COMPACTNESS OF MAXIMAL COMMUTATORS OF BILINEAR CALDERÓN-ZYGMUND SINGULAR INTEGRAL OPERATORS

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Abstract. Let $T$ be a bilinear Calderón-Zygmund singular integral operator and $T_*$ be its corresponding truncated maximal operator. The commutators in the $i$-th entry and the iterated commutators of $T_*$ are defined by

$$T_{*,b,1}(f,g)(x) = \sup_{\delta > 0} \left| \int_{|x-y|+|x-z| > \delta} K(x,y,z)(b(y) - b(x))f(y)g(z) dydz \right|,$$

$$T_{*,b,2}(f,g)(x) = \sup_{\delta > 0} \left| \int_{|x-y|+|x-z| > \delta} K(x,y,z)(b(z) - b(x))f(y)g(z) dydz \right|,$$

$$T_{*,(b_1,b_2)}(f,g)(x) = \sup_{\delta > 0} \left| \int_{|x-y|+|x-z| > \delta} K(x,y,z)(b_1(y) - b_1(x))(b_2(z) - b_2(x))f(y)g(z) dydz \right|.$$  

In this paper, the compactness of the commutators $T_{*,b,1}$, $T_{*,b,2}$ and $T_{*,(b_1,b_2)}$ on $L^p(\mathbb{R}^n)$ is established.

1. Introduction and Main Results

Let $T_\Omega$ be the well-known Calderón-Zygmund singular integral operator defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$  

In 1976, Coifman, Rochberg and Weiss [11] defined the following well-known commutator of $T_\Omega$ for smooth functions,

$$(1.1) \ [b, T_\Omega]f(x) = b(x)T_\Omega(f)(x) - T_\Omega(bf)(x) = p.v. \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$  

The authors of [11] proved that $[b, T_\Omega]$ is bounded on $L^p$ for $1 < p < \infty$ when $b \in BMO$ and $\Omega$ satisfies:

(i) $\Omega(\lambda x) = \Omega(x)$, for any $\lambda > 0$ and $x \neq 0$,
(ii) $\int_{S^{n-1}} \Omega(x') \ d\sigma(x') = 0$,
(iii) $|\Omega(x') - \Omega(y')| \leq |x' - y'|$, for any $x', y' \in S^{n-1}$.

The following characterization of $L^p$-compactness of $[b, T_\Omega]$ was given by Uchiyama in 1978.

Key words and phrases. Bilinear maximal Calderón-Zygmund singular integral, Iterated commutators, $C^\infty$ space, Compactness.

The first author is supported by NSFC (No.11371057) and SRFDP (No.20130003110003), the third author is supported by NCET-13-0065.
Formally, they can take the form (1.4)

\[ T \]

where (1.5)

\[ T \]

Let \( T \) be a bilinear Calderón-Zygmund operator (see (13)) and assume that the kernel \( K \) satisfies the usual conditions in such a theory, that is \( K \in 2-CZK(A, \gamma) \). Let \( T_\ast \) be the corresponding bilinear maximal singular integral operator defined by

\[
T_\ast(f, g)(x) = \sup_{\delta > 0} \left| \int \int_{|x-y|+|x-z|>\delta} K(x, y, z)f(y)g(z) \, dydz \right|.
\]

In 2002, Grafakos and Torres [14] obtained the following \( L^p \)-estimate of \( T_\ast \).

\[
\|T_\ast(f, g)\|_r \leq C\|f\|_p\|g\|_q,
\]

for \( 1 < p, q < \infty, 1/2 < r < \infty \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

Let \( b, b_1, b_2 \in BMO(\mathbb{R}^n) \). We are interested in the following three maximal commutators of bilinear Calderón-Zygmund operators:

\[
T_{\ast, b, 1}(f, g)(x) = \sup_{\delta > 0} \left| [T_\delta, b]_1(f, g)(x) \right| = \sup_{\delta > 0} \left| (T_\delta(bf, g) - bT_\delta(f, g))(x) \right|,
\]

\[
T_{\ast, b, 2}(f, g)(x) = \sup_{\delta > 0} \left| [T_\delta, b]_2(f, g)(x) \right| = \sup_{\delta > 0} \left| (T_\delta(f, bg) - bT_\delta(f, g))(x) \right|,
\]

\[
T_{\ast, (b_1, b_2)}(f, g)(x) = \sup_{\delta > 0} \left| ([T_\delta, b_1]_1, b_2]_2(f, g)(x) \right| = \sup_{\delta > 0} \left| ([T_\delta, b_1]_1(f, b_2g) - b_2[T_\delta, b_1]_1(f, g))(x) \right|,
\]

where \( T_\delta(f, g)(x) = \int \int_{|x-y|+|x-z|>\delta} K(x, y, z)f(y)g(z) \, dydz \).

