A FINITE DIMENSIONAL ALGEBRA
OF THE DIAGRAM OF A KNOT

CLAUDE CIBILS

(Communicated by Birge Huisgen-Zimmermann)

Abstract. To a regular projection of a knot we associate a finite dimensional
non-commutative associative algebra which is self-injective and special biserial.

1. Introduction

Let $D$ be an oriented knot diagram that is a regular projection of a knot to
the plane where each crossing is given with the information of which part of the
knot under-crosses the other one; this appears in $D$ by interrupting the part of the
diagram which is under-crossing.

In this note we associate to $D$ a finite dimensional algebra over a field $k$, presented
by a quiver with relations deduced from the knot. In other words we associate to $D$
a $k$-category given by a presentation; it has a finite number of objects corresponding
to the crosses, and finite dimensional vector spaces of morphisms. The algebra of
the diagram is the direct sum of all the morphisms of the $k$-category with product
induced by the composition of the category or, equivalently, is the path algebra of
the quiver modulo the two-sided ideal generated by the relations.

This algebra of the knot projection is Morita reduced, special biserial and self-
injective. As such it is not invariant under Reidemeister moves since its dimension
changes.

The main purpose of this note is the description of this family of algebras. They
can be of interest in order to test homological conjectures or to analyze repre-
sentation theory aspects. On the other hand one may expect information on the
knot via the algebra of a diagram, as well as links between knot theory and the
representation theory of finite dimensional algebras.

In some examples this algebra happens to admit a connected grading by the
fundamental group of the knot.

2. Quiver and relations of an oriented knot diagram

A quiver is an oriented graph $Q$ given by two finite sets $Q_0$ (vertices) and $Q_1$
(arrows) and two maps $s, t : Q_1 \to Q_0$ assigning to each arrow $a$ the source and
target vertices $s(a)$ and $t(a)$. Consider the vector space $kQ$ with basis the set of
oriented paths of $Q$ including the trivial ones given by the vertices. The quiver path
algebra is $kQ$ equipped with the product on paths induced by their concatenation

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if it can be performed and 0 otherwise. This way the vertices provide a complete
set of primitive orthogonal idempotents. Note that \( Q_0 \) needs to be non-empty in
order to get an algebra.

Consider \( F \) the two-sided ideal generated by the arrows. Clearly \( kQ/F \cong kQ_0 \),
where \( kQ_0 \) is the commutative semi-simple algebra of the set \( Q_0 \). In case \( Q \) has
no oriented cycles \( F \) is the Jacobson radical of \( kQ \). A well known theorem of
P. Gabriel states that for each finite dimensional Morita reduced algebra \( \Lambda \) over an
algebraically closed field there is a unique quiver \( Q_\Lambda \) such that \( \Lambda \) is isomorphic to a
quotient \( kQ_\Lambda /I \) where \( I \) is an admissible two-sided ideal; that is, \( I \) contains \( F^2 \) and
is contained in \( F^n \) for some positive integer \( n \). Note that \( I \) is not unique in general,
while \( Q_\Lambda \) is the \( \text{Ext} \) quiver with vertices the iso-classes of simple modules and as
many arrows between two simples than the dimension of \( \text{Ext}^1_\Lambda \) between them.

Let \( D \) be an oriented knot diagram. An arc of \( D \) is obtained by following the
diagram according to its orientation from an under-crossing to the next one; in
other words, an arc is a connected component of \( D \). A segment of the diagram is
obtained by following the diagram according to its orientation, from one crossing
to a crossing with no crossings in-between.

**Definition 2.1.** The quiver \( Q_D \) has set of vertices the crossings of \( D \). Note that
this requires having at least one crossing in \( D \) so that \( (Q_D)_0 \neq \emptyset \). Each segment
provides an arrow having source and target the corresponding crossings. We say
that the source of an arrow is negative or positive according to whether the segment
starts respectively by under-crossing or by over-crossing. Similarly the target vertex
of an arrow is also negative or positive.

**Remark 2.2.** There are two specific oriented cycles at each vertex \( e \) as follows:

1. \( \alpha_e \) starts at \( e \) by the arrow with positive source \( e \) and successive arrows
   by browsing the oriented diagram until reaching the arrow having target \( e \)
   with negative sign.

2. Similarly, \( \beta_e \) starts at \( e \), but by the negative source arrow, and ends with
   the arrow with positive target \( e \).

**Definition 2.3.** The two fundamental cycles at \( e \) are as follows: the over-
crossing (or positive) one, \( \gamma_e^+ = \beta_e \alpha_e \), and the under-crossing (or negative) one,
\( \gamma_e^- = \alpha_e \beta_e \).

**Lemma 2.4.** The length of the fundamental cycles does not depend on the vertex
and equals the number of arrows \( n_D \) of \( Q_D \).

