TOWARDS THE KPP–PROBLEM AND log t–FRONT SHIFT FOR HIGHER-ORDER NONLINEAR PDES III.
DISPERSION AND HYPERBOLIC EQUATIONS

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Abstract. Some aspects of extensions of ideas of Kolmogorov, Petrovskii, and Piskunov (1937) on travelling wave propagation in the reaction-diffusion equation

\[ u_t = u_{xx} + u(1-u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad u_0(x) = H(-x) \equiv \{1 \text{ for } x < 0; \ 0 \text{ for } x \geq 0\}, \]

are discussed. The present paper continues the study began in [9, 10] for higher-order parabolic semilinear and quasilinear bi-harmonic equations such as

\[ u_t = -u_{xxxx} + u(1-u), \quad u_t = -(|u|^n u)_{xxxx} + u(1-u) \quad (n > 0), \quad \text{etc.} \]

Here, higher-order dispersion equations such as \((D_x = \frac{\partial}{\partial x} \text{ and } D_t = \frac{\partial}{\partial t})\)

\( u_t = -D_x^{11} u + u(1-u) \quad \text{and up to} \quad D_x^9 u = -D_x^{11} u + u(1-u), \quad \text{etc.} \)

are studied. Some features of KPP-like results are also shown to exist for semilinear dispersion-parabolic equations such as

\[ u_{ttt} = -D_x^{10} u + u(1-u), \quad u_{tttt} = D_x^{10} u + u(1-u), \quad \text{and others,} \]

and for pure hyperbolic ones

\[ u_{tt} = -u_{xxxx} + u(1-u) \quad \text{and up to} \quad u_{tttt} = -D_x^{10} u + u(1-u), \quad \text{etc.} \]

As an example, we also treat a quasilinear PDE \(u_t = -D_x^{11} (|u|^n u) + u(1-u), \) with \(n > 0.\)

Two main questions are: (i) existence of travelling waves via any analytical/numerical methods, and (ii) their stability and derivation of the log \(t\)-shifting of moving fronts.

1. Introduction: The classic KPP-problem and other higher-order PDE models

1.1. The classic KPP-problem of 1937. The classic KPP-problem [13] (1937)

\[ u_t = u_{xx} + u(1-u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}, \]

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with the step (Heaviside) initial function

\[ u_0(x) = H(-x) \equiv \begin{cases} 1, & x < 0; \\ 0, & x \geq 0, \end{cases} \]

consists of studying large-time convergence of the solution of the Cauchy problem (1.1), (1.2) to the unique minimal travelling wave solution, with the minimal speed \( \lambda_0 = 2 \) of the standard form

\[
\begin{align*}
\{ & u_*(x,t) = f(y), \ y = x - \lambda_0 t, \text{ where} \\
& -\lambda_0 f' = f'' + f(1 - f), \ y \in \mathbb{R}; \ f(-\infty) = 1, \ f(+\infty) = 0.
\end{align*}
\]

The KPP paper [13] contains a number of pioneering remarkable results, which founded several new directions of modern nonlinear PDE theory. For further use, it suffices for us to refer to a survey and key references in [9].

Let us state the main result of [13]. The convergence to the minimal TW (1.3) was performed in the TW moving frame, proving that the TW front moves like

\[ x_f(t) = 2t - g(t) \quad \text{as} \quad t \to \infty, \quad \text{with} \quad g(t) = o(t), \]

where the front location \( x_f(t) \) is uniquely determined from the equation

\[ u(x_f(t),t) = \frac{1}{2} \quad \text{for all} \quad t \geq 0. \]

Then the convergence result of [13] takes the form:

\[ u(x_f(t) + y,t) \to f(y) \quad \text{as} \quad t \to +\infty \quad \text{uniformly in} \quad y \in \mathbb{R}. \]

In 1983, Bramson [1], using probabilistic techniques, proved that there exists \textit{unbounded} \( \log t \)-shift of the moving TW front

\[ g(t) = k \log t(1 + o(1)), \quad \text{with} \quad k = \frac{3}{2}. \]

Therefore, (1.7) implies eventual, as \( t \to +\infty, \) \textit{infinite} retarding of the solution \( u(x,t) \) from the corresponding minimal TW, thought the convergence (1.6) takes place in the TW frame.

1.2. \textbf{KPP-like problem to higher-order semilinear and quasilinear parabolic PDEs.} We dealt with such semilinear reaction-diffusion PDEs in [9], where the main basic model was the \textit{semilinear bi-harmonic equation}, i.e., a fourth-order semilinear heat equation (SHE–4)

\[ u_t = -u_{xxxxx} + u(1-u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+. \]

The corresponding TW with the speed of propagation \( \lambda \) is then governed by the following fourth-order ODE:

\[ u_*(x,t) = f(y), \ y = x - \lambda t \implies -\lambda f' = -f^{(4)} + f(1 - f), \]

with the singular boundary conditions at infinity:

\[ f(y) \to 0 \quad \text{and} \quad f(y) \to 1 \quad \text{as} \quad y \to \pm\infty \quad \text{“maximally” exponentially fast}. \]
In particular, we have found a “maximal speed \( \lambda_{\text{max}} = 1.27148 \) ... such that

\[(1.11) \quad \text{TW profiles } f(y; \lambda) \text{ exist for all } 0 < \lambda < \lambda_{\text{max}}, \text{ and nonexistent for } \lambda > \lambda_{\text{max}}.\]

In [10], we extend some of the above results to the quasilinear KPP–4n problem for

\[(1.12) \quad u_t = -(|u|^n u)_{xxxx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}+,\]

where \( n > 0 \) is a parameter, as well as to some other parabolic equations.

