ON INCREASING STABILITY OF THE CONTINUATION FOR
ELLIPTIC EQUATIONS OF SECOND ORDER WITHOUT
(PSEUDO)CONVEXITY ASSUMPTIONS

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Abstract. We derive bounds of solutions of the Cauchy problem for general elliptic partial differential equations of second order containing parameter (wave number) $k$ which are getting nearly Lipschitz for large $k$. Proofs use energy estimates combined with splitting solutions into low and high frequencies parts, an associated hyperbolic equation and the Fourier-Bros-Iagolnitzer transform to replace the hyperbolic equation with an elliptic equation without parameter $k$. The results suggest a better resolution in prospecting by various (acoustic, electromagnetic, etc) stationary waves with higher wave numbers without any geometric assumptions on domains and observation sites.

1. Introduction. The Cauchy Problem (or equivalently the continuation of solutions) for partial differential equations has a long and rich history, starting with the Holmgren-John theorem on uniqueness for equations with analytic coefficients. It is of great importance in the theories of boundary control and of inverse problems. In 1938 T. Carleman introduced a special exponentially weighted energy (Carleman type) estimates to handle non analytic coefficients. These estimates imply in addition some conditional Hölder type bounds for solutions of this problem. In 1960 [18] F. John showed that for the continuation of solutions to the Helmholtz equation from inside of the unit disk onto any larger disk the stability estimate which is uniform with respect to the wave numbers is still of logarithmic type. Logarithmic stability is quite damaging for numerical solution of many inverse problems. In recent papers [2], [3], [7], [8], [9], [13] it was shown that in a certain sense stability is always improving for larger $k$ under (pseudo) convexity conditions on the geometry of the domain and of the coefficients of the elliptic equation. These assumptions are not satisfied in the John’s example.

In this paper we attempt to eliminate any convexity type condition on the elliptic operator or the domain. Due to the John’ counterexample, one can not expect increasing stability for all solutions without suitable a priori constraint. We show that (near Lipschitz) stability holds on a subspace of (“low frequency”) solutions which is growing with the wave number $k$ under some mild boundedness constraints on complementary “high frequency” part. Preliminary results are given in [11], [12].

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Using new logarithmic stability estimates in the Cauchy problem for hyperbolic equations without any convexity assumptions we bound the norm of the “high frequency” part norm and obtain a conditional stability bound which is improving for larger wave numbers $k$.

We consider the Cauchy problem

\begin{equation}
(A - ia_0 k + k^2) u = f \text{ in } \Omega,
\end{equation}

with the Cauchy data

\begin{equation}
u = u_0, \partial_\nu u = u_1 \text{ on } \Gamma \subset \partial \Omega,
\end{equation}

where

\[ Au = \sum_{j,m=1}^n a_{jm} \partial_j \partial_m u + \sum_{j=1}^n a_j \partial_j u + au \]

is a general partial differential operator of second order satisfying the uniform ellipticity condition

\[ \varepsilon_0 |\xi|^2 \leq \sum_{j,l=1}^n a_{jl}(x) \xi_j \xi_l \leq E_0^2 |\xi|^2 \]

for some positive numbers $\varepsilon_0, E_0$ and all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. We assume that $a_{jm}, \partial_\nu a_{jm}, a_j, a \in L^\infty(\Omega)$.

We consider bounded open domain $\Omega$ in $\mathbb{R}^n$ with the boundary $\partial \Omega \in C^2$ and $\Gamma$ which is an open subset of $\partial \Omega$.

We use the classical Sobolev spaces $H^l(\Omega)$ with the standard norm $\| \cdot \|_{l(\Omega)}$.

We recall the common notation

\[ \|u\|_{L^2([-T,T); H^1(\Omega))} = \left( \int_{-T}^T \|u(t, \cdot)\|_{H^1(\Omega)}^2 \, dt \right)^{\frac{1}{2}}. \]

Let $\omega$ be an open subset of $\Omega$.

In what follows $C$ denote generic constants depending only on $\Omega, \Omega_0, \Gamma, \omega, A, a_0$.

Let

\[ \varepsilon = \|f\|_{(0)}(\Omega) + \|u_0\|_{(1)}(\Gamma) + \|u_1\|_{(0)}(\Gamma) + \|u\|_{(1)}(\omega). \]

We claim

**Theorem 1.1.** Let one of $\Gamma, \omega$ be not empty, and $\Omega_0$ be a sub domain of $\Omega$ with $\bar{\Omega}_0 \subset \Omega \cup \Gamma$.

There are positive constants $C, \kappa$ such that

\[ \|u\|_{(0)}(\Omega_0) \leq C \left( \|u\|_{(0)}(\Gamma) + k^{-1} (\|f\|_{(0)}(\Omega) + \|u_1\|_{(0)}(\Gamma) + \|u\|_{(1)}(\omega) + \varepsilon^6 \|u\|_{(1)}(\Omega)) + k^{-\frac{1}{2}} \|u\|_{(1)}(\Omega) |\ln \varepsilon|^{-\frac{1}{4}} \right) \]

for all $u \in H^2(\Omega)$ solving (1), (2).

A more complicated but more precise result implying Hölder (and near Lipschitz) stability in a subspace which grows with $k$ is the following

**Theorem 1.2.** Let one of $\Gamma, \omega$ be not empty, and $\Omega_0$ be a sub domain of $\Omega$ with $\bar{\Omega}_0 \subset \Omega \cup \Gamma$. Assume that $\Omega$ is $C^2$-diffeomorphic to the closed unit ball.

There are a monotone family of closed subspaces $H^1(\Omega; k)$ of $H^1(\Omega)$ such that $\bigcup_k H^1(\Omega; k) = H^1(\Omega)$, a semi norm $\| \cdot \|_{1;k}(\Omega)$ on $H^1(\Omega)$ which is zero on $H^1(\Omega; k)$ and decreasing with respect to $k$ and positive constants $C, \kappa$ such that
\[ \|u\|_0(\Omega_0) \leq C \left( \|u\|_0(\Gamma) + k^{-1}(\|f\|_0(\Omega) + \|u_0\|_0(\Gamma) + \|u_1\|_0(\Gamma) + \|u\|_1(\Omega)) + k^{-\frac{1}{2}} \|u\|_1(\Omega) \frac{1}{\|u\|_{1,k}(\Omega)} \ln(\frac{1}{\epsilon}) \right) \]

for all \( u \in H^2(\Omega) \) solving (1), (2).

As known [1], [12], there are Hölder stability estimates in the elliptic Cauchy problem, i.e. the bound (3) without logarithmic term. At fixed (low) \( k \) when \( \Omega_0 \) is close to \( \Omega \) the Hölder exponent \( \kappa \) is getting small which leads to a very low (numerical) resolution in the Cauchy and corresponding inverse problems and dramatically limits applications. On the other hand, as shown in [18], when \( k \) grows constants in Hölder stability estimates blow up (rapidly go to \( \infty \)). Since constants \( C, \kappa \) do not depend on \( k \), Theorems 1.1, 1.2 show that under a natural energy constraint \( (\|u\|_1(\Omega) \leq M) \) for any \( \Omega_0 \) stability is improving to a Lipschitz one for large \( k \), and hence so should do the numerical resolution. Stability (and hence numerical resolution and applied value) in inverse problems is directly linked to stability of the continuation which is in more detail discussed in [1]. In particular, in seismic prospecting (in geophysics) many numerical algorithms explicitly involve the continuation of solutions of the reduced wave equations (under name of migration).

To derive these main results we will use the following theorems.

**Theorem 1.3.** Let \( \Gamma \) be not empty set. Then there is a constant \( C \) such that
\[ \|u\|_0(\Omega) \leq C \left( \|u\|_0(\Gamma) + k^{-1}(\|f\|_0(\Omega) + \|u_0\|_0(\Gamma) + \|u_1\|_0(\Gamma) + \|u\|_1(\Omega)) \right) \]
for all \( u \in H^2(\Omega) \) solving (1), (2).

Moreover, if in addition \( \Omega \) is \( C^2 \) diffeomorphic to the closed unit ball, then there are a monotone family of closed subspaces \( H^1(\Omega; k) \) of \( H^1(\Omega) \) with \( \cup_k H^1(\Omega; k) = H^1(\Omega) \), a semi norm \( ||| \cdot |||_{1,k}(\Omega) \) on \( H^1(\Omega) \) which is zero on \( H^1(\Omega; k) \) and decreasing with respect to \( k \), and a constant \( C \) such that
\[ \|u\|_0(\Omega) \leq C \left( \|u\|_0(\Gamma) + k^{-1}(\|f\|_0(\Omega) + \|u_0\|_0(\Gamma) + \|u_1\|_0(\Gamma) + |||u|||_{1,k}(\Omega)) \right) \]
for all \( u \in H^2(\Omega) \) solving (1), (2).

Now we give a version of this result for an increasing stability of the continuation from an open subset \( \omega \) of \( \Omega \).

**Theorem 1.4.** Let \( \omega \) be non empty open subset of \( \Omega \). Then there is a constant \( C \) such that
\[ \|u\|_0(\Omega) \leq Ck^{-1}(\|f\|_0(\Omega) + \|u\|_1(\Omega) + \|u\|_1(\Omega)) \]
for all \( u \in H^2(\Omega) \) solving (1).

Moreover, if \( \Omega \) is a \( C^2 \)-diffeomorphic to the closed unit ball and \( \tilde{\omega} \) contains an open (in \( \partial \Omega \)) non empty subset of \( \partial \Omega \), then there are a monotone family of closed subspaces \( H^1(\Omega; k) \) of \( H^1(\Omega) \) with \( \cup_k H^1(\Omega; k) = H^1(\Omega) \), a semi norm \( ||| \cdot |||_{1,k}(\Omega) \) on \( H^1(\Omega) \) which is zero on \( H^1(\Omega; k) \) and decreasing with respect to \( k \), and a constant \( C \) such that
\[ \|u\|_0(\Omega) \leq Ck^{-1}(\|f\|_0(\Omega) + \|u\|_1(\Omega) + |||u|||_{1,k}(\Omega)) \]
for all \( u \in H^2(\Omega) \) solving (1).
Now, for particular $\Omega, \Gamma$ in $\mathbb{R}^n, n = 2, 3$, we will describe more explicitly the subspaces $H^1(\Omega; k)$ of Lipschitz stability and semi norms $\|u\|_{(1,k)}$. Let $\Omega = \{x : |x| < R\}$, $\Gamma = \{x : |x| = 1\}$, and $\Gamma_1 = \{x : |x| = R\}$. We will use the (angular) orthogonal series
\[ u(\sigma) = \sum_{1 \leq m, p \leq p(m)} u(m, p)e(\sigma; m, p), \]
where $\{e(\sigma; m, p)\}, m = 1, 2, \ldots, p = 1, \ldots, p(m)$ is an orthonormal basis of exponential functions or of spherical harmonics in $L^2(S^{n-1})$. We recall that in the polar coordinates $(r, \sigma), \sigma \in S^{n-1}$ the Laplace operator $\Delta = (\partial_r)^2 + r^{-2}\Delta_\sigma + (n-1)r^{-1}\partial_r$, where $\Delta_\sigma$ is the Beltrami operator on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. As known, $e(\sigma; m, p)$ are eigenfunctions of the Beltrami operator. If $n = 2$, then corresponding eigenvalues are $\lambda_m = -(m-1)^2$ and $p(1) = 1, p(m) = 2, m = 2, 3, \ldots$ if $n = 3$ eigenvalues are $\lambda_m = -m(m-1)$ and $p(m) = 2m - 1$. We introduce the low frequency part of $u$
\[ u_l(\sigma) = \sum_{E^2 \leq m^2 < (1 - \varepsilon)k^2, 1 \leq p \leq p(m)} u(m, p)e(\sigma; m, p). \]
Under a constraint on the high frequency component of $u$ we have

