Modules of infinite projective dimension

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Abstract

We characterize the modules of infinite projective dimension over the endomorphism algebras of Opperman-Thomas cluster tilting objects \( X \) in \((n + 2)\)-angulated categories \((C, \Sigma^n, \Theta)\). For an indecomposable object \( M \) of \( C \), we define in this article the ideal \( I_M \) of \( \text{End}_C(\Sigma^n X) \) given by all endomorphisms that factor through \( \text{add} M \), and show that the \( \text{End}_C(X) \)-module \( \text{Hom}_C(X, M) \) has infinite projective dimension precisely when \( I_M \) is non-zero. As an application, we generalize a recent result by Beaudet-Brüstle-Todorov for cluster-tilted algebras.

Key words: \((n + 2)\)-angulated categories; cluster tilting objects; projective dimension.

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1 Introduction

Cluster-tilting theory sprouted from the categorification of Fomin-Zelevinsky’s cluster algebras. It is used to construct abelian categories from some triangulated categories. By Buan-Marsh-Reiten [BMR] Theorem 2.2 in the case of cluster categories, by Keller-Reiten [KR] Proposition 2.1 in the 2-Calabi-Yau case, then by Koenig-Zhu [KZ] Theorem 3.3 and Iyama-Yoshino [IY] Corollary 6.5 in the general case, one can pass from triangulated categories to abelian categories by factoring out cluster tilting subcategories.

Recall that the notion of cluster-tilting subcategory which is due to Iyama [I]. Let \( C \) be a triangulated category with shift functor \( \Sigma \). A subcategory \( \mathcal{X} \) of \( C \) is called \textit{cluster tilting} if it satisfies the following conditions:

1. \( \mathcal{X} \) is contravariantly finite and covariantly finite in \( C \).
2. \( X \in \mathcal{X} \) if and only if \( \text{Hom}_C(X, \Sigma \mathcal{X}) = 0 \), i.e. \( \text{Hom}_C(X, \Sigma M) = 0 \), for any \( M \in \mathcal{X} \).
3. \( X \in \mathcal{X} \) if and only if \( \text{Hom}_C(\mathcal{X}, \Sigma X) = 0 \), i.e. \( \text{Hom}_C(M, \Sigma X) = 0 \), for any \( M \in \mathcal{X} \).

An object \( X \) is called \textit{cluster tilting}, if \( \text{add} X \) is cluster tilting, where \( \text{add} X \) is the subcategory of \( C \) consisting of direct summands of direct sum of finitely many copies of \( X \).

In fact, Koenig and Zhu [KZ] Lemma 3.2 show that \( \mathcal{X} \) is cluster tilting if and only if \( \text{Hom}_C(\mathcal{X}, \Sigma \mathcal{X}) = 0 \), i.e. \( \text{Hom}_C(X, \Sigma Y) = 0 \), for any \( X, Y \in \mathcal{X} \), and for any object \( C \in C \), there exists a triangle \( X_0 \rightarrow X_1 \rightarrow C \rightarrow \Sigma X_0 \) where \( X_0, X_1 \in \mathcal{X} \).

Let \( C \) be a triangulated category with a shift functor \( \Sigma \), and \( X \in C \) be a cluster tilting object. For any object \( M \in C \), we denote by \( I_M \) the ideal of \( \Gamma := \text{End}_C(\Sigma X) \) (\( \Gamma \) is called

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a cluster-tilted algebra) given by all endomorphisms that factor through $\text{add} M$ and call it \textit{factorization ideal} of $M$. Beaudet-Brüstle-Todorov showed the following.

\textbf{Theorem 1.1.} [BBT, Theorem 1.1] \textit{Let $C$ be a triangulated category with a shift functor $\Sigma$ and $X \in C$ be a cluster tilting object. Let $M$ be an indecomposable object in $C$ which does not belong to $\text{add} \Sigma X$. Then the $\text{End}_C(X)$-module $\text{Hom}_C(X, M)$ is of infinite projective dimension if and only if the factorization ideal $I_M$ is non-zero.}

Later, Liu and Xiu [LX, Theorem 3.1] generalized the work of Beaudet-Brüstle-Todorov for cluster-tilted algebras to the endomorphism algebras of maximal rigid objects in 2-Calabi-Yau triangulated categories.

