Abstract. Here is a sample of the results proved in this paper: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho] \to [0, +\infty]$ be a continuous increasing function such that $\lim_{\xi \to \rho^-} \int_0^\xi \omega(x)dx = +\infty$. Consider $C^0([0, 1]) \times C^0([0, 1])$ endowed with the norm

$$
\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt.
$$

Then, the following assertions are equivalent:

(a) the restriction of $f$ to $\left[-\sqrt{\frac{\rho}{2}}, \sqrt{\frac{\rho}{2}}\right]$ is not constant;

(b) for every convex set $S \subseteq C^0([0, 1]) \times C^0([0, 1])$ dense in $C^0([0, 1]) \times C^0([0, 1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$
\begin{cases}
-\omega \left(\int_0^1 |u'(t)|^2dt\right) u'' = \beta(t)f(u) + \alpha(t) & \text{in } [0, 1] \\
u(0) = u(1) = 0 \\
\int_0^1 |u'(t)|^2dt < \rho
\end{cases}
$$

has at least two classical solutions.

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1. - Introduction

Let $H$ be a real Hilbert space. A very classical result of Efimov and Stechkin ([3]) states that if $X$ is a non-convex sequentially weakly closed subset of $H$, then there exists $y_0 \in H$ such that the restriction to $X$ of the function $x \to \|x - y_0\|$ has at least two global minima. A more precise version of such a result was obtained by I. G. Tsar’kov in [10]. Actually, he proved that any convex set dense in $H$ contains a point $y_0$ with the above property.

In the present paper, as a by product of a more general result, we get the following:

THEOREM 1.1 - Let $X \subset H$ be a non-convex sequentially weakly closed set and let $u_0 \in \text{conv}(X) \setminus X$. Then, if we put

$$
\delta := \text{dist}(u_0, X)
$$

and, for each $r > 0$,

$$
\rho_r := \sup_{\|y\| < r} \left(\text{dist}(u_0 + y, X)^2 - \|y\|^2\right),
$$

then $H$ contains a point $y_0$ with the above property.
for every convex set $S \subseteq H$ dense in $H$, for every bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbb{R}$ and for every $r$ satisfying

$$r > \frac{\rho_e - \delta^2 + \sup_X \varphi - \inf_X \varphi}{2\delta},$$

there exists $y_0 \in S$, with $\|y_0 - u_0\| < r$, such that the function $x \to \|x - y_0\|^2 + \varphi(x)$ has at least two global minima in $X$.

So, with respect to the Efimov-Stechkin-Tsar’kov result, Theorem 1.1 gives us two remarkable additional informations: a precise localization of the point $y_0$ and the validity of the conclusion not only for the function $x \to \|x - y_0\|^2$, but also for suitable perturbations of it.

Let us recall the most famous open problem in this area: if $X$ is a subset of $H$ such that, for each $y \in H$, the restriction of the function $x \to \|x - y\|$ to $X$ has a unique global minimum, is it true that the set $X$ is convex? So, Efimov-Stechkin’s result provides an affirmative answer when $X$ is sequentially weakly closed. However, it is a quite common feeling that the answer, in general, should be negative ([1], [2], [5], [8]). In the light of Theorem 1.1, we posit the following problem:

**PROBLEM 1.1.** - Let $X$ be a subset of $H$ for which there exists a bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbb{R}$ such that, for each $y \in H$, the function $x \to \|x - y\|^2 + \varphi(x)$ has a unique global minimum in $X$. Then, must $X$ be convex?

What allows us to reach the advances presented in Theorem 1.1 is our particular approach which is entirely based on the minimax theorem established in [9]. So, also the present paper can be regarded as a further ring of the chain of applications and consequences of that minimax theorem.

2. - Results

In the sequel, $X$ is a topological space and $E$ is real normed space, with topological dual $E^*$. For each $S \subseteq E^*$, we denote by $\mathcal{A}(X, S)$ (resp. $\mathcal{A}_r(X, S)$) the class of all pairs $(I, \psi)$, whith $I : X \to \mathbb{R}$ and $\psi : X \to E$, such that, for each $\eta \in S$ and each $s \in \mathbb{R}$, the set

$$\{x \in X : I(x) + \eta(\psi(x)) \leq s\}$$

is closed and compact (resp. sequentially closed and sequentially compact).

Let us start establishing the following useful proposition. $E'$ denotes the algebraic dual of $E$.

