On the Capacity Region of Broadcast Packet Erasure Relay Networks With Feedback

Quan Geng  
CSL and Dept. of ECE  
UIUC, IL 61801  
Email: geng5@illinois.edu

Hieu T. Do  
School of EE and ACCESS Linnaeus Center  
KTH, Stockholm, Sweden  
Email: hieu.do@ee.kth.se

Abstract—We derive a new outer bound on the capacity region of broadcast traffic in multiple input broadcast packet erasure channels with feedback, and extend this outer bound to packet erasure relay networks with feedback. We show the tightness of the outer bound for various classes of networks. An important engineering implication of this work is that for network coding schemes for parallel broadcast channels, the “xor” packets should be sent over correlated broadcast subchannels.

I. INTRODUCTION

The broadcast packet erasure channel (PEC) was introduced in [1], which captures the broadcast nature of wireless communication. In packet erasure relay networks, each directed link connecting two nodes \(i\) and \(j\) is modeled as a packet erasure channel, which is a natural generalization of the binary erasure channel from binary symbol to packet. During each time slot, node \(i\) can send out a packet of fixed size to node \(j\). With probability \((1-\epsilon_{ij})\) node \(j\) receives the whole packet correctly, and with probability \(\epsilon_{ij}\) the packet is erased by the channel. Furthermore, due to the broadcast nature of wireless communication, during each transmission the transmitter sends out the same packet to all the nodes it is connected to. Additionally, in packet erasure relay networks we assume that there is no interference, i.e., each node can receive packets sent from different nodes simultaneously without interference. This assumption is valid when some mechanisms in practical systems are implemented to avoid interference, e.g., via frequency-division multiplexing. In PECs with feedback, each transmitter can get the channel output feedback immediately from the receivers after each transmission, i.e., the transmitter will know whether each receiver has got the packet or not. In practice, this type of feedback can be obtained using the Automatic Repeat-reQuest (ARQ) mechanism.

A. Existing Work

Reference [2] characterizes the broadcast capacity region of multiple input broadcast PECs without feedback, which can be achieved by the time sharing scheme. It turns out that feedback can significantly improve the capacity region, and [3] derives the capacity region of 1-to-2 broadcast PECs with feedback, and gives a simple networking coding scheme to achieve the capacity region. [4] and [5] independently study the general 1-to-\(K\) broadcast PECs with feedback, and characterize the capacity region for \(K = 3\). A recent work [6] derives the linear network coding capacity region of 2-receiver MIMO broadcast PECs with feedback. For broadcast packet erasure relay networks with feedback, [1] derives the capacity of unicast traffic and gives a random linear network coding scheme to achieve the capacity. [7] presents a simple capacity-achieving “dynamical routing” scheme, which can be written as a linear program, for unicast traffic on packet erasure networks with feedback. [8] shows that local network coding and global routing can achieve the cut-set bound within a factor of \(O(\log^3 k \log d_{\text{max}})\) for \(k\)-unicast traffic in broadcast packet relay networks with commensurate feedback, where \(d_{\text{max}}\) is the maximum degrees of nodes in the network. One direct extension of [8] is that for broadcast traffic, the same separation scheme can also achieve the cut-set bound within a factor of \(O(\log^3 k \log d_{\text{max}})\).

B. Our Contribution

In this work, we derive a new outer bound on the capacity region of broadcast traffic in multiple input broadcast packet erasure channels with feedback, and extend this outer bound to packet erasure relay networks with feedback. The new outer bound combines the standard cut-set bound technique with the capacity region of the degraded broadcast channel. We show the tightness of the outer bound for certain classes of networks. An important engineering implication of this work is that for linear network coding schemes for parallel broadcast channels, the “xor” packets should be sent over the correlated broadcast subchannels.

C. Organization

We describe the system model in Section II present the new outer bounds for multiple input PECs with feedback and packet erasure relay networks with feedback in Section III and Section IV respectively. We show the tightness of the outer bounds for certain classes of networks in Section V. Section VI concludes this paper.

II. SYSTEM MODEL

A. Wireless Broadcast Packet Erasure Channel

As introduced in Section I, the broadcast packet erasure channel (PEC) captures the broadcast nature of wireless communication by modeling each directed link between two nodes as a packet erasure channel. During each time slot, node \(i\)
sends out a packet of fixed size to all the nodes it is connected to. A node \( j \) connected to node \( i \) will receive the whole packet correctly with probability \((1 - \epsilon_{ij})\), and receive nothing with probability \( \epsilon_{ij} \). In the latter case the packet is said to be “erased” by the channel, and \( \epsilon_{ij} \) is called the erasure probability of link \( ij \). Equivalently, in each time slot a node broadcasts the same symbol from a large field \( GF(q) \) to all the nodes which it is connected to. In addition, we assume the channel is memoryless and time invariant, and erasure events over different links are independent.

More formally, consider a 1-to-\( K \) broadcast packet erasure channels, where a node \( s \) is connected to \( K \) nodes \( t_1, t_2, \ldots, t_K \). Let \([K]\) denote \( \{1, 2, \ldots, K\} \). For \( j \in [K] \), the channel between nodes \( s \) and \( t_j \) is a packet erasure channel with erasure probability \( \epsilon_{ij} \). During the \( n \)-th channel use, \( s \) sends out a packet \( X[n] \in GF(q) \). Let \( Y_j[n] \) be the symbol received by node \( t_j \). If channel erasure events are independent for all links, then for any subset \( A \subset [K] \),

\[
\text{Prob}(Y_j[n] = X[n], Y_{j'}[n] = *, \forall j \in A, j' \in A^C) = \prod_{j \in A} (1 - \epsilon_{ij}) \prod_{j' \in A^C} \epsilon_{j'},
\]

where * denotes that the packet has been erased by the channel, and \( A^C \) denotes the complement of \( A \) in \([K]\).

