Carleman estimates for a stochastic degenerate parabolic equation and applications to null controllability and an inverse random source problem

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Abstract
In this paper, we establish two Carleman estimates for a stochastic degenerate parabolic equation. The first one is for the backward stochastic degenerate parabolic equation with singular weight function. Combining this Carleman estimate and an approximate argument, we prove the null controllability of the forward stochastic degenerate parabolic equation with the gradient term. The second one is for the forward stochastic degenerate parabolic equation with regular weighted function, based on which we obtain the Lipschitz stability for an inverse problem of determining a random source depending only on time in the forward stochastic degenerate parabolic equation.

Keywords: stochastic degenerate parabolic equation, Carleman estimate, null controllability, inverse random source problem

1. Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\) is defined such that \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration generated by \(B(\cdot)\), augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Let \(I = (0, 1)\), \(Q_T = I \times (0, T)\). Then, we
consider the following forward stochastic degenerate parabolic equation:

\[
\begin{aligned}
&\frac{dy}{dt} - (x^\alpha y_x)_x = f dt + F dB(t), \\
&y(1,t) = 0, \\
&y(0,t) = 0 \quad \text{for } \alpha \in (0,1), \\
&y(x,0) = y_0(x), \\
&y(x,t) = y_T(x), \\
\end{aligned}
\]

and the following backward stochastic degenerate parabolic equation:

\[
\begin{aligned}
&\frac{dy}{dt} + (x^\alpha y_x)_x = f dt + F dB(t), \\
&y(1,t) = 0, \\
&y(0,t) = 0 \quad \text{for } \alpha \in (0,1), \\
&y(x,T) = y_T(x), \\
\end{aligned}
\]

Physically, \( f, F \) are source terms, \( F \) stands for the intensity of a random force of the white noise type. In next section we will give the well-posedness of (1.1) under suitable \( f \) and \( F \). Obviously, the equation is degenerate at the left-end point \( x = 0 \).

Throughout this paper, we denote by \( L^2_\mathcal{F}(0,T) \) the space of all progressively measurable stochastic process \( X \) such that \( \mathbb{E} \left( \int_0^T |X|^2 dt \right) < \infty \). For a Banach space \( H \), we denote by \( L^2_\mathcal{F}(0,T;H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t\geq0} \)-adapted processes \( X(\cdot) \) such that \( \mathbb{E}(\|X(\cdot)\|_{L^2(0,T;H)}) < \infty \), with the canonical norm; by \( L^2_\mathcal{F}(0,T;H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t\geq0} \)-adapted bounded processes; and by \( L^2_\mathcal{F}(\Omega;C([0,T];H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t\geq0} \)-adapted continuous processes \( X(\cdot) \) such that \( \mathbb{E}(\|X(\cdot)\|_{C([0,T];H)}) < \infty \), with the canonical norm.

The main objective of this paper is to obtain Carleman estimates for backward/forward stochastic degenerate equations. As applications, we then apply these Carleman estimates to study a null controllability problem and an inverse random source problem. More precisely, for given subdomain \( \omega = (x_1, x_2) \) such that \( 0 < x_1 < x_2 < 1 \), we consider the following two problems:

**Null controllability.** Find a pair control \( (g, G) \in L^2_\mathcal{F}(0,T;L^2(I)) \times L^2_\mathcal{F}(0,T;L^2(I)) \) such that the solution \( y \) of the following stochastic degenerate parabolic with the gradient term:

\[
\begin{aligned}
&\frac{dy}{dt} - (x^\alpha y_x)_x = (ay_x + by + g I_\omega) dt + (cy + G) dB(t), \\
&y(0,t) = y(1,t) = 0, \\
&y(x,0) = y_0(x), \\
&y(x,T) = y_T(x), \\
\end{aligned}
\]

satisfies

\[ y(x,T) = 0, \quad x \in I, \quad \mathbb{P} - \text{a.s.}, \]

where \( \alpha \in (0, \frac{1}{2}) \), \( a, b, c \in L^2_\mathcal{F}(0,T;L^\infty(I)) \) and \( I_\omega \) is the characteristic function of the set \( \omega \).
Remark 1.1. For deterministic case, in [1] the authors pointed out that the restriction \( \alpha \in \left(0, \frac{1}{2}\right) \) was optimal for establishing the Carleman estimate under \( a \in L^\infty(Q_T) \) according to the methods as [2, 3]. A further explanation about this restriction was given in [4]. In other words, \( \alpha \in \left(0, \frac{1}{2}\right) \) is essentially caused by the tool used to prove the null controllability, i.e. Carleman estimates, and is also the best result based on the method used in this paper.

Inverse random source problem. Determine \( h(t) \in L^2_T(0, T) \) in the following stochastic degenerate parabolic equation:

\[
\begin{aligned}
\frac{dy}{dt} - (x^\alpha y_x)_x = h(t)r(x, t)dB(t), \quad (x, t) \in Q_T, \\
y(1, t) = 0, \quad t \in (0, T), \\
y(0, t) = 0 & \quad \text{for } \alpha \in (0, 1), \\
(x^\alpha y_x)(0, t) = 0 & \quad \text{for } \alpha \in [1, 2), \\
y(x, 0) = y_0(x), & \quad x \in I,
\end{aligned}
\]

(1.4)

by the observation data

\[ y|_{\omega_T} \quad \text{and} \quad y(x, T), \]

where \( r \in L^2(0, T; W^{1, \infty}(I)) \) and \( \omega_T = \omega \times (0, T) \).

Carleman estimate is an important tool to study null controllability and inverse problems, which is a weighted estimate for a solution to a partial differential equation. There are rich references on Carleman estimates for deterministic partial differential equations, see [5–13]. Recently, Carleman estimates for stochastic partial differential equations are getting more and more attention. We refer to [14, 15] for stochastic parabolic equation, [16] for stochastic hyperbolic equation, [17] for stochastic Korteweg–de Vries equation, [18] for stochastic complex Ginzburg–Landau equations and so on. To the best of our knowledge, there is only one paper about Carleman estimate for stochastic degenerate equation [19], in which the global Carleman estimates for some forward and backward stochastic degenerate parabolic equations were established and then were applied to an insensitizing control problem.

One successful application of Carleman estimate in stochastic partial differential equations is to study related control problems for various mathematical models with stochastic effect [15, 20–23]. As for null controllability for the deterministic degenerate equations, we refer to [2, 3] for degenerate parabolic equation, [1, 4] for degenerate parabolic equation with the gradient terms, [24–27] for coupled degenerate systems and so on. On the other hand, there are few works on inverse problems for stochastic partial differential equations. We refer to [28] the uniqueness of an inverse source problem for the stochastic parabolic equation. An inverse source problem of determining two kinds of sources simultaneously for a stochastic wave equation was studied in [29]. Global uniqueness of an inverse problem of simultaneously determining random source and initial data for the stochastic hyperbolic equation in [30]. This method then was extended to stochastic Euler–Bernoulli beam equation [31]. As for applications of regularization techniques in the numerical methods for inverse random source problems, we refer to [32] or [33].

In this paper, we first focus on Carleman estimates for stochastic degenerate parabolic equations. More precisely, we will prove two Carleman estimates for backward/forward stochastic degenerate parabolic equation, respectively with singular/regular weight functions. We apply the first Carleman estimate with singular weight function to study the null controllability for stochastic degenerate parabolic equation with the gradient term (1.3), in whose proof we only assume that the coefficient of the first order term \( a \in L^\infty_T(0, T; L^\infty(I)) \). Since the
equation is degenerate, we could not apply directly the Carleman estimate to absorb the first order term, if $a \in L^\infty(0, T; L^\infty(I))$. To overcome this difficulty, we have to improve this Carleman estimate by using the method in [4, 34] for deterministic differential equations, also see [35] for stochastic differential equations. For this reason, we only obtain the null controllability result for $\alpha \in (0, \frac{1}{2})$. On the other hand, unlike the deterministic counterparts, the solution of a stochastic differential equation is not differentiable with respect to time variable, Carleman estimate with singular weight function could not be applied to inverse random source problem. Hence we would like to borrow some ideas from [30] to prove the second Carleman estimate with regular weight function. Applying this Carleman estimate, we obtain a Lipschitz stability for our inverse random source problem. In comparison with [19], on one hand we release the power of $x$ on the left-hand side of Carleman estimate, which leads to that we can deal with the null controllability of stochastic degenerate equation with the first order term, see theorem 4.2. On the other hand, since the weight function in [19] is singular in the Carleman estimate, which could not be applied to study our inverse problem.

The remainder of this paper is organized as follows. In next section, we prove the well-posedness of forward/backward stochastic degenerate parabolic equation with the first order term. In section 3, we show two Carleman estimates for backward/forward stochastic degenerate parabolic equations. In next two sections, based on these two Carleman estimates we study the null controllability and the inverse random source problem, respectively.

2. Well-posedness

In this section, we use an approximate argument to prove the well-posedness of the following stochastic degenerate parabolic equation:

$$\begin{cases}
\frac{dy}{dt} - (x^\alpha y_x)_x = f dt + F dB(t), & (x, t) \in \Omega_T, \\
y(1, t) = 0, & t \in (0, T), \\
y(0, t) = 0 \quad \text{for} \quad \alpha \in (0, 1), & t \in (0, T), \\
(x^\alpha y_x)(0, t) = 0 \quad \text{for} \quad \alpha \in [1, 2), & t \in (0, T), \\
y(x, 0) = y_0(x), & x \in I, 
\end{cases}
$$

(2.1)

To deal with degeneracy at $x = 0$, we have to introduce following weighted space:

$$H^1_{\alpha, 0}(I) := \left\{ \zeta \in L^2(I) \mid x^\alpha \zeta x \in L^2(I), \zeta(0) = \zeta(1) = 0 \right\}$$

$$H^1_{\alpha, 1}(I) := \left\{ \zeta \in L^2(I) \mid x^\alpha \zeta x \in L^2(I), \zeta(1) = 0 \right\},$$

and

$$H^1_{\alpha}(I) = \begin{cases}
H^1_{\alpha, 0}(I), & \alpha \in (0, 1), \\
H^1_{\alpha, 1}(I), & \alpha \in [1, 2). 
\end{cases}$$

We endow the space $H^1_{\alpha}(I)$ with the norm

$$\| \zeta \|^2_{H^1_{\alpha}} = \int_I (|\zeta|^2 + x^\alpha |\zeta x|^2) \, dx.$$
Further, we set
\[ H^1 = L^2_x(\Omega; C([0, T]; L^2(I))) \cap L^2_x(0, T; H^1(I)) \]
\[ H^1_\alpha = L^2_x(\Omega; C([0, T]; L^2(I))) \cap L^2_x(0, T; H^1_\alpha(I)). \]

**Definition.** A stochastic process \( y \) is said to be a weak solution of the forward stochastic degenerate parabolic equation (2.1) if \( y \in H^1_\alpha \) and \( y(0) = y_0 \) in \( I \), \( P \) - a.s. and it holds for all \( \phi \in C^0(\overline{I}) \) that
\[
\begin{align*}
\int_I y(x, t) \phi(x) dx - \int_I y_0(x) \phi(x) dx + \int_0^T x^n y_t \phi dx dt \\
&= \int_Q f \phi dx dt + \int_Q F \phi dx dB(t), \quad P - \text{a.s.}
\end{align*}
\]
where \( Q_t = I \times (0, t) \).

**Theorem 2.1.** Let \( f, F \in L^2(0, T; L^2(I)) \) and \( y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)) \). Then (2.1) admits a unique weak solution \( y \in H^1_\alpha \).

**Proof.** Letting \( \varepsilon \in (0, 1) \), we consider the following nondegenerate approximate problem:
\[
\begin{align*}
\begin{cases}
\frac{dy^\varepsilon}{dt} - \left( (x + \varepsilon) y^\varepsilon \right) x &= f dt + F dB(t), \\
y^\varepsilon(1, t) &= 0, \\
y^\varepsilon(0, t) &= 0 & &\text{for } \alpha \in (0, 1), \\
y^\varepsilon(x, 0) &= y_0^\varepsilon(x) & &\text{for } \alpha \in [1, 2), \\
y^\varepsilon(x, 0) &= y_0(x) & &\text{for } x \in I,
\end{cases}
\end{align*}
\]
where \( y^\varepsilon_0 \rightarrow y_0 \) in \( L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)) \).

Then by [36] or [37], it is easy to check that (2.2) admits a unique weak solution \( y^\varepsilon \in H^1_\alpha \).

Now we prove a uniform estimate in \( \varepsilon \) for \( y^\varepsilon \):
\[
E \sup_{t \in [0, T]} \int_I |y^\varepsilon|^2(x, t) dx + E \int_Q \left( (x + \varepsilon)^\alpha |y^\varepsilon|^2 \right) dx dt
\leq C E \int_Q |y_0^\varepsilon|^2 dx + C E \int_Q (|f|^2 + |F|^2) dx dt.
\]
where \( C \) is depending on \( I, T \) and \( \alpha \), but independent of \( \varepsilon \). By Itô formula and the equation of \( y^\varepsilon \), we obtain
\[
\begin{align*}
\int_I |y^\varepsilon|^2(x, t) dx + 2 \int_Q (x + \varepsilon)^\alpha |y^\varepsilon|^2 dx dt \\
&= \int_I |y_0^\varepsilon|^2 dx + 2 \int_Q F y^\varepsilon dx dt + \int_Q |y^\varepsilon|^2 dx dt + 2 \int_Q F y^\varepsilon dB(t) \\
&\leq \int_I |y_0^\varepsilon|^2 dx + \int_Q (|f|^2 + |F|^2) dx dt + \int_Q |y^\varepsilon|^2 dx dt + 2 \int_Q F y^\varepsilon dB(t).
\end{align*}
\]
Then taking mathematical expectation and applying Grönwall’s inequality yields that
\[
\sup_{r \in [0,T]} E \int y^2(x, t) dx \leq CE \int |y_0|^2 dx + CE \int_{Q_T} \left( |f|^2 + |F|^2 \right) dx dt.
\] (2.6)

Moreover, it follows from the Burkholder–Davis–Gundy inequality and (2.5) that
\[
E \sup_{r \in [0,T]} \int [y^1 + y^2]^2(x, t) dx + E \int_{Q_T} (x + \varepsilon)^\alpha |y^1|^2 dx dt
\leq E \int |y_0|^2 dx + CE \int_{Q_T} |y^1|^2 dx dt + CE \int_{Q_T} (|f|^2 + |F|^2) dx dt.
\] (2.7)

Substituting (2.6) into (2.7), we obtain (2.4).

Similarly, we have for any \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) that
\[
E \sup_{r \in [0,T]} \int [y^1 - y^2]^2(x, t) dx + E \int_{Q_T} (x + \varepsilon)^\alpha |y^1 - y^2|^2 dx dt
\leq CE \int |y_0|^2 - |y_0|^2 dx.
\]

Therefore, \( \{y^\varepsilon\} \) is a Cauchy sequence in \( H^1_{\alpha_0} \). Letting \( \varepsilon \to 0 \), we find that (2.1) admits a weak solution \( y \in H_{\alpha_0} \) (the limit of \( y^\varepsilon \) in \( H_{\alpha_0} \)). The uniqueness of solution could be directly deduced from (2.4).

Next we consider the stochastic degenerate parabolic equation with gradient term:
\[
\begin{cases}
\frac{dy}{dt} - (ax^a)_x dt = (ay + by + f) dt + (cy + F) dB(t), & (x, t) \in Q_T, \\
y(0, t) = y(1, t) = 0, & t \in (0, T), \\
y(x, 0) = y_0(x), & x \in I.
\end{cases}
\] (2.8)

In comparison with (2.1), the main difficulty is how to deal with the gradient term under \( a \in L^\infty_T(0, T; L^\infty(I)) \). Due to degeneracy, we could not control this term directly. However we could apply the same method used in [1] to overcome this difficulty and then obtain the well-posedness of (2.8) for \( \alpha \in (0, 1) \).

**Corollary 2.1.** Let \( \alpha \in (0, 1), a, b, c \in L^\infty_T(0, T; L^\infty(I)), f, F \in L^2_T(0, T; L^2(I)) \) and \( y_0 \in L^2(I), F_0, P; L^2(I) \). Then (2.8) admits a unique weak solution \( y \in H^1_{\alpha_0} \).

