TWO CATEGORICAL CHARACTERIZATIONS OF LOCAL FIELDS

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Abstract. — In the present paper, we discuss two categorical characterizations of local fields. We first prove that a certain full subcategory of the category of finite flat coverings of the spectrum of the ring of integers of a local field equipped with coherent modules completely determines the isomorphism class of the local field. Next, we also prove that a certain full subcategory of the category of irreducible schemes which are finite over the spectrum of the ring of integers of a local field completely determines the isomorphism class of the local field.

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Introduction

Let $K$ be a local field, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number $p$. Write $\mathcal{O}_K$ for the ring of integers of $K$ and

$$\mathcal{B}_K$$

for the category of irreducible normal schemes which are finite, flat, and generically étale over $\mathcal{O}_K$ [cf. Definition 1.2]. Then one may verify that the category $\mathcal{B}_K$ is, by the functor taking function fields, equivalent to the category of finite separable extensions of $K$ [cf. Lemma 1.4, (ii)]. Thus,

the category $\mathcal{B}_K$ completely determines and is completely determined by the absolute Galois group of $K$ [cf. Theorem 1.10]. In particular, one may conclude from [4], §2, Theorem, that the equivalence class of the category $\mathcal{B}_K$ does not determine the isomorphism class of the field $K$

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[cf. Corollary 1.12, (i)]. In the present paper, we introduce two categories which contain, as a full subcategory, the above category $B_K$ and prove that these categories completely determine the isomorphism class of the field $K$.

First, let us write

$$\mathcal{C}_K$$

for the category of pairs of objects of $B_K$ and coherent modules on the objects [cf. Definition 2.1] and take a full subcategory

$$\mathcal{C}_K$$

of $\mathcal{C}_K$ which satisfies the condition (C) [cf. Definition 2.3], i.e., such that, roughly speaking,

(C-a) $\mathcal{C}_K$ is closed under the operation of taking submodules, and

(C-b) $\mathcal{C}_K$ contains every object of $\mathcal{C}_K$ whose module is torsion and generated by a single element.

Then, by the conditions (C-a) and (C-b), one may regard the category $B_K$ as a full subcategory of $\mathcal{C}_K$ [cf. Lemma 2.4, (iii)].

Next, let us write

$$\mathcal{F}_K$$

for the category of irreducible schemes which are finite over $O_K$ [cf. Definition 3.1] and take a full subcategory

$$\mathcal{F}_K$$

of $\mathcal{F}_K$ which satisfies the condition (F) [cf. Definition 3.4], i.e., such that, roughly speaking,

(F-a) $\mathcal{F}_K$ contains the object $\text{Spec}(O_K)$,

(F-b) $\mathcal{F}_K$ is closed under the operation of taking normalizations of objects which are the spectra of integral domains of dimension one,

(F-c) $\mathcal{F}_K$ is closed under the operation of taking finite separable extensions and subfields of the function fields of objects which are the spectra of integral domains of dimension one, and

(F-d) $\mathcal{F}_K$ is closed under the operation of taking closed subschemes.

Then, by the conditions (F-a), (F-b), and (F-c), one may regard the category $B_K$ as a full subcategory of $\mathcal{F}_K$ [cf. Lemma 3.5, (v)].

The main result of the present paper is as follows [cf. Theorem 2.14; Theorem 3.20]:

**Theorem.** — Let $K_\circ, K_\ast$ be local fields. Then the following hold:

(i) Let $\mathcal{C}_{K_\circ}, \mathcal{C}_{K_\ast}$ be full subcategories of $\mathcal{C}_{K_\circ}, \mathcal{C}_{K_\ast}$ as above, respectively. Suppose that the category $\mathcal{C}_{K_\circ}$ is equivalent to the category $\mathcal{C}_{K_\ast}$. Then the field $K_\circ$ is isomorphic to the field $K_\ast$.

(ii) Let $\mathcal{F}_{K_\circ}, \mathcal{F}_{K_\ast}$ be full subcategories of $\mathcal{F}_{K_\circ}, \mathcal{F}_{K_\ast}$ as above, respectively. Suppose that the category $\mathcal{F}_{K_\circ}$ is equivalent to the category $\mathcal{F}_{K_\ast}$. Then the field $K_\circ$ is isomorphic to the field $K_\ast$.\]
In §1, we discuss the category $\mathcal{B}_K$. In §2, we prove Theorem, (i). In §3, we prove Theorem, (ii). In the proof of Theorem, the main result of [2] plays an important role. Here, let us recall that the main result of [2] was generalized in [1].

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1. CATEGORY OF FINITE FLAT COVERINGS

In the present §1, let us discuss a category of certain finite flat coverings of the spectrum of the ring of integers of a local field [cf. Definition 1.2].

**Definition 1.1.** — If $K$ is a local field, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number $p$, then we shall write

- $\mathcal{O}_K \subseteq K$ for the ring of integers of $K$,
- $m_K \subseteq \mathcal{O}_K$ for the maximal ideal of $\mathcal{O}_K$, and
- $K \overset{\text{def}}{=} \mathcal{O}_K/m_K$ for the residue field of $\mathcal{O}_K$.

In the remainder of the present §1, let $K$ be a local field.

**Definition 1.2.** — We shall write $\mathcal{B}_K$ for the category defined as follows:

- An object of $\mathcal{B}_K$ is a pair $(S, \phi)$ consisting of a (nonempty) irreducible normal scheme $S$ and a morphism $\phi: S \rightarrow \text{Spec}(\mathcal{O}_K)$ of schemes which is finite, flat, and generically étale. To simplify the exposition, we shall often refer to $S$ [i.e., just the domain of the morphism $\phi$] as an “object of $\mathcal{B}_K$”.
- Let $S, T$ be objects of $\mathcal{B}_K$. Then a morphism $S \rightarrow T$ in $\mathcal{B}_K$ is defined as a morphism of schemes from $S$ to $T$ lying over $\mathcal{O}_K$.

**Definition 1.3.** — Let $S$ be an object of $\mathcal{B}_K$. Then we shall write $K_S$ for the function field of $S$.

**Lemma 1.4.** — The following hold:

(i) A terminal object of $\mathcal{B}_K$ is given by the pair $(\text{Spec}(\mathcal{O}_K), \text{id}_{\text{Spec}(\mathcal{O}_K)})$.

(ii) The assignment “$S \mapsto K_S$” determines an equivalence of categories of $\mathcal{B}_K$ with the category defined as follows:

- An object of the category is a finite separable extension of $K$.
- A morphism in the category is a homomorphism of fields over $K$. 
PROOF. — These assertions follow immediately from the definition of the category $\mathcal{B}_K$. □

DEFINITION 1.5.

(i) We shall say that a morphism $f: S \to T$ in $\mathcal{B}_K$ is Galois if the finite separable extension $K_S/K_T$ determined by $f$ [cf. Lemma 1.4, (ii)] is Galois.

(ii) We shall say that an object $S$ of $\mathcal{B}_K$ is Galois if there exists a Galois morphism from $S$ to a terminal object of $\mathcal{B}_K$ [cf. Lemma 1.4, (i)].

(iii) We shall say that a projective system $(S_\lambda)_{\lambda \in \Lambda}$ consisting of objects and morphisms of $\mathcal{B}_K$ is a basepoint of $\mathcal{B}_K$ if $S_\lambda$ is Galois for each $\lambda \in \Lambda$, and, moreover, for each object $T$ of $\mathcal{B}_K$, there exist an element $\lambda_T \in \Lambda$ and a morphism $S_{\lambda_T} \to T$ in $\mathcal{B}_K$.

(iv) Let $\tilde{S} = (S_\lambda)_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{B}_K$. Then we shall write

$$K_{\tilde{S}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} K_{S_\lambda}$$

for the field obtained by forming the injective limit of the $K_{S_\lambda}$’s and

$$\Pi_{\tilde{S}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} \text{Aut}(S_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii)] group obtained by forming the projective limit of the $\text{Aut}(S_\lambda)$’s.

LEMMA 1.6. — The following hold:

(i) There exists a basepoint of $\mathcal{B}_K$.

(ii) Let $S$ be a Galois object of $\mathcal{B}_K$. Then $\text{Aut}(S)$ is isomorphic to $\text{Gal}(K_S/K)$.

(iii) Let $\tilde{S}$ be a basepoint of $\mathcal{B}_K$. Then the field $K_{\tilde{S}}$ is a separable closure of $K$. Moreover, the profinite group $\Pi_{\tilde{S}}$ is isomorphic to the absolute Galois group $\text{Gal}(K_{\tilde{S}}/K)$ of $K$.

PROOF. — These assertions follow, in light of Lemma 1.4, (ii), from elementary field theory. □

LEMMA 1.7. — Let $S, T$ be objects of $\mathcal{B}_K; f: S \to T$ a morphism in $\mathcal{B}_K$. Then it holds that $f$ is Galois if and only if, for each two morphisms $g_1, g_2: U \to S$ in $\mathcal{B}_K$ such that $f \circ g_1 = f \circ g_2$, there exists an automorphism $h$ of $S$ over $T$ such that $g_2 = h \circ g_1$.

