Gradient catastrophe and flutter in vortex filament dynamics

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Received 27 August 2011, in final form 13 September 2011
Published 7 October 2011
Online at stacks.iop.org/JPhysA/44/432001

Abstract

Gradient catastrophe and flutter instability in the motion of a vortex filament within the localized induction approximation are analyzed. It is shown that the origin of this phenomenon is in the gradient catastrophe for the dispersionless Da Rios system which describes the motion of a filament with slow varying curvature and torsion. Geometrically, this catastrophe manifests as a rapid oscillation of a filament curve in a point that resembles the flutter of airfoils. Analytically, it is the elliptic umbilic singularity in the terminology of the catastrophe theory. It is demonstrated that its double scaling regularization is governed by the Painlevé-I equation.

PACS numbers: 47.32.C, 02.30.Ik, 47.35.Jk
Mathematics Subject Classification: 76B47, 58K35, 35Q05

(Some figures may appear in colour only in the online journal)

1. Introduction

The motion of a thin vortex filament in an incompressible inviscid fluid in an infinite three-dimensional domain within the localized induction approximation (LIA) is governed by the simple equation for the induced velocity \( \vec{v} \):

\[ \vec{X}_t(s, t) \equiv \vec{v}(s, t) = \vec{X}_s \wedge \vec{X}_{ss} = K(s, t) \vec{b}, \]

which implies the following intrinsic equations for the curvature \( K \) and torsion \( \tau \):

\[ K_t = -2K\tau - K\tau_s, \]
\[ \tau_t = KK_s - 2\tau \tau_s + \left( \frac{K_{ss}}{K} \right) s. \]

(Please note the corrections made to the equations.)
Here, $\vec{X}(s, t)$ is a position vector for a point on the curve representing the filament, $t$ is the time, $s$ is the arclength parameter, the subscripts indicate the differentiation with respect to the indicated variables and $\vec{b}$ is the binormal.

Equations (1) and (2) were derived by Da Rios in 1906 [1] and were rediscovered 60 years later in [2, 3]. The detailed history of the Da Rios (DR) system is presented in [4]. For the LIA indicated variables and time, it is the system of quasilinear equations of elliptic type which in terms of the complex-valued vortex filament with curvature and torsion slowly varying along the filament and in the motion.

(2) directly. We first consider a dispersionless (dDR) system which describes the motion of a vortex filament dynamics. For this purpose it is convenient to deal with the DR system for the vortex filament behavior. For this system and solutions exhibit the gradient catastrophe-type behavior at finite time. The behavior of solutions of the dispersionless NLS equation near the point of gradient catastrophe and beyond it has also been studied in [36–38].

In 1972, Hasimoto observed [7] that equation (1) is equivalent to the focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^2\psi = 0$$

via the transformation

$$\psi(s, t) = K(s, t) \exp\left(i \int_{0}^{t} \tau(s', t') ds'\right).$$

One year previously, Zakharov and Shabat [8] had discovered that the NLS equation (3) is integrable by the inverse scattering transform (IST) method. Hasimoto’s result has demonstrated that the whole powerful machinery of the IST method (solitons, infinite sets of integrals of motion, symmetries, etc) is applicable to the vortex filament dynamics and has led to the explosion of interest in the vortex filament dynamics. Since that time various aspects of this dynamics have been analyzed by different methods (see e.g. [9–27]).

The stability of vortex filament motion and formation of singularities are two important problems partially addressed during this period (see e.g. [13, 15, 22, 26]). Modulation instability of the focusing NLS equation (3) (see e.g. [28–30]) is definitely a key element in such an analysis. Recently, it was shown that the dispersionless limit of the NLS equation (3) is quite relevant to the study of the modulation instability [31–38]. In this limit the focusing NLS equation is the elliptic system of quasilinear PDEs. The Cauchy problem is ill-posed for this system and solutions exhibit the gradient catastrophe-type behavior at finite time. The behavior of solutions of the dispersionless NLS equation near the point of gradient catastrophe and beyond it has also been studied in [36–38].

