ON THE NORMALIZER OF A GROUP IN THE CAYLEY REPRESENTATION

SURINDER K. SEHGAL

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210

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ABSTRACT If $G$ is embedded as a proper subgroup of $X$ in the Cayley representation of $G$, then the problem of "if $N_X(G)$ is always larger than $G$" is studied in this paper.

KEY WORDS AND PHRASES: Cayley representation, Wreath product, permutation group.

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Let $R$ be the Cayley representation (i.e., the right regular representation) of a group $G$ given by $R(g) = \{x \mapsto g^x\}$ for all $g \in G$ and $x \in G$. Under the mapping $R$, the group $G$ is embedded into a subgroup $R(G)$ of the symmetric group $S_\Omega$, the group of permutations on the set $\Omega$ consisting of the elements of the group $G$. We identify $R(G)$ with $G$ and say that $G$ is a subgroup of $S_\Omega$. The centralizer of $G$ in $S_\Omega$ consists precisely of the elements of the form $\{x \mapsto g^x\}$. (See Lemma 1.) In particular, if $G$ is abelian then $G$ is self centralizing in $S_\Omega$. Also, the normalizer of $G$ in $S_\Omega$ is equal to $G \cdot \text{Aut}(G)$ where $\text{Aut}(G)$ is the full automorphism group of $G$ (see Lemma 2).

Suppose that the group $G$ is nonabelian. If $X$ is a subgroup of $S_\Omega$, containing a permutation of the type $\{x \mapsto g^x\}$ for some $g \in G - Z(G)$ such that the property

$G \not\subseteq X \subseteq S_\Omega$

holds, then it follows that $N_X(G)$ contains $G$ properly. However, it is easy to see that any element of $S_\Omega$ which normalizes $G$ is not always a permutation of the form $\{x \mapsto g^x\}$.

When the group $G$ is abelian, the permutations $\{x \mapsto g^x\}$ all lie in $G$ and so $G$ is self centralizing in $S_\Omega$. In this way one cannot find a group $X$ satisfying $(\ast)$ by the above method. However, P. Bhattacharya [1] proved that if $G$ is any finite, abelian $p$ group satisfying $(\ast)$ then $N_X(G) \not\supseteq G$. P. Bhattacharya and N. Mukherjee [2] also prove that if $G$ is any finite, nilpotent, Hall subgroup of $X$ satisfying $(\ast)$ and the Sylow $p$ subgroups of $G$ are regular for all primes $p$ dividing the order of $G$, then $N_X(G) \not\supseteq G$. In other words, that $X$ must contain an element of the outer automorphism group of $G$. 
In this paper we will prove that if $G$ is any abelian Hall subgroup of $X$, satisfying the condition (*) then $G \not\leq N_X(G)$. We will also give an example to show that the condition of being Hall subgroup is necessary in the above theorem. We will also show that if $G$ is any nilpotent, Hall subgroup of $X$ satisfying the condition (*) and the Sylow $p$ subgroups $P$ of $G$ do not have a factor group that is isomorphic to the Wreath product of $Z_p \wr Z_p$ then $G \not\leq N_X(G)$. In particular it follows that if $G$ is any finite $p$-group and does not have a factor group isomorphic to $Z_p \wr Z_p$ then $G \not\leq N_X(G)$ [i.e., the condition being a Hall subgroup is not necessary]. As a corollary it also follows that if $G$ is any regular $p$-group satisfying the condition (*) then $G \not\leq N_X(G)$. We will give an example to show that the condition of $G$ having no factor group isomorphic to $Z_p \wr Z_p$ is necessary.

**Lemma 1.** Let $R$ be the right regular representation of a finite group $G$ and $L$, the left regular representation of $G$. Under the mappings $L$ and $R$, the groups $L(G)$ and $R(G)$ are subgroups of $S_G$ and $C_{S_G}(R(G)) = L(G)$.

