Twist deformations of Newton-Hooke Hopf algebras

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Abstract

We construct five new quantum Newton-Hooke Hopf algebras with the use of Abelian twist procedure. Further we demonstrate that the corresponding deformed space-times with quantum space and classical time are periodic or expanding in time.
1 Introduction

The two Newton-Hooke cosmological algebras \( NH_\pm \) (acting on commutative space-time) were introduced in [1] with the use of nonrelativistic contraction of the de Sitter and anti-de Sitter groups respectively (see also [2]-[4]). The characteristic time scale \( \tau \) present in both algebras can be interpreted in terms of the inverse of Hubble’s constant for the expanding universe \( (NH_+) \) or associated to the ”period” for the oscillating case \( (NH_-) \). Obviously, for time parameter \( \tau \) approaching infinity one gets the Galilean symmetry acting on the standard (flat) nonrelativistic space-time. The Newton-Hooke symmetries has found an application in nonrelativistic cosmology [5]-[7] as well as in M- and string theory [8].

In this article we discuss the role which can be played by Newton-Hooke symmetries in a context of noncommutative geometry\(^1\). The suggestion to use noncommutative coordinates goes back to Heisenberg and was formalized by Snyder in [9]. Recently, there were also found arguments based on Quantum Gravity [10], [11] and String Theory models [12], [13] indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature.

In our treatment we consider the Abelian (Reshetikhin) twist deformations (see [14], [15]) of the Newton-Hooke Hopf algebras \( U_0(NH_\pm) \). In such a way we get five new quantum groups providing noncommutative \( NH \) (Newton-Hooke) space-times. It should be noted, however, that similar investigation has been already performed in the case of standard relativistic (Poincare) and nonrelativistic (Galilei) groups. Consequently, in accordance with the Hopf-algebraic classification of all deformations of mentioned symmetries, one can distinguish three kinds of quantum spaces [18], [19]:

1) Canonical \( (\theta^{\mu\nu}-\text{deformed}) \) space-time

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu} ; \quad \theta_{\mu\nu} = \text{const},
\]  

introduced in [20], [21] in the case of Poincare quantum group, and in [22], [23] for its Galilean counterpart.

2) Lie-algebraic modification of classical space

\[
[x_\mu, x_\nu] = i\theta^{\rho}_{\mu\nu} x_\rho ,
\]

with particularly chosen coefficients \( \theta^{\rho}_{\mu\nu} \) being constants. This type of noncommutativity has been represented by \( \kappa \)-Poincare [25] and \( \kappa \)-Galilei [26] as well as by twisted relativistic [27] (see also [28]) and nonrelativistic [22], [23] symmetries respectively.

\(^1\)Particularly, we ask about the impact of \( \tau \) parameter on the form of quantum space.

\(^2\)The Newton-Hooke Hopf algebras \( U_0(NH_\pm) \) can be obtained by nonrelativistic contraction of the classical de Sitter and anti-de Sitter quantum groups respectively (see e.g. [16], [17]). They are given by algebraic commutation relations for \( NH_\pm \) groups supplemented by the trivial coproduct sector \( \Delta_0(a) = a \otimes 1 + 1 \otimes a \).
3) Quadratic deformation of Minkowski and Galilei space

\[ [x_\mu, x_\nu] = i\theta^{\rho\tau}_{\mu\nu} x_\rho x_\tau, \]

with coefficients \( \theta^{\rho\tau}_{\mu\nu} \) being constants. This kind of deformation has been proposed in [30], [29], [27] at relativistic and in [23] at nonrelativistic level.