PROOF. — This follows, in light of Lemma 1.4, (ii), from elementary field theory. □

DEFINITION 1.8. — Let $S$ be an object of $\mathcal{B}_K$ and $\tilde{S} = (S_\lambda)_{\lambda \in \Lambda}$ a basepoint of $\mathcal{B}_K$. Then we shall write

$$S(\tilde{S}) \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} \text{Hom}(S_\lambda, S).$$
LEMMA 1.9. — Let $\tilde{S} = (S_{\lambda})_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{B}_K$. Then the assignment “$S \mapsto S(\tilde{S})$” determines an equivalence of categories of $\mathcal{B}_K$ with the category defined as follows:

- An object of the category is a [nonempty] finite set equipped with a continuous transitive action of $\Pi_{\tilde{S}}$.
- Let $A, B$ be objects of the category. Then a morphism $A \rightarrow B$ in the category is defined as a $\Pi_{\tilde{S}}$-equivariant map from $A$ to $B$.

PROOF. — This follows from Lemma 1.4, (ii), and Lemma 1.6, (iii), together with elementary Galois theory. □

THEOREM 1.10. — Let $K_0, K_\star$ be local fields. Then it holds that the category $\mathcal{B}_{K_0}$ [cf. Definition 1.2] is equivalent to the category $\mathcal{B}_{K_\star}$ if and only if the absolute Galois group of the field $K_0$ is isomorphic, as a profinite group, to the absolute Galois group of the field $K_\star$.

PROOF. — The necessity follows, in light of Lemma 1.6, (i), from Lemma 1.6, (iii), and Lemma 1.7. The sufficiency follows, in light of Lemma 1.6, (i), (iii), from Lemma 1.9. □

LEMMA 1.11. — Let $G$ be a profinite group which is isomorphic to the absolute Galois group of $K$. Then the following hold:

(i) It holds that $K$ is of characteristic zero if and only if, for each prime number $l$, there exists an open subgroup of $G$ such that $l$ divides the cardinality of the [necessarily finite] module consisting of torsion elements of the abelianization of the open subgroup.

(ii) Suppose that $K$ is of positive characteristic. Then it holds that $\sharp K - 1$ coincides with the cardinality of the [necessarily finite] module consisting of torsion elements of the abelianization of $G$.

PROOF. — Let us first recall from local class field theory [cf., e.g., [3], §2], together with the well-known structure of the multiplicative group $K^\times$, that the abelianization of $G$ [i.e., as a profinite group] is isomorphic to the profinite module $O_K^\times \times \hat{\mathbb{Z}}$. Next, let us also recall that if $K$ is of positive characteristic, then, again by the well-known structure of the multiplicative group $K^\times$, the composite $\mu(K) \hookrightarrow O_K^\times \rightarrow K^\times$ — where we write $\mu(K) \subseteq O_K^\times$ for the group of roots of unity of $K$ — is an isomorphism. Thus, assertions (i), (ii) follow immediately from the [easily verified] fact that $\hat{\mathbb{Z}}$ is torsion-free. This completes the proof of Lemma 1.11. □

COROLLARY 1.12. — The following hold:

(i) There exist local fields $K_0$ and $K_\star$ such that the category $\mathcal{B}_{K_0}$ is equivalent to the category $\mathcal{B}_{K_\star}$, but the field $K_0$ is not isomorphic to the field $K_\star$.

(ii) Let $K_0, K_\star$ be local fields. Suppose that the category $\mathcal{B}_{K_0}$ is equivalent to the category $\mathcal{B}_{K_\star}$, and that either $K_0$ or $K_\star$ is of positive characteristic. Then the field $K_0$ is isomorphic to the field $K_\star$. 
Proof. — Assertion (i) follows from Theorem 1.10, together with [4], §2, Theorem. Finally, we verify assertion (ii). Suppose that $B_{K^o}$ is equivalent to $B_{K^*}$, and that $K^o$ is of positive characteristic. Then it follows from Theorem 1.10 that the absolute Galois group of $K^o$ is isomorphic to the absolute Galois group of $K^*$. Thus, it follows immediately from Lemma 1.11, (i), that $K^*$ is of positive characteristic. Moreover, it follows from Lemma 1.11, (ii), that $\sharp K^o = \sharp K^*$. Thus, since [one verifies easily that] the fields $K^o$, $K^*$ are isomorphic to the local fields “$\mathbb{F}_{\mathcal{K}^o}((t))$”, “$\mathbb{F}_{\mathcal{K}^*}((t))$”, respectively, we conclude that $K^o$ is isomorphic to $K^*$, as desired. This completes the proof of assertion (ii). □

2. Category of Finite Flat Coverings with Coherent Modules

In the present §2, let us discuss a certain full subcategory of the category of finite flat coverings of the spectrum of the ring of integers of a local field equipped with coherent modules [cf. Definition 2.1; Definition 2.3]. In the present §2, let $K$ be a local field, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number $p$.

**Definition 2.1.** — We shall write $\mathcal{C}_K$ for the category defined as follows:

- An object of $\mathcal{C}_K$ is a pair $X = (S_X, F_X)$ consisting of an object $S_X$ of $\mathcal{B}_K$ [cf. Definition 1.2] and a coherent $\mathcal{O}_{S_X}$-module $F_X$.

- Let $X = (S_X, F_X)$, $Y = (S_Y, F_Y)$ be objects of $\mathcal{C}_K$. Then a morphism $X \to Y$ in $\mathcal{C}_K$ is defined as a pair $f = (f_S, f_F)$ consisting of a morphism $f_S: S_X \to S_Y$ in $\mathcal{B}_K$ and a homomorphism $f_F: F_X \to f_S^*F_Y$ of $\mathcal{O}_{S_X}$-modules.

**Definition 2.2.** — Let $X, Y$ be objects of $\mathcal{C}_K$; $f: X \to Y$ a morphism in $\mathcal{C}_K$.

(i) We shall say that $X$ is scheme-like if $F_X(S_X) = \{0\}$.

(ii) We shall say that $f$ is a scheme-isomorphism if $f_S$ is an isomorphism of schemes. [Thus, a scheme-isomorphism is not necessarily an isomorphism in $\mathcal{C}_K$.]

(iii) Suppose that $X = Y$, and that $f$ is an automorphism. Then we shall say that $f$ is a scheme-identity if $f_S$ is the identity automorphism of $S_X$. We shall write $\text{Aut}_{\text{id}}(X)$ for the group of scheme-identities of $X$.

(iv) We shall say that $f$ is a rigidification [of $Y$] if $X$ is scheme-like, and $f$ is a scheme-isomorphism.

(v) We shall write $K_X$ for the function field of $S_X$.

**Definition 2.3.** — Let $\mathcal{C}_K$ be a full subcategory of $\mathcal{C}_K$. Then we shall say that $\mathcal{C}_K$ satisfies the condition ($\mathcal{C}$) if
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(a) the full subcategory $\mathcal{C}_K$ is closed under the operation of taking submodules, i.e., if $X$ is an object of $\mathcal{C}_K$, and $\mathcal{G} \subseteq \mathcal{F}_X$ is an $\mathcal{O}_{S_X}$-submodule of $\mathcal{F}_X$, then the object $(S_X, \mathcal{G})$ of $\overline{\mathcal{C}}_K$ is an object of $\mathcal{C}_K$, and

(b) the full subcategory $\mathcal{C}_K$ contains every object of $\overline{\mathcal{C}}_K$ whose module is torsion and generated by a single element, i.e., if an object $X$ of $\mathcal{C}_K$ satisfies the condition that the $\mathcal{O}_{K_X}$-module [cf. Definition 1.1; Definition 2.2, (v)] $\mathcal{F}_X(S_X)$ is torsion and generated by a single element, then $X$ is an object of $\mathcal{C}_K$.

In the remainder of the present §2, let $\mathcal{C}_K$ be a full subcategory of $\overline{\mathcal{C}}_K$ which satisfies the condition (C).

**Lemma 2.4.** — The following hold:

(i) Every scheme-like object of $\overline{\mathcal{C}}_K$ is an object of $\mathcal{C}_K$.

(ii) A terminal object of $\mathcal{C}_K$ is given by the pair $(\text{Spec}(\mathcal{O}_K), \{0\})$. Moreover, every terminal object of $\mathcal{C}_K$ is scheme-like.

(iii) There exists a (tautological) equivalence of categories of $\mathcal{B}_K$ with the full subcategory of $\mathcal{C}_K$ consisting of scheme-like objects of $\mathcal{C}_K$.

**Proof.** — These assertions follow immediately from the definition of the category $\mathcal{C}_K$ [cf. Definition 2.3, (a), (b)].

**Lemma 2.5.** — Let $X$, $Y$ be objects of $\mathcal{C}_K$; $f: X \rightarrow Y$ a morphism in $\mathcal{C}_K$. Then the following hold:

(i) It holds that $f$ is a monomorphism [i.e., in $\mathcal{C}_K$] if and only if $f$ is a scheme-isomorphism, and, moreover, the homomorphism $f_{\mathcal{F}}(S_X): \mathcal{F}_X(S_X) \rightarrow f_{\mathcal{F}}S_Y(S_X)$ of $\mathcal{O}_{K_X}$-modules is injective.

(ii) It holds that $f$ is a rigidification if and only if $f$ is a monomorphism, and, moreover, $f$ is an initial object among monomorphisms whose codomains are $Y$.

(iii) It holds that $X$ is scheme-like if and only if there exists a rigidification in $\mathcal{C}_K$ whose domain is $X$.

(iv) It holds that $f$ is a scheme-isomorphism if and only if there exist rigidifications $g: Z \rightarrow X$, $h: Z \rightarrow Y$ in $\mathcal{C}_K$ such that $f \circ g = h$.

(v) Suppose that $X = Y$, and that $f$ is an automorphism. Then it holds that $f$ is a scheme-identity if and only if there exists a rigidification $g: Z \rightarrow X$ in $\mathcal{C}_K$ such that $g = f \circ g$.

