Stable Hydrogen-burning Limits in Rapidly Rotating Very Low Mass Objects

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Abstract

We present novel effects of uniform rapid stellar rotation on the minimum mass of stable hydrogen burning in very low mass stars, using an analytic model and relaxing the assumption of spherical symmetry. We obtain an analytic formula for the minimum mass of hydrogen burning as a function of the angular speed of stellar rotation. Further, we show the existence of a maximum mass of stable hydrogen burning in such stars, which is purely an artifact of rapid rotation. The existence of this extremum in mass results in a minimum admissible value of the stellar rotation period of ~22 minutes, below which a very low mass object does not reach the main sequence, within the ambit of our model. For a given angular speed, we predict a mass range beyond which such an object will not evolve into a main-sequence star.

Unified Astronomy Thesaurus concepts: Stellar physics (1621); Brown dwarfs (185); Stellar rotation (1629)

1. Introduction

Brown dwarfs (BDs), which were theoretically predicted by Kumar (1963) and Hayashi & Nakano (1963), are substellar objects whose masses range between 13 times that of Jupiter (~10^-2 M_J) and those of stars at the bottom of the main sequence (~10^-1 M_Sun). During their lifetimes, these “failed stars” do not attain sustained nuclear fusion of hydrogen into helium, as their masses are less than a certain minimum value, dubbed as the minimum mass of hydrogen burning (M_{mmhb}) or sometimes as the minimum main-sequence mass. Burrows & Liebert (1993) review the works in the period from the mid-1960s to the early 1990s (see also D’Antona & Mazzitelli 1985; Burrows et al. 1989) and provide analytical models of BDs and very low mass (VLM) stars, although their observational aspects were still in the nascent stages at that time, given that BDs are particularly difficult to detect, due to their typically low luminosities. Later, Rebolo et al. (1995) announced the first observation of a BD in Pleiades, and this was closely followed by the similar discovery of Nakajima et al. (1995). The plethora of activities that followed immediately thereafter are well documented in the review articles by Chabrier & Baraffe (2000), Basri (2000), Burrows et al. (2001), and the textbook by Rebolo & Zapatero-Oorio (2000) (see also Burrows et al. 1997; Chabrier & Baraffe 1997; D’Antona & Mazzitelli 1997). The research carried out in the area in the next decade is outlined in the more recent textbook by Joergens (2014) (see also Allard et al. 2012; Chabrier et al. 2014; Marley & Robinson 2015).

What distinguishes BDs from VLM main-sequence stars (MSSs) is the M_{mmhb}, with the currently accepted range of ~0.08 M_Sun, assuming a static scenario (the recent review of Audty et al. 2016 quotes the range 0.064–0.087 M_Sun, based on some modifications of earlier analytical models). However, it is by now well known that various factors may affect the M_{mmhb}, one example being accretion in binary systems (Salpeter 1992). In this context, we show here that the M_{mmhb} can also be enhanced from its accepted value, via stellar rotation (the physics of rotating stars are described in sufficient details in the older literature, e.g., Kippenhahn & Thomas 1970, and in the recent monographs by Tassoul 2000 and Maeder 2009). Indeed, more than five decades ago, Kippenhahn (1970) showed a possible increase in the M_{mmhb} due to rotational effects. The basic physics may seem simple—namely, that with centrifugal forces effectively reducing gravity inside a stellar object, a rotating star can maintain hydrostatic equilibrium at lower core densities and temperatures, thus requiring more mass to achieve thermal stability than its nonrotating cousin. Here one has to keep in mind that in rotating stellar objects all stellar parameters, such as density, temperature, etc., depend on the angular speed of rotation Ω, and that there are several competing effects involving the degeneracy as well. One of the results in this paper is an analytic formula of M_{mmhb} as a function of Ω.

In particular, we consider rapid rotation, where the approximation of spherical symmetry needs to be abandoned. By rapid, we mean a rotation period much smaller than that of Jupiter, which has a period of ~10 hr. Such rapid rotations in cool dwarfs have been abundantly reported in the recent past. Clarke et al. (2008) presented photometric observations of a T6 dwarf with a rotation period of 1.41 hr. Metchev et al. (2015) presented data on a T7 dwarf with a rotation period of 1.55 hr. The analysis of Route & Wolszczan (2016) obtains a dramatically smaller period of ~17 minutes for a T6 dwarf, although the authors point out that this might be a subharmonic of a longer period. Follow-up observations of the same object by Williams et al. (2017), however, indicated that this period might in fact be closer to 1.93 hr, although these authors also mention the need for more data to confirm this. The most recent analysis appears in Tannock et al. (2021), who reported on the observation of photometric periods ranging from 1.08 to 1.23 hr. Clearly then, the latest available data on the rapidly rotating BDs point to the smallest period of 1.08 hr, and Tannock et al. (2021) claim that these are “unlikely” to rotate much faster, given the clustering of the BDs having the smallest rotation periods.

