DISCRETE SYMMETRIES AS AUTOMORPHISMS OF THE PROPER POINCARÉ GROUP

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We present the consistent approach to finding the discrete transformations in the representation spaces of the proper Poincaré group. To this end we use the possibility to establish a correspondence between involutory automorphisms of the proper Poincaré group and the discrete transformations. As a result, we derive rules of the discrete transformations for arbitrary spin-tensor fields without the use of relativistic wave equations. Besides, we construct explicitly fields carrying representations of the extended Poincaré group, which includes the discrete transformations as well.

I. INTRODUCTION

As it is well known, the Lorentz transformations in Minkowski space are divided into continuous and discrete ones. The transformations which can be obtained continuously from the identity form the proper Poincaré group. A classification of irreducible representations (irreps) of the the Poincaré group was given by Wigner \cite{1} (see also the books \cite{2–6}). The representation theory of the proper Poincaré group, in fact, provides us only by continuous transformations in the representation spaces. At the same time, there is no a regular way to describe the discrete transformations in such spaces on the ground of purely group-theoretical considerations. Moreover, it turns out that there is no one-to-one correspondence between the set \((P,T)\) of discrete transformations in Minkowski space and the complete set of discrete transformations in the representation spaces. The latter set is wider than the former one (it includes \(P,T,C,T_w,T_{sch})\).

As a rule, finding discrete transformations in the representation spaces demands an analysis of the corresponding wave equations, and has, in a sense, heuristic character. Besides, the possibility to have different wave equations for particles with the same spin results in a certain "fuzziness" of the definition of the discrete transformations in the representation spaces (see, e.g., \cite{11,12}). All that stresses the lack of a regular approach to the definition of such discrete transformations and, therefore, creates an uncertainty in using the discrete transformations as symmetry ones. More detail consideration have led authors of \cite{11} to the

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\textsuperscript{1}There are three different transformations related to the change of the sign of time: time reflection \(T\) considered in detail in \cite{7}, Wigner time reversal \(T_w\) \cite{8} and Schwinger time reversal \(T_{sch}\) \cite{9,10}.
that "the situation is clearly an unsatisfactory one from a fundamental point of view".

One ought to mention attempts to define the discrete transformations in the representation spaces without appealing to any relativistic wave equations or model assumptions. In particular, some features of the discrete transformations were considered on the base of their commutation relations with the generators of the Poincaré group [13]. Using such relations, it is possible to define the extended Poincaré group, which includes the discrete transformations as well, and then to consider irreps of the extended group. However, here, besides of the ambiguity of such extensions [14,15], the problem of the explicit construction of the discrete transformations in the representation spaces remains still open.

In the present work, we offer the consistent approach to the description of the discrete transformations which is completely based on the representation theory of the proper Poincaré group. Our consideration contains two key points:

First, we study a scalar field on the proper Poincaré group, which carries representations with all possible spins. This field depends on the coordinates $x$ on Minkowski space (which is a coset space of the Poincaré group with respect to the Lorentz subgroup) and the coordinates $z$ on the Lorentz group, which correspond to spin degrees of freedom. Some of the discrete transformations affect only the spacetime coordinates $x$, and some of them affect only the spin coordinates $z$. The consideration of the scalar field on the Poincaré group gives a possibility to describe "nongeometrical" transformations (i.e. ones that leave spacetime coordinates $x$ unchanged), and, in particular, charge conjugation, on an equal footing with the reflections in Minkowski space. Expanding the scalar field in powers of $z$, it is easy to obtain the usual spin-tensor fields as the corresponding coefficient functions.

Second, we identify the discrete transformations with the involutory automorphisms of the proper Poincaré group.

As it is known, there are two types of automorphisms. By definition, an inner automorphism of the group $G$ can be represented in the form $g \rightarrow g_0gg_0^{-1}$, where $g_0 \in G$. Other automorphisms are called outer ones. It is evident that the outer automorphisms of the proper Poincaré group can’t be reduced to the continuous transformations of the group (whereas for the inner automorphisms a supplementary consideration is required); as we will see below, they correspond to the reflections of the coordinate axes or to the dilatations. The connection between some discrete transformations and the outer automorphisms was also mentioned earlier [7,16–19]. In particular, [17] contains the idea that the outer automorphisms of internal-symmetry groups may correspond to discrete (possibly broken) symmetries. In this context it is necessary to point out the work [7], where an outer automorphism of the Lorentz group was taken as a starting point for the consideration of the space reflection transformation.

Studying involutory (both outer and inner) automorphisms of the proper Poincaré group, we describe all discrete transformations and give explicit formulae for the action of the discrete transformations on arbitrary spin-tensor fields without appealing to any relativistic wave equations.

One has to point out that there is a discussion in the literature about the sign of the mass term in the relativistic wave equations for half-integer spins (see, e.g., [20–24]). We apply the approach under consideration to present a full solution for such a problem.

The paper is organized as follows.
In Section 2 we show that the outer involutory automorphisms of the Poincaré group are generated by the reflections in Minkowski space and, thus, there is one-to-one correspondence between such automorphisms and the reflections.

In Section 3 we introduce the scalar field on the Poincaré group and we find transformation laws of the field under the outer and inner automorphisms. This allows us to describe the action of all the discrete transformations in terms of the field on the group.

In Section 4, decomposing the field on the group, we obtain formulae for the action of the discrete transformations on arbitrary spin-tensor fields.

In Section 5 we derive transformation laws under the automorphisms for the generators of the Poincaré group and for some other operators. On this base, we consider properties of the discrete transformations, and, in particular, we compare Wigner and Schwinger time reversals.

In Section 6 we extend the Poincaré group by the discrete transformations and describe characteristics of irreps of the extended group.

In Sections 7,8,9 we construct explicitly massive and massless fields with different characteristics corresponding to the discrete transformations and consider the relation between our construction and the theory of relativistic wave equations. Then we classify solutions of relativistic wave equations for arbitrary spin with respect to the representations of the extended Poincaré group.

II. REFLECTIONS IN MINKOWSKI SPACE AND OUTER AUTOMORPHISMS OF THE PROPER Poincaré GROUP

In this Section we consider how discrete transformations in Minkowski space may generate outer involutory automorphisms of the proper Poincaré group.

As is known, Poincaré group transformations

\[ x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \]

in Minkowski space (\( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \)) of coordinates \( x = (x^\mu, \mu = 0,1,2,3) \) are defined by the pairs \( (a, \Lambda) \), where \( a = (a^\mu) \) is arbitrary vector and the matrix \( \Lambda \in O(3,1) \). They obey the composition law

\[ (a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1). \]

Any matrix \( \Lambda \) can be presented in one of the four forms: \( \Lambda_0, \Lambda_s \Lambda_0, \Lambda_t \Lambda_0, \Lambda_s \Lambda_t \Lambda_0 \).

Here \( \Lambda_0 \in SO_0(3,1) \), where \( SO_0(3,1) \) is a connected component of \( O(3,1) \), and matrices \( \Lambda_s = \text{diag}(1,-1,-1,-1), \Lambda_t = \text{diag}(-1,1,1,1) \) correspond to space reflection \( P \) and time reflection \( T \). Then inversion \( I_x = PT = \Lambda_s \Lambda_t \).

Pairs \( (a, \Lambda_0) \) with the composition law \( (2.2) \) form a group, which is a semidirect product of the translation group \( T(4) \) and of the group \( SO_0(3,1) \). We denote the latter group by \( M_0(3,1) \).

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\(^2\)Recall that an automorphism \( A \) of a group \( G \) is a mapping of the group onto itself which preserves the group multiplication law; for an involutory automorphism \( A^2 \) is the identity mapping.
Under space reflection the equation \( x' = \Lambda_0 x + a \) takes the form
\[
\Lambda_s x' = \Lambda_s \Lambda_0 \Lambda_s^{-1} \Lambda_s x + \Lambda_s a, \quad \text{or} \quad \bar{x}' = \overline{\Lambda}_0 \bar{x} + \bar{a},
\]
where
\[
\bar{x} = \Lambda_s x = (x^0, -x^k), \quad \bar{a} = \Lambda_s a = (a^0, -a^k), \quad \overline{\Lambda}_0 = \Lambda_s \Lambda_0 \Lambda_s^{-1} = (\Lambda_0^T)^{-1}.
\] (2.3)

In a similar manner, using the operations \( T \) and \( I_x \), we obtain finally that \( P, T, I_x \) generate three outer involutory automorphisms of the group \( M_0(3,1) \):
\[
P : \quad (a, \Lambda_0) \to (\bar{a}, (\Lambda_0^T)^{-1}); \quad \text{(2.4)}
\]
\[
T : \quad (a, \Lambda_0) \to (-\bar{a}, (\Lambda_0^T)^{-1}); \quad \text{(2.5)}
\]
\[
I_x : \quad (a, \Lambda_0) \to (-a, \Lambda_0). \quad \text{(2.6)}
\]

Notice that \( P \) and \( T \) generate the same automorphism of the group \( SO_0(3,1) \).

Consider now an universal covering group for \( M_0(3,1) \). Such a group, which we denote by \( M(3,1) \), is the semidirect product of \( T(4) \) and \( SL(2,C) \). As is known, there is a one-to-one correspondence between any vectors \( v \) from Minkowski space and \( 2 \times 2 \) hermitian matrices \( V \) (see, e.g., \( [3,25,26] \)):
\[
v^\mu \leftrightarrow V = v^\mu \sigma_\mu, \quad v^\mu = \frac{1}{2} \text{Tr}(V \bar{\sigma}^\mu). \quad \text{(2.8)}
\]

Proper Poincare transformations \( x' = \Lambda_0 x + a \) can be rewritten in new terms as
\[
X' = UXU^\dagger + A, \quad \text{(2.9)}
\]
where \( X = x^\mu \sigma_\mu, \quad A = a^\mu \sigma_\mu, \) and \( U \in SL(2,C) \) (two different matrices \( \pm U \) correspond to any matrix \( \Lambda_0 \)). Elements of \( M(3,1) \) are now given by the pairs \( (A,U) \) with the composition law
\[
(A_2, U_2)(A_1, U_1) = (U_2 A_1 U_2^\dagger + A_2, \ U_2 U_1). \quad \text{(2.10)}
\]

Space reflection takes \( x = (x^0, x^k) \) into \( \bar{x} = (x^0, -x^k) \), or in terms of \( X = x^\mu \sigma_\mu \),
\[
P : \quad X \to \overline{X} = \bar{x}^\mu \sigma_\mu = x^\mu \bar{\sigma}_\mu. \quad \text{(2.11)}
\]

Using the relation \( \overline{X} = \sigma_2 X^T \sigma_2 \) and the identity \( \sigma_2 U \sigma_2 = (U^T)^{-1} \), we obtain as a consequence of (2.9)

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\(^3\) We use two sets of \( 2 \times 2 \) matrices \( \sigma_\mu = (\sigma_0, \sigma_k) \) and \( \bar{\sigma}_\mu = (\sigma_0, -\sigma_k) \).
\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(2.7)}
\]
\[
\overline{X'} = (U^\dagger)^{-1}\overline{X}U^{-1} + \overline{A}.
\]

Thus, \( \overline{X} \) is transformed by means of the element \( (\overline{A}, (U^\dagger)^{-1}) \) of \( M(3, 1) \). The relation

\[
P : (A, U) \rightarrow (\overline{A}, (U^\dagger)^{-1})
\]

defines an outer involutory automorphism of the proper Poincaré group. In a similar manner, we obtain automorphisms of the group \( M(3, 1) \) which are generated by \( T, I_x \),

\[
T : (A, U) \rightarrow (-\overline{A}, (U^\dagger)^{-1});
\]

\[
I_x : (A, U) \rightarrow (-A, U).
\]

The automorphisms corresponding to \( P \) and \( T \) exhaust all outer involutory automorphisms of the Poincaré group in the following sense. Any outer involutory automorphism can be presented as a composition of these two automorphisms and of an inner automorphism of the group.\(^4\) In particular, the automorphism of complex conjugation

\[
C : (A, U) \rightarrow (\crule{A}{A}, \crule{U}{U}).
\]

is the product of the outer automorphism (2.13) and of the inner automorphism

\[
(0, i\sigma^2)(\overline{A}, (U^\dagger)^{-1})(0, -i\sigma^2) = (\crule{A}{A}, \crule{U}{U}).
\]

As one can see from (2.13), (2.14), \( P \) and \( T \) generate the same automorphisms of the Lorentz group \( SL(2, C) \), namely, \( U \rightarrow (U^\dagger)^{-1} \), whereas \( PT \) generates the identity automorphism of \( SL(2, C) \) and outer automorphism \( A \rightarrow -A \) of the translation group.

Thus, we have demonstrated how discrete transformations in Minkowski space generate outer involutory automorphisms of the proper Poincare group. In the next Section we are going to relate the automorphisms of the proper Poincare group with disere transformations in the representation spaces of the group, which are of our main interest.

