Common Codebook Millimeter Wave Beam Design: Designing Beams for Both Sounding and Communication with Uniform Planar Arrays

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Abstract—Fifth generation (5G) wireless networks are expected to utilize wide bandwidths available at millimeter wave (mmWave) frequencies for enhancing system throughput. However, the unfavorable channel conditions of mmWave links, e.g., higher path loss and attenuation due to atmospheric gases or water vapor, hinder reliable communications. To compensate for these severe losses, it is essential to have a multitude of antennas to generate sharp and strong beams for directional transmission. In this paper, we consider mmWave systems using uniform planar array (UPA) antennas, which effectively place more antennas on a two-dimensional grid. A hybrid beamforming setup is also considered to generate beams by combining a multitude of antennas using only a few radio frequency chains. We focus on designing a set of transmit beamformers generating beams for the directional characteristics of mmWave links assuming a UPA and hybrid beamforming. We first define ideal beam patterns for UPA structures. Each beamformer is constructed to minimize the mean square error from the corresponding ideal beam pattern. Simulation results verify that the proposed codebooks enhance beamforming reliability and data rate in mmWave systems.

Index Terms—Millimeter wave communications, Hybrid beamforming, Codebook design algorithm, Uniform planar array.

I. INTRODUCTION

WIRELESS broadband systems operating in the millimeter wave (mmWave) spectrum are thought to be a prime candidate to provide the system throughput enhancements needed for fifth generation (5G) wireless networks [2]–[5]. The wide bandwidths available at mmWave frequencies can be an attractive alternative to the sub-6GHz frequencies employed in most of today’s cellular networks. Also, the directional characteristics of mmWave links are suitable for reducing interuser interference in multiuser channels. However, the higher expected path loss caused by the high carrier frequency, atmospheric gases, and water vapor absorption result in serve link quality degradation. The unfavorable channel conditions at mmWave frequencies necessitate utilizing highly directional transmission with a large beamforming gain.

Employing a multitude of antennas is necessary for mmWave systems to ensure good link quality with a large beamforming gain. The small wavelengths of mmWave frequencies enable a large number of antennas to be implemented in a small form factor. Phased array transmit/receive architectures, e.g., uniform linear arrays (ULAs) and uniform planar arrays (UPAs), are usually considered for mmWave systems to allow high-resolution beamforming [6]. Systems using ULAs have been widely studied for use at mmWave frequencies [7]–[15]. UPAs are being considered due to their higher space efficiency, obtained by packing antennas on a two-dimensional (2D) grid [16].

Millimeter wave systems having a large number of antennas may not be able to use baseband beamforming techniques requiring one radio frequency (RF) chain per antenna due to the high cost and power consumption [4], [5]. Therefore, mmWave systems based on analog beamforming relying upon a single RF chain have been reported in [7]–[12]. Also, hybrid beamforming techniques using a few RF chains wired to sets of phase shifters have been proposed in [12]–[15].

Current cellular systems construct the transmit beamformer based on the channel state information (CSI) at the transmitter, which often is only available through receiver feedback. In feedback-assisted frequency division duplexing architectures, it is essential for the receiver to estimate the CSI using downlink pilot signals [16]–[18]. In large-scale mmWave systems, it may be difficult to explicitly estimate the CSI due to the large number of resources required for training each antenna [17], [18]. Channel estimation algorithms relying on compressed sensing techniques [14], [19]–[21] could be suitable for mmWave downlink training. However, compressed sensing techniques need stringent sparsity requirements to be satisfied.

Millimeter wave systems may use beam alignment approaches that choose the transmit beamformer without estimating the channel matrix explicitly [7]–[11]. In the codebook-based beam alignment approaches, it is desirable to use a common codebook for both channel sounding and data transmission. However, both phases enforce conflicting design requirements on the beams in the codebook. Data transmission beams should ideally be narrow to allow for maximum beamforming gain when properly aligned, but channel sounding beams should ideally be wide to sound a wide geographic area with small overhead. In addition, the codebook size must be small to ensure minimal system overhead.

The design of a common codebook has been extensively studied for validating practical mmWave systems [1], [8], [14]. Previous codebook design algorithms have been mainly focusing on particular vector subspaces characterized by...
ULA structures. However, codebooks for mmWave systems employing UPAs should be designed to sound much wider geographic area as well as to facilitate large beamforming gain. In addition, adaptive beam alignment approaches mostly utilize a multitude of hierarchical codebooks [8], [10], [13]. This necessitates design guidelines for multi-resolution codebooks that capture channel characteristics of UPAs.

In this paper, we propose a practical codebook design algorithm that utilizes the strong directivity of mmWave links assuming a UPA structure. The codebook design algorithm is developed assuming hybrid beamforming at the transmitter. First, an ideal beam pattern is defined to develop a codebook design criterion. The codebook is designed in such a way that each beamformer minimizes the mean square error (MSE) between the codebook’s beam pattern and the corresponding ideal beam pattern. The orthogonal matching pursuit (OMP) algorithm [14], [15], [22], [23] is used to construct a set of candidate beamformers, satisfying a power constrained hybrid beamforming setup. The final beamformer accomplishing the MSE minimization objective will be chosen among the set of beamformer candidates for each ideal beam pattern. We also show that all other beamformers in the codebook can be generated from one optimized beamformer, which can expedite on/offline codebook construction.

The remainder of this paper is organized as follows. In Section II, we describe a mmWave system with hybrid beamforming and define the beam space. In Section III, we define the ideal beam pattern by considering a UPA structure and the directional characteristics of mmWave links. In Section IV, a practical codebook design algorithm is proposed for mmWave systems based on predefined ideal beam patterns. In Section V, simulation results are presented to verify the performance of the proposed codebook, and Section VI details our conclusions.

