Families of Small Regular Graphs of Girth 5

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Abstract

In this paper we obtain \((q + 3)\)-regular graphs of girth 5 with fewer vertices than
previously known ones for \(q = 13, 17, 19\) and for any prime \(q \geq 23\) performing operations
of reductions and amalgams on the Levi graph \(B_q\) of an elliptic semiplane of type \(C\).
We also obtain a \(13\)-regular graph of girth 5 on 236 vertices from \(B_{11}\) using the same

technique.

1 Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges).
For definitions and notations not explicitly stated the reader may refer to [12].

Let \(G = (V(G), E(G))\) be a graph with vertex set \(V = V(G)\) and edge set \(E = E(G)\).
The girth of a graph \(G\) is the length \(g = g(G)\) of its shortest circuit. The degree of a vertex
\(v \in V\) is the number of vertices adjacent to \(v\). A graph is called \(k\)-regular if all its vertices
have the same degree \(k\), and bi-regular or \((k_1, k_2)\)-regular if all its vertices have either degree
\(k_1\) or \(k_2\). A \((k, g)\)-graph is a \(k\)-regular graph of girth \(g\) and a \((k, g)\)-cage is a \((k, g)\)-graph
with the smallest possible number of vertices. The necessary condition obtained from the
distance partition with respect to a vertex yields a lower bound \(n_0(k, g)\) on the number of vertices
of a \((k, g)\)-graph, known as the Moore bound.

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Exoo and Jajcay \[18\] in the surveys by Wong \[32\], by Holton and Sheehan \[23, Chapter 6\], or the recent one by all the cages known so far appear. More details about constructions of cages can be found on distinct methods for constructing cubic cages \[11\]. Royle \[29\] keeps a web-site in which [1, 2, 4, 5, 6, 7, 9, 13, 17, 19, 22, 25, 27, 28, 32]. Biggs is the author of an impressive report [16] proved the existence of a (\(k, g\)) graphs \[32\]. Erdős and Sachs previously known ones (cf. \[24, 20\]) for \(k, g\) \([10\]) calls excess of a \((k, g)\)-graph \(G\) the difference \(|V(G)| - n_0(k, g)\). Cages have been intensely studied since they were introduced by Tutte \[30\] in 1947. Erdős and Sachs of the work carried out has been focused on constructing smallest \((k, g)\)-graphs (see e.g. 1 2 4 5 6 7 9 13 17 19 22 25 27 28 32). Biggs is the author of an impressive report on distinct methods for constructing cubic cages \[11\]. Royle \[29\] keeps a web-site in which all the cages known so far appear. More details about constructions on cages can be found in the surveys by Wong \[32\], by Holton and Sheehan \[23\] Chapter 6], or the recent one by Exoo and Jajcay \[18\].

A partial plane is an incidence structure \(I = (P, L, |\) in which two distinct points are incident with at most one line. In an incidence structure a flag is an incident point line pair \(p_1 | l_1\), an anti-flag is an non-incident point line pair \(p_1 \nmid l_1\), two lines are parallel if there is no point incident with both, and, dually, two points are parallel if there is no line incident with both.

A \(v_k\)-configuration or a configuration of type \(v_k\) is a partial plane consisting of \(v\) points and \(v\) lines such that each point and each line are incident with \(k\) lines and \(k\) points, respectively. A finite elliptic semiplane of order \(k - 1\) is a \(v_k\)-configuration satisfying the following axiom of parallels: for each anti-flag \(p_1 \nmid l_1\), there exists at most one line \(l_2\) incident with \(p_1\) and parallel to \(l_1\), and at most one point \(p_2\) incident with \(l_1\) and parallel to \(p_1\). \[15\] [21].

A Baer subset of a finite projective plane \(P\) is either a Baer subplane \(B\) or, for a distinguished point-line pair \((p, l)\), the union \(B(p, l)\) of all lines and points incident with \(p\) and \(l\), respectively. We write \(B(p | l)\) or \(B(p \nmid l)\), according to the incidence or non- incidence of \(p\) and \(l\). It was already known to Dembowski \[15\] that elliptic semiplanes are obtained by deleting a Baer subset from a projective plane. We call any such elliptic semiplane Desarguesian if the projective plane from which it is constructed is so. In \[15\] Dembowski classified elliptic semiplanes into five types. In this paper we will only be concerned with those of type \(C\), which are \(C_q = PG(2, q) - B(p, l)\), i.e. the complement of a Baer subset \(B(p | l)\) in a desarguesian projective plane \(PG(2, q)\), for each prime power \(q\). Hence the elliptic semiplane of type \(C_q\) is also a configuration of type \((q^2)_q\).

