Standardizing the gamma-ray bursts with the Amati $E_{p,i} - E_{iso}$ relation: the updated Hubble diagram and implications for cosmography

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ABSTRACT

The correlation between the peak photon energy of the internal spectrum $E_{p,i}$ and isotropic equivalent radiated energy $E_{iso}$ (the Amati relation) is explored in a scalar field model of dark energy. Using an updated data set of 109 high-redshift gamma-ray bursts (GRBs), we show that the correlation parameters only weakly depend on the cosmological model. Once the parameters of the Amati relation have been determined, we use this relation to construct a fiducial GRB Hubble diagram (HD) that extends up to redshifts $\sim 8$. Moreover, we apply a local regression technique to estimate, in a model-independent way, the distance modulus from the recently updated Union Type Ia supernova (SNIa) sample, containing 557 SNIa spanning the redshift range of $0.015 \leq z \leq 1.55$. The derived calibration parameters are used to construct an updated GRB HD, which we call the calibrated GRB HD. We also compare the fiducial and calibrated GRB HDs, which turned out to be fully statistically consistent, thus indicating that they are not affected by any systematic bias induced by the different calibration procedures. This means that the high-redshift GRBs can be used to test different models of dark energy settling the circularity problem. Furthermore, we investigate possible evolutionary effects that might have important influence on our results. Our analysis indicates that the presently available GRB data sets do not show statistically unambiguous evolutionary effect with the cosmological redshift. Finally, we propose another approach to calibrate the GRB relations, by using an approximate luminosity–distance relation, which holds in any cosmological model. We use this calibration of the Amati relation to construct an empirical approximate HD, which we compare with the calibrated GRB HD. We finally investigate the implications of this approach for the high-redshift cosmography.

Key words: gamma-ray burst: general – cosmological parameters – distance scale.

1 INTRODUCTION

At the end of the 1990s observations of high-redshift supernovae of Type Ia (SNIa) revealed that the Universe is now expanding at an accelerated rate. This surprising result has been independently confirmed by observations of small-scale temperature anisotropies of the cosmic microwave background (CMB) radiation (Astier et al. 2006; Riess et al. 2007; Spergel et al. 2007; Kowalski et al. 2008). It is usually assumed that the observed accelerated expansion is caused by a so-called dark energy with unusual properties. The pressure of dark energy $p_{de}$ is negative, and it is related to the positive energy density of dark energy $\epsilon_{de}$ by $p_{de} = w \epsilon_{de}$, where the proportionality coefficient $w < 0$. According to the present day estimates, about 75 per cent of matter energy in the Universe is in the form of dark energy, so that now the dark energy is the dominating component in the Universe. The nature of dark energy is not known. Proposed so far models of dark energy can be divided, at least, into three groups: (a) a non-zero cosmological constant, in this case $w = -1$, (b) a potential energy of some not yet discovered scalar field or (c) effects connected with non-homogeneous distribution of matter and averaging procedures. In the last two possibilities, in general, $w$ is not a constant and it depends on the redshift $z$. Observations of SNIa and small-scale anisotropies of the CMB radiation are consistent with the assumption that the observed accelerated expansion is due to the non-zero cosmological constant. However, so far the SNIa have been observed only at redshifts $z < 2$, while in order to test if $w$ is changing with redshift it is necessary to use more distant
objects. New possibilities opened up when the gamma ray bursts (GRBs) have been discovered at higher redshifts, the present record is at \( z = 8.26 \) (Greiner et al. 2009). GRBs are however enigmatic objects. First of all the mechanism that is responsible for releasing the incredible amounts of energy that a typical GRB emits is not yet known (see e.g. Meszaros 2006, for a recent review). It is also not yet definitely known if the energy is emitted isotropically or is beamed. Despite these difficulties, GRBs are promising objects that can be used to study the expansion rate of the Universe at high redshifts (Bloom, Frail & Kulkarni 2003; Bradley 2003; Schaefer 2003; Dai, Liang & Xu 2004; Firmani at al. 2005; Schaefer 2007; Amati et al. 2008; Li et al. 2008; Tsutsui et al. 2009). Using the observed spectrum and light curves it is possible to derive additional parameters, for example the peak photon energy \( E_{\text{p,i}} \), at which the burst is the brightest, and the variability parameter \( V \) which measures the smoothness of the light curve (for definitions of these and other parameters mentioned below, see Schaefer 2007). From observations of the afterglow, it is possible to derive another set of parameters; redshift is the most important, and also the jet opening angle \( \Theta_{\text{jet}} \), inferred from the achromatic break in the light curve of the afterglow, the time lag \( \tau_{\text{lag}} \), which measures the time offset between high- and low-energy GRB photons arriving at the detector, and \( \tau_{\text{GRB}} \) – the shortest time over which the light curve increases by half of the peak flux of the burst. Moreover, even though the most important parameters – the intrinsic luminosity \( L \) and the total (isotropic) radiated energy \( E_{\text{iso}} \) – are not directly observable, several correlations have been found between the \textit{additional parameters}, such as the peak photon energy \( E_{\text{p,i}} \), the variability \( V \), the time lag \( \tau_{\text{lag}} \), etc., and the GRBs radiated energy or luminosity. Assuming that GRBs emit radiation isotropically, it is possible to relate \( L \) to the observed bolometric peak flux \( P_{\text{bolo}} \) and the bolometric fluence \( S_{\text{bolo}} \), respectively, by

\[
L = 4\pi d_L^2(z, \text{cp}) P_{\text{bolo}}
\]

and

\[
E_{\text{iso}} = 4\pi d_L^2(z, \text{cp}) S_{\text{bolo}}(1 + z)^{-1},
\]

