Damping of Confined Excitations Modes of 1D Condensates in an Optical Lattice

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We study the damping of the collective excitations of Bose-Einstein condensates in a harmonic trap potential loaded in an optical lattice. In the presence of a confining potential the system is non-homogeneous and the collective excitations are characterized by a set of discrete confined phonon-like excitations. We derive a general convenient analytical description for the damping rate, which takes into account, the trapping potential and the optical lattice, for the Landau and Beliaev processes at any temperature, T. At high temperature or weak spatial confinement, we show that both mechanisms display linear dependence on T. In the quantum limit, we found that the Landau damping is exponentially suppressed at low temperatures and the total damping is independent of T. Our theoretical predictions for the damping rate under thermal regime is in completely correspondence with the experimental values reported for 1D condensate of sodium atoms. We show that the laser intensity can tune the collision process, allowing a resonant effect for the condensate lifetime. Also, we study the influence of the attractive or repulsive non-linear terms on the decay rate of the collective excitations. A general expression of the renormalized Goldstone frequency has been obtained as a function of the 1D non-linear self-interaction parameter, laser intensity and temperature.

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I. INTRODUCTION

The damping process plays a crucial role in the dynamics of Bose-Einstein condensates (BEC). Phenomena such as superfluid phase transitions,[1–7] Josephson effect,[8–9] quantized vortices,[10–12] Mott insulator transition,[13] among others, are limited by the finite lifetime of the collective excitations of the condensed atoms, i.e. by the damping mechanisms and their dependence with the temperature. After the experimental confirmation that the collective excitations are damped,[2–13] the behavior of the decay rates have been of major interest in the physics of BEC. Thus, several studies and theoretical calculations of the damping of excitations have been performed in three-dimensional (3D)[11–13] two-dimensional (2D)[11,22,23] as well as one-dimensional(1D) systems.[24,25] For a better understanding of the damping process, it becomes necessary to consider the contribution of the parabolic confining potential. The behavior and characteristics of the decay rate and its dependence with temperature, differer radically if we are dealing or not with homogeneous systems. The main assumption for homonegous system is to consider the condensate density constant in the all space.

In the present work we are dealing with a microscopic theory for the damping rate of collective oscillations, specifically for a quasi-one-dimensional (1D) condensate confined to a parabolic harmonic trap potential and loaded into an optical lattice. The existence of a non-negligible confining external potential breaks the invariance symmetry, which leads to the damping rate showing a different qualitative behavior in comparison with previous formalism, where the condensate is tackled as a homogeneous system (see for example Refs. [22,27,28] and references there in). The influence of both external interactions -the trap potential and laser intensity- must provide a physically richer scenario for the decay of collective excitations. The knowledge of excited states or Goldstone modes enables the characterization of the condensate dynamics in a general framework. In the case we are in presence of spatially non-homogeneous BEC system, the label spacing of the discrete spectrum (confined phonon-like modes) and also, the nature and symmetry of wavefunction of the collective modes, are required for the calculation of the collision scattering process.[23]

Cigar-shaped traps can be considered as quasi-one-dimensional systems. We select such platform of a condensate loaded simultaneously into a 1D harmonic potential and an optical lattice to characterize the phenomenon of damping and tackle the problem analytically. Within the framework of mean field theory, the physical characteristics of a BEC in such trapping profile are ruled by the temperature dependent non-linear Gross-Pitaevskii equation (GPE) in an external potential

\[ V_{\text{ext}}(x) = \frac{1}{2} m \omega_0^2 x^2 - V_L \cos^2 \left( \frac{2 \pi}{d} x \right), \]

where \( m \) is the alkaline atom mass, \( V_L \) the laser intensity, \( d \) its laser wavelength, and \( \omega_0 \) the frequency of the harmonic trap.
It becomes clear that the last diagram in Fig. 1b) does not contribute to the self-energy interaction at $T = 0$ K, since a thermal excited mode $\omega_i$ must be present in the system (see Eq. (3) below). We must recall that the thermal cloud in the present theory is assumed in thermal equilibrium.