**PROPOSITION 2.1.** - Let $I : X \to \mathbb{R}$, let $\psi : X \to E$ and let $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in [0, 1]$, with $\sum_{i=1}^n \lambda_i = 1$. Then, one has

$$\sup_{\eta \in E'} \inf_{x \in X} \left(I(x) + \eta \left(\psi(x) - \sum_{i=1}^n \lambda_i \psi(x_i)\right)\right) \leq \max_{1 \leq i \leq n} I(x_i).$$

**PROOF.** Fix $\eta \in E'$. Clearly, for some $j' \in \{1, \ldots, n\}$, we have

$$\eta \left(\psi(x_{j'}) - \sum_{i=1}^n \lambda_i \psi(x_i)\right) \leq 0.$$ (1)

Indeed, if not, we would have

$$\eta(\psi(x_j)) > \sum_{i=1}^n \lambda_i \eta(\psi(x_i))$$

for each $j \in \{1, \ldots, n\}$. So, multiplying by $\lambda_j$ and summing, we would obtain

$$\sum_{j=1}^n \lambda_j \eta(\psi(x_j)) > \sum_{i=1}^n \lambda_i \eta(\psi(x_i)),$$
Then, taking (4) and (5) into account, we have
\[ \inf_{x \in X} \left( I(x) + \eta \left( \psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \right) \leq I(x_j') + \eta \left( \psi(x_j') - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \leq I(x_j') \leq \max_{1 \leq i \leq n} I(x_i) \]
and so we get the conclusion due to the arbitrariness of $\eta$.

Our main result is as follows:

**THEOREM 2.1.** - Let $I : X \to \mathbb{R}$, let $\psi : X \to E$, let $S \subseteq E^*$ be a convex set dense in $E^*$ and let $u_0 \in E$.

Then, for every bounded function $\varphi : X \to \mathbb{R}$ such that $(I + \varphi, \psi) \in A(X, S)$ and for every $r$ satisfying
\[ \sup_{x \in X} \varphi - \inf_{x \in X} \varphi < \inf_{x \in X} (I(x) + \|\psi(x) - u_0\|_r) - \sup_{\|\eta\|_{E^*} < r} \inf_{x \in X} (I(x) + \eta(\psi(x) - u_0)) , \]
there exists $\tilde{\eta} \in S$, with $\|\tilde{\eta}\|_{E^*} < r$, such that the function $I + \tilde{\eta} \circ \psi + \varphi$ has at least two global minima in $X$.

**PROOF.** Consider the function $g : X \times E^* \to \mathbb{R}$ defined by
\[ g(x, \eta) = I(x) + \eta(\psi(x) - u_0) \]
for all $(x, \eta) \in X \times E^*$. Let $B_r$ denote the open ball in $E^*$, of radius $r$, centered at 0. Clearly, for each $x \in X$, we have
\[ \sup_{\eta \in B_r} \eta(\psi(x) - u_0)) = \|\psi(x) - u_0\|_r . \]
Then, from (2) and (3), it follows
\[ \sup_{x \in X} \varphi - \inf_{x \in X} \varphi < \inf_{x \in X} \sup_{\eta \in B_r} g - \sup_{\eta \in B_r} \inf_{x \in X} g . \]
Now, consider the function $f : X \times (S \cap B_r) \to \mathbb{R}$ defined by
\[ f(x, \eta) = g(x, \eta) + \varphi(x) \]
for all $(x, \eta) \in X \times (S \cap B_r)$. Since $S$ is dense in $E^*$, the set $S \cap B_r$ is dense in $B_r$. Hence, since $g(x, \cdot)$ is continuous, we obtain
\[ \inf_{X \times B_r} \sup_{X \times \eta \in S \cap B_r} g = \inf_{X \times \eta \in S \cap B_r} \sup_{X \times \eta \in S \cap B_r} g . \]
Then, taking (4) and (5) into account, we have
\[ \sup_{S \cap B_r} \inf_{X \times \eta \in S \cap B_r} g \leq \inf_{S \cap B_r} \sup_{X \times \eta \in S \cap B_r} \varphi \leq \inf_{S \cap B_r} \sup_{X \times \eta \in S \cap B_r} g \leq \inf_{\eta \in S \cap B_r} \left( \sup_{x \in X} g(x, \eta) + \varphi(x) \right) = \inf_{X \times \eta \in S \cap B_r} \sup_{X \times \eta \in S \cap B_r} f . \]
Now, since $(I + \varphi, \psi) \in A(X, S)$ and $f$ is concave in $S \cap B_r$, we can apply Theorem 1.1 of [9]. Therefore, since (by (6)) $\sup_{S \cap B_r} \inf_{X \times f} \leq \inf_{X \times \eta \sup_{S \cap B_r}} f$, there exists of $\tilde{\eta} \in S \cap B_r$ such that the function $f(\cdot, \tilde{\eta})$ has at least two global minima in $X$ which, of course, are global minima of the function $I + \tilde{\eta} \circ \psi + \varphi$. \(\triangle\)

If we renounce to the very detailed informations contained in its conclusion, we can state Theorem 2.1 in an extremely simplified form.