**Remark 2.1.** If \( a \) has the decomposition \( a = x^2 \tilde{a} \) with some \( \tilde{a} \in L^\infty_T(0, T; L^\infty(I)) \) as in [26], the term of \( ay \) could be absorbed directly by the diffusion term. Then we can obtain the well-posedness of (2.8) for all \( \alpha \in (0, 2) \). Or if \( a \in L^2_T(0, T; W^{1,\infty}(I)) \), we also obtain the well-posedness for \( \alpha \in (0, 2) \).

**3. Carleman estimates for stochastic degenerate parabolic equation**

In this section, we will show two Carleman estimates for stochastic degenerate parabolic equations. One is for the backward stochastic degenerate parabolic equation. We will apply this Carleman estimate to prove the null controllability result for the forward stochastic degenerate parabolic equation with the gradient term. So that we use a singular weight function in this Carleman estimate. The other one is for the forward stochastic degenerate parabolic equation,
which will be used to study our inverse random source problem. Unlike the deterministic case, we could not differentiate the stochastic equation with respect to time. For this reason, in order to prove the Lipschitz stability of our inverse problem, we have to introduce a regular weight function to put the term of unknown random source on the left-hand side of Carleman estimate.

### 3.1. Carleman estimate for backward stochastic degenerate equation

We first introduce some weight functions. For \( \omega = (x_1, x_2) \), we choose \( \omega^{(i)} := (x_1^{(i)}, x_2^{(i)})(i = 1, 2) \) such that \( \omega^{(2)} \equiv \omega^{(1)} \equiv \omega \). Let \( \eta \in C^2(\mathcal{I}) \) be a cut-off function such that \( 0 \leq \eta(x) \leq 1 \) for \( x \in I, \chi(x) \equiv 1 \) for \( x \in (0, x_1^{(2)}) \) and \( \chi(x) \equiv 0 \) for \( x \in (x_2^{(2)}, 1) \). For a suitable positive constant \( \beta \), we introduce

\[
\eta_1(x) = (x + \epsilon) - \epsilon \beta, \quad x \in I,
\]

and \( \eta_2 \in C^2(\mathcal{I}) \) such that

\[
\eta_2(x) > 0, \quad x \in I, \quad \eta_2(0) = \eta_2(1) = 0 \quad \text{and} \quad |\eta_{2,x}(x)| > 0, \quad x \in I^c_\omega^{(1)}
\]

and

\[
\eta_1(x) = \eta_2(x), \quad x \in (x_1^{(2)}, x_2^{(2)}).
\]

Let us define

\[
\xi_1(t) = \frac{1}{e^t}, \quad \psi_1(x) = e^{\lambda_1(x)} - e^{2\lambda M}, \quad \phi_1(x) = e^{\lambda_1(x)}, \quad i = 1, 2.
\]

where \( \lambda \) is a positive parameter and \( M \) is a sufficiently large constant such that

\[
M \geq \max \left\{ \|\eta_1\|_{C^2(\mathcal{I})}, \|\eta_2\|_{C^2(\mathcal{I})} \right\}.
\]

Now we introduce weight function in the first Carleman estimate

\[
\varphi(x, t) = \chi(x)\psi_1(x, t) + (1 - \chi(x))\psi_2(x, t), \quad (x, t) \in QT
\]

with

\[
\varphi_i(x, t) = \psi_i(x)\xi_1(t), \quad i = 1, 2.
\]

We easily see that

\[
\varphi(x, t) = \begin{cases}
\varphi_1(x, t), & (x, t) \in (0, x_1^{(2)}) \times (0, T), \\
\varphi_1(x, t) = \varphi_2(x, t), & (x, t) \in (x_1^{(2)}, x_2^{(2)}) \times (0, T), \\
\varphi_2(x, t), & (x, t) \in (x_2^{(2)}, 1) \times (0, T).
\end{cases}
\]

In order to deal with degeneracy, we first prove the following uniform Carleman estimate in \( \varepsilon \).

**Theorem 3.1.** Let \( \alpha \in (0, 2), \ f_1 \in L^2_x(0, T; L^2(I)), \ F_1 \in L^2_x(0, T; H^2(I)), \ u_0 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I)) \) and \( \beta, \beta_0 \) such that

\[
\begin{cases}
1 < \beta \leq 2 - \alpha, & \alpha \in (0, 1), \\
\beta = 1, & \alpha = 1, \\
\beta_0 < \beta \leq 2 - \alpha, & \alpha \in (1, 2),
\end{cases}
\]

\[ (3.2) \]
with
\[ \beta_0 = \max \left\{ 0, 3 - 2\alpha, \frac{14 - 9\alpha + \sqrt{17\alpha^2 - 44\alpha + 36}}{2} \right\}. \]

Then for any \( \varepsilon \in (0, 1) \), there exist positive constants \( \lambda_1 = \lambda_1(\omega, I, T, \alpha, M) \), \( s_1 = s_1(\omega, I, T, \alpha, M, \lambda) \) and \( C_1 = C_1(\omega, I, T, \alpha, M), C_2 = C_2(\omega, I, T, \alpha, M, \lambda) \) such that
\[
\begin{align*}
&\mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 (x + \varepsilon)^{2\alpha + 3 - 4} |u|^2 \varepsilon^{2\alpha} \, dx \, dt + \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 (x + \varepsilon)^{2\alpha + 3 - 2} |\dot{u}|^2 \varepsilon^{2\alpha} \, dx \, dt \\
&\leq C_1 \mathbb{E} \int_{\mathcal{Q}_T} |f|^2 \varepsilon^{2\alpha} \, dx \, dt + C_2(\lambda) \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 |\dot{f}|^2 \varepsilon^{2\alpha} \, dx \, dt \\
&+ C_2(\lambda) \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 |u|^2 \varepsilon^{2\alpha} \, dx \, dt
\end{align*}
\]
(3.3)

for all \( \lambda \geq \lambda_1 \), \( s \geq s_1 \) and all \( u \in \mathcal{H}_1 \) satisfying
\[
\begin{align*}
&\left\{ \begin{array}{l}
\dot{u} + \left( (x + \varepsilon)u \right)_{t} = f_{1} \, dt + f_{1} \, dB(t), \\
u'(1, t) = 0, \\
\text{and} \\
\left\{ \begin{array}{l}
\dot{u}(0, t) = 0 \quad \text{for} \quad \alpha \in (0, 1), \\
(1 + \varepsilon)^{\alpha} u(0, t) = 0 \quad \text{for} \quad \alpha \in [1, 2), \\
u'(x, T) = u_T(x),
\end{array} \right.
\end{array} \right. \\
&\quad \text{for} \quad (x, t) \in \mathcal{Q}_T,
\end{align*}
\]
(3.4)

\begin{remark}
Given any \( \varepsilon \in (0, 1) \), the equation (3.4) is not degenerate. Therefore, the regularity \( u \in \mathcal{H}_1 \) we assumed in theorem 3.1 is reasonable.
\end{remark}

\begin{remark}
For \( \alpha \in (1, 2), \beta \in (\beta_0, 2 - \alpha) \) is nonempty.

Letting \( \varepsilon \to 0 \) in theorem 3.1, we obtain the following Carleman estimate:
\end{remark}

\begin{theorem}
Let \( \alpha \in (0, 2), f_{1} \in L_{T}^{2}(0, T; L_{2}(I)), F_{1} \in L_{T}^{2}(0, T; L_{2}^{*}(I)), u_{T} \in L_{T}^{2}(\Omega, F_{T}, P; L_{2}^{*}(I)). \) Then for any \( \varepsilon \in (0, 1) \), there exist positive constants \( \lambda_1 = \lambda_1(\omega, I, T, \alpha, M) \), \( s_1 = s_1(\omega, I, T, \alpha, M, \lambda) \) and \( C_1 = C_1(\omega, I, T, \alpha, M), C_2 = C_2(\omega, I, T, \alpha, M, \lambda) \) such that
\[
\begin{align*}
&\mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 x^{2 - \alpha} |u|^2 \varepsilon^{2\alpha} \, dx \, dt + \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 x^{2 - \alpha} |\dot{u}|^2 \varepsilon^{2\alpha} \, dx \, dt \\
&\leq C_1 \mathbb{E} \int_{\mathcal{Q}_T} |f|^2 \varepsilon^{2\alpha} \, dx \, dt + C_2(\lambda) \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 |\dot{f}|^2 \varepsilon^{2\alpha} \, dx \, dt \\
&+ C_2(\lambda) \mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 |u|^2 \varepsilon^{2\alpha} \, dx \, dt
\end{align*}
\]
(3.5)
for all \( \lambda \geq \lambda_1, s \geq s_1 \) and all \( u \in \mathcal{H}_0^1 \) satisfying
\[
\begin{aligned}
dt + (x^\alpha u_x) dt &= f dt + F_1 dB(t), \\
(x, t) \in Q_T,
\end{aligned}
\]
\[
\begin{aligned}
u(1, t) &= 0, \\
& t \in (0, T),
\end{aligned}
\]
and
\[
\begin{aligned}
u(0, t) &= 0 \\
& \text{for } \alpha \in (0, 1), \\
(x^\alpha u_x)(0, t) &= 0 \\
& \text{for } \alpha \in [1, 2),
\end{aligned}
\]
\[
\begin{aligned}
u(x, T) &= u_T(x), \\
& x \in I.
\end{aligned}
\]

**Proof.** Letting \( u_T \rightarrow u_T \) in \( L^2(\Omega, F_T, \mathbb{P}; L^2(I)) \), we easily see that \( u^\varepsilon \rightarrow u \) in \( \mathcal{H}_0^1 \). Then for a.e. \( t \in (0, T) \), we have \( u^\varepsilon(t, t) \in \mathcal{H}_0^1 \), which implies that \( xu^\varepsilon \in W^{1, 1}(I) \) and \( xu^\varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) by lemma 3.5 in [3]. Then we set \( \beta = 2 - \alpha \) and choose \( \varepsilon \rightarrow 0 \) in (3.3) to obtain (3.5).

To prove theorem 3.1, we need the following two lemmas. One is Hardy–Poincaré inequality [38]. The detailed proof could be found in [39] or [2]. The other one is the Cacciopoli inequality for the stochastic parabolic equation, whose proof is detailed in appendix A and omitted here.

**Lemma 3.1.** Let \( \gamma \in [0, 1) \cup (1, 2] \) and \( z \in \mathcal{H}_0^1(I) \). Then for any \( \varepsilon \in (0, 1) \), we have
\[
\int_I (x + \varepsilon)^{-\gamma}|z|^2 dx \leq \frac{4}{(\gamma - 1)^2} \int_I (x + \varepsilon)^{2-\gamma}|z|^2 dx.
\]

**Lemma 3.2.** Let \( f_1 \in L^2(0, T; L^2(I)) \) and \( F_1 \in L^2(0, T; L^2(I)) \). Then there exist positive constants \( C_3 = C_3(\omega, I, T, \alpha, M) \) and \( C_4 = C_4(\omega, I, T, \alpha, M, \lambda) \) such that the solution \( u \in \mathcal{H} \) of the backward stochastic degenerate parabolic equation (3.4) satisfies
\[
\begin{aligned}
\mathbb{E} &\int_{\omega T} |u_t|^2 e^{2\varepsilon\gamma} dxdt \\
& \leq C_3(\lambda) \mathbb{E} \int_{\omega T} s^{2\gamma} |u|^2 e^{2\varepsilon\gamma} dxdt + C_4 \mathbb{E} \int_{\omega T} s^{-2} |f_1|^2 e^{2\varepsilon\gamma} dxdt \\
& \quad + C_4 \mathbb{E} \int_{\omega T} |F_1|^2 e^{2\varepsilon\gamma} dxdt.
\end{aligned}
\]

Now we prove theorem 3.1. To do this, we first give a weighted identity for backward stochastic degenerate parabolic operator. Based on this weighted identity, we establish two Carleman estimates for degenerate part and nondegenerate part, respectively. Finally we add up these two Carleman estimates to obtain (3.3).

**Proof of theorem 3.1.** We split the proof into the following four steps.

1. A weighted identity for backward stochastic degenerate parabolic operator.

Let \( l_1 = s \varphi_1, \theta_1 = e^{\Theta_1} \) and \( U = \theta_1 u^\varepsilon \). Then we have
\[
\theta_1 \left[ du^\varepsilon + ((x + \varepsilon)^\gamma u_x) dt \right] = I_1 + I_2 dt
\]
with
\[
\begin{aligned}
U(1, t) &= 0, \\
& t \in (0, T),
\end{aligned}
\]
\[
\begin{aligned}
U(0, t) &= 0, \\
& \text{for } \alpha \in (0, 1), \\
(t(x + \varepsilon)^\gamma U_x)(0, t) &= ((x + \varepsilon)^\gamma l_{1, x} U)(0, t) \\
& \text{for } \alpha \in [1, 2),
\end{aligned}
\]
\[
\begin{aligned}
U(x, 0) &= U(x, T) = 0, \\
& x \in I.
\end{aligned}
\]

(3.9)
where

\[
I_1 = dU - 2(x + \varepsilon)^\alpha l_{1,x} U_x dt - \left((x + \varepsilon)^\alpha l_{1,x}\right)_x U dt,
\]

\[
I_2 = ((x + \varepsilon)^\alpha U, x) + (x + \varepsilon)^\alpha l_{1,x} U - I_{1,x} U.
\]

Hence,

\[
\theta_{1} I_2 \left[ dt^x + ((x + \varepsilon)^\alpha u_x^x) dt \right] = I_1 I_2 + |I_2|^2 dt. \tag{3.10}
\]

Now we deal with the term \(I_1 I_2\). By applying Itô formula, we obtain

\[
I_2 dU = \left[(x + \varepsilon)^\alpha U^x dU_x\right] - \frac{1}{2} d \left[(x + \varepsilon)^\alpha U^2_x\right] + \frac{1}{2}(x + \varepsilon)^\alpha (dU^2_x)
\]

\[
+ \frac{1}{2} d \left[(x + \varepsilon)^\alpha l_{1,x} U^2\right] - (x + \varepsilon)^\alpha l_{1,x} U^2 dt - \frac{1}{2}(x + \varepsilon)^\alpha l_{1,x}^2 (dU)^2
\]

\[
- \frac{1}{2} d(l_{1,x} U^2) + \frac{1}{2} l_{1,x} U^2 dt + \frac{1}{2} l_{1,x}^2 (dU)^2. \tag{3.11}
\]

On the other hand, a direct calculation yields

\[
I_2 \left[-2(x + \varepsilon)^\alpha l_{1,x} U_x dt - \left((x + \varepsilon)^\alpha l_{1,x}\right)_x U dt\right]
\]

\[
= \left[(x + \varepsilon)^\alpha l_{1,x} U^2\right]_x + (x + \varepsilon)^\alpha l_{1,x} U^2 dt
\]

\[
+ \left(x + \varepsilon l_{1,x} U^2\right)_x dt + \left[2\alpha(x + \varepsilon)^{2\alpha - 1} l_{1,x}^2 + 3(x + \varepsilon)^{2\alpha l_{1,x}} l_{1,xx} U^2 dt
\]

\[
+ \left[(x + \varepsilon)^\alpha l_{1,x} l_{1,xx} U^2\right]_x dt - \left[(x + \varepsilon)^\alpha l_{1,x} l_{1,xx}\right] U^2 dt
\]

\[
- \left[(x + \varepsilon)^\alpha l_{1,x} U^2\right]_x dt + \left[(x + \varepsilon)^\alpha l_{1,x}\right]_x (x + \varepsilon)^\alpha U U_x dt
\]

\[
+ \left[(x + \varepsilon)^\alpha l_{1,x}\right]_x (x + \varepsilon)^\alpha U_x^2 dt - \left[(x + \varepsilon)^{2\alpha l_{1,x}} l_{1,xx} + \alpha(x + \varepsilon)^{2\alpha - 1} l_{1,x}\right] U^2 dt
\]

\[
+ \left[(x + \varepsilon)^\alpha l_{1,x}\right]_x l_{1,x} U^2 dt. \tag{3.12}
\]

Therefore, by (3.10)–(3.12), we obtain the following weighted identity

\[
\theta_{1} I_2 \left[ dt^x + ((x + \varepsilon)^\alpha u_x^x) dt \right] = |I_2|^2 dt + K_1 dt + (K_2)_x + dK_3 + K_4, \tag{3.13}
\]

where

\[
K_1 = \left[\alpha(x + \varepsilon)^{2\alpha - 1} l_{1,x}^2 + 2(x + \varepsilon)^{2\alpha l_{1,x} l_{1,xx}}\right] U^2
\]

\[
+ \left[2\alpha(x + \varepsilon)^{2\alpha l_{1,x}} + \alpha(x + \varepsilon)^{2\alpha - 1} l_{1,x}\right] l_{1,xx} U U_x
\]

\[
- (x + \varepsilon)^\alpha l_{1,x} l_{1,xx} U^2 + \frac{1}{2} l_{1,xx} U^2 - \left[(x + \varepsilon)^\alpha l_{1,x} l_{1,xx}\right] U^2
\]

\[
+ \left[(x + \varepsilon)^\alpha l_{1,x}\right] l_{1,x} U^2,
\]

\[
K_2 = (x + \varepsilon)^\alpha U dU - (x + \varepsilon)^{2\alpha l_{1,x}} U_x^2 dt - (x + \varepsilon)^{2\alpha l_{1,x}^2} U^2 dt
\]

\[
+ (x + \varepsilon)^\alpha l_{1,x} l_{1,xx} U^2 dt - \left[(x + \varepsilon)^\alpha l_{1,x}\right] (x + \varepsilon)^\alpha U U_x dt,
\]

\[
K_3 = - \frac{1}{2}(x + \varepsilon)^\alpha U^2 + \frac{1}{2} (x + \varepsilon)^{2\alpha l_{1,x}^2} U^2 - \frac{1}{2} l_{1,x}^2 U^2.
\]
\[ K_4 = \frac{1}{2} (x + \varepsilon)^\alpha (dU_\varepsilon)^2 - \frac{1}{2} (x + \varepsilon)^\beta l_{1,\varepsilon}^2 (dU)^2 + \frac{1}{2} l_{1,\varepsilon}^2 (dU)^2. \]

Step 2. Carleman estimate for degenerate part.

