The Shrinking Projection Method for Solving Split Best Proximity Point and Equilibrium Problems

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Abstract. In this paper, we propose a new explicit iteration method using shrinking projection for solving the split best proximity point and equilibrium problems. We prove its strong convergence under some suitable conditions in Hilbert spaces. A numerical example is given to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Let $H_1$ and $H_2$ be two real Banach spaces. Let $C$ and $D$ be two subsets of $H_1$ with $d(C, D) = \inf\{\|c - d\| : c \in C \text{ and } d \in D\}$, $K$ a closed convex subset of $H_2$, $A : H_1 \to H_2$ a bounded linear operator. Let $S : C \to D$ be a mapping and $f : K \times K \to \mathbb{R}$ be a bi-function. The SBPEP is

\begin{equation}
\|p - Sp\| = d(C, D),
\end{equation}

and

\begin{equation}
such that \quad u := Ap \in K \text{ solves } f(u, v) \geq 0, \forall v \in K.
\end{equation}

We denote the solution set of SBPEP by $\Omega = \{p \in \text{Best}_C S : Ap \in \text{EP}(f)\}$. If we consider only (1), then (1) is a classical best proximity point problem.

This problem was first introduced by Tiammee and Suantai [1]. This problem is a generalization of the common solution of best proximity point and equilibrium problem.

The best proximity point problem for nonlinear mappings is an interesting topic in the optimization theory (see [2–4]). It can be reduced to fixed point problem.

On the other hand, if we consider only (2), then (2) is a classical equilibrium point problem. Various problems arising in physics, optimization and economics can be modeled as equilibrium problems. So equilibrium problem plays very important role in solving existence of solution of these problems (see [5, 6]).
Some authors have proposed some methods to find the solution of the best proximity point problems (see [8, 11] and equilibrium problem (see [5–7]).

In 2008, Takahashi et al. introduced a new projection method which is called shrinking projection method by using the modification Mann’s iteration for obtaining strong convergence theorem for a countable family of nonexpansive mapping in real Hilbert spaces.

**Theorem 1.1.** Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $\{T_n\}$ and $\tau$ be a family of nonexpansive mapping $s$ of $C$ into $H$ such that $F := \cap_{n=0}^{\infty} F(T_n) = F(\tau) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\tau$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ in $C$ as follows:

\[
\begin{aligned}
  y_n &= \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\
  C_{n+1} &= \{ z \in C_n : \| y_n - z \| \leq \| y_n - z \| \}, \\
  u_{n+1} &= P_{C_{n+1}} x_0, \quad n \in \mathbb{N},
\end{aligned}
\]

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $u_n$ converges strongly to a point $z_0 = P_F x_0$.

In 2019, Tiammee and Suantai [1] introduced the following iterative process to approximate a solution of SBPEP in Hilbert space:

\[
\begin{aligned}
  \{ x_n \} \subset C_0, \\
  u_n &= (1 - \alpha_n) x_n + \alpha_n P_C S x_n, \quad \forall n \geq 1, \\
  x_{n+1} &= P_C \left[ u_n + \gamma A^T (T_n - I) A u_n \right], \quad n \in \mathbb{N},
\end{aligned}
\]

where $(\alpha_n) \subset (0, 1]$ with $\limsup_{n \to \infty} \alpha_n < 1$, $\tau_n \subset (0, \infty)$ with $\liminf_{n \to \infty} \tau_n > 0$ and $\gamma \in \left(0, \frac{1}{\| A \|} \right)$ is a constant. It was proved that the sequence $\{x_n\}$ generated by (4) converges weakly to $\Omega$.

In this paper, we construct some iterative algorithm which is the modified shrinking projection method for solving the SBPEP when the nonlinear mapping is best proximally nonexpansive in Hilbert spaces. Strong convergence theorem are established. The results obtained in this paper can be established as the common best proximity point problem and equilibrium problem. We also give an numerical example to support our main convergence theorem.

### 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Recall that a mapping $T : H \to H$ is said to be

1. **nonexpansive** if

\[ \| Tx - Ty \| \leq \| x - y \| \quad \text{for all} \quad x, y \in H; \]

2. **quasi-nonexpansive** if $F(T) \neq \emptyset$ and

\[ \| Tx - q \| \leq \| x - q \| \quad \text{for all} \quad x \in H, q \in F(T), \]

where $F(T) = \{ x \in C : Tx = x \}$. Observe that nonexpansive operators are quasi-nonexpansive.

