Evaporation and anti-evaporation instability of a Schwarzschild-de Sitter braneworld: the case of five-dimensional F(R) gravity

Andrea Addazi,1,2 Shin’ichi Nojiri,3,4 and Sergei Odintsov5,6

1 Dipartimento di Fisica, Università di L’Aquila, 67010 Coppito AQ, Italy
2 Laboratori Nazionali del Gran Sasso (INFN), 67010 Assergi AQ, Italy,
3 Department of Physics, Nagoya University, Nagoya 464-8602, Japan,
4 Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan
5 Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, Spain,
6 Institut de Ciencies de l’Espai (IEEC-CSIC), Campus UAB,
Carrer de Can Magrans, s/n 08193 Cerdanyola del Valles, Barcelona, Spain

We study the problem of a four-dimensional brane lying in the five-dimensional degenerate Schwarzschild-de Sitter (Nariai) black hole, in five-dimensional F(R)-gravity. We show that there cannot exist the brane in the Nariai bulk space except the case that the brane tension vanishes. We demonstrate that the five-dimensional Nariai bulk is unstable in a large region of the parameter space. In particular, the Nariai bulk can have classical (anti-)evaporation instabilities. The bulk instability back-reacts on the four-dimensional brane, in case that the brane tension vanishes, and the unstable modes propagate in their world-volume.

PACS numbers: 04.50.Kd,04.70.-s, 04.70.Dy, 04.62.+v, 05.05.Mt
Keywords: Alternative theories of gravity, black hole physics, quantum black holes, dS/CFT, brane-Worlds

1. INTRODUCTION

The possibility that our Universe could be a brane lying in a higher dimensional bulk is strongly motivated by string theory and was largely explored in literature (see Refs. [1, 2] for reviews on these subjects). Some observational experiments are currently searching for the manifestation of extra dimensions. On the other hand, the study of brane in a higher dimensional de Sitter bulk is strongly motivated as in context of string theory (see, for instance, Refs. [3–9]). In particular, the extension of the higher-dimensional theory of gravity from the Einstein-Hilbert formulation to ghost-free F(R)-gravity is highly motivated by O ((α′)n) perturbative corrections to string amplitude as well as by possible non-perturbative effects like (Euclidean) D-brane or world sheet instantons (see Ref. [10] for a useful review on these aspects). In this letter, we explore the dynamics of brane lying in the higher dimensional Nariai black hole bulk, in the context of five-dimensional F(R)-gravity.

We find some surprising results;

1. The Nariai black hole background is unstable in a large space of parameters of the F(R)-gravity. In particular, (anti-)evaporation instabilities and evaporation/anti-evaporation iterative oscillations studied in four dimensional are retried in this context [15, 23].

2. The background instabilities change the brane dynamics provoking worldsheet instabilities.

2. FIVE-DIMENSIONAL BULK INSTABILITIES

Let us consider the F(R)-gravity theory in five dimensions;

\[ S = \frac{1}{2\kappa_5^2} \int \sqrt{-g} \left[ F^{(5)}(R) + S_m \right], \]

where \( \kappa_5 \) is the five-dimensional gravitational constant and \( S_m \) is the action of the matter. The equations of motion in the vacuum are given by

\[ F^{(5)}_R(R) \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \frac{1}{2} g_{\mu\nu} \left[ F^{(5)}(R) - R F^{(5)}_R(R) \right] + \left[ \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right] F^{(5)}_R(R), \]

1 See also Refs. [11–13] for useful reviews in extended theories of gravity.
where \( F_R^{(5)} = dF^{(5)}/dR \). Especially if we assume that the metric is covariantly constant, that is, \( R_{\mu\nu} \propto Kg_{\mu\nu} \) with a constant \( K \), we find
\[
0 = RF_R^{(5)}(R) - \frac{5}{2} F^{(5)}(R) .
\] (3)

We denote the solution of Eq. (3) as \( R = R_0 \) and define the length parameter \( l \) by \( R_0 = 20/l^2 \). We should note that the metric of the Schwarzschild-de Sitter solution is covariantly constant and given by,
\[
ds^2_{SdS, (5)} = \frac{1}{h(a)} da^2 - h(a) dt^2 + a^2 d\Omega^2_{(3)}, \quad h(a) = 1 - \frac{a^2}{l^2} - \frac{16\pi G_{(5)} M}{3a^2}.
\] (4)

