The Coupled Modified Korteweg-de Vries Equations

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Generalization of the modified KdV equation to a multi-component system, that is expressed by
\[
\frac{\partial u_i}{\partial t} + 6 \left( \sum_{j,k=0}^{M-1} C_{jk} u_j u_k \right) \frac{\partial u_i}{\partial x} + \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 0, 1, \cdots, M - 1
\]

is studied. We apply a new extended version of the inverse scattering method to this system. It is shown that this system has an infinite number of conservation laws and multi-soliton solutions. Further, the initial value problem of the model is solved.

KEYWORDS: modified KdV equation, multi-component system, inverse scattering method, conservation law, initial value problem, soliton solution

§1. Introduction

Since the discovery of the inverse scattering method (ISM), it has been shown that the ISM is applicable to many soliton equations. Among the soliton equations, the modified Korteweg-de Vries (mKdV) equation has been studied extensively because of its simplicity and physical significance. Generalization of the mKdV equation to a multi-component system or a matrix equation has been studied by some authors. One example is a vector version of the mKdV equation proposed by Yajima and Oikawa. Sasa and Satsuma solved the initial value problem of the system, and constructed multi-soliton solution. Another example is a matrix version of the mKdV equation studied by Athorne and Fordy.

Recently, Iwao and Hirota discussed a simple coupled version of the modified KdV equation,
\[
\frac{\partial u_i}{\partial t} + 6 \left( \sum_{j,k=0}^{M-1} C_{jk} u_j u_k \right) \frac{\partial u_i}{\partial x} + \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 0, 1, \cdots, M - 1, \tag{1.1}
\]

where the constants \( C_{jk} \) are set to be symmetric with respect to the subscripts, \( C_{jk} = C_{kj} \), without any loss of generality. They obtained multi-soliton solution of this system with the condition

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$C_{ij} = 0$. Hirota \cite{Hirota} also studied solutions of semi-discrete version of the model. We call eq. (1.1) the coupled modified KdV (cmKdV) equations. The cmKdV equations for $M = 1, 2$ have been solved by the ISM. However, it has not been known whether the cmKdV equations for $M \geq 3$ and their hierarchy can be solved by the ISM or not.

In this paper, we propose a matrix generalization of the ISM that includes matrix mKdV equation, matrix nonlinear Schrödinger equation (NLS) equation and other integrable equations. This formulation also includes the cmKdV equations as a reduction of the matrix mKdV equation. It provides us a method to solve the initial value problem and obtain soliton solutions.

The outline of the paper is as follows. In §2, we introduce a Lax representation for the matrix mKdV equation and the matrix NLS equation. Further the reduction of the matrix mKdV equation to the cmKdV equations is given. In §3, we perform the ISM for the matrix mKdV equation and the matrix NLS equation. In §4, we cast the results in §3 into those for the cmKdV equations. The last section, §5, is devoted to the concluding remarks.

§2. General Formulation

2.1 Normalization

We introduce an $M$-component vector field $u$ and a constant $M \times M$ matrix $G$,

$$u = (u_0, u_1, \cdots, u_{M-1})^T, \quad G = (-C_{ij}),$$

where the symbol $T$ means the transposition. Using this notation, a system of the cmKdV equations (1.1) is expressed as

$$u_t - 6(u^T Gu)u_x + u_{xxx} = 0.$$  \hspace{1cm} (2.2)

We assume that $G$ is a real symmetric and regular matrix in what follows. Because a real symmetric matrix is diagonalized by a real orthogonal matrix, we can put

$$G = P^T \Lambda P, \quad P^T P = PP^T = I,$$

$$\Lambda = \text{diag}(\lambda_0, \cdots, \lambda_{M-1}), \quad \lambda_j \neq 0.$$  \hspace{1cm} (2.3a, 2.3b)

Thus, defining a new set of dependent variables $v = (v_0, v_1, \cdots, v_{M-1})^T$ as

$$v = Pu,$$

eq. (2.2) is cast into

$$v_t - 6(v^T \Lambda v)v_x + v_{xxx} = 0,$$

or more explicitly

$$\frac{\partial v_i}{\partial t} - 6 \left( \sum_{j=0}^{M-1} \lambda_j v_j^2 \right) \frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad \lambda_j \neq 0, \quad i = 0, 1, \cdots, M - 1.$$  \hspace{1cm} (2.6)
If we change a scale of $v_i$ by $\sqrt{|\lambda|} \cdot v_i \rightarrow v_i$, we finally obtain normalized cmKdV equations,
\[
\frac{\partial v_i}{\partial t} - 6 \left( \sum_{j=0}^{M-1} \varepsilon_j v_j^2 \right) \frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad \varepsilon_j = \text{sgn}(\lambda_j) = \pm 1, \quad i = 0, 1, \ldots, M - 1. \tag{2.7}
\]
The above equation (2.7) is more convenient than eq. (1.1) to perform the ISM. Thus, we mainly deal with eq. (2.7) as the cmKdV equations in the following. In addition, the dependent variables $\{v_i\}$ are assumed to be real.

2.2 Lax pair

We consider a set of auxiliary linear equations
\[
\Psi_x = U \Psi, \quad \Psi_t = V \Psi. \tag{2.8}
\]
Here $\Psi$ is a $(p + q)$-component vector and $U$, $V$ are $(p + q) \times (p + q)$ matrices. The compatibility condition of eq. (2.8) is given by
\[
U_t - V_x + UV - VU = O. \tag{2.9}
\]
We call $U$, $V$ Lax pair and eq. (2.9) a zero-curvature condition, or Lax equation. We introduce the following form of the Lax pair,
\[
U = i\zeta \begin{bmatrix} -I_1 & O \\ O & I_2 \end{bmatrix} + \begin{bmatrix} O & Q \\ R & O \end{bmatrix}, \tag{2.10}
\]
\[
V = i\zeta^3 \begin{bmatrix} -4I_1 & O \\ O & 4I_2 \end{bmatrix} + \zeta^2 \begin{bmatrix} O & 4Q \\ 4R & O \end{bmatrix} + i\zeta \begin{bmatrix} -2QR & 2Qx \\ -2Rx & 2RQ \end{bmatrix} + \begin{bmatrix} QxR - QRx & -Qxx + 2QQR \\ -RxQ + RQx + 2RQ & 4RQx \end{bmatrix}, \tag{2.11}
\]
where $\zeta$ is the spectral parameter which does not depend on time, $\zeta_t = 0$. $I_1$ and $I_2$ are respectively the $p \times p$ and $q \times q$ unit matrices; $Q$ is a $p \times q$ matrix (made up of $p$ rows and $q$ columns); $R$ is a $q \times p$ matrix.

Substituting eqs. (2.10) and (2.11) into eq. (2.9), we get a set of matrix equations
\[
Q_t + Q_{xxx} - 3Q_x RQ - 3Q R Q_x = O, \tag{2.12a}
\]
\[
R_t + R_{xxx} - 3R_x Q R - 3R R Q_x = O. \tag{2.12b}
\]
Suppose that $R$ is connected with the Hermitian conjugate of $Q$ by
\[
R = \varepsilon Q^1, \quad \varepsilon = \pm 1. \tag{2.13}
\]
Then, eq. (2.12) is reduced to
\[ Q_t + Q_{xxx} - 3\varepsilon(Q_x Q^\dagger Q + QQ^\dagger Q_x) = O, \quad \varepsilon = \pm 1. \] (2.14)

If we restrict \( Q \) to be a real matrix, eq. (2.14) becomes equivalent to what Athorne and Fordy studied. We call eq. (2.12) or eq. (2.14) the matrix mKdV equation. We consider the ISM for eq. (2.14) with \( \varepsilon = -1 \) in §3.

2.3 Lax pair for the matrix NLS equation

We employ another form of \( V \) as
\[ V = i\xi^2 \begin{bmatrix} -2I_1 & O \\ O & 2I_2 \end{bmatrix} + \zeta \begin{bmatrix} O & 2Q \\ 2R & O \end{bmatrix} + i \begin{bmatrix} -QR & Q_x \\ -R_x & RQ \end{bmatrix}. \] (2.15)

Substituting eqs. (2.10) and (2.15) into eq. (2.9), we get a set of matrix equations
\[ iQ_t + Q_{xx} - 2QRQ = O, \] (2.16a)
\[ iR_t - R_{xx} + 2RQR = O. \] (2.16b)

Under the reduction
\[ R = \varepsilon Q^\dagger, \quad \varepsilon = \pm 1, \] (2.17)
eq (2.16) is cast into
\[ iQ_t + Q_{xx} - 2\varepsilon QQ^\dagger Q = O, \quad \varepsilon = \pm 1. \] (2.18)

We call eq. (2.16) or eq. (2.18) the matrix NLS equation. By changing the time dependence of the ISM in §3, we can solve the initial value problem for this system (2.18) with \( \varepsilon = -1 \).

