Recurrence and transience of Rademacher series

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Abstract. We introduce the notion of an $a$-walk $S(n) = a_1X_1 + \cdots + a_nX_n$, based on a sequence of positive numbers $a = (a_1, a_2, \ldots)$ and a Rademacher sequence $X_1, X_2, \ldots$. We study recurrence/transience (properly defined) of such walks for various sequences of $a$. In particular, we establish the classification in the cases where $a_k = \lfloor k^\beta \rfloor$, $\beta > 0$, as well as in the case $a_k = \lceil \log \gamma k \rceil$ or $a_k = \log \gamma k$ for $\gamma > 1$.

1. Introduction

We will say that a random variable $X$ has a Rademacher distribution and write $X \sim \text{Rademacher}$, if $\mathbb{P}(X = +1) = \mathbb{P}(X = -1) = \frac{1}{2}$. Let $X_i \sim \text{Rademacher}$, $i = 1, 2, \ldots$, be i.i.d., and $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ be the sigma-algebra generated by the first $n$ members of this sequence. Let $a = (a_1, a_2, \ldots)$ be a non-random sequence of positive numbers. Define the $a$-walk as

$$S(n) = a_1X_1 + a_2X_2 + \cdots + a_nX_n = \sum_{k=1}^{n} a_kX_k$$

with the convention $S(0) = 0$.

Definition 1. Let $C \geq 0$. We call the $a$-walk $S$ defined above $C$-recurrent, if the event $\{|S(n)| \leq C\}$ occurs for infinitely many $n$. (In case when $C = 0$, this is equivalent to the usual recurrence, i.e., $S(n) = 0$ for infinitely many $n$, so we will call the walk just recurrent.)

We call the $a$-walk transient, if it is not $C$-recurrent for any $C \geq 0$.

Our aim is to determine the probability that the $a$-walk for given $a$ and $C$ is recurrent; in principle, this probability may be different from 0 and 1 (for example, if $a = (1, 1, 3, 3, 3, 3, \ldots)$ then the $a$-walk is recurrent with probability $1/2$). A simplest example of an $a$-walk is when all $a_i \equiv a \in \mathbb{R}_+$. Such a random walk is obviously a.s. recurrent since it is equivalent to the one-dimensional simple random walk.

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The question of recurrence is naturally related to the Littlewood-Offord problem which deals with the maximization of probability $\mathbb{P}(S(n) = v)$ over all $v$, subject to various hypotheses on $a$. In particular, in Tao and Vu (2009) the authors develop an inverse Littlewood-Offord theory, using which they show that this probability is large only when the elements of $a$ are contained in a generalized arithmetic progression; see also Nguyen (2012).

The study of $a$-walk is also somewhat relevant to the conjecture by Boguslaw Tomaszewski (1986), which says that $\mathbb{P} \left( \left| S(n) \right| \leq \sqrt{a_1^2 + \cdots + a_n^2} \right) \geq \frac{1}{2}$ for all sequences $a$ and all $n$. The conjecture was recently proved in Keller and Klein (2022).

Let us first start with some general statements. First, we show that the choice of $C > 0$ is sometimes unimportant for the definition of $C$-recurrence.

**Theorem 1.** Suppose that $a_n \to \infty$ and at the same time $|a_n - a_{n-1}| \to 0$ as $n \to \infty$. Then if an $a$-walk is $C$-recurrent with a positive probability for some $C > 0$ then it is $\tilde{C}$-recurrent with a positive probability for all $\tilde{C} > 0$.

**Proof:** Since the notion of $C$-recurrence is monotone in $C$, i.e. if an $a$-walk is $C_1$-recurrent for $C_1 > 0$ then it is $C_2$-recurrent for all $C_2 \geq C_1$, it suffices to prove that $C$-recurrence implies $\frac{2C}{3}$-recurrence.

Indeed, suppose the $a$-walk is $C$-recurrent; formally, if we define the events

$$E = \{ S(n) \in [-C,C] \text{ for infinitely many } n \},$$

$$\tilde{E} = \{ S(n) \in [-2C/3,2C/3] \text{ for infinitely many } n \}$$

then $\mathbb{P}(E) > 0$. We want to show that $\mathbb{P}(\tilde{E}) > 0$ as well.

Let $n_1$ be so large that $|a_i - a_{i-1}| < C/6$ for all $i \geq n_1$. Define the sequence $n_k$, $k \geq 2$, by setting

$$n_k = \min\{ i \geq n_{k-1} + 1 : a_i \geq a_{n_{k-1}} + C/6 \}$$

(which is well-defined since $a_i \to \infty$), then trivially

$$\frac{C}{6} \leq a_{n_{k+1}} - a_{n_k} \leq \frac{C}{3} \quad \text{for each } k = 1, 2, \ldots \quad (1.1)$$

Fix a positive integer $K$ and for $y = (y_1, y_2, \ldots, y_K) \in \Omega_K := \{-1, +1\}^K$ define

$$\bar{X}_K = \{ X_1, X_2, \ldots, X_K \};$$

$$s_y = a_1y_1 + a_2y_2 + \ldots a_ky_k.$$ 

Let $y \in \Omega_K$ be such that $\mathbb{P}(\{ \bar{X}_K = y \} \cap E) > 0$. Observe that

$$\{ \bar{X}_K = y \} \cap E = \{ \bar{X}_K = y \} \cap B_K(s_y)$$

where

$$B_K^+(u) = \{ \text{there exist } m_1 < m_2 < \ldots \text{ such that } u + \sum_{i=K+1}^{m_j} a_i X_i \in [0, C] \};$$

$$B_K^-(u) = \{ \text{there exist } m'_1 < m'_2 < \ldots \text{ such that } u + \sum_{i=K+1}^{m'_j} a_i X_i \in [-C, 0] \};$$

$$B_K(u) = B_K(u)^+ \cup B_K(u)^-.$$ 

Since $\{ \bar{X}_K = y \}$ and $B_K(u)$ are independent, we have

$$\mathbb{P}(\{ \bar{X}_K = y \} \cap B_K(s_y)) = \mathbb{P}(\bar{X}_K = y) \mathbb{P}(B_K(s_y)).$$

Consequently, $\mathbb{P}(B_K(s_y)) > 0$, and as a result, $\mathbb{P}(B_K^+(s_y)) > 0$ or $\mathbb{P}(B_K^-(s_y)) > 0$ (or both).
Let $\Omega^*_K \subseteq \Omega_K$ contain those $y$s for which there is an index $k$ such that $n_{k+2} \leq K$ and $y_{nk} = -1, y_{nk+1} = +1, y_{nk+2} = -1$; let $k$ be the smallest such index. For $y \in \Omega^*_K$ define the mappings $\sigma^+, \sigma^- : \Omega^*_K \to \Omega_K$ by

$$
\sigma^+(y) = \begin{cases} -y_i, & \text{if } i = n_k \text{ or } i = n_{k+1}; \\
y_i, & \text{otherwise}; 
\end{cases}
$$

$$
\sigma^-(y) = \begin{cases} -y_i, & \text{if } i = n_{k+1} \text{ or } i = n_{k+2}; \\
y_i, & \text{otherwise}. 
\end{cases}
$$

Then for $y \in \Omega^*_K$

$$
s_{\sigma^+(y)} = s_y + 2a_{n_k} - 2a_{n_{k+1}} \in [s_y - 2C/3, s_y - C/3],
$$

$$
s_{\sigma^-(y)} = s_y - 2a_{n_k} + 2a_{n_{k+1}} \in [s_y + C/3, s_y + 2C/3].
$$

As a result, it is not hard to see that

$$
\{ \tilde{X}_K = \sigma^+(y) \} \cap B^+_K(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } ms \right\} = \tilde{E},
$$

$$
\{ \tilde{X}_K = \sigma^-(y) \} \cap B^-_K(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } ms \right\} = \tilde{E}.
$$

Since at least one of $B^+_K(s_y)$ and $B^-_K(s_y)$ has a positive probability, $\mathbb{P}\left( \tilde{X}_K = \sigma^\pm(y) \right) = 2^{-K}$ and the events on the LHS are independent, we conclude that $\mathbb{P}(\tilde{E}) > 0$.

