Dynamic model of fine-pitch gear pair considering multi-clearance coupling

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Abstract. In this paper, the parallel shaft fine-pitch involute plastic-powder metallurgy engagement spur gear pair produced according to the recommended backlash value of DIN standard is taken as research object. The coupling between the backlash and the bearing radial clearance is considered. And the expression of the dynamic backlash is derived. A three-degree-of-freedom two-state dynamic mathematical model of the parallel shaft gear transmission system is established. The time-domain curves of gear displacement, velocity, engagement force and dynamic backlash are obtained. The results of mathematical model show that a larger backlash range than DIN standard recommended is more conducive to smooth transmission.

1. Introduction
With the development of polymer materials, plastic has been playing an increasingly important role in small electrical products and precision machinery as a new type of gear material. Compared with the industrial gear mechanism research, there are two main differences in dynamics modeling study of plastic fine-pitch gear system with multiple clearances.

The first difference is that the clearance between shaft and hole of plastic fine-pitch gears will be relatively obvious. When the clearance between shaft and hole causes the radial runout, the actual center distance between the gears will change, and the size of the meshing point and gear backlash will also change.

Due to the adverse effects of clearance on the dynamic characteristics of mechanisms, the research on the dynamics of mechanisms with clearance has gradually become the focus of scholars’ study. Tian et al.[1] presented a comprehensive survey of the literature of the most relevant analytical, numerical, and experimental approaches for the kinematic and dynamic analyses of multibody mechanical systems with clearance joints. However, as a result of the simplification of dynamic model, the research is basically focused on the single clearance problem. Backlash is often concerned by scholars in gear driving system. Yi and Yang et al.[2,3] established the dynamic model considering nonlinear backlash. Shi et al.[4] analyzed the model of light load spur gear pair considering multi-state mesh with backlash. Shen et al.[5] analyzed the model of light load spur gear pair considering multi-state mesh with backlash.

The second point is that industrial gear mechanisms generally adopt oil lubrication, and it is considered that the components at the radial clearance are always under the continuous action of oil film
force. However, for plastic fine-pitch gear mechanism, grease lubrication is generally adopted. As a result, there are two states of contact and separation at the radial clearance.

The two-state model can not only describe the dynamic characteristics of the mechanism with clearance well, but also the calculation process is relatively simple. Dubowsky[5] is the most representative of its research work. Liu et al.[6] explored the interactions between bearing clearance and the backlash.

In this paper, based on the fine-pitch parallel shaft gear pair designed by a company according to DIN standard, a three-degree-of-freedom two-state dynamic model considering clearance coupling is established. On this basis, the correctness of parameters such as dynamic backlash is discussed. It is verified that the backlash value recommended by DIN standard is not suitable for this fine-pitch gear pair.

2. Dynamical mathematical model of fine-pitch gear pair
This paper takes the first gear transmission of the domestic robot gearbox produced by a certain company as the research object. In order to reflect the coupling phenomenon between radial clearance and backlash, a three-degree-of-freedom dynamic model of fine-pitch involute spur gear pair is established considering multiple clearance coupling based on Huibo Zhang’s multi-clearance coupling model [7].

2.1. Dynamic backlash model
In the inertial coordinate system $xyz$, the relative motion between the gear shaft and the bearing can be described by the clearance vector. $a_a$ and $a_b$ are the displacement vectors of the geometric center point of the gear shaft and the bearing. The radial embedding quantity $\delta$ is introduced as

$$\delta = \sqrt{a_{bxs}^2 + a_{bys}^2} - (r_b - r_s) \tag{1}$$

In Formula (1), the first half is the eccentricity between the gear shaft and the bearing, and the second half is the magnitude of radial clearance. In order to determine the current state of the shaft and sleeve of the gear, the radial clearance function $f_r(\delta)$ is established, which can be expressed as

$$f_r(\delta) = \begin{cases} 
\sqrt{a_{bxs}^2 + a_{bys}^2} - (r_b - r_s) & \delta > 0 \\
0 & \delta \leq 0 
\end{cases} \tag{2}$$

In the equation, when $\delta > 0$, the shaft and the bearing are in a state of free movement. When $\delta = 0$, they are in a critical contact state, and the contact force is zero. When $\delta < 0$, they are in the contact state, generating contact collision force.