Formally, they can take the form

\[
T_{\ast, b, 1}(f, g)(x) = \sup_{\delta > 0} \left| \int \int_{|x-y|+|x-z|>\delta} K(x, y, z)(b(y) - b(x))f(y)g(z) \, dydz \right|,
\]

\[
T_{\ast, b, 2}(f, g)(x) = \sup_{\delta > 0} \left| \int \int_{|x-y|+|x-z|>\delta} K(x, y, z)(b(z) - b(x))f(y)g(z) \, dydz \right|,
\]

\[
T_{\ast, (b_1, b_2)}(f, g)(x) = \sup_{\delta > 0} \left| \int \int_{|x-y|+|x-z|>\delta} K(x, y, z)(b_1(y) - b_1(x))(b_2(z) - b_2(x))f(y)g(z) \, dydz \right|.
\]
By the results in [18], the third operator maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for all $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$, with the following estimate:

\begin{equation}
(1.7) \quad \|T_*(b_1, b_2)(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \prod_{j=1}^{2} \|b_j\|_{BMO} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.
\end{equation}

**Remark 1.1.** We can't find any results for the $L^p$-boundedness of the first two operators, but it is trivial. We also can use the ideal in [18]. Give the $L^p$-estimates of two maximal commutators controlling $T_{*,b,i}$ and obtain the following result.

\begin{equation}
(1.8) \quad \|T_{*,b,i}(f, g)\|_{L^r(\mathbb{R}^n)} \leq C\|b\|_{BMO} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)},
\end{equation}

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for all $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$.

Compare with the classical compact results in Theorem A, for the compactness of bilinear operators, recently, Árpád Bényi and R. H. Torres in [1] first studied the compactness for commutators of bilinear Calderón-Zygmund singular integral operators. Árpád Bényi et al. [2] also considered compactness properties of commutators of bilinear fractional integrals. Let us recall the definition of the compact bilinear operator (see [1]).

**Definition 1.2.** Let $B_{r,x} = \{ x \in X : \|x\| \leq r \}$ be the closed ball of radius $r$ centered at the origin in the normed space $X$. A bilinear operator $T : X \times Y \to Z$ is called compact if $T(B_{1,x} \times B_{1,y})$ is precompact in $Z$.

It is natural to ask whether the compact results still hold for the maximal commutators $T_{*,b,i}$, $T_{*,(b_1,b_2)}$ of the bilinear singular integral operators or not. We have found that there is no result for the compactness of the commutators of $T_*$ defined in (1.4), (1.5) and (1.6), even in the classical linear case.

The main purpose of the present paper is to show the compactness for the maximal commutators $T_{*,b,1}$, $T_{*,b,2}$ and $T_{*,(b_1,b_2)}$ of bilinear Calderón-Zygmund operators when the symbols $b, b_1, b_2 \in CMO(\mathbb{R}^n)$, which denotes the closure of $C^\infty_c(\mathbb{R}^n)$ in the $BMO(\mathbb{R}^n)$ topology. Now, the difficulty lies in that $T_*$ is a sub-linear operator, we can’t use the classical known method. The $L^p$-boundedness of the commutators of $T_*$ comes from the $L^p$-boundedness of two maximal commutators which control the commutators of $T_*$ (see [18]). In fact, we can get the compact results of the sum of the two maximal commutators controlling the commutators of $T_*$, but from this we can’t deduce the compact result for the commutators of $T_*$ (see also [6]).

Our main results are as follows.

**Theorem 1.1.** Let $1 \leq r < \infty$, $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For any $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, let $T_{*,b,1}(f, g)$, $T_{*,b,2}(f, g)$ be defined in (1.4) and (1.5). If $b \in CMO(\mathbb{R}^n)$, then $T_{*,b,1}$ and $T_{*,b,2}$ are compact operators from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

**Theorem 1.2.** Let $1 \leq r < \infty$, $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For any $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, let $T_{*,(b_1,b_2)}(f, g)$ be defined in (1.6). If $b_1, b_2 \in CMO(\mathbb{R}^n)$, then $T_{*,(b_1,b_2)}$ is a compact operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. 
Remark 1.3. Theorem 1.1 and 1.2 also hold for m-linear maximal Calderón-Zygmund singular integral operators (including linear case \( m = 1 \)). The essential ideas in the proof are similar, of course, with more complicated and delicate division in the main steps of the proof in Section 2,3. For simplicity, we omit the proof.

Remark 1.4. The compact results are new even for the linear case which can be proved by using similar ideas and steps as in the proof of Theorem 1.1.

2. The Proof of Theorem 1.1

In this part, we will give the proof of Theorem 1.1. We first give the following lemmas.

Lemma 2.1. (Frechet-Kolmogorov) [21] A subset \( G \) of \( L^p(\mathbb{R}^n)(1 \leq p < \infty) \) is strongly pre-compact if and only if \( G \) satisfies the following conditions:

(a) \( \sup_{f \in G} \| f \|_p < \infty; \)

(b) \( \lim_{\alpha \to \infty} \| f \chi_{E_\alpha} \|_p = 0 \), uniformly for \( f \in G \), where \( E_\alpha = \{ x \in \mathbb{R}^n : |x| > \alpha \} \);

(c) \( \lim_{|h| \to 0} \| f(\cdot + h) - f(\cdot) \|_p = 0 \), uniformly for \( f \in G \).