**Theorem 1.5.** Let $\theta > 0$.

There are $C, C(\theta)$ such that for a solution to the Cauchy problem (1), (2)
\[ k\|u\|_{(0)}(\Gamma_1) + \|\nabla u\|_{(0)}(\Gamma_1) + \|u\|_{(1)}(\Omega) \leq \]
\[ C(k\|u_0\|_{(0)}(\Gamma) + \|u_1\|_{(0)}(\Gamma) + \|f\|_{(0)}(\Omega)) + C(\theta)k^{-\frac{1}{2} + \theta}\|u - u_l\|_{(2)}(\Omega). \]

Moreover,
\[ \|u\|_{(1)}(\Omega) \leq C\left(k\|u_0\|_{(0)}(\Gamma) + \|u_1\|_{(0)}(\Gamma) + \|f\|_{(0)}(\Omega) + k^{-1}\|u - u_l\|_{(2)}(\Omega)\right), \]

and
\[ \|u\|_{(0)}(\Omega) \leq C\left(\|u_0\|_{(0)}(\Gamma) + k^{-1}(\|u_1\|_{(0)}(\Gamma) + \|f\|_{(0)}(\Omega) + \|u - u_l\|_{(1)}(\Omega))\right). \]

Observe, that Theorem 1.3 implies a (best possible) Lipschitz stability when $u \in H^1(\Omega; k)$, and, since the subspaces $H^1(\Omega; k)$ grow with respect to the wave number $k$ and exhaust the whole $H^1$, Theorem 1.3 can be viewed as an indication of the increasing stability in the Cauchy problem.

To bound the semi norm $\|u\|_{(1,k)}(\Omega)$ we will use a logarithmic stability result for the hyperbolic equation
\[ (A - a_0\partial_t - \partial_x^2)v = f, \text{ in } Q = \Omega \times (-T, T), \]
with the Cauchy data
\[ v = v_0, \partial_x v = v_1 \text{ on } \Gamma \times (-T, T), \]
where $\Gamma$ is an open subset of $\partial \Omega$. Sometimes it is more convenient to replace the Cauchy data (14) by $u$ on $\omega \times (-T, T)$ where $\omega = \Omega \cap \Omega_0$, $\Omega_0$ is a neighbourhood of a point of $\partial \Omega$. One of $\Gamma, \omega$ (but not the both) can be the empty set.

**Theorem 1.6.** Let open $\Omega_0 \subset \Omega$, $\Omega_0 \subset \Omega \cup \Gamma$ and $0 < T_0$.

Then there are positive constants $C, \kappa_1, T$ depending only on $A, a_0, \Omega, \Omega_0, \Gamma$, $T_0$ such that
\[ \|\partial^n v\|_{(0)}(\Omega_0 \times (-T_0, T_0)) \leq \]

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(15) \( C\left( \|v\|_{(1)}(Q) + \|\partial_t \partial^\alpha v\|_{(0)}(Q) \right) \frac{1}{\sqrt{| \ln \varepsilon |}} + C \varepsilon \),

when \( |\alpha| \leq 1, \alpha_0 = 0 \) and

\[
\|v\|_{(1)}(\Omega_0 \times (-T_0, T_0)) \leq C \varepsilon \]

for all \( v \in H^2(\Omega) \) solving (13), (14), where

\[
\varepsilon_1 = \|f_v\|_{(0)}(Q) + \|v_0\|_{L^2((-T,T);H^1(\Gamma_1))} + \|v_1\|_{(0)}(\Gamma \times (-T, T)) + \|v\|_{L^2((-T,T);H^1(\omega))}.
\]

To derive Theorems 1.1, 1.2 from Theorems 1.3, 1.4, and 1.6 we let \( v(x, t) = u(x) e^{ikt} \). Then \( f_v(x, t) = f(x)e^{ikt} \),

\[
\|u\|_{(1)}(\Omega_0) \leq C \sum_{|\alpha| \leq 1, \alpha_0 = 0} \|\partial^\alpha v\|_{(0)}(\Omega_0 \times (-T_0, T_0)),
\]

\[
\|v\|_{(1)}(Q) + \|\partial^\alpha \partial_t v\|_{(0)}(Q) \leq C k \|u\|_{(1)}(\Omega), \quad |\alpha| \leq 1, \alpha_0 = 0,
\]

\[
\|v_0\|_{L^2((-T,T);H^1(\Gamma_1))} \leq C \|u_0\|_{(1)}(\Gamma), \quad \|v_1\|_{(0)}(\Gamma \times (-T, T)) \leq C \|u_1\|_{(0)}(\Gamma),
\]

(17) \( \|v\|_{L^2((-T,T);H^1(\omega))} \leq C \|u\|_{(1)}(\omega), \|v\|_{L^2((-T,T);H^1(\Omega))} \leq C \|u\|_{(1)}(\Omega) \)

and (4), (3) follow from (6), (7), and (8) used for \( \Omega = \Omega_0 \) and (15). To explain it in more detail, we write

\[
k^{-1} |\|u\|_{(1,k)}(\Omega_0)| \leq C k^{-\frac{1}{2}} |\|u\|_{(1,k)}(\Omega_0)| k^{-1} |\|u\|_{(1)}(\Omega_0)| \]

and use that due to the first inequality in (17) and then (15) we have

\[
(k^{-1} |\|u\|_{(1)}(\Omega_0)|)^{\frac{1}{2}} \leq C k^{-1} \sum_{|\alpha| \leq 1, \alpha_0 = 0} |\|\partial^\alpha v\|_{(0)}(\Omega_0 \times (-T_0, T_0))|^{\frac{1}{2}} 
\]

\[
C(k^{-1} |\|v\|_{(1)}(Q)| + \sum_{|\alpha| \leq 1, \alpha_0 = 0} |\|\partial_t \partial^\alpha v\|_{(0)}(Q)|) \frac{1}{\sqrt{| \ln \varepsilon |}} + C \varepsilon \leq C \varepsilon.
\]

again because of the second and the last inequality in (17) and because \( \varepsilon_1 \leq C \varepsilon \) due to the third, fourth and fifth inequalities in (17). To get the last inequality we use that \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \). Thus (4) follows.

Similarly one gets (3).

To illustrate that these results (in particular, (4), (3)) are close to be sharp we recall the famous counterexample of Fritz John [18], [12]. Let \((r, \phi)\) be the polar coordinates in \( \mathbb{R}^2 \). It is shown in [18] that the functions \( u(r, \phi; k) = k^{-\frac{1}{2}} J_k(kr)e^{ik\phi} \) solve the homogeneous Helmholtz equation in the plane,

(18) \( k^{-1} \leq C \|u(k; \phi)\|_{(0)}, \quad \|u(k; \phi)\|_{(1)}(\Omega) \leq C, \quad \|u(k; \phi)\|_{(1)}(\Gamma) + |\partial_t u(k; \phi)\|_{(0)}(\Gamma) \leq C q^k \)

for some \( q, 0 < q < 1 \). Here \( J_k \) is the Bessel function of order \( k \). Moreover, as observed in [18],

\[
J_k(kr) = C_0 \sqrt{k} (r^2 - 1)^{-\frac{1}{2}} \cos \left( -\frac{\pi}{4} + k \sqrt{r^2 - 1 - arccos \frac{1}{r}} \right) + o(k^{-\frac{1}{2}}),
\]
where \( C_0 \) is some constant and \( \lim k^{\frac{1}{2}} a(k^{-\frac{1}{2}}) = 0 \) as \( k \to 0 \) uniformly on \([1.5, 2]\). Hence

\[
\int_{1.5}^{2} J^2_k(kr) r dr =
\]

\[
C_0^2 k^{-1} \int_{1.5}^{2} (r^2 - 1)^{-\frac{1}{4}} \cos^2\left(-\frac{\pi}{4} + k(\sqrt{r^2 - 1} - \arccos \frac{1}{r})\right) r dr + o(k^{-1}) =
\]

\[
C_0^2 k^{-1} \int_{\rho(1.5)}^{\rho(2)} \cos^2\left(-\frac{\pi}{4} + k\rho\right) r(r^2 - 1)^{-\frac{1}{2}} d\rho + o(k^{-1})
\]

where \( \rho(r) = \sqrt{r^2 - 1} - \arccos \frac{1}{r} \). It is easy to see that \( \frac{d\rho}{dr} = \frac{\sqrt{r^2 - 1}}{r} \). Therefore,

\[
\int_{1.5}^{2} J^2_k(kr) r dr \geq C^{-1} k^{-1} \int_{\rho(1.5)}^{\rho(2)} \cos^2\left(-\frac{\pi}{4} + k\rho\right) d\rho =
\]

\[
(Ck)^{-1} (\rho(2) - \rho(1.5)) - (2k)^{-1} (\sin(-\frac{\pi}{2} + 2k\rho(2)) - \sin(-\frac{\pi}{2} + 2k\rho(1.5))) \geq (Ck)^{-1}
\]

for large \( k \). Now from the definition of \( u(\cdot; k) \) we conclude that

\[
\|u(\cdot; k)\|_{(0)}(\Omega_0) \geq C^{-1} k^{-\frac{1}{2}},
\]

where \( \Omega_0 = \{ x : 1.5 < |x| < 2 \} \). (18) and (19) suggest that the bound (3) is nearly sharp (indeed, with the choice \( u = u(\cdot; k) \) the difference of powers of \( k \) on the left and right hand sides is \( \frac{A}{12} \)). It is feasible that precise (but fairly complicated) representations of \( J_k(kr) \) near \( r = 1 \) ([24], p.252) will imply that the bound (3) is sharp.

Now we comment on the further content of the paper.

In section 2 we derive Theorems 1.3, 1.4 by mapping \( \Omega \) onto special domains where one can make use of standard energy estimates for second order equations. For special domains and under additional conditions Theorems 1.3, 1.4 are derived in in [8], [9], [11]. The additional conditions for Theorem 1.3 require that \( \omega \) is not empty and hence exclude results when the data are given only on \( \Gamma \). Since our equations are of the elliptic type these energy estimates have (high order) terms which are not positive. To handle these terms we split \( u \) into “low frequency” and “high frequency” parts. “Low frequency” parts are dominated by positive terms containing \( k^2 \), while “high frequency” parts remain in the right hand sides of (5), (6), (7), (8) and can be viewed as a priori constraints. The special domains are diffeomorphic to the unit ball and are used to introduce semi norms \( ||u||_{(1,k)} \) and (“low frequency”) subspaces \( H^1(\Omega; k) \). In less precise forms of stability estimates (5), (7) these semi norms and subspaces are not needed and estimates can be obtained in general domains (not diffeomorphic to the unit ball) by the conventional partitioning into special domains. In section 3 we consider annular domains where a construction of a “low frequency” part is explicit by using spherical harmonics. Otherwise the arguments are very similar to section 2, we only have to combine them with some basic differentiation and harmonic analysis on the unit sphere. To bound “high frequency” part in (5)-(8) we transform the elliptic equation with the parameter \( k \) into a hyperbolic equation and derive logarithmic stability bounds in the lateral (non hyperbolic) Cauchy problem for this equation. To do so we use the Fourier-Bros-Iagolnitzer (FBI) transform to replace the hyperbolic equation with an elliptic equation without large parameter \( k \) as first proposed by Robbiano [19], [20]. For the resulting elliptic equation in a standard domain we use known Carleman estimates with a simple suitable weight function. The crucial step in the proof is

\[
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\]
the choice of parameters in the FBI transform and the Carleman weight function. Observe that the idea of converting an hyperbolic equation into an elliptic one (and back) was conceived and used already by Hadamard in 1930s. Theorem 1.6 has similarities with results in [20], however a crucial difference is that we do not assume any homogeneous boundary condition on $\partial \Omega \times (-T, T)$ as in [20] and derive a conditional logarithmic stability in a sub domain $\Omega_0$ but not in $\Omega$. As far as we know the closest bound on $\|v\|_{(0)}(\Omega_0 \times (-T_0, T_0))$ is obtained by similar methods in [4], however with smaller power of $|ln \varepsilon|$ and when $v_0 = v_1 = 0$. Also we obtain bounds (15) which are more suitable for applications to the Helmholtz type equations when we need bounds on $\|v\|_{(1)}$, not only on $\|v\|_{(0)}$ as in [4]. In [23] there are (different) logarithmic stability bounds in the whole $\Omega$ for the continuation problem under homogeneous boundary conditions on $\partial \Omega$. Finally, in concluding section 5 we comment on challenging questions and outline possible new directions and links to source and coefficients identification problems.