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathcal{CN}(m, \sigma^2)$ denotes the complex normal distribution with mean $m$ and variance $\sigma^2$, $[a, b]$ is the closed interval between $a$ and $b$, $U(a, b)$ denotes the uniform distribution in the closed interval $[a, b]$, $(a)_{\ell}$ is the $\ell$-th entry of the column vector $a$, $1_{a,b}$ is the $a \times b$ all ones matrix, $I_N$ is the $N \times N$ identity matrix, $\lceil \cdot \rceil$ is the ceiling function, $E[\cdot]$ is the expectation operator, $1$ is the indicator function, $\| \cdot \|_p$ is the $p$-norm, $\odot$ is the Hadamard product, and $\otimes$ is the Kronecker product. Also, $A^H$, $A^\ast$, $A_{a,b}$, $A_{a,b}^0$, $A_{a,b}^\max \{A\}$ denote the conjugate transpose, element-wise complex conjugate, $(a, b)^{th}$ entry, $a^{th}$ row, $b^{th}$ column, and principal eigenvector of the matrix $A$, respectively.

II. SYSTEM MODEL

A. System Model

We consider a multiple-input single-output (MISO) system operating in the mmWave spectrum. The transmitter employs $M = M_h M_r$ transmit antennas, which are controlled by $N$ RF chains ($N \leq M$), and the receiver has a single receive antenna [14], [15]. This hybrid beamforming configuration is shown in Fig. 1. The transmit array is laid out in a grid pattern with $M_h$ columns and $M_r$ rows, as shown in Fig. 2. The horizontal and vertical elements are spaced uniformly with separations $d_h$ and $d_v$, respectively [6].

The input-output expression for the mmWave system is

$$y = \sqrt{\rho} h^H cs + n,$$  \hspace{1cm} (1)

where $y$ is the received signal, $\rho$ is the transmit signal-to-noise ratio (SNR), $h \in \mathbb{C}^M$ is the block fading mmWave channel, $s \in \mathbb{C}$ is the transmit data symbol subject to the constraint $E[|s|^2] \leq 1$, and $n \sim \mathcal{CN}(0, 1)$ is the additive Gaussian noise.

The unit norm transmit beamformer $c = Fv \in \mathbb{C}^M$ is formed by combination of an analog beamsteering matrix $F = [f_1, \cdots, f_N] \in \mathbb{C}^{M \times N}$ consisting of $N$ unit norm beamsteering vectors and a baseband beamformer $v \in \mathbb{C}^N$. The analog beamsteering vector $f_{\ell} \in B_M$ is realized by a set of RF phase shifters and the beamsteering vector is subject to the equal gain subset

$$B_M = \{ f \in \mathbb{C}^M : (f)_{\ell} = e^{j\theta_\ell}/\sqrt{M}, \ \ell \in \{1, \cdots, M\} \}.$$  \hspace{1cm} (2)

We assume that a phase control register generates $2^B$ quantized phases in the analog beamforming hardware [8]. Each element phase is then chosen from the set of quantized phases

$$\theta_\ell \in \mathbb{Z}_{2^B} \doteq \{0, 2\pi/2^B, \cdots, 2\pi(2^B - 1)/2^B\}. \hspace{1cm} (3)$$

The weight vector $v$ combining $F$ performs beamforming at baseband without equal gain constraints, while the combination of $F$ and $v$ is subject to the constraint $\|Fv\|_2 = 1$.

In multiple-input multiple-output (MIMO) systems, we also need to perform beam alignment at the receiver as in [7], [8]. In this paper, we discuss only beam alignment at the transmitter for simplicity. Note that the proposed codebook design algorithm can be used to construct a codebook at the receiver as well.
The beamformer \( \mathbf{c} \) is chosen from a codebook \( \mathcal{C} = \{ \mathbf{c}_{1,1}, \ldots, \mathbf{c}_{Q_h, Q_v} \} \) consisting of \( Q = Q_h Q_v \) beamformers. The selected beamformer is then written as \( \mathbf{c} = \mathbf{c}_{q, p} \),

\[
(q, p) = \arg \max_{(q, p) \in \mathcal{Q}} \sqrt{\mathbf{h}^H \mathbf{c}_{q, p} + n_{q, p}},
\]

where \( n_{q, p} \) is the noise that interrupts correct beam alignment.

### B. Vector subspace for mmWave channels

For the UPA scenario, the mmWave channel is modeled by the combination of a line-of-sight (LOS) path and a few non-line-of-sight (NLOS) paths as \([24]–[26]\)

\[
\mathbf{h} = \sqrt{\frac{MK}{1 + K}} \alpha_d \mathbf{d}_M(\psi_0, \psi_0) + \sqrt{\frac{M}{R(1 + K)}} \sum_{r=1}^{R} \alpha_r \mathbf{d}_M(\psi_{hr}, \psi_{vr}),
\]

where \( K \) is the Ricean \( K \)-factor, \( \alpha_r \sim \mathcal{CN}(0, 1) \) is the complex channel gain, \( R \) is the number of NLOS paths, and

\[
\mathbf{d}_M(\psi_{hr}, \psi_{vr}) \doteq \mathbf{d}_{M_r}(\psi_{hr}) \otimes \mathbf{d}_{M_v}(\psi_{vr}) \in \mathbb{C}^M
\]

is the \( r \)-th normalized beam defined by the Kronecker product of array response vectors

\[
\mathbf{d}_{M_r}(\psi_a) \doteq \frac{1}{V_M} \left[ 1, e^{j\psi_a}, \ldots, e^{j(M_r-1)\psi_a} \right]^T \in \mathbb{C}^{M_r}
\]

with \( \psi_h = \frac{2\pi d_h}{\lambda} \cos \theta_h \), and \( \psi_v = \frac{2\pi d_v}{\lambda} \sin \theta_v \), where \( \theta_h \) is the angle of departure (AoD) \([6]\). Note that \( a \in \{h, v\} \) denote both horizontal and vertical domains.

Millimeter wave channels are expected to have a large Ricean \( K \)-factor that is matched with a strong channel directivity \([4]\), \([27]\), \([28]\). Therefore, we restrict a particular vector subspace defined by a single dominant beam, i.e., an array manifold

\[
\mathcal{A} = \{ \mathbf{a} : \mathbf{a} = \sqrt{M} \mathbf{d}_M(\psi_h, \psi_v), (\psi_h, \psi_v) \in \mathcal{B} \}.
\]

In \([9]\), \( \mathcal{B} \) denotes the set of beam directions in both horizontal and vertical domains and we may focus on a limited area, e.g., \( \mathcal{B} = [-\pi, \pi] \times [-\pi, \pi] \) as depicted in Fig. 3(a) because the array vector is periodic such as \( \mathbf{d}_{M_r}(\psi_a + 2\pi) = \mathbf{d}_{M_r}(\psi_a) \).