Lemma 2.2. For any \( \delta > 0 \), \( 0 < \epsilon < \frac{1}{2} \), we have the following inequalities

\[
\int_{|y| + |z| \leq \frac{\delta}{1 + 2\epsilon}} \frac{dydz}{(|y| + |z|)^{2n}} \leq C[1 - (1 + 2\epsilon)^{-n}],
\]

(2.1)

\[
\int_{|y| + |z| \leq \frac{\delta}{1 + 2\epsilon}} \frac{dydz}{(|y| + |z|)^{2n}} \leq C[(1 - 2\epsilon)^{-n} - 1],
\]

(2.2)

where the constant \( C \) is independent of \( \delta \) and \( \epsilon \).

Proof. We first give the estimate for (2.1). Simple computation and spherical coordinates transformations give that

\[
\int_{|y| + |z| \leq \delta} \frac{dydz}{(|y| + |z|)^{2n}} \leq \int_{|z| \leq \delta} \left( \int_{|y| + |z| \leq \delta} \frac{dy}{(|y| + |z|)^{2n}} \right) + \int_{|y| \leq |z|} \left( \int_{|y| + |z| \leq \delta} \frac{dy}{(|y| + |z|)^{2n}} \right) \leq C \int_{|z| \leq \delta} \left( \int_{|y| \leq |z|} \frac{dy}{(|y| + |z|)^{2n}} \right) + C \int_{|y| \leq |z|} \left( \int_{|z| \leq \delta} \frac{dy}{(|y| + |z|)^{2n}} \right) \leq C \int_{|z| \leq \delta} \int_{|y| \leq |z|} \left( \int_{|y| + |z| \leq \delta} \frac{dy}{(|y| + |z|)^{2n}} \right) \leq C[1 - (1 + 2\epsilon)^{-n}].
\]
Analogously, we also can obtain (2.2):
\[
\int \int_{\delta \leq |y| + |z| \leq \frac{\delta}{1+\delta}} \frac{dydz}{(|y| + |z|)^{2n}} 
\leq \int_{|z| \leq \delta} \int_{|y| \leq \frac{\delta}{1+\delta} - |z|} \frac{dydz}{(|y| + |z|)^{2n}} + \int_{|y| \leq \frac{\delta}{1+\delta} - |z|} \int_{|y| \leq \frac{\delta}{1+\delta} - |z|} \frac{dydz}{(|y| + |z|)^{2n}} 
\leq C \int_{|z| \leq \delta} \int_{|y| \leq \frac{\delta}{1+\delta} - |z|} [\delta^{-n} - \delta^{-n}(1 - 2\epsilon)^n]dz + C \int_{|y| \leq \frac{\delta}{1+\delta} - |z|} (|z|^{-n} - \delta^{-n}(1 - 2\epsilon)^n)dz 
\leq C[1 - (1 - 2\epsilon)^n] + C\delta^{-n}[\delta^n(1 - 2\epsilon)^{-n} - \delta^n] \leq C[(1 - 2\epsilon)^{-n} - 1].
\]

We complete the proof of the lemma.

Proof of Theorem 2.1. We only prove \( i = 1 \). Without loss of generality, let \( B_1, B_2 \) be unit balls in \( L^p(\mathbb{R}^n) \) and \( L^q(\mathbb{R}^n) \), respectively. We need to show the set \( \{T_{b,1}(f, g) : f \in B_1, g \in B_2\} \) is strongly pre-compact in \( L^*(\mathbb{R}^n) \) with \( b \in CMO(\mathbb{R}^n) \).

