Ultraviolet regularity for QED in $d=3$

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Abstract

We study the ultraviolet problem for QED in $d=3$ using Balaban’s formulation of the renormalization group. The model is defined on a fine toroidal lattice and we seek control as the lattice spacing goes to zero. As a first step we take a bounded field approximation and solve the renormalization problem. Namely we show that the bare energy density and the bare fermion mass can be chosen to depend on the lattice spacing, so that under the renormalization group flow they take preassigned values on a macroscopic scale. This is accomplished by a nonpertubative technique which is insensitive to whether the renormalizations are finite or infinite.

1 Introduction

1.1 overview

Constructive quantum field theory is concerned with the construction of mathematically rigorous quantum field theory models. It has had some notable successes in the case of super-renormalizable models in dimension $d \leq 3$, see the surveys [33], [36]. However quantum electrodynamics (QED) in $d = 3$ has so far resisted analysis. In this paper we start a program to gain control over this model. The immediate task is to control the ultraviolet (short distance) singularities in a finite volume. In a subsequent paper [30] the goal is to prove an ultraviolet stability bound.

The problem is formulated in a renormalization group language for lattice gauge theories. Originally due to Wilson, a precise version was developed by Balaban. [1] - [15]. The idea is to perform a series of block averaging renormalization group (RG) transformations on the action. Each transformation integrates out some short distance modes. One tracks the effective actions and hopes to control the flow by specifying that parameters like fermion mass and energy density end at certain specified values on a macroscopic scale. This involves renormalization: bare parameters are to be chosen to depend on the lattice spacing. In [29] we studied weakly coupled scalar QED in $d = 3$ and accomplished this using a non-perturbative technique. In the present we work we accomplish the same result for weakly coupled fermion QED in $d = 3$.

Generally in quantum field theory infinite renormalizations are required, i.e. some bare parameters must diverge as the lattice spacing goes to zero. The present technique is insensitive to whether the renormalizations are finite or infinite, and in fact we do not decide whether they are finite or infinite. However taking some wisdom from the perturbative analysis of continuum theories it is quite likely that the energy density requires an infinite renormalization and the fermion mass does not.

In Balaban’s approach each RG transformation features a division of the lattice into regions where the field is small (i.e bounded) and a region where the field is large. The large field region makes a

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tiny contribution to the effective actions and does not require renormalization. The renormalization problem is confined to the small field region, and that is what we study here. For simplicity we consider the case that the fields are small on the whole torus, anticipating that the analysis will work in arbitrary regions and provide the central ingredient for the full problem. A paradigm for the complete analysis is given for the scalar $\phi^4$ interaction in $d = 3$ in [25] - [27].

Some preliminary work on QED$_3$ was done by Balaban, O’Carroll, and Schor [20], [21] and also by the author [23], [24] (the latter with massive photons, something we avoid here). We also mention some work on the infrared problem with massive fermions integrated out [31].

### 1.2 the model

We work on the toroidal lattices

$$\mathbb{T}^{-N}_M = (L^{-N} \mathbb{Z}/L^M \mathbb{Z})^3$$

where $L$ is a (large) positive odd integer. To begin we take the lattice $\mathbb{T}^{-N}_0$ with unit volume and spacing $\epsilon = L^{-N}$ on this lattice the fermi fields are elements of a Grassman algebra $\bar{\psi}_\alpha(x), \psi_\alpha(x)$ indexed by points $x \in \mathbb{T}^{-N}_0$ and components $1 \leq \alpha \leq 4$ and satisfying

$$\psi_\alpha(x)\psi_\beta(y) = -\psi_\beta(y)\psi_\alpha(x)$$

and so forth. Alternatively we can take $x \in L^{-N} \mathbb{Z}^3$ and make the identification

$$\psi(x + e_\mu) = \psi(x)$$

where the $e_\mu$ are the unit basis vectors. We also want to consider anti-periodic boundary conditions and then impose instead that $\psi(x + \epsilon e_\mu) = -\psi(x)$. In the latter case elements of the algebra with an even number of fields (which is all we consider) have no change in sign and can be considered as indexed by points in $\mathbb{T}^{-N}_0$ as well.

There is also an abelian gauge field (electromagnetic potential, connection, one-form) $A$ mapping bonds in $\mathbb{T}^{-N}_0$ to $\mathbb{R}$. A bond from $x$ to a nearest neighbor $x'$ is the ordered pair $b = [x, x']$. We require that $A(b) = A(x, x') = -A(x', x)$. Oriented bonds have the form $[x, x'] = [x, x + \epsilon e_\mu]$, and we sometimes write $A_\mu(x) = A(x, x + \epsilon e_\mu)$

A covariant derivative with charge $\epsilon$ is defined by

$$(\partial A, \mu) f(x) = \left(e^{i\epsilon A(x, x + \epsilon e_\mu)} f(x + \epsilon e_\mu) - f(x) \right) \epsilon^{-1}$$

This is a forward derivative. The transpose is given by

$$(\partial^T A, \mu) f(x) = \left(e^{i\epsilon A(x, x - \epsilon e_\mu)} f(x - \epsilon e_\mu) - f(x) \right) \epsilon^{-1}$$

and is a backward derivative. We also consider the symmetric derivative

$$\nabla A, \mu = \frac{1}{2} (\partial A, \mu - \partial^T A, \mu)$$

and the covariant Laplacian

$$\Delta A = - (\partial A) \partial^T A$$

Let $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$ be a representation of the Clifford algebra for $d = 4$. We only use $\gamma_0, \gamma_1, \gamma_2$ in the three dimensional Dirac operator, but $\gamma_3$ will also play a roll since it anti-commutes with the others. The Dirac operator on spinors is

$$\mathcal{D}_A = \gamma \cdot \nabla A - \frac{1}{2} \epsilon \Delta A = \sum_{\mu=0}^2 \gamma_\mu \nabla A, \mu - \frac{1}{2} \epsilon \Delta A$$
The extra term $\frac{1}{2} \Delta A$ was added by Wilson to prevent doubling of fermion species. The operator can also be written

$$(\mathcal{D}_A f)(x) = -\epsilon^{-1} \sum \mu \left[ \left( \frac{1 - \gamma_\mu}{2} \right) e^{ie A(x, x+\epsilon e_\mu)} f(x + \epsilon e_\mu) + \left( \frac{1 + \gamma_\mu}{2} \right) e^{ie A(x, x-\epsilon e_\mu)} f(x - \epsilon e_\mu) - f(x) \right]$$

(8)

The gauge field $A$ has field strength $dA$ defined on plaquettes (squares) by

$$dA(p) = \sum_{b \in \partial p} A(b) \epsilon^{-1} \quad \text{or} \quad (dA)_{\mu\nu}(x) = dA \left( x, x + \epsilon e_\mu, x + \epsilon e_\mu, x + \epsilon e_\nu, x \right)$$

(9)

The action with fixed bare fermion mass $0 \leq \bar{m} \leq 1$

$$S(A, \bar{\psi}, \psi) = \frac{1}{2} ||dA||^2 + < \bar{\psi}, (\mathcal{D}_A + \bar{m}) \psi > + \bar{m}^N < \bar{\psi}, \psi > + \bar{v}^N$$

(10)

where

$$< \bar{\psi}, \psi > = \int \bar{\psi}(x) \psi(x) dx = \sum_\alpha \sum_x e^3 \bar{\psi}_\alpha(x) \psi_\alpha(x)$$

$$||dA||^2 = \int |dA(p)|^2 dp = \sum_{\mu<\nu} \sum_x e^3 |(dA)^{\mu\nu}(x)|^2$$

(11)

The vacuum energy density $\bar{v}^N$ and the mass $m^N$ are counter terms and will be chosen to depend on $N$. The $N \to \infty$ limit formally gives the standard continuum theory. We are interested in bounds uniform in $N$ on things like the partition function

$$\int \exp(-S(A, \bar{\psi}, \psi)) \ D\bar{\psi} \ D\psi \ D\bar{\psi} \ D\psi \ \prod_b d(A(b))$$

(12)

where the fermion integral is the standard Grassman integral. The integral will need gauge fixing to enable convergence.

1.3 symmetries

The action is invariant under lattice symmetries, that is translations, reflections, and rotations by multiples of $\pi/2$. Indeed let $a$ be a lattice point, let $r$ be such a reflection or rotation, and let $S$ be a corresponding element of $Spin(3)$ so that $S^{-1}_\mu S = \sum_\nu \epsilon_{\mu\nu} \gamma_\nu$. Then with

$$\psi_{a,r}(x) = S \psi(r^{-1}(x-a)) \quad \bar{\psi}_{a,r}(x) = (S^{-1})^T \bar{\psi}(r^{-1}(x-a)) \quad A_{a,r}(b) = A \left( r^{-1}(b - (a, a)) \right)$$

(13)

we have

$$S(A_{a,r}, \bar{\psi}_{a,r}, \psi_{a,r}) = S(A, \bar{\psi}, \psi)$$

(14)

The action is also gauge invariant. For $\lambda : \mathbb{T}_0^{-N} \to \mathbb{R}$ a gauge transformation is defined by

$$\psi^\lambda(x) = e^{ie\lambda(x)} \psi(x) \quad \bar{\psi}^\lambda(x) = e^{-ie\lambda(x)} \bar{\psi}(x) \quad A^\lambda(x, x') = A(x, x') - \partial \lambda(x, x')$$

(15)

Then $\mathcal{D}_A \psi^\lambda = (\mathcal{D}_A \psi)^\lambda$ and

$$S(A^\lambda, \bar{\psi}^\lambda, \psi^\lambda) = S(A, \bar{\psi}, \psi)$$

(16)
Another symmetry is charge conjugation invariance. We define a charge conjugation matrix \( C \) to satisfy \(-\gamma^T \mu = C^{-1}\gamma_\mu C \) and can choose a representation such that \( C^T = C^{-1} = -C \). Then since \((\nabla_A)^T = -\nabla_{-A} \) and \( \Delta^T_A = \Delta_{-A} \)

\[
(\mathcal{D}_A + m)^T = C^{-1}(\mathcal{D}_{-A} + m)C
\]  

Charge conjugation on the Grassman algebra is defined by

\[
\mathcal{C}\psi = C\bar{\psi} \quad \mathcal{C}\bar{\psi} = -C\psi
\]

and on the gauge field by \( A \rightarrow -A \). Then

\[
\langle C\bar{\psi}, (\mathcal{D}_{-A} + \bar{m})C\psi \rangle = -\langle \psi, (\mathcal{D}_A + \bar{m})^T \bar{\psi} \rangle = \langle \bar{\psi}, (\mathcal{D}_A + \bar{m})\psi \rangle
\]

where we used \( \bar{\psi}\bar{\psi} = -\bar{\psi}\psi \). It follows that the entire action has the symmetry

\[
S(-A, C\bar{\psi}, \bar{\psi}) = S(A, \bar{\psi}, \psi)
\]

Note also that since \( \gamma^T \mu = \gamma_\mu \) we also have \(-\gamma_\mu = C^{-1}\gamma_\mu C \) which implies \( \gamma_\mu = (\gamma_3 C)^{-1}\gamma_\mu (\gamma_3 C) \).

Assuming \( A \) is real we have \( \nabla_A = \nabla_{-A} \) and \( \Delta_A = \Delta_{-A} \) and therefore with \( \bar{m} \) real we have the complex conjugation

\[
(\mathcal{D}_A + \bar{m})^* = (\gamma_3 C)^{-1}(\mathcal{D}_{-A} + \bar{m})(\gamma_3 C)
\]

It follows from (17) and (21) that

\[
\det (\mathcal{D}_A + \bar{m})^* = \det (\mathcal{D}_A + \bar{m})
\]

from which one can deduce that (after gauge fixing) the partition function real and not zero.

1.4 the scaled model

The model has been formulated on a fine lattice with unit volume \( T_0^N \). But we immediately scale up to the large unit lattice \( T_N^0 \). Then the ultraviolet problem is recast as in infrared problem, the natural home of the renormalization group. Let \( \Psi_\alpha(x), \bar{\Psi}_\alpha(x) \) be elements of a Grassman algebra indexed by \( x \in T_N^0 \) and \( 1 \leq \alpha \leq 4, \) and let \( A : \{ \text{ bonds in } T_N^0 \} \rightarrow \mathbb{R} \) be a gauge field on this lattice. These scale down to fields on the original lattice \( T_0^N \)

\[
A_{L^{-N}}(b) = L^{N/2}A(L^Nb) \quad \Psi_{L^{-N}}(x) = L^N\Psi(L^Nx) \quad \bar{\Psi}_{L^{-N}}(x) = L^N\bar{\Psi}(L^Nx)
\]

The action on the new lattice is \( S_0(A, \bar{\Psi}, \Psi) = S(A_{L^{-N}}, \bar{\Psi}_{L^{-N}}, \Psi_{L^{-N}}) \) which is

\[
S_0(A, \bar{\Psi}, \Psi) = \frac{1}{2} dA + \langle \bar{\Psi}, (\mathcal{D}_A + \bar{m}_0^N)\Psi \rangle + m_0^N\langle \bar{\Psi}, \Psi \rangle + \varepsilon_0^N\text{Vol}(T_0^N)
\]

Now lattice sums are unweighted and derivatives are unit lattice derivatives such as

\[
(\partial_{A,\mu}\Psi)(x) = e^{i\varepsilon_0^N A(x, x + e_\mu)}\Psi(x + e_\mu) - \Psi(x)
\]

The scaled coupling constants are now tiny and given by

\[
\varepsilon_0^N = L^{-\frac{1}{2}N}e \quad \bar{m}_0 = L^{-N}\bar{m}
\]

The scaled counter terms are

\[
\varepsilon_0^N = L^{-3N}e^N \quad m_0^N = L^{-N}m^N
\]
In the following we omit the superscript $N$ writing $e_0, \bar{m}_0$ and $\varepsilon_0, m_0$.

As we proceed with the RG analysis the volume will shrink back down. After $k$ steps the torus will be $\mathbb{T}^0_{N-k}$ or $\mathbb{T}^{-k}_{N-k}$. The coupling constants will scale up to

$$e_k = L^{\frac{k}{2}} e_0 = L^{-\frac{k}{2}(N-k)} e $$

$$\bar{m}_k = L^k \bar{m}_0 = L^{-(N-k)} \bar{m}$$

The counterterms $\varepsilon_k, m_k$ will evolve in a more complicated manner.

**Conventions:**

- Throughout the paper $O(1)$ stands for a constant independent of all parameters. Also $c, C, \gamma$ are constants ($C \geq 1, c, \gamma \leq 1$) which may depend on $L$ and which may change from line to line.

- Distances are taken in a sup metric

$$d(x, y) = |x - y| = \sup_{\mu} |x_\mu - y_\mu|$$

2 **RG transformation for fermions**

We explain how the RG transformation is defined for fermions with a gauge field background. The analysis is originally due to Balaban, O’Carroll, and Schor [20], [21]. See also [23], [24], [29].

2.1 **Grassman variables**

We first review some facts and conventions about Grassman variables, see Appendix A for more details. General references are [35], [32].

We consider the Grassman algebra generated by $\Psi_\alpha(x), \bar{\Psi}_\alpha(x)$ where $(x, \alpha)$ are spacetime and spinor indices, $x \in \mathbb{T}_N^0$. Let $\xi$ stand for $(x, \alpha, \omega)$ with $\omega = (0, 1)$. Combine the two by defining

$$\Psi(\xi) = \begin{cases} 
\Psi_\alpha(x) & \xi = (x, \alpha, 0) \\
\bar{\Psi}_\alpha(x) & \xi = (x, \alpha, 1)
\end{cases}$$

Then the $\Psi(\xi)$ satisfy $\Psi(\xi)\Psi(\xi') = -\Psi(\xi')\Psi(\xi)$ and generate the algebra. Take some fixed ordering for the index set which is all such $\xi$. Any element of the algebra can be uniquely written as

$$F(\Psi) = \sum_{n=0}^{\infty} \sum_{\xi_1 < \ldots < \xi_n} F_n(\xi_1, \ldots, \xi_n) \Psi(\xi_1) \cdots \Psi(\xi_n)$$

If the kernel $F_n(\xi_1, \ldots, \xi_n)$ is extended to be an anti-symmetric function of unordered collections $(\xi_1, \ldots, \xi_n)$ then

$$F(\Psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\xi_1, \ldots, \xi_n} F_n(\xi_1, \ldots, \xi_n) \Psi(\xi_1) \cdots \Psi(\xi_n)$$

The size of the $F(\Psi)$ is measured by a norm of the kernel depending on a parameter $h > 0$ and defined by

$$\|F\|_h = \sum_{n=0}^{\infty} h^n \sum_{\xi_1 < \ldots < \xi_n} |F_n(\xi_1, \ldots, \xi_n)|$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} \sum_{\xi_1, \ldots, \xi_n} |F_n(\xi_1, \ldots, \xi_n)|$$
The integral of an element of the Grassman algebra is the projection onto the element of maximal degree which is identified with the complex numbers

$$\int F(\Psi)\,D\Psi = F_{\text{max}}$$  \hfill (34)

A Gaussian integral with non-singular covariance $\Gamma = D^{-1}$ is defined by

$$\int F(\Psi)d\mu_{\Gamma}(\Psi) = Z^{-1}\int F(\Psi)e^{-\langle \bar{\Psi},D\Psi \rangle}D\Psi$$

$$Z = \int e^{-\langle \bar{\Psi},D\Psi \rangle}D\Psi = \det D$$  \hfill (35)

In this formula $\Psi$ stands for the pair $\Psi_\alpha(x), \bar{\Psi}_\alpha(x)$ in $F(\Psi)$, but just $\Psi_\alpha(x)$ in $\langle \bar{\Psi},D\Psi \rangle$. This ambiguity shows up throughout the paper, but it should be clear from the context what is meant. If $\bar{J}_\alpha(x), J_\alpha(x)$ are additional Grassman variables the Gaussian measure can be characterized by

$$\int e^{\langle \bar{J},\Psi \rangle + \langle J,\bar{\Psi} \rangle}d\mu_{\Gamma}(\Psi) = e^{\langle \bar{J},\Gamma J \rangle}$$  \hfill (36)

### 2.2 block averages

Let $A$ be a background gauge field and let $\Psi$ be a spinor valued function on a unit lattice $T^0_N$ or a generator of the Grassman algebra. We define a covariant block averaging operator $Q(A)$ taking $\Psi$ to $Q(A)\Psi$ defined on the $L$-lattice $T^1_N$. In any square lattice let $B(y)$ be a cube with $L$ sites on a side centered on a point $y$. Here on the lattice $T^0_N$ for $y \in T^1_N$ we have

$$B(y) = \{ x \in T^0_N : |x - y| < L/2 \}$$  \hfill (37)

The $B(y)$ partition the lattice. For $x \in B(y)$ let $\pi$ be a permutation of $(1,2,3)$ and let $\Gamma^\pi(y,x)$ be that path from $y$ to $x$ obtained by varying each coordinate to its final value in the order $\pi$. There are $3!$ of these. For any path $\Gamma$ let $A(\Gamma) = \sum_{b \in \Gamma} A(b)$ and define an average over the various paths from $y$ to $x$ by

$$(\tau A)(y,x) = \frac{1}{3!} \sum_{\pi} A(\Gamma^\pi(y,x))$$  \hfill (38)

Then we define the averaging operator

$$(Q(A)\Psi)(y) = L^{-3} \sum_{x \in B(y)} e^{ie_0(\tau A)(y,x)}\Psi(x) \quad y \in T^1_N$$  \hfill (39)

and similarly on $\bar{\Psi}$.

The definition is covariant under symmetries of the lattice $T^1_N$. In particular if $r$ is a rotation by a multiple of $\pi/2$ or a reflection $Q(A_r)\Psi= (Q(A)\Psi)_r$  \hfill (40)

It is also is constructed to be gauge covariant:

$$Q(A^\lambda)\Psi^\lambda = (Q(A)\Psi)^{\lambda^{(1)}}$$  \hfill (41)

where $\lambda^{(1)}$ is $\lambda$ restricted to the lattice $T^1_N$. For the conjugate field we need to take $Q(-A)\bar{\Psi}$ to preserve the covariance.

The transpose operator $Q^T(A) \equiv (Q(A))^T$ maps functions $\Psi$ on $T^1_N$ to functions on $T^0_N$. It is computed with sums on $T^0_N$ weighted by $L^d$ and is given by

$$(Q^T(A)\Psi)(x) = e^{ie_0(\tau A)(y,x)}\Psi(y) \quad x \in B(y)$$  \hfill (42)
Then we have
\[ Q(A)Q^T(-A) = I \] (43)
while
\[ P(A) = Q^T(-A)Q(A) \] (44)
is a projection which satisfies \( P^T(A) = P(-A) \).

### 2.3 the transformation

Suppose we start with a density \( \rho(A, \psi) \) with fermion field \( \psi \) and background gauge field \( A \) on \( \mathbb{T}^{0-N} \). It scales up to a density
\[ \rho_0(A, \Psi) \equiv \rho_{L^N}(A, \Psi_0) \equiv \rho(A_{L-N}\Psi_{0,L-N}) \] (45)
where \( A, \Psi_0 \) are defined on \( \mathbb{T}^N_0 \). Starting with \( \rho_0(A, \Psi_0) \) we create a sequence of densities \( \rho_k(A, \Psi_k) \) defined for \( A \) on \( \mathbb{T}^{N-k}_k \) and \( \Psi_k \) on \( \mathbb{T}^N_{N-k} \). They are defined recursively first by

\[ \hat{\rho}_{k+1}(A, \Psi_{k+1}) = \int \delta_G\left( \Psi_{k+1} - Q(A)\Psi_k \right) \rho_k(A, \Psi_k) D\Psi_k \] (46)

where \( \Psi_{k+1} \) are new Grassman variables defined on the coarser lattice \( \mathbb{T}^{N-k}_N \). The \( \delta_G \) is a Gaussian approximation to the delta function. For a constant \( b = \mathcal{O}(1) \) it is defined by

\[ \delta_G\left( \Psi_{k+1} - Q(A)\Psi_k \right) = N_k \exp\left( -\frac{b}{L} \left( \Psi_{k+1} - Q(-A)\Psi_k, \Psi_{k+1} - Q(A)\Psi_k \right) \right) \] (47)

Here \( N_k = (bL^2)^{-4sN_{N-k}} \) where \( s_N = L^3N \) is the number of sites in a 3 dimensional lattice with \( L^N \) sites on a side. Also \( \langle \Psi_{k+1}, \Psi_{k+1} \rangle = \sum_x L^3 \Psi_{k+1}(x)\Psi_{k+1}(x) \), etc. The averaging operator \( Q(A) \) is taken to be a modification of \( \langle \Psi_{k+1}, \Psi_{k+1} \rangle \):

\[ (Q(A)\Psi_k)(y) = L^{-3} \sum_{x \in B(y)} e^{i\varepsilon_k\eta(\tau A)(y,x)} \Psi_k(x) \] (48)

Here \( (\tau A)(y,x) \) is still defined by \( \langle \Psi_{k+1}, \Psi_{k+1} \rangle \), but now in \( \mathcal{A}(\Gamma) \) the sum is over bonds of length \( \eta = L^{-k} \), hence the weighting factor \( \eta \) in the exponent. The normalization factor \( N_k \) is chosen so that

\[ \int D\Psi_{k+1} \delta_G\left( \Psi_{k+1} - Q(A)\Psi_k \right) = 1. \]

Therefore

\[ \int \hat{\rho}_{k+1}(A, \Psi_{k+1}) D\Psi_{k+1} = \int \rho_k(A, \Psi_k) D\Psi_k \] (49)

Next one scales back to the unit lattice. If \( A \) is a field on \( \mathbb{T}^{N-k-1}_{N-k-1} \) and \( \Psi_{k+1} \) is a field on \( \mathbb{T}^{N-k}_{N-k-1} \) then then

\[ A_L(b) = L^{-1/2}A(L^{-1}b) \quad \Psi_{k+1,L}(x) = L^{-1}\Psi_{k+1}(L^{-1}x) \] (50)

are fields on \( \mathbb{T}^{N-k}_k \) and \( \mathbb{T}^{N-k}_N \) respectively, and we define

\[ \rho_{k+1}(A, \Psi_{k+1}) = \hat{\rho}_{k+1}(A_L, \Psi_{k+1,L})L^{-8(s_N-s_{N-k-1})} \] (51)

If \( \Psi_{k+1} = \Psi_{k+1,L} \) we have by \( \langle \Psi_{k+1,L}, \Psi_{k+1,L} \rangle \)

\[ \int \rho_{k+1}(A, \Psi_{k+1,L}) D\Psi_{k+1} = L^{-8(s_N-s_{N-k-1})} \int \hat{\rho}_{k+1}(A_L, \Psi_{k+1,L}) D\Psi_{k+1} \]

\[ = L^{-8s_N} \int \hat{\rho}_{k+1}(A_L, \Psi_{k+1}'_{k+1}) D\Psi_{k+1}' = L^{-8s_N} \int \rho_k(A_L, \Psi_k) D\Psi_k \] (52)
Hence it is true for $k = 0$; suppose it is true for $k$. If $\psi = \psi'_L$ then $D\psi = L^{8N} D\psi'$ and so by (52)

$$
\int \rho_k(A, \Psi_{k+1}) D\Psi_{k+1} = L^{-8N} \int \rho_k(A_L, \Psi_k) D\Psi_k
$$

(54)

$$
= L^{-8N} \int \rho_0(A_{Lk+1}, \psi_{Lk}) D\psi = \int \rho_0(A_{Lk+1}, \psi'_{Lk+1}) D\psi'
$$

Hence it is true for $k + 1$.

For $k = N$ (53) says that for $A$ on $\mathbb{T}_0^{-N}$ and $\Psi_N$ on $\mathbb{T}_0^0$

$$
\int \rho_N(A, \Psi_N) D\Psi_N = \int \rho(A, \psi) D\psi
$$

(55)

where the integral is over $\psi \in \mathbb{T}_0^{-N}$. We are back to the original integral. The right side is the integral over a space with an unbounded number of dimensions, but can be computed as the left side which is the integral over a low dimensional space. This is the point of the renormalization group approach.

### 2.4 composition of averaging operators

To investigate the sequence $\rho_k(A, \Psi_k)$ we first study how averaging operators compose. Define

$$
Q_k(A) = Q(A) \circ \cdots \circ Q(A) \quad (k \text{ times})
$$

(56)

This maps fields on $\mathbb{T}_0^{-N}$ to fields on $\mathbb{T}_0^0$. We assume here that $A$ is defined on $\mathbb{T}_0^{-N}$ and each $Q(A)$ has coupling constant $e_k$. The identity $Q_k(A)Q_k^T(-A) = I$ is satisfied and

$$
P_k(A) = Q_k^T(-A)Q_k(A)
$$

(57)

is a projection operator.

Later we will need an explicit representation for $Q_k(A)$. In general let $B_k(y)$ be a cube with $L^k$ sites on a side centered on $y$. Suppose $x \in \mathbb{T}_0^{-N}$ and $y \in \mathbb{T}_0^{-N}$ satisfy $x \in B_k(y)$, which is the same as $|x - y| < \frac{1}{2}$. There is an associated sequence $x = y_0, y_1, y_2, \ldots y_k = y$ such that $y_j \in \mathbb{T}_0^{-N}$ and $x \in B_j(y_j)$. Define

$$
(\tau_k A)(y, x) = \sum_{j=0}^{k-1} (\tau A)(y_{j+1}, y_j)
$$

(58)

Then one can show (29) that for $A, \psi$ on $\mathbb{T}_0^{-N}$ and $\Psi$ on $\mathbb{T}_0^0$

$$
(Q_k(A)\psi)(y) = \int_{|x - y| < \frac{1}{2}} e^{i\pi \eta(\tau_k A)(y, x)} \psi(x) \, dx
$$

(59)

$$
(Q_k^T A\Psi)(x) = e^{i\pi \eta(\tau_k A)(y, x)} \Psi(y) \quad x \in B_k(y)
$$

Next we show that the individual RG transformations compose to give a transformation with averaging operator $Q_k(A)$:
Lemma 1. For $A, \psi$ on $\mathbb{T}^{N-k}_R$ and $\Psi_k$ on $\mathbb{T}^0_{N-k}$ the density $\rho_{k,A}(\Psi_k)$ can be written

$$\rho_k(A, \Psi_k) = N_k \int \exp \left( -b_k \left( \hat{\Psi} - Q_k(-A)\bar{\psi}, \Psi_k - Q_k(A)\psi \right) \right) \rho_0(A_{L_k}, \psi_{L_k}) \ D\psi$$

where

$$b_{k+1} = \frac{bb_k}{b_k + a} \quad \text{or} \quad b_k = b \left( \frac{1 - L^{-1}}{1 - L^{-k}} \right)$$

and $N_k = b^{-4s_N-k}_k$ is a normalizing constant.

Proof. It holds for $k = 1$ by [40, 51] at $k = 0$. Suppose it is true for $k$. Then for $k + 1$ we have

$$\hat{\rho}_{k+1}(A, \Psi_{k+1}) = \text{const} \int \exp \left( -\frac{b}{L} \left( \hat{\Psi}_{k+1} - Q(-A)\hat{\Psi}_k, \Psi_{k+1} - Q(A)\Psi_k \right) \right)$$

$$\exp \left( -b_k \left( \hat{\Psi}_k - Q_k(-A)\bar{\psi}, \Psi_k - Q_k(A)\psi \right) \right) \rho_0(A_{L_k}, \psi_{L_k}) D\Psi_k D\psi$$

We evaluate the integral by expanding around the critical point of the quantity in the exponential in $\Psi_k$. To find the critical point we treat the fields as real-valued rather than elements of a Grassman algebra. Taking derivatives in $\bar{\Psi}$ we have

$$-bL^{-1}Q^T(-A)\left( \Psi_{k+1} - Q(A)\Psi_k \right) + b_k \left( \Psi_k^{\text{crit}} - Q_k(A)\psi \right) = 0$$

$$-bL^{-1}Q^T(-A)\left( \bar{\Psi}_k - Q(-A)\bar{\psi} \right) + b_k \left( \bar{\Psi}_k^{\text{crit}} - Q(-A)\bar{\psi} \right) = 0$$

The first equation can be rewritten as

$$\left( b_k + bL^{-1}P(A) \right) \Psi_k^{\text{crit}} = bL^{-1}Q^T(-A)\Psi_{k+1} + b_k Q_k(A)\psi$$

But

$$\left( b_k + bL^{-1}P(A) \right)^{-1} = b_k^{-1} (1 - P(A)) + (b_k + bL^{-1})^{-1}P(A)$$

$$= b_k^{-1} + \left( (b_k + bL^{-1})^{-1} - b_k^{-1} \right) P(A)$$

$$= b_k^{-1} - \frac{bL^{-1}}{b_k + bL^{-1}} P(A)$$

We have $P(A)Q^T(-A) = Q^T(-A)$ so the $(1 - P(A))$ do not survive on $bL^{-1}Q^T(-A)\Psi_{k+1}$. Also $P(A)Q_k(A) = Q^T(-A)Q_{k+1}(A)$ so applying the inverse gives

$$\Psi_k^{\text{crit}} = \Psi_k^{\text{crit}}(\Psi_{k+1}, \psi)$$

$$\equiv Q_k(A)\psi + \frac{bL^{-1}}{b_k + bL^{-1}} Q^T(-A)\Psi_{k+1} - \frac{bL^{-1}}{b_k + bL^{-1}} Q_k^T(-A)Q_{k+1}(A)\psi$$

There is a similar expression for $\bar{\Psi}_k^{\text{crit}}$. We expand the quantity in the exponential around the critical point by $\Psi_k = \Psi_k^{\text{crit}} + W$ and $\bar{\Psi}_k = \bar{\Psi}_k^{\text{crit}} + W$ and then integrate over $W$ rather than $\Psi_k$. The linear term must vanish and the term quadratic in $W$ just contributes an overall constant. Thus we have

$$\hat{\rho}_{k+1}(A, \Psi_{k+1}) = \text{const} \int D\psi \rho_0(A_{L_k}, \psi_{L_k})$$

$$\exp \left( -bL^{-1} \left( \hat{\Psi}_{k+1} - Q(-A)\Psi_k^{\text{crit}}, \Psi_{k+1} - Q(A)\Psi_k^{\text{crit}} \right) \right) - b_k \left( \Psi_k^{\text{crit}} - Q_k(-A)\bar{\psi}, \Psi_k^{\text{crit}} - Q_k(A)\psi \right)$$

(67)
However
\[ Q(-A)\bar{\psi}^\text{crit}_k = Q_{k+1}(A)\psi_k + \frac{bL^{-1}}{b_k + bL^{-1}}\Psi_{k+1} - \frac{bL^{-1}}{b_k + bL^{-1}}Q_{k+1}(A)\psi_k \]  \tag{68}

and so
\[ \Psi_{k+1} - Q(-A)\bar{\psi}^\text{crit}_k = \left(1 - \frac{bL^{-1}}{b_k + bL^{-1}}\right)(\Psi_{k+1} - Q_{k+1}(A)\psi_k) = \frac{b_k}{b_k + bL^{-1}}(\Psi_{k+1} - Q_{k+1}(A)\psi_k) \]  \tag{69}

also
\[ \Psi^\text{crit}_k - Q_k(A)\psi = \frac{bL^{-1}}{b_k + bL^{-1}}Q^T(-A)(\Psi_{k+1} - Q_{k+1}(A)\psi) \]  \tag{70}

Combining these and using
\[ bL^{-1}\left(\frac{b_k}{b_k + bL^{-1}}\right)^2 + b_k\left(\frac{bL^{-1}}{b_k + bL^{-1}}\right)^2 = \frac{b_kbL^{-1}}{b_k + bL^{-1}} = b_{k+1}L^{-1} \]  \tag{71}

we have
\[ bL^{-1}\left(\bar{\psi}_{k+1} - Q(-A)\bar{\psi}^\text{crit}_k, \Psi_{k+1} - Q(A)\Psi^\text{crit}_k\right) + b_k\left(\bar{\psi}^\text{crit}_k - Q(-A)\bar{\psi}_k, \Psi^\text{crit}_k - Q_k(A)\psi\right) = b_{k+1}L^{-1}\left(\bar{\psi}_{k+1} - Q_{k+1}(-A)\bar{\psi}_k, \Psi_{k+1} - Q_{k+1}(A)\psi\right) \]  \tag{72}

Make this substitution in \(67\). Then scale with \(A \to \mathcal{A}_L, \psi_{k+1} \to \Psi_{k+1}L, \bar{\psi} \to \psi_L\) and use \(Q_{k+1}(A_L)\psi_L = (Q_{k+1}(A)\psi)L\) (now with coupling constant \(e_{k+1}\)). We obtain
\[ \rho_{k+1}(A, \Psi_{k+1}) = \text{const}\int \exp\left(-b_{k+1}\left(\bar{\psi}_{k+1} - Q_{k+1}(-A)\bar{\psi}_k, \Psi_{k+1} - Q_{k+1}(A)\psi\right)\right)\rho_0(\mathcal{A}_L, \psi_{L+k})D\psi \]  \tag{73}

But the constant must be \(\mathcal{N}_{k+1}\) in order that the identity \(63\) hold, so we have the result for \(k + 1\).

### 2.5 free flow

Now consider an initial density which is a perturbation of the free fermion action:
\[ \rho_0(A, \Psi_0) = F_0(\Psi_0)\exp\left(-\left\langle \bar{\psi}, (\mathcal{D}_A + \bar{m}_0)\psi\right\rangle\right) \]  \tag{74}

Insert this in \(60\) and use for \(A, \psi\) on \(\mathcal{T}_{N-k}^{-k}\)
\[ \left\langle \bar{\psi}_{L+k}, (\mathcal{D}_{A_{L+k}} + \bar{m}_0)\psi_{L+k}\right\rangle = \left\langle \bar{\psi}, (\mathcal{D}_A + \bar{m}_k)\psi\right\rangle \]  \tag{75}

where now \(\mathcal{D}_A\) is defined with coupling constant \(e_k\). Then with \(F_{0,L-k}(\psi) = F_0(\psi_{L+k})\) we have from \(60\)
\[ \rho_k(A, \Psi_k) = \mathcal{N}_k \int F_{0,L-k}(\psi) \exp\left(-b_k\left(\bar{\psi}_k - Q_k(-A)\bar{\psi}_k, \Psi_k - Q_k(A)\psi\right) - \left\langle \bar{\psi}, (\mathcal{D}_A + \bar{m}_k)\psi\right\rangle\right)D\psi \]  \tag{76}

We expand around the critical point in \(\psi\) for the expression in the exponential. The critical point satisfies the equations
\[ b_kQ^T_k(-A)\left(\Psi_k - Q_k(A)\psi^\text{crit}\right) - (\mathcal{D}_A + \bar{m}_k)\psi^\text{crit} = 0 \]
\[ b_kQ^T_k(A)\left(\Psi_k - Q_k(-A)\psi^\text{crit}\right) - (\mathcal{D}_A + \bar{m}_k)^T\psi^\text{crit} = 0 \]  \tag{77}
These are solved by $\psi^{\text{crit}} = \psi_k(A)$ and $\bar{\psi}^{\text{crit}} = \bar{\psi}_k(A)$ where on $\mathbb{T}^N_{N-k}$:

$$
\psi_k(A) = \psi_k(A, \Psi_k) \equiv b_k S_k(A) Q^T_k(-A) \Psi_k \\
\bar{\psi}_k(A) = \bar{\psi}_k(A, \bar{\Psi}_k) \equiv b_k S^T_k(A) Q_k^T(A) \bar{\Psi}_k
$$

(78)

and the propagator (Green’s function) is defined as:

$$
S_k(A) = \left( \mathcal{D}_A + \bar{m}_k + b_k P_k(A) \right)^{-1}
$$

This pair of equations (78) is abbreviated as

$$
\psi_k(A) = H_k(A) \Psi_k
$$

(80)

We expand around the critical point introducing by $\psi = \psi_k(A) + \mathcal{W}$ and $\bar{\psi} = \bar{\psi}_k(A) + \bar{\mathcal{W}}$ and integrating over new Grassman variables $\mathcal{W}, \bar{\mathcal{W}}$ instead of $\psi, \bar{\psi}$. The linear term must vanish and we have

$$
b_k \left\langle \bar{\Psi}_k - Q_k(-A) \bar{\psi}_k(A), \Psi_k - Q_k(A) \psi_k(A) \right\rangle + \left\langle \bar{\psi}, (\mathcal{D}_A + \bar{m}_k) \psi \right\rangle \\
= \mathcal{S}_k(A, \Psi_k, \psi_k(A)) + \left\langle \bar{\mathcal{W}}, \left( \mathcal{D}_A + \bar{m}_k + b_k P_k(A) \right) \mathcal{W} \right\rangle
$$

(81)

where

$$
\mathcal{S}_k(A, \Psi_k, \psi_k(A)) = b_k \left\langle \bar{\Psi}_k - Q_k(-A) \bar{\psi}_k(A), \Psi_k - Q_k(A) \psi_k(A) \right\rangle + \left\langle \bar{\psi}_k(A), (\mathcal{D}_A + \bar{m}_k) \psi_k(A) \right\rangle \\
= \left\langle \bar{\Psi}_k, \left[ b_k - b^2_k Q_k(A) S_k(A) Q_k^T(-A) \right] \Psi_k \right\rangle \\
= \left\langle \bar{\Psi}_k, D_k(A) \Psi_k \right\rangle
$$

(82)

Our expression becomes

$$
\rho_k(A, \Psi_k) = N_k Z_k(A) F_k(\psi_k(A)) \exp \left( - \mathcal{S}_k(A, \Psi_k, \psi_k(A)) \right)
$$

(83)

where

$$
F_k(\psi) = Z_k(A)^{-1} \int F_{0,L,-k}(\psi + \mathcal{W}) \exp \left( - \left\langle \bar{\mathcal{W}}, \left( \mathcal{D}_A + \bar{m}_k + b_k P_k(A) \right) \mathcal{W} \right\rangle \right) D\mathcal{W}
$$

$$
Z_k(A) = \int \exp \left( - \left\langle \bar{\mathcal{W}}, \left( \mathcal{D}_A + \bar{m}_k + b_k P_k(A) \right) \mathcal{W} \right\rangle \right) D\mathcal{Z} = \det(S_k(A))^{-1}
$$

(84)

### 2.6 the next step

If we start with the expression (83) for $\rho_k$ and apply another renormalization transformation we again get $\rho_{k+1}$. Working out the details will give us some useful identities. We have first

$$
\hat{\rho}_{k+1}(A, \Psi_{k+1}) = N_k N_k Z_k(A) \\
\int \exp \left( - b L^{-1} \left\langle \bar{\Psi}_{k+1} - Q(-A) \bar{\Psi}_k, \Psi_{k+1} - Q(A) \Psi_k \right\rangle - \mathcal{S}_k(A, \Psi_k, \psi_k(A)) \right) F_k(\psi_k(A)) \: D\Psi_k
$$

(85)

To evaluate the integral we want expand around the critical point for this quadratic form in the exponential. Using the representation (82) one can argue as in the previous section the critical point is

$$
\Psi^{\text{crit}}_k = b L^{-1} \Gamma_k(A) Q^T(-A) \Psi_{k+1} \\
\bar{\Psi}^{\text{crit}}_k = b L^{-1} \bar{\Gamma}^T_k(A) Q^T(A) \bar{\Psi}_{k+1}
$$

(86)
where
\[ \Gamma_k(A) = \left( D_k(A) + b L^{-1} P(A) \right)^{-1} \]  
(87)

However it is useful to find another expression for these critical points.

We define on \( T_{N-k} \) the operator
\[ S_{k+1}^0(A) = \left( \mathcal{D}_A + \bar{m}_k + L^{-1} b_{k+1} P_{k+1}(A) \right)^{-1} \]  
(88)

and the field
\[ \psi_{k+1}^0(A) = L^{-1} b_{k+1} S_{k+1}^0(A) Q^T_{k+1}(-A) \Psi_{k+1} \]  
(89)

These scale to \( S_{k+1}(A), \psi_{k+1}(A) \) on \( T_{-N-k-1} \).

**Lemma 2.**

\[ \Psi_k^{\text{crit}}(A) = \Psi^*(\Psi_{k+1}, \psi_{k+1}^0(A)) \]  
(90)

\[ \psi_{k+1}^0(A) = \psi_k(A, \Psi_k^{\text{crit}}(A)) \]

**Proof.** These are identities between operators and it suffices to treat the fields as real-valued functions rather than elements of the Grassman algebra.

