COMPACT OPERATORS ON VECTOR–VALUED BERGMAN SPACE
VIA THE BEREZIN TRANSFORM

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ABSTRACT. In this paper, we characterise compactness of finite sums of finite products of
Toeplitz operators acting on the \( \mathbb{C}^d \)-valued weighted Bergman Space, denoted \( A^p_\alpha(\mathbb{B}^n; \mathbb{C}^d) \).
The main result shows that a finite sum of finite product of Toeplitz operators acting on
\( A^p_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) is compact if and only if its Berezin transform vanishes on the boundary of the
ball.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Definition of the Spaces \( L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) and \( A^p_\alpha(\mathbb{B}^n; \mathbb{C}^d) \). Let \( \mathbb{B}^n \) denote the open unit
ball in \( \mathbb{C}^n \). For \( d \in \mathbb{N} \), a function, \( f \), defined on \( \mathbb{B}^n \) and taking values in \( \mathbb{C}^d \) (that is, \( f \) is
vector-valued) is said to be measurable if \( z \mapsto \langle f(z), e \rangle_{\mathbb{C}^d} \) is measurable for every \( e \in \mathbb{C}^d \).
Since \( \mathbb{C}^d \) is finite dimensional, this is the same as requiring that \( f \) be measurable in each
coordinate. For a vector, \( v \in \mathbb{C}^d \), let \( \|v\|_2 \) denote the usual euclidean norm on \( \mathbb{C}^d \). That is
\( \|v\|_2^2 := \langle v, v \rangle_{\mathbb{C}^d} \), where, of course, \( \langle \cdot, \cdot \rangle_{\mathbb{C}^d} \) is the standard inner product on \( \mathbb{C}^d \). For \( \alpha > -1 \), let
\[
dv_\alpha(z) := c_\alpha (1 - |z|^2)^\alpha dV(z)
\]
where \( dV \) is volume measure on \( \mathbb{B}^n \) and \( c_\alpha \) is a constant such that \( \int_{\mathbb{B}^n} dv_\alpha(z) = 1 \). Define
\( L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) to be the set of all measurable functions on \( \mathbb{B}^n \) taking values in \( \mathbb{C}^d \) such that
\[
\|f\|_{L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d)}^2 := \int_{\mathbb{B}^n} \langle f(z), f(z) \rangle_{\mathbb{C}^d} dv_\alpha(z) = \int_{\mathbb{B}^n} \|f(z)\|_2^2 dv_\alpha(z) < \infty.
\]
It should be noted that \( L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) is a Hilbert Space with inner product:
\[
\langle f, g \rangle_{L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d)} := \int_{\mathbb{B}^n} \langle f(z), g(z) \rangle_{\mathbb{C}^d} dv_\alpha(z).
\]
It is obvious that this is an inner product and that \( \|f\|_{L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d)}^2 = \langle f, f \rangle_{L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d)} \).
Similarly, a function \( f : \mathbb{B}^n \to \mathbb{C}^d \) is said to be holomorphic if \( z \mapsto \langle f(z), e \rangle_{\mathbb{C}^d} \) is a
holomorphic function for every \( e \in \mathbb{C}^d \). (Similarly, this is the same as requiring that \( f \) be
holomorphic in each coordinate.) Define \( A^p_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) to be the set of holomorphic functions
on \( \mathbb{B}^n \) that are also in \( L^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \). Additionally, define \( \mathcal{L}(A^2_\alpha(\mathbb{B}^n; \mathbb{C}^d)) \) be the set of bounded
linear operators from \( A^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) to \( A^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \).

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1.2. Background for the Scalar–Valued Case. For the moment, let \( d = 1 \). Recall the reproducing kernel:

\[
K_z^{(\alpha)}(w) = K^{(\alpha)}(z, w) := \frac{1}{(1 - \overline{z}w)^{n+1+\alpha}}.
\]

That is, if \( f \in A_{\alpha}^2(\mathbb{B}^n; \mathbb{C}) \) there holds:

\[
f(z) = \langle f, K_z \rangle_{A_{\alpha}^2(\mathbb{B}^n; \mathbb{C})} = \int_{\mathbb{B}^n} \frac{f(w)}{(1 - \overline{z}w)^{n+1+\alpha}} dv_\alpha(w).
\]

Recall also the normalized reproducing kernels, \( k_z^{(2,\alpha)} \), normalized so that \( \|k_z^{(2,\alpha)}\|_{L_2^d(\mathbb{B}^n; \mathbb{C})} = 1 \).

A simple calculation shows:

\[
k_z^{(2,\alpha)}(w) = k^{(2,\alpha)}(z, w) := \|K_z^{(\alpha)}\|^{-1}_{L_2^d(\mathbb{B}^n; \mathbb{C})} K_z^{(\alpha)}(w) = \frac{(1 - |z|^2)^{n+1+\alpha}}{(1 - \overline{z}w)^{n+1+\alpha}}.
\]

The reproducing kernels allow us to explicitly write the orthogonal projection, \( P_\alpha \), from \( L_2^d(\mathbb{B}^n; \mathbb{C}) \) to \( A_{\alpha}^2(\mathbb{B}^n; \mathbb{C}) \):

\[
(P_\alpha f)(z) = \langle f, K_z^{(\alpha)} \rangle_{L_2^d(\mathbb{B}^n; \mathbb{C})}.
\]

Let \( \varphi \in L^\infty(\mathbb{B}^n) \). The Toeplitz operator with symbol \( \varphi \) is defined to be:

\[
T_\varphi := P_\alpha M_\varphi,
\]

where \( M_\varphi \) is the multiplication operator. So there holds: \( (T_\varphi f)(z) = \langle \varphi f, K_z^{(\alpha)} \rangle_{L_2^d(\mathbb{B}^n; \mathbb{C})} \).

Recall that the Berezin Transform of \( T \), denoted \( \widetilde{T} \), is a function on \( \mathbb{B}^n \) defined by the formula:

\[
\widetilde{T}(\lambda) = \langle T k^{(2,\alpha)}_\lambda, k^{(2,\alpha)}_\lambda \rangle_{A_{\alpha}^2(\mathbb{B}^n; \mathbb{C})}.
\]

1.3. Generalization to Vector–Valued Case. Now, we consider \( d \in \mathbb{N} \) and \( d > 1 \). The preceding discussion can be carried over with only a few modifications. First, the reproducing kernels remain the same, but the function \( f \) is now \( \mathbb{C}^d \)–valued and the integrals must be interpreted as a vector–valued integrals (that is, integrate in each coordinate). To make this more precise, if \( f \) is a \( \mathbb{C}^d \)-valued function on \( \mathbb{B}^n \), and \( \{e_k\}_{k=1}^d \) is the standard orthonormal basis for \( \mathbb{C}^d \), define:

\[
\int_{\mathbb{B}^n} f(z) dv_\alpha(z) := \sum_{k=1}^d \int_{\mathbb{B}^n} \langle f(z), e_k \rangle_{\mathbb{C}^d} dv_\alpha(z) e_k.
\]

Let \( L_{M_d}^\infty \) denote the set of \( d \times d \) matrix–valued functions such that the function \( z \mapsto \|\varphi(z)\| \) is an \( L^\infty \) function. Note that it is not particularly important which matrix norm is used – since \( \mathbb{C}^d \) is finite dimensional all norms are equivalent. The second change is that our symbols are now matrix–valued functions in \( L_{M_d}^\infty \).