The Levi graph or incidence graph \(G\) of an incidence structure \(I = (P, L, |)\), is a bipartite graph with \(V(G) = V_1 \cup V_2\), where \(V_1 = P\) and \(V_2 = L\) and two vertices are adjacent in \(G\) if and only if the corresponding point and line are incident in \(I\). Recall that the Levi graph of a finite projective plane is a \((k, 6)\)-cage, attaining Moore’s bound, i.e. these are Moore graphs \[32\].

In this paper we obtain \((q + 3)\)-regular graphs of girth 5 with fewer vertices than previously known ones (cf. \[24\] [20]) for \(q = 13, 17, 19\) and for any prime \(q \geq 23\) performing

\[
n_0(k, g) = \begin{cases} 1 + k + k(k - 1) + \ldots + k(k - 1)^{(g - 3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k - 1) + \ldots + (k - 1)^{g/2 - 1}) & \text{if } g \text{ is even.} \end{cases}
\]
operations of reductions (cf. Section 3) on the Levi graph $B_q$ of $C_q$ and then amalgams with bi–regular graphs (cf. Section 4) into the obtained reduced graph or $B_q$ itself. We also obtain a new 13–regular graph of girth 5 on 236 vertices from $B_{11}$ using the same technique.

2 Preliminaries

Throughout the paper we will use the following notation when dealing with the elliptic semiplane of type $C_q$.

In $PG(2, q)$, choose $p$ and $l$ to be the point and line at infinity, respectively. Then, in $C_q$ it is possible to choose the affine coordinates $(x, y)$, for the points, and $[m, b]$ for the lines $\{x, y, m, b\} \in GF(q)$, which imply that the incidence between a point and a line is given by the equation $y = mx + b$. Recall that in $C_q$ vertical lines have been deleted from $PG(2, q)$ along with the point at infinity, the line at infinity and all its points.

Define the sets $P_i = \{(i, y) | y \in GF(q)\}$ for $i \in GF(q)$ and $L_j = \{[j, b] | b \in GF(q)\}$ for $j \in GF(q)$. These sets correspond to the partition of the points and lines of $C_q$ into parallel classes, according to the axiom of parallels for elliptic semiplanes. Note also that if $(x, y)[[m, b]$ then $(x, y + a)[[m, b + a]$ for any $a \in GF(q)$.

The following properties of the Levi graph $B_q$ of $C_q$ are well known and they will be fundamental throughout the paper.

**Proposition 2.1** Let $B_q$ be the Levi graph of $C_q$ then:

(i) It is $q$–regular, bipartite, vertex transitive, of order $2q^2$ and has girth 6;

(ii) It admits a partition $V_1 = \bigcup_{i=0}^{q-1} P_i$ and $V_2 = \bigcup_{j=0}^{q-1} L_j$ of its vertex set;

(iii) Each block $P_i$ is connected to each block $L_j$ by a perfect matching, for $i, j \in GF(q)$;

(iv) Each vertex in $P_0$ and $L_0$ is connected straight to all its neighbours in $B_q$, meaning that for $p = (0, y)$, $N(p) = \{[i, y] | i \in GF(q)\}$ and analogously for $l = [0, b]$, $N(l) = \{(j, b) | j \in GF(q)\}$;

(v) The other matchings between $P_i$ and $L_i$ are twisted and the rule can be defined algebraically in $GF(q)$.

For further information regarding these properties and for constructions of the adjacency matrix of $B_q$ as a block $(0, 1)$–matrix please refer to [3] [8].

3 Reductions

In this section we will describe two reduction operations that we perform on the graph $B_q$. 
Lemma 3.2 \[ \bigcup_{u \in S} \] for \( B \) graph

In this section we will describe amalgam operations that can be performed on the reduced graph. For \( \Gamma = (S, T, u) \in \mathbb{B} \) and \( (S, T, u) \) the graph obtained from \( S, T, u \) by deleting the last \( u \) pairs of blocks of vertices \( P_i, L_i \) and \( B_q(S, T, u) = B_q - S_0 - T_0 - \bigcup_{i=1}^{u} (P_{q-i} \cup L_{q-i}). \)

Lemma 3.2 Let \( u \in \{0, \ldots, q - 1\} \). Then, the graph \( B_q(u) \) is \((q - u)\)-regular of order \( 2(q^2 - qu) \) and the graph \( B_q(S, T, u) \) is bi-regular with degrees \((q - u - 1, q - u)\) and order \( 2(q^2 - qu) - |S| - |T| \). Moreover, the vertices \((i, t) \in V_1\) and \([j, s] \in V_2\), for each \( i, j \in GF(q) \), \( s \in S \) and \( t \in T \) are the only vertices of degree \( q - u - 1 \) in \( B_q(S, T, u) \), together with \([0, s] \in V_2\) for \( s \in S - T \) if \( T \not\subseteq S \).