The quadratic form in the exponential in (83) is \( J(\Psi_{k+1}, \Psi_k, \psi_k(A, \Psi_k)) \) is

\[ J(\Psi_{k+1}, \Psi_k, \psi) = b L^{-1} \left\{ \Psi_{k+1} - Q(-A) \Psi_k, \Psi_{k+1} - Q(A) \Psi_k \right\} \]

\[ + b_k \left\{ \Psi_k - Q(-A) \bar{\psi}, \Psi_k - Q(A) \psi \right\} + \left( \bar{\psi}_k, (\mathcal{D}_A + \bar{m}_k) \psi \right) \]  
(91)

The critical point \( \Psi_k^{\text{crit}}(A) \) is the unique solution of \( \partial / \partial \Psi_k(\Psi_{k+1}, \Psi_k, \psi_k(A, \Psi_k)) \) \( = 0 \) or

\[ \frac{\partial J}{\partial \Psi_k(x)} \left( \Psi_{k+1}, \Psi_k, \psi_k(A, \Psi_k) \right) + \int \mathcal{H}_k(A)(x,y) \frac{\partial J}{\partial \psi(y)} \left( \Psi_{k+1}, \Psi_k, \psi_k(A, \Psi_k) \right) dy = 0 \]  
(92)

To get at this we study the critical point of \( J(\Psi_{k+1}, \Psi_k, \psi) \) in both variables \( \Psi_k, \psi \). Taking derivatives in \( \Psi_k, \Psi_k, \psi, \bar{\psi} \) gives the equations for the critical point of \( J(\Psi_{k+1}, \Psi_k, \psi) \)

\[ \frac{\partial J}{\partial \Psi_k} = - b L^{-1} Q^T(-A) \left( \Psi_{k+1} - Q(A) \Psi_k' \right) + b_k \left( \Psi_k' - Q_k(A) \psi' \right) = 0 \]

\[ \frac{\partial J}{\partial \Psi_k} = - b L^{-1} Q^T(A) \left( \Psi_{k+1} - Q(-A) \Psi_k' \right) + b_k \left( \Psi_k' - Q_k(-A) \psi' \right) = 0 \]  
(93)

\[ \frac{\partial J}{\partial \psi} = - b Q^T_k(-A) \left( \Psi_k' - Q_k(A) \psi' \right) + (\mathcal{D}_A + \bar{m}_k) \psi' = 0 \]

\[ \frac{\partial J}{\partial \bar{\psi}} = - b Q^T_k(A) \left( \Psi_k' - Q_k(-A) \bar{\psi}' \right) - (\mathcal{D}_A + \bar{m}_k)^T \bar{\psi}' = 0 \]

We have seen these equations before, but now we have to solve then simultaneously. The first and third equations have the solutions

\[ \Psi_k' = \Psi^*(\Psi_{k+1}, \psi') \]

\[ \psi' = \psi_k(A, \Psi_k') \]  
(94)
Substitute the expression for \( \Psi'_k \) into the third equation which can be written

\[
(\mathcal{D}_A + \bar{m}_k + P_k(A))\psi' = b_k Q_k^T(A) \Psi'_k
\]  

(95)

We compute

\[
b_k Q_k^T(-A) \Psi'_k = b_k Q_k^T(-A) \left( Q_k(A) \psi' + \frac{bL^{-1}}{b_k + bL^{-1}} Q^T(-A) \Psi_{k+1} - \frac{bL^{-1}}{(b_k + bL^{-1})} Q^T(-A) Q_{k+1}(A) \psi \right)
\]

\[
= b_k P_k(A) \psi' + \frac{b b L^{-1}}{b_k + bL^{-1}} Q^T_{k+1}(-A) \Psi_{k+1} - \frac{b b L^{-1}}{(b_k + bL^{-1})} P_{k+1}(A) \psi'
\]  

(96)

Substitute this in (95) and use \( b_k b (b_k + bL^{-1})^{-1} = b_{k+1} \) to obtain

\[
(\mathcal{D}_A + \bar{m}_k + L^{-1} b_{k+1} P_{k+1}(A)) \psi' = L^{-1} b_{k+1} Q_{k+1}^T(-A) \Psi_{k+1}
\]  

(97)

which has the solution \( \psi' = \psi_{k+1}^0(A) \). With this identification the identities (94) become

\[
\Psi'_k = \Psi^*(\Psi_{k+1}, \psi_{k+1}^0(A))
\]

\[
\psi_{k+1}^0(A) = \psi_k(A, \Psi'_k)
\]  

(98)

Since \( \psi' = \psi_k(A, \Psi'_k) \) the first and third equations in (93) now read

\[
\frac{\partial J}{\partial \Psi_k} (\Psi_{k+1}, \psi_k(A, \Psi'_k)) = 0 \quad \frac{\partial J}{\partial \psi_k} (\Psi_{k+1}, \psi_k(A, \Psi'_k)) = 0
\]  

(99)

Thus \( \Psi_k = \Psi'_k \) solves the equation which which identifies \( \Psi^\text{crit}_k(A) = \Psi'_k \). With this the identities (98) become the identities (99) of the theorem. This completes the proof.

We expand around the critical point by

\[
\Psi_k = \Psi^\text{crit}_k(A) + W \quad \bar{\Psi}_k = \bar{\Psi}^\text{crit}_k(A) + \bar{W}
\]  

(100)

This also entails

\[
\psi_k(A) = \psi_{k+1}^0(A) + W_k(A) \quad \bar{\psi}_k(A) = \bar{\psi}_{k+1}^0(A) + \bar{W}_k(A)
\]  

(101)

where

\[
W_k(A) = \psi_k(A, W) = \mathcal{H}_k(A) W
\]  

(102)

We introduce

\[
\mathcal{S}^0_{k+1}(A, \Psi_{k+1}, \psi)
\]

\[
= b_{k+1} L^{-1} \langle \Psi_{k+1} - Q_{k+1}(-A) \psi, \Psi_{k+1} - Q_{k+1}(A) \psi \rangle + \langle \psi, (\mathcal{D}_A + \bar{m}_k) \psi \rangle
\]  

(103)

which scales to \( \mathcal{S}_{k+1}(A, \Psi_{k+1}, \psi) \)

**Lemma 3.** Under the transformation (100), (101) the quadratic form

\[
bL^{-1} \langle \Psi_{k+1} - Q(-A) \bar{\Psi}_k, \Psi_{k+1} - Q(A) \Psi_k \rangle + \mathcal{S}_k(A, \Psi_k, \psi_k(A))
\]  

(104)

becomes

\[
\mathcal{S}^0_{k+1}(A, \Psi_{k+1}, \psi_{k+1}^0(A)) + \langle W, \left( D_k(A) + bL^{-1} P(A) \right) W \rangle
\]  

(105)

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Proof. Since we are at the critical point the cross terms vanish. The terms in \( \tilde{\Psi}_k^{\text{crit}}(A) \) and \( \tilde{\psi}_k(A) \) are identified using (82) as

\[
bl^{-1} \langle \tilde{\Psi}_{k+1} - Q(-A) \tilde{\Psi}_k^{\text{crit}}(A), \Psi_{k+1} - Q(A) \Psi_k^{\text{crit}}(A) \rangle + \mathcal{S}_k \left( \mathcal{A}, \tilde{\Psi}_k^{\text{crit}}(A), \psi_{k+1}^0(\mathcal{A}) \right)
\]

(106)

The terms in \( W, \mathcal{W}_k(A) \) are identified using (82) as

\[
bl^{-1} \langle Q(-A)W, Q(A)W \rangle + \mathcal{S}_k \left( \mathcal{A}, W, \mathcal{W}_k(A) \right) = \langle \tilde{W}, \left( \mathcal{D}_k(\mathcal{A}) + bl^{-1}P(\mathcal{A}) \right)W \rangle
\]

(107)

This completes the proof.

Now in (85) we make the change of variables and integrate over the new Grassman variables \( \tilde{W}, \tilde{W} \) instead of \( \tilde{\Psi}_k, \tilde{\psi}_k \). This gives

\[
\tilde{\rho}_{k+1}(A, \Psi_{k+1}) = N_k N_k Z_k(A) \exp \left( -\mathcal{S}_k^{0+1} \left( A, \Psi_{k+1}, \psi_{k+1}^0(\mathcal{A}) \right) \right)
\]

\[
\int \exp \left( -\left( \tilde{W}, \left( \mathcal{D}_k(\mathcal{A}) + bl^{-1}P(\mathcal{A}) \right)W \right) \right) F_k \left( \psi_{k+1}^0(\mathcal{A}) + \mathcal{W}_k(\mathcal{A}) \right) DW
\]

(108)

Next identify the Gaussian integral \( \int [\cdots] d\mu_{\Gamma_k(A)} \) where

\[
d\mu_{\Gamma_k(A)}(W) = Z_k(A)^{-1} \exp \left( -\left( \tilde{W}, \left( \mathcal{D}_k(\mathcal{A}) + bl^{-1}P(\mathcal{A}) \right)W \right) \right) DW \]

(109)

\[
\delta Z_k(A) = \int \exp \left( -\left( \tilde{W}, \left( \mathcal{D}_k(\mathcal{A}) + bl^{-1}P(\mathcal{A}) \right)W \right) \right) DW = \det(\Gamma_k(A))^{-1}
\]

Then (108) is rewritten as

\[
\tilde{\rho}_{k+1}(A, \Psi_k) = N_k N_k Z_k(A) \delta Z_k(A) \exp \left( -\mathcal{S}_k^{0+1} \left( A, \Psi_{k+1}, \psi_{k+1}^0(\mathcal{A}) \right) \right)
\]

\[
\int F_k \left( \psi_{k+1}^0(\mathcal{A}) + \mathcal{W}_k(\mathcal{A}) \right) d\mu_{\Gamma_k(A)}(W)
\]

(110)

Next we scale by (81). Using \( \psi_{k+1}^0(A_L, \Psi_{k+1, L}) = [\psi_{k+1}(A)]_L \) and

\[
\mathcal{S}_k^{0+1} \left( A_L, \Psi_{k+1, L}, [\psi_{k+1}(A)]_L \right) = \mathcal{S}_k^{0+1} \left( A, \Psi_{k+1}, \psi_{k+1}(A) \right)
\]

(111)

we have

\[
\rho_{k+1}(A, \Psi_{k+1}) = N_k N_k Z_k(A_L) \delta Z_k(A_L) L^{-8(sN - sN - k - 1)} \exp \left( -\mathcal{S}_k^{0+1} \left( A, \Psi_{k+1, L}, \psi_{k+1}(A) \right) \right) \int F_k \left( [\psi_{k+1}(A)]_L + \mathcal{W}_k(A_L) \right) d\mu_{\Gamma_k(A_L)}(W)
\]

(112)

Comparing this expression with (83) for \( k + 1 \) we find that

\[
N_k Z_k(A) = N_k Z_k(A_L) L^{-8(sN - sN - k - 1)}
\]

(113)

and that

\[
F_k \left( \psi_{k+1}(A) \right) = \int F_k \left( [\psi_{k+1}(A)]_L + \mathcal{W}_k(A_L) \right) d\mu_{\Gamma_k(A)}(W)
\]

(114)

The latter is fluctuation integral of a type we investigate in detail for special \( F_k \).
We also note that $\Gamma_k(\mathcal{A})$ has the alternate representation:

$$
\Gamma_k(\mathcal{A}) = B_k(\mathcal{A}) + \tilde{b}_k^2 B_k(\mathcal{A}) Q_k(\mathcal{A}) S^0_{k+1}(\mathcal{A}) Q^T_k (-\mathcal{A}) B_k(\mathcal{A})
$$

(115)

where $B_k(\mathcal{A})$ is the operator \[^{65}\] and

$$
S^0_{k+1}(\mathcal{A}) = \left( \mathcal{D}_\mathcal{A} + \bar{m}_k + b_{k+1} L^{-1} P_{k+1}(\mathcal{A}) \right)^{-1}
$$

(116)

is the operator on $T_{-k}^N$, which scales to $S_{k+1}(\mathcal{A})$ on $T_{-k-1}^N$. See the Appendix \[^{13}\] with $y = 0$ for derivation of this representation.

2.7 propagators

We study the propagator

$$
S_k(\mathcal{A}) = \left( \mathcal{D}_\mathcal{A} + \bar{m}_k + b_k P_k(\mathcal{A}) \right)^{-1}
$$

(117)

an operator on functions on $T_{-k}^N$ defined for a background field $\mathcal{A}$ on $T_{-k}^N$. We first list some general properties.

(a.) With gauge transformation $\lambda$ on $T_{-k}^N$ defined as in \[^{15}\] we have

$$
\mathcal{D}_\mathcal{A}^\lambda \psi^\lambda = (\mathcal{D}_\mathcal{A} \psi)^\lambda \quad Q_k(\mathcal{A}^\lambda) \psi^\lambda = (Q_k(\mathcal{A}) \psi)^{\lambda(0)}
$$

(118)

where $\lambda^{(0)}$ is the restriction to the unit lattice $T_0^N$. It follows that

$$
S_k(\mathcal{A}^\lambda) \psi^\lambda = (S_k(\mathcal{A}) \psi)^\lambda \quad \mathcal{H}_k(\mathcal{A}^\lambda) \Psi_k^{\lambda(0)} = (\mathcal{H}_k(\mathcal{A}) \Psi_k)^\lambda
$$

(119)

Under charge conjugation

$$
C^{-1} S_k(-\mathcal{A}) C = S_k^T(\mathcal{A}) \quad C \mathcal{H}_k(\mathcal{A}) \Psi_k = \mathcal{H}_k(-\mathcal{A}) C \Psi_k
$$

(120)

and if $\mathcal{A}$ is real

$$
(\gamma_3 C)^{-1} S_k(-\mathcal{A}) (\gamma_3 C) = \overline{S_k(\mathcal{A})}
$$

(121)

(b.) At some points we will want to change the background field on $T_{-k}^N$ from $\mathcal{A} + \mathcal{Z}$ to a background field $\mathcal{A}$ by

$$
S_k(\mathcal{A}) - S_k(\mathcal{A} + \mathcal{Z}) = S_k(\mathcal{A} + \mathcal{Z}) V_k(\mathcal{A}, \mathcal{Z}) S_k(\mathcal{A})
$$

(122)

where

$$
V_k(\mathcal{A}, \mathcal{Z}) \equiv \left( \mathcal{D}_{\mathcal{A} + \mathcal{Z}} + \bar{m}_k + b_k P_k(\mathcal{A} + \mathcal{Z}) \right) - \left( \mathcal{D}_\mathcal{A} + \bar{m}_k + b_k P_k(\mathcal{A}) \right)
$$

(123)

We have explicitly with $\eta = L^{-k}$

$$
(V_k(\mathcal{A}, \mathcal{Z}) f)(x) = -\sum_\mu \left( \frac{1 - \gamma_\mu}{2} \right) e^{i x_\mu A(x, x + \eta \nu)} \left( \frac{e^{i x_\mu A(x, x + \eta \nu) \eta} - 1}{\eta} \right) f(x + \eta \nu)
$$

$$
-\sum_\mu \left( \frac{1 + \gamma_\mu}{2} \right) e^{i x_\mu A(x, x - \eta \nu)} \left( \frac{e^{i x_\mu A(x, x - \eta \nu) \eta} - 1}{\eta} \right) f(x - \eta \nu)
$$

$$
+ b_k \left( P_k(\mathcal{A} + \mathcal{Z}) - P_k(\mathcal{A}) \right)
$$

(124)

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(c.) Now some comments about boundary conditions. The propagator \( S_k(A) \) on the torus \( \mathbb{T}_{N-k}^{k} \) is the same as the propagator on the cube \([−\frac{1}{2}L^{N-k}, \frac{1}{2}L^{N-k}]^3\) with periodic boundary conditions. In the latter form it can be obtained from propagator \( S_{k,\eta\mathbb{Z}^3}(A) \) on \( \eta\mathbb{Z}^3 \) (defined with a periodic extension of \( A \)) by the statement that the kernels satisfy

\[
S_k(A, x, y) = \sum_{n \in \mathbb{Z}^3} S_{k,\eta\mathbb{Z}^3}(A, x, y + nL^{N-k})
\]

For anti-periodic boundary conditions this is modified to

\[
S_k(A, x, y) = \sum_{n \in \mathbb{Z}^3} (-1)^{n_1 + n_2} S_{k,\eta\mathbb{Z}^3}(A, x, y + nL^{N-k})
\]

This satisfies for all \( x, y \)

\[
S_k(A, x, y + L^{N-k}e_\mu) = -S_k(A, x, y) = S_k(A, x + L^{N-k}e_\mu, y)
\]

Hence when restricted to the cube it satisfies the anti-periodic boundary conditions in either variable.

### 2.8 local propagators

We develop a random walk expansion for \( S_k(A) \) following Balaban, O’Carroll, and Schor [21]. The first step is to find local inverses for the Dirac operator.

Partition the lattice \( \mathbb{T}_{N-k}^k \) into large cubes \( \square \) of linear size \( M = L^m \) centered on points in \( \mathbb{T}_{N-k}^m \) for some integer \( m > 1 \). Let \( \hat{\square} \) be cubes of linear size \( 3M \) centered on the same points, and more generally let \( \square^n \) be the cubes of linear size \( (2n + 1)M \) centered on the same points.

We seek local propagators \( S_k(\square, A) \) localized near \( \hat{\square} \) with the property that for \( x \in \hat{\square} \)

\[
\left( (\mathcal{D}A + \bar{m} + b_k P_k(A)) S_k(\square, A) f \right)(x) = f(x)
\]

We also want bounds on \( S_k(\square, A) \) and a certain Holder derivative \( \delta_{\alpha,A}S_k(\square, A) \). The Holder derivative for \( 0 < \alpha < 1 \) is defined for \( 0 < |x - y| < 1 \) by

\[
(\delta_{\alpha,A} f)(x, y) = \frac{e^{ik_s \eta A(\Gamma_{xy})} f(y) - f(x)}{|x - y|^{\alpha}}
\]

where \( \Gamma_{xy} \) is one of the standard paths from \( x \) to \( y \). Or one can replace \( A(\Gamma_{xy}) \) by the average over paths \( (\tau A)(x,y) \). There is an associated norm

\[
\|\delta_{\alpha,A} f\|_{\infty} = \sup_{0 < |x - y| < 1} |(\delta_{\alpha,A} f)(x, y)|
\]

**Lemma 4.** Let \( e_k \) be sufficiently small depending on \( L, M \). Let \( A \) on \( \hat{\square}^5 \) be real-valued and gauge equivalent to a field satisfying \(|A| < e_k^{-1+\epsilon} \) for some small positive constant \( \epsilon \). Then there is an operator \( S_k(\square,A) \) on functions on \( \hat{\square}^5 \) satisfying (128) and

\[
|S_k(\square,A)f|, \|\delta_{\alpha,A}S_k(\square,A)f\|_{\infty} \leq C\|f\|_{\infty}
\]

Furthermore let \( \Delta_y, \Delta_{y'} \) be unit squares centered on unit lattice points \( y, y' \in \hat{\square}^5 \), let \( \tilde{\Delta}_y \) be the enlargement of \( \Delta_y \) by a layer of unit cubes, and let \( \zeta_y \) be a smooth partition on unity with \( \text{supp} \zeta_y \subset \tilde{\Delta}_y \). Then

\[
|1_{\Delta_y}S_k(\square,A)1_{\Delta_y'} f|, \|\delta_{\alpha,A} \zeta_y S_k(\square,A)1_{\Delta_{y'}} f\|_{\infty} \leq Ce^{-\gamma d(y,y')}\|f\|_{\infty}
\]

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The condition on $A$ is satisfied if, for example, $|\partial A| < e_k^{-1+2\epsilon}$. Indeed by subtracting a constant the field $A$ is gauge equivalent to a field satisfying

$$|A| \leq \mathcal{O}(1) \text{ diameter } (\overline{\mathcal{D}}^5) ||\partial A||_\infty \leq \mathcal{O}(1) M e_k^{-1+2\epsilon} \leq e_k^{-1+\epsilon}$$

(133)

With a little more work this can be established with only a bound on the field strength $|dA| < e_k^{-1+2\epsilon}$.

For the proof of the lemma see [21], [24]. A candidate for $S_k(\Box,A)$ would be to restrict the operator $\mathcal{D}_A + \bar{m}_k + b_k P_k(A)$ to a neighborhood on $\Box$, say with Dirichlet boundary conditions, and then invert it. This would satisfy (128) but it is difficult to get good estimates. Instead one uses a rather complicated construction involving soft boundary conditions implemented by a multi-scale random walk expansion.

The previous result is extended to complex fields $A$ on $\mathbb{T}_{N-k}$ of the form

$$A = A_0 + A_1$$

$A_0$ is real and on each $\overline{\mathcal{D}}^5$ is gauge equivalent to a field satisfying $|A_0| < e_k^{-1+\epsilon}$, and $A_1$ is complex and satisfies $|A_1| < e_k^{-1+\epsilon}$.

(134)

Then $S_k(\Box,A)$ has an analytic extension to the region $\mathfrak{R}(134)$, and for such fields $S_k(\Box,A)$ again satisfies bounds of the form (131), (132). This follows by expanding $S_k(\Box,A) = S_k(\Box,A_0 + A_1)$ around $A_1 = 0$ and using the bounds for $S_k(\Box,A_0)$.

### 2.9 random walk expansion

The random walk expansion is based on the operators $S_k(\Box,A)$ just discussed. We assume that $A$ is in the domain $\mathfrak{R}(134)$ so that these have good estimates by lemma 2. Let $h_\Box$ be a partition of unity with $\sum_{\Box} h_\Box^2 = 1$ and supp $h_\Box$ well inside $\Box$. We define a parametrix

$$S_k^*(A) = \sum_{\Box} h_{\Box} S_k(\Box,A) h_\Box$$

(135)

On supp $h_\Box$ the identity (128) is applicable and so

$$\left( \mathcal{D}_A + m_k + a_k P_k(A) \right) S_k^*(A) = I - \sum_{\Box} K_{\Box}(A) S_k(\Box,A) h_\Box \equiv I - K$$

(136)

where

$$K_{\Box}(A) = - \left[ \left( \mathcal{D}_A + \bar{m}_k + a_k P_k(A) \right), h_\Box \right]$$

(137)

Then

$$S_k(A) = S_k^*(A)(I - K)^{-1} = S_k^*(A) \sum_{n=0}^{\infty} K^n$$

(138)

if the series converges. This can be written as the random walk expansion

$$S_k(A) = \sum_{\omega} S_{k,\omega}(A)$$

(139)

where a path $\omega$ is a sequence of cubes $\omega = (\Box_0, \Box_1, \ldots, \Box_n)$ in $\mathbb{T}_{N-k}$ such that $\Box_i, \Box_{i+1}$ are nearest neighbors (in a sup metric), and

$$S_{k,\omega}(A) = \left( h_{\Box_0} S_k(\Box_0,A) h_{\Box_0} \right) \left( K_{\Box_1}(A) S_k(\Box_1,A) h_{\Box_1} \right) \cdots \left( K_{\Box_n}(A) S_k(\Box_n,A) h_{\Box_n} \right)$$

(140)

Note that $S_{k,\omega}(A)$ only depends on $A$ in the set $\bigcup_{i=0}^n \mathcal{D}_i^5$.
Lemma 5. [21], [23] Let $M$ be sufficiently large (depending on $L$), and $\epsilon_k$ sufficiently small (depending on $L, M$), and let $A$ be in the domain $[174]$. Then the random walk expansion [18] for $S_k(A)$ converges to a function analytic in $A$ which satisfies

$$\|S_k(A)f\|, \|\delta_{\alpha,A}S_k(A)f\|_\infty \leq C\|f\|_\infty \tag{141}$$

Furthermore let $\Delta_y, \Delta_{y'}$ be unit squares centered on unit lattice points $y, y' \in \mathbb{T}_{N-k}$ and let $\zeta_y$ be a smooth partition on unity with $\text{supp } \zeta_y \subset \Delta_y$. Then

$$|1_{\Delta_y}S_k(A)1_{\Delta_{y'}}f|, \|\delta_{\alpha,A}\zeta_yS_k(A)1_{\Delta_{y'}}f\|_\infty \leq Ce^{-\gamma d(y,y')}\|f\|_\infty \tag{142}$$

Proof. We compute

$$\left(\mathcal{D}_A, h_\square \right)(x) = -\sum_\mu \left(\frac{1-\gamma \mu}{2}\right)(\partial h_\square)(x, x + \eta \epsilon_\mu)e^{i\epsilon_k \eta A(x,x+\eta \epsilon_\mu)}f(x + \eta \epsilon_\mu)$$

$$-\sum_\mu \left(\frac{1+\gamma \mu}{2}\right)(\partial h_\square)(x, x - \eta \epsilon_\mu)e^{i\epsilon_k \eta A(x,x-\eta \epsilon_\mu)}f(x - \eta \epsilon_\mu) \tag{143}$$

and with $x \in \Delta_y$

$$\left(\left[ P_k(A), h_\square \right] f \right)(x) = \int_{|x'-y|<\frac{1}{4}} e^{-i\epsilon_k \eta \gamma A(y,x)}e^{i\epsilon_k \eta A(y,x')}(h_\square(x') - h_\square(x))f(x')dx' \tag{144}$$

The functions $\{h_\square\}$ can be chosen so that $|\partial h_\square| \leq O(1)M^{-1}$. Then the representations (143), (144) lead to the bound

$$|K_\square(A)f| \leq O(1)M^{-1}\|f\|_\infty \tag{145}$$

and therefore by lemma 4

$$|K_\square(A)S_k(\square, A)f| \leq CM^{-1}\|f\|_\infty \tag{146}$$

These imply that

$$|S_k(\omega)(A)f| \leq C(CM^{-1})^\omega\|f\|_\infty \tag{147}$$

This is sufficient to establish the convergence of the expansion for $M$ sufficiently large, since the number of paths with a fixed length $n$ is bounded by $O(1)^n$. The bound on the Holder derivative follows as well.

For the local estimates use the locality of $K_\square(A)$ and (132) to obtain

$$|1_{\Delta_y}K_\square(A)S_k(A, \square)1_{\Delta_{y'}}f| \leq CM^{-1}e^{-\gamma d(y,y')}\|f\|_\infty \tag{148}$$

The decay factors combine to give an overall decay factor (with a smaller $\gamma$) and the result follows

Remarks. (1.) The same bounds (141), (142) also hold for $\mathcal{H}_k(A)$, for example

$$|\mathcal{H}_k(A)f|, \|\delta_{\alpha,A}\mathcal{H}_k(A)f\|_\infty \leq C\|f\|_\infty \tag{149}$$

We note that $S^0_k(A)$ also has a random walk expansion and satisfies the same bounds as $S_k(A)$. Then the representation (116) leads to the bound

$$|\Gamma_k(A)f| \leq C\|f\|_\infty \tag{150}$$
The operator $\mathcal{H}_k$ has a kernel $\mathcal{H}_k(\mathcal{A}, \xi, x)$ where $\xi = (x, \beta, \omega)$ with $x \in \mathbb{T}_{N-k}^-$ and $x = (x, \alpha, \omega)$ with $x \in \mathbb{T}_{N-k}^-$. The operator $\delta_{\alpha, A} \mathcal{H}_k$ has a kernel $(\delta_{\alpha, A} \mathcal{H}_k)(\mathcal{A}, \zeta, x)$ with $\zeta = (x, y, \beta, \omega)$ and $x, y \in \mathbb{T}_{N-k}^-$. We consider the $\ell^1 - \ell^\infty$ norms on the kernels

$$
\| \mathcal{H}_k(\mathcal{A}) \|_{1, \infty} = \sup_{\xi} \sum_x |\mathcal{H}_k(\mathcal{A}, \xi, x)| = \sup_{\xi, \|f\|_\infty \leq 1} |(\mathcal{H}_k(\mathcal{A})f)(\xi)| 
$$

$$
\| \delta_{\alpha, A} \mathcal{H}_k(\mathcal{A}) \|_{1, \infty} = \sup_{\zeta} \sum_x |\delta_{\alpha, A} \mathcal{H}_k(\mathcal{A}, \zeta, x)| = \sup_{\zeta, \|f\|_\infty \leq 1} |(\delta_{\alpha, A} \mathcal{H}_k(\mathcal{A})f)(\zeta)|
$$

(151)

The second form for the norms follows since on our finite measure space $\ell^\infty$ is the dual space to $\ell^1$. Then (149) implies

$$
\| \mathcal{H}_k(\mathcal{A}) \|_{1, \infty}, \| \delta_{\alpha, A} \mathcal{H}_k(\mathcal{A}) \|_{1, \infty} \leq C
$$

(152)

### 2.10 weakened propagators

The random walk expansion makes it possible to introduce weakened forms of the propagators. Note that if $|\omega| = 0$ then $S_{k, \omega}(\mathcal{A}) = S_k^0(\mathcal{A})$ and so

$$
S_k(s, \mathcal{A}) = S_k^0(\mathcal{A}) + \sum_{\omega:|\omega| \geq 1} S_{k, \omega}(\mathcal{A})
$$

(153)

For each $\omega = (\square_0, \square_1, \ldots, \square_n)$ with $n \geq 1$ define

$$
X_\omega = \bigcup_{i=1}^n \square_i^6
$$

(154)

We introduce weakening parameters $\{s_\square\}$ indexed by the $M$ cubes $\square$ with $0 \leq s_\square \leq 1$ and define

$$
s_\omega = \prod_{\square \subset X_\omega} s_\square
$$

(155)

Weakened propagators are defined by

$$
S_k(s, \mathcal{A}) = S_k^0(\mathcal{A}) + \sum_{\omega:|\omega| \geq 1} s_\omega S_{k, \omega}(\mathcal{A})
$$

(156)

The $S_k(s, \mathcal{A})$ interpolate between $S_k(\mathcal{A}) = S_k^0(1, \mathcal{A})$ and a strictly local operator $S_k(0, \mathcal{A})$. If $s_\square$ is small then the coupling through $\square$ is reduced. If $Y$ is a union of $M$ cubes and $s_\square = 0$ for $\square \subset Y^c$, then no path with $X_\omega$ intersecting $Y^c$ contributes to the sum (156). Then $S_k(s, \mathcal{A})$ only connects points in $Y$ and only depends on $\mathcal{A}$ in $Y$.

The bounds (141), (143) still hold for the weakened propagators $S_k(s, \mathcal{A})$, even if we allow $s_\square$ complex and rather large. In fact let $\alpha_0$ be a small parameter and take complex $s_\square$ satisfying

$$
|s_\square| \leq M^{\alpha_0}
$$

(157)

Then for $\alpha_0$ sufficiently small (independent of all parameters)

$$
|s_\omega| \leq \exp(\alpha_0 \log M |X_\omega|_M) \leq \exp(O(1)\alpha_0 \log M |\omega|) = M^{\frac{1}{2}|\omega|}
$$

(158)

Still assuming $M$ is sufficiently large this does not affect convergence of the random walk expansion which is driven by the factor $M^{-|\omega|}$.

The weakened propagator $S_k(s, \mathcal{A})$ has the analyticity, bounds, and symmetries of $S_k(\mathcal{A})$. The weakened propagator $S_k(s, \mathcal{A})$ also gives a weakened operator $\mathcal{H}_k(s, \mathcal{A})$ which satisfies the same bounds as $\mathcal{H}_k$. Similarly $S^0_{k+1}(\mathcal{A})$ weakens to $S^0_{k+1}(s, \mathcal{A})$ and $\Gamma_k(\mathcal{A})$ weakens to $\Gamma_k(s, \mathcal{A})$ with the same bounds.
3  RG transformations for gauge fields

For gauge fields we follow the analysis developed by Balaban [5, 6, 8], and Balaban, Imbrie, and Jaffe [18, 19]. The basic treatment is identical with [28, 29].

3.1  axial gauge

We are concerned with formal integrals over fields on $T_0^{-N}$ of the form

$$\int f(A) \exp \left(-\frac{1}{2} \|dA\|^2\right) DA$$

(159)

Scaling up by $L^N$ the integral is a constant times $\int \rho_0(A_0) DA_0$ where for $A_0$ on $T_0^{-N}$

$$\rho_0(A_0) = F_0(A_0) \exp \left(-\frac{1}{2} \|dA_0\|^2\right)$$

(160)

and $F_0(A_0) = f_L^{-N}(A_0) = f(A_{0L^{-N}})$. We seek control over the integral $\int \rho_0(A_0) DA_0$ with RG transformations, which at the same time supplies the gauge fixing necessary for convergence. Specifically we want to define a sequence of well-defined densities $\rho_0, \rho_1, \ldots, \rho_N$ where $\rho_k(A_k)$ is a function of $A_k$ on $T_0^{-N-k}$ and represents a partial integral of $\rho_0$.

Suppose $\rho_k$ is already defined. Then for $A_k$ on $T_0^{-N-k}$ define an averaged field on oriented bonds in $T_1^{-N-k}$ by (for reverse oriented bonds take minus this)

$$(QA)(y, y + Le_\mu) = \sum_{x \in B(y)} L^{-4} A(\Gamma_{x,x+Le_\mu})$$

(161)

where $\Gamma_{x,x+Le_\mu}$ is the straight line between the indicated points. First consider for $A_{k+1}$ on $T_1^{-N-k}$

$$\hat{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k) \rho_k(A_k) DA_k$$

(162)

This does not converge as it stands. For convergence we introduce an axial gauge fixing function (justified by a formal Fadeev-Popov argument)

$$\delta(\tau A_k) = \prod_{y \in T_1^{-N-k}} \prod_{x \in B(y), x \neq y} \delta\left((\tau A_k)(y, x)\right)$$

(163)

where $(\tau A_k)(y, x)$ is defined in [18]. Instead of (162) we define $\hat{\rho}_{k+1}(A_{k+1})$ for $A_{k+1}$ on $T_1^{-N-k}$ by

$$\hat{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k) \delta(\tau A_k) \rho_k(A_k) DA_k$$

(164)

Then we return to a unit lattice defining $\rho_{k+1}(A_{k+1})$ for $A_{k+1}$ on $T_0^{-N-k-1}$ by

$$\rho_{k+1}(A_{k+1}) = \hat{\rho}_{k+1}(A_{k+1}, L) L^{\frac{1}{2}(b_N - b_{N-k-1} - \frac{1}{2}(s_N - s_{N-k-1})}$$

(165)

Here $b_n = 3L^3N$ is the number of bonds in a three dimensional toroidal lattice with $L^N$ sites on a side, and again $s_N = L^3N$ is the number of sites.

The result of the iteration can be computed explicitly as

$$\rho_k(A_k) = \int \delta(A_k - QA_k) \delta(\tau_k A_k) \rho_{0, L^{-k}}(A_k) DA$$

$$= \int \delta(A_k - QA_k) \delta(\tau_k A_k) F_{0, L^{-k}}(A) \exp \left(-\frac{1}{2} \|dA\|^2\right) DA$$

(166)
where now $\mathcal{A}$ is defined on bonds in $\mathbb{T}_{N-k}^{-1}$ and the $k$-fold averaging operator is defined by $Q_k = Q \circ \cdots \circ Q$. Then $Q_k \mathcal{A}$ is given on oriented bonds in $\mathbb{T}_{N-k}^0$ by

$$
(Q_k \mathcal{A})(y, y + e_\mu) = \int_{|x-y| < \frac{1}{2}} L^{-k} \mathcal{A}(\Gamma_{x,x+e_\mu}) \, dx
$$

(167)

and the gauge fixing function is now

$$
\delta(\tau_k \mathcal{A}) = \prod_{j=0}^{k-1} \delta(\tau Q_j \mathcal{A})
$$

(168)

One can show that if $F_0$ is exponentially bounded then $\rho_k (A_k)$ is well-defined. Furthermore the final density $\rho_N$ is a constant given by

$$
\rho_N = \int \delta(Q_N \mathcal{A}) \delta(\tau_N \mathcal{A}) f(\mathcal{A}) \exp \left( -\frac{1}{2} \|d\mathcal{A}\|^2 \right) D\mathcal{A}
$$

(169)

This is a gauge fixed version of the original integral (159).

To evaluate $\rho_k(A_k)$ as given by (168) we expand around the minimum of $\|d\mathcal{A}\|^2$ subject to the constraints of the delta functions. We define $A_k^x = A_k^x(A_k)$ on $\mathbb{T}_{N-k}^{-1}$ by

$$
A_k^x \equiv H_k^x A_k = \text{minimizer of } \|d\mathcal{A}\|^2 \text{ subject to } Q_k \mathcal{A} = A_k, \tau_k \mathcal{A} = 0
$$

(170)

where the linear operator $H_k^x$ has a specific representation in terms of Green’s functions. Expanding around the minimizer by $A = A_k^x + Z$ the linear term vanishes and one finds

$$
\rho_k(A_k) = Z_k F_k(A_k^x) \exp \left( -\frac{1}{2} \|A_k^x\|^2 \right)
$$

(171)

where

$$
F_k(A_k^x) = Z_k^{-1} \int \delta(Q_k Z) \delta(\tau_k \mathcal{A}) F_0 L^{-k} (A_k^x + Z) \exp \left( -\frac{1}{2} \|dZ\|^2 \right) DZ
$$

$$
Z_k = \int \delta(Q_k Z) \delta(\tau_k \mathcal{A}) \exp \left( -\frac{1}{2} \|dZ\|^2 \right) DZ
$$

(172)

Next consider how one passes from the representation for $\rho_k$ to the representation for $\rho_{k+1}$. Suppose we are starting with the expression (171) for $\rho_k(A_k)$. In the next step generated by (164) we have

$$
\hat{\rho}_{k+1}(A_{k+1}) = Z_k \int \delta(A_{k+1} - QA_k) \delta(\tau A_k) F_k(A_k^x) \exp \left( -\frac{1}{2} \|dA_k^x\|^2 \right) DA_k
$$

(173)

Define the minimizer $A_k^{\text{min}} = A_k^{\text{min}}(A_{k+1})$ by

$$
A_k^{\text{min}} \equiv H_k^x A_{k+1} = \text{minimizer of } \|dA_k^x\|^2 \text{ in } A_k \text{ subject to } QA_k = A_{k+1}, \tau A_k = 0
$$

(174)

Expand around the minimizer by $A_k = A_k^{\text{min}} + Z$ and integrate over $Z$ instead of $A_k$ Then

$$
A_k^x = A_{k+1}^0 + Z_k
$$

(175)

where

$$
A_k^0 = H_k^0 A_k^{\text{min}} = H_k^x H_k^x A_{k+1}, \quad Z_k = H_k^x Z
$$

On the constrained subspace

$$
\frac{1}{2} \|A_k^x\|^2 = \frac{1}{2} \|dA_k^0\|^2 + \frac{1}{2} \|dZ_k\|^2
$$

(177)
and we also write with \( \delta = d^T \) on two-forms (functions on plaquettes)

\[
\|dZ_k\|^2 = \left\langle Z, \Delta_k Z \right\rangle
\]

where \( \Delta_k = \mathcal{H}_k^T \delta d\mathcal{H}_k \) (178)

We find

\[
\tilde{\rho}_{k+1} (A_{k+1}) = Z_k \exp \left( -\frac{1}{2} \|dA_{k+1}^{0,x}\|^2 \right) \int \delta(QZ) \delta(\tau Z) F_k \left( A_{k+1}^{0,x} + Z_k \right) \exp \left( -\frac{1}{2} \left\langle Z, \Delta_k Z \right\rangle \right) dZ (179)
\]

This scales to

\[
\rho_{k+1} (A_{k+1}) = L^{\frac{1}{2} (bn - bN - k - 1) - \frac{1}{2} (sN - sN - k - 1)} Z_k \exp \left( -\frac{1}{2} \|dA_{k+1}^{0,x}\|^2 \right) \int \delta(QZ) \delta(\tau Z) F_k \left( A_{k+1}^{0,x} (A_{k+1,L}) + Z_k \right) \exp \left( -\frac{1}{2} \left\langle Z, \Delta_k Z \right\rangle \right) dZ (180)
\]

Compare this with \( \rho_{k+1} (A_{k+1}) = Z_{k+1} \exp(-\frac{1}{2} \|dA_{k+1}^{0,x}\|^2) F_k (A_{k+1}^{0,x}) \) and we have the identifications (making special choices of \( F \))

\[
Z_{k+1} = Z_k \delta Z_k L^{\frac{1}{2} (bn - bN - k - 1) - \frac{1}{2} (sN - sN - k - 1)}
\]

\[
A_{k+1,L} = A_{k+1}^{0,x} (A_{k+1,L})
\]

\[
F_{k+1} (A_{k+1}^{0,x}) = \delta Z_k^{-1} \int \delta(QZ) \delta(\tau Z) F_k \left( A_{k+1,L} + Z_k \right) \exp \left( -\frac{1}{2} \left\langle Z, \Delta_k Z \right\rangle \right) dZ (181)
\]

\[
\delta Z_k = \int \delta(QZ) \delta(\tau Z) \exp \left( -\frac{1}{2} \left\langle Z, \Delta_k Z \right\rangle \right) dZ
\]

Note that if \( F_0 \) is gauge invariant then \( F_k \) is gauge invariant for any \( k \).

The fluctuation integral (181) can be parametrized as

\[
F_{k+1}(A) = \int F_k (A_L + H_k C \tilde{Z}) \exp \left( -\frac{1}{2} \left\langle C \tilde{Z}, \Delta_k C \tilde{Z} \right\rangle \right) d\mu_{C_k}(\tilde{Z}) \quad (182)
\]

where \( \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) \). The field \( \tilde{Z}_1 \) is defined on bonds within each block \( B(y) \) and satisfies \( \tilde{Z}_1 \in \ker \tau \). The field \( \tilde{Z}_2 \) is defined on bonds joining \( B(y), B(y') \) denoted \( B(y,y') \), but not the central bond on each face denoted \( b(y,y') \). The mapping \( Z = C \tilde{Z} \) is the identity on all bonds except the central bond and assigns a value to the central bond so that \( QZ = 0, \tau Z = 0 \). If we define \( C_k = (C^T \Delta_k C)^{-1} \) then the integral can be expressed with the Gaussian measure \( \mu_{C_k} \) with covariance \( C_k \) as

\[
F_{k+1}(A) = \int F_k (A_L + H_k C \tilde{Z}) d\mu_{C_k}(\tilde{Z}) \quad (183)
\]

Integrals of this type can be explicitly evaluated by choosing a basis for the functions \( \tilde{Z} \). We mention a particular class of bases which will be used in the following. For the \( \tilde{Z}_1 \) we take functions \( \{e^y_i\} \) on bonds in \( B(y) \) which are in \( \ker \tau \) and orthonormal with respect to the usual inner product. For the \( \tilde{Z}_2 \) we take delta functions \( \delta_b \) for \( b \in B(y,y') - b(y,y') \). Together they give an orthonormal basis

\[
\{Y_\alpha\} = \left( \bigcup_y \{e^y_i\} \right) \bigcup \left( \bigcup_{y,y'} \{\delta_b\} \right) \quad (184)
\]

For such a basis integrals over \( \tilde{Z} \) are evaluated by

\[
\int f(\tilde{Z}) d\mu_{C_k}(\tilde{Z}) = \int f \left( \sum_\alpha z_\alpha Y_\alpha \right) d\mu_{C_k}(z)
\]

\[
= (2\pi)^{-n/2} \det(\hat{C}_k^{-1})^{\frac{1}{2}} \int f \left( \sum_\alpha z_\alpha Y_\alpha \right) \exp \left( -\frac{1}{2} \left\langle z, \hat{C}_k^{-1} z \right\rangle \right) \prod_\alpha dz_\alpha \quad (185)
\]

where \( \hat{C}_{\alpha\beta} = C(Y_\alpha, Y_\beta) \) and \( n \) is the number of elements in the basis. The expression is independent of the basis.
3.2 Landau gauge

One also formulate the problem in the Landau gauge. Instead of axial gauge fixing one imposes that the divergence vanishes. Instead of (166) we define for $A_k$ on $T_{N-k}^0$ and $\mathcal{A}$ on $T_{N-k}^-$

$$\rho_k(A_k) = \text{const} \int \delta(A_k - QA)\delta(R_k\delta\mathcal{A})F_{0,L-k}(\mathcal{A}) \exp\left(-\frac{1}{2}||d\mathcal{A}||^2\right)D\mathcal{A}$$

(186)

Now $\delta = d^2$ on one-forms (functions on bonds) is the divergence operator and $\delta\mathcal{A}$ is a scalar. The operator $R_k$ is the projection onto the subspace $\Delta(\ker Q_k)$ where $Q_k$ is the averaging operator on scalars, and $\delta(R_k\delta\mathcal{A})$ denotes the delta function in this subspace. A Fadeev-Popov argument shows equivalence with the axial gauge expression.

Evaluation of such integrals depends on

$$A_k = \mathcal{H}_k A_k = \text{minimizer of } ||d\mathcal{A}||^2 \text{ subject to } QA = A_k \text{ and } R_k\delta\mathcal{A} = 0$$

(187)

There is an explicit expression for $\mathcal{H}_k$ in terms of Landau gauge Green’s functions defined for $a > 0$ by

$$\mathcal{G}_k = \left(\delta d + dR_k\delta + aQ_k^TQ_k\right)^{-1}$$

(188)

It is

$$\mathcal{H}_k = \mathcal{G}_k Q_k^T(Q_k\mathcal{G}_k Q_k^T)^{-1}$$

(189)

The minimizer $\mathcal{H}_k$ is gauge equivalent to the axial gauge minimizer in the sense that

$$\mathcal{H}_k = \mathcal{H}_k + \partial\mathcal{O}_k$$

(190)

for some operator $\mathcal{O}_k$. This means that in gauge invariant expression we can can replace $\mathcal{H}_k$ by $\mathcal{H}_k$. Hence we can make this replacement in the fluctuation integral (183), and in particular $||dA_k||^2$ can be replaced by $||dA_k||^2$ where $A_k = \mathcal{H}_k A_k$. This is useful because $\mathcal{H}_k$ is more regular than the axial $\mathcal{H}_x$. Also $\Delta_k$ can be expressed in the Landau gauge as

$$\langle Z, \Delta_k Z \rangle = ||d\mathcal{H}_k Z||^2$$

(191)

There is a bound below on $\Delta_k$. We have $Z = Q_k\mathcal{H}_k Z$, hence $dZ = Q_k^{(2)}d\mathcal{H}_k Z$ for a certain averaging operator $Q_k^{(2)}$ on two forms, and hence $||dZ||^2 \leq ||d\mathcal{H}_k Z||^2$. Furthermore Balaban [6] shows that there is a constant $c_0$ depending only on $L$ such that $||dZ||^2 \geq c_0||Z||^2$ on the constrained surface $QZ = 0, \tau Z = 0$. Thus on the same surface we have

$$\langle Z, \Delta_k Z \rangle \geq c_0||Z||^2$$

(192)

This shows that the fluctuation integrals of the previous section are well-defined.

