From quantum to quantum via decoherence*

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March 28, 2022

Abstract
Various physical effects resulting from decoherence are discussed in the algebraic framework. In particular, it is shown that the environment may induce not only classical properties like superselection rules, pointer states or even classical behavior of the quantum system, but, what is more, it also allows the transition from statistical description of infinite quantum systems to quantum mechanics of systems with a finite number of degrees of freedom. It is shown that such transition holds for the quantum spin system in the thermodynamic limit interacting with the phonon field.

1 INTRODUCTION
The problem of transition from microscopic to macroscopic description of Nature is a fundamental one in the discussion of the interpretation of quantum mechanics. In recent years decoherence has received much attention and has been accepted as the mechanism responsible for the appearance of classicality in quantum measurements and the absence in the real world of Schrödinger-cat-like states [1, 2, 3, 4, 5]. It was also shown that decoherence is a universal short time phenomenon independent of the character of the system and reservoir [6]. Different decoherence regimes that are important

*Dedicated to Gianfausto Dell’Antonio on his 70th birthday
for the experimental search of the transition between classical and quantum worlds were discussed in [7]. The intuitive idea of decoherence is rather clear: quantum interference effects for macroscopic systems are practically unobservable because superpositions of their quantum states are effectively destroyed by the surrounding environment. More precisely, it accepts the wave function description of such a system but contends that it is practically impossible to distinguish between vast majority of its pure states and the corresponding statistical mixtures. Therefore, this approach has been called by Bell a FAPP (for all practical purposes) solution to the measurement problem and to the Schrödinger cat paradox. However, in spite of the progress in the theoretical and experimental understanding of decoherence, its range of validity and its full meaning still need to be revealed [8, 9].

1.1 Algebraic framework

Everybody agrees that concepts of classical and quantum physics are opposite in many aspects. Therefore, in order to demonstrate how quanta become classical, it is necessary to express them in one mathematical framework. In a recent paper [10] such an algebraic framework which enables a general discussion of environmentally induced classical properties in quantum systems has been proposed. It is worth noting that the idea of using the same algebraic description of both quantum and classical mechanics was suggested in [11]. In this approach observables of any physical system are represented by self-adjoint elements of some operator algebra $\mathcal{M}$, the so-called von Neumann algebra, acting in a Hilbert space associated with the system. Genuine quantum systems are represented by factors i.e. algebras with a trivial center $Z(\mathcal{M}) = \mathbb{C} \cdot 1$, $1$ stands for the identity operator, whereas classical systems are represented by commutative algebras. Since a classical observable by definition commutes with all other observables so it belongs to the center of algebra $\mathcal{M}$. Hence the appearance of classical properties of a quantum system results in the emergence of an algebra with a nontrivial center, while transition from a noncommutative to commutative algebra corresponds to the passage from quantum to classical description of the system. Since automorphic evolutions preserves the center of each algebra so this program may be accomplished only if we admit the loss of quantum coherence, i.e. that quantum systems are open and interact with their environment.

In order to study decoherence, analysis of the evolution of the reduced density matrices obtained by tracing out the environmental variables is the most convenient strategy. More precisely, the joint system composed of a
quantum system and its environment evolves unitarily with the Hamiltonian $H$ consisting of three parts

$$H = H_S \otimes 1_E + 1_S \otimes H_E + H_I. \tag{1}$$

The time evolution of the reduced density matrix is then given by

$$\rho_t = \text{Tr}_E(e^{-\frac{i}{\hbar}Ht}(\rho_0 \otimes \omega_E)e^{\frac{i}{\hbar}Ht}), \tag{2}$$

where $\text{Tr}_E$ denotes the partial trace with respect to the environmental variables, and $\omega_E$ is a reference state of the environment. Alternatively, one may define the time evolution in the Heisenberg picture by

$$T_t(A) = P_E(e^{\frac{i}{\hbar}Ht}(A \otimes 1_E)e^{-\frac{i}{\hbar}Ht}), \tag{3}$$

where $A \in \mathcal{M}$ is an observable of the system and $P_E$ denotes the conditional expectation onto the algebra $\mathcal{M}$ with respect to the reference state $\omega_E$. In this paper we shall work in the Heisenberg picture. Superoperators $T_t$ being defined as the composition of a $^*$-automorphism and conditional expectation satisfy in general a complicated integro-differential equation. However, for a large class of models, this evolution can be approximated by a dynamical semigroup $T_t = e^{tL}$, whose generator $L$ is given by a Markovian master equation, see [12, 5, 13]. It represents on the algebraic level irreversible evolution of the system.

