Resolving Fock states near the Kerr-free point of a superconducting resonator

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We have designed a tunable nonlinear resonator terminated by a SNAIL (Superconducting Nonlinear Asymmetric Inductive eLement). Such a device possesses a sweet spot in which the external magnetic flux allows to suppress the Kerr interaction. We have excited photons near this Kerr-free point and characterized the device using a transmon qubit. The excitation spectrum of the qubit allows to observe photon-number-dependent frequency shifts about nine times larger than the qubit linewidth. Our study demonstrates a compact integrated platform for continuous-variable quantum processing that combines large couplings, considerable relaxation times and excellent control over the photon mode structure in the microwave domain.

Encoding quantum information in the infinite Hilbert space of a harmonic oscillator is a promising avenue for quantum computing. Recently, significant progress has been made by using three-dimensional (3D) microwave cavities [1–6]. Thanks to the strong-dispersive coupling, quantum states such as cat states [7, 8], GKP states [9] and the cubic phase state [10] have been engineered. However, currently, the scalability and connectivity is difficult, limited by the size of the cavity. Another simpler method is to use coplanar microwave resonators where resonators and qubits can be fabricated together in a single chip [11–13]. The drawback is the shorter relaxation time of coplanar resonators compared to 3D cavities.

Both 2D and 3D microwave resonators as well as acoustic resonators [14, 15] host linear modes. Therefore, an ancillary qubit is customarily used to introduce a nonlinearity for state preparation and operation. However, the limited coherence of the ancillary qubit and the imperfect operations on it will decrease the fidelity of the actual states [10, 16, 17]. To avoid operations on ancillary qubits, a Superconducting QUantum Interference Device (SQUID) can be used to terminate a coplanar resonator. This not only provides the tunability of the mode frequency by changing the external magnetic flux through the loop [18–23], it also induces a sufficiently nonlinearity. Using this nonlinearity, experiments in waveguide quantum electrodynamics have demonstrated the generation of entangled microwave photons by the parametrical pumping of a symmetrical SQUID loop [24, 25]. Non-Gaussian states, regarded as a resource for quantum computing, have also been realized [26–28]. Therefore, state preparation and operations can be implemented in a nonlinear resonator, even without ancillary qubits. However, in those experiments, the generated states can not be stored for a long time since the resonators are directly coupled to the waveguides.

Theoretically, it has been shown that pulsed operations on a novel tunable nonlinear resonator can be used to achieve a universal gate set for continuous-variable (CV) quantum computation [29]. The proposed device is similar to a parametric amplifier where the Josephson junction or SQUID loop is replaced by an asymmetric Josephson device known as the SNAIL (Superconducting Nonlinear Asymmetric Inductive eElement) [30, 31], is applied. Using a SNAIL or an asymmetrically threaded SQUID loop [32, 33], it is possible to realize three-wave mixing free of residual Kerr interactions by biasing the element at a certain external magnetic flux. In this work, we refer to this flux sweet spot as the Kerr-free point.

Therefore, it is meaningful to investigate a SNAIL-terminated resonator in circuit quantum electrodynamics (cQED) where the nonlinear resonator, decoupled to the waveguide, can be used for state preparation and storage. Strong dispersive coupling between oscillators and qubits was achieved a decade ago for photons [11] and recently for phonons [34–36], in linear resonators. Here, we observed the very well-resolved photon-number splitting up to the 9th-photon Fock state in a SNAI-terminated resonator coupled to a qubit. Our nonlinear resonator has a considerable relaxation time up to $T_1 = 20 \mu s$ under a few-photon drive, limited by (Two-Level Systems) TLSs. Our study opens the door to implement, operate and store quantum states on this scalable platform in the future. Moreover, compared to a linear resonator, our resonator has a non-negligible dephasing rate from its high sensivity to the magnetic flux noise due to the SNAIL loop.

Characterization of the SNAIL-terminated resonator. In order to characterize the parameters for the SNAIL-terminated resonator directly, we first fabricated a SNAIL-terminated $\lambda/2$ resonator capacitively coupled to a coplanar transmission line [not shown], which is the same as the nonlinear resonator in Fig. 1. By measuring the transmission
A superconducting qubit with a cross-shaped island (green) and a single Josephson junction (JJ, light blue in the middle inset), capacitively coupled to both a coplanar read-out resonator (red) and the nonlinear resonator (blue). The nonlinear resonator is formed by a linear coplanar resonator terminated by a superconducting nonlinear asymmetric inductive element (SNAIL) that has three big Josephson junctions and one small junction (orange in right inset). The charge line (pink) is used to operate the qubit and drive the nonlinear resonator. The qubit state read-out is implemented through the transmission of the feedline. The on-chip flux line is not used in this work, instead, a superconducting coil on the top of the chip [not shown] is used to generate the external magnetic flux \( \Phi_{\text{ext}} \) through the SNAIL. See more fabrication details and the measurement setup for this device in Supplementary.

### Table I. SNAIL-terminated resonator parameters at zero magnetic flux and near the Kerr-free point. The values are extracted from a transmission coefficient measurement on a SNAIL-terminated resonator coupled to a transmission line. We have \( \gamma_s = 1/T_s = \omega_s/Q_s \) where \( \gamma_s, T_s, \omega_s \), and \( Q_s \) are the resonator intrinsic decoherence rate, coherence time, frequency and the internal Q value, respectively. The intrinsic decoherence rate is given by \( \gamma_s = \Gamma_1 + \Gamma_2 \) where the intrinsic relaxation rate \( \Gamma_1 = \frac{1}{T_1} \) with the lifetime \( T_1 \) and the pure dephasing rate \( \Gamma_2 = \frac{1}{T_2} \) with the pure dephasing time \( T_2 \).