A natural extension of the broadcast packet erasure channel is the broadcast packet erasure channel with feedback. In this channel we assume the transmitter immediately receives a perfect feedback from the receivers after each transmission, which indicates whether each receiver has received the packet or not. A practical mechanism for this type of feedback is the Automatic Repeat-reQuest (ARQ) protocol.

**B. Wireless Broadcast Packet Erasure Relay Networks**

In wireless broadcast packet erasure relay networks, nodes are connected to other nodes via broadcast packet erasure channels. We can model a wireless broadcast erasure relay network by a directed graph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} \) denotes the set of nodes and \( \mathcal{E} \) denotes the set of links. A directed edge \((i, j) \in \mathcal{E}\) if node \( i \) is connected to \( j \), and the corresponding packet erasure probability is denoted as \( \epsilon_{ij} \). Wireless broadcast packet erasure relay networks capture the broadcast nature of wireless communication by forcing a node to send the same symbol to all the nodes it is connected to during each channel use, while we assume each node can receive packets sent from different nodes simultaneously without interference. This assumption is valid in practical systems when orthogonal schemes such as frequency-division multiplexing are implemented to avoid interference.

In this work, we consider the broadcast traffic in wireless broadcast packet erasure relay networks with channel output feedback, where there is a single source \( s \), which wants to send \( K \) independent messages to \( K \) different destinations \( t_1, t_2, \ldots, t_K \) through a network of relays. Let \((R_1, R_2, \ldots, R_K)\) denote the tuple of reliable transmission rates from \( s \) to the \( K \) destinations, whose definition is the same as in [2]. Our goal is to characterize the capacity region and the sum capacity of the network. Fig. 1 shows an example of a three-layer 1-to-\( K \) broadcast packet erasure relay network.

**III. OUTER BOUND FOR MULTIPLE INPUT PECs WITH FEEDBACK**

In this section we derive an outer bound on the capacity region of broadcast traffic in multiple input broadcast PECs with feedback.

Consider an \((M, K)\) multiple input broadcast PECs with feedback, where there is a single source \( s \) connected to \( K \) destinations via \( M \) parallel 1-to-\( K \) broadcast PECs with feedback. Throughout this paper, we assume the channel is memoryless and erasure events over different subchannels are independent.

Define \([K] \triangleq \{1, 2, 3, \ldots, K\}\) and let \( \pi \) be a permutation function from \([K]\) to \([K]\). Let \( \epsilon_{ij} \) denote the channel erasure probability on the link connecting \( s \) and \( t_j \) of the \( i \)-th subchannel. For any subset \( A \subset [K] \), let \( \epsilon_{iA} \) denotes the probability that for any \( j \in A, t_j \) does not receive the packet on the \( i \)-th subchannel. For example, if erasure events on all links are independent, then \( \epsilon_{iA} = \prod_{j \in A} \epsilon_{ij} \). Lastly, we define \( \pi(A) = \{\pi(j) | j \in A\} \), for any subset \( A \subset [K] \).

**Theorem 1.** For any achievable rate tuple \( \mathbf{R} \triangleq (R_1, R_2, \ldots, R_K) \), it must satisfy that for any permutation function \( \pi : [K] \rightarrow [K] \),

\[
\mathbf{R} \in \mathbf{C}_\pi,
\]

where

\[
\mathbf{C}_\pi \triangleq \{(R_1, R_2, \ldots, R_K) | \sum_{j=1}^{K} \frac{R_{\pi(j)}}{1 - \epsilon_{\pi(j)}} \leq 1, \quad R_k = \sum_{m=1}^{M} R_{mk}, \quad R_{\pi(k)} \geq 0, \forall i \in [M], k \in [K]\}.
\]

Theorem 1 gives a natural outer bound of the capacity region, and the proof idea is the same as the proof of Proposition 1 in [3] by introducing auxiliary pipes connecting destinations to create physically degraded subchannels. The proof of Theorem 1 is given in Appendix A.
The above bound can be tightened using Lemma 2.

**Lemma 2.** For an \((M, K)\) multiple input broadcast PECs with feedback, if each subchannel is physically degraded (different subchannels may be degraded in different ways), then feedback does not improve the capacity region and the capacity region can be achieved by time sharing among receivers for each subchannel.

Similar to the proof of Theorem 3 in [2], Lemma 2 can be proved by showing that the supporting functions of the time sharing rate region and the capacity region are the same. The proof of Lemma 2 is given in Appendix B.

Applying Lemma 2, we get a tighter outer bound on the capacity region of broadcast traffic in multiple input broadcast PECs with feedback.

**Theorem 3.** Consider an \((M, K)\) multiple input broadcast PECs with feedback. For any achievable rate tuple \(\mathbf{R} = (R_1, R_2, \ldots, R_M)\), it must satisfy that for any \(M\) permutation functions \(\pi^1, \pi^2, \ldots, \pi^M\), where \(\pi^i : [K] \rightarrow [K]\), for \(i \in [M]\),

\[R \in C(\pi^1, \pi^2, \ldots, \pi^M),\]

where

\[C(\pi^1, \pi^2, \ldots, \pi^M) \triangleq \left\{ (R_1, R_2, \ldots, R_K) \mid \sum_{j=1}^{K} \frac{R_{\pi^i(j)}}{1 - \epsilon_{\pi^i(j)}} \leq 1 \right\}.

\[R_k = \sum_{m=1}^{M} R_{mk}, \quad R_k \geq 0, \quad \forall i \in [M], k \in [K].\]