We first show that if the set \( \{T_{b,1}(f, g) : f \in B_1, g \in B_2\} \) is strongly pre-compact in \( L^*(\mathbb{R}^n) \) for \( b \in C^\infty(\mathbb{R}^n) \), then the set \( \{T_{b,1}(f, g) : f \in B_1, g \in B_2\} \) is also strongly pre-compact in \( L^*(\mathbb{R}^n) \) for \( b \in CMO \). In fact, suppose that \( b \in CMO \), then for any \( \epsilon > 0 \), there exists \( b' \in C^\infty(\mathbb{R}^n) \) such that \( \|b - b'\|_{BMO} < \epsilon \). It is easy to see that
\[
|T_{b,1}(f, g)(x) - T_{b',1}(f, g)(x)| 
\leq \sup_{\delta > 0} \left| \int_{|x - y| + |x - z| > \delta} \left[ (b(y) - b(x)) - (b'(y) - b'(x)) \right] K(x, y, z) f(y) g(z) dydz \right| 
\leq T_{b-b',1}(f, g)(x).
\]
Then combine with the above inequality and (1.8), we have
\[
(2.3) \quad \|T_{b,1}(f, g) - T_{b',1}(f, g)\|_r \leq \|T_{b-b',1}(f, g)\|_r \leq C\|b - b'\|_{BMO}\|f\|_p\|g\|_q \leq C\epsilon.
\]
Denote \( F_1 := \{T_{b-b',1}(f, g) : f \in B_1, g \in B_2\} \), then (a), (b) and (c) in Lemma 2.1 hold for \( F_1 \). We need to show that (a), (b) and (c) also hold for the set \( \tilde{F}_1 := \{T_{b,1}(f, g) : f \in B_1, g \in B_2\} \). (2.3) gives that
\[
(2.4) \quad \sup_{f \in B_1, g \in B_2}\|T_{b,1}(f, g)\|_r \leq \sup_{f \in B_1, g \in B_2}\|T_{b-b',1}(f, g)\|_r + C\epsilon < \infty.
\]
On the other hand,
\[
(2.5) \quad \lim_{\alpha \to \infty}\|T_{b,1}(f, g)\chi_{E_\alpha}\|_r \leq \lim_{\alpha \to \infty}\|T_{b-b',1}(f, g)\chi_{E_\alpha}\|_r + \|T_{b-b',1}(f, g)\|_r 
\leq C\epsilon \to 0, \quad (\epsilon \to 0).
\]
\[
(2.6) \quad \lim_{|h| \to 0}\|T_{b,1}(f, g)(\cdot + h) - T_{b,1}(f, g)(\cdot)\|_r 
\leq \lim_{|h| \to 0}\|T_{b-b',1}(f, g)(\cdot + h) - T_{b-b',1}(f, g)(\cdot)\|_r + 2\|T_{b-b',1}(f, g)\|_r 
\leq C\epsilon \to 0, \quad (\epsilon \to 0).
\]
It is obvious to see that the above limits hold uniformly in $F_1$. Therefore, we know $F_1$ is strongly pre-compact in $L^r(\mathbb{R}^n)$ for $b \in CMO$. Thus, to prove Theorem 1.1, it suffices to verify that the set $F_1 := \{T_{*,b,1}(f,g) : f \in B_1, g \in B_2\}$ is strongly pre-compact in $L^r(\mathbb{R}^n)$ for $b \in C_c^\infty(\mathbb{R}^n)$. By Lemma 2.1 we need only to prove (a), (b) and (c) hold uniformly in $F_1$.

For (a), by (2.7), we easily obtain

$$\sup_{f \in B_1, g \in B_2} \|T_{*,b,1}(f,g)\|_r \leq C\|b\|_{BMO} \sup_{f \in B_1, g \in B_2} \|f\|_p \|g\|_q < C < \infty.$$ 

Notice $b \in C_c^\infty(\mathbb{R}^n)$, without loss of generality, we can assume that $\text{supp} b \subset \{x \in \mathbb{R}^n : |x| \leq \beta\}$ with $\beta > 1$. For any $\epsilon > 0$, $0 < s < \frac{n}{q}$, we take $\alpha > 2\beta$ such that $\alpha^{-n-s+n/r} < \epsilon$, then we have

$$\|T_{*,b,1}(f,g)\chi_{E_\epsilon}\|_r < C\epsilon.$$ 

In fact, combine with the support set of $b$ and notice that $K \in 2-CZK(A,\gamma)$, for $|x| > \alpha$, Hölder’s inequality gives

$$T_{*,b,1}(f,g)(x) \leq C \int_{|y| \leq \beta} \frac{|b(y)|}{|x-y|^{n+s}} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x-y| + |x-z|)^{n-s}} \, dz \, dy$$

$$\leq C\|b\|_{\infty} \int_{|y| \leq \beta} \frac{|f(y)|}{|x-y|^{n+s}} \, dy \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |x-z|)^{(n-s)q}} \, dz \right)^{1/q'} \|g\|_q$$

$$\leq C\beta^{-\frac{n}{q'}} |x|^{-n-s} \|f\|_p \|g\|_q \leq C|x|^{-n-s}.$$ 

Thus, we have

$$\left( \int_{\mathbb{R}^n} |T_{*,b,1}(f,g)(x)|^r \chi_{E_\epsilon}(x) \, dx \right)^{1/r} \leq C \left( \int_{|x| > \alpha} |x|^{-(n+s)r} \, dx \right)^{\frac{1}{r}} \leq C\epsilon.$$ 

That is, (2.8) holds uniformly for $f \in B_1, g \in B_2$.

It remains to prove that (c) holds also for $T_{*,b,1}(f,g)$ uniformly with $f \in B_1, g \in B_2$. That is, we need to verify that for any $0 < \epsilon < \frac{1}{4}$, if $|h|$ is sufficiently small and dependent only on $\epsilon$, then

$$\|T_{*,b,1}(f,g)(\cdot + h) - T_{*,b,1}(f,g)(\cdot)\|_r < C\epsilon$$

holds uniformly for $f \in B_1, g \in B_2$.