where \( d_L \) is the luminosity distance and \( \text{cp} \) denotes the set of cosmological parameters that specify the background cosmological model. Equivalently, one can use the total collimation corrected energy. It is clear from these relations that, in order to get the intrinsic luminosity, it is necessary to specify the fiducial cosmological model and its basic parameters. But we want to use the observed properties of GRBs to derive the cosmological parameters. Several procedures to overcome this complicated circular situation have been proposed (see e.g. Schaefer 2007; Basilakos & Perivolaropoulos 2008; Cardone, Capozziello & Dainotti 2008; Demianski, Piedipalumbo & Rubano 2011). In this paper we apply a Bayesian-motivated technique, already implemented in Demianski et al. (2011) to standardize 109 long GRBs with the well-known Amati relation, i.e. to find the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation parameters, where \( E_{\text{p,i}} \) is the peak photon energy of the intrinsic spectrum and \( E_{\text{iso}} \) is the isotropic equivalent radiated energy. This correlation needs only two parameters that are directly inferred from observations (this fact minimizes the effects of systematics and increases the number of GRBs that can be used by a factor of \( \sim 3 \)) makes this correlation an interesting tool for cosmology. Previous analyses of the \( E_{\text{p,i}}-E_{\text{iso}} \) plane of GRB parameters showed that different classes of GRBs exhibit different behaviours, and while normal long GRBs and X-ray flashes (XRFs, i.e. particularly soft bursts) follow the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation, short GRBs and the peculiar very near and subenergetic GRBs do not (Amati et al. 2008). This fact may depend on the different emission mechanisms involved in different classes of GRBs and makes the \( E_{\text{p,i}}-E_{\text{iso}} \) relation a useful tool to distinguish between them (Antonelli 2009). The impact of selection and instrumental effects on the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation of long GRBs were investigated since 2005, mainly based on the large sample of Burst and Transient Source Experiment (BATSE) GRBs with unknown redshifts. Different authors came to different conclusions (see e.g. Ghirlanda, Ghisellini & Firmani 2005). In particular, Ghirlanda at al. (2005) showed that BATSE events potentially follow the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation and that the question to clarify is if, and how much, its measured dispersion is biased. There were also claims that a significant fraction of \textit{Swift} GRBs is inconsistent with this correlation (Butler et al. 2007). However, when considering those \textit{Swift} events with peak energy measured by broad-band instruments like, e.g. Konus–\textit{Wind} or the Fermi/Gamma-ray Burst Monitor or reported by the Burst Alarm Telescope (BAT) team in their catalog (Sakamoto et al. 2008), it is found that they are all consistent with the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation as determined with previous/other instruments (Amati, Frontera & Guidorzi 2009). In addition, it turns out that the slope and normalization of the correlation based on the data sets provided by GRB detectors with different sensitivities and energy bands are very similar. These facts further support the

2 STANDARDIZING THE GRBs AND CONSTRUCTING THE HUBBLE DIAGRAM

In this section, we investigate the possibility of constructing the HD from the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation; here \( E_{\text{p,i}} \) is the peak photon energy of the intrinsic spectrum and \( E_{\text{iso}} \) is the isotropic equivalent radiated energy. \( E_{\text{iso}} \) is defined by equation (2). This correlation was initially discovered in a small sample of \textit{BeppoSAX} GRBs with known redshifts (Amati et al. 2002) and confirmed afterwards by HETE-2 and \textit{Swift} observations (Lamb, Donaghy & Graziani 2005; Amati 2006). Although it was the first correlation discovered for GRB observables, it was never used for cosmology because of its significant ‘extrinsic’ scatter. However, the recent increase in the efficiency of GRB discoveries combined with the fact that \( E_{\text{p,i}}-E_{\text{iso}} \) correlation needs only two parameters that are directly inferred from observations (this fact minimizes the effects of systematics and increases the number of GRBs that can be used by a factor of \( \sim 3 \)) makes this correlation an interesting tool for cosmology. Previous analyses of the \( E_{\text{p,i}}-E_{\text{iso}} \) plane of GRB parameters showed that different classes of GRBs exhibit different behaviours, and while normal long GRBs and X-ray flashes (XRFs, i.e. particularly soft bursts) follow the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation, short GRBs and the peculiar very near and subenergetic GRBs do not (Amati et al. 2008). This fact may depend on the different emission mechanisms involved in different classes of GRBs and makes the \( E_{\text{p,i}}-E_{\text{iso}} \) relation a useful tool to distinguish between them (Antonelli 2009). The impact of selection and instrumental effects on the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation of long GRBs were investigated since 2005, mainly based on the large sample of Burst and Transient Source Experiment (BATSE) GRBs with unknown redshifts. Different authors came to different conclusions (see e.g. Ghirlanda, Ghisellini & Firmani 2005). In particular, Ghirlanda at al. (2005) showed that BATSE events potentially follow the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation and that the question to clarify is if, and how much, its measured dispersion is biased. There were also claims that a significant fraction of \textit{Swift} GRBs is inconsistent with this correlation (Butler et al. 2007). However, when considering those \textit{Swift} events with peak energy measured by broad-band instruments like, e.g. Konus–\textit{Wind} or the Fermi/Gamma-ray Burst Monitor or reported by the Burst Alarm Telescope (BAT) team in their catalog (Sakamoto et al. 2008), it is found that they are all consistent with the \( E_{\text{p,i}}-E_{\text{iso}} \) correlation as determined with previous/other instruments (Amati, Frontera & Guidorzi 2009). In addition, it turns out that the slope and normalization of the correlation based on the data sets provided by GRB detectors with different sensitivities and energy bands are very similar. These facts further support the
reliability of the Amati correlation (Amati et al. 2009). It is clear from equation (2) that, in order to get $E_{\text{iso}}$, it is necessary to specify the fiducial cosmological model. In Demianski et al. (2011), we fitted the Amati relation in a quintessence cosmological model, where the dark energy is described by the exponential potential of the scalar field discussed in Demianski et al. (2005). In our analysis, we consider a sample of 109 long GRBs/XRFs, adding to the sample of 95 long GRBs/XRFs compiled in Amati et al. (2008, 2009) data of 14 unpublished GRBs, kindly provided by Amati (private communication). Their redshift distribution covers a broad range of $z$, from 0.033 to 8.23, thus extending far beyond that of SNIa ($z < \sim 1.7$), and including GRB 092304, the new high-$z$ record holder of GRBs.