2.4 Conservation laws

In this subsection, we present a systematic method to construct local conservation laws for the matrix mKdV equation and the matrix NLS equation. We start from a special class, \( p = q = n \), of eq. (2.8),
\[ \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \] (2.19)
\[ \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \] (2.20)

where all the entries in eqs. (2.19) and (2.20) are assumed to be \( n \times n \) square matrices. The following is an extension of the method for the \( n = 1 \) case. If we define a square matrix \( \Gamma \) by
\[ \Gamma \equiv \Psi_2 \Psi_1^{-1}, \] (2.21)
we can prove the following relations from eqs. \( 2.9 \), \( 2.19 \) and \( 2.20 \),
\[
\{ \text{tr}(U_{12} \Gamma + U_{11}) \}_t = \{ \text{tr}(V_{12} \Gamma + V_{11}) \}_x,
\]
\[
\Gamma_x = U_{21} + U_{22} \Gamma - \Gamma U_{11} - \Gamma U_{12} \Gamma.
\]
Equation \( 2.23 \) is interpreted as the matrix Riccati equation. Assuming that \( U \) is expressed as eq. \( 2.10 \), we have
\[
U_{11} = -i\zeta I, \quad U_{22} = i\zeta I, \quad U_{12} = Q, \quad U_{21} = R,
\]
where \( Q \) and \( R \) are square matrices in this case. Then eqs. \( 2.22 \) and \( 2.23 \) are cast into the following equations,
\[
\{ \text{tr} (Q \Gamma) \}_t = \{ \text{some function of } \Gamma, Q, R \text{ and } \zeta \}_x, \quad \text{(2.25)}
\]
\[
2i\zeta Q \Gamma = -QR + Q(Q^{-1} \cdot Q \Gamma)_x + (Q \Gamma)^2. \quad \text{(2.26)}
\]
Equation \( 2.25 \) is nothing but a local conservation law. It shows that \( \text{tr}(Q \Gamma) \) is a generating function of the conserved densities. We expand \( Q \Gamma \) with respect to the spectral parameter \( \zeta \) as follows,
\[
Q \Gamma = \sum_{l=1}^{\infty} \frac{1}{(2i\zeta)^l} F_l. \quad \text{(2.27)}
\]
Substituting eq. \( 2.27 \) for eq. \( 2.26 \), we obtain a recursion formula,
\[
F_{l+1} = -\delta_{l,0} QR + Q(Q^{-1} F_l)_x + \sum_{k=1}^{l-1} F_k F_{l-k}, \quad l = 0, 1, \ldots. \quad \text{(2.28)}
\]
Each \( \text{tr } F_l \) is a conserved density for all positive integer \( l \). Using the above formula \( 2.28 \), first four conserved densities are given by
\[
F_1 = -QR, \quad \text{(2.29)}
\]
\[
\text{tr} F_2 = \text{tr}\{-QR_x\}, \quad \text{(2.30)}
\]
\[
\text{tr} F_3 = \text{tr}\{-QR_{xx} + QQR\}, \quad \text{(2.31)}
\]
\[
\text{tr} F_4 = \text{tr}\{-QR_{xxx} + 2QR_xQR + QRQ_xR + 2QRQR_x\}. \quad \text{(2.32)}
\]
It should be noted that all elements of \( F_1 = -QR \) are conserved densities for the matrix mKdV equation \( 2.12 \) and the matrix NLS equation \( 2.16 \). This fact can be proved simply by a direct calculation.

2.5 Hamiltonian structure of the matrix mKdV equation and the matrix NLS equation

In this subsection, we consider the matrix mKdV equation and the matrix NLS equation with the condition that \( Q \) and \( R \) are \( n \times n \) square matrices. The matrix mKdV equation \( 2.12 \) has the following Hamiltonian structure. A set of the Hamiltonian and the Poisson bracket is
\[
H = \text{tr} \left\{ i F_1 \right\} dx = \text{tr} \left\{ -iQR_{xxx} + \frac{3}{2}QR(QR_x - Q_x R) \right\} dx, \quad \text{(2.33)}
\]
\[ \{ Q(x) \otimes Q(y) \} = \{ R(x) \otimes R(y) \} = O, \]  
\[ \{ Q(x) \otimes R(y) \} = i\delta(x - y)\Pi, \]

where \( \{ X \otimes Y \}_{kl}^{ij} = \{ X_{ij}, Y_{kl} \} \) for matrices \( X, Y \) and \( \Pi \) denotes an \( n^2 \times n^2 \) permutation matrix.

For the matrix NLS equation, the Hamiltonian is
\[ H' = \text{tr} \int \{ -F_2 \} dx = \text{tr} \int \{ QR_{xx} - QRQR \} dx, \] instead of eq. (2.33), while the Poisson bracket is the same. We can rewrite Poisson bracket between each element of \( Q \) and \( R \) explicitly as follows,
\[ \{ Q_{ij}(x), Q_{kl}(y) \} = \{ R_{ij}(x), R_{kl}(y) \} = 0, \]
\[ \{ Q_{ij}(x), R_{kl}(y) \} = i\delta_{il}\delta_{jk}\delta(x - y) = i\Pi_{kl}^{ij}\delta(x - y). \]

Indeed, we can check the following relations,
\[ Q_t = \{ Q, H \} = -Q_{xxx} + 3Q_xRQ + 3QRQ_x, \]
\[ R_t = \{ R, H \} = -R_{xxx} + 3R_xQR + 3RQR_x, \]
which are nothing but eq. (2.12). It shows that eq. (2.33) and eq. (2.34) are respectively the Hamiltonian and the Poisson bracket for the matrix mKdV equation. The same thing is true for the matrix NLS equation.

2.6 r-matrix representation of the matrix mKdV equation and the matrix NLS equation

In §2.4, we have shown that the matrix mKdV equation (2.12) and the matrix NLS equation (2.16) have an infinite number of conservation laws. In this subsection, we show that all the integrals of motion are in involution. In the following, we consider the systems with an infinite interval and assume the rapidly decreasing boundary conditions,
\[ Q(x,t), \ R(x,t) \to O \quad \text{as} \ \ x \to \pm \infty. \]  

If we define a classical r-matrix by
\[ r(\zeta_1, \zeta_2) \equiv \frac{1}{2(\zeta_1 - \zeta_2)} \begin{bmatrix} \Pi & 0 & \Pi \\ O & \Pi & O \\ \Pi & O & \Pi \end{bmatrix}, \]  
the following relation
\[ \{ U(x; \zeta_1) \otimes U(y; \zeta_2) \} = \delta(x - y) \begin{bmatrix} r(\zeta_1, \zeta_2), \ U(x; \zeta_1) \otimes \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \ U(x; \zeta_2) \otimes \begin{bmatrix} I & O \\ O & I \end{bmatrix} \end{bmatrix} \]  
(2.41)

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is satisfied. Here \([A, B] \equiv AB - BA\) is a commutator and \(U\) is given by eq. (2.10). By means of eq. (2.41), we obtain a relation for transition matrices:

\[
\{T(x, y; \zeta_1) \otimes T(x, y; \zeta_2)\} = [r(\zeta_1, \zeta_2), \ T(x, y; \zeta_1) \otimes T(x, y; \zeta_2)]
\]

(2.42)

where \(T(x, y; \zeta)\) is the transition matrix defined by

\[
T(x, y; \zeta) = P \exp \left\{ \int_{y}^{x} U(z, \zeta)dz \right\},
\]

(2.43)

with \(P\) being the path ordering.

Taking the traces of both sides of eq. (2.42), we get

\[
\{\log \tau(\zeta_1), \log \tau(\zeta_2)\} = 0,
\]

(2.44)

where \(\tau(\zeta)\) is defined by

\[
\tau(\zeta) = \text{tr} \ T(\infty, -\infty; \zeta),
\]

(2.45)

for the systems with an infinite interval. Expanding (2.44) with respect to the spectral parameters \(\zeta_1\) and \(\zeta_2\), we have the involutiveness of conserved quantities \(\{J_n\}\)

\[
\{J_n, J_m\} = 0.
\]

(2.46)

This fact indicates the complete integrability of the matrix mKdV equation and the matrix NLS equation.

### 2.7 Reduction of the Lax pair for the coupled modified KdV equations

In this subsection, we show a method to reduce the matrix mKdV equation to the cmKdV equations. We recursively define \(2^{m-1} \times 2^{m-1}\) matrices \(Q^{(m)}\) and \(R^{(m)}\) by

\[
Q^{(1)} = \mu_0 v_0 + iv_1, \quad R^{(1)} = \varepsilon_1 (\mu_0 v_0 - iv_1),
\]

(2.47)

\[
Q^{(m+1)} = \begin{bmatrix}
Q^{(m)} & -\varepsilon_{2m+1}(\mu_{2m} v_{2m} + iv_{2m+1})I_{2^{m-1}} \\
-(\mu_{2m} v_{2m} - iv_{2m+1})I_{2^{m-1}} & -R^{(m)}
\end{bmatrix},
\]

(2.48)

\[
R^{(m+1)} = \begin{bmatrix}
R^{(m)} & -\varepsilon_{2m+1}(\mu_{2m} v_{2m} + iv_{2m+1})I_{2^{m-1}} \\
-(\mu_{2m} v_{2m} - iv_{2m+1})I_{2^{m-1}} & -Q^{(m)}
\end{bmatrix}.
\]

(2.49)

Here \(I_{2^{m-1}}\) is the \(2^{m-1} \times 2^{m-1}\) unit matrix. Each \(\varepsilon_i\) is a constant which is either 1 or \(-1\). \(\mu_{2m}\) is a constant that satisfies

\[
\mu_{2m}^2 = \frac{\varepsilon_{2m}}{\varepsilon_{2m+1}} = \varepsilon_{2m\varepsilon_{2m+1}}.
\]

(2.50)

For \(Q^{(m)}\) and \(R^{(m)}\) defined by eqs. (2.47)–(2.49), we can prove a simple relation,

\[
Q^{(m)} R^{(m)} = R^{(m)} Q^{(m)} = \sum_{j=0}^{2m-1} \varepsilon_j v_j^2 \cdot I_{2^{m-1}},
\]

(2.51)
by the induction method. Then substituting $Q^{(m)}$ and $R^{(m)}$ for $Q$ and $R$ in the matrix mKdV equation (2.12), we obtain

$$\frac{\partial v_i}{\partial t} - 6 \left( \sum_{j=0}^{M-1} \varepsilon_j v_j^2 \right) \frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad i = 0, 1, \cdots, M - 1,$$

(2.52)

where we set $M = 2m$.

