NASH STRATEGIES FOR THE INVERSE INCLUSION
CAUCHY-STOKES PROBLEM

ABDERRAHMANE HABBAL*
Université Côte d’Azur, Inria, CNRS, LJAD, UMR 7351
Parc Valrose, Nice 06108, France

MOEZ KALLEL AND MARWA OUNI
Université de Tunis El Manar
Ecole Nationale d’Ingénieurs de Tunis, LAMSIN
BP 37, 1002 Tunis Belvedere, Tunisia

(Communicated by Fioralba Cakoni)

Abstract. We introduce a new algorithm to solve the problem of detecting
unknown cavities immersed in a stationary viscous fluid, using partial boundary
measurements. The considered fluid obeys a steady Stokes regime, the cavities
are inclusions and the boundary measurements are a single compatible pair
of Dirichlet and Neumann data, available only on a partial accessible part of
the whole boundary. This inverse inclusion Cauchy-Stokes problem is ill-posed
for both the cavities and missing data reconstructions, and designing stable
and efficient algorithms is not straightforward. We reformulate the problem as
a three-player Nash game. Thanks to an identifiability result derived for the
Cauchy-Stokes inclusion problem, it is enough to set up two Stokes boundary
value problems, then use them as state equations. The Nash game is then
set between 3 players, the two first targeting the data completion while the
third one targets the inclusion detection. We used a level-set approach to get
rid of the tricky control dependence of functional spaces, and we provided the
third player with the level-set function as strategy, with a cost functional of
Kohn-Vogelius type. We propose an original algorithm, which we implemented
using Freefem++. We present 2D numerical experiments for three different
test-cases. The obtained results corroborate the efficiency of our 3-player Nash
game approach to solve parameter or shape identification for Cauchy problems.

1. Introduction. Fluid dynamics are central in many industrial, biological and
biomedical processes. The good functioning of the involved systems could be dra-
matically damaged in the presence of undesired small obstacles (impurities) or in-
closures (cavitation). For example, polymer material degradation is related to the
formation of inclusions during polymer extrusion [42]; as well, the mechanism of
joint cracking is related to cavity formation [34].

A large spectrum of the processes above can be considered as Stokes flows, though
they should be taken unsteady and anisotropic to render satisfactorily the complex
phenomenon of the formation of cavities [43]. The shape and location of the inclu-
sions is generally out of reach for direct observation, hence the need for effective
nondestructive monitoring solutions, known as geometric inverse problems when

2010 Mathematics Subject Classification. Primary: 49J20, 65K10; Secondary: 65N06, 90C30.
Key words and phrases. Data completion, Cauchy-Stokes problem, shape identification, Nash
games.

* Corresponding author: A. Habbal.
mathematics and algorithms are involved. Popular mathematical models build on the assumption that some specific measurements are available over the whole boundary of the structure under investigation, dealing with partial differential equations of boundary value -BVP- type. However, it should be noticed that from a technological point of view, when industrial devices are involved, the assumption above is in general impossible to fulfill, either because it is too expensive, or simply because part of the boundary is not accessible to probing, think of a heart valve [38]. Such restrictions lead to develop complex protocols like for the detection of flaws in metal melts in foundry industry [33]. Industrial solutions use in general protocols where emission and reception of the probing signals are set on the same location of the boundary. From a mathematical point of view, we have access to over-specified boundary data (e.g. temperature and thermal flux) on the probing location, and no data elsewhere. Thus, we deal with partial differential equations, having access to over specified boundary data, and missing data to recover as well as unknown inclusions to detect. We are then in the framework of geometric inverse problems for the so called Cauchy-Stokes system. We shall restrict ourselves to the case of steady and Newtonian Stokes flows.

Figure 1. An example of the geometric configuration of the problem: the whole domain including cavities is denoted by \( \Omega \). It contains an inclusion \( \omega^* \). The boundary of \( \Omega \) is composed of \( \Gamma_c \), an accessible part where over-specified data are available, and an inaccessible part \( \Gamma_i \) where the data are missing.

Let us introduce a preliminary mathematical description of the problem. Consider a bounded open domain \( \Omega \subset \mathbb{R}^d \) (d=2, 3) occupied by an incompressible viscous fluid, see Figure-1. We assume that the outer boundary of \( \Omega \) is sufficiently smooth and composed of two parts \( \Gamma_c \) and \( \Gamma_i \). Let \( \omega^* \subset \subset \Omega \) be an unknown inclusion immersed in \( \Omega \). The Cauchy-Stokes geometric inverse problem considered here consists, then, from given velocity \( f \) and fluid stress forces \( \Phi \) prescribed only on the accessible part \( \Gamma_c \) of the boundary, to identify \( \omega^* \in \mathcal{D}_{ad} \) (a set of admissible shapes defined later) such that the fluid velocity \( u \) and the pressure \( p \) are solution of the following Stokes problem:

\[
\begin{align*}
\nu \Delta u - \nabla p &= 0 \quad \text{in} \quad \Omega \setminus \omega^*, \\
\text{div} u &= 0 \quad \text{in} \quad \Omega \setminus \omega^*, \\
\sigma(u,p)n &= 0 \quad \text{on} \quad \partial \omega^*, \\
u u &= f \quad \text{on} \quad \Gamma_c, \\
\sigma(u,p)n &= \Phi \quad \text{on} \quad \Gamma_c,
\end{align*}
\]

(1)
where $n$ is the unit outward normal vector on the boundary, and $\sigma(u, p)$ the fluid stress tensor defined as follows:

$$
\sigma(u, p) = -pI_d + 2\nu D(u)
$$

with $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ being the linear strain tensor and $I_d$ denotes the $d \times d$ identity matrix. For the sake of simplicity, from now on, the viscosity $\nu$ of the fluid is set to $\nu = 1$.

Additionally to the geometric identification problem (i.e. detect the inclusions $\omega^*$) one has to complete the boundary data, that is to recover the missing traces of the velocity $u$ and of the normal stress $\sigma(u, p).n$ over $\Gamma_i$ the inaccessible part of the boundary. Remark that the difference between obstacles and inclusions amounts to which boundary condition is used: homogeneous Dirichlet one for the obstacles and homogeneous Neumann condition for the inclusions (considered as free surfaces).

Even when restricted to elliptic equations, mostly Laplace and Stokes systems, there exists a prolific literature dedicated to each of these two problems separately, and because of their well known ill-posedness (in the sense of Hadamard) [29], most of the literature addresses as well (if not exclusively) the ensuing stability and other computational issues. For the Cauchy problem, far from being exhaustive, an excerpt of popular approaches are the least-square penalty techniques used in [25] and in the earlier paper [26], Tikhonov regularization methods [21], quasi reversibility methods [12], alternating iterative methods [36] [31] and control type methods [7] [1]. Recently, an approach based on game theory, using decentralized strategies, was proposed in [28]. This work has been extended in [32], in the image inpainting problem for a nonlinear Cauchy problem and in [19] for the solution of coupled conductivity identification and data completion in cardiac electrophysiology.

Let us mention that many of the papers dedicated to data completion or to obstacle detection and based on control or optimization approaches, minimize a so-called Kohn-Vogelius type functional, an energy error function introduced in the framework of parameter identification in [35].

Regarding the obstacle identification problem, a more challenging geometric inverse problem, and again with a very partial on the existing literature, the authors in [4] address the obstacle detection problem for unsteady Stokes and Navier-Stokes flows, quasi reversibility coupled to a level set approach is used in [13] for the Laplace equation, shape optimization [15] and topological gradient [17] are used for the Stokes system, and in [10] stability issues are addressed for the inverse obstacle problem in a Stokes flow.

In contrast, rather a few papers address the joint geometric and data completion inverse problems, at least regarding its computational aspects. Close to our present work, the inverse obstacle problem for the Cauchy-Laplace equation is studied in [18] where a control-type approach is used and applied to a Kohn-Vogelius functional. In [14] the authors use quasi reversibility coupled to a level set approach to solve the inverse obstacle problem for the Cauchy-Stokes equations. A formulation based on nonlinear integral equations arising from the reciprocity gap [6] principle is used in [5].

Presently, we consider the inverse inclusion problem for the Cauchy-Stokes system. In order to solve the joint completion/detection problem, we reformulate it as a three players Nash game, following the ideas introduced earlier in [28] to solve the Cauchy-Laplace (completion) problem.

The game is defined as follows: first, the Cauchy-Stokes problem is formulated as two boundary value problems (BVP). The first BVP defines the first player, it
inherits the available Dirichlet data \( f \) specified on the boundary \( \Gamma_c \), and has control on a Neumann data set over the inaccessible boundary \( \Gamma_i \), the latter control being the first player’s strategy aimed at minimizing the gap over \( \Gamma_c \) between first player’s normal stress and the prescribed normal stress \( \Phi \). The second BVP defines the second player, as it inherits the available normal stress data \( \Phi \) set over \( \Gamma_c \), and uses Dirichlet data set over the inaccessible boundary \( \Gamma_i \) as strategy variables. The second player’s Dirichlet strategy is aimed at minimizing the gap over \( \Gamma_c \) between second player’s and the prescribed Dirichlet data \( f \). The fading and regularizing difference between the solutions to these two BVPs is shared by the two players. The third player has no own BVP, but has access to the two previous ones, and uses as control variable the shape of the inclusion(s). The third player’s criteria to minimize is a Kohn-Vogelius type functional. The three players play a static Nash game with complete information, whose relevant solution concept is the so-called Nash equilibrium (NE).

We shall present and prove some theoretical results for the Cauchy-Stokes problem, precisely that a Nash equilibrium exists and is unique, and coincides with the missing data as soon as the Cauchy problem has a solution (that is, when the over specified data are compatible). Then, we propose a new algorithm dedicated to the joint computation of the missing data and the obstacle shapes. In this algorithm, a Nash subgame is played by the completion first and second players in order to precondition the Cauchy problem and tackle its ill-posedness. A level set approach is used for the latter geometric identification problem. We lead a sensitivity analysis, and present several numerical experiments that corroborate the efficiency of our approach and its nice stability with respect to noisy data.

The paper is organized as follows. In Section 2, we extend our previous [28] Nash game approach to the data completion for the steady Stokes flows. In view of the formulation of the geometric inverse problem, we first recall in Section 3 a now classical identifiability proof [4] usually established for obstacles, so with homogeneous Dirichlet boundary condition, with a minor adaption to fit the case of inclusions, whose boundary conditions are of homogeneous Neumann type. Then, we formulate in Section 4 the Nash game approach to tackle the joint completion and geometric identification problem. We detail our algorithm, and some numerical aspects of the level set method, used to capture the inclusion boundary. Section 5 is devoted to the presentation of three numerical 2D test cases which assess the ability of our algorithm to jointly recover the missing boundary data and the location and shape of the inclusions as well. We finally draw some concluding remarks in Section 6.

2. Data completion for the Stokes problem. We consider in the present section the case where possible obstacles or inclusions are known, which amounts to simply not consider them, focusing solely on the data completion problem. In the following, we apply the Nash game formulation of [28] to the Stokes problem. Results and proofs in the cited reference extend easily to the present case.

With the previous notations, let be \( f \in H^1(\Omega)^d \) and \( \Phi \in H^1(\Gamma)^d \) given Cauchy data. The Cauchy-Stokes problem is stated as follows: find \( u \in H^1(\Omega)^d \) and \( p \in L^2(\Omega) \) such that

\[
\begin{align*}
\Delta u - \nabla p &= 0 \quad \text{in} \quad \Omega, \\
\text{div} u &= 0 \quad \text{in} \quad \Omega, \\
u &= f \quad \text{on} \quad \Gamma_c, \\
\sigma(u,p)n &= \Phi \quad \text{on} \quad \Gamma_c.
\end{align*}
\]
The data completion problem, which is simply a reformulation of the Cauchy-Stokes one, amounts to find \( \tau^* \in H^\frac{1}{2}(\Gamma)^d \) and \( \eta^* \in H^\frac{1}{2}(\Gamma)^d \) such that \( u = \tau^* \) and \( \sigma(u, p) n = \eta^* \) over \( \Gamma_1 \).

For any given \( \eta \in H^\frac{1}{2}(\Gamma)^d \) and \( \tau \in H^\frac{1}{2}(\Gamma)^d \), we define the states \((u_1(\eta), p_1(\eta)) \in H^1(\Omega)^d \times L^2(\Omega) \) and \((u_2(\tau), p_2(\tau)) \in H^1(\Omega)^d \times L^2(\Omega) \) as the unique weak solutions of the following Stokes mixed boundary value problems \((SP_1)\) and \((SP_2)\):

\[
(SP_1) \begin{cases}
\Delta u_1 - \nabla p_1 = 0 & \text{in } \Omega, \\
\text{div} u_1 = 0 & \text{in } \Omega, \\
u_1 = f & \text{on } \Gamma_c, \\
\sigma(u_1, p_1) n = \eta & \text{on } \Gamma_1,
\end{cases}
\]

\[
(SP_2) \begin{cases}
\Delta u_2 - \nabla p_2 = 0 & \text{in } \Omega, \\
\text{div} u_2 = 0 & \text{in } \Omega, \\
u_2 = \tau & \text{on } \Gamma_1, \\
\sigma(u_2, p_2) n = \Phi & \text{on } \Gamma_1.
\end{cases}
\]

The existence and uniqueness of solutions to \((SP_1)\) and \((SP_2)\) can be derived from the general theory on existence of solutions to the incompressible steady state Stokes equations, which can be found e.g. in [22, 45]. See also [16] annex A.1 which is suitable to the Cauchy-Stokes framework of the resent paper. Notice that, thanks to the assumption on Cauchy data, we have \((u_1(\eta), p_1(\eta)) \in H^2(\Omega)^d \times H^1(\Omega)^d \).

