A new high-order accuracy numerical method for numerical modeling of supernovae explosions

Victor Protasov$^1$, Ivan Ulyanichev$^1$, Irek Gubaydullin$^2$

$^1$ Institute of Computational Mathematics and Mathematical Geophysics SB RAS, 6-th Lavrentyev avenue, Novosibirsk, Russian Federation
$^2$ Institute of Petrochemistry and Catalysis of RAS, 141-th Oktyabria avenue, Ufa, Russian Federation
E-mail: wznonacomvn@mail.ru

Abstract. The paper presents a new numerical method of high accuracy order for simulation of supernova explosion. The numerical method is based on a combination of the method of separation of operators, the Godunov method, the HLL scheme, and the piecewise-parabolic method on local stencil. The results of the verification of the new method on shock tube problems, the Sedov problem, and the collapse of the cloud are presented. The results of modeling a non-central supernova explosion characteristic of SNIa are presented.

1. Introduction
The Type Ia supernova progenitor problem is one of the most exciting problems in astrophysics, requiring detailed numerical modeling to complement observations of these explosions. One possible progenitor is the white dwarf merger scenario, which has the potential to naturally explain many of the observed characteristics of SNIa [1]. The main future of SNIa is the formation of heavy elements in Universe to Fe.

The main method for solving the equations of hydrodynamics is the Godunov method, also high-order realizations have been made on the basis of the Godunov method-a monotonic counter-current scheme of the second order of accuracy MUSCL [2, 3], non-oscillating schemes with weights TVD-WENO [4, 5, 6, 7, 8], the piecewise parabolic method of PPM [9]. However, what is understood by the high order of accuracy in the case of discontinuous solutions is not very clear [10]. The general idea of such approaches is the construction of a piecewise polynomial function in each cell of the computational domain. A significant drawback of high accuracy methods is the need to use a different from a compact grid calculation template, which leads to the need to use more overlapping layers, which increases the load on the network infrastructure and, as a consequence, the performance drop. In this connection, a modification of the PPM method was developed - a piecewise parabolic method on the local template (PPML) [11, 12, 13].

The second section describes a mathematical model of gravitational hydrodynamics using the equation for entropy. The third section is devoted to the description of the numerical solution method. In the fourth section the verification of the numerical method and briefly describes its parallel implementation. The fifth section is devoted to the modeling of supernovae with a noncentral source of an explosion. In the sixth section, formulated a conclusion on the article.
2. Mathematical model
To describe the process of supernova explosion, an overdetermined system of the gravitational hydrodynamics equation is used in a completely divergent form using the entropy equation. Advantages of this approach are described in the discussion paper [13].

\[
\frac{\partial}{\partial t} \begin{pmatrix}
\rho \\
\rho \vec{u} \\
\rho S \\
\varepsilon + \rho \frac{\vec{u}^2}{2}
\end{pmatrix}
+ \nabla \cdot \begin{pmatrix}
\rho \vec{u} \\
\rho \vec{u} \otimes \vec{u} + p \\
\rho S \vec{u} \\
(\varepsilon + \rho \frac{\vec{u}^2}{2} + p) \vec{u}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
-\rho \nabla \Phi \\
0
\end{pmatrix}
\]

(1)

\[\Delta \Phi = 4\pi G \rho\]

where \(\rho\) is density of the gas, \(\vec{u}\) is velocity, \(S\) is entropy, \(p = p(\rho, S)\) is pressure, \(\varepsilon\) is internal energy, \(\gamma\) is adiabatic index, \(\Phi\) is gravity, \(G\) is gravity constant. For numerical simulation in this papers we should use next from of equation of state:

\[p = (\gamma - 1) \varepsilon = S \rho^\gamma\]

of course, equation (1) allow to take into account any kind of equation of state.