In the following, we choose

$$\varepsilon_i = -1, \quad i = 0, 1, \cdots, 2m - 1,$$

(2.53)

$$\mu_{2j} = 1, \quad j = 0, 1, \cdots, m - 1,$$

(2.54)

and consider a self-focusing type of the cmKdV equations (2.7), that is,

$$\frac{\partial v_i}{\partial t} + 6 \left( \sum_{j=0}^{M-1} v_j^2 \right) \frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad i = 0, 1, \cdots, M - 1.$$

(2.55)

In this case, the recursion relations of $Q^{(m)}$ and $R^{(m)}$ are

$$Q^{(m+1)} = \begin{bmatrix} Q^{(m)} & (v_{2m} + iv_{2m+1})I_{2m-1} \\ -(v_{2m} - iv_{2m+1})I_{2m-1} & -R^{(m)} \end{bmatrix},$$

(2.57)

$$R^{(m+1)} = \begin{bmatrix} R^{(m)} & (v_{2m} + iv_{2m+1})I_{2m-1} \\ -(v_{2m} - iv_{2m+1})I_{2m-1} & -Q^{(m)} \end{bmatrix}.$$}

(2.58)

We can show that a simple relation between $Q^{(m)}$ and $R^{(m)}$ holds,

$$R^{(m)} = -Q^{(m)\dagger}.$$}

(2.59)

It should be noted that the Hamiltonian and the Poisson bracket for the matrix mKdV equation (2.12) become invalid for the cmKdV equations (2.55) with $M \geq 3$. This is because the degree of freedom of $Q^{(m)}$ and $R^{(m)}$ for the cmKdV equations is less than that for the matrix mKdV equation.

³³. Inverse Scattering Method

In this section we consider the scattering problem associated with $2n \times 2n$ matrix (2.11) under the constraint (2.13) with $\varepsilon = -1$ and the boundary conditions (2.39), that is,

$$\Psi_x = U\Psi, \quad U = \begin{bmatrix} -i\zeta I & Q \\ R & i\zeta I \end{bmatrix}, \quad R = -Q^{\dagger},$$

(3.1)

$$Q, R(-Q^{\dagger}) \to O \quad \text{as} \quad x \to \pm\infty.$$}

(3.2)

The results of this section are applicable to the matrix mKdV equation, the matrix NLS equation and other members of the hierarchy only if $Q$ and $R$ are $n \times n$ square matrices. The main idea in what follows is a modification of the analysis in refs. 13, 20 for the matrix KdV equation.
3.1 Scattering problem

Let \( \Psi(\zeta) \) and \( \Phi(\zeta) \) be solutions of eq. (3.1) composed by \( 2n \) rows and \( n \) columns. We can show that

\[
\frac{d}{dx}\{\Psi^\dagger(\zeta^*)\Phi(\zeta)\} = O. \tag{3.3}
\]

Hence we employ the following definition of \( x \)-independent matrix Wronskian \( W[\Psi, \Phi] \),

\[
W[\Psi, \Phi] \equiv \Psi^\dagger(\zeta^*)\Phi(\zeta). \tag{3.4}
\]

We introduce Jost functions \( \phi, \bar{\phi} \) and \( \psi, \bar{\psi} \) which satisfy the boundary conditions,

\[
\phi \sim \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} \quad \text{as} \quad x \to -\infty, \tag{3.5a}
\]

\[
\bar{\phi} \sim \begin{bmatrix} O \\ -I \end{bmatrix} e^{i\zeta x} \quad \text{as} \quad x \to -\infty, \tag{3.5b}
\]

and

\[
\psi \sim \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x} \quad \text{as} \quad x \to +\infty, \tag{3.5c}
\]

\[
\bar{\psi} \sim \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} \quad \text{as} \quad x \to +\infty. \tag{3.5d}
\]

Here \( O \) and \( I \) are respectively the \( n \times n \) zero matrix and the \( n \times n \) unit matrix. It can be shown that \( \phi e^{i\zeta x}, \psi e^{-i\zeta x} \) are analytic in the upper half plane of \( \zeta \), and \( \bar{\phi} e^{-i\zeta x}, \bar{\psi} e^{i\zeta x} \) are analytic in the lower half plane of \( \zeta \) because \( Q, R \) are assumed to go to \( O \) sufficiently rapidly at \( x \to \pm\infty \). We assume the following integral representation of the Jost functions \( \psi \) and \( \bar{\psi} \),

\[
\psi = \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x} + \int_x^\infty K(x, s)e^{i\zeta s}ds, \tag{3.6}
\]

\[
\bar{\psi} = \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} + \int_x^\infty \bar{K}(x, s)e^{-i\zeta s}ds, \tag{3.7}
\]

where \( K(x, s) \) and \( \bar{K}(x, s) \) are column vectors whose elements are \( n \times n \) square matrices,

\[
K(x, s) = \begin{bmatrix} K_1(x, s) \\ K_2(x, s) \end{bmatrix}, \bar{K}(x, s) = \begin{bmatrix} \bar{K}_1(x, s) \\ \bar{K}_2(x, s) \end{bmatrix}. \tag{3.8}
\]

We substitute eq. (3.4) for eq. (3.1) and get the relations for \( K_1 \) and \( K_2 \),

\[
\lim_{s \to +\infty} \begin{bmatrix} K_1(x, s) \\ K_2(x, s) \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix}, \tag{3.9}
\]

\[-2K_1(x, x) = Q(x), \tag{3.10}\]
\[(\partial_x - \partial_s)K_1(x, s) = Q(x)K_2(x, s) \quad (s > x), \quad (3.11)\]
\[(\partial_x + \partial_s)K_2(x, s) = R(x)K_1(x, s) \quad (s > x). \quad (3.12)\]

Similarly, substituting eq. (3.7) for eq. (3.1), we get for \(\bar{K}_1\) and \(\bar{K}_2\),
\[
\lim_{s \to +\infty} \begin{bmatrix} \bar{K}_1(x, s) \\ \bar{K}_2(x, s) \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix}, \quad (3.13)
\]
\[-2\bar{K}_2(x, x) = R(x), \quad (3.14)\]
\[(\partial_x - \partial_s)\bar{K}_2(x, s) = R(x)\bar{K}_1(x, s) \quad (s > x), \quad (3.15)\]
\[(\partial_x + \partial_s)\bar{K}_1(x, s) = Q(x)\bar{K}_2(x, s) \quad (s > x). \quad (3.16)\]

Because a pair of the Jost functions \(\phi\) and \(\bar{\phi}\), or \(\psi\) and \(\bar{\psi}\) forms a fundamental system of solutions of eq. (3.1), we can set
\[
\phi(x, \zeta) = \bar{\psi}(x, \zeta)A(\zeta) + \psi(x, \zeta)B(\zeta), \quad (3.17a)
\]
\[
\bar{\phi}(x, \zeta) = \bar{\psi}(x, \zeta)\bar{B}(\zeta) - \psi(x, \zeta)\bar{A}(\zeta). \quad (3.17b)
\]

Here the coefficients \(A(\zeta), \bar{A}(\zeta), B(\zeta)\) and \(\bar{B}(\zeta)\) are \(x\)-independent \(n \times n\) matrices and called scattering data.

According to the asymptotic behaviors of the Jost functions (3.5a)–(3.5d), we get
\[
W[\phi, \bar{\phi}] = W[\bar{\phi}, \phi] = W[\psi, \psi] = W[\bar{\psi}, \bar{\psi}] = I, \quad (3.18a)
\]
\[
W[\bar{\phi}, \phi] = W[\bar{\psi}, \psi] = O, \quad (3.18b)
\]
\[
A(\zeta) = W[\bar{\psi}, \phi], \quad (3.18c)
\]
\[
\bar{A}(\zeta) = -W[\psi, \bar{\phi}], \quad (3.18d)
\]
\[
B(\zeta) = W[\psi, \phi], \quad (3.18e)
\]
\[
\bar{B}(\zeta) = W[\bar{\psi}, \bar{\phi}]. \quad (3.18f)
\]

The expressions (3.18a)–(3.18f) show that \(A(\zeta)\) and \(\bar{A}(\zeta)\) are analytic respectively in the upper half plane and in the lower half plane. Using the above relations (3.18a)–(3.18f), we obtain the following relations among \(A(\zeta), \bar{A}(\zeta), B(\zeta)\) and \(\bar{B}(\zeta)\),
\[
A^1(\zeta^*)A(\zeta) + B^1(\zeta^*)B(\zeta) = I, \quad (3.19a)
\]
\[
\bar{A}^1(\zeta^*)\bar{A}(\zeta) + \bar{B}^1(\zeta^*)\bar{B}(\zeta) = I, \quad (3.19b)
\]
\[
A^1(\zeta^*)\bar{B}(\zeta) - B^1(\zeta^*)\bar{A}(\zeta) = O. \quad (3.19c)
\]

These relations are written as
\[
\begin{bmatrix} A^1(\zeta^*) & B^1(\zeta^*) \\ \bar{B}^1(\zeta^*) & -\bar{A}^1(\zeta^*) \end{bmatrix} \begin{bmatrix} A(\zeta) & \bar{B}(\zeta) \\ B(\zeta) & -\bar{A}(\zeta) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad (3.20)
\]
which leads to the inversion of eq. (3.17):

\[
\bar{\psi}(x, \zeta) = \phi(x, \zeta) A^\dagger(\zeta^*) + \tilde{\phi}(x, \zeta) \tilde{B}^\dagger(\zeta^*), \tag{3.21a}
\]

\[
\psi(x, \zeta) = \phi(x, \zeta) B^\dagger(\zeta^*) - \tilde{\phi}(x, \zeta) \tilde{A}^\dagger(\zeta^*). \tag{3.21b}
\]

### 3.2 Gel’fand-Levitan-Marchenko equation

To derive the formula of the ISM concisely, we assume that \( A(\zeta), \tilde{A}(\zeta), B(\zeta) \) and \( \tilde{B}(\zeta) \) are entire functions. This assumption is true if the potentials \( Q \) and \( R \) decrease faster than any exponential function at \( x \to \pm \infty \). The result is, however, applicable to larger classes of potentials \( Q \) and \( R \).

