A Study on Exponential and Linear Density Stratification on Rectangular Channel Problems

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Abstract. In this paper we analyze viscous oscillatory MHD inhomogeneous fluid flow through the rectangular channel with one wall being porous through which suction or injection is taken place with constant rate and the other wall being rigid. The density function in the equation of incompressibility are considered in first case as exponentially distributed and that in the second case as linearly distributed. The governing equation of motion in both cases are solved using similarity transformation and the graphical representation in both cases are analyzed and compared for velocity profiles. Further for second-degree approximation, the exponential stratification reduces to the results of linear stratification results, which are already available in the literature.

1. Introduction

Fluid flows which has density as a function of various parameters like height and time are known as density stratified fluids. These fluids often tend to realign their original form once any disturbance applied to them. Such phenomena are very common in various environmental flows and chemical industry. The vertical layering in general are of two types, (a) Exponentially distributed (b) Linearly distributed which is an approximated version of case (a). It is to be noted that the linear distribution is degree two approximation of exponential case. C. K. Kirubhashankar et. al detailed “Homotopy Method for Solving Unsteady MHD Fluid Flow between Two Parallel Plates” by considering one of the plates to be stationary while the other moving at constant speed [4]. “The effect of suction and injection on the unsteady flow (between two parallel porous) with variable properties through rectangular channel” was studied by Attia [1]. “Magnetohydrodynamic flow and heat transfer of dusty viscoelastic stratified fluid down an inclined channel in porous medium under variable viscosity” was experimented by Chakraborty [3]. By assuming the pressure as a suitable expression of the variables x and y and unknown expression of t, exact solution was presented. “Heat transfer in magnetohydrodynamic flow of dusty viscoelastic stratified fluid in porous medium under variable viscosity” was presented by Prakash by considering both vertical and horizontal velocity expressions as an expression of y, t and by depicting the similarity solution to fluid velocity [8]. Zahir shah et al [12] discussed on “combined effect of magnetic and electric field on micropolar nanofluid between two parallel plates in the rotating system”.

Arianna Bonzanini et al [2] analyzed “velocity profiles description and shape factors included in a hyperbolic, one-dimensional, transient two-fluid model for stratified and slug flow simulations in pipes”
by computing velocity profiles in stratified conditions. Krechenikov elaborated the various uses and purposes of “lubrication approximations to non-unidirectional coating flows with clean and surfactant interfaces” [5]. Krishna and Shrama solved the problem on “motion of an axisymmetric body in a rotating stratified fluid confined between two parallel planes” by considering the direction of planes to be vertical of the perpendicular axis of circulations [6]. Naidu explained the detailed solution to “stratified viscous flow between two oscillating cylinders” by formulating density stratification as exponentially distributed with governing equations of motion of fluid in polar form [7].

Prasanna Venkatesh analysed the “magnetohydrodynamics viscous oscillatory flow in a narrow channel due to suction or injection through the porous wall” by considering the electrically conducting fluid in both axial and transverse direction and its effects on channels which are very thin in nature [11]. Prasanna Venkatesh presented effect of “Stratification in rectangular channel problem” with the assumption that the density when there is no disturbance in the fluid as a function of depth only and also the flow being time dependent [10]. Prasanna Venkatesh and Surya prabha studied the “effect of narrow channel approximation on viscous oscillatory density stratified fluid flow through rectangular channels” by comparing the cases of flow without applying the lubrication approximation with the that when it is applied [9]. In present problem the MHD flow of density stratified fluid which is oscillatory as well as viscous passing through a vertical narrow rectangular channel with a pervious wall along which the withdrawal of fluid is taken place with a constant velocity is modelled. The flow is assumed to be time dependent and narrow channel approximation is applied to the equation of motion considering the equation of incompressibility and density stratification which is linearly distributed in undisturbed state. The main aim of the research work is to compare how the stratification property on the basis of exponential and linear distribution of the fluid influence the fluid flow through rectangular channel and its comparison with that on various flow parameters.

2. Problem Formulation and Solution
The region of the flow is between two plates where the left side plate is placed on y – axis and the right-side plate is fixed parallel to y – axis which is separated by a distance ‘h’. The plate on the right side is porous and the other is nonporous. The fluid is removed or included with a periodic constant suction or injection velocity through the pervious plate. As the fluid flow considered is fully developed the initial velocity of the fluid is assumed to be constant. The fluid is considered to be density stratified in such a way that in an unperturbed state it is considered to be exponentially distributed as a function of y. In disturbed condition, density distribution is a function of all parameters pertaining to space and time. The representation of the flow in terms of mathematical equations are presented here.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(1)

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)
\]  

(2)

\[
\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \sigma B_0^2 v - \rho g
\]  