Let $A$ and $B$ be two nonempty closed convex subsets of $H$. We define $A_0$ and $B_0$ by the following sets:

\[
\begin{aligned}
  A_0 &= \{ x \in A : \| x - y \| = D(A, B), \text{ for some } y \in B \}, \\
  B_0 &= \{ y \in B : \| x - y \| = D(A, B), \text{ for some } x \in A \}.
\end{aligned}
\]

We recall some useful definitions and lemmas, which will be used in the later sections.

Let $C$ be a nonempty closed convex subset of Hilbert space $H$. For any $x \in H$, its projection onto $C$ is defined as

\[ P_C(x) = \text{argmin} \{ \| y - x \| : y \in C \} \]

The mapping $P_C : H \to C$ is called a **projection operator**, which has the well-known properties in the following lemma.
Lemma 2.1. Let $C$ be a nonempty closed convex subset of Hilbert space $H$. Then for all $x, y \in H$ and $z \in C$,

- $(P_C x - x, z - P_C x) \geq 0$;
- $\|P_C x - P_C y\|^2 \leq (P_C x - P_C y, x - y)$;
- $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$;
- $\|z - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - z\|^2$

A Banach space $(X, \| \cdot \|)$ said to satisfy Opial’s condition if, for each sequence $\{x_n\}$ in $X$ which converges weakly to a point $x \in X$, we have

$$\lim inf_{n \to \infty} \|x_n - x\| < \lim inf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$ 

It is well-known that each Hilbert space satisfies Opial’s condition.

Lemma 2.2 (8). Let $A, B$ be two nonempty subsets of a uniformly convex Banach spaces $X$ such that $A$ is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_A T|_{A_0}) = \text{Best}_A(T)$.

Definition 2.3 (8). Let $A$ and $B$ be two nonempty subsets of a real Hilbert space $H$ and $C$ a subset of $A$. A mapping $T : A \to B$ is said to be $C$-nonexpansive if

$$\|T x - T z\| \leq \|x - z\|$$

for all $x \in A$ and $z \in C$. If $C = \text{Best}_A T$, we say that $T$ is a best proximally nonexpansive mapping.

Definition 2.4 (see 10). Let $A$ and $B$ be closed subsets of a metric space $(X, d)$. Then, $A$ and $B$ are said to satisfy the $P$-property if, for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:

$$d(x_1, y_1) = d(x_2, y_2) = D(A, B) \to d(x_1, x_2) = d(y_1, y_2).$$

Notic that, for any pair $(A, B)$ of nonempty closed and convex subsets of a real Hilbert space, $H$ has the $P$-property.

Lemma 2.5 (see 11). Let $A, B$ be two nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed and convex. Suppose that $T : A \to B$ is mapping such that $T(A_0) \subseteq B_0$. Then, $T|_{A_0}$ satisfies the proximal property if and only if $1 - P_A T|_{A_0}$ is demiclosed at zero.

Lemma 2.6 (see 5). Let $K$ be a nonempty closed convex subset of $H$ and $F$ be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$;
(A2) is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;
(A3) for each $x, y \in K$,

$$\lim sup_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \text{for all } x, y \in K.$$

Lemma 2.7 (see 12). Let $K$ be a nonempty closed convex subset of $H$ and let $F$ be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to K$ as follows:

$$T_r^F(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in K \right\}$$

for all $x \in H$. Then the following hold:
1. $T^f_r$ is single-valued;
2. $T^f_r$ is firmly-nonexpansive, that is, for any $x, y \in H$,
   \[ \|T^f_r(x) - T^f_r(y)\| \leq \langle T^f_r(x) - T^f_r(y), x - y \rangle; \]
3. $F(T^f_r) = EP(F)$ for all $r > 0$;
4. EP(F) is closed and convex.

**Lemma 2.8 (see [13])**. Let $K$ be a nonempty closed convex subset of $H$. For $x \in H$, let the mapping $T^f_r$ be the same as in Lemma 2.7. Then for $r, s > 0$ and $x, y \in H$,
   \[ \|T^f_r(x) - T^f_r(y)\| \leq \|y - x\| + \frac{|s - r|}{s} \|T^f_r(y) - y\|. \]

**3. Main results**

In this section, by using shrinking projection method, we obtain a strong convergence theorem for finding the solution of the SBEP in real Hilbert spaces.

