Strength of Yukawa Potential for Elementary Masses Less than Meson Mass

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Abstract. Study of uncorrected and corrected Yukawa Potential show that at both Yukawa potential have same nature but strength is different. The strength of uncorrected Yukawa potential is identical at short range while corrected Yukawa potential strength is not identical at short range. Moreover, the strength of uncorrected and corrected Yukawa potential has the same strength at long distances when separated from each other. Uncorrected Yukawa potential is study gives more detail at a short distance while corrected not give such detail at the same distance, for considered elementary particles masses less than the mass of meson. The solution of corrected Yukawa potential show depends upon the quantum number and distance separation between two elementary particles masses less than meson.

1. Introduction

Yukawa was interested in the forces inside the nuclei of the atoms with arguing that the nuclear strong force is carried with an unknown particle whose mass is approximately 200 times that of an electron. This heavy mass is known as called the muon and later in 1947, three particles with masses approximately 270 times that of an electron were found in cosmic rays. The nature of this article is similar to what Yukawa had predicted, like positive charge, negative charge and neutral charge. These particles are also known as pi-mesons or pions [1]. Yukawa noticed that the interaction between the elementary particles in the nucleus can be described utilizing a force field, just as the interaction between the charged particles is described by the electromagnetic field. The potential of force between the neutron and the proton should not be of Coulomb type, but decreasing much more rapidly with distance and described by Yukawa interaction potential [2].

1.1. Screening factor and Pauli Exclusion Principle for Yukawa Potential

The density of nucleons in heavy nuclei is taken to be constant and argue the short-range attractive potential, Pauli’s exclusion principle and the repulsive hardcore potential of nucleons. The attractive interaction potential between a pair of nucleons is given by the short-range Yukawa Potential,
\[ V = -\frac{k e^{-\alpha r}}{r} \]  

(1)

Where \( k \) is the nuclear charge (\( \frac{q^2}{\hbar c} \) is the dimensionless nuclear fine structure constant) and \( \alpha \) characterizes the range of the attractive potential. Moreover, Yukawa potential is defined also define as \( V(r) = \frac{k}{r} e^{-\alpha r} \). Where \( \alpha = \frac{\hbar}{mc} \) and this equal to Compton wavelength, \( m \) is the mass of elementary particles [3]. This type of potential depends strongly screening parameter that is if screening parameters is zero then the potential is Coulomb, for the positive value of screening parameters potential reduce to a finite one, while for large screening parameters potential reach a critical value.

2. Literature Review

The Yukawa potential is used in various areas of physics like high energy physic (collision), atomic and molecular physics and so on. This potential is often used to compute bound-state normalizations and energy levels of neutral atoms which have been studied over the past years. The generalized Yukawa potential takes of form as,

\[ V = -V_0 \frac{e^{-\alpha r}}{r} + \frac{k}{r^2} \]  

(2)

Where \( k \) is the strength of the generalized Yukawa potential.

In spherical coordinates, for a given potential Schrodinger equation is reduced to a generalized hypergeometric-type equation with appropriate coordinate transformation [4] as,

\[ \psi''(x) + \frac{\sigma(x)}{\sigma(x)} \psi'(x) + \frac{\sigma(x)}{\sigma(x)} \psi(x) = 0 \]  

(3)

Where \( \sigma(x) \) is a polynomial of at most first degree \( \left( A x + B \right) \), \( \sigma(x) \) and \( \sigma(x) \) is polynomial of at most second-degree \( \left( A x^2 + B y = C \right) \) and \( \psi(x) \) is a function of hypergeometric-type. The criteria which are related to degrees of polynomial coefficients constitute boundary conditions of the method [5].

2.1. Yukawa potential with other different potential

The solutions of the Klein-Gordon equation with more general exponential screened Coulomb (MGESC), Yukawa potential (YP) and the sum of the mixed potential (MGESCY) using the Parametric Nikiforov-Uvarov Method (PNUM). The more general exponential screened Coulomb (MGESC) potential expressed as,

\[ V(r) = -\frac{V_0}{r} \left( 1 + (1 + \alpha r) e^{-2\alpha r} \right) \]  

(5)

is a potential of great interest which on expansion comprises of the sum of Coulomb potential, modified screened coulomb of the Yukawa potential and a modified exponential potential given as,

\[ V(r) = -\frac{V_0}{r} - \frac{V_0}{r} e^{-2\alpha r} - V_0 \alpha e^{-2\alpha r} \]  

(6)

This potential is known to describe adequately the effective potential of a many-body system of a variety of fields such as the atomic, solid-state, plasma and quantum field theory [6]. The problems arising from screened Coulomb potential is of indubitable importance in physics and chemistry of atomic incidence. Roy in 2013 carried out extensive studies on some exponential screened Coulomb potentials such as the Exponential Cosine Screened Coulomb (ECSC) and General Exponential Screened Coulomb (GESC) potential with special emphasis on higher states and stronger interactions.