Let \( \tau : Q_0 \to k^* \) be a map. For instance, \( \tau(e) = q^l(e) \), where \( q \in k^* \) and
\( l : Q_0 \to \mathbb{Z} \) is some map given for example by the length of \( \alpha_e \). We consider a
two-sided ideal \( I_\tau \) of \( kQ_D \) generated by two kinds of relations as follows:

- **Type I**: Let \( ba \) be a path of length 2 in \( Q_D \) and let \( e = t(a) = s(b) \) be its
  middle vertex. In case the signs at \( e \) of \( a \) and \( b \) are different (that is, if \( ba \)
is not a follow-up of two segments of the diagram), then \( ba \) is a generator.
  This way each vertex of the quiver provides two generators.

- **Type II**: The elements \( \alpha_e \beta_e - \tau(e) \beta_e \alpha_e \) for all the vertices \( e \).

**Definition 2.5.** The algebra of the diagram \( D \) with respect to \( \tau \) is \( \Lambda_{D,\tau} = kQ_D/I_\tau \).
Proposition 2.6. The algebra $\Lambda_{D, \tau}$ is finite dimensional. A basis is given by the positive fundamental cycles and the non-zero paths of length strictly less than $n_D$, that is, the paths made by following-up the segments and having strictly less than $n_D$ segments.

Proof. The proof relies on the observation that a path $\delta$ of length $n_D + 1$ is zero in the algebra. Indeed, let $\delta = a \gamma$ where $\gamma$ is a fundamental cycle. If the source sign of $a$ is not the same as the target sign of $\gamma$, then $\delta \in I_\tau$. Otherwise let $\gamma'$ be the other fundamental cycle; then $a \gamma$ and $a \gamma'$ are equal up to a non-zero element of $k$ and the latter is in the ideal. \qed

Remark 2.7. The number of vertices of $Q$ change through the first and second Reidemeister moves.

Let $J_D$ be the two-sided ideal generated by the relations of type I and type II', where type II' is the set of all paths of length $n_D + 1$. Instead of $\Lambda_{D, \tau}$ a monomial algebra $\Xi_D = kQ_D/J_D$ can be considered which we call the monomial algebra of the diagram. A larger basis than before is provided by all the non-zero paths of length strictly less than $n_D + 1$.

Example 2.8. Let $D$ be the diagram of the trivial knot with one crossing. Then the algebra $\Lambda_{D, \tau}$ is $k\{a, b\}/\langle ab - qba \rangle$ for $q \in k^\times$. This algebra provides the first example for a negative answer to Happel’s question; see [3]. More precisely its global dimension is infinite, but for $q$ not a root of unity it has zero Hochschild cohomology in degrees large enough (in fact starting at degree 3). Nevertheless this algebra verifies Han’s conjecture (see [16]); namely, its Hochschild homology is non-zero in arbitrarily large degrees (in fact in all degrees).

3. Properties

We recall first the definition of a family of algebras which arose in representation theory of finite dimensional algebras.

Definition 3.1. Let $Q$ be a quiver and $I$ a two-sided admissible ideal generated by a set of relation $\rho$. Then $(Q, \rho)$ is called special biserial (see for instance [13]) if it verifies the following conditions:

1. Any vertex of $Q$ is the source of at most two arrows and is the target of at most two arrows.
2. If two different arrows $c$ and $d$ start at the target of an arrow $a$, then at least one of the paths $ca$ or $da$ is in $\rho$.
3. If two different arrows $a$ and $b$ end at the source of an arrow $c$, then at least one of the two paths $ca$ or $cb$ is in $\rho$.

As mentioned by C.M. Ringel in [13] special biserial algebras were first considered by I.M. Gelfand and V.A. Ponomarev in [15]. Blocks of a group algebra with cyclic or dihedral defect group are special biserial. As a consequence of [15] special biserial algebras are of tame representation type; in other words their indecomposable modules can be classified (see also [12,23]). Precise conditions are given in [4] for the vanishing of the first Hochschild cohomology of a special biserial algebra (which in turn implies that the cohomology in degrees larger than 1 also vanishes).
The following result is immediate:

Proposition 3.2. The algebra (or the monomial algebra) of the diagram of a knot is special biserial.

We recall that an algebra is self-injective if it admits a non-degenerate bilinear form \( \beta : \Lambda \times \Lambda \rightarrow k \) which is associative, that is, \( \beta(xy, z) = \beta(x, yz) \) for any triple of elements \((x, y, z)\) in \(\Lambda\). Associated to \(\beta\) there is a linear map \(t : \Lambda \rightarrow k\) given by \(t(x) = \beta(x, 1)\) which is a free generator of the left \(\Lambda\)-module \(\text{Hom}_k(\Lambda, k)\).

Theorem 3.3. Algebras of diagrams of knots are self-injective.