We then define the following cost functionals:

\[
J_1(\eta, \tau) = \frac{1}{2} \|\sigma(u_1(\eta), p_1(\eta)) n - \Phi\|_{H^\frac{1}{2}(\Gamma)}^2 + \frac{1}{2} \|u_1(\eta) - u_2(\tau)\|_{H^\frac{1}{2}(\Gamma)}^2
\]

\[
J_2(\eta, \tau) = \frac{1}{2} \|u_2(\tau) - f\|_{H^\frac{1}{2}(\Gamma)}^2 + \frac{1}{2} \|u_1(\eta) - u_2(\tau)\|_{H^\frac{1}{2}(\Gamma)}^2
\]

We are now in a position to formulate the two-player Nash game. The first player is defined by its strategy \( \eta \in H^\frac{1}{2}(\Gamma)^d \) and cost \( J_1 \), while the second one has control on \( \tau \in H^\frac{1}{2}(\Gamma)^d \) and aims at minimizing the cost \( J_2 \). The two players play a static Nash game with complete information. The most popular solution concept for such games is the one of a Nash equilibrium (NE) given by the

**Definition 2.1.** A strategy pair \((\eta_N, \tau_N) \in H^\frac{1}{2}(\Gamma)^d \times H^\frac{1}{2}(\Gamma)^d \) is a Nash equilibrium if the following holds:

\[
\begin{cases}
J_1(\eta_N, \tau_N) \leq J_1(\eta, \tau_N), & \forall \eta \in H^\frac{1}{2}(\Gamma)^d, \\
J_2(\eta_N, \tau_N) \leq J_2(\eta_N, \tau), & \forall \tau \in H^\frac{1}{2}(\Gamma)^d.
\end{cases}
\]

Recall that, similarly as [28], this two players solve in parallel the associated BVP’s \((SP_1)\) and \((SP_2)\). Their respective objectives involve the gap between the non used Neumann/Dirichlet known data and the traces of the BVP’s solutions over the accessible boundary \( \Gamma_c \), plus a common coupling term \( \frac{1}{2} \|u_1(\eta) - u_2(\tau)\|_{H^\frac{1}{2}(\Gamma)}^2 \).

This term depends on both \( \eta \) and \( \tau \), has a regularizing effect in partial minimization, \( u_1 \) (resp. \( u_2 \)) is fixed in the partial minimization process of \( J_2 \) (resp. \( J_1 \)). Furthermore, the partial mapping \( \eta \mapsto J_1(\eta, \tau) \) (resp. \( \tau \mapsto J_2(\eta, \tau) \)) is a quadratic strongly convex functional over \( H^\frac{1}{2}(\Gamma)^d \) (resp. \( H^\frac{1}{2}(\Gamma)^d \)). This partial ellipticity property of \( J_1 \) holds uniformly w.r.t. \( \tau \), and conversely for \( J_2 \). It allows to restrict the search for Nash equilibrium in data completion to a bounded subsets of the strategy spaces, which remains consistent with the classical results of conditional stability of Cauchy problem (see e.g. [2]).

The recourse to a game formulation and to a NE solution finds its justification in the following result:
Proposition 1. Consider the Nash game defined above, with costs given by (2) and (3).

(i) There always exists a unique Nash equilibrium \((\eta_N, \tau_N) \in H^\frac{1}{2}(\Gamma)^d \times H^\frac{1}{2}(\Gamma)^d\), which is also the minimum of the potential 

\[
L(\eta, \tau) = \frac{1}{2}\|\sigma(u_1(\eta), p_1(\eta))n - \Phi\|^2_{H^\frac{1}{2}(\Gamma)^d} + \frac{1}{2}\|u_2(\tau) - f\|^2_{H^\frac{1}{2}(\Gamma)^d} + \frac{1}{2}\|u_1(\eta) - \tau\|^2_{H^\frac{1}{2}(\Gamma)^d}.
\]

(ii) If the Cauchy problem has a solution \((u,p)\), then \((u_1(\eta_N), p_1(\eta_N)) = (u_2(\tau_N), p_2(\tau_N)) = (u, p)\) and \((\eta_N, \tau_N)\) are the missing data, i.e. \(\eta_N = \sigma(u, p)n|_\Gamma\) and \(\tau_N = u|_\Gamma\).

Proof. (i). We first prove the uniqueness of a NE. It is easy to check that the potential \(L\) is strictly convex by computing its second order differential with respect to \((\eta, \tau)\), see [1]. Thus, \(L\) has at most a one minimum. Moreover, if it exists, the minimum of \(L\) is a Nash equilibrium, and conversely. Indeed, let be \((\eta_0, \tau_0)\) the minimum of \(L\), then, we have

\[
\begin{cases}
L(\eta_0, \tau_0) \leq L(\eta, \tau_0), & \forall \eta \in H^\frac{1}{2}(\Gamma)^d, \\
L(\eta_0, \tau_0) \leq L(\eta_0, \tau), & \forall \tau \in H^\frac{1}{2}(\Gamma)^d.
\end{cases}
\]

Thanks to the specific structure of \(L\), this is equivalent to write

\[
\begin{cases}
J_1(\eta_0, \tau_0) \leq J_1(\eta, \tau_0), & \forall \eta \in H^\frac{1}{2}(\Gamma)^d, \\
J_2(\eta_0, \tau_0) \leq J_2(\eta_0, \tau), & \forall \tau \in H^\frac{1}{2}(\Gamma)^d.
\end{cases}
\]

That is, \((\eta_0, \tau_0)\) is a Nash equilibrium. Conversely, if \((\eta_0, \tau_0)\) is a Nash equilibrium then,

\[
\begin{cases}
J_1(\eta_0, \tau_0) \leq J_1(\eta, \tau_0), & \forall \eta \in H^\frac{1}{2}(\Gamma)^d, \\
J_2(\eta_0, \tau_0) \leq J_2(\eta_0, \tau), & \forall \tau \in H^\frac{1}{2}(\Gamma)^d.
\end{cases}
\]

Adding the term \(\frac{1}{2}\|u_2(\tau) - f\|^2_{H^\frac{1}{2}(\Gamma)^d}\) in the first inequality and the term \(\frac{1}{2}\|\sigma(u_1, p_1)n - \Phi\|^2_{H^\frac{1}{2}(\Gamma)^d}\) in the second one, we get,

\[
\begin{cases}
L(\eta_0, \tau_0) \leq L(\eta, \tau_0), & \forall \eta \in H^\frac{1}{2}(\Gamma)^d, \\
L(\eta_0, \tau_0) \leq L(\eta_0, \tau), & \forall \tau \in H^\frac{1}{2}(\Gamma)^d.
\end{cases}
\]

By the optimality conditions, we have

\[
\begin{align*}
\frac{\partial L}{\partial \eta}(\eta_0, \tau_0) &= 0, \\
\frac{\partial L}{\partial \tau}(\eta_0, \tau_0) &= 0,
\end{align*}
\]

thus, \((\eta_0, \tau_0)\) is the minimum of \(L\), the uniqueness of which implies that of the Nash equilibrium.

The proof of existence follows the same lines as in [28], the main ingredient being the uniform ellipticity of the convex partial maps \(\eta \rightarrow J_1(\eta, \tau)\) and of \(\tau \rightarrow J_2(\eta, \tau)\) which allows for a direct application of the Nash Theorem, see ibidem references to the Nash games and theorem.

(ii). If we assume that the Cauchy-Stokes problem has a solution \((u, p)\), which is then unique by the unique continuation property, proved by Fabre and Lebeau.
In [24], then let us define the following \( \eta_C = \sigma(u,p)|_{\Gamma} \) and \( \tau_C = u|_{\Gamma} \). It is then straightforward to check that the solutions \((u_1(\eta_C), p_1(\eta_C))\) to \((SP_1)\) and \((u_2(\tau_C), p_2(\tau_C))\) to \((SP_2)\) coincide with the Cauchy solution \((u,p)\), thanks to the uniqueness of the solution of the boundary value Stokes problem. Thus, \( L(\eta_C, \tau_C) = 0 \) so that \((\eta_C, \tau_C)\) is a minimum of \( L \geq 0 \). Thanks to the uniqueness result above, \((\eta_N, \tau_N) = (\eta_C, \tau_C)\).

For the computation of the NE for the Cauchy-Stokes problem, we used a popular algorithm [8] which amounts basically to solve iteratively the following coupled problem, using gradient descent methods,

\[
(\eta_N, \tau_N) = \arg\min_{\eta, \tau} J_1(\eta, \tau), \\
(\eta_N, \tau_N) = \arg\min_{\eta, \tau} J_2(\eta, \tau).
\]

We describe in Algorithm 1 below the main steps of the method, with a version whose proof is given in Appendix A.1.

**Algorithm 1:** Computation of a Cauchy-Stokes Nash equilibrium

Given : \( \varepsilon > 0 \) a convergence tolerance, \( K_{\text{max}} \) a computational budget, \( \sigma \) a noise level and \( \rho(\sigma) \) a -tuned- function which depends on the noise.

Choose an initial guess \( S^{(0)} = (\eta^{(0)}, \tau^{(0)}) \in H^{\frac{1}{2}}(\Gamma_i)^d \times H^{\frac{1}{2}}(\Gamma_i)^d \). Set \( k = 1 \).

- **Step 1:** Compute \( \eta^{(k)} \) solution of \( \min_{\eta, \tau} J_1(\eta, \tau^{(k-1)}) \) and determine \( \eta^{(k)} = t\eta^{(k-1)} + (1-t)\eta^{(k)} \) with \( 0 \leq t < 1 \).

- **Step 2:** Compute \( \tau^{(k)} \) solution of \( \min_{\eta, \tau} J_2(\eta^{(k-1)}, \tau) \) and determine \( \tau^{(k)} = t\tau^{(k-1)} + (1-t)\tau^{(k)} \) with \( 0 \leq t < 1 \).

- **Step 3:** Compute \( s_k = \|u_2^{(k)} - f^*\|_{L^2(\Gamma_i)} \), where \((u_2^{(k)}, p_2^{(k)})\) is the solution of the following direct problem

\[
\begin{cases}
\Delta u_2^{(k)} - \nabla p_2^{(k)} = 0 & \text{in } \Omega, \\
\text{div} u_2^{(k)} = 0 & \text{in } \Omega, \\
u_2^{(k)} = \tau^{(k)} & \text{on } \Gamma_i, \\
\sigma(u_2^{(k)}, p_2^{(k)}) n = \Phi & \text{on } \Gamma_c.
\end{cases}
\]

While \( s_k \geq \rho(\sigma)\varepsilon \) and \( k < K_{\text{max}} \) set \( k = k + 1 \), return back to step 1.

The gradient descent methods used to solve steps 1 and 2 in the algorithm above do require the computation of the gradients of the costs \( J_1 \) and \( J_2 \), with respect to their respective strategies. The fast computation of the latter is classical, and led by means of an adjoint state method, as shown by the Proposition 2 below, whose proof is given in Appendix A.1.

We shall use the following classical notation:

\[
H^1_0(\Omega) = \{ \varphi \in H^1(\Omega)^d \mid /\varphi|_\Gamma = 0 \} \quad \text{and} \quad H^2_0(\Omega) = \{ \varphi \in H^2(\Omega)^d \mid /\varphi|_\Gamma = 0 \}
\]

whenever \( \Gamma \) is a non empty subset of the boundary of \( \Omega \).

**Proposition 2.** We have the following two partial derivatives:
Remark 1. The existence and uniqueness of the solutions to the problems \((AP_1)\) and \((AP_2)\), namely the adjoint states \((\lambda_1, \kappa_1) \in H^1_0(\Omega) \times L^2(\Omega)\) and \((\lambda_2, \kappa_2) \in H^1(\Omega)^d \times L^2(\Omega)\) is straightforward, thanks to the regularity assumption on the Cauchy data \((f, \Phi) \in H^2(\Omega)^d \times H^1(\Omega)^d\) and to the regularity \(H^2(\Omega)^d \times H^1(\Omega)^d\) results on the solutions to the Stokes problems \((SP_1)\) and \((SP_2)\), see e.g. [27] (or [11] Theorem 5.2).

Later on, Algorithm 1 described above will be embedded into an overall algorithm with the specific task of processing the data recovery problem. We shall then use the partial derivatives given by Proposition 2. The overall algorithm stems from the Nash game played by the data recovery problem against the inclusion inverse problem. Next section is then devoted to a mandatory preamble for geometric inverse problems, that is the identifiability question.