Here \( M \) corresponds to the mass of the black hole and \( G_{(5)} \) is defined by \( 8\pi G_{(5)} = \kappa^2 5^5 \). The space-time expressed by the metric (4) has two horizons at
\[
a^2 = a^2_{\pm} = \frac{l^2}{2} \left( 1 \pm \sqrt{1 - \frac{64\pi G_{(5)} M}{3l^2}} \right).
\] (5)

The two horizons degenerate in the limit,
\[
\frac{64\pi G_{(5)} M}{3l^2} \to 1,
\] (6)

and we obtain the degenerate Schwarzschild-de Sitter (Nariai) solution. The metric in the Nariai space-time is given by
\[
ds^2 = \frac{1}{\Lambda} \left( -\sin^2 \chi d\psi^2 + d\chi^2 + d\Omega^2_{(3)} \right),
\] (7)

where there are the horizons at \( \chi = 0, \pi \) and \( \Lambda = \frac{2}{\ell} \). Let us perform the coordinate transformation \( \chi = \arccos \zeta \),
\[
ds^2 = -\frac{1}{\Lambda} \left( 1 - \zeta^2 \right) d\psi^2 + \frac{d\zeta^2}{\Lambda (1 - \zeta^2)} + \frac{1}{\Lambda} d\Omega^2_{(3)},
\] (8)

which is singular at \( \zeta = \pm 1 \). By changing the coordinate \( \zeta = \tanh \xi \), the metric can be rewritten as,
\[
ds^2 = \frac{1}{\Lambda \cosh^2 \xi} \left( -d\psi^2 + d\xi^2 \right) + \frac{1}{\Lambda} d\Omega^2_{(3)}.
\] (9)

We often analytically continue the coordinates by
\[
\psi = ix, \quad \zeta = i\tau,
\] (10)

and we obtain the following metric
\[
ds^2 = -\frac{1}{\Lambda \cos^2 \tau} \left( -d\tau^2 + dx^2 \right) + \frac{1}{\Lambda} d\Omega^2_{(3)}.
\] (11)

Of course, after the analytic continuation, the obtained space is a solution of the equations although the topology is changed. This expression of the metric was used in [15].

In order to consider the perturbation, we now consider the general metric in the following form,
\[
ds^2 = e^{2\rho(x, \tau)} \left( -dt^2 + dx^2 \right) + e^{-2\phi(x, \tau)} d\Omega^2_{(3)},
\] (12)

which generalizes the Nariai’s metric in Eq. (11) with generic functions \( \rho(x, \tau), \phi(x, \tau) \).

Then the equation of motion can be decomposed in components as
\[
0 = -\frac{\rho''}{2} F^{(5)} - \left( -\rho + \frac{3}{2} \phi' + \rho'' - 3 \phi^2 - 3 \phi' \phi' - 3 \rho' \phi' \right) F_R^{(5)} + \tilde{F}_R^{(5)}
\]
\[-\rho' \tilde{F}_R^{(5)} - \rho' \left( F_R^{(5)} \right)' + e^{2\phi} \left[ -\frac{\partial}{\partial \tau} \left( e^{-2\phi} \tilde{F}_R^{(5)} \right) + \left( e^{-2\phi} (F_R^{(5)})' \right)' \right],
\] (13)
from (21), we can put $c$

Here

Therefore the four equations (17), (18), (19), and (20) include only two $c$

The perturbation of the scalar curvature $\delta R$

We consider the perturbations at the first order around the Nariai background Eq.(11) with $R_0 = \frac{20}{\tau}$,

where $F' = \frac{\partial F}{\partial x}$ and $\dot{F} = \frac{\partial F}{\partial \tau}$ and we have used the expressions of the curvatures [A3] in the Appendix A.

We consider the perturbations at the first order around the Nariai background Eq.(11) with $R_0 = \frac{20}{\tau}$,

The perturbation of the scalar curvature $\delta R$ is given in terms of $\delta \rho$ and $\delta \phi$ as follows,

Therefore the four equations (17), (18), (19), and (20) include only two $\delta \phi$ and $\delta \rho$, which tell that only two equations in the four equations (17), (18), (19), and (20) should be independent ones.

One can find that Eq. (19) can be easily integrated

Here $c_1(x)$ and $c_2(\tau)$ are arbitrary functions but because $\delta R$ should vanish when both of $\delta \rho$ and $\delta \phi$ vanish as seen from (24), we can put $c_1(x) = c_2(\tau) = 0$.

