Correspondence between genus expansion and $\alpha'$ expansion in string theory

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Abstract

In this paper, we demonstrate that locally, the $\alpha'$ expansion of a string propagating in AdS can be summed into a closed expression, where the $\alpha'$ dependence is manifested. The T-dual of this sum exactly matches the expression controlling all genus expansion in the Goparkumar-Vafa formula, which in turn also matches the loop expansion of the Chern-Simons gauge theory. We therefore find an exact correspondence between the $\alpha'$ expansion for a string moving in AdS and the genus expansion of a string propagating in four dimensional flat spacetime. We are then able to give a closed form of the $\alpha'$ expansion for all values of $\sqrt{\alpha'}/R_{AdS}$. Moreover, the correspondence makes it possible to conjecture the exact $g_s$ dependence of the strongly coupled theories.
1 Motivations

The celebrated AdS/CFT correspondence conjectures an equivalence of the gravitational theory in the bulk of AdS and the gauge theory on the AdS conformal boundary [1]. The parameters in these two theories are related with \( \lambda = g_{\text{YM}}^2 N = 4\pi g_s N = \left(\frac{R_{\text{AdS}}}{\sqrt{\alpha'}}\right)^4 \), where \( \lambda \) is the ’t Hooft coupling, \( g_{\text{YM}} \) is the Yang-Mills coupling, \( g_s \) is the string coupling and \( SU(N) \) is the gauge group. \( \frac{R_{\text{AdS}}}{\sqrt{\alpha'}} \) comes from string theory where \( R_{\text{AdS}} \) is the radius of the AdS background and \( \sqrt{\alpha'}/\ell_s \) is the string length. As one can easily see from the parameter relations, large \( \lambda \) maps to small \( \sqrt{\alpha'}/R_{\text{AdS}} \) and vice versa.

It is well known that string theory has a double perturbation expansion. One is the genus expansion, controlled by the string coupling \( g_s = \exp (\phi) \), where \( \phi \) is the dilaton. This expansion is sometimes interpreted as classical string interactions, counting worldsheets with different topologies characterized by the genus. In the ’t Hooft’s argument, this genus expansion is equivalent to the loop expansion of a Yang-Mills gauge theory in terms of the ’t Hooft coupling \( \lambda = g_{\text{YM}}^2 N = 4\pi g_s N \). The second one is the \( \alpha' \) correction, coming from the non-linear Sigma model when the string propagates in a curved background with a characteristic radius of curvature \( R_c \). The \( \alpha' \) expansion is controlled by the parameter \( \sqrt{\alpha'}/R_c \) and sometimes referred as a quantum expansion.

Since the controlling parameters of the \( \alpha' \) expansion and genus expansion are related by the AdS/CFT correspondence, it is appealing to ask if there exists some relation between these two expansions. It is obvious that the possibly existing relation cannot arise in a simple manner since we have a strong-weak correspondence \( g_1^4 \sim \sqrt{\alpha'}/R_{\text{AdS}} \). In order to have such a correspondence, there ought to be a mechanism somehow transfers a large \( \sqrt{\alpha'}/R_{\text{AdS}} \) theory to a small \( \sqrt{\alpha'}/R_{\text{AdS}} \) theory, or the same pattern for the \( g_s \) side (S-duality may play the role).

The purpose of this paper is to show there does exist a local exact correspondence between the \( \alpha' \) expansion of a string moving in AdS and genus expansion of a string propagating in flat spacetime. We will show that in AdS, T-duality exhibits in a different manner from the ordinary one. As we know, compactification is a prerequisite for the ordinary T-duality and the physics are identical for compactification radii \( R \) and \( \alpha'/R \). However, in AdS, using the Buscher’s rule [2], we are going to demonstrate that the lengths of T-dual strings are related by \( \ell \bar{\ell} = R_{\text{AdS}}^2 \). The difference here from the ordinary T-duality is that compactification is not necessary and two dual strings have different string lengths \( \ell \) and \( \bar{\ell} \) respectively. It seems that this is possible only for AdS geometry since it is maximally symmetric, conformally flat and the dual geometry is still an AdS with the same radius. Therefore, when performing \( \alpha' \) expansion, the dual strings are expanded in dual regimes, namely small/large respectively. It is conceivable that in order to justify this T-duality, there ought to be a correspondence between the \( \alpha' \) expansion and genus expansion. Or conversely speaking, the existence of the correspondence must predict a duality between large \( \sqrt{\alpha'}/R_{\text{AdS}} \) theory and a small \( \sqrt{\alpha'}/R_{\text{AdS}} \) one.

