Efficient DC Algorithm for Constrained Sparse Optimization

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Abstract

We address the minimization of a smooth objective function under an \( \ell_0 \)-constraint and simple convex constraints. When the problem has no constraints except the \( \ell_0 \)-constraint, some efficient algorithms are available; for example, Proximal DC (Difference of Convex functions) Algorithm (PDCA) repeatedly evaluates closed-form solutions of convex subproblems, leading to a stationary point of the \( \ell_0 \)-constrained problem. However, when the problem has additional convex constraints, they become inefficient because it is difficult to obtain closed-form solutions of the associated subproblems. In this paper, we reformulate the problem by employing a new DC representation of the \( \ell_0 \)-constraint, so that PDCA can retain the efficiency by reducing its subproblems to the projection operation onto a convex set. Moreover, inspired by the Nesterov’s acceleration technique for proximal methods, we propose the Accelerated PDCA (APDCA), which attains the optimal convergence rate if applied to convex programs, and performs well in numerical experiments.
1 Introduction

1.1 Background

In recent years, sparse optimization problems which include the $\ell_0$-norm of decision vector in their objectives or constraints have drawn significant attentions in many applications such as signal processing, bioinformatics, and machine learning. Since such problems are intractable due to the nonconvexity and discontinuity of the $\ell_0$-norm [16], many approaches have been proposed to approximate the $\ell_0$-norm. The $\ell_1$-norm regularization, initiated by Tibshirani [24] for linear regression, has been at the center of sparse optimization. However, the $\ell_1$-regularizer does not always capture the true relevant variables since it can be a loose relaxation of the $\ell_0$-norm [3]. To overcome this drawback, many regularizers which abandon the convexity have been proposed to approximate the $\ell_0$-norm in better ways. Typical examples are Smoothly Clipped Absolute Derivation (SCAD) [7], Log-Sum Penalty (LSP) [3], Minimax Concave Penalty (MCP) [28], and capped-$\ell_1$ penalty [30].

On the other hand, there are some approaches which do not approximate the $\ell_0$-norm; DC (Difference of Convex functions) optimization approaches, employed in [23, 9], replace the $\ell_0$-norm by a difference of two convex functions and then apply the DC algorithm (DCA) [19] (also known as Convex-ConCave Procedure (CCCP) [27] or the Multi-Stage (MS) convex relaxation [30]) to the resulting DC program. However, as some papers including [8, 13] pointed out, DCA requires solving a sequence of convex subproblems, often resulting in a large computation time.

When the problem has no additional constraints other than the $\ell_0$-norm constraint, this issue can be resolved; some algorithms whose subproblems have closed-form solutions have been proposed. Gotoh et al. [9] transformed the problem without convex constraints into an equivalent problem minimizing a DC objective function; a special DC decomposition is employed so that its subproblems can be reduced to the so-called soft-thresholding operations, which can be carried out in linear time. The resulting DCA is called the Proximal DC Algorithm (PDCA), which constitutes a special case of the framework of Sequential Convex Programming (SCP) [14]. Iterative Hard-Thresholding (IHT) algorithm [4] is another efficient method for the $\ell_0$-constrained optimization. In IHT algorithm, we repeat solving subproblems of minimizing a quadratic surrogate function under the $\ell_0$-norm constraint, whose solutions are simply obtained by the so-called hard-thresholding operation.

Some applications of sparse optimization have convex constraints such as the $\ell_2$-norm constraint and nonnegative constraint other than the $\ell_0$-norm constraint. For such constrained sparse optimization, all the algorithms mentioned above generate a sequence of convex subproblems whose closed-form
solutions cannot be readily available in general. To overcome this issue, we propose a new DC representation of the $\ell_0$-constraint, which leads to a PDCA whose subproblems have closed-form solutions.

1.2 Contributions

We propose an efficient approach to the constrained sparse optimization: the minimization of an objective function under the $\ell_0$-constraint and some convex constraints. Gotoh et al. [9] proposed to express the $\ell_0$-norm as a difference of two convex functions as $\phi_1 - \phi_2$, both of which are nonsmooth. However, in applying PDCA to such a constrained problem, the nonsmoothness of the first term $\phi_1$ collides with the convex constraints, resulting in making the subproblems difficult to have closed-form solutions. In this paper, we rewrite the $\ell_0$-norm constraint as another DC function so that the former convex function $\phi_1$ is smooth. In applying PDCA, the smoothness of the former term makes subproblems easily solvable by a projection operation onto the convex set.

To achieve faster convergence, we further propose the Accelerated version of PDCA (APDCA), inspired by the preceding work [12] on extending the Accelerated Proximal Gradient (APG) method (originally for convex program) to nonconvex program. We construct APDCA by employing techniques used in the nonmonotone APG [12] for nonconvex program, so the convergence results for the nonmonotone APG can be shown to hold for APDCA; (i) APDCA has the convergence rate of $O(1/t^2)$, if applied to convex program, where $t$ denotes the iteration counter; (ii) APDCA has the subsequential convergence to a stationary point.

In the numerical section, we demonstrate the numerical performance of our approach compared to the existing DC optimization approaches. The efficiency of APDCA applied to our reformulation is confirmed with three typical examples of the constrained sparse optimization, using both synthetic and real-world data.

The remainder of this paper is structured as follows. In Section 2, we define the constrained sparse optimization problem and review some existing approaches. In Section 3, we propose a DC representation of the $\ell_0$-norm constraint and then show how to apply PDCA to the transformed problem. In Section 4, we show a close relation between PDCA and Proximal Gradient Method (PGM), which motivates us to extend the framework of PDCA. In Section 5, we accelerate PDCA to achieve faster convergence and review some related algorithms. In Section 6, we demonstrate the efficiency of our methods in comparison with other DCA frameworks.
2 Preliminaries

2.1 Problem settings

In this paper, we address the following $\ell_0$-constrained problem:

$$\min_{x} \{ \phi(x) : \|x\|_0 \leq k, \ x \in C \},$$

(1)

where $\phi : \mathbb{R}^n \to \mathbb{R}$, $k \in \{1, \ldots, n\}$, $\|x\|_0$ denotes the number of nonzero elements (called the $\ell_0$-norm or the cardinality) of a vector $x$, and $C \subseteq \mathbb{R}^n$ is a nonempty closed convex set. A solution $x$ satisfying the $\ell_0$-constraint, $\|x\|_0 \leq k$, is said to be $k$-sparse.

Throughout the paper, we make the following assumptions.

Assumption 1. (a) $\phi$ is continuously differentiable with $L_\phi$-Lipschitz continuous gradient, i.e., there exists a constant $L_\phi$ such that

$$\|\nabla \phi(x) - \nabla \phi(y)\|_2 \leq L_\phi \|x - y\|_2 \quad (x, y \in \mathbb{R}^n),$$

where $\|x\|_2$ denotes the $\ell_2$-norm of $x$.

(b) The projection $\text{proj}_C(u)$ of a point $u \in \mathbb{R}^n$ onto $C$ can be evaluated efficiently:

$$\text{proj}_C(u) := \arg \min_{x \in C} \left\{ \frac{1}{2} \|x - u\|_2^2 \right\}.$$

(c) $\phi(x) + I_C(x)$ is bounded from below and coercive, i.e., $\phi(x) + I_C(x) \to \infty$ as $\|x\|_2 \to \infty$, where $I_C$ denotes the indicator function of $C$ defined as

$$I_C(x) := \begin{cases} 0, & (x \in C), \\ +\infty, & (x \notin C). \end{cases}$$

(d) The feasible region $\{x \in C : \|x\|_0 \leq k\}$ of (1) is nonempty.

Various problems in many application areas are formulated as (1).

Example 1 (sparse principal component analysis [23]). Let $V \in \mathbb{R}^{n \times n}$ be a covariance matrix. When

$$\phi(x) = -x^\top V x, \ C = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\},$$

problem (1) is called the sparse Principal Component Analysis (PCA). In sparse PCA, we seek a $k$-sparse vector that approximates the eigenvector which corresponds to the largest eigenvalue and regard it as the first principal component.
Example 2 (sparse portfolio selection). Let $V \in \mathbb{R}^{n \times n}$ be a covariance matrix, $r \in \mathbb{R}^n$ a mean vector of returns of investable assets, and $\alpha > 0$ a risk-aversion parameter. When

$$\phi(x) = \alpha x^\top V x - r^\top x, \quad C = \{x \in \mathbb{R}^n : 1^\top x = 1\},$$

where $1 \in \mathbb{R}^n$ denotes the all-one vector, problem (1) can be seen as a variant of the sparse portfolio selection (e.g., [10, 22]).

Example 3 (sparse nonnegative linear regression). Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $I \subseteq \{1, \ldots, n\}$. When

$$\phi(x) = \frac{1}{2} \|Ax - b\|_2^2, \quad C = \{x \in \mathbb{R}^n : x_i \geq 0 \ (i \in I)\},$$

problem (1) is the sparse nonnegative linear regression problem. This problem includes the following problems as special cases: the ordinary least squares problem with variable selection ($I = \emptyset$) and the sparse least squares problem with all variables nonnegative ($I = \{1, \ldots, n\}$) [21].