In a similar manner, we studied in [9] the semilinear tri-harmonic equation (SHE-6):

\[(1.13) \quad u_t = u_{xxxx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}+, \quad u(x, 0) = H(-x) \quad \text{in } \mathbb{R}.\]

The corresponding TW with the speed of propagation \( \lambda \) is then governed by the following fourth-order ODE:

\[(1.14) \quad u^*(x, t) = f(y), \quad y = x - \lambda t \quad \implies \quad -\lambda f' = f^{(6)} + f(1 - f), \quad \text{with (1.10)}.\]

The (1.11) remains true, with \( \lambda_{\text{max}} = 2.12110 \ldots \)

In general, in [9], we presented some numerical evidence on existence of various TWs and there properties for semilinear parabolic 2mth-order PDEs (SHE-2m) such as (here, \( D_x = \frac{d}{dx} \))

\[(1.15) \quad u_t = (-1)^{m+1}D_x^{2m}u + u(1 - u) \quad (D_x = \frac{d}{dt}), \quad \text{with the ODEs}\]

\[(1.16) \quad u^*(x, t) = f(y), \quad y = x - \lambda t \quad \implies \quad -\lambda f' = (-1)^{m+1}f^{(2m)} + f(1 - f)\]

(plus (1.10)), and rather sharply estimated \( \lambda_{\text{max}} = \lambda_{\text{max}}(m) > 0 \) for \( m = 3, 4, 5, \) i.e., up to the tenth-order parabolic equation as in (1.15).

1.3. Results I: dispersion PDEs (Section 2). In the present paper, we will deal with higher-order dispersion\(^1\) equations

\[(1.17) \quad u_t = -D_x^{11}u + u(1 - u) \quad \implies \quad -\lambda f'' = -f^{(11)} + f(1 - f);\]

\[(1.18) \quad u_{ttt} = -D_x^{11}u + u(1 - u) \quad \implies \quad -\lambda f''' = -f^{(11)} + f(1 - f);\]

\[(1.19) \quad u_{tttt} = -D_x^{11}u + u(1 - u) \quad \implies \quad -\lambda f^{(5)} = -f^{(11)} + f(1 - f);\]

\[(1.20) \quad D_t^{7}u = -D_x^{11}u + u(1 - u) \quad \implies \quad -\lambda^7 f^{(7)} = -f^{(11)} + f(1 - f);\]

\(^1\)Here, we use a PDE classification, associated with some a priori bounds admitted by the principal linear differential operators by multiplying by appropriate time derivatives \( D_t^{k} u \) in the \( L^2 \)-metric. Therefore, while, for most of parabolic and hyperbolic equations, such a classification coincides with the classic rigorous Petrovskii’s one [10] (recall: “parabolic in Petrovskii’s sense”), for others, our classification can be different and uses a “more applied” understanding of dispersion and related phenomena. Anyway, we do not think that, for truly higher-order (in both \( x \) and \( t \) up to 11th or 12th orders) semilinear PDEs, any formal classification may somehow essentially help to understand the nature of TW patterns obtained below (note that such “evolution” patterns have been obtained even for obviously elliptic PDEs, for which a standard evolution interpretation makes no sense, due to Hadamard’s example of an ill-posed Cauchy problem).
plus singular boundary conditions (1.10). We have chosen rather higher-order dispersion operator \( D_{x}^{11} \) on the right-hand side in order to avoid a kind of “temptation” to rely on linearized analysis, which was heavily used in [9, § 2] in studying the lower-order bi-harmonic KPP-problem for (1.8).

1.4. Results II: dispersion-hyperbolic PDEs (Section 3). We next consider existence of TW solutions of higher-order dispersion-hyperbolic equations

\[
(1.22) \quad u_{tt} = -D_{x}^{11}u + u(1 - u) \quad \implies \quad \lambda^2 f'' = -f^{(11)} + f(1 - f);
\]

\[
(1.23) \quad u_{ttt} = -D_{x}^{11}u + u(1 - u) \quad \implies \quad \lambda^4 f^{(4)} = -f^{(11)} + f(1 - f);
\]

\[
(1.24) \quad u_{tttt} = -D_{x}^{11}u + u(1 - u) \quad \implies \quad \lambda^6 f^{(6)} = -f^{(11)} + f(1 - f);
\]

\[
(1.25) \quad D_{x}^{8}u = -D_{x}^{11}u + u(1 - u) \quad \implies \quad \lambda^8 f^{(8)} = -f^{(11)} + f(1 - f);
\]

\[
(1.26) \quad D_{x}^{10}u = -D_{x}^{11}u + u(1 - u) \quad \implies \quad \lambda^{10} f^{(10)} = -f^{(11)} + f(1 - f);
\]

plus singular boundary conditions (1.10).

1.5. Results III: dispersion-parabolic PDEs (Section 4). We next consider, in the KPP setting, the following equations, which, in view of certain estimates, can be considered as dispersion-parabolic (the signs in front of \( D_{x}^{10} \) avoids “backward parabolic” features) models

\[
(1.27) \quad u_{ttt} = -D_{x}^{10}u + u(1 - u) \quad \implies \quad -\lambda^3 f''' = -f^{(10)} + f(1 - f);
\]

\[
(1.28) \quad u_{ttttt} = D_{x}^{10}u + u(1 - u) \quad \implies \quad -\lambda^5 f^{(5)} = f^{(10)} + f(1 - f);
\]

\[
(1.29) \quad D_{x}^{7}u = -D_{x}^{10}u + u(1 - u) \quad \implies \quad -\lambda^7 f^{(7)} = -f^{(10)} + f(1 - f);
\]

\[
(1.30) \quad D_{x}^{9}u = D_{x}^{10}u + u(1 - u) \quad \implies \quad -\lambda^9 f^{(9)} = f^{(10)} + f(1 - f);
\]

with the conditions (1.10).

