Deforming Convex Hypersurfaces to a Hypersurface with Prescribed Harmonic Mean Curvature *

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Abstract. Let $F$ be a smooth convex and positive function defined in $A = \{x \in \mathbb{R}^{n+1} : R_1 < |x| < R_2 \}$ satisfying $F(x) \geq nR_2$ on the sphere $|x| = R_2$ and $F(x) \leq nR_1$ on the sphere $|x| = R_1$. In this paper, a heat flow method is used to deform convex hypersurfaces in $A$ to a hypersurface whose harmonic mean curvature is the given function $F$.

Keywords: heat flows, prescribed curvature problems, hypersurfaces, convexity-preserving, parabolic equations on manifolds.

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1. Introduction

Let $M$ be a smooth embedded hypersurface in $\mathbb{R}^{n+1}$ and $k_1, k_2, \cdots, k_n$ be its principal curvatures. Then $H^{-1}$ is called the harmonic mean curvature of $M$ if

$$H = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}.$$  \hspace{1cm} (1.1)

The question which we are concerned with is that given a function $f$ defined in $\mathbb{R}^{n+1}$, under what conditions does the equation

$$H^{-1}(X) = f(X), X \in M$$

has a solution for a smooth, closed, convex and embedded hypersurface $M$, where $X$ is a position vector on $M$.

The kind of such question was proposed by S.T.Yau in his famous problem section [Y]. Many authors have studied the cases of mean curvature and Gauss curvature instead. See, for instance, [BK], [TW], [T], [CNS], and [TS1] for the mean curvature and [O1], [O2], [TS2] for the Gauss curvature, [CNS], [G1] and [G2] for general curvature functions.

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Let \( F = f^{-1} \), then the problem above is equivalent to looking for a smooth, closed, convex and embedded hypersurface \( M \) in \( \mathbb{R}^{n+1} \) such that
\[
H(X) = F(X), \quad X \in M
\]
where \( H \) is the inverse of harmonic mean curvature given by (1.1). We are interested only in the hypersurfaces \( M \subset A \), a ring domain defined by
\[
A = \{ X \in \mathbb{R}^{n+1} : R_1 < |X| < R_2 \}
\]
for some constants \( R_2 > R_1 > 0 \). For this purpose, we need to suppose that \( F \) is a smooth positive function defined in \( \mathbb{R}^{n+1} \) satisfying
\[
\text{(a)} \quad F(X) > nR_2 \quad \text{for} \quad |X| = R_2 \quad \text{and} \quad F(X) < nR_1 \quad \text{for} \quad |X| = R_1,
\]
and
\[
\text{(b)} \quad F \text{ is concave in } A.
\]

We will use a heat flow method to deform convex hypersurfaces to a solution to (1.2). That is, we consider the parabolic equation
\[
\frac{\partial X}{\partial t} = (H(X) - F(X))\nu(X), \quad X \in M_t, t \in (0, T)
\]
where \( X(x, t) : S^n \rightarrow \mathbb{R}^{n+1} \) is the parametrization of \( M_t \) given by the inverse Gauss map, which will be solved, and \( \nu(X) \) is the outer normal at \( X \in M_t \), so \( \nu(X(x, t)) = x \) by the definition. Of course, \( M_0 \) is a given initial hypersurface.

The following is our main result of this paper.

**Theorem 1.1.** Suppose that \( F \) is a smooth positive function satisfying conditions (a) and (b), and a initial hypersurface \( M_0 \subset A \) is smooth, uniformly convex and embedded, satisfying \( H(X_0) \geq F(X_0) \) for all \( X_0 \in M_0 \). Then equation (1.3) has a unique smooth solution for \( T = \infty \) which parametrizes a family of smooth, closed, uniformly convex and embedded hypersurfaces, \( \{ M_t : t \in [0, \infty) \} \). Moreover, there exists a subsequence \( t_k \rightarrow \infty \) such that \( M_{t_k} \) converges to a smooth, closed, uniformly convex and embedded hypersurface which lies in \( \bar{A} \) and solves problem (1.2).

We will have to meet two difficulties in proving the result above. One is the gradient estimate for the support function, \( u(x, t) \), of the hypersurfaces \( M_t \) which solves the equation (2.4) below; the other is the proof of convexity preserving (see the inequality (2.5)). In order to overcome the first difficulty, we use the well-known equality \( |X(x, t)|^2 = u^2 + |\nabla u|^2 \) and a general geometric result suggested by R. Bartnik to estimate the \( |X(x, t)|^2 \) for the equation (1.3). The proof of convexity-preserving is inspired by the computations in Hamilton [H1,H2] and Huisken [HU].