Then, one can directly consider Eq. (24): Substituting in it $\delta R(\delta \phi)$ obtained in Eq.(22), we find a simple equation

Here

Eq. (23) is nothing but a time-dependent Klein-Gordon equation for the $\delta \phi$ mode, with an effective oscillating mass term in time. An explicit solution of (23) is given by

$$\delta \phi = \phi_0 \cos (\beta x) \cos^\beta \tau .$$
Here $\beta$ is given by solving the equation $M^2 = \beta (\beta - 1)$. The anti-evaporation corresponds to the increasing of the radius of the apparent horizon, which is defined by the condition,
\[
\nabla \delta \phi \cdot \nabla \delta \phi = 0.
\]
(26)
In other words, it is imposed that the (flat) gradient of the two-sphere size is null. By using the solution in (25), we find $\tan \beta x = \tan \tau$, that is, $\beta x = \tau$. Therefore on the apparent horizon, we find
\[
\delta \phi = \phi_0 \cos^{\beta + 1} \tau.
\]
(27)
Because the horizon radius $r_H$ is given by $r_H = e^{-\phi}$, we find
\[
r_H = \frac{e^{-\phi_0 \cos^{\beta + 1} \tau}}{\sqrt{\Lambda}}.
\]
(28)
Then if $\beta < -1$, the horizon grows up, which corresponds to the anti-evaporation depending on the sign of $\phi_0$. The sign could be determined by the initial condition of the perturbation. On the other hand, it is also possible the case in which $\beta, \omega$ are complex parameters. In this case, solutions of perturbed equations read
\[
\delta \phi = \text{Re} \left\{ (C_1 e^{\beta t} + C_2 e^{-\beta t}) e^{\beta x} \right\},
\]
(29)
where $C_1, C_2$ are complex numbers. $\delta \phi$ always increase in time for $C_1 \neq 0$ because of $\text{Re} \beta > 0$. This means that the Nariai solution is unstable also in this region of parameters. A particular class among possible complex parameter solutions is
\[
\delta \phi = \phi_0 (-1)^n \left\{ e^{\frac{x}{\sqrt{2}} \tan \frac{\gamma}{2} \left( \frac{g(t-x)}{2} + \frac{1}{\gamma} \sin \frac{g(t-x)}{2} \right)} + e^{\frac{x}{\sqrt{2}} \tan \frac{\gamma}{2} \left( \frac{g(t+x)}{2} - \frac{1}{\gamma} \sin \frac{g(t+x)}{2} \right)} \right\},
\]
(30)
where $\beta \equiv \frac{1}{2}(1 + i\gamma)$ and $\gamma \equiv \pm \sqrt{\frac{2-9\alpha}{\alpha}}$.

On the horizon, the fluctuations must satisfy the condition $\frac{\delta \phi^2}{2} \gamma^2 \cos \frac{g(t-x)}{2} \sin \frac{g(t-x)}{2} = 0$, which corresponds to two classes of solutions with $x = \mp t + \frac{2\pi n}{\gamma}$,
\[
\delta \phi = \phi_0 (-1)^n \left\{ e^{\frac{x}{\sqrt{2}} \tan \frac{\gamma}{2} \left( \frac{g(t-x)}{2} + \frac{1}{\gamma} \sin \frac{g(t-x)}{2} \right)} + e^{\frac{x}{\sqrt{2}} \tan \frac{\gamma}{2} \left( \frac{g(t+x)}{2} - \frac{1}{\gamma} \sin \frac{g(t+x)}{2} \right)} \right\},
\]
(31)
which implies an oscillating horizon radius.

Let us consider a class of $F^{(5)}(R)$ models
\[
F^{(5)}(R) = \frac{R}{2\kappa^2} + f_2 R^2 + f_0 M^{5-2n} R^n.
\]
(32)
Here $f_2$ and $M$ are constants with a mass dimension and $f_0$ is a dimensionless constant. In this case, $\alpha$ is given by
\[
\alpha = \frac{4\Lambda \left( 2f_2 + n(n-1)f_0 M^{5-2n} R_0^{n-2} \right)}{1/2\kappa^2 + 2f_2 R_0 + n f_0 M^{5-2n} R_0^{n-1}}.
\]
(33)
Then $\beta$ is given by
\[
\beta^2 - \beta = \frac{1}{2\alpha} (4\alpha - 1),
\]
(34)
that is
\[
\beta = \frac{1}{2} \left( \frac{\sqrt{9\alpha - 2}}{2} \right).
\]
(35)
Then the condition of the anti-evaporation $\beta < -1$ (for $\phi_0 < 0$) can be satisfied only by $\beta_-$ and for $\alpha < 0$. On the other hand, for $\beta$ as a complex parameter in Eq. (31), the oscillation instabilities are obtained for $0 < \alpha < 2/9$. In this case, evaporation and antievaporation phases are iterated.
2.1. Brane dynamics in the bulk