We define the set of possible beam directions that characterizes the array manifold in \([9]\). For a given \( \theta_v \), in vertical domain, the range of horizontal domain is bounded as

\[
|\psi_h| \leq 2\pi d_h \cos \theta_v / \lambda,
\]

because the beam direction in \( \psi_h \) domain is a function of not only \( \theta_h \) but also \( \theta_v \). The beam space should be defined by considering the coupled ranges together. Under the assumption of \( d_h = d_v = d \), the beam space is defined as shown in Fig. 3(b)

\[
\mathcal{B}_c = \{ (\psi_h, \psi_v) : \psi_h^2 + \psi_v^2 \leq (2\pi d / \lambda)^2 \}.
\]

We consider AoD distributed as \( (\theta_h, \theta_v) \in [-\pi / 2, \pi / 2] \times [-\pi / 2, \pi / 2] \) assuming sectorized cellular system. The beam space is then defined as shown in Fig. 3(c)

\[
\mathcal{B}_s = \{ -2\pi d_h / \lambda, 2\pi d_h / \lambda \} \times \{ -\sqrt{2}\pi d_v / \lambda, \sqrt{2}\pi d_v / \lambda \}.
\]

The beam space, which is a feasible set of possible beams,

\[
\mathcal{B}_p = \mathcal{B} \cap \mathcal{B}_c \cap \mathcal{B}_s.
\]

is then characterized as depicted with solid line in Fig. 3(d).

We now take a closer look at the beam space \( \mathcal{B}_p \). In Fig. 3(d), the patterned areas \( (\mathcal{B} \cap \mathcal{B}_c) \setminus \mathcal{B}_s \) excluded from \( \mathcal{B}_p \) (which describes the coupling effect between the horizontal and vertical domains) are negligible compared to the main area. For simplicity, we develop a codebook design algorithm without taking into account the coupling between horizontal and vertical domains. In the assumption of \( d = \lambda / 2 \), the beam directions are simplified as \( \psi_{a,v} = \pi \sin \theta_{a,v} \) and the beam space is bounded such as

\[
\mathcal{B}_s = \{ -\psi_{a,v}^\prime, \psi_{a,v}^\prime \} \times \{ -\psi_{a,v}^\prime, \psi_{a,v}^\prime \}
\]

with \( \psi_{a,v}^\prime = \pi \) and \( \psi_{a,v}^\prime = \pi / \sqrt{2} \). Finally, we define the array manifold by plugging \( \mathcal{B}_s \) into \([9]\).

### III. Framework for Codebook Design Algorithm

To use a common codebook for both channel sounding and data transmission, beamformers should be designed to generate beam patterns satisfying the following criteria:

1) Having higher effective channel gain.
2) Covering wider geographic area uniformly.

We define ideal, but unachievable in practice, beam patterns that satisfy above criteria. Design guidelines for true beamformers will be developed based on the ideal beam patterns.
where the reference gain is defined as
\[
B(t) = -\psi_a' t + \Delta^a (b - 1), \Delta^a = 2\psi_a'/Q_a.
\]  

Note that \( q \in \{1, \ldots, Q_h\}, \) \( p \in \{1, \ldots, Q_v\}, \) and \( \Delta^a \) is the beam-width of a beamformer in domain \( a \in \{h, v\}. \)

To develop ideal beam patterns, we discuss a constraint that is subject to any beam patterns in the following.

\textbf{Lemma 1}: For any vector \( c \in \mathbb{C}^M \) that satisfies \( \|c\|_2^2 = 1 \), the integral with respect to the reference gain over \((\psi_h, \psi_v) \in \mathcal{B}\) is identical to any beam patterns, i.e.,
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(\psi_h, \psi_v, c)dh\psi db\psi = (2\pi)^2 \frac{\|c\|_2^2}{M},
\]
where the reference gain is defined as
\[
G(\psi_h, \psi_v, c) = \|\mathbf{d}_{M_h}(\psi_h) \otimes \mathbf{d}_{M_v}(\psi_v)\|_2^2.
\]

\textit{Proof}: The integration of the reference gain is
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(\psi_h, \psi_v, c)dh\psi db\psi
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \sum_{\ell=1}^{M_v} (\mathbf{d}_{M_h}(\psi_h)e^{j(1-\ell)\psi_v})^H \mathbf{c}_\ell \right)^2 \frac{dh\psi db\psi}{M_v}
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \sum_{m=1}^{M_h} c(m) e^{-j(1-\ell)\psi_v} (\mathbf{c}_\ell)_m \right)^2 \frac{dh\psi db\psi}{M_h M_v}
= \frac{2\pi}{M} \sum_{m=1}^{M_h} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-j(1-\ell)\psi_v} (\mathbf{c}_\ell)_m^2 \frac{dh\psi db\psi}{M_h M_v}
= \frac{(2\pi)^2}{M} \sum_{m=1}^{M_h} \sum_{\ell=1}^{M_v} |(\mathbf{c}_\ell)_m|^2 = \frac{(2\pi)^2}{M},
\]
where \(a\) is derived because \( c = [c_1^T, \ldots, c_{M_h}^T]^T, \) \( c_\ell \in \mathbb{C}^{M_h} \) and \(b, c\) are derived based on the Parseval’s theorem [8]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{m=1}^{M_h} (a)_m e^{j(m-1)\psi_v} \right)^2 d\psi = \|a\|_2^2
\]
for any vector \( a \in \mathbb{C}^M \).

