Local and global robustness in conjugate Bayesian analysis

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Abstract

This paper studies the influence of perturbations of conjugate priors in Bayesian inference. A perturbed prior is defined inside a larger family, local mixture models, and the effect on posterior inference is studied. The perturbation, in some sense, generalizes the linear perturbation studied in Gustafson (1996). It is intuitive, naturally normalized and is flexible for statistical applications. Both global and local sensitivity analyses are considered. A geometric approach is employed for optimizing the sensitivity direction function, the difference between posterior means and the divergence function between posterior predictive models. All the sensitivity measure functions are defined on a convex space with non-trivial boundary which is shown to be a smooth manifold.

Keywords: Bayesian sensitivity; Local mixture model; Perturbation space; Newton’s method; Smooth manifold.

1 Introduction

Statistical analyses are often performed under certain assumptions which are not directly validated. Hence, there is always interest in investigating the degree to which a statistical inference is sensitive to perturbations of the model and data. Specifically, in a Bayesian analysis for which conjugate priors have been chosen the sensitivity of the posterior to prior choice is an important issue. A rich literature on sensitivity to perturbations of data, prior and sampling distribution exists in Cook (1986), McCulloch (1989), Lavine (1991), Ruggeri and Wasserman (1993), Blyth (1994), Gustafson (1996), Critchley and Marriott (2004), Linde (2007) and Zhu, Ibrahim, and Tang (2011).

Sensitivity analysis with respect to a perturbation of the prior, which is the focus of this paper, is commonly called robustness analysis. A comprehensive literature and review of existing methods can be found in Insua and Ruggeri (2000). In robustness analysis it is customary to choose a base prior model and a plausible class of perturbations. The influence of a perturbation is assessed either locally, or globally, by measuring the divergence of certain features of the posterior distribution. For instance, Gustafson (1996) studies linear and non-linear model perturbations, and Weiss (1996) uses a multiplicative perturbation to the base prior and specifies the important perturbations using the posterior density of the parameter of interest. Common global measures of influence include divergence functions (Weiss, 1996) and relative sensitivity (Ruggeri and Sivaganesan, 2000). Note that any analysis highly depend on the selected influence measure, see in particular Sivaganesan (2000).

In local analysis, the rate at which a posterior quantity changes, relative to the prior, quantifies sensitivity (Gustafson, 1996, Linde, 2007, Berger et al., 2000, Gustafson, 1996), which we follow closely, obtains the direction in which a certain posterior expectation has the maximum sensitivity to prior perturbation by considering a mapping from the space of perturbations to the space of posterior expectations. In Linde (2007), the Kullback-Leibler and $\chi^2$ divergence functions are utilized for assessing local sensitivity with respect to a multiplicative perturbation of the base prior or likelihood model. They approximate the local sensitivity using the Fisher information of the mixing parameter in additive and geometric mixing.
In this paper we consider both local, and global, sensitivity analyses with respect to perturbations of a conjugate base prior. We aim for three important properties for our method. Firstly, a well-defined perturbation space whose structure is such that it allows the analyst to select the generality of the perturbation in a clear way. Secondly, we want the space to be tractable, hence we look at convex sets inside linear spaces. Finally we want, in order to allow for meaningful comparisons, the space to be consistent with elicited prior knowledge. Thus if a subject matter expert indicates a prior moment or quantile has a known value – or if a constraint such as symmetry is appropriate – then all perturbed priors should be consistent with respect to this information.

Such an approach to defining the perturbation space extends the linear perturbations studied in Gustafson (1996) in all three ways. We do not require the same positivity condition, rather use one which is more general and returns naturally normalized distributions. Further, our space is highly tractable, due to intrinsic linearity and convexity. Finally it is clear, with our formulation, how to remain consistent with prior information which may have been elicited from an expert. The cost associated with this generalisation is the boundary defined in by (1) in §2.1, and the methods we have developed to work with it. We also can compare our method with the geometric approach of Zhu et al. (2011) which uses a manifold based approach. Our, more linear, approach considerably improves interpretability and tractability while sharing an underlying geometric foundation.

In the examples of this paper we work with our perturbation space in three ways. Similarly to Gustafson (1996) and Zhu et al. (2011) in Example 1 we look for the worst possible perturbation, both locally and globally. In Example 2 we add constraints to the perturbation space, representing prior knowledge, and again look for maximally bad local and global perturbations. Finally, in Example 3, we marginalise over the perturbation space – rather than optimising over it – as a way of dealing with the uncertainty of the prior.

