Some arithmetic properties of an elliptic Dedekind sum

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Abstract

We give an explicit expression of the elliptic classical Dedekind sum which is a special case of multiple elliptic Dedekind sums introduced by Egami. We also determine the denominator of the rational part and zeros of the elliptic classical Dedekind sum.

1 Introduction

Let $a$ and $b$ be relatively prime positive integers, then we set

\[ s(a; b) := \frac{1}{4b} \sum_{\nu=1}^{b-1} \cot \left( \frac{\pi a \nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right), \quad (1.1) \]

\[ s(a; 1) := 0, \quad (1.2) \]

which is well known (classical) Dedekind sum. Although Dedekind sum has no closed form with some exceptions like

\[ s(1; a) = \frac{(a-1)(a-2)}{12a}, \quad s(2; a) = \frac{(a-1)(a-5)}{24a}, \]

the following results hold.

1. **parity**

\[ s(-a; b) = -s(a; b), \quad (1.3) \]

2. **reduction**

\[ s(a + b; b) = s(a; b), \quad (1.4) \]

3. **reciprocity**

\[ s(a; b) + s(b; a) = -\frac{1}{4} + \frac{a^2 + b^2 + 1}{12ab}. \quad (1.5) \]

More precisely, classical Dedekind sum $s(a; b)$ has some arithmetic properties \[3\]:

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1. parity
2. reduction
3. reciprocity
Determine denominator

\[(2b \cdot \gcd (3, b)) \cdot s(a; b) \in \mathbb{Z},\]  
\(1.6\)

(5) Determine zeros

\[a^2 + 1 \equiv 0 \pmod{b} \iff s(a; b) = 0.\]  
\(1.7\)

For Dedekind sum and its reciprocity law, there are various generalizations. In particular, Egami [1] introduced an elliptic analogue of multiple Dedekind sums and gave its reciprocity law which is different from Sczech’s elliptic Dedekind sum [4]. After his job, Bayad, Fukuhara-Yui, Asano, Machide, et al. have studied more generalization of the multiple elliptic Dedekind sums and their reciprocity laws. However, it seems that investigations of specializations of their results have not been yet even in an elliptic analogue of the classical Dedekind sum

\[s_\tau(a; b) := \frac{1}{4b} \sum_{0 \leq \mu, \nu \leq b-1 \atop (\mu, \nu) \neq (0, 0)} (-1)^\mu \cs \left(2K a \frac{\mu \tau + \nu}{b}, k\right) \cs \left(2K a \frac{\mu \tau + \nu}{b}, k\right),\]

\[s_\tau(a; 1) := 0,\]

which we call elliptic classical Dedekind sum. This elliptic sum is desired to obtain more precise results like the classical case \((1.6), (1.7)\). In this article, we give an explicit expression of the elliptic classical Dedekind sum and derive some arithmetic properties Theorem 8 and Theorem 11.

The content of this paper is as follows. In Section 2, we introduce the elliptic function \(cs(z, k)\) and list its fundamental properties. In Section 3, we mention an elliptic analogue of the classical Dedekind sum and its known results. Section 4 is the key part of this article. By using fundamental properties the elliptic function \(cs(z, k)\) and the elliptic classical Dedekind sum, we give an explicit expression of the elliptic classical Dedekind sum. In Section 5 and 6, we derive fundamental properties of the rational part for the elliptic classical Dedekind sum. In particular, we determine the denominator of rational part and zeros. Finally, in Section 7, we mention two problems related to our research.

2 The elliptic function \(cs(z, k)\)

Throughout the paper, we denote the ring of rational integers by \(\mathbb{Z}\), the field of real numbers by \(\mathbb{R}\), the field of complex numbers by \(\mathbb{C}\) and the upper half plane \(\mathcal{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}\). For \(\tau \in \mathcal{H}\), we put

\[e(x) := e^{2\pi \sqrt{-1} x}, \quad q := e(\tau).\]
First, we recall the Jacobi theta functions

\[ \theta_1(z, \tau) := 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \sin ((2n+1)\pi z) \]

\[ = 2q^{\frac{1}{8}} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e(z))(1 - q^n e(-z)), \]

\[ \theta_2(z, \tau) := 2 \sum_{n=0}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \cos ((2n+1)\pi z) \]

\[ = 2q^{\frac{1}{8}} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n e(z))(1 + q^n e(-z)), \]

\[ \theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos (2n\pi z) \]

\[ = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{-\frac{1}{2}} e(z))(1 + q^{-\frac{1}{2}} e(-z)), \]

\[ \theta_4(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \cos (2n\pi z) \]

\[ = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{-\frac{1}{2}} e(z))(1 - q^{-\frac{1}{2}} e(-z)). \]