In this article we demonstrate that in the case of Newton-Hooke Hopf algebras \( \mathcal{U}_0(NH_{\pm}) \) the twist deformation provides the new space-time noncommutativity which is expanding \((\mathcal{U}_0(NH_+))\) or periodic \((\mathcal{U}_0(NH_-))\) in time, i.e. it takes the form\(^3\) \(^4\)

\[ [t, x_i] = 0, \quad [x_i, x_j] = f_{\pm} \left( \frac{t}{\tau} \right) \theta_{ij}(x), \]

with time-dependent functions

\[ f_+ \left( \frac{t}{\tau} \right) = f \left( \sinh \left( \frac{t}{\tau} \right), \cosh \left( \frac{t}{\tau} \right) \right), \quad f_- \left( \frac{t}{\tau} \right) = f \left( \sin \left( \frac{t}{\tau} \right), \cos \left( \frac{t}{\tau} \right) \right), \]

and \( \theta_{ij}(x) \sim \theta_{ij} = \text{const} \) or \( \theta_{ij}(x) \sim \theta_{ij}^k x_k \). In such a way we indicate the impact of time scale \( \tau \) on the structure of quantum space, i.e. we show that the time parameter \( \tau \) is responsible for oscillation or expansion of space-time noncommutativity. Of course, for time scale \( \tau \) running to infinity we reproduce the canonical (1), Lie-algebraic (2) and quadratic (3) twisted Galilei space-times provided in [22] and [23].

It should be noted that similar investigations have been already performed in [17], [24] at relativistic level. Particularly, it has been shown that one can get from q-deformed (anti-)de Sitter space-time (containing cosmological constant) the Lie-algebraically \( \kappa \)-deformed Minkowski space. However, it should be also mentioned, that considered in [17] and [24] (anti-)de Sitter space-times have been obtained in different framework in comparison with the technique used in this paper, as the translation sector of corresponding dual Hopf structure.

Finally, let us note that the motivations for present studies are manyfold. First of all, such investigations are interesting because they provide five new explicit quantum groups and the corresponding noncommutative space-times. Besides, it should be noted, that the provided Newton-Hooke Hopf algebras permit to construct the corresponding phase-spaces in the framework of so-called Heisenberg double procedure [14]. Consequently, it permits us to discuss of Heisenberg uncertainty principle [33] associated with such deformed quantum space-times. Finally, one can consider corresponding classical and quantum nonrelativistic particle models. Such a construction has been already presented (see [34], [35]) in the case of classical particle moving in external constant force on the

\(^3x_0 = ct.\)

\(^4\)The mentioned space-times are defined as the Hopf modules of twisted Newton-Hooke Hopf algebras respectively (see e.g. [31], [32], [21]).
space-times (1)-(3). The studies of deformations (4) in a context of basic dynamical models seems quite interesting and is postponed for further investigations.

The paper is organized as follows. In second Section the new twisted Newton-Hooke Hopf algebras are provided. The corresponding deformed space-times with quantum space and classical time are derived in Section 3. The final remarks are presented in the last Section.

2 Twist deformed Newton-Hooke Hopf algebras

In this Section we provide five twisted Newton-Hooke Hopf algebras. All of them are described by the following (Abelian) classical r-matrices:

\[ r_{\alpha_1} = \frac{1}{2} \alpha_{1}^{kl} P_k \wedge P_l \quad \left[ \alpha_{1}^{kl} = -\alpha_{1}^{lk} \right], \]

\[ r_{\alpha_2} = \frac{1}{2} \alpha_{2}^{kl} K_k \wedge P_l \quad \left[ \alpha_{2}^{kl} = -\alpha_{2}^{lk} \right], \]

\[ r_{\alpha_3} = \frac{1}{2} \alpha_{3}^{kl} K_k \wedge K_l \quad \left[ \alpha_{3}^{kl} = -\alpha_{3}^{lk} \right], \]

\[ r_{\alpha_4} = \alpha_4 K_m \wedge M_{kl} \quad \left[ m, k, l - \text{fixed}, \ m \neq k, l \right], \]

\[ r_{\alpha_5} = \alpha_5 P_m \wedge M_{kl} \quad \left[ m, k, l - \text{fixed}, \ m \neq k, l \right], \]

satisfying the classical Yang-Baxter equation (CYBE)

\[ [ [ r, r ] ] = [ r_{12}, r_{13} + r_{23} ] + [ r_{13}, r_{23} ] = 0 , \]

where the symbol \([ [ \cdot, \cdot ] ]\) denotes the Schouten bracket, \(a \wedge b = a \otimes b - b \otimes a\), and for \(r = \sum_i a_i \otimes b_i\)

\[ r_{12} = \sum_i a_i \otimes b_i \otimes 1 , \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i , \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i . \]