Recently, Geiss, Keller and Oppermann introduced in [GKO] a new type of categories, called $(n+2)$-angulated categories, which generalize triangulated categories: the classical triangulated categories are the special case $n = 1$. These categories appear for instance when considering certain $n$-cluster tilting subcategories of triangulated categories.

The notion of cluster tilting objects can be generalised to $(n + 2)$-angulated categories, which is due to Oppermann and Thomas [OT, Definition 5.3].

\textbf{Definition 1.2.} [OT, Definition 5.3] \textit{Let $C$ be an $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$. An object $X \in C$ is called \textit{Opperman-Thomas cluster tilting} if}

\begin{enumerate}
  \item $\text{Hom}_C(X, \Sigma^n X) = 0$.
  \item For any object $C \in C$, there exists an $(n + 2)$-angle

$$X_0 \to X_1 \to \cdots \to X_n \to C \to \Sigma^n X_0$$

where $X_0, X_1, \cdots, X_n \in \text{add} X$.
\end{enumerate}

Let $C$ be an $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$, and $X \in C$ be an Opperman-Thomas cluster tilting object. For any object $M \in C$, we denote by $I_M$ the ideal of $\text{End}_C(\Sigma^n X)$ given by all endomorphisms that factor through $M$ and call it \textit{factorization ideal} of $M$. Our main result is the following, which is a generalization of Beaudet-Brüstle-Todorov result.

\textbf{Theorem 1.3.} (See Theorem 3.1 for details) \textit{Let $C$ be an $(n + 2)$-angulated category with an $n$-suspension functor $\Sigma^n$ and $X \in C$ be an Opperman-Thomas cluster tilting object. Let $M$ be an indecomposable object in $C$ which does not belong to $\text{add} \Sigma^n X$. Then the $\text{End}_C(X)$-module $\text{Hom}_C(X, M)$ is of infinite projective dimension if and only if the factorization ideal $I_M$ is non-zero.}

This article is organised as follows: In Section 2, we review some elementary definitions and facts of $(n + 2)$-angulated categories. In Section 3, we prove our main result.

\section{Preliminaries}

In this section, we briefly recall the definition and basic properties of $(n + 2)$-angulated categories from [GKO]. Let $C$ be an additive category with an automorphism $\Sigma^n : C \to C$, and $n$
an integer greater than or equal to one.

An \((n + 2)\)-\(\Sigma^n\)-sequence in \(C\) is a sequence of objects and morphisms

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma^n A_0.
\]

Its left rotation is the \((n + 2)\)-\(\Sigma^n\)-sequence

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma^n A_0 \xrightarrow{(\!\!-\!)^n f_0} \Sigma^n A_1.
\]

A morphism of \((n + 2)\)-\(\Sigma^n\)-sequences is a sequence of morphisms \(\varphi = (\varphi_0, \varphi_1, \cdots, \varphi_{n+1})\) such that the following diagram commutes

\[
\begin{array}{cccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & A_{n+1} & \xrightarrow{f_{n+1}} & \Sigma^n A_0 \\
\downarrow{\varphi_0} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \cdots & & \downarrow{\varphi_{n+1}} & & \downarrow{\Sigma^n \varphi_0} \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_n} & B_{n+1} & \xrightarrow{g_{n+1}} & \Sigma^n B_0
\end{array}
\]

where each row is an \((n + 2)\)-\(\Sigma^n\)-sequence. It is an isomorphism if \(\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{n+1}\) are all isomorphisms in \(C\).

**Definition 2.1.** [GKO, Definition 2.1] An \(n\)-angulated category is a triple \((C, \Sigma^n, \Theta)\), where \(C\) is an additive category, \(\Sigma^n\) is an automorphism of \(C\) (\(\Sigma^n\) is called the \(n\)-suspension functor), and \(\Theta\) is a class of \((n + 2)\)-\(\Sigma^n\)-sequences (whose elements are called \((n + 2)\)-angles), which satisfies the following axioms:

1. **(N1)** (a) The class \(\Theta\) is closed under isomorphisms, direct sums and direct summands.
   