**Proof.** — First, we verify assertion (i). The sufficiency follows immediately from the [easily verified] flatness of a morphism in $\mathcal{B}_K$. In the remainder of the proof of assertion (i), we verify the necessity.

First, suppose that $f_S$ is not an isomorphism. Then since the finite extension $K_X/K_Y$ determined by $f$ is nontrivial and separable [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)], it follows from elementary field theory that there exist a finite separable extension $L$ of
$K$ and two inclusions $i_1, i_2: K_X \hookrightarrow L$ such that $i_1 \neq i_2$ but $i_1|_{K_Y} = i_2|_{K_Y}$. Thus, by considering the two morphisms from the object $(\text{Spec}(O_L), \{0\})$ of $\mathcal{C}_K$ [cf. Lemma 2.4, (i)] to $X$ determined by $i_1, i_2$, respectively, we conclude that $f$ is not a monomorphism.

Next, suppose that $f_S$ is an isomorphism, but that the homomorphism $f_{\mathcal{F}}(S_X)$ of $\mathcal{O}_{K_X}$-modules is not injective, i.e., that $\{0\} \neq \text{Ker}(f_{\mathcal{F}}(S_X)) \subseteq \mathcal{F}_X(S_X)$. Then we have the natural inclusion $j_1 : \text{Ker}(f_{\mathcal{F}}(S_X)) \hookrightarrow \mathcal{F}_X(S_X)$ and the zero homomorphism $j_2 : \text{Ker}(f_{\mathcal{F}}(S_X)) \rightarrow (\{0\} \rightarrow) \mathcal{F}_X(S_X)$. Write $Z$ for the object of $\mathcal{C}_K$ determined by the pair $(S_X, \text{Ker}(f_{\mathcal{F}}(S_X)))$ [cf. Definition 2.3, (a)]. Then, by considering the natural two scheme-isomorphisms from $Z$ to $X$ determined by $j_1, j_2$, respectively, we conclude that $f$ is not a monomorphism. This completes the proof of the necessity, hence also of assertion (i).

Assertion (ii) follows immediately, in light of Lemma 2.4, (i), from assertion (i). Assertions (iii), (iv), and (v) follow immediately, in light of Lemma 2.4, (i), from the various definitions involved. 

\begin{definition}

(i) Let $X, Y$ be scheme-like objects of $\mathcal{C}_K$. Then we shall say that a morphism $f : X \rightarrow Y$ in $\mathcal{C}_K$ is Galois if the finite separable extension $K_X/K_Y$ determined by $f$ [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)] is Galois.

(ii) Let $X$ be a scheme-like object of $\mathcal{C}_K$. Then we shall say that $X$ is Galois if there exists a Galois morphism from $X$ to a terminal object of $\mathcal{C}_K$ [cf. Lemma 2.4, (ii)].

(iii) We shall say that a projective system $(X_\lambda)_{\lambda \in \Lambda}$ consisting of objects and morphisms of $\mathcal{C}_K$ is basepoint of $\mathcal{C}_K$ if $X_\lambda$ is Galois [hence also scheme-like] for each $\lambda \in \Lambda$, and, moreover, for each scheme-like object $Y$ of $\mathcal{C}_K$, there exist an element $\lambda_Y \in \Lambda$ and a morphism $X_\lambda \rightarrow Y$ in $\mathcal{C}_K$.

(iv) Let $\tilde{X} = (X_\lambda)_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{C}_K$. Then we shall write

$$K_{\tilde{X}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} K_{X_\lambda}$$

for the field obtained by forming the injective limit of the $K_{X_\lambda}$’s and

$$\Pi_{\tilde{X}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} \text{Aut}(X_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)] group obtained by forming the projective limit of the $\text{Aut}(X_\lambda)$’s.

\begin{lemma}

The following hold:

(i) There exists a basepoint of $\mathcal{C}_K$.

(ii) Let $X$ be a Galois object of $\mathcal{C}_K$. Then $\text{Aut}(X)$ is isomorphic to $\text{Gal}(K_X/K)$.

(iii) Let $\tilde{X}$ be a basepoint of $\mathcal{C}_K$. Then the field $K_{\tilde{X}}$ is a separable closure of $K$. Moreover, the profinite group $\Pi_{\tilde{X}}$ is isomorphic to the absolute Galois group $\text{Gal}(K_{\tilde{X}}/K)$ of $K$.

\end{lemma}
PROOF. — These assertions follow, in light of Lemma 2.4, (iii), from Lemma 1.6.

**Lemma 2.8.** — Let $X, Y$ be scheme-like objects of $\mathcal{C}_K$; $f : X \to Y$ a morphism in $\mathcal{C}_K$. Then it holds that $f$ is Galois if and only if, for each scheme-like object $Z$ in $\mathcal{C}_K$ and each two morphisms $g_1, g_2 : Z \to X$ in $\mathcal{C}_K$ such that $f \circ g_1 = f \circ g_2$, there exists an automorphism $h$ of $X$ over $Y$ such that $g_2 = h \circ g_1$.

PROOF. — This follows, in light of Lemma 2.4, (iii), from Lemma 1.7.

**Lemma 2.9.** — Let $X, Y$ be objects of $\mathcal{C}_K$; $f : X \to Y$ a rigidification in $\mathcal{C}_K$. Then the following hold:

(i) For each automorphism $g$ of $Y$, there exists a unique automorphism $\tilde{g}$ of $X$ such that $f \circ \tilde{g} = g \circ f$.

(ii) The assignment “$g \mapsto \tilde{g}$” of (i) determines an exact sequence of groups

$$1 \to \text{Aut}_{\text{id}}(Y) \to \text{Aut}(Y) \to \text{Aut}(X) \to 1.$$  

PROOF. — First, we verify assertion (i). Since $X$ is scheme-like, the automorphism of $S_X$ given by $f_S^{-1} \circ g_S \circ f_S$ determines an automorphism $\tilde{g}$ of $X$ such that $f \circ \tilde{g} = g \circ f$. Moreover, the uniqueness of such a “$\tilde{g}$” follows from the fact that a rigidification is a monomorphism [cf. Lemma 2.5, (ii)]. This completes the proof of assertion (i).

Finally, we verify assertion (ii). One verifies easily that, to verify assertion (ii), it suffices to verify the following two assertions:

1. For each $g \in \text{Aut}(Y)$, it holds that $\tilde{g}$ is the identity automorphism of $X$ if and only if $g$ is a scheme-identity.

2. For each $h \in \text{Aut}(X)$, there exists $g \in \text{Aut}(Y)$ such that $h = \tilde{g}$.

On the other hand, assertion (1) follows from the description of “$\tilde{g}$” given in the proof of assertion (i); assertion (2) is immediate. This completes the proof of Lemma 2.9.

**Definition 2.10.**

(i) Let $X, Y$ be objects of $\mathcal{C}_K$; $f : X \to Y$ a rigidification in $\mathcal{C}_K$. Then it follows from Lemma 2.9, (ii), that we have an exact sequence of groups

$$1 \to \text{Aut}_{\text{id}}(Y) \to \text{Aut}(Y) \to \text{Aut}(X) \to 1,$$

which thus determines an outer action of $\text{Aut}(X)$ on $\text{Aut}_{\text{id}}(Y)$:

$$\text{Aut}(X) \to \text{Out}(\text{Aut}_{\text{id}}(Y)).$$

We shall write

$$\text{Aut}(X)_f \overset{\text{def}}{=} \text{Ker}(\text{Aut}(X) \to \text{Out}(\text{Aut}_{\text{id}}(Y))) \subseteq \text{Aut}(X)$$

for the kernel of this action.

(ii) Let $X$ be an object of $\mathcal{C}_K$ and $n$ a nonnegative integer. Then we shall say that $X$ is $n$-simple if the $\mathcal{O}_K$-module $\mathcal{F}_X(S_X)$ is isomorphic to $\mathcal{O}_K/\mathfrak{m}_K^n$ [cf. Definition 1.1; Definition 2.2, (v)].
Lemma 2.11. — Let $X$ be a scheme-like object of $\mathcal{C}_K$ and $n$ a nonnegative integer. Then there exists a rigidification of an $n$-simple object whose domain is $X$.

Proof. — This is immediate [cf. Definition 2.3, (b)].

Lemma 2.12. — Let $X$ be an object of $\mathcal{C}_K$. Then the following hold:

(i) It holds that $X$ is $0$-simple if and only if $X$ is scheme-like.

(ii) Let $n$ be a positive integer. Then it holds that $X$ is $n$-simple if and only if there exists a morphism $f: Y \to X$ in $\mathcal{C}_K$ which satisfies the following conditions:

(1) The object $Y$ is $(n - 1)$-simple.

(2) The morphism $f$ is a monomorphism but not an isomorphism.

(3) Let $g: Y \to Z$, $h: Z \to X$ be morphisms in $\mathcal{C}_K$ such that $f = h \circ g$. If both $g$ and $h$ are monomorphisms, then either $g$ or $h$ is an isomorphism.

(4) The group $\text{Aut}_{id}(X)$ is abelian.

Proof. — Assertion (i) is immediate. In the remainder of the proof, we verify assertion (ii). The necessity follows immediately from Lemma 2.5, (i) [cf. Definition 2.3, (a)]. To verify the sufficiency, suppose that there exists a morphism $f: Y \to X$ in $\mathcal{C}_K$ which satisfies conditions (1), (2), (3), and (4). Then it follows immediately, in light of Lemma 2.5, (i), from conditions (1), (2), and (3) that the $\mathcal{O}_{K_X}$-module $F_X(S_X)$ is isomorphic to either $\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n$ or $(\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^{n-1}) \oplus (\mathcal{O}_{K_X}/\mathfrak{m}_{K_X})$. Thus, it follows from condition (4) that the $\mathcal{O}_{K_X}$-module $F_X(S_X)$ is isomorphic to $\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n$, as desired. This completes the proof of the sufficiency.