We now define ideal beam patterns by taking Lemma 1 into account. We assume the \((q, p)\)-th ideal beamformer generates an ideal beam pattern that is biased toward its desired beam space. For example, the \((q, p)\)-th beamformer generates an ideal beam pattern having a non-zero reference gain \( t(\psi_h, \psi_v) \) for the beam space \( \mathcal{B}_{q,p} \) and zero gain for the rest of the beam space \( \mathcal{B}_a \setminus \mathcal{B}_{q,p} \), such as
\[
G^{\text{ideal}}_{q,p}(\psi_h, \psi_v) = t(\psi_h, \psi_v)\mathbb{I}_{\mathcal{B}_{q,p}}(\psi_h, \psi_v).
\]

We also discuss an ideal distribution of the non-zero reference gains that maximizes beam alignment performance. We define the probability of beam misalignment as
\[
P_{\text{mis}} = \Pr(c_{q,p} \neq c_{q,p}^\ast),
\]
where \((q, p)\) is the index of the chosen beamformer in [4] and
\[
(q, p) = \arg \max_{(q, p) \in \mathcal{Q}} \left| H c_{q,p}^T \right|^2
\]
is the index of the optimally selected beamformer. The non-zero reference gain \( t(\psi_h, \psi_v) \) for the \((q, p)\)-th ideal beam pattern should be flattened as
\[
t(\psi_h, \psi_v) = \frac{Q}{M} \mathbb{I}_{\mathcal{B}_{q,p}}(\psi_h, \psi_v).
\]

\textbf{Proof}: The probability of beam misalignment is expressed in a closed form function of \( t(\psi_h, \psi_v) \) as in [10],
\[
P_{\text{mis}} = \sum_{q=1}^{Q-1} \left( \frac{Q}{M} \right) \left( 1 + q \right) \left( 1 - \exp \left( -\rho \frac{||h||_2^2 t(\psi_h, \psi_v)q}{1 + q} \right) \right),
\]
in an ideal setting that all beam patterns are orthogonal. The average probability of beam misalignment is then bounded as
\[
E[P_{\text{mis}}] = \sum_{q=1}^{Q-1} \left( \frac{Q}{M} \right) \left( 1 + q \right) \left( 1 + q \right) \left( 1 - \exp \left( -\rho \frac{||h||_2^2 t(\psi_h, \psi_v)q}{1 + q} \right) \right) \left( 1 + q \right)
\]
\[
= \sum_{q=1}^{Q-1} \left( \frac{Q}{M} \right) \left( 1 + q \right) \left( 1 + q \right) \left( 1 - \exp \left( -\rho \frac{||h||_2^2 (2\pi)^2 q}{1 + q} \right) \right) \left( 1 + q \right)
\]
where \(a\) is derived based on the Jensen’s inequality and \(b\) is derived based on Lemma 1. The average probability of beam misalignment is minimized when \(b\) becomes equality, and this condition holds if the non-zero reference gain is uniform over \( \mathcal{B}_{q,p} \) such as \( t(\psi_h, \psi_v) = g \mathbb{I}_{\mathcal{B}_{q,p}}(\psi_h, \psi_v) \) with \( g \in \mathbb{R}. \)

The integration of the ideal beam pattern over the beam space \( \mathcal{B}_a \) in [10] is defined based on Lemma 1 as
\[
\int_{\mathcal{B}_a} G_{\text{ideal}}(\psi_h, \psi_v)d\psi_h d\psi_v = \int_{\mathcal{B}_a} g d\psi_h d\psi_v
= \frac{(2\pi)^2 g}{Q} = \frac{(2\pi)^2}{M},
\]
which gives \( g = Q/M. \) The flattened reference gain is finalized at \( g = \min \left\{ 1, Q/M \right\}. \) Because of the assumption \( M \geq Q, \)
we have \( g \leq 1. \) Finally, the ideal beam pattern becomes
\[
G_{\text{ideal}}^{(q,p)}(\psi_h, \psi_v) = \frac{Q}{M} \mathbb{I}_{\mathcal{B}_{q,p}}(\psi_h, \psi_v),
\]
with \((\psi_h, \psi_v) \in \mathcal{B}_a. \)

\textbf{Lemma 2} intuitively makes sense assuming uniform distribution of users. The derived ideal beam patterns are depicted in Fig. [4].

\footnote{Any reference gain is upper bounded as \( G(\psi_h, \psi_v, c) \leq \|\mathbf{d}_{M_h}(\psi_h) \otimes \mathbf{d}_{M_v}(\psi_v)\|_2^2 = 1. \)
where the phase shifting function is defined as

\[ \psi_q = -\psi_a + \Delta^b (b - 1) + \Delta^a (\ell - 0.5) / L_a \]  

where \( b \in \{1, \cdots, Q_a \} \) and \( \ell \in \{1, \cdots, L_a \} \). In this paper, we define \( L = L_h L_v \) satisfying \( LQ \geq M \). Then, the optimization problem is redefined as

\[ \min_{F_v} \| G_{\text{ideal}}^{1,1} - G(F_v) \|_2^2. \]  

In (17), the column vector for the actual beam pattern is

\[ G(F_v) = [G(\psi_h^{1,1}, \psi_v^{1,1}, F_v) \cdots G(\psi_h^{Q_h L_h}, \psi_v^{Q_v L_v}, F_v)]^T \]

and that for the ideal beam pattern is written as

\[ G_{\text{ideal}}^{1,1} = \frac{Q_h}{M_h} (e_1^h \otimes 1_{L_h,1}) \otimes \frac{Q_v}{M_v} (e_1^v \otimes 1_{L_v,1}) \]  

where \( e_1^h \) is the first column vector of \( 1_{L_h} \).

