Model Predictions for the Phenomenology of Isoscalar Heavy Baryons

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Abstract

We study certain physical observables of isoscalar heavy baryons using potential models based on hadronic degrees of freedom. The goal is to compare these results with those from an effective theory obtained in a combined heavy quark and large $N_c$ expansion and thereby test the convergence of the effective theory. We study the excitation energy of $\Lambda_b$ and semileptonic decay form factors. The models tend to confirm the usefulness of the effective theory for most observables.
I. INTRODUCTION

In a series of previous papers [1, 2, 3, 4, 5], the phenomenology of isoscalar heavy baryons (where by heavy baryon we mean one containing a charm or bottom quark) and their excited states were studied in a combined heavy quark and large $N_c$ expansion where the natural counting parameter is in terms of $\lambda^2$ (where $\lambda = \frac{1}{N_c} \frac{\Lambda}{m_Q}$ with $N_c$ denoting the number of QCD colors, $m_Q$ represents the heavy quark mass, and $\Lambda$ is a typical hadronic scale). In this combined expansion, the ratio $N_c \Lambda m_Q$ is arbitrary. As $\lambda \to 0$ in the combined limit, QCD acquires a dynamical symmetry—a contracted $O(8)$ symmetry, which connects orbitally excited states of the heavy baryon to its ground state [2]. This contracted $O(8)$ symmetry group contains as a subgroup the generating algebra of a simple harmonic oscillator. In ref. [5], predictions at next-to-leading order (NLO) in this combined expansion have been made for the energy levels, semi-leptonic decays, and electromagnetic decay rates of $\Lambda_c$ and $\Lambda_b$ baryons. Although the effective theory with its approximate symmetry can be derived directly from QCD [1], these heavy baryons can be described in terms of models based on a bound state picture which describes the heavy baryon as a bound state of an ordinary baryon with a heavy meson [6, 7, 8, 9, 10, 11, 12, 13, 14]. These bound state models exhibit the same QCD contracted $O(8)$ symmetry with similar counting rules for the effective operators. While the predictions in ref. [5] based on the effective theory are model independent, they do depend on the convergence of the effective expansion. In this paper, we explore the issue of convergence of the expansion by comparing the theoretical predictions based on the expansion derived in ref. [4] with predictions based on “realistic” models. Our goal is to see whether for these models the contribution of higher order terms in the effective expansion are large enough to render the expansion useless or, conversely, whether the expansion proves to be relatively accurate. To do this, we study various potential models which are based on the bound state picture of heavy baryons. We will choose “reasonable” potential models with scales typical of hadronic physics. These models are supposed to describe the states of heavy baryons below threshold between the dissociation into an ordinary light baryon and a heavy meson. The effective theory ought to well describe the low-lying states below threshold provided the expansion is convergent. We will test the model predictions for the physical observables of low–lying $\Lambda_c$ and $\Lambda_b$ baryons against the effective theory at NLO as a way to get a qualitative sense on how convergent the expansion is expected to be in nature. Of course, these models are not nature and thus there is no guarantee that effective convergence in any given model insures that in nature the expansion ought to converge. However, if the validity of the expansion is robust in the sense that the effects of the higher order terms are small for a wide variety of sensible models, it is plausible that the expansion will be useful for the real world.

Heavy baryons can be envisioned in terms of two complementary bound state pictures; one picture describes the heavy baryon as the bound state of a heavy meson (containing the heavy quark) coupled to an ordinary light baryon. Models of this sort based on the Skyrme model were commonly studied in the early 1990s [6, 7, 8, 9, 10, 11]. We will refer to this particular picture as the bound state picture. Another way to describe the heavy baryon is in terms of a single heavy quark coupled to the brown-muck—a technical term for the light degrees of freedom which includes the light quarks and gluons. The connection of these pictures with QCD is straightforward. The key point is that the motion between the heavy quark and the brown-muck or, equivalently, between the heavy meson and the ordinary light baryon is collective. This motion is characteristically low energy with a
characteristic excitation energy of order $\lambda^{3/2}$ \[1\], while the internal excitations of the brown-muck (or equivalently of the ordinary light baryon) are of order $\lambda^{0}$. This scale separation is what permits the development of an effective theory.

In this paper, we will work with the bound state picture which involves the bound state of a heavy meson and light baryon. This picture has the advantage that the heavy meson and light baryon masses are well defined and experimentally accessible in an unambiguous way. This is not the case, however, for the heavy quark and brown-muck masses. In the combined limit, the heavy meson mass and the light baryon mass both scale as $\lambda^{-1}$ since the light baryon mass $m_{\text{bar}} \sim N_{c}$ (because of the large $N_{c}$ limit \[15, 16\]) and the heavy meson mass $m_{H} \sim m_{Q}$ (because of the heavy quark limit \[17, 18, 19, 20, 21, 22\]). Because both of these masses are heavy, the collective wave function describing the low-energy excitations is localized near the bottom of the effective potential and does not spread appreciably. The collective motion describing the low-lying states of heavy baryons is approximately harmonic since all non-singular potentials look harmonic at the bottom of any potential well. The excitation energy of these low-lying states at leading order (LO) is given by,

$$\omega = \sqrt{\frac{\kappa}{\mu_{Q}}},$$  \hspace{1cm} (1)

where $\kappa$ is the effective spring constant. By general large $N_{c}$ counting rules \[15, 16\], $\kappa$ scales as $N_{c}^{0} \sim \lambda^{0}$, and $\mu_{Q} = \frac{m_{\text{bar}}m_{H}}{m_{\text{bar}}+m_{H}} \sim \lambda^{-1}$ is the reduced mass of the bound state picture where $m_{\text{bar}}$ is the light baryon mass and $m_{H}$ denotes the heavy meson mass. As $\lambda \to 0$, the whole tower of harmonically excited states becomes degenerate with the ground state—a clear signature of an emergent symmetry. This symmetry is a contracted $O(8)$ symmetry which connects orbitally excited states of the heavy baryon to its ground state via raising and lowering operators. For small $\lambda$, the low-lying collective states of the heavy baryon are well described in terms of harmonic oscillator wave functions. This approximation, however, clearly breaks down near threshold where the states become anharmonic.

A generic potential model can have an arbitrary number of free parameters. However, since we are using the potential models to test our effective theory at NLO (at this order the physical observables of $\Lambda_{c}$ and $\Lambda_{b}$ baryons are described in terms of two relevant parameters), it is convenient to study potential models which also have two free parameters. We can use this freedom to fix the observables to experiment. We can then make predictions for other observables, and further test how well the expansion works. Accordingly, we chose potential models with two parameters—the potential depth and a length parameter. For simplicity, we begin with a study of single-channel potentials. Of course, any model based on physical hadrons couples the light baryon to a physical heavy meson. The physical heavy meson states are denoted by $H$ and $H^{*}$ (where $H$ is a heavy meson with spin $J = 0$, and $H^{*}$ is a heavy meson with $J = 1$.) Recall, however, that in the heavy quark limit $H$ and $H^{*}$ are degenerate and, from the point of view of the collective dynamics, one can construct a simple model where we neglect the splitting and treat the two states as a generic heavy meson. In the charmed sector, for example, the two mesons are the $D$ and $D^{*}$ mesons which, in these simple models, can be treated as a single meson with a mass taken to be the spin-averaged mass of the two states. Note that this single-channel model is consistent with the $\lambda^{3/2}$ expansion at NLO since the mass splitting between the heavy mesons contributes only at NNLO.

By using potential models, we can calculate a number of observables for $\Lambda_{Q}$ (whereby $Q$ we mean a charm or bottom quark) baryons. We first compute the spectroscopic observables for these baryons. We should note however that, since the effective theory clearly breaks
down beyond the first doublet of excited states, we will restrict our attention to the excitation energy of the first excited state. We also study semi-leptonic decays of $\Lambda_c$ and $\Lambda_b$ baryons. These decays involve transitions between the ground state of $\Lambda_b$ and the ground state of $\Lambda_c$ as well as transitions between the ground state of $\Lambda_b$ and the doublet of first excited states of $\Lambda_c$. These observables have the virtue that, at NLO in the $\lambda^2$ expansion, the currents associated with these observables are determined entirely by the symmetries of the effective theory with no free parameters [3]. Thus, these observables can be predicted at NLO without additional fitting. Of course, in a general potential model, there is no reason why these currents need to be exactly of these forms. However, we can take them to be of these simple forms in order to reduce the number of free parameters. With these currents, we can compare the model predictions for these observables to the predictions based on the effective theory at NLO in the expansion. Since we have made a number of arbitrary assumptions about the form of the potentials and choice of currents, this comparison cannot be taken as definitive in determining the degree of convergence of the expansion for the various observables. It should, however, provide a qualitative test of convergence.

We can also generalize the potential models to include coupled channels where the heavy meson states $H$ and $H^*$ are treated as two distinct states coupled separately to the light baryon. This occurs only at NNLO in the $\lambda$ expansion. Accordingly, the mass splitting effects of these meson states are beyond the order to which we have worked in the expansion. Thus, these models are a useful place to look for effects which may spoil the expansion. While these coupled-channel potential models produce more states than the single-channel potentials, these models should agree with the expansion (and hence with the single-channel models) for low-lying states in the limit where $\lambda \to 0$. Moreover, we find that there is no way to match these theories unless we simultaneously include at least one excitation of the light baryon (or equivalently the brown-muck). In doing this, we effectively obtain, in the $\lambda \to 0$ limit, two decoupled copies of the collective motion: one based on the ground state of the light baryon oscillating against the generic heavy meson, and one on its excited state. Indeed, in the general derivation of the effective theory, it was pointed out that, since the intrinsic structure of the brown-muck is associated with qualitatively faster dynamics than the collective motion, the collective motion can be built on either the ground state of the brown-muck or its excited states. In fact, as was shown in ref. [3], the collective motion is essentially the same for both cases. As $\lambda$ deviates from zero, these states mix due to the splitting of the heavy mesons.

