At the very origin of chaos in three-dimensional dynamical systems

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The mechanism responsible for the emergence of chaotic behavior has been identified analytically within a class of three-dimensional dynamical systems which generalize the well-known E.N. Lorenz 1963 system. The dynamics in the phase space has been reformulated in terms of a first-exit-time problem. Chaos emerges due to discontinuous solutions of a transcendental problem ruling the time for a particle to cross a potential wall. Numerical results point toward the genericity of the new found mechanism.

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Many chaotic phenomena in physical sciences and engineering can be satisfactorily described by low-dimensional autonomous dynamical systems. Among them the most celebrated example is the well-known E.N. Lorenz model

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -y + x + (r - 1)(1 - z)x \\
\dot{z} &= -b(z - xy),
\end{align*}
\]

here rewritten using rescaled variables \(x, y, z\) in the case \(r > 1\). Besides it attracted the attention of the scientific community mainly on account of its ability to illustrate how a simple model can produce very rich and varied dynamics in the phase space in terms of more familiar ideas such as mechanical properties of particles moving in one-dimensional potential fields subjected to viscous forcing. Such reformulation will be the starting point to generalize the original Lorenz system to a whole class of Lorenz-like systems. Among them we shall select a particular piecewise linearized model through which analytical results on the dynamics can be obtained. As we shall see, the dynamics generated by this model can be mapped into a first-exit-time problem. This will allow us to identify analytically the very origin of the system chaotic behavior. Finally, the robustness of the identified mechanism will be tested numerically for other fully nonlinear generalized Lorenz-like systems.

Let us start our analysis by noticing that, far from the initial transient, the system (1) is equivalent to the (integral-) differential equation

\[
\dot{x} = -(\sigma + 1)x - \frac{\partial U}{\partial x}. 
\]

Eq. (3) can be interpreted as a customary classical dynamical equation of motion for a (unit mass) particle subjected to a viscous force \(-(\sigma + 1)x\) in the potential field \(U(x, t)\). Here,

\[
U(x, t) = \frac{b}{2\sigma}U(x) + \left(1 - \frac{b}{2\sigma}\right)U_t(x)
\]

is a potential field resulting from a weighted average (in the chaotic regime \(b < 2\sigma\)) of a constant quartic potential \(U(x) \equiv \sigma(r - 1)(x^2 - 1)^2/4\) and a time depending quadratic potential \(U_t(x) \equiv \sigma(r - 1)(x^2 - 1)_b(x^2 - 1)/2\). We have defined

\[
[f]_b \equiv b\int_0^\infty d\tau e^{-b\tau} f(t - \tau)
\]

where \(f(t)\) is a generic function for which the integral exists. The latter integral clearly represents the memory of \(f(t)\) (at the time \(t\)), i.e., the present outcome of its exponentially weighted past evolution. Thus, the potential \(U_t(x)\) depends on time through the exponential memory of the past motion.

One can check that, given \(x\) from (3), the variables \(y\) and \(z\) are obtained from the relations \(y = \dot{x}/\sigma + x\) and \(z = b/(2\sigma)x^2 + (1 - b/(2\sigma))(x^2)_b\), respectively. In this formulation it is for instance evident that \(x\) and \(y\) are synchronizing coordinates while this is not for \(z\).

The above description immediately leads to the following considerations highlighting the role of the memory in the route to chaos. For \(U_t = 0\), the particle motion stops after some time in one of the two minima of \(U(x)\). This because of the viscous term \((\sigma + 1)x\). On the contrary, non trivial behaviors, including chaotic ones, may take place due to the statistical balance between dissipation and energy exchanges produced by \(U_t\). The bistable character of \(U\) plays a crucial role for the emergence of chaos. Indeed, initially very close trajectories starting in...
the same cell, may undergo to completely different evolutions if, at a certain time, one has sufficiently energy to cross the peak in \( x = 0 \), while this is not for the other. This is the first clue that chaos arises from the unpredictability of the instants when particles pass through the barrier in \( x = 0 \). We shall give in the sequel the analytical proof of this heuristic argument together with the reason at the very origin of such unpredictability. To do that, let us give a further reformulation of the system \( \Phi(x) \). This will make it possible to generalize and simplify \( \Phi(x) \) with the final goal to deal with chaos analytically. By suitably scaling the time: \( t \rightarrow \tau \equiv [(r-1)/2]^{1/2} t \), Eq. (2) can be recast in the form \( \Phi(x) \) and

\[
\dot{x} + \eta \dot{x} + (x^2 - 1) x = - \alpha [x^2 - 1] \beta x \quad (5)
\]

where \( \eta, \alpha \) and \( \beta \) are related to the original parameters \( \sigma, b \) and \( r \).