According to Yukawa, the expanded that interactions of particles are not always accompanied by the emission of light particles when heavy particles are transmitted from neutron state to proton state, but the liberated energy due to the transmission is taken up sometimes by another heavy particle, which will be transformed from proton state into neutron state. The Yukawa potential is a potential that decreases more rapidly with distance and can be expressed as the Coulomb potential when \( m \rightarrow 0 \). Since then, numerous researches had been conducted by various scientists to obtain a bound state of the potential by applying different scientific Methods.
Yukawa gained from these applications, and bearing in mind the significance of a reliable algebraic solution for bound and continuum regions, have been proven the success of the formalism. With the confidence additional/perturbed potential. The applications [8] of this novel treatment to different problems in both, bound and continuum regions, have been proven the success of the formalism. With the confidence gained from these applications, and bearing in mind the significance of a reliable algebraic solution for Yukawa-type potentials, that is reported in [9], we demonstrate here how such interaction potentials can be simply treated within the framework of the present formalism.

Sharma et al. calculated bound state for all angular momenta for superposed two static screened Coulomb potentials (SSCP) expressed as:

\[ V(r) = -\frac{g_1 e^{-ar} + g_2 e^{-sr}}{r} \]  \tag{7}

Where \( g_1 \) and \( g_2 \) are coupling constants, \( \alpha \) is the screening parameter and \( \gamma \) is the screening strength.\nonate and Ojunubah applied the supersymmetric shape invariance approach and formalism on a class of Yukawa potential is expressed,

\[ V(r) = \frac{-br^2 ce^{-ar} - ae^{-2ar}}{r^2} \]  \tag{8}

And obtained bound state energy eigenvalue calculations. Ita et al. obtained bound state solutions of the Schrödinger’s equation for Manning-Rosen plus a Class of Yukawa (MRCY) potential given as,

\[ V(r) = -\left[\frac{Ce^{-ar} + De^{-2ar}}{(1-e^{-ar})^2}\right] - \frac{v_0}{r} e^{-ar} - \frac{v'_0}{r^2} e^{-2ar} \]  \tag{9}

They deduced three different potentials such as the Manning-Rosen, Yukawa and inversely quadratic Yukawa potential and obtained bound state energy eigenvalues as well as wave functions for different principal quantum number \( n \) for the s-state.

Ikhdair obtained approximate analytical bound state solution of the Klein-Gordon equation with equal Scalar and Vector Eckart type potential given as,

\[ V(r, q) = 4V_1 \frac{e^{-2ar}}{(1-e^{-2ar})} - V_2 \frac{1+e^{-2ar}}{(1-qe^{-2ar})} \]  \tag{10}

Ikot et al. obtained approximate analytic solutions of the Klein-Gordon in D-dimension for any \( l \) state for a seven parameter type potential expressed as,

\[ V(r) = A + \frac{B}{(q+e^{-2ar})} + \frac{C}{(q+e^{-2ar})^2} + \frac{Fb e^{2ar}}{(q+e^{-2ar})} + \frac{Gb e^{2ar}}{(q+e^{-2ar})^2} \]  \tag{11}

Where A, B, C, F and G are potential parameters, \( b = e^{2ar}c, r_c \) is the distance from the equilibrium position and \( \alpha \) is the screening parameter.

3. Methodology

3.1. Yukawa potential from Klein-Gordon equation

For static potential in spherically symmetric goes to Klein Gordon equation is

\[ -\frac{1}{c^2} \frac{d^2V}{dr^2} + \nabla^2 V = \frac{m^2 c^2}{\hbar^2} V \]  \tag{12}

Using Laplacian for spherical coordinate system equation (13) become,

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = \frac{m^2 c^2}{\hbar^2} V \]  \tag{13}

Solving equation (14) using \( U = r V \), one get

\[ \frac{d^2U}{dr^2} - \frac{m^2 c^2}{\hbar^2} U = 0 \]  \tag{15}

Equation (15) has an exponential solution [15] as

\[ U = U_0 \exp \left( \pm \frac{mc}{\hbar} \frac{r}{a} \right) \]  \tag{16}

Choosing the decaying exponential for our solution becomes, \( V = -\frac{k}{r} e^{-\frac{r}{a}} \), where \( a = \frac{\hbar}{mc} \) the reduced Compton wavelength of a particle of mass \( m \).

More recently, a new methodology [7] has been introduced based on the decompose of the radial Schrödinger equation in two pieces having an exactly solvable part with an additional piece leading to either a closed analytical solution or an approximate treatment depending on the nature of the additional/perturbed potential. The applications [8] of this novel treatment to different problems in both, bound and continuum regions, have been proven the success of the formalism. With the confidence gained from these applications, and bearing in mind the significance of a reliable algebraic solution for Yukawa-type potentials, that is reported in [9], we demonstrate here how such interaction potentials can be simply treated within the framework of the present formalism.
3.2. Solution and Correction of Yukawa potential using Schrödinger equation

SE with Yukawa potential in the spherical coordinate system as

\[
\left\{ \frac{1}{R(r)} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) R(r) - \frac{2m r^2}{\hbar^2} \left[ V(r) - E \right] \right\} + \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) Y(\theta, \phi) \right\} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} Y(\theta, \phi) = 0
\]

(17)

The first term of curly bracket is dependent upon only radius or distance while the second and third term of the curly bracket is dependent upon only angles. Since the angular part is related with orbital quantum number \( l \) and given as \( l(l + 1) \)

Therefore from equation (17) we have two-equation one call radical equation and another angular equation written as

\[
\frac{1}{R(r)} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) R(r) - \frac{2m r^2}{\hbar^2} \left[ V(r) - E \right] = l(l + 1)
\]

(18a)

This equation is the radial equation for the hydrogen atom.