We are now in a position to discuss rigorously the dynamical emergence of classical observables. As was shown in [10] for each (up to some technical assumptions) Markov semigroup $T_t$ on $\mathcal{M}$ one may associate a decomposition

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \tag{4}$$

such that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are $T_t$-invariant and the following properties hold:

(i) $\mathcal{M}_1$ is a von Neumann subalgebra of $\mathcal{M}$ and the evolution $T_t$ when restricted to $\mathcal{M}_1$ is reversible, given by a one parameter group of $^*$-automorphisms of $\mathcal{M}_1$.

(ii) $\mathcal{M}_2$ is a linear space (closed in the norm topology) such that for any observable $B = B^* \in \mathcal{M}_2$ and any statistical state $\rho$ of the system there is

$$\lim_{t \to \infty} < T_tB >_\rho = 0, \tag{5}$$

where $< A >_\rho$ stands for the expectation value of an observable $A$ in state
The above result means that any observable $A$ of the system may be written as a sum $A = A_1 + A_2$, $A_i \in M_i$, $i = 1, 2$, and all expectation values of the second term $A_2$ are beyond experimental resolution after the decoherence time. Therefore, if decoherence is efficient then almost instantaneously what we can observe are observables contained in the subalgebra $M_1$. In other words we apply Borel’s 0th axiom: Events with very small probability never occur. Hence all possible outcomes of the process of decoherence can be directly expressed by the description of this subalgebra and its reversible evolution.

1.2 Four aspects of decoherence

One of the effects resulting from decoherence which has been widely discussed so far is the destruction of macroscopic interferences or, in other words, environmentally induced superselection rules. They arise when the phase factors between states belonging to two distinct subspaces of the Hilbert space of the quantum system are being continuously destroyed by the interaction with its environment. The loss of quantum coherence in the Markovian regime was established in a number of papers [14, 15] giving clear evidence of dynamical appearance of superselection rules. It was also shown that superselection rules may emerge through the interaction of a charged particle with electromagnetic fields [16]. Expressing these results in terms of the algebraic language we will say that decoherence induces superselection rules in the quantum system if the algebra $M_1$ is still noncommutative but has a nontrivial center $Z(M_1)$. Indeed, in such a case the algebra $M_1$ is a block algebra with respect to the decomposition of the Hilbert space $H = \bigoplus \mathcal{H}_\alpha$ associated with the central projections in $M_1$. The discreteness or continuity of the center $Z(M_1)$ corresponds therefore to the case of discrete or continuous superselection rules.

Another aspect of decoherence which was analyzed in a number of models is the selection of the preferred basis of pointer states, the so-called einselection, [5, 17, 18]. It occurs when the reduced density matrix of the system becomes approximately diagonal in a time much shorter than the relaxation time. Most models predict that these states exist and are orthogonal so they allow to define a unique set of alternative events with well definite probabilities. It follows that pointer states do not evolve at all, while all other pure states deteriorate in time to classical probability distributions over the one-dimensional projections corresponding to these states. However, it should
be pointed out that the algebra generated by these projections is always of a discrete type, and, as was shown in [19], the discreteness is unavoidable as long as we consider quantum systems with a finite number of degrees of freedom. A new perspective is opened when we consider quantum systems in the thermodynamic limit. In [20] it was shown that the interaction between an infinite quantum spin system linearly coupled to a phonon field yields a selection of a continuous family of pointer states corresponding to an apparatus with continuous readings. These results suggest the following definition. We will say that decoherence induces pointer states of the quantum system if $\mathcal{M}_1$ is commutative and the restriction of the evolution $T_t$ to $\mathcal{M}_1$ is trivial, i.e. $T_t(A) = A$ for any observable $A \in \mathcal{M}_1$ and all times $t$. The discreteness or continuity of the pointer states corresponds again to the same property of the algebra $\mathcal{M}_1$.

