Self-Averaging Identities for Random Spin Systems

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Abstract

We provide a systematic treatment of self-averaging identities for various spin systems. The method is quite general, basically not relying on the nature of the model, and as a special case recovers the Ghirlanda-Guerra and Aizenman-Contucci identities, which are therefore proven, together with their extension, to be valid in a vast class of spin models. We use the dilute spin glass as a guiding example.

Key words and phrases: spin glasses, diluted spin glasses, Ghirlanda-Guerra, self-averaging.

1 Introduction

Despite many years of intense work, and the much awaited proof of the validity of the Parisi ansatz for the free-energy of the Sherrington-Kirkpatrick (SK) and related models, the mathematical comprehension of thermodynamics of mean field spin glasses remains largely incomplete. We know from theoretical physics that in fully connected models, all the properites
of the low temperature spin glass phase can be encoded in the probability distribution of the overlap between two different copies of the system. The analysis of Parisi et al. predicts an ultrametric organization of the phases (see [12] and references therein). So far the rigorous proof (or disproof) of ultrametricity, and, more in general, the analysis of the structure of Gibbs measures at low temperature, turned out to be a very difficult task. A step in this direction was performed by Ghirlanda and Guerra in [8]. They found a simple and elegant way, based on the self-averaging of the internal energy, to prove a remarkable property of the overlaps. Given $s$ replicas, the Gibbs measure must be such that when one adds a further replica this is either identical to one these, or statistically independent of them; each case occurring with the same probability. More generally, various constraints on the distribution of the different overlaps have been found in the same spirit ([2, 14]). Such features have found several applications ([16, 4]) in the rigorous analysis of spin glass models. For example, the property of non-negativity of the overlap, which in some models plays a role in turning the cavity free-energy into a rigorous lower bound, turns out to be a consequence of the Ghirlanda-Guerra self-averaging identities ([16]). In the same way these identity have a role in the rigorous analysis of spin glasses close to the critical temperature ([1]).

In more general spin-glass systems, like finite dimensional systems or spin systems on random graphs, the statistics of the overlap are not enough to fully characterize the low temperature spin glass phase. For instance, in diluted models the statistics of the local cavity fields, or equivalently of all the multi-overlaps, is necessary to describe the low temperature thermodynamic properties. In this paper, we analyse two families of identities for the local fields and multi-overlap distributions that are a consequence of self-averaging relations. We will see that one of the two families is a consequence of the self-averaging with respect to the Gibbs measure or, equivalently, of stochastic stability, as the two phenomena turn out to be equivalent. The other family of identities is instead a consequence of self-
averaging with respect to the global measure (quenched after Gibbs). Our conclusions will not rely much on the specific form of the Hamiltonian of the model. We will however use the example of spin models on sparse random graphs (dilute spin glass models), where we expect that our results could provide hints for progresses in the mathematical analysis of the low temperature phases. Diluted mean field spin glasses have, in recent time, attracted a lot of attention in statistical physics, due to their intrinsic interest of spin glasses where each spin interacts with a finite number of variables, but more importantly because fundamental problems in computer science, such as the random K-SAT and graph coloring, the random X-OR-SAT, tree reconstruction [11] and others, admit a formulation in terms of spin glass systems on random graphs. The cavity approach to these problems has led in many cases to results believed to be exact, albeit for the moment several rigorous proofs are still lacking.

Some of the identities that we will discuss appeared already in [7] to discuss free energy bounds in diluted models with non-Poissonian connectivity. Here we re-derive with different methods this family of identities, and we exhibit a second family of new identities.

2 The notations

We will use the stereotypical dilute spin glass model, the Viana-Bray (VB), to introduce here the notations we need, and to derive our results in the next two sections.

Notations: $\alpha, \beta$ are non-negative real numbers (degree of connectivity and inverse temperature respectively); $P_\zeta$ is a Poisson random variable of mean $\zeta$; $\{i_\nu\}, \{j_\nu\}$, etc. are independent identically distributed random variables, uniformly distributed over the points $\{1, \ldots, N\}$; $\{J_\nu\}, J$, etc. are independent identically distributed random variables, with symmetric distribution; $\mathcal{J}$ is the set of all the quenched random variables above; the map $\sigma : i \rightarrow \sigma_i, i \in \{1, \ldots, N\}$ is a spin configuration from the configura-
tion space $\Sigma = \{-1, 1\}^N$: $\pi_\zeta(\cdot)$ is the Poisson measure of mean $\zeta$; $E$ is an average over all (or some of) the quenched variables; $\omega_J$ or simply $\omega$ is the Boltzmann-Gibbs average explicitly written below; $\Omega_N$ or simply $\Omega$ are a product of the needed number of independent identical copies (replicas) of $\omega_J$; $\langle \cdot \rangle$ will indicate the composition of an $E$-type average over quenched variables and the Boltzmann-Gibbs average over the spin variables (see below). We will often drop the dependance on some variables or indices or slightly change notations to lighten the expressions, when there is no ambiguity. As a main example, consider the Hamiltonian of the Viana-Bray model, defined as

$$H_{V B}^N(\sigma, \alpha; J) = -\sum_{\nu=1}^{\alpha N} J_{\nu} \sigma_{i_{\nu}} \sigma_{j_{\nu}}.$$ 

We will limit to the case $J = \pm 1$, without loss of generality [10]. We follow the usual basic definitions and notations of thermodynamics for the partition function $Z_N$, the pressure $p_N$, the free energy per site $f_N$ and its thermodynamic limit $f$, so to have in general

$$Z_N(\beta, \alpha) = Z(H_N; \beta, \alpha) = \sum_{\{\sigma\}} \exp(-\beta H_N(\sigma, \alpha)),$$

$$p_N(\beta, \alpha) = -\beta f_N(\beta, \alpha) = \frac{1}{N} E \ln Z_N(\beta, \alpha), \quad f(\beta, \alpha) = \lim_{N \to \infty} f_N(\beta, \alpha).$$

The Boltzmann-Gibbs average of an observable $O: \Sigma \to \mathbb{R}$ is

$$\omega(O) = Z_N(\beta, \alpha)^{-1} \sum_{\{\sigma\}} O(\sigma) \exp(-\beta H_N(\sigma, \alpha)),$$

$E$ denotes the average with respect to the quenched variables, and $\langle \cdot \rangle = E\omega(\cdot)$ is the global average.

