FRACTIONAL FOURIER TRANSFORMS, HARMONIC OSCILLATOR PROPAGATORS AND STRICHARTZ ESTIMATES ON PILIPOVIĆ AND MODULATION SPACES

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Abstract. We give a proof of that harmonic oscillator propagators and fractional Fourier transforms are essentially the same. We deduce continuity properties and fix time estimates for such operators on modulation spaces, and apply the results to prove Strichartz estimates for such propagators when acting on Pilipović and modulation spaces. Especially we extend some results by Balhara, Cordero, Nicola, Rodino and Thangavelu. We also show that general forms of fractional harmonic oscillator propagators are continuous on suitable Pilipović spaces.

0. Introduction

In the paper we investigate mapping properties for powers of harmonic oscillators, their propagators and fractional Fourier transforms on Pilipović spaces and modulation spaces. Especially we link fractional Fourier transforms with harmonic oscillator propagators, which opens up for full transitions of various properties between these operators.

We also deduce certain continuity properties for fractional Fourier transforms on modulation spaces (including Wiener amalgam spaces). By using the link between the fractional Fourier transform and harmonic oscillator propagators we extend at the same time certain continuity properties of harmonic oscillator propagators on modulation spaces, given in [8, 15, 16]. These investigations are also related to [13], where among others, $L^p$ and Hausdorff-Young estimates for fractional Fourier transforms are established. Thereafter we apply such continuity results to extend certain Strichartz estimates for harmonic oscillator propagators in [16] when acting on modulation spaces. In the end we apply our results to certain general classes of time-dependent equations, similar to Schrödinger and heat equations. We prove that several of these equations are ill-posed in the framework of Schwartz space, Gelfand-Shilov spaces and their distribution spaces, but well-posed in the framework of suitable Pilipović spaces.

Harmonic oscillators and their propagators are important in quantum mechanics, e.g. when investigating free particles in quantum systems. An important question concerns continuity for such operators. For the (standard) harmonic oscillator

$$H_x = |x|^2 - \Delta_x, \quad x \in \mathbb{R}^d,$$

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the corresponding propagator is given by
\[ P_\varrho = e^{-i\varrho H_x}, \tag{0.1} \]
which can also be formulated as
\[ P_\varrho = e^{-iH_{x,\varrho}}, \quad H_{x,\varrho} = \varrho(|x|^2 - \Delta_x). \tag{0.1}' \]
Here \( \varrho \in \mathbb{R} \). (See Sections 1–4 for more general operators of such forms.)

It is well-known that these operators are homeomorphisms on the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) and its dual \( \mathcal{S}'(\mathbb{R}^d) \), the set of tempered distributions. (See [35] and Section 1 for notations.) Similar continuity properties hold true with Pilipović spaces \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) and \( \mathcal{H}_s(\mathbb{R}^d) \), and their distribution spaces \( \mathcal{H}'_{0,s}(\mathbb{R}^d) \) and \( \mathcal{H}'_s(\mathbb{R}^d) \), in place of \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \). We recall that Fourier invariant (standard) Gelfand-Shilov are special cases of Pilipović spaces.

More precisely we have
\[ \mathcal{H}_s(\mathbb{R}^d) = \mathcal{S}_s(\mathbb{R}^d) \neq \{0\}, \quad s \geq \frac{1}{2}, \quad \mathcal{H}_s(\mathbb{R}^d) \neq \mathcal{S}_s(\mathbb{R}^d) = \{0\}, \quad s < \frac{1}{2}, \]
\[ \mathcal{H}_{0,s}(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d) \neq \{0\}, \quad s > \frac{1}{2}, \quad \mathcal{H}_{0,s}(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s \leq \frac{1}{2} \]
(see [48, 53]).

Harmonic oscillators and their propagators also possess convenient mapping properties in the background of suitable modulation spaces, a family of (quasi-)Banach spaces which were introduced in [23] by H. Feichtinger and further developed in [24–26, 28, 32]. For example, in [37–40], several continuity results for Schrödinger propagators including potential terms when acting on modulation spaces are deduced. Harmonic oscillator propagators are then obtained by choosing the potentials as \( c|x|^2 \) for some positive constant \( c \). For example, it is proved in [37–39] that for the Fourier invariant modulation space \( MP(\mathbb{R}^d) \), the map
\[ e^{iH_{x,\varrho}} : MP(\mathbb{R}^d) \to MP(\mathbb{R}^d) \tag{0.2} \]
is continuous. See also [8, 9, 15, 16] for other results concerning mapping properties of propagators on modulation spaces.

In Section 3 we give a strict proof, based on the Bargmann transform, of that the propagator in (0.1) can be identified with fractional Fourier transforms by the formula
\[ e^{-i\frac{\pi}{4} H_{x,\varrho}} = e^{-i\frac{\pi d \varrho}{2}} \mathcal{F}_\varrho, \tag{0.3} \]
for every \( \varrho \in \mathbb{R} \). Here recall that the fractional Fourier transform \( \mathcal{F}_\varrho \), acting on \( \mathcal{S}(\mathbb{R}^d) \) is given by
\[ \mathcal{F}_\varrho f(\xi) = \langle K_{\varrho,\xi}(\xi, \cdot), f \rangle, \]
where \( K_{\varrho,\xi}(\xi, x) \) is the distribution kernel, given by
\[ K_{\varrho,\xi} = \frac{1}{2} \sum_{j=1}^{d} K_{\varrho}, \tag{0.4} \]
with
\[
K_\varrho(\xi, x) = \begin{cases} 
\left(\frac{1-i\cot(\frac{\varrho \pi}{2})}{2\pi}\right)^{\frac{1}{2}} \exp \left(i \cdot \frac{(x^2+\xi^2)\cos(\frac{\varrho \pi}{2})-2\varrho x}{2\sin(\frac{\varrho \pi}{2})}\right), & \varrho \in \mathbb{R}\setminus 2\mathbb{Z}, \\
\delta(\xi - x), & \varrho \in 4\mathbb{Z}, \\
\delta(\xi + x), & \varrho \in 2 + 4\mathbb{Z},
\end{cases}
\] (0.5)

Here \(x, \xi \in \mathbb{R}\). (See e.g. [3, 4] and the references therein. See also Section 1 for more details on fractional Fourier transform.) We observe that
\[
(\mathcal{F}_1 f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx
\]
and
\[
(\mathcal{F}_3 f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{i(x, \xi)} \, dx
\]
are the (ordinary) Fourier transform and the inverse Fourier transform, respectively.

Ideas on fractional Fourier transforms goes back to at least 1929 (cf. historical notes and references in [12]). On the other hand, the first rigorous approach seems to be given at first 1939 by Kober in [11]. Since then numerous applications of fractional Fourier transforms have appeared. For example, they were explicitly introduced in quantum mechanics around 1980 and thereafter applied in optics (see e.g. [22, 41] and the references therein). In quantum mechanics, the formulae (1.3) and (1.5) appear naturally by considering certain rotations in the phase space and their induced actions on quantum observables (see e.g. [15, 16, 18–20] and Remark 1.10 in Section 1). There are also several applications in signal analysis e.g. when discussing rotation properties for time-frequency representations in time-frequency analysis, phase retrieval, optics and pattern recognition. (See e.g. [2, 3, 22, 45] and the references therein.) Here we also remark that in some aspects, the theory of the fractional Fourier transform is merely a special case of the metaplectic representation (see [20]).

In terms of the Bargmann transform \(\mathcal{B}_d\), fractional Fourier transforms take the convenient form
\[
(\mathcal{B}_d(\mathcal{F}_\varrho f))(z) = (\mathcal{B}_d f)(e^{-i\frac{\varrho \pi}{4}} z), \quad z \in \mathbb{C}^d.
\] (0.6)
(See e.g. [5, 53].) The relation (1.3) was obtained for \(\varrho = 1\) already in [3] by Bargmann.

A motivation of the identity (0.6) is given in e.g. [12], where several links between the harmonic oscillator and fractional Fourier transforms are established. We also remark that F. G. Mehler established a formula (afterwards named Mehler’s formula) for the operator \(e^{-\frac{1}{4}H_{x, y}}\) for \(\varrho > 0\), already in [43]. Observe that \(e^{-\frac{1}{2}H_{x, y}}\) is the canonical density operator in statistical physics, see [39] Section 3.4).

Analytic extensions of Mehler’s formula lead to that the kernel of \(e^{-\frac{1}{2}H_{x, y}}\) is essentially the same as the kernel of the fractional Fourier transform (0.4) and (0.5).
Proposition 0.1. Let $g \in \mathbb{R}$ and $p, q \in (0, \infty]$ be such that $q \leq p$. Then the following is true:

1. the map \[ \mathcal{F}_g = e^{-i \frac{\pi}{2} H_{x, g}} : M^{p, q}(\mathbb{R}^d) + W^{q, p}(\mathbb{R}^d) \to M^{q, p}(\mathbb{R}^d) + W^{p, q}(\mathbb{R}^d) \]

is continuous;

2. if in addition $g \notin \mathbb{Z}$, then the map \[ \mathcal{F}_g = e^{-i \frac{\pi}{2} H_{x, g}} : M^{p, q}(\mathbb{R}^d) + M^{q, p}(\mathbb{R}^d) \to M^{q, p}(\mathbb{R}^d) \cup W^{p, q}(\mathbb{R}^d) \]

is continuous.

There are several types of estimates behind the conclusions in the previous proposition, which are expected to be applicable in non-linear partial differential equations. Examples on such estimates are

\[ \| \mathcal{F}_g f \|_{M^{p, q}} = \| e^{-i \frac{\pi}{2} H_{x, g}} f \|_{M^{p, q}} \lesssim \left| \sin \left( \frac{\pi g}{2} \right) \right| = \frac{1}{2} p^{-\frac{1}{p}} \| f \|_{M^{p, q}}, \]

\[ \| \mathcal{F}_g f \|_{M^{q, p}} = \| e^{-i \frac{\pi}{2} H_{x, g}} f \|_{M^{q, p}} \lesssim \left| \cos \left( \frac{\pi g}{2} \right) \right| = \frac{1}{2} q^{-\frac{1}{q}} \| f \|_{W^{q, p}}, \]

\[ \| \mathcal{F}_g f \|_{W^{p, q}} = \| e^{-i \frac{\pi}{2} H_{x, g}} f \|_{W^{p, q}} \lesssim \left| \cos \left( \frac{\pi g}{2} \right) \right| = \frac{1}{2} q^{-\frac{1}{q}} \| f \|_{M^{q, p}}, \]

and

\[ \| \mathcal{F}_g f \|_{W^{p, q}} = \| e^{-i \frac{\pi}{2} H_{x, g}} f \|_{W^{p, q}} \lesssim \left| \sin \left( \frac{\pi g}{2} \right) \right| = \frac{1}{2} p^{-\frac{1}{p}} \| f \|_{W^{q, p}}, \]

still with $q \leq p$ (see Theorems 3.6 and 3.7).

Some of our investigations include more general propagators and fractional Fourier transforms, where $g$ in (0.1)' and (0.3) is allowed to be any complex number. If $\text{Im}(\zeta) > 0$, then $P_c$ does not make sense as a continuous operator on $\mathcal{F}(\mathbb{R}^d)$, nor on any Fourier invariant Gelfand-Shilov space and their duals. Consequently, in order to investigate such extended class of propagators, or, even more generally, powers of $H_{x, g}$ and their propagators, i.e. operators of the forms \[ H_{x, g}^r \text{ and } P_{g, r} = e^{-i H_{x, g}^r}, \quad g \in \mathbb{C}, \ r \in \mathbb{R}, \quad (0.7) \]
other families of function and distribution spaces are needed. It turns out
that such continuity discussions can be performed in the framework of certain
Pilipović spaces and their distribution spaces.

In order to shed some further lights, we present the following propositions,
which are immediate consequences of our investigations. For the fractional
Fourier transforms, these conclusions also follows from the analysis in [53].

**Proposition 0.2.** Let \( q \in \mathbb{C} \) and \( s \in \mathbb{R}_0 \). Then the following is true:

1. if \( s < \frac{1}{2} \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are homeomorphisms on \( \mathcal{H}_s(\mathbb{R}^d) \) and
   on \( \mathcal{H}'_s(\mathbb{R}^d) \);
2. if \( \text{Im}(q) < 0 \) and \( s \geq \frac{1}{2} \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are continuous
   injections but not surjections on \( \mathcal{H}_s(\mathbb{R}^d) \), \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}'(\mathbb{R}^d) \) and on
   \( \mathcal{H}'_s(\mathbb{R}^d) \);
3. if \( \text{Im}(q) = 0 \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are homeomorphisms on \( \mathcal{H}_s(\mathbb{R}^d) \),
   \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}'(\mathbb{R}^d) \) and on \( \mathcal{H}'_s(\mathbb{R}^d) \);
4. if \( \text{Im}(q) > 0 \) and \( s \geq \frac{1}{2} \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are discontinuous on
   \( \mathcal{H}_s(\mathbb{R}^d) \), \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}'(\mathbb{R}^d) \) and on \( \mathcal{H}'_s(\mathbb{R}^d) \).

The same holds true with \( s > \frac{1}{2} \), \( s \leq \frac{1}{2} \) and \( \mathcal{H}_0, s \) in place of \( s \geq \frac{1}{2} \), \( s < \frac{1}{2} \)
and \( \mathcal{H}_s \) at each occurrence.

**Proposition 0.3.** Let \( q \in \mathbb{C} \). Then the following is true:

1. if \( \text{Im}(q) < 0 \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are continuous from \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \) to
   \( \mathcal{S}_{1/2}(\mathbb{R}^d) \), and
   \[
   \mathcal{F}_q(\mathcal{S}'_{1/2}(\mathbb{R}^d)) = e^{-i \frac{q}{2} H x, e}(\mathcal{S}'_{1/2}(\mathbb{R}^d)) \subseteq \mathcal{S}_{1/2}(\mathbb{R}^d);
   \]
2. if \( \text{Im}(q) > 0 \), then \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are discontinuous from \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \)
to \( \mathcal{S}_{1/2}(\mathbb{R}^d) \), and
   \[
   \mathcal{S}'_{1/2}(\mathbb{R}^d) \subseteq \mathcal{F}_q(\mathcal{S}_{1/2}(\mathbb{R}^d)) = e^{-i \frac{q}{2} H x, e}(\mathcal{S}_{1/2}(\mathbb{R}^d)) \subseteq \mathcal{H}_0, 1/2(\mathbb{R}^d).
   \]

By usual inclusion relations for Pilipović spaces, it follows that Proposition
0.3 is a refinement of (2) and (4) in Proposition 0.2. In fact, consider the
inclusions
\[
\mathcal{S}_s(\mathbb{R}^d) \subseteq \Sigma_t(\mathbb{R}^d) \subseteq \mathcal{S}_t(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)
\]
\[
\mathcal{S}'(\mathbb{R}^d) \subseteq \Sigma'_t(\mathbb{R}^d) \subseteq \mathcal{S}_t(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d), \quad \frac{1}{2} \leq s < t, \quad (0.8)
\]
between the Schwartz space, its distribution space, and all (standard) Fourier
invariant Gelfand-Shilov spaces of functions and ultra-distributions. Then
Proposition 0.3 (1) shows that if \( \text{Im}(q) < 0 \), then the images of the spaces
in (0.8) under \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) are strict subspaces of \( \mathcal{S}_{1/2}(\mathbb{R}^d) \), the smallest
space in (0.8). If instead \( \text{Im}(q) > 0 \), then Proposition 0.3 (2) shows that the image of this smallest space \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) under \( \mathcal{F}_q \) and \( e^{-i \frac{q}{2} H x, e} \) is a superspace
of \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \), the largest space in (0.8).

This implies, roughly speaking, that the (standard) spaces in (0.8) are dis-
qualified when performing detailed continuity investigations of the canonical
density operator \( e^{-H x, e} \) in statistical physics, and that these spaces can not
be used in continuity investigations of the inverse $e^{H_{x,0}}$ of that operator. On the other hand, Proposition 0.2 (1) shows that Pilipović spaces and their distribution spaces, which are not Gelfand-Shilov spaces of functions and distributions, are suitable for continuity investigations of $e^{-H_{x,0}}$ and $e^{H_{x,0}}$.

In the most general case we allow $\rho \in C^d$, in which case $H_{x,0}$ is defined as

$$H_{x,0} = \sum_{j=1}^{d} \rho_j (x_j^2 - \xi_j^2),$$

and $\mathcal{F}_0$ is a fractional Fourier transform of the multiple order $\rho$ (see e.g. [20]). In Section 3 we show that (0.3) and (0.6) still hold true for such general $\rho$.