The construction of such coupled-channel potentials is subtle. The collective dynamics clearly favor s-waves for the ground state. Thus, the parity of the ground state collective wave function is even. The parity of the ground state heavy baryon is also even. Thus, the product of the intrinsic parities of the light baryon and the heavy meson in the model must also be even. The $H$ and $H^*$ mesons both have odd intrinsic parity. Thus, in order to obtain states consistent with the parity of $\Lambda_Q$ baryons, these heavy mesons have to be coupled to a light baryon with odd intrinsic parity. Accordingly, in such models, we will not use the nucleon since it has a positive intrinsic parity. Instead, we consider an excited state of the nucleon with negative—odd—parity. For concreteness, we will use the $N^*(1535)$ nucleon resonance state as our low-lying light baryon state and the $N^{**}(2090)$ nucleon resonance state as our excited light baryon. We note in passing that previous model treatments based on the bound state picture have used Skyrmions for the light baryon [1, 2, 8, 9, 10, 11, 12, 13, 14]; in these models, the Skyrmion implicitly includes baryons of both parities and excitations. Thus, the treatment here is in some ways analogous to previous work. The present models,
however, allow for detailed dynamical calculations.

Again, in principle, there are an infinite number of arbitrary parameters that arise in building the most general coupled-channel potential. Here we will pick arbitrary forms with two free parameters. Our strategy will be as follows: we start with a single-channel potential with two free parameters. We consider the limit of this potential where $\lambda \to 0$. In this limit, the form of the coupling of the two channels is fixed. We keep fixed the relative strengths of the two channels as found in this limit and then depart from the $\lambda = 0$ limit by using the full potential form and including the mass splitting of the heavy mesons. This strategy guarantees the correct behavior as the symmetry limit is approached. Again, this is not the most general approach one can take, but it should provide insights about convergence. One important limitation to this approach is that we do not build in any spin-orbit interactions. Thus, in these models $L$ is a good quantum number. Moreover, the absence of spin-orbit interactions means that the doublets of states for $L \neq 0$ are all unsplit. Of course, at the expense of an additional parameter, we could build in the spin-orbit interaction. However, doing this will not provide us with any additional qualitative insights about the convergence of the expansion, and thus we have chosen not to do this.

As will be shown in this paper, the model-independent predictions based on the effective expansion work rather well for these observables. Naively, since $\lambda \sim \frac{1}{N_c}$ and the expansion is in $\lambda^{\frac{1}{2}}$, one would expect typical errors at NLO to be of order $\frac{1}{3} \approx 33\%$. In fact, we see that most of the model-independent predictions based on the expansion \[\text{[5]}\] seem to work at least this well for all of the models studied. This gives us some confidence that, as experimental measurements of these quantities become available, the model-independent predictions of ref. \[\text{[5]}\] may give us a reliable guide. The exception to this general trend is predictions of the electromagnetic decay widths. The reason for the relatively poor behavior of these observables is largely due to the fact that charge assignments for quarks in the large $N_c$ limit differ substantially from those in the real world \[\text{[3]}\]. Thus, these observables are known to have very large $\frac{1}{N_c}$ corrections and are not expected to give accurate predictions. Accordingly, calculations of these observables will not be reported here.

We will also see, however, that there are large differences between the single and coupled-channel model predictions for the semi-leptonic decays. As these differences come entirely from effects at NNLO and beyond in the effective theory, these effects would ordinarily be small provided the expansion were well converged at NNLO. We suspect that the success of the NLO model-independent predictions for the semi-leptonic decays, despite the clear evidence of large NNLO effects, is due to correlations in the combinations of observables in our model-independent predictions which render them relatively insensitive to higher order effects.

This paper is organized as follows. In the next section, we provide a brief review of the model-independent predictions at NLO in the expansion. Next, we discuss the results based on a simple single-channel model. Following this, we discuss how to derive coupled-channel potentials consistent with our expansion. Finally, we compare the results based on the potential models to the model-independent predictions based on the effective theory at NLO. 
II. MODEL-INDEPENDENT PREDICTIONS BASED ON THE EFFECTIVE EXPANSION

In this section, the model-independent predictions of ref. \[4, 5\] will briefly be reviewed. The effective theory is expressed in terms of collective operators. These operators are defined unambiguously in QCD up to order $\lambda$ and describe the position and momentum of the brown-muck relative to the heavy quark as well as the total position and momentum of the system. At NLO, the relative variables are completely equivalent to the position and momentum of an ordinary light baryon relative to a heavy meson. The derivation is detailed in ref. [2]. Apart from the general QCD large $N_c$ and heavy quark counting rules, there is an additional symmetry issue. If the minimum of the effective potential is at zero separation (the symmetric case) which we assume here, then the following counting rules apply:

\[(\vec{x}, \vec{X}) \sim \lambda^{\frac{1}{2}},\]  
\[(\vec{p}, \vec{P}) \sim \lambda^{-\frac{1}{2}},\]  

where $\vec{x}$ and $\vec{p}$ are the relative position and momentum, and $\vec{X}$ and $\vec{P}$ are the total position and momentum of the heavy quark and brown-muck [2].

Including terms of $O(\lambda)$, the effective Hamiltonian based on the counting rules of eq. (2) was derived in ref. [2],

\[\mathcal{H}_{\text{eff}} = (m_H + m_{\text{bar}}) + c_0 + \frac{p^2}{2(m_{\text{bar}} + m_H)} + \frac{\mu_Q^2}{2\mu_Q} + \frac{\kappa}{2} |\vec{x}|^2 + \frac{\alpha}{4!} |\vec{x}|^4 + O(\lambda^2),\]

where $c_0$ is the potential depth relative to the dissociation threshold, $m_{\text{bar}}$ is the light baryon mass, $m_H$ is the spin-averaged heavy meson mass, and $\mu_Q = \frac{m_{\text{bar}} m_H}{m_{\text{bar}} + m_H}$ is the reduced mass of the bound state picture. (In the effective theory, $m_{\text{bar}}$ corresponds to the mass of the nucleon, while in the bound state picture this light baryon mass is taken to be the lowest odd parity resonance state of the nucleon. To this order in the expansion—$O(\lambda)$—the relative differences in these light baryon masses are NLO effects.) The notation $\mathcal{H}_\lambda^n$ indicates the piece of the Hamiltonian whose contribution to the low-lying states is of order $\lambda^n$. The term $\mathcal{H}_{\lambda^{\frac{1}{2}}}$ is referred to as leading order since the terms $\mathcal{H}_{\lambda^{-1}}$ and $\mathcal{H}_{\lambda^0}$ are constants and do not affect the dynamics; $\mathcal{H}_\lambda$ is the NLO contribution.

At $O(\lambda)$ in the effective expansion, the excitation energy of the heavy baryon is expressed in terms of two unknown phenomenological parameters $\kappa$ and $\alpha$,

\[\Delta m_{\lambda Q} = \sqrt{\frac{\kappa}{\mu_Q} + \frac{\alpha}{4!} \frac{5}{\mu_Q \kappa}} + O(\lambda^{\frac{3}{2}}),\]

where the second term is obtained using perturbation theory. Note that in nature, the first excited state of the heavy baryon is actually a doublet, which in the single-channel models is taken to be the spin-averaged mass (i.e., $m_{\lambda Q} = \frac{1}{3} m_{\lambda Q 1} + \frac{2}{3} m_{\lambda Q 2}$, where $m_{\lambda Q 1}$ is the mass of the excited state of the doublet with total angular momentum $J = 0$, and $m_{\lambda Q 2}$ is the mass corresponding to the state with $J = 1$). On the other hand, for coupled-channel models, these doublet of excited states are treated distinctly.
From the effective theory, the dominant semi-leptonic form factors for transitions between \( \Lambda_b \) and \( \Lambda_c \) were also derived in ref. [4]. Up to NLO, these observables are completely determined in terms of the parameters \( \kappa \) and \( \alpha \); there are no new parameters associated with the currents between these states at this order. The transitions between the ground states of \( \Lambda_b \) and \( \Lambda_c \) involve matrix elements of the form,

\[
\langle \Lambda_c(\vec{v}') | \bar{c} \gamma^\mu (1 - \gamma_5) b | \Lambda_b(\vec{v}) \rangle = \bar{u}_c(\vec{v}') (\Gamma_V - \Gamma_A) u_b(\vec{v})
\]

(6)

where,

\[
\Gamma_V = f_1 \gamma^\mu + i f_2 \sigma^{\mu\nu} q_\nu + f_3 q^\mu,
\]

(7)

\[
\Gamma_A = (g_1 \gamma^\mu + i g_2 \sigma^{\mu\nu} q_\nu + g_3 q^\mu) \gamma_5,
\]

(8)

where \( q = m_{\Lambda_c} v' - m_{\Lambda_b} v \) is the momentum transfer and \( u_b(\vec{v}) \), \( u_c(\vec{v}) \) are the Dirac spinors corresponding to \( \Lambda_c \) and \( \Lambda_b \) baryons.

