Consistency of Superspace Low Energy Equations of Motion of 4D Type II Superstring with Type II Sigma Model at Tree-Level

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We derive the torsion constraints and show the consistency of equations of motion of four-dimensional Type II supergravity in superspace, with Type II sigma model. This is achieved by coupling the four-dimensional compactified Type II superstring to an N=2 curved background and requiring that the sigma-model has superconformal invariance at tree-level. We compute this in a manifestly 4D N=2 supersymmetric way. The constraints break the target conformal and SU(2) invariances and the dilaton will be a conformal, SU(2) × U(1) compensator. For Type II superstring in four dimensions, worldsheet supersymmetry requires two different compensators. One type is described by chiral and anti-chiral superfields. This compensator can be identified with a vector multiplet. The other Type II compensator is described by twist-chiral and twist-anti-chiral superfields and can be identified with a tensor hypermultiplet. Also, the superconformal invariance at tree-level selects a particular gauge, where the matter is fixed, but not the compensators. After imposing the reality conditions, we show that the Type II sigma model at tree-level is consistent with the equations of motion for Type II supergravity in the string gauge.

I. INTRODUCTION

Low-energy effective actions play an important role in the study of string theory. Beyond phenomenological applications, they also provide important pieces of evidence for the existence of various dual descriptions of string theories.

One way to construct low-energy effective actions in string theory is looking for the low-energy equations of motion. This is achieved by defining the sigma-model for the string in a curved background and requesting conformal invariance [1]. The low-energy effective action for type II superstring is some N=2 supergravity theory. So, the structure of this supergravity theory is constrained by the dynamics of the two-dimensional sigma-model.

To derive the structure and equations of motion of the N=2 supergravity theory that represents the Type II low-energy effective action, we need to formulate the sigma-model directly in terms of a target superspace and which has manifest local target space supersymmetry. Without such a sigma-model description, one can only work in components and it is not possible to determine the off-shell description of the supergravity selected by string theory. In addition, it is in general quite difficult to obtain the fermionic part of the effective action without a manifestly supersymmetric sigma-model. In the RNS formalism, for example, the part of the effective action coming from the Ramond fields is much less understood than the part coming from the Neveu-Schwarz sector. In the particular case of Type II superstrings, where there are conjectures relating non-perturbative states with the Ramond-Ramond sector [2], this lack of understanding is especially bothersome. On the other hand, in the Green-Schwarz formalism, we have manifest SUSY, but the covariant quantization is rather impossible due to the kappa symmetry.

A new formalism for the superstring was discovered by Berkovits with local N=2 worldsheet superconformal invariance. This formalism is known as hybrid formalism and is related to the RNS formalism by a field-redefinition [3]; it has the advantage of being manifestly spacetime supersymmetric. It is especially well-suited for compactifications to four dimensions, where it allows manifestly 4D super-Poincaré invariant quantization [4]. The coupling of the theory to background fields was discussed in [5], where also the form of the low-energy effective action was proposed, based on indirect arguments. In [6], the low-energy effective equations of motion of heterotic superstring were derived directly in superspace by means of this formalism. Although Berkovits has discovered another formalism based on pure spinors, which allows manifestly 10D super-Poincaré covariant quantization, the hybrid formalism is best understood and it has been widely used [7]. In this paper, we adopt this approach to derive the constraint structure and show consistency of the equations of motion for the N = 2 supergravity theory that corresponds to the low-energy effective theory of Type II superstrings in four dimensions. In the next section, we describe the Type II sigma-model with local 4D manifest supersymmetry. In Section 3 we carry out a covariant background field expansion to check N = (2, 2) superconformal invariance of Type II worldsheet at tree-level. Finally, in Section 4, we present and discuss the results. The methods presented here are a generalization of the methods developed for the heterotic superstring in [6].

II. TYPE II SIGMA-MODEL IN THE HYBRID FORMALISM.

A critical N=1 string can be formulated as a critical N=2 string, without changing the physical content. This is achieved by twisting the ghost sector of the critical
\[ S = \frac{1}{\alpha'} \int d^2z \frac{1}{2} \partial x_m \overline{\partial x}_m + p_\alpha \overline{\partial \theta}^\alpha + p_\alpha \overline{\partial \phi} = \frac{1}{2} \partial \rho \right. \\
\left. + \frac{\alpha'}{2} \overline{\partial \rho} (\partial \rho + a_z) + \frac{\alpha' \rho}{2} \overline{\partial \rho} (\partial \rho + a_z) + S_c \right) \tag{1} \]

where \( S_c \) is the action for the compactification-dependent superconformal field theory. In this letter, we will not be worried about the fields that depend on compactification, so we need to concentrate just in the \( c = -3 \) sector. The four-dimensional part of the action contains the spacetime variables, \( x^m (m = 0 \text{ to } 3) \), the left-moving fermionic variables, \( \theta^\alpha \) and \( \phi \), the conjugate left-moving fermionic variables, \( p_\alpha \) and \( \overline{p}_\alpha \), and one left-moving boson, \( \rho \), with a 'wrong' sign for the kinetic term. The right-sector of the Type II superstring is described by the right-moving fermionic fields, \( \overline{\theta}^{\hat{\alpha}} \) and \( \overline{\phi} \), the conjugate \( \overline{p}_\hat{\alpha} \) and \( \overline{\phi} \), and one right-moving boson, \( \overline{\rho} \). The fields \( a_z \), \( \overline{a}_z \) are the worldsheet \( U(1) \times U(1) \) gauge fields (\( e^\rho \) carries \( U(1) \)) charge). The components \( a_z \), \( \overline{a}_z \) can be fixed since the four present gauge fields fix just two symmetries (\( \rho \rightarrow \rho + \text{constant} \), \( \overline{\rho} \rightarrow \overline{\rho} + \text{constant} \)) \footnote{We can see this by coupling the theory to N=(2,2) worldsheet supergravity, which contains two independent \( U(1) \) gauge world-sheets.}