### III. AUTOMORPHISMS OF PROPER POINCARÉ GROUP AND DESCRETE TRANSFORMATIONS IN REPRESENTATION SPACES

To introduce the scalar field on the proper Poincaré group we describe briefly the principal points of the corresponding technique \[19\]. It is well known \[27,3,28\] that any irrep of a group \( G \) is contained (up to the equivalence) in a decomposition of a generalized regular

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\(^4\)The Poincaré group \( M(3, 1) \) is a semidirect product of the Lorentz group \( SL(2, C) \) and the group of four-dimensional translations \( T(4) \). Any outer automorphism of \( SL(2, C) \) is a product of involutory automorphism \( U \rightarrow (U^\dagger)^{-1} \) and of an inner automorphism \[\[\]. Outer automorphisms of the translation group are generated by the dilatations \( x^\mu \rightarrow cx^\mu, c \neq 0, 1 \), and are involutory only at \( c = -1 \). Outer automorphisms of \( SL(2, C) \) and \( T(4) \) generate the following outer automorphisms of the Poincaré group: \( (A, U) \rightarrow (\overline{A}, (U^\dagger)^{-1}), (A, U) \rightarrow (cA, U) \).
representation. Consider the left generalized regular representation $T_L(g)$, which is defined in the space of functions $f(h)$, $h \in G$, on the group as

$$T_L(g)f(h) = f'(h) = f(g^{-1}h), \ g \in G.$$  \hspace{1cm} (3.1)

As a consequence of the relation (3.1) we can write

$$f'(h') = f(h), \ h' = gh.$$  \hspace{1cm} (3.2)

Let $G$ be the group $M(3,1)$, and we use the parametrization of its elements by two $2 \times 2$ matrices (one hermitian and another one from $SL(2,C)$), which was described in the previous Section. At the same time, using such a parametrization, we choose the following notations:

$$g \leftrightarrow (A,U), \ h \leftrightarrow (X,Z),$$  \hspace{1cm} (3.3)

where $A, X$ are $2 \times 2$ hermitian matrices and $U, Z \in SL(2,C)$. The map $h \leftrightarrow (X, Z)$ creates the correspondence

$$h \leftrightarrow (x, z, \underline{z}), \ \text{where} \ x = (x^\mu), \ z = (z_\alpha), \ \underline{z} = (\underline{z}_\alpha),$$

$$\mu = 0, 1, 2, 3, \ \alpha = 1, 2, \ z_1 \underline{z}_2 - z_2 \underline{z}_1 = 1,$$  \hspace{1cm} (3.4)

by virtue of the relations

$$X = x^\mu \sigma_\mu, \ Z = \begin{pmatrix} z_1 \underline{z}_1 \\ z_2 \underline{z}_2 \end{pmatrix} \in SL(2,C).$$  \hspace{1cm} (3.5)

On the other hand, we have the correspondence $h' \leftrightarrow (x', z', \underline{z}')$,

$h' = gh \leftrightarrow (X', Z') = (A,U)(X, Z) = (UXU^+ + A, UZ) \leftrightarrow (x', z', \underline{z}')$,

$$x'^\mu \sigma_\mu = X' = UXU^+ + A \ \implies \ x'^\mu = (A_0)^\mu_\nu x^\nu + a^\mu, \ A_0 \leftrightarrow U \in SL(2,C),$$  \hspace{1cm} (3.6)

$$\begin{pmatrix} z'_1 \\ z'_2 \\ \underline{z}'_1 \\ \underline{z}'_2 \end{pmatrix} = Z' = UZ \implies z'_\alpha = U_\alpha^\beta z_\beta, \ \underline{z}'_\alpha = U_\alpha^\beta \underline{z}_\beta, \ U = (U_\alpha^\beta), \ z'_1 \underline{z}'_2 - z'_2 \underline{z}'_1 = 1.$$  \hspace{1cm} (3.7)

Then the relation (3.2) takes the form

$$f'(x', z', \underline{z}') = f(x, z, \underline{z}),$$  \hspace{1cm} (3.8)

$$x'^\mu = (A_0)^\mu_\nu x^\nu + a^\mu, \ A_0 \leftrightarrow U \in SL(2,C),$$  \hspace{1cm} (3.9)

$$z'_\alpha = U_\alpha^\beta z_\beta, \ \underline{z}'_\alpha = U_\alpha^\beta \underline{z}_\beta, \ z_1 \underline{z}_2 - z_2 \underline{z}_1 = z'_1 \underline{z}'_2 - z'_2 \underline{z}'_1 = 1.$$  \hspace{1cm} (3.10)

The relations (3.8)-(3.10) admit a remarkable interpretation. We may treat $x$ and $x'$ in these relations as position coordinates in Minkowski space $M(3,1)/SL(2,C)$ (in different Lorentz reference frames) related by proper Poincare transformations, and the sets $z, \underline{z}$ and $z', \underline{z}'$ may be treated as spin coordinates in these Lorentz frames. They are transformed according to the formulae (3.10). Carrying two-dimensional spinor representation of the Lorentz group, the variables $z$ and $\underline{z}$ are invariant under translations as one can expect for spin degrees of freedom. Thus, we may treat sets $x, z, \underline{z}$ as points in a position-spin space with the transformation law (3.3), (3.10) under the change from one Lorentz reference
frame to another. In this case equations (3.8)-(3.10) present the transformation law for scalar functions on the position-spin space.

On the other hand, as we have seen, the set \((x, z, \bar{z})\) is in one-to-one correspondence to the group \(M(3, 1)\) elements. Thus, the functions \(f(x, \bar{z}, z)\) are still functions on this group. That is why we often call them scalar functions on the group as well, remembering that the term "scalar" came from the above interpretation.

Remember now that different functions of such type correspond to different representations of the group \(M(3, 1)\). Thus, the problem of classification of all irreps of this group is reduced to the problem of a classification of all scalar functions on position-spin space. It is natural to restrict ourselves by the scalar functions which are analytic both in \(z, \bar{z}\) and in \(\tilde{z}, \tilde{\bar{z}}\) (or, simply speaking, which are differentiable with respect to these arguments). Further, such functions are denoted by \(f(x, z, \tilde{z}, \tilde{\bar{z}}) = f(x, z)\), \(z = (z, \tilde{z}, \tilde{\bar{z}}, \tilde{z})\). In consequence of the unimodularity of matrices \(U\) there exist invariant antisymmetric tensors \(\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \varepsilon^{\tilde{\alpha}\tilde{\beta}} = -\varepsilon^{\tilde{\beta}\tilde{\alpha}}, \varepsilon_{12} = \varepsilon_{12} = 1, \varepsilon_{12} = \varepsilon_{12} = -1\). Spinor indices are lowered and raised according to the rules

\[
z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^\alpha = \varepsilon^{\alpha\beta} z_\beta, \quad \tilde{z}_\tilde{\alpha} = \varepsilon_{\tilde{\alpha}\tilde{\beta}} \tilde{z}^\tilde{\beta}, \quad \tilde{z}^\tilde{\alpha} = \varepsilon^{\tilde{\alpha}\tilde{\beta}} \tilde{z}_\tilde{\beta}.
\] (3.11)

The continuous transformations (3.10) corresponding to the Lorentz rotations are ones, which do not mix \(z^\alpha\) and \(\tilde{z}^\alpha\) (and their complex conjugate \(\tilde{z}^\alpha\), \(\tilde{z}^\alpha\)). Therefore four subspaces of functions \(f(x, z)\), \(f(x, \tilde{z})\), \(f(x, \tilde{z})\), \(f(x, \tilde{\bar{z}})\) are invariant with respect to \(M(3, 1)\) transformations.

In the framework of the scalar field theory on the Poincaré group [19] the standard spin description in terms of multicomponent functions arises under the separation of space and spin variables.

Since \(z\) is invariant under translations, any function \(\phi(z)\) carry a representation of the Lorentz group. Let a function \(f(h) = f(x, z)\) allows the representation

\[
f(x, z) = \phi^\alpha(z) \psi^\alpha(x),
\] (3.12)

where \(\phi^\alpha(z)\) form a basis in the representation space of the Lorentz group. The latter means that one may decompose the functions \(\phi^\alpha(z')\) of transformed argument \(z' = gz\) in terms of the functions \(\phi^\alpha(z)\):

\[
\phi^\alpha(z') = \phi^\beta(z) L_l^n(U).
\] (3.13)

Thus, the action of the Poincaré group on a line \(\phi(z) = (\phi^\alpha(z))\) is reduced to a multiplication by matrix \(L(U)\), where \(U \in SL(2, C)\): \(\phi(z') = \phi(z) L(U)\).

Comparing the decompositions of the function \(f'(x', z') = f(x, z)\) over the transformed basis \(\phi(z')\) and over the initial basis \(\phi(z)\),

\[
f'(x', z') = \phi(z') \psi'(x') = \phi(z) L(U) \psi'(x') = \phi(z) \psi(x),
\]

where \(\psi(x)\) is a column with components \(\psi^\alpha(x)\), one may obtain

\[
\psi'(x') = L(U^{-1}) \psi(x),
\] (3.14)
i.e. the transformation law of a tensor field on Minkowski space. This law corresponds to
the representation of the Poincaré group acting in a linear space of tensor fields as follows
\( T(g)\psi(x) = L(U^{-1})\psi(\Lambda^{-1}(x - a)) \). According to (\ref{eq:3.13}) and (\ref{eq:3.14}), the functions \( \phi(z) \) and 
\( \psi(x) \) are transformed under contragradient representations of the Lorentz group.

Consider now the action of automorphisms in the space of functions on the Poincaré
group. Automorphisms \( g \rightarrow IgI^{-1} \) (both inner and outer) generate the following transfor-
mations of the left generalized regular representation of the Poincaré group:

\[
T_L(g) \rightarrow IT_L(g)I^{-1} \equiv T_L(IgI^{-1}),
\]

(\ref{eq:3.15})

\[
f(h) \rightarrow If(h) \equiv f(IhI^{-1}),
\]

(\ref{eq:3.16})

where (\ref{eq:3.16}) defines the mapping of the space of functions \( f(h) \) into itself, corresponding to
the automorphism (\ref{eq:3.15}). Notice that, putting \( If(h) = f(Ih) \) instead of (\ref{eq:3.16}), we come to
a contradiction, since \( Ih \) is not an element of the group if \( I \) is an outer automorphism, and
corresponding transformation expels from the space of functions on the group.

Transformation rules of \((A,U)\) and \(X\) under automorphisms corresponding to space and
time reflections are given by the formulae (\ref{eq:2.13})-(\ref{eq:2.15}). In order to establish the transfor-
mation rule of \(Z\), it is sufficient to note that the composition law of the group is conserved
under automorphisms, and therefore \((X,Z)\) is transformed just as \((A,U)\):

\[
P : \quad (X,Z) \rightarrow (\overline{X},(Z^t)^{-1});
\]

(\ref{eq:3.17})

\[
T : \quad (X,Z) \rightarrow (-\overline{X},(Z^t)^{-1});
\]

(\ref{eq:3.18})

\[
I_x : \quad (X,Z) \rightarrow (-X,Z).
\]

(\ref{eq:3.19})

Thus, the automorphisms under consideration correspond to the replacement of the
arguments of scalar functions \(f(h)\) on the group according to the formulae (\ref{eq:3.17})-(\ref{eq:3.19}).

The replacement

\[
Z \xrightarrow{PT} (Z^t)^{-1}, \quad \text{or} \quad \begin{pmatrix} z_1^1 & z_2^1 \\
\ast z_1^2 & \ast z_2^2 \end{pmatrix} \xrightarrow{PT} \begin{pmatrix} \ast z_1^1 & \ast z_2^1 \\
\ast z_1^2 & \ast z_2^2 \end{pmatrix}
\]

(\ref{eq:3.20})

corresponds to space and time reflections. Transformation (\ref{eq:3.20}) maps functions of \( z^a \) into
functions of \( \ast z^a \). Thus, the space of scalar functions on the group contains two subspaces
of functions \( f(x,z,\ast z) \) and \( f(x,\ast z,\ast z) \), which are invariant ones with respect to both transfor-
mations of the proper Poincaré group and the discrete transformations under consideration
(space and time reflections). Below we will consider mainly these two subspaces, which we
denote by \(V_+\) and \(V_-\) respectively.

Complex conjugation

\[
C : \quad T(g) \rightarrow CT(g)C^{-1} \equiv T(g), \quad f(h) \rightarrow Cf(h) \equiv \ast f(h),
\]

(\ref{eq:3.21})

affects both functions and complex coordinates on the Lorentz group, and therefore it takes
subspaces \(V_+\) and \(V_-\) into one another. The transformation (\ref{eq:3.21}) of the field \( f(h) \) can be
identified with charge conjugation, which interchanges particle and antiparticle fields \([19]\);
below we consider this identification and various particular cases in detail.
Studying involutory outer automorphisms of the Poincaré group and complex conjugation in the space of functions on the group, we have obtained the description of three independent discrete transformations (space reflection $P$, time reflection $T$ and charge conjugation $C$). However, one can show that there exist two supplementary discrete transformations, which are not reduced to discussed above.

It is easy to see that we have two different transformation laws of arguments of functions $f(h)$ under Lorentz rotations and under inner automorphisms:

\[(0, U)(X, Z) = (UXU^\dagger, UZ),\]  
\[(0, U)(X, Z)(0, U^{-1}) = (UXU^\dagger, UZU^{-1}).\]  
\[(3.22)\]
\[(3.23)\]

In both cases the coordinates $x$ are transformed the same way, and therefore the action of inner automorphisms \[(3.23)\] in the space of the scalar functions $f(x)$ on Minkowski space is reduced to Lorentz rotations. But in general case of functions $f(x, z)$ it is necessary to consider the action of inner automorphisms more detail.

If inner automorphism \[(3.23)\] corresponds to some discrete transformation, then the conditions $U^2 = e^{i\phi}$ and $\det U = 1$ must be fulfilled. Diagonal matrices with elements $e^{i\phi/2}$ and in the special case $e^{i\phi} = -1$ also matrices of the form

\[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = -1.\]  
\[(3.24)\]

satisfy these conditions. Then, the square of the product of two different matrices of the form \[(3.24)\] also must be proportional to the identity matrix. The latter condition reduces (up to sign) the set \[(3.24)\] to the set of three matrices

\[i\sigma_1, i\sigma_2, i\sigma_3.\]

Matrix $U = i\sigma_2$ gives an explicit realization of inner involutory automorphism

\[(X, Z) \rightarrow (X^T, (Z^T)^{-1}).\]  
\[(3.25)\]

(This realization we have used above, see \[(2.17)\].) A straightforward consideration of the automorphism \[(3.25)\] is inconvenient, because two coordinates $x^1, x^3$ change sign and $x^2$ remains unaltered (this correspond to the rotation by the angle $\pi$ in Minkowski space). Therefore we consider a transformation that is a composition of the inner automorphism corresponding to the element $(0, U)$ and the Lorentz rotation corresponding to the element $(0, U^{-1})$:

\[(X, Z) \rightarrow (X, ZU^{-1}).\]  
\[(3.26)\]

For $U = i\sigma_2$ we obtain the transformation, which we denote by $I_z$, \[I_z : \quad (X, Z) \rightarrow (X, Z(-i\sigma_2)), \quad \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \rightarrow \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.\]  
\[(3.27)\]

This transformation maps the spaces of functions $f(x, z, \bar{z})$ and $f(x, \bar{z}, z)$ into one another like charge conjugation \[(3.21)\] but unlike the latter is reduced to replacement of arguments and does not conjugate function.
For $U = i \sigma_3$ we obtain the transformation

$$I_3: \quad (X, Z) \rightarrow (X, Z(-i \sigma_3)), \quad \left(\frac{z^1}{z^2}, \frac{\bar{z}^1}{\bar{z}^2}\right) \rightarrow \left(-iz^1, -iz^2\right).$$

(3.28)

The transformation associated with $U = i \sigma_1$ is the product of just considered transformations $I_z$ and $I_3$.

