A Simple Proof of a Theorem by Uhlenbeck and Yau

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Abstract. A subbundle of a Hermitian vector bundle \((E, h)\) can be metrically and differentiably defined by the orthogonal projection onto this subbundle. A weakly holomorphic subbundle of a Hermitian holomorphic bundle is, by definition, an orthogonal projection \(\pi\) lying in the Sobolev space \(L^2_1\) of \(L^2\) sections with \(L^2\) first order derivatives in the sense of distributions, which satisfies furthermore \((\Id - \pi) \circ D''\pi = 0\). We give a new simple proof of the fact that a weakly holomorphic subbundle of \((E, h)\) defines a coherent subsheaf of \(\mathcal{O}(E)\), that is a holomorphic subbundle of \(E\) in the complement of an analytic set of codimension \(\geq 2\). This result was the crucial technical argument in Uhlenbeck’s and Yau’s proof of the Kobayashi-Hitchin correspondence on compact Kähler manifolds. We give here a much simpler proof based on current theory. The idea is to construct local meromorphic sections of \(\text{Im} \pi\) which locally span the fibers. We first make this construction on every one-dimensional submanifold of \(X\) and subsequently extend it via a Hartogs-type theorem of Shiffman’s.

0.1 Introduction

Let \((E, h)\) be a holomorphic vector bundle of rank \(r\) equipped with a \(C^\infty\) Hermitian metric over a compact Kähler manifold \(X\), and let \(\mathcal{F} \subset \mathcal{O}(E)\) be a coherent analytic subsheaf of the locally free sheaf \(\mathcal{O}(E)\) associated to \(E\). Since \(\mathcal{F}\) is torsion-free (as a coherent subsheaf of a torsion-free sheaf), it is locally free outside an analytic subset of codimension \(\geq 2\) (see, for instance, [Kob87], V.5). Thus \(\mathcal{F}\) can be seen as a vector bundle with singularities. More precisely, there exists an analytic subset \(S \subset X\), \(\text{codim } S \geq 2\), and a holomorphic vector bundle \(F\) on \(X \setminus S\), such that

\[ \mathcal{F}|_{X \setminus S} = \mathcal{O}(F). \]

Equip the subbundle \(F \hookrightarrow E|_{X \setminus S}\) with the Hermitian metric induced by \(h\) and consider the orthogonal projection \(\pi : E|_{X \setminus S} \to F\). Then \(\pi\) can be seen as a \(C^\infty\) section on \(X \setminus S\) of the holomorphic vector bundle \(\text{End } E\) satisfying:

\[ (0.1) \quad \pi = \pi^* = \pi^2, \quad (\Id - \pi) \circ D''\pi = 0 \]
on $X \setminus S$, where $D''$ is the $(0, 1)$-component of the Chern connection on $\text{End } E$ associated to the metric induced by $h$. The latter equality above says that the holomorphic structure of $F$ is the restriction of the holomorphic structure of $E_{|X \setminus S}$. Let $Q$ be the quotient bundle of $E_{|X \setminus S}$ by $F$, equipped with the metric induced by $h$, and let $\det Q$ be the associated determinant line bundle equipped with the induced metric. Its curvature form $i\Theta(\det Q) = \text{Tr}_Q(i\Theta(Q)) = \text{Tr}_E(i\Theta(Q))$ is a $C^\infty$ $(1, 1)$-form on $X \setminus S$ given by the formula:

$$i\Theta(\det Q) = \text{Tr}_E(i\Theta(h)_{|Q}) + \text{Tr}_E(iD'\pi \wedge D''\pi) \quad \text{(see [Gri69]).}$$

As $\text{codim } S \geq 2$, the $C^\infty$ $(1, 1)$-form $\text{Tr}_E(iD'\pi \wedge D''\pi)$ has locally finite mass in the neighbourhood of $S$. In other words, every $x \in S$ has a neighbourhood $U \subset X$ such that

$$\int_U \text{Tr}_E(iD'\pi \wedge D''\pi) \wedge \omega^{n-1} < +\infty,$$

where $\omega$ is an arbitrary Hermitian metric on $X$. This is a consequence of a general current theory result stating that if $T$ is a closed positive current of bidegree $(p, p)$ (or equivalently of bidimension $(n-p, n-p)$) in the complement of an analytic subset $A$ of codimension $\geq p+1$, then the mass of $T$ is locally finite in the neighbourhood of $A$ (see [Sib85], p. 178, corollaire 3.2).

In particular, the $(1, 1)$-form $\text{Tr}_E(iD'\pi \wedge D''\pi)$ extended by $0$ across $S$ is $L^1$ on $X$. Since $|\text{Tr}_E(iD'\pi \wedge D''\pi)|$ dominates $|D'\pi|^2$ and $|D''\pi|^2$, the norms being considered in their respective bundles, we see that $D'\pi$ and $D''\pi$ are $L^2$ 1-forms on $X \setminus S$. Since every projection is $L^\infty$ and, thanks to the compacity of $X$, implicitly $L^2$, the projection $\pi$ belongs to the Sobolev space $L^2_1$ of $L^2$ sections of $\text{End } E$ whose first order derivatives in the sense of distributions are still $L^2$.

This discussion can be summed up as follows.

**Remark.** Every coherent analytic subsheaf $\mathcal{F}$ of $\mathcal{O}(E)$ defines a section $\pi \in L^2_1(X, \text{End } E)$ that is $C^\infty$ in the complement of an analytic set of codimension $\geq 2$ and satisfies relations (0.1).

The goal of the present paper is to prove, by relatively elementary methods, the reverse statement that was originally stated and proved in [UY 86, 89]. More precisely, we prove the following.

**Theorem 0.1.1** Let $(E, h)$ be a holomorphic vector bundle of rank $r$ equipped with a $C^\infty$ Hermitian metric over a compact complex Kähler manifold $X$, and let $\pi \in L^2_2(X, \text{End } E)$ such that $\pi = \pi^* = \pi^2$ and $(\text{Id}_E - \pi) \circ D''_{\text{End } E} \pi = 0$ almost everywhere.

Then there exist a coherent analytic subsheaf $\mathcal{F} \subset \mathcal{O}(E)$ and an analytic subset $S \subset X$ of codimension $\geq 2$ such that:
1) \( \pi|_{X\setminus S} \in C^\infty(X \setminus S, \text{End} E) \)

2) \( \pi = \pi^* = \pi^2 \) and \( \text{Id}_E - \pi \circ D''_{\text{End} E} \pi = 0 \) on \( X \setminus S \)

3) \( \mathcal{F}|_{X\setminus S} = \pi|_{X\setminus S}(E|_{X\setminus S}) \hookrightarrow E|_{X\setminus S} \) is a holomorphic subbundle of \( E|_{X\setminus S} \).

In what follows, \( L^2_1(X, \text{End} E) \) stands for the Sobolev space of \( L^2 \) sections of the holomorphic bundle \( \text{End} E \) whose first order derivatives in the sense of distributions are still \( L^2 \). We equip \( \text{End} E \) with the metric induced by \( h \) and denote \( D'_{\text{End} E}, D''_{\text{End} E} \) the \((1, 0)\) and respectively \((0, 1)\) components of the associated Chern connection.

A section \( \pi \in L^2_1(X, \text{End} E) \) satisfying the hypotheses of theorem 0.1.1 is called weakly holomorphic subbundle of \( E \).