3. An identifiability result for the inverse inclusion Cauchy-Stokes problem. In the present section, we adapt an identifiability result in [4], established for the case of obstacles, that is with a homogeneous Dirichlet condition, to the case of inclusions defined by Neumann (or free surface) boundary conditions.

The set of admissible inclusions is defined by:

\[ \mathcal{D}_{ad} = \{ \omega \subset \Omega \text{ is a } C^1 \text{-open set and } \Omega \setminus \omega \text{ is connected} \}. \]

We follow \textit{grosso modo} the same proof technique of [4], noticing that, differently from the obstacle (Dirichlet) case, inclusions are not identifiable in case of over specified data \(f\) of affine free divergence form. Consequently, even if the over specified fluid stress \(\Phi\) is identically zero, it is enough for the identifiability to hold, that the velocity data \(f\) be non affine.

Theorem 3.1. Let be \(\Omega \subset \mathbb{R}^2\) an open bounded Lipschitz domain and \(\Gamma_c\) a non-empty open subset of the boundary \(\partial \Omega\). Assume there exists a pair of compatible data \((f, \Phi) \in H^{1/2}(\Gamma_c)^d \times H^{1/2}(\Gamma_c)^d\) for the Cauchy-Stokes problem, such that either

\[ \frac{\partial J_1}{\partial \eta} \psi = -\int_{\Gamma_1} \psi \lambda_1 ds, \quad \forall \psi \in H^\bot(\Gamma_1)^d, \]

with \((\lambda_1, \kappa_1) \in H^1_0(\Omega) \times L^2(\Omega)\) solution of the adjoint problem:

\[ \begin{cases} 
\int_{\Gamma_1} (\sigma(u_1, p_1)n - \Phi)(\nabla \gamma + \nabla \gamma^T)n)ds + \int_{\Gamma_1} (u_1 - \tau)\gamma ds \\
+ \int_{\Omega} (\nabla \gamma + \nabla \gamma^T) : \nabla \lambda_1 dx - \int_{\Omega} \kappa_1 \text{div} \gamma dx = 0, \quad \forall \gamma \in H^1_0(\Omega), \\
- \int_{\Gamma_1} (\sigma(u_1, p_1)n - \Phi)\delta nds - \int_{\Omega} \text{div} \lambda_1 dx = 0, \quad \forall \delta \in H^1(\Omega)^d,
\end{cases} \]

\[ \frac{\partial J_2}{\partial \tau} \mu = \int_{\Gamma_1} (\sigma(\lambda_2, \kappa_2)n - (u_1(\eta) - u_2(\tau)))\mu ds, \quad \forall \mu \in H^{1/2}(\Gamma_1)^d, \]

with \((\lambda_2, \kappa_2) \in H^1(\Omega)^d \times L^2(\Omega)\) solution of the adjoint problem:

\[ \begin{cases} 
\int_{\Omega} (\nabla \lambda_2 + \nabla \lambda_2^T) : \nabla \varphi dx - \int_{\Omega} \kappa_2 \text{div} \varphi dx = \int_{\Gamma_c} (f - u_2(\tau))\varphi ds, \\
\int_{\Omega} \xi \text{div} \lambda_2 dx = 0, \quad \forall \xi \in L^2(\Omega),
\end{cases} \]

where, by a classical convention, \(\nabla u : \nabla v = Tr(\nabla u \nabla v^T) = \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_j}.\)
Inverse inclusion Cauchy-Stokes problem

\( \Phi \not\equiv 0 \) or \( f(x) \not\equiv Ax + b \) where \( A \) is a constant matrix with null diagonal. Consider two admissible open sets \( \omega_1 \) and \( \omega_2 \) in \( \mathcal{D}_{ad} \). For \( i = 1, 2 \), let \( (u_i, p_i) \) the solutions to the following Cauchy-Stokes inclusion problem:

\[
\begin{cases}
\Delta u_i - \nabla p_i = 0 \quad &\text{in} \quad \Omega \setminus \overline{\omega_i}, \\
\text{div} u_i = 0 \quad &\text{in} \quad \Omega \setminus \overline{\omega_i}, \\
\sigma(u_i, p_i)n = 0 \quad &\text{on} \quad \partial \omega_i, \\
u_i = f \quad &\text{on} \quad \Gamma_c, \\
\sigma(u_i, p_i)n = \Phi \quad &\text{on} \quad \Gamma_c.
\end{cases}
\]

(5)

Then \( \omega_1 = \omega_2 \).

**Proof.** Denote by \( \omega = \omega_1 \cup \omega_2 \) and define, over the set \( \Omega \setminus \overline{\omega} \), \( v = u_1 - u_2 \) and \( q = p_1 - p_2 \), where \( (u_1, p_1) \) and \( (u_2, p_2) \) are the solutions to the system (5).

One sees that \( (v, q) \) satisfies

\[
\begin{cases}
\Delta v - \nabla q = 0 \quad &\text{in} \quad \Omega \setminus \overline{\omega}, \\
\text{div} v = 0 \quad &\text{in} \quad \Omega \setminus \overline{\omega}, \\
v = 0 \quad &\text{on} \quad \Gamma_c, \\
\sigma(v, q)n = 0 \quad &\text{on} \quad \Gamma_c.
\end{cases}
\]

Thus, thanks to the unique continuation property for the Stokes system (see [24] or [4] Corollary 2.2), we have \( v = 0 \) in \( \Omega \setminus \overline{\omega} \) and then \( u_1 = u_2 \) in \( \Omega \setminus \overline{\omega} \).

Let us suppose that \( \omega_1 \not= \omega_2 \), and assume then (up to a swap in subscripts) that \( \omega_1 \setminus \overline{\omega_2} \) is an open non-empty subset of \( \Omega \). We know from system (5) that:

\( \Delta u_2 - \nabla p_2 = 0 \) in \( \omega_1 \setminus \overline{\omega_2} \).

First, let us consider the case where \( \omega_1 \setminus \overline{\omega_2} \) is Lipschitz. Then, we multiply the equation above by \( u_2 \) and take the integral over \( \omega_1 \setminus \overline{\omega_2} \). Observe then that, thanks to \( \text{div} u_2 = 0 \) in \( \omega_1 \setminus \overline{\omega_2} \), one has \( \Delta u_2 = 2 \text{div}(D(u_2)) \) where we recall that \( D(u_2) = 1/2(\nabla u_2 + \nabla u_2^T) \). By use of the Green formula, we obtain

\[
\int_{\omega_1 \setminus \overline{\omega_2}} D(u_2) : \nabla u_2 dx - \int_{\omega_1 \setminus \overline{\omega_2}} p_2 \text{div} u_2 dx = \int_{\partial(\omega_1 \setminus \overline{\omega_2})} (-p_2 I_d + D(u_2)) u_2 ds,
\]

which can be rewritten as follows:

\[
\frac{1}{2} \int_{\omega_1 \setminus \overline{\omega_2}} |D(u_2)|^2 dx = \int_{\partial(\omega_1 \setminus \overline{\omega_2})} \sigma(u_2, p_2) u_2 ds.
\]

Now, since \( \sigma(v, q) \) vanishes in \( \Omega \setminus \overline{\omega} \), and thanks to the continuity of the involved normal-traces, one has \( \sigma(v, q)n = 0 \) on \( \partial \omega \). From other part, one has \( \sigma(u_1, p_1)n = 0 \).

**Figure 2.** Different situations
on \( \partial \omega_1 \) thanks to equations (5). We then have \( \sigma(u_2, p_2)n = 0 \) on \( \partial \omega_1 \setminus \partial(\omega_1 \cap \omega_2) \). Now, since we know that \( \sigma(u_2, p_2)n = 0 \) on \( \partial \omega_2 \) thanks to equations (5), we obtain

\[
\int_{\partial(\omega_1 \setminus \omega_2)} \sigma(u_2, p_2)n u_2 ds = 0,
\]

that is,

\[
\frac{1}{2} \int_{\omega_1 \setminus \omega_2} |D(u_2)|^2 dx = 0.
\]

Since \( |D(u_2)|^2 \leq \omega_1 \setminus \omega_2 \) = 0, the components of the matrix of \( D(u_2) \) are a.e. zero. Consequently, the velocity field \( u_2 \) has an affine form in \( \omega_1 \setminus \omega_2 \), shortly given by \( u_2(x) = Ax + b \) where \( A \) is a constant matrix with null diagonal.

We know from above and from equations (5) that \( (u_2, p_2) \) satisfies the following system,

\[
\begin{aligned}
\Delta u_2 - \nabla p_2 &= 0 \quad \text{in} \quad \omega_1 \setminus \omega_2, \\
\text{div} u_2 &= 0 \quad \text{in} \quad \omega_1 \setminus \omega_2, \\
\sigma(u_2, p_2)n &= 0 \quad \text{on} \quad \partial(\omega_1 \setminus \omega_2). 
\end{aligned}
\]

Thus, by application to \( u_2(x) - (Ax + b) \) which fulfills the system (6) above, of the unique continuation theorem for the steady Stokes equation established in [24], we conclude that \( u_2(x) = Ax + b \) and \( p_2 = 0 \) in the whole domain \( \Omega \setminus \omega_2 \).

Finally, reasoning with the traces on the boundary of the domain \( \partial \Omega \), we observe that \( u_2(x) = Ax + b \) and \( p_2 = 0 \) in \( \Omega \setminus \omega_2 \) yields \( \sigma(u_2, p_2)n = \Phi = 0 \) and \( u_2(x) = f(x) = Ax + b \) over \( \Gamma \), which, by assumption, is impossible. We conclude that \( \omega_1 \setminus \omega_2 = \emptyset \), and so \( \omega_1 = \omega_2 \).

Now, when \( \omega_1 \setminus \omega_2 \) is not Lipschitz, then the use the Green formula is not justified. We recall and follow here the solution given by [16]: the author in the cited reference introduces an additional regularity assumption on \( \omega_1 \) and \( \omega_2 \), assuming that they have \( C^{1,1} \) boundary. We assumed more regularity on the Cauchy data in order to have more regularity for the solution \( (u_2, p_2) \), that is \( u_2 \in H^2(\Omega \setminus \omega_2) \) and \( p_2 \in H^1(\Omega \setminus \omega_2) \). Then, consider two Lipschitz open sets \( \Omega_1, \Omega_2 \in \Omega \setminus \omega_2 \), and \( \omega_1 \subset \Omega \setminus \omega_2 \). By application to \( u_2(x) - (Ax + b) \) which fulfills the system (6) above, of the unique continuation theorem for the steady Stokes equation established in [24], we conclude that \( u_2(x) = Ax + b \) and \( p_2 = 0 \) in the whole domain \( \Omega \setminus \omega_2 \).

Next, by use the Green formula on \( \omega_1 \) and \( \omega_2 \), we obtain

\[
\frac{1}{2} \int_{\omega_1 \setminus \omega_2} |D(u_2)|^2 dx = \int_{\partial \omega_1} \sigma(u_2, p_2)n u_2 ds \quad \text{and} \quad \frac{1}{2} \int_{\omega_1 \setminus \omega_2} |D(u_2)|^2 dx = \int_{\partial \omega_2} \sigma(u_2, p_2)n u_2 ds.
\]

Subtracting these two equalities, we get

\[
\frac{1}{2} \int_{\omega_1 \setminus \omega_2} |D(u_2)|^2 dx = 0.
\]

Hence, we conclude as previously. Finally, for the case where the domain \( \Omega \setminus \omega_1 \cup \omega_2 \) is not connected (fourth case in figure 2), we refer the reader to the same cited above [16] where this case is successfully handled. \( \square \)

The identifiability result suggests that there is no need for a third party state equation, the two state equations \( (SP_1) \) and \( (SP_2) \) formulated with inclusions and dedicated to the completion problem should suffice. Only a third player’s cost functional should be defined, playing with inclusions as strategies. Hence we enforced the data completion steps, by letting the first and second players lead a Nash subgame during the overall iterations, see next section. Numerical experiments show that this choice turned out to be efficient.
4. Coupled data completion and geometry identification for the Stokes problem. The aim of the present section is to introduce an algorithm dedicated to recover the missing boundary data while solving the inverse inclusion problem for steady Stokes flows. We extend the two-player Nash game set for the completion problem to a three-player Nash game, the third player being in charge of the inverse inclusion problem.

We recall that the inverse inclusion problem amounts to find \( \omega^* \in D_{ad} \) such that the fluid velocity \( u \) and the pressure \( p \) are solution to the following Cauchy-Stokes problem:

\[
\begin{cases}
\Delta u - \nabla p = 0 & \text{in } \Omega \setminus \overline{\omega^*}, \\
\text{div} u = 0 & \text{in } \Omega \setminus \overline{\omega^*}, \\
\sigma(u, p)n = 0 & \text{on } \partial \omega^*, \\
u = f & \text{on } \Gamma_c, \\
\sigma(u, p)n = \Phi & \text{on } \Gamma_c.
\end{cases}
\]

\[(7)\]

Thanks to the identifiability result stated in section 3, a single pair of compatible-measurements \((f, \Phi)\) is enough to recover the inclusion(s) as well as the missing data. Next, we shall set up a three-player Nash game following the same philosophy than in section 2 dedicated for the sole completion.

