THE STACK OF HIGHER INTERNAL CATEGORIES AND STACKS OF ITERATED SPANS.

DAVID LI-BLAND

Abstract. In this paper, we show that two constructions form stacks: Firstly, as one varies the $\infty$-topos, $\mathcal{X}$, Lurie’s homotopy theory of higher categories internal to $\mathcal{X}$ varies in such a way as to form a stack over the $\infty$-category of all $\infty$-topoi.

Secondly, we show that Haugseng’s construction of the higher category of iterated spans in a given $\infty$-topos (equipped with local systems) can be used to define various stacks over that $\infty$-topos.

As a prerequisite to these results, we discuss properties which limits of $\infty$-categories inherit from the $\infty$-categories comprising the diagram. For example, Riehl and Verity have shown that possessing (co)limits of a given shape is hereditary. Extending their result somewhat, we show that possessing Kan extensions of a given type is hereditary, and more generally that the adjointability of a functor is hereditary.

Contents

1. Introduction 1
1.1. Acknowledgements 4
1.2. Notation 4
2. Properties Inherited by Limit $\infty$-Categories 5
2.1. (co)Limits in Limit $\infty$-categories 7
2.2. Adjunctions and Kan Extensions 11
3. Complete $k$-fold Segal objects. 18
3.1. The Sheaf of Complete $k$-Fold Segal Objects 22
4. $\infty$-categories of spans. 27
4.1. Continuity of the formation of $\infty$-categories of iterated spans. 27
4.2. The sheaf of iterated spans with local systems. 30
References 38

1. INTRODUCTION

Building upon the ideas of Rezk [16] and Barwick [2,5], Lurie constructed a model for the homotopy theory of higher categories internal to an $\infty$-topos $\mathcal{X}$ [14]. More precisely,
he constructs an $\infty$-category $CSS_k(\mathcal{X})$ of complete $k$-fold Segal objects in an arbitrary $\infty$-topos $\mathcal{X}$. Our first result (cf. Theorem 3.1) in this paper is to show that the construction (1.1)

$$\mathcal{X} \mapsto CSS_k(\mathcal{X})$$

satisfies a certain descent condition: suppose that $\mathcal{X}_i \to \mathcal{X}$ is an ´ etale cover of $\mathcal{X}$ indexed by a small simplicial set $i \in I$, i.e.

$$\mathcal{X} \cong \text{co lim}_{i \in I} \mathcal{X}_i,$$

then

$$CSS_k(\mathcal{X}) \cong \text{lim}_{i \in I} CSS_k(\mathcal{X}_i).$$

In other words, (1.1) defines a stack (cf. [13, Notation 6.3.5.19]).

Given an $\infty$-category $\mathcal{C}$, Barwick [4] showed how to construct an $(\infty, 1)$-category $\text{Span}(\mathcal{C})$ which has the same space of objects as $\mathcal{C}$, but whose morphisms between two objects $c_0, c_1 \in \mathcal{C}$ is the space of diagrams in $\mathcal{C}$ of the form

$$x \Downarrow c_1 \quad \Downarrow \quad c_0$$

That is, spans $c_0 \to c_1$ in $\mathcal{C}$. Composition of two such morphisms is given by taking the fibred product. Given an $\infty$-topos $\mathcal{X}$ and a complete $k$-fold Segal object $X_{\bullet, \ldots, \bullet} \in CSS_k(\mathcal{X})$, Haugseng extends this construction in [8] to produce an $(\infty, k)$-category $\text{Span}_k(\mathcal{X}, X_{\bullet, \ldots, \bullet}) \in CSS_k(\hat{S})$ of iterated $k$-fold spans with local systems valued in $X_{\bullet, \ldots, \bullet} \in CSS_k(\mathcal{X})$ (here $\hat{S}$ is the $\infty$-category of (not necessary small) spaces).

Our second main result is to show that the assignment of $\text{Span}_k(\mathcal{X}, X_{\bullet, \ldots, \bullet})$ to $X_{\bullet, \ldots, \bullet} \in CSS_k(\mathcal{X})$ depends continuously on $\mathcal{X}$ and $X_{\bullet, \ldots, \bullet}$ (i.e. it preserves small limits). This result is somewhat more subtle than it first appears: for example, the functor

$$CSS_k(\mathcal{X}) \xrightarrow{X_{\bullet, \ldots, \bullet} \mapsto \text{Span}_k(\mathcal{X}, X_{\bullet, \ldots, \bullet})} CSS_k(\hat{S})$$

is not continuous - it preserves neither products nor the terminal object. To correctly understand the continuity of $\text{Span}_k$, we need to work in a larger context: we assemble all the $\infty$-categories $CSS_k(\mathcal{X})$ into one large $\infty$-category $\int CSS_k$, whose (roughly speaking)

- objects are pairs $(\mathcal{X}, X_{\bullet, \ldots, \bullet})$, where $\mathcal{X}$ is an arbitrary $\infty$-topos and $X_{\bullet, \ldots, \bullet} \in CSS_k(\mathcal{X})$ is a higher category internal to $\mathcal{X}$, and
- morphisms $(\mathcal{X}, X_{\bullet, \ldots, \bullet}) \to (\mathcal{Y}, Y_{\bullet, \ldots, \bullet})$ consist of a geometric morphism of $\infty$-topoi, $(f^* \dashv f_*) : \mathcal{X} \leftrightarrows \mathcal{Y}$, together with a morphism $X_{\bullet, \ldots, \bullet} \to (f_*Y_{\bullet, \ldots, \bullet}$ in $CSS_k(\mathcal{X})$.

In Theorem 4.2 we prove that the functor

$$\int CSS_k (\mathcal{X}, X_{\bullet, \ldots, \bullet}) \mapsto \text{Span}_k(\mathcal{X}, X_{\bullet, \ldots, \bullet}) \to CSS_k(\hat{S})$$

is continuous (preserves small limits).

1More precisely, $\int CSS_k$ is the lax colimit of the functor $\mathcal{X} \mapsto CSS_k(\mathcal{X})$, (cf. [13]); equivalently, $\int CSS_k$ is Lurie’s unstraightening of that functor. We provide a direct construction of $\int CSS_k$, however.
Consequently, suppose that for every $U \in \mathcal{X}$ we assign (in a natural way) a complete Segal object $\sigma(U) \in CSS_k(\mathcal{X}/U)$, in a manner which depends locally on $U \in \mathcal{X}$: that is, for any colimit diagram $U_i \to U$ in $\mathcal{X}$ indexed by a small simplicial set $i \in I$,

$$\sigma(U) = \lim_{i \in I} \sigma(U_i)$$

(where the latter limit is taken in $\int CSS_k$), then

$$(1.2a) \quad \mathcal{X}^{op} \xrightarrow{U \mapsto Span_k(\mathcal{X}/U, \sigma(U))} CSS_{k}(\hat{S})$$

defines an $(\infty, k)$-stack over $\mathcal{X}$ (cf. Theorem 4.3).

As a first example, $\text{Span}_k(\mathcal{X}, X_{\bullet, \ldots, \bullet})$ itself forms a stack

$$(1.2b) \quad \mathcal{X}^{op} \xrightarrow{U \mapsto \text{Span}_k(\mathcal{X}/U, U \times X_{\bullet, \ldots, \bullet})} CSS_k(\hat{S})$$

over $\mathcal{X}$. As a second example, taking $\mathcal{X} = \text{dSt}_K$ to be derived stacks over a field $K$ of characteristic zero, and $X_{\bullet, \ldots, \bullet}$ to be trivial, the fact that $(1.2)$ forms a stack implies that the derived composition of spans depends continuously (algebraically, in fact) on the spans involved.

Our motivation for these results comes from mathematical physics: the success of the Lagrangian Creed:

“everything is a Lagrangian correspondence”

places Lagrangian correspondences between symplectic manifolds at the centre of classical mechanics. Lagrangian correspondences have two major flaws however: firstly they fail to compose in general, i.e. given two Lagrangian correspondences

$$U \xhookleftarrow{L} V \text{ and } V \xhookleftarrow{L'} W,$$

their set theoretic composite

$$(1.3) \quad U \xhookleftarrow{L \circ L'} W,$$

often fails to be smooth, and - secondly - when the composite $(1.3)$ exists as a Lagrangian correspondence, it may not depend continuously on $L$ and $L'$.

The first of these issues was essentially resolved by Pantev, Toën, Vaquié, and Vessozi [15], and Calaque [6] using derived geometry. Building upon this, Haugseng [8] then gave an embedding of Weinstein’s symplectic ‘category’ [18] whose morphisms are the Lagrangian correspondences, as a subcategory of $\text{Span}_1(\text{dSt}_K, A_{cl}^2)$, spans of derived stacks with local systems valued in closed 2-forms. The fact that $(1.2)$ is a stack is a first step towards a deeper understanding of what it means to restore the continuity of composition using derived geometry.

Moreover, the second issue - the failure of composition to be continuous - is closely related to the failure to quantize classical mechanics functorially: After quantizing pairs where the composite $(1.3)$ fails to depend continuously on $L$ and $L'$, one is typically trying to multiply Dirac $\delta$-functions in the corresponding quantization. In work in progress with

---

2Paraphrased from [19] “everything is a Lagrangian submanifold”. 
Gwilliam, Haugseng, Johnson-Freyd, Scheimbauer, and Weinstein, we show that at least to first order (i.e. after linearizing),

- the derived composition of Lagrangian correspondences \( (1.3) \) depends continuously on \( L \) and \( L' \) (cf. [10]), and
- there is a functorial quantization.