In fact, for any $h \in \mathbb{R}^n$, we denote $\tilde{K}_\delta(x,y,z) = K(x,y,z)\chi_{[x-y|+|x-z|>\delta]}$, then

$$|T_{*,b,1}(f,g)(x + h) - T_{*,b,1}(f,g)(x)|$$

$$\leq \sup_{\delta > 0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{K}_\delta(x+h,y,z)(b(y) - b(x+h))f(y)g(z) \, dy \, dz \right.$$ 

$$- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{K}_\delta(x,y,z)(b(y) - b(x))f(y)g(z) \, dy \, dz \right|.$$
We can control the right hand side of the above inequality by the sum of the following four terms:

\[ J_1 := \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z)(b(x) - b(x + h))f(y)g(z)dydz \right|, \]

\[ J_2 := \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} (\tilde{K}_\delta(x + h, y, z) - \tilde{K}_\delta(x, y, z))(b(y) - b(x + h))f(y)g(z)dydz \right|, \]

\[ J_3 := \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|\leq\epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z)(b(y) - b(x))f(y)g(z)dydz \right|, \]

\[ J_4 := \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|\leq\epsilon^{-1}|h|} \tilde{K}_\delta(x + h, y, z)(b(y) - b(x + h))f(y)g(z)dydz \right|. \]

We will give the estimates for \( J_1, J_2, J_3, J_4 \), respectively in the following.

**Estimate for \( J_1 \).** It is easy to see that

\[ J_1 \leq |b(x + h) - b(x)| \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z)f(y)g(z)dydz \right| \]

\[ \leq |h| \|\nabla b\|_\infty \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} K(x, y, z)f(y)g(z)dydz \right| \]

\[ \leq C|h|T_*(f, g)(x). \]

Applying (1.13), we obtain

(2.11) \[ ||J_1||_{\infty} \leq C|h|. \]

**Estimate for \( J_2 \).** Notice that

\[ J_2 \leq \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} (K(x + h, y, z) - K(x, y, z))\chi_{|x+h-y|+|x+h-z|>\delta} \times (b(y) - b(x + h))f(y)g(z)dydz \right| \]

\[ + \sup_{\delta > 0} \left| \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} K(x, y, z)\chi_{|x+h-y|+|x+h-z|>\delta} - \chi_{|x-y|+|x-z|>\delta} \times (b(y) - b(x + h))f(y)g(z)dydz \right| \]

\[ = : J_{21} + J_{22}. \]

Observe that if \(|x-y|+|x-z|>\epsilon^{-1}|h|\) and \(0 < \epsilon < 1/4\), then \(|h| \leq \frac{1}{2} \max\{|x-y|, |x-z|, |y-z|\}\). Since \(K \in 2-CZK(A, \gamma)\), we have

\[ J_{21} \leq C||b||_{\infty} \iint_{|x-y|+|x-z|>\epsilon^{-1}|h|} \frac{|h|^\gamma}{(|x-y| + |x-z|)^{2n+\gamma}}|f(y)g(z)| dydz. \]
Minkowski’s inequality and Hölder’s inequality give that

$$\|J_{21}\|_r \leq C \int \int_{|y|+|z|>\epsilon^{-1}|h|} \frac{|h|^\gamma}{(|y|+|z|)^{2n+\gamma}} \left( \int_{\mathbb{R}^n} |f(x-y)g(x-z)|^r \, dx \right)^{\frac{1}{r}} \, dydz \leq C\epsilon^\gamma.$$ 

For $J_{22}$, it is easy to see that

$$J_{22} \leq \sup_{\delta>0} \left| \int \int_{|x-y|+|x-z|>\epsilon^{-1}|h|} K(x,y,z)(b(y)-b(x+h))f(y)g(z)dydz \right|$$

$$+ \sup_{\delta>0} \left| \int \int_{|x-y|+|x-z|>\epsilon^{-1}|h|} K(x,y,z)(b(y)-b(x+h))f(y)g(z)dydz \right|$$

$$= J_{221} + J_{222}.$$ 

For $J_{221}$, as $|x-y|+|x-z| > \epsilon^{-1}|h|$, $|x+h-y|+|x+h-z| > \delta$ and $0 < \epsilon < 1/4$, then $|x-y|+|x-z| \geq \frac{1}{2\epsilon+1}(|x+h-y|+|x+h-z|) \geq \frac{\delta}{2\epsilon+1}$. Then for any $1 < r_0 < \min\{p,q\}$, Hölder’s inequality and (2.1) give that

$$J_{221} \leq C\|b\|_{\infty} \sup_{\delta>0} \int \int_{\frac{1}{2\epsilon+1} \leq |x-y|+|x-z| \leq \delta} \frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n}} \, dydz$$

$$\leq C\sup_{\delta>0} \left( \int \int_{\frac{1}{2\epsilon+1} \leq |y|+|z| \leq \delta} \frac{|f(x-y)g(x-z)|^{r_0}}{(|y|+|z|)^{2n}} \, dydz \right)^{\frac{1}{r_0}}$$

$$\times \sup_{\delta>0} \left( \int \int_{\frac{1}{2\epsilon+1} \leq |y|+|z| \leq \delta} \frac{dydz}{(|y|+|z|)^{2n}} \right)^{\frac{1}{r_0}}$$

$$\leq C\sup_{\delta>0} \left( (2\epsilon+1)^{2n}\delta^{-2n} \int \int_{|y|+|z| \leq \delta} |f(x-y)g(x-z)|^{r_0} \, dydz \right)^{\frac{1}{r_0}} [1-(2\epsilon+1)^{-n}]^{-\frac{1}{r_0}}$$

$$\leq C\epsilon^{\frac{1}{r}} M(|f|^{r_0})(x)^{\frac{1}{r_0}} M(|g|^{r_0})(x)^{\frac{1}{r_0}}.$$ 