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) Y(\theta, \phi) - \frac{2m r^2}{\hbar^2} \left[ Y(\theta, \phi) - E \right] = -l(l + 1)
\]

(18b)

This equation (18b) is called the angular equation for the hydrogen atom. We have from equation (18a),

\[
V = -V_o \frac{e^{-ar}}{r} \quad \text{therefore} \quad \alpha = \frac{1}{a} = \frac{\hbar c}{m}
\]

\[
\frac{1}{R(r)} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) R(r) + \frac{2m r^2}{\hbar^2} \left[ E + V_o \frac{e^{-ar}}{r} \right] = l(l + 1)
\]

Since \( Z = 1 \) for the hydrogen atom,

\[
\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \frac{2m E + 2m V_o e^{-ar}}{\hbar^2} \frac{e^{-ar}}{r^2} R(r) = \frac{l(l + 1)}{r^2} R(r)
\]

(19)

Since SE with Yukawa potential has no exact solution therefore here we used approximation for a centrifugal term as

\[
\frac{1}{r^2} \approx 4 \alpha \left( \frac{e^{-2ar}}{(1 - e^{-2ar})^2} \right)
\]

Or equivalently

\[
\frac{1}{r} \approx 2 \alpha \left( \frac{e^{-ar}}{(1 - e^{-2ar})} \right)
\]

Which valid for \( ar \ll 1 \). Therefore Yukawa potential is reduced or corrected form

\[
V(r) = -4 \alpha V_o \frac{e^{-2ar}}{(1 - e^{-2ar})^2}
\]

(20)

Now equation (19) becomes,

\[
\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ \frac{2m E}{\hbar^2} + \frac{4m \alpha V_o}{\hbar^2} \frac{e^{-2ar}}{(1 - e^{-2ar})^2} - \frac{l(l + 1)}{r^2} \right] R(r) = 0
\]

Let \( -\frac{2m E}{\hbar^2} = \beta^2 \) as SE of hydrogen

\[
\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ -\beta^2 + \frac{4m \alpha V_o}{\hbar^2} \frac{e^{-2ar}}{(1 - e^{-2ar})^2} - \frac{l(l + 1)}{r^2} \right] R(r) = 0
\]

(21)

Let \( \rho = 2\beta r \), where \( \rho \) is a dimension-less variable therefore we can develop an expression

\[
\frac{d}{d\rho} \left( \rho^2 \frac{dR(r)}{d\rho} \right) = \frac{d}{d\rho} \left( \rho^2 \frac{dR}{d\rho} \right)
\]


Now equation this value and \( r = \frac{\rho}{2\beta} \) equation (21) becomes
\[
4\beta^2 \frac{d}{\rho^2} \frac{dR}{d\rho} \left( \frac{\rho^2}{d\rho} \right) + \left[ -\beta^2 + \frac{4m\alpha V_0}{\hbar^2} \frac{e^{-\frac{\rho}{\beta}}}{1 - e^{-\frac{\rho}{\beta}}} - \frac{4l(l + 1)\beta^2}{\rho^2} \right] R = 0
\]

Let \( \gamma = \frac{a\rho}{\beta} \) and solving we get,
\[
4\beta^2 \frac{d}{\rho^2} \frac{dR}{d\rho} \left( \frac{\rho^2}{d\rho} \right) + \left[ \frac{1}{4} + \frac{maV_0}{\beta^2\hbar^2} (1 - e^{-\gamma}) \right] R = 0
\]
\[
\frac{dR}{d\rho^2} + 2 \frac{dR}{d\rho} + \left[ \frac{maV_0}{\beta^2\hbar^2} (1 - e^{-\gamma}) \frac{1}{4} - \frac{l(l + 1)}{\rho^2} \right] R = 0
\]

Let \( n = \frac{maV_0}{\beta^2\hbar^2} e^{-\gamma} \) therefore above equation become,
\[
\frac{d^2 R}{d\rho^2} + \frac{2}{\rho} \frac{dR}{d\rho} + \left[ \frac{4n - 1}{\rho^2} - \frac{l(l + 1)}{\rho^2} \right] R = 0 \quad (22)
\]

Now to obtained asymptotic solution of equation (22) neglect \( 1/\rho^2 \) as \( \rho \to \infty \), and \( e^{-\gamma} = 0 \), therefore equation (22) becomes
\[
\frac{d^2 R}{d\rho^2} - \frac{R}{4} = 0
\]

Hence this equation has two solutions \( R = Ae^{\frac{\rho}{2}} \) and \( R = e^{-\frac{\rho}{2}} \) both solutions are acceptable. With this asymptotic solution, the possible solution of (22) is
\[
R = Ae^{\frac{\rho}{2}} F(\rho) \quad (23)
\]