In accordance with twist procedure \[14], \[15\] the algebraic sector of all algebras remains classical

\[ [ M_{ab}, M_{cd} ] = i ( \delta_{ad} M_{bc} - \delta_{bd} M_{ac} + \delta_{bc} M_{ad} - \delta_{ac} M_{bd} ) , \quad [ H, P_a ] = \pm \frac{i}{\tau^2} K_a , \]

\[ [ M_{ab}, K_c ] = i ( \delta_{bc} K_a - \delta_{ac} K_b ) , \quad [ M_{ab}, P_c ] = i ( \delta_{bc} P_a - \delta_{ac} P_b ) , \]

\[ [ M_{ab}, P_c ] = i ( \delta_{bc} P_a - \delta_{ac} P_b ) , \]

\[^5^\text{It seems quite interesting to check, for example, the possible connection of such deformed Newton-Hooke symmetries with Modified Newtonian Dynamics (MOND) model [30].}

\[^6^\text{The symbols } M_{ab}, K_a, P_a \text{ and } H \text{ denote rotations, boosts and space-time translation generators respectively.}\]
\[ [M_{ab}, H] = [K_a, K_b] = [K_a, P_b] = 0 \, , \, [K_a, H] = -iP_a \, , \, [P_a, P_b] = 0 \, , \]

while the (twisted) coproducts and antipodes take the form

\[ \Delta_0(a) \to \Delta_0(a) = \mathcal{F} \circ \Delta_0(a) \circ \mathcal{F}^{-1} \, , \, S(a) = u. S_0(a) u^{-1} \, , \tag{12} \]

with \( \Delta_0(a) = a \otimes 1 + 1 \otimes a \), \( S_0(a) = -a \) and \( u = \sum f_1 S_0(f_2) \) (we use Sweedler’s notation \( \mathcal{F} = \sum f_1 \otimes f_2 \)). Present in the commutation relations \([11]\) parameter \( \tau \) denotes the characteristic for Newton-Hooke algebra time scale. In the limit \( \tau \to \infty \) we get from \( U_0(NH_\pm) \) algebras the standard Galilei Hopf structure \( U_0(G) \). It should be also noted, that one can pass from \( U_0(NH_\pm) \) groups to \( U_0(NH_\pm) \) algebras by the multiplication of \( H \) and \( P_a \) generators by the imaginary element \( i = \sqrt{-1} \). The twist factors \( \mathcal{F} \in U_0(NH_\pm) \otimes U_0(NH_\pm) \) satisfy the classical cocycle condition

\[ \mathcal{F}_{12} \cdot (\Delta_0 \otimes 1) \mathcal{F} = \mathcal{F}_{23} \cdot (1 \otimes \Delta_0) \mathcal{F} \, , \tag{13} \]

and the normalization condition

\[ (\epsilon \otimes 1) \mathcal{F} = (1 \otimes \epsilon) \mathcal{F} = 1 \, , \tag{14} \]

with \( \mathcal{F}_{12} = \mathcal{F} \otimes 1 \) and \( \mathcal{F}_{23} = 1 \otimes \mathcal{F} \). They look as follows

