Method of convex rigid frames and applications in studies of multipartite quNit pure-states

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In this Letter we suggest a method of convex rigid frames in the studies of the multipartite quNit pure-states. We illustrate what are the convex rigid frames and what is the method of convex rigid frames. As the applications we use this method to solve some basic problems and give some new results (three theorems): The problem of the partial separability of the multipartite quNit pure-states and its geometric explanation; The problem of the classification of the multipartite quNit pure-states, and give a perfect explanation of the local unitary transformations; Thirdly, we discuss the invariants of classes and give a possible physical explanation.

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It is known that in quantum mechanics and quantum information, contrast the case of the bipartite quantum systems with the case of studies of the multipartite quantum systems, the latter is even more difficult. For instance, for the general multipartite quantum systems the problems of the criteria of various separability, of the entanglement measures, of the classification and invariants, etc., are all not solved better as yet. In the studies of the multipartite quantum pure-states, generally we always use the traditional way, i.e. we discuss the state vectors or the density matrices in the Hilbert space, etc.. However, sometimes this way is not quite effective, especially for some problems the results always are short of an explicit or geometric explanation. This urges us to find some non-traditional ways in quantum mechanics and quantum information. The purpose in this Letter is just to discuss some problems in this respect.

In this Letter, first we illustrate what is a convex rigid frame, and we suggest a new way, we call it the ‘method of convex rigid frames’ (see below), which associates a multipartite quNit pure-state to a convex polyhedron and its a point in the Hilbert-Schmidt (H-S) space (On the real number field all Hermitian operators acting upon a Hilbert space form a linear space, it is called the Hilbert-Schmidt space). Sometimes, this method is more effective. As some applications, in this Letter we use this method to study three basic problems and give some new results (three theorems): The first is the problem of the so-called partial separability of the multipartite quNit pure-states and its a perfect geometric explanation; Secondly we discuss the problem of the classification of the multipartite quNit pure-states, and give a perfect geometric explanation of the local unitary transformations; Thirdly, we discuss the invariants of classes and give a possible physical explanation.

Sometimes, we call a vector (operator) in the H-S space a ‘point’. In this Letter, the operators (vectors, points) considered by us all are the density matrixes. In the H-S space, the interior product between two vectors $A$ and $B$ is defined as $\langle A, B \rangle = \text{tr}(A^\dagger B)$, the modulus of a vector $A$ is defined by $\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\text{tr}(A^\dagger A)}$. The distance $d(A, B)$ between two points $A$ and $B$ is defined by $d(A, B) = \|A - B\|$. In the H-S space, if a $n$-convex polyhedron $P_n$ has $n$ vertexes $\sigma_i (i = 1, \ldots, n)$, then the convex sum $\sigma = \sum_{i=1}^{n} \lambda_i \sigma_i \left( 0 \leq \lambda_i \leq 1, \sum_{i=1}^{n} \lambda_i = 1 \right)$ denotes a point in $P_n$, we label this point $\sigma$ by $(\lambda_i) \equiv (\lambda_1, \ldots, \lambda_n)$. We denote the set of above $n$ vertexes $\{\sigma_i\}$ and the fixed point $\sigma$ together a symbol $\{(\sigma_i), (\lambda_i)\}$. In this Letter, for the study of the $M$-partite quNit pure-states, every related convex polyhedron $P_n$ and the corresponding point $\sigma$ only can moved as a rigid body as in the classical mechanics, so we call it a ‘$n$-convex rigid frame ($n$-CRF)’, simply read it the symbol $CRF = \{(\sigma_i), (\lambda_i)\}$.

Definition 1. Two $n$-convex rigid frames $CRF = \{(\sigma_1), (\lambda_1)\}$ and $CRF' = \{(\sigma'_1), (\lambda'_1)\}$ are called to be identical, if $d(\sigma_i, \sigma_j) = d(\sigma'_i, \sigma'_j)$ and $\lambda_i = \lambda'_i$ for any $i, j = 1, \ldots, n$. In this case we call the process $CRF \rightarrow CRF'$ a ‘motion from $CRF$ to $CRF'$’.