Further we put

\[ k = k(\tau) := \frac{\theta_2(0, \tau)^2}{\theta_3(0, \tau)^2}, \quad \lambda = \lambda(\tau) := k(\tau)^2, \quad K = K(\tau) := \frac{\pi}{2} \theta_3(0, \tau)^2 \]

and introduce the Jacobi elliptic functions

\[ \text{sn} \ (2Kz, k) := \frac{\theta_3(0, \tau) \theta_1(z, \tau)}{\theta_2(0, \tau) \theta_4(z, \tau)}, \]

\[ \text{cn} \ (2Kz, k) := \frac{\theta_4(0, \tau) \theta_2(z, \tau)}{\theta_2(0, \tau) \theta_4(z, \tau)}, \]

\[ \text{dn} \ (2Kz, k) := \frac{\theta_4(0, \tau) \theta_3(z, \tau)}{\theta_3(0, \tau) \theta_4(z, \tau)}. \]

As is well known, the Jacobi elliptic functions \( \text{sn} \ (2Kz, k) \), \( \text{cn} \ (2Kz, k) \) and \( \text{dn} \ (2Kz, k) \) only depend on \( \lambda(\tau) = k(\tau)^2 \) (elliptic lambda function) that is a modular function of the modular subgroup

\[ \Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \ (\text{mod} \ 2), \ b \equiv c \equiv 0 \ (\text{mod} \ 2) \right\}. \]
Therefore under the following we restrict \( \tau \) to the fundamental domain of \( \Gamma(2) \)

\[
\Gamma(2) \setminus \mathcal{H} \simeq \left\{ \tau \in \mathcal{H} \left| \Re \tau \leq 1, \left| \tau \pm \frac{1}{2} \right| \geq \frac{1}{2} \right. \right\}.
\]

We remark that for the generators of \( \text{SL}_2(\mathbb{Z}) \)

\[
S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ T := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

we have

\[
\lambda \left( -\frac{1}{\tau} \right) = 1 - \lambda(\tau), \quad \lambda(\tau + 1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \tag{2.1}
\]

In particular, since

\[
\lambda(\sqrt{-1}) = \lambda \left( -\frac{1}{\sqrt{-1}} \right) = 1 - \lambda(\sqrt{-1}),
\]

we have

\[
\lambda(\sqrt{-1}) = \frac{1}{2}. \tag{2.2}
\]

The elliptic function \( \text{cs}(2Kz,k) \) is defined by

\[
\text{cs}(2Kz,k) := \frac{\text{cn}(2Kz,k)}{\text{sn}(2Kz,k)},
\]

which is regarded as an elliptic analogue of \( \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} \). According to the wolfram functions site [7], we list fundamental properties of \( \text{cs} \).

**Lemma 1.** (1) (parity)

\[
\text{cs}(-2Kz,k) = -\text{cs}(2Kz,k). \tag{2.3}
\]

(2) (periodicity) For any \( \mu, \nu \in \mathbb{Z} \),

\[
\text{cs}(2K(z + \mu \tau + \nu),k) = (-1)^{\mu} \text{cs}(2Kz,k), \tag{2.4}
\]

(3) (Laurent expansion at \( z = 0 \))

\[
2K \text{cs}(2Kz,k) = \frac{1}{z} - \left( \frac{1}{3} - \frac{1}{6} \lambda \right)(2K)^2 z - \left( \frac{1}{45} - \frac{1}{45} \lambda - \frac{7}{360} \lambda^2 \right)(2K)^4 z^3 - \cdots. \tag{2.5}
\]

(4) (Derivation)

\[
\frac{\partial}{\partial z} 2K \text{cs}(2Kz,k) = -2K \text{ds}(2Kz,k) 2K \text{ns}(2Kz,k). \tag{2.6}
\]

\[\text{Page 4}\]
(5) (Relation between the Weierstrass φ function)

\[(2Kcs(2Kz,k))^2 = \varphi(z,\tau) - \varphi\left(\frac{1}{2},\tau\right).\]  

(2.7)

Here, \(\varphi(z,\tau)\) is the Weierstrass φ function defined by

\[
\varphi(z,\tau) := \frac{1}{z^2} + \sum_{n,m \in \mathbb{Z}} \left\{ \frac{1}{(m+n\tau+z)^2} - \frac{1}{(m+n\tau)^2} \right\}.
\]

(5) (Trigonometric degeneration)

\[
\lim_{\tau \to \sqrt{-1}\infty} k(\tau) = 0, \quad \lim_{\tau \to \sqrt{-1}\infty} 2K(\tau) = \pi,
\]

(2.8)  

(2.9)

\[
\lim_{\tau \to \sqrt{-1}\infty} \text{cs}(2K(z + w\tau),k) = \begin{cases} 
(-1)^w \cot(\pi z) & (w \in \mathbb{Z}) \\
-(-1)^{\lfloor w \rfloor i} & (w \notin \mathbb{Z}).
\end{cases}
\]

(2.10)

Here, \(\lfloor w \rfloor\) denotes the greatest integer not exceeding \(w\).