1.6. Results IV: higher-order hyperbolic PDEs (Section 5). Finally, we consider four purely hyperbolic higher-order equations:

\[
(1.31) \quad u_{tt} = D_{x}^{10}u + u(1 - u) \quad \implies \quad \lambda^2 f'' = f^{(10)} + f(1 - f);
\]

\[
(1.32) \quad u_{ttt} = -D_{x}^{10}u + u(1 - u) \quad \implies \quad \lambda^4 f^{(4)} = -f^{(10)} + f(1 - f);
\]

\[
(1.33) \quad D_{x}^{6}u = D_{x}^{10}u + u(1 - u) \quad \implies \quad \lambda^6 f^{(6)} = f^{(10)} + f(1 - f);
\]

\[
(1.34) \quad D_{x}^{8}u = -D_{x}^{10}u + u(1 - u) \quad \implies \quad \lambda^8 f^{(8)} = -f^{(10)} + f(1 - f);
\]
with the conditions (1.10). We also present some TW patterns for the corresponding elliptic PDEs such as

\[(1.35) \quad u_{tt} = -D_x^{10} u + u(1 - u) \quad \Rightarrow \quad \lambda^2 f'' = -f^{(10)} + f(1 - f);\]

\[(1.36) \quad u_{tttt} = D_x^{10} u + u(1 - u) \quad \Rightarrow \quad \lambda^4 f^{(4)} = f^{(10)} + f(1 - f);\]

\[(1.37) \quad D_t^6 u = -D_x^{10} u + u(1 - u) \quad \Rightarrow \quad \lambda^6 f^{(6)} = -f^{(10)} + f(1 - f);\]

Clearly, as evolution equations, these lead to unstable (and ill-posed, in Hadamard’s sense, 1906) problems, but as certain special TW patterns, such solutions make sense, though are highly oscillatory, as one can expect.

1.7. Results V: KPP–(10,11) and KPP–(11,12) (Section 6). These are most exotic KPP–models under consideration. As an example, we consider two of such PDEs, where the order in \(t\) exceeds the order in \(x\):

\[(1.38) \quad D_t^{11} u = -D_x^{10} u + u(1 - u) \quad \Rightarrow \quad -\lambda^{11} f^{(11)} = -f^{(10)} + f(1 - f),\]

\[(1.39) \quad D_t^{12} u = -D_x^{11} u + u(1 - u) \quad \Rightarrow \quad \lambda^{12} f^{(12)} = -f^{(11)} + f(1 - f),\]

with conditions (1.10).

1.8. Results VI: semilinear eleventh-order in time PDEs (Section 7). We end up with the following equations, which were not treated above and are induced by (1.38):

\[(1.40) \quad D_t^{11} u = -u_x + u(1 - u) \quad \Rightarrow \quad -\lambda^{11} f^{(11)} = -f' + f(1 - f),\]

\[(1.41) \quad D_t^{11} u = -u_{xx} + u(1 - u) \quad \Rightarrow \quad -\lambda^{11} f^{(11)} = -f'' + f(1 - f),\]

\[(1.42) \quad D_t^{11} u = -u_{xxx} + u(1 - u) \quad \Rightarrow \quad -\lambda^{11} f^{(11)} = -f''' + f(1 - f),\]

\[(1.43) \quad D_t^{11} u = -u_{xxxx} + u(1 - u) \quad \Rightarrow \quad -\lambda^{11} f^{(11)} = -f^{(4)} + f(1 - f).\]

Note that, according to a priori bounds, the models (1.41) and (1.43) belong to the parabolic type, while (1.40) and (1.43) can be treated as dispersion ones.

1.9. Results VII: a quasilinear dispersion KPP–(11,1) problem (Section 8). As in [10] for parabolic KPP–problems, we claim that many present results admit extensions to quasilinear PDEs. To this end, we treat the following quasilinear KPP–(11,1) (cf. (1.17)) problem

\[(1.44) \quad u_t = -D_x^{11}(|u|^n u) + u(1 - u) \quad \Rightarrow \quad -\lambda f' = -(|f|^n f)^{(11)} + f(1 - f),\]

where \(n > 0\) is a fixed parameter. The main distinguished feature of degenerate equations as in (1.44) is that the TW profiles \(f(y)\) have finite interface at some finite \(y_0 > 0\), so we need first to explain the local (periodic) structure of such solutions as \(y \to y_0\).
1.10. **The origin of \( \log t \)-front shift and general goals of the paper.** In Section 9 using a general third-order in \( t \) semilinear model, we show, via a kind of an “affine centre subspace expansion”, how \( \log t \)-front shift can appear for \( t \gg 1 \) in convergence to a TW pattern.

Concerning our general classification, using the KPP-setting, we refer to such problems as to the KPP–\((k,l)\), where \( k \) stands for the order of the differential operator in \( x \) and \( l \) for the order of the derivative in \( t \).