With this status of observational signatures, the question we ask here is whether there are any constraints on rotations of VLM objects set by theory. This is important and interesting for several reasons. First, it is not difficult to imagine that this
The novelty of our work is the implementation of the physics of VLM stars and BDs in a rapidly rotating scenario. In addition to the results mentioned above, we are also able to provide an analytical formula for the luminosity of a VLM object when it reaches the main sequence, as a function of its mass and angular speed, within the ambit of this model. These are the main results of this paper. We should mention that we are in effect considering a toy analytical model of VLM objects, where, apart from assuming a polytropic equation of state (EOS), atmospheric corrections are ignored. Modeling the atmospheres of VLM stars and BDs is indeed an active area of current research (see, e.g., Marley & Robinson 2015). Incorporating such models along with rapid rotations of VLM objects is indeed a formidable challenge. Our simplified treatment, on the other hand, brings out several novel physical features of rapidly rotating VLM objects.

The organization of this paper is as follows. In Section 2, we recall the basic features of nonrotating VLM objects and set up the analytical model. This is then used in Section 3 to include rotation, and we present our main results in the subsequent Section 4. The paper ends with a summary in Section 5.

2. Nonrotating VLM Objects

The basic assumptions that we use here are as follows. First, the VLM object is assumed to be fully convective, containing a helium and partially ionized hydrogen mixture, with partially degenerate electrons in the interior, and a helium and molecular hydrogen mixture at the photosphere. The pressure, which arises owing to both thermal effects (ions) and degeneracy (electrons), is considered to be nonrelativistic. Importantly, these last two assumptions imply that we can safely use a polytropic EOS (polytropic approximations are discussed in the textbook of Chandrasekhar 1939; for more discussions on the applicability of this approximation to VLM objects, see Rappaport & Joss 1984; Nelson et al. 1986). Further, it is assumed that the core temperature in VLM objects is not sufficient to produce He i. Hence, the truncated $p - p$ chain thermonuclear reactions take place in the stellar interior. Note that the original model of Burrows & Liebert (1993) assumes spherical symmetry, which we will relax when we consider rapid stellar rotation.

To set the stage, and to develop the notations used in the rest of this paper, we will now recall some known facts in the evolution of VLM objects. After its formation, during the initial stages, a VLM object keeps contracting owing to its self-gravity. In the process, it keeps radiating energy from its surface, which is referred to as surface luminosity $L_S$. Now, $L_S$ keeps decreasing as the object contracts with time. The contraction, however, initially leads to an increase in its core temperature and density. It is known that rates of thermonuclear reactions are dependent on both of these. Hence, at some stage, if the attained core temperature and density are sufficient, then thermonuclear reactions start taking place. The energy generated within the object due to this is referred to as hydrogen-burning luminosity $L_{HB}$. With further contraction of the object, $L_{HB}$ starts increasing. A stellar object is said to undergo stable/sustained hydrogen burning if the amount of energy liberated from the surface is balanced by that produced from thermonuclear reactions within the star, i.e., $L_S = L_{HB}$. Hence, at some point during the object’s contracting phase, if stable hydrogen burning is attained, then further contraction ceases and the object is said to become an MSS. However, if a
considerable amount of degeneracy sets in before the object attains stability, a part of the thermal energy of the object is used up in accommodating a large number of degenerate electrons in a smaller volume. This forbids the core temperature from rising further. The core temperature thus starts falling with further contraction. This eventually leads to a decrease in the $L_{\text{HB}}$, and hence the object does not stabilize thermally. The object is then said to become a BD.

Now, if the initial mass of the object, after formation, happens to be greater than a certain minimum value, then the object eventually stabilizes before the onset of considerable electron degeneracy. It is commonly believed that this minimum value, the $M_{\text{mmhb}}$ (in the nonrotating case), sets the boundary between an MSS and a BD. Recently, however, Forbes & Loeb (2019) have shown that theoretically over-massive BDs (mass $\geq M_{\text{mmhb}}$) are possible, via accretion effects. According to their analysis, $M_{\text{mmhb}}$ should no longer demarcate between MSS and BD. However, it is still the minimum main-sequence mass.