In this step, we will prove the Carleman estimate for degenerate part \( \left( 0, x_1^{(2)} \right) \times (0, T) \):

\[
\begin{align*}
\mathbb{E} \int_0^T \int_{x_1^{(2)}} \int_0^{x_2} & s^3 (x + \varepsilon)^{2\alpha + 3\beta - 4} |\mu|^2 e^{2s^2} \, dx \, ds \, dr + \mathbb{E} \int_0^T \int_0^{x_2} \int_0^{x_1^{(2)}} s(x + \varepsilon)^{2\alpha + \beta - 2} |\mu|^2 e^{2s^2} \, dx \, ds \, dr \\
& \leq C \mathbb{E} \int_0^T \int_0^{x_2} |\mu|^2 e^{2s^2} \, dx \, ds \, dr + C(\lambda) \mathbb{E} \int_0^T \int_0^{x_2} s^2 \xi^2 |\mu|^2 e^{2s^2} \, dx \, ds \, dr \\
& \quad + \mathbb{E} \int_{x_1^{(2)}} \int_0^{x_2} |\mu|^2 + |\alpha|^2 e^{2s^2} \, dx \, ds \, dr + C(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{\alpha + \beta - 1} \xi^3 |\mu|^2 e^{2s^2}]_{x=0} \, ds \, dr.
\end{align*}
\]  

(3.14)

By using

\[
\begin{align*}
l_{1,\varepsilon} &= \beta s \lambda (x + \varepsilon)^{\beta - 1} \phi_1 \xi, \\
l_{1,\varepsilon} &= \beta s \lambda (x + \varepsilon)^{\beta - 1} \phi_1 \xi, \\
l_{1,\varepsilon} &= s (\beta^2 \lambda^2 (x + \varepsilon)^{2\beta - 2} + \beta (\beta - 1) \lambda (x + \varepsilon)^{\beta - 2} ) \phi_1 \xi, \\
|l_{1,\varepsilon}| &= |s \phi_1 \xi| \leq C(\lambda) s \xi^2, \\
|l_{1,\varepsilon}| &= |s \phi_1 \xi| \leq C(\lambda) s \xi^2,
\end{align*}
\]  

(3.15)

we obtain

\[
\begin{align*}
K_1 dr & \geq (\alpha + 2\beta - 2) \beta^3 \lambda^3 (x + \varepsilon)^{2\alpha + 3\beta - 4} \phi_1^3 \xi^3 |U|^2 \, dr \\
& \quad + (\alpha + 2\beta - 2) \beta s \lambda (x + \varepsilon)^{2\alpha + \beta - 2} \phi_1 \xi |U_\varepsilon|^2 \, dr + X_1 dr + X_2 dr + X_3 dr,
\end{align*}
\]  

(3.16)

where

\[
\begin{align*}
X_1 &= -2x^2 (x + \varepsilon)^\alpha \varphi_{1}\varphi_{1}\xi |U|^2, \\
X_2 &= \frac{1}{2} s \varphi_{1,\varepsilon} |U|^2, \\
X_3 &= s(x + \varepsilon)^\alpha [(x + \varepsilon)^\beta \varphi_{1}]_{x} U U_\varepsilon.
\end{align*}
\]

Now we estimate \( X_1, X_2 \) and \( X_3 \). Obviously, by (3.15) we have

\[
\begin{align*}
X_1 & \geq -C \lambda^2 \xi^2 (x + \varepsilon)^{2\alpha + 2\beta - 2} \phi_1^2 \xi^2 |U|^2 \geq -C(\lambda) s^2 (x + \varepsilon)^{2\alpha + 3\beta - 4} \phi_1^3 \xi^3 |U|^2, \\
& \quad - C(\lambda) s^2 \xi^2 |U|^2,
\end{align*}
\]  

(3.17)

due to \( \beta \leq 2 - \alpha \). Obviously,

\[
X_2 \geq -C(\lambda) s \xi^2 |U|^2.
\]  

(3.18)

For \( X_3 \), we have
We then substitute (3.16)–(3.19) into (3.13), we find that

\begin{align*}
X_3 & \geq - \left[ C_{\alpha,\beta}^1 s\lambda (x + \epsilon)^{2\alpha + \beta - 3} + C \xi^2 (x + \epsilon)^{2\alpha + 3\beta - 3} \\
& + C \lambda^3 (x + \epsilon)^{2\alpha + 3\beta - 3} \phi_1 \xi \|U\| \|U_x\| \\
& \geq - C_{\alpha,\beta}^1 s\lambda (x + \epsilon)^{2\alpha + \beta - 3} \phi_1 \xi \|U\| \|U_x\| - C \lambda^3 (x + \epsilon)^{2\alpha + 2\beta - 3} \phi_1 \xi \|U\| \|U_x\| \\
& \geq - C_{\alpha,\beta}^1 s\lambda (x + \epsilon)^{2\alpha + \beta - 3} \phi_1 \xi \|U\| \|U_x\| - C \lambda^3 (x + \epsilon)^{2\alpha + 2\beta - 2} \phi_1 \xi \|U_x\|^2 \\
& \geq - C \lambda^3 (x + \epsilon)^{2\alpha + 3\beta - 4} \phi_1 \xi \|U\|^2.
\end{align*}

(3.19)

with

\[
C_{\alpha,\beta}^1 = \beta(\alpha + \beta - 1)(2 - \alpha - \beta) \geq 0.
\]

Then substituting (3.16)–(3.19) into (3.13), we find that

\begin{align*}
& \theta_1 I_2 \left[ du + ((x + \epsilon)^\nu u_x)_t \right] + C(\lambda) s \xi^2 \|U\|^2 \|U_t\| \\
& \geq |I_2|^2 \|\phi_1 \xi\| \|U_x\| + (\alpha + 2\beta - 2) \beta^3 \lambda^3 - C(\lambda) s^2 \xi^2 \|U_t\|^2 \\
& + [(\alpha + 2\beta - 2) \beta s\lambda - C s] (x + \epsilon)^{2\alpha + \beta - 2} \phi_1 \xi \|U_x\|^2 \\
& + dK_3 + K_4.
\end{align*}

(3.20)

Integrating both sides of (3.20) on $Q_T$ and taking mathematical expectation, we have

\begin{align*}
& \mathbb{E} \int_{Q_T} |I_2|^2 \, dx \, dt + \mathbb{E} \int_{Q_T} [(\alpha + 2\beta - 2) \beta^3 \lambda^3 - C(\lambda) s^2 \|U_t\|^2 \\
& + \mathbb{E} \int_{Q_T} [(\alpha + 2\beta - 2) \beta s\lambda - C s] (x + \epsilon)^{2\alpha + \beta - 2} \phi_1 \xi \|U_x\|^2 \\
& \leq Y_1 + Y_2 + Y_3 = \mathbb{E} \int_{Q_T} [K_2]_{t=0}^T - \mathbb{E} \int_{Q_T} K_3 \, dx - \mathbb{E} \int_{Q_T} K_4 \, dx.
\end{align*}

(3.21)

where

\begin{align*}
Y_1 &= \mathbb{E} \int_{Q_T} \theta_1 I_2 \left[ du + ((x + \epsilon)^\nu u_x)_t \right] \, dx, \\
Y_2 &= C_{\alpha,\beta}^1 \mathbb{E} \int_{Q_T} s\lambda (x + \epsilon)^{2\alpha + \beta - 3} \phi_1 \xi \|U_x\| \, dx, \\
Y_3 &= C(\lambda) \mathbb{E} \int_{Q_T} s \xi^2 \|U\|^2 \, dx.
\end{align*}

Now we estimate $Y_1, Y_2, Y_3$. For $Y_1$, by noting $\mathbb{E} \int_{Q_T} \theta_1 I_2 F_1 \, dB(t) = 0$, we obtain

\begin{align*}
Y_1 &= \mathbb{E} \int_{Q_T} \theta_1 I_2 (f_1 \, dt + F_1 \, dB(t)) \, dx \leq \frac{1}{2} \mathbb{E} \int_{Q_T} |I_2|^2 \, dx \, dt + \frac{1}{2} \mathbb{E} \int_{Q_T} \theta_1^2 |f_1|^2 \, dx \, dt.
\end{align*}

(3.22)

By using Young’s inequality and lemma 3.1, we have
\[
Y_2 \leq \frac{1}{4\epsilon_1^4} C_{\alpha,\beta}^{(1)} E \int_{Q_T} s\lambda(x + \epsilon)^{(4 - 2\alpha - \beta)}\phi_1(\xi)|U|^2 \, dx \, dt \\
+ \epsilon_1 C_{\alpha,\beta}^{(1)} \int_{Q_T} s\lambda(x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)|U_x|^2 \, dx \, dt \\
\leq \frac{1}{4\epsilon_1^4} C_{\alpha,\beta}^{(1)} C_{\alpha,\beta}^{(2)} \int_{Q_T} s\lambda(x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)\phi_1^+(\frac{\partial}{\partial x})_x U_x^2 \, dx \, dt \\
+ \epsilon_1 C_{\alpha,\beta}^{(1)} \int_{Q_T} s\lambda(x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)|U_x|^2 \, dx \, dt \\
\leq \left( \epsilon_1 C_{\alpha,\beta}^{(1)} + \frac{1}{4\epsilon_1^4} C_{\alpha,\beta}^{(1)} C_{\alpha,\beta}^{(2)} + \epsilon_2 \right) \int_{Q_T} s\lambda(x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)|U|^2 \, dx \, dt \\
+ C(\epsilon_1, \epsilon_2) \int_{Q_T} s^3(x + \epsilon)^{2\alpha + 3\beta - 4}\phi_1(\xi)|U|^2 \, dx \, dt,
\] (3.23)

with

\[
C_{\alpha,\beta}^{(2)} = \frac{4}{(3 - 2\alpha - \beta)^2}.
\]

Similarly,

\[
Y_3 \leq C E \int_{Q_T} (x + \epsilon)^{-(4 - 2\alpha - \beta)}\phi_1(\xi)|U|^2 \, dx \, dt + C(\lambda) E \int_{Q_T} s^2(x + \epsilon)^{4 - 2\alpha - \beta}\phi_1(\xi)^3|U|^2 \, dx \, dt \\
\leq C E \int_{Q_T} (x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)|U_x|^2 \, dx \, dt + C(\lambda) E \int_{Q_T} s^2(x + \epsilon)^{2\alpha + 3\beta - 4}\phi_1(\xi)^3|U|^2 \, dx \, dt.
\] (3.24)

From (3.21)–(3.24), it follows that

\[
E \int_{Q_T} \left[ (\alpha + 2\beta - 2)^3 x^3 - C(\lambda) s^3 - C(\epsilon_1, \epsilon_2) s x^3 \right] (x + \epsilon)^{2\alpha + 3\beta - 4}\phi_1(\xi)^3|U|^2 \, dx \, dt \\
+ E \int_{Q_T} \left( C_{\alpha,\beta}^{(3)} s x - C s - C \right) (x + \epsilon)^{2\alpha + \beta - 2}\phi_1(\xi)|U_x|^2 \, dx \, dt \\
\leq C E \int_{Q_T} \theta_1^2 |f| \, dx \, dt - E \int_{0}^{T} \left[ K_1 \right]_{t=0}^{t} - E \int_{Q_T} dK_3 \, dx - E \int_{Q_T} dK_4 \, dx,
\] (3.25)

where

\[
C_{\alpha,\beta}^{(3)} = (\alpha + 2\beta - 2)\beta - \epsilon_1 C_{\alpha,\beta}^{(1)} - \frac{1}{4\epsilon_1^4} C_{\alpha,\beta}^{(1)} C_{\alpha,\beta}^{(2)} - \epsilon_2.
\]

By using (3.2), we can prove for all $\alpha \in (0, 2)$ that

\[
\epsilon_1 C_{\alpha,\beta}^{(1)} + \frac{1}{4\epsilon_1^4} C_{\alpha,\beta}^{(1)} C_{\alpha,\beta}^{(2)} < (\alpha + 2\beta - 2)\beta.
\] (3.26)

We first fix $\epsilon_1 = \frac{1}{(3 - 2\alpha - \beta)}$. For $\alpha \in (0, 1)$, (3.26) can be simplifies as

\[
\alpha - \alpha \beta + 2\beta - 2 > 0,
\]

13
which holds for $\beta > 1$. For $\alpha \in (1, 2)$, since $\beta > 3 - 2\alpha$, (3.26) is equivalent to

$$4\beta^2 + (9\alpha - 14)\beta + 4\alpha^2 - 13\alpha + 10 > 0,$$

which holds for $\beta > \frac{14 - 9\alpha + \sqrt{17\alpha^2 - 44\alpha + 36}}{8\alpha}$. Moreover, when $\alpha = 1$, we easily see $C_{\alpha, 0}^{(1)} = 0$ and then (3.26). Therefore, (3.26) holds for all $\alpha \in (0, 2)$, if $\beta$ satisfies (3.2). Further for sufficiently small $\epsilon_2$ we have $C_{\alpha, 0}^{(3)} > 0$. Consequently, there exist $\lambda_1$ and $s_1$ such that for all $\lambda > \lambda_1$ and $s > s_1$, it holds that

$$\mathbb{E} \int_{Q_T} s^3 \lambda^2 (x + \epsilon)^{2\alpha - 3\beta - 4} \phi_1^3 \theta_1^2 |u|^2 \text{d}x \text{d}t + \mathbb{E} \int_{Q_T} s\lambda (x + \epsilon)^{2\alpha - \beta - 2} \phi_1 \theta_1^2 |u|^2 \text{d}x \text{d}t \leq C \mathbb{E} \int_{Q_T} \theta_1^2 |f_1|^2 \text{d}x \text{d}t - C \mathbb{E} \int_0^T [K_{2x}^{(1)}]_{x=0} - C \mathbb{E} \int_{Q_T} K_3 \text{d}x \text{d}t. \tag{3.27}$$

Now we deal with the boundary term of $K_2$. For $\alpha \in (0, 1)$, by using (3.9) we have

$$\mathbb{E} \int_0^T [K_{2x}^{(1)}]_{x=0} = \mathbb{E} \int_0^T [(x + \epsilon)^{2\alpha} l_{1x} |U_x|]_{x=0}^1 \text{d}t \leq C \mathbb{E} \int_0^T s\lambda [\phi_1 \theta_1^2 |u_x|^2]_{x=0}^1 \text{d}t. \tag{3.28}$$

