Ergodic Energy Management Leveraging Resource Variability in Distribution Grids

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Abstract—Distribution grids are critically challenged by the variability of renewable energy sources. Slow response times and long energy management periods cannot efficiently integrate intermittent renewable generation and demand. Yet stochasticity can be judiciously coupled with system flexibilities to improve efficiency of the grid operation. Voltage magnitudes for instance can transiently exceed regulation limits, while smart inverters can be overloaded over short time intervals. To implement such a mode of operation, an ergodic energy management framework is developed here. Considering a distribution grid with distributed energy sources and a feed-in tariff program, active power curtailment and reactive power compensation are formulated as a stochastic optimization problem. Tighter operational constraints are enforced in an average sense, while looser margins are satisfied at all times. Stochastic dual subgradient solvers are developed based on exact and approximate grid models of varying complexity. Numerical tests on a real-world 56-bus distribution grid relying on both grid models corroborate the advantages of the novel schemes over its deterministic alternative.

Index Terms—Energy management control, reactive power compensation, active power curtailment, stochastic optimization.

I. INTRODUCTION

Renewable generation systems (such as photovoltaics and micro-wind turbines) are being increasingly installed in medium- and low-voltage distribution grids. Distinct from transmission networks, where bus voltage magnitudes are relatively insensitive to active power injections, integrating renewable generation in distribution grids can cause voltage fluctuations [1], [2]. On a clear day, high solar generation could easily exceed local demand and give rise to reverse power flows. Intermittent photovoltaic (PV) generation on a cloudy day or overnight electric vehicle charging could lead to serious voltage fluctuations. Installing distributed generation (DG) units of various ratings at different locations in a distribution grid further challenges voltage profiles.

Energy management control (EMC) is viewed here as the task of using smart inverters for joint reactive power compensation and active power curtailment towards varying objectives (power loss minimization under voltage constraints, conservation voltage reduction, or voltage regulation). Traditionally, voltage has been regulated by under-load tap changers (ULTCs), switched capacitor banks, step voltage regulators, and static var compensators [3]. However, these devices exhibit slow response times and their lifetime is severely decreased by frequent adjustments. On the other hand, recent research efforts have focused on the possibility of engaging smart inverters in EMC of distribution grids [2], [4], [5], [6], [7].

Albeit prohibited by current standards [7], the power inverters found in distributed generators and electric vehicles can be commanded to provide reactive injections [4]. Using approximate grid models, voltage regulation is effected through a multi-agent scheme in [8], a learning approach is proposed in [9], and local control algorithms are devised in [5]. Building on the exact full AC grid model, reactive power control is an instance of the optimal power flow (OPF) problem, which is non-convex in general [10]. Recently, different convex relaxations have been proposed; see [11] for a review. In radial networks, the OPF can be relaxed into a second-order cone program (SOCP) via either polar coordinates [12], or the branch flow model [13]; or into a semidefinite program (SDP) [14]. Although the two relaxations have been shown to be equivalent, [15] advocates using the SOCP one due to its lower computational complexity. Leveraging the SOCP relaxation, a two timescale conventional and inverter-based reactive power control has been formulated in [16]. Accounting for stochasticity, an adaptive local control algorithm for single-branch radial grids is developed in [17], and a stochastic centralized approach has been pursued in [18].

Apart from exploiting the reactive power capabilities of smart inverters, active power curtailment has been advocated as an ancillary service as well [19], [20], [21], [22]. Droop-based active power curtailment has been proposed as an efficient means for overvoltage prevention [19]. In [20], an SDP-based relaxation has been devised for jointly commanding active and reactive power setpoints to inverters in multi-phase distribution grids. Leveraging joint reactive power compensation and active power curtailment, a multi-objective OPF is formulated for unbalanced four-wire distribution grids in [21]. Local voltage control strategies are developed for customers enrolled in a feed-in tariff (FIT) policy [22]. FIT is an energy supply policy that compensates DG owners for the surplus of renewable energy they inject into the grid. Such policies or similarly structured programs have been successful around the world, particularly in Europe and in several states in the US [23].

Existing EMC schemes enforce inverter- and voltage
regulation-related constraints at all times. Nevertheless, the operation of future grids could benefit from currently unexploited system flexibilities. Two possible options are the flexibility regarding the overloading capability of smart inverters, and the voltage regulation margins. Specifically, the inverters found in DG units and storage units are empirically designed to work at even 1.2-1.5 times higher their nameplate apparent power rating. This smart inverter overloading capability has already been exploited in panel designs [24]. Moreover, most voltage regulation standards, such as the American National Standard Institute (ANSI) C84.1 [25] and the EN 50160 standard [26], define two utilization voltage regions: one for normal operations and one whose use is limited in frequency and duration. According to the EN 50160 standard, for example, nodal voltage magnitudes are required to lie in the latter region for 95% of any 10-minute sample [26].