The continuous contraction of a VLM object, after its formation, leads to an increase in its degeneracy. Hence, one simulates the time evolution of an object of a given mass, by varying the degeneracy parameter (called $\eta$ hereafter). For an object with a given mass, we compute $L_{\text{HB}}$ and $L_{\text{G}}$ at every instant of its contracting phase (i.e., lower $\eta$ value to higher ones). In the process, for every $\eta$, we get the value of the ratio of the two luminosities, $L_{\text{ratio}} = \frac{L_{\text{HB}}}{L_{\text{G}}}$. We then plot $L_{\text{ratio}}$ versus $\eta$ for the given mass. From the above discussions, we see that $L_{\text{ratio}}$ first increases and then starts descending after attaining a maximum. If the maximum value is less than unity, then it indicates that the object can never reach the stable hydrogen-burning condition $L_{\text{HB}} = L_{\text{G}}$. Hence, we repeat the above numerical procedure for a mass higher than the one previously chosen and repeat the numerical procedure, until the maximum of the plot attains unity. This mass is then the $M_{\text{mmhb}}$. At the point where an object of a given mass attains stable hydrogen burning (i.e., $L_{\text{ratio}} = 1$), it has evolved into an MSS. Thus, from that point onward one needs to consider an MSS model to further track the evolutionary process. The discussion above is illustrated in Figure 1, from which one can see that objects having masses greater than $M_{\text{mmhb}}$ attain stability (i.e., $L_{\text{ratio}} = 1$) at lower $\eta$ values.

3. Effects of Rotation in the VLM Objects’ Evolutionary Process

We first consider the VLM object to be centered at the origin of a Cartesian coordinate system $\{x^1, x^2, x^3\}$. Now we consider uniform rotation of the object along the $x^2$-axis. Due to centrifugal forces, the object bulges near the equatorial plane (i.e., $x^2 = 0$ plane), deforming it into an oblate spheroid. Hence, an object under rapid rotation loses spherical symmetry. The essential features of the analytic model, chosen for studying effects of rapid rotation, are the same as those of Burrows & Liebert (1993), except for the assumption of spherical symmetry there, which will lead to crucial modifications as we discuss below.

3.1. The Stellar Equations and Numerical Recipe

We begin with the polytropic EOS, the Poisson equation, and the Euler equation corresponding to momentum conservation. The polytropic equation reads

\[ P = \kappa \rho (1 + \eta)^n, \]

where $P$ is the pressure, $\kappa$ is the polytropic constant, and $n$ is the polytropic index, which is taken to be 1.5, as appropriate for VLM objects. The Poisson equation reads

\[ \nabla^2 \phi = 4\pi G \rho, \]

where $\rho$ is the density and $\phi$ is the gravitational potential. Finally, the Euler equation corresponding to momentum conservation is

\[ \rho \frac{\partial v^j}{\partial t} + \rho v^j \frac{\partial v^i}{\partial x^j} = - \frac{\partial P}{\partial x^i} - \rho \frac{\partial \phi}{\partial x^i}, \]

where $t$ is the temporal coordinate and $v^i = \Omega \{x^3, 0, -x^1\}$ is the velocity field of the object, with $\Omega$ being its uniform angular speed.

We get the deformed equilibrium configuration of the rotating object of a fixed mass $M$, by numerically solving Equations (2) and (3), for a given $\eta$. Initially we solve the Lane–Emden equation to obtain the spherically symmetric density profile, which is then used in Equation (2), to obtain the gravitational potential $\phi$. Then, using this $\phi$ in Equation (3), we obtain the updated density profile $\rho$. We feed this updated $\rho$ back into Equation (2), to obtain an updated $\phi$, which in turn yields an updated $\rho$ from Equation (3). This iteration is repeated until a desired convergence is achieved for a given tuple $\{M, \Omega, \kappa\}$. For further details of the numerical procedure, the reader is referred to Ishii et al. (2005) and Banerjee et al. (2021).

From the converged solution, we also obtain the central density $\rho_c$ of the deformed object in equilibrium. Now, the polytropic constant is related to the degeneracy parameter ($\eta$) of

![Figure 1. $L_{\text{ratio}}$ vs. $\eta$ for nonrotating VLM objects. The red curve corresponds to $M_{\text{mmhb}}(0.081 M_\odot)$, the green one is for mass $= 0.085 M_\odot$, and the black one is for mass $= 0.078 M_\odot$.](image)
where $m_e$ and $m_H$ denote the electron and hydrogen mass, respectively, and $\alpha = 5\mu_e/2\mu_1$. Here $\mu_e$ is the number of baryons per electron ($\mu_e = 1.143$) and $\mu_1$ is the mean molecular weight of the helium and the partially ionized hydrogen mixture in the interior ($\mu_1 = 0.996$ for Model D of Chabrier et al. 1992; see Table 1 of Aududy et al. 2016).