The paper is organized as follows. In Section 2, the perturbation space is introduced and its properties are studied. Sections 3 and 4 develop the theory of local and global sensitivity analysis. Section 5 describes the geometry of the perturbation parameter space and proposes possible algorithms for quantifying local and global sensitivity. In Section 6 we examine three examples. The proofs are sketched in Appendix.

2 Perturbation Space

2.1 Theory and Geometry

We construct a perturbation space using the following definitions (Marriott, 2002; Marriott, 2006; Anaya-Izquierdo and Marriott, 2007). For more details about convex and differential geometry see Berger (1987) and Amari (1990).

Definition 1 For the family of mean parameterized models \( f(x; \theta) \) the perturbation space is defined by the family of models \( f(x; \theta, \lambda) \) such that,

(i) \( f(x; \theta, 0) = f(x; \theta) \) for all \( \theta \).

(ii) \( f(x; \theta_0, \lambda) − f(x; \theta_0) \) is Fisher orthogonal to the score of \( f(x; \theta) \) at \( \theta_0 \).

(iii) For fixed \( \theta \) the \( f(x; \theta_0, \lambda) \) space is affine in the mixture \(-1\) affine geometry defined in Marriott (2002).

A natural way to implement Definition 1 is to extend the family \( f(x; \theta) \) by attaching to it, at each \( \theta_0 \), the subfamily \( f(x; \theta_0, \lambda) \), which is finite dimensional and spanned by a set of linearly independent functions \( v_j(x; \theta_0) \), \( j = 1, \cdots, k \), all Fisher orthogonal to the score of \( f(x; \theta) \) at \( \theta_0 \). Thus, the subfamily \( f(x; \theta_0, \lambda) \) can be defined as the linear space \( f(x; \theta_0) + \sum \lambda_j v_j(x; \theta_0) \), where \( \lambda_j \) is a component of the vector \( \lambda \). For \( f(x; \theta_0, \lambda) \) to be a naturally normalized density, we need two further restrictions: (i) \( \int v_j dx = 0 \), and (ii) the \( \lambda \) parameters must be restricted such that each subfamily is non-negative for all \( x \). This defines the parameter space as

\[
\Lambda_{\theta_0} = \left\{ \lambda \mid f(x; \theta_0) + \sum \lambda_j v_j(x; \theta_0) \geq 0, \text{ for all } x \right\}.
\]
Note the space $\Lambda_\theta_0 \subset R^k$, is an intersection of half-spaces and consequently is convex [Berger [1987], Ch.11).

Clearly, to construct such a perturbation space, the functions $\nu_j$ must be selected. A particular form of Definition 1 with naturally specified $\nu_j$’s is the family of local mixture models. This family is introduced in [Marriott [2002]] as an asymptotic approximation to a subspace of continuous mixture models with small mixing variation relatively to the total variation. Because of this small, or local, assumption, all perturbations are, in some sense, close to the baseline prior, and so any correspondingly large changes in the posterior will be of interest, as we show in the examples.

**Definition 2** The local mixture of a regular exponential family $f(x; \theta)$ of order $k$ via its mean parameterization, $\theta$, is defined as

$$h(x; \lambda, \theta) = f(x; \theta) + \lambda_2 f^{(2)}(x; \theta) + \cdots + \lambda_k f^{(k)}(x; \theta), \quad \lambda \in \Lambda_\theta \subset R^{k-1} \quad (2)$$

where $\lambda = (\lambda_2, \cdots, \lambda_k) \in \Lambda_\theta$ and $f^{(j)}(x; \theta) = \frac{\partial^j}{\partial \theta^j} f(x; \theta)$, $(j = 1, \cdots, k)$. Also, $\Lambda_\theta$, for any fixed and known $\theta$, is a convex space defined by a set of supporting hyperplanes.

For regular exponential family $\int f^{(j)}(x; \theta_0) dx = 0$, and as shown in [Morris [1982]], for natural exponential family, $f^{(j)}(x; \theta_0)$’s are linearly independent and all Fisher orthogonal to the score function at $\theta_0$. This family is identifiable in all parameters, behaves locally similar to genuine mixture models, yet it is richer in the sense that compared to a regular density function with the same mean they can also produce smaller variance. Further properties of these models are studied in [Anaya-Izquierdo and Marriott [2007]].

### 2.2 Prior Perturbation

Suppose the base prior model is $\pi_0(\mu, \theta)$, the probability (density) function of a natural exponential family with the hyper-parameter $\theta$.