Remarkably, we find that if we take Riemann normal coordinate to locally expand a string moving in AdS,
the $\alpha'$ expansion can be added up to a closed expression. Furthermore, the closed form of the T-dual string is the same as the expression controlling all genus expansion in the Goparkumar-Vafa formula (GV). With this correspondence, it is possible to make predictions on $g_s$ dependence for strongly coupled GV, i.e. the strongly coupled gauge theory.

Since we are going to compare the $\alpha'$ expansion to the genus expansion of the GV formula, it is necessary to recall some results of the GV formula [3, 4, 5]. As we know, $F-$terms in a supersymmetric field theory or string theory play special roles since nonperturbative results can usually be extracted from $F-$terms only. Therefore, if there exists an exact relation between the $\alpha'$ expansion and genus expansion, this term should be interpreted as the four dimensional superspace effective action in a supersymmetric background with a constant anti-selfdual graviphoton. Moreover, instead of calculating the expansion and completely control the expansion. Indeed as we went to show, the $\alpha'$ expansion in AdS does give

$$\mathcal{I} = \sum_{g \geq 0} I^g = -\frac{1}{2\pi} \int_{\mathbb{R}^4} d^4x \, \theta \sum_{g \geq 0} \mathcal{F}_g(X^A)(\mathcal{W}_{AB}\mathcal{W}^{AB})^g,$$

where $\theta$ is the superspace chiral coordinate, $g$ is the worldsheet genus, $X^A(x, \theta) = X^A + \theta \Psi^A + \theta^2 F^A + \ldots$ are the $\mathcal{N} = 2$ chiral superfields associated to vector multiplets, $\mathcal{W}_{AB}$ with spinor indices $A,B$ is a superfield with the anti-selfdual graviphoton $W^-$ as its bottom component. A crucial property of this interaction is that $I^g$ only exists in genus $g$ order precisely and in the string frame proportional to $g_s^{2g-2}$. Gopakumar and Vafa proposed that this interaction $\mathcal{I}$ should be interpreted as the four dimensional superspace effective action in a supersymmetric background with a constant anti-selfdual graviphoton. Moreover, instead of calculating the holomorphic function $\mathcal{F}_g$ and then $\mathcal{I}_g$ order by order, as what did in topological string theory, we should directly calculate $\mathcal{I}$ as a whole. This idea is achieved by lifting a Type IIA superstring compactified on $\Sigma \subset \mathbb{R}^4 \times Y$ to an M2-brane wrapped on $\Sigma \times S^1 \subset \mathbb{R}^4 \times S^1 \times Y$. As follows, they give a remarkable (GV) formula

$$\mathcal{I} = -\int \frac{d^4x \, \theta}{(2\pi)^4} \sqrt{g^E} e^{2\pi i \sum_{i} q_i \, z_i} \left[ \frac{\text{4} g_s^2 W^2}{\sinh^2 \left( \frac{1}{2} g_s \mathcal{W} \right)} \right] \frac{\pi^2}{16} \frac{1}{g_s^2},$$

where $g^E$ is the Einstein metric. The compactification factor $e^{2\pi i \sum_{i} q_i \, z_i}$ determined by the mass and constant gauge field is irrelevant to our discussion. Without loss of generality, we take the degree $k = 1$. We used the conventions in [3], and replaced the $\chi_0$ (imaginary) there by the string coupling $g_s$. As explicitly showed in [5], this action includes the sum of contributions from all string worldsheets of genus $g \geq 0$, completely controlled by the expansion:

$$\frac{\text{4} g_s^2 W^2}{\sinh^2 \left( \frac{1}{2} g_s \mathcal{W} \right)} = 1 - \frac{g_s^2}{12} W^2 + \frac{g_s^4}{240} W^4 + O(W^6).$$