In order to show that (1.1) and (1.2) define stacks, we first need to examine limits of \( \infty \)-categories. Suppose that \( C_k, \ k \in K \) is some diagram of \( \infty \)-categories indexed by a simplicial set \( K \). In [17], Riehl and Verity show that if each \( C_k \) has all (co)limits of shape \( I \) (where \( I \) is some small simplicial set), and for each arrow \( k \to k' \) in \( K \), the corresponding functor \( C_k \to C_{k'} \) preserves all (co)limits of shape \( I \), then the limit \( \infty \)-category \( \lim_{k \in K} C_k \) also has all (co)limits of shape \( I \). After providing an alternate proof of this result (cf. Theorem 2.1), we extend their result to show that if each \( C_k \) possesses all Kan extensions along a functor \( \mathcal{I} \to \mathcal{I}' \), and each functor \( C_k \to C_{k'} \) preserves those Kan extensions, then the limit \( \infty \)-category \( \lim_{k \in K} C_k \) also has all Kan extensions along \( \mathcal{I} \to \mathcal{I}' \) (cf. Corollary 2.1).

More generally, suppose that \( F_k : C_k \rightleftarrows D_k : G_k, \ k \in K \) is diagram of adjunctions coherently indexed by a small simplicial set \( K \), then we prove there is an adjunction

\[
\lim_{k \in K} F_k : \lim_{k \in K} C_k \rightleftarrows \lim_{k \in K} D_k : \lim_{k \in K} G_k
\]

between the corresponding limit \( \infty \)-categories (cf. Theorem 2.2).

1.1. Acknowledgements. We would like to thank Owen Gwilliam, Theo Johnson-Freyd, Claudia Scheimbauer, and Alan Weinstein for many important discussions surrounding the content of this paper. We would also like to thank Thomas Nikolaus for a very helpful introduction to descent theory, and Omar Antolín Camarena for a number of helpful conversations about higher category theory. The author was supported by an NSF Postdoctoral Fellowship DMS-1204779.

1.2. Notation. We generally use the notation and terminology developed by Lurie (cf. [13]). In particular, by an \( \infty \)-category, we mean a quasicategory, i.e. a simplicial set satisfying certain horn filling conditions. In addition, we use the following notation, some of which differs from Lurie’s:

- \( \Delta \) denotes the simplicial indexing category whose objects are non empty finite totally ordered sets \( [n] := \{0, 1, ..., n\} \) and morphisms are order-preserving functions between them. \( \Delta^n : \Delta^{op} \to \text{Sets} \) is the simplicial set represented by \( [n] \).
- We denote generic \( \infty \)-categories by upper-case caligraphic letters, \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots \). We typically denote elements \( c \in \mathcal{C} \) of a generic \( \infty \)-category by lowercase versions of the same letter.
- We let \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) denote the \( \infty \)-category of functors between \( \infty \)-categories, and \( \text{Map}_{\mathcal{C}}(c, c') \) denote the mapping space between two objects \( c, c' \in \mathcal{C} \).
- If \( \mathcal{C} \) is an \( \infty \)-category, we write \( \mathcal{U} \mathcal{C} \) for the interior or classifying space of objects of \( \mathcal{C} \), i.e. the maximal Kan complex contained in \( \mathcal{C} \).
• If \( f : \mathcal{C} \to \mathcal{D} \) is left adjoint to a functor \( g : \mathcal{D} \to \mathcal{C} \), we will refer to the adjunction as \( f \dashv g \).
• \( \mathsf{Cat}_\infty \) denotes the \( \infty \)-category of small \( \infty \)-categories, and the \( \infty \)-category of spaces, \( \mathcal{S} \subset \mathsf{Cat}_\infty \), is the full subcategory spanned by the Kan complexes.
• If \( \mathcal{C} \) is an \( \infty \)-category, we let \( \yo : \mathcal{C} \to \mathcal{P}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}, \mathcal{S}) \)
denote the Yoneda embedding.

2. Properties Inherited by Limit \( \infty \)-Categories

Let \( \mathsf{Cat}_\infty \) denote the \( \infty \)-category of small \( \infty \)-categories, let \( K \) be a small simplicial set, and consider a diagram \( p' : K \to \mathsf{Cat}_\infty \). We will be interested in the limit \( \infty \)-category, \( \lim p' \in \mathsf{Cat}_\infty \).

To compute such limits, consider the functor
\[
\Delta : \mathsf{Cat}_\infty \to \mathsf{Fun}(K, \mathsf{Cat}_\infty),
\]
which sends an \( \infty \)-category \( \mathcal{C} \in \mathsf{Cat}_\infty \) to the constant diagram:
\[
\Delta \mathcal{C} : k \to \mathcal{C}, \quad \text{for any } k \in K.
\]
The right adjoint to \( \Delta \) is the functor which sends a diagram \( p' \in \mathsf{Fun}(K, \mathsf{Cat}_\infty) \) to the corresponding limit \( \infty \)-category \( \lim p' \in \mathsf{Cat}_\infty \).

Now, fix a second diagram \( p'_0 \in \mathsf{Fun}(K, \mathsf{Cat}_\infty) \) and consider the functor
\[
\Delta \times p'_0 : \mathsf{Cat}_\infty \to \mathsf{Fun}(K, \mathsf{Cat}_\infty),
\]
which sends any \( \infty \)-category \( \mathcal{C} \in \mathsf{Cat}_\infty \) to the functor
\[
\Delta \mathcal{C} \times p'_0 : k \to \mathcal{C} \times p'_0(k), \quad \text{for any } k \in K.
\]
The right adjoint to \( \Delta \times p'_0 \) sends any diagram \( p' \in \mathsf{Fun}(K, \mathsf{Cat}_\infty) \) to the \( \infty \)-category \( \mathsf{Nat}_K(p'_0, p') \in \mathsf{Cat}_\infty \) of natural transformations between \( p'_0 \) and \( p' \). Since in the special

3Following Theodore Johnson-Freyd’s suggestion, we denote the Yoneda embedding by the first Hiragana character - pronounced ‘yo’ - of his name, \( \yo \).
4Note that \( \mathsf{Fun}_K(X_0, X_1) \) is denoted by \( \mathsf{Map}_K(X_0, X_1) \) in [13], while \( \mathsf{Fun}_K^{(\text{co})\text{Cart}}(X_0, X_1) \) is denoted by \( \mathsf{Map}_K^{(\text{co})\text{Cart}}(X_0, X_1) \). We choose this alternate notation (in line with [7]) to emphasize that the resulting simplicial set is an \( \infty \)-category.
case that \( p'_0 = \Delta_* \) is the constant diagram at the terminal \( \infty \)-category, both functors (2.1b) and (2.1a) coincide, in particular, for any diagram \( p' \in \text{Fun}(K, \mathcal{C}) \), we have an equivalence

\[
\lim p' \cong \text{Nat}_K(\Delta_*, p')
\]

between the limit of \( p' \) and the \( \infty \)-category of natural transformations from the trivial diagram to \( p' \).

In practice, often the best description of diagrams in \( \mathcal{C} \) is in terms of (co)Cartesian fibrations, as developed by Lurie \cite{13, § 2.4]. Briefly, given an inner fibration between \( \infty \)-categories \( C \to D \) an edge \( f : c \to c' \) in \( C \) is called (co)Cartesian if

\[
\text{C}_{f/} \to \text{C}_{c/} \times_{\text{D}_{p(c)/}} \text{D}_{p(f)/}
\]

is an equivalence. The fibration \( C \to D \) is called (co)Cartesian if there is a (co)Cartesian edge over any edge in \( D \) starting at any vertex of \( C \). The fibration \( C \to D \) is called Cartesian if \( C^{op} \to D^{op} \) is coCartesian.

Functors \( p' : K \to \mathcal{C} \) correspond to coCartesian fibrations \( p : X \to K \) via the straightening/unstraightening construction \cite{13}; for every \( k \in K \), the fibre \( X_k := p^{-1}(k) \) is equivalent to \( p'(k) \), and for every edge \( k \to k' \) in \( K \), the corresponding functor \( X_k \to X_{k'} \) is equivalent to one sending any \( x \in X_k \) to the target of a coCartesian edge over \( k \to k' \) starting at \( x \).