Remark 2. (1) Egami and others use \(\phi(\tau,z) := \sqrt{\varphi(z,\tau) - \varphi\left(\frac{1}{2},\tau\right)} = \frac{1}{z} + O(z) \quad (z \to 0)\)

instead of \(2Kcs(2Kz,k)\). However, Egami and others did not mention that \(\phi(\tau,z)\) is the Jacobi elliptic function \(2Kcs(2Kz,k)\) exactly.

(2) R. Sczech \[1\] introduced another elliptic Dedekind sum. He considered a real analytic Eisenstein series

\[
G(s, z; \tau) := \sum_{m,n \in \mathbb{Z}} \frac{m\tau + n + z}{|m\tau + n + z|^{2s}}
\]

for \(\text{Re}(s) > 1\) and by analytic continuation for other values of the complex number \(s\). In particular, \(G(1, z; \tau)\) is real analytic and doubly periodic for \(z\)

\[
G(1, z + 1; \tau) = G(1, z + \tau; \tau) = G(1, z; \tau),
\]

which is regarded as another elliptic analogue of cotangent function. Actually, \(G(1, z; \tau)\) has the following explicit expression

\[
G(1, z; \tau) = \zeta(z, \tau) - zG_2(\tau) + \frac{2\pi i}{\tau - \overline{\tau}}(z - \overline{z}),
\]
where $\zeta(z, \tau)$ is Weierstrass $\zeta$ function

$$
\zeta(z, \tau) := \frac{1}{z} + \sum_{n,m \in \mathbb{Z}, (n,m) \neq (0,0)} \left\{ \frac{1}{m+n\tau+z} - \frac{1}{m+n\tau} + \frac{z}{(m+n\tau)^2} \right\}
$$

and $G_2(\tau)$ is Eisenstein series of weight 2

$$
G_2(\tau) := \frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{8\pi^2 q^n}{(1-q^n)^2}.
$$

### 3 The elliptic classical Dedekind sum

Let $a$ and $b$ are relatively prime positive integers. According to Egami [1], we introduce the elliptic classical Dedekind sum by

$$
s_\tau(a; b) := \frac{1}{4b} \sum_{\substack{0 \leq \mu, \nu < b-1 \atop (\mu, \nu) \neq (0,0)}} (-1)^\mu \cos \left( 2Ka\frac{\mu\tau + \nu}{b} \right) \cos \left( 2K\frac{\mu\tau + \nu}{b} \right),
$$

(3.1)

$$
s_\tau(a; 1) := 0.
$$

(3.2)

It is regarded as an elliptic analogue of the classical Dedekind sum

$$
s(a; b) := \frac{1}{4b} \sum_{\nu=1}^{b-1} \cot \left( \frac{\pi a\nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right),
$$

$$
s(a; 1) := 0.
$$

For convenience, we introduce the following notations.

$$
R(a; b) := \frac{a^2 + b^2 + 1}{4ab},
$$

$$
U^o := \{(a; b) \in \mathbb{Z} \setminus \{0\} \times \mathbb{Z}_{>0} \mid \gcd(a, b) = 1, \ a + b \text{ is odd}\}.
$$

For the elliptic classical Dedekind sum, the following properties are known.

**Theorem 3.** (1) (parity)

$$
s_\tau(-a; b) = -s_\tau(a; b).
$$

(3.3)

(2) (even reduction)

$$
s_\tau(a + 2b; b) = s_\tau(a; b).
$$

(3.4)

(3) (inversion formula) If $b$ is odd and $aa' \equiv \pm 1 \pmod{b}$, or $b$ is even and $aa' \equiv \pm 1 \pmod{2b}$, then

$$
s_\tau(a; b) = \pm s_\tau(a'; b).
$$

(3.5)
(4) (reciprocity)

\[ s_{\tau}(a; b) + s_{\tau}(b; a) = R(a; b) \left( \frac{1}{3} - \frac{1}{6} \lambda(\tau) \right) \quad ((a; b) \in U^o). \]  

(5) (rationality) For any \((a; b) \in U^o\), there exists a unique rational number \(Q(a; b)\) such that

\[ s_{\tau}(a; b) = Q(a; b) \left( \frac{1}{3} - \frac{1}{6} \lambda(\tau) \right). \]  

