Correlation and Entanglement of Multipartite States

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We derive a classification and a measure of classical- and quantum-correlation of multipartite qubit, qudit, and in general, \( n \)-level systems, in terms of SU\((n)\) representations of density matrices. We compare the measure for the case of bipartite correlation with concurrence and the entropy of entanglement. The characterization of correlation is in terms of the number of nonzero singular values of the correlation matrix, but that of mixed state entanglement requires additional invariant parameters in the density matrix. For the bipartite qubit case, the condition for mixed state entanglement is written explicitly in terms of the invariant parameters in the density matrix. For identical particle systems we analyze the effects of exchange symmetry on classical and quantum correlation.

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Quantum entanglement is an information resource; it plays an important role in many protocols for quantum-information processing, including quantum computation [1], quantum cryptography [2], teleportation [3], superdense coding [4], and quantum error correction protocols [5]. Techniques for characterizing the bipartite entanglement and correlation of pure and mixed quantum states have enabled many advances in quantum information and the study of decoherence [6]. Many quantum information protocols use bipartite entanglement, but multipartite entanglement [7], also has quantum-information applications, e.g., controlled secure direct communication [8], quantum error correction [9], controlled teleportation [10] and secret sharing [11]. It has been shown that any inseparable two-qubit states can be distilled to a singlet-state form with enough copies of the qubit-pairs [12] and algorithms for multi-copy entanglement distillation for pairs of qubits have been developed [13]. Moreover, multipartite entanglement offers a means of enhancing interferometric precision beyond the standard quantum limit and is therefore relevant to increasing the precision of atomic clocks by decreasing projection noise in spectroscopy [13].

Here we use a representation of the density matrix for qubit, qudit, and more generally, \( n \)-level systems containing 2, 3, \ldots, and \( N \)-parts, in terms of the correlations between the subsystems to quantify the classical and quantum correlation of multipartite systems. Our classification of correlation is in terms of the correlation matrix and its singular values, and our classification of entanglement of mixed states is associated with the geometric precision beyond the standard quantum limit and the Bloch vectors are \( \mathbf{n}_{ij} \), where \( \mathbf{n}_{ij} = \frac{1}{2} (\mathbf{1} + \mathbf{\sigma}_j \cdot \mathbf{\sigma}_i) \), for \( \mathbf{\sigma}_j \) = Pauli matrices for particle \( j \) and the Bloch vectors are \( \mathbf{n}_{ij} = \frac{1}{2} (\mathbf{1} + \mathbf{\sigma}_j \cdot \mathbf{\sigma}_i) \). For two correlated qubits, \( \rho_{AB} = \frac{1}{4} \left[ (1 + \mathbf{n}_A \cdot \mathbf{\sigma}_A) (1 + \mathbf{n}_B \cdot \mathbf{\sigma}_B) + \mathbf{\sigma}_A \cdot \mathbf{C}_{AB} \cdot \mathbf{\sigma}_B \right] \), where the tensor \( \mathbf{C}_{AB} \) specifies the qubit correlations, \( C_{ij}^{AB} \equiv \langle \sigma_{iA} \sigma_{jB} \rangle - \langle \sigma_{iA} \rangle \langle \sigma_{jB} \rangle = \langle \sigma_{iA} \sigma_{jB} \rangle - n_{i,A} n_{j,B} \).

Werner [15] defined a mixed state of an \( N \)-partite system as separable, i.e., classically-correlated, if it can be written as a convex sum,

\[
\rho = \sum_k \rho_k^A \rho_k^B \cdots \rho_k^N, \quad p_k > 0, \quad \sum_k p_k = 1, \quad (1)
\]

where \( \rho_k^A \) is a valid density matrix of subsystem \( A \), etc. Otherwise, Werner defined it to be entangled, i.e., quantum-correlated. Unfortunately, this definition of entanglement for mixed states is not constructive, since, in general, it cannot be used to decide whether a given density matrix is separable or entangled. Moreover, a quantitative measure of entanglement of multi-partite systems has proven to be difficult to devise. Note that studies of the best separable approximation to an arbitrary density matrix have been carried out and have led to a proposal of a measure for entanglement [16]. Furthermore, aspects of the geometry of separability and entanglement based on Schmidt decomposition have been studied and led to an analysis of the question of separability for the two-qubit case [17].

