Static charged perfect fluid with Weyl-Majumdar relation

Daisuke Ida

Department of Physics, Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan
E-mail address: ida@tap.scphys.kyoto-u.ac.jp

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Static charged perfect fluid distributions have been studied. It is shown that if the norm of the timelike Killing vector and the electrostatic potential have the Weyl-Majumdar relation, then the background spatial metric is the space of constant curvature, and the field equations reduce to a single non-linear partial differential equation. Furthermore, if the linear equation of state for the fluid is assumed, then this equation becomes a Helmholtz equation on the space of constant curvature. Some explicit solutions are given.

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I. INTRODUCTION

In Newtonian theory, there are static equilibrium configurations of charged particles in which the gravitational attractions and the electric repulsions among the particles exactly balance. In general relativity, it is well known that the Einstein equation has such a static solution corresponding to these Newtonian situations, namely the Majumdar-Papapetrou solution [2,3] describing an equilibrium state of many charged black holes. [4]

Majumdar [2] studied the static Einstein-Maxwell system that is described by the static metric

\[ g = f dt \otimes dt - f^{-1} h \]  

(1)

and the vector potential

\[ A = -\psi dt, \]  

(2)

subject to the coupled Einstein and source-free Maxwell equations. The function \( f < 0 \) is the square of the timelike Killing vector field \( \partial_t \), \( h \) is the Riemannian 3-metric, and the function \( \psi \) denotes the electrostatic potential. Majumdar [2] proved that if there is a functional relationship between \( f \) and \( \psi \), this must be of the form

\[ f = -\psi^2 + c \]  

(3)

with a constant \( c \), generalizing Weyl’s result [1] in the case of axial symmetry. Moreover, he showed that if

\[ f = -\psi^2, \]  

(4)

then the 3-metric \( h \) is a flat metric, i.e., \( h = dx^2 + dy^2 + dz^2 \), and the field equations reduce to a single Laplace equation in the 3-dimensional flat space. Solutions of this type were studied by Papapetrou [3] also. Later, Hartle and Hawking [4] investigated the global structure of the Majumdar-Papapetrou solution, and they showed that this can be interpreted as describing an equilibrium state of extremely highly charged black holes.

For a system of charged dust in Newtonian theory, a similar equilibrium between gravitational and Coulomb forces is possible, whenever the mass density \( \mu \) and charge density \( \sigma \) of the charged dust satisfy

\[ \sigma = \pm \mu. \]  

(5)

Das [5] studied the corresponding situation in general relativity, and he showed that the condition (3), as (3), implies the Weyl-Majumdar relation (1), and in this case the field equations reduce to a single non-linear equation: i.e., the balance condition (4) makes sense in precisely the same form in general relativity. Further discussion concerning the generality of the assumption (3) can be found in De and Raychaudhuri. [6] Explicit solutions for this system have been found by many authors, e.g., Refs. [9–12] with spherical symmetry, and a recent paper [13] without spatial symmetry.

The balance equation and the Weyl-Majumdar relationship are closely related, and moreover, it seems that the force balance condition greatly simplifies the Einstein equation. The most typical examples are the Weyl and Majumdar-Papapetrou metrics mentioned above, and the Gürses metric with static charged-dust source. [14] In Newtonian theory, the gravitational and electrostatic forces acting on an element of charged dust must be parallel. This is also the case in Einstein theory: namely, the Bianchi identity (Euler equation) implies that there is a functional relationship between the gravitational and electrostatic potentials (not necessarily in the form of (1) or (2)). Such an analogy between the Newtonian and Einstein theory might hold even in the presence of a third force, e.g., a magnetic field or a pressure gradient. A configuration of a magnetic field parallel to the electric field is easily obtained by a duality rotation of pure electric field. However, there will be a magnetic monopole in this case. The presence of non-parallel magnetic and electric fields generates a Poynting flux, contributing to the energy-flux component of the stress-energy tensor, which is a characteristic of a stationary field rather than static one. On the other hand, the (isotropic) matter pressure itself does not generate any energy (heat) fluxes. Rather, it effectively contributes only to the energy-density and momentum-flux parts of the stress-energy tensor by definition, and one may expect that there is a large class of static configurations generated by a charged perfect-fluid. Such a configuration may be considered as the interior metric of multiple Majumdar-Papapetrou black-holes. Since the Majumdar-Papapetrou metrics describe
multiple (extremely highly) charged black-holes, they inevitably have timelike singularities in each horizon, so that consideration of the regular interior metric might be interesting. Gautreau and Hoffman generalized the balance equation in the presence of matter pressure, assuming Eq. (3), and showed

\[ \sigma = \pm (\mu + 3p) \left(1 + \frac{c}{\psi^2}\right)^{\frac{1}{2}}, \tag{6} \]

where \( p \) is the isotropic pressure of the fluid, though they did not consider the rest of the field equations. The existence of such a simple balance equation seems to imply that there are many static configurations. In fact, Guilfoyle recently studied this system of a charged perfect fluid subject to (3) and obtained a large class of algebraically general (type I) metrics.