Multiplying \( A(\zeta) - 1 \) and \( \tilde{A}(\zeta) - 1 \) from the right to eqs. (3.17a) and (3.17b) respectively, we get

\[
\phi(x, \zeta) A(\zeta) - 1 = \tilde{\psi}(x, \zeta) + \psi(x, \zeta) B(\zeta) A(\zeta)^{-1}, \tag{3.22a}
\]

\[
\tilde{\phi}(x, \zeta) \tilde{A}(\zeta) - 1 = -\psi(x, \zeta) + \tilde{\psi}(x, \zeta) \tilde{B}(\zeta) \tilde{A}(\zeta)^{-1}. \tag{3.22b}
\]

We operate

\[
\frac{1}{2\pi} \int_C d\zeta e^{i\zeta y} \quad (y > x)
\]

to eq. (3.22a), where \( C \) is a semi-circle contour from \(-\infty + i0^+\) to \(+\infty + i0^+\) passing above all poles of \( \{\det A(\zeta)^{-1}\} \). After a standard calculation, we get the Gel’fand-Levitan-Marchenko equation,

\[
\tilde{K}(x, y) + \begin{bmatrix} O & I \end{bmatrix} F(x + y) + \int_x^\infty K(x, s) F(s + y) ds = \begin{bmatrix} O & I \end{bmatrix} \quad (y > x), \tag{3.24}
\]

where \( F(x) \) is defined by

\[
F(x) = \frac{1}{2\pi} \int_C d\zeta e^{i\zeta x} B(\zeta) A(\zeta)^{-1}. \tag{3.25}
\]

We remark that \( A(\zeta)^{-1} \) is given by

\[
A(\zeta)^{-1} = \frac{1}{\det A(\zeta)} \tilde{A}(\zeta), \tag{3.26}
\]

where \( \tilde{A} \) is the cofactor matrix of \( A \). We assume that \( 1/\det A(\zeta) \) has \( N \) isolated simple poles \( \{\zeta_1, \zeta_2, \ldots, \zeta_N\} \) in the upper half plane and is regular on the real axis. Each of these poles determines one bound state. Then, by the use of the residue theorem, we get an alternative expression of \( F \),

\[
F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta e^{i\zeta x} B(\zeta) A(\zeta)^{-1} - i \sum_{j=1}^{N} C_j e^{i\zeta_j x}. \tag{3.27}
\]

Here \( C_j \) is the residue matrix of \( B(\zeta) A(\zeta)^{-1} \) at \( \zeta = \zeta_j \).

Similarly, we operate

\[
\frac{1}{2\pi} \int_{\bar{C}} d\zeta e^{-i\zeta y} \quad (y > x)
\]

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to eq. (3.22b), where $\bar{C}$ is a semi-circle contour from $-\infty + i0^-$ to $+\infty + i0^-$ passing below all poles of $\{\det \bar{A}(\zeta)\}^{-1}$. We get the counterpart of the Gel’fand-Levitan-Marchenko equation,

$$K(x, y) - \begin{bmatrix} I & O \\ O & O \end{bmatrix} \tilde{F}(x + y) - \int_x^\infty \tilde{K}(x, s) \tilde{F}(s + y) ds = \begin{bmatrix} O \\ O \end{bmatrix} \quad (y > x), \quad (3.29)$$

where $\tilde{F}(x)$ is defined by

$$\tilde{F}(x) = \frac{1}{2\pi} \int_C d\zeta e^{-i\zeta x} \bar{B}(\zeta) \bar{A}(\zeta)^{-1}. \quad (3.30)$$

If we assume that $1/\det \bar{A}(\zeta)$ has $\bar{N}$ isolated simple poles $\{\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_\bar{N}\}$ in the lower half plane and is regular on the real axis, we get an alternative expression of $\tilde{F}$,

$$\tilde{F}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty d\xi e^{-i\xi x} \bar{B}(\xi) \bar{A}(\xi)^{-1} + i \sum_{k=1}^{\bar{N}} \bar{C}_k e^{-i\bar{\zeta}_k x}. \quad (3.31)$$

Here $\bar{C}_k$ is the residue matrix of $\bar{B}(\zeta) \bar{A}(\zeta)^{-1}$ at $\zeta = \bar{\zeta}_k$.

### 3.3 Time-dependence of the scattering data

Under the rapidly decreasing boundary conditions (3.2), the asymptotic form of the Lax matrix $V$ is given by

$$V \to \begin{bmatrix} -4i\zeta^3 I & O \\ O & 4i\zeta^3 I \end{bmatrix} \quad \text{as } x \to \pm \infty. \quad (3.32)$$

We define time-dependent Jost functions by

$$\phi(t) \equiv e^{-4i\zeta^3 t} \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x - 4i\zeta^3 t} \quad \text{as } x \to -\infty, \quad (3.33a)$$

$$\bar{\phi}(t) \equiv \bar{e}^{4i\zeta^3 t} \begin{bmatrix} O \\ -I \end{bmatrix} e^{i\zeta x + 4i\zeta^3 t} \quad \text{as } x \to -\infty, \quad (3.33b)$$

$$\psi(t) \equiv \psi e^{4i\zeta^3 t} \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x + 4i\zeta^3 t} \quad \text{as } x \to +\infty, \quad (3.33c)$$

$$\bar{\psi}(t) \equiv \bar{\psi} e^{-4i\zeta^3 t} \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x - 4i\zeta^3 t} \quad \text{as } x \to +\infty. \quad (3.33d)$$

From the relations

$$\frac{\partial \phi(t)}{\partial t} = V \phi(t), \quad \frac{\partial \bar{\phi}(t)}{\partial t} = V \bar{\phi}(t), \quad (3.34)$$

we get

$$\frac{\partial \phi}{\partial t} = (V + 4i\zeta^3 I)\phi, \quad \frac{\partial \bar{\phi}}{\partial t} = (V - 4i\zeta^3 I)\bar{\phi}. \quad (3.35)$$

We substitute the definitions of the scattering data,

$$\phi(x, \zeta) = \bar{\psi}(x, \zeta) A(\zeta, t) + \psi(x, \zeta) B(\zeta, t), \quad (3.36a)$$
\[
\bar{\psi}(x, \zeta) = \bar{\psi}(x, \zeta) \bar{B}(\zeta, t) - \psi(x, \zeta) \bar{A}(\zeta, t),
\] (3.36b)

into eq. (3.35). Then taking the limit \(x \to +\infty\), we get

\[
A_t(\zeta, t) = O,
\] (3.37a)

\[
B_t(\zeta, t) = 8i\zeta^3 B(\zeta, t),
\] (3.37b)

and

\[
\bar{A}_t(\zeta, t) = O,
\] (3.37c)

\[
\bar{B}_t(\zeta, t) = -8i\zeta^3 \bar{B}(\zeta, t).
\] (3.37d)

The above relations lead to the following time-dependence of the scattering data:

\[
A(\zeta, t) = A(\zeta, 0),
\] (3.38a)

\[
B(\zeta, t) = B(\zeta, 0)e^{8i\zeta^3 t},
\] (3.38b)

and

\[
\bar{A}(\zeta, t) = \bar{A}(\zeta, 0),
\] (3.38c)

\[
\bar{B}(\zeta, t) = \bar{B}(\zeta, 0)e^{-8i\zeta^3 t}.
\] (3.38d)

By eqs. (3.38a)–(3.38d), the time-dependences of \(BA^{-1}, C_j\) and \(\bar{B}\bar{A}^{-1}, \bar{C}_k\) are respectively given by

\[
B(\xi, t) A(\xi, t)^{-1} = B(\xi, 0) A(\xi, 0)^{-1} e^{8i\xi^3 t},
\] (3.39)

\[
C_j(t) = C_j(0)e^{8i\zeta_j^3 t},
\] (3.40)

and

\[
\bar{B}(\xi, t) \bar{A}(\xi, t)^{-1} = \bar{B}(\xi, 0) \bar{A}(\xi, 0)^{-1} e^{-8i\xi^3 t}.
\] (3.41)

\[
\bar{C}_k(t) = \bar{C}_k(0)e^{-8i\bar{\zeta}_k^3 t}.
\] (3.42)

To summarize, we obtain explicit time-dependent forms of \(F(x, t)\) and \(\bar{F}(x, t)\),

\[
F(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\zeta_x + 8i\zeta^3 t} B(\xi, 0) A(\xi, 0)^{-1} - \sum_{j=1}^{N} C_j(0)e^{i\zeta_j x + 8i\zeta_j^3 t},
\] (3.43)

\[
\bar{F}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{-i\zeta_x - 8i\zeta^3 t} \bar{B}(\xi, 0) \bar{A}(\xi, 0)^{-1} + \sum_{k=1}^{\bar{N}} \bar{C}_k(0)e^{-i\bar{\zeta}_k x - 8i\bar{\zeta}_k^3 t}.
\] (3.44)
3.4 Initial value problem

Because of the constraint \( R = -Q^\dagger \), we have some relations which make the further analysis simple.