2.2 Existing approaches to $\ell_0$-constrained optimization

2.2.1 Case for general $\ell_0$-constrained optimization

Gotoh et al. [9] proposed to express the $\ell_0$-norm constraint as a DC function:

$$\|x\|_0 \leq k \iff \|x\|_1 - \|x\|_{k,1} = 0,$$

where $\|x\|_{k,1}$, which we call top-$(k,1)$ norm, denotes the $\ell_1$-norm of a sub-vector composed of top-$k$ elements in absolute value. Precisely,

$$\|x\|_{k,1} := |x_{\pi(1)}| + \cdots + |x_{\pi(k)}|,$$  \hspace{1cm} (2)

where $\pi$ is an arbitrary permutation of $\{1, \ldots, n\}$ such that $|x_{\pi(1)}| \geq \cdots \geq |x_{\pi(n)}|$. Namely, $x_{\pi(i)}$ denotes the $i$-th largest element of $x$ in absolute value.

Then [9] considered the following penalized problem associated with (1):

$$\min_{x \in C} \{\phi(x) + \rho(\|x\|_1 - \|x\|_{k,1})\}. \quad \text{(3)}$$

and gave an exact penalty parameter under which problems (1) and (3) are equivalent for some examples, e.g., $C$ is $\mathbb{R}^n$ and $\{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$. Problems (1) and (3) are equivalent. Then the so-called DC Algorithm (DCA) is applied to the reformulation (3). In general, to minimize a DC function $\phi_1(x) - \phi_2(x)$, expressed by two convex functions $\phi_1$ and $\phi_2$, DCA solves the following subproblem repeatedly:

$$x^{(t+1)} \in \arg \min_{x \in \mathbb{R}^n} \left\{\phi_1(x) - x^\top s(x^{(t)})\right\}, \quad \text{(4)}$$

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where \( s(x^{(t)}) \) is a subgradient of \( \phi_2(x^{(t)}) \) at \( x^{(t)} \), i.e.,
\[
s(x^{(t)}) \in \partial \phi_2(x^{(t)}) := \{ y \in \mathbb{R}^n : \phi_2(x) \geq \phi_2(x^{(t)}) + \langle y, x - x^{(t)} \rangle \ (x \in \mathbb{R}^n) \}.
\]

When applying DCA to (3), \[9\] used the following decomposition for a DC function \( \phi = \gamma - \iota \):
\[
\phi_1(x) = \gamma(x) + \rho \| x \|_1 + I_{C}(x), \quad \phi_2(x) = \iota(x) + \rho \| x \|_{k,1}.
\]

The resulting subproblem (4) is a convex problem, but because it generally does not have a closed-form solution for (4), we need to repeatedly apply some convex optimization algorithm to solve the convex problem, which is often time-consuming.

Thiao et al. \[23\] gave another DC formulation, which is based on Mixed Integer Programming (MIP). They first rewrote the \( \ell_0 \)-norm using a binary vector \( u \) as
\[
\| x \|_0 \leq k \iff |x_i| \leq Mu_i \ (i = 1, \ldots, n), \quad 1^\top u \leq k, \quad u \in \{0, 1\}^n,
\]
where \( M \) is a so-called big-M constant, which is set to be sufficiently large. Then using the following equivalence:
\[
u \in \{0, 1\}^n \iff u \in [0, 1]^n, \quad (1 - u)^\top u \leq 0,
\]
they finally obtained a penalized DC formulation of (1):
\[
\min_{x \in C} \left\{ \phi(x) + \rho(1 - u)^\top u : |x_i| \leq Mu_i \ (i = 1, \ldots, n), \quad 1^\top u \leq k, \quad u \in [0, 1]^n \right\},
\]
which is solved by DCA (4). While this approach was originally proposed just for Example 1, it works also in our general settings. We need to use some convex optimization algorithms for the resulting convex subproblem as well as the above-mentioned DCA of \[9\].

### 2.2.2 Case for \( \ell_0 \)-constrained optimization without other constraints

For the case where \( C = \mathbb{R}^n \), paper \[9\] proposed a different DC decomposition, \( \phi_1 - \phi_2 \), where
\[
\phi_1(x) = \left( \frac{L_\phi}{2} \| x \|_2^2 + \rho \| x \|_1 \right), \quad \phi_2(x) = \left( \frac{L_\phi}{2} \| x \|_2^2 - \phi(x) + \rho \| x \|_{k,1} \right).
\]

The DC decomposition (7) gives a closed-form solution for the subproblem (4). We call the resulting algorithm the Proximal DC Algorithm (PDCA). The subproblem (7) of PDCA is written as
\[
x^{(t+1)} \in \arg\min_{x \in \mathbb{R}^n} \left\{ \frac{L_\phi}{2} \| x \|_2^2 + \rho \| x \|_1 - x^\top \left( L_\phi x^{(t)} - \nabla \phi(x^{(t)}) + s(x^{(t)}) \right) \right\},
\]

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where $s(x^{(t)}) \in \partial \rho \|x^{(t)}\|_{k,1}$. By using the proximal operator notation:

$$\text{prox}_g(u) := \arg\min_x \left\{ g(x) + \frac{1}{2} \|x - u\|^2 \right\},$$  \hfill (9)

we can further rewrite the subproblem (8) as

$$x^{(t+1)} \in \arg\min_{x \in \mathbb{R}^n} \left\{ \rho \frac{L_\phi}{L_\phi} \|x\|_1 + \frac{1}{2} \left\| x - \left( x^{(t)} - \frac{1}{L_\phi} \nabla \phi(x^{(t)}) + \frac{1}{L_\phi} s(x^{(t)}) \right) \right\|_2^2 \right\} = \text{prox}_{\frac{\rho}{L_\phi} \|\cdot\|_1} \left( x^{(t)} - \frac{1}{L_\phi} \nabla \phi(x^{(t)}) + \frac{1}{L_\phi} s(x^{(t)}) \right),$$

which is easily computed by using the so-called soft-thresholding [6], whose element is given as

$$\left[ \text{prox}_{\frac{\rho}{L_\phi} \|\cdot\|_1}(u) \right]_i = \text{sign}(u_i) \max\{u_i - \rho/L_\phi, 0\},$$  \hfill (10)

where $\text{sign}(u) = 1$ if $u > 0$; $-1$ if $u < 0$; $0$, otherwise.

Bertsimas et al. [4] addresses (1) without replacing the $\ell_0$-constraint by other terms. Since the function $\phi$ has a quadratic majorant at each point $x^{(t)}$ because of its $L_\phi$-smoothness, the paper proposes to iteratively solve the subproblems:

$$x^{(t+1)} \in \arg\min_{\|x\|_0 \leq k} \left\{ \phi(x^{(t)}) - (x - x^{(t)})^\top \nabla \phi(x^{(t)}) + \frac{L_\phi}{2} \|x - x^{(t)}\|_2^2 \right\}. \hfill (11)$$

The subproblem is computed by the so-called hard-thresholding operation, so repeating (11) is called the Iterative Hard-Thresholding (IHT) algorithm. They showed that the optimal solution $\hat{x}$ of $\min_{x \in \mathbb{R}^n} \{ \|x - u\|_2^2 : \|x\|_0 \leq k \}$ is obtained as follows: $\hat{x}$ retains the $k$ largest elements in absolute value of $u$ and sets the rest elements to zero. Since the hard-thresholding works only when $C = \mathbb{R}^n$, IHT algorithm is not applicable to (11) with $C \neq \mathbb{R}^n$.

3 DC representation for constrained sparse optimization

3.1 Main idea

PDCA with the DC decomposition [5] for [3] can perform poorly even if a simple convex constraint consists of $C$, since its subproblem has no closed-form solutions in general. To overcome this issue, we give another equivalent DC representation of the $\ell_0$-constraint. Let us start with the following equivalence results, which slightly generalize Theorem 1 of [3].
Proposition 1. Let \( \nu : \mathbb{R} \to \mathbb{R}_+ \) be a nonnegative function such that \( \nu(a) = 0 \) if and only if \( a = 0 \), and with a permutation \( \pi \) of \( \{1, \ldots, n\} \), denote by \( \nu(x_{\pi(i)}) \) the \( i \)-th largest element of \( \nu(x_1), \ldots, \nu(x_n) \), i.e., \( \nu(x_{\pi(1)}) \geq \cdots \geq \nu(x_{\pi(n)}) \). For any integers \( k, h \) such that \( 1 \leq k < h \leq n \), and \( \mathbf{x} \in \mathbb{R}^n \), the following three conditions are equivalent:

1. \( \|\mathbf{x}\|_0 \leq k \),
2. \( \sum_{i=1}^{h} \nu(x_{\pi(i)}) - \sum_{i=1}^{k} \nu(x_{\pi(i)}) = 0 \), and
3. \( \sum_{i=1}^{n} \nu(x_i) - \sum_{i=1}^{k} \nu(x_{\pi(i)}) = 0 \).

Furthermore, the following three conditions are equivalent:

4. \( \|\mathbf{x}\|_0 = k \),
5. \( k = \min \{ \kappa \in \{1, \ldots, h - 1\} : \sum_{i=1}^{h} \nu(x_{\pi(i)}) - \sum_{i=1}^{\kappa} \nu(x_{\pi(i)}) = 0 \} \), and
6. \( k = \min \{ \kappa \in \{1, \ldots, n - 1\} : \sum_{i=1}^{n} \nu(x_i) - \sum_{i=1}^{\kappa} \nu(x_{\pi(i)}) = 0 \} \).

