On a Visibility Representation for Graphs in Three Dimensions

by

Prosenjit Bose
Hazel Everett
Sándor P. Fekete
Anna Lubiw
Henk Meijer
Kathleen Romanik
Tom Shermer
Sue Whitesides

1995

Submitted to: Journal of Graph Algorithms and Applications, 1996
Prosenjit Bose  
Departement de Math. et Inform.  
Université du Québec à Trois-Rivières  
Trois-Rivières, Québec G9A 5H7  
CANADA

Hazel Everett  
Department d’informatique  
Université du Québec à Montréal  
Montréal, Québec  
CANADA

Sándor P. Fekete  
Center for Parallel Computing  
Universität zu Köln  
D–50923 Köln  
GERMANY

Anna Lubiw  
Department of Computer Science  
University of Waterloo  
Waterloo, Ontario N2L 3G1  
CANADA

Henk Meijer  
Department of Computer Science  
Queen’s University  
Kingston, Ontario K7L 3N6  
CANADA

Kathleen Romanik  
DIMACS  
Rutgers, The State University of New Jersey  
Piscataway, NJ 08855-1179  
USA

Tom Shermer  
Department of Computer Science  
Simon Fraser University  
Vancouver, British Columbia  
CANADA

Sue Whitesides  
School of Computer Science  
McGill University  
Montréal, Québec  
CANADA
1991 Mathematics Subject Classification: 05C10, 05C75, 68R05, 68R10
Keywords: graph representation, visibility, graph drawing, 3 dimensional geometry
On a Visibility Representation for Graphs in Three Dimensions*

Prosenjit Bose‡, Hazel Everett‡, Sándor Fekete‡, Anna Lubiw§, Henk Meijer‖, Kathleen Romanik**, Tom Shermer‖,†† and Sue Whitesides‖,††

Abstract

Visibility representations of graphs map vertices to sets in Euclidean space and express edges as visibility relations between these sets. Application areas such as VLSI wire routing and circuit board layout have stimulated research on visibility representations where the sets belong to $\mathbb{R}^2$. Here, motivated by the emerging research area of graph drawing, we study a 3-dimensional visibility representation.

In this representation, each vertex of the graph maps to a closed rectangle in $\mathbb{R}^3$ such that the rectangles are disjoint, the planes determined by the rectangles are perpendicular to the $z$-axis, and the sides of the rectangles are parallel to the $x$- or $y$-axes. Edges are expressed by the following visibility relation. Two rectangles $R_i$ and $R_j$ are considered visible provided that there exists a closed cylinder $C$ of non-zero radius such that the ends of $C$ are contained in $R_i$ and $R_j$, the axis of $C$ is parallel to the $z$-axis, and $C$ does not intersect any other rectangle.

Our main results are as follows. All planar graphs have a representation, as do many non-planar graphs. In particular, a complete graph $K_n$ has a representation for values of $n \leq 20$. However, a complete graph $K_n$ does not have a representation for $n \geq 103$. (Recently, this bound has been improved to 56 by Fekete, Houle, and Whitesides, using extensions of the techniques presented here.) The complete bipartite graph $K_{m,n}$ has a representation for all $m$ and $n$. Finally, we show that the family of graphs with a representation is not closed under graph minors.

---

*Research on this paper was supported by NSERC, NSF, and DFG.
‡Université du Québec à Trois-Rivières, Canada; jit@uqtr.uquebec.ca
§Université du Québec à Montréal, Canada; everett@math.uqam.ca
‖Universität zu Köln, Germany; sandor@zpr.uni-koeln.de
¶University of Waterloo, Canada; alubiw@uwaterloo.ca
‖Queen’s University, Canada; henk@qucis.queensu.ca
**Rutgers University; romanik@dimacs.rutgers.edu
††Simon Fraser University, Canada; shermer@cs.sfu.ca
‡‡McGill University, Canada; sue@cs.mcgill.ca
1 Introduction

The problem of drawing or representing graphs has been studied extensively in the literature (see the paper of Di Battista et al. [6] for a survey of graph drawing research and its applications). In particular, determining a visibility representation of a graph, where the vertices of the graph map to disjoint sets or objects in the plane and the edges are expressed as visibility relations between these sets, has received considerable attention recently due to the large number of applications in areas such as VLSI wire routing, algorithm animation, CASE tools, and circuit board layout. However, visibility representations in 3-dimensions have been relatively unstudied. In this paper, we define a 3-dimensional visibility representation and study its properties. Earlier conference and technical report versions of this paper ([2], [3]) have motivated a large number of conference papers – see [1], [9], [10], [11], [14], [15]. Closely related questions have been examined in [5], [13]. It is one of our objectives to present a formal version of this early paper of the area.