(6) (degeneration)

\[ \lim_{\tau \to \sqrt{-1} \infty} s_{\tau}(a; b) = s(a; b) + \frac{1}{4} S(a; b). \]  

Here, \(S(a; b)\) is the Hardy-Berndt sum defined by

\[ S(a; b) := \sum_{\mu=1}^{b-1} (-1)^{\mu+1} \left\lfloor \frac{a \mu}{b} \right\rfloor. \]

Proof. Actually, (3.3) and (3.4) follow from (2.3) and (2.4) respectively. For (3.5), in the case that \(b\) is odd,

\[
\begin{align*}
\sum_{0 \leq \mu, \nu \leq b-1 \atop (\mu, \nu) \neq (0, 0)} (-1)^{\mu} 
& \left( 2K a \frac{2a' \mu \tau + 2a' \nu}{b}, k \right) 
& \left( 2K a \frac{2a' \mu \tau + 2a' \nu}{b}, k \right) 
= \frac{1}{4b} \sum_{0 \leq \mu, \nu \leq b-1 \atop (\mu, \nu) \neq (0, 0)} (-1)^{\mu} 
& \left( 2K a \frac{2a' \mu \tau + 2a' \nu}{b}, k \right) 
& \left( 2K a \frac{2a' \mu \tau + 2a' \nu}{b}, k \right) 
= \pm s_{\tau}(a'; b).
\end{align*}
\]

Similarly, in the case that \(b\) is even,

\[
\begin{align*}
\sum_{0 \leq \mu, \nu \leq b-1 \atop (\mu, \nu) \neq (0, 0)} (-1)^{\mu} 
& \left( 2K a \frac{a' \mu \tau + a' \nu}{b}, k \right) 
& \left( 2K a \frac{a' \mu \tau + a' \nu}{b}, k \right) 
= \frac{1}{4b} \sum_{0 \leq \mu, \nu \leq b-1 \atop (\mu, \nu) \neq (0, 0)} (-1)^{\mu} 
& \left( 2K a \frac{a' \mu \tau + a' \nu}{b}, k \right) 
& \left( 2K a \frac{a' \mu \tau + a' \nu}{b}, k \right) 
= \pm s_{\tau}(a'; b).
\end{align*}
\]
The reciprocity (3.6) is a specialization of the Egami’s reciprocity [1]. Unfortunately, Egami’s original statement (Theorem 1 in [1]) is incorrect, which is pointed out by Fukuhara-Yui [2]. Hence, we refer the correct result from Lemma 3.1 in [2].

\[- \text{cs} \left( 2Kaz, k \right) \text{cs} \left( 2Kbz, k \right) + \frac{1}{a} \text{ds} \left( 2Kz, k \right) \text{ns} \left( 2Kz, k \right) \]
\[= \frac{1}{a} \sum_{\mu, \lambda = 0}^{a-1} (-1)^{\lambda} \text{cs} \left( 2Kb \frac{\mu \tau + \lambda}{a}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \lambda}{a} \right), k \right) \]
\[+ \frac{1}{b} \sum_{\nu = 1}^{b-1} (-1)^{\mu} \text{cs} \left( 2Ka \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right). \quad (3.9)\]

The reciprocity (3.6) follows from (2.5), (3.9), the derivation of \( \text{cs} \left( z, k \right) \) and its Laurent expansion at \( z = 0 \)

\[\text{ds} \left( 2Kz, k \right) \text{ns} \left( 2Kz, k \right) = - \frac{1}{2K} \frac{\partial}{\partial z} \text{cs} \left( 2Kz, k \right) \]
\[= \frac{1}{(2K)^2 z^2} + \left( \frac{1}{3} - \frac{1}{6} \lambda \right) + 3 \left( \frac{1}{45} - \frac{1}{45} \lambda - \frac{7}{360} \lambda^2 \right)(2K)^2 z^2 + \cdots.\]

Rationality of \( s_\tau(a; b) \) (3.7) follows from (3.3), (3.4) and (3.6) immediately.