Now it only remains to show that there exists $y \in \Omega^*_K$ such that $\mathbb{P}(\{ \tilde{X}_K = y \} \cap E) > 0$. Let $\kappa := \kappa(K) = \max\{k \in \mathbb{Z}_+ : n_k \leq K\}$; obviously, $\kappa(K) \to \infty$ as $K \to \infty$. If we choose $y$ from $\Omega_K$ uniformly, we can trivially bound the probability that $y \notin \Omega^*_K$ by\footnote{exact: see the sequence A005251 in the online encyclopedia of integer sequences (https://oeis.org/A005251), $\mathbb{P}(\tilde{X}_K \notin \Omega^*_K) \approx \lambda^n, \lambda = \sqrt[3]{100+12\sqrt{65}} + \frac{2}{3 \sqrt[3]{100+12\sqrt{65}}} + \frac{2}{3} = 0.877..., \kappa = \kappa(K)$}  

$$
\left( 1 - \frac{1}{8} \right)^{\lceil \kappa/3 \rceil} \to 0 \quad \text{as } K \to \infty
$$

by grouping together triples $(X_{n_1}, X_{n_2}, X_{n_3})$, $(X_{n_4}, X_{n_5}, X_{n_6})$, etc.; in each such a triple

$$
\mathbb{P}\left( (X_{n_k}, X_{n_{k+1}}, X_{n_{k+2}}) = (-1, +1, -1) \right) = 1/8.
$$

Hence

$$
\mathbb{P}(E) = \sum_{y \in \Omega_K} \mathbb{P}(\{ \tilde{X}_K = y \} \cap E) = \sum_{y \in \Omega_K} \mathbb{P}(\{ \tilde{X}_K = y \} \cap E) + \mathbb{P}\left( \{ \tilde{X}_K \in \Omega_K \setminus \Omega^*_K \} \cap E \right). \quad (1.2)
$$

Since $\mathbb{P}\left( \{ \tilde{X}_K \in \Omega_K \setminus \Omega^*_K \} \cap E \right) \leq \mathbb{P}\left( \tilde{X}_K \in \Omega_K \setminus \Omega^*_K \right)$, by making $K$ sufficiently large, we can ensure that the second term on the RHS of (1.2) is less than $\mathbb{P}(E)$, implying that there exist some $y \in \Omega^*_K$ such that $\mathbb{P}(\{ \tilde{X}_K = y \} \cap E) > 0$, as required.

Our next result shows that if the sequence $a$ is non-decreasing, then the walk will always “jump” over $0$ infinitely many times, even if the walk is not $C$-recurrent.

**Theorem 2.** Suppose that $a$ is a non-decreasing positive sequence. Then the event $\{S(n) > 0\}$ holds for infinitely many $n$ a.s. The same is true for the event $\{S(n) < 0\}$.

The theorem immediately follows from symmetry and the more general
Proposition 1. Suppose that $a_i$ is a non-decreasing sequence, $m$ is an integer such that $a_{m+1} > 0$, and $S(m) = A > 0$. Define $$\tau = \inf \{k \geq 0 : S(m + k) \leq 0\}.$$ Let $Y_j \sim \text{Rademacher}$ be i.i.d., and $$\tilde{\tau} = \inf \{k \geq 0 : Y_1 + Y_2 + \cdots + Y_k \leq -r\}$$ where $r = [A/a_{m+1}]$; note that $\tilde{\tau} < \infty$ a.s. and that, in fact, $Y_1 + \cdots + Y_{\tilde{\tau}} = -r$. Then $\tau$ is stochastically smaller than $\tilde{\tau}$, that is, $$\mathbb{P}(\tau > m) \leq \mathbb{P}(\tilde{\tau} > m), \quad m = 0, 1, 2, \ldots$$

Proof: We will use coupling. Indeed, we can write $$S(m + j) = A + a_{m+1}Y_1 + a_{m+2}Y_2 + \cdots + a_{m+j}Y_j, \quad j = 1, 2, \ldots$$ Suppose that $\tilde{\tau} = k$, that is $$Y_1 > -r, \quad Y_1 + Y_2 > -r, \quad \ldots, \quad Y_1 + Y_2 + \cdots + Y_{k-1} > -r; \quad Y_1 + Y_2 + \cdots + Y_k = -r.$$

Then, recalling that $a_i$ is a non-decreasing sequence,

$$S(m + k) = A + a_{m+1}Y_1 + \cdots + a_{m+k-1}Y_{k-1} + a_{m+k}Y_k$$

$$\leq A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-1}Y_{k-1} + a_{m+k-1}Y_k \quad \text{(since } Y_k = -1)$$

$$= A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-1}[Y_{k-1} + Y_k]$$

$$\leq A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-2}[Y_{k-1} + Y_k]$$

$$= A + a_{m+1}Y_1 + \cdots + a_{m+k-2}[Y_{k-2} + Y_{k-1} + Y_k]$$

$$\leq \cdots \leq A + a_{m+1}[Y_1 + \cdots + Y_k] = A - ra_{m+1} \leq 0,$$

since $Y_k, Y_{k-1} + Y_k, Y_{k-2} + Y_{k-1} + Y_k, \ldots, Y_1 + \cdots + Y_k$ are all negative. Therefore, $\tau \leq \tilde{\tau}$. \qed

Throughout the paper we will use a version of the Azuma-Hoeffding inequality; compare with the results of Montgomery-Smith (1990).

Lemma 1.1. Suppose that $b_1, b_2, \ldots, b_m$ is a sequence of non-negative numbers and $S = b_1Y_1 + b_2Y_2 + \cdots + b_mY_m$, where $Y_j \sim \text{Rademacher}$ are i.i.d. Then

$$\mathbb{P}(|S| \geq A) \leq 2 \exp \left(-\frac{A^2}{2(b_1^2 + \cdots + b_m^2)}\right) \quad \text{for all } A > 0. \quad (1.3)$$

We also state the following fairly standard result.

Lemma 1.2. Let $T_i = Y_1 + \cdots + Y_i$ be a simple random walk. Suppose that $L_k$ and $y_k$, $k = 1, 2, \ldots$, are two sequences such that $L_k \to \infty$, $y_k \to \infty$ and $y_k/\sqrt{L_k} \to r > 0$. Then

$$\lim_{k \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq L_k} T_i \geq y_k \right) = 2 \mathbb{P}(\eta \geq r) = 2 - 2 \Phi(r)$$

where $\eta \sim \mathcal{N}(0, 1)$ and $\Phi(\cdot)$ is its CDF.

Proof: Let $\tilde{y}_k = [y_k] \in \mathbb{Z}_+$. By the reflection principle,

$$\mathbb{P} \left( \max_{1 \leq i \leq L_k} T_i \geq y_k \right) = \mathbb{P} \left( \max_{1 \leq i \leq L_k} T_i \geq \tilde{y}_k \right) = 2 \mathbb{P}(T_{L_k} \geq \tilde{y}_k) - \mathbb{P}(T_{L_k} = \tilde{y}_k)$$

$$= 2 \mathbb{P} \left( \frac{T_{L_k}}{\sqrt{L_k}} \geq \frac{\tilde{y}_k}{\sqrt{L_k}} \right) + O \left( \frac{1}{\sqrt{L_k}} \right) \to 2 \mathbb{P}(\eta \geq r)$$

by the Central Limit Theorem, using also the fact that $\tilde{y}_k/y_k \to 1$. \qed
2. Integer-valued \(a\)-walks

Suppose that the sequence \(a\) contains only integers.

**Proposition 2.** Let \(z \in \mathbb{Z}\). Suppose that the sequence

\[
\int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt, \quad n = 1, 2, \ldots
\]

is summable. Then the events \(\{S(n) = z\}\) occur for finitely many \(n\) a.s.

**Proof:** The result follows from standard Fourier analysis. Indeed,

\[
\mathbb{E}e^{itS(n)} = \sum_{k \in \mathbb{Z}} e^{ikt} \mathbb{P}(S(n) = k)
\]

where the sum above goes, in fact, effectively over a finite number of \(k\)s (as \(|S(n)| \leq a_1 + \cdots + a_n\)).

At the same time,

\[
\int_{-\pi}^\pi e^{it(k-z)} dt = \begin{cases} 2\pi, & \text{if } k = z; \\ 0, & \text{if } k \in \mathbb{Z} \setminus \{z\}. \end{cases}
\]

By changing the order of summation and integration, we obtain

\[
\frac{1}{2\pi} \int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)} dt = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\pi}^\pi e^{it(k-z)} \mathbb{P}(S(n) = k) dt = \mathbb{P}(S(n) = z).
\]

On the other hand,

\[
\frac{1}{2\pi} \int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)} dt = \frac{1}{2\pi} \int_{-\pi}^\pi e^{-itz} \prod_{k=1}^n \mathbb{E}e^{ita_k X_k} dt = \frac{1}{\pi} \int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt
\]

by the symmetry of \(\cos\) and the fact that the imaginary part must equal zero. Now the result follows from the Borel-Cantelli lemma, since \(\sum_n \mathbb{P}(S(n) = z) < \infty\). \(\square\)

**Corollary 2.1.** Suppose that the sequence

\[
\int_0^\pi \left| \prod_{k=1}^n \cos(ta_k) \right| dt, \quad n = 1, 2, \ldots
\]

is summable. Then the \(a\)-walk is transient a.s.

**Proof:** From Proposition 2 we know that for each \(z \in \mathbb{Z}\)

\[
\pi \mathbb{P}(S(n) = z) = \int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt \leq \int_0^\pi \left| \cos(tz) \prod_{k=1}^n \cos(ta_k) \right| dt \leq \int_0^\pi \prod_{k=1}^n \cos(ta_k) dt.
\]

Hence the event \(\{S(n) = z\}\) occurs finitely often a.s. for each \(z\). Since for each \(C > 0\) there are only finitely many integers in \([-C, C]\) we conclude that the walk is not \(C\)-recurrent a.s. for every \(C\). \(\square\)

An interesting and quite natural example is when \(a = (1, 2, 3, \ldots)\), i.e., \(a_i = i\). It was previously published in the IMS Bulletin, 51(2), in the Student Puzzle Corner no. 37.