\[
\begin{align*}
\mathcal{F}_{\alpha_1} &= \exp \left( \frac{i}{2} \alpha_{1}^{kl} P_k \wedge P_l \right) \quad \text{[} \alpha_{1}^{kl} = -\alpha_{1}^{lk} \text{]} , \\
\mathcal{F}_{\alpha_2} &= \exp \left( \frac{i}{2} \alpha_{2}^{kl} K_k \wedge P_l \right) \quad \text{[} \alpha_{2}^{kl} = -\alpha_{2}^{lk} \text{]} , \\
\mathcal{F}_{\alpha_3} &= \exp \left( \frac{i}{2} \alpha_{3}^{kl} K_k \wedge K_l \right) \quad \text{[} \alpha_{3}^{kl} = -\alpha_{3}^{lk} \text{]} , \\
\mathcal{F}_{\alpha_4} &= \exp (i \alpha_4 M_k \wedge M_l) \quad \text{[} m, k, l \text{ fixed, } m \neq k, l \text{]} , \\
\mathcal{F}_{\alpha_5} &= \exp (i \alpha_5 P_m \wedge M_{kl}) \quad \text{[} m, k, l \text{ fixed, } m \neq k, l \text{]} .
\end{align*}
\]

Hence, in accordance with the formula \([12]\), the antipodes remain classical while the corresponding twisted coproducts take the form

\[
\begin{align*}
\Delta_{\alpha_1}(M_{ab}) &= \Delta_{0}(M_{ab}) - \alpha_{1}^{kl} \left[ (\delta_{ka} P_b - \delta_{kb} P_a) \otimes P_l + P_k \otimes (\delta_{la} P_b - \delta_{lb} P_a) \right] , \\
\Delta_{\alpha_1}(K_a) &= \Delta_{0}(K_a) \, , \, \Delta_{\alpha_1}(P_a) = \Delta_{0}(P_a) \, , \, \Delta_{\alpha_1}(H) = \Delta_{0}(H) \pm \frac{\alpha_{1}^{kl}}{\tau^2} K_k \wedge P_l \tag{21}
\end{align*}
\]

in the case of first deformation

\[
\begin{align*}
\Delta_{\alpha_2}(M_{ab}) &= \Delta_{0}(M_{ab}) - \frac{i}{2} \alpha_{2}^{kl} [M_{ab}, K_k] \wedge P_l - \frac{\alpha_{2}^{kl}}{2} K_k \wedge (\delta_{la} P_b - \delta_{lb} P_a) , \tag{22}
\end{align*}
\]
\[ \Delta_{\alpha_2}(K_a) = \Delta_0(K_a) , \quad \Delta_{\alpha_2}(P_a) = \Delta_0(P_a) , \] (23)

\[ \Delta_{\alpha_2}(H) = \Delta_0(H) \pm \frac{\alpha_{kl}^2}{2\tau^2} K_k \wedge K_l + \frac{\alpha_{k}^2}{2} P_k \wedge P_l , \] (24)

for the second Hopf structure

\[ \Delta_{\alpha_3}(M_{ab}) = \Delta_0(M_{ab}) - \alpha_{3}^{kl} \left[ (\delta_{ka} K_b - \delta_{kb} K_a) \otimes K_l + K_k \otimes (\delta_{la} K_b - \delta_{lb} K_a) \right] , \] (25)

\[ \Delta_{\alpha_3}(K_a) = \Delta_0(K_a) , \quad \Delta_{\alpha_3}(P_a) = \Delta_0(P_a) , \quad \Delta_{\alpha_3}(H) = \Delta_0(H) + \alpha_{3}^{kl} P_l \wedge K_k \] (26)

for third Newton-Hopf algebra

\[ \Delta_{\alpha_4}(M_{ab}) = \Delta_0(M_{ab}) + M_{kl} \wedge \alpha_4 (\delta_{am} K_b - \delta_{bm} K_a) \\
+ i [M_{ab}, M_{kl}] \wedge \sin(\alpha_4 K_m) \\
+ [[M_{ab}, M_{kl}], M_{kl}] \perp (\cos(\alpha_4 K_m) - 1) \\
+ M_{kl} \sin(\alpha_4 K_m) \perp \alpha_4 (\psi_m K_k - \chi_m K_l) \\
+ \alpha_4 (\psi_m K_l + \chi_m K_k) \wedge M_{kl} (\cos(\alpha_4 K_m) - 1) , \] (27)