We now consider the $F^{(d+1)}(R)$ gravity in the $d + 1$ dimensional space-time $M$ with $d$ dimensional boundary $B$, whose action is given by

$$ S = \frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{-g} F^{(d+1)}(R), $$

which can be rewritten in the scalar-tensor form. We begin by rewriting the action (36) by introducing the auxiliary field $A$ as follows,

$$ S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left\{ F^{(d+1)'}(A) (R - A) + F^{(d+1)}(A) \right\}. $$

By the variation of the action with respect to $A$, we obtain the equation $A = R$ and by substituting the obtained expression $A = R$ into the action (37), we find that the action in (36) is reproduced. If we rescale the metric by conformal transformation,

$$ g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}, \quad \sigma = -\ln F^{(d+1)'}(A), $$

we obtain the action in the Einstein frame,

$$ S_E = \frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{-g} \left( R - (d - 1)\Box \sigma - \frac{(d-2)(d-1)}{4} \partial^\mu \sigma \partial_\mu \sigma - V(\sigma) \right) $$

$$ = \frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{-g} \left( R - \frac{(d-2)(d-1)}{4} \partial^\mu \sigma \partial_\mu \sigma - V(\sigma) \right) + (d-1) \int_B d^d x \sqrt{-\hat{g}} n^\mu \partial_\mu \sigma, $$

$$ V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{F^{(d+1)'}(A)} - \frac{F^{(d+1)}(A)}{F^{(d+1)'}(A)^2}. $$

Here $g(e^{-\sigma})$ is given by solving the equation $\sigma = -\ln F^{(d+1)'}(A)$ as $A = g(e^{-\sigma})$. By the integration of the term $\Box \sigma$, there appears the boundary term, where $n^\mu$ is the unit vector perpendicular to the boundary and the direction of the vector is inside. Furthermore $\hat{g}_{\mu\nu}$ is the metric induced on the boundary, $g_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The existence of the boundary makes the variational principle with respect to $\sigma$ ill-defined, we cancel the term by introducing the boundary action

$$ S_B = -(d-1) \int_B d^d x \sqrt{-\hat{g}} n^\mu \partial_\mu \sigma. $$

Then one may forget the boundary term,

$$ S_E \rightarrow S_E + S_B = \frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{-g} \left( R - \frac{(d-2)(d-1)}{4} \partial^\mu \sigma \partial_\mu \sigma - V(\sigma) \right). $$

As is well-known, because the scalar curvature $R$ includes the second derivative term, the variational principle is still ill-defined in the space-time with boundary (28) (see also, Refs. 21, 27).

Because the variation of the scalar curvature with respect to the metric is given by

$$ R = -\delta g_{\mu\nu} R^{\mu\nu} + g^{\sigma\nu} \left( \nabla_\mu \delta \Gamma^\mu_{\sigma\nu} - \nabla_\sigma \delta \Gamma^\mu_{\mu\nu} \right), $$

the variation of the action with respect to the metric is given by,

$$ \delta S_E = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} Q^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2\kappa^2} \int_B d^d x \sqrt{-\hat{g}} g^{\sigma\nu} \left( -n_\mu \delta \Gamma^\mu_{\sigma\nu} + n_\sigma \delta \Gamma^\mu_{\mu\nu} \right). $$

