Graphical representation of marginal and underlying probabilities in quantum mechanics

Taksu Cheon

Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan

Abstract

Wigner’s marginal probability theory is revisited, and systematically applied to n-particle correlation measurements. A set of Bell inequalities whose corollaries are Hardy contradiction and its generalisation are derived with intuitive graphical analysis.

Keywords: Bell inequality, local reality, marginal probability

PACS: 03.65.Ud, 03.67.-a, 42.50.Ar

1. Introduction

An unambiguous signal of the departure from local causality in quantum mechanics is expressed as violation of Bell inequality [1] and its variants, the Bell theorems, such as CHSH inequalities [2], GHZ theorem [3] and Hardy’s arguments [4, 5], which has been verified by the experiments by use of various physical systems [6, 7, 8, 9, 10]. The relations among these Bell theorems have been uncovered gradually [11, 12]. The violation of Bell inequality can be viewed also as a sign of the breakdown of the classical concepts of joint probability. As independently pointed out by Wigner and Fine [13, 14, 15], the Bell inequality, in its original form devised by Bell, presupposes underlying joint probability distribution for all possible measurement setups and outcomes in the experiment, whose marginal probabilities yield the probability distributions of the actual measurement setup of each run.

In this article, we revisit the argument by Wigner and Fine [13, 14, 15], and try to make clear the relation between their arguments and the concept of local reality. In the process, we uncover the graphic structure of the underlying joint probability, which we utilise to systematically to search general form of Hardy’s equalities and associated inequalities.

This article is organized as follows. In Sec. 2, we introduce the concepts of the underlying probabilities which respects statistical independence and yield the measurable probability distributions as their marginal probabilities. In Sec. 3, we check consistency between the assumptions of statistical independence and the local reality. In Sec. 4, we turn to the graphical representation of the marginal probability for two-by-two experiment. Using this representation, we clarify that Hardy’s equalities can be thought of as an extremal case of CHSH inequality. In Sec. 5, we generalise the above observation for multi-setting case and multi-partite case. Sec. 5 has the discussion and our conclusion.
2. Marginal and Underlying Probabilities, Statistical Independence of Measurements

Consider a system of two 1/2-spins, each of which are measured separately by two observers Alice and Bob, both of whom can set up the measurement devices along two different axes, \( a = 0, 1 \) for Alice, and \( b = 0, 1 \) for Bob. We assume that the result of the dichotomic measurements is specified by \( A = 0, 1 \) for Alice, and \( B = 0, 1 \) for Bob. The usual sign notations of projection are recovered by identifications \( s_A = (-)^A \) and \( s_B = (-)^B \).

Let us denote the probability of Alice finding the system in the state \( A \) along the \( a \)-axis and Bob in \( B \) along \( b \)-axis by \( P_{ab}(A, B) \). The assumption of statistical independence is expressed as

\[
P_{ab}(A, B) = P_a(A)P_b(B).
\]

Let us now consider parallel measurements on an ensemble of the system in two combinations of axes pair \((a, b)\) and \((a', b')\). We write the joint probability of finding the results \((A, B)\) and \((A', B')\) as \( W([A, B]_{ab}; [A', B']_{a'b'}) \). If the parallel measurements are statistically independent, we should have

\[
W([A, B]_{ab}; [A', B']_{a'b'}) = P_{ab}(A, B)P_{a'b'}(A', B'),
\]

From the assumption (1), we have

\[
P_{00}(A_0, B_0)P_{11}(A_1, B_1) = P_{01}(A_0, B_1)P_{10}(A_1, B_0)
\]

since both are given by the product \( P_0(A_0)P_1(A_1)P_0(B_0)P_1(B_1) \). This immediately leads to the equivalence between \( W([A_0, B_0]_{00}; [A_1, B_1]_{11}) \) and \( W([A_0, B_1]_{01}; [A_1, B_0]_{10}) \). This means that we can define unconditional underlying probability \( \rho(A_0, B_0; A_1, B_1) \) which is the joint probability of Alice finding her system in state \( A_0 \) along \( 0 \)-axis, \( A_1 \) along \( 1 \)-axis and Bob finding his particle in state \( B_0 \) along \( 0 \)-axis, \( B_1 \) along \( 1 \)-axis;