2.1 Fitting the $E_{p,i}$-$E_{\text{iso}}$ relation and estimating its parameters

In this section, we present results of our analysis of the Amati correlation performed on a new updated data set, assuming that the background cosmological model is one of the quintessence models that we have studied some time ago (Demianski et al. 2005), and show that it is consistent with the basic cosmological tests, and which has been recently referred to as the fiducial model to standardize the GRBs and to construct the HD using previous samples of GRBs. First of all, we consider the $E_{p,i}$-$E_{\text{iso}}$ relation in the form

$$\log \left( \frac{E_{\text{iso}}}{1 \text{ erg}} \right) = b + a \log \left[ \frac{E_{p,i} (1+z)}{300 \text{ keV}} \right],$$

where $a$ and $b$ are constants. In fitting this relation, we need to fit a data array $\{x_i, y_i\}$ with uncertainties $\{\sigma_{x,i}, \sigma_{y,i}\}$ to a straight line

$$y = b + ax,$$

in order to determine the two fit parameters ($a, b$). Actually, the situation is not so simple since both the ($y, x$) variables are affected by measurement uncertainties ($\sigma_{y}, \sigma_{x}$) which cannot be neglected. Moreover, $\sigma_{y} \sim \sigma_{x}$ so that it is impossible to choose as independent variable the fit the one with the smallest relative error. Finally, the correlation we are fitting is not of theoretical nature, i.e. it is not (yet) derived from an underlying theoretical model determining the detailed features of the GRB explosion and afterglow phenomenology. Indeed, we do expect a certain amount of intrinsic scatter, $\sigma_{\text{int}}$, around the best-fitting line that has to be taken into account and determined together with ($a, b$) by the fitting procedure. Different statistical recipes are available to cope with these problems. As in Demianski et al. (2011), we apply a Bayesian-motivated technique (D’Agostini 2005) maximizing the likelihood function $L(a, b, \sigma_{\text{int}}) = \exp \left[ -L(a, b, \sigma_{\text{int}}) \right]$ with

$$L(a, b, \sigma_{\text{int}}) = \frac{1}{2} \sum \ln \left( \sigma_{\text{int}}^2 + \sigma_i^2 + a^2 \sigma_0^2 \right) + \frac{1}{2} \sum \frac{(y_i - ax_i - b)^2}{\sigma_{\text{int}}^2 + \sigma_i^2 + a^2 \sigma_0^2},$$

where the sum is over the $N$ objects in the sample. Note that, actually, this maximization is performed in the two-parameter space ($a, \sigma_{\text{int}}$) since $b$ may be estimated analytically by solving the equation $\frac{\partial}{\partial b} L(a, b, \sigma_{\text{int}}) = 0$, as

$$b = \left[ \sum \frac{y_i - ax_i}{\sigma_{\text{int}}^2 + \sigma_i^2 + a^2 \sigma_0^2} \right] \left[ \sum \frac{1}{\sigma_{\text{int}}^2 + \sigma_i^2 + a^2 \sigma_0^2} \right]^{-1}.$$

To quantitatively estimate the goodness of this fit, we use the median and root mean square of the best-fitting residuals, defined as $\delta = y_{\text{obs}} - y_{\text{th}}$. To quantify the uncertainties of some fit parameter $p_i$, we evaluate the marginalized likelihood $L(p_i)$ by integrating over the other parameter. The median value for the parameter $p_i$ is then found by solving

$$\int_{p_{i,\text{min}}}^{p_{i,\text{med}}} L(p_i) \, dp_i = \frac{1}{2} \int_{p_{i,\text{min}}}^{p_{i,\text{max}}} L(p_i) \, dp_i,$$

The 68 per cent (95 per cent) confidence range ($p_{i,\text{L}}, p_{i,\text{H}}$) are then found by solving (D’Agostini 2005)

$$\int_{p_{i,\text{med}}}^{p_{i,\text{L}}} L(p_i) \, dp_i = \frac{1 - \varepsilon}{2} \int_{p_{i,\text{min}}}^{p_{i,\text{med}}} L(p_i) \, dp_i,$$

$$\int_{p_{i,\text{med}}}^{p_{i,\text{H}}} L(p_i) \, dp_i = \frac{1 - \varepsilon}{2} \int_{p_{i,\text{med}}}^{p_{i,\text{max}}} L(p_i) \, dp_i,$$

with $\varepsilon = 0.68$ and 0.95 for the 68 and 95 per cent confidence levels, respectively. Just considering our correlation in equation (3) we find that the likelihood method gives $a = 1.52$, $b = 52.67$ and $\sigma_{\text{int}} = 0.41$. The marginalized likelihood functions are shown in Fig. 1. In Fig. 2, we show the likelihood contours in the ($a, \sigma_{\text{int}}$) plane and in Fig. 3 we show the correlation between the observed $\log E_{p,i}$ and derived $\log E_{\text{iso}}$ with our assumed background cosmological model. The solid line is the best fit obtained using the D’Agostini’s method (D’Agostini 2005) and the dashed line is the best fit obtained by the weighted $\chi^2$ method. If one marginalizes$^1$ with respect to $b$, then the likelihood values of $a$ and $\sigma_{\text{int}}$ are $a = 1.52 \pm 0.13$ and $\sigma_{\text{int}} = 0.45 \pm 0.05$. The performed statistical analysis shows that