The origin of deterministic laws that govern the classical domain of our everyday experience has also attracted much attention in recent years. In particular, the emergence of classical mechanics described by differential, and hence local, equations of motion from the evolution of delocalized quantum states was at the center of this issue. For example, the question in which asymptotic regime non-relativistic quantum mechanics reduces to its ancestor, i.e. Hamiltonian mechanics, was addressed in [21]. It was shown there that for very many bosons with weak two-body interactions there is a class of states for which time evolution of expectation values of certain operators in these states is approximately described by a non-linear Hartree equation. The problem under what circumstances such an equation reduces to the Newtonian mechanics of point particles was also discussed in that paper. A different point of view was taken in a seminal paper by Gell-Mann and Hartle [22]. They gave a thorough analysis of the role of decoherence in the derivation of phenomenological classical equations of motion. Various forms of decoherence (weak, strong) and realistic mechanisms for the emergence of various degrees of classicality were also presented. In the same spirit it was shown in [23] that an infinite quantum system subjected to a specific interaction with another quantum system may be effectively described as a simple classical dynamical system. More precisely, the effective observables of the system were parameterized by a single collective variable which underwent a continuous periodic evolution. These results lead us to the following definition. We will say that decoherence induces classical behavior of the quantum system if $\mathcal{M}_1$ is commutative and its evolution is given by a continuous flow on the configuration space of the algebra $\mathcal{M}_1$.

While the interaction of quantum systems with their environment con-
tributes a great deal to the appearance of classical reality like superselection rules, pointer states and classical dynamics, this is not the whole story. It is clear from the above discussion that something is missing in the presented effects of decoherence. Indeed, it may happen that phase factors are destroyed in such a specific way that the observables immune to decoherence form again a noncommutative algebra with a trivial center. In such a case, which, as far as we know, has never been addressed, one may speak of the appearance of a new genuine quantum system without any classical properties and with completely different quantum properties. The most interesting example of such an effect is of course the reduction of an infinite quantum system to a quantum system possessing only one degree of freedom. This would help in the understanding how it is possible that quantum mechanics is so efficient in the world, where almost all quantum objects should be described in terms of quantum field theory. The possibility of such transition is the main objective of the present paper. For its derivation we consider a completely solvable but simplified model of an infinite array of spin-$\frac{1}{2}$ particles. Since we neglect the position variables what we achieve is a toy model of quantum mechanics represented by a spin algebra of $2 \times 2$ matrices with the Hamiltonian evolution given by the third Pauli matrix. This simplified model suggests, however, the possibility of deriving the Schrödinger equation form quantum theory of infinite systems interacting with their environment.

2 DECOHERENCE INDUCED SPIN ALGEBRA

There are two approaches to the algebraic structure associated with a quantum system. In the first one one starts with the Hilbert space of states of the system and subsequently introduces the algebra of operators corresponding to physical observables. In the second approach of statistical mechanics one postulates certain structural features, like canonical commutation or anti-commutation relations, of an abstract algebra, and then recovers the traditional point of view by passing to a particular representation, the so-called Gelfand-Naimark-Segal (GNS in short) representation, of the algebra [24]. Clearly, the description of quantum systems in the thermodynamic limit by statistical mechanics is an idealization of a finite physical system with a huge number of degrees of freedom by an infinite theoretical model. Nevertheless, such an approach proved to be very efficient in many concrete problems. In this section we use this algebraic framework to discuss the transition of an infinite system of spin-$\frac{1}{2}$ particles, linearly coupled to a phonon field, to the
spin algebra.

2.1 The model

The infinite quantum spin system consists of a set of noninteracting spin-$\frac{1}{2}$ particles fixed at positions $n = 1, 2, \ldots$ and exposed to a magnetic field. The algebra $\mathcal{M}$ of its bounded observables is given by the $\sigma$-weak closure of $\pi_0(\otimes_1^\infty M_{2\times2})$, where $\pi_0$ is a (faithful) GNS representation with respect to a tracial state $\text{tr}$ on the Glimm algebra $\otimes_1^\infty M_{2\times2}$, and $M_{2\times2}$ is the algebra generated by Pauli matrices. Let us point out that $\mathcal{M}$ is not a "big" matrix algebra. It is a continuous algebra (factor of type II$_1$) in which there are no pure states. In fact, any projection $e \in \mathcal{M}$ contains a nontrivial subprojection $f \in \mathcal{M}$. It is worth noting that the absence of minimal projections is a new feature which may be present only in systems in the thermodynamic limit. Since the particles are noninteracting, their evolution is given by a free Hamiltonian which corresponds to the interaction of the spins with an external magnetic field parallel to the $z$-axis and of strength $H(n)$ at the site $n$