If we employ the absolute value for \( \nu \), i.e., \( \nu(a) = |a| \), it is valid that

\[
\sum_{i=1}^{h} \nu(x_{\pi(i)}) = \|(\nu(x_1), \ldots, \nu(x_n))\|_{h,1} \quad \text{and} \quad \sum_{i=1}^{n} \nu(x_i) = \|(\nu(x_1), \ldots, \nu(x_n))\|_{1},
\]

and the above statements result in Theorem 1 of [9]. We can prove Proposition 4 by just replacing the absolute value with the function \( \nu \) in the proof of Theorem 1 of [9], and thus omit the proof here.

With \( \nu(a) = a^2 \) instead of \( |a| \), we can attain a quadratic DC representation. To align with the notation of \( \|\mathbf{x}\|_{k,1} \), we denote \( \|(x_1^2, \ldots, x_n^2)\|_{k,1} \) by \( \|\mathbf{x}\|_{k,2} \). Based on the equivalence between items 1. and 3. in Proposition 4 we have another DC representation of the \( \ell_0 \)-constraint:

\[
\|\mathbf{x}\|_0 \leq k \iff \|\mathbf{x}\|_2^2 - \|\mathbf{x}\|_{k,2}^2 = 0.
\]  

Note that \( \|\mathbf{x}\|_{k,2}^2 \) is convex since it can be written as a pointwise maximum of convex functions:

\[
\|\mathbf{x}\|_{k,2}^2 = \max_{\mathbf{v}} \left\{ \sum_{i=1}^{n} v_i x_i^2 : \mathbf{v} \in \{0, 1\}^n, \|\mathbf{v}\|_1 = k \right\}.
\]

1In other words, \( \|\mathbf{x}\|_{k,2} \) equals the \( \ell_2 \)-norm of a subvector composed of top-\( k \) elements of \( \mathbf{x} \) in square value, i.e., \( \|\mathbf{x}\|_{k,2} = \sqrt{x_{\pi(1)}^2 + \cdots + x_{\pi(k)}^2} \) with permutation \( \pi \) such that \( x_{\pi(1)}^2 \geq \cdots \geq x_{\pi(n)}^2 \). Analogously to \( \|\mathbf{x}\|_{k,1} \), we may call \( \|\mathbf{x}\|_{k,2} \) \((k, 2)\) norm.

2More generally, \( \|(\nu(x_1), \ldots, \nu(x_n))\|_{k,1} \) (or \( \sum_{i=1}^{n} \nu(x_{\pi(i)}) \)) is convex if \( \nu \) is convex.
With the equivalence \((12)\), we consider the penalized problem associated with \((1)\):

\[
\min_{x \in C} \{ \phi(x) + \rho(\|x\|_2^2 - \|x\|_{k,2}^2) \},
\]

where \(\rho > 0\) denotes a penalty parameter. The next theorem, which can be proved similarly to Theorem 17.1 in [17], ensures that problem \((13)\) is essentially equivalent to the original problem \((1)\) if we take the limit of the penalty parameter \(\rho\).

**Theorem 1.** Let \(\{\rho_t\}\) be an increasing sequence with \(\lim_{t \to \infty} \rho_t = \infty\) and \(x_t\) be an optimal solution of \((13)\) with \(\rho = \rho_t\). Then any accumulation point \(x^*\) of \(\{x_t\}\) is also optimal to \((1)\).

**Proof.** Let \(\bar{x}\) be an optimal solution of \((1)\). Since \(x_t\) is a minimizer of \((13)\) with \(\rho = \rho_t\), we have

\[
\phi(x_t) + \rho_t(\|x_t\|_2^2 - \|x_t\|_{k,2}^2) \leq \phi(\bar{x}) + \rho_t(\|\bar{x}\|_2^2 - \|\bar{x}\|_{k,2}^2) = \phi(\bar{x}),
\]

which is transformed into

\[
\|x_t\|_2^2 - \|x_t\|_{k,2}^2 \leq \frac{1}{\rho_t}(\phi(\bar{x}) - \phi(x_t)).
\]

Let \(T\) be an infinite subsequence such that \(\lim_{t \to T \to \infty} x_t = x^*\). By taking the limit on both sides and considering the nonnegativity of the penalty, we have

\[
0 \leq \|x^*\|_2^2 - \|x^*\|_{k,2}^2 \leq \lim_{t \to T \to \infty} \frac{1}{\rho_t}(\phi(\bar{x}) - \phi(x^*)) = 0,
\]

which implies \(x^*\) is feasible to \((1)\). In addition, by taking the limit on both sides of \((14)\), we have

\[
\phi(x^*) \leq \phi(x^*) + \lim_{t \to T \to \infty} \rho_t(\|x_t\|_2^2 - \|x_t\|_{k,2}^2) \leq \phi(\bar{x}).
\]

Since \(\bar{x}\) is an optimal solution of \((1)\), \(x^*\) is also optimal to \((1)\). \(\square\)

As we see in the next subsection, the associated subproblems of the specialized PDCA can be efficiently solved owing to the smoothness of \(\|x\|_{k,2}^2\).

### 3.2 Proximal DC algorithm for the transformed problem

To apply PDCA to \((13)\), we consider the following DC decomposition:

\[
\phi_1(x) = \frac{L_1}{2} \|x\|_2^2 + \rho(\|x\|_2^2 - \|x\|_{k,2}^2) + I_C(x),
\]

\[
\phi_2(x) = \frac{L_2}{2} \|x\|_2^2 - \phi(x) + \rho(\|x\|_{k,2}^2).
\]
Then the corresponding PDCA subproblem becomes

\[
x^{(t+1)} \in \arg\min_{x \in \mathbb{R}^n} \left\{ I_C(x) + \frac{L\phi + 2\rho}{2} \left\| x - \frac{1}{L\phi + 2\rho} \left( L\phi x^{(t)} - \nabla\phi(x^{(t)}) + s^{(t)} \right) \right\|_2^2 \right\}
\]

\[
= \text{prox}_{\frac{L\phi}{L\phi + 2\rho}} \left( \frac{1}{L\phi + 2\rho} \left( L\phi x^{(t)} - \nabla\phi(x^{(t)}) + s^{(t)} \right) \right),
\]\n
where \( s^{(t)} \in \partial(\rho \| x \|_{k,2}^2) \). The subdifferential of \( \| x \|_{k,2}^2 \) is given as

\[
\partial(\| x \|_{k,2}^2) = \left\{ v : v_i = \begin{cases} 2x_i & (\pi(i) \leq k) \\ 0 & (\pi(i) > k) \end{cases} \right\}.
\]

Note that the proximal operator of \( I_C \) is nothing but the projection onto \( C \). Therefore, the subproblem (16) is easily solved for various feasible sets \( C \). We list below how to obtain \( \text{proj}_C \) for the three constraint sets in Examples 1–3.

(i) For \( C = \{ x \in \mathbb{R}^n : \| x \|_2 \leq 1 \} \), \( \text{proj}_C \) is given by

\[
\text{proj}_C(u) = \begin{cases} \frac{u}{\| u \|_2} & (\| u \|_2 \geq 1) \\ u & (\| u \|_2 < 1) \end{cases}.
\]

(ii) For \( C = \{ x \in \mathbb{R}^n : \mathbf{1}^\top x = 1 \} \), \( \text{proj}_C \) is given by

\[
\text{proj}_C(u) = u + (1 - \mathbf{1}^\top u)\mathbf{1}/n.
\]

(iii) For \( C = \{ x \in \mathbb{R}^n : x_i \geq 0 (i \in I \subseteq \{1, \ldots, n\}) \} \), \( \text{proj}_C(u) \) is obtained by setting the negative elements of \( u \) corresponding to \( I \) to zero and retaining the rest.

We summarize the procedure of PDCA for the transformed problem \( (13) \) in Algorithm 1. For practical use, the termination criterion of Algorithm 1 is replaced by \( \Phi(x^{(t)}) - \Phi(x^{(t+1)}) < \varepsilon \), where \( \Phi \) denotes the objective function in \( (13) \) and \( \varepsilon \) is a sufficiently small positive value. As we mentioned in Section 2, Algorithm 1 is a kind of DCA with the special DC decomposition. Since the global convergence of DCA is shown in \( [19] \) for a general problem setting including \( (13) \), the convergence property is also valid for Algorithm 1.

Theorem 2. Let \( \{ x^{(t)} \} \) be the sequence generated by Algorithm 1. Then \( \{ x^{(t)} \} \) globally converges to a stationary point \( x^* \in C \) of \( (13) \), i.e.,

\[
0 \in \partial\phi_1(x^*) - \partial\phi_2(x^*),
\]

where \( \phi_1(x) \) and \( \phi_2(x) \) are given by \( (15) \).
Algorithm 1 Proximal DC Algorithm (PDCA) for (13)
\[ x^{(0)} \in C. \]
for \( t = 0, 1, \ldots \) do
    Pick a subgradient \( s(x^{(t)}) \in \partial g_2(x^{(t)}) \) and compute
    \[ x^{(t+1)} = \text{proj}_C \left( \frac{1}{L_\phi + 2\rho} \left( L_\phi x^{(t)} - \nabla \phi(x^{(t)}) + s(x^{(t)}) \right) \right). \]
end for

4 PDCA with backtracking step size rule

In this section, we show a link between the Proximal DC Algorithm (PDCA) and the Proximal Gradient Method (PGM), and present a PDCA with backtracking step size rule. More specifically, we first discuss that the framework of PDCA can be extended to more general settings. Then we clarify that PDCA is a generalized version of PGM for DC optimization, which implies that some useful techniques to speed up PGM can also be employed in PDCA.

