THE IGUSA-TODOROV $\phi$ FUNCTION FOR TRUNCATED PATH ALGEBRAS

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Abstract. For a finite dimensional algebra $A$ with $0 < \phi \dim(A) = m < \infty$ we prove that there always exist modules $M$ and $N$ such that $\phi(M) = m - 1$ and $\phi(N) = 1$. On the other hand, we see that not every value between 1 and $m - 1$ will be reached by $\phi$. Also we prove that for $B$ a truncated path algebra $\phi \dim B = \phi \dim B^{\text{op}}$. And compute, when $Q$ has no sources nor sinks, the $\phi$-dimension of $B$ in function of the $\phi$-dimension of the radical square zero algebra with the same associated quiver.

1. Introduction

One of the most important conjectures in the Representation Theory of Artin Algebras is the finitistic conjecture. It states that $\sup\{\text{pd}(M) : M$ is a finitely generated module of finite projective dimension$\}$ is finite. In an attempt to prove the conjecture Igusa and Todorov defined in [IT] two functions from the objects of $\text{mod}A$ (the category of right finitely generated modules over an Artin algebra $A$) to the natural numbers, which generalizes the notion of projective dimension. Nowadays they are known as the Igusa-Todorov functions, $\phi$ and $\psi$. One of its nicest features is that they are finite for each module, and allow us to define the $\phi$-dimension and the $\psi$-dimension of an algebra. These are new homological measures in the module category. In particular it holds that:

$$\text{findim}(A) \leq \phi \dim(A) \leq \psi \dim(A) \leq \text{gldim}(A),$$

and they all agree in the case of algebras with finite global dimension.

Recently, various works were dedicated to study and generalize the properties of these functions. See for instance [HL], [LMM], [LM], in particular in [LMM] the authors compute $\phi$ and $\psi$ for radical square zero algebras.

This article is organized as follows: the introduction and the preliminary section are devoted to fixing the notation and recalling the basic facts needed in this work. In section 3 we prove that some values are admissible for the $\phi$ function for any algebra. Also we define the partial $\phi$-dimensions. In section 4 we concentrate on the case of truncated path algebras and radical square zero algebras. In 4.1 we generalize some results made in [LMM] for radical square zero algebras to truncated path algebras. Finally, in 4.2 we prove some results in the case of radical square zero algebras.

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2. PRELIMINARIES

Let $A$ be a finite dimensional basic algebra defined over a field $k$. The category of finite dimensional right $A$-modules will be denoted by $\text{mod}A$, the indecomposable modules of $A$ by $\text{ind}A$, and the set of isoclasses of simple $A$-modules by $\mathcal{S}(A)$. If $S \in \mathcal{S}(A)$, $P(S)$ denotes the indecomposable projective associated to $S$. For a $A$-module $M$ we denote by $\text{Soc}(M)$ its socle and by $\text{Top}(M)$ its top.

Given an $A$-module $M$ we denote its projective dimension by $\text{pd}(M)$ and the $n^{th}$-syzygy by $\Omega^n(M)$. We recall that the global dimension of $A$, which we denote by $\text{gldim}(A)$, is the supremum of the set of projective dimensions of $A$-modules. The global dimension can be a natural number or infinity. The finitistic dimension of $A$, denoted by $\text{findim}(A)$, is the supremum of the set of projective dimensions of $A$-modules with finite projective dimension.

If $Q$ is a finite connected quiver, $kQ$ denotes its path algebra. Given a path in $kQ$, $l(\rho)$, $s(\rho)$ and $t(\rho)$ denote the length, start and target of $\rho$ respectively. $C^n$ denotes the oriented cycle graph with $n$ vertices.

The following theorem will be useful.

**Theorem 2.1** ([N]). Let $A$ be a finite dimensional basic algebra. The following statements are equivalent:

1. $A$ is self-injective
2. The rule $S \rightarrow \text{Soc}(P(S))$ defines a permutation $\nu : \mathcal{S}(A) \rightarrow \mathcal{S}(A)$.

2.1. *Igusa-Todorov* $\phi$ function. We recall the definition of the Igusa-Todorov $\phi$ function and some basic properties. We also define the $\phi$-dimension of an algebra.

**Definition 2.2.** Let $K_0(A)$ be the abelian group generated by all symbols $[M]$, where $M$ is a f.g. $A$-module, modulo the relations:

1. $[M] - [M'] - [M'']$ if $M \cong M' \oplus M''$.
2. $[P]$ for each projective.

Let $\overline{\Omega} : K_0(A) \rightarrow K_0(A)$ be the group endomorphism induced by $\Omega$, and let $K_i(A) = \overline{\Omega}(K_{i-1}(A)) = \ldots = \overline{\Omega}^i(K_0(A))$. Now, if $M$ is a finitely generated $A$-module then $\langle \text{add}M \rangle$ denotes the subgroup of $K_0(A)$ generated by the classes of indecomposable summands of $M$.

**Definition 2.3.** The (right) *Igusa-Todorov function* $\phi$ of $M \in \text{mod}A$ is defined as

$$\phi_A(M) = \min \left\{ l : \overline{\Omega}^s_A|_{\overline{\Omega}^s_A(\text{add}M)} \text{ is a monomorphism for all } s \in \mathbb{N} \right\}.$$  

In case that there is no possible misinterpretation we denote by $\phi$ the Igusa-Todorov function $\phi_A$.

**Proposition 2.4** ([IT], [HLM]). Given $M, N \in \text{mod}A$.

1. $\phi(M) = \text{pd}(M)$ if $\text{pd}(M) = \infty$.
2. $\phi(M) = 0$ if $M \in \text{ind}A$ and $\text{pd}(M) = \infty$.
3. $\phi(M) \leq \phi(M \oplus N)$.
4. $\phi(M^k) = \phi(M)$ for $k \in \mathbb{N}$.
5. If $M \in \text{mod}A$, then $\phi(M) \leq \phi(\Omega(M)) + 1$.

**Proof.** For the statements 1-4 see [IT], and for 5 see [HLM]. □
Definition 2.5. The \( \phi \)-dimension of an algebra \( A \) is defined as follows

\[
\phi \text{dim}(A) = \sup \{ \phi(M) : M \in \text{mod}A \}.
\]

Theorem 2.6 (III). If \( A \) is an Artin algebra then \( A \) is self-injective if and only if \( \phi \text{dim}(A) = 0 \).

2.2. Radical square zero algebras. Given a quiver \( Q \) we denote by \( J \) the ideal generated by the arrows in \( kQ \). By a radical square zero algebra we mean an algebra which is isomorphic to an algebra of the type \( A = \frac{kQ}{I} \). If \( S(A) = \{ S_1, \ldots, S_n \} \) denotes a complete set of simple \( A \)-modules up to isomorphism, then we divide the set \( S(A) \) in the following three distinct sets:
- \( S_I \) the set of injective modules in \( S \).
- \( S_P \) the set of projective modules in \( S \).
- \( S_D = S \setminus (S_I \cup S_P) \).