Define the Toeplitz algebra, denoted by \( T_{p_\alpha} \), to be the SOT closure of finite sums of finite products of Toeplitz operators with \( L_{M_d}^\infty \) symbols.

Finally, we change the way that we define the Berezin transform of an operator. The Berezin transform will be a matrix–valued function, acting on \( \mathbb{C}^d \), given by the following relation (see also [1]):
\[ \langle \tilde{T}(z)e, h \rangle_{C^d} = \langle T(k_z e), k_z h \rangle_{A^2_\alpha} \]  

for \( e, h \in \mathbb{C}^d \).

We are now ready to state the main theorem of the paper, but first we need to introduce an auxiliary operator.

**Definition 1.1.** Let \( U_z \) be defined by:
\[
(U_z f)(w) = (f \circ \varphi_z)(w) k_z(w).
\]

**Definition 1.2.** For \( T \in \mathcal{L}(L^p_{\alpha}) \) set \( T_z = U_z T U_z \).

**Theorem 1.3.** Let \( T \) be an operator in \( \mathcal{L}(A^2_\alpha) \) which can be written as
\[
T = \sum_{j=1}^{m} \prod_{k=1}^{m_j} T_{u_{j,k}},
\]
where \( u_{j,k} \in L^\infty_{M^d} \). Then the following are equivalent:

1. \( T \) is compact;
2. \( \langle \tilde{T}(z)e, h \rangle_{C^d} \to 0 \) as \( z \to \partial B^n \) \( e, h \in \mathbb{C}^d \) with \( \|e\|_2 = \|h\|_2 = 1 \);
3. \( T_z e \to 0 \) weakly as \( z \to \partial B^n \), where \( e \in \mathbb{C}^d \);
4. \( \|T_z e\|_{L^p_{\alpha}} \to 0 \) as \( z \to \partial B^n \) for any \( p > 1 \).

Our main interest is the equivalence between (1) and (2) above, but it is easier to prove their equivalence by proving that all four statements in Theorem 1.3 are equivalent.

### 1.4. Discussion of the Theorem.

By now, there are many results that relate the compactness of an operator to its Berezin transform. For example, if \( T \in \mathcal{L}(A^2_0(B; \mathbb{C})) \) can be written as a finite sum of finite products of Toeplitz operators, Axler and Zheng prove in [2] that \( T \) is compact if and only if its Berezin transform vanishes on the boundary of \( B \). (Recall that \( (A^2_0(B; \mathbb{C})) \) is the standard unwighted Bergman space on the unit ball in \( \mathbb{C} \).) This was improved independently by Raimondo who extended the result to the spaces \( A^2_0(B^n; \mathbb{C}) \) in [11] and Engliš who extended the results in great generality to Bergman spaces on bounded symmetric domains in [4].

There are also several results along these lines for more general operators than those that can be written as finite sums of finite products of Toeplitz operators. In [5] Engliš proves that any compact operator is in the operator–norm topology closure of the set of finite sums of Toeplitz operators (this is called the Toeplitz algebra.) In [13], Suárez proves that an operator, \( T \), in \( \mathcal{L}(A^0_0(B^n; \mathbb{C})) \) is compact if and only if it is in the Toeplitz algebra and its Berezin transform vanishes on \( \partial B^n \). This was extended to the weighted Bergman spaces \( (A^2_\alpha(B^n; \mathbb{C})) \) in [7] By Suárez, Mitkovski, and Wick. Mitkovski and Wick achieve similar results for Bergman spaces on the polydisc in [8] and they extend these results to bounded symmetric domains in [9].

It is interesting to note that the hypothesis that \( T \) is a finite sum of finite products of Toeplitz operators is used only in Lemma 3.4 to obtain an easy estimate. (This is also true for proofs of several of the results mentioned above. See, for example, the proofs in [2] and [11].) In particular, it is shown that:

\[
\sup_{\|e\|_2 = 1} \sup_{z \in B^n} \|T_z e\|_{A^p_{\alpha}} < \infty. 
\]  

(1.2)
Therefore, instead of requiring that $T$ be a finite sum of finite products of Toeplitz operators, we can require that $T$ satisfy (1.2). Specifically, as a corollary of the proof of Theorem 1.3, the following holds:

**Corollary 1.4.** Let $T$ be an operator in $\mathcal{L}(A^2_\alpha)$ that satisfies (1.2). Then the following are equivalent:

1. $T$ is compact;
2. $\left< \tilde{T}(z)e, h \right>_{C^d} \to 0$ as $z \to \partial B^n$, $e, h \in C^d$ with $\|e\|_2 = \|h\|_2 = 1$;
3. $T_z e \to 0$ weakly as $z \to \partial B^n$, where $e \in C^d$;
4. $\|T_z e\|_{L^p_\alpha} \to 0$ as $z \to \partial B^n$ for any $p > 1$.

This is not a new observation. Indeed, in [6], the authors prove a similar result.

## 2. Preliminaries

We first fix notation that will last for the rest of the paper. The vectors $\{e_i\}$, etc. will denote the standard orthonormal basis vectors in $C^d$. The letter $e$ will always denote a unit vector in $C^d$. For vectors in $C^d$, $\|\cdot\|_p$ will denote the $l^p$ norm on $C^d$. If $M$ is a $d \times d$ matrix, $\|M\|$ will denote any convenient matrix norm. Since all norms of matrices are equivalent in finite dimensions, the exact norm used does not matter for quantitative considerations. Additionally, $M_{(i,j)}$ will denote the $(i, j)$ entry of $M$. Finally, to lighten notation, fix an integer $d > 1$, an integer $n \geq 1$ and a real $\alpha > -1$. Because of this, we will usually suppress these constants in our notation. For example, the reproducing kernels will be written as $K_z$ instead of $K^{(\alpha)}_z$ and $k^{(2,\alpha)}_z$ is simply $k_z$. Furthermore, we will write $L^2_\alpha$ and $A^2_\alpha$ instead of $L^2_\alpha(B^n; C^d)$ and $A^2_\alpha(B^n; C^d)$. (We keep the $\alpha$ in the notation for the spaces because this is customary).

### 2.1. Well-Known Results and Extensions to the Present Case

We will discuss several well-known results about the standard Bergman Spaces, $A^p_\alpha(B^n; C)$ and state and prove their generalizations to the present vector-valued Bergman Spaces, $A^p_\alpha$.

Let $\varphi_z$ be the automorphisms of the ball that interchange $z$ and 0. The automorphisms are used to define the following metrics:

$$\rho(z, w) := |\varphi_z(w)| \quad \text{and} \quad \beta(z, w) := \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

These metrics are invariant under the maps $\varphi_z$. Let $D(z, r)$ be the ball in the $\beta$ metric centered at $z$ with radius $r$. Recall the following identity:

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

The following change of variables formula is in [16, Prop 1.13]:

$$\int_{B^n} f(w)dv_\alpha(w) = \int_{B^n} (f \circ \varphi_z)(w) |k_z(w)|^2 dv_\alpha(w). \quad (2.1)$$

Straight-forward computations reveal:

$$k_z(\varphi_z(w))k_z(w) \equiv 1. \quad (2.2)$$

The following propositions appear in [16].
Proposition 2.1. If $a \in \mathbb{B}^n$ and $z \in D(a, r)$, there exists a constant depending only on $r$ such that $1 - |a|^2 \simeq 1 - |z|^2 \simeq |1 - \langle a, z \rangle|$.