While there exist in total six different form factors associated with the current operator in eq. (6), in the combined expansion at NLO only two form factors are dominant for ground-to-ground semi-leptonic decays,

\[
\langle \Lambda_c(\vec{v}') | \bar{c} \gamma^\mu (1 - \gamma_5) b | \Lambda_b(\vec{v}) \rangle = F_1 \bar{u}_c(\vec{v}') \gamma^\mu u_b(\vec{v})(1 + \mathcal{O}(\lambda^2)),
\]

(9)

\[
\langle \Lambda_c(\vec{v}') | \bar{c} \gamma^i \gamma_5 b | \Lambda_b(\vec{v}) \rangle = G_1 \bar{u}_c(\vec{v}') \gamma^i \gamma_5 u_b(\vec{v})(1 + \mathcal{O}(\lambda^2)),
\]

(10)

where \( F_1 \) and \( G_1 \) are respectively the scalar and vector form factors for ground-to-ground transitions, and \( \vec{v} \) and \( \vec{v}' \) are, respectively, the initial and final baryon velocities. These expressions are valid only for small velocities (i.e., of order \( \lambda^2 \)). As shown in ref. [4], since \( F_1 \) and \( G_1 \) are equal up to corrections of order \( \lambda \) in the combined expansion, these ground-to-ground transitions can be expressed in terms of a single form factor. At LO, the form factor is given by:

\[
\Theta = F_1 = \frac{2 \sqrt{2} \mu_b \mu_c}{(\sqrt{\mu_b} + \sqrt{\mu_c})^2} \exp \left( - \frac{m_{\bar{b}} |\delta \vec{v}|^2}{2(\sqrt{\mu_b} + \sqrt{\mu_c})^2} \right) (1 + \mathcal{O}(\lambda)),
\]

(11)

where \( |\delta \vec{v}| \sim \lambda^{3/2} \). Expressed as a function of the velocity transfer \( |\delta \vec{v}| \), \( \Theta \) has an essential singularity in the combined limit; it vanishes faster than any power of \( \lambda \). However, it can be re-expressed as a smooth function of a new dimensionless kinematic variable \( z \) where,

\[
z \equiv \frac{m_{\bar{b}} \sqrt{2(w - 1)}}{(\sqrt{\mu_b} + \sqrt{\mu_c})^{3/2}}.
\]

Accordingly, we will express all observables in terms of \( z \) instead of the velocity transfer parameter \( w = v \cdot v' \) (where \( v \) and \( v' \) are respectively the initial and final state baryon velocities) as is generally done in the heavy quark expansion. Of course, there is no difference
in the information content whether the function is written in terms of \( w \) or \( z \). At NLO, the semi-leptonic form factor for ground-to-ground transitions is given by,

\[
\Theta(z) = \frac{2\sqrt{2} \mu_b^3 \mu_c^3}{(\sqrt{\mu_b + \sqrt{\mu_c}})^4} \exp \left( \frac{-z^2}{2\sqrt{\kappa}} \right) \left( 1 + \frac{\alpha}{4! \kappa^4} \left( \frac{45(\sqrt{\mu_b} - \sqrt{\mu_c})^2}{16\kappa^2 \sqrt{\mu_b} \mu_c (\sqrt{\mu_b} + \sqrt{\mu_c})} \right) \right) \left( 1 + \mathcal{O}(\lambda^2) \right) .
\]

These expressions for the form factors are functions, and thus contain an infinite amount of information. Of particular interest are the values of \( \Theta(z) \) evaluated at zero recoil and the curvature of \( \Theta(z) \) at \( z = 0 \); by symmetry the slope vanishes at \( z = 0 \). At NLO, \( \Theta(z) \) evaluated at zero recoil is given by,

\[
\Theta_0 \equiv \Theta(z = 0) = \frac{2\sqrt{2} \mu_b^3 \mu_c^3}{(\sqrt{\mu_b + \sqrt{\mu_c}})^4} \left( 1 + \frac{\alpha}{4! \kappa^4} \frac{45(\sqrt{\mu_b} - \sqrt{\mu_c})^2}{16\kappa^2 \sqrt{\mu_b} \mu_c (\sqrt{\mu_b} + \sqrt{\mu_c})} \right) \left( 1 + \mathcal{O}(\lambda^2) \right) .
\]

The curvature of \( \Theta(z) \) evaluated at zero recoil is given by,

\[
\rho \equiv \frac{\partial^2 \Theta}{\partial z^2}(z = 0) = -\frac{2\sqrt{2} \mu_b^{3/8} \mu_c^{3/8}}{\sqrt{\kappa}(\sqrt{\mu_b} + \sqrt{\mu_c})^{3/2}} \left( 1 + \frac{\alpha}{4! \kappa^4} \frac{45(\sqrt{\mu_b} - \sqrt{\mu_c})^2 - 160\sqrt{\mu_b} \mu_c}{16\kappa^2 \sqrt{\mu_b} \mu_c (\sqrt{\mu_b} + \sqrt{\mu_c})} \right) \left( 1 + \mathcal{O}(\lambda^2) \right) .
\]

A \( \Lambda_b \) can also undergo a semi-leptonic decay into the first excited state of \( \Lambda_c \). For these ground-to-first excited state transitions, there exist two semi-leptonic decay channels which correspond to decays from the ground state of \( \Lambda_b \) to the doublet of the first excited states of \( \Lambda_c \): \( \Lambda_b \to \Lambda_{c1} \ell \bar{\nu} \) and \( \Lambda_b \to \Lambda_{c}^* \ell \bar{\nu} \). The dominant form factor which determines the hadronic amplitudes for these two channels is given in terms of a single form factor \( \Xi(z) \) [4, 5],

\[
\langle \Lambda_{c1}(\vec{v}') | \gamma^\mu (1 - \gamma_5) b | \Lambda_b(\vec{v}) \rangle = \sqrt{3} \Xi(z) \bar{u}_c(\vec{v}') \gamma^\mu (1 - \gamma_5) u_b(\vec{v}) \left( 1 + \mathcal{O}(\lambda) \right) ,
\]

\[
\langle \Lambda_{c}^*(\vec{v}') | \gamma^\mu (1 - \gamma_5) b | \Lambda_b(\vec{v}) \rangle = \Xi(z) \bar{u}_{c\nu}(\vec{v}') (\sigma^{\mu\nu} \gamma_5 - g^{\mu\nu}) u_b(\vec{v}) \left( 1 + \mathcal{O}(\lambda) \right) ,
\]

where the Rarita-Schwinger spinors are normalized according to \( \bar{u}_{c\nu}(\vec{v}, s) u^\nu(\vec{v}, s) = -1 \). For \( z \) of order unity (i.e., for velocity transfers of order \( \lambda^2 \)), the form factor \( \Xi(z) \) at NLO is,

\[
\Xi(z) = \frac{4 z^3 \mu_b^3 \mu_c^3}{\kappa^4 (\sqrt{\mu_b} + \sqrt{\mu_c})^2} \exp \left( -\frac{z^2}{2\sqrt{\kappa}} \right) \left( 1 + \frac{\alpha}{4! \kappa^4} \left( \frac{(105\mu_b - 230\sqrt{\mu_b} \mu_c + 45\mu_c)\kappa}{16\sqrt{\mu_b} \mu_c} + \frac{13}{2} \kappa^{1/2} z^2 - \frac{1}{4} z^4 \right) \right) \left( 1 + \mathcal{O}(\lambda^2) \right) .
\]

While the form factor \( \Xi(z) \) vanishes at zero recoil, the slope of \( \Xi(z) \) evaluated at zero recoil does not,

\[
\sigma \equiv \frac{\partial \Xi}{\partial z}(z = 0) = \frac{4 \mu_b^3 \mu_c^3}{\kappa^4 (\sqrt{\mu_b} + \sqrt{\mu_c})^2} \left( 1 + \frac{\alpha}{4! \kappa^4} \frac{105\mu_b - 230\sqrt{\mu_b} \mu_c + 45\mu_c}{16\kappa^2 \sqrt{\mu_b} \mu_c (\sqrt{\mu_b} + \sqrt{\mu_c})} \right) \left( 1 + \mathcal{O}(\lambda^2) \right) .
\]
We will compare results for these observables based on the effective theory to NLO to predictions obtained using potential models. For the single and coupled-channel models, we use model currents to describe transitions between states of $\Lambda_b$ and $\Lambda_c$. The semi-leptonic decays are obtained using model wave functions. The most general current operator in the effective theory derived in ref. [3] is given in terms of the relative position operator $\vec{x}$. The general boost operator can be expressed in terms of the velocity transfer $\delta \vec{v}$ as,

$$B(\delta \vec{v}) = \exp [-im_{\text{bar}} \delta \vec{v} \cdot \vec{x}]$$  \hspace{1cm} (18)

where $m_{\text{bar}}$ is the mass of the lowest $N^*$ state, and $\vec{x}$ is the relative position between $N^*(1535)$ and the heavy meson. In the effective theory, $m_{\text{bar}}$ in the boost operator corresponds to the nucleon mass.

The total radiative decay rates of excited $\Lambda_c$ and $\Lambda_b$ baryons were also calculated in the effective theory. The total decay rates in the charm and bottom sectors were derived up to NLO in ref. [4] for a single-channel transition where the doublet of first excited states are considered as one:

$$\Gamma(\Lambda_{c1} \rightarrow \Lambda_c \gamma) = \frac{1}{6} e^2 \kappa \left( \frac{m_D - m_{\text{bar}}}{m_{\text{bar}} m_{\text{bar}}} \right)^2 \left( 1 - \frac{\alpha}{4!} \frac{5}{\sqrt{\kappa^3 \mu_c}} \right) (1 + \mathcal{O}(\lambda))$$ \hspace{1cm} (19)

$$\Gamma(\Lambda_{b1} \rightarrow \Lambda_b \gamma) = \frac{1}{6} e^2 \kappa \left( \frac{m_B + m_{\text{bar}}}{m_{\text{bar}} m_{\text{bar}}} \right)^2 \left( 1 - \frac{\alpha}{4!} \frac{5}{\sqrt{\kappa^3 \mu_b}} \right) (1 + \mathcal{O}(\lambda))$$ \hspace{1cm} (20)

where $e$ is the electromagnetic coupling constant ($e^2 \approx \frac{1}{137}$).