In this paper, we study the geometrical and analytical implications of such an analysis for the vortex filament dynamics. For this purpose it is convenient to deal with the DR system (2) directly. We first consider a dispersionless (dDR) system which describes the motion of a vortex filament with curvature and torsion slowly varying along the filament and in the motion. It is the system of quasilinear equations of elliptic type which in terms of the complex-valued Riemann invariant $\beta = -\tau + iK$ and slow variables $x = \epsilon s, y = \epsilon t, \epsilon \ll 1$ is of the form

$$\beta_y = \frac{1}{2}(3\beta + \bar{\beta})\beta_x.$$  (5)

This system approximates well the DR system (2) until the derivatives $\beta_x$ and $\beta_y$ are not large. Hodograph equations for equation (5) describe critical points of the function $W$ which obey the Euler–Poisson–Darboux equation $E(\frac{1}{2}, \frac{1}{2})$.

Gradient catastrophe for the dDR system occurs in a point $(x_0, y_0)$ for given initial data. At this point $\beta_x$ and $\beta_y$ (or $K_x, K_y, \tau_x, \tau_y$) explode. The acceleration $\ddot{a} = \ddot{u}$ explodes too. So the filament becomes fast oscillating near the point $x_0$ at the ‘time’ $y_0$. Numerical analysis (borrowed from [33]) shows that such oscillations begin to expand along the filament. Intrinsically, the filament begins to oscillate around the rectifying plane near the point $x_0$. So, the gradient catastrophe for the system (5) gives rise to fast oscillation which can be referred to as filament flutter by analogy with airfoil flutter (see e.g. [39]).

Analysis of behavior near the point $(x_0, y_0)$ of gradient catastrophe shows that $\beta = \beta_0 + \epsilon \beta^*$, $x = x_0 + \epsilon x^*, y = y_0 + \epsilon y^*, \epsilon \ll 1$, and

$$W = W_0 + \epsilon^3 \left(\chi^*(fU + gV) + \frac{1}{3}U^3 - UV^2\right),$$  (6)
where \( U + iV \propto \beta \) and \( f, g \) are some constants. Thus, the function \( W \) exhibits an elliptic umbilic singularity behavior in the terminology of Thom [40]. Regularization of this singularity is one of the issues of this paper.

At the point of gradient catastrophe, approximation (5) becomes invalid and should be substituted by the full DR system (2). First-order corrections can be obtained by the double-scaling technique together with appropriate modification of the function \( W(\beta, \beta) \) to a functional \( W^* \) such that the equations of critical points for \( W \) are substituted by the Euler–Lagrange equation for \( W^* \). The resulting equation for small corrections is equivalent to the Painlevé-I (P-I) equation

\[
\Omega_{\xi\xi} = 6\Omega^2 - \xi. 
\] (7)

The result of the paper [36] allows us to conjecture that any generic solution of the gradient catastrophe behaves as \( \beta = \beta_0 + \epsilon \beta^* \) where the correction \( \beta^* \) is described by the tritronqué solution of the P-I equation.

The paper is organized as follows. The dispersionless DR system, its different forms, hodograph equation and other properties are discussed in section 2. The gradient catastrophe for the dDR system is analyzed in section 3. Geometrical implications of the gradient catastrophe and flutter of filament are considered in section 4. In section 5, it is shown that near to the singular point the filament exhibits the elliptic umbilic catastrophe behavior. Regularization of this singularity via the Painlevé-I equation is presented in section 6.

2. Dispersionless DR system

In order to describe the dispersionless (or quasiclassical) limit of the DR system one, in a standard manner, introduces slow variables \( x = \epsilon s, y = \epsilon t \) with \( \epsilon \ll 1 \) and assumes that curvature and torsion are smooth functions of slow variables \( K = K(x, y) \) and \( \tau = \tau(x, y) \). Under these assumptions equation (2) take the form

\[
\begin{align*}
K_y &= -2K\tau - K\tau_x, \\
\tau_y &= KK_x - 2\tau\tau_x + \epsilon^2 \left( \frac{K_x}{K} \right)_x.
\end{align*} 
\] (8)