Proof It is immediate from Remark 3.1 (i), (v) and Lemma 3.1

Note that, \( B_q(u) = B_q \) and \( B_q(S, T, u) = B_q(S, T) \) when \( u = 0 \).

4 Amalgams

In this section we will describe amalgam operations that can be performed on the reduced graph \( B_q(S, T, u) \) or on \( B_q \) itself.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be two graphs of the same order and with the same label on their vertices.

In general, an amalgam of \( \Gamma_1 \) into \( \Gamma_2 \) is a graph obtained adding all the edges of \( \Gamma_1 \) to \( \Gamma_2 \).

Let \( P_i \) and \( L_i \) be defined as in Section 2 Consider the graph \( B_q(S, T, u) \), for some \( T \subseteq S \subseteq GF(q) \) and \( u \in \{0, \ldots, q - 1\} \). Let \( S_0 \subseteq P_0, T_0 \subseteq L_0 \) as in Reduction 1, and let \( P'_0 := P_0 - S_0 \) and \( L'_0 := L_0 - T_0 \) be the blocks in \( B_q(S, T, u) \) of order \( q - |S| \) and \( q - |T| \), respectively.

Let \( H_1, H_2, G_i \), for \( i = 1, 2 \), be graphs of girth at least 5 and order \( q - |S|, q - |T| \) and \( q \), respectively. Let \( H_1 \) be a \( k \)-regular graph. If \( |S| = |T| \), let \( H_2 \) be \( k \)-regular and otherwise let it be \((k, k + 1)\)-regular, with \(|S - T| \) vertices of degree \( k + 1 \). If \( T = \emptyset \), let \( G_1 \) be a...
edges between $P$ Thus for the vertices in $H$

Therefore, $C = \{0, \ldots, q - u - 1\}$ and $u \in \{0, \ldots, q - 2\}$. We also define $B_q^*(S, T, q - 1)$ to be the amalgam of $H_1$ into $P_0', H_2$ into $L_0'$. To simplify notation in our results, we label $P_i$ and $L_i$ as in Section [2] but assume that the labellings of $H_1, H_2, G_1$ and $G_2$, correspond to the second coordinates of $P_0', L_0', P_i$ and $L_i$ respectively for $i \in \{1, \ldots, q - u - 1\}$ and $u \in \{0, \ldots, q - 2\}$. Suppose also that the vertices of degree $k + 1$, if any, in $H_2, G_1$ and $G_2$ are labelled in correspondence with the second coordinates of $S - T, T$ and $S$, respectively.

With such a labelling, let $ab$ be an edge in $H_1, H_2, G_1$ or $G_2$, and define the weight or the Cayley Color of $ab$ to be $\pm (b - a) \in \mathbb{Z}_q^*$. Let $P_\omega$ be the set of weights in $H_1$ and $G_1$, and let $L_\omega$ be the set of weights in $H_2$ and $G_2$.

The following result is a special case of [20, Theorem 2.8] for the coordinates we have chosen for $C_q$ (cf. Section [2]). On the other hand, it generalizes such a Theorem since we delete vertices from $P_0$ and $L_0$, pairs of blocks $P_i, L_i$ and amalgam with graphs which are not regular, but chosen in such a way that the obtained amalgam is regular.

**Theorem 4.1** Let $T \subseteq S \subseteq GF(q)$, $u \in \{0, \ldots, q - 1\}$. Let $H_1, H_2, G_1$ and $G_2$ be defined as above and suppose that the weights $P_\omega \cap L_\omega = \emptyset$. Then the amalgam $B_q^*(S, T, u)$ is a $(q + k - u)$-regular graph of girth at least 5 and order $2(q - u) - |S| - |T|$.

**Proof** The order and the regularity of $B_q^*(S, T, u)$ follow from Lemma [3.2] and the choice of $H_1, H_2, G_1$ and $G_2$. Note that the vertices of $L_i$, with degree $q - u - 1$ in $B_q(S, T, u)$, have degree $k + 1$ in $G_2$, which add up to to degree $q + k - u$ in $B_q^*(S, T, u)$, for $i \in \{1, \ldots, q - u - 1\}$. Similarly for the vertices in $L_0$ and for those in $P_i$, for $i \in \{1, \ldots, q - u - 1\}$. Let $C$ be the shortest circuit in $B_q^*(S, T, u)$ and suppose, by contradiction, that $|C| \leq 4$. Therefore, $C = (xyz)$ or $C = (wxyz)$. Since $B_q$ has girth 6 and $H_1, H_2, G_1, G_2$ have girth at least 5, then $C$ cannot be completely contained in $B_q$ or in some $H_i$ or $G_i$ for $i = 1, 2$. Then, w.l.o.g. the path $xyz$ in $C$ is such that $x, y \in P_i$ and $z \in L_m$ for some $i, m \in GF(q)$. Since the edges between $P_i$ and $L_m$ form a matching, then $xz \notin E(B_q)$ and hence $xz \notin E(B_q^*(S, T, u))$. Thus $|C| > 3$ and we can assume $|C| = 4$ and $C = (wxyz)$.