The contribution from world-sheets of genus $g$ is identified as that proportional to $W^{2g}$, which can be understood from (1.1). Therefore, if there exists an exact relation between the $\alpha'$ expansion and genus expansion, this term must also serve as a closed sum of the $\alpha'$ expansion and completely control the expansion. Indeed as we are going to show, the $\alpha'$ expansion in AdS does give

$$g^{ij} \left( \tilde{X} \right) = \left[ \frac{1}{\sinh^2 \left( \frac{1}{2} \pi \tilde{R} \right)} \right] \eta^{ij},$$

with definitions of the quantities in (3.20). Moreover, it turns out that our result makes it possible to make conjectures on the expression of the strong $g_s > 1$ theory.
2 T-duality in AdS background

The ordinary T-duality demands background compactifications and says that the physics are identical for compactified radii \( R \) and \( \alpha'/R \). But in AdS geometry, we are going to show that T-duality exhibits a different manner. We start with the usual AdS metric in Poincare coordinates

\[
\begin{align*}
\text{(2.5)} \\
\end{align*}
\]

where the constant \( R_{\text{AdS}} \) is the radius. For convenience and clarity, we use double coordinates \((X^i, \tilde{X}^j)\) to represent the T-dual fields. We take the convention that the worldsheet coordinate \((\tau, \sigma)\) are dimensionless. As introduced in [6], the T-duality transformations are

\[
\begin{align*}
\text{(2.6)} \quad g_{ij} \partial_\sigma X^j &= \partial_\tau \tilde{X}^i, \\
g^{ij} \partial_\sigma \tilde{X}^j &= \partial_\tau X^i, \\
\end{align*}
\]

combined with

\[
\tilde{Z} \equiv \frac{R^2_{\text{AdS}}}{Z}. \quad \text{(2.7)}
\]

Then the dual geometry of AdS is still an AdS with the same radius

\[
\begin{align*}
\text{(2.8)} \\
\end{align*}
\]

where the metric are defined by \( g_{\mu\nu} = \text{diag}\left\{ \frac{R^2_{\text{AdS}}}{Z^2}, \frac{R^2_{\text{AdS}}}{Z^2} \eta_{ij} \right\} \) and the dual metric \( g^{\mu\nu} = \text{diag}\left\{ \frac{R^2_{\text{AdS}}}{Z^2}, \frac{R^2_{\text{AdS}}}{Z^2} \eta^{ij} \right\} \).

The definitions are completely consistent as one can easily check by using \( \left( \frac{R^2_{\text{AdS}}}{Z^2} \right)^{-1} = \frac{R^2_{\text{AdS}}}{Z^2} \). Let us denote the characteristic length of \( X \) as \( \ell_s \equiv \sqrt{\alpha'} \) and write \( X^i = \ell_s X^i \), where \( X^i \) is dimensionless. Similarly, we set \( Z = \ell_s Z \). Therefore, from the T-duality (2.6), we can get

\[
\tilde{X}^j = \int d\sigma \frac{R^2_{\text{AdS}}}{Z^2} \partial_\tau X^i = \frac{R^2_{\text{AdS}}}{\ell_s} \int d\sigma Z^{-2} \partial_\tau X. \quad \text{(2.9)}
\]

Since \( \sigma, \tau, Z \) and \( X \) are all dimensionless, it is natural to identify the characteristic length of \( \tilde{X}^j \) as \( \ell_{\tilde{s}} = \frac{R^2_{\text{AdS}}}{\ell_s} \).

This identification is consistent for \( \tilde{X}_0 \) and \( \tilde{Z} \) too. Therefore, the T-duality in AdS geometry takes the form:

\[
\begin{align*}
(Z, X^0, X^i) : \quad & \ell_s, \\
(\tilde{Z}, \tilde{X}_0, \tilde{X}^j) : \quad & \ell_{\tilde{s}}, \\
\end{align*}
\]

under

\[
\ell \ell_{\tilde{s}} = R^2_{\text{AdS}}. \quad \text{(2.10)}
\]

We can see that this T-duality relation is quite different from the ordinary one. We do not perform any compactifications and there is only one radius but with two string lengths. An interesting feature is that as
we know, AdS geometry is non-compact. There might be some mathematical implications from this T-duality. Moreover, the dual relation (2.11) indicates a strong-weak correspondence, which is very useful when we \( \alpha' \) expand the string theory.