Thus, for several KPP–\((k,l)\) problems, with \( k \geq 3 \) and \( l \geq 1 \), the main questions to study, here, as in [9, 10], are:

(I) **The problem of TW existence:** existence of travelling waves by using analytical/numerical methods,

(II) **The problem of “maximal” speed:** defining, in a natural sense, the \( \omega \)-limit set \( \omega(H) \) of the properly shifter orbit (1.6), to discuss whether or not, at least, in some particular KPP–problems,

\[
\omega(H) = \{ f(\cdot; \lambda) : \lambda \in \Lambda \}, \quad \text{where } \Lambda \subset \mathbb{R} \text{ is bounded.}
\]

Two main questions arise:

\[
\text{(1.45)} \quad \text{what is the maximal speed } \lambda_{\text{max}} = \sup \Lambda, \quad \text{and whether } \Lambda = \{\lambda_{\text{max}}\}?
\]

Those questions are “remnants” of the KPP setting. It turns out that, unlike in [13], for several parabolic [9, 10] and dispersion models

\[
\text{(1.46)} \quad \Lambda = (0, \lambda_{\text{max}}), \quad \text{so that } \lambda_{\text{max}} = \sup \Lambda, \quad \lambda_{\text{max}} \not\in \Lambda, \quad \text{and } \lambda_{\min} = 0 \not\in \Lambda.
\]

However, for some PDEs under consideration, there exist stationary patterns corresponding to \( \lambda = 0 \). Therefore, for such a variety of semilinear PDEs of different orders, any clear general conclusions on the above speed sets are illusive, and each problem might have some individual features.

(III) **The log \( t \)-shift problem:** studying the stability of the TW \( f(y; \lambda_0) \) and derivation of the log \( t \)-shifting of the moving front in the problem (1.1), (1.2), connected with a kind of an “affine centre subspace behaviour” for the rescaled equation (Section 9).

Overall, as in [9 [10], we expect that the log \( t \)-shifting phenomenon is quite a generic property of many nonlinear KPP-type problems regardless their particular types.

Note that, here, we are not able to solve or even discuss one of the main problem about the TW velocity \( \lambda_0 \) (or velocities? – the \( \omega \)-limit set of the rescaled orbit might include a connected continuous curve of profiles \( \{ f(\cdot, \lambda), \lambda \in \hat{\Delta} \} \), which appear in the PDE setting with the Heaviside initial data (1.2) (cf. (II) above). Numerically, this would require a full set of hard PDE numerical experiments, which we do not perform in such a generality. The author believes that using such a full-scale of both the ODE and PDE numerics is too exhaustive and inevitable moves the research into a pure numerical area, where some important mathematical aspects of the KPP–ideology, which comprise the
main goal of the present study, could be lost would be essentially reduced to numerical aspects to appear for sure.

2. Dispersion equations of various orders in $t$

As in [9] for PDEs (1.8), (1.13), and (1.15) ($m = 4, 5$), for new classes of dispersion PDEs (1.17)–(1.21), it is convenient to present numerical results, which directly show the global structure of such TW profiles to be, at least partially, justified analytically. A more detailed description of using the *bv4p4c* solver of the *MatLab* is given in [9 § 2]. However, it is important to note that, as the initial data for further iterations, we always took the Heaviside function as in (1.2). This once more had to help us to converge to a proper “minimal” profile (indeed, there are many other TW profiles), though, of course this was not guaranteed a priori. We keep this rule for all other KPP–$(k, l)$ problems of interest, including the hyperbolic ones (1.31)–(1.34) and more higher-order ones, where initial velocity, acceleration, etc. were taken zero.

Thus, numerically, we observe existence of TWs for all the PDEs (1.17)–(1.21). In Figure 1, we show the TW profiles for the first-order (in $t$) dispersion equation (1.17) for the speeds $\lambda = -1, 0.5, 1$. In addition, a stationary solution (denoted by the dash line in Figure 1) exists for $\lambda = 0$, hence, satisfying the ODE (to be referred to later on a few times)

$$f^{(11)} = f(1 - f) \quad \text{in} \quad \mathbb{R}, \quad \text{plus} \quad (1.10).$$

Existence (but not uniqueness) of a solution to (2.1) (and of ones of arbitrary odd order $f^{(2m-1)}$, with any $m \geq 2$) is well known for a long time; see [2, 14]. Uniqueness of such stationary solutions of equations like (2.1) of various odd orders was observed numerically; see [7], where further related references were available. For convenience and for future use, in Figure 3 we present the solution of this stationary ($\lambda = 0$) problem (2.1).

Figure 2 shows a quite oscillatory TW profile for $\lambda = -3$.

Further numerical experiments confirm the following $\lambda$-range of existence of such TW profiles for (1.17) (cf. (1.11)):

$$1.2 \leq \lambda_{\text{max}} < 1.3.$$

The TW profile for the existence parameter $\lambda = 1.2$ is shown in Figure 4. Numerical integration is performed on a smaller interval $[-60, 60]$, that, in view of small oscillatory behaviour of $f(y)$ in Figure 4 cannot cause any problem (though the length of the interval might affect this critical speed value). Moreover, it is seen that, as $\lambda \rightarrow \lambda_{\text{max}}$, the profile $f(y)$ tends to be non-oscillatory at all, so that this $\lambda_{\text{max}}$ characterizes the case when the characteristic equation of the linearized operator admits complex roots with vanishing

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2In other words, the author’s position here as follows: (ODEs) numerics, indeed, helps to understand various features/aspects of modern nonlinear higher-order PDE theory, but should be used carefully and balanced, i.e., not shadowing (and not replacing!) any mathematical ideas/methods/tools/ideology/etc., which could appear from and connected with treating reliable numerical results (though the above is more relevant to [9] than to the present paper).
Figure 1. A TW profile $f(y)$ satisfying the ODE in (1.17) for $\lambda = -1$.

Figure 2. An oscillatory TW profile $f(y)$ satisfying the ODE in (1.17) and (1.10) for $\lambda = -3$.

imaginary part. Then this changes the dimension of the asymptotic bundles and makes matching impossible; cf. Theorem 2.1 in [9, § 2].