| \( \Phi_{\text{ext}}/\Phi_0 \) | \( \omega_s/2\pi \) (GHz) | \( Q_s \) (kHz) | \( \gamma_s/2\pi \) (s\(^{-1}\)) | \( T_s \) (\( \mu s \)) |
|---|---|---|---|---|
| 0 | 5.14 | 2.23 \times 10^3 | 23 | 6.92 |
| 0.386 | 4.31 | 3.86 \times 10^4 | 112 | 1.42 |

superconducting phase across the small junction, \( \phi_{\text{ext}} = 2\pi \Phi_{\text{ext}}/\Phi_0 \) is the reduced external magnetic flux, and \( E_1 \), related to the Josephson inductance \( L_1 \), is the Josephson energy of the big junctions in the SNAIL. Upon quantization, the Hamiltonian for the SNAIL-terminated resonator becomes

\[
H_s = \hbar \omega_s + g_3 (a + a^\dagger)^3 + g_4 (a + a^\dagger)^4, \tag{2}
\]

where \( g_3 \) (\( g_4 \)) is the couplings for the three (four)-wave mixing coupling strength, and \( \omega_s \) is the resonator frequency which follows the relation \( \omega_s / 2 = \omega_{r0} \tan (\pi / 2 \omega_{r0} / \omega_s) \) for the characteristic impedance of the resonator, and \( c_2 \) is a numerically determinable coefficient for the linear coupling whose specific value depends on \( \beta \) and \( \phi_{\text{ext}} \) in our case.
By sending a very weak probe with the average photon number in the resonator from the probe much less than one, we measure the corresponding transmission coefficient to extract the internal qubit frequency ω₂. For ω₀, as shown in Table I, we find that the coherence time in both cases is the SNAIL-terminated resonator frequency. As shown in Fig. 2a, at a fixed Φₐ, the coherence time is reduced by a factor of five due to the pure dephasing from the magnetic flux noise through the SNAIL.

A SNAIL-terminated resonator coupled to a qubit. In order to verify the nonlinear resonator performance in cQED in Fig. 1, the nonlinear resonator is dispersively coupled to a fixed frequency superconducting qubit with the bare qubit frequency \( ω₂ \approx 5.222 \) GHz. The effective Hamiltonian that contains the leading order corrections to the rotating wave approximation for describing the coupled SNAIL-terminated resonator and qubit in the dispersive regime \( (g₀ \ll \Delta₀) \) \(^{(4)}\) is

\[
\frac{H_{\text{eff}}}{\hbar} \approx ω_c a^\dagger a + K a^2 b^2 + \frac{ω_q}{2} b^\dagger b - \frac{χ₀}{2} a^\dagger a b^\dagger b^\dagger - \frac{α_q}{2} a^2 b^2,
\]

where the bare SNAIL-terminated resonator is dispersively shifted to be \( ω_c = ω_s - \frac{g₀^2}{2α_q} \), and the qubit frequency for the qubit mode b is \( ω_q = ω_q₀ + \frac{g₀^2}{2Δ₀} \) with the bare qubit frequency \( ω_q₀ \) and the Stark shift \( \frac{g₀^2}{2Δ₀} \).

\( g₀ \) is the coupling strength between the nonlinear resonator and the qubit, and \( \Delta₀ = ω_q₀ - ω_s \) is the detuning between the resonator and the bare qubit frequencies. The dispersive shift is \( χ₀ = \frac{g₀^2α_q}{Δ₀(Δ₀ - α_q)} \) with the qubit anharmonicity \( α_q \approx 450 \text{ MHz} \). Furthermore, near the Kerr-free point where the Kerr nonlinearity (strength \( K \)) is cancelled in the leading order, the residual Kerr nonlinearity is due to the interplay of four- and three-wave mixing processes \(^{(29)}\) as well as the qubit induced nonlinearity.

To experimentally find the qubit frequency, we sweep the frequency of a qubit-excitation pulse with the pulse length \( τ_p = 1 \) μs (spectroscopy pulse, see pulse generation in Supplementary). When the nonlinear resonator frequency is changed by the external flux \( Φ_{\text{ext}} \), the values of \( Δ₀ \) and the qubit frequency are also changed. In Fig. 2a, at a fixed \( Φ_{\text{ext}} \), we sweep the pulse frequency, and when the pulse is on resonance with the qubit, the qubit is excited. Then, we measure the qubit state by sending a readout pulse to obtain \( S(ρ_{ee}) \approx ρ_{ee} \), which is manifested as bright dots in Fig. 2a. After finding the qubit frequency and calibrating the corresponding π-pulses at different values of \( Φ_{\text{ext}} \), we send a continuous coherent drive to the nonlinear resonator meanwhile driving the qubit on resonance. However, when the drive is on resonance with the nonlinear resonator, the qubit is populated, leading to a change of the qubit frequency due to the dispersive interaction. Thus, the π-pulse will not excite the qubit anymore, resulting in a
FIG. 5. **Photon-number splitting and lifetime of the SNAIL-terminated resonator near the Kerr-free point.** a, the qubit-frequency distribution $P(\omega_q) = P(\omega_0) \times 10^6$ with the average photon number $|\alpha|^2 \approx 6$, of the coherent displacement in the nonlinear resonator. The envelope of the distribution matches well with a Poisson distribution with the corresponding coherent-state amplitude $|\alpha| \approx 2.4$ [red solid line]. b, Dispersive shift $\chi$ over different photon numbers $n$ inside the nonlinear resonator. The data is fitted to the equation $\chi = \chi_0 + \chi'(n-1)/2$ with $\chi_0/2\pi = 3.143 \pm 0.021$ MHz and the higher order correction $\chi'/2\pi = 35 \pm 11$ kHz. c, the qubit population $\rho_{ee}$ depending on the photon number $\langle n \rangle = |\alpha|^2$ in the nonlinear resonator. $\tau$ is the waiting time after the short displacement before the qubit read-out. d, The relaxation time $T_1$ of the nonlinear resonator vs. different photon number $\langle n \rangle$ inside the resonator. The error bars are for the standard deviation.

smaller signal $S(\rho_{ee})$ [Fig. 2b].