Theorem 0.1.1 provides the crucial technical argument in the proof given by Uhlenbeck and Yau ([UY 86, 89]) to the existence of a unique Hermitian-Einstein metric in every stable holomorphic vector bundle over a compact Kähler manifold. The comparatively easier converse, asserting that every Hermitian-Einstein vector bundle is semistable and splits into a direct sum of stable subbundles, had previously been proved by Kobayshi and Lübke ([Kob87, LT95]). Uhlenbeck and Yau were thus completing the proof of the Kobayashi-Hitchin correspondence over compact Kähler manifolds. The idea of their proof is the following. Having fixed the metric \( h \) of \( E \), every \( C^\infty \) metric \( h_1 \) on \( E \) is of the form:

\[
h_1(s, t) = h(f(s), t),
\]

for all sections \( s \) and \( t \) of \( E \), where \( f \in C^\infty(X, \text{End} E) \) is a positive definite and self-adjoint endomorphism (for \( h \)) of \( E \). Then \( h_1 \) is a Hermitian-Einstein metric if and only if \( f \) is a solution of a nonlinear partial differential equation.

The authors solve a perturbed equation depending on a parameter \( \varepsilon \) and find a solution \( f_\varepsilon \). One of the following two situations occurs. Either \( f_\varepsilon \) converges to an endomorphism \( f_0 \) when \( \varepsilon \) tends to 0, in which case they prove that \( f_0 \) actually defines a Hermitian-Einstein metric; or \( f_\varepsilon \) does not converge, in which case they prove that the stability hypothesis on \( E \) is violated by producing a destabilizing subsheaf of \( \mathcal{O}(E) \). It is in the construction of such a destabilizing subsheaf that theorem 0.1.1 plays a key role.

Their proof is, however, extremely technical and not very enlightening. This is why we wish to give a natural proof of theorem 0.1.1 by constructing local meromorphic sections which span \( \text{Im} \pi \). The coherent sheaf \( \mathcal{F} \) that we intend to construct will then be defined by its local sections.

### 0.2 Preliminaries

Let \( (X, \omega) \) be a compact Kähler manifold of dimension \( n \) and let \( (E, h) \) be a Hermitian holomorphic vector bundle of rank \( r \) over \( X \). Let \( \pi \in L^2_1(X, \text{End} E) \)
be a section such that $\pi = \pi^* = \pi^2$ and $(\text{Id} - \pi) \circ D''\pi = 0$ almost everywhere, the derivative $D''\pi$ being computed in the sense of distributions. Since $\pi$ is a projection, we even have

$$\pi \in L^2_1(X, \text{End} E) \cap L^\infty(X, \text{End} E).$$

The subbundle $F = \text{Im} \, \pi \subset E$ is defined almost everywhere as an $L^2$ bundle. This means that the fiber $F_x$ is defined as $\text{Im} \, \pi_x$ for almost all points $x \in X$ and the transition matrices have an $L^2$ dependence on $x$. Likewise, the quotient bundle $Q = E/F$ is defined almost everywhere as an $L^2$ bundle. Let $\beta$ and $\beta^*$ be the $(1,0)$ $L^2$ current with values in $\text{Hom} \,(F, Q)$, and respectively the $(0,1)$ $L^2$ current with values in $\text{Hom} \,(Q, F)$, uniquely determined by the following equalities:

$$D'\pi = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, \quad D''\pi = \begin{pmatrix} 0 & \beta^* \\ 0 & 0 \end{pmatrix},$$

where $D'\pi$ and $D''\pi$ are calculated in the sense of distributions. The current $\beta$ corresponds to the second fundamental form of the exact sequence $0 \to F \to E \to Q \to 0$ in the case where $\pi$ is $C^\infty$. We refer to [Gri69] for details concerning exact sequences of Hermitian vector bundles or, for a presentation using the same notation as in the present article, to chapter 5 of the book [Dem97].

**Remark 0.2.1** For all $\pi \in L^2_1(X, \text{End} E)$ satisfying $\pi = \pi^* = \pi^2$ almost everywhere, the following equalities in the sense of currents are equivalent:

\begin{align*}
(2.a) & \quad (\text{Id} - \pi) \circ D''\pi = 0;  \\
(2.b) & \quad D'\pi \circ (\text{Id} - \pi) = 0 \\
(2.c) & \quad \pi \circ D'\pi = 0;  \\
(2.d) & \quad D''\pi \circ \pi = 0.
\end{align*}

**Proof.** The equivalence of $(a)$ and $(b)$ is obtained by taking adjoints, for $\pi = \pi^*$. On the other hand, if we apply $D'$ to the equality $\pi = \pi^2$ we find $D'\pi = D'\pi \circ \pi + \pi \circ D'\pi$, which gives the equivalence of $(b)$ and $(c)$. Equality $(d)$ is inferred from $(c)$ by taking adjoints. All products are well-defined in the sense of currents since an $L^2$ form can be multiplied by an $L^\infty$ form. \qed

Proving theorem 0.1.1 amounts to proving that the $L^2$ bundle $F = \text{Im} \, \pi$ is holomorphic outside an analytic subset of codimension $\geq 2$. It is thus enough to construct local meromorphic sections of $F$ which span $F$ locally (for meromorphic sections are holomorphic in the complement of an analytic subset of codimension $\geq 2$). The idea is to construct local holomorphic sections of $F \otimes \text{det} \, Q$ which span $F \otimes \text{det} \, Q$ locally, in parallel with the construction of a local $\bar{\partial}$-closed section of $\text{det} \, Q$ which spans $\text{det} \, Q$ locally. A division of the local holomorphic sections of $F \otimes \text{det} \, Q$ by the local section of $\text{det} \, Q$ yields the meromorphic sections of $F$ that we wish to construct.

In the construction of a local holomorphic section of $\text{det} \, Q$ the $(1,1)$-current $\text{Tr}_E(i\beta \wedge \beta^* + i\Theta(E)|_Q)$, interpreted a posteriori as the curvature current of $\text{det} \, Q$, plays a key role.
Remark 0.2.2 The restriction of the current $\text{Tr}_E(i\beta \wedge \beta^* + (\text{Id} - \pi) \circ i\Theta(E)_{h^1} \circ (\text{Id} - \pi))$ to almost every complex line contained in a coordinate patch of $X$ defines a $d$-closed current.

Proof. The argument is almost trivial. The existence of the restriction of an $L^1$ function to almost every line is a consequence of the Fubini theorem (the restriction being $L^1$ on this line). To see this, we start by considering a system of lines parallel to a given direction. Now, every current of maximal bidegree is closed. In particular, bidegree $(1, 1)$ currents are closed on complex submanifolds of dimension 1. \qed

In view of theorem 0.1.1, since the problem is local, we can work on an open set $U \subset X$ such that $E|_U \simeq U \times \mathbb{C}^r$. After possibly shrinking $U$, the curvature of $E$ can be made positive on $U$ by a change of metric. Indeed, let $h_1(z) = h(z) \cdot e^{-m|z|^2}$ be a new metric on $E|_U$, $m$ being a positive scalar and $z = (z_1, \ldots, z_n)$ local coordinates on $U$. Since

$$i\Theta_{h_1}(E) = i\Theta_h(E) + m i d''|z|^2 \otimes \text{Id}_E,$$

and since the $(1,1)$-form $i d''|z|^2$ is $> 0$, we see that $i\Theta_{h_1}(E) \geq \varepsilon \omega \otimes \text{Id}_E$ for a certain $\varepsilon > 0$, provided that $m$ is sufficiently large. On the other hand, the product $\beta \wedge \beta^*$ defines an $L^1$ $(1, 1)$-current with values in $\text{End} E$ and $\text{Tr}_E(i\beta \wedge \beta^*) \geq 0$ in the sense of currents (see [Gri69] for the $C^\infty$ case and the proof given there still works for currents). Thus the $(1, 1)$-current

$$\text{Tr}_E(i\beta \wedge \beta^* + (\text{Id} - \pi) \circ i\Theta_{h_1}(E) \circ (\text{Id} - \pi))$$

is positive on $U$. Moreover, this scalar change of metric preserves the property of $\pi$ being self-adjoint. We can then make the following convention without loss of generality.

Convention. We assume from now on that, locally, the curvature of $E$ is positive.