Finally, in Section 4 we apply our results to deduce certain types of Strichartz estimates. We recall that Strichartz estimates appears when finding properties on solutions to Cauchy problems like the Schrödinger equation

$$i\partial_t u - H_x u = F, \quad u(0, x) = u_0(x), \quad (t, x) \in I \times \mathbb{R}^d. \quad (0.9)$$

Here $I = [0, \infty)$ or $I = [0, T]$ for some $T > 0$, $F$ is a suitable function or (ultra-)distribution on $I \times \mathbb{R}^d$ and $u_0$ is a suitable function or (ultra-)distribution on $\mathbb{R}^d$. It follows that continuity properties of the propagator

$$(E f)(t, x) = (e^{-itH_x} u_0)(x), \quad (t, x) \in I \times \mathbb{R}^d, \quad (0.10)$$

$$(S_1 F)(t, x) = \int_0^t (e^{-i(t-s)H_x} F(s, \cdot))(x) \, ds, \quad (t, x) \in I \times \mathbb{R}^d, \quad (0.11)$$

and for

$$(S_2 F)(t, x) = \int_I (e^{-i(t-s)H_x} F(s, \cdot))(x) \, ds, \quad (t, x) \in I \times \mathbb{R}^d, \quad (0.12)$$

are essential when finding estimates for solutions to (0.9) (see [16, 20, 50]). Such estimates are called Strichartz estimates (see also Subsection 1.8 for more details).

In Section 3 we deduce Strichartz estimates of the operators $E$, $S_1$ and $S_2$ when acting on modulation spaces or Lebesgue spaces with values in modulation spaces. For example by straight-forward applications of Proposition 0.1, we get the following result, which is a special case of Theorem 4.3 in Section 4.

**Theorem 0.4.** Let $p, q, r_0 \in (0, \infty]$ be such that $q \leq p$ and

$$\frac{1}{r_0} > d \left( \frac{1}{q} - \frac{1}{p} \right).$$

Then $E$ is uniquely extendable to a continuous map

$$E : M^{p,q}(\mathbb{R}^d) + W^{q,p}(\mathbb{R}^d) \to L^{r_0}([0, T]; M^{p,q}(\mathbb{R}^d)) \bigcap L^{r_0}([0, T]; W^{p,q}(\mathbb{R}^d)).$$

Another application of Proposition 0.1 in combination of the Hardy-Littlewood-Sobolev inequality leads to the following special case of Theorem 4.3 in Section 4.
Theorem 0.5. Let \( p, p_0, q \in (1, \infty)\) and \( r_0 \in (0, \infty) \) be such that
\[
0 \leq d \left( \frac{1}{q} - \frac{1}{p} \right) < 1, \quad d \left( \frac{1}{q} - \frac{1}{p_0} \right) \leq 1 + \frac{1}{r_0} - \frac{1}{p_0}.
\]
Then \( S_1 \) and \( S_2 \) from \( C([0, T]; M^1(\mathbb{R}^d)) \) to \( L^\infty([0, T]; M^1(\mathbb{R}^d)) \) are uniquely extendable to continuous mappings
\[
S_j : L^p([0, T]; M^{p,q}(\mathbb{R}^d) + W^{p,q}(\mathbb{R}^d)) \to L^{r_0}([0, T]; M^{q,p}(\mathbb{R}^d) \cap W^{p,q}(\mathbb{R}^d)),
\]
\( j = 1, 2 \).

In Section 4 we also discuss well-posed properties for more general equations, where \( H_x \) in (0.9) is replaced by \( \zeta H_{x,\breve{\varphi}} \) for some \( \zeta \in \mathbb{C}, \varphi \in \mathbb{C}^d \) and \( r > 0 \) which satisfy \( \text{Im}(\zeta \varphi_j) > 0 \) for some \( j \). Here we show that such equations are ill-posed, not only in the framework of Schwartz functions and tempered distributions, but also for Gelfand-Shilov functions and distributions, while the equation is well-posed for suitable Pilipović spaces and their distributions.

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1. Preliminaries

In this section we recall some basic facts. We start by discussing Gelfand-Shilov spaces, then modulation spaces and thereafter Pilipović spaces and some of their properties. Then we recall the Bargmann transform and some of its properties, and introduce suitable classes of power series expansions and entire functions on \( \mathbb{C}^d \). Finally, in Subsection 1.8 we recall some facts on Strichartz estimates.

1.1. Gelfand-Shilov spaces and their distribution spaces. We start by recalling definitions of Fourier invariant (standard) Gelfand-Shilov spaces and their distribution spaces (cf. e.g. [29, 47]). Let \( s \geq 0 \) and \( h \in \mathbb{R}_+ \) be fixed. Then \( S_{s,h}(\mathbb{R}^d) \) is the set of all \( f \in \mathcal{C}^\infty(\mathbb{R}^d) \) such that
\[
\|f\|_{S_{s,h}} = \sup_{h^{|\alpha| + |\beta|}(\alpha! \beta!)} \frac{|x^{\beta} \partial^\alpha f(x)|}{h^{|\alpha| + |\beta|}(\alpha! \beta!)} < \infty.
\]
is finite. Here the supremum is taken over all \( \alpha, \beta \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d \).

Obviously \( S_{s,h} \subset \mathcal{S} \) is a Banach space which increases with \( h \) and \( s \).

The Gelfand-Shilov space \( S_s(\mathbb{R}^d) \) (\( \Sigma_s(\mathbb{R}^d) \)) of Roumieu type (Beurling type) is the projective limit (projective limit) of \( S_{s,h}(\mathbb{R}^d) \) with respect to \( h \).

This implies that
\[
S_s(\mathbb{R}^d) = \bigcup_{h > 0} S_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h > 0} S_{s,h}(\mathbb{R}^d) \tag{1.1}
\]
We remark that (1.8) is true with dense embeddings, and that indexed by the extended set of power series expansions are based on certain spaces of sequences on \( s_1.2. \)

Fourier transforms (cf. \([14, 34, 53]\)).

Distribution spaces, e.g. by suitable estimates of their Fourier and short-time Fourier transforms. Let \( \phi \) defined by the formula

\[
\phi(x,\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx
\]

when \( f \in L^1(\mathbb{R}^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \).

The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( \mathcal{S}_s'(\mathbb{R}^d) \) and \( \mathcal{S}_s(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \mathcal{S}_s(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

Next we recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms. Let \( \phi \in \mathcal{S}(\mathbb{R}^d) \) be fixed. For every \( f \in \mathcal{S}'(\mathbb{R}^d) \), the short-time Fourier transform \( V_\phi f \) is the distribution on \( \mathbb{R}^{2d} \) defined by the formula

\[
(V_\phi f)(x,\xi) = \mathcal{F}(f \bar{\phi}((\cdot - x)))(\xi) = (f, \phi(\cdot - x)e^{i(x,\xi)})
\]

We recall that if \( T(f,\phi) \equiv V_\phi f \) when \( f,\phi \in \mathcal{S}_s(\mathbb{R}^d) \), then \( T \) is uniquely extendable to sequentially continuous mappings

\[
T : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}) \bigcap \mathcal{C}_c^\infty(\mathbb{R}^{2d}),
\]

\[
T : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s'(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}),
\]

and similarly with \( \mathcal{S}_s \) in place of \( \mathcal{S}_s \) at each occurrence (cf. \([17,53]\)). We also note that \( V_\phi f \) takes the form

\[
V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \bar{\phi}(y - x) e^{-i(y,\xi)} \, dy
\]

for admissible \( f \).

There are several characterizations of Gelfand-Shilov spaces and their distribution spaces, e.g. by suitable estimates of their Fourier and Short-time Fourier transforms (cf. \([14,34,53]\)).

1.2. Spaces of sequences. The definitions of Pilipović spaces and spaces of power series expansions are based on certain spaces of sequences on \( \mathbb{N}^d \), indexed by the extended set

\[
\mathbb{R}_\sigma = \mathbb{R}_+ \bigcup \{ b_\sigma : \sigma \in \mathbb{R}_+ \},
\]

of \( \mathbb{R}_+ \). We extend the ordering relation on \( \mathbb{R}_+ \) to the set \( \mathbb{R}_\sigma \), by letting

\[
s_1 < b_\sigma < s_2 \quad \text{and} \quad b_{\sigma_1} < b_{\sigma_2}
\]
when \( s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R}_+ \), satisfy \( s_1 < \frac{1}{2} \leq s_2 \) and \( \sigma_1 < \sigma_2 \). (Cf. [53].)

**Definition 1.1.** Let \( s \in \mathbb{R}_+ \) and \( r, \sigma \in \mathbb{R}_+ \).

1. The set \( \ell'_0(\mathbb{N}^d) \) consists of all formal sequences \( a = \{a(\alpha)\}_{\alpha \in \mathbb{N}^d} \subseteq \mathbb{C} \), and \( \ell_0(\mathbb{N}^d) \) is the set of all \( a \in \ell'_0(\mathbb{N}^d) \) such that \( a(\alpha) \neq 0 \) for at most finite numbers of \( \alpha \in \mathbb{N}^d \);

2. The Banach spaces \( \ell_{\infty,r}(\mathbb{N}^d) \) and \( \ell_{\infty,r}^\ast(\mathbb{N}^d) \) consist of all \( a \in \ell'_0(\mathbb{N}^d) \) such that their corresponding norms

\[
\|a\|_{\ell_{\infty,r}} = \begin{cases} 
\sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{r|\alpha|^{1/2}}, & s \in \mathbb{R}_+, \\
\sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{-r|\alpha|^{1/2}}, & s = \beta_\sigma,
\end{cases}
\]

and

\[
\|a\|_{\ell_{\infty,r}^\ast} = \begin{cases} 
\sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{-r|\alpha|^{1/2}}, & s \in \mathbb{R}_+, \\
\sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{r|\alpha|^{1/2}}, & s = \beta_\sigma,
\end{cases}
\]

respectively, are finite;

3. The space \( \ell_s(\mathbb{N}^d) \) \((\ell_{0,s}(\mathbb{N}^d))\) is the inductive limit (projective limit) of \( \ell_{\infty,r}(\mathbb{N}^d) \) with respect to \( r > 0 \), and \( \ell'_s(\mathbb{N}^d) \) \((\ell'_{0,s}(\mathbb{N}^d))\) is the projective limit (inductive limit) of \( \ell_{\infty,r}^\ast(\mathbb{N}^d) \) with respect to \( r > 0 \).

We also let \( \|\cdot\|_{\ell_{0,N}} \) be the semi-norm

\[
\|a\|_{\ell_{0,N}} = \sup_{|\alpha| \leq N} |a(\alpha)|, \quad a \in \ell'_0(\mathbb{N}^d),
\]

and \( \ell_{0,N}(\mathbb{N}^d) \) be the Banach space with norm (1.3) and consisting of all \( a \in \ell'_0(\mathbb{N}^d) \) such that \( a(\alpha) = 0 \) when \( |\alpha| \geq N \). Then \( \ell_0(\mathbb{N}^d) \) is the inductive limit of \( \ell_{0,N}(\mathbb{N}^d) \) with respect to \( N \), and \( \ell'_0(\mathbb{N}^d) \) is a Fréchet space under the semi-norms (Cf. [53].)

In what follows, \((\cdot, \cdot)_{\mathcal{H}}\) denotes the scalar product in the Hilbert space \( \mathcal{H} \).

**Remark 1.2.** Let \( s \in \overline{\mathbb{R}}_+ \). Then the duals of

\[
\ell_{0,s}(\mathbb{N}^d), \quad \ell_s(\mathbb{N}^d), \quad \ell'_s(\mathbb{N}^d) \quad \text{and} \quad \ell'_{0,s}(\mathbb{N}^d)
\]

are given by

\[
\ell'_0(\mathbb{N}^d), \quad \ell'_s(\mathbb{N}^d), \quad \ell_s(\mathbb{N}^d) \quad \text{and} \quad \ell_{0,s}(\mathbb{N}^d),
\]

respectively, with respect to unique extensions of the form \((\cdot, \cdot)_{\ell_s(\mathbb{N}^d)}\) on \( \ell_0(\mathbb{N}^d) \times \ell_0(\mathbb{N}^d) \). If \( s > 0 \), then \( \ell_0(\mathbb{N}^d) \) is dense in \( \ell'_0(\mathbb{N}^d) \) and the spaces in (1.4). (See e.g. [53].)

**1.3. Pilipović spaces and spaces of power series expansions on \( C^d \).**

We recall that the Hermite function of order \( \alpha \in \mathbb{N}^d \) is defined by

\[
h_{\alpha}(x) = \pi^{-\frac{d}{4}} (-1)^{|\alpha|} (2|\alpha|!)^{-\frac{1}{2}} e^{-\frac{1}{2}|x|^2} (\partial^\alpha e^{-|x|^2}).
\]

It follows that

\[
h_{\alpha}(x) = ((2\pi)^d |\alpha|!)^{-\frac{1}{2}} e^{-\frac{1}{2}|x|^2} p_{\alpha}(x),
\]

where \( p_{\alpha}(x) \) is the Hermite polynomial of order \( \alpha \).
for some polynomial \( p_\alpha \) of order \( \alpha \) on \( \mathbb{R}^d \), called the Hermite polynomial of order \( \alpha \). The Hermite functions are eigenfunctions to the Fourier transform, and to the Harmonic oscillator

\[
H_{x,c} = H_x + c, \quad H_x = |x|^2 - \Delta_x, \quad x \in \mathbb{R}^d,
\]

which acts on functions and (ultra-)distributions defined on \( \mathbb{R}^d \). Here \( c \in \mathbb{C} \) is fixed. More precisely, we have

\[
H_{x,c} h_\alpha = (2|\alpha| + d + c) h_\alpha.
\] (1.6)

More generally, for any \( c \in \mathbb{C} \) and \( \varrho = (\varrho_1, \ldots, \varrho_d) \in \mathbb{C}^d \), we let

\[
H_{x,\varrho,c} \equiv \left( \sum_{j=1}^d \varrho_j (x_j^2 - c_{x_j}) \right) + c = \left( \sum_{j=1}^d \varrho_j H_{x_j} \right) + c, \quad x \in \mathbb{R}^d.
\] (1.7)

Evidently, \( H_{x,\varrho,c} \) is positive definite when \( \varrho \in \mathbb{R}^d \) and \( c > - \sum_{j=1}^d \varrho_j \).

For convenience we put \( H_{x,\varrho,c} = H_{x,\varrho_0,c} \) when \( \varrho = (\varrho_0, \ldots, \varrho_0) \in \mathbb{C}^d \), and observe that

\[
H_{x,\varrho,c} = \varrho H_x + c \quad \text{when} \quad \varrho \in \mathbb{C}.
\]

It is well-known that the set of Hermite functions is a basis for \( \mathcal{F}(\mathbb{R}^d) \) and an orthonormal basis for \( L^2(\mathbb{R}^d) \) (cf. [48]). In particular, if \( f, g \in L^2(\mathbb{R}^d) \), then

\[
\|f\|^2_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} |c_h(f,\alpha)|^2 \quad \text{and} \quad (f, g)_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} c_h(f,\alpha) c_h(g,\alpha),
\]

where

\[
f(x) = \sum_{\alpha \in \mathbb{N}^d} c_h(f,\alpha) h_\alpha(x)
\] (1.8)

is the Hermite series expansion of \( f \), and

\[
c_h(f,\alpha) = (f, h_\alpha)_{L^2(\mathbb{R}^d)}
\] (1.9)

is the Hermite coefficient of \( f \) of order \( \alpha \in \mathbb{R}^d \).

We let \( \mathcal{H}_0(\mathbb{R}^d) \) be the set of all formal Hermite series expansions in (1.8), and \( \mathcal{A}_0(\mathbb{C}^d) \) be the set of all formal power series expansions

\[
F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F,\alpha) e_\alpha(z), \quad e_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}} \quad \alpha \in \mathbb{N}^d,
\] (1.10)

on \( \mathbb{C}^d \). Then the map

\[
T_H : \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \mapsto \sum_{\alpha \in \mathbb{N}^d} c(\alpha) h_\alpha
\] (1.11)

is bijective from \( \ell_0(\mathbb{N}^d) \) to \( \mathcal{H}_0(\mathbb{R}^d) \), and

\[
T_A : \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \mapsto \sum_{\alpha \in \mathbb{N}^d} c(\alpha) e_\alpha
\] (1.12)
is bijective from $\ell'_0(\mathbb{N}^d)$ to $A'_0(\mathbb{C}^d)$. We let the topologies of $H'_0(\mathbb{R}^d)$ and $A'_0(\mathbb{C}^d)$ be inherited from $\ell'_0(\mathbb{N}^d)$ through the mappings $T_H$ and $T_A$, respectively.

**Definition 1.3.** Let $s \in \mathbb{R}$. 

(1) The spaces 
\[ H_{0,s}(\mathbb{R}^d), \quad H_s(\mathbb{R}^d), \quad H'_s(\mathbb{R}^d) \quad \text{and} \quad H'_{0,s}(\mathbb{R}^d), \] 
and their topologies, are the images under the map $T_H$ of the spaces and their topologies in \((1.4)\), respectively.

(2) The spaces 
\[ A_{0,s}(\mathbb{C}^d), \quad A_s(\mathbb{C}^d), \quad A'_s(\mathbb{C}^d) \quad \text{and} \quad A'_{0,s}(\mathbb{C}^d), \] 
and their topologies, are the images under the map $T_A$ of the spaces and their topologies in \((1.4)\), respectively.

