Almost periodic solution of a discrete competitive system with delays and feedback controls

Yalong Xue*, Xiangdong Xie, and Qifa Lin

Abstract: A discrete nonlinear almost periodic multispecies competitive system with delays and feedback controls is proposed and investigated. We obtain sufficient conditions to ensure the permanence of the system. Also, we establish a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of the system. In additional, an example together with its numerical simulation are presented to illustrate the feasibility of the main result.

Keywords: Almost periodic solution, Competitive system, Lyapunov functional, Delays

MSC: 34C25, 92D25, 34D20, 34D40

1 Introduction

The importance of species competition in nature is obvious. For example, competition may be territory which is directly related to food resources. The widely used Lotka-Volterra system is considered as a disadvantage and that is the linearity. Ayala et al. [1] presented the following competitive system:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left[ r_1 - x_1(t) - a_1 x_2(t) - c_1 x_2^2(t) \right], \\
\dot{x}_2(t) &= x_2(t) \left[ r_2 - x_2(t) - a_2 x_1(t) - c_2 x_1^2(t) \right].
\end{align*}
\]

Besides, Gopalsamy [2] discussed the continuous version with discrete delays, Tan and Liao [3] established the discrete time version with discrete delays, Xue et al. [4] proposed the discrete time version with infinite delays and single feedback control. Recently, the almost periodic solutions of discrete system with feedback controls has more extensively investigated (see [5–10]). Motivated by above, we study the following system with delays and feedback controls:

\[
\begin{align*}
\dot{x}_i(k+1) &= x_i(k) \exp \left[ r_i(k) - a_i(k)x_i(k) - \sum_{j=1, j \neq i}^{n} b_{ij}(k)x_j(k - \tau_j) - \sum_{j=1, j \neq i}^{n} d_{ij}(k)x_j^2(k - \tau_j) - e_i(k)u_i(k) \right], \\
u_i(k+1) &= \left(1 - f_i(k)\right)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)x_j(k), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

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where \( \{r_i(k)\}, \{a_i(k)\}, \{b_i(k)\}, \{d_i(k)\}, \{e_i(k)\}, \{f_i(k)\} \) and \( \{g_{ij}(k)\} \) are bounded nonnegative almost periodic sequences such that
\[
\begin{align*}
0 < r_i^j \leq r_i(k) \leq r_i^{n_i}, & \quad 0 < a_i^j \leq a_i(n) \leq a_i^{n_i}, & \quad 0 < b_i^j \leq b_i(n) \leq b_i^{n_i} \\
0 < d_i^j \leq d_i(n) \leq d_i^{n_i}, & \quad 0 < e_i^j \leq e_i(n) \leq e_i^{n_i}, & \quad 0 < f_i^j \leq f_i(n) \leq f_i^{n_i} < 1, & \quad 0 < g_{ij}^j \leq g_{ij}(n) \leq g_{ij}^{n_i}.
\end{align*}
\]

For any almost periodic sequence \( \{f(k)\} \) defined on \( Z \), we use the notations \( f^l = \inf_{k \in Z} f(k) \) and \( f^u = \sup_{k \in Z} f(k) \).

We consider the solution of system (2) with the following initial conditions:
\[
\begin{align*}
x_i(\theta) &= \varphi_i(\theta) \geq 0, & \quad x_i(0) = \varphi_i(0) > 0, \\
u_i(\theta) &= \psi_i(\theta) \geq 0, & \quad u_i(0) = \psi_i(0) > 0,
\end{align*}
\]
\( \tau = \max \{ \tau_i : i = 1, 2, \cdots, n \}, \quad \theta \in \{-\tau, -\tau + 1, \cdots, 0\} \).

The main objective of this paper is to investigate the existence of the almost periodic solutions of system (2). The set-up of this paper is as follows. In the coming section, we present some useful definitions and lemmas. In the rest of this paper, we systematically explore the existence of a unique positive almost periodic solution, which is uniformly asymptotically stable. An example together with its numerical simulation are presented to show the feasibility of the main results. This study reveals that the feedback controls, to some extent, will destroy the stability of the system.

## 2 Preliminaries

In this section, first we will mention several foundational definitions and lemmas. Denote \([a, b]_Z = [a, b] \cap Z \) and \( K = [-\tau, +\infty)_Z \), where \( \tau \) is defined as before.