In what follows, we categorize classically-correlated and quantum-correlated states and characterize their correlation in terms of the number of nonzero singular values [18], \( \{ \mathcal{d}_{ij} \} \), of the correlation matrix \( \mathbf{C} \), and characterize entanglement of bipartite qubit systems using the Peres-Horodecki criterion [19] which is reformulated totally in terms of the parameters used in forming the density matrix.

First, let us consider a bipartite qubit system. For two uncorrelated qubits, call them \( A \) and \( B \), we can write the density matrix as a product, \( \rho_{AB} = \rho_A \rho_B \), where the individual qubit density matrices can be written as \( \rho_j = \frac{1}{2} (\mathbf{1} + \mathbf{n}_j \cdot \mathbf{\sigma}_j) \), where \( J = A, B \), the \( \mathbf{\sigma}_j \) are Pauli matrices for particle \( j \) and the Bloch vectors are \( \mathbf{n}_j = \frac{1}{2} (\mathbf{1} + \mathbf{\sigma}_j \cdot \mathbf{\sigma}_i) \). For two correlated qubits,

\[
\rho_{AB} = \frac{1}{4} \left[ (1 + \mathbf{n}_A \cdot \mathbf{\sigma}_A) (1 + \mathbf{n}_B \cdot \mathbf{\sigma}_B) + \mathbf{\sigma}_A \cdot \mathbf{C}_{AB} \cdot \mathbf{\sigma}_B \right], \quad (2)
\]

where the tensor \( \mathbf{C}_{AB} \) specifies the qubit correlations,

\[
C_{ij}^{AB} \equiv \langle \sigma_{iA} \sigma_{jB} \rangle - \langle \sigma_{iA} \rangle \langle \sigma_{jB} \rangle = \langle \sigma_{iA} \sigma_{jB} \rangle - n_{i,A} n_{j,B}. \quad (3)
\]
The density matrix $\rho_{AB}$ is a $4 \times 4$ Hermitian matrix with trace unity, so 15 parameters are required to parameterize it. The 3 components of $\mathbf{n}_A$, the 3 components of $\mathbf{n}_B$, and the 9 components $C_{ij}$ of the $3 \times 3$ matrix $C$, where we have no longer explicitly shown the subsystem superscripts, are sufficient for this purpose.

Similarly for the bipartite qutrit case. The $3 \times 3$ density matrix of a single qutrit can be written as $\rho = \frac{1}{2} \left( (1 + \frac{3}{2} \lambda_i \lambda_j) \right)$ where the $\lambda_i$ are the eight traceless Hermitian Gellman matrices familiar from SU(3) [21], and $\langle \lambda_i \rangle = \text{Tr} \lambda_i \rho$. A bipartite qutrit density matrix can be parameterized in the form

$$
\rho_{AB} = \frac{1}{9} \left[ (1 + \frac{3}{2} \lambda_i \lambda_j) (1 + \frac{3}{2} \lambda_j \lambda_i) + \frac{9}{4} \lambda_i \lambda_j C_{ij} \lambda_i \lambda_j, \right],$$

$$
C_{ij} \equiv \langle \lambda_i \lambda_j \rangle - \langle \lambda_i \rangle \langle \lambda_j \rangle, \quad \text{(5)}
$$

where $C_{ij}$ specifies the correlation between $\lambda_i$ and $\lambda_j$. Here, $\rho_{AB}$ is a $9 \times 9$ Hermitian matrix with trace unity, so 80 parameters are required to parameterize it. The eight components of $\langle \lambda_i \rangle$, eight components of $\langle \lambda_i \lambda_j \rangle$, and 64 components $C_{ij}$ of the $8 \times 8$ matrix $C$ are sufficient for this purpose. The same procedure can be used for bipartite 4-level systems using the 15 traceless $4 \times 4$ Hermitian generator matrices for SU(4), and bipartite $n$-level systems with the $n^2 - 1$ traceless $n \times n$ Hermitian matrices. Likewise, a general qubit-qutrit 6 $\times$ 6 density matrix takes the form $\rho_{ABC} = \frac{1}{2} \left[ (1 + \mathbf{n}_A \cdot \mathbf{\sigma}_A) (1 + \mathbf{n}_B \cdot \mathbf{\sigma}_B) (1 + \mathbf{n}_C \cdot \mathbf{\sigma}_C) \right] + \sum_{i,j} \sigma_{ij} \cdot C_{ijk} \lambda_i \lambda_j \lambda_k$ with $C_{ijk} \equiv \langle \mathbf{C}_{ijk} \rangle - \langle \mathbf{C}_i \rangle \langle \mathbf{C}_j \rangle$, $i = 1, 2, 3$ and $j = 1, \ldots, 8$.