**Proof:** Let $(\xi, \eta) \in R(G), (\xi', \eta') \in L(G)$

$$
\left( \begin{array}{c}
\xi \\
\xi' \\
\eta \\
\eta'
\end{array} \right) = \left( \begin{array}{c}
\left( \begin{array}{c}
x \\
\xi x \\
hz \\
hzx
\end{array} \right) \\
\left( \begin{array}{c}
\xi' \\
x \\
\eta' \\
hz
\end{array} \right)
\end{array} \right) = \left( \begin{array}{c}
\xi \\
\xi' \\
\eta \\
\eta'
\end{array} \right).
$$

Hence $L(G) \subseteq C_{S_G}(R(G))$.

Now suppose $(\xi, \eta) \in S_G$, and $(\xi, \eta)$ centralizes $R(G)$. So $(\xi, \eta)' = (\xi, \eta)'(\xi, \eta)'(\xi, \eta)'(\eta, \xi)'$. And $(\xi, \eta)'(\xi, \eta)' = (\xi, \eta)'(\xi, \eta)'$.

Since $(\xi, \eta) \in S_G$, so $x'g = (xg)'g^{-1}$ for all $x, g \in G$.

Hence $x' = (xg)'g^{-1}$. Now plug in $g = x^{-1}$. So $x' = 1' \cdot x$. Thus $(\xi, \eta)' = (1', \xi, \eta) \in L(G)$. Hence $C_{S_G}(R(G)) = L(G)$.

**Lemma 2:** With the same notation as in Lemma 1, we have $N_{S_G}(R(G)) = R(G) \cdot Aut(G)$.

**Proof:** Let $(\xi, \eta) \in Aut(G)$ then $(\xi, \eta) \in S_G$,

$$
\left( \begin{array}{c}
\xi \\
\xi' \\
\eta \\
\eta'
\end{array} \right) = \left( \begin{array}{c}
\xi' \xi \\
x \\
xg \\
xg
\end{array} \right) = \left( \begin{array}{c}
\xi' \\
x \\
xg \\
xg
\end{array} \right) = \left( \begin{array}{c}
\xi' \\
x \\
xg \\
xg
\end{array} \right).
$$

Hence $Aut(G) \subseteq N_{S_G}(R(G))$. Conversely, let $(\xi, \eta)$ be an arbitrary element of $N_{S_G}(G)$. Let $a = 1'$. So $(\xi, \eta^{-1})R(G)$. Let $O = (\xi, \eta^{-1})R(G)$. So $O$ sends 1 to 1. Now $(\xi, \eta^{-1})^{-1} = (\xi, \eta) \in R(G)$. So $(\xi, \eta^{-1}) = (\xi, \eta^{-1})(\xi, \eta) = (\xi, \eta^{-1})(\xi, \eta)$ since it lies in $R(G)$, i.e., $(xg)^O = x^O \cdot g^O$. Plug in $x = 1$, we get $g^O = (xg)^O = x^O \cdot g^O \Rightarrow O$ is an automorphism of $G \Rightarrow N_{S_G}(R(G)) = R(G) \cdot Aut(G)$.

**Lemma 3:** Let $G$ be any finite group satisfying the condition (*). Then for any $\alpha \in \Omega$

(i) $G \cap X_\alpha = \{e\}$

(ii) $X = G \cdot X_\alpha$

(iii) $X_\alpha$ is core free, i.e., it does not contain any non-identity normal subgroup of $X$.

**Proof:** Recall that here $G$ is identified with $R(G)$ in $G \leq X \leq S_G$. Since $R$ is the right regular representation of $G$, so $R(g)$ does not fix any $\alpha \in \Omega$ except when $g = e$. So $G \cap X_\alpha = \{e\}$.
\{e\}. Also \(X\) acts transitively on \(\Omega\), \(\{\alpha^X\} = |\Omega| = |G|\). Now \([X : X_\alpha] = |\alpha^X| = |G|\). So \(X = G \cdot X_\alpha\). For part (iii) suppose \(N \triangleleft X\) and \(N \subseteq X_\alpha\). So \(N \subseteq \cap_{x \in X} x^{-1}X_\alpha x\), i.e., if \(n\) is an arbitrary element of \(N\), then \(n\) can be written as \(n = x^{-1}ux\) for all \(x \in X\) and some \(u \in X_\alpha\). Here \(u\) depends on \(x\), i.e., \(x \cdot n = u \cdot x\) or \(\alpha^u n = \alpha^x\) since \(u\) fixes \(\alpha\), i.e., \(n\) fixes \(\alpha^x\) for all \(x \in X\), but \(X\) acts transitively on \(\Omega \Rightarrow n\) fixes every element of \(X \Rightarrow n = e \Rightarrow N = \{e\}\).

