ON THE CHAIN STRUCTURE IN THE DE BRANGES SPACES

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Abstract. We study the indivisible intervals and the monotonic-ity of the growth of the exponential type in the chains of de Branges subspaces in terms of the spectral measure. We prove that for spectral measures supported on \( \mathbb{Z} \), there exist at most two subspaces of the same type, which then bound an indivisible interval. Furthermore, in this case, we study possible locations of the indivisible intervals.

1. Introduction and main results

One of the main parts of the de Branges theory of Hilbert spaces of entire functions is the study of the chains of the de Branges subspaces of a given de Branges space. There are different equivalent definitions of de Branges spaces (axiomatic approach, in terms of Hermite–Biehler functions, in terms of the Hamiltonians, as the weighted Cauchy transforms, spaces isometrically imbedded into \( L^2 \) with respect to a measure on the real line). For more information on the de Branges theory see \([10, 14, 15]\) and the references therein. For some recent progress see, for example, \([6, 16]\).

In this paper we use the weighted Cauchy transform definition of the de Branges spaces. Let \( T = \{t_n\}_{n \in \mathbb{N}} \) be a discrete subset of the real
line and let \( \mu = \sum_{n \in \mathbb{N}} \mu_n \delta_{t_n} \) be a positive measure such that
\[
\sum_{n \in \mathbb{N}} \frac{\mu_n}{t_n^2 + 1} < \infty.
\]

Fix an entire function \( A \) real on the real line with simple zeros at \( T = \text{supp} \mu \) and define the corresponding de Branges space (in the weighted Cauchy transform form) \( \mathcal{HC}(\mu) \) as follows:
\[
\mathcal{HC}(\mu) = \mathcal{HC}(A, \mu) = \left\{ f(z) = A(z) \sum_{n \in \mathbb{N}} a_n \mu_n^{1/2} \frac{1}{z - t_n} : \{a_n\}_{n \in \mathbb{N}} \in \ell^2 \right\},
\]
\[
\|f\|_{\mathcal{HC}(\mu)} = \|a_n\|_{\ell^2}.
\]

We study an important class of de Branges spaces corresponding to the so called canonical systems on finite interval. Namely, given a \( 2 \times 2 \) real summable a.e. positively semi-defined matrix function \( H \) (Hamiltonian) on a finite interval \([0, L] \), we consider the system
\[
JY'(t) = zH(t)Y(t), \quad t \in [0, L], \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
where \( z \in \mathbb{C} \) is the so called spectral parameter, \( Y \) is an absolutely continuous column vector function such that \( Y(0)^T = (0, 1) \) and \( Y(L)^T = (A, B) \). The entire functions \( A \) and \( B \) are real on the real line with simple real interlacing zeros. We define a measure \( \mu \) supported on \( \mathbb{Z}_A \), where \( \mathbb{Z}_F \) is the zero set of an entire function \( F \), with masses \( B(t)/A'(t), t \in \mathbb{Z}_A \), and associate to the canonical system with Hamiltonian \( H \) the de Branges space \( \mathcal{HC}(A, \mu) \).

It is known that a de Branges space \( \mathcal{HC}(A, \mu) \) corresponds to a canonical system on a finite interval if and only if it is regular, that is,
\[
z \mapsto \frac{f(z) - f(w)}{z - w} \in \mathcal{HC}(A, \mu),
\]
whenever \( f \in \mathcal{HC}(A, \mu), w \in \mathbb{C} \). This is equivalent to the condition that the functions in our space are in the Cartwright class and
\[
\sum_{n \in \mathbb{N}} \frac{1}{\mu_n(t_n^2 + 1)A'(t_n)^2} < \infty,
\]
and from now on we assume that this condition is satisfied.

By the de Branges theory, the de Branges subspaces of the space \( \mathcal{HC}(\mu) \) constitute a chain \( \{\mathcal{HC}(\mu_s)\} \) ordered by inclusion. Every such
\( \mathcal{HC}(\mu_s) \) corresponds to a point \( s \) or to a subinterval \((a, s]\) of \((0, L]\) and to the restriction of our Hamiltonian \( H \) to \([0, s]\). We set \((A_s, B_s) = Y(s)^T\) and we associate to the space \( \mathcal{HC}(\mu_s) \) the corresponding entire function \( A_s \). We have \( \mathcal{Z}_{A_s} = \text{supp} \mu_s \). Given the de Branges space \( \mathcal{HC}(\mu) \) we denote the corresponding chain of the de Branges subspaces by \( \text{Chain}(\mu) \). We are interested in the so called \( H \)-indivisible intervals in this chain, that is the semi open maximal intervals \( I = (a, s] \subset [0, L] \) such that \( H \) is a degenerate constant matrix on \( I \) and, hence, the subspaces \( \mathcal{HC}(\mu_s) \) coincide for \( s \in I \).

If the spectrum of \( \mu \) is \( \mathbb{Z} \), then one can easily verify (see Section 2 below) that the chain does not contain indivisible end intervals \((0, a]\) and \((a, L]\). It is of interest to study the indivisible intervals inside the chain.

**Theorem 1.** If \( \text{supp}(\mu) = \mathbb{Z} \), and the space \( \mathcal{HC}(\mu) \) is regular, then the corresponding chain \( \text{Chain}(\mu) \) can contain one indivisible interval and cannot contain two contiguous indivisible intervals.

When the support of the measure \( \mu \) is \( \mathbb{Z} \), such chains may contain infinitely many non-contiguous indivisible intervals, see Theorem 2 below. On the other hand, these indivisible intervals are somewhat separated, see Theorem 6.

Furthermore, if the support of the measure \( \mu \) is similar enough to \( \mathbb{Z} \), then we get results analogous to Theorem 1, see Section 3.

1.1. **Exponential type.** Another important characteristic of a de Branges space \( \mathcal{HC}(\mu) \) is the exponential type \( \text{Type}(\mathcal{HC}(\mu)) \), that is the exponential type \( t(A) \) of the function \( A \).

**Theorem 2.** Let \( \Sigma \) be a countable subset of the interval \((0, \pi)\). There exists a measure \( \mu \) supported on \( \mathbb{Z} \) such that the space \( \mathcal{HC}(\mu) \) is regular, and the corresponding chain \( \text{Chain}(\mu) \) contains indivisible intervals \( J_s \) with \( \text{Type}(\mathcal{HC}(\mu_t)) = s \), for all \( t \in J_s, s \in \Sigma \).

One can express the exponential type of a de Branges space \( \mathcal{HC}(\mu) \) in terms of the Hamiltonian \( H \) of the corresponding canonical system. Namely, the Krein–de Branges formula (see e.g. [15, Theorem 11])
states that

\[
\text{Type}(\mathcal{HC}(\mu)) = \int_0^L \sqrt{\det H(t)} \, dt.
\]

Let us recall here some key facts from the de Branges theory which are necessary to formulate our results.