**Theorem 3.** The \(a\)-walk with \(a = (1, 2, 3, \ldots)\) is a.s. transient.

This statement follows from a much stronger Theorem 4, but for the sake of completeness, we present its self-contained proof.
Proof of Theorem 3: Let $A_n = \{S(n) = 0\} = \{X_1 + 2X_2 + \cdots + nX_n = 0\}$. Then $\mathbb{P}(A_n) = Q_n/2^n$, where

$$Q_n = \text{number of ways to put } \pm \text{ in the sequence } *1*2*3*\cdots*n \text{ such that the sum equals } 0.$$ 

For example, $Q_1 = Q_2 = 0$, $Q_3 = Q_4 = 2$, $Q_5 = Q_6 = 0$, $Q_7 = 8$, $Q_8 = 14$, etc. It was essentially shown in Sullivan (2013) that

$$Q_n \sim \sqrt{\frac{6}{\pi}} \frac{2^n}{n^{3/2}} \quad \text{when } n \bmod 4 \in \{0, 3\}$$

(and zero otherwise) as $n \to \infty$, meaning that the ratio of the RHS and the LHS converges to one. Consequently, $\sum_n \mathbb{P}(A_n) \sim \sum_n \text{const}_{n^{3/2}} < \infty$ and the events $A_n$ occur a.s. finitely often by the Borel-Cantelli lemma. Hence the walk is a.s. not recurrent.

Moreover, since for any $m \in \mathbb{Z}$

$$\mathbb{P}(S(n + 2|m|) = S(n) - m \mid F_n) \geq \frac{1}{2^m}$$

(by making the signs of $X_{n+1}, X_{n+2}, \ldots, X_{n+2|m|}$ alternate), we conclude that if the event $\{S(n) = m\}$ occurs infinitely often, then $A_n$ shall also occur infinitely often a.s., leading to contradiction. As a result, $\mathbb{P}(\{S(n) = m\} \text{ i.o.}) = 0$ for all integer $ms$, and thus the walk is a.s. not $C$-recurrent for any non-negative $C$.

Remark 1. Though the $(1, 2, 3, \ldots)$-walk is transient, it still can jump over zero infinitely many times, as it was shown by Theorem 2.

In fact, Theorem 3 can be generalized greatly, using the result from Sárközi and Szemerédi (1965), or even a weaker result of Erdős (1965), which provide the estimates for the maximum number of solutions of the equation $\sum_{i=1}^n \varepsilon_i a_i = t$ where $\varepsilon_i \in \{0, 1\}$ while $a_i$'s and $t$ are all integers.

Theorem 4. Let $a$ be such that all $a_i$'s are distinct integers. Then $a$-walk is a.s. transient.

Proof: The main result of Sárközi and Szemerédi (1965) implies that for any $\varepsilon > 0$

$$\text{card}(\{(x_1, x_2, \ldots, x_n) : \forall i \neq j, x_i = \pm 1, a_1x_1 + \cdots + a_nx_n = m\}) \leq \frac{(1 + \varepsilon)2^n + 3}{n^{3/2}\sqrt{\pi}}$$

for all $n \geq n_0(\varepsilon)$ and all $m$. Setting $\varepsilon = 1$, and fixing $m \in \mathbb{Z}$, we obtain that

$$\sum_{n=n_0(1)}^\infty \mathbb{P}(S(n) = m) \leq \sum_{n=n_0(1)}^\infty \frac{2 \cdot 2^n + 3}{n^{3/2}\sqrt{\pi}} \times \frac{1}{2^n} = \frac{16}{\sqrt{\pi}} \sum_{n=n_0(1)}^\infty \frac{1}{n^{3/2}} < \infty.$$ 

Therefore, by the Borel-Cantelli lemma, only finitely many events $\{S(n) = m\}$ occur a.s. Since $S(n)$ takes only integer values, this implies that $\{|S(n)| \leq C\}$ happens finitely often a.s. for any $C > 0$. \hfill \Box

Remark 2.
(a) It is not difficult to see that under the condition of Theorem 4 it suffices that all $a_k$'s are distinct only starting from some $k_0 \geq 1$.
(b) If $a_k = \lfloor k^\beta \rfloor$ with $\beta \geq 1$, then we immediately have a.s. transience by Theorem 4.
(c) In the proof of Theorem 4 we use the result of Sarkozy and Szemeredi from 1965. The constant in their bound can, in fact, be replaced by the constant $\sqrt{\frac{6}{\pi}}$ from Sullivan's result. Even though the value of the constant does not matter for our proof, it is worth mentioning the (1980) result of Stanley that the set $\{1, 2, 3, \ldots, n\}$ is extremal among sets of $n$ distinct integers for maximizing the maximum concentration probability of its Rademacher sum. This fact was proved by Stanley using some high-powered algebraic geometry but was then proved again soon afterwards in a simpler way using Lie algebras in Proctor (1982).
3. A non-trivial recurrent example

We assume here that \( a = (B_1, B_2, B_3, \ldots) \) where each \( B_k \) is a consecutive block of \( k \)'s of length precisely \( L_k \geq 1 \). Denote also by \( i_k = 1 + L_1 + L_2 + \cdots + L_{k-1} \) the index of the first element of the \( k \)-th block. For example, if \( L_k = 2^k \), then \( i_k = 2^k - 1 \) and

\[
a = (1, 1, \underbrace{2, 2, 2, 2}_{\text{L}_2 \ \text{times}}, \underbrace{3, 3, 3, 3}_{\text{L}_3 \ \text{times}}, \underbrace{3, 3, 3, 4, 4, 4, \ldots}_{\text{L}_4 \ \text{times}}),
\]

one can also notice that \( a_i = \lceil \log_2(i + 1) \rceil = \lceil \log_2(i + 2) \rceil - 1 \).

**Theorem 5.** Suppose that for some \( \varepsilon > 0, r > 0 \), and \( k_0 \) we have

\[
\frac{L_k}{L_1 + L_2 + \cdots + L_{k'}} \geq (2 + \varepsilon) \ln k;
\]

\[
\frac{L_k}{L_{k' + 1} + L_{k' + 2} + \cdots + L_{k-1}} \geq 2r;
\]

\[
L_k \geq k^4,
\]

whenever \( k - k' \geq \frac{k}{\ln k} - 2 \) and \( k, k' \geq k_0 \). Then the \( a \)-walk described above is a.s. recurrent.

**Remark 3.** One can easily check that the conditions of the theorem are satisfied if \( a_k = \lfloor (\log \gamma k)^\beta \rfloor \), where \( \gamma > 1 \) and \( \beta \in (0, 1] \).

**Proof of Theorem 5:** We will proceed in FIVE steps.

**Step 1: Preliminaries**

First, we need the following lemma, which is probably known.

**Lemma 3.1.** Let \( m \in \mathbb{Z}_+ \) and \( T_m \) be a simple symmetric random walk on \( \mathbb{Z}^1 \), that is, \( T_m = Y_1 + \cdots + Y_m \), where \( Y_i \sim \text{Rademacher} \) are i.i.d. There exists a universal constant \( c_1 > 0 \) such that for all integers \( z \) such that \( |z| \leq 2\sqrt{m} \), assuming that \( m \) is sufficiently large and \( m + z \) is even,

\[
\mathbb{P}(T_m = z) \geq \frac{c_1}{\sqrt{m}}.
\]

**Proof:** W.l.o.g. assume \( z \geq 0 \). We have

\[
\mathbb{P}(T_m = z) = \mathbb{P}\left( \frac{T_m + m}{2} = \frac{z + m}{2} \right) = \mathbb{P}(\tilde{T} = w)
\]

where \( \tilde{T} \sim \text{Bin}(m, 1/2) \) and \( w = \frac{z + m}{2} \in \mathbb{Z}_+ \). Note that \( \tilde{m} \leq w \leq \tilde{m} + \sqrt{m} \) where \( \tilde{m} = m/2 \). So

\[
\mathbb{P}(\tilde{T} = w) = \binom{m}{w} \frac{1}{2^m} \frac{\tilde{m}! \tilde{m}!}{(m-w)!} = \frac{1 + o(1)}{\sqrt{\pi \tilde{m}}} \frac{(2\tilde{m} - w + 1)(2\tilde{m} - w + 2) \cdots \tilde{m}}{(\tilde{m} + 1)(\tilde{m} + 2) \cdots w}
\]

\[
\geq \frac{1 + o(1)}{\sqrt{\pi \tilde{m}}} \left( 1 - \frac{w - \tilde{m}}{\tilde{m} + 1} \right) \cdots \left( 1 - \frac{w - \tilde{m}}{w} \right)
\]

\[
\geq \frac{1 + o(1)}{\sqrt{\pi \tilde{m}}} \left( 1 - \frac{w - \tilde{m}}{\tilde{m} + 1} \right) w^{-\tilde{m}} \geq \frac{1 + o(1)}{\sqrt{\pi \tilde{m}}} \left( 1 - \frac{\sqrt{2} + o(1)}{\sqrt{\tilde{m}}} \right)^{\sqrt{2m}} = e^{-2} + o(1) \geq \frac{0.1}{\sqrt{m}}
\]

for large enough \( m \).