For equations (1.18)–(1.21), TW profiles, with various negative and positive $\lambda$’s, are presented in Figure 5. In all the cases, there exists also the stationary profile for $\lambda = 0$ satisfying (2.1).
3. Dispersion-hyperbolic PDEs

For equations (1.22)–(1.25), TW profiles, with various speeds $\lambda$'s, are presented in Figure 6. In the last case (c), i.e., in the KPP–(11,8), numerics reveal existence of $\lambda_{\text{max}}$. 

Figure 3. The stationary TW profile $f(y)$ satisfying the ODE in (1.17) for $\lambda = 0$.

Figure 4. A TW profile $f(y)$ satisfying the ODE in (1.17) for $\lambda = 1.2 \approx \lambda_{\text{max}}$. 

Figure 5. TW profiles for dispersion equations (1.18)–(1.21).

satisfying

(3.1) \[ 1.0443 \leq \lambda_{\text{max}} < 1.0445. \]

When approaching \( \lambda_{\text{max}} \), the TW profiles \( f(y) \) remain essentially oscillatory for \( y \gg 1 \), so nonexistence for slightly \( \lambda \geq \lambda_{\text{max}} \) is not related to changing of the dimension of the linearized bundle (cf. [9, § 2.5] for the KPP–(4,1)), but means the impossibility of the corresponding “nonlinear matching” of such bundles.

For the most exotic KPP–(11,10) equation (1.26), we present TW profiles for \( \lambda = 0 \) (the stationary equilibrium), \( \lambda = 1 \), and \( \lambda = 1.1 \) in Figure 7. Even for \( \lambda \) slightly larger than 1, the profiles get very oscillatory about the equilibrium 1 as \( y \to -\infty \); cf. \( \lambda = 1.1 \) in Figure 7.
4. Further dispersion-parabolic equations

For equations (1.27)–(1.30), TW profiles, with various speeds $\lambda$'s, are presented in Figure 8. For equations (1.28) and (1.30), for $\lambda = 0$, the stationary equation

$$f^{(10)} = -f(1 - f) \quad \text{in} \quad \mathbb{R}$$

admits a solution with a periodic behaviour as $y \to +\infty$; see Figure 9.

Similarly, by symmetry, the stationary equations (1.27) and (1.29) admit analogous stationary profiles that are oscillatory about 1 as $y \to -\infty$.

5. Higher-order hyperbolic equations and elliptic patterns

5.1. Hyperbolic equations. For equations (1.31)–(1.34), TW profiles, with various speeds $\lambda$'s, are presented in Figure 10.
Figure 7. TW profiles for the KPP–(11,10) problem (1.26).

Note that, for such hyperbolic problems, the behaviour as $y \to +\infty$ often becomes non-decaying oscillatory (as a feature of hyperbolic flows), without a decay or with a very slow decaying algebraic envelope. However, as seen from some figures above, several TW profiles decay at $+\infty$, and hence satisfy the standard KPP setting.

Thus, here, we fix operators that are tenth-order in $x$ to avoid any questions on a possibility of a reliable and rigorous local and/or global ODE analysis similar to that performed in [9 § 2] for the parabolic KPP–4 problem. However, the asymptotic study of these ODEs as $y \to +\infty$ can be performed justifying that the dimensions of stable bundles as $y \to \pm\infty$ well-correspond to the existence of TW profiles in the sense of a multi-parameter shooting. Nevertheless, any rigorous proof of existence of such $f(y)$ remains hopeless and represents an open problem, as the existence of a heteroclinic path between the equilibria 0 and $\{1, 0, ..., 0\}$ in the tenth-dimensional phase space occurring for ODEs in (1.31)–(1.34).

Any stability analysis of such TWs in both hyperbolic and dispersion cases leads to very difficult spectral problems for pencils of non self-adjoint operators, which we are not going to treat here; see further “spectral” references and examples for hyperbolic/dispersion equations in [8 § 8.4].

5.2. Elliptic patterns. Though the corresponding elliptic equations (1.35)–(1.37) do not admit any evolution setting, in Figure [11] we present elliptic patterns, which are always highly oscillatory and “almost periodic” either as $y \to -\infty$ or $y \to +\infty$. 

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Figure 8. TW profiles for dispersion-parabolic equations (1.27)–(1.30).
6. KPP–(10,11) AND KPP–(11,12)

In Figure 12 we present TW profiles for the KPP–(10,11) problem (1.38) for $\lambda = 0$.2 (a) (after a proper reflection, this profile is very close to the stationary one in Figure 9 and for $\lambda = -1, -1.1$ (b).

In Figure 13 we show TW profiles for the PDE (1.39) for $\lambda = 1$ and 0.8.

7. SOME SEMILINEAR PDEs THAT ARE ELEVENTH-ORDER IN $t$

For equations (1.40)–(1.43), which all of different types, the TW profiles look rather similar and are presented in Figure 14. Note that, in all the cases, we fix negative $\lambda = -1$. Choosing positive speeds $\lambda = +1$ always led to highly oscillatory behaviour at the right-hand end point of the interval of integration and no convergence was observed.

Note also that the first equation (1.40) in this list admits obvious (and well known) explicit stationary solutions for $\lambda = 0$:

$$(7.1) \quad (1.40), \quad \lambda = 0 : \quad f' = -f(1 - f) \quad \Rightarrow \quad f(y) = \frac{e^{-y}}{1 + e^{-y}}.$$ 

Therefore, one can expect a branching of TW profiles for small $|\lambda| > 0$ from (1.40) at $\lambda = 0$. Then, if this is not a subcritical pitchfork $\lambda$-bifurcation, we expect existence $f(y)$, at least, for all small $\lambda > 0$ (thought, not extensible to $\lambda = +1$, as mentioned above).