**Theorem 3** ([15, Theorem 17]). Given a positive measure \(\nu\) on \(\mathbb{R}\) such that

\[
\int_{\mathbb{R}} \frac{d\nu(t)}{1 + t^2} < \infty,
\]

there exists a chain of regular de Branges spaces \(\mathcal{H}_{t,\nu}\), \(t \in (0, \infty)\) or \(t \in (0, L]\) such that \(\mathcal{H}_{t,\nu}\) is isometrically embedded in \(L^2(\nu)\), the set \(\bigcup_t \mathcal{H}_{t,\nu}\) is dense in \(L^2(\nu)\) and

\[
\mathcal{H}_{t_1,\nu} \subset \mathcal{H}_{t_2,\nu}, \quad t_1 \leq t_2.
\]

Every regular de Branges space isometrically embedded in \(L^2(\nu)\) belongs to this chain. Furthermore, if \(L < \infty\), then \(T = \text{supp} \nu\) is discrete, and for an entire function \(A\) real on the real line with simple zeros at \(T\), we have \(\mathcal{H}_{L,\nu} = \mathcal{HC}(A, \nu^*)\), where \(\nu^*(\{t\}) = 1/(\nu(\{t\})(A'(t))^2)\), \(t \in \text{supp} T\). Furthermore, \(\mathcal{H}_{L,\nu}\) restricted to \(T\) is equal to \(L^2(\nu)\).

For example if \(\nu\) is the Lebesgue measure, then the corresponding de Branges chain consists of the Paley–Wiener spaces \(\mathcal{PW}_a\), \(a \in (0, \infty)\).

The number \(T = \sup_t \text{Type}(\mathcal{H}_{t,\nu})\) is called the exponential type of the measure \(\nu\). One of the fundamental question of harmonic analysis is to determine \(T\) via \(\nu\), see [9], [13] and the references therein. We are interested in the closely related question about the regularity of growth of the exponential type.

**Definition 4.** Given a positive measure \(\nu\) on the real line satisfying (1.3), we say that \(\nu\) generates a thin chain if for any type \(t > 0\) there exists at most one element \(\mathcal{H}_{t,\nu}\) of the chain such that \(\text{Type}(\mathcal{H}_{t,\nu}) = t\).

In particular, Hamiltonians corresponding to thin chains satisfy the condition \(\det H \neq 0\) a.e. on any interval and vice versa.
Given a measure $\nu$ satisfying (1.3), we define the Hilbert transform of $\nu$ by the formula

$$\tilde{\nu}(z) = \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{z - t} + \frac{t}{t^2 + 1} \right) d\nu(t).$$

The function $\tilde{\nu}(z)$ is well defined for $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, it is well-known that for absolutely continuous measures $\nu$, $d\nu = w \, dt$, the Hilbert transform $\tilde{\nu}$ exists on $\mathbb{R}$ if we understand the right-hand side of (1.4) in the principal value sense. The Hilbert transform naturally appears in many problems of harmonic analysis.

If the weight $w$ has convergent logarithmic integral, then it is well-known that the type of $w(x) \, dx$ is infinite (see [9], [13]). We are able to show that under some additional regularity assumptions, the measure $w(x) \, dx$ generates a thin chain.

**Theorem 5.** Let $w$ be a $C^1$ smooth positive function such that $w \in L^1 \left( \frac{dx}{1 + x^2} \right)$, $\int_{\mathbb{R}} \frac{\log w(x)}{1 + x^2} \, dx > -\infty$, $(\log w)' \in L^\infty(\mathbb{R})$. Then the measure $w(x) \, dx$ generates a thin chain.

Sometimes the chain generated by a measure is not thin, but “almost thin”, that is, for every $t > 0$ the chain contains at most two subspaces of type $t$. In particular, this is the case if the support of the measure is $\mathbb{Z}$, as shows the following result.

**Theorem 6.** Let $\nu$ be a positive measure satisfying (1.3) such that $\text{supp } \nu = \mathbb{Z}$ and the corresponding chain of regular de Branges spaces $\mathcal{H}_{t,\nu}$ is defined on a finite interval $(0, \infty)$. Then for any $0 < t_2 < t_1 \leq L$ such that $\text{Type}(\mathcal{H}_{t_1,\nu}) = \text{Type}(\mathcal{H}_{t_2,\nu})$, we have

$$\dim(\mathcal{H}_{t_1,\nu} \ominus \mathcal{H}_{t_2,\nu}) \leq 1.$$
1.2. Notation and organization of the paper. In this text, $A \lesssim B$ means that $A \leq CB$ with a positive constant $C$, $A \gtrsim B$ means $A \geq cB$ with a constant $c > 0$, and $A \simeq B$ means that $A \lesssim B$ and $A \gtrsim B$ simultaneously.

Some function theoretic criteria for the existence of (contiguous) indivisible intervals in a chain of the Branges spaces are given in Section 2. In Section 3 we consider the indivisible intervals in the chains associated with the de Branges spaces represented as the Cauchy transforms with spectrum in $\mathbb{Z}$ or some perturbations of $\mathbb{Z}$. In Section 4 we deal with the de Branges subspaces of the same type in a chain. Section 5 describes some properties of different isometric Cauchy transform representations for the de Branges spaces.

2. INDIVISIBLE INTERVALS

In this section we deal with regular de Branges spaces. We start with some equivalent conditions for the existence of an indivisible interval in a de Branges chain.

Lemma 7. Given a de Branges space $\mathcal{HC}(\mu)$, the following assertions are equivalent:

(i) The chain $\text{Chain}(\mu)$ contains an indivisible interval.

(ii) For some subspaces in the chain, we have $\dim(\mathcal{HC}(\mu_s) \ominus \mathcal{HC}(\mu_a)) = 1$.

(iii) For some subspace $\mathcal{HC}(A, \nu) = \mathcal{HC}(\mu_s)$ in the chain, we have $A \sum_{n \in \mathbb{N}} \nu_n / (\cdot - t_n) \in \mathcal{HC}(A, \nu)$, where $\nu = \sum_{n \in \mathbb{N}} \nu_n \delta_{t_n}$.

(iv) There exists a function $G$ in $\mathcal{HC}(\mu)$ real on the real line with simple real zeros such that $G$ is orthogonal to $G/(\cdot - \lambda)$, $\lambda \in \mathbb{Z}_G$.

(v) For some subspace $\mathcal{HC}(\mu_s)$ in the chain, the measure $\mu_s$ is finite.

(vi) For some subspace $\mathcal{HC}(\mu_s)$ in the chain, the domain of the operator of multiplication by $z$ is not dense in $\mathcal{HC}(\mu_s)$.

Under the conditions of (iv), the de Branges space $\mathcal{H}$ spanned by $G$ and $G/(\cdot - \lambda)$, $\lambda \in \mathbb{Z}_G$, has exponential type equal to that of $G$.

Proof. For the implication (i) $\implies$ (ii) see [15, Section 4.3]. The implication (ii) $\implies$ (i) is evident, see [10, Problem 86]. By [10, Theorem 29], we obtain the implication (ii) $\implies$ (iii). Next, taking
\[ G = A \sum_{n \in \mathbb{N}} \nu_n / (\cdot - t_n), \] and using that \( G \perp G/(\cdot - \lambda), \lambda \in \mathcal{Z}_G \), we obtain the implication (iii) \( \implies \) (iv). The implication (iv) \( \implies \) (ii) is evident, because the closed space spanned by \( G/(\cdot - \lambda), \lambda \in \mathcal{Z}_G \), is a de Branges subspace of \( \mathcal{HC}(\mu) \). The equivalence (iii) \( \iff \) (v) follows because (iii) means that the sequence \( \{\nu_n^{1/2}\}_{n \in \mathbb{N}} \) is in \( \ell^2 \). The equivalence (iii) \( \iff \) (vi) follows from [10, Theorem 29].