**THEOREM 2.2.** - Let $I : X \to \mathbb{R}$, let $\psi : X \to E$ and let $S \subseteq E^*$ be a convex set weakly-star dense in $E^*$. Assume that $\psi(X)$ is not convex and that $(I, \psi) \in A(X, S)$.

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in $X$.

**PROOF.** Fix $u_0 \in \text{conv}(\psi(X)) \setminus \psi(X)$ and consider the function $g : X \times E^* \to \mathbb{R}$ defined by
\[ g(x, \eta) = I(x) + \eta(\psi(x) - u_0) \]
for all \((x, \eta) \in X \times E^*\). By Proposition 2.1, we know that
\[
\sup_{E^*} \inf_X g < +\infty .
\]
On the other hand, for each \(x \in X\), since \(\psi(x) \neq u_0\), we have
\[
\sup_{\eta \in E^*} \eta(\psi(x) - u_0) = +\infty .
\]
Hence, since \(S\) is weakly-star dense in \(E^*\) and \(g(x, \cdot)\) is weakly-star continuous, we have
\[
\sup_{\eta \in S} g(x, \eta) = +\infty .
\]
Therefore
\[
\sup_{S} \inf_X g < \inf_X \sup_{S} g . \tag{7}
\]
Now, taken into account that \((I, \psi) \in A(X, S)\), we can apply Theorem 1.1 of [9] to \(g|_{X \times S}\). So, in view of (7), there exists \(\tilde{\eta} \in S\) such that the function \(g(\cdot, \tilde{\eta})\) (and so \(I + \tilde{\eta} \circ \psi\)) has at least two global minima in \(X\), as claimed.

The next result is a sequential version of Theorem 1.1 of [9].

**THEOREM 2.3.** - Let \(X\) be a topological space, \(E\) a topological vector space, \(Y \subseteq E\) a non-empty separable convex set and \(f : X \times Y \to \mathbb{R}\) a function satisfying the following conditions:
(a) for each \(y \in Y\), the function \(f(\cdot, y)\) is sequentially lower semicontinuous, sequentially inf-compact and has a unique global minimum in \(X\);
(b) for each \(x \in X\), the function \(f(x, \cdot)\) is continuous and quasi-concave.

Then, one has
\[
\sup_{Y} \inf_{X} f = \inf_{Y} \sup_{X} f .
\]

**PROOF.** The pattern of the proof is the same as that of Theorem 1.1 of [9]. We limit ourselves to stress the needed changes. First, for every \(n \in \mathbb{N}\), one proves the result when \(E = \mathbb{R}^n\) and \(Y = S_n := \{(\lambda_1, ..., \lambda_n) \in ([0, +\infty[^n : \lambda_1 + ... + \lambda_n = 1\}.\) In this connection, the proof agrees exactly with that of Lemma 2.1 of [9], with the only difference of using the sequential version of Theorem 1.A of [9] instead of such a result itself (see Remark 2.1 of [9]). Next, we fix a sequence \(\{x_n\}\) dense in \(Y\). For each \(n \in \mathbb{N}\), set
\[
P_n = \text{conv}(\{x_1, ..., x_n\}) .
\]
Consider the function \(\eta : S_n \to P\) defined by
\[
\eta(\lambda_1, ..., \lambda_n) = \lambda_1 x_1 + ... + \lambda_n x_n
\]
for all \((\lambda_1, ..., \lambda_n) \in S_n\). Plainly, the function \((x, \lambda_1, ..., \lambda_n) \to f(x, \eta(\lambda_1, ..., \lambda_n))\) satisfies in \(X \times S_n\) the assumptions of Theorem A, and so, by the case previously proved, we have
\[
\sup_{(\lambda_1, ..., \lambda_n) \in S_n} \inf_{x \in X} f(x, \eta(\lambda_1, ..., \lambda_n)) = \inf_{x \in X} \sup_{(\lambda_1, ..., \lambda_n) \in S_n} f(x, \eta(\lambda_1, ..., \lambda_n)) .
\]
Since \(\eta(S_n) = P_n\), we then have
\[
\sup_{P_n} \inf_{X} f = \inf_{X} \sup_{P_n} f .
\]
Now, set
\[
D = \bigcup_{n \in \mathbb{N}} P_n .
\]
In view of Proposition 2.2 of [9], we have

\[
\sup_{D} \inf_{X} f = \inf_{X} \sup_{D} f .
\]