Our bipartite correlation measure for an $n$-level and $m$-level system is based on the $(n^2 - 1) \times (m^2 - 1)$ correlation matrix $C$:

$$
\mathcal{E}_C \equiv \frac{n^2}{4(n^2 - 1)} \text{Tr} \mathbf{C} \mathbf{C}^T = \frac{n^2}{4(n^2 - 1)} \sum_{i,j} C_{ij} C_{ji}^T, \quad \text{(6)}
$$

where $n = \min(n,m)$. $\mathcal{E}_C$ is a nonnegative real number. If $C$ is a normal matrix [18], $\text{Tr} \mathbf{C} \mathbf{C}^T$ equals to the sum of the squares of its eigenvalues, but $C$ need not be normal. $\mathcal{E}_C$ is basis-independent; any rotation in Hilbert space leaves it unchanged. The normalization factor $n^2/4(n^2 - 1)$ is such that the maximum possible value of $\mathcal{E}_C$ is unity. $\mathcal{E}_C$ measures both classical- and quantum-correlation. This measure of bipartite correlation was suggested in Ref. [22] for pure states and $n = m$.

The correlation matrix $C$ quantifies the correlation and the entanglement of bipartite states. For pure two-qubit states, the number of nonzero singular values (NSVs) of $C$ is zero for non-entangled states (the $C$ matrix vanishes), three, if only two basis states are present in the entangled state, five, if one of the qubits contains only two basis states but the other contains three, and eight if all three basis states are present. For classically correlated qutrit states, there are 1, 2, ..., 8 NSVs for 2, 3, ..., and 9 or more terms in the sum, etc. A similar classification in terms of the number of NSVs exists for qubit-qutrit and $n$-level systems.

A general three-qubit density matrix can be written as

$$
\rho_{ABC} = \frac{1}{8} \left[ (1 + \mathbf{n}_A \cdot \mathbf{\sigma}_A) (1 + \mathbf{n}_B \cdot \mathbf{\sigma}_B) (1 + \mathbf{n}_C \cdot \mathbf{\sigma}_C) \right. 
+ \mathbf{\sigma}_A \cdot C^{AB} \cdot \mathbf{\sigma}_B + \mathbf{\sigma}_A \cdot C^{AC} \cdot \mathbf{\sigma}_C + \mathbf{\sigma}_B \cdot C^{BC} \cdot \mathbf{\sigma}_C 
+ \sum_{i,j,k} \sigma_{ij} \sigma_{jk} \sigma_{ik} \mathbf{D}_{ijk}, \quad \text{(7)}
$$

where $C^{AB}, C^{AC},$ and $C^{BC}$ are the bipartite correlation matrices and the tensor that specifies the tripartite correlations is

$$
D_{ijk} \equiv \langle \sigma_{i,A} \sigma_{j,B} \sigma_{k,C} \rangle - \langle \sigma_{i,A} \rangle \langle \sigma_{j,B} \rangle \langle \sigma_{k,C} \rangle. \quad \text{(8)}
$$

A tripartite qutrit state can be similarly parameterized:

$$
\rho_{ABC} = \frac{1}{27} \left[ \prod_l (1 + \frac{3}{2} \lambda_i \lambda_j) \lambda_i \lambda_j + \frac{9}{4} \sum_{i,j} \lambda_i \lambda_j C_{ij,j} \lambda_j \lambda_j 
+ \frac{27}{8} \sum_{i,j,k} \lambda_i \lambda_j \lambda_k \lambda_k \mathbf{D}_{ijk}, \quad \text{(9)}
$$

where $\lambda_i, \lambda_j, \lambda_k$ are the three traceless 3-dimensional Hermitian matrices familiar from SU(3) [21].
D_{ij,k,l} ≡ \langle \lambda_i, \lambda_j, \lambda_k, \lambda_l \rangle - \langle \lambda_i, \lambda_j \rangle \langle \lambda_k, \lambda_l \rangle. \quad (10)