2. Proofs under “high frequency” energy a priori constraints. In this section we will prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3

Proof. We will first prove (6). Then $\tilde{\Omega}$ is the $C^2$ diffeomorphic image of some standard set $\Omega(1) = \{x : 0 < x_n < 1, |x'| \leq h(x_n)\}$, where $h$ is a function in $C(\mathbb{R})$ and in $C^2((0, 1))$ satisfying the conditions: $h(0) = h(1) = 1, h(\frac{1}{2}) = 2, 0 < h'$ on $(0, \frac{1}{2})$, the graph of $h$ is symmetric with respect to $\{x_n = \frac{1}{2}\}$ and such that its inverse $h_0$ (from $[0, \frac{1}{2}]$ onto $[1, 2]$ is in $C^2([1, 2])$, $h''_0(1) = h_0''(1) = 0$. Moreover, for the corresponding $C^2(\tilde{\Omega}(1))$-diffeomorphism $y(x)$ one has $y(\partial \Omega(1) \cap \{0 < x_n < 1\}) \subset \Gamma$. Since the form of the second order equation (1) (in particular the ellipticity) is invariant under such diffeomorphisms it suffices to give a proof only for $\Omega = \Omega(1)$ and $\Gamma = \partial \Omega(1) \cap \{x_n < 1\}$.

To explain how this diffeomorphism can be constructed we can assume that $\Omega$ is the unit ball $\{y : |y| < 1\}$. Combining the inversion with respect to a sphere $\partial B(x^0; 1)$ centred at $x^0$ which is outside $\Omega$ and is close to $\Gamma$, scaling, and a rotation/translation we obtain a diffeomorphism $X(y)$ and in new $X$-coordinates we can assume that $\Omega = \{X : |X - \frac{1}{2} e(n)| < \frac{1}{2}\}$ and $\partial \Omega \cap \{X_n < \frac{1}{2}\} \subset \Gamma$. Here $e(n) = (0, ..., 0, 1)$. To complete the construction it suffices to “flatten” $\partial \Omega \cap \{\frac{1}{2} < X_n\}$ and a symmetric lower part of $\partial \Omega$ which can be done using polar coordinates.

To introduce a semi norm $\|u\|_{(1, k)}$ we will extend $u$ from $\Omega$ onto $\mathbb{R}^{n-1} \times (0, 1)$ and use the (partial) Fourier transform $\mathcal{F}$ with respect to $x'$.

We recall the uniform ellipticity condition

$$\varepsilon_0 |\xi|^2 \leq \sum_{j,l=1}^n a_{jl}(x) \xi_j \xi_l \leq E_0^2 |\xi|^2$$

for some positive numbers $\varepsilon_0, E_0$ (possibly changed from the original ellipticity constants in the new coordinates) and all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Let $u^*(x', x_n) \in H^1(\mathbb{R}^{n-1})$ be an extension of the function $u(x', x_n)$ with respect to $x'$ from $B(x_n)$ onto $\mathbb{R}^{n-1}$ which admits the bound

$$\|u^*(x', x_n)\|_{(0)(\mathbb{R}^{n-1})} \leq C_\varepsilon \|u(x', x_n)\|_{(0)}(B(x_n)), 0 < x_n < 1.$$  \hspace{1cm} (20)

Since the radii of balls $B(x_n)$ are in $(1, 2)$ the standard extension operators satisfy the bound (20). We introduce low and high frequency projectors.
(21) \[ u_l(x) = \mathcal{F}^{-1} \chi_{k} \mathcal{F} u^*(x), \quad u_k = u - u_l, \]

where \( \mathcal{F} \) is the (partial) Fourier transformation with respect to \( x' = (x_1, \ldots, x_{n-1}, 0) \), \( \chi_k(\xi') = 1 \) when \( C_2^2 \mathcal{E}_0^2 |\xi'|^2 < (1 - \varepsilon_1) k^2 \) for some positive \( \varepsilon_1 \) and \( \chi_k(\xi') = 0 \) otherwise. We let

\[
||u||_{(1,k)}(\Omega) = \left( \sum_{j=1}^{n-1} ||\partial_j u_k||_{L_0^2}(\Omega) \right)^{\frac{1}{2}}
\]

Observe that

\[
a_{mn} (\partial_n^2 u \bar{u} + \partial_n \bar{u} \partial_n u) e^{-\tau x_n} = \\
\partial_n (a_{nn} |\partial_n u|^2 e^{-\tau x_n}) + \tau a_{nn} |\partial_n u|^2 e^{-\tau x_n} - (\partial_n a_{nn}) |\partial_n u|^2 e^{-\tau x_n}, \\
a_{jm} (\partial_j \partial_m u \bar{u} + \partial_j \bar{u} \partial_m u) e^{-\tau x_n} = \\
\partial_j (a_{jm} |\partial_m u|^2 e^{-\tau x_n}) - (\partial_j a_{jm}) |\partial_m u|^2 e^{-\tau x_n}, \quad j = 1, \ldots, n - 1.
\]

Let \( \Omega(\theta) = \Omega \cap \{0 < x_n < \theta\}, \Gamma(\theta) = \Gamma \cap \{x_n < \theta\} \). Integrating by parts with respect to \( x_j \), we yield

\[
\int_{\Omega(\theta)} \sum_{j=1}^{n-1} a_{jm} \partial_j \partial_m u \partial_n \bar{u} e^{-\tau x_n} = \int_{\Gamma(\theta)} \sum_{j=1}^{n-1} a_{jm} \partial_m u \partial_n \bar{u} e^{-\tau x_n} \nu_j d\Gamma - \\
\int_{\Omega(\theta)} \sum_{j=1}^{n-1} a_{jm} \partial_m u \partial_j \partial_n \bar{u} e^{-\tau x_n} - \int_{\Gamma(\theta)} \sum_{j=1}^{n-1} (\partial_j a_{jm}) \partial_m u \partial_n \bar{u} e^{-\tau x_n}.
\]

We have

\[
\sum_{j,m=1}^{n-1} a_{jm} (\partial_m u \partial_j \partial_n \bar{u} + \partial_m \bar{u} \partial_j \partial_n u) e^{-\tau x_n} = \sum_{j,m=1}^{n-1} \partial_n (a_{jm} \partial_m u \partial_j \bar{u} e^{-\tau x_n}) + \\
\tau \sum_{j,m=1}^{n-1} a_{jm} \partial_m u \partial_j \bar{u} e^{-\tau x_n} - \sum_{j,m=1}^{n-1} (\partial_n a_{jm}) \partial_m u \partial_j \bar{u} e^{-\tau x_n},
\]

due to the symmetry of \( a_{jm} \).

To form a standard energy integral we multiply the both sides of (1) by \( \partial_n \bar{u} e^{-\tau x_n} \), add its complex conjugate, and integrate by parts over \( \Omega(\theta), 0 < \theta < 1 \), with using (23), (24), (25), and the notation \( B(\theta) = \{x' : |x'| < h(\theta)\} \) to yield

\[
\int_{B(\theta)} a_{nn} |\partial_n u|^2 e^{-\tau \theta} + \int_{\Gamma(\theta)} a_{nn} |\partial_n u|^2 e^{-\tau x_n} \nu_n d\Gamma + \int_{\Omega(\theta)} a_{nn} |\partial_n u|^2 e^{-\tau x_n} + \\
2 \int_{\Gamma(\theta)} \sum_{j=1}^{n-1} a_{jm} \partial_m u \partial_n \bar{u} e^{-\tau x_n} d\Gamma + \int_{\Gamma(\theta)} \sum_{j=1}^{n-1} a_{jm} (\partial_m u \partial_n \bar{u} + \partial_m \bar{u} \partial_n u) e^{-\tau x_n} \nu_j d\Gamma - \\
\int_{B(\theta)} \sum_{j,m=1}^{n-1} a_{jm} \partial_m \partial_n \bar{u}(\theta) e^{-\tau \theta} - \int_{\Gamma(\theta)} \sum_{j,m=1}^{n-1} a_{jm} \partial_j u \partial_m \bar{u} e^{-\tau x_n} \nu_n d\Gamma - \\
\tau \int_{\Omega(\theta)} \sum_{j,m=1}^{n-1} a_{jm} \partial_j u \partial_m \bar{u} e^{-\tau x_n} + \\
k^2 \int_{B(\theta)} |u|^2 e^{-\tau \theta} + k^2 \int_{\Gamma(\theta)} |u|^2 e^{-\tau x_n} \nu_n d\Gamma + \tau k^2 \int_{\Omega(\theta)} |u|^2 e^{-\tau x_n} + \ldots =
\]
Therefore, using (27) and (28) we obtain

\[ C \int \sum_{j=1}^{n} |\partial_j u|^2 + k^2 |u|^2 e^{-\tau x_n}. \]

To bound the sixth and eighth integrals on the left hand side of (26) we need to split \( u \) into “low” and “high” frequencies parts.

We have

\[ \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u \partial_m \bar{u}(x_n) = \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j (u_l + u_h) \partial_m (\bar{u}_l + \bar{u}_h)(x_n) = \]

\[ \sum_{j,m=1}^{n-1} (a_{jm}(x_n) \partial_j u_l \partial_m \bar{u}_l(x_n) + 2a_{jm}(x_n) \partial_j u_l \partial_m \bar{u}_h(x_n) + \]

\[ a_{jm}(x_n) \partial_j u_h \partial_m \bar{u}_h(x_n)) \leq \]

\[ \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_l \partial_m \bar{u}_l(x_n) + C \delta \sum_{j=1}^{n-1} |\partial_j u_l|^2(x_n) + C \delta^{-1} \sum_{j=1}^{n-1} |\partial_j u_h|^2(x_n), \]

where we used the elementary inequality \( |AB| \leq \frac{1}{2} |A|^2 + \frac{1}{2} |B|^2 \) with \( A = \partial_j u_l, B = \partial_m u_h \) and \( \delta \in (0, 1) \) to be chosen later.