Finally, by continuity and density, we have

\[
\sup_{y \in D} f(x, y) = \sup_{y \in Y} f(x, y)
\]

for all \(x \in X\), and so

\[
\inf_{X} \sup_{Y} f = \inf_{X} \sup_{D} f = \inf_{X} \inf_{Y} f = \inf_{X} \inf_{Y} f
\]

and the proof is complete. \(\triangle\)

Reasoning as in the proof of Theorem 2.1 and using Theorem 2.3, we get

**THEOREM 2.4.** - Let the assumptions of Theorem 2.1 be satisfied. In addition, assume that \(E^*\) is separable.

Then, the conclusion of Theorem 2.1 holds with \(A_s(X, S)\) instead of \(A(X, S)\).

Analogously, the sequential version of Theorem 2.2 is as follows:

**THEOREM 2.5.** - Let \(I : X \to \mathbb{R}\), let \(\psi : X \to E\) and let \(S \subseteq E^*\) be a convex set weakly-star separable and weakly-star dense in \(E^*\). Assume that \(\psi(X)\) is not convex and that \((I, \psi) \in A_s(X, S)\).

Then, there exists \(\hat{\eta} \in S\) such that the function \(I + \hat{\eta} \circ \psi\) has at least two global minima in \(X\).

Here is a consequence of Theorem 2.1:

**THEOREM 2.6.** - Let \(E\) be a Hilbert space, let \(\psi : X \to E\) be a weakly continuous function and let \(S \subseteq E\) be a convex set dense in \(E\). Assume that \(\psi(X)\) is not convex and that the function \(\|\psi(\cdot)\|\) is inf-compact. Let \(u_0 \in \text{conv}(\psi(X)) \setminus \psi(X)\).

Then, for every bounded function \(\varphi : X \to \mathbb{R}\) such that \(\|\psi(\cdot)\| + \varphi(\cdot)\) is lower semicontinuous and for every \(r\) satisfying

\[
r > \frac{\sup_{\|y\| < r} ((\text{dist}(u_0 + y, \psi(X)))^2 - \|y\|^2) - (\text{dist}(u_0, \psi(X)))^2 + \sup_X \varphi - \inf_X \varphi}{2\text{dist}(u_0, \psi(X))},
\]

there exists \(\hat{y} \in S\), with \(\|\hat{y} - u_0\| < r\), such that the function \(\|\psi(\cdot) - \hat{y}\|^2 + \varphi(\cdot)\) has at least two global minima in \(X\).

**PROOF.** First, we observe that the set \(\psi(X)\) is sequentially weakly closed (and so norm closed). Indeed, let \(\{x_n\}\) be a sequence in \(X\) such that \(\psi(x_n)\) converges weakly to \(y \in E\). So, in particular, \(\psi(x_n)\) is bounded and hence, since \(\|\psi(\cdot)\|\) is inf-compact, there exists a compact set \(K \subseteq X\) such that \(x_n \in K\) for all \(n \in \mathbb{N}\). Since \(\psi\) is weakly continuous, the set \(\psi(K)\) is weakly compact and hence weakly closed. Therefore, \(y \in \psi(K)\), as claimed. This remark ensures that \(\text{dist}(u_0, \psi(X)) > 0\). Now, we apply Theorem 2.1 identifying \(E\) with \(E^*\) and taking

\[
I(x) = \frac{1}{2} \|\psi(x) - u_0\|^2
\]

for all \(x \in X\). Of course, we have

\[
I(x) + \langle \psi(x) - u_0, y \rangle = \frac{1}{2} \left( \|\psi(x) - u_0 + y\|^2 - \|y\|^2 \right)
\]

for all \(y \in E\). In view of (8) and (9), we have

\[
\frac{1}{2} \sup_X \varphi - \inf_X \varphi < \frac{1}{2} (\text{dist}(u_0, \psi(X)))^2 + r \text{dist}(u_0, \psi(X)) - \frac{1}{2} \sup_{\|y\| < r} ((\text{dist}(u_0 - y, \psi(X)))^2 - \|y\|^2)
\]
Let us show that $(I + \frac{1}{2} \varphi, \psi) \in \mathcal{A}(X, E)$. So, fix $y \in E$. Since $\psi$ is weakly continuous, $(\psi(\cdot), v)$ is continuous in $X$ for all $v \in E$. Observing that