**Remark 1.2.** Applying a usual approximation method, one can replace condition (a) in theorem 1.1 by
\[
\text{(a')} \quad F(X) \geq nR_2 \quad \text{for} \quad |X| = R_2 \quad \text{and} \quad F(X) \leq nR_1 \quad \text{for} \quad |X| = R_1.
\]

After completing the paper, the author was aware that similar results had been obtained by Gerhardt in [G1]. Although the results in [G1] are very general, they don’t
include the above theorem 1.1 because our curvature function $H$ does not belong the class $(K)$ in [G1] and our initial hypersurfaces $M_0$ may be arbitrary instead of the fixed barrier $M_1 = \{|X| = R_1\}$ in [G1]. Moreover, the arguments are absolutely different.

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2. Evolution equations, convexity-preserving and global solutions

In this section, we will at first reduce the equation (1.3) to an equivalent quasilinear parabolic equation on $S^n$ for the support function of $M_t$, then we will show that this parabolic problem is globally solvable and it preserves convexity.

We recall some facts in [U, p.97-98]. Let $e_1, e_2, \ldots, e_n$ be a smooth local orthonormal frame on $S^n$, and let $\nabla_i = \nabla_{e_i}$, $i = 1, 2, \ldots, n$ and $\nabla = (\nabla_1, \nabla_2, \ldots, \nabla_n)$ be the covariant derivatives and the gradient on $S^n$, respectively. Since $X(x, t)$ is the inverse Gauss map, the support function of $M_t$ is given by

$$u(x, t) = < x, X(x, t) >, x \in S^n \tag{2.1}$$

where $< \cdot , \cdot >$ denotes the usual inner product in $R^{n+1}$. The second fundamental form of $M_t$ is

$$h_{ij}(X(x, t)) = \nabla_i \nabla_j u(x, t) + \delta_{ij} u(x, t), i, j = 1, 2, \ldots, n.$$ 

If $M_t$ is uniformly convex, then $h_{ij}$ is invertible, and hence the inverse harmonic mean curvature is the sum of all the eigenvalues of the matrix

$$b_{ij} = \left[ g^{ik} h_{kj} \right]^{-1},$$

where $g_{ij}$ is the metric of $M_t$. But the Gauss-Weingarten relation

$$\nabla_i x = h_{ik} g^{kl} \nabla_l X$$

and the fact $< \nabla_i x, \nabla_j x > = \delta_{ij}$ imply $g_{ij} = h_{ik} h_{jk}$. Therefore, $b_{ij} = h_{ij}$ and

$$H = \frac{1}{k_1} + \cdots + \frac{1}{k_n} = \Delta u + nu, \tag{2.2}$$

where $\Delta = \sum_{i=1}^n \nabla_i \nabla_i$. Furthermore, since $x, \nabla_1 x, \nabla_2 x, \ldots, \nabla_n x$ form a standard orthonormal basis at point $X(x, t)$, so we have

$$X(x, t) = < X, x > + < X, \nabla_i x > \nabla_i x = ux + \nabla_i < X, x > \nabla_i x = ux + \nabla_i u \nabla_i x. \tag{2.3}$$

Using the results above and repeating the argument [U, p.98-101] in verbatim, one has obtained the following lemma.
Lemma 2.1. If for \( t \in [0,T) \) with \( T \leq \infty \) \( X(x, t) \) is a solution of (1.3) which parametrizes a smooth, closed, uniformly convex and embedded hypersurface, then the support functions \( u(x, t) \) of \( M_t \) satisfy

\[
\frac{\partial u}{\partial t} = \Delta u + nu - F(ux + \nabla_i u \nabla_i x), \quad (x, t) \in S^n \times (0, T)\tag{2.4}
\]

and

\[
\nabla^2 u + u I > 0 \text{ in } S^n \times (0, T), \tag{2.5}
\]

where \( I \) denotes the \( n \times n \) unit matrix. Conversely, if \( u \) is a smooth solution to (2.4) and satisfies (2.5), then the hypersurface \( M_t \), determined by its support function \( u(x, t) \), is a smooth, closed, uniformly convex, embedded hypersurface and solves (1.3) for \( t \in [0, T) \).