The computation of the model-independent results for the electromagnetic decays has an important subtlety. The charges of the quarks are not unambiguously defined. At first sight, it may seem obvious to use the physical charge assignments for the $N_c = 3$ world of $\frac{2}{3}$ and $\frac{1}{3}$. However, this presents a problem. Recall that to order $\lambda$, there are two complementary and equivalent pictures of the collective dynamics: one can view the heavy baryon as a heavy quark oscillating against the brown-muck or alternatively as a heavy meson oscillating against a light baryon. Dynamically, this makes little difference since, for example, the mass of the heavy quark is not so different from the mass of the heavy meson. Consider, however, the charges. The charge of a $c$-quark is $\frac{2}{3}$ but the charge of a D-meson is either 1 or zero depending on the isospin projection. Similarly, the charge of the recoiling brown-muck will be $-\frac{2}{3}$, while the charge of the recoiling light baryon is either -1 or zero. Thus, there exists an order unity differences between the two pictures. Formally, this issue can be resolved by considering the charges as given in the large $N_c$ limit. This is fully consistent with the derivation of the effective theory which is based on large $N_c$ arguments. Assigning charges so that the usual baryons have charge of unity (for the proton and $\Lambda_c$) and zero (for the neutron and $\Lambda_b$) requires that the quark charges be,

$$e_u = e_c = e_t = \frac{1}{2} + \frac{1}{N_c}$$

$$e_d = e_s = e_b = -\frac{1}{2} + \frac{1}{N_c}$$ \hspace{1cm} (21)

It is easy to see that, as $N_c \rightarrow \infty$, using these charge assignments, the two dynamical pictures agree up to $\frac{1}{N_c}$ corrections. Thus, at a formal level, there is no difficulty in predicting the
observables of eqs. (19) and (20). However, there is a problem phenomenologically. The very large difference between the two descriptions at $N_c = 3$ suggests that for the physical world one expects very large $\frac{1}{N_c}$ corrections; thus we do not expect the effective theory to be accurate for these observables and will not report on them in this paper.

At leading order in the effective theory, all observables depend on a single parameter, $\kappa$, and all other observables may be predicted once one observable is fixed. In practice, we already know the splitting between the ground state of $\Lambda_c$ and its first excited state. Hence, we can use this splitting to predict all remaining observables to this order. These observables were derived in [4], and are listed in Table [I]. At NLO in the expansion, the parameters $\kappa$ and $\alpha$ can be eliminated to give model-independent predictions for the physical observables of $\Lambda_c$ and $\Lambda_b$ baryons. At $O(\lambda)$, each observable can be expressed in terms of two additional observables [5]. Thus, once two of these observables are measured experimentally, the other observable can easily be predicted. At present, only one observable of $\Lambda_Q$ baryons has been measured—the excitation energy of $\Lambda_c$. Accordingly, it is useful to express all of the observables in terms of this known splitting and one of the other observables; when one measurement gets made, we will then be able to predict the other observables to this order. These relations were derived in [4] and are reported in Table [II]. The predictions of Table [II] require some explanation. We re-scale the constants $\kappa$ and $\alpha$ as well as the observables using the typical momentum scale of the collective degrees of freedom. This re-scaling allows us to parameterize the size of the NLO corrections in a simple way. It is defined as: $\Lambda \equiv (\mu_c \Delta m_{\Lambda_c}^2)^{\frac{1}{3}} \approx 410$ MeV.

III. SINGLE-CHANNEL POTENTIAL MODELS

In this section, we study the phenomenology of isoscalar heavy baryons based on the bound state picture of ref. [6]. We first examine the spectroscopy of $\Lambda_c$ and $\Lambda_b$ baryons using a variety of single-channel potentials. The bound states of these systems correspond to solutions of the Schrödinger equation in which no distinction is made between the $H$ and $H^*$ mesons. Our failure to distinguish between these two heavy mesons is reasonable in the context of the effective theory at NLO since the splitting between these states only occurs at NNLO. Thus, the single-channel models do not test all possible ways the effective theory can break down. In practice, however, what these models probe are effects due to anharmonicity in the collective motion. Recall that, in the effective theory at NLO, only the leading harmonic correction was included; in these models, however, harmonic effects are included to all orders. We study coupled-channel models later; these studies give insights into the role played by the splitting of the $H$ and $H^*$ mesons.

For single-channel potentials, the Schrödinger equation is given by,

$$u''(r) + 2\mu \left( E - V(r) - \frac{\ell(\ell + 1)}{2\mu r^2} \right) u(r) = 0, \quad (22)$$

where the reduced wave function is given by $u(r) = rR(r)$, $\ell$ represents the orbital angular momentum between the light baryon and heavy meson, $V(r)$ is the single-channel potential between the light baryon and heavy meson in the bound state picture, and $E$ is the bound state energy. The reduced mass $\mu$ of the bound state picture is given in terms of the mass of the light baryon and the spin-averaged mass of the heavy meson ($H$ and $H^*$),

$$\mu = \frac{m_{\text{bar}}m_H}{m_{\text{bar}} + m_H}, \quad (23)$$
In accordance with the bound state picture, the light baryon mass $m_{\text{bar}}$ is taken to be the mass of the lowest odd parity resonance state of the nucleon namely N*(1535). (In the quark model, this may be interpreted as a state where one quark is in an L=1 excited state.) In the charm sector, $\mu \approx 863$ MeV, while $\mu \approx 1191$ MeV in the bottom sector. The N*(1535) has the following quantum numbers: $I(J^P) = \frac{1}{2} \left( \frac{1}{2}^+ \right)$, where $I$ is the total isospin of N*, $J$ is the total spin, and $P$ is its parity. Thus, by coupling N* to the heavy meson H (with $I(J^P) = \frac{1}{2} (0^-)$), we obtain consistent quantum numbers with those of $\Lambda_Q$.

Equation (22) is solved by standard means. By fitting the ground and first excited state energy eigenvalue solutions of eq. (22) to the experimental values for the ground and first excited states of $\Lambda_c$, we can, in principle, predict additional excited states below threshold. However, since we don’t believe the higher-lying states—these states lie beyond the range where we believe the expansion to be valid—we only consider the excitation energy between the ground and first orbitally excited states in both the charm and bottom sectors. We use model wave functions (i.e., solutions to eq. (22)) as well as model currents given in eq. (18) to calculate the physical observables of eqs. (5), (13), (14), and (17).

While there exist infinitely many potentials that can be used to describe the low-lying bound states of heavy baryons, we explore a few; we hope these are enough to reach qualitative conclusions on the convergence of the effective expansion. As we will see, certain models work better than others. We choose simple potential models which describe the low-lying states of heavy baryons based on the effective theory. These potential models are simple in form in that they have a minimum at the origin, and die off at sufficiently large distances. For both the single and coupled-channel models, the integration limit (i.e., where the potential becomes weak,) lies between 3 and 4 fm. We will study the following potential models:

$$V(r) = -c_0 e^{-\left(\frac{r}{a_0}\right)^2},$$
$$V(r) = -c_0 e^{-\left(\frac{r}{a_0}\right)^2},$$
$$V(r) = -\frac{c_0}{2} \left( e^{-\left(\frac{r}{a_0}\right)^2} + e^{-\left(\frac{r}{a_0}\right)^4} \right),$$
$$V(r) = -\frac{c_0}{2} \left( e^{-\left(\frac{r}{a_0}\right)^2} e^{-\left(\frac{r}{a_0}\right)^4} \right),$$
$$V(r) = -c_0 e^{-\left(\frac{r}{a_0}\right)^2 + \left(\frac{r}{a_0}\right)^4}.$$  

In general, the parameters of these models are fixed so that the ground and first excited state energies are fitted to the mass of $\Lambda_c$ and the spin-averaged mass of the $\Lambda_c$ doublet.

IV. PREDICTIONS BASED ON SINGLE-CHANNEL MODELS

Using the various single-channel models, we obtain consistent predictions for the physical observables of $\Lambda_c$ and $\Lambda_b$ baryons. For these models, the first excited odd parity resonance mass of the nucleon is used in place of the nucleon mass, which is used in the effective theory, for parity reasons. This choice presents no ambiguity since the difference in mass between these two nucleon states occurs at NNLO in the effective expansion. For these models, the predicted excitation energy for $\Lambda_b$ differs from the LO prediction based on the effective theory by at most 20 MeV. The prediction for $\Theta_0$ (the form factor for ground-to-ground
transitions evaluated at zero recoil) is consistent with the LO prediction of unity obtained in the combined expansion and in HQET. The model prediction for $\rho$ (the curvature of the form factor for ground-to-ground transitions evaluated at zero recoil) is approximately $-1.50 \times 10^{-4}$ MeV$^{-3/2}$. This prediction differs from the LO value (where $\rho \approx -1.50 \times 10^{-4}$ MeV$^{-3/2}$) by 25%. The model prediction for $\sigma$ (the slope of the form factor for ground-to-first excited state transitions evaluated at zero recoil) is approximately $0.018$ MeV$^{-3/4}$. This prediction differs by 60% from the LO prediction based on the effective theory, where $\sigma \approx 0.011$ MeV$^{-3/4}$.