Lemma 2.13. — Let $X$, $Y$ be objects of $\mathcal{C}_K$; $f: X \to Y$ a morphism in $\mathcal{C}_K$; $n$ a nonnegative integer. Suppose that $X$ is Galois, that $Y$ is $n$-simple, and that $f$ is a rigidification. Then the subgroup $\text{Aut}(X)_f \subseteq \text{Aut}(X)$ corresponds, relative to the natural isomorphism of $\text{Aut}(X)$ with $\text{Gal}(K_X/K)$ [cf. Lemma 2.7, (ii)], to the kernel

$$\text{Ker}(\text{Gal}(K_X/K) \to \text{Aut}(\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n))$$

of the natural action of $\text{Gal}(K_X/K)$ on $\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n$.

Proof. — It follows immediately from the definition of an $n$-simple object that $\text{Aut}_{id}(Y)$ is naturally isomorphic to $(\mathcal{O}_{K_Y}/\mathfrak{m}_{K_Y}^n)^\times$. Thus, the subgroup $\text{Aut}(X)_f \subseteq \text{Aut}(X)$ corresponds, relative to the natural isomorphism of $\text{Aut}(X)$ with $\text{Gal}(K_X/K)$, to the kernel

$$\text{Ker}(\text{Gal}(K_X/K) \to \text{Aut}(\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n)^\times)).$$

In particular, Lemma 2.13 follows immediately from the [easily verified] fact that

$$\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n = \mathfrak{m}_{K_X}/\mathfrak{m}_{K_X}^n \cup (\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n)^\times,$$

$$1 + (\mathfrak{m}_{K_X}/\mathfrak{m}_{K_X}^n) \subseteq (\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n)^\times.$$

This completes the proof of Lemma 2.13. □
THEOREM 2.14. — Let $K_0, K_*$ be local fields; $C_{K_0}, C_{K_*}$ full subcategories of $\mathcal{C}_{K_0}, \mathcal{C}_{K_*}$ [cf. Definition 2.1] which satisfy the condition $(\mathcal{C})$ [cf. Definition 2.3], respectively. Suppose that the category $\mathcal{C}_{K_0}$ is equivalent to the category $\mathcal{C}_{K_*}$. Then the field $K_0$ is isomorphic to the field $K_*$. 

PROOF. — Suppose that there exists an equivalence of categories $\phi: \mathcal{C}_{K_0} \xrightarrow{\sim} \mathcal{C}_{K_*}$. Let $X_0, Y_0$ be objects of $\mathcal{C}_{K_0}$; $f_0: X_0 \to Y_0$ a morphism in $\mathcal{C}_{K_0}$. Write $X_*, Y_*$ for the objects of $\mathcal{C}_{K_*}$ corresponding, via $\phi$, to $X_0, Y_0$, respectively; $f_*: X_* \to Y_$ for the morphism in $\mathcal{C}_{K_*}$ corresponding, via $\phi$, to $f_0$. Then it follows from Lemma 2.5, (ii), that

(a) it holds that $f_0$ is a rigidification if and only if $f_*$ is a rigidification.

Thus, it follows from Lemma 2.5, (iii), that

(b) it holds that $X_0$ is scheme-like if and only if $X_*$ is scheme-like;

moreover, it follows from Lemma 2.5, (iv) (respectively, (v)), that

(c) it holds that $f_0$ is a scheme-isomorphism (respectively, scheme-identity) if and only if $f_*$ is a scheme-isomorphism (respectively, scheme-identity).

In particular, it follows from Lemma 2.12 that, for each nonnegative integer $n$,

(d) it holds that $X_0$ is $n$-simple if and only if $X_*$ is $n$-simple.

Next, let $\bar{X}_0 = ((X_0)_{\lambda})_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{C}_{K_0}$ [cf. Lemma 2.7, (i)]. Then it follows from Lemma 2.8, together with (b), that the projective system $\bar{X}_* = ((X_*)_{\lambda})_{\lambda \in \Lambda}$ consisting of objects and morphisms of $\mathcal{C}_{K_*}$ corresponding, via $\phi$, to $\bar{X}_0$ is a basepoint of $\mathcal{C}_{K_*}$. Thus, the equivalence $\phi$ determines an isomorphism of profinite groups

$$\Pi_{\phi}: \Pi_{\bar{X}_0} = \lim_{\lambda \in \Lambda} \text{Aut}((X_0)_{\lambda}) \xrightarrow{\sim} \Pi_{\bar{X}_*} = \lim_{\lambda \in \Lambda} \text{Aut}((X_*{\lambda})_{\lambda}).$$

In particular, if either $K_0$ or $K_*$ is of positive characteristic, then it follows, in light of Lemma 2.7, (iii), from Theorem 1.10 and Corollary 1.12, (ii), that $K_0$ is isomorphic to $K_*$, as desired. In the remainder of the proof, suppose that both $K_0$ and $K_*$ are of characteristic zero.

Next, let $\lambda$ be an element of $\Lambda$, $n$ a nonnegative integer, and $(f_0)_{\lambda}: (X_0)_{\lambda} \to (Y_0)_{\lambda}$ a rigidification of an $n$-simple object $(Y_0)_{\lambda}$ whose domain is the member $(X_0)_{\lambda}$ of $X_0$ [cf. Lemma 2.11]. Write

$$\Pi_{\phi, \lambda}: \text{Aut}((X_0)_{\lambda}) \xrightarrow{\sim} \text{Aut}((X_*)_{\lambda})$$

for the isomorphism induced by $\Pi_{\phi}$ and $(f_0)_{\lambda}: (X_0)_{\lambda} \to (Y_0)_{\lambda}$ for the rigidification [cf. (a)] of the $n$-simple object $(Y_0)_{\lambda}$ [cf. (d)] corresponding, via $\phi$, to $(f_0)_{\lambda}: (X_0)_{\lambda} \to (Y_0)_{\lambda}$. Then it follows from (c) that the isomorphism $\Pi_{\phi, \lambda}$ restricts to an isomorphism of subgroups

$$\text{Aut}((X_0)_{\lambda})_{(f_0)_{\lambda}} \xrightarrow{\sim} \text{Aut}((X_*)_{\lambda})_{(f_*)_{\lambda}}.$$ 

Thus, it follows from Lemma 2.13 that the isomorphism $\Pi_{\phi, \lambda}$ is compatible — relative to the natural identifications [cf. Lemma 2.7, (ii)] of $\text{Aut}((X_0)_{\lambda})$, $\text{Aut}((X_*)_{\lambda})$ with $\text{Gal}(K(X_0)_{\lambda}/K_0)$, $\text{Gal}(K(X_*)_{\lambda}/K_*)$, respectively — with the respective filtrations of higher ramification subgroups in the lower numbering, hence also [cf., e.g., (3), §4.1] in the upper numbering. In particular, the isomorphism $\Pi_{\phi}$ is compatible — relative to the natural
identifications [cf. Lemma 2.7, (iii)] of $\Pi_{K_0}$, $\Pi_{\tilde{K}_0}$ with $\operatorname{Gal}(K_{\tilde{X}}/K_0)$, $\operatorname{Gal}(K_{X}/K_0)$, respectively — with the respective filtrations of higher ramification subgroups in the upper numbering. Thus, it follows from [2], Theorem, that $K_0$ is isomorphic to $K_\circ$, as desired. This completes the proof of Theorem 2.14.

3. Category of Finite Schemes

In the present §3, let us discuss a certain full subcategory of the category of irreducible schemes which are finite over the spectrum of the ring of integers of a local field [cf. Definition 3.1; Definition 3.4]. In the present §3, let $K$ be a local field, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number $p$.

**Definition 3.1.** We shall write $\mathcal{F}_K$ for the category defined as follows:

- An object of $\mathcal{F}_K$ is a pair $(S, \phi)$ consisting of an [nonempty] irreducible scheme $S$ and a finite morphism $\phi: S \to \operatorname{Spec}(\mathcal{O}_K)$ of schemes. To simplify the exposition, we shall often refer to $S$ [i.e., just the domain of the morphism $\phi$] as an “object of $\mathcal{F}_K$”.

- Let $S, T$ be objects of $\mathcal{F}_K$. Then a morphism $S \to T$ in $\mathcal{F}_K$ is defined as a morphism of schemes from $S$ to $T$ lying over $\mathcal{O}_K$.

**Lemma 3.2.** The following hold:

(i) Every object of $\mathcal{F}_K$ is isomorphic to the spectrum of a noetherian complete local ring of dimension zero or one.

(ii) Every object of $\mathcal{F}_K$ is of cardinality one or two.

(iii) Every morphism in $\mathcal{F}_K$ is injective. In particular, if the domain (respectively, codomain) of a morphism in $\mathcal{F}_K$ is of cardinality two (respectively, one), then the morphism is bijective.