Similarly, for $\alpha \in [1, 2)$ we have

$$\mathbb{E} \int_0^T [K_{2x}^{(1)}]_{x=0} = \mathbb{E} \int_0^T [(x + \epsilon)^{\alpha} U_x \text{d}U]_{x=0} + \mathbb{E} \int_0^T [(x + \epsilon)^{2\alpha} l_{1x} |U_x|]_{x=0}^1 \text{d}t \leq C \mathbb{E} \int_0^T s\lambda [\phi_1 \theta_1^2 |u_x|^2]_{x=0}^1 \text{d}t.$$

By using Itô formula and (3.9) again, we have

$$\mathbb{E} \int_0^T [(x + \epsilon)^{\alpha} U_x \text{d}U]_{x=0} = \mathbb{E} \int_0^T [(x + \epsilon)^{\alpha} l_{1x} U \text{d}U]_{x=0}^1 = \frac{1}{2} \mathbb{E} \int_0^T \beta s \lambda d[(x + \epsilon)^{\alpha + \beta - 1} \phi_1 \xi |U|^2]_{x=0} - \frac{1}{2} \mathbb{E} \int_0^T \beta s \lambda [(x + \epsilon)^{\alpha + \beta - 1} \phi_1 \xi |U|^2]_{x=0} \text{d}t \leq C(\lambda) \mathbb{E} \int_0^T s [(x + \epsilon)^{\alpha + \beta - 1} \phi_1 \xi^2 \theta_1^2 |u_x|^2]_{x=0} \text{d}t. \tag{3.30}$$
Therefore, combining (3.28)–(3.30), we obtain for all \( \alpha \in (0, 2) \) that

\[
\begin{align*}
-\mathbb{E} \int_0^T [K_2]_{t=1=0} &= \mathcal{C} \mathbb{E} \int_0^T s \lambda \left[ \phi_1 \xi \theta_1^2 |u_1'|^2 \right]_{t=0} \, dt \\
&\quad + C(\lambda) \mathbb{E} \int_0^T s^2 \left[ (x + \varepsilon)^{\alpha+\beta-1} \phi_1 \xi^3 \theta_1^2 |u'|^2 \right]_{x=0} \, dt.
\end{align*}
\]  

(3.31)

By using \( U(x, 0) = U(x, T) = 0 \), we have

\[
-\mathbb{E} \int_{Q_T} dK_3 \, dx = 0.
\]  

(3.32)

Moreover, by (3.15), \( (dU)^2 = \theta_1^2 |F_1|^2 \, dx \) and \( \beta > 1 - \frac{\alpha}{2} \) for all \( \alpha \in (0, 2) \), we have the following estimate:

\[
\begin{align*}
-\mathbb{E} \int_{Q_T} K_3 \, dx &\leq C \mathbb{E} \int_{Q_T} s^2 \lambda^2 (x + \varepsilon)^{\alpha+\beta-2} \phi_1 \xi^2 (dU)^2 \, dx + C(\lambda) \mathbb{E} \int_{Q_T} s \xi \phi_1^2 (dU)^2 \, dx \\
&\quad \leq C(\lambda) \mathbb{E} \int_{Q_T} s^2 \phi_1^2 \xi^2 |F_1|^2 \, dx \, dr.
\end{align*}
\]  

(3.33)

Then substituting (3.32) and (3.33) into (3.27) yields

\[
\begin{align*}
\mathbb{E} \int_{Q_T} s^3 \lambda^3 (x + \varepsilon)^{2\alpha+3\beta-4} \phi_1^3 \xi^3 \theta_1^2 |u'|^2 \, dx \, dt &\quad + \mathbb{E} \int_{Q_T} s \lambda (x + \varepsilon)^{2\alpha+\beta-2} \phi_1 \xi^2 \theta_1^2 |u'|^2 \, dx \, dt \\
&\quad \leq C \mathbb{E} \int_{Q_T} \theta_1^2 |f_1|^2 \, dx \, dt + C(\lambda) \mathbb{E} \int_{Q_T} s^2 \phi_1^2 \xi^2 |F_1|^2 \, dx \, dt \\
&\quad + C(\lambda) \mathbb{E} \int_0^T s^2 \left[ (x + \varepsilon)^{\alpha+\beta-1} \phi_1 \xi^3 \theta_1^2 |u'|^2 \right]_{x=0} \, dt + C \mathbb{E} \int_0^T s \lambda \left[ \phi_1 \xi \theta_1^2 |u_1'|^2 \right]_{x=1=0} \, dt.
\end{align*}
\]  

(3.34)

Next, we eliminate the boundary term on \( x = 1 \). We consider the following stochastic parabolic equation of \( \bar{u}' = \chi u' \):

\[
\begin{align*}
\begin{cases}
\frac{d\bar{u}}{dt} + ((x + \varepsilon)^\alpha \bar{u})_x = \bar{f}_1 \, dt + \bar{F}_1 \, dB(t), & (x, t) \in Q_T, \\
\bar{u}'(1, t) = 0, & t \in (0, T), \\
\bar{u}(0, t) = 0 & \text{for } \alpha \in (0, 1), \\
((x + \varepsilon)^\alpha \bar{u})_x(0, t) = 0 & \text{for } \alpha \in [1, 2),
\end{cases}
\end{align*}
\]

where

\[
\bar{f}_1 = ((x + \varepsilon)^\alpha \chi_x u')_x + (x + \varepsilon)^\alpha \chi_x u' + \chi f_1, \quad \bar{F}_1 = \chi F_1.
\]

Applying (3.34) to \( \bar{u} \) and using the definition of \( \chi \), we find that
Together with $\varphi_1 = \varphi$ for $x \in (0, x_\epsilon^{(2)})$, we deduce (3.14) from (3.35).

Step 3. Carleman estimate for nondegenerate part.

Now we derive the Carleman estimate for nondegenerate part $(x_\epsilon^{(2)}, 1) \times (0, T)$:

$$
\mathbb{E} \int_0^T \int_{x_\epsilon^{(2)}}^1 \lambda_s \xi^2 (x + \varepsilon)^{2n+3-4\xi} |u|^2 e^{2\varepsilon} \, dx \, dt + \mathbb{E} \int_0^T \int_{x_\epsilon^{(2)}}^1 \lambda_s \xi (x + \varepsilon)^{2n+3-2\xi} |u|^2 e^{2\varepsilon} \, dx \, dt
\leq C \mathbb{E} \int_{x_\epsilon^{(2)}}^1 |f_1|^2 e^{2\varepsilon} \, dx \, dt + C(\lambda) \mathbb{E} \int_0^T \int_{x_\epsilon^{(2)}}^1 \lambda_s \xi^2 |F_1|^2 e^{2\varepsilon} \, dx \, dt
+ C(\lambda) \mathbb{E} \int_{x_\epsilon^{(2)}}^1 (|u|^2 + s^3 \xi^3 |u|^2) e^{2\varepsilon} \, dx \, dt.
$$

(3.36)

To do this, letting $\overline{\pi} = (1 - \chi)u^\alpha$, then we have

$$
\begin{cases}
\frac{d\overline{\pi}}{dt} + ((x + \varepsilon)^{\alpha} \overline{\pi})_\alpha = \overline{\pi}_1 \alpha dB(t), & (x, t) \in Q_T, \\
\overline{\pi}(1, t) = 0, & t \in (0, T), \\
\overline{\pi}(0, t) = 0 & \text{for } \alpha \in (0, 1), \\
((x + \varepsilon)^{\alpha} \overline{\pi})_\alpha(0, t) = 0 & \text{for } \alpha \in [1, 2),
\end{cases}
$$

where

$$
\overline{\pi}_1 = (1 - \chi)f_1 - ((x + \varepsilon)^{\alpha} \chi_s u^\alpha)_s - (x + \varepsilon)^{\alpha} \chi_s u^\alpha, \quad \overline{F}_1 = (1 - \chi)F_1.
$$

By the classic Carleman estimate for stochastic nondegenerate parabolic equation, e.g. [35] or [29], we have

$$
\mathbb{E} \int_{Q_T} s^3 \lambda \xi^3 |\overline{\pi}|^2 e^{2\varepsilon} \, dx \, dt + \mathbb{E} \int_{Q_T} s \lambda \xi |\overline{\pi}_1|^2 e^{2\varepsilon} \, dx \, dt
\leq C \mathbb{E} \int_{Q_T} |f_1|^2 e^{2\varepsilon} \, dx \, dt + C \mathbb{E} \int_{Q_T} s^3 \lambda \xi^2 |F_1|^2 e^{2\varepsilon} \, dx \, dt + C \mathbb{E} \int_{Q_T} s^3 \lambda \xi^3 |\overline{\pi}|^2 e^{2\varepsilon} \, dx \, dt
\leq C \mathbb{E} \int_{Q_T} (1 - \chi)^2 |f_1|^2 e^{2\varepsilon} \, dx \, dt + C \mathbb{E} \int_0^T \int_{x_\epsilon^{(2)}}^1 (|u|^2 + |u|^2) e^{2\varepsilon} \, dx \, dt
+ C \mathbb{E} \int_{Q_T} s^2 \lambda^2 (1 - \chi)^2 \xi^2 |F_1|^2 e^{2\varepsilon} \, dx \, dt + C \mathbb{E} \int_0^T \int_{x_\epsilon^{(2)}}^1 s^3 \lambda^3 \xi^3 |u|^2 e^{2\varepsilon} \, dx \, dt.
$$

(3.37)
Since $\varphi_2 = \varphi$ for $x \in (x_i^{(2)}, 1)$ and $\min\{(x + \varepsilon)^{2\alpha + 3\beta - 4}, (x + \varepsilon)^{2\alpha + \beta - 2}\} \geq C > 0$ for $x \in (x_i^{(2)}, 1)$, together with (3.37), we obtain (3.36).

Step 4. End of the proof.

Combining (3.14) and (3.36) and adding to both sides of the inequality the term

$$
E \int_0^T \int_{x_1^{(2)}}^{x_2^{(2)}} 3\xi^2(x + \varepsilon)^{2\alpha + 3\beta - 4} |u|^2 e^{2\alpha r} dx dr + E \int_0^T \int_{x_1^{(2)}}^{x_2^{(2)}} s\xi^2(x + \varepsilon)^{2\alpha + \beta - 2} |u|^2 e^{2\alpha r} dx dr,
$$

we obtain

$$
E \int_{Q_T} s^2 \lambda \xi^3(x + \varepsilon)^{2\alpha + 3\beta - 4} |u|^2 e^{2\alpha r} dx dr + E \int_{Q_T} s\lambda \xi(x + \varepsilon)^{2\alpha + \beta - 2} |u|^2 e^{2\alpha r} dx dr
\leq C E \int_{Q_T} f_1(x)^2 e^{2\alpha r} dx dr + C(\lambda) E \int_{Q_T} s^2 \xi^2 |u|^2 e^{2\alpha r} dx dr
+ C(\lambda) E \int_{Q_T} s^\xi |u|^2 e^{2\alpha r} dx dr + C(\lambda) E \int_0^T s^2 [(x + \varepsilon)^{\alpha + \beta - 1} \xi^3 |u|^2 e^{2\alpha r}]_{x=0} dt.
$$

(3.38)

Finally, by lemma 3.2, we obtain (3.3). This completes the proof of theorem 3.1.

3.2. Carleman estimate for forward stochastic degenerate equation

In this subsection, we will introduce a regular weight function into a new Carleman estimate for the backward stochastic degenerate equation, in which the random source and the initial data are put on the left-hand side. This allows us to prove the stability for our inverse random source problem.

We set

$$
\varphi(x, t) = \eta_i(x) - (\lambda - i)^2 + \lambda^2, \quad \Phi_i(x, t) = e^{\lambda \varphi_i(x, t)}, \quad i = 1, 2,
$$

where $\eta_i(i = 1, 2)$ are same as the ones in section 3.1. We introduce regular weight function

$$
\Phi(x, t) = \chi(x)\Phi_1(x, t) + (1 - \chi(x))\Phi_2(x, t), \quad (x, t) \in Q_T.
$$

So that, similar to (3.1) we also have

$$
\Phi(x, t) =
\begin{cases}
\Phi_1(x, t), & (x, t) \in (0, x_1^{(2)}) \times (0, T), \\
\Phi_1(x, t) = \Phi_2(x, t), & (x, t) \in (x_1^{(2)}, x_2^{(2)}) \times (0, T), \\
\Phi_2(x, t), & (x, t) \in (x_2^{(2)}, 1) \times (0, T).
\end{cases}
$$

(3.39)

Theorem 3.3. Let $\alpha \in (0, 2)$, $f_1 \in L^2(0, T; L^2(I))$, $F_2 \in L^2(0, T; H^1(I))$ and $\beta$ such that (3.2). Then for any $\varepsilon \in (0, 1)$, there exist positive constants $\lambda_2 = \lambda_2(\omega, I, T, \alpha)$, $s_2 = s_2(\omega, I, T, \alpha, \lambda)$ and $C_5 = C_5(\omega, I, T, \alpha)$, $C_6 = C_6(\omega, I, T, \alpha, \lambda)$ such that
\[
\mathbb{E} \int_{Q_T} s \lambda \Phi|F_2|^2 e^{2\varphi} \, dx \, dt + \mathbb{E} \int_{Q_T} s^3 \lambda^3 \Phi^4 (x + \varepsilon)^{2\alpha + 3\beta - 4} |\varphi|^2 e^{2\varphi} \, dx \, dt \\
+ \mathbb{E} \int_{Q_T} s \lambda \Phi (x + \varepsilon)^{2\alpha + 3\beta - 2} |\varphi|^2 e^{2\varphi} \, dx \, dt \\
\leq C_5 \mathbb{E} \int_{Q_T} |F_1|^2 e^{2\varphi} \, dx \, dt + C_5 \mathbb{E} \int_{Q_T} s\Phi|F_2|^2 e^{2\varphi} \, dx \, dt \\
+ C_6(\lambda) \mathbb{E} \int_{Q_T} s^3 \Phi^3 |\varphi|^2 e^{2\varphi} \, dx \, dt + C_6(\lambda) s^2 e^{C(\lambda)\varepsilon} \|v^\varepsilon(\cdot, T)\|^2_{L_2(\Omega, F_T, L_2(0))} \\
+ C_6(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{\alpha + \beta - 1} (|\varphi|^2 + |F_2|^2) e^{2\varphi}] \, dt \\
\] (3.40)

for all \( \lambda \geq \lambda_2, s \geq s_2 \) and all \( \varphi \in H^1 \) satisfying

\[
\begin{aligned}
\frac{d\varphi}{dt} - ((x + \varepsilon)^\alpha \varphi)_x & = f_2 dt + F_2 dB(t), \quad (x, t) \in Q_T, \\
\varphi(1, t) & = 0, \quad t \in (0, T), \\
\text{and} \quad \varphi(0, t) & = 0 \quad \text{for} \quad \alpha \in (0, 1), \quad t \in (0, T), \\
\varphi(x, 0) & = 0 \quad \text{for} \quad \alpha \in [1, 2), \quad x \in I. 
\end{aligned}
\] (3.41)

**Remark 3.3.** The second large parameter \( \lambda \) in the proof of null controllability could be omitted. However, in inverse random source problem it plays a very important role.