During the engagement process, the backlash can be considered as a dynamic value, which is represented by the center distance [8]. The backlash of the external meshing gear pair is shown in the Fig. 2 $BC$ is the dynamic backlash $b_t$. According to the geometric relation of gear meshing, the three points
According to Hertz contact theory, the basic form of radial contact force can be expressed as Equation (3), \( b_0 \) is initial backlash.

\[
b_t = (\overline{AB} - AC) + b_0 = [r_{b1}(\tan \alpha' - \tan \alpha_B) - r_{b2}(\tan \alpha_C - \tan \alpha')] + b_0
\]  

(3)

According to the angle relation in the figure, it can be known that

\[
\alpha_B = \alpha' - (\Phi_{AE} - \Phi_{BE}) = \alpha' - \frac{t - s_1'}{r_1'} + (\text{inv} \alpha' - \text{inv} \alpha_B)
\]  

(4)

In the equation, \( \alpha' \) is the circular pitch of the actual engagement. \( s_i \) is the tooth thickness on pitch circle of driving gear. And \( r_i \) is the radius of driving gear pitch circle. Then Equation (5) can be derived from above

\[
\tan \alpha' - \tan \alpha_B = \frac{t - s_1'}{r_1'} = \frac{\cos \alpha_0}{\cos \alpha'} \left[ \frac{s_1'}{r_1'} - 2r_1(\text{inv} \alpha' - \text{inv} \alpha_0) \right] = \frac{\cos \alpha_0}{\cos \alpha'} \frac{r_1'}{r_1'}
\]  

(5)

After reorganization and simplification, the expression of \( AB \) can be obtained as follows

\[
\overline{AB} = r_{b1}(\tan \alpha' - \tan \alpha_B) = r_{b1} \frac{[\cos \alpha_0 + 2r_1(\text{inv} \alpha' - \text{inv} \alpha_0)]}{\cos \alpha'}
\]  

(6)

In a similar way, \( a_c \) can be expressed. By sorting out Equations (3) to (6), the expression of dynamic backlash \( b_i \) can be deduced as follows. Where \( a_0 \) is the ideal center distance, \( \alpha' \) is the actual center distance, \( b_0 \) is the initial backlash, and \( a_0 \) is the standard pressure angle.

\[
b_t = [r_{b1}(\tan \alpha' - \tan \alpha_B) - r_{b2}(\tan \alpha_C - \tan \alpha')] + b_0
\]  

(7)

2.2. Multiple clearance coupling dynamics model

In the global coordinate system \( x_1y_2z \), \( x_1 \) and \( x_2 \) are the displacements of the gears along the \( x \) direction, \( y_1 \) and \( y_2 \) are the displacements along the \( y \) direction in a similar way, \( \theta_1 \) and \( \theta_2 \) are the rotation angles of the gears around the \( z \) axis. \( r_1 \) and \( r_2 \) are the pitch radius of the driving and driven gear. And \( \alpha'' \) is the pressure angle. Thus, the relative displacement \( s_r \) of the gears in the direction of the path of contact can be expressed as

\[
s_r = r_1\theta_1 - r_2\theta_2 + (x_1 - x_2) \sin \alpha' + (y_1 - y_2) \cos \alpha' + e
\]  

(8)

When the gear teeth are engaged, the gear meshing pair will show the contact-detach-contact impact phenomenon as a result of the backlash’s influence. According to the meaning of backlash and its specific manifestation, backlash is generally expressed as piecewise function \( f_i(s_r) \).

\[
f_i(s_r) = \begin{cases} s_r - b & s_r > b \\ 0 & -b < s_r < b \\ s_r + b & s_r < b \end{cases}
\]  

(9)

The gear shaft and the sliding bearing will produce contact and collision forces in the process of radial displacement. According to Hertz contact theory, the basic form of radial contact force can be expressed as a function of the embedding quantity \( \delta \)

\[
F_r = K_r\delta^n + C_r\dot{\delta}
\]  

(10)
In the equation, $K_r$ is the nonlinear contact stiffness between gear shaft and sliding bearing. $n$ represents power index, which is related to the properties of the colliding material. $C_r$ is the nonlinear damping coefficient, which is used to describe the energy loss in the process of contact collision.

$$F_t = K_r s_r + C_r \dot{s}_r$$  \hspace{1cm} (11)

In the equation, $K_r$ is the time-varying engagement stiffness between gear pair. $C_r$ is the engagement damping coefficient. And $s_r$ is the relative velocity along the meshing line.