In this paper, we investigate a system of a pure-electrically charged perfect fluid with the Weyl-Majumdar relation from a general point of view, and we show that there exists a large class of exact solutions, which can be algebraically general (type I) and need not have any spatial symmetries. As shown below, if the Weyl-Majumdar relation is assumed, then the 3-metric \( g \) must be that of a space of constant curvature, and the field equation reduces to a single non-linear equation. Moreover, if the fluid obeys a linear equation of state \( (p = \text{const.} \times \mu) \), then the field equation becomes a Helmholtz equation on the space of constant curvature. Therefore, solutions are expressed in terms of eigenfunctions of the Laplace operator. Some explicit solutions are given in [11].

II. FIELD EQUATIONS

The metric of static spacetimes can be expressed in the form

\[ g = -\frac{1}{\phi^2} dt \otimes dt + \phi^2 h, \tag{7} \]

where neither the function \( \phi > 0 \) nor the Riemannian 3-metric \( h \) depends on the coordinate \( t \); i.e., the vector field \( \xi = \partial_t \) satisfies the Killing equation

\[ \mathcal{L}_\xi g = 0. \tag{8} \]

The Killing vector field \( \xi \) of a static spacetime is hypersurface orthogonal, or twist-free, and thus it satisfies the Frobenius condition

\[ \xi \wedge d\xi = 0, \tag{9} \]

where \( \xi = -\phi^{-2} dt \) is the Killing 1-form. (Here and in what follows, we use the same symbol for vector fields and their corresponding 1-forms.)

The time-space components of the Ricci tensor are obtained from the equation \( \text{see e.g., Ref. [18]} \)

\[ 16\pi \hat{j} = -\ast d \ast d\xi, \tag{10} \]

where

\[ 8\pi \hat{\rho} = -\mathcal{R}(\xi, \partial_\mu) dx^\mu, \tag{11} \]

and \( \ast \) represents the Hodge operator. The Lichnerowicz theorem states that \( j \) is proportional to \( \xi \) in the static case. This can be shown by

\[ 0 = d \ast (\xi \wedge d\xi) = d\iota_\xi \ast d\xi \]

\[ = (\mathcal{L}_\xi - \iota_\xi d) \ast d\xi = 16\pi \iota_\xi \ast \hat{j} \]

\[ = -16\pi \xi \wedge \hat{j}, \tag{12} \]

where \( \iota_\xi \) denotes the interior product with respect to \( \xi \), and the identity \( \mathcal{L}_\xi = d\iota_\xi + \iota_\xi d \) has been used.

The time-space component of the Ricci tensor is obtained by calculating the Laplacian of \( f = -\phi^{-2} \):

\[ d \ast df = d \ast d\iota_\xi \xi = d \ast (\mathcal{L}_\xi - \iota_\xi d) \xi \]

\[ = d(\xi \wedge \ast d\xi) = \ast (d\xi : d\xi) - \xi \wedge d \ast d\xi \]

\[ = \ast (d(\iota_\xi d\xi, d\xi) - d\iota_\xi \ast d\xi) \]

\[ = \ast \left( \frac{1}{f} df \otimes df \right) + 16\pi \ast g^{-1}(\xi, \hat{j}). \tag{13} \]

Here the colon denotes the inner product of differential forms \( (i.e., \alpha : \beta = (1/p！)\alpha_{\mu_1, \mu_2, ..., \mu_p} \beta^{\mu_1, \mu_2, ..., \mu_p}) \) for \( p \)-forms \( \alpha \) and \( \beta \). Therefore, we have

\[ 8\pi \hat{\rho} = \mathcal{R}(\xi, \xi) = \phi^{-2} \ast d \ast d\phi + \phi^{-4} g^{-1}(d\phi, d\phi). \tag{14} \]