According to the definition of \( E_0 \),

\[ -\sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_l \partial_m \bar{u}_l(x_n) \geq -\int_{B(x_n)} E_0^{n-1} \sum_{j=1}^{n-1} |\partial_j u_l|^2(x_n) \geq \]

\[ -\int_{B(x_n)} E_0^{n-1} \sum_{j=1}^{n-1} |\partial_j u_l|^2(x_n) = -\int_{B(x_n)} E_0^{n-1} \sum_{j=1}^{n-1} \xi_j^2 |\mathcal{F}u_l|^2(x_n) \geq \]

\[ -\int_{B(x_n)} k^2 (1 - \varepsilon_1) C^{-1} |\mathcal{F}u_l|^2(x_n) = -(1 - \varepsilon_1) k^2 C^{-2} \int_{B(x_n)} |u_l|^2(x_n) \geq \]

\[ -(1 - \varepsilon_1) k^2 C^{-2} \int_{B(x_n)} |u_l|^2(x_n), \]

where we utilized the Parseval’s identity and used that \( \mathcal{F}u_l(\xi', x_n) = 0 \) when \(-C_2^2 E_0^{n-1} |\xi'|^2 < -(1 - \varepsilon_1) k^2 \), due to (21), and (20). Similarly,

\[ \int_{B(x_n)} \sum_{j=1}^{n-1} |\partial_j u_l|^2(x_n) \leq C k^2 \int_{B(x_n)} |u_l|^2(x_n). \]

Therefore, using (27) and (28) we obtain

\[ -\sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_l \partial_m \bar{u}_l(x_n) \geq \]

\[ -(1 - \varepsilon_1) k^2 \int_{B(x_n)} |u_l|^2(x_n) - C \delta k^2 \int_{B(x_n)} |u_l|^2(x_n) - \]

\[ C \delta \int_{B(x_n)} \sum_{j=1}^{n-1} |\partial_j u_h|^2(x_n). \]
Hence from (26) and (30) by using the inequalities $2bc \leq b^2 + c^2$ and $\frac{1}{2} < a_{nn}$ (due to the ellipticity of $A$) we conclude that

$$\frac{1}{C} \int_{B(\theta)} |\partial_n u|^2(\theta)e^{-\tau\theta} + \frac{\tau}{C} \int_{\Omega(\theta)} |\partial_n u|^2 e^{-\tau x_n} + (\varepsilon_1 - C\delta)k^2 \int_{B(\theta)} |u|^2(\theta)e^{-\tau\theta} + \tau(\varepsilon_1 - C\delta)k^2 \int_{\Omega(\theta)} |u|^2 e^{-\tau x_n} \leq$$

$$C(\int_{\Gamma} (|\nabla u|^2 + k^2|u|^2) d\Gamma + \int_{\Omega} |f|^2 + \int_{B(\theta)} \frac{1}{\delta} \sum_{j=1}^{n-1} |\partial_j u_k|^2(\theta)e^{-\tau\theta} + \int_{\Omega(\theta)} (\frac{\tau}{\delta} \sum_{j=1}^{n-1} |\partial_j u_k|^2 + \sum_{j=1}^{n} |\partial_j u|^2 + |\partial_n u|^2 + k^2|u|^2)e^{-\tau x_n}).$$

Let $\delta = \frac{1}{2k}$ and use that $u = u_1 + u_h$, then we yield the inequality

$$\int_{B(\theta)} |\partial_n u|^2(\theta)e^{-\tau\theta} + \int_{\Omega(\theta)} |\partial_n u|^2 e^{-\tau x_n} + k^2 \int_{B(\theta)} |u|^2(\theta)e^{-\tau\theta} + \int_{\Omega(\theta)} |u|^2 e^{-\tau x_n} \leq$$

$$C\left(\int_{\Gamma} (|\nabla u|^2 + k^2|u|^2) d\Gamma + \int_{\Omega} |f|^2 + \int_{B(\theta)} \sum_{j=1}^{n-1} |\partial_j u_k|^2(\theta)e^{-\tau\theta} + \int_{\Omega(\theta)} \sum_{j=1}^{n-1} |\partial_j u_k|^2(\theta) + \int_{\Omega(\theta)} \sum_{j=1}^{n-1} |\partial_j u_k|^2\right).$$

Choosing and fixing sufficiently large $\tau$ (depending on the same parameters as $C$) to absorb the three last terms on the right hand side in (31) (with the help of (29)) by the left hand side we obtain

$$\int_{B(\theta)} |\partial_n u|^2(\theta) + \int_{\Omega(\theta)} |\partial_n u|^2 + k^2 \int_{B(\theta)} |u|^2(\theta) + k^2 \int_{\Omega(\theta)} |u|^2 \leq$$

$$C\left(\int_{\Gamma} (|\nabla u|^2 + k^2|u|^2) d\Gamma + \int_{\Omega} |f|^2 + \int_{B(\theta)} \sum_{j=1}^{n-1} |\partial_j u_k|^2(\theta) + \int_{\Omega(\theta)} \sum_{j=1}^{n-1} |\partial_j u_k|^2\right).$$

Integrating the inequality (32) with respect $\theta$ over $(0, 1)$ and dropping the first two terms on the left side, we yield

$$k^2\|u\|_{l_1(\delta)}^2(\Omega) \leq$$

$$C(\|u_1\|_{l_1(\delta)}^2(\Gamma) + \|u_0\|_{l_1(\delta)}^2(\Gamma) + k^2\|u_0\|_{r_1(\delta)}^2(\Gamma) + \|f\|_{l_1(\delta)}^2(\Omega) + \sum_{j=1}^{n-1} \|\partial_j u_k\|_{r_1(\delta)}^2(\Omega)).$$

Recalling the definition of the semi norm (22) and dividing by $k^2$ we obtain (6).

The bound (5) follows from (6) by a suitable partitioning of $\Omega$. We claim that there is a finite covering $\Omega_1, ..., \Omega_j$ of $\Omega$ such that $\Omega_j$ is the $C^2(\Omega(1))$-diffeomorphic image of $\Omega(1)$ and $\partial \Omega_j \cap \Gamma$ is a non empty open (in $\partial \Omega$) subset of $\partial \Omega$.

Indeed, if $x \in \partial \Omega$ there is $x$ which is the $C^2(\Omega(1))$-diffeomorphic image of $\Omega(1)$ such that $\partial \Omega^x \cap \Gamma$ is a non empty open (in $\partial \Omega$) set and $\partial \Omega^x \cap \partial \Omega$ is also open set containing $x$. If $x \in \Omega$ there is $\Omega^x$ which is the $C^2(\Omega(1))$-diffeomorphic image of $\Omega(1)$ such that $\partial \Omega^x \cap \Gamma$ is a non empty open (in $\partial \Omega$) set and $x \in \Omega^x$. $\partial \Omega^x \cap \partial \Omega$ form an open covering of compact set $\partial \Omega$, so there is a finite sub covering $\Omega_1, ..., \Omega_j(1)$. Then
by some $\Omega$. As in the proof of Theorem 1.3, we first will show (8). Observe, that then second order equation (1) (in particular the ellipticity) does not change under such $\partial$ of $x$. Then for some neighbourhood $U$ of $\gamma$ there is a $C^2(U)$-diffeomorphism $X(x)$ such that

$$X(U \cap \Omega) = \{ X : |X'| < \varepsilon_0, g_1(X') < X_n < g_2(X') \}$$

for some $C^2$ functions $g_1, g_2$ and a positive number $\varepsilon_0$. Using the $C^2$-diffeomorphims $(X', (g_2(X') - g_1(X'))^{-1}(X_n - g_1(X'))) \) we transform $X(U \cap \Omega)$ into the standard cylinder $Cy_l = \{ y : |y'| < \varepsilon_0, 0 < y_n < 1 \}$. One can (explicitly) give a subdomain $Cy_l$ of $Cy_l$ which is $C^2$ diffeomorphic to the unit ball with $Cy_l \cap \{ |y'| < 0.5\varepsilon_0 \} = \{ y : |y'| < 0.5\varepsilon_0, 0 < y_n < 1 \}$.

As $\Omega'$ one can take $Cy_l$ in the original coordinates (i.e. the inverse image of $Cy_l$ under the composition of the two described above diffeomorphisms).

Similarly, we can find $\Omega'$ when $x \in \Omega$.

Obviously,

$$\|u\|_{(0)}(\Omega')^2 \leq \|u\|_{(0)}(\Omega_1)^2 + \sum \|u\|_{(0)}(\Omega_i)^2 \leq C(\|u_0\|_{(0)}^2(\Omega) + \|u_0\|_{(0)}^2(\Gamma) + \|u_1\|_{(0)}^2(\Gamma) + \|u_1\|_{(1)}^2(\Omega)$$

since

$$\|u\|_{(0)}(\Omega_i)^2 \leq C(\|u_0\|_{(0)}^2(\Gamma \cap \partial\Omega_i) + \|u_0\|_{(1)}^2(\Gamma \cap \partial\Omega_i) + \|u_1\|_{(0)}^2(\Gamma \cap \partial\Omega_i) + \|u_1\|_{(1)}^2(\Omega_i)) \leq C(\|u_0\|_{(0)}^2(\Gamma) + \|u_0\|_{(1)}^2(\Gamma) + \|u_1\|_{(0)}^2(\Gamma) + \|u_1\|_{(1)}^2(\Omega))$$

due to (6).

The proof is complete.

**Proof of Theorem 1.4**

*Proof.* As in the proof of Theorem 1.3, we first will show (8). Observe, that then $\Omega$ is the image of $\Omega(2) \subset \{ x : 0 < x_n < 1 \}$ under a $C^2(\Omega(2))$-diffeomorphism $y(x)$, so that for some open set $\omega(2) \subset \Omega(2)$ we have $y(\omega(2)) \subset \omega$ and the closure of $\partial\Omega(2) \setminus \omega(2)$ is in the interior (in $\mathbb{R}^{n-1}$) of $\partial\Omega(2) \cap \{ x_n = 1 \}$. Essentially, $\omega(2) = \Omega\setminus \Omega(2)$ where $\Omega$ is a neighbourhood of the closure of $\partial\Omega(2) \cap \{ x_n < 1 \}$. It follows that $dist(\Omega(2) \setminus \omega(2), \{ 0 < x_n < 1 \} \setminus \Omega(2)) > 0$. One can choose $\Omega(2)$ to be invariant with respect to rotations around the $x_n$-axis. Since the form of the second order equation (1) (in particular the ellipticity) does not change under such diffeomorphisms it suffices to give a proof only for $\Omega = \Omega(2)$ and $\omega = \omega(2)$.