Thus, if in Minkowski space there exist only two independent discrete transformations corresponding to outer automorphisms of the Poincaré group, then for the scalar field on the group there exist five independent discrete transformations corresponding to both outer and inner automorphisms, which are not reduced to transformations of the proper Poincaré group. Charge conjugation is associated with complex conjugation of the functions on the group, and other four transformations are associated with following replacements of the arguments of the scalar functions on the group:

| Transformation | $x^0$ | $x$ | $z^\alpha$ | $\bar{z}_\alpha$ | $z^\alpha$ | $\bar{z}_\alpha$ |
|---------------|------|-----|-----------|-----------------|-----------|-----------------|
| $P$           | $x^0$ | $-x$ | $-z^\alpha$ | $z^\alpha$       | $z^\alpha$ | $-z^\alpha$     |
| $I_x$         | $-x^0$ | $-x$ | $z^\alpha$ | $\bar{z}_\alpha$ | $z^\alpha$ | $\bar{z}_\alpha$ |
| $I_z$         | $x^0$ | $x$  | $z^\alpha$ | $\bar{z}_\alpha$ | $-z^\alpha$ | $-\bar{z}_\alpha$ |
| $I_3$         | $x^0$ | $x$  | $-iz^\alpha$ | $i\bar{z}_\alpha$ | $iz^\alpha$ | $i\bar{z}_\alpha$ |

(3.29)

IV. ACTION OF THE AUTOMORPHISMS ON SPIN-TENSOR FIELDS

Decomposing the scalar field on the Poincaré group in powers of $z = (z, \bar{z}, \bar{z}^*, \bar{z}^*)$, it is easy to obtain transformation laws for spin-tensor fields, which are coefficient functions and depend on the coordinates in Minkowski space only. At the same time one must take into account that in comparison with corresponding scalar fields on the group it is necessary to use two sets of indices (dotted and undotted, which for fields on the group simply duplicate the sign of complex conjugation of the coordinates on the Lorentz group) and stipulate what kind of object (particle or antiparticle) is described by the function (instead of using underlined and non-underlined coordinates on the Lorentz group).

As a simple example we consider linear in $z$ functions, which correspond to spin 1/2. If particle field is described by a function $f(x, z, \bar{z}^*) \in V_+$,

$$f(x, z, \bar{z}^*) = \chi_{\alpha}(x)z^\alpha + \psi^{\alpha}(x)\bar{z}_\alpha = Z_D \Psi(x), \quad Z_D = (z^\alpha \bar{z}_\alpha), \quad \Psi(x) = \left(\begin{array}{c} \chi_{\alpha}(x) \\ \psi^{\alpha}(x) \end{array}\right),$$

(4.1)

then antiparticle field is described by a function $f(x, \bar{z}, \bar{z}^*) \in V_-$,

$$f(x, \bar{z}, \bar{z}^*) = \chi_{\alpha}(x)\bar{z}^\alpha + \psi^{\alpha}(x)\bar{z}_\alpha = \bar{Z}_D \bar{\Psi}(x), \quad \bar{Z}_D = (\bar{z}^\alpha \bar{z}_\alpha),$$

(4.2)

where $Z_D$ and $\bar{Z}_D$ (and therefore bispinor $\Psi(x)$ in both formulae) have the same transformation law under the proper Poincaré group $M(3, 1)$.  

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According to (3.17), (3.20) we get for space reflection

\[ P : \quad Z_D \Psi(x) \rightarrow Z_D \Psi^*(\bar{x}), \quad \Psi^*(\bar{x}) = -\left( \frac{\psi^{\alpha}(\bar{x})}{\chi_{\alpha}(\bar{x})} \right) = \gamma^0 \Psi(\bar{x}). \]

Thus, for time reflection we have \( \Psi^*(\bar{x}) = \gamma^0 \Psi(-\bar{x}) \). Charge conjugation corresponds to the complex conjugation in the space of scalar functions, and, according to (3.21), we may write

\[ C : \quad Z_D \Psi(x) \rightarrow \bar{Z}_D \Psi^*(x) = Z_D \Psi(c)(x), \quad \Psi(c)(x) = -\left( \frac{\psi_\alpha(x)}{\chi_{\alpha}(x)} \right) = i \gamma^2 \Psi(x). \quad (4.3) \]

Finally, using formulae (3.27) and (3.28), we obtain for the transformations \( I_z \) and \( I_3 \)

\[ I_z : \quad Z_D \Psi(x) \rightarrow Z_D \gamma^5 \Psi(x), \quad (4.4) \]
\[ I_3 : \quad Z_D \Psi(x) \rightarrow -Z_D i \Psi(x). \quad (4.5) \]

Notice that both transformations \( I_z \) and \( C \) interchange particles and antiparticles. The transformation \( I_3 \) produces only a phase factor.

In order to find transformation laws for spin-tensor fields we need the explicit form of bases of the Lorentz group irreps. Consider the monomial basis

\[ (z^1)^a(z^2)^b(\bar{z}^1)^c(\bar{z}^2)^d \]

in the space of functions \( \phi(z, \bar{z}) \). The values \( j_1 = (a + b)/2 \) and \( j_2 = (c + d)/2 \) are invariant under the action of generators of the Lorentz group (3.14). Hence the space of irrep \((j_1, j_2)\) is the space of homogeneous functions depending on two pairs of complex variables of power \((2j_1, 2j_2)\). We denote these functions as \( \varphi_{j_1j_2}(z) \).

For finite-dimensional nonunitary irreps of \( SL(2, C) \), \( a, b, c, d \) are integer nonnegative, therefore \( j_1, j_2 \) are integer or half-integer nonnegative numbers. One can write functions \( f_s(x, z) \), which are polynomials of the power \( 2s = 2j_1 + 2j_2 \) in \( z, \bar{z} \), in the form

\[ f_s(x, z, \bar{z}) = \sum_{j_1 + j_2 = s} \sum_{m_1, m_2} \psi_{j_1j_2}^{m_1m_2}(x) \varphi_{j_1j_2}^{m_1m_2}(z, \bar{z}). \quad (4.6) \]

Here the functions

\[ \varphi_{j_1j_2}^{m_1m_2}(z, \bar{z}) = N^j(z^1)^{j_1}z^{j_2}z^{j_1-m_1}(\bar{z})^{j_2-m_2}, \quad (4.7) \]
\[ N = (2s)!/(j_1 + m_1)!/(j_1 - m_1)!/(j_2 + m_2)!/(j_2 - m_2)! \quad (4.8) \]

form a basis of the irrep of the Lorentz group. This basis corresponds to a chiral representation. On the other hand, one can write a decomposition of the same functions in terms of symmetric spin-tensors \( \psi_{\alpha_1...\alpha_2j_1}^{\beta_1...\beta_2j_2}(x) = \psi_{\alpha_1...\alpha_2j_1}^{\beta_1...\beta_2j_2}(x) \):

\[ f_s(x, z) = \sum_{j_1 + j_2 = s} f_{j_1j_2}(x, z), \quad f_{j_1j_2}(x, z) = \psi_{\alpha_1...\alpha_2j_1}^{\beta_1...\beta_2j_2}(x)z^{\alpha_1}...z^{\alpha_2j_1}z^{\beta_1}...z^{\beta_2j_2} \quad (4.9) \]
Comparing these decompositions, we obtain the relation

\[ N^{1 \frac{1}{2}} \psi_{j_1 j_2}^{m_1 m_2}(x) = \psi_{1 \ldots 1 \frac{2}{2} \frac{2}{2} \ldots \ldots 1}^{j_2 + m_2 \ j_2 - m_2 \ j_1 + m_1 \ j_1 - m_1}(x). \]  (4.10)

Consider now the action of the discrete transformations on the functions \( \psi(x) \). According to (3.17) and (3.20), the automorphism, which is related to \( P \), allows one to write (see (4.6), (4.7))

\[ f(x, z, \bar{z}) \xrightarrow{P} f(\bar{x}, -\bar{z}, -z) = \varphi(-\bar{z}^* - z)\psi(\bar{x}) = \varphi(\bar{z}, z)\psi^*(\bar{x}). \]  (4.11)

It follows from (4.7) that

\[ \varphi_{j_{1j_2}}^{m_1 m_2}(-\bar{z}, -z) = (-1)^{2(j_1 + j_2)}\varphi_{j_{2j_1}}^{m_2 m_1}(z, \bar{z}), \]  (4.12)

thus we get

\[ \psi^*(s)^{m_1 m_2}(\bar{x}) = (-1)^{2(j_1 + j_2)}\psi^{m_2 m_1}(\bar{x}). \]

Finally, in terms of spin-tensor fields, we can write

\[ \psi_{\alpha_{1 \ldots \alpha_{2j_1}}}^{\beta_1 \ldots \beta_{2j_2}}(x) \xrightarrow{P} (-1)^{2(j_1 + j_2)}\psi^{\alpha_1 \ldots \alpha_{2j_1}}_{\beta_1 \ldots \beta_{2j_2}}(\bar{x}). \]  (4.13)

Charge conjugation \( C \) maps functions \( f(x, z, \bar{z}) \in V_+ \) into functions \( f(x, \bar{z}, z) \in V_- \):

\[ f(x, z, \bar{z}) \xrightarrow{C} f(x, \bar{z}, z) = \varphi^*(\bar{z}, z)\psi^*(x) = \varphi(\bar{z}, z)\psi^*(x). \]  (4.14)

Using again (4.7) to write

\[ \varphi_{j_{1j_2}}^{m_1 m_2}(z, \bar{z}) = \varphi_{j_{1j_2}}^{m_1 m_2}(\bar{z}, z) = (-1)^{(j_1 - m_1) + (j_2 + m_2)}\varphi_{j_{2j_1}}^{m_2 m_1}(z, \bar{z}), \]  (4.15)

we obtain

\[ \psi^{(c) m_1 m_2}(x) = (-1)^{(j_1 - m_1) + (j_2 + m_2)}\psi^{m_2 m_1}(x). \]

The latter results in the following relations for spin-tensor fields

\[ \psi_{\alpha_{1 \ldots \alpha_{2j_1}}}^{\beta_1 \ldots \beta_{2j_2}}(x) \xrightarrow{C} \psi^{\beta_1 \ldots \beta_{2j_2}}_{\alpha_1 \ldots \alpha_{2j_1}}(x) = (-1)^{(j_1 - m_1) + (j_2 + m_2)}\psi^{\alpha_1 \ldots \alpha_{2j_1}}_{\beta_1 \ldots \beta_{2j_2}}(x). \]  (4.16)

The action of discrete transformations on the functions \( f(x, z, \bar{z}) \in V_- \), which correspond to antiparticle fields, can be obtained in a similar way.

The obtained formulae are summarized in two tables, where we give transformation laws both for scalar fields on the Poincaré group and for spin-tensors fields in Minkowski space.
Table 1. Discrete transformations for particle fields.

|    | $f(x, z, \bar{z})$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $\Psi(x)$ |
|----|-------------------|-------------------------------------------------|--------|
| $P$ | $f(\bar{x}, -\bar{z}, -z)$ | \((-1)^{2(j_1+j_2)}\psi_{\bar{\beta}_1...\bar{\beta}_{2j_2}}^{\bar{\alpha}_1...\bar{\alpha}_{2j_1}}(\bar{x})$ | $\gamma^0\Psi(\bar{x})$ |
| $T$ | $f(-\bar{x}, -\bar{z}, -z)$ | \((-1)^{2(j_1+j_2)}\psi_{\bar{\beta}_1...\bar{\beta}_{2j_2}}^{\bar{\alpha}_1...\bar{\alpha}_{2j_1}}(-\bar{x})$ | $\gamma^0\Psi(-\bar{x})$ |
| $I_x = PT$ | $f(-x, z, \bar{z})$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(-x)$ | $\Psi(-x)$ |
| $C$ | $f(x, z, \bar{z})$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $i\gamma^2\Psi(x)$ |
| $T_{sch} = CT$ | $f(-x, -\bar{z}, z)$ | \((-1)^{2(j_1+j_2)}\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(x)$ | $i\gamma^0\gamma^2\Psi(-\bar{x})$ |
| $I_z$ | $f(x, z, -\bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $\gamma^5\Psi(x)$ |
| $T_w = I_z CT$ | $f(-\bar{x}, z, -\bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-\bar{x})$ | $-i\gamma^5\gamma^0\gamma^2\Psi(-\bar{x})$ |
| $PCT_w = I_z I_x$ | $f(-x, z, -\bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-x)$ | $\gamma^5\Psi(-x)$ |
| $I_3$ | $f(x, -i\bar{z}, -i\bar{z})$ | \((-1)^{2(j_1+j_2)}\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $-i\Psi(x)$ |

Table 2. Discrete transformations for antiparticle fields.

|    | $f(x, z, \bar{z})$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $\Psi(x)$ |
|----|-------------------|-------------------------------------------------|--------|
| $P$ | $f(\bar{x}, \bar{z}, z)$ | $\psi_{\bar{\beta}_1...\bar{\beta}_{2j_2}}^{\bar{\alpha}_1...\bar{\alpha}_{2j_1}}(\bar{x})$ | $-\gamma^0\Psi(\bar{x})$ |
| $T$ | $f(\bar{x}, z, \bar{z})$ | $\psi_{\bar{\beta}_1...\bar{\beta}_{2j_2}}^{\bar{\alpha}_1...\bar{\alpha}_{2j_1}}(-\bar{x})$ | $-\gamma^0\Psi(-\bar{x})$ |
| $I_x = PT$ | $f(-x, \bar{z}, z)$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-x)$ | $\Psi(-x)$ |
| $C$ | $f(x, z, \bar{z})$ | $\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(x)$ | $i\gamma^2\Psi(x)$ |
| $T_{sch} = CT$ | $f(-x, \bar{z}, z)$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-x)$ | $-i\gamma^0\gamma^2\Psi(-\bar{x})$ |
| $I_z$ | $f(x, -z, \bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $\gamma^5\Psi(x)$ |
| $T_w = I_z CT$ | $f(-\bar{x}, z, \bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-\bar{x})$ | $-i\gamma^5\gamma^0\gamma^2\Psi(-\bar{x})$ |
| $PCT_w = I_z I_x$ | $f(-x, -z, \bar{z})$ | \((-1)^{2}j_2\psi_{\alpha_1...\alpha_{2j_1}}^{\bar{\beta}_1...\bar{\beta}_{2j_2}}(-x)$ | $\gamma^5\Psi(-x)$ |
| $I_3$ | $f(x, i\bar{z}, i\bar{z})$ | \((-1)^{2(j_1+j_2)}\psi_{\alpha_1...\alpha_{2j_1}}^{\beta_1...\beta_{2j_2}}(x)$ | $i\Psi(x)$ |

Besides of the five independent transformations $P, T, C, I_z, I_3$, we include in these tables two operations related to the change of sign of time (Wigner time reversal $T_w$ and Schwinger time reversal $T_{sch}$), inversion $I_x$ (which affects only spacetime coordinates $x^\mu$), and $PCT_w$-transformation.