Given a second coCartesian fibration \( p_0 : X_0 \to K \) corresponding to a diagram \( p'_0 : K \to \mathcal{C} \), Gepner, Haugseng, and Nikolaus identify the \( \infty \)-category of natural transformations

\[
\text{Nat}_K(p'_0, p') \cong \text{Fun}_K^{\text{coCart}}(X_0, X)
\]

with the \( \infty \)-category of coCartesian maps \( X_0 \to X \), i.e. those maps \( X_0 \to X \) over \( K \) which preserve the coCartesian edges (cf. \cite[Proposition 6.9]{7}). In particular, Lurie shows that an elegant model of \( \lim p' \) is the \( \infty \)-category

\[
\lim p' \cong \text{Nat}_K(\Delta_*, p') \cong \text{Fun}_K^{\text{coCart}}(K, X)
\]

of coCartesian sections of \( p \) \cite[Corollary 3.3.3.2]{13} (note that every edge in the trivial fibration \( K \to K \) is coCartesian).

\begin{example}
Let \( \mathcal{C} \) denote the ordinary category whose objects are (small) \( \infty \)-categories, and suppose that \( C \) is a (small) category. Given a strict functor \( p' : C \to \mathcal{C} \), the corresponding coCartesian fibration can be computed via the relative nerve construction (cf. \cite[§ 3.2.5]{13}). Forming the resulting \( \infty \)-category of coCartesian sections, one sees that a model for \( \lim p' \) is the simplicial set whose \( k \) simplices consist of the following data:

\footnote{The terminal \( \infty \)-category, \( * \), has exactly one object, one 1-morphism (the identity) and one \( n \)-morphism for every \( n \).}

\footnote{Recall that for any diagram \( q : I \to \mathcal{C} \), the undercategory \( \mathcal{C}^{q/} \) satisfies the universal property that the space of maps \( Y \to \mathcal{C}^{q/} \) classifies maps of the form \( I \to Y \) (which restrict to \( q \) along \( I \)), where

\[
I \to Y = \coprod_{I \times Y \times \{0\}} (I 	imes Y \times \Delta^1) \coprod_{I \times Y \times \{1\}} Y.
\]}

\end{example}
• for every functor \((x, y) : [n] \to C \times [k]\), a choice of \(n\)-simplex, \(\tau_{(x, y)} : \Delta^n \to p'(x(n))\), such that

1. for every \(f : [m] \to [n]\), the following diagram commutes:

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{\tau f^*(x,y)} & p'(x(f(m))) \\
| & | & | \\
\Delta^n & \xrightarrow{\tau(x,y)} & p'(x(n)) \\
\end{array}
\]

and

2. whenever \(y(i) = y(j)\), then \(\tau_{(x, y)}(\Delta^\{i,j\}) \in p'(x(n))\) is an equivalence.

2.1. (co)Limits in Limit ∞-categories. Let \(I\) be a second (small) simplicial set, and suppose that

1. for each vertex \(k \in K\), the ∞-category \(p'(k)\) admits (co)limits for all diagrams indexed by \(I\).

2. for each edge \((k \to k') \in K\), the functor \(p'(k) \to p'(k')\) preserves (co)limits for all diagrams indexed by \(I\).

then Riehl and Verity [17] have shown that the limit ∞-category \(\operatorname{lim} p' \in \mathcal{C}at_\infty\) admits (co)limits for all diagrams indexed by \(I\), and that those (co)limits are preserved by the functors in the limit cone. We now provide an alternate proof of this result, based on Lurie’s (co)Cartesian fibrations.

To disambiguate our presentation, we will prove our results only for colimits (rather than limits) in the limit ∞-category \(\operatorname{lim} p'\); the duality between colimits in \(\operatorname{lim} p'\) and limits in \((\operatorname{lim} p')^{op}\) imply that the corresponding results hold equally for limits as well.

We begin with the special case where \(I\) is the empty set, in which case we have the following variant of [13, Proposition 2.4.4.9]:

**Proposition 2.1.** Given a functor \(p' : K \to \mathcal{C}at_\infty\), if

1. for each \(k \in K\), the ∞-category \(p'(k)\) admits an initial object \(t \in X_k\), and

2. for each edge \(k \to k' \in K\) the functor \(p'(k) \to p'(k')\) preserves initial objects,

then

**A:** the limit ∞-category \(\operatorname{lim} p'\) admits an initial object \(t_\infty\), and

**B:** an object \(t \in \operatorname{lim} p'\) is initial if and only if for each \(k \in K\), the object \(\pi_k(t) \in X_k\) is initial where \(\pi_k : \operatorname{lim} p' \to p'(k)\) is the functor appearing in the limit cone.

**Proof.** We will find it easier to model our functor \(p' : K \to \mathcal{C}at_\infty\) in terms of a Cartesian fibration \(p : X \to K^{op}\) (rather than a coCartesian fibration).

Let \(p : X \to K^{op}\) be a Cartesian fibration of simplicial sets classified by the functor \(p' : K \to \mathcal{C}at_\infty\). By assumption

1. for each \(k \in K\), the ∞-category \(X_k \cong p'(k)\) admits an initial object \(t \in X_k\), and
(2') for each $p$-Cartesian edge $f : t' \to t$ over $p(f) : k' \to k$ the object $t' \in X_{k'}$ is initial whenever $t \in X_k$ is,

Let $X' \subseteq X$ be the simplical subset spanned by those vertices $t \in X$ which are initial objects of $X_{p(t)} \cong p' \circ p(t)$. Then (as we shall show), every edge $f : t' \to t$ in $X'$ is $p$-Cartesian (when seen as an edge of $X$). To see this suppose that $f : t' \to t$ is such an edge. Let $f' : t'' \to t$ be a $p$-Cartesian edge in $X$ over $p(f)$; then (cf. [13 Remark 2.4.1.4.]) there exists a 2-simplex $\sigma : \Delta^2 \to X$ such that

$$\sigma(\Delta^{[1,2]}) = f', \quad \sigma(\Delta^{[0,2]}) = f, \quad \text{and} \quad p(\sigma(\Delta^{[0,1]})) = p(s_0(t')),$$

where $s_0 : K_0 \to K_1$ is the degeneracy map.

By assumption $t'' \in X_{p(s)}$ are both initial, and hence $\sigma(\Delta^{[0,1]}) \in X_{p(t')}$ is an equivalence. In particular $\sigma(\Delta^{[0,1]}) \in X$ is a $p$-Cartesian morphism. It follows from [13 Proposition 2.4.1.7.] that $f = \sigma(\Delta^{[0,2]})$ is $p$-Cartesian.

Now, by [13 Proposition 2.4.4.9.], there exists a section $t_\infty : K^{\text{op}} \to X'$; and by the previous discussion, $t_\infty$ is a Cartesian section.

Now $\lim p' \cong \text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$ is the $\infty$-category of Cartesian sections of $p$ (cf. [13 Corollary 3.3.3.2]). Thus, we can identify $t_\infty$ with an element of $\lim p'$. We now claim that $t_\infty \in \lim p'$ is an initial object: Notice that $\lim p'$ is the full subcategory of $\text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$ spanned by the Cartesian sections. Suppose we have a diagram

$$\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{f} & \text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X) \\
\Delta^n \downarrow \quad & & \quad \downarrow \quad \\
& \text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X) & \leftarrow \text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)
\end{array}$$

such that $f|_{\{0\}} = t_\infty$. Then by [13 Proposition 2.4.4.9.], the arrow $\tilde{f}$ exists (making the diagram commute), but since $\text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$ is the full subcategory of $\text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$ spanned by the Cartesian sections, and all the vertices of $\tilde{f}$ lie in the image of $f$, and hence in $\text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$, it follows that $\tilde{f}$ factors through a map $\tilde{f}' : \Delta^n \to \text{Fun}_{K_{\text{op}}}(K^{\text{op}}, X)$.

Thus $t_\infty \in \lim p'$ is an initial object.

By construction, for every $k \in K$, $\pi_k(t_\infty) \in p'(k)$ is the image of the initial object $t_\infty(k) \in X_k$ under the equivalence of $\infty$-categories $X_k \xrightarrow{\simeq} p'(k)$. Thus (2) follows from the uniqueness of initial objects. \qed

Now, we can interpret arbitrary colimits in terms of initial objects using the concept of an undercategory, as follows: Suppose that $C \in \text{Cat}_\infty$ is an $\infty$-category and $q : I \to C$ is a diagram of shape $I$ (where $I$ is a small simplicial set). Then a colimit diagram for $q$ is equivalent to an initial object of the undercategory, $C/q$.

Suppose that $K$ is a (small) simplicial set and $p' : K^{\Delta^\circ} \to \text{Cat}_\infty$ is a diagram with cone point $C \in \text{Cat}_\infty$. For any vertex $k \in K^{\Delta^\circ}$, let $\pi_k : C \to p'(k)$ denote the corresponding functor in the cone. Given any diagram $q : I \to C$, since the formation of undercategories
is natural, there exists a diagram \((\bar{p}')^q/ : K^< \to \mathsf{Cat}_\infty\) indexing the undercategories:

\[(\bar{p}')^q/(k) \cong (\bar{p}'(k))^{q_k}_{K^<},\]

for all \(k \in K^<\), along with a natural transformation \((\bar{p}')^q/ \to \bar{p}'\) which restricts at every \(k \in K^<\) to the canonical functors \((\bar{p}'(k))^{q_k}_{K^<} \to \bar{p}'(k)\).

**Lemma 2.1.** If \(\bar{p}' : K^< \to \mathsf{Cat}_\infty\) is a limit diagram, then so is \((\bar{p}')^q/ : K^< \to \mathsf{Cat}_\infty\).

**Proof.** Let \(\bar{p} : \bar{X} \to K^<\) be a coCartesian fibration classified by \(\bar{p}'\); and let \(* \in (K^{op})^<\) denote the cone point. Since \(\{*\}^{\sharp} \subseteq (K^<)^{\sharp}\) is marked anodyne, we have a natural equivalence of the \(\infty\)-category of coCartesian sections:

\[
\text{Fun}^{\text{coCart}}_{K^<}(K^<, \bar{X}) \cong \text{Fun}^{\text{coCart}}_{K^<}(\{*\}, \bar{X}) \cong \bar{X} \times_{K^< \times \{*\}} C.
\]

Therefore, we may lift \(q : I \to C\) to a diagram

\[
\tilde{q} : I \to \text{Fun}^{\text{coCart}}_{K^<}(K^<, \bar{X})
\]

in a homotopically unique way. Choose a factorization \(I \to I' \to \text{Fun}^{\text{coCart}}_{K^<}(K^<, \bar{X})\) of \(\tilde{q}\), where \(I \to I'\) is inner anodyne (and therefore a categorical equivalence) and \(I' \to \text{Fun}^{\text{coCart}}_{K^<}(K^<, \bar{X})\) is an inner fibration (so that \(I'\) is an \(\infty\)-category). The map \(I \to I'\) is a categorical equivalence, and therefore cofinal. We are free to replace \(I\) by \(I'\), and may thereby assume that \(I\) is an \(\infty\)-category.