We comment on a possible choice of such a diffeomorphism. We can assume that $\Omega$ is the ball centred at $(0, \ldots, 0, \frac{1+\delta}{2})$ of the radius $\frac{1-\delta}{2}$ and $\omega$ contains $\Omega \cap B_1$ where $B_1$ is some ball centred at $(0, 0, \ldots, \delta)$. Choosing small $\delta$ we can achieve that the image of $\Omega$ under the inversion $|x|^{-2}x$, a translation in the $x_n$-direction and some scaling is the unit ball $B$ and the closure of the image of $\omega$ contains $\partial B \cap \{ x_n < 0.5 \}$. Now we use the same notation $x$ for variables after a transformation and “flattening” the part of $\partial B$ outside the image of $\omega$. To do so we can use the map $x^*(x) = (x', (1 - \chi_n(x_n)\chi_1(|x'|))x_n + \chi_n(x_n)\chi_1(|x'|)|x|)$,

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where \( x' = (x_1, \ldots, x_{n-1}) \), \( \chi_n \in C^\infty(\mathbb{R}), \chi_n(x_n) = 1 \) when \( 0.5 < x_n < 1 \), \( \chi_n(x_n) = 0 \) when \( x_n < 0 \) or \( x_n > 1 \). To define the function \( \chi_1 \) we pick up positive numbers \( \delta_1, \delta_2, \delta_3, \) then \( \delta_1 < \delta_2 < \delta_3 < 1 \) and let \( \chi_1 \in C^\infty(\mathbb{R}), \chi_1(r) = 1 \) when \( r < \delta_1 \), \( \chi_1(r) = 0 \) when \( \delta_1 < r \) and \( 0 \leq \chi_1 \leq 1 \). Then

\[
\frac{\partial x_n^s}{\partial x_n} = 1 - \chi_1 \chi_n + \chi_1 \chi_n |x|^{-1} x_n + \chi_1 \chi_n^r(|x| - x_n).
\]

Considering the cases when \( 0.5 < x_n \) (then \( \frac{\partial x_n^s}{\partial x_n} = 1 - \chi_1 + \chi_1 |x|^{-1} x_n \)), when \( 0.3 \leq x_n \leq 0.5 \) and when \( x_n < 0.3 \) (then \( \frac{\partial x_n^s}{\partial x_n} = 1 \)), we conclude that \( 0 < \frac{\partial x_n^s}{\partial x_n} \) on \( \{ x : |x| \leq 1 \} \), and hence \( x^*(x) \) is a \( C^\infty \)-diffeomorphism of the closed unit ball transforming the part of \( \partial B \) outside \( \omega \) into the hyperplane \( \{ x_n^* = 1 \} \). Obviously we have the rotational invariance around the \( x_n \)-axis.

Let \( \chi \) be a \( C^2(\mathbb{R}^n) \) cut off function, such that \( \chi = 1 \) on \( \Omega \setminus \omega \), \( \chi = 0 \) on \( (\mathbb{R}^{n-1} \times (0,1)) \setminus \Omega \). Let \( u_* = \chi u \).

To introduce a semi norm \( |||u|||(1,k) \) in the case of Theorem 1.4 we will extend \( u_* \) from \( \Omega \) onto \( (\mathbb{R}^{n-1} \times (0,1)) \setminus \Omega \) as zero and use the (partial) Fourier transform \( \mathcal{F} \) with respect to \( x' \). We introduce low and high frequency projectors

\[
(33) \quad u_{s\ell}(x) = \mathcal{F}^{-1} \chi_k \mathcal{F} u_*(x), \quad u_{s\ell} = u_* - u_{s\ell},
\]

where \( \chi_k(\xi') = 1 \) when \( E_k^2|\xi'|^2 < (1 - \varepsilon_1)k^2 \) for some positive \( \varepsilon_1 \) and \( \chi_k(\xi') = 0 \) otherwise. Now we let

\[
(34) \quad |||u|||(1,k)(\Omega) = \left( \sum_{j=1}^{n-1} |||\partial_j u_k|||_{L^2(0)}^2 \right)^{\frac{1}{2}}.
\]

From (1) by using the Leibniz formula we yield

\[
(35) \quad (\sum_{j,m=1}^n a_{jm} \partial_j \partial_m + \sum_{j=1}^n a_j \partial_j + a - ika_0 + k^2)u_* = f_*,
\]

where

\[
f_* = \chi f + 2 \sum_{j,m=1}^n a_{jm} \partial_j \chi \partial_m u + (\sum_{j,m=1}^n a_{jm} \partial_j \partial_m \chi + \sum_{j=1}^n a_j \partial_j \chi)u.
\]

To form an energy integral we multiply the both sides of (35) by \( \partial_\ell u_* e^{-\tau x_n} \), add its complex conjugate, and integrate by parts over \( \{ 0 < x_n < \theta \} \) with using (23), (24), (25), and as in the proof of Theorem 1.3 we yield

\[
\begin{align*}
&\int_{\mathbb{R}^{n-1}} a_{mn} |\partial_n u_*|^2(\ell,\theta)e^{-\tau \theta} + \tau \int_{\mathbb{R}^{n-1} \times (0,\theta)} a_{nn} |\partial_n u_*|^2 e^{-\tau x_n} - \\
&\int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^n a_{jm} \partial_j u_* \partial_m \bar{u}_*(\ell,\theta)e^{-\tau \theta} - \tau \int_{\mathbb{R}^{n-1} \times (0,\theta)} \sum_{j,m=1}^n a_{jm} \partial_j u_* \partial_m \bar{u}_* e^{-\tau x_n} + \\
&k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(\ell,\theta)e^{-\tau \theta} + \tau k^2 \int_{\mathbb{R}^{n-1} \times (0,\theta)} |u_*|^2 e^{-\tau x_n} + \ldots = \\
&\left( \int_{\mathbb{R}^{n-1} \times (0,\theta)} (\partial_\ell \bar{u}_* f_* + \partial_n u_* f_*)e^{-\tau x_n} ,
\right)
\end{align*}
\]

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From (37), (38) and (39) we obtain

\[ C \int_{\mathbb{R}^{n-1} \times (0,1)} \left( \sum_{j=1}^{n} |\partial_j u_*|^2 + k^2 |u_*|^2 \right) e^{-\tau x_n}. \]

As in the proof of Theorem 1.3 we split \( u_* \) into “low” and “high” frequencies.

As in (27),

\[ \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_* \partial_m \bar{u}_*(x_n) \leq \]

(37)

\[ \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_* \partial_m \bar{u}_*(x_n) + C\delta \sum_{j=1}^{n-1} |\partial_j u_*|^2(x_n) + C\delta^{-1} \sum_{j=1}^{n-1} |\partial_j u_*|^2(x_n). \]

By using (33), similarly to (28), (29) we have

(38)

\[ - \int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j u_* \partial_m \bar{u}_*(x_n) \geq -(1 - \varepsilon_1)k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(x_n), \]

(39)

\[ \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} |\partial_j u_*|^2 \leq Ck^2 \int_{\mathbb{R}^{n-1}} |u_*|^2. \]

From (37), (38) and (39) we obtain

(40)

\[ - (1 - \varepsilon_1)k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(x_n) - C\delta k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(x_n) - \frac{C}{\delta} \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} |\partial_j u_*|^2(x_n). \]

From (36) and (40) we conclude that

\[ \frac{1}{C} \int_{\mathbb{R}^{n-1}} |\partial_n u_*|^2(\theta)e^{-\tau \theta} + \frac{\tau}{C} \int_{\mathbb{R}^{n-1} \times (0,\theta)} |\partial_n u_*|^2 e^{-\tau x_n} + \]

\[ (\varepsilon_1 - C\delta)k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(\theta)e^{-\tau \theta} + \tau(\varepsilon_1 - C\delta)k^2 \int_{\mathbb{R}^{n-1} \times (0,\theta)} |u_*|^2 e^{-\tau x_n} \leq \]

\[ C(\int_{\Omega} |f_*|^2 e^{-\tau x_n} + \int_{\mathbb{R}^{n-1} \times (0,\theta)} \frac{1}{\delta} \sum_{j=1}^{n-1} |\partial_j u_*|^2(\theta)e^{-\tau +} \]

\[ \int_{\mathbb{R}^{n-1} \times (0,\theta)} \frac{1}{\delta} \sum_{j=1}^{n-1} |\partial_j u_*|^2 + |\partial_n u_*|^2 + k^2 |u_*|^2) e^{-\tau x_n}). \]

Let \( \delta = \frac{\varepsilon_1}{2k} \), then we yield the inequality

\[ \int_{\mathbb{R}^{n-1}} |\partial_n u_*|^2(\theta)e^{-\tau \theta} + \tau \int_{\mathbb{R}^{n-1} \times (0,\theta)} |\partial_n u_*|^2 e^{-\tau x_n} + \]

\[ k^2 \int_{\mathbb{R}^{n-1}} |u_*|^2(\theta)e^{-\tau \theta} + \tau k^2 \int_{\mathbb{R}^{n-1} \times (0,1)} |u_*|^2 e^{-\tau x_n} \leq \]
two terms on the left hand side, and recalling that $u$ates one can see that

$\Omega$ 

where $e$ of $\Omega(2)$ as in the proof of Theorem 1.3.

$(44)$ 

parts formula on 

vector field $V$ gradient onto the tangent space. We also will use the tangential divergence of the $a$ where

$(42)$ 

Choosing and fixing sufficiently large $\tau$ (depending on the same parameters as $C$) to absorb the two last terms on the right hand side in $(41)$ by the left hand side we obtain

\[
\int_{\mathbb{R}^{n-1}} |\partial_1 u_\ast|^2 + |\partial_n u_\ast|^2 + k^2 \int_{\mathbb{R}^{n-1}} |u_\ast|^2 \leq
\]

\[
C(\int_{\Omega} |f|^2 e^{-\tau x_\ast} + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} |\partial_j u_\ast|^2 + k^2 |u_\ast|^2 e^{-\tau x_\ast}.
\]

Integrating the inequality (42) with respect to $\theta$ over $(0, 1)$, dropping the first two terms on the left hand side, and recalling that $u_\ast = \chi u$ we yield

\[
k^2 \|u\|_{L^2(\Omega)}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n-1} \|\partial_j u_\ast\|_{L^2(\Omega)}^2((\mathbb{R}^{n-1} \times (0, 1)).
\]

Recalling the definition (34) of the semi norm $\|\cdot\|_{(1, k)}$, using that $u = v$ on $\Omega \setminus \omega$, and dividing by $k^2$ we obtain (8).

(7) follows from (8) by partitioning $\Omega$ into the union of $C^2$ diffeomorphic images of $\Omega(2)$ as in the proof of Theorem 1.3.

3. Proof for annular domains. In this section we will prove Theorem 1.5. We will use polar coordinates $(r, \sigma), \sigma \in S^{n-1}$, and the operator $A$ in these coordinates:

$(43)$ 

$Au = a^{nn} \partial_r^2 u + A_{1,\sigma} \partial_r u + A_{2,\sigma} u$

where $A_{1,\sigma}$ is a $j$-th order linear partial differential operator on $S^{n-1}$.