**Lemma 4:** (Core Theorem): Let \(H\) be any subgroup of \(G\) with \([G : H] = n\), then \(G/\text{core } H\) is isomorphic to a subgroup of \(S_n\) where \(\text{core } H\) is the largest normal subgroup of \(G\) which is contained in \(H\).

**Proof:** Let \(\Omega\) be the set of distinct right cosets of \(H\) in \(G\), i.e.,
\[
\Omega = \{Hg_1, Hg_2, \ldots, Hg_n\}
\]
Then the mapping \(\sigma\) defined by \(\sigma(g) = \left(\begin{array}{c}Hg_1 \\ Hg_2 \end{array} \right)\) is a transitive permutation representation of \(G\) of degree \(n\) with Kernel of \(\sigma = \text{core } H\).

**Theorem 5:** Let \(G\) be a finite abelian, Hall subgroup of \(X\), satisfying the condition (*). Then \(N_X(G) \supseteq G\).

**Proof:** Suppose the result is false, i.e., there exists a subgroup \(X\) of \(S_\Omega\) satisfying \(G \leq X \leq S_\Omega\) and \(N_X(G) = G\). Amongst all subgroups of \(S_\Omega\) containing \(G\) properly, pick \(X\) to be smallest. In other words, \(G\) is a maximal subgroup of \(X\). Let \(|G| = p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_t^{\alpha_t}\) with \(p_i\) distinct primes. Let \(P_i\) be Sylow \(p_i\) subgroups of \(G\) for \(i = 1, 2, \ldots, t\). Since \(G\) is a maximal subgroup of \(X\), so \(N_X(P_i) = G\) or \(N_X(P_i) = X\). Renumber the \(p_i\)'s if necessary and say \(N_X(P_i) = G\) for \(i = 1, \ldots, t\) and \(N_X(P_i) = X\) for \(i = t + 1, \ldots, t\). For \(i = 1, \ldots, t\), \(N_X(P_i) = C_X(P_i) = G\). So by Burnside Lemma \(X\) has a normal \(P_i\) complement. For \(j = t + 1, \ldots, t\), \(P_j \triangleleft X \Rightarrow C_X(P_j) \triangleleft X \Rightarrow C_X(P_j) = X = N_X(P_j)\). So \(X\) has a normal \(P_i\) complement \(M_i\) for all \(i \Rightarrow X_\alpha = \bigcap_{i=1}^t M_i\). \(X_\alpha \triangleleft S\) which is a contradiction to Lemma 3.

In the case where \(G\) is abelian, but not Hall subgroup of \(X\), the result is not true as illustrated by the following example.

**Example 6:** Let \(X = Z_3 \times Z_3 = (a) \times (b, c) | b^3 = c^2 = 1, c^{-1}bc = b^{-1}\).

Let \(G = Z_3 \times Z_2 = (a) \times (c) \cong Z_6\). Let \(H\) be the subgroup of \(X\) of order 3 generated by the ordered pair \((a, b)\). Then \(H\) is not normal in \(X\) since \((e, c)\) does not normalize \(H\). So \(H\) is core free, of index 6 in \(X\). By Lemma 4, \(G \not\leq X \leq S_6\). Now \(G\) is abelian, not Hall subgroup of \(X\) and \(N_X(G) = G\).

**Theorem 7:** Let \(G\) be a finite, nilpotent, Hall subgroup of \(X\), satisfying the condition (*). Suppose that the Sylow \(p_i\) subgroups \(P_i\) of \(G\) do not have a factor group isomorphic to the Wreath product of \(Z_p \wr Z_p\) for all primes \(p\) dividing the order of \(G\). Then \(N_X(G) \supseteq G\).