The chain \( \text{Chain}(\mu) \) starts with an indivisible interval \((0, a] \) (or several contiguous indivisible intervals \((0, a_1], (a_1, a_2], \ldots, (a_{k-1}, a_k]\)) if and only if \( 1 \in \mathcal{HC}(\mu) \) or, correspondingly, \( 1, \ldots, z^{k-1} \in \mathcal{HC}(\mu) \) if and only if

\[
\sum_{n \in \mathbb{N}} \frac{1}{\mu_n A'(t_n)^2} < \infty
\]

or, correspondingly,

\[
\sum_{n \in \mathbb{N}} \frac{t_n^{2(k-1)}}{\mu_n A'(t_n)^2} < \infty.
\]

The chain ends with \( k \) contiguous indivisible intervals \((a_1, a_2], (a_2, a_3], \ldots, (a_k, L]\) if and only if

\[
\sum_{n \in \mathbb{N}} \mu_n t_n^{2(k-1)} < \infty.
\]

Furthermore, the chain \( \text{Chain}(\mu) \) contains \( k \) contiguous indivisible intervals if and only if we can find an entire function \( G \) real on the real line with simple real zeros such that \( G \) is orthogonal to \( G/(\cdot - \lambda), \lambda \in \mathcal{Z}_G \), and \( z^{k-1}G \in \mathcal{HC}(\mu) \).

In this article, we are mainly interested in indivisible intervals inside the chain.

The reproducing kernel \( K_{t_n} \) of \( \mathcal{HC}(\mu) \) at \( t_n \in T \),

\[
\langle F, K_{t_n} \rangle = F(t_n), \quad F \in \mathcal{HC}(\mu),
\]

is given by

\[
K_{t_n}(z) = \mu_n A'(t_n) \frac{A(z)}{z - t_n}, \quad n \in \mathbb{N}.
\]

Therefore,

\[
\|K_{t_n}\| = \mu_n^{1/2} |A'(t_n)|, \quad n \in \mathbb{N},
\]
and
\[ \frac{K_{t_n}(z)}{\|K_{t_n}\|} = \frac{\|K_{t_n}\|}{A'(t_n)} z - t_n, \quad n \in \mathbb{N}. \]

**Lemma 8.** If the chain \( \text{Chain}(\mu) \) contains an indivisible interval, and \( A \) is associated to \( \mathcal{H}_C(\mu) \), then there exist entire functions \( S \) and \( G \) real on the real line such that \( G \in \mathcal{H}_C(\mu) \), \( S \in \ell^2(\mu) \), and
\[
\frac{GS}{A} = \sum_{n \in \mathbb{N}} \frac{a_n^2}{\cdot - t_n},
\]
where \( a_n = \mu_n^{-1/2} G(t_n)/A'(t_n) \).

In the opposite direction, if there exist two entire functions \( S \) and \( G \) real on the real line such that \( \lim_{|y| \to \infty} y^{k-1} G(iy)/A(iy) = 0 \) and
\[
\begin{align*}
&(2.1) \quad S \in \ell^2(\mu), \\
&G \frac{A}{A'} \in \ell^2(1/\mu), \\
&(2.3) \quad \frac{GS}{A} = \sum_{n \in \mathbb{N}} \frac{c_n}{\cdot - t_n},
\end{align*}
\]
where \( \{c_n\}_{n \in \mathbb{N}} \in \ell^1 \), \( \sum_{n \in \mathbb{N}} c_n \neq 0 \), then the chain \( \text{Chain}(\mu) \) contains an indivisible interval corresponding to a subspace of exponential type coinciding with that of \( G \).

**Proof.** Let \( G \in \mathcal{H}_C(A, \mu) \) be an entire function real on the real line with simple real zeros, orthogonal to \( G/(\cdot - \lambda) \), \( \lambda \in \mathbb{Z}_G \). Then we have
\[
G = A \sum_{n \in \mathbb{N}} a_n \frac{\mu_n^{1/2}}{\cdot - t_n} = \sum_{n \in \mathbb{N}} a_n \frac{K_{t_n}(z)}{\|K_{t_n}\|} \cdot \frac{A'(t_n)}{|A'(t_n)|}
\]
with real coefficients \( a_n \). Then
\[
G \perp \frac{G}{\lambda - \cdot} \quad \lambda \in \mathbb{Z}_G \quad \iff \\
\left\langle \frac{G}{\lambda - \cdot}, \sum_{n \in \mathbb{N}} a_n \frac{K_{t_n}(z)}{\|K_{t_n}\|} \cdot \frac{A'(t_n)}{|A'(t_n)|} \right\rangle = 0, \quad \lambda \in \mathbb{Z}_G \quad \iff \\
(2.4) \quad \sum_{n \in \mathbb{N}} \frac{a_n^2}{\lambda - t_n} = 0, \quad \lambda \in \mathbb{Z}_G,
\]
because
\[
(2.5) \quad G(t_n) = a_n A'(t_n) \mu_n^{1/2}.
\]
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Next, (2.4) is equivalent to the existence of an entire function $S$ real on the real line such that

$$\sum_{n \in \mathcal{N}} \frac{a_n^2}{-t_n} = \frac{GS}{A}.$$  

Comparing the residues on $T$, we obtain that

$$a_n^2 = \frac{G(t_n)S(t_n)}{A'(t_n)}, \quad n \in \mathcal{N},$$

and hence,

$$S(t_n) = a_n\mu_n^{-1/2}, \quad n \in \mathcal{N}.$$  

Finally, (2.5)–(2.7) yield (2.1)–(2.3) with $c_n \geq 0, 0 < \sum_{n \in \mathcal{N}} c_n < \infty$.

In the opposite direction, suppose that we can find two entire function $S$ and $G$ real on the real line such that $\lim_{|y| \to \infty} G(iy)/A(iy) = 0$, and relations (2.1)–(2.3) hold with $\{c_n\}_{n \in \mathcal{N}} \in \ell^1, \sum_{n \in \mathcal{N}} c_n \neq 0$.

Set $b_n = S(t_n)\mu_n^{-1/2}, n \in \mathcal{N}$, and consider

$$H = \sum_{n \in \mathcal{N}} b_n \frac{K_{t_n}}{\|K_{t_n}\|} \cdot \frac{A'(t_n)}{|A'(t_n)|} \in \mathcal{HC}(A, \mu).$$

Since $G/A' \in \ell^2(1/\mu)$ and $\lim_{|y| \to \infty} G(iy)/A(iy) = 0$, a result from the de Branges theory [10, Theorem 26] yields that $G \in \mathcal{HC}(A, \mu)$.

Set

$$a_n = \frac{G(t_n)}{A'(t_n)\mu_n^{1/2}}, \quad n \in \mathcal{N}.$$

Then $a_n b_n = c_n, n \in \mathcal{N}$,

$$\langle G, H \rangle = \sum_{n \in \mathcal{N}} a_n b_n = \sum_{n \in \mathcal{N}} c_n \neq 0,$$

and

$$\left\langle \frac{G}{\lambda - \cdot}, H \right\rangle = \sum_{n \in \mathcal{N}} \frac{G(t_n)b_n}{(\lambda - t_n)\|K_{t_n}\|} \frac{A'(t_n)}{|A'(t_n)|}$$

$$= \sum_{n \in \mathcal{N}} \frac{a_n b_n}{\lambda - t_n} = \frac{GS}{A} = 0, \quad \lambda \in \mathbb{Z}_G.$$

Thus $G \notin \text{Span}\{G/(\cdot - \lambda)\}_{\lambda \in \mathbb{Z}_G}$, and, by Lemma 7, we get an indivisible interval. \qed
If $S$ and $G$ in the formulation of Lemma 8 are not polynomials, then the indivisible interval we obtain is inside the chain.