Given two morphisms of simplicial sets \(Y \to K^<\), and \(Z \to K^<\), recall that \(Z \circ_{K^<} Y\) denotes the relative (alternate) join of the simplicial set \(Z\) and \(Y\),

\[
Z \circ_{K^<} Y := Z \coprod_{Z \times_K Y \times \{0\}} (Z \times_K Y \times \Delta^1) \coprod_{Z \times_K Y \times \{1\}} Y
\]

(cf. [13, § 4.2.2] and [9]).

Let \(q_{K^<} : I \times K^< \to \bar{X}\) denote the composite

\[
I \times K^< \xrightarrow{q \times \text{id}_{K^<}} \text{Fun}^{\text{coCart}}_{K^<}(K^<, \bar{X}) \times K^< \xrightarrow{\text{ev}} \bar{X}.
\]

As in [13, § 4.2.2] we define \(X^{q_{K^<}/} \to K^<\) to be the simplicial set satisfying the universal property that for any morphism of simplicial sets \(Y \to K^<\), commutative diagrams of the form

\[
\begin{tikzcd}
I \times K^< \ar[hookrightarrow]{d} \ar[rightarrow]{r}[swap]{q_{K^<}} & \bar{X} \ar[rightarrow]{d} \\
(I \times K^<) \circ_{K^<} Y \ar[rightarrow]{r} & K^<
\end{tikzcd}
\]

correspond to diagrams of the form

\[
\begin{tikzcd}
Y \ar[rightarrow]{r} & X^{q_{K^<}/} \ar[rightarrow]{d} \\
& K^<
\end{tikzcd}
\]
Note that \( p \circ q_K : I \times K \rightarrow K \) is just the projection, so \( p \circ q_K \) is a Cartesian fibration; and by [13, Proposition 4.2.4.1] \( X_{q_K/} \rightarrow K \) is a coCartesian fibration classified by \( (\tilde{p}')_{q/} : (K)^{q/} \rightarrow \mathcal{C}at_\infty \). In particular, the fibre of \( X_{q_K/} \) over any \( k \in K \) may be identified with the undercategory \( (\tilde{p}')_{q/}^{k/} \) (cf. [13, § 4.2.2]).

Let \( X = \tilde{X} \times_{K^{q/}} K \), and \( q_K = q_K I \times K : I \times K \rightarrow X \). Then \( X_{q_K/} \times_{K^{q/}} K \rightarrow K \) is canonically isomorphic to \( X_{q_K/} \rightarrow K \). Consequently, by [13, Proposition 3.3.3.1], it suffices to show that whenever (2.2a)

\[
\theta : \text{Fun}_{K^{q/}}^{\text{coCart}}(K^{q/}, \tilde{X}) \rightarrow \text{Fun}_{K}^{\text{coCart}}(K, X)
\]

is an equivalence of \( \infty \)-categories, so is (2.2b)

\[
\text{Fun}_{K^{q/}}^{\text{coCart}}(K^{q/}, \tilde{X}_{q_K/}) \rightarrow \text{Fun}_{K}^{\text{coCart}}(K^{q/}, X_{q_K/}).
\]

Using the identification \( (I \times K)^{q/} \circ_{K^{q/}} (\Delta^n \times K^{q/}) \cong (I \circ \Delta^n) \times K^{q/} \), one sees that the \( n \) simplices of \( \text{Fun}_{K^{q/}}^{\text{coCart}}(K^{q/}, \tilde{X}_{q_K/}) \) are lifting diagrams of the form

\[
\begin{array}{ccc}
I \times K^{q/} & \xrightarrow{q_{K^{q/}}} & \tilde{X} \\
\downarrow & \sigma \downarrow & \downarrow p \\
(I \circ \Delta^n) \times K^{q/} & \rightarrow & K^{q/}
\end{array}
\]

such that for each vertex \( v \) of \( I \circ \Delta^n \), the restriction \( \sigma|_{(v) \times K^{q/}} : K^{q/} \rightarrow \tilde{X} \) is coCartesian. Thus, \( \text{Fun}_{K^{q/}}^{\text{coCart}}(K^{q/}, \tilde{X}_{q_K/}) \cong \left( \text{Fun}_{K^{q/}}^{\text{coCart}}(K^{q/}, \tilde{X}) \right)^{q/} \).

Similarly, \( \text{Fun}_{K}^{\text{coCart}}(K, X_{q_K/}) \cong \left( \text{Fun}_{K}^{\text{coCart}}(K, X) \right)^{q/} \). It follows that (2.2b) is an equivalence whenever (2.2a) is.

Combining Proposition 2.1 and Lemma 2.1 yields the general case:

**Theorem 2.1.** Let \( I \) and \( K \) be small simplicial sets, and suppose \( p' : K \rightarrow \mathcal{C}at_\infty \) is a functor. Let \( \lim p' \in \mathcal{C}at_\infty \) denote the limit \( \infty \)-category and for each \( k \in K \), let \( \pi_k : \lim p' \rightarrow p'(k) \) denote the corresponding functor in the limit cone. Suppose that \( q : I \rightarrow \lim p' \) is a diagram indexed by \( I \), and that

1. For each vertex \( k \in K \), the composite diagram \( \pi_k \circ q : I \rightarrow p'(k) \) has a (co)limit diagram, and
2. For each edge \( f : k \rightarrow k' \) of \( K \), the functor \( p'(f) : p'(k) \rightarrow p'(k') \) takes (co)limit diagrams extending \( \pi_k \circ q \) to (co)limit diagrams extending \( \pi_{k'} \circ q \).

Then:

- **A:** there exists a map \( \tilde{q} : I^\circ \rightarrow \lim p' \) which extends \( q \) and such that each composite \( \pi_{k'} \circ \tilde{q} : I^\circ \rightarrow p'(k) \) is a (co)limit (co)cone, and
- **B:** an arbitrary extension \( \tilde{q} : I^\circ \rightarrow \lim p' \) of \( q \) is a (co)limit diagram extending \( q \) if and only if each composite \( \pi_{k'} \circ \tilde{q} : I^\circ \rightarrow p'(k) \) is a (co)limit diagram extending \( \pi_{k'} \circ q \).
In particular, if

1. for each vertex \( k \in K \) the \( \infty \)-category \( p'(k) \) admits (co)limits for all diagrams indexed by \( I \), and
2. for each edge \( f : k \rightarrow k' \) of \( K \), the functor \( p'(f) : p'(k) \rightarrow p'(k') \) preserves (co)limits for all diagrams indexed by \( I \).

then the limit \( \infty \)-category \( \lim p' \) admits all (co)limits of shape \( I \), and the functors \( \pi_k : \lim p' \rightarrow p'(k) \) fitting into the limit cone preserve all (co)limits of shape \( I \).

 Proof. Let \( \bar{p}' : K^\triangleright \rightarrow \text{Cat}_\infty \) be a limit cone extending \( p' \) which maps the cone point \( \infty \in K^\triangleright \) to \( \lim p' \) and the cone edge \( \infty \rightarrow k \) to \( \pi_k \) for each \( k \in K^\triangleright \). Let \( (\bar{p}')^{q/} : K^\triangleright \rightarrow \text{Cat}_\infty \) denote corresponding diagram of undercategories, as in Lemma 2.1. Then \( (\bar{p}')^{q/} : K^\triangleright \rightarrow \text{Cat}_\infty \) is a limit cone which (by assumptions (1) and (2)) satisfies the assumptions for Proposition 2.1.

Now for any \( k \in K^\triangleright \), the \( \infty \)-category of diagrams \( I^\triangleright \rightarrow \bar{p}'(k) \) extending \( \pi_k \circ q \) is equivalent to the undercategory \( (\bar{p}'(k))^{\pi_k \circ q} \), and this equivalence identifies colimit diagrams with initial objects of the undercategory. Moreover, for any edge \( k \rightarrow k' \) in \( K^\triangleright \), the functor \( (\bar{p}'(k))^{\pi_k \circ q/} \rightarrow (\bar{p}'(k'))^{\pi_{k'} \circ q/} \) preserves initial objects if and only if the functor \( \bar{p}'(k) \rightarrow \bar{p}'(k') \) takes colimits diagrams extending \( \pi_k \circ q \) to colimit diagrams extending \( \pi_{k'} \circ q \).

Therefore statements A and B follow from Proposition 2.1.