**Theorem 3.1 (Strong convergence theorem)**. Let $H_1$ and $H_2$ be two real Hilbert spaces and $C, D \subset H_1, K \subset H_2$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $\text{Best}_C S \neq \emptyset$ and $f : K \times K \to \mathbb{R}$ a bi-function with EP($f$) $\neq \emptyset$. Suppose that $S$ satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by

\[
\begin{align*}
  x_0 & \in C_0, \\
  u_n &= (1 - \alpha_n)x_n + \alpha_n P_C Sx_n, \quad \forall n \geq 1, \\
  y_n &= P_C \left[ u_n + \gamma A^*(T^f_r - I)Au_n \right], \\
  C_{n+1} &= \{ v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\| \}, \\
  x_{n+1} &= P_{C_{n+1}}(x_n), \quad n \in \mathbb{N},
\end{align*}
\]

where $\{\alpha_n\} \subset (0, 1]$ with $\limsup_{n \to \infty} \alpha_n < 1$ and $\gamma \in \left(0, \frac{1}{\text{diam}(F)}\right)$ is a constant. Suppose that $\Omega = \{p \in \text{Best}_C S : Ap \in EP(f) \neq \emptyset\}$, then the sequence $\{x_n\}$ converges strongly to an element $x^* \in \Omega$.

**Proof.** It is clear that $C_{n+1}$ is closed and convex for all $n \in \mathbb{N}$. Let $p \in \Omega$. Since $\|P_C Sx_n - Sx_n\| = D(A, B)$ and $\|p - Sp\| = D(A, B)$, using P-property, we have

\[ \|P_C Sx_n - p\| = \|Sx_n - Sp\|. \]

Since $S$ is best proximally nonexpansive, and (7) we obtain

\[ \|u_n - p\| = \|(1 - \alpha_n)x_n + \alpha_n P_C Sx_n - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(P_C Sx_n - p)\| \leq \|(1 - \alpha_n)||x_n - p|| + \alpha_n||P_C Sx_n - p\| \leq (1 - \alpha_n)||x_n - p|| + \alpha_n||Sx_n - Sp|| = ||x_n - p|| \]

Next, it follows from Lemma 2.7 that

\[ 2\gamma(u_n - p, A^*(T^f_r - I)Au_n) \leq -\gamma \|(T^f_r - I)Au_n\|^2 \]
From (9) we have
\[ \|y_n - p\|^2 = \|P_C [u_n + yA^*(T_{r_n} - I)Au_n] - p\|^2 \]
\[ = \|P_C [u_n + yA^*(T_{r_n} - I)Au_n] - PCp\|^2 \]
\[ \leq \|u_n + yA^*(T_{r_n} - I)Au_n - p\|^2 \]
\[ = \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T_{r_n} - I)Au_n\|^2 + 2\gamma\langle u_n - p, A^*(T_{r_n} - I)Au_n \rangle \]
\[ \leq \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T_{r_n} - I)Au_n\|^2 - \gamma\|(T_{r_n} - I)Au_n\|^2 \]
\[ = \|u_n - p\|^2 - \gamma(1 - \gamma\|A^*\|^2)\|(T_{r_n} - I)Au_n\|^2. \] (10)

Since \( \gamma \in (0, \frac{1}{\|A^*\|^2}) \), \( \gamma(1 - \gamma\|A^*\|^2) > 0 \). It follows from (8) and (10) that
\[ \|y_n - p\| \leq \|u_n - p\| \leq \|x_n - p\| \text{ for all } n \in \mathbb{N}, \] (11)
this show \( \Omega \subseteq C_n \) and \( C_n \neq \emptyset \) for all \( n \in \mathbb{N} \). It is easy to see that \( \Omega \) is a closed convex set, so there exists a unique element \( q = P_C(x_0) \in \Omega \subseteq C_n \). Because \( x_n = P_{C_n}(x_0) \), then \( \|x_n - x_0\| \leq \|q - x_0\| \) for all \( n \in \mathbb{N} \). It follows that \( \{x_n - x_0\} \) is bounded. So are \( \{u_n\} \) and \( \{y_n\} \). Since \( C_{n+1} \subseteq C_n \) and \( x_{n+1} = P_{C_{n+1}} \in C_{n+1} \), then
\[ \|x_{n+1} - x_0\| \geq \|x_n - x_0\|, \text{ for all } n \in \mathbb{N}. \] (12)