The Ricci tensor of the 3-metric \( h \) encodes the remaining components of the Ricci tensor, and with a direct calculation, we obtain the expression

\[ \mathcal{R}(h) = \mathcal{R} - 8\pi \phi^2 (\hat{\rho} g + 2 \xi \otimes \hat{j}) + 2 \frac{d\phi \otimes d\phi}{\phi^2}. \tag{15} \]

Next, consider the static electro-magnetic field \( F = dA, \mathcal{L}_\xi F = 0 \). Here we assume that only the electric field exists for static observers, which implies

\[ \xi \wedge F = 0. \tag{16} \]

In the source-free Einstein-Maxwell static case, this assumption is justified (up to duality rotation) by the Einstein equation, while in the charged perfect fluid case, this condition is equivalent to imposing the condition that the fluid element as well as the gravitational field is static. Moreover, there is no freedom of the duality rotation in this case. Here, the Maxwell equation \( dF = 0 \) becomes

\[ d\iota_\xi F = (\mathcal{L}_\xi - \iota_\xi d) F = 0, \tag{17} \]

while the other Maxwell equation, \( d \ast F = 4\pi \sigma \ast u \), reduces to

\[ d \ast \iota_\xi F = -d(\xi \wedge \ast F) = -\ast (d\xi : F) + 4\pi \sigma \ast g^{-1}(\xi, u) \]

\[ = -2 \ast \frac{g^{-1}(\xi, d\phi)}{\phi} + 4\pi \sigma \ast g^{-1}(\xi, u), \tag{18} \]
where \( u \) and \( \sigma \) denote the velocity and the charge density of the fluid, respectively. Equation (17) implies that locally there exists a real function (electrostatic potential) \( \psi \) such that \( d\psi = i\xi F \) holds. Then the full Maxwell equations reduce to a single Poisson equation,

\[
* d* d\psi = 2g^{-1}(d\psi, d\phi) - 4\pi \sigma g^{-1}(\xi, u). \tag{19}
\]

On the other hand, the stress-energy tensor of the Maxwell field \( T_F \) is

\[
8\pi T_F(\partial_\mu, \partial_\nu) = g^{-1}(i_{\partial_\mu} F, i_{\partial_\nu} F) + g^{-1}(i_{\partial_\mu} * F, i_{\partial_\nu} * F) \tag{20}
\]

In pure electric cases, \( j_F = -T_F(\xi, \partial_\mu)dx^\mu \) is proportional to \( \xi \). \( \xi \) Thus, the energy-flux components of the stress-energy tensor vanish,

\[
\xi \wedge j_F = 0, \tag{21}
\]

while the energy-density component becomes

\[
8\pi \rho_F = 8\pi T_F(\xi, \xi) = g^{-1}(d\psi, d\psi). \tag{22}
\]

In terms of \( j_F \) and \( \rho_F \), Eq. (20) can be written as

\[
8\pi T_F = 8\pi \phi^2(\rho_F g + 2\xi \otimes j_F) - 2\phi^2 d\psi \otimes d\psi. \tag{23}
\]

Next, consider a perfect fluid. Its stress-energy tensor has the form

\[
T_m = (\mu + p)u \otimes u + pg, \tag{24}
\]

or equivalently

\[
\tilde{T}_m = T_m - \frac{1}{2} \text{tr}(T_m)g = (\mu + p)u \otimes u + \frac{1}{2}(\mu - p)g. \tag{25}
\]

The fluid velocity \( u \) must be proportional to \( \xi \); i.e., \( u = \phi \xi \). This is a consequence of the time-space components of the Einstein equation. In analogy to the electric field, we define

\[
\tilde{j}_m = -\tilde{T}_m(\xi, \partial_\mu)dx^\mu = \frac{1}{2}(\mu + 3p)\xi \tag{26}
\]

and

\[
\tilde{\rho}_m = \tilde{T}_m(\xi, \xi) = \frac{\mu + 3p}{2\phi^2}. \tag{27}
\]

The equation of hydrostatic equilibrium (Euler equation) is obtained from the Bianchi identity for the stress-energy tensor \( \nabla \cdot (T_F + T_m) = 0 \), which gives

\[
d\ln \phi = \frac{1}{\mu + p}(dp + \sigma d\psi). \tag{28}
\]

Now we consider the Einstein equation

\[
Ric = 8\pi(T_F + \tilde{T}_m). \tag{29}
\]