Remark 2.7. For radical square zero algebras it holds that \( \Omega(S_i) = \bigoplus_{\alpha: i \to j} S_j \), i.e.:

\[
\Omega(S_i) \text{ is a direct sum of simple modules, and the number of summands isomorphic to } S_j \text{ coincides with the number of arrows starting in the vertex } i \text{ and ending at the vertex } j.
\]

Given a radical square zero algebra \( A \) with \( n \) vertices and finite global dimension, it is easy to compute its global dimension using the last fact. Explicitly we have

\[
\text{gldim}(A) = \sup \{ l(\rho) : s(\rho) \text{ is a source, and } t(\rho) \text{ is a sink} \},
\]

concluding that \( \text{gldim}(A) \) must be less or equal to \( n - 1 \).

Remark 2.8. For a radical square zero algebra \( A = \frac{kQ}{I} \), \( \Omega(M) \) is a semisimple \( A \)-module for every \( A \)-module \( M \), and \( K_1(A) \) has basis \( \{ [S]_{S \in S_D} \} \). In particular if \( Q \) has no sink nor source, then \( K_1(A) \) has basis \( \{ [S]_{S \in S} \} \).

Definition 2.9. Let \( A = \frac{kQ}{I} \) be a radical square zero algebra, where \( Q \) is finite with \( n \) vertices without sources nor sinks. We define \( T : Q^n \to Q^n \) as the linear transformation given by \( T(e_i) = \sum_{j=1}^n |\{ \alpha : i \to j \}| e_j \).

Remark 2.10. Given a radical square zero algebra \( A \) with \( Q \) as the previous definition, the matrix of \( T \) in the canonical basis and the matrix of \( \Omega|_{K_1(A)} \) in the basis \( \{ [S]_{S \in S} \} \) agree.

3. Admissible values for \( \phi \) and partial \( \phi \)-dimensions

3.1. Admissible values. Given an algebra \( A \) we say that a value \( t \in \mathbb{N} \), with \( t \leq \phi \text{dim}(A) \), is admissible if there exists an \( A \)-module \( M \) such that \( \phi(M) = t \). We prove that for any algebra \( A \) the value 1 is always admissible. And if \( A \) is of finite \( \phi \)-dimension \( m \) then the value \( m - 1 \) is admissible.

Lemma 3.1. If \( [M] \in K_1(A) \) then for \( M' \subseteq M \), \( [M'] \in K_1(A) \).

Proof. Let \( M \) be \( A \)-module such that \( [M] \in K_1(A) \), then there exists \( N \) such that \( \Omega_A(N) = M \), this means that we have a short exact sequence:

\[
0 \to M \to P \to N \to 0
\]

where \( P \) is a projective module. Let \( M' \) be a submodule of \( M \), given that the composition \( M' \hookrightarrow M \twoheadrightarrow P \) is a monomorphism, we can consider the next short exact sequence:
Remark 3.2. Given an algebra $A$, if there exist $M \in \text{mod} A$ such that $\text{pd}(M) = m \in \mathbb{N}^+$ then $\phi(\Omega^{m-i}(M)) = i$, for $1 \leq i < m$. In particular, if $A$ has finite global dimension then $\phi(M) = \text{pd}(M)$ for all $M \in \text{mod} A$. Therefore, for all $0 \leq i \leq \text{gldim}(A)$ there exists $M_i \in \text{mod}(A)$ such that $\phi(M_i) = i$.

Theorem 3.3. Let $A$ be a finite dimensional algebra. If $\phi(\text{dim}(A)) > 0$ then there exist $M \in \text{mod} A$ such that $\phi(M) = 1$. In the case that $\phi(\text{dim}(A))$ is finite, then there exist $N \in \text{mod} A$ such that $\phi(N) = \phi(\text{dim}(A)) - 1$.

Proof. Suppose that $A$ is an infinite global dimensional algebra, in the other case the proof is trivial by 3.2.

Recall the Nakayama rule $\nu$ from Theorem 2.1.

- Assume that $\text{pd}(M) = \infty$ or $0$. By Theorem 2.6 $A$ is not a self-injective algebra, then applying Theorem 2.1 we get that $\nu$ is not a permutation. In this situation there are two possible cases:

  (1) $\nu$ is a function but is not injective. In this case there exist simple modules $S_1$ and $S_2$ such that $\nu(S_1) = \nu(S_2)$.

  Let $M_1$ and $M_2$ be the indecomposable modules ($\text{Top}(M_i)$ is simple) given by:

  $$
  0 \rightarrow \nu(S_1) \rightarrow P_1 \rightarrow M_1 \rightarrow 0
  $$

  $$
  0 \rightarrow \nu(S_2) \rightarrow P_2 \rightarrow M_2 \rightarrow 0
  $$

  therefore $\phi(S_1 \oplus S_2) = 1$.

  (2) Now, if $\nu$ is not a function, then there is a simple module $S$ such that $\text{Soc}(P(S))$ is not a simple module.

  If $S_1 \oplus S_2$ is a direct summand of $\text{Soc}(P(S))$ with $S_1$ and $S_2$ non-isomorphic simple modules, then there exist two indecomposable projective modules $P_1$ and $P_2$ such that $S \subseteq \text{Soc}(P_1) \cap \text{Soc}(P_2)$. Therefore we can construct a pair of non-isomorphic modules, $M_1$ and $M_2$, such that $\phi(M_1 \oplus M_2) = 1$, similarly to the previous case.

  Otherwise $\text{Soc}(P(S)) = S^k$ with $S'$ a simple module and $k \geq 2$. Let $M_1$ and $M_2$ be the following indecomposable modules given by:

  $$
  0 \rightarrow S' \rightarrow P_1 \rightarrow M_1 \rightarrow 0
  $$

  $$
  0 \rightarrow S'^2 \rightarrow P_2 \rightarrow M_2 \rightarrow 0
  $$

  It is clear that $M_1$ and $M_2$ are indecomposable non-isomorphic modules and $\phi(M_1 \oplus M_2) = 1$. 


Let \( m = \phi \dim(A) \) and \( N \) be an \( A \)-module such that \( \phi(N) = m > 0 \), then we will prove \( \phi(\Omega(N)) \geq m - 1 \).

Suppose that \( \phi(\Omega(N)) > m - 1 \). Because \( \phi \dim(A) = m \) we have that \( \phi(\Omega(N)) = m \). Consider the decomposition into indecomposable modules \( \oplus_i M_i = \Omega(N) \). Using Lemma 3.1 we obtain that \( M_i \in K_1 \) and for each \( i + 1, \ldots, s \) there exists an indecomposable module \( N_i \in \mod A \) such that \( \Omega(N_i) = M_i \). In this case the \( A \)-module \( N' = \oplus_i N_i \) would have \( \phi(N') = m + 1 \) which is absurd.