For \( \eta \in H^ \frac{1}{2}(\Gamma)^d \), \( \tau \in H^ \frac{1}{2}(\Gamma)^d \) and \( \omega \in D_{ad} \), let us define the following three cost functionals:

\[
\begin{align*}
J_1(\eta, \tau; \omega) &= \frac{1}{2} |\sigma(u_\eta^\tau(\eta), p_\eta^\tau(\eta))n - \Phi|^2_{H^ \frac{1}{2}(\Gamma)^d} + \frac{1}{2} |u_\eta^\tau(\eta) - u_\tau^\phi(\tau)|^2_{H^ \frac{1}{2}(\Gamma) \ast}, \\
J_2(\eta, \tau; \omega) &= \frac{1}{2} |u_\tau^\phi(\tau) - f|^2_{H^ \frac{1}{2}(\Gamma) \ast} + \frac{1}{2} |u_\eta^\tau(\eta) - u_\tau^\phi(\tau)|^2_{H^ \frac{1}{2}(\Gamma) \ast}, \\
J_3(\eta, \tau; \omega) &= ||\sigma(u_\eta^\tau(\eta), p_\eta^\tau(\eta)) - \sigma(u_\tau^\phi(\tau), p_\tau^\phi(\tau))||^2_{L^2(\Omega \setminus \overline{\omega})} + \mu |\partial \omega|,
\end{align*}
\]

where the parameter \( \mu > 0 \) is a penalization of the perimeter \( |\partial \omega| \), defined as the Hausdorff measure \( H^1(\partial \omega) \), \((u_\eta^\tau(\eta), p_\eta^\tau(\eta))\) and \((u_\tau^\phi(\tau), p_\tau^\phi(\tau))\) are the solutions of the respective BVP \((P_1)\) and \((P_2)\):

\[
\begin{align*}
\begin{cases}
\Delta u_\eta^\tau - \nabla p_\eta^\tau = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\text{div} u_\eta^\tau = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\sigma(u_\eta^\tau, p_\eta^\tau)n = f & \text{on } \partial \omega, \\
\sigma(u_\eta^\tau, p_\eta^\tau)n = \eta & \text{on } \Gamma_c, \\
\end{cases} \quad \begin{cases}
\Delta u_\tau^\phi - \nabla p_\tau^\phi = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\text{div} u_\tau^\phi = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\sigma(u_\tau^\phi, p_\tau^\phi)n = \Phi & \text{on } \Gamma_c, \\
w_\tau^\phi = \tau & \text{on } \Gamma_c.
\end{cases}
\end{align*}
\]

\[(P_1) \quad \text{and} \quad (P_2)\]

In a few words, there are three players: Player (1) controls the strategy variable \( \eta \in H^ \frac{1}{2}(\Gamma)^d \) and aims at minimizing the cost \( J_1 \) and Player (2) controls the strategy variable \( \tau \in H^ \frac{1}{2}(\Gamma)^d \) and aims at minimizing the cost \( J_2 \). These two players may be interpreted exactly the same way than in the completion game stated section 2: they are given Dirichlet (resp. Neumann) data and try to minimize the gap with the Neumann (resp. Dirichlet) remaining condition. The player (3) controls the strategy variable \( \omega \in D_{ad} \) and aims at minimizing the Kohn-Vogelius type functional \( J_3 \), to which we added the regularizing term \( \mu |\partial \omega| \) which prevents from obtaining too irregular contours, as classical from Mumford-Shah functionals, see e.g. [9].

Notice that the state variables \((u_\eta^\tau(\eta), p_\eta^\tau(\eta))\) and \((u_\tau^\phi(\tau), p_\tau^\phi(\tau))\) belong to the space \( (H^1(\Omega \setminus \overline{\omega}))^d \times L^2(\Omega \setminus \overline{\omega}) \), which obviously depends on \( \omega \), a variable intended to be a control. In order to circumvent this tricky dependence, we recourse to a
level-set formulation, before stating the actual three-player Nash game effectively implemented.

4.1. A level-set formulation. The level-set approach is a very convenient tool in shape identification, see [39] for a general introduction, or [41] where the approach is applied to detect obstacles in a Stokes flow. The boundary of the shape to be identified is postulated to be a zero level-set of a smooth enough (say Lipschitz) function \( \phi : \Omega \to \mathbb{R} \). In other words, when \( \phi \) varies in some -admissible- functional space, admissible open subsets \( \omega \in \Omega \) are those defined by the following

\[
\begin{cases}
\phi(x) < 0 & \text{in } \omega, \\
\phi(x) > 0 & \text{in } \Omega \setminus \omega, \\
\phi(x) = 0 & \text{on } \partial \omega.
\end{cases}
\]

The open set \( \Omega \setminus \omega \) is then given in terms of the level-set function as follows:

\[
\Omega \setminus \omega = \{ x \in \Omega \text{ such that } H(\phi(x)) = 1 \},
\]

where \( H(\cdot) \) is the Heaviside function. The perimeter of \( \omega \) can then be formally given by

\[
|\partial \omega| = \int_\Omega |\nabla H(\phi)| dx = \int_\Omega |\delta(\phi)| \nabla \phi | dx,
\]

where \( \delta \) is the Dirac distribution.

For regularity reasons, and for ensuring the well-posedness of the modified Stokes system as well, it is usual to use smoothed versions of the Heaviside and Dirac distributions. Given two small enough parameters \( \varepsilon > 0 \) and \( \beta > 0 \), we used smoothed versions denoted respectively by \( H_{\varepsilon,\beta}(\cdot) \) and \( \delta_{\varepsilon,\beta}(\cdot) \), expressed as follows, for \( s \in \mathbb{R} \),

\[
H_{\varepsilon,\beta}(s) = \begin{cases}
\frac{1}{\beta} & \text{if } s > \varepsilon, \\
\frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{s}{\varepsilon} \right) \right) & \text{if } |s| \leq \varepsilon, \\
\frac{1}{\varepsilon} \left( \frac{s}{\varepsilon^2 + s^2} \right) & \text{if } |s| > \varepsilon,
\end{cases}
\]

\[
\delta_{\varepsilon,\beta}(s) = \begin{cases}
\frac{1}{\beta} & \text{if } |s| \leq \varepsilon, \\
\frac{\varepsilon}{\pi} & \text{if } |s| > \varepsilon.
\end{cases}
\]

Let us define the -control free- Sobolev state spaces :

Given \( g \in H^{\frac{7}{2}}(\Gamma_c)^d, \psi \in H^{\frac{7}{2}}(\Gamma)^d, \) \( V_g = \{ v \in H^1(\Omega)^d / \text{div} v = 0; v|_{\Gamma_c} = g \} \)

and \( W_\psi = \{ v \in H^1(\Omega)^d / \text{div} v = 0 \quad \forall v|_{\Gamma} = \psi \} \).

Problems \( (P_1) \) and \( (P_2) \) are then rephrased in terms of the level-set, yielding the modified weak form :

\((P_{1,\varepsilon,\beta})\) \quad \begin{aligned}
\text{Find } (u^\varepsilon_1, p^\varepsilon_1) \in V_f \times L^2(\Omega) \text{ such that } \\
\int_\Omega (\sigma(u^\varepsilon_1, p^\varepsilon_1) : \nabla v_1) H_{\varepsilon,\beta}(\phi) dx = \int_\Omega \eta v_1 ds, \quad \forall v_1 \in H^1_0(\Omega),
\end{aligned}

\((P_{2,\varepsilon,\beta})\) \quad \begin{aligned}
\text{Find } (u^\varepsilon_2, p^\varepsilon_2) \in W_f \times L^2(\Omega) \text{ such that } \\
\int_\Omega (\sigma(u^\varepsilon_2, p^\varepsilon_2) : \nabla v_2) H_{\varepsilon,\beta}(\phi) dx = \int_\Omega \Phi v_2 ds, \quad \forall v_2 \in H^1_0(\Omega).
\end{aligned}

The existence and uniqueness of solutions to the problems \( (P_{1,\varepsilon,\beta}) \) and \( (P_{2,\varepsilon,\beta}) \) follows from the fact that \( H_{\varepsilon,\beta}(\phi) \geq \beta > 0 \) is bounded, and thanks to the well-posedness of the Stokes system with mixed boundary conditions (see Remark 1).
It is not the scope of the present paper to discuss the dependence of the modified Stokes problems with respect to \((\varepsilon, \beta)\), which is known to behave consistently \cite{[23] [44]}, so we still refer to problems \((P_{1,\varepsilon,\beta})\) and \((P_{2,\varepsilon,\beta})\) as \((P_1)\) and \((P_2)\), and we omit to underline the dependence of the state variables w.r.t. \((\varepsilon, \beta)\) as well.

4.2. **Level-set sensitivity and optimality condition.** The player (3) in charge of the inverse inclusion problem has now control on the level-set function \(\phi\) instead of the open subset \(\omega \in \mathcal{D}_{ad}\). The new form of the third player’s cost functional is now as follows:

\[
J_3(\eta, \tau; \phi) = \int_{\Omega} |\sigma(u_2^\phi, p_2^\phi) - \sigma(u_1^\phi, p_1^\phi)|^2 H_{\varepsilon,\beta}(\phi)dx + \mu \int_{\Omega} \delta_{\varepsilon,\beta}(\phi)|\nabla \phi|dx
\]

where \((u_1^\phi, p_1^\phi)\) and \((u_2^\phi, p_2^\phi)\) solve respectively problems \((P_{1,\varepsilon,\beta})\) and \((P_{2,\varepsilon,\beta})\). We choose as convenient space for the level-set variables the Sobolev space \(S = H^1(\Omega)\) though it is not optimal (in the sense that it may introduce too much regularity requirement, hampering the capture of non \(H^1\) inclusions).

In order to perform the partial optimization of \(J_3(\eta, \tau; \phi)\) w.r.t. \(\phi\) for \((\eta, \tau)\) given by players (1) and (2), one needs to compute the derivative of \(J_3\) w.r.t. \(\phi\). We have the following:

**Proposition 3.** Assume \(\phi \in S\) satisfies \(\Delta \phi \in L^2(\Omega)\) and \(|\nabla \phi| \neq 0\) with the boundary condition \(\frac{\partial \phi}{\partial n} = 0\) over \(\partial \Omega\). Then \(J_3(\eta, \tau; \phi)\) is Fréchet-differentiable with respect to \(\phi\), and the partial derivative of \(J_3(\eta, \tau; \phi)\) with respect to \(\phi\), in any direction \(\psi \in S\), is given by

\[
\left(\frac{\partial J_3}{\partial \phi}(\eta, \tau; \phi), \psi\right) = \int_{\Omega} \delta_{\varepsilon,\beta}(\phi) \left[|\sigma(u_2^\phi, p_2^\phi) - \sigma(u_1^\phi, p_1^\phi)|^2 - \mu \text{div} \left(\frac{\nabla \phi}{|\nabla \phi|}\right) + \sigma(u_1^\phi, p_1^\phi) : \nabla \lambda_1 + \sigma(u_2^\phi, p_2^\phi) : \nabla \lambda_2\right] \psi dx,
\]

where \((\lambda_1, \pi_1) \in H^1_{\varepsilon,\beta}(\Omega) \times L^2(\Omega)\) and \((\lambda_2, \pi_2) \in H^1_{\varepsilon,\beta}(\Omega) \times L^2(\Omega)\) are respective solutions of the adjoints problems,

\[
(12) \quad \begin{cases}
-2 \int_{\Omega} \left(\sigma(u_2^\phi, p_2^\phi) - \sigma(u_1^\phi, p_1^\phi)\right) : (\nabla h_1 + \nabla h_1^T) H_{\varepsilon,\beta}(\phi)dx - \int_{\Omega} \pi_1 \text{div} h_1 H_{\varepsilon,\beta}(\phi)dx \\
+ \int_{\Omega} ((\nabla h_1 + \nabla h_1^T) : \nabla \lambda_1) H_{\varepsilon,\beta}(\phi)dx = 0, \quad \forall h_1 \in H^1_{\varepsilon,\beta}(\Omega), \\
2 \int_{\Omega} \left(\sigma(u_2^\phi, p_2^\phi) - \sigma(u_1^\phi, p_1^\phi)\right) : (k_1 I_d) H_{\varepsilon,\beta}(\phi)dx - \int_{\Omega} k_1 \text{div} \lambda_1 H_{\varepsilon,\beta}(\phi)dx = 0, \quad \forall k_1 \in L^2(\Omega),
\end{cases}
\]