\[
\rho(A_0, A_1; B_0, B_1) = W([A_0, B_0]_{00}; [A_1, B_1]_{11}) = W([A_0, B_1]_{01}; [A_1, B_0]_{10}),
\]

which signify the “reality” of the set of physical observables \( (A_0, B_0; A_1, B_1) \) irrespective to the measurement. There are \( 2^4 = 16 \) of \( U \)s. As probabilities, they are all non-negative real numbers, which add up to unity. The observable probabilities \( P_{ij} \) are obtained from \( \rho \) by partial sum as

\[
P_{00}(A_0, B_0) = \sum_{A_1, B_1} \rho(A_0, B_0; A_1, B_1), \quad P_{10}(A_1, B_0) = \sum_{A_0, B_1} \rho(A_0, B_0; A_1, B_1),
\]

\[
P_{01}(A_0, B_1) = \sum_{A_1, B_0} \rho(A_0, B_0; A_1, B_1), \quad P_{11}(A_1, B_1) = \sum_{A_0, B_0} \rho(A_0, B_0; A_1, B_1).
\]

Following Wigner, we call the probability \( P_{ab}s \) that are obtainable from direct measurements as marginal probabilities. Although the existence of unconditional probability \( \rho \) which is guaranteed by the relations (4), or equivalently (3), are derivable from the statistical independence between Alice’s and Bob’s observables, (1), the former is a looser assumption, from which the latter may not necessarily follow. So irrespective to the logic we have followed until now, we shall use the existence of unconditional probability \( U \) expressed in relations (1) and (5) as the basic assumption.
3. Derivation of Joint Probability Based on Locally Realistic Theory with Hidden Variable

Although in the derivation of \( P(A_0, B_0; A_1, B_1) \), the cornerstone of our approach, we have pretended that it is derived from the statistical independence assumption of marginal probabilities, it can be derived by quite a different set of assumptions based on the concept of hidden variable theory with local realism. Of course, this is how the Bell inequality has been historically derived, and this is exactly the reason that Bell theorem is termed one of the most profound theorems in physics. We need, however, to put it in perspective in view of the fact that there are vocal dissenting views [17] on the significance of Bell experiment that what is proven is just the statistical separability of probabilities, not the negation of local realism.

Here, we detail how Wigner has arrived at conditional and unconditional underlying probabilities \( W \) and \( \rho \), stating from deterministic theory with hidden variables. Although we base our argument on the specific case of two-by-two Bell experiment, readers will see its generality and this is exactly the reason that Bell theorem is termed one of the most profound theorems in physics. We need, however, to put it in perspective in view of the fact that there are vocal dissenting views [17] on the significance of Bell experiment that what is proven is just the statistical separability of probabilities, not the negation of local realism.

We now consider an ensemble of this deterministic system whose density of distribution on the space of \( \Lambda \) is given by

\[
W(\Lambda) = W(A_{00}, B_{00}; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11})
\]

(8)

The observable probability \( P_{ab}(A, B) \) are given by

\[
P_{00}(A, B) = \sum_{A_{00}, B_{00}} \sum_{A_{10}, B_{10}} \sum_{A_{01}, B_{01}} \sum_{A_{11}, B_{11}} W(A, B; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11}),
\]

\[
P_{10}(A, B) = \sum_{A_{00}, B_{00}} \sum_{A_{10}, B_{10}} \sum_{A_{01}, B_{01}} \sum_{A_{11}, B_{11}} W(A_{00}, B_{00}; A, B; A_{01}, B_{01}; A_{11}, B_{11}), \quad \text{etc.,}
\]