A more expressive form of Eq. (5) is

\[
\dot{x} + \eta \dot{x} + \left( q(x) + \alpha \sigma q(x) \right) \Phi'(x) = 0, \quad (6)
\]

where \( \Phi(x) = 1/2 x^2 \) and \( q(x) = x^2 - 1 = 2\Phi(x) - 1 \). This equation describes the motion of a (unit mass) particle subjected to a viscous force \(-\eta \dot{x}\) and interacting with a fixed potential field \( \Phi(x) \) through a ‘dynamically varying charge’ \( q(x) = q(x) + \alpha \sigma q(x) \). It is constituted by a fixed ‘core’ charge, a ‘locally acquired’ charge, related to the local potential, and an exponentially vanishing ‘memory’ charge, continuously depending on the previous instantaneous charge history. This scheme can be used to mimic the instantaneous effective charge of a particle moving in a background (structured) particle bath. Indeed, the coupling of \( q(x) \) with the background potential \( \Phi(x) \) yields an endogenous forcing term which allows a self-sustained unceasing motion, even in the presence of friction, and causes a corresponding unceasing inner transfer of the bath charge.

At this point the Lorenz system (1) can be easily generalized in the form

\[
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -y + x + (r - 1)(1 - z)\Phi'(x) \\
\dot{z} &= -b \left[ z - q(x)(y - x) - q(x) - 1 \right],
\end{aligned}
\quad (7)
\]

from which Eq. (4) follows after some algebraic manipulations.

As already observed, the chaotic behavior of Lorenz’s system essentially depends on the unpredictability of the instants when \( x \) change its sign: as long as it keeps constant sign the system evolution is certainly nonlinear, and nevertheless not chaotic at all. This fact suggests a slight modification of the original form, in order to single out analytically the origin of chaos without to be faced with the difficulties arising from nonlinear problems. We thus set in Eqs. (4) and (5) \( \Phi(x) = |x| \) and \( q(x) = \Phi(x) - 1 \) obtaining:

\[
\dot{x} + \eta \dot{x} + \{ |x| - 1 + \alpha |x| - 1 \} \text{sgn}(x) = 0 \quad (8)
\]

and the corresponding piecewise linearized system

\[
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -y + x + (r - 1)(1 - z) \text{sgn}(x) \\
\dot{z} &= -bz + b \text{sgn}(x) \frac{x^2 + y}{2},
\end{aligned}
\quad (9)
\]

where \( \text{sgn}(x) \equiv |x|/x \).

Our assumptions for \( \Phi \) and \( q \) correspond in Eq. (2) to

\[
U(x) = \sigma(r - 1)/(|x| - 1)^2/2 \quad \text{and} \quad U_1(x) = \sigma(r - 1)/|x| - 1 \beta(|x| - 1).
\]

The fundamental bistable character of \( U \) is thus maintained.

In order to solve Eq. (3), we exploit the fact that it is left invariant under the transformations \( x \rightarrow -x, \dot{x} \rightarrow -\dot{x} \) and \( \tau \rightarrow -\tau \). We can thus focus on one of the two regions \( x < 0 \) and \( x > 0 \). Let us consider, e.g., \( x > 0 \).

In this case Eq. (3) is equivalent to the third order linear equation

\[
\frac{d^3 \xi}{d\tau^3} + (\beta + \eta) \frac{d^2 \xi}{d\tau^2} + (1 + \beta \eta) \frac{d \xi}{d\tau} + \beta(1 + \alpha)\xi = 0 \quad (10)
\]

where \( \xi \equiv x - 1 \).

Notice that when the particles cross the barrier in \( x = 0 \), the evolution described by Eq. (10) has to restart with new initial conditions corresponding to an elastic collision against a rigid wall posed in \( x = 0 \).

It can be checked that, in the case of chaotic motion we are interested in, the general solution of Eq. (10) can always be written in the form

\[
\xi(\tau) = e^{\lambda_3 \tau} \left( C_1 \cos(\lambda_1 \tau) + C_2 \sin(\lambda_1 \tau) \right) + C_3 e^{-\lambda_0 \tau} \quad (11)
\]

with \( \lambda_0, \lambda_1, \lambda_1 \geq 0 \) and \( C_1, C_2 \) and \( C_3 \) real coefficients determined from the initial conditions on \( \xi, \xi' \) and \( \xi'' \).