From now on, we assume that the initial hypersurface \( M_0 \) is smooth, closed, uniformly convex. That is \( u_0 \in C^\infty(S^n) \) and for some positive constant \( C_0 \),

\[
C_0 I \leq \nabla^2 u_0 + u_0 I. \tag{2.6}
\]

Noting that (2.4) is a quasilinear parabolic equation on the compact manifold \( S^n \), by standard result for short-time existence (see, for example, [H3]), we have

Lemma 2.2. There exists a maximal existence time \( T = T(u_0) \in (0, \infty] \) such that (2.4) has a unique smooth solution \( u \in C^\infty(S^n \times (0, T)) \cap C([0, T); C^\infty(S^n)) \). If \( T < \infty \), then

\[
\lim_{t \to T^-} \|u(\cdot, t)\|_{C^1(S^n)} = \infty.
\]

Remark 2.3. One can use the contraction principle and repeat the same argument as in proving the short-time existence for harmonic heat flows to give a direct proof of this lemma. See [ES] or [D].

For the sake of deriving an apriori estimates, we need the following geometric result which was suggested by R. Bartnik.

Lemma 2.4. Let \( X \) be the positive vector of a smooth, closed hypersurface \( M \) in \( \mathbb{R}^{n+1} \) with outer normal \( \nu(X) \) at \( X \in M \). Then if \( |X| = \langle X, X \rangle^{\frac{1}{2}} \) attains a maximum \( R \) at a point \( X_0 \in M \), then \( X_0 = R\nu(X_0) \) and

\[
\Pi(w, w) \geq \frac{1}{R} g(w, w), \forall w \in T_{X_0} M;
\]

if \( |X| \) attains a minimum \( r \) at a point \( X_0 \in M \), then \( X_0 = r\nu(X_0) \) and

\[
\Pi(w, w) \leq \frac{1}{r} g(w, w), \forall w \in T_{X_0} M_0,
\]

where \( g \) is the metric on \( M \) and \( \Pi \) is the second fundamental form of \( M \) with respect to the direction \(-\nu\).

Proof. We consider only the first case, because the latter is completely analogous. For \( X_0 \in M \) and any \( w \in T_{X_0} M \), choose a curve \( \gamma(s) \) on \( M, \gamma : [0, 1] \to M \), such that

\[
\gamma(0) = X_0, \quad \dot{\gamma} = w.
\]
Let \( \rho(X) = |X| \). Since \( \rho^2(X) \) attains a maximum at \( X_0 \in M \), then at this point
\[
\nabla \rho^2 = 0 \text{ and } \nabla^2 \rho^2 \leq 0.
\]
Therefore, we have
\[
\frac{d}{ds} \rho^2(\gamma(0)) = \nabla \rho^2(X_0) \cdot \dot{\gamma}(0) = 0
\]
and
\[
\frac{d^2}{ds^2} \rho^2(\gamma(0)) = \ddot{\gamma}(0) \cdot \nabla^2 \rho^2(X_0) \cdot \dot{\gamma}(0) + \nabla \rho^2(X_0) \cdot \dddot{\gamma}(0) \leq 0.
\]
On the other hand,
\[
\frac{d\rho^2}{ds} = \frac{d}{ds} |\gamma(s)|^2 = 2\gamma(s) \cdot \dot{\gamma}(s)
\]
and
\[
\frac{1}{2} \frac{d^2 \rho^2}{ds^2} = |\dot{\gamma}(s)|^2 + \gamma(s) \cdot \ddot{\gamma}(s).
\]
Thus,
\[
0 = 2\gamma(0) \cdot \dot{\gamma}(0) = 2X_0 \cdot w \tag{2.7}
\]
and
\[
0 \geq |\dot{\gamma}(0)|^2 + X_0 \cdot \ddot{\gamma}(0). \tag{2.8}
\]
Since \( w \in T_{X_0}M \) can be arbitrary, (2.7) implies
\[
X_0 = |X_0|\nu(X_0) = R\nu(X_0),
\]
and so,
\[
X_0 \cdot \dddot{\gamma}(0) = R < \nu(X_0), \dddot{\gamma}(0) > = R < \nu(X_0), D_{\dddot{\gamma}(0)} \dddot{\gamma}(0) > = -R \Pi (\dddot{\gamma}(0), \dddot{\gamma}(0)) = -R \Pi (w, w),
\]
which, together with (2.8), gives us that
\[
\Pi(w, w) \geq \frac{1}{R} |\dot{\gamma}(0)|^2 = \frac{1}{R} < \dot{\gamma}(0), \dot{\gamma}(0) > = \frac{1}{R} < D_{\dot{\gamma}(0)}X, D_{\dot{\gamma}(0)}X > = \frac{1}{R} g(\dot{\gamma}(0), \dot{\gamma}(0)) = \frac{1}{R} g(w, w).
\]