$$
I(x) + \frac{1}{2} \varphi(x) + \langle \psi(x), y \rangle = \frac{1}{2} \langle \psi(x) \rangle^2 + \varphi(x) + \langle \psi(x), y - u_0 \rangle + \frac{1}{2} \|u_0\|^2 ,
$$

we infer that $I(\cdot) + \frac{1}{2} \varphi(\cdot) + (\psi(\cdot), y)$ is lower semicontinuous since $\|\psi(\cdot)\|^2 + \varphi(\cdot)$ is so by assumption. Now, let $s \in \mathbb{R}$. We readily have

$$
\left\{ x \in X : I(x) + \frac{1}{2} \varphi(x) + \langle \psi(x), y \rangle \leq s \right\} \subseteq \left\{ x \in X : \|\psi(x)\|^2 - 2\|y - u_0\| \|\psi(x)\| \leq 2s - \inf_{X} \|\psi(x)\|^2 \right\} .
$$

(11)

Since $\|\psi(\cdot)\|$ is inf-compact, the set in the right-hand side of (11) is compact and hence so is the set in left-hand right, as claimed. Since the set $u_0 - S$ is convex and dense in $E$, in view of (10), Theorem 2.1 ensures the existence of $\tilde{v} \in u_0 - S$, with $\|\tilde{v}\| < r$, such that the function $I(\cdot) + \langle \psi(\cdot), \tilde{v} \rangle + \frac{1}{2} \varphi(\cdot)$ has at least two global minima in $X$. Consequently, since

$$
I(x) + \langle \psi(x), \tilde{v} \rangle + \frac{1}{2} \varphi(x) = \frac{1}{2} \langle \psi(x) + \tilde{v} - u_0 \rangle^2 + \varphi(x) - \frac{1}{2} \langle u_0 \rangle^2 - \|\tilde{v} - u_0\|^2 ,
$$

if we put

$$
\tilde{y} := u_0 - \tilde{v} ,
$$

we have $\tilde{y} \in S$, $\|\tilde{y} - u_0\| < r$ and the function $\|\psi(\cdot) - \tilde{y}\|^2 + \varphi(\cdot)$ has at least two global minima in $X$. The proof is complete.

\textbf{Remark 2.1.} - Of course, Theorem 1.1 is an immediate corollary of Theorem 2.6: take $E = H$, consider $X$ equipped with the relative weak topology, take $\psi(x) = x$ and observe that if $\varphi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous, then $\|\cdot\|^2 + \varphi(\cdot)$ is weakly lower semicontinuous in view of the Eberlein-Šmulyan theorem.

Here is an application of Theorem 2.2. An operator $T$ between two Banach spaces $F_1, F_2$ is said to be sequentially weakly continuous if, for every sequence $\{x_n\}$ in $F_1$ weakly convergent to $x \in F_1$, the sequence $\{T(x_n)\}$ converges weakly to $T(x)$ in $F_2$.

\textbf{Theorem 2.7.} - Let $V$ be a reflexive real Banach space, let $x_0 \in V$, let $r > 0$, let $X$ be the open ball in $V$, of radius $r$, centered at $x_0$, let $\gamma : [0, r] \to \mathbb{R}$, with $\lim_{\xi \to -} \gamma(\xi) = +\infty$, let $I : X \to \mathbb{R}$ and $\psi : X \to E$ be two Gâteaux differentiable functions. Moreover, assume that $I$ is sequentially weakly lower semicontinuous, that $\psi$ is sequentially weakly continuous, that $\psi(X)$ is bounded and non-convex, and that

$$
\gamma(\|x - x_0\|) \leq I(x)
$$

for all $x \in X$.

Then, for every convex set $S \subseteq E^*$ weakly-star dense in $E^*$, there exists $\tilde{y} \in S$ such that the equation

$$
I'(x) + (\tilde{y} \circ \psi)'(x) = 0
$$

has at least two solutions in $X$.