The instant \( \tau_1 \) at which the first particle collision against the wall in \( x = 0 \) occurs is thus defined by the equation \( \xi(\tau_1) = -1 \). From \( (11) \), \( \tau_1 \) is thus the smallest positive solution of the transcendental equation

\[
C_1 e^{\lambda_3 \tau_1} \cos(\lambda_1 \tau_1) + C_2 e^{\lambda_2 \tau_1} \sin(\lambda_1 \tau_1) + C_3 e^{-\lambda_0 \tau_1} = -1. \quad (12)
\]

Geometrically, we can interpret \( \tau_1 \) as the first intersection of \( g(\tau) \equiv -C_3 e^{-\lambda_0 \tau} - 1 \) with \( h(\tau) \equiv C_1 e^{\lambda_3 \tau} \cos(\lambda_1 \tau) + C_2 e^{\lambda_2 \tau} \sin(\lambda_1 \tau) \). The function \( g \) is a decreasing exponential and \( h \) an oscillating function with growing amplitude. It is thus easily understood why even a little modification of initial conditions can produce a discontinuous variation of \( \tau_1 \) (see Fig. 1). As we shall see, the same reason applies also for the class of systems (4).
FIG. 1. Graphical interpretation of the discontinuous character of \( \tau_1 \) for small changes of the initial conditions. Full line represents \( g(\tau) \), dashed lines \( h(\tau) \) for the parameters \( \alpha = 6.50, \beta = 0.19, \eta = 0.78 \) and for initial conditions (a): \( \xi_0 = 2.26, \frac{\dot{\xi}_0}{\tau} = -2.00 \); (b): \( \xi_0 = 2.30, \frac{\dot{\xi}_0}{\tau} = -2.00 \). Bullets denote the first intersection between \( g \) and \( h \) defining the first collision time against the wall (a): \( \tau_1 = 3.91 \); (b) \( \tau_1 = 8.76 \).

The dependence of \( \tau_1 \) on the initial conditions \( \xi_0 \) and \( \xi_0 \) (\( \xi_0 = -1 \)) is implicitly contained in Eq. (12). Coefficients \( C_1 \), \( C_2 \) and \( C_3 \) are indeed linearly related to the initial conditions. The way to describe the global behavior of \( \tau_1 \) in terms of the pair \( (\xi_0, \xi_0) \) although simple results quite lengthy. We thus confine our attention on the corresponding graphical shape presented in Fig. 2.

FIG. 2. The behavior of the first collision time, \( \tau_1 \) defined by Eq. (12), as the initial conditions \( \xi_0 \) and \( \xi_0 \) for \( \alpha = 6.50, \beta = 0.19, \eta = 0.78 \).

From the figure it appears that \( \tau_1 \) shows sensitivity with respect to the initial conditions only in a limited subset of the half-plane \( \xi_0 \geq 0 \). It is natural to expect that in the chaotic regime the system is quickly attracted inside this region. It is now clear why chaotic behaviors can arise also for apparently simple nonlinearities as isolated (non eliminable) discontinuities (see, e.g., Refs. [10–14] for other examples of piecewise linearized systems displaying varied form of chaotic behaviors). Notice that the same behavior showed in Fig. 2 for \( \tau_1 \) holds for the \( n \)-th collision time \( \tau_n \) as a function of the \((n - 1)\)-th initial conditions \( (\xi_0^{(n-1)}, \xi_0^{(n-1)}) \).

For \( (\xi_0, \xi_0) = (\lambda_0, -\lambda^2_0) \) one has \( \tau_n \to \infty \) (the peak in Fig. 2). In this case, it is possible to show that the system configuration, described by \( \xi \equiv (\xi(\tau), \dot{\xi}(\tau), \ddot{\xi}(\tau)) \), exactly lies on the stable manifold \( W^s = \{ t \mathbf{w}_3 \mid t > -1 \} \) with \( \mathbf{w}_3 \equiv (1, -\lambda_0, \lambda^2_0) \) and its motion is an exponential decay on the fixed point \( \xi = 0 \).

For \( (\xi_0, \xi_0) \neq (\lambda_0, -\lambda^2_0) \) the evolution in the phase space \( \xi, \xi, \ddot{\xi} \) consists of both a ‘rapid’ decay towards \( \xi = 0 \) along the stable manifold \( W^s \) and a ‘slow’ amplified oscillation on the two-dimensional unstable manifold, \( W^u \), generated by \( \mathbf{w}_1 \equiv (1, \lambda_0, \lambda^2_0) \) and \( \mathbf{w}_2 \equiv (1, \lambda_0, \lambda^2_0) \), with \( \lambda_0 = \lambda_0 + i\lambda_i \) and \( \lambda_0 = \lambda_0 - i\lambda_i \).

