Quasi $\mathcal{PT}$-symmetry in passive photonic lattices

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Abstract
The concept of quasi-$\mathcal{PT}$ symmetry in an optical wave-guiding system is elaborated by comparing the evolution dynamics of a $\mathcal{PT}$-symmetric directional coupler and a passive directional coupler. In particular, we show that in the low-loss regime, apart from an overall exponentially damping factor that can be compensated via a dynamical renormalization of the power flow in the system along the propagation direction, the dynamics of the passive coupler fully reproduce the one in the $\mathcal{PT}$-symmetric system.

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1. Introduction
Quantum mechanics is one of the brightest scientific theories of the last century. When facing this theory for the first time, one of the first things that one learns is that only Hermitian operators (that possess, by definition, a real spectrum of eigenvalues) are associated with physical quantities, and that non-Hermitian ones have, in general, no physical meaning [1]. In 1998, however, Bender and co-workers [2] introduced the concept of $\mathcal{PT}$-symmetry, pointing out that Hermitian Hamiltonians do not represent all the possible physically meaning quantum systems, but they are only a special subset of a more general theory. The so-called $\mathcal{PT}$-symmetric systems are characterized by a complex potential, which per se possesses neither parity symmetry ($\mathcal{P}$) nor time-reversal symmetry ($\mathcal{T}$), but the Hamiltonians of these systems share the same eigenstates with the parity-time operator $\mathcal{PT}$. Under these conditions, the eigenvalues of the Hamiltonian are real (up to a certain threshold below which $\mathcal{PT}$-symmetry is preserved) even if that the potential is complex-valued. Bender and Mannheim [3] have also shown that any Hamiltonian with real or complex conjugate pairs of eigenvalues possesses a generalized $\mathcal{PT}$-symmetry, making generalized $\mathcal{PT}$-symmetry a necessary (but not sufficient) condition for a real eigenvalue spectrum. In addition to this, it has also been proven that a $\mathcal{PT}$-symmetric system can be mapped onto an Hermitian system under a proper redefinition of the scalar product [4], only in the case in which the $\mathcal{PT}$-Hamiltonian possesses only real eigenvalues, the case that Bender calls ‘unbroken $\mathcal{PT}$-symmetry’ [2].

Despite the interesting nature of these works, and the significant amount of discussion on the impact of $\mathcal{PT}$-symmetry in quantum mechanics itself [5], quantum field theory [6], open quantum systems [7] and Anderson localization phenomena [8], this extension of ordinary quantum theories remained, however, a mere speculative curiosity, mainly because of the difficulty of finding in nature a system that exhibits such a peculiar Hamiltonian.

As the development of $\mathcal{PT}$-symmetric quantum mechanics, optics and waveguiding structures has progressed, interest has grown in a laboratory tool that mimicks the evolution of quantum systems. This interest is a result of the mathematical correspondence between the paraxial wave equation that describes the propagation of light in a guiding structure and the Schrödinger equation [9]. The analogy, moreover, does not limit itself only to the non-relativistic case, and, very recently, optical analogues of relativistic effects, such as Klein tunneling [10], Zitterbewegung [11] and pair production in a vacuum [12] have been proposed. In 2007, the concept of $\mathcal{PT}$-symmetry was brought into optics [13], revealing that optical systems are the natural candidates to realize in an easy way $\mathcal{PT}$-symmetric systems. In the last years, $\mathcal{PT}$-symmetric optical systems attracted a large
amount of interest, and this topic was extensively studied [16–22].

A common feature of all these $\mathcal{PT}$-symmetric optical systems is that in order to achieve $\mathcal{PT}$-symmetry, an equal amount of gain and loss must be present in the system. However, in several works quasi-$\mathcal{PT}$-symmetric systems were considered, in which only loss is present in the individual waveguides and no gain [15, 23, 24]. Such a scheme of an optical system partly lossless and partly lossy was also studied well before the introduction of $\mathcal{PT}$-symmetry as a way to reduce soliton jitter and noise power in optical fibers [25] and, more recently, as an efficient way to realize ‘non-reciprocal’ light propagation in silicon photonic circuits [26] and unidirectional reflectionless metamaterials, where a detailed comparison between $\mathcal{PT}$ and quasi-$\mathcal{PT}$ Bragg gratings is reported [27].

It is the aim of this paper to provide a detailed proof that in such passive systems the evolution dynamics are in fact the same as in systems exhibiting gain and loss, up to a global exponential damping factor and for sufficiently small losses. We base our considerations on the prototypical example of a $\mathcal{PT}$-symmetric directional coupler, but the extension to general lattices is straightforward.

This work is organized as follows: in section 2, we briefly review the salient features of a $\mathcal{PT}$-symmetric directional coupler with balanced gain and loss. In section 3, we introduce the concept of quasi-$\mathcal{PT}$-symmetry by analyzing the evolution of a light field in a directional coupler with unbalanced loss and show that the dynamical features are exactly the same as in the full-$\mathcal{PT}$-case. The end of section 3, will be then dedicated to the study of how the amount of loss in the system contribute to spoil this correspondence. Conclusions are then drawn in section 4.