The time-space component equations of the field equation \( \xi \wedge (\tilde{j}_F - \tilde{j}_m) = 0 \) are identically satisfied, while the time-time component \( \tilde{\rho} = \rho_F + \tilde{\rho}_m \) leads to

\[
* d* d\phi = -\frac{g^{-1}(d\phi, d\phi)}{\phi} + \phi^3 g^{-1}(d\psi, d\psi) + 4\pi \phi(\mu + 3p). \tag{30}
\]

From Eqs. (20), (23) and (25), we obtain the remaining equation,

\[
Ric[h] + 16\pi \phi^2ph = 2\phi^{-2}d\phi \otimes d\phi - 2\phi^2 d\psi \otimes d\psi. \tag{31}
\]

Our task is to solve Eqs. (19), (28), (30) and (31). However, we need additional conditions, such as an equation of state, to make this system deterministic.

To solve Eq. (31), we assume the Weyl-Majumdar relation (4), or

\[
\psi + c\phi^{-1} = 0, \quad (c = \pm 1). \tag{32}
\]

Then the r.h.s. of Eq. (31) vanishes, and the 3-dimensional manifold with metric \( h \) becomes an Einstein space; i.e., the Ricci tensor has only a trace part (scalar curvature). Since in three or higher dimensional Einstein space, the scalar curvature must be constant (which can be seen from the contracted Bianchi identity), the matter pressure \( p \) is proportional to \( \phi^{-2} \) with a constant ratio. Without loss of generality, we may take \( p = -\kappa/8\pi \phi^2 \) (\( \kappa = 0, \pm 1 \)). Since a 3-dimensional Einstein space is a space of constant curvature (namely, the metric as well as the curvature is characterized by a constant scalar curvature), we have

\[
h = \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{1}{4}\kappa(x^2 + y^2 + z^2)\right]^2}. \tag{33}
\]

Then, Eqs. (28) and (30) reduce to

\[
(\Delta + \lambda)\phi = 0 \tag{34}
\]

and

\[
\sigma = \epsilon(\mu + 3p), \tag{35}
\]

where \( \Delta \) denotes the Laplacian with respect to \( h \), and \( \lambda = 4\pi \phi^2(\mu + 3p) \). It is easily checked that the Euler equation (20) is consistent with (35).

Equation (34) is non-linear. However, it can be made linear if we assume a linear (isothermal-type) equation of state for the fluid, i.e. \( \mu = \text{const} \times p (= \text{const} \times \phi^{-2}) \). Then, \( \lambda \) becomes constant, and Eq. (34) is simply a Helmholtz equation in the space of constant curvature, whose solutions are known (see, e.g., Ref. [20]).

III. EXAMPLES

Here we study some explicit solutions of Eq. (34). We mainly consider the spherically symmetric solutions as simple examples. We assume the dominant energy condition, which requires \( \mu \geq |p| \).
A. Positive pressure \((\kappa = -1)\)

In this case, the dominant energy condition requires \(\lambda \geq 2\). In the spherical polar coordinate system \(\{r, \theta, \varphi\}\), the 3-metric \(h\) can be written as

\[
h = \eta(r)[dr^2 + r^2(d\theta^2 + \sin^2 \varphi d\varphi^2)],
\]

where \(\eta = (1 - r^2/4)^{-1}\) \((0 < r < 2)\). Equation (44) for the function \(\phi(r)\) becomes

\[
\frac{1}{\eta^{-1/2}r^2} \frac{d}{dr}(\eta^2 r^2 \frac{d\phi}{dr}) + \lambda \phi = 0. \tag{37}
\]

The general solution of this equation is

\[
\phi = l \frac{1 - r^2/4}{r} \sin \left(\frac{(\lambda - 1)^{1/2} \ln \frac{2 + r}{2 - r} + \delta}{2}\right), \tag{38}
\]

where \(l > 0\) and \(0 \leq \delta \leq \pi\) are integration constants. In general, the set on which \(\phi = 0\) is a singularity, since, for example, the field invariant \(F : F = -\tilde{\phi}^{-2}g^{-1}(d\phi, d\phi)\) blows up there. Hence this solution must have a singularity, and the situation is the same even without spherical symmetry. The locations of the singularities are given by

\[
r = 2\tan \frac{(n\pi - \delta)}{2(\lambda - 1)^{1/2}}, \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{39}
\]

at which the area radius \(R = |\phi\eta r|\) vanishes, so that these singularities are actually point-like, and each connected component of the spacetime has in general the topology \(S^3\) minus two points. (Without spherical symmetry, the singularity might be line-like or surface-like, and the spacetime topology would be \(S^3\) minus several such sets.)