(3)

\[
\rho = \rho_0(y) + \rho'(x,y,t)
\]  

(4)

Case (a)

\[
\rho_0(y) = \rho_0 e^{-\beta y}
\]  

(5)

\[
\frac{\partial \rho}{\partial t} = \rho_0 \beta e^{-\beta y} v
\]  

(6)
Case (b)
\[ \rho_0(y) = \rho_0'(1 - \beta y) \]  
(7)
\[ \frac{\partial \rho'}{\partial t} = \rho_0' \beta v \]  
(8)

The following explains the notations used in the above system of equations. (i) \( \mu \) - coefficient of viscosity (ii) \( \rho \) - density of the fluid (iii) \( \sigma \) - electrical conductivity (iv) \( B_0 \) - electromagnetic induction (v) \( \mu_e \) - magnetic permeability (vi) \( \rho_0' \) - constant density (vii) \( \rho'_t(y, t) \) - perturbation density (viii) \( \beta \) - stratification parameter (ix) \( N \) - Brunt – Vaisala frequency \( N \)  
\[ N^2 = \beta g \]  
(x) \( g \) – gravity (xii) \( v_0 \) - initial average velocity , \( u_0 = v_0 h_0/L \). Using mathematical simplification process on (2), (3) and (6) we get

For Case (a)
\[ \rho_0' \left( \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) = \mu \frac{\partial}{\partial t} \nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \rho_0' N^2 e^{-\beta y} \frac{\partial v}{\partial x} \]  
(9)

For Case (b)
\[ \rho_0' \left( \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) = \mu \frac{\partial}{\partial t} \nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \rho_0' N^2 \frac{\partial v}{\partial x} \]  
(10)

The time component in both velocity and pressure function are chosen based of the fluid oscillation in the following manner

\[ u(x, y, t) = u(x, y)e^{i\omega t}, v(x, y, t) = V(x, y)e^{i\omega t}, \text{and } p(x, y, t) = p(x, y)e^{i\omega t} \]

For Case (a)
\[ -\rho_0' \omega^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = \mu i \omega \nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \rho_0' N^2 e^{-\beta y} \frac{\partial v}{\partial x} \]  
(11)

For Case (b)
\[ -\rho_0' \omega^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = \mu i \omega \nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \rho_0' N^2 \frac{\partial v}{\partial x} \]  
(12)

We define Stream Function \( \psi \) such that

\[ u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \]  
(13)

For Case (a)
\[ \left( \nabla^2 - \frac{i \rho_0 \omega}{\mu} \right) \nabla^2 \psi - \frac{\rho_0 N^2}{\mu i \omega} e^{-\beta y} \frac{\partial^2 \psi}{\partial x^2} = 0 \]  
(14)

For Case (b)
\[ \left( \nabla^2 - \frac{i \rho_0 \omega}{\mu} \right) \nabla^2 \psi - \frac{\rho_0 N^2}{\mu i \omega} \frac{\partial^2 \psi}{\partial x^2} = 0 \]  
(15)

Where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( \psi(x, y, t) = \psi(x, y)e^{i\omega t} \)

The values of velocity on the boundary are assumed to be

\[ u = 0 \text{ at } x=0, u = u_1 \text{ at } x = h \]
\[ v = 0 \text{ at } x=0, v = 0 \text{ at } x = h \]
\[ \Psi = \left( u_0 - \frac{v_0 y}{h} \right) f(\eta) \]  

(16)

Where \( \eta = \frac{x}{h} \) and \( u_0 \) is the average entrance velocity. Equation (10) becomes

For Case (a)

\[ \left( D^4 - \frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2}{\omega} e^{-\beta y_0} \right) D^2 \right) f(\eta) = 0 \]  

(17)

Where \( D^2 \frac{d^2}{d\eta^2} \alpha^2 = \frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2 e^{-\beta y_0}}{\omega} \right) \Rightarrow \alpha = \sqrt{\frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2 e^{-\beta y_0}}{\omega} \right)} \)

For Case (b)

\[ \left( D^4 - \frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2}{\omega} \right) D^2 \right) f(\eta) = 0 \]  

(18)

Where \( D^2 \frac{d^2}{d\eta^2} \alpha^2 = \frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2}{\omega} \right) \Rightarrow \alpha = \sqrt{\frac{1}{\mu} \left( \rho_0' - \frac{i \rho_0 N^2}{\omega} \right)} \)

Equation (17) by considering particular values for \( y_0 \) and (18) reduces to a linear differential equation with constant complex coefficient. Here \( \alpha^2 \) is a complex number, hence there is only one possibility.

