Generalizations and Variants of the Largest Non-crossing Matching Problem in Random Bipartite Graphs

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Abstract

A two-rowed array \( \alpha_n = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \) is said to be in lexicographic order if \( a_k \leq a_{k+1} \) and \( b_k \leq b_{k+1} \) if \( a_k = a_{k+1} \). A length \( \ell \) (strictly) increasing subsequence of \( \alpha_n \) is a set of indices \( i_1 < i_2 < \cdots < i_\ell \) such that \( b_{i_1} < b_{i_2} < \cdots < b_{i_\ell} \). We are interested in the statistics of the length of the longest increasing subsequence of \( \alpha_n \) chosen according to \( D_n \), for distinct families of distributions \( D = (D_n)_{n \in \mathbb{N}} \), and when \( n \) goes to infinity. This general framework encompasses well studied problems such as the so called Longest Increasing Subsequence problem, the Longest Common Subsequence problem, problems concerning directed bond percolation models, among others. We define several natural families of distinct distributions and characterize the asymptotic behavior of the length of a longest increasing subsequence chosen according to them. In particular, we consider generalizations to \( d \)-rowed arrays as well as symmetry restricted two-rowed arrays.

1 Introduction

Suppose that we select uniformly at random a permutation \( \pi \) of \( [n] \) def \( \{1, \ldots, n\} \). We can associate to \( \pi \) the two-rowed lexicographically sorted array \( \alpha_\pi = \begin{pmatrix} \pi(1) & 2 & \cdots & n \\ \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix} \). We denote by \( lis(\pi) \) the length of a longest increasing subsequence of \( \alpha_\pi \). The determination, as \( n \to \infty \), of the first moments of \( lis(\pi) \) has been a problem of much interest for a long time (for surveys see [AD99, OR98, Sta02] and references therein). This line of research led to what is considered a major breakthrough: the determination by Baik, Deift and Johansson [BDJ99] of, after proper scaling, the distribution of \( lis(\cdot) \). In [BR01], variations are studied where instead of permutations of \( [n] \), random involutions, signed permutations, and signed involutions are selected at random. Generalizations where \( d - 1 \) random permutations are selected can be restated as problems concerning longest increasing subsequences of \( d \)-rowed arrays.

Suppose now that we select uniformly at random two words \( \mu \) and \( \nu \) from \( \Sigma^n \), where \( \Sigma \) is some finite alphabet of size \( k \). We can associate to \( (\mu, \nu) \) the two-rowed lexicographically sorted array \( \alpha_{\mu,\nu} \) where \( \begin{pmatrix} i \\ j \end{pmatrix} \) is a column of \( \alpha_{\mu,\nu} \) if and only if the \( i \)-th character of \( \mu \) is the same as the \( j \)-th character of \( \nu \) (for an example, see Figure 1). The length of a longest common subsequence of \( \mu \) and \( \nu \), denoted \( lcs(\mu, \nu) \), equals

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the length of a longest increasing subsequence of $\alpha_{\mu, \nu}$. Since the mid 70’s, it has been known [CS75] that the expectation of $\text{les}(\mu, \nu)$ when normalized by $n$, converges to a constant $\gamma_k$ (the so called Chvatal-Sankoff constant). The determination of the exact value of $\gamma_k$, for $k$ fixed, remains a challenging open problem. To the best of our knowledge, the asymptotic distribution theory of the longest increasing subsequence problem is essentially uncharted territory. Generalizations where $d$ random length $n$ words are chosen from a finite alphabet $\Sigma$ can also be restated as problems concerning longest increasing subsequences of $d$-rowed arrays.

We now discuss yet one more relevant instance, previously considered by Seppäläinen [Sep97], and encompassed by the framework described above. Fix a parameter $0 < p < 1$ and let $n$ be a positive integer. For each site of the lattice $[n]^2$, let a point be present (the site is occupied) with probability $p$ and absent (the site is empty) with probability $q = 1 - p$, independently of all the other sites. Let $\omega : [n]^2 \rightarrow \{0, 1\}$ be an encoding of the occupied/empty sites (1 representing an occupied site and 0 a vacant one). We can associate to $\omega$ a two-rowed lexicographically sorted array $\alpha_\omega$ where \[\left(\begin{array}{c} i \\ j \end{array}\right)\] is a column of $\alpha_\omega$ if and only if site $(i, j) \in [n]^2$ is occupied. Let $L(\omega)$ equal the number of sites on a longest strictly increasing path of occupied sites according to $\omega$, where a path $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ of points on $[n]^2$ is strictly increasing if $x_1 < x_2 < \ldots < x_m$ and $y_1 < y_2 < \ldots < y_m$. Observe that $L(\omega)$ equals the length of a longest increasing subsequence of $\alpha_\omega$. Subadditivity arguments easily imply that the expected value of $L(\omega)$, when normalized by $n$, converges to a constant $\gamma_p$. Via a reformulation of the problem as one of interacting particle systems, Seppäläinen [Sep97] shows that $\gamma_p = 2\sqrt{p}/(1 + \sqrt{p})$. Also worth noting is that the same object $\alpha_\omega$ arises in the study of the asymptotic shape of a directed bond percolation model (see [Sep97] §1 for details). Symmetric variants, where for example site $(i, j)$ is occupied if and only if $(j, i)$ is occupied, can be easily formulated. Generalizations where $d$-dimensional lattices are considered can also be restated as problems concerning longest increasing subsequences of $d$-rowed arrays. However, to the best of our knowledge, neither of the latter two variants has been considered in the literature.

Thus far, we have described well studied scenarios where the general problem formulated in the abstract naturally arises. This motivates our work. However, for the sake of clarity of exposition and in order to use more convenient notation, it will be preferable to reformulate the issues we are interested in as one concerning hyper-graphs. To carry out this reformulation, below we introduce some useful terminology and then address in this language the problem of determining the statistics of the length of a longest increasing subsequence of a randomly chosen lexicographically sorted $d$-rowed array.

Let $A_1, \ldots, A_d$ be $d$ disjoint (finite) sets, also called color classes. We assume that over each $A_i$ there is a total order relation, which abusing notation, we denote $\leq$ in all cases. When we consider subsets of a totally ordered color class we always assume the subset inherits, and thus respects, the original order. A $d$-partite hyper-graph over totally ordered color classes $A_1, \ldots, A_d$ with edge set $E \subseteq A_1 \times \ldots \times A_d$ is a tuple $G = (A_1, \ldots, A_d; E)$, and its edge set is denoted by $E(G)$. For $A'_i \subseteq A_i$ with $1 \leq i \leq d$ and hyper-graph $G = (A_1, \ldots, A_d; E)$, we denote by $G|_{A'_1 \times \ldots \times A'_d}$ the hyper-subgraph induced by $G$ in $A'_1 \times \ldots \times A'_d$, i.e. the hyper-graph with node set $V' = A'_1 \times \ldots \times A'_d$ and edge set $E \cap V'$. We say that two hyper-graphs are disjoint if their corresponding vertex sets are disjoint. Let $K_{A_1, \ldots, A_d}$ denote the complete $d$-partite hyper-graph over color classes $A_1, \ldots, A_d$ whose edge set is $A_1 \times A_2 \times \ldots \times A_d$. Henceforth, we
denote the cardinality of $A_i$ by $n_i$. If we identify $A_i$ with $[n_i]$, then we write $K_{n_1,\ldots,n_d}$ instead of $K_{A_1,\ldots,A_d}$. If $n_1 = \ldots = n_d$, then we write $K_n^{(d)}$ instead of $K_{n_1,\ldots,n_d}$. Over the edge set of $K_{A_1,\ldots,A_d}$ we consider the natural partial order relation $\preceq$ defined by
\[(v_1,\ldots,v_d) \preceq (v'_1,\ldots,v'_d) \iff v_i \leq v'_i \text{ for all } 1 \leq i \leq d.\]

We say that a collection of node-disjoint edges $M \subseteq E(G)$ is a non-crossing hyper-matching if for every pair of edges $e, f \in M$ it holds that $e \preceq f$ or $f \preceq e$. When $G = (A_1,\ldots,A_d; E)$ is such that $E(G)$ is a non-crossing hyper-matching we will simply say that $G$ is a non-crossing $d$-partite hyper-graph, or simply a non-crossing hyper-matching. Furthermore, we will denote by $L(G)$ the size of a largest non-crossing hyper-matching of $G$ and by $L(F)$ the random variable $L(G)$ when $G$ is chosen according to a distribution $F$ over $d$-partite hyper-graphs. When we want to stress that we are dealing with only two color classes, we will speak of graphs and matchings instead of hyper-graphs and hyper-matchings.

Now, consider a family of distributions $\mathcal{D} = (\mathcal{D}(K_{A_1,\ldots,A_d}))$ where each $\mathcal{D}(K_{A_1,\ldots,A_d})$ is a probability distribution over subgraphs of $K_{A_1,\ldots,A_d}$. In this work we are interested in understanding what we refer to as the Longest Non-crossing Matching problem, i.e. the behavior of the expectation of $L(G)$ when $G$ is chosen according to various distinct families of distributions $\mathcal{D} = (\mathcal{D}(K_n^{(d)}))$ and $n$ goes to infinity. Of course, in order to be able to derive some meaningful results we will need some assumptions on the distributions $\mathcal{D}(K_n^{(d)})$. Below, we encompass in a definition a minimal set of assumptions that are both easy to establish and general enough to capture several relevant scenarios.

**Definition 1** Let $\mathcal{D} = (\mathcal{D}(K_{A_1,\ldots,A_d}))$ be a family of distributions where each $\mathcal{D}(K_{A_1,\ldots,A_d})$ is a probability distribution over the collection of hyper-subgraphs of $K_{A_1,\ldots,A_d}$. We say that $\mathcal{D}$ is a random $d$-partite hyper-graph model if for $H$ chosen according to $\mathcal{D}(K_{A_1,\ldots,A_d})$ the following two conditions hold:

1. **Monotonicity:** If $A'_i \subseteq A_i$ with $1 \leq i \leq d$ and $n'_i = |A'_i|$, then the distribution of $H|_{A'_1 \times \ldots \times A'_d}$ is $\mathcal{D}(K_{n'_1,\ldots,n'_d})$.

2. **Block independence:** If $A'_i, A''_i \subseteq A_i$ are disjoint with $1 \leq i \leq d$, then $H' = H|_{A'_1 \times \ldots \times A'_d}$ and $H'' = H|_{A''_1 \times \ldots \times A''_d}$ are independent (and so, $L(H')$ and $L(H'')$ are also independent).

For some of the results we will establish, the following weaker notion will suffice.

**Definition 2** Let $\mathcal{D} = (\mathcal{D}(K_n^{(d)}))$ be a family of distributions where each $\mathcal{D}(K_n^{(d)})$ is a probability distribution over the collection of hyper-subgraphs of $K_n^{(d)}$. We say that $\mathcal{D}$ is a weak random $d$-partite hyper-graph model if for $H$ chosen according to $\mathcal{D}(K_n^{(d)})$ the following two conditions hold:

1. **Weak monotonicity:** If $A' \subseteq [n], |A'| = n'$, then the distribution of $H|_{A' \times \ldots \times A'}$ is $\mathcal{D}(K_n^{(d)})$.

2. **Weak block independence:** If $A', A'' \subseteq [n]$ are disjoint with $1 \leq i \leq d$, then $H' = H|_{A' \times \ldots \times A'}$ and $H'' = H|_{A'' \times \ldots \times A''}$ are independent (and so, $L(H')$ and $L(H'')$ are also independent).

The reader may easily verify that the following distributions (on which we will focus attention) give rise to random $d$-partite hyper-graph models:

- $\Sigma(K_{n_1,\ldots,n_d}, k)$ (the random $d$-word model) — the distribution over the set of hyper-subgraphs obtained from $K_{n_1,\ldots,n_d}$ when each element in the vertex set of $K_{n_1,\ldots,n_d}$ is uniformly and independently randomly assigned one of $k$ letters and where edges, for which not all of its nodes end up being assigned the same letter, are discarded.
Figure 2: A bipartite graph in the support of $S(K_{12,12}, p)$.

Figure 3: A bipartite graph in the support of $A(K_{12,12}, p)$.

- $G(K_{n_1,\ldots,n_d}, p)$ (the $d$-dimensional binomial random hyper-graph model) — the distribution over the set of hyper-subgraphs $H$ of $K_{n_1,\ldots,n_d}$ where the events $\{H \mid e \in E(H)\}$ for $e \in E(K_{n_1,\ldots,n_d})$ have probability $p$ and are mutually independent.

The model $\Sigma(K_n^{(d)}, k)$ is referred to as the random word model because it arises when one considers the letters of $d$ words $\omega_1, \ldots, \omega_d$ of length $n_1, \ldots, n_d$, respectively. The letters in each word are chosen uniformly and independently from a finite alphabet of size $k$. Then, each word is identified with a color class of a hyper-subgraph $H$ of $K_{n_1,\ldots,n_d}$ whose hyper-edges are the $(v_1, \ldots, v_d) \in V(H)$ for which $v_1, \ldots, v_d$ have been assigned the same letter. It is easy to see that the longest common subsequence of $\omega_1, \ldots, \omega_d$ equals $\ell$ if and only if $L(H) = \ell$. The random word model thus encompasses the Longest Common Subsequence problem discussed above. Similarly, the attentive reader probably already noticed that the binomial random graph model also encompasses the already discussed point lattice process considered by Seppäläinen [Sep97].

Inspired in the work of Baik and Rains [BR01] cited above, where symmetric variants of the Longest Increasing Subsequence problem were considered, we will also study the following two symmetric variants of the binomial random graph model:

- $S(K_{n,n}, p)$ (the symmetric binomial random graph model) — the distribution over the set of subgraphs $H$ of $K_{n,n}$ where the events $\{H \mid (i,j),(j,i) \in E(H)\}$ for $1 \leq i < j \leq n$, have probability $p$ and are mutually independent.

- $A(K_{2n,2n}, p)$ (the anti-symmetric binomial random graph model) — the distribution over the set of subgraphs $H$ of $K_{2n,2n}$ where the events $\{H \mid (i,j),(2n-i+1,2n-j+1) \in E(H)\}$ for $1 \leq i,j \leq 2n$ have probability $p$ and are mutually independent.

Note that $(S(K_{n,n}, p))_{n \in \mathbb{N}}$ is not a random model according to Definition 1 but it is a weak random model according to Definition 2. On the other hand, $(A(K_{2n,2n}, p))_{n \in \mathbb{N}}$ is not even a weak random model.

Henceforth, given a random bipartite graph model $D = (\mathcal{D}(\cdot))$, any value that is constant across the distributions $\mathcal{D}(\cdot)$ will be called internal parameter of the model — e.g. $1/p$ and $k$ in $G(\cdot,p)$ and $\Sigma(\cdot,k)$, respectively.