**Definition 1** (see [11]). A sequence \( x : Z \to \mathbb{R}^k \) is called an almost periodic sequence if the \( \varepsilon \)-translation set of \( x \)
\[
E_x(\varepsilon, x) = \{ \tau \in Z : |x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z \}
\] is a relatively dense set in \( Z \) for all \( \varepsilon > 0 \); that is, for any given \( \varepsilon > 0 \), there exists an integer \( l(\varepsilon) > 0 \) such that each discrete interval of length \( l(\varepsilon) \) contains an integer \( \tau = \tau(\varepsilon) \in E_x(\varepsilon, x) \) such that
\[
|x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z.
\]

**Definition 2** (see [11]). Let \( f : Z \times D \to \mathbb{R}^k \), where \( D \) is an open set in \( C = \{ \varphi : [-\tau, 0]_Z \to \mathbb{R}^k \} \). \( f(n, \varphi) \) is said to be almost periodic in \( n \) uniformly for \( \varphi \in D \), if for any \( \varepsilon > 0 \) and any compact set \( S \subset D \), there exists a positive integer \( l = l(\varepsilon, S) \), such that any interval of length \( l = l(\varepsilon, S) \) contains an integer \( \tau \), for which
\[
|f(n + \tau, \varphi) - f(n, \varphi)| < \varepsilon, \forall (n, \varphi) \in Z \times S.
\]

**Definition 3** (see [12]). The hull of \( f \), denoted by \( H(f) \), is defined by
\[
H(f) = \{ g(n, x) : \lim_{k \to \infty} f(n + \tau_k, x) = g(n, x), \text{uniformly on } Z \times S \},
\]
for some sequence \( \tau_k \), where \( S \) is any compact set in \( D \).

**Lemma 4** (see [13]). \( \{x(n)\} \) is an almost periodic sequence if and only if for any integer sequence \( \{k_i\} \), there exists a subsequence \( \{k_i\} \subset \{k'_i\} \) such that \( x(n + k_i) \) converges uniformly on \( n \in Z \) as \( i \to \infty \). Furthermore, the limit sequence is also an almost periodic sequence.

**Lemma 5** (see [14]). Assume that \( r(n) > 0 \), \( \{x(n)\} \) satisfies \( x(n) > 0 \), and
\[
x(n + 1) \leq x(n) \exp \left\{ r(n)(1 - ax(n)) \right\},
\]
for \( n \in [n_1, +\infty) \), where \( a \) is a positive constant. Then
\[
\limsup_{n \to +\infty} x(n) \leq \frac{1}{ar} \exp(r^u - 1). 
\] (10)

**Lemma 6** (see [14]). Assume that \( r(n) > 0 \), \( \{x(n)\} \) satisfies \( x(n) > 0 \), and
\[
x(n + 1) \geq x(n) \exp \left\{ r(n)(1 - ax(n)) \right\},
\] (11)
for \( n \in [n_1, +\infty) \), \( \limsup_{n \to +\infty} x(n) \leq x^* \), and \( x(n_1) > 0 \), where \( a \) and \( x^* \) are positive constants such that \( ax^* > 1 \). Then
\[
\liminf_{n \to +\infty} x(n) \geq \frac{1}{a} \exp \left( r^u(1 - ax^*) \right).
\] (12)

**Lemma 7** (see [15]). Assume that \( A > 0 \) and \( x(0) > 0 \), and further suppose that
\[
x(n + 1) \leq Ax(n) + B(n),
\] (13)
then, for any integer \( k \leq n \),
\[
x(n) \leq A^k x(n - k) + \sum_{i=0}^{k-1} A^i B(n - i - 1).
\] (14)
Specifically, if \( A < 1 \) and \( B(n) \) is bounded above with respect to \( M \), then
\[
\limsup_{n \to +\infty} x(n) \leq \frac{M}{1 - A}.
\] (15)

**Lemma 8** (see [15]). Assume that \( A > 0 \) and \( x(0) > 0 \), and further suppose that
\[
x(n + 1) \geq Ax(n) + B(n),
\] (16)
then, for any integer \( k \leq n \),
\[
x(n) \geq A^k x(n - k) + \sum_{i=0}^{k-1} A^i B(n - i - 1).
\] (17)
Specifically, if \( A < 1 \) and \( B(n) \) is bounded below with respect to \( m \), then
\[
\liminf_{n \to +\infty} x(n) \leq \frac{m}{1 - A}.
\] (18)

**Lemma 9** (see [11]). Suppose that there exists a Lyapunov function \( V(n, \phi, \psi) \) satisfying the following conditions:
(i) \( a(\langle \phi(0) - \psi(0) \rangle) \leq V(n, \phi, \psi) \leq b(\|\phi - \psi\|) \), where \( a, b \in P \) with \( P = \{a : [0, +\infty) \to [0, +\infty), a(0) = 0 \) and \( a(u) \) is continuous, increasing in \( u \).\)
(ii) \( V(n, \phi_1, \psi_1 - V(n, \phi_2, \psi_2) \leq L(\|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\|) \), where \( L > 0 \) is a constant.
(iii) \( \triangle V(n, \phi, \psi) \leq -\alpha V(n, \phi, \psi) \), where \( 0 < \alpha < 1 \) is a constant.

**Remark 10** (see [16]). Condition (iii) of Lemma 9 can be replaced by:
(iii) \( \triangle V(n, \phi, \psi) \leq -\beta V(n, \phi, \psi) \), where \( \beta \in \{c : [0, +\infty) \to [0, +\infty), c \) is continuous \(, c(0) = 0 \) and \( c(s) > 0 \) for \( s > 0 \).