**Definition 3** The perturbed prior model corresponding to $\pi_0(\mu, \theta)$ is defined by

$$\pi(\mu, \lambda, \theta) := \pi_0(\mu, \theta) + \sum_{j=2}^{k} \lambda_j \pi_0^{(j)}(\mu, \theta)$$

$$= \pi_0(\mu, \theta) \left\{ 1 + \sum_{j=2}^{k} \lambda_j q_j(\mu, \theta) \right\}, \quad \lambda \in \Lambda_\theta \quad (3)$$

where $\lambda = (\lambda_2, \lambda_3, \cdots, \lambda_k)$ is the perturbation parameter vector, and $q_j(\mu, \theta) = \frac{\pi_0^{(j)}(\mu, \theta)}{\pi_0(\mu, \theta)}$ are polynomials of degree $j$.

In Definition (3), $\pi_0$ is perturbed linearly, similar to the linear perturbation

$$\tau(\cdot, \pi_0, u^*) = \pi_0(\cdot) + u^*(\cdot), \quad u^*(\cdot) > 0$$

studied in [Gustafson [1996]], but with a different positivity condition, and is, as we shall show, very interpretable for applications. Definition (3) can also be seen as the multiplicative perturbation model $\pi(\mu, \lambda, \theta) = \pi_0(\mu, \theta) h^*(\mu, \lambda, \theta)$ studied in [Linde [2007]].

As shown in [Anaya-Izquierdo and Marriott [2007]], the base and perturbed models share the same mean $\theta$; however, the perturbation is implemented through changing the higher order moments by adding linear combinations of $\lambda$. This fact guarantees the properties mentioned in Section (1).

### 3 Local Sensitivity

In this section we study the influence of local perturbations, defined inside the perturbation space, on the posterior mean. Similar to [Gustafson [1996]] we obtain the direction of sensitivity using the Fréchet derivative of a mapping between two normed spaces. Throughout the rest of the paper we denote the sampling density and base prior by $f(x; \mu)$ and $\pi_0(\mu, \theta)$, respectively, and $x = (x_1, \cdots, x_n)$ represents the vector of observations.
Lemma 1 Under the prior perturbation $\xi$, the perturbed posterior model is
\[
\pi_p(\mu, \lambda|x; \theta) = \frac{\pi^0_p(\mu|x, \theta)}{\xi(\lambda, \theta)} \left\{ 1 + \sum_{j=2}^k \lambda_j q_j(\mu, \theta) \right\}, \quad \lambda \in \Lambda_\theta
\]
with $\xi(\lambda, \theta) = 1 + \sum_{j=2}^k \lambda_j E^p_p[q_j(\mu, \theta)] > 0$, where $\pi^0_p(\mu|x, \theta)$ and $E^p_p(\cdot|x)$ are the posterior density and posterior mean of the base model.

The following lemma characterizes the $l^{th}$ moment of the perturbed posterior model. Note that, throughout the rest of the paper, for simplicity of exposition, we suppress the explicit dependence of $\xi, q_j, \pi^0_p$ and $\pi_p$ on $\theta$.

Lemma 2 The moments of the perturbed posterior distribution are given by
\[
E_p(\mu^l|x, \lambda) = \frac{1}{\xi(\lambda)} \left\{ E^0_p(\mu^l) + \sum_{j=2}^k \lambda_j A_j^l(x) \right\}, \quad \lambda \in \Lambda_\theta.
\]
where $A_j^l(x) = E^0_p(\mu^l q_j(\mu)|x)$.

To quantify the magnitude of perturbation we exploit the size function as defined in Gustafson (1996), i.e., the $L^p$ norm of the ratio $\frac{\psi}{\psi_0}$, for $p < \infty$, with respect to the induced measure by $\pi_0$. Accordingly, the size function for $u(\cdot)$ is
\[
\text{size}(u) = \left[ E_{\pi_0} \left( \left| \sum_{j=2}^k \lambda_j q_j(\mu) \right| \right)^p \right]^{\frac{1}{p}},
\]
which, (i) is a finite norm and (ii) is invariant with respect to change of the dominating measure and also with respect to any one-to-one transformation on the sample space. Clearly, size$(u)$ is finite if the first $k + p$ moments of $\pi_0(\mu, \theta)$ exist. In addition, property (ii) holds by use of change of variable formula and the fact that for any one-to-one transformation $m = \nu(\mu)$ we have $\frac{\pi^0(\mu, \theta)}{\pi_0(\mu, \theta)} = \frac{\pi^0(\mu, \theta)}{\pi_0(\mu, \theta)}$.