Thus, in the limit \( \epsilon \to 0 \) one has the dDR system

\[
\begin{align*}
K_y &= -2K\tau - K\tau_x, \\
\tau_y &= KK_x - 2\tau\tau_x.
\end{align*} 
\] (9)

Analogously to the dispersionless NLS equation [34–37] the solutions of the dDR system (9) well approximate solutions of the DR system in points where \( K_0 \) and \( \tau_0 \) are finite. As with any two-component system it is linearizable by hodograph transformation \((x, y) \leftrightarrow (K, \tau)\). The characteristic velocities for the dDR system are complex \( \lambda = -2\tau + iK \). Riemann invariant \( \beta = -\tau + iK \) and in terms of \( \beta \) the dDR system is of the form (9). We note that solutions of the dNLS system discussed in [36] and those of the dDR system (9) are connected by the simple relations \( u = K^2, v = 2\tau \). For the geometrical consideration the system (9) is more convenient.

The dDR system is known to be integrable similar to its hyperbolic version, i.e. the one-layer Benney system [41]. It has an infinite set of symmetries and integrals of motion. One of the forms of the dDR hierarchy is given by the set of equations (see e.g. [42])

\[
p_{yn} = \left( \left( \frac{\omega}{p} \right)_x^+ \right)_x, \quad n = 1, 2, 3, \ldots ,
\] (10)
where $p = z + p_1(x, y)z^{-1} + p_2(x, y)z^{-3} + p_3(x, y)z^{-5} + \cdots$ is a formal Laurent series defined by the equations

$$p^2 = (z - \beta)(z - \overline{\beta}) = (z + \tau)^2 + K^2,$$

where $\tau_n$ are time variables and $f_+^+$ denotes the polynomial part of $f$. The dDR system (9) is the first flow of the hierarchy (10) at $n = 1$. All $p_{2k+1}(x, y)$ are densities of integrals of motion for the dDR system. They are the dispersionless limit of the densities of integrals of motion for the full DR system (2) found in [14, 17].

Solutions of the dDR system can be calculated via the standard hodograph equation. It was shown in [43, 44] that these hodograph equations are, in fact, the equations

$$W_\beta = 0, \quad W_\overline{\beta} = 0,$$

which define the critical points of the function of the form

$$W = \frac{x}{2}(\beta + \overline{\beta}) + \frac{y}{8}(3\beta^2 + 2\overline{\beta}^2 + 3\overline{\beta}^2) + \tilde{W}(\beta, \overline{\beta}),$$

where the function $\tilde{W}(\beta, \overline{\beta}) = \tilde{W}(\overline{\beta}, \beta)$ is defined by the initial data for $\beta$ and it is such that $W$ obeys the Euler–Poisson–Darboux equation $E \left( \frac{1}{2}, \frac{1}{2} \right)$, i.e.

$$2(\beta - \overline{\beta})W_{\beta \overline{\beta}} = W_{\beta} - W_{\overline{\beta}},$$

Since the function $W$ is real valued, the second equation of (12) is the complex conjugated to the first one. The hodograph equation is

$$W_\beta = \frac{x}{2} + \frac{y}{4}(3\beta + \overline{\beta}) + \tilde{W}_\beta = 0.$$

In terms of $K$ and $\tau$ the function $W$ is

$$W = -x\tau + y(\tau^2 - \frac{1}{4}K^2) + \tilde{W}(\tau, K),$$

the hodograph equations (15) are given by

$$W_K = -yK + \tilde{W}_K = 0 \quad W_\tau = -x + 2y\tau + \tilde{W}_\tau = 0,$$

while the Euler–Poisson–Darboux equation is of the form

$$K(W_{KK} + W_{\tau\tau}) + W_K = 0.$$

It is the axisymmetric three-dimensional Laplace equation studied by Beltrami [45] (see also [46]). Here, $K$ and $\tau$ play the role of the radial and axial coordinates, respectively, in the cylindrical system of coordinates.