We begin by reviewing some of the results in 2-dimensional visibility representations. To do this, we first discuss in more detail the most common types of visibility representations that have been studied.

A visibility representation is defined by specifying a class of objects to represent the vertices and the visibility relation between the objects. In two dimensions, common choices for objects are axis aligned line segments or rectangles to represent the objects. Two visibility relations that have been considered are defined as follows:

- Two objects are mutually visible or visible if and only if they can be joined by a line segment that does not intersect any other object. Often, the direction of the line segment is restricted to be axis parallel.

- Two objects are \(\epsilon\)-visible if and only if they can be joined by two distinct parallel line segments such that neither the line segments nor the region between them intersects another object. Again, the direction of the line segments is often restricted to be axis parallel.

For example, in Figure 1, line segments 1 and 2 are visible but not \(\epsilon\)-visible. Line segments 2 and 3 as well as line segments 1 and 3 are \(\epsilon\)-visible. However, line segments 1 and 3 are not vertically \(\epsilon\)-visible. Line segments 2 and 4 are not visible by either of the above notions of visibility. Although these two different types of visibility seem closely related, the restrictions imposed by \(\epsilon\)-visibility often radically change the class of graphs that admit visibility representations. See [16] for a discussion of the different definitions of visibility and the sensitivity of results to the choice of visibility definition.

Now we can describe some previous results concerning 2-dimensional visibility representations. Wismath [17] and Tamassia and Tollis [16] independently showed that every 2-connected planar graph admits a visibility representation where the
vertices are represented by closed disjoint horizontal line segments in the plane and where two vertices are adjacent if and only if their corresponding segments are vertically \( e \)-visible. (More precisely, they showed that such a representation exists for a graph if and only if the graph is planar and has an embedding with all cut vertices on the exterior face.) For the same model, Kant et al. [12] studied the visibility representations of trees. When the vertices are represented by disjoint axis-aligned rectangles in the plane and visibility is defined as \( e \)-visibility in the horizontal and vertical directions, Wismath [17] showed that every planar graph admits a visibility representation. Dean and Hutchinson [5] proved that \( K_8 \) is the largest complete graph that admits a visibility representation in this model. For a comprehensive overview of the various visibility representations studied, see [6].

Motivated by the results on 2-dimensional visibility representations such as the ones described above, we study a 3-dimensional visibility representation defined as follows. Consider an arrangement of closed, disjoint rectangles in \( \mathbb{R}^3 \) such that the planes determined by the rectangles are perpendicular to the \( z \)-axis, and the sides of the rectangles are parallel to the \( x \)- or \( y \)-axes. Two rectangles \( R_i \) and \( R_j \) are \( e \)-visible restricted to the vertical direction if and only if between the two rectangles there is a closed cylinder \( C \) of non-zero length and radius such that the ends of \( C \) are contained in \( R_i \) and \( R_j \), the axis of \( C \) is parallel to the \( z \)-axis, and the intersection of \( C \) with any other rectangle in the arrangement is empty. When such a line of sight exists, we say that the two rectangles are \( z \)-visible.

Given a graph \( G = (V, E) \), we say that \( G \) admits a visibility representation by axis parallel rectangles in \( \mathbb{R}^3 \) if and only if the following hold:

- there exists a 1-to-1 correspondence between the rectangles and the vertices of \( G \), and
- vertices \( v_i \) and \( v_j \) are adjacent if and only if their corresponding rectangles \( R_i \) and \( R_j \) are \( z \)-visible.

If a graph can be represented in this way, we say that the graph has a repres-
representation. Even though one can also think of representations by other objects, we will concentrate on the case of axis-aligned rectangles that are orthogonal to the $z$-axis; throughout this paper, this is implied by the term “representation”.