**Proof:** Suppose the result is false, i.e., there exists a subgroup \(X\) of \(S_\Omega\) satisfying \(G \leq X \leq S_\Omega\) and \(N_X(G) = G\). Amongst all subgroups of \(S_\Omega\) containing \(G\) properly, pick \(X\) to be smallest. In other words \(G\) is a maximal subgroup of \(X\). Let \(|G| = p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_t^{\alpha_t}\), here \(p_i\) are all distinct primes. Since \(G\) is nilpotent, so \(G = P_1 \times P_2 \times \ldots \times P_t\) where \(P_i\) are Sylow \(p_i\) subgroups of \(G\). So we have either \(N_X(P_i) = G\) or \(N_X(P_i) = X\). Renumber the \(p_i\)'s if necessary and say \(N_X(P_i) = G\) for \(i = 1, \ldots, t\) and \(N_X(P_i) = X\) for \(i = t + 1, \ldots, t\).

Let us look at the case \(i = 1, \ldots, t\). We have \(N = N_X(P_i) = G\). By Yoshida's transfer theorem [3], \(X\) has normal \(p_i\) complement \(M_i\).

Let \(M = \cap_{i=1}^t M_i\). So \(p_i \mid |M|\) for \(i = 1, \ldots, t\). Now for \(j = t + 1, \ldots, t\), \(N_X(P_j) = X\). So \(P_j \triangleleft X\) which implies that \(C_X(P_j) \triangleleft X\) and \(P_jC_X(P_j) \triangle X\) and \(G \subseteq P_j \cdot C_X(P_j) \Rightarrow P_jC_X(P_j) = X\).
For $\alpha \in \Omega$, by Lemma 3 $X = G \cdot X_\alpha$; $G \cap X_\alpha = 1$; $|G|, |X_\alpha| = 1 \Rightarrow X_\alpha \subseteq C_X(P_\ell) \cap M \Rightarrow X_\alpha \subseteq C_M(P_\ell)$ for $f = \ell + 1, \ldots, t$. $|M| = p_\ell^{a_\ell+1} \cdots p_t^{a_t} \cdot |X_\alpha| \Rightarrow X_\alpha \Delta M \Rightarrow X_\alpha$ is a characteristic subgroup of $M \Delta G \Rightarrow X_\alpha \Delta G$, which is a contradiction to Lemma 3.

As an immediate corollary to the theorem, we get the result of P. Bhattacharya and N. Mukherjee [2].

Corollary 8: Let $G$ be a finite, regular $p$ subgroup of $X$ and satisfies the condition (*), then $N_X(G) \not\geq G$.

Proof: If $G$ is not a Hall subgroup of $X$ then $G$ is properly contained in a Sylow $p$ subgroup of $X$ and so $N_X(G) \not\geq G$. So we can assume that $G$ is a Hall subgroup of $X$. Now $G$ being a regular $p$ group $\Rightarrow G$ does not have a factor group isomorphic to $Z_p \wr Z_p$. So Theorem 7 proves the result.

Corollary 9: Let $G$ be a finite, nilpotent, Hall subgroup of $X$, satisfying the condition (*). Suppose further that Sylow $p$ subgroups of $G$ are regular for all primes $p$ dividing the order of $G$ then $N_X(G) \not\geq G$.

Corollary 10: Let $G$ be a finite $p$ group, satisfying the condition (*). Suppose that $G$ does not have a factor group isomorphic to $Z_p \wr Z_p$, then $G \not\leq N_X(G)$.

The condition that the Sylow $p$ subgroups of $G$ in Theorem 6 have the property that it has no homomorphic isomorphic to $Z_p \wr Z_p$ is necessary. See example below.

Example: Let $X$ be the simple group of order 168. Let $G \in Syl_2(X)$. Then $G \cong Z_2 \wr Z_2$ so $G$ is nilpotent, Hall subgroup of $X$. Since $H = \text{the normalizer of a Sylow 7 subgroup has index 8}$, so by Lemma 4, $G \subseteq X \subseteq S_6$, i.e., $G$ satisfies the condition (*) but $N_X(G) = G$.

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