Unfortunately, there exists no closed form solution for the minimization problem in (17) because entries in \( G(F_v) \) and \( G_{\text{ideal}}^{1,1} \) are found in the form of absolute square of a complex number. To get insights on the structure of the beam pattern, we decompose each entry into an arbitrary complex number and its complex conjugate. Then, the ideal beam pattern vector \( G_{\text{ideal}}^{1,1} \) is decomposed as

\[ G_{\text{ideal}}^{1,1} = (\psi_a \otimes (g_h \circ g_h^*)) \otimes (\psi_v \otimes (g_v \circ g_v^*)) \]

\[ = (\sqrt{Q/M} e_1^h \otimes g_h) \otimes (e_1^v \otimes g_v^*) \]

\[ \otimes (\sqrt{Q/M} (e_1^h \otimes g_h) \otimes (e_1^v \otimes g_v)^*), \]

where \( (\cdot) \) is derived because the all ones vector in (18) can be manipulated into any equal gain vector and its element-wise complex conjugate as \( 1_{L_h} = g_h \circ g_h^* \). Note that \( g_h \in G_{L_h}^{1_h} \) is the equal gain vector subject to the constrained set

\[ G_{L_h}^{1_h} = \{ g \in C^{L_h} : (e_1^h)_{\ell} = e_1^z, \ell \in Z \} \]

with \( (e_1^z)_{\ell} = 1 \). Similarly, the actual beam pattern \( G(F_v) \) is rewritten as

\[ G(F_v) = ((D_h \otimes D_v)^H F_v) \otimes ((D_h \otimes D_v)^H F_v)^* \]

with the set of array vectors for quantized directions in (16)

\[ D_h = [D_{a,1}^h, \cdots, D_{a,2}^h] \in C^{M_a \times L_h Q_h}, \]

\[ D_v = [d_{Ma,(\psi_a^{(1,1)}, \cdots, d_{Ma,(\psi_a^{(b,1)}, \cdots, d_{Ma,(\psi_a^{(b,L}_a)} \in C^{M_a \times L_v}. \]

We focus on comparing the decomposed vectors for a given \((g_h, g_v)\) to compute a set of \( F \) and \( v \) that satisfies

\[ \beta(D_h \otimes D_v)^H F_v = \sqrt{Q/M} (e_1^h \otimes g_h) \otimes (e_1^v \otimes g_v), \]

where \( \beta \in C \) is a normalization constant. The combination of \( F \) and \( v \) may not be able to construct \((e_1^h \otimes g_h) \otimes (e_1^v \otimes g_v)\) because it may not be in the column space of \((D_h \otimes D_v)^H F \). Therefore, \( F_{\{g_h, g_v\}} \) and \( v_{\{g_h, g_v\}} \) are computed to minimize the MSE between the decomposed vectors as

\[ \min_{F_v} \| \beta (D_h \otimes D_v)^H F_v - \sqrt{Q/M} (e_1^h \otimes g_h) \otimes (e_1^v \otimes g_v) \|_2^2. \]

Any set of equal gain vectors \((g_h, g_v) \in G_{L_h}^{1_h} \times G_{L_v}^{1_v} \) construct \( G_{\text{ideal}}^{1,1} \). The entries in \( g_h \) have the fixed absolute value, while the phase can be arbitrary in \( Z_1 \), defined in (3).
We compute the normalization constant $\beta$ by differentiating the object function over $\beta^*$ based on Wirtinger derivatives [10, 29]. Then, the derived complex gain is given by
\[
\hat{\beta} = \sqrt{\frac{Q}{M}} (Fv)^H (D_h (e_i \otimes g_h) \otimes D_v (e_i \otimes g_v)) \| (D_h \otimes D_v)^* Fv \|^2_2
\]
\[
\| (D_h \otimes D_v)^* Fv \|^2_2 = \sqrt{\frac{Q}{M}} (Fv)^H (D_h (e_i \otimes g_h) \otimes D_v (e_i \otimes g_v)) \| (D_h \otimes D_v)^* Fv \|^2_2
\]
where (a) is derived because $D_h (e_i \otimes g_h) = D_h^1 g_h$. By plugging $\hat{\beta}$ into (22), a beamformer candidate is computed as
\[
(F_{g_h, g_v} v_{g_h, g_v}) = \arg\max_{F, v} \left| (D_h^1 g_h \otimes D_v^1 g_v)^H Fv \right|^2 \| (D_h \otimes D_v)^* Fv \|^2_2
\]
\[
= \arg\max_{F, v} \left| (D_h^1 g_h \otimes D_v^1 g_v)^H Fv \right|^2 \| (D_h \otimes D_v)^* Fv \|^2_2
\]
where (a) is derived because $D_h D_h^H = \frac{LQ}{M} I_{M_h}$ and the denominator is fixed to $\| (D_h \otimes D_v)^* Fv \|^2_2 = \frac{LQ}{M}$ for any $Fv$ satisfying the power constraint $\|Fv\|^2_2 = 1$.

**B. OMP algorithm constructing beamformer candidates**

We now solve the maximization problem as in (23) to generate each beamformer candidate for a given $(g_h, g_v)$ as
\[
c_{g_h, g_v} = F_{g_h, g_v} v_{g_h, g_v}
\]
Each beamformer candidate is a feasible solution accomplishing the minimization objective in (15). For a given $(g_h, g_v)$, an optimal solution to the problem in (23) is computed as
\[
c_{g_h, g_v} = \frac{D_h^1 g_h \otimes D_v^1 g_v}{\|D_h^1 g_h \otimes D_v^1 g_v\|^2_2}
\]
However, $F$ and $v$ may not be able to construct the optimal beamformer because column vectors in $F$ are subject to the equal gain subset $B_M$ in (2). To compute a beamformer $c_{g_h, g_v}$ that satisfies the hybrid beamforming setups, we solve the maximization problem based on the OMP algorithm in [14, 15, 22, 23]. The problem formulation in [22, 23] is equivalent to our problem in (23), which is designed to minimize the residual vector
\[
r = c_{g_h, g_v} - F_{g_h, g_v} v_{g_h, g_v}
\]
In the OMP algorithm, the beamsteering matrix $F_{g_h, g_v}$ and beamformer $v_{g_h, g_v}$ are updated $N$ times to minimize $\|r\|_2$. We first compute a beamsteering vector $f_n$ at the $n$-th update to remove the residual vector $r_n$.
\[
r_n = c_{g_h, g_v} - F_{g_h, g_v} v_{g_h, g_v}
\]
which is not suppressed until the previous update. The beamsteering vector $f_n$ should be aligned to the residual vector $r_n$. We select the $n$-th beamsteering vector
\[
f_n = \exp (j \angle r_{n-1}) / \sqrt{M}
\]
because the beamsteering vectors are subject to the equal gain subset as $f_n \in B_M$ in (2). Each element phase in $f_n$ is quantized with the set of quantized phases $\mathcal{Q}_\mathcal{B}$ [8]. The $n$-th beamsteering matrix is then updated as $F_n = [f_{n-1}, f_n] \in \mathbb{C}^{M \times n}$.