Based on theorem 3.3, letting \( \beta = 2 - \alpha \) and \( \varepsilon \to 0 \), we could drop the boundary term in (3.40) as in theorem 3.2. Then we obtain the following result:

**Theorem 3.4.** Let \( \alpha \in (0, 2), f_2 \in L_2^2(0, T; L_2(I)), F_2 \in L_2^2(0, T; H^1(I)) \) and \( \beta \) such that (3.2). Then for any \( \varepsilon \in (0, 1) \), there exist positive constants \( \lambda_2 = \lambda_2(\omega, I, T, \alpha), s_2 = s_2(\omega, I, T, \alpha, \lambda) \) and \( C_5 = C_5(\omega, I, T, \alpha), C_6 = C_6(\omega, I, T, \alpha, \lambda) \) such that

\[
\mathbb{E} \int_{Q_T} s \lambda \Phi|F_2|^2 e^{2\varphi} \, dx \, dt + \mathbb{E} \int_{Q_T} s^3 \lambda^3 \Phi^4 (x + \varepsilon)^{2\alpha + 3\beta - 4} |\varphi|^2 e^{2\varphi} \, dx \, dt \\
+ \mathbb{E} \int_{Q_T} s \lambda \Phi (x + \varepsilon)^{2\alpha + 3\beta - 2} |\varphi|^2 e^{2\varphi} \, dx \, dt \\
\leq C_5 \mathbb{E} \int_{Q_T} |F_1|^2 e^{2\varphi} \, dx \, dt + C_5 \mathbb{E} \int_{Q_T} s\Phi|F_2|^2 e^{2\varphi} \, dx \, dt \\
+ C_6(\lambda) \mathbb{E} \int_{Q_T} s^3 \Phi^3 |\varphi|^2 e^{2\varphi} \, dx \, dt + C_6(\lambda) s^2 e^{C(\lambda)\varepsilon} \|v^\varepsilon(\cdot, T)\|^2_{L_2(\Omega, F_T, L_2(0))} \\
+ C_6(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{\alpha + \beta - 1} (|\varphi|^2 + |F_2|^2) e^{2\varphi}] \, dt \\
\] (3.42)

for all \( \lambda \geq \lambda_2, s \geq s_2 \) and all \( v \in H^1_0 \) satisfying
\[
\begin{cases}
    dv = (x^n v_2)_x dt = f_2 dt + F_2 dB(t), & (x, t) \in Q_T, \\
v(1, t) = 0, & t \in (0, T), \\
and (x^n v_2)(0, t) = 0 & \text{for } \alpha \in (0, 1), \\
v(x, 0) = 0, & x \in I.
\end{cases}
\] (3.43)

Now we prove theorem 3.3. The proof of this theorem is similar to theorem 3.1. Based on a weighted identity, we first deal with Carleman estimate for degenerate part. Then we establish the other one for nondegenerate part. Finally, combining these two inequalities, we obtain (3.40).

**Proof of Theorem 3.3.** Let \( L_1 = s \Phi_1, \Theta_1 = e^{\delta t} \) and \( V = \Theta_1 v^\tau \). Then we have
\[
\Theta_1 \left[ d\tau - \left( (x + \varepsilon)^n v_2^\tau \right)_x dt \right] = J_1 + J_2 dt
\] (3.44)

with
\[
\begin{cases}
    V(1, t) = 0, & t \in (0, T), \\
\text{and } \left\{ 
    \begin{array}{ll}
        V(0, t) = 0 & \text{for } \alpha \in (0, 1), \\
        ((x + \varepsilon)^n V_x)(0, t) = ((x + \varepsilon)^\alpha L_{1,x} U)(0, t) & \text{for } \alpha \in [1, 2), \\
        V(x, 0) = 0, & x \in I,
    \end{array}
\right.
\end{cases}
\] (3.45)

where
\[
J_1 = dV + 2(x + \varepsilon)^\alpha L_{1,x} V_x dt + ((x + \varepsilon)^\alpha L_{1,x})_x V dt,
\]
\[
J_2 = -((x + \varepsilon)^n V_x)_x - (x + \varepsilon)^\alpha L_{1,x}^2 V - L_{1,x} V.
\]

Hence,
\[
\Theta_1 J_2 \left[ d\tau - \left( (x + \varepsilon)^n v_2^\tau \right)_x dt \right] = J_1 J_2 + |J_2|^2 dt.
\] (3.46)

Then by a similar argument to (3.13), we have
\[
\Theta_1 J_2 \left[ d\tau - \left( (x + \varepsilon)^n v_2^\tau \right)_x dt \right] = |J_2|^2 dt + R_1 dt + (R_2)_x + dR_3 + R_4,
\]
where
\[
R_1 = \left[ \alpha(x + \varepsilon)^{2\alpha-1} L_{1,x}^3 + 2(x + \varepsilon)^{2\alpha} L_{1,x}^2 L_{1,xx} \right] V^2 + 2(x + \varepsilon)^{2\alpha} L_{1,xx} + \alpha(x + \varepsilon)^{2\alpha-1} L_{1,xx} \right] V_x^2 + \left[ (x + \varepsilon)^\alpha L_{1,x} \right]_{xx} (x + \varepsilon)^\alpha V V_x + (x + \varepsilon)^\alpha L_{1,x} L_{1,xx} V^2 + \frac{1}{2} L_{1,x} V^2 + \left[ (x + \varepsilon)^\alpha L_{1,x} L_{1,xx} \right]_x V^2 dt - \left[ (x + \varepsilon)^\alpha L_{1,x} \right]_x L_{1,x} V^2,
\]
\[
R_2 = - (x + \varepsilon)^\alpha V_x dV - (x + \varepsilon)^{2\alpha} L_{1,x} V^2 dt - (x + \varepsilon)^{2\alpha} L_{1,x}^3 V^2 dt
\]
Then by a similar process to obtain (3.27), we can prove that there exist \( \lambda_2 \) and \( s_2 \) such that for all \( \lambda > \lambda_2 \) and \( s > s_2 \), it holds that
\[
\mathbb{E} \int_{Q_T} s^2 \lambda^2 (x + \varepsilon)^{2\alpha + \beta - 2} |\Phi_t|^2 |v|^2 dx dt + \mathbb{E} \int_{Q_T} s \lambda (x + \varepsilon)^{2\alpha + \beta - 2} |\Phi_t|^2 |v|^2 dx dt \\
\leq C \mathbb{E} \int_{Q_T} |\Phi_t|^2 |\ell|^2 dx dt - \mathbb{E} \int_{Q_T} \left[ \frac{R_2(t)}{s} \right]_{t=0}^{T} - \mathbb{E} \int_{Q_T} \frac{dR_3}{dx} - \mathbb{E} \int_{Q_T} R_4 dx. \tag{3.47}
\]
Now we analyze the terms of \( R_2, R_3 \) and \( R_4 \). For the boundary term of \( R_2 \), noticing that
\[
- \mathbb{E} \int_{0}^{T} [(x + \varepsilon)^\alpha V_x V]_{x=0} = - \mathbb{E} \int_{\Omega} [(x + \varepsilon)^\alpha L_{1x} V]_{x=0} \\
= - \frac{1}{2} \mathbb{E} \int_{0}^{T} \beta s \lambda d [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |V|^2]_{x=0} + \mathbb{E} \int_{0}^{T} \beta s \lambda^2 (\lambda - t) [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |V|^2]_{x=0} dt \\
+ \frac{1}{2} \mathbb{E} \int_{0}^{T} \beta s \lambda [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t (dV)^2]_{x=0} \\
\leq C(\lambda) \mathbb{E} \int_{0}^{T} s [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |\ell|^2]_{x=0} dt + C \mathbb{E} \int_{0}^{T} s \lambda [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |\ell|^2 |\ell|^2]_{x=0} dt,
\]
similar to (3.31), we obtain
\[
- \mathbb{E} \int_{0}^{T} \left[ \frac{R_2(t)}{s} \right]_{t=0}^{T} \leq C \mathbb{E} \int_{0}^{T} s \lambda [\Phi_t |\ell|^2]_{x=0} dt + C(\lambda) \mathbb{E} \\
\times \int_{0}^{T} s^2 [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |\ell|^2]_{x=0} dt \\
+ C \mathbb{E} \int_{0}^{T} s \lambda [(x + \varepsilon)^{\alpha + \beta - 1} \Phi_t |\ell|^2 |\ell|^2]_{x=0} dt. \tag{3.48}
\]
By \( V(x, 0) = 0, \mathbb{P} \)-a.s. in \( I \), we have
\[
- \mathbb{E} \int_{Q_T} dR_3 dx = \mathbb{E} \int_{I} \left[ - \frac{1}{2} (x + \varepsilon)^\alpha V_x^2 + \frac{1}{2} (x + \varepsilon)^\beta L_{1x} V^2 + \frac{1}{2} L_{1x} V^2 \right]_{j=0}^{J} dx \\
\leq C(\lambda) s^2 e^{C(\lambda) t} \|v(t, \cdot)\|^2_{L^2(\Omega, F_I, F_T^2(\Omega))}. \tag{3.49}
\]
For the term of \( R_4 \), by \((dV)^2 = \Theta_1^2 |F_2|^2 \)dr and
\[
(dV)^2 = (s \Phi_1, \Theta_1 d\nu^r + \Theta_1 d\nu^e)^2
\]
\[
= \Theta_1^2 |F_{2,r}|^2 dr + 2s \Phi_1, \Theta_1^2 F_2 F_{2,r} dr + s^2 \Phi_1, \Theta_1^2 |F_{2,e}|^2 dr,
\]
we have
\[
-\mathbb{E} \int_{Q_T} R_4 dx = \frac{1}{2} \mathbb{E} \int_{Q_T} (x + \varepsilon)^r (dV)^2 dx - \frac{1}{2} \mathbb{E} \int_{Q_T} (x + \varepsilon)^r L_1^2 (dV)^2 dx - \mathbb{E} \int_{Q_T} \frac{1}{2} L_2 (dV)^2 dx
\]
\[
= \frac{1}{2} \mathbb{E} \int_{Q_T} (x + \varepsilon)^r \Theta_1^2 |F_{2,r}|^2 dx + \mathbb{E} \int_{Q_T} \beta s \lambda (x + \varepsilon)^{\alpha + \beta - 1} \Phi_1, \Theta_1^2 F_2 F_{2,r} dx
\]
\[
- \mathbb{E} \int_{Q_T} s \lambda (\lambda - t) \Phi_1, \Theta_1^2 |F_{2,e}|^2 dx
\]
\[
\leq \frac{1}{2} \mathbb{E} \int_{Q_T} (x + \varepsilon)^r \Theta_1^2 |F_{2,r}|^2 dx + \mathcal{C} \mathbb{E} \int_{Q_T} s \Phi_1, \Theta_1^2 |F_{2,r}|^2 dx
\]
\[
- \mathbb{E} \int_{Q_T} s \lambda [(1 - \varepsilon) \lambda - t] \Phi_1, \Theta_1^2 |F_{2,e}|^2 dx,
\]
which implies
\[
-\mathbb{E} \int_{Q_T} R_4 dx \leq \mathcal{C} \mathbb{E} \int_{Q_T} s \Phi_1, \Theta_1^2 |F_{2,r}|^2 dx - \mathbb{E} \int_{Q_T} s \lambda^2 \Phi_1, \Theta_1^2 |F_{2,e}|^2 dx.
\] (3.50)

for sufficiently small \( \varepsilon \) and sufficiently large \( \lambda \) and \( s \).

Then substituting (3.48)–(3.50) into (3.47) yields that
\[
\mathbb{E} \int_{Q_T} s^2 \Phi_1, \Theta_1^2 |F_{2,r}|^2 dx + \mathbb{E} \int_{Q_T} s^3 \lambda^3 (x + \varepsilon)^{2\alpha + 3\beta - 4} \Phi_1, \Theta_1^2 |\nu^e|^2 dx
\]
\[
+ \mathbb{E} \int_{Q_T} s \lambda (x + \varepsilon)^{2\alpha + \beta - 2} \Phi_1, \Theta_1^2 |\nu^e|^2 dx
\]
\[
\leq \mathcal{C} \mathbb{E} \int_{Q_T} \Theta_1^2 |F_{2,r}|^2 dx + \mathcal{C} \mathbb{E} \int_{Q_T} s \Phi_1, \Theta_1^2 |F_{2,r}|^2 dx + C(\lambda) \mathbb{E} \int_0^T s \Theta_1^2 |\nu^e|^2 \| s \Theta_1^2 |\nu^e|^2 \|_{L^2_2(\Omega, \mathcal{F}, \mathcal{P}, L^2_2(\Omega))}
\]
\[
+ C(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{\alpha + \beta - 1} \Theta_1^2 |(\nu^e)^2 + |F_{2,e}|^2]_{x=0}^1 dt
\] (3.51)

In order to deal with Carleman estimate for nondegenerate part, we use \( \Phi_2 \) as weight function in Carleman estimate. Letting \( L_2 = s \Phi_2, \Theta_2 = e^{L_2} \) and repeating the above process, we have the following estimate:
Therefore, by applying (3.51) to \( \tilde{v} \) and using (3.39), we have

\[
\begin{aligned}
\mathbb{E} \int_{Q_T} s^3 \lambda^3 |F_2|^2 e^{2s^3} \Phi^3 |v^2|^2 dx dt + \mathbb{E} \int_0^T \int_0^{q(2)} s^3 \lambda^3 (x + \varepsilon)^{2\alpha+2\beta-4} \Phi^3 |v^2|^2 e^{2s^3} dx dt \\
+ \mathbb{E} \int_0^T \int_0^{q(2)} s \lambda (x + \varepsilon)^{2\alpha+2\beta-4} \Phi^3 |v^2|^2 e^{2s^3} dx dt \\
\leq C \mathbb{E} \int_{Q_T} |F_2|^2 e^{2s^3} \Phi^3 |v^2|^2 dx dt + C \mathbb{E} \int_0^T \int_0^{q(2)} s \Phi(|F_{2x}|^2 + |F_2|^2) e^{2s^3} dx dt \\
+ C \mathbb{E} \int_0^T \int_0^{q(2)} (|v|^2 + |v_x|^2) e^{2s^3} dx dt + C(\lambda) s^2 e^{C(\lambda)} \|v\|_{L^2(\Omega, F_T, \mathcal{P}, L^2(\Omega))}^2 \\
+ C(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{2\alpha+2\beta-4} |v^2|^2 + |F_2|^2]_x^2 dt.
\end{aligned}
\]  

(3.53)

for sufficient large \( \lambda \) and \( s \) such that \( s \geq C(\lambda) \), where \( \gamma = \min \{2\alpha - 2, \alpha - 1\} \).

Now we apply (3.51) and (3.52) to obtain two Carleman estimates for degenerate part and nondegenerate part, respectively. Obviously, \( \tilde{v} = \chi v^\gamma \) satisfies

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
d\tilde{v}^\gamma + ((x + \varepsilon)^\gamma \tilde{v})_x = \tilde{f}_2 dt + F_2 dB(t), & (x, t) \in Q_T, \\
\tilde{v}^\gamma (1, t) = 0, & t \in (0, T), \\
\tilde{v}^\gamma (0, t) = 0 & \text{for } \alpha \in (0, 1), \\
((x + \varepsilon)^\gamma \tilde{v})_x (0, t) = 0 & \text{for } \alpha \in [1, 2),
\end{array} \right.
\]

where

\[
\tilde{f}_2 = -((x + \varepsilon)^\alpha \chi_x v^\gamma)_x - (x + \varepsilon)^\alpha \chi_x v^\gamma + \chi f_2, \quad \tilde{F}_2 = \chi F_2.
\]

Therefore, by applying (3.51) to \( \tilde{v} \) and using (3.39), we have

\[
\begin{aligned}
\mathbb{E} \int_{Q_T} s^3 \lambda^3 |F_2|^2 e^{2s^3} \Phi^3 |v^2|^2 dx dt + \mathbb{E} \int_0^T \int_0^{q(2)} s^3 \lambda^3 (x + \varepsilon)^{2\alpha+2\beta-4} \Phi^3 |v^2|^2 e^{2s^3} dx dt \\
+ \mathbb{E} \int_0^T \int_0^{q(2)} s \lambda (x + \varepsilon)^{2\alpha+2\beta-4} \Phi^3 |v^2|^2 e^{2s^3} dx dt \\
\leq C \mathbb{E} \int_{Q_T} |F_2|^2 e^{2s^3} \Phi^3 |v^2|^2 dx dt + C \mathbb{E} \int_0^T \int_0^{q(2)} s \Phi(|F_{2x}|^2 + |F_2|^2) e^{2s^3} dx dt \\
+ C \mathbb{E} \int_0^T \int_0^{q(2)} (|v|^2 + |v_x|^2) e^{2s^3} dx dt + C(\lambda) s^2 e^{C(\lambda)} \|v\|_{L^2(\Omega, F_T, \mathcal{P}, L^2(\Omega))}^2 \\
+ C(\lambda) \mathbb{E} \int_0^T s^2 [(x + \varepsilon)^{2\alpha+2\beta-4} |v^2|^2 + |F_2|^2]_x^2 dt.
\end{aligned}
\]  

(3.52)

Similarly, applying (3.52) to \( \bar{v} = (1 - \chi) v^\gamma \) yields that
Obviously, \((1 - \chi)^2(x + \varepsilon)^{-2\alpha}(x + \varepsilon)^{-2\alpha}\) in \(Q_T\), where \(C\) is not depending on \(\varepsilon\). Then, we further obtain for any \(\varepsilon > 0\) that

\[
\mathbb{E} \int_{Q_T} s\lambda^2 \Phi_2(1 - \chi)^2 |F_2|^2 e^{2\lambda \Phi_2} dx \, dt + \mathbb{E} \int_0^T \int_{\Delta_1^{(2)}} s\lambda^4 \Phi_3(x + \varepsilon)^{2\alpha+3\beta-4} |\psi|^2 e^{2\lambda \Phi_2} \, dx \, dt
\]

\[
+ \mathbb{E} \int_0^T \int_{\Delta_1^{(2)}} s\lambda^2 \Phi(x + \varepsilon)^{2\alpha+\beta-2} |s\lambda|^2 e^{2\lambda \Phi_2} \, dx \, dt
\]

\[
\leq C \mathbb{E} \int_{Q_T} s\lambda^2 \Phi_2(1 - \chi)^2 |F_2|^2 e^{2\lambda \Phi_2} dx \, dt + C \mathbb{E} \int_0^T \int_{\Delta_1^{(2)}} s\Phi(1 - \chi)^2 + |F_2|^2 e^{2\lambda \Phi_2} dx \, dt
\]

\[
+ C(\lambda)s^2 e^{C(\lambda)\|\omega(x, T)\|_{L^2(\Omega,F_2)}^2} + C \mathbb{E} \int_{\Delta_0^{(1)}} (s\lambda^3 \Phi_3) |\psi|^2 + \varepsilon s\lambda^3 \Phi_3 |s\lambda|^2 e^{2\lambda \Phi_2} dx \, dt
\]  \hspace{1cm} (3.54)

for sufficient large \(s\) and \(\lambda\).