For the dDR hierarchy (10) the function $W$ has the form

$$W = \sum_{l=0}^{\infty} y_l \oint_{\gamma} \frac{dz}{2\pi i p(z)},$$

where $\gamma$ denotes a small circle around infinity. The Euler–Poisson–Darboux equation (14) is quite useful in the study of a singular sector of the one-layer Benney and dDR hierarchies [43, 44].
3. Gradient catastrophe

The hodograph equations (12) provide us with a unique solution of the dDR system if the standard condition
\[ \Delta \equiv \det \begin{pmatrix} W_{\beta\beta} & W_{\beta\gamma} \\ W_{\gamma\beta} & W_{\gamma\gamma} \end{pmatrix} \neq 0 \] is satisfied. Due to the Euler–Poisson–Darboux equation (14) on the solutions of the dDR system with \( K \neq 0 \) the function \( W \) obeys the equation \( W_{\beta\beta} = 0 \). Hence,
\[ \Delta = |W_{\beta\beta}|^2 \] and, consequently, the hodograph equations (15) are uniquely solvable if \( W_{\beta\beta} \neq 0 \). In this situation, the derivatives \( \beta_x \) and \( \beta_y \) are bounded together with \( \beta \). Indeed, differentiating equation (15) with respect to \( x \), one obtains the system
\[ \begin{pmatrix} W_{\beta\beta} & W_{\beta\gamma} \\ W_{\gamma\beta} & W_{\gamma\gamma} \end{pmatrix} \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0. \] (22)

So, for the solutions of the dDR system one has
\[ \beta_x = -\frac{1}{2} \frac{1}{W_{\beta\beta}}. \] (23)

Thus, the regular sector of the dDR system (9) is characterized by the condition [44]
\[ W_\beta = 0, \quad W_{\beta\beta} \neq 0. \] (24)

In this sector, \( \beta \) and the derivatives \( \beta_x, \beta_y \), i.e. the curvature \( K \), torsion \( \tau \), and their derivatives, are bounded.

The singular sector of the dDR system is composed by solutions for which \( W_{\beta\beta} = 0 \). For these solutions \( \beta \) (i.e. curvature and torsion) remain bounded while their derivatives explode. Such a situation is usually referred to as a gradient catastrophe (see e.g. [47, 48]). A generic gradient catastrophe is characterized by the conditions [44]
\[ W_\beta = 0, \quad W_{\beta\beta} = 0, \quad W_{\beta\beta\beta} \neq 0. \] (25)

At \( K \neq 0 (\beta \neq \beta) \) one also has \( W_{\beta\beta} = 0 \) and conditions (25), in terms of curvature and torsion are,
\[ W_K = 0, \quad W_\tau = 0, \quad W_{KK} = 0, \quad W_{K\tau} = 0, \quad W_{K\tau\tau} \neq 0, \quad W_{\tau\tau\tau} \neq 0. \] (26)

Equations \( W_{\tau\tau} = 0, W_{KK\tau} = 0, W_{K\tau\tau} = 0, \) are consequences of (26).

Lastly, conditions (26) allow us to solve the conditions \( W_{KK} = W_{\tau\tau} = 0 \) with respect to \( K \) and \( \tau \). Substituting these expressions into the first two conditions (26), one obtains two equations
\[ f_1(x, y) = 0, \quad f_2(x, y) = 0. \] (27)

Thus, a generic gradient catastrophe for the dDR system occurs in a single point \( (x_0, y_0) \) for given \( W(\beta, \beta) \), i.e. initial data for \( K \) and \( \tau \) [44]. For the focusing dNLS equation this fact has been first noted in [36]. We would like to emphasize that in contrast to the hyperbolic case where the catastrophe locus is a curve in \( x, y \) here the catastrophe occurs in a point.
4. Flutter of filament

A gradient catastrophe for the dDR system means a special behavior of a filament at the point $x_0$ at time $y_0$. First one observes, using formula (20) in [16] with $U = W = 0$ and $V = K$, that the acceleration $\vec{a}$ for the velocity (1)

$$\underline{\vec{a}} = \vec{v}_t = \epsilon (K_2 \vec{b} + K \vec{h}_n) = \epsilon \{(-2K_3 \tau - K_4 \tau^2)\vec{b} - K_5 \vec{t} + \tau^2 \vec{n}\}$$  \hspace{0.5cm} (28)

explodes at $x_0$. Thus, at the moment $t_0 = \epsilon y_0$, a sharp bump is formed at the point $s_0 = \epsilon x_0$ of the filament.