Hence we get the following

Corollary 0.2.3 The current $\text{Tr}_E(i\beta \wedge \beta^* + (\text{Id} - \pi) \circ i\Theta_{h_1}(E) \circ (\text{Id} - \pi))$ of bidegree $(1, 1)$ admits a local subharmonic potential on almost every complex line contained in a coordinate patch of $X$.

This means that for every point $x \in X$ and for almost every complex line $L$ with respect to a system of local coordinates in the neighbourhood of $x$, there exists a subharmonic function $\varphi_L$ such that

$$i\partial\bar{\partial}\varphi_L = \text{Tr}_E(i\beta \wedge \beta^* + (\text{Id} - \pi) \circ i\Theta_{h_1}(E) \circ (\text{Id} - \pi)),$$
Proof. By the Poincaré lemma, every $d$-closed current is locally $d$-exact. It is therefore also $\partial\bar{\partial}$-exact by the $\partial\bar{\partial}$ lemma. After possibly shrinking the trivializing open set $U$, we may assume that there exists a function $\varphi_L$ as in the statement of the corollary. Since the above current is positive, the potential $\varphi_L$ is subharmonic. \hfill \Box

The main difficulty in the proof of theorem 0.1.1 comes from the insufficient regularity of $\pi$. Certain wedge-products of currents are not well-defined, and distributions cannot be multiplied. The following lemma states an elementary distribution theory result that will enable subsequent computations to make sense. For every real number $s$, $L_s^2$ stands for the space of temperate distributions $u \in S'(\mathbb{R}^n)$ such that the Fourier transform $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{\frac{s}{2}} \cdot \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. It can be easily seen that if $u$ is a compactly supported element of $L_{s,\text{loc}}^2$, there exists a decomposition $u = \sum_j D^j v_j + v$, where $v_j \in L_{s+1,\text{loc}}^2$ and $v \in L_{s+2,\text{loc}}^2$ are compactly supported. In particular, for $s = -1$, every $L_{-1}^2$ function can be locally written as a sum of partial derivatives of order 1, in the sense of distributions, of $L_1^2$ functions. In an analogous way we define the space $(L_1^1)'$ of temperate distributions arising locally as a sum of partial derivatives of order 1, in the sense of distributions, of $L_1^1$ functions. We thus have the inclusions:

$$ L^1 \hookrightarrow (L^1)' \hookrightarrow \mathcal{D}'_1, $$

where $\mathcal{D}'_1$ stands for the space of order 1 distributions. The topology of $(L^1)'$ is defined to be the restriction of the topology of $\mathcal{D}'_1$.

**Lemma 0.2.4** The map

$$(f, g) \mapsto u_{fg} \quad \text{acting from} \quad L_{1,\text{loc}}^2 \times L_{-1,\text{loc}}^2 \quad \text{to} \quad (L^1)'$$

is well-defined, bilinear and continuous, where $u_{fg}$ stands for the distribution defined as:

$$ < u_{fg}, \varphi > = -\sum_j \int g_j D^j(\theta_j f \varphi) + \int f \psi h \varphi, $$

for every test function $\varphi$ and every local decomposition $g = \sum_j \theta_j \cdot D^j g_j + \psi h$, with $g_j \in L^2$, $h \in L^2_1$ and $\theta_j, \psi$ test functions.

The standard proof of this lemma can well be left to the reader.

**0.3 Proof of Theorem 0.1.1**

The overall idea of proof is to achieve enough regularity on expressions containing $\pi$ and derivatives of $\pi$ enabling us to retrieve the classical $C^\infty$ situation outside an analytic set of codimension $\geq 2$. Frequent side-glances
at the $C^\infty$ situation will show us the way. We shall proceed in several steps.

- **First step**: reduction to the case of zero curvature

  In order to massively simplify subsequent computations, we start off by showing that we can locally reduce the problem to the case where the curvature of $E$ vanishes identically. The following elementary lemma will be of use.

**Lemma 0.3.1** Let $E$ be a complex vector space of dimension $r$ and $F$ a vector subspace of dimension $p$. Consider two Hermitian metrics $h$ and $h_0$ on $E$ and let $\pi$, $\pi_0$ be the orthogonal projections, for $h$ and respectively $h_0$, of $E$ on $F$.

  If $E = F \oplus F^\perp_h$ (respectively $E = F \oplus F^\perp_{h_0}$) is the orthogonal decomposition of $E$ for $h$ (respectively $h_0$), then there exists an automorphism $v : E \to E$ such that $v(F) = F$, $v(F^\perp_h) = F^\perp_{h_0}$, and $h(s, t) = h_0(vs, vt)$, for all $s, t \in E$.

  Moreover, for every such $v$ the projections $\pi$ and $\pi_0$ are related by the formula $\pi_0 = v\pi v^{-1}$.

  The elementary proof of this lemma is left to the reader.

**Corollary 0.3.2** Let $(E, h)$ be a holomorphic vector bundle of rank $r$ equipped with a $C^\infty$ Hermitian metric over a complex manifold $X$, and let $\pi \in L^2_1(X, \text{End } E)$ be such that $\pi = \pi^* = \pi^2$ and $(\text{Id}_E - \pi) \circ D''_{\text{End } E} \pi = 0$ almost everywhere. Set $F = \text{Im } \pi$. Let $U$ be a trivializing open set for the bundle $E$ and let $h_0$ be the trivial flat metric on $E|_U \cong U \times \mathbb{C}^r$. Let $\pi_0 \in L^2_1(U, \text{End } E)$ be the orthogonal projection of $E|_U$ onto $F|_U$ with respect to the metric $h_0$.

  Then there exists $v \in C^\infty(U, \text{End } E)$ such that $(\text{Id} - \pi_0) \circ v \circ \pi = 0$ (or equivalently $(\text{Id} - \pi_0) \circ v \circ \pi_0 = 0$), $\pi_0 \circ v \circ (\text{Id} - \pi) = 0$ almost everywhere on $U$, and $h(s, t) = h_0(vs, vt)$ for all $s, t \in E|_U$. Furthermore, $\pi_0 = v\pi v^{-1}$ almost everywhere on $U$.

**Lemma 0.3.3** Under the hypotheses of corollary 0.3.2 the projection $\pi_0$ satisfies moreover : $(\text{Id} - \pi_0) \circ D'' \pi_0 = 0$ almost everywhere on $U$.

  This result corresponds a posteriori to the holomorphic structure of $F$, viewed as a holomorphic subbundle of $E$ in the complement of an analytic set, being independent of the choice of metric.

**Proof.** As $\pi_0 = v\pi v^{-1}$, we infer:

\[
(\text{Id} - \pi_0) \circ D'' \pi_0 = (\text{Id} - \pi_0) \circ D'' v \circ \pi \circ v^{-1} + (\text{Id} - v\pi v^{-1}) \circ v \circ D'' \pi \circ v^{-1} + (\text{Id} - v\pi v^{-1}) \circ v \circ \pi \circ D''(v^{-1}).
\]

The above expressions are well-defined in the sense of distributions since $v$ is $C^\infty$. The second term in the above sum is equal to :
\( v \circ D''\pi \circ v^{-1} - v \circ \pi \circ D''\pi \circ v^{-1} = 0, \)

for \( D''\pi = \pi \circ D''\pi \) and the subtraction of two \( L^2 \) expressions is well-defined. The third term in the above sum is equal to:

\[
v \circ \pi \circ D''(v^{-1}) - v \circ \pi \circ v^{-1} \circ v \circ \pi \circ D''(v^{-1}) = 0,
\]

for \( \pi \circ v^{-1} \circ v \circ \pi = \pi^2 = \pi \) and two \( L^2 \) expressions can be subtracted. The sum is thus reduced to its first term. Hence:

(a) \( (\text{Id} - \pi_0) \circ D''\pi_0 = (\text{Id} - \pi_0) \circ D''v \circ \pi \circ v^{-1}. \)

Apply the operator \( D'' \) to the equality \( (\text{Id} - \pi) \circ v \circ \pi = 0 \) and get:

(b) \( -D''\pi \circ v \circ \pi + (\text{Id} - \pi) \circ D''v \circ \pi + (\text{Id} - \pi) \circ v \circ D''\pi = 0. \)

Since \( D''\pi = \pi \circ D''\pi \), we see that:

\[
(\text{Id} - \pi) \circ v \circ D''\pi = \left( (\text{Id} - \pi) \circ v \circ \pi \right) \circ D''\pi = 0,
\]

for the expression in brackets vanishes. Equality (b) becomes:

(c) \( D''\pi \circ v \circ \pi = (\text{Id} - \pi) \circ D''v \circ \pi. \)

On the other hand, for all \( \xi \in E \) there exists \( \eta \in E \) such that \( v(\pi \xi) = \pi \eta \), for \( v \) preserves \( \text{Im} \pi \). Since \( D''\pi \circ \pi = 0 \) thanks to relation (2.d), we get:

\[
(D''\pi \circ v \circ \pi)(\xi) = D''\pi(v(\pi \xi)) = (D''\pi \circ \pi)(\eta) = 0,
\]

for all \( \xi \in E \). Consequently, \( D''\pi \circ v \circ \pi = 0 \) and relation (c) implies:

(\text{Id} - \pi) \circ D''v \circ \pi = 0. \) This is equivalent to: \( D''v \circ \pi = \pi \circ D''v \circ \pi \). If we apply the operator \( \text{Id} - \pi_0 \) to this last equality we get:

\[
(\text{Id} - \pi_0) \circ D''v \circ \pi = (\text{Id} - \pi_0) \circ \pi \circ D''v \circ \pi = 0,
\]

for \( (\text{Id} - \pi_0) \circ \pi = 0 \) (\( \text{Im} \pi = \text{Im} \pi_0 \) and \( (\text{Id} - \pi_0) \circ \pi_0 = 0 \)). Equality (a) finally gives: \( (\text{Id} - \pi_0) \circ D''\pi_0 = 0 \), and this is what we wanted to prove. \( \square \)

Lemma 0.3.3 enables us to reduce locally to the case of a flat vector bundle. Indeed, since the problem is local we may assume from now on, possibly after replacing locally the original metric \( h \) of \( E \) with the trivial flat metric \( h_0 \), that \( i\Theta(E)_h = 0 \) on the trivializing open set \( U \).

- **Second step**: reinterpretation of \( \text{Im} \pi \)
In order to prove theorem 0.1.1 we will show that the $L^2$ bundle $F = \text{Im} \ \pi$ is locally generated by its local meromorphic sections. As before, we will draw on the $C^\infty$ situation considered in the very simple lemma below.

**Lemma 0.3.4** Let $(E, h)$ be a Hermitian holomorphic vector bundle of rank $r$ and let $\pi \in C^\infty(X, \text{End} E)$ be such that $\pi = \pi^* = \pi^2$ and $(\text{Id} - \pi) \circ D'' \pi = 0$. We assume that the curvature form of $(E, h)$ vanishes identically. Let $p$ be the rank of $\pi$ and $q = r - p$. Consider the holomorphic subbundle $F = \text{Im} \ \pi$ of $E$ and the exact sequence of holomorphic vector bundles:

$$0 \longrightarrow F \overset{j}{\longrightarrow} E \overset{g}{\longrightarrow} Q \longrightarrow 0,$$

where $j$ is the inclusion and $g = \text{Id} - \pi$ is the projection onto the quotient bundle $Q$. Then there exists a holomorphic bundle morphism:

$$\Lambda^{q+1} E \otimes \Lambda^q Q^* \overset{\sigma}{\longrightarrow} E,$$

whose image is $F$. More precisely, if $e_1, \ldots, e_r$ is a local orthonormal holomorphic frame of $E$ and $K = (k_1 < \cdots < k_q)$ is a multiindex, consider the local holomorphic section of $\text{det} Q = \Lambda^q Q$ defined as:

$$v_K = (\text{Id} - \pi) e_{k_1} \wedge \cdots \wedge (\text{Id} - \pi) e_{k_q} = \sum_{|J|=q} D_{JK} \cdot e_J,$$

where $D_{JK}$ is the minor corresponding to the lines $J = (j_1 < \cdots < j_q)$ and the columns $K = (k_1 < \cdots < k_q)$ in the matrix representing $\text{Id} - \pi$ in the frame under consideration, and $e_J := e_{j_1} \wedge \cdots \wedge e_{j_q}$ for all $J = (j_1 < \cdots < j_q)$. Associate to $v_K$ the local holomorphic section of $\Lambda^q Q^*$ defined as:

$$v_{K}^{-1} = \frac{\sum_{|J|=q} \bar{D}_{JK} \cdot e_J^*}{\sum_{|J|=q} |D_{JK}|^2}.$$

Then for all multiindices $I = (i_1 < \cdots < i_{q+1})$ and $K = (k_1 < \cdots < k_q)$, the morphism $\sigma$ is locally defined by the relation:

$$(3.1) \quad \sigma(e_I \otimes v_{K}^{-1}) = \sum_{l=1}^{q+1} (-1)^l \cdot \frac{\sum_{|J|=q} \bar{D}_{JK} \cdot e_J^*(e_{I \setminus \{i_l\}})}{\sum_{|J|=q} |D_{JK}|^2} \cdot e_{i_l}.$$ 

In particular, there exists a holomorphic bundle morphism:

$$\Lambda^{q+1} E \overset{u}{\longrightarrow} E \otimes \text{det} Q$$

whose image is $F \otimes \text{det} Q$, which is induced by $\sigma$ after tensorizing to the right by $\text{det} Q = \Lambda^q Q$. Morphisms $\sigma$ and $u$ are locally related by:
\[ \sigma(e_I \otimes v^{-1}_K) = \frac{u(e_I)}{v_K}, \]

the division being performed in the line bundle \( \det Q \).

This lemma shows that the vector bundle \( F \otimes \det Q \) can be realized as the image of a holomorphic projection from \( \Lambda^{q+1}E \). It will prove useful later on since the projection of \( E \) on \( F \) is not holomorphic in general.

**Proof.** The quotient bundle \( Q \) can be seen as a \( C^\infty \) subbundle of \( E \) via the \( C^\infty \) inclusion \( Q \to \pi^{-1} E \). This defines the \( C^\infty \) inclusion \( \det Q = \Lambda^q Q \to \Lambda^q E \) and the orthogonal decomposition \( \Lambda^q E = \Lambda^q Q \oplus (\Lambda^q Q)^{\perp} \). The element \( v^{-1}_K \) of \( \Lambda^q E^* \) satisfies the identities:

\[ v^{-1}_K(v_K) = 1 \quad \text{and} \quad v^{-1}_K(\xi) = 0, \quad \text{for all} \ \xi \in (\Lambda^q Q)^{\perp}. \]