If $w \in P_i$, by the same argument, $wz \notin E(B_q^*(S, T, u))$ and we have a contradiction. There are no edges between $P_i$ and $P_j$ in $B_q^*(S, T, u)$, so $w \notin P_j$ for $j \in GF(q) - \{i\}$, which implies that $w \in L_n$ for some $n \in GF(q)$. If $n \neq m$, we have a contradiction since there are no edges between $L_m$ and $L_n$ in $B_q^*(S, T, u)$. Therefore $x, y \in P_i$, and $w, z \in L_m$. Let $x = (i, a)$, $y = (i, b)$, $z = [m, c]$ and $w = [m, d]$ as in the labelling chosen in Section [2]. Then $wx, yz \in E(B_q^*(S, T, u))$ imply that $a = m \cdot i + d$ and $b = m \cdot i + c$, respectively, which give
The graph $B_q^*(S, T, u)$ has girth exactly 5. We describe two cases that we will use in Sections 5 and 6.

(i) If some $H_i$ or $G_i$ contains a 5–circuit, for $i \in \{1, 2\}$, then so does $B_q^*(S, T, u)$.

(ii) Let $t \in GF(q)$ be the smallest weight of an edge in some $H_i$, say w.l.o.g. in $H_1$. If $t = 1$ and $u < q - 1$ then $\{(0, i), (0, j), [1, j], (1, i), [0, i]\}$ is a 5–circuit in $B_q^*(S, T, u)$. If $t > 1$ then $\{(0, i), (0, j), [1, j], (t, i), [0, i]\}$ is a 5–circuit in $B_q^*(S, T, u)$ as long as the $t^{th}$–pair of blocks from $B_q^*(S, T)$ is not deleted, i.e. $u < q - t$.

All the graphs constructed in the next sections have girth exactly 5 since either some $H_i$ or $G_i$ contains a 5–circuit, for $i \in \{1, 2\}$, or 1 $\in P_\omega$.

5 New Regular Graphs of Girth 5

In this section we will construct new $(q + 3)$–regular graphs of girth 5, for any prime $q \geq 23$, applying reductions and amalgams to the graph $B_q$. In each case we will specify the sets $S$ and $T$ of vertices to be deleted from $P_0$ and $L_0$ and the graphs $H_1, H_2, G_1, G_2$ to be used for the amalgam into $B_q^*(S, T, u)$. For $u = 0$, all the graphs $B_q^*(S, T, u)$ constructed in this section have two vertices less than the ones that appear in $[24, 20]$.

Recall that every prime $q$ is either congruent to 1 or 5 modulo 6. We will now treat these two cases separately, when $q = 6n + 1$ or $q = 6n + 5$ is a prime.

5.1 Construction for primes $q = 6n + 1$

Throughout this subsection we will consider $n \geq 5$. The smaller cases will be treated in Section 6. Let $H_1$ and $H_2$ be two graphs of order $q - 1$ with the vertices labeled from 1 through $6n$, and partitioned into $W_1 = \{1, 2, \ldots, 3n\}$ and $W_2 = \{3n + 1, \ldots, 6n\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows:

| Set | Edges | Description |
|-----|-------|-------------|
| $A_1$ | $\{(i, i + 1) | i = 1, \ldots, 3n - 1\} \cup \{(3n, 1)\}$ | $(3n)$–circuit with weights 1 and $3n - 1$ |
| $B_1$ | $\{(i, i + 2) | i = 3n + 1, \ldots, 6n - 2\}$ \cup $\{(6n - 1, 3n + 1), (6n, 3n + 2)\}$ | one or two circuits according to the parity of $n$, with weights 2 and $3n - 2$ |
| $C_1$ | $\{(i, 3n + i) | i = 1, \ldots, 3n\}$ | Prismatic edges between $W_1$ and $W_2$ of weight $3n$ |

The graph $H_1$ is cubic and has weights $\pm\{1, 2, 3n - 2, 3n - 1, 3n\}$.
Lemma 5.1  The graph $H_1$ has girth 5.