Next, let us consider the solution (48) in the neighborhood of \(r = 0\). We may assume that \(0 \leq \delta \leq \pi/2\), since the transformation \(r \to -r\) corresponds to \(\delta \to \pi - \delta\). Then, let us consider the region \(r < 0\). Whenever \(\delta \neq 0\), this solution diverges at \(r = 0\) like \(l\sin \delta/r\). However, this is only a coordinate singularity. The hypersurface \(r = 0\) is a null hypersurface, on which the Killing vector becomes null, and it is generated by spheres of radius \(R = l\sin \sigma\). In fact, whenever \(\delta \neq 0, \pi/2\), the region \(r < 0\) is extended analytically to the region \(r > 0\) across this null hypersurface. To see this, let us consider the coordinate transformation

\[
v = t + \int \tilde{r}^2 \eta dr. \tag{40}
\]

Then the metric transforms to

\[
g = -\frac{dv^2}{\tilde{\phi}^2} + \frac{2}{1 - r^2/4} dv dr + \tilde{r}^2 \sin^2 \left[\frac{(\lambda - 1)^{1/2} \ln \frac{2 + r}{2 - r}}{2}\right] (d\theta^2 + \sin^2 \varphi d\varphi^2), \tag{41}
\]

which is manifestly non-singular at \(r = 0\).

Moreover, we can consider a class \(C^1\) extension of the region \(r < 0\) to the exterior field of a black-hole spacetime. The Misner-Sharp mass \(m_H\) (which coincides with the Hawking mass or Kodama mass in the case of spherical symmetry) with respect to the surface \(r = 0\) is

\[
m_H = \frac{l \sin \delta}{2}. \tag{42}
\]

On the other hand, the electric charge inside a two-surface \(S\) can be evaluated as

\[
q = -\frac{1}{4\pi} \int_S *F. \tag{43}
\]

In particular, when a surface defined by \(r = \text{const}\) is chosen as \(S\), we find

\[
q = \epsilon \frac{r^2}{1 - r^2/4} \frac{d|\phi|}{dr}, \tag{44}
\]

where ‘inside’ corresponds to smaller \(r\), so that the electric charge inside the surface \(r = 0\) is

\[
q_H = \epsilon l \sin \delta = 2\epsilon m_H. \tag{45}
\]

This relationship is identical to that satisfied by the extreme Reissner-Nordström spacetime on the horizon. Furthermore, the mass density and matter pressure of the fluid vanish on the surface \(r = 0\). These evaluations suggest that the region \(r < 0\) with the metric characterized by the Eq. (38) can be matched at \(r = 0\) to the exterior field with the extreme Reissner-Nordström metric. By rescaling the coordinates according to

\[
\tilde{r} = l(\lambda - 1)^{1/2} \cos \delta r, \quad \tilde{t} = \frac{t}{l(\lambda - 1)^{1/2} \cos \delta}, \tag{46}
\]

the metric transforms to

\[
g = -\frac{dv^2}{\tilde{\phi}^2} + \tilde{\phi}^2 \eta^2 \left[dr^2 + \tilde{r}^2 (d\theta^2 + \sin^2 \varphi d\varphi^2)\right], \tag{47}
\]

where

\[
\tilde{\phi} = \frac{l}{r \eta} \sin \left[\frac{(\lambda - 1)^{1/2} \ln \frac{2(\lambda - 1)^{1/2} \cos \delta + \tilde{r}}{2(\lambda - 1)^{1/2} \cos \delta - \tilde{r}}}{2}\right], \tag{48}
\]

and

\[
\tilde{\eta} = \left[1 - \frac{\tilde{r}}{4l^2(\lambda - 1) \cos^2 \delta}\right]^{-1}. \tag{49}
\]

Note that \(\tilde{\phi}\) behaves near \(\tilde{r} = 0\) like

\[
\tilde{\phi} = 1 + \frac{l \sin \delta}{\tilde{r}} + O(\tilde{r}^2). \tag{50}
\]