\\

2.2. Adjunctions and Kan Extensions.

2.2.1. Limits of adjunctions. Our first application of Theorem 2.1 is to prove the following result:

Theorem 2.2 (A limit of adjunctions is an adjunction). Suppose

\[ F_k : C_k \rightleftarrows D_k : G_k, \quad k \in K \]

is diagram of adjunctions coherently indexed by a small simplicial set \( K \), i.e. given by a diagram \( (f \dashv g) : K \rightarrow \text{Adj} \) into the \( \infty \)-category of adjunctions. Let \( C = \lim_{k \in K} C_k \) and \( D = \lim_{k \in K} D_k \), and suppose that

- there is a functor \( F : C \rightarrow D \) which fits into the cone edge of a diagram \( \bar{f} : K^\triangleright \times \Delta^1 \rightarrow \text{Cat}_\infty \) extending \( f : K \times \Delta^1 \rightarrow \text{Cat}_\infty \), and
- there is a functor \( G : D \rightarrow C \) which fits into the cone edge of a diagram \( \bar{g} : K^\triangleright \times \Delta^1 \rightarrow \text{Cat}_\infty \), extending \( g : K \times \Delta^1 \rightarrow \text{Cat}_\infty \),

such that

1. the restrictions \( \bar{f}|_{K^\triangleright \times \{0\}} \) and \( \bar{g}|_{K^\triangleright \times \{1\}} \) are limit cones for \( C \), and
2. the restrictions \( \bar{f}|_{K^\triangleright \times \{1\}} \) and \( \bar{g}|_{K^\triangleright \times \{0\}} \) are limit cones for \( D \).

Then \( F \) and \( G \) form a pair of adjoint functors

\[ F : C \rightleftarrows D : G \]

We defer the proof until later: we will first need to give a precise definition of the \( \infty \)-category, \( \text{Adj} \), of adjunctions. To do this, we will use the framework for adjunctions of
∞-categories in terms of pairing of ∞-categories, as developed in [11]. For now we give an immediate corollary:

**Corollary 2.1.** Suppose that \( \delta : I \to I' \) is a morphism of simplicial sets. Let \( \text{Cat}^\delta_\infty \subset \text{Cat}_\infty \) be the subcategory consisting of

- those ∞-categories \( C \) which admit left (right) Kan extensions along \( \delta \) for any functor \( f : I \to C \), and
- those functors \( C \to C' \) which preserve left (right) Kan extensions along \( \delta \).

Then \( \text{Cat}^\delta_\infty \subset \text{Cat}_\infty \) is closed under (small) limits.

**Proof.** Let \( \phi : \text{Cat}_\infty \to \text{Fun}(\Delta^1, \text{Cat}_\infty) \) be the functor which sends an ∞-category \( C \) to the pullback-functor

\[
(2.3) \quad \text{Fun}(I, C) \leftarrow \text{Fun}(I', C) : \delta^*.
\]
Evaluating \( \phi \) at either endpoint of \( \Delta^1 \)

\[
ev_{(0)} \circ \phi : \text{Cat}_\infty \xrightarrow{C \to \text{Fun}(I', C)} \text{Cat}_\infty
\]

\[
ev_{(1)} \circ \phi : \text{Cat}_\infty \xrightarrow{C \to \text{Fun}(I, C)} \text{Cat}_\infty
\]

results in continuous functors (they are right adjoints). Therefore [13, Corollary 5.1.2.3] implies that \( \phi \) is continuous.

Now \( C \) admits left Kan extensions along \( \delta \) if and only if (2.3) is a right adjoint. In particular, we may identify \( \text{Cat}^\delta_\infty \) with the pullback

\[
\begin{array}{ccc}
\text{Cat}_\infty & \xrightarrow{\phi} & \text{Cat}_\infty \\
\text{Adj} & \xrightarrow{R} & \text{Fun}(\Delta^1, \text{Cat}_\infty)
\end{array}
\]

Where \( R : \text{Adj} \to \text{Fun}(\Delta^1, \text{Cat}_\infty) \) is the functor which sends an adjunction \((F \dashv G)\) to its right adjoint \( G \). By Theorem 2.2 the functor \( R \) is continuous. Therefore, by Theorem 2.1 \( \text{Cat}^\delta_\infty \) admits all small limits and the functor \( \text{Cat}^\delta_\infty \to \text{Cat}_\infty \) is continuous. \( \square \)

2.2.2. **Pairings of ∞-categories.** We recall the theory of pairings of ∞-categories; essentially all this material is taken from [11], though we provide proofs for certain details that will be important to us when discussing adjunctions. Recall that the ∞-category of pairings

\( \text{CPair} \subset \text{Fun}(\Lambda^2_0, \text{Cat}_\infty) \)

is the full subcategory consisting of diagrams

\[
\begin{array}{ccc}
\lambda_C & \xrightarrow{\text{id}} & \lambda_D \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\lambda} & \mathcal{D}^{op}
\end{array}
\]

such that \( \lambda : \mathcal{M} \xrightarrow{\lambda_C \times \lambda_D} \mathcal{C} \times \mathcal{D}^{op} \) is equivalent to a right fibration.
Given such a right fibration, $\lambda$ is classified by a functor (cf. [13, § 2.2.1])

$$\mathcal{D} \times \mathcal{C}^{op} \to \mathcal{S}$$

to the $\infty$-category, $\mathcal{S}$, of spaces; or equivalently a functor

$$(2.4) \quad \lambda' : \mathcal{D} \to \text{Fun}(\mathcal{C}^{op}, \mathcal{S}) =: \mathcal{P}(\mathcal{C})$$

to the $\infty$-category of presheaves over $\mathcal{C}$. Here $\lambda'$ takes each vertex $d \in \mathcal{D}$ to the right fibration

$$(2.5) \quad \mathcal{M} \times_{\mathcal{D}} \{d\} \to \mathcal{C}.$$  

As in [11], we call an object of $m \in \mathcal{M}$ right universal if it is a terminal object of $\mathcal{M} \times_{\mathcal{D}} \{\lambda_D(m)\}$ and we call a right fibration

$$(2.6) \quad \lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{op}$$ a right representable pairing, if for each $d \in \mathcal{D}^{op}$, there exists a right universal object in the fibre $\mathcal{M} \times_{\mathcal{D}} \{d\}$ over $d$. In this case, for each $d \in \mathcal{D}$, the right fibration (2.5) is representable (cf. [13, Proposition 4.4.4.5]). The Yoneda embedding $\check{\kappa} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ identifies $\mathcal{C}$ with the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the representable presheaves (cf [13, Proposition 5.1.3.1]); whence it follows that $\lambda'$ factors through $\mathcal{C}$,

$$\xymatrix{ \mathcal{D} \ar[r]^-{\lambda^R} \ar[dr]_{\lambda'} & \mathcal{C} \ar[d]^\check{\kappa} \\
& \mathcal{P}(\mathcal{C}) }$$

The Yoneda lemma implies that we have a weak equivalence of spaces

$$\text{Hom}_\mathcal{C}(c, \lambda^R(d)) \cong \{c\} \times_{\mathcal{C}} \mathcal{M} \times_{\mathcal{D}^{op}} \{d\},$$

which depend naturally on $(c, d) \in \mathcal{C}^{op} \times \mathcal{D}$.

Similarly, an object of $m \in \mathcal{M}$ is called left universal if it is a terminal object of $\mathcal{M} \times_{\mathcal{C}} \{\lambda_C(m)\}$, and the right fibration (2.6) is called a left representable pairing, if for each $c \in \mathcal{C}$, there exists a left universal object in the fibre $\mathcal{M} \times_{\mathcal{C}} \{c\}$ over $c$. As before, this determines a functor $\lambda^L : \mathcal{C} \to \mathcal{D}$; and the yoneda Lemma implies that we have weak equivalences of spaces

$$\text{Hom}_\mathcal{C}(\lambda^L(c), d) \cong \{c\} \times_{\mathcal{C}} \mathcal{M} \times_{\mathcal{D}^{op}} \{d\} \cong \text{Hom}_\mathcal{C}(c, \lambda^R(d))$$

depending naturally on $(c, d) \in \mathcal{C}^{op} \times \mathcal{D}$. Indeed, $\lambda^R$ is a right adjoint to $\lambda^L$ (cf. [11] or [13, § 5.2.6] for more details).

following Theodore Johnson-Freyd’s suggestion, we denote the Yoneda embedding by the first Hiragana character of his name, よねだ
Suppose that \( M \to C \times D \) and \( M' \to C' \times D' \) are two right representable right fibrations of \( \infty \)-categories, then a morphism of diagrams

\[
\begin{array}{ccc}
M & \xrightarrow{\gamma} & M' \\
\downarrow & & \downarrow \\
C \times D \mathbf{op} & \xrightarrow{\alpha \times \beta} & C' \times D' \mathbf{op}
\end{array}
\]

is called \emph{right representable} if it takes right universal objects to right universal objects.

The \( \infty \)-category of right-representable pairings \( \mathsf{CPair}^R \subseteq \mathsf{CPair} \) is defined to be the subcategory whose objects are equivalent to right representable pairings, and whose morphisms are equivalent to right representable morphisms. The \( \infty \)-category of left-representable pairings \( \mathsf{CPair}^L \subseteq \mathsf{CPair} \) is defined analogously.

\textbf{Lemma 2.2.} Let \( \mathsf{CPair}^R \subseteq \mathsf{CPair} \subseteq \mathsf{Fun}(\Lambda^2_0, \mathsf{Cat}_\infty) \) be the \( \infty \)-categories defined in \cite{11}. Then both subcategories are closed under small limits.

\textbf{Proof.} Since \( \mathsf{CPair} \subseteq \mathsf{Fun}(\Lambda^2_0, \mathsf{Cat}_\infty) \) is a reflective localization (cf. \cite{11} Remark 4.2.9), it is closed under small limits; so we need only show that \( \mathsf{CPair}^R \subseteq \mathsf{CPair} \) is also closed under small limits.