$^1$ It is worth noting that in the marginalization procedure we have to take into account also equation (6).
at the 65 per cent confidence level, and the $y = \log (1 + z)$ of the residuals, $\delta = y_{\text{obs}} - y_{\text{fit}}$, for the $E_{p,i} - E_{\text{iso}}$ correlation. We see that the figure does not exhibit any evidence of correlation.

Figure 2. Regions of 68, 95 and 99 per cent of confidence in the space of parameters $a, \sigma_{\text{int}}$.

Figure 3. Best-fitting curves for the $E_{p,i} - E_{\text{iso}}$ correlation superimposed on the data. The solid and dashed lines refer to the results obtained with the maximum likelihood and weighted $\chi^2$ estimator, respectively.

relation (3) has a statistical weight similar to the one exhibited by the other relations previously studied in Demianski et al. (2011), since both $\delta_{\text{int}}$ and $\delta_{\text{in}}$ have almost the same values over the full set (of relations), where $\delta = y_{\text{obs}} - y_{\text{fit}}$. In our analysis, we have assumed that the fit parameters do not change with the redshift, which indeed spans a quite large range (from $z = 0.0331$ up to $z \simeq 8$). The limited number of GRBs prevents detailed exploration of the validity of this usually adopted working hypothesis, which we tested somewhat investigating if the residuals correlate with the redshift. We have not found any significant correlation, as shown in Fig. 4. Moreover, we tested the fit of the $E_{p,i} - E_{\text{iso}}$ correlation with respect to the evolution with redshift, separating the GRB samples into four groups corresponding to the following redshift bins: $z \in [0, 1], z \in (1, 2], z \in (2, 3] \text{ and } z > 3$. We thus maximized the likelihood in each group of redshifts and determined the best-fitting parameters $a, b$, and the intrinsic dispersion $\sigma_{\text{int}}$, as summarized in Table 1. It turns out that no statistical evidence of a dependence of the $(a, b, \sigma_{\text{int}})$ parameters on the redshift exists. This is in agreement with what has recently been found by Ghirlanda et al. (2008) and Wang, Deng & Qu (2008), or, as regards to other correlations, by Cardone et al. (2008) and Basilakos & Perivolaropoulos (2008), and also confirmed in our previous paper (Demianski et al. 2011).

2.2 Constructing the Hubble diagram

Once the Amati correlation has been fitted, and its parameters have been estimated, we can now use them to construct the estimated fiducial GRB HD. Actually, let us remind that the luminosity distance of a GRB with the redshift $z$ may be computed as

$$d_L(z)^2 = \left( \frac{E_{\text{iso}}(1 + z)}{4\pi S_{\text{bolo}}} \right).$$ (10)

The uncertainty of $d_L(z)$ is then estimated through the propagation of the measurement errors on the involved quantities. In particular, recalling that our correlation can be written as a linear relation, as in equation (4), where $y$ is the distance-dependent quantity, while $x$ is not, the error on the distance-dependent quantity $y$ is estimated as

$$\sigma(y) = \sqrt{\sigma^2(x) + \sigma_{\text{int}}^2},$$ (11)

and is then added in quadrature to the uncertainties of the other terms entering equation (10) to get the total uncertainty. The distance modulus $\mu(z)$ is easily obtained from its definition:

$$\mu(z) = 25 + 5 \log d_L(z),$$ (12)

with its uncertainty obtained again by error propagation. We finally estimate the distance modulus for each $i$th GRB in our sample at redshift $z_i$ to build the HD plotted in Fig. 5. In what follows, we will refer to this data set as the fiducial GRB HD since to compute the distances it relies on the calibration based on the fiducial quintessential model. We also investigated the impact of varying the parameters of our fiducial cosmological model, fitting the Amati correlation on a regular grid in the space of parameters of our quintessential model, as allowed by the data (Demianski et al. 2005). For each point in the grid, we repeated all the steps described above to get the distance modulus to each GRB in the sample. We then collect these values and evaluate, for each GRB, the root mean square of the percentage deviation from the fiducial $\mu$ value. In this way, we are performing a sort of average of the absolute percentage deviation, which allows us to quantify the order of magnitude of the studied effect. Actually, it turns out that the distance modulus may be underestimated or overestimated by a modest 0.3 per cent with values never larger than 1 per cent.
**2.2.1 Cosmological-independent calibration: local regression on SNIa**