One main motivation for considering this type of 3-dimensional representation is the fact that many non-planar graphs can be represented. The rest of this paper is organized as follows: In Section 3, we show that all planar graphs have a representation. In Section 4, we show that $K_n$ has a representation for values of $n \leq 20$. On the other hand, we show in Section 5 that $K_n$ does not have a representation for $n \geq 103$. (Recently, this bound has been improved to 56, using extended techniques. See [9].) In Section 6 we demonstrate that any complete bipartite graph can be represented. Finally, we show in Section 7 that the family of graphs with representations is not closed under graph minors.

## 2 Notation and preliminaries

We begin by defining some of the conventions used in this paper. Unless stated otherwise, $\epsilon$-visibility will be restricted to the $z$-direction, the planes containing rectangles will be orthogonal to the $z$-axis and the edges of the rectangles will be parallel to the $x$- or $y$-axes. For ease of reference, all figures will be drawn as their projections into the plane $z = 0$. We will use the terms up and down to refer to increasing and decreasing $z$-coordinates, the terms above and below to refer to increasing and decreasing $y$-coordinates, and the terms right and left to refer to increasing and decreasing $x$-coordinates. Similarly, upmost and downmost refer to $z$-coordinates, top and bottom refer to $y$-coordinates, and rightmost and leftmost refer to $x$-coordinates. Without loss of generality, we may assume that all rectangles in a representation have distinct, non-negative integer $z$-coordinates.

(a) \hspace{1cm} (b)

![Figure 2: Examples of $R_i <_l R_j$ and $R_i <_r R_j$](image)

## 3 Representations for planar graphs

In this section, we show that all planar graphs have a representation. There are two main ingredients in the proof. The first is the result due independently to
Wismath [17] and to Tamassia and Tollis [16] that any 2-connected planar graph has what [16] calls an $e$-visibility representation. (Vertices correspond to closed, disjoint, horizontal line segments in the plane, and two vertices are adjacent in the graph if and only if their corresponding segments are $e$-visible in the vertical direction.) The second ingredient is the use of the third dimension to deal with cut vertices. This is similar to an idea of [17] for obtaining a visibility representation for all planar graphs by rectangles in $\mathbb{R}^2$ that have $e$-visibility in both the $x$ and $y$ directions. For ease of notation, let $P_{-\infty}$ represent the plane $z = -\infty$ and let $P_{\infty}$ represent the plane defined by $z = +\infty$.

**Theorem 3.1** Every planar graph has a representation.

**Proof:** We will prove, by induction on the number of 2-connected components, the following stronger result:

**Claim:** Let $G$ be a connected planar graph and let $v$ be a vertex of $G$. Then $G$ has a representation such that each rectangle is visible from $P_{\infty}$, but only the rectangle representing $v$ is visible from $P_{-\infty}$.

**Base case for the induction:** $G$ is 2-connected. We say a line segment is a $y$-segment if it is parallel to the $y$-axis and lies in the $y,z$-plane. By the result of Wismath [17] and Tamassia and Tollis [16], $G$ can be represented by (planar) $e$-visibility restricted to the $z$ direction of $y$-segments, with only the segment representing the vertex $v$ visible from below. Place the $y,z$-plane containing this configuration in 3-space at $x = 0$. Number the segments in order of decreasing $z$-coordinate. Expand each segment to an $x,y$-rectangle by pulling it out until its $x$-length is equal to its $z$-coordinate. The rectangles then form a “staircase” shape. Each rectangle is visible from $P_{\infty}$, and only the rectangle representing $v$ is visible from $P_{-\infty}$.

Now assume the result is true for graphs with at most $k$ 2-connected components. Let the number of 2-connected components of $G$ be $k + 1$. Let $x$ be a cut vertex of $G$, and break $G$ at $x$ into two subgraphs $G_1$ and $G_2$. (Vertex $x$ may still be a cut vertex in these subgraphs.) Suppose that $v$ lies in $G_1$. By induction $G_1$ has a representation with all rectangles visible from $P_{\infty}$ and only the rectangle representing $v$ visible from $P_{-\infty}$. Identify a rectangular area $A$ of the rectangle corresponding to the vertex $x$, such that $A$ is visible from $P_{\infty}$. By induction $G_2$ has a representation with all rectangles visible from $P_{\infty}$ and only the rectangle representing $x$ visible from $P_0$. Scale the representation for $G_2$, so that it can be placed upward of the rectangular area $A$ in the $z$-direction. After the representation of $G_2$ is in place, remove the rectangle corresponding to $x$ from the representation of $G_2$. The result is a representation of $G$; all rectangles are visible from $P_{\infty}$, and only the rectangle representing $v$ is visible from $P_{-\infty}$. This completes the proof. □
4 Representations for complete graphs

In this section, we present the construction of a representation of the complete graph on 20 vertices. First, we introduce some additional notation, used to discuss representations of cliques in this section and the next.