Combining (3.53) and (3.54) and adding to both sides of the inequality the term

\[
\mathbb{E} \int_0^T \int_{\Delta_1^{(2)}} s\lambda^4 \Phi_3(x + \varepsilon)^{2\alpha+3\beta-4} |\psi|^2 e^{2\lambda \Phi_2} \, dx \, dt
\]

\[
+ \mathbb{E} \int_0^T \int_{\Delta_1^{(2)}} s\lambda^2 \Phi(x + \varepsilon)^{2\alpha+\beta-2} |s\lambda|^2 e^{2\lambda \Phi_2} \, dx \, dt,
\]

we obtain

\[
\mathbb{E} \int_{Q_T} s\lambda^2 [\Phi_1 \lambda^2 e^{2\lambda \Phi_1} + \Phi_2(1 - \chi)^2 e^{2\lambda \Phi_2}] |F_2|^2 dx \, dt
\]

\[
+ \mathbb{E} \int_{Q_T} s\lambda^4 \Phi_3(x + \varepsilon)^{2\alpha+3\beta-4} |\psi|^2 e^{2\lambda \Phi_2} dx \, dt + \mathbb{E} \int_{Q_T} s\lambda^4 \Phi_3(x + \varepsilon)^{2\alpha+\beta-2} |s\lambda|^2 e^{2\lambda \Phi_2} dx \, dt
\]
Theorem 4.1. Let \( \text{stochastic degenerate parabolic equation (1.3)}, \) i.e. the following theorem 4.1. In this section, we will apply theorem 3.1 to prove the null controllability result for the forward stochastic degenerate parabolic equation:

\[
\begin{align*}
\text{E} & \int_{0}^{T} s^2 \lambda^2 \Phi \left( |F_2| e^{2\omega} \right) \, dt + CE \int_{0}^{T} s^3 \lambda^3 \Phi \left( |F_2| e^{2\omega} \right) \, dt \\
& + C \int_{0}^{T} \left( s^3 \lambda^3 \Phi \left( \epsilon^2 |v|^2 + \epsilon s^2 \Phi \left( |v|^2 \right) e^{2\omega} \right) \, dt + C(\lambda) \left( s^3 \lambda^3 \Phi \left( |v|^2 \right) e^{2\omega} \right) \int_{0}^{T} \lambda |F_2| e^{2\omega} \, dt \\
& + C(\lambda) \int_{0}^{T} s^2 \left( (x + \epsilon)^{\alpha+\beta-1} (|v|^2 + |F_2|^2) e^{2\omega} \right) \, dt.
\end{align*}
\]

Noticing that \( \Phi_1 \lambda^2 e^{2\omega} \Phi + \Phi_2 (1 - \lambda^2) e^{2\omega} \geq C \Phi e^{2\omega} \) in \( \mathcal{Q}_T, \) we further have

\[
\text{E} \int_{0}^{T} s^2 \lambda^2 \Phi |F_2|^2 e^{2\omega} \, dt + CE \int_{0}^{T} s^3 \lambda^3 \Phi (x + \epsilon)^{2\alpha+3\beta-4} |v|^2 e^{2\omega} \, dt \\
+ CE \int_{0}^{T} \lambda |F_2|^2 e^{2\omega} \, dt + CE \int_{0}^{T} s^2 \lambda^2 \Phi |F_2| e^{2\omega} \, dt \\
+ CE \int_{0}^{T} s^3 \lambda^3 \Phi \left( |v|^2 + \epsilon s^2 \Phi \left( |v|^2 \right) e^{2\omega} \right) \, dt + C(\lambda) \left( s^3 \lambda^3 \Phi \left( |v|^2 \right) e^{2\omega} \right) \int_{0}^{T} \lambda |F_2| e^{2\omega} \, dt \\
+ C(\lambda) \int_{0}^{T} s^2 \left( (x + \epsilon)^{\alpha+\beta-1} (|v|^2 + |F_2|^2) e^{2\omega} \right) \, dt.
\]

(3.55)

Similar to lemma 3.2, we have the following Cacciopoli inequality for forward stochastic degenerate parabolic equation:

\[
\text{E} \int_{0}^{T} \Phi |v|^2 e^{2\omega} \, dt \leq C(\lambda) \text{E} \int_{0}^{T} s^2 \Phi \left( |v|^2 \right) e^{2\omega} \, dt + CE \int_{0}^{T} s^3 \Phi \left( |F_2| e^{2\omega} \right) \, dt \\
+ CE \int_{0}^{T} \Phi |F_2|^2 e^{2\omega} \, dt.
\]

(3.56)

Finally, substituting (3.56) into (3.55) and choosing \( \epsilon \) sufficiently small, we can absorb the term of \( F_2 \) on the right-hand side of (3.56) and then obtain (3.40). This completes the proof of theorem 3.3.

\[ \square \]

4. Null controllability

In this section, we will apply theorem 3.1 to prove the null controllability result for the forward stochastic degenerate parabolic equation (1.3), i.e. the following theorem 4.1.

Theorem 4.1. Let \( \alpha \in (0, 1) \) and \( a, b, c \in L^\infty(0, T; L^\infty(I)) \). Then for any \( y_0 \in L^2(\Omega, F_0; P; L^2(G)) \), there exists a pair of controls \( (g, G) \in L^2_\beta(0, T; L^2(I)) \times L^2_\beta(0, T; L^2(I)) \) such that the solution \( y \) of (1.3) satisfies \( y(x, T) = 0 \) in \( I, \mathbb{P} \)-a.s.

Since the system (1.3) is degenerate, we first transfer to study a uniform null controllability in \( \epsilon \) for a nondegenerate approximate system. More precisely, letting \( 0 < \epsilon < 1 \), we consider...
\[
\begin{aligned}
\begin{cases}
\frac{dy}{dt} = ((x + \varepsilon)^\alpha y)_x, & \text{for } (x, t) \in Q_T, \\
y(0, t) = y(1, t) = 0, & \text{for } t \in (0, T), \\
y(x, 0) = y_0(x), & \text{for } x \in I,
\end{cases}
\end{aligned}
\]

(4.1)

where

\[y_0 \rightarrow y_0 \text{ in } L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)).\]

(4.2)

It is well known that the key ingredient for studying the null controllability is to obtain observation inequality for the corresponding adjoint equation. An important tool is Carleman estimate, in whose proof the main difficulty is how to deal with the first order term in the stochastic degenerate parabolic system. In order to use the terms on the left-hand side of Carleman estimate to absorb this term directly, we need \(x^{-\frac{\alpha}{2}}a \in L^\infty_\mathbb{F}(0, T; L^\infty(I))\), which means that the coefficient \(a\) of the first order term goes to zero at some polynomial rate as \(x \to 0\). More reasonable condition is \(a \in L^\infty_\mathbb{F}(0, T; L^\infty(I))\). For this condition on \(a\), we will apply a duality technique to establish a new Carleman estimate for the stochastic degenerate parabolic equation with convection term.

In next subsection we first prove a Carleman estimate for the following corresponding adjoint system of (4.1):

\[
\begin{aligned}
\begin{cases}
dz + ((x + \varepsilon)^\alpha z)_x \, dt = ((az)_x - bz - cz) \, dt + zdB(t), & \text{for } (x, t) \in Q_T, \\
z(0, t) = z(1, t) = 0, & \text{for } t \in (0, T), \\
z(x, T) = z_T(x), & \text{for } x \in I.
\end{cases}
\end{aligned}
\]

(4.3)

Next based on this Carleman estimate, we obtain observation inequality and then prove the null controllability result, i.e. theorem 4.1.

### 4.1. Carleman estimate for a backward stochastic degenerate equation with convection term

Our main result in this subsection is the following estimate, whose proof is based on a duality argument introduced by Imanuvilov and Yamamoto [34] for deterministic parabolic equation, or introduced by Liu [35] or Yan [23] for stochastic parabolic equation.

**Theorem 4.2.** Let \(\alpha \in (0, \frac{1}{2})\), \(a, b, c \in L^\infty_\mathbb{F}(0, T; L^\infty(I))\) and \(z_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))\). Then for any \(\varepsilon \in (0, \nu)\) with a sufficiently small \(\nu > 0\), there exist positive constants \(\lambda_3 = \lambda_3(\omega, I, T, \alpha, M, \nu)\), \(s_3 = s_3(\omega, I, T, \alpha, M, \nu, \lambda)\) and \(C = C(\omega, I, T, \alpha, M, \nu, \lambda)\) such that

\[
\mathbb{E} \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu t} \, dx \, dt + \mathbb{E} \int_{Q_T} s\xi |x + \varepsilon|^\alpha |z_x|^2 e^{2\nu t} \, dx \, dt \\
\leq C \mathbb{E} \int_{Q_T} s^2 \xi^2 |z|^2 e^{2\nu t} \, dx \, dt + C \mathbb{E} \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu t} \, dx \, dt \tag{4.4}
\]

for all \(\lambda \geq \lambda_3\), \(s \geq s_3\) and all \(z \in \mathcal{H}_1^4\) satisfying (4.3).
In order to prove theorem 4.2, we consider the following controlled forward stochastic parabolic equation:

\[
\begin{aligned}
&\begin{cases}
    dw - ((x + \varepsilon)^{\alpha}w) \, dt = \left( s^3 \xi^3 e^{2\varphi} + h \mathbf{1}_{\omega} \right) \, dt + HdB(t), & (x, t) \in Q_T, \\
    w(0, t) = w(1, t) = 0, & t \in (0, T), \\
    w(x, 0) = 0, & x \in I,
\end{cases}
\end{aligned}
\]

where \((h, H) \in L^2_2(0, T; L^2(\omega)) \times L^2_2(0, T; L^2(I))\) is a pair control. Then, we have the following controllability result, whose proof will be put in the appendix A.

**Lemma 4.1.** Let \(\alpha \in (0, \frac{1}{2})\). Then for any \(\varepsilon \in (0, 1)\), there exists a pair of controls \((h, H) \in L^2_2(0, T; L^2(\omega)) \times L^2_2(0, T; L^2(I))\) such that \((4.5)\) admits a solution \(w \in H^1\) corresponding to \((h, H)\) satisfying \(w(x, T) = 0\) in \(I\), P-a.s. Moreover, there exists a positive constant \(C = C(\omega, I, T, \alpha, M)\) such that

\[
\begin{aligned}
&\mathbb{E} \int_{Q_T} |w|^2 e^{-2\varphi} \, dx \, dt + \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2}(x + \varepsilon)^{\alpha} |w_x|^2 e^{-2\varphi} \, dx \, dt \\
&\quad + \mathbb{E} \int_{Q_T} s^{-3} \xi^{-3} |h|^2 e^{-2\varphi} \, dx \, dt + \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} |H|^2 e^{-2\varphi} \, dx \, dt \\
&\leq C \mathbb{E} \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\varphi} \, dx \, dt. \tag{4.6}
\end{aligned}
\]

Now we prove theorem 4.2.

**Proof of theorem 4.2.** By lemma 4.1, we know that there exists a pair of controls \((h, H)\) such that the solution \(w\) of \((4.5)\) corresponding to \((h, H)\) satisfies \(w(x, T) = 0\) in \(I\), P-a.s. Then by using Itô formula and integrating by parts, we obtain the following duality between \(w\) and \(z\):

\[
\begin{aligned}
&\mathbb{E} \int_{Q_T} \left( s^3 \xi^3 |z|^2 e^{2\varphi} + zh|\omega| \right) \, dx \, dt \\
&= \mathbb{E} \int_{Q_T} (azw) \, dx \, dt + \mathbb{E} \int_{Q_T} (bzw + cZw) \, dx \, dt - \mathbb{E} \int_{Q_T} ZH \, dx \, dt.
\end{aligned}
\]

By Young’s inequality, we further find that

\[
\begin{aligned}
&\mathbb{E} \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\varphi} \, dx \, dt \\
&\leq \epsilon \mathbb{E} \int_{Q_T} s^{-3} \xi^{-3} |h|^2 e^{-2\varphi} \, dx \, dt + \epsilon \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2}(x + \varepsilon)^{\alpha} |w_x|^2 e^{-2\varphi} \, dx \, dt \\
&\quad + \epsilon \mathbb{E} \int_{Q_T} |w|^2 e^{-2\varphi} \, dx \, dt + \epsilon \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} |H|^2 e^{-2\varphi} \, dx \, dt \\
&\quad + C(\epsilon) \mathbb{E} \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\varphi} \, dx \, dt + C(\epsilon) \mathbb{E} \int_{Q_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |z|^2 e^{2\varphi} \, dx \, dt \\
&\quad + C(\epsilon) \mathbb{E} \int_{Q_T} |z|^2 e^{2\varphi} \, dx \, dt + C(\epsilon) \mathbb{E} \int_{Q_T} s^2 \xi^2 |Z|^2 e^{2\varphi} \, dx \, dt. \tag{4.7}
\end{aligned}
\]
Substituting (4.6) into (4.7) and choosing \( \epsilon \) sufficiently small, we obtain

\[
E \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt \leq CE \int_{Q_T} s^2 \xi^2 (x + \epsilon)^{-\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt + CE \int_{Q_T} s^2 \xi^2 |Z|^2 e^{2\nu \epsilon} \, dx \, dt \\
+ CE \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt.
\]  
(4.8)

Now we estimate \( \mathbb{E} \int_{Q_T} s \xi (x + \epsilon)^\alpha |z| |z|^2 e^{2\nu \epsilon} \, dx \, dt \). To do this, we use Itô formula again and the equation of \( z \) to obtain

\[
2E \int_{Q_T} s \xi (x + \epsilon)^\alpha |z| |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
= -E \int_{Q_T} s \xi (x + \epsilon)^\alpha |z|^2 e^{2\nu \epsilon} \, dx \, dt - 2E \int_{Q_T} s \xi (x + \epsilon)^\alpha (e^{2\nu \epsilon})_z z \, dx \, dt \\
+ 2E \int_{Q_T} s \xi a \xi (x + \epsilon)^\alpha \, dx \, dt + 2E \int_{Q_T} s \xi (b z + c Z) e^{2\nu \epsilon} \, dx \, dt - \mathbb{E} \int_{Q_T} s \xi |Z|^2 e^{2\nu \epsilon} \, dx \, dt \\
\leq CE \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt + E \int_{Q_T} s \xi (x + \epsilon)^\alpha |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
+ CE \int_{Q_T} s \xi (x + \epsilon)^{-\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt + CE \int_{Q_T} s \xi |Z|^2 e^{2\nu \epsilon} \, dx \, dt,
\]

which implies

\[
E \int_{Q_T} s \xi (x + \epsilon)^\alpha |z| |z|^2 e^{2\nu \epsilon} \, dx \, dt \leq CE \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt + CE \int_{Q_T} s \xi (x + \epsilon)^{-\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
+ CE \int_{Q_T} s \xi |Z|^2 e^{2\nu \epsilon} \, dx \, dt.
\]  
(4.9)