For a mapping $T : U \rightarrow V$, where $U$ and $V$ are, respectively, the perturbations space normed with size$(\cdot)$, and the space of posterior expectations normed with absolute value, the Fréchet derivative at $u_0 \in U$ is defined by the linear functional $T'(u_0) : U \rightarrow V$ satisfying
\[
||T(u_0 + u) - T(u_0) - T'(u_0)u||_V = o(||u||_U),
\]
in which $T'(u_0)u$ is the rate of change of $T$ at $u_0$ in direction $u$. Let $Cov^0_p(\cdot, \cdot)$ be the posterior covariance with respect to the base model. Theorem 1 expresses $T'(u_0)u$ as a linear function of $\lambda$, at $u_0 = 0$ which corresponds to the base prior model.

Theorem 1 $T'(0)u$ is a linear function of $\lambda$ as
\[
\varphi(\lambda) = \sum_{j=2}^k \lambda_j Cov^0_p(\mu, q_j(\mu)), \quad \lambda \in \Lambda_\theta.
\]

4 Global sensitivity

Here we use two commonly applied measures of sensitivity – the posterior mean difference and Kullback-Leibler divergence function – for assessing the global influence of prior perturbation on posterior mean and prediction, respectively. The following theorem characterizes the difference between the posterior mean of the base and perturbed models as a function of $\lambda$.

Theorem 2 Let $\Psi(\lambda) = E_p(\mu|x, \lambda) - E^0_p(\mu|x)$ represents the difference between the posterior expectations, then
\[
\Psi(\lambda) = \frac{1}{\xi(\lambda)} \varphi(\lambda), \quad \lambda \in \Lambda_\theta.
\]
The function in (8) behaves in a intuitively natural way, for as \( \lambda \to 0 \) we have \( \xi(\lambda) \to 1 \), and consequently \( \Psi(\lambda) \) behaves locally in a similar way to \( \varphi(\lambda) \).

To assess the influence of the prior perturbation on prediction, we quantify the change in the divergence in the posterior predictive distribution.

As a illustrative example, suppose the sampling distribution and the base prior model are respectively \( N(\mu, \sigma^2) \) and \( N(\theta, \sigma_0^2) \). The posterior predictive distribution for the base model is \( N(\mu_{\pi}, \sigma_{\pi}^2 + \sigma^2) \), where

\[
\mu_{\pi} = \frac{\theta \sigma^2 + n \sigma_0^2 \bar{x}}{n \sigma_0^2 + \sigma^2}, \quad \sigma_{\pi}^2 = \frac{\sigma^2 \sigma_0^2}{n \sigma_0^2 + \sigma^2}.
\]

**Lemma 3** The posterior predictive distribution for the perturbed model is

\[
g_p(y) = \frac{1}{\xi(\lambda)} \left\{ g_p^0(y) + \Gamma \sum_{j=2}^k \lambda_j E^*[q_j(\mu)] \right\}
\]

in which, \( g_p^0(y) \) is the posterior predictive density for the base model, \( \Gamma \) is a function of \((y, x, n, \theta_0, \sigma_0^2, \sigma^2)\) and \( E^*[\cdot] \) is expectation with respect to a normal distribution.

For probability measures \( P_0 \) and \( P_1 \) with the same support space, \( S \), and densities \( g_p^0(\cdot) \) and \( g_p(\cdot) \), respectively, Kullback-Leibler divergence functional is defined by,

\[
D_{KL}(P_0, P_1) = \int_S \log \left( \frac{g_p^0(y)}{g_p(y)} \right) g_p^0(y) dy
\]

which satisfies the following conditions (see [Amari](1990)),

1. \( D_{KL}(P_0, P_1) \geq 0 \), with equality if and only if \( P_0 \equiv P_1 \).
2. \( D_{KL}(P_0, P_1) \) is invariant under any transformation of the sample space.

**Theorem 3** Kullback-Leibler divergence between \( g_p^0(\cdot) \) and \( g_p(\cdot) \) as a function of \( \lambda \) is

\[
D_{KL}(\lambda) = \int_S \log \left( \frac{g_p^0(y)}{g_p(y)} \right) g_p^0(y) dy + \log[\xi(\lambda)]
\]

\[- \int_S \log \left( g_p^0(y) + \Gamma \sum_{j=2}^k \lambda_j E^*[q_j(\mu)] \right) g_p^0(y) dy, \quad \lambda \in \Lambda_\theta \]