3.3 random walk expansions

We quote some results on random walk expansions, essentially all due to Balaban.

3.3.1 expansion for $\mathcal{G}_k$

First consider the Green’s function $\mathcal{G}_k$. With the projection operator $P_k = I - R_k$ it can also be written

$$\mathcal{G}_k = (\Delta - dP_k\delta + aQ_k^TQ_k)^{-1}$$

(193)
Similarly define enlargements $\hat{M}$ of little torus. These satisfy the bounds (195).

Here we identify $\text{supp} \hat{\zeta}$ that

The first term is the identity operator. For the second term we write

Proof. We sketch the proof. We use a covering of $\mathbb{T}_{N-k}^n$ by cubes $\Box$ of width $2M$ centered on the points of the $M$ lattice $\mathbb{T}^n_{N-k}$. Let $\hat{\Box}$ be the union of all such cubes whose distance to $\Box$ is zero. Similarly define enlargements $\hat{\Box}^2, \hat{\Box}^3, \ldots$.

For each cube $\Box$ one introduces the inverse on the 3-fold enlargement $\hat{\Box}^3$:

$$\hat{G}_{k,\Box} = \left[ \Delta - dP_k \hat{\Box} + aQ_k^T Q_k \right]^{-1}$$

(196)

Here we take the inverse with periodic boundary conditions, so we are regarding the cube $\hat{\Box}^3$ as a little torus. These satisfy the bounds (195).

Take a partition of unity $\sum_{\Box} h^2_{\Box} = 1$ with $\text{supp} h_{\Box} \subset \Box$, and define a parametrix

$$\hat{G}_k = \sum_{\Box} h_{\Box} \hat{G}_{k,\Box} h_{\Box}$$

(197)

Here we identify $\text{supp} h_{\Box}$ with a subset of the torus $\hat{\Box}^3$ so that $h_{\Box} \hat{G}_{k,\Box} h_{\Box}$ can be regarded as an operator on the full torus $\mathbb{T}_{N-k}^n$. Let $D_k = \Delta - dP_k h_{\Box} + aQ_k^T Q_k$ and let $D_{k,\Box} = [\Delta - dP_k \hat{\Box} + aQ_k^T Q_k]_{\Box}^3$ so that $\hat{G}_k = D_k^{-1}$ and $\hat{G}_{k,\Box} = D_{k,\Box}^{-1}$. Then we compute

$$D_k \hat{G}_k = \sum_{\Box} h_{\Box} D_{k,\Box} \hat{G}_{k,\Box} h_{\Box} - \sum_{\Box} K_{\Box} \hat{G}_{k,\Box} h_{\Box}$$

(198)

The first term is the identity operator. For the second term we write

$$K_{\Box} = - \left[ \Delta + aQ_k^T Q_k, h_{\Box} \right] + \left( h_{\Box} (dP_k h_{\Box} - (dP_k h_{\Box}) h_{\Box} ) \right)$$

(199)

Let $\zeta_{\Box}$ be a smooth approximation to the characteristic function of $\Box$ with $\zeta_{\Box} = 1$ on $\Box$ and all points a distance $\frac{1}{2} M$ from $\Box$, and with $\zeta_{\Box} = 0$ on points greater than $\frac{3}{2} M$ from $\Box$. In the second term in $K_{\Box}$ multiply by $\zeta_{\Box} + (1 - \zeta_{\Box})$. Since $(1 - \zeta_{\Box}) h_{\Box} = 0$ this term can be written

$$\zeta_{\Box} \left( (dP_k h_{\Box} - (dP_k h_{\Box}) h_{\Box} ) - (1 - \zeta_{\Box}) (dP_k h_{\Box}) h_{\Box} \right)$$

(200)

All the terms in (199), (200) are well localized except the last and we insert here $1 = \sum_{\Box'} h^2_{\Box'}$. Only $\Box \neq \Box'$ contributes. Now we can write

$$D_k \hat{G}_k^* = I - K = I - \sum_{\Box' \neq \Box} K_{\Box'} \hat{G}_{k,\Box} h_{\Box}$$

(201)
where

\[
K_{\boxtimes,\boxdiamond} = \begin{cases} 
- \left[ \Delta + a Q_k^T Q_k, \ h_{\boxtimes} \right] + \zeta \left( (dP_k, \delta, h_{\boxtimes}) - d(P_k - P_{\boxtimes}) \delta h_{\boxtimes} \right) & \boxdiam = \boxtimes \\
- h_{\boxtimes}^2 (1 - \zeta) (dP_k, \delta) h_{\boxtimes} & \boxdiam' \neq \boxtimes 
\end{cases} (202)
\]

Then

\[
G_k = G_k^T(I - K)^{-1} = G_k^T \sum_{n=0}^{\infty} K^n = \sum_{\omega} G_{k,\omega} (203)
\]

where for a sequence \(\omega = (\square_0, \square_1, \square_2 \ldots, \square_{2n-1}, \square_{2n})\) of \(2M\) cubes

\[
G_{k,\omega} = \left( h_{\square_0}, g_{k,\square_0, h_{\square_0}} \right) \left( k_{\square_1}, g_{k,\square_1, h_{\square_1}} \right) \cdots \left( k_{\square_{2n-1}, \square_{2n}}, g_{k,\square_{2n}, h_{\square_{2n}}} \right) (204)
\]

Now we claim that

\[
|1_{\Delta_y} K_{\boxtimes,\boxdiam} G_{k,\omega} 1_{\Delta_y'} f| \leq C M^{-1} e^{-\gamma d(y, y') \|f\|_\infty} (205)
\]

For \(\boxtimes \neq \boxdiam'\) the first term \(\left[ \Delta + a Q_k^T Q_k, \ h_{\boxtimes} \right]\) in (202) is local and can be expressed in terms of derivatives of \(h\) which are \(O(M^{-1})\). Combined with the exponential decay for \(G_{k,\boxtimes}\) and its derivatives this gives a bound of the form (205). The other terms require some rather detailed knowledge about \(P_k, P_{k,\boxtimes}\) which are given by

\[
P_k = G_k Q_k^T N_k Q_k G_k \quad N_k = (Q_k^T G_k^2 Q_k)^{-1} (206)
\]

\[
P_{k,\boxtimes} = G_{k,\boxtimes} Q_k^T N_{k,\boxtimes} Q_k G_{k,\boxtimes} \quad N_{k,\boxtimes} = (Q_k^T G_{k,\boxtimes}^2 Q_k)^{-1}
\]

Here \(Q_k\) is the averaging operator on scalars and \(G_k = (\Delta + a Q_k^T Q_k)^{-1}\) on scalars. The operators \(G_{k,\boxtimes}, N_{k,\boxtimes}\) are defined on the torus \(\bigboxtimes\). Both \(G_k, G_{k,\boxtimes}\) satisfy bounds of the form \(|1_{\Delta_y} G_k 1_{\Delta_y'} f| \leq C e^{-\gamma d(y, y') \|f\|_\infty}\). The same is true for \(N_k, N_{k,\boxtimes}\) (pointwise bounds for these unit lattice operators) and hence for \(P_k, P_{k,\boxtimes}\). The term \([dP_k, \delta, h_{\boxtimes}]\) in (202) can be expressed in derivatives of \(h_{\boxtimes}\) and together with the bounds on \(P_{k,\boxtimes}\) yields a bound of the form (205). Furthermore both \(G_k, G_{k,\boxtimes}\) have random walk expansions based on the fundamental cubes \(\boxtimes\). The expansions are locally the same and differ only in the global topology. In particular the leading terms are the same and so when localized near \(\boxtimes\) the difference \(G_k - G_{k,\boxtimes}\) is \(O(M^{-1})\). The same is true for the pair \(N_k, N_{k,\boxtimes}\) and hence for \(P_k, P_{k,\boxtimes}\). This leads to a bound of the form (205) for the term \(\zeta \delta h_{\boxtimes} d(P_k - P_{k,\boxtimes})\) in (202). Finally for the term \(h_{\boxtimes}^2 (1 - \zeta) (dP_k, \delta) h_{\boxtimes}\) in (202) we use the fact that the supports of \((1 - \zeta)\) and \(h_{\boxtimes}\) are separated by \(\frac{1}{M}\). The exponential decay of \(P_k\) then gives a factor \(O(e^{-\frac{1}{2} \gamma M})\). Hence this term is \(O(M^{-1})\) and satisfies (205). All this is a rather long story for which we refer to Balaban [5, 16].

Now we write

\[
1_{\Delta_y} G_k 1_{\Delta_y'} = \sum_{\omega} \sum_{y_1, y_3 \ldots y_{2n-1}} 1_{\Delta_y} \left( h_{\square_0, g_{k,\square_0, h_{\square_0}}} \right) 1_{\Delta_y} \left( k_{\square_1, g_{k,\square_1, h_{\square_1}}} \right) \cdots 1_{\Delta_y} \left( k_{\square_{2n-1}, g_{k,\square_{2n}, h_{\square_{2n}}} \right) 1_{\Delta_y'} (207)
\]

The sums are restricted by the conditions \(y \in \square_0, y' \in \square_n\) and for \(i\) odd \(y_i \in \square_{i-1} \cap \square_i \neq \emptyset\). Then (205) gives the bound

\[
|1_{\Delta_y} G_k 1_{\Delta_y'} f| \leq \sum_{\omega} \sum_{y_1, y_3 \ldots y_{2n-1}} C (CM^{-1})^{\omega} \exp \left( - \gamma \left( d(y_1, y_1) + d(y_1, y_3) + \cdots + d(y_{2n-1}, y') \right) \right) \|f\|_\infty (208)
\]

\(^2\)Note that \(\delta h_{\boxtimes} = h_{\boxtimes} [\delta, h_{\boxtimes}]\) does not necessarily involve derivatives of \(h_{\boxtimes}\).
Split the exponent into thirds. In the first third we use \(d(y, y_1) + d(y_1, y_3) + \cdots + d(y_{2n-1}, y') \geq d(y, y')\). The second third is used for convergence of the sum over the \(y_i\). For the last third let \(z_i \in T_{N-k}^\alpha\) be the center of the cube \(\square_i\) which we could then label as \(\square_{z_i}\). We claim that for \(i\) odd
\[
d(y_i, y_{i+2}) \geq \frac{1}{3} d'(z_i, z_{i+2})
\] (209)
where \(d'(z, z')\) is the usual distance but set to zero if \(\square_{z_i} \cap \square_{z_i'}\) are neighbors, i.e. if \(d(z, z') \leq 2M\). Indeed if \(d(z_i, z_{i+2}) \geq 3M\) then \(d(y_i, y_{i+2}) \geq d(z_i, z_{i+2}) - 2M \geq \frac{1}{3} d(z_i, z_{i+2})\), while if \(d(z_i, z_{i+2}) \leq 2M\) the inequality is trivial. Similarly \(d(y_{2n-1}, y') \geq \frac{1}{3} d'(z_{2n-1}, z_{2n})\). Now write the sum over \(\omega\) as a sum over \(n\) and \(z_0, z_1, \ldots, z_{2n}\) with \(z_0, z_{2n}\) constrained by the conditions \(\square_{z_0} \ni y\) and \(\square_{z_{2n}} \ni y'\) and for \(i\) odd \(\square_{z_{i-1}} \cap \square_{z_i} \neq \emptyset\). We have then
\[
|1_{\Delta_y} \mathcal{G}_k 1_{\Delta_y'} f| \leq e^{-\frac{1}{3} \gamma d(y, y')} \sum_{n=0}^\infty \sum_{z_0, \ldots, z_{2n}} C(CM^{-1})^n \exp \left( -\frac{1}{9} \gamma \left( d'(z_1, z_3) + d'(z_3, z_5) + \cdots + d'(z_{2n-1}, z_{2n}) \right) \right) \|f\|_\infty
\] (210)
The sums over \(z_i\) for \(i\) even gives a factor \(O(1)^n\), the sum over \(z_i\) for \(i\) odd converges by the exponential decay, and the sum over \(n\) converges for \(M\) sufficiently large by the factor \(M^{-n}\). We obtain
\[
|1_{\Delta_y} \mathcal{G}_k 1_{\Delta_y'} f| \leq Ce^{-\frac{1}{3} \gamma d(y, y')} \|f\|_\infty
\] (211)
which is the first estimate in (199) with a new \(\gamma\). The derivatives are treated similarly. The bounds (194) follow by summing over \(y'\). The results stated in the lemma now follow.

However because of the presence of long jumps the expansion is not as local as we would like. The long jumps arise from the term \(K_{\square, \square} = h_\square^2(1 - c_\square)(dP_k \delta h_\square\delta)\) in the case \(\square' \neq \square\). The remedy is to insert the random walk expansion for \(P_k\). After some rearrangements one ends with the following result [8]. The expansion is not based just on cubes but on small connected unions of cubes \(X\), called localization domains. There is a constant \(r_0 = O(1)\) such that the number of \(M\) cubes in \(X\) satisfies \(|X|_M \leq r_0\). A walk is a sequence of localization domains \(X_0, X_1, \ldots, X_n\) with the property that \(X_i \cap X_{i+1} \neq \emptyset\). For each \(X\) there are operators \(R_\alpha(X)\) localized in \(X\) and indexed by \(\alpha\) which ranges over a finite set including 0. If \(\alpha = 0\) then the only localization domain possible is \(X = \square\) and
\[
R_0(\square) = h_\square \mathcal{G}_k h_\square
\] (212)
We have the bounds
\[
|1_{\Delta_y} R_\alpha(X) 1_{\Delta_y'} f|, |1_{\Delta_y} \partial R_\alpha(X) 1_{\Delta_y'} f|, \|\delta_y \zeta_\gamma \partial R_\alpha(X) 1_{\Delta_y'} f\|_\infty \leq C e^{-\gamma d(y, y')} \|f\|_\infty
\] (213)
\(\alpha = 0\)
\[
(CM^{-1} e^{-\gamma d(y, y')} \|f\|_\infty \alpha \neq 0
\]
The expansion has the form \(\mathcal{G}_k = \sum \mathcal{G}_{k, \omega}\) where the sum is over indexed walks
\[
\omega = \left( (0, X_0), (\alpha_1, X_1), \ldots, (\alpha_n, X_n) \right)
\] (214)
with \(\alpha_i \neq 0\) if \(i \neq 0\) and
\[
\mathcal{G}_{k, \omega} = R_0(X_0) R_{\alpha_1}(X_1) \cdots R_{\alpha_n}(X_n)
\] (215)
The walk can also be written with \(|\omega| = n\)
\[
\mathcal{G}_k = \sum_{\square} h_\square \mathcal{G}_k h_\square + \sum_{|\omega| \geq 1} \mathcal{G}_{k, \omega}
\] (216)
The results stated in the lemma also follow from this form of the random walk expansion.

Expansions of this form will be called generalized random walk expansions. We do not require that all possible combinations of the \((\alpha, X)\) actually occur in the sum over \(\omega\).

In our expansion we can introduce weakening parameters. Given a walk \(\omega\) of the form \((214)\) let \(X_\omega\) be the connected set \(X_\omega = \bigcup_{i=0}^{n} X_i\). Then \(G_{k,\omega}\) is localized in \(X_\omega\). For the weakened propagator we associate with each \(M\) cube \(\square\) a variable \(s_\square\) and define

\[
G_k(s) = \sum_{\square} h_{\square} G_{k,\square} h_{\square} + \sum_{|\omega| \geq 1} s_{\omega} G_{k,\omega} \quad s_\omega = \prod_{\square \subset X_\omega} s_{\square}
\] (217)

If \(Y\) is a union of \(M\) cubes and \(s_{\square} = 0\) for \(\square \subset Y^c\), then no path with \(X_\omega\) intersecting \(Y^c\) contributes to \(G_k(s)\) and so \(G_k(s)\) is localized in \(Y\).

**Lemma 7.** There is a small constant \(\alpha_0 = \mathcal{O}(1)\) such that for \(|s_{\square}| \leq M^{\alpha_0}\) the weakened propagator \(G_k(s)\) satisfies \((194), (195)\).

**Proof.** We have \(|X_i|_M \leq r_0 = \mathcal{O}(1)\) and so for \(|\omega| = n\)

\[
|X_\omega| \leq \sum_{i=0}^{n} |X_i|_M \leq r_0(|\omega| + 1)
\] (218)

Therefore if \(|s_{\square}| \leq M^{\alpha_0}\) and \(\alpha_0\) insufficently small

\[
|s_\omega| \leq \exp \left(\alpha_0 \log M|X_\omega|_M\right) \leq \exp \left(\alpha_0 r_0 \log M(|\omega| + 1)\right) \leq \mathcal{O}(1)M^{\frac{1}{2}|\omega|}
\] (219)

This changes the convergence factor \(M^{-|\omega|}\) to \(M^{-\frac{1}{2}|\omega|}\), but for \(M\) sufficiently large this is still small enough to guarantee convergence of the expansion.

**Remarks.**

1. The operator \(G_{k,\square}\) itself has a random walk expansion of the form of the lemma. The sums over \(\square\) would then be over the much smaller set \(\tilde{\square}^3\) rather than \(T_{N-k}\). The expansions are locally the same and only differ with the global topology. In particular the leading term \(G^*_k\) is the same in each case. It follows that say for \(y, y' \in \square\)

\[
|1_{\Delta^y}(G_k - G_{k,\square})1_{\Delta^y}f| \leq CM^{-1}e^{-\gamma d(y,y')}\|f\|_{\infty}
\] (220)

2. The operator \(G_{k+1}^0\) on \(T_{N-k-1}\) scales up to an operator \(G_{k+1}^0\) on \(T_{N-k-1}^k\). It is given for any \(a > 0\) by

\[
G_{k+1}^0 = \left(\delta d + dR_{k+1}^0 + aQ_{k+1}^T\right)^{-1}
\] (221)

Here \(R_{k+1}^0\) is the projection onto \(\Delta(\ker Q_{k+1})\). All the results stated for \(G_k\) hold for \(G_{k+1}^0\) as well.
3.3.2 expansion for \( N_k = (Q_k G_k Q_k^T)^{-1} \)

We consider the operators

\[
N_k \equiv (Q_k G_k Q_k^T)^{-1} \quad \text{on } \mathbb{T}_{N-k}^0
\]

\[
N_{k+1}^0 \equiv (Q_{k+1} G_{k+1}^0 Q_{k+1}^T)^{-1} \quad \text{on } \mathbb{T}_{N-k}^1
\]

(222)

**Lemma 8.** \([2], [3], [5]\) The operator \( N_k \) has a random walk expansion \( N_k = \sum_\omega N_k, \omega \) based on blocks of size \( M \), convergent for \( M \) sufficiently large which yields a bound

\[
|N_k(b, b')| \leq C e^{-\gamma d(b,b')}
\]

(223)

The same is true for \( N_{k+1}^0 \).

**Proof.** The proofs are essentially the same and we give the proof for \( N_{k+1}^0 \). First consider the operator \( Q_{k+1} G_{k+1}^0 Q_{k+1}^T \) defined on \( \hat{\mathbb{T}}^3 \) again regarded as a small torus. This is bounded above independent of the volume by our estimates on \( G_{k+1}^0 \). We claim that it is also bounded below. This relies on the identity for \( A \) on the \( L \)-lattice in \( \hat{\mathbb{T}}^3 \)

\[
\langle A, Q_{k+1} G_{k+1}^0 Q_{k+1}^T A \rangle = \langle [Q^T A]^{-1}, \tilde{C}_k, \square [Q^T A]^{-1} \rangle
\]

(224)

where \( \tilde{C}_k, \square \) is the inverse of \( \Delta_k + a Q^T Q \) on the subspace \( \text{ker} \tau \) of the unit lattice in \( \hat{\mathbb{T}}^3 \) and \( [Q^T F]^\square \) is the projection of \( Q^T F \) onto this subspace. For the proof on a general torus see \([6]\) or Appendix \([C]\)

Now \( \Delta_k + a Q^T Q \) is bounded above as a quadratic form and so \( C_k, \square \) is bounded below. Hence the right side of (224) is bounded below by constant times \( \| [Q^T A]^{-1} \|^2 \). However we show in appendix \([D]\) that \( \| [Q^T A]^{-1} \|^2 \geq L^{-1} \| A \|^2 \). Thus we have

\[
c\| A \|^2 \leq \langle A, Q_{k+1} G_{k+1}^0 Q_{k+1}^T A \rangle \leq C \| A \|^2
\]

(225)

Hence the inverse has the same bounds. But before we invert it we restrict from \( \hat{\mathbb{T}}^3 \) to smaller cube \( \square \), that is to functions on bonds with at least one end in \( \square \). The restriction satisfies the same bounds and if we define

\[
N_{k+1, \square}^0 = \left[ Q_{k+1} G_{k+1, \square}^0 Q_{k+1}^T \right]^{-1}
\]

(226)

then with new constants for \( A \) on \( \square \)

\[
c\| A \|^2 \leq \langle A, N_{k+1, \square}^0 A \rangle \leq C \| A \|^2
\]

(227)

Furthermore by \([15b]\) we have the bound \( |(Q_{k+1} G_{k+1, \square}^0 Q_{k+1}^T)(b, b')| \leq C e^{-\gamma d(b,b')} \). It follows by Balaban’s theorem on unit lattice operators (section 5 in \([6]\)) that the same is true for the inverse and so

\[
|N_{k+1}^0(b, b')| \leq C e^{-\gamma d(b,b')}
\]

(228)

To generate the random walk for \( N_k \) we again take the partition of unity \( \sum \square h\square = 1 \), identify \( h\square N_{k+1, \square}^0 h\square \) with an operator on \( \mathbb{T}_{N-k}^0 \) and define the parametrix

\[
N_{k+1}^* = \sum \square h\square N_{k+1, \square}^0 h\square
\]

(229)
Then we compute
\[ Q_{k+1}G^0_{k+1}Q^T_{k+1}N_{k+1} = \sum_{\Box} h_{\Box} \left[ 1_{\Box} Q_{k+1} G^0_{k+1,\Box} Q^T_{k+1} 1_{\Box} N_{k+1,\Box} \right] h_{\Box} \]
\[ - \sum_{\Box} \left[ h_{\Box}, 1_{\Box} Q_{k+1} G^0_{k+1,\Box} Q^T_{k+1} 1_{\Box} N_{k+1,\Box} h_{\Box} \right] + \sum_{\Box} 1_{\Box} Q_{k+1} \left( G^0_{k+1,\Box} - G^0_{k+1,\Box} \right) Q^T_{k+1} h_{\Box} N_{k+1,\Box} h_{\Box} \]
\[ = I - \sum_{\Box} K_{\Box,\Box} N_{k+1,\Box} h_{\Box} = I - K \]  
(230)

Here we have identified the first term as the identity. Now we have
\[ N_{k+1}^0 = N_{k+1}^*(I - K)^{-1} = N_{k+1}^* \sum_{n=0}^{\infty} K^n = \sum_{\omega} N_{k,\omega} \]
(231)

where for a sequence \( \omega = (\Box_0, \Box_1, \Box_2, \ldots, \Box_{2n-1}, \Box_{2n}) \)
\[ N_{k+1,\omega} = \left( h_{\Box_0} N_{k+1,\Box_0} h_{\Box_1} \right) \left( K_{\Box_1,\Box_2} N_{k+1,\Box_2} h_{\Box_3} \right) \cdots \left( K_{\Box_{2n-1},\Box_{2n}} N_{k+1,\Box_{2n}} h_{\Box_{2n}} \right) \]
(232)

Now we claim that
\[ |K_{\Box,\Box} N_{k+1,\Box} h(b, b')| \leq CM^{-1} e^{-\gamma d(b, b')} \]
(233)

This is argued as follows. The second term in (230) can be expressed in terms of derivatives of \( h_{\Box} \) and so is \( O(M^{-1}) \). For the third term in (230) we use that \( G^0_{k+1,\Box} - G^0_{k+1,\Box} \) satisfies a bound like (220) to get an estimate \( O(M^{-1}) \). For the fourth term in (230) we can assume that \( \Box' \) and \( \text{supp } h_{\Box} \) are separated by at least \( \frac{1}{2} M \). By the exponential decay of \( G_k \) this term has a decay factor \( O(e^{-\gamma M}) \) and hence \( O(M^{-1}) \) as well.

The estimate (233) gives convergence of the expansion and the stated estimates just as in the previous lemma. However again the long jumps are unwelcome. They come from the term \( K_{\Box,\Box} = h_{\Box_0} 1_{\Box} Q_{k+1} G^0_{k+1,\Box} Q^T_{k+1} h_{\Box} N_{k+1,\Box} h_{\Box} \) for \( \Box \neq \Box' \). The remedy in this case is to replace the \( G^0_{k+1} \) by its random walk expansion. Again the result is a generalized random walk expansion of the form
\[ N_{k+1}^0 = \sum_{\Box} h_{\Box} N_{k+1,\Box} h_{\Box} + \sum_{|\omega| \geq 1} N_{k+1,\omega} \]
(234)

where
\[ N_{k+1,\omega} = \sum R_0(X_0) R_{\alpha_1}(X_1) \cdots R_{\alpha_1}(X_1) \]
(235)

and for \( \alpha \neq 0 \), \( R_\alpha(X) \) satisfies
\[ |R_\alpha(X; b, b')| \leq CM^{-1} e^{-\gamma d(b, b')} \]
(236)

3.3.3 expansion for \( \mathcal{H}_k \)

From the representation \( \mathcal{H}_k = G_k Q_k^T \left( Q_k G_k Q_k^T \right)^{-1} \) and the last two results we have

**Lemma 9.** [3], [4], [5] The Landau gauge minimizer \( \mathcal{H}_k \) has a generalized random walk expansion based on blocks of size \( M \), convergent for \( M \) sufficiently large. This yields the bounds
\[ |\mathcal{H}_k f|, |\partial \mathcal{H}_k f|, \|\delta_\alpha \partial \mathcal{H}_k f\|_\infty \leq C \| f \|_\infty \]
(237)

The local version is for \( y, y' \) on \( T_{N-k}^N \)
\[ |1_{\Delta_y} \mathcal{H}_k 1_{\Delta_y'} f|, |1_{\Delta_y} \partial \mathcal{H}_k 1_{\Delta_y'} f|, \|\delta_\alpha \zeta_y \partial \mathcal{H}_k 1_{\Delta_y'} f\|_\infty \leq C e^{-\gamma d(y, y')} \| f \|_\infty \]
(238)
One can introduce weakening parameters as before and define \( \mathcal{H}_k(s) \). For \( |s| \leq M^{\alpha_0} \) and \( \alpha_0 \) sufficiently small these satisfy the same bounds as \( \mathcal{H}_k \).

### 3.3.4 expansion for \( \tilde{\mathcal{G}}_k \)

The Green’s function \( \mathcal{G}_k \) can be defined by stating that \( A = \mathcal{G}_kJ \) is the minimizer of the quadratic form \( \frac{1}{2} < A, (\delta d + dR_\delta + aQ_k^TQ_k)A > - < A, J > \). We are also interested in a modified Green’s function \( \tilde{\mathcal{G}}_k \) defined by stating that \( A = \tilde{\mathcal{G}}_kJ \) is the minimizer of the same quadratic form subject to the constraint \( Q_kA = 0 \) (in which case the term \( aQ_k^TQ_k \) is optional). These turn out to be related by

\[
\tilde{\mathcal{G}}_k = \mathcal{G}_k - G_kQ_k^TN_kQ_kG_k
\]  

(239)

On the small torus \( \mathbb{T}^3 \) this takes the form

\[
\tilde{\mathcal{G}}_{k,\square} = \mathcal{G}_{k,\square} - G_{k,\square}Q_k^TN_kQ_kG_{k,\square}
\]  

(240)

**Lemma 10.** The operator \( \tilde{\mathcal{G}}_k \) can has a generalized random walk expansion of the form

\[
\tilde{\mathcal{G}}_k = \sum_{\square} h_{\square}\tilde{\mathcal{G}}_{k,\square} + \sum_{\omega: |\omega| \geq 1} \tilde{\mathcal{G}}_{k,\omega}
\]  

(241)

This yields the bounds (195) for \( \tilde{\mathcal{G}}_k \). The second term in (241) is \( O(M^{-1}) \).

We have seen that the operators \( \mathcal{G}_k \) and \( N_k \) have a random walk expansions. We can insert these in the definition of \( \tilde{\mathcal{G}}_k \) and get a random walk expansion. However this would not have the leading term we want. We need to modify the procedure and we sketch the idea below. If we had supplied full details in the previous theorems we would already be familiar with this strategy taken from [8].

In (239) insert the expansion for \( \mathcal{G}_k \) in the first term and the expansion for \( N_k \) in the second term. This yields

\[
\tilde{\mathcal{G}}_k = \sum_{\square} h_{\square}\mathcal{G}_{k,\square} \mathcal{G}_k - \sum_{\square} \mathcal{G}_kQ_k^TN_kQ_k\mathcal{G}_{k,\square} + O(M^{-1})
\]  

(242)

where as before the sum is over a covering of the \( M \)-lattice by \( 2M \) cubes. For each fixed \( \square \) we introduce a new cover defined from the old cover by replacing \( \square \) by \( \square_0 \), then deleting all cubes contained in \( \square \), and leaving the other cubes in the old cover alone. Denote the cubes of the new cover by \( \square_0 \). Make a random walk expansion for \( \mathcal{G}_k \) based on the new cover. This will have the form \( \tilde{\mathcal{G}}_k = \sum_{\square_0} h_{\square_0}\mathcal{G}_{k,\square_0}h_{\square_0} + O(M^{-1}) \). When this is inserted in two places in the second term in (242) only the term with \( \square_0 \supset \square \) gives an \( O(1) \) contribution which was the goal. Using also \( h_{\square_0}Q_k^TQ_kh_{\square_0} = Q_k^TQ_kh_{\square_0} \) the second term in (242) can be written with \( \square_0 \supset \square \)

\[
\sum_{\square_0} h_{\square_0}\mathcal{G}_{k,\square_0}Q_k^TN_kQ_k\mathcal{G}_{k,\square_0}h_{\square_0} + O(M^{-1})
\]  

(243)

Next move the \( h_{\square_0} \) to the outside using \([h_{\square_0}, Q_k] = O(M^{-1}) \) and \([h_{\square_0}, \mathcal{G}_{k,\square_0}] = O(M^{-1}) \) and use \( h_{\square_0}h_{\square} = h_{\square} \). Then the last expression becomes

\[
\sum_{\square} h_{\square}\mathcal{G}_{k,\square}Q_k^TN_kQ_k\mathcal{G}_{k,\square}h_{\square} + O(M^{-1})
\]  

(244)

Now write

\[
h_{\square}\mathcal{G}_{k,\square} = h_{\square}\mathcal{G}_{k,\square} + h_{\square}(\mathcal{G}_{k,\square} - \mathcal{G}_{k,\square})\zeta_{\square} + h_{\square}(\mathcal{G}_{k,\square} - \mathcal{G}_{k,\square})(1 - \zeta_{\square})
\]  

(245)
The last term here is $O(e^{-\frac{1}{2} M}) = O(M^{-1})$ since $h_\Box$ is separated from $1 - \zeta_\Box$ by at least $\frac{1}{2} M$ and both $G_{k,\Box}$ and $G_{k,\Box}$ have exponential decay. The second term is localized near $\Box$ and $O(M^{-1})$ as in (220). Insert (245) in (244) and then back in (242). This gives the desired result

$$\tilde{G}_k = \sum_{\Box} h_\Box \left( G_{k,\Box} - G_{k,\Box} Q_k^T \right) h_\Box + O(M^{-1})$$

(246)

$\Box$

$3.3.5$ expansions for $C_k, C_k^\frac{1}{2}$

We also need a random walk expansion for $C_k = (C^T \Delta_k C)^{-1}$ and $C_k^\frac{1}{2}$. We have already noted that $C^T \Delta_k C$ is bounded below, and it is also bounded above by the bound on $\Delta_k$. Thus the same is true for the inverse:

\[ c\|\tilde{Z}\|^2 \leq \left\langle \tilde{Z}, C_k \tilde{Z} \right\rangle \leq C\|\tilde{Z}\|^2 \]

(247)

Here $\tilde{Z}$ is a restricted unit lattice variable defined as in section 3.1. We also consider $C_{k,\Box} = (C^T \Delta_{k,\Box} C)^{-1}$ defined on the small torus $\Box^3$ which satisfies

\[ c\|\tilde{Z}\|^2 \leq \left\langle \tilde{Z}, C_{k,\Box} \tilde{Z} \right\rangle \leq C\|\tilde{Z}\|^2 \]

(248)

**Lemma 11.** [6], [8], [12] The operators $C_k, C_k^\frac{1}{2}$ have generalized random walk expansions based on blocks of size $M$, convergent for $M$ sufficiently large. For $C_k$ it has the form

\[ C_k = \sum_{\Box} h_\Box C_k h_\Box + \sum_{\omega : |\omega| \geq 1} C_k,\omega \]

(249)

The expansions yield the bounds:

\[ |C_k(\Upsilon, \Upsilon')|, |C_k^\frac{1}{2}(\Upsilon, \Upsilon')| \leq C e^{-\gamma d(\Upsilon, \Upsilon')} \]

(250)

where $d(\Upsilon, \Upsilon') \equiv d(\text{supp} \Upsilon, \text{supp} \Upsilon')$ and $\Upsilon, \Upsilon'$ are taken from the basis $\{\Upsilon_\alpha\}$ defined in (184). The first term in (249) is bounded above and below (by (245)) and the second term in (246) is $O(M^{-1})$.

**Proof.** We give details for $C_k$. The expansion is based on the representation [8], [28].

\[ CC_k C^T = (1 + dM) Q_k \tilde{G}_{k+1} Q_k^T (1 + dM)^T \]

(251)

where

\[ \tilde{G}_{k+1} = G_{k+1}^0 - G_{k+1}^0 Q_{k+1}^T \nabla_{k+1}^0 Q_{k+1}^0 \]

(252)

The operator $\tilde{G}_{k+1}$ has a random walk expansion like $\tilde{G}_k$ and satisfies the same bounds (245). Also $\mathcal{M}$ maps one-forms to scalars by

\[ (\mathcal{M} Z)(x) = - (\tau Z)(y, x) + L^{-3} \sum_{x' \in B(y), x' \neq y} (\tau Z)(y, x') \]

(253)

Insert the expansion (241) for $\tilde{G}_{k+1}$ in (251) and obtain

\[ CC_k C^T = \sum_{\Box} (1 + dM) Q_k h_\Box \tilde{G}_{k+1} h_\Box Q_k^T (1 + dM)^T \]

\[ + \sum_{\omega : |\omega| \geq 1} (1 + dM) Q_\omega \tilde{G}_{k+1,\omega} Q_k^T (1 + dM)^T \]

(254)
In the first term move the $h_{\square}$ to the outside and identify
\[
CC_{k,\square} C^T = (1 + dM) Q_k \tilde{G}_{k+1,\square} Q_k^T (1 + dM)^T
\]  
(255)
to write it as as
\[
\sum_{\square} h_{\square} (CC_{k,\square} C^T) h_{\square} + \sum_{\square} L_{4,\square}
\]  
(256)
Here $L_4(\square)$ involves the commutators $[Q_k, h_{\square}] = \mathcal{O}(M^{-1})$ and $[dM, h_{\square}] = \mathcal{O}(M^{-1})$. Using these and the exponential decay for $\tilde{G}_{k+1,\square}$ one can show
\[
|L_{4,\square}(Y, Y')| \leq CM^{-1} e^{-\gamma d(Y, Y')}
\]  
(257)
Now in the first term in (256) move the $h_{\square}$ back inside and write it as
\[
C \left( \sum_{\square} h_{\square} C_{k,\square} h_{\square} \right) C^T + \sum_{\square} L_{5,\square}
\]  
(258)
The operator $L_{5,\square}$ involves the commutator $[C, h_{\square}] = \mathcal{O}(M^{-1})$ (see Appendix A in [29]) and also satisfies the bound (257). Now (254) becomes
\[
CC_k C^T = C \left( \sum_{\square} h_{\square} C_{k,\square} h_{\square} \right) C^T + \sum_{\square} (L_{4,\square} + L_{5,\square}) + \sum_{\omega | |\omega| \geq 1} (1 + dM) Q_k \tilde{G}_{k+1,\omega} Q_k^T (1 + dM)^T
\]  
(259)
The operator $C^T \Delta_k C$ has an exponentially decaying kernel also inherited from the bounds on $H_k$. By (a slight modification of) Balaban’s theorem on unit lattice operators (section 5 in [4]) the same is true for the inverse and so
\[
|C_{k,\square}(Y, Y')| \leq C e^{-\gamma d(Y, Y')}
\]  
(260)
The second and third terms satisfy the same bound. Hence $CC_k C^T$ has a random walk expansion and satisfies an exponential decay bound.

Now we argue that the results for $CC_k C^T$ imply the same for $C_k$. Recall that $C$ is a map on $V = V_1 \oplus V_2$ where $V_1$ is functions on bonds in $\cup_y B(y)$ in ker $\tau$ and $V_2$ is functions on bonds in $\cup_{y', y''} B(y, y') - b(y, y')$. And it maps to $V' = V \oplus V_3$ where $V_3$ is functions on the central bonds $b(y, y')$. As a map from $V$ to $V$ it is the identity. The transpose $C^T$ goes from $V'$ to $V$, but if we restrict it to functions on $V$ it is again the identity. (For $Z, \tilde{Z} \in V$ we have $< C^T Z, \tilde{Z} > = < Z, C \tilde{Z} > = < Z, Z >$.) Thus $CC_k C^T$ on $V \times V$ is the same as $C_k$ on $V \times V$. This gives the result for $C_k(Y, Y') = < Y, C_k Y' >$. (These remarks, which mirror similar observations in [8] render the discussion of $C^{-1}$ in [29] unnecessary.)

The result for $C^\dagger$ is based on the representation
\[
C_k^\dagger = \frac{1}{\pi} \int_0^\infty \frac{dx}{\sqrt{x}} (C^T \Delta_k C + x)^{-1} dx
\]  
(261)
and representation like (251) for $C(C^T \Delta_k C + x)^{-1} C^T$. For details see [8], [28].

### 3.3.6 expansion for $C_k(Y)$

We will also need an expansion for $C_k(Y) = [C^T \Delta_k C]_Y^{-1}$ where $Y$ is a union of $M$-cubes in $T_{N-k}^\infty$. The notation means we restrict to functions on bonds with at least one end in $Y$ before taking the inverse.

This is a special case of a multi-scale situation which arises discussed in detail in [8], [28]. We will make extensive use of it in [30], here we just sketch the situation. There one has a decreasing sequence
of regions $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_k)$ each $\Omega_j$ a union of $L^{-(k-j)}M$-cubes in $\mathbb{T}^{k}_{N-k}$, and associated with it an operator $\Delta_k, \Omega$. One is interested in

$$C_{k,\Omega}(\Omega_{k+1}) \equiv [T^T \Delta_{k,\Omega} T]_{\Omega_{k+1}}^{-1}$$

(262)

where $\Omega_{k+1} \subset \Omega_k$ is a union of $LM$ or $M$-cubes. For this operator one has a representation like (251) which has the form with $\Omega^+ = (\Omega, \Omega_{k+1})$

$$CC_{k,\Omega}(\Omega_{k+1}) T = \left[ (1 + dM)Q_k \tilde{\mathcal{G}}_{k+1,\Omega^+} Q_k^T (1 + dM)^T \right]_{\Omega_{k+1}}$$

(263)

where

$$\tilde{\mathcal{G}}_{k+1,\Omega^+} = \mathcal{G}_{k+1,\Omega^+}^0 - \mathcal{G}_{k+1,\Omega^+}^0 Q_{k+1}^T \left( Q_{k+1} \mathcal{G}_{k+1,\Omega^+} Q_{k+1}^T \right)^{-1} Q_{k+1} \mathcal{G}_{k+1,\Omega^+}^0$$

(264)

and where on $\mathbb{T}^{k}_{N-k}$

$$\mathcal{G}_{k+1,\Omega^+}^0 = \left( \delta d + dR_{k+1,\Omega^+} \delta + Q_{k+1,\Omega^+}^a Q_{k+1,\Omega^+}^{-1} \right)^{-1}$$

(265)

Both $\mathcal{G}_{k+1,\Omega^+}^0$ and $(Q_{k+1} \mathcal{G}_{k+1,\Omega^+} Q_{k+1}^T)^{-1}$ have (multiscale) random walk expansions and this generates expansions for $\tilde{\mathcal{G}}_{k+1,\Omega^+}$ and $C_{k,\Omega}(\Omega_{k+1})$ as in lemmas 10-11.

In the case at hand we have $\Omega_j = \mathbb{T}^{k}_{N-k}$ for $1 \leq j \leq k$ and $\Omega_{k+1} = \Omega$. Then $C_k(\Omega) = [T^T \Delta_k T]_\Omega^{-1}$ and the identity (263) reads

$$CC_k(\Omega) T = \left[ (1 + dM)Q_k \tilde{\mathcal{G}}_{k+1,\Omega^+} Q_k^T (1 + dM)^T \right]_{\Omega}$$

(266)

where $\tilde{\mathcal{G}}_{k+1,\Omega^+}$ is defined from $\mathcal{G}_{k+1,\Omega^+}^0$ as in (264) and the latter is an operator of the form

$$\mathcal{G}_{k+1,\Omega^+}^0 = \left( \delta d + dR_{k+1,\Omega^+} \delta + Q_{k+1,\Omega^+}^a Q_{k+1,\Omega^+}^{-1} \right)^{-1}$$

(267)

Also for $\square \subset \Omega$ we define on the small torus $\mathbb{T}^3$ the operator $C_{k,\square}(\Omega) = [T^T \Delta_k, \square T]_\Omega^{-1}$. This is bounded above and below and satisfies an identity like (266). Then we have the following variation of lemma 11.

**Lemma 12.** [9], [8], [12] The operators $C_k(\Omega)$ have generalized random walk expansions of the form

$$C_k(\Omega) = \sum_{\square} h_{\square} C_{k,\square}(\Omega) h_{\square} + \sum_{|\omega| \geq 1} C_{k,\omega}(\Omega)$$

(268)

The expansions yield the bounds:

$$|C_k(\Omega; \Omega', \Omega'')| \leq C e^{-\gamma d(\Omega, \Omega')}$$

(269)

The first term in (268) is bounded above and below, and the second term in (268) is $O(M^{-1})$. 

33
3.3.7 resummed random walk

The random walk expansions can be resummed so that any particular union of $M$-blocks can be treated as a unit [8], [12]. This can be done for any of the random walks discussed so far, but we discuss the details for $G_k$. Recall that the generalized random walk expansion can be written in the form

$$G_k = \sum_{\omega} \sum_{X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n} R_0(X_0) R_{\alpha_1}(X_1) \cdots R_{\alpha_n}(X_n)$$

(270)

The sum is restricted to sequences where $X_i \cap X_{i+1} \neq \emptyset$, but we can regard it as an unrestricted sum since the summand vanishes if the constraint is violated.

Let $Y$ be a connected union of $M$ cubes. Define

$$\tilde{R}_0(Y) = \sum_{X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n : X_i \cap Y} R_0(X_0) R_{\alpha_1}(X_1) \cdots R_{\alpha_n}(X_n)$$

$$\tilde{R}(Y) = \sum_{\alpha_1, X_1, \ldots, \alpha_n, X_n : X_i \subset Y} R_{\alpha_1}(X_1) \cdots R_{\alpha_n}(X_n)$$

(271)

These converge and satisfy the same bounds [213] as $R_\alpha(X)$. The proof is the same as the proof for the convergence of the overall series.