Notice that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $\frac{p}{r_0}, \frac{q}{r_0} > 1$. Thus, Hölder’s inequality and the $L^p$-boundedness of Hardy-Littlewood maximal operator $M$ give

$$\|J_{221}\|_r \leq C\epsilon^{\frac{1}{r}} M(|f|^{r_0})^{\frac{1}{r_0}} M(|g|^{r_0})^{\frac{1}{r_0}} \|_r \leq C\epsilon^{\frac{1}{r}} M(|f|^{r_0})^{\frac{1}{r_0}} \|_{p\cdot q} M(|g|^{r_0})^{\frac{1}{r_0}} \|_q$$

$$\leq C\epsilon^{\frac{1}{r}} \|f|^{r_0}\|_{p\cdot q} \|g|^{r_0}\|_{q\cdot r_0} \leq C\epsilon.$$

The estimate for $J_{222}$ is completely similar. It is easy to see that as $|x-y|+|x-z| > \epsilon^{-1}|h|$, $|x+h-y|+|x+h-z| \leq \delta$ and $0 < \epsilon < 1/4$, then $|x-y|+|x-z| \leq \frac{\delta}{1-2\epsilon}(|x+h-y|+|x+h-z|) \leq \frac{\delta}{1-2\epsilon}$. Therefore, analogous to $J_{221}$, using Hölder’s
inequality and \((2.2)\), then for any \(1 < r_0 < \min\{p, q\},\)
\[
J_{222} \leq C\|b\|_{\infty} \sup_{\delta > 0} \int_{|y| + |z| - \delta \leq \frac{1}{2M}} \frac{|f(y)g(z)|}{(|y| + |z|)^{2n}} \, dydz 
\]
\[
\leq CM(\|f\|_{r_0}(x)^{\frac{1}{r_0}})M(\|g\|_{r}(x)^{\frac{1}{r}})(1 - 2\epsilon)^{-n} - 1)^{\frac{1}{r_0}} 
\]
\[
\leq C\epsilon^{r_0}M(\|f\|_{r_0}(x)^{\frac{1}{r_0}})M(\|g\|_{r}(x)^{\frac{1}{r}}). 
\]

Thus, we also can obtain
\[
\|J_{222}\|_r \leq C\epsilon. 
\]

Combine with the estimates for \(J_{21}, J_{221}\) and \(J_{222}\), then
\[
(2.12) \quad \|J_2\|_r \leq C\epsilon. 
\]

**Estimate for \(J_3\).** Note that, \(|b(x) - b(y)| \leq \|\nabla b\|_{\infty}|x - y|\) and \(K \in 2-CZK(A, \gamma)\), we have
\[
J_3 \leq C\|\nabla b\|_{\infty} \int_{|x - y| + |x - z| \leq \epsilon^{-1}|h|} \frac{|f(y)g(z)|}{(|x - y| + |x - z|)^{2n-1}} \, dydz. 
\]

Therefore, Minkowski’s inequality and Hölder’s inequality give that
\[
(2.13) \quad \|J_3\|_r \leq C\int_{|x - z| \leq \epsilon^{-1}|h|} \left( \int_{\mathbb{R}^n} |f(x - y)g(x - z)|^r \, dx \right)^{\frac{1}{r}} \frac{dydz}{(|y| + |z|)^{2n-1}} \leq C\epsilon^{-1}|h|. 
\]

**Estimate for \(J_4\).** With the same way, we have
\[
(2.14) \quad \|J_4\|_r \leq C(2 + \epsilon^{-1})|h|. 
\]

Note that the constants \(C\) in \((2.11)-(2.14)\) are independent of \(h\) and \(\epsilon\). Taking \(|h|\) to be sufficiently small, we obtain \((2.9)\). Therefore, (c) holds for \(T_{*,b,1}(f, g)\) uniformly for \(f \in B_1, g \in B_2\). We complete the proof.