First, we have

\[
\det \tilde{A}(\zeta) = \{ \det A(\zeta^*) \}^*,
\]

which is proved in Appendix A. This relation gives us a useful information about the total number and the positions of the poles of \( A(\zeta)^{-1} \) and \( \tilde{A}(\zeta)^{-1} \),

\[
\tilde{N} = N, \quad \tilde{\zeta}_k = \zeta_k^*.
\]

Second, due to eq. (3.19c), we have

\[
\tilde{B}(\zeta) \tilde{A}(\zeta)^{-1} = \{ \tilde{B}(\zeta^*) A(\zeta^*)^{-1} \}^\dagger,
\]

which leads to

\[
\tilde{B}(\xi) \tilde{A}(\xi)^{-1} = \{ \tilde{B}(\xi) A(\xi)^{-1} \}^\dagger (\xi : \text{real}),
\]

\[
\tilde{C}_k = C_k^*.
\]

The relations (3.46) and (3.48) give a connection between \( \tilde{F}(x,t) \) and \( F(x,t) \),

\[
\tilde{F}(x,t) = F(x,t)^\dagger.
\]

Combining the above results, we arrive at

\[
K_1(x,y;t) = F(x+y,t)^\dagger - \int_x^\infty ds_1 \int_x^\infty ds_2 K_1(x,s_2; t) F(s_2 + s_1, t) F(s_1 + y, t)^\dagger,
\]

\[
K_2(x,y;t) = -F(x+y,t) - \int_x^\infty ds_1 \int_x^\infty ds_2 K_2(x,s_2; t) F(s_2 + s_1, t)^\dagger F(s_1 + y, t),
\]

where \( F(x,t) \) is given by eq. (3.43).

We can solve the initial value problem of the matrix mKdV equation as follows.

(1) For given potentials at \( t = 0 \), \( Q(x,0) \) and \( R(x,0) \) which satisfy \( R(x,0) = -Q(x,0)^\dagger \), we solve the scattering problem (3.1), and obtain scattering data \( \{ B(\xi) A(\xi)^{-1}, \zeta_j, iC_j \} \).

(2) The time-dependence of the scattering data is determined by eqs. (3.39) and (3.40).

(3) We substitute the time-dependent scattering data into the Gel’fand-Levitan-Marchenko equations (3.50) and (3.51). Solving the equations, we reconstruct the time-dependent potentials,

\[
Q(x,t) = -2K_1(x,x;t),
\]

\[
R(x,t) = -2\tilde{K}_2(x,x;t).
\]

In this way, we obtain the solution \( Q(x,t) \) and \( R(x,t) \) from the initial condition \( Q(x,0) \) and \( R(x,0) \). This procedure proves directly the complete integrability of the matrix mKdV equation (2.14) with \( \varepsilon = -1 \).
If we employ other time-dependences of the scattering data, for instance,

\[ B(\xi, t)A(\xi, t)^{-1} = B(\xi, 0)A(\xi, 0)^{-1}e^{4i\xi^2t}, \]

(3.54)

\[ C_j(t) = C_j(0)e^{4i\xi_j^2t}, \]

(3.55)
the initial value problem of the matrix NLS equation (2.18) with \( \varepsilon = -1 \) can be solved.

As for the constraint \( R = -Q^\dagger \), one comment is in order. Because \( \bar{F} \) is connected with \( F \) by eq. (3.49), we can prove by the Neumann-Liouville expansion (see Appendix B) that the solution of eqs. (3.50) and (3.51) satisfies

\[ \bar{K}_2(x, x; t) = -K_1(x, x; t)^\dagger. \]

(3.56)

This relation assures that the relation \( R(x, t) = -Q(x, t)^\dagger \) holds at any time \( t \).

### 3.5 Soliton solutions

Assuming the reflection-free potentials which satisfy

\[ B(\xi) = \bar{B}(\xi) = O \quad (\xi : \text{real}), \]

(3.57)
we can construct soliton solutions of the matrix mKdV equation. In this case \( F(x) \) is given by

\[ F(x, t) = -i \sum_{j=1}^{N} C_j(t)e^{i\xi_j x}, \]

(3.58)

\[ C_j(t) = C_j(0)e^{8i\xi_j^3t}. \]

(3.59)
To solve eq. (3.50) with eq. (3.58), we set

\[ K_1(x, y; t) = i \sum_{k=1}^{N} P_k(x, t)C_k(t)^\dagger e^{-i\xi_k^3(x+y)}. \]

(3.60)
Introducing eq. (3.60) into eq. (3.50), we have a set of algebraic equations,

\[ P_k(x, t) - \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(\xi_j - \xi_k^*)(\xi_j - \xi_i^*)} P_l(x, t)C_l(t)^\dagger C_j(t)e^{2i(\xi_j - \xi_l^*)x} = I. \]

(3.61)

We define a matrix \( S \) by

\[ S_{lk} = \delta_{lk}I - \sum_{j=1}^{N} \frac{e^{2i(\xi_j - \xi_l^*)x}}{(\xi_j - \xi_k^*)(\xi_j - \xi_l^*)} C_l(t)^\dagger C_j(t), \quad 1 \leq l, k \leq N. \]

(3.62)
Then eq. (3.61) is expressed as

\[
\begin{pmatrix}
P_1 & P_2 & \cdots & P_N
\end{pmatrix}
\begin{pmatrix}
S_{11} & \cdots & S_{1N} \\
\vdots & \ddots & \vdots \\
S_{N1} & \cdots & S_{NN}
\end{pmatrix}
= \begin{pmatrix}
I & \cdots & I
\end{pmatrix}_{N}.
\]

(3.63)
Thus the $N$-soliton solution of the matrix mKdV equation (2.14) with $\varepsilon = -1$ is given by

$$Q(x, t) = -2K_1(x, x; t)$$

$$= -2i \sum_{k=1}^{N} P_k(x, t) C_k(t) \hat{e}^{-2i\zeta_k x}$$

$$= -2i \left( \prod_{N}I \right) S^{-1} \begin{pmatrix} C_1(t) \hat{e}^{-2i\zeta_1 x} \\ C_2(t) \hat{e}^{-2i\zeta_2 x} \\ \vdots \\ C_N(t) \hat{e}^{-2i\zeta_N x} \end{pmatrix}. \tag{3.64}$$

As a special case $N = 1$, we get 1-soliton solution of the matrix mKdV equation,

$$Q(x, t) = -2i \left\{ I - \frac{e^{8i(\zeta_1^3 - \zeta_1^3) t}}{(\zeta_1 - \zeta_1^3)^2} C_1(0) \hat{e}^{2i(\zeta_1 - \zeta_1^3) x} \right\}^{-1} C_1(0) \hat{e}^{2i(\zeta_1 - \zeta_1^3) x - 8i(\zeta_1^3 - \zeta_1^3) t}$$

$$= -2i \left\{ e^{-i(\zeta_1 - \zeta_1^3) x - 4i(\zeta_1^3 - \zeta_1^3) t} I - \frac{1}{(\zeta_1 - \zeta_1^3)^2} C_1(0) \hat{e}^{i(\zeta_1 - \zeta_1^3) x + 4i(\zeta_1^3 - \zeta_1^3) t} \right\}^{-1} C_1(0) \hat{e}^{-i(\zeta_1 + \zeta_1^3) x - 4i(\zeta_1^3 + \zeta_1^3) t}. \tag{3.65}$$

If we replace the time dependence (3.59) in eq. (3.58) with eq. (3.55), we obtain the $N$-soliton solution of the matrix NLS equation (2.18) with $\varepsilon = -1$.

§4. Reduction of the ISM for the Coupled Modified KdV Equations

In order to make the ISM in §3 applicable to the cmKdV equations, we have to take into account the internal symmetry of $Q$ and $R$ defined by eqs. (2.56)–(2.58).