It is easy to see that $C^2 = P^2 = T^2 = 1$. Operators $I_z, I_3$ correspond to products of involutory inner automorphisms and the rotation by the angle $\pi$ (see (3.25)). Hence $I_z^2 = I_3^2 = T_w^2 = R_{2\pi}$, where $R_{2\pi}$ is the operator of rotation by $2\pi$. It changes the signs of the spin variables, $f(x, z, \bar{z}) \xrightarrow{R_{2\pi}} f(x, -z, \bar{z})$. The latter corresponds to the multiplication by the phase factor $(-1)^{2(j_1+j_2)}$ only.

In the general case the transformation laws for particle and antiparticle spin-tensor fields are distinguished by signs (for space reflection this fact is pointed out, in particular, in [23]). This signs play an important role, because their change leads to non-commutativity of discrete transformations.

There are two different transformations $C$ and $I_z$ which interchange particle and antiparticle fields. The operator $I_z$ is a spin part of $PCT_w$-transformation. Indeed, the relation
\( PCT_w = I_x I_z \) means that \( PCT_w \)-transformation is factorized in inversion \( I_x \), affecting only spacetime coordinates \( x^\mu \) and in \( I_z \)-transformation, affecting only spin coordinates \( z \).

Consider now scalar fields which are eigenfunctions for \( C \); they describe neutral particles. Such fields obey the condition \( C f(h) = \overline{f(h)} = e^{i\phi} f(h) \). Multiplying them by the phase factor \( e^{i\phi/2} \) we transform them to real fields obeying the condition \( C f(h) = f(h) \). The charge conjugation \( C \) maps \( z, \bar{z} \) into the complex conjugate pair. Thus, there are two invariant (with respect to \( C \)) subspaces of the scalar functions, namely, spaces of real functions \( f(x, z, \bar{z}) \) and \( f(x, \bar{z}, \bar{z}) \), which we denote by \( V_z \) and \( V_{\bar{z}} \) respectively. They are mapped into one another under space reflection, \( V_z \leftrightarrow V_{\bar{z}} \).

Linear in \( z, \bar{z} \) eigenfunctions of \( C \) (with the eigenvalue 1) have the form

\[
\psi \alpha(x) z^\alpha - \psi^\dot{\alpha}(x) \bar{z}^{\dot{\alpha}} = Z_M \Psi_M(x),
\]

(4.17)

where \( \Psi_M(x) \) is a Majorana spinor, \( \psi_M(x) = i\gamma^2 \Psi_M(x) \). The space reflection maps functions from \( V_z \) into functions from \( V_{\bar{z}} \),

\[
P : \quad Z_M \Psi_M(x) \rightarrow Z_M \Psi_M(x), \quad \bar{Z}_M = (z^\alpha \bar{z}^{\dot{\alpha}}),
\]

\[
\Psi_M(x) = -\left( \psi^\alpha(x), -\psi^{\dot{\alpha}}(x) \right), \quad \bar{\Psi}_M(x) = -\gamma^0 \gamma^5 \Psi_M(x) = i\gamma^0 \gamma^5 \gamma^2 \Psi_M(x).
\]

(4.18)

Therefore, the spaces \( V_z \) and \( V_{\bar{z}} \) (in contrast to the spaces \( V_+ \) and \( V_- \)) do not contain eigenfunctions of \( P \) (i.e. states with definite parity). According to (3.27) and (3.28) we obtain

\[
I_z : \quad Z_M \Psi(x) \rightarrow Z_M \Psi(x), \quad (4.19)
\]

\[
I_3 : \quad Z_M \Psi(x) \rightarrow -Z_M i\gamma^5 \Psi(x). \quad (4.20)
\]

Thus, there are four nontrivial independent discrete transformations for the fields under consideration. These transformations for bispinors \( \Psi(x) \) and \( \psi_M(x) \) are performed by matrices from the same set. However, one and the same discrete symmetry operation induces different operations with bispinors \( \Psi(x) \) and \( \Psi_M(x) \).

The \( PCT_w \)-transformation maps the spaces of functions \( f(x, z, \bar{z}) \) and \( f(x, \bar{z}, \bar{z}) \) into themselves. Such functions can be used, in particular, for describing ”physical” Majorana particle defined as \( PCT_w \)-self-conjugate particle with spin \( 1/2 \) [30].

V. TRANSFORMATIONAL LAWS OF OPERATORS

Consider the action of the involutory automorphisms, which correspond to the discrete transformations, in terms of Lee algebra of the Poincaré group. Generators of the Poincaré group in the left generalized regular representation have the form

\[
\hat{p}_\mu = -i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu}, \quad (5.1)
\]
where $\hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ are orbital momentum operators and $\hat{S}_{\mu\nu}$ are spin operators depending on $z$ and $\partial/\partial z$. Explicit form of the spin operators is given in the Appendix. The generators (5.1) obey the commutation relations

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\eta_{\nu\rho}\hat{J}_{\mu\sigma} - i\eta_{\mu\rho}\hat{J}_{\nu\sigma} - i\eta_{\nu\sigma}\hat{J}_{\mu\rho} + i\eta_{\mu\sigma}\hat{J}_{\nu\rho}. \quad (5.2)$$

Fields on the Poincaré group depend on 10 independent variables. For the classification of these fields one can use a complete set of commuting operators on the group, which along with the left generators (5.1) includes generators in the right generalized regular representation

$$T_R(g)f(x, z) = f(xg, zg), \quad xg \leftrightarrow X + ZAZ^\dagger, \quad zg \leftrightarrow ZU. \quad (5.3)$$

As a consequence of the formula (5.3) one can obtain

$$\hat{p}^R_\mu \sigma^\mu = -Z^{-1}\hat{p}_\nu \sigma^\nu (Z^\dagger)^{-1}, \quad \hat{J}^R_{\mu\nu} = \hat{S}^R_{\mu\nu}. \quad (5.4)$$

Operators $\hat{S}_{\mu\nu}$ and $\hat{S}^R_{\mu\nu}$ from (5.1) and (5.4) are the left and right generators of $SL(2, C)$ and do not depend on $x$. All the right generators (5.4) commute with all the left generators (5.1) and obey the same commutation relations (5.2). Below for spin projection operators we use three-dimensional vector notation $\hat{S}_k = \frac{i}{2}\epsilon_{ijk}\hat{S}^{ij}$, $\hat{B}_k = \hat{S}_{0k}$. The explicit form of the spin operators is given by formulae (A1)-(A3).

According to theory of harmonic analysis on Lie groups [28,3] there exists a complete set of commuting operators, which includes Casimir operators, a set of the left generators and a set of right generators (both sets contain the same number of the generators). The total number of commuting operators is equal to the number of parameters of the group. In a decomposition of the left GRR the nonequivalent representations are distinguished by eigenvalues of the Casimir operators, equivalent representations are distinguished by eigenvalues of the right generators, and the states inside the irrep are distinguished by eigenvalues of the left generators.

The physical meaning of right generators usually is not so transparent as of left ones. However, the right generators of $SO(3)$ in the nonrelativistic rotator theory are interpreted as angular momentum operators in a rotating body-fixed reference frame [31,33]. Since the right transformations commute with the left ones, they define quantum numbers, which are independent of the choice of the laboratory reference frame.

The right generators $\hat{S}^3_R$ and $\hat{B}^3_R$ of the Poincaré group can be used to distinguish functions from the subspaces $V_+, V_-$ and $V_{\underline{z}}, V_{\underline{\bar{z}}}$. Polynomials of power $2s$ belonging to $V_+$ and $V_-$ are eigenfunctions of the operator $\hat{S}^3_R$ with eigenvalues $\mp s$ respectively. Polynomials of power $2s$ belonging to $V_{\tilde{z}}$ and $V_{\tilde{\bar{z}}}$ are eigenfunctions of the operator $i\hat{B}^3_R$ with eigenvalues $\mp s$ respectively.

The explicit form of the generators (see (5.1),(5.4) and (A1),(A2)) allows us easily to establish their transformation properties under involutory automorphisms, and thus under discrete transformations. The transformations $P, T$ correspond to the outer automorphisms of the algebra. Therefore the left and the right generators have identical transformation rules under $P$ and $T$; in particular,

$$\hat{p}_\mu \rightarrow \mp(-1)^{\delta_{0\mu}}\hat{p}_\mu, \quad \hat{S} \rightarrow \hat{S}, \quad \hat{B} \rightarrow -\hat{B},$$
where the upper sign corresponds to $P$. Obviously, spatial and boost components of total and orbital angular momenta have the same transformation rules that $\hat{S}$ and $\hat{B}$.

Complex conjugation $C$ leads to the change of sign of all generators, as it follows from their explicit form. Signs in the commutation relations are also changed, and for their restoring it is necessary to replace $i$ by $-i$ in (5.2).

The transformations $I_z, I_3$, connected with inner automorphisms, according to (3.26) are defined as right finite transformations of the Poincaré group. They do not affect the left generators because the right transformations commute with the left ones. Thus the transformations $I_z, I_3$ induce automorphisms of the algebra of the right generators: $I_z$ changes signs of the first and the third components of $\hat{S}_R$ and $\hat{B}_R$, and $I_3$ changes signs of the first and the second components of $\hat{S}_R$ and $\hat{B}_R$.

An intrinsic parity of a massive particle is defined as the an eigenvalue of the operator $P$ in the rest frame, $P f(h) = \eta f(h)$, $\eta = \pm 1$. Since the operator $P$ commutes with $T, C, I_z$, the intrinsic parity is not changed under corresponding discrete transformations.

Transformation properties of physical quantities under the discrete transformations are represented in Table 3. To compose this table we have used Tables 1,2 and the explicit form of corresponding operators given in the Appendix. The intrinsic parity $\eta$ and the sign of $p_0$ label irreps of the improper Poincaré group, which include space reflection; farther the table includes the left generators $\hat{p}_\mu$, the spin parts $\hat{S}, \hat{B}$ of the left Lorentz generators, and two right Lorentz generators.

We also include a current four-vector $j_\mu$ for the first order equation (A6) (the Dirac and Duffin-Kemmer equations are the particular cases of this equation for $s = 1/2$ and $s = 1$ respectively). In the space of functions on the group this current is described by the operators $\hat{\Gamma}_\mu$, see (A7). The particle and antiparticle fields are distinguished by the sign of charge, i.e. by the sign of a component $j_0$ of the current vector. One can see from the table, that for scalar fields on the group the sign of the right generator $S^3_R$ can be used to distinguish particles and antiparticles, because this sign and the sign of $j_0$ have identical transformation rules under the discrete transformations. As we will show, the sign of the mass term in the equation (A6) is changed with the sign of the product $p_0S^3_R$ under discrete transformations, see the next to last column of Table 3.

Table 3. The action of the discrete transformations on the signs of physical quantities.

| $\eta$ | $p_0$ | $\hat{p}$ | $\hat{S}$ | $\hat{B}$ | $S^3_R$ | $B^3_R$ | $j_0$ | $j_1$ | $p_0S^3_R$ | L-R |
|-------|-------|----------|----------|----------|--------|--------|-------|-------|-------------|------|
| $P$   | +     | +        | -        | +        | -       | -       | +     | +     | -           | -    |
| $T$   | +     | +        | +        | +        | +       | -       | +     | -     | -           | -    |
| $I_x = PT$ | +     | -        | -        | +        | +       | +       | +     | -     | -           | +    |
| $C$   | +     | -        | -        | -        | -       | -       | -     | +     | -           | +    |
| $CP$  | +     | -        | +        | -        | +       | -       | +     | +     | +           | -    |
| $T_{sch} = CT$ | +     | +        | -        | -        | +       | -       | -     | +     | -           | +    |
| $PCT = I_x C$ | +     | +        | -        | -        | -       | -       | -     | -     | -           | -    |
| $I_z$ | +     | +        | +        | +        | -       | -       | -     | -     | +           | -    |
| $T_w = I_z CT$ | +     | +        | -        | -        | +       | -       | +     | +     | +           | +    |
| $PT_w = I_z I_x C$ | +     | +        | -        | -        | +       | +       | +     | +     | +           | +    |
| $PCT_w = I_z I_x$ | +     | -        | +        | -        | +       | -       | -     | +     | -           | -    |

1) for the states described by the first order equation (A6)
The last column of the table ("L-R") describes the passage between two types of spinors (left $\dot{z}_a$, $\dot{\bar{z}}_a$ and right $z^\alpha$, $\bar{z}^\alpha$) labelled by dotted and undotted indices. If the transformation interchanges dotted and undotted spinors, then this column contains sign "−", and contains sign "+") in the opposite case. If we define a chirality as the difference between the number of dotted and undotted indices, then the last column of the table corresponds to the sign of the chirality. In the space of functions on the group the chirality is described by the operator $\hat{\Gamma}_5$, see (A9).

The time reflection transformation $T$ (in context of the Lorentz group it was considered in detail in [7]) maps positive energy states into negative energy ones. On the other hand, time reversal is defined usually by the relation $x \rightarrow -\bar{x}$ with supplementary condition of conservation of the energy sign. Obviously, the product of time reflection and charge conjugation $CT$, which we denote by $T_{sch}$, obeys this condition. The transformation $T_{sch}$ was introduced by Schwinger [9], see also [10]. This transformation interchanges particle and antiparticle fields with opposite sign of the component $j_0$ of the current vector.