Although the above analysis has shown that the choice of the underlying cosmological model has only a modest impact on the final estimate of the distance modulus, we compared our estimated fiducial HD with a model-independent calibrated HD, carried out using SNIa as distance indicators. We apply local regression to estimate the distance modulus \( \mu(z) \) from the recently updated SNIa sample, the Supernova Cosmology Project (SCP) Union2 compilation (Amanullah et al. 2010), which is an update of the original Union compilation, now bringing together data for 719 SNe, drawn from 17 data sets. Of these, 557 SNe, spanning the range \( 0.015 \leq z \leq 1.55 \), form the final sample considered in our analysis. To use this large data set as an input to the local regression estimate of \( \mu(z) \), we have firstly to set a redshift \( z \), where \( \mu(z) \) has to be recovered, order the SNIa data set according to the increasing value of \( |z - z_i| \) and select the first \( n = aN_{\text{SNIa}} \), where \( a \) is a user-selected value and \( N_{\text{SNIa}} \) is the total number of SNIa. Then we can fit a first-order polynomial to the previously selected data, weighting each SNIa with the corresponding value of an appropriate weight function, like, for instance

\[
W(u) = \begin{cases} 
(1 - |u|^2)^2 & |u| \leq 1, \\
0 & |u| \geq 1, 
\end{cases}
\]

and take the zeroth-order term as the best estimate of \( \mu(z) \). Here, \( u = |z - z_i|/\Delta \) and \( \Delta \) is the maximum value of the \( |z - z_i| \) over the subset chosen before. To estimate the error on \( \mu(z) \), we use the root mean square of the weighted residuals with respect to the best-fitting zeroth-order term.\(^2\) Having found an efficient way of estimating the distance modulus at redshift \( z \) in a model-independent way, we can now fit the \( E_{\text{iso}} - E_{\text{peak}} \) correlation, using the local regression reconstructed \( \mu(z) \) in equation (2). We consider only GRBs with \( z \leq 1.55 \) in order to cover the same redshift range spanned by the SNIa data. For such subset of GRBs we apply the Bayesian-fitting procedure described above to estimate the correlation parameters. We use other GRBs to construct a new GRB HD that we call the calibrated GRB HD. In Fig. 6, we show the likelihood contours in the \( (a, \sigma_{\text{int}}) \) plane, and the marginalized likelihood functions are shown in Fig. 7.

Moreover, it turns out that the estimated and calibrated HDs are fully statistically consistent, as shown in Fig. 8.

\(^2\) It is worth stressing that both the choice of the weight function and the order of the fitting polynomial are somewhat arbitrary. Similarly, the value of \( a \) to be used must not be too small in order to make up a statistically valuable sample, but also not too large to prevent the use of a low-order polynomial. In our local regression routine, we have performed an extensive set of simulations, ending up with a mock catalogue having the same redshift and error distribution of the actual SNIa one. This mock catalogue is used as an input to the routine sketched above, and finally the reconstructed \( \mu(z) \) value for each point in the catalogue is compared to the input one (see Cardone et al. 2008 for details).
In equation (15), $v$ is the affine parameter, $T_{\mu\nu}$ is the matter density tensor, $k^\alpha = \Sigma^\alpha_{\mu\nu} = -\Sigma_{\nu\mu}$ is the vector field tangent to the light ray congruence and $\Sigma$ is the null surface along which the light rays propagate from the source. In the general form equation (15) is very complicated. General properties of this equation have been extensively studied (see e.g. Kantowski 1998; Kantowski, Kao & Thomas 2000; Kantowski & Thomas 2001; Demianski et al. 2003). In most cases, equation (15) does not have analytical solution, and from the mathematical point of view it can be reduced to a Fuchsian type with several regular singular points and a regular singular point at infinity. The solutions near each of these singular points can be expanded in a series of hypergeometric functions. When we introduced the dimensionless angular diameter distance $r = DH_0/c$, we discovered (Demianski et al. 2003) that there is a simple function $r(z)$, which quite accurately reproduces the exact numerical solutions of equation (15) for $z$ up to very high values; it has the form

$$r(z) = \frac{z}{\sqrt{d_1 z^2 + (1 + d_2 z + d_3 z^2)^2}},$$

(17)

where $d_1$, $d_2$ and $d_3$ are constants. Moreover, function (17) automatically satisfies the imposed initial conditions, so $r(0) = 0$ and $\frac{dr}{dz}(0) = 1$. This approximate expression immediately provides an empirical formula for the luminosity–distance relation of the SNIa, through the equation (1). Fitting relation (17) to the SNIa Union data set, we obtained the following best-fitting values for the fitting parameters $d_i$:

$$d_1 = -3.87,$$

(18)

$$d_2 = 1.51,$$

(19)

$$d_3 = 0.70.$$

(20)

In Fig. 9, we show the approximate distance modulus with the SNIa data superimposed. The approximate function agrees with the real data within a relative error not larger than a few per cent in the SNIa redshift range, as shown in Fig. 10.

From the empirical approximate luminosity distance we can construct the empirical approximate distance modulus $\mu_{\text{approx}}(z)$, which we use to calibrate in a cosmological-independent way, alternative to the one described above, the $E_{p,i} - E_{\text{iso}}$ (Amati) correlation, following our standard procedure, and considering only the GRBs with $z \leq 1.55$ in order to cover the same redshift range spanned by the SNIa data. We use other GRBs to construct a new GRB HD that we call the approximate calibrated GRB HD. In Fig. 11, we show the correlation between the observed $\log E_{p,i}$ and derived $\log E_{\text{iso}}$ with

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**Figure 7.** Marginalized likelihood functions constructed by the local regression on SNeIa: the likelihood function, $\mathcal{L}_a$, is obtained marginalizing over $\sigma_{\text{int}}$, and the likelihood function, $\mathcal{L}_{\text{int}}$, is obtained marginalizing over $a$.

**Figure 8.** Comparison of the distance modulus $\mu(z)$ for the calibrated and estimated GRB HD made up fitting the Amati correlation. The red solid line represents the graph of the function $F(\mu_{\text{SN}}) = \mu_{\text{SN}}$, and it turns out that it is also the fitting function if we do not include a constant term in the list of basis functions (needed to perform the fit procedure). The dashed green line is the best-fitting function.