The spaces $H_s(\mathbb{R}^d)$ and $H_{0,s}(\mathbb{R}^d)$ in Definition 1.3 are called Pilipović spaces of Roumieu respectively Beurling types of order $s$, and $H'_s(\mathbb{R}^d)$ and $H'_{0,s}(\mathbb{R}^d)$ are called Pilipović distribution spaces of Roumieu respectively Beurling types of order $s$.

There are several characterizations of Pilipović spaces, e.g. in terms of estimates of powers of the harmonic oscillator on the involved functions (see e.g. \([4, 27, 53]\)).

**Remark 1.4.** Let $s_1, s_2 \in \mathbb{R}$. For future references we recall that 
\[ H_{s_1}(\mathbb{R}^d) = S_{s_1}(\mathbb{R}^d), \quad H'_{s_1}(\mathbb{R}^d) = S'_{s_1}(\mathbb{R}^d), \quad s_1 \geq \frac{1}{2}, \] 
\[ H_{0,s_2}(\mathbb{R}^d) = \Sigma_{s_2}(\mathbb{R}^d), \quad H'_{0,s_2}(\mathbb{R}^d) = \Sigma'_{s_2}(\mathbb{R}^d), \quad s_2 > \frac{1}{2}, \] 
while for the other choices of $s_1, s_2$ we have 
\[ H_{s_1}(\mathbb{R}^d) \neq S_{s_1}(\mathbb{R}^d) = \{0\}, \quad s_1 < \frac{1}{2}, \] 
\[ H_{0,s_2}(\mathbb{R}^d) \neq \Sigma_{s_2}(\mathbb{R}^d) = \{0\}, \quad 0 < s_2 \leq \frac{1}{2}, \] 
and that $H_{s_1}(\mathbb{R}^d)$ and $H_{0,s_2}(\mathbb{R}^d)$ in \((1.15)\) and \((1.16)\) are dense in $\mathcal{S}(\mathbb{R}^d)$. (See e.g. \([4, 27, 53]\).)

Hence, any non-trivial Gelfand-Shilov space and its distribution space, agree with corresponding Pilipović space and its distribution space. In particular, Gelfand-Shilov spaces and their distribution spaces can be characterized in similar ways as Pilipović spaces and their distribution spaces in terms of estimates of their coefficients in their Hermite function expansions.

In this context we also recall that the Schwartz space and the set of tempered distributions can be characterized as 
\[ f \in \mathcal{S}(\mathbb{R}^d) \quad \Leftrightarrow \quad |c_h(f, \alpha)| \lesssim \langle \alpha \rangle^{-N} \quad \text{for every } N \geq 0 \] 
\[ (1.17) \] and 
\[ f \in \mathcal{S}'(\mathbb{R}^d) \quad \Leftrightarrow \quad |c_h(f, \alpha)| \lesssim \langle \alpha \rangle^{N} \quad \text{for some } N \geq 0. \] 
\[ (1.18) \]
1.4. **Weight functions.** Next we recall some facts on weight functions. A weight on \( \mathbb{R}^d \) is a positive function \( \omega \in L_{loc}^\infty (\mathbb{R}^d) \) such that \( 1/\omega \in L_{loc}^\infty (\mathbb{R}^d) \). The set of weights on \( \mathbb{R}^d \) is denoted by \( \mathcal{P}_A (\mathbb{R}^d) \). In the sequel we usually assume that \( \omega \) is moderate, or \( v \)-moderate for some positive function \( v \in L_{loc}^\infty (\mathbb{R}^d) \). This means that
\[
\omega (x + y) \leq \omega (x)v(y), \quad x, y \in \mathbb{R}^d.
\]
(1.19)

Here \( A \leq B \) means that \( A \leq cB \) for a suitable constant \( c > 0 \), and we write \( A = B \) when \( A \leq B \) and \( B \leq A \). We note that (1.19) implies that \( \omega \) fulfills the estimates
\[
v (-x)^{-1} \leq \omega (x) \leq v (x), \quad x \in \mathbb{R}^d.
\]
(1.20)

We let \( \mathcal{P}_E (\mathbb{R}^d) \) be the sets of all moderate weights on \( \mathbb{R}^d \).

In several situations we also deal with weights which are radial symmetric in each phase space variable \( (x_j, \xi_j) \). The set of such weights is denoted by \( \mathcal{P}_{A,r} (\mathbb{R}^{2d}) \). That is, \( \mathcal{P}_{A,r} (\mathbb{R}^{2d}) \) consists of all \( \omega \in \mathcal{P}_A (\mathbb{R}^{2d}) \) such that \( \omega (x, \xi) = \omega_0 (\rho) \) for some \( \omega_0 \in \mathcal{P}_A (\mathbb{R}^d) \), where \( \rho_j = x_j^2 + \xi_j^2 \).

It can be proved that if \( \omega \in \mathcal{P}_E (\mathbb{R}^d) \), then \( \omega \) is \( v \)-moderate for some \( v (x) = e^{r|x|} \), provided the positive constant \( r \) is chosen large enough (cf. [33]). In particular, (1.20) shows that for any \( \omega \in \mathcal{P}_E (\mathbb{R}^d) \), there is a constant \( r > 0 \) such that
\[
e^{-r|x|} \leq \omega (x) \leq e^{r|x|}, \quad x \in \mathbb{R}^d.
\]
(1.21)

We also let \( \mathcal{P} (\mathbb{R}^d) \) be the set of all weights \( \omega \) on \( \mathbb{R}^d \) such that \( \omega \) is moderated by \( v (x, \xi) = (1 + |x| + |\xi|)^r \), for some \( r \geq 0 \). Evidently, \( \mathcal{P} (\mathbb{R}^d) \subseteq \mathcal{P}_E (\mathbb{R}^d) \).

We say that \( v \) is submultiplicative if \( v \) is even and (1.19) holds with \( \omega = v \). In the sequel, %v always stand for a submultiplicative weight if nothing else is stated.

1.5. **Modulation spaces and Wiener amalgam spaces.** Before defining modulation spaces we first address some notions on mixed norm spaces of Lebesgue types. Let \( p, q \in (0, \infty] \) and \( r = \min (p, q) \). For any \( f \in L_{loc}^r (\mathbb{R}^{2d}) \), let
\[
\| f \|_{L_{p,q}^r (\mathbb{R}^{2d})} = \| f \|_{L_{p,q}^r (\mathbb{R}^{2d})} = \| g_1 f, \xi \|_{L_p (\mathbb{R}^d)}, \quad \text{where} \quad g_1 f, \xi (\xi) = \| f (\cdot, \xi) \|_{L_p (\mathbb{R}^d)}
\]
and
\[
\| f \|_{L_{p,q}^r (\mathbb{R}^{2d})} = \| f \|_{L_{p,q}^r (\mathbb{R}^{2d})} = \| g_2 f, q \|_{L_p (\mathbb{R}^d)}, \quad \text{where} \quad g_2 f, q (x) = \| f (x, \cdot) \|_{L_q (\mathbb{R}^d)}.
\]

We also let \( L_{p,q}^r (\mathbb{R}^{2d}) \) and \( L_{p,q}^r (\mathbb{R}^{2d}) \) be the quasi-Banach spaces which consist of all \( f \in L_{loc}^r (\mathbb{R}^{2d}) \) such that \( \| f \|_{L_{p,q}^r} \) and \( \| f \|_{L_{p,q}^r} \) are finite, respectively.

Let \( \phi \in \Sigma_1 (\mathbb{R}^d) \setminus \{ 0 \}, p, q \in (0, \infty] \) and \( \omega \in \mathcal{P}_E (\mathbb{R}^{2d}) \). Then the modulation spaces \( M_{p,q}^\phi (\mathbb{R}^d) \) and \( W_{p,q}^\phi (\mathbb{R}^d) \) consist of all \( f \in \Sigma'_1 (\mathbb{R}^d) \) such that \( V_\phi f \cdot \omega \) belongs to \( L_{p,q}^r (\mathbb{R}^{2d}) \) respectively \( L_{p,q}^r (\mathbb{R}^{2d}) \). We equip \( M_{p,q}^\phi (\mathbb{R}^d) \) and \( W_{p,q}^\phi (\mathbb{R}^d) \) with the quasi-norms
\[
f \mapsto \| f \|_{M_{p,q}^\phi} = \| V_\phi f \cdot \omega \|_{L_{p,q}^r} \quad \text{and} \quad f \mapsto \| f \|_{W_{p,q}^\phi} = \| V_\phi f \cdot \omega \|_{L_{p,q}^r}, \quad (1.22)
\]
Remark that \( \omega \) is fixed and equal to the Gaussian \( \phi(x) = e^{-\frac{1}{2}x^2} \). For any \( \omega \in \mathcal{S}(\mathbb{R}^d) \), the modulation spaces \( M^p_{(\omega)}(\mathbb{R}^d) \) and \( W^p_{(\omega)}(\mathbb{R}^d) \) consist of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that corresponding quasi-norms in (1.22) are finite. It is proved in [53] that \( M^p_{(\omega)}(\mathbb{R}^d) \) and \( W^p_{(\omega)}(\mathbb{R}^d) \) are quasi-Banach spaces. If in addition \( p, q \geq 1 \), then these spaces are Banach spaces.

1.6. Spaces of entire functions and the Bargmann transform. Let \( \Omega \subseteq \mathbb{C}^d \) be open. Then \( A(\Omega) \) denotes the set of all analytic functions in \( \Omega \).

Next we recall some properties of the Bargmann transform (cf. [53, 54]). We set

\[
\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j \quad \text{and} \quad \langle z, w \rangle = \langle z, w \rangle,
\]

when \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) and \( w = (w_1, \ldots, w_d) \in \mathbb{C}^d \), and otherwise \( \langle \cdot, \cdot \rangle \) denotes the duality between test function spaces and their corresponding duals. The Bargmann transform \( \mathcal{F}_d f \) of \( f \in L^2(\mathbb{R}^d) \) is
defined by the formula

\[(\mathfrak{V}_d f)(z) = \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2}(\langle z, z \rangle + |y|^2) + 2z^\dagger \langle z, y \rangle \right) f(y) \, dy \]  

(1.23)  

(cf. [5]). We note that if \( f \in L^2(\mathbb{R}^d) \), then the Bargmann transform \( \mathfrak{V}_d f \) of \( f \) is the entire function on \( \mathbb{C}^d \), given by

\[(\mathfrak{V}_d f)(z) = \int_{\mathbb{R}^d} \mathfrak{A}_d(z, y) f(y) \, dy, \]  

or

\[(\mathfrak{V}_d f)(z) = \langle f, \mathfrak{A}_d(z, \cdot) \rangle, \]  

(1.24)  

where the Bargmann kernel \( \mathfrak{A}_d \) is given by

\[\mathfrak{A}_d(z, y) = \pi^{-\frac{d}{2}} \exp \left( -\frac{1}{2}(\langle z, z \rangle + |y|^2) + 2z^\dagger \langle z, y \rangle \right).\]

Evidently, the right-hand side in (1.24) makes sense when \( f \in S^1_{1/2}(\mathbb{R}^d) \) and defines an element in \( A(\mathbb{C}^d) \), since \( y \mapsto \mathfrak{A}_d(z, y) \) can be interpreted as an element in \( S^1_{1/2}(\mathbb{R}^d) \) with values in \( A(\mathbb{C}^d) \).

It was proved in [5] that \( f \mapsto \mathfrak{V}_d f \) is a bijective and isometric map from \( L^2(\mathbb{R}^d) \) to the Hilbert space \( A^2(\mathbb{C}^d) \equiv B^2(\mathbb{C}^d) \cap A(\mathbb{C}^d) \), where \( B^2(\mathbb{C}^d) \) consists of all measurable functions \( F \) on \( \mathbb{C}^d \) such that

\[\| F \|_{B^2} \equiv \left( \int_{\mathbb{C}^d} |F(z)|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty. \]  

(1.25)  

Here \( d\mu(z) = \pi^{-d}e^{-|z|^2} \, d\lambda(z) \), where \( d\lambda(z) \) is the Lebesgue measure on \( \mathbb{C}^d \).

We recall that \( A^2(\mathbb{C}^d) \) and \( B^2(\mathbb{C}^d) \) are Hilbert spaces, where the scalar product are given by

\[(F, G)_{B^2} = \int_{\mathbb{C}^d} F(z)\overline{G(z)} \, d\mu(z), \quad F, G \in B^2(\mathbb{C}^d). \]  

(1.26)  

If \( F, G \in A^2(\mathbb{C}^d) \), then we set \( \| F \|_{A^2} = \| F \|_{B^2} \) and \( (F, G)_{A^2} = (F, G)_{B^2} \).

In [5] it is also proved that

\[\mathfrak{V}_d h_\alpha = e_\alpha, \quad \text{where} \quad e_\alpha(z) \equiv \frac{z^\alpha}{\sqrt{\alpha!}}, \quad z \in \mathbb{C}^d. \]  

(1.27)  

In particular, the Bargmann transform maps the orthonormal basis \( \{h_\alpha\}_{\alpha \in \mathbb{N}^d} \) in \( L^2(\mathbb{R}^d) \) bijectively into the orthonormal basis \( \{e_\alpha\}_{\alpha \in \mathbb{N}^d} \) of monomials in \( A^2(\mathbb{C}^d) \).

For general \( f \in \mathcal{H}_0'(\mathbb{R}^d) \) we now set

\[\mathfrak{V}_d f = (T_A \circ T_H^{-1}) f, \quad f \in \mathcal{H}_0'(\mathbb{R}^d), \]  

(1.28)  

where \( T_H \) and \( T_A \) are given by (1.11) and (1.12). It follows from (1.27) that \( \mathfrak{V}_d f \) in (1.28) agrees with \( \mathfrak{V}_d f \) in (1.23) when \( f \in L^2(\mathbb{R}^d) \), and that this is the only way to extend the Bargmann transform continuously to a continuous map from \( \mathcal{H}_0'(\mathbb{R}^d) \) to \( \mathcal{A}'(\mathbb{C}^d) \). From these observations and definitions, we get the following. The details are left for the reader.

**Proposition 1.7.** Let \( s \in \mathbb{R}_+ \). Then \( \mathfrak{V}_d \) is a homeomorphism from \( \mathcal{H}_0'(\mathbb{R}^d) \) to \( \mathcal{A}'(\mathbb{C}^d) \), and restricts to homeomorphisms from the spaces in (1.13) to the spaces in (1.14), respectively.
It follows that if \( f, g \in L^2(\mathbb{R}^d) \) and \( F, G \in A^2(\mathbb{C}^d) \), then
\[
(f, g)_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha)c_h(g, \alpha),
\]
\[
(F, G)_{A^2(\mathbb{C}^d)} = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha)c(G, \alpha).
\] (1.29)

By the definitions we get the following proposition on duality for Pilipović spaces and their Bargmann images. The details are left for the reader.

**Proposition 1.8.** Let \( s_1 \in \mathbb{R}_+ \) and \( s_2 \in \overline{\mathbb{R}_+} \). Then the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \) on \( \mathcal{H}_0(\mathbb{R}^d) \times \mathcal{H}_0(\mathbb{R}^d) \) is uniquely extendable to sesqui-linear forms on
\[
\mathcal{H}'_{s_2}(\mathbb{R}^d) \times \mathcal{H}_{s_2}(\mathbb{R}^d), \quad \mathcal{H}_{s_2}(\mathbb{R}^d) \times \mathcal{H}'_{s_2}(\mathbb{R}^d),
\]
\[
\mathcal{H}'_{s_1}(\mathbb{R}^d) \times \mathcal{H}_{s_1}(\mathbb{R}^d) \quad \text{and on} \quad \mathcal{H}_{s_1}(\mathbb{R}^d) \times \mathcal{H}'_{s_1}(\mathbb{R}^d).
\]
The (strong) duals of \( \mathcal{H}_{s_2}(\mathbb{R}^d) \) and \( \mathcal{H}_{s_1}(\mathbb{R}^d) \) are equal to \( \mathcal{H}'_{s_2}(\mathbb{R}^d) \) and \( \mathcal{H}'_{s_1}(\mathbb{R}^d) \), respectively, through the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \).

The same holds true if the spaces in (1.13) and the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \) are replaced by corresponding spaces in (1.14) and the form \( (\cdot, \cdot)_{A^2(\mathbb{C}^d)} \), at each occurrence.

If \( s \in \overline{\mathbb{R}} \), \( f \in \mathcal{H}_s(\mathbb{R}^d) \), \( g \in \mathcal{H}'_s(\mathbb{R}^d) \), \( F \in A'_s(\mathbb{C}^d) \) and \( G \in A''_s(\mathbb{C}^d) \), then \( (f, g)_{L^2(\mathbb{R}^d)} \) and \( (F, G)_{A^2(\mathbb{C}^d)} \) are defined by the formula (1.29). It follows that
\[
c_h(f, \alpha) = c(F, \alpha) \quad \text{when} \quad F = \mathfrak{U}_d f, \quad G = \mathfrak{U}_d g.
\] (1.30)
holds for such choices of \( f \) and \( g \).