8. QUASILINEAR DISPERSION KPP–(11,1) PROBLEM

Consider the quasilinear ODE in (1.44), with $n > 0$. First of all, we justify that the TW profiles are assumed to have finite interfaces at some finite point $y = y_0 > 0$; see extra
Figure 10. TW profiles for hyperbolic equations (1.31)–(1.34).

Details for such a functional setting in [10] (the singular conditions as $y \to -\infty$ remain principally the same as in (1.10)). It is clear that, as $y \to y_0^-$, the source term $f(1 - f)$ can be neglected, so that the asymptotics of $f(y)$ near this finite interface is described by both leading higher-order terms, so one needs to study the following asymptotic ODE:

$$\frac{d}{ds} \lambda \varphi = - (|\varphi|^n \varphi)^{(11)} \quad \Rightarrow \quad \lambda \varphi = (|\varphi|^n \varphi)^{(10)} \quad (\lambda > 0).$$

Similar to the approach in [10] for quasilinear parabolic equations, we claim that this asymptotic behaviour is given by

$$f(y) = (y_0 - y)^\gamma \varphi(s), \quad \text{where} \quad s = \ln(y_0 - y) \to -\infty \quad \text{as} \quad y \to y_0^-; \quad \gamma = \frac{10}{n},$$

where the oscillatory component $\varphi(s)$ satisfies another complicated tenth-order ODE:

$$P_{10}[|\varphi|^n \varphi] = \lambda \varphi \quad \text{in} \quad \mathbb{R}.$$
Here, the linear polynomial operator $P_{10}[\phi]$ belongs to the family $\{P_k[\phi], k \geq 0\}$ of operators that are constructed by the iteration

$$P_{k+1}[\phi] = (P_k[\phi])' + (\gamma - k)P_k[\phi] \quad \text{for} \quad k = 0, 1, \ldots, \quad P_0[\phi] = \phi.$$  

In particular, this yields:

$$P_3[\phi] = \phi''' + 3(\gamma - 1)\phi'' + (3\gamma^2 - 6\gamma + 2)\phi' + \gamma(\gamma - 1)(\gamma - 2)\phi;$$

$$P_4[\phi] = \phi^{(4)} + 2(2\gamma - 3)\phi''' + (6\gamma^2 - 18\gamma + 11)\phi'' + 2(2\gamma^3 - 9\gamma^2 + 11\gamma - 3)\phi' + \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)\phi;$$
Figure 12. Oscillatory one-sided “periodic” TW patterns for elliptic equations (1.35)–(1.37).

Figure 13. A TW profile for the KPP–(11,12), (1.39), for $\lambda = 1$ and 0.8.

\[
P_5[\phi] = 5(\gamma - 2)\phi^{(4)} + 5(2\gamma^2 - 8\gamma + 7)\phi''' + 5(\gamma - 2)(2\gamma^2 - 8\gamma + 5)\phi'' + (5\gamma^4 - 40\gamma^3 + 105\gamma^2 - 100\gamma + 24)\phi' + \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)(\gamma - 4)\phi, \quad \text{etc.}
\]

The operator $P_{10}$ in (8.3) is too ambiguous to present it here.

The next main point now is that the ODE (8.3) for $\varphi$ admits a periodic solution $\varphi_\ast(s)$, which, together with its stability set as $s \to -\infty$ (including the obvious translations in
Figure 14. TW profiles for equations (1.40)–(1.43).
s), represents the actual asymptotic bundle of all admissible solutions of the form (8.2) near the interface.

Examples of such periodic solutions $\varphi_*(s)$ for operators $P_6$ in (8.3) can be found in [11, p. 192]; see also similar examples for $P_5$ in [3]. The higher-order case of $P_9$ (the parabolic one) is given in [11, p. 143], etc. The $P_{10}$ case is no much different, though, since $\varphi_*(s)$ can be more unstable as $s \to -\infty$ (its unstability manifold $s \to -\infty$, i.e., approaching the interface, becomes even more dimensional), so numerics may get more difficult.

Examples of full global solutions of the ODE in (8.1) are presented in a number of figures below, where, for convenience and by obvious reasons, we represent the function (8.8) $F(y) = |f(y)|^n f(y) \implies F^{(11)} = \frac{n+1}{n} |F|^{-\frac{n}{n+1}} F' + |F|^{-\frac{n}{n+1}} (1 - |F|^{-\frac{n}{n+1}} F)$.

Namely, Figure 15 shows the TW profile for $\lambda = 1$ and $n = 1$. The next Figure 16 shows TW profiles $F(y)$ again for $n = 1$ for $\lambda = 0.5$ and $\lambda = 1.96$. The last value turns out to be close to the maximal value $\lambda_{\text{max}}$ and we get the estimate (8.9) $n = 1 : \quad 1.196 \leq \lambda_{\text{max}}(1) < 1.197$.

Recall that, for the semilinear case $n = 0$, $\lambda_{\text{max}}(0)$ is slightly larger; see (2.2).

TW profiles for some negative velocities $\lambda$ and $n = 1$ are presented in Figure 17. For the sake of comparison, we also indicate therein the stationary profile, with $\lambda = 0$, for $n = 0$.

In Figure 18 the TW profiles $F(y)$ correspond to a larger $n = 2$ and $\lambda = 0, 1, 2$. The last value is not that far from the maximal value: our computations show that (8.10) $n = 2 : \quad 2.25 \leq \lambda_{\text{max}}(2) < 2.26$.