We may assume without loss of generality that all rectangles have distinct, non-negative integer $z$-coordinates. We use $R_i$ to denote a rectangle with $z$-coordinate $i$. When we speak of a rectangle $R_\infty$, we refer to the upmost rectangle in the collection, i.e., the rectangle with the greatest $z$-coordinate in the collection.

Our representation of $K_{20}$ consists of rectangles labeled $R_0, R_1, \ldots, R_{19}=R_\infty$. Note that we may as well assume that the extremal rectangles $R_0$ and $R_\infty$ have greater extent in the $x$- and $y$-directions than any of the intermediate rectangles $R_1, \ldots, R_{19}$, as extending the extremal rectangles does not change any of the visibility relationships.

Given the projections of two rectangles $R_i$ and $R_j$, we will write $R_i <_l R_j$ if the left edge of $R_i$ is to the left of the left edge of $R_j$, and we will write $R_i <_r R_j$ if the right edge of $R_i$ is to the left of the right edge of $R_j$ (see Figure 2). Similarly, we will say that $R_i >_t R_j$ if the top edge of $R_i$ is above the top edge of $R_j$, and we will say $R_i >_b R_j$ if the bottom edge of $R_i$ is above the bottom of $R_j$. The relations $<_l, <_r, <_t$ and $<_b$ are partial orders on the set of vertices of a given representation.

Our construction is based on two main building blocks. Each of the blocks has some special properties that are exploited in order to achieve the construction.

![Figure 3: $K_5$](image)

The first building block consists of five rectangles arranged as in Figure 3. Notice that the visibility graph of this configuration is the complete graph $K_5$ and that $R_1 >_t R_2 >_t R_3 >_t R_4 >_t R_5$, and $R_1 >_r R_2 >_r R_3 >_r R_4 >_r R_5$, and all five rectangles see $R_0$ and $R_\infty$.

The other building block is the arrangement of four rectangles shown in Figure 4. Not only is the visibility graph of this configuration the complete graph
Figure 4: $K_4$

$K_4$, but also $R_6 >_1 R_7 >_1 R_8 >_1 R_9$, and $R_6 >_b R_7 >_b R_8 >_b R_9$, and all four rectangles see $R_0$ and $R_\infty$.

Figure 5: $K_9$

A $K_9$ can be built by placing these two blocks one upon the other as in Figure 5. The right edges of the $K_5$ and the bottom edges of the $K_4$ provide the visibility needed to achieve a $K_9$. Note that this arrangement of nine rectangles, whose visibility graph is complete, satisfies the relation $R_1 >_1 R_2 >_1 R_3 \cdots >_1 R_9$, and all rectangles $R_1, \ldots, R_9$ see $R_0$ and $R_\infty$.

A $K_{18}$ is constructed as a composite of two $K_9$ configurations. One $K_9$, referred to as the “down” $K_9$, is placed as in Figure 5. The other $K_9$, referred to as the “up” $K_9$, is placed upon the down $K_9$. The up $K_9$ is an exact copy of the down one, but flipped over and rotated $90^\circ$ so that its rectangles have $z$-coordinates $10, \ldots, 18$ and the relation $R_{18} <_t R_{17} <_t R_{16} \cdots <_t R_{10}$ holds. The top edges of the down $K_9$ and the left edges of the up $K_9$ provide the visibility needed to achieve $K_{18}$. Notice that the $K_{18}$ is constructed such that all its rectangles see $R_0$ and $R_\infty$. Therefore, by adding $R_0$ and $R_\infty$, we have a representation of $K_{20}$ (see Figure 6). Hence, we have the following theorem.

**Theorem 4.1** The complete graph on 20 vertices has a representation.

Notice that all the rectangles in Figure 4 and Figure 3 can be enlarged into squares without removing any of the properties of either building block. Figure 7
Figure 6: $K_{20}$ (See Figure 5 for details of the $K_9$’s.)

Figure 7: Square building blocks.
shows the two main building blocks of squares that can be used to form a complete visibility graph of size 20. In Figure 8 we have the two building blocks placed as

in the case of rectangles, to form a $K_9$. The construction for $K_{20}$ is the same as for rectangles. Therefore, we have the following corollary.