**Figure 9.** Behaviour of the approximate modulus of distance with the redshift with the SNIa Union data set superimposed.
Figure 10. Behaviour with the redshift of the relative residuals, \( \delta = \frac{r_{\text{obs}} - r_{\text{fit}}}{r_{\text{obs}}} \), of the fit procedure which provides our empirical formula in equation (17), which agrees with the real data within a relative error of magnitude not more than few per cent.

Figure 11. Best-fitting curves for the \( E_p - E_{\text{iso}} \) correlation, obtained using our empirical approximate luminosity–distance formula, superimposed on the data. The solid and dashed lines refer to the results obtained with the Bayesian and Levenberg–Marquardt estimator, respectively.

our approximate luminosity distance. The solid line is the best fit obtained using the D’Agostini’s method (D’Agostini 2005) and the dashed line is the best fit obtained by the weighted \( \chi^2 \) method. In order to check the reliability of the approximate calibrated GRB HD, we compare it with the calibrated (SNIa) HD. It turns out that these HDs are fully statistically consistent, as shown in Figs 12 and 13, and the resulting distances are strongly correlated with the Spearman’s correlation \( \rho = 0.99 \).

3.1 Implications for high-redshift cosmography

Recently the cosmographic approach to cosmology gained increasing interest for catching as much information as possible directly from observations, retaining the minimal priors of isotropy and homogeneity and leaving aside any other assumptions. Actually, the only ingredient taken into account a priori in this approach is the Friedmann–Lemaître–Robertson–Walker (FLRW) line element obtained from kinematical requirements:

\[
ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right].
\]

Using this metric, it is possible to express the luminosity distance \( d_L \) as a power series in the redshift parameter \( z \), the coefficients of the expansion being functions of the scale factor \( a(t) \) and its higher order derivatives. Such an expansion leads to a distance–redshift relation which only relies on the assumption of the Robertson–Walker metric thus being fully model independent since it does not depend on the particular form of the solution of cosmic evolution equations. To this aim, it is convenient to introduce the following parameters (Visser 2004):

\[
H = \frac{1}{a} \frac{da}{dt},
\]

(22)

\[
q = -\frac{1}{a} \frac{d^2 a}{dt^2},
\]

(23)

\[
j = \frac{1}{a} \frac{d^3 a}{dt^3}.
\]

(24)
\[ s = \frac{a^6}{H^2}. \]

which are usually referred to as the Hubble, deceleration, jerk and snap parameters, respectively. Their present day values (which we will denote with a subscript 0) may be used to characterize the evolutionary status of the Universe. For instance, \( q_0 < 0 \) denotes an accelerated expansion, while \( j_0 \) allows us to discriminate among different accelerating models. It is worth noting that it is possible to infer implications for cosmography just using our empirical formula (17) of the luminosity distance in our analysis. Actually, we first recast our approximate \( d_L \) as a function of a new variable \( y = z/(1+z) \) (Capozziello & Izzo 2010; Vitagliano et al. 2010) in such a way that \( z \in (0, \infty) \) is mapped into \( y \in (0, 1) \), obtaining

\[ d_L^{\text{approx}}(y) = \frac{c}{H_0} \left( \frac{y}{y-1} \right)^{\frac{1}{2}} \sqrt{\frac{d_L^2}{y-1} + \left( -\frac{d_L^2}{y-1} + \frac{d_L^2}{y-1} + 1 \right)^2}. \] (26)

Expanding our approximate luminosity distance up to the fourth order in the \( y \) parameter, we get

\[
d_L(y) = \frac{c}{H_0} \left( y^3 \left( -\frac{d_1}{2} + d_2^2 - 4d_2 - d_3 + 6 \right) + y^4 \left( \frac{d_1}{2}(3d_2 - 5) - d_2^2 + 5d_2^2 + 2d_2(d_3 - 5) - 5(d_3 - 2) \right) - (d_3 - 3)y^2 + y \right). \] (27)

It is now possible to relate the \( d_L \) to the cosmographic parameters \( q_0, j_0 \) and \( s_0 \) by comparing the expansion in equation (27) with the standard expansion to the fourth order:

\[
d_L(y) = \frac{c}{H_0} \left( y^3 \left( -\frac{1}{2}q_0 - 3 \right)y^2 + \frac{1}{6} \left[ 12 - 5q_0 + 3q_0^2 - j_0 \right] y^3 \right.
\]
\[ + \frac{1}{24} \left[ 60 - 7j_0 - 10 - 32q_0 + 10q_0j_0 + 6q_0 \right] y^4 + O(y^5) \].
\] (28)

It turns out that

\[ d_1 = 1 + j_0 + q_0 + 6j_0q_0 - 2q_0^2(1 + 3q_0) + s_0, \] (29)
\[ d_2 = \frac{3 + q_0}{2}, \] (30)
\[ d_3 = \frac{14 + j_0(5 - 4q_0) + q_0(16 + 3(-1 + q_0)q_0) - s_0}{12(3 + q_0)}. \] (31)

Inverting the systems of equations (29)–(31), it is possible to recover the cosmographic parameters \( q_0, j_0 \) and \( s_0 \) as the functions of our fitted parameters \( d_L \). Actually, we get

\[ q_0 = -3 + 2d_2, \] (32)
\[ j_0 = 17 + 3d_1 - 22d_2 + 6d_2^2 + 6d_3, \] (33)
\[ s_0 = d_1(51 - 24d_2) + 158d_2^2 - 24d_2^3 - 8d_2(35 + 9d_3) + 3(49 + 34d_3). \] (34)