   (b) For each object \(A \in C\) the trivial sequence
   
   \[
   A \xrightarrow{1_A} A \to 0 \to 0 \to \cdots \to 0 \to \Sigma^n A
   \]
   
   belongs to \(\Theta\).

   (c) Each morphism \(f_0: A_0 \to A_1\) in \(C\) can be extended to an \((n + 2)\)-angle:
   
   \[
   A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma^n A_0.
   \]

2. **(N2)** An \((n + 2)\)-\(\Sigma^n\)-sequence belongs to \(\Theta\) if and only if its left rotation belongs to \(\Theta\).

3. **(N3)** For each solid commutative diagram

   \[
   \begin{array}{cccccccccc}
   A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & A_{n+1} & \xrightarrow{f_{n+1}} & \Sigma^n A_0 \\
   \downarrow{\varphi_0} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \cdots & & \downarrow{\varphi_{n+1}} & & \downarrow{\Sigma^n \varphi_0} \\
   B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_n} & B_{n+1} & \xrightarrow{g_{n+1}} & \Sigma^n B_0
   \end{array}
   \]

   with rows in \(\Theta\), the dotted morphisms exist and give a morphism of \((n + 2)\)-angles

4. **(N4)** In the situation of (N3), the morphisms \(\varphi_2, \varphi_3, \cdots, \varphi_{n+1}\) can be chosen such that the
mapping cone

\[ A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \phi_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \phi_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n+1} & 0 \\ \phi_{n+1} & g_n \end{pmatrix}} \Sigma^n A_0 \oplus B_{n+1} \xrightarrow{\begin{pmatrix} -\Sigma^n f_0 & 0 \\ \Sigma^n \phi_1 & g_{n+1} \end{pmatrix}} \Sigma^n A_1 \oplus \Sigma^n B_0 \]

belongs to \( \Theta \).

Now we give an example of \((n+2)\)-angulated categories.

**Example 2.2.** We recall the standard construction of an \((n+2)\)-angulated category given by Geiss-Keller-Oppermann [GKO] Theorem 1]. Let \( C \) be a triangulated category and \( T \) an \( n \)-cluster tilting subcategory which is closed under \( \Sigma^n \), where \( \Sigma \) is the shift functor of \( C \). Then \((T, \Sigma^n, \Theta)\) is an \((n+2)\)-angulated category, where \( \Theta \) is the class of all sequences

\[ A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma^n A_0 \]

such that there exists a diagram

\[ \begin{array}{ccc}
A_0 & \xrightarrow{f_0} & A_1 \\
& \searrow & \searrow \\
& A_2 & \cdots & A_n \xrightarrow{f_n} A_{n+1}
\end{array} \]

with \( A_i \in T \) for all \( i \in \mathbb{Z} \), such that all oriented triangles are triangles in \( C \), all non-oriented triangles commute, and \( f_{n+1} \) is the composition along the lower edge of the diagram.

From now on to the end of this section, we assume that \( C \) is an \((n+2)\)-angulated category with an \( n \)-suspension functor \( \Sigma^n \), \( X \in C \) is an Opperman-Thomas cluster tilting object and \( \Gamma := \text{End}_C(X) \) is the endomorphism algebra of \( X \). Recall the following result.

**Theorem 2.3.** [JJ1, Theorem 0.5] and [ZZ, Theorem 3.8] Consider the essential image \( \mathcal{D} \) of the functor \( \text{Hom}_C(X, -) : C \to \text{mod} \Gamma \). Then \( \mathcal{D} \) is an \( n \)-cluster tilting subcategory of \( \text{mod} \Gamma \) where \( \text{mod} \Gamma \) is the category of finite dimensional right \( \Gamma \)-modules. There exists a commutative diagram, as shown below, where the vertical arrow is the quotient functor and the diagonal arrow is an equivalence of categories:

\[ \begin{array}{ccc}
C & \xrightarrow{\text{Hom}_C(X, -)} & \mathcal{D} \\
\downarrow & \searrow \downarrow & \\
C/\text{add} \Sigma^n X. & \xrightarrow{?} & \\
\end{array} \]

Moreover, \( \Gamma \) is an \( n \)-Gorenstein algebra, that is, each injective module has projective dimension \( \leq n \), and each projective module has injective dimension \( \leq n \).