Next, we introduce the advanced time coordinate

\[
v = \tilde{t} + \int \tilde{r}^2 \tilde{\eta} d\tilde{r}. \tag{51}
\]
Then the metric (47) becomes
\[ g = -\frac{dv^2}{\phi^2} + 2\eta vd\tilde{r} + \phi^2\eta^2\tilde{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2), \] (52)

while the metric of the extreme Reissner-Nordström has the form
\[ g_{RN} = -\left(1 + \frac{m}{r}\right)^{-2}dv^2 + 2vd\tilde{r} + \left(1 + \frac{m}{r}\right)^2\tilde{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2). \] (53)

If the mass parameter (ADM mass) of the metric (53) is
\[ m = l \sin \delta, \] (54)

then these two metrics are matched on the hypersurface \( \tilde{r} = 0 \), and it is easily seen from (44) that this extension is class \( C^1 \). In this case, the exterior metric (53) and the interior metric (52) correspond to \( \tilde{r} > 0 \) and \( \tilde{r} < 0 \), respectively. We can similarly verify that the region \( r > 0 \) for (58) can be matched to the \( m = l \sin \delta \) interior metric (i.e., the metric (53) for \( m = -l \sin \delta < 0, \tilde{r} > 0 \)), and if \( l \sin \delta < 0 \), then the regions \( r < 0 \) and \( r > 0 \) for (58) can be matched to the interior \( (m = -|l \sin \delta|) \) and exterior \( (m = |l \sin \delta|) \) metric (53), respectively.

Of course, we may also obtain solutions without spherical symmetry using the method of multipole expansion. Instead, here we give an example by superposing monopole solutions with different centers \((x_i, y_i, z_i)\):
\[ \phi = \sum_i l_i \frac{1 - r_i^2/4}{r_i} \sin \left(\frac{(\lambda - 1)\pi}{2} \ln \frac{2 + r_i}{2 - r_i} + \delta_i\right), \] (55)
where
\[ r_i = [(x - x_i)^2 + (y - y_i)^2 + (y - y_i)^2]^{1/2}. \] (56)

Here, the quantities \( l_i \) and \( \delta_i \) are constants. Each hole \( r_i = 0 \) can be matched to the Reissner-Nordström metric with \( m = l_i \sin \delta_i \).

**B. Dust \( \kappa = 0 \)**

For completeness, we briefly discuss this special case. We only give some explicit solutions here, since the properties of these solutions are quite similar to those in the positive pressure case. Recently, Gürses [13] also found this class. The field equation is the ordinary Helmholtz equation,
\[ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \lambda\right)\phi = 0, \] (57)

and the energy condition requires \( \lambda \geq 0 \). The general spherically symmetric solution is
\[ \phi = l \sin(\lambda r/2 + \delta), \] (58)

where \( l > 0 \) and \( 0 \leq \delta \leq \pi \) are integration constants. As in the positive pressure case, the solution vibrates around \( \phi = 0 \) in general, so that there always exists a singularity. The solution is matched to the exterior Reissner-Nordström metric with \( m = l \sin \delta \) at \( r = 0 \) with class \( C^1 \) differentiability. Moreover, in the case of spherically symmetric static dust solutions, the metric can be matched to an extreme Reissner-Nordström metric on any symmetric sphere by virtue of the absence of matter pressure.

The solution without spatial symmetry can be expressed in terms of the spherical harmonics and the spherical Bessel functions. We simply give a multi-center solution
\[ \phi = \sum_i \frac{\sin(\lambda r_i/2 + \delta)}{r_i}. \] (59)

For more information, see Ref. [13].

**C. Negative pressure \( \kappa = 1 \)**

In this case the dominant energy condition requires \( \lambda \geq -1 \). Consider the solution of Eq. (55) with \( \eta = (1 + r^2/4)^{-1} \). The general solution is
\[ \phi = \frac{1 + r^2/4}{r} \sin \left[(1 + \lambda)\frac{\pi}{4} \arcsin \frac{r}{1 + r^2/4} + \delta\right]. \] (60)

where \( l > 0 \) and \( 0 \leq \delta \leq \pi \) are integration constants, and the arcsin above should be regarded as
\[ \arcsin \frac{r}{1 + r^2/4} = \begin{cases} -
\pi - \text{Arcsin} \frac{r}{1 + r^2/4} & (r < -2) \\
\text{Arcsin} \frac{r}{1 + r^2/4} & (-2 \leq r \leq 2) \\
\pi - \text{Arcsin} \frac{r}{1 + r^2/4} & (r > 2) \end{cases} \] (61)

in terms of the Arcsin (which takes the principal value). Unlike the case of positive pressure or dust, these solutions need not have any singularity in the appropriate range of \( \lambda \). In fact, if \(-1 \leq \lambda \leq -3/4 (|p| \leq 3/2|p|)\) and \((1 + \lambda)^{1/2} \pi \leq \delta \leq \pi - (1 + \lambda)^{1/2} \pi\), then \( \phi \) gives a regular solution for all \( r \), and if \( \delta = 0 \) and \(-1 \leq \lambda \leq 0 (|p| \leq \mu \leq 3|p|)\), then it gives a regular solution for \( r \geq 0 \).