(9)

and we should always be able to construct a deterministic theory whose initial ensemble of variable \( \Lambda(t = 0) \), that corresponds to the initial setup of the experiment, evolve into the distribution \( W(\Lambda) \) with \( \Lambda = \Lambda(t) \) at the time of the measurement, that reproduces the observed \( P_{ab}(A, B) \). It has to be remarked that the existence of this supposed underlying theory itself is rather unremarkable. The fact, that it requires an eight-bits variables capable of \( 2^8 = 256 \) values to reproduce 16 experimental numbers, makes this theory an example of “cost-ineffective” generalization.
Now, we consider only those deterministic theories that respect local realism à la J.S.Bell. Then, the result of the Alice’s measurement should not depend on how Bob’s measurement device is placed, and Bob’s result should not depend on how Alice’s measurement device is placed. The values of the variables, in this type of theory, have to be limited in such way to reflect this fact, namely

\[ A_{00} = A_{01} \equiv A_0, \quad A_{10} = A_{11} \equiv A_1, \quad B_{00} = B_{10} \equiv B_0, \quad B_{01} = B_{11} \equiv B_1. \]  

(10)

So half of the bits in the variable \( A \) are redundant. In other word, local realistic theories underlying the observable \( P_{ab}(A, B) \) is to be described by a half byte variable \( \lambda \), which we define in the form

\[ \lambda = [A_0, B_0, A_1, B_1]. \]  

(11)

The ensemble-averaged system we observe in experiments is to be specified by the density of distribution

\[ \rho(\lambda) = \rho(A_0, B_0; A_1, B_1), \]  

(12)

which gives the observable probability \( P_{ab} \) in the form

\[ P_{00}(A, B) = \sum_{A_0, B_0} \rho(A, B; A_0, B_0), \quad P_{10}(A, B) = \sum_{A_0, B_1} \rho(A, B; A_0, B_1), \quad \text{etc.} \]  

(13)

Note that the variable \( \lambda \) can take 16 discrete values, as opposed to 256 for \( \Lambda \). Consequently, there are 16 of \( \rho(\lambda) \) functions as in contrast to 256 of \( W(\Lambda) \) functions.

In formal term, the local realistic subclass of hidden variable theory is obtained from all possible theory by the reduction of density distribution

\[ W(A_{00}, B_{00}; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11}) = \rho(A_{00}, B_{00}; A_{11}, B_{11})\delta_{A_{00}A_{01}}\delta_{B_{00}B_{01}}\delta_{B_{01}B_{11}}. \]  

(14)

or conversely,

\[ \rho(A_0, B_0; A_1, B_1) = W(A_0, B_0; A_1, B_1; A_0, B_0; A_1, B_1). \]  

(15)

We see that the “hidden variables” \( A_{ab} \), \( B_{ab} \) and their reductions \( A_a \), \( B_b \) are indeed hidden behind their guise as projection indices in \( W \) and \( U \).

4. Diagrammatical Proof of Bell inequalities and Hardy Contradiction

The assumption of the existence of underlying unconditional probability \( \rho(A_0, B_0; A_1, B_1) \) leads to several relations among marginal probabilities \( P_{ab}(A, B) \).

Let us first reconfirm our notational conventions. The index \( a \), that takes two values 0 and 1, stands for the axis of choice by Alice, along which the first particle is projectively measured to yield the value \( s_A = + \) or \(-\), which is alternatively expressed as \( A = 0 \) or 1. Similarly, the index \( b \), that can be 0 or 1, stands for the axis of Bob’s measurement, which yields the value \( s_B = + \) or \(-\) that is alternatively expressed as \( B = 0 \) or 1. We define inter-particle axis specific projections

\[ i = A_0 + 2B_0, \quad j = A_1 + 2B_1. \]  

(16)
The value of \( i \) represents the combined projection along the “0” axes of Alice and Bob, and \( j \) along their “1”- axes. We now reorder \( \rho(A_0, A_1; B_0, B_1) \) by the indices \( i \) and \( j \) with the use of a matrix \( V \) defined by

\[
V_{i,j} = \rho(A_0, B_0; A_1, B_1) |_{i = A_0 + 2B_0, j = A_1 + 2B_1}.
\] (17)