This solution is mass-dependent which is considered in this work is less than meson masses, here we select the negative part because of the short-range and differentiate equation (23) we get,
\[
\frac{dR}{d\rho} = -\frac{\rho}{2} Ae^{\frac{\rho}{2}} F(\rho) + Ae^{\frac{\rho}{2}} \frac{dF}{d\rho}
\]
\[
\frac{d^2 R}{d\rho^2} = Ae^{\frac{\rho}{2}} \frac{d^2 F}{d\rho^2} - Ae^{\frac{\rho}{2}} \frac{dF}{d\rho} + \frac{Ae^{\frac{\rho}{2}}}{4} \frac{dF}{d\rho}
\]

On substituting and solving equation (22) become,
\[
Ae^{\frac{\rho}{2}} \frac{d^2 F}{d\rho^2} - Ae^{\frac{\rho}{2}} \frac{dF}{d\rho} + \frac{A}{4} e^{\frac{\rho}{2}} F - \frac{Ae^{\frac{\rho}{2}}}{\rho} F + \frac{2}{\rho} Ae^{\frac{\rho}{2}} \frac{dF}{d\rho} + \left[ \frac{4n - 1}{\rho^2} - \frac{l(l + 1)}{\rho^2} \right] Ae^{\frac{\rho}{2}} F = 0
\]
\[
\frac{d^2 F}{d\rho^2} - \frac{dF}{d\rho} + \frac{1}{4} - \frac{1}{\rho} F + \frac{2}{\rho} F + \left[ \frac{4n - 1}{\rho^2} - \frac{l(l + 1)}{\rho^2} \right] F = 0
\]
\[
\frac{d^2 F}{d\rho^2} + \frac{2}{(\rho - 1)} \frac{dF}{d\rho} + \left[ \frac{4n - 1}{\rho^2} - \frac{l(l + 1)}{\rho^2} \right] F = 0 \quad (24)
\]

This is an equation representing the radial part with corrected Yukawa potential using the Centrifugal approximation method. This equation depends upon the masses of elementary particles less than meson masses which is considered in this work.
3.3. Corrected Yukawa Potential with Centrifugal term approximation at k=0.5, 1, 1.5, and 2

Table 1: Masses have taken into consideration for our study

| S.N. | Masses Notation | Mass of Elementary Particles in a.m.u. |
|------|-----------------|---------------------------------------|
| 1.   | M1              | 1$m_e$                                |
| 2.   | M2              | 50$m_e$                               |
| 3.   | M3              | 100$m_e$                              |
| 4.   | M4              | 150$m_e$                              |
| 5.   | M5              | 200$m_e$                              |

3.4. Corrected Yukawa Potential with Centrifugal term approximation at k=0.5, 1, 1.5, and 2.
An appendix gives the detail of the numerical value of equation (20) and shows the strength of Yukawa potential at strength k=0.5, 1.0, 1.5, and 2. The calculation is based on the atomic unit that is $h = 1$, $m_e = 1$, $c = 1$. The calculation is considered to form elementary masses of particles whose masses is equal to the mass of electron that is 1times, 50times, 150times and 200times of mass of the electron. Here YPC1, YPC50, YPC100, YPC150, and YPC200 represent Corrected Yukawa potential for 1$m_e$, 50$m_e$, 100$m_e$, 150$m_e$ and 200$m_e$, respectively. This distance separation between two elementary particles is whose masses are less than meson. The separation of the two elementary particles whose masses are less than meson show that the strength of corrected Yukawa potential is infinity at zero separation. The strength of potential goes decrease with distance separation increase between elementary particles. Appendix table 1, Appendix table 2, Appendix table 3, and Appendix table 4 tabulated the numerical calculation based on equation (20) for different masses less than meson.

3.5. Old Yukawa Potential without Centrifugal term approximation at k = 0.5, 1.0, 1.5, and 2.0
An appendix gives the detail of the numerical value of equation (1) and shows the strength of Old Yukawa potential at strength k=0.5, 1.0, 1.5, and 2. The calculation is based on the atomic unit that is $h = 1$, $m_e = 1$, $c = 1$. The calculation is considered to form elementary masses of particles whose masses is equal to the mass of electron that is 1times, 50times, 150times and 200times of mass of the electron. Here YPO1, YPO50, YPO100, YPO150, and YPO200 represent Corrected Yukawa potential for 1$m_e$, 50$m_e$, 100$m_e$, 150$m_e$ and 200$m_e$, respectively. This distance separation between two elementary particles is whose masses are less than meson. The separation of the two elementary particles whose masses are less than meson show that the strength of corrected Yukawa potential is infinity at zero separation. The strength of potential goes decrease with distance separation increase between elementary particles. Appendix table 5, Appendix table 6, Appendix table 7, and Appendix table 8 tabulated the numerical calculation based on equation (20) for different masses less than meson.