With the values of $\omega_{q0}$ and the dressed-qubit frequencies [Fig. 2a], we can obtain the values of the Stark shift at different external magnetic fluxes. Therefore, we can compensate the Stark shift from the qubit onto the SNAIL-terminated resonator to obtain the bare resonator frequency $\omega_s$ [blue dots in Fig. 3a from Fig. 2b]. With the parameters extracted from Fig. 3a, we can calculate the Kerr coefficient as a function of external magnetic flux [pink curve in Fig. 3b, see Supplementary]. At the device operation point $\Phi_{\text{ext}} = 0.386 \Phi_0$ [green dot in Fig. 3b], close to the Kerr-free point $\Phi_{\text{ext}} = 0.386 \Phi_0$, we deduce a residual Kerr at the operation point of $K = -0.31 \pm 0.04$ MHz with $g_3/2\pi = -11.6 \pm 0.4$ MHz, and $g_4/2\pi = -0.128 \pm 0.004$ MHz.

Due to the dispersive coupling to the qubit, the qubit can be used as a very efficient probe to determine the frequency of the SNAIL-terminated resonator. In our case, the linewidth of the nonlinear resonator is only a few kHz. Thus, it will be extremely difficult to find the resonator frequency based on the reflection coefficient measurement [42–45] through the charge line. However, since our qubit is coupled to the resonator with a dispersive shift up to a few MHz as shown in Fig. 5c, a few photons inside the resonator will shift the qubit frequency significantly. Consequently, even though the frequency detuning between the pump frequency and the resonator frequency is much larger than the linewidth of the resonator, as soon as the intensity of the pump is strong enough to inject a few photons inside the resonator, we can perform qubit spectroscopy to roughly find the resonator frequency. Afterwards, we can find the resonator frequency more accurately by decreasing the pump intensity [Fig. 4].

**Photon-number splitting near the Kerr-free point.** Once we have determined the nonlinear resonator frequency, we perform a pump-probe measurement consisting of a short pulse (50 ns) to excite the resonator
followed by a long qubit excitation pulse with \( \tau_p = 2 \mu s \) and frequency \( \omega_q \), along with a read-out pulse at the end to infer the qubit excited-state population. Due to the weak hybridization between the qubit and the nonlinear resonator near the Kerr-free point, the short coherent pulse drives the resonator into an approximately coherent state with amplitude \( |\alpha| \). We observed the qubit frequency splitting with the Fock states up to 9 photons where each peak has a Gaussian shape due to the quantum fluctuation of the photon number inside the resonator [46] [Fig. 5a]. The separation of each peak is about 3 MHz, ten times larger than the qubit linewidth \( \frac{2\pi}{\tau} \approx 280 \text{ kHz} \), leading to well-resolved peaks. The pulse length satisfies \( \tau_p \gg 1/\gamma \), resulting in a good dispersion resolution.

From the peak difference of the qubit spectroscopy [see Supplementary], we can extract the dispersive shift which increases with the photon number [Fig. 5b]. According to the dispersive shift \( \chi_0 \) and the qubit anharmonicity \( \alpha_q \), we can obtain the coupling strength \( g_0/2\pi \approx 53 \text{ MHz} \). Moreover, near the Kerr-free point, the self-Kerr \( K_{\alpha q} \) of the SNAIL-resonator induced by the qubit-resonator coupling [41, 47] is \( K_{\alpha q} = -24g_0^2/\Delta_0^2 \approx -2\pi \times 9 \text{ kHz} \), which is more than three hundred times smaller than the dispersive shift and more than 10 times smaller than \( \gamma_\alpha \). Thus, the residual Kerr coefficient from the qubit has a small influence on the states stored in our device, which can be also completely suppressed by slightly shifting the external flux \( \Phi_{\text{ext}} \).

**Lifetime of a single photon in the nonlinear resonator near the Kerr-free point.** Even though a statistic study can be done by measuring multiple resonators coupled to waveguides to infer the average lifetime, it is still important to study the relaxation time directly on a specific device in cQED since in reality the device performance varies over different devices due to the imperfect fabrication process.

Here, for the device shown in Fig. 1, utilizing the dispersive coupling to the qubit, we can determine the energy lifetime of the nonlinear resonator \( T_1 \) at the single-photon level. Again, we send a 50-ns pulse to displace the resonator with the initial amplitude \( |\alpha(\tau = 0)| \), and after a time delay \( \tau \), we apply a conditional \( \pi \)-pulse where the pulse is on resonance with the qubit frequency when the resonator is empty. Therefore, the qubit-excitation population depends on the photon number \( \langle n \rangle \). In Fig. 5c at \( \tau \sim 0 \), when \( \langle n \rangle \ll 1 \), the qubit frequency is still mainly centered at \( \omega_0/2\pi = 5.228 \text{ GHz} \), leading to the qubit population \( \rho_{ee} \approx 1 \). However when \( \langle n \rangle \gg 1 \), the qubit frequency starts to split, leading to a decreased \( \rho_{ee} \) where the population goes to zero smoothly and reaches \( \rho_{ee} \approx 0 \) with \( \langle n \rangle \approx 5.8 \). However, with increasing \( \tau \), the coherent field in the resonator decreases as \( \alpha(\tau) = \alpha(0) \exp(-\tau/(2T_1)) \) and the qubit frequency moves back. Thus, the conditional pulse can excite the qubit again.