For any union of $M$ cubes $Y$ let $\{\beta\}$ be the connected components. We resum the expansion [274] grouping together parts of the walk which stay in the same $Y_\beta$. Given a general sequence $(X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n)$ replace each chain of localization domains staying in some $Y_\beta$ by $Y_\beta$. We generate a new sequence $(X_0, X_1, \ldots, X_m)$ where $X_0$ is either some $Y_\beta$ or $(0, \emptyset)$ with $\emptyset \subset Y^c$, and $X_i$ for $i \neq 0$ is either some $Y_\beta$ or a pair $(\alpha, X)$ with $\alpha \neq 0$ and $X \cap Y^c \neq \emptyset$. We associate with $X_0$ an operator

$$R_\alpha'(X_0) = \begin{cases} \tilde{R}_0(Y_\beta) & X_0 = Y_\beta \\ R_0(\emptyset) & X_0 = (0, \emptyset) \end{cases}$$

(272)

and with $\chi_i$

$$R'(X_i) = \begin{cases} \tilde{R}(Y_\beta) & X_i = Y_\beta \\ R_\alpha(X) & X_i = (\alpha, X) \end{cases}$$

(273)

These are localized in the associated region and satisfy

$$|1_{\Delta_y} R_\alpha'(X_0) 1_{\Delta_y} f| \leq C e^{-\gamma (y,y')} \|f\|_\infty$$

$$|1_{\Delta_y} R'(X_i) 1_{\Delta_y} f| \leq C M^{-1} e^{-\gamma (y,y')} \|f\|_\infty$$

(274)

together with the bounds on derivatives as in [213].

Classify the terms in the original sum by the sequence $(X_1, \ldots, X_m)$ that they generate. This gives an new expansion

$$G_k = \sum_{(X_0, X_1, \ldots, X_m)} \sum_{(X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n) \to (X_0, X_1, \ldots, X_m)} R_0(X_0) R(X_2) \cdots R(X_n)$$

$$= \sum_{(X_0, X_1, \ldots, X_m)} \left( \sum_{(X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n) \to X_0} R_0(X_0) R(X_1) \cdots R(X_n) \right)$$

$$\prod_{i=1}^m \left( \sum_{(\alpha_1, X_1, \ldots, \alpha_m, X_n) \to X_i} R_{\alpha_1}(X_1) \cdots R_{\alpha_m}(X_n) \right)$$

$$= \sum_{(X_0, X_1, \ldots, X_m)} R_\alpha'(X_0) R'(X_1) \cdots R'(X_m)$$

(275)

We must have $X_i \cap X_{i+1} \neq \emptyset$ for a non-zero contribution. This is our new random walk expansion.
4 Polymers

4.1 dressed Grassman variables

As we track the flow of the RG transformations the densities will be expressed in terms of localized elements of the Grassman algebra depending on the fundamental variables $\Psi_k$ on $T_{N,k}^{-1}$. We assume $\mathcal{A}$ is in the domain so we have good estimates on $\mathcal{H}_k(\mathcal{A})$.

Let $\xi = (x, \beta, \omega)$ with $x \in T_{N,k}^{-1}$, $1 \leq \beta \leq 4$, and $\omega = 0, 1$. We treat $\psi, \bar{\psi}$ as a single field by

$$
\psi_k(A, \xi) = \begin{cases}
\psi_{k,\beta}(A, x) & \xi = (x, \beta, 0) \\
\bar{\psi}_{k,\beta}(A, x) & \xi = (x, \beta, 1)
\end{cases}
$$

(276)

We also define fields depending on two variables $x, y \in T_{N,k}^{-1}$. Let $\zeta = (x, y, \beta, \omega)$ and define

$$
\chi_k(A, \zeta) = \left(\delta_{\alpha,\mathcal{A}} \psi_k(A)\right)(\zeta) = \begin{cases}
|x - y|^{-\alpha} \left(e^{ie_k \eta(x,y)} \psi_{k,\beta}(y) - \bar{\psi}_{k,\beta}(y)\right) & \zeta = (x, y, \beta, 0) \\
|x - y|^{-\alpha} \left(e^{-ie_k \eta(x,y)} \bar{\psi}_{k,\beta}(y) - \psi_{k,\beta}(y)\right) & \zeta = (x, y, \beta, 1)
\end{cases}
$$

(277)

We consider elements of the Grassman algebra generated by $\Psi_k$ of the form

$$
E(A, \psi_k(A), \chi_k(A)) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int E_{nm}(A, \xi_1, \ldots, \xi_n; \zeta_1, \ldots, \zeta_m)
$$

$$
\psi_k(A, \xi_1) \cdots \psi_k(A, \xi_n) \chi_k(A, \zeta_1) \cdots \chi_k(A, \zeta_n) d\xi_1 \cdots d\xi_n d\zeta_1 \cdots d\zeta_m
$$

(278)

Here with $\eta = L^{-k}$

$$
\int d\xi = \sum_{x, \beta, \omega} \eta^3 \quad \int d\zeta = \sum_{x, y, \beta, \omega} \eta^6
$$

(279)

The kernel is the collection of functions $\{E_{nm}(A)\}$. The $E_{nm}(A, \xi_1, \ldots, \xi_n; \zeta_1, \ldots, \zeta_m)$ are taken to be anti-symmetric the $\xi_i$ and the $\zeta_j$ separately. Note that the $\psi_k(A), \chi_k(A)$ are not independent and different kernels may give the same algebra element.

We define a norm on the $E(A)$ by

$$
\|E_{nm}(A)\| = \int |E_{nm}(A, \xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)| d\xi_1 \cdots d\xi_n d\zeta_1 \cdots d\zeta_m
$$

(280)

and for a pair of positive real numbers $\mathbf{h} = (h_1, h_2)$

$$
\|E(A)\|_h = \sum_{n,m=0}^{\infty} \frac{h_1^n h_2^m}{n!m!} \|E_{nm}(A)\|
$$

(281)

This is not a norm on the algebra element but rather on the representation, i.e., on the kernels.

The norm has the property that if $G(A) = E(A) F(A)$ then the kernels satisfy

$$
\|G(A)\|_h \leq \|E(A)\|_h \|F(A)\|_h
$$

(282)

This is a special case of from Appendix A.3. In the terminology there the two measure spaces are $(T_1, \nu_1)$ with $T_1$ equal to all $\xi$ and $\nu_1(\xi) = \eta^3$ and $(T_2, \nu_2)$ with $T_2$ equal to all $\zeta$ and $\nu_2(\zeta) = \eta^6$.

We also want to bound the true norm by the dressed norm. This again is a special case of a result from in Appendix A.3. In that terminology our fields are

$$
\psi_1(\xi) = (\mathcal{H}_1 \Psi)(\xi) = (\mathcal{H}_k(\mathcal{A}) \Psi)(x, \beta, \omega)
$$

$$
\psi_2(\zeta) = (\mathcal{H}_2 \Psi)(\zeta) = |x - y|^{-\alpha} \left(e^{ie_k \eta(x,y)} (\mathcal{H}_k(\mathcal{A}) \Psi)(y, \beta, \omega) - (\mathcal{H}_k(\mathcal{A}) \Psi)(x, \beta, \omega)\right)
$$

(283)

35
By (152)

\[
\|H_1\|_{1,\infty} = \|H_k(A)\|_{1,\infty} \leq C \\
\|H_2\|_{1,\infty} \leq O(1)\|H_k(A)\|_{1,\infty} + \|\delta_n H_k(A)\|_{1,\infty} \leq C
\]  
(284)

Here for the second estimate we consider separately the two cases \(|x-y| \leq 1\) and \(|x-y| \geq 1\). In the case \(|x-y| \leq 1\) we identify the norm \(\|\delta_n H_k(A)\|_{1,\infty}\). In the case \(|x-y| \geq 1\) we bound the parallel translation by \(O(1)\) and then estimate each term by \(\|H_k(A)\|_{1,\infty}\). It follows by lemma 20 in Appendix A.3 that if \(E'(A, \Psi) = E(A, \psi_k(A), \chi_k(A))\), then the norm \(\|E'(A)\|_h\) as defined in (83) satisfies

\[
\|E'(A)\|_h \leq \|E(A)\|_{C_h, C_h}
\]  
(285)

### 4.2 a domain for the gauge field

Before proceeding we make some further restrictions on the gauge field. Let \(\epsilon > 0\) be a fixed small number. A domain our fields is defined by the following (notation slightly different from [29])

**Definition 1.** \(\mathcal{R}_k\) is all complex-valued fields \(A\) on \(\mathbb{T}^{-k}_{N-k}\) satisfying

\[
|A| < e_k^{-\frac{4}{3} + \epsilon} \quad |\partial A| < e_k^{-\frac{4}{3} + 2\epsilon} \quad |\delta_n \partial A| < e_k^{-\frac{4}{3} + 3\epsilon}
\]  
(286)

We also define an extended domain with the property that the fields are locally gauge equivalent a field in \(\mathcal{R}_k\). Choose a constant \(c_0 = O(1)\) and let \(\square^{\dagger} = \tilde{\square}^{(c_0L)}\) be the enlarged union of \(M\)-cubes with \(c_0L\) cubes on a side. (\(\square^{\dagger}\) was called \(\square^3\) in [29])

**Definition 2.** \(\tilde{\mathcal{R}}_k\) is all fields \(A = A_0 + A_1\) on \(\mathbb{T}^{-k}_{N-k}\) where \(A_0\) is real and on each \(\square^{\dagger}\) is gauge equivalent to a field in \(\mathcal{R}_k\) and \(A_1\) is complex and in \(\mathcal{R}_k\).

Assuming \(c_0L \geq 5\) these conditions are stronger than the conditions (134) which we needed for the treatment of the \(S_k(A), H_k(A)\). Indeed if \(A \in \tilde{\mathcal{R}}_k\) then \(e_k^{-\frac{4}{3}}\mathcal{A}\) satisfies these conditions. Also in [29] it is established that in each \(\square^{\dagger}\) the real field \(A_k = H_kA_k\) is gauge equivalent to a field \(A\) satisfying

\[
|A|, |\partial A|, |\delta_n \partial A| \leq CM\|dA_k\|_{\infty}
\]  
(287)

Thus we only need control over the field strength \(dA_k\) to conclude that \(A_k \in \tilde{\mathcal{R}}_k\)

### 4.3 polymer functions

Next we localize. A *polymer* \(X\) in \(\mathbb{T}^{-k}_{N-k}\) is defined to be a connected union of \(M\) cubes, with the convention that two cubes are connected if they have an entire face in common. The set of all polymers is denoted \(\mathcal{D}_k\). A *polymer function* depending on a gauge field \(A\) in \(\tilde{\mathcal{R}}_k\) and a polymer \(X\) is an element of the Grassman algebra of the form

\[
E(X, A, \psi_k(A), \chi_k(A)) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int E_{nm}(X, A, \xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)
\]  
(288)

\[
\psi_k(A, \xi_1) \cdots \psi_k(A, \xi_n) \chi_k(A, \zeta_1) \cdots \chi_k(A, \zeta_m) d\xi_1 \cdots d\xi_n d\zeta_1 \cdots d\zeta_m
\]

We require that only terms with equal numbers of \(\Psi, \bar{\Psi}\) contribute.

The kernels \(E_{nm}(X, A, \xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)\) are required to vanish unless all \(\xi_i \in X\) and all \(\zeta_i \cap X \neq \emptyset\), and to depend on \(A(b)\) only if \(b \cap X \neq \emptyset\). We also require that \(E(X, A)\) is bounded and analytic on the domain \(A \in \tilde{\mathcal{R}}_k\). Norms \(\|E_{nm}(X, A)\|\) and \(\|E(X, A)\|_h\) are defined as in (280), (281) and now we define

\[
\|E_{nm}(X)\|_{\tilde{\mathcal{R}}_k} = \sup_{A \in \tilde{\mathcal{R}}_k} \|E_{nm}(X, A)\| \quad \|E(X)\|_{h, \tilde{\mathcal{R}}_k} = \sup_{A \in \tilde{\mathcal{R}}_k} \|E(X, A)\|_h
\]  
(289)
We have $\|E(X)\|_{\bar{R}_k, \bar{h}} \leq \|E(X)\|_{\bar{R}_k, \bar{h}}$, where the latter is defined by

$$\|E(X)\|_{\bar{R}_k, \bar{h}} = \sum_{n,m=0}^{\infty} \frac{h^n h^m}{n!m!} \|E_{nm}(X)\|_{\bar{R}_k}$$  \hspace{1cm} (290)

The polymer functions are required to have tree decay in the polymer $X$. Size is measured on the $M$-scale and we define $d_M(X)$ by

$$Md_M(X) = \text{length of the shortest continuum tree joining the $M$-cubes in $X$}.$$  \hspace{1cm} (291)

If $|X|_M$ is the number of $M$ cubes in $X$, then

$$d_M(X) \leq |X|_M \leq O(1)(d_M(X) + 1)$$  \hspace{1cm} (292)

Also there are constants $\kappa_0, K_0 = O(1)$ such that for any $M$-cube $\Box$

$$\sum_{X \in D_k, X \supset \Box} e^{-\kappa_0 d_M(X)} \leq K_0$$  \hspace{1cm} (293)

We assume $\kappa = O(1)$ and $\kappa \geq \kappa_0$. We define an associated norm

$$\|E\|_{\bar{R}_k, h, \kappa, \kappa} = \sup_X \|E(X)\|_{\bar{R}_k, h} e^{\kappa d_M(X)}$$  \hspace{1cm} (294)

It is useful to let $h$ depend on the running coupling constant $e_k$. Pick a fixed $\epsilon > 0$ small. Then define $h_k = e_k^{-\frac{1}{4}}$ and

$$h_k = (h_k,1, h_k,2) = (h_k, e_k^{-\frac{1}{4}}, e_k^{-\frac{1}{4}+\epsilon})$$  \hspace{1cm} (295)

The basic norm after $k$ steps is then

$$\|E\|_k \equiv \|E\|_{\bar{R}_k, h_k, \kappa}$$  \hspace{1cm} (296)

The space of all polymer functions with this norm is a Banach space called $K_k$.

### 4.4 scaling

At this point drop the reference to the specific fields $\psi_k(A, \xi), \chi_k(A, \zeta)$ and consider general Grassman variables $\psi(\xi), \chi(\zeta)$ on $\mathbb{T}_{N-k}^{-k}$ of the same type, but do not assume any relation between them. Let $E(X, A, \psi, \chi)$ be a polymer function of these variables as above.

We will want to scale the polymer function. Since $M$-cubes do not scale to $M$-cubes we first need a blocking operation. If $X \in D_k$ let $X^L$ be the smallest union of $LM$ blocks containing $X$. Then if $Z$ is a connected union of $LM$ blocks we define

$$(BE)(Z, \psi, \chi) = \sum_{X: X^{L} = Z} E(X, \psi, \chi)$$  \hspace{1cm} (297)

We define a scaled polymer function $(BE)_{L^{-1}}$ on $\mathbb{T}_{N-k-1}^{-k-1}$ as follows. For $Y, A, \psi, \chi$ on $\mathbb{T}_{N-k-1}^{-k-1}$

$$(BE)_{L^{-1}}(Y, A, \psi, \chi) \equiv (BE)(LY, A_L, \psi_L, \chi_L) = \sum_{X: X^{L} = LY} E(X, A_L, \psi_L, \chi_L)$$  \hspace{1cm} (298)

Then we have

$$\sum_{X \in D_k} E_k(X, A_L, \psi_L, \chi_L) = \sum_{Y \in D_{k+1}} (BE)_{L^{-1}}(Y, A, \psi, \chi)$$  \hspace{1cm} (299)
The scaled fields on $T_{N-k}$ are

$$A_L(b) = L^{-\frac{1}{2}} A(L^{-1}b)$$

$$\psi_L(x) = \psi_L(x, \beta, \omega) = L^{-1} \psi(L^{-1}x, \beta, \omega)$$

$$\chi_L(x) = \chi_L(x, y, \beta, \omega) = L^{-1-\alpha} \chi(L^{-1}x, L^{-1}y, \beta, \omega)$$

(300)

If we define $E(X, A_L, \psi_L, \chi_L) = (S_L E)(X, A, \psi, \chi)$, then the kernel of $S_L E$ is

$$(S_L E)_{nm}(X, A, \xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m) = L^{3n+6m} L^{-n} L^{-(1+\alpha)m} E_{nm}(X, A_L, L\xi_1, \ldots, L\xi_n, L\zeta_1, \ldots, L\zeta_m)$$

(301)

and it has the norm

$$\|(S_L E)_{nm}(X, A)\| = \int L^{3n+6m} L^{-n} L^{-(1+\alpha)m} |E_{nm}(X, A_L, L\xi_1, \ldots, L\xi_n, L\zeta_1, \ldots, L\zeta_m)| d\xi_1 \cdots d\xi_n \ d\zeta_1 \cdots d\zeta_m$$

(302)

$$= \int L^{-n} L^{-(1+\alpha)m} |E_{nm}(X, A_L, \xi'_1, \ldots, \xi'_n, \zeta'_1, \ldots, \zeta'_m)| d\xi'_1 \cdots d\xi'_n \ d\zeta'_1 \cdots d\zeta'_m$$

$$ = L^{-n} L^{-(1+\alpha)m} \|E_{nm}(X, A_L)\|$$

If $A \in R_{k+1}$ then since $e_{k+1} = L^2 \varepsilon_k$

$$|A_L| < L^{-\frac{1}{2}} e_{k+1} < L^{-\frac{1}{2}+\frac{\alpha}{3}+\frac{\alpha}{2}+\varepsilon}$$

$$|\partial A_L| < L^{-\frac{1}{2}+\frac{\alpha}{3}+\frac{\alpha}{2}+\varepsilon} e_{k+1}$$

$$|\delta_{\alpha} \partial A_L| < L^{-\frac{1}{2}+\frac{\alpha}{3}+\frac{\alpha}{2}+\varepsilon} e_{k+1}$$

(303)

The $L$ factors are all less than $L^{-\frac{3}{2}}$ so $A_L \in L^{-\frac{3}{2}} R_k$. It follows also that $A \in R_{k+1}$ implies $A_L \in L^{-\frac{3}{2}} \tilde{R}_k$. Thus we have

$$\|(S_L E)_{nm}(X)\|_{\tilde{R}_{k+1}} \leq L^{-n} L^{-(1+\alpha)m} \|E_{nm}(X)\|_{L^{-\frac{3}{2}} \tilde{R}_k}$$

(304)

For the moment we throw away the contracting factors $L^{-n} L^{-(1+\alpha)m}$, enlarge $L^{-\frac{3}{2}} \tilde{R}_k$ to $\tilde{R}_k$, and take $h_{k+1} < h_k$ to get

$$\|(S_L E)(X)\|_{\tilde{R}_{k+1}, h_{k+1}} \leq \|E(X)\|_{\tilde{R}_k, h_k} \leq \|E\|_{k} e^{-\kappa d_M(X)}$$

(305)

and so

$$\|(BE)_{L^{-1}}(X)\|_{\tilde{R}_{k+1}, h_{k+1}} \leq \|E\|_{k} \sum_{X : X = L Y} e^{-\kappa d_M(X)}$$

(306)

If $X = L Y$ then $L d_M(Y) \leq d_M(X)$ so we can extract a factor $e^{-L(\kappa - \kappa_0) d_M(Y)}$ leaving $e^{-\kappa_0 d_M(X)}$. Then using (303) the sum over $X$ is bounded by $O(1)|L Y|_M \leq O(1) L^3 e^{d_M(Y)}$. Therefore

$$\sum_{X : X = L Y} e^{-\kappa d_M(X)} \leq O(1) L^3 e^{-L(\kappa - \kappa_0 - 1) d_M(Y)} \leq O(1) L^3 e^{-\kappa d_M(Y)}$$

(307)

where the second inequality holds for $L$ sufficiently large. Thus we get the crude bound

$$\|(BE)_{L^{-1}}\|_{k+1} \leq O(1) L^3 \|E\|_{k}$$

(308)
4.5 symmetries

We assume the polymer functions $E(X,\mathcal{A},\psi,\chi)$ have the following symmetries:

- Invariance under $T_{N-k}$ lattice symmetries
  \[ E(rX + a,\mathcal{A}_{a,r},\psi_{a,r},\chi_{a,r}) = E(X,\mathcal{A},\psi,\chi) \]  
  \[ (309) \]

- Gauge invariance
  \[ E(X,\mathcal{A}^\lambda,\psi^\lambda,\chi^\lambda) = E(X,\mathcal{A},\psi,\chi) \]  
  \[ (310) \]

- Charge conjugation invariance
  \[ E(X,-\mathcal{A},C\psi,C\chi) = E(X,\mathcal{A},\psi,\chi) \]  
  \[ (311) \]

- Complex conjugation. If $\mathcal{A}$ is real the kernels satisfy (now distinguishing $\psi,\bar{\psi}$ and $\chi,\bar{\chi}$)
  \[ \text{ker} E\left(X,-\mathcal{A},\gamma_3C\psi,\gamma_3C\chi,[(\gamma_3C)^{-1}]^T\bar{\psi},\gamma_3C\chi,[(\gamma_3C)^{-1}]^T\bar{\chi}\right) = \text{ker} E\left(X,\mathcal{A},\psi,\bar{\psi},\chi,\bar{\chi}\right) \]  
  \[ (312) \]

For the first three we also require the corresponding transformation properties for the the kernels. In particular the gauge invariance says that $E_{nm}(X,\mathcal{A} - \partial \lambda)$ and $E_{nm}(X,\mathcal{A})$ differ by a phase factor. It follows that the the norm is gauge invariant:

\[ \|E_{nm}(X,\mathcal{A} - \partial \lambda)\| = \|E_{nm}(X,\mathcal{A})\| \]  
\[ (313) \]

Here are some consequences for the piece $E_{00}(X,\mathcal{A})$ with no fermion fields. The $p^{th}$ derivative in $\mathcal{A}$ is the multilinear functional

\[ \frac{\delta^p E_{00}}{\delta \mathcal{A}^p} \left(X,\mathcal{A}; f_1,\ldots,f_p\right) = \frac{\partial^p}{\partial t_1 \ldots \partial t_p} \left[E_{00}(X,\mathcal{A} + t_1 f_1 + \cdots + t_p f_p,0)\right]_{t=0} \]  
\[ (314) \]

If one of the functions has the form $f_i = \partial \lambda$, then by gauge invariance there is no dependence on $t_i$ and the derivative vanishes. Thus we have the Ward identity

\[ \frac{\delta^p E_{00}}{\delta \mathcal{A}^p} \left(X,\mathcal{A}; f_1,\ldots,\partial \lambda,\ldots,f_p\right) = 0 \]  
\[ (315) \]

Charge conjugation invariance gives $E_{00}(X,-\mathcal{A}) = E_{00}(X,\mathcal{A})$ and this implies

\[ \frac{\delta^p E_{00}}{\delta \mathcal{A}^p} (X,0) = 0 \quad \text{if } p \text{ is odd} \]  
\[ (316) \]

4.6 normalization

As we iterate the RG transformations the scaling operation can increase the size of the polymer functions by as much as $O(L^3)$ as is evident from (308). We have to watch this carefully and start by introducing a criterion to avoid the growth. The following is similar to the analysis in [22, 25, 29].

**Definition 3.** A polymer function $E(X,\mathcal{A},\psi_k(\mathcal{A}),\chi_k(\mathcal{A}))$ with kernels $E_{nm}(X,\mathcal{A})$ satisfying the stated symmetries is said to be normalized if in addition to the vanishing derivatives (315), (316) we have

\[ E_{00}(X,0) = 0 \quad \int E_{20}(X,0; (x,\alpha,1),(y,\beta,0)) \, dx dy = 0 \]  
\[ (317) \]

We generally only require normalization small polymers.

**Definition 4.** A polymer $X$ is small if $d_M(X) \leq L$ and large if $d_M(X) > L$. The set of all small polymers in denoted $\mathcal{S}$. 

39
4.6.1 extraction

Normalization is achieved by extracting certain relevant terms from the polymer function. Given $E(X, A, \psi, \chi)$ with kernels $E_{nm}(X, A)$ on $\mathbb{T}^{N-k}_-$ satisfying lattice, gauge, and charge conjugation symmetries we define $(\mathcal{RE})(X, A, \psi, \chi)$ as follows. If $X$ is large then $(\mathcal{RE})(X, A) = E(X, A)$. If $X$ is small ($X \in S$) then $(\mathcal{RE})(X, A)$ is defined by

\[
E(X, A, \psi, \chi) = \alpha_0(E, X) \text{Vol}(X) + \int_X \bar{\psi} \left[ \alpha_2(E, X) \right] \psi + \mathcal{R}E(X, A, \psi, \chi) \quad (318)
\]

where

\[
\alpha_0(E, X) = \frac{1}{\text{Vol}(X)} E_00(X, 0) \quad \quad \alpha_2(E, X)_{\alpha \beta} = \frac{1}{\text{Vol}(X)} \int E_{20}(X, 0; (x, \alpha, 1), (y, \beta, 0)) dxdy \quad (319)
\]

Lemma 13. $\mathcal{RE}$ is invariant under lattice, gauge, and charge symmetries. $\mathcal{RE}$ is normalized for small polymers and satisfies

\[
\|\mathcal{RE}\|_k \leq \mathcal{O}(1) \|E\|_k \quad (320)
\]

Proof. The invariance follows since everything else in (318) is invariant. The derivatives (317) match on the left and right except for the term $\mathcal{RE}$, hence its derivatives vanish. The bound holds since everything else in (318) satisfies the bound. We omit the details.

For global quantities we only have to remove energy and mass terms.

Corollary 1.

\[
\sum_X E(X) = -\varepsilon(E) \text{Vol}(\mathbb{T}^{N-k}) - m(E) \int \bar{\psi} \psi + \sum_X \mathcal{RE}(X) \quad (321)
\]

where

\[
\varepsilon(E) = - \sum_{X \supset \square, X \in S} \alpha_0(E, X) \quad \quad m(E)\delta_{\alpha \beta} = [m(E)]_{\alpha \beta} = - \sum_{X \supset \square, X \in S} [\alpha_2(E, X)]_{\alpha \beta} \quad (322)
\]

are real and satisfy

\[
|\varepsilon(E)| \leq \mathcal{O}(1) \|E\|_k \quad (323)
\]

\[
|m(E)| \leq \mathcal{O}(1) \hbar^{-2} \|E\|_k = \mathcal{O}(1) \varepsilon^{\frac{1}{2}} \|E\|_k
\]

Proof. The constant term in $\sum_X E(X)$ is $\sum_{X \in S} \alpha_0(E, X) \text{Vol}(X)$. Insert $\text{Vol}(X) = \sum_{\square \subset X} \text{Vol}(\square)$ and change the order of the sums to write this as

\[
- \sum_{\square} \varepsilon(E, \square) \text{Vol}(\square) \quad \quad \varepsilon(E, \square) = - \sum_{X \supset \square, X \in S} \alpha_0(E, X) \quad (324)
\]

But $\varepsilon(E, \square)$ is independent of $\square$ and is denoted $\varepsilon(E)$ to give the first term in (321). The bounds on $\varepsilon(E)$ follows directly.
The mass term in $\sum_X E(X)$ is $\sum_X \int_X \bar{\psi} \left[ \alpha_2(E,X) \right] \psi$. Write $\int_X = \sum_{\square \subset X} \int_{\square}$ and change the order of the sums to write this as

$$- \sum_{\square} \int_{\square} \bar{\psi} m(E,\square) \psi \quad \text{where} \quad m(E,\square) = - \sum_{X \supset \square, X \in S} \alpha_2(E,X) \quad (325)$$

But $m(E,\square)$ is independent of $\square$ and is denoted $m(E)$ and the expression becomes $\int \bar{\psi} m(E) \psi$.

Next we explain why the matrix $m(E)$ is a multiple of the identity. Our assumption that $E$ is invariant under lattice symmetries implies if $S$ is a spinor representation of a rotation or reflection $r$ then $S^{-1} \alpha(E,rX)S = \alpha(E,X)$. Specialize to $r$ leaving the center of $\square$ invariant and sum over $X \in S, X \supset \square$ to get $S^{-1} m(E)S = m(E)$. Take $S = \gamma_\mu$ for $0 \leq \mu \leq 3$ which induce reflections. We conclude that $[\gamma_\mu, m(E)] = 0$ and hence that $m(E)$ is a multiple of the identity.

We also need to show that $m(E)$ is real. This follows since $E_{2,0}(X,0) = (\gamma_3 C)^{-1} E_{2,0}(X,0) \gamma_3 C$ by $\Box$ and hence $m(E) = (\gamma_3 C)^{-1} m(E) \gamma_3 C = m(E)$. Similarly $\epsilon(E)$ is real.

The bounds on $\epsilon(E), m(E)$ follow from

$$\text{Vol}(X)|\alpha_0(E,X)| \leq e^{-kd_M(X)} \|E\|_k$$

$$\text{Vol}(X)||\alpha_2(E,X)\alpha_\beta| \leq \|E_k(X,0)\| \leq h_k^{-2} \|E_k(X,0)\|_h \leq h_k^{-2} e^{-kd_M(X)} \|E\|_k$$

and $\sum_{X \supset \square} e^{-kd_M(X)} \leq O(1)$

**Remark.** For the subsequent paper we generalize this construction to the case where instead of $\sum_X E(X)$ we have a restricted sum $\sum_{X \subset \Lambda} E(X)$ where $\Lambda$ is a union of $M$ blocks. In this case $\epsilon(E,\square)$ and $m(E,\square)$ are replaced by

$$\epsilon_\Lambda(E,\square) = - \sum_{X \in S, \square \subset X \subset \Lambda} \alpha_0(E,X)$$

$$m_\Lambda(E,\square) = - \sum_{X \in S, \square \subset X \subset \Lambda} \alpha_2(E,X) \quad (327)$$

which vanishes unless $\square \subset \Lambda$. Now we have

$$\sum_{X \subset \Lambda} E(X) = - \epsilon(E) \text{Vol}(\Lambda) - m(E) \int_\Lambda \bar{\psi} \psi + \sum_{X \subset \Lambda} RE(X) + B_\Lambda \quad (328)$$

where the extra term is

$$B_\Lambda = - \sum_{\square \subset \Lambda} (\epsilon_\Lambda(E,\square) - \epsilon(E)) \text{Vol}(\square) - \sum_{\square \subset \Lambda} \int_{\square} \bar{\psi} \left( m_\Lambda(E,\square) - m(E) \right) \psi$$

Inserting the definitions $\Box$ only polymers $X$ which intersect both $\Lambda$ and $\Lambda^c$ contribute, denoted $X \# \Lambda$, and this can be rearranged to

$$B_\Lambda = \sum_{X \in S, X \# \Lambda} B_\Lambda(X) \quad (330)$$

where

$$B_\Lambda(X) = - \alpha_0(E,X) \text{Vol}(\Lambda \cap X) - \int_{X \cap \Lambda} \bar{\psi} \alpha_2(E,X) \psi$$

This correction term is localized around the boundary of $\Lambda$ and we have

$$\|B_\Lambda\|_k \leq O(1) \|E\|_k$$

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4.6.2 adjustment

The next result shows that if \( E(X, A, \psi_k(A), \chi_k(A)) \) is normalized for small \( X \) then we can adjust the kernels so that both \( E_{00}(X, 0) = 0 \) (there is no energy term) and \( E_{20}(X, 0) = 0 \) (there is no mass term).

**Lemma 14.** If \( E(X, A_k, \psi_k(A)) \) with kernels \( E_{nn}(X, A) \) is normalized for \( X \in S \) then it can be written with new kernels \( E_{nn}^2(X, A) \) modified for \( X \in S \) and \( n + m = 2 \) such that \( E_{20}^2(X, 0) = 0 \). Furthermore

1. For \( A \in \tilde{R}_k \)

\[
\|E_{20}^2(X, A)\| \leq O(1)\|E_{20}(X, A)\|
\]

\[
\|E_{11}^2(X, A)\| \leq O(1)ML\|E_{11}(X, A)\| + \|E_{11}(X, A)\|\]

\[
\|E_{02}^2(X, A)\| \leq O(1)(ML)^2\|E_{20}(X, A)\| + \|E_{02}(X, A)\|
\]

2. Let \( e_k \) be sufficiently small (depending on \( L, M \)) then

\[
\|E^2(X, A)\|_h \leq O(1)\|E(X, A)\|_h
\]

**Proof.** The modification comes in the piece with two \( \psi \) fields which we write in an abbreviated notation as

\[
\int_{X \times X} \tilde{\psi}_k(A, x)E_{20}(X, A, x, y)\psi_k(A, y)dxdy
\]

We introduce dummy variables \( x_0, y_0 \) and write this as

\[
\frac{1}{\text{Vol}(X)^2} \int_{X \times X \times X \times X} \tilde{\psi}_k(A, x)E_{20}(X, A, x, y)\psi_k(A, y)dxdydx_0dy_0
\]

Now make the substitution

\[
\psi_k(A, y) = e^{i\epsilon_k \eta(\tau, A)(y, y_0)}\psi_k(A, y_0) - \chi_k(A, y, y_0)|y - y_0|^\alpha
\]

\[
\tilde{\psi}_k(A, x) = e^{-i\epsilon_k \eta(\tau, A)(x, x_0)}\tilde{\psi}_k(A, x_0) - \chi_k(A, x, x_0)|x - x_0|^\alpha
\]

This yields four terms (all gauge invariant, all integrals over \( X \))

\[
\frac{1}{\text{Vol}(X)^2} \int \tilde{\psi}_k(A, x_0)\left[ \int e^{-i\epsilon_k \eta(\tau, A)(x, x_0)}E_{20}(X, A, x, y)e^{i\epsilon_k \eta(\tau, A)(y, y_0)}dxdy \right]\psi_k(A, y_0)dxdy_0
\]

\[
\frac{1}{\text{Vol}(X)^2} \int \tilde{\psi}_k(A, x_0)\left[ \int e^{-i\epsilon_k \eta(\tau, A)(x, x_0)}E_{20}(X, A, x, y)|y - y_0|^\alpha dx \right]\chi_k(A, y, y_0)dxdy_0dy_0
\]

\[
\frac{1}{\text{Vol}(X)^2} \int \tilde{\chi}_k(A, x, x_0)\left[ \int |x - x_0|^\alpha E_{20}(X, A, x, y)e^{i\epsilon_k \eta(\tau, A)(y, y_0)}dxdy \right]\tilde{\psi}_k(A, y_0)dxdx_0dy_0
\]

\[
\frac{1}{\text{Vol}(X)^2} \int \tilde{\chi}_k(A, x, x_0)\left[ |x - x_0|^\alpha E_{20}(X, A, x, y)|y - y_0|^\alpha \right]\chi_k(A, y, y_0)dxdy_0dx_0
\]

For the first term has a kernel

\[
E_{20}^2(X, A, x_0, y_0) = \frac{1}{\text{Vol}(X)^2} \int e^{-i\epsilon_k \eta(\tau, A)(x, x_0)}E_{20}(X, A, x, y)e^{i\epsilon_k \eta(\tau, A)(y, y_0)}dxdy
\]

This does vanish at \( A = 0 \) by the normalization assumption. We want to bound the norm \( \|E_{20}^2(X, A)\| \).

Since \( X \in S \) it is contained in some \( \square \) and so the field \( A \in \tilde{R}_k \) is gauge equivalent to a field
\[ A \in \mathcal{R}_k. \] Since \( \| E_{20}^k(X,A) \| \) is gauge invariant it suffices to assume \( A \in \mathcal{R}_k. \) Since \( x, x_0 \in X \) we have \( |x - x_0| \leq O(1)ML \) and so

\[
|e_k \eta(x,x_0)| \leq O(1)e_kML\|A\|_\infty \leq O(1)MLe_k^{1+\epsilon} \leq 1 \tag{340}
\]

This yields the desired bound

\[
\| E_{20}^k(X,A) \| \leq O(1) \frac{1}{\text{Vol}(X)^2} \int |E_{20}(X,A,x,y)| dx dy dx_0 dy_0 \leq O(1) \int |E_{20}(X,A,x,y)| dx dy = O(1)\| E_{20}(X,A) \| \tag{341}
\]

The second term has the kernel

\[
E_{11}^{(1)}(X,A,x_0,y_0,y) = \frac{1}{\text{Vol}(X)^2} \int e^{-ie_k \eta(x,x_0)} E_{20}(X,A,x,y) |y - y_0|^\alpha dx \tag{342}
\]

and we estimate

\[
\| E_{11}^{(1)}(X,A) \| \leq O(1) \frac{1}{\text{Vol}(X)^2} ML \int |E_{20}(X,A,x,y)| dx_0 dy_0 dx dy = O(1)ML\| E_{20}(X,A) \| \tag{343}
\]

This gives a contribution to \( E_{11}^k(X,A) \) as does the third term in \( (338) \), as well as the original term \( E_{11}(X,A) \). The stated bound on \( E_{11}^k(X,A) \) follows.

The last term has the kernel

\[
E_{02}^{(1)}(X,A,x_0,x,y_0,y) = \frac{1}{\text{Vol}(X)^2} |x - x_0|^\alpha E_{20}(X,A,x,y) |y - y_0|^\alpha \tag{344}
\]

with norm bounded by \( O(1)(ML)^2\| E_{20}(X,A) \| \). This contributes to \( E_{02}^k(X,A) \) as does the original term \( E_{02}(X,A) \) and the stated bound follows. This completes the proof of part 1.

Part 2 is where we use the fact that \( \psi \) and \( \chi \) are weighted differently. It suffices to look at the terms \( E_{11}^k(X,A) \) and \( E_{02}^k(X,A) \) since \( E_{20}^k(X,A) = 0 \) and all the others are unchanged. In the first case we

\[
h_{k,1}h_{k,2}\| E_{11}^k(X,A) \| \leq h_{k,1}h_{k,2} \left( O(1)ML\| E_{20}(X,A) \| + \| E_{11}(X,A) \| \right)
\]

\[
\leq \frac{1}{2} h_{k,1}^2 \| E_{20}(X,A) \| + h_{k,1}h_{k,2} \| E_{11}(X) \|
\]

\[
\leq \| E(X,A) \| h_k
\]

Here we used \( h_{k,2} = h_{k,1}e_k \) and then \( O(1)MLe_k^\epsilon < \frac{1}{2} \) for \( e_k \) sufficiently small. The term \( E_{02}^k(X,A) \) is treated similarly and the result follows.

4.6.3 improved scaling

After the adjustments of the last two sections are made we have improved scaling.

**Lemma 15.** Let \( L \) be sufficiently large and \( e_k \) sufficiently small (depending on \( L,M \)). Suppose \( E(X,A,\psi,\chi) \) has all the symmetries and satisfies for small sets \( X \):

\[
E_{00}(X,0) = 0 \quad E_{20}(X,0) = 0 \tag{346}
\]

Then

\[
\|(BE)_{L^{-1}}\|_{k+1} \leq O(1)L^{-\frac{1}{2}+2\epsilon}\| E \|_k \tag{347}
\]

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**Proof.** This follows a similar proof in [29]. Refer to the previous bound in section 4.1 and consider separately small polymers \( X \in S \) and large polymers \( X \notin S \). For \( X \notin S \) we have \( d_M(X) > L \) and so in the sum \( (307) \) restricted to \( X \notin S \) we can take \( e^{-Ld_M(X)} \leq e^{-L(e^{-(n-1)d_M(X)} - \epsilon)} \). The rest of the estimate proceeds essentially as before and we find that these terms are bounded by \( O(1) L^3 e^{-L\|E_k\|} \) which is more than enough.

Thus we consider small sets. It suffices to show that \( (305) \) can be improved for \( X \in S \) to

\[
\|(SLE)(X)\|_{\mathcal{R}_{k+1}, h_{k+1}} \leq O(1) L^{-4} \sum_{n} \|h_{n+1}h_{k+1,2}\|_{L^{-2} X} \leq O(1) L^{-4} \sum_{n} \|h_{n+1}h_{k+1,2}\|_{L^{-2} X} \quad (348)
\]

The extra factor \( L^{-\frac{4}{2}} \) beats the factor \( L^3 \) in the blocking operation and we have the result. As noted in the proof of lemma 14 for small polymers we can replace the extended domain \( \mathcal{R}_k \) by the more manageable \( \hat{\mathcal{R}}_k \). So it suffices to prove for \( X \in S \)

\[
\|(SLE)(X)\|_{\mathcal{R}_{k+1}, h_{k+1}} \leq O(1) L^{-4} \sum_{n} \|h_{n+1}h_{k+1,2}\|_{L^{-2} X} \quad (349)
\]

Taking account the total number of fermi fields is even we write

\[
\|SLE(X)\|_{\mathcal{R}_{k+1}, h_{k+1}} = \|(SLE)_{00}(X)\|_{\mathcal{R}_{k+1}} + \frac{1}{2} h_{k+1,1} h_{k+1,2} \|SLE(11)(X)\|_{\mathcal{R}_{k+1}} + \frac{1}{2} h_{k+1,2} \|SLE(02)(X)\|_{\mathcal{R}_{k+1}} + \sum_{n+m \geq 4} \frac{h_{k+1,1} h_{k+1,2}}{n!m!} \|SLE(0m)(X)\|_{\mathcal{R}_{k+1}} \quad (350)
\]

and look at each term separately.

The last term in (350) is bounded by (304) by

\[
\sum_{n+m \geq 4} \frac{h_{k+1,1} h_{k+1,2}}{n!m!} L^{-n} L^{-1} E_{nm}(X) \leq L^{-4} \sum_{n+m \geq 4} \frac{h_{k+1,1} h_{k+1,2}}{n!m!} E_{nm}(X) \quad (351)
\]

which suffices.

For the third term in (350) we have by (304) and \( h_{k+1,i} \leq L^{-\frac{4}{2} + \frac{3}{2}} h_{k,i} \)

\[
h_{k+1,1} h_{k+1,2} \|(SLE)(11)(X)\|_{\mathcal{R}_{k+1}} \leq h_{k+1,1} h_{k+1,2} L^{-1} E_{11}(X) \|_{L^{-2} \mathcal{R}_k} \leq L^{-\frac{4}{2} - \epsilon} \|E_{11}(X)\|_{\mathcal{R}_k} \leq L^{-\frac{4}{2} + 2\epsilon} \|E_{11}(X)\|_{\mathcal{R}_k} \quad (352)
\]

Here we have assumed \( 1 - \epsilon < \alpha < 1 \) so \( \alpha - \epsilon > 1 - 2\epsilon \). The fourth term in (350) is even easier since we have \( L^{-2(1+\alpha)} \) instead of \( L^{-1} L^{-1(1+\alpha)} \). But for the second term we only have \( L^{-2} \) which is not enough.