$$H_S = \pi_0 \left( -g\mu_B \sum_{n=1}^\infty H(n)\sigma^3_n \right),$$

(6)

where $g$ is the Landé factor, $\mu_B$ is the Bohr magneton and $\sigma^3_n$ is the third Pauli matrix in the $n$th site. We assume that the magnetic field decreases as $H(n) \sim (\frac{1}{q})^n$ for some $q \geq 2$. Since the coefficients $H(n)$ are summable, the Hamiltonian $H_0$ is bounded. Moreover, its eigenvalues are nondegenerate.

The reservoir is chosen to consist of noninteracting phonons of an infinitely extended one dimensional harmonic crystal at the inverse temperature $\beta = \frac{1}{kT}$. The Hilbert space $\mathcal{H}$ representing pure states of a single phonon is (in the momentum representation) $\mathcal{H} = L^2(\mathbf{R}, dk)$. A phonon energy operator is given by the dispersion relation $\omega(k) = |k|$ ($\hbar = 1$, $c = 1$). It follows that the Hilbert space of the reservoir is $\mathcal{F} \otimes \mathcal{F}$, where $\mathcal{F}$ is the symmetric Fock space over $\mathcal{H}$. A phonon field $\phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))$, where $a^*(f)$ and $a(f)$ are given by the Araki-Woods representation [25]:

$$a^*(f) = a_F^*(1 + \rho)^{1/2} f \otimes I + I \otimes a_F(\rho^{1/2} f),$$

(7)

$$a(f) = a_F((1 + \rho)^{1/2} f) \otimes I + I \otimes a_F^*(\rho^{1/2} f).$$

(8)

Here $a_F^*(a_F)$ denotes respectively creation (annihilation) operators in the Fock space, and $\rho$ is the thermal equilibrium distribution related to the
phonons energy according to the Planck law
\[ \rho(k) = \frac{1}{e^{\beta \omega(k)} - 1}. \]  
(9)

Since the phonons are noninteracting, their dynamics is completely determined by the energy operator
\[ H_E = H_0 \otimes I - I \otimes H_0, \]  
(10)

where \( H_0 = d\Gamma(\omega) = \int \omega(k)a_F^*(k)a_F(k)dk \) describes dynamics of the reservoir at zero temperature. The reference state of the reservoir is taken to be a gauge-invariant quasi-free thermal state given by
\[ \omega_E(a^*(f)a(g)) = \int \rho(k)g(k)f(k)dk. \]  
(11)

Clearly, \( \omega_E \) is invariant with respect to the free dynamics of the environment.

The Hamiltonian \( H \) of the joint system consists of the three parts \( H = H_S + H_E + H_I \), where \( H_I \) is the interacting Hamiltonian. We assume that the coupling is linear (as in the spin-boson model), i.e. \( H_I = \lambda Q \otimes \phi(g) \), where
\[ Q = \pi_0 \left( \sum_{n=1}^{\infty} a_n \sigma^1_n \right), \]  
(12)

\( \sigma^1_n \) stands for the first Pauli matrix in the \( n \)th site, \( \lambda > 0 \) is a coupling constant, and \( a_n \sim \left( \frac{1}{n} \right)^p \) for some \( p \geq 2 \). Again, since the coefficients \( a_n \) are summable, the coupling operator \( Q \) is bounded and has a nondegenerate spectrum. Finally, we impose some restriction on the test function \( g(k) \) of the phonon field. We assume that \( g(k) = |k|^{1/2}\chi(k) \), where \( \chi(k) \) is an even and real valued function such that: (i) \( \chi \) is differentiable with bounded derivative, (ii) for large \( |k|, |\chi(k)| \leq C |k|^{-\epsilon}, \) \( C > 0, \) \( \epsilon > 0, \) and \( \chi(0) = 1. \) The behavior of the test function \( g \) at the origin and its asymptotic bound are taken to ensure that \( H \) is essentially self-adjoint. Hence it induces a unitary evolution of the compound system.