### 3 Permanence

**Theorem 11.** Assume that
\[
\triangle_i = \left( r_i - \sum_{j=1, j \neq i}^n b_{ij}^u M_j - \sum_{j=1, j \neq i}^n a_{ij}^u M_j^2 - c_i^u H_i \right) > 0
\] (19)
hold, then the system (2) is permanent. i.e., there exist positive constants \( m_i, M_i, h_i \) and \( H_i \), such that for any positive solution \( (x_1(k), \cdots, x_n(k), u_1(k), \cdots, u_n(k)) \) of system (2), one has

\[
\begin{align*}
m_i & \leq \liminf_{k \to +\infty} x_i(k) \leq \limsup_{k \to +\infty} x_i(k) \leq M_i, \\
h_i & \leq \liminf_{k \to +\infty} u_i(k) \leq \limsup_{k \to +\infty} u_i(k) \leq H_i, \quad i = 1, 2, \cdots, n.
\end{align*}
\]

(20)

Proof. From system (2),

\[
x_i(k + 1) \leq x_i(k) \exp \left[ r_i(k) \left( 1 - \frac{a_i^u}{M_i} x_i(k) \right) \right].
\]

(21)

By Lemma 5, we know that

\[
\limsup_{k \to +\infty} x_i(k) \leq \frac{1}{a_i^l} \exp(r_i^u - 1) \overset{\text{def}}{=} M_i.
\]

(22)

We can choose a sufficiently small \( \varepsilon \) such that for large enough \( K_1 > 0 \), we have

\[
x_i(k) \leq M_i + \varepsilon, \quad \forall k > K_1.
\]

(23)

For \( k > K_1 \), we have

\[
u_i(k + 1) \leq \left( 1 - f_i^l \right) u_i(k) + \sum_{j=1}^{n} g_{ij}(k)(M_j + \varepsilon).
\]

(24)

As a direction corollary of Lemma 7, one has

\[
\limsup_{k \to +\infty} u_i(k) \leq \frac{1}{f_i^l} \sum_{j=1}^{n} g_{ij}^u(M_j + \varepsilon).
\]

(25)

Setting \( \varepsilon \to 0 \),

\[
\limsup_{k \to +\infty} u_i(k) \leq \frac{1}{f_i^l} \sum_{j=1}^{n} g_{ij}^u M_j \overset{\text{def}}{=} H_i.
\]

(26)

For above \( \varepsilon \), there exists an integer \( K_2 > K_1 \) such that

\[
u_i(k) \leq H_i + \varepsilon, \quad \forall k > K_2.
\]

(27)

From (23), (27) and system (2),

\[
x_i(k + 1) \geq x_i(k) \exp \left[ \triangle_i^e - a_i^e x_i(k) \right], \quad k > K_2 + \tau,
\]

(28)

where \( \triangle_i^e = r_i^l - \sum_{j=1, j \neq i}^{n} b_{ij}^u(M_j + \varepsilon) - \sum_{j=1, j \neq i}^{n} d_{ij}^u(M_j + \varepsilon)^2 - e_i^u(H_i + \varepsilon) > 0 \). Noting the fact that \( \exp(x - 1) > x \), for \( x > 0 \), we have

\[
\frac{a_i^e}{\Delta_i^e} M_j = a_i^e, \quad \frac{\exp(r_i^u - 1)}{r_i^l - \sum_{j=1, j \neq i}^{n} b_{ij}^u(M_j + \varepsilon) - \sum_{j=1, j \neq i}^{n} d_{ij}^u(M_j + \varepsilon)^2 - e_i^u(H_i + \varepsilon)} > 1.
\]

(29)

Then by Lemma 6, one has

\[
\liminf_{k \to +\infty} x_i(k) \geq x_i(k) \frac{\Delta_i^e}{a_i^l} \exp[\Delta_i^e - a_i^e M_j].
\]

(30)

Setting \( \varepsilon \to 0 \),

\[
\liminf_{k \to +\infty} x_i(k) \geq \frac{\Delta_i}{a_i^l} \exp[\Delta_i - a_i^e M_i] \overset{\text{def}}{=} m_i.
\]

(31)

There exists a positive integer \( K_3 > K_2 + \tau \) such that

\[
x_i(k) \geq m_i - \varepsilon, \quad \forall k > K_3.
\]

(32)
From (32) and system (2), we have
\[ u_i(k + 1) \geq \left(1 - f_i^p\right) u_i(k) + \sum_{j=1}^{n} g_{ij}(k)(m_j - \epsilon). \] (33)

By Lemma 8, one has
\[ \liminf_{k \to +\infty} u_i(k) \geq \frac{1}{p} \sum_{j=1}^{n} g_{ij}(m_j - \epsilon). \] (34)

Setting \( \epsilon \to 0 \), it follows that
\[ \liminf_{k \to +\infty} u_i(k) \geq \frac{1}{p} \sum_{j=1}^{n} g_{ij} m_j \overset{\text{def}}{=} h_i. \] (35)

Denoting \( \Omega = \{ \text{every solution of system (2)} \text{ satisfying} \}
\[ m_i \leq x_i(k) \leq M_i, \quad h_i \leq u_i(k) \leq H_i, \quad \forall k \in Z^+ \}. \]