\[
(13) \quad \begin{cases}
2 \int_{\Omega} \left(\sigma(u_2^\phi, p_2^\phi) - \sigma(u_1^\phi, p_1^\phi)\right) : (\nabla h_2 + \nabla h_2^T) H_{\varepsilon,\beta}(\phi)dx - \int_{\Omega} \pi_2 \text{div} h_2 H_{\varepsilon,\beta}(\phi)dx \\
+ \int_{\Omega} ((\nabla h_2 + \nabla h_2^T) : \nabla \lambda_2) H_{\varepsilon,\beta}(\phi)dx = 0, \quad \forall h_2 \in H^1_{\varepsilon,\beta}(\Omega), \\
-2 \int_{\Omega} \left(\sigma(u_2^\phi, p_2^\phi)dx - \sigma(u_1^\phi, p_1^\phi)\right) : (k_2 I_d) H_{\varepsilon,\beta}(\phi)dx - \int_{\Omega} k_2 \text{div} \lambda_2 H_{\varepsilon,\beta}(\phi)dx = 0, \quad \forall k_2 \in L^2(\Omega),
\end{cases}
\]

and where \((u_1^\phi, p_1^\phi)\) and \((u_2^\phi, p_2^\phi)\) are the solutions to respectively \((P_{1,\varepsilon,\beta})\) and \((P_{2,\varepsilon,\beta})\).
Proof. To prove Proposition 3 above, let us first remark that we are in a classical elliptic case: the states \((u_1^0, p_1^0)\) and \((u_2^0, p_2^0)\) solve respectively problems \((P_{1,\varepsilon,\beta})\) and \((P_{2,\varepsilon,\beta})\) which are elliptic, and with a smooth dependence of energy functionals on \(\phi\), namely on the regularized parameter \(H_{\varepsilon,\beta}(\phi)\). It is then well known that the states \((u_1^0, p_1^0)\) and \((u_2^0, p_2^0)\) are Frechet-differentiable with respect to \(\phi\), see [40] or page 107, the assumptions made therein are straightforward in our case. From other part, the assumption \(|\nabla \phi| \neq 0\) is fulfilled by the reinitialisation step (17) which forces \(|\nabla \phi|\) to be close to 1. The cost function \(J_3(\eta, \tau; \phi)\) is then partially Frechet-differentiable with respect to \(\phi\), as it is partially with respect to a generic state \((v_1, q_1)\) and \((v_2, q_2)\) (dismissing the dependence of these w.r.t. \(\phi\), the functional is quadratic w.r.t. the states \((v_i, q_i), i = 1, 2\). The cited references state then that the reduced cost function (taking into account the implicit dependence of the states w.r.t. the control \(\phi\)) is Frechet-differentiable with respect to \(\phi\), and the derivative is computable by means of an adjoint state method, as stated in Proposition 3. The derivation of the adjoint state equations and of the formula for the derivative is given in Appendix A.2.

\[ \Box \]

Remark 2. The assumption that \(\phi\) satisfies \(\Delta \phi \in L^2(\Omega)\) is necessary for the existence of the Neumann trace of \(\phi\). Indeed, from one part, stating regularity results for the nonlinear implicit equation (14) fulfilled by \(\phi\) below is not straightforward; and from other part, the discretized scheme (16) of the latter equation provides itertes \(\phi^{(n)}\) which are in \(H^2(\Omega)\).

The necessary optimality condition for the minimization problem \(\min_{\phi \in S} J_3(\eta, \tau; \phi)\) is then formulated as the following Euler-Lagrange equation:

\[
\begin{align*}
\delta_{\varepsilon,\beta}(\phi) &\left[ |\sigma(u_2^0, p_2^0) - \sigma(u_1^0, p_1^0)|^2 - \mu \text{div}(\nabla \phi) \right] \\
&= 0, \quad \text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \phi}{\partial n} &\equiv 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

The strongly nonlinear equation above -with implicit terms- is solved iteratively as the stationary state of the following evolution equation:

\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \delta_{\varepsilon,\beta}(\phi) &\left[ |\sigma(u_2^0, p_2^0) - \sigma(u_1^0, p_1^0)|^2 - \mu \text{div}(\nabla \phi) \right] \\
&= 0, \quad \text{in } \mathbb{R}^+ \times \Omega,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \phi}{\partial n} &\equiv 0, \quad \text{on } \mathbb{R}^+ \times \partial \Omega,
\end{align*}
\]

\[
\phi(0, x) = \phi_0(x), \quad \text{in } \Omega,
\]

where \(\phi_0 \in S\) is a given initial condition.

The variational formulation associated to the problem (15) above reads

\[
\int_{\Omega} \frac{\partial \phi}{\partial t} \psi dx = - \int_{\Omega} \delta_{\varepsilon,\beta}(\phi) \left| \sigma(u_2^0, p_2^0) - \sigma(u_1^0, p_1^0) \right|^2 \psi dx + \mu \int_{\Omega} \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi) \psi dx \\
- \int_{\Omega} \delta_{\varepsilon,\beta}(\phi) \left[ |\sigma(u_2^0, p_2^0)| \cdot \nabla \phi + \sigma(u_1^0, p_1^0) \cdot \nabla \phi \right] \psi dx, \quad \forall \psi \in H^1(\Omega).
\]

But, since one has

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi) + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi)
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

\[
\text{div}(\delta_{\varepsilon,\beta}(\phi) \nabla \phi) = \delta_{\varepsilon,\beta}(\phi) \nabla \phi \cdot \nabla \phi + \delta_{\varepsilon,\beta}(\phi) \text{div}(\nabla \phi),
\]

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we get,
\[
\int_{\Omega} \frac{\partial \phi}{\partial t} \psi \, dx = - \int_{\Omega} \delta_{\varepsilon, \beta}(\phi) \sigma(u_2^0, p_2^0) - \sigma(u_1^0, p_1^0) |^2 \psi \, dx - \mu \int_{\Omega} \frac{\delta_{\varepsilon, \beta}(\phi)}{|\nabla \phi|} \nabla \phi \nabla \psi \, dx
\]
\[
- \int_{\Omega} \delta_{\varepsilon, \beta}(\phi) \left[ \sigma(u_1^0, p_1^0) : \nabla \lambda_1 + \sigma(u_2^0, p_2^0) : \nabla \lambda_2 \right] \psi \, dx
\]
\[
- \mu \int_{\Omega} \delta_{\varepsilon, \beta}(\phi) |\nabla \phi| \psi \, dx, \quad \forall \psi \in H^1(\Omega).
\]

The problem above is solved numerically by means of a semi-implicit Euler scheme with \(\frac{\partial \phi}{\partial t}\) approximated by \(\frac{\phi^{n+1} - \phi^n}{\delta t}\), where \(\phi^n(\cdot) = \phi(t_n, \cdot)\) and \(t_n = n\delta t\), with \(\delta t > 0\) a given time step.

We obtain the following iterative scheme:
\[
\text{(16)} \quad \begin{cases}
\text{Given } \phi^n, \text{ Find } \phi^{n+1} \in H^1(\Omega) \text{ such that:} \\
a(\phi^{n+1}, \psi) = l(\psi), \quad \forall \psi \in H^1(\Omega),
\end{cases}
\]
where
\[
\begin{align*}
\text{Given } \phi^n, \text{ Find } \phi^{n+1} = H^1(\Omega) \text{ such that:} \\
a(\phi^{n+1}, \psi) & = \int_{\Omega} \phi^{n+1} \psi \, dx + \mu \delta t \int_{\Omega} \frac{\delta_{\varepsilon, \beta}(\phi^n)}{|\nabla \phi^n|} \nabla \phi^{n+1} \nabla \psi \, dx, \\
l(\psi) & = -\delta t \int_{\Omega} \delta_{\varepsilon, \beta}(\phi^n) \sigma(u_2^n, p_2^n) - \sigma(u_1^n, p_1^n) |^2 \psi \, dx - \mu \delta t \int_{\Omega} \delta_{\varepsilon, \beta}(\phi^n) |\nabla \phi^n| \psi \, dx \\
& \quad + \int_{\Omega} \phi^n \psi \, dx - \delta t \int_{\Omega} \delta_{\varepsilon, \beta}(\phi^n) \left[ \sigma(u_1^n, p_1^n) : \nabla \lambda_1^n + \sigma(u_2^n, p_2^n) : \nabla \lambda_2^n \right] \psi \, dx
\end{align*}
\]
with \((u_1^n, p_1^n) = (u_1(\eta, \phi^n), p_1(\eta, \phi^n)), (u_2^n, p_2^n) = (u_2(\tau, \phi^n), p_2(\tau, \phi^n))\) are solutions to \((P_{1, \varepsilon, \beta})\) and \((P_{2, \varepsilon, \beta})\) for a given \(\phi^n\), and \(\lambda_{i=1,2} = \lambda_{i=1,2}(\phi^n)\) are the adjoint state solutions of the problems \((12)\) and \((13)\).

In order to prevent the level set iterates from being too flat or too steep, we trigger from time to time a regularization pass that reinitializes the level set to a signed distance (see e.g. [3]). This step is mandatory in order to keep the iterated level sets smooth enough, but is also necessary to have a non vanishing \(|\nabla \phi^n|\) that ensures the ellipticity of \(a(\cdot, \cdot)\) in \((16)\). The update is performed by solving the following equation:
\[
\begin{align*}
\text{(17)} \quad \begin{cases}
\frac{\partial \psi}{\partial t} + \text{sign}(\phi^n)(|\nabla \psi| - 1) & = 0 \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\frac{\partial \psi}{\partial n} & = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega, \\
\psi(0, x) & = \phi^n(x) \quad \text{in } \Omega.
\end{cases}
\end{align*}
\]
In practice, equation \((17)\) above is solved for a few time steps (typically 5 or 6) then one reassigns the last computed \(\psi\) to \(\phi^n(x)\).

The six variational problems \((P_{1, \varepsilon, \beta}), (P_{2, \varepsilon, \beta}), (12), (13), (16)\) and \((17)\) are solved by means of \textit{ad hoc} Finite Element methods (see Section 5 below).

4.3. The three-player Nash algorithm. We are now ready to state the three-player identification/completion Nash game. As aforementioned in section 4, players (1) and (2) aim at solving the Cauchy problem, while player (3) is aimed at minimizing a Kohn-Vogelius type energy, intended to capture the shape of the inclusion. The game is of Nash type, which means that it is static with complete information [28] and hence its solution is a Nash equilibrium (NE), see Definition 4.

Given a triplet \((\eta, \tau, \phi) \in H^{\frac{1}{2}}(\Gamma)^d \times H^{\frac{1}{2}}(\Gamma)^d \times S\), let \((u_1^0(\eta), p_1^0(\eta))\) be the solution to the approximate Stokes problem \((P_{1, \varepsilon, \beta})\) and \((u_2^0(\tau), p_2^0(\tau))\) the solution to the
approximate Stokes problem ($\mathcal{P}_{2,\beta}$), then the three players and their respective costs are defined as follows:

- Player (1) has control on the Neumann strategies $\eta \in H^\frac{1}{2}(\Gamma)^d$, and its cost functional is given by

$$J_1(\eta, \tau, \phi) = \frac{1}{2} ||\sigma(u_1^\phi(\eta), p_1^\phi(\eta))n - \Phi||_{H^\frac{1}{2}(\Gamma)}^2 + \frac{1}{2} ||u_1^\phi(\eta) - u_2^\phi(\tau)||_{H^\frac{1}{2}(\Gamma)}^2$$

- Player (2) has control on the Dirichlet strategies $\tau \in H^\frac{1}{2}(\Gamma)^d$, and its cost functional is given by

$$J_2(\eta, \tau, \phi) = \frac{1}{2} ||u_2^\phi(\tau) - f||_{H^\frac{1}{2}(\Gamma)^d}^2 + \frac{1}{2} ||u_1^\phi(\eta) - u_2^\phi(\tau)||_{H^\frac{1}{2}(\Gamma)^d}^2$$

- Player (3) has control on the inclusion level-set strategies $\phi \in \mathcal{S}$, and its cost functional is given by

$$J_3(\eta, \tau; \phi) = \int_{\Omega} |\sigma(u_2^\phi(\tau), p_2^\phi(\tau)) - \sigma(u_1^\phi(\eta), p_1^\phi(\eta))| H_{\epsilon,\beta}(\phi) dx + \mu \int_{\Omega} \delta_{\epsilon,\beta}(\phi) \nabla \phi |dx$$

In Algorithm 2 below, we describe the main steps in computing the Nash equilibrium. This algorithm is unusual in the sense that, first, it introduces a completion-oriented Nash subgame, solved incompletely ($K_{\text{max}}$ is small, around ten iterations), and second, it processes the third player’s minimization step by iterating on the necessary optimality condition. Classical algorithms compute Nash equilibria with $K_{\text{max}} = 1$. It is easy to check (by writing down the stationarity equations) that when the two algorithms converge, they lead to the same limit point, which is a Nash equilibrium of the three-player’s game defined above.

As we shall see in section 5 below, Algorithm 2 outperforms the classical one.

**Algorithm 2:** Computation of the coupled inclusion-completion Nash equilibrium

Given: convergence tolerances $\varepsilon_N > 0$, $\varepsilon_S > 0$, $K_{\text{max}}$ a computational budget per Nash iteration, $N_{\text{max}}$ a maximum Nash iterations, $\sigma$ a noise level and $\rho(\sigma)$ a -tuned- function which depends on the noise.