Our tripartite correlation measure $E_D$ is based on the correlation matrix $D$, $E_D ≡ K \sum_{i,j} D_{ij,k,l}$, which can also be written as

$$E_D = K \text{Tr}(\rho_{ABCD} - \rho_A \rho_B \rho_C - \sum_{I,J(I \neq J)} \sum_{i,j} C_{ij}^{I,J} \sigma_i \sigma_j)^2,$$

where $K = 1/4$ for qubits, and $K = 27/160$ for qutrits with $\sigma$ replaced by $\lambda$. $E_D$ is also a basis-independent nonnegative real number; any rotation in Hilbert space leaves it unchanged. A tripartite system may have bipartite-as well as tripartite-correlation. The bipartite correlation of a tripartite system is the sum of the correlation for the three bipartite pairs,

$$E_C = \frac{n^2}{4(n^2 - 1)} \sum_{I,J(I \neq J)} \text{Tr} C_{I,J}^{I,J}(C^{I,J})^T,$$

where $I, J = A, B, C$. The density matrices of four-partite and higher qubit, qutrits, and $n$-level system states can be constructed similarly, but with increased complexity. For example, it is clear from Eq. (11) how to generalize and obtain the four-partite correlation of four-partite systems: $E_E ≡ K' \sum_{i,j,k,l} E_{ijkl}^E$, where the four-partite-correlation term of the four-qubit density matrix $\rho_{ABCD}$ is $\sum_{i,j,k,l} \sigma_i \sigma_j \sigma_k \sigma_l \rho_{ijkl}$ and $K' = 1/8$.

We now present some examples of qubit and qutrit bipartite and tripartite correlated states. The maximally entangled bipartite qubit states are the Bell states, 

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle), \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle). \quad (13)$$

For all these states, $\langle \sigma_i A \rangle = \langle \sigma_i B \rangle = 0$, i.e., $n_A = n_B = 0$. For the singlet, $\langle \sigma_i A \sigma_i B \rangle = -\delta_{ij}$ (the spins are oppositely polarized). The density matrices of the Bell states are:

$$\rho_{\Psi^-} = \frac{1}{4} (|1\rangle \langle 1| B - \sigma_A \cdot \sigma_B), \quad \rho_{\Psi^+} = \frac{1}{4} (|1\rangle \langle 1| B + \sigma_A \cdot \sigma_B - 2\sigma_z A \sigma_z B), \quad \rho_{\Phi^-} = \frac{1}{4} (|1\rangle \langle 1| B + \sigma_A \cdot \sigma_B - 2\sigma_y A \sigma_y B), \quad \rho_{\Phi^+} = \frac{1}{4} (|1\rangle \langle 1| B + \sigma_A \cdot \sigma_B - 2\sigma_x A \sigma_x B). \quad (14)$$

The correlation matrices of the Bell’s states are diagonal and the correlation measure is $E_C = 1$, i.e., they are maximally entangled.

Let us now consider the Rashid pure states $|\phi^\pm\rangle = (2 \cosh(2\theta))^{-1/2} (e^{-\theta} |\uparrow \uparrow\rangle \pm e^{\theta} |\downarrow \downarrow\rangle)$, which density matrix is $\rho_{\phi^+} = \frac{1}{4} \{ |1\rangle \langle 1| B - \tanh(2\theta) \sigma_z A |1\rangle \langle 1| B + \tanh(2\theta) \sigma_z B + \text{sech}(2\theta)(\sigma_x A \sigma_x B - \sigma_y A \sigma_y B) + \text{sech}^2(2\theta) \sigma_z A \sigma_z B \}$. When $\theta = 0$,

$$|\phi^+\rangle = |\Phi^+\rangle, \quad \text{and as } \theta \to \pm \infty, \text{ an unentangled state results.}$$