If we set $Q^{(m)}$ and $R^{(m)}$ for $m \geq 2$ as

$$Q^{(m)} = v_0 I + \sum_{k=1}^{2m-1} v_k e_k, \quad R^{(m)} = -v_0 I + \sum_{k=1}^{2m-1} v_k e_k, \tag{4.1}$$

the following important relations for $2^{m-1} \times 2^{m-1}$ matrices $\{e_i\}$ hold,

$$\{e_i, e_j\}_+ = -2 \delta_{ij} I, \tag{4.2}$$

$$e_k^\dagger = -e_k, \tag{4.3}$$

$$\text{tr} e_k = 0. \tag{4.4}$$

Here $\{\cdot, \cdot\}_+$ denotes the anti-commutator. $I$ is the $2^{m-1} \times 2^{m-1}$ unit matrix. Equation (4.2) leads to

$$\text{tr}(e_i e_j) = -2^{m-1} \delta_{ij}. \tag{4.5}$$

The results in §2.4 assures that the cmKdV equations have an infinite number of conservation laws. From the explicit forms of the conserved quantities, we find that the first four conserved
densities for the original cmKdV equations (4.1) are given by
\[ \sum_{j,k} C_{jk} u_j u_k, \] (4.6)
\[ u_{jk}, \quad \forall j, k \quad (j \neq k), \] (4.7)
\[ \left( \sum_{j,k} C_{jk} u_j u_k \right)^2 - \sum_{j,k} C_{jk} u_{jk,x} u_{jk,x}, \] (4.8)
\[ \left( \sum_{j,k} C_{jk} u_j u_k \right)^3 - 3 \sum_{j,k} C_{jk} u_j u_k \cdot \sum_{j,k} C_{jk} u_{jk,x} u_{jk,x} + \frac{1}{2} \sum_{j,k} C_{jk} u_{jk,xx} u_{jk,xx} - \frac{1}{2} \left\{ \left( \sum_{j,k} C_{jk} u_j u_k \right)_x \right\}^2. \] (4.9)

We remark that the method in §2.4 does not give the quantity (4.7).

Next, we discuss the initial value problem and the soliton solutions of the cmKdV equations. Considering the scattering problem (3.1) with the potentials \( Q^{(m)} \) and \( R^{(m)} \) for \( m \geq 2 \), we can show that there are following restrictions on the scattering data.

**Proposition 4.1**

1. The determinant of \( A(\zeta) \) satisfies
   \[ \det A(\zeta) = \{ \det A(-\zeta^*) \}^*, \] (4.10)
   as a function of complex \( \zeta \). Thus the poles of \( 1/\det A(\zeta) \) in the upper half plane should appear on the imaginary axis or as pairs which are situated symmetric with respect to the imaginary axis. Therefore, we can set the values of \( 2N \) poles as
   \[ \zeta_{2j-1} = \xi_j + i\eta_j, \quad j = 1, 2, \cdots, N, \] (4.11a)
   \[ \zeta_{2j} = -\zeta_{2j-1}^* = -\xi_j + i\eta_j, \quad j = 1, 2, \cdots, N, \] (4.11b)
   for \( \eta_j > 0 \). The condition (4.11) should be interpreted as follows; if \( \zeta_i \) is pure imaginary, it does not need its counterpart.

2. The reflection coefficient \( B(\xi)A(\xi)^{-1} \) for real \( \xi \) should be expressed as
   \[ B(\xi)A(\xi)^{-1} = r^{(0)} \mathbb{I} + \sum_{k=1}^{2m-1} r^{(k)} e_k. \] (4.12)
   Here \( r^{(0)} \) and \( r^{(k)} \) are complex functions of \( \xi, t \) which satisfy
   \[ r^{(0)}(-\xi) = r^{(0)}(\xi)^*, \quad r^{(k)}(-\xi) = r^{(k)}(\xi)^*. \] (4.13)

3. The residue matrices \( \{ C_1, C_2, \cdots, C_{2N-1}, C_{2N} \} \) should be expressed as
   \[ iC_{2j-1} = c_j^{(0)} \mathbb{I} + \sum_{k=1}^{2m-1} c_j^{(k)} e_k, \quad j = 1, 2, \cdots, N, \] (4.14a)
the time-dependences of \( r^0 \), \( r^k \) and \( c^0_j \), \( c^k_j \) are complex constants. For example, \( \{ C_1, C_2, \cdots, C_{2N} \} \) for the 4-component cmKdV equations are given by

\[
\begin{align*}
\alpha_j & \quad \beta_j \\
-\gamma_j & \quad \delta_j
\end{align*}
\]

\[
iC_{2j-1} = \begin{bmatrix} \alpha_j & \beta_j \\ -\gamma_j & \delta_j \end{bmatrix},
\]

\[
iC_{2j} = \begin{bmatrix} \delta^*_j & \gamma^*_j \\ -\beta^*_j & \alpha^*_j \end{bmatrix},
\]

in a different notation. A proof of the statements is given in Appendix C.

Considering the above conditions, we have explicit expressions of \( F \) and \( \tilde{F} \) in terms of \( \mathbf{I} \) and \( e_k \),

\[
F(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\xi, t)A(\xi, t)^{-1} e^{i\xi x} d\xi - i \sum_{j=1}^{2N} C_j(t)e^{i\xi_j x}
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} \left\{ \left( r^{(0)}e^{i\xi x} + r^{(0)*}e^{-i\xi x} \right)\mathbf{I} + \sum_{k=1}^{2m-1} \left( r^{(k)}e^{i\xi x} + r^{(k)*}e^{-i\xi x} \right)e_k \right\} d\xi
\]

\[
- \sum_{j=1}^{N} \left\{ \left( c^{(0)}_j e^{i\xi_j x} + c^{(0)*}_j e^{-i\xi_j x} \right)\mathbf{I} + \sum_{k=1}^{2m-1} \left( c^{(k)}_j e^{i\xi_j x} + c^{(k)*}_j e^{-i\xi_j x} \right)e_k \right\},
\]

\[
\tilde{F}(x, t) = F(x, t)^\dagger
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{B}(\xi, t)\bar{A}(\xi, t)^{-1} e^{-i\xi x} d\xi + i \sum_{j=1}^{2N} C_j(t)^\dagger e^{-i\xi_j x}
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} \left\{ \left( r^{(0)*}e^{i\xi x} + r^{(0)}e^{-i\xi x} \right)\mathbf{I} - \sum_{k=1}^{2m-1} \left( r^{(k)*}e^{i\xi x} + r^{(k)}e^{-i\xi x} \right)e_k \right\} d\xi
\]

\[
- \sum_{j=1}^{N} \left\{ \left( c^{(0)*}_j e^{i\xi_j x} + c^{(0)}_j e^{-i\xi_j x} \right)\mathbf{I} - \sum_{k=1}^{2m-1} \left( c^{(k)*}_j e^{i\xi_j x} + c^{(k)}_j e^{-i\xi_j x} \right)e_k \right\}.
\]

Because \( B(\xi, t)A(\xi, t)^{-1} \) and \( C_j(t) \) depend on \( t \) as

\[
B(\xi, t)A(\xi, t)^{-1} = B(\xi, 0)A(\xi, 0)^{-1} e^{8i\xi^3 t},
\]

\[
C_j(t) = C_j(0)e^{8i\xi_j^3 t},
\]

the time-dependences of \( r^{(0)} \), \( r^{(k)} \) and \( c^{(0)}_j \), \( c^{(k)}_j \) are given by

\[
r^{(0)}(\xi, t) = r^{(0)}(\xi, 0)e^{8i\xi^3 t}, \quad r^{(k)}(\xi, t) = r^{(k)}(\xi, 0)e^{8i\xi^3 t},
\]

\[
c^{(0)}_j(t) = c^{(0)}_j(0)e^{8i\xi_j^3 t}, \quad c^{(k)}_j(t) = c^{(k)}_j(0)e^{8i\xi_j^3 t}.
\]
It should be noted that $F(x,t)$ and $\bar{F}(x,t)$ are expressed as

\begin{align}
F(x,t) &= f^{(0)}(x,t) \mathbb{I} + \sum_{k=1}^{2m-1} f^{(k)}(x,t) e_k, \\
\bar{F}(x,t) &= f^{(0)}(x,t) \mathbb{I} - \sum_{k=1}^{2m-1} f^{(k)}(x,t) e_k,
\end{align}

where the real functions $f^{(0)}(x,t)$ and $f^{(k)}(x,t)$ satisfy

\begin{align}
(\partial_t + \partial_{xxx}) f^{(0)}(2x,t) &= 0, \\
(\partial_t + \partial_{xxx}) f^{(k)}(2x,t) &= 0.
\end{align}

(4.26)

Taking into account the conditions (4.11)–(4.14), we can advance the analysis in parallel with the discussion in §3 for the matrix mKdV equation. Thus the initial value problem of the cmKdV equations can be solved by the ISM, as has been shown in §3.

We replace $N$ in §3 by $2N$ and find the $N$-soliton solution of the cmKdV equations (2.55) with $M = 2m$ components,

\begin{equation}
Q^{(m)}(x,t) = -2K_1(x,x; t) = -2i \sum_{k=1}^{2N} P_k(x,t) C_k(t) e^{-2i\zeta_k^* x},
\end{equation}

\begin{equation}
= -2i \left( \begin{array}{cccc}
I & I & \cdots & I
\end{array} \right)_{2N} S^{-1} \left( \begin{array}{c}
C_1(t) e^{-2i\zeta_1^* x} \\
C_2(t) e^{-2i\zeta_2^* x} \\
\vdots \\
C_{2N}(t) e^{-2i\zeta_{2N}^* x}
\end{array} \right),
\end{equation}

(4.27)

where the matrix $S$ is given by

\begin{equation}
S_{lk} \equiv \delta_{lk} I - \sum_{j=1}^{2N} \frac{e^{2i(\zeta_j^* - \zeta_l^*) x}}{(\zeta_j - \zeta_k^*)(\zeta_j - \zeta_l^*)} C_l(t)^\dagger C_j(t), \quad 1 \leq l, k \leq 2N.
\end{equation}

(4.28)

It is not evident whether eq. (4.27) can be expressed as

\begin{equation}
Q^{(m)}(x,t) = v_0(x,t) \mathbb{I} + \sum_{k=1}^{2m-1} v_k(x,t) e_k,
\end{equation}

(4.29)

without using $e_i e_j, e_i e_j e_k,$ etc. But, noting the fact that summations and products of real quaternions are real quaternions, this can be proved for $m = 2$ (4-component cmKdV equations) by using Neumann-Liouville expansion (see Appendix B). It is an open problem to prove eq. (4.29) for general $M = 2m$.