In this context, this work proposes an EMC scheme where voltage regulation and inverter capacity constraints are imposed in a probabilistic rather than a deterministic sense. Our contribution is a stochastic framework exploiting the two aforementioned sources of flexibility to lower energy management costs, and thus, to better integrate the renewables. Assuming an FIT program, reactive power compensation and active power curtailment are considered as ancillary services. Per control interval, a central controller collects predictions for prices, loads, and the available renewable generation. After modeling predictions as ergodic random processes, our novel stochastic dual subgradient schemes operate on either exact or approximate grid models depending on the computational complexity that can be afforded. The schemes rely solely on data to command DG unit set-points with well-studied convergence and feasibility guarantees. Numerical tests using synthetic and real data on an industrial 56-bus distribution grid corroborate the efficacy of the novel EMC schemes.

Paper outline. The rest of the paper is outlined as follows. The novel EMC problem is formulated in Section II. An ergodic optimization approach is presented in Section III while a stochastic approximation solver is developed in Section IV. The implementation of the solver depending on two grid models is presented in Section V while performance advantages over the deterministic alternatives are supported by the numerical tests of Section VI. Concluding remarks are drawn in Section VII.

Notation. Lower- (upper-) case boldface letters denote column vectors (matrices), with the exception of power flow vectors (P, Q). Calligraphic symbols are reserved for sets, while $\mathbb{R}^N$ denotes the set of all nonnegative N-dimensional vectors; the symbol $\top$ stands for transposition; and $\mathbf{0}$ and $\mathbf{1}$ denote the all-zeros and all-ones vectors, respectively.

II. Problem Formulation

Consider a radial distribution grid consisting of $N + 1$ buses, modeled by a tree graph $T := (\mathcal{N}_o, \mathcal{L})$, where $\mathcal{N}_o := \{0, 1, \ldots, N\}$ is the set of nodes (buses) and $|\mathcal{L}| = N$ denotes the cardinality of the edge set $\mathcal{L}$. The tree is rooted at the substation bus indexed by $n = 0$, and all non-root buses comprise the set $\mathcal{N} := \{1, \ldots, N\}$. Let $v_n$ be the squared voltage magnitude at bus $n$, and $p_n + jq_n$ the complex power injection into bus $n$ for all $n \in \mathcal{N}_o$. For notational brevity, nodal quantities related to non-root buses are stacked to form column vectors $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{q}$.

Active and reactive power injections are split into their generation and consumption components as $p_n := p_n^g - p_n^c$ and $q_n := q_n^g - q_n^c$. For a purely load bus, the consumption components $(p_n^c, q_n^c)$ are oftentimes related via a constant power factor, whereas $p_n^g = q_n^g = 0$. A DG bus in addition to the nonnegative components $(p_n^g, q_n^g)$, it also provides renewable generation $p_n^g \geq 0$ and reactive support $q_n^g$. For buses having only a shunt capacitor, it holds that $p_n^g = p_n^c = q_n^g = 0$ and $q_n^g > 0$. Generation and consumption components are also collected in vectors $\mathbf{p}_r^g$, $\mathbf{p}_r^c$, $\mathbf{q}_r^g$, and $\mathbf{q}_r^c$, accordingly.

In the envisioned scenario, the grid operation is divided into short control periods indexed by $t$. The duration of these periods depends on the variability of active and reactive power consumption and the availability of data predictions. Without loss of generality, a 30-sec control interval will be presumed hereafter. During time period $t$, the grid operator can buy or sell energy $p_{0,t}$ from or to the main grid through the substation bus. The price for this energy exchange is $\pi_{0,t}$, and it is assumed to be positive at all times. Internally in the distribution grid, customers with renewable generation units, such as PVs or wind micro-turbines, can subscribe to a so termed FIT program [22]. According to this program, the surplus of renewable energy a customer may have can be bought if deemed appropriate by the utility company at the FIT price $\pi_{f,t}$. FIT prices are adjusted on a monthly basis, and are also assumed to be positive. Energy consumption from both FIT and regular customers is charged at a retail price $\pi_{r,t}$. The energy cost for customer $n$ during period $t$ is:

$$\pi_{r,t}[p_{n,t} - p_{n,t}] - \pi_{f,t}[p_{n,t} - p_{n,t}]^+$$

where the operators $[a]^+ := \max\{a, 0\}$ and $[a]^− := \max\{0, -a\}$ [27]. Apparently, at most one of the two summands in (1) is nonzero per slot $t$.

From the utility side, the energy cost for slot $t$ is

$$\pi_{0,t} p_{0,t} + \pi_{f,t} [p_t^g - p_t^c]^+ + \pi_{r,t} [p_t^g - p_t^c]^−$$

where $[a]^+$ and $[a]^−$ for vectors are applied entry-wise now. (1)

Note that the energy exchange with the main grid $p_{0,t}$ depends on the internal consumption and generation and the associated power losses on distribution lines. Thus, the energy exchange $p_{0,t}$ can be thought of as a function of the control variables $(p_t^g, q_t^g)$, while its dependence on $(p_t^c, q_t^c)$ and grid power losses is implicitly indicated by the subscript $t$.

The EMC task is to minimize the operational cost for the utility by performing joint reactive power compensation and active power curtailment. The energy management controller is run centrally by the utility, and communicates set-points to DG units. If the EMC were to minimize the utility’s cost in (2), it would force the minimum possible local generation. To see that, consider a node $n$ where at time $t$ the demand is higher than the installed solar capacity; hence, $[p_{n,t} - p_{n,t}]^− = p_{n,t} - p_{n,t} < 0$. Then, the utility EMC would command $p_{n,t} = 0$ unless there is an under-voltage condition. Such a policy contradicts the purpose of a FIT program. The FIT
program should curtail renewable power only if a customer has a surplus and the surplus cannot be bought due to either financial, or voltage regulation reasons. To accommodate the FIT terms, the utility does not curtail renewable generation when the net injection is negative. Thus, the cost to be minimized by the EMC is $\pi_{0,t} p_{0,t}(p^g_t, q^g_t) + \pi_{f,t} 1^T [p^c - p^c_t]_+$ rather than that in (4).