Let $\nabla_{\sigma}$ be the tangential gradient on $S^{n-1}$, i.e. the orthogonal projection of the gradient onto the tangent space. We also will use the tangential divergence of the vector field $V = (V_1, \ldots, V_n)$ defined as

\[
\nabla_{\sigma} \cdot V = \sum_{j=1}^{n} \nabla_{\sigma} V_j \cdot e(j),
\]

where $e(1), \ldots, e(n)$ is the standard orthonormal base in $\mathbb{R}^n$. By using local coordinates one can see that

\[
A_{2,\sigma} u = r^{-2} \nabla_{\sigma} \cdot (a \nabla_{\sigma} u) + \ldots,
\]

where $a$ is the matrix $(a_{jk})$ of the partial differential operator $A$ and ... are terms with the absolute value bounded by $C(|\nabla u| + |u|)$. We recall the integration by parts formula on $S^{n-1}$:

$(44)$ 

\[
\int_{S^{n-1}} g \nabla_{\sigma} \cdot V dS = - \int_{S^{n-1}} V \cdot \nabla_{\sigma} g dS
\]

for a vector field $V$ and a function $g$ on $S^{n-1}$.
Proof. From (1), (43) we yield

\begin{equation}
\tag{45}
 a^{nn}\partial_t^2 u + A_{1,\sigma}\partial_t u + A_{2,\sigma} u - ika_0 u + k^2 u = f \text{ in } S^{n-1} \times (1, R).
\end{equation}

By using (44) we have

\begin{equation*}
\int_{S^{n-1}} (A_{2,\sigma} u \partial_t \tilde{u} + A_{2,\sigma} \tilde{u} \partial_t u) dS =
\end{equation*}

\begin{equation*}
\int_{S^{n-1}} r^{-2}((\nabla_\sigma \cdot (a \nabla_\sigma u))\partial_t \tilde{u} + (\nabla_\sigma \cdot (a \nabla_\sigma \tilde{u}))\partial_t u + \ldots) dS =
\end{equation*}

\begin{equation*}
- \int_{S^{n-1}} r^{-2}((a \nabla_\sigma u) \cdot \nabla_\sigma \partial_t \tilde{u} + (a \nabla_\sigma \tilde{u}) \cdot \nabla_\sigma \partial_t u + \ldots) dS =
\end{equation*}

\begin{equation*}
- \int_{S^{n-1}} r^{-2}((a \nabla_\sigma u) \cdot \nabla_\sigma \partial_t \tilde{u} + (a \nabla_\sigma \tilde{u}) \cdot \nabla_\sigma \partial_t u + \ldots) dS =
\end{equation*}

\begin{equation}
\tag{46}
- \int_{S^{n-1}} r^{-2}(\partial_t(a \nabla_\sigma \cdot \nabla_\sigma \tilde{u}) + \ldots) dS,
\end{equation}

Repeating the argument from the proof of Theorem 1.3 (multiplying the both parts of (45) by \(\partial_t \tilde{u} e^{-\tau r}\), adding its complex conjugate, and integrating by parts over \(S^{n-1} \times (1, \rho), 1 < \rho \leq R\), we will have

\begin{equation*}
\int_{S^{n-1}} a^{nn} |\partial_t u|^2(\rho) e^{-\tau r} \rho^{n-1} dS - \int_{S^{n-1}} a^{nn} |\partial_t u|^2(1) e^{-\tau r} dS +
\end{equation*}

\begin{equation*}
\tau \int_{S^{n-1} \times (1, \rho)} a^{nn} |\partial_t u|^2 e^{-\tau r} \rho^{n-1} dr dS -
\end{equation*}

\begin{equation*}
\int_{S^{n-1}} a \nabla_\sigma u \cdot \nabla_\sigma \tilde{u}(\rho) e^{-\tau r} \rho^{n-3} dS + \int_{S^{n-1}} a \nabla_\sigma u \cdot \nabla_\sigma \tilde{u}(1) e^{-\tau r} dS -
\end{equation*}

\begin{equation*}
\tau \int_{S^{n-1} \times (1, \rho)} a \nabla_\sigma u \cdot \nabla_\sigma \tilde{u} e^{-\tau r} \rho^{n-3} dr dS +
\end{equation*}

\begin{equation*}
k^2 \int_{S^{n-1}} |u|^2(\rho) e^{-\tau r} \rho^{n-1} dS - k^2 \int_{S^{n-1}} |u|^2(1) dS +
\end{equation*}

\begin{equation*}
\tau k^2 \int_{S^{n-1} \times (1, \rho)} |u|^2 e^{-\tau r} \rho^{n-1} dr dS + \ldots =
\end{equation*}

\begin{equation}
\tag{47}
\int_{S^{n-1} \times (1, \rho)} (\partial_t \tilde{u} f + \partial_t u \bar{f}) e^{-\tau r} \rho^{n-1} dr dS,
\end{equation}

where \(\ldots\) denotes the sum of terms bounded by

\begin{equation*}
C \int_{\Omega} (|\nabla u|^2 + k^2 |u|^2) e^{-\tau r}.
\end{equation*}

To handle the fourth and sixth terms on the left hand side of (47) we use that

\begin{equation*}
- \int_{S^{n-1}} a \nabla_\sigma u \cdot \nabla_\sigma \tilde{u}(r) dS \geq
\end{equation*}

\begin{equation*}
- \int_{S^{n-1}} a \nabla_\sigma u l \cdot \nabla_\sigma \tilde{u}(r) dS - \delta \int_{S^{n-1}} |\nabla_\sigma u l|^2 r dS - \frac{C}{\delta} \int_{S^{n-1}} |\nabla_\sigma u_k|^2(r) dS.
\end{equation*}

As in the proof of Theorem 1.3, using (9) and that \(e(\sigma; m, p)\) are eigenfunctions of the Beltrami operator we yield

\begin{equation*}
- r^{-2} \int_{S^{n-1}} a \nabla_\sigma u l \cdot \nabla_\sigma \tilde{u}(r) dS \geq -E_0^2 \int_{S^{n-1}} |\nabla_\sigma u l|^2(r) dS =
\end{equation*}
\[ E_0^2 \int_{S^{n-1}} \Delta_\sigma u \bar{u} (r) dS \geq - (1 - \varepsilon) k^2 \int_{S^{n-1}} |u|^2 (r) dS \geq - (1 - \varepsilon) k^2 \int_{S^{n-1}} |u|^2 (r) dS, \]

when \( 1 < r \) and similarly

\[ - \int_{S^{n-1}} |\nabla_\sigma u|^2 (r) \geq - C k^2 \int_{S^{n-1}} |u|^2 (r) dS, \quad 1 < r. \]

Hence from (47) we conclude that

\[ \int_{S^{n-1}} a^{nn} |\partial_\rho u|^2 (\rho) e^{-\tau \rho} \rho^{n-1} dS + \tau \int_{S^{n-1} \times (1, R)} a^{nn} |\partial_{\rho \rho} u|^2 e^{-\tau \rho} \rho^{n-1} dS \]

\[ (\varepsilon - C \delta) k^2 \int_{S^{n-1}} |u|^2 (\rho) e^{-\tau \rho} \rho^{n-1} dS + \tau (\varepsilon - C \delta) k^2 \int_{S^{n-1} \times (1, R)} \rho^{n-1} dS \leq \]

\[ C \left( \int_{S^{n-1}} (|\partial_\rho u|^2 (1) + k^2 |u|^2 (1)) dS + \int_{S^{n-1} \times (1, R)} |f|^2 e^{-\tau \rho} \rho^{n-1} dS + \right) \]

\[ \frac{C}{\delta} \left( \int_{S^{n-1}} (|\nabla_\sigma u|^2 (\rho) e^{-\tau \rho} \rho^{n-1} dS + \right) \]

\[ \tau \int_{S^{n-1} \times (1, R)} (|\partial_\rho u|^2 + |\nabla_\sigma u|^2) e^{-\tau \rho} \rho^{n-1} dS + \int_{S^{n-1} \times (1, R)} (|\nabla_\sigma u|^2 + k^2 |u|^2) e^{-\tau \rho} \rho^{n-1} dS). \]

Choosing \( \delta = \frac{\varepsilon}{2C} \) and using that \( \frac{1}{C} < a^{nn} \) we yield

\[ \int_{S^{n-1}} (|\partial_\rho u|^2 (\rho) + k^2 |u|^2 (\rho)) e^{-\tau \rho} \rho^{n-1} dS + \]

\[ \tau \int_{S^{n-1} \times (1, R)} (|\partial_\rho u|^2 + k^2 |u|^2) e^{-\tau \rho} \rho^{n-1} dS \leq \]

\[ C \left( \int_{S^{n-1}} (|\partial_\rho u|^2 (1) + k^2 |u|^2 (1)) dS + \int_{S^{n-1} \times (1, R)} |f|^2 e^{-\tau \rho} \rho^{n-1} dS + \int_{S^{n-1}} |\nabla_\sigma u|^2 (\rho) e^{-\tau \rho} \rho^{n-1} dS + \tau \int_{S^{n-1} \times (1, R)} |\nabla_\sigma u|^2 e^{-\tau \rho} \rho^{n-1} dS + \right) \]

\[ \int_{S^{n-1} \times (1, R)} (|\nabla u|^2 + k^2 |u|^2) e^{-\tau \rho} \rho^{n-1} dS). \]

Since \( u = u_l + u_h \), from the triangle inequality we have

\[ \int_{S^{n-1} \times (1, R)} |\nabla_\sigma u|^2 e^{-\tau \rho} \rho^{n-1} dS \leq \]

\[ 2 \int_{S^{n-1} \times (1, R)} (|\nabla_\sigma u_l|^2 + |\nabla_\sigma u_h|^2) e^{-\tau \rho} \rho^{n-1} dS \leq \]

\[ C k^2 \int_{S^{n-1} \times (1, R)} |u|^2 + \int_{S^{n-1} \times (1, R)} |\nabla_\sigma u_h|^2 e^{-\tau \rho} \rho^{n-1} dS, \]

when we apply (48). So from (49) we obtain

\[ \int_{S^{n-1}} (|\partial_\rho u|^2 (\rho) + k^2 |u|^2 (\rho)) e^{-\tau \rho} \rho^{n-1} dS + \]

\[ \tau \int_{S^{n-1} \times (1, R)} (|\partial_\rho u|^2 + k^2 |u|^2) e^{-\tau \rho} \rho^{n-1} dS \leq \]
brevity we return to the old notation replacing (50)

It suffices to consider two cases: a) 

Proof.

4. Proof for hyperbolic equations. In this section we will prove Theorem 1.6.

Proof. By using compactness of \( \Omega_0 \) we can cover this set by finitely many \( \Omega_{0;j} \), which are the images of an open set \( \Omega_0 = \{ \frac{1}{2} < y_n < 1, |y| < 1 \} \) under \( C^2 \)-diffeomorphisms \( x(j) \) of the semi ball \( B^+ = \{ y : |y| \leq 1, 0 \leq y_n \} \) with \( x(j)(\overline{B^+}) \subset \Omega_0 \cap \Gamma \cap (V \cap \partial \Omega) \).

It suffices to consider two cases: a) \( x(j)(\omega_0) \subset \omega \), where \( \omega_0 = \{ y : 1 - \delta < |y| < 1 \} \cap \{ 0 < y_n \} \) with some positive \( \delta \) and b) \( x(j)(\partial B^+ \cap \{ 0 < y_n < 1 \}) \subset \Gamma \). For brevity we return to the old notation replacing \( y \) by \( x \), so it is sufficient to prove Theorem 1.6 when \( \Omega = B^+, \Omega_0 = B^+ \cap \{ \frac{1}{2} < x_n \} \), \( \omega = \omega_0 \) or \( \Gamma = \partial B^+ \cap \{ 0 < x_n < 1 \} \).

First we handle the case a).

Let \( \chi_0 \) be a \( C^\infty(\mathbb{R}^{n+1}) \) function, \( \chi_0 = 1 \) on \( (\Omega \setminus \omega) \times \mathbb{R} \), \( \chi_0 = 0 \) on \( \{ 0 < x_n < 1 \} \setminus \Omega \times \mathbb{R} \), \( 0 \leq \chi_0 \leq 1 \), and \( \partial_t \chi_0 = 0 \). Introducing

\[
w = \chi_0 v,\]

from (13) we will have

\[
(A - a_0 \partial_t - \partial_t^2)w = f_0, \quad \text{in } Q = \Omega \times (-T,T),
\]

where \( f_0 = \chi_0 f + 2 \sum_{j,m=1}^n a_{jm} \partial_j \chi_0 \partial_m v + (A \chi_0 - a \chi_0)v.\)
Let \( \chi_T(t) \) be a \( C^\infty(\mathbb{R}) \) function, \( \chi_T(t) = 1 \) when \( |t| < T - 1 \), \( 0 \leq \chi_T(t) \leq 1 \), and \( \chi_T(t) = 0 \) when \( T < |t| < +\infty \). As first suggested in [19], we will make use of the FBI transform

\[
W(x, s; \lambda, t_1) = \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} e^{-\frac{t}{\lambda}(is + t_1 - t)} \chi_T(t) w(x, t) dt
\]

of a function \( w(x, t) \).