Proof  Let $C$ be the shortest circuit in $H_1$. If $C$ is a subgraph of either $H_1[W_1]$ or $H_1[W_2]$ then $|C| \geq 5$, since $H_1[W_1]$ has girth at least 15 and $H_1[W_2]$ has girth at least 9. Otherwise, there is a path $xyz$ in $C$ is such that either $x, y \in W_1$ and $z \in W_2$ or $x \in W_1$ and $y, z \in W_2$.

The first case has the following subcases:

(i) $x = 1, y = 3n, z = 6n$
(ii) $x = i, y = i - 1, z = 3n + i - 1$, for $i = 2, \ldots, 3n$
(iii) $x = i, y = i + 1, z = 3n + i + 1$, for $i = 1, \ldots, 3n - 1$
(iv) $x = 3n, y = 1, z = 3n + 1$

The second case has similar subcases. If we show that $z \notin N_{H_1}(x)$ then $|C| \neq 3$, and if $y = N_{H_1}(x) \cap N_{H_1}(z)$ then $|C| \neq 4$. In subcase (i) the neighbourhoods of $x$ and $z$ in $H_1$ are $N_{H_1}(x) = \{2, 3n, 3n + 1\}$ and $N_{H_1}(z) = \{3n, 3n + 2, 6n - 2\}$, respectively. Thus, $z \notin N_{H_1}(x)$ and $y = N_{H_1}(x) \cap N_{H_1}(z)$. Hence, $|C| \geq 5$. All the other cases are analogous. The circuit $(1, 2, 3, 3n + 3, 3n + 1)$ is a 5–circuit in $H_1$.

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows:

| Set   | Edges                                                                 | Description                                                                 |
|-------|-----------------------------------------------------------------------|-----------------------------------------------------------------------------|
| $A_2$ | $\{(i, i + 3)|i = 1, \ldots, 3n - 3\}$                               | Three $n$–circuit with weights                                              |
|       | $\cup(3n - 2, 1), (3n - 1, 2), (3n, 3)$                              | $3$ and $3n - 3$                                                           |
| $B_2$ | $\{(i, i + 4)|i = 3n + 1, \ldots, 6n - 4\}$                         | One, two or four circuits                                                   |
|       | $\cup(6n - 3, 3n + 1), (6n - 2, 3n + 2), (6n - 1, 3n + 3), (6n, 3n + 4)$ | according to the congruency of $3n$ modulo 4,                               |
|       |                                                                        | with weights $4$ and $3n - 4$                                              |
| $C_2$ | $\{(i, 3n + 4 + i)|i = 1, \ldots, 3n - 4\}$                         | Prismatic edges between $W_1$ and $W_2                                    |
|       | $\cup(3n - 3, 3n + 1), (3n - 2, 3n + 2), (3n - 1, 3n + 3), (3n, 3n + 4)$ | $\equiv 3n - 3 \pmod q$                                                   |

The graph $H_2$ is cubic and has weights $\pm\{3, 4, 3n - 4, 3n - 3\}$.

Lemma 5.2  The graph $H_2$ has girth at least 5.

Proof  Similar to the proof of Lemma 5.1

Lemma 5.3  Let $G$ be a graph of girth at least 5. Let $x_1x_2, x_3x_4 \in E(G)$ be two independent edges of $G$ such that $N(x_i) \cap N(x_j) = \emptyset$, for all $i, j \in \{1, 2, 3, 4\}, i \neq j$. Let $G' = G - \{x_1x_2, x_3x_4\} \cup \{(v, x_i)|i = 1, 2, 3, 4\}$ be the graph of order $|V(G)| + 1$, where $v = V(G') - V(G)$. Then $G'$ has girth at least 5.

Proof  Let $C$ be the shortest circuit in $G'$. If $E(C) \subseteq E(G)$ then, by hypothesis, $|C| > 4$. Otherwise $v \in V(C)$ and $x_ix_j$ is a path in $C$ for some $i, j \in \{1, 2, 3, 4\}, i \neq j$. In $G'$ the set
\{x_i | i = 1, 2, 3, 4\} is independent, so |C| > 3. By hypothesis, \(N(x_i) \cap N(x_j) = \emptyset\) in \(G'\) and hence |C| > 4.  

Let \(G_1\) be a graph on \(q\) vertices labelled from 0 through \(q - 1\) and defined as follows
\[G_1 := H_1 - \{(1, 3n), ([3n+1 \over 2], 3n) + [3n+1 \over 2] \} \cup \{(0, 1), (0, [3n+1 \over 2]), (0, 3n), (0, 3n + [3n+1 \over 2])\}\n
**Lemma 5.4** The graph \(G_1\) has girth at least 5.