\[ \Delta_{\alpha_4}(K_a) = \Delta_0(K_a) + i [K_a, M_{kl}] \wedge \sin(\alpha_4 K_m) \\
+ [[K_a, M_{kl}], M_{kl}] \perp (\cos(\alpha_4 K_m) - 1) , \] (28)

\[ \Delta_{\alpha_4}(P_a) = \Delta_0(P_a) + \sin(\alpha_4 K_m) \wedge (\delta_{ka} P_k - \delta_{la} P_l) \\
+ (\cos(\alpha_4 K_m) - 1) \perp (\delta_{ka} P_k + \delta_{la} P_l) , \] (29)

\[ \Delta_{\alpha_4}(H) = \Delta_0(H) - \alpha_4 M_{kl} \wedge P_m , \]

in the case of forth deformation, and finally,

\[ \Delta_{\alpha_5}(M_{ab}) = \Delta_0(M_{ab}) + M_{kl} \wedge \alpha_5 (\delta_{am} P_b - \delta_{bm} P_a) \\
+ i [M_{ab}, M_{kl}] \wedge \sin(\alpha_5 P_m) \\
+ [[M_{ab}, M_{kl}], M_{kl}] \perp (\cos(\alpha_5 P_m) - 1) \\
+ M_{kl} \sin(\alpha_5 P_m) \perp \alpha_5 (\psi_m P_k - \chi_m P_l) \\
+ \alpha_5 (\psi_m P_l + \chi_m P_k) \wedge M_{kl} (\cos(\alpha_5 P_m) - 1) , \] (30)

\[ \Delta_{\alpha_5}(K_a) = \Delta_0(K_a) - i [K_a, M_{kl}] \wedge \sin(\alpha_5 P_m) \\
+ [[K_a, M_{kl}], K_{kl}] \perp (\cos(\alpha_5 P_m) - 1) , \] (31)

\[ \Delta_{\alpha_5}(P_a) = \Delta_0(P_a) + \sin(\alpha_5 P_m) \wedge (\delta_{ka} P_k - \delta_{la} P_l) \\
+ (\cos(\alpha_5 P_m) - 1) \perp (\delta_{ka} P_k + \delta_{la} P_l) , \] (32)

in the case of forth deformation, and finally,
\[ \Delta_{\alpha_5}(H) = \Delta_0(H) \pm \frac{\alpha_5}{\tau^2} K_m \wedge M_{kl}, \]  

(33)

for the last case, where

\[ \psi_m = \delta_{bm}\delta_{la} - \delta_{am}\delta_{lb}, \quad \chi_m = \delta_{bm}\delta_{ka} - \delta_{am}\delta_{kb}, \quad a \perp b = a \otimes b + b \otimes a. \]  

(34)

The algebraic sector (11) together with classical antipodes and coalgebraic relations (21)-(33) define the \( \alpha_i \)-deformed Newton-Hooke Hopf algebras \( U_{\alpha_i}(NH_{\pm}) \) respectively \((i = 1, 2, ..., 5)\). It should be noted that for all parameters \( \alpha_i \) running to zero the above Hopf structures become classical. Besides, one can also observe that for time scale \( \tau \) approaching infinity we get \( U_{\alpha_i}(G) \) Galilei quantum groups constructed in \[22\].

3 Quantum Newton-Hooke space-times

Let us now turn to the deformed space-times corresponding to the Hopf algebras provided in previous Section. They are defined as the quantum representation spaces (Hopf modules) for quantum Newton-Hooke algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules \[31\], \[32\], \[21\].