**Remark 1.9.** In [27, 53], the spaces in (1.14), contained in \( A'_{0,s_1}(\mathbb{C}^d) = A(\mathbb{C}^d) \) are identified as canonical spaces of analytic functions. For example it is here shown that if \( \sigma_1 > 0 \) and \( \sigma_2 > 1 \), then
\[
A'_{s_1}(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d) : |F(z)| \leq e^{r|z|^{2\sigma_1}} \text{ for some } r > 0 \}
\]
and
\[
A''_{s_2}(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d) : |F(z)| \leq e^{r|z|^{2\sigma_2}} \text{ for every } r > 0 \}.\]

**1.7. Fractional Fourier transforms.** We recall that (multiple ordered) fractional Fourier transform \( \mathcal{F}_\varrho \) with respect to \( \varrho = (\varrho_1, \ldots, \varrho_d) \in \mathbb{R}^d \) is the operator with kernel given by \( K_{d,\varrho} \) in (0.4) and (0.5) (see e.g. [20]). Evidently,
\[
\mathcal{F}_\varrho = \mathcal{F}_{\varrho_1} \otimes \cdots \otimes \mathcal{F}_{\varrho_d}, \quad \varrho = (\varrho_1, \ldots, \varrho_d) \in \mathbb{R}^d,
\] (1.31)
and it follows that \( \mathcal{F}_\varrho \) makes sense as homeomorphisms on
\[
\mathcal{H}_{s_1}(\mathbb{R}^d), \quad \mathcal{H}_s(\mathbb{R}^d), \quad \mathcal{F}(\mathbb{R}^d), \quad \mathcal{F}'(\mathbb{R}^d), \quad \mathcal{H}'_s(\mathbb{R}^d) \quad \text{and} \quad \mathcal{H}'_{s_1}(\mathbb{R}^d),
\]
and to a unitary operators on \( L^2(\mathbb{R}^d) \) (see e.g. [34, 53]). For convenience we put \( \mathcal{F}_{00} = \mathcal{F}_\varrho \) when \( \varrho = (\varrho_0, \ldots, \varrho_0) \in \mathbb{R}^d \).
Remark 1.10. Apart from the cases when $q$ in $\mathcal{F}_q$ are (multiple) integers, the formula for the fractional Fourier transform might not look like a visual eye candy, because the kernel $K_{q}(\xi, x)$ is rather complex. However the formula appear naturally by using suitable changes of symplectic coordinates in quantum mechanics. In fact, the symplectic map which rotates $(x, \xi)$ in the phase space $\mathbb{R}^{2d}$ with angle $-\frac{q}{2}$ into $(\xi, -x)$, induces that the observables in quantum mechanics (which are operators) should be conjugated by the Fourier transform $\mathcal{F}$ (see e.g. [15,16,18–20]). It might then be natural to define the fractional Fourier transform $\mathcal{F}_q$ to be the operator which should conjugate the quantum observables, when the phase space is rotated with the angle $-\frac{q}{2}$. That is,

\begin{align*}
\text{Symplectic map} & \quad \text{Conjugation of quantum observables} \\
\text{Rotation with angle } -\frac{q}{2} & \quad \mathcal{F}_1 \\
\text{Rotation with angle } -\frac{q}{2} & \quad \mathcal{F}_q
\end{align*}

This gives a unique definition of $\mathcal{F}_q$, and after some computations, it follows that the kernel of $\mathcal{F}_q$ is given by $K_{q}(\xi, x)$ in (0.4) and (0.5). An equivalent approach and which leads to the same formulae, consist of using the metaplectic representation of symplectic group and considering metaplectic operators. (See e.g. [15, 16, 18–20] for more facts on metaplectic representations and corresponding operators.)

A slightly equivalent way to reach the fractional Fourier transform consists of investigating mapping properties of the Bargmann transform, $\mathfrak{B}_d$. It is proved already in [1] that the Bargmann image of $\mathcal{F} f$ is given by

$$\mathfrak{B}_d(\mathcal{F} f)(z) = \langle \mathfrak{B}_d f \rangle(z) = \langle \mathfrak{B}_d \hat{f} \rangle(-iz) = \langle \mathfrak{B}_d \hat{f} \rangle(e^{-i\frac{3}{2}z}).$$

That is, the Bargmann image of $\mathcal{F} f$ is obtained by retrieving corresponding image of $f$ and then (again) rotating the argument with the angle $-\frac{q}{2}$. In particular, the Fourier transform can be evaluated as

$$\mathcal{F}_1 = \mathfrak{B}_d^{-1} \circ U_1 \circ \mathfrak{B}_d, \quad (U_1 F)(z) = F(e^{-i\frac{3}{2}z}).$$

It might then be natural to define the fractional Fourier transform as

$$\mathcal{F}_q = \mathfrak{B}_d^{-1} \circ U_q \circ \mathfrak{B}_d, \quad (U_q F)(z) = F(e^{-i\frac{q}{2}z}),$$

and a straight-forward computations show that we attain the same formula of the kernel $K_{d,q}$ of $\mathcal{F}_q$ as before.

Due to the previous remark, the Bargmann image of $\mathcal{F}_q$ in (0.4), (0.5) and (1.31) takes the form

$$\langle \mathfrak{B}_d(\mathcal{F}_q f) \rangle(z) = \langle \mathfrak{B}_d f \rangle(e^{-\frac{q}{4}z_1^2 + \cdots + e^{-\frac{q}{4}z_d^2} z_d), \quad f \in \mathcal{H}_0(\mathbb{R}^d). \tag{1.32}$$

Let $\phi(x) = -\frac{d}{4} x e^{-\frac{1}{4}|x|^2}$. Then we recall that the Bargmann transform and the short-time Fourier transform can be linked as

$$V_{\phi}f(x, \xi) = (2\pi)^{-d/2} e^{-\frac{1}{4}|z|^2} e^{-\frac{i}{4}(x, \xi)}(\mathfrak{B}_d f)(2^{-\frac{1}{4}z^2}), \quad z = x + i\xi, \quad x, \xi \in \mathbb{R}^d \tag{1.33}$$
(see (1.28) in [52]). A combination of (1.32) and (1.33) gives that if \( \varrho \in \mathbb{R}^d \),
\[
R_{1,\varrho}(x, \xi) = (\cos \theta_j x_j + (\sin \theta_j) \xi_j),
\]
\[
R_{2,\varrho}(x, \xi) = -(\sin \theta_j) x_j + (\cos \theta_j) \xi_j,
\]
\[
R_{k,\varrho}(x, \xi) = (R_{k,\varrho_1}(x_1, \xi_1), \ldots, R_{k,\varrho_d}(x_d, \xi_d)), \quad k = 1, 2, \quad (1.34)
\]
\[
A_{d,\varrho}(x, \xi) = (R_{1,\varrho}(x, \xi), R_{2,\varrho}(x, \xi)),
\]
\[
U_{d,\varrho}(x, \xi) = R_{1,\varrho}(x, \xi) + iR_{2,\varrho}(x, \xi),
\]
then
\[
(V_0(\mathcal{F}_\varrho f))(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{i}{4} |z|^2} e^{-\frac{i}{2} \langle x, \xi \rangle} (\mathcal{G}_d f)(2^{-\frac{i}{2}} U_{d,\varrho}(z))
\]
\[
= e^{i \frac{1}{2} \varphi_\varrho(x, \xi)} V_0 f(A_{d,\varrho}(x, \xi)), \quad z = x + i\xi \in \mathbb{C}^d.
\]
(1.35)

For convenience we set
\[
A_{d,\varrho_0}(x, \xi) = A_{d,\varrho}(x, \xi) \quad \text{and} \quad U_{d,\varrho_0}(x, \xi) = U_{d,\varrho}(x, \xi)
\]
when
\[
\varrho = (\varrho_0, \ldots, \varrho_0) \in \mathbb{R}^d \quad \text{and} \quad x, \xi \in \mathbb{R}^d.
\]
It then clearly that (1.35) still hold true when \( \mathcal{F}_\varrho \) is the fractional Fourier transform on \( \mathcal{H}_0(\mathbb{R}^d) \) of order \( \varrho \in \mathbb{R} \).

As in [53] we observe that if, more generally, \( \varrho \in \mathbb{C}^d \), then the map
\[
f \mapsto (z \mapsto (\mathcal{G}_d f(e^{-\frac{i}{2} \varphi_\varrho_1 z_1}, \ldots, e^{-\frac{i}{2} \varphi_\varrho_d z_d}))
\]
makes sense as a homeomorphism from \( \mathcal{H}_0(\mathbb{R}^d) \) into \( \mathcal{A}_0'(\mathbb{C}^d) \). In similar ways as in [53], we define the fractional Fourier transform \( \mathcal{F}_\varrho \) by (1.32) when \( \varrho \in \mathbb{C}^d \). Then it follows that still \( \mathcal{F}_\varrho \) is continuous on \( \mathcal{H}_0(\mathbb{R}^d) \).

1.8. Strichartz estimates. We recall that for a linear operator \( R \) acting on suitable functions or (ultra-)distributions on \( \mathbb{R}^d \), Strichartz estimates appears when finding properties on solutions to Cauchy problems like the generalized inhomogeneous Schrödinger equation
\[
\begin{cases}
    i\partial_t u - Ru = F, \\
    u(0, x) = u_0(x), \quad (t, x) \in I \times \mathbb{R}^d.
\end{cases}
\]
(1.36)

(See e.g. [50] and the references therein.) Here \( I = [0, \infty) \) or \( I = [0, T] \) for some \( T > 0 \), \( F \) is a suitable function or (ultra-)distribution on \( I \times \mathbb{R}^d \), \( R \) is a linear operator acting on functions or distributions on \( \mathbb{R}^d \), and \( u_0 \) is a suitable function or (ultra-)distribution on \( \mathbb{R}^d \). The solution of (1.36) is formally given by
\[
u(t, x) = (e^{-itR} u_0)(x) - i \int_0^t (e^{i(t-s)R} F(t, \cdot))(x) \, ds.
\]
(1.37)

In particular it follows that continuity properties of the propagator
\[
(ERf)(t, x) \equiv (e^{-itR} u_0)(x), \quad (t, x) \in I \times \mathbb{R}^d.
\]
(1.38)
as well as for the operator
\[ (S_{1,R}F)(t,x) = \int_0^t (e^{-it(s-t)}R F(s, \cdot))(x) \, ds, \quad (t,x) \in I \times \mathbb{R}^d, \quad (1.39) \]
are essential for finding estimates for solutions to (1.36). We observe that the \( L^2(I \times \mathbb{R}^d) \) adjoint \( E_R^* \) of \( E_R \) is given by
\[ (E_R^*F)(x) = \int_I (e^{isR} F(s, \cdot))(x) \, ds, \quad x \in \mathbb{R}^d, \quad (1.40) \]
and that the composition \( E_R \circ E_R^* \) of \( E_R \) and \( E_R^* \) is the operator \( S_{2,R} \), similar to \( S_{1,R} \), and given by
\[ (S_{2,R}F)(t,x) = \int_I (e^{-it(s-t)}R F(s, \cdot))(x) \, ds, \quad (t,x) \in I \times \mathbb{R}^d. \quad (1.41) \]
We recall that continuity properties of \( E_R \) (or \( E_R^* \)) are strongly linked to continuity properties for \( S_{2,R} \) (see [30]). Estimates on the operator \( E_R^* \) in (1.38) is called \textit{homogeneous Strichartz estimates}, while estimates for \( S_{1,R} \) in (1.39), or even for \( S_{2,R} \) in (1.41), are called \textit{inhomogeneous Strichartz estimates}.

In our situation, the operator \( R \) is given by the operator \( H_{x,\varrho,c} \) for some \( c \in C \) and \( \varrho \in \mathbb{R}^d \) or \( \varrho \in \mathbb{C}^d \), and for such choice of \( R \), we put \( E = E_R \) and \( S_j = S_{j,R}, \ j = 1, 2 \). Hence
\[ (Ef)(t,x) = (e^{-itH_{x,\varrho,c}u_0})(x), \quad (t,x) \in I \times \mathbb{R}^d, \quad (1.38)' \]
\[ (E^*F)(x) = \int_I (e^{isH_{x,\varrho,c}} F(s, \cdot))(x) \, ds, \quad x \in \mathbb{R}^d. \quad (1.40)' \]
\[ (S_1F)(t,x) = \int_0^t (e^{-i(t-s)H_{x,\varrho,c}} F(s, \cdot))(x) \, ds, \quad (t,x) \in I \times \mathbb{R}^d, \quad (1.39)' \]
and
\[ (S_2F)(t,x) = \int_I (e^{-i(t-s)H_{x,\varrho,c}} F(s, \cdot))(x) \, ds, \quad (t,x) \in I \times \mathbb{R}^d. \quad (1.41)' \]

In Section 4 we deduce continuity properties for the operators \( E \) when acting on modulation spaces, and for \( S_1 \) and \( S_2 \) when acting on Lebesgue spaces with values in modulation spaces.

2. Powers of generalized harmonic oscillator propagators on Pilipović spaces

In this section we show that powers of harmonic oscillators, \( H_{x,c} \), or more generally \( H_{x,\varrho,c} \), are continuous on Pilipović spaces. If in addition \( H_{x,\varrho,c} \) is injective, then we show that powers of \( H_{x,\varrho,c} \) are in fact homeomorphisms on Pilipović spaces and their distribution spaces. We also consider harmonic oscillator propagators and deduce homeomorphism properties of such operators on Pilipović spaces.

In the last part we show that powers of harmonic oscillators are continuous on Hilbert modulation spaces of the form \( M^{2,2}_{(\vartheta_r)}(\mathbb{R}^d) \), where \( \vartheta_r(x, \xi) = \langle (x, \xi) \rangle^r \).
2.1. Continuity of powers of $H_{x,c}$ and their propagators. For any $c \in C$, it follows that $H_{x,c}$ is continuous on $\mathcal{H}_0(\mathbb{R}^d)$, and that
\[
H_{x,c}(x) = \sum_{\alpha \in \mathbb{N}^d} (2|\alpha| + d + c) c_h(f, \alpha) h_\alpha(x), \tag{2.1}
\]
when $f \in \mathcal{H}_0(\mathbb{R}^d)$ is given by (1.8). By duality it follows that $H_{x,c}$ on $\mathcal{H}_0(\mathbb{R}^d)$ is uniquely extendable to a continuous map on $\mathcal{H}'(\mathbb{R}^d)$, and that (2.1) still holds true when $f \in \mathcal{H}'(\mathbb{R}^d)$ is given by (1.8).

In the same way it follows that if $r \geq 0$ is real, then
\[
H_{x,c}^r : \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha) h_\alpha \mapsto \sum_{\alpha \in \mathbb{N}^d} (2|\alpha| + d + c)^r c_h(f, \alpha) h_\alpha \tag{2.2}
\]
and
\[
e^{\xi H_{x,c}} : \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha) h_\alpha \mapsto \sum_{\alpha \in \mathbb{N}^d} e^{(2|\alpha|+d+c)^r} c_h(f, \alpha) h_\alpha \tag{2.3}
\]
are continuous on $\mathcal{H}_0(\mathbb{R}^d)$, and uniquely extendable to continuous mappings on $\mathcal{H}'(\mathbb{R}^d)$. More generally, if $r \in \mathbb{R}$, and
\[
c \in C \setminus \{ -2n - d ; n \in \mathbb{N} \} \quad \text{and} \quad r \in \mathbb{R} \tag{2.4}
\]
or
\[
c \in C \quad \text{and} \quad r \in \overline{\mathbb{R}}_+, \tag{2.5}
\]
then the operators (2.2) and (2.3) are continuous on $\mathcal{H}_0(\mathbb{R}^d)$ and on $\mathcal{H}'(\mathbb{R}^d)$. The following result extends these continuity properties to other Pilipović spaces.

**Proposition 2.1.** Let $\zeta \in C$, $r \in \mathbb{R}$ and $c \in C$ be as in (2.4) or as in (2.5), and let $s, s_1, s_2 \in \overline{\mathbb{R}}_0$ be such that $0 < s_1 \leq \frac{1}{2r}$ and $s_2 < \frac{1}{2r}$. Then the following is true:

1. the map (2.2) on $\mathcal{H}_0(\mathbb{R}^d)$ restricts to continuous mappings on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$ and on the spaces in (1.13). If (2.1) holds true, then these mappings are homeomorphisms;

2. the map (2.3) on $\mathcal{H}_0(\mathbb{R}^d)$ restricts to homeomorphisms on $\mathcal{H}_{0,s_1}(\mathbb{R}^d)$, $\mathcal{H}_{s_2}(\mathbb{R}^d)$, $\mathcal{H}'_{s_2}(\mathbb{R}^d)$ and on $\mathcal{H}'_{0,s_1}(\mathbb{R}^d)$. \hspace{1cm} (2.6)

If in addition $\zeta \in i\mathbb{R}$, then the map (2.3) is homeomorphic on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$ and the spaces in (1.13).