Here the Einstein equation in the bulk is given by $Q_{\mu\nu} = 0$. Then the variational principle becomes well-defined if we add the following boundary term,

$$ \tilde{S}_b = -\frac{1}{2\kappa^2} \int_B d^d x \sqrt{-\hat{g}} g^{\sigma\nu} \left( -n_\mu \Gamma^\mu_{\sigma\nu} + n_\sigma \Gamma^\mu_{\mu\nu} \right). $$
Although the above boundary term (44) is not invariant under the reparametrization, because
\[ \nabla_\mu n_\nu = \partial_\mu n_\nu - \Gamma^\lambda_{\mu\nu} n_\lambda, \quad \nabla_\mu n^\nu = \partial_\mu n^\nu + \Gamma^\nu_{\mu\lambda} n^\lambda, \] (45)
we find
\[ g^{\sigma\nu} \left( -n_\mu \Gamma^\nu_{\sigma\mu} + n_\sigma \Gamma^\nu_{\mu\mu} \right) = -\partial_\mu n_\mu - 2g^{\delta\rho} \partial_\delta n_\rho \nabla_\mu n^\mu, \] (46)
which is just equal to \( \nabla_\mu n^\mu \) on the boundary [24–28]. Therefore we can replace the boundary term (44) by the Gibbons-Hawking boundary term,
\[ S_{GH} = \frac{1}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}} \nabla_\mu n^\mu. \] (47)

Let the boundary is defined by a function \( f(x^\mu) \) as \( f(x^\mu) = 0 \). Then by the analogy of the relation between the electric field and the electric potential in the electromagnetism, we find that the vector \( (\partial_\mu f(x^\mu)) \) is perpendicular to the boundary because \( dx^\mu \partial_\mu f(x^\mu) = 0 \) on the boundary, which gives an expression for \( n_\mu \) as
\[ n_\mu = \frac{\partial_\mu f}{\sqrt{g^{\rho\sigma} \partial_\rho f \partial_\sigma f}}. \] (48)

Then with respect to the variation of the metric, the variation of \( n^\mu \) is given by
\[ \delta n^\mu = \frac{1}{2} \frac{\partial_\mu f}{(g^{\rho\sigma} \partial_\rho f \partial_\sigma f)^{\frac{3}{2}}} \partial^\sigma f \partial^\mu f = \frac{1}{2} n_\mu n^\rho \delta g^{\rho\sigma}. \] (49)

By using the expression in (49), one finds the variation of \( \nabla_\mu n^\mu \) with respect to the metric,
\[ \delta (2\nabla_\mu n^\mu) = -2\delta g^{\mu\nu} n^\mu n^\nu - n^\mu \nabla_\nu \delta g^{\mu\nu} - g^{\mu\nu} n_\mu \delta \Gamma^\nu_{\mu\rho} + n^\nu \delta \Gamma^\mu_{\rho\nu} . \] (50)
The last two terms in (50) are necessary to make the variational principle well-defined but the second term \( n^\mu \nabla_\nu \delta g^{\mu\nu} \) also may violate the variational principle. By using the reparametrization invariance, however, we can choose the gauge condition so that \( \nabla_\nu \delta g^{\mu\nu} = 0 \).

We may also add the following boundary term,
\[ S_{BD} = \int_B d^d x \sqrt{-\hat{g}} \mathcal{L}_B, \] (51)
The variation of the total action
\[ S_{\text{total}} = S_E + S_B + S_{GH} + S_{BD}, \] (52)
is given by
\[ \delta S_{\text{total}} = \frac{1}{2\kappa^2} \int d^{d+1} x \sqrt{-\hat{g}} Q^{\mu\nu} \delta g_{\mu\nu} + \int_B d^d x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \left( \frac{1}{2} \kappa \hat{g}^{\mu\nu} - K^{\mu\nu} \right) + \frac{1}{2} T^{\mu\nu}_B \right] \delta g_{\mu\nu}. \] (53)
Here we have defined the extrinsic curvature by \( K_{\mu\nu} \equiv \nabla_\mu n_\nu \) and \( K \equiv g^{\mu\nu} K_{\mu\nu} \). We also wrote the variation of \( S_{BD} \) as
\[ \delta S_{BD} = \frac{1}{2} \int_B d^d x \sqrt{-\hat{g}} T^{\mu\nu}_B \delta g_{\mu\nu}. \] (54)
Then on the boundary, we obtain the following equation,
\[ 0 = \frac{1}{2} K \hat{g}^{\mu\nu} - K^{\mu\nu} + \kappa^2 T^{\mu\nu}_B, \] (55)
which may be called the brane equation. Especially if the boundary action \( S_{BD} \) consists of only the brane tension \( \kappa \),
\[ S_B = \frac{\kappa}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}}, \] (56)
we find
\[ 0 = \frac{1}{2} \mathcal{K} \tilde{g}^{\mu\nu} - K^{\mu\nu} + \tilde{\kappa} g^{\mu\nu}, \]  
which can be rewritten as,
\[ 0 = \frac{2}{d - 2} \tilde{\kappa} \tilde{g}^{\mu\nu} - K^{\mu\nu}. \]