**Corollary 3.2.** Let \( T_m, m = 0, 1, 2, \ldots, \) be as simple symmetric random walk as in Lemma 3.1. Assume that \( m \) and \( k \) are positive integers such that \( k^2 \leq m \). Let \( u \in \mathbb{Z} \), and either \( k \) is odd, or both \( k \) and \( m - u \) are even. Then for large \( ks \)

\[
\mathbb{P}(T_m - u \mod k = 0) \geq \frac{c_1}{2k}
\]

where \( c_1 \) is the constant from Lemma 3.1.
Proof of Corollary 3.2: First, assume that \( m \), and hence \( u \), are both even. Since \( (T_m - u) \mod k = 0 \iff T_m = \tilde{u} \mod k \), where \( \tilde{u} = (u \mod k) \in \{0, 1, \ldots, k-1\} \), it suffices to show the statement for \( \tilde{u} \).

Let \( M = \lfloor 2\sqrt{m} \rfloor \in (2\sqrt{m} - 1, 2\sqrt{m}] \) and define
\[
\mathbb{I} = [-M, -M + 1, \ldots, -1, 0, 1, \ldots, M] = \mathbb{I}_0 \cup \mathbb{I}_1; \\
\mathbb{I}_0 = \{z \in \mathbb{I} : z \text{ is even}\}; \\
\mathbb{I}_1 = \{z \in \mathbb{I} : z \text{ is odd}\}.
\]

There are at least \( M \) elements in each \( \mathbb{I}_0 \) and \( \mathbb{I}_1 \).

If \( k \) is odd, then each of these two sets contains at least \( \lfloor M/k \rfloor \) elements \( z \) such that \( z = \tilde{u} \mod k \). If \( m \) is even (odd, resp.) for all \( z \) either in \( \mathbb{I}_0 \) (in \( \mathbb{I}_1 \), resp.) by Lemma 3.1 for large \( ks \) (and hence large \( m \)) we have \( \mathbb{P}(T_m = z) \geq c_1/\sqrt{m} \). Consequently,
\[
\mathbb{P}(T_m = \tilde{u} \mod k) \geq \sum_{z \in \mathbb{I}, z = \tilde{u} \mod k} \mathbb{P}(T_m = z) \geq \left\lfloor \frac{M}{k} \right\rfloor \times \frac{c_1}{\sqrt{m}} \geq \left( \frac{M}{k} - 1 \right) \times \frac{c_1}{\sqrt{m}} \geq \frac{c_1}{k} - O(k^{-2})
\]
since \( m \geq k^2 \).

If \( k \) is even, then if \( m \) is even (and thus \( a \) is also even) then \( \mathbb{I}_0 \) contains at least \( \lfloor M/k \rfloor \) elements \( z \) such that \( z = \tilde{u} \mod k \) and at the same time Lemma 3.1 is applicable for \( z \in \mathbb{I}_0 \). On the other hand, if \( m \) (and so \( u \)) is odd then \( \mathbb{I}_1 \) contains at least \( \lfloor M/k \rfloor \) elements \( z \) such that \( z = \tilde{u} \mod k \) and Lemma 3.1 is applicable for \( z \in \mathbb{I}_1 \). The rest of the proof is the same as for the case when \( k \) is odd. \( \square \)

Step 2: Splitting \( S(n) \)
Recall that that \( i_k \) denotes the first index of block \( k \) and note that the sum of all the steps within block \( k \) can be represented as
\[
S(i_k + 1) - S(i_k) = k \cdot T_k, \quad T_k = X^{(k)}_1 + \cdots + X^{(k)}_{L_k}
\]
where \( X^{(k)}_j \)'s are i.i.d. Rademacher random variables.

For \( m = 2, \ldots, \), let
\[
k_m = \begin{cases} \lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is odd;} \\ \lfloor m \ln m \rfloor + 1 & \text{if } \lfloor m \ln m \rfloor \text{ is even.} \end{cases} \tag{3.2}
\]
Thus \( k_m \) is always odd; \( k_m, m = 2, 3, \ldots \) equal 1, 3, 5, 7, 9, 13, 15, 19, 23, etc. Define also
\[
A_m = \{S(j) = 0 \text{ for some } i_{k_m} \leq j < i_{k_m+1}\},
\]
the event that \( S(j) \) hits zero for the steps within block \( B_{k_m} \), and the sequence of sigma-algebras
\[
\mathcal{G}_m = \mathcal{F}_{i_{k_m+1} - 1} = \sigma \left( \bigcup_{\ell=1}^{k_m} \sigma \left( X^{(\ell)}_1, X^{(\ell)}_2, \ldots, X^{(\ell)}_{L_\ell} \right) \right).
\]
Intuitively, \( \mathcal{G}_m \) contains all the information about the walk during its steps corresponding to the first \( k_m \) blocks.

To simplify notations, let us now write \( k = k_m \) and \( k' = k_{m-1} \), and observe that
\[
k - k' = k_m - k_{m-1} \geq m \ln m - (m - 1) \ln(m - 1) - 2 = \ln m - 1 + O\left( \frac{1}{m} \right) \geq \ln m - 2 \tag{3.3}
\]
for large \( m \).
Let us split $S(j)$ where $j \in [i_k, i_{k+1})$, as follows:

$$S(j) = S(i_{k'}) + \sum_{n=k'+1}^{k-1} (X_1^{(n)} + \cdots + X_{L_n}^{(n)}) + k \cdot (X_1^{(k)} + \cdots + X_{j-i_k}^{(k)})$$

$$= S(i_{k'}) + \left[ \sum_{n=k'+1}^{k-2} (X_1^{(n)} + \cdots + X_{L_n}^{(n)}) + (k-1) \sum_{\ell=1}^{i_k-2k-1} X_{\ell}^{(k-1)} \right]$$

$$+ (k-1) \cdot \Sigma_3 + k \cdot (X_1^{(k)} + \cdots + X_{j-i_k}^{(k)})$$

$$= \Sigma_1 + \Sigma_2 + (k-1) \cdot \Sigma_3 + k \cdot \Sigma_4$$

where $\Sigma_1 = S(i_{k'})$ and

$$\Sigma_2 = \sum_{n=k'+1}^{k-2} nT_n + (k-1)T'_{k-1}, \quad T'_{k-1} = \sum_{\ell=i_k-1}^{i_k-2k-1} X_{\ell}^{(k-1)},$$

$$\Sigma_3 = X_{i_k-2k}^{(k-1)} + X_{i_k-2k+1}^{(k-1)} + \cdots + X_{i_k-2}^{(k-1)} + X_{i_k-1}^{(k-1)},$$

$$\Sigma_4 = X_1^{(k)} + X_2^{(k)} + \cdots + X_{j-i_k}^{(k)}.$$ 

Note that $\Sigma_i$, $i = 1, 2, 3, 4$ are independent, and $\Sigma_3$ has precisely $2k^2$ terms.

**Step 3: Estimating $\Sigma_1$**

Recall that $k = k_m$, $k' = k_{m-1}$ and let

$$E_{m-1} = \{ |\Sigma_1| < k\sqrt{L_k} \} \in G_{m-1}.$$

By Lemma 1.1 and (3.1), assuming $k$ is large,

$$\mathbb{P}(E_{m-1}^c) \leq \mathbb{P}(|S(k')| \geq k'\sqrt{L_k}) \leq 2 \exp \left( -\frac{k'^2 \cdot L_k}{2(L_1 + 2^2 \cdot L_2 + 3^2 \cdot L_3 + \cdots + k'^2 \cdot L_{k'})} \right)$$

$$\leq 2 \exp \left( -\frac{L_k}{2(L_1 + L_2 + L_3 + \cdots + L_{k'})} \right) \leq 2 \exp (-1 + \epsilon/2) \ln k) \leq \frac{2}{k^{1+\epsilon/2}} =: \varepsilon_m.$$ 

**Step 4: Estimating $\Sigma_2$**

Again, by Lemma 1.1 and (3.1), assuming that $k$ is sufficiently large,

$$\mathbb{P} \left( \left| \Sigma_2 \right| \geq k \sqrt{\frac{L_k}{r}} \right) \leq 2 \exp \left( -\frac{k'^2 \cdot L_k}{2((k'+1)^2 L_{k'+1} + \cdots + (k-2)^2 L_{k-2} + (k-1)^2 (L_{k-1} - 2k^2))} \right)$$

$$\leq 2 \exp \left( -\frac{r^{-1} L_k}{2(L_{k'+1} + \cdots + L_{k-1} - 2k^2)} \right) \leq 2 \exp (-1) = 0.7357588824 \ldots$$

Consequently,

$$\mathbb{P} \left( \left| \Sigma_2 \right| < k \sqrt{\frac{L_k}{r}} \right) \geq 0.2 \quad \text{for large } k.$$ 

(3.5)
Step 5: Finishing the proof

We have a trivial lower bound

$$P(A_m \mid E_{m-1}, G_{m-1}) \geq P\left(A_m \mid \Sigma_2 < k\sqrt{\frac{L_k}{r}}, E_{m-1}, G_{m-1}\right) \times P\left(\Sigma_2 < k\sqrt{\frac{L_k}{r}} \mid E_{m-1}, G_{m-1}\right)$$

$$=: (*) \times 0.2 \quad \text{for large } k \quad (3.6)$$

by (3.5), since the second multiplier equals $P\left(\Sigma_2 < k\sqrt{L_k/r}\right)$ by independence.