Figure 1 presents the leading diagrams contributing to the self-energy $\pi_p$. Accordingly, the complex frequency correction can be factorized into two main processes as $\pi_p = \Delta\omega_L + \Delta\omega_B$ and, in the Hartree-Fock-Bogoliubov approximation we obtain that

$$\pi_p = \frac{2\pi g_1^2}{\hbar^2} \sum_{i,j} \left[ \frac{2(f_i - f_j)|A_{ij}|^2}{\omega_p + \omega_i - \omega_j + i\varepsilon} + \frac{(1 + f_i + f_j)|B_{ij}|^2}{\omega_p - \omega_i - \omega_j + i\varepsilon} \right],$$

where $g_1$ is the coupling constant, $A_{ij}$ ($B_{ij}$) represents the matrix elements for the Landau (Beliaev) process $\omega_p + \omega_i \rightarrow \omega_j$ ($\omega_p \rightarrow \omega_i + \omega_j$). In Eq. (3) the sum $\sum_{i,j}$ takes into account all possible virtual transitions $|\omega_i\rangle$ and $|\omega_j\rangle$, contributing to the decay rate, while the term $1 + f_i + f_j$ gives us the Bose-Einstein statistical factor of the phonon $|\omega_p\rangle$ decaying into two confined phonon modes (first diagram in Fig. 1c) $|\omega_i\rangle$ and $|\omega_j\rangle$, and $f_i - f_j$ corresponds to the thermal correction for the annihilation and creation of phonons with frequencies $\omega_i$ and $\omega_j$ (second diagram in Fig. 1c), respectively.

### II. THEORETICAL BACKGROUND

In the framework of the Green function formalism, the spectrum of the excited states is obtained by the poles of the dressed Green function $G_p$. The solution of the Dyson equation, shown diagrammatically in Fig. 1a), is the renormalized Green function $G_p$ given by

$$G_p^{-1} = G_0^{-1} - \pi_p. \quad (2)$$

In absence of interaction $G_p \rightarrow G_0 = [\omega - \omega_p + i\varepsilon]^{-1}$, $\varepsilon > 0$, $\omega_p$ is the eigenfrequency of the excited state, and $\pi_p$ is the self-energy contribution. The solution of Eq. (2) leads to the complex frequency $\omega = \omega_p + \pi_p$. Here, $Re(\pi_p)$ represents the renormalized contribution to the eigenfrequency $\omega_p$, while $Im(-\pi_p)$ corresponds to the damping rate.

We assume that the decay processes are associated to the collision between confined phonon states. In first order, the collision term is described by the interaction between three interacting phonon modes, giving rise to the vertices in the self-energy diagrams shown in Fig. 1c). It becomes clear that the last diagram in Fig. 1c) does not contribute to the self-energy interaction at $T = 0$ K, since a thermal excited mode $\omega_i$ must be present in the system (see Eq. (3) below). We must recall that the thermal cloud in the present theory is assumed in thermal equilibrium.

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#### A. Bogoliubov type excitations

Bogoliubov type excitations can be searched by applying a small deviation from the GPE stationary solutions $|\psi_0\rangle \exp(-i\mu t/\hbar)$, i.e.

$$|\Psi(t)\rangle = \exp(-i\mu t/\hbar) \left[ |\psi_0\rangle + |u\rangle \exp(-i\omega t) + |v^*\rangle \exp(i\omega t) \right], \quad (4)$$

with $\mu$ being the chemical potential. By linearizing the time-dependent non-linear GPE, we obtain the Bogoliubov-de Gennes equations (B-dGE) for the eigenfrequencies $\omega$ and amplitudes $|u|$ and $|v|$. As stated above, if the harmonic trap potential is switched off, we are in presence of the homogeneous case, where the phonon wavevector $q = q_x e_x$ is a good quantum number and with excited frequency $\omega(q_x)$ depending on the phonon wavevector. Thus, the Bogoliubov’s low-lying excitation spectrum shows a linear phonon dispersion in $q_x$. On the other hand when the condensate is loaded into a harmonic potential, $\omega_0 \neq 0$ in Eq. (1), the system becomes inhomogeneous and the wavevector $q$ is no longer a good quantum number. In such case the B-dGE provides a set of discrete excited state frequencies $\omega_p$ ($p = 1, 2, ...$).