**Proof.** The assertion (1) follows from (2.2), the definitions of the spaces in (1.13) and the fact that
\[
\{ c(\alpha) \}_{\mathbb{N}^d} \mapsto \{ (2|\alpha| + d + c)^r c(\alpha) \}_{\mathbb{N}^d} \tag{2.7}
\]
is continuous on the sequence spaces in Definition 1.1 (3). The homeomorphism property in the case $c \neq -d$ follows from the fact that the map (2.7) is then a continuous bijection.

In the same way, (2) follows from (2.3) and the fact that
\[
\{ c(\alpha) \}_{\mathbb{N}^d} \mapsto \{ e^{\zeta(2|\alpha|+d+c)^r} c(\alpha) \}_{\mathbb{N}^d} \tag{2.8}
\]
is a homeomorphism on the sequence spaces which correspond to the spaces in (2.6).

We also have the following negative result concerning continuity for the operator in (2.3).

**Proposition 2.2.** Let \( \zeta \in \mathbb{C} \) be such that Re(\( \zeta \)) > 0, \( r > 0 \), \( c \in \mathbb{C} \), and let \( s, s_1, s_2 \in \mathbb{R} \) be such that \( s_1 > \frac{1}{2r} \) and \( s_2 \geq \frac{1}{2r} \). Then the following is true:

1. the map (2.3) is discontinuous from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \);
2. the map (2.3) is discontinuous from \( \mathcal{H}_{0,s_1}(\mathbb{R}^d) \) to \( \mathcal{H}'_{0,s_1}(\mathbb{R}^d) \);
3. the map (2.3) is discontinuous from \( \mathcal{H}_{s_2}(\mathbb{R}^d) \) to \( \mathcal{H}'_{s_2}(\mathbb{R}^d) \).

**Proof.** We only prove (1) and (2), and then in the case when \( s_1 \in \mathbb{R} \) and \( c = 0 \). The other cases follow by similar arguments and are left for the reader.

Since \( e^{itH_{\zeta}} \) is a homeomorphism on all involved spaces when \( t \) is real, we may assume that \( \zeta > 0 \) is real. Let

\[
f = \sum_{\alpha \in \mathbb{N}^d} e^{-(\log(1+|\alpha|))^2} h_\alpha,
\]

i.e. the Hermite coefficients for \( f \) are given by \( c_h(f, \alpha) = e^{-(\log(1+|\alpha|))^2} \). Since

\[
e^{-(\log(1+|\alpha|))^2} \leq \langle \alpha \rangle^{-N}
\]

for every \( N \geq 1 \), it follows that \( f \in \mathcal{S}(\mathbb{R}^d) \) in view of (1.17).

On the other hand, by (2.3) we get

\[
c_h(e^{iH_{\zeta}}f, \alpha) = e^{\zeta |\alpha| + d^r} e^{-(\log(1+|\alpha|))^2} \geq e^{\zeta |\alpha| + d^r} \geq \langle \alpha \rangle^{N},
\]

for every \( N \geq 1 \). Hence (1.18) shows that \( e^{iH_{\zeta}}f \notin \mathcal{S}'(\mathbb{R}^d) \), while \( f \in \mathcal{S}(\mathbb{R}^d) \). This gives (1).

In order to prove (2), let \( s \in \mathbb{R} \) be such that \( \frac{1}{2r} < s < s_1 \), and let

\[
f = \sum_{\alpha \in \mathbb{N}^d} e^{-(1+|\alpha|)^2} h_\alpha,
\]

i.e. the Hermite coefficients for \( f \) are given by \( c_h(f, \alpha) = e^{-(1+|\alpha|)^2} \). Since

\[
e^{-(1+|\alpha|)^2} \leq e^{-r(1+|\alpha|)^\frac{1}{2r}}
\]

for every \( r > 0 \), it follows that \( f \in \mathcal{H}_{0,s_1}(\mathbb{R}^d) \) in view of the definitions of \( \mathcal{H}_{0,s_1}(\mathbb{R}^d) \).

On the other hand, by (2.3) we get

\[
c_h(e^{iH_{\zeta}}f, \alpha) = e^{\zeta |\alpha| + d^r} e^{-(1+|\alpha|)^2} \geq e^{\zeta |\alpha| + d^r} \geq e^{r(1+|\alpha|)^\frac{1}{2r}},
\]

for every \( r > 0 \). Hence \( e^{iH_{\zeta}}f \notin \mathcal{H}'_{0,s_1}(\mathbb{R}^d) \), due to the definitions, while \( f \in \mathcal{H}_{0,s_1}(\mathbb{R}^d) \). This gives (2), and the result follows. \( \square \)
2.2. Extensions to powers of $H_{x,\varrho,c}$ and their propagators. The previous results can be extended to allow $H_{x,\varrho,c}$ in place of $H_{x,c}$, for more general choices of $\varrho \in \mathbb{C}^d$. We notice that

$$H^r_{x,\varrho,c} : \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha)h_{\alpha} \rightarrow \sum_{\alpha \in \mathbb{N}^d}(2\alpha, \varrho) + \text{sum}(\varrho) + c)^r c_h(f, \alpha)h_{\alpha}$$

(2.2)

and

$$e^{\zeta H^r_{x,\varrho,c}} : \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha)h_{\alpha} \rightarrow \sum_{\alpha \in \mathbb{N}^d} e^{(2\alpha, \varrho) + \text{sum}(\varrho) + c)^r} c_h(f, \alpha)h_{\alpha}$$

(2.3)

when $f \in \mathcal{H}_0^0(\mathbb{R}^d)$, and

$$c \in \mathcal{C}\{ -2(\alpha, \varrho) - \text{sum}(\varrho); \alpha \in \mathbb{N}^d \} \quad \text{and} \quad r \in \mathbb{R}$$

(2.4)

or

$$c \in \mathcal{C} \quad \text{and} \quad r \in \overline{\mathbb{R}}_+,$$

(2.5)

Here and in what follows we let

$$\text{sum}(\varrho) = \sum_{j=1}^d \varrho_j, \quad \text{when} \quad \varrho = (\varrho_1, \ldots, \varrho_d) \in \mathbb{C}^d.$$  

By similar arguments as in the proof of Proposition 2.1 we get the following extension. The details are left for the reader.

**Proposition 2.1** Let $\zeta \in \mathcal{C}, r \in \mathbb{R}, c \in \mathbb{C}^d$ and $c \in \mathcal{C}$ be as in (2.2) or as in (2.3), and let $s, s_1, s_2 \in \overline{\mathbb{R}}_+$ be such that $0 < s_1 < \frac{1}{2r}$ and $s_2 < \frac{1}{2r}$. Then the following is true:

1. the map (2.2) on $\mathcal{H}_0^0(\mathbb{R}^d)$ restricts to continuous mappings on $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$ and on the spaces in (1.13). If in addition (2.3) holds true, then these mappings are homeomorphisms;

2. the map (2.3) on $\mathcal{H}_0^0(\mathbb{R}^d)$ restricts to homeomorphisms on

$$\mathcal{H}_{0,s_1}(\mathbb{R}^d), \quad \mathcal{H}_{s_2}(\mathbb{R}^d), \quad \mathcal{H}'_{s_2}(\mathbb{R}^d) \quad \text{and on} \quad \mathcal{H}_{0,s_2}(\mathbb{R}^d).$$

(2.6)

If in addition $\zeta \varrho_j \in i\mathbb{R}$ for every $j$, then the map (2.3) is homeomorphic on $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$ and the spaces in (1.13).

In the same way, similar arguments as in the proof of Proposition 2.2 gives the following extension. The details are left for the reader.

**Proposition 2.2**. Let $r > 0, \varrho \in \mathbb{C}^d$ be such that $\text{Re}(\zeta \varrho_j^r) > 0$ for every $j = 1, \ldots, d, c \in \mathcal{C}$, and let $s, s_1, s_2 \in \overline{\mathbb{R}}_+$ be such that $s_1 > \frac{1}{2r}$ and $s_2 \geq \frac{1}{2r}$. Then the following is true:

1. the map (2.3) is discontinuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$;

2. the map (2.3) is discontinuous from $\mathcal{H}_{0,s_1}(\mathbb{R}^d)$ to $\mathcal{H}_{0,s_1}'(\mathbb{R}^d)$;

3. the map (2.3) is discontinuous from $\mathcal{H}_{s_2}(\mathbb{R}^d)$ to $\mathcal{H}_{s_2}'(\mathbb{R}^d)$.  

21
2.3. Powers of harmonic oscillator on Hilbert modulations spaces. Let $r \in \mathbb{R}$, $p,q \in (0, \infty]$, $s > 1$, $\omega \in \mathcal{P}_E(s(\mathbb{R}^{2d}))$ and $\vartheta_r(x,\xi) = \langle x,\xi \rangle^r$. Then it follows from [1, Proposition 1.34] that $H^*_r$ is homeomorphic from $M^p_{\omega \vartheta_N}(\mathbb{R}^d)$ to $M^p_{\omega}(\mathbb{R}^d)$ when $N$ is an integer. So far, we are not able to prove any such result when the integer $N$ is replaced by a general real number for such general modulation spaces. On the other hand we have the following for certain Hilbert modulation spaces. See also [11, 21] for similar results. Here we restrict ourself to weights which are rotational invariant with respect to each phase space, or complex variable. That is, we assume that

$$\omega(x,\xi) = \omega_0(\rho), \quad \rho_j = x_j^2 + \xi_j^2,$$

for some positive function $\omega_0$ on $\mathbb{R}_+^d$.

**Theorem 2.3.** Let $r,t \in \mathbb{R}$, $\vartheta_r(x,\xi) = \langle x,\xi \rangle^r$ and suppose that $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ satisfies (2.9) for some positive function $\omega_0$ on $\mathbb{R}_+^d$. Then the following is true:

1. $H^*_r$ on $\mathcal{H}_0(\mathbb{R}^d)$ is uniquely extendable to a homeomorphism from $M^2_{\omega}(\mathbb{R}^d)$ to $M^2_{\omega \vartheta_r}(\mathbb{R}^d)$, with bound of $H^*_r$ which is independent of $r$;

2. $e^{itH^*_r}$ on $\mathcal{H}_0(\mathbb{R}^d)$ is uniquely extendable to a homeomorphism on $M^2_{\omega}(\mathbb{R}^d)$, with bound of $e^{itH^*_r}$ which is independent of $t$ and $r$.

The following lemma shows that $M^2(\omega)(\mathbb{R}^d)$ with weights of the form (2.9) can be expressed as norm estimates of Hermite series expansions of the involved functions and distributions. Here we extract the weight function

$$\nu_\omega : \mathbb{N}^d \to \mathbb{R}_+,$$

from $\omega$ by the formula

$$\nu_\omega(\alpha) = \left( \alpha!^{-1} \int_{\mathbb{R}_+^d} r^\alpha \omega_0(r)^2 e^{-(r_1^2 + \cdots + r_d^2)} \, dr \right)^{\frac{1}{2}}, \quad \alpha \in \mathbb{N}^d. \quad (2.10)$$

**Lemma 2.4.** Let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that (2.9) holds for some positive function $\omega_0$ on $\mathbb{R}_+^d$. Also let $\nu_\omega$ be given by (2.10). Then $M^2(\omega)(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|f\|_\omega = \left( \sum_{\alpha \in \mathbb{N}^d} |c_h(f,\alpha)|^2 \nu_\omega(\alpha) \right)^{\frac{1}{2}} \quad (2.11)$$

is finite. The Hilbert norm $\| \cdot \|_\omega$ is equivalent to $\| \cdot \|_{M^2(\omega)}$.

**Proof.** The result follows by straight-forward applications of [53, Theorem 3.5], and that the Bargmann image of $M^2(\omega)(\mathbb{R}^d)$ is equal to $A^2(\omega)(\mathbb{C}^d)$. The details are left for the reader. $\Box$

**Proof of Theorem 2.3.** Let $\| \cdot \|_\omega$ be the norm given in (2.11). Then (2.2) shows that

$$c_h(H^*_r f,\alpha) = (2|\alpha| + d)^r c(f,\alpha)$$
and
\[ c_h(e^{itH x} f, \alpha) = e^{it(2|\alpha|+d)r} c(f, \alpha). \]

Since \(|e^{it(2|\alpha|+d)r}| = 1\) and that \(\omega\) satisfies (2.9), the assertion follows from Lemma [2.4]. □

Remark 2.5. So far we are not able to extend Theorem 2.3 to more general modulation spaces \(M_p^q(R^d)\), without the assumption (2.9) on the weight \(\omega\). Suppose that \(\omega\) is the same as in Theorem 2.3. Then using standard embeddings for modulation spaces like
\[ M_{\omega\theta_{\tau_1}}(R^d) \hookrightarrow M^2_{\omega}(R^d) \hookrightarrow M_{\omega\theta_{\tau_2}}(R^d), \]

\[ \tau_1 \geq d \max \left( \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{q} \right), \]

\[ \tau_2 \geq d \max \left( \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{q} \right), \]

with strict inequalities if \(p \neq 2\) or \(q \neq 2\), it follows that \(H^{\tau_0} e^{itH x} : M^p_q(\omega)(R^d) \rightarrow M^p_q(\omega/\theta_\theta)(R^d), \theta = \tau_0 + \tau_1 + \tau_2\) (2.13) is continuous, with bounds which are independent of \(r\). In fact, by Theorem 2.3 and (2.12) we have
\[ H^{\tau_0} e^{itH x} M^p_q(\omega)(R^d) \hookrightarrow H^{\tau_0} e^{itH x} M^2_{\omega}(\cap_{\tau_1})(R^d) \hookrightarrow M_{\omega\theta_{\tau_0+\tau_1}}(R^d) \hookrightarrow M_{\omega\theta}(R^d). \]

3. General harmonic oscillator propagators and fractional Fourier transforms

In this section we prove that generalized harmonic oscillator propagators of the forms \(e^{iH x\theta c}\) are essentially fractional Fourier transforms of multiple orders. We use such identities to link some results in \([8]\) with results in \([53]\), especially when such operators act on (weighted) modulation spaces.

3.1. Identifications between fractional Fourier transforms and harmonic oscillator type propagators. Let \(\theta \in C^d\) and \(c \in C\). Then the operator \(H_{\theta,\theta,c}\) is transformed into the operator
\[ 2H_{\theta,\theta,c} = 2H_{\theta,\theta} + \text{sum}(\theta) + c, \] where \(H_{\theta,\theta} = \sum_{j=1}^{d} \theta_j z_j \vec{c}_j.\)

That is,
\[ \mathfrak{N}_d \circ H_{\theta,\theta,c} = (2H_{\theta,\theta} + \text{sum}(\theta) + c) \circ \mathfrak{N}_d, \hspace{1cm} H_{\theta,\theta} = \sum_{j=1}^{d} \theta_j z_j \vec{c}_j, \] (3.1)

or equivalently,
\[ \mathfrak{N}_d H_{\theta,\theta,c} f = (2H_{\theta,\theta} + \text{sum}(\theta) + c) F, \hspace{1cm} F = \mathfrak{N}_d f. \]
For convenience we put

\[ H_{\varrho_0,\varrho} = H_{\varrho,c,\varrho_0} \quad \text{and} \quad H_{\varrho,\varrho} = H_{\varrho,\varrho}, \]

when \( \varrho = (\varrho_0, \ldots, \varrho_0) \in \mathbb{C}^d \), and we put

\[ H_{\varrho} = H_{1,\varrho}. \]

In particular, it follows from (3.1) that \( \frac{1}{2}H_{x,-d} \) is transformed into \( H_{\varrho} \). It also follows from (3.1) that if \( r,\zeta \in \mathbb{C} \), then

\[ \mathfrak{M}_d \circ e^{iH_{x,\varrho}} = e^{(2H_{x,\varrho} + \text{sum}(\varrho) + c)r} \circ \mathfrak{M}_d, \quad (3.2) \]

as continuous operators from \( \mathcal{H}_0'(\mathbb{R}^d) \) to \( \mathcal{A}_0'(\mathbb{C}^d) \).

By straight-forward computations we get

\[ H_{\varrho,\varrho} c_\alpha(z) = \langle \varrho, \alpha \rangle c_\alpha(z) \]

(see e.g. [2]). This implies that

\[ e^{\varrho H_{x,\varrho}} c_\alpha(z) = e^{\langle \varrho, \alpha \rangle} c_\alpha(z) = e_\alpha(e^{\varrho_1 z_1}, \ldots, e^{\varrho_d z_d}), \quad (3.3) \]

for every \( \zeta \in \mathbb{C} \), which gives

\[ (e^{\varrho H_{x,\varrho}} F)(z) = F(e^{\varrho_1 z_1}, \ldots, e^{\varrho_d z_d}), \quad F \in \mathcal{A}_0'(\mathbb{C}^d). \quad (3.4) \]

Remark 3.1. We observe that the map

\[ F(z) \mapsto (e^{\varrho H_{x,\varrho}} F)(z) = F(e^{\varrho_1 z_1}, \ldots, e^{\varrho_d z_d}) \]

is a continuous bijection on the spaces

\[ \mathcal{A}_{s_1}(\mathbb{C}^d), \quad \mathcal{A}_{0,s_2}(\mathbb{C}^d), \quad \mathcal{A}_1'(\mathbb{C}^d), \quad \mathcal{A}_2'(\mathbb{C}^d), \quad (3.5) \]

when \( s_1 < \frac{1}{2} \) and \( s_2 \leq \frac{1}{2} \). See [33] for definition and some characterizations of the spaces in (3.5). This is also a consequence of Proposition 2.1.