The partial sum (5) then becomes

\[
P_{11}(A, B) = \sum_{j=0}^{3} V_{A+2B,j} \quad P_{10}(A, B) = \sum_{i=0}^{1} (V_{2B+iA} + V_{2B+iA+2}),
\]

\[
P_{01}(A, B) = \sum_{i=0}^{1} (V_{A,2B+j} + V_{A+22B+j}), \quad P_{11}(A, B) = \sum_{i=0}^{3} V_{iA+2B}.
\] (18)

These expression has a very intuitive graphical representation Fig. 1. If we place \( \rho \) on a 4-by-4 grid according to the indices \( i \) and \( j \), \( P_{00} \) are given by sums along horizontal lines, \( P_{11} \) by sums along vertical lines, while \( P_{10} \) and \( P_{01} \) are given by sums ribbon-shaped lines vertically and horizontally placed.

![Figure 1: Marginal probabilities \( P_{ab}(A, B) \) represented as a line or linked lines which cover the the grids that represent the underlying probabilities \( V_{ij} \) to be summed up.](image)

From the compositions (18), it follows, for example, that

\[
P_{10}(0, 0) + P_{01}(0, 0) + P_{11}(1, 1) = 2V_{0,0} + V_{0,1} + V_{0,2} + V_{0,3} + V_{1,0} + V_{1,2} + V_{1,3} + V_{2,0} + V_{2,1} + V_{2,3} + V_{3,3}
\] (19)

which contains

\[
P_{00}(0, 0) = V_{0,0} + V_{0,1} + V_{0,2} + V_{0,3}
\] (20)

and some more positive quantities. Thus we have a Bell inequality in the form

\[
P_{10}(0, 0) + P_{01}(0, 0) + P_{11}(1, 1) - P_{00}(0, 0) \geq 0
\] (21)

A graphical representation of this inequality is shown in Fig. 2.

A corollary immediately obtained is that

if \( P_{10}(0, 0) = P_{01}(0, 0) = P_{11}(1, 1) = 0 \) hold, then \( P_{00}(0, 0) = 0 \) follows. (22)
This is nothing but the equality obtained by Lucien Hardy. A similar construction to (19), (20) leads to

\[
P_{10}(1, 1) + P_{01}(1, 1) + P_{11}(0, 0) - P_{00}(1, 1) \geq 0,
\]

\[
P_{10}(1, 0) + P_{01}(1, 0) + P_{11}(0, 1) - P_{00}(1, 0) \geq 0,
\]

\[
P_{10}(0, 1) + P_{01}(0, 1) + P_{11}(1, 0) - P_{00}(0, 1) \geq 0.
\]

With the definition of correlation \( C_{ab} \),

\[
C_{ab} = P_{ab}(0, 0) - P_{ab}(1, 0) - P_{ab}(0, 1) + P_{ab}(1, 1)
\]

inequalities (21) and (23) lead to

\[
|C_{10} + C_{01} + C_{11} - C_{00}| \leq 2,
\]

which is a celebrated CHSH inequality.

With the interchanges of axis indices \( a \) and \( b \), and with interchanges of projections 0 and 1 for both \( A \) and \( B \) in \( P_{ab}(A, B) \), there are 64 inequalities of the type (21). Each four of them sharing the same leg \( ab \) form a single CHSH inequality. As an obvious corollary, there are 64 variants of Hardy equality.

5. Three Particle Generalization and More

The extension to \( n \)-particle case is rather straightforward. We illustrate it with three particle case. Let us now consider a system consisting of three spin 1/2 particles that are respectively measured by Alice, Bob and Chris, all of whom have two choices each for the orientation of measurement represented by \( a = 0 \) or 1, \( b = 0 \) or 1 and \( c = 0 \) or 1, respectively. Alice’s measured projection is represented by \( A_a = 0 \) or 1, Bob’s by \( B_b = 0 \) or 1 and Chris’ by \( C_c = 0 \) or 1. If there is an underlying probability \( \rho(A_0, B_0, C_0; A_1, B_1, C_1) \), that presuppose the simultaneous existence of all projections for all three observers, the marginal probabilities \( P_{abc} \)s which are the
joint probabilities of direct observables are given by the partial sums

\[ P_{000}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \quad P_{100}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \]

\[ P_{010}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \quad P_{110}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \]

\[ P_{001}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \quad P_{101}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \]

\[ P_{011}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \quad P_{111}(A, B, C) = \sum_{A', B', C'} \rho(A, B, C; A', B', C'). \] (26)