Proposition 2.2. Suppose $r > 0$, $p > 0$, and $\alpha > -1$. Then there exists a positive constant that depends only on $\alpha$ and $r$ such that

$$|f(\lambda)|^p \lesssim \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,r)} |f(w)|^p \, dv_\alpha(w)$$

for all holomorphic $f : \mathbb{B}^n \to \mathbb{C}$ and all $\lambda \in \mathbb{B}^n$.

The following vector-valued analogue will be used:

Proposition 2.3. Suppose $r > 0$, $p \geq 1$, and $\alpha > -1$. Then there exists a positive constant that depends only on $\alpha$, $r$, and $d$ such that

$$\sup_{z \in D(\lambda,r)} \|f(z)\|^p_p \lesssim \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,2r)} \|f(w)\|^p_p \, dv_\alpha(w)$$

for all holomorphic $f : \mathbb{B}^n \to \mathbb{C}^d$ and all $\lambda \in \mathbb{B}^n$.

Proof. Let $q$ be conjugate exponent to $p$. Then

$$\sup_{z \in D(\lambda,r)} \|f(z)\|^p_p = \sup_{z \in D(\lambda,r)} \sup_{\|e\|_q = 1} |\langle e, f(z) \rangle|_C^p.$$ 

By definition, $\langle e, f(z) \rangle$ is holomorphic for all $e \in \mathbb{C}^d$. By Proposition 2.2 and Proposition 2.1, for $e \in \mathbb{C}^d$ and $z \in D(\lambda, r)$ there holds:

$$|\langle e, f(z) \rangle|_C^p \lesssim \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,r)} |\langle e, f(w) \rangle|_C^p \, dv_\alpha(w)$$

$$\lesssim \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,r)} \|f(w)\|^p_p \, dv_\alpha(w)$$

$$\lesssim \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,r)} \|f(w)\|^p_p \, dv_\alpha(w)$$

$$\leq C \frac{1}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda,2r)} \|f(w)\|^p_p \, dv_\alpha(w).$$

Which completes the proof. \qed

The next lemma is in [16]:

Lemma 2.4. For $z \in \mathbb{B}^n$, $s$ real and $t > -1$, let

$$F_{s,t}(z) := \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^t}{|1 - \overline{w}z|^s} \, dv(w).$$

Then $F_{s,t}$ is bounded if $s < n+1+t$ and grows as $(1 - |z|^2)^{n+1+t-s}$ when $|z| \to 1$ if $s > n+1+t$.

The next lemma is the version of Schur’s Test that we will use. Note that it is essentially the same as the “usual” Schur’s Test. The proof is omitted.
Lemma 2.5. Suppose \((X, \mu)\) is a measure space, \(1 < p < \infty\), and \(q\) conjugate exponent to \(p\). Let \(T\) be an integral operator with (matrix-valued) kernel \(M(x, y)\) and let \(f\) be vector-valued. That is,

\[
(Tf)(x) = \int_X M(x, y)f(y)d\mu(y).
\]

If there is a \(C_1\) and a \(C_2\) and a positive function \(h\) such that the following is true:

\[
\int_X \|M(x, y)\|h(y)^pd\mu(y) \leq C_1 h(x)^q
\]

for almost every \(x \in X\), and

\[
\int_X \|M(x, y)\|h(x)^p d\mu(x) \leq C_2 h(y)^p
\]

for almost every \(y \in X\) then \(T : L^p(X, \mu) \to L^p(X, \mu)\) is bounded with norm at most \(C_1^{1/q}C_2^{1/p}\).

There is also a vector–valued test to determine an operator's membership in the Hilbert-Schmidt Class.

Definition 2.6. The matrix-valued function, \(M(z, w)\), is in \(L^2(\mathbb{B}^n \times \mathbb{B}^n, dv_\alpha \times dv_\alpha)\) if

\[
\|M\|^2_{L^2(\mathbb{B}^n \times \mathbb{B}^n, dv_\alpha \times dv_\alpha)} := \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \|M(z, w)\|^2 dv_\alpha(w)dv_\alpha(z) < \infty.
\]

Lemma 2.7 (Vector–Valued Hilbert Schmidt Test). A linear operator \(T \in \mathcal{L}(L^2_\alpha)\) is Hilbert-Schmidt if there is a matrix-valued function \(M\) in \(L^2(\mathbb{B}^n \times \mathbb{B}^n, dv_\alpha \times dv_\alpha)\) such that

\[
(Tf)(z) = \int_{\mathbb{B}^n} M(z, w)f(w)dv_\alpha(w).
\]

Proof. Let \(M^*(z, w)\) be the adjoint of the matrix \(M(z, w)\). Let \(\{\varphi_n\}\) be an orthonormal basis for \(L^2_\alpha\), let \(e \in \mathbb{C}^d\) with \(\|e\|_2 = 1\) and let \(M_z(w) = M(z, w)\). Then there holds:

\[
\|M^* e\|^2_{L^2_\alpha} = \sum_{n=1}^{+\infty} \left| \langle M^*_e, \varphi_n \rangle_{A^2_\alpha} \right|^2
\]

\[
= \sum_{n=1}^{+\infty} \left| \int_{\mathbb{B}^n} \langle M^*_e(w), \varphi_n(w) \rangle_{\mathbb{C}^d} dv_\alpha(w) \right|^2
\]

\[
= \sum_{n=1}^{+\infty} \left| \int_{\mathbb{B}^n} \langle e, M(w)\varphi_n(w) \rangle_{\mathbb{C}^d} dv_\alpha(w) \right|^2
\]

\[
= \sum_{n=1}^{+\infty} \left| \int_{\mathbb{B}^n} M_z(w)\varphi_n(w)dv_\alpha(w) \right|_{\mathbb{C}^d}^2
\]

\[
= \sum_{n=1}^{+\infty} \left| \langle e, (T\varphi_n)(z) \rangle_{\mathbb{C}^d} \right|^2.
\]

Using this computation, there holds:

\[
\sum_{n=1}^{+\infty} \|T\varphi_n\|^2_{L^2_\alpha} = \sum_{n=1}^{+\infty} \int_{\mathbb{B}^n} \|T\varphi_n(z)\|^2_2 dv_\alpha(z)
\]
\[
= \sum_{i=1}^{d} \int_{\mathbb{B}^n} \sum_{n=1}^{+\infty} |\langle e_i, (T\varphi_n)(z) \rangle_{\mathbb{C}^d}|^2 \, dv_{\alpha}(z)
\]
\[
= \sum_{i=1}^{d} \int_{\mathbb{B}^n} \|M^*_z e_i\|_{L^2_{\alpha}}^2 \, dv_{\alpha}(z)
\]
\[
= \sum_{i=1}^{d} \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \|M^*(z, w)e_i\|^2 \, dv_{\alpha}(w) \, dv_{\alpha}(z)
\]
\[
\leq \sum_{i=1}^{d} \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \|M(z, w)\|^2 \, dv_{\alpha}(w) \, dv_{\alpha}(z)
\]
\[
= d < \infty.
\]
Therefore, \( T \) is a Hilbert-Schmidt Operator. \( \square \)

**Lemma 2.8.** For any \( z \in \mathbb{B}^n \), the operator \( U_z \) is self-adjoint, idempotent, and isometric on \( A^2_{\alpha} \).