Set $n = 0$, choose an initial level-set $\phi^{(0)} \in \mathcal{S}$.

- **Step I:** (a completion Nash subgame) Set $k = 1$.
  - Choose an initial guess $S^{(k-1)} = (\eta^{(k-1)}, \tau^{(k-1)}) \in H^\frac{1}{2}(\Gamma)^d \times H^\frac{1}{2}(\Gamma)^d$.
  - **Step 1:** Compute $\bar{\pi}^{(k)}$ solution of $\min_\eta J_1(\eta, \tau^{(k-1)}, \phi^{(n)})$ and set $\eta^{(k)} = \alpha \eta^{(k-1)} + (1 - \alpha) \bar{\pi}^{(k)}$ with $0 \leq \alpha < 1$.
  - **Step 2:** Compute $\bar{\pi}^{(k)}$ solution of $\min_\tau J_2(\eta^{(k-1)}, \tau, \phi^{(n)})$ and set $\tau^{(k)} = \alpha \tau^{(k-1)} + (1 - \alpha) \bar{\pi}^{(k)}$ with $0 \leq \alpha < 1$.
  - **Step 3:** While $||S^{(k)} - S^{(k-1)}|| > \varepsilon_S$ and $k < K_{\text{max}}$, set $k = k + 1$, return back to step 1.

- **Step II:** Compute $r_k = ||u_2^{(k)} - f||_{L^2(\Omega)}$, where $(u_2^{(k)}, p_2^{(k)})$ is the solution of the problem ($\mathcal{P}_{2,\beta}$) with the level-set $\phi = \phi^{(n)}$ and with the Dirichlet condition $u_2^{(k)} = \tau^{(k)}$ over $\Gamma$.

- **Step III:** While $r_k \geq \rho(\sigma)\varepsilon$ and $n < N_{\text{max}}$ update the level-set : compute $\phi^{(n+1)}$ solution to the variational problem (16) and set $n = n + 1$, go back to step I.
5. Numerical experiments. In this section, we provide and discuss the numerical results of experiments led for three test cases, named A, B and C. These 3 test-cases share the following common settings:

The domain: \( \Omega = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \)

The boundaries: \( \Gamma = \{ \frac{1}{2} \} \times [-\frac{1}{2}, \frac{1}{2}] ; \quad \Gamma = \partial \Omega \setminus \Gamma \)

Normal stress: \( \Phi(x, y) = -2(y^2 - 1/4; 0) \) prescribed over \( \partial \Omega \)

Initial strategies: for Step I, we always used \( S^{(0)} = (\eta^{(0)}, \tau^{(0)}) = (0, 0) \) and took \( \alpha = 0.1 \)

Parameters: for Step III: we took \( \delta t = 0.02 \) for solving equation (16).

The test-cases differ in the shape and/or number of connected components of the inclusions.

Given a known shape and location of the inclusion \( \omega^* \in \mathcal{D}_{ad} \), we solve the following Stokes problem:

\[
\begin{aligned}
\Delta u - \nabla p &= 0 \quad \text{in } \Omega \setminus \overline{\omega^*}, \\
\text{div} u &= 0 \quad \text{in } \Omega \setminus \overline{\omega^*}, \\
\sigma(u, p)n &= 0 \quad \text{on } \partial \omega^*, \\
\sigma(u, p)n &= \Phi \quad \text{on } \partial \Omega,
\end{aligned}
\]

where the (phantom) exact solution \((u, p)\) is used to build the remaining Cauchy data \( f = u|_\Gamma \), and the exact missing data \( u|_\Gamma \) and \( \sigma(u, p)n|_\Gamma \). The two latter data together with the known inclusion shape \( \omega^* \) are used to compute the following relative errors:

\[
err_D = \frac{||\tau_N - u|\Gamma_i||_{L^2(\Gamma_i)}}{||u|\Gamma_i||_{L^2(\Gamma_i)}}, \quad err_N = \frac{||\eta_N - \sigma(u, p)n|\Gamma_i||_{L^2(\Gamma_i)}}{||\sigma(u, p)n|\Gamma_i||_{L^2(\Gamma_i)}},
\]

\[
err_O = \frac{\text{mes}(\omega^* \cup \omega_N) - \text{mes}(\omega^* \cap \omega_N)}{\text{mes}(\omega^*)},
\]

where \((\eta_N, \tau_N, \phi_N)\) is the approximate Nash equilibrium output from Algorithm 2, and \( \omega_N = 1(\phi_N < 0) \). These metrics are used to assess the efficiency of our approach. The stability w.r.t. noise was stressed by solving the joint inverse inclusion/completion problem with noisy perturbations of the Dirichlet data \( f^\sigma = f + \sigma N \) with \( N \) being a Gaussian white noise.

Two different initial level-sets were used: \( \phi_1^{(0)} \) has as zero level-set the disk \( B(c_0, r_0) \) where \( c_0 = (0, 0) \) and \( r_0 = 0.3 \), and \( \phi_2^{(0)} \) is a periodic function with as zero level-set 30 small ellipsoids with uniformly distributed centers, see Figures 5(a) and 6(e).

The solvers for Stokes, equation (16) and adjoint systems, the sensitivity routines, and the minimization algorithms as well, were implemented using the Finite Element package FreeFem++ [30].

Test-case A. The exact inclusion is a disk \( \omega^* = B(c, r) \) centered at \( c \) and with a radius \( r \) where \( c = (0, 0) \) and \( r = 0.1 \).

The FreeFem++ implementation of Algorithm 2 was ran for two different initial contours, leading to very close results, both of them in good accordance with the exact solutions (inclusion and missing data). It can be however observed from Figure 5(c)(e) and Figure 6(c)(e) that the initial contour \( \phi_2^{(0)} \) outperforms \( \phi_1^{(0)} \) as the computed first component of the fluid velocity and normal stress are more accurate with the initial contour \( \phi_2^{(0)} \). Indeed, in all our subsequent numerical
experiments, the initial contour $\phi_2^{(0)}$ outperformed $\phi_1^{(0)}$, so we shall later on present only those results obtained with $\phi_2^{(0)}$.

For the case of noisy Dirichlet data $f^\sigma$ given over $\Gamma_\epsilon$, it can be seen from the profiles presented in Figure 7 that the boundary data recovery is remarkably stable with respect to the noise magnitude, and even more striking is the stability of the detected inclusion.

The relative errors defined by formulas (21) are summarized in Table 1 for the test-case A.

| Noise level | $\sigma = 0\%$ | $\sigma = 1\%$ | $\sigma = 3\%$ | $\sigma = 5\%$ |
|-------------|----------------|----------------|----------------|----------------|
| $err_D$     | 0.010          | 0.015          | 0.039          | 0.063          |
| $err_N$     | 0.031          | 0.033          | 0.051          | 0.07           |
| $err_O$     | 0.032          | 0.043          | 0.066          | 0.117          |

Table 1. Test-case A. $L^2$ relative errors on missing data on $\Gamma_\epsilon$ (on Dirichlet and Neumann data), and the error between the reconstructed and the real shape of the inclusion for various noise levels.

**Test-case B.** The exact inclusion $\omega^*$ has a peanut-like shape, with a boundary parameterized as follows:

$$\partial \omega = \left\{ \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) + r(\theta) \left( \begin{array}{c} a \sin \theta \\ b \cos \theta \end{array} \right) : \theta \in [0, 2\pi) \right\},$$

where $r(\theta) = \sqrt{\sin^2 \theta + 0.25 \cos^2 \theta}$, $(x_0, y_0) = (0, 0)$ and $(a, b) = (0.15, 0.18)$.

In this test-case, the shape of the inclusion is nonconvex. We observe from Figure 8(b) that while the computed zero level-set is in good accordance with the exact one, it is however unable to accurately capture the nonconvex features of the real inclusion. This is not very surprising in view of the different smoothed approximations used for the inclusion Stokes problems as well as for the level-set equation. The data completion results are however very satisfactory, as shown by the velocity and normal stress profiles in Figures 8(c)-(e). There is also a remarkable stability with respect to noisy data of both the inclusion detected and the recovered boundary data, see Figure 9.

**Test-case C.** The inclusion to be detected is the union of two separate disks $\omega_1^* = B(c_1, r_1)$ centered at $c_1 = (0.2, 0.2)$ and with a radius $r_1 = 0.10$ and $\omega_2^* = B(c_2, r_2)$ centered at $c_2 = (-0.2, -0.2)$ and with a radius $r_2 = 0.12$.

This third and last test-case was set up to assess the ability of our algorithm to identify inclusions with several components. One observes from Figure 10(b) that the locations and the shapes of the two components of the inclusion are well detected, as well as the recovered data Figure 10(c)-(f). The recovery of missing boundary data is stable with respect to noisy Dirichlet measurements while there is a barely slight shift in the location of the detected approximation of $\omega_1^*$ for the noise levels 3% and 5%, as shown Figure 11.

The relative errors presented in Table 2 corroborate the stability of the detected contours and missing data with respect to noisy Dirichlet measurements.
Inverse inclusion Cauchy-Stokes problem

Table 2. Test-case C. $L^2$-errors on missing data over $\Gamma_i$ (on Dirichlet and Neumann data), and the error between the reconstructed and the real shape for various noise levels.

| Noise level | $\sigma = 0\%$ | $\sigma = 1\%$ | $\sigma = 3\%$ | $\sigma = 5\%$ |
|-------------|-----------------|-----------------|-----------------|-----------------|
| $\text{err}_D$ | 0.042 | 0.044 | 0.046 | 0.08 |
| $\text{err}_N$ | 0.095 | 0.1 | 0.13 | 0.16 |
| $\text{err}_O$ | 0.099 | 0.11 | 0.13 | 0.15 |

Algorithm 2 vs classical. In a classical algorithm [37] dedicated to the computation of a Nash equilibrium, there would be no inner do loop as performed during step I in Algorithm 2, or in other words, $K_{max} = 1$. We have compared these two approaches, for two noise free test-cases. We used the same number of total calls (400) to the Stokes Finite Element solvers. We see from Table 3 that, for both test-cases, Algorithm 2 outperforms the classical one. Iterations in step I compute a two-player Nash equilibrium for a fixed level-set. This step, which we call a preconditioning Nash subgame, is dedicated to enforce the data completion part does indeed enforce the identifiability property as well, since from the result established in Proposition 3, it is enough for a candidate velocity $u(\omega)$, for some inclusion $\omega$, to be a Cauchy solution for the pair of boundary measurements $(f, \Phi)$, to ensure that $\omega = \omega^*$, the real inclusion.

Table 3. Relative errors on the reconstructed missing data and inclusion shape for the Stokes problem (with noise free measurements), compared for a classical Nash algorithm and Algorithm 2: (left) test-case A (right) test-case C.

| Case A | Classical algorithm | Algorithm 2 |
|--------|---------------------|-------------|
| $\text{err}_D$ | 0.058 | 0.033 |
| $\text{err}_N$ | 0.106 | 0.032 |
| $\text{err}_O$ | 0.358 | 0.140 |

| Case C | Classical algorithm | Algorithm 2 |
|--------|---------------------|-------------|
| $\text{err}_D$ | 0.067 | 0.058 |
| $\text{err}_N$ | 0.208 | 0.122 |
| $\text{err}_O$ | 0.566 | 0.167 |

Sensitivity to the mesh size, to closeness to the inaccessible boundary and inverse crime. The coupled completion/detection problem for the Cauchy-Stokes system involves many numerical tricks of more or less severity. In the lack of theoretical convergence results, which are not easy to establish in the present framework, we led three kinds of numerical experiments.

The first one is related to the behaviour of the overall coupled algorithm with respect to finite element discretization of the domain $\Omega$. The second one is related to the study of the efficiency of the algorithm in detecting obstacles located near the inaccessible boundary $\Gamma_i$. Figure-3 shows that the relative errors decrease w.r.t. the mesh size, with different convergence rates; it also shows that the errors increase dramatically when the inclusion becomes too close to the inaccessible boundary $\Gamma_i$. 
Figure 3. Test case A. (Left) sensitivity of the reconstruction w.r.t. the mesh size (on abscissae: the number of F.E. nodes on the boundary \( \partial \Omega \)). (Right) sensitivity of the reconstruction w.r.t. the distance to the inaccessible boundary \( \Gamma_i \) (on abscissae: the distance of the center of the circular inclusion from \( \Gamma_i \)).

Figure 4. Test case C. (Left) Mesh used for solving the direct problem with the P1 bubble-P1 finite element, in order to construct the synthetic data. (Right) Mesh used for solving the coupled inverse problem with P2-P1 finite element, using the P1 bubble-P1 synthetic data.