V. COUPLED-CHANNEL POTENTIAL MODELS

In this section, we describe the formalism for obtaining coupled-channel models which can be used to calculate the physical observables of $\Lambda_c$ and $\Lambda_b$ baryons. These potential models are of particular interest because they allow us to test another possible source of error in the predictions at NLO in the effective theory—namely, the effects due to the splitting of the heavy mesons. The coupled-channel models requires an excited state of the heavy baryon. In some models (see [7]), the bound state picture is based on a soliton-heavy meson model in which the soliton incorporates the internal excitations of the brown-muck. In our bound state picture, however, these internal excitations correspond to excited states of the nucleon. Here we will consider an explicit excited state of the light baryon which lies above the lowest-lying odd parity $N^*(1535)$ state. We choose the $N^{**}(2090)$ state for two reasons: it has the right quantum numbers as those of $N^*(1535)$ (i.e., $I(J^P) \rightarrow \frac{1}{2}(1^-)$ in accordance with the usual spectroscopic notation where $I$ is the particle isospin, $J$ is the total spin of the particle, and $P$ is its parity). In addition, its energy lies approximately 550 MeV above the $N^*(1535)$ state. Thus, by choosing this particular excited nucleon resonance state, we maintain an energy splitting characteristic of typical internal excitations of the brown-muck. Moreover, when we couple the $N^*(1535)$ and the $N^{**}(2090)$ states to the heavy mesons $H$ and $H^*$ in the bound state picture, we obtain states with consistent quantum numbers as those of the heavy baryon.

For coupled-channel models, we need to map the bound state picture onto the heavy quark and brown-muck picture in order to obtain two pictures with consistent quantum numbers as those of the heavy baryon $\Lambda_Q$. We set up the mapping in such a way that, in the limit where $\lambda = 0$, we obtain two identical displaced copies of the excitation energy. This displaced energy is due to the splitting between the states of the light baryon (i.e., $N^*(1535)$ and $N^{**}(2090)$). When the splitting between the $H$ and $H^*$ states goes to zero, we retrieve the single-channel results. While this mapping strategy is not the most general approach to define coupled-channel models, it is appropriate in our case because it retrieves the single-channel model-independent results in the $\lambda \rightarrow 0$ limit. Thus, in this limit, the heavy quark–brown-muck and bound state pictures become equivalent.

In the heavy quark and brown-muck picture, the heavy baryon ground state can be written as some collective wave function times

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle_Q |0, 0\rangle_{bm},$$

(24)

where $Q$ denotes the heavy quark state and $bm$ denotes the brown-muck state. First, in order to obtain a hadronic picture with quantum numbers consistent with the heavy quark
and brown-muck picture (which has the brown-muck in an isosinglet state with total spin zero), we need to remove a $q\bar{q}$ (quark and anti-quark) pair with total spin zero from the vacuum. This action leaves the brown-muck in a total spin zero state. We then couple $\bar{q}$ to the heavy quark spin to make a meson with total spin $J = 0$ or 1. Finally, we couple $q$ to the brown-muck to make an excited nucleon (in this case, an $N^*$ or an $N^{**}$) with total spin $\frac{1}{2}$. By using appropriate Clebsch-Gordan coefficients, the total wave function is expressed in terms of a superposition of $H$ and $H^*$ and $N^*$ and $N^{**}$ state corresponding to the coupling of the heavy meson to excited nucleons is given by,

$$
\frac{1}{\sqrt{2}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
1 \\
\end{array} \right|_\bar{q} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
1 \\
\end{array} \right|_\bar{q} \right) \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^*} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^{**}} \right) +
$$

$$
\frac{1}{\sqrt{2}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} - \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} \right) \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^*} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^{**}} \right)
$$

times the appropriate collective wave function. In eq. (25), the first element in the brackets denotes the total spin of the particle, and the second element is the projection. The symmetric wave function given by $\frac{1}{\sqrt{2}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} \right)$ corresponds to the heavy meson state H, while the anti-symmetric state $\frac{1}{\sqrt{2}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} - \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_Q \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_\bar{q} \right)$ is the state of $H^*$.

In the bound state picture, the heavy mesons $H$ and $H^*$ correspond to the D and $D^*$ mesons and to the B and $B^*$ mesons, respectively. These states have the following quantum numbers: $D^0 \rightarrow \frac{1}{2}(0^-)$, $D^* \rightarrow \frac{1}{2}(1^-)$, $B^0 \rightarrow \frac{1}{2}(0^-)$, and $B^* \rightarrow \frac{1}{2}(1^-)$. The states of the bound state picture, which include the heavy meson states and nucleon resonance states, can all be expressed in terms of states of well-defined angular momentum,

$$
|s, \ell, J, M_J\rangle = \sum_{m_s m_\ell} \langle j_1 j_2, m_1, m_2 | J, M; j_1 j_2 \rangle |s, m_s \rangle \ell, m_\ell),
$$

where $s$ is the total spin of the light baryon and heavy meson ($H$ or $H^*$) and $m_s$ represents its projection, $\ell$ is the relative orbital angular momentum between the light baryon and the heavy meson ($H$ or $H^*$) and $m_\ell$ is its projection, $J = |s + \ell|$, $|s + \ell - 1|, \ldots |s - \ell|$ is the total spin, and $M = m_s + m_\ell$ represents the projection of the total spin. Note that, as discussed earlier, there is no spin-orbit contribution in this description. Using this notation, the ground state (i.e., the $\ell = 0$ state) of the bound state picture can be expressed in terms of a particular linear combination of the states $H$ and $H^*$ and $N^*$ and $N^{**}$ as follows:

$$
\alpha \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^*} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^{**}} \right) |0, 0\rangle_H +
$$

$$
\beta \left( \sqrt{\frac{2}{3}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^*} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^{**}} \right) |1, 1\rangle_{H^*} - \sqrt{\frac{1}{3}} \left( \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^*} + \left| \begin{array}{c} 1 \\
2 \\
2 \\
\end{array} \right|_{N^{**}} \right) |1, 0\rangle_{H^*} \right)
$$

times the collective wave function; $\alpha$ and $\beta$ are coefficients specifying the correct superposition with the obvious constraint that $\alpha^2 + \beta^2 = 1$. The bra-kets in eq. (27) are expressed, respectively, in terms of the total spin of the particle and its projection. The coefficients $\alpha$ and $\beta$ can easily be obtained by matching the ground state of the bound state picture in
eq. (27) to the ground state of the heavy quark and brown-muck picture in eq. (25). This yields,

\[ \alpha = \frac{1}{2} \text{ and } \beta = \frac{\sqrt{3}}{2}. \]  

(28)

Because we take the heavy meson states H and \( H^* \) to be nondegenerate for coupled-channel potentials, we need to include these two orthogonal states in our mapping procedure. We should note that at order \( \lambda \) in the effective expansion, the orbital contribution decouples so that the orthogonal states of the bound state picture are given by,

\[ \alpha \left( \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^{**}} \right) |0, 0\rangle_H + \]

(29)

\[ \beta \left( \sqrt{\frac{2}{3}} \left( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^{**}} \right) |1, 1\rangle_{H^*} - \sqrt{\frac{1}{3}} \left( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{N^{**}} \right) |1, 0\rangle_{H^*} \right) \]

and,

\[ -\beta \left( \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^{**}} \right) |0, 0\rangle_H + \]

(30)

\[ \alpha \left( \sqrt{\frac{2}{3}} \left( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^{**}} \right) |1, 1\rangle_{H^*} - \sqrt{\frac{1}{3}} \left( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{N^*} + \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{N^{**}} \right) |1, 0\rangle_{H^*} \right) \]

times the collective wave function. Next, we transform the Hamiltonian from the heavy quark and brown-muck basis (where at least at low orders in the expansion we know its form) to a hadronic basis where the states are given in terms of the coefficients of the orthogonal basis states of the bound state picture in eqs. (29) and (30). Note that, as discussed above, in the heavy quark and brown-muck basis, we include two copies of the dynamics—one based on the ground state of the brown-muck and the other based on an excited state of the brown-muck. We will treat these two states as having the same collective dynamics (apart from small reduced mass effects which contribute beyond NLO in the effective theory). Moreover, these states are displaced from each other by the excitation energy of the brown-muck. The reason for doing this should be clear—once we have made this transformation to the hadronic basis we are then in a position to add perturbations due to the splitting of the two heavy meson states. The transformation is

\[ \mathcal{H}' = \Phi^{-1} \mathcal{H} \Phi \]

(31)

where \( \Phi \) contains the coefficients of the orthogonal states (see eq. (28)) of the transformation from the heavy quark and brown-muck picture to the bound state picture:

\[ \Phi = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \]

(32)

The effective Hamiltonian, \( \mathcal{H} \), can be written as follows:

\[ \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix} \]

14
where,

\[ H_1 = H_S, \quad \text{and} \]

\[ H_2 = H_S + \Delta + (m_H - m_{\pi\pi}), \]

where \( H_S \) denotes the single-channel Hamiltonian, \( \Delta \) is the mass splitting between the \( N^*(1535) \) and \( N^{**}(2090) \), and \( \bar{H} \) is the spin-averaged mass of the heavy meson. Thus, the bound state Hamiltonian in the coupled channel can be written as,

\[
\mathcal{H}' = \left( \begin{array}{cc} \alpha & -\beta \\ \beta & \alpha \end{array} \right) \left( \begin{array}{cc} H_1 & 0 \\ 0 & H_2 \end{array} \right) \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right)
\]

\[ = \left( \begin{array}{cc} \alpha^2 H_1 + \beta^2 H_2 \alpha \beta (H_1 - H_2) \\ \alpha \beta (H_1 - H_2) \beta^2 H_1 + \alpha^2 H_2 \end{array} \right). \]  