Based on the Euler equation and Lagrange equation of Newton's second law, the dynamic equation of vibration system can be established. This model uses the second Lagrange equation to derive the dynamics equation of the gear transmission system. The multi-clearance coupling model is applied to the dynamics model of gear transmission system. Based on the coupling relationship between the radial clearance and the dynamic backlash, a dynamic model of the multi-clearance coupling gear engagement system is established.

$$\begin{align*}
J_1 \cdot \ddot{\theta}_1 + F_t(t) \cdot r_1 &= T_1 \\
m_1 \cdot \ddot{x}_1 - F_{rx1}(t) - F_{rx1}(t) &= 0 \\
m_1 \cdot \ddot{y}_1 - F_{ry1}(t) - F_{ry1}(t) &= 0 \\
m_2 \cdot \ddot{x}_2 - F_{rx2}(t) - F_{rx2}(t) &= 0 \\
m_2 \cdot \ddot{y}_2 - F_{ry2}(t) - F_{ry2}(t) &= 0
\end{align*}$$

$$\begin{align*}
J_2 \cdot \ddot{\theta}_2 - F_t(t) \cdot r_2 &= T_2 \\
m_2 \cdot \ddot{x}_2 + (K_r f_{abs1x})^n + D_r f_{abs1x} \cdot s_r + (K_t f_{abs1y}) \cdot \dot{s}_r \cdot \sin \alpha' &= 0 \\
m_2 \cdot \ddot{y}_2 + (K_r f_{abs1y})^n + D_r f_{abs1y} \cdot \dot{s}_r + (K_t f_{abs1y}) \cdot \dot{s}_r \cdot \cos \alpha' &= 0 \\
m_2 \cdot \ddot{x}_2 - (K_r f_{abs1x})^n + D_r f_{abs1x} \cdot \dot{s}_r \cdot \sin \alpha' &= 0 \\
m_2 \cdot \ddot{y}_2 - (K_r f_{abs1y})^n + D_r f_{abs1y} \cdot \dot{s}_r \cdot \cos \alpha' &= 0
\end{align*}$$

(12)

$T_1$ is the driving torque of driving gear. $T_2$ is the load torque applied on driven gear. $F_{rx1}(t)$, $F_{ry1}(t)$, $F_{rx2}(t)$ and $F_{ry2}(t)$ are the radial contact force components of gear pair in $x$ and $y$ directions respectively. $F_{rx1}(t)$, $F_{ry1}(t)$, $F_{rx2}(t)$ and $F_{ry2}(t)$ are the dynamic engagement force components in $x$ and $y$ directions respectively.

### 3. Results and discussion

It is very difficult to solve the nonlinear dynamic equation of the gear transmission system because of its strong nonlinear characteristics. In this paper, an adaptive Runge-Kutta numerical integration method with 5-order variable step size is used to solve the dynamics model of a small module gear transmission system. The main structural parameters are shown in Table 1.

| Parameter                  | Numeric value | Parameter                  | Numeric value |
|----------------------------|---------------|----------------------------|---------------|
| Modulus                    | 0.35          | Mass of driving gear (g)   | 1.0245        |
| Pressure angle (°)         | 20            | Mass of driven gear (g)    | 0.6773        |
The backlash of the fine-pitch gear is designed according to the German standard DIN 58405. The upper and lower allowance value of base tangent length of fine-pitch plastic gear refer to the type h in Table 7 of DIN 58405-2. The corresponding backlash value of fine-pitch gear pair can be calculated by the base tangent length. After calculation, the corresponding backlash range should be 0.006mm.

The steady-state response of the system is obtained by using the fifth-order variable-step-size adaptive Runge-Kutta numerical integration method (ode45) to solve the dynamic equations.