Focusing now on the \( n \)-th collision, we thus have from Eq. (12) the behavior of \( \tau_n \) vs the pair \( (\xi_0^{(n-1)}, \xi_0^{(n-1)}) \).

Taking the first and the second time derivative of Eq. (12), we can relate \( \dot{\xi}_0^{(n)} \) and \( \ddot{\xi}_0^{(n)} \) to the initial conditions \( \dot{\xi}_0^{(n-1)} \) and \( \ddot{\xi}_0^{(n-1)} \) (remember that \( C_1, C_2 \) and \( C_3 \) are linearly related to such initial conditions). The result is a two-dimensional Poincaré map connecting \( (\xi_0^{(n-1)}, \dot{\xi}_0^{(n-1)}) \) to \( (\xi_0^{(n)}, \dot{\xi}_0^{(n)}) \). The nonlinear character of the map is entirely contained in the transcendental and discontinuous relation between \( \tau_n \) and the initial conditions.

FIG. 3. The one-dimensional map \( \tau_n = \tau_n(\xi_0^{(n-1)}) \); (a): for the piecewise linearized system (10); (b): for the Lorenz model (4).

Let us now assume that the attraction towards the unstable manifold \( W^u \) is sufficiently fast (the goodness of this approximation is controlled by \( \beta_1 + \eta \)) for the system to be considered as belonging to \( W^u \) at the collision time against the plane \( \xi = -1 \). If this is the case, ‘immediately’ after each collision the system thus necessary lies on a straight line. This implies a linear relationship between velocity and acceleration immediately after the \( n \)-th collision. The functional dependence of \( \tau_n \) on \( \dot{\xi}_0^{(n-1)} \) and \( \xi_0^{(n-1)} \) is thus restricted on a straight line in the plane \( \dot{\xi}_0^{(n-1)} - \xi_0^{(n-1)} \). This amounts to say that \( \tau_n \) is a function of one variable alone, say, \( \xi_0^{(n-1)} \). The map \( \tau_n = \tau_n(\xi_0^{(n-1)}) \) is shown in Fig. 3(a).
Furthermore, the Poincaré map connecting \((\xi_0^{(n-1)}, \dot{\xi}_0^{(n-1)})\) to \((\xi_0^{(n)}, \dot{\xi}_0^{(n)})\) reduces to a one-dimensional map, say, between \(\xi_0^{(n-1)}\) and \(\xi_0^{(n)}\). Fig. 4(a) shows this map for \(\alpha = 6.50, \beta = 0.19, \eta = 0.78\). Its behavior is evidently chaotic: the discontinuities derive from the analogous ones in the function \(\tau_n = \tau_n(\xi_0^{(n-1)})\).

![Figure 4](image)

**FIG. 4.** The one-dimensional map between \(\dot{\xi}_0^{(n)}\) and \(\xi_0^{(n)}\); (a): for the piecewise linearized system; (b): for the Lorenz model.

The final question to be addressed concerns the relation of our results with the original Lorenz system and more generally with the class of dynamical systems. One immediately realizes that, under the condition \(\Phi (0) = 0\), the linear relationship between velocity and acceleration in \(x = 0\) follows:

\[
x_0^{(n)} + \eta \dot{x}_0^{(n)} = 0.
\]

(13)

As for the piecewise linearized system, the generalized Lorenz systems are thus described by one-dimensional maps. Focusing in particular on the Lorenz model, the analogous of the maps reported in Figs. 3(a) and 4(a) are shown in Figs. 3(b) and 4(b). Similar behaviors have been obtained for other choices of \(\Phi\) and \(q\) in [1]. These maps have been derived by numerical integration of (1) by a standard Runge–Kutta scheme, whereas we remember that all results relative to the piecewise linearized system have been obtained analytically. The resemblance of Figs. 3(a), 4(a) with Figs. 3(b), 4(b) points toward the robustness of the mechanism we have identified as cause of chaos in the linearized system [1].

In conclusion, the very origin of chaos for a whole class of three-dimensional autonomous dynamical systems has been identified. Chaos is entirely contained in a transcendental equation ruling a first-exit-time problem whose solutions appear discontinuous for small changes in the initial conditions. Results have been obtained analytically for a piecewise linearized model belonging to a more general class of dynamical systems. We however showed numerically that the basic reason for the chaos to emerge applies also for the general case.

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