We recall that the fractional Fourier transform \( \mathfrak{F}_\varrho \) of (multiple) order \( \varrho \in \mathbb{C}^d \) satisfies

\[ (\mathfrak{M}_d(\mathfrak{F}_\varrho f))(z) = (\mathfrak{M}_d f)(e^{-i\varrho_1 \frac{\pi}{2}} z_1, \ldots, e^{-i\varrho_d \frac{\pi}{2}} z_d), \quad f \in \mathcal{H}_0'(\mathbb{R}^d). \quad (3.6) \]

Hence, a combination of Proposition 2.4, (3.1), (3.2) and (3.4) with

\[ e^{-iH_{x,\varrho} c_1} = e^{(c_1 - c_2)} e^{-iH_{x,\varrho} c_2}, \quad c_1, c_2 \in \mathbb{C}, \]

gives the following extension of results given in [42, p. 161].

**Theorem 3.2.** Let \( \varrho \in \mathbb{C}^d \), \( c \in \mathbb{C} \) and \( s, s_1, s_2 \leq \mathbb{R}_0 \) be such that \( 0 < s_1 \leq \frac{1}{2} \) and \( s_2 < \frac{1}{2} \). Then

\[ e^{-i\frac{\pi}{2} H_{x,\varrho} c} = e^{-i\frac{\pi}{2} (\text{sum}(\varrho) + c)} \mathfrak{F}_\varrho \]

as operators on \( \mathcal{H}_0'(\mathbb{R}^d) \). The operators in (3.7) restrict to homeomorphisms on the spaces in (2.6).

Evidently, (3.7) is the same as

\[ \mathfrak{F}_\varrho = e^{i\frac{\pi}{2} (\text{sum}(\varrho) + c)} e^{-i\frac{\pi}{2} H_{x,\varrho} c}, \quad (3.7) \]

and can be used to transfer properties between fractional Fourier transform and harmonic oscillator propagators.

By combining with Theorem 3.2 we get the following extensions of Propositions 0.2 and 0.3 from the introduction. The details are left for the reader.
Proposition 0.2 Let $I_d = \{1, \ldots, d\}$, $g \in \mathbb{C}^d$ and $s \in \mathbb{R}$. Then the following is true:

1. if $s < \frac{1}{2}$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are homeomorphisms on $\mathcal{H}_s(\mathbb{R}^d)$ and on $\mathcal{H}'_s(\mathbb{R}^d)$;

2. if $\text{Im}(\varrho_j) \leq 0$ for every $j \in I_d$, $\text{Im}(\varrho_{j0}) < 0$ for some $j_0 \in I_d$ and $s > \frac{1}{2}$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are continuous injections but not surjections on $\mathcal{H}_s(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{F}'(\mathbb{R}^d)$ and on $\mathcal{H}'_s(\mathbb{R}^d)$;

3. if $\text{Im}(\varrho_j) = 0$ for every $j \in I_d$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are homeomorphisms on $\mathcal{H}_s(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{F}'(\mathbb{R}^d)$ and on $\mathcal{H}'_s(\mathbb{R}^d)$;

4. if $\text{Im}(\varrho_j) > 0$ for some $j \in I_d$ and $s \geq \frac{1}{2}$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are discontinuous on $\mathcal{H}_s(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{F}'(\mathbb{R}^d)$ and on $\mathcal{H}'_s(\mathbb{R}^d)$.

The same holds true with $s > \frac{1}{2}$, $s < \frac{1}{2}$ and $\mathcal{H}_0, s$ in place of $s \geq \frac{1}{2}$, $s < \frac{1}{2}$ and $\mathcal{H}_s$ at each occurrence.

Proposition 0.3 Let $I_d = \{1, \ldots, d\}$ and $g \in \mathbb{C}^d$. Then the following is true:

1. if $\text{Im}(\varrho_j) < 0$ for every $j \in I_d$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are continuous from $S'_{1/2}(\mathbb{R}^d)$ to $S_{1/2}(\mathbb{R}^d)$, and

\[
\mathcal{F}_g(S'_{1/2}(\mathbb{R}^d)) = e^{-i \frac{\pi}{2} H_{x,g}}(S'_{1/2}(\mathbb{R}^d)) \subseteq S_{1/2}(\mathbb{R}^d);
\]

2. if $\text{Im}(\varrho_j) > 0$ for every $j \in I_d$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are discontinuous from $S_{1/2}(\mathbb{R}^d)$ to $S'_{1/2}(\mathbb{R}^d)$, and

\[
S_{1/2}(\mathbb{R}^d) \subseteq \mathcal{F}_g(S'_{1/2}(\mathbb{R}^d)) = e^{-i \frac{\pi}{2} H_{x,g}}(S_{1/2}(\mathbb{R}^d)) \subseteq \mathcal{H}'_{0,1/2}(\mathbb{R}^d);
\]

3. if $\text{Im}(\varrho_j) > 0$ for some $j \in I_d$, then $\mathcal{F}_g$ and $e^{-i \frac{\pi}{2} H_{x,g}}$ are discontinuous from $S_{1/2}(\mathbb{R}^d)$ to $S'_{1/2}(\mathbb{R}^d)$, and

\[
\mathcal{F}_g f \in \mathcal{H}'_{0,1/2}(\mathbb{R}^d) \setminus S'_{1/2}(\mathbb{R}^d) \quad \text{and} \quad e^{-i \frac{\pi}{2} H_{x,g}} f \in \mathcal{H}'_{0,1/2}(\mathbb{R}^d) \setminus S_{1/2}(\mathbb{R}^d)
\]

for some $f \in S_{1/2}(\mathbb{R}^d)$.

In the following proposition we point out some auxiliary group properties for fractional Fourier transforms of complex orders, which extends similar results in [12] for fractional Fourier transforms of real orders. The result follows by straight-forward applications of [3.6] and the fact that the Bargmann transform is injective. The details are left for the reader.

Proposition 3.3. For any $g \in \mathbb{C}^d$, let $\mathcal{F}_0$ be acting on $\mathcal{H}'_0(\mathbb{R}^d)$. Then $\{\mathcal{F}_g\}_{g \in \mathbb{C}^d}$ is a commutative group under composition, with identity element $\mathcal{F}_0 = \text{Id}_{\mathcal{H}'_0(\mathbb{R}^d)}$, and

\[
\mathcal{F}_{g_1} \circ \mathcal{F}_{g_2} = \mathcal{F}_{g_1 + g_2}, \quad \mathcal{F}_g^{-1} = \mathcal{F}_{-g}, \quad \mathcal{F}_{g + g_0} = \mathcal{F}_g,
\]

\[
\forall g, g_1, g_2 \in \mathbb{C}^d, \; g_0 \in 4\mathbb{Z}^d.
\]
3.2. Continuity for one-parameters fractional Fourier transforms and harmonic oscillator propagators on modulation spaces. As an example we shall next transfer mapping properties of \( \mathcal{F}_\varrho \), \( \varrho \in \mathbb{R} \), when acting on certain classes of modulation spaces into analogous properties for harmonic oscillator propagators.

For fractional Fourier transforms on modulation spaces we recall the following special case of [53] Proposition 7.1. We refer to Subsection 1.3 for notations on weight classes.

Proposition 3.4. Let \( \varrho \in \mathbb{R} \), \( p \in (0, \infty] \) and \( \omega \in \mathcal{P}_{\omega}(\mathbb{R}^{2d}) \). Then \( \mathcal{F}_\varrho \) is an isometric homeomorphism on \( M_{\omega}^{p}(\mathbb{R}^{d}) \).

A combination of Theorem 3.2 and Proposition 3.4 now gives the following. (See \([8\text{, Section 2.6]}\) and \([9\text{, Theorem 1.7]}\) in the case when \( p \geq 1 \) and \( \omega = 1 \). See also \([15]\) in the case when \( p \geq 1 \) and \( \omega \) is moderated by polynomially bounded weights.)

Proposition 3.5. Let \( c \in \mathbb{C} \), \( \varrho \in \mathbb{R} \), \( p \in (0, \infty] \) and \( \omega \in \mathcal{P}_{\omega}(\mathbb{R}^{2d}) \). Then \( e^{-i\frac{c}{\pi}H_{\omega}} \) is a homeomorphism on \( M_{\omega}^{p}(\mathbb{R}^{d}) \).

We also have the following extensions of \([16\text{, Proposition 5.1]}\) and the previous propositions. Here the involved weights are allowed to belong to the general class \( \mathcal{P}_{\omega}(\mathbb{R}^{2d}) \), and are linked as

\[
\omega_{\varrho}(x, \xi) = \omega(A_{d, \varrho}(x, \xi)) \tag{3.9}
\]

(see Subsections 1.3, 1.4 and Remark 1.6).

Theorem 3.6. Let \( \varrho \in \mathbb{R}\setminus\{2\mathbb{Z} + 1\} \), \( c \in \mathbb{C} \), \( \omega, \omega_{\varrho} \in \mathcal{P}_{\omega}(\mathbb{R}^{2d}) \) and \( p, q \in (0, \infty] \) be such that \( q \leq p \) and \( \omega_{\varrho}(x, \xi) \) hold. Then \( \mathcal{F}_\varrho \) and \( e^{-i\frac{c}{\pi}H_{\omega}} \) on \( \mathcal{H}_{\omega}^{p}(\mathbb{R}^{d}) \) restrict to continuous mappings from \( M_{\omega}^{p,q}(\mathbb{R}^{d}) \) to \( M_{\omega_{\varrho}}^{p,q}(\mathbb{R}^{d}) \), and from \( W_{\omega}^{q,p}(\mathbb{R}^{d}) \) to \( W_{\omega_{\varrho}}^{q,p}(\mathbb{R}^{d}) \), and

\[
\| \mathcal{F}_\varrho f \|_{M_{\omega_{\varrho}}^{p,q}} = e^{-\frac{c}{\pi} \text{Im}(c)} \| e^{-i\frac{c}{\pi}H_{\omega}} f \|_{M_{\omega}^{p,q}} \lesssim \left| \sin \left( \frac{\pi c}{2} \right) \right| \left( \frac{1}{\pi} \right)^{\frac{d}{p} - \frac{1}{2}} \| f \|_{M_{\omega}^{p,q}}, \quad f \in M_{\omega}^{p,q}(\mathbb{R}^{d}) \tag{3.10}
\]

and

\[
\| \mathcal{F}_\varrho f \|_{W_{\omega_{\varrho}}^{q,p}} = e^{-\frac{c}{\pi} \text{Im}(c)} \| e^{-i\frac{c}{\pi}H_{\omega}} f \|_{W_{\omega}^{q,p}} \lesssim \left| \sin \left( \frac{\pi c}{2} \right) \right| \left( \frac{1}{\pi} \right)^{\frac{d}{p} - \frac{1}{2}} \| f \|_{W_{\omega}^{q,p}}, \quad f \in W_{\omega}^{q,p}(\mathbb{R}^{d}) \tag{3.11}
\]

Theorem 3.7. Let \( \varrho \in \mathbb{R}\setminus\{2\mathbb{Z} + 1\} \), \( \omega, \omega_{\varrho} \in \mathcal{P}_{\omega}(\mathbb{R}^{2d}) \) and \( p, q \in (0, \infty] \) be such that \( q \leq p \) and \( \omega_{\varrho}(x, \xi) \) hold. Then \( \mathcal{F}_\varrho \) and \( e^{-i\frac{c}{\pi}H_{\omega}} \) on \( \mathcal{H}_{\omega}^{p}(\mathbb{R}^{d}) \) restrict to continuous mappings from \( M_{\omega}^{p,q}(\mathbb{R}^{d}) \) to \( M_{\omega_{\varrho}}^{p,q}(\mathbb{R}^{d}) \), and from \( W_{\omega}^{q,p}(\mathbb{R}^{d}) \) to \( W_{\omega_{\varrho}}^{q,p}(\mathbb{R}^{d}) \), and

\[
\| \mathcal{F}_\varrho f \|_{W_{\omega_{\varrho}}^{q,p}} = e^{-\frac{c}{\pi} \text{Im}(c)} \| e^{-i\frac{c}{\pi}H_{\omega}} f \|_{W_{\omega}^{q,p}} \lesssim \left| \cos \left( \frac{\pi c}{2} \right) \right| \left( \frac{1}{\pi} \right)^{\frac{d}{p} - \frac{1}{2}} \| f \|_{M_{\omega}^{p,q}}, \quad f \in M_{\omega}^{p,q}(\mathbb{R}^{d}) \tag{3.12}
\]
and
\[
\|\mathcal{F}_\theta f\|_{M_{\omega(\ell)}^q} = e^{-\frac{i}{4}\text{Im}(c)}\|e^{-i\frac{\theta}{2}H_{\omega,\theta}} f\|_{M_{\omega(\ell)}^q}
\]
\[
\leq |\cos(\frac{\pi\theta}{2})\|d(\frac{1}{p} - \frac{1}{q})\|f\|_{W_{\omega(\ell)}^p}, \quad f \in W_{\omega(\ell)}^p(\mathbb{R}^d). \quad (3.13)
\]

For the proofs of Theorems 3.3 and 5.7, we need the following version of Minkowski’s inequality.

**Lemma 3.8.** Let \( q \in \mathbb{R}, p, q \in (0, \infty] \) be such that \( q \leq p \), \( A_{d, q} \) be as in Subsection 1.7, and let \( T_\theta \) from \( \Sigma_1(\mathbb{R}^{2d}) \) to \( \Sigma_1(\mathbb{R}^{2d}) \) be given by
\[
(T_\theta f)(x, \xi) = f(A_{d, q}(x, \xi)), \quad f \in L_{\text{loc}}^q(\mathbb{R}^{2d}).
\]

Then the following is true:
1. if \( q \neq 2\mathbb{Z} \), then \( T_\theta \) extends uniquely to a continuous map from \( L_{p,q}^p(\mathbb{R}^{2d}) \) to \( L_{p,q}^q(\mathbb{R}^{2d}) \) and from \( L_{p,q}^q(\mathbb{R}^{2d}) \) to \( L_{p,q}^p(\mathbb{R}^{2d}) \), and
\[
\|T_\theta f\|_{L_{p,q}^p} \leq \|f\|_{L_{p,q}^q}, \quad f \in L_{\text{loc}}^q(\mathbb{R}^{2d}); \quad (3.14)
\]
2. if \( q \neq 2\mathbb{Z} + 1 \), then \( T_\theta \) extends uniquely to a continuous map from \( L_{p,q}^p(\mathbb{R}^{2d}) \) to \( L_{p,q}^q(\mathbb{R}^{2d}) \) and from \( L_{p,q}^q(\mathbb{R}^{2d}) \) to \( L_{p,q}^p(\mathbb{R}^{2d}) \), and
\[
\|T_\theta f\|_{L_{p,q}^q} \leq \|f\|_{L_{p,q}^p}, \quad f \in L_{\text{loc}}^q(\mathbb{R}^{2d}); \quad (3.15)
\]
The same holds true with \( L_{p,q}^p \) and \( L_{p,q}^q \) in place of \( L_{p,q}^q \) and \( L_{p,q}^p \), respectively, at each occurrence.

**Proof.** We only prove (1). The assertion (2) follows by similar arguments and is left for the reader.

First suppose that \( p < \infty \). Then \( q < \infty \). Let \( \theta = \frac{\pi}{2} \), \( f \in \mathcal{S}(\mathbb{R}^d) \), \( p_0 = \frac{p}{q} \geq 1 \), \( h \in \mathcal{S}(\mathbb{R}^d) \) be such that \( \|h\|_{L_{p_0}} \leq 1 \),
\[
f_\theta(x, \xi) = f(A_{d, q}(x, \xi)) \quad \text{and} \quad g_\theta(\xi) = \int_{\mathbb{R}^d} |f_\theta(x, \xi)|^q dx.
\]
Since \( f_0 = f \) and
\[
(y, \eta) = A_{d, q}(x, \xi) \Leftrightarrow (x, \xi) = A_{d, -q}(y, \eta),
\]
we get
\[
|(g_\theta, h)_{L^2}| = \left\|\int_{\mathbb{R}^d} |f_0(A_{d, q}(x, \xi))|^q h(\xi) \, dx d\xi \right\|
\]
\[
= \left\|\int_{\mathbb{R}^d} |f_0(y, \eta)|^q h((\sin \theta) y + (\cos \theta) \eta) \, dx d\xi \right\|
\]
\[
\leq \int_{\mathbb{R}^d} \|f_0(\cdot, \eta)|^q \|_{L_{p_0}} \|h((\sin \theta) \cdot + (\cos \theta) \eta)\|_{L_{p_0}} \, d\xi
\]
\[
= |\sin \theta| \|h\|_{L_{p_0}} \int_{\mathbb{R}^d} \|f_0(\cdot, \eta)|^q \|_{L_{p_0}} \, d\xi \leq |\sin \theta| \|f_0\|_{L^q_{\omega, \theta}}.
\]
By taking the supremum over all possible \( h \) with \( \|h\|_{L^p_0} \leq 1 \) we obtain
\[
\|g_1\|_{L^p_0} \leq |\sin \theta|^{-\frac{d}{p_0}} \|f_0\|_{L^p_0}^{q} = |\sin \theta|^{-\frac{d}{p_0}} \|f\|_{L^p_0}^{q},
\]
which is the same as (3.14) when \( f \in \mathcal{S}(\mathbb{R}^d) \). Since \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( L^{p,q}(\mathbb{R}^d) \) when \( p, q < \infty \), (3.14) follows when \( p < \infty \).