**Proof.** First, we verify assertion (i). Let $S$ be an object of $\mathcal{F}_K$. Let us first observe that since $S$ is finite over $\mathcal{O}_K$, the scheme $S$ is isomorphic to the spectrum of a finite, hence also noetherian, $\mathcal{O}_K$-algebra $A$. Thus, since $A$ is finite over the complete [hence also henselian] local ring $\mathcal{O}_K$, and $S$ is irreducible, it holds that $A$ is a complete local ring. Finally, since $A$ is finite over the local ring $\mathcal{O}_K$ of dimension one, it holds that $A$ is of dimension zero or one. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $S$ be an object of $\mathcal{F}_K$. Then since the scheme $S$ is irreducible and finite over the complete [hence also henselian] local ring $\mathcal{O}_K$, the fiber of the structure morphism $S \to \operatorname{Spec}(\mathcal{O}_K)$ at the [uniquely determined] closed (respectively, generic) point of $\operatorname{Spec}(\mathcal{O}_K)$ is of cardinality one (respectively, of cardinality zero or one). In particular, the scheme $S$ is of cardinality one or two. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let us first observe that it is immediate that, to verify assertion (iii), it suffices to verify that the structure morphism “$\phi$” of each object “$(S, \phi)$” of $\mathcal{F}_K$ is injective. On the other hand, this injectivity follows from the proof of assertion (ii). This completes the proof of assertion (iii).
**Definition 3.3.** — Let $S$ be an object of $\mathcal{F}_K$.

(i) We shall say that $S$ is point-like if $S$ is of cardinality one, or, alternatively, is of dimension zero; we shall say that $S$ is non-point-like if $S$ is not of cardinality one [i.e., is of cardinality two, or, alternatively, is of dimension one — cf. Lemma 3.2, (i), (ii)].

(ii) We shall say that $S$ is a trait (respectively, quasi-trait) if $S$ is normal (respectively, integral) and non-point-like.

(iii) Suppose that $S$ is a quasi-trait. Then we shall write $K_S$ for the function field of $S$.

**Remark 3.3.1.** — One verifies easily from Lemma 3.2, (i), that it holds that an object of $\mathcal{F}_K$ is a trait if and only if the object is isomorphic to the spectrum of a complete discrete valuation ring.

**Definition 3.4.** — Let $\mathcal{F}_K$ be a full subcategory of $\mathcal{F}_K$. Then we shall say that $\mathcal{F}_K$ satisfies the condition ($\mathfrak{F}$) if

(a) the full subcategory $\mathcal{F}_K$ contains the object $(\text{Spec}(\mathcal{O}_K), \text{id}_{\text{Spec}(\mathcal{O}_K)})$,

(b) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking normalizations of quasi-traits, i.e., if $S$ is a quasi-trait of $\mathcal{F}_K$, then the trait of $\mathcal{F}_K$ obtained by forming the normalization of $S$ is an object of $\mathcal{F}_K$,

(c) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking finite separable extensions and subfields of the function fields of quasi-traits, i.e., if $S$ is a quasi-trait of $\mathcal{F}_K$, and $L$ is a finite separable extension of $K_S$ (respectively, an intermediate extension of $K_{S}/K$), then there exist a quasi-trait $T$ of $\mathcal{F}_K$ and a morphism $T \to S$ (respectively, $S \to T$) in $\mathcal{F}_K$ such that $K_T$ is isomorphic, over $K_S$ (respectively, as an intermediate extension of $K_S/K$), to $L$,

(d) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking closed subschemes, i.e., if $S$ is an object of $\mathcal{F}_K$, then every closed immersion in $\mathcal{F}_K$ whose codomain is $S$ is a morphism in $\mathcal{F}_K$.

In the remainder of the present §3, let $\mathcal{F}_K$ be a full subcategory of $\mathcal{F}_K$ which satisfies the condition ($\mathfrak{F}$).

**Lemma 3.5.** — The following hold:

(i) A terminal object of $\mathcal{F}_K$ is given by the pair $(\text{Spec}(\mathcal{O}_K), \text{id}_{\text{Spec}(\mathcal{O}_K)})$. Moreover, every terminal object of $\mathcal{F}_K$ is a trait.

(ii) The assignment “$S \mapsto K_S$” determines a faithful functor from the full subcategory of $\mathcal{F}_K$ consisting of quasi-traits of $\mathcal{F}_K$ to the category defined as follows:

- An object of the category is a finite extension of $K$.
- A morphism in the category is a homomorphism of fields over $K$. 
Let $S$ be a trait of $\mathcal{F}_K$, $T$ a quasi-trait of $\mathcal{F}_K$, and $i: K_T \hookrightarrow K_S$ a homomorphism of fields over $K$. Then there exists a unique morphism $S \to T$ in $\mathcal{F}_K$ which induces, via the functor of (ii), the homomorphism $i$.

The restriction of the functor of (ii) to the full subcategory of $\mathcal{F}_K$ consisting of traits of $\mathcal{F}_K$ is full.

There exists a tautological equivalence of categories of $\mathcal{B}_K$ [cf. Definition 1.2] with the full subcategory of $\mathcal{F}_K$ consisting of traits of $\mathcal{F}_K$ which are generically étale over $\mathcal{O}_K$.

Proof. — Assertions (i), (ii), and (iii) follow immediately from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (a)]. Assertion (iv) follows from assertion (iii). Assertion (v) follows from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (a), (b), (c)]. □

Definition 3.6. — We shall say that $\mathcal{F}_K$ is separable if the essential image of the functor of Lemma 3.5, (ii), consists of finite separable extensions of $K$.

Definition 3.7.

(i) Let $S$, $T$ be traits of $\mathcal{F}_K$. Then we shall say that a morphism $f: S \to T$ in $\mathcal{F}_K$ is Galois if the finite extension $K_S/K_T$ determined by $f$ [cf. Lemma 3.5, (ii)] is Galois.

(ii) Let $S$ be a trait of $\mathcal{F}_K$. Then we shall say that $S$ is Galois if there exists a Galois morphism from $S$ to a terminal object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)].

(iii) We shall say that a projective system $(S_\lambda)_{\lambda \in \Lambda}$ consisting of objects and morphisms of $\mathcal{F}_K$ is basepoint of $\mathcal{F}_K$ if $S_\lambda$ is Galois [hence also a trait which is generically étale over $\mathcal{O}_K$] for each $\lambda \in \Lambda$, and, moreover, for each trait $T$ of $\mathcal{F}_K$ which is generically étale over $\mathcal{O}_K$, there exist an element $\lambda_T \in \Lambda$ and a morphism $S_{\lambda_T} \to T$ in $\mathcal{F}_K$.

(iv) Let $\tilde{S} = (S_\lambda)_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{F}_K$. Then we shall write

$$K_{\tilde{S}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} K_{S_\lambda}$$

for the field obtained by forming the injective limit of the $K_{S_\lambda}$’s and

$$\Pi_{\tilde{S}} \overset{\text{def}}{=} \lim_{\lambda \in \Lambda} \text{Aut}(S_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii); Lemma 3.5, (v)] group obtained by forming the projective limit of the Aut$(S_\lambda)$’s.

Lemma 3.8. — The following hold:

(i) There exists a basepoint of $\mathcal{F}_K$.

(ii) Let $S$ be a Galois object of $\mathcal{F}_K$. Then $\text{Aut}(S)$ is isomorphic to $\text{Gal}(K_S/K)$.

(iii) Let $\tilde{S}$ be a basepoint of $\mathcal{F}_K$. Then the field $K_{\tilde{S}}$ is a separable closure of $K$. Moreover, the profinite group $\Pi_{\tilde{S}}$ is isomorphic to the absolute Galois group $\text{Gal}(K_{\tilde{S}}/K)$ of $K$. 
Lemma 3.9. — Let $S$, $T$ be objects of $\mathcal{F}_K$; $f: S \to T$ a morphism in $\mathcal{F}_K$. Then the following hold:

(i) It holds that $f$ is a monomorphism [i.e., in $\mathcal{F}_K$] if and only if $f$ is a closed immersion.

(ii) It holds that $S$ is point-like if and only if there exists a morphism $S \to O$, where $O$ is a terminal object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)], which satisfies the following condition: The morphism $S \to O$ factors through a closed immersion $U \to O$ in $\mathcal{F}_K$ which is not an isomorphism.

(iii) It holds that $S$ is non-point-like if and only if $S$ is not point-like.

(iv) It holds that $S$ is integral and point-like if and only if there exists a closed immersion $S \to U$ in $\mathcal{F}_K$ which is an initial object among closed immersions whose codomains are $U$.

(v) Suppose that $S$ is non-point-like. Then it holds that $S$ is a quasi-trait if and only if there exists a closed immersion $S \to U$ in $\mathcal{F}_K$ which is an initial object among closed immersions whose codomains are $U$ and whose domains are non-point-like.

(vi) Suppose that $S$ is a quasi-trait. Then it holds that $S$ is a trait if and only if there exists a birational morphism $S \to U$ in $\mathcal{F}_K$ which is an initial object among birational morphisms whose codomains are $U$ and whose domains are quasi-trait objects of $\mathcal{F}_K$.

Proof. — First, we verify assertion (i). The sufficiency is immediate. To verify the necessity, suppose that $f$ is a monomorphism. Write $A_S \overset{\text{def}}{=} O_S(S)$ and $A_T \overset{\text{def}}{=} O_T(T)$. Then it follows from elementary field theory that there exist a finite extension $M$ of $A_S$ and two inclusions $i_1, i_2: A_S \hookrightarrow M$ such that $i_1 \neq i_2$ but $i_1|_{A_T} = i_2|_{A_T}$. In particular, since $f$ is a monomorphism [which thus implies that the morphism in $\mathcal{F}_K$ from the spectrum of $A_S$ to $T$ determined by the composite $A_T \to A_S \to A_S$ is a monomorphism], we obtain a contradiction [cf. Definition 3.4, (c), (d)]. Thus, the composite $A_T \to A_S \to A_S$ is an isomorphism. In particular, we conclude that the morphism $f: S \to T$ has a splitting, i.e., a morphism $s: T \to S$ such that $f \circ s = \text{id}_T$.