The multi-overlaps $q_{1 \cdots m}: \Sigma^m \to [-1, 1]$, where we use the notation $\Sigma^n = \Sigma^{(1)} \times \cdots \times \Sigma^{(n)}$, among the “replicas” $\Sigma^{(r_1)} \ni \sigma^{(r_1)}, \ldots, \Sigma^{(r_n)} \ni \sigma^{(r_n)}$ is defined by

$$q_{r_1 \cdots r_n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(r_1)} \cdots \sigma_i^{(r_n)},$$
but sometimes we will just write \( q_n \); \( q_1 \) can be identified with the magnetization \( m \)

\[
m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i .
\]

Dealing with binary spins, we will not be using powers of the spins, so we will often drop the brackets (\( () \)) in the replica index for the spins, so that \( \sigma_i^s \equiv \sigma_i^{(s)} \) will mean the \( i\)-th spin from the replica \( s \), \( \Sigma^{(s)} \), not the \( s\)-th power of \( \sigma_i \). Notice

\[
\mathbb{E}_{\omega^2n}(\sigma_i) = \langle q_1 \cdots q_n \rangle , \quad \mathbb{E}_{\omega}(\sigma_i) = \mathbb{E}_{\omega}(m) = \langle m \rangle .
\]

### 3 Stochastic Stability and self-averaging of the Gibbs measure

In the study of finite connectivity models it emerged that in a suitable probability space it is possible to formulate an exact variational principle for the computation of the free energy. This was obtained with the introduction of Random Multi-Overlap Structures (RaMOSt). We refer to [6] for details. The ROSSt approach is based on the use of generic random weights to average the “cavity” part and the relative “internal correction” in the free energy (these are the numerator and the denominator of the trial free energy \( G_N \) introduced in (4)). See [6] for details). Here we are not interested in a detailed discussion of the RaMOSt approach, but we study the effect of a perturbation to the measure of our model, which does not need to be the Gibbs measure. That is why introduce this more general weighting scheme, although the reader may keep in mind the Gibbs measure as a guiding example.

#### 3.1 Random Multi-Overlap Structures

The proper framework for the calculation of the free energy per spin is that of the Random Multi-Overlap Structures (RaMOST, see [6] for more
details).  

**Definition 1** Given a probability space \( \{ \Omega, \mu(d\omega) \} \), a Random Multi-Overlap Structure \( \mathcal{R} \) is a triple \((\tilde{\Sigma}, \{\tilde{q}_{2n}\}, \xi)\) where

- \( \tilde{\Sigma} \) is a discrete space;
- \( \xi : \tilde{\Sigma} \to \mathbb{R}_+ \) is a system of random weights, such that \( \sum_{\gamma \in \tilde{\Sigma}} \xi_\gamma \leq \infty \) \( \mu \)-almost surely;
- \( \tilde{q}_{2n} : \tilde{\Sigma}^{2n} \to \mathbb{R}, n \in \mathbb{N} \) is a positive semi-definite Multi-Overlap Kernel (equal to 1 on the diagonal of \( \tilde{\Sigma}^{2n} \), so that by Schwartz inequality \( |\tilde{q}_{2n}| \leq 1 \)).

A RaMOSt needs to be equipped with \( N \) independent copies of a random field \( \{\tilde{h}_i^\gamma(\alpha; \tilde{J})\}_{i=1}^N \) and with another random field \( \tilde{H}_\gamma(\alpha; \tilde{J}) \) such that

\[
\frac{d}{d\alpha} \mathbb{E} \ln \sum_{\gamma \in \tilde{\Sigma}} \xi_\gamma \exp(-\beta \tilde{h}_i^\gamma) = 2 \sum_{n>0} \frac{1}{2n} \tanh^{2n}(\beta)(1 - \langle \tilde{q}_{2n} \rangle), \tag{2}
\]

\[
\frac{d}{d\alpha} \mathbb{E} \ln \sum_{\gamma \in \tilde{\Sigma}} \xi_\gamma \exp(-\beta \tilde{H}_\gamma) = \sum_{n>0} \frac{1}{2n} \tanh^{2n}(\beta)(1 - \langle \tilde{q}_{2n}^2 \rangle). \tag{3}
\]

These two fields are employed in the definition of the trial pressure

\[
G_N(\mathcal{R}; \beta) = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\gamma,\sigma} \xi_\gamma \exp(-\beta \sum_{i=1}^N \tilde{h}_i^\sigma \sigma_i)}{\sum_{\gamma} \xi_\gamma \exp(-\beta H_\gamma)}. \tag{4}
\]

The reason why this is the proper framework for the calculation of the free energy is explained by the next \[6\]

**Theorem 1 (Extended Variational Principle)** Taking the infimum for each \( N \) separately of the trial function \( G_N(\mathcal{R}; \beta) \) over the space of all RaMOSt’s, the resulting sequence tends to the limiting pressure \(-\beta f(\beta)\) of the VB model as \( N \) tends to infinity:

\[
-\beta f(\beta) = \lim_{N \to \infty} \inf_{\mathcal{R}} G_N(\mathcal{R}; \beta). \]
A RaMOSt $\mathcal{R}$ is said to be optimal if $G(\mathcal{R}; \beta) = -\beta f(\beta) \forall \beta$. We will denote by $\Omega$ the measure associated to the RaMOSt weights $\xi$ as well.

The Boltzmann RaMOSt [6] is optimal, and constructed by thinking of a reservoir of $M$ spins $\tau$

$$\Sigma = \{-1, 1\}^M \ni \tau, \quad \xi_{\tau} = \exp(-\beta H_M(\tau)), \quad \tilde{q}_{1\ldots2n} = \frac{1}{M} \sum_{k=1}^{M} \tau_{k}^{(1)} \cdots \tau_{k}^{(2n)}$$

with

$$\tilde{h}_r^i(\alpha) = \sum_{\nu=1}^{P_{2\alpha}} J_{r}^{i}, \quad \hat{H}_r(\alpha N) = - \sum_{\nu=1}^{P_{\alpha N}} \hat{J}_{r},$$

and $\tilde{J}, \hat{J}$ all independent copies of $J$.

Let $c_i = 2 \cosh(\beta \tilde{h}^i)$. It is possible to show [6] that optimal RaMOSt’s enjoy the same factorization property enjoyed by the Boltzmann RaMOSt and described in the next [6].

**Theorem 2 (Factorization of optimal RaMOSt’s)** With the possible exception of a zero measure set of values of the degree of connectivity, the following Cesàro limit is linear in $N$ and $\bar{\alpha}$

$$\mathbf{Clim}_M \mathbb{E} \ln \Omega_M\{c_1 \cdots c_N \exp[-\beta \hat{H}(\bar{\alpha})]\} = N(-\beta f + \alpha A) + \bar{\alpha} A,$$

where

$$A = \sum_{n=1}^{\infty} \frac{1}{2n} \mathbb{E} \tanh^{2n}(\beta J)(1 - \langle q_{2n}^2 \rangle).$$

(5)

This factorization property is called *invariance with respect to the cavity step*, or *Quasi-Stationarity*, and it is found in the hierarchical Parisi ansatz as well. When $\bar{\alpha}$ is zero, the theorem above states the factorization of the cavity fields, and it is possible to show that from this property one can deduce the family of identities we will discuss in the next subsection [3].