Also note that in Equation (4) $\eta$ is defined to be the ratio of the Fermi energy to $k_BT$, where $T$ is the temperature and $k_B$ is the Boltzmann constant. Thus, fixing $\eta$ inherently determines the corresponding $\kappa$. We then compute the central temperature $T_c$ from Equation (4), using the obtained value of $\rho_c$ for the converged equilibrium configuration of the deformed object. Using these, we numerically calculate the hydrogen-burning luminosity of the object,

$$L_{\text{HB}} = \int_V \rho \epsilon \, dx_1 \, dx_2 \, dx_3,$$

where $\rho$ and $\epsilon$ are functions of the spatial coordinates $(x_1, x_2, x_3)$, with $\epsilon$ being the energy generation rate per unit mass. The integration is performed numerically over the deformed volume $V$. For VLM objects, with typical values of the core temperature $T_c \sim 3 \times 10^6$ K and core density $\rho_c \sim 10^3$ gm cm$^{-3}$, we can fit $\epsilon$ with a power law in $T$ and $\rho$ following Burrows & Liebert (1993) and obtain

$$\epsilon = \epsilon_c \left( \frac{T}{T_c} \right) \left( \frac{\rho}{\rho_c} \right)^{\alpha - 1}, \quad \epsilon_c = \epsilon_0 T_c^4 \rho_c^{\alpha - 1} \text{erg g}^{-1} \text{s}^{-1},$$

with $s \approx 6.31$, $\alpha \approx 2.28$, and $\epsilon_0 = 1.66 \times 10^{-46}$. Next, we compute the luminosity at the photosphere (i.e., surface luminosity) of the deformed object. At any point near the surface, the atmosphere can be locally approximated to be plane parallel, irrespective of rotation. Thus, we use the following definition of optical depth ($\tau(z)$) for a planar atmosphere to determine the location of the photosphere in a deformed object:

$$\tau(z) = \int_z^\infty \kappa_R \rho dz,$$

where $z$ is the local vertical depth of the atmosphere and $\kappa_R$ is the Rosseland mean opacity, taken here to be 0.01 cm$^2$ g$^{-1}$, which is an order-of-magnitude estimate, being roughly 1/10 of the free electron opacity (see Burrows & Liebert 1993; Forbes & Loeb 2019). The photosphere is then defined to be located at $z_P$ for which $\tau(z_P) = 2/3$. The temperature $T_c$ and the density $\rho_c$ at the photosphere are related through

$$T_c = \frac{b_1 \times 10^6}{\eta^\nu} \left( \frac{\rho_c}{\text{g cm}^{-3}} \right)^{0.4},$$

where $b_1 = 2.0$ and $\nu = 1.60$ for Model D of Chabrier et al. (1992); see again Table 1 of Aududy et al. (2016). Now, using Equation (8) and the ideal gas law and assuming approximate constancy of the acceleration due to gravity near the surface, we obtain a local expression for the temperature at the photosphere:

$$T_c = \left( \frac{2 \mu_2 m_H}{3 \kappa_R k_B} \right)^{\frac{2}{5}} \left( \frac{b_1 \times 10^6}{\eta^\nu} \right)^{0.4},$$

where $\mu_2$ is the mean molecular weight of the helium and molecular hydrogen mixture at the photosphere ($\mu_2 = 2.285$). For a deformed object, the relative position of the photosphere with respect to the surface of the object does not remain constant throughout, i.e., it varies from one surface point to another, unlike the case for a spherically symmetric object. This would also be the case with $T_c$. Now, applying the Stefan–Boltzmann law, we compute the total surface luminosity as

$$L_S = \int_S \sigma T_c^4 \, dA,$$

where $dA$ is the elemental surface area and $\sigma$ is the Stefan–Boltzmann constant. The integration is performed over the entire surface of the deformed object. Finally, we compute $L_{\text{ratio}}$ for the given $\{M, \Omega, \eta\}$, which would be of fundamental importance to decide on the fate of a rotating object’s evolution. It should be noted that in the limiting case of $\Omega \to 0$ we recover the original model due to Burrows & Liebert (1993).

4. Results and Analysis

We now present the main results obtained from our computational scheme discussed above.

4.1. $M_{\text{mmhb}}$ as a Function of $\Omega$

Rotation tends to reduce the strength of gravity inside a star. This effectively makes a rotating star achieve hydrostatic equilibrium at a lower core temperature and density. As a result, total nuclear energy production is reduced, so a higher mass is required to achieve thermal stability ($L_{\text{ratio}} = 1$). Hence, we find the $M_{\text{mmhb}}$ in the presence of rotation to be larger than that in the nonrotating case. In order to find $M_{\text{mmhb}}$ corresponding to a given stellar rotation $\Omega$, we carry out a similar algorithm described in Section 2, using the numerical prescription mentioned in Section 3.1. We perform this numerical procedure for different $\Omega$ values, to obtain a fitted formula for $M_{\text{mmhb}}$ as a function of $\Omega$. We find

$$M_{\text{mmhb}}(\Omega) = 0.0814 + 0.2302 \Omega + 245.58 \Omega^2 + 67646 \Omega^3,$$