This set of construction may or may not result in the assumption of statistical independence

\[ P_{abc}(A, B, C) = P_a(A)P_b(B)P_c(C), \quad \text{(27)} \]

although the latter necessarily results in the former. In case (27) holds, we have

\[ P_{000}(A, B, C)P_{111}(A', B', C') = P_{001}(A, B, C)P_{110}(A', B', C), \]

\[ P_{000}(A, B, C)P_{110}(A', B', C') = P_{100}(A', B', C)P_{010}(A, B', C'), \]

\[ P_{111}(A, B, C)P_{001}(A', B', C') = P_{011}(A', B', C)P_{101}(A, B', C'), \] (28)

etc.. With the three-digits grouping of indices

\[ V_{i,j} = \rho(A_0, B_0, C_0; A_1, B_1, C_1)_{\mid i=0+2B_0+4C_0, j=A_1+2B_1+4C_1}, \quad \text{(29)} \]
the partial sums become

\[ P_{000}(A, B, C) = \sum_{j=0}^{7} V_{A+2B+4C,j}, \quad P_{111}(A, B, C) = \sum_{j=0}^{7} V_{i+A+2B+4C,j}, \]

\[ P_{110}(A, B, C) = \sum_{j=0}^{3} (V_{4C+i,A+2B} + V_{4C+i,A+2B+4}), \quad P_{001}(A, B, C) = \sum_{j=0}^{3} (V_{A+2B,4C+j} + V_{A+2B+4C+j}), \]

\[ P_{100}(A, B, C) = \sum_{j=0}^{1} (V_{2B+4C+i,A} + V_{2B+4C+i,A+2} + V_{2B+4C+i,A+4} + V_{2B+4C+i,A+6}), \]

\[ P_{011}(A, B, C) = \sum_{j=0}^{1} (V_{A+2B+4C+j} + V_{A+2B+4C+j} + V_{A+2B+4C+j} + V_{A+2B+4C+j}), \]

\[ P_{010}(A, B, C) = \sum_{j=0}^{1} (V_{A+4C,2B+j} + V_{A+4C,2B+j} + V_{A+4C,2B+j} + V_{A+4C,2B+j}), \]

\[ P_{010}(A, B, C) = \sum_{j=0}^{1} (V_{2B+i,A+4C} + V_{2B+i,A+4C} + V_{2B+i,A+4C} + V_{2B+i,A+4C}). \]

This extends the previous graphical expression of 4-by-4 matrix grid for two-particle case with 8-by-8 matrix grid on which \( \rho_s \) are placed with indices \( i = A_0 + 2B_0 + 4C_0 \) and \( j = A_1 + 2B_1 + 4C_1 \). \( P_{000} \) and \( P_{111} \) are sums over horizontal and vertical lines, while \( P_{100}, P_{010} \) and \( P_{001} \) are sums over variously shaped ribbons, and \( P_{011}, P_{101} \) and \( P_{110} \) their respective mirror images with respect to the diagonal lines, all indicated in Fig.3. Comparison between Fig. 1 and Fig. 3 reveals how the \( n = 3 \) graphs are made out of \( n = 2 \) graphs; \( P_{000}, P_{100} \) and \( P_{010} \) are just the “sideway doubling” of \( P_{100}, P_{101} \) and \( P_{110} \), and the remaining \( P_{111}, P_{011} \) and \( P_{101} \) are obtained by mirroring with respect to diagonal line that corresponds to the reversing axis indices \( (a, b, c): 0 \leftrightarrow 1 \). This construction carries over to any \( n \to n + 1 \) extension. Drawing various ribbons on the matrix grid as before, the following inequality is shown to hold

\[ P_{100}(0, 0, 0) + P_{010}(0, 0, 0) + P_{001}(0, 0, 0) + P_{111}(1, 1, 1) - P_{000}(0, 0, 0) \geq 0 \]