Two of the roots are distinct and complex and the other two are zero

\( i.e., m^2(m + \alpha)(m - \alpha) = 0 \).

\[ f(\eta) = c_1 + c_2 \eta + c_3 e^{\alpha \eta} + c_4 e^{-\alpha \eta} \]  

(19)

The transformed velocity values at the boundary in terms of \( f(\eta) \) are

\[ f(0) = 0 \; ; \; f(1) = -1 \; ; \; f'(0) = 0 \; ; \; f'(1) = 0 \]  

(20)

The equations using (20) to determine the arbitrary constants in \( f(\eta) \) are

\[ c_1 + c_3 + c_4 = 0 \]  

(21)

\[ c_1 + c_2 + c_3 e^{\alpha} + c_4 e^{-\alpha} = -1 \]  

(22)

\[ c_2 + \alpha c_3 - \alpha c_4 = 0 \]  

(23)

\[ c_2 + c_3 e^{\alpha} - \alpha c_4 e^{-\alpha} = 0 \]  

(24)

Using the expression for \( \psi \) and equation (9) after substituting the constants, the velocity components are

\[ u(x, y) = -v_1 \left( e^{\alpha - 1 - \alpha (1 + e^{\alpha})} x + e^{\frac{\alpha e^{\alpha}}{(a+2)+(a-2)e^{\alpha}}} \right) \]  

(25)

\[ v(x, y) = -\left( u_0 - \frac{v_0 y}{h} \right) \left( -\alpha (1 + e^{\alpha}) + \alpha e^{\frac{\alpha e^{\alpha}}{(a+2)+(a-2)e^{\alpha}}} \right) \]  

(26)
3. Results and Discussion

The effects of various parameters for exponential stratification and linear stratification problems on velocity profiles are detailed through graphs. The parameter values are chosen so that the results are comparable with real time results. Water is a well-known stratified fluid, hence stratification values are considered for soft water to hard water. For convenience the graphical representation of velocity are plotted in such a way that the velocity values are taken in horizontal line while the varying space parameter is considered in vertical line.

Figure 1(a): Transverse Velocity for varying Time parameter for exponential stratification case
Figure 1(b): Transverse Velocity for varying Time parameter for linear stratification case

Figure 1(a) and 1(b) explains the transverse velocity variations for time parameter increased from 0 to \( \pi \). Both Exponential stratification as well as linear stratification makes transverse velocity to decrease with time. This confirms that the flow decreases with time in vertical direction. Also, the figure indicates that the flow is comparatively faster in exponential distribution than in case of linear distribution of density. It clearly suggests us that for exponential stratification to happen we require the value of ‘h’ to be significantly larger and hence the corresponding effects can influence the flow towards the gravitational forces.

Figure 2(a): Transverse Velocity for varying Height parameter for exponential stratification case
Figure 2(b): Transverse Velocity for varying Height parameter for linear stratification case

Figure 2(a) and 2(b) depicts how transverse velocity profile is affected for different values of \( x = 0 \) to 1, while values of \( \rho = 1, \mu = 0.894 \times 10^{-3} \text{ s}^{-1} \text{ kgm}^{-1} \text{ s}^{-1}, \omega t = 0, u_0 = 10, h = 10, N = 0.5 \times 10^{-4} \text{ s}^{-1} \) and \( v_1 = 2 \). By allowing the height parameter \( y \) to vary from 0 to 0.6 with values of \( x \) substituted from 0 to 1, the figures show that as the height increases the flow decreases. As exponential stratification allows the fluid to realign a little later than linear stratification, we see that the velocity is more distributed in case (a) than that in case (b).
Figure 3(a) and (b) represents transverse velocity profiles for a large range of stratification values which are very close to each other. Clearly it is noted that in both the cases the influence on velocity happens on the centre part of the channel by reversing the flow where as nearly one third of the portion of the channel on either extreme boundary, the velocity profiles are in positive direction.

Figure 4(a) and (b) are drawn for axial velocity for varying stratification values which describes the impact of stratification on axial velocity by dividing the channel in to two diametrically opposite parts with reference to the center point, The velocity profiles are decreasing above this point whereas increasing below the same point.

4. Conclusion

The effects of Exponential stratification in comparison with Linear stratification on rectangular channel problem with a porous wall through which suction and injection takes place is presented in this paper. The momentum equation for both the cases are solved using similarity variable separable transformation. The following observations are obtained as a result of the analysis. Transverse velocity profiles have an inversing tendency with respect to time parameter. Increase in stratification parameter N results in reverse flow around the center of the channel for both cases. In case of linear stratification there is an increased nonlinear transverse flow nearer to the boundaries. In both exponential as well as linear cases axial velocity profiles are symmetric about the center point of the channel and increases with N values below the point region and decreases with N values above the point region. By omitting the second degree terms involving stratification parameter in exponential stratification case all the results reduces to that of linear stratification case which are already available in the literature.
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