The main purpose of this work is to establish a general result, referred to as Main Theorem, with a minimal set of easily verifiable hypothesis, that characterizes the limit behavior, when properly normalized,
of $\mathbb{E} \left[ L(D(K_n^{(d)}, p)) \right]$ when $d$ is fixed and both $n$ and the internal parameter $t$ go to infinity. We also show several applications of our Main Theorem. Specifically, we characterize aspects of the limiting behavior for the four previously introduced random hyper-graphs models. In the following section we formally state our Main Theorem and the results of its application.

1.1 Main contributions

A straightforward application of Talagrand’s inequality (as stated in [JLR00, Theorem 2.29]) yields that both $L(\Sigma(K_n^{(d)}, k))$ and $L(G(K_n^{(d)}, p))$ are concentrated around any one of their (potentially not unique) medians. As we shall see, the same is true for $L(\Sigma(K_n^{(d)}, k))$ and $L(G(K_n^{(d)}, p))$. Somewhat equivalent statements hold for the the symmetric and anti-symmetric binomial random graph models. The following general notion will encompass the concentration type requirement the random hyper-graph models will need to satisfy in order for our Main Theorem to be applicable.

**Definition 3** Let $F$ be a distribution over bipartite hyper-graphs and $Med$ be a median of $L(F)$. We say that $F$ has concentration constant $h$ if for all $s \geq 0$,

$$
\Pr \left[ L(F) \leq (1 - s)Med \right] \leq 2 \exp \left( -hs^2Med \right),
$$

$$
\Pr \left[ L(F) \geq (1 + s)Med \right] \leq 2 \exp \left( -h \frac{s^2}{1 + s}Med \right).
$$

We say that the random bipartite hyper-graph model $D = (D(\cdot))$ has concentration constant $h$ if each $D(\cdot)$ has concentration constant $h$.

Note that if one can estimate a median of $L(F)$ for some distribution $F$, show that the median and mean are close, and establish that $F$ has a concentration constant, then one can derive a concentration (around its mean) result for $L(F)$. Unfortunately, it is not in general easy to estimate a median of $L(D(K_{n_1, \ldots, n_d}))$ for the distributions $D(K_{n_1, \ldots, n_d})$ we consider. However, we will be able to approximate them under some assumptions on $n_1, \ldots, n_d$. In particular, we will show that there is a median that is proportional to the geometric mean of $n_1, \ldots, n_d$. The following definition captures the aforementioned assumptions we will need, and the sort of approximation guarantee that we will be able to establish.

**Definition 4** Let $D = (D(K_{n_1, \ldots, n_d}))$ be a random $d$-partite hyper-graph model with internal parameter $t$. Fix $n_1, \ldots, n_d$ and let $N = \left( \prod_{i=1}^{d} n_i \right)^{1/d}$ and $S = \sum_{i=1}^{d} n_i$ denote the geometric mean and sum of $n_1, \ldots, n_d$, respectively. We say that $D$ admits a $(c, \lambda, \theta)$-approximate median (or simply a $(c, \lambda, \theta)$-median) if for all $\delta > 0$ there are sufficiently large constants $a(\delta), b(\delta)$, and $t'(\delta)$, such that for all $t \geq t'$, for which

- **Size lower bound condition**: $N \geq at^\lambda$,
- **Size upper bound condition**: $Sb \leq t^\theta$,

it holds that

$$
(1 - \delta)\frac{cN}{t^\lambda} \leq Med \left[ L(D(K_{n_1, \ldots, n_d})) \right] \leq (1 + \delta)\frac{cN}{t^\lambda}.
$$

In other words, if $D = (D(K_{n_1, \ldots, n_d}))$ is a a random $d$-partite hyper-graph model with internal parameter $t$ that admits a $(c, \lambda, \theta)$-median and the geometric mean (respectively sum) of $n_1, \ldots, n_d$ is $N$ (respectively $S$) are such that $N = \Omega(t^\lambda)$ (respectively $S = O(t^\theta)$), then for sufficiently large $t$, every median of
\(L(D(n_1, \ldots, n_d))\) will be close to \(cNt^{-\lambda}\). Although the above defined approximate median notion might at first glance sound artificial, we will see that it is possible to obtain such type of approximations for the random hyper-graph models we are interested on.

Returning to our discussion, the relevance of the notion of approximate median is, when the random hyper-graph model admits a concentration constant, that it allows us to derive concentration bounds around an approximation of the median which in turn will be closed to the mean. Endowed with such estimates of the mean, we can easily derive the thought after limiting behavior of such expected values. This in essence, is the crux of our approach to attacking all variants of the Largest Non-crossing Matching problem.

Unfortunately, the approximation of \(\text{Med} \left[ L(D(n_1, \ldots, n_d)) \right] \) guaranteed by the existence of a \((c, \lambda, \theta)\)-median, as in Definition 4, holds for the rather restrictive condition \(b \sum_{i=1}^{d} n_i \leq t^\theta\). However, the monotonicity and block independence properties of random hyper-graph models allow us to relax the restriction and still obtain essentially the same conclusion. More precisely, it will be possible to obtain the same guarantee, but requiring only that the sum of the \(n_i\)’s is not too large in comparison with the geometric mean of the \(n_i\)’s. Moreover, and of crucial importance, under the same conditions one can show that the median and mean of \(L(D(n_1, \ldots, n_d))\) are close to each other. The following result, which is the main result of this work, precisely states the claims made in the preceding informal discussion.

**Theorem 5** [Main Theorem] Let \(D = (D(K_{n_1}, \ldots, n_d))\) be a random hyper-graph model with internal parameter \(t\) and concentration constant \(h\) which admits a \((c, \lambda, \theta)\)-median. Fix \(n_1, \ldots, n_d\) and let \(N\) and \(S\) denote the geometric mean and sum of \(n_1, \ldots, n_d\), respectively. Let \(0 \leq \eta \leq \min \{ \lambda/(d-1), \theta - \lambda \} \) and \(g = O(t^\eta)\).

For all \(\epsilon > 0\) there exists \(t_0\) and \(A\) sufficiently large such that if \(t \geq t_0\) is such that \(N \geq t^\lambda A\) (size constraint) and \(S \leq g(t)N\) (balance condition), then

\[
(1 - \epsilon) \frac{cN}{t^\lambda} \leq \mathbb{E} \left[ L(D(K_{n_1}, \ldots, n_d)) \right] \leq (1 + \epsilon) \frac{cN}{t^\lambda},
\]

and the following hold:

- If \(\text{Med}\) is a median of \(\text{Med} \left[ L(D(K_{n_1}, \ldots, n_d)) \right] \), then

\[
(1 - \epsilon) \frac{cN}{t^\lambda} \leq \text{Med} \leq (1 + \epsilon) \frac{cN}{t^\lambda}.
\]

- There is a constant \(K > 0\) such that

\[
\text{Pr} \left[ L(D(K_{n_1}, \ldots, n_d)) \leq (1 - \epsilon) \frac{cN}{t^\lambda} \right] \leq \exp \left( -Kh \frac{cN}{t^\lambda} \right),
\]

\[
\text{Pr} \left[ L(D(K_{n_1}, \ldots, n_d)) \geq (1 + \epsilon) \frac{cN}{t^\lambda} \right] \leq \exp \left( -Kh \frac{cN}{t^\lambda} \right).
\]

Moreover, if \(n_1 = \ldots = n_d = n\) and \(D = (D(K_n^{(d)}))\) is just a weak random hyper-graph model, then the the lower bounds in (1) and (2), and inequality (3), still hold.

As a consequence of the previously stated Main Theorem, with some additional work, we can derive several results concerning the asymptotic behavior of the expected length of a largest non-crossing matching for all of the random models introduced above. Our first two applications of the Main Theorem concern the random binomial hyper-graph model \((\mathcal{G}(K_n^{(d)}, p))_{n \in \mathbb{N}}\) and the random word model \((\Sigma(K_n^{(d)}, k))_{n \in \mathbb{N}}\). The
asymptotic behavior of the length of a largest non-crossing hyper-matching for both of these models is (interestingly!) related to a constant $c_d$ that arises in the work of Bollobás and Winkler [BW88] concerning the height of a largest chain among random points independently chosen in the $d$-dimensional unit cube $[0,1]^d$. Specifically, for the random binomial hyper-graph model, we show:

**Theorem 6** For $0 < p < 1$, there exists a constant $\delta_p$ such that

$$
\lim_{n \to \infty} \frac{1}{n} E \left[ L(G(K_n^{(d)}, p)) \right] = \inf_{n \in \mathbb{N}} \frac{1}{n} E \left[ L(G(K_n^{(d)}, p)) \right] = \delta_p,
$$

and $\delta_p / \sqrt{p} \to c_d$ when $p \to 0$.

For the case where the underlying model is the one that arises when interested in the length of a longest common subsequence of $d$ randomly chosen words over a finite alphabet, i.e. the random $d$-word model, we establish:

**Theorem 7** For $k \in \mathbb{N}$, there exists a constant $\gamma_k$ such that

$$
\lim_{n \to \infty} \frac{1}{n} E \left[ L(\Sigma(K_n^{(d)}, k)) \right] = \inf_{n \in \mathbb{N}} \frac{1}{n} E \left[ L(\Sigma(K_n^{(d)}, k)) \right] = \gamma_k,
$$

and $k^{1-1/d} \gamma_k \to c_d$ when $k \to \infty$.

The $d = 2$ case of Theorems 6 and 7 were already established by Kiwi, Loebl, Matoušek [5]. This work generalizes and strengthens the arguments developed in [5], as well as elicits new connections with other previously studied problems (most notably in [BW88]).

Finally, we consider the symmetric versions of random graph models introduced above and show how the Main Theorem, plus some additional observations, allows one to characterize some aspects of the asymptotic behavior of the length of a longest non-crossing matching. Specifically, we prove the following two results.

**Theorem 8** For $0 < p < 1$, there exists a constant $\sigma_p$ such that

$$
\lim_{n \to \infty} \frac{1}{n} E \left[ L(S(K_{n,n}, p)) \right] = \inf_{n \in \mathbb{N}} \frac{1}{n} E \left[ L(S(K_{n,n}, p)) \right] = \sigma_p,
$$

and $\sigma_p / \sqrt{p} \to 2$ when $p \to 0$.

**Theorem 9** For $0 < p < 1$, there exists a constant $\alpha_p$ such that

$$
\lim_{n \to \infty} \frac{1}{2n} E \left[ L(A(K_{2n,2n}, p)) \right] = \inf_{n \in \mathbb{N}} \frac{1}{2n} E \left[ L(A(K_{2n,2n}, p)) \right] = \alpha_p,
$$

and $\alpha_p / \sqrt{p} \to 2$ when $p \to 0$. 

7
1.2 Preliminaries

For future reference we determine below concentration constants for the binomial and word models.

**Proposition 10** The $d$-dimensional binomial random hyper-graph model admits a concentration constant of $1/4$. The random $d$-word model admits a concentration constant of $1/(4d)$.

**Proof:** Let $H$ be chosen according to $G(K_{n_1,\ldots,n_d}, p)$. Since $L(H)$ depends exclusively on whether or not an edge appears in $H$ (and by independence among these events), it follows that $L(H)$ is 1-Lipschitz, i.e. $|L(H) - L(H \triangle \{e\})| \leq 1$. Moreover, if $L(H) \geq r$, then there is a set of $r$ edges that are a witness for the fact that $L(H) \geq r$, for every $H$ containing such a set of $r$ edges. A direct application of Talagrand’s inequality (as stated in [ILR00, Theorem 2.29]) proves the claim about the concentration constant for the $d$-dimensional binomial random hyper-graph model. The case of the random $d$-word model is similar and left to the reader to verify. ■

1.3 Organization:

For the sake of clarity of exposition and given that the arguments employed are different, we prove in separate sections the lower and upper bounds (as well as lower and upper tail bounds) of the Main Theorem’s statement. Specifically, in Section 2, we establish all the lower bounds and lower tail bounds claimed in the Main Theorem. In Section 3, we prove the upper bounds and upper tail bounds stated in the Main Theorem, thence completing its proof. Finally, in Section 4, we apply the Main Theorem to four distinct scenarios. Specifically, we consider the cases where the underlying random model is the binomial random hyper-graph model, the random word model, the symmetric binomial random graph model, and the anti-symmetric binomial random graph model.

2 Lower bounds

In this section we will establish the lower bounds claimed in the statement of the Main Theorem, i.e. the lower bounds in (1) and (2), and inequality (3).

Let $D, c, \lambda, \theta, \eta,$ and $\epsilon$ be as in the statement of the Main Theorem. Let $\delta > 0$ be sufficiently small so that

$$(1-\delta)^2(1-2\delta) \geq 1 - \frac{\epsilon}{2}, \quad \text{(Definition of $\delta$)}$$

and let $a = a(\delta), b = b(\delta)$ and $t' = t'(\delta)$ as guaranteed by the definition of $(c, \lambda, \theta)$-median.

Since $g = O(t^n)$, there are constants $C_g > 1$ and $t_g \geq 0$ such that $g(t) \leq C_g t^n$ for all $t \geq t_g$. Choose $A$ large enough so

$$A \geq \max \left\{ \frac{2a}{1-\delta}, \frac{2}{1 - (1-\delta)^d C_g^d - 1 t_g^{(d-1)} - \lambda}, \frac{2}{h\delta^2 c}, \frac{16 \ln(2)}{hce^2} \right\}.$$  \quad (5)

Choose $t_0 > \max \{t_g, t'(\delta)\}$ sufficiently large so that for all $t \geq t_0$,

$$g(t) \leq C_g t^n \quad \text{and} \quad C_g b A t^n \leq t^{\theta-\lambda}. \quad \text{(6)}$$

Now, assume $t > t_0$ and that the geometric mean $N$ and sum $S$ of $n_1, \ldots, n_d$ satisfy the size and balance conditions. Thus, the size constraint and balance condition guarantee that

$$N \geq A t^\lambda \quad \text{and} \quad S \leq g(t) N \leq C_g N t^n. \quad \text{(7)}$$
Finally, assume $H$ is chosen according to $\mathcal{D}(K_{n_1,\ldots,n_d})$.

If the $n_i$'s satisfy the size conditions of the definition of a $\{c,\lambda,\theta\}$-median and since the model admits a concentration constant, then we would have a concentration bound around $cNt^\lambda$ for $L(H)$. Unfortunately, when some of the $n_i$'s are large, then $S$ will be large, and the size upper bound condition need not be satisfied, leaving us without the desired concentration bound. To overcome this situation, we break apart $H$ into hyper-subgraphs $H_1, \ldots, H_q$ of roughly the same size which we will refer to as blocks. The blocks will be vertex disjoint, the proportion between the sizes of the color classes in each $H_i$ will be roughly the same than the one in $H$. However, the crucial new aspect is that the size upper bound condition will be satisfied in each block $H_i$ allowing us to derive a concentration bound for $L(H_i)$. This will later allow us to obtain a concentration bound for $L(H)$, details follow.