The action of Newton-Hooke groups \( U_{\alpha_i}(NH_{\pm}) \) on a Hopf module of functions depending on space-time coordinates \((t,x_a)\) is given by

\[ H \triangleright f(t,\bar{x}) = i \partial_t f(t,\bar{x}), \quad P_a \triangleright f(t,\bar{x}) = i C_{\pm} \left( \frac{t}{\tau} \right) \partial_a f(t,\bar{x}), \]  

(35)

and

\[ M_{ab} \triangleright f(t,\bar{x}) = i (x_a \partial_b - x_b \partial_a) f(t,\bar{x}), \quad K_a \triangleright f(t,\bar{x}) = i \tau S_{\pm} \left( \frac{t}{\tau} \right) \partial_a f(t,\bar{x}), \]  

(36)

with \( C_{\pm} \left( \frac{t}{\tau} \right) = \cosh \left[ \frac{t}{\tau} \right], \quad C_{\mp} \left( \frac{t}{\tau} \right) = \cos \left[ \frac{t}{\tau} \right], \quad S_{\pm} \left( \frac{t}{\tau} \right) = \sinh \left[ \frac{t}{\tau} \right], \quad S_{\mp} \left( \frac{t}{\tau} \right) = \sin \left[ \frac{t}{\tau} \right]. \)

Moreover, the \( \star \)-multiplication of arbitrary two functions is defined as follows

\[ f(t,\bar{x}) \star g(t,\bar{x}) := \omega \circ (F^{-1} \triangleright f(t,\bar{x}) \otimes g(t,\bar{x})), \]  

(37)

where symbol \( F \) denotes the twist factor corresponding to a proper Newton-Hooke group and \( \omega \circ (a \otimes b) = a \cdot b. \)

Consequently, we get:

1) The Newton-Hooke space-times corresponding to the quantum groups \( U_{\alpha_1}(NH_{\pm}) \). Then, the proper twist factors look as follows

\[ F_{\alpha_1} = \exp \left( i \alpha_1^{kl} C_{\pm} \left( \frac{t}{\tau} \right) \partial_l \wedge \partial_k \right), \]  

(38)

while the nonrelativistic space-times take the form

\[ [t, x_a]_{\alpha_1} = 0, \quad [x_a, x_b]_{\alpha_1} = i \alpha_1^{kl} C_{\pm} \left( \frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}). \]
2) The quantum spaces associated with \( U_{\alpha_2}(NH_{\pm}) \) Hopf algebras. In such a case the twist factors and corresponding space-times are given by

\[
\mathcal{F}_{\alpha_2} = \exp \left( \frac{i}{2} \alpha_2^{kl} \tau C_{\pm} \left( \frac{t}{\tau} \right) S_{\pm} \left( \frac{t}{\tau} \right) \partial_l \land \partial_k \right),
\]

and

\[
[t, x_a]_{\ast_{\alpha_2}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_2}} = i\alpha_2^{kl} \tau C_{\pm} \left( \frac{t}{\tau} \right) S_{\pm} \left( \frac{t}{\tau} \right) (\delta_{ak} \delta_{bl} - \delta_{al} \delta_{bk}),
\]

respectively.

3) The deformed space-times associated with \( U_{\alpha_3}(NH_{\pm}) \) quantum groups. Then, we get the following twist factors

\[
\mathcal{F}_{\alpha_3} = \exp \left( \frac{i}{2} \alpha_3^{kl} \tau^2 S_{\pm}^2 \left( \frac{t}{\tau} \right) \partial_l \land \partial_k \right),
\]

and the proper Newton-Hooke spaces

\[
[t, x_a]_{\ast_{\alpha_3}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_3}} = i\alpha_3^{kl} \tau^2 S_{\pm}^2 \left( \frac{t}{\tau} \right) (\delta_{ak} \delta_{bl} - \delta_{al} \delta_{bk}).
\]

4) The fourth class of space-times \( (U_{\alpha_4}(NH_{\pm})) \)

\[
[t, x_a]_{\ast_{\alpha_4}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_4}} = 2i\alpha_4 \tau S_{\pm} \left( \frac{t}{\tau} \right) \left[ \delta_{ma}(x_k \delta_{bl} - x_l \delta_{bk}) - \delta_{mb}(x_k \delta_{al} - x_l \delta_{ak}) \right],
\]

corresponding to the following twist factors

\[
\mathcal{F}_{\alpha_4} = \exp \left( i\alpha_4 \tau S_{\pm} \left( \frac{t}{\tau} \right) (x_k \partial_l - x_l \partial_k) \land \partial_m \right).
\]