As in the other cases, we can match the solution to the exterior metric \( (\tilde{r} > 0) \) of the extreme Reissner-Nordström spacetime \( [13] \) at \( r = 0 \) whenever \( \delta \neq 0, \pi/2, \pi \). Note that we may assume \( 0 < \delta < \pi/2 \), since \( \delta \) is transformed to \( \pi - \delta \) by \( r \rightarrow -r \). Let us consider the coordinate transformation
\[ \tilde{r} = l(1 + \lambda)^{1/2} \cos \delta \, r, \quad \tilde{t} = \frac{t}{l(1 + \lambda)^{1/2} \cos \delta}. \] (62)
Then the metric can be written

\[ g = \frac{dt^2}{\phi^2} + \delta^2 \eta^2 \left[ d\vartheta^2 + \tilde{r}^2 (d\phi^2 + \sin^2 \vartheta d\varphi^2) \right], \quad (63) \]

where

\[ \tilde{\phi} = \frac{l}{r\tilde{\eta}} \sin \left[ (1 + \lambda) \theta \arcsin \frac{\tilde{r} \hat{\eta}}{l(1 + \lambda) \theta \cos \delta} + \delta \right], \quad (64) \]

and

\[ \tilde{\eta} = \left[ 1 + \frac{\tilde{r}^2}{4l^2(1 + \lambda) \cos^2 \delta} \right]^{-1}. \quad (65) \]

Furthermore, the transformation

\[ v = \tilde{t} + \int_{\tilde{r}}^{r} \tilde{\phi}^2 \tilde{\eta} d\tilde{r} \quad (66) \]

gives a metric in the form

\[ g = -\tilde{\phi}^{-2} dv^2 + 2\tilde{\eta} dv d\tilde{r} + \tilde{\phi}^2 \tilde{\eta}^2 d(2\theta^2 + \sin^2 \vartheta d\varphi^2). \quad (67) \]

The region \( \tilde{r} < 0 \) with this metric can be matched to the region \( r > 0 \) with the metric (53) with

\[ m = l \sin \delta. \quad (68) \]

In this case also, the extension is class \( C^1 \), since

\[ \tilde{\phi} = 1 + \frac{l \sin \delta}{\tilde{r}} + O(\tilde{r}^2), \quad \tilde{\eta} = 1 + O(\tilde{r}^2). \quad (69) \]

Unlike the case of a positive pressure fluid, the metric (64) or (73) can be regular for all \( r < 0 (\tilde{r} < 0) \) if \( \delta \geq (1 + \lambda) \frac{\pi}{2} \) is satisfied. In this case, the surface \( r = -\infty \) is neither infinity nor a singularity. In fact, it can be shown that this surface is a null hypersurface and that the spacetime can be extended beyond \( r = -\infty \), unless \( \delta = (1 + \lambda) \frac{\pi}{2} \), in which case \( r = -\infty \) simply corresponds to a regular center. To see this, consider the inverse of the radial coordinate,

\[ \tilde{r} = 4/r. \quad (70) \]

In terms of this coordinate, the metric transforms to

\[ g = \frac{dt^2}{\phi^2} + \delta^2 \eta^2 \left[ d\vartheta^2 + \tilde{r}^2 (d\phi^2 + \sin^2 \vartheta d\varphi^2) \right], \quad (71) \]

where

\[ \tilde{\phi} = \frac{1 + \tilde{r}^2/4}{\tilde{r}} \sin \left[ (1 + \lambda) \theta \arcsin \frac{\tilde{r}}{1 + \tilde{r}^2/4} \right. \]

\[ + \pi + (1 + \lambda) \frac{\pi}{2} - \delta \left], \quad (72) \]

and

\[ \tilde{\eta} = \left( 1 + \frac{1}{4} \tilde{r}^2 \right)^{-1}. \quad (73) \]

Hence, the effect of this transformation is simply

\[ \delta \mapsto \pi + (1 + \lambda) \frac{\pi}{2} - \delta, \quad (74) \]

so that the metric can be analytically extended across the null hypersurface \( \tilde{r} = 0 \) in a manner similar explained above, and it can also be matched to the interior region \( (\tilde{r} < 0) \) with the Reissner-Nordström metric (53) with \( m = l \sin[\delta - (1 + \lambda) \frac{\pi}{2} \pi] \).