**Theorem 12.** Assume that the condition (19) holds, then \( \Omega \neq \emptyset \).

**Proof.** Since the coefficients are almost periodic sequences, there exists an integer valued sequence \( \{t_p\} \) with \( t_p \to \infty \) as \( p \to \infty \) such that
\[ r_i(k + t_p) \to r_i(k), \quad a_i(k + t_p) \to a_i(k), \quad b_{ij}(k + t_p) \to b_{ij}(k), \]
\[ d_{ij}(k + t_p) \to d_{ij}(k), \quad e_i(k + t_p) \to e_i(k), \quad f_i(k + t_p) \to f_i(k), \]
\[ g_{ij}(k + t_p) \to g_{ij}(k). \] (36)

We can choose a sufficiently small \( \epsilon \). From Theorem 11, there exists a positive integer \( N_0 \) such that
\[ m_i - \epsilon \leq x_i(k) \leq M_i + \epsilon, \quad h_i - \epsilon \leq u_i(k) \leq H_i + \epsilon, \quad k > N_0. \] (37)

Denoting \( x_{ip}(k) = x_i(k + t_p), \ u_{ip}(k) = u_i(k + t_p) \) for \( k > N_0 - t_p \) and \( p = 1, 2, \ldots \). For any positive integer \( q \), it is easy to see that there exists a sequence \( \{x_{ip}(k) : p \geq q\} \) such that the sequence \( \{x_{ip}(k)\} \) has a subsequence, also denoted by \( \{x_{ip}(k)\} \), converging on any finite interval \( A \) of \( Z^+ \) as \( p \to \infty \).

In fact, for any finite subset \( A = \{l_1, l_2, \cdots, l_m\} \subseteq Z^+ \), where \( m \) is a finite number, \( t_p + l_j > N_0 (j = 1, 2, \cdots, m) \), when \( p \) is large enough. Therefore \( m_i - \epsilon \leq x_i(k + t_p) \leq M_i + \epsilon (i = 1, 2, \cdots, n) \); that is, \( x_i(k + t_p) \) are uniformly bounded when \( p \) is sufficiently large. Next, for \( l_j \in A \), we choose a subsequence \( \{t_{ip}^{(1)}\} \) of \( \{t_p\} \) such that \( x_i(k + t_{ip}^{(1)}) \) uniformly converge on \( Z^+ \) for \( p \) sufficiently large. Similar to the arguments of \( l_1 \), for \( l_2 \in A \), one can select a subsequence \( \{t_{ip}^{(2)}\} \) of \( \{t_{ip}^{(1)}\} \) such that \( x_i(k + t_{ip}^{(2)}) \) uniformly converge on \( Z^+ \) for \( p \) sufficiently large. Repeating above-mentioned process, for \( l_m \in A \), one obtains a subsequence \( \{t_{ip}^{(m)}\} \) of \( \{t_{ip}^{(m-1)}\} \) such that \( x_i(k + t_{ip}^{(m)}) \) uniformly converge on \( Z^+ \) for \( p \) sufficiently large.

Based on the above, one selects the sequence \( \{t_{ip}^{(m)}\} \) which is a subsequence of \( \{t_p\} \), still denoted by \( \{t_p\} \), then one gets \( x_i(k + t_p) \to x_i^* \) uniformly in \( k \in A \) as \( k \to \infty \). So the conclusion holds truly due to the arbitrariness of \( A \). Thus we have a sequence \( \{y_i(k)\} \) such that for \( k \in Z^+ \),
\[ x_{ip}(k) \to y_i(k), \quad u_{ip}(k) \to v_i(k), \quad \text{as} \quad p \to \infty. \] (38)

which, together with (36), yields that
\[
\begin{cases}
  x_{ip}(k + 1) = x_{ip}(k) \exp \left[ r_i(k + t_p) - a_i(k + t_p) x_{ip}(k) - \sum_{j=1, j \neq i}^{n} b_{ij}(k + t_p) x_{ip}(k - \tau_j) \right. \\
  \quad - \sum_{j=1, j \neq i}^{n} d_{ij}(k + t_p) x_{ip}^2(k - \tau_j) - e_i(k + t_p) u_{ip}(k) \bigg], \\
  u_{ip}(k + 1) = \left(1 - f_i(k + t_p)\right) u_{ip}(k) + \sum_{j=1}^{n} g_{ij}(k + t_p) x_{ip}(k),
\end{cases}
\] (39)
It follows from (36), (38) and (39) that
\[
\begin{align*}
  y_i(k + 1) &= y_i(k) \exp \left[ r_i(k) - a_i(k) y_i(k) - \sum_{j=1,j\neq i}^{n} b_{ij}(k) y_j(k - \tau_j) - \sum_{j=1}^{n} d_{ij}(k) y_j^2(k - \tau_j) - e_i(k) y_i(k) \right], \\
  v_i(k + 1) &= (1 - f_i(k)) v_i(k) + \sum_{j=1}^{n} g_{ij}(k) y_j(k),
\end{align*}
\]
(40)

It is easy to see that \((y_1(k), \ldots, y_n(k), v_1(k), \ldots, v_n(k))\) is a solution of system (2) and \(m_i - \varepsilon \leq y_i(k) \leq M_i + \varepsilon, \quad h_i - \varepsilon \leq v_i(k) \leq H_i + \varepsilon\) for \(k \in [-\tau, 0)\).