In Example 3.16 we show a finite dimensional algebra \( A \) with \( \phi \dim(A) = m > 3 \), such that there exists \( 1 < t < m - 1 \) that is not admissible.

### 3.2. Partial \( \phi \)-dimensions

We define partial \( \phi \)-dimension for a natural \( l \) and restate, in Proposition 3.6, the main result of [HL] in our context. We also give examples in which we compute the partial \( \phi \)-dimensions.

**Definition 3.4.** Given \( l \in \mathbb{Z}^+ \) we define the \( l \) \( \phi \)-dimension of an algebra \( A \) as:

\[
\phi \dim_l(A) = \sup \{ \phi(M_1 \oplus \ldots \oplus M_l) : M_i \in \text{ind} A \forall i = 1, \ldots, l \}.
\]

**Remark 3.5.** Given an algebra \( A \) we have:

1. \( \text{findim}(A) = \phi \dim_1(A) \).
2. For all \( l \in \mathbb{Z}^+ \) \( \phi \dim_l(A) \leq \phi \dim_{l+1}(A) \leq \phi \dim(A) \).
3. If \( \phi \dim(A) \) is finite, then there exists \( l \in \mathbb{Z}^+ \) such that: \( \phi \dim_l(A) = \phi \dim(A) \).

**Proposition 3.6.** Let \( A \) be a finite dimensional algebra. The following statements are equivalent:

1. \( A \) is a self-injective algebra.
2. \( \phi \dim(A) = 0 \).
3. There exists \( l \in \mathbb{Z}^+ \) such that \( \phi \dim_l(A) = 0 \).
4. \( \phi \dim_2(A) = 0 \).

**Proof.** By Corollary 6 of [HL] the first two statements are equivalent. By Remark 3.5 part (2) we have that the second statement implies the third one, and also that the third one implies the fourth one.

To see that the fourth statement implies the second one, by the proof of Theorem 3.3 if \( \phi \dim(A) > 0 \) then there exist two indecomposable modules \( M_1, M_2 \) such that \( \phi(M_1 \oplus M_2) = 1 \). □

**Example 3.7.** Let \( \Gamma^m \) be the following quiver with \( |(\Gamma^m)_0| = n \) and without sinks nor sources.
then $\phi \dim_2 \left( \frac{H_{m+1}}{H_m} \right) = \phi \dim \left( \frac{H_m}{H_{m-1}} \right) = m - 1$ for $0 < m \leq n$.

For a proof of the following result see Theorem 4.7 of [LM].

**Example 3.8.** If $A$ is an $m$-Gorenstein algebra, then:

$$\phi \dim_1(A) = \phi \dim_2(A) = \cdots = \phi \dim(A) \leq m.$$

The following example shows that the inequalities in Remark 3.5 could be strict.

**Example 3.9.** Consider the radical square zero algebra $A$, whose quiver $Q$ is the following:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow m$$

Then $K_1(A)$ has basis $\{[S_1], [S_2], [S_3], [S_4]\}$, where $S_i$ is the simple module associated to the vertex $i$. The values of $\Omega$ in the simple modules are the following:

- $\Omega(S_1) = S_1 \oplus S_2$.
- $\Omega(S_2) = S_3$.
- $\Omega(S_3) = S_4$.
- $\Omega(S_4) = S_3 \oplus S_4$.

From this we see that $rk(K_1(A)) = 4 > 3 = rk(K_2(A)) = rk(K_3(A))$, which implies that $\phi \dim(A) = 2$. If we assume that $M = M_1 \oplus M_2$ is the decomposition of a module $M$ such that $\phi(M) = \phi \dim A$, we see that $\{[\Omega(M_1)], [\Omega(M_2)]\}$ must be linearly independent and $\{[\Omega^2(M_1)], [\Omega^2(M_2)]\}$ must be colinear. Since $[S_2] + [S_3] - [S_4]$ is the basis of the kernel of $\Omega$ we get, without loss of generality, that:

- $S_2 \oplus S_3$ is a direct summand of $\Omega(M_1)$.
- $S_4$ is direct summand of $\Omega(M_2)$.

Since $P_1$ is the only projective indecomposable module, up to isomorphism, which has $S_2$ as a submodule and, on the other side, $S_3$ is a submodule of the projective modules $P_2$ and $P_4$, analysing the quiver $Q$, we can infer that $M_1$ cannot be indecomposable.

On the other hand, it is easy to see that $\phi(S_2 \oplus S_3 \oplus S_4) = 1$. Because $S_2, S_3, S_4$ belongs to $K_1$, there exist indecomposable modules $M_1, M_2, M_3$, such that $\Omega(M_i) = S_i$, thus $\phi(M_1 \oplus M_2 \oplus M_3) = 2$.

Finally we conclude that $1 = \phi \dim_2 \left( \frac{\omega_1}{\omega_2} \right) < \phi \dim_3 \left( \frac{\omega_1}{\omega_2} \right) = 2$. 
3.3. **Radical square zero algebras.** We study the partial $\phi$-dimensions for radical square zero algebras. The following two results can be found in [LMM]. We finalize this section giving an example of an algebra $A$ such that there exists values that are not admissible.

**Proposition 3.10.** If $A = \frac{Q_0}{J}$ is a radical square zero algebra with $|Q_0| = n$, then $\phi\dim(A) = \phi\dim_n(A)$.

**Proof.** See Proposition 4.14 of [LMM]. □

**Proposition 3.11.** If $A = \frac{Q_0}{J}$ is a radical square zero algebra with $|Q_0| = n$, and $\phi\dim(A) = n$ then $\phi\dim_2(A) = n$.

**Proof.** See Corollary 5.7 of [LMM]. □

Now, we give a generalization of Proposition 3.11.

**Proposition 3.12.** Let $A = \frac{Q_0}{J}$ be a radical square zero algebra with $|Q_0| = n$ and without sinks nor sources. If $\phi\dim(A) = k \leq n$ then $\phi\dim_{n-k+2}(A) = k$.