Obviously, this identical relation is an equivalence relation, therefore all $n$-CRFs can be classified by this identical relation.

Now, we consider a multipartite quantum system $H = \bigotimes_{i=1}^{M} H_i$ with $M$ parties, all local Hilbert spaces $H_i$ have the same dimension $N$, then the total dimensionality of $H$ is $N^M$. Under the standard natural basis $\{|i_1 \cdots i_M\rangle \} (i_k = 0, 1, \ldots, N - 1$ and $k = 1, \ldots, M)$, a normalized $M$-partite quNit state vector $|\Psi\rangle \in H$ is in form as

$$|\Psi\rangle := \sum_{i_1, \cdots, i_M = 0}^{N-1} c_{i_1 \cdots i_M} |i_1 \cdots i_M\rangle, \quad c_{i_1 \cdots i_M} \in \mathbb{C}, \quad \sum_{i_1, \cdots, i_M = 0}^{N-1} |c_{i_1 \cdots i_M}|^2 = 1 \quad (1)$$

In the following, we denote the set of all $M$-partite quNit pure-state density matrixes $\rho = |\Psi\rangle \langle \Psi|$ by the symbol $\mathcal{P}_{M \times N}$, then $\mathcal{P}_{M \times N}$ is a set of points in the $N^{2M}$-dimensional H-S space. For a given $\rho = |\Psi\rangle \langle \Psi|$, by the
following way we at once can obtain a set of CRFs. In the following, $\mathbb{Z}_M$ denotes the integer set $\{1, \ldots, M\}$, and $(r)_P$ denotes a non-null, proper and naturally ordered subset in $\mathbb{Z}_M$, $(r)_P \subset \mathbb{Z}_M$, $(r)_P \equiv \{r_1, \ldots, r_P\}$, where $1 \leq P \leq M - 1$, $r_1 < \cdots < r_P$, and we denote the set $[i(r)_P] \equiv \{i_{r_1}, \ldots, i_{r_P}\}$ \(\{i_{r_1}, \ldots, i_{r_P} = 0, \ldots, N - 1\}\). Now, for a $|\Psi\rangle >$ as in Eq.(1) and any fixed set $[i(r)_P]$, we define a $(M - P)$-partite quNit pure-state $|\Psi [i(r)_P]\rangle >$ by

$$|\Psi [i(r)_P]\rangle > = \sum_{i_{s_1} \cdots i_{s_{M-P}} = 0}^{N-1} c_{i_{s_1} \cdots i_{s_{M-P}}} |i_{s_1} \cdots i_{s_{M-P}} - i_1 \cdots i_M\rangle \tag{2}$$

i.e. for $|\Psi [i(r)_P]\rangle >$, the indexes $i_{r_1}, \ldots, i_{r_P}$ are fixed, sum up only for the others, $i_{s_1}, \ldots, i_{s_{M-P}}$. Notice that $|\Psi [i(r)_P]\rangle >$, generally, is not normalized, we make the normalization

$$|\varphi [i(r)_P]\rangle > = \left(\eta_{i(r)_P}^\rho\right)^{-1} |\Psi [i(r)_P]\rangle >, \quad \eta_{i(r)_P}^\rho = \sqrt{\sum_{i_{s_1} \cdots i_{s_{M-P}} = 0}^{N-1} |c_{i_{s_1} \cdots i_{s_{M-P}}}|^2} \tag{3}$$

where $\eta_{i(r)_P}^\rho$ is the normalization factor. We write the pure-state density matrix $\sigma_{i(r)_P}^\rho(\rho) \equiv |\varphi [i(r)_P]\rangle <\varphi [i(r)_P]\rangle |$, of them (for all possible $[i(r)_P]$) the total is $N^P$.