**Proof sketch:** Given the permutation functions \(\pi^1, \pi^2, \ldots, \pi^M\), we construct a new multiple input broadcast erasure channel with feedback in the following way: \(\forall i \in [M]\), we create information pipes connecting node \(t_{\pi^i(k)}\) to node \(t_{\pi^i(k+1)}\) in the \(i\)th subchannel, \(\forall k \in [K - 1]\), so that \(t_{\pi^i(k+1)}\) will get all packets node \(t_{\pi^i(k)}\) receives in the \(i\)th subchannel. Each subchannel of this new channel is physically degraded, and thus the capacity region of this new channel is \(C(\pi^1, \pi^2, \ldots, \pi^M)\) due to Lemma 2. Since this new channel is no worse than the original channel, we conclude

\[R \in C(\pi^1, \pi^2, \ldots, \pi^M).\]

IV. OUTER BOUND FOR BROADCAST PACKET ERASURE RELAY NETWORKS

In this section we give a new outer bound on the capacity region of broadcast traffic in broadcast packet erasure relay networks with feedback. Although the construction of the new outer bound appears complicated, the idea is relatively simple. We use the standard cut-set bound technique by dividing the network of nodes into two parts. While the cut-set bound assumes nodes in each part can fully cooperate, to derive the new outer bound, we only allow nodes in the source part to fully cooperate, and allow nodes in the destination part to only partially cooperate. In this way, we obtain a multiple input broadcast PEC, the capacity region of which upper bounds the capacity region of the original packet erasure relay networks.

By applying Theorem 3 to the new multiple input broadcast PEC, we get the new outer bound for the original network.

Consider a wireless packet erasure relay network modeled as a directed graph \(G = (V, \mathcal{E})\), with a single source \(s\) and \(K\) destinations \(t_1, \ldots, t_K\). As described in Section III-B, each directed edge \((i, j) \in \mathcal{E}\) has an associated packet erasure probability \(\epsilon_{ij}\). Let \(A \subset V\), which does not contain the source node \(s\), i.e., \(s \notin A\). Let \(J(A) = \{j \mid (i, j) \in \mathcal{E}\}\), the set of indices of destinations contained in \(A\). Let \(\mathcal{E}_A\) denote the edge cut corresponding to \(A\). More precisely,

\[\mathcal{E}_A \triangleq \{ (v, w) | (v, w) \in \mathcal{E}, v \notin A, w \in A \}.

Let \(\mathcal{V}_A \triangleq \{ v \in \mathcal{V} | \exists w \in A, s.t. (v, w) \in \mathcal{E}_A \}\). For any subset \(\mathcal{W}_A \subset \mathcal{V}_A\), we will derive an upper bound on the achievable rates tuple \(R_{J(A)} \triangleq \{ R_i \}_{i \in J(A)}\) in terms of \(\mathcal{E}_A\) and \(\mathcal{W}_A\).

We use the following algorithm to construct an edge set \(\mathcal{E}^*\) and a vertex set \(\mathcal{V}^*\).

1. Initialization: set \(\mathcal{E}^* = \emptyset, \mathcal{V}^* = \emptyset\).
2. For each node \(w \in A\), if all edges which go to \(w\) come from nodes in \(\mathcal{W}_A\), then a) add these edges to \(\mathcal{E}^*\), b) if \(w\) is one of the \(K\) destinations, add \(w\) to \(\mathcal{V}^*\), c) delete all edges that go to \(w\) and leave from \(w\), and delete \(w\).
3. For each node \(w \in A\), if no edges go to \(w\), then a) delete all edges that leave from \(w\), b) add \(w\) to \(\mathcal{V}^*\) if \(w\) is one of the \(K\) destinations, c) delete \(w\).
4. Repeat step 2 and step 3 until no node will be deleted.
5. Consider the subgraph consisting of only nodes in \(A\) which have not been deleted. Make the subgraph undirected, and then find out all the connected components. Delete the connected components which does not contain any destinations. Denote the remaining connected components by \(C_1, C_2, \ldots, C_p\), where \(C_i\) is the set of vertices in the corresponding components. Then merge each component to a single super node, denoted by \(v_1, v_2, \ldots, v_p\).

Now we use \(\mathcal{E}_A, \mathcal{V}_A, \mathcal{W}_A, \mathcal{E}^*, \mathcal{V}^*\) to construct multiple input broadcast PECs. The capacity region of this new network will yield an outer bound on the rate tuples \(R_{J(A)}\).

For each node \(v \in \mathcal{V}_A\), we process \(v\) as follows:

1. Consider the edges which leave from \(v\) and are in \(\mathcal{E}_A\), and then divide them into two subsets \(\mathcal{E}_1\) and \(\mathcal{E}_2\); \(\mathcal{E}_1\) is the set of edges in \(\mathcal{E}^*\), and \(\mathcal{E}_2\) is the set of edges in \(\mathcal{E}_A \setminus \mathcal{E}^*\).
2. Split node \(v\) into two nodes \(v_1\) and \(v_2\).
3. Connect \(v_1\) to each node in \(\mathcal{V}^*\) and the super nodes \(v_1, v_2, \ldots, v_p\) via broadcast PECs with feedback, with erasure probability \(\prod_{(i,j) \in \mathcal{E}_1} \epsilon_{ij}\). If \(\mathcal{E}_1\) is an empty set, set the erasure probability to be 1 (or equivalently, we can delete node \(v_1\)). Further, we assume the erasure events over all these links are the same, i.e., if the packet over some link is erased, then packets over all links are erased.
4. Connect \(v_2\) to each super node \(v_1, v_2, \ldots, v_p\) via a broadcast packet erasure channel with feedback. The erasure probability of each link is set as follows.
each \( i \in [p] \), let \( \mathcal{E}_i \) denote the set of edges in \( E_2 \) which leave from \( v \) and go to some node in \( C_i \). If \( \mathcal{E}_i \) is an empty set, then set the erasure probability to be 1; otherwise, set the erasure probability to be \( \prod_{(i,j) \in \mathcal{E}_i} \epsilon_{ij} \).