The degenerate limit (3.8) corresponds to trigonometric degeneration (2.10). Actually,

\[\lim_{\tau \to \sqrt{-1} \infty} s_\tau(a; b) = \lim_{\tau \to \sqrt{-1} \infty} \frac{1}{4b} \sum_{\nu = 1}^{b-1} (-1)^{\mu} \text{cs} \left( 2Ka \frac{\nu}{b}, k \right) \text{cs} \left( 2K \frac{\nu}{b}, k \right) \]
\[+ \frac{1}{4b} \sum_{0 \leq \mu, \nu \leq b-1} (-1)^{\mu} \text{cs} \left( 2Ka \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) \]
\[= \frac{1}{4b} \sum_{\nu = 1}^{b-1} (-1)^{\mu} \cot \left( \frac{\pi a \nu}{b}, k \right) \cot \left( \frac{\pi \nu}{b}, k \right) \]
\[+ \frac{1}{4b} \sum_{0 \leq \mu, \nu \leq b-1} (-1)^{\mu} (-i)^2 (-1)^{\left\lfloor \frac{\mu}{2} \right\rfloor} (-1)^{\left\lfloor \frac{\nu}{2} \right\rfloor} \]
\[= s(a; b) + \frac{1}{4b} b S(a; b).\]

We point out the values of \( s_\tau(a; b) \) on \( U^o \) are determined by (3.3), (3.4), (3.6) and the Euclidean algorithm exactly. Hence we need not recall the original definition of \( \text{cs}(z, k) \) to evaluate \( s_\tau(a; b) \) on \( U^o \). As a corollary of (3.7), we have the following result. \[\Box\]
Corollary 4.

\[ s_{-\frac{1}{2}}(a; b) = Q(a; b) \left( \frac{1}{6} + \frac{1}{6} \lambda(\tau) \right), \quad (3.10) \]
\[ s_{\tau + 1}(a; b) = Q(a; b) \frac{\lambda(\tau) - 2}{6(\lambda(\tau) - 1)}, \quad (3.11) \]
\[ s_{-\frac{1}{\tau + 1}}(a; b) = Q(a; b) \frac{1 - 2\lambda(\tau)}{6(\lambda(\tau) - 1)}, \quad (3.12) \]
\[ s_{\frac{1}{\tau + 1}}(a; b) = Q(a; b) \frac{\lambda(\tau) + 1}{6\lambda(\tau)}, \quad (3.13) \]
\[ s_{\frac{1}{\tau + 1}}(a; b) = Q(a; b) \frac{2\lambda(\tau) - 1}{6\lambda(\tau)}. \quad (3.14) \]

In particular,

\[ s_{\frac{1}{\tau + 1}}(a; b) = s_{\frac{1}{\tau + 1}}(a; b) = 0. \quad (3.15) \]

Proof. From modular transform (2.1),

\[ \lambda \left( \frac{1}{\tau + 1} \right) = \frac{1}{1 - \lambda(\tau)}, \quad \lambda \left( \frac{\tau - 1}{\tau} \right) = \frac{\lambda(\tau) - 1}{\lambda(\tau)}, \quad \lambda \left( \frac{\tau}{\tau + 1} \right) = \frac{1}{\lambda(\tau)}. \]

Hence we have (3.10) - (3.14). Recalling a special values of \( \lambda \) (2.2),

\[ s_{\frac{1}{\tau + 1}}(a; b) = s_{\frac{1}{\tau + 1}}(a;b) = Q(a; b) \left( \frac{1 - 2\lambda(\sqrt{-1})}{6(\lambda(\sqrt{-1}) - 1)} \right) = 0, \]
\[ s_{\frac{1}{\tau + 1}}(a; b) = s_{\frac{1}{\tau + 1}}(a; b) = Q(a; b) \left( \frac{2\lambda(\sqrt{-1}) - 1}{6\lambda(\sqrt{-1})} \right) = 0. \]

Under the following we assume \( \tau \neq \frac{-1 + \sqrt{-1}}{2} \) or \( \frac{1 + \sqrt{-1}}{2} \). In these cases, since \( \frac{1}{3} - \frac{1}{2} \lambda(\tau) \neq 0 \), our elliptic classical Dedekind sum \( s_\tau(a; b) \) on \( U^o \) is equivalent to the rational part \( Q(a; b) \). Under the following sections, we assume \( (a; b) \in U^o \).

**Theorem 5.** (1) *parity*

\[ Q(-a; b) = -Q(a; b). \quad (3.16) \]

(2) *even reduction*

\[ Q(a + 2b; b) = Q(a; b). \quad (3.17) \]

(3) *inversion formula* If \( b \) is odd and \( a, a' \) are even such that \( aa' \equiv \pm 1 \pmod{b} \), or \( b \) is even and \( a, a' \) are even such that \( aa' \equiv \pm 1 \pmod{2b} \), then

\[ Q(a; b) = \pm Q(a'; b). \quad (3.18) \]

(4) *reciprocity*

\[ Q(a; b) + Q(b; a) = R(a; b) \quad ((a; b) \in U^o). \quad (3.19) \]