Now we have the identity automorphism $\text{id}_S$ of $S$ and the composite $S \xrightarrow{\text{id}_S} T$ $\xrightarrow{s} S$. Since $f$ is a monomorphism, we conclude that $\text{id}_S = s \circ f$, i.e., that $f$ is a closed immersion. This completes the proof of the necessity, hence also of assertion (i).

Assertion (ii) follows immediately from Lemma 3.2, (iii); Lemma 3.5, (i) [cf. Definition 3.4, (d)]. Assertion (iii) is immediate.

Next, we verify assertion (iv). The necessity follows from the observation that if $S$ is integral and point-like, then the identity automorphism of $S$ satisfies the condition
in the statement of assertion (iv). Next, to verify the \textit{sufficiency}, suppose that there exists a closed immersion \( S \to U \) in \( \mathcal{F}_K \) that satisfies the condition in the statement of assertion (iv). Write \( T \to U \) for the closed immersion determined by the residue field [cf. Lemma 3.2, (i)] of \( \mathcal{O}_U(U) \) [cf. Definition 3.4, (d)]. Then it follows from our assumption that the closed immersion \( S \to U \) factors through the closed immersion \( T \to U \), which thus implies that we obtain a closed immersion \( S \to T \). Now observe that since \( T \) is the spectrum of a field, the closed immersion \( S \to T \) is an \textit{isomorphism}, which thus implies that \( S \) is \textit{integral} and \textit{point-like}, as desired. This completes the proof of the \textit{sufficiency}, hence also of assertion (iv).

Next, we verify assertion (v). The \textit{necessity} follows from the observation that if \( S \) is a \textit{quasi-trait}, then the identity automorphism of \( S \) satisfies the condition in the statement of assertion (v). Next, to verify the \textit{sufficiency}, suppose that there exists a closed immersion \( S \to U \) in \( \mathcal{F}_K \) that satisfies the condition in the statement of assertion (v). Write \( T \to U \) for the closed immersion defined by the ideal of \( \mathcal{O}_U(U) \) of nilpotent elements [cf. Definition 3.4, (d)]. Note that since \( U \) is \textit{non-point-like} [cf. Lemma 3.2, (iii)], and the closed immersion \( T \to U \) is \textit{bijective} [cf. Lemma 3.2, (iii)], it follows that \( T \) is \textit{non-point-like}, hence also a \textit{quasi-trait}. Thus, it follows from our assumption that the closed immersion \( S \to U \) factors through the closed immersion \( T \to U \), which thus implies that we obtain a closed immersion \( S \to T \). Now observe that since \( T \) is a \textit{quasi-trait}, the [necessarily \textit{bijective} — cf. Lemma 3.2, (iii)] closed immersion \( S \to T \) is an \textit{isomorphism}, which thus implies that \( S \) is a \textit{quasi-trait}, as desired. This completes the proof of the \textit{sufficiency}, hence also of assertion (v).

Finally, we verify assertion (vi). The \textit{necessity} follows from the observation that if \( S \) is a \textit{trait}, then, by the \textit{Zariski main theorem}, the identity automorphism of \( S \) satisfies the condition in the statement of assertion (vi). Next, to verify the \textit{sufficiency}, suppose that there exists a birational morphism \( S \to U \) in \( \mathcal{F}_K \) that satisfies the condition in the statement of assertion (vi). Write \( T \to U \) for the normalization of \( U \) [cf. Definition 3.4, (b)]. Then it follows from our assumption that the birational morphism \( S \to U \) factors through the birational morphism \( T \to U \), which thus implies that we obtain a birational morphism \( S \to T \). Now observe that since \( T \) is a \textit{trait}, it follows from the \textit{Zariski main theorem} that the birational morphism \( S \to T \) is an \textit{isomorphism}, which thus implies that \( S \) is a \textit{trait}, as desired. This completes the proof of the \textit{sufficiency}, hence also of assertion (vi).

\[ \square \]

**Definition 3.10.** — Let \( S, T \) be quasi-traits of \( \mathcal{F}_K \); \( f : S \to T \) a morphism in \( \mathcal{F}_K \).

(i) We shall say that \( f \) is \textit{purely inseparable} (respectively, \textit{quasi-Galois}) if the finite extension \( K_S/K_T \) determined by \( f \) [cf. Lemma 3.5, (ii)] is purely inseparable (respectively, quasi-Galois, or, alternatively, normal, i.e., \( K_S \) is Galois over the purely inseparable closure of \( K_T \) in \( K_S \)).

(ii) Suppose that \( f \) is quasi-Galois. Then we shall write \( \text{qGal}(f) \defeq \text{Gal}(K_S/L) \) (= \( \text{Aut}_{K_T}(K_S) \)), where we write \( L \subseteq K_S \) for the purely inseparable closure of \( K_T \) in \( K_S \) [which thus implies that the finite extension \( K_S/L \) is Galois].
LEMMA 3.11. — Let $S, T$ be quasi-traits of $\mathcal{F}_K$; $f : S \to T$ a morphism in $\mathcal{F}_K$. Then the following hold:

(i) It holds that $f$ is either birational or purely inseparable if and only if the following condition is satisfied: For each quasi-trait $U$ of $\mathcal{F}_K$ and each two morphisms $g_1, g_2 : U \to S$ in $\mathcal{F}_K$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

(ii) Consider the following conditions:

(1) For each quasi-trait $U$ of $\mathcal{F}_K$ and each two morphisms $g : S \to U$, $h : U \to T$ such that $f = h \circ g$, if every automorphism of $S$ over $T$ is an automorphism over $U$ [i.e., relative to $g$], then $h$ is either birational or purely inseparable.

(2) The morphism $f$ is quasi-Galois.

Then (1) implies (2). If, moreover, $S$ is a trait, then (1) is equivalent to (2).

PROOF. — First, we verify assertion (i). The necessity follows, in light of Lemma 3.5, (ii), from elementary field theory. Next, we verify the sufficiency. Suppose that $f$ is neither birational nor purely inseparable. Then it follows from elementary field theory that there exist a finite separable extension $L$ of $K_S$ and two inclusions $i_1, i_2 : K_S \hookrightarrow L$ such that $i_1 \neq i_2$ but $i_1|_{K_T} = i_2|_{K_T}$. Thus, by considering suitable two morphisms from a trait whose generic point is isomorphic to the spectrum of $L$ [cf. Definition 3.4, (b), (c)] to $S$, we conclude from Lemma 3.5, (ii), that $f$ does not satisfy the condition in the statement of assertion (i). This completes the proof of the sufficiency, hence also of assertion (i).

Finally, we verify assertion (ii). Let us first observe that if condition (1) is satisfied, then it follows immediately from Lemma 3.5, (ii), that the intermediate extension of $K_S/K_T$ consisting of $\text{Aut}_T(S)$-invariants in $K_S$ is \text{either the trivial extension or} a purely inseparable extension of $K_T$. Thus, the implication (1) $\Rightarrow$ (2) follows from Lemma 3.5, (ii), together with elementary field theory. The implication (2) $\Rightarrow$ (1) in the case where $S$ is a trait follows immediately, in light of Lemma 3.5, (ii), (iv), from elementary field theory. This completes the proof of assertion (ii). \hfill \Box

LEMMA 3.12. — Let $S, T$ be quasi-traits of $\mathcal{F}_K$; $f : S \to T$ a quasi-Galois morphism in $\mathcal{F}_K$. Then the following hold:

(i) For each quasi-trait $U$ of $\mathcal{F}_K$ and each morphism $g : U \to S$ in $\mathcal{F}_K$ which is either birational or purely inseparable, it holds that $\text{Aut}_T(U)$ is isomorphic to a subgroup of $\text{qGal}(f)$.

(ii) There exist a quasi-trait $U$ of $\mathcal{F}_K$ and a birational morphism $g : U \to S$ in $\mathcal{F}_K$ such that $\text{Aut}_T(U)$ is isomorphic to $\text{qGal}(f)$.

PROOF. — Assertion (i) follows from Lemma 3.5, (ii), together with elementary field theory. Assertion (ii) follows from Lemma 3.5, (ii), (iv) [cf. Definition 3.4, (b)]. \hfill \Box

LEMMA 3.13. — Let $O$ be a terminal object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)]. Then the following hold:

(i) Consider the following conditions:
(i-1) The category $\mathcal{F}_K$ is separable.

(i-2) For each quasi-trait $S$ of $\mathcal{F}_K$, there exists a morphism in $\mathcal{F}_K$ whose codomain is $S$ and whose domain is Galois.

(i-3) For each quasi-trait $S$ of $\mathcal{F}_K$, there exist a morphism from a quasi-trait $T$ to $S$ and a quasi-Galois morphism $T \to O$ in $\mathcal{F}_K$.

(i-4) For each quasi-trait $S$ of $\mathcal{F}_K$ and each morphism $f: S \to O$ in $\mathcal{F}_K$, if $f$ is either birational or purely inseparable, then $f$ is an isomorphism.

Then the following equivalences hold:

\[(i-1) \iff (i-2) + (i-4) \iff (i-3) + (i-4).\]

(ii) Let $p$ be a prime number. Then the following conditions are equivalent:

(ii-1) It holds that $\overline{K}$ is of characteristic $p$.