**Lemma 2.5.** Suppose that \( M_0 \subset A \) in addition to (2.6). Let \( u \) be a smooth solution to (2.4) and satisfies (2.5) on \( S^n \times (0, T) \) with \( T \leq \infty \). Then for all \((x, t) \in S^n \times (0, T)\), we have
\[
R_1^2 < u^2(x, t) + |\nabla u(x, t)|^2 < R_2^2.
\]
Proof. It follows from lemma 2.1 that the position vector $X(\cdot,t)$ of $M_t$ determined by the support function $u(x,t)$ satisfies (1.3), i.e.

$$\frac{\partial X}{\partial t} = (H(X) - F(X))\nu(X), X(x,t) \in M_t, (x,t) \in S^n \times (0,T) \} \quad (2.9)$$

Moreover, (2.3) and the fact that $x, \nabla_1 x, \cdots, \nabla_n x$ form a standard orthonormal basis imply that

$$|X|^2 = u^2 + |\nabla u|^2, \forall (x,t) \in S^n \times (0,T). \quad (2.10)$$

Thus, it is sufficient to prove that for $X$ solving (2.9),

$$R_1^2 < |X(x,t)|^2 < R_2^2, \forall (x,t) \in S^n \times (0,T). \quad (2.11)$$

For each $t \in [0,T)$, let

$$P_{\text{min}}(t) = \min_{X \in M_t} |X|^2 = \min_{x \in S^n} |X(x,t)|^2,$$

and

$$P_{\text{max}}(t) = \max_{X \in M_t} |X|^2 = \max_{x \in S^n} |X(x,t)|^2.$$ 

By virtue of the assumption for $M_0$, we have (2.6) and

$$R_1^2 < P_{\text{min}}(0) \leq P_{\text{max}}(0) < R_2^2. \quad (2.12)$$

Since $M_t$ is smooth, $P_{\text{min}}$ and $P_{\text{max}}$ are obviously Lipschitzian on $[0,T)$. Were the inequality (2.11) not true, then by (2.12) we could find $t_1$ and $t_2$ in $[0,T)$ such that either

$$P_{\text{min}}(t_1) = R_1^2 \quad (2.13)$$

or

$$P_{\text{max}}(t_2) = R_2^2. \quad (2.14)$$

Without loss of generality, we assume that the case (2.13) happens, and the case (2.14) is completely similar. Let

$$t^* = \inf \{ t \in (0,T) : P_{\text{min}}(t) = R_1^2 \},$$

and choose $x^* \in S^n$ such that

$$P_{\text{min}}(t^*) = |X(x^*,t^*)|^2.$$ 

In order to compute the principal curvatures $k_1, \cdots, k_n$ of $M_{t^*}$ at $X(x^*,t^*)$, we use the principal direction $\xi_1, \cdots, \xi_n$ to obtain

$$\Pi(\xi_i, \xi_i) = k_i < \xi_i, \xi_i > = k_i g(\xi_i, \xi_i),$$ 

so

$$k_i = \Pi(\xi_i, \xi_i) g^{-1}(\xi_i, \xi_i), i = 1, 2, \cdots, n.$$
Thus, lemma 2.4 implies that $X(x^*, t^*) = R_1\nu(X(x^*, t^*))$ and
\[ k_i \leq \frac{1}{R_1}, \quad i = 1, 2, \ldots, n \]
at $X(x^*, t^*)$. Therefore, we have
\[ H(X(x^*, t^*)) \geq nR_1, \]
and
\[ \frac{\partial |X|^2}{\partial t}(x^*, t^*) = X \cdot \frac{\partial X}{\partial t} = R_1
\]
\[ = R_1 \nu \cdot (H - F)\nu \]
\[ \geq R_1[nR_1 - F(X(x^*, t^*))] \]
\[ > 0, \]
where we have used the condition (a) for $X(x^*, t^*) = R_1$.