\textbf{Proof.} We apply Theorem 2.2 considering $X$ equipped with the relative weak topology. Let $\eta \in E^*$. Since $\psi(X)$ is bounded, we have $c := \inf_{x \in X} \eta(\psi(x)) > -\infty$. Let $s \in \mathbb{R}$. We have

$$
\{x \in X : I(x) + \eta(\psi(x)) \leq s\} \subseteq \{x \in X : I(x) \leq s - c\} \subseteq \{x \in X : \gamma(\|x - x_0\|) \leq s - c\} .
$$

(12)
Since \( \lim_{t \to \delta} \gamma(t) = +\infty \), there is \( \delta \in [0, r] \), such that \( \gamma(\xi) > s - c \) for all \( \xi \in [\delta, r] \). Consequently, from (12), we obtain

\[
\{ x \in X : I(x) + \eta(\psi(x)) \leq s \} \subseteq \{ x \in V : \|x - x_0\| \leq \delta \}.
\] (13)

From the assumptions, it follows that the function \( I + \eta \circ \psi \) is sequentially weakly lower semicontinuous in \( X \). Hence, from (13), since \( \delta < r \) and \( V \) is reflexive, we infer that the set \( \{ x \in X : I(x) + \eta(\psi(x)) \leq s \} \) is sequentially weakly compact and hence weakly compact, by the Eberlein-Smulian theorem. In other words, \( (I, \psi) \in \mathcal{A}(X, E^*) \). Therefore, we can apply Theorem 2.2. Accordingly, there exists \( \tilde{\eta} \in S \) such that the function \( I + \tilde{\eta} \circ \psi \) has at least two global minima in \( X \) which are critical points of it since \( X \) is open. \( \triangle \)

Here is an application of Theorem 1.1:

**THEOREM 2.8.** - Let \( H \) be a Hilbert space and let \( I, J : H \to \mathbf{R} \) be two \( C^1 \) functionals with compact derivative such that \( 2I - J^2 \) is bounded. Moreover, assume that \( J(0) \neq 0 \) and that there is \( \hat{x} \in H \) such that \( J(-\hat{x}) = -J(\hat{x}) \).

Then, for every convex set \( S \subseteq H \times \mathbf{R} \) dense in \( H \times \mathbf{R} \) and for every \( r \) satisfying

\[
r > \frac{\|\hat{x}\|^2 + |J(\hat{x})|^2 - \inf_{x \in H}(\|x\|^2 + |J(x)|^2) + \sup_{x \in H}(2I - J^2) - \inf_X(2I - J^2)}{2 \inf_{x \in H} \sqrt{\|x\|^2 + |J(x)|^2}},
\]

there exists \( (y_0, \mu_0) \in S \), with \( \|y_0\|^2 + |\mu_0|^2 < r^2 \), such that the equation

\[
x + I'(x) + \mu_0 J'(x) = y_0
\]

has at least three solutions.

**PROOF.** We consider the Hilbert space \( E := H \times \mathbf{R} \) with the scalar product

\[
\langle (x, \lambda), (y, \mu) \rangle_E = \langle x, y \rangle + \lambda \mu
\]

for all \( (x, \lambda), (y, \mu) \in E \). Take

\[
X = \{ (x, \lambda) \in E : \lambda = J(x) \}.
\]

Since \( J' \) is compact, the functional \( J \) turns out to be sequentially weakly continuous ([11], Corollary 41.9). So, the set \( X \) is sequentially weakly closed. Moreover, notice that \( (0, 0) \notin X \), while the antipodal points \( (\hat{x}, J(\hat{x})) \) and \(- (\hat{x}, J(\hat{x})) \) lie in \( X \). So, \( (0, 0) \in \text{conv}(X) \). Now, with the notations of Theorem 1.1, taking, of course, \( u_0 = (0, 0) \), we have

\[
\delta = \inf_{x \in X} \sqrt{\|x\|^2 + |J(x)|^2}
\]

and

\[
\rho_r = \sup_{\|y\|^2 + |\mu|^2 < r^2} \inf_{x \in X} (\|x\|^2 + |J(x)|^2 - 2\langle (x, J(x)), (y, \mu) \rangle_E).
\]

Then, from Proposition 2.1, we infer that

\[
\rho_r \leq \|\hat{x}\|^2 + |J(\hat{x})|^2.
\]

Now, consider the function \( \varphi : X \to \mathbf{R} \) defined by

\[
\varphi(x, \lambda) = 2I(x) - \lambda^2
\]

for all \( (x, \lambda) \in X \). Notice that \( \varphi \) is sequentially weakly continuous and \( r \) satisfies the inequality of Theorem 1.1. Consequently, there exists \( (y_0, \mu_0) \in S \) such that the functional

\[
(x, \lambda) \to \| (x, \lambda) \|^2_E - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2
\]

has at least two global minima in \( X \). Of course, if \( (x, \lambda) \in X \), we have

\[
\| (x, \lambda) \|^2_E - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2 = \|x\|^2 + J^2(x) - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) - J^2(x).
\]
In other words, the functional \( x \mapsto \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) \) has at least three global minima in \( H \). Since the functional \( x \mapsto -2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) \) has a compact derivative, a well known result ([11], Example 3.25) ensures that the functional \( x \mapsto \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) \) has the Palais-Smale property and so, by Corollary 1 of [6], it possesses at least three critical points. The proof is complete. \( \triangle \)