(ii-2) There exists a finite subquotient of the absolute Galois group of $K$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(ii-3) There exist traits $S, T$ of $\mathcal{F}_K$ and a Galois morphism $S \to T$ in $\mathcal{F}_K$ such that $\text{Aut}_T(S)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(ii-4) There exist quasi-traits $S, T$ of $\mathcal{F}_K$ and a quasi-Galois morphism $f: S \to T$ in $\mathcal{F}_K$ such that $q\text{Gal}(f)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(iii) Let $q_\times$ be a positive integer. Then the following conditions are equivalent:

(iii-1) It holds that $\sharp \overline{K}^\times = q_\times$, i.e., that $\sharp \overline{K} = q_\times + 1$.

(iii-2) The positive integer $q_\times$ is the maximum positive integer such that $q_\times$ is not divisible by the characteristic of $\overline{K}$, and, moreover, there exists a finite quotient of the absolute Galois group which is isomorphic to $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

(iii-3) The positive integer $q_\times$ is the maximum positive integer such that $q_\times$ is not divisible by the characteristic of $\overline{K}$, and, moreover, there exists a Galois object of $\mathcal{F}_K$ whose automorphism group is isomorphic to $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

(iii-4) The positive integer $q_\times$ is the maximum positive integer such that $q_\times$ is not divisible by the characteristic of $\overline{K}$, and, moreover, there exist a quasi-trait $S$ of $\mathcal{F}_K$ and a quasi-Galois morphism $f: S \to O$ in $\mathcal{F}_K$ such that $q\text{Gal}(f)$ is isomorphic to $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

PROOF. — First, we verify assertion (i). The implication (i-1) $\Rightarrow$ (i-2) follows immediately from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (b), (c)]. The implication (i-2) $\Rightarrow$ (i-3) is immediate. Next, we verify the implication (i-1) $\Rightarrow$ (i-4). Let us first observe that it follows from Lemma 3.5, (i), that $O$ is a trait. Thus, it follows from Lemma 3.9, (vi), that the identity automorphism of $O$ is an initial object among birational morphisms whose codomains are $O$ and whose domains are quasi-trait of $\mathcal{F}_K$. On the other hand, it follows from (i-1) that $f$ is birational. Thus, the morphism $f$ is an isomorphism, as desired. This completes the proof of the implication (i-1) $\Rightarrow$ (i-4).

Thus, to complete the verification of assertion (i), it suffices to verify that if $\mathcal{F}_K$ satisfies condition (i-3) but does not satisfy condition (i-1), then $\mathcal{F}_K$ does not satisfy condition (i-4). On the other hand, this follows immediately from the definition of the category.
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$F_K$, together with elementary field theory [cf. Definition 3.4, (c)]. This completes the proof of assertion (i).

Next, we verify assertions (ii), (iii). First, we verify the equivalences (ii-1) $\Leftrightarrow$ (ii-2) and (iii-1) $\Leftrightarrow$ (iii-2). Write $G$ for the absolute Galois group of $K$ and $p_K$ for the characteristic of $K$. Then it follows from local class field theory [cf., e.g., [3], §2], together with the well-known structure of the multiplicative group $K^\times$, that there exist a cyclic $p_K$-group $M_{cyc}$ and a free $\mathbb{Z}_{p_K}$-module $M_{free}$ of rank $[K : \mathbb{Q}_p]$ (respectively, of infinite rank) if $K$ is of characteristic zero (respectively, of positive characteristic) such that the abelianization of $G$ [i.e., as a profinite group] is isomorphic to the profinite module $K^\times \times M_{cyc} \times M_{free} \times \hat{\mathbb{Z}}$.

Thus, the equivalences (ii-1) $\Leftrightarrow$ (ii-2) and (iii-1) $\Leftrightarrow$ (iii-2) hold, as desired.

Moreover, the equivalences (ii-2) $\Leftrightarrow$ (ii-3) $\Leftrightarrow$ (ii-4) and (iii-2) $\Leftrightarrow$ (iii-3) $\Leftrightarrow$ (iii-4) follow immediately from Lemma 3.5, (iv), together with elementary field theory [cf. Definition 3.4, (b), (c)]. This completes the proofs of assertions (ii), (iii).

**Definition 3.14.** — Let $S, T$ be objects of $F_K$; $f : S \to T$ a morphism in $F_K$; $n$ a positive integer. Then we shall say that $f$ is $n$-simple if $T$ is Galois [hence also a trait which is generically étale over $\mathcal{O}_K$], $f$ is a closed immersion, and, moreover, the object $(T, f_*\mathcal{O}_S)$ of $\mathcal{C}_K$ [cf. Definition 2.1; Lemma 3.5, (v)] is $n$-simple in the sense of Definition 2.1; (v)

**Lemma 3.15.** — Let $S, T$ be objects of $F_K$; $f : S \to T$ a morphism in $F_K$; $n$ a positive integer. Suppose that $f$ is $n$-simple. Then, for each automorphism $g$ of $T$, there exists a unique automorphism $\tilde{g}$ of $S$ such that $f \circ \tilde{g} = g \circ f$. Moreover, the assignment "$g \mapsto \tilde{g}$" determines a homomorphism of groups

$$\text{Aut}(T) \longrightarrow \text{Aut}(S).$$

**Proof.** — The existence of such a "$\tilde{g}$" is immediate from the definition of an $n$-simple morphism. Moreover, the uniqueness of such a "$\tilde{g}$" follows from the fact that an $n$-simple morphism is a monomorphism [cf. Lemma 3.9, (i)]. Finally, the final assertion is immediate. This completes the proof of Lemma 3.15.

**Definition 3.16.** — Let $S, T$ be objects of $F_K$; $f : S \to T$ a morphism in $F_K$; $n$ a positive integer. Suppose that $f$ is $n$-simple. Then it follows from Lemma 3.15 that we have a homomorphism of groups

$$\text{Aut}(T) \longrightarrow \text{Aut}(S).$$

We shall write

$$\text{Aut}(T)_f \overset{\text{def}}{=} \ker(\text{Aut}(T) \to \text{Aut}(S)) \subseteq \text{Aut}(T)$$

for the kernel of this homomorphism.

**Lemma 3.17.** — Let $S$ be a Galois object of $F_K$ and $n$ a positive integer. Then there exists an $n$-simple morphism in $F_K$ whose codomain is $S$. 

Proof. — This is immediate [cf. Definition 3.4, (d)].

Lemma 3.18. — Let $S$, $T$ be objects of $\mathcal{F}_K$; $f : S \to T$ a morphism in $\mathcal{F}_K$. Suppose that $T$ is Galois, and that $f$ is a closed immersion. Then the following hold:

(i) It holds that $f$ is 1-simple if and only if $S$ is integral and point-like.

(ii) Let $n \geq 2$ be an integer. Then it holds that $f$ is $n$-simple if and only if there exists a closed immersion $g : U \to S$ in $\mathcal{F}_K$ which satisfies the following conditions:

(1) The composite $f \circ g : U \to T$ is $(n - 1)$-simple.

(2) The morphism $g$ is not an isomorphism.

(3) Let $h : U \to V$, $i : V \to S$ be morphisms in $\mathcal{F}_K$ such that $g = i \circ h$. If both $h$ and $i$ are closed immersions, then either $h$ or $i$ is an isomorphism.

Proof. — This is immediate [cf. Definition 3.4, (d)].

Lemma 3.19. — Let $S$, $T$ be objects of $\mathcal{F}_K$; $f : S \to T$ a morphism in $\mathcal{F}_K$; $n$ a positive integer. Suppose that $f$ is $n$-simple. Then the subgroup $\text{Aut}(T)_f \subseteq \text{Aut}(T)$ corresponds, relative to the natural isomorphism of $\text{Aut}(T)$ with $\text{Gal}(K_T/K)$ [cf. Lemma 3.8, (ii)], to the kernel

$$\text{Ker} \left( \text{Gal}(K_T/K) \to \text{Aut}(O_{K_T}/m_{K_T}^n) \right)$$

of the natural action of $\text{Gal}(K_T/K)$ on $O_{K_T}/m_{K_T}^n$.

Proof. — This is immediate.

Theorem 3.20. — Let $K_0$, $K_\bullet$ be local fields; $\mathcal{F}_{K_0}$, $\mathcal{F}_{K_\bullet}$ full subcategories of $\overline{\mathcal{F}}_{K_0}$, $\overline{\mathcal{F}}_{K_\bullet}$ [cf. Definition 3.1] which satisfy the condition $(\mathfrak{F})$ [cf. Definition 3.4], respectively. Suppose that the category $\mathcal{F}_{K_0}$ is equivalent to the category $\mathcal{F}_{K_\bullet}$. Then the field $K_0$ is isomorphic to the field $K_\bullet$.

Proof. — Suppose that there exists an equivalence of categories $\phi : \mathcal{F}_{K_0} \xrightarrow{\sim} \mathcal{F}_{K_\bullet}$. Let $S_0$, $T_0$ be objects of $\mathcal{F}_{K_0}$; $f_0 : S_0 \to T_0$ a morphism in $\mathcal{F}_{K_0}$. Write $S_\bullet$, $T_\bullet$ for the objects of $\mathcal{F}_{K_\bullet}$ corresponding, via $\phi$, to $S_0$, $T_0$, respectively; $f_\bullet : S_\bullet \to T_\bullet$ for the morphism in $\mathcal{F}_{K_\bullet}$ corresponding, via $\phi$, to $f_0$. Then it follows from Lemma 3.9, (i), (ii), (iii), (v), that

(a) it holds that $S_0$ is a quasi-trait if and only if $S_\bullet$ is a quasi-trait.

In particular, it follows from Lemma 3.11, (i), that

(b) if both $S_0$ and $T_0$ [hence also both $S_\bullet$ and $T_\bullet$ — cf. (a)] are quasi-trait, then it holds that $f_0$ is either birational or purely inseparable if and only if $f_\bullet$ is either birational or purely inseparable.