The nonvanishing correlation matrix elements are:

$$C_{xx} = \text{sech}(2\theta), C_{yy} = -\text{sech}(2\theta), C_{zz} = \text{sech}^2(2\theta).$$

Using (10), we obtain the correlation measure $E_C(|\phi^+\rangle) = \frac{1}{4} \text{Tr} \rho_{\phi^+} = \frac{1}{4} (2 \text{sech}^2(2\theta) + \text{sech}^4(2\theta)).$ The concurrence $C \equiv \frac{1}{2} \sqrt{2(1 - \text{Tr} [\rho_A^2])}$ is $C(|\phi^+\rangle) = \sqrt{2(1 - \frac{1}{4} (2 \text{sech}^2(2\theta) + \text{sech}^4(2\theta)))}$, since

$$\rho_A = \frac{1}{2} \left( \begin{array}{cc} 1 - \tanh(2\theta) & 0 \\ 0 & 1 + \tanh(2\theta) \end{array} \right),$$

and the entanglement entropy is $S \equiv -\text{Tr} [\rho_A \log_2 \rho_A].$ These results are graphically presented in Fig. 2. All the measures equal unity for $\theta = 0$ and decrease rapidly vs. $\theta$.

![FIG. 2: (color online) Comparison of the $E_C$ measure of correlation, the concurrence $C$ and the entanglement entropy $S$ for the Rashid pure states.](image)

Two-qubit classically-correlated states take the form

$$\rho^{CC} = \frac{1}{4} \sum_{k \geq 2} p_k (1 + n_{A,k} \cdot \sigma_A) (1 + n_{B,k} \cdot \sigma_B), \quad (16)$$

with $\sum_k p_k = 1$ and $p_k > 0$. The density matrix for the classically-correlated state can be written in the form of Eq. (2) with Bloch vectors

$$n_A = \sum_k p_k n_{A,k}, \quad n_B = \sum_k p_k n_{B,k}, \quad (17)$$

and correlation matrix

$$C_{ij} = \sum_k p_k n_{i,A,k} \left[ n_{j,B,k} - \sum_l p_l n_{j,B,l} \right]. \quad (18)$$

For example, for classically-correlated mixed states of the form $\rho^{CC} = (2 \text{sech}^2(2\theta))^{-1/2} (e^{-\theta} |\uparrow \uparrow\rangle \langle \downarrow \downarrow| + e^{\theta} |\downarrow \downarrow\rangle \langle \uparrow \uparrow|)$, the correlation matrices are diagonal and the correlation measure $E_C = 1$, i.e., they are maximally entangled.
we find that all the correlation coefficients vanish, except for 

$$C_{zz} = -\text{sech}^2(2\theta),$$

the density matrix in representation (24) is

$$\rho^{CC} = \frac{1}{2} \left( 1_{A} 1_{B} - \text{sech}^2(2\theta) \sigma_z A \sigma_z B \right),$$

and the classical-correlation measure is

$$\mathcal{E}^{CC} = \frac{1}{2} \text{sech}^4(2\theta).$$

It is elucidating to consider the Werner two-qubit density matrix composed of a sum of a singlet state and the

$$\rho^{W} = p|\psi^\prime\rangle\langle\psi^\prime| + \frac{1-p}{4} \mathbf{1},$$

where

$$|\psi^\prime\rangle = (2\cosh(2\theta))^{-1/2} \left( e^{-\theta} |\uparrow\rangle - e^{\theta} |\downarrow\rangle \right).$$

$$\rho^{GW}$$

reduces to

$$\rho^{W}$$

for \( \theta = 0 \). For \( \rho^{GW} \),

$$n_A = -n_B = p \tanh(2\theta) \hat{z},$$

and

$$C^{GW} = -p \begin{pmatrix} \text{sech}(2\theta) & 0 & 0 \\ 0 & \text{sech}(2\theta) & 0 \\ 0 & 0 & 1 - p + p \text{sech}^2(2\theta) \end{pmatrix}. \tag{21}$$