**Proof.** We can assume that $k \geq 3$ because the other cases were considered in Proposition 3.10. Since $\phi\dim(A) = k$ and $\phi\dim(A) = \phi(\oplus_{S \in S} S) + 1$, then:

$$rkT^{k-2} > rkT^{k-1} = m_0 \leq n - (k - 1).$$

This implies that there exists a set $\{T^{k-1}(e_i), T^{k-1}(e_{i_2}), \ldots, T^{k-1}(e_{i_m})\}$ that is linearly dependent, and $\{T^{k-2}(e_i), T^{k-2}(e_{i_2}), \ldots, T^{k-2}(e_{i_m})\}$ that is linearly independent. Thus $\phi(S_{i_1} \oplus S_{i_2} \oplus \ldots \oplus S_{i_m}) \geq k - 1$. Finally, if we consider $M_j \in \text{mod}A$ such that $\Omega(M_j) = S_j$, then $\phi(\oplus_{j=1}^{m+1} M_j) = k$. □

**Proposition 3.13.** Let $A = \frac{Q_0}{J}$ be a radical square zero algebra such that $\phi\dim(A) \geq 2$, then $\phi(\oplus_{i=1}^{l} M_i) \geq k \geq 2$ if and only if there exist $v \in \text{Ker}T^{l-1} \backslash \text{Ker}T^{l-2}$ such that $v \in \langle \{\Omega([M_i])\}_{i=1}^{l} \rangle$ and $l \geq k$.

**Proof.** If there exist $v \in \mathbb{Q}^n$ such that $v \in (\text{Ker}T^{k-1} \backslash \text{Ker}T^{k-2}) \cap \langle \{\Omega([M_i])\}_{i=1}^{l} \rangle$, this implies that:

$$rk\Omega^k(\langle [M_i] \rangle_{i=1}^{l}) = rkT^{k-1}(\langle \Omega([M_i]) \rangle_{i=1}^{l}) < rkT^{k-2}(\langle \Omega([M_i]) \rangle_{i=1}^{l}) = rk\Omega^{k-1}(\langle [M_i] \rangle_{i=1}^{l}),$$

because

- $\sum_{i=1}^{l} \alpha_i T^{k-1}(\Omega[M_i]) = 0$ and
- $\sum_{i=1}^{l} \alpha_i T^{k-2}(\Omega[M_i]) \neq 0$,

where $v = \sum_{i=1}^{l} \alpha_i (\Omega[M_i])$. Therefore $\phi(\oplus M_i) \geq k$.

Now suppose that $\phi(M) = \langle [\Omega_M] \rangle_{i=1}^{l} \geq 2$, then there exist $u = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Q}^n$ such that:

- $\sum_{i=1}^{l} \alpha_i (\Omega_i[M_i]) = \sum_{i=1}^{l} \alpha_i T^{l-1}(\Omega_i[M_i]) = 0$ and
- $\sum_{i=1}^{l} \alpha_i (\Omega_i^{-1}[M_i]) = \sum_{i=1}^{l} \alpha_i T^{l-2}(\Omega_i[M_i]) \neq 0$,

then $v = \sum_{i=1}^{l} \alpha_i (\Omega_i[M_i]) \in \text{Ker}T^{l-1} \backslash \text{Ker}T^{l-2}$. □

**Corollary 3.14.** Let $A = \frac{Q_0}{J}$ be a radical square zero algebra such that $\phi\dim(A) \geq 2$. If $\phi(\oplus_{i=1}^{l} M_i) \geq 2$, then

$$\phi(\oplus_{i=1}^{l} M_i) = \sup \{k : \text{there exist } v \in (\text{Ker}T^{k-1} \backslash \text{Ker}T^{k-2}) \cap \langle \{\Omega([M_i])\}_{i=1}^{l} \rangle\}$$
The example below shows us that for all \( l \geq 2 \) there exist radical square zero algebras \( A \) such that:

\[
\phi \dim_l(A) < \phi \dim_{l+1}(A) = \phi \dim(A).
\]

**Example 3.15.** Let \( Q \) be the following quiver:

\[
\begin{array}{ccccccc}
(0,1) & \rightarrow & (0,2) & \rightarrow & \ldots & \rightarrow & (0,l) & \rightarrow & (0,l+1) \\
(1,1) & \rightarrow & (1,2) & \rightarrow & \ldots & \rightarrow & (1,l) & \rightarrow & (1,l+1) \\
(2,1) & \rightarrow & (2,2) & \rightarrow & \ldots & \rightarrow & (2,l) & \rightarrow & (2,l+1) \\
\vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots \\
(k,1) & \rightarrow & (k,2) & \rightarrow & \ldots & \rightarrow & (k,l) & \rightarrow & (k,l+1) \\
(k+1,1) & \rightarrow & (k+1,2) & \rightarrow & \ldots & \rightarrow & (k+1,l) & \rightarrow & \ldots \\
(0,1) & \rightarrow & (0,2) & \rightarrow & \ldots & \rightarrow & (0,l) & \rightarrow & (0,l+1) \\
\end{array}
\]

If \( A = \frac{kQ}{J} \), then \( \phi \dim_s(A) < \phi \dim_{s+1}(A) = \phi \dim(A) \) for \( s = 1, \ldots, l \).

Let \( V_i \) be the following subspaces of \( \mathbb{Q}^n \):

- \( V_i \) has basis \( B_i = \{ e_{(i,j)} : j \in \{0, \ldots, l+1\} \} \) for all \( i \leq k \)
- \( V_{k+1} \) has basis \( B_{k+1} = \{ e_{(k+1,j)} : j \in \{0, \ldots, l\} \} \).

Then \( T(V_i) = V_{i+1} \) for all \( i \leq k \) and \( T(V_{k+1}) = V_0 \). Consider the linear transformation \( S = T^{k+1}_{|V_{k+1}} : V_{k+1} \to V_{k+1} \), then its associated matrix to the basis \( B_{k+1} \) is

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \vdots & & \\
\vdots & 0 & 1 & 0 & \vdots & \\
\vdots & \vdots & 0 & \ddots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 2
\end{pmatrix}
\]

Hence \( S^{k+2} \) is invertible and we deduce that \( \phi \left( \bigoplus_{j=1}^{l} S_{(k+1,j)} \right) = 0 \).

Now \( T^{k+1-i}(V_i) \subset V_{k+1} \), and by Proposition 2.4 we have \( \phi \left( \bigoplus_{j=1}^{l} S_{(i,j)} \right) \leq k+1-i \) for all \( i < k + 1 \). In fact the previous inequality is an equality, it follows from the existence of the vector
\[ w_i = \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} e_{(i,2j-1)} - \sum_{j=1}^{\lfloor \frac{l-1}{2} \rfloor} e_{(i,2j)} \in \text{Ker}(T^{k+1-i}) \]

and because \( \text{Ker}(T^{k+1-i}) = \langle w_i \rangle \oplus \text{Ker}(T^{k-i}) \).

On the other hand, the quiver \( Q \) has no source nor sink, so we conclude that \( \phi \text{dim}(A) = k + 2 \).

Let \( M = \oplus_{j \in J} M_j \) be an \( A \)-module where each \( M_j \) is indecomposable for \( j \in J \) and \( \phi(M) = k + 2 \). By Proposition 3.13 we have a vector \( \alpha w_0 + v \in \langle \Omega[M_j]_{j \in J} \rangle \) with \( \alpha \neq 0 \) and \( v \in \text{Ker}T^k \).