Now for every pure-state density matrix $\rho = |\Psi <\Psi \rangle$, from the normalization condition of $|\Psi >$ we have

$$\sum_{\text{for all possible } (r)_P} \eta_{i(r)_P}^\rho(\rho) = 1 \tag{4}$$

then $\sigma_{i(r)_P}^\rho(\rho) = \sum_{\text{for all possible } (r)_P} \lambda_{i(r)_P}^\rho(\rho)\sigma_{i(r)_P}^\rho(\rho)$ is a point in the $N^P$-convex polyhedron with vertexes $\{\sigma_{i(r)_P}\}$, where $\lambda_{i(r)_P}^\rho(\rho) = \eta_{i(r)_P}^\rho(\rho)$. Thus, for every pur-state density matrix $\rho$ we always give a corresponding $N^P$-CRF as

$$\text{CRF}_{(r)_P}(\rho) = \left\{\left(\sigma_{i(r)_P}^\rho(\rho), \lambda_{i(r)_P}^\rho(\rho)\right) \right\} \text{ (for all possible } [i(r)_P]) \tag{5}$$

Here, we notice an interesting fact that every $\text{CRF}_{(r)_P}(\rho)$, as a matrix, is just equal to the partial trace $tr_{(r)_P}(\rho) \equiv tr_{r_1 \cdots r_P}(\rho)$ . In fact, from the definition of the partial traces and Eq.(5), this conclusion is obvious, however in the method of convex rigid frames, $\text{CRF}_{(r)_P}(\rho)$ always is regarded as a CRF. Of course, for the distinct $\rho$ and $\rho'$, generally, $\text{CRF}_{(r)_P}(\rho)$ and $\text{CRF}_{(r)_P}(\rho')$ may be distinct. In the following, for a fixed $(r)_P$, we use the symbol $\text{CRF}_{(r)_P} = \{\text{CRF}_{(r)_P}(\rho) \mid \rho \in \mathbb{P}_{M \times N}\}$ which is a set of CRFs corresponding to various pure-states density matrixes $\rho$.

**Theorem 1.** For each fixed proper subset $(r)_P \equiv \{r_1, \ldots, r_P\} \subset \mathbb{Z}_M$, $(r_1 < \cdots < r_P, 1 \leq P \leq M - 1)$, there is a 1-1 correspondence $T_{(r)_P}$ between the set $\mathbb{P}_{M \times N}$ and the set $\text{CRF}_{(r)_P}$, symbolize this by $T_{(r)_P} : \mathbb{P}_{M \times N} \rightarrow \text{CRF}_{(r)_P}$.

**Proof.** As in the above, by using of Eq.(5) for every pure-state $\rho$ there always is a corresponding $\text{CRF}_{(r)_P}(\rho)$, now we define the mapping $T_{(r)_P}$ by

$$T_{(r)_P} : \mathbb{P}_{M \times N} \rightarrow \text{CRF}_{(r)_P}, T_{(r)_P}(\rho) = \text{CRF}_{(r)_P}(\rho) \text{ for } \rho \in \mathbb{P}_{M \times N} \tag{6}$$

If $\rho' \neq \rho$, then $|\Psi >\neq \pm |\Psi >$, this means that there is at least one of $N^P$ real numbers $\lambda_{i(r)_P}^\rho(\rho')$, or of $N^P$ matrixes $\sigma_{i(r)_P}^\rho(\rho)$ which is different from one of $\lambda_{i(r)_P}^\rho(\rho')$, or of matrixes $\sigma_{i(r)_P}^\rho(\rho')$, thus $\text{CRF}_{(r)_P}(\rho) \neq \text{CRF}_{(r)_P}(\rho')$.