After processing all the nodes \( v \) in \( V_A \), we allow each newly created node to fully cooperate with each other and the source node \( s \). Let \( N \) be the total number of nodes in \( V^* \) plus \( p \). Therefore we get a multiple input 1-to-2 broadcast packet erasure channel with feedback. Note that, by construction, the erasure events over all links in the subchannel are not independent for some subchannels. (Indeed they are completely correlated.)

Next we upper bound \( R_{J(A)} \) by using an outer bound on the capacity region of this multiple input broadcast PEC with feedback. By our construction, for each \( j \in J(A) \), the destination \( t_j \) is contained in either \( V^* \) or \( C_i \), for some \( i \in [p] \). For each \( i \in [p] \), let \( J_i \) denote the set of indices of destinations which are in \( C_i \). More precisely, \( J_i \triangleq \{ j \in J(A) | t_j \in C_i \} \).

Relabel the \( N \) sink nodes in the new multiple input PECs by \( d_1, d_2, \ldots, d_N \). Define \( N \) auxiliary variables \( Q_1, Q_2, \ldots, Q_N \), where \( Q_i \triangleq R_j \) if \( d_i \) is the single destination \( t_j \), or \( Q_i \triangleq \sum_{j \in J_i} R_j \) if \( d_i \) is the super node \( v \).

We can use Theorem 3 to upper bound the capacity region of this new 1-to-2 multiple input broadcast PEC with feedback. Let \( N(A, W_A) \) denote this new multiple input broadcast PEC with feedback, and let \( R_{outer}(N(A, W_A)) \) denote the outer bound of the capacity region of \( N(A, W_A) \) given in Theorem 3.

Theorem 4.

\( (Q_1, Q_2, \ldots, Q_N) \in R_{outer}(N(A, W_A)). \) (1)

Therefore, Equation (1) gives an outer bound on \( R_{J(A)} \).

Proof sketch: We only need to argue that the new channel \( N(A, W_A) \) is no worse than the original network for destinations \( d_1, d_2, \ldots, d_N \). First note that, there is no loss by deleting nodes of which the incoming edges come from \( W_A \), since in the last step we connect these edges directly to all destinations. Second, there is no loss by merging the connected component to a super node, since in this way all the nodes in the connected component are assumed to fully cooperate with each other and behave like a single node. Third, there is no loss by assuming the new created nodes can cooperate and share the same message with source \( s \). Therefore, the capacity region of the new constructed multiple input PEC upper bounds the achievable rates tuple \( R_{J(A)} \).

V. TIGHTNESS OF OUTERBOUNDS

Section III and Section IV give outer bounds on the capacity region of broadcast traffic in multiple input PECs with feedback and broadcast packet erasure relay networks with feedback. In this section, we show that for certain classes of networks, these outer bounds are tight in terms of sum rate.

Theorem 5 gives an outer bound on the capacity region of multiple input broadcast erasure channels with feedback. A natural inner bound can be derived by adding the capacity region of each subchannel. In general, this inner bound without coding across subchannels does not match the outer bound. In the following, we show that for certain simple two-input 1-to-2 broadcast PECs with feedback, the above inner bound does not match the outer bound, while the maximum sum rate of the outer bound can be achieved by coding across the two subchannels.

Consider a two-input 1-to-2 broadcast PEC with feedback where there are two destinations \( t_1 \) and \( t_2 \) and two subchannels. In the first subchannel, the packet erasure probabilities are \( \epsilon_{11} = \epsilon_{12} = \epsilon_1 \) and erasure events on the two links are independent of each other. In the second subchannel, \( \epsilon_{21} = \epsilon_{22} = \epsilon_2 \) and erasure events on the two links are the same, i.e., at any time, \( t_1 \) receives a packet if and only if \( t_2 \) also receives the packet. We assume erasure events on different subchannels are independent.

Theorem 5. If \( \epsilon_2 \geq 1 - \frac{(1-\epsilon_1)\epsilon_1}{2} \) and \( 0 < \epsilon_1, \epsilon_2 < 1 \), then the maximum sum rate of the outer bound in Theorem 3 is tight and can be achieved by coding across subchannels, which thus characterizes the sum capacity of this channel. In addition, the inner bound without coding across subchannels is strictly suboptimal.

Proof sketch: The maximum sum rate of the inner bound is

\[
R_{inner}^{sum} = \frac{2(1 - \epsilon_1)}{2 + \epsilon_1} + 1 - \epsilon_2,
\]

and the maximum sum rate of the outer bound is

\[
R_{outer}^{sum} = \frac{2(2 - \epsilon_1^2 + \epsilon_1 - \epsilon_2 - \epsilon_1\epsilon_2)}{2 + \epsilon_1}.
\] (2)

It is easy to verify that there is a nonzero gap between the inner bound and outer bound:

\[
R_{outer}^{sum} - R_{inner}^{sum} = \frac{2(2 - \epsilon_1^2 + \epsilon_1 - \epsilon_2 - \epsilon_1\epsilon_2)}{2 + \epsilon_1} - \frac{2(1 - \epsilon_1^2)}{2 + \epsilon_1} + 1 - \epsilon_2
\]

\[
= \epsilon_1(1 - \epsilon_2) > 0.
\]

for \( 0 < \epsilon_1, \epsilon_2 < 1 \).

