Wess-Zumino-Witten model off criticality

D. C. Cabra *

Departamento de Física
Universidad Nacional de La Plata
C.C.67 - 1900 La Plata, Argentina

Abstract

We study the renormalization group flow properties of the Wess-Zumino-Witten model in the region of couplings between $g^2 = 0$ and $g^2 = 4\pi/k$, by evaluating the two-loop Zamolodchikov’s $c$-function. We also discuss the region of negative couplings.

*CONICET-Argentina
In this work we are going to study the Wess-Zumino model by perturbation theory on a manifold with the topology of the plane.

We construct the Zamolodchikov’s $c$-function from its original definition up to two loops and show that it fulfills the expected properties except the stationarity condition at the trivial fixed point (FP).

In the region of couplings between the trivial FP $g^2 = 0$ and the well known infrared stable FP $g^2 = \frac{4\pi}{k}$ our $c$-function is monotonically decreasing along a Renormalization Group trajectory (for large values of $k$) and takes the value of the Virasoro central charge (VCC) of the critical theory at the corresponding FP’s. It is stationary at the points $g^2 = \frac{4\pi}{k}$ but it is not stationary at the point $g^2 = 0$. In the region of negative couplings between $g^2 = -\frac{4\pi}{k}$ and $g^2 = 0$, where the theory is non-unitary, the $c$-function is increasing to the infrared, takes the values of the VCC at the FP’s, is stationary at the point $g^2 = -\frac{4\pi}{k}$ but not at the trivial one.

Besides, we show that our expression for the $c$-function coincides with the one that can be obtained from the generalized definition proposed by Cardy for curved manifolds when applied to the Wess-Zumino model in $S^2$. This is a non-trivial check since it is not proven in general that the generalization of the $c$-function to curved manifolds does fulfill the expected properties.

Using the result for the $\beta$-function obtained in Ref. we show that the $c$-function is monotonically decreasing to the infrared and we find the particular form of the coefficient which relates the $\beta$-function with the derivative of the $c$-function in our regularization scheme. (This coefficient cannot be set equal to one in an arbitrary regularization scheme.) We find that in our case this coefficient is positive definite.

The action for the Wess-Zumino-Witten (WZW) model is given by:

$$W[h] = \frac{1}{2g^2} \int d^2x \text{tr} \left( \partial_{\mu} h^{-1} \partial_{\mu} h \right) + \frac{k}{12\pi} \int d^3y \varepsilon_{ijk} \text{tr} \left( h^{-1} \partial_i hh^{-1} \partial_j hh^{-1} \partial_k h \right),$$

where $h$ takes values in some compact Lie group $G$.

Besides the trivial fixed point, $g^2 = 0$, we know for this model that it has
an exact non-trivial fixed point (IR stable) at:

\[ g^2 = \frac{4\pi}{k}, \]  

(2)
as was stated in Refs. [6], [7] using symmetry arguments.

Perturbative evaluations of the \( \beta \)-function have been done by several authors [4]–[8], and the results show the existence of this fixed point at each order in perturbation theory. Also, the VCC of the model at this point is known to be exactly:

\[ c = \frac{k \text{dim } G}{k + C_G}, \]  

(3)

where \( k \) is the level of the Kac-Moody algebra and \( C_G \) is the quadratic Casimir of \( G \).

In the region of negative couplings, there is another point in which the theory becomes conformally invariant [6] which corresponds to:

\[ g^2 = -\frac{4\pi}{k}, \quad k > 0, \]  

(4)

with VCC given by [8]:

\[ c = \frac{-k \text{dim } G}{-k + C_G}. \]  

(5)

The corresponding theory is non-unitary and must be quantized with an indefinite metric.

We will include this point in our study of the model since in the study of coset models (both for bosonic [9], [10] and fermionic [11] descriptions), WZW models with negative Kac-Moody central charge do appear. (In those cases one has a BRST quantization condition which avoids the appearance of negative norm states from the physical spectrum [10].)

We are going to evaluate the Zamolodchikov’s \( c \)-function perturbatively using its original definition and study the two regions corresponding to \( g^2 \) in \([-4\pi/k, 0]\) and \( g^2 \) in \([0, 4\pi/k]\).

In order to make contact with Zamolodchikov’s original construction, we are going to evaluate:

\[ c_{Zam}(g^2) = \left[ 2z^4 < T(x)T(0) > +4z^2x^2 < T(x)\Theta(0) > -6x^4 < \Theta(x)\Theta(0) > \right]_{x^2=R^2}, \]  

(6)
where $z = x_0 + ix_1$, $\overline{z} = x_0 - ix_1$, $T = T_{zz}$ and $\Theta = 4T_{z\overline{z}}$ are the two independent components of the energy momentum tensor in these coordinates ($R$ is a normalization point), by formulating the theory on a curved world sheet in a generally covariant way. We then have to take the functional derivatives with respect to the background metric $\gamma_{\alpha\beta}$ and finally take the limit $\gamma_{\alpha\beta} \rightarrow \delta_{\alpha\beta}$.