Plugging eqs. (33) and (34) into eq. (35), the Hamiltonian for the bound state picture reduces to:

\[
\mathcal{H}' = \left( \begin{array}{cc} H_{s\bar{H}} + \beta^2 \Delta + (m_H - m_{\pi\pi}) & -\alpha \beta \Delta \\ -\alpha \beta \Delta & H_{s\bar{H}}^* + \alpha^2 \Delta + (m_{H^*} - m_{\pi\pi}) \end{array} \right)
\]

(36)

We can now write down the corresponding coupled Schrödinger equations,

\[
u''(r) + 2\mu_{\bar{H}} \left( E - V_{\bar{H}}(r) - \frac{\ell(\ell + 1)}{2\mu_{\bar{H}} r^2} - \frac{3}{4} \Delta - (m_H - m_{\pi\pi}) \right) u(r) + 2\alpha \beta \Delta \mu_{\bar{H}} w(r) = 0,
\]

(37)

\[
w''(r) + 2\mu_{H^*} \left( E - V_{H^*}(r) - \frac{\ell(\ell + 1)}{2\mu_{H^*} r^2} - \frac{1}{4} \Delta - (m_{H^*} - m_{\pi\pi}) \right) w(r) + 2\alpha \beta \Delta \mu_{H^*} u(r) = 0,
\]

(38)

where \( u(r) \) is the bound state reduced wave function, with reduced mass \( \mu_{\bar{H}} \), of the light baryon and \( H \), and \( w(r) \) is the bound state reduced wave function, with reduced mass \( \mu_{H^*} \), of the light baryon and \( H^* \). In eqs. (37) and (38), the light baryon corresponds to the lowest odd parity nucleon resonance state \( N^*(1535) \).

There exists an ambiguity in the reduced masses in the coupled-channel formalism. It is unclear as to what masses one should use since the orthogonal states of the bound state picture involve linear combinations of \( H \) and \( H^* \) as well as \( N^* \) and \( N^{**} \) (as can be seen in eqs. (29) and (30)). For concreteness, we express the reduced masses in the coupled channels in terms of bound states of \( H \) with \( N^*(1535) \) and \( H^* \) with \( N^*(1535) \) because the difference between these reduced masses is small, in accordance with the effective theory (i.e., the corrections are NNLO effects.) Again, we remind the reader that the error in the reduced masses in the coupled channel should be of the same order as that in the effective
theory since the effective theory is supposed to describe the “real world”. These masses are
distinguished for no other reason than to take into account mass splitting effects and to fit
the form of the potentials.

To determine the collective wave functions of heavy baryons, we solve eqs. (37) and (38)
by imposing standard boundary conditions such that:

\begin{align*}
  u_{r \to 0}(r) & \to r^{\ell+1} \to 0 \text{ as } r \to 0 \text{ and } \\
  u_{r \to \infty}(r) & \to \sqrt{r} (J_k(z) + Y_k(z)) \to 0 \text{ as } \\
  z & = \left( \sqrt{2\mu_{N^*H}}E_r \text{ or } \sqrt{2\mu_{N^*H}}E_r \right) \to \infty,
\end{align*}

where \( J_k(z) \) is the Bessel function of the first kind, and \( Y_k(z) \) is the Bessel function
of the second kind. Here \( k = \frac{1}{2}(1 + 2\ell) \), where \( \ell \) is the relative orbital angular momentum
between the excited nucleon and the heavy meson. The values of the potential depth \( c_0 \) and
the length parameter \( a_0 \) are fitted by requiring that the predicted energies of the ground
and first excited states of the bound state picture be matched to the experimental ground
and first doublet of excited state energies of \( \Lambda_c \),

\begin{align*}
  m_{\Lambda_c} &= \Lambda_c - (m_{\text{bar}} + m_{\text{D}}) \approx -1221 \text{ MeV}, \\
  m_{\Lambda_c'} &= \Lambda_c' - (m_{\text{bar}} + m_{\text{D}}) \approx -892 \text{ MeV},
\end{align*}

where \( m_{\text{bar}} \) is the mass the \( N^*(1535) \) state, \( m_{\text{D}} \) is the spin-averaged
mass of the D and \( D^* \) mesons, and where \( m_{\Lambda_c'} \) is the spin-averaged mass of the doublet of first excited states of \( \Lambda_c \).
The ground and first doublet of excited states of heavy baryons are, in general, written as:

\begin{align*}
  |\Lambda_Q\rangle & \equiv |\Lambda_Q; 0, \frac{1}{2}, J_z \rangle \sim \Lambda_c, \, \Lambda_b, \\
  |\Lambda_Q\rangle & \equiv |\Lambda_Q; 1, \frac{1}{2}, J_z \rangle \sim \Lambda_c(2593), \, \Lambda_b(?) \,, \\
  |\Lambda_{Q}^*\rangle & \equiv |\Lambda_Q; 1, \frac{3}{2}, J_z \rangle \sim \Lambda_c(2625), \, \Lambda_b(?) \,.
\end{align*}

While the reduced masses of \( \Lambda_c \) and \( \Lambda_b \) are not equal in the combined limit where \( \lambda \to 0 \),
these two sectors are modeled to have the same potential forms. This is valid up to NLO
in the effective theory. Thus, higher-order corrections of the difference between the two
potential forms will be ignored. This is not done for any deep reason—just for simplicity.
Again, it is rigorously correct in the \( \lambda \to 0 \) limit. This allows us to substitute the values of \( \kappa \)
and \( \alpha \) extracted from the charm sector using coupled-channel models to make quantitative
predictions for the observables in both the charm and bottom sectors. Once again, we
remind the reader that we have chosen simple potentials with two free parameters—the
potential depth \( c_0 \) and a length parameter \( a_0 \), characteristic of hadronic scales. These
potentials describe the low-lying states of heavy baryons based on the effective theory. The
parameters \( \kappa \) and \( \alpha \) in the effective Hamiltonian of eq. (4) are calculated directly from the
potential energy once \( c_0 \) and \( a_0 \) are determined for a given potential. At \( \mathcal{O}(\lambda) \) in the effective
expansion, the total potential energy of the heavy baryon is given by,

\begin{equation}
  V(\vec{x}) = -c_0 + m_{\text{bar}} + m_{\overline{\text{D}}} + \kappa \frac{x^2}{2!} + \alpha \frac{x^4}{4!} + \mathcal{O}(\lambda^2).
\end{equation}
The value of $\kappa$ in the effective Hamiltonian is obtained in the bound state picture by determining the value of $\frac{dV(r)}{dr^2}|_{r=0}$. Similarly, $\alpha$ can be measured by evaluating $\frac{dV(r)}{dr^4}|_{r=0}$. Using these parameters extracted from the potential models, the observables obtained in the effective theory at NLO in eqs. (5), (13), (14), and (17) can be calculated. In addition to the Hamiltonian, the currents are taken to be identical in structure to those obtained in the effective theory with no new parameters (valid only up to order $\lambda$ in the expansion.) Higher-order terms are neglected for simplicity. Although the excitation energy of $\Lambda_b$ has not yet been measured experimentally, it can be predicted by substituting the values of $\kappa$ and $\alpha$ obtained from the $\Lambda_c$ fit into the differential equation for the bound state picture of $\Lambda_b$ (see eqs. (37) and (38)) with reduced mass given by,

$$\mu_b = \frac{m_{\text{bar}} m_B}{m_{\text{bar}} + m_B},$$

where $m_{\overline{\text{B}}}$ is the spin-averaged mass of the B-meson.

VI. PREDICTIONS BASED ON COUPLED-CHANNEL MODELS

We use the coupled-channel model derived in section V to compute the physical observables discussed previously. Tables [IX-XIV] are arranged as follows. The second column lists the model predictions for the physical observables of $\Lambda_c$ and $\Lambda_b$ baryons. The third column contains the model-independent predictions based on the effective theory at LO, and the last two columns give model-independent predictions (with parameters extracted from models) based on the effective theory to NLO.

Our results based on the coupled-channel models show that the predicted excitation energy in the bottom sector differs at most by 40 MeV from the LO prediction based on the effective theory of 300 MeV (a 13% error which lies within the expected error at this order.) We also notice agreement in the model predictions with typical errors for the semi-leptonic decays. The model prediction for $\Theta_0$ is consistent with the LO prediction of unity obtained from the effective theory. The model prediction for $\rho$ is approximately $-0.90 \times 10^{-4}$ MeV$^{-\frac{1}{2}}$—a 25% difference from the LO prediction, where $\rho \approx -1.20 \times 10^{-4}$ MeV$^{-\frac{1}{2}}$. The coupled-channel model prediction for $\sigma$ is approximately $0.0080$ MeV$^{-\frac{3}{2}}$—a 27% difference from the LO prediction, where $\sigma \approx 0.011$ MeV$^{-\frac{3}{2}}$.

VII. DISCUSSION

In this paper, we tested the model independent predictions for the observables of isoscalar heavy baryons (which include $\Lambda_c$ and $\Lambda_b$ baryons) obtained from an effective theory based on a combined heavy quark and large $N_c$ expansion (derived in ref. [3]) against model predictions based on the bound state picture of heavy baryons. The model predictions (obtained using “reasonable” potentials) are compared with LO and NLO predictions obtained from the effective theory. The model-independent expressions for the observables of $\Lambda_c$ and $\Lambda_b$ baryons were derived to order $\lambda$ in the effective theory, where $\lambda$ is the natural expansion parameter. By fitting the excitation energy of $\Lambda_c$ to experiment for both the single and coupled-channel models, we were able to predict the excitation energy of $\Lambda_b$ as well as the semi-leptonic decays of $\Lambda_c$ and $\Lambda_b$ baryons.
By calculating the properties of $\Lambda_c$ and $\Lambda_b$ baryons using potential models, we explored the usefulness of the effective theory in the combined expansion. In principle, we want to test the convergence of the effective theory. Naively, one would expect the predictions based on the effective theory at LO to be rather crude since the expansion is in powers of $\lambda \approx \frac{1}{\sqrt{\Lambda}}$—a 60% error. At NLO, we might expect the error in the predictions for the physical observables to be approximately 30%.