We describe a coding scheme to achieve the outer bound \( R_{outer}^{sum} \). The scheme is essentially the same as the two-phase scheme introduced in [3], except that we use the second subchannel only to send the “xor” packets. For the first subchannel:

1) In the first \( N \) transmissions, the source \( s \) sends packets for destination \( t_1 \). The source resends the same packet if and only if neither \( t_1 \) nor \( t_2 \) receives the packet. After the first \( N \) transmissions, on average, \( t_1 \) receives \( N(1 - \epsilon_1) \) packets, and \( t_2 \) receives \( N(1 - \epsilon_1) \) packets which are for \( t_1 \) but not received by \( t_1 \). Denote these packets received by \( t_2 \) only by \( P_1 \).

2) Similarly, during the second \( N \) transmissions, the source \( s \) sends packets for destination \( t_2 \). The source resends the same packet if and only if neither \( t_1 \) nor \( t_2 \) receives the packet. After the second \( N \) transmissions, on average, \( t_2 \)
and thus after all the transmissions, on average, both $t_1$ and $t_2$ will receive $N(1 - \epsilon_1)$ packets, and $t_1$ receives $N(1 - \epsilon_1)\epsilon_1$ packets which are for $t_2$ but not received by $t_2$. Denote these packets received by $t_1$ only by $P_2$.

3) In the following $N^*$ transmissions, where $N^*$ is to be determined soon, $s$ sends out packets which are random linear combinations of $P_1$ and $P_2$.

For the second subchannel, during the $N + N + N^*$ transmissions, $s$ only sends out random linear combinations of $P_1$ and $P_2$. Note that one can use block coding to resolve the noncausality issue.

The destination $t_1$ needs $N(1 - \epsilon_1)^2$ packets which are random linear combinations of $P_1$ and $P_2$ to decode $P_1$. Similarly, the destination $t_2$ needs $N(1 - \epsilon_1)^2$ packets which are random linear combinations of $P_1$ and $P_2$ to decode $P_2$. After all the transmissions, on average, both $t_1$ and $t_2$ will receive

$$N^*(1 - \epsilon_1) + (2N + N^*)(1 - \epsilon_2)$$

random linear combinations of $P_1$ and $P_2$.

Therefore, $N^*(1 - \epsilon_1) + (2N + N^*)(1 - \epsilon_2) = N(1 - \epsilon_1)\epsilon_1$, and thus $N^* = \frac{1 - \epsilon_1}{1 - \epsilon_1 - 2(1 - \epsilon_2)} \geq 0$.

After these $N + N + N^*$ transmissions, $t_1$ can decode $N(1 - \epsilon_1)^2 + N(1 - \epsilon_1)\epsilon_1 = N(1 - \epsilon_1^2)$ packets for it, and similarly $t_2$ can also decode $N(1 - \epsilon_1)^2 + N(1 - \epsilon_1)\epsilon_1 = N(1 - \epsilon_2^2)$ packets for it. Therefore, the achieved sum rate is

$$\frac{N(1 - \epsilon_1^2) + N(1 - \epsilon_2^2)}{N + N + N^*} = \frac{2(1 - \epsilon_1^2)}{2 + (1 - \epsilon_1)(1 - 2\epsilon_1 - \epsilon_2)}$$

$$= \frac{2(1 + \epsilon_1)(2 - \epsilon_1 - \epsilon_2)}{2 + \epsilon_1} = R_{\text{sum}}^\text{outer}.$$

Therefore this coding scheme achieves $R_{\text{sum}}^\text{outer}$, which is thus the sum capacity of this channel.

Based on the above two input 1-to-2 broadcast PECs with feedback, we can construct a packet erasure relay network with feedback where the new outer bound in Theorem 4 is tight in terms of sum rate.

Consider the broadcast packet erasure relay network in Fig. 2, where all links are independent and the packet erasure probabilities are zero for all links except the links $(r_c, t_1), (r_c, t_2)$ and $(r_b, r_d)$ with erasure probability $\epsilon_1, \epsilon_1$ and $\epsilon_2$, respectively. In addition, $\epsilon_1$ and $\epsilon_2$ satisfy the assumption in Theorem 3. We can derive the sum capacity of this network by applying Theorem 4 as follows. In the algorithm given in Section IV, take $A = \{t_1, t_2, r_d\}$ and $V_A = \{r_c, r_b\}$. Take $W_A = \{r_b\}$. Then the new network $N(A, W_A)$ constructed by the algorithm is exactly the channel described in Theorem 5 with the same channel parameters. Therefore, Theorem 4 upper bounds the sum rate of the network in Fig. 2 by $R_{\text{sum}}^\text{outer}$ in Equation (2). Furthermore, it is achievable by using the corresponding scheme in the proof of Theorem 5.

VI. CONCLUSION

We derive a new outer bound on the capacity region of broadcast traffic in multiple input broadcast packet erasure channels with feedback, and extend this outer bound to packet erasure relay networks with feedback. The new outer bound involves the standard cut-set bound technique and the capacity region of the degraded broadcast channel. We show the tightness of the outer bound for certain classes of networks. One important engineering implication of this work is that for network coding schemes for parallel broadcast channels, the “xor” packets should be sent over the correlated broadcast subchannels.

APPENDIX A

PROOF OF THEOREM 1

Proof of Theorem 1. For any permutation function $\pi$, we construct new multiple input broadcast packet erasure channels with feedback by creating information pipes connecting node $t_{\pi(i)}$ to node $t_{\pi(i+1)}$, so that $t_{\pi(i+1)}$ will get all packets node $t_{\pi(i)}$ receives, for all $i \in [K - 1]$. Therefore, the probability that node $t_{\pi(i)}$ receives the packet sent by the transmitter on the $m$th parallel broadcast channel is $1 - \epsilon_m\pi([i])$. The new multiple input broadcast packet erasure channels are physically degraded, so feedback does not change the capacity region [9]. Therefore, we can assume there is no feedback. Since the capacity region of the broadcast channel without feedback only depends on the marginal distribution of channel output, we can further assume that the erasure events over all links in the new model are independent. Lastly, applying Theorem 3 of [2] on the capacity region of multiple input broadcast packet erasure channels without feedback, we conclude that $C_\pi$ is the capacity region of the new model. Therefore, for any achievable rates tuple $R$ on the original channel model, we have $R \in C_\pi$.