The time reversal transformation $T_w$ was considered for the first time by Wigner [8]. Wigner time reversal conserves the sign of $j_0$. Connecting different states of the same particle, this transformation is an analog of time reversal in nonrelativistic quantum mechanics. Changing signs of the vectors $p, S, j$, Wigner time reversal corresponds to the reversal of the direction of motion. (Notice that sometimes the term ”time reversal” is used instead of ”time reflection” also for transformations changing the sign of energy, which can lead to misunderstanding.)

The transformation $I_z$ is the right finite transformation of the proper Poincaré group, see (3.26). This transformation leaves signs of the left generators unaltered (because the right transformations commute with the left ones) but changes signs of the current vector and of some right generators. Hence, the left generators have identical transformation rules under $T_{sch}$ and $T_w = I_z T_{sch}$. The transformation $I_3$, as was mentioned above, change the sign of the first and second components of vectors $S_R$ and $B_R$ only and conserve signs of all quantities contained in Table 3.

VI. REPRESENTATIONS OF THE EXTENDED POINCARÉ GROUP

Consider the problem of extending the Poincaré group by means of the discrete transformations.

Different fields on the Poincaré group with identical transformation rule under left transformations (i.e. under rotations and translations of the reference frame) carry equivalent subrepresentations of the left generalized regular representation (3.1), even if they have different transformation rules under right transformations (5.3). The discrete transformations (automorphisms) also act in the space of functions on the group, and functions carrying equivalent representations of the proper Poincaré group can be transformed differently under discrete automorphisms. Therefore, these functions carry non-equivalent representations of the Poincaré group extended by the discrete transformations.

The operator $I, I^2 = 1$, corresponding to a discrete transformation and the identity operator form finite group $Z_2$ consisting of two elements. The operator $I$ allows us to distinguish two types of states, one with definite ”charge” and another with definite ”charge parity”. Two states with opposite ”charges”, which we denote by $\psi_+$ and $\psi_-$, are interchanged under
the reflection $I$: $\psi_+ \leftrightarrow \psi_-$. The states $\psi_+ \pm \psi_-$ with definite "charge parity" are eigenfunctions of operator $I$ with the eigenvalues $\pm 1$ and form the bases of one-dimensional irreps of the group $Z_2$. The operators $(1 \pm I)/2$ obey the condition $((1 \pm I)/2)^2 = (1 \pm I)/2$ and thus they are projection operators on the states with definite "charge parity".

The operators of the discrete transformations (automorphisms) commute with each other and commute (sign "+") in Table 3) or anticommute (sign "−") in Table 3) with the generators of the Poincaré group. The latter means that the discrete transformation interchanges eigenfunctions of the generator with opposite eigenvalues (opposite "charges").

Here for clearness we adduce the table with parameters labelling the Poincaré group irreps which correspond to finite-component (with respect to spin) massive and massless fields.

Table 4. Parameters labelling irreps of the proper and of improper Poincaré groups.

| proper Poincaré group | improper Poincaré group with space reflection |
|-----------------------|---------------------------------------------|
| massive case          | $m, \text{sign } p_0, s$                    |
| massless case         | $\text{sign } p_0, \lambda$                |




\(1\) for $\lambda \neq 0$ massless irreps with $\eta = \pm 1$ are equivalent \[34,4\].

Here the mass $m > 0$, the spin $s = 0, 1/2, 1, \ldots$, the intrinsic parity $\eta = \pm 1$, and the helicity $\lambda = 0, \pm 1/2, \pm 1, \ldots$. Mass $m$ and sign of $p_0$ label the orbit (the upper or lower sheet of hyperboloid or cone), $s$ and $\lambda$ label irreps of the little groups $SO(3)$ and $SO(2)$. $s, \eta$ and $|\lambda|, \eta$ label irreps of the little groups $O(3)$ and $O(2)$ respectively (see \[2,4\] for details). The mass and the spin can by also defined as eigenvalues of the Casimir operators:

$$\hat{p}^2 f(x, z) = m^2 f(x, z), \quad \hat{W}^2 f(x, z) = -m^2 s(s + 1)f(x, z), \quad (6.1)$$

where $\hat{W}^\mu$ is Lubanski-Pauli four-vector, $z = (z, z, z, z)$ are coordinates on the Lorentz group.

In order to find parameters labelling irreps of the extended Poincaré group, we consider four independent discrete transformations: $P, I_x, I_z, C$.

1. Irreps of the improper Poincaré group including space reflection $P$ can be classified (as it was mentioned above) with the help of the little group method. Space reflection allows us to distinguish two types of the states: ones with definite intrinsic parity $\eta$ and ones with definite (left or right) charge, or chiral states. In the space of functions on the group the chirality operator $\hat{\Gamma}^5$ is given by the formula (A9). Fields with zero chirality can be considered as pure neutral ones with respect to space reflection.

2. Inversion $I_x$ affects only spacetime coordinates $x$. It couples two irreps of the proper (or improper) Poincaré group characterized by sign $p_0 = \pm 1$ into one representation of extended group. Eigenstates of $I_x$ are the states with definite "energy parity". However, since the sign of energy $p_0$ is already used to label irreps of the proper group, this extension does not lead to appearing a supplementary characteristic.

3. As mentioned earlier, the operator $I_z$ is the spin part of $PCT_w$-transformation, $PCT_w = I_x I_z$, and affects only spin coordinates $z$. Operator $I_z$ commutes with all the left generators and with space reflection $P$, and thus can’t change the parameters labelling
irreps of the proper or improper Poincaré group. $I_z$ interchanges the states with opposite eigenvalues of $S_3^R$; a charge parity $\eta_c = \pm 1$ arises as its eigenvalue. Corresponding extension of the group, conserving all characteristics from Table 4, gives, in addition, the charge parity $\eta_c$ as a characteristic of irreps. Below, taking into account the simple relation of $I_z$ to $PCT_w$-transformation, we will call corresponding quantities by $PCT_w$-charge and $PCT_w$-parity.

4. Charge conjugation $C$ changes signs of all the generators. In fact, this means that any extension connected with different sets of the discrete transformations including $C$ must be considered separately. Here we note that $C$ does not change $\eta$ and $\eta_c$ and like $I_z$ changes sign of the charge $S_3^R$. However, if the Poincaré group is already extended by $P, T, I_z$, then instead of $C$ one can consider Wigner time reversal $T_w = I_z CT$ as fourth independent transformation. The latter corresponds to the reversal of the direction of motion and does not change $P$-charge (chirality) and $I_z(CPT_w)$-charge.

As a result, considering four independent discrete transformations, we have established that irreps of the extended Poincaré group have two supplementary characteristics with respect to irreps of the proper group: the intrinsic parity $\eta$ and $PCT_w$-parity $\eta_c$. These characteristics are associated with $P$-charge (chirality) and $PCT_w$-charge (in particular it distinguishes particles and antiparticles), which in the space of functions on the group are defined as eigenvalues of the operators $\Gamma^5$ and $S_3^R$. It is necessary to note that for the particles with half-integer spins these charges also are half-integer. This means that for half-integer spins there are no pure neutral (with respect to the discrete transformations under consideration) particles, which have zero chirality or zero $PCT_w$-charge. On the other hand, both for integer and half-integer spins it is possible to construct states with definite intrinsic parity (e.g., the Dirac field) or with definite $PCT_w$-parity (e.g., "physical" Majorana field), which are mapped into themselves under $P$ and $PCT_w$ respectively.

VII. DISCRETE SYMMETRIES OF THE RELATIVISTIC WAVE EQUATIONS. MASSIVE CASE

Here we explicitly construct the massive fields on the Poincaré group and analyze their characteristics associated with the discrete transformations. On this base we give a compact group-theoretical derivation of basic relativistic wave equations and consider their discrete symmetries. In particular, this allow us to solve an old problem concerning two possible signs of a mass term in the Dirac equation. We also give a classification of the solutions of various types of higher spin relativistic wave equations with respect to characteristics of the extended Poincaré group irreps; this classification is turned out to be nontrivial, especially for the case of first order equations.

Consider eigenfunctions of operators $\hat{p}_\mu$ (plane waves). For $m \neq 0$ there exists the rest frame, where the dependence on $x$ is reduced to the factor $e^{\pm imx^0}$. Linear functions of coordinates $z$ on the Lorentz group correspond to spin $1/2$. For fixed mass $m$ there are 16 linearly independent functions:

\begin{align}
L: & V_+ = e^{\pm imx^0} z^\alpha \\
R: & V_- = e^{\pm imx^0} \bar{z}^\alpha
\end{align}

(7.1)
They can by classified (labelled) by means of left generators of the Poincaré group and the operators of the discrete transformations \( P, C \). The eigenvalues of the left generators \( \hat{J}^3 \) (in the rest frame \( \hat{J}^3 = \hat{S}^3 \)) and \( \hat{p}^0 \) give the spin projection (for \( \alpha, \hat{\alpha} = 1 \) and \( \alpha, \hat{\alpha} = 2 \) we have \( s^3 = \pm 1/2 \) respectively) and the sign of \( p_0 \). This sign, along with the mass \( m \) and the spin \( s \), characterizes nonequivalent irreps of the proper Poincaré group. The operator \( I_\eta \) interchanges the states with opposite signs of \( p_0 \), the operator \( P \) interchanges \( L- \) and \( R- \) states (states with opposite chiralities), and the operators \( C \) and \( I_\eta \) interchange the particle and antiparticle states, belonging to the spaces \( V_+ \) and \( V_- \) respectively. Unlike the charge conjugation operator \( C \) the operator \( I_\eta \) conserves signs of energy and chirality.

However, the states (7.4) with definite chirality are transformed under reducible representation of the improper Poincaré group, whose irreps are characterized by intrinsic parity \( \eta \). In the rest frame the states with definite \( \eta \) are eigenfunctions of the operator \( P, P f(x, z) = \eta f(x, z) \):

\[
\begin{align*}
\eta = -1 : & \quad e^{\pm imx^\alpha} (z^\alpha + \*z^\alpha) \quad \quad e^{\pm imx^\alpha} (z^\alpha - \*z^\alpha) \\
\eta = 1 : & \quad e^{\pm imx^\alpha} (z^\alpha - \*z^\alpha) \quad \quad e^{\pm imx^\alpha} (z^\alpha + \*z^\alpha)
\end{align*}
\]

As in the case of the states (7.1), the operators \( C \) and \( I_\eta \) interchange functions from the spaces \( V_+ \) and \( V_- \). On the other hand, the states with different intrinsic parity \( \eta = \pm 1 \) (unlike the states with different chirality) are not interchanged by the operators of the discrete transformations.

Both the states (7.1) and (7.2) are eigenstates of the Casimir operators \( \hat{p}^2 \) and \( \hat{W}^2 \) of the Poincaré group with the eigenvalues \( m^2 \) and \(-3/4m^2\). But only the states (7.2) are transformed under irrep of the improper Poincaré group. Besides, it is easy to check that the states (7.2) (unlike the states (7.1)) are the solutions of the equations

\[
(\hat{p}_\mu \hat{\Gamma}^\mu \pm ms) f(x, z, \*z) = 0, \quad (\hat{p}_\mu \hat{\Gamma}^\mu \mp ms) f(x, \*z, z) = 0,
\]

where \( s = 1/2 \), upper sign corresponds to \( \eta \) \( \text{sign} \) \( p_0 = 1 \), and lower sign corresponds to \( \eta \) \( \text{sign} \) \( p_0 = -1 \). The operator \( \hat{p}_\mu \hat{\Gamma}^\mu \) (explicit form of \( \hat{\Gamma}^\mu \) is given by (A7)) is not affected by space inversion and charge conjugation. The spaces \( V_+ \) and \( V_- \) also are invariant under space reflection, but they are interchanged under charge conjugation.

Considering the action of the discrete transformations on the component \( j_0 \) of current four-vector of free equation, we have established above that if particle is described by the function from the space \( V_+ \), then antiparticle is described by the function from the space \( V_- \). This can be also shown on the base of the equations in an external field. Acting by the operator of charge conjugation \( C \) (which in the space of functions on the group acts as the operator of complex conjugation) on the equation

\[
((\hat{p}_\mu - eA_\mu(x)) \hat{\Gamma}^\mu \pm ms) f(x, z, \*z) = 0
\]

for functions \( f(x, z, \*z) \in V_+ \), we obtain that the functions \( \*f(x, z, \*z) \in V_- \) obey the equation with opposite charge:

\[
((\hat{p}_\mu + eA_\mu(x)) \hat{\Gamma}^\mu \pm ms) \*f(x, z, \*z) = 0.
\]
Substituting the functions \( f(x, z, z) = Z_D \Psi(x) \) and \( \hat{f}(x, \hat{z}, \hat{z}) = Z_D \Psi(c)(x) \) into equations (7.4) and (7.3) (see also (1.1), (1.2)), we obtain two Dirac equations for charge-conjugate bispinors \( \Psi(x) \) and \( \Psi(c)(x) \):

\[
((\hat{p}_\mu - eA_\mu(x))\gamma^\mu \pm m)\Psi(x) = 0, \quad ((\hat{p}_\mu + eA_\mu(x))\gamma^\mu \pm m)\Psi(c)(x) = 0.
\]

Thus we have to use the different scalar functions on the group to describe particles and antiparticles and hence two Dirac equations for both signs of charge respectively. That matches completely with the results of the article [35]. It was shown there that in the course of a consistent quantization of a classical model of spinning particle namely such (charge symmetric) quantum mechanics appears. At the same time it is completely equivalent to the one-particle sector of the corresponding quantum field theory.

In Section 8 we continue to consider spin-1/2 case and give an exact group-theoretical formulation of conditions which lead to the Dirac equation.