3.6. Comparison of corrected and old Yukawa potential at k=1 form elementary particle masses less than meson.
The compassion of old and corrected Yukawa potential for masses 150$m_e$ and 200$m_e$ at k=1, listed in Appendix table 9. This listed numerical value is based on equation (1) and equation (20) for the same masses and distance separation both have different potential strength. The strength of corrected potential is greater than the strength of old Yukawa potential at the same condition that is the same elementary mass and distance separation.

3.7. Energy eigenvalue and solution of corrected Yukawa potential
The solution and development of the equation for corrected Yukawa potential are shown in equation (23) and (24). The solution is similar to the hydrogen atom radial part of SE for corrected Yukawa potential is dependent upon distance and quantum number.

4. Result and Discussion
The derived relation (20) is corrected Yukawa potential using centrifugal term approximation and relation (1) is old or uncorrected Yukawa potential. To study the comparison between corrected and uncorrected Yukawa potential different mass of unknown elementary particles less than meson mass is calculated. The equation beyond (20) gives the solution and develop the model for energy eigenvalue for corrected Yukawa potential.

4.1. Old Yukawa Potential at \( k = 0.5, 1.0, 1.5, \) and 2.0 for different masses less than meson

Figures 1 shows the nature of old Yukawa potential with different masses of particles less than meson mass. The shifting nature of old Yukawa potential towards the origin, as the particles go to closer to each other. For the particles, whose masses are equal to electron mass has high strength, as well as strength, goes shifted way from the origin in comparison to other masses like \( 50m_e, 100m_e, 150m_e, \) and \( 200m_e \). One more interesting observation is seen that as a mass of elementary particles goes increase the strength of Yukawa potential shifts away from origin as separation distance increase between elementary particles. With increasing the masses of elementary particles the shifting of strength is slow in the case of old Yukawa potential.

![Old Yukawa Potential for M1, M2, M3, M4, and M5 at k=0.5](Figure 1(a))

![Old Yukawa Potential for M1, M2, M3, M4, and M5 at k=1](Figure 1(b))
But increase with the value of \(k\) the nature of Yukawa potential are separated that is when the distance between two elementary particles is very close the strength of potential is mixed which is inseparable. Moreover, at a short distance, the strength of potential is independent of the mass of elementary mass and charge.

4.2. Corrected Yukawa Potential at \(k=, 0.5, 1.0, 1.5, \text{ and } 2.0\) for different masses less than meson

Figures 2 shows the nature of corrected Yukawa potential with different masses of particles less than meson mass. The shifting nature of corrected Yukawa potential towards origin different from the nature of old Yukawa potential, as the particles go closer to each other. For the particles, whose masses are equal to electron mass has high strength as well as strength goes shifted way from the origin in comparison to other masses like \(50m_e, 100m_e, 150m_e, \text{ and } 200m_e\). One more interesting observation
is seen that as a mass of elementary particles goes increase the strength of corrected Yukawa potential shifted sarapatel from the origin as separation distance increase between elementary particles. With increasing the masses of elementary particles the shifting and separation of the strength of potential for elementary particles with distance separation between the elementary particles.

![Corrected Yukawa Potential for M1, M2, M3, M4, and M5 at k=0.5](image1)

**Figure 2(a)**

![Corrected Yukawa Potential for M1, M2, M3, M4, and M5 at k=1](image2)

**Figure 2(b)**

![Corrected Yukawa Potential for M1, M2, M3, M4, and M5 at k=1](image3)
But increase with the value of $k$ the nature of corrected Yukawa potential is separated clearly from that of old Yukawa potential. In corrected Yukawa potential at a short distance or when the particle is very close strength of potential are not mixed but in the case of old Yukawa potential the strength is mixed. This is the major difference between old and corrected potential is that one can study each separate nature of potential strength using equation (20) while studying the nature of potential using equation (1) is very difficult to equation.

4.3. Comparison of old and corrected Yukawa potential at $200m_e$ and $150m_e$ elementary particles at $k = 1$

Figure’s 3, show the compassion of Yukawa potential for the same elementary mass and $k$. The nature of potential is the same but the separation between them is about 0.0035fm for $k=1$ and mass 200me, while for $k=1$ and mass 150me is about 0.004fm. The nature of uncorrected and corrected Yukawa
potential shows that uncorrected Yukawa potential detail at very close distance while corrected Yukawa potential is not given detail at same close distance. But the strength of corrected Yukawa potential is higher than uncorrected Yukawa potential at an ideal condition that is the same mass and same value of $k$ at a different distance. Corrected Yukawa potential uniformly decreases with asymptotic nature while uncorrected is sharply decrease and become constant as the separation distance between the particles is increased. Moreover, the strength of potential is the same at a long separation distance but different strength at a short distance as shown in figure 3 (a) and (b).