Next, a beamformer $v_n$ is computed by solving the maximization problem (23) for a given $F_n$ over $v \in \mathbb{C}^n$. To compute the beamformer that satisfies the power constraint of hybrid beamforming system $\|Fv\|^2_2 = 1$, the maximization problem is rewritten by changing dummy variables as $v = u/\|F_n u\|_2$ for $u \in \mathbb{C}^n$. For a given $(g_h, g_v)$ and $F_n$, the $n$-th beamformer is computed based on the generalized Rayleigh quotient solution in [30], such as $v_n = \arg\max_{u \in \mathbb{C}^n} \frac{u^H (f_n^H (\Gamma_h \otimes \Gamma_v) F_n) u}{u^H (f_n^H F_n) u}$
\[
= \hat{u} + \sum_{i=1}^n \left( (f_n^H F_n) \right)_{ii} \frac{u_i}{\|f_n u\|_2}
\]
with $\Gamma_n = \left( (f_n^H F_n) \right)_{ii} (\Gamma_h \otimes \Gamma_v) F_n$.

Finally, the $n$-th beamformer becomes $c_n = F_n v_n$. The residual vector $r_n = c_{g_h, g_v} - F_n v_n$ is updated for the following update steps. The iterative process is summarized in Algorithm 1. Each beamformer candidate is formed by combination of the updated solution, i.e., $F_{g_h, g_v}$ and $v_{g_h, g_v}$.
\[
c_{g_h, g_v} = F_{g_h, g_v} v_{g_h, g_v}
\]
for a given $(g_h, g_v) \in \mathcal{G}_{L_h} \times \mathcal{G}_{L_v}$.

**C. Final solution**

The final beamformer that generates beam pattern close to the $(1, 1)$-ideal beam pattern is computed as
\[
c_{1,1} = F_{g_h, g_v} v_{g_h, g_v}
\]
where the index of the beamsteering matrix and the baseband beamformer is selected by brute force
\[
(g_h, g_v) = \arg\min_{g_h, g_v} ||G_{\text{ideal}} - G(F_{g_h, g_v} v_{g_h, g_v})||_2^2
\]
over $(g_h, g_v) \in \mathcal{G}_{L_h} \times \mathcal{G}_{L_v}$. Note that $L_{L_h+L_v-2}$ beamformer candidates are computed because each pair of equal gain

7The initial residual vector is defined as $r_0 = c_{g_h, g_v}$. 
where \( \Delta \) is the non-overlapped beam-width of each beamformer in (13), and \( \gamma \Delta^a \) is the overlapped beam-width of the guard band, which is defined by the design parameter \( \gamma \). Because each beamformer covers the widened beam space including the guard band, an ideal beam pattern may have a lower non-zero reference gain. Thus, the ideal beam pattern is redefined as

\[
\tilde{G}_{q,p}^{\text{ideal}}(\psi_h, \psi_v) = \frac{Q}{M(1 + 2\gamma)^2} \mathbb{B}_{q,p}(\psi_h, \psi_v).
\]

Beamformers alleviating the sharp dips are also computed based on the proposed codebook design algorithm by plugging the redefined beam pattern into the optimization problem.

V. SIMULATION RESULTS

In this section, numerical results are presented to verify the beamforming performances of proposed codebook design algorithm. In this paper, four RF chains and six bits phase control register, i.e., \( N = 4 \), \( B = 6 \), are considered for hybrid beamforming architectures. The beamforming codebook \( \mathcal{C} = \{\mathbf{c}_{1,1} \cdots \mathbf{c}_{Q_h Q_v}\} \) consisting of \( Q = Q_h Q_v \) beamformers is designed as in (27). For the minimization problem in (26), the equal gain sets \( \mathcal{G}_L \) and \( \mathcal{G}_r \) are defined with parameters \( I = 3 \) and \( L_h = L_v = 8 \). In addition, we consider 30 directions in each beam-width \( \psi \) to compute the MSE between the beamformer candidate’s beam pattern and the ideal beam pattern.

A. Beam patterns of codebook examples

In Figs. 5 and 6 we compare the beam patterns of the proposed codebook and the codebooks in [8], [14]. The ULA codebook in [8] is extended to a 2D UPA codebook by maximizing a minimum reference gain in each target beam space \( \mathbb{B}_{q,p} \). The codebook in [14] is extended to UPA structures with a single set of equal gain vectors

\[
\langle \mathbf{g}_1', \mathbf{g}_1'' \rangle = \left( \mathbf{1} \left[ \frac{\pi}{8} \right], \frac{\pi}{8} \right). \tag{28}
\]

In Fig. 5 we compare the beam patterns of a single beamformer by using the reference gain defined as

\[
G_R(\theta_h, \theta_v, \mathbf{c}_{q,p}) = |\mathbf{c}_{q,p}^H \mathbf{d}_M(\pi \sin \theta_h \cos \theta_v, \pi \sin \theta_v)|^2
\]
Over beam directions $(\theta_h, \theta_v) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$. In Fig. 6, we plot beam patterns in each target beam space based on the reference gain $G(\theta_h, \theta_v, c(q, p))_{\theta_h, \theta_v}$, such that

$$
(q, p)_{\theta_h, \theta_v} = \arg \max_{(q, p) \in Q} \mathcal{H}^H_{q, p} \mathcal{d}_M \left( \pi \sin \theta_h \cos \theta_v, \pi \sin \theta_h \right)^2
$$

where $Q$ is defined in [5]. As shown in Figs. 5 and 6, the proposed codebook can generate flattened and higher reference gains in each beam space. It is also shown that the proposed codebook considering the guard band in Section IV-D alleviates sharp dips between consecutive beams.

