. Owing to the theoretical background from which self-similarity processes emerged, this method is named the Poincaré plot.

The Poincaré plot is a geometric method that analyzes the dynamic behavior of time series. This method represents a time series in a Cartesian plane, where each measurement result is plotted as a function of a previous result, and the result is similar to Figure 1 (Tulppo et al. 1998). Our study applies the procedure used by Tulppo et al. (1998) to compute the Poincaré plot.

Strictly speaking, a Poincaré plot can be analyzed by adjusting an ellipse to the diagram formed by the attractor with its center at \((0, 0)\), as shown in Figure 1. The SD1 line indicates the dispersion of data perpendicular to the identity line \((x_{n+1} = x_n)\), whereas the standard deviation along the identity line is represented by SD2.

SD2 is measured by the length histogram, which is obtained by projecting the points onto the identity line (Brennan et al. 2001, 2002; Khandoker et al. 2013). According to the traditional Poincaré analysis, SD1 is often used as a measure of the short-term variability, and SD2 is a measure of the long-term variability. However, these aspects are only correct when the time series have slow linear trends; thus, the SD1 and SD2 descriptors are linear statistics.

As mentioned by Khandoker et al. (2013), the SD1 and SD2 descriptors are related to the distance from the major and minor axes of the ellipse, respectively (for further details, see Equations (2.5) and (2.6) from Khandoker et al. 2013). SD2 is a weighted combination of low and intermediate frequencies, which portray the long-term characteristics of the signal, and SD1 is strictly a measure of short-term variability. Specifically, using time series data for M dwarf stars, SD1 is related to the magnetic slope or to the photon noise. The contributions of granulation and oscillations are already above the Nyquist limit and clearly below the Kepler photon noise. On the other hand, SD2 is related to the dominant period, within which are global trends but not necessarily the trend of the rotation period.

2.2. Multifractal Indices of Magnetic Activity

In the literature, there are several complexity testing techniques available to explore time series (Mandelbrot & Wallis 1969; Feder 1988; Kantelhardt et al. 2002; Ihlen 2012). In general, time series are measured with a wider spectrum of complexity measurements, including nonstationarity, nonlinearity, fractality, stochasticity, periodicity, and chaos (Tang et al. 2015). In this context, linear techniques such as standard deviation do not include the temporal variation at the point-to-point level or multiple lag correlations. For example, the lag-1
Poincaré plot does not provide more information about the intermediate timescales that affect the short-lived active regions on characteristic scales of a few hours or days (Lanza et al., 2003, 2004). A powerful technique to address these assumptions is multifractal analysis (Gu & Zhou 2010; Tang et al., 2015). As de Freitas et al. (2016) proposed, we focus on the multifractal detrending moving average (MFDMA) algorithm.

2.2.1. Methodology

According to Gu & Zhou (2010), we can summarize the MFDMA algorithm in the following steps:

1. Step 1: calculating the time series profile:
   First, we consider a time series \( x(t) \) defined over time \( t = 1, 2, 3, \ldots, N \), from which we construct the sequence of cumulative sums given by
   \[
   y(t) = \sum_{i=1}^{t} x(i), \quad t = 1, 2, 3, \ldots, N. \tag{1}
   \]

2. Step 2: calculating the moving average function of Equation (1) in a moving window:
   \[
   \tilde{y}(t) = \frac{1}{s} \sum_{k=-[\alpha-1]s}^{[\alpha-1]s} y(t-k), \tag{2}
   \]
   where \( s \) is the window size; \([\alpha]\) is the smallest integer not smaller than argument \( \alpha \); \([\alpha]\) is the largest integer not larger than argument \( \alpha \); and \( \theta \) is the position index with a range between 0 and 1, i.e., it describes the delay between the moving average function and the original time series. In the present work, \( \theta = 0 \), which refers to a backward-moving average, i.e., \( \tilde{y}(t) \) is calculated over all past \( s - 1 \) data of the time series.

3. Step 3: detrending the series by removing the moving average function \( \tilde{y}(i) \) and obtaining the residual sequence \( \epsilon(i) \) through
   \[
   \epsilon(i) = y(i) - \tilde{y}(i). \tag{3}
   \]

4. Step 4: calculating the rms function \( F_{\epsilon}(s) \) for a segment of size \( s \):
   \[
   F_{\epsilon}(s) = \left\{ \frac{1}{s} \sum_{i=1}^{s} \epsilon_{\tilde{y}}(i)^2 \right\}^{\frac{1}{2}}, \tag{4}
   \]

5. Step 5: generating the function of \( F_{\epsilon}(s) \) on the \( q \)th order:
   \[
   F_{\epsilon}(q) = \left\{ \frac{1}{N_s} \sum_{s=1}^{N_s} F_{\epsilon}^q(s) \right\}^{\frac{1}{q}}, \tag{5}
   \]
   for all \( q \neq 0 \), where the \( q \)-order is the statistical moment (e.g., for \( q = 2 \), we have the variance), and for \( q = 0 \),
   \[
   \ln[F_{\epsilon}(s)] = \frac{1}{N_s} \sum_{s=1}^{N_s} \ln[F_{\epsilon}(s)], \tag{6}
   \]
   where the scaling behavior of \( F_{\epsilon}(s) \) follows a relationship given by \( F_{\epsilon}(s) \sim s^{h(q)} \), and \( h(q) \) denotes the Holder exponent or generalized Hurst exponent.