The vibration of the driven gear in the torsional and radial(x) directions is shown in Fig. 4 and Fig. 5. According to the time-displacement response in Fig. 4, it can be seen that the gear has a strong vibration at the beginning of meshing, and then gradually becomes periodic and stable. Fig. 5 shows a section of radial displacement curve in the x direction after stabilization. The radial displacement curve in the y direction is similar. It can be seen that the maximum value of radial displacement fluctuation is around 0.01mm. The radial clearance value set in the model is also 0.01mm. Thus the gear is channeling within the range of radial clearance, which can prove the availability and rationality of the model.

Furthermore, the calculation result of the gear dynamic engagement force of the model is given. As shown in Fig. 6, the engagement force of gear conforms to the vibration law of gear. The figure also shows the interval after the fluctuation of engagement force is stable. The average value agrees with the theoretical calculation value.

At the same time, the calculation results of dynamic backlash of the model are also given. As can be seen from Fig. 7, the dynamic backlash varies with the radial displacement of the gear and fluctuates

| Number of driving gear teeth | 11 |
|-------------------------------|----|
| Number of driven gear teeth   | 20 |

![Fig.4 Time-domain curve of driven gear’s torsional displacement](image)

![Fig.5 Time-domain curve of driven gear’s radial displacement](image)

![Fig.6 Time-domain curve of dynamic engagement force](image)

![Fig.7 Time-domain curve of dynamic backlash](image)
within the range of 36um to 44.5um. The fluctuation range is larger than the fluctuation range of backlash value derived from the recommended base tangent length of DIN standard.

Because the fine-pitch gear is smaller than the conventional gear, it is more sensitive to the clearance. In the coupling model, the range of the backlash value exceeds the recommended value of DIN standard. This indicates that the reserved tooth clearance is not enough, which will lead to interference in the meshing process and cause unnecessary vibration and noise. The results of mathematical model show that a larger backlash range than DIN standard recommended is more conducive to achieve better transmission effect.

4. Conclusions
(1) Considering the coupling between the backlash and the bearing radial clearance, the expression of the dynamic backlash of fine-pitch gear pair is derived.

(2) A three-degree-of-freedom two-state dynamic mathematical model of the parallel shaft gear transmission system is established based on the lubrication mode of fine-pitch gear pair.

(3) The time-domain curves of gear displacement, velocity, engagement force and dynamic backlash are obtained, which provides guidance for the later improvement of the fine-pitch gear pair.

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References
[1] Qiang T, Paulo F., Hamid M. (2018) A comprehensive survey of the analytical, numerical and experimental methodologies for dynamics of multibody mechanical systems with clearance or imperfect joints. Mechanism and Machine Theory, 122: 1-57.

[2] Yi Y, Huang K, Xiong YS, Sang M. (2019) Nonlinear dynamic modelling and analysis for a spur gear system with time-varying pressure angle and gear backlash. Mechanical Systems and Signal Processing, 132: 18-34.

[3] Yang Y, Cao LY, Li H, Dai YP. (2019) Nonlinear dynamic response of a spur gear pair based on the modeling of periodic mesh stiffness and static transmission error. Applied Mathematical Modelling, 72: 444-469.

[4] Shi JF, Gou XF, Zhu LY. (2019) Modeling and analysis of a spur gear pair considering multi-state mesh with time-varying parameters and backlash. Mechanism and Machine Theory, 314: 857-870.

[5] Dubowsky S., Gardner TN. (1977) Design and Analysis of Multilink Flexible Mechanisms with Multiple Clearance Connections, Trans. ASME, J. Eng. Ind., 99 (2): 88-96.

[6] Liu ZX, Liu ZS, Zhao JM, Zhang GH. (2017) Study on interactions between tooth backlash and journal bearing clearance nonlinearity in spur gear pair system. Mechanism and Machine Theory, 107: 229-245.

[7] Zhang HB, Qi CQ, Fan JZ, Dai SJ, You BD. (2018) Vibration characteristics analysis of planetary gears with a multi-clearance coupling in space mechanism. Energies, 11 (10): 2687.

[8] Han JC, Liang L, Zhao Y. (2021) Dynamic Performance of Planetary Gear Joint for Satellite Antenna Driving Mechanism Considering Multi-Clearance Coupling. Energies, 14 (4): 815.