Combining (4.8) and (4.9) yields that

\[
E \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt + E \int_{Q_T} s \xi (x + \epsilon)^\alpha |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
\leq CE \int_{Q_T} s^2 \xi^2 (x + \epsilon)^{-\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt + CE \int_{Q_T} s^2 \xi^2 |Z|^2 e^{2\nu \epsilon} \, dx \, dt \\
+ CE \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt.
\]  
(4.10)

Applying Young’s equality and Hardy–Poincaré inequality (3.7), we obtain

\[
E \int_{Q_T} s^2 \xi^2 (x + \epsilon)^{-\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
\leq E \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt + CE \int_{Q_T} s \xi (x + \epsilon)^{-2\alpha} |z|^2 e^{2\nu \epsilon} \, dx \, dt \\
\leq E \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \epsilon} \, dx \, dt + C(\epsilon) \int_{Q_T} s \xi (x + \epsilon)^2 (s^2 |z|^2 + s^3 \xi^3 |z|^2) e^{2\nu \epsilon} \, dx \, dt.
\]
(4.11)
Choosing $\epsilon$ sufficiently small and substituting (4.11) into (4.10), we find that
\[
\mathbb{E} \int_{Q_T} s \xi^3 |z|^2 e^{2w} \, dx \, dt + \mathbb{E} \int_{Q_T} s \xi(x + \epsilon)^n |\tilde{z}|^2 e^{2w} \, dx \, dt \\
\leq C \mathbb{E} \int_{Q_T} s \xi^2 |Z|^2 e^{2w} \, dx \, dt + C \mathbb{E} \int_{-T}^T s \xi^3 |z|^2 e^{2w} \, dx \, dt \\
+ C \mathbb{E} \int_{Q_T} (x + \epsilon)^{2-2n} (s\xi|\tilde{z}|^2 + s\xi^3|z|^2) e^{2w} \, dx \, dt. \tag{4.12}
\]

The remainder of the proof is to eliminate the last term on the right-hand side of (4.12). In order to overcome the degeneracy in this term, we transfer to consider the equation of $z$ in a interval outside of $x = 0$. For some given $0 < \nu < \frac{1}{4}$, we set $I_\nu = (\nu, 1)$ and $Q_{\nu,T} = I_\nu \times (0, T)$. Further we introduce a cut-off function $\rho \in C^2(\bar{I})$ such that $0 \leq \rho(x) \leq 1$ for $x \in I$, $\rho(x) \equiv 1$ for $x \in (3\nu, 1)$ and $\rho(x) \equiv 0$ for $x \in (0, 2\nu)$. Additionally, we choose a weight function $\tilde{\varphi}$ such that
\[
\tilde{\varphi}(x, t) = e^{\tilde{\eta}(x) \tilde{t}} - e^{2\lambda \tilde{M}} \frac{t}{(T - \tilde{t})^2}, \quad (x, t) \in Q_{\nu,T},
\]
where $\tilde{\eta} \in C^2(\bar{I})$ satisfies $\tilde{\eta} > 0$ in $I_\nu$ and
\[
\tilde{\eta}(x) = \begin{cases} 
0, & x = \nu, \\
\eta(x), & x \in (\nu, 2\nu), \\
\eta_1(x), & x \in (2\nu, 1), \\
\eta_2(x), & x \in (1, 2), \\
\eta_3(x), & x \in (2, 1).
\end{cases} \tag{4.13}
\]
Then we easily see that

\[
\tilde{\eta} > 0, \quad x \in I_\nu, \quad \tilde{\eta}(\nu) = \tilde{\eta}(1) = 0 \quad \text{and} \quad |\tilde{\eta}_x(x)| > 0, \quad x \in \bar{I}_\nu \setminus \{0,1\},
\]

and
\[
\tilde{\varphi}(x, t) = \varphi(x, t), \quad (x, t) \in (2\nu, 1) \times (0, T). \tag{4.14}
\]

Letting $\tilde{z} = z \rho$, we now consider
\[
\begin{cases} 
\mathrm{d}\tilde{z} + ((x + \epsilon)^n \tilde{z}_x) \mathrm{d}t = \left((a\tilde{z}_x - b\tilde{z} - c\tilde{Z} + \tilde{f}) + \tilde{Z} \mathrm{d}B(t)\right), & (x, t) \in Q_{\nu,T}, \\
\chi(x, t) = \chi(1, t) = 0, & t \in (0, T), \\
\chi(x, T) = \chi_1(x), & x \in I_\nu,
\end{cases}
\tag{4.15}
\]
where
\[
\tilde{f} = ((x + \epsilon)^n \rho_x z)_x + (x + \epsilon)^n \rho_x z \mathcal{Z}_x - a \rho_x z, \quad \tilde{Z} = \rho \mathcal{Z}.
\]
By the Carleman estimate for stochastic nondegenerate parabolic equation, e.g. theorem 6.1 in [15], we obtain that there exists a constant $C$ depending on $\omega, I, T, \alpha$ and $\nu$, but independent of $\varepsilon$ such that

$$
\mathbb{E} \int_{Q_{0,T}} s^2 |z_1|^2 e^{2s^2} \, dx \, dt + \mathbb{E} \int_{Q_{0,T}} s^3 \xi_1^3 |z|^2 e^{2s^2} \, dx \, dt \\
\leq C(\nu) \mathbb{E} \int_{0,T} (|\bar{Z}|^2 + |\bar{Z}|^2) e^{2s^2} \, dx \, dt + C(\nu) \mathbb{E} \int_{Q_{0,T}} s^2 \xi_1^2 |\bar{Z}|^2 e^{2s^2} \, dx \, dt \\
+ C(\nu) \mathbb{E} \int_{Q_{0,T}} s^3 \xi_1^3 |z|^2 e^{2s^2} \, dx \, dt
$$

Using the definition of $\rho$ and (4.14), we further obtain

$$
\mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} s^2 |z_1|^2 e^{2s^2} \, dx \, dt + \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} s^3 \xi_1^3 |z|^2 e^{2s^2} \, dx \, dt \\
\leq C(\nu) \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (|z|^2 + |z_1|^2) e^{2s^2} \, dx \, dt + C(\nu) \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} s^2 \xi_1^2 |Z|^2 e^{2s^2} \, dx \, dt \\
+ C(\nu) \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} s^3 \xi_1^3 |z|^2 e^{2s^2} \, dx \, dt. 
$$

(4.16)

On the other hand, we easily obtain that

$$
\mathbb{E} \int_{Q_T} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt \\
\leq \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt \\
+ \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt \\
\leq (4\nu)^{2-3\alpha} \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt \\
+ 2^{2-2\alpha} \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2) e^{2s^2} \, dx \, dt
$$

(4.17)

for $\nu \in (0, \frac{1}{2})$ and $\varepsilon \in (0, \nu)$. Then from (4.16) and (4.17) it follows that

$$
\mathbb{E} \int_{Q_T} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt \\
\leq (4\nu)^{2-3\alpha} \mathbb{E} \int_{0,T} \int_{\mathbb{R}^3} (x + \varepsilon)^{2-2\alpha} \left( s^2 |z_1|^2 + s^3 \xi_1^3 |z|^2 \right) e^{2s^2} \, dx \, dt
$$
By substituting (4.18) into (4.12), we obtain

\[ E \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \tau} \, dx \, dt + E \int_{Q_T} s \xi (x + \varepsilon)^9 |z|_T^2 e^{2\nu \tau} \, dx \, dt \]
\[ \leq C(\nu) \int_{Q_T} s^3 \xi^2 |z|^2 e^{2\nu \tau} \, dx \, dt + C(\nu) \int_{Q_T} s^3 \xi^3 |z|^2 e^{2\nu \tau} \, dx \, dt \]
\[ + (4\nu)^{2-3\alpha} \, C \, E \int_{Q_T} \left( s \xi (x + \varepsilon)^9 |z|_T^2 + s^3 \xi^3 |z|^2 \right) e^{2\nu \tau} \, dx \, dt \]
\[ + C(\nu) \int_{Q_T} |z|^2 e^{2\nu \tau} \, dx \, dt + \frac{C(\nu)}{(2\nu)^{\alpha}} \int_{Q_T} (x + \varepsilon)^9 |z|_T^2 e^{2\nu \tau} \, dx \, dt. \]  

(4.19)

Finally, choosing \( \nu \) sufficiently small such that \( (4\nu)^{2-3\alpha} C \leq \frac{1}{4} \) and then \( \varepsilon \) sufficiently large such that \( \frac{1}{\varepsilon} \min_{[0,T]} \xi > \max \left\{ C(\nu) \left( \frac{c(\nu)}{(2\nu)^{\alpha}} \right) \right\} \), we can absorb the last three terms on the right-hand side of (4.19) and then obtain (4.4). This completes the proof of theorem 4.2.

4.2. Proof of theorem 4.1

In this section, we will show the null controllability result for system (1.3), i.e. theorem 4.1. To do this, we first prove the following observation inequality.

**Lemma 4.2.** Let \( \alpha \in (0, \frac{1}{2}) \), \( a, b, c \in L^\infty(0, T; L^\infty(I)) \) and \( z \in L^2(\Omega, F_t; \mathbb{P}; L^2(I)) \). Then for any \( \varepsilon \in (0, \nu) \) with a sufficiently small \( \nu > 0 \), there exist positive constant \( C = C(\omega, I, T, \alpha, M, \nu) \) such that the solution \( z \) of the adjoint system (4.3) satisfies

\[ E \int_I |z|^2(x, 0) \, dx \leq C \int_{\Omega_T} |Z|^2 \, dx + C \int_I |z|^2 \, dx. \]  

(4.20)

**Proof.** By Itô formula, we obtain for \( 0 \leq \tau < \bar{\tau} \leq T \) that

\[ E \int_I |z|^2(x, \tau) \, dx + 2E \int_{\tau}^{\bar{\tau}} (x + \varepsilon)^9 |z|_T^2 \, dx \, dt + E \int_{\tau}^{\bar{\tau}} |z|^2 \, dx \, dt \]
\[ = E \int_I |z|^2(x, \bar{\tau}) \, dx + 2E \int_{\tau}^{\bar{\tau}} az |z|_x \, dx \, dt + 2E \int_{\tau}^{\bar{\tau}} |b| |z|^2 \, dx \, dt \]
\[ + 2E \int_{\tau}^{\bar{\tau}} \int_I cz |z|_z \, dx \, dt. \]  

(4.21)

By the same method using in lemma 2.1 in [1], we have the following estimate

\[ E \int_{\tau}^{\bar{\tau}} \int_I a |z|_x \, dx \, dt \leq \frac{1}{2} E \int_{\tau}^{\bar{\tau}} \int_I (x + \varepsilon)^9 |z|_T^2 \, dx \, dt + C E \int_{\tau}^{\bar{\tau}} \int_I |z|^2 \, dx \, dt. \]  

(4.22)

Substituting (4.22) into (4.21) yields that

\[ E \int_I |z|^2(x, \tau) \, dx \leq E \int_I |z|^2(x, \bar{\tau}) \, dx + C E \int_{\tau}^{\bar{\tau}} \int_I |z|^2 \, dx \, dt, \quad 0 \leq \tau < \bar{\tau} \leq T. \]
Then applying Grönwall’s inequality yields that
\[ E \int I |\xi|^2(x, \tau)dx \leq e^{C(\tau - \tau_0)} E \int I |\xi|^2(x, \tau_0)dx, \quad 0 \leq \tau < \tau \leq T. \] (4.23)

Letting \( \tau = 0 \) and integrating over \( \left[ \frac{T}{3}, \frac{2T}{3} \right] \) with respect to \( \tau \), we find that
\[ \frac{T}{3} E \int I |\xi|^2(x, 0)dx \leq CE \int_{\frac{T}{4}}^{\frac{2T}{3}} \int I |\xi|^2dxdt. \] (4.24)

On the other hand, by theorem 4.2 we obtain
\[
E \int_{Q_T} s^3 \xi^3 |\xi|^2 e^{2s^2} dxdt + E \int_{Q_T} s^3 (x + \varepsilon)|\xi|^2 e^{2s^2} dxdt 
\leq CE \int_{Q_T} s^3 \xi^2 |\xi|^2 e^{2s^2} dxdt + CE \int_{\omega_T} s^3 \xi^3 |\xi|^2 e^{2s^2} dxdt
\]
for all \( \lambda \geq \lambda_3, s \geq s_3 \). We fix \( \lambda = \lambda_3 \) and \( s = s_3 \). By
\[
\xi^3 e^{2s^2} \geq \left( \frac{4}{T^2} \right)^6 \exp \left( 2s \left( \frac{9}{2T^2} \right)^2 e^{-2sM} \right), \quad t \in \left[ \frac{T}{3}, \frac{2T}{3} \right],
\]
we further have
\[
E \int_{\frac{T}{4}}^{\frac{2T}{3}} \int I |\xi|^2 dxdt \leq C(\lambda_3, s_3) E \int_{Q_T} \xi^3 |\xi|^2 e^{2s^2} dxdt + C(\lambda_3, s_3) E \int_{\omega_T} \xi^3 |\xi|^2 e^{2s^2} dxdt.
\] (4.25)

Since \( \max_{(x, t) \in Q_T} \xi^3(t) e^{2(t-L, t)} < \infty \), we deduce from (4.25) that
\[
E \int_{\frac{T}{4}}^{\frac{2T}{3}} \int I |\xi|^2 dxdt \leq CE \int_{Q_T} |\xi|^2 dxdt + CE \int_{\omega_T} |\xi|^2 dxdt.
\] (4.26)

Finally, we obtain the desired estimate (4.20) from (4.24) and (4.26) and then complete the proof of lemma 4.2.

Now we prove theorem 4.1

**Proof of Theorem 4.1.** The proof is based on a classical dual argument and an approximate method. We introduce a linear subspace of \( L^2_F(0, T; L^2(\omega)) \times L^2_F(0, T; L^2(I)) \):
\[ X = \{ (\zeta, Z) \mid (\zeta, Z) \text{ solves the system (4.3) with some } \zeta^T \in L^2(\Omega, F_T, P; L^2(I)) \} \]
endowed with the norm
\[ \| (\zeta, Z) \|_X^2 = \int_{\omega_T} |\zeta|^2 dxdt + \int_{Q_T} |Z|^2 dxdt. \]

We further define a linear functional on \( X \) as follows:
\[ \mathcal{L}(\zeta, Z) = -E \int_I \zeta^T(x, 0)\zeta(x, 0)dx. \]
By lemma 4.2, we see that for any $\varepsilon \in (0, \nu)$, there exists constant $C$ independent of $\varepsilon$ such that
\[
|\mathcal{L}(z, Z)| \leq \left( E \int_{\omega} |y^\varepsilon(x, 0)|^2 \, dx \right)^{\frac{1}{2}} \left( E \int_{\omega} |z(x, 0)|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( E \int_{\omega} |y^\varepsilon(x, 0)|^2 \, dx \right)^{\frac{1}{2}} \|(z, Z)\|_X,
\]
which means that $\mathcal{L}$ is a bounded linear functional on $X$. We can extend $\mathcal{L}$ to be a bounded linear functional on $L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(I))$ and use the same notation for this extension.

Now by Riesz representation, we know that for any $\varepsilon \in (0, \nu)$, there exists a unique pair of controls $(g^\varepsilon, G^\varepsilon) \in L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(I))$ such that
\[
-E \int_{\omega} y_0^\varepsilon(x) z(x, 0) \, dx = \int_{\omega} g^\varepsilon z \, dx + \int_{QT} G^\varepsilon Z \, dxdt,
\]
and
\[
\|(g^\varepsilon, G^\varepsilon)\|_X \leq C \left( E \int_{\omega} |y_0^\varepsilon(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]
By the duality between $z$ and $y^\varepsilon$,
\[
E \int_{\omega} y^\varepsilon(x, T) z(x, T) \, dx - E \int_{\omega} y_0^\varepsilon(x) z(x, 0) \, dx = E \int_{QT} (g^\varepsilon 1 + G^\varepsilon Z) \, dxdt,
\]
we see that for any $\varepsilon \in (0, \nu)$, there exists a pair of controls $(g^\varepsilon, G^\varepsilon) \in L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(I))$ such that $y^\varepsilon(x, T) = 0$, $\mathbb{P}$-a.s. Since the equation (4.1) is linear, we could further obtain that \( \{(g^\varepsilon, G^\varepsilon)\} \) is Cauchy sequence such that
\[
\|(g^{\varepsilon_1} - g^{\varepsilon_2}, G^{\varepsilon_1} - G^{\varepsilon_2})\|_X \leq C \left( E \int_{\omega} |(y_0^{\varepsilon_1} - y_0^{\varepsilon_2})(x)|^2 \, dx \right)^{\frac{1}{2}}
\]
for any $\varepsilon_1, \varepsilon_2 \in (0, \nu)$. Notice that the constant $C$ in (4.27) is independent of $\varepsilon$. Therefore together with $y_0^\varepsilon \to y_0$ in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I))$, letting $\varepsilon \to 0$, we obtain a control $(g, G) \in L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(I))$ that drives the corresponding solution $y$ to zero at time $T$.