The strange \( \alpha'- \)dependence of \( \rho \) in (1) will later be shown to be related to the Fradkin-Tseytlin term. Also this dependence will permit to get the equations for the dilaton at tree level. The left-moving \( c = -3 \) generators for this N=(2,2) string are:

\[ T = \left( -\frac{1}{2} \overline{\Pi}^{\hat{a}} \Pi_{\alpha \hat{a}} - d_\alpha \overline{\partial \theta}^\alpha - d_\hat{a} \overline{\partial \phi} + \frac{\alpha'}{2} \partial \rho \partial \rho + \partial^2 \rho \right) \]

\[ G = \frac{1}{i \alpha' \sqrt{8} \alpha'} \exp (i \rho) d^\alpha d_{\alpha'} \]

\[ \mathcal{G} = \frac{1}{i \alpha' \sqrt{8} \alpha'} \exp (-i \rho) d^\alpha d_{\alpha'} \]

\[ J = -i \partial \rho \]

where we have used Pauli matrix to write vectors in terms of bi-espins and we have defined:

\[ d_\alpha = p_\alpha + i \overline{\theta}^{\hat{\alpha}} \partial x_{\alpha \hat{\alpha}} + \frac{1}{2} \overline{\partial} \partial \theta = \frac{1}{4} \theta_\alpha \partial (\overline{\partial})^2 \]

\[ d_\hat{a} = p_\alpha + i \overline{\phi} \partial x_{\alpha \hat{\alpha}} + \frac{1}{2} \overline{\partial} \partial \phi = \frac{1}{4} \theta_\alpha \partial (\overline{\partial})^2 \]

\[ \Pi_{\alpha \hat{a}} = d_\alpha - \theta^\alpha \overline{\partial} \theta + i \theta_\hat{a} \overline{\partial} \phi \]

The right-moving \( c = -3 \) \( \text{N}=(2,2) \) generators are:

\[ \hat{G} = \frac{1}{i \alpha' \sqrt{8} \alpha'} e^{i \overline{\phi} d^\alpha d_{\hat{a}}} \]

\[ \hat{T} = T \left( -\frac{1}{2} \Pi_{\alpha \hat{a}} - d_\alpha \overline{\partial \theta}^\alpha - d_\hat{a} \overline{\partial \phi} + \frac{\alpha'}{2} \partial \rho \partial \rho \right) \]

where \( \hat{d}_\alpha \) and \( \hat{d}_{\alpha} \) are obtained from (3) by using hatted variables and replacing \( \partial \) by \( \overline{\partial} \). Using the free-field OPE's the holomorphic (or left-moving) part of the N=(2,2), \( c = -3 \) algebra can be written as

\[ T \left( z \right) T \left( w \right) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \]

\[ T \left( z \right) G \left( w \right) = \frac{\hat{T} G \left( w \right)}{(z-w)^4} + \frac{\overline{G} \left( w \right)}{(z-w)^2} \]

\[ T \left( z \right) \overline{G} \left( w \right) = \frac{\overline{T} \left( z \right) \overline{G} \left( w \right)}{(z-w)^4} + \frac{\overline{G} \left( w \right)}{(z-w)^2} \]

\[ J \left( z \right) T \left( w \right) = \frac{G \left( w \right)}{(z-w)^4} \]

\[ J \left( z \right) G \left( w \right) = \frac{G \left( w \right)}{(z-w)^4} \]

\[ \frac{c/3}{(z-w)^2} \]

\[ G \left( z \right) \overline{G} \left( w \right) = \frac{\hat{T} c}{(z-w)^4} + \frac{2J \left( w \right)}{(z-w)^2} + \frac{2T \left( w \right)}{(z-w)^2} + \frac{\partial J \left( w \right)}{(z-w)^2} \]

The anti-holomorphic (or right-moving) generators satisfy the same algebra changing \( (z, w) \) for \( (\overline{z}, \overline{w}) \). The advantage of the variables \( d \) and \( \Pi \) over \( p \) and \( x \) is that they commute with the target space supersymmetry generators [5].

The action also becomes manifestly supersymmetric when expressed in terms of \( d \) and \( \Pi \). This is achieved by writing the coordinate of the N=2 flat superspace as \( Z^A \rightarrow \left( x^\alpha, \theta^\alpha, \overline{\theta}^{\hat{\alpha}}, \overline{\phi} \right) \) and defining variables \( \Pi^A, \Pi^\hat{A} \) using vielbeins \( E_M^A \) to convert curved \( (M) \) into flat indices \( (A) \): \( \Pi^A = \partial x^M E_M^A \) and \( \overline{\Pi}^\hat{A} = \partial \overline{\theta}^{\hat{\alpha}} E^{\hat{\alpha}}_M \), where
the indices of the flat and curved N=2 superspace can be written using the SU(2) notation: $A = (a, \alpha_j, \tilde{\alpha}_j)$ and $M = (m, \mu_j, \nu_j)$. The $j = \pm$ is an SU(2)-index and can be raised and lowered using the anti-symmetric $\epsilon_{jk}$ tensor. Comparing with the previous notation

$$\theta^{\alpha+} = \theta^\alpha, \quad \theta^{\alpha-} = \tilde{\theta}^\alpha, \quad \bar{\theta}^{\dot{i}+} = \bar{\theta}^{\dot{i}}.$$ (6)

From the previous section, $\Pi^a$ reduces to (3) when $E_M^A$ is the vielbein of flat superspace, which is one on the diagonal and has off-diagonal components: $E_{\mu_j}^j = \sigma_{\mu_j}^a \tilde{\theta}^a$.