Two-level systems (TLSs) are investigated extensively using conventional coplanar resonators [38, 48–53] along waveguides, where the photon number inside the resonator can be only roughly estimated according to the attenuation of the setup, and there is no TLS study in cavity quantum electrodynamics. Here, thanks to the accurate calibration of the photon number inside the nonlinear resonator, it is possible to directly study the TLS effect on our nonlinear resonator. The value of \( T_1 \) in Fig. 5d, extracted from the data in Fig. 5c [see Supplementary], increases from 8 \( \mu s \) to 20 \( \mu s \), which can be explained by saturation of two-level systems (TLS). According to the TLS model [50, 53, 54], the resonator internal \( Q_1 \) is given by

\[
\frac{1}{Q_1} = \frac{1}{\omega_1 T_{1}} = F \delta_{\text{TLS}} \frac{\tan^2 \left( \frac{\omega_0}{2\omega_1 T_{1}} \right)}{1 + \left( \frac{\langle n \rangle}{n_e} \right)} + \delta_{\text{other}},
\]

where \( F \) is the filling factor describing the ratio of electrical field in the TLS host volume to the total volume. \( \delta_{\text{TLS}} \) is the TLS loss tangent of the dielectric hosting the TLSs, \( n_e \) is the critical photon number within the resonator to saturate one TLS. \( \delta_{\text{other}} \) is the contribution from non-TLS loss mechanisms. A fit to Eq. (5) gives \( F \delta_{\text{TLS}} \approx 4.5 \times 10^{-6} \), \( \delta_{\text{other}} \approx 1.3 \times 10^{-6} \) and \( n_e \approx 0.1 \) photons [red solid curve in Fig. 5d], which is several orders of magnitude smaller than typical nonlinear resonators [37–39, 50, 52, 53, 55]. This can be explained by long-lived TLSs due to \( n_e \approx \sqrt{T_{1,\text{TLS}}/T_{2,\text{TLS}}} \) [54]. Additionally, we also analyse the non-TLS loss where the loss could be possibly from both the chip design and chip modes [See supplementary].

Compared to the coherence time of the SNAIL-terminated resonator coupled to a waveguide near the Kerr-free point, the relaxation time here is significantly larger, indicating that the coherence time is limited by the pure dephasing. Assuming that the resonator coupled to the waveguide has the same internal relaxation time as the nonlinear resonator coupled to a qubit, according to \( T_\phi = 1/(2T_1) + 1/T_0 \) with \( T_1 \approx 8 \mu s \) at \( \langle n \rangle \approx 0 \) from Fig. 5d, we can infer the pure dephasing time \( T_\phi \approx 12 \mu s \) and \( 2 \mu s \) at \( \Phi_{\text{ext}} = 0 \) and \( \Phi_{\text{ext}} = 0.386 \Phi_0 \), respectively. Especially, at \( \Phi_{\text{ext}} = 0.386 \Phi_0 \), the resonator becomes much more sensitive to the magnetic flux noise, leading to a shorter coherence time which could be increased by reducing the SNAIL parameter \( \beta \) or adding more magnetic shielding to suppress the magnetic-flux noise or decrease the loop size.

In this study, we developed an efficient method to characterize a SNAIL-terminated nonlinear resonator and observed well-resolved photon-number splitting up to nine photons, where the relaxation time can be up to 20 \( \mu s \). The parameters of our device are suitable for implementing universal gate sets for quantum computing [29]. Compared to linear modes coupled to qubits, in our case, the Kerr effect from the ancilla qubit can be suppressed, leading to a better storage of bosonic modes where the Kerr coefficient makes the quantum state collapse [56]. Moreover, by taking the advantages of the nonlinearity of our resonator, we could also generate and store entangled photon modes and non-Gaussian...
The dispersive coupling between the qubit and the resonator could also enable us to make quantum state tomography very efficient. Finally, compared to 3D cavities, our chip-scale structure is more compact for integration and scalability required by the large-scale CV quantum processing in the future.

SUPPLEMENTARY

Experimental setup. The complete experimental setup is shown in Fig. 6. The sample, as shown in Fig.1 in the main text, is wire-bonded in a nonmagnetic oxygen-free cooper sample box, mounted to the 10 mK stage of a dilution refrigerator, and shielded by a μ-metal can which is used for suppressing the static and fluctuating magnetic field. The signal to the qubit and the SNAIL-terminated resonator, with heavily attenuation, is combined to the charge line on the sample. A pulse is sent to the feedline on the sample to read out the state of the qubit with amplification using a traveling-wave parametric amplifier (TWPA) at the 10 mK stage, a high electron mobility transistor (HEMT) amplifier at 3 K and a room-temperature amplifier.

Sample Fabrication. We first clean a high-resistance (10 kΩ) 2-inch silicon wafer by hydrofluoric acid with 2% concentration. Then, we evaporate aluminum on top of the silicon substrate, followed by direct laser writing, and etching using wet chemistry to obtain all the sample details except the Josephson junction. The Josephson junctions for the qubit and the SNAIL are defined in a bi-layer resist stack using electron-beam lithography. Later, we deposit aluminum again by using a two-angle evaporation technique. In order to ensure a superconducting contact between the junctions and the rest of circuit, an argon ion mill is used to remove native aluminium oxide before the junction aluminium deposition. Finally, the wafer is diced into individual chips and cleaned properly using both wet and dry chemistry.

Device parameters for the SNAIL-terminated resonator coupled to the waveguide. By sending a microwave probe to the SNAIL-terminated resonator coupled to the waveguide [the device for Table I in the main text], we measure the transmission coefficient of a SNAIL-terminated resonator to obtain the bare resonator frequency [blue stars in Fig. 7(a)]. Then, we get $\beta = 0.095$ and $E_J/(2\pi\hbar) = \frac{\Phi_0}{2\pi L_J} \approx 830 \text{ GHz}$ with the Josephson inductance $L_J = 600 \text{ pH}$ by fitting the data to Eq.(3) in the main text [solid curve]. In Fig. 7(b), based on the values of $E_J$ and $\beta$, we numerically calculate the Kerr coefficient $K$ due to the four-wave mixing coupling, which $K$ can be suppressed to zero at the flux point $\Phi_{\text{ext}}/\Phi_0 = 0.39$.