Conversely, for any $\text{CRF}_{N^P} = \{(\mu_k, \omega_k) \mid k = 1, \ldots, N^P\} \subset \text{CRF}_{(r)_P}$, we can take the set $\{i_1, \ldots, i_P\}$ \(\{i_1, \ldots, i_P = 0, \ldots, N - 1\}\) to substitute the set $\{i, \ldots, i_P\}$ of indexes in the nature order of $\{k\}$, and we can rewrite $\text{CRF}_{N^P}$ as $\text{CRF}_{N^P} = \{(\mu_{[i(r)_P]}(\mu_k), \omega_{[i(r)_P]}(\omega_k))\}$. Suppose that the pure-state $\mu_{[i(r)_P]} = |\xi [i(r)_P]\rangle > < \xi [i(r)_P]\rangle |$, $\xi [i(r)_P] > = \sum_{i_{s_1} \cdots i_{s_{M-P}} = 0}^{N-1} d_{i_{s_1} \cdots i_{s_{M-P}}} |i_{s_1} \cdots i_{s_{M-P}}\rangle >$, then we write $|\Phi > = \sum_{j_1 \cdots j_M = 0}^{N-1} f_{j_1 \cdots j_M} \xi [i(r)_P]\rangle > < \xi [i(r)_P]\rangle |$, where $f_{j_1 \cdots j_M}$ is determined by

$$f_{i_1 \cdots i_M} = (\mu_{[i(r)_P]}(\mu_k))_{[i_1 \cdots i_{M-P}] = 0}^{N-1} \text{ when as a set } (i_1 \cdots i_M) = (i_{r_1} \cdots i_{r_P}) \cup (i_{s_1} \cdots i_{s_{M-P}}) \tag{7}$$

It can be verified directly that $|\Phi >$ is a $M$-partite quNit normalized pure-state, and we just have $T_{(r)_P}(|\Phi > < \Phi |) = \text{CRF}_{N^P}$. $\square$
Since in the above discussion, \((r)_p\) and \((s)_{M-P}\) are completely symmetric in status, thus by the similar way, for the subset \((s)_{M-P}\) we have yet a \(T_{(s)_{M-P}} : \mathbb{P}_{M \times N} \cong CRF_{(s)_{M-P}}\). From the theorem 1, \(\mathbb{P}_{M \times N}\) and the set \(\{CRF_{(r)_p}\}\) are 1-1 corresponding, therefore some studies of the multipartite quNit pure-states can be returned into the studies about \(\{CRF_{(r)_p}\}\). In this Letter, we call this way the method of convex rigid frames. Sometimes, this method is a more effective means. As the examples of applications, in the following we use this method to study some basic problems.

The first is the partial separability problem. Generally, the common so-called separability, in fact, is the ‘full-separability’. For the general multipartite systems, the problem becomes even more complex. In fact, there yet is other concept of separability weaker than full-separability, i.e. the partial separability, e.g. for a tripartite qubit pure-state \(\rho_{ABC}\), there are the A-BC-separability, B-AC-separability, C-AB-separability, etc.[2,3]. Related to Bell-type inequalities and some criteria of partial separability of the multipartite systems, see [4-6].

In the first place, we need to define strictly what is the partial separability of a multipartite quNit pure-state. About this, we must consider the order numbered by us of the particles. If two ordered proper subsets \((r)_p = \{r_1, \cdots, r_p\} (1 \leq r_1 < \cdots < r_p \leq M)\) and \((s)_{M-P} = \{s_1, \cdots, s_{M-P}\} (1 \leq s_1 < \cdots < s_{M-P} \leq M)\) in \(\mathbb{Z}_M\) obey

\[
(r)_p \cup (s)_{M-P} = \mathbb{Z}_M, \quad (r)_p \cap (s)_{M-P} = \emptyset
\]

where \(P\) is an integer, \(1 \leq P \leq M - 1\), then the set \(\{ (r)_p, (s)_{M-P}\} \) forms a partition of \(\mathbb{Z}_M\), in the following for the sake of stress, we denote by the symbol \((r)_p \parallel (s)_{M-P}\). Now, for a given partition \((r)_p \parallel (s)_{M-P}\), we use the natural basis \(\{|i_{r_1} \cdots i_{r_p} i_{s_1} \cdots i_{s_{M-P}}\}\) and write

\[
|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle = \sum_{i_1, \cdots, i_{M-P}} d_{i_1 i_2 \cdots i_{M-P}} |i_{i_1} \cdots i_{r_p} i_{s_1} \cdots i_{s_{M-P}}\rangle.
\]