Let \( p : K^{op} \to \mathsf{CPair}^R \) be a diagram (for which we wish to compute the limit). The composite functor \( K^{op} \to \mathsf{CPair}^R \to \mathsf{CPair} \) is classified by a diagram of simplicial sets

\[
\begin{array}{ccc}
\mathcal{M}_p & \xrightarrow{\lambda_C} & D^{op}_p \\
\lambda_D \downarrow & & \downarrow \tilde{p}_D \\
C_p & \xrightarrow{\tilde{p}_C} & K^{op}
\end{array}
\]

where \( \lambda_p = \lambda_C \times_K \lambda_D : \mathcal{M}_p \to C_p \times_K D_p \) is a right fibration and \( \tilde{p}_C \) and \( \tilde{p}_D \) are Cartesian fibrations. The limit of \( p \) is a right fibration (cf. \cite{11} Remark 4.2.9.)

\[
(2.7) \quad M := \lim p_M \to \lim p_C \times \lim p_D =: C \times D,
\]

where \( p_M : K \to \mathsf{Cat}_\infty \) is the functor classified by \((\tilde{p}_C \times_K \tilde{p}_D) \circ \lambda_p\), and \( p_C \) and \( p_D \) are classified by \( \tilde{p}_C \) and \( \tilde{p}_D \) respectively. We need to show that \( (2.7) \) is right representable and that the canonical morphisms to \( (2.7) \) are right representable.

Now a vertex \( d \in \lim p_D = D \) can be identified with a Cartesian section \( \tilde{d} : K \to D_p \) of \( \tilde{p}_D \) (cf. \cite{13} Corollary 3.3.3.2.). Let \( \mathcal{M}_{p,d} \xrightarrow{\lambda_d} K \) be the Cartesian fibration which fits into
the pullback square

\[ \begin{array}{ccc}
\mathcal{M}_{p,d} & \longrightarrow & \mathcal{M}_p \\
\downarrow & & \downarrow \\
K & \overset{d}{\longrightarrow} & D_p
\end{array} \]

and let \( q : K^{op} \to \mathsf{Cat}_\infty \) be the corresponding functor. Then \( \lim q \cong \mathcal{M}_d := \mathcal{M} \times_D \{d\} \) (since taking pullbacks commutes with taking limits). To show that (2.7) is right representable, we need to show that \( \mathcal{M}_d \) has a final object. However, since \( p \) takes values in the \( \infty \)-category \( \mathsf{CPair}^R \) of right representable pairings, for each \( k \in K \), the pullback

\[ \mathcal{M}_{p,d,k} := \mathcal{M}_{p,d} \times_K \{k\} \]

has a final object, and for each morphisms \((k \to k')\) in \( K^{op} \), the corresponding functor \( \mathcal{M}_{p,d,k} \to \mathcal{M}_{p,d,k'} \) takes final objects to final objects. Thus, by [17, Theorem 3.16] (or Proposition 2.1), the limit \( \mathcal{M}_d \cong \lim q \) has a final object, and moreover the canonical morphisms \( \mathcal{M}_d \to \mathcal{M}_{p,d,k} \) preserve final objects.

It follows that \( \lim p \in \mathsf{CPair} \) is in fact an element of \( \mathsf{CPair}^R \) and that the limit cone is a diagram in \( \mathsf{CPair}^R \); i.e. \( \mathsf{CPair}^R \subseteq \mathsf{CPair} \) is closed under small limits.

\[ \square \]

**Proposition 2.2.** There are equivalences of \( \infty \)-categories

(2.8) \[ \mathsf{CPair}^L \cong \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty), \]

(2.9) \[ \mathsf{CPair}^R \cong \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty), \]

which associate a left representable pairing \( \lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{op} \) to the functor \( \lambda^L : \mathcal{C} \to \mathcal{D} \), and a right representable pairing \( \lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{op} \) to the functor \( \lambda^R : \mathcal{D} \to \mathcal{C} \).

**Proof.** As in [11], we say that a right fibration (2.6) is a perfect pairing if it is both left and right representable, and an object \( m \in \mathcal{M} \) is left universal if and only if it is right universal. Let \( \mathsf{CPair}^{perf} \subseteq \mathsf{CPair}^L \) be the full subcategory spanned by the perfect pairings. Let \( \phi : \mathsf{CPair}^{perf} \to \mathsf{Cat}_\infty \) denote the forgetful functor which sends a perfect pairing (2.6) to \( \mathcal{C} \); and let \( \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty) \) denote the \( \infty \)-category fitting into the pullback square

\[ \begin{array}{ccc}
\mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty) & \longrightarrow & \mathsf{CPair}^{perf} \\
\downarrow & & \downarrow \\
\mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty) & \overset{\phi}{\longrightarrow} & \mathsf{Cat}_\infty
\end{array} \]

Note: since the bottom arrow is a Cartesian fibration (cf. [13 Corollary 2.4.7.11]), this homotopy pullback can be computed as a pullback of simplicial sets (cf. [13 Corollary 3.3.1.4]). Since \( \phi \) is an equivalence of \( \infty \)-categories (cf. [11 Remark 4.2.12]), the left arrow defines an equivalence between \( \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty) \) and \( \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty) \).
The inclusion $\text{CPair}^{\text{perf}} \subset \text{Fun}(\Delta_0^2, \text{Cat}_\infty)$ allows us to identify $\text{Fun}(\Delta^1, \text{Cat}_\infty)$ with diagrams of the form

$$
\begin{array}{c}
\mathcal{C} \\
\downarrow f \\
\mathcal{D} \\
\mathcal{P} \\
\downarrow \mathcal{D}^{\text{op}}
\end{array}
$$

where $\mathcal{P} \to \mathcal{D} \times \mathcal{D}^{\text{op}}$ is a perfect pairing. Taking the limit of such a diagram yields

$$
\begin{array}{c}
\mathcal{C} \\
\downarrow f \\
\mathcal{D} \\
\mathcal{P} \\
\downarrow \mathcal{D}^{\text{op}}
\end{array}
\xrightarrow{\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{\text{op}}}
\begin{array}{c}
\mathcal{M} \\
\downarrow \mathcal{D} \\
\mathcal{D}^{\text{op}}
\end{array}
$$

(2.10)

where

$$
\begin{array}{c}
\mathcal{C} \\
\downarrow f \\
\mathcal{D} \\
\mathcal{P} \\
\downarrow \mathcal{D}^{\text{op}}
\end{array}
\xrightarrow{\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{\text{op}}}
\begin{array}{c}
\mathcal{M} \\
\downarrow \mathcal{D} \\
\mathcal{D}^{\text{op}}
\end{array}
$$

(2.11)

is a left-representable pairing. Thus we get a functor

$$
A : \text{Fun}(\Delta^1, \text{Cat}_\infty) \to \text{CPair}^L,
$$

sending an object of the form (2.10) to the left representable pairing (2.11). Notice that, by construction, the functor $f$ appearing in (2.10) is equivalent to $\lambda^L : \mathcal{C} \to \mathcal{D}$.

It remains to show that $A$ is an equivalence of categories. The essential surjectivity of $A$ is explained in [11, Remark 4.2.13]. We argue that $A$ is fully faithful: Suppose that

$$
\tilde{f} = \begin{array}{c}
\mathcal{C} \\
\downarrow f \\
\mathcal{D} \\
\mathcal{P} \\
\downarrow \mathcal{D}^{\text{op}}
\end{array} \quad \text{and} \quad \tilde{f}' = \begin{array}{c}
\mathcal{C}' \\
\downarrow f' \\
\mathcal{D}' \\
\mathcal{P}' \\
\downarrow \mathcal{D}'^{\text{op}}
\end{array}
$$

are a pair of objects in $\text{Fun}(\Delta^1, \text{Cat}_\infty)$, with

$$
A(\tilde{f}) = (\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{\text{op}}), \quad \text{and} \quad A(\tilde{f}') = (\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'^{\text{op}}).
$$

We need to show that the natural map between the mapping spaces

(2.12)

$$
A : \text{Map}_{\text{Fun}(\Delta^1, \text{Cat}_\infty)}(\tilde{f}, \tilde{f}') \to \text{Map}_{\text{CPair}^L}(\mathcal{M}, \mathcal{M}')
$$

is a homotopy equivalence.\footnote{For two left-representable pairings $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}^{\text{op}}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'^{\text{op}}$, the mapping space $\text{Map}_{\text{CPair}^L}(\mathcal{M}, \mathcal{M}')$ is the subspace of

$$
\text{Map}_{\text{Cat}_\infty}(\mathcal{C}, \mathcal{C}') \times_{\text{Map}_{\text{Cat}_\infty}(\mathcal{M}, \mathcal{C}')} \text{Map}_{\text{Cat}_\infty}(\mathcal{M}, \mathcal{M}') \times_{\text{Map}_{\text{Cat}_\infty}(\mathcal{M}, \mathcal{D}')} \text{Map}_{\text{Cat}_\infty}(\mathcal{D}, \mathcal{D}')
$$}
On the one hand, 
\[ \text{Map}_{\text{Fun}(\Delta^1, \text{Cat}_\infty)}(\tilde{f}, \tilde{f}') \cong \text{Map}_{\text{Cat}_\infty}(C, C') \times h_{\text{Map}_{\text{Cat}_\infty}(C, D')} \text{Map}_{\text{CPair}^L}(\mathcal{P}, \mathcal{P}') \]