where $\Omega$ is in s$^{-1}$ and the formula gives $M_{\text{mmhb}}(\Omega)$ in units of the solar mass $M_{\odot}$. From Equation (11), we find that $M_{\text{mmhb}}$ increases monotonically with $\Omega$ as depicted by the red curve in Figure 4, but that for small $\Omega$ the change from the case $\Omega = 0$ is maximally by a few percent. For example, for the smallest observed period of 1.08 hr of Tannock et al. (2021), the increase in $M_{\text{mmhb}}$ is by $\sim 1.6\%$. Faster rotations can, however, significantly change the result. Using the period of 17 minutes from Route & Wolszczan (2016), the increase in $M_{\text{mmhb}}$ is $\sim 33\%$. However, such a small value of the period is ruled out in our analysis, as we will momentarily see. We find that the minimum period of a VLM star can be $\sim 22$ minutes, and hence the maximal increase in $M_{\text{mmhb}}$ is $\sim 17\%$. Importantly, we have
Figure 2. \( L_{\text{ratio}} \) vs. \( \eta \) plot for \( \Omega = 0.003 \text{ s}^{-1} \). The red curve corresponds to \( M_{\text{mmhb}} \). As explained in the text, it starts from \( \eta = \eta_{\text{crit}} \) and nonzero critical \( L_{\text{ratio}} \), represented by the corresponding filled black circle. It reaches main sequence when \( L_{\text{ratio}} = 1 \). For higher masses, \( \eta_{\text{crit}} \) is lower and critical \( L_{\text{ratio}} \) is higher. Higher-mass objects reach the main sequence at lower values of \( \eta \). The minimum permissible period for Model A is \( \sim 28 \) minutes. For Model H, \( \sim 1.4\% \) for Model A and \( \sim 3.1\% \) for Model H, respectively. However, such a small value of \( \eta \) corresponds to critical points corresponding to different masses ranging from \( M_{\text{mmhb}} \) to \( M_{\text{max}} \) for this \( \Omega \). \( M_{\text{max}} \) is marked by the filled magenta circle.

As an aside, we note the behavior of \( M_{\text{mmhb}} \) with \( \Omega \) for the two extreme models of Chabrier et al. (1992)—Model A and Model H in Auddy et al. (2016). For Model A we obtain \( M_{\text{mmhb}}(\Omega) = 0.0879 + 0.2441 \Omega + 220.53 \Omega^2 + 60446 \Omega^3 \), while for Model H, we obtain \( M_{\text{mmhb}}(\Omega) = 0.0636 + 0.2113 \Omega + 445.97 \Omega^2 + 117094 \Omega^3 \). The increase in \( M_{\text{mmhb}} \), corresponding to the smallest observed rotation period of 1.08 hr from Tannock et al. (2021), is by \( \sim 1.4\% \) for Model A and \( \sim 3.1\% \) for Model H. The smaller 17 minutes from Route & Wolszczan (2016) corresponds to an increase in \( M_{\text{mmhb}} \) by \( \sim 27\% \) and \( \sim 72\% \) for Model A and Model H, respectively. However, such a small value of the rotation period, \( \sim 17 \) minutes, falls below the minimum permissible periods corresponding to both Model A and Model H and is thus ruled out in our analysis. The minimum permissible period for Model A is \( \sim 20 \) minutes, which corresponds to a maximal increase in \( M_{\text{mmhb}} \) by \( \sim 19\% \). For Model H, the maximal increase in \( M_{\text{mmhb}} \) is by \( \sim 21\% \), corresponding to the minimum permissible period of \( \sim 28 \) minutes.

4.2. Behavior of \( L_{\text{ratio}} \) versus \( \eta \) Plot for Nonzero Stellar Rotation

The \( L_{\text{ratio}} \) versus \( \eta \) plot for VLM objects corresponding to a given nonzero rotation \( \Omega \) is very different from its nonrotating counterpart (compare Figure 1 with Figure 2).

Here we observe that for a given mass and rotation the \( L_{\text{ratio}} \) versus \( \eta \) curve starts from a particular point (let us call it the critical point). The particular values of \( \eta \) and \( L_{\text{ratio}} \) corresponding to the critical point are referred to as the \( \eta_{\text{crit}} \) and critical \( L_{\text{ratio}} \). We also know that during the contraction phase of a VLM object its central density \( \rho \) increases with an increase in degeneracy \( \eta \). Hence, the central density at \( \eta_{\text{crit}} \) for any VLM object of a given mass and rotation denotes the minimum value of central density below which the object cannot sustain the applied rotation (i.e., for central densities corresponding to \( \eta < \eta_{\text{crit}} \) no model solution exists). As Figure 2 indicates, for a particular \( \Omega \), higher-mass VLM objects attain the corresponding critical density at lower \( \eta_{\text{crit}} \) values, and the corresponding critical \( L_{\text{ratio}} \) are higher. It is observed from that figure that for a given rotation higher-mass objects reach the main sequence at a lower value of \( \eta \). The mass value for which critical \( L_{\text{ratio}} = 1 \) will be called \( M_{\text{max}} \) for the given \( \Omega \). For example, \( M_{\text{max}} = 0.29 M_\odot \) for \( \Omega = 0.003 \text{ s}^{-1} \) (see Figure 2).