6. Step 6: knowing \( h(q) \), the multifractal scaling exponent \( \tau(q) \) can be computed:
   \[
   \tau(q) = qh(q) - 1. \tag{7}
   \]
   Finally, the singularity strength function \( \alpha(q) \) and the multifractal spectrum \( f(\alpha) \) are obtained via a Legendre transform, respectively:
   \[
   \alpha(q) = \frac{d\tau(q)}{dq} \tag{8}
   \]
   and
   \[
   f(\alpha) = q\alpha - \tau(q). \tag{9}
   \]
   For a detailed theoretical description of the MFDMA theoretical background, we strongly recommend the papers by Gu & Zhou (2010) and de Freitas et al. (2016).

2.2.2. Multifractal Indices

Facilitating our discussion of the results, we begin by presenting a typical multifractal spectrum of a time series as the inverse parabolic shape in Figure 2. For a monofractal time series, \( \tau(q) \) is a linear function given by \( qH - 1 \), where \( H \) is the global Hurst exponent. For a multifractal signal, \( \tau(q) \) is nonlinear, and the multifractal spectrum takes the form presented in Figure 2. We aim to propose a set of four multifractal descriptors extracted from the spectrum \( f(\alpha) \) as new stellar magnetic activity proxies, as follows:

1. The degree of asymmetry (A), or the skewness in the shape of the \( f(\alpha) \) spectrum, may be quantified by the following ratio:
   \[
   A = \frac{\alpha_{\text{max}} - \alpha_{\text{0}}}{\alpha_{\text{0}} - \alpha_{\text{min}}}, \tag{10}
   \]
   where \( \alpha_{\text{0}} \) is the value of \( \alpha \) when \( f(\alpha) \) is maximal. This index presents three shapes according to the value of \( A \), which represents asymmetry as right-skewed (\( A > 1 \)), left-skewed (\( 0 < A < 1 \)), or symmetric (\( A = 1 \)) as illustrated in Figure 2. The right endpoint \( \alpha_{\text{max}} \) and the left endpoint \( \alpha_{\text{min}} \) denote the extremal values of the singularity exponent and are associated with the minimum and maximum fluctuation of signal, respectively.

2. The degree of multifractality (\( \Delta \alpha \)) is represented by
   \[
   \Delta \alpha = \alpha_{\text{max}}(q) - \alpha_{\text{min}}(q), \tag{11}
   \]
   where \( \alpha_{\text{max}} \) and \( \alpha_{\text{max}} \) are the maximum and minimum Holder exponents, respectively, of the statistical distribution of \( \alpha \) when \( q \rightarrow \pm \infty \) (see Figure 2). A high value of \( \alpha(q) \) indicates that the time series is smooth in that region, and the multifractal strength is consequently lower (de Freitas & De Medeiros 2009).
3. The singularity ratio $C$ is characterized by the ratio between $\Delta f_{\text{left}}(\alpha)$ and $\Delta f_{\text{right}}(\alpha)$ measured in relation to the maximum fractal dimension $f^\text{max}[\alpha(q = 0)]$. The shape of multifractal spectrum, as shown in Figure 2, represents an asymmetric spectrum. Furthermore, this spectrum can also have either a left or right truncation, as indicated by parameters $\Delta f_{\text{left}}(\alpha)$ and $\Delta f_{\text{right}}(\alpha)$, and the truncations originate from a leveling of the $q$-order Hurst exponent for negative or positive $q$ values, respectively. In this sense, the index $C$ can be interpreted as a direct measure of truncation, where for $C < 1$ the right-hand side is truncated, while for $C > 1$ the truncation occurs on the left-hand side. As mentioned by Ihlen (2012), a long left tail implies that the time series have a multifractal structure that is insensitive to local fluctuations with small magnitudes. On the other hand, a long right tail indicates that the time series have a multifractal structure that is insensitive to local fluctuations with large magnitudes. As shown in Figure 2, the ratio between the width of the left- and right-hand sides $f(\alpha)$ denotes the degree of the strong and weak singularities as follows:

$$C = \frac{\Delta f_{\text{left}}(\alpha)}{\Delta f_{\text{right}}(\alpha)},$$

where the singularity strength $\alpha$ is inversely proportional to the multifractal spectrum strength (Hampson & Mallen 2011). Moreover, $h$ is a measure of the rate of decay of the fluctuation amplitude, i.e., high values of this exponent denote smoother fluctuations; therefore, the singularity strength is lower. For a time series, the maximum value of $f(\alpha)$ is unity. Indeed, this feature reveals that the singularity indicated by $\alpha(q)$ is present everywhere in the time series.