5 Estimating \( \lambda \)

To obtain the values of \( \lambda \) which find the most sensitive local and global perturbations, as described in Section 1 we apply an optimization approach to the functions (7), (8) and (11). \( \varphi(\lambda) \) is a linear function of \( \lambda \) on the space \( \Lambda_\theta \) which presents the directional derivative of the mapping \( T \) at \( \lambda = 0 \). Thus, for obtaining the maximum direction of sensitivity, called the worst local sensitivity direction in [Gustafson](1996), we need to maximize \( \varphi(\lambda) \) over all the possible directions at \( \lambda = 0 \) restricted by the boundary of \( \Lambda_\theta \). However, \( \Psi(\lambda) \) and \( D_{KL}(\lambda) \) are smooth objective functions on the convex space \( \Lambda_\theta \), for which we propose a suitable gradient based constraint optimization method that exploits the geometry of the parameter space. By Definition 2 for a fixed known \( \theta \), the space \( \Lambda_\theta \) is a non-empty convex subspace in \( R^{k-1} \) with its boundary obtained by the following infinite set of hyperplanes

\[
\mathcal{H} = \left\{ \lambda \mid 1 + \sum_{j=2}^k \lambda_j q_j(\mu) = 0 ; \mu \in R \right\}.
\]

Specifically, for the normal example with order \( k = 4 \), \( \mathcal{H} \) is the infinite set of planes of the form

\[
P_\lambda(z) = \left( z^2 - \frac{1}{\sigma_0^2} \right) \lambda_2 + \left( z^3 - \frac{3z}{\sigma_0^2} \right) \lambda_3 + \left( z^4 - \frac{6z^2}{\sigma_0^2} + \frac{3}{\sigma_0^4} \right) \lambda_4 + 1.
\]

where \( z = \frac{\mu - \theta}{\sigma_0} \). Lemma 4 describes the boundary of \( \Lambda_\theta \) as a smooth manifold.
Lemma 4 The boundary of $\Lambda_\theta$ is a manifold (smooth surface) embedded in $\mathbb{R}^3$ Euclidean space. In addition, the interior of $\Lambda_\theta$, which guarantees positivity of $\pi(\mu, \lambda; \theta)$ for all $\mu \in \mathbb{R}$, can be characterized by the necessary and sufficient positivity conditions of general polynomials of degree four. The corresponding polynomial to Equation (12) is a quartic with highest degree coefficient $\lambda_4$; hence, the necessary positivity condition is $\lambda_4 > 0$. Also the comprehensive necessary and sufficient conditions are given in Barnard and Child (1936) and Bandy (1966). Throughout the rest of the paper we let $k = 4$, as it gives a perturbation space which is flexible enough for our analysis and it has been illustrated in Marriott (2002), through examples, that simply increasing the order of local mixture models does not significantly increase flexibility. However, all the results and algorithms can be generalized to higher dimensions with possible generalization of the positivity conditions on polynomials with higher degrees.

Lemma 5 $\varphi(\lambda)$ attains its maximum value at the gradient direction $\nabla \varphi$ if it is feasible; otherwise, the maximum direction is the direction of the orthogonal projection of $\nabla \varphi$ onto the boundary plane corresponding to $\lambda_4 = 0$.

$D_{KL}(\lambda)$ and $\Psi(\lambda)$ are smooth functions which can achieve their maximum either in the interior or on the smooth boundary of $\Lambda_\theta$. Therefore, optimization shall be implemented in two steps: searching the interior using regular Newton-Raphson algorithm, and then searching the boundary using a generalized form of Newton-Raphson algorithm on smooth manifolds, see Shub (1986) and also Maroufy and Marriott (2015).

6 Examples

We consider three examples, where the first two study local and global sensitivity in the normal conjugate model using the optimization approaches developed earlier to address the questions in Section 1. In the last example, we address sensitivity analysis in finite mixture models with independent conjugate prior models for all parameters of interest. Rather than using an optimization approach, for this example a Markov Chain Monte Carlo method is used and sensitivity of the posterior distribution of each parameter is assessed. For demonstrating the effect of the perturbation obtained in each example we compare the posterior distributions before and after perturbation and also use the relative difference between the Bayes estimates defined by

$$d = \frac{|E_p^0(\mu) - E_p^\lambda(\mu)|}{std_p^0(\mu)}$$

in which $E_p^0(\mu)$ and $E_p^\lambda(\mu)$ are the Bayes estimates with respect to the base and perturbed models, respectively, and $std_p^0(\mu)$ is the posterior standard deviation under the base model. Since $\Psi(\lambda)$ also allows negative values, care must be taken as we may need to minimize this function instead of maximizing it for achieving the maximum discrepancy between the posterior distributions.