These equations, together with the values of the fitting parameters in equations (18)–(20) with the corresponding physically acceptable regions of confidence,\(^4\) allow us to estimate the corresponding parameter confidence intervals for \( q_0, j_0 \) and \( s_0 \). We actually obtain that,

\[
\begin{align*}
q_0 & = -1.11 \quad 0.27 \\
j_0 & = -23.9 \quad 1.70 \\
s_0 & = -256.4 \quad -34.7
\end{align*}
\]

the best fit being \( q_0 = -0.45, j_0 = -12.3 \) and \( s_0 = -99.3 \), which agree with the values found in the literature (see e.g. Capozziello & Izzo 2010; Vitagliano et al. 2010). In Fig. 14, we plot our approximate cosmographic distance moduli together with the SNIa Union data set. In addition, we investigate the possibility of using high-redshift GRBs to determine parameters of our approximate cosmography. Therefore, in the following, we use the calibrated (with SNIa) GRB HD, described above, to fit the \( d_L \) parameters and then to derive the cosmographic parameters \( q_0, j_0 \) and \( s_0 \). We obtain the following (2σ) parameter confidence intervals:

\[
\begin{align*}
\frac{d_1}{d_2} & \approx 0.950028 \quad 11.2294 \\
\frac{d_2}{d_3} & \approx -2.76299 \quad 1.92514 \\
\frac{d_3}{d_0} & \approx 0.140305 \quad 0.836675
\end{align*}
\]

and it turns out that the corresponding parameter confidence intervals for \( q_0, j_0 \) and \( s_0 \) (through equations 29–31) are

\[
\begin{align*}
q_0 & = -1.09 \quad 0.10 \\
j_0 & = -0.3 \quad 2.71 \\
s_0 & = -1.23716 \quad 8.34
\end{align*}
\]

\(^4\) It is worth noting that such physically acceptable regions are obtained imposing on the confidence regions provided by the standard statistical fitting procedure, some priors, which have to guarantee that the approximate function in equation (17) preserves the special shape typical of the angular diameter distance. Instead, since the SNIa data set used to fit the parameters \( d_L \) is limited in redshift by \( z \leq 1.39 \), the approximate function in equation (17) is not sampled at higher values of the redshift, and its behaviour could result mis-shaped.
the best fit being \( q_0 = -0.51, j_0 = 0.69 \) and \( s_0 = -0.48 \), in agreement with other results in the literature (see e.g. Gao, Liang & Zhu 2010; Vitagliano et al. 2010). In Fig. 15, we plot our approximate cosmographic distance modulus together with the SN Ia Union data set and the calibrated GBR HD. The reliability of the reconstruction is measured by the relative residuals, \( \epsilon = \frac{\mu_{\text{obs}} - \mu_{\text{fit}}}{\mu_{\text{obs}}} \), shown in Fig. 16. We finally perform our approximate cosmographic analysis, considering a whole data set containing both the SN Ia Union data set and the calibrated GBR HD, which we call the cosmographic data set. We obtain the following parameter confidence (at 2\(\sigma\)) intervals for the \( d_i \) and the cosmographic parameters, respectively:

\[
\begin{pmatrix}
 d_1 & -1.35 & 2.26 \\
 d_2 & 0.71 & 1.30 \\
 d_3 & 0.39 & 0.59 \\
 q_0 & -1.58 & -0.39 \\
 j_0 & 2.67 & 8.87 \\
 s_0 & -6.88 & 46.9376
\end{pmatrix}
\]

The best fit being \( q_0 = -0.9, j_0 = 5.25 \) and \( s_0 = 27.25 \), which again agree with the values found in the literature. In Fig. 17, we show the approximate cosmographic distance modulus together with the cosmographic data set. Moreover, we observe that since the cosmographic data set also includes the GBR HD, which spans a quite large range of redshift up to \( z \approx 8 \), the approximate function in equation (17) is sampled also at higher values of the redshift, and its behaviour is not misshaped, even without any prior on the confidence regions, as needed above. In order to further check the reliability of our procedure, we compare the results summarized above with that provided by the standard cosmography. We apply this procedure to the cosmographic data set only and obtain the following parameter confidence (2\(\sigma\)) intervals:

\[
\begin{pmatrix}
 q_0 & -1.09 & -0.08 \\
 j_0 & -0.30 & 2.71 \\
 s_0 & -1.24 & 8.34
\end{pmatrix},
\]

which fully agree with the previous results. Also, from the point of view of the analysis of residuals it turns out that our approximate cosmography is statistically fully consistent with the standard cosmography; actually, we obtain a fully compatible values for the root mean square and the correlations in both cases. In Fig. 18, we show the standard cosmographic distance modulus together with the cosmographic data set. In short, let us note that we implemented a cosmographic analysis, using both the SNe and the GRB data, which allow us to obtain constraints on the parameters of cosmography (starting from our approximated luminosity distance), which we also compare with the results of a standard approach. Therefore, our approach is very different from the one used in Capozziello & Izzo (2010), where they obtain a cosmographic luminosity distance in the y redshift, which is used to calibrate the \( E_{p,i} - E_{p,0} \) relation using a weighted \( \chi^2 \) estimator, but without making up the GRB HD. Moreover, in that case the constraints on the cosmographic parameters are obtained from the SN Ia data only.
4 DISCUSSION AND CONCLUSIONS