4. The Hurst index ($H$) can be obtained from the multifractal spectrum through the second-order generalized Hurst exponent $h(q = 2)$ (as shown in Figure 2) (see Hurst 1951; Hurst et al. 1965). In particular, $H = h(2)$ for a stationary signal (i.e., with a constant mean and variance expected over time), which is called fractional Gaussian noise (fGn). For the nonstationary case (i.e., with time-dependent variance) with the fractional Brownian motion (fBm), the relationship is $H = h(2) + 1$ (Movahed et al. 2006; Seuront 2010). As Seuront (2010) proposed, a relevant procedure to distinguish between these two types of processes is to identify the nature of fBm noises, where $\beta$ is a scaling exponent of the Fourier power spectrum. To perform this procedure, we can estimate $\beta$ by determining the slope of a linear trend and identify whether the calculated slope is in an interval of $1 < \beta < 3$, which is characterized as fBm. We can also estimate $\beta$ from the relationship $\beta = 2 + \tau(2)$, where $\tau(2)$ is the second-order statistical moment mass exponent (Ivanov et al. 1999). For example, for $\beta = 1$ and 2, we have a flicker noise and random walk, respectively (for further details, see Table 1, p. 162, in Donner & Barbosa 2008).

As de Freitas et al. (2013) reported, when $H$ is between 0.5 and 1, it describes a long-range dependence (LRD) and memory effects on all timescales according to...
the level of persistence, wherein the time series becomes increasingly periodic as \( H \) approaches 1. In contrast, values of \( H \) close to zero indicate that the series must change the direction of every sample as white noise. If \( H = 0.5 \), the time series is truly random and uncorrelated data. In particular, the values of \( H \) near 0.5 imply a short-range dependence. A time series with \( H < 0.5 \) can be characterized as anti-persistent, i.e., the signal tends not to continue in the same direction but turns back on itself and gives a less smooth time series (Hampson & Mallen 2011).

2.3. Correlation Methods: Spearman and Pearson Coefficients

To understand the degree of correlation among different parameters (e.g., \( H \) and \( P_{\text{rot}} \)), we used two bivariate analysis techniques: (i) Pearson’s product moment correlation coefficient and (ii) Spearman’s rank correlation coefficient (Press et al. 2007). Qualitatively, the Pearson method measures the strength of the linear relationship between normally distributed variables. However, when the variables are not normally distributed or the relationship between the variables is not linear, the Spearman method is more appropriate (Mukaka 2012). Quantitatively, Spearman and Pearson coefficients are, respectively, given by

\[
r_s = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2 - 1)},
\]

where \( d_i \) represents the difference between ranks of variables \( x \) and \( y \) (for example, \( x = A \) and \( y = (S_{\text{ph},k}) \)) and \( n \) is the number of observations, and

\[
r_p = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}},
\]

where \( S \) denotes the covariance.

Additionally, if \( |r_s| > |r_p| \), this simply means that there is a stronger monotonic than linear relationship. Specifically, a monotonic behavior between the variables can imply a linear relationship and, consequently, the analysis of the results becomes more complex; the presence of outliers may cause this discrepancy between the values of the coefficients. For this case, it is necessary to exclude the outliers, recalculate the coefficients, and verify any change.

In this sense, we want to evaluate whether a correlation exists between \( x \) and \( y \). To that end, the significance of the correlation coefficient can be estimated using a \( t \)-statistic. First, we specify the null and alternative hypotheses: (i) the null hypothesis \( H_0: \rho = 0 \) (there is no association), and (ii) the alternative hypothesis \( H_a: \rho > 0 \) (a nonzero correlation could exist) for the two-tailed test or \( H_a: \rho < 0 \) (a negative correlation could exist) and \( H_0: \rho > 0 \) (a positive correlation could exist) as the results for the left- and right-tailed tests, respectively. Second, we calculate the value of the \( t \)-statistic using the following equation:

\[
t_{\text{calculated}} = \frac{r \sqrt{n - 2}}{\sqrt{1 - r^2}},
\]

where \( n \) represents the sample size and \( r \) is the Pearson’s correlation coefficient. In addition to the Gaussian distribution, this test is reasonably robust to non-Gaussian data. Third, we use a \( t \)-table to find the critical value \( (t_{\text{critical}}) \), considering a 95% confidence level and, consequently, a significance level \( \alpha \) equal to 0.05 and 0.025 for each tail related to one- and two-sided tests, respectively (Trauth 2006). In hypothesis testing, a critical value is a point on the \( t \)-distribution that is compared to the calculated \( t \)-statistic to determine whether the null hypothesis is rejected or not. Finally, we compare the calculated \( t \)-statistic (see Equation (15)) to the critical value. In general, if the absolute value of the calculated \( t \)-statistic is greater than the critical value, then the null hypothesis can be rejected at the 95% level of confidence in favor of the alternative hypothesis (Trauth 2006; Press et al. 2007).

2.4. Timeout

In Sections 2.1 and 2.2, we highlighted two words: trend and LRD. We put these special words in bold text to help us understand their meaning in this paper. In addition, we believe that a more rigorous definition of this terminology is necessary because of their relationship with the variability in different frequency ranges. A trend is accepted as a part of the time series that changes slowly over time and is broadly defined as a “long-term change in the mean level” (Donner & Barbosa 2008). An alternative to the notion of trends in the time series analysis is to consider the inverse concept of stationarity, such as an fGn-like signal. In fact, a stationary time series is identified as having “no trend.” However, it is necessary to distinguish between deterministic and stochastic trend-like components. Specifically, an LRD can affect the analysis of this distinction. Several works (e.g., Donner & Barbosa 2008; Seuront 2010; Pascual-Granado 2011) have emphasized that LRD is found in low-frequency variability, which indicates that the autocorrelation function slowly decays and exhibits a scale invariance with the scaling exponent \( \beta \). In the astrophysics, LRD can be identified as long-lived features (spots or active regions) and effects because of the differential rotation (see Das Chagas et al. 2016).