**Proof** The edges \(e_1 = (1, 3n)\) and \(e_2 = ([3n+1 \over 2], 3n + [3n+1 \over 2])\) are independent in \(H_1\). The neighbourhoods of the endvertices of \(e_1\) and \(e_2\) are:
- \(N(1) = \{2, 3n, 3n + 1\}\)
- \(N([3n+1 \over 2]) = \{[3n+1 \over 2] - 1, [3n+1 \over 2] + 1, 3n + [3n+1 \over 2]\}\)
- \(N(3n) = \{1, 3n - 1, 6n\}\)
- \(N(3n + [3n+1 \over 2]) = \{3n + [3n+1 \over 2] - 1, 3n + [3n+1 \over 2] + 1, \lfloor 3n+1 \over 2 \rfloor\}\)

which satisfy the hypothesis of Lemma 5.3. Since \(G_1\) is constructed from \(H_1\) as \(G'\) from \(G\) in Lemma 5.3 we can conclude that \(G_1\) has girth at least 5.

All together the weights of \(H_1\) and \(G_1\) modulo \(p\) give
\[
\mathcal{P}_w := \begin{cases} 
\pm 1, 2, \frac{3n+1}{2}, 3n - 2, 3n - 1, 3n & \text{if } n \text{ is odd} \\
\pm 1, 2, \frac{3n+1}{2}, \frac{3n+1}{2}, 3n - 2, 3n - 1, 3n & \text{if } n \text{ is even}
\end{cases}
\]

Let \(G_2\) be a graph on \(q\) vertices labelled from 0 through \(q - 1\) and defined as follows:
\[
G_2 := \begin{cases} 
H_2 - \{(3, 22), (5, 24)\} \cup \{(0, 3), (0, 5), (0, 22), (0, 24)\} & \text{if } n = 5 \\
H_2 - \{(3, 3n + 7), (4, 3n + 8)\} & \text{if } n \geq 6 \\
\cup \{(0, 3), (0, 4), (0, 3n + 7), (0, 3n + 8)\}
\end{cases}
\]

Note that for \(n = 5\) the edge \((0, 3n + 8) = (0, 23)\) has weight \(-8\) which lies already in \(\mathcal{P}_w\) and Theorem 4.1 cannot be applied. This is why, in the definition of \(G_2\), we choose to delete the edge \((5, 24)\) from \(H_2\), instead of \((4, 3n + 8) = (4, 23)\).

**Lemma 5.5** The graph \(G_2\) has girth at least 5.

**Proof** First suppose \(n \geq 6\). As in Lemma 5.4 the edges \((3, 3n + 7), (4, 3n + 8)\) are independent in \(H_2\) and the neighbourhoods
- \(N(3) = \{6, 3n, 3n + 7\}\)
- \(N([3n+1 \over 2]) = \{[3n+1 \over 2] - 1, [3n+1 \over 2] + 1, 3n + [3n+1 \over 2]\}\)
- \(N(3n) = \{1, 3n - 1, 6n\}\)
- \(N(3n + [3n+1 \over 2]) = \{3n + [3n+1 \over 2] - 1, 3n + [3n+1 \over 2] + 1, \lfloor 3n+1 \over 2 \rfloor\}\)
which satisfy the hypothesis of Lemma 5.3. Since $G_2$ is constructed from $H_2$ as $G'$ from $G$ in Lemma 5.3, $G_2$ has girth at least 5.

Similarly for $n = 5$.

All together the weights of $H_2$ and $G_2$ modulo $q$ give

$$w := \begin{cases} \pm\{3, 4, 7, 9, 11, 12\} & \text{if } n = 5 \\ \pm\{3, 4, 3n - 7, 3n - 6, 3n - 4, 3n - 3\} & \text{if } n \geq 6 \end{cases} \quad (2)$$

**Theorem 5.6** Let $q$ be a prime such that $q = 6n + 1$, $n \geq 2$. Then, there is a $(q + 3 - u)$-regular graph of girth 5 and order $2(q^2 - u - 1)$, for each $0 \leq u \leq q - 1$.

**Proof** We treat the cases $n = 2, 3$ in Section 6. For $n = 4$, $q = 6n + 1 = 25$ is not a prime, therefore we can assume that $n \geq 5$.

Let $S = T = \{0\}$ and choose $H_i, G_i$ for $i = 1, 2$ as previously described in this subsection. Lemmas 5.1, 5.2, 5.4, 5.5 together with (1) and (2) imply that the hypothesis of Theorem 4.1 are satisfied. Therefore, the graphs $B_q^*(S, T, u)$ are $(q + 3 - u)$-regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q - 1$. Note that the girth of $B_q^*(S, T, u)$ is exactly 5 (cf. Remark 4.2).