But, since \( \mathcal{P}' \to \mathcal{D}' \times \mathcal{D}'^{\text{op}} \) is a perfect pairing, [11, Proposition 4.2.10] shows we have homotopy equivalences of mapping spaces

\[ \text{Map}_{\text{CPair}^L}(\mathcal{P}, \mathcal{P}') \cong \text{Map}_{\text{Cat}_\infty}(\tilde{D}, \tilde{D}') \cong \text{Map}_{\text{CPair}^L}(\mathcal{M}, \mathcal{P}'). \]

Consequently,

\[ (2.13a) \quad \text{Map}_{\text{Fun}(\Delta^1, \text{Cat}_\infty)}(\tilde{f}, \tilde{f}') \cong \text{Map}_{\text{Cat}_\infty}(C, C') \times h_{\text{Map}_{\text{Cat}_\infty}(C, D')} \text{Map}_{\text{CPair}^L}(\mathcal{P}, \mathcal{P}') \]

On the other hand, since \( \mathcal{M}' \) is a pullback of \( \mathcal{P}' \), we have a homotopy equivalence of mapping spaces

\[ (2.13b) \quad \text{Map}_{\text{CPair}^L}(\mathcal{M}, \mathcal{M}') \cong \text{Map}_{\text{Cat}_\infty}(C, C') \times h_{\text{Map}_{\text{Cat}_\infty}(C, D')} \text{Map}_{\text{CPair}^L}(\mathcal{M}, \mathcal{P}'). \]

It follows from (2.13) that (2.12) is a homotopy equivalence. In particular, \( A \) is fully faithful. \[\square\]

2.2.3. The \( \infty \)-category of Adjunctions, and the proof of Theorem 2.2. We are now in a position to define the \( \infty \)-category of adjunctions and to prove Theorem 2.2.

**Definition 2.1.** The \( \infty \)-category of adjunctions, \( \text{Adj} \) is defined as the pullback of \( \infty \)-categories

\[ \text{Fun}(\Delta^1, \text{Cat}_\infty) \quad \cong \quad \text{CPair}^R \]

\[ \text{Adj} \quad \cong \quad \text{CPair} \]

\[ \text{Fun}(\Delta^1, \text{Cat}_\infty) \quad \cong \quad \text{CPair}^L \]

(2.14)

**Proof of Theorem 2.2**. By Lemma 2.2 and Proposition 2.1, each of the categories in the diagram (2.14) are complete, and each of the functors in the diagram preserve small limits. Thus, any diagram \( (f \dashv g) : K \to \text{Adj} \) admits a limit. Moreover, the limiting left adjoint \( F : \mathcal{C} \to \mathcal{D} \) is a limiting functor for the diagram \( f : K \to \text{Fun}(\Delta^1, \text{Cat}_\infty) \). In particular, \( F \) can be characterized as in the statement of the theorem (cf. [13, Corollary 5.1.2.3]).

Similarly, the limiting right adjoint \( G : \mathcal{D} \to \mathcal{C} \) is a limiting functor for the diagram \( g : K \to \text{Fun}(\Delta^1, \text{Cat}_\infty) \); so \( G \) can be characterized as in the statement of the theorem. \( \square \)

which preserves left universal objects. Notice that the homotopy pullbacks can be taken to be strict pullbacks when \( \lambda \) and \( \lambda' \) are right fibrations.
3. Complete \( k \)-fold Segal objects.

Let \( \Delta \) denote the simplex category, and for any simplicial set \( K \), let \( \Delta/K \rightarrow \Delta \) denote the corresponding category of simplices\(^9\) of \( K \). The spine of the standard \( n \)-simplex is the subsimplicial set

\[
\text{Sp}(n) = \Delta^{[0,1]} \times_{\Delta^{[1]}} \Delta^{[n]} \subseteq \Delta^n.
\]

generated by the 1-simplices \( \Delta_{[i,i+1]} \subseteq \Delta^n \). The inclusion \( \text{Sp}(n) \subseteq \Delta^n \) is a categorical equivalence\(^{10}\) and a simplicial object \( X \) in \( X \) is called a category object if it satisfies the so-called Segal conditions (cf. [16]): i.e. for each \( n \geq 0 \), the natural map

\[
(3.1) \quad X_n \rightarrow \lim_{/\text{Sp}(n)} X \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

is an equivalence.

Given a category object \( X \) in \( X \), one should think of \( X_0 \in X \) as describing the objects of an \((\infty,1)\)-category internal to \( X \), \( X_1 \in X \) as describing the morphisms of an \((\infty,1)\)-category internal to \( X \), \( X_i \in X \) as describing the object classifying composable \( i \)-tuples of morphisms, and the various structural maps between the \( X_i \)'s as describing the homotopy-associative composition and units.

Now suppose that \( X \) is an \( \infty \)-topos. We let \( \text{Cat}(X) \subseteq \text{Fun}(\Delta^{\text{op}}, X) \) denote the full subcategory spanned by the category objects. Unfortunately, \( \text{Cat}(X) \) doesn’t describe the correct homotopy theory of \((\infty,1)\)-categories internal to \( X \); one must localize with respect to an appropriate class of “fully faithful and essentially surjective functors”. In order to describe this phenomena in more detail, we recall that a category object \( X \) is called a groupoid object if all it’s morphisms are invertible, i.e.

\[
X_2 \rightarrow \lim_{/\text{Sp}(n)} \text{Cat}(X).
\]

\(^9\) The objects of \( \Delta/K \) over \([n] \) are simplicial maps \( \Delta^n \rightarrow K \) from the standard \( n \)-simplex, and morphisms in \( \Delta/K \) over a morphism \( f : [m] \rightarrow [n] \) are commutative diagrams

\[
\Delta^n \xrightarrow{f} \Delta^n \xleftarrow{K}
\]

Equivalently, \( (\Delta/K) \) is the Grothendieck fibration (or category of elements) associated the functor \( K : \Delta^{\text{op}} \rightarrow \text{Sets} \).

\(^{10}\) In fact, the model structure on simplicial sets for \( \infty \)-category is the Cisinski model structure induced by the localizer which consists of the inclusions \( \text{Sp}(n) \subseteq \Delta^n \) (cf. [1]).
is an equivalence, where
\[ \Lambda^2_0 = \Delta_{0,1} \coprod_{\Delta_0} \Delta_{0,2} \subset \Delta^2. \]

We let \( \text{Gpd}(\mathcal{X}) \subseteq \text{Cat}(\mathcal{X}) \) denote the full subcategory spanned by the groupoid objects. The underlying groupoid functor \( \text{Gp} : \text{Cat}(\mathcal{X}) \to \text{Gpd}(\mathcal{X}) \) is any right adjoint to the inclusion. For a category object \( X_\bullet \), one should think of \( \text{Gp} X_\bullet \) as describing the “maximal groupoid contained in \( X_\bullet \)”, which classifies the “objects” of the internal \((\infty, 1)\)-category \( X_\bullet \).

The \textbf{fully faithful and essentially surjective morphisms} (cf. [14, Definition 1.2.12]), are those morphisms of category objects \( X_\bullet \to Y_\bullet \) in \( \mathcal{X} \) which are

- \textbf{fully faithful:} the diagram

\[
\begin{array}{ccc}
X_1 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
X_0 \times X_0 & \longrightarrow & Y_0 \times Y_0
\end{array}
\]

is a pullback square, and

- \textbf{essentially surjective:} the map

\[ |\text{Gp} X_\bullet| \to |\text{Gp} Y_\bullet| \]

between the classifying spaces of objects is an equivalence, where \(|-|\) denotes the geometric realization:

\[ |Z_\bullet| = \operatorname{co lim}_{\text{op}} Z_\bullet \]

for any \( Z_\bullet : \Delta_{\text{op}} \to \mathcal{X} \).

Localizing along the fully faithful and essentially surjective morphisms of category objects, one obtains \( \text{CSS}(\mathcal{X}) \subseteq \text{Cat}(\mathcal{X}) \), the correct homotopy theory of \((\infty, 1)\)-category objects in \( \mathcal{X} \). Following Rezk [16] Lurie proves [14, Theorem 1.2.13] that \( \text{CSS}(\mathcal{X}) \subseteq \text{Cat}(\mathcal{X}) \) is equivalent to the full subcategory spanned by the \textit{complete Segal objects}: those category objects \( X_\bullet \in \text{Cat}(\mathcal{X}) \) such that \( \text{Gp} X_\bullet \) is essentially constant (i.e. \( \text{Gp} X_\bullet : \Delta_{\text{op}} \to \mathcal{X} \) is equivalent to a constant functor).