Pulse generation and calibration. The microwave control for the qubit and the nonlinear resonator is achieved by up-converting the inphase (I) and the quadrature (Q) components of a low frequency pulse generated by four AWG channels (two each for the qubit and the resonator). The qubit read-out pulse is up-converted to the sample and then down-converted after the amplification with the same local oscillator (yellow area). The pump pulse for the displacement is generated by an IQ mixer with an input pulse from an Arbitrary Waveform Generator (AWG) with amplitude. In Fig. 8, we vary the amplitude of the pump pulse by changing the voltage amplitude $A$, and then obtain the corresponding displacement in the resonator according to the photon-number splitting as described in the main text. The data (black dots) is fitted to the equation $|\alpha| = kA$ with $k \approx 8$. This measurement is useful to calibrate the pump pulse with different amplitudes. We also notice that the residual thermal population with $A = 0$ is $|\alpha| = 0.16 \langle \bar{n} \rangle = |\alpha|^2 \approx 0.03$ photons. The thermal
and Poisson distributions are almost the same when the thermal photon number is small. In this case, we only take into account the single photon $P_1$ and the vacuum $P_0$ populations. Thus, we can calculate the corresponding thermal temperature as $T_{th} = \hbar \omega_c / [k_B \ln (1 + 1/\langle n \rangle)] \approx 58 \text{ mK}$. **Kerr coefficient calculation.** According the extracted parameters $\beta$ and $L_J$ from Fig. 3a in the main text, we can calculate the coefficients $g_3$ and $g_4$ for the three-wave mixing and four-wave mixing terms, respectively [Fig. 9]. Then, we can infer the coupling strength for the Kerr coefficient $K$ according to $K = 12(g_4 - 5g_3^2/\omega_c)$ [Fig.3b in the main text]. **Qubit spectroscopy.** In order to increase the signal-to-noise ratio, here, we average the qubit spectroscopy with different driven voltages [Fig. 10a] to obtain Fig. 10b.

![FIG. 7. Frequency and Kerr strength for a SNAIL-terminated resonator coupled to a waveguide. a, The frequency of the nonlinear resonator vs. the external magnetic flux through the SNAIL. Stars are the experimental data determined from the transmission coefficient of the vector analyzer network. The solid blue curve is calculated numerically. b, The Kerr-frequency shift per photon, $K$, vs. the external magnetic flux through the SNAIL. The Kerr-free point occurs at $\Phi_{\text{ext}} = 0.39 \Phi_0$.](#)

![FIG. 8. Pulse amplitude calibration. Calibration of the coherent displacement $\alpha$ with the voltage output $A$, of the arbitrary-wavefunction generator at the room-temperature.](#)

![FIG. 9. $g_3$ and $g_4$ numerical values for the SNAIL-terminated resonator coupled to a qubit. The numerical values of $g_3$ and $g_4$ for the SNAIL-terminated resonator coupled to a qubit.](#)

![FIG. 10. Qubit spectroscopy under different drive voltages. a, The qubit-frequency distribution $P(\omega_q) = P(\omega_q) \times 10^6$ Vs. different drive voltages. b, The averaged qubit-frequency distribution.](#)

![FIG. 11. Extract the value of $T_1$ from the qubit population. The qubit population $\rho_{ee}$ vs. the waiting time $\tau$ with $\langle n \rangle = 5.8$.](#)
the main text with the average photon number $\langle n \rangle = 5.8$. Here, we show an example to obtain the value of $T_1$ from Fig.5c in the main text. The qubit excitation probability is $\rho_{ee} = 1 - \exp[-|\alpha(0)|^2 \exp(-\gamma/\Omega)]$. By fitting the rising curve [Fig. 11] with the calibrated $|\alpha(0)| = \sqrt{5.8}$ ($\langle n \rangle = 5.8$), we extract $T_1 = 19.2 \pm 0.2 \mu s$, corresponding to a quality factor of $Q = \omega_0 T_1 \approx 5.18 \times 10^5$ which is close to the $Q_1$ value for a coplanar linear resonator in the few-photon limit [39, 55]. In order to improve $T_1$ further in the future, we also analyse the non-TLS loss. The non-TLS loss limits the lifetime to be $T_{\text{other}} = 1/(\delta_{\text{other}} \omega_c) \approx 28 \mu s$ comparable to our qubit lifetime $T_{1,q} \approx 20 \mu s$. In our sample, the nonlinear resonator couples to the qubit, charge line and a flux line where they contribute to the non-TLS loss as $\gamma_c/2\pi = (g_0/\Delta_0)^2 \frac{1}{2\gamma_1} \approx 170 \text{ Hz}$, $\gamma_c/2\pi \approx 2.07 \text{ kHz}$ and $\gamma_f/2\pi \approx 477 \text{ Hz}$, respectively (the values of $\gamma_c$ and $\gamma_f$ are from the simulation). In total, the loss from the chip design is $\gamma_{\text{total}} = \gamma_a + \gamma_c + \gamma_f \approx 2\pi \times 2.717$kHz corresponding to $T_{\text{total}} = 1/\gamma_{\text{total}} \approx 59 \mu s \approx 2T_{\text{other}}$, meaning that other unknown losses such as the chip modes are similar to the loss from the chip design.

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AUTHOR CONTRIBUTIONS

Y.L. planned the project. Y.L. performed the measurements with help from M.K. and J.Y.Y. Y.L. designed and fabricated the sample. T.H. and F.Q. helped to make the numerical calculation. Y.L. wrote the manuscript with input from all the authors. Y.L. analyzed the data. P.D. supervised this project.

COMPETING INTERESTS

The authors declare no competing interests.

REFERENCES

[1] Yuwei Ma, Yuan Xu, Xianghao Mu, Weizhou Cai, Ling Hu, Weiting Wang, Xiaojuan Pan, Haiyan Wang, YP Song, C-L Zou, and L Sun, “Error-transparent operations on a logical qubit protected by quantum error correction,” Nature Physics 16, 827 (2020).