Operation of the EMC is detailed next. Before control period $t$, the EMC collects predictions for prices ($\pi_{0,t}$, $\pi_{f,t}$), loads ($p^c_t$, $q^c_t$), and the maximum renewable generation $P^g_t$. At every period $t$, buses are partitioned to those having a renewable energy surplus comprising the set

$$\mathcal{N}_t := \{ n \in \mathcal{N} : \mathcal{P}^g_{n,t} \geq p^c_{n,t} \}$$

and to those buses belonging to the complementary set of $\mathcal{N}_t$ denoted by $\overline{\mathcal{N}}_t$. To jointly perform active curtailment and reactive power management, the EMC could solve the ensuing problem for each time interval $t$:

$$J^*_t := \min_{p^c_t, q^c_t} \pi_{0,t} p_{0,t}(p^g_t, q^g_t) + \pi_{f,t} 1^T [p^c - p^c_t]_+$$

s.t. $0 \leq p^g_{n,t} \leq p^g_{n,t}, \forall n \in \mathcal{N}_t$ (4b)

$\mathcal{P}^g_{n,t} = p^g_{n,t}, \forall n \in \overline{\mathcal{N}}_t$ (4c)

$\{(p^g_{n,t})^2 + (q^g_{n,t})^2 \leq s^2_n, \forall n \}$ (4d)

$|v| \leq \nu_{n,t} (p^c_t, q^c_t) \leq \nu_0, \forall n$ (4e)

Power injections $\{(p^g_{n,t}, q^g_{n,t})\}_n$ are constrained in the feasible set defined by (4b)-(4e). Constraints (4b)-(4d) apply locally per bus $n$, whereas the voltage constraints in (4e) couple multiple injections across the grid. Specifically, the term $\mathcal{P}^g_{n,t} - p^g_{n,t}$ in (4b) is the active power curtailed by inverter $n \in \mathcal{N}_t$. Constraint (4d) originates from the maximum apparent power capability (nameplate rating) $s_n$ of inverter $n$. Constraint (4e) maintains the squared voltage magnitudes in the prescribed interval $\mathcal{V} := [v_0, \nu_0]$ at all nodes. Even though apparent power limits on distribution lines could be imposed as well, they are not considered here to simplify the exposition. Similar to the energy exchange $p_{0,t}(p^g_t, q^g_t)$, voltage magnitudes are expressed as implicit functions of $(p^g_t, q^g_t)$; the actual forms of these functions depend on the postulated grid model and will be elaborated in Section V.

Problem (4) guarantees that all power and voltage constraints are satisfied at all times. Nevertheless, future distribution grids will afford flexibilities that can be leveraged to lower operational costs and better integrate renewables. Two possible sources of flexibility are the overload capability of smart inverters and the voltage regulation ranges. Regarding the former, a grid-tied power inverter can exceed its apparent power capacity for a short period of time. Indeed, power electronics are empirically designed to operate at even 1.2-1.5 times higher than their nameplate rating [24]. For the latter, instead of requiring the squared voltage magnitudes to lie in $\mathcal{V}$ at every $t$, it suffices for their time-averages to lie in $\mathcal{V}$, and the instantaneous values to lie within a wider range. For instance, according to the standard EN 50160, voltages are required to stay in $\mathcal{V}$ for 95\% of any 10-minute sample [26]. Additionally, heed that problem (4) depends on predictions ($p^k_t, q^k_t, p^k_t, \pi_{0,t}, \pi_{f,t}$), and prices ($\pi_{0,t}, \pi_{f,t}$). It is therefore optimal only if load demand, renewable generation, and prices are perfectly known. In practice though ($p^k_t, q^k_t, p^k_t, \pi_{0,t}, \pi_{f,t}$) involve uncertainties (e.g. measurement noise, time-delay, and system variability) rendering the solution of (4) hardly optimal.

To leverage operational flexibilities and cope with uncertainties, a stochastic rather than the deterministic EMC formulation of (4) is pursued next. To that end, the parameters $\{p^k_t, q^k_t, p^k_t, \pi_{0,t}, \pi_{f,t}\}_t$ are modeled as stochastic processes [28, 29, 30]. To capture ensemble averages via time averages, the aforementioned stochastic processes are assumed stationary and ergodic, yet not necessarily independent across time; see [31] and [32]. The novel EMC scheme entails solving the following stochastic optimization:

$$J_2^* := \min_{p^c_t, q^c_t} \mathbb{E} \left[\pi_{0,t} p_{0,t}(p^g_t, q^g_t) + \pi_{f,t} 1^T [p^c - p^c_t]_+\right]$$

s.t. $0 \leq p^g_{n,t} \leq p^g_{n,t}, \forall n \in \mathcal{N}_t$ (5b)