**Lemma 9.** If the chain $\text{Chain}(\mu)$ contains $k$ contiguous indivisible intervals, and $A$ is associated to $\mathcal{HC}(\mu)$, then there exist entire functions $S$ and $G$ real on the real line such that $z^{k-1}G \in \mathcal{HC}(A, \mu)$,

\[ S \in \ell^2(\mu), \]

\[ \frac{GS}{A} = \sum_{n \in \mathbb{N}} \frac{a_n^2}{-t_n}, \]

where $a_n = \mu_n^{-1/2}G(t_n)/A'(t_n)$.

In the opposite direction, if there exist two entire functions $S$ and $G$ real on the real line such that $\lim_{|y| \to \infty} y^{k-1}G(iy)/A(iy) = 0$ and

\[ S \in \ell^2(\mu), \]

\[ \frac{z^{k-1}G}{A'} \in \ell^2(1/\mu), \]

\[ \frac{GS}{A} = \sum_{n \in \mathbb{N}} \frac{c_n}{-t_n}, \]

where $\{c_n\}_{n \in \mathbb{N}} \in \ell^1$, $\sum_{n \in \mathbb{N}} c_n \neq 0$, then the chain $\text{Chain}(\mu)$ contains $k$ contiguous indivisible intervals.

The proof is analogous to that of Lemma 8.

Again, if $S$ and $G$ are not polynomials, then the contiguous indivisible intervals we obtain are inside the chain.

3. The spectrum $\mathbb{Z}$ and its perturbations

Here, we start with the case when the spectrum $T$ of the de Branges space is $\mathbb{Z}$, and, correspondingly, $A(z) = A_0(z) = \sin(\pi z)$.

3.1. Indivisible interval inside the chain; proof of Theorem 1

**Proof of Theorem 1** By Lemma 8 we know that the existence of $k$ contiguous intervals is equivalent to the existence of two non-zero entire functions $S$ and $G$ real on the real line and such that $z^{k-1}G \in \mathcal{HC}(A, \mu)$. 

...
\( \mathcal{HC}(A_0, \mu) \),

\[
\begin{cases}
S \in \ell^2(\mu), \\
z^{k-1}G \in \ell^2(1/\mu), \\
\frac{GS}{A_0} = \sum_{n \in \mathbb{Z}} \frac{c_n}{n - n} .
\end{cases}
\]

\( \{c_n\}_{n \in \mathbb{Z}} \in \ell^1 \). Additionally, we could impose the restriction \( c_n \geq 0, n \in \mathbb{Z} \).

Now for \( k = 1 \), choose entire functions \( G \) and \( S \) real on the real line such that \( G(z)S(z) = z^{-1}A_0(z), |G(x)| \asymp \text{dist}(x, \mathbb{Z}_G)(1 + |x|)^{-1/2}, |S(x)| \asymp \text{dist}(x, \mathbb{Z}_S)(1 + |x|)^{-1/2} \). (For example, we can take \( G(z) = \prod_{n \geq 1}(1 - z^2n^{-1})(1 + z^2n^{-1}) \)). Then \( \lim_{|y| \to \infty} G(iy)/A_0(iy) = 0 \).

Set

\[
\mu_n = \begin{cases}
|n|^{-1/2}, & n \in \mathbb{Z}_G, \\
|n|^{1/2}, & n \in \mathbb{Z}_S,
\end{cases}
\]

and \( \mu_0 = 1 \). The measure \( \mu = \sum_{n \in \mathbb{Z}} \mu_n \delta_n \) satisfies conditions (1.1) and (1.2). Furthermore, conditions (3.1) are satisfied and the space \( \mathcal{HC}(\mu) \) contains an indivisible interval inside the chain.

In the opposite direction, suppose that there are two contiguous indivisible intervals. Conditions (1.1), (1.2) and (3.1) imply that

\[
\sum_{n \in \mathbb{Z}} (S^2(n) + (1 + |n|)^{-2})\mu_n < \infty,
\]

\[
\sum_{n \in \mathbb{Z}} (n^2G^2(n) + (1 + |n|)^{-2})\mu_n^{-1} < \infty.
\]

Therefore,

\[
\sum_{n \in \mathbb{Z}} (|S(n)| + (1 + |n|)^{-1})(n|G(n)| + (1 + |n|)^{-1}) < \infty.
\]

By the Cartwright theorem \([8]\), \( S \) and \( G \) have strictly positive exponential types. Since \( t(S) + t(G) \leq \pi \), these exponential types are smaller than \( \pi \).

Since \( G \in \ell^1(\mathbb{Z}) \), we conclude that \( G \in L^1(\mathbb{R}) \) \([8\text{ Section 10.6}]\). In a similar way, since \( S(n)(1 + |n|)^{-1} \in \ell^1(\mathbb{Z}) \), we have \( S(x)(1 + |x|)^{-1} \in \)
\[ L^1(\mathbb{R}) \text{ [1, Theorem 3a]} \]. Next we use that
\[ GS_{A_0} = \sum_{n \in \mathbb{Z}} a_n^2 \cdot n, \quad c_n \geq 0, \quad n \in \mathbb{Z}, \quad \{c_n\}_{n \in \mathbb{Z}} \in \ell^1. \]

Using a version of Boole’s lemma by Khrushchev–Vinogradov [11], we obtain that
\[ |G(x)S(x)| \asymp |x|^{-1} \text{ on a set } E \subset \mathbb{R} \text{ of infinite logarithmic length.} \]

As a result,
\[ \infty = \left( \int_E \frac{dt}{1 + |t|} \right)^2 \leq \int_E |G(t)| \, dt \int_E \frac{|S(t)| \, dt}{1 + |t|} \]
\[ \leq \int_{\mathbb{R}} |G(t)| \, dt \int_{\mathbb{R}} \frac{|S(t)| \, dt}{1 + |t|} < \infty. \]

This contradiction shows that no de Branges space with spectrum \( \mathbb{Z} \) possesses two contiguous indivisible intervals. \( \square \)

Next, we consider some situations where the spectrum of our de Branges space is a perturbation of \( \mathbb{Z} \).

**Proposition 10.** Suppose that for some \( \gamma \in \mathbb{R} \) and \( T \subset \mathbb{R} \) we have
\[ |A(z)| \asymp \min(1, \text{dist}(z, T))(1 + |z|)^\gamma \exp(\pi|\Im z|), \quad z \in \mathbb{C}. \]

If a de Branges space \( \mathcal{H}C(A, \mu) \) is regular, then it can contain \( k \) contiguous indivisible intervals if and only if \( k < 2 + \gamma \).

In the proof, we use a possibility to factorize such entire functions \( A \) into factors of precise asymptotics. For a similar arguments, see Lemmas [13] and [14] below.

**Proof.** Let \( 1 \leq k < 2 + \gamma \). Choose \( \alpha \in (k - \gamma - 1, 1) \). Then choose \( \beta \in (-1-2\gamma, \min(1, 1-2\alpha-2\gamma)) \), \( \delta \in (\max(-1-2\gamma, 2k-2\alpha-2\gamma-1), 1) \) and define entire functions \( G \) and \( S \) real on the real line such that \( GS = A/(\cdot - \lambda) \) for some \( \lambda \in T \), \( |G(t)| \asymp \text{dist}(t, Z_G)(1 + |t|)^{-\alpha}, |S(t)| \asymp \text{dist}(t, Z_S)(1 + |t|)^{\gamma-1+\alpha}, t \in \mathbb{R} \), and a measure \( \mu = \sum_{t \in T} \mu_t \delta_t \) with \( \mu_t = (1 + |t|)^\beta, t \in Z_G, \mu_t = (1 + |t|)^\delta, t \in Z_S, \mu_\lambda = 1 \). It remains to apply Lemma [9]. In the opposite direction, we argue by analogy with the proof of Theorem [1]. \( \square \)

**Proposition 11.** Given \( \beta > 0 \), set \( T = \cup_{n \in \mathbb{Z}} \{n, n + (2 + |n|)^{-\beta}\} \). If the space \( \mathcal{H}C(T, \mu) \) is regular, then \( \beta < 1 \), and the corresponding de
Branges subspaces chain $\text{Chain}(\mu)$ can contain an indivisible interval and cannot contain two contiguous indivisible intervals.