![Comparsion of Yukawa Potential for M5 at k=1](image1)

![Comparsion of Yukawa Potential for M4 at k=1](image2)

**Figure 3**: Comparison of Yukawa potential of elementary masses at $k=1$

5. **Conclusion**

On using equation (1) and equation (20) study of old and corrected Yukawa potential is shown in figure’s (1) and figure’s 2. But the comparison of these two potential studies using figure’s 3. The study is based on the mass of elementary particles less than masses of meson elementary particle. The mass considers in this work is 50times, 100times, 150times, and 200times mass of electron and $k$ value 0.5, 1.0, 1.5 and 2.0. At long-distance separation, the potential is the same but at short distance strength is potential is not the same. Moreover, at a short distance, uncorrected potential strength is identical for
all elementary mass less than meson and at a large distance also identical. For corrected potential, the strength of potential is not identical that is strength of potential is different for different elementary masses less than meson, but the same at large distance separation. In addition, the solution for the corrected Yukawa potential equation gives a new result that is it depends upon the quantum number and distance separation between elementary masses less than meson.

6. Declaration statement
Availability of data and materials: The data was generated using MATLAB software for this work based on the equation derived above, the plot above is drawn using MATLAB code.

Competing interest: No competing interest

Authors Contribution: Equally, contribute

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8. References
[1] Yukawa H 1935. Proc. Phys. and Math. Soc. 17 48-57.
[2] Anisiu MC 2015. Dida. Math. 33 1–9.
[3] Azam M and Gowda R S 2005. A simple analytical calculation of mean-field potential in Heavy Nuclei. Arxiv 1-10.
[4] Nikiforov A V and Uvarov V B 1998. Special Functions of Mathematical Physics, Birkhauser, Boston.
[5] Karayer H, Demirhan D and Buyukli F 2015. Extension of Nikiforov-Uvarov Method for the Solution of Heun Equation. Arxiv 1-10.
[6] Hitler L, Iserom I B, Tchoua P and Etta A A 2018. J. Phys. and Math. 9 261 12-30
[7] Gonul L 2004. Chi. Phys. Lett. 21 1685 23-30.
[8] Ozer O and Gonul B 2003. Mod. Phys. Lett. A. 18 2581 45-50.
[9] Chakrabarti B and Das T K 2001. Phys. Lett. A. 11 285 12-15.
Appendix

Appendix table 1: Corrected Yukawa potential at $k=0.5$ for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YPC1 (fm$^{-1}$) | YPC50 (fm$^{-1}$) | YPC100 (fm$^{-1}$) | YPC150 (fm$^{-1}$) | YPC200 (fm$^{-1}$) |
|-------------------------|------------------|------------------|--------------------|--------------------|--------------------|
| 0                       | $-\infty$        | $-\infty$        | $-\infty$          | $-\infty$          | $-\infty$          |
| 0.001                   | -12481.4         | -458766          | -670275            | -734472            | -715395            |
| 0.01                    | -1179.55         | -2718.16         | -235.299           | -15.2766           | -0.88162           |
| 0.02                    | -553.876         | -58.8248         | -0.2204            | -0.00062           | -1.55E-06          |
| 0.03                    | -346.775         | -1.6974          | -0.00028           | -3.35E-08          | -3.62E-12          |
| 0.04                    | -244.25          | -0.0551          | -3.87E-07          | -2.04E-12          | -9.53E-18          |
| 0.05                    | -183.506         | -0.00191         | -5.80E-10          | -1.32E-16          | -2.68E-23          |
| 0.06                    | -143.614         | -6.88E-05        | -9.05E-13          | -8.92E-21          | -7.82E-29          |
| 0.07                    | -115.605         | -2.55E-06        | -1.45E-15          | -6.20E-25          | -2.35E-34          |
| 0.08                    | -94.997          | -9.67E-08        | -2.38E-18          | -4.40E-29          | -7.23E-40          |
| 0.09                    | -79.302          | -3.72E-09        | -3.97E-21          | -3.17E-33          | -2.25E-45          |
| 0.1                     | -67.0275         | -1.45E-10        | -6.69E-24          | -2.31E-37          | -7.12E-51          |

Appendix table 2: Corrected Yukawa potential at $k=1$ for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YP1 (fm$^{-1}$) | YP50 (fm$^{-1}$) | YP100 (fm$^{-1}$) | YP150 (fm$^{-1}$) | YP200 (fm$^{-1}$) |
|-------------------------|-----------------|-----------------|------------------|------------------|------------------|
| 0                       |                 |                 |                  |                  |                  |
| 0.001                   | -24962.7        | -917532         | -1340549         | -1468945         | -1430790         |
| 0.005                   | -4868.7         | -52260.9        | -21745.3         | -6786            | -1882.39         |
| 0.01                    | -2359.1         | -5436.32        | -470.598         | -30.5332         | -1.76324         |
| 0.02                    | -1107.75        | -117.65         | -0.44081         | -1.24E-03        | -3.09E-06        |
| 0.03                    | -6.94E+02       | -3.39E+00       | -5.51E-04        | -6.70E-08        | -7.24E-12        |
| 0.04                    | -4.89E+02       | -1.10E-01       | -7.74E-07        | -4.07E-12        | -1.91E-17        |
| 0.05                    | -3.67E+02       | -3.82E-03       | -1.16E-09        | -2.64E-16        | -5.35E-23        |
| 0.06                    | -2.87E+02       | -1.38E-04       | -1.81E-12        | -1.78E-20        | -1.56E-28        |
| 0.07                    | -2.31E+02       | -5.11E-06       | -2.91E-15        | -1.24E-24        | -4.71E-34        |
| 0.08                    | -1.90E+02       | -1.93E-07       | -4.76E-18        | -8.80E-29        | -1.45E-39        |
| 0.09                    | -1.59E+02       | -7.44E-09       | -7.93E-21        | -6.34E-33        | -4.51E-45        |
| 0.1                     | -1.34E+02       | -2.90E-10       | -1.34E-23        | -4.63E-37        | -1.42E-50        |