$p^g_{n,t} = p^g_{n,t}, \forall n \in \overline{\mathcal{N}}_t$ (5c)

$|p^g_{n,t}|^2 + |q^g_{n,t}|^2 \leq \pi^2_n, \forall n \in \mathcal{N}_t$ (5d)

$|v| \leq \pi_{n,t} (p^c_t, q^c_t) \leq \nu_0, \forall n$ (5e)

$\mathbb{E} \left[(p^g_{n,t})^2 + (q^g_{n,t})^2\right] \leq s^2_n, \forall n$ (5f)

$|v| \leq \mathbb{E} [v_{n,t}(p^c_t, q^c_t)] \leq \nu_0, \forall n$ (5g)

where the optimization variables consist of $(p^g_t, q^g_t)$ for all periods $t$, and the expectations are taken over the joint distribution of $(p^k_t, q^k_t, p^k_t, \pi_{0,t}, \pi_{f,t})$ across all periods $t$. Constraint (5i) guarantees that the average apparent power complies with the nameplate inverter capacity for all buses; while constraint (5a) enforces a hard limit $\pi_n (\pi_n \geq s_n)$ on the instantaneous apparent power for all $n$. Similarly, the averages of squared voltage magnitudes are maintained in $\mathcal{V}$ according to (5g), whereas constraint (5a) ensures that their instantaneous values lie in a region $\mathcal{V}' := [\nu_0, \pi_0]$ with $\mathcal{V} \subseteq \mathcal{V}'$. For example, the ANSI C84.1 requires voltage magnitudes to lie within $\mathcal{V} = [0.95^2, 1.05^2]$ per unit (p.u.) of normal operation, but within $\mathcal{V}' = [0.917^2, 1.058]^2$ p.u. over short durations [25].

Let us compare the solution of (5) to the minimizers obtained from (4) at every time $t$. Note that constraint (4d) implies constraints (5d) and (5i), but not the converse. Likewise, constraints (4e) guarantees (5c) and (5g). Therefore, the stochastic scheme in (5) constitutes a relaxation of the deterministic problem (4) solved over time $t$. As such, the minimizers of (5) could yield a lower average operational cost, i.e., $J^*_t^2 \leq \mathbb{E}[J^*_t^1]$, where the expectation is taken over time $t$.

The stochastic problem in (5) involves infinitely many variables $(p^g_t, q^g_t)_t$. Nodal power injections at time $t$ should satisfy the instantaneous constraints (5a)-(5f). Further, the infinitely many variables are coupled across time via the objective function and the average constraints (5a) and (5g), hence challenging the solution of (5). A stochastic optimization approach for tackling (5) is pursued in the next section.

III. Ergodic Energy Management

The task of ergodic energy management (EEM) is to design algorithms that sequentially observe predictions $(p^k_t, q^k_t, p^k_t, \pi_{0,t}, \pi_{f,t})$, and solve near optimally the stochastic problem in (5). The EEM is inspired by related ideas from
resource allocation in wireless networking, where due to propagation channel uncertainties and variabilities, one optimizes the average rather than the instantaneous system behavior [33]. The key assumption is that only realizations of those stochastic processes are available, while their joint probability density function is typically unknown.

Since optimization variables, henceforth collectively denoted by $\mathbf{x} := \{(p^s_n, q^s_n)\}_n$, are coupled via expectations, constraints (5) and (6) are dualized. Let $\boldsymbol{\nu} \in \mathbb{R}_+^N$, $\boldsymbol{\xi} \in \mathbb{R}_+^N$, and $\overline{\boldsymbol{\xi}} \in \mathbb{R}_+^N$ denote the dual variables corresponding to (5), and the lower and upper voltage bounds in (5), respectively. All other constraints are kept explicit. Using these definitions, the Lagrangian function of (5) is readily written as

$$
\mathcal{L}(\mathbf{x}, \boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) := \mathbb{E}\left[\pi_0, t p_0, t \mathbf{p}^s_n, \mathbf{q}^s_n + \pi_f, t \mathbf{1}^T \left[\mathbf{p}^s_n - \mathbf{p}^f_n\right]\right] + \sum_{n=1}^N \nu_n \left\{ \mathbb{E} \left[\left(p^s_n\right)^2 + (q^s_n)^2\right] - s_n^2 \right\} + \sum_{n=1}^N \xi_n \left\{ v_{n,t} - \mathbb{E}\left[v_{n,t}(\mathbf{p}^s_n, \mathbf{q}^s_n)\right] \right\} + \sum_{n=1}^N \bar{\xi}_n \left\{ \mathbb{E}\left[v_{n,t}(\mathbf{p}^s_n, \mathbf{q}^s_n)\right] - v_{u,n} \right\}.
$$

(6)