**Proof.** We first show idempotency: For \( f \in A^2_{\alpha} \),
\[
U_z^2 f = U_z (U_z f)
\]
\[
= ((U_z f) \circ \varphi_z)(k_z)
\]
\[
= ((f \circ \varphi_z)(k_z) \circ \varphi_z)(k_z)
\]
\[
= (f \circ \varphi_z \circ \varphi_z)(k_z)
\]
\[
= f1
\]
\[
= f. \tag{2.3}
\]
The equality in (2.3) follows from (2.2). Next we show isometry:
\[
\|U_z f\|_{A^2_{\alpha}}^2 = \int_{\mathbb{B}^n} \|(U_z f)(w)\|_{\mathbb{C}^d}^2 \, dv_{\alpha}(w)
\]
\[
= \int_{\mathbb{B}^n} \sum_{k=1}^{d} |\langle (U_z f)(w), e_k \rangle_{\mathbb{C}^d}|^2 \, dv_{\alpha}(w)
\]
\[
= \sum_{k=1}^{d} \int_{\mathbb{B}^n} |\langle (f \circ \varphi_z)(w)k_z(w), e_k \rangle_{\mathbb{C}^d}|^2 \, dv_{\alpha}(w)
\]
\[
= \sum_{k=1}^{d} \int_{\mathbb{B}^n} |\langle (f \circ \varphi_z)(w), e_k \rangle_{\mathbb{C}^d}|^2 |k_z(w)|^2 \, dv_{\alpha}(w)
\]
\[
= \sum_{k=1}^{d} \int_{\mathbb{B}^n} |\langle f(w), e_k \rangle_{\mathbb{C}^d}|^2 \, dv_{\alpha}(w) \tag{2.4}
\]
where (2.4) is due to the change of variables formula (2.1). To show that \( U_z \) is self-adjoint, we note the two previous conditions imply that:
\[
\langle U_z f, g \rangle_{A^2_{\alpha}} = \langle (U_z U_z)f, U_z g \rangle_{A^2_{\alpha}} = \langle f, U_z g \rangle_{A^2_{\alpha}}.
\]
3. Basic Lemmas

We begin by investigating the relationship between the Berezin Transform and the maps \( \varphi_z \).

**Lemma 3.1.** If \( T \in \mathcal{L}(L^2_\alpha) \) and \( z \in \mathbb{B}^n \), then \( \widetilde{T} \circ \varphi_z = \widetilde{T}_z \).

**Proof.** Suppose that \( T \in \mathcal{L}(L^2_\alpha) \) and \( z, w \in \mathbb{C}^n \), then there holds

\[
U_z(K_w e) = \langle k_z(w)\rangle(K^{(\alpha)}_{\varphi(w)})e.
\]

Indeed, if \( f \) is in \( L^2_\alpha \) then there holds, for \( \|e\| \in \mathbb{C}^d \):

\[
\langle f, U_z(K_w e) \rangle_{A^2_\alpha} = \langle U_z f, K_w e \rangle_{A^2_\alpha} = \langle (U_z f)(w), e \rangle_{\mathbb{C}^d} = \langle (f \circ \varphi_z)(w) \times k_z(w), e \rangle_{\mathbb{C}^d} = \langle (f \circ \varphi_z)(w), k_z(w)e \rangle_{\mathbb{C}^d} = \langle f, k_z(w)K_{\varphi_z(w)e} \rangle_{A^2_\alpha},
\]

which implies the claim. By writing this equality as:

\[
U_z(K_w e) = \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha}^{-1} (U_z K_w e)
\]

It follows that

\[
U_z(K_w e) = \left( \big\|K_w\big\|_{A^2_\alpha}^{-1} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \overline{k_z(w)} \right) (k_{\varphi_z(w)} e).
\]

(3.1)

Since \( U_z \) is an isometry, the \( A^2_\alpha \) norm of \( K_w = \|K_w\|_{A^2_\alpha} k_w \) is equal to the \( A^2_\alpha \) norm of the extreme right side of of (3.1). Computing this norm and using the fact that \( \langle k_w e, k_w e \rangle_{A^2_\alpha} = \langle k_{\varphi_z(w)} e, k_{\varphi_z(w)} e \rangle_{A^2_\alpha} = 1 \), gives:

\[
\big\|K_w\big\|_{A^2_\alpha}^{-1} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \overline{k_z(w)} = \big\|K_w\big\|_{A^2_\alpha}.
\]

(3.2)

Therefore there holds:

\[
\big\|K_w\big\|_{A^2_\alpha}^{-1} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \overline{k_z(w)} = 1.
\]

We use this fact to make the following computation, for \( \|e_1\| = \|e_2\| = 1 \):

\[
\langle \widetilde{T} \circ \varphi_z(w)e_1, e_2 \rangle_{\mathbb{C}^d} = \langle \widetilde{T}(\varphi_z(w))e_1, e_2 \rangle_{\mathbb{C}^d}
\]

\[
= \langle T(k_{\varphi_z(w)}e_1), k_{\varphi_z(w)}e_2 \rangle_{A^2_\alpha}
\]

\[
= \left( \big\|K_w\big\|_{A^2_\alpha}^{-1} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \big\|K_z(w)\big\| \right)^{-2} \langle TU_z(K_w e_1), U_z(K_w e_2) \rangle_{A^2_\alpha}
\]

\[
= \left( \big\|K_w\big\|_{A^2_\alpha}^{-2} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \big\|K_w(z)\big\| \right)^{-2} \langle TU_z(k_w e_1), U_z(k_w e_2) \rangle_{A^2_\alpha}
\]

\[
= \left( \big\|K_w\big\|_{A^2_\alpha}^{-1} \big\|K_{\varphi_z(w)}\big\|_{A^2_\alpha} \big\|k_w(z)\big\| \right)^{-2} \langle TU_z(k_w e_1), U_z(k_w e_2) \rangle_{A^2_\alpha}
\]
Now, we calculate

\[
\begin{aligned}
&= \langle TU_z(k_w e_1), U_z(k_w e_2) \rangle_{A^2_\alpha} \\
&= \langle U_z TU_z(k_w e_1), k_w e_2 \rangle_{A^2_\alpha} \\
&= \langle T_z(k_w e_1), k_w e_2 \rangle_{A^2_\alpha} \\
&= \langle \tilde{T}_z e_1, e_2 \rangle_{\mathbb{C}^d}.
\end{aligned}
\]  

(3.3)

In the equality in (3.3) we used (3.2). This shows that \( \tilde{T} \circ \varphi_z = \tilde{T}_z \) as maps on \( \mathbb{C}^d \).

\[\square\]

**Lemma 3.2.** For every \( u \in L^\infty_M \) and for every \( z \in \mathbb{B}^n \), there holds:

\[ U_z T_u U_z = T_{u \circ \varphi_z} \]

\[.\]

**Proof.** Since \( U_z \) is idempotent, it is enough to prove that \( T_u U_z = U_z T_{u \circ \varphi_z} \). To this end, we compute \( T_u U_z \). Let \( f \) be in \( A^2_\alpha \) and \( e \in \mathbb{C}^d \), then:

\[
\begin{aligned}
\langle T_u U_z f, k_w e \rangle_{A^2_\alpha} &= \langle T_u ((f \circ \varphi_z) k_z), k_w e \rangle_{A^2_\alpha} \\
&= \langle PM_u ((f \circ \varphi_z) k_z), k_w e \rangle_{A^2_\alpha} \\
&= \langle u (f \circ \varphi_z) k_z, k_w e \rangle_{A^2_\alpha} \\
&= \int_{\mathbb{B}^n} \langle u(\eta) (f \circ \varphi_z)(\eta) k_z(\eta), k_w(\eta)e \rangle_{\mathbb{C}^d} dv_\alpha(\eta).
\end{aligned}
\]

