THE REAL TEST OF “TRIVIALITY”

ON THE LATTICE

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The generally accepted “triviality” of $\lambda \Phi^4$ theories does not forbid Spontaneous Symmetry Breaking but implies a trivially free shifted field which becomes effectively governed by a quadratic hamiltonian. As a consequence, one expects the one-loop potential to be exact. We present a lattice computation of the effective potential for massless $\lambda \Phi^4$ theory which nicely confirms the expectations based on “triviality”. Our results imply that the magnitude of the Higgs boson mass, beyond perturbation theory, does not represent a measure of the observable interactions in the scalar sector of the standard model.

Recently it has been emphasized that the generally accepted “triviality” of $(\lambda \Phi^4)_4$ is not in contradiction with Spontaneous Symmetry Breaking (SSB). Indeed, “triviality” means that the renormalized Green’s functions of the continuum theory can be expressed in terms of the first two moments of a gaussian distribution. This statement, by its very nature, does not forbid SSB but implies a trivially free shifted field which becomes effectively controlled by a quadratic hamiltonian.

Therefore, it is not surprising that precisely this result has been obtained in by analyzing the effective potential of massless $\lambda \Phi^4$ theories where SSB is discovered in those approximations to the effective potential (one-loop and gaussian) where the “Higgs” field $h(x)$ is free by definition. In fact, at one-loop the effects from the shifted field self-interaction are consistently neglected whereas in the gaussian approximation they are fully reabsorbed into the Higgs mass and the vacuum expectation value by obtaining a normal-ordered quadratic Hamiltonian which is equivalent to the Hartree-Fock-Bogolubov approximation.

“Triviality” implies the inconsistency of the usual perturbative approach to massless $\lambda \Phi^4$, which is based on the attempt to generate a cutoff-independent and non vanishing renormalized coupling at non-zero external momenta. One should beware of spurious contradictions which arise by using quasi-perturbative approximation methods to the effective potential which are inherently incompatible with the “generalized free field” nature of the field $h(x)$. Thus, perturbation theory, the loop expansion (beyond one-loop), and leading-log re-summation are all wholly misleading because they insist upon having a finite connected 4-point function at non-zero external momenta. The “triviality” of $(\lambda \Phi^4)_4$ theory implies that the bare interaction hamiltonian for the shifted field (the terms proportional to $h^3$ and $h^4$) does not produce any observable interactions. One can neglect it completely or treat it exactly. However, it is disastrous to take it into account in a perturbative or quasi-perturbative manner.

To provide further evidence on the validity of our picture we have undertaken the lattice computation of the effective potential along the same lines of Huang et al. In terms of the bare vacuum field $\phi_B = \langle \Phi_B \rangle$ and of the bare coupling $\lambda_B$ we obtain the well-known result (1) ($\omega^2(\phi_B) = \frac{\lambda_B \phi_B^2}{2}$ and $\Lambda$ is the euclidean ultraviolet cutoff):

$$ V^{1-\text{loop}}(\phi_B) = \frac{\Lambda^2}{4!} \phi_B^4 + \frac{\omega^4(\phi_B)}{64\pi^2} \left( \frac{\omega^2(\phi_B)}{\Lambda^2} - \frac{1}{2} \right). $$

This represents the classical potential plus the zero-point energy of the free $h$ field. “Triviality” implies that there are no $h$-interaction effects, so that this result should be exact. $V^{1-\text{loop}}$ can be re-expressed in the form

$$ V^{1-\text{loop}}(\phi_B) = \frac{\pi^2 \phi_B^4}{Z_\phi^2} \left( \frac{\phi_B}{v_B^2} - \frac{1}{2} \right), $$

(2)

where $Z_\phi = \frac{16\pi^2}{\lambda_B}$ from the minimum condition

$$ m_h^2 = \omega^2(v_B) = \frac{1}{2} \lambda_B v_B^2 = \Lambda^2 \exp(-\frac{2\pi^2}{3\lambda_B}), $$

(3)