3 \( \alpha' \) expansion of strings propagating in AdS

In conformal gauge, the action describing a string moving in a curved background, namely the non-linear Sigma model, is

\[
S = -\frac{1}{4\pi\alpha'} \int g_{ij}(X) \partial_{\alpha} X^i \partial^{\alpha} X^j
\]

(3.12)

3.1 Case \( \ell_s/R_{AdS} = \sqrt{\alpha'}/R_{AdS} < 1 \)

In this regime, we can expand \( X \) but not the dual \( \tilde{X} \). Consider expanding the field at some point \( \bar{x} \),

\[
X^i(\tau, \sigma) = \bar{x}^i + \ell_s Y^i(\tau, \sigma),
\]

(3.13)

where \( Y^i \)'s are dimensionless fluctuations. Note the \( \alpha' \) in front of the integral in the action is canceled under this expansion. Locally around any point, we can always pick Riemann normal coordinates such that the metric expansion is greatly simplified

\[
g_{ij}(X) = \eta_{ij} + \frac{\ell_s^2}{3} R_{ijkl} Y^k Y^l + \frac{\ell_s^4}{6} D_k R_{ilmj} Y^k Y^l Y^m + \ldots
\]

(3.14)

We now set the background as a \( D \)-dimensional AdS spacetime, which is a maximally symmetric space with

\[
D_m R_{ijkl} = 0 \quad \text{and} \quad R_{ijkl} = -\frac{1}{R_{AdS}} (g_{ij} g_{kl} - g_{ik} g_{jl}).
\]

It is remarkable that, with some careful calculation, the expansion can be summed into a closed form,

\[
g_{ij}(X) = \eta_{ij} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{n+2}}{(2n+2)!} \left( \frac{\lambda}{8\pi} \right)^{2n} \eta_{mn} (R^2)^m a_1 (R^2)^{a_2} \cdots (R^2)^{a_{n-1}} j
\]

(3.15)

where we defined

\[
\lambda \equiv \frac{\ell_s^4}{R_{AdS}^4}, \quad (R^2)^a, \quad (R^2)^a b \equiv 64\pi^2 \left( \frac{R_{AdS}}{\ell_s} \right)^6 R_{AdS}^2 R^a_{ij} Y^i Y^j = 64\pi^2 \left( \frac{R_{AdS}}{\ell_s} \right)^6 (\delta_{ab} Y^2 - Y^a Y^b).
\]

(3.16)

Note that this \( \alpha' \) expansion for \( X \) is valid only for \( \ell_s/R_{AdS} < 1 \). The indices are raised and lowered by \( \eta_{ab} \). Clearly it is not legal to expand the dual field \( \tilde{X} \), since the coupling \( \sqrt{\alpha'}/R_{AdS} = \hat{\ell}/R_{AdS} = R_{AdS}/\ell_s > 1 \).
However, the T-duality of the metric $g^{ij}(\tilde{X}) = (g_{ij}(X))^{-1}$ enables us to get a closed expression for the strongly coupled dual theory, though in terms of $\lambda = \ell_s^4/R_{AdS}^4$ and variables $Y_i$.

$$g^{ij}(\tilde{X}) = (g_{ij}(X))^{-1} = \left[ \frac{\frac{1}{4} \left( \frac{s}{r} \right)^2 \mathcal{R}^2}{\sinh^2 \left( \frac{1}{2} \frac{\lambda}{4\pi} \mathcal{R} \right)} \right]^{i} a \eta^{aj}. \tag{3.17}$$

Apparently, it looks similar to the genus expansion (1.3). The only problem is that it is not expressed by the native dual coupling $\tilde{\lambda} = \ell_s^4/R_{AdS}^4$ and variables $\tilde{Y}_i$. Actually, we can substitute $\lambda = 1/\tilde{\lambda}$. To replace $Y$ by $\tilde{Y}$, we can make use of the T-duality transformation (2.6). We can expect after these substitutions, the form will change and may not be exactly the same as the genus expansion (1.3). But we do get a closed expression for a strongly coupled theory. To have a clear comparison of the two expansions, let us consider another regime.