On the other hand, we obviously have
\[ |X(x^*, t)|^2 > |X(x^*, t^*)|^2, \forall t \in [0, t^*) \]
thus
\[ \frac{\partial |X|^2}{\partial t}(x^*, t^*) \leq 0 \]
which yields a contradiction.

Next, we will prove that the convexity of $M_t$ is preserved. That is, (2.5) remains true for all $t \in (0, T)$ if it is so at $t = 0$.

Let
\[ h_{kl} = \nabla_k \nabla_l u + \delta_{kl}u, \quad k, l = 1, 2, \ldots, n. \]
It follows directly from (2.4) and (2.3) that
\[ \frac{\partial}{\partial t}(\delta_{kl}u) = \delta_{kl} \left[ \sum_{i=1}^n h_{ii} - F(ux + X(x, t)) \right]. \quad (2.15) \]
On the other hand, we differentiate the equation (2.4) twice to get
\[ \frac{\partial}{\partial t} \nabla_k \nabla_l u = \nabla_k \nabla_l \Delta u + n \nabla_k \nabla_l u - F_j(X) \nabla_k X^j - F_{jh}(X) \nabla_k X^h \nabla_l X^j. \quad (2.16) \]
Using the standard formula for interchanging the order of covariant differentiation with respect to the orthonormal frame on $S^n$, we have
\[ \nabla_k \nabla_l \Delta u = \Delta \nabla_k \nabla_l u - 2n \nabla_k \nabla_l u + 2\delta_{kl} \Delta u, \quad (2.17) \]
see, for instance, [CLT, p.85]. Thus, combining (2.15)-(2.17), we get
\[ \frac{\partial}{\partial t} h_{kl} = \Delta h_{kl} + \delta_{kl} \Delta u - n \nabla_k \nabla_l u + \delta_{kl} \left[ \sum_{i=1}^n h_{ii} - F(ux + X(x, t)) \right] \]
\[ - F_j(X(x, t)) \nabla_k \nabla_l X^j - F_{jh}(X) \nabla_k X^h \nabla_l X^j. \quad (2.18) \]
Set
\[ G(x, t) = \sum_{i=1}^{n} h_{ii}(x, t) - F(X(x, t)) \].

By virtue of (2.18), (2.3), and (2.4), we see that
\[ \frac{\partial G}{\partial t} = \frac{\partial}{\partial t} \sum_{i=1}^{n} h_{ii} - \frac{\partial F}{\partial t} \]
\[ = \Delta G + nG - \frac{\partial F}{\partial t} \]
\[ = \Delta G + nG - F_j(X)(\frac{\partial u}{\partial t} x^j + \nabla_i \frac{\partial u}{\partial t} \nabla_i x^j) \]
\[ = \Delta G + nG - F_j(X)(Gx^j + \nabla_i G \nabla_i x^j). \tag{2.19} \]

**Lemma 2.6.** Suppose that in addition to (2.6), the support function \( u_0 \) of the initial hypersurface \( M_0 \) satisfies
\[ G(x, 0) = \Delta u_0(x) + nu_0(x) - F(u_0 + \nabla_i u_0 \nabla_i x) \geq 0, \quad x \in S^n \tag{2.20} \]
and let \( u(x, t) \) be a smooth solution to (2.4) on \( S^n \times (0, T) \) with \( T \leq \infty \). Then for all \( (x, t) \in S^n \times (0, T) \),
\[ G(x, t) = \Delta u(x, t) + nu(x, t) - F(u x + \nabla_i u \nabla_i x) \geq 0. \tag{2.21} \]

**Proof.** It is sufficient to prove that for each \( T_0 < T \), (2.21) holds true for all \( (x, t) \in S^n \times [0, T_0] \). Let
\[ C_1 = \max \{ |u(x, t)| + |\nabla u(x, t)| : (x, t) \in S^n \times [0, T_0] \}, \]
\[ C_2 = \max \left\{ \sum_{j=1}^{n} |\frac{\partial F}{\partial y_j}| : |y| \leq C_1 \right\}, \]
and
\[ G_{\min}(t) = \min \{ G(x, t) : x \in S^n \}. \]
If \( G_{\min}(t) \leq 0 \) for some \( t \in (0, T_0) \) we could find \( x_t \in S^n \) such that
\[ G(x_t, t) = G_{\min}(t) \leq 0, \quad \Delta G(x_t, t) \geq 0, \quad \nabla G(x_t, t) = 0. \]
Thus, (2.19) implies that at \( (x_t, t) \)
\[ \frac{\partial G}{\partial t} \geq nG - F_j(X)Gx^j \geq (n + C_2)G. \]
This yields
\[ \frac{\partial}{\partial t} \left( G e^{-(n+C_2)t} \right)(x_t, t) \geq 0. \]