3. Observations and Data Preparation

In our paper, we use photometric data recorded by the VIRGO/\textit{SOHO}\(^7\) and \textit{Kepler}\(^8\) missions. Our sample is composed of time series of the Sun (see Figure 3) and a sample of 34 M dwarf stars, which were observed by the \textit{Kepler} mission at a cadence of ~30 minutes and previously analyzed by Mathur et al. (2014b). These stars are M dwarf stars with log \( T_{\text{eff}} \) lower than 3.6, log \( g \) greater than 4.0, and well-defined rotational periods \( P_{\text{rot}} < 15 \) days, as shown in Table 1.

The first part of our sample is based on a data set of Sun continuous observations obtained by Variability of solar Irradiance and Gravity Oscillations (VIRGO; Fröhlich et al. 1995, 1997). The VIRGO experiment is a component of the payload of the \textit{SOHO} spacecraft and is based on four instruments: DIARAD, LOI, PMO6, and SPM. In the present paper, we use the VIRGO data in the green (500 nm) and red (862 nm) bandwidths of the SPM (Sun PhotoMeters) instrument, as proposed by Basri et al. (2013), due to a good...
Figure 3. Time series based on the VIRGO/SPM (Green + Red channels) instrument obtained as described in Section 3.

Table 1

| Star     | $A$  | $\Delta \alpha$ | $C$  | $H$   | log SD1 [ppt] | log SD2 [ppt] | $P_{\text{rot}}$ (days) | $\langle \delta P \rangle$ |
|----------|------|---------------|------|------|---------------|----------------|--------------------------|---------------------|
| KIC 2157356 | 3.025 | 0.699      | 0.233 | 0.444 | 0.06          | 0.91           | 12.9                     | 4109.1              |
| KIC 2302851 | 3.066 | 0.768      | 0.303 | 0.404 | −0.72         | 0.81           | 12.2                     | 3934.1              |
| KIC 2570846 | 3.827 | 0.845      | 0.215 | 0.393 | −0.12         | 1.28           | 10.9                     | 11871.9             |
| KIC 2574427 | 2.376 | 0.687      | 0.353 | 0.409 | −0.3          | 0.37           | 13.4                     | 1196.9              |
| KIC 2692704 | 3.228 | 0.670      | 0.259 | 0.475 | 0.02          | 1.16           | 14.8                     | 8858.5              |
| KIC 2832398 | 3.946 | 0.781      | 0.181 | 0.463 | −0.24         | 0.88           | 15                       | 4688.7              |
| KIC 2834612 | 3.309 | 0.812      | 0.237 | 0.428 | −0.11         | 1              | 13.3                     | 6391.3              |
| KIC 2835393 | 3.100 | 0.690      | 0.262 | 0.446 | −0.01         | 0.81           | 15                       | 3597.7              |
| KIC 3012763 | 2.977 | 0.714      | 0.272 | 0.454 | −0.04         | 1.19           | 14.4                     | 9310.4              |
| KIC 3232393 | 1.613 | 0.425      | 0.541 | 0.444 | −0.2          | 0.04           | 14.5                     | 404.9               |
| KIC 3634308 | 2.864 | 0.720      | 0.283 | 0.421 | −0.08         | 0.8            | 12.9                     | 3708                |
| KIC 3935499 | 4.272 | 0.842      | 0.216 | 0.257 | 0.08          | 1.45           | 5.2                      | 17429.4             |
| KIC 4833367 | 2.541 | 0.571      | 0.302 | 0.438 | −0.14         | 0.37           | 14.2                     | 1135.2              |
| KIC 5041192 | 3.464 | 0.767      | 0.207 | 0.318 | −0.08         | 0.95           | 10.8                     | 5483.9              |
| KIC 5096204 | 3.177 | 0.732      | 0.247 | 0.443 | −0.62         | 0.32           | 14.8                     | 1272.1              |
| KIC 5210507 | 3.374 | 0.867      | 0.253 | 0.333 | −0.13         | 1.3            | 8.8                      | 12345.3             |
| KIC 5611092 | 2.260 | 0.584      | 0.344 | 0.428 | −0.43         | 0.15           | 14.4                     | 711.4               |
| KIC 5900600 | 3.3958 | 0.627     | 0.208 | 0.460 | −0.26         | 0.64           | 14                       | 2194.3              |
| KIC 5950024 | 3.465 | 0.860      | 0.241 | 0.417 | −0.63         | 0.61           | 14.1                     | 2454.5              |
| KIC 5954552 | 4.185 | 0.702      | 0.169 | 0.470 | −0.44         | 1.02           | 14.9                     | 7121.1              |
| KIC 5956957 | 2.813 | 0.680      | 0.264 | 0.443 | −0.44         | 0.76           | 14.9                     | 3825.4              |
| KIC 6070686 | 3.650 | 0.827      | 0.214 | 0.427 | −0.44         | 0.82           | 13.3                     | 3807.7              |
| KIC 6464396 | 3.412 | 0.707      | 0.209 | 0.416 | −0.21         | 0.93           | 13.2                     | 4146.5              |
| KIC 6545415 | 2.831 | 0.865      | 0.266 | 0.324 | −0.03         | 1.2            | 5.5                      | 9018.6              |
| KIC 6600771 | 3.632 | 0.802      | 0.210 | 0.445 | −0.12         | 1.03           | 13.1                     | 6494.6              |
| KIC 7091787 | 2.532 | 0.556      | 0.332 | 0.452 | −0.08         | 0.39           | 14.1                     | 3313.1              |
| KIC 7106306 | 3.513 | 0.900      | 0.233 | 0.416 | −0.52         | 0.71           | 14.2                     | 3194.6              |
| KIC 7174385 | 2.748 | 0.630      | 0.283 | 0.406 | −0.19         | 0.54           | 14.5                     | 2190.2              |
| KIC 7190459 | 2.618 | 0.671      | 0.303 | 0.318 | −0.27         | 0.64           | 6.8                      | 2435.6              |
| KIC 7282705 | 2.486 | 0.661      | 0.337 | 0.428 | −0.1          | 0.54           | 14.5                     | 1842.5              |
| KIC 7285617 | 2.391 | 0.624      | 0.347 | 0.412 | −0.16         | 0.48           | 13.7                     | 1698.3              |
| KIC 7534455 | 3.323 | 0.607      | 0.189 | 0.433 | −0.32         | 0.56           | 12.1                     | 2087.3              |
| KIC 7620399 | 2.299 | 0.555      | 0.343 | 0.440 | −0.27         | 0.35           | 13.7                     | 1170.5              |
| KIC 7673428 | 3.907 | 0.737      | 0.167 | 0.4532 | −0.12        | 1.07           | 15                       | 7213.6              |