When one removes instead the cavity terms $c_1, \ldots, c_N$ from the previous theorem, the statement becomes what is usually referred to as Stochastic Stability. We will show that the latter too implies the same family of identities. We will have in mind the case of a small perturbation of our
spin system, but what we find holds for more general RaMOSt’s, provided the previous theorem holds, that is for Quasi-Stationary RaMOSt’s.

3.2 The first family of identities

We will now prove a lemma that expresses the stability of the Gibbs measure of our model against a macroscopic but small stochastic perturbation. In different terms, the lemma expresses the linear response of the free energy to the connectivity shift the perturbation consists of. The lemma we are about to prove will be used to show that from stochastic stability one can deduce a certain self-averaging which in turn imposes a family of constraints on the distribution of the overlaps.

Lemma 1 Let $\Omega, \langle \cdot \rangle$ be the usual Gibbs and quenched Gibbs expectations at inverse temperature $\beta$, associated with the Hamiltonian $H_N(\sigma, \alpha; J)$. Then, with the possible exception of a zero measure set of values of the degree of connectivity,

$$\lim_{N \to \infty} E \ln \Omega \exp \left( \beta \sum_{\nu=1}^{P_{\alpha'}} J'_\nu \sigma_{i'\nu} \sigma_{j'\nu} \right) = \alpha' \sum_{n=1}^{\infty} \frac{1}{2n} \tanh^{2n}(\beta') (1 - \langle q_{2n}^2 \rangle), \quad (6)$$

where the random variables $P_{\alpha'}$, $\{J'_\nu\}$, $\{i'\nu\}$, $\{j'\nu\}$ are independent copies of the analogous random variables in the Hamiltonian contained in $\Omega$.

Notice that, in distribution

$$\beta \sum_{\nu=1}^{P_{\alpha_N}} J_\nu \sigma_{i\nu} \sigma_{j\nu} + \beta' \sum_{\nu=1}^{P_{\alpha'}\beta' / \beta} J'_\nu \sigma_{i'\nu} \sigma_{j'\nu} \sim \beta \sum_{\nu=1}^{P_{\alpha + \alpha' / N}} J''_\nu \sigma_{i\nu} \sigma_{j\nu}, \quad (7)$$

where $\{J''_\nu\}$ are independent copies of $J$ with probability $\alpha N / (\alpha N + \alpha')$ and independent copies of $J' / \beta$ with probability $\alpha' / (\alpha N + \alpha')$. In the right hand side above, the quenched random variables will be collectively denoted by $J''$. Notice also that the sum of Poisson random variables is a Poisson random variable with mean equal to the sum of the means, and
hence we can write

\[ A_t \equiv \mathbb{E} \ln \Omega \exp \left( \beta' \sum_{\nu=1}^{P_{\alpha'}} J_{\nu}' \sigma_{i_{\nu}'} \sigma_{j_{\nu}'} \right) = \mathbb{E} \ln \frac{Z_N(\alpha_t; J'')}{Z_N(\alpha; J)} , \tag{8} \]

where we defined, for \( t \in [0,1] \),

\[ \alpha_t = \alpha + \alpha' \frac{t}{N} \tag{9} \]

so that \( \alpha_t \to \alpha \ \forall \ t \) as \( N \to \infty \).

**Proof.** Let us compute the \( t \)-derivative of \( A_t \), as defined in (8)

\[ \frac{d}{dt} A_t = \mathbb{E} \sum_{m=1}^{\infty} \frac{d}{dt} \pi_{\alpha'(t)}(m) \ln \sum_{\sigma} \exp \left( \beta' \sum_{\nu=1}^{m} J_{\nu}' \sigma_{i_{\nu}'} \sigma_{j_{\nu}'} \right) . \]

Using the following elementary property of the Poisson measure

\[ \frac{d}{dt} \pi_{\zeta}(m) = \zeta(\pi_{\zeta}(m-1) - \pi_{\zeta}(m)) \tag{10} \]

we get

\[ \frac{d}{dt} A_t = \alpha' \mathbb{E} \sum_{m=0}^{\infty} [\pi_{\alpha'(t)}(m-1) - \pi_{\alpha'(t)}(m)] \ln \sum_{\sigma} \exp(\beta' \sum_{\nu=1}^{m} J_{\nu}' \sigma_{i_{\nu}'} \sigma_{j_{\nu}'}) \]

\[ = \alpha' \mathbb{E} \ln \sum_{\sigma} \exp(\beta' J' \sigma_{i_{m}'} \sigma_{j_{m}'}) \exp(\beta' \sum_{\nu=1}^{P_{\alpha'}} J_{\nu}' \sigma_{i_{\nu}'} \sigma_{j_{\nu}'}) - \alpha' \mathbb{E} \ln \sum_{\sigma} \exp(\beta' \sum_{\nu=1}^{P_{\alpha'}} J_{\nu}' \sigma_{i_{\nu}'} \sigma_{j_{\nu}'}) \]

\[ = \alpha' \mathbb{E} \ln \Omega_t \exp(\beta' J' \sigma_{i_{m}'} \sigma_{j_{m}'}) , \]

where we included the \( t \)-dependent weights in the average \( \Omega_t \). Now use the following identity

\[ \exp(\beta' J' \sigma_i \sigma_j) = \cosh(\beta' J') + \sigma_i \sigma_j \sinh(\beta' J') \]

to get

\[ \frac{d}{dt} A_t = \alpha' \mathbb{E} \ln \Omega_t \left[ \cosh(\beta' J')(1 + \tanh(\beta' J') \sigma_{i_{m}'} \sigma_{j_{m}'}) \right] . \]
It is clear that
\[ E \omega_t^{2n}(\sigma_{i_m}\sigma_{j_m}) = \langle q_{2n}^2 \rangle_t , \]
so we now expand the logarithm in power series and see that, in the limit of large \( N \), as \( \alpha_t \to \alpha \) the result does not depend on \( t \), everywhere the expectation \( \langle \cdot \rangle_t \) is continuous as a function of the parameter \( t \) (or equivalently as a function of the degree of connectivity). From the comments that preceded the current proof, formalized in (7)-(8)-(9), this is the same as assuming that \( \Omega \) is regular as a function of \( \alpha \), because \( J'' \to J \) in the sense that in the large \( N \) limit \( J'' \) can only take the usual values \( \pm 1 \) since the probability of being \( \pm \beta' / \beta \) becomes zero. Therefore integrating over \( t \) from 0 to 1 is the same as multiplying by 1. Due to the symmetric distribution of \( J \), the expansion of the logarithm yields the right hand side of (6), where the odd powers are missing.