While the potential models we have chosen do not necessarily describe the “real world,” they do describe the low-lying states of heavy baryons in a manner consistent with the effective theory. Using various single-channel models, consistent predictions for the low-lying excitation energy of $\Lambda_b$ were obtained (i.e., $\Delta m_{\Lambda_b} \approx 288$ MeV). This predicted energy differs from the LO prediction of 300 MeV by approximately 4%. For coupled-channel models, on the other hand, the predicted excitation energy of $\Lambda_b$ differs by at most 13% from the LO prediction. Moreover, when we compare the single-channel model predictions for the excitation energy of $\Lambda_b$ to the coupled-channel model predictions, we notice that these models differ from each other by at most 10%.

Based on our calculations, the single and coupled-channel model predictions for the semi-leptonic decays of $\Lambda_c$ and $\Lambda_b$ baryons are in close agreement with the NLO model-independent predictions (with parameters extracted from the models). The model prediction for $\Theta_0$ (the semi-leptonic form factor for ground-to-ground transitions evaluated at zero recoil given in eq. (13) to NLO) is consistent for both the single and coupled-channel models with the LO prediction for unity obtained in the combined expansion and in HQET.

Comparing the single and coupled-channel model predictions for $\rho$ (the curvature of the semi-leptonic form factor for ground-to-ground transitions evaluated at zero recoil given in eq. (14)) with the NLO model-independent predictions for $\rho$, we notice close agreement with a degree of accuracy expected from an NLO expansion. For the single-channel models, the prediction for $\rho$ (where $\rho \approx -1.50 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$) differ from the model-independent predictions (expressed in terms of the excitation energy of $\Lambda_c$ and $\sigma$) (where $\rho \approx -1.26 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$) by approximately 20%. This prediction agrees with the LO prediction of $-1.20 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$ to within 25%—an error which lies well within the expected error. Moreover, for the coupled-channel models, the prediction for $\rho$ is approximately $-0.90 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$. This prediction differs from the LO prediction based on the effective expansion (where $\rho \approx -1.20 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$) by 25%. The NLO model-independent predictions for $\rho$ also agree for all models (i.e., $\rho \approx -1.18 \times 10^{-4}$ MeV$^{-\frac{3}{2}}$) to within the expected accuracy. This prediction lies within 2% of the LO prediction for $\rho$. Thus, based on these predictions, the expansion for $\rho$ appears to be useful at NLO.

For the single-channel models, $\sigma \approx 0.018$ MeV$^{-\frac{3}{2}}$—a 60% difference from the LO prediction based on the effective expansion (where $\sigma \approx 0.011$ MeV$^{-\frac{3}{2}}$). The NLO model-independent prediction for $\sigma$ (expressed in terms of the excitation energy of $\Lambda_c$ and $\rho$) is approximately 0.013 MeV$^{-\frac{3}{2}}$. This prediction differs from the LO prediction based on the effective theory by 18%, and agrees with the model to within 28%. Similar agreement for the $\sigma$ observables is seen for the coupled-channel models. The prediction for $\sigma$ is approximately 0.0080 MeV$^{-\frac{3}{2}}$, which differs from the LO prediction of 0.011 MeV$^{-\frac{3}{2}}$ by 27%. Again, this error lies within the expected error. The prediction for $\sigma$ is also seen to agree with the model-independent predictions for $\sigma$ (where $\sigma \approx 0.0087$ MeV$^{-\frac{3}{2}}$) to the expected level of accuracy. This prediction differs from the LO prediction by about 21%. Again, based on these predictions, the expansion for $\sigma$ also appears to be useful at NLO.

By comparing the single and coupled-channel model predictions for $\Theta_0$, $\rho$, and $\sigma$ in Ta-
bles [III-XIV] to the respective model-independent predictions (with parameters based on model results), we notice that the expansion seems to work rather well for these observables. Thus, it is plausible that these model independent predictions might be useful in describing the phenomenology of isoscalar heavy baryons in the real world. However, there is a caveat which should be made. While the model-independent predictions based on the $\lambda$ expansion appear to work for our models to the expected accuracy, the difference in results between the single and coupled-channel models gives a cause for concern. Note that if the logic underlying our expansion is correct, the difference between these two types of models should appear at NNLO or higher. To the extent that the expansion at NLO gives reliable results, we would expect the difference between the predictions of the two types of models to be modest—of order of the uncertainties and thus typically of the same order as the discrepancies between the model-independent NLO predictions and the results of the models. Instead, the disagreements between the predictions of the two models are in fact much larger. For example, one sees that the single-channel model predictions for $\rho$ are typically $-1.50 \times 10^{-4}$ MeV$^{-4}$, while the coupled channel models had $\rho \approx 0.090 \times 10^{-4}$ MeV$^{-4}$. Thus, the single-channel results are nearly 1.7 times the coupled-channel results. Moreover, the discrepancy for $\sigma$ is even larger; the single-channel model predictions are more than double those of the coupled channel models.

The discrepancy between the single and coupled-channel model predictions seems to suggest that the parameters in the models and (by inference presumably those of the real world) are such that we may well be beyond the region of useful convergence of the expansion, at least for these observables. How can we reconcile the apparent success of the model-independent predictions based on the expansion with the differences between the two classes of models? A reasonable conjecture is that the expansion is rather marginal \textit{(i.e., whether the expansion is useful at NLO depends on details.)} We see that the expansion is more robust for some observables, or combinations of observables, than for others. This will happen if certain combinations of observables lead to effects of higher order terms largely canceling, while other combinations lead to these effects adding more coherently. Presumably, the combinations of observables in our model-independent predictions relating $\rho$ and $\sigma$ and the splitting of the $\Lambda_c$ states are robust in this sense at least in the context of these models. This is not too surprising. In fact, one might well expect correlations between the effects of higher-order terms on the form factor for decays to the ground state of $\Lambda_c$ and its excited states. It is not unreasonable to hope that a similar robustness may apply for the real world so that the $\lambda$ expansion may be useful in making predictions. It will be interesting to see if this is the case when measurements of the semi-leptonic decay form-factors become available.

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TABLE I: These are the model-independent predictions based on the effective theory at LO. Here \( \Delta m_{\Lambda_c} \) is the spin-average excitation energy of \( \Lambda_c \) obtained from experiment, \( \Delta m_{\Lambda_b} \) is the predicted excitation energy of \( \Lambda_b \) to LO (i.e, \( \Delta m_{\Lambda_b} \approx \sqrt{\frac{2\kappa}{\mu_b}} \) where \( \kappa \approx (410 \text{ MeV})^3 \) to LO and where \( \mu_b \) is the reduced mass of \( \Lambda_b \)). \( \Theta_0 \) is the form factor evaluated at zero recoil \((z = 0)\) for transitions between the ground state of \( \Lambda_b \) to the ground state of \( \Lambda_c \) (see eq.(13)). The second derivative of the form factor for ground-to-ground transitions evaluated at zero recoil is given in eq.(14) by \( \rho \), and \( \sigma \) is the derivative of the form factor evaluated at zero recoil for transitions between the ground state of \( \Lambda_b \) and first excited state of \( \Lambda_c \) (see eq.(17)).

| \( \Delta m_{\Lambda_c} \) (MeV) | \( \Delta m_{\Lambda_b} \) (MeV) | \( \Theta_0 \) | \( \rho \) (MeV\(^{-3/2}\)) | \( \sigma \) (MeV\(^{-3/4}\)) |
|-----------------|-----------------|--------|----------------|----------------|
| 330             | 300             | 0.99   | \(-1.20 \times 10^{-4}\) | 0.011          |

TABLE II: These are the model independent predictions based on the effective theory for the observables \( \Delta m_{\Lambda_b}, \Theta_0, \rho, \text{ and } \sigma \) given in eqs.(5),(13),(14), and (17) at NLO. These observables are rescaled in terms of \( \Lambda \) (where \( \Lambda \equiv (\mu_c(\Delta m_{\Lambda_c})^2)^{1/3} = 410 \text{ MeV} \) and are expressed at NLO in terms of the known excitation energy of \( \Lambda_c \) and one additional observable.

| Predictions | \( \Delta m_{\Lambda_c} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|----------------|----------------|----------------|
| \( \Delta m_{\Lambda_b} \Lambda^{-1} \) | \( 1.29 + 0.57 (\rho \Lambda^{3/2}) \) | \( 0.77 - 0.05 (\sigma \Lambda^{3/4}) \) |
| \( \Theta_0 \) | \( 0.95 - 0.05 (\rho \Lambda^{3/2}) \) | \( 0.99 + 0.004 (\sigma \Lambda^{3/4}) \) |
| \( \rho \Lambda^{3/2} \) | \( -0.92 - 0.08 (\sigma \Lambda^{3/4}) \) | \( \sigma \Lambda^{3/4} \) \( 0.25 - 0.72 (\rho \Lambda^{3/2}) \) |