The PH entanglement criterion [19] shows that this state is entangled if

$$p[1 + 2\cosh(2\theta)] \geq 1.$$ 

Figure 3 plots the PH criterion limit and the correlation measure,

$$\mathcal{E}_C(p, \theta) = \sum_i d_i^2 = 1 - p + (2p^2 + 2p)\text{sech}^2(2\theta),$$

for the generalized Werner state. Note that the PH criterion is not obtainable from \( C \) alone, but can be obtained using the invariant parameters

$$\xi = \sum_i d_i - n_A \cdot n_B$$

and

$$n_A \cdot n_B.$$ 

More explicitly, \( p[1 + 2\cosh(2\theta)] = -\xi + \sqrt{\xi^2/4 - n_A \cdot n_B} \), so the PH condition reads

$$-\frac{\xi}{2} + \xi + \sqrt{\xi^2/4 - n_A \cdot n_B} \geq 1,$$ \tag{22}

which can be written as the condition: the largest root of the quadratic equation, \((x + \xi/2)^2 + \xi(x + \xi/2) + n_A \cdot n_B = 0\), is greater than unity. Thus, mixed state entanglement is determined not only by \( C \) but by additional invariant characteristics of the density matrix, i.e., invariant characteristics composed of the parameters \( C, n_A \) and \( n_B \) used to form the density matrix (whereas the correlation is determined only in terms of \( C \)). The physical significance of the scalar product \( n_A \cdot n_B \) as the projection of the expectation value of the spin of one qubit on the other, is clear, as is the physical significance of \( \xi \) as a specific projection of the singular values of the correlation matrix that depends on the average spins \( n_A \) and \( n_B \) [23]. However, the physical significance of the PH entanglement criterion is not yet clear; i.e., the physical interpretation of Eq. (22) [or the quadratic equation] remains to be uncovered. But at least the PH condition is now expressed only in terms of the physical parameters appearing in the density matrix, rather than by the partial transposition condition, which is more removed from physical interpretation.

As an example of a tripartite pure qutrit state, consider

$$|\psi^{E3}\rangle = \frac{e^{\theta_1} e^{\theta_2} |v_1 v_2 v_3\rangle + e^{-\theta_1} |v_2 v_1 v_3\rangle + e^{-\theta_2} |v_3 v_1 v_2\rangle}{\sqrt{e^{2\theta_1} e^{2\theta_2} + e^{-2\theta_1} + e^{-2\theta_2}}}.$$ \tag{23}

FIG. 3: (color online) \( \mathcal{E}_C(p, \theta) \) versus \( p \) and \( \theta \) for the generalized Werner density matrix \( \rho^{GW} \), and the Peres-Horodecki entanglement criterion limit, \( p[1 + 2\cosh(2\theta)] = 1 \), drawn on the \( p-\theta \) plane and projected onto the \( \mathcal{E}_C \) surface.

FIG. 4: (color online) The bipartite and tripartite correlation measures, \( \mathcal{E}_C \) and \( \mathcal{E}_D \) for the tripartite-qutrit pure state [23].
In most quantum information systems, qubits are distinguishable, but for identical bosonic or fermionic systems, the density matrix must be properly symmetrized, e.g., $\rho_{AB}^{\text{sym}} = S\rho_{AB}S$. For two qubits, the symmetrization operator is $S = \frac{1}{2}(1 + P_{AB}) = \frac{1}{2} + \frac{1}{2}\sigma_A \cdot \sigma_B$ and the antisymmetrization operator is $A = \frac{1}{2}(1 - P_{AB}) = \frac{1}{2} - \frac{1}{2}\sigma_A \cdot \sigma_B$, hence, $\rho_{AB}^{\text{sym}} = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\sigma_A \cdot \sigma_B) \rho_{AB} (\frac{1}{2} + \frac{1}{2}\sigma_A \cdot \sigma_B)$, $\rho_{AB}^{\text{anti}} = \frac{1}{2}(\frac{1}{2} - \frac{1}{2}\sigma_A \cdot \sigma_B) \rho_{AB} (\frac{1}{2} - \frac{1}{2}\sigma_A \cdot \sigma_B)$. If the qubits are antisymmetric, $\rho_{AB} = \rho_{AB}^{\text{anti}} = A\rho_{AB}A$, the state must be pure singlet, $\rho_{AB}^{\text{anti}} = \frac{1}{2}(1 - \sigma_A \cdot \sigma_B)$; it cannot be a mixed state, as opposed to $\rho_{AB}^{\text{sym}}$ which can be mixed. If spatial degrees of freedom need to be included in the description, in addition to the internal degrees of freedom, a bipartite density matrix can always be written as a product of an internal (i.e., spin) part and an external (i.e., space) part. Hence, a symmetric density matrix $\rho_{AB}^{\text{sym}}$ for the internal degrees of freedom must be multiplied by a symmetric [antisymmetric] spatial density matrix for the spatial degrees of freedom $\{r_A, r_B\}$ for bosons [fermions], and $\rho_{AB}^{\text{anti}}$ must be multiplied by an antisymmetric [symmetric] spatial density matrix for bosons [fermions], so that the full density matrix has the right exchange symmetry. We show elsewhere that this has relevance to collisional shifts in atomic clocks.