APPENDIX B

PROOF OF LEMMA 2

Before presenting the proof, we give some preliminaries on the characterization of closed convex sets.

For any $u, v \in \mathbb{R}^n$, we use $(u, v)$ to denote the inner product of $u$ and $v$.

**Definition 3** ([10]). Given a closed convex set $V \subseteq \mathbb{R}^n$, the supporting functional $\phi_V$ of $V$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$, which...
which is defined as
\[ \phi_v(\mu) \triangleq \max_{v \in \mathcal{V}} \langle \mu, v \rangle, \quad \forall \mu \in \mathbb{R}^n. \]

The supporting functional of a closed convex set gives an alternative characterization of the convex set in the following sense.

**Lemma 7.** Given two closed and bounded convex sets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) in \( \mathbb{R}^n \), \( \mathcal{V}_1 = \mathcal{V}_2 \) if and only if \( \phi_{\mathcal{V}_1} \equiv \phi_{\mathcal{V}_2} \).

**Proof:** The if direction is straightforward, which is true by the definition of \( \phi_{\mathcal{V}} \). For the only if part, suppose \( \mathcal{V}_1 \neq \mathcal{V}_2 \), then without loss of generality, we can assume that there exists a point \( p \in \mathbb{R}^n \) such that \( p \in \mathcal{V}_1 \) and \( p \notin \mathcal{V}_2 \). We will construct \( \mu \in \mathbb{R}^n \) such that \( \phi_{\mathcal{V}_1}(\mu) > \phi_{\mathcal{V}_2}(\mu) \).

Define the set \( \mathcal{W} \triangleq \{ q - p \mid q \in \mathcal{V}_2 \} \), which is also a closed and bounded convex set and \( 0 \notin \mathcal{W} \). Consider the minimization \( \min_{w \in \mathcal{W}} \|w\| \), where \( \|w\| \) denotes the \( \ell^2 \) norm of \( w \). Since the norm is a continuous function and \( \mathcal{W} \) is a closed and bounded set and thus compact, there exists an optimal solution \( w^* \in \mathcal{W} \). Clearly, \( w^* \neq 0 \), since \( 0 \notin \mathcal{W} \). We claim that for any \( x \in \mathcal{W} \), \( \langle x, w^* \rangle \geq \langle w^*, w^* \rangle \). Suppose not, then there exists \( x \in \mathcal{W} \) such that \( \langle w^*, x - w^* \rangle < 0 \). We can easily compute the derivative of the objective function at \( w^* \) in the direction \( x - w^* \):

\[ \lim_{t \to 0} \frac{\|w^* + t(x - w^*)\| - \|w^*\|}{t} = \frac{\langle w^*, x - w^* \rangle}{\|w^*\|} < 0. \]

Consider the point \( w(t) \triangleq w^* + t(x - w^*) \). Since \( \mathcal{W} \) is a convex set, for \( 0 \leq t \leq 1, w(t) \in \mathcal{W} \). By Taylor expansion,

\[ \|w(t)\| = \|w^*\| + t\langle w^*, x - w^* \rangle + O(t^2). \]

Since \( \langle w^*, x - w^* \rangle < 0 \), when \( t \) is sufficiently small, \( \|w(t)\| < \|w^*\| \), which contradicts the assumption that \( w^* \) is the solution to \( \min_{w \in \mathcal{W}} \|w\| \). Therefore, for any \( x \in \mathcal{W} \), \( \langle x, w^* \rangle \geq \langle w^*, w^* \rangle \). Since \( w^* \neq 0 \), \( \langle x, w^* \rangle > 0 \). Hence, for any \( v_2 \in \mathcal{V}_2 \), \( \langle w^*, v_2 - p \rangle > 0 \), i.e., \( \langle w^*, v_2 \rangle > \langle w^*, p \rangle \). Therefore, by letting \( \mu = -w^* \), we have

\[ \langle \mu, v_2 \rangle < \langle \mu, p \rangle, \]

for any \( v_2 \in \mathcal{V}_2 \). Hence, \( \phi_{\mathcal{V}_2}(\mu) < \langle \mu, p \rangle \leq \phi_{\mathcal{V}_1}(\mu) \), which contradicts the assumption that \( \phi_{\mathcal{V}_2} \equiv \phi_{\mathcal{V}_1} \). Therefore, we have \( \mathcal{V}_1 = \mathcal{V}_2 \).

Due to Lemma 7, we only need to prove that the supporting functionals of the time sharing rate region and the capacity region are the same.

**Proof of Lemma 2.** Let \( C \) denote the capacity region of this channel, and let \( \mathcal{R}_T \) denote the rate region achieved by time sharing. We will prove that for all tuples \( \mu \triangleq (\mu_1, \mu_2, \ldots, \mu_K) \in \mathbb{R}^K, \phi_C(\mu) = \phi_{\mathcal{R}_T}(\mu) \).

For now, we assume all \( \mu_i \) are nonnegative. First, we compute \( \phi_{\mathcal{R}_T}(\mu) \). It is easy to see the rate region \( \mathcal{R}_T \) achieved by time sharing is

\[ \mathcal{R}_T = \{ (R_1, R_2, \ldots, R_K) \mid 0 \leq R_j \leq \sum_{i=1}^{M} \alpha_{ij}(1 - \epsilon_{ij}), \alpha_{ij} \geq 0, \sum_{j=1}^{K} \alpha_{ij} = 1 \}, \]

where \( i \) is the index of subchannels and \( j \) is the index of destinations.