5) The last kind of quantum spaces associated with \( U_{\alpha_5}(NH_{\pm}) \) groups. In such a case the twist factors take the form

\[
\mathcal{F}_{\alpha_5} = \exp \left( i\alpha_5 C_{\pm} \left( \frac{t}{\tau} \right) (x_k \partial_l - x_l \partial_k) \land \partial_m \right),
\]

while the noncommutative space-times look as follows

\[
[t, x_a]_{\ast_{\alpha_5}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_5}} = 2i\alpha_5 C_{\pm} \left( \frac{t}{\tau} \right) \left[ \delta_{ma}(x_k \delta_{bl} - x_l \delta_{bk}) - \delta_{mb}(x_k \delta_{al} - x_l \delta_{ak}) \right].
\]
Let us note that due to the form of functions $C_{\pm}(t/\tau)$ and $S_{\pm}(t/\tau)$ the spatial noncommutativities 1) - 5) are expanding or periodic in time for $U_{\alpha_i}(NH_{\pm})$ and $U_{\alpha_i}(NH_{\mp})$ Hopf algebras respectively. It should be also noted that for deformation parameters $\alpha_i$ approaching zero all above space-times become classical. Besides, as it was mentioned in Introduction, for time parameter $\tau$ running to infinity we recover the Galilei quantum space-times provided in [22] and [23], i.e. we get

\begin{align*}
\text{i)} & \quad [t, x_a]_{\ast_{\alpha_1}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_1}} = i\alpha_1^{kl}(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}), \\
\text{ii)} & \quad [t, x_a]_{\ast_{\alpha_2}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_2}} = i\alpha_2^{kl} t(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}), \\
\text{iii)} & \quad [t, x_a]_{\ast_{\alpha_3}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_3}} = i\alpha_3^{kl} t^2(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}), \\
\text{iv)} & \quad [t, x_a]_{\ast_{\alpha_4}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_4}} = 2i\alpha_4 t [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})], \\
\text{v)} & \quad [t, x_1]_{\ast_{\alpha_5}} = 0, \quad [x_a, x_b]_{\ast_{\alpha_5}} = 2i\alpha_5 [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})],
\end{align*}

respectively. The quantum space i) corresponds to the canonical type of noncommutativity (1), the space-times ii) and v) - to the Lie-algebraic class (2), while the quantum spaces iii) and iv) belong to the quadratic type of space-time deformation (3).

4 Final remarks

In this article we introduce five twisted Newton-Hooke Hopf algebras and the corresponding deformed space-times with quantum space and classical time (see 1) - 5)). We demonstrate that derived quantum spaces are respectively periodic and expanding in time, with $U_{\alpha_i}(NH_{\pm})$ and $U_{\alpha_i}(NH_{\mp})$ quantum groups as symmetries. In the limit $\tau \to \infty$ we also recover five twisted Galilei quantum spaces i) - v) proposed in [22] and [23].

It should be noted that present studies can be extended in various ways. First of all, one can find the dual Hopf structures $D_{\alpha_i}(NH_{\pm})$ with the use of FRT procedure [37] or by canonical quantization of the proper Poisson-Lie structures [38]. Besides, as it was mentioned in Introduction, one can look for dynamical models corresponding to the Newton-Hooke space-times 1) - 5). In the case of twisted Galilei Hopf algebras such investigations have been performed in [34] and [35]. Finally, one can consider the more complicated (non-Abelian) twist deformations of Newton-Hooke Hopf algebras, and find subsequently the twisted coproducts, corresponding noncommutative space-times and dual Newton-Hooke Hopf structures. The mentioned problems are now under consideration.

\[7\] There exist yet one (omitted in this article) Abelian twist factor with carrier $M_{ab} \wedge H$. However, due to the form of representation [69], [80], it provides well known, less interesting type of noncommutativity [2].
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