By introducing the physical vacuum field $\phi_R^2 = \frac{\phi_B}{Z_\phi}$ which reabsorbs into its normalization all zero-momentum interaction effects, i.e. such that
\[
\frac{d^2 V^{1\text{-loop}}}{d\phi_R^2} |_{\phi_R=v_R} = m_h^2
\]

we obtain

\[
V^{1\text{-loop}}(\phi_R) = \pi^2 \phi_R^4 (\ln \frac{\phi_R^2}{v_R^2} - \frac{1}{2})
\]

and \(m_h^2 = 8\pi^2 v_R^2\). The non-perturbative nature of the vacuum field renormalization \(Z_\phi \sim 1/\lambda_B\), first discovered in the gaussian approximation by Stevenson and Tarrach [9], should not be confused with the \(\hbar\)-field wave function renormalization for which at one loop, where \(\hbar\) is a free field with mass \(\omega(\phi_B)\), one has trivially \(Z_\hbar = 1\). The structure \(Z_\phi \neq Z_\hbar\), allowed by the Lorentz-invariant nature of the field decomposition into \(p_\mu = 0\) and \(p_\mu \neq 0\) components [1], is more general than in perturbation theory and is the essential ingredient which allows SSB to coexist with “triviality”.

Differentiating Eq.(2), we obtain the bare “source”

\[
J(\phi_B) = \frac{dV^{1\text{-loop}}(\phi_B)}{d\phi_B} = \frac{4\pi^2 \phi_B^3}{Z_\phi^2} \ln \frac{\phi_B^2}{v_B^2},
\]

which can be compared with the lattice results for \(J = J(\phi_B)\) along the same lines as Ref. [4]. Strictly speaking, just as the effective potential is the convex envelope of Eq.(2) [2,3], Eq.(6) is valid only for \(|\phi_B| > v_B\) and \(J(\phi_B)\) should vanish in the presence of SSB in the range \(-v_B \leq \phi_B \leq v_B\). This means that the average bare field

\[
\phi_B(J) = \langle \Phi_B \rangle_J,
\]

which, for any \(J \neq 0\), should satisfy the relation

\[
\phi_B(J) = -\phi_B(-J),
\]

should tend to the limits

\[
\pm v_B = \lim_{J \to 0^\pm} \phi_B(J).
\]

In our case (we run on a \(10^4\) lattice), Eq.(8) is not well reproduced numerically at low values of \(J\) (for \(|J|a^3 \sim 0.01\) or smaller, \(a\) denoting the lattice spacing). As a consequence, the values of \(\phi_B\) (as well as of the higher-order Green’s functions) extracted from the direct computation in the broken case at \(J \sim 0\) are not reliable. Thus, we have to restrict to a “safe” region of \(J\)-values (in our case \(|J|a^3 \geq 0.05\) and extrapolate the values of \(v_B\) and \(Z_\phi\) from a fit to the data by using Eq.(6) once we have identified on the lattice the “massless” regime, i.e. corresponding to a renormalized theory with no intrinsic scale in its symmetric phase \((\Phi_B) = 0\). Usually, this would require to find the value of the bare mass-squared \(r_o\) in the euclidean lattice action

\[
a^4 \sum_x \left[ \sum_{\mu=1}^4 (\Phi_B(x+a e_\mu) - \Phi_B(x))^2 - \frac{1}{2} r_o \Phi_B(x)^2 + \frac{\lambda_o}{4} \Phi_B^4(x) \right]
\]

for which \(\frac{dV}{dx}|_{x=a} = 0\) [1] (in Eq.(9) we are using the same notations of [10] and replace in the fourth order coupling \(\frac{\lambda_B}{4} \rightarrow \frac{\lambda_o}{4}\) such that \(\lambda_o = 1\) implies \(\lambda_B = 6\)). However, the region around \(\phi_B = 0\) being not directly accessible, we have argued as follows. We start with the general expression [3]

\[
J(\phi_B) = \alpha \phi_B^2 \ln(\phi_B^2/v_B^2) + \beta v_B^2 \phi_B (1 - \phi_B^2/v_B^2),
\]