Let $q = \lceil N/(At^\lambda) \rceil$. For each $1 \leq j \leq d$, let $n'_j = \lfloor n_j/q \rfloor$. Henceforth, let $N'$ and $S'$ denote the geometric mean and the sum of the $n'_j$'s. Denote the $j$-th color class of $H$ by $A_j$. Recall that $A_j$ is totally ordered. Let $A_{j,1}$ be the first $n'_j$ elements of $A_j$, $A_{j,2}$ be the following $n'_j$ elements of $A_j$, so on and so forth up to defining $A_{j,q}$. Clearly, the $A_{j,i}$'s are disjoint, but do not necessarily cover all of $A_j$. Now, for $1 \leq i \leq q$, define $H_i$ as the hyper-subgraph induced by $H$ in $A_{1,i} \times \ldots \times A_{d,i}$ (for an illustration, see Figure 4). Observe that the proportion between the sizes of the color classes of $H_i$ is roughly the same as the one among the color classes of $H$.

Note that by monotonicity, the distribution of $H_i$ is $\mathcal{D}(K_{n'_1,\ldots,n'_d})$. Moreover, since the $H_i$'s are disjoint, by block independence, their distributions. It follows that $L(H_1), \ldots, L(H_q)$ are independent random variables. A crucial, although trivial, observation is that

$$L(H) \geq \sum_{i=1}^{q} L(H_i).$$

On the other hand, by definition of $q$ and the size constraint condition,

$$\frac{N}{At^\lambda} \leq q \leq \frac{N}{At^\lambda} + 1 \leq \frac{2N}{At^\lambda}.$$  (Estimate of $q$)

In order to estimate the geometric mean $N'$ of $n'_1, \ldots, n'_d$, the following result will be useful.

**Lemma 11** If $x_1, \ldots, x_d$ are positive real numbers, then

$$\prod_{j=1}^{d} (x_j - 1) \geq \prod_{j=1}^{d} x_j - \left( \sum_{j=1}^{d} x_j \right)^{d-1}.$$
Proof: By induction on $d$. ■

It follows, by the preceding lemma, the estimate of $q$, and the balance condition, that

$$\frac{N}{q} = \prod_{j=1}^{d} \left( \frac{n_j}{q} \right)^{1/d} \geq \frac{1}{\frac{N}{q}} \geq \left( \prod_{j=1}^{d} \left( \frac{n_j}{q} - 1 \right) \right)^{1/d} \geq \left( \prod_{j=1}^{d} \left( \frac{n_j}{q} \right) - \left( \sum_{j=1}^{d} \frac{n_j}{q} \right) \right)^{1/d} = \frac{N}{q} \left( 1 - q \frac{S^{d-1}}{N^d} \right)^{1/d}.$$ 

By (7), our estimate of $q$, and since $\eta(d-1) < \lambda$,

$$q \frac{S^{d-1}}{N^d} \leq \frac{2}{A} C_g^d \frac{\eta(d-1) - \lambda}{\lambda^d}.$$ 

Given the way we have chosen $A$, we have that $(1 - q \frac{S^{d-1}}{N^d})^{1/d} \geq 1 - \delta$ and thus

$$\frac{N}{q} \geq \frac{N}{q(1 - \delta)} \geq \frac{1}{2} At^{\lambda(1 - \delta)} \geq a t^{\lambda}.$$ 

Based on the preceding estimate of $N'$ and the estimate for $q$ we will now show that $n'_1, \ldots, n'_d$ satisfy the size conditions required by the definition of $(c, \lambda, \theta)$-median. Indeed, by our estimate of $N'$ and $q$, and (5)

$$N' \geq \frac{N}{q(1 - \delta)} \geq 1 - \lambda a t^{\lambda}.$$ 

Moreover, by definition of $S'$, our estimate of $q$, (7), and (6),

$$S'b \leq \frac{Sb}{q} \leq \frac{Sb}{C_g At^{\lambda(1 - \delta)}} \leq C_g b At^{\lambda + \eta} \leq t^\theta.$$ 

Now, let $H'$ be chosen according to $D(K_{n'_1, \ldots, n'_d})$ and let $\text{Med}'$ be a median of $L(D(K_{n'_1, \ldots, n'_d}))$. By definition of $(c, \lambda, \theta)$-median, we get that $cN't^{-\lambda}(1 - \delta) \leq \text{Med}' \leq cN't^{-\lambda}(1 + \delta)$. Moreover, by definition of constant of concentration and approximate median, applying Markov’s inequality yields,

$$E \left[ L(H') \right] \geq (1 - 2\delta) \frac{cN'}{t^\lambda} \Pr \left[ L(H') \geq \left( 1 - 2\delta \right) \frac{cN'}{t^\lambda} \right] \geq (1 - 2\delta) \frac{cN'}{t^\lambda} \Pr \left[ L(H') \geq \left( 1 - \frac{\delta}{1 - \delta} \right) \text{Med}' \right] \geq (1 - 2\delta) \frac{cN'}{t^\lambda} \left( 1 - 2 \exp \left( -h \frac{\delta^2}{(1 - \delta)^2 \text{Med}'} \right) \right) \geq (1 - 2\delta) \frac{cN'}{t^\lambda} \left( 1 - 2 \exp \left( -h \frac{\delta^2}{1 - \delta} \text{Med}' \right) \right).$$

As observed above, $N' \geq At^{\lambda(1 - \delta)/2}$, so by choice of $A$, we get that $E \left[ L(H') \right] \geq (1 - 2\delta)(1 - \delta)cN't^{-\lambda}$. Hence, given that $L(H) \geq \sum_{i=1}^{q} L(H_i)$, the estimate of $N'$, the definition of $\delta$, and elementary algebra,

$$E \left[ L(H) \right] \geq \sum_{i=1}^{q} E \left[ L(H_i) \right] \geq (1 - 2\delta)(1 - \delta)q \frac{cN'}{t^\lambda} \geq (1 - 2\delta)(1 - \delta)^2 \frac{cN}{t^\lambda} \geq (1 - \epsilon/2) \frac{cN}{t^\lambda}.$$
We have thus established the lower bound claimed in (1). Now, we proceed to show (3). Note that

\[
\text{Pr} \left[ L(H) \leq (1 - \epsilon) \frac{cN}{t^{\lambda}} \right] \leq \sum_{(s_1, \ldots, s_q) \in \mathbb{N}^q} \text{Pr} \left[ L(H_i) = s_i, i = 1, \ldots, q \right].
\]

(9)

Let \( T \) be the set of indices of the summation in the preceding displayed equation. Also, for \( T = (s_1, \ldots, s_q) \) belonging to \( T \) let \( P_T \) denote \( \text{Pr} \left[ L(H_i) = s_i, i = 1, \ldots, q \right] \). We will show that \( P_T \) is exponentially small with respect to \( cN t^{-\lambda} \). Recalling that the \( L(H_i) \)'s are independent and distributed as \( L(H') \) when \( H' \) is chosen according to \( D(K_{n_1, \ldots, n_d}) \),

\[
P_T = \prod_{i=1}^{q} \text{Pr} \left[ L(H_i) = s_i \right] \leq (\text{Pr} \left[ L(H') \leq s_i \right])^q.
\]

Again, by the way in which \( H' \) is chosen, the definition of \( \text{Med}' \), and the definition of \((c, \lambda, \theta)\)-median, for all \( i \) such that \( s_i \leq (1 - \delta)cN t^{-\lambda} \leq \text{Med}' \leq (1 + \delta)cN t^{-\lambda} \leq 2cN t^{-\lambda} \), it holds that

\[
\text{Pr} \left[ L(H') \leq s_i \right] = \text{Pr} \left[ L(H') \leq \left(1 - \frac{\text{Med}' - s_i}{\text{Med}'}\right) \text{Med}' \right] \leq 2 \exp \left(-\frac{h}{2cN} ((1 - \delta)cN t^{-\lambda} - s_i)^2 \right).
\]

Hence, for all \( 1 \leq i \leq q \),

\[
\text{Pr} \left[ L(H') \leq s_i \right] \leq 2 \exp \left(-\frac{ht^\lambda}{2cN} \max \left\{0, ((1 - \delta)cN t^{-\lambda} - s_i)^2 \right\} \right),
\]

and then

\[
-\ln P_T \geq -\sum_{i=1}^{q} \ln \text{Pr} \left[ L(H') \leq s_i \right] \geq -q \ln(2) + \frac{ht^\lambda}{2cN} \sum_{i=1}^{q} \left(\max \left\{0, (1 - \delta)cN t^{-\lambda} - s_i \right\}\right)^2.
\]

By Cauchy-Schwartz’s inequality, our estimate of \( N' \), the fact that \( s_1 + \ldots + s_q \leq (1 - \epsilon)cN t^{-\lambda} \), and since by definition of \( \delta \) we know that \((1 - \delta)^2 \geq 1 - \epsilon/2\),

\[
\sqrt{q \sum_{i=1}^{q} \left(\max \left\{0, (1 - \delta)cN t^{-\lambda} - s_i \right\}\right)^2} \geq \sum_{i=1}^{q} \max \left\{0, (1 - \delta)cN t^{-\lambda} - s_i \right\} \geq (1 - \delta)cN qt^{-\lambda} - \sum_{i=1}^{q} s_i \geq (1 - \delta^2)cN t^{-\lambda} - (1 - \epsilon)cN t^{-\lambda} \geq \frac{cN \epsilon}{2t^{\lambda}}.
\]

Combining the last two displayed inequalities and recalling our estimate of \( N' \), we get

\[
-\ln P_T \geq -q \ln(2) + \frac{ht^\lambda}{2cNq} \cdot \frac{c^2N^2 \epsilon^2}{4t^{2\lambda}} \geq -q \ln(2) + \frac{hcN \epsilon^2}{8t^{\lambda}}.
\]
By (9) and using the standard estimate \( (\frac{a}{b}) \leq (ea/b)^k \), we have
\[
\Pr \left[ L(H) \leq (1 - \epsilon) \frac{cN}{t^\lambda} \right] \leq \sum_{T \in T} P_T \leq |T| \cdot \max_{T \in T} P_T \\
\leq \left( \frac{((1 - \epsilon)cNt^{-\lambda}) + q}{q} \right) \cdot \max_{T \in T} P_T \leq \exp \left( q \ln \left( 2e[1 + (1 - \epsilon)cNt^{-\lambda}/q] \right) - \frac{hcNe^2}{8t^\lambda} \right).
\]
Now, by q’s estimate we know that \( N \leq qAt^\lambda \leq 2N \). Thus, if we require that \( A \) is large enough so that \( \ln(2e[1 + (1 - \epsilon)cA]) \leq Ahc^2/32 \), we get that
\[
\Pr \left[ L(H) \leq (1 - \epsilon) \frac{cN}{t^\lambda} \right] \leq \exp \left( 2NAt^\lambda \ln(2e[1 + (1 - \epsilon)cA]) - \frac{hcNe^2}{8t^\lambda} \right) \leq \exp \left( -\frac{hc^2cN}{16t^\lambda} \right).
\]
This proves the lower bound claimed in (3). What remains is to show the lower bound in (2). By q’s estimate we have \( N \geq At^\lambda \) which together with our choice of \( A \) (see (5)), imply that
\[
\exp \left( -\frac{hc^2cN}{16t^\lambda} \right) \leq \exp \left( -\frac{hc^2c}{16A} \right) \leq \frac{1}{2}.
\]
Combining the last two displayed equations, it follows that \( \Pr \left[ L(H) \leq (1 - \epsilon) \frac{cN}{t^\lambda} \right] \leq 1/2 \), implying that any median of \( L(H) \) must be at least \((1 - \epsilon)\frac{cN}{t^\lambda}\).

**Remark 12** The reader may check that all claims proved in this section still hold if instead of \( D = (D(K_{n_1},...,n_d)) \) we had worked with a weak random hyper-graph model \( D = (D(K_n^{(d)})) \). Indeed, if this would have been the case, then for \( H \) chosen according to \( D(K_n^{(d)}) \), the hyper-graphs \( H_1, \ldots, H_q \) obtained above from \( H \) would have all their color classes of equal size, and the weak random hyper-graph model assumption is all that is all that is need to carry forth the arguments laid out in this section.

### 3 Upper bounds

In this section we will establish the upper bounds claimed in the statement of the Main Theorem, i.e. the upper bounds in (1) and (2), and inequality (4). The proof of the latter of these bounds, the upper tail bound, is rather long. For sake of clarity of exposition, we have divided its proof in three parts. First, in Section 3.1 we introduce some useful variables. In Section 3.2 we establish (4) for not to large values of the geometric mean \( N \). Then, in Section 3.3 we consider the case where \( N \) is large. Finally, in Section 3.4 we conclude the proof of the bounds claimed in the Main Theorem.

#### 3.1 Basic variable definitions

For the rest of this section, let \( D, c, \lambda, \theta, \eta, \) and \( \epsilon \) be as in the statement of the Main Theorem. Define
\[
\delta = \min \left\{ 1, \frac{\epsilon^2}{1 + \epsilon}, \frac{\epsilon}{6} \right\}.
\]
(Definition of \( \delta \))

Let \( a = a(\delta) \), \( b = b(\delta) \) and \( t' = t'(\delta) \) as guaranteed by the definition of \((c, \lambda, \theta)\)-median. Choose \( A \) so
\[
A = \max \left\{ \frac{a}{\delta^2}, \frac{8 \ln(2)}{h \delta c} \right\}.
\] (10)
For technical reasons, it will be convenient to fix constants $\alpha$ and $\beta$ such that

$$\lambda < \alpha < \beta < \theta - \eta. \quad (11)$$

We shall also encounter two constants $K_1$ and $K_2$, depending solely on $d$. Since $g = O(t^\eta)$, there are constants $C_g > 1$ and $t_g \geq t'$ such that for all $t \geq t_g$ it holds that $g(t) \leq C_g t^\eta$, and

$$\max \left\{ 9At^\lambda, e \right\} \leq t^\alpha \leq t^\beta \leq \frac{1}{bdC_g} t^{\theta - \eta}, \quad (12)$$

$$\frac{2t^\lambda}{c} \leq \frac{2\alpha K_1}{\delta c K_2 h} t^\lambda \leq \frac{t^\alpha}{\ln(t)}. \quad (13)$$

Consider now $t \geq t_g$ and the positive integers $n_1, n_2, \ldots, n_d$ with geometric mean $N$, summing up to $S$, and satisfying both the size constraint condition ($N \geq At^\lambda$) and balance condition ($S \leq g(t)N$). Furthermore, define $M = cNt^{-\lambda}$ and choose $H$ according to $D(K_{n_1, \ldots, n_d})$. In the following two sections, we separately consider the case where $N$ is less than and at least $t^\beta$.