In general case for the classification of functions corresponding to higher spins it is necessary to use a complete set of ten commuting operators (including also right generators) on the group, for example

\[
\hat{p}_\mu, \hat{W}^2, \hat{p}\hat{S}, \hat{S}^2 - \hat{B}^2, \hat{S}^3_R, \hat{B}^3_R.
\]

In the rest frame \( \hat{p}\hat{S} = 0 \), and the complete set can be obtained from (7.6) by changing \( \hat{p}\hat{S} \) to \( \hat{S}^3 \). Functions from the spaces \( V_+ \) and \( V_- \) depend on eight real parameters, and therefore one can consider only eight operators; our choice is

\[
\hat{p}_\mu, \hat{W}^2, \hat{p}\hat{S} (\hat{S}^3 \text{ in the rest frame}), \hat{p}_\mu\hat{\Gamma}^\mu, \hat{S}^3_R.
\]

The problem of constructing the complete sets of the commuting operators on the Poincaré group was discussed in [36,3,19].

Consider eigenfunctions of the operators (7.7). For functions from the spaces \( V_+ \) and \( V_- \) one can show that if the eigenvalue of \( \hat{p}_\mu\hat{\Gamma}^\mu \) is equal to \( \pm ms \), where \( 2s \) is the power of polynomial (the eigenvalue of \( \hat{S}^3_R \)), then the eigenvalue of the operator \( \hat{W}^2 \) is also fixed and corresponds to spin \( s \) [19]. Thus, the system

\[
\hat{p}^2 f(x, z) = m^2 f(x, z), \quad \hat{p}_\mu\hat{\Gamma}^\mu f(x, z) = \pm ms f(x, z), \quad \hat{S}^3_R f(x, z) = \pm s f(x, z),
\]

picks out the states with definite mass and spin.

Depending on the choice of the functional space and of the sign of the mass term, the second equation of the system (7.8) can be written in one of four forms:

\[
V_+: \quad (\hat{p}_\mu\hat{\Gamma}^\mu + ms) f(x, z, \hat{z}) = 0 \quad (\hat{p}_\mu\hat{\Gamma}^\mu - ms) f(x, \hat{z}, \hat{z}) = 0 \quad (7.9)
\]

\[
V_-: \quad (\hat{p}_\mu\hat{\Gamma}^\mu - ms) f(x, z, \hat{z}) = 0 \quad (\hat{p}_\mu\hat{\Gamma}^\mu + ms) f(x, \hat{z}, \hat{z}) = 0 \quad (7.10)
\]

In the rest frame for definite \( m \) and \( s \) the solutions of equations (7.9) and (7.10) are given by (7.11) and (7.12) respectively:
\begin{align}
\eta \text{ sign } p_0 &= -1 : \quad e^{\pm imx^0}(z^1 \pm \frac{s}{2})s^{3}\eta(x^2 \pm \frac{s}{2})s^{3} \\
\eta \text{ sign } p_0 &= 1 : \quad e^{\mp imx^0}(z^1 \pm \frac{s}{2})s^{3}\eta(x^2 \pm \frac{s}{2})s^{3} \\
\end{align}

Here the sign of \( \eta p_0 \) is specified for half-integer spins; for integer spins all solutions are characterized by \( \eta = 1 \). These solutions are eigenfunctions of the Casimir operators \( \hat{p}^2 \) and \( \hat{W}^2 \) with the eigenvalues \( m^2 \) and \( -s(s+1)m^2 \) and of spin projection operator \( \hat{S}^3 \) with eigenvalue \( s^3 \), \(-s \leq s^3 \leq s\).

For half-integer spin a general solution of the system (7.8) with definite sign of the mass term in the second equation possesses definite sign of \( \eta p_0 \). This general solution carries a reducible representation of the improper Poincaré group, which is direct sum of two irreps with opposite signs of \( \eta \) and \( p_0 \). Hence, the general solution contains \( 2(2s+1) \) independent components. Since \( \eta \) is invariant under the discrete transformations, the representation carried by the solution remains reducible under the extended Poincaré group.

Thus, for half-integer spin \( s \) the sign in the equations (7.9), (7.10) coincides with the sign of the product

\[ \eta p_0 S^R_3, \quad (7.13) \]

where the sign of \( S^R_3 \) distinguishes particles and antiparticles; this sign is fixed by the choice of the space \( V_+ \) or \( V_- \). In each space the general solution carries the direct sum of two irreps of the improper Poincaré group characterized by sign \( p_0, \eta \) or \( - \eta \). For integer spin in each space the general solution carries the direct sum of two irreps characterized by fixed intrinsic parity \( \eta = 1 \) and different signs of \( p_0 \).

As we saw, the formulation on the base of the set (7.4) including the first order in \( \partial / \partial x \) operator \( \hat{p}_\mu \Gamma^\mu \) allows us to fix some characteristics of representations of the extended Poincaré group. As it was shown in [14], for the case of finite-dimensional representations of the Lorentz group the system (7.8) is equivalent to the Bargmann-Wigner equations. In turn, for half-integer spins the latter equations are equivalent to the Rarita-Schwinger equations [13]. Hence the above conclusions concerning the structure of the solutions of the system (7.8) are also valid for mentioned equations.

It is obvious that for the equations fixing not only \( m \) and \( s \) (like the Casimir operators (5.1)), but also such characteristics as signs of the energy or of the charge, only a part of the discrete transformations forms the symmetry transformations. For example, a discrete symmetry group of equations (7.9), (7.10) with definite sign of the mass term (and therefore discrete symmetry groups of the Dirac and Duffin-Kemmer equations) includes only the transformations that conserve the sign of \( p_0 S^3_R \).

The transformations \( P, C, T_w \) conserve the sign of \( p_0 S^3_R \) and therefore the sign of the mass term in the first order equations under consideration. Fourth independent transformation (one can take, for example, inversion \( I_x \) or Schwinger time reversal \( T_{sch} \)) changes the sign of \( p_0 S^3_R \) and correspondingly the sign of the mass term.

The Majorana equations associated with infinite-dimensional irreps of \( SL(2, C) \) [37,38] are more restrictive with respect to possible symmetry transformations, since allow only the transformations conserving the sign of \( p_0 \) [39,40].

On the other hand, a formulation on the base of the set (7.6) of commuting operators and the use of representations \( (s0) \oplus (0s) \) of the Lorentz group allows all four independent discrete
transformations as symmetry transformations. To pick out the representation \((s0) \oplus (0s)\) one can use the Casimir operators of the Lorentz group or (for the subspaces \(V_+\) and \(V_-\) the operators \(\hat{B}_R^3, \hat{S}_R^3\), all these operators are contained in the set \((7.6)\). One can show that for subspaces \(V_+\) and \(V_-\) the system of equations
\[
\hat{p}^2 f(x, z) = m^2 f(x, z), \quad \hat{S}_R^3 f(x, z) = \pm s f(x, z), \quad i\hat{B}_R^3 f(x, z) = \pm s f(x, z),
\]
(7.14)
fixes the spin \(s\) \([19]\). For definite spin projection \(s^3\), solutions of the system in the rest frame have the form (in contrast to \((7.11)\), \((7.12)\) signs in exponents and in brackets can be taken independently):
\[
V_+: \quad e^{\pm imx^0} \left( (z^1)^s + s^3 (z^2)^s - s^3 \right) \pm \left( (z_1^*)^s + s^3 (z_2^*)^s - s^3 \right), \tag{7.15}
\]
\[
V_-: \quad e^{\pm imx^0} \left( (z_1)^s + s^3 (z_2)^s - s^3 \right) \pm \left( (z_1^*)^s + s^3 (z_2^*)^s - s^3 \right). \tag{7.16}
\]
The sign in brackets defines the intrinsic parity of the solution. For half-integer spins the upper sign corresponds to \(\eta = 1\) and the lower sign corresponds to \(\eta = -1\). For integer spins the upper sign in \((7.15)\) and the lower sign in \((7.16)\) correspond to \(\eta = 1\), and the opposite signs correspond to \(\eta = -1\). Thus, for each space \((V_+\) or \(V_-)\) the general solution of the system has \(4(2s + 1)\) independent components and carries the reducible representation of the improper Poincaré group. This representation splits into four irreps labelled by different signs of \(\eta\) and \(p_0\).

The formulation under consideration allows the coupling of higher spin with electromagnetic field. This connected with the fact that unlike the system \((7.8)\) the system \((7.14)\) contain only one equation with operator depending on \(\partial/\partial x\). (For the system \((7.8)\) the first equation is a consequence of other two only in the cases \(s = 1/2\) and \(s = 1\) \([19]\), which correspond to the Dirac and the Duffin-Kemmer equations.) Particles with definite spin \(s\) and mass \(m\) are described by Klein-Gordon equation with polarization,
\[
((\hat{p} - A(x))^2 - \frac{e}{2s} \hat{F}_{\mu\nu} F^{\mu\nu} - m^2)\psi(x) = 0,
\]
where \(\psi(x)\) is transformed under the representation \((s0) \oplus (0s)\) of the Lorentz group \([11-15]\). For \(s = 1/2\) this equation is the squared Dirac equation. Solutions of the Klein-Gordon equation with polarization are casual, but in contrast to the Dirac and Duffin-Kemmer equations, whose solutions have \(2(2s + 1)\) independent components, these solutions have \(4(2s + 1)\) independent components (for any sign of energy there are solutions corresponding to the intrinsic parity \(\eta = \pm 1\)).

Irrespective of the specific form of relativistic wave equations, the above analysis shows that in the massive case two of four nontrivial discrete transformations map any irrep of the improper Poincaré group into itself; these transformations are \(P\) and \(T_w\). The operator \(P\) labels irreps of the improper group. Wigner time reversal \(T_w\) corresponds to the reversal of the direction of motion and does not change characteristics of representations of the Poincaré group extended by other discrete transformations (\(\eta\) and signs of energy and \(PCT\)-charge). For example, for spin-1/2 particle at rest (see \((7.1)\)) we have \(e^{imx^0} z_\alpha T_w^\alpha e^{imx^0} z_\alpha\), and transformation reduces to the rotation by the angle \(\pi\). In general case \(T_w\) is not reduced to continuos or other discrete transformations. \(T_w\) changes signs both of momentum vector and spin pseudovector, unlike \(P\) changing only the sign of vectors.
Two discrete transformations interchange nonequivalent representations of the improper group extended by the operator $I_z$, which distinguishes particles and antiparticles. As such transformations one can choose $I_x$ and $I_z$ or $I_x$ and $C$, in correspondence with that done below, where the first sign is one of $PCT_w$-charge and the second sign is one of $p_0$:

$$
\begin{align*}
C &\uparrow & \eta = 1 & \quad & \eta = -1 & \downarrow C \\
++ &\quad & ++ & \quad & ++ \\
\downarrow I_x & \quad & \downarrow I_x & \quad & \downarrow I_x \\
-- &\quad & -+ & \quad & -- \\
\end{align*}
$$

(7.17)

A problem of relative parity of particle and antiparticle admits different treatments. For the first time it was pointed out in [46] for spin-1/2 particle; some other cases was studied in [18,47,23]. Consider the problem in the framework of the representation theory of the extended Poincaré group.

As mentioned above, charge conjugation or $PCT_w$-transformation can’t change the intrinsic parity $\eta$, since $C$ and $I_z$ commute with $P$. Therefore, if one suppose that (i) a particle is described by an irrep of improper Poincaré group and (ii) corresponding antiparticle is described by $PCT_w$-conjugate (or charge-conjugate) irrep, then parities of particle and antiparticle must coincide for any spin. An alternative possibility is instead of (ii) to suppose that antiparticle is described by the irrep labelled not only by opposite $PCT_w$-charge, but also by opposite parity $\eta$. In the latter case the irreps describing particle and antiparticle are not connected by the discrete transformations $C$ or $PCT_w$.

Usually the relation between parities of particle and antiparticle is discussed on the base of various wave equations. Consider some relativistic wave equation describing field with definite spin and mass. As a rule, a general solution of the equation carries a reducible representation of the improper Poincaré group; irreducible subrepresentations (or their charge conjugate) are identified with particle and antiparticle fields. Since different equations have different structure of the solutions, both possibilities mentioned above can be realized in such an approach.

Consider some examples. One can suppose that for $s = 1/2$ ”wave function of antiparticle is a bispinor charge-conjugate to some ”negative-frequency” solution of the Dirac equation” [18]. Free Dirac equation have solutions corresponding to two nonequivalent irreps of the improper Poincaré group; these irreps are characterized by opposite signs of $\eta$ and $p_0$. If ”positive-frequency” solution has intrinsic parity $\eta$, then ”negative-frequency” solution has opposite intrinsic parity $-\eta$, which is not changed under charge conjugation and intrinsic parities of particle and antiparticle are opposite. Solutions of the Duffin-Kemmer equation with different signs of energy have identical intrinsic parity, and similarly we come to the conclusion that for spin 1 intrinsic parities of particle and antiparticle are identical. This consideration corresponds to the standard point of view. However, the study of a class of equations associated with the representations $(s0) \oplus (0s)$ of the Lorentz group leads to alternative conclusion that for integer spin the intrinsic parities of particle and antiparticle are opposite [47,23].
Let us consider a pure group-theoretical derivation of the Dirac equation in detail. The derivation is based on fixing of quantities characterizing the representations of the extended Poincaré group. In addition to the evident conditions (fixing the mass and spinor representation of the Lorentz group) it is necessary to demand that states with definite energy possess definite parity, and also that the states possess definite $PCT_w$-charge. This formulation shows that the sign of mass term in the Dirac equation coincides with the sign of the product of three characteristics of the extended Poincaré group representations, namely, the intrinsic parity, the sign of $PCT_w$-charge, and the sign of energy. Notice that the consideration and attempts of physical interpretation of two possible signs of the mass term in the Dirac equation have a long history, see, in particular, [20–22,24] and references therein.

Consider a representation of the extended Poincaré group with the following characteristics: (i) definite mass $m > 0$; (ii) definite $PCT_w$-charge; (iii) states with definite sign of energy possess definite intrinsic parity $\eta = \pm 1$; suppose then that (iv) the field $f(x,z)$ with above characteristics is linear in $z$ (or, that is the same, we fix the representation $(\frac{1}{2} 0) \oplus (0 \frac{1}{2})$ of the spin Lorentz subgroup).

According to (iii), this reducible representation of the extended Poincaré group, which we denote by $T_D$, is the direct sum of two representations labelled by opposite signs of energy and intrinsic parity.