For $n = 4$, Figure 19 shows the profiles $F$ for $\lambda = 0$, 0.3, and 0.5.
Figure 16. A TW profile $F(y)$ satisfying (8.8) for $\lambda = 0$, 0.5, 1.196, and $n = 1$.

Figure 17. TW profiles $F(y)$ satisfying (8.8) for $\lambda = 0$, $-0.1$, $-0.25$, and $n = 1$. 
Figure 18. TW profiles $F(y)$ satisfying (8.8) for $\lambda = 0, 1, 2$, and $n = 2$.

Figure 19. TW profiles $F(y)$ satisfying (8.8) for $\lambda = 0$ (stationary), 0.3, 0.5, and $n = 4$.

9. A log $t$-shift in the dispersion KPP–(11,3) problem: a Centre subspace pattern

Thus, we begin with a formal analysis of a kind of a “centre subspace behaviour”, which generates a necessary log $t$-shift of the wave front. We restrict to a semilinear PDE. For similar applications to quasilinear (parabolic) ones, see [10].
As a typical PDE, bearing in mind (1.18), consider a semilinear KPP-type problem for a PDE,

\[ u_{ttt} = A u + u(1 - u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \]

where \( A \) is a proper homogeneous isotropic and translational invariant linear differential operator satisfying some extra conditions specified below. For instance, we can fix the dispersion operator

\[ A = -D_{x}^{11} \quad \text{(cf. (1.17))}. \]

We assume that the corresponding ODE problem

\[ -\lambda^3 f'' = A f + f(1 - f), \]

with the conditions (1.10) admits a unique solution \( f \).

Attaching the solution \( u(x,t) \) to the front moving and setting, as usual, \( x_f(t) \equiv \lambda_0 t - g(t) \), the PDE reads

\[ u(x,t) = v(y,t), \quad y = x - \lambda_0 t + g(t). \]

Then \( v \) satisfies the following perturbed equation:

\[ v_{ttt} + 3v_{tty}(-\lambda_0 + g') + 3v_{tyy}(-\lambda_0 + g')^2 + 3v_{t}g'' = A v + v(1 - v) - 3\lambda_0 f g'' - v_y g''. \]

Thus, again, as usual, we assume that \( g'(t) \to 0 \) as \( t \to +\infty \) sufficiently fast, i.e., at least algebraically, so that

\[ |g''(t)| \ll |g'(t)| \quad \text{and} \quad |g'''(t)| \ll |g''(t)|, \quad \text{etc. for} \quad t \gg 1. \]

We next “linearize” (9.5) by setting

\[ v(y,t) = f(y) + w(y,t). \]

Then, using (9.3), yields the following perturbed equation, where, according to (9.6), we keep the leading terms only:

\[ w_{ttt} - 3\lambda_0 w_{tty} + 3\lambda_0^2 w_{tyy} + 3w_{t}g'' = B w - 3\lambda_0 f g'' - f' g''' - w^2, \]

where \( B w = A w + (1 - 2f)w + \lambda_0^3 w_{yyy} \).

Note that here we face an essentially non-autonomously perturbed flow, so we cannot use advanced semigroup theory; cf. [15]. Instead, we will apply formal asymptotic expansion techniques.

Now, assuming that, in this \( g(t) \)-moving frame, there exists the convergence as in (1.6), so that \( w(t) \to 0 \) as \( t \to +\infty \), one can see that, under the hypothesis (9.6), the leading non-autonomous perturbations in (9.8) are those of order \( O(g''(t)) \), since the rest of the terms are negligible as \( t \to +\infty \). Therefore, one needs to balance these major terms, but then the actual behaviour of \( g(t) \) for \( t \gg 1 \) (and, hence, proper \( \log t \)-shifts of the front) will depend on the next matching.
Thus, under the hypothesis (9.6), the only possible way to balance all the terms therein (including the quadratic one \(-w^2\)) for \(t \gg 1\) is to assume the following expansion:

\[
(9.9) \quad w(y, t) = g'(t)\psi(y) + \varepsilon(t)\varphi(y) \quad \text{as} \quad t \to +\infty, \quad \text{where} \quad |\varepsilon(t)| \ll |g'(t)|
\]
is still an unknown coefficient. Substituting (9.9) into (9.8) yields

\[
ge^{(4)}\psi + \varepsilon''\varphi + 3(g''\psi' + \varepsilon'\varphi'(-\lambda_0 + g')
+ 3(g''\psi' + \varepsilon'\varphi'\lambda_2 - 2\lambda_0g' + (g')^2) + 3g''\psi' + \varepsilon'\varphi')g'' + ...
\]

(9.10)

\[
ge = g'B\psi + \varepsilon B\varphi - (g')^2\psi^2 - 2\varepsilon g'\varphi\psi
+ \varepsilon^2\varphi^2 - (f'' + g'\varphi'' + \varepsilon\varphi''(-\lambda_0 + 3\lambda_0^2g' - 3\lambda_0(g')^2 + (g')^3)
- 3f'' + g'\varphi'' + \varepsilon\varphi''(-\lambda_0 + g')g'' - (f' + g'\varphi' + \varepsilon\varphi')g'' + ...,
\]

where we have omitted some obviously negligible terms.