By considering the periodic potential in (1) and the non-linear term, $g_1 |\phi_0|^2$, in the GPE as a perturbation...
with respect to the harmonic trap potential, \( \frac{1}{2}m\omega_0^2x^2 \), phonon frequencies \( \omega_p \) up to the second order in \( g_1 \) are given by\(^{39} \)

\[
\frac{\omega_p}{\omega_0} = p + \frac{\Lambda}{\sqrt{2\pi}} \left[ -1 + \frac{2\Gamma(p+1/2)}{\sqrt{\pi p!}} \right] - \frac{V_0}{2} \exp \left( -\alpha^2 \right) \times \left[ L_i(2\alpha^2) - 1 \right] - \frac{\Lambda V_0}{\sqrt{2\pi}} \exp \left( -\alpha^2 \right) \left[ Ei(2\alpha^2) - C - \ln \frac{\alpha^2}{2} + \frac{\delta_p(\alpha)}{\sqrt{\pi}} \right] + \frac{V_0^2}{4} \exp \left( -2\alpha^2 \right) \left[ Chi(2\alpha^2) - C - \ln 2\alpha^2 + \rho_p(\alpha) \right] + \Lambda^2 \left[ \frac{\gamma_p}{2\pi^2} + 0.033106 \right], \quad p = 1, 2, ...., (5)
\]

with \( \Lambda = g_1N/(\hbar_0\omega_0) \), \( N \) the number of atoms, \( \hbar_0 = \sqrt{\hbar/m_0} \) defining the characteristic unit length, \( \alpha = 2\pi\hbar_0/d \) and \( V_0 = V_L/\omega_0 \). In addition \( L_i(z), \Gamma(z), Ei(z), Chi(z) \), and \( C \) are the Laguerre polynomials, the gamma function, the exponential integral, the cosine hyperbolic integral and the Euler’s constant, respectively. The parameters \( \gamma_p, \delta_p \) and \( \rho_p \) are reported elsewhere\(^{39} \).

**B. Symmetry of the excited states**

Owing to the inversion symmetry, the space of solutions can be decoupled into two independent subspaces, \( O \) and \( \mathcal{E} \) for \( p = 1, 3, .... \) and \( p = 2, 4, .... \) modes, respectively. Hence, the components \( |u_p\rangle \) and \( |v_p\rangle \) are expanded over the complete set 1D oscillator wave functions \( \{\phi_{2p+1}\} \) or \( \{\phi_{2p}\} \) for the odd, \( O \) and even, \( \mathcal{E} \) Hilbert subspaces. The normalized eigenvectors \( |\Phi_p\rangle = [|u_p\rangle, |v_p\rangle] \), up to first order in \( \Lambda \) and \( V_0 \), can be cast as\(^{10} \)

\[
|\Phi_p\rangle = \left( |\phi_p\rangle + \sum_{m\neq p} \frac{4\lambda f_{p,m} - V_0g_{p,m}}{2(p-m)} |\phi_m\rangle \right) \exp \left(-\alpha^2\right) \left[ L_i(2\alpha^2) - 1 \right] - \frac{\Lambda V_0}{\sqrt{2\pi}} \exp \left(-\alpha^2\right) \left[ Ei(2\alpha^2) - C - \ln \frac{\alpha^2}{2} + \frac{\delta_p(\alpha)}{\sqrt{\pi}} \right] + \frac{V_0^2}{4} \exp \left(-2\alpha^2\right) \left[ Chi(2\alpha^2) - C - \ln 2\alpha^2 + \rho_p(\alpha) \right] + \Lambda^2 \left[ \frac{\gamma_p}{2\pi^2} + 0.033106 \right], \quad p = 1, 2, ...., (6)
\]