REMARK 2.2. - In Theorem 2.8, apart from being \( C^1 \) with compact derivative, the truly essential assumption on \( J \) is, of course, that its graph is not convex. This amounts to say that \( J \) is not affine. The current assumptions are made to simplify the constants appearing in the conclusion. Actually, from the proof of Theorem 2.8, the following can be obtained:

THEOREM 2.9. - Let \( H \) be a Hilbert space and let \( I, J : H \to \mathbb{R} \) be two \( C^1 \) functionals with compact derivative such that \( 2I - J^2 \) is bounded. Moreover, assume that \( J \) is not affine.

Then, for every convex set \( S \subseteq H \times \mathbb{R} \) dense in \( H \times \mathbb{R} \), there exists \((y_0, \lambda_0) \in S\) such that the equation
\[
x + I'(x) + \lambda_0 J'(x) = y_0
\]
has at least three solutions.

REMARK 2.3. - For \( I = 0 \), the conclusion of Theorem 2.9 can be obtained from Theorem 4 of [7] (see also [4]) provided that, for some \( r \in \mathbb{R} \), the set \( J^{-1}(r) \) is not convex. Therefore, for instance, the fact that, for any non-constant bounded \( C^1 \) function \( J : \mathbb{R} \to \mathbb{R} \), there are \( a, b \in \mathbb{R} \) such that the equation
\[
x + aJ'(x) = b
\]
has at least three solutions, follows, in any case, from Theorem 2.9, while it follows from Theorem 4 of [7] only if \( J \) is not monotone.

We conclude presenting an application of Theorem 2.7 to a class of Kirchhoff-type problems.

THEOREM 2.10. - Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function, let \( \rho > 0 \) and let \( \omega : [0, \rho] \to [0, +\infty[ \) be a continuous increasing function such that \( \lim_{\xi \to \rho^-} \int_0^\xi \omega(x)dx = +\infty \). Consider \( C^0([0,1]) \times C^0([0,1]) \) endowed with the norm
\[
\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt .
\]

Then, the following assertions are equivalent:

(a) the restriction of \( f \) to \([-\frac{1}{\omega}, \frac{1}{\omega}] \) is not constant ;

(b) for every convex set \( S \subseteq C^0([0,1]) \times C^0([0,1]) \) dense in \( C^0([0,1]) \times C^0([0,1]) \), there exists \( (\alpha, \beta) \in S \) such that the problem
\[
\begin{cases}
-\omega \int_0^1 |u'(t)|^2dt u'' = \beta(t)f(u) + \alpha(t) & \text{in } [0,1] \\
u(0) = u(1) = 0 \\
\int_0^1 |u'(t)|^2dt < \rho
\end{cases}
\]
has at least two classical solutions.

PROOF. Consider the Sobolev space \( H^1_0([0,1]) \) with the usual scalar product
\[
\langle u, v \rangle = \int_0^1 u'(t)v'(t)dt .
\]
Let \( B_{\sqrt{\rho}} \) be the open ball in \( H^1_0([0,1]) \), of radius \( \sqrt{\rho} \), centered at \( 0 \). Let \( g : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Consider the functionals \( I, J_g : B_{\sqrt{\rho}} \to \mathbb{R} \) defined by
\[
I(u) = \frac{1}{2} \omega \int_0^1 |u'(t)|^2dt ,
\]
and
On the other hand, we have
\[ C = E \alpha, \beta \]\nfor all \( \alpha, \beta \). Notice that
\[ g(\xi) = f(\xi) \]
for all \( \xi \). Indeed, let
\[ g(\xi) = f(\xi) \]
and hence, thanks to Theorem 2.7, there exists \( \alpha, \beta \) such that, if we put
\[ g(t, \xi) = \alpha(t) + \beta(t) f(\xi), \]
for all \( t \in [0,1] \). By classical results, taking into account that if \( \omega(x) = 0 \) then \( x = 0 \), it follows that the classical solutions of the problem
\[
\begin{cases}
-\omega \left( \int_0^1 |u'(t)|^2 dt \right) u'' = g(t, u) & \text{in } [0,1] \\
u(0) = u(1) = 0 \\
\int_0^1 |u'(t)|^2 dt < \rho
\end{cases}
\]
are exactly the critical points in \( B_{\gamma} \) of the functional \( I - J_g \).