**Remark 2.4.** [JJ1, Lemma 2.1] The classic \text{add-proj}-correspondence holds, as the functor
Hom\(_C(X, -)\) restricts to an equivalence \(\text{add} X \rightarrow \text{proj} \Gamma\) where \(\text{proj} \Gamma\) is the category of projective finite dimensional right \(\Gamma\)-modules.

**Corollary 2.5.** Each \(\Gamma\)-module is either of infinite projective dimension or of projective dimension at most \(n\).

**Proof.** Suppose that \(M\) is a \(\Gamma\)-module of finite projective dimension and \(M\) is of projective dimension \(m\). Then there exists an exact sequence:

\[
0 \longrightarrow \Omega^m M \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0
\]

where \(P_0, P_1, \ldots, P_m\) are projective modules. Since \(M\) is of projective dimension \(m\), we have that \(\Omega^m M\) is projective module. By Theorem 2.3, we know that \(\Gamma\) is an \(n\)-Gorenstein algebra. It follows that \(\Omega^m M\) and \(P_m\) are of injective dimension at most \(n\). Thus \(\Omega^{m-1} M\) is of injective dimension at most \(n\). By induction on \(t \geq 0\), we find that \(\Omega^{m-t} M\) is of injective dimension at most \(n\). In particular, \(M\) is of injective dimension at most \(n\). Dually, one shows that if \(N\) is a \(\Gamma\)-module of finite injective dimension, then \(N\) is of projective dimension at most \(n\). Hence we get that \(M\) is of projective dimension at most \(n\). \(\square\)

**Lemma 2.6.** [JJ2, Lemma 2.2] If \(M \in \mathcal{C}\) has no direct summands in \(\text{add} \Sigma^n X\), then there exists an \((n+2)\)-angle

\[
X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow M \rightarrow \Sigma^n X_0
\]

in \(\mathcal{C}\) with the following properties: Each \(X_i\) is in \(\text{add} X\), and applying the functor \(\text{Hom}_{\mathcal{C}}(X, -)\) gives a complex

\[
\text{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \text{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{C}}(X, X_n) \rightarrow \text{Hom}_{\mathcal{C}}(X, M) \rightarrow 0
\]

which is the start of the augmented minimal projective resolution of \(\text{Hom}_{\mathcal{C}}(X, M)\).

## 3 Proof of the Main Theorem

Now we begin to prove the main result of this article.

**Theorem 3.1.** Let \(\mathcal{C}\) be an \((n+2)\)-angulated category with an Opperman-Thomas cluster tilting object \(X\), and \(\Gamma\) be the endomorphism algebra of \(X\). Let \(M\) be an indecomposable object in \(\mathcal{C}\) which does not belong to \(\text{add} \Sigma^n X\). Then the \(\Gamma\)-module \(\text{Hom}_{\mathcal{C}}(X, M)\) is of infinite projective dimension if and only if the factorization ideal \(I_M\) is non-zero.

**Proof.** Since \(M\) is an indecomposable object in \(\mathcal{C}\) which does not belong to \(\text{add} \Sigma^n X\), then there exists an \((n+2)\)-angle

\[
X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \rightarrow \cdots \rightarrow X_n \xrightarrow{\theta} M \xrightarrow{\beta} \Sigma^n X_0
\]  \(\text{(3.1)}\)
which satisfies the properties of Lemma 2.6.

First, we assume $\Gamma$-module $\text{Hom}_C(X, M)$ has infinite projective dimension. Applying the functor $\text{Hom}_C(X, -)$ to the $(n + 2)$-angle (3.1), we have the following exact sequence in $\text{mod}\Gamma$:

$$
(X, \Sigma^{-n}M) \xrightarrow{(X, (-1)^n\Sigma^{-n}\beta)} (X, X_0) \to (X, X_1) \to \cdots \to (X, X_n) \to (X, M) \to 0
$$

where we omitted $\text{Hom}_C$ because of lack of space. It follows that the morphism