Let us define
\[ \hat{H}(\alpha'; J) = \sum_{\nu=1}^{P'} J_{\nu} \sigma_{i_{\nu}} \sigma_{j_{\nu}} \sim H(\alpha'/N; J) \]
Let us now consider the statement of Lemma 1 in the case of two independent perturbations (the quenched variables in the perturbations, denoted by \( J'_1, J'_2 \), are independent one another and independent from those in the Hamiltonian of the Boltzmann factor). Then the fundamental theorem of calculus can be used twice to extend the statement of the previous lemma to
\[ E \ln \Omega[\exp(-\beta'_1 \hat{H}(\alpha'_1; J'_1) - \beta'_2 \hat{H}(\alpha'_2; J'_2))] = (\alpha'_1 + \alpha'_2)A , \quad (11) \]
where \( A \) again does not depend, in the thermodynamic limit, on \( \alpha'_1, \alpha'_2 \), and incidentally has the same form as the right hand side of (6). In the equation above, assumed to be taken in the thermodynamic limit, \( \Omega \) is the Gibbs measure associated with the unperturbed Hamiltonian of the original model, and the same holds for the averages appearing in \( A \), just like in the previous lemma. Clearly we then have (omitting the dependence on the
independent quenched random variables)

\[ \frac{\partial^2}{\partial \alpha_1' \partial \alpha_2'} \mathbb{E} \ln \Omega[\exp(-\beta_1' \hat{H}(\alpha_1') - \beta_2' \hat{H}(\alpha_2'))] = 0 , \]

and again in the thermodynamic limit \( \Omega \) does not include any perturbation with \( \alpha_1', \alpha_2', \beta_1', \beta_2' \). A simple computation yields

\[ \frac{\partial^2}{\partial \alpha_1' \partial \alpha_2'} \mathbb{E} \ln \Omega[\exp(-\beta_1' \hat{H}(\alpha_1') - \beta_2' \hat{H}(\alpha_2'))] = 0 \]

\[ = \mathbb{E} \ln \Omega[\exp(\beta_1' J_1' \sigma_{i_1} \sigma_{j_1} + \beta_2' J_2' \sigma_{i_2} \sigma_{j_2})] \]

\[ - \mathbb{E} \ln \Omega[\exp(\beta_1' J_1' \sigma_{i_1} \sigma_{j_1}, \Omega[\exp(\beta_2' J_2' \sigma_{i_2} \sigma_{j_2})] \]

Every time a derivative with respect to a perturbing parameter is taken, the relative perturbation is added to the weights of the measure \( \Omega \), but if the perturbation is small (like in our case, as explained in the previous lemma) it disappears from the measure in the thermodynamic limit. This is true for almost all values of the perturbing parameters. Hence we may assume that both in the equation above and in the next calculation \( \beta_1', \beta_2' \) are not in the measure \( \Omega \), and we get

\[ \frac{\partial^2}{\partial (\beta_1' J_1') \partial (\beta_2' J_2')} \mathbb{E} \ln \Omega[\exp(\beta_1' J_1' \sigma_{i_1} \sigma_{j_1} + \beta_2' J_2' \sigma_{i_2} \sigma_{j_2})] \]

\[ = \mathbb{E} \Omega(\sigma_{i_1} \sigma_{j_1}) - \mathbb{E} \Omega(\sigma_{i_1}) \Omega(\sigma_{j_1}) = 0 , \quad (12) \]

at the price of a zero measure set of values of the parameters (which allows us to use always the unperturbed expectation \( \Omega \)). The first line of this equation gives us the generator of a family of relations that we will obtain by means of an expansion in powers of \( \beta_1', \beta_2' \). The second line of the equation formulates the self-averaging (with respect to the Gibbs measure) implied by the stochastic stability.

So we proceed starting from the next lemma and the next theorem, summarizing what we just discussed.
Lemma 2 Let $\Omega'$ be the Gibbs measure including two independent perturbations of the form

$$\hat{H}(\alpha') = \sum_{\nu=1}^{P_{\alpha'}} J'_\nu \sigma_i \sigma_j,$$

with parameters $\alpha'_1, \alpha'_2, \beta'_1, \beta'_2$ like in (11). Then, recalling that $m$ is the magnetization, the following self-averaging (with respect to the Gibbs measure) identity

$$\lim_{N \to \infty} \mathbb{E}\{\Omega'(m^2) - [\Omega'(m)]^2\} = 0$$

holds for almost all values of the two perturbing parameters $\alpha'_1, \alpha'_2$.

We will see again that in the first line of equation (12) the expression remains zero even without the derivative. In fact the generator of the identities we want to prove is expressed in the following

Theorem 3 In the thermodynamic limit the following holds for almost all values of $\alpha'_1$ and $\alpha'_2$:

$$\mathbb{E} \ln \Omega'(\exp(\beta'_1 J'_1 \sigma_i \sigma_j + \beta'_2 J'_2 \sigma_i \sigma_j)) =$$

$$\mathbb{E} \ln \Omega'(\exp(\beta'_1 J'_1 \sigma_i \sigma_j)) + \mathbb{E} \ln \Omega'(\exp(\beta'_2 J'_2 \sigma_i \sigma_j)) .$$

The relations we will derive are a simple consequence of this theorem, and formalized in the next

Corollary 1 In the thermodynamic limit, for almost all values of the perturbing parameters $\alpha'_1, \alpha'_2$ we have

$$\sum_{a=0}^{\min\{r,s\}} (-)^{a+1} \frac{(2r + 2s - a - 1)!}{a!(2r - a)!(2s - a)!} (q_{2r} q_{2s}' a) = 0 \quad \forall \ r, s \in \mathbb{N} ,$$

where the subscript $a$ in the global average $\langle \cdot \rangle'_a = \mathbb{E} \Omega'_a$ means that a replicas are in common among those in $q_r$ and those in $q_s$, so that in particular $\Omega_a$ is (in a given term) the product measure of only $2r + 2s - a$ copies of $\omega'$. 
The “prime” superscript indicates as usual that the measure contains the perturbations, which vanish in the thermodynamic limit but allows us “almost sure” statements only.