### 5.2 Construction for primes $q = 6n + 5$

We consider $n \geq 3$ throughout this subsection and we treat smaller cases in Section 6. Let $H_1$ and $H_2$ be two graphs of order $q - 1$ with the vertices labelled from 1 through $6n + 4$, and partitioned into $W_1 = \{1, 2, .., 3n + 2\}$ and $W_2 = \{3n + 3, .., 6n + 4\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows:

| Set | Edges | Description |
|-----|-------|-------------|
| $A_1$ | $\{(i, i + 1) | i = 1, \ldots, 3n + 1\} \cup \{(3n + 2, i)\}$ | $(3n + 2)$-circuit with weights 1 and $3n + 1$ |
| $B_1$ | $\{(i, i + 2) | i = 3n + 3, \ldots, 6n + 2\} \cup \{(6n + 3, 3n + 3), (6n + 4, 3n + 4)\}$ | one or two circuits according to the parity of $n$, with weights 2 and 3n |
| $C_1$ | $\{(i, 3n + i + 2) | i = 1, \ldots, 3n + 2\}$ | Prismatic edges between $W_1$ and $W_2$ of weight $3n + 2$ |

The graph $H_1$ is cubic and has weights $\pm\{1, 2, 3n, 3n + 1, 3n + 2\}$.

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows:
Lemma 5.7

Theorem 5.8

Let \( q \) be a prime such that \( q = 6n + 5 \), for \( n \geq 3 \). Then, there is a \((q+3-u)\)–regular graph of girth 5 and order \( 2(q^2 - u - 1) \) for each \( 0 \leq u \leq q - 1 \).
\textbf{Proof} Let $S = T = \{0\}$ and choose $H_i, G_i$ for $i = 1, 2$ as previously described in this subsection. By (3), (4) and Lemma 5.7 all the hypothesis of Theorem 4.1 are satisfied. Thus, the graphs $B_q^*(S, T, u)$ are $(q + 3 - u)$–regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q - 1$. Note that the girth of $B_q^*(S, T, u)$ is exactly 5 because $H_1$ has girth 5 (cf. Remark 4.2).

6 Small Cases

We now present some constructions of graphs $B_q^*(S, T, u)$ for small prime values of $q$. The first two constructions complete the proof of Theorem 5.6.

6.1 $q = 13$

In this case, let $S = T = \{0\}$, $H_1, H_2, G_1$ and $G_2$ as in Figure 1. The graphs $G_i$ are obtained from $H_i$ deleting two independent edges satisfying the hypothesis of Lemma 5.3 and joining all their end–vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (3, 12)\} \cup \{(0, 1), (0, 3), (0, 10), (0, 12)\}$, $G_2 = H_2 - \{(2, 8), (5, 11)\} \cup \{(0, 2), (0, 8), (0, 5), (0, 11)\}$ and as unlabeled graphs $G_1$ is isomorphic to $G_2$. Hence, the graphs $G_1$ and $G_2$ have order 13, girth 5 and are bi–regular with one vertex of degree four and all other vertices of degree three.

![Figure 1: The graphs $H_i$ and $G_i$ for $i \in 1, 2$ and $q = 13$.](image-url)
Note that as unlabeled graphs $H_1$ is isomorphic to $H_2$ and they are both isomorphic to one of the two cubic graphs on 12 vertices of girth 5, specifically 12 cubic graph 84 from [26, 31].

Lemma 6.1 Let $S = T = \{0\}$, $H_1, H_2, G_1$ and $G_2$ as described above. Then the graph $B^*_13(0, 0, u)$ is a $(16 - u)$–regular graph of girth 5 and order $336 - 26u$, for $0 \leq u \leq q - 1$.

Proof The weights of these graphs are $P_\omega = \pm\{1, 3, 4\}$ and $L_\omega = \pm\{2, 5, 6\}$. Thus, by Theorem 4.1 the graph $B^*_13(0, 0, u)$ is a $(16 - u)$–regular graph of girth 5 and order $26(13 - u) - 2 = 336 - 26u$, for $0 \leq u \leq q - 1$.

- For $u = 0$, we obtain a 16–regular graph of girth 5 and order 336, with exactly the same order as the $(16, 5)$–graphs that appear in [24, 20];
- for $u = 1$, we obtain a 15–regular graph of girth 5 and 310 vertices which has two vertices less than the $(15, 5)$–graphs that appear in [24, 20];
- for $u = 2$ we obtain a 14–regular graph of girth 5 and 284 vertices which has four vertices less than the $(14, 5)$–graphs in [24];

6.2 $q = 19$

Let $S = T = \{0\}$ and let $H_1, H_2, G_1$ and $G_2$ be as in Figure 2. The graphs $G_i$ are obtained from $H_1$ deleting two independent edges satisfying the hypothesis of Lemma 5.3 and joining all their end–vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (9, 16)\} \cup \{(0, 1), (0, 9), (0, 10), (0, 16)\}$ and $G_2 = H_2 - \{(8, 13), (11, 15)\} \cup \{(0, 8), (0, 13), (0, 11), (0, 15)\}$. Hence, the graphs $G_1$ and $G_2$ have order 19, girth 5 and are bi–regular with one vertex of degree four and all other vertices of degree 3.