The conditions (ii) and (iv) restrict our consideration to two functions

$$f_+(x,z) = z\psi_R + \bar{z}\psi_L, \quad f_-(x,z) = \bar{z}\psi_R + z\psi_L,$$

where we have introduced the columns $\psi_L = (\psi^\alpha)$, $\psi_R = (\psi_{\dot{\alpha}})$. These functions correspond to two possible signs of $PCT_w$-charge. According to (i) there exist functions $f(x,z)$ such that $\hat{p}_0 f(x,z) = p_0 f(x,z)$, $\hat{p}_k f(x,z) = 0$, corresponding to particles in the rest frame, where the energy $p_0 = \varepsilon_E m$, $\varepsilon_E = \text{sign } p_0$. According to (iii) these functions are characterized by the parity $\eta$, defined as an eigenvalue of the space inversion operator, $P f(x,z) = \eta f(x,z)$. Using the latter equation and the relation (3.23), we obtain for functions $f_+(x,z)$ and $f_-(x,z)$ that $\psi_R(\hat{p}) = -\eta \psi_L(\hat{p})$ and $\psi_R(\hat{p}) = \eta \psi_L(\hat{p})$ respectively, where $\hat{p} = (\varepsilon_E m, 0)$. Both the cases can be combined into one equation

$$\psi_R(\hat{p}) = \varepsilon_c \eta \psi_L(\hat{p}),$$

where $\varepsilon_c = \text{sign } S^R_3$ is the sign of charge.

The Lorentz transformation of the spinors $\psi_R(p) = U \psi_R(\hat{p})$, $\psi_L(p) = (U^\dagger)^{-1} \psi_L(\hat{p})$, corresponds to the transition to the state characterized by momentum $P = U p_0 U^\dagger$, where $P = p_\mu \sigma^\mu$, $P_0 = \varepsilon_E m \sigma_0$, whence we obtain

$$\varepsilon_E m U U^\dagger = p_\mu \sigma^\mu.$$  

\[5\] An heuristic discussion of the problem can be found in [49,47,51,23].
Taking into account the transformation law of spinors, one can rewrite the relation (8.2) in the form
\[ \psi_R = \varepsilon_c \eta UU^\dagger \psi_L, \quad \psi_L = \varepsilon_c \eta (UU^\dagger)^{-1} \psi_R. \]
Using (8.3), we express \( UU^\dagger \) in terms of momentum,
\[ m\psi_R = \varepsilon_c \varepsilon E \eta p_\mu \sigma^\mu \psi_L, \quad m\psi_L = \varepsilon_c \varepsilon E \eta \bar{p}_\mu \bar{\sigma}^\mu \psi_R, \]
and combine these two equations into one:
\[ (p_\mu \gamma^\mu - \varepsilon_c \varepsilon E \eta m) \Psi = 0, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \quad (8.4) \]
Finally, for the plane waves under consideration one can change the momentum \( p_\mu \) to the corresponding differential operator \( \hat{p}_\mu \). Since the plane waves form the basis of the representation \( T_D \) and the superposition principle is valid for differential equation obtained, the states belonging to \( T_D \) are subjected to the equation
\[ (\hat{p}_\mu \gamma^\mu - \varepsilon_c \varepsilon E \eta m) \Psi = 0. \quad (8.5) \]
In the above derivation one can use instead of (iii) more restrictive condition of irreducibility of the representation of the improper Poincaré group. But solutions of the equation obtained will be transformed under reducible representation obeying the condition (iii) anyway. The above derivation also shows the impossibility of the derivation of the Dirac equation only in terms of characteristics of irreps of the proper or improper Poincaré group, since the Dirac equation connects signs of the energy \( \varepsilon_E \), of the parity \( \eta \), and of the charge \( \varepsilon_c \), which characterize representations of the extended Poincaré group.

**IX. DISCRETE SYMMETRIES OF THE RELATIVISTIC WAVE EQUATIONS. MASSLESS CASE**

For standard massless fields with discrete spin the eigenvalues of the Casimir operators \( \hat{p}^2 \) and \( \hat{W}^2 \) are equal to zero (see, e.g., [4]). As a consequence such massless fields obey the conditions
\[ \hat{W}_\mu f(x, z) = \lambda \hat{p}_\mu f(x, z), \quad \lambda = \pm 1 \]
where \( \lambda \) is the helicity. For the component \( \hat{W}_0 \) we have
\[ \hat{p}\hat{S} f(x, z) = \lambda \hat{p}_0 f(x, z). \quad (9.2) \]
The transformations \( P \) and \( C \) change the sign in equation (9.2); on the other hand, the transformations \( I_x, I_z, T_{\text{sch}} \), which change the sign of mass term of the Dirac equation, are symmetry transformations of equation (9.2). Discrete symmetries of equation (9.2) are generated by three independent transformations, for example, \( I_x, CP, T_w \), the first of which is not a symmetry transformation of the Dirac equation.

The Weyl equation \( \hat{p}\sigma \Psi(x) = \pm \hat{p}_0 \Psi(x) \) is the particular case of (9.2) corresponding to helicity \( \pm 1/2 \); they can be obtained by the substitution of the function \( f(x, z) = \Psi_\alpha(x)z^\alpha \) into (9.2).
Massless irreps of the proper Poincaré group (see Table 4) are labelled by two numbers, namely, by the helicity $\lambda = pS/p_0$ and by the sign of $p_0$. If we do not claim that the states possess definite parity, then instead of the subspaces $V_+$ and $V_-$ it is natural to consider four subspaces of functions $f(x, z)$, $f(x, z^\dagger)$, $f(x, z^\ddagger)$, $f(x, z^\ddagger)$.

For definite chirality $s$ in any subspace equation (9.2) has four solutions which correspond to the motion along the axis $x^3$. These solutions are labelled by the signs of helicity and $p_0$. Considering the action of the operators $C$ and $I_z$, it is easy to see that these functions describe particles which are not coincide with their antiparticles.

For $p_0 > 0$ we have for particles

$$\lambda = s : \quad e^{i(px^0 + px^3)}(z^1)^2s, \quad e^{i(px^0 + px^3)}(z^2)^2s,$$

and for antiparticles

$$\lambda = s : \quad e^{i(px^0 + px^3)}(z^1)^2s, \quad e^{i(px^0 + px^3)}(z^2)^2s,$$

The operators $P$ and $C$ interchange the states with opposite chirality. The operator $I_z$, interchanging the states with opposite $PCT_w$-charge, conserves the signs of the chirality and of the energy.

The signs of the helicity and of the chirality are changed simultaneously under the discrete transformations. But unlike the parity $\eta$, which also is not changed under the discrete transformations, the sign connecting the helicity and the chirality characterizes equivalent representations of the extended Poincaré group.

Above we have developed the description of particles which differ from their antiparticles. As an example let us consider now the description of pure neutral massless spin-1 particle (photon) in terms of field on the Poincaré group. Such a particle is its own antiparticle (i.e. it has zero $PCT_w$-charge) and possesses chirality $\pm 1$. For the quadratic in $z = (z, z^\dagger, z^\ddagger)$ functions on the group these conditions are satisfied only by the fields depending on $z^\alpha z^\beta$, $z^\alpha z^\beta$, and being eigenfunctions of $\hat{S}_{3}^{3}$ with zero eigenvalue.

Thus, a pure neutral massless spin-1 particle should be described by the function

$$f(x, z) = \chi_{\alpha\beta}(x)z^\alpha z^\beta + \psi_{\alpha\beta}(x)z^\alpha z^\beta = \frac{1}{2}F_{\mu\nu}(x)q^{\mu\nu},$$

where

$$q_{\mu\nu} = -q_{\nu\mu} = \frac{1}{2}\left((\sigma_{\mu\nu})_{\alpha\beta}z^\alpha z^\beta + (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}z^{\dot{\alpha}}z^{\dot{\beta}}\right), \quad q^{\mu\nu} = q_{\mu\nu},$$

$$F_{\mu\nu}(x) = -2\left((\sigma_{\mu\nu})_{\alpha\beta}\chi_{\alpha\beta}(x) + (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}\psi_{\dot{\alpha}\dot{\beta}}(x)\right).$$

The functions $\chi_{\alpha\beta}(x)$ and $\psi_{\dot{\alpha}\dot{\beta}}(x)$ must be symmetric in their indices; in opposite case by virtue of the constraint $z^1z^2z^3 = 1$ (which is a consequence of unimodularity of $SL(2, C)$) the field (9.7) will contain components $\chi_{[\alpha\beta]}(x)$ and $\psi_{[\dot{\alpha}\dot{\beta}]}(x)$ of zero spin. Therefore the formulations in terms of $\chi_{\alpha\beta}(x)$, $\psi_{\dot{\alpha}\dot{\beta}}(x)$ and $F_{\mu\nu}(x)$ are equivalent. Left and right fields can be described by
\[ F^L_{\mu\nu}(x) = F_{\mu\nu}(x) - i\tilde{F}_{\mu\nu}(x) = -4(\sigma_{\mu\nu})_{\dot{\alpha}\dot{\beta}}\psi_{\dot{\alpha}\dot{\beta}}(x), \] (9.10)

\[ F^R_{\mu\nu}(x) = F_{\mu\nu}(x) + i\tilde{F}_{\mu\nu}(x) = -4(\sigma_{\mu\nu})_{\alpha\beta}\chi_{\alpha\beta}(x), \] (9.11)

where \( \tilde{F}_{\mu\nu}(x) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \).

To describe the states with definite helicity, the function \((9.7)\) should obey equation \((9.2)\) for \( \lambda = \pm 1 \),

\[ (\hat{p}\hat{S}\mp\hat{p}^0)f(x,z) = 0. \] (9.12)

For \( p_0 > 0 \) equation \((9.12)\) has four solutions which correspond to the motion along the axis \( x^3 \). These solutions are distinguished by signs of helicity \( \lambda \) and chirality:

\[ \lambda = 1 : \quad e^{i(px^0+px^3)}z_{\dot{1}}^{1,1}, \quad e^{i(px^0+px^3)}z_{\dot{1}}^{1,1}, \] (9.13)
\[ \lambda = -1 : \quad e^{i(px^0+px^3)}z_{\dot{2}}^{2,2}, \quad e^{i(px^0+px^3)}z_{\dot{2}}^{2,2}. \] (9.14)

Fixing the sign connecting helicity and chirality (this sign distinguishes the equivalent representations of the Poincaré group), we obtain two solutions corresponding to two polarization states.

Substituting the functions \( f_L(x,z) = \psi_{\dot{\alpha}\dot{\beta}}(x)z^{\dot{1}}\dot{z}^{\dot{2}} \) and \( f_R(x,z) = \chi_{\alpha\beta}(x)z^{\dot{1}}\dot{z}^{\dot{2}} \) into \((9.12)\) (for \( \lambda = \pm 1 \) respectively) and in accordance with \((9.10)\) \((9.11)\) going over to the vector notation, we obtain equations for \( F^L_{\mu\nu}(x) \) and \( F^R_{\mu\nu}(x) \),

\[ \partial^\mu F^L_{\mu\nu}(x) = 0, \quad \partial^\mu F^R_{\mu\nu}(x) = 0; \] (9.15)

then, constructing their linear combinations, we come to the Maxwell equations:

\[ \partial^\mu F_{\mu\nu}(x) = 0, \quad \partial^\mu \tilde{F}_{\mu\nu}(x) = 0. \] (9.16)

If we introduce complex potentials \( A_\mu, F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu, \) then the second equation is satisfied identically.

Taking into account the action of operators of the discrete transformations on \( z \) (see \((3.22)\)), we find for \( q^{\mu\nu} : \quad q^{\mu\nu} \overset{P}{\rightarrow} (1)\delta_{\mu\dot{\nu}} + \delta_{\mu\nu} q^{\mu\nu}, \quad q^{\mu\nu} \overset{I_3}{\rightarrow} -q^{\mu\nu}, \quad q^{\mu\nu} \overset{C}{\rightarrow} q^{\mu\nu}, \) whence as a consequence of \((9.7)\) we obtain

\[ P : \quad F_{\mu\nu}(x) \rightarrow (1)\delta_{\mu\dot{\nu}} + \delta_{\mu\nu} F_{\mu\nu}(\bar{x}), \quad A_\mu(x) \rightarrow -(1)\delta_{\mu\nu} A_\mu(\bar{x}); \] (9.17)
\[ I_3 : \quad F_{\mu\nu}(x) \rightarrow -F_{\mu\nu}(x), \quad A_\mu(x) \rightarrow -A_\mu(x), \] (9.18)
\[ C : \quad F_{\mu\nu}(x) \rightarrow \ast F_{\mu\nu}(x), \quad A_\mu(x) \rightarrow \ast A_\mu(x). \] (9.19)

It is easy to see that the charge conjugation transformation, acting on the function \((9.7)\) as complex conjugation, interchanges states with opposite helicities and thus can’t be considered for left and right fields separately (as it was pointed out, e.g., in [3]). The transformation \( I_3 \) for functions \((9.7)\) is the identity transformation.

Unlike the initial equations \((9.12)\), where the sign at \( p_0 \) is changed under space reflection and charge conjugation, equations \((9.10)\) are invariant under these transformations, since the left and right fields are contained in \( F_{\mu\nu}(x) \) on an equal footing. Thus four discrete transformations \( P, I_3, C, T_w \) are symmetry transformations of equations \((9.16)\).
Formally one can define two real fields, $F_{\mu\nu}^{(1)}(x)$ and $F_{\mu\nu}^{(2)}(x)$, as real and imaginary parts of $F_{\mu\nu}(x)$, which satisfy the same equations (9.16) and are characterized by opposite parities with respect to charge conjugation $C$. However, the real field $F_{\mu\nu}^{(i)}(x)$ can't describe the states with definite helicity, since according to (9.9) includes both left and right components. It is necessary also to note that $F_{\mu\nu}^{L}(x)$ and $F_{\mu\nu}^{R}(x)$ are no classical electromagnetic fields themselves. They can be treated as wave functions of left-handed and right-handed photons [6,51,52]. This shows that, as for all other cases considered above, the explicit realization of the representations of the Poincaré group in the space of functions on the group corresponds to the construction of one-particle sector of the quantum field theory.

**X. CONCLUSION**

We have shown that the representation theory of the proper Poincaré group implies the existence of five nontrivial independent discrete transformations corresponding to involutory automorphisms of the group. As such transformations one can choose space reflection $P$, inversion $I_x$, charge conjugation $C$, Wigner time reversal $T_w$; the fifth transformation for most fields of physical interest (except the Majorana field) is reduced to the multiplication by a phase factor.