To continue we have to take advantage of the scaling in \( \mathcal{A} \). Let \( x_0 \) be a point in \( X \) and for \( \mathcal{A} \in \mathcal{R}_{k+1} \) define \( A'(x) = A(x) - A(x_0) \). Then this is a gauge transformation with \( \lambda(x) = A(x_0) \cdot (x - x_0) \). For any fixed \( n \) and \( x \in X \) we have for \( e_k \) sufficiently small

\[
|A'(x)| < O(1) M L \max(\|A\|_\infty) \leq O(1) M L e^{-\frac{4}{2} + 2\epsilon} < L^n e_k^{-\frac{4}{2} + \epsilon} \quad (353)
\]

and the same holds for \( A'_L \). Also \( \|\delta_n A'_L\| < L^{-\frac{4}{2} e_k^{-\frac{4}{2} + 2\epsilon}} \) and \( \|\partial_n A'_L\| < L^{-\frac{4}{2} + \epsilon} e_k^{-\frac{4}{2} + 3\epsilon} \) from (303) and we conclude that \( A'_L \in T^{-\frac{4}{2}} \mathcal{R}_k \) which improves the original \( A_L \in T^{-\frac{4}{2}} \mathcal{R}_k \). Since the gauge transformation is complex we no longer have that \( \|E_{nm}(X, A)\| \) is gauge invariant. The kernels are transformed by
phase factors $\varepsilon^{\imath k \lambda}$. However $|\varepsilon^{\imath k \lambda}| \leq O(1) M L \varepsilon^k \leq 1$ and so there is a constant $c = O(1)$ (depending on $n, m$) so
\[
\frac{1}{c} \| \mathcal{E}_{nm}(X, \mathcal{A}_L) \| \leq \| \mathcal{E}_{nm}(X, \mathcal{A}_L') \| \leq c \| \mathcal{E}_{nm}(X, \mathcal{A}_L) \| \tag{354}
\]

Now for the second term in (350) we have
\[
\frac{1}{2} \mathcal{H}_k L^{-2} \| \mathcal{E}_{20}(X, \mathcal{A}_L) \| = \frac{1}{2} \mathcal{H}_k L^{-2} \| \mathcal{E}_{20}(X, \mathcal{A}_L') \| \leq O(1) \mathcal{H}_k L^{-2} \| \mathcal{E}_{20}(X, \mathcal{A}_L') \| \tag{355}
\]

Since $\mathcal{A}_L' \in L^{-\frac{2}{3}} \mathcal{R}_k$ we have that $t \rightarrow t \mathcal{A}_L'$ is an analytic function from complex $t$ satisfying $|t| < L^\frac{2}{3}$ to $\mathcal{R}_k$. Hence $t \rightarrow \mathcal{E}_{20}(X, t \mathcal{A}_L')$ is analytic with norm bounded by $\| \mathcal{E}_{20}(X) \|_{\mathcal{R}_k}$. Since $\mathcal{E}_{20}(X, 0) = 0$ we have
\[
\mathcal{E}_{20}(X, \mathcal{A}_L') = \frac{1}{2\pi i} \int_{|t| = L^\frac{2}{3}} \frac{dt}{t(t - 1)} \mathcal{E}_{20}(X, t \mathcal{A}_L') dt
\]
and this gives the estimate for $\mathcal{A}_L \in \mathcal{R}_{k+1}$
\[
\| \mathcal{E}_{20}(X, \mathcal{A}_L') \| \leq L^{-\frac{2}{3}} \| \mathcal{E}_{20}(X) \|_{\mathcal{R}_k}
\]

Put this in (355) and the second term in (350) is bounded by
\[
O(1) \mathcal{H}_k L^{-\frac{2}{3}} \| \mathcal{E}_{20}(X) \|_{\mathcal{R}_k} \leq O(1) L^{-\frac{2}{3}} \| \mathcal{E}(X) \|_{\mathcal{R}_{k,n_k}}
\]

Finally consider the first term in (350). Since $\mathcal{E}_{00}(X, \mathcal{A})$ vanishes at zero and is even in $\mathcal{A}$ the expansion around $\mathcal{A} = 0$ starts with the second order term. We have for $\mathcal{A}_L \in \mathcal{R}_{k+1}$, again making a gauge transformation to $\mathcal{A}'$
\[
(\mathcal{S}_L \mathcal{E})_{00}(X, \mathcal{A}) = \mathcal{E}_{00}(X, \mathcal{A}_L) = \mathcal{E}_{00}(X, \mathcal{A}_L') = \frac{1}{2\pi i} \int_{|t| = L^\frac{2}{3}} \frac{dt}{t(t - 1)} \mathcal{E}_{00}(X, t \mathcal{A}_L')
\]
which gives for
\[
(\mathcal{S}_L \mathcal{E})_{00}(X, \mathcal{A}) \| \leq O(1) L^{-\frac{2}{3}} \| \mathcal{E}_{00}(X) \|_{\mathcal{R}_k} \leq O(1) L^{-\frac{2}{3}} \| \mathcal{E}(X) \|_{\mathcal{R}_{k,n_k}}
\]

### 4.7 polymer propagators

We can also localize the fermion propagators with polymers using the random walk expansion (139). Assume $\mathcal{A}$ is in the domain (134) or the smaller $\mathcal{R}_k$, and for a walk $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ define $X^{\omega}_L = \cup_{i=0}^n X_{\omega_i}$. Then write
\[
S_k(\mathcal{A}) = \sum_{X \in D_k} S_k(X, \mathcal{A})
\]
where
\[
S_k(X, \mathcal{A}) = \sum_{\omega, X_{\omega} = X} S_k(\omega) = \sum_{n=0}^{\infty} \sum_{\omega, |\omega| = X, X_{\omega} = X} S_k(\omega, \mathcal{A})
\]

Then $S_k(X, \mathcal{A})$ only depends on $\mathcal{A}$ in $X$, and the kernel $S_k(X, \mathcal{A}, x, y)$ vanishes unless $x, y \in X$.

Recall that if $|\omega| = n$ then $|S_k(\omega, \mathcal{A})f| \leq C(M^{-1})^n \|f\|_{\infty}$. But $d_M(X) \leq |X_M' - X_M| \leq 27(n + 1)$ so we can make the estimate
\[
(CM^{-\frac{1}{3}})^n \leq O(1) (CM^{-\frac{1}{3}})^{d_M(X)/27} \leq O(1) e^{-\kappa d_M(X)}
\]
for $M$ sufficiently large. The remaining factor $(CM^{-\frac{1}{3}})^n$ still gives the overall convergence of the series and we have the bound
\[
|S_k(X, \mathcal{A})f| \leq Ce^{-\kappa d_M(X)} \|f\|_{\infty}
\]

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5 Fermion determinant

5.1 a determinant identity

In [16] Balaban establishes the following identity for the determinant of a positive self-adjoint matrix T. It is

\[ \det T = \exp( \text{Tr} \log T) \]  

where for any \( R_0 > 0 \)

\[
\log T = T \int_{R_0}^{\infty} \frac{dx}{x} (T + x)^{-1} - \int_0^{R_0} dx (T + x)^{-1} + \log R_0
\]

Here we prove a generalization for the case where T has negative spectrum as well. We take the branch of the logarithm with the cut on the negative imaginary axis, so \( \log \) is defined on the entire real axis except the origin. Then the identity \( \det T = \exp( \text{Tr} \log T) \) still holds and we have

Lemma 16. Let T be an invertible self-adjoint matrix. Then

\[ \det T = \exp( \text{Tr} \log T) \]

where for any \( R_0 > 0 \)

\[
\log T = T \int_{R_0}^{\infty} \frac{dy}{y} (T + iy)^{-1} - i \int_0^{R_0} dy (T + iy)^{-1} + \log R_0 + \frac{i\pi}{2}
\]

Proof. Consider simple closed curve \( \Gamma \) traversed counterclockwise and made up of the pieces

- \( \Gamma_R = \{ z \in \mathbb{C} : |z| = R, \ -\frac{\pi}{2} + \epsilon \leq \arg z \leq \frac{\pi}{2} - \epsilon \} \)
- \( \Gamma_- = \{ z \in \mathbb{C} : r \leq |z| \leq R, \ \arg z = \frac{\pi}{2} - \epsilon \} \)
- \( \Gamma_r = \{ z \in \mathbb{C} : |z| = r, \ -\frac{\pi}{2} + \epsilon \leq \arg z \leq \frac{\pi}{2} - \epsilon \} \)
- \( \Gamma_+ = \{ z \in \mathbb{C} : |z| \leq r, \ \arg z = -\frac{\pi}{2} + \epsilon \} \)

For \( R \) sufficiently large and \( r \) sufficiently small this encloses the spectrum of \( T \). The function \( \log z \) is analytic inside \( \Gamma \) and so

\[
\log T = \frac{1}{2\pi i} \int_{\Gamma} \log z \ (z - T)^{-1}
\]

Now for any \( r < R_0 < R \) split the contour by \( \Gamma = \Gamma_- + \Gamma_+ \) where \( \Gamma_- = \Gamma \cap \{ z : |z| \leq R_0 \} \) and \( \Gamma_+ = \Gamma \cap \{ z : |z| \geq R_0 \} \). In the integral over \( \Gamma_+ \) we insert the identity

\[
(z - T)^{-1} = z^{-1} + z^{-1}T(z - T)^{-1}
\]

We take the limit \( \epsilon \to 0 \). For the first term the discontinuity in \( \log z \) across the negative imaginary axis contributes \(-2\pi i\) and we get

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_>} \frac{dz}{z} \log z = -\int_{R_0}^{R} \frac{dy}{y} + \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z} \log z
\]

\[
= -\log R + \log R_0 + \frac{1}{2\pi i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta (\log R + i\theta)
\]

\[
= \log R_0 + \frac{i\pi}{2}
\]

For the integral of the second term we have

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_>} \frac{dz}{z} \log z (z - T)^{-1} = T \int_{R_0}^{R} \frac{dy}{y} (iy + T)^{-1} + \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z} \log z (z - T)^{-1}
\]
Lemma 17. For the first term the integrand is \(O(y^{-2})\) and it converges to the integral over \([0, R_0]\). Finally there is the integral over \(\Gamma_c\) which is

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_c} dz \log z (z - T)^{-1} = -i \int_{r}^{R_0} dy(iy + T)^{-1} + \frac{1}{2\pi i} \int_{|z| = r} dz \log z (z - T)^{-1} \tag{372}\]

Take the limit \(r \to 0\). Since zero is not an eigenvalue the second term is \(O(\log r)\) and converges to zero. For the first term the integrand is bounded and it converges to the integral over \([0, R_0]\).

5.2 determinant of the fluctuation operator

We want to apply this to \(D_k(A) + bL^{-1}P(A)\) where \(D_k(A) = b_k - b_k^2Q_k(A)S_k(A)Q_k^T(-A)\) and \(S_k(A) = (\mathcal{D}_A + \tilde{m} + b_kP(A))^{-1} = \mathcal{D}_A - \gamma \cdot \nabla A - \frac{1}{2} \eta \Delta A \tag{373}\)

At first we assume \(A\) is real. Then \((\gamma \cdot \nabla A)^* = -\gamma \cdot \nabla A\) while \(\Delta^*_A = \Delta_A\), so \(\mathcal{D}_A\) is not self-adjoint. However \((\gamma_3 \gamma \cdot \nabla A)^* = \gamma_3 \gamma \cdot \nabla A\) and \(\gamma_3 \Delta^*_A = \gamma_3 \Delta_A\), so \(\gamma_3 \mathcal{D}_A\) is self-adjoint. The same is true for \(\gamma_3 P_k(A)\) and hence for \(S_k(A)\gamma_3\) and \(D_k(A)\gamma_3\). Therefore \((D_k(A) + bL^{-1}P(A))\gamma_3\) is self-adjoint. Since \(\det \gamma_3 = 1\) this has the same determinant as \(D_k(A) + bL^{-1}P(A)\) namely \(\delta Z_k(A)\).

Lemma 17. For \(A\) real and in the domain \([134]\) (or \(\tilde{R}_k\))

\[
\delta Z_k(A) = \det((D_k(A) + bL^{-1}P(A))\gamma_3) = \exp \left( 4T^R_{N-k} \left( (1 - L^{-3}) \log b_k + L^{-3} \log (b_k + bL^{-1}) \right) - i\gamma_3 b_k^2 \int_0^\infty Tr \left[ B_{k,y}(A)Q_k(A)S_k(A)Q_k^T(-A)B_{k,y}(A) \right] dy \right) \tag{374}
\]

where

\[
B_{k,y}(A) = \frac{1}{b_k + i\gamma_3 y} (I - P(A)) + \frac{1}{b_k + bL^{-1} + i\gamma_3 y} P(A) \quad S_{k,y}(A) = \left( \mathcal{D}_A + \tilde{m} + \alpha_{k,y} P_k(A) + \beta_{k,y} P_{k+1}(A) \right)^{-1} \tag{375}
\]

\[
\alpha_{k,y} = \frac{b_k \gamma_3 y}{b_k + i\gamma_3 y} \quad \beta_{k,y} = \frac{b_k^2 bL^{-1}}{(b_k + bL^{-1} + i\gamma_3 y)(b_k + i\gamma_3 y)}
\]

Remark. Note that \(S_{k,y}(A)\) interpolates between \(S_k(A)\) at \(y = \infty\) and \(S^0_{k+1}(A)\) at \(y = 0\) (use \([61]\)).

Proof. By lemma \([16]\)

\[
\delta Z_k(A) = \exp \left( Tr \log \left( (D_k(A) + bL^{-1}P(A))\gamma_3 \right) \right) \tag{376}
\]

where for any \(R_0\)

\[
\log \left( (D_k(A) + P_k(A))\gamma_3 \right) = \left( D_k(A) + \frac{b}{L} P_k(A) \right) \gamma_3 \int_{R_0}^\infty dy \left( (D_k(A) + bL^{-1}P(A))\gamma_3 + iy \right)^{-1} - i \int_0^{R_0} dy \left( (D_k(A) + bL^{-1}P(A))\gamma_3 + iy \right)^{-1} + \log R_0 + \frac{i\pi}{2} \tag{377}
\]

\[
= \left( D_k(A) + bL^{-1}P(A) \right) \int_{R_0}^\infty \frac{dy}{y} \Gamma_{k,y}(A) - i\gamma_3 \int_0^{R_0} dy \Gamma_{k,y}(A) + \log R_0 + \frac{i\pi}{2}
\]
Here we defined
\[ \Gamma_{k,y}(A) = \left( D_k(A) + bL^{-1}P(A) + i\gamma_3y \right)^{-1} \]  
(378)

From appendix [I] we have the representation
\[ \Gamma_{k,y}(A) = B_{k,y}(A) + b_2B_{k,y}(A)Q_k(A)S_{k,y}(A)Q_k^T(-A)B_{k,y}(A) \]  
(379)

Now in (377) take the limit \( R_0 \to \infty \). We have \( \Gamma_{k,y}(A) = O(y^{-1}) \), hence the first term in (377) goes to zero. The second term in (377) is
\[ \int_0^{R_0} \frac{-i\gamma_3dy}{b_k + i\gamma_3y} \left( I - P(A) \right) + \int_0^{R_0} \frac{-i\gamma_3dy}{b_k + bL^{-1} + i\gamma_3y} P(A) \]
\[ - i\gamma_3b_k^2 \int_0^{R_0} B_{k,y}(A)Q_k(A)S_{k,y}(A)Q_k^T(-A)B_{k,y}(A) \, dy \]
(380)

As \( y \to \infty \) we have \( B_{k,y}(A) = O(y^{-1}) \) and we show below that \( S_{k,y}(A) = O(1) \). Hence last term is \( O(y^{-2}) \) so we can take the limit \( R_0 \to \infty \). For the first term we compute
\[ \int_0^{R_0} \frac{-i\gamma_3dy}{b_k + i\gamma_3y} = \int_0^{R_0} dy \frac{-i\gamma_3(b_k - i\gamma_3y)}{b_k^2 + y^2} dy = -i\gamma_3 \tan^{-1} \left( \frac{R_0}{b_k} \right) + \frac{1}{2} \left( \log b_k^2 - \log(b_k^2 + R_0^2) \right) \]
(381)

and similarly for the second term. Now use \( \tan^{-1}(R_0/b_k) \to \pi/2 \) and \(-\frac{1}{2} \log(b_k^2 + R_0^2) + \log R_0 \to 0\) and obtain
\[ \log \left( (D_k(A) + \frac{b}{L}P(A))\gamma_3 \right) = \log b_k(I - P(A)) + \log(b_k + bL^{-1})P(A) \]
\[ - i\gamma_3b_k^2 \int_0^\infty B_{k,y}(A)Q_k(A)S_{k,y}(A)Q_k^T(-A)B_{k,y}(A) \, dy + i\frac{\pi}{2}(1 - \gamma_3) \]
(382)

For the determinant we need to take the trace of this and exponentiate. The trace of the projection is
\[ \text{Tr } P(A) = \text{Tr } (Q^T(-A)Q(A)) = \text{Tr } (Q(A)Q^T(-A)) = 4|\mathbb{T}_{N-k}^1| = 4L^{-3}|\mathbb{T}_{N-k}^0| \]
(383)
independent of \( A \). Similarly \( \text{Tr } (I - P(A)) = 4(1 - L^{-3})|\mathbb{T}_{N-k}^0| \). Furthermore
\[ \text{Tr } i\frac{\pi}{2}(1 - \gamma_3) = \text{Tr } i\frac{\pi}{2} = i\frac{\pi}{2} 4|\mathbb{T}_{N-k}^0| = 2\pi i|\mathbb{T}_{N-k}^0| \]
(384)
does not contribute when exponentiated. Hence we have the result (374).

We will also need a random walk expansion for \( S_{k,y}(A) \). As for \( S_k(A) \) the main ingredient is control over a local inverses for the modified Dirac operator. Instead of lemma [4] we have:

**Lemma 18.** Under the hypotheses of lemma [2] and for \( A \) in the domain (134) there is an operator \( S_{k,y}(\mathbb{D}, A) \) on functions on \( \mathbb{D}(5) \) satisfying
\[ \left( \left( D_A + \bar{m}_k + \alpha_k y P_k(A) + \beta_{k,y} P_{k+1}(A) \right) S_{k,y}(\mathbb{D}, A) f \right)(x) = f(x) \quad x \in \mathbb{D} \]
(385)

and
\[ |S_{k,y}(\mathbb{D}, A) f|, \leq C\|f\|_\infty \]
\[ |1_{\Delta_+} S_{k,y}(\mathbb{D}, A) 1_{\Delta'_+} f|, \leq C e^{-\gamma d(y,y')}\|f\|_\infty \]
(386)

The proof is similar to the analysis of [21], and is postponed to [30] where multi-scale random walk expansions are discussed in detail. Assuming this result we have instead of lemma [5]
Lemma 19. Under the hypotheses of lemma [5] and for $A$ in the domain $\{\tilde{A}_k\}$ (or $\tilde{R}_k$) there is a random walk expansion

$$S_{k,y}(A) = \sum_{\omega} S_{k,y,\omega}(A)$$

(387)

converging to a function analytic in $A$ which satisfies

$$|S_k(A)f|, \leq C\|f\|_\infty \quad |1_{\Delta,\nu}S_k(A)1_{\Delta,\nu}f|, \leq C e^{-\gamma d(y,y')}\|f\|_\infty$$

(388)

The proof follows the proof of lemma [5].

As in section 4.7 there is an associated polymer expansion

$$S_{k,y}(A) = \sum_\chi S_{k,y}(\chi,\chi)$$

(389)

with

$$|S_{k,y}(\chi,\chi)f| \leq C e^{-\kappa d_M(\chi)}\|f\|_\infty$$

(390)

6 The main theorem

6.1 the theorem

The starting density on $\mathbb{T}_N^0$ from (24) is

$$\rho_0(A_0, \Psi_0) = \exp \left( -\frac{1}{2} \|dA_0\|^2 - \langle \bar{\Psi}_0, (\mathcal{D}A_0 + m_0)\Psi_0 \rangle - m_0 \langle \bar{\Psi}_0, \Psi_0 \rangle - \varepsilon_0 \right)$$

(391)

For the full analysis of the model we define a sequence of densities $\rho_k(A_k, \Psi_k)$ for fields on $\mathbb{T}_{N-k}^0$ by successive RG transformations. Given $\rho_k$ we first define as in (46) and (164) for fields on $\mathbb{T}_{N-k}^0$

$$\rho_{k+1}(A_{k+1}, \Psi_{k+1}) = - \int \delta(A_{k+1} - QA_k) \delta(\tau A_k) \delta_G \left( \Psi_{k+1} - Q(\tilde{A}_{k+1})\Psi_k \right) \rho_k(A_k, \Psi_k) \mathcal{D}A_k \mathcal{D}\Psi_k$$

(392)

We chose the background field $\tilde{A}_{k+1}$ on $\mathbb{T}_{N-k}^0$ to be a smeared out version of $A_{k+1}$ defined precisely later on. Then we scale to fields on $\mathbb{T}_{N-k}^0$ as in (51), (165) by

$$\rho_{k+1}(A_{k+1}, \Psi_{k+1}) = \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) L^{(b_N^N - b_{N-k-1}^N) - \frac{s_N}{s_{N-k-1}}}$$

(393)

In this paper we consider a bounded field approximation in which $\rho_k$ is replaced by

$$\tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = - \int \chi_{k+1} \, (\tilde{A}_{k+1} - QA_k) \delta(\tau A_k) \delta_G \left( \Psi_{k+1} - Q(\tilde{A}_{k+1})\Psi_k \right) \rho_k(A_k, \Psi_k) \mathcal{D}A_k \mathcal{D}\Psi_k$$

(394)

and scaling is the same. New are the characteristic functions $\chi_k\tilde{\chi}_k$ enforcing bounds on the fields. The bounds are logarithmic in the coupling constant and depend on the quantities

$$p_k = (-\log \epsilon_k)^p \quad p_{0,k} = (-\log \epsilon_k)^{p_0}$$

(395)

where $p, p_0$ are sufficiently large positive integers satisfying $p_0 < p$. Since $\epsilon_k$ is small these are somewhat large. The bounds are on the real minimizer $A_k = H_k A_k$ on $\mathbb{T}_{N-k}^0$ and on the real fluctuation field $(A_k - H_k^* A_{k+1})$ on $\mathbb{T}_{N-k}^0$

$$\chi_k = \chi(\|dA_k\| \leq p_k)$$

$$\tilde{\chi}_k = \chi(\|A_k - H_k^* A_{k+1}\| \leq p_{0,k})$$

(396)
We have $A_k = Q_k A_k$ and $dA_k = Q_k^{(2)} dA_k$ where $Q_k^{(2)}$ is a certain averaging operator on functions on plaquettes. Hence $\chi_k$ also enforces that $|dA_k| \leq p_k$. These restrictions are natural in Balaban’s formulation of the renormalization group. Our goal is to study the flow of these modified transformations. As noted earlier this is the location of the renormalization problem.

We are going to assert that after $k$ steps for real $A_k$ with $|dA_k| \leq p_k$ we have a density $\rho_k(A_k, \Psi_k)$ essentially of the form

$$\rho_k(A_k, \Psi_k) = N_k Z_k Z_k(0) \exp \left( -\frac{1}{2} \|dA_k\|^2 - \Theta_k(A_k, \psi_k(A_k)) - m_k \left( \tilde{\psi}_k(A_k), \psi_k(A_k) \right) - \varepsilon_k \exp(\varepsilon_k A_k) + E_k(A_k, \psi_k^\#(A_k)) \right)$$

(397)

Here $\psi_k(A) = \mathcal{H}_k(A) \Psi_k$ and

$$\psi_k^\#(A) = \left( \psi_k(A), \chi_k(A) \right) = \left( \psi_k(A), \delta_{\alpha,A} \psi_k(A) \right)$$

(398)

The free fermi action $\Theta_k(A_k, \psi_k(A_k))$ is defined in (82), the determinant $Z_k(A_k)$ is defined in (84) and the determinant $Z_k$ is defined in (1.72). The function $E_k(A, \psi^\#(A))$ is a sum over polymer functions

$$E_k(A, \psi^\#(A)) = \sum_{X \in D_k} E_k(X, A, \psi^\#(A))$$

(399)

These assumptions are true for $k = 0$ with $Z_0 = Z_0(A) = 1, E_0 = 0$, and the convention that $A_0 = A_0$ and and $\psi_0(A_0) = \Psi_0$ and $D_0(A_0) = D_0 + \delta_0$.

**Theorem 1.** Let $L$ be sufficiently large, let $M$ be sufficiently large (depending on $L$), and let $\varepsilon$ be sufficiently small (depending on $L, M$). Suppose that $\rho_k(A_k, \Psi_k)$ has the representation (397) for $k$ such that $|dA_k| \leq p_k$. Suppose the polymer function $E_k(X, A, \psi^\#(A_k))$ has kernels $E_k(X, A)$ defined and analytic in $A \in \mathcal{R}_k$ with all the symmetries of section 1.3. Suppose also that

$$|m_k| \leq \varepsilon_k^4 \quad \|E_k\| \leq 1$$

(400)

Then up to a phase shift $\rho_{k+1}(A_{k+1}, \Psi_{k+1})$ has a representation of the same form for $A_{k+1}$ such that $|dA_{k+1}| \leq p_{k+1}$, now with $\varepsilon_{k+1} = L^{1/2} \varepsilon_k$. The bounds (400) do not necessarily hold for $k+1$, but we do have

$$\varepsilon_{k+1} = L^3 \left( \varepsilon_k + \varepsilon(E_k) + \varepsilon_k^0 \right)$$

$$m_{k+1} = L \left( m_k + m(E_k) \right)$$

$$E_{k+1} = L \left( R E_k + E_k^{\text{det}} + E_k^\#(m_k, E_k) \right)$$

(401)

where $\mathcal{L} : \mathcal{K}_k \rightarrow \mathcal{K}_{k+1}$ is the linear reblocking and scaling operator $\mathcal{L} E = (BE)_{L^{-1}}$. The polymer function $E_k^{\text{det}} = E_k^{\text{det}}(X, A)$ depends only on $A$, vanishes at $A = 0$, and satisfies $\|E_k^{\text{det}}\| \leq \varepsilon_k^{1/4-\varepsilon}$. The polymer function $E_k^\# = \mathcal{L}(E_k^\#)$ satisfies

$$\|E_k^\#\|_{k+1} \leq \varepsilon_k^{1/4-6\varepsilon}$$

(402)

Finally $\varepsilon_k^0 = O(\varepsilon_k^n)$ for any $n$. 

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Remarks.

1. The quantities $\varepsilon(E_k), m(E_k)$ are the extracted corrections from $E_k$ to the energy density and mass as defined in (322), and $\mathcal{R}E_k$ is the remaining irrelevant part. They are linear in $E_k$ and satisfy

$$|\varepsilon(E_k)| \leq O(1)||E_k||_{k} \quad m(E_k) \leq O(1)\rho_k^\frac{k}{2}||E_k||_{k}$$  \(403\)

For the kernel of $\mathcal{LRE}_k$ we allow the adjustment of lemma [14] and take $\mathcal{LRE}_k = (\mathcal{B}(\mathcal{R}E_k)^{2})_{L-1}$. Combining the results of lemma [13] lemma [14] we then have the key contactive estimate

$$||\mathcal{LRE}_k||_{k+1} \leq O(1)L^{-\frac{k}{2}+2}||E_k||_{k}$$  \(404\)

2. The reference to the phase shift means we are actually showing that $\rho_{k+1}(A_{k+1}, e^{i\theta}\Psi_{k+1})$ has the stated form for some real $\theta = \theta(A_{k+1})$. For iterating the transformation we redefine it as $\rho_{k+1}(A_{k+1}, \Psi_{k+1})$. This is allowed since it does not change the crucial normalization property [55].

3. The bound $|dA_k| \leq p_k$ and the local estimate [287] imply that $A_k \in \mathcal{R}_k$, in fact is well inside. Therefore $\psi_k^\#(A_k) = (\psi_k(A_k), \chi_k(A_k))$, the kernels $E_k(X, A_k)$, and hence $E_k(A_k, \psi_k^\#(A_k))$ are all well-defined under our assumptions.

4. The proof follows strategies developed in [11], [12], [19], [25], [29].

6.2 proof of the theorem

6.2.1 extraction

First we normalize by extracting the relevant terms from $E_k$. We have by (321)

$$E_k(A, \psi_k(A), \chi_k(A)) = -\varepsilon(E_k)\text{Vol}(\mathbb{T}_{N-k}) - m(E_k)\left(\hat{\psi}_k(A), \psi_k(A)\right) + E'_k(A, \psi_k(A), \chi_k(A))$$  \(405\)

The last term $E'_k \equiv \mathcal{RE}_k$ has a local expansion $E'_k(A, \psi, \chi) = \sum X E'_k(X, A, \psi, \chi)$ with $E'_k(X, A)$ which are normalized and satisfy

$$||E'_k||_{k} \leq O(1)||E_k||_{k} \leq O(1)$$  \(406\)

Then the representation [321] holds with $E_k$ replaced by $E'_k$ and $\varepsilon_k, m_k$ replaced by

$$\varepsilon'_k = \varepsilon_k + \varepsilon(E_k) \quad m'_k = m_k + m(E_k)$$  \(407\)

A renormalized free action is defined as

$$\mathcal{S}_k^\dagger(A_k, \psi_k(A_k)) = \mathcal{S}_k(A_k, \psi_k(A_k)) + m'_k\left(\hat{\psi}_k(A_k), \psi_k(A_k)\right) + \varepsilon'_k\text{Vol}(\mathbb{T}_{N-k})$$  \(408\)

The block averaging now has the form

$$\hat{\rho}_{k+1}(A_{k+1}, \Psi_{k+1})$$
$$= \mathcal{N}_k Z_k Z_k(0) \int D\dot{A}_k D\Psi_k \chi_k \delta \left(A_{k+1} - QA_k\right) \delta(\tau A_k)\delta_G \left(\Psi_{k+1} - Q(\tilde{A}_{k+1})\Psi_k\right)$$
$$\exp \left(-\frac{1}{2}||dA_k||^2 - \mathcal{S}_k^\dagger(A_k, \psi_k(A_k)) + E'_k(A_k, \psi_k^\#(A_k))\right)$$  \(409\)

What restrictions should we put on $A_{k+1}$ on $\mathbb{T}_{N-k}^1$ here? Later when we scale by $A_{k+1} = A'_{k+1,L}$ we will require $A'_{k+1}$ on $\mathbb{T}_{N-k}^1$ to satisfy $|dA'_{k+1} \leq p_{k+1}$ where $A'_{k+1} = \mathcal{H}_{k+1}A'_{k+1}$. So define an operator $\mathcal{H}_{k+1}^0$ on $\mathbb{T}_{N-k}^1$ and a field $A'_{k+1}$ on $\mathbb{T}_{N-k}^1$ by

$$A'_{k+1} = \mathcal{H}_{k+1}^0A_{k+1} = (\mathcal{H}_{k+1}A_{k+1,L})_{L} = A'_{k+1,L}$$  \(410\)
\[ |dA_{k+1}^0| = |d(A'_{k+1,L})| \leq L^{-\frac{3}{2}} \|dA_{k+1}^0\|_\infty \leq L^{-\frac{3}{2}} p_{k+1} \quad (411) \]

Conversely this condition will scale to \(|dA_{k+1}^0| \leq p_{k+1}\). Thus we impose (411) as the condition on \(A_{k+1}\). Furthermore the statement that \(A_{k+1}^0\) is well inside \(\tilde{R}_{k+1}\) translates to the statement that \(A_{k+1}^0\) is well inside \(\tilde{R}_k\).

### 6.2.2 Gauge field translation

We translate to the minimum of \(\|dA_k\|^2\) on the surface \(\mathcal{Q}A_k = A_{k+1}, \tau A_k = 0\) as before. Write \(A_k = H_k^x A_{k+1} + Z\) and integrate over \(Z\) instead of \(A_k\). Then \(A_k = \mathcal{H}_k A_k\) becomes \(A_{k+1} + Z_k\) where

\[ \hat{A}_{k+1} = \mathcal{H}_k H_k^x A_{k+1} \quad Z_k = \mathcal{H}_k Z \quad (412) \]

This is the \(\hat{A}_{k+1}\) that appears in (392). With this translation \(\frac{1}{2}\|dA_k\|^2\) becomes \(\frac{1}{2}\|d\hat{A}_{k+1}\|^2 + \frac{1}{2} \langle Z, \Delta_k Z \rangle\) as before and our expression becomes

\[ \hat{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = N_k Z_k Z(0) \exp \left( -\frac{1}{2} \|d\hat{A}_{k+1}\|^2 \right) \int DZ \, D\Psi_k \, \chi_k \hat{\chi}_k \delta(QZ) \, \delta(\hat{\tau}Z) \]

\[ \delta_G \left( \Psi_{k+1} - Q(\hat{A}_{k+1})\Psi_k \right) \exp \left( -\frac{1}{2} \left( Z, \Delta_k Z \right) - \mathcal{G}_k^+ \left( A_{k+1} + Z_k, \psi_k(A_{k+1} + Z_k) \right) \right) + E_k^\prime \left( \hat{A}_{k+1} + Z_k, \psi_k(\hat{A}_{k+1} + Z_k) \right) \quad (413) \]

Next note that \(\hat{A}_{k+1}\) differs from \(A_{k+1}^0\) by a gauge transformation. To see this we use \(\mathcal{H}_k^x = \mathcal{H}_k + \partial O_k\) connecting the axial and Landau gauges. Also define \(\mathcal{H}_{k+1}^{0,x}\) by scaling \(\mathcal{H}_{k+1}^x\) and then \(\mathcal{H}_{k+1}^0 = \mathcal{H}_{k+1}^{0,x} + \partial O_{k+1}^0\). But by (161) we also have \(\mathcal{H}_{k+1}^{0,x} A_{k+1} = \mathcal{H}_k^x H_k^x A_{k+1}\). Using these facts

\[ \hat{A}_{k+1} = H_k^x H_k^x A_{k+1} \]

\[ = \mathcal{H}_k H_k^x A_{k+1} - \partial O_k H_k^x A_{k+1} \]

\[ = \mathcal{H}_{k+1}^{0,x} A_{k+1} - \partial O_k H_k^x A_{k+1} \]

\[ = \mathcal{H}_{k+1}^0 A_{k+1} - \partial \left( O_k H_k^x A_{k+1} - O_{k+1}^0 A_{k+1} \right) \]

\[ = A_{k+1}^0 - \partial \omega \quad (414) \]

where the last line defines \(\omega = \omega(A_{k+1})\).

We use this identity to replace \(A_{k+1}\) by \(A_{k+1}^0\) in (413). If \(\omega^{(0)}\) the restriction of \(\omega\) to the unit lattice \(T^N_{\Lambda - k}\) then by (119)

\[ \psi_k(A - \partial \omega) = \mathcal{H}_k (A - \partial \omega) \Psi_k = e^{i\omega_k} \mathcal{H}_k (A) e^{-i\omega^{(0)}_k} \Psi_k \quad (415) \]

We also change variables by \(\Psi_k \rightarrow e^{i\omega^{(0)}_k} \Psi_k\). This is a rotation so the Jacobian is one. Then \(\psi_k(A - \partial \omega)\) becomes \(e^{i\omega^{(0)}_k} \psi_k(A)\) and

\[ \mathcal{G}_k^+ \left( A - \partial \omega + Z, e^{i\omega^{(0)}_k} \psi_k(A + Z) \right) = \mathcal{G}_k^+ \left( A + Z, \psi_k(A + Z) \right) \quad (416) \]

We also have by (411)

\[ \delta_G \left( \Psi_{k+1} - Q(A_{k+1}^0 - \partial \omega) e^{i\omega^{(0)}_k} \Psi_k \right) = \delta_G \left( \Psi_{k+1} - e^{i\omega^{(1)}_k} Q(A_{k+1}^0) \Psi_k \right) \quad (417) \]
where $\omega^{(1)}$ is the restriction of $\omega$ to $\mathbb{T}^N_{N-k}$. We replace $\Psi_{k+1}$ by $e^{i\omega^{(1)} \Psi_{k+1}}$ so the phase factor here disappears as well. The terms $\|\partial A\|^2$ and $Z_k(A)$ and $E_k(A, \psi^k(A))$ are all gauge invariant, as are the characteristic functions.

Finally as in section 3 we parametrize the integral replacing $Z$ by $C\tilde{Z}$, and identify a Gaussian integral by

$$\int f(Z)\delta(QZ)\delta(\tau Z)\exp \left(-\frac{1}{2}\langle Z, \Delta_k Z \rangle\right) = \delta Z_k \int f(C\tilde{Z})\, d\mu_{C_k}(\tilde{Z}) \tag{418}$$

So now we understand $Z_k$ as $Z_k = \mathcal{H}_k C\tilde{Z}$. The characteristic functions have become

$$\chi_k = \chi(\{dA_{k+1} + Z_k\} \le p_k)$$

$$\hat{\chi}_k = \chi(\{C\tilde{Z}\} \le p_{0,k}) \tag{419}$$

But we are assuming $|dA_{k+1}^0| \le L^{-\frac{1}{2}}p_{k+1} \le L^{-\frac{1}{2}}p_k \le \frac{1}{2}p_k$ and by the bounds (237) on $\mathcal{H}_k$ the fluctuation field $Z_k = \mathcal{H}_k C\tilde{Z}$ satisfies $|dZ_k| \le C\|C\tilde{Z}\|_{\infty} \le C|p_{0,k}| \le \frac{1}{2}p_k$. Thus the first characteristic function is always one (and we could have omitted it from the start).

With all these changes:

$$\tilde{\rho}_{k+1}(A_{k+1}, e^{i\omega^{(1)}} \Psi_{k+1}) = N_k Z_k \delta Z_k(0) \exp \left(-\frac{1}{2}\|dA\|^2\right) \int d\mu_{C_k}(\tilde{Z})\, D\Psi_k$$

$$\tilde{\chi}_k \delta_{G} \left(\Psi_{k+1} - Q(A)\Psi_k\right) \exp \left(-\frac{1}{2}\langle Z, \Delta_k Z \rangle - G_k^+(A + Z_k, \psi_k(A + Z_k))\right) + E_k(A + Z_k, \psi^k(A + Z_k)) \tag{420}$$

$$E_k^1(A, Z, \psi_k(A)) = G_k(A, \psi_k(A)) - G_k(A + Z_k, \psi_k(A + Z_k))$$

$$E_k^2(A, Z, \psi^k(A), \Psi_k) = E_k(A + Z, \psi^k(A + Z)) - E_k(A, \psi^k(A))$$

$$E_k^3(A, Z, \psi_k(A), \Psi_k) = m_k\left(\tilde{\psi}_k(A), \tilde{\psi}_k(A)\right) - m_k\left(\tilde{\psi}_k(A + Z), \tilde{\psi}_k(A + Z)\right)$$

These functions are naturally functions of $\Psi_k$ but we eventually change back to functions of $\psi_k(A)$ or $\psi^k(A)$ using the identity $\Psi_k = T_k(A)\psi_k(A)$ where $T_k(A)$ is the left inverse of $\mathcal{H}_k(A)$:

$$T_k(A)\psi \equiv b_{k-1}^{-1} Q_k(A) \left(\mathcal{D}_A + \tilde{m}_k + b_k Q_k(A) Q_k(A)\right)\psi \tag{422}$$

In appendix C in [29] it is shown that $\|Q_k(A)\|_{\infty} \le C\|f\|_{\infty}$ and that $\|Q_k(A)(\eta \Delta A)f\|_{\infty} \le C\|\partial A f\|_{\infty} \le C\|f\|_{\infty}$ and hence $|\mathcal{D}_A f| \le C\|f\|_{\infty}$. Since also $\|Q_k(A)f\|_{\infty} \le C\|f\|_{\infty}$ we have for some constant $C_T$

$$|T_k(A)f| \le C_T\|f\|_{\infty} \tag{423}$$

Now with $E^{(3)} = E^{(1)} + E^{(2)} + E^{(3)}$ we have the representation

$$\tilde{\rho}_{k+1}(A_{k+1}, e^{i\omega^{(1)}} \Psi_{k+1}) = N_k Z_k \delta Z_k(0) \exp \left(-\frac{1}{2}\|dA\|^2\right) \int d\mu_{C_k}(\tilde{Z})\, D\Psi_k$$

$$\tilde{\chi}_k \delta_{G} \left(\Psi_{k+1} - Q(A)\Psi_k\right) \exp \left(-\frac{1}{2}\langle Z, \Delta_k Z \rangle + E_k^1(A, \psi_k(A)) + E_k^{(3)}(A, Z_k, \psi_k^k(A))\right)\bigg|_{A=A_{k+1}^0} \tag{424}$$
6.2.3 first localization

We want to localize the terms contributing to $E^{(\leq 3)}_k(\mathcal{A}, \mathcal{Z}, \Psi_k)$. These will be treated in the region

$$\mathcal{A} \in \frac{1}{2} \tilde{R}_k \quad |\mathcal{Z}| \leq e^{-3\epsilon} \quad |\partial \mathcal{Z}| \leq e^{-2\epsilon} \quad \|\delta, \partial \mathcal{Z}\| \leq e^{-\epsilon}$$

(425)

The conditions on $\mathcal{Z}$ are stronger than the condition $\mathcal{Z} \in \frac{1}{7} R_k$. Thus we have $\mathcal{A} + \mathcal{Z} \in \tilde{R}_k$ so the $E^{(i)}_k(\mathcal{A}, \mathcal{Z}, \psi, \Psi_k)$ as given by (421) are well-defined.

We are particularly interested in the case $\mathcal{A} = A_{k+1}^0$ and $\mathcal{Z} = Z_k$. We have already noted that $A_{k+1}^0 \in \frac{1}{2} \tilde{R}_k$, but what about $Z_k = \mathcal{H}_k C \tilde{Z}$? We take as a condition on $\tilde{Z}$ that $|\tilde{Z}| \leq e^{-2\epsilon}$ and consider the domain

$$\mathcal{A} \in \frac{1}{2} \tilde{R}_k \quad |\tilde{Z}| \leq e^{-2\epsilon}$$

(426)

On this domain $|Z| \leq C e^{-2\epsilon} \leq e^{-\epsilon}$ and similarly for derivatives so we are in the domain (425).

Furthermore the characteristic function $\chi_k$ enforces that $|\tilde{Z}| \leq p_{0, k}$ which puts us in the domain (426).

**Lemma 20.** The function $E^{(1)}_k(\mathcal{A}, \mathcal{Z}, \Psi_k)$ has a local expansion

$$E^{(1)}_k(\mathcal{A}, \mathcal{Z}, \Psi_k) = \sum_{X} E^{(1)}_k(X)$$

where $E^{(1)}_k(X, \mathcal{A}, \tilde{Z}, \psi_k(\mathcal{A}))$ depends on these fields only in $X$, is analytic in (426) and satisfies there

$$\|E^{(1)}_k(X, \mathcal{A}, \tilde{Z})\|_{\mathcal{A}} \leq e^{-\epsilon} e^{-(\kappa_{X} - 2) d_M(X)}$$

(427)

**Proof.** First we study $E^{(1)}_k(\mathcal{A}, \mathcal{Z}, \Psi_k)$ in the region (425). By (82)

$$E^{(1)}_k(\mathcal{A}, \mathcal{Z}, \Psi_k) = \left\langle \Psi_k, \left( D_k(\mathcal{A} + \mathcal{Z}) - D_k(\mathcal{A}) \right) \Psi_k \right\rangle = \left\langle \Psi_k, \left( M_k(\mathcal{A} + \mathcal{Z}) - M_k(\mathcal{A}) \right) \Psi_k \right\rangle$$

(428)

where $M_k$ is the operator on spinors on $T^0_{N-k}$

$$M_k(\mathcal{A}) = -b_k^2 Q_k(\mathcal{A}) S_k(\mathcal{A}) Q_k^T(\mathcal{A})$$

(429)

Next insert $S_k(\mathcal{A}) = \sum_{X} S_k(X, \mathcal{A})$ from (361) and get $M_k(X) = \sum_{X} M_k(X, \mathcal{A})$ where

$$M_k(X, \mathcal{A}) = -b_k^2 Q_k(\mathcal{A}) S_k(X, \mathcal{A}) Q_k^T(\mathcal{A})$$

(430)

Then $E^{(1)}_k = \sum_{X} \tilde{E}^{(1)}_k(X)$ where

$$\tilde{E}^{(1)}_k(X, \mathcal{A}, \mathcal{Z}, \Psi_k) = \left\langle \Psi_k, \left( M_k(X, \mathcal{A} + \mathcal{Z}) - M_k(X, \mathcal{A}) \right) \Psi_k \right\rangle$$

(431)

This only depends on $\Psi_k$ in $X$ since $S_k(X, \mathcal{A})$ only connects points in $X$ and $Q_k(\mathcal{A})$ is local on $M$ scale.