with

\[
f_{p,m} = \frac{(-1)^{p-m}/2}{\sqrt{2\pi m!}} \Gamma \left( \frac{p + m + 1}{2} \right), \quad (7)
\]

\[
g_{p,m} = \frac{(-1)^{p-m}/2}{\sqrt{2\pi m!}} \frac{h!}{(2\alpha^2)^{p-m}/2} \exp \left(-\alpha^2\right) L_h^{p-m}(2\alpha^2), \quad (8)
\]

\( L_h^{\alpha}(2\alpha^2) \) the Askey Laguerre polynomials, \( h = (p + m - |p-m|)/2 \) and \( m + p \) is an even number.

The parity of the function \( |\Phi_p\rangle \) is linked to the index \( p \), if \( p \) is even or odd the eigenstate \( |\Phi_p\rangle \) is symmetric or antisymmetric. The decay process of a certain phonon \( p \) is restricted by the symmetry property of the matrix elements in Eq. \((3)\). The amplitudes \( A_{ij}(p) \) and \( B_{ij}(p) \) impose a parity selection rule for the involved states \( |\Phi_p\rangle, |u_i\rangle \) and \( |v_i\rangle \). As shown in Eqs. \((9)\) and \((11)\), for a symmetric (antisymmetric) state \( |\Phi_p\rangle \), the amplitudes \( |u_i\rangle \) and \( |v_i\rangle \) must fulfill the parity condition \( i + j = \) even (odd) number, therefore limiting the possible number process for Beliaev, \( \omega_p \rightarrow \omega_i + \omega_j \), and Landau, \( \omega_p + \omega_i \rightarrow \omega_j \) decay rates. Besides the symmetry of the matrix elements \( A_{ij}(p) \) and \( B_{ij}(p) \), for certain eigenmode \( |\Phi_p\rangle \) with frequency \( \omega_p \), a key role in the damping process is ruled by the label spacing between the Bogoliubov collective oscillations \( \Delta^p_{i,j} = (\omega_p - \omega_i - \omega_j)/\omega_0 \) (see Eq. \((3)\)) on the reduced laser intensity \( V_0 \) for Beliaev and Landau damping rates, respectively. Critical values, \( V_0^{(p,i,j)} \), where \( \Delta^p_{i,j} \) approaches zero, are shown by arrows.

![FIG. 2: (Color online) Dependence of the label spacing \( \Delta^p_{i,j} \) on the reduced laser intensity \( V_0 \) for Beliaev and Landau damping rates, respectively. Critical values, \( V_0^{(p,i,j)} \), where \( \Delta^p_{i,j} \) approaches zero, are shown by arrows.](image-url)
III. DECAY RATE

In the sequel, we consider that the damping is originated by a collision process, and in first-order approximation, it is described by the interaction of the three confined phonons, giving rise to a cubic interaction in the bare phonon amplitude. This mechanism is represented by the vertex diagrams of the self-energy part shown in Fig. 1b).

A. Beliaev damping

In first-order loop approximation, the Beliaev mechanism arises from the collision of three particles where one phonon with frequency \( \omega_p \) is annihilated decaying into two confined excitations \( \omega_i \) and \( \omega_j \). Therefore, the allowed processes for the confined modes \( \omega_p \) are those with \( p = 2, 3, \ldots \). Following Feynman diagrams of Fig. 1b) and using the eigenfunction amplitudes given in Eq. (6), we find that the decay amplitude \( B_{ij} \) can be cast as:

\[
B_{ij} = \int dx \psi_0 \left[ u_p (u_i^* u_j^* + u_j^* u_i^*) + v_p (v_i^* v_j^* + v_j^* v_i^*) \right] . \tag{9}
\]