To this end we define the effective action in a background metric as usual:

$$e^{-S_{\text{eff}}[\gamma]} \equiv \int Dh e^{-W[h, \gamma]} ,$$

and evaluate its finite part perturbatively up to two loops using dimensional regularization in an arbitrary metric [12]. In order to avoid infrared divergences one must include a mass term [4] but, as was shown in Ref.[13] this does not affect the $c$-function.

We just quote the result here (for details see Ref.[13]).

Our expression for the finite effective action is:

$$S_{\text{eff}}[\gamma] = (N^2 - 1)D[\gamma] \left[ 1 - \frac{Ng^2}{8\pi} \left( 3 - \left( \frac{g^2 k}{4\pi} \right)^2 \right) + \ldots \right] ,$$

where:

$$D[\gamma] = \frac{1}{2} \left[ \ln \det \left( -\nabla^2 \right) \right].$$

(The finite effective action has no essential differences with the case when the manifold has the topology of the sphere [3].)

In the case at hand, the determinant of the laplacian operator is given by [14]:

$$D[\gamma] = -\frac{1}{48\pi} \int d^2x d^2y \sqrt{\gamma(x)} \sqrt{\gamma(y)} R(x)R(y)G(x, y) + c \int d^2x \sqrt{\gamma(x)},$$

where $G(x, y)$ is the Green function of the covariant Laplacian:

$$\partial_\mu \left( \frac{1}{\sqrt{\gamma}} \gamma^{\mu\nu} \partial_\nu \right) G(x, y) = \delta(x, y).$$

The connected part of the general correlator of two energy momentum operators is defined through:
\[ \langle T_{\mu\nu}(x)T_{\rho\sigma}(y) \rangle - \langle T_{\mu\nu}(x) \rangle \langle T_{\rho\sigma}(y) \rangle = \]
\[ - \frac{2}{\sqrt{\gamma(x)}} \frac{2}{\sqrt{\gamma(y)}} \delta^{(2)} S_{\text{eff}}[\gamma] \bigg|_{\gamma_{\mu\nu} = \delta_{\mu\nu}} \bigg( \delta_{\gamma_{\mu\nu}}(x) \delta_{\gamma_{\rho\sigma}}(y) \bigg) \bigg|_{\gamma_{\mu\nu} = \delta_{\mu\nu}}. \] (12)

where:
\[ \langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta_{\gamma_{\mu\nu}}} S_{\text{eff}}[\gamma]. \] (13)

From eqs. (6)-(13) it follows that (up to contact terms):
\[ \langle T(x)T(0) \rangle = \frac{1}{2} c(g^2, k), \]
\[ \langle T(x)\Theta(0) \rangle = 0, \]
\[ \langle \Theta(x)\Theta(0) \rangle = 0, \] (14)

where:
\[ c(g^2, k) = (N^2 - 1) \left[ 1 - \frac{Ng^2}{8\pi} \left( 3 - \left( \frac{g^2k}{4\pi} \right)^2 \right) + \ldots \right] \] (15)

and hence:
\[ c_{\text{Zam}} \equiv c(g^2, k). \] (16)

It must be pointed out that there are no additional contributions to these quantities coming from the divergent part of the effective action \[13\].

Expression (16) has the expected properties at the fixed points given in eqs. (2), (4). That is:
\[ c(g^2, k) \bigg|_{g^2 = \pm \frac{4\pi}{k}} = (N^2 - 1) \left[ 1 \mp \frac{N}{k} + \ldots \right], \]
\[ \frac{\partial c(g^2, k)}{\partial g^2} \bigg|_{g^2 = \pm \frac{4\pi}{k}} = 0. \] (17)

Since we are making a perturbative expansion with \(1/k\) as a small parameter, we can study with the desired approximation the region of couplings...
between \( g^2 = 0 \) and \( g^2 = \frac{4\pi}{k} \). (We can also study separately the region of negative couplings between \( g^2 = -\frac{4\pi}{k} \) and \( g^2 = 0 \).)

We see that in the trivial fixed point, \( g^2 = 0 \), we have:

\[
c(g^2, k)|_{g^2=0} = N^2 - 1, \tag{18}
\]

which corresponds to the value of the Virasoro central charge of \( N^2 - 1 \) free massless bosons, but the stationarity condition is not fulfilled at this point:

\[
\frac{\partial c(g^2, k)}{\partial g^2}|_{g^2=0} \neq 0. \tag{19}
\]

In order to study the renormalization group behaviour of the \( c \)-function we use the result for the \( \beta \)-function quoted in Ref.[4]:

\[
\beta(g^2) = -\frac{N g^4}{2\pi} \left[ 1 - \left( \frac{g^2 k}{2\pi} \right)^2 \right] + ..., \tag{20}
\]

which shows that \( g^2(\mu) \) is increasing to the IR so the Zamolodchikov’s \( c \)-function is decreasing along a RG-trajectory for \( g^2 > 0 \) as expected.