Now, we calculate \( U_z T_{u \circ \varphi_z} \). For \( e \in \mathbb{C}^d \):

\[
\begin{aligned}
\langle U_z T_{u \circ \varphi_z} f, k_w e \rangle_{A^2_\alpha} &= \langle T_{u \circ \varphi_z} f, U_z (k_w e) \rangle_{A^2_\alpha} \\
&= \langle PM_{u \circ \varphi_z} f, U_z (k_w e) \rangle_{A^2_\alpha} \\
&= \langle (u \circ \varphi_z) f, (k_w \circ \varphi_z) k_z e \rangle_{A^2_\alpha} \\
&= \int_{\mathbb{B}^n} \langle (u \circ \varphi_z)(\eta) f(\eta), k_z(\eta)(k_w \circ \varphi_z)(\eta)e \rangle_{\mathbb{C}^d} dv_\alpha(\eta) =: \mathcal{A}.
\end{aligned}
\]

Make the substitution \( \eta = \varphi_z(\xi) \). Then there holds, for \( e \in \mathbb{C}^d \), using Lemma 2.1, and

the fact that \( k_z \circ \varphi_z(\xi)k_z(\xi) \equiv 1 \),

\[
\mathcal{A} = \int_{\mathbb{B}^n} \langle u(\xi)(f \circ \varphi_z)(\xi), (k_z \circ \varphi_z)(\xi)k_w(\xi)e \rangle_{\mathbb{C}^d} |k_z(\xi)|^2 dv_\alpha(\xi)
\]

\[
= \int_{\mathbb{B}^n} \langle u(\xi)(f \circ \varphi_z)(\xi), (k_z \circ \varphi_z)(\xi)k_z(\xi)k_w(\xi)k_z(\xi)e \rangle_{\mathbb{C}^d} dv_\alpha(\xi)
\]

\[
= \int_{\mathbb{B}^n} \langle u(\xi)(f \circ \varphi_z)(\xi), k_w(\xi)k_z(\xi)e \rangle_{\mathbb{C}^d} dv_\alpha(\xi)
\]

\[
= \int_{\mathbb{B}^n} \langle k_z(\xi) u(\xi)(f \circ \varphi_z)(\xi), k_w(\xi)e \rangle_{\mathbb{C}^d} dv_\alpha(\xi).
\]

This gives, \( \langle T_u U_z f, k_w e \rangle_{A^2_\alpha} = \langle U_z T_{u \circ \varphi_z} f, k_w e \rangle_{A^2_\alpha} \) for every \( w \in \mathbb{B}^n \) and \( e \in \mathbb{C}^d \). This completes the proof. \[\square\]

Before going on, we introduce a new operator on \( A^2_\alpha \): \( (U_{\mathcal{R}} f)(w) = f(-w) \). Let

\[
\mathcal{J}_{\mathcal{R}z}(z) = \int_{\mathbb{B}^n} \frac{(1 - |w|)^t dv_\alpha(w)}{|1 - z w|^{n+1+t+c}}.
\]
By Lemma 2.4, for $c < 0$ and $t > -1$, the function $J_{c,t}$ is bounded on $\mathbb{B}^n$. We will state a proposition that will be used in conjunction with Schur’s Test later on. The proof can be easily deduced from the proof in [11] and is omitted.

**Lemma 3.3.** Given $p \in \mathbb{R}$ with $0 < p - 1 < (n+1)^{-1}$, and $T \in \mathcal{L}(A^2_{\alpha})$ and $e \in \mathbb{C}^d$, then

$$
\int_{\mathbb{B}^n} \|U_T U_R (K_z e)(w)\|_2 \|K_w\|_{A^2_{\alpha}}^p\, dv_{\alpha}(w) \leq \|K_z\|_{A^2_{\alpha}}^p \left( \sup_{z \in \mathbb{B}^n} \|T_{-z} e\|_{A^2_{\alpha}} \right) \sup_{z \in \mathbb{B}^n} |J_{a,b}(z)|^{1/p}.
$$

$$
\int_{\mathbb{B}^n} \|U_T U_R (K_z e)(w)\|_2 \|K_w\|_{A^2_{\alpha}}^p\, dv_{\alpha}(z) \leq \|K_z\|_{A^2_{\alpha}}^p \left( \sup_{w \in \mathbb{B}^n} \|T_{-w} e\|_{A^2_{\alpha}} \right) \sup_{w \in \mathbb{B}^n} |J_{a,b}(w)|^{1/p}.
$$

where: $2(p - 1)/p < \epsilon < 2/(n+1)^2$, $a = (p - 1)(n + 1) - (n + 1)\epsilon p/2$ and $b = -(n + 1)\epsilon p/2$ and $p^{-1} + q^{-1} = 1$. Moreover, the quantity,

$$
\sup_{z \in \mathbb{B}^n} |J_{a,b}(z)|^{1/p}
$$

is finite.

**Lemma 3.4.** Let $T$ be an operator in $\mathcal{L}(A^2_{\alpha})$ which can be written as $T = \prod_{k=1}^{m} T_{u_{j,k}}$, where $u_{j,k} \in L^\infty_{M^d}$. Then, for every $1 < p < \infty$, $\sup_{\|f\|_2 = 1} \sup_{z \in \mathbb{B}^n} \|T_z e\|_{A^2_{\alpha}} < \infty$.

**Proof.** We can assume that $T = \prod_{k=1}^{m} T_{u_j}$. Using Lemma 3.2, we have that $T_z = \prod_{k=1}^{m} T_{u_k \circ \varphi_z}$. Since $P_{\alpha}$ is bounded from $L^p_{\alpha} \to A^2_{\alpha}$, we have $\|P_{\alpha} f\|_{A^2_{\alpha}} \lesssim \|f\|_{L^p_{\alpha}}$. Therefore, since $\|u \circ \varphi_z\|_{\infty} = \|u\|_{\infty}$, we have $\|T_{u \circ \varphi_z} f\|_{L^p_{\alpha}} \lesssim \|u\|_{\infty} \|f\|_{L^p_{\alpha}}$. This implies that $\|T_z e\|_{L^p_{\alpha}} \lesssim \prod_{k=1}^{m} \|T_{u_k \circ \varphi_z} e\|_{L^p_{\alpha}} \lesssim \prod_{k=1}^{m} \|u_k\|_{\infty}$. Since the right hand side of this estimate is independent of $z$ (the implied constant depends only on $p$), we are done.

## 4. The Main Theorem

For convenience, we remind the reader of the main theorem.

**Theorem 4.1.** Let $T$ be an operator in $\mathcal{L}(A^2_{\alpha})$ which can be written as

$$
T = \sum_{j=1}^{m} \prod_{k=1}^{m_j} T_{u_{j,k}},
$$

where $u_{j,k} \in L^\infty_{M^d}$. Then the following are equivalent:

1. $T$ is compact;
2. $\langle \tilde{T}(z)e, h \rangle_{\mathbb{C}^d} \to 0$ as $z \to \partial \mathbb{B}^n$, $e, h \in \mathbb{C}^d$ with $\|e\|_2 = \|h\|_2 = 1$;
3. $T_z e \to 0$ weakly as $z \to \partial \mathbb{B}^n$;
4. $\|T_z e\|_{L^p_{\alpha}} \to 0$ as $z \to \partial \mathbb{B}^n$ for any $p > 1$.