Thus, for the Beliaev damping rate we obtain

\[
\gamma^{(B)} = \frac{\gamma^{(0)}}{2} |A| \mathcal{M}^{(B)}(\Lambda, V_0) , \tag{10}
\]

with \( \gamma^{(0)} = 4\pi g_1/(l_0 \hbar) \) and \( \mathcal{M}^{(B)}(\Lambda, V_0) \) being defined in the Appendix A. Notice that Beliaev mechanism is forbidden if \( \epsilon \to 0 \) in Eq. (3). The energy conservation limits the real phonon transitions \( \omega_p \to \omega_i + \omega_j \).

Figure 3 displays the behavior of the Beliaev damping rate, \( \gamma^{(B)} \), in units of \( \gamma^{(0)} \), as a function of the reduced parameter \( \Lambda \) for the first seven allowed confined modes \( p = 2, \ldots, 8 \). In this calculation we used \( T = 0 \) and laser intensity \( V_L = 0 \). For small values of \( \Lambda \), all the normalized damping seen in Fig. 3 presents a linear behavior, while for increasing values of \( \Lambda \), the function \( \gamma^{(B)}/\gamma^{(0)} \) behaves non-monotonically and reaching a maximum. For a given excited state \( |\Phi_p\rangle \), the position of the maximum is not symmetric with respect of the type of non-linear interaction (repulsive \( g_1 > 0 \) or attractive \( g_1 < 0 \)). In Fig. 4 we show the dependence of \( \gamma^{(B)} \) on the dimensionless laser intensity \( V_0 \), and calculated for \( \Lambda = 2, d/l_0 = 0.25 \) and \( T = 0 \) K. It can be observed that the excited states \( p = 4, 5 \) and 6 show sharp peaks at certain values of \( V_0 \). These features are linked to the zeros of the frequency label spacing \( \Delta_p^{(ij)}(V_0) \) as represented in Fig. 2 while the number of transitions \( \omega_p \to \omega_i + \omega_j \) and the strength of the matrix elements, \( \overline{B}_{ij} \), dictate the relative intensity of the peaks.

B. Landau damping

Here, in the damping process a phonon mode with frequency \( \omega_p \) and a thermal excitation \( \omega_i \) are annihilated and confined phonon is created. Thus, the Landau mechanism is a thermal process at finite temperature. In present case the vertices phonon-phonon interaction (see

\[
\text{FIG. 3: (Color online) Reduced Beliaev damping } \gamma^{(B)}/\gamma^{(0)} \text{ for the Goldstone modes } p=2,\ldots,8 \text{ versus the dimensionless self-interaction parameter } \Lambda \text{ at laser intensity } V_0 = 0 \text{ and } T = 0 \text{ K.}
\]

\[
\text{FIG. 4: (Color online) Influence of the laser intensity } V_0 \text{ on the reduced Beliaev damping } \gamma^{(B)}/\gamma^{(0)} . \text{ Resonant peaks are related to the zeros of the label spacing } \Delta_p^{(ij)}(V_0) \text{ in Eq. (3).}
\]
\[ A_{ij} = \int dx \phi_0 \left[ u_p \left( u_i u_j^* + v_i v_j^* + v_j u_i^* \right) + v_p \left( u_i u_j^* + v_i v_j^* + v_i u_j^* \right) \right] . \]  

Thus, we have

\[ \gamma_L^{(p)}(\Lambda) = \gamma^{(0)}(\Lambda) \left| M_p^{(L)}(\Lambda, V_0) \right|, \]  

where \( M_p^{(L)}(\Lambda, V_0) \) is defined in the Appendix B. Hence, the total damping can be cast as

\[ \gamma^{(p)} = \gamma^{(0)}(\Lambda) \left( M_p^{(L)} + \frac{1}{2} M_p^{(B)} \right). \]  