The action can now be written directly in the target superspace:

$$S = \frac{1}{2\alpha'} \int d^2 z \left[ \frac{1}{2} \Pi^{\alpha a} \Pi_{\alpha a} + d_{\alpha} \bar{\Pi}^{a+} + d_{\bar{\alpha}} \bar{\Pi}^{\bar{a}-} + \bar{d}_{\bar{\alpha}} \Pi^{\bar{a}+} + \frac{1}{2} \Pi^A \Pi^B B_{AB} - \frac{\alpha'}{2} \bar{\theta}_{\alpha} (\partial \bar{\theta} + a_z) \right],$$ (7)

where we have introduced an anti-symmetric tensor field $B_{Aa}$ whose only non-zero components are:

$$B_{\alpha \alpha_j} = iC_{\alpha \beta} \theta_{\alpha_j} \quad B_{\alpha \beta, \alpha} = \theta_{\alpha} \dot{\theta}_{\beta} \quad B_{\alpha, \beta+} = \theta_{\alpha} \dot{\theta}_{\beta}$$ (8)

### III. SIGMA-MODEL IN CURVED N=2 SUPERSPACE

To formulate the action in a curved background, we can assume that the vielbein and the B-field in (7) have a general form. When expanded to first order around a flat background, one should recover the massless vertex operators of a flat background and the complete set of massless physical states of type II superstring should be presented.

The massless vertex operators were discussed in [5]. Due to manifest supersymmetry, the vertex operators do not distinguish Ramond and Neveu-Schwarz sectors and may be written in terms of a superfield that has both sectors. For type II superstring, the massless compactification-independent vertex operators have the form:

$$V = \int d^2 z \{ \tilde{G}, \{ \hat{G}, [G, G, U]\} \}$$ (9)

where $G = \int d^2 z G(z) = G_{-1/2}$ and similarly $\hat{G}$ are the $N = (2,2)$ world-sheet supersymmetry generators; they have precisely the form as one would expect for a theory with $N = (2,2)$ world-sheet supersymmetry. Furthermore, from the requirement that the vertex operator produces a state in the BRST cohomology, it follows that $U$ must be an $N = (2,2)$ primary field of conformal weight zero and $U(1) \times U(1)$ charge zero, i.e., $U$ should have only single poles in the OPE with $T, G, \hat{G}, \bar{T}, \bar{G}$, and have a regular OPE with $J$ and $\bar{J}$. These conditions do not yet completely classify the inequivalent vertex operators of the theory, because it is possible to perform certain gauge transformations that do not change the form of $V$:

$$\delta U = \nabla^2 \Lambda + \nabla^2 \bar{X} + \tilde{\nabla}^2 \Lambda + \tilde{\nabla}^2 \bar{X}$$ (10)

The $N=(2,2)$ primary field conditions and the gauge conditions imply that $U$ is a pre-potential for an $N=2$ conformal supergravity coupled to a hyper-tensorial multiplet. The gauge fields of supergravity sit in a Weyl multiplet with 24 bosonic and 24 fermionic off-shell components, while the matter fields are described by a hyper-tensorial multiplet with 8 bosonic and 8 fermionic off-shell components. This pre-potential represents the massless compactification-independent fields of the Type II superstring, without the dilaton. The dilaton does not couple classically in the action; it is part of the compensator fields and not part of the hypertensorial multiplet. To know the precise form of the off-shell N=2 Poincaré supergravity that describes the low-energy effective action for Type II superstrings, we need to know the compensator and the complete set of supergravity constraints, in particular the torsion constraints that break the conformal invariance. In a more conventional formulation, one would use the conformal invariance to gauge-fix the conformal compensator rather than the tensor hypermultiplet, but we will show that in the sigma-model it is the other way around. The action in (7) and the vertex operators provide all necessary ingredients to write down the classical part of Type II sigma-model. Here, by classical one means zero-order in $\alpha'$. The coupling of the dilaton in the sigma-model is described by the Fradkin-Tseytlin term, that is, the direct generalization of the dilaton coupling $\int d^2 z \sqrt{\bar{G}} \bar{G} \Phi$ of the bosonic string. So, we need to couple the dilaton with the N=(2,2) supercurvatures of the worldsheet; this coupling selects the particular target N=2 superfield that the dilaton will represent. In this case, worldsheet supersymmetry of the Fradkin-Tseytlin term requires two different compensators. One type is described by chiral and anti-chiral superfields, $\Phi_c$ and $\Phi_c$, satisfying $\nabla_{\alpha} \Phi_c = \nabla_{\alpha} \bar{\Phi}_c = 0$. After imposing the reality condition $(\nabla)^2 \Phi_c = (\nabla)^2 \Phi_c$, $\Phi_c$ and $\Phi_c$ can be identified with the chiral and anti-chiral field strength of a vector multiplet. This reality condition is not required by tree-level worldsheet symmetries of the sigma-model, but it is necessary for constructing superspace effective actions and can be derived at one loop level.\footnote{We are using the notation: $\nabla^2 = \nabla^\alpha \nabla_{\alpha}$, $\bar{\nabla}^2 = \nabla^\alpha \nabla_{\alpha}$}