### 3. The proof of Theorem 1.2

**Proof of Theorem 1.2.** \(T_{*,(b_1,b_2)}\) is bounded from \(L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)\) to \(L^r(\mathbb{R}^n)\) by \((1.7)\). Let \(B_1, B_2\) be unit balls in \(L^p(\mathbb{R}^n)\) and \(L^q(\mathbb{R}^n)\), respectively. We need to prove that the set \(G = \{T_{*,(b_1,b_2)}(f, g) : f \in B_1, g \in B_2\}\) is strongly pre-compact in \(L^r(\mathbb{R}^n)\). Notice that
\[
|T_{*,(b_1,b_2)}(f, g)(x) - T_{*,(b_1,b_2)}(f, g)(x)| 
\]
\[
\leq \sup_{\delta > 0} \left| \int_{|y| + |z| > \delta} \left[ (b_1(y) - b_1(x))(b_2(z) - b_2(x)) - (b'_1(y) - b'_1(x))(b'_2(z) - b'_2(x)) \right] 
\]
\[
\times K(x, y, z) f(y)g(z) \, dydz \right| 
\]
\[
\leq T_{*,(b_1-b_1',b_2)}(f, g)(x) + T_{*,(b_1,b_2-b_2')}(f, g)(x). 
\]
Therefore, if \( b_j' \in C_c^\infty(\mathbb{R}^n) \), such that \( \| b_j - b_j' \|_{BMO} < \epsilon \) \((j = 1, 2)\), then \((1.7)\) gives that
\[
(3.1) \quad \| T_{(b_1, b_2)}(f, g) - T_{(b_1', b_2')}(f, g) \|_r \leq \| T_{(b_1 - b_1', b_2 - b_2')}(f, g) \|_r + \| T_{(b_1', b_2 - b_2')}(f, g) \|_r \\
\leq C(\| b_2 \|_{BMO} \| b_1 - b_1' \|_{BMO} + \| b_1' \|_{BMO} \| b_2 - b_2' \|_{BMO}) \| f \|_p \| g \|_q \leq C\epsilon.
\]
Therefore, we only need to prove that \( G \) is strongly pre-compact in \( L^r(\mathbb{R}^n) \) for \( b_j \in C_c^\infty(\mathbb{R}^n) \). By Lemma 2.1, we only need to show that \((a), (b)\) and \((c)\) in Lemma 2.1 hold for \( T_{(b_1, b_2)}(f, g) \) uniformly in \( G \) with \( b_1, b_2 \in C_c^\infty(\mathbb{R}^n) \).

By \((1.7)\), we easily obtain that \((a)\) holds for \( T_{(b_1, b_2)}(f, g) \) uniformly with \( f \in B_1, g \in B_2 \). Notice \( b_j \in C_c^\infty(\mathbb{R}^n) \), without loss of generality, we can also assume that \( \text{supp } b_j \subset \{ x \in \mathbb{R}^n : |x| \leq \beta \} \) with \( \beta > 1 \). For any \( \epsilon > 0 \), we take \( \alpha > 2\beta \) such that \((\alpha - \beta)^{-2n+n/r} < \epsilon \), then we have
\[
(3.2) \quad \left( \int_{\mathbb{R}^n} |T_{(b_1, b_2)}(f, g)(x)|^r \chi_{E_\alpha}(x) \, dx \right)^{r^{-1}} \\
\leq C \left( \int_{\mathbb{R}^n} \left( \int_{B(0, \beta)} \left( \int_{B(0, \beta)} \frac{|b_1(y)||b_2(z)|}{|x-y| + |x-z|} |f(y)||g(z)| \, dy \, dz \right)^r \, dx \right)^{1/r} \\
\leq C \| b_1 \|_{\infty} \| b_2 \|_{\infty} \int_{B(0, \beta)} \left( \int_{|x| > \alpha - \beta} \frac{dx}{|x|^{2nr}} \right)^{1/r} \| f \|_r \| g \|_r \, dy \, dz \\
\leq C \| b_1 \|_{\infty} \| b_2 \|_{\infty}^{3n(2-1/r)} \epsilon,
\]
which shows that \((b)\) holds for \( T_{(b_1, b_2)}(f, g) \) uniformly with \( f \in B_1, g \in B_2 \). It remains to prove that for any \( 0 < \epsilon < 1/4 \), if \( |h| \) is sufficiently small and dependent only on \( \epsilon \), then
\[
(3.3) \quad \| T_{(b_1, b_2)}(f, g)(\cdot + h) - T_{(b_1, b_2)}(f, g)(\cdot) \|_r < C\epsilon
\]
holds uniformly for \( f \in B_1, g \in B_2 \).

In fact, for any \( h \in \mathbb{R}^n \), we denote \( \bar{b}(x, y, z) = (b_1(y) - b_1(x))(b_2(z) - b_2(x)) \) and \( \bar{h}(x, y, z) = \bar{b}(x + h, y, z) - \bar{b}(x, y, z) \), using the notation in the proof of Theorem 1.1, we have
\[
|T_{(b_1, b_2)}(f, g)(x + h) - T_{(b_1, b_2)}(f, g)(x)| \\
\leq \sup_{\delta > 0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{K}_\delta(x + h, y, z) \bar{b}(x + h, y, z) f(y) g(z) dy dz \\
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{K}_\delta(x, y, z) \bar{b}(x, y, z) f(y) g(z) dy dz \right|.
\]
Similar to the decomposition for \((2.10)\), we can control the right hand side of the above inequality by \( L_1 + L_2 + L_3 + L_4 \), where
\[
L_1 := \sup_{\delta > 0} \left| \int_{|x-y| + |x-z| > \epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z) \bar{h}(x, y, z) f(y) g(z) dy dz \right| \\
L_2 := \sup_{\delta > 0} \left| \int_{|x-y| + |x-z| > \epsilon^{-1}|h|} (\tilde{K}_\delta(x + h, y, z) - \tilde{K}_\delta(x, y, z)) \bar{b}(x + h, y, z) f(y) g(z) dy dz \right|,
\]
and...
In the following, we will give the estimates for $L$. Note that (3.5)