Let \( Q^s = \Omega \times (-1, 1) \) be the domain in the \((x, s)\)-space. Using the time independence of the coefficients of \( A \), applying the FBI transform to the both sides of (54) and integrating by parts we obtain

\[
AW + a_0 i \partial_s W + \partial_s^2 W = F_0 - F(\cdot; T) \text{ on } Q^s,
\]

where \( F_0 \) is the FBI transform of \( f_0 \) (as in (55) for \( w \)),

\[
F(x, s; T) = \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} e^{-\frac{t}{\lambda}(is + t_1 - t)} (a_0 \partial_s \chi_T(t) w(x, t) + 2 \partial_t \chi_T(t) \partial_t w(x, t) + \partial_t^2 \chi_T(t) w(x, t)) dt.
\]

Observe that, as in [21],

\[
\|W\|_{L^2(Q)}^2 \leq C e^\lambda \|w\|_{L^2(Q)}^2,
\]

(57)

\[
\|\partial_j W\|_{L^2(Q)}^2 \leq C e^\lambda \|\partial_j w\|_{L^2(Q)}^2 + (1 + \lambda) \|w\|_{L^2(Q)}^2,
\]

\[
\|\partial_t W\|_{L^2(Q)}^2 \leq C e^\lambda \|w\|_{L^2(Q)}^2,
\]

\[
\|F(\cdot; T)\|_{L^2(Q)}^2 \leq C e^{\lambda(T - 1 - T_1)} \|w\|_{L^2(Q)}^2,
\]

provided \(|t_1| < T_1 < T - 1\), where \( T_1 \) is to be chosen later. Here we let \( \partial_0 = \partial_s \) in the \((x, s)\) space and \( \partial_0 = \partial_t \) in the \((x, t)\)-space. Observe that to get the last bound (57) we integrated by parts in the integral defining \( F(\cdot; T) \) to eliminate \( \partial_t w \). Similar bounds hold when we replace \( \omega \) by \( \Gamma \).

We will use the Carleman weight function

\[
\varphi(x, s) = e^{\gamma(x_n - s^2)} - 1.
\]

As known [12], section 3.2, there is \( \gamma \) (depending only on \( A, a_0, \Omega \)) such that the following Carleman type estimate holds

\[
\int_{Q^s} \tau^{3 - 2|\alpha|} \|\partial^\alpha W\|_{L^2}^2 e^{2\tau \varphi} \leq C \tau^{3 - 2|\alpha|} \|\partial^\alpha W\|_{L^2}^2 e^{2\tau \varphi} + \int_{\partial Q^s} (\tau^3 |W|^2 + \tau |\nabla_{x,s} W|^2) e^{2\tau \varphi},
\]

(59)

provided \(|\alpha| \leq 1, C < \tau\).

We will denote \( Q^s(\delta) = Q^s \cap \{ \delta < \varphi \} \) and use another cut off function \( \chi(\cdot; \delta) = 0 \) on \( Q^s \setminus Q^s(\frac{\delta}{2}) \) and \( \chi(\cdot; \delta) = 1 \) on \( Q^s(\delta) \) with \( |\partial^\alpha \chi(\cdot; \delta)| \leq C \delta^{-|\alpha|}, |\alpha| \leq 2. \) Observe that \( \chi(\cdot; \delta) = 0 \) near \( \varphi = -1 \) and \( \{x_n = 0\} \).

Since

\[
(A + a_0 i \partial_s + \partial_s^2)(\chi W) = \chi(A + a_0 i \partial_s + \partial_s^2) W + 2 \partial_s \chi \partial_s W + \sum_{j,k=1}^n a^{jk} \partial_j \chi \partial_k W + ((A - a + a_0 i \partial_s + \partial_s^2) \chi) W,
\]

we have

\[
\int_{Q^s(\delta)} \tau^{3 - 2|\alpha|} \|\partial^\alpha (\chi W)\|_{L^2}^2 e^{2\tau \varphi} \leq C \tau^{3 - 2|\alpha|} \|\partial^\alpha (\chi W)\|_{L^2}^2 e^{2\tau \varphi} + \int_{\partial Q^s(\delta)} (\tau^3 |W|^2 + \tau |\nabla_{x,s} W|^2) e^{2\tau \varphi},
\]

provided \(|\alpha| \leq 1, C < \tau\).
using $\chi(\delta_1)W$ instead of $W$ in (59), (54), and (56) we yield
\[
\int_{Q^*}(2\delta_1) \tau^{3-2\alpha}|\partial^\alpha W|^2 e^{2\tau\varphi} \leq C\left(\int_{Q^*} |F|^2 e^{2\tau\varphi} + \int_{\omega^*}(|V|^2 + |\nabla_x V|^2) e^{2\tau\varphi} + \int_{Q^*} |F(\tau)|^2 e^{2\tau\varphi} + \int_{Q^* \backslash Q^*{(2\delta_1)}}(|W|^2 + |\nabla_x W|^2) e^{2\tau\varphi}, |\alpha| \leq 1.\right)
\]
Here $V$ is the FBI transform (55) of $v$, $\omega^* = \omega \times (-1, 1)$ and $\delta_1$ is a positive number to be chosen later, after (64). Observe that here we used that due to (54) and the choice of $\chi_0$ before it we have $|F_0|^2 \leq C(|F|^2 + |V|^2 + |\nabla V|^2)$ on $\omega^*$ and $F_0 = F$ on $Q^* \backslash \omega^*$.

Let $\Phi = \sup \varphi$ over $Q^*$. Since $\varphi \leq \Phi$ on $Q^*$ and $\delta < \varphi$ on $Q^*(\delta)$, it follows that
\[
e^{4\tau\delta_1} \int_{Q^*{(2\delta_1)}} \tau^{3-2\alpha}|\partial^\alpha W|^2 \leq C(e^{2\tau\Phi}) (\|F\|_0(\Phi^*) + \|V\|_0^2(\omega^*) + \|\nabla_x V\|_0^2(\omega^*) + e^{2\tau\Phi}\|F(\tau)\|_0^2(\Phi^*) + e^{2\tau\delta} \|W\|_0^2(\Phi^*)).
\]
Using the bounds (57) and recalling that
\[
\varepsilon_1 = \|f\|_0(Q) + \|v_0\|_{L^2((-T,T);H^1(\Gamma))} + \|v_1\|_0(\Gamma_0 \times (-T,T)) + \|v\|_{L^2((-T,T);H^1(\omega))}
\]
we obtain
\[
\int_{Q^*{(2\delta_1)}} |\partial^\alpha W(\lambda, t_2)|^2 \leq C(e^{2\tau(\Phi+1)}) e^{2\lambda \varepsilon_1^2 + 2\lambda e^{-\lambda(T-1-T_1)^2}\|y\|_0^2(Q) + e^{-2\tau\delta} \|x\|_{L^2((-T,T);H^{1}(\omega))},
\]
provided $|t_2| < T_1 < T - 1$.

From the mean value bounds for the complex analytic function $(\partial^\alpha W(x, s; \lambda, t_2))^2$ of $z = t_2 + is$ we have
\[
|\partial^\alpha W(x, 0; \lambda, t_1)|^2 \leq \frac{4}{\pi} \int_{\{(t_2-t_1)+is|<\frac{1}{4}\}} |\partial^\alpha W(x, s; \lambda, t_2)|^2 dsdt_2 \leq \frac{4}{\pi} \int_{\{|t_2-t_1|<\frac{1}{4},|s|<\frac{1}{4}\}} |\partial^\alpha W(x, s; \lambda, t_2)|^2 dsdt_2,
\]
when we assume that $|t_1| < T_1 - \frac{1}{2}$.

Now we make a crucial choice of
\[
(62) \quad \tau = -\mu \ln \varepsilon_1, \quad \lambda = -\beta \mu \ln \varepsilon_1, \varepsilon_1 < 1,
\]
where positive $\beta, \mu, T, T_1, \kappa_1$ (depending only on $\Omega, T_0, A, a_0$) are selected so that
\[
(63) \quad \kappa_1 + \beta < \delta_1, \quad \kappa_1 + \beta((\Phi + 1) + \beta) < 2(\kappa_1 + \Phi) < \beta((T-1-T_1)^2 - 2).
\]
Due to this choice,
\[
e^{2\tau(\Phi+1)+2\lambda} \varepsilon_1^2 \leq e^{-2\mu(\Phi+1)\ln \varepsilon_1 - 2\beta \mu \ln \varepsilon_1 + 2\ln \varepsilon_1} \leq e^{2\kappa_1 \ln \varepsilon_1} = \varepsilon_1^2 \varepsilon_1^2,
\]
\[
e^{-2\tau(\Phi+1)+2\lambda} \varepsilon_1^2 \leq e^{2\mu(\Phi+1)\ln \varepsilon_1} \leq e^{2\kappa_1 \ln \varepsilon_1} = \varepsilon_1^2 \varepsilon_1^2,
\]
\[
e^{2\tau(\Phi+1)+2\alpha} \varepsilon_1^2 \leq e^{2\mu(\Phi+1)\ln \varepsilon_1} \leq e^{2\kappa_1 \ln \varepsilon_1} \leq e^{2\kappa_1 \ln \varepsilon_1} = \varepsilon_1^2 \varepsilon_1^2,
\]
so from (60) we yield
\[
(64) \quad \int_{\{\frac{1}{4}<\varepsilon_1,|s|<\frac{1}{4}\}} |\partial^\alpha W(\lambda, t_2)|^2 \leq C(\varepsilon_1^{2\kappa_1} + \varepsilon_1^{2\kappa_1} \|v\|_0^2(\Omega))
\]
where we assumed that $|t_2| < T_1 < T - 1$, let $\frac{T}{2} = 2^{\frac{ln(1+2\delta_1)}{\gamma}}$ (or $\delta_1 = \frac{\pi}{2} - 1$) and used that because of this choice \{ $\frac{1}{2} < x_n, |s| < \frac{1}{2}$ \} $Q^s(2\delta_1)$. After choosing $\delta_1$ we will comment on the choice of $\beta, \mu, \kappa, T_1, T$ to satisfy (63). First we choose $\beta, \kappa_1$ to satisfy the first inequality (63) (for example, $\beta = \frac{\delta_1}{2}, \kappa_1 = min(\frac{1}{2}, \frac{\delta_1}{4})$). Then we select $T_1 = T_0 + 1$ (to fit the later condition at the end of the proof in the case a). After that we choose (large $T$) to satisfy the third inequality (63) and then (small positive) $\mu$ to satisfy the second inequality. To satisfy the condition $C < \tau$ in (59) with our choice of $\tau$ in (62) we can assume that $\varepsilon < \frac{1}{C}$, otherwise the bounds (15), (16) are obvious.