\footnotetext{1}{In general we use $\Phi = e^\Phi$ to construct the low-energy effec-}
other Type II compensator is described by twisted-chiral and twisted-anti-chiral superfields, \( \Phi_{tc} \) and \( \bar{\Phi}_{tc} \), satisfying
\[
\nabla_a \Phi_{tc} = \nabla_a \bar{\Phi}_{tc} = \nabla_{a+} \bar{\Phi}_{tc} = \nabla_{a-} \Phi_{tc} = 0.
\]
These fields are related with a tensor hypermultiplet, that is usually used to fix the \( SU(2) \) symmetry of the \( N=2 \) conformal supergravity. Although the twisted-chirality condition on \( \Phi_{tc} \) does not look \( SU(2) \) covariant, it can be made covariant by identifying \( \Phi_{tc} \) and \( \bar{\Phi}_{tc} \) with \( L_{++} \) and \( L_{--} \) where \( L_{jk} \) is the linear field strength of a tensor hypermultiplet satisfying \( \nabla_{\alpha_i} L_{jk} = 0 \). This equation also implies that \( \Phi_{tc} \) satisfies the reality condition
\[
(\nabla)^2 \Phi_{tc} = (\nabla)^2 \bar{\Phi}_{tc}.
\]
In the conformal gauge, the dilaton couples to the \( \rho \)-field. In this gauge, the sigma-model is:
\[
S = \frac{1}{\alpha'} \int d^2z \left[ \frac{1}{2} P^{\alpha\beta} \partial_\alpha \Pi^B_{\beta} + d_\alpha \Pi^{\alpha+} + d_\alpha \Pi^{\alpha-} + \hat{a}_\alpha \Pi^{\alpha-} + \hat{d}_\alpha \Pi^{\alpha+} + \frac{1}{2} P^{\alpha\beta} B_{AB} + d_\alpha P^{\alpha\beta} \hat{d}_\beta + d_\alpha P^{\alpha\beta} \hat{d}_\beta + d_\alpha Q^{\alpha\beta} \hat{d}_\beta + d_\alpha \bar{Q}^{\alpha\beta} \hat{d}_\beta - \frac{\alpha'}{2} \left( \bar{\rho} + i \partial \left( \phi_c - \bar{\phi}_c + \phi_{tc} - \bar{\phi}_{tc} \right) \right) \times \left( \partial \bar{\rho} + i \partial \left( \phi_c - \bar{\phi}_c + \phi_{tc} - \bar{\phi}_{tc} \right) \right) - \frac{\alpha'}{2} \left( \bar{\rho} + i \partial \left( \phi_c - \bar{\phi}_c - \phi_{tc} + \bar{\phi}_{tc} \right) \right) \times \left( \partial \bar{\rho} + i \partial \left( \phi_c - \bar{\phi}_c - \phi_{tc} + \bar{\phi}_{tc} \right) \right) \right].
\]
with the previous definitions for the \( \Pi \)-field, but now with arbitrary vielbeins and anti-symmetric tensor fields. \( P^{\alpha\beta} \) and \( Q^{\alpha\beta} \) are chiral and twisted-chiral field strengths of \( N=2 \) conformal supergravity whose lowest components are the Type II Ramond-Ramond field strengths. From the \( d^\alpha d^\alpha \) and \( d^\alpha \bar{d}^\alpha \) terms in the supergravity vertex operator, one sees that at linearized level, \( P^{\alpha\beta} = (\nabla)^2 \nabla_{\alpha} (\nabla)^2 \nabla_{\beta} U \) and \( Q^{\alpha\beta} = (\nabla)^2 \nabla_{\alpha} (\nabla)^2 \nabla_{\beta} U \). The sigma-model action contains potentials \( B_{AB} \) and \( E_{M}^{\alpha} \) rather than prepotentials like \( U \). Of course, the antisymmetric tensor field, the \( P^{\alpha\beta} \), \( Q^{\alpha\beta} \) and the vielbeins contain more degrees of freedom than the prepotential \( U \). As usual in supergravity theories, there are torsion constraints which relate the gauge fields and field strengths to their prepotentials. In general, these constraints are imposed by hand; here we will derive these constraints imposing \( N=(2,2) \) superconformal invariance at tree-level in the sigma-model.

**IV. Perturbative Check of the \( N=(2,2) \) Algebra**

The \( N=(2,2) \) algebra derived in (5) for Type II superstring coupled to flat superspace must be satisfied in the curved sigma-model. However, in this case we do not have any longer worldsheet fields satisfying free OPE’s and we need a perturbative approach to check the \( N=(2,2) \) algebra. As usual in string theory, \( \alpha' \) counts the number of loops in the two-dimensional quantum theory. Here, we have an immediate problem caused by the fields \( \rho \) and \( \bar{\rho} \). Its kinetic term does not have an explicit factor of \( \frac{1}{\alpha'} \) in front, and therefore the \( \alpha' \)-perturbation theory does not make sense for these fields. In addition, the worldsheet Lagrange multipliers, \( a_z \) and \( \bar{a}_z \), impose the constraints \( \partial (\rho - i (\phi_c - \bar{\phi}_c)) = 0 \) and \( \partial (\bar{\rho} + i (\phi_c - \bar{\phi}_c)) = 0 \), which are difficult to handle. These last two problems disappear if we make the field-redefinition:
\[
\rho \rightarrow \rho - i (\phi_c - \bar{\phi}_c + \phi_{tc} - \bar{\phi}_{tc}) ,
\bar{\rho} \rightarrow \bar{\rho} - i (\phi_c - \bar{\phi}_c - \phi_{tc} + \bar{\phi}_{tc}) ;
\]
after that, \( \rho \) and \( \bar{\rho} \) become chiral and anti-chiral bosons, which can be quantized exactly, and which do not interact with the other fields of the theory. So, after this redefinition, \( \rho \) and \( \bar{\rho} \) obey the same free fields OPE’s that we have used to derive the algebra \( N=2 \) in (5); for the other fields, we will use perturbation theory. Surprisingly, this redefinition allows to derive information about the dilaton at tree-level. It is so because the fermionic generators have now the form:
\[
G \rightarrow \frac{1}{\alpha' \sqrt{8a'}} \left( e^{i \rho} e^{i \bar{\rho}} d^\alpha d_\alpha + \alpha' (...) \right)
\]
\[
\bar{G} \rightarrow \frac{1}{\alpha' \sqrt{8a'}} \left( e^{-i \rho} e^{-i \bar{\rho}} d^\alpha d_\alpha + \alpha' (...) \right)
\]
\[
\tilde{G} \rightarrow \frac{1}{\alpha' \sqrt{8a'}} \left( e^{i \phi} e^{i \bar{\phi}} d^\alpha d_\alpha + \alpha' (...) \right)
\]
\[
\tilde{\bar{G}} \rightarrow \frac{1}{\alpha' \sqrt{8a'}} \left( e^{-i \phi} e^{-i \bar{\phi}} d^\alpha d_\alpha + \alpha' (...) \right)
\]
the dots are terms that come from the Fradkin-Tseytlin term and do not contribute at tree-level. Next, we describe the covariant background formalism that we intend to perform. A typical beta-function calculation does not guarantee the full \( N=(2,2) \) superconformal invariance. The latter would only follow from a standard supersymmetric beta-function calculation if the model could be formulated in \( N=(2,2) \) superspace on the worldsheet, which does not seem possible. So, we need to check the \( N=(2,2) \) algebra by calculating the OPE’s perturbatively. At tree-level, there are no double contractions and it is necessary just to verify the part of the
\( N=2(2,2) \) algebra that depends on simple contractions \(^{\ddagger}\). To perform these calculations, we will use a background covariant expansion that preserves manifestly all the local symmetries of the target superspace. In our case, these symmetries are local Lorentz transformations and the following \( U(1) \times U(1) \) transformations:

\[
\delta \phi_c = \frac{1}{2} \left( \lambda + \tilde{\lambda} \right) \quad \delta \tilde{\phi}_c = -\frac{1}{2} \left( \lambda + \tilde{\lambda} \right)
\]

\[
\delta \Pi^{\alpha^+} = -\frac{1}{2} \Pi^{\alpha^+} \quad \delta \Pi^{\alpha^-} = \frac{1}{2} \lambda \Pi^{\alpha^-}
\]

\[
\delta \lambda_d = \frac{1}{2} \lambda d_a \quad \delta \lambda_{\tilde{d}} = -\frac{1}{2} \tilde{\lambda} d_a \quad \delta \rho = -2i \lambda
\]

The traditional way to achieve such an expansion is to use a tangent vector that relates two points in target superspace, the classical field and the quantum fluctuations, then expand all the tensors in potentials of this vector and use Riemann normal coordinates to covariantize the expansion. In this letter, we will not discuss the details of this expansion and will just put the results. The covariant derivative in the tangent superspace is:

\[
\nabla_A = E_A^\alpha \partial_M + \omega_A \beta^\gamma \gamma M_\beta^\gamma + \omega_A \beta^\gamma \gamma M_\beta^\gamma
\]

\[+ \omega_A \beta^\gamma \gamma M_\beta^\gamma + \omega_A \beta^\gamma \gamma M_\beta^\gamma + \Gamma_A Y + \tilde{\Gamma}_A Y
\]

(15)

where \( \omega, \Gamma, \tilde{\Gamma} \) are the Lorentz and \( U(1) \times U(1) \) connections, \( M \) are the Lorentz generators and \( Y, \tilde{Y} \) the \( U(1) \times U(1) \) generators. We must observe that there are two independent spacetime spinors in the Type II sigma-model, so one has two independent fermionic structure groups. Thus, just like the two independent \( U(1) \) connections one has two independent sets of irreducible spin connections: \( \omega_{\alpha^\beta}, \omega_{\alpha^\beta}, \omega_{\alpha^\beta}, \omega_{\alpha^\beta} \). The covariant derivative satisfies the algebra

\[
[\nabla_C, \nabla_A] = T_{CA} B \nabla_B + R_{CAE} D M_D E + F_{CA} Y + \tilde{F}_{CA} Y
\]

(16)

where \( F \) and \( \tilde{F} \) are the super \( U(1) \times U(1) \) field strengths and \( T, R \) are the superpotorsions and supercurvatures.

For the purpose of our tree-level calculation, we only need to go up to two background fields in the quadratic part in the quantum fields of action. With the covariant derivative and the algebra, by using the algorithm developed in \([10,6]\), with the notation: \( \hat{A}_{\alpha^\beta} = (A^{\alpha^+ \alpha}, A^{\alpha^- \alpha}) \) and \( \hat{A}_{\alpha^\beta} = (A^{\alpha^+ \alpha}, A^{\alpha^- \alpha}) \), we have the action expanded up to this order:

\[
S^2 = \frac{1}{2} \nabla y^a \nabla y^a + \frac{1}{2} \nabla y^a \delta y^a + \frac{1}{2} \nabla \delta y^a + \frac{1}{2} \nabla \delta y^a
\]

\[+ \frac{1}{2} \nabla y^2 \left( \Pi^B T_{BC} \right) + \frac{1}{2} \nabla y^2 \left( \Pi^B T_{BC} \right)
\]

\[- \frac{1}{4} \nabla y^2 \left( \Pi^B T_{BC} \right) + \Pi^B H_{ABC}
\]

\[+ \frac{1}{2} \nabla y^2 \left( \Pi^B T_{BC} \right) + \frac{1}{2} \nabla y^2 \left( \Pi^B T_{BC} \right)
\]

\[+ \frac{1}{2} \nabla y^2 \left( \Pi^B T_{BC} \right) + \Pi^B H_{ABC}
\]

(17)

where:

\[
T_{DCB}^A = R_{DCB}^A + \omega (A) F_{DCB}^B + \omega (A) \tilde{F}_{DCB}^B
\]

\[+ T_{DC}^E T_{EB}^A + \left(-1\right)^{CD} \nabla_v T_{DB}^A
\]

\[
H_{DCB}^A = \nabla_C H_{DBA} + \left(-1\right)^{CD} \nabla_v C_{TDB}^A
\]

\[- T_{CA}^E H_{EBD} + \left(-1\right)^{AB + D + CD} + T_{DCB}^E H_{ABC}
\]

\[
\omega (A) \text{ and } \tilde{\omega} (A) \text{ are the } U(1) \times U(1) \text{ weights of } A. \text{ The only ones different from zero are: }\omega (a^+ \alpha) = \tilde{\omega} (a^+ \alpha) = \frac{1}{2} \omega (a^+ \alpha) = -\frac{1}{2} \omega (a^- \beta) = \frac{1}{2} \omega (a^- \beta) = -\frac{1}{2} \omega (a^- \beta)
\]

In the expanded action, we have the quantum fields: \( Y^A, \hat{d}_a, \hat{d}_{\tilde{a}} \) and the background fields: \( \Pi^A, D_a, \tilde{D}_a \). Since the \( d_a \) and \( \tilde{d}_a \) fields are spacetime independent, we have chosen a simple expansion for these fields, that preserves the superspace symmetries: \( d_a = d_a + D_a \) and similarly for \( \tilde{d}_{\tilde{a}} \).