Let

$$\text{Div}_k = \{\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3 = 0 \mod k\} = \{\Sigma_1 + \Sigma_2 - \Sigma_3 = 0 \mod k\}.$$ 

Since only on the event $\text{Div}_k$, it is possible that $S(j) = 0$ for some $j$ (since the step sizes are $\pm k$ in the block $B_k$), we conclude that for large $k$

$$(* = P\left(A_m \mid \text{Div}_k, \Sigma_2 < k\sqrt{\frac{L_k}{r}}, E_{m-1}, G_{m-1}\right) \times P\left(\text{Div}_k \mid \Sigma_2 < k\sqrt{\frac{L_k}{r}}, E_{m-1}, G_{m-1}\right)$$

$$\geq P\left(A_m \mid \text{Div}_k, \Sigma_2 < k\sqrt{\frac{L_k}{r}}, E_{m-1}, G_{m-1}\right) \times \frac{c_1}{2k}\quad (3.7)$$

due to the fact that by Corollary 3.2, $P(\text{Div}_k \mid F_{ik-2k^2-1}) \geq c_1/(2k)$. On the other hand,

$$P\left(A_m \mid \text{Div}_k, \Sigma_2 < k\sqrt{\frac{L_k}{r}}, E_{m-1}, G_{m-1}\right) \geq \min_{z \in \mathbb{Z}_k} P(z + T_m = 0 \text{ for some } m \in [0, L_k])$$

$$\geq \beta := 1 - \Phi\left(r^{-1/2} + 3\right) > 0 \quad (3.8)$$

where $z + T_m$ is a simple random walk starting at $z$ (see Lemma 3.1), and

$$Z_k = \left\{z \in \mathbb{Z} : |z| \leq (r^{-1/2} + 3) \sqrt{L_k}\right\}.$$ 

Indeed, using the last part of (3.1), and the conditions we imposed, we have

$$|\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3| \leq k\sqrt{L_k} + k\sqrt{L_k/r} + 2(k-1)k^2 < (1 + r^{-1/2} + 2)k\sqrt{L_k}$$

for large $k$, $S(j) = [\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3] + k \cdot \Sigma_4$, and by Lemma 1.2

$$\liminf_{k \to \infty} \min_{z \in \mathbb{Z}_k} P(z + T_m = 0 \text{ for some } m \in [0, L_k]) \geq 2P(\eta > r^{-1/2} + 3) = 2\beta,$$

so the minimum in (3.8) is $\geq \beta$ for all sufficiently large $k$.

Finally, from (3.6), (3.7), and (3.8) we get that

$$\sum_m P(A_m \mid E_{m-1}, G_{m-1}) \geq \sum_m \frac{0.2c_1\beta}{2m \log m} = +\infty$$

and $P(E^c_m)$ is summable by (3.4), so we can apply Lemma 5.1 of Appendix 1 to conclude that events $A_m$ occur infinitely often and thus our $a$-walk is recurrent.

4. Continuous example

The example of $a$-walk described in Theorem 5 roughly corresponds to the case $a_k = \lceil \log_{\gamma} k \rceil$, $k = 1, 2, \ldots$. But what if $a_k$’s take non-integer values, but, for example, equal

$$a_k = \log_{\gamma} k \equiv c \ln k, \quad k = 1, 2, \ldots,$$
where \( \gamma = e^{1/c} > 1 \)? In this Section we will study this example. It is unreasonable to assume that such \( \mathbf{a} \)-walk is recurrent, because of the irrationality of the step sizes, however, we might want to investigate if this walk is \( C \)-recurrent for some \( C > 0 \). Our main result is the following

**Theorem 6.** Let \( c > 0 \) and \( a_k = c \ln k \). Then the \( \mathbf{a} \)-walk is a.s. \( C \)-recurrent for every \( C > 0 \).

To prove this theorem, it is sufficient to show that whatever the value \( c > 0 \) is, \( \{|S(n)| \leq 3\} \) happens i.o. almost surely. Indeed, take any \( C > 0 \). Then the statement that \( \mathbf{a}' \)-walk with \( a_k' = \frac{3c}{c} \ln k, k = 1, 2, \ldots \), is 3-recurrent is equivalent to the statement that \( \mathbf{a} \)-walk with \( a_k = c \ln k \) is \( C \)-recurrent.

The proof will proceed similarly to that of Theorem 5. Let us define \( k_m \) slightly differently from (3.2); namely, let

\[
k_m = \begin{cases} 
\lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is even;} \\
\lfloor m \ln m \rfloor - 1 & \text{if } \lfloor m \ln m \rfloor \text{ is odd.}
\end{cases}
\]

Thus now \( k_m \) are always even. As before, set \( k = k_m \), and \( k' = k_{m-1} \), and define

\[
i_k = \lfloor \gamma^k \rfloor = \max\{i \geq 1 : a_i < k\} + 1 = \min\{i \geq 1 : a_i \geq k\} \in [\gamma^k, \gamma^k + 1),
\]

i.e., the first index when \( a_i \) starts exceeding \( k \). For \( i \in J_k := [i_k, i_{k+1}) \) write

\[
S(i) = S(i_k' - 1) + [S(i_k - 1) - S(i_k' - 1)] + \Sigma_3 + [S(i) - S(i_k - 1)] + \Sigma_4 (i)
\]

where

\[
\Sigma_3 = [S(i_k - 1 + k^2 - 1) - S(i_k - 1)] + [S(i_k) - S(i_k - k^2)].
\]

Note that \( \Sigma_i, i = 1, 2, 3, 4 \) are independent, and \( \Sigma_3 \) has \( 2 \cdot k^2 \) terms, and contains the first \( k^2 \) and the last \( k^2 \) steps of the walk, when the step sizes lie in \([k, k+1]\).

Let

\[
E_{m-1} = \left\{ |\Sigma_1| < k \sqrt{i_k} \right\} = \left\{ S(i_{k_{m-1}}) < k_m \sqrt{i_{k_m}} \right\}
\]

By Lemma 1.1, since \( a_i < k' < k \) for \( i < i_{k'} \),

\[
\mathbb{P}(E_{m-1}^c) = \mathbb{P}\left( |\Sigma_1| \geq k \sqrt{i_k} \right) \leq 2 \exp\left( -\frac{i_k k^2}{2 \sum_{j=1}^{i_k} a_j^2} \right) = 2 \exp\left( -\frac{i_k}{2 i_{k'}} \right) \leq 2 \exp\left( -\frac{c}{2 \gamma^k} \right) = 2 \exp\left( -\gamma^{k-1} \right) = c_{m-1}
\]

using (3.3) for \( k \) sufficiently large\(^2\). Observe that \( c_m \) is summable.

Similarly, by Lemma 1.1

\[
\mathbb{P}\left( |\Sigma_2| \geq 2k \sqrt{i_k} \right) \leq 2 \exp\left( -\frac{4k^2 i_k}{2k^2 (i_k - i_{k'}) - 2k^2} \right) < 2 e^{-2} = 0.27\ldots
\]

Hence,

\[
\mathbb{P}(F_m) \geq 0.72 \text{ where } F_m = \left\{ |\Sigma_2| < 2k \sqrt{i_k} \right\}.
\]

\(^2\)Note that (3.3) was stated for \( k_m \) defined slightly differently; however, it holds here as well.
Lemma 4.1. Let $n = k^2$ where $k$ is an even positive integer, and assume also that $k$ is sufficiently large. Suppose that $X_i, Y_i, i = 1, 2, \ldots, n$, are i.i.d. Rademacher. Let

$$T = (k - 1)(X_1 + X_2 + \cdots + X_n) + k(Y_1 + Y_2 + \cdots + Y_n). \tag{4.5}$$

Then

$$\mathbb{P}(T = j) \geq \frac{c_1^2}{4n} \text{ for each } j = 0, \pm 2, \pm 4, \ldots, \pm n$$

where $c_1$ is the constant from Lemma 3.1.

Proof: It follows from Corollary 3.2 that

$$\mathbb{P}(X_1 + \cdots + X_n = \ell) \geq \frac{c_1}{2k}, \quad \mathbb{P}(Y_1 + \cdots + Y_n = \ell) \geq \frac{c_1}{2k} \text{ for all } \ell \text{ such that } |\ell| \leq 2k. \tag{4.6}$$

Let $j = 2\tilde{j} \in \{0, 2, 4, \ldots, n - 2, n\}$. Consider the sequence of $k - 1$ numbers

$$\tilde{j}, \tilde{j} - k, \tilde{j} - 2k, \tilde{j} - 3k, \ldots, \tilde{j} - (k - 2)k;$$

they all give different remainders when divided by $k - 1$. Hence there must be an $m \in \{0, 1, \ldots, k - 2\}$ such that $\tilde{j} - mk = b(k - 1)$ and $b$ is an integer; moreover, since $0 \leq \tilde{j} \leq n/2$, we have $b \in \left[-\frac{k(k-2)}{k-1}, \frac{n}{2(k-1)}\right]$. For such $m$ and $b$ we have $j = 2\tilde{j} = (2m)k + (2b)(k - 1)$, and, since both $|2m|$ and $|2b| \leq 2k$,

$$\mathbb{P}(T = j) \geq \mathbb{P}(X_1 + \cdots + X_n = 2b) \cdot \mathbb{P}(Y_1 + \cdots + Y_n = 2m) \geq \left(\frac{c_1}{2k}\right)^2 = \frac{c_1^2}{4n}$$

by (4.6). The result for negative $j$ follows by symmetry. \hfill \Box

Corollary 4.2. Let $\varepsilon = \frac{2ck^4}{\gamma^2}$. Then for large even $k$

$$\mathbb{P}(\Sigma_3 \in [j - \varepsilon, j + \varepsilon]) \geq \frac{c_1^2}{4k^2} \text{ for each } j = 0, \pm 2, \pm 4, \ldots, \pm k^2.$$