The dual function for problem (5) is the minimum of the Lagrangian function over all primal variables. Due to the linearity of the expectation operator, the minimization and the expectation operators can be interchanged [34]. After rearranging terms, the dual function is thus expressed as

$$
g(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) := \mathbb{E}\left[g(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}})\right] - \sum_{n=1}^N \left( \nu_n s_n^2 - \xi_n v_{n,t} + \bar{\xi}_n v_{u,n} \right)
$$

where functions $g_t(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}})$ are defined as

$$
g_t(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) := \min_{(\mathbf{p}^s_n, \mathbf{q}^s_n) \in \Omega_t} \left\{ \pi_0, t p_0, t \mathbf{p}^s_n, \mathbf{q}^s_n + \pi_f, t \mathbf{1}^T \left[\mathbf{p}^s_n - \mathbf{p}^f_n\right]\right. + \sum_{n=1}^N \nu_n \left\{ \left(p^s_n\right)^2 + (q^s_n)^2\right\} + \sum_{n=1}^N \left( \xi_n - \xi_{u,n} \right) v_{n,t}(\mathbf{p}^s_n, \mathbf{q}^s_n) \right\}.
$$

(7)

and the feasible set $\Omega_t$ is given by the instantaneous constraints in (5) as

$$
\Omega_t := \{(p^s_n, q^s_n)\} \text{ satisfying (5a) - (5c)}.
$$

(8)

The dual problem is obtained by maximizing the dual function over the dual variables, that is

$$
g(\boldsymbol{\nu}^*, \boldsymbol{\xi}^*, \bar{\boldsymbol{\xi}}^*) := \max_{\boldsymbol{\nu}, \boldsymbol{\xi} \geq 0} g(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}).
$$

(9)

Evaluating $g(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}})$ requires solving infinitely many problems of the form in (7), and then averaging the optimal costs over the joint probability density function (pdf) of $\{\mathbf{p}^s_n, \mathbf{q}^s_n, \mathbf{p}^f_n, \pi_0, t, \pi_f, t\}$. Even if the joint pdf were available, finding the expectations would be non-trivial. Hence, even evaluating the dual function becomes challenging. To maximize the dual function in a feasible manner, a stochastic optimization solver is proposed next.

### IV. Stochastic Approximation Solver

The problem at hand is tackled using a stochastic dual subgradient method [33], [35]. To maximize $g(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}})$, the Lagrange multipliers are updated using the projected subgradient iterations for some step size $\mu > 0$, as

$$
\nu_t := [\nu_{t-1} + \mu \delta^*_t]_+,
$$

(10a)

$$
\xi_t := [\xi_{t-1} + \mu \delta^*_t]_+,
$$

(10b)

$$
\bar{\xi}_t := [\bar{\xi}_{t-1} + \mu \delta^*_t]_+.
$$

(10c)

where the vector $\delta^*_t := [\delta^*_t, \delta^*_t, \delta^*_t]^T$ is a subgradient of $g_t(\boldsymbol{\nu}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}})$ evaluated at the previous iterate $(\nu_{t-1}, \xi_{t-1}, \bar{\xi}_{t-1})$. The entries of the subgradient vector, denoted by $[\delta^*_t]_n$, can be found as

$$
[\delta^*_t]_n := (p^s_{n,t})^2 + (q^s_{n,t})^2 - s_n^2
$$

(11)

$$
[\delta^*_t]_n := v_{n,t} - v_{n,t}(\mathbf{p}^s_n, \mathbf{q}^s_n)
$$

(12)

$$
[\delta^*_t]_n := v_{n,t}(\mathbf{p}^s_n, \mathbf{q}^s_n) - v_{u,n}
$$

(13)

for all $n$, where $(p^s_{n,t}, q^s_{n,t})$ are the minimizers of the problem in (7) for $g_t(\nu_{t-1}, \xi_{t-1}, \bar{\xi}_{t-1})$. Note that the Lagrange multipliers are updated at every control interval, that is when predictions are collected and before every control period starts. Table I summarizes the EEM algorithm consisting of the iterative application of two steps: the primal update that involves solving (7) for the current value of the Lagrange multipliers, and the dual subgradient update (10). It is worth stressing that the proposed energy management scheme does not require any distributional knowledge on the input data $(\mathbf{p}^s_n, \mathbf{q}^s_n, \mathbf{p}^f_n, \pi_0, t, \pi_f, t)$. Moreover, although the focus here is on utility cost minimization, other energy management tasks, such as voltage regulation and conservation voltage reduction, could be cast under this framework.

As far as convergence is concerned, note first that all primal and dual iterates depend on the realizations $(\mathbf{p}^s_n, \mathbf{q}^s_n, \mathbf{p}^f_n, \pi_0, t, \pi_f, t)$, and are thus random. For that reason, convergence claims are in probability. Upon defining $h_t := [\delta^*_t, \delta^*_t, \delta^*_t]^T$, it is easy to show first that there exists a finite $\tilde{H}$ such that $\mathbb{E}[\|h_t\|^2_2] \leq H^2$ for all $t$, i.e., the subgradient $\delta_t$ in (11) is bounded at all times. In particular, it holds that $[\delta^*_t]_n^2 \leq s_n^2$, while $[\delta^*_t]_n^2$ and $[\delta^*_t]_n^2$ are both upper bounded by $(v_{n,t} - v_{u,n})^2$. Therefore, the bound $H$ can be selected as

$$
H := \sum_{n=1}^N \left( s_n^2 + 2(v_{n,t} - v_{u,n})^2 \right).
$$

(14)
Adopting Th. 1, the following result characterizes the almost sure feasibility and optimality of the EEM algorithm.