In Figure 2, the black dotted–dashed line represents the locus of critical points (marked by filled circles) of the curves corresponding to different masses ranging from \( M_{\text{mmhb}} \) to \( M_{\text{max}} \) and is a portion of the critical \( L_{\text{ratio}} \) versus \( \eta \) curve in Figure 3, which we now explain.

4.3. The Existence of \( M_{\text{max}} \)

As we have already indicated, for central densities less than \( \rho_{\text{crit}} \) no model solution is possible for a VLM object of given mass and angular speed for which we are referring the \( \rho_{\text{crit}} \). We call the stellar configuration a critical configuration when the central density equals \( \rho_{\text{crit}} \). For a given \( \Omega \), we find the critical configuration for each value of the degeneracy \( \eta \). For each of these critical stellar configurations, we record the critical
density \( \rho_{\text{crit}} \) and compute the critical value of \( L_{\text{ratio}} \), as well as the critical mass, which we call \( M_{\text{crit}} \). While we find that \( \rho_{\text{crit}} \) remains constant with \( \eta \), Figure 3 indicates that both critical \( L_{\text{ratio}} \) and \( M_{\text{crit}} \) fall with increasing degeneracy, as is not difficult to justify physically from the following:

1. For a given \( \Omega \), we have seen that the critical density remains constant with an increase in \( \eta \). We also know that with an increase in \( \eta \) the \( \kappa \) value decreases according to Equation (4). Hence, with an increase in \( \eta \) the central pressure for the corresponding critical configurations keeps decreasing according to the formula for a polytropic EOS. We know that reduced central pressure can support lower stellar mass. Hence, for a given \( \Omega \), an increase in \( \eta \) leads to a decrease in \( M_{\text{crit}} \).

2. For a given \( \Omega \), with an increase in \( \eta \) value, the central temperature corresponding to the critical configuration keeps decreasing according to Equation (4), due to constancy of \( \rho_{\text{crit}} \). This leads to a reduction in \( L_{\text{ratio}} \) and hence \( L_{\text{ratio}} \) for the corresponding critical configurations.

The information that one obtains from the plot in Figure 3 is the particular minimum value of \( \eta = \eta_{\text{crit}} \) at which a VLM object of given mass \( M_{\text{crit}} \) has just the sufficient central density \( \rho_{\text{crit}} \) in order to sustain the given rotation \( \Omega \). One also obtains the value of the corresponding critical \( L_{\text{ratio}} \). From \( \eta_{\text{crit}} \) onward, as the object of that particular mass \( M_{\text{crit}} \) keeps contracting, the systematic behavior of \( L_{\text{ratio}} \) versus \( \eta \) follows. At this point one can draw complete correspondence between Figure 3 and Figure 2. For example, in Figure 3, we see that a VLM object of mass 0.1 \( M_\odot \) attains the critical density at \( \eta_{\text{crit}} = 5.18 \) and the corresponding critical \( L_{\text{ratio}} = 0.098 \). This particular mass object will not possess any model solution below the minimum \( \eta_{\text{crit}} \) under the given rotation. Hence, we see from Figure 2 that the \( L_{\text{ratio}} \) versus \( \eta \) curve for an object of mass 0.1 \( M_\odot \) starts from the point \( (\eta = 5.18, L_{\text{ratio}} = 0.098) \). From that point onward, as the given object contracts, the central density keeps increasing along with an increase in degeneracy \( \eta \). Thus, the model solutions for the object will exist under the given rotation for \( \eta \geq \eta_{\text{crit}} \), and we get the systematic plot of \( L_{\text{ratio}} \) versus \( \eta \) for that particular object starting from \( \eta = \eta_{\text{crit}} \).

For a given \( \Omega \), the critical \( L_{\text{ratio}} \) value corresponding to \( \eta_{\text{crit}} \) of a specified mass object reveals the relative magnitude of hydrogen-burning luminosity and surface luminosity, at the initial stage of its evolution, when it has attained sufficient degeneracy to sustain the given rotation. Three situations can arise at this particular critical point. First, if critical \( L_{\text{ratio}} < 1.0 \), the object will attain stability with further evolution only if \( M \geq M_{\text{mmb}}(\Omega) \). Second, for critical \( L_{\text{ratio}} = 1.0 \), the object has already turned into a MS. We label the corresponding mass as \( M_{\text{max}} \) for the given \( \Omega \). Finally, for critical \( L_{\text{ratio}} > 1.0 \), the hydrogen-burning luminosity exceeds surface luminosity and our VLM object can never stabilize while maintaining the given uniform rotation. We will show this explicitly in a moment. Hence, for a given \( \Omega \), the valid range of mass \( M \) for which a VLM object can eventually evolve into an MS is \( M_{\text{mmb}} \leq M \leq M_{\text{max}} \). We call this \( M_{\text{max}} \) the maximum mass of stable hydrogen burning for this \( \Omega \), and as is clear from the context, the quantity \( M_{\text{max}} \) is purely an artifact of rotational effects. We shall refer to this mass range as the transition mass range.