At this point, we pause to discuss the difference between each of the beam pattern. First, we consider the codebook in [8]. A beamformer in the codebook of [8] is designed to maximize a minimum reference gain in each target beam space, defined primarily for the corresponding beamformer. The codebook in [8] can generate beam patterns with flattened reference gains, while the value of reference gains are much lower than that of the proposed codebook. Next, we consider the codebook in [14]. Each beamformer in the proposed codebook is optimized over $L_h L_v - 2$ beamformer candidates generated using equal gain vectors in the sets $G^L_h$ and $G^L_v$, while the codebook in [14] is designed by using a single set of all ones vectors $(\hat{g}_h, \hat{g}_v)$ in [28]. The proposed codebook suppresses the MSE between the ideal beam patterns and the actual beam patterns better than that in [14].

## B. Beamforming performance

We now evaluate the performance of the codebooks based on the normalized beamforming gain defined as

$$
G_{BF} = \mathbb{E} \left[ \| \mathbf{h} \|_2^2 \right],
$$

where $\mathbf{c}$ is the selected beamformer for data transmissions. The preferred beamformer $\mathbf{c}$ in [29] is chosen based on hard-decision beam alignment algorithms [7]–[10]. The beamforming performances of the codebooks are evaluated from Monte Carlo simulations with 10,000 independent channel realizations. For demonstrations, we consider two channel scenarios based on the street geometry conditions under ray-like propagation assumptions [26]. In the first scenario, we consider channels consisting of a LOS path and three NLOS paths. The mmWave channel model is proposed in [8], [9]. Based on the channel measurements in [27], the Ricean $K$-factor is set to 13.5 dB.

In the second scenario, the channel vector is characterized by three NLOS paths [25], [26]. We assume that $\| \mathbf{h} \|^2 = M$ for fair comparison between two channel scenarios.

We first consider beam alignment approach that utilizes a single codebook. The transmit beamformer is chosen as $\mathbf{c} = \hat{c}_{\hat{q}, \hat{p}}$, where $(\hat{q}, \hat{p})$ is the index of the selected beamformer in [4]. In our beam alignment approach, we compare the ratio between the first and second largest test samples

$$
\tau = \left| \sqrt{\mathbf{h}^H \mathbf{c}_{\hat{q}, \hat{p}}} + n_{\hat{q}, \hat{p}} \right|^2 / \left| \sqrt{\mathbf{h}^H \mathbf{c}_{\hat{q}, \hat{p}}} + n_{\hat{q}, \hat{p}} \right|^2
$$

with $(\hat{q}, \hat{p}) = \arg \max_{(q, p) \in Q} \mathbb{E} |\mathbf{h}^H \mathbf{c}_{q, p} + n_{q, p}|^2$, and $(\hat{q}, \hat{p})$ in [4]. To avoid beam misalignment, the receiver asks the transmitter to perform an additional cycle of channel sounding if the ratio is smaller than a design parameter $\tau_0$. In this case, each test sample on two cycles of channel sounding is combined together. The selected transmit beamformer is then communicated to the transmitter via a feedback link employing overheads of $B_{tot} = \log_2 Q_h Q_v$ bits.

In Fig. 7 the beamforming gains of different codebooks are compared in UPA structures $(M_h, M_v) = (8, 4), (12, 6)$, (16, 8) with different sizes of codebooks $(Q_h, Q_v) = (4, 4), (8, 4), (8, 4)$, respectively. In both channel scenarios, it is shown that the proposed codebook scans the mmWave channels better and provide higher beamforming gains than the codebooks in [8], [14]. However, the small-sized codebooks provide limited beamforming gains because beamformers have wide beamwidths generating lower gains.

We next consider beam alignment utilizing hierarchical codebooks [7], [8]. We review the beam alignment algorithm with two rounds of sounding. In this approach, the preliminary beamformer is selected from the first level codebook $C_1 = \{c_{1,1} \cdots c_{Q_h,1}\}$ based on $S_1 = Q_h Q_v$ sounding samples as in [4]. The selected beamformer is then communicated to the transmitter via a feedback link employing overheads of $\log_2 S_1$ bits. In the following round, the transmitter sounds the beamformers in the hierarchical codebook $C_2 = \{c_{1,1} \cdots c_{Q_h,2}\}$ of selected beamformer.

For simulation, the design parameter is set to $\tau_0 = 2$.\footnote{For simulation, the design parameter is set to $\tau_0 = 2$.}
The beamformers in $C_2$ have narrower beamwidths and higher beamforming gains compared to those in $C_1$. Finally, the transmit beamformer is selected and communicated to the transmitter by employing feedback overheads of $\log_2 S_2$ bits.

In Fig. 8 the beamforming gain based on different codebooks are compared in UPA structures, i.e., $(M_h, M_v) = (8, 4), (12, 6), (16, 8)$. The size of the codebooks and the channel sounding times are summarized in Table I. It is shown that the proposed codebooks provide higher beamforming performance than the previously reported codebooks [8], [14].

VI. CONCLUSIONS

In this paper, we proposed a beam pattern design algorithm suited to the directional characteristics of the mmWave channels corresponding to UPAs. We proposed an iterative algorithm to construct small-sized beam alignment codebooks for mmWave systems. Hybrid beamforming architectures using a mixture of analog and digital beamforming was considered to design effective beams suitable to large-scale mmWave systems with limited RF chains. In the proposed algorithm, each beamformer is constructed to minimize the MSE between its actual beam pattern and the corresponding ideal beam pattern. We developed a simplified approach to solve the approximated MSE minimization problem. In addition, we used the OMP algorithm to design beamformers satisfying the hybrid beamforming setup. The beamforming performance of proposed codebooks is verified through Monte Carlo simulations. Numerical results show that our codebooks outperform previously reported codebooks in mmWave channels using UPA structures.