[2] Ling Hu, Yuwei Ma, Weizhou Cai, Xianghao Mu, Yuan Xu, Weiting Wang, Yukai Wu, Haiyan Wang, YP Song, C-L Zou, et al., “Quantum error correction and universal gate set operation on a bosonic logical qubit,” Nature Physics 15, 503–508 (2019).

[3] Philip Reinhold, Serge Rosenblum, Wen-Long Ma, Luigi Frunzio, Liang Jiang, and Robert J Schoelkopf, “Error-corrected gates on an encoded qubit,” Nature Physics 16, 822 (2020).

[4] Alexander Grimm, Nicholas E Frattini, Shrutik Pur, Shantanu O Mundhada, Steven Touzard, Mazyar Mirrahimi, Steven M Girvin, Shyam Shankar, and Michel H Devoret, “Stabilization and operation of a kerr cat qubit,” Nature 584, 205 (2020).

[5] Jeffrey M Gertler, Brian Baker, Jiulang Li, Shrutik Shiroli, Jens Koch, and Chen Wang, “Protecting a bosonic qubit with autonomous quantum error correction,” Nature 590, 243 (2021).

[6] Yvonne Y Gao, Brian J Lester, Yaxing Zhang, Chen Wang, Serge Rosenblum, Luigi Frunzio, Liang Jiang, SM Girvin, and Robert J Schoelkopf, “Programmable interference between two microwave quantum memories,” Physical Review X 8, 021073 (2018).

[7] Brian Vlastakis, Gerhard Kirchmair, Zaki Leghtas, Simon E Nigg, Luigi Frunzio, Steven M Girvin, Mazyar Mirrahimi, Michel H Devoret, and Robert J Schoelkopf, “Deterministically encoding quantum information using 100-photon schrödinger cat states,” Science 342, 607 (2013).

[8] Chen Wang, Yvonne Y Gao, Philip Reinhold, Reinier W Heeres, Nissim Ofek, Kevin Chou, Christopher Axline, Matthew Reagor, Jacob Blumoff, KM Sliwa, et al., “A schrödinger cat living in two boxes,” Science 352, 1087 (2016).

[9] Philippe Campagne-Ibarcq, Alec Eickbusch, Steven Touzard, Evan Zalys-Geller, Nicholas E Frattini, Volodymyr V Sivak, Philip Reinhold, Shrutik Pur, Shyam Shankar, Robert J Schoelkopf, et al., “Quantum error correction of a qubit encoded in grid states of an oscillator,” Nature 584, 368 (2020).

[10] Marina Kudra, Mikael Kervinen, Ingrid Strandberg, Shahnawaz Ahmed, Marco Scigliuzzo, Amr Osman, Daniel Pérez Lozano, Mats O Tholén, Riccardo Bongarv, David B. Haviland, Giulia Ferrari, Jonas Bylander, Anton Frisk Kockum, Fernando Quijandria, Per Delsing,
and Simone Gasparinetti, “Robust preparation of wigner-negative states with optimized snap-displacement sequences,” PRX Quantum 3, 030301 (2022).

[11] DI Schuster, Andrew Addidon Houck, JA Schreier, Yiwen Chu, Prashanta Kharel, Taekwan Yoon, Luigi O.W. Kennedy, J. Burnett, J.C. Fenton, N.G.N. Sumedh Mahashabde, Ernst Otto, Domenico Reinier W. Heeres, Brian Vlastakis, Eric Holland, Stefan Max Hofheinz, EM Weig, M Ansmann, Radoslaw C H. Wang, M. Hofheinz, M. Ansmann, R. C. Bialczak, Margareta Wallquist, VS Shumeiko, and Göran Wendin, “Universal gate set for continuous-variable quantum computation,” Nature 544, 310 (2008).

[12] Yiwen Chu, Prashanta Kharel, Taekwan Yoon, Luigi Frunzio, Peter T Rakich, and Robert J Schoelkopf, “Creation and control of multi-phonon fock states in a bulk acoustic-wave resonator,” Nature 563, 666 (2018).

[13] Gustav Andersson, Baladitya Suri, Lingzhen Guo, Thomas Aref, and Per Delsing, “Non-exponential decay of a giant artificial atom,” Nature Physics 15, 1123 (2019).

[14] Reinier W. Heeres, Brian Vlastakis, Eric Holland, Stefan Krastanov, Victor V. Albert, Luigi Frunzio, Liang Jiang, and Robert J. Schoelkopf, “Cavity state manipulation using photon-number selective phase gates,” Phys. Rev. Lett. 115, 130502 (2015).

[15] Reinier W Heeres, Philip Reinhold, Nissim Ofek, Luigi Frunzio, Liang Jiang, Michel H Devoret, and Robert J Schoelkopf, “Implementing a universal gate set on a logical qubit encoded in an oscillator,” Nat. Commun. 8, 1 (2017).

[16] Margareta Wallquist, VS Shumeiko, and Göran Wendin, “Selective coupling of superconducting charge qubits mediated by a tunable stripline cavity,” Phys. Rev. B 74, 224506 (2006).

[17] Sunned Mahashabde, Ernst Otto, Domenico Montemurro, Sebastian de Graaf, Sergey Kubatkin, and Andrey Danilov, “Fast tunable high-$q$-factor superconducting microwave resonators,” Phys. Rev. Applied 14, 044040 (2020).

[18] O.W. Kennedy, J. Burnett, J.C. Fenton, N.G.N. Constantino, P.A. Warburton, J.J.L. Morton, and E. Dupont-Ferrier, “Tunable Nb superconducting resonator based on a constriction nano-squid fabricated with a Ne focused ion beam,” Phys. Rev. Applied 11, 014006 (2019).

[19] Agustín Palacios-Laloy, Francois Nguyen, Francois Mallet, Patrice Bertet, Denis Vion, and Daniel Esteve, “Tunable resonators for quantum circuits,” Journal of Low Temperature Physics 151, 1034 (2008).

