Global existence and uniqueness of the solution to a nonlinear parabolic equation

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Abstract

Consider the equation

\[ u'(t) - \Delta u + |u|^\rho u = 0, \quad u(0) = u_0(x), \quad (1) \]

where \( u' := \frac{du}{dt} \), \( \rho = \text{const} > 0 \), \( x \in \mathbb{R}^3 \), \( t > 0 \).

Assume that \( u_0 \) is a smooth and decaying function,

\[ \|u_0\| = \sup_{x \in \mathbb{R}^3, t \in \mathbb{R}_+} |u(x, t)|. \]

It is proved that problem (1) has a unique global solution and this solution satisfies the following estimate

\[ \|u(x, t)\| < c, \]

where \( c > 0 \) does not depend on \( x, t \).

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1 Introduction

Let

\[ u' - \Delta u + |u|\rho u = 0, \quad u(0) = u_0; \quad u' := \frac{du}{dt}, \quad (1) \]

where \( \rho > 0 \), \( t \in \mathbb{R}_+ = [0, \infty) \), \( x \in \mathbb{R}^3 \), \( X \) is a Banach space of real-valued functions with the norm \( \|u(x, t)\| := \sup_{x \in \mathbb{R}^3, t \in \mathbb{R}_+} |u(x, t)| \). We assume that

\[ \|u\| \leq c. \quad (2) \]

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We say that $u$ is a global solution to (1) if $u$ exists $\forall t \geq 0$.

Our result is formulated in Theorem 1. Our method is simple and differs from the published results, see [1], [2] and references there.

The novel points in this work are:

a) There is no restriction on the upper bound of $\rho$.

In [1], (Section 1.1) a nonlinear hyperbolic equation with the same nonlinearity is studied in a bounded domain, uniqueness of the solution is proved only for $\rho \leq 2/(n - 2)$, and existence is proved by a different method. The contraction mapping theorem is not used.

In [2] the quasi-linear problems for parabolic equations are studied in Chapter 5 in a bounded domain and under the assumptions different from ours. There are many papers and books on non-linear problems for parabolic equations (see the bibliography in [1], [2].

b) Existence of the global solution is proved.

c) Method of the proof differs from the methods in the cited literature.

Our result is formulated in Theorem 1:

**Theorem 1.** Problem (1) has a unique global solution in $X$ for any $u_0 \in X$.

### 2 Proofs

Let $g(x, t) = \frac{e^{-|x|^2}}{(4\pi t)^{3/2}}$. If $u$ solves (1) then

$$u(t) = -\int_0^t d\tau \int g(x - y, t - \tau)|u|^\rho u dy + \int g(x - y, t)u_0(y)dy := A(u) + F := Q(u),$$

where $\int := \int_{\mathbb{R}^3}$. Let $X$ be the Banach space of continuous in $\mathbb{R}^3 \times R_+$ functions, $\mathbb{R}_+ := [0, \infty)$, $\|u\| := \max_{x \in \mathbb{R}^3, t \in [0, T]} |u(x, t)|$. If $\|u\| \leq R$ then $\|A(u)\| \leq T R^\rho + 1$, where the identity $\int g(x - y, t - \tau)dy = 1$ was used. From (3) one gets

$$\|u\| \leq T\|u\|^\rho + \|F\|.$$

Thus, $Q$ maps the ball $B(R) = \{u : \|u\| \leq R\}$ into itself if $T$ is such that

$$TR^\rho + 1 + \|F\| \leq R.$$

The $Q$ is a contraction on $B(R)$ if

$$\|Q(u) - Q(v)\| \leq T(\rho + 1)R^\rho \|u - v\| \leq q\|u - v\|, \quad 0 < q < 1.$$

Thus, if

$$T(\rho + 1)R^\rho \leq q < 1,$$
then \( Q \) is a contraction in \( B(R) \) in the Banach space \( X_T \) with the norm \( \| \cdot \| \), \( t \in [0, T] \). We use the same notations for the norms in \( X_T \) and in \( X_\infty \).

We have proved that

For \( T \) satisfying (5) - (6) there exists and is unique the solution to (1), and this solution can be obtained from (3) by iterations.

The problem now is:

Does this solution exist and is unique on \( R^+ \)?

From our proof it follows that if the solution exists and is unique in \( X_T \), then the solution exists and is unique in \( X_{T_1} \) for some \( T_1 > T \).

To prove that the solution \( u(x, t) \) to (1) exists on \( R^+ \), assume the contrary: this solution does not exist on any interval \( [0, T_1) \), \( T_1 > T \), where \( T \) is the maximal interval of the existence of the continuous solution. Then \( \lim_{t \to T^-} u(x, t) = \infty \), because otherwise there is a sequence \( t_n \to T - 0 \) such that \( u(x, t_n) \to u(x, T) \) and one may construct the solution defined on \( [T, T_1] \), \( T_1 > T \), by using the local existence and uniqueness of the solution to (1) with the initial value \( u(x, T) \) for \( t \in [T, T_1] \). This contradicts the assumption that \( T \) is the maximal interval of the existence of the continuous solution \( u \).

Thus, if \( T < \infty \) then one has \( \lim_{t \to T^-} u(x, t) = \infty \). Let us prove that this also leads to a contradiction. Then we have to conclude that \( T = \infty \) and Theorem 1 is proved.

We need some estimates. Multiply (1) by \( u \), integrate over \( R^3 \) with respect to \( x \), and then integrate by parts the second term. The result is:

\[
0.5 \frac{dN(u)}{dt} + N(\text{grad}u) + \int |u|^{\rho+2}dy = 0,
\]

where \( N(u) := \int u^2dy \). Integrate (7) with respect to time over \( [0, T] \) and get

\[
0.5N(u(T)) + \int_0^T \left( N(\text{grad}u) + \int |u|^{\rho+2}dy \right) d\tau = 0.5N(u(0)).
\]

Therefore,

\[
N(u(t)) \leq c, \quad \forall t \in [0, T], \quad \int_0^T N(\text{grad}u)d\tau \leq c, \quad \int_0^T d\tau \int |u|^{\rho+2}dy \leq c,
\]

where \( c = 0.5N(u_0) \).

**Lemma 1.** From (9) and (3) it follows that

\[
\|u(x, t)\| < \infty \quad \forall t \in [0, T].
\]

If (10) is proved then \( T \) is not the maximal interval of the existence of the solution to (1). This contradiction proves Theorem 1.
Proof of Lemma 1. One uses the Hölder inequality twice and gets

\[
\begin{aligned}
\int_0^T d\tau \int g(x-y,t-\tau)|u|^\rho dy & \leq \left( \int_0^T d\tau \int |u|^{\rho+2} dy \right)^{(\rho+1)/(\rho+2)} \left( \int_0^T d\tau \int g^{\rho+2} dy \right)^{1/(\rho+2)} \\
& \leq \left( \int_0^T d\tau \int |u|^{\rho+2} dy \right)^{(\rho+1)/(\rho+2)} \left( \int_0^T d\tau \int g^{\rho+2} dy \right)^{1/(\rho+2)}.
\end{aligned}
\]  

(11)

By the last inequality (9) it follows that \( \int_0^T d\tau \int |u|^{\rho+2} dy < c \forall T > 0 \), where \( c > 0 \) is a constant independent of \( T \). The last integral in (11) is also bounded independently of \( T \). It can be calculated analytically.

Thus, inequalities (11), (9) and equation (3) imply (10).

Lemma 1 is proved. \( \square \)

Therefore Theorem 1 is proved. \( \square \)

The ideas related to the ones used in this paper were developed and used in [3]–[5].

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