2.2 Description of effective observables

The reduced (irreversible) dynamics of the system is given by Eq. (3) with the Hamiltonian \( H \) introduced in the previous subsection. Because neither \( H_S \) and \( H_I \), nor \( H_E \) and \( H_I \) commute, it is a nontrivial step to derive an explicit formula for the superoperators \( T_t \). However, as was mentioned in
the Introduction, one may apply the Markovian approximation to simplify
the problem. Because the thermal correlation function

\[
< \phi_t(g) \phi(g) > = \omega_E (e^{itH_E} \phi(g) e^{-itH_E} \phi(g)) \\
= \omega_E (\phi(e^{it\omega} g) \phi(g))
\]  

(13)
is integrable, we use the so-called singular coupling limit to conclude that

\[ T_t = e^{tL} \]
is a quantum Markov semigroup with the generator \( L \) given by the
following master equation, see [20],

\[
L(A) = i[H_S - bQ^2, A] + L_D(A),
\]  

(14)
where

\[
L_D(A) = \frac{2\pi \lambda}{\beta} (QAQ - \frac{1}{2} (Q^2, A)),
\]  

(15)
and \( b = \int_0^\infty \chi^2(k) dk > 0 \). The first part in Eq. (14) is the commutator with
a new collective Hamiltonian \( H_C = H_S - bQ^2 \), while the second term is a
dissipative operator. The collective Hamiltonian

\[
H_C = -\pi_0 \left( g\mu_B \sum_{n=1}^{\infty} H(n)\sigma_n^3 \right) \\
- \pi_0 \left( b \sum_{n,m=1}^{\infty} J(n+m)\sigma_n^1\sigma_m^1 \right),
\]  

(16)
where \( J(n) = a_n \), is similar to that of the anisotropic Heisenberg model
with an infinite range interaction. However, the potentials \( H(n) \) and \( J(n) \)
are not translationally invariant.

We are now in a position to formulate our main result (its proof will be
given in the Appendix).

THEOREM: For the semigroup \( T_t = e^{tL} \) the decomposition (4) holds with
\( \mathcal{M}_1 = C \cdot 1_S \). If we put in Eq. (12) \( a_1 = 0 \), then \( \mathcal{M}_1 = M_{2x2} \) and for any
\( A \in \mathcal{M}_1 \)

\[
T_t(A) = e^{ith_1 \sigma^3} A e^{-ith_1 \sigma^3},
\]  

(17)
where \( h_1 = H(1) \).
This result shows that the infinite quantum spin system, subjected to a
specific interaction with the phonon field, after the decoherence time may
be effectively described as a quantum system with only one degree of freedom
(generalization to a finite number of degrees of freedom is straightforward).
In other words, the environment forces the spin particles to behave in a collective way what allows introduction of three collective observables which satisfy the standard commutation relations of spin momenta. Although, the presented model neglects position variables and so is not complex enough to allow derivation of the Schrödinger equation, it suggests that decoherence induced reduction of quantum statistical mechanics of many body systems to quantum mechanics of wave functions is possible.

A Proof of theorem

Step 1. It is clear from the form of the generator $L$, see Eq. (14), that it generates a semigroup of completely positive and normal superoperators which are contractive in the operator norm. Moreover, $\text{tr}L(A) = 0$, which implies that the semigroup $T_t$ is trace preserving. Hence the decomposition (4) follows from Theorem 11 in [10].

Step 2. The subalgebra $\mathcal{M}_1$ is defined by the property $T_t^*T_t x = T_t T_t^* x = x$ for all $t \geq 0$ [23]. Hence

$$
\bigcap_{l=0}^{\infty} \ker L_D \circ \delta_{HC}^l \subset \mathcal{M}_1,
$$

where $\delta_{HC}(\cdot) = i[H_C, \cdot]$. We prove now the reverse inclusion. Suppose that $x \in \mathcal{M}_1$. Then, by differentiating the equation $T_t^*T_t x = x$ at time $t = 0$, we get $\mathcal{M}_1 \subset \ker L_D$. Assume that