### 3.2 Upper tail bound for not too large values of $N$

Throughout this section, we assume $N < t^\beta$. We will show that $H$ satisfies the size lower bound restriction in the definition of $(c, \lambda, \theta)$-median. The fact that $D$ admits a concentration constant $h$ will allow us obtain a bound on the upper tail of $L(H)$.

Let $t \geq t_g$. Since $N$ satisfies both the size constraint and balance condition, by (10), (12), and the definition of $\delta$,

$$N \geq At^\lambda \geq \frac{at^\lambda}{\delta} \geq at^\lambda,$$

$$Sb \leq g(t)bN \leq C_g bt^\eta \leq \frac{t^\theta}{d} \leq t^\theta.$$

Thus, $n_1, \ldots, n_d$ satisfy both the size lower and upper bound conditions of the definition of $(c, \lambda, \theta)$-median. Hence, if $H$ is chosen according to $D(K_{n_1, \ldots, n_d})$, then every median $\text{Med}$ of $L(H)$ is $\delta M$ close to $M$. Simple algebra, the definitions of concentration constant and $(c, \lambda, \theta)$-median, and given that by definition of $\delta$ we know that $\delta < \epsilon$, we have

$$\Pr \left[ L(H) \geq (1 + \epsilon)M \right] = \Pr \left[ L(H) \geq \left( 1 + \frac{(1 + \epsilon)M - \text{Med}}{\text{Med}} \right) \text{Med} \right] \leq 2 \exp \left( -h \frac{(1 + \epsilon)M - \text{Med}^2}{(1 + \epsilon)M} \right) \leq \exp \left( \frac{\ln(2) - h(\epsilon - \delta)^2}{1 + \epsilon} M \right).$$

By (10), since $N \geq At^\lambda$, the fact that by definition of $\delta$ we know that $\delta \leq \epsilon^2/(1 + \epsilon)$, and recalling that $M = cNt^{-\lambda}$,

$$\Pr \left[ L(H) \geq (1 + \epsilon)M \right] \leq \exp \left( \frac{Ah\delta c}{8} - \frac{he^2}{4(1 + \epsilon)M} \right) \leq \exp \left( h\delta Nc - \frac{he^2}{8(1 + \epsilon)M} \right) \leq \exp \left( -\frac{he^2}{8(1 + \epsilon)M} \right).$$

We have thus established (1) for $N < t^\beta$. 

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3.3 Upper tail bound for large values of $N$

We now consider the case where $N \geq t^\beta$. The magnitude of $N$ is such that we can not directly apply the definition of $(c, \lambda, \theta)$-median to a hyper-graph generated according to $D(K_{n_1,...,n_d})$, and thus derive the sought after exponentially small tail bound. We again resort to the block partitioning technique introduced in the proof of the lower bound. However, both the block partitioning and the analysis are more delicate and involved in the case of the upper bound.

3.3.1 Block partition

Let $l = t^\alpha$, $L = C_l t^{n+\alpha}$ and 

$$m_{\text{max}} = \lceil (1 + \epsilon)M \rceil. \quad (14)$$

In what follows, we shall upper bound the probability that $H$ chosen according to $D(K_{n_1,...,n_d})$ has a non-crossing hyper-matching of size at least $m_{\text{max}}$, i.e. the probability that $L(H) \geq m_{\text{max}}$.

We begin with a simple observation; since distinct edges of a non-crossing hyper-matching of $H$ can not have vertices in common, $L(H) \leq n_i$ for all $i$. It immediately follows that $L(H)$ is upper bounded by the geometric mean of the $n_i$’s, i.e. $L(H) \leq N$. Thus, if $m_{\text{max}} > N$, then $\Pr[L(H) \geq m_{\text{max}}] = 0$. This justifies why, in the ensuing discussion, we assume that $m_{\text{max}} \leq N$.

Let $J$ be a non-crossing hyper-subgraph of $K_{n_1,...,n_d}$ such that the number of edges of $J$ is (exactly equal) $m_{\text{max}}$. We shall partition the edge set of $J$ into consecutive sets of edges to which we will refer as blocks. The partition will be such that for any color class, the set of vertices appearing in a block are “not to far apart”, the precise meaning being clarified shortly. The maximum number of edges in any block will be $s_{\text{max}}$, where:

$$s_{\text{max}} = \left\lfloor \frac{l}{N} m_{\text{max}} \right\rfloor. \quad (15)$$

Given two edges $e$ and $\tilde{e}$ of $J$, such that $e \leq \tilde{e}$, we denote by $[e, \tilde{e}]$ the collection of edges $f$ of $J$ such that $e \leq f \leq \tilde{e}$. We now define a partition into blocks of the edge set of $J$, denoted $\mathcal{P}(J)$, as follows: $\mathcal{P}(J) = \{[e_i, \tilde{e}_i] : 1 \leq i \leq q \}$ where the $e_i$’s, the $\tilde{e}_i$’s, and $q$ are determined through the following process:

- $e_1$ is the first (smallest according to $\leq$) edge of $J$.

- Assuming $e_i = (v_1^{(i)}, v_2^{(i)}, ..., v_d^{(i)})$ has already been defined, $\tilde{e}_i = (\tilde{v}_1^{(i)}, \tilde{v}_2^{(i)}, ..., \tilde{v}_d^{(i)})$ is the last edge of $J$ satisfying the following two conditions (see Figure 3.3.1 for an illustration):

  - $[e_i, \tilde{e}_i]$ has at most $s_{\text{max}}$ elements.
  - $v_j^{(i)} - \tilde{v}_j^{(i)} \leq L$ for all $1 \leq j \leq d$ (where we have relied on the abuse of notation entailed by our identification of the $j$-th color class of $K_{n_1,...,n_d}$ with the set $\{1,2,...,n_j\}$ endowed with the natural order).

- Assuming $\tilde{e}_i$ has already been defined and provided there are edges $e$ of $J$ strictly larger than $\tilde{e}_i$, we define $e_{i+1}$ to be the smallest such $e$.

Clearly, the value taken by $q$ above depends on $J$. Nevertheless, we will show that the following estimate of $q = |\mathcal{P}(J)|$ holds for all $J$ non-crossing hyper-subgraphs of $K_{n_1,...,n_d}$:

$$\frac{N}{l} \leq |\mathcal{P}(J)| \leq \frac{3N}{l}. \quad \text{(Estimate of } q)$$
we know that
Now, say a block is short if it is either \([e_q, \bar{e}_q]\) or a block with exactly \(s_{\text{max}}\) edges. Let \(I_0\) be the collection of indices of short blocks. It follows that \(m_{\text{max}} \geq (|I_0| - 1)s_{\text{max}}\). However, since \(m_{\text{max}} \geq M = cNt^{-\lambda}\), we know that
\[
\frac{m_{\text{max}}}{s_{\text{max}}} = \frac{N}{T} \cdot \frac{1}{1 - N/(m_{\text{max}})} \leq \frac{N}{T} \cdot \frac{1}{1 - t^{\lambda-\alpha}/c}.
\]
We thus have, since (13) implies that \(t^{\lambda-\alpha} < c/2\), that \(|I_0| \leq 2N/l\).

Say a block is regular if it is not short, and let \(I_1 = [\bar{q}] \setminus I_0\) be the set of indices of such blocks. We shall call block cover the collection of all nodes between the first edge of the block (inclusive) and the first edge of the next block (exclusive). By definition of block partition, if the \(i\)-th block is regular, then for some color class \(j\), we must have \(v_j^{(i+1)} - v_j^{(i)} > L\). Hence, \(\sum_{j=1}^d (v_j^{(i+1)} - v_j^{(i)}) > L\). In other words, a regular block gives rise to a block cover of cardinality at least \(L\). Since every node belongs to at most one block cover, \(|I_1| \leq S/L\). Recalling that \(L = C_g t^n/l\) and that \(S\) satisfies the balance condition (hence, \(S \leq C_g t^n N\) for \(t \geq t_g\)), we conclude that \(|I_1| \leq N/l\).

Putting together the conclusions reached in the last two paragraphs, we see that \(q = |I_0| + |I_1| \leq 3N/l\), which establishes the claimed estimate of \(q\).

### 3.3.2 Partition types

Let \(s_i\) be the number of edges of \(J\) in the \(i\)-th block \([e_i, \bar{e}_i]\) of the partition \(P(J)\). Let \(q\) be the number of blocks of \(P(J)\). We refer to the \((3q)\)-tuple \(T = (e_1, \bar{e}_1, v_1, \ldots, e_q, \bar{e}_q, s_q)\) as the type of partition \(P(J)\), and denote it \(T(P(J))\). Furthermore, let \(T\) be the collection of all possible types of partitions of hyper-subgraphs of \(K_{n_1, \ldots, n_d}\) with exactly \(m_{\text{max}}\) edges.

**Lemma 13** There is a constant \(K_1\), depending only on \(d\), such that \(|T| \leq \exp \left(K_1 \frac{N}{l} \ln(l)\right)\).

**Proof:** Observe that each \(e_i\) is completely determined by specifying its vertices. Hence, the number of ways of choosing \(e_1, \ldots, e_q\) is at most the number of ways of choosing \(q\) elements from each of the node color classes, i.e. at most \(\prod_{j=1}^d |v_j|\). The number of choices for \(\bar{e}_1, \ldots, \bar{e}_q\) is bounded by the same amount. On the other hand, since \(J\) has exactly \(m_{\text{max}}\) edges, the number of choices for \(s_1, \ldots, s_q\) is at most the number of ways of summing up to \(m_{\text{max}}\) with \(q\) positive integer summands. Since we are assuming that \(m_{\text{max}} \leq N\) (see comment in this section’s second paragraph), we have that the aforementioned quantity can be bounded.
by \( \binom{N}{q} \). Using that \( \binom{N}{q} \leq (ea/b)^q \) we obtain, for fixed \( q \), that the number of types is bounded by

\[
\left( \frac{N}{q} \right) \left( \prod_{i=1}^{d} \left( \frac{n_i}{q} \right) \right)^2 \leq \left( \frac{eN}{q} \right)^q \left( \prod_{i=1}^{d} \left( \frac{en_i}{q} \right) \right)^{2q} = \left( \frac{eN}{q} \right)^{q+2qd}.
\]

Recalling our estimate for \( q \), we get that

\[
|\mathcal{T}| \leq \sum_{q=\lfloor N/l \rfloor}^{\lceil 3N/l \rceil} \left( \frac{eN}{q} \right)^{q(1+2d)} \leq \frac{3N}{l} (e) l (1+2d) |\mathcal{T}| N / l.
\]

Since \( \ln(x) \leq x \) for all \( x > 0 \) and by (12) we know that \( l = t^\alpha \geq e \),

\[
\ln |\mathcal{T}| \leq \ln \left( \frac{3N}{l} \right) + (1 + 2d) \frac{3N}{l} (1 + \ln(l)) \leq (2 + 2d) \frac{3N}{l} (1 + \ln(l)) \leq 12 (1 + d) N / l \ln(l).
\]

The desired conclusion follows choosing \( K_1 = 12 (1 + d) \). \( \blacksquare \)

### 3.3.3 Probability of a block partition occurring

The purpose of this section is to show that for a given fixed type \( T \), with exponentially small in \( M \) probability a hyper-graph chosen according to \( \mathcal{D}(K_{n_1, \ldots, n_d}) \) contains a hyper-subgraph of type \( T \) with \( m_{max} \) edges. Specifically, we will prove the following result.

**Lemma 14** For \( T \in \mathcal{T} \), let \( P_T \) denote the probability that a hyper-subgraph randomly chosen according to \( \mathcal{D}(K_{n_1, \ldots, n_d}) \) contains a non-crossing hyper-subgraph \( J \) with \( m_{max} \) edges such that \( T(\mathcal{P}(J)) = T \). Then, for some absolute constant \( K_2 > 0 \),

\[
P_T \leq \exp \left( -K_2h \frac{e^2}{1 + \epsilon} M \right).
\]

We now proceed with the proof of the preceding result. Let \( T = (e_1, e_2, s_1, \ldots, e_q, e_q, s_q) \). As before, for all \( i \), let \( e_i = (v_{1}^{(i)}, v_{2}^{(i)}, \ldots, v_{q}^{(i)}) \) and \( e_i = (\tilde{v}_{1}^{(i)}, \tilde{v}_{2}^{(i)}, \ldots, \tilde{v}_{q}^{(i)}) \). Let \( H \) be chosen according to \( \mathcal{D}(K_{n_1, \ldots, n_d}) \), and let \( H_i \) be the hyper-subgraph of \( H \) induced by the nodes between \( e_i \) and \( \tilde{e}_i \), i.e.,

\[
v_{1}^{(i)}, v_{1}^{(i)} + 1, \ldots, v_{1}^{(i)}, v_{2}^{(i)}, v_{2}^{(i)} + 1, \ldots, v_{2}^{(i)}, v_{d}^{(i)}, v_{d}^{(i)} + 1, \ldots, v_{d}^{(i)}.
\]

Note that \( H_i \) is distributed according to \( \mathcal{D}(K_{n_{1}^{(i)}, n_{2}^{(i)}, \ldots, n_{d}^{(i)}}) \), where \( n_{j}^{(i)} = \tilde{v}_{j}^{(i)} - v_{j}^{(i)} + 1 \) is the size of the \( j \)-th color class of \( H_i \). Moreover, if there is a hyper-subgraph \( J \) of \( H \) such that \( T(\mathcal{P}(J)) = T \), then it must hold that \( L(H_i) \geq s_i \), for all \( i = 1, \ldots, q \). Since by hypothesis, \( \mathcal{D} \) satisfies the block independence property, the events \( L(H_i) \geq s_i, i = 1, \ldots, q \), are independent, so

\[
P_T \leq \prod_{i=1}^{q} \mathbb{P} \left[ L \left( \mathcal{D}(K_{n_{1}^{(i)}, n_{2}^{(i)}, \ldots, n_{d}^{(i)}}) \right) \geq s_i \right].
\]

Now, let \( N_i \) and \( S_i \) denote the geometric mean and sum of \( n_{1}^{(i)}, \ldots, n_{d}^{(i)} \), respectively. The \( i \)-th term in the product of the last displayed equation will be small provided the sizes of the color classes of \( H_i \), i.e. the
\(n_j^{(i)}\)'s, satisfy the size constraints of the definition of a \((c, \lambda, \theta)\)-median. Unfortunately, this may not occur for every \(i\), somewhat complicating the analysis. Below we see how to handle this situation.