### 4 Stability of almost periodic solution

**Theorem 13.** Assume that the condition (19) and
\[
\begin{align*}
  \lambda_i^{(1)} &= 1 - \max(1 - a_i^n M_i, 1 - a_i\|M_i\| - \sum_{j=1,j\neq i}^{n} (b_{ij}^n M_i + 2d_{ij}^n M_i^2) - \sum_{j=1}^{n} g_{ij}^n M_i > 0, \\
  \lambda_i^{(2)} &= f_i^j - e_i^j,
\end{align*}
\]
(41)

hold, where \(i, j = 1, 2, \ldots, n\). Then there exists a unique uniformly asymptotically stable almost periodic solution \((x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_n(k))\) of system (2) which is bounded by \(\Omega\) for all \(k \in \mathbb{N}^+\).

**Proof.** Let \(\omega_i(k) = \ln x_i(k), \quad i = 1, 2, \ldots, n\), then system (2) can be rewritten as
\[
\begin{align*}
  \omega_i(k + 1) &= \omega_i(k) + r_i(k) - a_i(k)e^{\omega_i(k)} - e_i(k)u_i(k) - \sum_{j=1,j\neq i}^{n} b_{ij}(k)e^{\omega_j(k - \tau_j)} - \sum_{j=1}^{n} d_{ij}(k)e^{2\omega_j(k - \tau_j)}, \\
  u_i(k + 1) &= (1 - f_i(k))u_i(k) + \sum_{j=1}^{n} g_{ij}(k)e^{\omega_j(k)}. 
\end{align*}
\]
(42)

From Theorem 12, there exists a solution \((\omega_1(k), \ldots, \omega_n(k), u_1(k), \ldots, u_n(k))\) of system (42) such that
\[
\ln m_i \leq \omega_i(k) \leq \ln M_i, \quad h_i \leq u_i(k) \leq H_i, \quad \forall k \in K, 
\]
(43)

which implies that \(|\omega_i(k)| \leq B_i = \max(\|\ln m_i\|, \|\ln M_i\|)\) and \(|u_i(k)| \leq C_i = \max(h_i, H_i), \quad i = 1, 2, \ldots, n\). For \(k \in \mathbb{Z}^+\) and \(s \in [-\tau, 0)\), assign
\[
W_k(s) = (\omega_1(k + s), \ldots, \omega_n(k + s), u_1(k), \ldots, u_n(k)), \\
Z_k(s) = (z_1(k + s), \ldots, z_n(k + s), v_1(k), \ldots, v_n(k)).
\]
(44)

are two solutions of system (42) defined on \(D\),
\[
D = \{(\omega_1(k), \ldots, \omega_n(k), u_1(k), u_n(k)) : \\
m_i \leq \omega_i(k) \leq \ln M_i, \quad h_i \leq u_i(k) \leq H_i, \quad i = 1, 2, \ldots, n, \quad \forall k \in K\}.
\]
(45)

Defining
\[
\|W_k(s)\| = \sup_{s \in [-\tau, 0)} \sum_{i=1}^{n} |\omega_i(k + s)| + |u_i(k)|,
\]
(46)
then \( ||W_k(s)|| < E = \sum_{i=1}^{n} [B_i + C_i] \).

Consider the product system of (42)

\[
\begin{align*}
\omega_i(k+1) &= \omega_i(k) + r_i(k) - a_i(k)e_{\omega_i(k)} - e_i(k)u_i(k) - \sum_{j=1,j\neq i}^{n} b_{ij}(k)e_{\omega_j(k-r_i)} - \sum_{j=1,j\neq i}^{n} d_{ij}(k)e_{2\omega_j(k-r_j)}, \\
u_i(k+1) &= (1 - f_i(k))u_i(k) + \sum_{j=1}^{n} g_{ij}(k)e_{\omega_j(k)}, \\
z_i(k+1) &= z_i(k) + r_i(k) - a_i(k)e_{\omega_i(k)} - e_i(k)\nu_i(k) - \sum_{j=1,j\neq i}^{n} b_{ij}(k)e_{\omega_j(k-r_i)} - \sum_{j=1,j\neq i}^{n} d_{ij}(k)e_{2\omega_j(k-r_j)}, \\
v_i(k+1) &= (1 - f_i(k))\nu_i(k) + \sum_{j=1}^{n} g_{ij}(k)e_{\omega_j(k)}. \\
\end{align*}
\]

(47)

