On the energy of a charged dilaton black hole

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ABSTRACT

Employing energy-momentum pseudotensor of Einstein, we obtain the energy distribution of a dyonic dilaton black hole. The energy distribution of this black hole depends on mass $M$, electric charge $Q_e$, magnetic charge $Q_m$ and asymptotic value of the dilaton $\phi_0$. We also make some comparisons between the results of Virbhadra et. al. and ours.
Charged dilaton black hole solutions have been obtained by Grafinkle, Horowitz and Strominger (GHS solutions) [1]. They are static spherical symmetric black hole solutions of the dilaton gravity theory. In this theory the gravity is coupled to electromagnetic and dilaton fields and can be described by the four-dimensional effective string action. The action can be expresses as

\[ I = \int d^4x \sqrt{-g} \left[ -R + 2(\nabla \phi)^2 + e^{-2\phi} F^2 \right]. \tag{1} \]

In a peculiar coordinate form, GHS solutions have a singularity whose area is zero at \( r^* = \frac{Q^2}{M} e^{2\phi_0} \). To characterize a gravitational point source, their choice of the coordinate system may not be the most appropriate one. Thus, Cheng, Lin and Hsu [2] use the standard spherical coordinate system which is more suitable for describing the structure of the charged dilaton black hole. They obtained the more general solutions which named the dyonic dilaton black hole solutions. GHS solutions were found to be the special cases of dyonic dilaton black hole solutions when electric or magnetic charges are switched off. For these special cases, two solutions are related by some coordinate transformations [2]. For example, The singularity \( r^* = \frac{Q^2}{M} e^{2\phi_0} \) of GHS solutions corresponds to the singularity \( r = 0 \) of the dyonic dilaton black hole solutions.

Recently, Virbhadra et. al. [3] get the gravitational energy of charged dilaton black hole solutions. Based on the GHS solutions, they found a
charge independent result

\[ E(r) = M, \tag{2} \]

in which the positive energy is confined to the interior of the black hole. Their result is different from the gravitational energy of Reissner-Nordström (RN) solutions \[ E(r) = M - \frac{Q^2}{2r}, \tag{3} \]

where the energy is shared by the interior as well as the exterior of the black hole and becomes negative for \( r < \frac{Q^2}{2M} \). Moreover, Chamorro and Virbhadra \[ I = \int d^4x \sqrt{-g} \left[ -R + 2(\nabla \phi)^2 + e^{-2\alpha \phi} F^2 \right], \tag{4} \]

where \( \alpha \) is a dimensionless parameter which controls the coupling between the dilaton and the Maxwell fields. They found the gravitational energy of charged black hole is

\[ E(r) = M - \frac{Q^2}{2r} \left( 1 - \alpha^2 \right). \tag{5} \]

The energy remains positive for all \( r \) as long as \( \alpha^2 > 1 \), and the total energy is also shared by both regions.

In this paper, we will investigate the gravitational energy of dyonic dilaton black hole in the standard spherical coordinate instead. We will also make some comments on the results of Virbhadra et. al. and ours.
The dyonic dilaton black holes are static, spherical symmetric solutions of the dilaton gravity theory. In terms of the standard spherical coordinate [2], the dyonic dilaton black hole solutions are

\[ ds^2 = \Delta^2 dt^2 - \frac{\sigma^2}{\Delta^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \]  
(6)

\[ \sigma^2 = \frac{r^2}{r^2 + \lambda^2}, \]  
(7)

\[ \Delta^2 = 1 - \frac{2M}{r^2} \sqrt{r^2 + \lambda^2} + \frac{\beta}{r^2}, \]  
(8)

\[ \lambda = \frac{1}{2M} \left( Q_e^2 e^{2\phi_0} - Q_m^2 e^{-2\phi_0} \right), \]  
(9)

\[ \beta = Q_e^2 e^{2\phi_0} + Q_m^2 e^{-2\phi_0}, \]  
(10)

\[ e^{2\phi} = e^{-2\phi_0} \left( 1 - \frac{2\lambda}{\sqrt{r^2 + \lambda^2} + \lambda} \right), \]  
(11)

\[ F_{01} = \frac{Q_e}{r^2} e^{2\phi}, \]  
(12)

\[ F_{23} = \frac{Q_m}{r^2}. \]  
(13)

The properties of the dyonic dilaton black holes are characterized by mass \( M \), electric charge \( Q_e \), magnetic charge \( Q_m \) and asymptotic value of the dilaton \( \phi_0 \). The structures of the dyonic dilaton black holes are similar to that of the RN [3] black holes.

The well known energy - momentum complex of Einstein [6] is defined as

\[ \Theta^\nu_\mu = \frac{1}{16\pi} \frac{\partial H^{\nu\sigma}_\mu}{\partial x^\sigma}, \]  
(14)

where

\[ H^{\nu\sigma}_\mu = \frac{g_{\mu\rho}}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[ (-g) \left( g^{\nu\rho} g^{\sigma\eta} - g^{\sigma\rho} g^{\nu\eta} \right) \right]. \]  
(15)
The Greek indices run from 0 to 3 and $x^0$ is the time coordinate. Then, the energy component $E$ is given by

\[ E = \int \int \Theta_0^0 dx^1 dx^2 dx^3 \]
\[ = \frac{1}{16\pi} \int \int \int \frac{\partial H_0^{0l}}{\partial x^l} dx^1 dx^2 dx^3, \]

where the Latin index takes values from 1 to 3.