which is still consistent with “triviality” (corresponding to an effective potential given by the sum of a classical background and the zero point energy of a free field) but allows for an explicit scale-breaking term \(\beta\). Setting \(\alpha = 0\) one obtains a good description of the data in the “extreme double well” limit \(r_o\) much more negative than \(r_c\), where \(r_c\) corresponds to the onset of SSB) where SSB is a semi-classical phenomenon and the zero-point energy represents a small perturbation. Then we start to increase \(r_o\) at fixed \(\lambda_o\), toward the unknown value \(r_c\) and look at the quality of the fit with \((\alpha, \beta, v_B)\) as free parameters. The minimum allowed value of \(r_o\) such that the quality of the 2- parameter fit \((\alpha, \beta = 0, v_B)\) is exactly the same as in the more general 3- parameter case will define the “massless” case so that
we can fit the data for $J = J(\phi_B)$ with our Eq.(6). Finally, Eq.(3) suggests that the vacuum field $v_B$, in lattice units, vanishes extremely fast for $\lambda_B \to 0$. Thus, to avoid that noise and signal become comparable, we cannot run the lattice simulation at very small values of $\lambda_o$ but have to restrict to values $\lambda_o \sim 1$ such that still $\frac{\lambda}{\pi} \ll 1$ but $v_B$, in lattice units, is not too small. Smaller values of $\lambda_o$ ($\sim 0.3-0.4$), however, should become accessible with the largest lattices available today ($\sim 10^4$) where one should safely reach values $|J|a^3 \sim 0.001$ being still in agreement with Eq.(8). At $\lambda_o = 1$ we have identified the massless regime at a value $r_o = r_o$ where $r_o a^2 \sim -0.45$. By using the accurate weak-coupling relation between the bare mass and the euclidean cutoff [1]

$$r_o = \frac{\lambda_B}{32\pi^2} \Lambda^2 = \frac{3\lambda_o}{16\pi^2} \Lambda^2$$  \hspace{1cm} (11)

and using the relation $\Lambda = \frac{\pi y_Q}{\omega}$ (where $y_Q$ is expected to be $O(1)$) we obtain $y_Q \sim 1.55$. Also, the massless relation (3) predicts,

$$(avB)^{1\text{-loop}} = \frac{\pi y_L}{\sqrt{3\lambda_o}} \exp\left(-\frac{8\pi^2}{9\lambda_o}\right)$$ \hspace{1cm} (12)

and

$$Z_\phi^{1\text{-loop}} = \frac{8\pi^2}{3\lambda_o}.$$ \hspace{1cm} (13)

In Eq.(12) we have used $y_L$ rather than $y_Q$ since one does not expect precisely the same coefficient to govern the relation between euclidean cutoff and lattice spacing for both quadratic and logarithmic divergences (see below).

We have used an ALPHA-VAX to compute with the Metropolis algorithm on a $10^4$ lattice. We started by comparing our results with those obtained by Huang et al. [10] and we found excellent agreement (better than 1%) with their curves $J = J(\phi_B)$ in the range $|J|a^3 \geq 0.05$. The other numerical results of [10], however, should be carefully reanalyzed due to two reasons : 1) they have directly evaluated the 1- and 2- point Green’s functions at $\lambda = 0$. As discussed above, the computations at low $J$ have large errors due to the finite size of the lattice producing unphysical violations of Eq.(8) and invalidating all calculations at $J = 0$ ; 2) there are uncontrolled errors from their assumption $Z_\phi = Z_h$ in the evaluation of the shifted field propagator.

Our numerical values for $\phi_B(J)$ and the results of the 2-parameter fits to the data by using Eq.(6) for $\lambda_o = 0.8,1.0$ and 1.2 are shown in Table I together with the one-loop prediction (13). The change of $r_o$ with $\lambda_o$ is computed by using Eq.(11) and our result $r_o a^2 \sim -0.45$ for $\lambda_o = 1$. After 60,000 iterations, which corresponds to the results of Table I, the average field $\phi_B(J)$ is stable at the level of the first three significant digits for all $J$-values. At large $J$, the stability is at the level of a few $10^{-4}$ and finite size effects are negligible. This has been checked with a few runs on a $16^4$ lattice and by comparing with Eq.(8). At lower $J$, comparison with Eq.(8) suggests that the third digit is affected by finite size effects. Many more details of the lattice calculation, including a critical comparison between Monte Carlo and the Langevin formulation of the lattice theory, will be reported elsewhere [14].