4.1 Proximal DC algorithm for composite nonconvex optimization

We consider the following composite nonconvex optimization problem:
\[
\min_{x \in \mathbb{R}^n} F(x) := f(x) + g(x),
\]
where \( f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). We make the following assumptions on (17).

Assumption 2. (a) \( f(x) \) is continuously differentiable with \( L \)-Lipschitz continuous gradient.

(b) \( g(x) \) is decomposed into a DC function as
\[
g(x) = g_1(x) - g_2(x),
\]
where \( g_1(x) \) is proper, lower semicontinuous and convex, and \( g_2(x) \) is continuous and convex.

(c) \( F(x) \) is bounded from below and coercive.

The penalized formulation (13) of the sparse constrained problem can be regarded as problem (17) satisfying Assumption 2:

- \( \phi(x) + \rho \|x\|_2^2 \) corresponds to the smooth term \( f(x) \) in (17) which has \( L = L_\phi + 2\rho \),
- the indicator function \( I_C(x) \) of \( C \) corresponds to \( g_1(x) \), and
- \( \rho \|x\|_2^2 \) corresponds to \( g_2(x) \), which is continuous and convex.
Table 1: DC decompositions of sparse regularizers (λ and θ denote nonnegative hyperparameters), given in [8] except for ℓ_{1-2}.

| name of regularizer | g_1 | g_2 |
|---------------------|-----|-----|
| ℓ_1 norm [24]       | λ∥x∥_1 | 0 |
| capped-ℓ_1 [30]     | λ∥x∥_1 | λ \sum_{i=1}^n \max\{|x_i| - θ, 0\} |
| LSP (Log Sum Penalty) [5] | λ∥x∥_1 | λ \sum_{i=1}^n (|x_i| - \log(1 + |x_i|/θ)) |
| SCAD (Smoothly Clipped Absolute Deviation) [7] | λ∥x∥_1 | \sum_{i=1}^n \left\{ \begin{array}{ll} 0 & \text{if } |x_i| \leq \lambda \\ \frac{x_i^2 - 2λ|x_i| + λ^2}{2(θ - 1)} & \text{if } \lambda < |x_i| \leq θλ \\ (λ|x_i| - \frac{θ+1}{2}λ^2) & \text{if } |x_i| > θλ \end{array} \right. |
| MCP (Minimax Concave Penalty) [28] | λ∥x∥_1 | \sum_{i=1}^n \left\{ \begin{array}{ll} 0 & \text{if } |x_i| \leq θλ \\ \frac{x_i^2}{2θ} & \text{if } |x_i| \leq θλ \\ λ|x_i| - \frac{θλ^2}{2} & \text{if } |x_i| > θλ \end{array} \right. |
| ℓ_{1-2} [26]       | λ∥x∥_1 | \rho∥x∥_2^{k,2} |

• g_2(x) = ρ∥x∥_2^{k,2}.

In addition, many nonconvex regularized problems are included in this setting, as shown in Table 1.

We can naturally extend our PDCA to (17), which is originally proposed for (3) in [9]. Similarly to (7), we consider the following DC decomposition of F:

\[
F(x) = \left( \frac{L}{2}∥x∥_2^2 + g_1(x) \right) - \left( \frac{L}{2}∥x∥_2^2 - f(x) + g_2(x) \right).
\]

Then the subproblem of DCA for (19) becomes

\[
\begin{align*}
\mathbf{x}^{(t+1)} & \in \arg\min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \frac{L}{2}∥\mathbf{x}∥_2^2 + g_1(\mathbf{x}) - \mathbf{x}^\top \left( L\mathbf{x}^{(t)} - \nabla f(\mathbf{x}^{(t)}) + s(\mathbf{x}^{(t)}) \right) \right\} \\
& = \text{prox}_{\frac{g_1}{L}} \left( \mathbf{x}^{(t)} - \frac{1}{L}\nabla f(\mathbf{x}^{(t)}) + \frac{1}{L}s(\mathbf{x}^{(t)}) \right), \tag{20}
\end{align*}
\]

where s^{(t)} ∈ ∂g_2(\mathbf{x}^{(t)}). The subproblem (20) of PDCA is reduced to calculating the proximal operator of g_1/L, which leads to closed-form solutions for various g_1.

Now we recall the Sequential Convex Programming (SCP) [14] as a related work. SCP solves problem (17) by generating a sequence \{\mathbf{x}^{(t)}\} obtained via

\[
\begin{align*}
\mathbf{x}^{(t+1)} & \in \arg\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}^{(t)}) + \langle \nabla f(\mathbf{x}^{(t)}), \mathbf{x} - \mathbf{x}^{(t)} \rangle \\
& \quad + \frac{L}{2}∥\mathbf{x} - \mathbf{x}^{(t)}∥_2^2 + g_1(\mathbf{x}) - g_2(\mathbf{x}^{(t)}) - \langle s(\mathbf{x}^{(t)}), \mathbf{x} - \mathbf{x}^{(t)} \rangle.
\end{align*}
\]
This problem is essentially the same as (20), but the paper does not mention how to solve such convex subproblems, nor the way of computing closed-form solutions. They derive this algorithm and analyze its convergence independently of the theory of DC programming. Our algorithm sheds a new light on SCP. Namely, SCP can be viewed as a variant of DC algorithm and thus its convergence property such as global convergence is automatically satisfied.

4.2 Relation to PGM variants

Especially for convex functions $f$ and $g$, we can see that PDCA reduces to the renowned Proximal Gradient Method (PGM):

$$x^{(t+1)} = \text{prox}_{g/L}(x^{(t)} - \frac{1}{L} \nabla f(x^{(t)})). \tag{21}$$

PGMs for convex optimization problems have been investigated in a different research stream from DCA for nonconvex optimization problems, but we can find a similarity of the resulting subproblems: (20) and (21). In recent years, developing efficient algorithms for solving convex cases of (17) has become a topic of intense research especially in the machine learning community and various techniques for obtaining faster convergence were proposed for PGMs. We also can use such techniques including the backtracking and the acceleration for our method, namely, PDCA.

Currently, popular research directions regarding PGMs include applying PGMs to nonconvex optimization problems. For example, General Iterative-Shrinkage-Thresholding (GIST) algorithm [8] was proposed for nonconvex (17). GIST generates a sequence $\{x^{(t)}\}$ by

$$x^{(t+1)} = \text{prox}_{g/l(t)}\left(x^{(t)} - \frac{1}{l(t)} \nabla f(x^{(t)})\right), \tag{22}$$

where $1/l(t)$ is a proper step size. The paper [8] showed closed-form solutions of (22) for the regularizers in Table I except for $\ell_{1,2}$. From applying GIST to the reformulations (3) and (13) for constrained sparse optimization problems (1) seems difficult because of the term $I_C(x)$. The sequence $\{x^{(t)}\}$ of GIST subsequentially converges to a stationary point of (17), as far as $l^{(t)}$ is fixed to an arbitrary value larger than a Lipschitz constant $L$ of $\nabla f(x)$. We will describe how to determine $l^{(t)}$ in practice, later when elaborating on our method.

4.3 Backtracking

To achieve faster convergence, several techniques such as the backtracking and the acceleration have been proposed for PGM and its variants, the latter

\footnote{For the $\ell_{1,2}$ regularizer, Liu and Pong [13] showed closed-form solutions of (22).}
of which is mentioned in Section 5. The backtracking line search initialized by Barzilai-Borwein (BB) rule [2] is employed in GIST [8] to use a larger step size $1/l(t)$ instead of $1/L$. In the backtracking, we accept $l(t)$ if the following criterion is satisfied for $\sigma \in (0, 1)$:

$$F(x^{(t+1)}) \leq F(x^{(t)}) - \frac{\sigma}{2} \|x^{(t+1)} - x^{(t)}\|_2^2,$$

(23)

otherwise $l(t) \leftarrow \eta l(t)$ with $\eta > 1$ and check the above inequality again. The initial $l(t)$ at each iteration $t$ is given by the BB rule [2] as

$$l(t) = \frac{\langle x^{(t)} - x^{(t-1)}, x^{(t)} - x^{(t-1)} \rangle}{\langle x^{(t)} - x^{(t-1)}, \nabla f(x^{(t)}) - \nabla f(x^{(t-1)}) \rangle},$$

(24)

For convergence and practical use, $l(t)$ is projected onto the interval $[l_{\text{min}}, l_{\text{max}}]$ with $0 < l_{\text{min}} < l_{\text{max}}$.