Note that \( e^{-(n+C_2)t}G_{\min}(t) \) is a Lipchitizian function of \( t \). Denote it by \( B(t) \) for simplicity. Then the result above gives us that
\[ \lim_{\tau \to 0^+} \inf \frac{B(t + \tau) - B(t)}{\tau} \geq 0 \text{ when } B(t) \leq 0 \text{ for } t \in (0, T_0). \]
Also see [U, p.107]. Now by a result of Hamilton [H2, lemma 3.1] we conclude that
\[ B(t) \geq 0, \quad \forall t \in [0, T_0]. \]
This proves the lemma.

**Lemma 2.7.** Assume that the support function \( u_0 \) of the initial hypersurface \( M_0 \) satisfies (2.6) and (2.20). Let \( u(x, t) \) be a smooth solution to (2.4) on \( S^n \times (0, T) \) with \( T \leq \infty \). Then for all \( (x, t) \in S^n \times (0, T) \), we have
\[ \nabla^2 u(x, t) + u(x, t)I > 0. \]

**Proof.** If the conclusion were not true, we could find a finite number \( t_0 \in (0, T) \) such that the minimum eigenvalue of the matrix \( [h_{kl}(x, t_0)] \) is zero, but \( [h_{kl}(x, t)] \) is positive definite for all \( (x, t) \in S^n \times [0, t_0) \). Thus the inverse matrix \( [h^{pq}(x, t)] \) exists for all such \( (x, t) \).

Let \( g_{kl}(x, t) = \langle \nabla_k X(x, t), \nabla_l X(x, t) \rangle \) be the metric of \( M_t \). Then Gauss-Weingarten relation and the fact \( \delta_{kl} = \langle \nabla_i x, \nabla_j x \rangle \) gives us that
\[ \nabla_k \nabla_l X^j = \nabla_k \left( g_{mi} h^{mi} \nabla_j x^j \right) \]
\[ = \nabla_k \left( h_{ii} \nabla_i x^j \right) \]
\[ = \nabla_k \nabla_i x^j h_{il} + \nabla_i x^j \nabla_k h_{il}. \]

On the other hand, it easily follows from the standard formula for commuting the order of covariant differentiation on \( S^n \) that
\[ \nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} \quad \text{for all} \quad i, j = 1, 2, \ldots, n \]
see [U, p.98]. Therefore, for each \( (x, t) \in S^n \times [0, t_0) \), we have
\[ \frac{\partial}{\partial t} h_{kl} = \Delta h_{kl} + \delta_{kl} \Delta u - n \nabla_k \nabla_l u + \delta_{kl} G \]
\[ - F_j(X) \left( \nabla_k \nabla_i x^j h_{il} + \nabla_i x^j \nabla_k h_{ik} \right) - F_{jh}(X) \nabla_k X^h \nabla_i X^j. \] (2.22)

Now let us suppose that the minimum eigenvalue of \( [h_{kl}] \) over \( S^n \) at time \( t \in [0, t_0) \) attains at a point \( x_t \in S^n \). By rotating the frame \( e_1, e_2, \ldots, e_n \), we may assume that \( h_{11}(x_t, t) \) is the minimum eigenvalue and \( h_{j1}(x_t, t) = 0 \) for \( j = 2, 3, \ldots, n \). (Also see [CLT, p.89].) Combining (2.22) and lemma 2.6 together, at \( (x_t, t) \) we have
\[ \frac{\partial h_{11}}{\partial t} \geq \Delta u - n \nabla_1 \nabla_1 u - F_j(X) \nabla_1 \nabla_1 x^j h_{11} - F_{jh} \nabla_1 X^j \nabla_1 X^h, \] (2.23)
where we have used $\Delta h_{11} \geq 0$ and $\nabla h_{11} = 0$ at $(x_t, t)$. Since at $(x_t, t)$