**Corollary 4.2** The complete graph on 20 vertices has a visibility representation by isothetic squares in 3-space.

It is interesting to note that although rectangles are more flexible objects than squares, we are unable to capitalize on this to achieve better bounds for rectangles than for squares.
5 Large complete graphs without representation

In this section, we provide an upper bound on the size of the largest clique that can be represented. We show that $K_n$ does not have a representation for any $n > 102$. We will make use of the following theorem of Erdős and Szekeres [8]:

**Theorem 5.1** Given any two positive integers $j$ and $k$, any sequence of more than $jk$ distinct integers has a (not necessarily contiguous) ascending subsequence of length $j+1$ or a descending subsequence of length $k+1$.

Consider an arrangement $\mathcal{R}$ of rectangles that has a clique for its visibility graph. As before, the upmost rectangle $R_\infty$, and the downmost $R_0$, are extended until their projections on the plane $z = 0$ enclose all of the other rectangles of $\mathcal{R}$. Let $\mathcal{R}' = \mathcal{R} \setminus \{R_\infty, R_0\}$. The collection $\mathcal{R}'$ must itself form a clique, and each of its rectangles must be able to see upwards past all other elements of $\mathcal{R}'$ (in order to see $R_\infty$) and downwards past all other elements of $\mathcal{R}'$ (in order to see $R_0$).

We begin by proving two key lemmas.

**Lemma 5.2** Let $R_i$, $R_j$, $R_k$, and $R_l$ be four rectangles of $\mathcal{R}'$ with $i < j < k < l$. It cannot be the case that $R_i <_l R_j <_l R_k <_l R_l$ and that $R_l <_r R_k <_r R_j <_r R_i$.

**Proof:** Suppose the lemma is false and that there are four rectangles as described. As $R_i$ must see the downmost rectangle $R_0$, either the bottom edge of $R_j$ lies below the bottom edge of $R_i$, or the top edge of $R_j$ lies above the top edge of $R_i$, or both. Note, however, that it is impossible for both conditions to hold since then $R_k$ would not see $R_i$. We assume without loss of generality that the bottom edge of $R_j$ lies below the bottom edge of $R_i$ (see Figure 9a, where the top edge of $R_j$ is drawn below the top of $R_i$).

Similarly, $R_k$ must see $R_0$. Thus, either the top edge of $R_k$ lies above the top edges of $R_i$ and $R_j$, or the bottom edge of $R_k$ lies below the bottom edge of
Lemma 5.3 Let \( R_i, R_j, R_k, R_l, \) and \( R_m \) be five mutually visible rectangles (chosen from \( \mathcal{R} \) or not) with \( i < j < k < l < m \). It cannot be the case that \( R_i <_{r} R_j <_{l} R_k <_{l} R_l <_{l} R_m \) and that \( R_i <_{r} R_j <_{r} R_k <_{r} R_l <_{r} R_m \).

Proof: Suppose the lemma is false and that there are five rectangles as described. We consider visibilities amongst \( \{R_i, R_j, R_k, R_l, R_m\} \). We consider cases based on the layout of \( R_j \) and \( R_k \). In order for the rectangle \( R_k \) to see \( R_i \), \( R_k \) must extend either above or below \( R_j \), or both. If \( R_k \) extends both above and below \( R_j \), then \( R_j \) will not see \( R_i \) or \( R_m \) (see Figure 10a). Suppose without loss of generality that \( R_k \) extends only below \( R_j \). Then, for \( R_l \) to see \( R_i \), \( R_l \) must extend above \( R_j \) (case (I)) and/or below \( R_k \) (case (II)). In case (I), \( R_j \) is blocked from seeing \( R_m \) (see Figure 10b). In case (II), \( R_l \) has to extend above \( R_k \) in order to see \( R_j \) (as shown in Figure 10c); therefore, \( R_k \) is blocked from seeing \( R_m \). As every case has led to a contradiction, the lemma is proved.

We can now show an upper bound of 102 on the size of \( \mathcal{R} \):

Theorem 5.4 No clique of size greater than 102 admits a representation.
Proof: Suppose that there were some collection $\mathcal{R}$ of 103 (or more) rectangles that represent a clique. We remove $R_{\infty}$ and $R_{0}$ and concentrate on $\mathcal{R}'$ which has at least 101 elements.