TABLE III: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single channel model potential \( V(r) = -c_0 \exp[-(r/a_0)^2] \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| \( \Delta m_{\Lambda_b} \) (MeV) | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_c} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|-----------------|-----------------|----------|----------------|----------------|
| 288             | 300             | 236      | 282            |
| \( \Theta_0 \) | 0.99            | 0.99     | 1.01           | 0.99          |
| \( \rho \) (MeV\(^{-3/2}\)) | \(-1.51 \times 10^{-4}\) | \(-1.20 \times 10^{-4}\) | \(-1.27 \times 10^{-4}\) |
| \( \sigma \) (MeV\(^{-3/4}\)) | 0.018           | 0.011    | 0.013          |
| Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-------------------|----------|-------------------------------|-------------------------------|
| $\Delta m_{\Lambda_b}$ (MeV) | 289 | 300 | 237 | 281 |
| $\Theta_0$ | 0.99 | 0.99 | 1.012 | 0.99 |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -1.50 x 10^{-4} | -1.20 x 10^{-4} | -1.27 x 10^{-4} |  |
| $\sigma$ (MeV$^{-\frac{1}{4}}$) | 0.019 | 0.011 | 0.013 |  |

**TABLE IV:** This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single-channel model potential $V(r) = -c_0 \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right]$. The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-------------------|----------|-------------------------------|-------------------------------|
| $\Delta m_{\Lambda_b}$ (MeV) | 280 | 300 | 231 | 284 |
| $\Theta_0$ | 0.99 | 0.99 | 1.014 | 0.99 |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -1.54 x 10^{-4} | -1.20 x 10^{-4} | -1.26 x 10^{-4} |  |
| $\sigma$ (MeV$^{-\frac{1}{4}}$) | 0.017 | 0.011 | 0.013 |  |

**TABLE V:** This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single-channel model potential $V(r) = -a_0 \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right] + \exp \left[ -\left( \frac{r}{a_0} \right)^4 \right]$. The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-------------------|----------|-------------------------------|-------------------------------|
| $\Delta m_{\Lambda_b}$ (MeV) | 285 | 300 | 234 | 283 |
| $\Theta_0$ | 0.99 | 0.99 | 1.013 | 0.99 |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -1.52 x 10^{-4} | -1.20 x 10^{-4} | -1.26 x 10^{-4} |  |
| $\sigma$ (MeV$^{-\frac{1}{4}}$) | 0.018 | 0.011 | 0.013 |  |

**TABLE VI:** This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single-channel model potential $V(r) = -\frac{a_0}{2} \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right] + \exp \left[ -\left( \frac{r}{a_0} \right)^4 \right] + \frac{a_0}{2} \exp \left[ -\left( \frac{r}{a_0} \right)^8 \right]$. The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).
TABLE VII: This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single-channel model potential $V(r) = -\frac{c_0}{2}(\exp[-\left(\frac{r}{a_0}\right)^2] + \exp[-\left(\frac{2r}{a_0}\right)^2])$. The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| $\Delta m_{\Lambda_b}$ (MeV) | Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-----------------------------|-------------------|-----------|-------------------------------|-------------------------------|
|                             | 292               | 300       | 238                           | 280                           |
| $\Theta_0$                  | 0.99              | 0.99      | 1.012                         | 0.99                          |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -1.50×10$^{-4}$   | -1.20×10$^{-4}$ | -1.28×10$^{-4}$ |
| $\sigma$ (MeV$^{-\frac{3}{2}}$) | 0.019             | 0.011     | 0.013                         |                               |

TABLE VIII: This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the single-channel model potential $V(r) = -\frac{c_0}{2}(\exp[-[1 + \left(\frac{r}{a_0}\right)^2]^{1/2} + 1])$. The third column give the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| $\Delta m_{\Lambda_b}$ (MeV) | Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-----------------------------|-------------------|-----------|-------------------------------|-------------------------------|
|                             | 293               | 300       | 239                           | 279                           |
| $\Theta_0$                  | 0.99              | 0.99      | 0.99                          | 0.99                          |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -1.50×10$^{-4}$   | -1.20×10$^{-4}$ | -1.28×10$^{-4}$ |
| $\sigma$ (MeV$^{-\frac{3}{2}}$) | 0.020             | 0.011     | 0.013                         |                               |

TABLE IX: This table gives the predictions for the excitation energy of $\Lambda_b$ and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential $V(r) = -c_0 \exp[-(\frac{r}{a_0})^2]$. The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| $\Delta m_{\Lambda_b}$ (MeV) | Model Predictions | Fitted LO | $\Delta m_{\Lambda_c}$ and $\rho$ | $\Delta m_{\Lambda_c}$ and $\sigma$ |
|-----------------------------|-------------------|-----------|-------------------------------|-------------------------------|
|                             | 306               | 300       | 353                           | 300                           |
| $\Theta_0$                  | 0.98              | 0.99      | 0.99                          | 0.99                          |
| $\rho$ (MeV$^{-\frac{3}{2}}$) | -0.91×10$^{-4}$   | -1.20×10$^{-4}$ | -1.18×10$^{-4}$ |
| $\sigma$ (MeV$^{-\frac{3}{2}}$) | 0.0082            | 0.011     | 0.0087                        |                               |
| \( \Delta m_{\Lambda_b} \) (MeV) | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_c} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|-----------------|-----------------|---|----------------|----------------|
| 324             | 300             | 353| 300           | 300           |

| \( \Theta_0 \) | 0.97 | 0.99 | 0.99 | 0.99 |
|----------------|------|------|------|------|
| \( \rho \) (MeV^{-\frac{3}{2}}) | -0.88 \times 10^{-4} | -1.20 \times 10^{-4} | -1.18 \times 10^{-4} |
| \( \sigma \) (MeV^{-\frac{4}{2}}) | 0.011 | 0.0076 | 0.0085 |

TABLE X: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential \( V(r) = -c_0 \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right] \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| \( \Delta m_{\Lambda_b} \) (MeV) | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_c} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|-----------------|-----------------|---|----------------|----------------|
| 290             | 300             | 352| 300           | 300           |

| \( \Theta_0 \) | 0.96 | 0.99 | 0.99 | 0.99 |
|----------------|------|------|------|------|
| \( \rho \) (MeV^{-\frac{3}{2}}) | -0.92 \times 10^{-4} | -1.20 \times 10^{-4} | -1.18 \times 10^{-4} |
| \( \sigma \) (MeV^{-\frac{4}{2}}) | 0.0084 | 0.011 | 0.0088 |

TABLE XI: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential \( V(r) = -c_0 \left( \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right] + \exp \left[ -\left( \frac{r}{a_0} \right)^4 \right] \right) \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

| \( \Delta m_{\Lambda_b} \) (MeV) | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_c} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|-----------------|-----------------|---|----------------|----------------|
| 290             | 300             | 351| 301           | 301           |

| \( \Theta_0 \) | 0.98 | 0.99 | 0.99 | 0.99 |
|----------------|------|------|------|------|
| \( \rho \) (MeV^{-\frac{3}{2}}) | -0.92 \times 10^{-4} | -1.20 \times 10^{-4} | -1.18 \times 10^{-4} |
| \( \sigma \) (MeV^{-\frac{4}{2}}) | 0.0081 | 0.011 | 0.0088 |

TABLE XII: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential \( V(r) = -c_0 \left( \exp \left[ -\left( \frac{r}{a_0} \right)^2 \right] + \exp \left[ -\left( \frac{r}{a_0} \right)^4 \right] \right) \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).
|                      | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_b} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|----------------------|-------------------|-----------|------------------------------------------|------------------------------------------|
| \( \Delta m_{\Lambda_b} \) (MeV) | 318               | 300       | 357                                      | 302                                      |
| \( \Theta_0 \)       | 0.97              | 0.99      | 0.99                                     | 0.99                                     |
| \( \rho \) (MeV\(^{-\frac{3}{2}}\)) | -0.89\(\times\)10\(^{-4}\) | -1.20\(\times\)10\(^{-4}\) | -1.18\(\times\)10\(^{-4}\)              |
| \( \sigma \) (MeV\(^{-\frac{4}{3}}\)) | 0.0076            | 0.011     | 0.0086                                    |

TABLE XIII: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential \( V(r) = -\frac{c_0^2}{2}(\exp[-(\frac{r}{a_0})^2] + \exp[-(2\frac{r}{a_0})^2]) \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).

|                      | Model Predictions | Fitted LO | \( \Delta m_{\Lambda_b} \) and \( \rho \) | \( \Delta m_{\Lambda_c} \) and \( \sigma \) |
|----------------------|-------------------|-----------|------------------------------------------|------------------------------------------|
| \( \Delta m_{\Lambda_b} \) (MeV) | 261               | 300       | 360                                      | 302                                      |
| \( \Theta_0 \)       | 0.97              | 0.99      | 0.99                                     | 0.99                                     |
| \( \rho \) (MeV\(^{-\frac{3}{2}}\)) | -0.87\(\times\)10\(^{-4}\) | -1.20\(\times\)10\(^{-4}\) | -1.18\(\times\)10\(^{-4}\)              |
| \( \sigma \) (MeV\(^{-\frac{4}{3}}\)) | 0.0076            | 0.011     | 0.0085                                    |

TABLE XIV: This table gives the predictions for the excitation energy of \( \Lambda_b \) and the semileptonic form factors given to NLO in eqs. (5), (13), (14) and (17). The second column gives predictions for the observables based on the coupled-channel model potential \( V(r) = -\frac{c_0^2}{2}(\exp[-[1+(\frac{r}{a_0})^2]^{1/2}+1]) \). The third column gives the leading-order predictions for the observables based on the effective theory. The last two columns give model predictions for the observables based on the model-independent predictions obtained from the effective theory at NLO (see Table [II]).