Note. The indices $A$, $\Delta \alpha$, $C$, and $H$ are in the multifractal analysis, whereas log SD1 [ppt] and log SD2 [ppt] were extracted from the Poincaré plot. These indices were compared to the rotational period $P_{\text{rot}}$ and $\langle \delta P \rangle$ in Mathur et al. (2014b), as shown in the last two columns.

approximation with the Kepler data. The VIRGO data analyzed in the present work consist of SSI (Spectral Solar Irradiance) time series with a temporal cadence of 1 minute and a date range from 1996 April 11 to 2014 March 30, corresponding to solar cycles 23 and 24. To properly compare these results to the stellar case, the time series were averaged into 30-minute cadences to match the Kepler measurements. This data set consists of ∼18 yr of continuous observations; however, as the temporal window of the Kepler data is ∼4 yr, we chose a region in the VIRGO/SPM data with few large gaps from 1999 April 22 to 2003 February 20, within the Sun’s active phase. The VIRGO/SPM data treatment followed the same procedure adopted by Garcfa et al. (2005, 2011) and Mathur et al. (2014b).
On the other hand, the *Kepler* sample was selected from the calibrated time series processed by the PDC-MAP pipeline (Jenkins et al. 2010); a careful treatment was applied to light curves using the so-called co-trending basis vectors provided by the *Kepler* archive (Twicken et al. 2010; Smith et al. 2012) to remove systematic long-term trends originating from the instruments, detector, or effects caused by reorientation of the spacecraft every ~90 days. To detect discontinuities and outliers and detrend the data on such short timescales, we applied the method developed by De Medeiros et al. (2013). We also recalculated the rotation periods and found the same results as those reported by Mathur et al. (2014b).

### 4. Results and Discussion

As previously mentioned in the introduction, Mathur et al. (2014a, 2014b) measured the magnetic activity indices using the standard deviation ($S_{ph}$) of the entire time series and the average standard deviation of the subseries defined by $k$-rotation periods (hereafter $S_{(ph,k)}$), where $k = 5$ is used. The values of $S_{(ph,k)}$ extracted from Mathur et al. (2014b) are reported in Table 1. This procedure divides a time series into macroscopic scales of order or higher than the rotational period. According to the authors, the standard deviation is a good indicator of the global magnetic activity based on photometric modulation and is consequently a classifier of the stellar activity cycle. Specifically, the standard deviation as a function of factor $k$ is only a useful measure to quantify the amount of variation or dispersion in a statistical data set (Feigelson & Jogesh Babu 2012). However, the long-cadence data used here are filtered, and no signatures of oscillations or granulation are observed above the photon noise. The data are also corrected for photon noise to remove this dependence, taking into account the magnitude of the star as described by Mathur et al. (2014a, 2014b). As emphasized by these authors, the $S_{ph}$ index, as calculated, is dominated by the timescales related to rotation and magnetism (for more details, see Salabert et al. 2016a, 2016b).