For \( 1 \leq s \leq l - 1 \) there is an unique indecomposable \( A \)-module \( N_s \) such that \( S_{(0,s)} \subset \Omega(N_s) \). On the other hand, \( \Omega(S_{(k+1,l)}) = S_{(0,l)} \oplus S_{(0,l+1)} \), and there exist indecomposable modules \( N_1, N_{l+1} \) such that \( \Omega(N_1) = S_{(0,l)} \), \( \Omega(N_{l+1}) = S_{(0,l+1)} \), and for any other indecomposable module \( N \), \( \Omega(N) \cap (S_{(0,l)} \oplus S_{(0,l+1)}) = \{0\} \). Finally, by the last facts \( M \) must have at least \( l + 1 \) non-isomorphic direct summands.

The following example shows that if a finite dimensional algebra \( A \) has \( \phi \text{dim}(A) = m > 3 \), then might exists \( 1 < h < m - 1 \) such that there is no \( A \)-module \( M \) with \( \phi(M) = h \). In this case we say that \( A \) has a gap.

Example 3.16. Let \( A = \frac{kQ}{\mathcal{J}} \) where \( Q \) is the following quiver:

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {0};
    \node (2) at (2,2) {2};
    \node (3) at (2,-2) {3};
    \node (4) at (4,0) {4};
    \node (5) at (4,-2) {5};
    \node (6) at (6,2) {6};
    \node (7) at (6,-2) {7};
    \node (8) at (8,0) {8};
    \node (9) at (8,-2) {9};
    \node (10) at (10,0) {10};
    \node (11) at (10,-2) {11};
    \node (12) at (12,2) {12};
    \node (13) at (12,-2) {13};
    \node (14) at (14,0) {14};

    \draw[->] (1) -- (2);
    \draw[->] (2) -- (3);
    \draw[->] (3) -- (4);
    \draw[->] (4) -- (5);
    \draw[->] (5) -- (6);
    \draw[->] (6) -- (7);
    \draw[->] (7) -- (8);
    \draw[->] (8) -- (9);
    \draw[->] (9) -- (10);
    \draw[->] (10) -- (11);
    \draw[->] (11) -- (12);
    \draw[->] (12) -- (13);
    \draw[->] (13) -- (14);

    \draw[->] (1) -- (13);
    \draw[->] (2) -- (14);
    \draw[->] (3) -- (12);
    \draw[->] (4) -- (11);
    \draw[->] (5) -- (10);
    \draw[->] (6) -- (9);
    \draw[->] (7) -- (8);
    \draw[->] (8) -- (7);
    \draw[->] (9) -- (8);
    \draw[->] (10) -- (9);
    \draw[->] (11) -- (10);
    \draw[->] (12) -- (11);
    \draw[->] (13) -- (12);
    \draw[->] (14) -- (13);
\end{tikzpicture}
\end{center}
Using Theorem 4.32 from [LMM] and the following facts, one can conclude that \( \phi \dim(A) = 8 \):

\[
\begin{align*}
\ker T &= \langle w_7 \rangle, \\
\ker T^2 &= \langle w_6 - w_2 \rangle + \ker T, \\
\ker T^3 &= \langle w_5 - w_1 \rangle + \ker T^2, \\
\ker T^4 &= \langle w_4 \rangle + \ker T^3, \\
\ker T^5 &= \langle w_3 \rangle + \ker T^4, \\
\ker T^6 &= \langle w_2 \rangle + \ker T^5, \\
\ker T^7 &= \langle w_1 \rangle + \ker T^6,
\end{align*}
\]

where \( w_i = e_{2i-1} - e_{2i} \) for \( i = 1, \ldots, 7 \) and \( v_i = e_{2i} + e_{2i-1} \) for \( i = 1, \ldots, 7 \), but there is no \( A \)-module \( M \) such that \( \phi(M) = 3 \).

Suppose that \( M \) is an \( A \)-module such that \( \phi(M) = 3 \), then there exist \( v \in \ker T^2 \backslash \ker T \cap \text{add} M \). This means that \( M = M_1 \oplus M_2 \) and \( v = \alpha \Omega([M_1]) - \beta \Omega([M_2]) \) with \( \alpha, \beta \in \mathbb{Q}^+ \). Because \( v \in \ker T^2 \backslash \ker T \) then \( \Omega([M_1]) = e_{11} + e_4 + [S] \) and \( \Omega([M_2]) = e_{12} + e_3 + [S'] \).

On the other hand, \( \Omega(N) = S_1 \oplus S \) implies that \( S_{-2} \) is a direct summand of \( N \) for \( i = 3, 4, 11, 12 \). Therefore \( S_1 \oplus S_2 \oplus S_9 \oplus S_{10} \) is a direct summand of \( M \) and this is absurd, because \( e_1 - e_2 \in \ker T^7 \backslash \ker T^6 \cap \text{add} M \).

4. Truncated path algebras

An Artin algebra \( A \) is called monomial if \( A \cong \frac{kQ}{\langle L \rangle} \), where \( I \) is generated by paths of \( Q \). Truncated path algebras are a special case of monomial algebras. An Artin algebra \( A \) is a truncated path algebra if there exists a quiver \( Q \) and \( k \in \mathbb{N} \) such that \( A \cong \frac{kQ}{\langle L \rangle} \).

**Remark 4.1.** If \( A \) is a truncated path algebra and \( \rho, \nu \) paths of \( kQ \), then:

- \( \rho A = \nu A \) if \( l(\rho) = l(\nu) \) and \( t(\rho) = t(\nu) \).
- If \( Q \) has no sinks nor sources then \( \rho A = \nu A \) if and only if \( l(\rho) = l(\nu) \) and \( t(\rho) = t(\nu) \).

We denote by \( M_i^l(A) \) the ideal \( \rho A \), where \( l(\rho) = l, t(\rho) = v \) and \( M^l(A) = \bigoplus_{v \in Q_0} M_i^l(A) \). Given \( 0 < l \leq k - 1 \), it is easy to see that \( M_i^l(A) \) exists if and only if \( \text{id}_{A} S_v \geq l \). Also, \( M_i^l(A) \) is a projective \( A \)-module if and only if \( \text{pd}_{A} S_v \leq k - l - 1 \). We also denote by \( V_i(A) \) the subgroup of \( K_0(A) \) with basis \( \{ [M_i^l(A)] : v \in Q_0, [M_i^l(A)] \neq 0 \} \).