Local consideration reveals another peculiarity of the behavior of a filament around the point $x_0$. It is well known that the coordinates of a curve in a neighborhood of a point in the reference system formed by the tangent vector $\vec{t}$, normal $\vec{n}$, and binormal $\vec{b}$ and origin in the point (see e.g. formula (6.7) [49]) are

$$x_1 = s - \frac{K^2}{6} s^3 + \cdots,$$
$$x_2 = \frac{K^2}{2} s^2 - \frac{K_4}{6} s^3 + \cdots,$$
$$x_3 = \frac{K \tau}{6} s^3 + \frac{1}{24} (2K_5 \tau + K_6 \tau^2) s^4 + \cdots.$$  \hspace{0.5cm} (29)

Passing to the slow variable $x$ and slow coordinates $\tilde{x}_1 = \epsilon x_1$, $\tilde{x}_2 = \epsilon x_2$, $\tilde{x}_3 = \epsilon x_3$, one obtains the same series (29) for slow variables with substitution $s \to x$. At an ordinary point where $K$, $\tau$ and $K_6$, $\tau$ are bounded the first terms in the rhs of (29) dominate and the curve, locally, is a twisted cubic. Its projection on the osculating plane (spanned by $\vec{t}$ and $\vec{n}$), normal plane (spanned by $\vec{b}$ and $\vec{n}$) and rectifying plane (spanned by $\vec{t}$ and $\vec{b}$) are parabola ($x_2 - \frac{K^2}{6} x_3^2 = 0$), cusp ($9Kx_2^2 - 2\tau^2 x_3^3 = 0$) and cubic ($x_3 + \frac{K_6}{6} x_3^2 = 0$), respectively (figure 1, see e.g. [49]).

The characteristic feature of a curve (and a filament) in an ordinary point (i.e. outside the points of gradient catastrophe) is that it lies always on one side of the positive direction of the normal and on one side of the osculating plane (depending on the sign of the torsion $\tau$).

At the point $x_0$ of gradient catastrophe, the behavior of a curve changes drastically. Indeed, when $K_6$ and $\tau$ become large, the second terms in $\tilde{x}_2$ and $\tilde{x}_3$ become relevant. So, parabola in the osculating plane may change sign or even convert into a cubic curve (figure 2), while in the normal plane it could be a plane (3, 4) curve and so on. So, around the point $x_0$ of gradient catastrophe a filament oscillates from one side of the rectifying planes to another and back.

![Figure 1](image-url)
Figure 2. Possible different forms of the projection on the osculating plane of a vortex filament near the catastrophe point.

Such oscillation is quite similar to that of an airfoil (see e.g. [39]) and one can refer to this oscillation of a filament in the point of gradient catastrophe as a flutter.

At each point of a curve there is a sphere which has contact of third order at this point. It is called the osculating sphere (see e.g. [49]). It has the radius

\[ R^2 = \frac{K^2 \tau^2 + K_s^2}{K^4 \tau^2}, \]

and its center has the coordinates

\[ \vec{X}_0 = \vec{X} + \frac{1}{K} \vec{n} - \frac{K_s}{K^2 \tau} \vec{b}, \]

where \( \vec{X} \) is the position vector of a point on the curve.

At the point of gradient catastrophe the radius of the osculating sphere and its center blow up to infinity.

Oscillation of a curve around the rectifying plane and blowup of the osculating sphere are geometrical features of the gradient catastrophe and flutter of a filament. At the point \( x_0, t_0 \) of gradient catastrophe derivatives \( K_x, \tau_x \) are so large that the terms of higher order derivatives in the DR equation (8) become relevant for finite \( \epsilon \). Solutions of the full DR system (8) at small \( \epsilon \) are almost indistinguishable from those of the dDR system at \( t < t_0 = \epsilon y_0 \) (for dNLS equation see [31–37]). At \( t \geq t_0 \) the behavior of solutions of the dDR and DR systems is completely different. Solutions of the DR system at \( t \geq t_0 \) develop a zone of rapid oscillations which expand around the point \( x_0 \). For the equivalent dNLS equation this fact has been observed both analytically and numerically (see e.g. [31–37]).