Now, we can describe the kind of calculation we intend to do. The kinetic part of the action provides the worldsheet propagators:

\[
\langle \delta d a y^a \rangle = \frac{\delta a}{z - w} \quad \langle \delta \tilde{d} a y^a \rangle = \frac{\delta \tilde{a} a}{z - w}
\]

(18)

and the same for the dot spinors. The bosonic fields have the two dimensional propagator: \( \langle y^a y^b \rangle = \eta y \ln |z - w| \).
The other part of the action provides the vertices. By expanding the generators using the same background covariant expansion we can calculate the tree-level OPE’s contracting the fields with the vertices in the action. By demanding that the N=(2,2) algebra be satisfied, we get the torsion constraints of N=2 supergravity. All the results we will get from the fermionic part of the algebra.

Besides Lorentz and U(1) × U(1) invariance, the background field expansion of the action has an additional set of symmetries, which we denote by ‘shift symmetries’. These originate from the fact that the original action depends only on the vielbeins, not on torsions and curvatures. In our case the shift symmetry has the form:

\[ \delta \omega_{AB}^C = Y_{AB}^C \]
\[ \delta \Gamma_A = X_A \]
\[ \delta \bar{X}_A = \bar{X}_A \]
\[ \delta T_{AB}^C = Y_{[AB]}^C + \omega (C) X_{[A} \delta B)^C + \bar{\omega} (C) \bar{X}_{[A} \delta B)^C \]
\[ \delta y^A = \frac{1}{2} y^B y^C \left( Y_{BC} A^A + \omega (A) X_C \delta B^A + \bar{\omega} (A) \bar{X}_C \delta B^A \right) \]
\[ \delta a_\alpha = \left( M Y_{\alpha \beta} + \frac{1}{2} y^A X_\alpha \delta \beta \right) (d_{\beta} + D_{\beta}) \]
\[ \delta \bar{a}_\alpha = \left( M \bar{Y}_{\alpha \beta} + \frac{1}{2} y^A \bar{X}_\alpha \delta \beta \right) \left( \bar{d}_{\beta} + \bar{D}_{\beta} \right). \]

This symmetry must be manifest in the OPE’s and we will use it to fix some components of the torsions, providing the ‘conventional constraints’.

We are now ready to check the superconformal invariance at tree-level. The generators must satisfy the tree-level part of the OPE’s (simple contractions) and in addition the left-moving generators should be holomorphic and the right-moving anti-holomorphic. By requesting the conditions \( \bar{\partial} G = \partial \bar{G} = 0 \) with help of the equations of motion,

\[ \Pi^{\alpha^+} + P^{\alpha \beta} \bar{d}_{\beta} + Q^{\alpha \beta} \bar{d}_{\beta} = 0 \]
\[ \bar{\Pi}^{\dot{\alpha}^+} + \bar{P}^{\dot{\alpha} \dot{\beta}} \bar{d}_{\dot{\beta}} + \bar{Q}^{\dot{\alpha} \dot{\beta}} \bar{d}_{\dot{\beta}} = 0 \]
\[ \Pi^{\alpha^-} - d_{\beta} P^{\beta \alpha} - d_{\dot{\beta}} Q^{\beta \dot{\alpha}} = 0 \]
\[ \bar{\Pi}^{\dot{\alpha}^-} - \bar{d}_{\dot{\beta}} \bar{P}^{\dot{\beta} \dot{\alpha}} - \bar{d}_{\dot{\beta}} \bar{Q}^{\dot{\beta} \dot{\alpha}} = 0 \]

\[ \nabla d_{\alpha} + \frac{1}{2} \left( \Pi^C T_{C_{\alpha \beta}} \Pi_{\beta} + \bar{\Pi}^C T_{\bar{C}_{\alpha \beta}} \bar{\Pi}_{\beta} \right) = \Pi^C \left( T_{C_{\alpha \beta}} d_{\beta} \right) - \Pi^C \left( T_{C_{\alpha \beta}} \bar{d}_{\beta} \right) \]
\[ - \Pi^C \left( T_{C_{\alpha \beta}} \bar{d}_{\beta} \right) - \Pi^C \left( \bar{P}^{\beta \alpha} \bar{d}_{\beta} \right) + d_{\beta} \left( \nabla_{\alpha} P^{\beta \gamma} \bar{d}_{\gamma} + \nabla_{\alpha} \bar{Q}^{\beta \gamma} \bar{d}_{\gamma} \right) + d_{\beta} \left( \nabla_{\alpha} P^{\beta \gamma} \bar{d}_{\gamma} + \nabla_{\alpha} \bar{Q}^{\beta \gamma} \bar{d}_{\gamma} \right) = 0 \]

we get

\[ T_{\dot{\beta} \dot{\alpha} \dot{c}} = 2 H_{\dot{\beta} \dot{c} \dot{\alpha}} = 0 \]
\[ T_{\dot{\beta} \dot{\alpha} \dot{c}} = 2 H_{\dot{\beta} \dot{c} \dot{\alpha}} = 0 \]

In addition to these constraints, we have the equations of motion for the dilaton:

\[ \nabla_\beta P^{\beta \gamma} - \bar{\Lambda}^{\gamma} \bar{\Lambda}_\beta - \Pi^{\alpha \beta} \Lambda_{\alpha \beta} = 0 \]
\[ \nabla_{\dot{\beta}} P^{\dot{\beta} \gamma} - \Pi^{\dot{\alpha} \dot{\beta}} \Lambda_{\dot{\alpha} \dot{\beta}} = 0 \]