Integrating (64) over $\{t_2 : |t_2 - t_1| < \frac{1}{2}\}$, and using (61) we obtain

\[ \int_{\{\frac{1}{2} < x_n\}} |\partial^\alpha W(x,0;\lambda,t_1)|^2 dx \leq C(\varepsilon^2), \quad \text{provided} \quad |t_1| < T_1 - 1/2. \]

Let $\theta = \frac{1}{\kappa}$. As known, e.g. [20], Lemma 6, for solutions $H(t_1, \theta)$ to the heat equation $\partial_\theta H - \partial_{\bar{k}}^2 H = 0$ we have

\[ \|H(0) - H(\theta)\|_{(0)}(\mathbb{R}) \leq C\sqrt{\theta}\|H(0)\|_{(3)}(\mathbb{R}). \]

Let $\chi_3$ be a $C^\infty(\mathbb{R})$-function, $\chi_3(t) = 1$ if $|t| < T_0$, $\chi_3(t) = 0$ if $T_0 + \frac{1}{2} < |t|$, $0 \leq \chi_3 \leq 1$. Due to the argument at the beginning of the proof $\Omega_0 \subset \{\frac{1}{2} < x_n\}$. Using that $H(t_1, \theta; x) = \partial^\alpha W(x,0;\theta^{-1},t_1)$ solves the heat equation and $H(t_1, 0; x) = \partial^\alpha(\chi_T(t_1)w(x,t_1))$ we yield

\[ \|\partial^\alpha v\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times (-T_0, T_0)) = \|\partial^\alpha w\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times (-T_0, T_0)) \leq \]

\[ \|\chi_3\partial^\alpha(\chi_T w)\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times \mathbb{R}) \leq \]

\[ \|\chi_3(\partial^\alpha W(0,0; \lambda, 0) - \partial^\alpha(\chi_T w))\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times \mathbb{R}) + \]

\[ \|\chi_3(\partial^\alpha W(0,0; \lambda, 0))\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times \mathbb{R}) \leq \]

\[ \|\partial^\alpha W(0,0; \lambda, 0) - \partial^\alpha(\chi_T w)\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times \mathbb{R}) + \|\chi_3(\partial^\alpha W(0,0; \lambda, 0))\|_{(0)}((\Omega_0 \setminus \bar{\omega}) \times \mathbb{R}) \]

provided $T_0 = T_1 - 1, |\alpha| = 1, \alpha_0 = 0$. If $\alpha_0 = 1$ the same argument holds when we substitute $\partial^\alpha(\chi_T v)$ by $\partial_\theta(\chi_T v)$. Using (65) and (66) we conclude that

\[ \|\partial^\alpha v\|_{(0)}((\Omega_0 \setminus \omega) \times (-T_0, T_0)) \leq \]

\[ C \sqrt{\lambda}(\|\partial^\alpha(\chi_T w)\|_{(0)}(\Omega \times \mathbb{R}) + \|\partial_\theta \partial^\alpha(\chi_T w)\|_{(0)}(\Omega \times \mathbb{R})) + \]

\[ C(\varepsilon_{\alpha} + \varepsilon_{\alpha} \lambda\|v\|_{L^2((-T,T);H^1(\Omega)))} \]

provided $T_0 = T_1 - 1 < T - 2$. Recalling (53), (62) we obtain (16) and complete the proof of Theorem 1.6 in the case a).

The proof in the case b) is similar. We only indicate few needed changes. Instead of (54) we will have the partial differential equation and the Cauchy data

\[ (A - a_0 \partial_\theta - \partial_\theta^2)w = f_0, \text{ in } Q = \Omega \times (-T, T), \]

\[ f_0 = \chi_0 f + 2 \sum_{j,m=1}^n a_{jm} \partial_j \chi_0 \partial_m v + (A\chi_0 - a\chi_0)v, \]

\[ w = \chi_0 v_0, \partial_\theta w = \chi_0 v_1 + \partial_\theta \chi_0 v_0 \text{ on } \Gamma_0 \times (-T, T). \]

and accordingly instead of (56)

\[ AW + a_0 i\partial_\theta W + \partial_\theta^2 W = F_0 - F(; T) \text{ on } Q^s, \]

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It is feasible that a priori bounds on the most natural energy norms \( ||\cdot|| \) do not have a meaningful physical interpretation. Besides, the resulting bounds on higher order Sobolev norms (like in (62)) to let \( \lambda \) of (62) to let 

\[
\int_{Q^*} \tau^{3-2|\alpha|} |\partial^\alpha W|^2 e^{2\tau \varphi} \leq 
\]

\[
C \left( \int_Q |F|^2 e^{2\tau \varphi} + \int_{\Gamma \times (-1,1)} \left( \tau^2 |V_0|^2 + \tau(\|\nabla_{x,s} V_0\|^2 + |V_1|^2) e^{2\tau \varphi} \right) + \int_{Q^* \backslash Q^* \backslash (\delta_1)} \left( |W|^2 + \|\nabla_{x,s} W\|^2 \right) e^{2\tau \varphi}, |\alpha| \leq 1. \right.
\]

and consequently

\[
e^{4\tau \delta_1} \int_{Q^* \backslash (2\delta_1)} \tau^{3-2|\alpha|} |\partial^\alpha W|^2 \leq C(e^{2\tau(\Phi+1)} ||F||_{(0)}^2(Q^*) + 
\]

\[
\tau^3 ||V_0||_{(1)}^2(\Gamma \times (-1,1)) + \tau ||V_1||_{(0)}^2(\Gamma \times (-1,1)) + e^{2\tau \Phi} ||F(\cdot;T)||_{(0)}^2(Q^*) + e^{2\tau \delta_1} ||W||_{(1)}^2(Q^*). \]

By using (57) with \( \Gamma \times (-T, T) \) instead of \( Q \) as in the case a) we arrive at (60), and the rest of the proof in the case b) is the same as in the case a).

The proof is complete. \( \square \)

5. Conclusion. We demonstrated that the solution of the Cauchy or continuation problem is improving with growing wave number disregard of geometry of a domain or (non)trapping properties of the metric corresponding to the principal part of the elliptic equation. These results suggest much better controllability of the higher frequency waves and resolution in the inverse problems (for sources, obstacles, or medium properties) for any site of (boundary) controls or sensors.

We think that the results of this paper can be extended onto higher order elliptic equations and systems. So far a conditional Hölder stability is obtained for hyperbolic principally diagonal systems of second order (including isotropic elasticity and Maxwell systems) under (pseudo)convexity conditions and sharp uniqueness of the continuation results are obtained without these conditions [6]. Despite recent results [4], [23] stability bounds in the whole sharp uniqueness domain \( \Omega_0 \), which is the union of all non characteristic \( C^1 \) deformations of \( \Gamma \times (-T, T) \) (given by Tataru [22]) when \( T \) is finite are still not available even in the scalar case. In particular, the bounds in [4] are for any domain \( \Omega_a \) with \( \Omega_a \subset \Omega_0 \cup \Gamma \), and these bounds depend on the distance to \( \partial \Omega_0 \backslash \Gamma \) in a non controllable way.

An interesting question concerns stability bounds in the whole domain (i.e. when \( \Omega_0 = \Omega \)). Due to available results (Theorems 1.9, 7.1 in [1]), it is feasible that one get bounds (3) in the whole \( \Omega \) with \( ||u||_{(1)}(\Omega)||\log \varepsilon||^{-\kappa} \) term replaced by \( ||u||_{(2)}(\Omega)||\log|\log \varepsilon||^{-\kappa} \) with some \( \kappa > 0 \). A possible way to derive it is to use in the proof of Theorem 1.6 logarithmic bounds of \( ||W||_{(1)}(Q^*) \) instead of (64) and instead of (62) to let \( \lambda = ||\log|\log \varepsilon| ||^{\kappa} \). Currently, in [1] there are logarithmic bounds on \( ||W||_{(1)}(Q^*) \). To get bounds in \( H^1(Q^*) \) one needs to assume a priori constraints on higher order Sobolev norms (like in \( H^2(Q^*) \), or in our situation, equivalently, on \( ||u||_{(2)}(\Omega) \)). While the norm in \( H^1(\Omega) \) has a clear meaning of energy, higher order norms do not have a meaningful physical interpretation. Besides, the resulting Hölder exponent \( \kappa \) of the double logarithm is hard to evaluate. An important question is about minimal a priori constraints (on the high frequency part) of a solution. It is feasible that a priori bounds on the most natural energy norms \( ||\cdot||_{(1)}(\Omega) \) are
sufficient, but now it is not clear now how to derive double logarithmic bounds in the whole \( \Omega \). It is the author’s intention to derive bounds in the whole \( \Omega \) under minimal constraints on \( u \).

Another useful confirmation of increasing stability can be obtained by proving that there are growing invariant subspaces where the solution of the Cauchy problem (1), (2) is Lipschitz stable. We will give one of corresponding conjectures.

Let \( \Omega \) be a Lipschitz bounded domain. Let us assume that there is an unique solution \( u \in H^{1}(\Omega) \) of the following Neumann problem

\[
Au + cku + k^{2}u = 0 \text{ in } \Omega,
\]

\[
\partial_{\nu}u = 0 \text{ on } \partial\Omega \setminus \Gamma_{1}, \quad \partial_{\nu}u = g \in H^{(-\frac{1}{2})}(\Gamma_{1}), \text{ on } \Gamma_{1}.
\]

The operator \( B \) mapping \( g \) into \( u_{0} = u \) on \( \Gamma_{0} \) is compact from \( L^{2}(\Gamma_{1}) \) into \( L^{2}(\Gamma_{0}) \). Hence it admits the singular value decomposition consisting of a complete orthonormal system of functions \( g_{m}, m = 1, 2, \ldots \) in \( L^{2}(\Gamma_{1}) \) and corresponding singular values \( \sigma_{m} \geq \sigma_{m+1} > 0 \) (eigenfunction and square roots of eigenvalues of \( B^{*}B \)). The conjecture is that there are positive numbers \( \delta_{1}, \delta_{2} \) depending only on \( A, c \) and \( \Omega \) (but not on \( k \)) such that \( \sigma_{m} > \delta_{1} \) when \( m < \delta_{2}k \). This conjecture for some interesting plane \( \Omega \) was numerically confirmed in [13].

Use of a low frequency zone does not need any (pseudo)convexity type assumptions on \( \Omega, \Gamma, A \) and for this reason is very promising for applications since the observation sites \( \Gamma \) or \( \omega \) can be chosen arbitrarily at convenient locations. In the paper [13] we studied this phenomenon on more detail and gave regularization schemes for numerical solution incorporating the increasing stability. We gave several numerical examples of increasing stability for the Helmholtz equation in some interesting plane domains, admitting or not admitting explicit analytical solution and complete analytic justification. It is important to collect numerical evidence of the increasing stability for more complicated geometries and for systems.

An increasing stability is expected in the inverse source problem, where one looks for \( f \) in the Helmholtz equation \( (\Delta + k^{2})u = f \) (not depending on \( k \)) in \( \Omega = \{ x : 1 < |x| < R \} \) from the Cauchy data \( u, \partial_{\nu}u \) on \( \Gamma = \{ x : |x| = 1 \}, k_{*} < k < k^{*} \). General uniqueness results and convincing numerical examples of increasing stability are given in [14]. One needs to obtain stability estimates improving with growing \( k^{*} \) and to give more of numerical evidence of better resolution for larger \( k^{*} \). Such stability estimates and numerical results under (pseudo)convexity conditions were obtained in [5], [15] by using sharp bounds on analytic continuation to higher wave numbers and exact observability for corresponding hyperbolic equations.

It was (numerically) observed, that use of only low frequency zone can produce a stable solution of the inverse problem, where one looks for a speed of the propagation from all possible boundary measurements. As above this low frequency zone grows with the wave number. Currently, there are no analytic proofs of it. One can look at the linearised problem: find \( f \) (supported in \( \Omega \subset \mathbb{R}^{3} \)) from

\[
\int_{\Omega} f(y) e^{ki|x-y|} e^{ki|z-y|} \frac{dy}{|x-y| |z-y|}
\]
given for \( x, z \in \Gamma \subset \partial\Omega \). The closest analytic results on improving stability are obtained in [10], [17] for the Schrödinger potential and in [16] for the attenuation and conductivity coefficients. The analytic results in [16] suggest a possibility of better resolution in the electrical impedance tomography when using the data from electromagnetic waves of higher frequencies.
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