Let us prove that \( (a) \rightarrow (b) \). We are going to apply Theorem 2.7 taking \( V = H_0^1([0,1]) \), \( x_0 = 0, r = \sqrt{p} \), \( I \) as above, \( \gamma(x) = \frac{1}{2} \omega(\xi^2) \), \( E = C^0([0,1]) \times C^0([0,1]) \) and \( \psi : B_{\gamma} \rightarrow E \) defined by
\[ \psi(u)(\cdot) = (u(\cdot), \tilde{f}(u(\cdot))) \]
for all \( u \in B_{\gamma} \), where \( \tilde{f}(\xi) = \int_0^\xi f(x) dx \). Clearly, the functional \( I \) is continuous and strictly convex (and so weakly lower semicontinuous), while the operator \( \psi \) is Gâteaux differentiable and sequentially weakly continuous due to the compact embedding of \( H_0^1([0,1]) \) into \( C^0([0,1]) \). Recall that
\[
\max_{[0,1]} |u| \leq \frac{1}{2} \sqrt{\int_0^1 |u'(t)|^2 dt}
\]
for all \( u \in H_0^1([0,1]) \). As a consequence, the set \( B_{\gamma} \) is bounded and, in view of \( (a) \), non-convex. Hence, each assumption of Theorem 2.7 is satisfied. Now, consider the operator \( T : E \rightarrow E^* \) defined by
\[
T(\alpha, \beta)(u,v) = \int_0^1 \alpha(t) u(t) dt + \int_0^1 \beta(t) v(t) dt
\]
for all \( (\alpha, \beta), (u,v) \in E \). Of course, \( T \) is linear and the linear subspace \( T(E) \) is total over \( E \). Hence, \( T(E) \) is weakly-star dense in \( E^* \). Moreover, notice that \( T \) is continuous with respect to the weak-star topology of \( E^* \). Indeed, let \( (\alpha_n, \beta_n) \) be a sequence in \( E \) converging to some \( (\alpha, \beta) \in E \). Fix \( (u,v) \in E \). We have to show that
\[
\lim_{n \to \infty} T(\alpha_n, \beta_n)(u,v) = T(\alpha, \beta)(u,v). \tag{14}
\]
Notice that
\[
\lim_{n \to \infty} \left( \int_0^1 |\alpha_n(t) - \alpha(t)| dt + \int_0^1 |\beta_n(t) - \beta(t)| dt \right) = 0. \tag{15}
\]
On the other hand, we have
\[
|T(\alpha_n, \beta_n)(u,v) - T(\alpha, \beta)(u,v)| = \left| \int_0^1 (\alpha_n(t) - \alpha(t)) u(t) dt + \int_0^1 (\beta_n(t) - \beta(t)) v(t) dt \right|
\leq \left( \int_0^1 |\alpha_n(t) - \alpha(t)| dt + \int_0^1 |\beta_n(t) - \beta(t)| dt \right) \max \left\{ \max_{[0,1]} |u|, \max_{[0,1]} |v| \right\}
\]
and hence (14) follows in view of (15). Finally, fix a convex set \( S \subseteq C^0([0,1]) \times C^0([0,1]) \) dense in \( C^0([0,1]) \times C^0([0,1]) \). Then, by the kind of continuity of \( T \) just now proved, the convex set \( T(-S) \) is weakly-star dense in \( E^* \) and hence, thanks to Theorem 2.7, there exists \( (\alpha_0, \beta_0) \in -S \) such that, if we put
\[
g(t, \xi) = \alpha_0(t) + \beta_0(t) f(\xi),
\]
the functional $I - J_g$ has at least two critical points in $B_{\sqrt{\rho}}$ which are the claimed solutions of the problem in (b), with $\alpha = -\alpha_0$ and $\beta = -\beta_0$.

Now, let us prove that (b) $\rightarrow$ (a). Assume that the restriction of $f$ to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is constant. Let $c$ be such a value. So, the classical solutions of the problem

\[
\begin{array}{l}
-\omega \left( \int_0^1 |u'(t)|^2 dt \right) u'' = c\beta(t) + \alpha(t) \quad \text{in } [0, 1] \\
u(0) = u(1) = 0 \\
\int_0^1 |u'(t)|^2 dt < \rho
\end{array}
\]

are the critical points in $B_{\sqrt{\rho}}$ of the functional $u \rightarrow \frac{1}{2} \omega \left( \int_0^1 |u'(t)|^2 dt \right) - \int_0^1 (c\alpha(t) + \beta(t))u(t)dt$. But, since $\omega$ is increasing and non-negative, this functional is strictly convex and so it possesses a unique critical point. The proof is complete.

\[\triangle\]

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