Now we consider employing the backtracking technique in PDCA. We use a larger step size $1/l(t)$ instead of $1/L$:

$$x^{(t+1)} \in \arg\min_{x \in \mathbb{R}^n} \left\{ \frac{l(t)}{2} \|x\|_2^2 + g_1(x) - x^\top \left( l(t) x^{(t)} - \nabla f(x^{(t)}) + s(x^{(t)}) \right) \right\}$$

$$= \text{prox}_{g_1/l(t)} \left( x^{(t)} - \frac{1}{l(t)} \nabla f(x^{(t)}) + \frac{1}{l(t)} s(x^{(t)}) \right),$$

(25)

The resulting algorithm is summarized in Algorithm 2 whose convergence is analyzed similarly to GIST algorithm in [8].

**Algorithm 2** Proximal DC Algorithm for (17) with backtracking line search

$x^{(0)} \in \text{dom} g_1$.

for $t = 0, 1, \ldots$ do

Compute $x^{(t+1)}$ by (25), where the step size $1/l(t)$ is dynamically computed by (23)–(24).

end for

**Theorem 3.** The sequence $\{x^{(t)}\}$ generated by Algorithm 2 converges to a stationary point of (17).

See Appendix A.1 for the Proof of Theorem 3. The next theorem ensures the convergence rate of Algorithm 2 with respect to $\|x^{(t+1)} - x^{(t)}\|_2^2$. The proof is almost the same as Theorem 2 in [8].

**Theorem 4.** For the sequence $\{x^{(t)}\}$ generated by Algorithm 2 and its accumulation point $x^*$, the following holds for any $\tau \geq 1$;

$$\min_{0 \leq t \leq \tau} \|x^{(t+1)} - x^{(t)}\|_2^2 \leq \frac{2(F(x^{(0)}) - F(x^*))}{\tau \sigma}.$$
Proof. It follows from the criterion (23) that

$$\frac{\sigma}{2} \|x^{(t+1)} - x^{(t)}\|_2^2 \leq F(x^{(t)}) - F(x^{(t+1)}).$$

Summing the above inequality over $t = 0, \ldots, \tau$, we have

$$\frac{\sigma}{2} \sum_{t=0}^{\tau} \|x^{(t+1)} - x^{(t)}\|_2^2 \leq F(x^{(0)}) - F(x^{(\tau+1)}).$$

Thus we have

$$\min_{0 \leq t \leq n} \|x^{(t+1)} - x^{(t)}\|_2^2 \leq \frac{2(F(x^{(0)}) - F(x^{(\tau+1)}))}{\tau \sigma} \leq \frac{2(F(x^{(0)}) - F(x^{*}))}{\tau \sigma}.$$

\[\square\]

5 Accelerated algorithm for constrained sparse optimization

In this section, we provide an accelerated version of PDCA for (17).

5.1 Overview of accelerated methods for nonconvex optimization

For convex $f$ and $g$, the so-called Nesterov’s acceleration technique helps PGM accelerate practically and theoretically; the resulting method is known as Accelerated Proximal Gradient (APG) method [3]. APG is guaranteed to have $O(1/t^2)$ convergence rate, which is optimal among all the first-order methods. However, APG in [3] is for convex problems and has no guarantees to yield stationary points for the nonconvex case until quite recently.

Recently, Li and Lin [12] has proposed two APGs for nonconvex optimization: monotone APG and nonmonotone APG, each of which uses a different type of acceleration. Their nonmonotone APG with fixed step size is summarized in Algorithm 3. This procedure generates a sequence $\{x^{(t)}\}$ that subsequentially converges to a stationary point, retaining $O(1/t^2)$ convergence rate for the convex case.

We can take advantage of the similarity of PDCA and PGM for developing an accelerated version of PDCA, as we did for developing PDCA with backtracking step size rule in the previous section. The accelerated version of PDCA, which will be discussed in Section 5.2, utilizes the acceleration technique of Li and Lin [12].

Quite recently, for the case where $f$ is convex, Wen et al. [25] has developed a modified PDCA by adding an extrapolation step to speed up its convergence. They call it the proximal Difference-of-Convex Algorithm with extrapolation (pDCA$_e$), which is summarized in Algorithm 4. We can see
Algorithm 3 nonmonotone Accelerated Proximal Gradient (nm-APG) method [12]
\[ x^{(0)} = x^{(1)} = z^{(1)} \in \text{dom } g_1, \theta^{(0)} = 0, \theta^{(1)} = 1, \delta > 0, \text{ and } \eta \in (0, 1]. \]

for \( t = 1, 2, \ldots \) do
\[ y^{(t)} = x^{(t)} + \frac{\theta^{(t-1)}}{\theta^{(0)}}(z^{(t)} - x^{(t)}) + \frac{\theta^{(t-1)} - 1}{\theta^{(0)}}(x^{(t)} - x^{(t-1)}), \]
\[ z^{(t+1)} = \text{prox}_{g_1/L}(y^{(t)} - \frac{1}{L} \nabla f(y^{(t)})). \]

if \( F(z^{(t+1)}) + \delta \|z^{(t+1)} - y^{(t)}\|_2^2 \leq \sum_{j=1}^{t} \eta^{t-j} F(x^{(j)}) \) then
\[ x^{(t+1)} = z^{(t+1)}. \]
else
\[ v^{(t+1)} = \text{prox}_{g_1/L}(x^{(t)} - \frac{1}{L} \nabla f(x^{(t)})). \]
end if
\[ \theta^{(t+1)} = \sqrt{1/(\theta^{(t)})^2 + 1} + 1. \]
end for

Algorithm 4 Proximal DC Algorithm with extrapolation (pDCAe) [25]
\[ x^{(0)} = x^{(1)} \in \text{dom } g_1, \{\beta_t\} \subseteq [0, 1) \text{ with } \sup_t \beta_t < 1. \]

for \( t = 1, 2, \ldots \) do
Pick any \( s(x^{(t)}) \in \partial g_2(x^{(t)}) \) and compute
\[ y^{(t)} = x^{(t)} + \beta_t (x^{(t)} - x^{(t-1)}), \]
\[ x^{(t+1)} = \text{prox}_{g_1/L}(y^{(t)} - \frac{1}{L} \nabla f(y^{(t)}) + \frac{1}{L} s^{(t)}). \]
end for
that pDCA is general enough to include many algorithms. It reduces to PDCA for convex \( f \) by setting \( \beta_t \equiv 0 \) in Algorithm 4, and to FISTA with the fixed or adaptive restart \[18\] for convex \( f \) and \( g \) by choosing \( \{ \beta_t \} \) appropriately. In other words, Wen et al. \[25\] developed another type of acceleration for PDCA, while we utilized the acceleration technique of Li and Lin \[12\] for accelerating PDCA. Both works were done in parallel at almost the same time, and we added comparison of two acceleration methods to our numerical experiment.

5.2 Proposed Algorithm

We propose the Accelerated Proximal DC Algorithm (APDCA) for \((17)\) by applying the Nesterov’s acceleration technique to PDCA. In order to establish good convergence properties, we employ the techniques used in Algorithm 3. The procedure is summarized in Algorithm 5, using the following procedure:

\[
y^{(t)} = x^{(t)} + \frac{\theta^{(t-1)}}{\theta^{(t)}} (z^{(t)} - x^{(t)}) + \frac{\theta^{(t-1)} - 1}{\theta^{(t)}} (x^{(t)} - x^{(t-1)}),
\]

\[
z^{(t+1)} = \text{prox}_{g_2/l_y^{(t)}} \left( y^{(t)} - \frac{1}{l_y^{(t)}} \nabla f(y^{(t)}) + \frac{1}{l_y^{(t)}} s(y^{(t)}) \right),
\]

\[
v^{(t+1)} = \text{prox}_{g_1/l_x^{(t)}} \left( x^{(t)} - \frac{1}{l_x^{(t)}} \nabla f(x^{(t)}) + \frac{1}{l_x^{(t)}} s(x^{(t)}) \right),
\]

\[
x^{(t+1)} = \begin{cases} 
    z^{(t+1)}, & (F(z^{(t+1)}) \leq F(v^{(t+1)})), \\
    v^{(t+1)}, & \text{(otherwise)},
\end{cases}
\]

\[
\theta^{(t+1)} = \frac{\sqrt{4(\theta^{(t)})^2 + 1} + 1}{2},
\]

where \( s(x^{(t)}) \) and \( s(y^{(t)}) \) denote subgradients of \( g_2 \) at \( x^{(t)} \) and \( y^{(t)} \), respectively.