Figure 5 presents the total damping \( \gamma^{(p)} \) (solid lines) as a function of \( \Lambda \) for the first fifth excited states. The Landau contribution \( \gamma_L^{(p)} \) is represented by dashed lines. As in the case of the Beliaev process, \( \gamma^{(p)} / \gamma^{(0)} \sim |\Lambda| \) for small values of the self-interaction atom-atom parameter, while for large values of |\( \Lambda \)|, the function \( \gamma^{(p)} / \gamma^{(0)} \) has a maximum at certain \( \Lambda_p \) value. We note that \( \gamma_L^{(p)} \) is smaller than \( \gamma_B^{(p)} \) for all excited states \( p = 1, ..., 5 \). For the Beliaev damping, the first excited state \( p = 1 \) is forbidden at any temperature, while for \( T \neq 0 \) K this mode becomes allowed for the Landau process.

The dependence of \( \gamma^{(p)} \) (solid lines) for \( p = 1, ..., 5 \) on the laser intensity is shown in Fig. 6. For sake of comparison the Landau damping contribution is represented by dashed lines. In the figure it is observed that the total damping presents the same behavior as the Beliaev decay (see Fig. 4), also, that \( \gamma_L^{(p)} \) shows resonant transitions for \( V_0 \sim 140 \).
where the Landau, the thermal regime supported by Eq. (A5), is a better aproximation with the exact result in Fig. 7, we can see that relatively. In the case of the Beliaev damping and by comparison with the experiment of Ref. [4] the condensate was loaded in a trap where the transversal frequency $\omega_r >> \omega_0$. Hence, we can argue that we are in presence of a quasi-1D condensate. The excitation frequency employed in the experiment was of $\omega_{ex} = 1.58\omega_0$ and $\omega_0 = 2\times19.3$ Hz. Following the Bogoliubov excitation spectrum of Eq. (5), the excitation frequency $\omega_{ex}$ corresponds to the $p = 2$ confined phonon mode with a dimensional non-linear parameter $\Lambda = 3.42$. Using the asymptotic expression for the high temperature regime, Eq. (14), we obtain $\gamma^{(2)} = 4.4$ s$^{-1}$ for $T = 200$ nK and 17.6 s$^{-1}$ for $T = 800$ nK which agree quite well with the reported experimental values of 4 s$^{-1}$ and 18 s$^{-1}$. In the evaluation was assumed a condensate of 3500 atoms and from the value of $\Lambda = 3.42$ we extract an effective 1D coupling constant $g_1 = 3.7 \times 10^{-25}$ eVm.

At very low temperature or strong confined regime, i.e. $k_B T << \hbar \omega_0$, from Eqs. (A7) and (B4) follows that the total damping of the exited mode $p$ can be cast as

$$\gamma^{(p)} = \left[A_p^{(1)}(T) + \frac{1}{2} (B_p^{(0)} + B_p^{(1)}(T))\right],$$

Here, the coefficients $A_p^{(1)}$ and $B_p^{(1)}$ decay exponentially with $T$ and $\gamma^{(p)}$ is almost constant independent of the temperature. Comparing the results of Eq. (15) with the theoretical calculations for 3D or 2D homogeneous systems, where the trap potential and confined effect are neglected, we found for the Landau damping a different behavior. Reference [22] reports the law $\gamma_L \sim T^2$, while the limit of $\gamma_L \sim T^4$ is predicted in Ref. [10][13][19]. The quantum limit or very low temperature for Beliaev damping and the total decay as a function of reduced temperature, are represented by open diamonds in Figs. 7 and 8. From the figures it can be notice that, for $k_B T << \hbar \omega_0$, the asymptotic approach given by Eq. (15) reproduces quite well the decay processes.