Let us now comment on the case of critical \( L_{\text{ratio}} > 1.0 \). To understand this, we use the standard stellar energy equation

\[
\dot{\varepsilon} - \frac{\partial L}{\partial M} = T dS/dt, \quad \text{where } L \text{ denotes the luminosity, } M \text{ the mass, } T \text{ the temperature, and } S \text{ is the entropy per unit mass.}
\]

Integrating this equation, we get, after a little bit of algebra,

\[
L_5(1 - L_{\text{ratio}}) = K \frac{d\eta}{\eta^2} dt \int \rho^{5/3} dV, \quad K = 5.25 \times 10^5 \frac{N_A k_B}{\rho_c^{5/3}},
\]

(12)

with \( N_A \) being Avogadro’s number and \( k_B \) Boltzmann’s constant. Clearly, then, at \( \eta_{\text{crit}} \), if critical \( L_{\text{ratio}} > 1 \), Equation (12) dictates that \( d\eta/dt < 0 \), and in this case the object will not possess a model solution below the minimum degeneracy \( \eta_{\text{crit}} \) for the given \( \Omega \).

4.4. \( M_{\text{max}} \) as a Function of \( \Omega \)

The concept of a critical density is only valid for a rotating stellar object. For a nonrotating stellar object, \( M_{\text{max}} \) is not defined. This means that any nonrotating object of mass \( M \geq M_{\text{mmb}} \) can in principle evolve to the main sequence. In order to find \( M_{\text{max}} \) corresponding to a given nonzero stellar rotation \( \Omega \), we numerically compute the \( M_{\text{crit}} \) value corresponding to critical \( L_{\text{ratio}} = 1.0 \). For example, from Figure 3 one can see that the \( M_{\text{crit}} \) value corresponding to critical \( L_{\text{ratio}} = 1.0 \) is 0.29 \( M_\odot \), which is the \( M_{\text{max}} \) for \( \Omega = 0.003 \text{ s}^{-1} \). We perform this numerical procedure for different \( \Omega \) values, to obtain a fitted formula for \( M_{\text{max}} \) as a function of \( \Omega \). We find, in units of \( M_\odot \),

\[
M_{\text{max}}(\Omega) = -1.2621 + 0.0176 \Omega^{-1} - 7.7873 \times 10^{-5} \Omega^{-2} + 1.1648 \times 10^{-3} \Omega^{-3},
\]

(13)

where \( \Omega \) is in \( \text{s}^{-1} \).

From Figure 4, which depicts this behavior, one can see that with an increase in \( \Omega \), \( M_{\text{max}} \) decreases. This can be explained as follows. We already know that for a given \( \Omega \), \( M_{\text{max}} \) corresponds to the mass of the particular critical configuration, for which stable hydrogen burning takes place at the critical point. Now, for a higher value of \( \Omega \), the corresponding critical configuration maintains hydrostatic equilibrium at higher central density and

![Figure 4](image-url)
temperature. As a consequence, total nuclear energy production gets magnified, resulting in critical $L_{\text{ratio}} > 1.0$. So a lower mass is needed to attain thermal stability at the critical point (i.e., critical $L_{\text{ratio}} = 1.0$). As a consequence, $M_{\text{max}}$ decreases with an increase in $\Omega$.

For $\Omega$ values less than 0.003 $\text{s}^{-1}$, the corresponding $M_{\text{max}}$ values are larger than 0.3 $M_\odot$, beyond which an object is no longer in fully convective equilibrium. Hence, those points are not shown in Figure 4.

From the behavior of $M_{\text{mnhb}}$ and $M_{\text{max}}$ with $\Omega$, we see a gradual decrease in the transition mass range [$M_{\text{mnhb}}(\Omega)$, $M_{\text{max}}(\Omega)$] with an increase in $\Omega$. There exists a certain $\Omega = 0.0047$ $\text{s}^{-1}$ where the two curves $M_{\text{mnhb}}(\Omega)$ and $M_{\text{max}}(\Omega)$ meet (this is not the one obtained by visual inspection in Figure 4, where the two quantities are drawn with different scales). Consequently, the distinctive transition mass range reduces to a single point at this particular $\Omega$. We shall call this angular speed $\Omega_{\text{max}}$. Thus, according to our model, for stellar rotations with angular speeds more than $\Omega_{\text{max}} \sim 0.0047$ $\text{s}^{-1}$ (or rotation periods less than $\sim 22$ minutes), a VLM object cannot evolve into an MSS.