[20] Michael R Vissers, Johannes Hubmayer, Martin Sandberg, Saptarshi Chaudhuri, Clint Bockstiegel, and Jiansong Gao, “Frequency-tunable superconducting resonators via nonlinear kinetic inductance,” Appl. Phys. Lett. 107, 062601 (2015).

[21] Martin Sandberg, CM Wilson, Fredrik Persson, Thilo Bauch, Göran Johansson, Vitaly Shumeiko, Tim Duty, and Per Delsing, “Tuning the field in a microwave resonator faster than the photon lifetime,” Appl. Phys. Lett. 92, 203501 (2008).

[22] B. H. Schneider, A. Bengtsson, I. M. Svensson, T. Are, G. Johansson, Jonas Bylander, and P. Delsing, “Observation of broadband entanglement in microwave radiation from a single time-varying boundary condition,” Phys. Rev. Lett. 124, 140503 (2020).

[23] C. W. Sandbo Chang, M. Simoen, José Aumentado, Carlos Sabin, P. Forn-Díaz, A. M. Vadraj, Fernando Quijandría, G. Johansson, I. Fuentes, and C. M. Wilson, “Generating multimode entangled microwaves with a superconducting parametric cavity,” Phys. Rev. Applied 10, 044019 (2018).

[24] C. W. Sandbo Chang, Carlos Sabin, P. Forn-Díaz, Fernando Quijandría, A. M. Vadraj, I. Nsanzieze, G. Johansson, and C. M. Wilson, “Observation of three-photon spontaneous parametric down-conversion in a superconducting parametric cavity,” Phys. Rev. X 10, 011011 (2020).

[25] A. Agustí, C. W. Sandbo Chang, F. Quijandría, G. Johansson, C. M. Wilson, and C. Sabin, “Tripartite genuine non-gaussian entanglement in three-mode spontaneous parametric down-conversion,” Phys. Rev. Lett. 125, 020502 (2020).

[26] Zhouyou Wang, Marek Pechal, E. Alex Wollack, Patricio Arrangoiz-Arriola, Maodong Gao, Nathan R. Lee, and Amir H. Safavi-Naeini, “Quantum dynamics of a few-photon parametric oscillator,” Phys. Rev. X 9, 021049 (2019).

[27] Timo Hillmann, Fernando Quijandría, Göran Johansson, Alessandro Ferraro, Simone Gasparinetti, and Giulia Ferrini, “Universal gate set for continuous-variable quantum computation with microwave circuits,” Phys. Rev. Lett. 125, 160501 (2020).

[28] N. E. Frattini, V. V. Sikva, A. Lingenfelter, S. Shankar, and M. H. Devoret, “Optimizing the nonlinearity and dissipation of a snail parametric amplifier for dynamic range,” Phys. Rev. Applied 10, 054020 (2018).

[29] V.V. Sikva, N.E. Frattini, V.R. Joshi, A. Lingenfelter, S. Shankar, and M.H. Devoret, “Kerr-free three-wave mixing in superconducting quantum circuits,” Phys. Rev. Applied 11, 054060 (2019).

[30] Raphael Lescanne, Marius Villiers, Théau Peronnin, Alain Sarlette, Matthieu Delbecq, Benjamin Huard, Takis Kontos, Mazyar Mirrahimi, and Zaki Leghtas, “Exponential suppression of bit-flips in a qubit encoded in an oscillator,” Nature Physics 16, 509 (2020).

[31] A Miano, G Liu, VV Sikva, NE Frattini, VR Joshi, W Dai, L Frunzio, and MH Devoret, “Frequency-tunable kerr-free three-wave mixing with a gradiometric snail,” Applied Physics Letters 120, 184002 (2022).

[32] L. R. Sletten, B. A. Moores, J. J. Viennot, and K. W. Lehnert, “Resolving phonon fock states in a multimode cavity with a double-slit qubit,” Phys. Rev. X 9, 021056 (2019).

[33] Patricio Arrangoiz-Arriola, E Alex Wollack, Zhouyou Wang, Marek Pechal, Wentao Jiang, Timothy P McKenna, Jeremy D Witmer, Raphaël Van Laer, and Amir H Safavi-Naeini, “Resolving the energy levels of a nanomechanical oscillator,” Nature 571, 537 (2019).

[34] Uwe von Lüpek, Yu Yang, Marius Bild, Laurent Michaud, and Simone Gasparinetti, “Robust preparation of wigner-negative states with optimized snap-displacement sequences,” PRX Quantum 3, 030301 (2022).
[37] J. Verjauw, A. Potočnik, M. Mongillo, R. Acharya, F. Mohiyaddin, G. Simion, A. Pacco, Ts. Ivanov, D. Wán, A. Vanleenehove, L. Souriau, J. Jussot, A. Thiam, J. Swerts, X. Piao, S. Couet, M. Heyns, B. Govoreanu, and I. Radu, “Investigation of microwave loss induced by oxide regrowth in high-q niobium resonators,” Phys. Rev. Applied 16, 014018 (2021).

[38] Greg Calusine, Alexander Melville, Wayne Woods, Rabindra Das, Corey Stull, Vlad Bolkhovsky, Danielle Braje, David Hover, David K Kim, Xhovalin Miloshi, et al., “Analysis and mitigation of interface losses in trenched superconducting coplanar waveguide resonators,” Appl. Phys. Lett. 112, 062601 (2018).

[39] D Kowsari, K Zheng, JT Monroe, NJ Thobaben, X Du, PM Harrington, EA Henriksen, DS Wisbey, and KW Murch, “Fabrication and surface treatment of electron-beam evaporated niobium for low-loss coplanar waveguide resonators,” Appl. Phys. Lett. 119, 132601 (2021).

[40] Sebastian Probst, FB Song, Pavel A Bushev, Alexey V Ustinov, and Martin Weides, “Efficient and robust analysis of complex scattering data under noise in microwave resonators,” Rev. Sci. Instrum. 86, 062408 (2015).