Now I claim that

(c) if both $S_0$ and $T_0$ are traits [which thus implies that both $S_\bullet$ and $T_\bullet$ are quasi-trait — cf. (a)], and $f_0$ is Galois, then $f_\bullet$ is quasi-Galois.
To this end, let us first observe that since \( S_o \) is a trait, it follows from Lemma 3.11, (ii), that \( f_o \) satisfies condition (1) of Lemma 3.11, (ii). Thus, it follows from (a), (b) that \( f_o \) satisfies condition (1) of Lemma 3.11, (ii). In particular, it follows from Lemma 3.11, (ii), that the morphism \( f_o \) is quasi-Galois, as desired. This completes the proof of (c).

Next, I claim that

(d) in the situation of (c), the four finite groups \( \text{Aut}_{T_o}(S_o) \), \( q\text{Gal}(f_o) \), \( \text{Aut}_{T_o}(S_o) \), and \( q\text{Gal}(f_o) \) are isomorphic.

To this end, let us first observe that since \( S_o \) is a trait, it is immediate [cf. Lemma 3.5, (iv)] that the three finite groups \( \text{Aut}_{T_o}(S_o) \), \( q\text{Gal}(f_o) \), and \( \text{Aut}_{T_o}(S_o) \) are isomorphic. In particular, it follows from Lemma 3.12, (i), that, for each quasi-trait \( U_o \) of \( F_{K_o} \) and each morphism \( g_o : U_o \to S_o \) in \( F_{K_o} \), which is either birational or purely inseparable, it holds that \( \text{Aut}_{T_o}(U_o) \) is isomorphic to a subgroup of \( \text{Aut}_{T_o}(S_o) \). Thus, it follows from (a), (b) that, for each quasi-trait \( U_o \) of \( F_{K_o} \), and each morphism \( g_o : U_o \to S_o \) in \( F_{K_o} \), which is either birational or purely inseparable, it holds that \( \text{Aut}_{T_o}(U_o) \) is isomorphic to a subgroup of \( \text{Aut}_{T_o}(S_o) \). In particular, it follows from Lemma 3.12, (ii), that \( q\text{Gal}(f_o) \) is isomorphic to a subgroup of \( \text{Aut}_{T_o}(S_o) \), which thus implies [cf. Lemma 3.5, (ii)] that \( q\text{Gal}(f_o) \) is isomorphic to \( \text{Aut}_{T_o}(S_o) \). This completes the proof of (d).

Next, I claim that

(e) it holds that \( (\text{char}(K_o), \sharp K_o^\times) = (\text{char}(K_o), \sharp K_o^\times) \).

To verify the equality \( \text{char}(K_o) = \text{char}(K_o) \), let us first observe that it follows from the implication (ii-1) \( \Rightarrow \) (ii-3) of Lemma 3.13, (ii), that there exists a Galois morphism between traits of \( F_{K_o} \) whose automorphism group is isomorphic to the direct product of three copies of \( \mathbb{Z}/\text{char}(K_o)\mathbb{Z} \). Thus, it follows from the implication (ii-4) \( \Rightarrow \) (ii-1) of Lemma 3.13, (ii), together with (d), that the equality \( \text{char}(K_o) = \text{char}(K_o) \) holds. Next, to verify the equality \( \sharp K_o^\times = \sharp K_o^\times \), observe that it follows from the implication (iii-1) \( \Rightarrow \) (iii-3) of Lemma 3.13, (iii), that there exists a Galois object of \( F_{K_o} \) whose automorphism group is isomorphic to the direct product of two copies of \( \mathbb{Z}/\sharp K_o^\times \mathbb{Z} \). Thus, it follows, in light of the equality \( \text{char}(K_o) = \text{char}(K_o) \), from the implication (iii-4) \( \Rightarrow \) (iii-1) of Lemma 3.13, (iii), together with (d), that the inequality \( \sharp K_o^\times \leq \sharp K_o^\times \) holds. Thus, by applying, \( \phi^{-1} \), a similar argument to the argument applied in the proof of the inequality \( \sharp K_o^\times \leq \sharp K_o^\times \), we conclude that \( \sharp K_o^\times = \sharp K_o^\times \). This completes the proof of (e).

Next, I claim that

(f) it holds that \( F_{K_o} \) is separable if and only if \( F_{K_o} \) is separable.

To this end, suppose that \( F_{K_o} \) is separable. Then it follows from the implication (i-1) \( \Rightarrow \) (i-4) of Lemma 3.13, (i), together with (a), (b), that \( F_{K_o} \) satisfies condition (i-4) of Lemma 3.13, (i). Moreover, it follows from the implication (i-1) \( \Rightarrow \) (i-2) of Lemma 3.13, (i), that \( F_{K_o} \) satisfies condition (i-2) of Lemma 3.13, (i). Thus, it follows from (a), (c) that \( F_{K_o} \) satisfies condition (i-3) of Lemma 3.13, (i). In particular, it follows from Lemma 3.13, (i), that \( F_{K_o} \) satisfies condition (i-1) of Lemma 3.13, (i), i.e., that \( F_{K_o} \) is separable, as desired. This completes the proof of (f).

Now suppose that either \( F_{K_o} \) or \( F_{K_o} \) is not separable. Then it follows from (f) that both \( K_o \) and \( K_o \) are of positive characteristic. Thus, it follows immediately from (e) that \( K_o \) is isomorphic to \( K_o \), as desired. In the remainder of the proof, suppose that both \( F_{K_o} \) and \( F_{K_o} \) are separable.
Then it follows from Lemma 3.9, (vi), together with (a), (b), that

(g) it holds that $S_0$ is a trait if and only if $S_\bullet$ is a trait.

Thus, it follows, in light of (c), from Lemma 3.18, together with Lemma 3.9, (i), (iv), that, for each positive integer $n$,

(h) it holds that $f_0$ is $n$-simple if and only if $f_\bullet$ is $n$-simple.

Next, let $\widetilde{S}_0 = ((S_0(\lambda))_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{F}_{K_0}$ [cf. Lemma 3.8, (i)]. Then it follows from (c), (g), that the projective system $\widetilde{S}_\bullet = ((S_\bullet(\lambda))_{\lambda \in \Lambda}$ consisting of objects and morphisms of $\mathcal{F}_{K_\bullet}$, corresponding, via $\phi$, to $\widetilde{S}_0$ is a basepoint of $\mathcal{F}_{K_\bullet}$. Thus, the equivalence $\phi$ determines an isomorphism of profinite groups

$$\Pi_\phi : \Pi_{\widetilde{S}_0} \cong \lim_{\lambda \in \Lambda} \text{Aut}((S_0(\lambda)) \xrightarrow{\sim} \Pi_{\widetilde{S}_\bullet} = \lim_{\lambda \in \Lambda} \text{Aut}((S_\bullet(\lambda)).$$

In particular, if either $K_0$ or $K_\bullet$ is of positive characteristic, then it follows, in light of Lemma 3.8, (iii), from Theorem 1.10 and Corollary 1.12, (ii), that $K_0$ is isomorphic to $K_\bullet$, as desired. In the remainder of the proof,

suppose that both $K_0$ and $K_\bullet$ are of characteristic zero.

Let $\lambda$ be an element of $\Lambda$, $n$ a positive integer, and $(f_0)_\lambda : (T_0)_\lambda \rightarrow (S_0)_\lambda$ an $n$-simple morphism whose codomain is the member $(S_\lambda)_\lambda$ of $\widetilde{S}_0$ [cf. Lemma 3.17]. Write

$$\Pi_{\phi,\lambda} : \text{Aut}((S_0)_\lambda) \cong \text{Aut}((S_\lambda)_\lambda)$$

for the isomorphism induced by $\Pi_\phi$ and $(f_\bullet)_\lambda : (T_\lambda)_\lambda \rightarrow (S_\lambda)_\lambda$ for the $n$-simple [cf. (h)] morphism corresponding, via $\phi$, to $(f_0)_\lambda : (T_0)_\lambda \rightarrow (S_\lambda)_\lambda$. Then one verifies easily that the isomorphism $\Pi_{\phi,\lambda}$ restricts to an isomorphism of subgroups

$$\text{Aut}((S_0)_\lambda(f_0)_\lambda) \cong \text{Aut}((S_\lambda)_\lambda(f_\bullet)_\lambda).$$

Thus, it follows from Lemma 3.19 that the isomorphism $\Pi_{\phi,\lambda}$ is compatible — relative to the natural identifications [cf.Lemma 3.8, (ii)] of $\text{Aut}((S_0)_\lambda)$, $\text{Aut}((S_\lambda)_\lambda)$ with $\text{Gal}(K(S_0)_\lambda/K_0)$, $\text{Gal}(K(S_\lambda)_\lambda/K_\bullet)$, respectively — with the respective filtrations of higher ramification subgroups in the lower numbering, hence also [cf., e.g., [3], §4.1] in the upper numbering. In particular, the isomorphism $\Pi_\phi$ is compatible — relative to the natural identifications [cf. Lemma 3.8, (iii)] of $\Pi_{\widetilde{S}_0}$ and $\Pi_{\widetilde{S}_\bullet}$ with $\text{Gal}(K_{\widetilde{S}_0}/K_0)$, $\text{Gal}(K_{\widetilde{S}_\bullet}/K_\bullet)$, respectively — with the respective filtrations of higher ramification subgroups in the upper numbering. Thus, it follows from [2], Theorem, that $K_0$ is isomorphic to $K_\bullet$, as desired. This completes the proof of Theorem 3.20.

\[\square\]

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