### 3.2 Case $\ell_s/R_{AdS} = \sqrt{\alpha'}/R_{AdS} > 1$

This is the regime $\tilde{\ell}/R_{AdS} < 1$, so we can expand the dual string $\tilde{X}$ by the dual length $\tilde{\ell}_s$

$$\tilde{X}_i(\tau, \sigma) = \tilde{x}_i + \tilde{\ell}_s \tilde{Y}_i(\tau, \sigma), \tag{3.18}$$

It is worthwhile to emphasize that the native dual metric to expand is $g^{ij}(\tilde{X})$. Follow the same pattern, the metric of the T-dual theory can also be put into a closed form

$$g^{ij}(\tilde{X}) = \left[ \frac{\frac{1}{4} \left( \frac{s}{r} \right)^2 \mathcal{R}^2}{\sinh^2 \left( \frac{1}{2} \frac{\lambda}{4\pi} \mathcal{R} \right)} \right]^{i} a \eta^{aj}. \tag{3.19}$$

where we set

$$\tilde{\lambda} \equiv \frac{\ell_s^4}{R_{AdS}^4} = \lambda^{-1}, \quad \left( \mathcal{R}^2 \right)^{a} b \equiv 64\pi^2 \left( \frac{R_{AdS}}{\ell_s} \right)^{6} R_{AdS}^2 R^{a} i j b \tilde{Y}^i \tilde{Y}^j = 64\pi^2 \left( \frac{R_{AdS}}{\ell_s} \right)^{6} \left( \delta_{a}^{b} \tilde{Y}^2 - \tilde{Y}^a \tilde{Y}^b \right). \tag{3.20}$$

It is really nice to see that this $\alpha'$ expansion matches the genus expansion (1.3) perfectly under the identifications

$$4\pi g_s = \tilde{\lambda} = \lambda^{-1} = \left( \frac{R_{AdS}}{\sqrt{\alpha'}} \right)^{4}, \quad \mathcal{W} = \mathcal{R}, \tag{3.21}$$

which is in perfect agreement with AdS/CFT! Again, the strongly coupled $X$ theory can be obtained by the T-duality

$$g_{ij}(X) = \left( g^{ij}(\tilde{X}) \right)^{-1} = \left[ \frac{\sinh^2 \left( \frac{1}{2} \frac{\lambda}{4\pi} \mathcal{R} \right)}{\frac{1}{4} \left( \frac{s}{r} \right)^2 \mathcal{R}^2} \right]^{i} a \eta^{aj}. \tag{3.22}$$

Moreover, our derivation also gives a simple way to manage the strong $g_s$ coupling theories. With the identifications (3.21), the $g_s < 1$ theory is represented by (1.3) or (3.19), while the strong $g_s > 1$ theory is described by (3.17), after substituting $\lambda = 1/\tilde{\lambda}$ and $\tilde{Y} = Y(\tilde{Y})$ from the T-duality (2.6).
Or a even better choice is to consider the X theory. Still with the identifications (3.21), from the analysis above, the \( g_s < 1 \) theory must be (3.22), though it looks quite different from the weak \( g_s < 1 \) expression (1.3) since it is expressed by \( \tilde{\lambda} = 1/\lambda \) and \( \tilde{\mathcal{Y}} \). The good news is that the strong \( g_s > 1 \) theory is just (3.15), already expressed in its native arguments. One can immediately conclude that it is controlled by \( \frac{1}{\sqrt{g_s}} \). Therefore, as one may already conceive, we really should expand the \( g_s > 1 \) theory by \( \frac{1}{\sqrt{g_s}} \), in contrast to expanding \( g_s < 1 \) theory by \( g_s \).

### 4 Some discussions

There are several remarks we want to address

#### Strong genus expansion and supersymmetry

From our derivations, the perturbative genus expansion (1.3) for \( g_s \ll 1 \) is in agreement with the dual \( \alpha' \) expansion (3.19) for \( \sqrt{\alpha'}/R_{AdS} \gg 1 \). It is wonderful that this is in perfect agreement with the prediction of AdS/CFT. Moreover, we can predict the closed form of the strong genus expansion for \( g_s \gg 1 \)! We can see in order to have the correspondence between the two distinct expansions, supersymmetry is a must in the genus expansion (1.3). The GV in non-supersymmetric background [3, 4] takes the form

\[
\mathcal{F} = \frac{1}{4\pi} \int_{\Sigma} \partial Y^i \partial Y^j \left[ \sin^2 \left( \frac{1}{4} \left( \frac{\alpha'}{\sqrt{\alpha'}} \frac{R}{L} \right) \right) \right] a_i \eta_{aj}, \quad \lambda \equiv \left( \frac{\sqrt{\alpha'}}{R_{dS}} \right)^4. \quad (4.23)
\]

which still maintains some similarity with the \( \alpha' \) expansion but we certainly need extra labor to arrive the non-supersymmetric gauge theory. Another interesting problem is to extend the \( \alpha' \) expansion to supersymmetric case and more accurate field identifications may be derived.