This accounts for the notation \( v^{-1}_K \) and shows that \( v^{-1}_K \in \Lambda^q Q^* = (\det Q)^{-1} \). We have to prove that \( \text{Im} \ \sigma = F \). The inclusion \( F \subseteq \text{Im} \ \sigma \) is obvious. Indeed, for all \( s \in F \), \( \sigma(s \wedge v^{-1}_K \otimes v^{-1}_K) = v^{-1}_K(v_K) \cdot s = s \). Let us now prove the inclusion \( \text{Im} \ \sigma \subseteq F \). We have : \( v^{-1}_K \in \Lambda^q Q^* \to \Lambda^q E^* \), and the inclusion is holomorphic. Viewed as an element of \( \Lambda^q E^* \), \( v^{-1}_K \) is defined by:

\[ v^{-1}_K(e_{j_1} \wedge \cdots \wedge e_{j_q}) = v^{-1}_K((\text{Id} - \pi)e_{j_1} \wedge \cdots \wedge (\text{Id} - \pi)e_{j_q}), \]

for all \( 1 \leq j_1 < \cdots < j_q \leq r \). Consequently, for all multiindex \( I = (i_1 < \cdots < i_{q+1}) \) we have:

\[
\sigma(e_I \otimes v^{-1}_K) = \sum_{l=1}^{q+1} (-1)^l v^{-1}_K((\text{Id} - \pi)e_{i_1} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_l} \wedge \cdots \wedge e_{i_{q+1}}) e_{i_l}
\]

\[
= \sum_{l=1}^{q+1} (-1)^l v^{-1}_K(\text{Id} - \pi)e_{i_1} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_l} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_{q+1}} e_{i_l}
\]

\[
= \sum_{l=1}^{q+1} (-1)^l \sigma((\text{Id} - \pi)e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_{q+1}} \otimes v^{-1}_K)
\]

\[
= \sum_{l=1}^{q+1} (-1)^l \sigma((\text{Id} - \pi)e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_{q+1}} \otimes v^{-1}_K)
\]

\[
= \sum_{l=1}^{q+1} (-1)^l v^{-1}_K((\text{Id} - \pi)e_{i_1} \wedge \cdots \wedge (\text{Id} - \pi)e_{i_l} \wedge \cdots \wedge \text{Id} - \pi) e_{i_{q+1}} \pi e_{i_l}.
\]

This proves that \( \sigma(e_I \otimes v^{-1}_K) \in F \), for all multiindices \( I \) and \( K \), since \( \pi e_{i_l} \in F \) for all \( i_l \). Therefore, \( \text{Im} \ \sigma \subseteq F \). \[ \square \]

Let us now turn back to the situation in theorem \[ \text{0.1.1} \]. The section \( \pi \) under consideration is \( L^2_L \). Fix a local holomorphic frame \( e_1, \ldots, e_r \) of \( E \)
on an open set $U$ and let $p$ be, as above, the rank almost everywhere of $F = \text{Im} \pi$, and $q = r - p$. For a fixed point $x_0 \in U$ we may assume that $e_1(x_0), \ldots, e_q(x_0)$ is a basis of $Q_{x_0}$ and that $e_{q+1}(x_0), \ldots, e_r(x_0)$ is a basis of $F_{x_0}$. It is then obvious that $(\text{Id} - \pi)e_j(x_0) = e_j(x_0)$ for $j \in \{1, \ldots, q\}$, and that $(\text{Id} - \pi)e_j(x_0) = 0$ for $j \in \{q+1, \ldots, r\}$. For every matrix $a = (a_{k,j})_{1 \leq k \leq q, 1 \leq j \leq p}$ with $a_{k,j} \in \mathbb{C}$ such that $(a_{k,j})_{1 \leq k \leq q, 1 \leq j \leq p} = \text{Id}_\mathbb{C}^q$, let us define the following local holomorphic sections of $E$ on $U$:

$$s_k = \sum_{j=1}^{r} a_{k,j} e_j, \quad \text{for } k = 1, \ldots, q,$$

and the local section of $\Lambda^q E$ on $U$:

$$\tau_a = (\text{Id} - \pi)s_1 \wedge \cdots \wedge (\text{Id} - \pi)s_q \in L^2_1 \cap L^\infty.$$

The section $\tau_a$ is a linear combination of the sections $v_K$ of $\det Q$ considered in lemma 0.3.4. A posteriori $\tau_a$ will be a local holomorphic section of $\det Q$. Furthermore, $\tau_a(x_0) = e_1(x_0) \wedge \cdots \wedge e_q(x_0)$, and therefore $|\tau_a(x_0)| \neq 0$. Imitating formula (3.1) of lemma 0.3.4, we thus get

**Corollary 0.3.5** For a Hermitian vector bundle $(E, h)$ and a section $\pi \in L^2(X, \text{End} E)$ satisfying the hypotheses of theorem 0.1.1, the local bundle morphism $\Lambda^{q+1}E_{|U} \xrightarrow{v} E_{|U}$ defined by the formula:

$$(3.2) \quad e_I := e_{i_1} \wedge \cdots \wedge e_{i_{q+1}} \xrightarrow{v} \sigma(e_I \otimes \tau_a^{-1}) = \frac{u(e_I)}{\tau_a},$$

for all $I = (1 \leq i_1 < \ldots < i_{q+1} \leq r)$, satisfies $\text{Im} \pi_{|U} = \text{Im} v$. Here $\sigma$ and $u$ are defined by the same formulae as in lemma 0.3.4.

- **Third step** : a Lelong-Poincaré-type lemma

  This is the key step of the proof. Since $\text{Im} \pi = \text{Im} v$ locally, it is enough to prove that, for every multiindex $I$ such that $|I| = q+1$ we have $D''(v(e_I)) = 0$ in the sense of currents, in order to conclude that $\text{Im} \pi$ defines a holomorphic bundle outside an analytic subset of codimension $\geq 2$. Although formula $D''(v(e_I)) = 0$ is formally true, this makes a priori no sense for $\frac{1}{\tau_a}$ does not necessarily define a distribution. (The coefficients of $\tau_a$ are $L^2$ functions and their inverses are only measurable functions.) What is therefore at stake in this approach is to achieve enough regularity enabling us to apply the operator $D''$ in the sense of distributions.

  To begin with, let us notice that a posteriori the Lelong-Poincaré formula applied to the holomorphic section $\tau_a$ of the (a posteriori) holomorphic line bundle $\det Q$ gives:

$$\frac{i}{2\pi} \partial \bar{\partial} \log |\tau_a| = [Z_a] - \frac{i}{2\pi} \Theta(\det Q) = [Z_a] - \frac{1}{2\pi} \text{Tr}_E(i \beta \wedge \beta^*),$$

11
where \( [Z_a] \) stands for the current of integration along the zero divisor \( Z_a \) of \( \tau_a \) and \( |\tau_a| \) designates the quotient norm of \( \tau_a \) in \( \det Q \) (equal, as a matter of fact, to the norm of \( \tau_a \) in \( \Lambda^q E \)). Since the curvature of \( E \) is assumed to vanish identically, \( i\Theta(\det Q) = \text{Tr}_E(i\beta \wedge \beta^*) \). Since \( [Z_a] \) is a \( (1, 1) \) positive current, we get:

\[
i\bar{\partial}\partial\log|\tau_a|^2 \geq -\text{Tr}_E(i\beta \wedge \beta^*).
\]

This a posteriori situation can be retrieved by direct computation of \( i\bar{\partial}\partial\log|\tau_a|^2 \) (the norm being taken in \( \Lambda^q E \)). This is the goal of the present step of the proof. Consider the following operators:

\[
D'_Q := (\text{Id} - \pi) \circ D'_E, \quad D''_Q := (\text{Id} - \pi) \circ D''_E,
\]

representing the projections of \( D'_E \) et respectively \( D''_E \). A posteriori, \( D'_Q \) and \( D''_Q \) will be the \( (1, 0) \) and respectively \( (0, 1) \) components of the Chern connection associated to the quotient metric of the vector bundle \( Q \) that we will construct. In order to avoid complications with denominators we will compute \( i\bar{\partial}\partial\log(|\tau_a|^2 + \delta^2) \) for real numbers \( \delta > 0 \) that we shall subsequently make converge to 0. The following lemma yields the key argument to the proof of theorem \( \text{[0.1.1]} \).