(31)

whose corollary is a three body extension of Hardy equality, which states

if \( P_{100}(0, 0, 0) = P_{010}(0, 0, 0) = P_{001}(0, 0, 0) = P_{111}(1, 1, 1) = 0 \) hold,

then \( P_{000}(0, 0, 0) = 0 \) follows.

(32)

The generalization of this result

\[ P_{000}(0, 0, ..., 0) + P_{010}(0, 0, ..., 0) + ... + P_{001}(0, 0, ..., 0) + P_{111}(1, 1, ..., 1) - P_{000}(0, 0, ..., 0) \geq 0, \]

(33)

and its associated Hardy equality is not hard to prove. Quite separately from above extension, we can prove a Zukowski inequality

\[ C_{111} - C_{001} - C_{010} - C_{100} \geq -2 \]

(34)
in a similar fashion with the ribbons on 8-by-8 grid, whose corollary

\[ \text{if } C_{001} = C_{010} = C_{100} = 1 \text{ then } C_{111} = 1 \]  

(35)

is, of course, the negation of GHZ contradiction.

Another set of extension exists in the form of two party \( m \)-axes Bell experiments. Take for example, a system of two spin 1/2 particle each measured by Alice and Bob both of whom now has a choice of three projection axes \( a = 0, 1, 2 \) and \( b = 0, 1, 2 \). Experimental results are specified by 64 marginal probabilities \( P_{ab}(A, B) \). The assumption of underlying unconditional probability distribution now involves 64 quantities \( \rho(A_0, B_0; A_1, B_1; A_2, B_2) \). The proper tabulation of \( \rho \) now requires the definition of three two-digits indices

\[ i_k = A_k + 2B_k, \quad k = 0, 2, 3, \]  

(36)

thus is a cube grid of size 4-by-4-by-4. As depicted in Fig. 5, \( P_{kk} \) is the sum over a slice parallel to the cube surfaces, and \( P_{kl} \) with \( k \neq l \) half sums of two non adjacent surfaces.

In a similar fashion to the previous cases, although now requiring some 3-dimensional recognition of patterns, we can prove a set of three-axes nonlocality inequalities. One such example is depicted in Fig. 6, showing an inequality

\[ P_{00}(1, 1) + P_{10}(0, 0) + P_{02}(0, 0) - P_{12}(0, 0) \geq 0. \]  

(37)

This is a three-axes version of inequalities of the type (21). In a sense, it can be regarded as physically identical to them, since just by renaming the axis ”2” of Bob as ”1”, this simply reduces back to one of the two-axis type inequality. But this inequality does involve genuinely different three axes. We may obtain a three-axes version of Hardy type equality again by setting the first three term of (37) to be zero. Instead, we show another interesting face of this inequality by it requiring only the first term to be zero. We then have

\[ \text{if } P_{00}(1, 1) = 0, \text{ then } P_{10}(0, 0) + P_{02}(0, 0) - P_{12}(0, 0) \geq 0. \]  

(38)

This is nothing but the original Bell inequality in the form devised by J. S. Bell expressed in marginal probabilities instead of correlation functions, and the proof shown here is just the graphical dressing of Wigner’s proof. Note that the original requirement of “being in singlet state” is
Figure 5: Graphical depiction of two-body three-axes marginal probabilities $P_{ab}(A, B)$ represented as planes painted in identical colors, which cover the three-dimensional grids that represent the underlying probabilities $V_{ijk}$ to be summed up.
Figure 6: Graphical depiction of two-body three-axes Bell inequality which leads to “Bell’s” Bell inequality. Marginal probabilities $P_{ab}(A, B)$ are represented as identically colored planes for terms in LHS, a transparent square for the RHS, which are superimposed on the grid representing the underlying probabilities $V_{ijk}$ to be summed up.

loosened to $P_{00}(1, 1) = 0$. We can now see that “Bell’s” Bell inequality occupies a midpoint between CHSH type inequality and Hardy type equality.