#### Other geometries

With the same pattern, it is straightforward to get the closed form of local \( \alpha' \) expansion for other non-trivial maximally symmetric spaces. For De-Sitter space, one only needs to replace the sinh functions by sine functions,

\[
S_{dS} = \frac{1}{4\pi} \int_{\Sigma} \partial Y^i \partial Y^j \left[ \frac{\sin^2 \left( \frac{1}{4} \left( \frac{\alpha'}{\sqrt{\alpha'}} \frac{R}{L} \right) \right)}{\frac{1}{4} \left( \frac{\alpha'}{\sqrt{\alpha'}} \frac{R}{L} \right)^2 \tilde{\mathcal{R}}^2} \right] a_i \eta_{aj}, \quad \tilde{\lambda} \equiv \left( \frac{\sqrt{\alpha'}}{R_{dS}} \right)^4. \quad (4.24)
\]

For sphere \( S_n \) and hyperbola \( H_n \), we simply replace the minkovski metric \( \eta_{ij} \) by the Euclidean metric \( \delta_{ij} \). However, since there is no such a nice T-duality as (2.6), it is tricky to talk about the dual theory. This might be the reason why we have AdS/CFT only but not others. Nevertheless, we do have a closed expression for the \( \sqrt{\alpha'}/R_c \) expansion, it is then natural to conjecture that for other maximally symmetric spaces with \( R_c \neq 0 \), the strongly coupled \( \sqrt{\alpha'}/R_c \gg 1 \) worldsheet CFT still takes the form

\[
S_{dS} = -\frac{1}{4\pi \alpha'} \int_{\Sigma} \partial X^i \partial X_j \left[ \frac{\sin^2 \left( \frac{1}{4} \left( \frac{\alpha'}{\sqrt{\alpha'}} \frac{R}{L} \right) \right)}{\frac{1}{4} \left( \frac{\alpha'}{\sqrt{\alpha'}} \frac{R}{L} \right)^2 \tilde{\mathcal{R}}^2} \right] a_i \eta_{aj}, \quad \tilde{\lambda} \equiv \left( \frac{R_c}{\sqrt{\alpha'}} \right)^4 = \frac{1}{\lambda}. \quad (4.25)
\]

\(^1\)To get \( 1/\sqrt{g_s} \), we used an identification \( \mathcal{Y} \sim \mathcal{W} \). It is possible to have other negative powers of \( g_s \) depending on the specific identifications. But the power of \( g_s \) should be negative.
Note the $\alpha'$ in front of the integral, $X$ but not $\bar{X}$ outside the bracket and the barred quantities in the bracket. Though it may not be possible to relate the parameters and fields in two regimes, this closed expression for strongly coupled system may have some applications. Moreover, even for asymmetric spaces, we may be able to solve the strongly coupled systems by perturbation on the corresponding maximally symmetric spaces.

**Ooguri-Vafa formula and open string**

We addressed the GV formula and closed string in this paper. In Ooguri-Vafa formula, which is an open string extension of GV, an extra D4-brane in $\mathbb{R}^2 \times L$, $L \subset Y$ as a special Lagrangian manifold, is introduced and the supersymmetry is broken to leave at most four supercharges. It would be of great important to generalize our discussion in this paper to that situation since one can expect more information about the non-abelian gauge theory can be extracted.

**Moduli calculation**

Determining the moduli space of higher genus surfaces is a very hard problem and no systematical method is available. The equivalence between the $\alpha'$ expansion and genus expansion provides a possible way to attack this problem. It is quite interesting that the closed expression of the $\alpha'$ expansion is obtained locally, while the genus expansion is not restricted to local region. It would be of interest to test it by calculating $n$-point functions on non-trivial surfaces, say, annulus or torus.