Actually, (3.16), (3.17), (3.18) and (3.19) correspond to (3.3), (3.4), (3.5) and (3.6) respectively. We remark that \( Q(a; b) \) on \( U^o \) are determined by (3.16), (3.17) and (3.19) exactly. Hence, using (3.16), (3.17) and (3.19), we give tables of \( 4bs_{\sqrt{-1}}(a; b) = bQ(a; b) \).
Table 1: $bQ(a; b)$

4 An explicit formula of the rational part $Q(a; b)$

Theorem 6.

$$Q(a; b) = 3 \left(s(a; b) + \frac{1}{4} S(a; b)\right). \quad (4.1)$$

Proof. From (3.7),

$$s_\tau(a; b) = Q(a; b) \left(\frac{1}{3} - \frac{1}{6}\lambda(\tau)\right).$$

Since $Q(a; b)$ does not depend on $\tau$, we have

$$Q(a; b) = \lim_{\tau \to \sqrt{-1}} \frac{s_\tau(a; b)}{\frac{1}{3} - \frac{1}{6}\lambda(\tau)}.$$
Recalling trigonometric degenerations (2.8) and (3.8), we obtain the conclusion.

The Hardy-Berndt sum is written by the classical Dedekind sum. Actually, Sitaramachandra Rao prove the following formula.

Lemma 7 (Sitaramachandra Rao [3]). If \((a; b) \in U^{o}\), then

\[
S(a; b) = 8s(a; 2b) + 8s(2a; b) - 20s(a; b).
\] (4.2)

Using this formula (4.2), we have

\[
Q(a; b) = 6(s(a; 2b) + s(2a; b) - 2s(a; b)).
\] (4.3)

5 Denominator

We determine the denominator of \(Q(a; b)\).

Theorem 8. For any \((a; b) \in U^{o}\), there exists an integer \(M(a; b) \in \mathbb{Z}\) such that

\[
bQ(a; b) = \frac{a(1 - 3b)}{2} + M(a; b).
\] (5.1)

In particular,

\[
bQ(a; b) \in \begin{cases} \frac{1}{2} + \mathbb{Z} & \text{(if } a \text{ is odd and } b \text{ is even)} \\ \mathbb{Z} & \text{(if } a \text{ is even and } b \text{ is odd).} \end{cases}
\] (5.2)

Proof. It is well known that the classical Dedekind sum has the following expression [3].

\[
s(a; b) = \sum_{\nu=1}^{b-1} \left( \frac{av}{b} - \left\lfloor \frac{av}{b} \right\rfloor - \frac{1}{2} \right) \left( \frac{\nu}{b} - \frac{1}{2} \right).
\] (5.3)

From (4.3) and (5.3), we have

\[
Q(a; b) = ab + \frac{a(1 - 3b)}{2b} + 3 \sum_{\nu=1}^{b-1} \left\lfloor \frac{av}{2b} \right\rfloor (b - \nu) + 3 \sum_{\nu=1}^{b-1} \left\{ \left\lfloor \frac{2av}{b} \right\rfloor - 2 \left\lfloor \frac{av}{b} \right\rfloor \right\} (b - 2\nu).
\]

Hence if we put

\[
M(a; b) := ab^2 + 3 \sum_{\nu=1}^{b-1} \left\lfloor \frac{av}{2b} \right\rfloor (b - \nu) + 3 \sum_{\nu=1}^{b-1} \left\{ \left\lfloor \frac{2av}{b} \right\rfloor - 2 \left\lfloor \frac{av}{b} \right\rfloor \right\} (b - 2\nu)
\]

then

\[
bQ(a; b) = \frac{a(1 - 3b)}{2} + M(a; b).
\]
As a corollary of Theorem 6 and Lemma 8, we obtain some properties for the integral part \( M(a; b) \).

**Corollary 9.** (1) \textit{(parity)}

\[
M(-a; b) = -M(a; b).
\]

(5.4)

(2) \textit{(even reduction)}

\[
M(a + 2b; b) = M(a; b) + b(1 - 3b).
\]

(5.5)

(3) \textit{(reciprocity)}

\[
aM(a; b) + bM(b; a) = \frac{1 - a^2 - b^2 + 6ab(a + b)}{4}.
\]

(5.6)

6 Zeros

We determine the denominator of zeros for \( Q(a; b) \).