**Proposition 1**. For the sequences \( \{\hat{p}_n^t, q_n^t\}_n \), generated by the algorithm in Table I, the next hold with probability 1 for all \( n \in N \)

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \left[ (\hat{p}_{n,\tau}^0)^2 + (\hat{q}_{n,\tau}^0)^2 \right] \leq s_n^2 \tag{15a}
\]

\[
v_t \leq \lim_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} v_{n,\tau} (\hat{p}_{n,\tau}^0, \hat{q}_{n,\tau}^0) \leq v_u. \tag{15b}
\]

Furthermore, the incurred operational costs satisfy

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \left[ \pi_{0,\tau} p_{0,\tau} (\hat{p}_n^0, q_n^0) + \pi_{f,t}^0 \left[ (\hat{p}_n^0)^2 - \mu \right] \right] - J_n^2 \leq \frac{\mu H^2}{2}
\]

almost surely for \( H = \sum_{n=1}^{N} (s_n^2 + 2(\pi_n - \pi_n^*)^2) \).

Proposition 1 asserts that the ensembles of primal sequences \( \{\hat{p}_n^t, q_n^t\}_n \) are feasible almost surely, meaning that constraints (15) and (5g) are satisfied almost surely. Moreover, the ergodic limit of the objective is at most \( \mu H^2/2 \) away from the optimal \( J_n^2 \) [cf. (5)]. The aforementioned claims hold even if the stochastic processes involved are correlated across time \( \{33\} \).

The EEM problem in (5) and its stochastic approximation solver of Table I involve the power losses \( p_{0,t}(\hat{p}_n^0, q_n^0) \) and the squared voltage magnitudes \( \{v_{n,t}(\hat{p}_n^0, q_n^0)\}_n \). So far, both quantities have been expressed as functions of the control variables \( (\hat{p}_n^0, q_n^0) \). To practically implement the EEM algorithm (in particular Step 5 of Table I), the actual form of these two functions is delineated next.

### V. GRID MODELING AND ALGORITHMS

This section presents two ways for describing \( p_{0,t}(\hat{p}_n^0, q_n^0) \) and \( \{v_{n,t}(\hat{p}_n^0, q_n^0)\}_n \), corresponding to two different distribution grid models. The obtained forms for \( p_{0,t}(\hat{p}_n^0, q_n^0) \) and \( \{v_{n,t}(\hat{p}_n^0, q_n^0)\}_n \) are then integrated into the EEM algorithm. The two models are an exact AC grid model and a linear approximation one. Selecting between the two models depends on the computational capabilities of the utility server. The AC model-based EEM can be formulated as an SOCP, while the linear approximation-based model yields a linearly constrained quadratic program. Though the former is more accurate, it is computationally more demanding. The two distribution grid models are detailed next.

#### A. Branch Flow Model-based EEM

Due to the radial structure of distribution grids, every non-root bus \( n \in N \) has a unique parent bus, which will be denoted by \( \pi_n \). The directed edge \((\pi_n, n) \in L\) corresponding to the distribution line feeding bus \( n \) will be indexed by \( n \); see Fig. 1. Without loss of generality, buses can be indexed such that \( \pi_n < n \) for all \( n \in N \).

Let \( z_n \) be the line impedance of line \( n \), and \( \ell_n \) the squared current magnitude on line \( n \) at time \( t \). If \( P_{n,t} + jQ_{n,t} \) is the complex power flow on line \( n \) seen at the sending end bus \( \pi_n \) at time \( t \), the grid can be described by the branch-flow model \([36, 37]\):

\[
p_{n,t} = \sum_{k \in C_n} P_{k,t} - (P_{n,t} - \ell_n \ell_{n,t}) \tag{16a}
\]

\[
q_{n,t} = \sum_{k \in C_n} Q_{k,t} - (Q_{n,t} - x_n \ell_{n,t}) \tag{16b}
\]

\[
v_{n,t} = v_{\pi_n,t} - 2(r_n P_{n,t} + x_n Q_{n,t}) + (r_n^2 + x_n^2) \ell_{n,t} \tag{16c}
\]

\[
\ell_{n,t} = \frac{P_{n,t}^2 + Q_{n,t}^2}{v_{n,t}} \tag{16d}
\]

for all \( n \in N \), where \( C_n \) is the set of the children nodes of bus \( n \). The power injections at the substation bus are \( p_{0,t} = \sum_{k \in C_0} P_{k,t}, q_{0,t} = \sum_{k \in C_0} Q_{k,t} \), and its squared voltage magnitude \( v_{0,t} \) is controlled at a desirable value. Similar to \( (p_t, q_t) \), the vectors \( r_t, P_t, Q_t, v_t, \ell_t \) collect the entries of \( r_n, P_{n,t}, Q_{n,t}, v_{n,t}, \ell_{n,t} \), accordingly.