**Hodge integrals and topological string theory**

It is obvious to see that our results are related to the generating function for Hodge integrals:

$$F(t, k) \equiv 1 + \sum_{g \geq 1} t^{2g} \sum_{i=0}^{g} k^i \int \bar{M}_{g, 1} \psi^{2g-2+i} \lambda_{g-i} = \left( \frac{t/2}{\sin(t/2)} \right)^{k+1},$$

where $t$ and $k$ are some parameters, $\bar{M}_{g, n}$ is the moduli space with genus $g$ and $n$ distinct marked points, $\psi^i$ is the first Chern class for a marked $i$ cotangent line bundle, and $\lambda_j$ is $j$th Chern class of Hodge bundle. This Hodge integrals play a central role in the Gromov-Witten theory and topological string theory. It is readily to see that $F(t, 1)$ is T-dual to $F(t, -3)$ on the basis of our results.

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A Appendix

A Closed Formula of Metric Expansion for Maximally Symmetric Spaces in RNC

In this appendix, we will calculate the metric expansion in Riemann normal coordinates (RNC). This normal coordinate system on a Riemannian manifold centered at $\bar{x}$ is defined by

$$g_{ij}(\bar{x}) = \eta_{ij}, \quad \Gamma^k_{ij}(\bar{x}) = 0. \quad (A.27)$$

Consider a point $X$ in a neighborhood of $\bar{x}$, whose coordinates are $\bar{x} + y$ in the RNC. The geodesic connecting $\bar{x}$ with $X$ is then given by $\gamma^i(s) = \bar{x}^i + y^i s$ with $\gamma(0) = \bar{x}$ and $\gamma(1) = X$. By substituting it into the geodesic equation, this implies that

$$\Gamma^k_{ij}(X) y^i y^j = 0. \quad (A.28)$$

Taylor expanding LHS of eqn. $(A.28)$ in terms of $y$ at $\bar{x}$ leads to

$$\Gamma^k_{i(j,i_1 \ldots i_n)}(\bar{x}) = 0, \text{ for } n \geq 0. \quad (A.29)$$

Since the metric $g_{ij}$ can be expressed with respect to the vielbein $e^a_i$:

$$g_{ij}(X) = e^a_i(X) e^b_j(X) \eta_{ab},$$

we could first find an expression for $e^a_i(X)$ in terms of the curvature. To do so, we can choose synchronous gauge for the vielbein:

$$i_y e^a_i(X) = \delta^a_i y^i, \quad (A.30)$$

where the radial vector $y$ is $y = y^i \partial_i$, and $i_y$ is the interior product. In RNC, the gauge condition $(A.30)$ implies a condition for the vielbein connection:

$$i_y \omega^{ab}(X) = 0. \quad (A.31)$$
In fact, differentiating both sides of eqn. (A.30) gives

\[ y^i y^j \partial_i e^a_j (X) = 0, \]

where we use \( e^a_j (0) = \delta^a_j \). Since the connection \( \Gamma^i_{jk} \) is

\[ \Gamma^i_{jk} (X) = e^i_a (X) e^a_{k,j} (X) + e^i_a (X) \omega^a_j (X) e^a_k (X), \]

eqn. (A.28) leads to

\[ \delta^b_j y^i (y^j \omega^a_i) = 0 \Rightarrow y^i \omega^{ab} = y^i \omega^{ab} = 0. \]

By Taylor expanding \( i_y \omega^{ab} (X) = 0 \) at \( \bar{x} \), we find

\[ \omega^{ab} (i, i_1, \ldots, i_n) (0) = 0. \]

Using \( L_y = i_y d + di_y \), one finds

\[ L_y e^a = d (i_y e^a) + (i_y e^b) \omega^a_b, \]

where we use \( de^a + \omega^a_b \wedge e^b = 0 \) since the torsion is absent. The gauge condition (A.30) gives

\[ L_y di_y e^a = d (i_y e^a), \]

and hence subtracting eqn. (A.30) from eqn. (A.35) leads to

\[ (L_y - 1) L_y e^a = (i_y e^b) L_y \omega^a_b. \]

Then using the gauge condition \( i_y \omega^{ab} = 0 \) and

\[ R^a_b = \frac{1}{2} R^a_{bij} dx^i \wedge dx^j = d \omega^a_b + \omega^a_c \wedge \omega^c_b, \]

we have

\[ (y \cdot \partial) (y \cdot \partial + 1) e^a_i (y) = y^j y^k R^a_j_{kb} (y) e^b_i (y), \]

where \( e^a_i (y) = e^a_i (X) \). Noting that

\[ \frac{d^n}{dv^n} e^a_i (vy) = y^{i_1} \cdots y^{i_n} \partial_{i_1} \cdots \partial_{i_n} e^a_i (vy), \]

we can rewrite eqn. (A.39) as a differential equation in terms of \( v \)

\[ \frac{d}{dv} \left[ v^2 \frac{d}{dv} e^a_i (vy) \right] = v^2 y^j y^k R^a_j_{kb} (vy) e^b_i (vy), \]

which can be integrated to

\[ e^a_i (y) = \delta^a_i + \int_0^1 (1 - v) y^j y^k R^a_j_{kb} (vy) e^b_i (vy) dv. \]
It is noteworthy that the solution (A.42) indeed satisfies the gauge condition (A.30) since \( R^{a}_{jki} y^k y^i = 0 \).