Appendix table 3: Corrected Yukawa potential at $k=1.5$ for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YP1 (fm$^{-1}$) | YP50 (fm$^{-1}$) | YP100 (fm$^{-1}$) | YP150 (fm$^{-1}$) | YP200 (fm$^{-1}$) |
|-------------------------|-----------------|-----------------|------------------|------------------|------------------|
| 0                       |                 |                 |                  |                  |                  |
| 0.001                   | -37444.1        | -1376298        | -2010824         | -2203417         | -2146184         |
| 0.01                    | -3538.65        | -8154.48        | -705.897         | -45.8298         | -2.64486         |
different separation distance of elementary particles of mass less than meson.

Appendix 8.1.

Appendix table 4: Corrected Yukawa potential at k=2 for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YP1 (fm$^{-1}$) | YP50 (fm$^{-1}$) | YP100 (fm$^{-1}$) | YP150 (fm$^{-1}$) | YP200 (fm$^{-1}$) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0                       |                 |                 |                 |                 |                 |
| 0.001                   | -49925.5        | -1835064        | -2681098        | -2937890        | -2861579        |
| 0.01                    | -47182.2        | -10872.6        | -941.196        | -61.064         | -3.52648        |
| 0.02                    | -2215.5         | -235.299        | -0.88162        | -0.00248        | -6.19E-06       |
| 0.03                    | -1.39E+03       | -6.79E+00       | -1.10E-03       | -1.34E-07       | -1.45E-11       |
| 0.04                    | -9.77E+02       | -2.20E-01       | -1.55E-06       | -8.14E-12       | -3.81E-17       |
| 0.05                    | -7.34E+02       | -7.63E-03       | -2.32E-09       | -5.28E-16       | -1.07E-22       |
| 0.06                    | -5.74E+02       | -2.75E-04       | -3.62E-12       | -3.57E-20       | -3.13E-28       |
| 0.07                    | -4.62E+02       | -1.02E-05       | -5.81E-15       | -2.48E-24       | -9.41E-34       |
| 0.08                    | -3.80E+02       | -3.87E-07       | -9.53E-18       | -1.76E-28       | -2.89E-39       |
| 0.09                    | -3.17E+02       | -1.49E-08       | -1.59E-20       | -1.27E-32       | -9.02E-45       |
| 0.1                     | -2.68E+02       | -5.80E-10       | -2.68E-23       | -9.26E-37       | -2.85E-50       |

8.1.

Appendix table 5: Old Yukawa potential at k=0.5 for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YPO1 (fm$^{-1}$) | YPO50 (fm$^{-1}$) | YPO100 (fm$^{-1}$) | YPO150 (fm$^{-1}$) | YPO200 (fm$^{-1}$) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0                       |                 |                 |                 |                 |                 |
| 0.001                   | -496.87         | -365.26         | -266.829        | -194.924        | -142.395        |
| 0.01                    | -46.9566        | -2.16414        | -0.09367        | -4.05E-03       | -1.75E-04       |
| 0.02                    | -2.20E+01       | -4.68E-02       | -8.77E-05       | -1.64E-07       | -3.08E-10       |
| 0.03                    | -1.38E+01       | -1.35E-03       | -1.10E-07       | -8.89E-12       | -7.20E-16       |
| 0.04                    | -9.72E+00       | -4.39E-05       | -1.54E-10       | -5.40E-16       | -1.90E-21       |
| 0.05                    | -7.31E+00       | -1.52E-06       | -2.31E-13       | -3.51E-20       | -5.32E-27       |
| 0.06                    | -5.72E+00       | -5.48E-08       | -3.60E-16       | -2.37E-24       | -1.56E-32       |
| 0.07                    | -4.60E+00       | -2.03E-09       | -5.78E-19       | -1.65E-28       | -4.68E-38       |
Appendix table 6: Old Yukawa potential at k=1 for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YPO1 $(fm^{-1})$ | YPO50 $(fm^{-1})$ | YPO100 $(fm^{-1})$ | YPO150 $(fm^{-1})$ | YPO200 $(fm^{-1})$ |
|--------------------------|------------------|------------------|------------------|------------------|------------------|
| 0.08                     | -3.78E+00        | -7.70E-11        | -9.48E-22        | -1.17E-32        | -1.44E-43        |
| 0.09                     | -3.16E+00        | -2.96E-12        | -1.58E-24        | -8.42E-37        | -4.49E-49        |
| 0.1                      | -2.66E89         | -1.15E-13        | -2.66E-27        | -6.14E-41        | -1.42E-54        |