**Proof.** The following shorthand will be employed

\[ t_1 = \tanh(\beta_1' J_1') , \ t_2 = \tanh(\beta_2' J_2') , \]

\[ \Omega_1' = \Omega'(\sigma_i \sigma_j) , \ \Omega_2' = \Omega'(\sigma_i' \sigma_j') , \ \Omega_{12}' = \Omega'(\sigma_i, \sigma_j, \sigma_i' \sigma_j') \]

and

\[ W = \Omega'(\exp(\beta_1' J_1' \sigma_i \sigma_j + \beta_2' J_2' \sigma_{i'} \sigma_{j'})) , \]

Observe that, if we let \( \delta = 1, 2 \),

\[ \frac{\partial}{\partial \beta J_\delta'} = (1 - t_\delta) \frac{\partial}{\partial t_\delta} . \]  

(15)

Now,

\[ \ln W = \ln(1 + t_1 \Omega_1 + t_2 \Omega_2 + t_1 t_2 \Omega_{12}) + \ln \cosh \beta J_1' + \ln \cosh \beta J_2' \]

and

\[ \ln(1 + t_1 \Omega_1 + t_2 \Omega_2 + t_1 t_2 \Omega_{12}) = \]

\[ \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{(-1)^{n+1}}{n} \binom{n}{l} \binom{l}{m} t_1^{n-l+m} t_2^{n-m} \Omega_1^m \Omega_2^{l-m} \Omega_{12}^{n-l} \]

\[ = \sum_{n,l,m} (-1)^{n+1} \frac{(n-1)!}{(n-l)! (l-m)! m!} t_1^{n-l+m} t_2^{n-m} \Omega_1^m \Omega_2^{l-m} \Omega_{12}^{n-l} . \]

The derivatives in (12) kill the two terms with the hyperbolic cosines, and from (15) we know that we can replace the derivatives with respect to \( \beta J_\delta' \) with the derivatives with respect to \( t_\delta \), \( \delta = 1, 2 \). Notice that the logarithm just expanded is zero for \( t_1 = 0 \) and for \( t_2 = 0 \), therefore as its derivative like in (12) is zero, the logarithm itself is zero. This is why Theorem 3 holds, being (14) just the integral of the second line in (12).

Thanks to (11), if we put

\[ n-l+m = r , \ n-m = s , \ n-l = a \]
we get
\[
\sum_{r,s} E[t_r^1 t_s^2] \sum_{a=0}^{\min\{r,s\}} (-)^{a+1} \frac{(r + s - a - 1)!}{a!(r-a)!(s-a)!} \langle q_r^2 q_s^2 \rangle'_a = 0
\]
where \(\langle \cdot \rangle_a\) means that \(a\) replicas are in common among those in \(q_r\) and those in \(q_s\). Hence the statement of the theorem to be proven
\[
\sum_{a=0}^{\min(2r,2s)} (-)^{a+1} \frac{(2r + 2s - a - 1)!}{a!(2r-a)!(2s-a)!} \langle q_r^2 q_s^2 \rangle'_a = 0
\]

### 3.3 Generalization to smooth functions of multi-overlaps

The fact that in our formulas we always got the square power of the overlaps is due to the fact that the Hamiltonian has 2-spin interactions. Everything we did so far could then be reproduced in the case of \(p\)-spin interactions, and we would obtain the same relations just derived, except the overlaps would appear in the power \(p\) instead of 2. Clearly the perturbation needed in this case is a \(p\)-spin perturbation too. More in general, we could consider a Hamiltonian consisting of the sum (over \(p\)) of \(p\)-spin Hamiltonians for any integer \(p\). Then we could perturb each of the \(p\)-spin Hamiltonians with its proper small \(p\)-spin perturbation, and add all these perturbations to the system. Clearly we have to make sure that all the terms in this whole Hamiltonian are weighted with sufficiently small weights so to have the necessary convergence. More explicitly, the perturbed Hamiltonian is
\[
H_N(\sigma, \alpha; J) = -\sum_p \left[ a_p \sum_{\nu=1}^{\nu_p} J_{\nu} \sigma_{i_1}^\nu \cdots \sigma_{i_p}^\nu + b_p \lambda_p \sum_{\nu=1}^{\nu_p'} J'_{\nu} \sigma_{j_1}^\nu \cdots \sigma_{j_p}^\nu \right],
\]
where \(\sum_p |a_p|^2 = \sum_p |b_p|^2 = 1\), the notation for all the quenched variables is the usual one, and \(\{\lambda_p\}\) are the independent perturbing real parameters.

It is not surprising then that we can state

**Corollary 2** With the possible exception of a zero measure set in the space
of all perturbing parameters, we have

\[
\min \{2r, 2s\} \sum_{a=0}^{\min \{2r, 2s\}} (-)^{a+1} \frac{(2r + 2s - a - 1)!}{a!(2r-a)!(2s-a)!} \langle q_m^r q_n^s \rangle_a = 0 \quad \forall \, r, s, m, n \in \mathbb{N}.
\]

Again, this corollary can be seen as a consequence of a self-averaging property, namely

\[
\mathbb{E}\Omega(\sigma_i \cdots \sigma_i^m \sigma_j \cdots \sigma_j^n) - \mathbb{E}[\Omega(\sigma_i \cdots \sigma_i^m)\Omega(\sigma_j \cdots \sigma_j^n)] = 0.
\]

Therefore we can replace each overlap by any smooth function of the relative replicas in the statement of the corollaries.

\section{Self-averaging of the quenched-Gibbs measure}

Roughly speaking, if a convex random function does not fluctuate much, then its derivative does not fluctuate much either, with the exception of bad cases. This is well explained in Proposition 4.3 of \cite{15} and Lemma 8.10 of \cite{5}. We are not interested in general theorems, in our case the convex function we are interested in is the free energy density, and we only need to know that it is self-averaging (in the sense that the random free energy density does not fluctuate around its quenched expectation, in the thermodynamic limit). In the case of finite connectivity random spin systems, a detailed proof of this can be found in \cite{10}. The derivative of the free energy density (times $-\beta$) with respect to $-\beta$ is the expectation of the internal energy density $u_N = H_N/N$. Like in \cite{9} and in section 2 of \cite{8}, we have therefore this further self-averaging

\[
\lim_{N \to \infty} [\langle u_N^2 \rangle - \langle u_N \rangle^2] = 0
\]

which implies (due to Schwartz inequality)

\[
\lim_{N \to \infty} \langle u_N^{(1)} \phi_s \rangle = \lim_{N \to \infty} \langle u_N \rangle \langle \phi_s \rangle
\]

(16)
for any bounded function $\phi_s$ of $s$ replicas, and $u^{(1)}_N$ is the internal energy density in the configuration space of the replica 1. More precisely, let us call the spin-configuration space $\{-1,1\}^N = \Sigma$, and consider a bounded function $\phi_s$ of $s$ replicas, i.e. $\phi_s : \Sigma^s \rightarrow \mathbb{R}$. The spin-configuration space $\Sigma$ is equipped with the Gibbs measure $\omega$, and the product space $\Sigma^s$ (“the space of the replicas”) is equipped with the product measure (“replica measure”) $\omega^\otimes s = \Omega$. The quenched variables are the same in each factor of the product space, and this means that the measure $\langle \cdot \rangle = \mathbb{E}\Omega(\cdot) = \mathbb{E}\omega^\otimes s(\cdot)$ on the product space $\Sigma^s$ is not a product measure. We will use for simplicity $\Omega$ for any value of $s$. So $f^{(1)}_N$ is the free energy in the space which is the first factor in the product space $\Sigma^s$. Notice that $\Sigma$ has the cardinality of the continuum in the thermodynamic limit $N \to \infty$. Apices will denumerate replicas for the spins and the Hamiltonian, while they are just regular exponents in the case of overlaps, where the replicas are counted or listed in the sub-index.