An important result is the knowledge of the excited frequency shift as a function of the condensate parameters and the applied laser intensity. The real part of the self-energy in Eq. (6) allows to an analytical expression for the renormalized excited frequency, $Re\{\pi_p\} = \Delta \tilde{\omega}_p$, as a
function of $\Lambda$, $V_0$ and $T$. Thus,
\[
\Delta \tilde{\omega}_p = \gamma_0 |\Lambda| \sum_{i,j} \left[ (1 + f_i + f_j) |B_{ij}|^2 \right.
\frac{(\omega_p - \omega_i - \omega_j)\omega_0}{(\omega_p - \omega_i - \omega_j)^2 + \varepsilon^2} +
\left. \frac{1}{2} (f_i - f_j) |A_{ij}|^2 \frac{(\omega_j - \omega_i - \omega_p)\omega_0}{(\omega_j - \omega_i - \omega_p)^2 + \varepsilon^2} \right]. \quad (16)
\]

Figures 9 shows the dependence of the dimensionless self-interaction parameter $\Lambda$ on the renormalized discrete phonon frequencies $\Delta \tilde{\omega}_p = \omega - \omega_p$ for the reduced temperature values $k_B T/\hbar \omega_0 = 0, 1$ and 2. We conclude that for the attractive regime ($\Lambda < 0$), the renormalized shift $\Delta \tilde{\omega}_p > 0$, while the opposite result is obtained for the repulsive interaction, i.e. $\Delta \tilde{\omega}_p < 0$ if $\Lambda < 0$. This behavior is understood by the dependence of $\Delta \tilde{\omega}_p$ in Eq. (16) on the label spacing $\Delta_{(i,j)}^{(1)} = (\omega_p - \omega_i - \omega_j)/\omega_0$ and $\Delta_{(i,p)}^{(j)} = (\omega_j - \omega_i - \omega_p)/\omega_0$ as a function of $\Lambda$. According to the results of Appendixes A and B, we have that in the thermal regime, $\Delta \tilde{\omega}_p$ is proportional to $k_B T/\hbar \omega_0$. Thus, a linear increase or decrease of the excited frequency with the temperature is predicted for attractive or repulsive interaction between atoms, respectively.

In conclusion, we evaluated the damping rates of confined phonon modes of 1D condensates in a harmonic trap potential loaded in an optical lattice. We remarked the influence of the spatial confinement potential on the collective oscillations and on the damping rates as a function of the temperature. The presence of an optical lattice as an external field, allows to manipulate the decay rate of the condensate. The damping $\gamma^{(p)}$ can be turned on or turned off as a function of the laser intensity. Also, for a given excited state $p$ and tuning the laser intensity, it is possible to get a set of transitions, $\omega_p \to \omega_j \pm \varepsilon$, reaching to a resonant effect for the total lifetime $1/\gamma^{(p)}$.

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**Appendix A: Beliaev matrix element**

After substituting the perturbed wave function $|u_p\rangle$ and $|v_p\rangle$ in the matrix element (9), and neglecting terms of the second order or higher in $\Lambda$ and $V_0$, for the function $\mathcal{M}^{(B)}_p$ we get
\[
\mathcal{M}^{(B)}_p(\Lambda, V_0) = \sum_{i,j} (1 + f_i + f_j) |B_{ij}|^2 \mathcal{L}^{(i)}_p + \mathcal{L}^{(j)}_p, \quad (A1)
\]
where $B_{ij} = T_{0ji} - \Delta F_{pij} + V_0 H_{pij}, T_{pij} (l + p + i + j = \text{even number})$ is reported elsewhere,$^{30}$
\[
F_{pij} = a_{pij} + b_{pij}^+ + b_{ijp}^+ + 2(b_{pij}^- + b_{jip}^- + b_{jpi}^-), \quad (A2)
\]
\[
H_{pij} = c_{pij} + d_{pij} + d_{jip} + d_{jpi}, \quad (A3)
\]
\[
a_{pij} = \sum_{m \neq 0} \frac{T_{2m000} T_{2mpij}}{2m} ; \quad b_{pij}^+ = \sum_{m} \frac{T_{00pm} T_{0mij}}{m \pm p}, \quad (A4)
\]
\[
c_{pij} = \sum_{m \neq 0} \frac{g_{0,2m} T_{0mij}}{2m} ; \quad d_{pij} = \sum_{m \neq 0} \frac{g_{p,m} T_{0mij}}{2(m - p)}, \quad (A4)
\]
with the parity condition $p + i + j = \text{even number}$, and the Lorenzian function,
\[
\mathcal{L}^{(\pm)}_p = \frac{1}{\pi} \frac{\omega_0 \varepsilon}{(\omega_p \mp \omega_i - \omega_j)^2 + \varepsilon^2}. \quad (A5)
\]