**Proof.** (1) $\implies$ (2). First, suppose that $T$ is compact. Observe that $k_z e \to 0$ weakly in $L^2_{\alpha}$ as $z \to \partial \mathbb{B}^n$. Indeed, if $f \in L^2_{\alpha}$ then

$$
\left| \langle f, k_z e \rangle_{A^2_{\alpha}} \right| \leq \sum_{k=1}^{d} \left| \langle e, e_k \rangle_{\mathbb{C}^d} \right| \left| \langle f, k_z e_k \rangle_{A^2_{\alpha}} \right|
$$

which goes to zero as $z \to \partial \mathbb{B}^n$. (Here we used the fact that $k_z \to 0$ weakly as $z \to \partial \mathbb{B}^n$ in $L^2_{\alpha}(\mathbb{B}^n; \mathbb{C})$). Since $T$ is compact, a well-known result about compact operators implies that $Tk_z e \to 0$ strongly, that is $\|Tk_z e\|_{A^2_{\alpha}} \to 0$ as $z \to \partial \mathbb{B}^n$. Then by the Cauchy-Schwarz
inequality, there holds \[ \left| \langle \tilde{T}(z)e, h \rangle_{A^2_\alpha} \right| = \left| \langle Tk_ze, k_zh \rangle_{A^2_\alpha} \right| \leq \|Tk_ze\|_{A^2_\alpha} \rightarrow 0 \text{ as } z \rightarrow \partial \mathbb{B}^n. \] This gives (2).

\[ \square \]

Proof. (2) \implies (3). If \{f_k\} is a countable orthonormal basis for \( A^2_\alpha(\mathbb{B}^n; \mathbb{C}) \) and \{e_i\} is an orthonormal basis for \( \mathbb{C}^d \), then \( \{f_k e_i\}_{k,i} \) is a countable orthonormal basis for \( A^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \). Let \( \beta = (\beta_1, \ldots, \beta_n) \) be a multi-index and define \( p_\beta(\lambda) = \lambda^\beta = (\lambda_1^{\beta_1}, \ldots, \lambda_n^{\beta_n}). \) Since this is an orthonormal basis for \( A^2_\alpha(\mathbb{B}^n; \mathbb{C}^d) \) (up to a normalization constant which does not matter to us) it is enough to show that \( \langle T_z e, p_\beta h \rangle_{A^2_\alpha} \rightarrow 0 \) as \( z \rightarrow \mathbb{B}^n \) for every \( \beta \in \mathbb{N}^n \) and for every \( \|h\|_2 = 1. \) We begin by observing that
\[ \left\langle \tilde{T}(\varphi_z(\lambda))e, h \right\rangle_{\mathbb{C}^d} = \left\langle \tilde{T}(\lambda)e, h \right\rangle_{\mathbb{C}^d} = \left\langle T_z(\lambda)e, k_\lambda h \right\rangle_{A^2_\alpha}. \]

Expanding \( k_\lambda \) in the orthonormal basis \( \{p_\beta\} \) we next observe that
\[ k_\lambda = (1 - |\lambda|^2)^{(n+1+\alpha)/2} \sum_{\beta \in \mathbb{N}^n} (\lambda^\beta p_\beta) / \gamma_\beta. \]

Combining these two observations, we deduce that
\[ \left\langle \left( \tilde{T} \circ \varphi_z \right)(\lambda)e, h \right\rangle_{\mathbb{C}^d} = \left\langle T_z(k_\lambda e), k_\lambda h \right\rangle_{A^2_\alpha} \]
\[ \quad = \left\langle (T_z)(1 - |\lambda|^2)^{(n+1+\alpha)/2} \sum_{\beta \in \mathbb{N}^n} (\lambda^\beta p_\beta e) / \gamma_\beta, \sum_{\tau \in \mathbb{N}^n} (\lambda^\tau p_\tau h) / \gamma_\tau \right\rangle_{A^2_\alpha} \]
\[ \quad = (1 - |\lambda|^2)^{n+1+\alpha} \sum_{\beta, \tau \in \mathbb{N}^n} \frac{\left\langle T_z(\lambda^\beta p_\beta e), \lambda^\tau p_\tau h \right\rangle_{A^2_\alpha}}{\gamma_\beta \gamma_\tau} \]
\[ \quad = (1 - |\lambda|^2)^{n+1+\alpha} \sum_{\beta, \tau \in \mathbb{N}^n} \frac{\langle T_z(p_\beta e), (p_\tau h) \rangle_{A^2_\alpha}}{\gamma_\beta \gamma_\tau} \lambda^\beta \lambda^\tau. \]

Now, we multiply both sides of this equation by \( \overline{p_{\lambda(\beta)}(\lambda)} / k_\lambda(\lambda) \) and integrate over \( r \mathbb{B}^n \) in the variable \( \lambda: \)
\[ \int_{r \mathbb{B}^n} \left\langle \tilde{T}(\varphi_z(\lambda)) \overline{p_\eta(\lambda)} / k_\lambda(\lambda), e, h \right\rangle_{\mathbb{C}^d} d\nu_\alpha(\lambda) = \sum_{\beta, \tau \in \mathbb{N}^n} \frac{\langle T_z(p_\beta e), (p_\tau h) \rangle_{A^2_\alpha}}{\gamma_\beta \gamma_\tau} \int_{r \mathbb{B}^n} p_{\eta+\beta}(\lambda) p_\tau(\lambda) d\nu_\alpha(\lambda). \]

Computing the integral on the right hand side, and re-writing the left hand side gives:
\[ \int_{r \mathbb{B}^n} \frac{p_\eta(\lambda)(k_\lambda)^{-1}(\lambda)}{r^{2|\eta|+2}} \left\langle \left( \tilde{T} \circ \varphi_z \right)(\lambda)e, h \right\rangle_{\mathbb{C}^d} d\nu_\alpha(\lambda) = r^{2|\eta|+2} \sum_{\beta \in \mathbb{N}^n} \frac{\langle T_z(p_\beta e), p_{\eta+\beta} h \rangle_{A^2_\alpha}}{\gamma_\beta} r^{2|\beta|}. \]

Now, since (2) holds, the left hand side goes to zero as \( z \rightarrow \partial \mathbb{B}^n \) for fixed \( 0 < r < \infty \). Divide both sides by \( r^{2|\eta|+2} \). This means the right hand side becomes:
\[ \frac{\langle T_z e, p_\eta h \rangle_{A^2_\alpha}}{\gamma_0} + \sum_{\beta \in \mathbb{N}^n \setminus \{0\}} \frac{\langle T_z(p_\beta e), p_{\eta+\beta} h \rangle_{A^2_\alpha}}{\gamma_\beta} r^{2|\beta|} \]

Thus, we conclude that for fixed \( r \in (0, 1) \) we have
\[ \frac{\langle T_z e, p_\eta h \rangle_{A^2_\alpha}}{\gamma_0} + \sum_{\beta \in \mathbb{N}^n \setminus \{0\}} \frac{\langle T_z(p_\beta e), p_{\eta+\beta} h \rangle_{A^2_\alpha}}{\gamma_\beta} r^{2|\beta|} \rightarrow 0. \]
as $z \to \partial \mathbb{B}^n$. Easy estimates also give:

$$\left| \sum_{\beta \in \mathbb{N}^n \setminus 0} \frac{\langle T_z(p_\beta e), p_{\gamma_\beta} h \rangle_{A_\beta^2}}{r^{2|\beta|}} \right| \leq \|T\| \left( \sum_{|\beta|=1} r^{2|\beta|} + \sum_{|\beta|>n} r^{2|\beta|} \right).$$

Observe two things. First, the quantity

$$\frac{\langle T_z e, p_\gamma h \rangle_{A_\beta^2}}{\gamma_0}$$

is independent of $r$, and second the quantity:

$$\|T\|_{L^p(U)} \left( \sum_{|\beta|=1} r^{2|\beta|} + \sum_{|\beta|>n} r^{2|\beta|} \right)$$

is the exact same quantity as in the proof the corresponding theorem in [11], where it is proven that for $r$ small enough, this quantity can be made smaller than $\epsilon$ and thus

$$\limsup_{z \to \partial \mathbb{B}^n} \left| \frac{\langle T_z e, p_\gamma h \rangle_{A_\beta^2}}{\gamma_0} \right| < \epsilon.$$

Since this is true for every $\epsilon$ we conclude that $\left| \frac{\langle T_z e, p_\gamma h \rangle_{A_\beta^2}}{\gamma_0} \right| \to 0$ as $z \to \partial \mathbb{B}^n$. Since $\eta$ and $h$ are arbitrary, our claim has been proven.