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\[ \nabla_{\dot{\beta}} P^{\dot{\beta} \gamma} - \Pi^{\dot{\alpha} \dot{\beta}} \Lambda_{\dot{\alpha} \dot{\beta}} = 0 \]

The tree-level OPE between the holomorphic generators and anti-holomorphic ones must be regular. Using the background expansion for the generators, we can write the G(z)̂G(w) OPE as

\[ G(z) \hat{G}(w) = D_{\alpha} (z) \left< \bar{d}^\alpha (z) \bar{d}^\beta (w) \right> \hat{D}_{\beta} (w) \]
\[ + D_{\alpha} (z) \left< d^\alpha (z) y^A (w) \right> \hat{D}^2 (w) \nabla_A (\phi - \bar{\phi}) \]
\[ + D^2 (z) \nabla_A (\phi - \bar{\phi}) \left< y^A (z) \bar{d}^\beta (w) \right> \hat{D}_{\alpha} \]
\[ + D^2 (z) \left< y^A (z) y^B (w) \right> \hat{D}^2 (w) \nabla_A (\phi - \bar{\phi}) \]

In this expression we did not write the exponential terms involving the \( \rho \) and \( \phi \) fields. By contracting the quantum fields in the expectation values with vertices that came from the action (17), we have many non-regular terms with 3 and 4 background fields. We can show that, when we use the previous constraints and equations of motion, all the non-regular terms are cancelled, as we should expect. However, the tree-level part of the \( \left< G(z) G(w) \right> \) is

\[ \left< G(z) G(w) \right> = \frac{1}{(z - w)} \left< \ldots \right> \]

where we have used the e^{i\rho(z)}e^{i\rho(w)} OPE and the dots are terms similar to (23), with the hatted fields replaced by
non-hatted fields. Here, we get new information. To get the right N=(2,2) algebra, the expectation values must be of order $O(z-w)$. Again, we contract the quantum fields with the vertices of the action and the anti-holomorphic part of these OPE’s are cancelled by the conditions (22). From the holomorphic part, we get a regular term in the expectation values that give us the constraints

$$T_{\gamma_+\alpha_+} = T_{\gamma^-\alpha} = H_{\gamma_+\alpha_+} = H_{\gamma^-\alpha} = 0$$

$$T_{\gamma_+\alpha_+} + 2\nabla_\alpha (\phi - \bar{\phi}) = 0$$  \hspace{1cm} (25)

This new information comes from the fact that we have exact conformal theory for the $\rho$ fields. The results for $\langle G(z)G(w) \rangle$ are similar, changing $\gamma_+, \alpha_+$ for $\gamma_-, \alpha_-.$

The OPE’s must be shift-invariant. This is true for all the derived constraints and equations of motion, except for the last equation in (25). This is another consequence of the $\rho$-field. Fortunately, there is a systematic way to cancel this equation. We can redefine the expansion for the $d_\alpha$-field, showing that the shift invariance fixes the expansion for this field or add the zero-term to the action:

$$\int d^2z (\nabla D_\gamma + \ldots) y^{a+y}_{\beta} + S_{\beta_+\alpha_+} \gamma^+,$$

where $\nabla D_\gamma + \ldots$ is the field equation with $d$ replaced by $D$. Since $D$ is on-shell, the extra term is always zero. If one partially integrates it, one gets a series of vertices with two background field, that does not give any contribution and precisely one vertex with one background field, namely $\int d^2z (-2D_\gamma y^{a+y}_{\beta} + S_{\beta_+\alpha_+} \gamma^+).$ We can choose $S_{\alpha_+} \beta$ to cancel the equation and as it may be verified, this term in the action does not give any new contribution; the same trick was used in [6] and [11].

Before writing down the constraints and equation of motion, we can check that no information came from the OPE’S $\langle G(z)G(w) \rangle$ and $\langle \bar{G}(z)\bar{G}(w) \rangle$. While the tree-level OPE’S $\langle d^2(z)d^2(w) \rangle$ and $\langle \bar{d}^2(z)\bar{d}^2(w) \rangle$ must be of order $O((z-w))$ and $O((\bar{z}-\bar{w}))$, the OPE’S $\langle d^2(z)d^2(w) \rangle$ and $\langle \bar{d}^2(z)\bar{d}^2(w) \rangle$ must be of order $O((z-w)^{-1})$ and $O((\bar{z}-\bar{w})^{-1})$. Since terms with more than two backgrounds fields are always of order $O((z-w)^{-1})$, they are not relevant and we have just terms with two background fields: $\langle d^2(z)d^2(w) \rangle$ and $\langle \bar{d}^2(z)\bar{d}^2(w) \rangle$, that are trivially zero since we do not have propagators from $d_\alpha$ to $\bar{d}_\alpha$ and from $d_\alpha$ to $\bar{d}_\alpha.$ The OPE’s involving $T$ and $\hat{T}$ are satisfied by using the equations of motion (20) and identities like $\nabla T_{\alpha_\beta} = \nabla T_{\beta_\alpha} = \Pi^A \Pi^B T_{AB} \alpha_\beta.$ We will discuss first the constraints; they are divided into the following categories:

A. Representation-preserving constraints:

$$T_{\alpha_\beta} \gamma^- = T_{\alpha_\beta} \gamma^+ = 0$$

$$T_{\alpha_\beta} \gamma^- = T_{\alpha_\beta} \gamma^+ = 0$$  \hspace{1cm} (26)

B. Conformal Constraints:

$$H_{\alpha_\beta} c = -i \delta_{ij} \delta^\lambda_{\alpha} \delta^\lambda_{\beta}$$

$$T_{\alpha_\beta} \gamma^- = -T_{\alpha_\beta} \gamma^+ = T_{\alpha_\beta} \gamma^- = T_{\alpha_\beta} \gamma^+ = 0$$