2. $\mathcal{PT}$-symmetric optical coupler

We begin our analysis by considering a monochromatic scalar electric field $\psi(x, z)$ characterized by a frequency $\omega_0$ and a wavelength $\lambda = 2\pi/\kappa$ that propagates inside a directional coupler. We assume that the mode field inside the coupler is strongly confined along the $y$-direction, so that the beam dynamics in such a system can be taken to be one-dimensional in the transverse plane. The evolution of the light field $\psi(x, z)$ inside such a structure is governed by the following adimensional paraxial equation:

$$i \frac{\partial \psi}{\partial z} = - \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \equiv \hat{H} \psi,$$

where $z$ and $x$ are suitably chosen dimensional coordinates and $V(x) = V_0(x) + iV_i(x)$ is the complex optical potential that implements $\mathcal{PT}$-symmetry in the system. Following [13], in order for $V(x)$ to be $\mathcal{PT}$-symmetric the condition $V^*(x) = V(-x)$ must be fulfilled. This condition arises from the necessity for the Hamiltonian in equation (1) to commute with the parity-time operator $\hat{PT}$ in such a way that $\psi$ is a common eigenstate of both $\hat{H}$ and $\hat{PT}$. The parity operator $\hat{P}$ is defined by the operations $\hat{z} \rightarrow - \hat{z}$ and $\hat{\rho} \rightarrow - \hat{\rho}$, while the time reversal operator $\hat{T}$ consists of the operations $\hat{\rho} \rightarrow - \hat{\rho}$ and $\hat{t} \rightarrow - \hat{t}$ [28]. Practically, this means that the optical complex potential $V(x)$ consists of a symmetric refractive index profile $V_0(x)$ and an antisymmetric gain/loss profile $V_i(x)$. A sketch of the refractive index profile of such a coupler is given in figure 1(a). By expanding the scalar electric field $\psi(x, z)$ onto the eigenmodes of the coupler as

$$\psi(x, z) = [a_1(z)u_1(x) + a_2(z)u_2(x)]e^{i\kappa z},$$

where $\kappa$ is the real (due to $\mathcal{PT}$-symmetry) propagation constant and the eigenmodes $u_i(x)$ are normalized according to [13]

$$\int dx \ u_m^*(-x) \ u_n(x) = \delta_{mn},$$

we can describe the evolution of light in such a system via the following coupled mode equations:

$$i \frac{d}{dz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \Delta + i\gamma & \kappa \\ \kappa & \Delta - i\gamma \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where $\kappa$ is the usual coupling coefficient and $\Delta + i\gamma$ is the shift in the propagation constant due to the coupling interaction. Note that here instead of following the design of [13] where a single waveguide experiences both gain and loss in equal amounts, we employ the dimer idea developed in [14], where $\mathcal{PT}$-symmetry is obtained by inserting gain in the first waveguide and loss in the second one in equal amounts. We note, moreover, that the real part $\Delta$ can be neglected because it can be eliminated by a gauge transformation, that corresponds to consider the two waveguides to have no relative detuning [29]. Without loss of generality we then set $\Delta = 0$ in equation (4). According to [15], as long as $\kappa/\gamma > 1$ the $\mathcal{PT}$-symmetry is unbroken and light is periodically exchanged between the two waveguides. On the other hand, $\mathcal{PT}$-symmetry is said to be broken when $\kappa/\gamma < 1$ and the light dynamics begin exponentially growing in one waveguide and exponentially damping in the other one [15]. Light evolution in these two regimes is depicted in figure 2.

3. Quasi-$\mathcal{PT}$-symmetric optical coupler

The directional coupler described in the previous section fully implements a $\mathcal{PT}$-symmetric system, and it is realized according to the rule described in [14], namely, to insert gain in one waveguide and loss in the other one in equal measure. Although this appears to be the normal way of building a $\mathcal{PT}$-symmetric structure in optics, it appears to be quite complicated to be realized experimentally, as a full control of gain and loss is a very difficult task. It is therefore interesting to study whether a similar physical problem as the one described in the previous section can be obtained by exploiting only passive systems, and create a loss imbalance between the two waveguides instead of a gain and loss structure. Let us then
consider a directional coupler in which each waveguide experiences a different level of loss, and no gain is inserted in the system. The complex refractive index profile is depicted in figure 1(b). Using a standard coupled-mode theory [29], the propagation of a light beam in such a structure can be described as follows:

\[ \gamma_1 \kappa \gamma_2 = - \frac{i \gamma_1}{\kappa} - i \gamma_2 \left( a_1, a_2 \right) \]  \( (5) \)