At this point we want to perturb the Hamiltonian and consider the derivative with respect to the perturbing parameter, as we did in the previous section:

$$-\beta H_N(\sigma) \rightarrow -\beta H_N(\sigma) + \beta' \sum_{\nu=1}^{P'} \mathcal{J}^{\nu}_{\alpha} \sigma^{\nu}_{i^{\nu}} \sigma^{j^{\nu}}_j,$$

in order to obtain an expansion in powers $\beta'$ with coefficients which do not depend on $\beta'$ in the thermodynamic limit.

We are going to prove, first of all, the following

**Theorem 4** For a given bounded function $\phi_s$ of $s$ replicas, the following relation constrains the distribution of the 4-overlap

$$\frac{s(s+1)(s+2)}{3!} \langle q_{1,s+1,s+2,s+3}^2 \phi_s \rangle - \frac{s(s+1)}{2!} \sum_a \langle q_{1,a,s+1,s+2}^2 \phi_s \rangle + s \sum_{a<b} \langle q_{1,a,b,s+1} \phi_s \rangle - \sum_{a<b<c} \langle q_{1,a,b,c}^2 \phi_s \rangle = \langle q_{1234}^2 \rangle \langle \phi_s \rangle.$$
The proof is straightforward but long, and it will be split into several steps.

Let us consider the right hand side of (16). Put $t = \tanh(\beta')$, $q_0 = 1$, and let us just indicate the number of replicas in the overlaps, rather than denumerating them all. Recall also that $p_N = -\beta f_N$, which here “contains” the perturbed Hamiltonian. Let us prove the next

**Lemma 3** The derivative of the (perturbed) pressure $p_N(\beta, \beta')$ with respect to the perturbing parameter $\beta'$ has the following form as a series in powers of $t = \tanh(\beta')$

$$
\partial_{\beta'} p_N(\beta, \beta') = -\alpha \sum_{n=0}^{\infty} t^{2n+1} (\langle q_{2n}^2 \rangle - \langle q_{2n+2}^2 \rangle).
$$

**Proof.** We have

$$
\partial_{\beta'} p_N(\beta, \beta') = -\alpha \sum_{m=1}^{\infty} \pi_\alpha(m) \sum_{\nu=1}^{m} \langle J'_\nu \sigma_{i'_\nu} \sigma_{j'_\nu} \rangle_m
$$

$$
= -\alpha \sum_{m=1}^{\infty} m \pi_\alpha(m) \langle J'_m \sigma_{i'_m} \sigma_{j'_m} \rangle_m
$$

$$
= -\alpha \sum_{m=1}^{\infty} (m-1) \langle J'_m \sigma_{i'_m} \sigma_{j'_m} \rangle_m
$$

where the sub $m$ indicates that the variable $P'_\alpha$ has been fixed to $m$. It is easy to see that

$$
\langle J'_m \sigma_{i'_m} \sigma_{j'_m} \rangle_m = \mathbb{E} \frac{\omega(J'_m \sigma_{i'_m} \sigma_{j'_m} \exp(\beta J'_m \sigma_{i'_m} \sigma_{j'_m}))_{m-1}}{\omega(\exp(\beta J'_m \sigma_{i'_m} \sigma_{j'_m}))_{m-1}}.
$$

Hence

$$
\partial_{\beta'} p_N(\beta, \beta') = -\alpha \mathbb{E} J'_m \frac{t + w}{1 + tw}, \ w \equiv \omega(\sigma_{i'_m} \sigma_{j'_m}),
$$

according to the usual notations. Now a simple expansion (that we will explicitly write in the next lemma) of $(1 + tw)^{-1}$ in powers of $t$ yields

$$
\partial_{\beta'} p_N(\beta, \beta') = -\alpha \sum_{n=0}^{\infty} t^{2n+1} (\langle q_{2n}^2 \rangle - \langle q_{2n+2}^2 \rangle).
$$
So the lemma is proven and we have an expression for the right hand side of (16), if we just multiply the average of the multi-overlaps by the average of $\phi_s$.

Let us now consider the left hand side of (16), recalling that $\phi_s$ is a function of $s$ replicas, that indices in the spins indicate which factor of the product space $\Sigma^s$ (which replica) the spin belongs to, and that the energy density is assumed to be taken in the first replica. We will henceforth omit the prime symbol in all the quenched variables, but still assume that they are independent of any other quenched variable implicitly contained in the averages.

**Lemma 4** Recalling that $w \equiv \omega(\sigma^i_m, \sigma^j_m)$, we have

$$
\langle u^{(1)}_N \phi_s \rangle = -\alpha t \mathbb{E}\{\Omega_s(1 + J t^{-1} \sigma^i_1 \sigma^j_1) \times \\
(1 + J \sum_a^{2,s} \sigma^a_i \sigma^a_j t + \sum_{a < b}^{2,s} \sigma^a_i \sigma^a_i \sigma^a_j \sigma^a_j t^2 + \sum_{a < b < c}^{2,s} \sigma^a_i \sigma^a_i \sigma^a_i \sigma^a_j \sigma^a_j \sigma^a_j t^3 + \cdots) \times \\
(1 - J s t w + \frac{s(s + 1)}{2!} t^2 w^2 - J \frac{s(s + 1)(s + 2)}{3!} t^3 w^3 + \frac{s(s + 1)(s + 2)(s + 3)}{4!} t^4 w^4 - \cdots)\}.
$$

**Proof.** From the proof of the previous lemma, in particular equations (17)-(18), and by definition of replica measure, we immediately get

$$
\langle u^{(1)}_N \phi_s \rangle = -\alpha \mathbb{E}_t \frac{\Omega_s[J \sigma^i_1 \sigma^j_1 \exp(\beta J (\sigma^1_1 \sigma^1_1 + \cdots + \sigma^s_1 \sigma^s_1)) \phi_s]}{\Omega_s(\exp(\beta J \sigma_1 \sigma_1))},
$$

that we rewrite as

$$
\langle u^{(1)}_N \phi_s \rangle = \frac{\Omega_s[(1 + J t^{-1} \sigma^1_1 \sigma^1_1) \prod_{a=2}^s (1 + J t \sigma^a_1 \sigma^a_1) \phi_s]}{(1 + J t w)^s}.
$$

Let us write explicitly the power expansion of the denominator, that we omitted in the previous lemma

$$
\frac{1}{(1 + J t w)^s} = 1 - J s t w + \frac{s(s + 1)}{2!} t^2 w^2 - J \frac{s(s + 1)(s + 2)}{3!} t^3 w^3 + \frac{s(s + 1)(s + 2)(s + 3)}{4!} t^4 w^4 - \cdots.
$$
It is also clear that
\[
\prod_{a=2}^{s} (1 + J t \sigma_i^a \sigma_j^a) = 1 + J \sum_{a} \sigma_i^a \sigma_j^a t + \sum_{a} \sigma_i^a \sigma_j^a t^2 + \sum_{a<b} \sigma_i^a \sigma_j^a \sigma_i^b \sigma_j^b \sigma_i^b \sigma_j^b t^3 + \cdots.
\]

Gathering all the ingredients completes the proof of the lemma.