Since \(T(\mathcal{P}(J)) = T\), we know that \(n_1^{(i)}, n_2^{(i)}, \ldots, n_d^{(i)} \leq L\). Recalling that \(\alpha < \beta\) and applying (13) we conclude that \(S_i b \leq dbL = C_g dbt^{\eta + \alpha} \leq C_g dbt^{\eta + \beta} \leq t^\theta\), so the size upper bound condition of the definition of a \((c, \lambda, \theta)\)-median holds. However, the same might not be true regarding the size lower bound condition \(N_i \geq at^\lambda\). In order to handle this situation, we artificially augment the size of the blocks where the condition fails. Specifically, for all \(i = 1, \ldots, q\) and \(j = 1, \ldots, d\) we define:

\[
\overline{n}_j^{(i)} = \max \left\{ \delta n_j At^\lambda / N, n_j^{(i)} \right\}.
\]

As usual, let \(\overline{N}_i\) and \(\overline{S}_i\) denote the geometric mean and sum of the \(\overline{n}_j^{(i)}\)'s. Now observe that when we augment the sizes of the color classes of the hyper-graphs chosen, by the monotonicity property of random hyper-graph models, the probability of finding a non-crossing hyper-subgraph of size at least \(s_i\) increases. Hence,

\[
Pr_T \leq \prod_{i=1}^{q} \Pr \left[ L \left( \mathcal{D} \left( K_{\overline{n}_j^{(1)}, \ldots, \overline{n}_j^{(d)}} \right) \right) \geq s_i \right] \leq \prod_{i=1}^{q} \Pr \left[ L \left( \mathcal{D} \left( K_{\overline{n}_j^{(1)}, \ldots, \overline{n}_j^{(d)}} \right) \right) \geq s_i \right] .
\]

We claim that the \(\overline{N}_i\)'s and \(\overline{S}_i\)'s satisfy the size conditions in the definition of a \((c, \lambda, \theta)\)-median. Indeed, by definition of \(\overline{n}_j^{(i)}\), since \(n_j^{(i)} \leq L\), and

\[
\frac{\delta n_j At^\lambda}{N} \leq \frac{\delta n_j t^\alpha}{N} \leq \frac{\delta S_j t^\alpha}{N} \leq \delta g(t)t^\alpha \leq \delta C_g t^{\alpha + \eta} = \delta L \leq L ,
\]

it follows that \(\overline{n}_j^{(i)} \leq L\), and hence, as before augmenting the block sizes, \(\overline{S}_i b \leq t^\theta\). On the other hand, by definition of \(\overline{n}_j^{(i)}\), given that \(N \geq At^\lambda\), and since by (10) we know that \(A \geq a / \delta\),

\[
\overline{N}_i = \left( \prod_{j=1}^{d} \overline{n}_j^{(i)} \right)^{1/d} \geq \delta At^\lambda \left( \prod_{j=1}^{d} n_j \right)^{1/d} = \delta At^\lambda \geq at^\lambda .
\]

This concludes the proof of the stated claim.

Now, let \(\text{Med}_i\) be a median of \(L(\mathcal{D}(K_{n_1^{(i)}, \ldots, n_d^{(i)}}))\). By definition of \((c, \lambda, \theta)\)-median,

\[
(1 - \delta)c\overline{N}_i t^{-\lambda} \leq \text{Med}_i \leq (1 + \delta)c\overline{N}_i t^{-\lambda} .
\]

Hence, for all \(i\) such that \(s_i \geq (1 + \delta)c\overline{N}_i t^{-\alpha} \geq \overline{\text{Med}}_i\), and using that \(h\) is a concentration constant for the random model \(\mathcal{D}\), we get

\[
\Pr \left[ L \left( \mathcal{D} \left( K_{n_1^{(i)}, \ldots, n_d^{(i)}} \right) \right) \geq s_i \right] \leq \exp \left( -h \left( s_i - \overline{\text{Med}}_i \right)^2 \right) \leq \exp \left( -h s_i (1 + \delta) c\overline{N}_i t^{-\lambda} )^2 \right) .
\]

Since \(s_i \leq s_{\text{max}}\) for all \(i\),

\[
\Pr \left[ L \left( \mathcal{D} \left( K_{n_1^{(i)}, \ldots, n_d^{(i)}} \right) \right) \geq s_i \right] \leq \exp \left( -h s_i (1 + \delta) c\overline{N}_i t^{-\lambda} )^2 \right) .
\]
Combining some of the previously derived bounds

\[-\ln P_T \geq -\ln \left( \prod_{i=1}^{q} \Pr \left[ L \left( D \left( K_{i_1}, \ldots, K_{i_d} \right) \right) \geq s_i \right] \right) \]

\[\geq -q \ln(2) + \frac{h}{s_{\max}} \sum_{i=1}^{q} \left( \max \left\{ 0, s_i - (1 + \delta) c N t^{-\lambda} \right\} \right)^2.\]

We now focus on the summation in the last term in the preceding displayed equation. We lower bound it, via the following generalization of Hölder’s Inequality.

**Lemma 15** [Generalization of Hölder’s Inequality] For any collection of positive real numbers \( x_{i,j} \), \( 1 \leq i \leq q, 1 \leq j \leq d \),

\[\left( \sum_{i=1}^{q} \prod_{j=1}^{d} x_{i,j} \right)^{1/d} \leq \prod_{j=1}^{d} \sum_{i=1}^{q} x_{i,j}^{d}.\]

Setting \( x_{i,j} = (n_{j}^{(i)})^{1/d} \) in the aforementioned stated inequality, observing that by definition of \( n_{j}^{(i)} \) we have \( n_{j}^{(i)} \leq n_j^{(i)} + \delta n_j A t^{\lambda}/N \), and recalling that the sum of \( n_1^{(i)}, \ldots, n_d^{(i)} \) is at most \( n_j \),

\[\sum_{i=1}^{q} N_i \leq \left( \prod_{j=1}^{d} \sum_{i=1}^{q} n_{j}^{(i)} \right)^{1/d} \leq \left( \prod_{j=1}^{d} \sum_{i=1}^{q} \left( n_{j}^{(i)} + \delta n_j A t^{\lambda}/N \right) \right)^{1/d} \leq N(1 + \delta q A t^{\lambda}/N).\]

Because of our estimate for \( q \) and \((12)\), we conclude that

\[\sum_{i=1}^{q} N_i \leq N(1 + 3\delta A t^{\lambda-\alpha}) \leq N(1 + \delta/3) \leq N(1 + \delta).\]

By Cauchy-Schwartz’s inequality and recalling that the sum of the \( s_i \)’s is exactly equal to \( m_{\max} = \lfloor (1 + \epsilon) M \rfloor \),

\[\sqrt{q \sum_{i=1}^{q} \left( \max \left\{ 0, s_i - (1 + \delta) c N_i t^{-\lambda} \right\} \right)^2} \geq \sum_{i=1}^{q} \max \left\{ 0, s_i - (1 + \delta) c N_i t^{-\lambda} \right\}
\]

\[\geq m_{\max} - (1 + \delta) c t^{-\lambda} \sum_{i=1}^{q} N_i \geq M(1 + \epsilon) - M(1 + \delta)^2.\]

Let’s now see that the just derived lower bound is actually positive. Recall, that by definition of \( \delta \) we know that \( \delta \leq \epsilon/6 \) and \( \delta \leq 1 \), so

\[(1 + \epsilon) - (1 + \delta)^2 = \epsilon - 2\delta - \delta^2 \geq \epsilon - 3\delta \geq \epsilon/2.\]

We then have,

\[\sqrt{q \sum_{i=1}^{q} \left( \max \left\{ 0, s_i - (1 + \delta) c N_i t^{-\lambda} \right\} \right)^2} \geq \frac{\epsilon M}{2}.\]
Putting things together, and since \( s_{\text{max}} \leq (l/N)(1 + \epsilon)M \), we find that

\[
-\ln P_T \geq -q \ln(2) + \frac{h}{s_{\text{max}}} \sum_{i=1}^{q} \left( \max \left\{ 0, s_i - (1 + \delta)eN_i t^{-\lambda} \right\} \right)^2 \\
\leq -q \ln(2) + \frac{h}{q s_{\text{max}}} \cdot \frac{e^2 M^2}{4} \geq -q \ln(2) + \frac{hN e^2 M}{4q(1 + \epsilon)l}.
\]

Finally, recall that by our estimate for \( q \) we know that \( q \leq 3N/l \) and by (12) we have that \( l = t^\alpha \geq 9A \lambda \), so

\[
-\ln P_T \geq -\frac{N \ln(2)}{3A l^\lambda} + \frac{h e^2 M}{12(1 + \epsilon)} = \frac{cN}{t^\lambda} \left( \frac{h e^2}{12(1 + \epsilon)} - \frac{\ln(2)}{3A c} \right).
\]

By (10) we know that \( A \geq 8 \ln(2)/(h c \delta) \), by definition of \( \delta \) we have that \( \delta \leq \epsilon^2/(1 + \epsilon) \), implying that

\[
-\ln P_T \geq \frac{cN}{t^\lambda} \left( \frac{h e^2}{12(1 + \epsilon)} - \frac{h \delta}{24} \right) \geq \frac{\epsilon^2}{1 + \epsilon} \cdot \frac{h M}{24}.
\]

We have thus shown that Lemma 14 holds taking \( K_2 = 1/24 \).

3.3.4 Upper tail bound

We are now ready to finally prove (4) for \( N \geq t^\beta \). First, note that

\[
\Pr \left[ \mathcal{L}(\mathcal{D}(K_{11},...,n_d)) \geq m_{\text{max}} \right] \leq \sum_{T \in \mathcal{T}} P_T \leq |\mathcal{T}| \cdot \max_{T \in \mathcal{T}} P_T.
\]

By Lemmas 13 and 14, the fact that \( l = t^\alpha \), by our choice of \( t_g \) so (13) would hold, recalling that by definition of \( \delta \) we have that \( \delta \leq \epsilon^2/(1 + \epsilon) \) and given that \( M = cN t^{-\lambda} \),

\[
\Pr \left[ \mathcal{L}(\mathcal{D}(K_{11},...,n_d)) \geq m_{\text{max}} \right] \leq \exp \left( K_1 \frac{N}{l} \ln(l) - K_2 h \frac{e^2}{1 + \epsilon} M \right) = \exp \left( K_1 \frac{N}{t^\lambda} \ln(t) - K_2 h \frac{e^2}{1 + \epsilon} M \right) \leq \exp \left( \frac{K_2 h e^2}{2(1 + \epsilon)} M \right).
\]

We thus conclude that (4) holds for any constant \( K \leq K_2/2 \) (since \( K_2 = 1/24 \), any \( K \leq 1/48 \) would do).

3.4 Upper bounds for the mean and median

We will now establish the two remaining unproved bounds claimed in the Main Theorem, i.e. (1) and (2).

Fix \( \epsilon = \epsilon_0 > 0 \) and choose \( \delta, \alpha, \beta, C_g, t_g, K_1 \) and \( K_2 \) as in Section 3.1. We can view \( \delta \) as a function of \( \epsilon \), henceforth denoted \( \delta(\epsilon) \). Similarly, we can view \( A \) and \( t_g \) as functions of \( \delta \), denoted \( A(\delta) \) and \( t_g(\delta) \) respectively. Let \( A' \) be a sufficiently large constant so

\[
\exp \left( -K h \frac{\epsilon_0^2}{4(1 + \epsilon_0/2)} cA' \right) \leq \frac{\epsilon_0}{28} \quad \text{and} \quad \frac{7}{6K h} \leq \frac{\epsilon_0}{4} cA'.
\]

(Definition of \( A' \))

Also, let \( \delta_0 = \delta(\epsilon_0/2) \). Observe that by definition of \( \delta \), for every \( \epsilon \geq 6 \) we have that \( \delta(\epsilon) = 1 \). Define now \( A = \max \{ A(\delta_0), A(1), A' \} \), and \( t_g = \max \{ t_g(\delta_0), t_g(1) \} \).
Let $t \geq \bar{t}_g$ and consider the positive integers $n_1, \ldots, n_d$ with geometric mean $N$ and summing $S$ satisfying the size and balance conditions in the statement of the Main Theorem, i.e.

$$N \geq \bar{A}t^\lambda, \quad \text{and} \quad Sb \leq g(t)N.$$ 

The choice of $\bar{t}_g$ and $\bar{A}$ guarantee that (4) holds for $\epsilon = \epsilon_0/2$ and for all $\epsilon \geq 6$.

As usual, let $H$ be chosen according to $\mathcal{D}(K_{n_1}, \ldots, n_d)$ and let $M = cNt^{-\lambda}$. Let $\mathbb{I}_A(x)$ denote the function that takes the value 1 if $x \in A$ and 0 otherwise. Observe that

$$\mathbb{E}[L(H)] = \mathbb{E}[L(H)\mathbb{I}_{[0,(1+\epsilon_0/2)M]}(L(H))] + \mathbb{E}[L(H)\mathbb{I}_{[(1+\epsilon_0/2)M, 7M]}(L(H))] + \mathbb{E}[L(H)\mathbb{I}_{[7M, +\infty]}(L(H))] .$$

Let's now upper bound separately each of the terms in the right hand side of the preceding displayed equation. The first one is trivially upper bounded by $(1 + \epsilon_0/2)M$. Thanks to (4), since $N \geq \bar{A}t^\lambda \geq \bar{A}'t^\lambda$, and by definition of $\bar{A}'$,

$$\mathbb{E}[L(H)\mathbb{I}_{[(1+\epsilon_0/2)M, 7M]}(L(H))] \leq 7M\mathbb{Pr}[L(H) > (1 + \epsilon_0/2)M] \leq 7M \exp\left(-Kh\frac{\epsilon_0^2}{4(1+\epsilon_0/2)} \cdot \frac{cN}{t^\lambda}\right) \leq \frac{M\epsilon_0}{4} .$$

Now, let's consider the third term. By (4), since for $\epsilon \geq 6$ it holds that $\epsilon/(1 + \epsilon) \geq 6/7$, given that $M = cNt^{-\lambda} \geq c\bar{A} \geq \bar{A}'$, and by definition of $\bar{A}'$,

$$\mathbb{E}[L(H)\mathbb{I}_{[7M, +\infty]}(L(H))] = \int_{7M}^{\infty} \mathbb{Pr}[L(H) > t] \, dt = M \int_6^{\infty} \mathbb{Pr}[L(H) > (1 + \epsilon)M] \, d\epsilon \leq M \int_6^{\infty} \exp\left(-\frac{6Kh}{7} \cdot M\epsilon\right) \, d\epsilon = M\left(\frac{6Kh}{7} M\right)^{-1} \exp\left(-\frac{36Kh}{7} \cdot M\right) \leq M\left(\frac{6Kh}{7} \cdot c\bar{A}'\right)^{-1} \leq \frac{M\epsilon_0}{4} .$$

Summarizing, we have that $\mathbb{E}[L(H)] \leq (1 + \epsilon_0)M$ which proves (1).