This completes the proof of theorem 4.1.

\section{5. Stability for inverse problem}

In this section, we apply Carleman estimate (3.42) to prove the Lipschitz stability for our inverse random source problem, i.e. the following theorem 5.1.

\textbf{Theorem 5.1.} Let $\alpha \in (0, 2)$, $r \in L_p^2(0, T; W^{1,\infty}(I))$ such that $|r(x, t)| \geq r_0 > 0$ for $(x, t) \in Q_T$, $\mathbb{P}$ – a.s., $h^{(i)} \in L^2(0, T)$ for $i = 1, 2$. Then there exists a positive constant $C = C(\omega, I, T, \alpha, r_0)$ such that
\[
\|h^{(1)} - h^{(2)}\|_{L^2(0, T)} \leq C \left( \|y^{(1)} - y^{(2)}\|_{L^2(0, T; L^2(\omega))} + \|y^{(1)}(\cdot, T) - y^{(2)}(\cdot, T)\|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))} \right),
\]
(5.1)
where $y^{(i)}$ is the solutions to (1.4) corresponding to $h^{(i)}$ for $i = 1, 2$, respectively.

**Proof.** Letting $\tilde{y} = y^{(1)} - y^{(2)}$ and $\tilde{h} = h^{(1)} - h^{(2)}$, we have

\[
\begin{cases}
d\tilde{y} = (x^n \tilde{y}_x)_{t} \, dt = \tilde{h}(t) \, r(x, t) \, dB(t), & (x, t) \in Q_T, \\
\tilde{y}(1, t) = 0, & t \in (0, T), \\
\text{and} & \\
\tilde{y}(0, t) = 0 & \text{for } \alpha \in (0, 1), \quad t \in (0, T), \\
\tilde{y}(x, 0) = 0 & x \in I.
\end{cases}
\]

Then applying theorem 3.4 to $\tilde{y}$, we obtain

\[
\mathbb{E} \int_{Q_T} s \lambda \Phi |\tilde{h}|^2 e^{2 \Phi} \, dx \, dt 
\leq C \mathbb{E} \int_{Q_T} s \Phi |\tilde{h}|^2 e^{2 \Phi} \, dx \, dt + C(\lambda) \mathbb{E} \int_{\omega_T} s^3 \Phi^3 |\tilde{y}|^2 e^{2 \Phi} \, dx \, dt 
+ C(\lambda)s^2 e^{C(\lambda)\|\tilde{y}\|_{L^2(\Omega; L^2(\omega_T))}} 
\leq C \mathbb{E} \int_{Q_T} s \Phi |\tilde{h}|^2 e^{2 \Phi} \, dx \, dt + C(\lambda) \mathbb{E} \int_{\omega_T} s^3 \Phi^3 |\tilde{y}|^2 e^{2 \Phi} \, dx \, dt 
+ C(\lambda)s^2 e^{C(\lambda)\|\tilde{y}\|_{L^2(\Omega; L^2(\omega_T))}} 
\leq C(\lambda) \mathbb{E} \int_{\omega_T} s^3 \Phi^3 |\tilde{y}|^2 e^{2 \Phi} \, dx \, dt + C(\lambda)s^2 e^{C(\lambda)\|\tilde{y}\|_{L^2(\Omega; L^2(\omega_T))}}.
\]

By means of $|r(x, t)| \geq r_0 > 0$ for $(x, t) \in Q_T$, $\mathbb{P}$ - a.s. and choosing $\lambda$ sufficiently large to absorb the first term on the right-hand side of (5.2), we have

\[
\mathbb{E} \int_{Q_T} s \lambda \Phi |\tilde{h}|^2 e^{2 \Phi} \, dx \, dt 
\leq C(\lambda) \mathbb{E} \int_{\omega_T} s^3 \Phi^3 |\tilde{y}|^2 e^{2 \Phi} \, dx \, dt + C(\lambda)s^2 e^{C(\lambda)\|\tilde{y}\|_{L^2(\Omega; L^2(\omega_T))}}.
\]

Finally, using $0 < \Phi e^{2\Phi} < C(\lambda, s)$ due to the regular weight function, we deduce (5.1) from (5.3) and complete the proof of theorem 5.1.

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**Appendix A**

Here, we prove lemmas 3.2 and 4.1.

**Proof of lemma 3.2.** Let $\rho_1 \in C^2(I)$ be a cut-function such that $0 \leq \rho_1(x) \leq 1$ for $x \in I$, $\rho_1(x) \equiv 1$ for $x \in \omega^{(1)}$ and $\rho_1(x) \equiv 0$ for $x \in \omega \setminus \omega^{(1)}$. By using Itô formula, $(du^r)^2 = F_t^2 \, dt$ and the equation of $u^r$, we have
Here we have used and study our null controllability. So that we list a detailed process here. Those papers, here we need that the estimate (4.6) is not depending on $\varepsilon$ of lemma 3.2.

Then integrating both side of (A.1) in $Q_T$ and taking mathematical expectation in $\Omega$, we find

$$2\mathbb{E}\int_{Q_T} \rho_1 \xi(x + \varepsilon)^n |u'|^2 e^{2\varepsilon} \, dx \, dt$$

$$= 2\mathbb{E}\int_{Q_T} \xi \left(\rho_1 e^{2\varepsilon} \right)_x (x + \varepsilon)^n |u'|^2 \, dx \, dt - 2\mathbb{E}\int_{Q_T} \rho_1 \xi u' e^{2\varepsilon} \, dx \, dt$$

$$- \mathbb{E}\int_{Q_T} \rho_1 (2s \xi \varphi_t + \xi_t) |u'|^2 e^{2\varepsilon} \, dx \, dt - \mathbb{E}\int_{Q_T} \rho_1 |F_1|^2 e^{2\varepsilon} \, dx \, dt$$

$$\leq C(\lambda) \mathbb{E}\int_{Q_T} s^2 \xi^3 |u'|^2 e^{2\varepsilon} \, dx \, dt + CE\int_{Q_T} s^{-2} |f_1|^2 e^{2\varepsilon} \, dx \, dt$$

$$+ CE\int_{Q_T} \xi |F_1|^2 e^{2\varepsilon} \, dx \, dr. \quad (A.2)$$

Here we have used $\left|\left(\rho_1 e^{2\varepsilon}\right)_x (x + \varepsilon)^n\right| \leq C(\lambda)s^2 \xi^2 e^{2\varepsilon}$ in $\omega^{(1)}$, where $C$ is independent of $\varepsilon$. Noting that $\rho_1 \equiv 1$ in $\omega^{(1)}$, we immediately deduce (3.8) from (A.2) and complete the proof of lemma 3.2. \hfill $\square$

Now we prove lemma 4.1, whose proof is similar to the one in [35] or [23]. Different from those papers, here we need that the estimate (4.6) is not depending on $\varepsilon$, which is important to study our null controllability. So that we list a detailed process here.

**Proof of lemma 4.1.** As [23], for any $\tau > 0$ we set

$$\varphi_t(x, t) = \frac{\chi(x) \psi_1(x) + (1 - \chi(x)) \psi_2(x)}{(t + \tau)^2(T - t + \tau)^2}$$

and

$$U = \left\{ (h, H) \mid \mathbb{E}\int_{Q_T} s^{-3} \xi^{-1} |h|^2 e^{-2\varepsilon} \, dx \, dt + \mathbb{E}\int_{Q_T} s^{-2} \xi^{-2} |H|^2 e^{-2\varepsilon} \, dx \, dt < \infty \right\}.$$

Then we consider the following constrained extremal problem:

$$\mathcal{J} = \frac{1}{2} \min_{(h, H) \in U} \left( \mathbb{E}\int_{Q_T} s^{-3} \xi^{-1} |h|^2 e^{-2\varepsilon} \, dx \, dt + \mathbb{E}\int_{Q_T} s^{-2} \xi^{-2} |H|^2 e^{-2\varepsilon} \, dx \, dt + \mathbb{E}\int_{Q_T} |w|^2 e^{-2\varepsilon} \, dx + \frac{1}{\tau} \mathbb{E}\int_{J} |w(x, T)|^2 \, dx \right), \quad (A.3)$$

where $w$ is the solution of (4.5) corresponding to $(h, H)$. By the variational method in [40], we see that for any given $\tau$, the control problem (A.3) admits a unique optimal solution $(h_\tau, H_\tau) \in U$ such that

$$h_\tau = -s^3 \xi^3 p_\tau e^{2\varepsilon}, \quad H_\tau = -s^2 \xi^2 p_\tau e^{2\varepsilon} \quad (A.4)$$
where \((p_\tau, P_\tau)\) is the solution of the following backward stochastic equation:

\[
\begin{aligned}
dp_\tau + (2\alpha + \epsilon) dp_{\tau,x} \, dt &= -w_\tau e^{-2\alpha x} \, dt + P_\tau \, dB(t), \\
p_\tau(0, t) &= p_\tau(1, t) = 0, \\
p_\tau(x, T) &= \frac{1}{\tau} w_\tau(x, T),
\end{aligned}
\tag{A.5}
\]

where \(w_\tau \in \mathcal{H}\) is the solution of \((4.5)\) corresponding to \((h_\tau, H_\tau)\).

Now we prove a uniform estimate for \((w_\tau, h_\tau, H_\tau)\) in \(\tau\) and \(\epsilon\). By Itô formula, \((4.5), (A.4), (A.5)\) and Young’s inequality, we find that

\[
\begin{aligned}
\mathbb{E} \int_I p_\tau(x, T) w_\tau(x, T) \, dx &= \mathbb{E} \int_I p_\tau(x, 0) w_\tau(x, 0) \, dx \\
+ \mathbb{E} \int_Q p_\tau \left[ (\alpha + \epsilon)^\alpha w_{\tau,x} \right] \, dx + \left( s^3 \xi^3 e^{2\alpha x} + h_\tau, 1_\omega \right) \, dx \\
+ \mathbb{E} \int_Q w_\tau \left[ - (\alpha + \epsilon)^\alpha p_{\tau,x} \right] \, dx - w_\tau e^{-2\alpha x} \, dx + \mathbb{E} \int_Q P_\tau H_\tau \, dx \, dt \\
\leq \epsilon \mathbb{E} \int_Q s^3 \xi^3 |p_\tau|^2 e^{2\alpha x} \, dx + C(\epsilon) \mathbb{E} \int_Q s^3 \xi^3 |\xi|^2 e^{2\alpha x} \, dx \\
- \mathbb{E} \int_Q |w_\tau|^2 e^{-2\alpha x} \, dx - \mathbb{E} \int_Q s^2 \xi^2 |p_\tau|^2 e^{2\alpha x} \, dx. 
\end{aligned}
\tag{A.6}
\]

On the other hand, since \(\alpha \in (0, 1), \) we can choose \(\beta = \frac{2\alpha + 2}{3\beta - 4} \in (1, 2 - \alpha)\) such that \(2\alpha + 3\beta - 4 = 0.\) Then applying theorem 3.1 to \(p_\tau\), yields that

\[
\mathbb{E} \int_Q s^3 \xi^3 |p_\tau|^2 e^{2\alpha x} \, dx \leq C \mathbb{E} \int_Q |w_\tau|^2 e^{-4\alpha x + 2\alpha x} \, dx + C \mathbb{E} \int_Q s^2 \xi^3 |p_\tau|^2 e^{2\alpha x} \, dx \\
+ C \mathbb{E} \int_Q s^2 \xi^3 |\xi|^2 e^{2\alpha x} \, dx. 
\tag{A.7}
\]

Together with \(\varphi_\tau \geq \varphi,\) we deduce from \((A.6)\) and \((A.7)\) that

\[
\begin{aligned}
\frac{1}{\tau} \mathbb{E} \int_I |w_\tau|^2(x, T) \, dx + \mathbb{E} \int_Q |w_\tau|^2 e^{-2\alpha x} \, dx \\
+ E \int_{\tau T} s^{-3} \xi^{-3} |h_\tau|^2 e^{-2\alpha x} \, dx + E \int_{\tau T} s^{-2} \xi^{-2} |H_\tau|^2 e^{-2\alpha x} \, dx \\
\leq C \mathbb{E} \int_Q s^3 \xi^3 |\xi|^2 e^{2\alpha x} \, dx, 
\end{aligned}
\tag{A.8}
\]

if we choose \(\epsilon\) sufficiently small. Notice that

\[
\begin{aligned}
d \left( s^{-2} \xi^{-2} |w_\tau|^2 e^{-2\alpha x} \right) &= s^{-2} \left( \xi^{-2} e^{-2\alpha x} \right) |w_\tau|^2 \, dt \\
&+ 2s^{-2} \xi^{-2} w_\tau e^{-2\alpha x} \, dw_\tau + s^{-2} \xi^{-2} e^{-2\alpha x} (dw_\tau)^2. 
\end{aligned}
\tag{A.9}
\]
Integrating both side of (A.9) in \( Q_T \), taking mathematical expectation and using the equation of \( w_r \), we then obtain

\[
2\mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^n |w_{r,x}|^2 e^{-2s\varphi_s} \, dx \, dt \\
= -\mathbb{E} \int_{s^{-2} \xi^{-2} (x + \varepsilon)^n} s^{-2} \xi^{-2} e^{-2s\varphi_s} \, dx + \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} e^{-2s\varphi_s} |w_r|^2 \, dx \, dt \\
- 2\mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^n (e^{-2s\varphi_s})_x w_{r,x} \, dx \, dt + 2\mathbb{E} \int_{Q_T} s\xi w_r e^{-2s\varphi_s} \, dx \, dt \\
+ 2\mathbb{E} \int_{\omega_T} s^{-2} \xi^{-2} w_r h_r e^{-2s\varphi_s} \, dx \, dt + \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} |H_r|^2 e^{-2s\varphi_s} \, dx \, dt \\
\leq CE \int_{Q_T} |w_r|^2 e^{-2s\varphi_s} \, dx \, dt + \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^n |w_{r,x}|^2 e^{-2s\varphi_s} \, dx \, dt \\
+ CE \int_{Q_T} s^2 \xi^2 |z|^2 e^{-2s\varphi_s} + 4s \varphi_s \, dx \, dt + CE \int_{Q_T} s^{-3} \xi^{-3} |H_r|^2 e^{-2s\varphi_s} \, dx \, dt \\
+ CE \int_{Q_T} s^{-2} \xi^{-2} |H_r|^2 e^{-2s\varphi_s} \, dx \, dt. \tag{A.10}
\]

Therefore, it follows from (A.8) and (A.10) that

\[
\frac{1}{\tau} \mathbb{E} \int_{I} |w_r|^2 (x, T) \, dx + \mathbb{E} \int_{Q_T} |w_r|^2 e^{-2s\varphi_s} \, dx \, dt \\
+ \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^n |w_{r,x}|^2 e^{-2s\varphi_s} \, dx \, dt + \mathbb{E} \int_{Q_T} s^{-3} \xi^{-3} |H_r|^2 e^{-2s\varphi_s} \, dx \, dt \\
+ \mathbb{E} \int_{Q_T} s^{-2} \xi^{-2} |H_r|^2 e^{-2s\varphi_s} \, dx \, dt \\
\leq CE \int_{Q_T} s^2 \xi^2 |z|^2 e^{2s\varphi_s} \, dx \, dt, \tag{A.11}
\]

where \( C \) is independent of \( \varepsilon \) and \( \tau \). Then there exist \( w \in H^1 \) and \( (h, H) \in \mathcal{U} \) such that

\[
(w_r, h_r, H_r) \rightarrow (w, h, H) \quad \text{in} \quad H^1 \times \mathcal{U}.
\]

As [35], by letting \( \tau \rightarrow 0 \), we can obtain (4.6) and \( w(x, T) = 0 \) in \( I, \ P \text{ - a.s.} \) This completes the proof of this lemma. \( \square \)

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