Example 1 (Normal conjugate) A sample of size $n = 15$ is taken from $\mathcal{N}(1, 1)$, and the base prior is $\mathcal{N}(2, 1)$. The estimate $\hat{\lambda}_D = (1.821, -0.014, 0.482)$ and $\hat{\lambda}_\Psi = (1.817, -0.009, 0.486)$ are obtained from maximizing $D_{KL}(\lambda)$ and minimizing $\Psi(\lambda)$, respectively. The corresponding relative discrepancies in Bayes estimate are respectively $d = 1.19, 1.2$; that is, the resulted biases in estimating posterior expectation are respectively 119% and 120% of the posterior standard deviation of the base model. Also, the corresponding posterior distributions are plotted in Figure 1. Considering the fact that we construct the perturbation space as a family of local mixture models which are close to the base prior model, these maximum global perturbations are obtained by searching over a reasonably small space of prior distributions which only different from the base prior by their tail behaviour. Hence, these results imply that although conjugate priors are convenient in applications, they might cause significant bias in estimation as a result of even plausibly small prior perturbations.
Example 1, where the estimate of $\lambda$ restricted the perturbation space further, there are still noticeable discrepancies in posterior den-

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tion returns $d_\theta$ for different values of $\alpha > 0$, as well as the boundary point $\lambda_0$ in direction of $\hat{\lambda}_\varphi$. The correspond-

ing relative differences in posterior expectation are $d = 0.1, 0.16, 0.25, 0.38, 0.49, 0.56$. Hence, additional to obtaining the worst direction, these values suggest that how far one can perturb the base prior along the worst direction so that relative discrepancy in posterior mean estimation is less than, say 50%.

![Figure 1](image1.png)

Figure 1: (a) sample, (b) and (c) posterior densities of based models and perturbed model (dashed) corresponding to $\lambda_D$ and $\hat{\lambda}_\varphi$ respectively.

For local analysis, we obtained the unit vector $\hat{\lambda}_\varphi$ which maximizes the directional derivative $\varphi(\lambda)$. Figure 2 presents the posterior density displacement by perturbation parameter $\lambda_\alpha = \alpha \hat{\lambda}_\varphi$ for different values of $\alpha > 0$, as well as the boundary point $\lambda_0$ in direction of $\hat{\lambda}_\varphi$. The corresponding relative differences in posterior expectation are $d = 0.1, 0.16, 0.25, 0.38, 0.49, 0.56$. Hence, additional to obtaining the worst direction, these values suggest that how far one can perturb the base prior along the worst direction so that relative discrepancy in posterior mean estimation is less than, say 50%.

![Figure 2](image2.png)

Figure 2: (a)-(e) posterior densities of based models and perturbed model (dashed) corresponding to $\lambda = \alpha \hat{\lambda}_\varphi$ where $\alpha = 0.05, 0.07, 0.1, 0.13, 0.15$ and (f) for boundary point in direction of $\hat{\lambda}_\varphi$.

Example 2 The central moments of the perturbed prior model, in Definition (3), are linearly related to the perturbation parameter $\lambda$. Specifically, for the normal model we can check that

$$
\hat{\mu}_2^{(2)} = \sigma^2 + 2\lambda_2, \quad \hat{\mu}_3^{(3)} = 6\lambda_3, \quad \hat{\mu}_4^{(4)} = \hat{\mu}_4^{(0)} + 12\sigma^2\lambda_2 + 24\lambda_4
$$

(13)

where $\hat{\mu}_j^{(j)}$ represents the $j$th central moment with respect to density $\pi$. Clearly, $\lambda_2$ modifies variance, $\lambda_3$ adds skewness, and $\lambda_4$ adjusts the tails.

Suppose that elicited prior knowledge tells us that the perturbed prior is required to stay symmetric, then the perturbation space must be modified by the extra restriction $\lambda_3 = 0$, which gives zero skewness. Consequently, we should be exploring the restricted space, say $\Lambda_\theta^0$, instead of $\Lambda_\theta$, for the worst direction and maximum global perturbation. $\Lambda_\theta^0$ is a 2-dimensional cross section obtained from intersection of $\Lambda_\theta$ with the plane defined by $\lambda_3 = 0$. Hence the boundary properties are preserved. For the same data in Example 2 sensitivity in the worst direction returns $d = 0.1, 0.16, 0.26, 0.42, 0.57, 0.64$ (Figure 3). Also, minimizing $\Psi(\lambda)|_{\lambda_3=0}$ returns $\hat{\lambda}_\varphi^0 = (1.837, 0.494)$.