Proof: $\Sigma_3$ has the same distribution as

$$\sum_{\ell=1}^{k^2} c \ln(i_{k-1} - 1 + \ell) X_\ell + \sum_{\ell=1}^{k^2} c \ln(i_k - \ell) Y_\ell$$

for some i.i.d. $X_\ell, Y_\ell \sim \text{Rademacher}$. At the same time, for $\ell \geq 1$,

$$|c \ln(i_{k-1} - 1 + \ell) - (k - 1)| = |c \ln(\lceil \gamma^{k-1} \rceil + \ell - 1) - (k - 1)|$$

$$\leq |c \ln(\gamma^{k-1} + \ell) - (k - 1)| = c \ln \left(1 + \frac{\ell}{\gamma^{k-1}}\right) \leq \frac{c\ell}{\gamma^{k-1}}$$

Similarly,

$$k - c \ln(i_k - \ell) = k - c \ln(\lceil \gamma^k \rceil - \ell) = k - c \ln(\gamma^k - \ell') = -c \ln \left(1 - \frac{\ell'}{\gamma^k}\right) \in \left[0, \frac{c\ell'}{\gamma^k}\right]$$

for some $\ell' \in [\ell - 1, \ell]$, assuming $\ell = o(\gamma^k)$. As a result, for $T$ defined by (4.5),

$$|\Sigma_3 - T| \leq \sum_{\ell=1}^{k^2} \frac{2c\ell}{\gamma^{k-1}} = \frac{ck^2(k^2 + 1)}{\gamma^{k-1}} \leq \frac{2ck^4}{\gamma^{k-1}}.$$

Now the result follows from Lemma 4.1. \hfill \Box
Proof of Theorem 6: Recall that \(J_k = [i_k, i_{k+1})\) and define

\[ A_m = \{S(i) = 0 \text{ for some } i \in J_{km}\}. \]

Then

\[ \mathbb{P}(A_m \mid E_{m-1}, G_{m-1}) \geq 0.72 \times \mathbb{P}(A_m \mid F_m, E_{m-1}, G_{m-1}) \quad (4.7) \]

(please see the definition of \(F_m\) in (4.4)). Recall formula (4.1) and write

\[ \tilde{S}(i) = S(i) - \Sigma_3 = \Sigma_1 + \Sigma_2 + \Sigma_4(i). \]

From now on assume that \(|\Sigma_1| < k\sqrt{i_k}\) and \(|\Sigma_2| < 2k\sqrt{i_k}\), that is, \(E_{m-1}\) and \(F_m\) occur. Also assume w.l.o.g. that \(\Sigma_1 + \Sigma_2 \geq 0\). Let

\[ L_k = i_{k+1} - i_k - k^2 = (\gamma - 1)\gamma^k + o(\gamma^k). \]

Consider a simple random walk with steps \(Y_t \sim \text{Rademacher}\) during its first \(L_k\) steps. The probability that its minimum will be equal to or below the level \(-3\sqrt{i_k} = -\frac{3+o(1)}{\sqrt{\gamma - 1}} \sqrt{L_k}\) converges by Lemma 1.2 to

\[ 2 \mathbb{P}\left( \eta > \frac{3}{\sqrt{\gamma - 1}} \right) = 2 - 2 \Phi \left( \frac{3}{\sqrt{\gamma - 1}} \right) =: 2c_2 \in (0, 1) \]

as \(k \to \infty\) (recall that \(\eta \sim \mathcal{N}(0, 1)\)). As a result, by Proposition 1, the probability that for some \(j_0 \in \{i_k, i_k + 1, i_k + 2, \ldots, i_k + L_k - 1\}\) we have the down-crossing, that is,

\[ \tilde{S}(j_0 - 1) \geq 0 > \tilde{S}(j_0) \]

is bounded below by \(c_2\) for \(k\) sufficiently large. Formally, let

\[ j_0 = \inf\{j > i_k : \tilde{S}(j) < 0\}; \]

\[ C_0 = \{i_k \leq j_0 \leq i_k + L_k - 1\}, \]

so we have showed that on \(E_{m-1} \cap F_m \cap \{\Sigma_1 + \Sigma_2 > 0\}\) we have \(\mathbb{P}(C_0) > c_2\) for large \(k\).

Now assume that event \(C_0\) occurred and define additionally

\[ b_0 = \tilde{S}(j_0) \in (-k - 1, 0]; \]

\[ \mathcal{C} = \left\{ \max_{0 \leq h \leq k^2} \sum_{g=1}^{h} X_{j_0+g} \geq k \right\}. \]

Again, from Lemma 1.2,

\[ \mathbb{P}(\mathcal{C}) = 2 \mathbb{P}(X_{j_0+1} + X_{j_0+2} + \cdots + X_{j_0+k^2} \geq k) \to 2(1 - \Phi(1)) = 0.3173\ldots \quad \text{as } k \to \infty. \]

From now on assume that \(k\) is so large that \(\mathbb{P}(\mathcal{C}) > 0.2\). On the event \(\mathcal{C}\) there exists an increasing sequence \(j_1, j_2, \ldots, j_k\) such that

\[ j_0 < j_1 < j_2 < \cdots < j_k \leq j_0 + k^2 < i_{k+1} \]

such that \(X_{j_0+1} + X_{j_0+2} + \cdots + X_{j_\ell} = \ell\) for each \(\ell = 1, 2, \ldots, k\), since the random walk must pass through each integer in \(\{1, 2, \ldots, k\}\) in order to reach level \(k\).

For \(\ell = 1, 2, \ldots, k\), define

\[ b_\ell := \tilde{S}(j_\ell) = b_0 + \sum_{h=j_0+1}^{j_\ell} a_h X_h = b_0 + a_{j_0} \left[ \sum_{h=j_0+1}^{j_\ell} X_h \right] + \sum_{h=j_0+1}^{j_\ell} (a_h - a_{j_0}) X_h \]

\[ = b_0 + a_{j_0} \ell + O \left( \frac{k^4}{\gamma^k} \right) \]
and consider now only those \( n \) for which \( a_n = M \), and that the cardinality of \( I_M \) is of order \( M^{1/\beta - 1} \). Next, fix some \( z \in \mathbb{Z} \) and define
\[
E_M = E_M(z) = \{ S(n) = z \text{ for some } n \in I_M \}.
\]
For each $z$ we will show that $\sum_M \mathbb{P}(E_M) < \infty$, and so by the Borel-Cantelli lemma a.s. only finitely many events $E_M$ occur. Since $S(n)$ takes only integer values, this will imply that the walk is not $C$-recurrent for any $C \geq 0$.

So, fix $z$ from now on, write $S(n) = S(k_M) + R(n)$ where

$$R(n) = \sum_{i=k_M}^n a_i X_i = M \sum_{i=k_M}^n X_i,$$

Observe also that $S(k_M)$ and $R(n)$ are independent. In order $S(n) = z$ for some $n \in I_M$, we need that $S(k_M) = z \mod M$. Let $Q = M^{\frac{1}{2\pi} + 1 - \varepsilon}$ for an $\varepsilon \in (0, 1/2)$. Assuming $M$ is so large that $Q \geq 2|z|$, 

$$\mathbb{P}(|R(n)| \geq Q - |z|) \leq 2 \exp \left( -\frac{(Q/2)^2}{2M^2 \cdot (k_M + 1 - k_M)} \right) = 2 \exp \left( -\frac{\beta Q^2}{(8 + o(1))M^2 \cdot M^{\frac{1}{2\pi} - 1}} \right)$$

by Lemma 5.1; hence

$$\mathbb{P}(\max_{n \in I_M} |R(n)| \geq Q - |z|) \leq |I_M| \times 2 \exp \left( -\frac{\beta M^{1 - 2\varepsilon}}{8 + o(1)} \right) =: \alpha_M = O \left( M^{\frac{1}{2\pi} - 1} e^{-\frac{\beta M^{1 - 2\varepsilon}}{8 + o(1)}} \right),$$

which is summable in $M$. So,

$$\mathbb{P}(E_M) \leq \mathbb{P} \left( E_M, \max_{n \in I_M} |R(n)| < Q - |z| \right) + \mathbb{P} \left( \max_{n \in I_M} |R(n)| \geq Q - |z| \right)$$

$$= \mathbb{P} \left( E_M, S(k_M) = z \mod M, \max_{n \in I_M} |R(n)| < Q - |z| \right) + \alpha_M = (*) + \alpha_M$$

where the term $\alpha_M$ is summable since $1 - 2\varepsilon > 0$. Since $E_M$ implies implies $-S(k_M) = R(n) - z$ for some $n \in I_M$, 

$$(*) \leq \mathbb{P}(|S(k_M)| < Q, S(k_M) = z \mod M) = \sum_{j: |j| < Q, j = z \mod M} \mathbb{P}(|S(k_M)| = j)$$

$$\leq \frac{\nu}{k_M^{1/2 + \beta}} \times |\{j: |j| < Q, j = z \mod M\}| \leq \frac{\nu + o(1)}{M^{1 + \frac{1}{2\pi}}} \times \left[ \frac{2Q + 1}{M} + 1 \right] = \frac{(\nu + o(1))}{\pi M^{1 + \varepsilon}}$$

by Proposition 3. The RHS is summable in $M$, which concludes the proof. \[\square\]

Remark 4. By setting $\varepsilon = 1/2 - \delta/2$, where $\delta > 0$ is very close to zero, we can ensure that

$$\mathbb{P}(|S(n)| < M^{1/2 - \delta} \text{ for some } n \in I_M) \leq \sum_{z:|z|<M^{1/2-\delta}} \mathbb{P}(E_M(z)) \leq \left[ \frac{\text{const}}{M^{1+\varepsilon} + \alpha_M} \right] \times 2M^{1/2-\delta}$$

$$= \frac{2 \text{const}}{M^{1+\delta/2}} + 2M^{1/2-\delta}\alpha_M$$

is summable. Hence, a.s. eventually $|S(n)|$ will be larger that $n^{\beta/2-\delta}$ for any $\delta > 0$.