An example of such a behavior is given by figure 13 of [33]. In the terms of \( K \) and \( \tau \) the initial data from [33] are \( K(x, 0) = 2e^{-x^2} \) and \( \tau(x, 0) = -\tanh(x) \). The gradient catastrophe occurs in \( x_0 = 0 \) and \( y_0 \approx 0.25 \). At this point the corresponding solution of the DR system oscillates. As time \( y \) grows a zone of rapid oscillations expands as \( \sqrt{y - y_0} \) symmetrically around the point \( x_0 \). For the filament dynamics this effect is seen as the creation of rapid oscillations at the point of gradient catastrophe and their subsequent expansion along the growing in time piece of filament (see figure 3).
Figure 3. Typical behavior of a flutter vortex line.

Such a behavior is evidence of the flutter-type instability of the vortex filament motion.

5. Elliptic umbilic catastrophe

In order to understand this phenomenon better one should analyze in more detail the structure of the singularity and the regularizing mechanism for the gradient catastrophe. For the focusing dNLS equation such an analysis based on the $\epsilon$-expansion of the integrals of motions of the NLS/Toda equation has been performed in [36]. Here, we will follow a different approach discussed recently in [50].

Thus, we consider a neighborhood of the gradient catastrophe point $x_0, y_0$ and denote the values of $\beta$ at this point by $\beta_0$. Following the double scaling limit method (see e.g. [51, 52]) we will look for solutions of the DR system near the point of gradient catastrophe of the form

$$x = x_0 + \epsilon^\alpha x^*, \quad y = y_0 + \epsilon^\sigma y^*, \quad \beta = \beta_0 + \epsilon^\gamma \beta^*,$$

where $\epsilon \ll 1$ and numbers $\alpha, \sigma, \gamma$ should be fixed by further consideration. For the sake of simplicity, we restrict ourselves to the case $y^* = 0$. We first consider the function $W(x, y, \beta)$ (13). One has

$$W(x_0 + \epsilon^\alpha x^*, y_0, \beta_0 + \epsilon^\alpha \beta^*) = W^0 + \frac{1}{2} \epsilon^\alpha (\beta_0 + \tilde{\beta}_0) x^*$$

$$+ \epsilon^\gamma \left[ \left( \frac{1}{2} x_0 + \frac{1}{4} (3\beta_0 + \tilde{\beta}_0) + \tilde{W}_\beta^0 \right) \beta^* + \left( \frac{1}{2} x_0 + \frac{1}{4} (\beta_0 + 3\tilde{\beta}_0) + \tilde{W}_\beta^0 \right) \beta^* \right]$$

$$+ \frac{1}{2} \epsilon^{2\gamma} x^* (\beta^* + \tilde{\beta})$$

$$+ \epsilon^{2\gamma} \left[ \left( \frac{3}{4} y_0 + \tilde{W}_\beta^0 \right) \beta^* + \left( \frac{3}{4} y_0 + \tilde{W}_\beta^0 \right) \beta^* \right]$$

$$+ \left( \frac{1}{2} + 2\tilde{W}_0 \right) \tilde{\beta}^* \beta^* + \frac{1}{6} \epsilon^{3\gamma} \left[ \tilde{W}_{\beta\beta\beta\beta} \beta^3 + 2\tilde{W}_{\beta\beta\beta\beta} \beta^3 \tilde{\beta} + 2\tilde{W}_{\beta\beta\beta\beta} \beta^3 \tilde{\beta} \right]$$

$$+ \tilde{W}_{\beta\beta\beta\beta} \beta^3 \tilde{\beta}^2 + \cdots,$$

where $W^0 = W|_{\beta=\beta_0}$, $\tilde{W}_\beta^0 = \frac{\partial W}{\partial \beta}|_{\beta=\beta_0}$, $\tilde{W}_\beta^0 = \frac{\partial^2 W}{\partial \beta^2}|_{\beta=\beta_0}$, and so on.