We are now able to compare the two sides of (16), and see what the self-averaging of the internal energy density in the thermodynamic limit brings.

Equating the expressions computed in the last two lemmas gives
\[
\sum_{n=0}^{\infty} t^{2n} (\langle q_{2n}^2 \rangle - \langle q_{2n+2}^2 \rangle) \langle \phi_s \rangle = \mathbb{E} \{ \phi_s (1 + J t^{-1} \sigma_i^1 \sigma_j^1) 
\]
\[
(1 + J \sum_{a} \sigma_i^a \sigma_j^a t + \sum_{a<b} \sigma_i^a \sigma_j^a \sigma_i^b \sigma_j^b t^2 + \sum_{a<b<c} \sigma_i^a \sigma_j^a \sigma_i^b \sigma_j^b \sigma_i^c \sigma_j^c t^3 + \cdots + J^{s-1} t^{s-1} \sigma_i^2 \cdots \sigma_j^2) \}
\]
\[
(1 - J s t w + \frac{s(s+1)}{2!} t^2 w^2 - J s(s+1)(s+2) t^3 w^3 + \frac{s(s+1)(s+2)(s+3)}{4!} t^4 w^4 - \cdots) \}.
\]

The equality holds for any smooth function $\phi_s$ (typical interesting information is obtained for $\phi_s \equiv 1$ or $\phi_s = 2 = q_{2n}^2$), so that we get equalities between expressions involving averages of (squared) overlaps.

Let us see in detail what information we can get from the lowest orders.

Denote by $\mathbb{E}(\cdot | A_s)$ the conditional expectation with respect to the sigma-algebra $A_s$ generated by the overlaps of $s$ replicas. Let us show that the usual [8] Ghirlanda-Guerra identities for the overlap hold in our quite general case too (as well known):

**Proposition 1** *The Ghirlanda-Guerra relation holds*

\[
\mathbb{E}(q_{a,s+1}^2 | A_s) = \frac{1}{s} \langle q_{12}^2 \rangle + \frac{1}{s} \sum_{b \neq a} q_{ab}^2.
\]
Proof. In the expansion \((21)\), where only the terms of even order survive due to the symmetry of the variables \(J\), at the lowest order in \(t\) one gets

\[
\langle \phi_s \rangle - \langle q_{12}^2 \rangle \langle \phi_s \rangle = \langle \phi_s \rangle - sE[\omega(\sigma_i^1, \sigma_j^1)]w\phi_s] + \sum_a 2s \langle \Omega(\sigma_i^1, \sigma_j^1, \sigma_j^1, \sigma_j^1)\phi_s \rangle
\]

\[
= \langle \phi_s \rangle - s(q_{1,s+1}^2 \phi_s) + \sum_a (q_{1a}^2 \phi_s),
\]

which is precisely what is stated in \((22)\), (see [16]), immediately completing the proof of the proposition.

So the usual Ghirlanda-Guerra identities for 2-overlaps are recovered (and proven to hold in dilute spin glasses too, for instance).

At the next order we get instead

\[
\langle q_{12}^2 \rangle \langle \phi_s \rangle - \langle q_{1234}^2 \rangle \langle \phi_s \rangle = \sum_{a<b}^2 (q_{ab}^2 \phi_s) + \frac{s(s+1)}{2!} (q_{s+1,s+2}^2 \phi_s)
\]

\[
- s \sum_a^2 (q_{a,s+1}^2 \phi_s) - \frac{s(s+1)(s+2)}{3!} (q_{1,s+1,s+2,s+3}^2 \phi_s)
\]

\[
+ \frac{s(s+1)}{2!} \sum_a^2 (q_{1,a,s+1,s+2}^2 \phi_s) - s \sum_{a<b}^2 (q_{1,a,b,s+1}^2 \phi_s) + \sum_{a<b} (q_{1,a,b,c}^2 \phi_s).
\]

(23)

Now consider the four 2-overlaps terms. A simple generalization of the usual Ghirlanda-Guerra relations [3] to the case when two replicas are added to a previously assigned set of other replicas, tells us that these terms cancel out. Let us check that explicitly.

Corollary 3 Relation \((22)\) implies

\[
E(q_{s+1,s+2}^2 | A_s) = \frac{2}{s+1} \langle q_{12}^2 \rangle + \frac{2}{s(s+1)} \sum_{a<b}^1 q_{ab}^2.
\]

(24)

Proof. Let us re-write \((22)\) in the case of \(s+1\) given replicas

\[
E(q_{s+1,s+2}^2 | A_{s+1}) = \frac{1}{s+1} \langle q_{12}^2 \rangle + \frac{1}{s+1} \sum_{b}^1 q_{b,s+1}^2.
\]
Now use
\[ E(\mathcal{E}(\cdot|A_{s+1})|A_s) = E(\cdot|A_s) \] (25)
to get
\[
E(q_{s+1,s+2}^2|A_s) = \frac{1}{s+1} \langle q_{12}^2 \rangle + \frac{1}{s+1} \sum_b^1 s \ E(q_{0,s+1}^2|A_s)
\]
\[
= \frac{1}{s+1} \langle q_{12}^2 \rangle + \frac{1}{s+1} \left( \langle q_{12}^2 \rangle + \frac{1}{s} \sum_b^1 s \sum_{c \neq b} q_{bc}^2 \right) .
\]
That is
\[
E(q_{s+1,s+2}^2|A_s) = \frac{2}{s+1} \langle q_{12}^2 \rangle + \frac{2}{s(s+1)} \sum_{a<b} q_{ab}^2 ,
\]
which is what we wanted to prove.

Now with (22) and (24) in our hands, let us take the three 2-overlap terms in the right hand side of (23)
\[
\frac{s(s+1)}{2} \langle q_{s+1,s+2}^2 \phi_s \rangle = s \langle q_{12}^2 \rangle \langle \phi_s \rangle + \sum_{a<b}^1 s \langle q_{ab}^2 \phi_s \rangle \\
- s \sum_a^2 \langle q_{a,s+1}^2 \phi_s \rangle = - s \sum_{a}^1 \langle q_{a,s+1}^2 \phi_s \rangle + s \langle q_{1,s+1}^2 \phi_s \rangle \\
\sum_{a<b}^2 \langle q_{ab}^2 \phi_s \rangle = \sum_{a<b}^1 \langle q_{ab}^2 \phi_s \rangle - \sum_{a}^2 \langle q_{1a}^2 \phi_s \rangle .
\]
The sum of these three terms clearly reduces to \( \langle q_{12}^2 \rangle \langle \phi_s \rangle \), which is precisely what we find in the left hand side of (23). The 2-overlap terms thus cancel out from (23). We are hence left with a new relation for 4-overlaps:
\[
\frac{s(s+1)(s+2)}{3!} \langle q_{s+1,s+2,s+3}^2 \phi_s \rangle = \frac{s(s+1)}{2!} \sum_{a}^2 \langle q_{1,a,s+1,s+2}^2 \phi_s \rangle \\
+ s \sum_{a<b}^2 \langle q_{1,a,b}^2 \phi_s \rangle = \langle q_{1234}^2 \rangle \langle \phi_s \rangle + \sum_{a<b}^2 \langle q_{1,a,b,c}^2 \phi_s \rangle ,
\]
and the proof of Theorem 4 is now complete.