Finally, we establish (2). Again, let $\epsilon > 0$ and choose $\delta, A, \alpha, \beta, C_g, t_g, K_1$ and $K_2$ as in Section 3.1. Let

$$\bar{A}' = \max\left\{A, \frac{(1 + \epsilon)\ln(2)}{Khc^2}\right\}. \quad \text{(Definition of $\bar{A}'$)}$$

Now, let $t \geq t_g$ and $n_1, \ldots, n_d$ be positive integers with geometric mean $N$ and summing up to $S$ satisfying the size and balance conditions with respect to the just defined constant $\bar{A}'$, i.e.

$$N \geq \bar{A}'t^\lambda, \quad \text{and} \quad Sb \leq g(t)N.$$ 

By (4) and definition of $\bar{A}'$, it follows that

$$\mathbb{Pr}[L(H) \geq (1 + \epsilon)M] \leq \exp\left(-Kh\frac{\epsilon^2}{1 + \epsilon} \cdot \frac{cN}{t^\lambda}\right) \leq \exp\left(-Kh\frac{\epsilon^2}{1 + \epsilon} \cdot c\bar{A}'\right) \leq \frac{1}{2} .$$

Hence, every median of $L(H)$ is at most $(1 + \epsilon)M$, thus establishing (2) and completing the proof of the Main Theorem.
4 Applications

4.1 Random binomial hyper-graph model

In this section, we show how to apply the Main Theorem to the $d$-partite random binomial hyper-graph model.

We will show that the constant $c$ of the definition of a $(c, \lambda, \theta)$-median for this model is related to a constant that arises in the study of the asymptotic behavior of the length of a longest increasing subsequence of $d-1$ randomly chosen permutations of $[n]$, when $n$ goes to infinity. We first recall some known facts about this problem. Given a positive integers $d$ and $n$, consider $d$ permutations of $\pi_1, \ldots, \pi_d$ of $[n]$. We say that $L = \{i_j, \pi_1(i_j), \ldots, \pi_d(i_j) | 1 \leq j \leq \ell\}$ is an increasing sequence of $(\pi_1, \ldots, \pi_d)$ of length $\ell$ if $i_1 < i_2 < \ldots < i_\ell$ and $\pi_i(i_1) < \pi_i(i_2) < \ldots < \pi_i(i_\ell)$ for $1 \leq i \leq d$. We denote by $lis_{d+1}(n)$ the random variable corresponding to the length of a longest increasing subsequence of $(\pi_1, \ldots, \pi_d)$ when $\pi_1, \ldots, \pi_d$ are randomly and uniformly chosen. The study of the asymptotic characteristics of the distribution of $lis_d(n)$ will be henceforth referred to as Ulam’s problem in $d$ dimensions (note that the $d = 2$ case corresponds precisely to the setting discussed in the first paragraph of the introductory section of this work).

Ulam’s problem in $d$-dimensions can be restated geometrically. Indeed, consider $\vec{x}(1), \ldots, \vec{x}(n)$ uniformly and independently chosen in the $d$-dimensional unit cube $[0,1]^d$ endowed with the natural component wise partial order. Let $H_d(n)$ be the length of a largest chain $C \subseteq \{\vec{x}(1), \ldots, \vec{x}(n)\}$. It is not hard to see that $H_d(n)$ and $lis_d(n)$ follow the same distribution. Bollobás and Winkler [BW88] showed that for every $d$ there exists a constant $c_d$ such that $H_d(n)/\sqrt{n}$ (and thus also $lis_d(n)/\sqrt{n}$) goes to $c_d$ as $n \to \infty$. Only the values $c_1 = 1$ and $c_2 = 2$ are known for these constants. However, in [BW88] it is shown that $c_1 \leq c_{i+1}$ and $c_i < e$ for all $i$, and that the $\lim_{d \to \infty} c_d = e$.

Now, back to our problem. Our immediate goal is to estimate a median of $L(\mathcal{G}(K_{n_1, \ldots, n_d}, p))$. Consider $H$ chosen according to $\mathcal{G}(K_{n_1, \ldots, n_d}, p)$ and let $H'$ be the hyper-subgraph of $H$ obtained from $H$ after removal of all edges incident to nodes of degree at least 2. Let $E = E(H)$ and $E' = E(H')$. In order to approximate a median of $L(\mathcal{G}(K_{n_1, \ldots, n_d}, p))$ it will be useful to estimate first the expected value of $L(H')$. We now come to a crucial observation: $L(H')$ is precisely the length of a largest chain (for the natural order among edges) contained in $E'$, or equivalently the length of a longest increasing subsequence of $d-1$ permutations of $\{1, \ldots, |E'|\}$. The preceding observation will enable us to build on the known results concerning Ulam’s problem and use them in the analysis of the Longest Non-crossing Matching problem for the random binomial hyper-graph model. In particular, the following concentration result due to Bollobás and Brightwell [BB92] for the length of a $d$-dimensional longest increasing subsequence will be useful for our purposes.

**Theorem 16** [Bollobás and Brightwell [BB92], Theorem 8] For every $d \geq 2$, there is a constant $D_d$ such that for $m$ sufficiently large and $2 < \lambda < m^{1/2d}/\log \log m$,

$$\Pr \left[ |lis_d(m) - E[lis_d(m)]| > \frac{\lambda D_d m^{1/2d} \log(m)}{\log \log(m)} \right] \leq 80\lambda^2 e^{-\lambda^2}.$$ 

We will not directly apply the preceding result. Instead, we rely on the following:

**Corollary 17** For every $d \geq 2$, $t > 0$ and $\alpha > 0$, there is a $m_0(t, \alpha, d)$ sufficiently large such that if $m \geq m_0$, then

$$\Pr \left[ |lis_d(m) - c_d m^{1/d}| > tc_d m^{1/d} \right] \leq \alpha.$$
Proof: Let $D_d$ be the constant in the statement of Theorem 16. By definition of Ulam’s constant, we know that $\lim_{m \to \infty} \frac{\Pr[|lis_d(m)|]}{m} = c_d$. Hence, we can choose $m_0 = m_0(t, \alpha, d)$ sufficiently large so that for all $m \geq m_0$, Theorem 16 holds and in addition the following conditions are satisfied:

- $|E[|lis_d(m)|] - c_d m^{1/d}| < tc_d m^{1/d}/2$.
- $\lambda = \frac{tc_d}{27D_d} \cdot \frac{m^{1/2d} \log \log(m)}{\log(m)} \leq \frac{m^{1/2d}}{\log \log(m)}$ and $80\lambda^2 e^{-\lambda^2} \leq \alpha$.

(Both conditions can be satisfied since $(\log \log(m))^2 = o(\log(m))$ and given that $\lambda(m) \to \infty$ when $m \to \infty$.) It follows that for all $m > m_0$,

$$\Pr \left[ |lis_d(m) - c_d m^{1/d}| > tc_d m^{1/d} \right] \leq \Pr \left[ |lis_d(m) - E[lis_d(m)] + E[lis_d(m)] - c_d m^{1/d}| > tc_d m^{1/d} \right]$$

$$\leq \Pr \left[ |lis_d(m) - E[lis_d(m)]| + |E[lis_d(m)] - c_d m^{1/d}| > \frac{1}{2} tc_d m^{1/d} \right]$$

$$= \Pr \left[ |lis_d(m) - E[lis_d(m)]| > \frac{\lambda D_d m^{1/2d} \log(m)}{\log \log(m)} \right]$$

$$\leq 80\lambda^2 e^{-\lambda^2}.$$ 

\[\Box\]

For future reference, we recall a well known variant of Chebyshev’s inequality.

**Proposition 18** (Chebyshev’s inequality for indicator random variables) Let $X_1, \ldots, X_m$ be random variables taking values in $\{0, 1\}$ and let $X$ denote $X_1 + \ldots + X_m$. Also, let $\Delta = \sum_{i,j:i \neq j} E[|X_i X_j|]$. Then, for all $t \geq 0$,

$$\Pr \left[ |X - E[X]| \geq t \right] \leq \frac{1}{t^2} \left( E[X] (1 - E[X]) + \Delta \right).$$

Moreover, if $X_1, \ldots, X_m$ are independent, then

$$\Pr \left[ |X - E[X]| \geq t \right] \leq \frac{E[X]}{t^2}.$$ 

**Proof:** Observe that since $X_i$ is an indicator variable, then $E[X_i^2] = E[X_i]$. Thus, if we let $V[X]$ denote the variance of $X$,

$$V[X] = E[X^2] - (E[X])^2 = \sum_{i=1}^m E[X_i^2] + \Delta - (E[X])^2 = E[X] (1 - E[X]) + \Delta.$$ 

A direct application of Chebyshev’s inequality yields the first bound claimed. The second stated bound, follows from the first one and the fact that if $X_1, \ldots, X_m$ are independent, then $\Delta \leq (E[X])^2$. \[\Box\]

We will also need the following two lemmas.

**Lemma 19** Let $N$ and $S$ denote the geometric mean and sum of $n_1, \ldots, n_d$. If $\tilde{N} = \left( \prod_{j=1}^d (n_j - 1) \right)^{1/d}$, then $\tilde{N}^d - \tilde{N}^d \leq S^{d-1}$. 

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Proof: Direct application of Lemma\textsuperscript{15} \hfill □

**Lemma 20** Let \( N \) and \( S \) denote the geometric mean and sum of \( n_1, \ldots, n_d \). If \( \tilde{N} = \left( \prod_{j=1}^{d} (n_j - 1) \right)^{\frac{1}{d}} \), then the following hold:

\[
\begin{align*}
\mathbb{E}[|E|] &= N^d p, \\
\mathbb{E}[|E'|] &= N^d p(1 - p)^{N^d - \tilde{N}^d} \geq N^d p(1 - S^{d-1}p), \\
\mathbb{E}[|E \setminus E'|] &\leq N^d S^{d-1} p^2.
\end{align*}
\]

Moreover, for all \( \eta > 0 \),

\[
\Pr[|E| - \mathbb{E}[|E|] \geq \eta \mathbb{E}[|E|]] \leq \frac{1}{\eta^2 \mathbb{E}[|E|]}.
\]

**Proof:** Let \( K = K_{n_1, \ldots, n_d} \), and for each \( e \in E(K) \) let \( X_e \) and \( Y_e \) denote the indicators of the events \( e \in E \) and \( e \in E' \), respectively. Note that \( |E| = \sum_{e \in E(K)} X_e \) and \( |E'| = \sum_{e \in E(K)} Y_e \). Clearly, \( \mathbb{E}[X_e] = p \) for all \( e \in E(K) \). Moreover, \( e \in E' \) if and only if \( e \in E \) and no edge \( f \in E \setminus \{e\} \) intersects \( e \). Since the number of edges in \( E(K) \) that intersect any given \( e \in E(K) \) is exactly \( N^d - \tilde{N}^d \), we have that \( \mathbb{E}[Y_e] = p(1 - p)^{N^d - \tilde{N}^d} \). Observing that \( |E(K)| = N^d \) we obtain (16) and the first equality in (17). On the other hand, since \( (1 - p)^m \geq 1 - pm \) and by Lemma\textsuperscript{19} we can finish the proof of (17) by noting that

\[
\mathbb{E}[|E'|] = N^d p(1 - p)^{N^d - \tilde{N}^d} \geq N^d p(1 - (N^d - \tilde{N}^d)p) \geq N^d p(1 - S^{d-1}p).
\]

Inequality (18) is a consequence of (16), (17), and the fact that \( E' \subseteq E \), as follows:

\[
\mathbb{E}[|E \setminus E'|] = \mathbb{E}[|E| - |E'|] \leq N^d S^{d-1} p^2,
\]

Applying Chebyshev’s inequality for independent indicator random variables \( \{ X_e \mid e \in E(K) \} \) yields (19).

\hfill □

We are now ready to exploit the fact, already mentioned, that \( L(H') \) equals the length of a longest increasing subsequence of \( d - 1 \) permutations of \( \{1, \ldots, |E'|\} \), and then apply Corollary\textsuperscript{17} in order to estimate its value. Formally, we prove the following claim.

**Proposition 21** Let \( \delta > 0 \), \( d \geq 2 \), and \( N \) and \( S \) be the geometric mean and sum of positive integers \( n_1, \ldots, n_d \), respectively. Moreover, let \( M = c_d N p^{1/d} \), where \( c_d \) is the \( d \)-dimensional Ulam constant. Then, there is a constant \( C = C(\delta) \) sufficiently large such that:

- If \( N p^{1/d} \geq C \) and \( 12 S^{2d - 2} p^{2 - 1/d} \leq d^{-1} \delta c_d \), then every median of \( L(G(K_{n_1, \ldots, n_d}, p)) \) is at most \( (1 + \delta)M \).
- If \( N p^{1/d} \geq C \) and \( 12 S^{d-1} p \leq \delta \), then every median of \( L(G(K_{n_1, \ldots, n_d}, p)) \) is at least \( (1 - \delta)M \).

**Proof:** To prove that every median of \( L(G(K_{n_1, \ldots, n_d}, p)) \) is at most \( (1 + \delta)M \), it suffices to show that \( \Pr[ L(H) \geq (1 + \delta)M ] \) is at most 1/2. To establish the latter, note that \( L(H) \leq L(H') + |E \setminus E'| \), hence

\[
\Pr[ L(H) \geq (1 + \delta)M ] \leq \Pr\left[ |E \setminus E'| \geq \frac{M\delta}{2} \right] + \Pr\left[ L(H') \geq (1 + \delta/2)M \right] \\
\leq \Pr\left[ |E \setminus E'| \geq \frac{M\delta}{2} \right] + \Pr\left[ |E'| \geq (1 + \delta/2) \frac{M^d}{c_d^d} \right] + \Pr\left[ L(H') \geq (1 + \delta/2)M, |E'| < (1 + \delta/2) \frac{M^d}{c_d^d} \right].
\]

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We now separately upper bound each of the latter three terms. For the first one, we rely on Markov’s inequality, inequality (17) of Lemma 17, the fact that $N \leq S/d$, and our hypothesis, to conclude that:

$$
\Pr \left[ |E \setminus E'| \geq \frac{M \delta}{2} \right] \leq \frac{2}{M \delta} E \left[ |E \setminus E'| \right] \leq \frac{2N^d S^{d-1} p^2}{\delta c_d N^d p^{1/d}}
$$

$$
= \frac{2N^d S^{d-1} p^2 - 1/d}{\delta c_d} \leq \frac{2S^2 - 2 p^2 - 1/d}{d^{d-1} \delta c_d} \leq \frac{1}{6}.
$$

To bound the second term, note that $|E| \geq |E'|$, and recall (16) and (19) of Lemma 20, so

$$
\Pr \left[ |E'| \geq (1 + \delta/2) \frac{M^d}{c_d} \right] = \Pr \left[ |E| \geq (1 + \delta/2) E \left[ |E| \right] \right] \leq \frac{4d}{\delta^2 E \left[ |E| \right]} = \frac{4}{\delta^2 N^d p}.
$$

Since by assumption $N^d p \geq C^d$, it suffices to take $C^d \geq 24/\delta^2$ in order to derive an upper bound of $1/6$ for the second term.