List the $z$-coordinates of the rectangles in $\mathcal{R}'$ in increasing $<_{z}$ order. By Theorem 5.1 (with $j = k = 10$), there must be an ascending or descending subsequence of length 11. Without loss of generality, we assume that this sequence is ascending. This sequence corresponds to a collection $\mathcal{T}$ of rectangles that have the same order in both $z$-coordinate and $<_{z}$.

Let $T_{11}$ be the upmost rectangle in $\mathcal{T}$. We consider visibilities amongst $\mathcal{T} \cup \{R_{\infty}, R_{0}\}$. The rectangle $T_{11}$ could interfere only with visibility between the rectangle $R_{\infty}$ and the remainder of $\mathcal{T}$; however, $R_{\infty}$ sees every element of $\mathcal{T}$ along the left edge of that element. Thus, we can extend $T_{11}$ so that it has the rightmost edge in $\mathcal{T}$. Let $\mathcal{T}'$ be $\mathcal{T} \setminus \{T_{11}\}$; $\mathcal{T}'$ has ten elements.

List the $x$-coordinates of the right edges of the rectangles of $\mathcal{T}'$ in increasing $<_{x}$ order. By Theorem 5.1 (with $j = k = 3$), there is an ascending or descending subsequence of length four. A descending subsequence of length four corresponds to rectangles in the situation prohibited by Lemma 5.2, where the rectangles were any set of five mutually visible rectangles, not necessarily chosen from $\mathcal{R}'$. An ascending subsequence of length four may be augmented with $T_{11}$ (as its last element) to create an ascending subsequence of length five. This corresponds to rectangles in the situation prohibited by Lemma 5.3. As neither case is permitted, our assumption that some $\mathcal{R}$ has more than 102 elements is false.

There is a large gap between the upper and lower bounds. While we have constructed an explicit representation for $K_{20}$, and hence for $K_{n}$, $n \leq 20$, we have also shown that $K_{n}$ has no representation for $n \geq 103$. (In the meantime, this upper bound has been improved to 56 in [9], using extensions of our techniques.) We conjecture that $n = 20$ is close to the correct bound. The upper bound presented in this section also holds for squares. It is unclear whether the additional conditions for squares can be used to give a tighter upper bound than for general rectangles.

6 Representations for bipartite graphs

In this section, we are concerned with representations of complete bipartite graphs $K_{m,n}$. It is not hard to see that any $K_{m,n}$ admits a representation (see Figure 11).

Theorem 6.1 $K_{m,n}$ admits a representation for any $m,n$. 

12
Nonclosure under graph minors

The family of graphs with a representation is not closed under graph minors. To prove this, consider the complete bipartite graph $K_{103,103}$, which has a representation as illustrated in Figure 11. When an edge of a complete bipartite graph is contracted, it yields a vertex that is adjacent to all the vertices of the graph. If we contract the 103 edges of a perfect matching in $K_{103,103}$, then we obtain the complete graph $K_{103}$, which does not have a representation by Theorem 5.4. This gives us the following result:

**Theorem 7.1** The class of graphs with a representation is not closed under graph minors.

Conclusions and open problems

We have studied the visibility representations of graphs where each vertex of the graph maps to a closed, disjoint rectangle in $\mathbb{R}^3$ such that the planes determined by the rectangles are perpendicular to the $z$-axis, the sides of the rectangles are parallel to the $x$ or $y$ directions, and the edges correspond to $\epsilon$-visibilities restricted to the $z$-direction.

We showed that all planar graphs have a representation, $K_n$ has a representation for $n \leq 20$, and $K_n$ does not have a representation for $n \geq 103$. Concerning bipartite graphs, we showed that $K_{m,n}$ is representable for all $m$ and $n$. Finally, we showed that the family of graphs with a representation is not closed under graph minors.

Many directions for further research have been generated from this initial investigation. For example, we have shown in [3] that $K_{5,5}$ minus a perfect matching has a representation. We conjecture that $K_{5,6}$ minus a perfect matching does not have a representation. We can also show that the complete tripartite graphs $K_{m,n,2}$ and $K_{m,4,3}$ can be represented.
9 Acknowledgments

Our study of visibility representations began at Bellairs Research Institute of McGill University during the Workshop on Visibility Representations organized by S. Whitesides and J. Hutchinson, February 12-19, 1993. We are grateful to the other conference participants Joan Hutchinson, Goos Kant, Marc van Kreveld, Beppe Liotta, J. R., Steve Skiena, Roberto Tamassia, Yanni Tollis, and Godfried Toussaint for their encouragement and comments.