4.5. Stable Luminosity Formula

Finally, we deduce a formula for the stellar luminosity ($L_{\text{HB}}$ due to H burning) at the point when the object reaches the main sequence, after initial evolution. This is denoted by $L_{\text{HB}}$ and is a function of both the stellar mass and the stellar rotation. The lowest-order polynomial that best fits our generated data is represented as $L_{\text{HB}}(M, \Omega) / L_\odot = \sum_{i,j} C_{i,j} (M/M_\odot)^i (\Omega/\Omega_\odot)^j$, and the coefficients $C_{i,j}$ are listed in Table 1.

Figure 5 represents the contour plot of $L_{\text{HB}}$. All the objects, having particular masses and rotations ($M, \Omega$) tuples, constituting any given contour, will end up in the main sequence, with the same luminosity. The results from this plot are not to be extrapolated beyond the valid range of transition mass ($M_{\text{mnhb}}(\Omega) \leq M \leq M_{\text{max}}(\Omega)$), where $\Omega \in (0, \Omega_{\text{max}})$, since beyond this range an object never evolves into an MSS.

It has been observed that our model parameters (corresponding to Model D of Chabrier et al. 1992) succeed in reproducing reasonable estimates of stellar luminosities, characteristic of the VLM objects up to a maximal mass of $\sim 0.1 M_\odot$. Hence, the confidence in our model lies precisely in the region

$$M_{\text{mnhb}}(\Omega) \leq M \leq M_{\text{max}}(\Omega),$$

with $M \leq 0.1 M_\odot$, and $0 \leq \Omega \leq \Omega_{\text{max}}$. (14)

In Figure 5, we have shown the $L_{\text{HB}}$ contours within the above-mentioned region. The blue curve represents $M_{\text{max}}(\Omega)$, while the red one corresponds to $M_{\text{mnhb}}(\Omega)$.

5. Discussions

In this paper, we have used a simplified analytical model to study the effects of rapid rotation on the $M_{\text{mnhb}}$. Our model is inspired by the one due to Burrows & Liebert (1993) to which it reduces, in the limiting nonrotating case. Following Auddy et al. (2016), we have chosen the values of some of the model parameters from Model D of Chabrier et al. (1992) in order to obtain reasonable estimates for the physical parameters. There are four main results that we have obtained:

1. We have found an analytical formula of the $M_{\text{mnhb}}$ as a function of the angular speed $\Omega$.
2. For a given $\Omega$, we have obtained the mass range $M_{\text{mnhb}}(\Omega) \leq M \leq M_{\text{max}}(\Omega)$ for VLM objects to evolve into MSSs.
3. We have obtained an upper bound $\Omega_{\text{max}} = 0.0047$ $\text{s}^{-1}$ beyond which a VLM object will not evolve into an MSS.
4. As a by-product of our analysis, we obtained the luminosity of a VLM object at the point where it reaches the main sequence, as a function of $M$ and $\Omega$.

A schematic diagram of the main results in the paper is given in Figure 6. Here, the red curve AB represents $M_{\text{mnhb}}(\Omega)$. The blue curve DG represents the maximal mass of $0.1 M_\odot$. The blue curve GB represents the portion of the $M_{\text{max}}(\Omega)$ up to a maximal mass of $0.1 M_\odot$. The two curves AB and GB intersect at point B. Point A corresponds to the $M_{\text{mnhb}}$ value for the nonrotating case, labeled as $M_{\text{mnhb}}(0)$ in the figure. The
horizontal line AC denotes a constant mass curve corresponding to the $M_{\text{mmhb}}$ for the nonrotating case. Also, O corresponds to origin, while E denotes $\Omega_{\text{max}}$. Overmassive BDs (in the region ABC) have been shown to exist purely as a result of uniform stellar rotation. In the absence of rotation, BDs lie in the region ACEO, labeled as “Normal BDs.” The VLM objects in the region DGBA can evolve into MSSs.

Here, we have used a toy model, with a number of assumptions. First, all the thermodynamic relations have been assumed to remain unaltered in the presence of rotation. This can be justified, as the rotational kinetic energy $I\Omega^2/2$, with $I$ being the moment of inertia of the deformed object computed numerically, can always be shown to be two orders of magnitude lower than the gravitational potential energy. Second, we have considered the effect of uniform stellar rotation on the VLM object’s evolution. A more realistic situation with differential and time-varying rotation is left for a future study. Third, in our analysis of stellar evolution under constant uniform rotation, the nonconservation of angular momentum has been inherently assumed. Finally, our model is polytropic and does not take account of atmospheric corrections and related details.

However, this simplistic toy model has successfully been able to decode the underlying physics of a rapidly rotating VLM object and has revealed several important limits.

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**Figure 6.** Schematic diagram of the results obtained in this paper. $M_{\text{mmhb}}(0)$ corresponds to $M_{\text{mmhb}}(\Omega = 0)$. See discussion in text.