**Lemma 0.3.6** If \( (E, h) \) is a holomorphic vector bundle of rank \( r \) equipped with a \( C^\infty \) Hermitian metric of zero curvature form, then for all \( \delta > 0 \) we have

\[
i\bar{\partial}\partial\log(|\tau_a|^2 + \delta^2) = \frac{i\{D'_{\det Q}\tau_a, D'_{\det Q}\tau_a\}}{|\tau_a|^2 + \delta^2} - \frac{i\{\{\tau_a, D'_{\det Q}\tau_a\}\tau_a\}}{|\tau_a|^2 + \delta^2} \geq -\frac{|\tau_a|^2}{|\tau_a|^2 + \delta^2} \cdot \text{Tr}_E(i\beta \wedge \beta^*),
\]

where

\[
D'_{\det Q}\tau_a := \sum_{k=1}^q (\text{Id} - \pi)s_1 \wedge \cdots \wedge D'_Q ((\text{Id} - \pi)s_k) \wedge \cdots \wedge (\text{Id} - \pi)s_q,
\]

\[
D''_{\det Q}\tau_a := \sum_{k=1}^q (\text{Id} - \pi)s_1 \wedge \cdots \wedge D''_Q ((\text{Id} - \pi)s_k) \wedge \cdots \wedge (\text{Id} - \pi)s_q.
\]

A word of explanation is in order here to justify the well-definedness of the above expressions. Recall that \( \pi \in L^\infty(X, \text{End} E) \cap L^2(X, \text{End} E) \). The currents \( \{D'_{\det Q}\tau_a, D'_{\det Q}\tau_a\} \) and \( \{\tau_a, D'_{\det Q}\tau_a\} \wedge \{\tau_a, D'_{\det Q}\tau_a\} \) are well-defined \((1, 1)\)-currents with \( L^1 \) coefficients because \( D'_{\det Q}\tau_a \) is a current with \( L^2 \) coefficients (arising as products of an \( L^2 \) function by \( q - 1 \) \( L^\infty \) functions). On the other hand, \( \frac{\tau_a}{|\tau_a|^2 + \delta^2} \) is easily seen to be an \( L^2_1 \) section of \( \Lambda^q E \). Then lemma \( \text{[0.2.4]} \) implies that the \((1, 1)\)-current:

\[
\frac{\tau_a}{|\tau_a|^2 + \delta^2} \cdot \text{Tr}_E(i\beta \wedge \beta^*),
\]

is well-defined. The proof is complete.
is well-defined and has \((L^1)^r\) distributions as coefficients obtained as products of an \(L^2\) distribution by an \(L^1\) function.

**Proof of lemma 0.3.4.** Since the curvature form of \(E\) is assumed to be zero (see the first step), the local frame \(e_1, \ldots, e_r\) can be chosen to be parallel for the Chern connection of \((E, h)\). This means that \(D'_E e_j = 0\) and \(D''_E e_j = 0\), for all \(j = 1, \ldots, r\). The section \(\tau_a\) of \(\Lambda^q E\) can then be locally written with respect to this frame as :

\[
\tau_a = \sum_{j_1, \ldots, j_q} a_{1j_1} \ldots a_{jq} (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q}.
\]

The operators \(D'_{\Lambda^q E}\) and \(D'_{\det Q}\), as well as the operators \(D''_{\Lambda^q E}\) and \(D''_{\det Q}\), when applied to \(\tau_a\), yield :

\[
D'_{\Lambda^q E} \tau_a = - \sum_{j_1, \ldots, j_q} a_{1j_1} \ldots a_{jq} \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge D' \pi(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q};
\]

\[
D'_{\det Q} \tau_a = - \sum_{j_1, \ldots, j_q} a_{1j_1} \ldots a_{jq} \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge (\text{Id} - \pi) \circ D' \pi(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q};
\]

\[
D''_{\Lambda^q E} \tau_a = - \sum_{j_1, \ldots, j_q} a_{1j_1} \ldots a_{jq} \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge D'' \pi(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q};
\]

\[
D''_{\det Q} \tau_a = - \sum_{j_1, \ldots, j_q} a_{1j_1} \ldots a_{jq} \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge (\text{Id} - \pi) \circ D'' \pi(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q}.
\]

The currents \(D'_{\Lambda^q E} \tau_a, D''_{\Lambda^q E} \tau_a, D'_{\det Q} \tau_a\) and \(D''_{\det Q} \tau_a\) are well-defined currents with \(L^2\) coefficients. Since \((\text{Id} - \pi) \circ D' = D' \pi\), as implied by relation (2. c), we see that

\[D'_{\det Q} \tau_a = D'_{\Lambda^q E} \tau_a.\]

Let us denote from now on this common value by \(D' \tau_a\). Relation (2. a) shows that \((\text{Id} - \pi) \circ D'' \pi = 0\), which entails that \(D''_{\det Q} \tau_a = 0\).

Let us start off by proving the inequality featured by lemma 0.3.6. The \((1, 1)\)-current \(i\{D' \tau_a, D' \tau_a\}\) is positive and the Cauchy-Schwarz inequality shows that :

\[
i\{D' \tau_a, \tau_a\} \wedge \{\tau_a, D' \tau_a\} \leq |\tau_a|^2 \cdot i\{D' \tau_a, D' \tau_a\},
\]

in the sense of currents. Consequently, \(i\{D' \tau_a, D' \tau_a\} = i\{D' \tau_a\} \wedge \{\tau_a, D' \tau_a\} \leq 0\), in the sense of currents. We will now show the following equality :
\( \{ D''_{\text{det}Q} D'_{\text{det}Q} \tau_a, \tau_a \} = |\tau_a|^2 \cdot \text{Tr}_E(\beta \wedge \beta^*), \)

and this will prove the inequality stated in the lemma. Since:

\[ D''_{\text{det}Q}((\text{Id} - \pi)e_j) = - (\text{Id} - \pi) \circ D''\pi (e_j) = 0, \quad \text{for all } j, \]

we get:

\[ D''_{\text{det}Q} D'_{\text{det}Q} \tau_a = - \sum_{j_1, \ldots, j_q} a_{1j_1} \cdots a_{qj_q} \cdot \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge (\text{Id} - \pi) \circ D'' D' \pi (e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi) e_{j_q} . \]

On the other hand, upon applying the operator \( D' \) to the identity \( (\text{Id} - \pi) \circ D''\pi = 0 \) we get:

\[-D' \pi \wedge D'' \pi + (\text{Id} - \pi) \circ D' D'' \pi = 0, \]

in the sense of currents. The current \( D' \pi \wedge D'' \pi \) has \( L^1 \) coefficients since it is the product of two currents with \( L^2 \) coefficients. The current \( D' D'' \pi \) has \( L^2_1 \) coefficients and lemma \( \text{0.2.4} \) allows its multiplication by the current \( \text{Id} - \pi \) with \( L^2_1 \) coefficients. Furthermore, since the curvature of \( E \) is assumed to be zero, the curvature of \( \text{End} E \), equipped with the metric induced from the metric of \( E \), is zero as well. Thus \( D' D'' \pi = -D'' D' \pi \), and the above equality entails:

\[ (\text{Id} - \pi) \circ D'' D' \pi = -D' \pi \wedge D'' \pi = -D' \pi \wedge D'' \pi \circ (\text{Id} - \pi), \]

in the sense of currents. This finally gives the following formula:

\[ D''_{\text{det}Q} D'_{\text{det}Q} \tau_a = \sum_{j_1, \ldots, j_q} a_{1j_1} \cdots a_{qj_q} \cdot \sum_k (\text{Id} - \pi) e_{j_1} \wedge \cdots \wedge (D' \pi \wedge D'' \pi) \circ (\text{Id} - \pi)(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi)e_{j_q}, \]

and since \( \text{Tr}_E(D' \pi \wedge D'' \pi) = \text{Tr}_E(\beta \wedge \beta^*), \) we get:

\[ D''_{\text{det}Q} D'_{\text{det}Q} \tau_a = \text{Tr}_E(\beta \wedge \beta^*) \cdot \tau_a. \]

This trivially implies identity \((\ast)\). The inequality stated in lemma \( \text{0.3.6} \) is thus proved.