Considering discrete automorphisms as operators acting in the space of the functions on the Poincaré group, we have obtained the explicit form for the discrete transformations of arbitrary spin fields without the use of any relativistic wave equations or special assumptions. The examination of the action of automorphisms on the operators, in particular, on the generators of the Poincaré group, ensures the possibility to get transformation laws of corresponding physical quantities. The analysis of the scalar field on the group allows us to construct explicitly the states corresponding to representations of the extended Poincaré group, and also to give the classification of the solutions of various types of relativistic wave equations with respect to representations of the extended group.

Since in the general case a relativistic wave equation can fix some characteristics of the extended Poincaré group representation, which are changed under the discrete transformations, only a part of the discrete transformations forms symmetry transformations of the equation. In particular, discrete symmetries of the Dirac equation and of the Weyl equation are generated by two different sets of the discrete transformations operators, $P, C, T_w$ and $PC, I_x, T_w$ respectively.

Being based on the concept of the field on the group and on the consideration of the group automorphisms, the approach developed can be applied to the analysis of discrete symmetries in other dimensions and also to other spacetime symmetry groups.

**ACKNOWLEDGMENTS**

The authors would like to thank A.Grishkov for useful discussions. I.L.B. and A.L.Sh. are grateful to the Institute of Physics at the University of São Paulo for hospitality. This work was partially supported by Brazilian Agencies CNPq (D.M.G.) and FAPESP (I.L.B., D.M.G. and A.L.Sh.).
APPENDIX A: OPERATORS IN THE SPACE OF FUNCTIONS ON THE POINCARE GROUP

1. Left and right generators of $SL(2, C)$ in the space of the functions on the group.

The left and right spin operators are given in the form

$$\hat{S}_k = \frac{1}{2}(z\sigma_k \partial_z - \bar{z}^* \sigma_k \partial_{\bar{z}}) + ...,$$

$$\hat{B}_k = \frac{i}{2}(z\sigma_k \partial_z + \bar{z}^* \sigma_k \partial_{\bar{z}}) + ...,$$ \quad $z = (z^1 z^2), \quad \partial_z = (\partial/\partial z^1 \partial/\partial z^2)^T; \quad (A1)$

$$\hat{S}_R^k = -\frac{1}{2}(\chi^* \sigma_k \partial_{\chi} - \chi \sigma_k \partial_{\bar{\chi}}) + ...,$$

$$\hat{B}_R^k = -\frac{i}{2}(\chi^* \sigma_k \partial_{\chi} + \chi \sigma_k \partial_{\bar{\chi}}) + ...,$$ \quad $\chi = (z^1 \bar{z}^2), \quad \partial_{\chi} = (\partial/\partial \bar{z}^1 \partial/\partial \bar{z}^2)^T; \quad (A2)$

Dots in the formulae mean analogous expressions obtained by the substitutions $z \to \bar{z} = (\bar{z}^1 \bar{z}^2), \chi \to \chi' = (\bar{z}^2 \bar{z}^2)$. One can rewrite two first formulae in four-dimensional notation:

$$\hat{S}^{\mu\nu} = \frac{1}{2}((\sigma^{\mu\nu})^{\beta}_{\alpha} z^\alpha \partial_{\beta} + (\bar{\sigma}^{\mu\nu})^{\bar{\beta}}_{\bar{\alpha}} \bar{z}^{\bar{\alpha}} \partial^{\bar{\beta}}) - \text{c.c.}, \quad (A3)$$

where \( \partial_\alpha = \partial/\partial z^\alpha, \: \bar{\partial}_{\alpha} = \partial/\partial \bar{z}^{\bar{\alpha}} \),

$$\sigma^{\mu\nu})^{\beta}_{\alpha} = -\frac{i}{4}(\sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu)^{\beta}_{\alpha}, \quad (\bar{\sigma}^{\mu\nu})^{\bar{\beta}}_{\bar{\alpha}} = -\frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \sigma^\nu \sigma^\mu)^{\bar{\beta}}_{\bar{\alpha}}, \quad (A4)$$

and c.c. is complex conjugate term corresponding to the action in the space of polynomials in $z^\alpha, \bar{z}^{\bar{\alpha}}$.

2. Operators $\hat{\Gamma}^\mu$ and equations for definite mass and spin.

The equations for functions on the Poincaré group

$$\hat{p}^2 f(x, z, \bar{z}) = m^2 f(x, z, \bar{z}), \quad (A5)$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, z, \bar{z}) = m s f(x, z, \bar{z}), \quad (A6)$$

where

$$\hat{\Gamma}^\mu = \frac{1}{2} \left( \bar{\sigma}^{\mu\bar{\alpha}} \bar{z}_\bar{\alpha} \partial_\alpha + \sigma^{\mu}_{\alpha\bar{\alpha}} z^{\alpha} \partial^{\bar{\alpha}} \right) - \text{c.c.}, \quad (A7)$$

describe a particle with fixed mass $m > 0$ and spin $s$, if we suppose that $f(x, z, \bar{z})$ is a polynomial of the power $2s$ in $z, \bar{z}$ [19]. Analogous statement also holds for polynomial in $\bar{z}, z$ functions $f(x, \bar{z}, z)$. Operators $\hat{\Gamma}^\mu$ and $\hat{S}^{\mu\nu}$ obey the commutation relations

$$[\hat{S}^{\lambda\mu}, \hat{\Gamma}^\nu] = i(\eta^{\mu\nu} \hat{\Gamma}^\lambda - \eta^{\lambda\nu} \hat{\Gamma}^\mu), \quad [\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = -i \hat{S}^{\mu\nu}, \quad (A8)$$

of the group $SO(3, 2)$, which coincide with the commutation relations of matrices $\gamma^\mu/2$.

These operators together with the chirality operator

$$\hat{\Gamma}^5 = \frac{1}{2} \left( z^\alpha \partial_\alpha + \bar{z}^{\bar{\alpha}} \partial^{\bar{\alpha}} \right) - \text{c.c.}, \quad (A9)$$
and the operators $\hat{\Gamma}^\mu = i[\hat{\Gamma}^\mu, \hat{\Gamma}^5]$, $\hat{S}_3^R$ form the set of 16 operators, which conserve the power of polynomials $f(x, z, 1/z)$ in $z, 1/z$.

Going over to spin-tensor notation, one can find that equation (A6) for $s = 1/2$ is transformed to the Dirac equation and for $s = 1$ is transformed to the Duffin-Kemmer equation. In general case, going over to spin-tensor notation, we obtain that the system (A3)-(A6) for polynomial of power $2s$ in $z, 1/z$ functions $f(x, z, 1/z)$ is transformed to the system of the Klein-Gordon equation and symmetric Bhabha equation [19]. The latter system is equivalent to the Bargmann-Wigner equations [53,19].

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[1] E.P. Wigner, “On unitary representations of the inhomogeneous Lorentz group,” Ann. Math. **40**, 149–204 (1939).
[2] G. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York, 1968).
[3] A.O. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN, Warszawa, 1977).
[4] W.-K. Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).
[5] Y.S. Kim and M.E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).
[6] Y. Ohnuki, *Unitary Representations of the Poincaré Group and Relativistic Wave Equations* (World Scientific, Singapore, 1988).
[7] I.M. Gel’fand, R.A. Minlos, and Z.Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon press, Oxford, 1963).
[8] E.P. Wigner, “Über die operation der zeitumkehr in der quantenmechanik,” Nachr. Ges. Wiss. Göt., 546–559 (1932).
[9] J. Schwinger, “The theory of quantized fields. I,” Phys. Rev. **82**, 914–927 (1951).
[10] H. Umezawa, S. Kamefuchi, and S. Tanaka, “On the time reversal in the quantized field theory,” Prog. Theor. Phys. **12**, 383–400 (1954).
[11] T.D. Lee and G.C. Wick, “Space inversion, time reversal, and other discrete symmetries in local field theories,” Phys. Rev. **148**, 1385–1404 (1966).
[12] I.M. Benn and R.W. Tucker, “Extended Lorentz invariance and field theory,” J. Phys. A **14**, 1745–1759 (1981).
[13] Yu.M. Shirokov, “A group-theoretical consideration of the basis of relativistic quantum mechanics. IV. Space reflections in quantum theory,” Sov. Phys. – JETP **7**, 493–498 (1958).
[14] Yu. M. Shirokov, “Space and time reflections in relativistic theory,” Nucl. Phys. **15**, 1–12 (1960).
[15] E.P. Wigner, “Unitary representations of the inhomogeneous Lorentz group including reflections,” in F. Gürsey, editor, *Group Theoretical Concepts and Methods in Elementary Particle Physics*, pages 37–80, (Gordon and Breach, New York, 1964).
[16] L. Michel, “Invariance in quantum mechanics and group extension,” in F. Gürsey, editor, *Group Theoretical Concepts and Methods in Elementary Particle Physics*, pages 135–200, (Gordon and Breach, New York, 1964).
[17] Kuo T.K., “Internal-symmetry groups and their automorphisms,” Phys. Rev. D 4, 3620–3637 (1971).
[18] Z.K. Silagadze, “On the internal parity of antiparticles,” Sov. J. Nucl. Phys. 55, 392–396 (1992).
[19] D.M. Gitman and A.L. Shelepin, “Fields on the Poincaré group: Arbitrary spin description and relativistic wave equations,” hep-th/0003146.
[20] M.A. Markov, Neutrino (Nauka, Moscow, 1964).
[21] J. Brana and K. Ljolje, “Dual symmetry and the Dirac field theory,” Fizika 12, 287–319 (1980).
[22] A.O. Barut and G. Ziino, “On parity conservation and the question of the ‘missing’ (right-handed) neutrino,” Mod. Phys. Lett. A 8, 1011–1020 (1993).
[23] Ahluwalia D.V., “Theory of neutral particles: McLennan-Case construct for neutrino, its generalization, and new wave equation,” Int. J. Mod. Phys. A 11, 1855–1874 (1996).
[24] V.V. Dvoeglazov, “Extra Dirac equations,” Nuovo Cimento B 111, 483–496 (1996).
[25] R.F. Streater and A.S. Wightman, PCT, Spin and Statistics, and all that (Reading, Massachusetts, 1964).
[26] I.L. Buchbinder and S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity (IOP Publishing Ltd, Bristol, 1995).
[27] N.Ya. Vilenkin, Special Functions and the Theory of Group Representations (AMS, Providence, 1968).
[28] D.P. Zhelobenko and A.I. Schtern, Representations of Lie Groups (Nauka, Moscow, 1983).
[29] R.G. Sachs, The Physics of Time Reversal (The University of Chicago Press, Chicago, 1987).
[30] B. Kayser and A.S. Goldhaber, “CPT and CP properties of Majorana particles, and the consequences,” Phys. Rev. D 28, 2341–2344 (1983).
[31] E.P. Wigner, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra (Academic Press, New York, 1959).
[32] L.D. Landau and E.M. Lifschitz, Quantum Mechanics, volume 3 of Course of Theoretical Physics (Pergamon, Oxford, 1977).
[33] L.S. Biedenharn and J.D. Louck, Angular Momentum in Quantum Physics (Addison-Wesley, Reading, Massachusetts, 1981).
[34] R. Shaw and J. Lever, “Irreducible multiplier corepresentations of the extended Poincaré group,” Commun. Math. Phys. 38, 279–297 (1974).
[35] S.P. Gavrilov and D.M. Gitman, “Quantization of point-like particles and consistent relativistic quantum mechanics,” hep-th/0003114.
[36] N.X. Hai, “Harmonic analysis on the Poincaré group, I. Generalized matrix elements,” Commun. Math. Phys. 12, 331–350 (1969).
[37] E. Majorana, “Teoria relativistica di particelle con momento intrinseco arbitrario,” Nuovo Cimento 9, 335–344 (1932).
[38] D.Tz. Stoyanov and I.T. Todorov, “Majorana representations of the Lorentz group and infinite-component fields,” J. Math. Phys. 9, 2146–2167 (1968).
[39] A.I. Oksak and I.T. Todorov, “Invalidity of TCP-theorem for infinite-component fields,” Commun. Math. Phys. 11, 125–130 (1968).
[40] S. Naka and T. Gotô, “C, P and T in c-number infinite component wave function,” Prog. Theor. Phys. 45, 1979–1986 (1971).
[41] R.P. Feynman and M. Gell-Mann, “Theory of Fermi interaction,” Phys. Rev. 109, 193–198 (1958).
[42] N.J. Ionesco-Pallas, “Relativistic Schrödinger equation for a particle with arbitrary spin,” J. of the Franklin Inst. 284, 243–250 (1967).
[43] W.J. Hurley, “Relativistic wave equations for particles with arbitrary spin,” Phys. Rev. D 4, 3605–3616 (1971).
[44] W.J. Hurley, “Invariant bilinear forms and the discrete symmetries for relativistic arbitrary-spin fields,” Phys. Rev. D 10, 1185–1200 (1974).
[45] S.I. Kruglov. “Pair production and solutions of the wave equation for particles with arbitrary spin,” hep-ph/9908410.
[46] Nigam B.P. and Foldy L.L., “Representation of charge conjugation for Dirac fields,” Phys. Rev. 102, 1410–1412 (1956).
[47] Ahluwalia D.V., Jonson M.B., and Goldman T., “A Bargmann-Wightman-Wigner-type quantum field theory,” Phys. Lett. B 316, 102–108 (1993).
[48] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Relativistic Quantum Theory (Pergamon, New York, 1971).
[49] L. Ryder, Quantum Field Theory (Cambridge University, Cambridge, 1988).
[50] F.H. Gaioli and E.T.G. Alvarez, “Some remarks about intrinsic parity in Ryder’s derivation of the Dirac equation,” Am. J. Phys. 63, 177–178 (1995).
[51] A.I. Akhiezer and V.B. Berestetskii, Quantum Electrodynamics (Nauka, Moscow, 1981).
[52] I. Białynicki-Birula, “On the wave function of the photon,” Acta Phys. Pol., Ser. B 86, 97–116 (1994).
[53] R.-K. Loide, I. Ots, and R. Saar, “Bhabha relativistic wave equations,” J. Phys. A 30, 4005–4017 (1997).