Construct the Lyapunov functional \( V(k) \),

\[
V(k) = V(k, W_k, Z_k) = \sum_{i=1}^{n} (|\omega_i(k) - z_i(k)| + |u_i(k) - \nu_i(k)|) + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{m=k-r_i}^{k-1} (b_{ij}^M M_j + 2d_{ij}^{M^2})|\omega_j(m) - z_j(m)|.
\]

(48)

Apparently,

\[
|W_k(0) - Z_k(0)| = \left( \sum_{i=1}^{n} (\omega_i(k) - z_i(k))^2 + (u_i(k) - \nu_i(k))^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{n} (|\omega_i(k) - z_i(k)| + |u_i(k) - \nu_i(k)|) \right) \leq V(k).
\]

(49)

Denoting

\[
\delta_i = \max \left\{ 1, \sum_{j=1,j\neq i}^{n} (b_{ij}^M M_j + 2d_{ij}^{M^2}) \right\}, \quad \delta = \max \{\delta_i\}, \quad \rho = \delta (\tau + 1).
\]

(50)

On the other side,

\[
\begin{align*}
V(k) &\leq \sum_{i=1}^{n} (|\omega_i(k) - z_i(k)| + |u_i(k) - \nu_i(k)|) + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{m=k-r_i}^{k-1} (b_{ij}^M M_j + 2d_{ij}^{M^2})|\omega_j(m) - z_j(m)| \\
&= \sum_{i=1}^{n} (|\omega_i(k) - z_i(k)| + |u_i(k) - \nu_i(k)|) + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{m=k-r_i}^{k-1} (b_{ij}^M M_j + 2d_{ij}^{M^2})|\omega_j(m) - z_j(m)| \\
&= \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{s=k-r_i}^{k-1} (b_{ij}^M M_j + 2d_{ij}^{M^2})|\omega_j(k + s) - z_j(k + s)| \\
&\leq \sum_{i=1}^{n} \delta_i \sum_{s=r}^{0} |\omega_i(k + s) - z_i(k + s)| + \sum_{i=1}^{n} |u_i(k) - \nu_i(k)| \\
&\leq \delta \sum_{s=r}^{0} \sum_{i=1}^{n} |\omega_i(k + s) - z_i(k + s)| + \sum_{i=1}^{n} |u_i(k) - \nu_i(k)| \\
&\leq \delta (\tau + 1) \sup_{s \in [-r, 0]} \sum_{i=1}^{n} (|\omega_i(k + s) - z_i(k + s)| + |u_i(k) - \nu_i(k)|) \\
&= \rho \|W_k - Z_k\|.
\end{align*}
\]

(51)
Thus condition (i) in Lemma 9 is satisfied if we take \( a(x) = x \) and \( b(x) = px \), where \( a, b \in C(R^*, R^*) \).

By using (3.2) in Ref. [16], for any \( W_k, Z_k, W_k, \bar{Z}_k \in D \), we have
\[
|V(k, W_k, Z_k) - V(k, W_k, \bar{Z}_k)|
\leq \sum_{i=1}^{n} \left\{ \left| \omega_i(k) - z_i(k) \right| - \left| \bar{\omega}_i(k) - \bar{z}_i(k) \right| + \left| u_i(k) - v_i(k) \right| - \left| \bar{u}_i(k) - \bar{v}_i(k) \right| \right\} \\
+ \sum_{i=1,i \neq j}^{n} \sum_{m=k-\tau_i}^{k-1} \left( b_{ij}^n M_j + 2d_{ij}^n M_j^2 \right) \left| \omega_i(m) - z_i(m) \right| - \left| \bar{\omega}_i(m) - \bar{z}_i(m) \right| \right\}
\leq \sum_{i=1}^{n} \left\{ \left( \omega_i(k) - \bar{\omega}_i(k) \right) + \left| z_i(k) - \bar{z}_i(k) \right| \right\} + \left( \left| u_i(k) - \bar{u}_i(k) \right| + \left| v_i(k) - \bar{v}_i(k) \right| \right) \\
+ \sum_{i=1,i \neq j}^{n} \sum_{m=k-\tau_i}^{k-1} \left( b_{ij}^n M_j + 2d_{ij}^n M_j^2 \right) \left( \omega_i(m) - \bar{\omega}_i(m) \right) + \left| z_i(m) - \bar{z}_i(m) \right| \right\}
\leq \rho \left( \| W_k - \bar{W}_k \| + \| Z_k - \bar{Z}_k \| \right),
\] (52)

so, condition (ii) in Lemma 9 is also satisfied.