We will now prove the equality featured by lemma \( \text{0.3.6} \). For \( \tau_a \) viewed as an \( L^2_1 \) section of the bundle \( \Lambda^q E \) we get by differentiation:
\[ i \partial \overline{\partial} \log(|\tau_a|^2 + \delta^2) = i \left\{ D'_{AqE}^{\tau_a}, D''_{AqE}^{\tau_a} \right\} - i \left\{ D'_{AqE}^{\tau_a}, D_{AqE}^{\tau_a} \right\} - i \left\{ D''_{AqE}^{\tau_a}, D_{AqE}^{\tau_a} \right\} \]

The above formulae of \( \tau_a \) and \( D''_{AqE}^{\tau_a} \) imply, for all \( k, l \), the identity

\[ \{ D''_E(e_{jk}), (\text{Id} - \pi) e_{jl} \} = 0. \]

Indeed, \( D''_E(e_{jk}) \) is an \( L^2 \) section of \( \text{Im} \pi \), and \( \text{Im} \pi \) and \( \text{Im}(\text{Id} - \pi) \) are orthogonal. This readily implies that:

\[ \{ D''_{AqE}^{\tau_a}, \tau_a \} = 0 \quad \text{and} \quad \{ \tau_a, D''_{AqE}^{\tau_a} \} = 0. \]

The formula of \( i \partial \overline{\partial} \log(|\tau_a|^2 + \delta^2) \) is then reduced to:

\[ i \partial \overline{\partial} \log(|\tau_a|^2 + \delta^2) = i \left\{ D'_{AqE}^{\tau_a}, D''_{AqE}^{\tau_a} \right\} - i \left\{ D'_{AqE}^{\tau_a}, D_{AqE}^{\tau_a} \right\} - i \left\{ D''_{AqE}^{\tau_a}, D_{AqE}^{\tau_a} \right\} \]

We will now prove the following three identities:

\[ (**): \quad i \left\{ D'_{AqE}^{\tau_a}, D''_{AqE}^{\tau_a} \right\} = -|\tau_a|^2 \cdot \text{Tr}_E(i\beta \wedge \beta^*), \]

\[ (***): \quad i \left\{ \tau_a, D''_{AqE}^{\tau_a} \right\} = -|\tau_a|^2 \cdot \text{Tr}_E(i\beta \wedge \beta^*), \]

\[ (****): \quad i \left\{ D''_{AqE}^{\tau_a}, D'_{AqE}^{\tau_a} \right\} = -|\tau_a|^2 \cdot \text{Tr}_E(i\beta \wedge \beta^*), \]

and this will prove the equality in the lemma. Let us first prove (**). The operator \( D'_{AqE} \), when applied to the previously obtained formula of \( D''_{AqE}^{\tau_a} \), gives:

\[ D'_{AqE}^{\tau_a} = \sum_{j_1, \ldots, j_q} A_{j_1, \ldots, j_q}, \quad \text{where} \]

\[ A_{j_1, \ldots, j_q} = \]

\[ = \sum_{l < k} (\text{Id} - \pi)e_{j_1} \wedge \cdots \wedge D'_{E}^{\pi}(e_{j_l}) \wedge \cdots \wedge D''_{E}^{\pi}(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi)e_{j_q} \]

\[ - \sum_{k} (\text{Id} - \pi)e_{j_1} \wedge \cdots \wedge D'D''_{E}^{\pi}(e_{j_k}) \wedge \cdots \wedge (\text{Id} - \pi)e_{j_q} \]

\[ - \sum_{k < l} (\text{Id} - \pi)e_{j_1} \wedge \cdots \wedge D''_{E}^{\pi}(e_{j_k}) \wedge \cdots \wedge D'_{E}^{\pi}(e_{j_l}) \wedge \cdots \wedge (\text{Id} - \pi)e_{j_q}. \]

Lemma \([224]\) (applied to the second sum above) implies that \( A_{j_1, \ldots, j_q} \) is a
well-defined current of bidegree \((1, 1)\) with values in \(E\).

Since every factor \(D''\pi(e_{jk})\) is orthogonal to every factor \((\text{Id} - \pi)e_{ji}\) occurring in the expression of \(\tau_a\), we get without difficulty:

\[
i\{D'_{\Lambda q}^{\prime}D''_{\Lambda q}^{\prime}E\tau_a, \tau_a\} = -i\{\text{Tr}_E(D'\pi \wedge D''\pi) \cdot \tau_a, \tau_a\} = -|\tau_a|^2 \cdot \text{Tr}_E(iD'\pi \wedge D''\pi),
\]

and this proves (**).

Let us now prove (***) The computations and arguments are very similar to those of the previous case. These computations show that \(D''_{\Lambda q}^{\prime}E\tau_a\) is equal, as a \((1, 1)\)-current, to \(D''_{\text{det}Q}^{\prime}D'_{\text{det}Q}^{\prime}E\tau_a\) plus some terms which have no contribution in the calculation of \(i\{\tau_a; D''_{\Lambda q}^{\prime}E\tau_a\}\). We thus finally get, while taking (3.3) into account as well:

\[
i\{\tau_a, D''_{\Lambda q}^{\prime}E\tau_a\} = i\{\tau_a, D''_{\text{det}Q}^{\prime}D'_{\text{det}Q}^{\prime}E\tau_a\} = i\{\tau_a, \text{Tr}_E(\beta \wedge \beta^*) \cdot \tau_a\} = -\{\tau_a, \text{Tr}_E(i\beta \wedge \beta^*) \cdot \tau_a\} = -|\tau_a|^2 \cdot \text{Tr}_E(i\beta \wedge \beta^*).
\]

This proves (***) It remains to prove (****). We have already proved that \(\{D''_{\Lambda q}^{\prime}E\tau_a, \tau_a\} = 0\) as a \((0, 1)\)-current with scalar values. If we apply the operator \(\partial\) we get:

\[
0 = \partial\{D''_{\Lambda q}^{\prime}E\tau_a, \tau_a\} = \{D''_{\Lambda q}^{\prime}D''_{\Lambda q}^{\prime}E\tau_a, \tau_a\} - \{D''_{\Lambda q}^{\prime}E\tau_a, D''_{\Lambda q}^{\prime}E\tau_a\},
\]

which implies, owing to (**):

\[
i\{D''_{\Lambda q}^{\prime}E\tau_a, D''_{\Lambda q}^{\prime}E\tau_a\} = i\{D''_{\Lambda q}^{\prime}E\tau_a, \tau_a\} = -|\tau_a|^2 \cdot \text{Tr}_E(i\beta \wedge \beta^*).
\]

This proves (****). Relations (⋆), (**) (*** and (****) imply the equality stated in lemma [0.3.6] Lemma [0.3.6] is thus completely proved.

- **Fourth step**: Construction of the bundle in dimension 1

We are now in a position to prove that \(\text{Im} \pi\) defines almost everywhere a holomorphic subbundle of \(E\) in restriction to almost every complex line considered locally in a coordinate patch. As a matter of fact, the proof still works for every complex subspace such that the restriction of the current \(\text{Tr}_E(i\beta \wedge \beta^*)\) is a well-defined \(d\)-closed current. Let us fix an arbitrary point \(x_0 \in X\) and a trivializing open set \(U \ni x_0\) of \(E\) contained in a coordinate patch with local coordinates \(z = (z_1, \ldots, z_n)\). Let us also fix a complex line \(L\) in this coordinate patch such that the restriction of \(\text{Tr}_E(i\beta \wedge \beta^*)\) to \(L\) is a well-defined \((1, 1)\)-current. This is the case for almost every choice of \(L\). Thanks to corollary [0.2.3], there exists a subharmonic potential \(\varphi = \varphi_L\) on \(U \cap L\) such that \(i\partial \bar{\partial} \varphi = \text{Tr}_E(i\beta \wedge \beta^*)_{U \cap L}\) (the curvature of \(E\) is assumed to be zero according to the first step). Then lemma [0.3.6] implies:
\[ i \partial \bar{\partial} \log(|\tau_a|^2 + \delta^2) \geq -\frac{|\tau_a|^2}{|\tau_a|^2 + \delta^2}(i \partial \bar{\partial} \varphi) \geq -i \partial \bar{\partial} \varphi, \quad \text{for all } \delta > 0, \]
on $U \cap L$, for $i \partial \bar{\partial} \varphi \geq 0$. This shows that the function \( \log(|\tau_a|^2 e^\varphi + \delta^2 e^\varphi) \) is subharmonic on $U \cap L$ for all $\delta > 0$, and consequently $\log(|\tau_a|^2 e^\varphi)$ is subharmonic on $U \cap L$ as a decreasing limit of subharmonic functions. In particular, the function \[
abla = \log(|\tau_a| e^\varphi) \]
is subharmonic and not identically $-\infty$ on $U \cap L$.