The first type of constraints is the usual representation-preserving constraints which allow a consistent definition of chiral and twisted-chiral superfields. The second type of constraints is conformal-breaking constraints, which are necessary since the sigma-model action is not invariant under the spacetime scale transformations that transform $\delta E_a^M = \pi E_a^M$. The relation between the $P^{a\beta}$ and $Q^{a\beta}$ with the torsiions shows us a clear geometrical meaning for the Ramond-Ramond fields in the superspace. In general, we can write the hypertensorial multiplet $H_{ABC}$ in terms of a linear multiplet $\varphi_{a\beta} = \sigma^{a\beta} L_{ij}$, where the lowest component of the linear multiplet is a $SU(2)$ triplet $l_{ij}.$ So, also the $SU(2)/U(1)$ invariance is broken in the Type II sigma-model by gauge-fixing $l_{jk} = \delta_{jk}.$ In addition, the Bianchi identities imply that $T_{abc} = -2H_{abc}.$ So, the Type II superstring selects a particular gauge, where the matter is fixed and breaks the scale and $SU(2)/U(1)$ symmetries of the conformal supergravity. Part of the hypermultiplet that is not fixed goes to the supergravity multiplet which, after imposing the conventional constraints, presents 32 + 32 off-shell degrees of freedom. In general, when passing from conformal supergravity to Poincaré supergravity, the compensator fields are fixed. For a historical view of the development of the $N=2, 4d$ sugra in superspace see ref [12]. In particular, the third reference discusses for the first time the change of the constraints that could lead to the string frame formulation derived here. In the last reference the vector and tensorial compensators are discussed *.* In [13], a “minimal multiplet” is employed as a starting point, consisting of $N=2$ conformal supergravity coupled to a vector multiplet, and by coupling this minimal multiplet to chiral, a non-linear, or a hypertensorial multiplet we have 4 different off-shell Poincaré supergravity. In particular, in the last one, the vector multiplet fixes the scale and $U(1)$ invariance while the hypermultiplet fixes $SU(2)/U(1)$. Here, we show that the Type II supergravity is a $U(1) \times U(1)$ version of this $N=2$ Poincaré supergravity, when the matter is fixed and the compensators are dynamical. Let us now determine a maximal set of conventional constraints. From our sigma-model point of view, part of the conventional constraints can be derived from the OPE’S and the other part can be viewed as a gauge fixing of the shift symmetry discussed previously, choosing $X_A, Y_{AB} C, \bar{X}_A$ and $\bar{Y}_{AB} C$ properly.

**I would like to thank Professor Jim Gates for clarifying this point to me**
C. Conventional constraints:

\[ T_{\alpha, \beta}^{\gamma \lambda} \delta^\gamma_\beta = -2 i \varepsilon_{ij} \delta^\lambda_\alpha \delta^\gamma_\beta \]

\[ T_{\alpha, (bc)}^{\gamma} = 0. \]

\[ T_{\alpha, \beta}^{\gamma +} = T_{\alpha, \beta}^{\gamma ^+} = T_{\alpha, \beta^-}^{\gamma} = T_{\alpha, \beta^-}^{\gamma ^-} = 0 \]

\[ T_{\alpha, \beta}^{\gamma -} = T_{\alpha, \beta}^{\gamma ^-} = T_{\alpha, \beta}^{\gamma ^-} = T_{\alpha, \beta}^{\gamma ^-} = 0 \]

These constraints define the vector components of the super-vielbein in terms of the spinor components, and the spin connections in terms of the super-vielbein. The equations of motion for the compensators become

\[ \nabla_\gamma (\phi_c - \phi_e + \phi_{tc} - \phi_{te}) = 0 \]

\[ \nabla_a (\phi_c - \phi_e + \phi_{tc} - \phi_{te}) = 0 \]

\[ \nabla_\gamma (\phi_i - \phi_c - \phi_{tc} + \phi_{te}) = 0 \]

\[ \nabla_a (\phi_i - \phi_c - \phi_{tc} + \phi_{te}) = 0. \]

(29)

With the help of Bianchi identities the equation involving \( \nabla_\beta P^{\beta \gamma} \) can be written as

\[ [\nabla_a, \nabla_\gamma] (\phi_c - \phi_e + \phi_{tc} - \phi_{te}) = 0, \]

so it is cancelled by (29). These equations of motion are precisely the equations of motion that describe the 16+16 degrees of freedom of the N=2 supergravity in the string gauge. At this point we need to emphasize that this is true if the dilaton fields satisfy the reality conditions discussed previously, which are not derived at tree level. So, strictly speaking, we just showed that the equations of motion obtained here are consistent with the equations of motion for Type II supergravity in the string gauge. To these equations make sense, we need to impose the reality conditions, that appear in one loop level [14]. These results shall soon be reported elsewhere. The corrections discussed in [15] will appear in higher loops.

It is unusual to get any information about the dilaton at tree level. This is other immediate consequence of the \( \rho \) field behavior. We can see that the \( \rho \) field has no \( \alpha' \)-dependence in the action, but appears with the same order of \( \alpha' \) as the \( d \) fields in the fermionic generators in (2); this is a particularity of this formalism. In addition, to have a consistent perturbation theory in the superconformal gauge we need to make the redefinition (12). So, we have a new dependence of the tree level part of the generators on \( e^{\phi_c} \) and \( e^{\phi_{tc}} \). This generates the conditions (29) but not the reality conditions, which appear at one loop.

The low-energy effective action for Type II superstring in four dimensions must reproduce these equations of motion. In the work of ref. [5], by using indirect arguments and not by checking directly the superconformal invariance, a Type II low-energy effective action was proposed in the harmonic superspace. We can check that this action generates the equations of motion showed here.

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