Obviously, \(|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle > 0\) and \(|\Psi\rangle > 0\) in Eq.(1), in fact, are completely same in physics, the difference only is the order numbered by us of particles. For instance, \(\Psi_{ABC} = \Psi_{ABC} \parallel BD = \Psi_{ACBD} \parallel BD = \Psi_{ACBD}\), and \(\Psi_{AC\parallel BD} = \sum c_{jkl} |i_{jkl}\rangle > 0\). However, we notice that, generally, \(\rho_{(r)_p \parallel (s)_{M-P}} > 0\), \(|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle > 0\), and \(|\Psi_{(s)_{M-P}}\rangle > 0\). In this case, \(\rho_{(r)_p \parallel (s)_{M-P}} = \rho_{(r)_p} \otimes \rho_{(s)_{M-P}}\).

**Definition 2.** For the partition \((r)_p \parallel (s)_{M-P}\), a M-partite quNit pure-state \(|\Psi\rangle > 0\) is called to be \((r)_p - (s)_{M-P}\)-separable, if the corresponding \(|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle > 0\) can be decomposed as a product of two pure-states as

\[
|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle = |\Psi_{(r)_p}\rangle \otimes |\Psi_{(s)_{M-P}}\rangle > 0 \quad \text{or} \quad \rho_{(r)_p \parallel (s)_{M-P}} = \rho_{(r)_p} \otimes \rho_{(s)_{M-P}}.
\]

where \(|\Psi_{(r)_p}\rangle > 0\) and \(|\Psi_{(s)_{M-P}}\rangle > 0\). If \(|\Psi\rangle > 0\) is not \((r)_p - (s)_{M-P}\)-separable, then we call it to be \((r)_p - (s)_{M-P}\)-inseparable.

We notice that for the distinct partitions, \(\rho\) can have different partial separability. Of course, if a pure-state \(\rho\) is partially inseparable with respect to any partition, then it must be entangled. Conversely, if a pure-state always is completely partially separable with respect to all possible partitions \((r)_p \parallel (s)_{M-P}\), then it is separable (disentangled, full-separable). By using the above method of CRFs, we can obtain the following theorem, which, in fact, is a geometric explanation of the partial separability of the M-partite quNit pure-states.

**Theorem 2.** The sufficient and necessary conditions of the M-partite quNit pure-state \(\rho = |\Psi\rangle < |\Psi\rangle\) to be \((r)_p - (s)_{M-P}\)-separable is that CRF\((r)_p\) \((\rho)\) or CRF\((s)_{M-P}\) \((\rho)\) shrinks to one point (pure-state vertex), i.e. all \(d (\sigma_{i(r)_p} - \sigma_{i'(r)_p}) = 0\) or all \(d (\sigma_{i(r)_p} - \sigma_{i'(s)_{M-P}}) = 0\) for any \(i, i' = 0, \cdots, N - 1\).

**Proof. Necessity.** Suppose that the pure-state \(\rho = |\Psi\rangle < |\Psi\rangle\) is \((r)_p - (s)_{M-P}\)-separable, according to the Definition 2, this means that (see Eq. (9)) \(|\Psi_{(r)_p \parallel (s)_{M-P}}\rangle > 0\) and \(|\Psi_{(s)_{M-P}}\rangle > 0\). If the normalized \(|\Psi_{(r)_p}\rangle > 0\) and \(|\Psi_{(s)_{M-P}}\rangle > 0\), respectively, are \(|\Psi_{(r)_p}\rangle = \sum_{i_{r_1} \cdots i_{r_p} = 0}^{N-1} d_{i_{r_1} \cdots i_{r_p}} |i_{r_1} \cdots i_{r_p}\rangle > 0\) and \(|\Psi_{(s)_{M-P}}\rangle = \sum_{i_{s_1} \cdots i_{s_{M-P}} = 0}^{N-1} e_{i_{s_1} \cdots i_{s_{M-P}}} |i_{s_1} \cdots i_{s_{M-P}}\rangle > 0\), then by a direct calculation, in CRF\((r)_p\) \((\rho)\) we have

\[
\lambda_{i(r)_p} (\rho) = |d_{i_{r_1} \cdots i_{r_p}}|^2 \quad \text{and all} \quad \sigma_{i(r)_p} (\rho) = |\Psi_{(r)_p}\rangle < |\Psi_{(r)_p}\rangle
\]
this means that $CRF_{(r)_P}(\rho)$ will indeed shrink to a point. Similarly, for $CRF_{(s)_M-P}(\rho)$.