If we consider the model which is given by gluing two space-time as in the Randall-Sundrum model \[29, 30\], the contribution from the bulk doubles and therefore the Gibbons-Hawking term also doubles,
\[ 0 = \frac{2}{d - 2} \tilde{\kappa} \tilde{g}^{\mu\nu} - 2K^{\mu\nu}. \]

Let us consider the following five-dimensional geometry,
\[ ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 d\Omega_3^2. \]  

Here \(d\Omega_3^2 = \tilde{g}_{ij} dx^i dx^j\) expresses the metric of the unit sphere in two dimensions. We now introduce a new time variable \(\tau\) so that the following condition is satisfied,
\[ -e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial a}{\partial \tau} \right)^2 = -1. \]  

Then we obtain the following FRW metric
\[ ds_4^2 = \tilde{g}_{ij} dx^i dx^j = -d\tau^2 + a^2 d\Omega_3^2. \]

Then
\[ n^\mu = \left( -e^{-2\rho} \frac{\partial a}{\partial \tau}, -e^{2\rho} \frac{\partial t}{\partial \tau}, 0, 0, 0 \right). \]

Because
\[ K_{ij} = \frac{\kappa}{2} e^{4\rho} a \tilde{g}_{ij} \frac{dt}{d\tau}, \]
from Eq. (57), we obtain,
\[ e^{2\rho} \frac{dt}{d\tau} = \tilde{\kappa} a. \]

Using (61) and defining the Hubble rate by \(H = \frac{1}{a} \frac{da}{d\tau}\), one finds the following FRW equation for the brane,
\[ H^2 = -\frac{e^{2\rho(s)}}{a^2} + \frac{\kappa^2}{4}. \]

Then in case of the Schwarzschild-de Sitter black hole,
\[ e^{2\rho} = \frac{1}{a^2} \left( -\mu + a^2 - \frac{a^4}{l_{\text{dS}}^2} \right), \]  
we obtain
\[ H^2 = \frac{1}{l_{\text{dS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^2} + \frac{\kappa^2}{4}. \]  

Here \(l_{\text{dS}}\) is the curvature radius of the de Sitter space-time and \(\mu\) is the black hole mass. On the other hand, in the Schwarzschild-AdS black hole,
\[ e^{2\rho} = \frac{1}{a^2} \left( -\mu + a^2 + \frac{a^4}{l_{\text{AdS}}^2} \right). \]
we obtain,

$$H^2 = -\frac{1}{l^2_{\text{AdS}}} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \quad (70)$$

In the Jordan frame, the metric is given by

$$ds^2_{J4} = F^{(5)'}(R)ds^2_4 = (-d\tau^2 + a^2 d\Omega^2_3). \quad (71)$$

Because the scalar curvature is a constant in the Schwarzschild-(anti-)de Sitter space-time, $F^{(5)'}(R)$ can be absorbed into the redefinition of $\tau$ and $a$,

$$d\tilde{\tau} \equiv dt \sqrt{F^{(5)'}(R)}, \quad \tilde{a} \equiv a \sqrt{F^{(5)'}(R)}. \quad (72)$$

Then the qualitative properties are not changed in the Jordan frame compared with the Einstein frame. We should also note that the motion of the brane does not depend on the detailed structure of $F^{(5)'}(R)$.

In the Nariai space, the radius $a$ is a constant and therefore $H = 0$. Furthermore in the Nariai space, we find $e^{2\rho(a)} = 0$ and therefore Eq. (60) shows that the brane tension $\tilde{\kappa}$ should vanish. That is, if and only if the tension vanished, the brane can exist. The non-vanishing tension might be cancelled with the contribution from the trace anomaly by tuning the brane tension. We should note, however, that there should not be any (FRW) dynamics of the brane in the Nariai space.