In summary, we have developed a classification of correlation for multipartite $N$-level quantum systems by writing their density matrices in terms of $SU(N)$ generators, and we defined a measure of correlation for such systems, based upon their correlation matrices. The entanglement involves not just the correlation matrix but also other invariant parameters in the density matrix for the system. This formulation can now be used for a variety of applications, e.g., in the optimization of quantum gates and in the calculation of collisional clock shifts.

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[1] P. W. Shor, SIAM J. Comput. 26, 1484 (1997); L. K. Grover, Phys. Rev. Lett. 79, 325 (1997).
[2] A. K. Ekert, Phys. Rev. Lett. 67, 661-663 (1991).
[3] C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).
[4] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[5] P. Shor, Phys. Rev. A52, R2493 (1995); A. Steane, Proc. Roy. Soc. Lond. A452, 2551 (1996).
[6] A. G. White, D. F. V. James, W. J. Munro, and P. G. Kwiat, Phys. Rev. A65, 012301 (2001).
[7] D. M. Greenberger, M. A. Horne and A. Zeilinger, “Going beyond Bell’s theorem”, in Bell’s theorem, quantum theory and conceptions of the universe, M. Kafatos, ed., (Kluwer Academic, Dordrecht) 73-76 (1989).
[8] C. Han, P. Xue and G.-C. Guo, Phys. Rev. A72, 034301 (2005).
[9] A.R. Calderbank and P.W. Shor, Phys. Rev. A 54, 1098 (1996).
[10] A. Karlsson and M. Bourennane, Phys. Rev. A 58, 4394 (1998).
[11] M. Hillery, V. Buzek and A. Berthiaume, Phys. Rev. A59, 1829 (1999).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[13] J. Dehaene M. Van den Nest and B. De Moor, Phys. Rev. A67, 022310 (2003).
[14] D. J. Wineland, J. J. Bollinger, W. M. Itano and D. J. Heinzen, Phys. Rev. A50, 67 (1994).
[15] R. F. Werner, Phys. Rev. A40, 4277 (1989).
[16] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261 (1998); S. Karnas and M. Lewenstein, J. Phys. A34, 6919 (2001); J. Sperling and W. Vogel, Phys. Rev. A79, 052313 (2009).
[17] J. M. Leinaas, J. Myrheim and E. Ovrum, Phys. Rev. A74, 012313 (2006).
[18] $C = U d V^\dagger$ where $U$ and $V$ are orthogonal and $d$ is diagonal. G. Strang, Linear Algebra and Its Applications, (Brooks Cole, 2005).
[19] A. Peres, Phys. Rev. Lett. 77, 1413 (1996); M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A223, 1 (1996); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[20] U. Fano, Rev. Mod. Phys. 55, 855(1983).
[21] H. Georgi, Lie Algebras in Particle Physics, (Westview Press, 1999).
[22] J. Schlienz and G. Mahler, Phys. Rev. A52, 4396 (1995).
[23] M. A. Rashid, J. Math. Phys. 19, 1391 (1978).
[24] W. K. Wootters, Phil. Trans. R. Soc. Lond A356, 1117-1131 (1998).
[25] In the case of the generalized Werner state, $\xi$ projects out the sum of two nonzero singular values $-p\sech^{2}(\theta)$.
[26] Y. B. Band and I. Osherov, “Collisionally Induced Atomic Clock Shifts and Correlations”, (to be published).