Finally, we obtain a multi-center solution in an obvious way:

\[ \phi = \sum_i l_i \frac{1 + \tilde{r}_i^2/4}{r_i} \sin \left[ (1 + \lambda) \theta \arcsin \frac{r_i}{1 + \tilde{r}_i^2/4} + \delta_i \right]. \quad (75) \]

IV. CONCLUSIONS

We have investigated the static field of a charged perfect fluid, and shown that only two assumptions, the Weyl-Majumdar relation (4) and a linear equation of state, allow us to reduce the field equations to a single linear equation. Thus, any solution of this type can be described by an eigenfunction of the Laplacian in the 3-dimensional space of constant curvature, and the eigenvalue determines the equation of state, i.e., the ratio of the energy density to pressure. These solutions have some desirable properties. For example, they obey a simple equation of state, as required, neither the mass density nor the matter pressure changes sign, and they need not have any spatial symmetry.

The mechanism responsible for this simplification of field equations may be understood in terms of force balance among the gravitational force, the Coulomb force and the pressure gradient. The function \( \phi^{-2} \) is the norm of the static Killing vector, and thus it can be interpreted as the gravitational potential. Then Eq. (4) gives a functional relationship between the gravitational and electrostatic potential, which implies that the gravitational and Coulomb forces are parallel. This property characterizes the Weyl (4) and Majumdar-Papapetrou (24) solutions.

One main point of this paper is that the pressure gradient automatically becomes parallel to both the gravitational and Coulomb forces as seen from the Einstein equation, even in the presence of a perfect fluid, which is implicit in the spherically symmetric case. Thus, the balance equation among the three relevant forces, which is originally a vector equation, becomes a scalar equation (53), explaining why a large class of inhomogeneous metrics are obtained. Though we have discussed only the situation in which equipotential surfaces of the gravitational potential, the electrostatic potential, and the matter pressure coincide, it is not a necessary condition for the static configuration; there would be a situation in which the vector balance equation holds at each point. However, such a
spacetime might have some spatial symmetry, such as axial or cylindrical symmetry, or the equation of state might depend on spacetime points. The static configuration considered here may be regarded as a final state of a non-rotating object composed of charged fluid. Through the realistic contraction of such an object with a given equation of state, the scalar balance equation might be satisfied in the final equilibrium state, even if it is not spherically symmetric.

Another characterization of the solution can be given in terms of the Ernst potentials [21]. In analogy to the stationary Einstein-Maxwell system, we can define the Ernst potential by $E = 1/\phi^2 - \psi^2$. Then Eq. (30) can be replaced by the Poisson equations for $E$. However, the assumption (4) implies that $E = 0$, so that this equation reduces to an algebraic equation, which is just the balance equation (35).

We have also shown that solutions corresponding to Eqs. (38), (58) and (60) can be matched to the extreme Reissner-Nordström metric at the horizon. However, this interior solution has a central singularity unless the matter pressure is negative. In the negative pressure case, the interior solution (75) can be regular for an appropriate range of parameters, and then $r \to -\infty$ does not represent conformal infinity but, rather, represents another ‘hole’ connected by another interior metric, or regular center. Since the matching conditions are determined by the behavior near the horizon, solutions (72), (56) and (55) could be matched to an appropriate Majumdar-Papapetrou metric [2] at $r_i = 0$ in a manner similar to that described here.

The existence of a regular interior metric has some connection with the Buchdahl theorem [22]. Roughly speaking, the Buchdahl theorem states that a relativistic star composed of perfect fluid with mass $M$ and radius $R$ can exist only if $R > (9/4)M$. The charged generalization of the Buchdahl theorem has been proved by Yu and Liu [24]. They showed that the radius of a star of charged fluid has a similar lower bound if the total charge of the star does not exceed the extreme limit $Q = M$. In the extreme case, there is a series of interior solutions in which the surface of the star can be arbitrarily close to a black-hole horizon [23,25] showing that $\inf R = M$. On the other hand, our example shows that there actually exists an $R = M$ star.

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