The matrix elements of $M_k(X, \mathcal{A})$ are

$$[M_k(X, \mathcal{A})]_{xy} = \left\langle \delta_x, M_k(X, \mathcal{A}) \delta_y \right\rangle = \left( Q_k^T(\mathcal{A}) \delta_x, S_k(X, \mathcal{A}) Q_k^T(\mathcal{A}) \delta_y \right)$$

(432)

Then by (361)

$$\|M_k(X, \mathcal{A})\|_{xy} \leq \|Q_k^T(\mathcal{A}) \delta_x\|_1 \|S_k(X, \mathcal{A}) Q_k^T(\mathcal{A}) \delta_y\|_\infty \leq C \|Q_k^T(\mathcal{A}) \delta_x\|_1 \|Q_k^T(\mathcal{A}) \delta_y\|_\infty e^{-\kappa_{d_M(X)}}$$

(433)

$$\leq C e^{-\kappa_{d_M(X)}}$$
To estimate \( \tilde{E}^{(1)}_{k}(X) \) note that under our assumptions \([125]\) on \( Z \) if we take complex \(|t| \leq \frac{1}{2} e^{-\frac{\kappa}{4} + 4\epsilon} \) then \( tZ \in \frac{1}{2} \mathcal{R}_k \), hence \( A + tZ \) is in \( \tilde{\mathcal{R}}_k \). Hence we are in the analyticity domain of \( t \to M_k(A + tZ) \) and can write

\[
[M_k(X, A + Z) - M_k(X, A)]_{xy} = \frac{1}{2\pi i} \int_{|t| = \frac{1}{2} e^{-\frac{\kappa}{4} + 4\epsilon}} \frac{dt}{t(t - 1)} [M_k(A + tZ)]_{xy} \tag{434}
\]

which yields

\[
||M_k(X, A + Z) - M_k(X, A)||_{xy} \leq C e^{\frac{\kappa}{4} - 4\epsilon} e^{-\kappa d_M} e^{-\kappa d}(X) \tag{435}
\]

The kernel of \( \tilde{E}^{(1)}_{k}(X, A, Z, \Psi_k) \) has the single non-zero entry (suppressing spin indices)

\[
\tilde{E}^{(1)}_{k,2}(X, A, Z, (1, x), (0, y)) = [M_k(X, A + Z) - M_k(X, A)]_{xy} \tag{436}
\]

Since \( \text{Vol}(X) = M^3|X| \leq \mathcal{O}(1)M^3 e^{\frac{\kappa}{4} - 4\epsilon} \) and \( h_k^2 = e^{-\frac{\kappa}{2}} \) we have

\[
\| \tilde{E}^{(1)}_{k}(X, A, Z) \|_{h_k} \leq C h_k^2 e^{\frac{\kappa}{4} - 4\epsilon} \text{Vol}(X)^2 e^{-\kappa d_M} \leq C M^6 e^{\frac{\kappa}{4} - 4\epsilon} e^{-(\kappa - 1)d_M}(X) \tag{437}
\]

Specialize to the case \( Z = Z_k = \mathcal{H}_k \mathcal{C} \mathcal{Z} \) and take the domain \([126]\). The \( \tilde{E}^{(1)}_{k}(X, A, Z, \Psi_k) \) is not local in \( \mathcal{Z} \). To localize we use the random walk expansion for \( \mathcal{H}_k \) to introduce weakening parameters \( s = \{s_\square\} \) and define \( Z_k(s) = \mathcal{H}_k(s, A) \mathcal{C} \mathcal{Z} \) and

\[
\tilde{E}^{(1)}_{k}(s, X, A, \mathcal{Z}, \Psi_k) \equiv \tilde{E}^{(1)}_{k}(X, A, Z_k(s), \Psi_k) \tag{438}
\]

For \(|s_\square| \leq M^{\alpha_0}\) and \( \alpha_0 \) sufficiently small these \( s \) dependent quantities satisfy bounds of the same form as the original case \( s_\square = 1 \) so

\[
\| \tilde{E}^{(1)}_{k}(s, X, A, \mathcal{Z}) \|_{h_k} \leq C M^6 e^{\frac{\kappa}{4} - 4\epsilon} e^{-(\kappa - 1)d_M} \tag{439}
\]

In each variable \( s_\square \) for \( \square \subset X \) we interpolate between \( s_\square = 1 \) and \( s_\square = 0 \) by

\[
f(s_\square = 1) = f(s_\square = 0) + \int_0^1 ds_\square \frac{\partial f}{\partial s_\square} \tag{440}
\]

This yields a new expansion \( \tilde{E}^{(1)}_{k}(Y) \) where

\[
\tilde{E}^{(1)}_{k}(Y, A, \mathcal{Z}, \Psi_k) = \sum_{X \subset Y} \int ds_{Y \setminus X} \frac{\partial}{\partial s_{Y \setminus X}} \left[ \tilde{E}^{(1)}_{k}(s, X, A, \mathcal{Z}, \Psi_k) \right]_{s_Y = 0, s_X = 1} \tag{441}
\]

and \( s_X = \{s_\square \}_{\square \subset X} \). Then \( \tilde{E}^{(1)}_{k}(Y) \) only depends on the indicated fields in \( Y \) since there is no coupling through \( Y^c \).

Now \( \tilde{E}^{(1)}_{k}(s, X) \) is analytic in \(|s_\square| \leq M^{\alpha_0}\) and we estimate the derivatives for \( 0 \leq s_\square \leq 1 \) by Cauchy inequalities. Each derivative then contributes a factor \( M^{-\alpha_0} \) and \( M^{-\alpha_0} \leq e^{-\kappa} \) for \( M \) sufficiently large. Hence we gain a factor \( e^{-\kappa |Y - X|} \) from the derivatives in \( s \) and have

\[
\| \tilde{E}^{(1)}_{k}(Y, A, \mathcal{Z}) \|_{h_k} \leq C M^6 e^{\frac{\kappa}{4} - 4\epsilon} \sum_{X \subset Y} e^{-\kappa |Y - X|} \tag{442}
\]

But one can show that \( |Y - X| \leq d_M(X) \geq d_M(Y) \) (see for example \([25]\)). Hence one can extract a factor \( e^{-\kappa (\kappa_0 - 1)d_M(X)} \) leaving a factor \( e^{-\kappa_0 d_M(X)} \) for the convergence of the sum over \( X \). The sum is bounded by \( \mathcal{O}(1)|Y| \leq \mathcal{O}(1)e^{d_M(Y)} \) and so we have the announced bound

\[
\| \tilde{E}^{(1)}_{k}(Y, A, \mathcal{Z}) \|_{h_k} \leq C M^6 e^{\frac{\kappa}{4} - 4\epsilon} e^{-(\kappa - \kappa_0 - 2)d_M(Y)} \tag{443}
\]

55
Finally as in (422) insert \( \Psi_k(A) = T_k(A)\psi_k(A) \) defining
\[
E^{(1)}(X,A,\tilde{Z},\psi_k(A)) = \tilde{E}^{(1)}(X,A,\tilde{Z},T_k(A)\psi_k(A))
\] (444)
From (423) we have \( \|T_k(A)\|_{t,\infty} \leq C_T \). Then by (455) in Appendix A and taking account that the function is quadratic in the fields we have
\[
\|E^{(1)}_k(X,A,\tilde{Z})\|_{h_k} \leq \|\tilde{E}^{(1)}_k(X,A,\tilde{Z})\|_{C_T h_k} \leq C_T^2 \|\tilde{E}^{(1)}_k(X,A,\tilde{Z})\|_{h_k} 
\]
(445)
In the last step we used \( CM^6e^c_k \leq 1 \). This completes the proof.

**Lemma 21.** The function \( E^{(2)}_k = E^{(2)}_k(A,Z,\psi^\#_k(A),\Psi_k) \) has a local expansion \( E^{(2)}_k = \sum X E^{(2)}_k(X) \) where \( E^{(2)}_k(X) = E^{(2)}_k(X,A,Z,\psi^\#_k(A)) \) depends on these fields only in \( X \), is analytic in (429) and satisfies there
\[
\|E^{(2)}_k(X,A,\tilde{Z})\|_{h_k} \leq O(1)e^{\frac{3}{2} - 5r}_k e^{-(\kappa - \kappa_0 - 1)d_M(X)}
\] (446)

**Proof.** First we study \( \tilde{E}^{(2)}_k(A,Z,\psi^\#_k(A),\Psi_k) \) in the domain (423). We have \( E^{(2)}_k = \sum X \tilde{E}^{(2)}_k(X) \) where
\[
\tilde{E}^{(2)}_k(X,A,Z,\psi^\#_k(A),\Psi_k) = E^{(2)}_k(X,A + Z,\psi^\#_k(A + Z)) - E^{(2)}_k(X,A,\psi^\#_k(A))
\]
\[
= E^{(2)}_k(X,A + Z,\psi^\#_k(A) + J^\#_k(A,Z)\Psi_k) - E^{(2)}_k(X,A,\psi^\#_k(A))
\]
(447)
Here we defined
\[
J_k(A,Z)\Psi_k = \psi_k(A + Z) - \psi_k(A) = (H_k(A + Z) - H_k(A))\Psi_k
\] (448)
and
\[
J^\#_k(A,Z)\Psi_k = \psi^\#_k(A + Z) - \psi^\#_k(A) = (J_k(A,Z)\Psi_k,\delta_\alpha J_k(A,Z)\Psi_k)
\] (449)
and
\[
E^{(2)}_k(t,X,A,Z,\psi^\#_k(A),\Psi_k) = E^{(2)}_k(X,A + tZ,\psi^\#_k(A) + tJ^\#_k(A,Z)\Psi_k)
\] (450)

We need to justify the representation (447) and use it for estimates. As in the previous lemma if we take complex \( |u| < c^\frac{3}{2} + 4r \) then \( uZ \in \frac{1}{4}R_k \) and \( A + uZ \) is in \( \tilde{R}_k \). Hence we are in the analyticity domain of \( u \to H_k(A + uZ) \). and can write
\[
J_k(A,Z)f = \frac{1}{2\pi i} \int_{|u|=c^\frac{3}{2} + 4r} \frac{du}{u(u-1)} H_k(A + uZ)f
\] (451)
Using the bounds (152) on \( H_k(A) \) we get
\[
|J_k(A,Z)f| \leq Ce^{\frac{3}{2} - 4r}_k \|f\|_{\infty}
\] (452)
Then for $|t| \leq \epsilon_k^{\frac{3}{4}+\epsilon}$ we have $|t|\|\mathcal{J}_k(A,Z)f| \leq C\epsilon_k^\frac{3}{4}\|f\|_\infty$. The same bound holds with the Holder derivative so $|t|\|\mathcal{J}_k^\#(A,Z)f| \leq C\epsilon_k^\frac{3}{4}\|f\|_\infty$. Hence by (655) in appendix $E_k^t(t,X,A,Z,\psi^\#(A),\Psi_k)$ satisfies (we take $C_T h_k$ for later purposes)

$$
\|E_k^t(t,X,A,Z)\|_{h_k,C_T h_k} \leq \|E_k^t(X,A+tZ)\|_{h_k+C_C h_k} \leq \|E_k^t(X,A+tZ)\|_{h_k} \leq O(1)e^{-\kappa_d M(X)}
$$

Hence the representation (447) holds and we have the bound

$$
\|\tilde{E}_k^{(2)}(X,A,Z)\|_{\tilde{h}_k,C_T h_k} \leq O(1)e^{\frac{3}{4}-5\epsilon}e^{-\kappa_d M(X)}
$$

Now specialize to the case $Z = Z_k = H_k C \tilde{Z}$ and study the function $\tilde{E}_k^{(2)}(X,A,Z_k,\psi^\#(A),\Psi_k)$ on the domain (426). This is not local in $\tilde{Z}$. To localize introduce weakening parameters $s = \{s_\square\}$ based on the random walk expansions and define

$$
\mathcal{J}_k(s,A,Z)\Psi_k = \left(\mathcal{H}_k(s,A+Z) - \mathcal{H}_k(s,A)\right)\Psi_k \\
\mathcal{J}_k^\#(s,A,Z)\Psi_k = \left(\mathcal{J}_k(s,A,Z)\Psi_k, \delta_0 \mathcal{J}_k(s,A,Z)\Psi_k\right)
$$

and

$$
\tilde{E}_k^{(2)}(s,X,A,\tilde{Z},\psi^\#(A),\Psi_k) = \frac{1}{2\pi i} \int_{|t|=\epsilon_k^{\frac{3}{4}+\epsilon}} \frac{dt}{t(t-1)} E_k^t(s,t,X,A,\tilde{Z},\psi^\#(A),\Psi_k)
$$

$$
E_k^t(s,t,X,A,\tilde{Z},\psi^\#(A),\Psi_k) = E_k^t(X,A+tZ(s),\psi^\#(A) + t\mathcal{J}_k^\#(s,A,Z_k(s),\Psi_k)
$$

For complex $|s_\square| \leq M^{\alpha_0}$ we get bounds of the same form and so

$$
\|\tilde{E}_k^{(2)}(s,X,A,\tilde{Z})\|_{\tilde{h}_k,C_T h_k} \leq O(1)e^{\frac{3}{4}-5\epsilon}e^{-\kappa_d M(X)}
$$

Again interpolate between $s_\square = 1$ and $s_\square = 0$ and get a new expansion $E_k^{(2)} = \sum_Y \tilde{E}_k^{(2)}(Y)$ where

$$
\tilde{E}_k^{(2)}(Y,A,\tilde{Z},\psi^\#(A),\Psi_k) = \sum_{X \subset Y} \int ds \frac{\partial}{\partial s} \left[ \tilde{E}_k^{(2)}(s,X,A,\tilde{Z},\psi^\#(A),\Psi_k)\right]_{s=0, s=1}
$$

and $\tilde{E}_k^{(2)}(Y)$ only depends on the indicated fields in $Y$. The derivatives are again estimated by Cauchy inequalities which gives a factor $e^{-\kappa|Y-X|\kappa_d}$. Then

$$
\|E_k^{(2)}(Y,A,\tilde{Z})\|_{h_k,C_T h_k} \leq O(1)e^{\frac{3}{4}-5\epsilon} \sum_{X \subset Y} e^{-\kappa|Y-X|\kappa_d M(X)} \leq O(1)e^{\frac{3}{4}-5\epsilon}e^{-(\kappa-\kappa_0-1)d_M(Y)}
$$

Finally from (422) we define

$$
E_k^{(2)}(Y,A,\tilde{Z},\psi^\#(A)) = \tilde{E}_k^{(2)}(Y,A,\tilde{Z},\psi^\#(A),T_k(A)\psi(A))
$$

which yields the desired bound

$$
\|E_k^{(2)}(Y,A,\tilde{Z})\|_{h_k} \leq \|\tilde{E}_k^{(2)}(Y,A,\tilde{Z})\|_{h_k} \leq O(1)e^{\frac{3}{4}-5\epsilon}e^{-(\kappa-\kappa_0-1)d_M(Y)}
$$
Lemma 22. The function $E^{(3)}_k = E^{(3)}_k(\mathcal{A}, Z_k, \psi_k(\mathcal{A}), \Psi_k)$ has a local expansion $E^{(3)}_k = \sum_X E^{(3)}_k(X)$ where $E^{(3)}_k(X, \mathcal{A}, Z, \psi_k(\mathcal{A}))$ depends on these fields only in $X$, is analytic in $\{\xi\}$ and satisfies there

$$\|E^{(3)}_k(X, \mathcal{A}, Z)\|_{\frac{1}{2}b_k} \leq e^{\frac{1}{4}-6e^{-((\kappa-\kappa_0-1)d_M(X)}}$$

**Proof.** First write

$$m_k\langle \tilde{\psi}_k(\mathcal{A}), \psi_k(\mathcal{A}) \rangle = m_k' \sum \langle \tilde{\psi}_k(\mathcal{A}), 1_\square \psi_k(\mathcal{A}) \rangle \equiv m_k' \sum \mathcal{E}(\square, \psi_k(\mathcal{A}))$$

The kernel $\mathcal{E}(\square)$ has the single non-vanishing element $\mathcal{E}_2(\square, (x, 1), (y, 0)) = m_k' 1_\square (y) \delta(x-y)$ which satisfies $\|\mathcal{E}_2(\square)\| \leq M^3m_k'$ and so

$$\|\mathcal{E}(\square)\|_{b_k} = h_k^2 \mathcal{E}(\square) \leq h_k^2 M^3 m_k'$$

But we are assuming $m_k \leq e^{\frac{1}{2}}$ and by \[323\] $m(E_k) \leq O(1)m_k^{\frac{1}{2}}$. Therefore $m_k' \leq O(1)m_k^{\frac{1}{2}}$. Since also $h_k^2 \leq e^{\frac{1}{2}}$ we have

$$\|\mathcal{E}(\square)\|_{b_k} \leq O(1) M^3$$

Now we are in the situation of lemma \[21\] except that our starting bound is worse by the factor $M^3$. Hence we get the result with a constant $O(1) M^3 e^\frac{1}{4}-5e \leq e^\frac{1}{4}-6e$.

### 6.2.4 fermi field translation

Now in \[424\] with $\mathcal{A} = \mathcal{A}^0_{k+1}$, we diagonalize the quadratic form

$$bL^{-1} \langle \tilde{\Psi}_{k+1} - Q(-\mathcal{A})\tilde{\Psi}_k, \tilde{\Psi}_{k+1} - Q(\mathcal{A})\tilde{\Psi}_k \rangle + \mathcal{S}(\mathcal{A}, \psi_k(\mathcal{A}))$$

which sits in the exponential. As in section \[2.6\] this is accomplished by the transformations

$$\Psi_k = \tilde{\Psi}_k(\mathcal{A}) + W$$

$$\psi_k(\mathcal{A}) = \psi^0_{k+1}(\mathcal{A}) + \mathcal{W}_k(\mathcal{A})$$

Also define $\psi^0_{k+1}(\mathcal{A}) = (\psi^0_{k+1}(\mathcal{A}), \delta_\alpha \psi^0_{k+1}(\mathcal{A}))$ and $\mathcal{W}^{\#}_k(\mathcal{A}) = (\mathcal{W}_k(\mathcal{A}), \delta_\alpha \mathcal{W}_k(\mathcal{A}))$ and then the transformation is

$$\psi^{\#}(\mathcal{A}) = \psi^0_{k+1}(\mathcal{A}) + \mathcal{W}^{\#}_k(\mathcal{A})$$

By lemma \[3\] the expression \[469\] becomes

$$\mathcal{S}^0_{k+1}(\mathcal{A}, \psi_{k+1}, \psi^0_{k+1}(\mathcal{A})) + \langle W, \text{K} \rangle \left(D_k(\mathcal{A}) + bL^{-1}P(\mathcal{A})\right)$$

Again we identify the Gaussian measure

$$\delta Z_k(\mathcal{A}) d\mu_k(\mathcal{A})(W) = \exp \left(-\langle W, \left(D_k(\mathcal{A}) + bL^{-1}P(\mathcal{A})\right)\rangle\right)DW$$

We also define $E^{(4)}_k, E^{(5)}_k$ with $\psi = \psi^0_{k+1}(\mathcal{A}), \mathcal{W} = \mathcal{W}_k(\mathcal{A})$ etc. by

$$E^{(4)}_k(\mathcal{A}, \psi^{\#}, \mathcal{W}^{\#}) = E'_{k}(\mathcal{A}, \psi^{\#} + \mathcal{W}^{\#}) - E'_{k}(\mathcal{A}, \psi^{\#})$$

$$E^{(5)}_k(\mathcal{A}, \psi, \mathcal{W}) = m_k' \langle \tilde{\psi}, \psi \rangle - m_k' \langle \tilde{\psi} + \tilde{\mathcal{W}}, \psi + \mathcal{W} \rangle$$

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Lemma 23. The function $E_k^{(\leq 5)}(A, \tilde{Z}, \psi^#)\) has a polymer expansion $E_k^{(\leq 5)} = \sum_X E_k^{(\leq 5)}(X, A, \tilde{Z}, \psi^#(A))$ where $E_k^{(\leq 5)}(X) = E_k^{(\leq 5)}(X, A, \tilde{Z}, \psi^#(A), W^#(A))$ depends fields only in $X$, is analytic in $X$, and satisfies there

$$\|E_k^{(\leq 5)}(X, A, \tilde{Z})\|_{L^1} \leq e^{\frac{1}{4} - 5\epsilon \kappa - 3\beta_0 - 3\beta_1}d_M(X)$$

Eventually we do this for the gauge field also, but the presence of the characteristic function $\tilde{\chi}$ is a temporary obstable here.
Proof. There are five terms in $E_k^{(≤5)}$ as defined in (472). We first consider the term $E_k^{(4)} = \sum_X E_k^{(4)}(X)$ where $E_k^{(4)}(X)$ can be written

$$E_k^{(4)}(X, A, \psi_{k+1}, W^\#(A)) = \frac{1}{2\pi i} \int_{|t|=\frac{1}{4}e^{-\frac{1}{2}+\varepsilon} \frac{dt}{t(t-1)} E_k^{(4)}(t, X, A, \psi_{k+1}, W^\#(A))}$$

(481)

where

$$E_k^{(4)}(t, X, A, \psi^\#, W^\#(A)) = E_k^+ \left( X, A, \psi^\#(A), tW^\#(A) \right)$$

(482)

$$E_k^+ \left( X, A, \psi^\#(A), W^\#(A) \right) = E_k' \left( X, A, \psi^\#(A) + W^\#(A) \right)$$

Define $h_{0,k} = (e_{k+i}^-, e_{k+i}^+)$. If $|t| \leq \frac{1}{4}e^{-\frac{1}{2}+\varepsilon}$ then $|t|h_{0,k} \leq \frac{1}{4}h_k$. By (355) in Appendix A, then (651), and then $\|E_k\| \leq O(1)$ we have

$$\|E_k^{(4)}(t, X, A)\| \leq \|E_k^+(X, A)\| \leq \|E_k'(X, A)\| \leq O(1)e^{-\kappa dM(X)}$$

Hence (482) gives us for $e_k$ sufficiently small

$$\|E_k^{(4)}(X, A)\| \leq O(1)e^{\frac{1}{2}−2\varepsilon} e^{-\kappa dM(X)}$$

(483)

Next define

$$\tilde{E}_k^{(4)} \left( X, A, \tilde{Z}, \psi_{k+1}^\#, W' \right) = E_k^{(4)} \left( X, A, \tilde{Z}, \psi_{k+1}^\#, W^\#(A) \right)$$

(485)

Now $W_k(A) = H_k(A) T_k(A) W'$ and by (149), (150)

$$|H_k(A) T_k(A)f|, |\delta_{\alpha} H_k(A) T_k(A)f| \leq C\|f\|_\infty$$

(486)

Then by a generalization of lemma (29) in Appendix A to four fields and (484) we have

$$\|\tilde{E}_k^{(4)}(X, A)\| \leq \|E_k^{(4)}(X, A)\| \leq \|E_k'(X, A)\| \leq O(1)e^{\frac{1}{2}−2\varepsilon} e^{-\kappa dM(X)}$$

(487)

To localize in $W'$ we use the random walk expansion described in section 2.10 to introduce weakened operators $\tilde{H}_k(s, A), \Gamma_k(s, A)$ and hence $W_k(s, A) = H_k(s, A) T_k(s, A) W'$. Then define the modified version of (482)

$$\tilde{E}_k^{(4)} \left( s, X, A, \tilde{Z}, \psi_{k+1}^\#, W' \right) = E_k^{(4)} \left( X, A, \tilde{Z}, \psi_{k+1}^\#, W^\#(s, A) \right)$$

(488)

which also satisfies for $|s| \leq M^{\alpha_0}$

$$\|\tilde{E}_k^{(4)}(s, X, A, \tilde{Z})\| \leq O(1)e^{\frac{1}{2}−2\varepsilon} e^{-\kappa dM(X)}$$

(489)

Again interpolate between $s = 1$ and $s = 0$ and get a new expansion $E_k^{(4)} = \sum_Y \tilde{E}_k^{(4)}(Y)$ where

$$\tilde{E}_k^{(4)}(Y, A, \tilde{Z}, \psi_k^\#, W') = \sum_{X \subset Y} \int_{s_Y} d_{s_Y} \frac{\partial}{\partial s_{s_{Y}}} \left[ \tilde{E}_k^{(4)}(s, X, A, \tilde{Z}, \psi_k^\#, W') \right]_{s_Y = 0, s_X = 1}$$

(490)

and $\tilde{E}_k^{(4)}(Y)$ only depends on the indicated fields in $Y$. The derivatives are again estimated by Cauchy inequalities which gives a factor $e^{-\kappa |Y| X_M}$ and then

$$\|\tilde{E}_k^{(4)}(Y, A)\| \leq O(1)e^{\frac{1}{2}−2\varepsilon} \sum_{X \subset Y} e^{-\kappa |Y| X_M} \leq O(1)e^{\frac{1}{2}−2\varepsilon} e^{-(\kappa - \kappa_0 - 1)|dM(Y)}$$

(491)
This is better than we need for the theorem, and so \( \tilde{E}_k^{(4)}(X) \) contributes to \( E_k^{\leq 5}(X) \).

The treatment of \( E_k^{(5)} \) is similar.

Now consider the term \( E_k^{(1),+} = \sum_X E_k^{(1),+}(X) \) where
\[
E_k^{(1),+}(X, A, \tilde{Z}, \psi_{k+1}^0(A), W_k^#(A)) = E_k^{(1)}(X, A, \tilde{Z}, \psi_{k+1}^0(A) + W_k^#(A))
\] (492)

Since \( h_{0,k} \leq \frac{1}{4} h_k \) we have by (554) in Appendix A and (327)
\[
\| E_k^{(1),+}(X, A, \tilde{Z}) \|_{\frac{1}{4} h_k + h_{0,k}} \leq \| E_k^{(1)}(X, A, \tilde{Z}) \|_{\frac{1}{4} h_k} \leq \text{O}(1) e^{\frac{k}{4} (5 - 5 k)} e^{-((\kappa - 2 k_0 - 3))d_M(X)}
\] (493)

Now we are in the same situation as we were at (484) in the analysis of \( E_k^{(4)} \), but with slightly weaker bounds. Localizing in \( W' \) as we did then find a new expansion \( \tilde{E}_k^{(1),+}(Y) \) where
\[
\| \tilde{E}_k^{(1),+}(Y, A, \tilde{Z}) \|_{\frac{1}{4} h_k, 1} \leq \text{O}(1) e^{\frac{k}{4} (5 - 5 k)} e^{-((\kappa - 2 k_0 - 3))d_M(Y)}
\] (494)

This is again better than we need so \( \tilde{E}_k^{(4)}(X) \) contributes to \( E_k^{\leq 5}(X) \).

The treatment of \( E_k^{(2),+} \) and \( E_k^{(3),+} \) is similar.

### 6.2.6 cluster expansion

We study the fluctuation integral which can now be written
\[
\Xi_k(A, \psi_{k+1}^0(A)) = \int \exp \left( \sum_X E_k^{(\leq 5)}(X, A, \tilde{Z}, \psi_{k+1}^0(A), W') \right) \tilde{\chi}_k(CZ) d\mu C_k \tilde{Z} d\mu \left( W' \right)
\] (495)

The cluster expansion gives this a local structure. The most straightforward way to proceed would be to mimic the fermion treatment and make a change of variables \( \tilde{Z} = C_k^{\frac{1}{2}} Z \) which changes the Gaussian measure to \( d\mu(Z') \). Then localize \( E_k^{\leq 5}(X) \) in the new variable \( Z' \). With strictly local \( E_k^{\leq 5}(X) \) and both ultralocal measures one might now contemplate using a standard cluster expansion. The trouble is that the characteristic function \( \chi_k \) is messed up by the change of variables. For the purposes of this paper one could fix this by altering the definition of \( \chi_k \). This is the approach taken in [29] and earlier versions of this paper. However this does not generalize very well when we consider the full model in [90]. Instead we closely follow the approach used by Balaban [12].

**Theorem 2. (cluster expansion)** For \( A \in \frac{1}{2} \mathcal{R}_k \)
\[
\Xi_k(A, \psi_{k+1}^0(A)) = \exp \left( \sum_X E_k^#(X, A, \psi_{k+1}^0(A)) \right)
\] (496)

where \( E_k^#(X, A, \psi_{k+1}^0(A)) \) satisfies
\[
\| \tilde{E}_k^#(X, A) \|_{\frac{1}{4} h_k} \leq \text{O}(1) e^{\frac{k}{4} (5 - 5 k)} e^{-(\kappa - 10 k_0 - 10)} d_M(X)
\] (497)
Proof. step I: First make a Mayer expansion writing
\[
\exp \left( \sum_{X} E^{(\leq 5)}_k (X) \right) = \prod_{X} \left( \exp E^{(\leq 5)}_k (X) - 1 \right) = \sum_{Y} K(Y) \tag{498}
\]
Here in the second step we expand the product and classify the terms in the resulting sum by the union of the polymers. Thus
\[
K(Y) = \sum_{\{X_i \cup i = Y\}} \prod_{i} \left( \exp E^{(\leq 5)}_k (X_i) - 1 \right) \tag{499}
\]
where \(\{X_i\}\) is a collection of distinct polymers. Two polymers are connected if they intersect or have a face in common. If \(\{Y_j\}\) are the connected components of \(Y\) then \(K(Y)\) factors as
\[
K(Y) = \prod_{j} K(Y_j) \tag{500}
\]
and each \(K(Y_j)\) is again given by \(499\). Instead of distinct unordered \(\{X_i\}\) we can write \(K(Y)\) as a sum over distinct ordered sets \((X_1, \ldots, X_n)\) by
\[
K(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1, \ldots, X_n) : \cup_i X_i = Y} \prod_{i} \left( e^{E^{(\leq 5)}_k (X_i)} - 1 \right) \tag{501}
\]
The partition function is now
\[
\Xi_k = \sum_{Y} \int K(Y) \chi_k (\tilde{C} \tilde{Z}) d\mu_{C_k} (\tilde{Z}) d\mu (W') \tag{502}
\]
To estimate \(K(Y)\) for \(Y\) connected we use the estimate 150 to obtain
\[
\| e^{E^{(\leq 5)}_k (X)} - 1 \|_{\frac{1}{2} h_{k,1}} \leq \sum_{n=1}^{\infty} \| E^{(\leq 5)}_k (X) \|_{\frac{1}{2} h_{k,1}} \leq 2 \| E^{(\leq 5)}_k (X) \|_{\frac{1}{4} h_{k,1}} \leq O(1) e^{\frac{1}{4} h_{k,1} - 5} e^{-(\kappa - 3\kappa_0 - 3) d_M (X)}
\]
and so
\[
\| K(Y) \|_{\frac{1}{4} h_{k,1}} \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1, \ldots, X_n) : \cup_i X_i = Y} \prod_{i=1}^{n} O(1) e^{\frac{1}{4} h_{k,1} - 5} e^{-(\kappa - 3\kappa_0 - 3) d_M (X)}
\]
Next extract a factor \(\exp \left( - (\kappa - \kappa_0) (d_M (Y) - (n - 1)) \right)\) and drop all conditions on the \(X_i\) except \(X_i \subset Y\) to obtain (see Appendix B in 25 for details)
\[
\| K(Y) \|_{\frac{1}{4} h_{k,1}} \leq O(1) e^{\frac{1}{4} h_{k,1} - 5} e^{-(\kappa - 5\kappa_0 - 5) d_M (X)}
\]
Now in 502 do the integral over \(W'\) defining
\[
K'(Y, A, \tilde{Z}, \psi_{k+1}^{0, #} (A)) = \int K(Y, A, \tilde{Z}, \psi_{k+1}^{0, #} (A), W') d\mu (W') \tag{506}
\]
Since the integral is strictly local this also factors over its connected components. By lemma 31 in Appendix A we have for \(Y\) connected
\[
\| K'(Y) \|_{\frac{1}{4} h_{k}} \leq \| K(Y) \|_{\frac{1}{4} h_{k,1}} \leq O(1) e^{\frac{1}{4} h_{k} - 5} e^{-(\kappa - 5\kappa_0 - 5) d_M (X)}
\]
The fluctuation integral is now written (relabelling $Y$ as $X$)

$$
\Xi_k = \sum_X \int K'(X) \hat{\chi}_k(C\hat{Z}) \, d\mu_{C_k}(\hat{Z})
$$

(508)

The fermion field $\psi_{k+1}^0(A)$ is a spectator for the rest of the proof.

**Step II:** Next take a fixed $X$ and remove characteristic functions from $X^c$ as follows. Define

$$
\hat{\chi}_k(X) = \prod_{\square \subset X} \hat{\chi}_k(\square) \quad \text{where} \quad \hat{\chi}_k(\square, C\hat{Z}) = \chi \left( \sup_{b' \in \square \neq \emptyset} |(C\hat{Z})(b)| \leq p_{0,k} \right)
$$

(509)

Then define $\hat{\zeta}_k(\square) = 1 - \hat{\chi}_k(\square)$ and write

$$
\hat{\chi}_k(X^c) = \prod_{\square \subset X^c} \hat{\chi}_k(\square) = \prod_{\square \subset X^c} \left( 1 - \hat{\zeta}_k(\square) \right) = \sum_{P \subset X^c} (-1)^{|P|} \hat{\zeta}_k(P)
$$

(510)

where $\hat{\zeta}(P) = \prod_{\square \subset P} \hat{\zeta}(\square)$. Then

$$
\hat{\chi}_k = \hat{\chi}_k(X)\hat{\zeta}_k(X^c) = \sum_{P \subset X^c} (-1)^{|P|} \hat{\chi}_k(X)\hat{\zeta}_k(P)
$$

(511)

Insert this back into (508) and classify the terms in the double sum over $X, P$ by $Y = X \cup P$. Thus we have

$$
\Xi_k = \sum_Y \int F(Y, \hat{Z})d\mu_{C_k}(\hat{Z})
$$

(512)

where

$$
F(Y, \hat{Z}) = \sum_{X, P, X \cup P = Y} (-1)^{|P|} \hat{\zeta}_k(P)\hat{\chi}_k(X)K'(X)
$$

(513)

Then $F(Y)$ is local in its variables and it factors over its connected components. The characteristic function $\hat{\zeta}_k(P)$ forces that there is a point in every cube $\square$ in $P$ where $|C\hat{Z}| \geq p_{0,k}$. Hence for $\square \subset P$ we have $p_{0,k}^2 \leq \|C\hat{Z}\|_{\square}^2 \leq C_0\|\hat{Z}\|_{\square}^2$ and so

$$
\hat{\zeta}(\square) \leq \exp \left( -p_{0,k} + p_{0,k}^{-1}C_0\|\hat{Z}\|_{\square}^2 \right)
$$

(514)

If $|P|_M = M^{-3}\text{Vol}(P)$ is the number of $M$ cubes in $P$ then

$$
|\hat{\zeta}_k(P)| \leq \exp \left( -p_{0,k}|P|_M + p_{0,k}^{-1}C_0\|\hat{Z}\|_{P}^2 \right)
$$

(515)

Using this we have the estimate for connected $Y$

$$
\| \hat{F}(Y) \|_{L^2} \leq \mathcal{O}(1)\varepsilon_k^{-5\varepsilon} \sum_{X, P, X \cup P = Y} \exp \left( p_{0,k}^{-1}C_0\|\hat{Z}\|_{P}^2 \right) e^{-(\kappa - 5\kappa_0 - 5)d_M(X) - p_{0,k}|P|_M}
$$

$$
\leq \mathcal{O}(1)\varepsilon_k^{-5\varepsilon} \exp \left( p_{0,k}^{-1}C_0\|\hat{Z}\|_{P}^2 \right) e^{-(\kappa - 6\kappa_0 - 5)d_M(Y)} \sum_{X, P \subset Y} e^{-\kappa_0 d_M(X)} e^{-\frac{1}{2}p_{0,k}|P|_M}
$$

(516)

$$
\leq \mathcal{O}(1)\varepsilon_k^{-5\varepsilon} \exp \left( p_{0,k}^{-1}C_0\|\hat{Z}\|_{P}^2 \right) e^{-(\kappa - 6\kappa_0 - 6)d_M(Y)}
$$

In the second step we used first that $\frac{1}{2}p_{0,k} \geq \kappa - 6\kappa_0 - 5$ and then $d_M(X) + |P|_M \geq d_M(Y)$. In the
third step the sum over \( Y \) is bounded by \( \mathcal{O}(1)|Y|_M \leq \mathcal{O}(1)(d_M(Y) + 1) \leq \mathcal{O}(1) \exp(\frac{1}{2} d_M(Y)) \). The sum over \( P \) (which may not be connected) is bounded taking \( e^{-\frac{1}{2} P r_0 k} \leq \lambda_k^n \) for any integer \( n \) and then

\[
\sum_{P \subseteq Y} \lambda_k^n |P|_M \leq (1 + \lambda_k^n)|Y|_M \leq e^{\lambda_k^n}|Y|_M \leq \mathcal{O}(1)e^{\frac{1}{2} d_M(Y)}
\]  

(517)

**Step III**: We study the integral in (512) by conditioning on the value of \( \tilde{Z} \) on \( Y^c \). This has the form (see [12] and appendix C)

\[
\int d\mu_{C_k}(Z) F(Y, \tilde{Z}) = \int d\mu_{C_k}(Z') \exp \left( -\frac{1}{2} \left\langle Z', \left[ \tilde{\Delta}_k C_k(Y) \tilde{\Delta}_k \right]_{Y^c} \right\rangle Z' \right) \left[ \int d\mu_{C_k}(Y) \tilde{Z} \exp \left( -\left\langle Z', [\tilde{\Delta}_k \tilde{Z}]_{Y^c} \right\rangle F(Y, \tilde{Z}) \right) \right]
\]

(518)

Here we have defined \( \tilde{\Delta}_k = C^T \Delta_k C \) so that \( C_k = \tilde{\Delta}_k^{-1} \) and defined \( C_k(Y) = [\tilde{\Delta}_k]_{Y^c} \) as in section 3.3.6. In the outside integral make the change of variables \( Z' = (C_k)^T Z'' \). Note that this does not affect the characteristic functions in \( F(Y, \tilde{Z}) \) which was our goal. Then we have

\[
\Xi_k = \sum_Y \int d\mu_I(Z'') G(Y, Z'')
\]

(519)

where

\[
G(Y, Z'') = \exp \left( -\frac{1}{2} \left\langle Z'', C_k^T [\tilde{\Delta}_k C_k(Y) \tilde{\Delta}_k]_{Y^c} C_k^T Z'' \right\rangle \right) \int d\mu_{C_k}(Y) \tilde{Z} \exp \left( -\left\langle Z'', [\tilde{\Delta}_k \tilde{Z}]_{Y^c} \right\rangle F(Y, \tilde{Z}) \right)
\]

(520)

**Step IV**: Next localize the expression \( G(Y; Z'') \) using generalized random walk expansions. First consider \( C_k \) which has the expansion (249):

\[
C_k = \sum_{\Box} h_{\Box} C_k_{\Box} + \sum_{\omega: |\omega| \geq 1} C_k_{\omega}
\]

(521)

where \( \omega = (X_0, \alpha_1, X_1, \ldots, \alpha_n, X_n) \) has localization domains with \( |X_i|_M \leq r_0 = \mathcal{O}(1) \). Let \( r = r_0 + 1 \). We resum the expansion so that the basic units include connected components of the enlargement \( \tilde{Y}_r \), denoted \( \tilde{Y}_r^{\beta} \). This operation is explained in section 3.3.7. Resum the nonleading terms (resummation of the leading term is optional) and get a similar expansion now with \( \omega = (X_0, \ldots, X_n) \) where each \( X_i \) is either a \( \tilde{Y}_r^{\beta} \) or an \( (\alpha_i, X_i) \) such that \( X_i \) intersects \( (\tilde{Y}_r^{\beta})^c \). The latter satisfy \( d(X_i, Y) \geq M \).

We introduce weakening parameters in \( (\tilde{Y}_r^{\beta})^c \) defining

\[
C_k(s) = \sum_{\Box} h_{\Box} C_k_{\Box} + \sum_{\omega: |\omega| \geq 1} s_{\omega} C_k_{\omega}
\]

(522)

Here as before \( s = \{s_{\Box}\} \) are parameters \( 0 \leq s_{\Box} \leq 1 \) indexed by a partition into \( M \) cubes \( \Box \), but now

\[
s_{\omega} = \prod_{\Box \subseteq X_\omega \cap (\tilde{Y}_r)^c} s_{\Box}
\]

(523)

and if \( X_\omega \cap (\tilde{Y}_r)^c = \emptyset \) then \( s_{\omega} \equiv 1 \). Only walks leaving \( \tilde{Y}_r^{\beta} \) are suppressed. If \( s = 0 \) then the only terms in the second sum which contribute are those with \( X_\omega \subseteq \tilde{Y}_r^{\beta} \). These have the form \( \omega = (\tilde{Y}_r^{\beta}, \ldots, \tilde{Y}_r^{\beta}) \) for some \( \beta \).
Similarly one can use resummed random walks to define a weakened version $C_k^\ast(s)$ of $C_k$ and a weakened version $C_k(s, Y)$ of $C_k(Y).$ ($C_k(Y)$ is an operator on $Y$, but the random walk expansion involves the whole space.) We also have a weakened version $\Delta_k(s) = \mathcal{H}_k^T(s) \delta \mathcal{H}_k(s)$ of $\Delta_k = \mathcal{H}_k^T \delta \mathcal{H}_k$ and define $\tilde{\Delta}_k(s) = C^T \Delta_k(s) C$.

Making these substitutions in $G(Y; Z'')$ we get a new function

$$G(s, Y, Z'') = \exp \left( -\frac{1}{2} \left( Z'', C_k^\ast(s) [\tilde{\Delta}_k(s) C_k(s, Y) \tilde{\Delta}_k(s)]_{Y', Y} C_k^\ast(s) Z'' \right) \right)$$

$$\int d\mu_{C_k(s, Y)}(\tilde{Z}) \exp \left( -\left( Z'', C_k^\ast(s) [\tilde{\Delta}_k(s) \tilde{Z}]_{Y', Y} \right) \right) F(Y, \tilde{Z})$$

(524)

The covariance of the Gaussian measure is still positive definite since $C_k(s)$ is a small $\mathcal{O}(M^{-1})$ perturbation of $C_k(0)$ which in turn is a small $\mathcal{O}(M^{-1})$ perturbation of the positive definite $\sum_{Y} h_{\square} C_k h_{\square}.$

Expanding around $s_\square = 1$ we find

$$\sum_{Y} G(Y; Z'') = \sum_{Y'} \tilde{G}(Y'; Z'')$$

(525)

where

$$\tilde{G}(Y'; Z'') = \sum_{Y: Y' \subset Y'} \int d\gamma_{Y' - Y'} \frac{\partial}{\partial \gamma_{Y' - Y'}} [G(s, Y, Z'')]_{s(Y') = 0}$$

(526)

Since $s_{(Y') < 0}$ no operator in (526) connects points in different connected components of $Y'.$ Hence $\tilde{G}(Y'; Z'')$ factors over its connected components and only depends on fields in $Y'.$ (To see the measure factors look at the characteristic function).

Now we can write

$$\Xi_k = \sum_{Y'} H(Y')$$

(527)

where

$$H(Y') \equiv \int \tilde{G}(Y', Z'') d\mu_{Y''}(\tilde{Z}'')$$

(528)

also factors over its connected components and depends on fields only in $Y'.$

**Step V:** We would like to estimate the $s$ derivatives in (526) by Cauchy bounds. This requires considering $G(s, Y, Z'')$ for complex $s_\square.$ In particular we need control over the complex covariance $C(s, Y)$. We collect some relevant bounds.

First we note that for $\Upsilon, \Upsilon'$ in the basis (184) and $\Upsilon, \Upsilon' \in Y$ (i.e. $\text{supp } \Upsilon, \text{supp } \Upsilon' \subset Y$)

$$|C_k(0, Y)(\Upsilon, \Upsilon')| \leq C e^{-\gamma d_M(\Upsilon, \Upsilon')}$$

(529)

This follows just as the bound (269) on $C_k(Y).$ It is also positive definite and then by (a modification of) Balaban’s lemma on unit lattice operators [3], we have a bound of the same form for the inverse:

$$|C_k(0, Y)^{-1}(\Upsilon, \Upsilon')| \leq C e^{-\gamma d_M(\Upsilon, \Upsilon')}$$

(530)

Next define $\delta C_k(s, Y)$ and $\delta C_k^{-1}(s, Y)$ by

$$C(s, Y) = C(0, Y) + \delta C(s, Y)$$

$$C(s, Y)^{-1} = C(0, Y)^{-1} + \delta C^{-1}(s, Y)$$

(531)
We claim that for $M$ sufficiently large, $a_0$ sufficiently small, and $|s_0| \leq M^{a_0}$ and $Y, Y' \in Y$

\[
\begin{align*}
|\delta C_k(s, y)(Y, Y')| & \leq CM^{-\frac{\gamma}{2}}e^{-\gamma d_M(Y, Y')} \\
|C_k(s, y)(Y, Y')| & \leq Ce^{-\gamma d_M(Y, Y')} \\
|C_k^{-1}(s, y)(Y, Y')| & \leq Ce^{-\gamma d_M(Y, Y')}
\end{align*}
\]

(532)

Indeed the operator $\delta C_k(s, y)$ only involves walks with $|\omega| \geq 1$ so if $|s_0| \leq 1$ the first bound holds with $M^{-\frac{\gamma}{2}}$ rather than $M^{-\frac{\gamma}{4}}$. For $|s_0| \leq M^{a_0}$ the bound erodes to $M^{-\frac{\gamma}{4}}$ just as in lemma 7 (since $s_0$ only involves the localization domains $X_i$, not the $\tilde{Y}_\beta^{n+1}$). The second bound follows from the first and (529). For the third bound make the expansion

\[
C_k^{-1}(s, Y) = (C_k(0, Y) + \delta C_k(s, Y))^{-1} = C(0, Y)^{-1}(I + \delta C_k(s, Y)C(0, Y)^{-1})^{-1}
\]

\[
= \sum_{n=0}^{\infty} C(0, Y)^{-1}(-\delta C_k(s, Y)C(0, Y)^{-1})^n
\]

(533)

This converges for $M$ sufficiently large, and the result follows by the bounds on $\delta C_k(s, Y)$ and $C(0, Y)^{-1}$. The expansion also gives the fourth bound.