[41] Atsushi Noguchi, Alto Melville, Wayne Woods, Rabindra Das, Corey Stull, Vlad Bolkhovsky, Danielle Braje, David Hover, David K Kim, Xhovalin Miloshi, et al., “Analysis and mitigation of interface losses in trenched superconducting coplanar waveguide resonators,” Appl. Phys. Lett. 112, 062601 (2018).

[42] Yong Lu, Ingrid Strandberg, Fernando Quijandría, Göran Johansson, Simone Gasparinetti, and Per Delsing, “Propagating wigner-negative states generated from the steady-state emission of a superconducting qubit,” Phys. Rev. Lett. 126, 253602 (2021).

[43] Yong Lu, Andreas Bengtsson, Jonathan J Burnett, Emely Wiegang, Baladitya Suri, Philip Krantz, Anita Fadavi Roudsari, Anton Frisk Kockum, Simone Gasparinetti, Göran Johansson, et al., “Characterizing decoherence rates of a superconducting qubit by direct microwave scattering,” Npj Quantum Inf. 7, 35 (2021).

[44] Yong Lu, Neill Lambert, Anton Frisk Kockum, Ken Funo, Andreas Bengtsson, Simone Gasparinetti, Franco Nori, and Per Delsing, “Favorable transport and work with a single artificial atom coupled to a waveguide: Emission without external driving,” PRX Quantum 3, 020305 (2022).

[45] W-J Lin, Y Lu, PY Wen, Y-T Cheng, C-P Lee, K-T Lin, K-H Chiang, MC Hsieh, JC Chen, C-S Chuu, et al., “Deterministic loading and phase shaping of microwaves onto a single artificial atom,” https://arxiv.org/abs/2012.15084 (2020).

[46] Jay Gambetta, Alexandre Blais, D. I. Schuster, A. Wallraff, L. Frunzio, J. Majer, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, “Qubit-photon interactions in a cavity: Measurement-induced dephasing and number splitting,” Phys. Rev. A 74, 042318 (2006).

[47] Timo Hillmann and Fernando Quijandría, “Designing kerr interactions for quantum information processing via counterrotating terms of asymmetric josephson-junction loops,” Phys. Rev. Applied 17, 064018 (2022).

[48] W. Woods, G. Calusine, A. Melville, A. Sevi, E. Golden, D.K. Kim, D. Rosenberg, J.L. Yoder, and W.D. Oliver, “Determining interface dielectric losses in superconducting coplanar-waveguide resonators,” Phys. Rev. Applied 12, 014012 (2019).

[49] James Wenner, R Barends, RC Bialczak, Yu Chen, J Kelly, Erik Lucero, Matteo Mariantoni, A Megrant, PJJ O’Malley, D Sank, et al., “Surface loss simulations of superconducting coplanar waveguide resonators,” Appl. Phys. Lett. 99, 113513 (2011).

[50] Jonathan Burnett, Lara Faoro, I Wisby, VL Guertovoi, AV Chernykh, GM Mikhailov, VA Tulin, R Shaikhdaidarov, V Antonov, PJ Meeson, et al., “Evidence for interacting two-level systems from the 1/f noise of a superconducting resonator,” Nat. Commun. 5, 4119 (2014).

[51] SE de Graaf, L Faoro, LB Ioffe, S Mahashabde, JJ Burnett, T Lindström, SE Kubatkin, AV Danilov, and A Ya Tzalenchuk, “Two-level systems in superconducting quantum devices due to trapped quasiparticles,” Sci. Adv. 6, eabc5055 (2020).

[52] Jan David Brehm, Alexander Blimes, Georg Weiss, Alexey V Ustinov, and Jürgen Lisenfeld, “Transmission-line resonators for the study of individual two-level tunneling systems,” Appl. Phys. Lett. 111, 112601 (2017).

[53] Corey Rae Harrington McRae, Haozhi Wang, Jiansong Gao, Michael R Vissers, Teresa Brecht, Andrew Dunsworth, David P Pappas, and Josh Mutus, “Materials loss measurements using superconducting microwave resonators,” Rev. Sci. Instrum. 91, 091101 (2020).

[54] Jiansong Gao, The physics of superconducting microwave resonators (PhD thesis, California Institute of Technology, 2008).

[55] Jonathan Burnett, Andreas Bengtsson, David Niepce, and Jonas Bylander, “Noise and loss of superconducting aluminium reservoirs at single photon energies,” in J. Phys.: Conf. Ser., Vol. 969 (2018) p. 012131.

[56] Gerhard Kirchmair, Brian Vlastakis, Zaki Leghtas, Simon E Nigg, Hanhee Paik, Eran Ginossar, Mazyar Fadavi Roudsari, Anton Frisk Kockum, Simone Gasparinetti, and Per Delsing, “Transmission-line resonators for the study of individual two-level systems,” Appl. Phys. Lett. 111, 112601 (2017).

[57] Corey Rae Harrington McRae, Haozhi Wang, Jiansong Gao, Michael R Vissers, Teresa Brecht, Andrew Dunsworth, David P Pappas, and Josh Mutus, “Materials loss measurements using superconducting microwave resonators,” Rev. Sci. Instrum. 91, 091101 (2020).

[58] Jiansong Gao, The physics of superconducting microwave resonators (PhD thesis, California Institute of Technology, 2008).

[59] Jonathan Burnett, Andreas Bengtsson, David Niepce, and Jonas Bylander, “Noise and loss of superconducting aluminium reservoirs at single photon energies,” in J. Phys.: Conf. Ser., Vol. 969 (2018) p. 012131.

[60] Gerhard Kirchmair, Brian Vlastakis, Zaki Leghtas, Simon E Nigg, Hanhee Paik, Eran Ginossar, Mazyar Mirrahimi, Luigi Frunzio, Steven M Girvin, and Robert J Schoelkopf, “Observation of quantum state collapse and revival due to the single-photon Kerr effect,” Nature 495, 205 (2013).