$$nu + n\nabla_1 \nabla u \leq \sum_{i=1}^{n} h_{ii} = \Delta u + nu,$$

we have

$$\Delta u - n\nabla_1 \nabla u \geq 0.$$ 

Applying this equality and the concavity of $F$ to (2.23), we obtain

$$\frac{\partial h_{11}}{\partial t} \geq -F_j(X) \nabla_1 \nabla^j h_{11}.$$ 

Denote

$$C_3 = \max \{|u(x, t)| + |\nabla u(x, t)| : (x, t) \in S^n \times [0, t_0]\},$$

$$C_4 = \max \left\{ \sum_{j=1}^{n} |F_j(Y)| : |Y| \leq C_3 \right\},$$

and

$$C_5 = C_4 \max \{|\nabla_1 \nabla x| : x \in S^n\}.$$ 

Noting $h_{11}(x_t, t)$ is positive for all $t \in [0, t_0)$, we have

$$\frac{\partial}{\partial t} h_{11}(x_t, t) \geq -C_5 h_{11}(x_t, t), \quad t \in [0, t_0).$$

That is

$$\frac{\partial}{\partial t} \left( h_{11}(x_t, t) e^{C_5 t} \right) \geq 0, \quad t \in [0, t_0).$$

By the maximum principle and the condition (2.6), we see that

$$h_{11}(x_t, t) \geq h_{11}(x_0, 0) e^{-C_5 t} \geq C_6 e^{-C_5 t} \quad \forall t \in [0, t_0).$$

Letting $t \to t_0^-$, we obtain

$$h_{11}(x_0, t_0) \geq C_6 e^{-C_5 t_0},$$

which is contradictory with the assumption that the minimum eigenvalue of $[h_{kl}]$ is zero. This proves the lemma.

**Theorem 2.8.** Suppose that the initial hypersurface $M_0 \subset A$ and its support function $u_0$ satisfies (2.6) and (2.20). Then there exists a unique smooth solution $u$ to the following problem:

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + nu - F(u x + \nabla_i u \nabla^i x) \quad \text{in} \quad S^n \times (0, \infty) \\
u(\cdot, 0) = u_0(\cdot) \quad \text{on} \quad S^n.
\end{cases} \quad (2.24)$$

Moreover, for all $(x, t) \in S^n \times (0, \infty), \ u(x, t)$ satisfies

$$\nabla^2 u + u I > 0, \quad (2.25)$$

$$\frac{\partial u}{\partial t} = \Delta u + nu - F(u x + \nabla_i u \nabla^i x) \geq 0, \quad (2.26)$$

where $F$ is a smooth function on $S^n \times [0, \infty)$, and $u_I$ is the identity matrix.
and

\[ R_2^2 < u^2 + |\nabla u|^2 < R_2^2. \] (2.27)

**Proof.** By virtue of lemma 2.2, we know that the problem (2.24) has a unique smooth solution on \( S^n \times (0, T) \) with some \( T \leq \infty \). Moreover, lemma 2.6 and 2.7 tell us that both (2.25) and (2.26) are satisfied on \( S^n \times (0, T) \). Thus, by lemma 2.5, we see that (2.27) is true for all \((x, t) \in S^n \times (0, T)\). Using lemma 2.2 again, we know that \( T \) is nothing but \( \infty \). This completes the theorem.

### 3. Converging to a convex hypersurface

In this section, we will use the theorem 2.8 in last section to prove the main result of this paper, theorem 1.1.

We begin with choosing a smooth, closed, uniformly convex hypersurface \( M_0 \) such that its support function satisfies (2.6) and (2.20). (The existence of such \( M_0 \) is obvious due to the condition (a)). By theorem 2.8 and lemma 2.1 we obtain a family of smooth, closed, uniformly convex hypersurfaces \( M_t \) whose position vectors are

\[ X(x, t) = u(x, t)x + \nabla_i u(x, t) \nabla_i x, \quad (x, t) \in S^n \times [0, \infty), \]

where \( u(x, t) \) are the support functions of \( M_t \) and satisfy (2.24)-(2.27) in theorem 2.8. It follows from (2.26) and (2.24) that

\[ \left( \frac{\partial u}{\partial t} \right)^2 = \frac{\partial u}{\partial t} (\Delta u + nu - F(X(x, t))) \leq \frac{\partial u}{\partial t} (\Delta u + nu). \]