#### 4.1. Results Based on the Poincaré Plot

As shown in Figure 4 (left panel), we analyze the sample using the Poincaré plot method and find a stronger relationship between the SD2 descriptor and the index $S_{(ph,k)}$ with Spearman and Pearson rank correlations $\sim 1$ (the data were tested at a significance level of 5%). The values of SD2 are summarized in Table 1 (fifth column). In contrast, the SD1 descriptor related to short-range variability is weakly correlated with the $S_{(ph,k)}$ index (see left panel of Figure 4). The values of SD1 are summarized in Table 1 (sixth column). The fact that the $S_{ph}$ index is highly correlated with the SD2 descriptor and not with SD1 provides strong evidence that this index is dominated by long-trend variations and not by short-term variations. With regard to the *Kepler* time series, these short-term variations are the photon noise. All Spearman and Pearson rank correlation coefficient values are presented in Table 2.

#### 4.2. Results Based on the Multifractal Method

In this paper, we propose a set of multifractal measures as magnetic indices for each star. As a counterpart to the different indices in the literature, our set of indices was defined by the $(A, \Delta \alpha, C, H)$-quartet mentioned in Section 2.2 and investigated in de Freitas et al. (2013, 2016). We also apply the multifractal indices proposed in this section to the sample defined by Mathur et al. (2014b).

In Figure 5, we indicate the solar values as horizontal dashed lines. We observe that semi-sinusoidal variations, those illustrated by rotational modulation, do not clearly depend on the indices $A$, $\Delta \alpha$, and $C$. In Figure 5 (bottom right panel), we fit the same analytical relationship between the $H$ index and the rotational period proposed by de Freitas et al. (2013) (see Equation (1) from this paper) at the 5% significance level. As we observe, the global Hurst exponent $H$ grows with increases in $P_{rot}$. In addition, this strong correlation between $H$ and $P_{rot}$ supports the results of de Freitas et al. (2013), i.e., the $H$ index is a powerful classifier for semi-sinusoidal time series. Indeed, the first step in the Hurst analysis is to identify whether the data set is fGn or fBm based on the $\beta$-exponent, as mentioned in
Section 2.2.2. For all stars, we found that \( \tau(2) \approx -0.02 \); hence, \( \beta \approx 2 \). As a result, all time series can be described as random-walk-like signals with fluctuations evolving more slowly than in noise-like time series. Because our values of \( h(2) \) are consistently within \( 1.5\sigma \) of the values of \( H \) measured by the rescaled range \( R/S \)-method in de Freitas et al. (2013), we assume that \( H = h(2) \) regardless of the type of signal, although the series is not stationary.

In Mathur et al. (2014a, 2014b), the authors suggest that the photometric index \( S_{ph} \) is a measure of the magnetic activity. They conclude that a slight anticorrelation between \( \langle S_{ph} \rangle \) and the rotational period can be used to distinguish different levels of magnetic activity and that it shows evidence of long-lived features. We calculate this anticorrelation using the Spearman and Pearson correlation coefficients and find that \( r_S = -0.27 \) and \( r_P = -0.57 \), i.e., \( r_P \approx 2r_S \) is likely due to outliers.

As stated in Section 2.3, the Spearman coefficient indicates that anticorrelations are negligible, whereas the Pearson coefficient indicates that such values are low anticorrelation. Our null hypothesis admits that the correlation between \( P_{rot} \) and \( S_{ph} \) is zero. For the left-tailed test, the \( t \)-statistic reveals that, because \( t_{\text{calculated}} = -3.888 \) is outside the range of \( -2.035 < t_{\text{critical}} < 2.035 \), we can reject the null hypothesis in favor of the alternative hypothesis, i.e., there is a negative correlation between the parameters mentioned. Similarly, for the two-tailed test, the \( t \)-statistic reveals that because \( t_{\text{calculated}} = -3.888 \) is outside the range of \( -2.035 < t_{\text{critical}} < 2.035 \), we can also reject the null hypothesis in favor of the alternative hypothesis. As shown in Table 3, the relationships among the different parameters extracted from multifractal analysis, the Poincaré plot, and Mathur et al.’s (2014b) analysis were determined through Pearson’s correlation coefficients, and 10 variables were included in the correlation matrix, including the relationship between \( P_{rot} \) and \( \langle S_{ph} \rangle \). However, it is not the anticorrelation between \( P_{rot} \) and \( S_{ph} \) index that can be used to distinguish different levels of magnetic activity, but rather the value of the index itself, as indicated by Mathur et al. (2014a, 2014b) and Salabert et al. (2016a, 2016b).

As Mukaka (2012) suggested, the \( \langle S_{ph} \rangle \) index was log-transformed. The reason for this transformation is to build normally distributed observables so that we can use Pearson’s coefficient. However, this does not change the results of the previous paragraph. In Figure 6, we defined a point with the vector \( \langle A, \log \langle S_{ph,k} \rangle, \Delta \alpha \rangle \) (left panel) and \( \langle C, \log \langle S_{ph,k} \rangle, H \rangle \) (right panel) from the projections on the planes. Thus, we constructed a 3D plot in which a point represents a time

![Figure 5. Plotted values of the multifractal indices including the asymmetry parameter \( A \), degree of multifractality \( \Delta \alpha \), rate of singularity \( C \), and global Hurst exponent \( H \), derived from the analysis of the multifractal spectrum as described in Section 2.2 as a function of rotational period. The Sun was analyzed in its active phase. The solar values are represented for the horizontal dashed lines.](image-url)
series with a particular rotational period, as indicated in Figure 5. According to the projections \((A, \log (\langle S_{ph,k} \rangle))\) and \((\log (\langle S_{ph,k} \rangle), \Delta \alpha)\) of the left panel of Figure 6, the \((\langle S_{ph} \rangle)\) index is strongly linked to the geometric properties of the multifractal spectrum \(f(\alpha)\) by parameters \(A\), \(\Delta \alpha\), and \(C\).