**Sufficiency.** If all $\sigma_{[i(r)_P]}(\rho)$ shrink to a point $\sigma = |\varphi><\varphi|$, $|\varphi> = \sum_{i_1 \cdots i_{k-M-P}} f_{k_1 \cdots k_P} |k_1 \cdots k_P>$, then according to Eqs.(9) and (10), this means that $|\Psi_{(r)_P}\rangle\langle(s)_{M-P}| = \sum_{i_1 \cdots i_{k-M-P}} g_{k_1 \cdots k_P} |i_1 \cdots i_{k-M-P}| > \sum_{i_1 \cdots i_{k-M-P}} g_{k_1 \cdots k_P} |i_1 \cdots i_{k-M-P}| > |i_1 \cdots i_{k-M-P}| > |\varphi > > |\varphi >$, where $|\psi> = \sum_{i_1 \cdots i_{k-M-P}} g_{i_1 \cdots i_{k-M-P}} |i_1 \cdots i_{k-M-P}| > i_1 \cdots i_{k-M-P}$, $g_{i_1 \cdots i_{k-M-P}}$ are some coefficients, and $k_1 \cdots k_P >$ has been substituted by $i_1 \cdots i_{k-M-P}$. Therefore $\rho$ is $(r)_{P} - (s)_{M-P}$-separable. □

**Corollary.** For a $M$-partite quNit pure-state $\rho$, $CRF_{(r)_{P}}(\rho)$ and $CRF_{(s)_{M-P}}(\rho)$ both shrink to points, or both not.

The proof is evident from the proof of the Theorem 2.

Therefore in view of method of CRFs, every separable multipartite quNit (disentangled) pure-states is an extremely special state, i.e. of which all CRFs must be shrink to a point. As an simple example, we consider the normalized tri-partite qutrit pure-state $\rho_{3 \times 3} = |\Psi_{3 \times 3} > < \Psi_{3 \times 3}| = \sum_{i,j,k=0} c_{ijk} |i_A j_B k_C > > (\sum_{i,j,k=0} |c_{ijk}|^2 = 1)$ and the partition $B \parallel AC$, by a direct calculation we obtain a 3-CRF

$$CRF_{(B)}(\rho_{3 \times 3}) = \{ (\sigma_{[i(j)_{B}]}(\rho_{3 \times 3})) , (\lambda_{[i(j)_{B}]}(\rho_{3 \times 3})) \} (j = 0, 1, 2)$$

(12)

where $\lambda_{[i(j)_{B}]}(\rho_{3 \times 3}) = \sum_{i\neq j} c_{ijk} |i_A j_B k_C > > 0 (j, j' = 0, 1, 2)$ leads to that for any $i, k = 0, 1, 2$ all rates $c_{0k} : c_{1k} : c_{2k}$ are equal, this is indeed the sufficient and necessary conditions of $|\Psi^{(3)} >$ to be B-AC-separable.

Secondly, we study the problem of classification of the $M$-partite quNit pure-states. In view of the method of CRFs, a very natural way of classification is to use the motions of the CRFs.

**Definition 3.** We call two $M$-partite quNit pure-states $\rho$ and $\rho'$ are ‘equivalent by motion’, symbolize by $\rho \sim \rho'$, if and only if $CRF_{(r)_{P}}(\rho)$ and $CRF_{(r)'_{P}}(\rho')$ are identical (see the Definition 1) with respect to all possible non-null proper subset $(r)_P (1 \leq P \leq M - 1)$.