Actually the first and last bounds in (532) can be made even stronger. We have excluded from $\delta C_i(s, Y)$ walks which stay in $\tilde{Y}_\beta^{n+1}$. We are starting (and finishing) in $Y$ so some step must then go to a localization domain $X_i$ which necessarily satisfies $d(X_i, Y) \geq M$. The accumulated exponent factors in a bound like (208) result in any overall factor $e^{-\gamma d(X_i, Y)} \leq e^{-\gamma M}$. Thus we have the improved bounds

\[
|\delta C_k(s, y)(Y, Y')| \leq Ce^{-\gamma M}e^{-\gamma d_M(Y, Y')}
\]

\[
|\delta C_k^{-1}(s, y)(Y, Y')| \leq Ce^{-\gamma M}e^{-\gamma d_M(Y, Y')}
\]

(534)

**Step VI:** We proceed with the estimate on $G(s, Y, Z'')$ for $|s_0| \leq M^{a_0}$ and $s(Y') = 0$. Write as

\[
G(s, Y, Z'') = \exp \left( -\frac{1}{2} \left( Z'', \Gamma_k^T(s, Y)C_k(s, Y)\Gamma_k(s, Y)Z'' \right) \right)
\]

\[
\int d\mu_{C_k(s, Y)}(\tilde{Z}) \exp \left( -\left( \tilde{Z}, \Gamma_k(s, Y)Z'' \right) \right) F(Y, \tilde{Z})
\]

(535)

where

\[
\Gamma_k(s, Y) = [\tilde{\Delta}_k(s)]_{Y, Y'} C_k^+(s)
\]

(536)

For any $f(\tilde{Z})$

\[
|\int f(\tilde{Z})d\mu_{C_k(s, Y)}(\tilde{Z})| \leq \left| \frac{\text{det} \left( C_k(s, Y)^{-1} \right)}{\text{det} \left( \text{Re} \ C_k(s, Y)^{-1} \right)} \right|^{\frac{1}{2}} \int |f(\tilde{Z})|d\mu(\text{Re} \ C_k(s, Y)^{-1})^{-1}(\tilde{Z})
\]

(537)

Thus we get the bound

\[
|G(s, Y, Z'')| \leq \exp \left( -\frac{1}{2} \left( Z'', \text{Re} \left( \Gamma_k^T(s, Y)C_k(s, Y)\Gamma_k(s, Y) \right) Z'' \right) \right)
\]

\[
\leq \left| \frac{\text{det} \left( C_k(s, Y)^{-1} \right)}{\text{det} \left( \text{Re} \ C_k(s, Y)^{-1} \right)} \right|^{\frac{1}{2}} \int d\mu(\text{Re} \ C_k(s, Y)^{-1})^{-1}(\tilde{Z}) \exp \left( -\left( \tilde{Z}, \text{Re} \left( \Gamma_k(s, Y)Z'' \right) \right) \right) |F(Y, \tilde{Z})|
\]

(538)
To estimate this we start by replacing every object with weakening parameters $s$ by the same with $s = 0$. There are several steps.

(1.) First consider the ratio of determinants. We have

$$\det(C_k(s, Y)^{-1}) = \det(C_k(0, Y)^{-1} \det \left(I + C_k(0, Y)\delta C_k^{-1}(s, Y)\right)$$

$$\det(\text{Re } C_k(s, Y)^{-1}) = \det(C_k(0, Y)^{-1} \det \left(I + C_k(0, Y)\text{Re } \delta C_k^{-1}(s, Y)\right)$$

The terms $\det(C_k(0, Y)^{-1}$ cancel. In general we have for $\|T\| < 1$

$$e^{-O(1)\|T\|} \leq |\det(I + T)| \leq e^{O(1)\|T\|}$$

where the trace norm is $\|T\|_1 = \text{Tr } (T^TT)^{1/2}$. Therefore

$$|\det \left(I + C_k(0, Y)\delta C_k^{-1}(s, Y)\right) | \leq \exp \left(O(1)\|C_k(0, Y)\delta C_k^{-1}(s, Y)\|_1\right)$$

We dominate the trace norm by Hilbert-Schmidt norms and use (529), (534)

$$O(1)\|C_k(0, Y)\delta C_k^{-1}(s, Y)\|_1 \leq O(1)\|C_k(0, Y)\|_{HS}\|\delta C_k^{-1}(s, Y)\|_{HS}$$

$$= O(1) \left(\sum_{T, T' \in T} |C_k(0, Y)(T, T')|^2\right)^{1/2} \left(\sum_{T, T' \in T} |\delta C_k^{-1}(s, Y)(T, T')|^2\right)^{1/2}$$

$$\leq Ce^{-\gamma M}\text{Vol}(Y) = Ce^{-\gamma M}M^3|Y|_M \leq O(M^{-1})|Y|_M$$

Then we have

$$|\det \left(I + C_k(0, Y)\delta C_k^{-1}(s, Y)\right) | \leq e^{O(M^{-1})|Y|_M}$$

Similarly $\det(I + C_k(0, Y)\text{Re } \delta C_k^{-1}(s, Y)^{-1}$ is bounded by $e^{O(M^{-1})|Y|_M}$. Hence the ratio of determinants in (538) is bounded by $e^{O(M^{-1})|Y|_M}$.

(2.) Next consider the term in the first exponential in (538). We define a quadratic form $R_1$ by

$$\langle Z'', R_1 Z'' \rangle = \langle Z'', \text{Re } \Gamma_k^T(s, Y)C_k(s, Y)\Gamma_k(s, Y) - \Gamma_k^T(0, Y)C_k(0, Y)\Gamma_k(0, Y) \rangle Z''$$

We have the estimate (534) for $\delta C_k(s, Y)$. By similar arguments the same holds for $\Gamma_k(s, Y) - \Gamma_k(0, Y)$ and so

$$|\Gamma_k(s, Y)(T, T') - \Gamma_k(0, Y)(T, T')| \leq Ce^{-\gamma M}e^{-\gamma d(T, T')}$$

These imply that $|R_1(T, T')| \leq Ce^{-\gamma M}e^{-\gamma d(T, T')}$. Since $\Gamma_k(s, Y)$ only connects to $Y'$ this gives

$$|\langle Z'', R_1 Z'' \rangle | \leq Ce^{-\gamma M}||Z''||_{Y'}^2$$

We also have in the integrand

$$\|\dot{Z}, \text{Re } \Gamma_k(s, Y) - \Gamma_k(0, Y) \rangle Z''\| \leq Ce^{-\gamma M}||Z''\|_{Y'} ||\dot{Z}\|_{Y} \leq Ce^{-\gamma M}(||Z''||_{Y'}^2 + ||\dot{Z}\|_{Y}^2)$$

We change the Gaussian measure by

$$\int f(\tilde{Z})d\mu_{\text{Re } C_k(s, Y)^{-1}}(\tilde{Z})$$

$$= \frac{\det \left(\text{Re } C_k(s, Y)^{-1}\right)^{1/2}}{\det \left(\text{Re } C_k(0, Y)^{-1}\right)^{1/2}} \int f(\tilde{Z}) \exp \left(-\frac{1}{2} \langle \tilde{Z}, \text{Re } \delta C_k^{-1}(s, Y)\tilde{Z}\rangle \right)d\mu_{C_k(0, Y)}(\tilde{Z})$$

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The ratio of determinants is again bounded by $e^{O(M^{-1})|Y|_M}$ and

$$|\langle \tilde{Z}, \text{Re} \delta C_k^{-1}(s,Y)\tilde{Z} \rangle| \leq C e^{-\gamma M}\|\tilde{Z}\|_{\mathcal{Y}}^2.$$  \hspace{1cm} (549)

We also bound $F(Y)$ which is a product over its connected components $F(Y) = \prod_{\alpha} F(Y_\alpha)$ and each $F(Y_\alpha)$ is bounded by (518). Combining all of this we have

$$\|G(s,Y,Z'')\|_{\mathcal{Y}_k} \leq e^{O(M^{-1})|Y|_M} \prod_{\alpha} O(1)e_k^{\frac{1}{2}s} e^{-(\kappa - 6\kappa_0 - 6)d_M(Y_\alpha)}$$

$$\exp \left(-\frac{1}{2} \left[Z'', \Gamma_k^T(0,Y)C_k(0,Y)\Gamma_k(0,Y)Z'' \right] + C e^{-\gamma M}\|Z''\|_{\mathcal{Y}}^2, \right)$$

$$\int d\mu_{C_k(0,Y)}(\tilde{Z}) \exp \left(-\langle \tilde{Z}, \Gamma_k(0,Y)Z'' \rangle + (C_0p_0^{-1} + C e^{-\gamma M})\|\tilde{Z}\|_{\mathcal{Y}}^2 \right)$$

(3.) Let $\frac{1}{2}\beta_k = C_0p_0^{-1} + C e^{-\gamma M}$ which is tiny. The integral is evaluated as

$$\int d\mu_{C_k(0,Y)}(\tilde{Z}) \exp \left(-\langle \tilde{Z}, \Gamma_k(0,Y)Z'' \rangle + \beta_k\|\tilde{Z}\|_{\mathcal{Y}}^2 \right)$$

$$= \left[ \frac{\det (C_k(0,Y)^{-1})}{\det (C_k(0,Y)^{-1} - \beta_k 1_Y)} \right]^{\frac{1}{2}} \exp \left(\frac{1}{2} \left[Z'', \Gamma_k^T(0,Y) \left( C_k(0,Y)^{-1} - \beta_k 1_Y \right)^{-1} \Gamma_k(0,Y)Z'' \right] \right)$$

The ratio of the determinants is bounded by

$$\left[ \det(I - \beta_k C_k(0,Y)) \right]^{-1} \leq \exp \left(\beta_k\|C_k(0,Y)\|_{\mathcal{Y}} \right) \leq \exp \left(\beta_k CM^2 |Y|_M \right) \leq e^{O(M^{-1})|Y|_M}$$  \hspace{1cm} (552)

Next consider the quadratic form $\frac{1}{2} \left[Z'', \Gamma_k^T(0,Y) \left( C_k(0,Y)^{-1} - \beta_k 1_Y \right)^{-1} \Gamma_k(0,Y)Z'' \right]$. Expand it in powers of the small parameter $\alpha_k$. The leading term $\frac{1}{2} \left[Z'', \Gamma_k^T(0,Y)C_k(0,Y)\Gamma_k(0,Y)Z'' \right]$ is canceled by a corresponding term in (550). The remainder has the form

$$\langle Z'', R_2Z'' \rangle = \langle Z'', \Gamma_k^T(0,Y) \left[ \sum_{n=1}^\infty \beta_k^n C_k(0,Y)^{n+1} \right] \Gamma_k(0,Y)Z'' \rangle$$

Arguing as before this satisfies $|R_2(T,T')| \leq C\beta_k e^{-\gamma d(T,T')}$ and so $|< Z'', R_2Z'' >| \leq C\beta_k\|Z''\|^2$.

We also take in (550)

$$O(M^{-1})|Y|_M = \sum_{\alpha} O(M^{-1})|Y_\alpha|_M \leq \sum_{\alpha} d_M(Y_\alpha) + 1$$

Let $\beta_k' \equiv C e^{-\gamma M} + C\beta_k$ which is tiny. The bound is now for $Y$ with connected components $\{Y_\alpha\}$ and $|s_{\square}| \leq M^{\alpha_0}$ and $s(Y_{\square}) = 0$:

$$\|G(s,Y,Z'')\|_{\mathcal{Y}_k} \leq e^{\beta_k'\|Z''\|_{\mathcal{Y}}^2} \prod_{\alpha} O(1)e_k^{\frac{1}{2}s} e^{-(\kappa - 6\kappa_0 - 7)d_M(Y_\alpha)}$$  \hspace{1cm} (555)

(4.) The function $\tilde{G}(Y'; Z'')$ is given in terms of derivatives of $G(s,Y,Z'')$ for $|s_{\square}| \leq 1$ in (520). Having established analyticity in the larger domain $|s_{\square}| \leq M^{\alpha_0}$ we can estimate these derivatives by Cauchy bounds. Each derivative contributes a factor $M^{-\alpha_0}$. Using also the bound (555) yields

$$\|\tilde{G}(Y'; Z'')\|_{\mathcal{Y}_k} \leq e^{\beta_k'\|Z''\|_{\mathcal{Y}}^2} \sum_{Y':Y' \subset Y'} M^{-\alpha_0}|Y' - Y'|_M \prod_{\alpha} O(1)e_k^{\frac{1}{2}s} e^{-(\kappa - 6\kappa_0 - 7)d_M(Y_\alpha)}$$  \hspace{1cm} (556)

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For $Y'$ connected we want to extract a factor $e^{-(\kappa - 7\kappa_0 - 7)d_M(Y')}$ from the exponentials in this expression. A connected component $Y'_\beta$ or $\tilde{Y}'_\beta$ is the union of the $\tilde{Y}_\alpha'$ contained in it so
\[
d_M(\tilde{Y}'_\beta) \leq \sum_{\alpha: Y_\alpha \subset Y'_\beta} (d_M(\tilde{Y}'_\alpha) + 2) \tag{557}
\]
and therefore
\[
\sum_{\beta} d_M(\tilde{Y}'_\beta) \leq \sum_{\alpha} (d_M(\tilde{Y}'_\alpha) + 2) \tag{558}
\]
However for connected $Y$ we have $d_M(\tilde{Y}) \leq O(1)(d_M(Y) + 1)$ (see (364) in [29] ) and hence $d_M(\tilde{Y}'_\beta) \leq O(1)(d_M(Y) + 1)$. Therefore
\[
\sum_{\beta} d_M(\tilde{Y}'_\beta) \leq O(1) \sum_{\alpha} (d_M(Y_\alpha) + 2) \tag{559}
\]
The same bound holds with $\sum_{\beta}(d_M(\tilde{Y}'_\beta) + 2)$ on the left. Also suppose we divide the connected $Y'$ up into pieces $Y'_\beta$ each containing exactly one $\tilde{Y}'_\beta$. Then
\[
d_M(Y') \leq \sum_{\beta} (d_M(Y'_\beta) + 2)
\leq \sum_{\beta} (|Y'_\beta - \tilde{Y}'_\beta|_M + d_M(\tilde{Y}'_\beta) + 2)
= |Y' - \tilde{Y}'|_M + \sum_{\beta} (d_M(\tilde{Y}'_\beta) + 2) \tag{560}
\]
Combining (559) and (560) we have that there is a constant $c_1 = O(1), c_1 < 1$ such that
\[
c_1d_M(Y') \leq |Y' - \tilde{Y}'|_M + \sum_{\alpha} (d_M(Y_\alpha) + 2) \tag{561}
\]
Assuming $M^{-\frac{1}{2} \alpha_0} \leq e^{-(\kappa - 7\kappa_0 - 7)}$ the last estimate allows us to pull a factor $e^{-c_1(\kappa - 7\kappa_0 - 7)d_M(Y')}$ out of the sum (557) leaving
\[
\|\tilde{G}(Y', Z'')\|_{\mathcal{H}_k} \leq e^{-c_1(\kappa - 7\kappa_0 - 7)d_M(Y')} e^{\beta(Y')_Y''} \sum_{Y: \tilde{Y}' \subset Y'} \sum_{\alpha} M^{-\frac{1}{2} \alpha_0}|Y' - \tilde{Y}'|_M \prod_{\alpha} O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{-\kappa_0d_M(Y_\alpha)} \tag{562}
\]
The sum here can also be written
\[
\sum_{Z \subset Y'} \sum_{Y: \tilde{Y}' \subset Z} M^{-\frac{1}{2} \alpha_0}|Y' - Z|_M \prod_{\alpha} O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{-\kappa_0d_M(Y_\alpha)} \tag{563}
\]
The sum over $Y = \{Y_\alpha\}$ is estimated by
\[
\sum_{\{Y_\alpha: Y_\alpha \subset Z\}} \prod_{\alpha} O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{-\kappa_0d_M(Y_\alpha)} \leq \sum_{n=1}^\infty \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n): Y_n \subset Z} \prod_{\alpha} O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{-\kappa_0d_M(Y_\alpha)}
= \sum_{n=1}^\infty \frac{1}{n!} \left( \sum_{Y \subset Z} O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{-\kappa_0d_M(Y)} \right)^n \leq \sum_{n=1}^\infty \frac{1}{n!} \left( O(1)e_k^{\frac{1}{2} - 5\epsilon} |Z|_M \right)^n \leq O(1)e_k^{\frac{1}{2} - 5\epsilon} |Z|_M \exp \left( O(1)e_k^{\frac{1}{2} - 5\epsilon} |Z|_M \right) \leq O(1)e_k^{\frac{1}{2} - 5\epsilon} e^{1/2 d_M(Y')} \tag{564}
\]
The sum over $Z$ is estimated by
\[
\sum_{Z \subseteq Y'} M^{-\frac{n+1}{2}|Y' - Z|} M \leq \left(1 + M^{-\frac{n+1}{2}}\right)^{|Y' - M|} \leq \exp \left(M^{-\frac{n+1}{2}|Y'|} M\right) \leq \mathcal{O}(1) e^{\frac{1}{2}d_M(Y')}
\]

This yields for $Y'$ connected
\[
\| \tilde{G}(Y'; Z'') \|_{\mathcal{H}_k} \leq \mathcal{O}(1) e^{\frac{1}{2} - 5e - c_1(\kappa - 7n - 8)d_M(Y')} e^{\beta_k} \| Z'' \|_Y^2
\]

For $H(Y) = \int \tilde{G}(Y'; Z'') d\mu_1(Z'')$ the integral over $Z''$ is estimated by
\[
\int e^{\beta_k} \| Z'' \|_Y^2 d\mu_1(Z'') \leq e^\beta_k \text{Vol} |Y'| \leq e^{\beta_k} M^3 |Y'| M \leq \mathcal{O}(1) e^{c_1 d_M(Y')}
\]

and so for $Y'$ connected we can take
\[
\| H(Y') \|_{\mathcal{H}_k} \leq \mathcal{O}(1) e^{\frac{1}{2} - 5e - c_1(\kappa - 7n - 8)d_M(Y')}
\]

**Step VII:** Now $H(Y)$ factors over its connected components, $H(Y) = \prod_i H(Y_i)$, and is small so we can exponentiate $\Xi_k = \sum_Y H(Y)$ by a standard formula (see for example [25]). This yields
\[
\Xi_k = \exp \left( E_k^\# \right) = \exp \left( \sum_X E_k^\# (X) \right)
\]

where the sum is over connected $X$ and
\[
E_k^\# (X) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \ldots, Y_n) \cup X} \rho^T(Y_1, \ldots, Y_n) \prod_i H(Y_i)
\]

for a certain function $\rho^T(Y_1, \ldots, Y_n)$ enforcing connectedness. This satisfies
\[
\| E_k^\# (X) \|_{\mathcal{H}_k} \leq \mathcal{O}(1) e^{\frac{1}{2} - 5e - c_1(\kappa - 10n - 10)d_M(X)}
\]

It is straightforward, but somewhat tedious, to check that $E_k^\# (X, A, \psi)$ still has all the symmetries. This generally comes down to a statement about the covariance. For example for charge conjugation invariance we need $C^{-1} \Gamma (-A) C = \Gamma (A)^T$. This follows from the representation $\Xi_{115}$ and the same property for $S_{k+1}^0(A)$ as in $\Xi_{120}$. This completes the proof of theorem $\Xi$.

We are still working on the proof of theorem $\Xi$. Updating the expression (474) we have
\[
\dot{\rho}_{k+1}(A_{k+1}, e^{i\epsilon_k \omega(1)} \Psi_{k+1}) = N_k Z_k \delta Z_{k+1} Z_k (0) \delta Z_k (A)
\]
\[
\exp \left( -\frac{1}{2} \| dA \|^2 - \mathcal{E}_{k+1}^{0, +} (A, \Psi_{k+1}, \psi^{0, +}_{k+1} (A)) + E_k (\alpha, \psi^0_k (A)) + E_k^\# (\alpha, \psi^{0, \#}_k (A)) \right) \bigg|_{A=A^0_{k+1}}
\]
6.2.7 fermion determinant

We now remove the gauge field from the fermion determinant

**Lemma 24.** For \( A \in \tilde{R}_k \)

\[
\frac{\delta Z_k(A)}{\delta Z_k(0)} = \exp \left( \sum_X E_k^{\text{det}}(X, A) \right) \tag{573}
\]

where \( E_k^{\text{det}}(X, A) \) vanishes at \( A = 0 \) and satisfies

\[
|E_k^{\text{det}}(X, A)| \leq e_k^{4-\epsilon} e^{-\kappa d_M(X)} \tag{574}
\]

**Proof.** First take \( A \) in the larger domain \( e_k^{-\frac{1}{2}} R_k \). This is still in the region of analyticity for propagators \( S_k(A), S_{y}(A) \). Take the expression for \( \delta Z_k(A) \) from (374) and insert the polymer expansion for \( S_{k,y}(A) \) from (389). Then we have

\[
\delta Z_k(A) = \exp \left( 4|T^0_{N-k}| \log b_k + \sum_X E_k^d(X, A) \right) \tag{575}
\]

where

\[
E_k^d(X, A) = -i\gamma_3 b_k^2 \int_0^\infty \text{Tr} \left[ B_{k,y}(A) Q_k(A) S_{k,y}(X, A) Q_k^T(-A) B_{k,y}(A) \right] dy \tag{576}
\]

The bound (390) is easily modified to \(|S_{k,y}(X, A) f| \leq e^{-(\kappa+1)d_M(X)} \|f\|_\infty\) and then as in (432), (433)

\[
\left| \left[ Q_k(A) S_{k,y}(X, A) Q_k^T(-A) \right]_{xx'} \right| \leq C e^{-(\kappa+1)d_M(X)} \tag{577}
\]

The factors \( B_{k,y}(A) \) are local and each supplies a factor \( O(y^{-1}) \) for convergence of the integral. The trace is only over the region \( X \) and gives a factor \( M^3 |X|_M \leq M^3 e^{d_M(X)} \). Therefore

\[
|E_k^d(X, A)| \leq C M^3 e^{-\kappa d_M(X)} \tag{578}
\]

For the ratio \( \delta Z_k(A)/\delta Z_k(0) \) the volume factors cancel and we have the expression (573) with

\[
E_k^{\text{det}}(X, A) = E_k^d(X, A) - E_k^d(X, 0) \tag{579}
\]

which again satisfies the bound (578).

Now take the smaller domain \( \tilde{R}_k \). Since \( E_k^{\text{det}}(X, 0) = 0 \) and \( e_k^{-\frac{1}{2}} A \) is in the larger domain we have

\[
E_k^{\text{det}}(X, A) = \frac{1}{2\pi i} \int_{|t| = e_k^{-\frac{1}{2}}} \frac{dt}{t(t-1)} E_k^{\text{det}}(X, tA) \tag{580}
\]

and this gives

\[
|E_k^d(X, A)| \leq C M^3 e_k^{\frac{1}{2}} e^{-\kappa d_M(X)} \leq e_k^{\frac{1}{2} - \epsilon} e^{-\kappa d_M(X)} \tag{581}
\]

to complete the proof.

Now we have

\[
\tilde{\rho}_{k+1}(A_{k+1}, e^{i\epsilon_k d_M} \Psi_{k+1}) = N_{k} N_{k} Z_k \delta Z_{k+1} Z_k(0) \delta Z_k(0)
\]

\[
\exp \left( -\frac{1}{2} \|dA\|^2 - \mathcal{E}_{k+1}^0 \left( A, \Psi_{k+1}, \psi_{k+1}^0(\Psi_{k+1} \right) \right) - (\epsilon_k + c_k) \text{Vol}(T^0_{N-k}) \right) \tag{582}
\]

and

\[
\exp \left( -m t_k \left< \bar{\psi}_{k+1}^0(A), \psi_{k+1}^0(\Psi_{k+1} \right) + E_k^0 \left( A, \psi_{k+1}^0(\Psi_{k+1} \right) + E_k^{\text{det}}(A) \right) \bigg|_{A = A_{k+1}^0} \right)
\]

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6.2.8 scaling

The last step is scaling. We reblock $E_k' = R E_k$ to $B R E_k$, $E_k^{\text{det}}$ to $B E_k^{\text{det}}$, and $E_k^\#$ to $B E_k^\#$, all defined on $LM$-polymers. Then we scale $\rho_{k+1}^{(308)}$ according to $\frac{309}{(309)}$. Taking account that $A_{k+1}^0$ scales to $A_{k+1}$, that $\psi_{k+1}^0 (A_{k+1}^0)$ scales to $\psi_{k+1} (A_{k+1})$, and $\frac{\epsilon}{(311)}$ we obtain the desired form

$$\rho_{k+1} (A_{k+1}, \epsilon^0_k) = N_{k+1} (A_{k+1}, \epsilon_k^0) = N_{k+1} Z_{k+1} (0)$$

$$\exp \left( - \frac{1}{2} \left[ \|dA_{k+1}\|^2 - \delta_{k+1}^2 (A, \Psi_{k+1} (A_k)) - \epsilon_{k+1} \text{Vol}(T^0_{N-k-1}) \right] \right)$$

$$\exp \left( - m_{k+1} \left[ \psi_{k+1} (A_{k+1}), \psi_{k+1} (A_{k+1}) \right] \right) E_{k+1} (A_{k+1}, \psi_{k+1}^0 (A_{k+1}))$$

(583)

Here we have made the identification from $\frac{113}{(113)}$, $\frac{181}{(181)}$

$$\left( N_k N_k Z_k (0) \delta Z_k (0) L^{-8(s_N-s_{N-k-1})} \right) \left( Z_k \delta Z_k L^{\frac{4}{k}} (b_N-b_{N-k-1}) \right)^{\frac{8}{k}}$$

$$N_{k+1} Z_{k+1} (0) Z_{k+1}$$

(584)

As announced in the theorem we have defined $\epsilon_{k+1} = L^3 (\epsilon_k^0 + \epsilon_k^0) = L^3 (\epsilon_k + \epsilon_k (E_k) + \epsilon_k^0)$, and $m_{k+1} = L m_k = L (m_k + m (E_k))$, and with $\frac{\cal L}{(585)} = \frac{(BE)}{L_{N-k-1}}$

$$E_{k+1} = \frac{\cal L}{(RE_k + E_k^{\text{det}} + E_k^\#)}$$

(585)

We already have a bound on $\epsilon_k^0$ so all that remains is a bound on the kernel of $E_k^\# = \frac{\cal L}{(E_k^\#)}$. We would like to use the bound $\frac{308}{(308)}$ and $\frac{\|E_k\|}{k} \leq O(1) \epsilon_k^{\frac{4}{k}-5\epsilon}$ to obtain

$$\|\frac{\cal L}{(E_k^\#)}\|_{k+1} \leq O(1) L^3 \|E_k^\#\|_{k} \leq O(1) L^3 \epsilon_k^{\frac{4}{k}-5\epsilon} \leq \epsilon_k^{\frac{4}{k}-6\epsilon}$$

(586)

But this is not quite correct since our bound $\frac{497}{(497)}$ does not give the bound on $\|E_k^\#\|_{k}$ but on a somewhat different quantity. We have to revisit the proof of $\frac{308}{(308)}$ using the actual bound $\frac{497}{(497)}$. If $A \in R_{k+1}$ then $A_L = \frac{1}{L} R_k$ so after scaling we are in the domain needed for $\frac{497}{(497)}$. The fermion field parameter in $\frac{497}{(497)}$ is $\frac{1}{L} h_k$ not $h_k$. But this does not affect the derivation of $\frac{308}{(308)}$ since we can take $h_{k+1} \leq \frac{1}{L} h_k$ rather than $h_{k+1} \leq h_k$. Finally the decay factor in $\frac{497}{(497)}$ is $e^{-\epsilon_{k-1} (k-10\sigma_0-10\sigma_{M}(X))}$ rather than $e^{-\epsilon_{k-k_M}(X)}$. This means that in the bound $\frac{497}{(497)}$ we get $O(1) L^3 e^{-L_{k-1} (k-10\sigma_0-10\sigma_{M}(X))}$ instead of $O(1) L^3 e^{-L_{k-1} (k-10\sigma_0-10\sigma_{M}(X))}$. For $L$ large this is still dominated by $O(1) L^3 e^{-L_{k_1} (k-10\sigma_0-10\sigma_{M}(X))}$. Thus the conclusion of $\frac{586}{(586)}$ is still valid.

This completes the proof of theorem $\frac{1}{1}$

7 The flow

We seek well-behaved solutions of the RG equations $\frac{201}{(201)}$. Thus we study

$$\epsilon_{k+1} = L^3 \left( \epsilon_k + \epsilon_k (E_k) + \epsilon_k^0 \right)$$

$$m_{k+1} = L (m_k + m (E_k))$$

$$E_{k+1} = \frac{\cal L}{(RE_k + E_k^{\text{det}} + E_k^\# (m_k, E_k))}$$

(587)

This can be iterated at most up to $k = K = N - m$ since at this point we are on the torus $T^0_{N-M} = T^0_m$ which consists of a single $M = L^m$ cube. Our goal is to show that for any $N$ we can choose the initial point so that the solution exists for $k = 0, 1, \ldots, K$ and finishes at preassigned values $\left( \epsilon_K, m_K \right) =$
(\varepsilon_K^N, m_K^N) independent of \( N \). This procedure is nonperturbative renormalization - the initial values for

\((\varepsilon_0, m_0) = (\varepsilon_0^N, m_0^N)\) will depend \( N \). Our proof follows the analysis in \cite{25, 29}.

Arbitrarily fixing the final values at zero, and starting with \( E_0 = 0 \) as dictated by the model, we
look for solutions \( \varepsilon_k, m_k, E_k \) for \( k = 0, 1, 2, \ldots, K \) satisfying

\[
\varepsilon_K = 0 \quad m_K = 0 \quad E_0 = 0
\]

This makes the effective mass \( \bar{m}_K + m_K = \bar{m}_K = L^{-m} \bar{m} \). At this point we temporarily drop the
equation for the energy density \( \varepsilon_k \) and just study

\[
m_{k+1} = L \left( m_k + m(E_k) \right) \\
E_{k+1} = L \left( R E_k + E_k^{\text{det}} + E_k^\#(m_k, E_k) \right)
\]

Let \( \xi_k = (m_k, E_k) \) be an element of the complete metric space \( \mathbb{R} \times K_k \) where \( K_k \) is the Banach
space defined in section 4.3. Consider sequences

\[
\underline{\xi} = (\xi_0, \ldots, \xi_K)
\]

Let \( \mathcal{B} \) be the space of all such sequences with norm

\[
\|\underline{\xi}\| = \sup_{0 \leq k \leq K} \left\{ e^{-\frac{4}{3} + 3e} |m_k|, e^{-\frac{4}{3} + 7e} \|E_k\|_k \right\}
\]

Let \( \mathcal{B}_0 \) be the subset of all sequences satisfying the boundary conditions. Thus

\[
\mathcal{B}_0 = \{ \underline{\xi} \in \mathcal{B} : m_K = 0, E_0 = 0 \}
\]

This is a complete metric space with distance \( \|\underline{\xi} - \underline{\xi}'\| \). Finally let

\[
\mathcal{B}_1 = \mathcal{B}_0 \cap \{ \underline{\xi} \in \mathcal{B} : \|\underline{\xi}\| < 1 \}
\]

Note that the condition \( \|\underline{\xi}\| \leq 1 \) implies

\[
|m_k| < e^{-\frac{4}{3} + 3e} \quad \|E_k\|_k < e^{-\frac{4}{3} - 7e}
\]

so we are well within the domain of validity for the main theorem.

Next define an operator \( \underline{\xi}' = T \underline{\xi} \) by

\[
m'_k = L^{-1} m_{k+1} - m(E_k) \\
E'_k = L \left( R E_{k-1} + E_{k-1}^{\text{det}} + E_{k-1}^\#(m_{k-1}, E_{k-1}) \right)
\]

Then \( \underline{\xi} \) is a solution of (589) and the boundary conditions iff it is a fixed point for \( T \) on \( \mathcal{B}_0 \). We look for
such fixed points in \( \mathcal{B}_1 \).

**Lemma 25.** Let \( L \) be sufficiently large and \( e \) sufficiently small. Then for all \( N \)

1. The transformation \( T \) maps the set \( \mathcal{B}_1 \) to itself.
2. There is a unique fixed point \( T \underline{\xi} = \underline{\xi} \) in this set.

**Proof.** We use the bound from (589)

\[
|m(E_k)| \leq \mathcal{O}(1) e^{-\frac{4}{3}} \|E_k\|_k
\]
We also use
\[
\|LRE_{k-1}\|_k \leq O(1)L^{-\frac{1}{4}+\varepsilon}E_{k-1}\|_{k-1} \quad \|LE_{k-1}\|_k \leq e^{\frac{1}{4}-2\varepsilon}
\]
(597)
\[
\|E_{k-1}\|_k = \|LE_{k-1}\|_k \leq e^{\frac{1}{4}-6\varepsilon}
\]

The first is the key contractive estimate \((\text{1})\). The second follows since \(\|E_{k-1}\|_{k-1} \leq e^{\frac{1}{4}-\varepsilon}\) so \(\|LE_{k-1}\|_k \leq O(1)L^3e^{\frac{1}{4}-\varepsilon} \leq e^{\frac{1}{4}-2\varepsilon}\). The third is \((\text{102})\).

(1.) To show the map sends \(B_1\) to itself we estimate using \(e_{k+1} = L^\frac{1}{2}e_k\)
\[
e_{k}^{-\frac{1}{4}+8\varepsilon}\|m'_{k+1}\| \leq e_{k}^{-\frac{1}{4}+8\varepsilon}\left(L^{-1}|m_{k+1}| + O(1)e_{k}^{\frac{1}{4}}||E_k||_k\right)
\leq L^{-\frac{1}{4}}\left[e_{k+1}^{-\frac{1}{4}+8\varepsilon}\|m_{k+1}\|\right] + O(1)e_{k}^{\frac{1}{4}+7\varepsilon}||E_k||_k
\leq \|\xi\| < 1
\]
(598)

We also have
\[
e_{k}^{-\frac{1}{4}+7\varepsilon}\|E'_{k}\|_k \leq e_{k}^{-\frac{1}{4}+7\varepsilon}\left(O(1)L^{-\frac{1}{4}+2\varepsilon}\|E_{k-1}\|_{k-1} + e_{k}^{-\frac{1}{2}+2\varepsilon} + e_{k-1}^{-\frac{1}{4}}\right)
\leq O(1)L^{-\frac{1}{4}+6\varepsilon}\left[e_{k-1}^{-\frac{1}{4}+7\varepsilon}\|E_{k-1}\|_{k-1}\right] + e_{k}
\leq \frac{1}{2}\|\xi\| + \frac{1}{2} < 1
\]
Hence \(\|T(\xi)\| < 1\) as required.

(2.) It suffices to show that the mapping is a contraction. We show that under our assumptions
\[
\|\xi' - \xi\| = \|T(\xi_1) - T(\xi_2)\| \leq \frac{1}{2}\|\xi_1 - \xi_2\|
\]
(600)

First consider the \(m\) terms. Since \(m(E)\) is linear \(|m(E_{1,k}) - m(E_{2,k})| \leq O(1)e_{k}^{\frac{1}{4}}||E_{1,k} - E_{2,k}||_k\) so
\[
e_{k}^{-\frac{1}{4}+8\varepsilon}|m'_{1,k} - m'_{2,k}| \leq e_{k}^{-\frac{1}{4}+8\varepsilon}\left(L^{-1}|m_{1,k+1} - m_{2,k+1}| + |m(E_{1,k}) - m(E_{2,k})|\right)
\leq L^{-\frac{1}{4}}\left[e_{k+1}^{-\frac{1}{4}+8\varepsilon}|m_{1,k+1} - m_{2,k+1}|\right] + O(1)e_{k}^{\frac{1}{4}+7\varepsilon}||E_{k-1} - E_{2,k}||_k
\leq \frac{1}{2}\|\xi_1 - \xi_2\|
\]
(601)

Now consider the \(E\) terms. The term \(LE_{k-1}\) cancels and we have with \(E_{k-1}^* = LE_{k-1}^\#\)
\[
E'_{1,k} - E'_{2,k} = LR(E_{1,k-1} - E_{2,k-1}) + (E_{k-1}^*(m_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{2,k-1}))
\]
(602)
Then
\[
e_{k}^{-\frac{1}{4}+7\varepsilon}\|E'_{1,k} - E'_{2,k}\|_k \leq O(1)e_{k}^{\frac{1}{4}+7\varepsilon}L^{-\frac{1}{4}+2\varepsilon}\|E_{1,k-1} - E_{2,k-1}\|_{k-1}
+ e_{k}^{\frac{1}{4}+7\varepsilon}\|E_{k-1}^*(m_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{2,k-1})\|_k
\]
(603)
For the first term in \((\text{603})\) we have
\[
O(1)e_{k}^{\frac{1}{4}+7\varepsilon}L^{-\frac{1}{4}+2\varepsilon}\|E_{1,k-1} - E_{2,k-1}\|_{k-1} \leq O(1)L^{-\frac{1}{4}+6\varepsilon}\left[e_{k-1}^{-\frac{1}{4}+7\varepsilon}\|E_{1,k-1} - E_{2,k-1}\|_{k-1}\right] \leq \frac{1}{6}\|\xi_1 - \xi_2\|
\]
(604)
For the second term in (603) we write
\[ e_k^{-\frac{1}{2} + 7\epsilon} \|E_{k-1}^*(m_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{2,k-1})\| \]
\[ \leq e_k^{-\frac{1}{2} + 7\epsilon} \|E_{k-1}^*(m_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{1,k-1})\| \]
\[ + e_k^{-\frac{1}{2} + 7\epsilon} \|E_{k-1}^*(m_{2,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{2,k-1})\| \]  (605)

Now \( E_{k}^*(m,E) \) is actually an analytic function of its arguments so for the first term in (604) we can write for \( r > 1 \)
\[ E_{k-1}^*(m_{1,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{1,k-1}) \]
\[ = \frac{1}{2\pi i} \int_{|t|=\pi} \frac{dt}{t(t-1)} E_{k-1}^*(m_{2,k-1} + t(m_{1,k-1} - m_{2,k-1}), E_{1,k-1}) \]  (606)

We can assume \( m_{1,k-1} \neq m_{2,k-1} \) and take \( r = 3e_k^{-\frac{2}{6} - \epsilon} |m_{1,k-1} - m_{2,k-1}|^{-1} \). This is greater than one since \( |m_{1,k-1} - m_{2,k-1}| \leq e_k^{-\frac{2}{6} - \epsilon} \|\xi_1 - \xi_2\| \leq 2e_k^{-\frac{1}{2} - \epsilon} \). Also it keeps us well inside the domain for \( E_{k-1}^* \) as given by the main theorem. Hence we can use the estimate \( \|E_{k-1}^*\|_k \leq e_k^{-\frac{1}{2} - \epsilon} \) from (607). Hence the first term in (603) is bounded by
\[ O(1)e_k^{-\frac{1}{2} + 7\epsilon} \| e_k^{-\frac{2}{6} - \epsilon} \| \xi_1 - \xi_2\| \leq \frac{1}{6} \|\xi_1 - \xi_2\| \]  (607)

For the second term in (603) we write for \( r > 1 \)
\[ E_{k-1}^*(m_{2,k-1}, E_{1,k-1}) - E_{k-1}^*(m_{2,k-1}, E_{2,k-1}) \]
\[ = \frac{1}{2\pi i} \int_{|t|=\pi} \frac{dt}{t(t-1)} E_{k-1}^*(m_{2,k-1}, E_{2,k-1} + t(E_{1,k-1} - E_{2,k-1})) \]  (608)

Now we take \( r = 3e_k^{-\frac{1}{2} + 7\epsilon} \|E_{1,k-1} - E_{2,k-1}\|_{k-1}^{-1} \) which is bigger than one since \( \|E_{1,k-1} - E_{2,k-1}\|_{k-1} \leq 2e_k^{-\frac{1}{2} - \epsilon} \), and it keeps us well within the domain of \( E_{k-1}^* \). Again using \( \|E_{k-1}^*\|_k \leq e_k^{-\frac{1}{2} - \epsilon} \) this term is bounded by
\[ O(1)e_k^{-\frac{1}{2} + 7\epsilon} \| e_k^{-\frac{2}{6} - \epsilon} \| \xi_1 - \xi_2\| \leq \frac{1}{6} \|\xi_1 - \xi_2\| \]  (609)

Combining (604), (607), (609) yields \( e_k^{-\frac{1}{2} + 7\epsilon} \|E_{1,k} - E_{k}^*\| \leq \frac{1}{2} \|\xi_1 - \xi_2\| \) which together with (601) gives the desired result \( \|\xi_1 - \xi_2\| \leq \frac{1}{2} \|\xi_1 - \xi_2\| \).

Now we can state:

**Theorem 3.** Let \( L \) be sufficiently large and \( \epsilon \) be sufficiently small. Then for each \( N \) there is a unique sequence \( \xi_k, m_k, E_k \) for \( k = 0, 1, 2, \ldots, K = N - m \) satisfying of the dynamical equation (587) and the boundary conditions (588), and with \( e_k = L^{-\frac{1}{2}(N-k)} \epsilon \)
\[ |m_k| \leq e_k^{\frac{3}{2} - 8\epsilon} \quad \|E_k\|_k \leq e_k^{-\frac{1}{2} - 7\epsilon} \]  (610)

Furthermore
\[ |\xi_k| \leq 2e_k^{-\frac{1}{2} - 7\epsilon} \]  (611)
Proof. This solution \((m_k, E_k)\) is the fixed point from the previous lemma and the bounds \((610)\) are a consequence.

The energy density \(\varepsilon_k\) is determined by
\[
\varepsilon_k = L^{-3} \varepsilon_{k+1} - \varepsilon(E_k) - \varepsilon_k^0
\]
with the final condition \(\varepsilon_K = 0\), which however we now treat as an initial condition. From \((610)\) we have \(|\varepsilon(E_k)| \leq c\|E_k\| \leq ce_k^{\frac{1}{4}}\varepsilon_k - e\varepsilon_k(E_k) - e_k^0\). We claim that
\[
|\varepsilon_k| \leq 2ce_k^{\frac{1}{4}}\varepsilon_k - e\varepsilon_k(E_k) - e_k^0 \leq 2ce_k^{\frac{1}{4}}\varepsilon_k - e\varepsilon_k(E_k) - e_k^0
\]
It is true for \(k = K\). We suppose it is true for \(k + 1\) and prove it for \(k\). This follows for \(L\) large by
\[
|\varepsilon_k| \leq L^{-3}2ce_k^{\frac{1}{4}}\varepsilon_k - e\varepsilon_k(E_k) - e_k^0 \leq 2ce_k^{\frac{1}{4}}\varepsilon_k - e\varepsilon_k(E_k) - e_k^0
\]
This completes the proof.

Remark. The treatment of scalar QED in \([29]\) featured a more awkward treatment of the matter determinant (as did an earlier version of this paper). This could be improved by adopting the present strategy for determinants. Also the iteration in that paper should have been stopped at \(K = N - m\) rather than \(N\).

A Grassman integrals

We develop facts about Grassman algebras and associated integrals. General references are \([35], [32]\).