It follows that \( \lim_{n \to \infty} \|x_n - x_0\| \) exists. Next, we will show that \( \{x_n\} \) is a Cauchy sequence. Let \( m, n \in \mathbb{N} \) with \( m > n \). Since \( x_m = P_{C_m}(x_0) \subseteq C_n \) and Lemma (2.1), we have
\[ \|x_n - x_m\|^2 + \|x_m - x_0\|^2 = \|x_n - P_{C_m}(x_0)\|^2 + \|x_0 - P_{C_m}(x_0)\|^2 \leq \|x_n - x_0\|^2. \]

It follows that \( \lim_{n \to \infty} \|x_n - x_m\| = 0 \), so \( \{x_n\} \) is a Cauchy sequence. Let \( x_n \to x^* \). Next we will show that \( x^* \in \Omega \). Since \( x_{n+1} = P_{C_{n+1}} \in C_{n+1} \), we obtain
\[ \|y_n - x_n\| \leq \|y_n - x_m\| + \|x_m - x_n\| \leq 2\|x_n - x_{n+1}\| \to 0, \]
\[ \|u_n - x_n\| \leq \|u_n - x_m\| + \|x_m - x_n\| \leq 2\|x_n - x_{n+1}\| \to 0, \] (13)
\[ \|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \to 0. \]

Moreover, from (10), we obtain
\[ \|(T_{r_n} - I)Au_n\|^2 \leq \frac{\gamma}{(1 - \gamma\|A^*\|^2)}(\|u_n - p\|^2 - \|y_n - p\|^2) \]
\[ = \frac{\gamma}{(1 - \gamma\|A^*\|^2)}(\|u_n - y_n\| + \|y_n - p\|) \]
\[ = \frac{\gamma}{(1 - \gamma\|A^*\|^2)}(\|u_n - y_n\| + \|y_n - p\|), \] (14)

which implies, by (13), that
\[ \lim_{n \to \infty} \|(T_{r_n} - I)Au_n\| = 0. \] (15)

Since \( x \to x^* \), \( A \) is a bounded linear operator and (13), we have
\[ \lim_{n \to \infty} \|Au_n - Ax^*\| = 0 \] (16)

So, by (15), (16) and Lemma 2.8, we have that for \( r > 0 \)
\[ \|T_{r_n}Ax^* - Ax^*\| \leq \|T_{r_n}Ax^* - T_{r_n}Au_n\| + \|T_{r_n}Au_n - Au_n\| + \|Au_n - Ax^*\| \]
\[ \leq \|Au_n - Ax^*\| + \frac{r_n - r}{r_n} \|(T_{r_n} - I)Au_n\| + \|T_{r_n}Au_n - Au_n\| \]
\[ + \|Au_n - Ax^*\| \to 0, \]
which implies that \(Ax^* \in F(T^r_f) = EP(f)\) for \(r > 0\).

By (6) and (13), we obtain
\[
\|P_CS_n x_n - x_n\| = \frac{1}{\alpha_n} \|x_n - u_n\| \to 0.
\]

(17)

Since \(S\) satisfies the proximal property, by Lemma 2.5, we have \(I - P_CS_n\) is demiclosed at zero. It follows that \(x^* \in F(P_CS_n) = \text{Best}_CS\). The proof is completed. \(\Box\)

By setting \(H_1 = H_2, A := I\) in Theorem 3.1, we have immediately the following collaries.

**Corollary 3.2.** Let \(H\) be a real Hilbert spaces, and \(C, D\) be nonempty closed convex subsets of \(H\). Let \(S : C \to D\) be best proximally nonexpansive mapping such that \(S(C_0) \subseteq D_0\) with \(\text{Best}_CS \neq \emptyset\) and \(f : C \times C \to \mathbb{R}\) a bi-function satisfying \((A1 - A4)\) with \(EP(f) \neq \emptyset\). Suppose that \(S\) satisfies the proximal property. Let \(\{x_n\}\) be a sequence generated by

\[
\begin{align*}
\begin{cases}
  x_0 \in C_0, \\
  u_n = (1 - \alpha_n)x_n + \alpha_n P_CS_n x_n, \\
  x_{n+1} = (1 - \gamma)u_n + \gamma T_n f u_n, \\
  n \in \mathbb{N},
\end{cases}
\end{align*}
\]

where \(\alpha_n \in (0, 1]\) with \(\limsup_{n \to \infty} \alpha_n < 1\) and \(\gamma \in \left(0, \frac{1}{\|f\|}\right)\) is a constant. Suppose that \(\text{Best}_CS \cap EP(f) \neq \emptyset\), then the sequence \(\{x_n\}\) converges weakly to an element \(x^* \in \text{Best}_CS \cap EP(f)\).