By the mean value theorem, it derives that
\[
e^{\omega_i(k)} - e^{z_i(k)} = e^{\theta_i(k)}(\omega_i(k) - z_i(k)),
\]
\[
e^{\omega_i(k-\tau_i)} - e^{z_i(k-\tau_i)} = e^{\eta_i(k-\tau_i)}(\omega_i(k-\tau_i) - z_i(k-\tau_i)),
\]
\[
e^{2\omega_i(k-\tau_i)} - e^{2z_i(k-\tau_i)} = 2e^{2\xi_i(k-\tau_i)}(\omega_i(k-\tau_i) - z_i(k-\tau_i)),
\]
where \( \theta_i(k) \) lie between \( \omega_i(k) \) and \( z_i(k) \), and \( \eta_i(k-\tau_i), \xi_i(k-\tau_i) \) all lie between \( \omega_i(k-\tau_i) \) and \( z_i(k-\tau_i) \), respectively. Then
\[
\ln m_i \leq \omega_i(k), \eta_i(k-\tau_i), \xi_i(k-\tau_i) \leq M_i, \quad k \in Z^*.
\] (53)

Calculating the \( \Delta V(k) \) along with the solution of system (47), we have
\[
\Delta V_{(47)}(k) = \sum_{i=1}^{n} \left\{ \left| \omega_i(k) - z_i(k) - a_i(k) e^{\theta_i(k)}(\omega_i(k) - z_i(k)) - e_i(k)(u_i(k) - v_i(k)) \right| \right.
\]
\[
- \sum_{j=1,j \neq i}^{n} \left( b_{ij}^n e^{\theta_i(k-\tau_j)} + 2d_{ij}^n e^{\xi_i(k-\tau_j)} \right) (\omega_i(k-\tau_j) - z_i(k-\tau_j)) \right\}
\]
\[
+ \left| 1 - f_i(k)(u_i(k) - v_i(k)) + \sum_{j=1}^{n} g_{ij}(k) e^{\theta_j(k)}(\omega_j(k) - z_j(k)) \right| \right\} - \sum_{i=1}^{n} \left\{ \left| \omega_i(k) - z_i(k) \right| + \left| u_i(k) - v_i(k) \right| \right\}
\]
\[
+ \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{n} \left( b_{ij}^n M_j + 2d_{ij}^n M_j^2 \right) \left( \omega_i(k) - z_i(k) - |\omega_j(k-\tau_j) - z_j(k-\tau_j)| \right) \right\}
\]
\[
\leq \sum_{i=1}^{n} \left\{ \left| 1 - a_i(k) e^{\theta_i(k)} \right| - 1 + \sum_{j=1,j \neq i}^{n} \left( b_{ij}^n M_j + 2d_{ij}^n M_j^2 \right) + \sum_{j=1}^{n} g_{ij}(k) e^{\theta_j(k)} \right\} |\omega_i(k) - z_i(k)|
\]
\[
+ \sum_{i=1}^{n} \left( e_i(k) - f_i(k) \right) |u_i(k) - v_i(k)|
\]
\[
\leq - \sum_{i=1}^{n} \left\{ \lambda_i^{(1)} |\omega_i(k) - z_i(k)| + \lambda_i^{(2)} |u_i(k) - v_i(k)| \right\}
\]
\[
\leq -A \sum_{i=1}^{n} \left\{ |\omega_i(k) - z_i(k)| + |u_i(k) - v_i(k)| \right\}
\]
\[
\leq -A \sum_{i=1}^{n} \left( (\omega_i(k) - z_i(k))^2 + (u_i(k) - v_i(k))^2 \right)^{\frac{1}{2}}
\]
\[
\leq -A \sum_{i=1}^{n} \left( |W_k(0) - Z_k(0)| \right),
\] (54)

where $\lambda = \min \{ \min_{i \in \mathbb{N}} \{ \lambda_i^{(1)}, \lambda_i^{(2)} \} \} > 0$. Denoting $c(x) = \lambda x \in C(R^+, R^+)$, also, the condition in Remark 10 is satisfied. Therefore, system (42) has a unique uniformly asymptotically stable almost periodic solution denoted by $(\omega_1^*(k), \cdots, \omega_n^*(k), u_1^*(k), \cdots, u_n^*(k))$, which is equivalent to saying that the system (2) has a unique uniformly asymptotically stable almost periodic solution denoted by $(x_1^*(k), \cdots, x_n^*(k), u_1^*(k), \cdots, u_n^*(k))$.

If the coefficients are bounded positive periodic sequences, we have the following corollary.

**Corollary 14.** System (2) shows a unique positive periodic solution which is uniformly asymptotically stable under the same assumptions of Theorem 13.

## 5 Numerical Simulations

We give an example to check the feasibility of our results.

**Example 15.** Consider the following system:

\[
\begin{align*}
    x_1(k+1) &= x_1(k) \exp \left[ 0.78 + 0.02 \sin(\sqrt{2}k\pi) - x_1(k) ight] \\
    &\quad - (0.011 - 0.001 \cos(\sqrt{2}k\pi))x_2(k-2) - (0.012 - 0.002 \sin(\sqrt{2}k\pi))x_2^2(k-2) \\
    &\quad - (0.025 - 0.005 \cos(\sqrt{2}k\pi))u_1(k), \\
    x_2(k+1) &= x_2(k) \exp \left[ 0.93 - 0.03 \cos(\sqrt{3}k\pi) - x_2(k) ight] \\
    &\quad - (0.021 - 0.001 \sin(\sqrt{3}k\pi))x_1(k-1) - (0.022 - 0.002 \sin(\sqrt{3}k\pi))x_1^2(k-1) \\
    &\quad - (0.015 - 0.005 \sin(\sqrt{2}k\pi))u_2(k), \\
    u_1(k+1) &= (0.073 + 0.007 \sin(\sqrt{2}k\pi))u_1(k) \\
    &\quad + (0.013 - 0.003 \sin(\sqrt{2}k\pi))x_1(k) + (0.015 - 0.005 \cos(\sqrt{2}k\pi))x_2(k), \\
    u_2(k+1) &= (0.035 + 0.005 \cos(\sqrt{3}k\pi))u_2(k) \\
    &\quad + (0.017 - 0.007 \sin(\sqrt{3}k\pi))x_1(k) + (0.014 - 0.004 \cos(\sqrt{3}k\pi))x_2(k).
\end{align*}
\]