Appendix 1: Modified conditional Borel-Cantelli lemma

**Lemma 5.1.** Suppose that we have an increasing sequence of sigma-algebras $\mathcal{G}_m$ and a sequence of $\mathcal{G}_m$-measurable events $A_m$ and $E_m$ such that

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq \alpha_m, \quad \mathbb{P}(E_m^c) \leq \varepsilon_m \quad \text{a.s.}$$
where the non-negative $\alpha_n$ and $\varepsilon_n$ satisfy
\[ \sum_m \alpha_m = \infty, \quad \sum_m \varepsilon_m < \infty. \] (5.1)

Then $\mathbb{P}(A_m \text{ i.o.}) = 1$.

**Proof:** Let $m > \ell \geq 1$ and $B_{\ell,m} = \bigcap_{i=\ell}^m A_i^c$. We need to show that for any $\ell \geq 1$
\[ \mathbb{P}(B_{\ell,\infty}) = \mathbb{P}(A_{\ell}^c \cap A_{\ell+1}^c \cap A_{\ell+2}^c \cap \ldots ) = 0. \] (5.2)

We have for $m \geq \ell + 1$
\[ \mathbb{P}(B_{\ell,m}) = \mathbb{P}(A_m^c \cap B_{\ell,m-1}) \leq \mathbb{P}(A_m^c \cap B_{\ell,m-1} \cap E_{m-1}) + \mathbb{P}(E_{m-1}) \]
\[ = \mathbb{P}(A_m^c | B_{\ell,m-1} \cap E_{m-1}) \mathbb{P}(B_{\ell,m-1} \cap E_{m-1}) + \mathbb{P}(E_{m-1}) \]
\[ \leq \mathbb{P}(A_m^c | B_{\ell,m-1} \cap E_{m-1}) \mathbb{P}(B_{\ell,m-1}^c) + \mathbb{P}(B_{\ell,m-1}^c \cap E_{m-1}). \quad (5.3) \]

By induction over $m, m - 1, m - 2, \ldots, \ell + 1$ in (5.3), we get that
\[ \mathbb{P}(B_{\ell,m}) \leq \varepsilon_{m-1} + (1 - \alpha_m) \mathbb{P}(B_{\ell,m-2}^c | \mathcal{G}_m) + \mathbb{P}(B_{\ell,m-2}^c \cap \mathcal{G}_m) \]
\[ \leq \varepsilon_{m-1} + (1 - \alpha_m) \varepsilon_{m-2} + (1 - \alpha_m)(1 - \alpha_{m-1}) \varepsilon_{m-3} + \ldots \]
\[ + (1 - \alpha_m)(1 - \alpha_{m-1}) \ldots (1 - \alpha_{\ell+1}) \varepsilon_{\ell} + (1 - \alpha_m)(1 - \alpha_m - 1) \ldots (1 - \alpha_{\ell+2}) \varepsilon_{\ell}. \]

Hence, for any integer $M \in (\ell, m)$
\[ \mathbb{P}(B_{\ell,m}) \leq \varepsilon_{m-1} + \varepsilon_{m-2} + \ldots + \varepsilon_{M} \]
\[ + (1 - \alpha_m)(1 - \alpha_{M-1}) \ldots (1 - \alpha_{\ell+1}) \varepsilon_{\ell} \leq \frac{\varepsilon_{m-1} + \varepsilon_{m-2} + \ldots + \varepsilon_{M}}{1 - \alpha_k} \]
\[ + \frac{(1 - \alpha_m)(1 - \alpha_{M-1}) \ldots (1 - \alpha_{\ell+1}) \varepsilon_{\ell}}{1 - \alpha_k}. \]

Fix any $\delta > 0$. By (5.1) we can find an $M$ be so large that $\sum_{i=M}^{\infty} \varepsilon_i < \delta/2$. Then, again by (5.1), there exists an $m_0 > M$ such that $\prod_{i=M+1}^{m_0} (1 - \alpha_i) < \frac{\delta}{2(1 + \sum_{i=\ell}^{M-1} \varepsilon_i)}$. Hence, for all $m \geq m_0$ we have $\mathbb{P}(B_{\ell,m}) \leq \delta/2 + \delta/2 = \delta$. Since $\delta > 0$ is arbitrary, and $B_{\ell,m}$ is a decreasing sequence of events in $m$, we conclude that $\mathbb{P}(B_{\ell,\infty}) = 0$, as required. \hfill \Box

**Appendix 2: Generalization of Blair Sullivan’s results**

Let $a_k = [k^\beta]$, where $0 < \beta < 1$.

**Lemma 5.2.** Let $F_n(t) = \prod_{k=1}^{n} |\cos(ta_k)|$. Then
\[ \int_{-\pi}^{\pi} F_n(t)dt = \frac{\sqrt{8\pi(1 + 2\beta)} + o(1)}{n^{\beta + 1/2}} \quad \text{as } n \to \infty. \]

**Remark 5.** Note that for $\beta = 1$ we would have obtained the same result as in Sullivan (2013).

**Proof:** We will proceed in the spirit of Sullivan (2013). Note that by symmetry
\[ \int_{-\pi}^{\pi} F_n(t)dt = 2 \int_0^{\pi/2} F_n(t)dt = 2 \int_0^{\pi/2} F_n(t)dt + 2 \int_0^{\pi/2} F_n(\pi - t)dt = 4 \int_0^{\pi/2} F_n(t)dt \]
since $|\cos((\pi - t)a_k)| = |\cos(\pi a_k - ta_k)| = |\cos(ta_k)|$ as $a_k$ is an integer. Let $\varepsilon > 0$ be very small and define
\[ I_0 = \left[ 0, \frac{1}{n^{\beta + 1/2 - \varepsilon}} \right], \quad I_1 = \left[ \frac{1}{n^{\beta + 1/2 - \varepsilon}}, \frac{1}{n^\beta} \right], \quad I_2 = \left[ \frac{1}{n^\beta}, \frac{c_1}{n^\beta} \right], \quad I_3 = \left[ \frac{c_1}{n^\beta}, \frac{\pi}{2} \right]. \]
for some $c_1 > 1$ to be determined later. Then

\[
\int_0^{\pi/2} F_n(t)dt = \int_0^a F_n(t)dt + \int_{I_1} F_n(t)dt + \int_{I_2} F_n(t)dt + \int_{I_3} F_n(t)dt.
\]

We will show that the contribution of all the integrals, except the first one, is negligible, and estimate then the value of the first one.

First, observe that when $0 \leq ta_k \leq \pi/2$ for all $k \leq n$, by the elementary inequality $|\cos u| \leq e^{-u^2/2}$ valid for $|u| \leq \pi/2$, we have

\[
F_n(t) \leq \prod_{k=1}^n \exp \left( -\frac{t^2 a_k^2}{2} \right) = \exp \left( -\frac{t^2}{2} \sum_{k=1}^n a_k^2 \right) = \exp \left( -\frac{t^2 n^{2\beta+1}(1+o(1))}{2(1+2\beta)} \right) \quad (5.4)
\]
since $a_k^2 = k^{2\beta}(1+o(1))$.

**Case 0: $t \in I_0$**

Here $ta_k \leq \frac{1}{n^{1/2} - \varepsilon} \ll 1$, hence for $n$ large enough

\[
\frac{(ta_k)^2}{2} \leq -\ln \cos(ta_k) = \frac{(ta_k)^2}{2} + O\left((ta_k)^4\right) \leq 1 + o(1)\frac{(ta_k)^2}{2}
\]
yielding $F_n(t) = \exp \left( -\frac{t^2 n^{2\beta+1}\rho_{n,t}}{2(1+2\beta)} \right)$ where $\rho_{n,t} = 1 + o(1)$ for large $n$ (compare with (5.4)). Since for any $r > 0$ we have

\[
\int_0^{n^{-\beta-1/2+\varepsilon}} \exp \left( -\frac{t^2 n^{2\beta+1}r}{2(1+2\beta)} \right) dt = \frac{1}{n^{1/2+\beta}} \int_0^{n^\varepsilon} \exp \left( -\frac{s^2 r}{2(1+2\beta)} \right) ds
\]

\[
= \frac{1}{n^{1/2+\beta}} \left[ \sqrt{\frac{\pi(1+2\beta)}{2r}} + o(1) \right]
\]

where the main term is monotone in $r$, by substituting $r = \rho_{n,t} = 1 + o(1)$ we conclude that

\[
\int_{I_0} F_n(t)dt = \frac{1}{n^{1/2+\beta}} \left[ \sqrt{\pi(1/2+\beta)} + o(1) \right].
\]

**Case 1: $t \in I_1$**

Since $ta_k \leq 1 < \pi/2$, by (5.4) for some $C_2 > 0$ we have $F_n(t) \leq \exp \left( -\frac{n^{2\varepsilon}(1+o(1))}{2(1+2\beta)} \right) \leq e^{-C_2 n^{2\varepsilon}}$, so

\[
\int_{I_1} F_n(t)dt \leq e^{-C_2 n^{2\varepsilon}}
\]

which decays faster than polynomially.