where \( \gamma_{1,2} \) accounts for the losses in the first and second waveguides, respectively, and \( \kappa \) has been defined before as the coupling coefficient. It is interesting to compare this equation with equation (4) with \( \Delta = 0 \). While in equation (4) there is only a sign difference between the two diagonal elements, here in principle \( \gamma_1 \neq \gamma_2 \) but the sign is the same. The sign discrepancy is due to the fact that while equation (4) describes a gain/lossy system, equation (5) describes a lossy system only. Note, moreover, that since the diagonal elements of the matrix in equation (5) are purely imaginary, they cannot be removed via a simple phase transformation. We can, however, exploit a common trick used in quantum field theory known as Wick rotation [30] that consists of rotating the time axis (in this case the propagation axis) by \( \pi/2 \) in the complex plane, thus employing a complex time (in our case a complex propagation direction) instead of a real one. This trick is very useful especially in lattice quantum field theory, to transform the Minkowski metric to an Euclidean metric, allowing methods of statistical mechanics to be used for evaluating path integrals on a lattice [31]. If we define \( \zeta = iz \) and substitute this ansatz into equation (5), we obtain

\[ -\frac{d}{dz} \left( a_1 \right) = -i \gamma_1 \kappa \left( a_1, a_2 \right) \]  \( (6) \)

We can now make the following phase transformation

\[ \left( a_1, a_2 \right) = \left( \tilde{a}_1, \tilde{a}_2 \right) e^{i\phi}, \]  \( (7) \)

and then transform back to the real propagation axis \( z \) to...
obtain

\[ i \frac{d}{dz} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ \kappa - i (\gamma_1 - \gamma_2) & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}. \]  

Before proceeding with the analysis of equation (8), an explanation about this procedure is needed. First, note that thanks to equation (7), to restore the initial amplitudes one needs to multiply the solutions of equation (8) by an exponentially damping factor \( \exp(-\gamma z) \). While theoretically this only accounts to a gauge transformation in Wick space, experimentally the presence of this extra damping term is not a problem because once the amount of losses \( \gamma \) is known, this term can be easily eliminated via post-processing of the acquired image.

It is now instructive to calculate the eigenvalues of the matrix that appears in equation (8) and compare them with the ones from equation (4). If we call \( \mu_{1,2} \) the eigenvalues of the \( \mathcal{PT} \)-symmetric system (4) and \( \lambda_{1,2} \) the ones for the passive system (8), we have the following result:

\[ \mu_{1,2} = \pm \sqrt{\kappa^2 - \gamma^2}, \]  

\[ \lambda_{1,2} = -i \left( \frac{\gamma_2 - \gamma_1}{2} \right) \pm \sqrt{\left( \frac{\gamma_2 - \gamma_1}{2} \right)^2 - \frac{k^2}{4}}. \]

If we now choose \( \gamma_2 - \gamma_1 = 2\gamma \), where \( \gamma \) is the same value of gain/loss that appears in the \( \mathcal{PT} \)-symmetric case, then, apart from a common imaginary part (that will result in an exponential damping factor), the dynamics of the passive system is the \( \mathcal{PT} \)-symmetric case, as \( \lambda_{1,2} = -i\gamma \pm \mu_{1,2} \). This is the main result of this paper: the dynamics of a \( \mathcal{PT} \)-symmetric system are exactly the same as the dynamics of a passive one (i.e., a system with only losses) provided that the losses of the system are chosen in such a way that \( \gamma_2 - \gamma_1 = 2\gamma \).

A comparison between the dynamics of a passive system as described by equation (8) and the correspondent \( \mathcal{PT} \)-case is depicted in figure 3 for unbroken \( \mathcal{PT} \)-symmetry and in figure 4 for the broken \( \mathcal{PT} \)-symmetry case. This is true for the dynamics of the passive system in terms of the amplitudes \( \hat{a}_{1,2} \). However, while below threshold (figure 3(b)), the presence of the exponential damping factor only affects the intensity of the light that propagates in the quasi-\( \mathcal{PT} \)-symmetry. By employing a Wick rotation of the system and applied a gauge transformation in Wick space to eliminate the losses in one of the waveguides and thus define the loss unbalance, that is, the governing parameter of the system. Our results show that

4. Conclusions

In conclusion, we have shown that a gain/loss structure as the one described in [14] is not a necessary condition for an optical system to show \( \mathcal{PT} \)-symmetry. By employing a passive (i.e., lossy) system, we proved that although this latter system follows non-Hermitian dynamics, for small enough losses the dynamical behavior of such a system, up to an overall exponential damping factor, can perfectly reproduce the characteristic dynamics of a \( \mathcal{PT} \)-system. To prove this we performed a Wick rotation of the system and applied a gauge transformation in Wick space to eliminate the losses in one of the waveguides and thus define the loss unbalance, that is, the governing parameter of the system. Our results show that
below the $\mathcal{PT}$-symmetry breaking threshold, the dynamics of the two systems are fully equivalent (in the low-loss regime), while above this threshold, a dynamical power renormalization as a function of the propagation distance is needed in order to extract the fully $\mathcal{PT}$-dynamics.

As a final remark we note that our results can be of high importance for experimental realizations of $\mathcal{PT}$-symmetric optical systems, as controlling the amount of loss in a passive system is surely an easier task than inserting a gain structure in an optical system.

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