Proof. Let \(\delta\) be a positive length basis path of \(\Lambda_{D,\tau}\) (according to Proposition 2.6) with source vertex \(e\). We define \(\delta'\) to be the path such that \(\delta' \delta = \gamma_e^\epsilon\), where \(\gamma_e^\epsilon\) is the fundamental cycle and \(\epsilon\) is the sign of \(e\) at the first arrow of \(\delta\). Note that \((\gamma_e^\epsilon)' = e\). Moreover, for each vertex \(e\) we put \(e' = \gamma_e^\epsilon\).

In case \(\delta_1\) and \(\delta_2\) are basis paths such that \(\delta_2 \neq \delta_1'\), we put \(\beta(\delta_2, \delta_1) = 0\).

If \(\delta_2 = \delta_1'\) we consider two cases:

1. In case \(\delta_1\) is a vertex or if the source of the first arrow of \(\delta_1\) is positive, then \(\beta(\delta_1', \delta_1) = 1\).
2. If the sign of the source \(e\) of the first arrow of \(\delta_1\) is negative, then \(\beta(\delta_1', \delta_1) = \tau(\epsilon)\).

The only difficulty for verifying that \(\beta\) is associative arises when \(\delta_1\) has a first arrow with negative source \(e\). We need to prove that

\[
\beta(\delta_1', \delta_1) = \beta(\delta_1' \delta_1, e).
\]

We have defined \(\beta(\delta_1', \delta_1) = \tau(\epsilon)\) while

\[
\beta(\delta_1' \delta_1, e) = \beta(\gamma_{e^-}^\epsilon, e) = \beta(\tau(\epsilon) \gamma_{e^-}^\epsilon, e) = \tau(\epsilon) \beta(\gamma_{e^-}^\epsilon, e) = \tau(\epsilon).
\]

There is no difficulty for showing that \(\beta\) is non-degenerated.

Remark 3.4. The class of special biserial self-injective algebras has been studied by K. Erdmann and A. Skowroński with respect to Euclidian components of the stable Auslander-Reiten quiver; see [14]. See also the work by Z. Pogorzaly [21] concerning stable equivalence of this class of algebras. Precise computations of the Hochschild cohomology of certain self-injective special biserial algebras are performed in [22].

4. Gradings

The fundamental group \(\text{al la Grothendieck}\) of a \(k\)-category has been considered in [7], [10], [11]. Previously a fundamental group depending on a presentation by a quiver with relations has been studied in relation with representation theory; see for instance [2, 3, 5, 17, 20]. The main tool for the theory of the intrinsic fundamental group are the connected gradings as follows.

Definition 4.1. Let \(\mathcal{B}\) be a small \(k\)-category. A grading \(X\) of \(\mathcal{B}\) with structural group \(\Gamma(X)\) is firstly a direct sum decomposition of each vector space of morphisms indexed by elements of \(\Gamma(X)\) – a direct summand with index \(g\) of this decomposition is called a homogeneous component of degree \(g\), and a non-zero morphism in this component is said to be homogeneous of degree \(g\). Secondly, the composition of two homogeneous morphisms is homogeneous with degree the product of the degrees.
The precise definition of homogeneous walks is given for instance in [III]. Roughly each homogeneous morphism $\varphi$ of degree $g$ provides a virtual one $(\varphi, -1)$ with reversed source and target vertices and of settled degree $g^{-1}$. A **homogeneous walk** is a sequence of concatenated (virtual or not) homogeneous morphisms; its degree is the product of the degrees.

The grading $X$ is **connected** if between two fixed objects and for any element of $g \in \Gamma(X)$ there exists a homogenous walk relying on the objects having degree $g$. In this case the smash product provides a connected category which is a Galois covering of $B$; see [6].

The fundamental group is obtained by considering all the connected gradings of $B$ and coherent families of elements of the structure groups with respect to morphisms of gradings; see [III].

We recall the following convention: a crossing of $D$ is positive if following the diagram according to the orientation the under line of the diagram goes from right to left. It is negative otherwise.

Let $D$ be the diagram of an oriented knot. Recall that the fundamental group of the complement of the knot with base point at the infinity is generated by the loop which passes just under each portion of the knot corresponding to an arc of the projection.

**Definition 4.2.** Let $Q_D$ be the diagram of a knot. The grading of $Q_D$ by the fundamental group of the knot is as follows: given an arrow, consider the corresponding segment and the arc to which it belongs. In case the crossings at the vertices of the arrow have the same sign, the degree of the arrow is the generator of the fundamental group corresponding to the arc. Otherwise the degree is trivial.

The verification of the following result is not difficult using the Wirtinger presentation of the fundamental group of the knot.

**Example 4.3.** For the usual diagram of the trefoil knot or of the figure-height knot the two-sided ideal $I_\tau$ is homogeneous, and the resulting grading for the algebra of the diagram is connected.

Nevertheless it seems that for the diagram of the knot 6_3 (see for instance the Knot Atlas) the ideal is not homogeneous.

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