**Remark 4.2.** If \( A = \frac{kQ}{\langle L \rangle} \) is a truncated path algebra then:

\[
\begin{align*}
\Omega(M_i^l(A)) &= \bigoplus_{\rho: s(\rho) = \nu} M_{w-i}^k(A), \\
\Omega^2(M_i^l(A)) &= \bigoplus_{\rho: s(\rho) = \nu} M_i^l(A).
\end{align*}
\]

For a proof of the following result see Theorem 5 of [DH-ZL].
Proposition 4.3. [DH-ZL]

Let \( A \cong \frac{kQ}{J} \) be a truncated path algebra. Then the following statements hold.

1. \( \text{gldim} A < \infty \) if and only if \( Q \) has no oriented cycles.
2. If \( A \) has finite global dimension, then
   \[
   \text{gldim} A = \begin{cases} 
   2 \left\lfloor \frac{l}{k} \right\rfloor + 1 & \text{if } l \equiv 0 \pmod{k} \\
   2 \left\lfloor \frac{l}{k} \right\rfloor + 1 & \text{otherwise},
   \end{cases}
   \]
   where \( l \) is the length of the largest path of \( kQ \).

For a proof of the next theorem see Theorem 5.11 of [BH-ZR], and for definitions of skeleton and \( \sigma \)-critical see [DH-ZL].

Theorem 4.4. [BH-ZR]

Let \( A \) be a truncated path algebra. If \( M \) is any nonzero left \( A \)-module with skeleton \( \sigma \), then:

\[
\Omega(M) \cong \bigoplus_{\rho \text{ is } \sigma\text{-critical}} \rho A
\]

The next result follows from the above theorem.

Proposition 4.5. If \( A = \frac{kQ}{J} \) is a truncated path algebra, then the following set \( \{[\rho A] : l(\rho) \geq 1, \ [\rho A] \neq 0 \} \) is a basis for \( K_1(A) \).

Proof. As a consequence of Theorem 4.4 \( K_1(A) \subset \{[\rho A] \}_{l(\rho) \geq 1} \). On the other hand, given \( \rho \) a path of length bigger or equal to 1, if \( \rho A \) is non-projective then \( \rho A \) is a submodule of \( s(\rho)A \). Thus \( \rho A = \Omega \left( \frac{s(\rho)A}{\rho A} \right) \).

□

For a non self-injective truncated path algebra we can compute its \( \phi \)-dimension as follows.

Corollary 4.6. If \( A = \frac{kQ}{J} \) is a non self-injective truncated path algebra, then

\[
\phi \text{dim}(A) = 1 + \phi \left( \bigoplus_{l(\rho) \geq 1} \rho A \right).
\]

Now if \( A \) is a self-injective truncated path algebra, we have the following result that generalize Proposition 4.13 of [LMM].

Proposition 4.7. For a connected non-simple truncated path algebra \( A = \frac{kQ}{J} \) the following statements are equivalent:

1. \( M_v^l \) exists, is not a projective module and \( \Omega(M_v^l) \) is indecomposable for all \( v \in Q_0 \) and \( 1 \leq l \leq k - 1 \).
2. \( Q \) is a cycle, i.e., \( Q = C^n \).
3. \( A \) is a Nakayama algebra without injective simple modules.
4. \( A \) is a Nakayama algebra without projective simple modules.
5. \( A \) is a selfinjective algebra.
6. All indecomposable projective modules have length \( k \).
7. All indecomposable injective modules have length \( k \).
8. The \( \phi \)-dimension of \( A \) is zero.

Proof. We start by showing that the first statement implies the second one. Since we are assuming that \( M_v^l \) exist for all \( v \in Q_0 \) then \( v \) is not a source for all \( v \in Q_0 \).

In an analogous way, since \( M_v^l \) is not projective for all \( v \in Q_0 \) then \( v \) is not a sink for all \( v \in Q_0 \). On the other hand, for every vertex \( v \) there is only one arrow with start
Remark 4.11. For a truncated path algebra $V$ where $\Omega$ is metric, i.e., $\phi = \exp$, the same is true for truncated path algebras.

We now show that the third statement implies the second one. This follows from the fact that the quiver of a Nakayama algebra is either a linearly ordered $A_n$ or a cycle $C_\infty$. If there is no simple injective it must be a cycle.

Equivalence between two and four is similar to the equivalence between two and three.

Finally we show that the fifth statement implies the second one. The hypothesis implies that there is exactly one arrow starting and one arrow ending at any vertex. Therefore $Q$ is a cycle $C_n$.

The sixth and seventh statements easily implies the second one.

And finally, the eighth and fifth statements are equivalent following Corollary 6 from [H].

Notation 4.8. Let $A = \frac{\mathbb{k}Q}{\mathbb{I}}$ be a truncated path algebra with quiver $Q$. If $Q_0 = \{1, \ldots, n\}$ and $1 \leq l \leq k - 1$, we denote by $I_l$ the following set of vertices of $Q$:

$$I_l = \{i \in Q_0 : \text{id}_{Q_0} S_i \leq l - 1 \text{ and } \text{pd}_{Q_0} S_i \leq k - l - 1\}$$

Notation 4.9. Let $\mathcal{M}$ be a square matrix in $M_n(\mathbb{k})$ and $I, J \subseteq \{1, \ldots, n\}$. We denote by $\mathcal{M}_{I,J}$ the matrix in $M_{n-|I|\times n-|J|}(\mathbb{k})$ obtained from $\mathcal{M}$ removing the rows from $I$ and the columns from $J$.

Remark 4.10. If $A = \frac{\mathbb{k}Q}{\mathbb{I}}$ is a truncated path algebra, then the associated matrix of $\Omega|_{V_1 \oplus V_2 \oplus \ldots \oplus V_{k-1}}$ is

$$
\begin{pmatrix}
0 & \ldots & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ldots & \ldots & \ldots \\
(\mathcal{M}^1)_{I_{k-1},I_k} & \cdots & (\mathcal{M}^{k-2})_{I_2,I_{k-2}} & 0 \\
(\mathcal{M}^{k-1})_{I_1,I_{k-1}} & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

where $V_i = \left\{\left(M_s^i(A)\right)_{v \in Q_0 \setminus I_i}\right\}$ and $\mathcal{M}$ is the adjacency matrix of the quiver $Q$.

Remark 4.11. For a truncated path algebra $A = \frac{\mathbb{k}Q}{\mathbb{I}}$ we have the following equalities:

$$\phi(\bigoplus_{l(\rho) \geq 1} \rho A) = \max_{1 \leq l \leq m^{-1}} \{\phi(M_s^l(A))\} = \max_{1 \leq l \leq m^{-1}} \{\phi(M_s^l(A) \oplus M^{k-l}(A))\}.$$

Remark 4.12. If $N = M^s$ then $\text{rkM}^s = \text{rkN}^s$ for all $s \in \mathbb{N}$.

For a radical square zero algebras $A$ it is known that the $\phi$-dimension is symmetric, i.e., $\phi\dim(A) = \phi\dim(A^{op})$, see [LMN]. The following result shows that the same is true for truncated path algebras.