$$
\mathcal{M}_1 \subset \bigcap_{l=0}^{n-1} \ker L_D \circ \delta_{HC}^l
$$

for some $n \geq 1$. Because

$$
\frac{d^{n+1}}{dt^{n+1}} T_t^*T_t x|_{t=0} = 0
$$

so

$$
\frac{d^{n+1}}{dt^{n+1}} T_t^*T_t x|_{t=0} = (-\delta_{HC} + L_D)^{n+1}(x) + \sum_{m=1}^{n} \binom{n+1}{m} (-\delta_{HC} + L_D)^{n+1-m} \circ (\delta_{HC} + L_D)^m(x)
$$
\[(\delta_{HC} + L_D)^n(x)\] 
\[= (-1)^n n! \delta_{HC}^n(x) + (-1)^n L_D \circ \delta_{HC}^n(x) + \delta_{HC}^n(x) + L_D \circ \delta_{HC}^n(x) + \]
\[\sum_{m=1}^{n} \left( \binom{n+1}{m} \delta_{HC}^m \right) \cdot L_D = (\delta_{HC} + L_D)^{n+1}(x)\]
\[= (-1)^n n! \delta_{HC}^n(x) + (-1)^n L_D \circ \delta_{HC}^n(x) + \delta_{HC}^n(x) + L_D \circ \delta_{HC}^n(x) + \]
\[\sum_{m=1}^{n} \left( \binom{n+1}{m} \delta_{HC}^m \right) (-1)^n n! \delta_{HC}^n(x) + \delta_{HC}^n(x) + L_D \circ \delta_{HC}^n(x) + \]
\[\sum_{m=1}^{n} \left( \binom{n+1}{m} \delta_{HC}^m \right) (-1)^n n! \delta_{HC}^n(x) + \delta_{HC}^n(x) + L_D \circ \delta_{HC}^n(x) + \]
\[+ \left\{ 1 + (-1)^n \left( \sum_{m=0}^{n} \binom{n+1}{m} (-1)^m \right) \right\} L_D = 2L_D \circ \delta_{HC}^n(x) = 0.\]

Hence, by induction,
\[\mathcal{M}_1 \subset \bigcap_{i=0}^{\infty} \ker L_D \circ \delta_{HC}^i.\]

Step 3. Let \(C_1\) (respectively \(C_3\)) be a \(C^*\)-subalgebra in the Glimm algebra generated by \(\{\sigma_1^{i_1}...\sigma_n^{i_n}\}\), where \(i_k = 0, 1, 3\) respectively, and \(n \in \mathbb{N}\). Then both \(\pi_0(C_1)\) and \(\pi_0(C_3)\) are maximal Abelian self-adjoint algebras (m.a.s.a in short) in \(\mathcal{M}\) such that \(\pi_0(C_1) \cap \pi_0(C_3) = \mathbf{C} \cdot 1_S\). The choice of coefficients \((H(n))\) and \((a_n)\) guarantees that \(L^\infty(Q) = \pi_0(C_1)\) and \(L^\infty(H_S) = \pi_0(C_3)\), where \(L^\infty(Q)\) is the von Neumann algebra generated by operator \(Q\). Hence \(L^\infty(Q) \cap L^\infty(H_S) = \mathbf{C} \cdot 1_S\).

Step 4. We show now that if \([Q, [Q, x]] = 0\) for some \(x \in \mathcal{M}\), then \(x \in L^\infty(Q)\). Let us define the derivation \(\delta_x(\cdot) = i[\cdot, x]\). If \([Q, [Q, x]] = 0\), then \([Q, x] \in L^\infty(Q)\) since, by step 3, \(L^\infty(Q)\) is a m.a.s.a. Suppose that \(P\) is a polynomial. Then
\[\delta_x(P(Q)) = i[Q, x]P'(Q) \in L^\infty(Q).\]

This implies that \(\delta_x(L^\infty(Q)) \subset L^\infty(Q)\) since \(\delta_x\) is continuous in the weak operator topology. Because \(L^\infty(Q)\) is commutative so \(\delta_x|_{L^\infty(Q)} = 0\), and hence \([Q, x] = 0\). Because \(L^\infty(Q)\) is a m.a.s.a so \(x \in L^\infty(Q)\).