If \( g^2 \) is allowed to take negative values then our expression for the \( c \)-function shows that it is increasing to the IR, and it is stationary at the non-trivial fixed point \( g^2 = -\frac{4\pi}{k} \). (The increasing of the \( c \)-function does not contradict the \( c \)-theorem since for negative couplings the model is non-unitary and hence Zamolodchikov’s proof does not hold.)

We can also calculate the coefficient which relates the \( \beta \)-function with the derivative of the \( c \)-function, \( \beta(g^2) = F(g^2) \frac{\partial c(g^2, k)}{\partial g^2} \), which up to this order is given by:

\[
F[g^2] = \frac{2}{3} g^4 (N^2 - 1)^{-1}, \tag{21}
\]

and is positive definite as predicted in Ref.[4]. This explains why for \( g^2 = 0 \) the vanishing of the \( \beta \)-function does not necessarily lead to the stationarity of \( c \).

As mentioned in the introduction, in Ref.[2], a generalization of the \( c \)-theorem to curved manifolds was proposed. In this paper it has been suggested that a natural definition, which could be useful in the generalization
of the c-theorem to four dimensions, is given by:

$$\tilde{c} = -3 \int_{S^2} < \Theta > \sqrt{d^2x}$$

(22)

where the integration is done over the sphere $S^2$ endowed with the metric induced by embedding into $R^3$.

The v.e.v $< \Theta(x) >$ has been evaluated exactly in [3] at the fixed point (2). In order to evaluate it off criticality one has to look not only at the finite part of the effective action, but also at its divergent part (which behaves as $1/d - 2$, where $d$ is the dimensionality of space-time), since it can give a non-trivial, finite contribution to the trace. However, in this case it is easy to show that the trace has no additional contributions arising from the divergent terms in the effective action. The result is simply:

$$< \Theta > \equiv \gamma^{\mu\nu}(x) < T_{\mu\nu}(x) > = c(g^2, k) \frac{R(x)}{24\pi},$$

(23)

where $R(x)$ is the scalar curvature, and $c(g^2, k)$ is given by eq.(15).

We then have simply that:

$$\tilde{c} \equiv c(g^2, k).$$

(24)

This result must be taken with a grain of salt since there does not exist a complete proof that this generalized “c-function” fulfills the conditions that the Zamolodchikov’s c-function does. In particular the decreasing property of this function has not been proven, although it was verified to lowest order in perturbation theory [3]. However, we see that the explicit calculation of the Zamolodchikov’s c-function, (eq.(16)), coincides with the one obtained from (22).

As a final comment it is interesting to note that the finite effective action can be written as:

$$e^{-S_{eff}[\gamma]} = [det(-\nabla)]^{-\frac{1}{2}c(g^2,k)},$$

(25)

which is the $c(g^2, k)$-power of the partition function for a massless free boson. This fact fits with the intuition that the c-function is a measure of the massless degrees of freedom.

I would like to thank C.Naón, E.Moreno and G.Rossini for helpful discussions and for carefully reading the manuscript.
References

[1] A.B.Zamolodchikov, Sov.Phys.JETP Lett.43, 731, (1986).

[2] J.L.Cardy, Phys. Lett. 215B, 749 (1988).

[3] H.Leutwyler and M.Shifman, Preprint BUTP-91/6, March 91.

[4] M.Bos, Phys. Lett. 189B, 435 (1987).

[5] D.Boyanovsky and R.Holman, Phys. Rev. D 40, 1964 (1989).

[6] E.Witten, Comm. Math. Phys. 92, 455 (1984).

[7] V.G.Knizhnik and A.B.Zamolodchikov, Nucl. Phys. B247, 83 (1984).

[8] Zheng-Min Xi, CCAST preprint, 1988.

[9] K.Gawedzki and A.Kupiainen, Phys. Lett. 215B, 119 (1988).

[10] D.Karabali, Q-Han Park, H.Schnitzer and Z.Yang, Phys. Lett. 216B, 307 (1989).

[11] K.Bardacki, E.Rabinovici and B.Saring, Nucl. Phys. B229, 151 (1988); D.Cabra, E.Moreno and C.von Reichenbach, Int. J. Mod. Phys. A 5, 2313, (1990).

[12] M.Luscher, Ann.Phys. (N.Y.)142, 359, (1982).

[13] D.C.Cabra, Phys.Rev.D, in press.

[14] A.M.Polyakov, Phys. Lett. 103B, 207 (1981).