Two observations can be made from these results. First, as in Example 2 although we have restricted the perturbation space further, there are still noticeable discrepancies in posterior densities caused by perturbation along the worst direction. Second, the results agree with that in Example 2, where the estimate of $\lambda_3$ does not seem to be significantly different from zero, and the rest of two parameter estimates are quite close in both examples.
Example 3 (Finite Mixture) Using a missing value formulation, the likelihood function of the mixture model \( \rho \mathcal{N}(x; \mu_1, \sigma_1) + (1 - \rho) \mathcal{N}(x; \mu_2, \sigma_2) \) can be written as follows

\[
L = \prod_{j=1}^{2} \rho^{n_j} \prod_{i \in A_j} \phi(x_i; \mu_j, \sigma_j),
\]

where \( A_j = \{i | w_i = j\} \), and \( w_i \) is the latent missing variable for \( x_i \) such that \( p(w_i = 1) = \rho \), and \( p(w_i = 2) = 1 - \rho \). The marginal conjugate base prior models are \( \mu_j \sim \mathcal{N}(\theta_j, \sigma_{0j}) \), \( \sigma_j^{-2} \sim \Gamma(k_j, \tau_j) \), and \( \rho \sim \text{Beta}(\alpha, \beta) \), \( (j = 1, 2) \).

In this example the base prior model can be split into five independent components and, correspondingly, five independent perturbation spaces are naturally defined. Unlike previous examples, where we find the maximum local and global perturbations, here we explore each marginalized perturbation space by generating perturbation parameters and observing their influence on the posterior model of parameters of interest.

Specifically, we use Markov Chain Monte Carlo Gibbs sampling for estimating the marginal posterior distribution of all parameters of interest, corresponding to the base and perturbed models. Each perturbation parameter is generated, independently from the rest, through a Metropolis algorithm with a uniform proposal distribution. Figure 4 shows the histograms of generated samples for an observed data set of size \( n = 15 \) from \( 0.4 \mathcal{N}(x; -1, 1) + 0.6 \mathcal{N}(x; 1, 1) \), and the hyper-parameters are set to be \( \theta_1 = -1.5 \), \( \theta_2 = 0.5 \), \( \tau_1 = \tau_2 = 1 \), \( k = 2 \) and \( \alpha = \beta = 1 \). Comparing the two histograms for each parameter, the posterior models for \( \mu_1 \) and \( \rho \) show significant differences between the base and perturbed models. The marginal relative differences are \( d = 0.49, 0.11, 0.40, 0.59, 0.71 \), respectively for \( (\rho, \mu_1, \mu_2, \sigma_1, \sigma_2) \). These differences are not as significant as those in the previous examples for since they do not correspond to maximum perturbations; instead, they return the average influences over all generated perturbation parameter values.

The examples of this paper have explored the perturbation space in three ways. In Example 1 we look for the worst possible perturbation, both locally and globally. In Example 2 we add constraints to the perturbation space, representing prior knowledge, and again look for maximally bad local and global perturbations. Finally, in Example 3, we marginalise over the perturbation space – rather than optimising over it – as a way of dealing with the uncertainty of the prior.

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Figure 4: First row: estimates from the base model; second row: estimates form the perturbed model

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**Appendix**

**Proof 1 (Lemma 1)**

\[
\pi_p(\mu | x, \lambda) = \frac{\pi(\mu, \lambda) f(x; \mu)}{g(x, \lambda)}
\]  

where

\[
g(x, \lambda) = \int \pi(\mu, \lambda; \theta) f(x; \mu) d\mu = \int f(x; \mu) \pi_0(\mu; \theta) d\mu + \sum_{i=2}^k \lambda_j \int q_j(\mu, \theta) f(x; \mu) \pi_0(\mu; \theta) d\mu
\]

\[
= g(x) \left\{1 + \sum_{i=2}^k \lambda_j E_0^p[q_j(\mu, \theta)]\right\}
\]  

(15)

Since \(f(x; \mu) \pi_0(\mu; \theta) = g(x) p_0^0(\mu | x, \theta)\) and \(g(x) = \int f(x; \mu) \pi_0(\mu; \theta) d\mu\) where, \(g(x)\) is the marginal density of sample in the base model. Hence,

\[
\pi_p(\mu, \lambda | x; \theta) = \frac{f(x; \mu) \pi_0(\mu; \theta) \left\{1 + \sum_{i=2}^k \lambda_j q_j(\mu, \theta)\right\}}{g(x) \left\{1 + \sum_{i=2}^k \lambda_j E_0^p[q_j(\mu, \theta)]\right\}}
\]

\[
= \frac{\pi_0^0(\mu | x, \theta)}{\xi(\lambda, \theta)} \left\{1 + \sum_{i=2}^k \lambda_j q_j(\mu, \theta)\right\}, \quad \lambda \in \Lambda_{\theta}
\]

with

\[
\xi(\lambda, \theta) = 1 + \sum_{i=2}^k \lambda_j E_0^p[q_j(\mu, \theta)].
\]