A notable advantage of this definition is that this equivalence relation do not break the partial separability of the $M$-partite quNit pure-states. In fact, we have the following

**Corollary.** If two $M$-partite quNit pure-states $\rho$ and $\rho'$ are equivalent by motion, then $\rho$ and $\rho'$ both are $(r)_{P} - (s)_{M-P}$-separable (or both $(r)_{P} - (s)_{M-P}$-inseparable), with respect to any $(r)_P \parallel (s)_{M-P}$, i.e. the partial separability is an invariant of class.

The proof is obvious.

Notice that about the above way of classification, we must still solve the problem of reasonableness in physics, because in quantum information a pure-states $\rho$, generally, represents some information status. It is known that, generally, for the indistinguishability of multipartite quNit states we must use the local operation and classical communications (LOCC)[7-9]. Now we prove that our way is reasonable, i.e. we prove that the above classification by motions, in fact, is just the classification of $PM \times N$ under the local unitary transformations (LUTs). In order to prove this, in fact, we only need to prove the following theorem.

**Theorem 3.** Two $M$-partite quNit pure-states $\rho$ and $\rho'$ are equivalent by motion (see the Definition 2), if and only if there are $M$ unitary matrices $u_i(N) \in U(N) i = 1, \cdots, M$ that the

$$\rho' = u_1(N) \otimes \cdots \otimes u_M(N) \rho u_M^\dagger(N) \otimes \cdots \otimes u_1^\dagger(N)$$

(13)

**Proof.** In the first place, we notice that in the H-S space only the unitary transformations of operators can keep the invariances of distances and modulus of the vectors, and a tensor product of some unitary matrices is still a unitary matrix. Now, if Eq.(13) holds, according to Eqs.(2,3,5), the change from $CRF_{(r)_{P}}(\rho)$ to $CRF_{(r)'_{P}}(\rho')$ for each $(r)_P$ is determined by a unitary matrix $u_{s_1}(NM-P) \otimes \cdots \otimes u_{s_{M-P}}(NM-P)$, which acts upon every ‘part’ $|\Psi[i(r)_P] > > |\Psi >$ in Eq.(2) and keeps $\eta_{[i(r)_P]}(\rho)$ to be invariant, thus the identical relation between $CRF_{(r)_{P}}(\rho)$ and $CRF_{(r)'_{P}}(\rho')$ is quite obvious.

Conversely, since $\lambda_{[i(r)_P]}(\rho) = \lambda_{[i(r)'_P]}(\rho')$ for all possible $[i(r)_P]$ , we know that for every $[i(r)_P]$ there must be a unitary matrix $u_{[i(r)_P]}(NM-P)$ which acts upon the ‘part’ $|\Psi[i(r)_P] > > |\Psi >$, and keeps that all equations $d(\sigma_{[i(r)_P]}(\rho), \sigma_{[i'(r)_P]}(\rho)) = d(\sigma_{[i(r)_P]}(\rho'), \sigma_{[i'(r)'_P]}(\rho'))$ always hold. This fact must hold for arbitrary $(r)_P$.
and arbitrary set \( [i_{(r)}] \) of indexes, the unique possibility is that there are some \( u_1 (N), \cdots, u_M (N), u_k (N) \in U (N) \) \((k = 1, \cdots, M)\) and \(| \Psi' > = u_1 (N) \otimes \cdots \otimes u_M (N)|\Psi > \). □

This theorem gives us a perfect explanation of the LUS, i.e. a LU acting upon \( \rho \), in fact, is a motion of CRFs, as a motion of a rigid body as in the classical mechanics.

Thirdly, we discuss the invariants of the classification and a possible explanation. Since the distance between two points (vectors) in the H-S space is invariant under any motion (LU), evidently there are at least two kinds of invariants of motions (LUs): The the volumes of the convex polyhedrons propped up by the CRFs, others are the angles of intersections of any two ‘props’ in every CRF.