Appendix table 7: Old Yukawa potential at k=1.5 for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YPO1 $(fm^{-1})$ | YPO50 $(fm^{-1})$ | YPO100 $(fm^{-1})$ | YPO150 $(fm^{-1})$ | YPO200 $(fm^{-1})$ |
|--------------------------|------------------|------------------|------------------|------------------|------------------|
| 0                        | -8.99E+00        | -4.59E+00        | -2.09E+00        | -1.09E+00        | -5.49E+00        |
| 0.01                     | -4.19E+00        | -2.19E+00        | -1.09E+00        | -5.49E+00        | -2.74E+00        |
| 0.05                     | -2.19E+00        | -1.09E+00        | -5.49E+00        | -2.74E+00        | -1.37E+00        |
| 0.1                      | -5.33E+00        | -2.31E+00        | -1.23E+00        | -6.49E+00        | -3.24E+00        |

Appendix table 8: Old Yukawa potential at k=2 for different mass $1m_e, 50m_e, 150m_e, 200m_e$ with a different separation distance of elementary particles of mass less than meson.

| Distance Separation (fm) | YPO1 $(fm^{-1})$ | YPO50 $(fm^{-1})$ | YPO100 $(fm^{-1})$ | YPO150 $(fm^{-1})$ | YPO200 $(fm^{-1})$ |
|--------------------------|------------------|------------------|------------------|------------------|------------------|
| 0.01                     | -2.19E+00        | -1.09E+00        | -5.49E+00        | -2.74E+00        | -1.37E+00        |
| 0.05                     | -2.19E+00        | -1.09E+00        | -5.49E+00        | -2.74E+00        | -1.37E+00        |
| 0.1                      | -5.33E+00        | -2.31E+00        | -1.23E+00        | -6.49E+00        | -3.24E+00        |
### Old Yukawa Potential at k=2

| Distance Separation (fm) | YPO1 (fm⁻¹) | YPO50 (fm⁻¹) | YPO100 (fm⁻¹) | YPO150 (fm⁻¹) | YPO200 (fm⁻¹) |
|-------------------------|-------------|-------------|-------------|-------------|-------------|
| 0                       |             |             |             |             |             |
| 0.001                   | -1987.48    | -1461.04    | -1067.32    | -779.695    | -569.582    |
| 0.01                    | -187.826    | -8.65656    | -0.37468    | -0.01622    | -0.0007     |
| 0.02                    | -88.1968    | -0.18734    | -0.00035    | -0.657E-07  | -1.23E-09   |
| 0.03                    | -55.2189    | -0.00541    | -4.38E-07   | -3.55E-11   | -2.88E-15   |
| 0.04                    | -38.8933    | -0.00018    | -6.16E-10   | -2.16E-15   | -7.59E-21   |
| 0.05                    | -29.2208    | -6.08E-06   | -9.23E-13   | -1.40E-19   | -2.13E-26   |
| 0.06                    | -22.8684    | -2.19E-07   | -1.44E-15   | -9.47E-24   | -6.23E-32   |
| 0.07                    | -18.4084    | -8.13E-09   | -2.31E-18   | -6.58E-28   | -1.87E-37   |
| 0.08                    | -15.1269    | -3.08E-10   | -3.79E-21   | -4.67E-32   | -5.75E-43   |
| 0.09                    | -12.6277    | -1.18E-11   | -6.32E-24   | -3.37E-36   | -1.80E-48   |
| 0.1                     | -10.6732    | -4.62E-13   | -1.06E-26   | -2.46E-40   | -5.67E-54   |

Appendix table 9: Strength of Yukawa potential at k=1 for elementary masses less than meson mass

### Comparison of Old and Corrected Yukawa Potential at k=1

| Separation Distance (fm) | YPO200 (fm⁻¹) | YPOC200 (fm⁻¹) | YPO150 (fm⁻¹) | YPOC150 (fm⁻¹) |
|-------------------------|---------------|---------------|---------------|---------------|
| 0                       |               |               |               |               |
| 0.0001                  | -8819.68      | -4.4E+07      | -9101.01      | -3.4E+07      |
| 0.0005                  | -1067.32      | -5362196      | -1248.76      | -4705309      |
| 0.001                   | -284.791      | -1430790      | -389.847      | -1468945      |
| 0.002                   | -40.5529      | -203738       | -75.9905      | -286332       |
| 0.003                   | -7.6994       | -38681.8      | -19.7498      | -74417.2      |
| 0.004                   | -1.64454      | -8262.17      | -5.77455      | -21758.5      |
| 0.005                   | -0.37468      | -1882.39      | -1.80096      | -6786         |
| 0.006                   | -0.08892      | -446.74       | -0.58508      | -2204.59      |
| 0.007                   | -0.02171      | -109.052      | -0.19551      | -736.673      |
| 0.008                   | -0.00541      | -27.1749      | -0.06669      | -251.291      |
| 0.009                   | -0.00137      | -6.87927      | -0.02311      | -87.0803      |
| 0.01                    | -0.0004       | -2.0194       | -0.009        | -33.9103      |