$$L_3 := \sup_{\delta > 0} \int \int_{|x-y|+|x-z| \leq \epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z) \tilde{b}(x, y, z) f(y) g(z) dydz,$$

$$L_4 := \sup_{\delta > 0} \int \int_{|x-y|+|x-z| \leq \epsilon^{-1}|h|} \tilde{K}_\delta(x + h, y, z) \tilde{b}(x + h, y, z) f(y) g(z) dydz.$$

In the following, we will give the estimates for $L_1, L_2, L_3, L_4$, respectively.

**Estimate for $L_1$.** Observe

(3.4) \[ \tilde{b}_h(x, y, z) = (b_1(x) - b_1(x + h))(b_2(x) - b_2(x + h)) \]

\[ + (b_1(x) - b_1(x + h))(b_2(z) - b_2(x)) + (b_1(y) - b_1(x))(b_2(x) - b_2(x + h)). \]

Note that $|b_j(x) - b_j(x + h)| \leq |h|\|\nabla b_j\|_{\infty}$, then we have

$$L_1 \leq |h|^2 \|\nabla b_1\|_{\infty} \|\nabla b_2\|_{\infty} \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z) f(y) g(z) dydz$$

$$+ |h| \|\nabla b_1\|_{\infty} \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z) (b_2(z) - b_2(x)) f(y) g(z) dydz$$

$$+ |h| \|\nabla b_2\|_{\infty} \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} \tilde{K}_\delta(x, y, z) (b_1(y) - b_1(x)) f(y) g(z) dydz$$

$$\leq C|h|^2 T_s(f, g)(x) + C|h| T_s, b_2, 2(f, g)(x) + C|h| T_s, b_1, 1(f, g)(x).$$

Then (1.3) and (1.8) give that

(3.5) \[ \|L_1\| \leq C(|h|^2 + |h| \|b_1\|_{BMO} + |h| \|b_2\|_{BMO}) \|f\|_p \|g\|_q \leq C(|h| + |h|^2). \]

**Estimate for $L_2$.** The estimate for $L_2$ is similar to $J_2$. We have

$$L_2 \leq \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} (K(x + h, y, z) - K(x, y, z)) \chi_{|x+h-y|+|x+h-z| > \delta} \tilde{b}(x + h, y, z) f(y) g(z) dydz$$

$$+ \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} K(x, y, z) \tilde{b}(x + h, y, z) f(y) g(z) dydz$$

$$+ \sup_{\delta > 0} \int \int_{|x-y|+|x-z| > \epsilon^{-1}|h|} K(x, y, z) \tilde{b}(x, y, z) f(y) g(z) dydz$$

$$= : L_{21} + L_{22} + L_{23}.$$

The estimates for $L_{21}, L_{22}, L_{23}$ are completely analogous to $J_{21}, J_{221}, J_{222}$. Then

(3.6) \[ \|L_2\| \leq C\epsilon. \]

**Estimate for $L_3$.** Note that, $|b_j(x) - b_j(y)| \leq \|\nabla b_j\|_{\infty} |x - y|$ $(j = 1, 2)$ and $K \in 2-CZK(A, \gamma)$, we have

$$L_3 \leq C \|\nabla b_1\|_{\infty} \|\nabla b_2\|_{\infty} \int \int_{|x-y|+|x-z| \leq \epsilon^{-1}|h|} \frac{|f(y) g(z)|}{(|x-y| + |x-z|)^{2n-2}} dydz.$$
Therefore, by Minkowski’s inequality and Hölder’s inequality, we have

\begin{equation}
\|L_3\|_r \leq C\|\nabla b_1\|_{\infty}\|\nabla b_2\|_{\infty} \int \int_{|y|+|z| \leq \varepsilon^{-1}|h|} \left( \int \int_{\mathbb{R}^n} |f(x-y)g(x-z)|^r \,dx \right)^{\frac{1}{r}} \,dydz \\
\leq C(\varepsilon^{-1}|h|)^2 \|f\|_p \|g\|_q \leq C(\varepsilon^{-1}|h|)^2.
\end{equation}

**Estimate for** $M_4$. With the same way, we have

\begin{equation}
\|L_4 \|_r \leq C(2+\varepsilon^{-1})^2|h|^2.
\end{equation}

Note that the constants $C$ in (3.5)-(3.8) are independent of $h$ and $\varepsilon$. Taking $|h| < (2 + \varepsilon^{-1})^{-1}\varepsilon$, we obtain (3.3). Therefore, (c) holds for $T_{*,(b_1,b_2)}(f,g)$ uniformly for $f \in B_1, g \in B_2$. We complete the proof.

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