Introducing the abbreviation

\[ R^a_b (z, y) = R^a_{ijb} (z) y^i y^j, \]

we can express \( e^a_i \) in terms of \( R^a_b \) by iterating eqn. (A.42)

\[ e^a_i (y) = \delta^a_i + \sum_{n=1}^{\infty} \int_0^1 \! dv_1 (1 - v_1) \int_0^1 \! dv_2 (1 - v_2) \cdots \int_0^1 \! dv_n (1 - v_n) \]

\[ \left( v_1^{2n-1} \cdots v_n^{1} \right) R^a_{i_1} (v_1 y, y) R^a_{l_2} (v_1 v_2 y, y) \cdots R^a_{l_{n-1}} (v_1 \cdots v_n y, y) \delta^b_i. \] (A.44)

On the other hand, Taylor expanding the curvature tensor in \( R^a_b (ty, y) \) gives

\[ R^a_b (ty, y) = \sum_{p=0}^{\infty} \frac{t^p (y^i \partial_{i'})^p}{p!} R^a_b (z, y) |_{z=0}, \]

\[ = \sum_{p=0}^{\infty} \frac{t^p (y^i \nabla_i')^p}{p!} R^a_b (z, y) |_{z=0}, \] (A.45)

where we use eqns. (A.29) and (A.34) to replace \( \partial \) with \( \nabla \) in the second line.

We now focus on maximally symmetric spaces, where \( \nabla_i R^m_{ijl} = 0 \). Using eqn. (A.45), we find that

\[ R^a_b (ty, y) = R^a_b (0, y) \]

in maximally symmetric spaces, and hence

\[ e^a_i (y) = \sum_{n=0}^{\infty} \frac{[R^a (0, y)]^a_b \delta^b_i}{(2n + 1)!}. \] (A.47)

Then, the result for the metric \( g_{ij} = e^a_i e^b_j \delta_{ab} \) becomes

\[ g_{ij} (X) = \eta_{ij} + \frac{1}{2} \sum_{n=1}^{2n+2} \eta_{im} R (0, y)^m_{a_1} R (0, y)^{a_1}_{a_2} \cdots R (0, y)^{a_{n-1}}_{a_n} (2n + 2)!. \] (A.48)

For maximally symmetric spaces, the Riemann curvature tensor is

\[ R_{ikjl} = L^{-2} (g_{ij} g_{kl} - g_{il} g_{kj}). \] (A.49)

If \( X = \bar{x} + \ell_A \mathbb{Y} \) where \( \mathbb{Y} \)'s are dimensionless, we find

\[ R^a_b (0, \ell_A \mathbb{Y}) = (-1)^{a - \frac{\text{sgn}(L^2)+1}{2}} \frac{\lambda^2}{4 \pi^2} (\mathcal{R}^2)^a_b, \]

where we define

\[ \lambda = \frac{\ell_A^4}{L^4}, \quad (\mathcal{R}^2)^a_b = 64 \pi^2 \left( \frac{|L|}{\ell_A} \right)^6 (\delta^a_b \mathbb{Y}^2 - \mathbb{Y}^a \mathbb{Y}^b). \] (A.51)

Summing up the series (A.48), we find

\[ g_{ij} (X) = \left[ \frac{\sinh^2 \left( \frac{\text{sgn}(L^2)+1}{2} \frac{\lambda^2}{4 \pi^2} \mathcal{R} \right)}{(-1)^{\frac{\text{sgn}(L^2)+1}{2}}} \right]^a_{\eta_{ij}}. \] (A.52)

For example, one has \( L^2 = -R^2_{AdS} \) in a D-dimensional AdS spacetime, and eqn. (A.52) becomes eqn. (3.16).