§5. Concluding Remarks

In this paper, we have constructed an extension of the ISM to solve the matrix mKdV equation and the matrix NLS equation. We get the coupled mKdV equations (2.7) as a reduction of the
matrix mKdV equation. Through the extension, we have shown that the coupled mKdV equations have an infinite number of conservation laws and multi-soliton solutions and that its initial value problem is solvable.

The existence of conserved quantities for the coupled mKdV equations has been proved by Svinolupov in a different approach. Iwao and Hirota obtained Pfaffian representation of N-soliton solution for the model by means of so-called Hirota’s method. We stress that the initial value problem of the coupled mKdV equations has been solved in the present paper for the first time. In addition, it directly proves the complete integrability of the model. Our scheme enables us to construct more general solution than the already known solutions.

We can transform the coupled mKdV equations to a new integrable coupled version of the Hirota equation, by changes of dependent variables $v_j$ and independent variables $x, t$. The new coupled Hirota equations describe interactions among different modes in optical fibers and seem to be physically significant. Wide applicability of our extension of the ISM will be reported in the succeeding papers. Further, the problem of integrable boundaries at $x = 0$ for the coupled mKdV equations will be studied elsewhere.

After completing writing the paper, the authors were informed by Hisakado that the similar Lax formulation was used in the work of Eichenherr and Pohlmeyer.

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Appendix A: Proof of eq. (3.45)

We begin with a counterpart of the linear equations, 

$$\Psi_x = U \Psi,$$  \hspace{1cm} (A-1)

where $\Psi$ is assumed to be a square matrix. Using this equation, we get a chain of identities,

$$\Psi_x \Psi^{-1} = U, \quad \text{tr} (\log \Psi)_x = \text{tr} U,$$

$$\log \det \Psi)_x = \text{tr} U,$$

$$\det \Psi = \det \Psi(x_0) \cdot e^{\text{tr} \int_{x_0}^x U \, dx}.$$  \hspace{1cm} (A-2)
In case that $U$ is given by

$$
U = \begin{bmatrix}
-i\zeta I & Q \\
R & i\zeta I
\end{bmatrix},
$$

(A.3)

with square matrices $I$, $Q$ and $R$, eq. (A.2) leads to

$$
\det\Psi = \text{const}.
$$

(A.4)

If we take $[\bar{\phi} \bar{\psi}]$ as $\Psi$, we get

$$
\det[\bar{\phi} \bar{\psi}] = \det A^\dagger(\zeta^*) = \det \bar{A}(\zeta),
$$

(A.5)

that means,

$$
\det \bar{A}(\zeta) = \{\det A(\zeta^*)\}^*.
$$

(A.6)

**Appendix B: Neumann-Liouville Expansion**

Due to the Gel’fand-Levitan-Marchenko equations (3.24) and (3.29), we get closed integral equations for $K_1$, $\bar{K}_2$ and $K_2$, $\bar{K}_1$,

$$K_1(x, y) = \bar{F}(x + y) - \int_x^\infty ds_1 \int_x^\infty ds_2 K_1(x, s_2) F(s_2 + s_1) \bar{F}(s_1 + y),
$$

(B.1)

$$\bar{K}_2(x, y) = -F(x + y) - \int_x^\infty ds_1 \int_x^\infty ds_2 \bar{K}_2(x, s_2) F(s_2 + s_1) F(s_1 + y),
$$

(B.2)

$$K_2(x, y) = -\int_x^\infty ds F(x + s) \bar{F}(s + y) - \int_x^\infty ds_1 \int_x^\infty ds_2 K_2(x, s_2) F(s_2 + s_1) \bar{F}(s_1 + y),
$$

(B.3)

$$\bar{K}_1(x, y) = -\int_x^\infty ds \bar{F}(x + s) F(s + y) - \int_x^\infty ds_1 \int_x^\infty ds_2 \bar{K}_1(x, s_2) F(s_2 + s_1) F(s_1 + y).
$$

(B.4)

By successive approximations, we obtain the Neumann-Liouville expansions for $K_1$, $\bar{K}_2$ and $K_2$, $\bar{K}_1$,

$$K_1(x, x) = \bar{F}(2x)
$$

$$- \int_x^\infty ds_1 \int_x^\infty ds_2 \bar{F}(x + s_2) F(s_2 + s_1) \bar{F}(s_1 + x)
$$

$$+ \cdots
$$

$$+ (-1)^n \int_x^\infty ds_1 \int_x^\infty ds_2 \cdots \int_x^\infty ds_{2n} \bar{F}(x + s_{2n}) F(s_{2n} + s_{2n-1}) \bar{F}(s_{2n-1} + s_{2n-2})
$$

$$\cdots F(s_2 + s_1) \bar{F}(s_1 + x)
$$

$$+ \cdots,
$$

(B.5)

$$\bar{K}_2(x, x) = -F(2x)
$$

$$+ \int_x^\infty ds_1 \int_x^\infty ds_2 F(x + s_2) \bar{F}(s_2 + s_1) F(s_1 + x)
$$

$$+ \cdots
$$
\[ (+1)^{n-1} \int_{x}^{\infty} ds_{1} \int_{x}^{\infty} ds_{2} \cdots \int_{x}^{\infty} ds_{2n} F(x + s_{2n}) F(s_{2n} + s_{2n-1}) F(s_{2n-1} + s_{2n-2}) \]
\[ \cdots F(s_{2} + s_{1}) F(s_{1} + x) \]
\[ + \cdots, \]  
(B.6)

\[ K_{2}(x, x) = - \int_{x}^{\infty} ds_{1} F(x + s_{1}) F(s_{1} + x) \]
\[ + \int_{x}^{\infty} ds_{1} \int_{x}^{\infty} ds_{2} \int_{x}^{\infty} ds_{3} F(x + s_{3}) F(s_{3} + s_{2}) F(s_{2} + s_{1}) F(s_{1} + x) \]
\[ + \cdots \]
\[ + (-1)^{n-1} \int_{x}^{\infty} ds_{1} \int_{x}^{\infty} ds_{2} \cdots \int_{x}^{\infty} ds_{2n+1} F(x + s_{2n+1} + s_{2n}) F(s_{2n} + s_{2n-1}) \]
\[ \cdots F(s_{2} + s_{1}) F(s_{1} + x) \]
\[ + \cdots, \]  
(B.7)

\[ K_{1}(x, x) = - \int_{x}^{\infty} ds_{1} F(x + s_{1}) F(s_{1} + x) \]
\[ + \int_{x}^{\infty} ds_{1} \int_{x}^{\infty} ds_{2} \int_{x}^{\infty} ds_{3} F(x + s_{3}) F(s_{3} + s_{2}) F(s_{2} + s_{1}) F(s_{1} + x) \]
\[ + \cdots \]
\[ + (-1)^{n-1} \int_{x}^{\infty} ds_{1} \int_{x}^{\infty} ds_{2} \cdots \int_{x}^{\infty} ds_{2n+1} F(x + s_{2n+1} + s_{2n}) F(s_{2n} + s_{2n-1}) \]
\[ \cdots F(s_{2} + s_{1}) F(s_{1} + x) \]
\[ + \cdots. \]  
(B.8)

**Appendix C: Proof of Proposition 4.1**

We remember \( \Gamma = \Psi_{2} \Psi_{1}^{-1} \) defined in \( \S \) 2.4. The relation (2.23) with eq. (2.24) becomes

\[ \Gamma_{x} = 2i \zeta \Gamma + R^{(m)} - \Gamma Q^{(m)} \Gamma, \]  
(C.1)

for the cmKdV equations. We assume

\[ \lim_{|\zeta| \to \infty} \Gamma = O \quad (\text{Im} \, \zeta > 0), \]  
(C.2)

and expand \( \Gamma \) as

\[ \Gamma = \sum_{l=1}^{\infty} \frac{1}{(2i \zeta)^{l}} G_{l}, \]  
(C.3)

instead of eq. (2.27). Substituting eq. (C.3) for eq. (C.1), we obtain a recursion formula,

\[ G_{l+1} = -d_{0,0} R^{(m)} + (G_{l})_{x} + \sum_{j=1}^{l-1} G_{j} Q^{(m)} G_{l-j}, \quad l = 0, 1, \cdots, \]  
(C.4)
where \( Q^{(m)} \) and \( R^{(m)} \) are given by

\[
Q^{(m)} = v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k, \quad R^{(m)} = -v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k.
\]

We first show the following theorem.