A.1 basic estimates

Consider the Grassman algebra generated by \(\{\Psi(s)\}_{s \in S}\) where \((S, \mu)\) is a finite measure space. The general element has the form
\[
E = E(\Psi) = \sum_n \frac{1}{n!} \sum_{s_1, \ldots, s_n} E_n(s_1, \ldots, s_n)\Psi(s_1) \cdots \Psi(s_n)\mu(s_1) \cdots \mu(s_n)
\]
where \(E_n(s_1, \ldots, s_n)\) is totally anti-symmetric in its arguments and \(\Psi(s)\Psi(s') = -\Psi(s')\Psi(s)\). Pick a fixed ordering for \(S\). Then we can also write this as a sum over ordered subsets \(I = (s_1, \ldots, s_n)\) of variable length as
\[
E(\Psi) = \sum_I E(I)\Psi(I)\mu(I)
\]
where
\[
E(I) = E(s_1, \ldots, s_n) \quad \Psi(I) = \Psi(s_1) \cdots \Psi(s_n) \quad \mu(I) = \mu(s_1) \cdots \mu(s_n)
\]

We define a norm
\[
\|E\|_h = \sum_n \frac{h^n}{n!} \sum_{s_1, \ldots, s_n} |E_n(s_1, \ldots, s_n)|\mu(s_1) \cdots \mu(s_n)
\]
which can also be written
\[
\|E\|_h = \sum_I h^{|I|} |E(I)|\mu(I)
\]

Lemma 26.
\[
\|EF\|_h \leq \|E\|_h \|F\|_h
\]
Proof. We have

\[ EF = \sum_{I,J} E(I)F(J)\Psi(I)\Psi(J)\mu(I)\mu(J) \quad (621) \]

Only terms with \( I \cap J = \emptyset \) contribute and we classify them by \( I \cup J \) so

\[ EF = \sum_{K} \left[ \sum_{I \cup J = K, I \cap J = \emptyset} E(I)F(J)\text{sgn}((I, J) \to K) \right] \Psi(K) \mu(K) \quad (622) \]

where \( \text{sgn}((I, J) \to K) \) is the sign of the permutation taking \((I, J)\) to \(K\). Then dropping the condition \( I \cap J = \emptyset \)

\[ \|EF\|_{h} = \sum_{I,J} h^{\|I\|} E(I)F(J)\mu(I)\mu(J) \quad (623) \]

Next consider the Grassman algebra generated by \( \Psi \) indexed by \((S_1, \mu_1)\) and \( \chi \) indexed by \((S_2, \mu_2)\). The general element can be written

\[ E = E(\Psi, \chi) = \sum_{I \subset S_1, J \subset S_2} E(I, J)\Psi(I)\chi(J)\mu_1(I)\mu_2(J) \quad (624) \]

An associated norm depends on two parameters \( h, k \) and is

\[ \|E\|_{h,k} = \sum_{I,J} h^{\|I\|} k^{\|J\|} E(I, J)\mu_1(I)\mu_2(J) \quad (625) \]

Lemma 27. Let \( \Psi, \Psi' \) be indexed by the same \((S, \mu)\) and define

\[ E^+ = E^+(\Psi, \Psi') = E(\Psi + \Psi') \quad (626) \]

Then

\[ \|E^+\|_{h,h'} \leq \|E\|_{h+h'} \quad (627) \]

Proof. With \( E(\Psi) \) of the form \( \text{(616)} \)

\[ E^+(\Psi, \Psi') = \sum_{K} E(K)((\Psi + \Psi')(K))\mu(K) \]

\[ = \sum_{K} E(K) \left[ \sum_{I \cup J = K, I \cap J = \emptyset} \text{sgn}((I, J) \to K)\Psi(I)\Psi'(J) \right] \mu(K) \quad (628) \]

\[ = \sum_{I \cup J = K, I \cap J = \emptyset} E(I \cup J)\Psi(I)\Psi'(J)\mu(I)\mu(J) \]

\[ = \sum_{I \cup J = K, I \cap J = \emptyset} E^+(I, J)\Psi(I)\Psi'(J)\mu(I)\mu(J) \]

77
where \(\text{sgn}(I,J)\) is the sign of the permutation that puts \((I,J)\) is standard order and \(E^+(I,J) = E(I \cup J)\text{sgn}(I,J)\) if \(I \cap J = \emptyset\) and is zero otherwise. Therefore

\[
\|E^+\|_{h,h'} = \sum_{I,J} h^{[I]}(h')^{[J]}|E^+(I,J)|\mu(I)\mu(J)
\]

\[
\leq \sum_{I \cap J = \emptyset} h^{[I]}(h')^{[J]}|E(I \cup J)|\mu(I)\mu(J)
\]

\[
= \sum_{K \cup J = K, I \cap J = \emptyset} h^{[I]}(h')^{[J]}|E(K)|\mu(K)
\]

\[
= \sum_K (h+h')^{[K]}|E(K)|\mu(K)
\]

\[
= \|E\|_{h+h'}
\]

\[
(629)
\]

A.2 dressed Grassman variables

Now suppose we introduce dressed fields \(\psi(t)\) defined on a new measure space \((T, \nu)\) and defined by

\[
\psi(t) = (H \Psi)(t) \equiv \sum_s H(t,s)\Psi(s)\mu(s)
\]

(630)

We consider elements of the algebra of the form

\[
E = E(\psi) = \sum_n \frac{1}{n!} \sum_{t_1, \ldots, t_n} E_n(t_1, \ldots, t_n)\psi(t_1) \cdots \psi(t_n)\nu(t_1) \cdots \nu(t_n)
\]

(631)

or for ordered subsets \(I \subset T\) in some fixed ordering

\[
E(\psi) = \sum_I E(I)\psi(I)\nu(I)
\]

(632)

We define a norm on the kernel by

\[
\|E\|_h = \sum_n \frac{h^n}{n!} \sum_{t_1, \ldots, t_n} |E_n(t_1, \ldots, t_n)|\nu(t_1) \cdots \nu(t_n)
\]

(633)

which can also be written

\[
\|E\|_h = \sum_I h^{[I]}|E(I)|\nu(I)
\]

(634)

Since \(\psi(t)\psi(t') = -\psi(t')\psi(t)\) we have as in lemma [29] that if \(G = EF\) then the kernels satisfy

\[
\|G\|_h \leq \|E\|_h\|F\|_h
\]

(635)

Next we estimate the basic norm in terms of the kernel norm. We use the norm

\[
\|H\|_{1,\infty} \equiv \sup_{t \in T} \sum_{s \in S} |H(t,s)|\mu(s)
\]

(636)

**Lemma 28.** If \(E'(\Psi) = E(H \Psi)\) as in \([630], [631]\) then

\[
\|E'\|_h \leq \|E\|_{H_{1,\infty}h}
\]

(637)
Proof. Start with

\[ E'(\Psi) = \sum_n \frac{1}{n!} \sum_{s_1, \ldots, s_n} E_n^s(s_1, \ldots, s_n) \Psi(s_1) \cdots \Psi(s_n) \mu(s_1) \cdots \mu(s_n) \]  

(638)

where

\[ E_n^s(s_1, \ldots, s_n) = \sum_{t_1, \ldots, t_n} E_n(t_1, \ldots, t_n) \prod_{i=1}^n \nu(t_i) \mathcal{H}(t_i, s_i) \]  

(639)

is totally anti-symmetric under permutations. Then

\[
\|E'\|_h \leq \sum_{n=0}^\infty \frac{h^n}{n!} \sum_{s_1, \ldots, s_n} \left[ \sum_{t_1, \ldots, t_n} |E_n(t_1, \ldots, t_n)| \prod_{i=1}^n \nu(t_i) \mathcal{H}(t_i, s_i) \right] \mu(s_1) \cdots \mu(s_n) \\
\leq \sum_{n=0}^\infty \frac{(\|\mathcal{H}\|_{1,\infty} h)^n}{n!} \sum_{t_1, \ldots, t_n} |E_n(t_1, \ldots, t_n)| \nu(t_1) \cdots \nu(t_n) \\
= \|E\|_{\mathcal{H}_{1,\infty} h}
\]

(640)

A.3 several dressed Grassman variables

The previous results generalize directly to the case of two or more dressed fields. Again consider the Grassman algebra generated by \( \Psi \) on measure spaces \((S, \mu)\). Define new fields \( \psi_1, \psi_2 \) on measure spaces \((T_1, \nu_1), (T_2, \nu_2)\) by

\[ \psi_i(t_i) = (\mathcal{H}_i \Psi)(t_i) = \sum_{s \in S} \mathcal{H}_i(t_i, s) \Psi(s) \mu(s) \quad i = 1, 2 \]  

(641)

We consider elements of the algebra of the form

\[ E = E(\psi_1, \psi_2) = \sum_{I,J} E(I, J) \psi_1(I) \psi_2(J) \nu_1(I) \nu_2(J) \]  

(642)

where \( I \) is an ordered subset from \( T_1 \) and \( J \) is an ordered subset from \( T_2 \). This can also be written without ordering as

\[
E(\psi_1, \psi_2) = \sum_{n,m} \frac{1}{n! m!} \sum_{t_1, \ldots, t_n, t'_1, \ldots, t'_m} E_{nm}(t_1, \ldots, t_n, t'_1, \ldots, t'_m) \\
\psi_1(t_1) \cdots \psi_1(t_n) \psi_2(t'_1) \cdots \psi_2(t'_m) \prod_{i=1}^n \nu_1(t_i) \prod_{i=1}^m \nu_2(t'_i) \\
\]

(643)

The associated norm on the kernels depends on parameters \( h = (h_1, h_2) \) and is

\[
\|E\|_h = \sum_{I,J} h_1^{|I|} h_2^{|J|} |E(I, J)\nu_1(I)\nu_2(J)\]

(644)

which is also written

\[
\|E\|_h = \sum_{n,m} \frac{h_1^n h_2^m}{n! m!} \sum_{t_1, \ldots, t_n, t'_1, \ldots, t'_m} |E_{nm}(t_1, \ldots, t_n, t'_1, \ldots, t'_m)| \prod_{i=1}^n \nu_1(t_i) \prod_{i=1}^m \nu_2(t'_i) \\
\]

(645)

As in lemma \ref{lemma26} we have again that if \( G = EF \) then

\[
\|G\|_h \leq \|E\|_h \|F\|_h
\]

(646)
Lemma 29. If \( E'(\Psi) = E(\mathcal{H}_1 \Psi, \mathcal{H}_2 \Psi) \) as above then

\[
\|E'\|_h \leq \|E\|_{\mathcal{H}_1,1,\infty h, \mathcal{H}_2,1,\infty h} \tag{647}
\]

Proof. We have

\[
E'(\Psi) = \sum_{n,m} \frac{1}{n!m!} \sum_{s_1, \ldots, s_n, s'_1, \ldots, s'_m} E'_{nm}(s_1, \ldots, s_n, s'_1, \ldots, s'_m) 
\]

\[
\Psi(s_1) \cdots \Psi(s_n) \Psi(s'_1) \cdots \Psi(s'_m) \prod_i \mu(s_i) \prod_j \mu(s'_j) \tag{648}
\]

where

\[
E'_{nm}(s_1, \ldots, s_n, s'_1, \ldots, s'_m)
\]

\[
= \sum_{t_1, \ldots, t_n, t'_1, \ldots, t'_m} E_{nm}(t_1, \ldots, t_n, t'_1, \ldots, t'_m) \prod_{i=1}^n \nu(t_i) \mathcal{H}_1(t_i, s_i) \prod_{j=1}^m \nu(t'_j) \mathcal{H}_2(t'_j, s'_j) \tag{649}
\]

is anti-symmetric under permutations within each group. We can rewrite (648) as

\[
E'(\Psi) = \sum_{I \cap J = \emptyset} E'(I, J) \Psi(I) \Psi(J) \mu(I) \mu(J)
\]

\[
= \sum_K \left[ \sum_{I \cup J = K, I \cap J = \emptyset} E'(I, J) \operatorname{sgn}(I, J \rightarrow K) \right] \Psi(K) \mu(K) \tag{650}
\]

\[
= \sum_K E'(K) \psi(K) \mu(K)
\]

Then

\[
\|E'\|_h = \sum_K h^{\|K\|} \|E'(K)\| \mu(K)
\]

\[
\leq \sum_{I \cup J = K, I \cap J = \emptyset} h^{\|I\|} h^{\|J\|} \|E'(I, J)\| \mu(I) \mu(J)
\]

\[
\leq \sum_{I, J} h^{\|I\|} h^{\|J\|} \|E'(I, J)\| \mu(I) \mu(J)
\]

which can be rewritten as

\[
\|E'\|_h \leq \sum_{n,m} \frac{h^{n+m}}{n!m!} \left[ \sum_{s_1, \ldots, s_n, s'_1, \ldots, s'_m} |E'(s_1, \ldots, s_n, s'_1, \ldots, s'_m)| \prod_i \mu(s_i) \prod_j \mu(s'_j) \right]
\]

\[
\leq \sum_{n,m} \frac{h^{n+m}}{n!m!} \|E_{nm}\| \|\mathcal{H}_1\|_{1,\infty} \|\mathcal{H}_2\|_{1,\infty} \tag{652}
\]

A.4 further results

We list some further results. Theses are mostly variations of earlier results, but now involve fields \( \psi_1, \psi_2 \) which can both be dressed fields. They are straightforward to check.
• If \( \psi, \psi' \) are defined on the same space \((T, \nu)\) and \( E^+(\psi, \psi') = E(\psi + \psi') \) then the kernels satisfy

\[ ||E^+||_{h,h'} \leq ||E||_{h+h'} \]  

(653)

More generally if \((\psi_1, \psi_2)\) and \((\psi'_1, \psi'_2)\) are fields such that \(\psi_1, \psi'_1\) are defined on the same space and \(\psi_2, \psi'_2\) are defined on the same space and \( E^+((\psi_1, \psi_2), (\psi'_1, \psi'_2)) = E(\psi_1 +, \psi'_1, \psi_2 + \psi'_2) \) then with \( h = (h_1, h_2) \) and \( h' = (h'_1, h'_2) \) the kernels satisfy

\[ ||E^+||_{h,h'} \leq ||E||_{h+h'} \]  

(654)

• Let \( A \) be an operator from functions on \((T_2, \nu_2)\) to \((T_1, \nu_1)\). Let \( E \) be defined on fields indexed by \((T_1, \nu_1)\) and define \( E' \) on fields indexed by \((T_2, \nu_2)\) by \( E'(\psi) = E(A\psi) \). Then the kernels satisfy

\[ ||E'||_h \leq ||E||_{A_{1,\infty}h} \]  

(655)

More generally let \( \psi_1, \psi_2 \) be indexed by \((T_1, \nu_1)\) and \((T_2, \nu_2)\) and suppose

\[ E'(\psi_1, \psi_2) = E(A_{11}\psi_1 + A_{12}\psi_2, A_{21}\psi_1 + A_{22}\psi_2) \]  

(656)

where \( A_{ij} \) is an operator mapping functions on \((T_j, \nu_j)\) to functions on \((T_i, \nu_i)\). If

\[ ||A_{ij}||_{1,\infty} \leq C_{ij} \]  

(657)

then the kernels satisfy

\[ ||E'||_{h_1,h_2} \leq ||E||_{C_{11}h_1 + C_{12}h_2, C_{21}h_1 + C_{22}h_2} \]  

(658)

### A.5 Gaussian integrals

Until now we have implicitly treated \( \Psi, \bar{\Psi} \) as different components of the same field. Now we distinguish them and consider our Grassman algebra as generated by \( \Psi, \bar{\Psi} \) each indexed by \((S, \mu)\). The general element now has the form

\[
E(\Psi, \bar{\Psi}) = \sum_{n,m} \frac{1}{n!m!} \sum_{s_1, \ldots, s_n, t_1, \ldots, t_m} E(s_1, \ldots, s_n, t_1, \ldots, t_m) \Psi(s_1) \cdots \Psi(s_n) \bar{\Psi}(t_1) \cdots \bar{\Psi}(t_m) \prod_i \mu(s_i) \prod_j \mu(t_j)
\]

(659)

which can also be written

\[
E(\Psi, \bar{\Psi}) = \sum_{I, \bar{I}} E(I, \bar{I}) \Psi(I) \bar{\Psi}(\bar{I}) \mu(I) \mu(\bar{I})
\]

(660)

where \( I, \bar{I} \) are ordered sequences of points. There is an associated norm

\[
||E||_h = \sum_{n,m} \frac{h^{n+m}}{n!m!} \sum_{s_1, \ldots, s_n, t_1, \ldots, t_m} |E(s_1, \ldots, s_n, t_1, \ldots, t_m)| \prod_i \mu(s_i) \prod_j \mu(t_j)
\]

(661)

or

\[
||E||_h = \sum_{I, \bar{I}} h^{|I|+|\bar{I}|} |E(I, \bar{I})| \mu(I) \mu(\bar{I})
\]

(662)

This norm agrees with the norm used when treating \( \Psi, \bar{\Psi} \) on the same footing.

The Gaussian integral with covariance \( \Gamma \) satisfies

\[
\int \Psi(s_1) \cdots \Psi(s_n) \bar{\Psi}(t_1) \cdots \bar{\Psi}(t_m) d\mu_\Gamma(\Psi) = \begin{cases} \det \{ \Gamma(s_i, t_j) \} & n = m \\ 0 & n \neq m \end{cases}
\]

(663)
If we have the identity covariance
\[
\int \Psi(s_1) \cdots \Psi(s_n) \bar{\Psi}(t_1) \cdots \bar{\Psi}(t_m) d\mu_I(\Psi) = \begin{cases} 
\det \{ \delta_{s_i,t_j} \} & n = m \\
0 & n \neq m 
\end{cases}
\] (664)
which is also written
\[
\int \Psi(I) \bar{\Psi}(\bar{I}) d\mu_I(\Psi) = \begin{cases} 
1 & I = \bar{I} \\
0 & I \neq \bar{I} 
\end{cases}
\] (665)
It follows that
\[
\int E(\Psi, \bar{\Psi}) d\mu_I(\Psi) = \sum_I E(I, I) \mu(I)^2 
\] (666)
and so
\[
| \int E(\Psi, \bar{\Psi}) d\mu_I(\Psi) | \leq \| E \|_1 
\] (667)

Here is a variation. Suppose that in addition to \( \Psi, \bar{\Psi} \) there are independent fields \( \psi, \bar{\psi} \) indexed by \((T, \nu)\). (Or there could be more extra fields). Consider elements of the form
\[
E(\psi, \bar{\psi}, \Psi, \bar{\Psi}) = \sum_{I, \bar{I}, J, \bar{J}} E(I, \bar{I}, J, \bar{J}) \psi(I) \bar{\psi}(\bar{I}) \Psi(J) \bar{\Psi}(\bar{J}) \nu(I) \nu(\bar{I}) \mu(J) \mu(\bar{J}) 
\] (668)
and define
\[
E^\#(\psi, \bar{\psi}) = \int E(\psi, \bar{\psi}, \Psi, \bar{\Psi}) d\mu_I(\Psi) 
\] (669)
Note that if contributions to \( E \) have equal numbers of \((\psi, \Psi)\) and \((\bar{\psi}, \bar{\Psi})\) variables, the integral selects terms with equal numbers of \( \Psi, \bar{\Psi} \) variables and hence \( E^\# \) must have equal numbers of \( \psi, \bar{\psi} \) variables.

**Lemma 30.**
\[
\| E^\# \|_h \leq \| E \|_{h,1} 
\] (670)

**Proof.** We have
\[
E^\#(\psi, \bar{\psi}) = \sum_{I, \bar{I}, J, \bar{J}} E(I, \bar{I}, J, \bar{J}) \psi(I) \bar{\psi}(\bar{I}) \nu(I) \nu(\bar{I}) \mu(J) \mu(\bar{J}) \int \Psi(J) \bar{\Psi}(\bar{J}) d\mu_I(\Psi) 
\] 
\[
= \sum_{I, \bar{I}} \left[ \sum_J E(I, \bar{I}, J, \bar{J}) \mu(J)^2 \psi(I) \bar{\psi}(\bar{I}) \nu(I) \nu(\bar{I}) \right] 
\] 
\[
= \sum_{I, \bar{I}} \left[ E^\#(I, \bar{I}) \psi(I) \bar{\psi}(\bar{I}) \nu(I) \nu(\bar{I}) \right] 
\] (671)
Then
\[
\| E^\# \|_h = \sum_{I, \bar{I}} h^{|I| + |\bar{I}|} \| E^\#(I, \bar{I}) \nu(I) \nu(\bar{I}) \| 
\leq \sum_{I, \bar{I}, J} h^{|I| + |\bar{I}|} \| E(I, \bar{I}, J, \bar{J}) \nu(I) \nu(\bar{I}) \mu(J)^2 \| 
\leq \sum_{I, \bar{I}, J} h^{|I| + |\bar{I}|} \| E(I, \bar{I}, J, \bar{J}) \nu(I) \nu(\bar{I}) \mu(J) \mu(\bar{J}) \| 
\equiv \| E \|_{h,1} 
\] (672)
B an identity

Lemma 31. \( \Gamma_{k,y}(A) = (D_k(A) + bL^{-1}P(A) + i\gamma_3 y)^{-1} \) has the representation

\[
\Gamma_{k,y}(A) = B_{k,y}(A) + b_k^2 B_{k,y}(A)Q_k(A)S_{k,y}(A)Q_k^T(-A)B_{k,y}(A)
\]

where

\[
\begin{align*}
B_{k,y}(A) &= \left( b_k + bL^{-1}P(A) + i\gamma_3 y \right)^{-1} = \frac{1}{b_k + i\gamma_3 y} \left( I - P(A) \right) + \frac{1}{b_k + bL^{-1} + i\gamma_3 y} P(A) \\
S_{k,y}(A) &= \left( D_A + \bar{m}_k + \frac{b_k i\gamma_3 y}{b_k + i\gamma_3 y} P_k(A) + \frac{b_k^2 bL^{-1}}{(b_k + i\gamma_3 y)(b_k + bL^{-1} + i\gamma_3 y)} P_{k+1}(A) \right)^{-1}
\end{align*}
\]

Proof. Start with

\[
\exp \left( < \bar{J} , \Gamma_k(A) J > \right) = \text{const} \int d\Psi \exp \left( \left< \bar{\Psi}, J \right> + \left< J, \Psi \right> - \left< \bar{\Psi}, (bL^{-1}P(A) + i\gamma_3 y)\Psi \right> - \left< \bar{\Psi}, D_k(A)\Psi \right> \right)
\]

and from section 2.5

\[
\exp \left( - \left< \bar{\Psi}, D_k(A)\Psi \right> \right) = \text{const} \int \exp \left( -b_k \left< \bar{\Psi} - Q_k(-A)\bar{\psi}, \Psi - Q_k(A)\psi \right> - \left< \bar{\psi}, (D_A + \bar{m}_k)\psi \right> \right) d\psi
\]

Insert the second into the first and do the integral over \( \Psi \) which is

\[
\begin{align*}
\int d\Psi \exp \left( \left< \bar{\Psi}, J \right> + \left< J, \Psi \right> - \left< \bar{\Psi}, (bL^{-1}P(A) + i\gamma_3 y)\Psi \right> - b_k \left< \bar{\Psi} - Q_k(-A)\bar{\psi}, \Psi - Q_k(A)\psi \right> \right) \\
= \int d\Psi \exp \left( \left< \bar{\Psi}, J + b_k Q_k(A)\psi \right> + \left< J + b_k Q_k(-A)\bar{\psi}, \Psi \right> \\
- \left< \bar{\Psi}, \left( b_k + bL^{-1}P(A) + i\gamma_3 y \right)\Psi \right> - b_k \left< \bar{\psi}, P_k(A)\psi \right> \right) \\
= \text{const} \exp \left( \left< J + b_k Q_k(-A)\bar{\psi}, B_{k,y}(A)(J + b_k Q_k(A)\psi) \right> - b_k \left< \bar{\psi}, P_k(A)\psi \right> \right)
\end{align*}
\]

This gives

\[
\exp \left( < \bar{J} , \Gamma_k(A) J > \right) = \text{const} \int \exp \left( \left( J + b_k Q_k(-A)\bar{\psi}, B_{k,y}(A)(J + b_k Q_k(A)\psi) \right) - \left< \bar{\psi}, (D_A + \bar{m}_k + b_k P_k(A))\psi \right> \right) d\psi
\]

\[
\text{const} \exp \left( < \bar{J} , B_{k,y}(A) J > \right) \int \exp \left( b_k Q_k^T(A)B_{k,y}(A)\bar{\psi}, b_k Q_k^T(-A)B_{k,y}(A)J \right) - \left< \bar{\psi}, S_{k,y}(A)^{-1}\psi \right> \right) d\psi
\]

\[
\text{const} \exp \left( < \bar{J} , B_{k,y}(A) J > + b_k^2 \left< J, B_{k,y}(A)Q_k(A)S_{k,y}(A)Q_k^T(-A)B_{k,y}(A)J \right> \right)
\]

where we defined

\[
S_{k,y}(A) = \left( D_A + \bar{m}_k + b_k P_k(A) - b_k^2 Q_k^T(-A)B_{k,y}(A)Q_k(A) \right)^{-1}
\]
This is the same as the definition in \[674\] since we can write
\[
B_{k,y}(A) = \frac{1}{b_k + i\gamma_3 y} - \left( \frac{bL^{-1}}{b_k + bL^{-1} + i\gamma_3 y} \right) P(A)
\]
(680)
and this yields
\[
b_k P_k(A) - b_k^2 Q_k^T(-A)B_{k,y}(A) Q_k(A)
\]
(681)
\[
= \left( b_k - \frac{b_k^2}{b_k + i\gamma_3 y} \right) P_k(A) + \left( \frac{b_k^2 bL^{-1}}{b_k + bL^{-1} + i\gamma_3 y} \right) P_{k+1}(A)
\]
C another identity

**Theorem 4.** For \( F \) on bonds on \( T_{N-k}^1 \) we have \( Q^T F \) on \( T_{N-k}^0 \) and \( Q^T_{k+1} F \) on \( T_{N-k}^{-1} \) and the identify
\[
\left( Q^T_{k+1} F, Q^T_{k+1} Q^T F \right) = \left( [Q^T F]^x, \hat{C}_k [Q^T F]^x \right)
\]
(682)
Here \( \hat{C}_k \) is the operator on the subspace \( \{ Z \text{ on } T_{N-k}^0 : \tau Z = 0 \} \) defined by
\[
e^{\frac{1}{2} <J, \hat{C}_k J>} = \int DZ \delta(\tau Z) \exp \left( -\frac{1}{2} \left< Z, [\Delta_k + aQ^T Q] Z \right> \right) e^{<Z,J>/\{J = 0 \}}
\]
(683)
and \([Q^T F]^x\) is the projection of \( Q^T F \) onto \( \ker \tau \).

**Proof.** We sketch the proof and refer to [5], [6] for more details. Let \( J = Q^T_{k+1} F \). Start with
\[
e^{\frac{1}{2} <J, \varphi_{k+1}^0 J>} = \int DA \left( -\frac{1}{2} \left< A, \left( \delta d + dR_{k+1} + aQ^T_{k+1} Q_{k+1} \right) A \right> \right) e^{<A,J>/\{J = 0 \}}
\]
(684)
Let \( \Delta = \delta d + d\delta \) and write \( \delta d + dR_{k+1} \delta = \Delta - dP_{k+1} \delta \) where \( P_{k+1} = I - R_{k+1} \) is a projection. We change from \( P_{k+1} \) to \( P_k \) using the identities for \( \lambda, \lambda' \) on \( T_{N-k}^{-1} \)
\[
\exp \left( -\frac{1}{2} \left< \delta A, P_k \delta A \right> \right) = \int DX \delta(K,\lambda') \exp \left( -\frac{1}{2} \| \delta A - \Delta \lambda' \|^2 \right)/\{ \delta A = 0 \}
\]
(685)
\[
\exp \left( -\frac{1}{2} \left< \delta A, P_{k+1} \delta A \right> \right) = \int DL \delta(K_{k+1},\lambda) \exp \left( -\frac{1}{2} \| \delta A - \Delta \lambda \|^2 \right)/\{ \delta A = 0 \}
\]
Also insert for \( \omega \) on \( T_{N-k}^0 \) insert the basic axial gauge identity
\[
\int D\omega \delta(Q\omega) \delta(\tau(Q_k A + d\omega)) = \text{const}
\]
(686)
This yields
\[
e^{\frac{1}{2} <J, \varphi_{k+1}^0 J>} = \text{const} \int D\omega \delta(Q\omega) \int DA \exp \left( -\frac{1}{2} \left< A, \left( \Delta - dP_k \delta + aQ^T_{k+1} Q_{k+1} \right) A \right> \right)
\]
(687)
\[
\delta \left( \tau(Q_k A + d\omega) \right) \int DX \delta(K,\lambda') \exp \left( -\frac{1}{2} \| \delta A - \Delta \lambda' \|^2 \right) e^{<A,J>}
\]
Next define $\lambda = \mathcal{H}_k'\mu$ to be the minimizer of $\|\Delta \lambda\|^2$ subject to $Q_k\lambda = \mu$ and make the change of variables $A \rightarrow A - d\mathcal{H}_k'\omega$. This is constructed to leave the quadratic form in $A$ invariant. It can be written $<A, \delta(A)dR_k\delta + a\mathcal{Q}_k^T Q_{k+1} >$. To see the invariance note that $Q_kd\mathcal{H}_k'\omega = dQ_k\mathcal{H}_k'\omega = d\omega$ so $Q_kA \rightarrow Q_kA - d\omega$. Then $Q_{k+1}A \rightarrow Q_{k+1}A - dQ_kA$, but $Q_kA = Q_{k+1}A$ so $Q_{k+1}A$ is invariant. Furthermore $\delta A \rightarrow \delta A - \Delta \mathcal{H}_k'\omega$ since $\delta d = \Delta$ on scalars. But one has the explicit representations $\mathcal{H}_k' = \Delta^{-2}Q_k^T(\Delta^{-2}Q_k^T)^{-1}$ as well as $P_k = \Delta^{-1}Q_k^T(\Delta^{-2}Q_k^T)^{-1}Q_k\Delta^{-1}$ and these combine to give $P_k\Delta \mathcal{H}_k'\omega = \Delta \mathcal{H}_k'\omega$ and hence $R_k\Delta \mathcal{H}_k'\omega = 0$ and so $R_k\delta A$ is invariant. And of course $dA$ is invariant so the form is invariant.

Other changes are that $\delta(\tau(Q_kA + d\omega))$ becomes $\delta(\tau(Q_kA))$ and that $<A, J> = <Q_{k+1}A, F>$ is invariant. The numerator in the big fraction becomes

$$\int D\lambda' \delta(Q_k\lambda')\exp\left(-\frac{1}{2}\|\delta A - \Delta(\mathcal{H}_k'\omega + \lambda')\|^2\right)$$

(688)

Make make the further change of variables $\lambda' \rightarrow \lambda' - \mathcal{H}_k'\omega$ which gives

$$\int D\lambda' \delta(Q_k\lambda' - \omega)\exp\left(-\frac{1}{2}\|\delta A - \Delta\lambda'\|^2\right)$$

(689)

Similarly the denominator in the big fraction is invariant since the change in $Q_{k+1}\lambda$ is $Q_{k+1}\mathcal{H}_k'\omega = Q\omega = 0$.

Now we have

$$e^{\frac{1}{2}<J, \mathcal{Q}_{k+1}^J >} = \text{const} \int D\omega \delta(Q\omega) \int DA \exp\left(-\frac{1}{2} <A, (\delta d + dR_k\delta + a\mathcal{Q}_k^T Q_{k+1})A >\right)$$

$$\delta(\tau Q_kA)\frac{\int D\lambda' \delta(Q_k\lambda' - \omega)\exp\left(-\frac{1}{2}\|\delta A - \Delta\lambda'\|^2\right)e^{<A, J>}}{\int D\lambda \delta(Q_{k+1}\lambda)\exp\left(-\frac{1}{2}\|\delta A - \Delta\lambda\|^2\right)}$$

(690)

$$= \text{const} \int DA \exp\left(-\frac{1}{2} <A, (\delta d + dR_k\delta + a\mathcal{Q}_k^T Q_{k+1})A >\right)\delta(\tau Q_kA)e^{<A, J>}$$

In the second step we did the integral over $\omega$ and canceled out the big fraction.

Next insert $1 = \int \delta(\mathcal{Q}_kA - Z)DZ$ and obtain

$$e^{\frac{1}{2}<J, \mathcal{Q}_{k+1}^J >}$$

$$= \text{const} \int DA \int DZ \delta(\mathcal{Q}_kA - Z)\delta(\tau Z)\exp\left(-\frac{1}{2} <A, (\delta d + dR_k\delta)A > - \frac{1}{2}a\|QZ\|^2\right)e^{<A, J>}$$

(691)

Let $\mathcal{H}_k$ be the minimizer in $A$ of $<A, (\delta d + dR_k\delta)A >$ subject to to $\mathcal{Q}_kA = Z$. In the integral over $A$ make the translation $A \rightarrow A + \mathcal{H}_kZ$. Then the integral factors into

$$e^{\frac{1}{2}<J, \mathcal{Q}_{k+1}^J >} = \text{const} \int DA \delta(Q_kA)\exp\left(-\frac{1}{2} <A, (\delta d + dR_k\delta)A >\right)e^{<A, J>}$$

$$\int DZ \delta(\tau Z)\exp\left(-\frac{1}{2}<\mathcal{H}_kZ, (\delta d + dR_k\delta)\mathcal{H}_kZ > - \frac{1}{2}a\|QZ\|^2\right)e^{<\mathcal{H}_kZ, J>}$$

(692)

But in the first integral $<A, J> = <Q_{k+1} A, F > = 0$ so this integral is a constant. In the second integral we have $<\mathcal{H}_k Z, J > = <Q_{k+1} \mathcal{H}_k Z, F > = <QZ, F >$. Also in the second integral we change from exponential gauge fixing to delta function gauge fixing by a Fadeev-Popov procedure and identify
the Landau gauge minimizer $\mathcal{H}_k$ to obtain
\[
\exp \left( -\frac{1}{2} \langle \hat{\mathcal{H}}_k Z, (\delta d + dR_k \delta) \hat{\mathcal{H}}_k Z \rangle \right)
= \text{const} \int DA \\delta(Q_k A - Z) \exp \left( -\frac{1}{2} \langle A, (\delta d + dR_k \delta) A \rangle \right)
= \text{const} \int DA \\delta(Q_k A - Z) \delta(R_k \delta A) \exp \left( -\frac{1}{2} \|dA\|^2 \right)
= \exp \left( -\frac{1}{2} \|d\mathcal{H}_k Z\|^2 \right) = \exp \left( -\frac{1}{2} \langle Z, \Delta_k Z \rangle \right)
\]
(693)

This yields
\[
e^{\frac{1}{2} \langle J, g_{k+1}^i J \rangle} = \text{const} \int DZ \delta(\tau Z) \exp \left( -\frac{1}{2} \langle Z, [\Delta_k + aQ^T Q] Z \rangle \right) \exp \left( \langle Z, Q^T F \rangle \right)
= \text{const} \exp \left( \frac{1}{2} \langle [Q^T F]^*, \tilde{C}_k [Q^T F]^* \rangle \right)
\]
(694)

Setting $F = J = 0$ we see that the constant must be one. This gives the result.

D an estimate on $Q^T$

**Lemma 32.** For $A$ on any $L$-lattice $T^1_{N-k}$
\[
\|Q^T A\|^2 \geq L^{-1} \|A\|^2
\]
(695)

Also if $[Q^T A]^*$ is the projection of $Q^T A$ onto $\ker \tau$
\[
\|[Q^T A]^*\|^2 \geq L^{-1} \|A\|^2
\]
(696)

**Proof.** Let $Q_s$ be the operator the averages over surface bonds of the cubes $B(y)$, see [18] for the precise definition. This satisfies $Q_s Q^T = L$ so $L^{-1} Q^T Q_s$ is an orthogonal projection. Also since $Q Q^T = I$ implies $Q_s Q^T = I$ we have
\[
\|Q^T A\|^2 \geq L^{-2} \|Q^T Q_s Q^T A\|^2 = L^{-2} \|Q_s A\|^2 = L^{-1} \|A\|^2
\]
(697)

The projection $L^{-1} Q^T Q_s$ annihilates functions on the interiors of the cubes $B(y)$. The projection $Q^T A \rightarrow [Q^T A]^*$ only affects the functions on the interiors. Thus $L^{-1} Q^T Q_s [Q^T A]^* = L^{-2} Q^T Q_s Q^T A$ and the same proof can be carried out.

E conditioning

Consider integrals of the form
\[
I = \int d\mu_{T-1}(A) F(A_\Lambda)
\]
(698)

where the integral is over functions $A$ on bonds in a unit lattice, $\mu_{T-1}$ is a Gaussian measure with covariance $T^{-1}$, $\Lambda$ is a subset of the lattice, and $A_\Lambda = 1_\Lambda A$ is the restriction to $\Lambda$, i.e. to bonds intersecting $\Lambda$. We want to express the integral in terms of a conditional expectation of $F(A_\Lambda)$ given the values $A_{\Lambda^c}$. This comes in two different forms. In general we define $T_\Lambda = 1_\Lambda T 1_\Lambda$ and $T_{\Lambda,\Lambda^c} = 1_\Lambda T 1_{\Lambda^c}$.
Lemma 33. The integral $I$ can be expressed as

$$I = \int d\mu_{T^{-1}}(A') \left[ \int d\mu_{T^{-1}}(A_L) F \left( A_{\Lambda} - T_{\Lambda}^{-1} T_{\Lambda} A'_{\Lambda} \right) \right]$$

(699)

or as

$$I = \int d\mu_{T^{-1}}(A') \exp \left( -\frac{1}{2} \left< A'_{\Lambda'}, T_{\Lambda} A_{\Lambda'} \right> \right) \left[ \int d\mu_{T^{-1}}(A_L) \exp \left( -\left< A'_{\Lambda}, T_{\Lambda} A_{\Lambda} \right> \right) F(A_L) \right]$$

(700)

Remark. These formulas were first used in [2] and [12] respectively.

Proof. We have

$$I = Z^{-1} \int DA \exp \left( -\frac{1}{2} \left< A, TA \right> \right) F(A_L) \quad Z = \int DA \exp \left( -\frac{1}{2} \left< A, TA \right> \right)$$

(701)

We write

$$\frac{1}{2} \left< A, TA \right> = \frac{1}{2} \left< A_{\Lambda'}, T_{\Lambda} A_{\Lambda} \right> + \left< A_{\Lambda'}, TA_{\Lambda} \right> + \frac{1}{2} \left< A_{\Lambda}, TA_{\Lambda} \right>$$

(702)

Diagonalize the quadratic form by the change of variables $A_{\Lambda} \to A_{\Lambda} - T_{\Lambda}^{-1} T_{\Lambda} A'_{\Lambda}$. This yields

$$I = Z^{-1} \int DA_{\Lambda'} \exp \left( -\frac{1}{2} \left< A_{\Lambda'}, \left( T_{\Lambda} - T_{\Lambda} A_{\Lambda} T_{\Lambda}^{-1} T_{\Lambda} A_{\Lambda} \right) A_{\Lambda'} \right> \right) \left[ \int DA_{\Lambda} \exp \left( -\frac{1}{2} \left< A_{\Lambda}, TA_{\Lambda} \right> \right) F \left( A_{\Lambda} - T_{\Lambda}^{-1} T_{\Lambda} A'_{\Lambda} \right) \right]$$

(703)

Now relabel $A_{\Lambda'}$ as $A'_{\Lambda}$ and insert

$$1 = Z_{\Lambda}^{-1} \int DA_{\Lambda}' \exp \left( -\frac{1}{2} \left< A'_{\Lambda}, TA'_{\Lambda} \right> \right)$$

(704)

Then the change of variables $A_{\Lambda}' \to A_{\Lambda}' + T_{\Lambda}^{-1} T_{\Lambda} A_{\Lambda} A'_{\Lambda}$ restores the quadratic form $\frac{1}{2} \left< A', TA' \right>$ and does not affect the interior integral. Thus

$$I = Z^{-1} \int DA' \exp \left( -\frac{1}{2} \left< A', TA' \right> \right) \left[ Z_{\Lambda}^{-1} \int DA_{\Lambda} \exp \left( -\frac{1}{2} \left< A_{\Lambda}, TA_{\Lambda} \right> \right) F \left( A_{\Lambda} - T_{\Lambda}^{-1} T_{\Lambda} A'_{\Lambda} \right) \right]$$

(705)

This is (699). In the interior integral make the inverse change of variables $A_{\Lambda} \to A_{\Lambda} + T_{\Lambda}^{-1} T_{\Lambda} A_{\Lambda}$ to regain $F(A_{\Lambda})$. The quadratic form $\frac{1}{2} \left< A_{\Lambda}, TA_{\Lambda} \right>$ becomes

$$\frac{1}{2} \left< A_{\Lambda}, TA_{\Lambda} \right> + \left< A_{\Lambda}', T_{\Lambda} A_{\Lambda} \right> + \frac{1}{2} \left< A_{\Lambda}', T_{\Lambda} A_{\Lambda} T_{\Lambda}^{-1} T_{\Lambda} A_{\Lambda} \right>$$

(706)

which gives (700).
References

[1] T. Balaban (Higgs)$_{2,3}$ Quantum fields in a finite volume- I Commun. Math. Phys., 85: 603-636, 1982.

[2] T. Balaban (Higgs)$_{2,3}$ Quantum fields in a finite volume- II Commun. Math. Phys., 86: 555-594, 1982.

[3] T. Balaban (Higgs)$_{2,3}$ Quantum fields in a finite volume- III Commun. Math. Phys., 88: 411-445, 1983.

[4] T. Balaban, Regularity and decay of lattice Green's functions, Commun. Math. Phys., 89: 571-597, 1983.

[5] T. Balaban, Propagators and renormalization transformations for lattice gauge field theories -I, Commun. Math. Phys. 95 (1984) 17-40.

[6] T. Balaban, Propagators and renormalization transformations for lattice gauge field theories- II, Commun. Math. Phys. 96 (1984) 223-250.

[7] T. Balaban, Averaging operators for lattice gauge field theories. Commun. Math. Phys. 98 (1985) 17-51.

[8] T. Balaban, Propagators for lattice gauge field theories in a background field, Commun. Math. Phys. 99 (1985) 389-434.

[9] T. Balaban, Ultraviolet stability of three-dimensional lattice pure gauge field theories. Commun. Math. Phys. 102 (1985) 255-275.

[10] T. Balaban, Variational problem and background field in renormalization group method for lattice gauge field theories. Commun. Math. Phys. 102 (1985) 277-309.

[11] T. Balaban, Renormalization group approach to lattice gauge field theories- I, Commun. Math. Phys. 109 (1987) 249-301.

[12] T. Balaban, Renormalization group approach to lattice gauge field theories- II, Commun. Math. Phys. 116 (1988) 1-22.

[13] T. Balaban, Convergent renormalization expansions for lattice gauge field theories, Commun. Math. Phys. 119 (1988) 243-285.

[14] T. Balaban, Large field renormalization-I, Commun. Math. Phys. 122 (1989) 175-202.

[15] T. Balaban, Large field renormalization-II, Commun. Math. Phys. 122 (1989) 355-392.

[16] T. Balaban, Localization expansions I. function of the background configurations. Commun. Math. Phys. 182: 33-82, 1996.

[17] T. Balaban, A. Jaffe, Constructive gauge theory. In Fundamental problems of gauge field theory. Erice, 1985, G. Velo and A. Wightman, eds., Plenum Press, 1986.

[18] T. Balaban, J. Imbrie, A. Jaffe, Renormalization of the Higgs model: minimizer, propagators, and the stability of mean field theory, Commun. Math. Phys. 97: 299-329, 1985.

[19] T. Balaban, J. Imbrie, A. Jaffe, Effective action and cluster properties of the abelian Higgs model, Commun. Math. Phys. 114: 257-315, 1988.
[20] T. Balaban, M. O’Carroll, R. Schor, Block renormalization group for Euclidean fermions, Commun. Math. Phys. 122: 233-247, 1989.

[21] T. Balaban, M. O’Carroll, R. Schor, Properties of block renormalization group operators for Euclidean fermions in an external field, J. Math Phys 32: 3199-3208, 1991.

[22] D. Brydges, J. Dimock, T. R. Hurd. A non-Gaussian fixed point for $\phi^4$ in $4 - \epsilon$ dimensions, Commun. Math. Phys. 198: 111-156, 1998.

[23] J. Dimock, Quantum electrodynamics on the 3-torus - I. arXiv: math-phys/0210020.

[24] J. Dimock, Quantum electrodynamics on the 3-torus - II. arXiv: math-phys/0407063.

[25] J. Dimock, The renormalization group according to Balaban - I. small fields, Rev. Math. Phys. 25, 1330010, 1-64, 2013.

[26] J. Dimock, The renormalization group according to Balaban - II. large fields, J. Math. Phys. 54, 092301, 1-85, 2013.

[27] J. Dimock, The renormalization group according to Balaban - III. convergence, Annales Henri Poincaré 15, 2133-2175, 2014.

[28] J. Dimock, Covariant axial gauge, Letters in Mathematical Physics 105, 959-987, 2015.

[29] J. Dimock, Nonperturbative renormalization of scalar QED in $d=3$, J. Math. Phys. 56, 102304, 1-78, 2015.

[30] J. Dimock, Ultraviolet stability for QED in $d=3$, to appear.

[31] J. Dimock, T. Hurd, A renormalization group analysis of infrared QED, J. Math. Phys. 33, 814-821, 1992.

[32] J. Feldman, H. Knörer, E. Trubowitz, Fermion functional integrals and the renormalization group, American Mathematical Society, 2002.

[33] J. Glimm, A. Jaffe, Quantum physics: a functional integral point of view, Springer, 1987.

[34] J. Imbrie, Renormalization group methods in gauge field theories. In K. Osterwalder and R. Stora, editors, Critical phenomena, random systems, gauge theories. North-Holland, 1986.

[35] M. Salmhofer, Renormalization: An Introduction, Springer, 1999.

[36] S. Summers, A Perspective on Constructive Quantum Field Theory, ArXiv: 1203.1991

[37] S. Weinberg, The quantum theory of fields- I., Cambridge, 1995