Also \(\xi(\lambda, \theta) > 0\), since \(h^*(\mu; \lambda, \theta) > 0\), for all \(\mu \in \mathbb{R}\) and \(\lambda \in \Lambda_{\theta}\), and \(\xi(\lambda, \theta) = E_0^p(h^*(\mu; \lambda, \theta))\).

**Proof 2 (Lemma 2)** Result follows by direct calculation and using the fact that,

\[
A_{j}^l(x) := \int \mu^l q_j(\mu) \pi_{\text{post}}^0(\mu | x) d\mu = E_{p}^0[\mu^l q_j(\mu)]
\]  

(16)

**Proof 3 (Theorem 1)** Substitute \(u^*(\cdot)\) by \(u(\cdot)\) in (Gustafson, 1996, Result 8).

**Proof 4 (Theorem 2)** By direct calculation and use of equation (16).
Proof 5 (Lemma 3)\\
\[ g_p(y) = \int f(y; \mu) \pi_p(\mu, \lambda | x) \, d\mu \] \quad (17)\\
is the convolution of \( N(\mu, \sigma^2) \) and \( N(\mu_\pi, \sigma_\pi^2) \). Since,

\[
\left( \frac{y - \mu}{\sigma^2} \right)^2 + \left( \frac{\mu - \mu_\pi}{\sigma_\pi^2} \right)^2 = \left( \frac{\mu \sigma_\pi^2 + \sigma^2 \mu_\pi}{\sigma^2 + \sigma_\pi^2} \right)^2 + \left( \frac{y - \mu_\pi}{\sigma^2 + \sigma_\pi^2} \right)^2
\]
hence, the posterior predictive distribution for base model is \( N(\mu_\pi, \sigma^2_\pi + \sigma^2) \) and (9) is obtained by direct calculation, where,

\[ \Gamma = \frac{1}{\sqrt{2\pi(\sigma^2_\pi + \sigma^2)}} \exp \left\{ -\frac{(y - \mu_\pi)^2}{2(\sigma^2_\pi + \sigma^2)} \right\} \]
and \( E^* (\cdot) \) is expectation with respect to \( \mu \) according to the following normal distribution

\[ N \left( \frac{\sigma^2_\pi y + \sigma^2 \mu_\pi}{\sigma^2_\pi + \sigma^2} , \frac{\sigma^2_\pi \sigma^2}{\sigma^2_\pi + \sigma^2} \right) \]

Proof 6 (Theorem 3) Use of Lemma 3 and direct calculation finishes the proof.

Proof 7 (Lemma 4) Let \( \sigma_0 = 1 \) in equation (12) for convenience and fix \( \lambda_4 \). From solving \( P_\lambda(z) = 0 \) and \( P_\lambda'(z) = 0 \), simultaneously for \( \lambda_2 \) and \( \lambda_3 \), we get a smooth parametrization for the boundary as follows

\[
\begin{align*}
\lambda_2(z) &= \lambda_4 \frac{z^6 - 3z^4 + 6z^2 + 9}{3z^3 + 3} \\
\lambda_3(z) &= \frac{2z[1 - (z^3 - 2z^2 + 3)\lambda_4]}{z^4 + 3}
\end{align*}
\]
(18)

Hence, by implicit function theorem (Rudin 1976, p.225) the boundary of \( \Lambda_{\theta_0} \) is a smooth surface (Manifold) embedded in \( R^3 \) by

\[
S_1 (z, \lambda_4) = [\lambda_2(z, \lambda_4), \lambda_3(z, \lambda_4), \lambda_4]
\]
(19)

Proof 8 (Lemma 5) \( \nabla \varphi = (a_2, a_3, a_4) \), is a vector originated at \( \lambda = 0 \), where \( a_j = \text{Cov}_p(\mu, q_j(\mu)) \). If it is feasible then clearly gives the maximum direction. However, if it is not feasible then \( a_4 \leq 0 \) since the condition \( a_4 > 0 \) is necessary for feasibility. Thus, the direction of the orthogonal projection of \( \nabla \varphi \) onto the boundary plane corresponding to \( \lambda_4 = 0 \) is the closest we get to a maximum and feasible direction.