**Proposition 10.** If \( a \) is even and \( b \) is odd such that

\[
a^2 \equiv -1 \pmod{b},
\]

then we have

\[
Q(a; b) = 0.
\]

(6.1)

**Proof.** From assumptions of Proposition 10 if \( a' = a \) then

\[
aa' \equiv -1 \pmod{b}.
\]

We recall (3) of Theorem 5 and have

\[
Q(a; b) = -Q(a'; b) = -Q(a; b).
\]

\(\square\)

**Theorem 11.** Let \( a \) be even and \( b \) be odd.

\[
Q(a; b) \in \mathbb{Z} \implies a^2 \equiv -1 \pmod{b}.
\]

In particular,

\[
Q(a; b) = 0 \iff a^2 \equiv -1 \pmod{b}.
\]

**Proof.** Since \( a \) is even and \( b \) is odd, \( 2aQ(b; a) = b(1 - 3a) + 2M(b; a) \in \mathbb{Z} \) and \( 2abQ(b; a) \in b\mathbb{Z} \). If \( Q(a; b) \in \mathbb{Z} \), then \( 4abQ(a; b) \in b\mathbb{Z} \). The reciprocity (3.19) multiplied by \( 4ab \) gives

\[
4abQ(a; b) + 4abQ(b; a) = a^2 + b^2 + 1.
\]

(6.2)

Hence we have

\[
a^2 + 1 \equiv 0 \pmod{b}.
\]

From Proposition 10 \( Q(a; b) = 0 \). \(\square\)
Next, we construct zeros pair \((a; b)\) of \(Q(a; b)\) explicitly. Let \(M\) be a non negative integer and \(N\) be a positive integer. We consider the following positive integer sequence defined by

\[
P^{(N)}_{M+2} = N P^{(N)}_{M+1} + P^{(N)}_M, \quad P^{(N)}_0 = 0, \quad P^{(N)}_1 = 1,
\]

which is a generalization Fibonacci sequence

\[
F_{M+2} = F_{M+1} + F_M, \quad F_0 = 0, \quad F_1 = 1
\]
or Pell sequence

\[
P_{M+2} = 2P_{M+1} + P_M, \quad P_0 = 0, \quad P_1 = 1.
\]

We remark that the Cassini type formula

\[
P^{(N)}_{M+1} P^{(N)}_{M-1} - P^{(N)}_M^2 = (-1)^M
\]

holds for this sequence \(\{P^{(N)}_M\}_{M=0,1,2,...}\). Therefore by applying Theorem 5 (3) to \(\{P^{(N)}_M\}_{M=0,1,2,...}\), we construct some zeros pairs explicitly.

**Theorem 12.** (1) We have

\[
P^{(2n)}_{2m+1} \in 1 + 2\mathbb{Z}, \quad P^{(2n)}_{2m} \in 2\mathbb{Z}
\]

and

\[
Q(P^{(2n)}_{2m}; P^{(2n)}_{2m+1}) = 0,
\]

\[
Q(P^{(2n)}_{2m+1}; P^{(2n)}_{2m}) = Q(P^{(2n)}_{2m}; P^{(2n)}_{2m+1}). \tag{6.3}
\]

(2) We have

\[
P^{(2n+1)}_{3m+1} \in 1 + 2\mathbb{Z}, \quad P^{(2n+1)}_{3m} \in 2\mathbb{Z}
\]

and

\[
Q(P^{(2n+1)}_{3m}; P^{(2n+1)}_{3m+1}) = (-1)^n Q(P^{(2n+1)}_{3m}; P^{(2n+1)}_{3m+1}), \tag{6.5}
\]

\[
Q(P^{(2n+1)}_{3m+1}; P^{(2n+1)}_{3m}) = (-1)^n Q(P^{(2n+1)}_{3m}; P^{(2n+1)}_{3m+1}). \tag{6.6}
\]

In particular, we obtain

\[
Q(P^{(2n+1)}_{6m+3}; P^{(2n+1)}_{6m+3+1}) = 0. \tag{6.7}
\]

7 Concluding remarks

We raise two problems related to our investigation. First, we desire to give more precisely result than Lemma 8. Actually, from the Table 2 of \(4bs\sqrt{-1}(a; b) = bQ(a; b)\), we consider the following conjecture.
Conjecture 13. If $a$ is even and $b$ is odd then

$$4bs\sqrt{-1}(a; b) = bQ(a; b) \in 4\mathbb{Z}.$$  

More precisely,

$$(6m \pm 1)Q(a; 6m \pm 1) \in 12\mathbb{Z}.$$  

The next problem is to generalize our results to the elliptic Dedekind-Apostol sum \[\text{[2]}\]

$$s_{2n+1,\tau}(a; b) := \frac{1}{4b} \sum_{\mu, \nu=0}^{b-1} (-1)^\mu \text{cs} \left( 2Ka^{\mu\tau} + \nu \right) \text{cs}^{(2n)} \left( 2K \frac{a^{\mu\tau} + \nu}{b} \right) \quad (n \in \mathbb{Z}_{\geq 0})$$

(remark that $s_{2n,\tau}(a; b)$ is identically zero). The elliptic Dedekind-Apostol sum $s_{2n+1,\tau}(a; b)$ has the following properties, similar to the elliptic classical Dedekind sum $s_\tau(a; b)$.