The third set of experiments was led to assess our algorithm and results against the so-called inverse crime, that arises when the same model is used to synthesize Cauchy data and to solve the corresponding inverse problem, resulting in possible artificial outperformance. To prevent such a bias, we synthesized Cauchy data using different meshes, and solving the direct problem with the well known P1 bubble - P1 finite element, see one example figure 4. All our numerical experiments produced results, in data completion as well as in obstacle detection, of the same good quality than the results presented in figures 5–11, see figure 12.
The Lagrange function $\mathcal{L}$ is defined as follows:

$$
\mathcal{L}(\eta, \tau, u_1, p_1, u_2, p_2, v_1, q_1, v_2, q_2, \pi) = \frac{1}{2}||u_1 - \tau||^2_{H^\frac{1}{2}(\Gamma_\tau)^d} + \frac{1}{2}||u_2 - f||^2_{H^\frac{1}{2}(\Gamma_\pi)^d}
$$

$$
+ \frac{1}{2}||\sigma(u_1, p_1)n - \Phi||^2_{H^\frac{1}{2}(\Gamma_\Phi)^d} + \int_{\Omega} \sigma(u_1, p_1) : \nabla v_1 dx - \int_{\Gamma_1} \eta v_1 ds
$$

$$
- \int_{\Omega} q_1 \text{div} u_1 dx + \int_{\Omega} \sigma(u_2, p_2) : \nabla v_2 dx - \int_{\Gamma_1} \Phi v_2 ds - \int_{\Omega} q_2 \text{div} u_2 dx
$$

$$
- \int_{\Gamma_1} \sigma(u_2, p_2) n v_2 ds - \int_{\Gamma_1} \pi(u_2 - \tau) ds
$$

for every $(\eta, \tau) \in H^\frac{1}{2}(\Gamma_\tau)^d \times H^\frac{1}{2}(\Gamma_\pi)^d$, $(u_1, u_2, v_1, v_2) \in H^2(\Omega)^d \times H^1(\Omega)^d \times H^1_0(\Omega)^d \times H^1(\Omega)^d$, $\pi \in H^\frac{1}{2}(\Gamma_\Phi)^d$, and $(p_1, p_2, q_1, q_2) \in H^1(\Omega)^d \times L^2(\Omega)^3$.
Figure 5. Test case A. Reconstruction of the inclusion shape and missing boundary data with noise free Dirichlet data over $\Gamma_c$. (a) Initial contour is $\phi^{(0)}_1$ (b) Exact inclusion shape - green line- and computed one - blue dashed- (c) Exact -line- and computed -dashed line- first component of the velocity over $\Gamma_i$ (d) Exact -line- and computed -dashed line- second component of the velocity over $\Gamma_i$ (e) Exact -line- and computed -dashed line- first component of the normal stress over $\Gamma_i$ (f) Exact -line- and computed -dashed line- second component of the normal stress over $\Gamma_i$. 
Figure 6. Test case A. Reconstruction of the inclusion shape and missing boundary data with noise free Dirichlet data over $\Gamma_c$. (a) initial contour is $\phi_2^{(0)}$ (b) exact inclusion shape -green line- and computed one - blue dashed- (c) exact -line- and computed -dashed line- first component of the velocity over $\Gamma_i$ (d) exact -line- and computed -dashed line- second component of the velocity over $\Gamma_i$ (e) exact -line- and computed -dashed line- first component of the normal stress over $\Gamma_i$ (f) exact -line- and computed -dashed line- second component of the normal stress over $\Gamma_i$. 
Figure 7. Test case A. Reconstruction of the inclusion shape and missing boundary data with noisy Dirichlet data over $\Gamma$ with noise levels $\sigma = \{1\%, 3\%, 5\\%\}$. (a) initial contour is $\phi^{(0)}_\Gamma$ (b) exact inclusion shape -green line- and computed ones for different noise levels (c) exact and computed first components of the velocity over $\Gamma$ (d) exact and computed second components of the velocity over $\Gamma$ (e) exact and computed first components of the normal stress over $\Gamma$ (f) exact and computed second components of the normal stress over $\Gamma$. 

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Figure 8. Test case B. Reconstruction of the inclusion shape and missing boundary data with noise free Dirichlet data over $\Gamma_c$. (a) initial contour is $\phi_2^{(0)}$ (b) exact inclusion shape -green line- and computed one - blue dashed- (c) exact -line- and computed -dashed line- first component of the velocity over $\Gamma_i$ (d) exact -line- and computed -dashed line- second component of the velocity over $\Gamma_i$ (e) exact -line- and computed -dashed line- first component of the normal stress over $\Gamma_i$ (f) exact -line- and computed -dashed line- second component of the normal stress over $\Gamma_i$. 

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Figure 9. Test case B. Reconstruction of the inclusion shape and missing boundary data with noisy Dirichlet data over $\Gamma_i$ with levels $\sigma = \{1\%, 3\%, 5\%\}$. (a) initial contour is $\phi_2^{(0)}$. (b) exact inclusion shape -green line- and computed ones for different noise levels (c) exact and computed first components of the velocity over $\Gamma_i$ (d) exact and computed second components of the velocity over $\Gamma_i$ (e) exact and computed first components of the normal stress over $\Gamma_i$ (f) exact and computed second components of the normal stress over $\Gamma_i$. 
Figure 10. Test case C. Reconstruction of the inclusion shape and missing boundary data with noise free Dirichlet data over $\Gamma_c$. (a) Initial contour is $\phi^{(0)}_2$. (b) Exact inclusion shape -green line- and computed one - blue dashed- (c) Exact -line- and computed -dashed line- first component of the velocity over $\Gamma_i$ (d) Exact -line- and computed -dashed line- second component of the velocity over $\Gamma_i$ (e) Exact -line- and computed -dashed line- first component of the normal stress over $\Gamma_i$ (f) Exact -line- and computed -dashed line- second component of the normal stress over $\Gamma_i$. 
Figure 11. Test case C. Reconstruction of the inclusion shape and missing boundary data with noisy Dirichlet data over $\Gamma_c$ with levels $\sigma = \{1\%, 3\%, 5\%\}$. (a) initial contour is $\phi_2^{(0)}$ (b) exact inclusion shape -green line- and computed ones for different noise levels (c) exact and computed first components of the velocity over $\Gamma_i$ (d) exact and computed second components of the velocity over $\Gamma_i$ (e) exact and computed first components of the normal stress over $\Gamma_i$ (f) exact and computed second components of the normal stress over $\Gamma_i$. 

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Figure 12. Assessing Inverse-Crime-Free reconstruction. Test case C. Top: initial and optimal contour. Middle: the two components of the velocity on $\Gamma_i$. Bottom: the two components of the normal stress on $\Gamma_i$ ($err_D = 0.0615048$, $err_N = 0.124296$, and $err_O = 0.113156$).
First, we compute the derivative of $J_1$ with respect to $\eta$ in some direction $\psi \in H^\frac{1}{2}(\Gamma_i)^d$, we have then

$$
\frac{\partial J_1}{\partial \eta}(\eta, \tau).\psi = \int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\sigma(u_1', p_1')nds + \int_{\Gamma_i} (u_1 - \tau)u_1' ds,
$$

where $u_1' = \frac{\partial u_1}{\partial \eta}.\psi$ and $p_1' = \frac{\partial p_1}{\partial \eta}.\psi$. We now derive the Lagrangian with respect to $u_1$ and w.r.t. $p_1$, we obtain

$$
\frac{\partial L}{\partial u_1} = \int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)(\nabla \Phi + \nabla \gamma)nds + \int_{\Gamma_i} (u_1 - \tau)u_1' ds
$$

$$
+ \int_{\Omega} (\nabla \gamma + \nabla \gamma^T) : \nabla v_1 dx - \int_{\Omega} q_1 \text{div} \gamma dx = 0, \quad \forall \gamma \in H^1(\Omega),
$$

(22)

$$
\frac{\partial L}{\partial p_1} = -\int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\gamma nds - \int_{\Omega} \text{div} v_1 dx = 0, \quad \forall \delta \in H^1(\Omega)^d.
$$

(23)

By replacing $(\gamma, \delta)$ with $(u_1', p_1')$ in (22) and (23), then by summing them, we get

$$
\int_{\Gamma_c} \sigma(u_1', p_1') : \nabla v_1 dx = -\int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\sigma(u_1', p_1') nds
$$

$$
- \int_{\Gamma_i} (u_1 - \tau)u_1' ds + \int_{\Omega} q_1 \text{div} u_1' dx,
$$

which is equivalent to,

$$
\int_{\Omega} \sigma(u_1', p_1') : \nabla v_1 dx = -\int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\sigma(u_1', p_1') nds
$$

$$
- \int_{\Gamma_i} (u_1 - \tau)u_1' ds + \int_{\Omega} q_1 \text{div} u_1' dx,
$$

(24)

with $(u_1', p_1')$ solves the following problem

$$
\begin{cases}
\Delta u_1' - \nabla p_1' = 0 & \text{in } \Omega, \\
\text{div} u_1' = 0 & \text{in } \Omega, \\
u_1' = 0 & \text{on } \Gamma_c, \\
\sigma(u_1', p_1')n = \psi & \text{on } \Gamma_i.
\end{cases}
$$

The weak formulation associated to this problem is

$$
\int_{\Omega} \sigma(u_1', p_1') : \nabla \varphi dx = \int_{\Gamma_i} \psi \varphi ds + \int_{\Gamma_c} \sigma(u_1', p_1') n \varphi ds, \quad \forall \varphi \in H^1(\Omega)^d,
$$

in particular for $\varphi = v_1$, we have

$$
\int_{\Omega} \sigma(u_1', p_1') : \nabla v_1 dx = \int_{\Gamma_i} \psi v_1 ds, \quad \text{because } v_1 \in H^1_c(\Omega).
$$

(25)

Now since $\text{div} u_1' = 0$ in $\Omega$, using (24) and (25), we find

$$
- \int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\sigma(u_1', p_1') nds - \int_{\Gamma_i} (u_1 - \tau)u_1' ds = \int_{\Gamma_i} \psi v_1 ds,
$$

then, we deduce that

$$
\frac{\partial J_1}{\partial \eta}(\eta, \tau).\psi = -\int_{\Gamma_i} \psi v_1 ds,
$$

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with \((v_1, q_1)\) solves the adjoint problem
\[
\left\{ \begin{array}{l}
\int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)(\nabla \gamma + \nabla \gamma^T)nds + \int_{\Gamma_i} (u_1 - \tau)\gamma ds \\
+ \int_{\Omega} (\nabla \gamma + \nabla \gamma^T) : \nabla v_1 dx - \int_{\Omega} q_1 \text{div} dx = 0, \quad \forall \gamma \in H^1_0(\Omega), \\
- \int_{\Gamma_c} (\sigma(u_1, p_1)n - \Phi)\delta nds - \int_{\Omega} \delta \text{div} v_1 dx = 0, \quad \forall \delta \in H^1(\Omega)^d.
\end{array} \right.
\]

Now we compute the other derivative, the derivative of the cost function \(J_2\) with respect to the variable \(\tau\). We have,
\[
\frac{\partial J_2}{\partial \tau} (\eta, \tau, \mu) = \int_{\Gamma_c} (u_2 - f)u'_2 ds - \int_{\Gamma_i} (u_1 - \tau)\mu ds,
\]
where \(u'_2 = \frac{\partial u_2}{\partial \tau}\) and \(p'_2 = \frac{\partial p_2}{\partial \tau}\). Let
\[
W_1 = \{v \in H^1(\Omega)^d / v|_{\Gamma_i} = 0\}.
\]

We have then,
\[
\frac{\partial L}{\partial u_2} . h = \int_{\Gamma_c} (u_2 - f)h ds + \int_{\Omega} (\nabla h + \nabla h^T) : \nabla v_2 dx \\
+ \int_{\Gamma_i} ((\nabla h + \nabla h^T)n)v_2 ds - \int_{\Omega} q_2 \text{div} h dx = 0, \quad \forall h \in W_1,
\]

Using the Green’s formula, and summing (27) and (28), we obtain
\[
\int_{\Gamma_c} (u_2 - f)h ds - \int_{\Omega} \text{div}(\nabla v_2 + \nabla v_2^T)h dx + \int_{\Gamma_c} ((\nabla v_2 + \nabla v_2^T)n)h ds \\
+ \int_{\Gamma_i} \sigma(h, k)n v_2 ds + \int_{\Omega} \nabla q_2 h dx - \int_{\Omega} (q_2 n)h ds - \int_{\Omega} k \text{div} v_2 dx = 0.
\]

Therefore, we get for all \((h, k)\) \(\in W_1 \times L^2(\Omega),\)
\[
- \int_{\Omega} \text{div}(\nabla v_2 + \nabla v_2^T)h dx + \int_{\Omega} \nabla q_2 h dx + \int_{\Omega} ((\sigma(v_2, q_2)n - (f - u_2))h ds \\
+ \int_{\Gamma_i} \sigma(h, k)n v_2 ds - \int_{\Omega} k \text{div} v_2 dx = 0.
\]

Finally, from this equality, we deduce the following problem
\[
\left\{ \begin{array}{l}
\Delta v_2 - \nabla q_2 = 0 \quad \text{in} \quad \Omega, \\
\text{div} v_2 = 0 \quad \text{in} \quad \Omega, \\
v_2 = 0 \quad \text{on} \quad \Gamma_i, \\
\sigma(v_2, q_2)n = f - u_2 \quad \text{on} \quad \Gamma_c.
\end{array} \right.
\]