3. Numerical solution method
3.1. A numerical method for solving the equations of hydrodynamics
Equations of hydrodynamics can be written in vector form:

\[\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0\]

To solve the equations, a numerical method is used, based on the combination of the operator splitting approach, the Godunov method, the HLL scheme, and the piecewise-parabolic method on local stencil. The flow through the boundary between the left (L) and right (R) cells is calculated with the help of equation:

\[F = \frac{F(-\lambda_L \tau) + F(\lambda_R \tau)}{2} + \frac{c + \|\vec{u}\|}{2} (U (-\lambda_L \tau) - U (\lambda_R \tau))\]

where

\[\lambda_L = c - \|\vec{u}\| \quad \lambda_R = c + \|\vec{u}\|\]

where \(c = \sqrt{\frac{2p}{\rho}}\) – sound speed. The integration over parabolas is carried out by the equations described in the paper [13].

3.2. Regularization of the solution
At the final stage of the hydrodynamic equations, a solution adjustment procedure is provided. In the case of a gas vacuum border:

\[\|\vec{u}\| = \sqrt{2(E - \varepsilon) / (E - \vec{u}^2/2)} / E \geq 10^{-3},\]

In the other area, an adjustment is used to guarantee non-decreasing entropy:

\[\rho \varepsilon = \left( \rho E - \rho \frac{\vec{u}^2}{2} \right) / (E - \vec{u}^2/2) / E < 10^{-3},\]

This modification provides a detailed balance of energy and guarantees non-decreasing entropy.
3.3. The method of solving the Poisson equation

After solving the hydrodynamic equations, it is necessary to restore the gravitational potential with respect to the gas density. To do this, we will use a 27-point template to approximate the Poisson equation. The algorithm for solving the Poisson equation will consist of three stages:

(i) Setting the boundary conditions for the gravitational potential at the boundary of the region.

(ii) The transformation of density to the harmonics space. For this, a fast Fourier transform is used.

(iii) Solution of the Poisson equation in harmonics space. After that it is necessary to perform the inverse fast Fourier transformation of the potential of harmonics into the functional space of the harmonics.

Details of the method are given in the work [13].

3.4. Description of parallel implementation

Using a uniform grid in Cartesian coordinates to solve the equations of hydrodynamics makes it possible to use an arbitrary Cartesian topology for decomposition of the computational domain. In the implementation uses one-dimensional decomposition of the computation area by means of FFTW. By one coordinate, the external one-dimensional cutting takes place using the MPI technology. For computational experiments the cluster ”Angara-K1” was used [14]. A study was made of the scalability of the implementation on the calculated grid $128^3$, where $p$ – is the number of nuclei used. Thus, each core has to size subregion $128^3$. For the study of scalability, the time was measured (Total) for a different number of used cores (Cores). The scalability of $T$ (Scalability) was calculated using the formula

$$T = \frac{Total_1}{Total_p}$$

where $Total_1$ – time of computations on a single kernel using a single core, $Total_p$ – time of computations on one nucleus when using $p$ cores. Each core was placed on a separate node of the cluster ”Angara-K1”. The results of the scalability studies are shown in the table 1. As can be seen from the table 1 there is a high degree of scalability. Here we will not dwell on the studies of the communication network, we will carry out this research in subsequent works.

4. Verification of the numerical method

In this section, we describe three main tests for codes created to simulate the explosion of supernovae. In our opinion, such tests should be:

| Cores | Scalability |
|-------|-------------|
| 1     | 1.00000     |
| 2     | 1.00001     |
| 3     | 0.99986     |
| 4     | 0.99945     |
| 8     | 0.99886     |
| 12    | 0.99727     |
| 16    | 0.99682     |

Table 1. The scalability of code
Figure 1. Graphs of density (top left), velocity (top right), pressure (bottom left) and internal energy (bottom right) in the shock tube problem.

(i) The problem of a shock tube for demonstrating the possibility of a numerical method to reproduce a shock wave, a contact discontinuity, and a rarefaction wave with small dissipation.

(ii) The task of Sedov for demonstration the possibility of the method reproduces infinitely large Mach numbers.

(iii) The task of the collapse of a gas cloud for demonstration the possibility of the method to reproduce the processes of strong compression with the fulfillment of the law of conservation of total energy.