### 4. Numerical Example

We give an example and numerical result for supporting our main theorem. Moreover, we compare convergence behavior and efficiency of our algorithms with the modified Mann algorithm, introduced by Tiammee and Suantai [1]. All numerical experimental results are performed on Intel Core-i5 with 4.00 GB RAM, MacOS Catalina 10.15, under MATLAB computing environment.

**Example 4.1.** Let \(H_1 = \mathbb{R}^2, H_2 = \mathbb{R}, C = \{-1, 0\} \times [0, 1], D = [3, 7] \times [0, 1]\) and \(K = [-3, 0]\). Define two mappings \(A : \mathbb{R}^2 \to \mathbb{R}\) and \(S : C \to D\) by \(A(x^{(1)}, x^{(2)}) = 3x^{(1)}\) for all \((x^{(1)}, x^{(2)}) \in \mathbb{R}^2\) and \(S(x^{(1)}, x^{(2)}) = (3 - x^{(1)}, \frac{x^{(2)}}{2})\) for all \((x^{(1)}, x^{(2)}) \in C\). Then \(C_0 = \{(0, z) : 0 \leq z \leq 1\}\). Let \(f(u, v) = (u - 1)(v - u)\) for all \(u, v \in K\). Choose \(\alpha_n = \frac{n}{2n + 1}\) and \(\gamma = \frac{1}{20}\). It is easy to check that \(f\) satisfies all conditions in Theorem 3.1 such that \(EP(f) = \{0\}\) and \(S\) is a best proximally nonexpansive mappings such that \(S(C_0) \subseteq D_0\) and \(\text{Best}_CS = \{(0, 0)\}\).

Then Algorithm (6) can be simplified as

\[
\begin{align*}
\begin{cases}
  x_0 = (0, 0) \\
  u_n = \left(0, \frac{(3n + 2)x_n^{(2)}}{4n + 2}\right), \\
  y_n = \left(0, u_n^{(2)}\right), \\
  x_{n+1} = \left(0, \frac{y_n^{(2)} + x_n^{(2)}}{2}\right)
\end{cases}
\end{align*}
\]

(18)

Next, choosing the initial point \(x_0 = (0, 1)\) and the stopping criterion for our testing method is \(E_n = \|x_{n+1} - x_n\| \leq 1 \times 10^{-9}\). The following table shows the numerical experimental of the proposed algorithm. From Table 1, we observe that the sequence \(\{x_n\}\) converges to \((0, 0)\) which is a best proximity point of \(S\) and \(A(0, 0) = 0\) is an equilibrium point of \(f\).

Moreover, we compare the performance of Algorithm 6 (SPM-iter) and Algorithm in [1] (Mann-iter), all controllers are setting in Table 2. In numerical experiment, it is revealed that the sequence generated by Mann-iter of Suantai and Tiammee [1] converges more quickly than by Algorithm 6 do.
| $n$ | $x_n$ | $E_n$ |
|-----|-------|-------|
| 0   | (0, 1) | -     |
| 1   | (0, 0.9167) | 0.0833|
| 2   | (0, 0.8250)  | 0.0917|
| 3   | (0, 0.7366)  | 0.0884|
| ... | ...     | ...   |
| 143 | (0, 8.2596e-09) | 1.1752e-09|
| 144 | (0, 7.2307e-09) | 1.0288e-09|
| 145 | (0, 6.3300e-09) | 9.0071e-10|

Table 1: Numerical results for Algorithm 6

| Method               | Setting                                                                 |
|----------------------|-------------------------------------------------------------------------|
| SPM-iter (Algorithm 6) | $\alpha_n = \frac{n}{2n+1}, r_n = \frac{n}{n+1}, \gamma = \frac{1}{20}$ and $x_0 = (0, 1)$ |
| Mann-iter [1]        | $\alpha_n = \frac{n}{2n+1}, r_n = \frac{n}{n+1}, \gamma = \frac{1}{20}$ and $x_0 = (0, 1)$ |

Table 2: Algorithms and their setting controls

Figure 1: The error plotting of $E_n = \|x_{n+1} - x_n\|$ 

Acknowledgements

This research was supported by Chiang Mai University. The second author would like to thank the Thailand Research Fund and Office of the Higher Education Commission under Grant No. MRG6180050 for the financial support.

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