1. **parity**

$$s_{2n+1,\tau}(-a; b) = -s_{2n+1,\tau}(a; b).$$

2. **even reduction**

$$s_{2n+1,\tau}(a + 2b; b) = s_{2n+1,\tau}(a; b).$$

3. **reciprocity** (Fukuhara-Yui) If $a + b$,

$$s_{2n+1,\tau}(a; b) + s_{2n+1,\tau}(b; a) = R_{2n+1,0}(a, b)g_{2n+1}(k) - \sum_{l=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} R_{2n+1,l}(a, b)g_{2n+1-l}(k)g_{2l-1}(k),$$

where $g_{2n+1}(k)$ is the coefficient of Laurent expansion for $\text{cs}(z, k)$ at $z = 0$

$$\text{cs}(z, k) = \frac{1}{z} + \sum_{n=0}^{\infty} g_{2n+1}(k)z^{2n+1}$$

and

$$R_{2n+1,l}(a, b) := \frac{(2n)!}{4} \left( a^{2l-1}b^{n+1-2l} + a^{n+1-2l}b^{2l-1} + \frac{2n + 1}{ab} \delta_{l,0} - a^n b^n \delta_{n,2l-1} \right).$$

It is easy to show that if $a + b$ is odd then there exists a rational number $Q_{2n+1,l}(a; b)$ independent of $k$ such that

$$s_{2n+1,\tau}(a; b) = Q_{2n+1,0}(a; b)g_{2n+1}(k) + \sum_{l=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} Q_{2n+1,l}(a; b)g_{2n+1-l}(k)g_{2l-1}(k).$$
Our main result is $Q_{1,0}(a; b) = 6(s(a; 2b) + s(2a; b) - 2s(a; b))$, and corresponds to the $n = 0$ case. We desire to obtain explicit formulas using Dedekind-Apostol sums

$$s_{2n+1}(a; b) := \frac{1}{4b} \sum_{\nu=1}^{b-1} \cot\left(\frac{\pi a \nu}{b}\right) \cot\left(\frac{\pi \nu}{b}\right),$$

$$s_{2n+1}(a; 1) := 0$$

for $Q_{2n+1,l}(a, b)$ ($n > 1$).

| $a \setminus b$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 |
|-----------------|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 3               | 1 |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |
| 5               | 0 | 3 |   |   |    |    |    |    |    |    |    |    |    |    |    |    |
| 7               | 3 | 3 | 6 |   |    |    |    |    |    |    |    |    |    |    |    |    |
| 9               | 1 | -1 | * | 10 |    |    |    |    |    |    |    |    |    |    |    |    |
| 11              | 6 | -3 | 6 | 3 | 15 |    |    |    |    |    |    |    |    |    |    |    |
| 13              | 3 | 9 | -3 | 0 | 9 | 21 |    |    |    |    |    |    |    |    |    |    |
| 15              | 10 | 8 | * | 10 | * | * | 28 |    |    |    |    |    |    |    |    |    |
| 17              | 6 | 0 | -9 | -6 | 12 | 12 | 9 | 36 |    |    |    |    |    |    |    |    |
| 19              | 15 | -3 | 18 | 6 | 15 | 6 | 3 | 18 | 45 |    |    |    |    |    |    |    |
| 21              | 10 | 17 | * | 1 | -10 | * | * | 17 | * | 55 |    |    |    |    |    |    |    |
| 23              | 21 | 15 | 15 | -18 | -9 | 21 | -6 | 9 | 6 | 18 | 66 |    |    |    |    |    |    |
| 25              | 15 | 3 | -3 | 30 | * | -15 | 3 | -3 | 0 | * | 30 | 78 |    |    |    |    |    |
| 27              | 28 | -1 | * | 10 | -10 | * | 28 | 26 | * | 1 | 26 | * | 91 |    |    |    |    |
| 29              | 21 | 27 | -12 | 3 | -30 | 0 | -21 | 24 | -3 | 24 | 27 | 12 | 30 | 105 |    |    |
| 31              | 36 | 24 | 33 | 24 | 45 | -12 | 6 | 36 | 12 | 6 | 30 | 30 | 33 | 45 | 120 |    |    |
| 33              | 28 | 8 | * | -8 | 26 | * | 19 | -28 | * | -17 | * | * | 19 | 17 | * | 136 | |

Table 2: $\frac{b}{4}Q(a; b)$

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