The weak formulation associated to this problem is,
\[
- \int_{\Omega} \sigma(v_2, q_2) : \nabla \varphi dx = \int_{\Gamma_c} (f - u_2)\varphi ds + \int_{\Gamma_i} \sigma(v_2, q_2)n\varphi ds, \quad \forall \varphi \in H^1(\Omega),
\]
Taking \(\varphi = u'_2\), we obtain,
\[
\int_{\Omega} \sigma(v_2, q_2) : \nabla u'_2 dx = \int_{\Gamma_c} (f - u_2)u'_2 ds + \int_{\Gamma_i} (\sigma(v_2, q_2)n)u'_2 ds,
\]
with the solution \((u'_2, p'_2)\) verify
\[
\begin{align*}
\Delta u'_2 - \nabla p'_2 &= 0 \quad \text{in} \quad \Omega, \\
\text{div} u'_2 &= 0 \quad \text{in} \quad \Omega, \\
u'_2 &= \mu \quad \text{on} \quad \Gamma_i, \\
\sigma(u'_2, p'_2)n &= 0 \quad \text{on} \quad \Gamma_c.
\end{align*}
\]

The weak formulation associated of this problem is
\[
\int_\Omega \sigma(u'_2, p'_2) \cdot \nabla \vartheta d\Omega = \int_{\Gamma_i} (\sigma(u'_2, p'_2)n) \vartheta ds, \quad \forall \vartheta \in H^1(\Omega)^d,
\]
if we replace \(\vartheta\) by \(v_2\), we obtain
\[
(30) \quad \int_\Omega \sigma(u'_2, p'_2) \cdot \nabla v_2 d\Omega = \int_{\Gamma_i} (\sigma(u'_2, p'_2)n)v_2 ds.
\]
From (29) and (30), and the boundary condition \(v_2|_{\Gamma_i} = 0\), we have
\[
\int_{\Gamma_c} (u_2 - f)u'_2 ds = \int_{\Gamma_i} (\sigma(v_2, q_2)n)\mu ds.
\]
Thus,
\[
\frac{\partial J_2}{\partial \tau}(\eta, \tau, \mu) = \int_{\Gamma_i} (\sigma(v_2, q_2)n)\mu ds - \int_{\Gamma_i} (u_1 - \tau)\mu ds, \quad \forall \mu \in H^{2/3}(\Gamma)^d.
\]

**A.2 Proposition 3 : Derivation of the sensitivity formula.** For fixed \((\eta, \tau) \in H^{2/3}(\Gamma)^d \times H^{2/3}(\Gamma)^d\), let us define the Lagrangian \(\mathcal{L}'\) by :
\[
\mathcal{L}'(\phi, \lambda_1, \pi_1, \lambda_2, \pi_2, u_1, p_1, u_2, p_2) = \int_\Omega |\sigma(u_2, p_2) - \sigma(u_1, p_1)|^2 H_{\epsilon, \beta}(\phi) d\Omega
\]
\[
+ \mu \int_\Omega \delta_{\epsilon, \beta}(\phi) \nabla \phi d\Omega + \int_\Omega (\sigma(u_1, p_1) : \nabla \lambda_1) H_{\epsilon, \beta}(\phi) d\Omega - \int_{\Gamma_i} \eta \lambda_1 ds
\]
\[
- \int_\Omega \pi_1 \text{div} u_1 H_{\epsilon, \beta}(\phi) d\Omega + \int_\Omega (\sigma(u_2, p_2) : \nabla \lambda_2) H_{\epsilon, \beta}(\phi) d\Omega - \int_{\Gamma_c} \Phi \lambda_2 ds
\]
\[
- \int_\Omega \pi_2 \text{div} u_2 H_{\epsilon, \beta}(\phi) d\Omega,
\]
where the control \(\phi \in \mathcal{S}\), the state variables \((u_1, u_2, p_1, p_2) \in V_f \times W_f \times L^2(\Omega) \times L^2(\Omega)\), the adjoint variables \(\lambda_1, \lambda_2 \in H^1_0(\Omega) \times H^1_0(\Omega)\) and \(\pi_1, \pi_2 \in L^2(\Omega) \times L^2(\Omega)\).

The differentiation of the functional \(\mathcal{J}_3(\eta, \tau; \phi)\) with respect to \(\phi\) in some direction \(\psi \in H^1(\Omega)\) yields
\[
\frac{\partial \mathcal{J}_3}{\partial \phi}(\eta, \tau; \phi)\psi = \int_\Omega |\sigma(u_2, p_2) - \sigma(u_1, p_1)|^2 \delta_{\epsilon, \beta}(\phi) \psi d\Omega
\]
\[
+ 2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (\sigma(u'_2, p'_2) - \sigma(u'_1, p'_1)) H_{\epsilon, \beta}(\phi) d\Omega
\]
\[
+ \mu \int_\Omega \delta'_{\epsilon, \beta}(\phi) \nabla \phi \psi d\Omega + \mu \int_\Omega \delta_{\epsilon, \beta}(\phi) \frac{\nabla \phi \nabla \psi}{|\nabla \phi|} d\Omega, \quad \forall \psi \in H^1(\Omega)^d,
\]
where we have used the notations
\[
(u'_1, p'_1) = \left( \frac{\partial u_1}{\partial \phi}, \frac{\partial p_1}{\partial \phi}, \psi \right) \quad \text{and} \quad (u'_2, p'_2) = \left( \frac{\partial u_2}{\partial \phi}, \frac{\partial p_2}{\partial \phi}, \psi \right).
\]

We know that \((u_1, p_1)\) solves the variational equation
\[
\int_{\Gamma_c} (u_1, p_1) : \nabla v_1 H_{\epsilon, \beta}(\phi) d\Omega = \int_{\Gamma_i} \eta v_1 ds, \quad \forall v_1 \in H^1_0(\Omega).
\]
Then, \((u'_1, p'_1)\) fulfills the following weak formulation
\[
\int_\Omega (\sigma(u'_1, p'_1) : \nabla v_1) H_{\varepsilon, \beta}(\phi) dx + \int_\Omega (\sigma(u_1, p_1) : \nabla v_1) \delta_{\varepsilon, \beta}(\phi) dx = 0, \quad \forall v_1 \in H^1_0(\Omega).
\]

Now, we derive the Lagrangian \(\mathcal{L}'\) with respect to \(u_1\) and with respect to \(p_1\), we get
\[
\begin{align*}
\frac{\partial \mathcal{L}'}{\partial u_1} h_1 &= -2 \int_\Omega (\nabla h_1 + \nabla h_1^T) : (\sigma(u_2, p_2) - \sigma(u_1, p_1)) H_{\varepsilon, \beta}(\phi) dx \\
&\quad + \int_\Omega (\nabla h_1 + \nabla h_1^T) : \nabla \lambda_1 H_{\varepsilon, \beta}(\phi) dx - \int_\Omega \pi_1 \text{div} h_1 dx = 0, \quad \forall h_1 \in H^1_0(\Omega), \\
\frac{\partial \mathcal{L}'}{\partial p_1} k_1 &= 2 \int_\Omega (k_1 I_d) : (\sigma(u_2, p_2) - \sigma(u_1, p_1)) H_{\varepsilon, \beta}(\phi) dx - \int_\Omega k_1 \text{div} \lambda_1 H_{\varepsilon, \beta}(\phi) dx \\
&= 0, \quad \forall k_1 \in L^2(\Omega).
\end{align*}
\]

If the pair \((h_1, k_1)\) is replaced by \((u'_1, p'_1)\) in (32) and because of \(\text{div} u_1 = 0\) implies \(\text{div} u'_1 = 0\), using the weak formulation for the couple \((u'_1, p'_1)\), we get
\[
-2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (\sigma(u'_1, p'_1)) H_{\varepsilon, \beta}(\phi) dx = \int_\Omega (\sigma(u_1, p_1) : \nabla \lambda_1) \delta_{\varepsilon, \beta}(\phi) dx,
\]
where \((\lambda_1, \pi_1)\) solves the adjoint state problem
\[
\begin{align*}
-2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (\nabla h_1 + \nabla h_1^T) H_{\varepsilon, \beta}(\phi) dx &+ \int_\Omega \pi_1 \text{div} h_1 H_{\varepsilon, \beta}(\phi) dx \\
&+ \int_\Omega ((\nabla h_1 + \nabla h_1^T) : \nabla \lambda_1) H_{\varepsilon, \beta}(\phi) dx = 0, \quad \forall h_1 \in H^1_0(\Omega), \\
2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (k_1 I_d) H_{\varepsilon, \beta}(\phi) dx &- \int_\Omega k_1 \text{div} \lambda_1 H_{\varepsilon, \beta}(\phi) dx = 0, \quad \forall k_1 \in L^2(\Omega).
\end{align*}
\]

In the same way, we find that
\[
2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (\sigma(u'_2, p'_2)) H_{\varepsilon, \beta}(\phi) dx = \int_\Omega (\sigma(u_2, p_2) : \nabla \lambda_2) \delta_{\varepsilon, \beta}(\phi) dx,
\]
where \((\lambda_2, \pi_2)\) solves the adjoint problem
\[
\begin{align*}
2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (\nabla h_2 + \nabla h_2^T) H_{\varepsilon, \beta}(\phi) dx &- \int_\Omega \pi_2 \text{div} h_2 H_{\varepsilon, \beta}(\phi) dx \\
&+ \int_\Omega ((\nabla h_2 + \nabla h_2^T) : \nabla \lambda_2) H_{\varepsilon, \beta}(\phi) dx = 0, \quad \forall h_2 \in H^1_0(\Omega), \\
-2 \int_\Omega (\sigma(u_2, p_2) - \sigma(u_1, p_1)) : (k_2 I_d) H_{\varepsilon, \beta}(\phi) dx &- \int_\Omega k_2 \text{div} \lambda_2 H_{\varepsilon, \beta}(\phi) dx = 0, \quad \forall k_2 \in L^2(\Omega).
\end{align*}
\]

Using (33) and (34), we obtain
\[
\frac{\partial J_3}{\partial \phi}(\eta, \tau; \phi) = \int_\Omega |\sigma(u_2, p_2) - \sigma(u_1, p_1)|^2 \delta_{\varepsilon, \beta}(\phi) \psi dx + \mu \int_\Omega \delta_{\varepsilon, \beta}(\phi) |\nabla \phi| \psi dx \\
+ \mu \int_\Omega \delta_{\varepsilon, \beta}(\phi) \frac{\nabla \phi \nabla \psi}{|\nabla \phi|} dx + \int_\Omega (\sigma(u_1, p_1) : \nabla \lambda_1) \delta_{\varepsilon, \beta}(\phi) \psi dx \\
+ \int_\Omega (\sigma(u_2, p_2) : \nabla \lambda_2) \delta_{\varepsilon, \beta}(\phi) \psi dx.
\]
On the one hand, we have
\[
\int_{\Omega} \text{div}(\delta_{\epsilon,\beta}(\phi) \frac{\nabla \phi}{|\nabla \phi|}) \psi \, dx = -\int_{\Omega} \delta_{\epsilon,\beta}(\phi) \frac{\nabla \phi}{|\nabla \phi|} \nabla \psi \, dx + \int_{\partial \Omega} \frac{\delta_{\epsilon,\beta}(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial n} \psi \, ds,
\]
and on the other hand,
\[
\int_{\Omega} \text{div}(\delta_{\epsilon,\beta}(\phi) \frac{\nabla \phi}{|\nabla \phi|}) \psi \, dx = \int_{\Omega} \delta_{\epsilon,\beta}(\phi) |\nabla \phi| \psi \, dx + \int_{\partial \Omega} \delta_{\epsilon,\beta}(\phi) \text{div}(\frac{\nabla \phi}{|\nabla \phi|}) \psi \, ds.
\]
Then, if \( \phi \) satisfies the boundary condition
\[
\frac{\delta_{\epsilon,\beta}(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial n} = 0, \quad \text{on} \quad \partial \Omega,
\]
one can conclude that
\[
\frac{\partial J_3}{\partial \phi}(\eta, \tau; \phi) \psi = \int_{\Omega} |\sigma(u_2, p_2) - \sigma(u_1, p_1)|^2 \delta_{\epsilon,\beta}(\phi) \psi \, dx
\]
\[
+ \int_{\Omega} \left[ \sigma(u_1, p_1) : \nabla \lambda_1 + (\sigma(u_2, p_2) : \nabla \lambda_2) \delta_{\epsilon,\beta}(\phi) \psi \, dx
\]
\[
- \mu \int_{\Omega} \delta_{\epsilon,\beta}(\phi) \text{div}(\frac{\nabla \phi}{|\nabla \phi|}) \psi \, dx.
\]

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Received October 2018; 1st revision November 2018; 2nd revision March 2019.

E-mail address: habbal@unice.fr
E-mail address: moez.kallel@enit.utm.tn
E-mail address: marwa1ouni@gmail.com