**TABLE I**

| Codebooks size | Sounding time | Antennas | Codebooks size | Sounding time |
|----------------|---------------|----------|----------------|---------------|
| $(M_h, M_v)$ | $(S_1, S_2)$ | $(M_h, M_v)$ | $(S_1, S_2)$ | $(M_h, M_v)$ | $(S_1, S_2)$ |
| $(8, 4)$ | $(4, 4)$ | $(8, 4)$ | $(16, 4)$ | $(8, 4)$ | $(16, 4)$ |
| $(12, 6)$ | $(4, 6)$ | $(16, 4)$ | $(16, 8)$ | $(8, 4)$ | $(16, 8)$ |
| $(16, 8)$ | $(4, 8)$ | $(16, 8)$ | $(32, 8)$ | $(16, 8)$ | $(32, 8)$ |

**Fig. 8.** Beamforming gain based on the multiple rounds beam alignment.

**APPENDIX A**

**PROOF OF LEMMA 3**

Any ideal beam pattern can be defined by shifting directions of the $(1,1)$-th ideal beam pattern as

$$C_{q,p}^{\text{ideal}}(\psi_h, \psi_v) = G_{1,1}^{\text{ideal}}(\psi_h - \Delta_h^q, \psi_v - \Delta_v^p).$$

(30)

Also, directions of the actual beam pattern can also be shifted based on the following proposition as depicted in Fig. 9.

**Proposition 1:** The reference gain can be rewritten by using the phase shifting function $T(F, \Delta_h^q, \Delta_v^p)$ in (15), such as

$$G(\psi_h, \psi_v, Fv) = G(\psi_h + \Delta_h^q, \psi_v + \Delta_v^p, T(F, \Delta_h^q, \Delta_v^p)v).$$

The proof of the proposition 1 is summarized in Appendix B. The minimization problem for the $(q,p)$-th beamformer is rewritten in (31). Note that (a) is derived based on (30) and (b) is derived based on the proposition 1.

We now modify the rewritten minimization problem in (31) to discuss the link with the solution for $(1,1)$-th beamformer $(F_{0,1}^R, \psi_{0,1})$. First, variables for the integration in (31) are changed by defining new variables as $\phi_h = \psi_h - \Delta_h^q$ in each domain of the double integral. We next change a dummy variable by defining the phase shifted matrix as

$$\tilde{F} = T(F, -\Delta_h^q, -\Delta_v^p).$$

By plugging redefined variables $\phi_h, \phi_v$, and $\tilde{F}$ into (31), the minimization problem is rewritten as in (32). Note that (a) is
\[
(F_{q,p}^{\text{opt}}, v_{q,p}^{\text{opt}}) = \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\psi_h - \Delta_q^h, \psi_v - \Delta_p^v) - G (\psi_h, \psi_v, F v)|^2 d\psi_v d\psi_h
\]
\[
= \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\psi_h - \Delta_q^h, \psi_v - \Delta_p^v) - G (\psi_h - \Delta_q^h, \psi_v - \Delta_p^v, T(F, -\Delta_q^h, -\Delta_p^v))v|^2 d\psi_v d\psi_h.
\]
\[
(F_{q,p}^{\text{opt}}, v_{q,p}^{\text{opt}}) = \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v d\phi_h
\]
\[
= \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v
d\phi_h
\]
\[
= \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v
d\phi_h
\]
\[
= \arg \min_{F, v} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G_{1,1}^{\text{ideal}} (\phi_h, \phi_v) - G (\phi_h, \phi_v, F v)|^2 d\phi_v d\phi_h = (F_{q,p}^{\text{opt}}, v_{q,p}^{\text{opt}}).
\]

\[
G(\psi_h, \psi_v, F v) = (d_M (\psi_h + \Delta_q^h, \psi_v + \Delta_p^v) \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v))^H (F \odot 1_{M,N}) v^2
\]
\[
= (d_M (\psi_h + \Delta_q^h, \psi_v + \Delta_p^v) \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v))^H (F \odot (d_M (\Delta_q^h, \Delta_p^v)1_{1,N}) \odot (d_M (-\Delta_q^h, -\Delta_p^v)1_{1,N})) v^2
\]
\[
= (\tilde{d}_M (\psi_h + \Delta_q^h, \psi_v + \Delta_p^v) \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v)1_{1,N}) v^2
\]
\[
= \left[ \tilde{d}_M (\psi_h + \Delta_q^h, \psi_v + \Delta_p^v) \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v) \right]^H
\]
\[
T(F, \Delta_q^h, \Delta_p^v) \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v)1_{1,N} \odot \tilde{d}_M (-\Delta_q^h, -\Delta_p^v)) v^2
\]
\[
= \tilde{d}_M (\psi_h + \Delta_q^h, \psi_v + \Delta_p^v)T(F, \Delta_q^h, \Delta_p^v) v^2 = G(\psi_h + \Delta_q^h, \psi_v + \Delta_p^v, T(F, \Delta_q^h, \Delta_p^v) v).
\]

\[
(b) \text{ is derived because } 1_{M,N} \text{ is decomposed as } 1_{M,N} = (\tilde{d}_M (\psi_1, \psi_2) \odot \tilde{d}_M (-\psi_1, -\psi_2))1_{1,N}
\]
\[
= (\tilde{d}_M (\psi_1, \psi_2)1_{1,N}) \odot (\tilde{d}_M (-\psi_1, -\psi_2)1_{1,N}),
\]
and \(c\) is rewritten by the definition of the phase shifting function \(T(F, \Delta_q^h, \Delta_p^v)\) in Lemma 3 and the associative property of the Hadamard product
\[
A \odot (B \odot C) = (A \odot B) \odot C
\]
for arbitrary matrices of the same size. Finally, \(d\) is derived based on the formulation between vectors \(a, b, c \in \mathbb{R}^M\),
\[
(a \odot b) c = \sum_{m=1}^{M} (a_m^*) (c_m) |(b)_m|^2
\]
\[
= a^H c,
\]
which holds in the condition of \(|(b)_m|^2 = 1\) for all \(m \in \{1, \cdots, M\}\). Note that \((d)\) satisfies this condition because each element of \(\tilde{d}_M (-\Delta_q^h, -\Delta_p^v)\) has a unit gain, i.e., \(|(\tilde{d}_M (-\Delta_q^h, -\Delta_p^v))_m|^2 = 1\) for all \(m \in \{1, \cdots, M\}\).

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