Hence

\[ \int_{S^n} \left( \frac{\partial u}{\partial t} \right)^2 dx \leq \frac{1}{2} \frac{\partial}{\partial t} \int_{S^n} (nu^2 - |\nabla u|^2) dx, \forall t \in (0, \infty). \]

This, together with (2.27), implies that

\[ \int_0^\infty \int_{S^n} \left( \frac{\partial u}{\partial t} \right)^2 dx dt \leq (n + 2)R_2^2 \cdot \text{vol}(S^n). \] (3.1)

Since (2.27) implies that for each \( \alpha \in (0, 1) \)

\[ nu - F(ux + \nabla_i u \nabla_i x) \in L^\infty(S^n \times [0, \infty)), \]

it follows from a property of heat equation that

\[ ||u(\cdot, t)||_{C^1,\alpha(S^n)} \leq C \] (3.2)

uniformly in \( t \in [1, \infty) \). See [LSU, ch.4] or [D]. Moreover, by (3.2) and the same argument, we have

\[ ||u(\cdot, t)||_{C^3,\alpha(S^n)} \leq C \] (3.3)

uniformly in \( t \in [1, \infty) \). Using (3.1) and (3.3), we can find a sequence \( t_k \to \infty \) as \( k \to \infty \) such that

\[ \int_{S^n} \left( \frac{\partial u}{\partial t_k} \right)^2 dx \to 0, \] (3.4)
and

\[ u(\cdot, t_k) \to U_0 \text{ in } C^{3,\alpha}(S^n). \]  

(3.5)

Furthermore, (2.24), (3.4) and (3.5) gives us

\[ \int_{S^n} [\Delta U_0 + nU_0 - F(U_0x + \nabla_i U_0 \nabla_i x)]^2 dx = 0, \]

which yields

\[ \Delta U_0 + nU_0 - F(U_0x + \nabla_i U_0 \nabla_i x) = 0, \quad x \in S^n. \]  

(3.6)

Since \( U_0 \in C^2(S^n) \), applying a standard elliptic theory to the equation (3.6), we know \( U_0 \in C^\infty(S^n) \).

On the other hand, it easily follows from (3.4), (3.5) and (3.6) that as \( k \to \infty \)

\[ \frac{\partial u}{\partial t_k}(\cdot, t_k) \to 0 \text{ in } C(S^n). \]

Hence

\[ \nabla \frac{\partial u}{\partial t_k}(\cdot, t_k) \to 0 \text{ in } C(S^n). \]  

(3.7)

Therefore, (3.5) and (3.7) implies that

\[ X(x, t_k) = u(x, t_k) x + \nabla_i u(x, t_k) \nabla_i x \to U_0x + \nabla_i U_0 \nabla_i x \text{ in } C^1(S^n), \]  

(3.8)

and

\[ \frac{\partial X}{\partial t_k}(x, t_k) = \frac{\partial u}{\partial t_k}(x, t_k) x + \nabla_i \frac{\partial u}{\partial t_k}(x, t_k) \nabla_i x \to 0 \text{ in } C(S^n). \]  

(3.9)

Since each hypersurface \( M_t \) is smooth, closed, uniformly convex, lemma 2.1 says that \( X(x, t) \) satisfy (1.3) for all \( t \in [0, \infty) \). Thus, using (2.2), (3.5), (3.8) and (3.9), we know that

\[ Y_0(x) = U_0(x) x + \nabla_i U_0 \nabla_i x \quad x \in S^n \]  

(3.10)

satisfies the equation (1.2).

Now let \( M_y \) be the hypersurface determined by the position vector \( Y_0(x), x \in S^n \). Obviously, \( M_y \) is closed and encloses the origin, and is convex because of (3.5), (3.8) and the uniformly convexity of hypersurfaces \( M_t \) for any \( t \in [0, \infty) \). So \( M_y \) is uniformly convex because \( Y_0 \) satisfies (1.2). Furthermore, due to the fact \( U_0 \in C^\infty(S^n) \), \( M_y \) is smooth, too. Finally, using (2.27), (3.5) and (3.10), we obtain

\[ R_1^2 \leq |X_0(x)|^2 \leq R_2^2, \quad x \in S^n, \]

which means that the hypersurface \( M_y \) lies in \( \bar{A} \). This prove theorem 1.1.
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