$$
\text{Hom}_C(X, (-1)^n\Sigma^{-n}\beta) \neq 0,
$$

since otherwise the projective dimension of $\text{Hom}_C(X, M)$ would be at most $n$. Choose a morphism $(-1)^n\Sigma^{-n}\gamma$ in $\text{Hom}_C(X, \Sigma^{-n}M)$ whose image under $\text{Hom}_C(X, (-1)^n\Sigma^{-n}\beta)$ is non-zero, that is, the composition

$$
X \xrightarrow{(-1)^n\Sigma^{-n}\gamma} \Sigma^{-n}M \xrightarrow{(-1)^n\Sigma^{-n}\beta} X
$$

is non-zero. This yields the non-zero composition

$$
\Sigma^nX \xrightarrow{\gamma} M \xrightarrow{\beta} \Sigma^nX.
$$

Hence there exists a non-zero element $\beta\gamma$ in the factorization ideal $I_M$.

Conversely, if the factorization ideal $I_M$ is non-zero, we prove that the $\Gamma$-module $\text{Hom}_C(X, M)$ is not of projective dimension at most $n$. Otherwise, applying the functor $\text{Hom}_C(X, -)$ to the $(n + 2)$-angle (3.1), we have the following exact sequence in $\text{mod}\Gamma$:

$$
0 \to \text{Hom}_C(X, X_0) \xrightarrow{\text{Hom}_C(X, \alpha_0)} \text{Hom}_C(X, X_1) \to \cdots \to \text{Hom}_C(X, X_n) \to \text{Hom}_C(X, M) \to 0.
$$

Let $ba: \Sigma^nX \xrightarrow{a} M \xrightarrow{b} \Sigma^nX$ be any element in $I_M$. Since $\text{Hom}_C(X, \Sigma^nX) = 0$, we obtain that the morphism $b\theta \in \text{Hom}_C(X_n, \Sigma^nX)$ is zero. Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{\alpha} & \Sigma^nX \\
\downarrow & & \downarrow a \\
X_n & \xrightarrow{\theta} & M \\
\downarrow b & & \downarrow \\
0 & \xrightarrow{\beta} & \Sigma^nX \\
\end{array}
$$

By (N3), there exist morphisms $c: X \to X_0$ and $d: X_0 \to X$ which make the following diagram commutative:

$$
\begin{array}{cccccccccccc}
X & \xrightarrow{0} & \cdots & \xrightarrow{0} & \xrightarrow{0} & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & X \\
\downarrow c & \cdots & \cdots & \cdots & \downarrow a & \cdots & \cdots & \cdots & \cdots & \downarrow \Sigma^n\alpha & \cdots & \cdots & \cdots & \downarrow \\
X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{X_{n-1}} & X_n & \xrightarrow{\theta} & M & \xrightarrow{\beta} & \Sigma^nX_0 \\
\downarrow d & \cdots & \cdots & \cdots & \downarrow b & \cdots & \cdots & \cdots & \cdots & \downarrow \Sigma^n\beta & \cdots & \cdots & \cdots & \downarrow \\
X & \xrightarrow{0} & \cdots & \xrightarrow{0} & \xrightarrow{0} & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & \cdots & \xrightarrow{\Sigma^nX} & X
\end{array}
$$
Thus $\alpha_0c = 0$ and then $\text{Hom}_C(X, c) = 0$ since $\text{Hom}_C(X, \alpha_0)$ is a monomorphism. Since the functor $\text{Hom}_C(X, -)$ is faithful, we have $c = 0$ implies $\Sigma^n c = 0$. Therefore $ba = \Sigma^n d \circ \Sigma^n c = 0$.

It follows that $I_M = 0$, which is a contradiction to our assumption. By Corollary 2.5 we get that the $\Gamma$-module $\text{Hom}_C(X, M)$ has infinite projective dimension. This finishes the proof.

As a special case of Theorem 3.1 when $n = 1$, we have the following.

**Corollary 3.2.** [BBT] Theorem 1.1] Let $C$ be a triangulated category with a cluster tilting object $X$, and $\Gamma$ be the endomorphism algebra of $X$. Let $M$ be an indecomposable object in $C$ which does not belong to $\text{add}\Sigma X$. Then the $\Gamma$-module $\text{Hom}_C(X, M)$ is of infinite projective dimension if and only if the factorization ideal $I_M$ is non-zero.

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