Proof. Set

$$A(z) = \sin(\pi z) \cdot \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{n + (2 + |n|)^{-\beta}}\right).$$

Then

$$|A'(t)| \asymp (1 + |t|)^{-\beta}, \quad t \in T.$$  

Since the space $\mathcal{HC}(T, \mu)$ is regular, by (1.1) and (1.2) we obtain that

$$\sum_{t \in T} (1 + |t|)^{\beta - 2} < \infty,$$

and, hence, $\beta < 1$.

Next, if $\text{Chain}(\mu)$ contains two contiguous indivisible intervals, then, as in the proof of Theorem 1, relations (1.1), (1.2), and (3.1) imply that

$$\sum_{t \in T} (S^2(t) + (1 + |t|)^{-2})\mu_t < \infty,$$

and we conclude as in the proof of Theorem 1.

If now $\beta \in (0, 1)$, let us verify that $\text{Chain}(\mu)$ can contain an indivisible interval. Given $n \in \mathbb{Z}$, set $n^* = n + (2 + |n|)^{-\beta}$. Choose $\Lambda \subset \mathbb{Z}$ such that $0 \not\in \Lambda$ and

$$S(x) = \prod_{n \in \Lambda} \left(1 - \frac{x}{n}\right) \asymp (1 + |x|)^{-(\beta + 1)/2}, \quad x \in \mathbb{Z} \setminus \Lambda,$$

and set $G(z) = A(z)/(zS(z))$. Denote $\Lambda^* = \{n^* : n \in \Lambda\}$. We have

$$|S(t)| \asymp \begin{cases} 
(1 + |t|)^{-(\beta + 1)/2}, & t \in \Lambda^*, \\
(1 + |t|)^{-(\beta + 1)/2}, & t \in T \setminus (\Lambda \cup \Lambda^*),
\end{cases}$$

$$|G(t)| \asymp (1 + |t|)^{-(\beta + 1)/2}, \quad t \in \Lambda \cup \{0\}.$$  

Now, we set

$$\mu_t = \begin{cases} 
(1 + |t|)^{(\beta + 1)/2}, & t \in \Lambda \cup \{0\}, \\
(1 + |t|)^{(3\beta - 1)/2}, & t \in T \setminus (\Lambda \cup \{0\}),
\end{cases}$$
A direct calculation shows that the measure $\mu = \sum_{t \in T} \mu_t \delta_t$, satisfies conditions (1.1) and (1.2). Furthermore, $\lim_{|y| \to \infty} G(iy)/A(iy) = 0$, $S \in \ell^2(\mu)$, $G/A' \in \ell^2(1/\mu)$, and we conclude by applying Lemma 8. □

3.2. Infinite number of indivisible intervals. Proof of Theorem 2

In the following result we consider lacunary canonical products constructed by rapidly growing zeros $\{z_k\}_{k \geq 1}$, $|z_{k+1}/z_k| \geq q > 1$, $k \geq 1$.

Proposition 12. Let $U$ be a lacunary canonical product $\Lambda \subset \mathbb{R}$, $\text{dist}(\Lambda, \mathbb{Z}) > 0$, $T = \Lambda \cup \mathbb{Z}$, and let $A(z) = \sin(\pi z) U(z)$. Then we can find a measure $\mu$ on $T$ such that the corresponding space $HC(A, T)$ contains a two sided sequence of infinitely many contiguous indivisible intervals.

Proof. Here we just choose entire functions $G$ and $S$ real on the real line such that $zG(z)S(z) = A(z)$, $|G(t)| \asymp \text{dist}(t, \mathbb{Z}_G) \psi(t)(1 + |t|)^{-1/2}$, $|S(t)| \asymp \text{dist}(t, \mathbb{Z}_S) \psi(t)(1 + |t|)^{-1/2}$, where $\psi(t) = 1 + \max_{|z|=t} |U(z)|^{1/2}$, $t \in \mathbb{R}$. Then set $\mu_t = \psi(t)^{-3}$, $t \in \mathbb{Z}_G$, $\mu_t = 1$, $t \in \mathbb{Z}_A \setminus \mathbb{Z}_G$. Then we obtain that $z^k S \in \ell^2(\mu)$, $z^k G/A' \in \ell^2(1/\mu)$, for any $k \geq 0$ and apply a natural analog of Lemma 9. □

Next we need some standard information on the asymptotics of canonical products associated with very regular sequences on the real line.

Given a countable symmetric $\Lambda \subset \mathbb{R} \setminus \{0\}$ of finite linear density, we set

$$C_\Lambda(z) = \prod_{t \in \Lambda} \left(1 - \frac{z}{t} \right) = \prod_{t \in \Lambda_+} \left(1 - \frac{z^2}{t^2} \right),$$

where $\Lambda_+ = \Lambda \cap \mathbb{R}_+$. Denote by $n_\Lambda$ the counting function of $\Lambda$,

$$n_\Lambda(x) = \text{card}(\Lambda \cap (0, x]), \quad x > 0.$$ We say that a symmetric $\Lambda \subset \mathbb{R} \setminus \{0\}$ has strong linear asymptotics $a_\Lambda x + b_\Lambda$ if the function

$$\psi_\Lambda(t) = \int_0^t \left(n_\Lambda(x) - [a_\Lambda x + b_\Lambda] \cdot 1_{[1, \infty)}(x) \right) dx$$
is bounded on \((0, \infty)\). We say that \(\Lambda \subset \mathbb{R}\) is uniformly discrete if
\[\inf\{|t_1 - t_2| : t_1, t_2 \in \Lambda, t_1 \neq t_2\} > 0.\]

**Lemma 13.** Given an infinite symmetric uniformly discrete \(\Lambda \subset \mathbb{R} \setminus \{0\}\) with strong linear asymptotics \(ax + b\), we have
\[|C_\Lambda(z)| \asymp \min(1, \text{dist}(z, \Lambda))(1 + |z|)^{-1 - 2b}e^{a\pi|\mathcal{I}z|}, \quad z \in \mathbb{C}.\]

**Proof.** Since the linear density of \(\lambda\) is \(a\), we need only to verify that
\[|C_\Lambda(x + i)| \asymp (1 + |x|)^{-1 - 2b}, \quad x \in \mathbb{R}.\]
It is easily seen that for \(\Lambda_{a,b} = \{\pm (n - b)/a\}_{n \geq 1}\) (with trivial modifications for small \(n\)) we have
\[n_{\Lambda_{a,b}}(x) = |ax + b| \cdot 1_{[1, \infty)}(x),\]
\[|C_{\Lambda_{a,b}}(x + i)| \asymp (1 + |x|)^{-1 - 2b}, \quad x \in \mathbb{R}.\]
Therefore, we need only to check that the function \(W\),
\[W(x) = \log \left| \frac{C_\Lambda(x + i)}{C_{\Lambda_{a,b}}(x + i)} \right| = \int_0^\infty \log \left| 1 - \left(\frac{x + i}{t}\right)^2 \right| (dn_\Lambda(t) - dn_{\Lambda_{a,b}}(t)),\]
is bounded on the real line. Integrating by parts twice and using that \(n_\Lambda - n_{\Lambda_{a,b}}\) and \(\psi_\Lambda\) are bounded, we obtain that
\[W(x) = \int_0^\infty \psi_\Lambda(t) \cdot \Re \left[ \frac{2}{t^2 - (t - x + i)^2} - \frac{1}{(t - x + i)^2} - \frac{1}{(t + x + i)^2} \right] dt.\]
The function in the right hand side is bounded because \(\psi_\Lambda\) are bounded. \(\square\)