In the limit of thermal regime, the probability $\mathcal{M}^{(B)}_p$ is reduced to
\[
\mathcal{M}^{(B)}_p(T) = \frac{k_B T}{\hbar \omega_0} \left[ B^{(1)}_p + \frac{1}{12} \left( \frac{\hbar \omega_0}{k_B T} \right)^2 B^{(0)}_p \right], \quad (A5)
\]
with
\[ B_p^{(r)} = \sum_{i,j} \omega_i + \omega_j \left( \frac{\omega_0^2}{\omega_i \omega_j} \right)^r |B_{ij}|^2 \mathcal{L}_{p}^{(+)} ; \quad (r = 0, 1) . \]

For low temperature we have
\[ \mathcal{M}_{\mathcal{B}}^{(B)}(T) = B_p^{(0)} + B_p^{(1)}(T) . \] (A7)

where
\[ B_p^{(r)} = \sum_{i,j} \left( \exp(-\omega_i/k_B T) + \exp(-\omega_j/k_B T) \right)^r \times |B_{ij}|^2 \mathcal{L}_{p}^{(+)} ; \quad (r = 0, 1) . \] (A8)

**Appendix B: Landau matrix element**

Using the wave function \(|u_p\rangle, |v_p\rangle\) and Eq. (11) and neglecting terms higher than \(A\) and \(V_0\) we have for \(\mathcal{M}_{\mathcal{B}}^{(B)}\)
\[ \mathcal{M}_{\mathcal{B}}^{(L)} = \sum_{i,j} (f_i - f_j) |A_{ij}|^2 \mathcal{L}_{p}^{(-)} , \] (B1)

where
\[ A_{ij} = T_{0p_{ij}} - \Lambda D_{p_{ij}} + V_0 G_{p_{ij}} , \]
\[ D_{p_{ij}} = d_{p_{ij}} + b_{p_{ij}}^+ + b_{p_{ij}}^{-1} + 2(b_{p_{ij}}^- + b_{p_{ij}}^+ + b_{p_{ij}}^-) , \]
\[ G_{p_{ij}} = c_{p_{ij}} + d_{p_{ij}} + d_{ijp} + d_{ji} . \]

If \(k_B T < \hbar \omega_0\), the Landau probability process can be approached to
\[ \mathcal{M}_{\mathcal{B}}^{(L)}(T) = \frac{k_B T}{\hbar \omega_0} \left[ A_p^{(1)} - \frac{1}{12} \left( \frac{\hbar \omega_0}{k_B T} \right)^2 A_p^{(0)} \right] \] (B2)

with
\[ A_p^{(r)} = \sum_{i,j} \omega_i - \omega_j \left( \frac{\omega_0^2}{\omega_i \omega_j} \right)^r |A_{ij}|^2 \mathcal{L}_{p}^{(-)} ; \quad (i = 0, 1) . \] (B3)

While, for the weak confinement, \(k_B T >> \hbar \omega_0\), it is obtained that
\[ \mathcal{M}_{\mathcal{B}}^{(L)}(T) = \frac{k_B T}{\hbar \omega_0} \sum_{i,j} \exp(-\omega_i/k_B T) \times \frac{\hbar \omega_0}{k_B T} \mathcal{L}_{p}^{(-)} . \] (B4)

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