However, the anti-evaporation may induce the dynamics of the brane. For the metric (12), one gets the expressions of the connection in (A2). We introduce a new time coordinate $\tilde{t}$ in the metric (12) as follows,

$$d\tilde{t}^2 = e^{\mu} (d\tau^2 - dx^2). \quad (73)$$

Then the metric (12) reduces to the form of the FRW-like metric,

$$ds^2 = -d\tau^2 + e^{-2\phi(x, \tau)} d\Omega^2_{(3)}, \quad (74)$$

if we identify $e^{-\phi(x, \tau)}$ with the scale factor $a$, $a = e^{-\phi(x, \tau)}$. Then the unit vector perpendicular to the brane is given by

$$n^\mu = \left(-e^{-2\phi} \partial_x, -e^{-2\phi} \partial_t, 0, 0, 0\right), \quad (75)$$

and the $(i, j)$ ($i, j = 1, 2, 3$) components Eq. (57) give

$$-e^{-2\phi} \partial_x \partial_x - e^{-2\phi} \partial_\tau \partial_t = \tilde{\kappa}. \quad (76)$$

As we discussed, in order that the brane exists in the Nariai space-time, we find $\tilde{\kappa} = 0$. By using the solution in (25), and analytically recontinuing the coordinates $x \rightarrow -ix$, $\tau \rightarrow -i\tau$, if we assume

$$\phi = \ln \Lambda + \phi_0 \cosh \omega \tau \cosh^\beta x, \quad (77)$$

with $\omega^2 = \beta^2$, we find

$$-\omega \sinh \omega \tau \cosh^\beta x \partial_x - \beta \cosh \omega \tau \cosh^{\beta - 1} x \sinh x \partial_\tau = 0, \quad (78)$$

that is,

$$\frac{\partial x}{\partial \tilde{t}} = -\frac{\beta \tanh x}{\omega \tanh \omega \tau} \frac{\partial \tau}{\partial \tilde{t}}. \quad (79)$$

Assuming that $x$ and $\tau$ only depend on $\tilde{t}$ on the brane,

$$0 = \frac{1}{\beta \tanh x} \frac{dx}{d \tilde{t}} + \frac{1}{\omega \tanh \omega \tau} \frac{d\tau}{d \tilde{t}} = \frac{d}{d \tilde{t}} \left(\frac{1}{\beta} \ln \sinh x + \ln \tanh \omega \tau\right), \quad (80)$$

that is $\frac{1}{\beta} \ln \sinh x + \ln \sinh \omega \tau$ is a constant, which gives the trajectory of the brane,

$$\sinh x = \frac{C}{\sinh^\beta \omega \tau}. \quad (81)$$

Here $C$ is a constant. Of course, the expression in (51) is valid as long as the perturbation $\delta \phi = \phi_0 \cosh \omega \tau \cosh^\beta x$ is small enough. We should also note that because $F^{(5)'}(R)$ is not a constant due to the perturbation, Eq. (72) also gives another source of the dynamics of the brane. However, that Eq. (72) gives only small correction to Eq. (51).
3. CONCLUSION

In this paper, we have studied the FRW brane lying in the degenerate Schwarzschild-de Sitter (Nariai) black hole, in a five dimensional $F(R)$-gravity. We have found that there cannot exist the brane in the Nariai bulk space except the case that the brane tension vanishes. We have shown how the Nariai bulk is unstable in a large variety of $F(R)$-gravity models. These instabilities are divided in three classes; classical evaporation, classical anti-evaporation and infinite iterations of evaporations/anti-evaporations. We have studied how these instabilities change the brane world-sheet dynamics.

Acknowledgments

AA would like to thank University of Fudan for hospitality during the preparation of this letter. This work is partially supported by the MIUR research grant “Theoretical Astroparticle Physics” PRIN 2012CPPYP7 and by SdC Progetto speciale Multiasse “La Societ`a della Conoscenza” in Abruzzo PO FSE Abruzzo 2007-2013(AA), by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas “Cosmic Acceleration” (No. 15H05890)(SN), by MINECO (Spain), project FIS2013-44881 (SDO) and by the CSIC I-LINK1019 Project (SDO and SN).

Appendix A: Components of the Ricci tensors and Ricci scalar

For the metric (12), we write $\eta_{ab} = \text{diag}(-1,0)$, $(a,b = \tau,x)$. For the metric $d\Omega^2_{(3)}$ of three dimensional unit sphere, we also write as

$$d\Omega^2_{(3)} = \hat{g}_{ij} dx^i dx^j , \quad (i,j = 1,2,3). \quad (A1)$$

Then we obtain $\hat{R}_{ij} = 2\hat{g}_{ij}$. Here $\hat{R}_{ij}$ is the Ricci curvature given by $\hat{g}_{ij}$.