We report for sake of completeness the general expression of the generic order in the power series expansion (21). From the explicit calculation in Lemma 4 we get

\[
\langle q^2_{2n} \rangle \langle \phi_s \rangle - \langle q^2_{2n+2} \rangle \langle \phi_s \rangle = \sum_{m=2n-s+1}^{2n} \sum_{l=0}^{s-1} \sum_{a_1 < \cdots < a_l} (-)^m \binom{s + m + 1}{m} E[w^m \Omega(\phi_s \sigma_{i_1}^{a_1} \cdots \sigma_{i_l}^{a_l} \sigma_{j_1}^{a_1} \cdots \sigma_{j_l}^{a_l})] \delta_{2n,m+l} 
\]

\[
+ \sum_{m=2n-s+2}^{2n+1} \sum_{l=0}^{s-1} \sum_{a_1 < \cdots < a_l} (-)^m \binom{s + m + 1}{m} E[w^m \Omega(\phi_s \sigma_{i_1}^{1} \sigma_{i_1}^{a_1} \cdots \sigma_{i_l}^{a_l} \sigma_{j_1}^{a_1} \cdots \sigma_{j_l}^{a_l})] \delta_{2n,m+l-1} 
\]

which becomes

\[
\langle q^2_{2n} \rangle \langle \phi_s \rangle - \langle q^2_{2n+2} \rangle \langle \phi_s \rangle = \sum_{l=0}^{2n-s-1} \sum_{a_1 < \cdots < a_l} (-)^{2n-l} \binom{2n + s - l + 1}{2n - l} \times \\
\left[ \langle \phi_s q_{a_1}^{2} \cdots a_l q_{s+1}^{2} \cdots s+2n-l \rangle - \frac{2n - l + s + 2}{2n - l + 1} \langle \phi_s q_{a_1}^{2} \cdots a_l q_{s+1}^{2} \cdots s+2n-l+1 \rangle \right]. 
\]

(26)

In both the expressions above the term for \( l = 0 \) is understood to be one.

The right hand side of (26), due to the presence of \( 1 + J^{-1} \sigma \) in the right hand side of (21) - along with the symmetry of \( J \), makes the expansion somewhat recursive. This means that at each order we find some terms already found in the previous order. More precisely, we claim without proving that at each \( 2n \)-th order of the expansion, all the terms involving \( 2m \)-overlaps with \( 2m \leq 2n \) cancel out thanks to a repeated use of (25) with the relations coming from the lower orders. Hence from the \( 2n \)-th order we get new relations involving \( 2n + 2 \)-overlaps only. This is what we explicitly verified only for 4-overlaps in the previous pages. More explicitly, if we re-write the difference in the right hand side of (26) as

\[
\langle q_{2n}^2 \rangle \langle \phi_s \rangle - \langle q_{2n+2}^2 \rangle \langle \phi_s \rangle = c_{2n} - d_{2n+2}, 
\]

we have

\[
\langle q_{2n}^2 \rangle \langle \phi_s \rangle = c_{2n}, \quad \langle q_{2n+2}^2 \rangle \langle \phi_s \rangle = d_{2n+2}, \quad c_{2n} = d_{2n}.
\]
So that the final formula becomes

\[
\langle q_{2n}^2 \rangle \langle \phi_s \rangle = 
\sum_{l=0}^{2n+s-1} \sum_{a_1 < \ldots < a_l} (-)^{2n-l} \binom{2n + s - l + 1}{2n - l} \langle q_{a_1} \cdots q_{a_l} q_{s+1} \cdots s+2n-l \phi_s \rangle.
\]

### 4.1 Generalization to Smooth Functions of Multi-Overlaps

Just like for the family of identities discussed in the previous section, we started our analysis with the most natural quantity: the energy of our model with 2-spin interactions. And so we got again some relations for the squared multi-overlaps. But we already know how to generalize these formulas to smooth functions of the overlaps. We can consider \(p\)-spin interactions, and the procedure would provide us with the same relations for the \(p\)-th power of the overlaps. Then, as already explained, we can take a convergent sum over all integer \(p\) of \(p\)-spin Hamiltonians, and consider the self-averaging of the desired one among them. The perturbed Hamiltonian is again

\[
H_N(\sigma, \alpha; J) = - \sum_p \left[ a_p \sum_{\nu=1}^{P_p} J_{\nu} \sigma_{i_\nu}^p \cdots \sigma_{i_p}^p + b_p \lambda_p \sum_{\nu=1}^{P'_p} J'_{\nu} \sigma_{j_\nu}^p \cdots \sigma_{j_p}^p \right],
\]

where \(\sum_p |a_p|^2 = \sum_p |b_p|^2 = 1\), the notation for all the quenched variables is the usual one, and \(\{\lambda_p\}\) are the independent perturbing real parameters.

As a side remark, we just point out that (like in [8]), in the case of this second family of identities it is not necessary to consider a Hamiltonian consisting of the sum of all possible \(p\)-spin Hamiltonians: only the perturbation must be so.

### Concluding remarks

Notice that while we derived our identities having as reference diluted spin glasses, all that matters in the derivation are the properties of the perturbing Hamiltonian, and they are therefore generically valid.
The Ghirlanda-Guerra identities for the overlap have been useful to prove non trivial properties of mean-field spin glasses. For instance Talagrand could prove that for all models where the identities are valid, the support of the overlap probability function has positive support. This positivity property is important as it enters in the the Guerra free-energy bounds in spin system without spin reversal symmetry. The corresponding bounds for diluted systems involve all possible multioverlap. It has been proved [7] that the cavity method provides free-energy lower bounds for the random K-SAT problem for even K. Due to the difficulty of proving the positivity of the multioverlap, the bound does not apply to the odd K case. Proving the positivity would therefore allow to extend the bound to this case and in particular to the symbolic case K=3. Unfortunately the derivation of Talagrand for the overlap does not extend immediately to the multi-overlap case. We believe however that the self-averaging identity will be useful in the mathematical analysis of diluted spin models.

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