**Case 2: $t \in I_2$**

As in Case 2 in Sullivan (2013), we will use monotonicity of $F_n(t)$ in $n$. Let $r = c_1^{-1/\beta} \in (0,1)$ then $|rn|^{\beta} \leq (rn)^{\beta} = \frac{n^\beta}{c_1},$ consequently by (5.4), since $t \leq \frac{c_1}{n^{\beta}},$

\[
F_{|rn|}(t) \leq \exp \left( -\frac{t^2 (rn)^{2\beta+1}(1+o(1))}{2(1+2\beta)} \right)
\]

and since $F_n(t) \leq F_{|rn|}(t)$, we get a similar bound as in Case 1.

**Case 3: $t \in I_3$**

Let

\[
k_m = \inf\{k \in \mathbb{Z}_+ : k^\beta \geq m\} = \lceil m^{1/\beta} \rceil, \quad m = 1, 2, \ldots;
\]

\[
\Delta_m = k_{m+1} - k_m = \beta^{-1} m^\gamma + O(m^{1/\beta-2}) + \rho_0, \quad \gamma := \frac{1-\beta}{\beta},
\]

\[
correction{\text{}}
where $|\rho_0| \leq 1$. Then

$$a_k = m \quad \text{if and only if} \quad k \in \{k_m, k_m + 1, \ldots, k_m + \Delta_m - 1 (\equiv k_{m+1} - 1)\}.$$ 

For $n \in \mathbb{Z}_+$ let

$$m_n = \max\{m : k_m \leq n\} = n^3 (1 + o(1)), \quad n \in [k_{m_n}, k_{m_n} + \Delta_{m_n} - 1].$$

By the inequality between the mean geometric and the mean arithmetic,

$$F_n(t) = \left(\frac{1}{n} \sum_{k=1}^{n} \cos^2(ta_k)\right)^{n/2} \leq \left(\frac{\sum_{k=1}^{n} \cos^2(ta_k)}{n}\right)^{n/2} \leq \left(\frac{\sum_{k=1}^{n} \cos^2(ta_k)}{n}\right)^{n/2} = \left(\frac{1}{2} + \frac{U_n(t)}{2n}\right)^{n/2}$$

where $U_n(t) = \sum_{k=1}^{n} \cos(2ta_k)$.

We will show that if $t$ is not too small, then for some $0 \leq c < 1$ we have $U_n(t) \leq cn$ and hence $F_n(t) \leq (\frac{1+c}{2})^{n/2}$. In order to do that, first note that

$$U_n(t) \leq \sum_{k=1}^{k_{m_n}} \cos(2ta_k) + (n - k_{m_n}) = \sum_{m=1}^{m_n} \Delta_m \cos(2tm) + (n - k_{m_n}).$$

Let $r \in (0, 1)$ and assume w.l.o.g. that $rm_n$ is an integer. For $m \in [rm_n + 1, m_n]$ we have

$$A \leq \Delta_m \leq A$$

where $A = \beta^{-1}(rm_n)^\gamma + O(1)$, $A = \beta^{-1}m_n^\gamma + O(1)$. Consequently,

$$\sum_{m=rm_n + 1}^{m_n} \Delta_m \cos(2tm) \leq \sum_{m=rm_n + 1}^{m_n} \left[\bar{A} \cdot 1_{\cos(2tm) \geq 0} + A \cdot 1_{\cos(2tm) < 0}\right] \cos(2tm)$$

$$= \sum_{m=rm_n + 1}^{m_n} [\bar{A} - A] \cos(2tm) \cdot 1_{\cos(2tm) \geq 0} + \sum_{m=rm_n + 1}^{m_n} A \cos(2tm)$$

$$\leq (1 - r)m_n (\bar{A} - A) + A \sum_{m=rm_n + 1}^{m_n} \cos(2tm) = (1 - r)m_n (\bar{A} - A)$$

$$+ A \left(\frac{\cos^2(rtm_n + t) - \cos^2(tm_n + t) + \cos t}{2 \sin t} \left[\sin(2t(m_n + 1)) - \sin(2t(rm_n + 1))\right]\right)$$

$$\leq \frac{m_n}{\beta} \left(1 - r\right) \left[1 - r^\gamma + O(m_n^{-\gamma})\right] + m_n^\gamma \left(\frac{r^\gamma}{\beta} + O(m_n^{-\gamma})\right) \left(1 + \frac{1}{|\sin t|}\right)$$

Hence, since $m_n^{1/\beta} = n + o(n),$

$$U_n(t) \leq \sum_{m=1}^{rm_n} \Delta_m + \sum_{m=rm_n + 1}^{m_n} \Delta_m \cos(2tm) + (n - k_{m_n}) = k_{rm_n + 1} + \sum_{m=rm_n + 1}^{m_n} \Delta_m \cos(2tm) + O(\Delta_m)$$

$$\leq r^{1/\beta}n + \frac{m_n^{1/\beta}}{\beta} \left(1 - r\right) \left[1 - r^\gamma + O(m_n^{-\gamma})\right] + m_n^\gamma \left(\frac{r^\gamma}{\beta} + O(m_n^{-\gamma})\right) \left(1 + \frac{1}{|\sin t|}\right) + O(m_n^\gamma)$$

$$\leq n \left(r^{1/\beta} + \frac{(1 - r)(1 - r^\gamma)}{\beta}\right) + \frac{4n^{1-\beta}\beta^{-1}r^\gamma}{|\sin t|} + o(n)$$

Consider now the the function

$$h(r, \beta) := r^{1/\beta} + \frac{(1 - r)(1 - r^\gamma)}{\beta} = r^{1/\beta} + \frac{(1 - r)(1 - r^{1/\beta - 1})}{\beta}$$
and note that
\[ h(1 - \beta, \beta) = (1 - \beta)^{1/\beta} + 1 - (1 - \beta)^{1/\beta - 1} = 1 - \beta(1 - \beta)^{1/\beta - 1} \leq 1 - e^{-1} \beta < 1 - \beta/3 \]
since \( \sup_{\beta \in (0,1)} (1 - \beta)^{1/\beta - 1} = e^{-1} \) by elementary calculus. So if we set \( r = 1 - \beta \in (0,1) \), by noting \( t \leq 2 \sin t \) for \( t \in [0, \pi/2] \), we conclude that \( U_n(t) \leq \left( 1 - \frac{\beta}{1} \right) n \) provided that \( t \geq \frac{c_1}{n^\beta} \) for some \( c_1 > 0 \). Consequently, \( \int_I F_n(t) dt \leq \left( 1 - \frac{\beta}{1} \right)^{n/2} \) for large \( n \), which converges to zero exponentially fast. \( \square \)

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References

Erdős, P. Extremal problems in number theory. In Proc. Sympos. Pure Math., Vol. VIII, pp. 181–189. Amer. Math. Soc., Providence, R.I. (1965). MR0174539.

IMS Bulletin, 51(2) (March 2022). Available at https://imstat.org/archive-vol-51-2022/.

Keller, N. and Klein, O. Proof of Tomaszewski’s conjecture on randomly signed sums. Adv. Math., 407, Paper No. 108558, 39 (2022). MR4452672.

Montgomery-Smith, S. J. The distribution of Rademacher sums. Proc. Amer. Math. Soc., 109 (2), 517–522 (1990). MR1013975.

Nguyen, H. H. A new approach to an old problem of Erdős and Moser. J. Combin. Theory Ser. A, 119 (5), 977–993 (2012). MR2891377.

Proctor, R. A. Solution of two difficult combinatorial problems with linear algebra. Amer. Math. Monthly, 89 (10), 721–734 (1982). MR683197.

Sárközi, A. and Szemerédi, E. Über ein Problem von Erdős und Moser. Acta Arith., 11, 205–208 (1965). MR182619.

Stanley, R. P. Weyl groups, the hard Lefschetz theorem, and the Sperner property. SIAM J. Algebraic Discrete Methods, 1 (2), 168–184 (1980). MR578321.

Sullivan, B. D. On a conjecture of Andrica and Tomescu. J. Integer Seq., 16 (3), Article 13.3.1, 6 (2013). MR3033734.

Tao, T. and Vu, V. H. Inverse Littlewood-Offord theorems and the condition number of random discrete matrices. Ann. of Math. (2), 169 (2), 595–632 (2009). MR2480613.