References

[1] H. Alt, M. Godau, and S. Whitesides. Universal 3-dimensional visibility representations for graphs. To appear in Proc. Graph Drawing ’95, Passau 1996. Lecture Notes in Computer Science, Springer-Verlag, 1996.

[2] P. Bose, H. Everett, S. P. Fekete, A. Lubiw, H. Meijer, K. Romanik, T. Shermer, and S. Whitesides. On a visibility representation for graphs in three dimensions. Proc. Graph Drawing ’93, Paris (Sèvres), 1993, pp. 38-39.

[3] P. Bose, H. Everett, S. P. Fekete, A. Lubiw, H. Meijer, K. Romanik, T. Shermer, and S. Whitesides. On a visibility representation for graphs in three dimensions. Snapshots of Computational and Discrete Geometry, 3, eds. D. Avis and P. Bose, McGill University School of Computer Science Technical Report SOCS-94.05, July 1994, pp. 2-25.

[4] R. Cohen, P. Eades, T. Lin, and F. Ruskey. Three-dimensional graph drawing. Proc. Graph Drawing ’94, Princeton NJ, 1994. Lecture Notes in Computer Science LNCS #894, Springer-Verlag, 1995, pp. 1-11.

[5] A. Dean and J. Hutchinson. Rectangle visibility representations of bipartite graphs. Proc. Graph Drawing ’94, Princeton NJ, 1994. Lecture Notes in Computer Science LNCS #894, Springer-Verlag, 1995, pp. 159-166.

[6] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis, Algorithms for automatic graph drawing: an annotated bibliography. Comput. Geometry: Theory and Applications, 4, 1994, pp. 235-282. Also available from wilma.cs.brown.edu by ftp.

[7] G. Di Battista and R. Tamassia. Algorithms for plane representations of acyclic digraphs. Theoretical Computer Science, 61, 1988, pp. 175-198.

[8] P. Erdős and A. Szekeres. A combinatorial problem in geometry. Compositio Mathematica 2, 1935, pp. 463-470.
[9] S. P. Fekete, M. E. Houle, and S. Whitesides. New results on a visibility representation of graphs in 3D. To appear in *Proc. Graph Drawing 95*, Passau 1996. Lecture Notes in Computer Science, Springer-Verlag, 1996.

[10] S. P. Fekete and H. Meijer. Rectangle and box visibility graphs in 3D. Manuscript. To be submitted to *Graph Drawing ’96*.

[11] J. Hutchinson, T. Shermer, and A. Vince. On representations of some thickness-two graphs. To appear in *Proc. Graph Drawing 95*, Passau 1996. Lecture Notes in Computer Science, Springer-Verlag, 1996.

[12] G. Kant, G. Liotta, R. Tamassia, and I. G. Tollis. Area requirements of visibility representations of trees. *Proceedings of the 5th Canadian Conf. on Comp. Geom.*, Waterloo, Ontario, 1993, pp. 192-197.

[13] E. Kranakis, D. Krizanc, and J. Urrutia. On the number of directions in visibility representations of graphs. *Proc. Graph Drawing ’94*, Princeton NJ, 1994, Lecture Notes in Computer Science LNCS #894, Springer-Verlag, 1995, pp. 167-176.

[14] K. Romanik. Directed VR-representable graphs have unbounded dimension. *Proc. Graph Drawing ’94*, Princeton, NJ, 1994, Lecture Notes in Computer Science LNCS #894, Springer-Verlag, 1995, pp. 177-181.

[15] F. J. Cobos, J. C. Dana, F. Hurtado, A. Marquez, and F. Mateos. On a visibility representation of graphs. To appear in *Proc. Graph Drawing ’95*, Passau 1996. Lecture Notes in Computer Science, Springer-Verlag, 1996.

[16] R. Tamassia and I. G. Tollis. A unified approach to visibility representations of planar graphs. *Discrete Comput. Geom.* 1, 1986, pp. 321-341.

[17] S. K. Wismath. Characterizing bar line-of-sight graphs. *Proc. ACM Symp. on Comp. Geometry*, 1985, pp. 147-152.