Equations (16a)-(16d) are linear with respect to the system variables \( (p_t, q_t, P_t, Q_t, v_t, \ell_t) \). The equations in (16d) are quadratic yielding a non-convex feasible set. Nonetheless, the latter equations have been recently relaxed to convex inequalities described by the hyperbolic constraints [13]

\[
P_{n,t}^2 + Q_{n,t}^2 \leq v_{n,t} \ell_{n,t}, \ \forall \ n \tag{17}
\]

which can be equivalently expressed as the convex second-order cone constraints

\[
\begin{bmatrix}
2P_{n,t} \\
2Q_{n,t} \\
v_{n,t} \ell_{n,t}
\end{bmatrix} \leq \begin{bmatrix}
\ell_{n,t} \\
v_{n,t} \ell_{n,t}
\end{bmatrix}, \ \forall \ n. \tag{18}
\]

Equations (16a)-(16d) and (18) represent now a convex feasible set. Recent works suggest using this relaxed feasible set to perform several grid optimization tasks. Under different conditions, the relaxation has been shown to be exact, i.e., the obtained minimizer attains (18) with equality; see [10] and references therein. Henceforth, all SOCP relaxations are assumed exact, which will be numerically verified in Section VI.

Based on (16a)-(16c) and (18), the active power injection at the substation bus \( p_{0,t}(\hat{p}_n^0, q_n^0) \) can be expressed as

\[
p_{0,t}(\hat{p}_n^0, q_n^0) = \sum_{n=1}^{N} (P_{n,t}^0 - P_{n,t}) + \sum_{n=1}^{N} r_n \ell_{n,t} = 1^T (\hat{p}_n^0 - P_{n,t}^0) + r^T \ell_t
\]

where the second summand represents the total power losses on distribution lines. Hence, under the aforementioned relaxed
grid model, the primal update (Step 4 in Table I) involves solving the optimization problem

$$\min \pi_{0,t} 1^T (p^t - p^0_t) + \pi_{0,t} \ell_t + \pi_{f,t} 1^T [p^t - p^f_t] + \sum_{n=1}^N \nu_{n,t-1} [\epsilon_{n,t}^q]^2 + (\epsilon_{n,t}^q)^2]$$

$$+ \sum_{n=1}^N (\xi_{n,t-1} - \xi_{n,t-1}) v_{n,t} \tag{19} \label{eq:19}$$

s.t. (5b) - (5e), (16a) - (16c), (18).

In addition to the original variables \((p^t, q^t)\), the primal update now involves the variables \((P_t, Q_t, v_t, \ell_t)\) too. Problem (19) can be reformulated to an SOCP. All instances of (19) solved in Section VI were exact. Nevertheless, solving (19) could be computationally demanding for large-scale distribution grids. This motivates our next instantiation of the EEM algorithm under an approximate grid model.

B. LinDistFlow-Based EEM

The so termed LinDistFlow model can be derived as follows. Because the line parameters \(\{r_n, x_n\}_{n \in N}\) have relatively small entries, the last summands in (16a) - (16c) can be ignored yielding the linear equations for all \(n \in N\) [38]

$$p_{n,t} = \sum_{k \in C_n} P_{k,t} - P_{n,t} \tag{20a}$$

$$g_{n,t} = \sum_{k \in C_n} Q_{k,t} - Q_{n,t} \tag{20b}$$

$$v_{n,t} = v_{n,t} - 2(r_n P_{n,t} + x_n Q_{n,t}) \tag{20c}$$

In this way, squared voltage magnitudes are now approximated as linear functions of \((p_t, q_t)\). Albeit (20) was obtained by dropping all terms related to \(\ell_t\), squared line current magnitudes can be approximated as [5]

$$\ell_{n,t} = \frac{P_{n,t}^2 + Q_{n,t}^2}{v_{n,t}^2} \approx p_{n,t}^2 + Q_{n,t}^2 \tag{21} \label{eq:21}$$

assuming that squared voltage magnitudes are close to unity. The active power injection at the substation bus can be thus approximated as

$$p_{0,t}(p^0_t, q^0_t) = 1^T (p^t - p^0_t) + \sum_{n=1}^N r_n (P_{n,t}^2 + Q_{n,t}^2)$$

Building on the approximate model of (20) - (21), the primal update of the EEM algorithm (Step 5 of Table I) for control period \(t\) entails solving the linearly-constrained quadratic program

$$\min \pi_{0,t} 1^T (p^t - p^0_t) + \pi_{0,t} \sum_{n=1}^N r_n (P_{n,t}^2 + Q_{n,t}^2)$$

$$+ \pi_{f,t} 1^T [p^t - p^f_t] + \sum_{n=1}^N (\xi_{n,t-1} - \xi_{n,t-1}) v_{n,t} + \sum_{n=1}^N \nu_{n,t-1} [\epsilon_{n,t}^q]^2 + (\epsilon_{n,t}^q)^2] \tag{22} \label{eq:22}$$

s.t. (5b) - (5e), (20a) - (20c).

From (20a) - (20b), the line flow variables \((P_t, Q_t)\) can be substituted as linear functions of \((p_t, q_t)\).

VI. NUMERICAL TESTS

The novel EMC schemes were numerically tested on a real-world 56-bus distribution grid from Southern California Edison [10]. In addition to the PV generator originally located on bus 45, a PV was installed on bus 19. Modeling high penetration of renewable energy, the capacity of each PV is set to 6 MW. At each 30-sec control period, the EEM controller collects power demands from load buses and solar generation predictions from PV buses. Subsequently, active and reactive power injections by PV inverters are determined by: i) solving the deterministic energy management scheme in (34); and ii) the novel stochastic energy management scheme of Table I that is initialized to zero.