Next suppose that \( p = \infty \). The result is obviously true when \( q = \infty \). Therefore suppose that \( q < \infty \), and let \( f \in L^q_p(\mathbb{R}^d) \) and
\[
g(\xi) = \|f(\cdot, \xi)\|_{L^p}.\]

Then
\[
\int_{\mathbb{R}^d} |f_g(x, \xi)|^q \, dx \leq \int_{\mathbb{R}^d} |g((-\sin \theta)x + (\cos \theta)\xi)|^q \, dx = |\sin \theta|^{-d} \|g\|_{L^q_0}^q.
\]
If we take the supremum over \( \xi \in \mathbb{R}^d \), then we obtain (3.14) for \( p = \infty \), and we have proved (3.14) for any \( p \in (0, \infty) \). This gives (1), and the result follows.

\section*{Proof of Theorems 3.6 and 3.7}

By Proposition 5.1 \[16\] by Cordero and Nicola.

Then
\[
\mathcal{F}(M^{p,q}_{\omega}(\mathbb{R}^d)) = W^{q,p}_{\omega}(\mathbb{R}^d), \quad \text{when } \omega_1(x, \xi) = \omega_2(\xi, -x),
\]
and that \( |\cos(\frac{\pi (q+1)}{2})| = |\sin(\frac{\pi q}{2})| \), it follows that Theorem 3.7 is the Fourier version of Theorem 3.6. Hence it suffices to prove Theorem 3.6. By (3.7) it also follows that it suffices to prove the norm estimates for \( \mathcal{F}_g f \) in (3.10) and (3.11).

Let \( \phi(x) = \pi^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2} \). Then (1.35) gives
\[
F_g(x, \xi) = |(V_\phi(\mathcal{F}_g f))(x, \xi)\omega_g(x, \xi)| = |(V_\phi f)(A_{d,\phi}(x, \xi))\omega(A_{d,\phi}(x, \xi))|.
\]
We also have
\[
F_0(x, \xi) = |(V_\phi f)(x, \xi)\omega(x, \xi)| \quad \text{and} \quad F_g(x, \xi) = F_0(A_{d,\phi}(x, \xi)).
\]

By Lemma 3.8 and the identities above we obtain
\[
\|\mathcal{F}_g f\|_{M_{\omega}^{p,q}} \asymp \|F_g\|_{L^p} \leq |\sin(\frac{\pi q}{2})|^{d(\frac{1}{p} - \frac{1}{q})} \|F_0\|_{L^p} \asymp |\sin(\frac{\pi q}{2})|^{d(\frac{1}{p} - \frac{1}{q})} \|f\|_{M_{\omega}^{p,q}},
\]
and (3.10) follows. In similar ways one obtains (3.11). The details are left for the reader, and the result follows.

\section*{Remark 3.9}

By choosing \( \omega = 1 \) and \( p = q' \geq 2 \), Theorem 3.7 agrees with [16] Proposition 5.1 by Cordero and Nicola.

It is evident that Theorems 3.6 and 3.7 implies the following weighted version of Proposition 0.1 in the introduction.

\section*{Proposition 0.1}

Let \( q \in \mathbb{R}, c \in \mathbb{C}, \omega, \omega_0 \in \mathcal{P}(\mathbb{R}^d) \) and \( p, q \in (0, \infty) \) be such that \( q \leq p \) and (3.9) holds. Then the following is true:

1. \( \mathcal{F}_g = e^{-i\pi \mathcal{H}_{x,\omega,\omega_0}} : M_{\omega}^{p,q}(\mathbb{R}^d) + W^{q,p}_{\omega}(\mathbb{R}^d) \rightarrow M_{\omega_0}^{q,p}(\mathbb{R}^d) + W^{p,q}_{\omega_0}(\mathbb{R}^d) \) is continuous;
(2) if in addition \( q \notin \mathbb{Z} \), then the map
\[
\mathcal{F}_q = e^{-i m H_{x,e} c} : M_{(p,q)}^q(\mathbb{R}^d) + W_{(\omega)}^q(\mathbb{R}^d) \rightarrow M_{(p,q)}^q(\mathbb{R}^d) \cap W_{(\omega)}^q(\mathbb{R}^d)
\]
is continuous.

By choosing \( p = q \) in (1) in previous proposition, we get Propositions 3.4 and 3.5.

3.3. Extensions to multiple ordered fractional Fourier transforms.

By using similar arguments as in the proofs of Theorems 3.6 and 3.7, it follows that the following extensions hold true. Again we refer to Subsections 1.2 and 2.3 for notations. The details are left for the reader.

**Theorem 3.6.** Let \( q \in \mathbb{R}^d \setminus 2\mathbb{Z}^d \), \( c \in \mathbb{C} \), \( \omega, \omega_q \in \partial_A(\mathbb{R}^d) \) and \( p, q \in (0, \infty) \) be such that \( q \leq p \) and (3.9) hold. Then \( \mathcal{F}_q \) and \( e^{-i m H_{x,e} c} \) on \( \mathcal{H}_{\omega}^q(\mathbb{R}^d) \) restrict to continuous mappings from \( M_{(p,q)}^q(\mathbb{R}^d) \) to \( M_{(p,q)}^q(\mathbb{R}^d) \), and from \( W_{(p,q)}^q(\mathbb{R}^d) \) to \( W_{(p,q)}^q(\mathbb{R}^d) \),
\[
\| \mathcal{F}_q f \|_{M_{(p,q)}^q} = e^{-\frac{\pi}{2} \text{Im}(c)} \| e^{-i m H_{x,e} c} f \|_{M_{(p,q)}^q} \leq \left( \prod_{j=1}^{d} \sin \left( \frac{\pi q}{2} \right) \right) \| f \|_{M_{(p,q)}^q}, \quad f \in M_{(p,q)}^q(\mathbb{R}^d) \quad (3.10)
\]
and
\[
\| \mathcal{F}_q f \|_{W_{(p,q)}^q} = e^{-\frac{\pi}{2} \text{Im}(c)} \| e^{-i m H_{x,e} c} f \|_{W_{(p,q)}^q} \leq \left( \prod_{j=1}^{d} \sin \left( \frac{\pi q}{2} \right) \right) \| f \|_{W_{(p,q)}^q}, \quad f \in W_{(p,q)}^q(\mathbb{R}^d). \quad (3.11)
\]

**Theorem 3.7.** Let \( q \in \mathbb{R}^d \setminus (2\mathbb{Z}^d + 1)^d \), \( \omega, \omega_q \in \partial_A(\mathbb{R}^d) \) and \( p, q \in (0, \infty) \) be such that \( q \leq p \) and (3.9) hold. Then \( \mathcal{F}_q \) and \( e^{-i m H_{x,e} c} \) on \( \mathcal{H}_{\omega}^q(\mathbb{R}^d) \) restrict to continuous mappings from \( M_{(p,q)}^q(\mathbb{R}^d) \) to \( M_{(p,q)}^q(\mathbb{R}^d) \), and from \( W_{(p,q)}^q(\mathbb{R}^d) \) to \( W_{(p,q)}^q(\mathbb{R}^d) \),
\[
\| \mathcal{F}_q f \|_{W_{(p,q)}^q} = e^{-\frac{\pi}{2} \text{Im}(c)} \| e^{-i m H_{x,e} c} f \|_{W_{(p,q)}^q} \leq \left( \prod_{j=1}^{d} \cos \left( \frac{\pi q}{2} \right) \right) \| f \|_{M_{(p,q)}^q}, \quad f \in M_{(p,q)}^q(\mathbb{R}^d) \quad (3.12)
\]
and
\[
\| \mathcal{F}_q f \|_{M_{(p,q)}^q} = e^{-\frac{\pi}{2} \text{Im}(c)} \| e^{-i m H_{x,e} c} f \|_{M_{(p,q)}^q} \leq \left( \prod_{j=1}^{d} \cos \left( \frac{\pi q}{2} \right) \right) \| f \|_{W_{(p,q)}^q}, \quad f \in W_{(p,q)}^q(\mathbb{R}^d). \quad (3.13)
\]

4. Applications to Strichartz estimates, and some further continuity properties for certain partial differential equations

In this section we apply results from the previous sections to extend certain Strichartz estimates in [16], with initial data in suitable Wiener amalgam spaces. Thereafter we deduce further continuity properties for a family of equations involving certain Schrödinger equations and heat equations.
4.1. Strichartz estimates for certain Schrödinger equations. We shall deduce Strichartz estimates in the framework of the operator $E$, $S_1$ and $S_2$ in Subsection 1.8 (see (1.38)). If $T > 0$, then it follows by straightforward estimates that $S_1$ and $S_2$ are continuous from $C([0, T]; M^1(\mathbb{R}^d))$ to $L^\infty([0, T]; M^1(\mathbb{R}^d))$.

By Proposition 3.5 it follows that $E$, $S_1$ and $S_2$ are uniquely defined and continuous. In the following result we extend the operator $S_j$ to act between spaces of the form $L^p([0, T]; W^{p,q}(\mathbb{R}^d))$.

**Theorem 4.1.** Let $p, p_0, q \in [1, \infty]$ and $r_0 \in (0, \infty)$ be such that

$$0 < d \left( \frac{1}{q} - \frac{1}{p} \right) < 1, \quad d \left( \frac{1}{q} - \frac{1}{p} \right) \leq 1 + \frac{1}{r_0} - \frac{1}{p_0},$$

(4.1)

with strict inequalities when $q < p$ and $p_0 = 1$, or when $q < p$ and $r_0 = \infty$. Also let $S_1$ and $S_2$ from $C([0, T]; M^1(\mathbb{R}^d))$ to $L^\infty([0, T]; M^1(\mathbb{R}^d))$ be given by (1.39) and (1.31), and let $\omega \in \mathcal{P}_{A,r}(\mathbb{R}^{2d})$. Then $S_1$ and $S_2$ are uniquely extendable to a continuous mappings

$$S_j : L^{p_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d)) \rightarrow L^{r_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d)),$$

(4.2)

$$S_j : L^{p_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d)) \rightarrow L^{r_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d)),$$

(4.3)

$$S_j : L^{p_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d)) \rightarrow L^{r_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d)),$$

(4.4)

and

$$S_j : L^{p_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d)) \rightarrow L^{r_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d)),$$

(4.5)

$j = 1, 2$.

Theorem 4.1 can also be formulated in the following way (cf. Theorem 0.5 in the introduction).

**Theorem 0.5.** Let $p, p_0, q \in [1, \infty]$ and $r_0 \in (0, \infty)$ be such that

$$0 < d \left( \frac{1}{q} - \frac{1}{p} \right) < 1, \quad d \left( \frac{1}{q} - \frac{1}{p} \right) \leq 1 + \frac{1}{r_0} - \frac{1}{p_0},$$

with strict inequalities when $q < p$ and $p_0 = 1$, or when $q < p$ and $r_0 = \infty$. Also let $\omega \in \mathcal{P}_{A,r}(\mathbb{R}^{2d})$. Then $S_1$ and $S_2$ from $C([0, T]; M^1(\mathbb{R}^d))$ to $L^\infty([0, T]; M^1(\mathbb{R}^d))$ are uniquely extendable to continuous mappings

$$S_j : L^{p_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d) + W^{p,q}_{(\omega)}(\mathbb{R}^d)) \rightarrow L^{r_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d) \cap W^{p,q}_{(\omega)}(\mathbb{R}^d)),$$

$j = 1, 2$, and

$$\|S_j F\|_{L^{r_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d) + W^{p,q}_{(\omega)}(\mathbb{R}^d))} + \|S_j F\|_{L^{r_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d))}$$

$$\leq \min \left( \|F\|_{L^{p_0}([0, T]; M^{p,q}_{(\omega)}(\mathbb{R}^d))}, \|F\|_{L^{p_0}([0, T]; W^{p,q}_{(\omega)}(\mathbb{R}^d))} \right), \quad j = 1, 2.$$

(4.6)

We need some preparations for the proof of Theorem 4.1 and start with the following.
Lemma 4.2. Suppose that $p_0, q_0, r_0 \in [1, \infty]$ satisfy
\[
\frac{1}{p_0} + \frac{1}{q_0} \leq 1 + \frac{1}{r_0} \quad \text{and} \quad q_0 > 1, \tag{4.7}
\]
with strict inequality when $q_0 < \infty$ and $p_0 = 1$, or when $q_0 < \infty$ and $r_0 = \infty$. Let $T > 0$ and $\phi_{q_0}(t) = |\sin t|^{-\frac{1}{q_0}}$ or $\phi_{q_0}(t) = |\cos t|^{-\frac{1}{q_0}}$, $t \in R$. Also let $T_1$ and $T_2$ be the mappings from $C[0, T]$ to $C[0, T]$, given by
\[
(T_1h)(t) = \int_0^T \phi_{q_0}(t - s)h(s) \, ds \quad \text{and} \quad (T_2h)(t) = \int_0^t \phi_{q_0}(t - s)h(s) \, ds,
\]
when $0 \leq t \leq T$. Then $T_1$ and $T_2$ are uniquely extendable to continuous mappings from $L^{p_0}[0, T]$ to $L^{q_0}[0, T]$.

Proof. We only prove the assertions for $T_1$. The assertions for $T_2$ follows by similar arguments and is left for the reader.

First suppose that $1 < p_0$ and $r_0 < \infty$, and let $\psi$ be a measurable complex-valued function on $R$ such that
\[
|\psi(t)| \leq |t|^{-\frac{1}{p_0}}.
\]
We observe that if $k$ is an integer, then
\[
|\phi_{q_0}(t)| \leq |t - k\pi|^{-\frac{1}{q_0}} \quad \text{when} \quad |t - k\pi| \leq \frac{\pi}{2},
\]
or
\[
|\phi_{q_0}(t)| \leq |t - (k + \frac{1}{2})\pi|^{-\frac{1}{q_0}} \quad \text{when} \quad |t - (k + \frac{1}{2})\pi| \leq \frac{\pi}{2}.
\]
Since $L^p[0, T]$ decreases with $p$, we may assume that equality is attained in the first inequality in (4.7). Then it follows from Lebesgue’s theorem and Hardy-Littlewood-Sobolev inequality in [35, Theorem 4.5] that $f \mapsto \psi * f$ from $C_0(R)$ to $C(R)$ is uniquely extendable to a continuous map from $L^{p_0}(R)$ to $L^{q_0}(R)$.

We may now divide $T_1$ into a finite sum
\[
T_1 = \sum_{j=1}^N T_{1,N},
\]
where
\[
(T_{1,N}f)(t) = \int_{a_j}^{b_j} f(t - s)\psi_j(s) \, ds,
\]
with $\psi_j$ being measurable functions on $R$ which satisfy
\[
|\psi_j(t)| \leq |t - a_j|^{-\frac{1}{q_0}} \quad \text{or} \quad |\psi_j(t)| \leq |t - b_j|^{-\frac{1}{q_0}},
\]
for some $a_j \leq b_j$, $j = 1, \ldots, N$. Since each $T_{1,N}$ is uniquely extendable to a continuous map from $L^{p_0}(R)$ to $L^{q_0}(R)$, in view of the previous part of the proof, the asserted continuity assertions for $T_1$ follows in the case $1 < p_0$ and $r_0 < \infty$.

Next suppose that $p_0 = 1$. Then $r_0 < q_0$ and $q_0 < \infty$, or $r_0 = q_0 = \infty$. This implies that $\phi_{q_0} \in L^{q_0}[0, T]$. By Minkowski’s inequality we obtain
\[
\|T_1h\|_{L^{q_0}[0,T]} \leq \int_0^T \|\phi_{q_0}(\cdot - s)\|_{L^{q_0}[0,T]} |h(s)| \, ds \leq 2\|\phi_{q_0}\|_{L^{q_0}[0,T]} \|h\|_{L^1[0,T]},
\]
31
and the result follows in the case \( p_0 = 1 \).