Step 5. Next we show that \(\ker L_D \cap L^\infty(H_C) = \mathbf{C} \cdot 1_S\). Here \(L^\infty(H_C)'\) stands for the commutant in \(\mathcal{M}\) of the algebra \(L^\infty(H_C)\). Suppose that \(x \in \ker L_D \cap L^\infty(H_C)\). Then \([Q, [Q, x]] = 0\) and \([H_C, x] = 0\). By step 4, \(x \in L^\infty(Q)\) which implies that \([H_S, x] = [H_C + bQ^2, x] = 0\). Hence
$x \in L^\infty(H_S)$. Because, by step 3, $L^\infty(Q) \cap L^\infty(H_S) = C \cdot 1_S$ so $x = z 1_S$, where $z \in C$.

Step 6. By step 2, $\delta_{H_C}(M_1) \subset M_1$. Hence, the derivation $\delta_1 := \delta_{H_C}|_{M_1}$ is well defined and bounded. Thus $\delta_1(\cdot) = i[H_1, \cdot]$, where $H_1 = H_1^* \in M_1$ [26]. By step 2 again, $H_1 \in \ker L_D$. On the other hand

$[H_C, H_1] = -i\delta_1(H_1) = [H_1, H_1] = 0$.

Hence $H_1 \in L^\infty(H_C)'$ and so, by step 5, $H_1$ is proportional to the identity operator. Suppose now that $x \in M_1$. Then $[H_C, x] = -i\delta_1(x) = 0$, and so $x \in L^\infty(H_C)'$. Because $x \in \ker L_D$ so, by step 5, $x$ is proportional to the identity operator. Hence $M_1 = C \cdot 1_S$.

Step 7. Finally, suppose that in Eq. (12) the coefficient $a_1 = 0$. The corresponding semigroup we shall denote by $T^1_t$. Let $\mathcal{A}$ be a subalgebra in $\mathcal{M}$ generated by \{ $\pi_0(\sigma^k_1) : k = 0, 1, 2, 3$ \}. Suppose that $x \in \mathcal{M}$. Then

$x = \sum_{k=0}^3 \pi_0(\sigma^k_1)x_k$,

where operators $x_k$ belong to $\mathcal{A}'$, the commutant in $\mathcal{M}$ of algebra $\mathcal{A}$. Let $S_t$ be a semigroup on $\mathcal{M}$ with a generator $L_0$ given by the following Markov master equation

$L_0(A) = i[H^0_S - bQ^2, A] + L_D(A),$

where $L_D$ is defined in Eq. (15), and

$H^0_S = \pi_0 \left( -g\mu_B \sum_{n=2}^{\infty} H(n)\sigma^3_n \right)$.

Note that the summation index ranges from 2 to infinity. Then

$T^1_t(x) = \sum_{k=0}^3 \pi_0(U^*_t \sigma^k_1 U_t)S_t(x_k),$

where $U_t = e^{-ith_1 \sigma^3_1}$. Let $L^2(\mathcal{M})$ be the noncommutative Hilbert space of square integrable (with respect to the trace tr) operators. Since operators $\pi_0(U^*_t \sigma^k_1 U_t)$, $k = 0, 1, 2, 3$, are orthogonal in $L^2(\mathcal{M})$ so

$\|T^1_t(x)\|_{L^2}^2 = \sum_{k=0}^3 \|S_t(x_k)\|_{L^2}^2$. 

12
Let us notice that the semigroup $S_t$ restricted to the commutant $\mathcal{A}'$ has the same properties as the semigroup $T_t$. Hence, if any of $x_k$ is not proportional to the identity operator, then, by step 6, $\|S_t(x_k)\|_{L^2} < \|x_k\|_{L^2}$ for all $t > 0$. Thus $\|T_t^1(x)\|_{L^2} < \|x\|_{L^2}$, too, which implies that such an operator cannot belong to $\mathcal{M}_1$. Hence, if $x \in \mathcal{M}_1$, then $x_k = z_k 1_S$, $z_k \in \mathbb{C}$, for all $k = 0, 1, 2, 3$, and so $x \in \mathcal{A}$. It follows that $\mathcal{M}_1 = \mathcal{A} = M_{2 \times 2}$, and the dynamics on it is given by unitary operators $U_t$.

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