Recently several interesting correlations among the GRB observables have been identified. Proper evaluation and calibration of these correlations are needed to use the GRBs as standard candles to constrain the expansion history of the Universe up to redshifts of \( z \sim 8 \). Here we used the GRB data set recently compiled by Amati et al. (2008, 2009) to investigate, in a quintessential cosmological scenario, the \( E_{p,i} - E_{iso} \) correlation. Marginalizing over the normalization, \( b \), our Bayesian analysis provides the following parameter confidence (at 3\( \sigma \)) intervals for the best-fitting power-law and the intrinsic dispersion, respectively:

\[
\begin{pmatrix}
    a \\
    \sigma_{int}
\end{pmatrix}
\begin{pmatrix}
    1.263 \\
    0.36
\end{pmatrix}
\begin{pmatrix}
    1.77 \\
    0.5
\end{pmatrix}
\]

The maximum likelihood value for the normalization coefficient turns out to be \( b = 52.67 \). We used the fitted parameters to construct an estimated fiducial HD that extends up to redshifts \( z \sim 8 \). In the first part of our analysis, we have assumed that the fit parameters do not change with the redshift. The limited number of GRBs prevents detailed exploration of the validity of this usually adopted working hypothesis, which we tested somewhat investigating if the residuals correlate with the redshift. We have not found any significant correlation. Moreover, we tested the fit of the \( E_{p,i} - E_{iso} \) correlation parameters with respect to the evolution with redshift, binning the GRBs into four groups with redshift from low to high, each group containing a reasonable number of GRBs. We thus maximized the likelihood in each group of redshifts and determined the best-fitting calibration parameters. Our analysis indicates that the presently available GRB data sets do not show statistically unambiguous evolutionary effect with the cosmological redshift, but this should be tested further with larger GRB samples. We also investigated the impact of varying the parameters of our fiducial cosmological model, fitting the Amati correlation on a regular grid in the space of parameters of our quintessential model and evaluating, for each GRB, the root mean square of the percentage deviation from the fiducial \( \mu \) value. In this way, we performed a sort of average of the absolute percentage deviation which provides the variation of the fitting parameters \( a \) and \( b \). It turns out that the distance modulus may be underestimated or overestimated by a modest 0.3 per cent with values never larger than 1 per cent, so that the underlying cosmological model (in terms of varying the values of its characteristic parameters) has only a modest impact on the final estimate of the distance modulus. However, this fact does not imply that the cosmological constraints that can be obtained from GRB data with this method are also marginal; already in Demianski et al. (2011) we tested the stability of values of the fitted correlation parameters by considering an ad hoc definition of the luminosity distance that gives much larger distances to objects at \( z > 2 \) than either of the fiducial model. It turned out that such artificial \( \text{crazy} \) luminosity distance is changing the values of the correlation parameters, but the difference is not dramatic at the 1\( \sigma \) confidence level. However the situation changes when we consider, in the space of parameters, the regions of 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) of confidence for our \( \text{crazy} \) model. It turns out that with respect to the same regions constructed for the fiducial model they overlap only at 1\( \sigma \), but differ consistently at higher levels of confidence. As a consequence, when we made up the GRB HD, the number of GRBs deviating from the fiducial \( \mu \) more than 2\( \sigma \) increases from \( N = 14 \), in the case of the quintessential cosmological model, to \( N = 51 \) in the case of the \( \text{crazy} \) model.5 The calibration of the Amati correlation, as well as the other known correlations, in several cosmological scenarios is therefore already needed to fully use the GRBs as a cosmological probe. Actually, quite recently, in Diaferio, Ostorero & Cardone (2011) the GRBs are used to test the \( \Lambda \) cold dark matter (LCDM) versus conformal gravity, and in a forthcoming paper we are standardizing the GRBs with the Amati relation to test cosmological models based on extended theories of gravity. Moreover, we apply a local regression technique to estimate, in a model-independent way, the distance modulus from the recently updated SNIa sample. The derived calibration parameters are used to construct an updated GRB HD, which we call the calibrated GRB HD. We also compare the fiducial and calibrated GRB HDs, which turned out to be fully statistically consistent, thus indicating that they are not affected by any systematic bias induced by the different calibration procedures. This means that the high-redshift GRBs can be used to test different models of dark energy settling the circularity problem. Finally, we propose a new approach to calibrate the GRB relations in a cosmologically independent way, by using an approximate luminosity–distance relation, which holds in any cosmological model. We use this calibration of the Amati relation to construct an empirical approximate HD, which is fully consistent with the calibrated GRB HD, probing in such a way the reliability of our approach, which could provide a robust procedure to calibrate in a cosmological-independent way different GRB correlations, especially if the available data set is poorly populated in the SNIa range of redshift, so that, in such a case, the SNIa fitting procedure fails. We finally investigated the implications of such an approach for the high-redshift cosmography. Actually, starting from the estimation of the constant \( d_j \), \( d_k \) and \( s_0 \) present in our approximate luminosity–distance relation, we constructed the map which connects our \( d_j \)s to the parameters describing the kinematical state of the Universe \( q_0 \), jerk \( j_0 \) and snap \( s_0 \). This map is a core for our approximate cosmography, which we actually applied to a whole data set containing both the SNIa Union data set and the calibrated GRB HD, which we call the cosmographic data set. Using the cosmographic data set, we show that the deceleration parameter, \( q_0 \), up to the 3\( \sigma \) confidence level is definitively negative. The constraints for the jerk and snap parameters are, instead, less strong. Finally, it turns out that our results are fully consistent with the results obtained using the standard approach, showing that our new approximate cosmography allows robust results in a wide range of redshifts.

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5 In Demianski et al. (2011), the results concerning the \( \text{crazy} \) model are inferred from fitting five correlations different from the Amati relation. However, we tested that for the Amati relation we also obtain the same kind of behaviour.