Lemma 6.2 Let $S = T = \{0\}$, $H_1, H_2, G_1$ and $G_2$ be as described above. Then the graph $B^*_19(0, 0, u)$ is a $(22 - u)$–regular graph of girth 5 and order $720 - 38u$, for $0 \leq u \leq q - 1$.

Proof The weights of these graphs are $P_\omega = \pm\{1, 2, 3, 7, 9\}$ and $L_\omega = \pm\{4, 5, 6, 8\}$. Thus, by Theorem 4.1 the graph $B^*_19(0, 0, u)$ is a $(22 - u)$–regular graph of girth 5 and order $38(19 - u) - 2 = 720 - 38u$, for $0 \leq u \leq q - 1$.

- For $u = 0$, we obtain a 22–regular graph of girth 5 and order 720, with exactly the same order as the $(22, 5)$–graphs that appear in [24, 20];
- for $u = 1$, we obtain a 21–regular graph of girth 5 and 682 vertices which has two vertices less than the $(21, 5)$–graphs that appear in [24, 20];

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6.3 $q = 11$

For $q = 11$ we are going to remove 6 vertices from $B_{11}$ instead of 2, but we will construct a $(q + 2)$–regular graph instead of a $(q + 3)$–regular one.

**Lemma 6.3** Let $S = \{0, 1, 2, 4, 6, 8\}$ and $T = \emptyset$. Let $H_1 = (3, 5, 10, 7, 9)$ be a 5–circuit with weights $\pm\{2, 3, 5\}$, $G_1 = (0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$ be a 11–circuit with weight $\{\pm2\}$, and $H_2 = G_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \cup (0, 4) \cup (2, 6) \cup (1, 8)$ be a 11–circuit with three chords and weights $\pm\{1, 4\}$ (see Figure 3). Then the graph $B_{11}^*(S, T, u)$ is a $(13 – u)$–regular graph of girth 5 and order $22(11 – u) – 6 = 236 – 22u$, for $u \leq q – 1$. In particular, we obtain a 13-regular graph of girth 5 and order 236 for $u = 0$.

Figure 3: The graphs $H_2 = G_2$ for $q = 11$. 
Proof Since $P_\omega = \pm\{2, 3, 5\} \text{ and } L_\omega = \pm\{1, 4\}$, the thesis follows by Theorem 4.1.

Note that the graph $B_{17}^*(S, T, 0)$ has four vertices less than those constructed in [24] [20].

6.4 $q = 17$

For $q = 17$ we are going to remove 6 vertices instead of 2 and construct a $(q + 3)$-regular graph, obtaining a better result than the one obtained in [11].

Lemma 6.4 Let $S = T = \{7, 10, 12\}$, $H_1, H_2, G_1$ and $G_2$ as in Figure 4. The graphs $G_1$ and $G_2$ have order 17, girth 5 and are bi-regular with three vertices of degree four and all other vertices of degree 3. Then the graph $B_{17}^*(S, T, u)$ is a $(20 - u)$-regular graph of girth 5 and order $572 - 34u$, for $u \geq q - 1$.

Proof In this case $P_w = \pm\{1, 3, 4, 5\}$ and $L_w = \pm\{2, 6, 7, 8\}$, thus, by Theorem 4.1 the graph $B_{17}^*(S, T, u)$ is a $(20 - u)$-regular graph of girth 5 and order $34(17 - u) - 6 = 572 - 34u$ for $u \geq q - 1$.

In [24] the author constructs $(k, 5)$-graphs of order $32(k - 2)$, while we have constructed $(k, 5)$-graphs of order $34(k - 3)$ which have $44 - 2k$ fewer vertices, for $k \in \{4, \ldots, 20\}$. In particular, we obtain a 20-regular graph of girth 5 and order 572 for $u = 0$ which has four vertices less than the one constructed in [24]. Note also that as unlabeled graphs $H_1 \cong H_2$ and they are both isomorphic to the Heawood graph.

![Graphs H_i and G_i](image)

Figure 4: The graphs $H_i$ and $G_i$ for $i \in 1, 2$ and $q = 17$. 14
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