By fixing $Z_\phi$ to its one-loop value (13) and using Eq.(12) for $avB$ we can determine the value of $y_L$ from the data at $\lambda_o = 1$. We obtain $y_L = 2.067 \pm 0.007$ with a $\chi^2 \over \text{d.o.f.} = 21$. Once we know $y_L$ we can predict the value of $avB = (avB)^{\text{Th}}$ at $\lambda_o=0.8$ and 1.2 . The comparison with the 1-parameter fits in Table II shows a remarkable agreement between one loop predictions and Monte Carlo data. This result is not a trivial test of perturbation theory but, rather, a non perturbative test of “triviality”. In fact, we have compared with the leading-logarithmically improved version of Eq.(6)

$$J^{\text{LL}}(\phi_B) = \frac{\lambda_o \phi_B^3}{1 + \frac{9\lambda_o^2}{8\pi} \ln \frac{x_{\text{LL}}}{a|\phi_B|}}$$

($x_{\text{LL}}$ denoting an adjustable parameter). We find, for 21 degrees of freedom, $(\chi^2)_{\text{LL}} = 53, 163, 365$ for $\lambda_o = 0.8,1.0,1.2$ respectively, with an unacceptably low confidence level (less than 0.001 in the best case). Also, the low $\chi^2$- values of the one-loop 1-parameter fits suggest that, probably, we are still overestimating the errors and the importance of the finite size effects.

In conclusion, the lattice computations nicely confirm the conjecture [14][15] that, as a consequence of “triviality”, the one-loop potential is exact. Indeed, there is a well defined region in the space of the bare parameters $(r_o, \lambda_o)$, controlled by Eq.(11), where the effective potential is described by its one-loop approximation to very high accuracy. By increasing the lattice size, our analysis can be further extended towards the physically relevant point $\frac{\lambda}{\pi} \to 0^+$,
\( r_o a^2 \to 0^- \) which corresponds, in the limit of quantum field theory, to “dimensional transmutation” \[11\] from the classically scale invariant case. The massless version of \( \lambda \Phi^4 \) theories, although “trivial” is not entirely trivial providing, at the same time, SSB and a meaningful continuum limit \( \Lambda \to \infty, \lambda_B \to 0^+ \) such that the mass of the free shifted field in Eq.(3) is cutoff independent. As such it represents the real candidate to generate SSB in the standard model from the pure scalar sector with \( \nu_R \) in Eqs.(4,5) related to the Fermi constant. The consequences of our results are substantial. As discussed in \[1,2,5–7\] and first pointed out in \[10\], the magnitude of the Higgs mass, beyond perturbation theory, does not represent a measure of the observable interactions in the scalar sector of the theory.

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TABLE I. The values of $\phi_B(J) = \langle \Phi_B \rangle_J$ for the massless case are reported as discussed in the text. At the various values of $\lambda_o$ and $r_o$ we also show the results of the 2-parameter fit with Eq.(6) and the one loop prediction (13).