(56)

By calculating, one has

\[
\begin{align*}
    M_1 &= 0.8187, \quad m_1 = 0.675, \quad M_2 = 0.9608, \quad m_2 = 0.7868, \\
    H_1 &= 0.0351, \quad h_1 = 0.0157, \quad H_2 = 0.0385, \quad h_2 = 0.0151, \\
    \lambda_1^{(1)} &= 0.5921, \quad \lambda_1^{(2)} = 0.89, \quad \lambda_2^{(1)} = 0.7129, \quad \lambda_2^{(2)} = 0.95, \\
    \lambda &= 0.5921 > 0, \quad \Delta_1 = 0.744 > 0, \quad \Delta_2 = 0.8659 > 0.
\end{align*}
\]

(57)

Clearly, the assumption of Theorem 13 is satisfied, i.e., system (56) admits a unique uniformly asymptotically stable positive almost periodic solution. The numerical simulations support our results (see Figure 1).

## 6 Discussion

In this paper we consider a discrete competitive system with delays and feedback controls. By constructing Lyapunov functional and using mean value theorem, the conditions on the asymptotical stability of the
positive almost periodic solution are established. Compared with the Theorem 10 in [3], it is easy to see that the feedback controls, to some extent, destroy the stability of the system.

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References

[1] Ayala F.J., Gilpin M.E., Ehrenfeld J.G., Competition between species: theoretical models and experimental tests, Theoretical Population Biology, 4(3)(1973), 331-356.

[2] Gopalsamy K., Stability and Oscillation in Delay Differential Equations of Population Dynamics, 1992, Dordrecht: Springer.

[3] Tan R.H., Liao L.L., A delay almost periodic competitive system in discrete time, Discrete Dynamics in Nature and Society, 2015, Article ID 896816, 11 pp.

[4] Xue Y.L., Xie X.D., Lin Q.F., et al., Global attractivity and extinction of a discrete competitive system with infinite delays and single feedback control, Discrete Dynamics in Nature and Society, 2018, Article ID 1893181, 14 pp.
[5] Chen L.J., Chen L.J., Li Z., Permanence of a delayed discrete mutualism model with feedback controls, Mathematical and Computer Modelling, 50(7-8)(2009), 1083-1089.
[6] Wang Z., Li Y.K., Almost periodic solutions of a discrete mutualism model with feedback controls, Discrete Dynamics in Nature and Society, 2010, Article ID 286031, 18 pp.
[7] Zhang T.W., Gan X.R., Almost periodic solutions for a discrete fishing model with feedback control and time delays, Communications in Nonlinear Science and Numerical Simulation, 19(1)(2014), 150-163.
[8] Lei C.Q., Dynamic behaviors of a non-selective harvesting May cooperative system incorporating partial closure for the populations. Communications in Mathematical Biology and Neuroscience, 2018(2018), Article ID 12.
[9] Chen F.D., Permanence of a discrete n-species cooperation system with time delays and feedback controls, Applied Mathematics and Computation, 186(1)(2007), 23-29.
[10] Yang K., Miao Z.S., Chen F.D., Influence of single feedback control variable on an autonomous Holling-II type cooperative system, Journal of Mathematical Analysis and Applications, 435(1)(2016), 874-888.
[11] Zhang S.N., Zheng G., Almost periodic solutions of delay difference systems, Applied Mathematics and Computation, 131(2-3)(2002), 497-516.
[12] Yuan R., Hong J.L., The existence of almost periodic solutions for a class of differential equations with piecewise constant argument, Nonlinear Analysis: Theory, Methods and Applications, 28(8)(1997), 1439-1450.
[13] Zhang S.N., Existence of almost periodic solutions for difference system, Annals of Differential Equations, 16(2)(2000), 184-206.
[14] Yang X.T., Uniform persistence and periodic solutions for a discrete predator-prey system with delays, Journal of Mathematical Analysis and Applications, 316(1)(2006), 161-177.
[15] Chen F.D., Xie X.D., Dynamic Behaviors of Cooperation Populations Models, 2014, Beijing: Science Press.
[16] Li Y.K., Zhang T.W., Almost periodic solution for a discrete hematopoiesis model with time delay, International Journal of Biomathematics, 1(2012), 35-43.