4.1. The problem of the shock tube
The initial configuration of the shock tube problem $x_0 = 0.5$ – is the initial position of the separator between two neighboring states ($L$ – is the left, $R$ – is the right). For computational experiments, 100 calculation cells were used. Modeling time $t = 0.2$. Gas parameters on the left – $\rho_L = 1$, $p_L = 1$, gas parameters on the right $\rho_R = 0.125$, $p_R = 0.1$. The initial gas velocity is zero. The purpose of the test is to determine the correctness of the description of the contact discontinuity. Most methods for solving gasdynamic equations yield either oscillation or diffusion (“smearing” of shock waves). The author’s method of high accuracy gives a small (into two cells)smearing of the solution in the region of the shock wave (figure 1).
4.2. The Sedov Problem
Sedov’s problem of a point explosion in astrophysics is formulated as the problem of a supernova explosion. To model the problem of a point explosion, we will consider the region \([-0.5; 0.5]^3\), the adiabatic exponent \(\gamma = 5/3\), the initial density in the region \(\rho_0 = 1\), and the initial pressure \(p_0 = 10^{-5}\). At time \(t = 0\) released internal energy \(E_0 = 0.6\). The area of the explosion is limited by the radius \(r_{\text{central}} = 0.02\). For the computational experiment we used the computational grid \(100^3\). The simulated profile of the density and angular momentum at time \(t = 0.05\) is shown in figure (2). Sedov’s problem of a point explosion is a standard test that verifies the ability of a method and its realization to reproduce strong shock waves with large Mach numbers. The author’s numerical method reproduces quite well the position of the shock wave, as well as the density and momentum profile.

4.3. The problem of the collapse of a gas cloud
To test SPH methods, a model problem is frequently used - the collapse of Everard. The problem is interesting because at first there is a short process of compression of the center, its rapid heating and further expansion with the formation of a shock wave. To solve this problem, we will simulate a nonrotating cloud with a dimensionless radius \(R_0 = 1\), a density distribution \(\rho(r) = 1/(2\pi r)\), an adiabatic exponent \(\gamma = 5/3\), the total internal energy \(u = 0.05\). In the results of the collapse test of Everard, it is worth noting that the total energy of the system is conserved.

5. Simulation of supernovae with a noncentral source of an explosion
As a model problem of a supernova explosion, let us consider a hydrostatically equilibrium density profile:

\[
\rho(r) = \begin{cases} 
2r^3 - 3r^2 + 1, & r \leq 1, \\
0, & r > 1.
\end{cases}
\]

and pressure profile:

\[
p(r) = \begin{cases} 
-\pi r^8/3 + 44\pi r^7/35 - 6\pi r^6/5 - 4\pi r^5/5 \\
+8\pi r^4/5 - 2\pi r^2/3 + \pi/7, & r \leq 1, \\
0, & r > 1.
\end{cases}
\]
Figure 3. Behavior of various types of energy.

Figure 4. The nondimensional density in time is equal to $t = 0.02$ for central (left), internal (center), external (right) explosions.

In cloud on x-axis injected pressure for $x = 0, 0.25, 0.5$ is equal to $P = 1000$ and normal distribution for the velocity vector. The results of the simulation are shown in the figure (4). Supernovae of Ia type are characterized by a noncentral point of detonation. At a distance from the center, a one-sided shock wave is formed, which propagates through the center of a supernova. Such a mechanism is one of the possible scenarios for the formation of objects with an envelope of the type NGC 6888. In the following, we will describe in more detail the mechanism of detonation based on the merger of two CO dwarfs.

Scenarios for the explosion of SNIa are based on the merger of two dwarfs, and the non-central explosion is a simulation of merger with the point of an explosion in a point of accretion the satellite dwarf to the main dwarf. The using equation of state with constant entropy is equal to EOS for degenerate gas. The temperature effects from chemistry/nuclear reactions should be include by means the right part in entropy equation. It’s not the subject of this paper.
6. Conclusion
A new numerical method of high accuracy for simulating a supernova explosion is described, based on a combination of the method of separation of operators, the Godunov method, the HLL scheme, and the piecewise-parabolic method on local stencil. The verification of the new method on shock tube problems, the Sedov problem, and the collapse of the cloud has been carried out. Various scenarios of a non-central supernova explosion characteristic of SNIa has been modeled.

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