The hodograph equation (15) and conditions (25) imply that

$$\frac{x_0}{2} + \frac{y_0}{4} (3\beta_0 + \tilde{\beta}_0) + \tilde{W}_\beta^0 = 0,$$

$$\frac{3}{4} y_0 + \tilde{W}_\beta^0 = 0.$$
At curvature $K \neq 0$ the Euler–Poisson–Darboux equations (14) and its differential consequences

$$2W_{\rho \rho} + 2(\beta - \bar{\beta})W_{\rho \bar{\rho}} = W_{\rho \beta} - W_{\bar{\rho} \bar{\beta}}$$

(35)

imply that

$$W_{\rho \rho}^0 = 0, \quad W_{\rho \bar{\rho}}^0 = W_{\rho \bar{\rho}}^0 = 0,$$

(36)
i.e.

$$\frac{y_0}{4} + \tilde{W}_{\rho \rho}^0 = 0, \quad \tilde{W}_{\rho \bar{\rho}}^0 = \tilde{W}_{\rho \bar{\rho}}^0 = 0.$$  

(37)

Taking into account equations (34) and (37), one finally obtains

$$W(x_0 + \epsilon^x x^*, y_0, \beta_0 + \epsilon^x \bar{\beta}^*) = W_0 + \frac{1}{2} \epsilon^x (\beta_0 + \bar{\beta}_0) x^* + \frac{1}{2} \epsilon^x \gamma (\beta^* + \bar{\beta}^*) x^* + \frac{1}{3} \epsilon^x \gamma (a \beta^* x^* + b \bar{\beta}^* x^*) + \cdots,$$

(38)

where $a = \frac{1}{2} \tilde{W}_{\rho \rho}^0$. Consequently, near the point $x_0, y_0$ the hodograph equations take the form

$$W_{\beta} = \frac{1}{2} \epsilon^x \gamma x^* + \epsilon^x \gamma a \beta^* x^* = 0.$$  

(39)

These equations readily imply that $\alpha = 2 \gamma$ and $\beta^* = x^*^{1/2}$. So, near the point $x_0$

$$\beta^* \sim (x - x_0)^{-1/2}.$$  

(40)

Thus, near the point of gradient catastrophe the function $W$ is of the form

$$W = W_0 + \frac{1}{2} \epsilon^x \gamma (\beta_0 + \bar{\beta}_0) x^* + \epsilon^x \gamma W^*,$$

(41)

where

$$W^* = \frac{1}{2} x^*(\beta^* + \bar{\beta}^*) + \frac{1}{3} (a \beta^* x^* + b \bar{\beta}^* x^*)$$

(42)

and $\beta = -\tau + iK$. Denoting $a^{1/3} \beta^* = U + iV$, one obtains

$$W^* = x^*(fU + gV) + \frac{1}{3} U^3 - UV^2,$$

(43)

where $f$ and $g$ are certain constants. In Thom’s catastrophe theory a function of this form is known as describing elliptic umbilic singularity (e.g. [40, 53]). So the gradient catastrophe and flutter for vortex filament motion enter into the general scheme of catastrophe theory. For the focusing NLS equation the appearance of elliptic umbilic singularity has been observed for the first time in [36].

6. Double scaling regularization and Painlevé-I equation

The next step is to regularize this singularity. Within the double scaling limit method adopted also in [36] one begins with the full dispersive system. Performing the appropriate double scaling limit one calculates the required regularizing terms. An approach discussed in [50] suggests to proceed in opposite direction, namely to begin with the original (say dispersionless system), to modify the corresponding function $W$ adding to it the differential terms of lowest order with appropriate scaling, and then to require that the critical point equations (12) for $W$ are substituted by the Euler Lagrange equations for the modified $W_q$, i.e.

$$\frac{\delta W_q}{\delta \beta} = 0, \quad \frac{\delta W_q}{\delta \bar{\beta}} = 0.$$  

(44)