**Lemma 14.** Let \(\Lambda_1 \subset \Lambda_2\) be two symmetric subsets of \(\mathbb{R} \setminus \{0\}\) with strong linear asymptotics, correspondingly, \(a_1x + b_1\) and \(a_2x + b_2\). Given \(a \in (a_1, a_2)\) and \(b \in (b_1, b_2)\), there exists a symmetric subset \(\Lambda\) of \(\mathbb{R} \setminus \{0\}\) with linear asymptotics \(ax + b\) such that \(\Lambda_1 \subset \Lambda \subset \Lambda_2\).

**Proof.** By observation. \(\square\)

**Proof of Theorem** We consider just the case of infinite \(\Sigma\). The other case is much simpler. Let \(\Sigma = \{\pi s_k\}_{k \geq 1}\). By induction in \(k \geq 1\), we construct a disjoint system of intervals \((a_k, b_k) \subset (0, 1)\) such that \(s_k < s_m \implies b_k < a_m, k, m \geq 1\). Set \(r_k = (a_k + b_k)/2, k \geq 1\). Also by induction in \(k \geq 1\), we construct, using Lemma symmetric sets
This completes the proof.

If (3.2), we have corresponding to the exponential type \( s \) verify that \( S \) Now, we set \( \mu \). Then the measure \( \mu \). By construction, if \( s \) \( n \), then \( n \in \Lambda_k \), and if \( s \) \( n \). If \( s(n) = s_m \in \Sigma \) and \( n \in \Lambda_m \), then we set \( u(n) = a_m \), otherwise set \( u(n) = \sup \{ b_k : s_k < s(n) \} \).

Given \( n \in \mathbb{Z} \setminus \{0\} \), we set
\[
s(n) = \sup \{ s_k : k \geq 1, \, n \notin \Lambda_k \}.
\]

By construction, if \( s_k > s(n) \), then \( n \in \Lambda_k \), and if \( s_k < s(n) \), then \( n \notin \Lambda_k \). If \( s(n) = s_m \in \Sigma \) and \( n \in \Lambda_m \), then we set \( u(n) = a_m \), otherwise set \( u(n) = \sup \{ b_k : s_k < s(n) \} \).

Now, we set \( \mu_0 = 1 \),
\[
\mu_n = (1 + |n|) \left( \frac{1}{2} - \frac{1}{n} \right), \quad n \in \mathbb{Z} \setminus \{0\}.
\]

Then the measure \( \mu = \sum_{n \in \mathbb{Z}} \mu_n \delta_n \) satisfies conditions (1.1) and (1.2).

Fix \( k \geq 1 \). To prove the existence of an indivisible interval corresponding to the exponential type \( s_k \), by Lemma [8] we need only to verify that \( S_k \in \ell^2(\mu) \) and \( G_k \in \ell^2(1/\mu) \).

If \( n \in \Lambda_k \), then \( s(n) \leq s_k \) and \( u(n) \leq a_k \). Therefore, by (3.3), we have
\[
\sum_{n \in \Lambda_k} |S_k(n)|^2 \mu_n \leq \sum_{n \in \Lambda_k} (1 + |n|)^{-2k/3} \sum_{n \in \mathbb{Z}} (1 + |n|)^{-1} \sum_{n \in \mathbb{Z}} (1 + |n|)^{-3a_k} < \infty.
\]

If \( n \in \mathbb{Z} \setminus (\Lambda_k \cup \{0\}) \), then \( s(n) \geq s_k \) and \( u(n) \geq b_k \). Therefore, by (3.2), we have
\[
\sum_{n \in \mathbb{Z} \setminus (\Lambda_k \cup \{0\})} |G_k(n)|^2 \mu_n^{-1} \leq \sum_{n \in \mathbb{Z} \setminus (\Lambda_k \cup \{0\})} (1 + |n|)^{-2k/3} \sum_{n \in \mathbb{Z}} (1 + |n|)^{-2k/3} \sum_{n \in \mathbb{Z}} (1 + |n|)^{-3b_k} < \infty.
\]

This completes the proof. \( \square \)
4. The same type subspaces

4.1. Regularity of growth of exponential type. Proof of Theorem 5. The proof of Theorem 5 is based on a combination of an atomization result in [5] and some fact on the completeness of mixed systems in the Paley–Wiener spaces from [2].

Proof of Theorem 5 We start with the following simple fact. If two positive weights are comparable, that is \( w_1(x) \asymp w_2(x) \), then the chains of the de Branges subspaces are the same, i.e. the de Branges subspaces from different chains coincide as sets with equivalent norms. Therefore, it is sufficient to consider any weight comparable to \( w \).

Now we apply Theorem 2.6 from [5] (with sufficiently large \( \sigma > 0 \)) and construct an entire function \( H \) of finite exponential type \( b \) with simple zeros such that

\[
|H(x)|^2 \asymp w(x), \quad x \in \mathbb{R}.
\]

It remains to prove that the measure \( |H(x)|^2 \, dm \) generates a thin chain. Assume the contrary. Then there exist two different de Branges spaces \( \mathcal{H}_1, \mathcal{H}_2 \) from the chain of the same exponential type. Let us fix some non-trivial function \( F_1 \) from \( \mathcal{H}_1 \ominus \mathcal{H}_2 \). Let \( F_2 \) be an \( A \)-function corresponding to \( \mathcal{H}_2 \) such that \( \mathcal{Z}_{F_2} \cap \mathcal{Z}_G = \emptyset \). Set \( a = t(F_1) = t(F_2) \). We have

\[
F_1 \perp \frac{F_2(z)}{z - \lambda}, \quad \lambda \in \mathcal{Z}_{F_2},
\]

where \( \perp \) means orthogonality in \( \mathcal{H}_1 \). We recall that the space \( \mathcal{H}_1 \) is isometrically embedded in \( L^2(|H|^2 \, dm) \). Hence,

\[
\int_{\mathbb{R}} \frac{F_1(x)H(x)F_2(x)\overline{H(x)}}{x - \lambda} \, dx = 0, \quad \lambda \in \mathcal{Z}_{F_2}.
\]

Since \( F_1H, F_2H/(\cdot - \lambda) \in L^2(\mathbb{R}), \lambda \in \mathcal{Z}_{F_2}, \) the functions \( F_1G, F_2H/(\cdot - \lambda), \lambda \in \mathcal{Z}_{F_2}, \) belong to the Paley–Wiener space \( \mathcal{PW}_{a+b} \). Thus, equation (4.1) can be considered as orthogonality of some vectors from \( \mathcal{PW}_{a+b} \).