In Table 2, we verify that the Spearman and Pearson coefficients between \(\langle S_{ph,k} \rangle\) and the \(A\) and \(\Delta \alpha\) indices are significantly higher than those for the index \(C\). This disparity occurs because there is an outlier, identified as star KIC 3232393, which presents the highest singularity ratio \(C\) but does not affect the other correlations. After excluding this star, we recalculate the coefficients and obtain \(r_s = -0.62\) and \(r_p = -0.64\).

According to Noyes et al. (1984), there is a correlation between the observed flux indices (Ca II HK flux and \(K_{HK}\)) and the rotation period. We verified that the multifractal \(H\) index has the strongest correlation with \(P_{rot}\). As proposed by de Freitas et al. (2013), the strong correlation between the \(H\) index and the rotation period would define \(H\) as a measure of the intrinsic memory in the light curve affected by semi-sinusoidal variations. On the other hand, the panels of Figure 5 are in agreement with the results of Mathur et al. (2014b), in which faster rotators are more active. Qualitatively, in the left panel of Figure 6 and in the \((\log (\langle S_{ph} \rangle), C)\)-plane from the right panel of the same figure, the correlation between \(\langle S_{ph} \rangle\) and the \((A, \Delta \alpha, C)\)-triplet reveals that our multifractal indices are related to the magnetic activity of the M dwarfs analyzed here.

Table 1 shows that all \(H\) values are below 0.5, which indicates that the fluctuations in the amplitude of the photometric flux are anticorrelated. In other words, an amplitude that has decreased in the past is more likely to increase than decrease in the future. According to the central limit theorem, for long time series, we expect values of \(H \sim 0.5\) because the memory between two points decreases with an increasing number of data points. These results can give us a clearer idea about the physical implications of the photometric index that Mathur et al. (2014b) used; consequently, we suggest that a multifractal framework can give us a new approach to Mathur et al.’s (2014b) index.

4.3. The \(\langle S_{ph,k} \rangle\) Index and the Source of Its Geometric Dependence

Because the \(\langle S_{ph,k} \rangle\) index is strongly correlated with the geometric indices \(A, \Delta \alpha\), and \(C\), we decided to investigate the changes in the curve of the \(f(\alpha)\) spectrum by comparing the indices calculated from the original series with those
obtained from the shuffled and surrogate series. This procedure finds the possible source(s) that affect the profiles of $A$, $\Delta \alpha$, and $C$ and consequently suggest the above-mentioned strong correlation. In general, the shuffled series methods remove any temporal correlations that eliminate the memory of the system, but the method does not affect the probability distribution function (PDF). The surrogate procedure eliminates nonlinearity and preserves only the linear properties of the actual series (Norouzzadeh et al. 2007; de Freitas et al. 2016).

In all multifractal spectra, the $(A, \Delta \alpha, C)$-triplets in the original time series are higher than their shuffled partners (see the red circles in Figure 7). Only a portion of the spectra from our sample are represented in Figure 7. In contrast, there is no difference between the indices from the original and surrogate time series, which indicates that the structural properties of the signal are essentially linear, i.e., $(A, \Delta \alpha, C)_{\text{surrogate}} \sim (A, \Delta h, C)_{\text{original}}$. In this context, we observe a meaningful effect of random shuffling in the spectra $f(\alpha)$. We observe a weak multifractal effect in the shuffled series from the generated $f(\alpha)$ spectra, i.e., the triplet $(A, \Delta \alpha, C)_{\text{shuffled}} < (A, \Delta \alpha, C)_{\text{original}}$. This behavior probably arises from the heavy-tailed distribution of the time series data.

In conclusion, the geometric dependence of the $\langle S_{\text{ph,k}} \rangle$ index originates from the multifractality of both the correlations and the PDF. This result reinforces the strong correlation between $\langle S_{\text{ph,k}} \rangle$ and the SD2 descriptor.

### 4.4. Possible Effects Derived of the Inclination Angle

We have not taken into account the possible effects due to the inclination angle of the rotation axis in relation to the line of sight. As investigated by Vázquez Ramió et al. (2011) and reported by Mathur et al. (2014a), the angle of inclination of the star, which can be estimated using a combination of the stellar radius ($R$), projected equatorial velocity ($\sin i$), and rotation period, is a relevant parameter owing to effects it can have on observations of magnetic cycles. Unfortunately, we can only derive the $\sin i$ value for KIC 6464396. For this star, $i \sim 90^0$ (Prša et al. 2011).