Finally, we focus on the third term. Let $m = \lfloor (1 + \delta/2)M^d/c_d \rfloor$. Recall that conditioned on $|E'| = n'$, the random variable $L(H')$ follows the same distribution as lis$(n')$. Thus, since $n \geq n'$ implies that lis$(n)$ dominates lis$(n')$, and given that $(1 + x)^a \leq 1 + ax$ for $x \geq -1$ and $0 < a < 1$,

$$
\Pr \left[ L(H') \geq (1 + \delta/2)M, |E'| < (1 + \delta/2) \frac{M^d}{c_d} \right] \leq \Pr \left[ \text{lis}_d(m) \geq \frac{1 + \delta/2}{(1 + \delta/2)^{1/d} c_d m^{1/d}} \right]
$$

$$
\leq \Pr \left[ \text{lis}_d(m) \geq \frac{1 + \delta/2}{1 + \delta/(2d) c_d m^{1/d}} \right] = \Pr \left[ \text{lis}_d(m) \geq \left( 1 + \frac{d-1}{2d} \delta \right) c_d m^{1/d} \right].
$$

Setting $t = (d-1)\delta/(2d + \delta)$ and requiring that $C^d \geq m_0 + 1$ with $m_0 = m_0(t, 1/6, d)$ as in Corollary 17 and since by assumption $N^d p \geq C^d$, we have

$$
m = \lfloor (1 + \delta/2)M^d/c_d \rfloor = \lfloor (1 + \delta/2)N^d p \rfloor \geq C^d \geq m_0.
$$

Thus, we can apply Corollary 17 and conclude that

$$
\Pr \left[ L(H') \geq (1 + \delta/2)M, |E'| < (1 + \delta/2) \frac{M^d}{c_d} \right] \leq \frac{1}{6}.
$$

In summary, $\Pr \left[ L(H) \geq (1 + \delta)M \right] \leq 3(1/6) = 1/2$ as we wanted to show.

Now, to prove that every median of $L(G(K_{n_1, \ldots, n_d}, p))$ is at least $(1 - \delta)M$, it suffices to show that $\Pr \left[ L(H) \leq (1 - \delta)M \right]$ is at most $1/2$. Note that $L(\cdot)$ is non-negative, so we can always assume that $\delta \leq 1$. Since $L(H') \leq L(H)$,

$$
\Pr \left[ L(H) \leq (1 - \delta)M \right] \leq \Pr \left[ |E| \leq (1 - \delta) \frac{M^d}{c_d} \right] + \Pr \left[ L(H') \leq (1 - \delta)M, |E| > (1 - \delta) \frac{M^d}{c_d} \right]
$$

$$
\leq \Pr \left[ |E| \leq (1 - \delta) \frac{M^d}{c_d} \right] + \Pr \left[ |E \setminus E'| \geq (\delta/2) \frac{M^d}{c_d} \right]
$$

$$
+ \Pr \left[ L(H') \leq (1 - \delta)M, |E'| > (1 - \delta/2) \frac{M^d}{c_d} \right].
$$

As above, we separately bound each of the two latter terms. In the case of the first term, by (16) and (19) of Lemma 20,

$$
\Pr \left[ |E| \leq (1 - \delta) \frac{M^d}{c_d} \right] = \Pr \left[ |E| \leq (1 - \delta) E \left[ |E| \right] \right] \leq \frac{1}{\delta^2 E \left[ |E| \right]} = \frac{1}{\delta^2 N^d p}.
$$
Since by assumption \(N^dp \geq C^d\), it suffices to take \(C^d \geq 6/\delta^2\) in order to establish an upper bound of \(1/6\) for the term under consideration.

To bound the second term, simply apply Markov’s inequality, use 18 of Lemma 20 and recall that by assumption \(12pS^{d-1} \leq \delta\) — an upper bound of \(1/6\) follows for the term under consideration.

Now, for the third term, let \(m = \lceil (1-\delta/2)M^d/c_d^d \rceil\). Recall that conditioned on \(|E'| = n'\), the random variable \(L(H')\) follows the same distribution as \(\text{lis}(n')\). Thus, since \(n' \geq n\) implies that \(\text{lis}(n')\) dominates \(\text{lis}(n)\), some basic arithmetic and given that \((1+x)^a \leq 1 + ax\) for \(x \geq -1\) and \(0 < a < 1\),

\[
\Pr \left[ L(H') \leq (1-\delta)M, |E'| > (1-\delta/2)\frac{M^d}{c_d^d} \right] \leq \Pr \left[ \text{lis}_d(m) \leq (1-\delta)M \right]
\leq \Pr \left[ \text{lis}_d(m) \leq \frac{1-\delta}{(1-\delta/2)^{1/d} c_d m^{1/d}} \right] \leq \Pr \left[ \text{lis}_d(m) \leq (1-\delta/2)^{1-1/d} c_d m^{1/d} \right]
\leq \Pr \left[ \text{lis}_d(m) \leq \left( \frac{1 - \delta}{2} \left( 1 - \frac{1}{d} \right) \right) c_d m^{1/d} \right].
\]

Setting \(t = (\delta/2)(1-1/d)\), requiring that \(C \geq (m_0/(1-\delta/2))^{1/d}\) with \(m_0 = m_0(t, 1, 6, d)\) as in Corollary 17 and since by assumption \(N^dp \geq C^d\), we get

\[
m \geq (1-\delta/2)\frac{M^d}{c_d^d} = (1-\delta/2)N^dp \geq (1-\delta/2)C^d \geq m_0.
\]

Thus, we can apply Corollary 17 and conclude that the third term is also upper bounded by \(1/6\).

Summarizing, \(\Pr \left[ L(H) \leq (1-\delta)M \right] \leq 3(1/6) = 1/2\) as we wanted to show. ■

**Corollary 22** Let \(d \geq 2\). If \(t = 1/p\), then the model \((G(K_{n_1, \ldots, n_d}, p))\) of internal parameter \(t\) admits a \((c, \lambda, \theta)\)-median where

\[
(c, \lambda, \theta) = \left( c_d, \frac{1}{d}, \frac{2d - 1}{2d(d - 1)} \right).
\]

**Proof:** As usual, let \(N\) and \(S\) denote the geometric mean and sum of \(n_1, \ldots, n_d\). Let \(H\) be chosen according to \(G(K_{n_1, \ldots, n_d}, p)\), \(M = cN/p^d = c_d N^{1/d}, \delta > 0\), and \(C(\delta)\) be as in Proposition 21. Define \(a(\delta) = C(\delta), b(\delta) = (12/(\delta d^{d-1} c_d))^{1/(2d-2)}\) and \(t'(\delta)\) sufficiently large so \(t > t'(\delta)\) and \(t^{1-1/(2d)} < \delta/(12) t(b(\delta))^{d-1}\). Note that if \(t > t'(\delta), N \geq a(\delta) t^{1/d}, \) and \(Sb(\delta) \leq (t^{2d-1})/(2d(d-1))\), then the hypothesis of Proposition 21 will be satisfied, and hence every median of \(L(H)\) will be between \((1-\delta)M\) and \((1+\delta)M\). ■

Recalling that by Proposition 10 we know that \(h = 1/4\) is a concentration constant for the \(d\)-dimensional binomial random hyper-graph model, by Corollary 22 and the Main Theorem, we obtain the following:

**Theorem 23** Let \(\epsilon > 0\) and \(g: \mathbb{R} \to \mathbb{R}\) be such that \(g(t) = O(t^\eta)\) for a given \(0 \leq \eta < 1/(2d(d-1))\). Fix \(n_1, \ldots, n_d\) and let \(N\) and \(S\) denote their geometric mean and sum, respectively. There exists a sufficiently small \(p_0\) and sufficiently large \(A\) such that if \(p \leq p_0, N p^{1/d} \leq A\) and \(S \leq g(1/p)\), then for \(M = c_d N p^{1/d}\) where \(c_d\) is the \(d\)-dimensional Ulam constant,

\[
(1-\epsilon)M \leq \mathbb{E}[L(G(K_{n_1, \ldots, n_d}, p))] \leq (1+\epsilon)M,
\]

and the following hold:
• If \( \text{Med} [L(G(K_{n_1, \ldots, n_d}, p))] \) is a median of \( L(G(K_{n_1, \ldots, n_d}, p)) \),
\[
(1 - \epsilon)M \leq \text{Med} [L(G(K_{n_1, \ldots, n_d}, p))] \leq (1 + \epsilon)M.
\]

• There is an absolute constant \( C > 0 \) such that
\[
\Pr [L(G(K_{n_1, \ldots, n_d}, p)) \leq (1 - \epsilon)M] \leq \exp \left(-C\epsilon^2M\right),
\]
\[
\Pr [L(G(K_{n_1, \ldots, n_d}, p)) \geq (1 + \epsilon)M] \leq \exp \left(-C\frac{\epsilon^2}{1 + \epsilon}M\right).
\]

We are now ready to prove Theorem 6, which is this section’s main result, and was already stated in the main contributions section.

**Proof of Theorem 6:** Let \( n, n', n'' \) be positive integers such that \( n = n' + n'' \). Clearly,
\[
E \left[ L(G(K_n^{(d)}, p)) \right] \geq E \left[ L(G(K_{n'}^{(d)}, p)) \right] + E \left[ L(G(K_{n''}^{(d)}, p)) \right].
\]

By subadditivity, it follows that the limit of \( E \left[ L(G(K_n^{(d)}, p)) \right] \) when normalized by \( n \) exists and equals \( \delta_p = \inf_{n \in \mathbb{N}} E \left[ L(G(K_n^{(d)}, p))/n \right] \). A direct application of Theorem 23 yields that \( \delta_p/\sqrt{n} \to c_d \) when \( p \to 0 \).

### 4.2 Random word model

In this section, we consider the random \( d \)-word model. The structure, arguments and type of derived results are similar to those obtained in the preceding section. However, the intermediate calculations are somewhat longer and more involved. We omit the proofs of this section’s results from the current draft.

As in the preceding section, we first show that the random model under consideration admits a \((c, \lambda, \theta)\)-median. Now consider \( H \) chosen according to \( \Sigma(K_{n_1, \ldots, n_d}, k) \) and let \( H' \) be the hyper-subgraph of \( H \) obtained from \( H \) as in the preceding section (i.e. by removal of all edges incident to nodes of degree at least 2). Let \( E = E(H) \) and \( E' = E(H') \). For the random word model, the analogue of Lemma 20 is the following:

**Lemma 24** Let \( N \) and \( S \) be the geometric mean and sum of positive integers \( n_1, \ldots, n_d \), respectively. Then,
\[
E \left[ |E| \right] = \frac{N^d}{k^{d-1}},
\]
\[
E \left[ |E'| \right] = \frac{N^d}{k^{d-1}} \left( \frac{k - 1}{k} \right)^{S-d} \geq \frac{N^d}{k^{d-1}} \left( 1 - \frac{S}{k} \right),
\]
\[
E \left[ |E \setminus E'| \right] \leq \frac{N^d S}{k^d}.
\]

Moreover, for all \( \eta > 0 \),
\[
\Pr \left[ |E'| - E \left[ |E'| \right] \geq \eta E \left[ |E'| \right] \right] \leq \frac{1}{\eta^2 E \left[ |E'| \right]} + \frac{1}{\eta^2} \left( \frac{k - 1}{k} \right)^{2d-1} - 1 \right).
\]

We can now determine an estimate the median of \( L(\Sigma(K_{n_1, \ldots, n_d}, k)) \).
Corollary 26  The model \((\Sigma(K_{n_1, \ldots, n_d}, k))\) of internal parameter \(k\) admits a \((c, \lambda, \theta)\)-median where

\[
(c, \lambda, \theta) = \left( \frac{c_d}{d}, 1 - \frac{1}{d} + \frac{1}{d^2} \right).
\]

Recalling that by Proposition 10 we have that \(h = 1/(4d)\) is a concentration constant for the random \(d\)-word model, by the preceding corollary and the Main Theorem, we obtain the following:

Theorem 27  Let \(\epsilon > 0\) and \(g : \mathbb{R} \to \mathbb{R}\) be such that \(g(k) = O(k^\eta)\) for a given \(0 \leq \eta < 1/d^2\). Fix \(n_1, \ldots, n_d\) and let \(N\) and \(S\) denote their geometric mean and sum, respectively. There exists sufficiently large constants \(k_0\) and \(A\) such that if \(k \leq k_0\), \(Nk^{1-1/d} \geq A\) and \(S \leq g(k)N\), then for \(M = c_dN/k^{1-1/d}\) where \(c_d\) is the \(d\)-dimensional Ulam constant,

\[
(1 - \epsilon)M \leq \mathbb{E} \left[ L(\Sigma(K_{n_1, \ldots, n_d}, k)) \right] \leq (1 + \epsilon)M,
\]

and the following hold:

- If \(\text{Med} \left[ L(\Sigma(K_{n_1, \ldots, n_d}, k)) \right]\) is a median of \(L(\Sigma(K_{n_1, \ldots, n_d}, k))\),

\[
(1 - \epsilon)M \leq \text{Med} \left[ L(\Sigma(K_{n_1, \ldots, n_d}, k)) \right] \leq (1 + \epsilon)M.
\]

- There is an absolute constant \(C > 0\) such that

\[
\Pr \left[ L(\Sigma(K_{n_1, \ldots, n_d}, k)) \leq (1 - \epsilon)M \right] \leq \exp \left( \frac{-C}{d} \epsilon^2 M \right),
\]

\[
\Pr \left[ L(\Sigma(K_{n_1, \ldots, n_d}, k)) \geq (1 + \epsilon)M \right] \leq \exp \left( \frac{-C}{d} \frac{\epsilon^2}{1 + \epsilon} M \right).
\]

We are now ready to prove Theorem 7, which is this section’s main result, and was already stated in the main contributions section.

Proof of Theorem 7: Let \(n, n', n''\) be positive integers such that \(n = n' + n''\). Clearly,

\[
\mathbb{E} \left[ L(\Sigma(K_n^{(d)}, k)) \right] \geq \mathbb{E} \left[ L(\Sigma(K_{n'}^{(d)}, k)) \right] + \mathbb{E} \left[ L(\Sigma(K_{n''}^{(d)}, k)) \right].
\]

By subadditivity, it follows that the limit of \(\mathbb{E} \left[ L(\Sigma(K_n^{(d)}, k)) \right]\) when normalized by \(n\) exists and equals \(\gamma_k = \inf_{n \in \mathbb{N}} \mathbb{E} \left[ L(\Sigma(K_n^{(d)}, k)) \right]/n\). A direct application of Theorem 27 yields that \(k^{1-1/d} \gamma_k \to c_d\) when \(k \to \infty\).  

\[\square\]
4.3 Symmetric and anti-symmetric binomial random graph models

Throughout this section we focus on the study of $L(D)$ when $D$ is either $S(K_{n,n}, p))$ or $A(K_{2n,2n}, p)$ as defined in the introduction to this work.