6. Discussion

The derivation of Bell inequalities with the use of underlying probabilities examined here is quite general. The argument is very straightforward, and this approach, in principle, is extendable to Bell inequalities with four or more choices and also to four or more players. But actual graphical representation quickly becomes messy and intractable for higher number of choices and players, and therefore, is not expected to be competitive against traditional systematic approaches [20]. There is, however, an advantage in our approach of having intuitive graphical representation, which should not be missed in pedagogical settings.

We thank Prof. I. Tsutsui and Dr. T. Ichikawa for stimulating discussions. This research was supported by the Japan Ministry of Education, Culture, Sports, Science and Technology under the Grant number 15K05216.

References

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics 1 (1965) 195-200.
[2] J. F. Clauser, M.A. Horne, A. Shimony and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. 23 (1969) 880-884.
[3] D. M. Greenberger, M.A. Horne, A. Shimony and A. Zeilinger, A. Bell’s theorem without inequalities, Am. J. Phys. 28 (1990) 1131-1143.
[4] L. Hardy, Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories, Phys. Rev. Lett. 68 (1992) 2981-2984.
[5] L. Hardy, Nonlocality for two particles without inequalities for almost all entangled states, Phys. Rev. Lett. 71 (1993) 1665-1668.
[6] A. Aspect, J. Dalibard and G. Roger, Experimental Test of Bell’s Inequalities Using Time-Varying Analyzers, Phys. Rev. Lett. 49 (1982) 1804-1807.
[7] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter and A. Zeilinger, Violation of Bell’s Inequality under Strict Einstein Locality Conditions, Phys. Rev. Lett. 81, 5039-5043 (1998).
[8] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe and D. J. Wineland, Experimental violation of a Bell’s inequality with efficient detection, Nature 409, 791-794 (2001).
[9] H. Sakai et al., Spin Correlations of Strongly Interacting Massive Fermion Pairs as a Test of Bell’s Inequality, Phys. Rev. Lett. 97, 150405 (2006).
[10] A. Go et al. (Belle Collaboration), Measurement of Einstein-Podolsky-Rosen-Type Flavor Entanglement in $\Upsilon(4S) \rightarrow B^0\bar{B}^0$ Decays, Phys. Rev. Lett. 99, 131802 (2007).
[11] J. L. Cereceda, Identification of all Hardy-type correlations for two photons or particles with spin 1/2, Found. Phys. Lett. 14, 401-424 (2001).
[12] J. L. Cereceda, Hardy’s nonlocality for generalized n-prticle GHZ states, Phys. Lett. A 327, 433-437 (2004).
[13] E. P. Wigner, On hidden variables and quantum mechanical probabilities, Am. J. Phys. 38, 1005-1009 (1970).
[14] A. Fine, Hidden Variables, Joint Probability, and the Bell Inequalities Phys. Rev. Lett. 48, 291-295 (1982).
[15] A. Fine, Joint distributions, quantum correlations, and commuting observables J. Math. Phys. 23, 1306-1310 (1982).
[16] J. S. Bell, On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. 38, 447-452 (1966).
[17] A. Fine, The Shaky Game: Einstein, Realism, and the Quantum Theory, 2nd Ed. (U. Chicago Press, 1996).
[18] D. Bohm and J. Bub, A proposed solution of the measurement problem in quantum mechanics by a hidden variable theory Rev. Mod. Phys. 38, 453-469 (1966).
[19] B. S. Tsirelson, Quantum Generalizations of Bell’s Inequality, Lett. Math. Phys. 4, 93-100 (1980).
[20] M. Zukowski and C. Brukner, Bells Theorem for General N-Qubit States Phys. Rev. Lett. 88, 210401 (2002).