We can estimate the median rotation velocity values from $R_{\text{KIC}}$, the Kepler stellar radius, and $P_{\text{rot}}$ through the relationship $\bar{v} = 2\pi (R_{\text{KIC}}/P_{\text{rot}})$. In addition, as the triplet $(A, \Delta \alpha, C)$ is linked to $\langle S_{\text{ph,k}} \rangle$, we can assume that the triplet also represents a lower limit of the photospheric activity (Salabert et al. 2016a).

In this sense, spectroscopic measurements are necessary for estimating the effect of $i$ over our sample to reach a conclusive result.

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**Figure 7.** Multifractal spectra of $f(\alpha)$ vs. $h$ of the original (red), shuffled (green), and surrogate (blue) time series. The spectra were created by the MFDMA algorithm.
Figure 8. Hurst exponent measured via multifractal procedure as a function of rotation period. The black solid line gives us the following log-log relationship: \( \log H = (0.45 \pm 0.04) \log P_{\text{rot}} - (0.88 \pm 0.04) \). The dotted line shows the log–log relationship from Skumanich (1972). Considering all the samples (34 stars + the Sun), the red solid line gives us the following log–log relationship: \( \log H = (0.47 \pm 0.03) \log P_{\text{rot}} - (0.90 \pm 0.03) \). The Sun is represented by the symbol \( \circ \). For both the reduced \( \chi^2 \), the value of the fit is \( \sim 10^{-1} \).

4.5. Age–Rotation–Activity Relationship and the Astrophysical Meaning of the H Index

Since the pioneering study by Skumanich (1972), the age–rotation–activity connection has been used to investigate the evolution of the stellar angular momentum and its implications for magnetic activity levels driven by the stellar dynamo (Kawaler 1988; de Freitas & De Medeiros 2013). In the present context, the rotation–H relationship suggests a clear similarity with a typical relationship known as the age–rotation relationship. Instead, this type of connection is expected in models that involve a link between rotation and a magnetic activity index proposed by Karoff et al. (2013b) and Metcalfe et al. (2014). As shown in Figure 8 and inspired by the age–rotation relationship, we suggest that the rotation–H relationship is better described as a power-like law of \( H = aP_{\text{rot}}^b \), where \( a \) is a normalization constant and \( b \) is the scaling exponent. In Figure 8, the \( \log H-\log P_{\text{rot}} \) relationship is represented by the dark solid line given by

\[
\log H = (0.45 \pm 0.04) \log P_{\text{rot}} - (0.88 \pm 0.04). \tag{16}
\]

The important parameter to compare here is the exponent \( 0.45 \pm 0.04 \). This slope is consistent with the original result of 0.5 in Skumanich (1972). Including the Sun in the plot, we verify that the relationship is maintained:

\[
\log H = (0.47 \pm 0.03) \log P_{\text{rot}} - (0.90 \pm 0.03). \tag{17}
\]

It is worth noting that the reason for this relationship is the different spin-down timescales for stars of different masses, which contradicts the results based on the \( \langle S_{\text{ph,k}} \rangle \) index (see Figure 3 in Mathur et al. 2014b). Compared to the multifractal theoretical background, this result shows that the evolution of the angular momentum is a function of the age or mass, as well as the dynamics of magnetic activity over different timescales, which is characterized by the H index here.

5. Concluding Remarks

In the previous section, after proving the correlation between the H index and the rotation period, we verified the possible correlations between the \( (A, \Delta \alpha, C, H) \)-quartet and the index defined by Mathur et al. (2014b). This analysis suggests that the multifractal properties of this quartet are highly correlated with the \( \langle S_{\text{ph}} \rangle \) index, though the H index is strongly linked to the rotational period. Moreover, the \( (A, \Delta \alpha, C) \)-triplet is strongly correlated with this index. Because of this inter-relationship, we conclude with full confidence that the \( \langle S_{\text{ph}} \rangle \) index is a measure of the effects of both temporal correlation and a heavy-tailed PDF, both of which affect the geometry of the multifractal spectrum.

We have compared the measures of both the \( \langle S_{\text{ph}} \rangle \) and H indices. Our result shows a strong relationship between the stellar rotation period from the analysis of the light curve and the values of the H index, as suggested by de Freitas et al. (2013). It is worth noting that this result strengthens the general proposition that the H index is related to the evolution of stellar angular momentum and magnetic activity. Because of the properties of H, the stellar rotation is associated with the degree of persistence of the signal.

From a physical viewpoint, perhaps because of this universal feature, we understand that multifractality does not imply a consequence of particular physical properties but rather represents a more general behavior of complex systems that have similar entities. In conclusion, the universality of the multifractal nature provides us with a more robust and deeper statistical ensemble than the traditional and more conservative approaches that dominate the usual methods in the astrophysical literature, which characterize stellar photometric variability in a notably limited manner.

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References

Aschwanden, M. J. 2011, Self-Organized Criticality in Astrophysics. The Statistics of Nonlinear Processes in the Universe (New York: Springer-Praxis)

Aschwanden, M. J., & Parnell, C. E. 2002, ApJ, 572, 1048

Bastien, F. A., Walkowicz, L. M., & Reiners, A. 2013, ApJ, 769, 37

Bastien, P. A., Stassun, K. G., Basti, G., & Pepper, J. 2013, Nat, 500, 427