We carry out the energy component calculation in the quasi-Cartesian coordinate $(t, x, y, z)$. The conversion of spherical coordinate into Cartesian coordinate is

\[
\begin{align*}
    x &= r \sin \theta \cos \varphi \\
    y &= r \sin \theta \sin \varphi \\
    z &= r \cos \theta
\end{align*}
\]

and it transforms the line element (6) into

\[ ds^2 = \Delta^2 dt^2 - (dx^2 + dy^2 + dz^2) - \frac{\sigma^2/\Delta^2 - 1}{r^2}(x dx + y dy + z dz)^2. \]

Thus, we obtain the required components $H_{0l}^{0l}$ in Eq.(16),

\[
\begin{align*}
    H_{01}^{01} &= \frac{2x\sigma}{r^2} \left(1 - \frac{\Delta^2}{\sigma^2}\right), \\
    H_{02}^{02} &= \frac{2y\sigma}{r^2} \left(1 - \frac{\Delta^2}{\sigma^2}\right), \\
    H_{03}^{03} &= \frac{2z\sigma}{r^2} \left(1 - \frac{\Delta^2}{\sigma^2}\right).
\end{align*}
\]

After plugging Eqs.(7),(8) and (19)-(21) into (16), and applying the Gauss theorem, we evaluate the integral over the surface of a sphere with radius $r$.

\[
E(r) = \frac{1}{16\pi} \int \int \int \frac{\partial H_0^{0l}}{\partial x^l} dx dy dz \\
= \frac{1}{16\pi} \int \frac{2\sigma}{r} \left(1 - \frac{\Delta^2}{\sigma^2}\right) r^2 \sin \theta d\theta d\varphi.
\]
Finally, we find the energy within a sphere with radius $r$ is

$$E(r) = M + \frac{M \lambda^2}{r^2} - \frac{1}{2\sqrt{r^2 + \lambda^2}} \left[ \frac{\beta \lambda^2}{r^2} + \lambda^2 + \beta \right].$$

(23)

The energy is shared both by the interior and by the exterior of the black hole. We plot the energy distributions of the dyonic black holes or the extremal dyonic black holes by ”GNUPLOT”. For the dyonic black hole or the extremal dyonic black hole, see Fig. 1 and Fig. 4, we find the energy distribution can be positive or negative. However, they are both positive in the region $r > r_H$. For the pure electric or pure magnetic charged black hole, ie $Q_e = 0$ or $Q_m = 0$, we find the remarkable property that the energy distribution is always positive except at singular point $r = 0$, see Fig. 2,3,5,6.

Here, we note that we can reparametrize $Q_e$ and $Q_m$ by new parameters $Q$ and $\overline{Q}$,

$$Q = \frac{1}{\sqrt{2}} (Q_e + Q_m),$$

$$\overline{Q} = \frac{1}{\sqrt{2}} (Q_e - Q_m).$$

(24)

After this reparametrization, Eqs.(23), will give the same energy distribution of RN black hole, as $\phi_0 = 0$ and $\overline{Q} = 0$. The correspondence of the energy distribution between the dyonic dilaton black hole solutions and RN solutions can be understood by observing the action Eqs.(1) or the solutions Eqs.(6)-(13). When dilaton field was suppressed, that is $\phi_0 = 0$ and $\lambda = 0$ (or $\overline{Q} = 0$), the dilaton gravity, Eqs.(1), will reduce to Einstein - Maxwell theory and the
dyonic dilaton black hole solutions will become to be RN solutions. Again, the results are same as the case of Schwarzschild black holes, as we switch off $Q_e$, $Q_m$ and $\phi_0$ [7].

As expected, the energy distribution of the dyonic dilaton black hole, depends on those parameters $M$, $Q_e$, $Q_m$ and $\phi_0$. They are different from the results of Virbhadra et. al. The differences do come from the different choices of coordinate representations in which energy distribution were evaluated. It can be understood that the local gravitation energy is a component of 4 momentum vector and it will change as different coordinate is chosen. It seems that different coordinates will respect to the different reference points of energy distribution. However, if we are only interested in the ADM mass [8] - total energy which is the limiting value of a certain flux integral as a spherical surface expands to spatial infinity, we find

$$M_{ADM} = E(r) \big|_{r\to\infty} = M.$$ \hspace{1cm} (25)

The result is same as the result of Virbhadra. It means that the ADM mass is independent of coordinate representation of the black hole.
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The position of horizon at $r_H = 2.971$

Figure 1: The energy distribution of dyonic black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 1$ and $Q_m = 1$.

The position of horizon at $r_H = 3.742$

Figure 2: The energy distribution of pure electric black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 1$ and $Q_m = 0$. 
Figure 3: The energy distribution of pure magnetic black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 0$ and $Q_m = 1$.

Figure 4: The energy distribution of extremal dyonic black hole with $\phi_0 = 0$, $M = 2$, $Q_e = \sqrt{2}$ and $Q_m = \sqrt{2}$. 
Figure 5: The energy distribution of extremal electrically charged solution with $\phi_0 = 0$, $M = 2$, $Q_e = 2\sqrt{2}$ and $Q_m = 0$.

Figure 6: The energy distribution of extremal magnetically charged solution with $\phi_0 = 0$, $M = 2$, $Q_e = 0$ and $Q_m = 2\sqrt{2}$. 

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