By following \[12\], we take \( y^{(t)} \) as a good extrapolation and omit to compute the second proximal operator \[28\] if the following criterion given by \[23\] is satisfied:

\[
F(z^{(t+1)}) + \delta \|z^{(t+1)} - y^{(t)}\|_2^2 \leq c^{(t)} := \frac{\sum_{j=1}^{t} \eta^{t-j} F(x^{(j)})}{\sum_{j=1}^{t} \eta^{t-j}},
\]

where \( \delta > 0 \) and \( \eta \in (0,1] \) controls the weights of the convex combination. Note that \( c^{(t)} \) is computed step by step as

\[
q^{(t+1)} = \eta q^{(t)} + 1,
\]

\[
c^{(t+1)} = \frac{\eta q^{(t)} + F(x^{(t+1)})}{q^{(t+1)}},
\]

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Algorithm 5 Accelerated Proximal DC Algorithm (APDCA)

$x^{(0)} = z^{(1)} \in \text{dom } g_1$, $\theta^{(0)} = 0$, and $\theta^{(1)} = 1$.

for $t = 1, 2, \ldots$ do

Compute $y^{(t)}$ and $z^{(t+1)}$ by (26)–(27), where the step size $1/l_x^{(t)}$ is fixed smaller than $1/L$ or dynamically computed by (23)–(24).

if (31) holds then

$x^{(t+1)} = z^{(t+1)}$.

else

Compute $x^{(t+1)}$ by (28) and (29), where the step size $1/l_x^{(t)}$ is fixed smaller than $1/L$ or dynamically computed by (23) and (32).

end if

Compute $\theta^{(t+1)}$ by (30).

end for

with $q^{(1)} = 1$ and $c^{(1)} = F(x^{(0)})$. Since $v^{(t)}$ is not necessarily computed in each iteration, the following initialization rule for $l_x^{(t)}$ is used instead of (24):

$$l_x^{(t)} = \frac{\langle x^{(t)} - y^{(t-1)}, x^{(t)} - y^{(t-1)} \rangle - \langle x^{(t)} - y^{(t-1)}, \nabla f(x^{(t)}) - \nabla f(y^{(t-1)}) \rangle}{\langle x^{(t)} - y^{(t-1)}, \nabla f(x^{(t)}) - \nabla f(y^{(t-1)}) \rangle},$$

(32)
after which we project $l_x^{(t)}$ onto $[l_{\min}, l_{\max}]$.

The convergence of Algorithm 5 is guaranteed by the next theorem, which is proved similarly to Theorem 4 in [12].

Theorem 5. Let $\Omega_1$ be the set of every $t$ at which (31) is satisfied and $\Omega_2$ be the set of the rest. Then the sequences $\{x^{(t)}\}$, $\{v^{(t)}\}$, and $\{y^{(t)}\}$ generated by Algorithm 5 are bounded and

1. if $\Omega_1$ or $\Omega_2$ is finite, then any accumulation point $x^*$ of $\{x^{(t)}\}$ is a stationary point of (17);

2. otherwise, any accumulation points $x^*$ of $\{x^{(t_1+1)}\}_{t_1 \in \Omega_1}$, $y^*$ of $\{y^{(t_1)}\}_{t_1 \in \Omega_1}$, $v^*$ of $\{v^{(t_2+1)}\}_{t_2 \in \Omega_2}$, and $x^*$ of $\{x^{(t_2)}\}_{t_2 \in \Omega_2}$ are stationary points of (17).

See Appendix A.2 for the proof of Theorem 5.

Remark 1. We can ensure the optimal convergence rate of Algorithm 5 with the fixed step size for convex optimization (17), though this is not the case with the $\ell_0$-constraint problem (13). Since Algorithm 5 with the fixed step size is identical to Algorithm 3 [12] if both $f$ and $g$ are convex, the following convergence rate is guaranteed exactly the same as that of Algorithm 3. Let $\{x^{(t)}\}$ be the sequence generated by Algorithm 5 with the fixed step size $1/L$ and assume that $f$ and $g$ are convex. Then for any $\tau \geq 1$, we have

$$F(x^{(\tau+1)}) - F(x^*) \leq \frac{2}{L(\tau + 1)^2} \|x^{(0)} - x^*\|^2_2,$$

where $x^*$ is a global minimizer of (17).
6 Numerical experiments

In this section, we demonstrate the numerical performance of our algorithm. All the computations were executed on a PC with 2.4GHz Intel CPU Core i7 and 16GB of memory.

6.1 Comparison of two accelerations for PDCA on unconstrained sparse optimization

We compared two types of acceleration for PDCA: APDCA (Algorithm 5) and pDCA \(_{e}^{[25]}\) (Algorithm 4). For pDCA \(_{e}\), we used the program code of [25], which is available at http://www.mypolyuweb.hk/~tkpong/pDCAe_final_codes/. In their code, \(1/L = 1/\lambda_{\text{max}}(A^T A)\) is employed as a step size and the extrapolation parameter \(\{\beta_t\}\) is set to perform both the fixed and the adaptive restart strategy (for the details on how to choose the parameters, see Sections 3 and 5 in [25]). Since their code is designed to solve the \(\ell_1 - \ell_2\) regularized linear regression problem:

\[
\min_{x \in \mathbb{R}^n} F(x) = \frac{1}{2} \|Ax - b\|_2^2 + \rho(\|x\|_1 - \|x\|_2),
\]

(33)

where \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\), and \(\rho > 0\), the comparison was made on this problem. We generated synthetic data following [25]. An \(m \times n\) matrix \(A\) was generated with i.i.d. standard Gaussian entries and then normalized so that each column \(a_i\) of \(A\) has unit norm, i.e., \(\|a_i\|_2 = 1\). Then a \(k\)-sparse vector \(\bar{x}\) was generated to have i.i.d. standard Gaussian entries on an index subset of size \(k\), which is chosen uniformly randomly from \(\{1, \ldots, n\}\). Finally, \(b \in \mathbb{R}^m\) was generated by \(b = Ax - 0.01 \cdot \varepsilon\), where \(\varepsilon \in \mathbb{R}^m\) is a random vector with i.i.d. Gaussian entries.

We implemented two methods: APDCA\(_{\text{fix}}\) and APDCA\(_{\text{bt}}\). APDCA\(_{\text{fix}}\) represents Algorithm 5 with the fixed step size \(1/L = 1/\lambda_{\text{max}}(A^T A)\), while APDCA\(_{\text{bt}}\) denotes Algorithm 5 with the backtracking line search with \(\sigma = 1.0 \times 10^{-5}\). We terminated these algorithms if the relative difference of the two successive objective values is less than \(10^{-5}\).

The computational results on the synthetic data with \((m, n, k) = (720i, 2560i, 80i)\) for \(i = 1, \ldots, 5\) are summarized in Table 2 where \(t_L\) denotes the time for computing \(L = \lambda_{\text{max}}(A^T A)\). We can see that all the APDCAs require much fewer iterations than pDCA\(_{e}\), while pDCA\(_{e}\) achieves the best objective value. pDCA\(_{e}\) converges in 1002 iterations for all the instances, which can be attributed to the fixed restart strategy employed at every 200 iterations of pDCA\(_{e}\). The CPU time per iteration seems to depend on how many times the objective value is evaluated. Actually, pDCA\(_{e}\) requires no evaluation of \(F\), APDCA\(_{\text{fix}}\) requires once, and APDCA\(_{\text{bt}}\) requires a couple of times. APDCA\(_{\text{fix}}\) converges the fastest on the four out of five instances.

\[^4\] The CPU times for APDCA\(_{\text{fix}}\) and pDCA\(_{e}\) include \(t_L\).
6.2 Results for constrained sparse optimization

We compared the performance of our algorithms with various DCAs for the $\ell_0$-constrained optimization problem having some convex constraints. In this section, we compare the following four methods for Examples 1–3.

- $\ell_2$-APDCA: We apply Algorithm 5 with backtracking line search to the penalized problem (13). Since we cannot set $\rho$ in (13) as an exact penalty parameter, some errors tend to remain in the $(n-k)$ smallest components (in absolute value) of the output of APDCA. Thus we round the output of APDCA to a $k$-sparse one by solving the small problem with $k$ variables obtained by fixing $(n-k)$ smallest components to 0.

- $\ell_2$-PDCA: We apply Algorithm 2 to the penalized problem (13) and round the solution as in $\ell_2$-APDCA.

- $\ell_1$-DCA [9]: We apply DC algorithm with the DC decomposition (5) to the top-$(k,1)$ penalized formulation (3). In the algorithm, the sub-problems are solved by an optimization solver, IBM ILOG CPLEX 12.

- MIP-DCA [23]: We apply DCA to the transformed problem (6), where the big-M constant is fixed to 100. The objective function in (6) is
decomposed as a DC function in the same way as (5). The DCA subproblems are solved by CPLEX.

Since a proper magnitude of penalty parameter $\rho$ depends on how we solve (1), we tested $\rho = 10^i$ ($i = 0, \pm 1, \ldots, \pm 4$) for each method and chose the one which attained the minimum objective value among $\rho$s that gave a $k$-sparse solution. We again terminated all the algorithms if the relative difference of the two successive objective values is less than $10^{-5}$.

6.2.1 Sparse principal component analysis

We consider a sparse PCA in Example 1:

$$\min_{x \in \mathbb{R}^n} \left\{ -x^\top A x : \|x\|_0 \leq k, \|x\|_2 \leq 1 \right\},$$

where $A$ is an $n \times n$ positive semidefinite matrix. We first examined the dependency on the initial solution $x^{(0)}$ with the pit props data [11], a standard benchmark to test the performance of algorithms for sparse PCA, whose correlation matrix has $n = 13$. We randomly generated 100 initial points where $x^{(0)}_i \sim N(0, 1)$ for $i = 1, \ldots, n$. Figure 1 shows the box plot of the objective values obtained by four algorithms with $k = 5$, where $\rho = 1$ was selected for all the algorithms. We can see that $\ell_2$-PDCA and $\ell_2$-APDCA tend to achieve better objective values and less dependency on the initial solution than the other DCAs.