Hence,

\[ \phi_{\mathcal{R}_T}(\mu) = \max_{R=(R_1, R_2, \ldots, R_K) \in \mathcal{R}_T} \sum_{j=1}^{K} \mu_j R_j \]

\[ = \max_{\alpha_{ij} \geq 0, \sum_{j=1}^{K} \mu_j (\sum_{i=1}^{M} \alpha_{ij} (1 - \epsilon_{ij}))} \sum_{j=1}^{K} \mu_j R_j \]

\[ = \max_{\alpha_{ij} \geq 0, \sum_{j=1}^{K} \mu_j (\sum_{i=1}^{M} \alpha_{ij} (1 - \epsilon_{ij}))} \sum_{i=1}^{M} \mu_j (1 - \epsilon_{ij}) \]

If for all \( j, k \in [K] \) such that \( \mu_j \leq \mu_k \), \( \epsilon_{ij} \leq \epsilon_{ik} \) for all \( i \in [M] \), then this channel is physically degraded broadcast channel (not only for each subchannel), and thus feedback does not change the capacity region. Therefore, we can assume that there is no feedback, and all erasure events to be independent while maintaining the same marginal distribution. These assumptions do not change the capacity region. Then applying Theorem 3 of [2] on the capacity region of multiple input broadcast erasure channels without feedback, we conclude that time sharing scheme achieves the capacity region.

Otherwise, there exist \( i_0 \in [M], j_0, k_0 \in [K] \) such that \( \mu_{j_0} \leq \mu_{k_0} \) and \( \epsilon_{i_0 j_0} > \epsilon_{i_0 k_0} \). Now we construct a better multiple input PECs with feedback than the current one. The new channel is the same as the current one, except in the \( i_0 \)th subchannel, we replace the link between \( s \) and \( t_{j_0} \) by the link between \( s \) and \( t_{k_0} \). Let \( \epsilon'_{ij} \) denote the packet erasure probability between \( s \) and \( t_{i} \) in the \( i \)th subchannel of the new channel. Then \( \epsilon'_{ij} = \epsilon_{ij} \) for \( (i, j) \neq (i_0, j_0) \), and \( \epsilon'_{i_0 j_0} = \epsilon_{i_0 k_0} \). Note that each subchannel is still physically degraded broadcast channel. Denote the capacity region of this new channel by \( C_1 \). Then, \( C \subset C_1 \), since we only enhance a link in the original channel. Let \( \mathcal{R}_T^1 \) denote time sharing rate region of this new channel. Then we have \( \phi_{\mathcal{R}_T}(\mu) = \phi_{\mathcal{R}_T^1}(\mu) \).

Indeed, for \( i \neq i_0 \),

\[ \max_{j \in [K]} \mu_j (1 - \epsilon'_{ij}) = \max_{j \in [K]} \mu_j (1 - \epsilon_{ij}) \]

and for \( i = i_0 \),

\[ \max_{j \in [K]} \mu_j (1 - \epsilon'_{i_0 j}) = \max_{j \neq j_0} \mu_j (1 - \epsilon_{i_0 j}) = \max_{j \neq j_0} \mu_j (1 - \epsilon_{i_0 j}) \]

(4)
where (4) holds since
\[ \phi_{b,j_0}(1 - \epsilon_{i_k,j_0}) = \mu_{b,j_0}(1 - \epsilon_{i_k,j_0}) = \mu_{b,j_0}(1 - \epsilon_{i_k,j_0}), \]
and (5) holds since
\[ \mu_{b,j_0}(1 - \epsilon_{i_k,j_0}) \leq \mu_{b,k_0}(1 - \epsilon_{i_k,j_0}). \]
Now we have constructed a better multiple input PECs with feedback, where each subchannel is still physically degraded and \( \phi_{T,R}^+(\mu) \) is the same as \( \phi_{R^+_T}(\mu) \).

We can continue this process until we arrive at a multiple input PECs with feedback where for all \( j, k \in [K] \) such that \( \mu_j \leq \mu_k, \epsilon_{i,j} \leq \epsilon_{i,k} \) for all \( i \in [M] \) (we used \( \epsilon^* \) to denote the packet erasure probabilities). Therefore, this multiple input broadcast PECs with feedback is physically degraded, and thus the capacity region of this channel is the same as the time sharing rate region. Let \( \mathcal{C}^* \) denote the capacity region of this physically degraded channel, and let \( \mathcal{R}^+_T \) denote the rate region achieved by time sharing. Then \( \mathcal{C}^* = \mathcal{R}^+_T \). Furthermore, due to the way we construct the new channel, \( \phi_{T,R}^+(\mu) = \phi_{R^+_T}(\mu) \).

Therefore, \( \phi_{T,R}^+(\mu) = \phi_{R^+_T}(\mu) = \phi_{R_T}(\mu) \). On the other hand, since \( \mathcal{R}_T \subset \mathcal{C} \), \( \phi_{R_T}(\mu) \geq \phi_{R_T}(\mu) \). Therefore, we conclude that
\[ \phi_{R_T}(\mu) = \phi_{R_T}(\mu), \]
for any nonnegative tuples \( \mu = (\mu_1, \mu_2, \ldots, \mu_K) \in \mathbb{R}^K \).

For general \( \mu = (\mu_1, \mu_2, \ldots, \mu_K) \in \mathbb{R}^K, \mu_i \) can be negative for some \( i \in [K] \). Define \( \tilde{\mu}_i \triangleq \max(\mu_i, 0) \),
\[ \tilde{\mu}_i \triangleq \max(\mu_i, 0), \]
for \( i \in [K] \).

We claim that \( \phi_{R_T}(\mu) = \phi_{R_T}(\tilde{\mu}) \).