Denote by \( k_\lambda \) the reproducing kernel in the space \( \mathcal{PW}_{a+b} \) at the point \( \lambda \in \mathbb{C} \). From (4.1) we obtain that the system

\[
\{k\lambda\}_{\lambda \in \mathcal{Z}_{F_1}} \cup \left\{ \frac{F_2H}{\cdot - \lambda} \right\}_{\lambda \in \mathcal{Z}_{F_2}}
\]
is not complete in $\mathcal{PW}_{a+b}$. This contradicts the following lemma:

**Lemma 15.** Let $T = T_1T_2$ be an entire function in the Paley–Wiener space $\mathcal{PW}_\pi$ with the conjugate indicator diagram $[-\pi i, \pi i]$ and with simple zeroes. Then the mixed system

$$\{k\lambda\}_{\lambda \in \mathbb{Z}_{T_1}} \cup \left\{ \frac{T_1T_2}{\cdot - \lambda} \right\}_{\lambda \in \mathbb{Z}_{T_2}}$$

is always complete in $\mathcal{PW}_\pi$.

This lemma follows immediately from [2, Proposition 2.1]. For other versions of this result see [3, 4]. □

4.2. **Spectrum $\mathbb{Z}$. Proof of Theorem 6.** The proof of Theorem 6 is based on a combination of Theorem 1, some results on the classical Pólya problem, and the second Beurling–Malliavin theorem.

**Proof of Theorem 6** First of all, since $\mathcal{H} = \mathcal{H}_{L,\nu} = \mathcal{H}(\sin(\pi z), \mu)$ is regular and the support of $\nu$ and $\mu$ is $\mathbb{Z}$, by the Cartwright theorem [8], the space $\mathcal{H}$ contains no entire functions of zero exponential type except 0.

Set $\mathcal{H}_1 = \mathcal{H}_{t_1,\nu}$, $\mathcal{H}_2 = \mathcal{H}_{t_2,\nu}$. Using Theorem 1 we obtain that if $\dim(\mathcal{H}_1 \ominus \mathcal{H}_2) < \infty$, then $\dim(\mathcal{H}_1 \ominus \mathcal{H}_2) \leq 1$. Thus, it remains to consider the case $\dim(\mathcal{H}_1 \ominus \mathcal{H}_2) = \infty$, $\text{Type}(\mathcal{H}_1) = \text{Type}(\mathcal{H}_2)$.

Choose a function $F \in \mathcal{H}_1 \setminus \{0\}$ such that $F \perp \mathcal{H}_2$. Let $A_2$ be an $A$-function corresponding to the space $\mathcal{H}_2$ such that $\mathcal{Z}_{A_2} \cap \mathbb{Z} = \emptyset$. We have

$$F \perp \frac{A_2}{\cdot - s_n}, \quad s_n \in \mathcal{Z}_{A_2}. \tag{4.2}$$

Now, relation (4.2) is equivalent to the interpolation formula

$$\sum_{n \in \mathbb{Z}} \frac{F(n)A_2(n)}{\mu_n(z - n)} = \frac{A_2(z)S(z)}{\sin(\pi z)}, \tag{4.3}$$

for some entire function $S$. Since there exists infinitely many linear independent functions $F$ satisfying (4.3) we can assume that the functions $F$ and $S$ have at least 100 common non-integer zeroes $\lambda_1, \ldots, \lambda_{100}$. 


Set \( P(z) = \prod_{k=1}^{100} (z - \lambda_k) \). From (4.3) we conclude that
\[
\sum_{n \in \mathbb{Z}} \frac{F(n)A_2(n)}{(n - \lambda_1) \mu_n (z - n)} = \frac{A_2(z)S(z)}{(z - \lambda_1) \sin(\pi z)},
\]
and then, by induction,
\[
\sum_{n \in \mathbb{Z}} \frac{F(n)A_2(n)}{P(n) \mu_n (z - n)} = \frac{A_2(z)S(z)}{P(z) \sin(\pi z)}.
\]
Hence,
\[(4.4) \quad \frac{F}{P} \perp A_2 \cdot \cdot \cdot s_n, \quad s_n \in \mathcal{Z}_{A_2}.
\]
Thus, we can assume that our function \( F \) satisfies the inequality \(|F(x)| \leq |x|^{-100}\), and is real on the real line. Using Lemmas 17,18 we find such a representation \( \mathcal{HC}(T, \gamma) \) of our space that the zeroes of \( F \) on the real line are away from the support \((x_n)_{n \in \mathbb{Z}}\) of \( \gamma = \sum_{n \in \mathbb{Z}} \gamma_n \delta_{x_n} \):
\[(4.5) \quad \text{the set } \mathcal{Z}_F \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} [x_n - |n|^{-10}, x_n + |n|^{-10}] \text{ is bounded.}
\]
By (4.4), using this representation \( \mathcal{HC}(T, \nu) \), we obtain that
\[
\sum_{n \in \mathbb{Z}} \frac{F(x_n)A_2(x_n)}{\gamma_n (\cdot - x_n)} = \frac{A_2U}{T}
\]
for some entire function \( U \). Moreover, \( t(A_2) + t(U) \leq t(T) = 1 \) and \( 0 < t(F) \leq t(A_2) \). By comparing residues we obtain
\[(4.6) \quad |U(x_n)| \lesssim (1 + |x_n|)^{-10}, \quad n \in \mathbb{Z}.
\]
Hence,
\[
\frac{UF}{T} = \sum_{n \in \mathbb{Z}} \frac{U(x_n)F(x_n)}{T'(x_n)(\cdot - x_n)} + R = \sum_{n \in \mathbb{Z}} \frac{F^2(n)}{\gamma_n (\cdot - x_n)} + R,
\]
for some entire function \( R \) of zero exponential type which is real on the real line.

**Case 1.** \( R \) is a polynomial. Then the zeroes of the product \( UF \) are sufficiently close to the support of \( \gamma \), which contradicts to (4.5).
Case 2. $R$ is a transcendental entire function of zero exponential type. The product $UF$ has at least one zero on every interval $(n, n+1)$. By (4.5) and (4.6), $R$ is bounded on $\Sigma = Z_F \cap \mathbb{R}$.

As in [3], we use now some information on the classical Pólya problem and the second Beurling–Malliavin theorem.

A sequence $X = \{x_n\} \subset \mathbb{R}$ is a Pólya sequence if any entire function of zero exponential type which is bounded on $X$ is a constant. We say that a disjoint sequence of intervals $\{I_n\}$ on the real line is a long sequence of intervals if

$$\sum_n \frac{|I_n|^2}{1 + \text{dist}^2(0, I_n)} = +\infty.$$ 

Since $\Sigma$ is not a Pólya sequence and is a union of two separated sequences, a theorem by Mitkovski–Poltoratski [12] (see also the discussion in [3]) gives that there exists a long sequence of intervals $\{I_n\}$ such that

$$\frac{\text{card}(\Sigma \cap I_n)}{|I_n|} \rightarrow 0.$$

Therefore, if $\Sigma' = Z_U \cap \mathbb{R}$, then

$$\frac{\text{card}(\Sigma' \cap I_n)}{|I_n|} \rightarrow 1.$$

By the second Beurling–Malliavin theorem [7], we obtain that $t(U) \geq 1$, and, hence, $t(F) = 0$. This contradiction completes the proof. □

5. Isometric Cauchy transform representations for de Branges spaces

We start with two standard results. For reader’s convenience we formulate them here and give the proofs.