Using (9.6) and (9.9) in balancing first the leading terms of the order \(O(g'(t))\) yields the inhomogeneous equation for \(\psi\):

\[
(9.11) \quad O(g'(t)) \quad (= O(\frac{1}{t}), \text{see below}) : \quad B\psi - 3\lambda_0^2f'' = 0.
\]

Then balancing the rest of the terms in (9.10) requires

\[
(9.12) \quad g''(t) \sim -(g'(t))^2 \sim \varepsilon(t), \quad \text{i.e.,} \quad g(t) = k\log t, \quad g'(t) = \frac{k}{t}, \quad g''(t) = -\frac{k}{t^2}, \quad \varepsilon(t) = \frac{1}{t^2}.
\]

Then, we obtain the second inhomogeneous singular Sturm–Liouville problem for \(\varphi\):

\[
(9.13) \quad O(\frac{1}{t^2}) : \quad B\varphi = k(3\lambda_0f'' - 3\lambda_0^2\psi'' - f') + k^2(\psi^2 + 3\lambda_0^2\psi'' - 3\lambda_0f'').
\]

Thus, the first simple asymptotic ODE in (9.12) gives the log \(t\)-dependence as in (1.7). Finally, we arrive at the following system for \(\{\psi, \varphi\}\):

\[
(9.14) \quad \begin{cases}
B\psi = 3\lambda_0^2f'', \\
B\varphi = k(3\lambda_0f'' - 3\lambda_0^2\psi'' - f') + k^2(\psi^2 + 3\lambda_0^2\psi'' - 3\lambda_0f'').
\end{cases}
\]

Solving this system, with typical boundary conditions as in (1.10), allows then continue the expansion of the solutions of (9.8) close to an “affine centre subspace” of \(B\) governed by the obvious (by translation) spectral pair

\[
(9.15) \quad \hat{\lambda}_0 = 0 \quad \text{and} \quad \hat{\psi}_0(y) = f'(y).
\]

The asymptotic expansion for \(t \gg 1\) then takes the form

\[
(9.16) \quad w(y, t) = \frac{k}{t}\psi(y) + \frac{1}{t^2}\varphi(y) + ...,
\]

which can be easily extended by introducing further terms, with similar inhomogeneous Sturm–Liouville problems for the expansion coefficients.

Since \(B\) does not have a discrete spectrum, we cannot derive a simple algebraic equation for \(k\) by demanding the standard orthogonality of the right-hand side in the second equation in (9.14) to the adjoint eigenvector \(\hat{\psi}_0^*\) of \(B^*\) in the \(L^2\)-metric (in which the adjoint operator \(B^*\) is obtained), like

\[
(9.17) \quad k : \quad \langle k(3\lambda_0f'' - 3\lambda_0^2\psi'' - f') + k^2(\psi^2 + 3\lambda_0^2\psi'' - 3\lambda_0f''), \hat{\psi}_0^* \rangle = 0.
\]
Therefore, the system (9.14) cannot itself determine the actual value of $k$ therein. As we have mentioned, the latter requires a difficult matching analysis in Inner and Outer Regions, which, for all present KPP–problems, remains an open problem.

One can see that the above elementary conclusion well corresponds to a “centre subspace analysis” of the non-autonomous PDE (9.8), and then $\tau = \log t$ naturally becomes the corresponding “slow” time variable; see various examples in [12] of such slow motion along centre subspaces in nonlinear parabolic problems with global and blow-up solutions. In the latter case, the slow time variable is $\tau = -\ln(T - t) \to +\infty$ as $t$ approaches finite blow-up time $T^-$. For the semilinear higher-order reaction-absorption equations such as

(9.18) \[ u_t = -u_{xxxx} - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad \text{with} \quad p > 1, \]

existence of log $t$-perturbed global asymptotics was established in [5]. For finite-time extinction, with $-1 < p < 1$ in (9.18), this was done in [6]. For the corresponding blow-up problem with the combustion source

(9.18) \[ u_t = -u_{xxxx} + |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad \text{with} \quad p > 1, \]

log($T - t$)-dependent blow-up singularities were constructed in [4]. We must admit that any justification of such log $t$-front corrections (and finding corresponding classes of initial data) in any of KPP–($k,l$) problems with $k,l > 2$ is much more difficult and remains an open problem.

10. Final conclusions

1. In all of KPP–($k,l$) problems, there exist TW solutions for various values of the speeds $\lambda$. Most of them satisfy singular boundary conditions at $\pm \infty$. However, for some types of PDEs involved, these can be oscillatory and/or periodic either as $y \to +\infty$, or $y \to -\infty$, and hence require special setting.

2. In all of the higher-order KPP–($k,l$) problems, we did not observe the classic KPP–2 phenomenon for (1.1), (1.3) [13] of existence of the minimal speed $\lambda_0 = 2$, such that TWs exist for $\lambda \geq 2$ only. It seems that this phenomenon is directly connected with the Maximum Principle (and other features of order-preserving flows), and becomes non-generic and non-existent when this fails.

3. Moreover, in several KPP–($k,l$) problems, on the contrary, we observed existence of a maximal speed $\lambda_{\text{max}}$, so that for slightly $\lambda \geq \lambda_{\text{max}}$, TW profiles do not exist. In particular, this always happens for parabolic problems, [9].

4. In all the KPP–($k,l$) problems, there exists a formal justification of existence of log $t$-drift of the propagating fronts in PDE setting along an “(affine) centre subspace” of semilinear rescaled operators involved. Then $\tau = \log t$ naturally appears as the corresponding “slow” time variable.

5. We must admit that the important PDE problem on the actual structure of the omega-limit set $\omega(H)$ of the rescaled (properly shifter) solution $u(x,t)$ of the various KPP-problems with the Heaviside data $H(-x)$ remains open. In particular, it is not still
known whether $\omega(H) = \{f(\cdot; \lambda_0)\}$ for some $\lambda_0 \in \Lambda$, i.e., whether $\omega(H)$ consists of a single TW profile. We believe that this problem deserves further study by analytical and PDE numerical methods, but expect it to be very difficult.

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