First, we study the behavior of $L(G)$ when $G$ is chosen according $S(K_{n,n}, p)$. Recall that in this case, the collection of events $\{(x, y), (y, x)\} \subseteq E(G)$ are independent, and each one occurs with probability $p$. Also note that $(x, y) \in E(G)$ if and only if $(y, x) \in E(G)$ — any graph for which this equivalence holds will be said to be symmetric, thus motivating the use of the word “symmetric” in naming the random graph model. As usual, we begin our study with the determination of the concentration constant for the random model under study.

**Lemma 28** The concentration constant for $(S(K_{n,n}, p))_{n \in \mathbb{N}}$ is $1/4$.

**Proof:** Direct application of Talagrand’s inequality (as stated in [LR00] Theorem 2.29).

As in the study of the binomial model (Section 4.1) and the word model (Section 4.2), given a graph $G$ chosen according to $S(K_{n,n}, p)$ we will consider a reduced graph $G'$ obtained from $G$ by removal of all edges incident to nodes of degree at least 2. An important observation is that the graph $G'$ thus obtained is also symmetric. Since $G'$ is symmetric, the number of vertices of degree 1 in each of the two color classes of $G'$ must be even, say $2m$. Thus, the arcs between nodes of degree 1 in $G'$ can be thought of as an involution of $[2m]$ without fix points. In fact, given that the distribution of $G'$ is invariant under permutation of its nodes, the distribution of $G'$ is also invariant under such permutation, and the resulting associated involution is distributed as a random involution of $[2m]$ without fix points. We shall see that under proper assumptions $L(G)$ and $L(G')$ are essentially equal — thus, $L(G)$ behaves (approximately) like the length of a longest increasing subsequence of a randomly chosen involution of $[2m]$ without fix points. This partly explains our recollection below of some results about the length of a longest increasing subsequence of randomly chosen involutions.

Let $I_{2m}$ be the distribution of a uniformly chosen involution of $[2m]$ without fix points. Let $L(I_{2m})$ denote the length of the longest increasing subsequence of an involution chosen according to $I_{2m}$. Baik and Rains [BR01] showed that the expected value of $L(I_{2m})$ is roughly $2\sqrt{2m}$, for $m$ large. Moreover, Kiwi [Theorem 5] established the following concentration result for $L(I_{2m})$ (we state the result in a weaker form):

**Theorem 29** For $m$ sufficiently large and every $0 \leq s \leq 2\sqrt{2m}$,

$$\Pr \left[ |L(I_{2m}) - E[L(I_{2m})]| \geq s + 32(2m)^{1/4} \right] \leq 4e^{-s^2/16e^{3/2}\sqrt{2m}}.$$  

**Corollary 30** For every $0 \leq t \leq 1$ and $\alpha > 0$ there exists a $m_0 = m_0(t, \alpha)$ sufficiently large such that for all $m \geq m_0$,

$$\Pr \left[ |L(I_{2m}) - 2\sqrt{2m}| \geq 2t\sqrt{2m} \right] \leq \alpha.$$  

**Proof:** Let $m_0 = m_0(t, \alpha)$ be sufficiently large such that Theorem 29 and the following conditions hold for all $m > m_0$:

- $|E[L(I_{2m})] - 2\sqrt{2m}| + 32(2m)^{1/4} \leq t\sqrt{2m}$.
- $4e^{-t^2\sqrt{2m}/16e^{3/2}} \leq \alpha$.  


Moreover, for \( n \) and \( \delta \), we need the following analogues of Lemmas 20 and 24.

Proposition 32 Let \( n \) be a positive integer. If \( E \) and \( E' \) denote \( E(G) \) and \( E(G') \), respectively, then

\[
\begin{align*}
\mathbb{E} \left[ |E| \right] & = pn(n-1), \quad (23) \\
\mathbb{E} \left[ |E'| \right] & = pn(n-1)(1-p)^{2n-4}, \quad (24) \\
\mathbb{E} \left[ |E \setminus E'| \right] & \leq 2p^2n(n-1)(n-2). \quad (25)
\end{align*}
\]

Moreover, for \( \eta > 0 \),

\[
\Pr \left[ ||E| - \mathbb{E}[|E|]|| \geq \eta \mathbb{E}[|E|] \right] \leq \frac{2}{\eta^2 \mathbb{E}[|E|]} . \quad (26)
\]

**Proof:** [Sketch] For \( i \neq j \), let \( X_{i,j} \) and \( Y_{i,j} \) denote the indicator of the event \((i,j) \in E \) and \((i,j) \in E' \), respectively. Observing that \( \mathbb{E}[X_{i,j}] = p \), \( \mathbb{E}[Y_{i,j}] = p(1-p)^{2n-4} \), \( |E| = \sum_{i,j: i \neq j} X_{i,j} \) and \( |E'| = \sum_{i,j: i \neq j} Y_{i,j} \), we observe that (23) and (24). Since \( E' \subseteq E \), it follows that \( |E \setminus E'| = |E| - |E'| \). Identity (26) follows from (23) and (24) observing that \( (1-p)^{2n-4} \geq 1 - (2n - 4)p \).

To establish (26), we observe that \( |E| \) can also be expressed as \( 2 \sum_{i<j} X_{i,j} \) and that \( \{ X_{i,j} \mid i < j \} \) is a collection of independent random variables. To conclude, note that

\[
\Delta \overset{\text{def}}{=} \sum_{(i,j),(k,l): i<j, k<l \atop (i,j) \neq (k,l)} \mathbb{E} \left[ X_{i,j}X_{k,l} \right] = \binom{n}{2} \left( \binom{n}{2} - 1 \right) p^2 \leq \frac{\mathbb{E}[|E|]^2}{4},
\]

and apply Chebyshev’s inequality for indicator random variables to conclude (26). \( \blacksquare \)

**Proposition 33** Let \( \delta > 0 \), \( 0 < p \leq 1 \) and \( n \) be a positive integer. There is a sufficiently large constant \( C_1 = C_1(\delta) \), and sufficiently small constants \( C_2 \) and \( C_3 \), such that

- If \( C_1/p \leq n^2 \leq C_2 \delta/p^{3/2} \), then every median of \( L(S(K_{n,n}, p)) \) is at most \( 2(1+\delta)n\sqrt{p} \).
- If \( C_1/p \leq n^2 \leq C_3 \delta^2/p^2 \), then every median of \( L(S(K_{n,n}, p)) \) is at least \( 2(1-\delta)n\sqrt{p} \).

**Proof:** Similar to the proof of Proposition 21. \( \blacksquare \)

We immediately have the following:
Figure 6: An illustration of a graph $G$ in the support of $\mathcal{S}(K_{12,12}, p)$ (top), and the graph $O$ in the support of $\mathcal{O}(K_{12,12}, p)$ obtained from $G$ by removal of all edges $(x, y)$ such that $x \geq y$ (bottom). Thicker edges represent a non-crossing matching $M$ of $G$ (top), and the associated non-crossing matching $N$ of $O$ with edge set $\{(\min \{x, y\}, \max \{x, y\}) | (x, y) \in E(M)\}$ (bottom).

**Corollary 33** The model $(\mathcal{S}(K_{n,n}, p))_{n \in \mathbb{N}}$ of internal parameter $t = 1/p$ admits a $(2, 1/2, 3/4)$-median.

We now define an auxiliary distribution which will be useful for our study:

- $\mathcal{O}(K_{n,n}, p)$ (the oriented symmetric binomial random graph model) — the distribution over the set of subgraphs $H$ of $K_{n,n}$ where the events $\{ H | (i, j) \in E(H) \}$ for $1 \leq i < j \leq n$, have probability $p$ and are mutually independent, and the events $\{ H | (i, j) \in E(H) \}$, $1 \leq j \leq i \leq n$, have probability 0.

(See Figure 4.3 for an illustration of the distinction between distributions $\mathcal{S}(K_{n,n}, p)$ and $\mathcal{O}(K_{n,n}, p)$.)

The following result justifies why we can henceforth work either with $L(\mathcal{S}(K_{n,n}, p))$ or $L(\mathcal{O}(K_{n,n}, p))$.

**Lemma 34** The random variables $L(\mathcal{S}(K_{n,n}, p))$ and $L(\mathcal{O}(K_{n,n}, p))$ are identically distributed.

**Proof:** Let $O$ be a graph in the support of $\mathcal{O}(K_{n,n}, p)$. We can associate to $O$ a graph $G$ over the same collection of vertices and having edge set $\{(x, y) | (x, y) \in E(O) \wedge (y, x) \not\in E(O)\}$. Clearly, $G$ is a symmetric subgraph of $K_{n,n}$ and hence it belongs to the support of $\mathcal{S}(K_{n,n}, p)$. It is easy to see that the mapping from $O$ to $G$ is one-to-one. Moreover, the probability of $G$ being chosen under $\mathcal{S}(K_{n,n}, p)$ is exactly equal to the probability of occurrence of $O$ under $\mathcal{O}(K_{n,n}, p)$.

On the other hand, if $M$ is a non-crossing subgraph of $G$, then there is a non-crossing subgraph of $O$ (and hence of $G$), say $N$, whose size is the same as the one of $M$. Indeed, it suffices to take as the collection of edges of $N$ the set $\{(\min \{x, y\}, \max \{x, y\}) | (x, y) \in E(M)\}$. (See Figure 4.3 for an illustration of the relation between $M$ and $N$.) We get that $L(G) = L(O)$, which concludes the proof.

We are now ready to prove the main result of this section.

**Theorem 35** For every $\epsilon > 0$ there is a sufficiently small constant $p_0$ and a sufficiently large constant $A$ such that for all $p \leq p_0$ and $n \geq A/\sqrt{p}$,

$$ (1 - \epsilon)2n\sqrt{p} \leq \mathbb{E}[L(\mathcal{S}(K_{n,n}, p))] \leq (1 + \epsilon)2n\sqrt{p}, \quad (27) $$

and the following hold
• If \( \text{Med} [L(S(K_n,n,p)) \] is a median of \( L(S(K_n,n,p)) \),
\[
(1 - \epsilon)2n\sqrt{p} \leq \text{Med} [L(S(K_n,n,p))] \leq (1 + \epsilon)2n\sqrt{p}.
\] (28)

• There is an absolute constant \( C > 0 \), such that
\[
\Pr [L(S(K_n,n,p)) \leq (1 - \epsilon)2n\sqrt{p}] \leq \exp \left( -C\epsilon^2n\sqrt{p} \right),
\] (29)

\[
\Pr [L(S(K_n,n,p)) \geq (1 + \epsilon)2n\sqrt{p}] \leq \exp \left( -C\frac{\epsilon^2}{1 + \epsilon}n\sqrt{p} \right).
\] (30)

Proof: Unfortunately, \( (S(K_n,n,p))_{n \in \mathbb{N}} \) is not a random hyper-graph model, so we can not immediately apply the Main Theorem. However, it is a weak random hyper-graph model. Hence, to prove the lower bound in \( (27) \) and \( (28) \), and inequality \( (29) \), we use the fact that the model \( S(K_n,n,p) \) with internal parameter \( t = 1/p \) has a concentration constant \( n = 1/4 \) \( (\text{Lemma} \ 28) \) admits a \( (2,1/2,3/4) \)-median \( (\text{Corollary 33}) \), and apply the Main Theorem.

To prove the remaining bounds, consider a bipartite graph \( H \) chosen according to \( G(K_n,n,p) \), and let \( O \) be the graph obtained from \( H \) by deletion of all its edges \( (x,y) \) such that \( x \geq y \). Since \( O \) is a subgraph of \( H \), it immediately follows that \( L(O) \leq L(H) \). Note that \( O \) follows the distribution \( O(K_n,n,p) \). By Lemma 34, \( L(O) \) has the same distribution as \( L(S(K_n,n,p)) \). Hence, if \( n \) and \( p \) satisfy the hypothesis of Theorem 23
\[
\text{E} [L(S(K_n,n,p))] = \text{E} [L(O)] \leq \text{E} [L(H)] \leq (1 + \epsilon)2n\sqrt{p},
\]
\[
\text{Med} [L(S(K_n,n,p))] = \text{Med} [L(O)] \leq \text{Med} [L(H)] \leq (1 + \epsilon)2n\sqrt{p},
\]
and provided \( C \) is as in Theorem 23
\[
\Pr [L(S(K_n,n,p)) \geq (1 + \epsilon)2n\sqrt{p}] = \Pr [L(O) \geq (1 + \epsilon)2n\sqrt{p}]
\]
\[
\leq \Pr [L(H) \geq (1 + \epsilon)2n\sqrt{p}] \leq \exp \left( -C\frac{\epsilon^2}{1 + \epsilon}2n\sqrt{p} \right).
\]

This concludes the proof of the stated result.

We can now establish Theorem 8

Proof of Theorem 8: Let \( n, n', n'' \) be positive integers such that \( n = n' + n'' \). Clearly,
\[
\text{E} [L(S(K_n,n,p))] \geq \text{E} [L(S(K_{n',n''},p))] + \text{E} [L(S(K_{n'',n''},k))].
\]

By subadditivity, it follows that the limit of \( \text{E} [L(S(K_n,n,p))] \) when normalized by \( n \) exists and equals \( \sigma_p = \inf_{n \in \mathbb{N}} \text{E} [L(S(K_n,n,p))/n] \). A direct application of Theorem 8 yields that \( \sigma_p/\sqrt{p} \to 2 \) when \( p \to 0 \).

One can also show, although not as straightforward as for the case of the symmetric binomial random graph model, that the following analogue of Theorem 35 holds for the anti-symmetric case.

Theorem 36 For every \( \epsilon > 0 \) there is a sufficiently small constant \( p_0 \) and a sufficiently large constant \( A \) such that for all \( p \leq p_0 \) and \( n \geq A/\sqrt{p} \),
\[
(1 - \epsilon)4n\sqrt{p} \leq \text{E} [L(A(K_{2n,2n},p))] \leq (1 + \epsilon)4n\sqrt{p},
\] (31)
and the following hold
• If \(Med[L(A(K_{2n,2n}, p))]\) is a median of \(L(A(K_{2n,2n}, p))\),
\[
(1 - \epsilon)4n\sqrt{p} \leq Med[L(A(K_{2n,2n}, p))] \leq (1 + \epsilon)4n\sqrt{p}.
\]

• There is an absolute constant \(C > 0\), such that
\[
\Pr[L(A(K_{2n,2n}, p)) \leq (1 - \epsilon)4n\sqrt{p}] \leq \exp\left(-C\epsilon^2n\sqrt{p}\right),
\]
\[
\Pr[L(A(K_{2n,2n}, p)) \geq (1 + \epsilon)4n\sqrt{p}] \leq \exp\left(-C\frac{\epsilon^2}{1 + \epsilon}n\sqrt{p}\right).
\]

Proof: Omitted from current draft.

Theorem 9 can now be established much in the same way as Theorem 8 was derived.

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