Following this approach we first observe that the contributions of $\beta^*$ and $\bar{\beta}^*$ into the perturbation $W^*$ of $W$ are separated. It is quite natural to modify the function (33) in such a way that this separation and reality property $W(\beta, \bar{\beta}) = W(\beta, \bar{\beta})$ are preserved. Thus, omitting
the inessential second term in (41), and choosing without loss of generality \( \gamma = 1 \), one has the following modified \( W \):

\[
W_q = W_0 + \epsilon^3 \left[ \frac{1}{2} x^* (\beta^* + \overline{\beta}^*) + \frac{1}{2} (a \beta^* + \frac{1}{2} \overline{a} \overline{\beta}^3) + \frac{1}{2} \epsilon \left[ b \beta^* \overline{\beta}^2 + \frac{1}{2} \overline{b} \overline{\beta}^2 \right] \right].
\]

(45)

where \( \delta \) and \( b \) are appropriate constants. The corresponding Euler–Lagrange equation for \( \beta^* \) and \( \overline{\beta}^* \) is

\[
\frac{\partial W_q}{\partial \beta^*} - \left( \frac{\partial W_q}{\partial \overline{\beta}^*} \right) x^* = 0,
\]

(46)

i.e.

\[
\epsilon^3 \left[ \frac{1}{2} x^* + a \beta^* \right] - \epsilon^3 [b \beta^* x^*] = 0.
\]

(47)

To balance the first term one should choose \( \delta = 3 \).

Thus, the modified \( W_q \) is of the form

\[
W_q = W_0 + \epsilon^3 \left[ \frac{1}{2} x^* (\beta^* + \overline{\beta}^*) + \frac{1}{2} (a \beta^* + \frac{1}{2} \overline{a} \overline{\beta}^3) + \frac{1}{2} b \beta^* \overline{\beta}^2 + \frac{1}{2} \overline{b} \overline{\beta}^2 \right] + \frac{1}{2} \epsilon \left[ b \beta^* \overline{\beta}^2 + \frac{1}{2} \overline{b} \overline{\beta}^2 \right],
\]

(48)

and the corresponding equation for \( \beta^* \) is

\[
b \beta^* x^* = a \beta^* + \frac{1}{2} x^*.
\]

(49)

This equation is converted into the classical Painlevé-I equation (7) by simple change of variables

\[
x^* = \lambda \xi, \quad \beta^* = -\frac{\lambda^3}{2b} \Omega
\]

(50)

with \( a \lambda^5 = -12b^2 \).

So, the regularized behavior at the point of the gradient catastrophone of the DR system, i.e. for the vortex filament dynamics, is governed by the Painlevé-I equation. The relevance of this equation for the NLS/Toda system near the critical point has been observed in a different way in [36]. The correspondence between the solution of equation (49) and those given by the formula (5.21) in [36] is

\[
\beta(s, t_0, \xi) \approx \beta_0 - \epsilon \left( -\frac{36}{a^2 b^2} \right)^{1/5} \Omega_0 \left( \left( -\frac{a}{12b^2} \right)^{1/5} \frac{x - s_0}{\epsilon^3} \right),
\]

(52)

where \( \Omega_0 \) is the tritronquée solution of the Painlevé-I equation. Extension of this observation to our case allows us to formulate

**Conjecture 6.1.** Generic behavior of the vortex filament near the point \( x_0, y_0 \) of gradient catastrophone is given by

\[
\beta(s, t_0, \xi) \approx \beta_0 - \epsilon \left( -\frac{36}{a^2 b^2} \right)^{1/5} \Omega_0 \left( \left( -\frac{a}{12b^2} \right)^{1/5} \frac{x - s_0}{\epsilon^3} \right),
\]

where \( \Omega_0 \) is the tritronquée solution of the Painlevé-I equation (7).

It was shown in [38] that the amplitude of the solution of the NLS equation near the catastrophone point \( (x_0, y_0) \) is three times more than the background amplitude in correspondence with the poles of the tritronquée solution. The geometrical meaning of the factor ‘3’ in terms of the curvature of the filament as well as other features of flutter for vortex filaments will be discussed elsewhere.
Acknowledgments

This work has been partially supported by PRIN grant no 28002K9KXZ and by FAR 2009 (Sistemi dinamici Integrabili e Interazioni fra campi e particelle) of the University of Milano Bicocca. We also thank the referee for his interesting remarks.

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