![Box plot of the objective values for 100 random initial solutions on the pit props data.](image)

Figure 1: Box plot of the objective values for 100 random initial solutions on the pit props data.

We show in Table 3 the results on the colon cancer data [1], which consists of 62 tissue samples with the gene expression profiles of $n = 2000$ genes extracted from DNA micro-array data. The parameters were fixed to $k = 100$ and $x^{(0)} = 1/n$. We can see that $\ell_2$-PDCA and $\ell_2$-APDCA converge faster owing to the light projection computations for the subproblems, while MIP-DCA achieves the best objective value.
Table 3: Results for sparse PCA with colon cancer data: the chosen $\rho$, the cardinality of the found solution, the attained objective value, CPU time (sec.), and the number of iterations.

| method   | $\rho$ | cardinality | objective value | time (s) | iteration |
|----------|--------|--------------|-----------------|----------|-----------|
| $\ell_2$-PDCA | 1000   | 100          | -45.70          | 0.4      | 28        |
| $\ell_2$-APDCA | 1000   | 100          | -48.24          | 0.2      | 26        |
| $\ell_1$-DCA   | 1000   | 100          | -45.65          | 3.3      | 3         |
| MIP-DCA        | 1000   | 100          | -80.68          | 17.3     | 8         |

6.2.2 Sparse portfolio selection

We consider a sparse portfolio selection problem in Example 2:

$$
\min_{x \in \mathbb{R}^n} \left\{ \alpha x^\top V x - r^\top x : \|x\|_0 \leq k, \ 1^\top x = 1 \right\},
$$

where $V$ is a covariance matrix, $r$ is a mean return vector, and $\alpha (>0)$ is a risk-aversion parameter.

We used the 2148 daily return vectors of 1338 stocks listed in the first section of Tokyo Stock Exchange (TSE) through February 2008 to November 2016.\footnote{This data set was collected through NEEDS-FinancialQUEST, a databank service provided by Nikkei Media Marketing, Inc., and was modified by deleting series of data which include missing values for the period.} We fixed the parameters as $\alpha = 10$, $k = 10$, and $x^{(0)} = 1/n$. Table 4 reports the results on the TSE return data. Our algorithms tend to require much more iterations but attain better objective values and converge much faster.

Table 4: Results for sparse portfolio selection with TSE return data: the chosen $\rho$, the cardinality of the found solution, the attained objective value, the obtained return, CPU time (sec.) and the number of iterations.

| method   | $\rho$ | cardinality | objective value | return | time (s) | iteration |
|----------|--------|--------------|-----------------|--------|----------|-----------|
| $\ell_2$-PDCA | 1      | 10           | -1.15e-04       | 8.05e-04 | 10.2     | 2523      |
| $\ell_2$-APDCA | 1      | 10           | -3.91e-04       | 8.11e-04 | 2.3      | 262       |
| $\ell_1$-DCA   | 1      | 9            | 7.75e-04        | 5.48e-04 | 45.8     | 2         |
| MIP-DCA        | 1      | 10           | 3.62e-04        | 6.03e-04 | 105.8    | 4         |
6.2.3 Sparse nonnegative least squares

We consider a sparse nonnegative least squares problem in Example 3:

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b\|_2^2 : \|x\|_0 \leq k, \ x_i \geq 0 \ (i \in I) \right\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $I \subseteq \{1, \ldots, n\}$.

We report the results on synthetic data generated as follows. Each column $a_i$ of the matrix $A^\top = (a_1, \ldots, a_m)$ was drawn independently from the normal distribution $N(0, \Sigma)$, where $\Sigma = (\sigma_{ij}) = (0.5 |i-j|)$, and each column of $A$ was then standardized, i.e., $\|a_i\|_2 = 1$; $b$ was generated by $b = A\bar{x} + \varepsilon$, where $\bar{x}_i \sim U(-1, 1)$ and $\varepsilon_i \sim N(0, 1)$.

Table 5 shows the results on synthetic data with various sizes. We use $k = n/10$, $x^{(0)} = 1/n$, and $I = \{1, \ldots, \lfloor n/10 \rfloor\}$. We can see that $\ell_2$-PDCA is the fastest, and $\ell_2$-APDCA tends to find better solutions with smaller objective function values than the others. The additional steps to accelerate PDCA also contribute to find better solutions. Based on the above observations, we may conclude that $\ell_2$-PDCA and $\ell_2$-APDCA find a good solution for constrained sparse optimization problems with a small amount of computation time.

7 Conclusions

In this paper, we have proposed an efficient DCA to solve $\ell_0$-constrained optimization problems having simple convex constraints. By introducing a new DC representation of the $\ell_0$-constraint, we have reduced the associated subproblem to the projection operation onto the convex constraint set, where the availability of closed-form solutions enables us to implement the operations very efficiently. Consequently, the resulting DCA, called PDCA, still retains the efficiency even if $\ell_0$-constrained optimization problems have some convex constraints. Moreover, we have shown a link between PDCA and Proximal Gradient Method (PGM), which leads to improvement of PDCA; the speed-up techniques proposed for PGM such as the backtracking step size rule and the Nesterov’s acceleration can be applied to PDCA. Indeed, the improved PDCA works very well in numerical experiments, while retaining theoretical properties such as the convergence to a stationary point of the input problem.

The techniques of PGMs have helped to speed up PDCA. There are still a lot of issues that need to be addressed in the future. Among those, some theoretical guarantee such as the convergence rate discussed in Remark is the foremost one that needs to be investigated for nonconvex problem settings including $\ell_0$-constrained optimization problems. Other speed-up techniques for PGMs such as adaptive restart strategy possibly improve the performance of APDCA.
Table 5: Results for sparse nonnegative least squares with synthetic data: the chosen \( \rho \), the cardinality of the found solution, the obtained objective value, CPU time (sec.) and the number of iterations.

| problem size | method | \( \rho \) | cardinality | objective value | time (s) | iteration |
|--------------|--------|------------|-------------|-----------------|---------|-----------|
| 640 \times 180 | \( \ell_2 \)-PDCA | 1 | 20 | 1.48e-01 | 0.1 | 68 |
|                | \( \ell_2 \)-APDCA | 1 | 20 | 1.27e-01 | 0.1 | 102 |
|                | \( \ell_1 \)-DCA | 0.1 | 20 | 1.45e-01 | 1.4 | 3 |
|                | MIP-DCA | 100 | 20 | 4.8e-01 | 3.2 | 3 |
| 1280 \times 360 | \( \ell_2 \)-PDCA | 1 | 40 | 1.36e-01 | 0.2 | 123 |
|                | \( \ell_2 \)-APDCA | 1 | 40 | 1.10e-01 | 0.3 | 125 |
|                | \( \ell_1 \)-DCA | 0.1 | 38 | 1.44e-01 | 7.8 | 4 |
|                | MIP-DCA | 100 | 40 | 4.40e-01 | 17.0 | 3 |
| 1920 \times 540 | \( \ell_2 \)-PDCA | 1 | 60 | 1.36e-01 | 0.4 | 97 |
|                | \( \ell_2 \)-APDCA | 1 | 60 | 1.12e-01 | 0.7 | 146 |
|                | \( \ell_1 \)-DCA | 0.1 | 54 | 1.41e-01 | 22.1 | 4 |
|                | MIP-DCA | 100 | 60 | 2.20e-01 | 71.4 | 3 |
| 2560 \times 720 | \( \ell_2 \)-PDCA | 1 | 80 | 1.33e-01 | 0.9 | 129 |
|                | \( \ell_2 \)-APDCA | 1 | 80 | 1.04e-01 | 1.3 | 156 |
|                | \( \ell_1 \)-DCA | 0.1 | 66 | 1.64e-01 | 43.2 | 3 |
|                | MIP-DCA | 100 | 80 | 2.07e-02 | 123.6 | 2 |
| 3200 \times 900 | \( \ell_2 \)-PDCA | 1 | 100 | 1.16e-01 | 1.8 | 173 |
|                | \( \ell_2 \)-APDCA | 1 | 100 | 9.46e-02 | 2.4 | 184 |
|                | \( \ell_1 \)-DCA | 0.1 | 80 | 1.50e-01 | 78.3 | 3 |
|                | MIP-DCA | 100 | 100 | 4.41e-01 | 201.3 | 2 |

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A Proofs of Propositions

A.1 Proof of Theorem 3

To prove Theorem 3 we provide the following lemma and proposition.