**Theorem C. 1** Let \( X, Y, Z \) be \( 2^{m-1} \times 2^{m-1} \) matrices given by

\[
X = x_0 \mathbb{1} + \sum_{k=1}^{2m-1} x_k e_k, \quad Y = y_0 \mathbb{1} + \sum_{k=1}^{2m-1} y_k e_k, \quad Z = z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k,
\]

where the coefficients \( x_0, x_k; y_0, y_k; z_0, z_k \) are real. Then there exist real numbers \( w_0, w_k \) that satisfy

\[
W \equiv XYZ + ZYX = w_0 \mathbb{1} + \sum_{k=1}^{2m-1} w_k e_k.
\]

**Proof** By the use of eq. (4.2), a direct calculation gives

\[
W = \left( x_0 \mathbb{1} + \sum_{i=1}^{2m-1} x_i e_i \right) \left( y_0 \mathbb{1} + \sum_{j=1}^{2m-1} y_j e_j \right) \left( z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k \right)
+ \left( z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k \right) \left( y_0 \mathbb{1} + \sum_{j=1}^{2m-1} y_j e_j \right) \left( x_0 \mathbb{1} + \sum_{i=1}^{2m-1} x_i e_i \right)
\]

\[
= 2x_0y_0z_0 \mathbb{1} + 2x_0y_0 \sum_{k=1}^{2m-1} z_k e_k + 2x_0z_0 \sum_{j=1}^{2m-1} y_j e_j + 2y_0z_0 \sum_{i=1}^{2m-1} x_i e_i
\]

\[
+ x_0 \left\{ \sum_{j,k=1}^{2m-1} y_j z_k (e_j e_k + e_k e_j) \right\} + y_0 \left\{ \sum_{i,k=1}^{2m-1} x_i z_k (e_i e_k + e_k e_i) \right\} + z_0 \left\{ \sum_{i,j=1}^{2m-1} x_i y_j (e_i e_j + e_j e_i) \right\}
\]

\[
+ \sum_{i,j,k=1}^{2m-1} x_i y_j z_k (e_i e_j e_k + e_j e_k e_i)
\]

\[
= 2x_0y_0z_0 \mathbb{1} + 2x_0y_0 \sum_{k=1}^{2m-1} z_k e_k + 2x_0z_0 \sum_{j=1}^{2m-1} y_j e_j + 2y_0z_0 \sum_{i=1}^{2m-1} x_i e_i
- 2x_0 \sum_{i=1}^{2m-1} y_i z_i \mathbb{1} - 2y_0 \sum_{i=1}^{2m-1} x_i z_i \mathbb{1} - 2z_0 \sum_{i=1}^{2m-1} x_i y_i \mathbb{1}
- 2 \sum_{i=1}^{2m-1} x_i y_i \sum_{k=1}^{2m-1} z_k e_k + 2 \sum_{k=1}^{2m-1} x_k z_k \sum_{j=1}^{2m-1} y_j e_j - 2 \sum_{j=1}^{2m-1} y_j z_j \sum_{i=1}^{2m-1} x_i e_i.
\]

This result shows that \( W \) does not include terms like \( e_i e_j \) or \( e_j e_k e_k \) and can be expressed in the form of eq. (C.7). \( \square \)
Using Theorem C. 1, we can prove by the inductive method that \( \{G_l\} \) are expressed by

\[
G_l = g_l^{(0)} I + \sum_{k=1}^{2m-1} g_l^{(k)} e_k ,
\]

(C.9)

where \( g_l^{(0)} \) and \( g_l^{(k)} \) are real coefficients. Therefore, \( \Gamma \) is given by

\[
\Gamma = \sum_{l=1}^{\infty} \frac{1}{(2i\zeta)^l} \left( g_l^{(0)} I + \sum_{k=1}^{2m-1} g_l^{(k)} e_k \right)
\]

\[
= \sum_{l=1}^{\infty} \frac{g_l^{(0)}}{(2i\zeta)^l} I + \sum_{k=1}^{2m-1} \left( \sum_{l=1}^{\infty} \frac{g_l^{(k)}}{(2i\zeta)^l} \right) e_k
\]

\[
= \gamma^{(0)}(\zeta) I + \sum_{k=1}^{2m-1} \gamma^{(k)}(\zeta) e_k ,
\]

(C.10)

where \( \gamma^{(0)}(\zeta) \) and \( \gamma^{(k)}(\zeta) \) satisfy

\[
\gamma^{(0)}(\zeta) = \{(\gamma^{(0)}(-\zeta^*))^* \}, \quad \gamma^{(k)}(\zeta) = \{(\gamma^{(k)}(-\zeta^*))^* \}.
\]

(C.11)

We recall the asymptotic behavior of the Jost function \( \phi \) at \( x \to \pm \infty \),

\[
\phi \sim \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} \quad \text{as} \quad x \to -\infty ,
\]

(C.12)

\[
\sim \begin{bmatrix} A(\zeta)e^{-i\zeta x} \\ B(\zeta)e^{i\zeta x} \end{bmatrix} \quad \text{as} \quad x \to +\infty .
\]

(C.13)

These relations yield

\[
\lim_{x \to -\infty} \phi_1 e^{i\zeta x} = I, \quad \lim_{x \to +\infty} \phi_1 e^{i\zeta x} = A(\zeta),
\]

(C.14)

where we have defined \( \phi_1 \) and \( \phi_2 \) by

\[
\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.
\]

(C.15)

We easily see that

\[
\lim_{|\zeta| \to \infty} \phi_2 \phi_1^{-1} = O \ (\text{Im} \ \zeta > 0),
\]

(C.16)

then we can replace \( \Gamma \) in eq. (C.10) with \( \phi_2 \phi_1^{-1} \). Thus \( \det A(\zeta) \) is expressed as

\[
\det A(\zeta) = \exp \{ \text{tr} \log A(\zeta) \}
\]

\[
= \exp \left\{ \text{tr} \int_{-\infty}^{\infty} \log (\phi_1 e^{i\zeta x})_x \, dx \right\}
\]

\[
= \exp \left\{ \text{tr} \int_{-\infty}^{\infty} (\phi_1 e^{i\zeta x})_x (\phi_1 e^{i\zeta x})^{-1} \, dx \right\}
\]

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\[ = \exp \left\{ \operatorname{tr} \int_{-\infty}^{\infty} Q^{(m)} \phi_2 \phi_1^{-1} dx \right\} \]
\[ = \exp \left\{ \operatorname{tr} \int_{-\infty}^{\infty} \left( v_0 \gamma^{(0)}(\zeta) - \sum_{k=1}^{2m-1} v_k \gamma^{(k)}(\zeta) \right) I dx \right\} \]
\[ = \exp \left\{ 2^{m-1} \int_{-\infty}^{\infty} \left( v_0 \gamma^{(0)}(\zeta) - \sum_{k=1}^{2m-1} v_k \gamma^{(k)}(\zeta) \right) dx \right\}, \quad \text{(C.17)} \]

where we have used eqs. (4.4) and (4.5). Due to eq. (C.11), \( \det A(\zeta) \), as a function of \( \zeta \), satisfies
\[ \det A(\zeta) = \{ \det A(-\zeta^*) \}^*. \quad \text{(C.18)} \]

This is the proof of Proposition 4.1 (1). Further, we obtain
\[ B(\zeta) A(\zeta)^{-1} = \lim_{x \to +\infty} \phi_2 \phi_1^{-1} e^{-2i\zeta x} \]
\[ = \lim_{x \to +\infty} \left[ \gamma^{(0)}(\zeta) e^{-2i\zeta x} I + \sum_{k=1}^{2m-1} \gamma^{(k)}(\zeta) e^{-2i\zeta x} e_k \right] \]
\[ = r^{(0)}(\zeta) I + \sum_{k=1}^{2m-1} r^{(k)}(\zeta) e_k, \quad \text{(C.19)} \]

with conditions
\[ r^{(0)}(\zeta) = \{ r^{(0)}(-\zeta^*) \}^*, \quad r^{(k)}(\zeta) = \{ r^{(k)}(-\zeta^*) \}^*. \quad \text{(C.20)} \]

Using eqs. (C.19) and (C.20), it is straightforward to prove Proposition 4.1 (2)(3).
Note added in proof—Equation (4.27) includes not only pure soliton solutions but also breather solutions. We should impose appropriate conditions on the arbitrary parameters in the residue matrices, e.g., \( C_{2j-1} \bar{C}_{2j} = \bar{C}_{2j} C_{2j-1} = C_{2j-1} \bar{C}_{2j} = 0 \), or equivalently
\[
\sum_{i=0}^{2m-1} (c_j^{(i)})^2 = 0,
\]
to obtain pure soliton solutions. In this case, the 1-soliton solution for the cmKdV equations (2.55) is given by
\[
Q^{(m)}(x,t) = 2\eta_1 \text{sech}\{2\eta_1 x - 8\eta_1 (\eta_1^2 - 3\xi_1^2)t - x_0\} \left( 2 \sum_{l=0}^{2m-1} |c_1^{(l)}|^2 \right)^{-\frac{1}{2}} \cdot \{-i \bar{C}_1(0)e^{-2i\xi_1 x - 8i\xi_1 (\xi_1^2 - 3\eta_1^2)t} - i \bar{C}_2(0)e^{2i\xi_1 x + 8i\xi_1 (\xi_1^2 - 3\eta_1^2)t} \},
\]
where \( x_0 \) is defined by
\[
e^{-x_0} = 2\eta_1 \left( 2 \sum_{l=0}^{2m-1} |c_1^{(l)}|^2 \right)^{-\frac{1}{2}} .
\]
For instance, \( v_0(x,t) \) (cf. (2.53)) is given by
\[
v_0(x,t) = 2\eta_1 \text{sech}\{2\eta_1 x - 8\eta_1 (\eta_1^2 - 3\xi_1^2)t - x_0\} \left( 2 \sum_{l=0}^{2m-1} |c_1^{(l)}|^2 \right)^{-\frac{1}{2}} \cdot \{c_1^{(0)*}e^{-2i\xi_1 x - 8i\xi_1 (\xi_1^2 - 3\eta_1^2)t} + c_1^{(0)}e^{2i\xi_1 x + 8i\xi_1 (\xi_1^2 - 3\eta_1^2)t} \}.
\]