**Proof.** (3) $\implies$ (4) We need to show that if $T_z e \to 0$ weakly as $z \to \partial \mathbb{B}^n$, then $\|T_z e\|_{L^p_U} \to 0$ as $z \to \partial \mathbb{B}^n$ for any $1 < p < \infty$. If $r \in (0,1)$, there holds:

$$\|T_z e\|_{L^p_U} = \int_{\mathbb{B}^n \setminus r \mathbb{B}^n} \langle (T_z e)(w), (T_z e)(w) \rangle_{C^d} dv_{\alpha}(w) + \int_{r \mathbb{B}^n} \langle (T_z e)(w), (T_z e)(w) \rangle_{C^d} dv_{\alpha}(w)$$

$$\leq \nu(\mathbb{B}^n \setminus r \mathbb{B}^n)^{1/2} \|T_z e\|_{L^p} + \int_{r \mathbb{B}^n} \langle (T_z e)(w), (T_z e)(w) \rangle_{C^d} dv_{\alpha}(w).$$

Choose $r$ close enough to 1 so that the first term on the right hand side smaller than any $\delta > 0$ (this is possible since $\|T_z e\|_{L^p}^2$ is bounded independent of $z$, by Lemma 3.4). Now, a sequence of holomorphic functions which converges weakly to zero also converges to zero in norm on compact sets. Thus, the second term on the right hand side goes to zero as $z \to \partial \mathbb{B}^n$. This proves our claim for the case $p = 2$. We now assume that $p \in (2, \infty)$. We have that:

$$\|T_z e\|_{L^p} \leq \|T_z e\|_{L^p}^{1/p} \|T_z e\|_{L^{2p-2p}}^{(p-1)/p}.$$
We then study radial truncations of the kernel, use Lemma 2.7 to prove the truncations induce compact operators, and then make a limiting argument to show that $T_R$ is compact. Let $e \in \mathbb{C}^d$. First, there holds:

\[
(T_R^* K_w e)(z) = \int_{\mathbb{B}^n} (T_R^* K_w e)(\lambda)(\overline{K_z(\lambda)})dv_\alpha(\lambda)
\]

\[
= \int_{\mathbb{B}^n} \sum_{i=1}^d \left\langle (T_R^* K_w e)(\lambda)(\overline{K_z(\lambda)}), e_i \right\rangle_{\mathbb{C}^d} e_i dv_\alpha(\lambda)
\]

\[
= \sum_{i=1}^d \int_{\mathbb{B}^n} \left\langle (T_R^* K_w e)(\lambda), K_z(\lambda)(e_i) \right\rangle_{\mathbb{C}^d} dv_\alpha(\lambda) e_i
\]

\[
= \sum_{i=1}^d \langle (T_R^* K_w e), K_z e_i \rangle_{A^2_\alpha} e_i
\]

\[
= \sum_{i=1}^d \langle K_w e, (T_R K_z e_i) \rangle_{A^2_\alpha} e_i
\]

\[
= \sum_{i=1}^d \langle (T_R K_z e_i)(w), e \rangle_{\mathbb{C}^d} e_i
\]

By uniqueness of expansion in orthonormal bases, this implies that:

\[
\langle (T_R^* K_w e)(z), e_i \rangle_{\mathbb{C}^d} = \langle e, (T_R K_z e_i)(w) \rangle_{\mathbb{C}^d}.
\]

This computation yields:

\[
\langle (T_R f)(w), e \rangle_{\mathbb{C}^d} = \langle T_R f, K_w e \rangle_{A^2_\alpha}
\]

\[
= \langle f, (T_R^* K_w e) \rangle_{A^2_\alpha}
\]

\[
= \int_{\mathbb{B}^n} \langle f(z), (T_R^* K_w e)(z) \rangle_{\mathbb{C}^d} dv_\alpha(z)
\]

\[
= \int_{\mathbb{B}^n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle e_j, (T_R^* K_w e)(z) \rangle_{\mathbb{C}^d} dv_\alpha(z)
\]

\[
= \int_{\mathbb{B}^n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle (T_R K_z e_j)(w), e \rangle_{\mathbb{C}^d} dv_\alpha(z)
\]

Finally, this computation gives:

\[
(T_R f)(w) = \sum_{i=1}^d \langle (T_R f)(w), e_i \rangle_{\mathbb{C}^d} e_i
\]
This shows us that \( T_R \) is an integral operator, with matrix–valued kernel \( M(z, w) \) given by the following relation

\[
\langle M(z, w)e_i, e_j \rangle_{C^d} = \langle (T_R K z e_j)(w), e_i \rangle_{C^d}.
\]

That is,

\[
(T_R f)(w) = \int_{\mathbb{B}^n} M(z, w) f(z) dv_{\alpha}(z).
\]

We now define the truncations of this operator. For \( t \in (0, 1) \), we define the operator \( (T_R)_t \) on \( A_\alpha^2 \) by:

\[
\left( (T_R)_t f \right)(w) = \int_{t \mathbb{B}^n} M(z, w) f(z) dv_{\alpha}(z).
\]

So that \( (T_R)_t \) is an integral operator with kernel given by: \( M_{[t]}(z, w) = 1_{t \mathbb{B}^n}(z) M(z, w) \) Let \( \| \cdot \|_F \) denote the Frobenius norm of a \( d \times d \) matrix. We make the following estimation:

\[
\|M(z, w)\|_F^2 = \sum_{i=1}^{d} \sum_{k=1}^{d} | \langle M(z, w)e_k, e_i \rangle_{C^d} |^2
\]

\[
= \sum_{i=1}^{d} \sum_{k=1}^{d} | \langle (T_R K z e_i)(w), e_k \rangle_{C^d} |^2
\]

\[
\leq \sum_{i=1}^{d} \| (T_R K z e_i)(w) \|_{C^d}^2.
\]