Finally suppose that \( r_0 = \infty \) and \( q_0 < \infty \). Then \( p_0' < q_0 \), giving that \( \phi_{q_0} \in L^{p_0'}[0, T] \). Hence, Hölder’s inequality gives

\[
|T_1 h(t)| \leq \| \phi_{q_0}(t - \cdot) \|_{L^{p_0'}[0, T]} \| h \|_{L^{p_0}[0, T]} \leq 2 \| \phi_{q_0} \|_{L^{p_0'}[0, T]} \| h \|_{L^{p_0}[0, T]},
\]

and the result follows.

**Proof of Theorem 4.4** We only prove the continuity for (1.2). The other cases follow by similar arguments and are left for the reader.

Since \( L^{r_0}[0, T] \) is decreasing with \( r_0 \) and that (4.1) is obviously true for some \( r_0 \geq 1 \), we may assume that \( r_0 \geq 1 \). We also observe that (4.1) implies that \( q \leq p \), which makes it possible to apply Theorem 3.6. Let \( q_0 \in \mathbb{R} \cup \{ \infty \} \) be defined by

\[
\frac{1}{q_0} = d \left( \frac{1}{q} - \frac{1}{p} \right)
\]

and let \( \phi_{q_0} \) be the same as in Lemma 4.2. Then \( q_0 \in (1, \infty) \). A combination of Theorem 3.6 and Minkowski’s inequality gives

\[
\|(SF)(t, \cdot)\|_{M^{p,q}_\omega} \leq \int_0^T \| (e^{-i\pi(t-s)}H_{s,c} F(s, \cdot)) \|_{M^{p,q}_\omega} ds \
\leq \int_0^T \| \phi_{q_0}(t - s)\|_{M^{p,q}_\omega} ds.
\]

By applying the \( L^{r_0}[0, T] \) norm on the last inequality and using Lemma 4.2 we get

\[
\| SF \|_{L^{r_0}(0, T):M^{p,q}_\omega} \leq \| SF \|_{L^{p_0}(0, T):M^{p,q}_\omega},
\]

and the result follows.

Next we shall apply Theorems 3.6 and 4.1 to deduce continuity properties for the operator \( E \) in (1.40), and thereby obtain Strichartz estimates for the harmonic oscillator propagator \( e^{-itH_{s,c}} \). The first result is the following and is a straightforward consequence of Theorem 3.6. The details are left for the reader. Here we let \( L^p_w(\Omega) \) be the weak \( L^p(\Omega) \) space when \( p \in (0, \infty) \), which is often denoted by \( L^{p,:}(\Omega) \) or \( L^{p,:}(\Omega) \) in the literature (see e.g. [4]).

**Theorem 4.3.** Let \( c \in C \), \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( p, q, r_0 \in (0, \infty) \) be such that \( q \leq p \) and

\[
\frac{1}{r_0} = d \left( \frac{1}{q} - \frac{1}{p} \right). \tag{4.8}
\]

Then \( E \) in (1.40) is uniquely extendable to a continuous map

\[
E : M^{p,q}_\omega(\mathbb{R}^d) + W^{q,p}_\omega(\mathbb{R}^d) \to L^{r_0}_w(\mathbb{R}^d) \cap L^{r_0}_w(\mathbb{R}^d),
\]

and

\[
\| Ef \|_{L^{r_0}_w(\mathbb{R}^d)} + \| Ef \|_{L^{r_0}_w(\mathbb{R}^d)} \leq \| f \|_{M^{p,q}_\omega + W^{q,p}_\omega},
\]

\[
f \in M^{p,q}_\omega(\mathbb{R}^d) + W^{q,p}_\omega(\mathbb{R}^d). \tag{4.9}
\]
Here recall that if \( B_1 \) and \( B_2 \) are quasi-Banach spaces, then \( B_1 + B_2 \) is the quasi-Banach space

\[
\{ f_1 + f_2 : f_1 \in B_1, f_2 \in B_2 \}
\]
equipped with the quasi-norm

\[
\| f \|_{B_1 + B_2} = \inf_{f = f_1 + f_2} (\| f_1 \|_{B_1} + \| f_2 \|_{B_2}).
\]

**Remark 4.4.** If \( p = q \) in Theorem 4.3 then \( r_0 = \infty \) in (4.8). By Theorems 3.6 and 3.7 it follows that \( E \) in (1.40)' is uniquely extendable to a continuous map

\[
E : M_{(\omega)}^p(\mathbb{R}^d) \to L^\infty(\mathbb{R}; M_{(\omega)}^p(\mathbb{R}^d)),
\]
and

\[
\| Ef \|_{L^\infty(\mathbb{R}; M_{(\omega)}^p)} \lesssim \| f \|_{M_{(\omega)}^p}, \quad f \in M_{(\omega)}^p(\mathbb{R}^d). \quad (4.9)
\]

The next result follows by combining Theorem 4.1 with general techniques in Section 2 in [30] for Strichartz estimates in order to deduce further continuity properties for \( E \) and \( E^* \).

**Theorem 4.5.** Let \( p, p_0 \in [1, \infty] \) be such that

\[
2 \leq p < \frac{2d}{d-1} \quad \text{and} \quad d \left( 1 - \frac{2}{p} \right) = \frac{2}{p_0}.
\]

Then \( E^* \) in (1.40)' is uniquely extendable to continuous mappings

\[
E^* : L^{p_0}([0, T]; M^{p,p'}(\mathbb{R}^d)) \to L^2(\mathbb{R}^d), \quad (4.11)
\]

\[
E^* : L^{p_0}([0, T]; W^{p,p'}(\mathbb{R}^d)) \to L^2(\mathbb{R}^d), \quad (4.12)
\]

and \( E \) in (1.38)' is uniquely extendable to continuous mappings

\[
E : L^2(\mathbb{R}^d) \to L^{p_0}([0, T]; M^{p',p}(\mathbb{R}^d)), \quad (4.13)
\]

\[
E : L^2(\mathbb{R}^d) \to L^{p_0}([0, T]; W^{p,p'}(\mathbb{R}^d)), \quad (4.14)
\]

**Proof.** By choosing \( r_0 = p'_0, q = p' \) and \( \omega = 1 \), it follows that (4.2) and (4.5) takes the forms

\[
S_2 : L^{p_0}([0, T]; M^{p,p'}(\mathbb{R}^d)) \to L^{p_0}([0, T]; M^{p',p}(\mathbb{R}^d)) \quad (4.2)'
\]

and

\[
S_2 : L^{p_0}([0, T]; W^{p,p'}(\mathbb{R}^d)) \to L^{p_0}([0, T]; W^{p',p}(\mathbb{R}^d)) \quad (4.5)'
\]

Since the ranks in (4.2)' and (4.5)' are the duals to their domains, the result follows by straight-forward applications of the equivalences (2.1)–(2.3) in [30]. □

By Theorem 4.5 and the fact that \( S_2 = E \circ E^* \), we get the following (see also [30, Corollary 2.1]).

**Theorem 4.6.** Let \( p_j, p_{0,j} \in [1, \infty] \) be such that

\[
2 \leq p_j < \frac{2d}{d-1} \quad \text{and} \quad d \left( 1 - \frac{2}{p_j} \right) = \frac{2}{p_{0,j}}.
\]
\( j = 1, 2 \). Then \( S_2 \) in (\textbf{4.11})\' is uniquely extendable to continuous mappings

\[
S_2 : L^{p_0,1}(0, T]; M^{p_1, p_1}(\mathbb{R}^d)) \to L^{p_0,2}(0, T]; M^{p_1, p_2}(\mathbb{R}^d)), \tag{4.16}
\]

\[
S_2 : L^{p_0,1}(0, T]; M^{p_1, p_1}(\mathbb{R}^d)) \to L^{p_0,2}(0, T]; M^{p_2, p_2}(\mathbb{R}^d)), \tag{4.17}
\]

\[
S_2 : L^{p_0,1}(0, T]; W^{p_1, p_1}(\mathbb{R}^d)) \to L^{p_0,2}(0, T]; M^{p_1, p_2}(\mathbb{R}^d)), \tag{4.18}
\]

and

\[
S_2 : L^{p_0,1}(0, T]; W^{p_1, p_1}(\mathbb{R}^d)) \to L^{p_0,2}(0, T]; M^{p_2, p_2}(\mathbb{R}^d)). \tag{4.19}
\]

The same holds true with \( S_1 \) in place of \( S \) at each occurrence.

Remark 4.7. We observe that (\textbf{4.19}) is the same as (\textbf{4.5}) in \textbf{16}.

Remark 4.8. We observe that the conditions (\textbf{4.10}) and (\textbf{4.15}) imply that \( p_0 < 2 \) and \( p_{0,j} < 2 \).

4.2. Continuity properties for a family of equations related to Schrödinger and heat equations. Next we consider more general equations, given by (\textbf{4.39}), where \( u_0 \) is a suitable function or ultra-distribution on \( \mathbb{R}^d \), \( F \) is a suitable function or ultra-distribution on \( \mathbb{R}^{d+1} \) and \( R = H^r_{x,q,c} \) for some \( r \geq 0 \), \( q \in \mathbb{C}^d \) and \( c \in \mathbb{C} \). That is, we consider

\[
\begin{cases}
i\partial_t u - H^r_{x,q,c} u = F, \\u(0, x) = u_0(x), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\end{cases}
\]

where \( T > 0 \) is fixed. Here we observe that (\textbf{4.20}) is a partial differential equation when \( r \) is an integer. For general \( r \), (\textbf{4.20}) becomes a pseudo-differential equation.

By (\textbf{4.37}) it follows that the formal solution is given by

\[
u(t, x) = (e^{-itH^r_{x,q,c} u_0})(x) - i \int_0^t (e^{i(t-s)H^r_{x,q,c} F}(t, \, \cdot\,))(x) \, ds. \tag{4.21}\]

Hence, questions on well-posed properties for the equation (\textbf{4.20}) rely completely on continuity properties of the propagator

\[
(E_{x,q,c} f)(t, x) = (e^{-itH^r_{x,q,c} u_0})(x), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{4.22}\]

as well as for the operators

\[
(S_{1,q,r,c} F)(t, x) = \int_0^t (e^{-i(t-s)H^r_{x,q,c} F}(s, \, \cdot\,))(x) \, ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{4.23}\]

and

\[
(S_{2,q,r,c} F)(t, x) = \int_0^T (e^{-i(t-s)H^r_{x,q,c} F}(s, \, \cdot\,))(x) \, ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{4.24}\]

Here we remark that there are different definitions of well-posed problems in the literature. We say that the problem (\textbf{4.20}) is well-posed if the solution \( u = u(t, x) \) depends continuously on the initial data \( u_0 \). If (\textbf{4.20}) fails to be well-posed, then (\textbf{4.20}) is called ill-posed.
By Proposition 4.9 it follows that the following is true, which shows that the operator (4.22) easily become discontinuous in the framework of classical functions and (ultra-)distribution spaces. The details are left for the reader.

**Proposition 4.9.** Let \( r, T > 0, c \in \mathbb{C} \) and \( q \in \mathbb{C}^d \) be such that \( \text{Im}(g_j') > 0 \) for some \( j \in \{1, \ldots, d\} \). Then following is true:

1. \( E_{q,r,c} \) in (4.22) is discontinuous from \( \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \);
2. if in addition \( r \geq 1 \), then \( E_{q,r,c} \) in (4.22) is discontinuous from \( S_{1/2}(\mathbb{R}^d) \) to \( S'_{1/2}(\mathbb{R}^d) \).

On the other hand, by Proposition 2.1, it follows that the mappings (4.22), (4.23) and (4.24) are continuous on suitable Pilipović spaces, which is explained in the following result. The details are left for the reader. Here we let

\[
L^1([0, T]; \mathcal{H}_{0,s}(\mathbb{R}^d)) = \text{proj lim}_{r > 0} L^1([0, T]; \mathcal{H}_{s,r}(\mathbb{R}^d)),
\]

\[
L^1([0, T]; \mathcal{H}_s(\mathbb{R}^d)) = \text{ind lim}_{r > 0} L^1([0, T]; \mathcal{H}_{s,r}(\mathbb{R}^d)),
\]

\[
L^1([0, T]; \mathcal{H}_s'(\mathbb{R}^d)) = \text{proj lim}_{r > 0} L^1([0, T]; \mathcal{H}_{s,r}'(\mathbb{R}^d)),
\]

and

\[
L^1([0, T]; \mathcal{H}_{s,r}'(\mathbb{R}^d)) = \text{ind lim}_{r > 0} L^1([0, T]; \mathcal{H}_{s,r}'(\mathbb{R}^d)),
\]

where \( \mathcal{H}_{s,r}(\mathbb{R}^d) \) and \( \mathcal{H}_{s,r}'(\mathbb{R}^d) \) are the images of \( \ell_{s,r}^\infty(\mathbb{N}^d) \) and \( \ell_{s,r}^{\infty,*}((\mathbb{N}^d) \), respectively under the map \( T_H \) in (1.11), also in topological sense. (See also Definition 1.1)

**Theorem 4.10.** Let \( r, T > 0, c \in \mathbb{C} \), \( q \in \mathbb{C}^d \) and let \( s, s_1, s_2 \in \mathbb{R}_0^+ \) be such that \( 0 < s_1 \leq \frac{1}{r} \) and \( s_2 < \frac{1}{s} \). Then following is true:

1. \( E_{q,r,c} \) in (4.22) is a homeomorphism on the spaces in (2.3);
2. \( S_{q,r,c} \) in (4.23) and (4.24) are homeomorphisms on the spaces

\[
L^1([0, T]; \mathcal{H}_{0,s_1}(\mathbb{R}^d)), \quad L^1([0, T]; \mathcal{H}_{s_2}(\mathbb{R}^d)),
\]

\[
L^1([0, T]; \mathcal{H}_{s_2}'(\mathbb{R}^d)) \quad \text{and} \quad L^1([0, T]; \mathcal{H}_{0,s_1}'(\mathbb{R}^d)).
\]

As consequences of Proposition 4.9 and Theorem 4.10 we get the following, concerning well-posed properties of the equation (4.20).

**Corollary 4.11.** Suppose that \( r \geq 1, c \in \mathbb{C} \) and \( q \in \mathbb{C}^d \) are such that \( \text{Im}(g_j') > 0 \) for some \( j \in \{1, \ldots, d\} \). Then the following is true:

1. the equation (4.20) is ill-posed in the framework of Schwartz functions, Gelfand-Shilov spaces, and their dual spaces;
2. the equation (4.20) is well-posed for the Pilipović spaces and their dual spaces in (2.6).
Remark 4.12. If \( c = 0, r = 1, F = 0 \) and \( \varrho_j = i \) for every \( j \) in Corollary 4.11, then (4.20) takes the form

\[
\begin{aligned}
\partial_t u &= -\Delta_x + |x|^2, \\
u(0, x) &= u_0(x), \\
(t, x) &\in [0, T] \times \mathbb{R}^d.
\end{aligned}
\]

That is we obtain a sort of heat equation, where the potential term \( |x|^2 \) is included. The minus sign in front of the Laplace operator \( \Delta_x \) implies that we are searching for a solution when moving backwards in time.

Corollary 4.11 then shows that it is not meaningful to investigate (4.20) in the framework of Schwartz spaces, Gelfand-Shilov spaces and their distribution spaces. On the other hand, it follows from the same corollary that it always makes sense to investigate such problems in the framework of Pilipović spaces which are not Gelfand-Shilov spaces, and their distributions.

4.3. Some further applications and remarks. A question which appears is whether our results are applicable to problems like

\[
\begin{aligned}
\partial_t u - \text{Op}_w^a(u) &= F, \\
u(0, x) &= u_0(x), \\
(t, x) &\in [0, T] \times \mathbb{R}^d,
\end{aligned}
\]

when \( a \) is a positive definite quadratic form on \( \mathbb{R}^{2d} \). Here \( \text{Op}_w^a(u) \) is the Weyl quantization of \( a \), i.e. the operator on \( \mathcal{S}(\mathbb{R}^d) \), given by

\[
\text{Op}_w^a(u) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(\frac{x+y}{2}, \xi) f(y) e^{i(x-y, \xi)} dyd\xi.
\]

The operator \( \text{Op}_w^a(u) \) is continuous on \( \mathcal{S}(\mathbb{R}^d) \) and on any Pilipović space, which extends uniquely to a continuous map on \( \mathcal{S}'(\mathbb{R}^d) \), and to any Pilipović distribution space. (See e.g. [35, 54].)

By introducing suitable new symplectic coordinates it follows that \( \text{Op}_w^a(u) \) takes the form

\[
\text{Op}_w^a(u) = \sum_{j=1}^{d} \varrho_j (x_j^2 - c^2_{x_j}) + c,
\]

for some \( \varrho_j > 0, j = 1, \ldots, d \), and some real constant \( c \), in these new coordinates. (See Section 18.6 in [55].)

By Proposition 4.9 and Theorem 4.10 it follows that (4.26) is ill-posed in the framework of Schwartz spaces, Gelfand-Shilov spaces and their distribution spaces, but well-posed for other types of Pilipović spaces of functions and distributions. Here we notice that we need to keep staying in these new symplectic coordinates, since Pilipović spaces which are not Gelfand-Shilov spaces, are not invariant under general changes of symplectic coordinates. (See e.g. [53].)

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