As for the problem how to calculate the volume of a convex polyhedron in the H-S space, see [10] and its references. For a \( M \)-partite quNit pure-state \( \rho \) and a given \( (r)_p \| (s)_{M-p} \), let \( V\left( CRF_{(r)} (\rho) \right) \) denote the volumes of the convex polyhedron with \( N^P \) vertices \( \left\{ \sigma_{[i_{(r)}]} \right\} \) for all possible \([i_{(r)}]\) as in Eq.(5). Similarly, \( V\left( CRF_{(s)} (\rho) \right) \). Now we denote the pair of volumes by

\[
V_{(r)_p\| (s)_{M-p}} (\rho) = \left[ V\left( CRF_{(r)} (\rho) \right), V\left( CRF_{(s)} (\rho) \right) \right]
\]

Obviously, \( V_{(r)_p\| (s)_{M-p}} (\rho) \) is a invariant under motions (LUs) of \( \rho \). In addition, in \( CRF_{(r)} (\rho) \), the direction from the point \( \sigma_{[i_{(r)}]} (\rho) = \left\{ \sigma_{[i_{(r)}]} \right\} \) to a fixed vertex \( \sigma \left( [k_{(r)}] \right) \) \((k = 0, \cdots, N - 1)\) of \( CRF_{(r)} (\rho) \) can be expressed by the vector

\[
\omega_{[k_{(r)}, l_{(r)}, \rho]} = \sum_{\text{all possible } [i_{(r)}]} \left( \lambda_{[i_{(r)}]} (\rho) - \delta_{[i_{(r)}]} [k_{(r)}] \right) \sigma_{[i_{(r)}]} (\rho)
\]

Therefore the angle (we label it by \( \theta (\rho, k_{(r)}; l_{(r)}; \rho) \)) of intersection of two direct \( \omega_{[k_{(r)}]} (\rho) \) and \( \omega_{[l_{(r)}]} (\rho) \) can be determined by

\[
\cos \theta \left( [k_{(r)}], [l_{(r)}], \rho \right) = \frac{\omega_{[k_{(r)}]} (\rho) \cdot \omega_{[l_{(r)}]} (\rho)}{\left\| \omega_{[k_{(r)}]} (\rho) \right\| \cdot \left\| \omega_{[l_{(r)}]} (\rho) \right\|}
\]

Obviously, \( \cos \theta \left( [k_{(r)}], [l_{(r)}], \rho \right) \) is yet an invariant under motions (LUs) of \( \rho \).

At present, we cannot yet understand what is the meaning of \( \cos \theta (\rho, k_{(r)}; l_{(r)}; \rho) \) in quantum information. However, we find a quite natural explanation of \( V_{(r)_p\| (s)_{M-p}} (\rho) \) as follows. From the Theorem 2, its corollary and the fact that a convex polyhedron shrinks to a point if and only if its volume vanishes, then we know that \( \rho \) is \((r)_p - (s)_{M-p}\)-separable if and only if \( V_{(r)_p\| (s)_{M-p}} (\rho) = (0, 0) \). Conversely, if \( V_{(r)_p\| (s)_{M-p}} (\rho) \neq (0, 0) \) then \( \rho \) is \((r)_p - (s)_{M-p}\)-inseparable, where the value of \( V\left( CRF_{(r)} (\rho) \right) \) means that the degree of the difficulty of the factor \( \rho_{(r)_p} \) to be separated out from \( \rho \). Similarly, for \( V\left( CRF_{(s)} (\rho) \right) \). Therefore we can regard that \( V_{(r)_p\| (s)_{M-p}} (\rho) \) denotes the degree of the measure of the \((r)_p - (s)_{M-p}\) inseparability. It is quite interesting that, generally, \( V\left( CRF_{(r)} (\rho) \right) \neq V\left( CRF_{(s)} (\rho) \right) \) unless they both vanish, this means that the above two degrees of the difficulties can be different. In addition, it is known that if \( \rho \) is \((r)_p - (s)_{M-p}\)-inseparable, then there must be the so-called partial entanglement[4,6]. What a pity, an entanglement measure, generally, should be in form as a Neumann entropy and it must at least obey some limits[11], however \( V_{(r)_p\| (s)_{M-p}} (\rho) \) has no these properties, so we cannot taken it as a measure of the partial entanglement.

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