Let us now consider a holomorphic function $f : U \cap L \to \mathbb{C}$ such that \[
abla \int_{U \cap L} |f|^2 e^{-2\psi} d\lambda < +\infty, \quad \text{where } d\lambda \text{ is the Lebesgue measure. The function } |f| e^{-\psi} \text{ is thus } L^2 \text{ on } U \cap L. \]
In particular, $\frac{f}{\tau_a e^{\frac{\psi}{2}}}$ is an $L^2$ section of $(\det Q)^{-1}$ on $U \cap L$. Since $e^{\frac{\psi}{2}}$ is subharmonic and, moreover, $L^\infty$ on $U \cap L$, we get that
\[
\frac{f}{\tau_a} = e^{\frac{\psi}{2}} \frac{f}{\tau_a e^{\frac{\psi}{2}}}
\]
is an $L^2$ section of $(\det Q)^{-1}$ on $U \cap L$. In particular, it defines a distribution and the expression $D'' \left( \frac{f}{\tau_a} \right)$ is well-defined in the sense of distributions. We have thus obtained the regularity needed for the application of the operator $D''$ (as explained at the beginning of the third step).

Now the arguments enabling us to conclude are purely formal. Indeed, $D'' \left( \frac{f}{\tau_a} \right) = 0$ at all points where this is well-defined. The bundle morphism $v$ defined by (3.2) can then be redefined on $U \cap L$ as :
\[
\Lambda^{q+1} E \xrightarrow{v} E, \quad e_I \mapsto \frac{f u(e_I)}{\tau_a},
\]
for all multiindex $I$ such that $|I| = q+1$. Since $u(e_I)$ is a $D''$-closed $L^2$ section of $E \otimes \det Q$ on $U \cap L$, we get that $\frac{f u(e_I)}{\tau_a} \in L^1(U \cap L, E)$ and
\[
D'' \left( \frac{f u(e_I)}{\tau_a} \right) = D'' \left( \frac{f}{\tau_a} \right) u(e_I) + \frac{f}{\tau_a} D'' u(e_I) = 0,
\]
for all $I$. Consequently, the $L^2$ bundle defined by $F = \text{Im } v = \text{Im } \pi$ is locally generated by its local meromorphic sections $\frac{f u(e_I)}{\tau_a}$ on almost every complex line $L$ contained in a coordinate patch.

**Fifth step:** application of a theorem of Shiffman’s

This step would be superfluous if we were able to prove that the current
Tr_{E}(i\beta \wedge \beta^*) is d-closed (see the explanation at the beginning of the previous step). Recall that p is the rank almost everywhere of \( \pi \). Let us now consider the following map relative to the trivializing open set \( U \) of \( E \):

\[
U \ni x \mapsto \Phi G(p, r)
\]

defined almost everywhere as \( \Phi(x) = \text{Im} \pi_x \). This is a \( p \)-dimensional vector subspace of \( E_x \), and it can therefore be viewed as an element in the Grassmannian \( G(p, r) \) of \( p \)-dimensional vector subspaces of \( \mathbb{C}^r \). As the Grassmannian is a projective manifold, there exists an isometric embedding of \( G(p, r) \) into the complex projective space \( \mathbb{P}^K \), which can in its turn be embedded into a Euclidian space \( \mathbb{R}^N \). The vector-valued map

\[
\Phi = (\Phi_1, \ldots, \Phi_N) : U \rightarrow G(p, r) \hookrightarrow \mathbb{R}^N
\]

is \( L^2_1 \) for it is defined by \( \pi \) which is assumed to be \( L^2_1 \). What we have proved above amounts to the component \( \Phi_j : U \cap L \rightarrow \mathbb{R} \) of \( \Phi \) being meromorphic almost everywhere for almost all complex line \( L \) and all \( j = 1, \ldots, N \). The following Hartogs-type theorem is due to B. Shiffman (see [Shi86], corollary 2, page 240). It states that a measurable function which is separately meromorphic almost everywhere is in fact meromorphic almost everywhere.

**Theorem (Shiffman, 1986).** Let \( \Delta \) be the unit disc of \( \mathbb{C} \) and let \( f : \Delta^n \rightarrow \mathbb{C} \) be a measurable function such that for all \( 1 \leq j \leq n \) and almost all \( (z_1, \ldots, \hat{z}_j, \ldots, z_n) \in \Delta^{n-1} \), the map \( \Delta \ni z_j \mapsto f(z_1, \ldots, z_n) \) is equal almost everywhere to a meromorphic function on \( \Delta \). Then \( f \) is equal almost everywhere to a meromorphic function.

It is noteworthy that the hypotheses of this theorem of Shiffman’s are quite loose. The function \( f \) is merely assumed to be measurable and meromorphic almost everywhere along almost all directions parallel to the coordinate axes. Our above-defined functions \( \Phi_j \) satisfy much stronger hypotheses. They are not only measurable but also \( L^2_1 \). They are meromorphic almost everywhere along almost all directions as well.

This result implies that the components \( \Phi_j \) of \( \Phi \) are meromorphic almost everywhere. The map \( \Phi \) is thus meromorphic almost everywhere. Since every meromorphic map is holomorphic outside an analytic subset of codimension \( \geq 2 \), we get that \( F = \text{Im} \pi \) is a holomorphic subbundle of \( E \) outside an analytic subset \( S \subset X \) of codimension \( \geq 2 \).

**Acknowledgements.** I am grateful to my thesis supervisor Jean-Pierre Demailly for his unflinching support and great scientific expertise.

**References**
[Dem 97] J.-P. Demailly — *Complex Analytic and Algebraic Geometry*—
http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.ps.gz

[Gri69] P. A. Griffiths — *Hermitian Differential Geometry, Chern Classes
and Positive Vector Bundles*— Global Analysis, papers in honour of K. Kodaira,
Princeton Univ. Press, Princeton, 1969, 181-251.

[Kob87] S. Kobayashi — *Differential Geometry of Complex Vector Bundles*
— Princeton University Press, 1987.

[LT95] M. Lübke, A. Teleman — *The Kobayashi-Hitchin Correspondence*—
World Scientific, 1995.

[Pop03] D. Popovici — *Quelques applications des méthodes effectives en
géométrie analytique* — PhD thesis, Université de Grenoble, 2003.

[Shi86] B. Shiffman — *Complete Characterization of Holomorphic Chains
of Codimension One*— Math. Ann., 274, (1986) P. 233-256.

[Sib85] N. Sibony — *Quelques problèmes de prolongement de courants en
analyse complexe* — Duke Math. J., 52, No. 1 (1985), P. 157 - 197.

[UY 86] K. Uhlenbeck, S.T. Yau — *On the Existence of Hermitian-Yang-
Mills Connections in Stable Vector Bundles* — Communications in Pure and
Applied Mathematics, Vol. XXXIX, 1986, Supplement, pp. S257 - S293.

[UY 89] K. Uhlenbeck, S.T. Yau — *A Note on Our Previous Paper : On the
Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles* —
Communications on Pure and Applied Mathematics, Vol. XLII, pp. 703 -
707.

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