Squared voltage regulation margins are set as \([v_u, v_d] = [0.9604, 1.0404]\) p.u. and \([\theta_u, \theta_d] = [0.9409, 1.0609]\) p.u., with nominal voltage \(v_0 = 1\) p.u. The time-dependent apparent capability for smart inverters is set to 1.3 times their nameplate capacity with a power factor of 0.8. Performance is evaluated in terms of the energy management cost and the instantaneous counterpart of the cost in (5). All algorithms were implemented using MATLAB and CVX on an Intel CPU @ 3.4 GHz (32 GB RAM) computer [39]. Every run for the full AC and the linear approximation model-based algorithms was performed within 1.5 and 1.3 seconds, respectively. It is worth mentioning that all SOCP relaxations encountered in the ensuing experiments were feasible and exact.

To verify the almost sure optimality, the first experiment simulates synthetic load consumption and solar generation as \(p^t = p^0 + \epsilon^p_t\) and \(p^0_t = p^0 + \epsilon^p_t\), respectively. The nominal values \(p^0\) and \(p^0\) are set to 40% of the peak demand values and 80% of the maximum PV generation, accordingly. Vectors \(\epsilon^p_t\) and \(\epsilon^q_t\) capture fluctuations modeled as independent zero-mean Gaussian vectors with standard deviations equal to 5% of the corresponding nominal values. Given that current FIT prices change on a monthly basis and are oftentimes 1/2 of
consumption prices \(^{22}\), prices were set to \(\pi_{0,t} = 30\$\)/kWh and \(\pi_{f,t} = 15\$\)/kWh for all \(t\).

Using the branch flow model, Fig. 2 depicts the energy management cost for the novel scheme and its deterministic counterpart over a single system realization. The step size for the stochastic scheme is set to \(\mu = 0.1\), while the time-average energy management cost per time slot \(t\) is defined as 

\[
\frac{1}{\tau_t} \sum_{t=1}^{\tau_t} \left[ \pi_{0,t} \{\underline{p}_t, \underline{q}_t\} + \pi_{f,t} \{\bar{p}_t - \bar{q}_t\}^\top \right].
\]

The novel scheme achieves a considerably lower energy management cost than the deterministic scheme.

The second experiment studies the effect of step size \(\mu\). The AC-based stochastic scheme was simulated for \(\mu \in \{0.1, 0.2, 0.3\}\) along with the deterministic scheme. Twenty Monte Carlo system realizations are averaged for each step size value, while the corresponding average energy management cost is plotted in Fig. 3. The curves demonstrate that larger step sizes incur higher energy management costs, an observation that agrees with the optimality gap of Prop. 1.

To test the proposed schemes in real-world conditions, the ensuing two experiments entail real data from the Smart project \(^{40}\). Consumption data involved the electricity usage at minute-level samples from 443 homes on April 2, 2011; and the power outputs of 3 residential PVs collected at 5-second intervals over August 12, 2011. Preprocessing the data entailed removing the minimum daily value, and normalizing the daily data curves to 1. To simulate a 30-sec control period, consumption data were interpolated using two-point averages. Consumption and solar generation curves were scaled to the nominal capacity of the corresponding buses.

A single system realization was simulated over the period 9:30 am – 2:30 pm with a control interval of 30 seconds for both the AC- and the linear approximation-based schemes. Figure 4 presents the cost for \(\mu = 0.25\). Using either the AC or the linear approximation model, the novel scheme achieves a lower cost than the deterministic one. It is also interesting to find that the linear approximation-based scheme attains relatively higher costs than the AC model-based schemes.

Figure 5 depicts the evolution of the squared voltage magnitude for bus 45, and the associated dual variable \(\xi_{45,t}\) using the AC model-based schemes.

\(\xi_{45,t}\) for the tight voltage margin constraint in (5g). The voltage magnitude for the deterministic scheme remains in the tight region \([v_l, v_u] = [0.9604, 1.0404]\) throughout the operation horizon. The voltage magnitude obtained from the stochastic scheme lies occasionally beyond the voltage margin \(v_u = 1.0404\). Nonetheless, over-voltage effects have short duration. At around 10:25 am, when the voltage magnitude violates the tight voltage constraint for the first time, the dual variable becomes positive and starts increasing. As long as the voltage magnitude fluctuates above the tight margin, the dual variable keeps increasing. After roughly 12:20 pm, the voltage magnitude drops and remains consistently below the upper margin, while the dual variable decreases and eventually becomes zero for the rest of the day.

VII. CONCLUDING REMARKS

To exploit the flexibilities in power system operation, an EEM framework has been introduced. Smart inverters were engaged in active power curtailment and reactive power support
in a stochastic sense. Stochastic dual subgradient algorithms enforce tighter operational margins at all times, yet letting system characteristics deviate over short time intervals. The developed algorithms are guaranteed to converge to the optimal operational point, while the feasibility is satisfied almost surely. Numerical tests using the exact full AC grid model and the linear approximation model showed the viability of such an approach in real-world conditions. In particular, the grid was operated within the regulated margins at all times, while during extreme conditions local variables could fluctuate over lesser ranges. The suggested flexible grid operation brings up several interesting questions. Decentralized and local implementations are timely and pertinent. Enforcing probabilistic rather than average constraints constitutes another challenging direction. Integrating uncertainties and multi-stage schemes are worthwhile future directions too.

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