Theorem 4.13. Let $A = \frac{\mathbb{k}Q}{\mathbb{I}}$ and $A^{op} = \frac{\mathbb{k}Q^{op}}{\mathbb{I}}$ be truncated path algebras, then:

$$\phi\dim(A) = \phi\dim(A^{op}).$$
Remark 4.15. \( \Omega \) is a quiver with \( |Q_0| = n \). If \( A = \frac{kQ}{\phi} \) is a truncated path algebra, then there exist a truncated path algebra \( B = \frac{kQ}{\phi} \) such that \( \phi \dim(A) \leq \phi \dim(B) \), where \( \bar{Q} \) is a quiver with \( |\bar{Q}| = n \), such that \( \bar{Q} \) is a subquiver of \( Q \) and it has neither sources nor sinks.

Proof. Suppose that the quiver \( Q \) has at least one source. Let \( Q^1 \) be the quiver formed by adding one loop in each vertex of \( Q \) which is a source. If \( B^1 = \frac{kQ^1}{\phi} \), then \( M^1(A) \oplus M^{k-1}(A) \) is a direct summand of \( M^1(B^1) \oplus M^{k-1}(B^1) \) as a \( B^1 \)-module and \( \Omega_{A|V_{1}(A)\oplus V_{k-1}(A)} = \Omega_{B|V_{1}(A)\oplus V_{k-1}(A)} \). Therefore \( \phi \dim(A) \oplus M^{k-1}(A) = \phi \dim(B^1) \oplus M^{k-1}(B^1) \), and finally \( \phi \dim(A) \leq \phi \dim(B^1) \).

If \( Q^1 \) has no sinks then \( B = B^1 \). Complementary, if \( Q^1 \) has no sources but at least has one sink, then \( (Q^1)^{\text{op}} \) has no sinks but has at least one source and \( \phi \dim(B^1) = \phi \dim((B^1)^{\text{op}}) \) by Theorem 4.13 and the result follows by the previous case.

4.1. Truncated path algebras of quivers without sources nor sinks. We now relate the \( \phi \)-dimensions of a truncated path algebra and a radical square zero algebra associated to the same quiver, when the quiver has no sinks or sources.

In this case the projective and injective dimensions of every simple modules is infinite. Then \( I_l \) is empty for every \( l \). Therefore the following remarks.

Remark 4.15. If \( Q \) is a quiver without sources nor sinks and \( A = \frac{kQ}{\phi} \) is a truncated path algebra, then

\[
\begin{pmatrix}
0 & \ldots & \ldots & 0 & M^{-1}_Q \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & M^2_Q & \ddots & \ddots & \ddots \\
M_1^Q & 0 & \ldots & \ldots & 0
\end{pmatrix}
\]

where \( M_Q \) is the adjacency matrix of the quiver \( Q \).

Remark 4.16. If \( Q \) is a quiver without sources nor sinks and \( A = \frac{kQ}{\phi} \) is a truncated path algebra, then
The results from the previous section allows us to give bounds on the \( \phi \) \( \phi \)-dimension of arbitrary truncated path algebras. We also characterize the quivers associated to the algebras with \( \phi \)-dimension 1.

The next result follows from Theorem 4.14 and Theorem 4.17.

**Corollary 4.20.** If \( A = \frac{kQ}{ J^k} \) is a truncated path algebra with \( |Q_0| = n \), then:

\[ \phi \text{dim}(A) \leq f_k(n). \]

Also there exists some algebra \( A \) such that the bound is reached.

**Remark 4.21.** Let \( A = \frac{kQ}{ J^k} \) be a truncated path algebra with \( |Q_0| = n \).

- If \( Q \) has neither sources nor sinks and \( \phi \text{dim}\frac{kQ}{ J^k} = 1 \), then \( \phi \text{dim}A = 1. \)
- If \( k \geq n \), \( Q \) has neither sources nor sinks, and \( \phi \text{dim}\frac{kQ}{ J^k} = 2 \), then:

\[ \phi \text{dim}A = 2. \]

**Theorem 4.17.** Let \( A = \frac{kQ}{ J^k} \) be a non self-injective truncated path algebra. If \( Q \) has no sources nor sinks, then:

- If \( \phi \text{dim}(\frac{kQ}{ J^k}) - 1 \equiv 0 \pmod{k} \) then \( \phi \text{dim}(A) = 2 \left( \frac{\phi \text{dim}(\frac{kQ}{ J^k}) - 1}{k} \right) + 1. \)
- If \( \phi \text{dim}(\frac{kQ}{ J^k}) - 1 \equiv 1 \pmod{k} \) then \( \phi \text{dim}(A) = 2 \left( \frac{\phi \text{dim}(\frac{kQ}{ J^k}) - 2}{k} \right) + 2. \)
- If \( \phi \text{dim}(\frac{kQ}{ J^k}) - 1 \not\equiv 0, 1 \pmod{k} \) then \( \phi \text{dim}(A) = 2 \left( \frac{\phi \text{dim}(\frac{kQ}{ J^k}) - 2}{k} \right) + 1. \)

**Proof.** It follows from Remark 4.16 and Corollary 4.16. \( \square \)

The above theorem induces the next definition.

**Definition 4.18.** Let \( k \) be a natural such that \( k \geq 2 \). We define \( f_k : \mathbb{N} \rightarrow \mathbb{N} \) as

\[
f_k(m) = \begin{cases} 
0 & \text{if } m = 0 \\
2 \left( \frac{m - 1}{k} \right) + 1 & \text{if } m \equiv 1 \pmod{k} \\
2 \left( \frac{m - 2}{k} \right) + 2 & \text{if } m \equiv 2 \pmod{k} \\
2 \left( \frac{m - 2}{k} \right) + 1 & \text{otherwise}
\end{cases}
\]

**Remark 4.19.** Note that \( f_k \) is an increasing function.

4.2. \( \phi \)-dimensions for truncated path algebras. The results from the previous section allows us to give bounds on the \( \phi \)-dimension of arbitrary truncated path algebras. We also characterize the quivers associated to the algebras with \( \phi \)-dimension 1.

\[
\bar{\Omega}^{2s+1} |_{V_1 \oplus V_2 \oplus \ldots \oplus V_k = 1} = \begin{pmatrix}
M^{sk}_k & 0 & \ldots & 0 \\
0 & M^{sk}_k & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & M^{sk}_k & 0 \\
0 & \ldots & 0 & M^{sk}_k
\end{pmatrix}
\]

where \( M_Q \) is the adjacency matrix of the quiver \( Q \).

\[
\bar{\Phi}^{2s+1} |_{V_1 \oplus V_2 \oplus \ldots \oplus V_k = 1} = \begin{pmatrix}
0 & \ldots & 0 & M^{sk+k+1}_Q \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & M^{sk+2}_Q & \ddots & 0 \\
M^{sk+1}_Q & \ldots & 0 & 0
\end{pmatrix}
\]
• If \( k \geq n - 1 \geq 2 \), then \( \phi \dim(A) \leq 3 \).