Then we find the following expression of the connections,

$$\Gamma^a_{bc} = \delta^a_{b} \rho_{,c} + \delta^a_{c} \rho_{,b} - \eta_{bc} \rho^a_{\,a}, \quad \Gamma^i_{ij} = e^{-2(\rho + \phi)} \hat{g}_{ij} \phi^a_{\,a}, \quad \Gamma^i_{aj} = \Gamma^i_{ja} = -\delta^i_j \phi_{,a}, \quad \Gamma^i_{jk} = \hat{\Gamma}^i_{jk}. \quad (A2)$$

Here $\hat{\Gamma}^i_{jk}$ is the connection given by $\hat{g}_{ij}$. By using the expressions in (A2), the curvatures are given by

$$R_{ab} = 3\phi_{,ab} - \eta_{ab} \Box \rho - 3(\phi_{,a}\rho_{,b} + \phi_{,b}\rho_{,a}) + 3\eta_{abc}\rho_{,c} - 3\phi_{,a}\phi_{,b},$$

$$R_{ij} = \hat{R}_{ij} + \hat{g}_{ij} e^{-2(\rho + \phi)} (\Box \phi - 3\phi_{,a}\phi^a_{\,a}), \quad R_{ai} = R_{ai} = 0,$$

$$R = e^{2\rho} \hat{R} + e^{-2\rho} (6\Box \phi - 2\Box \rho - 12\phi_{,a}\phi^a_{\,a}). \quad (A3)$$

We should note $\hat{R} = 6$ because $\hat{R}_{ij} = 2\hat{g}_{ij}$.
[13] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513 (2012) 1 doi:10.1016/j.physrep.2012.01.001 [arXiv:1106.2476 [astro-ph.CO]].
[14] S. Capozziello and M. De Laurentis, Phys. Rept. 509 (2011) 167 doi:10.1016/j.physrep.2011.09.003 [arXiv:1108.6266 [gr-qc]].
[15] R. Bousso and S. W. Hawking, Phys. Rev. D 57 (1998) 2436 [hep-th/9709224].
[16] S. Nojiri and S. D. Odintsov, Phys. Rev. D 59 (1999) 044026 doi:10.1103/PhysRevD.59.044026 [hep-th/9804033].
[17] S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A 14 (1999) 1293 doi:10.1142/S0217751X9900066X [hep-th/9802160].
[18] S. Nojiri and S. D. Odintsov, Class. Quant. Grav. 30 (2013) 125003 [arXiv:1301.2775 [hep-th]].
[19] S. Nojiri and S. D. Odintsov, Phys. Lett. B 735 (2014) 376 [arXiv:1405.2439 [gr-qc]].
[20] L. Sebastiani, D. Momeni, R. Myrzakulov and S. D. Odintsov, Phys. Rev. D 88 (2013) no.10, 104022 doi:10.1103/PhysRevD.88.104022 [arXiv:1305.4231 [gr-qc]].
[21] V. K. Oikonomou, Int. J. Mod. Phys. D 25 (2016) no.07, 1650078 doi:10.1142/S0218271816500784 [arXiv:1605.00583 [gr-qc]].
[22] A. Addazi. [arXiv:1610.04094] [gr-qc].
[23] A. Addazi and S. Capozziello, Mod. Phys. Lett. A 31 (2016) no.09, 1650054 doi:10.1142/S0217732316500541 [arXiv:1602.00485 [gr-qc]].
[24] S. Chakraborty and S. SenGupta, Eur. Phys. J. C 75 (2015) no.1, 11 doi:10.1140/epjc/s10052-014-3234-3 [arXiv:1409.4115 [gr-qc]].
[25] S. Chakraborty and S. SenGupta, Eur. Phys. J. C 75 (2015) no.11, 538 doi:10.1140/epjc/s10052-015-3768-z [arXiv:1504.07519 [gr-qc]].
[26] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. 37 (2005) 1419 doi:10.1007/s10714-005-0126-8 [hep-th/0409244].
[27] N. Deruelle, M. Sasaki and Y. Sendouda, Prog. Theor. Phys. 119 (2008) 237 doi:10.1143/PTP.119.237 [arXiv:0711.1150 [gr-qc]].
[28] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15 (1977) 2752. doi:10.1103/PhysRevD.15.2752
[29] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 doi:10.1103/PhysRevLett.83.3370 [hep-ph/9905221].
[30] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 doi:10.1103/PhysRevLett.83.4690 [hep-th/9906064].