Lemma 16. Given a de Branges space $HC(A, \mu)$, its reproducing kernel is

$$K_w(z) = A(z)A(w) \sum_{n \in \mathbb{N}} \frac{\mu_n}{(z - t_n)(w - t_n)}.$$

If $w_1, w_2 \not\in \text{supp} \mu$, then

$$K_{w_1}(w_2) = A(w_2)\overline{A(w_1)} \frac{\psi(w_2) - \psi(w_1)}{\overline{w_1} - w_2},$$

$$t(U) \geq 1,$$
where
\[
\psi(z) = \sum_{n \in \mathbb{N}} \mu_n \left( \frac{1}{z - t_n} + \frac{t_n}{t_n^2 + 1} \right).
\]

**Proof.** By observation. \(\square\)

**Lemma 17.** Given a regular de Branges space \(\mathcal{H} = \mathcal{HC}(A_0, \mu)\), \(A_0(z) = \sin(\pi z)\), and \(u \in \mathbb{R}\), set
\[
T = A_0 \cdot (\psi - u).
\]
Then \(T\) is an entire function of exponential type real on the real line, with the conjugate indicator diagram \([-\pi i, \pi i]\). For every \(n \in \mathbb{Z}\), \(T\) has exactly one simple zero \(x_n\) on \((n, n+1)\), \(\psi(x_n) = u\), and \(Z_T = \{x_n\}_{n \in \mathbb{Z}}\).

Next, \(T \notin \mathcal{H}\), \(\{K_{x_n}\}_{n \in \mathbb{Z}}\) is an orthogonal basis in \(\mathcal{H}\), and
\[
\mathcal{H} = \mathcal{HC}(T, \nu),
\]
where
\[
\nu = \sum_{n \in \mathbb{Z}} \|T/(\cdot - x_n)\|^2 \delta_{x_n}.
\]

**Proof.** Since the zeros of \(T\) and \(A_0\) are interlacing, \(T\) is of exponential type with the conjugate indicator diagram \([-\pi i, \pi i]\), and
\[
|A_0(iy)| = O(|yT(iy)|), \quad |y| \to \infty.
\]

If \(T = A_0 \cdot (\psi - u) \in \mathcal{H}\), then
\[
A_0(z)(\psi(z) - u) = A_0(z) \sum_{n \in \mathbb{Z}} \frac{a_n \mu_n^{1/2}}{z - n}
\]
for some sequence \(\{a_n\}_{n \in \mathbb{Z}} \in \ell^2\). Comparing the values at the integer points we obtain that \(a_n = \mu_n^{1/2}, n \in \mathbb{Z}\), and, hence, \(\sum_{n \in \mathbb{Z}} \mu_n < \infty\), which contradicts to \((1.2)\).

By formula \((5.1)\), \(\{K_{x_n}\}_{n \in \mathbb{Z}}\) is an orthogonal system in \(\mathcal{H}\). If \(F \in \mathcal{H} \setminus \{0\}\) is orthogonal to \(\{K_{x_n}\}_{n \in \mathbb{Z}}\), then \(F = TS\) for some entire function \(S\). By \((5.2)\), we obtain that \(|S(iy)| = O(|y|), |y| \to \infty\). Furthermore,
\[
S = \frac{F}{A_0} \cdot \frac{A_0}{T}.
\]
is of zero exponential type. Therefore, $S$ is a polynomial of order at most 1. Dividing $F$ by $S$ we obtain that $T \in \mathcal{H}$, which is impossible. Thus, $\{K_{x_n}\}_{n \in \mathbb{Z}}$ is an orthogonal basis in $\mathcal{H}$.

Denote
\[
\nu_n = \|T/(\cdot - x_n)\|_\mathcal{H}^{-2}.
\]
Since $\{T/(\cdot - x_n)\}_{n \in \mathbb{Z}}$ is a biorthogonal system to $\{K_{x_n}\}_{n \in \mathbb{Z}}$, it is an orthogonal basis in $\mathcal{H}$. Hence, for every $f \in \mathcal{H}$ we have
\[
f(z) = T(z) \sum_{n \in \mathbb{Z}} \frac{1}{z - x_n} \langle f, T/(\cdot - x_n) \rangle = T(z) \sum_{n \in \mathbb{Z}} \frac{a_n \nu_n^{1/2}}{z - x_n},
\]
where $a_n = \langle f, T/(\cdot - x_n) \rangle \nu_n^{1/2}$, $n \in \mathbb{Z}$, and $\{a_n\}_{n \in \mathbb{Z}} \in \ell^2$. Therefore,
\[
\mathcal{H} \subset \mathcal{HC}(T, \nu),
\]
where $\nu = \sum_{n \in \mathbb{Z}} \nu_n \delta_{x_n}$, and the inclusion is isometric. Finally, again since $\{T/(\cdot - x_n)\}_{n \in \mathbb{Z}}$ is an orthogonal basis in $\mathcal{H}$, we have $\mathcal{H} = \mathcal{HC}(T, \nu)$. \qed

Next we show that for every subset $\Lambda$ of $\mathbb{R}$ of finite upper linear density, we can find an isometric representation of our space with respect to a measure somewhat separated from $\Lambda$.

**Lemma 18.** In the conditions of Lemma 17, given a sequence of points $\{y_k\}_{k \geq 1}$ of finite upper linear density, we can find $u \in \mathbb{R}$ such that the intersection
\[
\{y_k\}_{k \geq 1} \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} \left[ x_n - |n|^{-10}, x_n + |n|^{-10} \right]
\]
is bounded.

**Proof.** Set $h = \arctan \psi$. We have
\[
|h'(t)| = \frac{\sum_{n \in \mathbb{Z}} \frac{\mu_n}{(t-n)^2}}{\left( \sum_{n \in \mathbb{Z}} \mu_n \left( \frac{1}{t-n} + \frac{n}{n^2+1} \right) \right)^2 + 1}.
\]
Let $t \in (m, m + 1)$. Without loss of regularity we can assume that $m \geq 1$, $s = t - m \leq 1/2$. Since
\[
\sum_{n \in \mathbb{Z}} \frac{\mu_n}{n^2 + 1} < \infty,
\]

we have
\[
\sum_{n \in \mathbb{Z}} \frac{\mu_n}{(t-n)^2} \lesssim m^2 + \mu_ms^{-2},
\]
\[
\sum_{n \in \mathbb{Z}} \mu_n \left( \frac{1}{t-n} + \frac{n}{n^2+1} \right) \geq \mu_ms^{-1} - O(m^2).
\]
Hence, since
\[
\sum_{n \in \mathbb{Z}} \frac{1}{(n^2+1)\mu_n} < \infty,
\]
we obtain
\[
(5.3) \quad |h'(t)| = O(t^6), \quad |t| \to \infty.
\]
Given \( k \geq 1 \), choose \( n \) such that \( y_k \in [n, n+1) \) and denote by \( \ell_k \) the length of the set
\[
J_k = h\left([y_k - 2|n|^{-10}, y_k + 2|n|^{-10}]\right).
\]
By \((5.3)\), \( \sum_{k \geq 1} \ell_k < \infty \). Therefore, we can find \( u \in \mathbb{R} \) which belongs to at most finitely many sets \( J_k \). Then for sufficiently large \( k \),
\[
\tan u \not\in \psi\left([y_k - 2|n|^{-10}, y_k + 2|n|^{-10}]\right).
\]
If \( \psi(s) = \tan u \), \( s \in (n, n+1) \), then \( s \not\in [y_k - 2|n|^{-10}, y_k + 2|n|^{-10}] \) and, hence, \( y_k \not\in [s - |n|^{-10}, s + |n|^{-10}] \). The same is true for \( s \in (n - 1, n) \cup (n + 1, n+2) \). This completes the proof. \( \square \)

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