The following example shows that there exists truncated path algebras with maximal \( \phi \)-dimension such that its associated quiver has neither sources nor sinks, unlike the case of radical square zero algebras. See Corollary 4.40 of [LMM].

**Example 4.22.** Let \( \Gamma \) be a quiver with \( \Gamma_0 = \{v_1, \ldots, v_m\} \) where \( m \geq 3 \) and \( \phi \dim_{\Gamma} = m \). Consider the quiver \( Q \) with

1. set of vertices \( Q_0 = \Gamma_0 \cup \{w_1, \ldots, w_{n-m+1}\} \)
2. set of arrows \( Q_1 = \Gamma_1 \cup \{\alpha_m, \ldots, \alpha_n\} \)

where \( s(\alpha_i) = w_{i+1}, \tau(\alpha_i) = v_i \) and \( \Gamma_0 \cap \{w_1, \ldots, w_{n-m+1}\} = \{w_{i_0}\} \) with \( 1 < i_0 < n - m + 1 \). Then \( Q \) has source \( w_1 \) and sink \( w_{n-m+1} \) and \( A = \frac{kQ}{J} \) has \( \phi \dim(A) = 3 \).

By Remark 4.21 we see that the \( \phi \)-dimension of \( A \) is maximal.

**Proposition 4.23.** Given \( n \in \mathbb{N} \) and \( k \geq 2 \) then there exist a family of truncated algebras \( \{A_l \mid l \in \{0, 1, \ldots, f_k(n)\}\} \) such that \( \phi \dim(A_l) = l \).

**Proof.** For \( l = 0 \) consider \( A_0 = \frac{kC^n}{J} \). And for \( l = f_k(n) \) consider \( A_l = \frac{kQ}{J} \), where \( Q \) verifies \( \phi \dim \left( \frac{kQ}{J} \right) = n \).

For the case \( k = 2 \) and the rest of the possible values of \( l \) see Example 3.7 or Proposition 4.60 of [LMM].

Now, let \( k \) be greater or equal to 3. Recall the quiver \( \Gamma^m \) from Example 3.7

1. For \( l = 1 \), consider \( A_1 = \frac{kC^3}{J} \).
2. For \( l \) even, consider \( A_l = \frac{kC^{l'}}{J} \), where \( l' = \frac{k(l-2)}{2} + 2 \).
3. For \( l \) odd, \( l > 1 \), consider \( A_l = \frac{kC^{l'}}{J} \), where \( l' = \frac{k(l-3)}{2} + 3 \).

Finally we give a characterization of truncated path algebras with \( \phi \)-dimension equal to 1.

**Theorem 4.24.** Let \( A = \frac{kQ}{J} \) be a truncated algebra, then the following statements are equivalent:

1. \( \phi \dim(A) = 1 \)
2. \( Q \neq C^n \) and
   - \( J^k = 0 \), or
   - \( \det(2\mathcal{M}_Q) \neq 0 \), where \( Q \) is the quiver obtained from \( Q \) by deleting its sources and sinks.

**Proof.** An easy computation proves that the second statement implies the first one. Assume now that the first statement is true, then there are two possible cases.

1. \( Q \) has neither sinks nor sources.

Then, by Theorem 4.17 \( \phi \dim(A) = 1 \) if and only if \( \phi \dim \left( \frac{kQ}{J} \right) = 1 \) and only if \( \det(2\mathcal{M}_Q) \neq 0 \) and \( Q \neq C^n \).

2. \( Q \) has sinks or sources.

If every path in \( Q \) has length less than \( k \), then \( J^k = 0 \) hence \( \phi \dim(A) = \text{gldim}(A) = 1 \), because \( A \) is a hereditary algebra. Suppose there is a path with length bigger or equal to \( k \). This is equivalent to the existence of a non-projective module \( M^l_0 \). Consider \( Q \) the full subquiver of \( Q \) generated by \( \{v \in Q_0 \mid [M^l_0] \neq 0 \text{ for some } l \in \{1, \ldots, k - 1\}\} \). Consider the
truncated algebra $B = \frac{\tilde{Q}}{\bar{Q}}$. Since $\tilde{\Omega}|_{K_1(A)} = \tilde{\Omega}|_{K_1(B)}$ then $\tilde{\Omega}|_{K_1(B)}$ is injective and it follows that $\tilde{Q}$ has no sinks nor sources and $\det(M_{Q}) \neq 0$.

We show now that the quiver $\tilde{Q}$ is obtained from $Q$ by deleting every source and sink. Consider $v_1, v_2, v_3 \in Q_0$ and $\rho, \gamma \in Q_1$ such that $\rho : v_1 \to v_2$, $\gamma : v_2 \to v_3$.

If $v_3 \in \tilde{Q}_0$ ($v_1 \in \tilde{Q}_0$) then $v_2 \in \tilde{Q}_0$, because the module $M_{v_2}^{k-1}$ is not projective ($M_{v_2}^k$ is not projective).

□

References

[ARS] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics 36 Cambridge University Press (1997).

[BH-ZR] E. Babson, B. Huisgen-Zimmermann, and R. Thomas, Generic representation theory of quivers with relations, J. Algebra 322 (6), pp. 1877-1918 (2009).

[DH-ZL] A. Dugas, B. Huisgen-Zimmermann and J. Learned, Truncated path algebras are homologically transparent. Part I, Models, Modules and Abelian Groups (R. Goldsmi th, eds.), de Gruyter, Berlin, pp. 445-461 (2008).

[HL] F. Huard, M. Lanzilotta, Self-injective right artinian rings and Igusa-Todorov functions, Algebras and Representation Theory, 16 (3), pp. 765-770 (2012).

[HLM] F. Huard, M. Lanzilotta, O. Mendoza. An approach to the finitistic dimension conjecture, J. Algebra 319 (9), pp. 3916-3934 (2008).

[LM] M. Lanzilotta, G. Mata, Igusa-Todorov functions for Artin algebras, Journal of pure and applied algebra, DOI: 10.1016/j.jpaa.2017.03.012 (2017).

[LMM] M. Lanzilotta, E. Marcos, G. Mata: Igusa-Todorov functions for radical square zero algebras, J. Algebra 487, pp. 357-385 (2017).

[IT] K. Igusa, G. Todorov, On finitistic global dimension conjecture for artin algebras, Representations of algebras and related topics, pp. 201-204, Fields Inst. Commun., 45, American Mathematical Society, Providence, RI, (2005).

[N] T. Nakayama. On Frobeniusean algebras I. Ann. of Math., 40, pp. 611-633 (1939)

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