Let \( I \triangleq \{ i \in [K] \mid \mu_i < 0 \} \). Let
\[ \mathbb{R}^* \triangleq (R_1^*, R_2^*, \ldots, R_K^*) = \arg \max_{\mathcal{R} \subseteq \mathcal{C}} \sum_{i=1}^{K} \mu_i R_i. \]
Since transmission rates cannot be negative, we have \( R_i^* \geq 0, \forall i \in [K] \). In addition, it is easy to see that for \( i \in I, R_i^* = 0 \), since otherwise we can get a strictly better solution by setting \( R_i^* = 0 \) for all \( i \in I \). Note that \( \phi_{R_T}(\tilde{\mu}) \geq \sum_{i=1}^{K} \tilde{\mu}_i R_i^* = \sum_{i \in [K] \setminus I} \mu_i R_i^* \). We claim that
\[ \phi_{R_T}(\tilde{\mu}) = \sum_{i \in [K] \setminus I} \mu_i R_i^*. \]

Suppose \( \phi_{R_T}(\tilde{\mu}) = \sum_{i=1}^{K} \tilde{\mu}_i \tilde{R}_i, \) i.e., \( \tilde{R} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_K) \) is the solution to \( \max_{\mathcal{R} \subseteq \mathcal{C}} \sum_{i=1}^{K} \tilde{\mu}_i \tilde{R}_i \) and \( \phi_{R_T}(\tilde{\mu}) > \sum_{i \in [K] \setminus I} \mu_i R_i^* \). We can assume that \( \tilde{R}_i = 0, \) for \( i \in I \), since \( \tilde{\mu}_i = 0 \). Then
\[
\sum_{i=1}^{K} \mu_i \tilde{R}_i = \sum_{i \in [K] \setminus I} \mu_i \tilde{R}_i = \sum_{i \in [K] \setminus I} \tilde{\mu}_i \tilde{R}_i = \sum_{i \in [K] \setminus I} \mu_i R_i^* = \phi_{R_T}(\mu). \]

Hence, \( \phi_{R_T}(\mu) < \sum_{i=1}^{K} \mu_i \tilde{R}_i, \) which contradicts the fact that
\[ \phi_{R_T}(\mu) = \max_{\mathcal{R} \subseteq \mathcal{C}} \sum_{i=1}^{K} \mu_i R_i. \]
Therefore, we conclude that \( \phi_{R_T}(\mu) = \phi_{R_T}^+(\mu) \).

Similarly, we can prove that \( \phi_{R_T}(\mu) = \phi_{R_T}^+(\mu) \).

Therefore, we have \( \phi_{R_T}(\mu) = \phi_{R_T}^+(\tilde{\mu}) = \phi_{C}(\tilde{\mu}) = \phi_{C}(\mu) \).

Hence for all \( \mu \in \mathbb{R}^K, \phi_{R_T}(\mu) = \phi_{C}(\mu) \).

We have proved that the supporting functionals of the closed and bounded convex sets \( \mathcal{C} \) and \( \mathcal{R}_T \) are the same. Due to Lemma 7, \( \mathcal{C} = \mathcal{R}_T \), i.e., the capacity region is the same as the rate region achieved by time sharing. This completes the proof of Lemma 2.

\section*{Acknowledgment}

This work was partially performed while Hieu Do was visiting the Coordinated Science Laboratory at UIUC in spring 2012.

The authors would like to thank Prof. Pramod Viswanath and Dr. Sreearun Kannan for the helpful discussions.

Research of Quan Geng was supported in part by Prof. Pramod Viswanath’s National Science Foundation grant No. CCF-1017430. Research of Hieu Do was supported in part by grants from Swedish Research Council and Ericsson Research Foundation.

\section*{References}

[1] A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, “Capacity of wireless erasure networks,” \textit{IEEE Transactions on Information Theory}, vol. 52, pp. 789–804, 2006.

[2] A. Dana and B. Hassibi, “The capacity region of multiple input erasure broadcast channels,” in \textit{Proceedings of IEEE International Symposium on Information Theory}, 2005, pp. 2315–2319.

[3] L. Georgiadis and L. Tassiulas, “Broadcast erasure channel with feedback - Capacity and algorithms,” in \textit{Workshop on Network Coding, Theory and Applications}, 2000.

[4] M. Gatzianas, L. Georgiadis, and L. Tassiulas, “Multiuser broadcast erasure channel with feedback - Capacity and algorithms,” \textit{IEEE Transactions on Information Theory}, vol. 59, no. 9, pp. 5779–5804, 2013.

[5] C.-C. Wang, “On the capacity of 1-to-k broadcast packet erasure channels with channel output feedback,” \textit{IEEE Transactions on Information Theory}, vol. 58, no. 2, pp. 931–956, February 2012.

[6] C.-C. Wang and D. Love, “Linear network coding capacity region of 2-receiver MIMO broadcast packet erasure channels with feedback,” in \textit{Proceedings of IEEE International Symposium on Information Theory}, July 2012, pp. 2062–2066.

[7] R. Gummadi, L. Massoulie, and R. Sreenivas, “The Role of Feedback in the Choice between Routing and Coding for Wireless Unicast,” in \textit{Workshop on Network Coding, Theory and Applications}, 2010.

[8] S. Kannan, A. Raja, and P. Voswanath, “Local phy + global flow: A layering principle for wireless networks,” in \textit{Proceedings of IEEE International Symposium on Information Theory}, 2011, pp. 1633–1637.

[9] A. El Gamal, “The feedback capacity of degraded broadcast channels (corresp.),” \textit{IEEE Transactions on Information Theory}, vol. 24, no. 3, pp. 379–381, 1978.

[10] J. Borwein and A. Lewis, \textit{Convex Analysis and Nonlinear Optimization: Theory and Examples}. Springer, 2006, vol. 3.