| $Ja^3$ | $\lambda_o = 0.8 \ r_o a^2 = -0.36$ | $\lambda_o = 1.0 \ r_o a^2 = -0.45$ | $\lambda_o = 1.2 \ r_o a^2 = -0.54$ |
|--------|---------------------------------|---------------------------------|---------------------------------|
| -0.700 | -1.0024 ± 0.3 · 10^{-3}         | -0.9401 ± 0.3 · 10^{-3}         | -0.8935 ± 0.3 · 10^{-3}         |
| -0.600 | -0.9540 ± 0.3 · 10^{-3}         | -0.8950 ± 0.3 · 10^{-3}         | -0.8512 ± 0.3 · 10^{-3}         |
| -0.500 | -0.8997 ± 0.5 · 10^{-3}         | -0.8444 ± 0.3 · 10^{-3}         | -0.8037 ± 0.3 · 10^{-3}         |
| -0.400 | -0.8371 ± 0.5 · 10^{-3}         | -0.7867 ± 0.5 · 10^{-3}         | -0.7493 ± 0.5 · 10^{-3}         |
| -0.300 | -0.763 ± 1.0 · 10^{-3}          | -0.718 ± 1.0 · 10^{-3}          | -0.685 ± 1.0 · 10^{-3}          |
| -0.200 | -0.670 ± 1.0 · 10^{-3}          | -0.631 ± 1.0 · 10^{-3}          | -0.603 ± 1.0 · 10^{-3}          |
| -0.150 | -0.611 ± 1.5 · 10^{-3}          | -0.576 ± 1.0 · 10^{-3}          | -0.551 ± 1.0 · 10^{-3}          |
| -0.125 | -0.576 ± 1.5 · 10^{-3}          | -0.544 ± 1.5 · 10^{-3}          | -0.521 ± 1.0 · 10^{-3}          |
| -0.100 | -0.537 ± 2.0 · 10^{-3}          | -0.508 ± 2.0 · 10^{-3}          | -0.486 ± 2.4 · 10^{-3}          |
| -0.075 | -0.490 ± 2.4 · 10^{-3}          | -0.464 ± 2.0 · 10^{-3}          | -0.446 ± 2.4 · 10^{-3}          |
| -0.050 | -0.430 ± 2.6 · 10^{-3}          | -0.409 ± 3.0 · 10^{-3}          | -0.393 ± 2.4 · 10^{-3}          |
| 0.050  | 0.427 ± 2.6 · 10^{-3}           | 0.406 ± 3.0 · 10^{-3}           | 0.391 ± 2.4 · 10^{-3}           |
| 0.075  | 0.488 ± 2.4 · 10^{-3}           | 0.462 ± 2.0 · 10^{-3}           | 0.444 ± 2.4 · 10^{-3}           |
| 0.100  | 0.535 ± 2.0 · 10^{-3}           | 0.506 ± 2.0 · 10^{-3}           | 0.484 ± 2.4 · 10^{-3}           |
| 0.125  | 0.575 ± 1.5 · 10^{-3}           | 0.543 ± 1.5 · 10^{-3}           | 0.520 ± 1.0 · 10^{-3}           |
| 0.150  | 0.609 ± 1.5 · 10^{-3}           | 0.575 ± 1.0 · 10^{-3}           | 0.550 ± 1.0 · 10^{-3}           |
| 0.200  | 0.669 ± 1.0 · 10^{-3}           | 0.631 ± 1.0 · 10^{-3}           | 0.602 ± 1.0 · 10^{-3}           |
| 0.300  | 0.762 ± 1.0 · 10^{-3}           | 0.717 ± 1.0 · 10^{-3}           | 0.684 ± 1.0 · 10^{-3}           |
| 0.400  | 0.8367 ± 0.5 · 10^{-3}          | 0.7863 ± 0.5 · 10^{-3}          | 0.7488 ± 0.5 · 10^{-3}          |
| 0.500  | 0.893 ± 0.5 · 10^{-3}           | 0.8445 ± 0.3 · 10^{-3}          | 0.8035 ± 0.3 · 10^{-3}          |
| 0.600  | 0.9540 ± 0.3 · 10^{-3}          | 0.8951 ± 0.3 · 10^{-3}          | 0.8513 ± 0.3 · 10^{-3}          |
| 0.700  | 1.0024 ± 0.3 · 10^{-3}          | 0.9402 ± 0.3 · 10^{-3}          | 0.8937 ± 0.3 · 10^{-3}          |

$Z_0 = 33.3 ± 0.6$  $Z_0 = 26.0 ± 0.4$  $Z_0 = 21.3 ± 0.3$

$Z_{\phi}^{1\text{-loop}} = 32.9$  $Z_{\phi}^{1\text{-loop}} = 26.3$  $Z_{\phi}^{1\text{-loop}} = 21.9$

TABLE II. By using Eq.(6), we show the results of the 1-parameter fits for $av_B$ at $\lambda_o = 0.8$ and 1.2 when $Z_\phi$ is constrained to its one-loop value in Eq.(13). We also show the predictions from Eq.(12), $(av_B)^{Th}$, for $y_L = 2.067 ± 0.007$ as determined from the fit to the data at $\lambda_o = 1$.

| $\lambda_o = 0.8 \ r_o a^2 = -0.36$ | $\lambda_o = 1.2 \ r_o a^2 = -0.54$ |
|---------------------------------|---------------------------------|
| $Z_\phi = 32.90 = fixed$        | $Z_\phi = 21.93 = fixed$        |
| $av_B = (7.30 ± 0.03)10^{-5}$   | $av_B = (2.28 ± 0.01)10^{-4}$   |
| $\chi^2 / df = 3 / 27$         | $\chi^2 / df = 13 / 27$         |

$(av_B)^{Th} = (7.24 ± 0.03)10^{-5}$  $(av_B)^{Th} = (2.29 ± 0.01)10^{-3}$