This gives for any \( t \in [0, 1) \):

\[
\int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \| M_{[t]}(z, w) \|_F^2 dv_{\alpha}(z) dv_{\alpha}(w) = \int_{\mathbb{B}^n} \int_{t \mathbb{B}^n} \| M(z, w) \|_F^2 dv_{\alpha}(z) dv_{\alpha}(w)
\]

\[
\leq \int_{\mathbb{B}^n} \int_{t \mathbb{B}^n} d \sum_{k=1}^{d} \| (T_R K z e_k)(w) \|_{C^d}^2 dv_{\alpha}(z) dv_{\alpha}(w)
\]

\[
\leq d \sum_{k=1}^{d} \int_{t \mathbb{B}^n} \int_{\mathbb{B}^n} \| (T_R K z e_k)(w) \|_{C^d}^2 dv_{\alpha}(z) dv_{\alpha}(z)
\]

\[
= d \sum_{k=1}^{d} \int_{t \mathbb{B}^n} \| T_R (K z e_k) \|_{A_\alpha^2}^2 dv_{\alpha}(z)
\]

\[
\leq d \sum_{k=1}^{d} \int_{t \mathbb{B}^n} \| T_R \|^2 \| K z e_k \|_{A_\alpha^2}^2 dv_{\alpha}(z)
\]

\[
\leq d^2 \| T_R \|^2 \int_{t \mathbb{B}^n} \| K z \|_{A_\alpha^2}^2 dv_{\alpha}(z)
\]

\[
< \infty.
\]
Thus, by Lemma 2.7 \((T_R)_{[t]}\) is Hilbert-Schmidt, and therefore compact. So, to show that \(T_R\) is compact, all we need to prove is that:

\[
\lim_{t \to 1^{-}} \| T_R - (T_R)_{[t]} \|_{\mathcal{L}(L^2)} = 0.
\]

Note that \(T_R - (T_R)_{[t]}\) is an integral operator with kernel given by \(1_{B^n \setminus B^n}(z)M(z, w)\). We will use Schur’s Test and Lemma 3.3 to estimate the norm of this operator. For Schur’s test, choose \(\| K_z \|_{A^2}^{p/2} \) as our test function. If we choose \(p\) such that \(0 < (p - 1) < (n + 1)^{-1} \) and \(q\) as conjugate exponent, and \(\epsilon \in (2(p - 1)^{-1}, 2(n + 1)^{-1}p^{-1})\) then we can apply Lemma 3.3. Also, let \(G(t, w) = 1_{B^n \setminus B^n}(w)\). Then

\[
\int_{B^n} \| G(t, w)M(z, w) \| \| K_w \|_{A^2}^{\epsilon} dv_\alpha(w) \simeq \int_{B^n} \| G(t, w) \| \sum_{i,j=1}^{d} \| \langle M(z, w)e_i, e_j \rangle_{C^d} \| K_w \|_{A^2}^{\epsilon} dv_\alpha(w)
\]

\[
\leq d \sum_{i=1}^{d} \int_{B^n} \| G(t, w) \| \| (T_R K_z e_i)(w) \|_{C^d} \| K_w \|_{A^2}^{\epsilon} dv_\alpha(w)
\]

\[
\leq \sum_{i=1}^{d} \| K_z \|_{A^2}^{\epsilon} \left( \sup_{z \in B^n} \| T_{-z} e_i \|_{A^2} \right) \sup_{z \in B^n} | J_{a, b}(z) |^{1/p} 
\]

\[
= \| K_z \|_{A^2}^{\epsilon} \sum_{i=1}^{d} \left( \sup_{z \in B^n} \| T_{-z} e_i \|_{A^2} \right) \sup_{z \in B^n} | J_{a, b}(z) |^{1/p}
\]

and

\[
\int_{B^n} \| G(t, z)M(z, w) \| \| K_z \|_{A^2}^{\epsilon} dv_\alpha(z) \simeq \int_{B^n} \| G(t, z) \| \sum_{i,j=1}^{d} \| \langle M(z, w)e_i, e_j \rangle_{C^d} \| K_z \|_{A^2}^{\epsilon} dv_\alpha(z)
\]

\[
= \int_{B^n} \| G(t, z) \| \sum_{i,j=1}^{d} \| \langle e_j, (T_R K_z e_i)(w) \rangle_{C^d} \| K_z \|_{A^2}^{\epsilon} dv_\alpha(z)
\]

\[
\leq d \sum_{i=1}^{d} \int_{B^n} \| G(t, z) \| \| (T_R K_z e_i)(w) \|_{C^d} \| K_z \|_{A^2}^{\epsilon} dv_\alpha(z)
\]

\[
\leq \sum_{i=1}^{d} \| K_w \|_{A^2}^{\epsilon} \left( \sup_{z \in B^n} \| T_{-z}^* e_i \|_{A^2} \right) \sup_{w \in B^n} | J_{a, b}(w) |^{1/p} 
\]

\[
= \| K_w \|_{A^2}^{\epsilon} \sum_{i=1}^{d} \left( \sup_{z \in B^n} \| T_{-z}^* e_i \|_{A^2} \right) \sup_{w \in B^n} | J_{a, b}(w) |^{1/p}.
\]

If we choose \(a, b\) as in Lemma 3.3, we have that \(\sup_{w \in B^n} | J_{a, b}(w) |^{1/p} < \infty\) and our hypotheses on \(\| T_{-z} e \|_{A^2}\), an application of Schur’s test gives that \(\lim_{t \to 1^{-}} \| T_R - (T_R)_{[t]} \|_{\mathcal{L}(L^2)} = 0\). This gives that \(T_R\) is compact, and therefore, \(T\) is compact.

\[ \square \]

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References

[1] S. Twareque Ali and M. Engliš, *Berezin-Toeplitz quantization over matrix domains*, Contributions in mathematical physics, Hindustan Book Agency, New Delhi, 2007, pp. 1–36. MR2423653 (2010e:47041)

[2] Sheldon Axler and Dechao Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. 47 (1998), no. 2, 387–400.

[3] Željko Ćučković and Sönmez Şahutoğlu, *Axler-Zheng type theorem on a class of domains in $\mathbb{C}^n$*, Integral Equations Operator Theory 77 (2013), no. 3, 397–405.

[4] Miroslav Engliš, *Compact Toeplitz operators via the Berezin transform on bounded symmetric domains*, Integral Equations Operator Theory 33 (1999), no. 4, 426–455.

[5] ——, *Density of algebras generated by Toeplitz operator on Bergman spaces*, Ark. Mat. 30 (1992), no. 2, 227–243.

[6] Jie Miao and Dechao Zheng, *Compact operators on Bergman spaces*, Integral Equations Operator Theory 48 (2004), no. 1, 61–79.

[7] Mishko Mitkovski, Daniel Suárez, and Brett D. Wick, *The essential norm of operators on $A^p(\mathbb{B}_n)$*, Integral Equations Operator Theory 75 (2013), no. 2, 197–233, DOI 10.1007/s00020-012-2025-1. MR3008923

[8] Mishko Mitkovski and Brett Wick, *The essential norm of operators on $A_n(\mathbb{D}^n)$*, available at http://arxiv.org/abs/1208.5819v3.

[9] ——, *A reproducing kernel thesis for operators on bergman-type function spaces*, available at http://arxiv.org/abs/1212.0507v3.

[10] Robert Rahm and Brett Wick, *Essential Norm of Operators on Vector-Valued Bergman Space.*

[11] Roberto Raimondo, *Toeplitz operators on the Bergman space of the unit ball*, Bull. Austral. Math. Soc. 62 (2000), no. 2, 273–285.

[12] Walter Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980.

[13] Daniel Suárez, *The essential norm of operators in the Toeplitz algebra on $A^p(\mathbb{B}_n)$*, Indiana Univ. Math. J. 56 (2007), no. 5, 2185–2232.

[14] Daniel Suárez, *Approximation and the $n$-Berezin transform of operators on the Bergman space*, J. Reine Angew. Math. 581 (2005), 175–192.

[15] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005.

[16] ——, *Operator theory in function spaces*, 2nd ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.

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