**Lemma 1.** In Algorithm 2, \( l(t) \) is bounded for any \( t \geq 0 \).

**Proof.** From Assumption 2 (a), we have

\[
f(x^{(t+1)}) \leq f(x^{(t)}) + \langle \nabla f(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle + \frac{L}{2} \| x^{(t+1)} - x^{(t)} \|_2^2. \tag{34}\]
Since \( x^{(t+1)} \) is obtained by computing (25), we have

\[
g_1(x^{(t+1)}) \leq g_1(x^{(t)}) - \langle \nabla f(x^{(t)}) - s(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle - \frac{l(t)}{2} \| x^{(t+1)} - x^{(t)} \|^2_2. \tag{35}
\]

It follows from the definition of the subgradient that

\[
g_2(x^{(t+1)}) \geq g_2(x^{(t)}) + \langle s(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle. \tag{36}
\]

Combining (34)–(35), we have

\[
F(x^{(t+1)}) \leq F(x^{(t)}) - \frac{l(t) - L}{2} \| x^{(t+1)} - x^{(t)} \|^2_2. \tag{37}
\]

Therefore, the criterion (23) is satisfied when \( l(t) \geq L + \sigma \) and thus \( l(t) \) is bounded. \( \square \)

**Proposition 2** (20), Proposition 1 in the supplemental of (12). Let \( \{x^{(t)}\} \) and \( \{u^{(t)}\} \) be sequences such that \( x^{(t)} \to x^* \), \( u^{(t)} \to u^* \), \( g_1(x^{(t)}) \to g_1(x^*) \), and \( u^{(t)} \in \partial g_1(x^{(t)}) \). Then we have \( u^* \in \partial g_1(x^*) \).

Now we are ready to prove Theorem 3. It follows from (25) that the sequence \( \{F(x^{(t)})\} \) is nonincreasing. This, together with Assumption 2 (c), implies that \( \lim_{t \to \infty} F(x^{(t)}) \) exists. Thus, by taking limits on both sides of (25), we have

\[
\lim_{t \to \infty} \| x^{(t+1)} - x^{(t)} \|_2 = 0. \tag{38}
\]

In addition, from Assumption 2 (c), the sequence \( \{x^{(t)}\} \) is bounded. Therefore, \( \{x^{(t)}\} \) is a converging sequence, whose limit is denoted by \( x^* \).

From the optimality condition of (25), we have

\[
0 \in \nabla f(x^{(t)}) + l(t)(x^{(t+1)} - x^{(t)}) + \partial g_1(x^{(t+1)}) - s(x^{(t)}),
\]

which is equivalent to

\[
-\nabla f(x^{(t)}) - l(t)(x^{(t+1)} - x^{(t)}) + s(x^{(t)}) \in \partial g_1(x^{(t+1)}). \tag{39}
\]

Since the sequence \( \{s(x^{(t)})\} \) is bounded due to the continuity and convexity of \( g_2 \) and the boundedness of \( \{x^{(t)}\} \), there exists a subsequence \( T \) such that \( s^* := \lim_{t \in T \to \infty} s(x^{(t)}) \) exists. Note that \( s^* \in \partial g_2(x^*) \) due to the closedness of \( \partial g_2 \).

Now we consider the sequence \( \{-\nabla f(x^{(t)}) - l(t)(x^{(t+1)} - x^{(t)}) + s(x^{(t)})\} \) of the left-hand side of (39). From the continuity of \( \nabla f \), the subsequential convergence of \( s(x^{(t)}) \) with respect to \( T \), (38), and Lemma 1, we have

\[
\lim_{t \in T \to \infty} \left( -\nabla f(x^{(t)}) - l(t)(x^{(t+1)} - x^{(t)}) + s(x^{(t)}) \right) = -\nabla f(x^*) + s^*. \tag{40}
\]
Then we will show that \( g_1(x^{(t)}) \to g_1(x^*) \) to apply Proposition 2. We have from (25) that

\[
\langle \nabla f(x^{(t)}) - s(x^{(t)}), x^{(t+1)} \rangle + \frac{l(t)}{2} \| x^{(t+1)} - x^{(t)} \|^2 + g_1(x^{(t+1)}) \leq \langle \nabla f(x^{(t)}) - s(x^{(t)}), x^* \rangle + \frac{l(t)}{2} \| x^* - x^{(t)} \|^2 + g_1(x^*). \tag{41}
\]

Using (41), Lemma 1, the convergence of \( \{x^{(t)}\} \), and the boundedness of \( \{\nabla f(x^{(t)})\} \) and \( \{s(x^{(t)})\} \), we have

\[
\limsup_{t \to \infty} g_1(x^{(t+1)}) \leq g_1(x^*).
\]

Since \( g_1 \) is lower semicontinuous, i.e.,

\[
\liminf_{t \to \infty} g_1(x^{(t+1)}) \geq g_1(x^*),
\]

we have

\[
\lim_{t \to \infty} g_1(x^{(t+1)}) = g_1(x^*). \tag{42}
\]

Finally, using (42), (40), and Proposition 2 for (39), we have

\[
-\nabla f(x^*) + s^* \in \partial g_1(x^*),
\]

which implies

\[
0 \in \{\nabla f(x^*)\} + \partial g_1(x^*) - \{s^*\} \subseteq \{\nabla f(x^*)\} + \partial g_1(x^*) - \partial g_2(x^*). \tag{43}
\]

(End of Proof of Theorem 3)

A.2 Proof of Theorem 5

In the same manner to the proof of Theorem 4 in [12], \( \{x^{(t)}\} \) and \( \{v^{(t)}\} \) can be proven to be bounded and

\[
\sum_{t_1 \in \Omega_1} \|x^{(t_1+1)} - y^{(t_1)}\|^2 + \sum_{t_2 \in \Omega_2} \|v^{(t_2+1)} - x^{(t_2)}\|^2 \leq \frac{c^{(1)} - F^*}{\delta(1-\eta)} < \infty, \tag{44}
\]

where \( F^* \) denotes the optimal value of \( F(x) \). We consider the following three cases.

Case 1: \( \Omega_2 \) is finite. In this case, there exists \( T \) such that (31) is satisfied for all \( t > T \). Thus we have from (44) that

\[
\sum_{t=T}^{\infty} \|x^{(t+1)} - y^{(t)}\|^2 < \infty, \; \|x^{(t+1)} - y^{(t)}\|^2 \to 0. \tag{45}
\]
Since \( \{x^{(t)}\} \) is bounded, we have that \( \{y^{(t)}\} \) is bounded and thus has accumulation points, one of which is denoted by \( y^* \), i.e., there exists a subsequence \( T \) such that

\[
\lim_{t \in T \to \infty} y^{(t)} = y^*.
\]

Then from (13), we have \( x^{(t+1)} \to y^* \) as \( t \in T \to \infty \). From the optimality condition of (27) and \( x^{(t+1)} = z^{(t+1)} \),

\[ 0 \in \{ \nabla f(y^{(t)}) + I_y^{(t)}(x^{(t+1)} - y^{(t)}) \} + \partial g_1(x^{(t+1)}) - \{ s(y^{(t)}) \}. \]

which is equivalent to

\[
-\nabla f(y^{(t)}) - I_y^{(t)}(x^{(t+1)} - y^{(t)}) + s(y^{(t)}) \in \partial g_1(x^{(t+1)}).
\]

(46)

Now similarly to the proof of Theorem 3, we can take a subsequence \( T' \) of \( T \) such that \( s^* := \lim_{t \in T' \to \infty} s(y^{(t)}) \) exists and \( s^* \in \partial g_2(y^*) \). Using this fact, in the same way as the derivation of (13), we have

\[ 0 \in \{ \nabla f(y^*) \} + \partial g_1(y^*) - \partial g_2(y^*). \]

Since \( \|x^{(t+1)} - y^{(t)}\|^2 < 0 \), \( \{x^{(t)}\} \) and \( \{y^{(t)}\} \) have the same accumulation points and thus

\[ 0 \in \{ \nabla f(x^*) \} + \partial g_1(x^*) - \partial g_2(x^*). \]

Case 2: \( \Omega_1 \) is finite. In this case, there exists \( T \) such that (31) is not satisfied for all \( t > T \). Thus we have from (44) that

\[ \sum_{t=1}^{\infty} \|v^{(t+1)} - x^{(t)}\|^2 < \infty, \|v^{(t+1)} - x^{(t)}\|^2 \to 0. \]

(47)

Then similarly to Case 1, for any accumulation point \( x^* \) of \( \{x^{(t)}\} \), we have

\[ 0 \in \{ \nabla f(x^*) \} + \partial g_1(x^*) - \partial g_2(x^*). \]

Case 3: \( \Omega_1 \) and \( \Omega_2 \) are both infinite. In this case, we have

\[ \|x^{(t+1)} - y^{(t)}\|^2 \to 0, \|v^{(t+1)} - x^{(t)}\|^2 \to 0, \]

where \( t_1 \in \Omega_1 \) and \( t_2 \in \Omega_2 \). Since \( \{x^{(t)}\} \) is bounded, \( \{y^{(t_1)}\}_{t_1 \in \Omega_1} \) is also bounded. Now similarly to Cases 1 and 2, any accumulation point \( y^* \) of \( \{y^{(t_1)}\}_{t_1 \in \Omega_1} \) and any accumulation point \( x^* \) of \( \{x^{(n_i)}\}_{n_i \in \Omega_2} \) are stationary points of (17). In addition, \( \{x^{(t+1)}\}_{t_1 \in \Omega_1} \) and \( \{y^{(t_1)}\}_{t_1 \in \Omega_1} \) have the same accumulation point and thus any accumulation point \( x^* \) of \( \{x^{(t+1)}\}_{t_1 \in \Omega_1} \) is also a stationary points of (17). Similarly, any accumulation point \( v^* \) of \( \{v^{(t+1)}\}_{t_2 \in \Omega_2} \) is a stationary points of (17).

(End of Proof of Theorem 5)
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