To describe \((\infty, k)\)-category objects in \( \mathcal{X} \), will be interested in the following full subcategories of multisimplicial objects

\[
\text{CSS}_k(\mathcal{X}) \subseteq \text{Seg}_k(\mathcal{X}) \subseteq \text{Cat}^k(\mathcal{X}) \subseteq \text{Fun}((\Delta^k_{\text{op}}), \mathcal{X}).
\]

Here \( \text{Cat}^k(\mathcal{X}) \) is spanned by the \( k \)-uple category objects, i.e those multisimplicial objects \( X_{\bullet, \ldots, \bullet} \) such that for any \( 1 \leq i \leq k \) and any \( n_1, \ldots, \hat{n}_i, \ldots, n_k \geq 0 \), the simplicial object

\[
X_{n_1, \ldots, n_{i-1}, \bullet, n_{i+1}, \ldots, n_k} : \Delta_{\text{op}} \to \mathcal{X}
\]

is a category object. As before \( X_{0, \ldots, 0} \) encodes the objects of the \( k \)-uple category internal to \( \mathcal{X} \), but now each of \( X_{1,0,\ldots,0}, X_{0,1,0,\ldots,0}, \ldots, X_{0,\ldots,0,1} \) encodes a different type of 1-morphism;
while each of $X_{i_1,\ldots,i_k}$ (with $0 \leq i_1, \ldots, i_k \leq 1$) represents a different type of $(i_1 + \cdots + i_k)$-morphism. As before, $\text{Cat}^k(\mathcal{X})$ does not model the correct homotopy theory of $k$-uple categories internal to $\mathcal{X}$; one must localize with respect to an appropriate class of “fully faithful and essentially surjective functors”.

Next, $\text{Seg}_k(\mathcal{X})$ is spanned by the $k$-fold Segal objects (cf. [2]), i.e. those $k$-uple category objects $X_{\bullet,\ldots,\bullet}$ such that for every $1 \leq i \leq k$, and any $n_1,\ldots,n_{i-1} \geq 0$ the multisimplicial object

\[ X_{n_1,\ldots,n_{i-1},0,\ldots,\bullet} \]

is equivalent to a constant functor. The idea behind this condition is that while a $k$-uple category object has \( \binom{k}{i} \) different types of $i$-morphisms, there is only one non-trivial type of $i$-morphism in a $k$-fold Segal object. More specifically, $X_{(\bullet,\ldots,\bullet)} : (\Delta^{op})^n \to \mathcal{X}$ encodes the data of an $(\infty,k)$-category as follows:

- $X_{(0,\ldots,0)}$ encodes the objects,
- $X_{(1,0,\ldots,0)}$ encodes the 1-morphisms,
- $X_{(1,1,0,\ldots,0)}$ encodes the 2-morphisms,
- 
- and $X_{(1,\ldots,1)}$ encodes the $k$-morphisms.

the remaining objects $X_{(n_1,\ldots,n_k)}$ encode composable configurations of morphisms, while the homotopy coherent associative composition and unit are encoded in the various structural maps between the spaces $X_{(n_1,\ldots,n_k)}$.

Note that $\text{Seg}_k(\mathcal{X})$ does not model the correct homotopy theory of $k$-fold categories internal to $\mathcal{X}$. However, when $\mathcal{X}$ is an $\infty$-topos (e.g. $\mathcal{X} = \mathcal{S}$), we may localize $\text{Seg}_k(\mathcal{X})$ with respect to an appropriate class of “fully faithful and essentially surjective functors”, to obtain $CSS_k(\mathcal{X})$, which is spanned by those $k$-fold Segal objects which satisfy a certain completeness condition; we refer the reader to [2, 5, 8, 14] for more details.

**Lemma 3.1.** Suppose $\mathcal{D}$ is an $\infty$-category, $\mathcal{X}$ is a presentable $\infty$-category, and $\tilde{\mathcal{X}}$ is any reflective localization of $\text{Cat}^k(\mathcal{X})$ (the two main examples being $\tilde{\mathcal{X}} = \text{Seg}_k(\mathcal{X})$, or when $\mathcal{X}$ is an $\infty$-topos, $\tilde{\mathcal{X}} = CSS_k(\mathcal{X})$). Then a functor

\[ \mathcal{D} \xrightarrow{d \mapsto F(d)(\bullet,\ldots,\bullet)} \tilde{\mathcal{X}} \]

is continuous if and only if each of the composite functors

\[ \mathcal{D} \xrightarrow{d \mapsto F(d)(i_1,\ldots,i_k)} \mathcal{X} \]

obtained by evaluating at $(i_1,\ldots,i_k) \in \Delta^k$ for $0 \leq i_1,\ldots,i_k \leq 1$, are continuous.

**Proof.** Without loss of generality, we may take $\tilde{\mathcal{X}} = \text{Cat}^k(\mathcal{X})$. Recall that $\text{Cat}^k(\mathcal{X})$ is a reflective localization of $\text{Fun}((\Delta^{k})^{op},\mathcal{X})$ (cf. [14]), so (3.4) is continuous if and only if the composite functor

\[ \mathcal{D} \xrightarrow{d \mapsto F(d)(\bullet,\ldots,\bullet)} \text{Cat}^k(\mathcal{X}) \xrightarrow{\text{Fun}((\Delta^k)^{op},\mathcal{X})} \]
is continuous. Since limits in functor $\infty$-categories are detected pointwise (cf. [13, Coro-
lary 5.1.2.3]), it follows that (3.4) is continuous if and only if the composite functors

\[ D \xrightarrow{d \mapsto F(d)(n_1, \ldots, n_k)} X \]

obtained by evaluating at any $(n_1, \ldots, n_k) \in \Delta^k$ are continuous. This proves the only if part of the statement.

Now we prove the if part of the statement. Let $i : \operatorname{Morph}^k \hookrightarrow \Delta^k$ denote the inclusion of the full subcategory spanned by $(i_1, \ldots, i_k) \in \Delta^k$, where $0 \leq i_1, \ldots, i_k \leq 1$. Then by assumption, (3.5) and hence the restricted functor

\[ D \xrightarrow{d \mapsto F(d)(\bullet, \ldots, \bullet)} \operatorname{Cat}^k(\mathcal{X}) \]

is continuous.

A map of simplices $\phi : [n] \to [m] \subseteq \Delta$ is said to be inert if it is the inclusion of a full sub-interval, i.e. $\phi(i + 1) = \phi(i) + 1$ for every $i \in [n]$ (cf. [3, 8]). We let $j : \Delta_{\text{int}} \hookrightarrow \Delta$ denote the inclusion of the wide subcategory containing only the inert maps. For any $(n_1, \ldots, n_k) \in \Delta^k$, let

\[ \operatorname{Spine}(n_1, \ldots, n_k) = \operatorname{Morph}^k \times_{\Delta_{\text{int}}} \Delta^k / (n_1, \ldots, n_k). \]

Then the Segal conditions imply that for any $(n_1, \ldots, n_k) \in \Delta^k$, and $d \in D$, the object $F(d)(n_1, \ldots, n_k) \in \mathcal{X}$ is a limit for the composite functor

\[ \operatorname{Spine}(n_1, \ldots, n_k)^{op} \rightarrow (\Delta^k)^{op} \xrightarrow{F(d)} \mathcal{X} \]

(cf. [8, Lemma 2.27]). Equivalently, the restriction $F(d)|_{(\Delta_{\text{int}}^k)^{op}}$ is a right Kan extension along $\delta : \operatorname{Spine}(n_1, \ldots, n_k)^{op} \rightarrow (\Delta_{\text{int}}^k)^{op}$.

Let

\[ \operatorname{Fun}((\Delta^k)^{op}, \mathcal{X}) \xrightarrow{\delta^*} \operatorname{Fun}((\operatorname{Morph}^k)^{op}, \mathcal{X}) \]

denote the right adjoint (the global right Kan extension) to the pullback $\delta^*$, then the composite

\[ D \xrightarrow{d \mapsto F(d)(\bullet, \ldots, \bullet)} \operatorname{Cat}^k(\mathcal{X}) \]

is continuous. However, by assumption, this functor is equivalent to the restricted functor

\[ D \xrightarrow{d \mapsto F(d)(\bullet, \ldots, \bullet)} \operatorname{Cat}^k(\mathcal{X}) \]

It follows that each (3.6) is continuous, whence (3.4) is continuous.
3.1. The Sheaf of Complete \( k \)-Fold Segal Objects. Given two morphisms of simplicial sets \( X \to S \) and \( Y \to S \), we let \( Y^X \to S \) denote the simplicial set satisfying the universal property that for any morphism of simplicial sets \( K \to S \), commutative diagrams of the form

\[
\begin{array}{ccc}
K \times_S X & \to & Y \\
\downarrow & & \downarrow \\
S & \to & S
\end{array}
\]

correspond to diagrams of the form

\[
\begin{array}{ccc}
K & \to & Y^X \\
\downarrow & & \downarrow \\
S & \to & S
\end{array}
\]

In particular, when \( Y \to S \) is a coCartesian fibration, and \( X \to S \) is a Cartesian fibration, then \( Y^X \to S \) is a coCartesian fibration satisfying

\[
\text{Fun}_S(K, Y^X) \cong \text{Fun}_S(K \times_S X, Y),
\]

(see [13, Corollary 3.2.2.13] for more details).

Let \( \mathcal{C}_{\infty} \) denote the \( \infty \)-category of (not necessarily small) \( \infty \)-categories, and \( \iota : \mathcal{L}\text{Top} \hookrightarrow \mathcal{C}_{\infty} \) denote the subcategory consisting of \( \infty \)-topoi and geometric morphisms (functors which preserve small colimits and finite limits). Notice that \( \iota \) factors through the subcategory of presentable \( \infty \)-categories and left adjoints. Let \( \iota^* \mathcal{Z} \to \mathcal{L}\text{Top} \) denote the (canonical) presentable fibration classified by \( \iota \) (cf. [13, Proposition 5.5.3.3]).\(^{11}\) We define a presentable fibration

\[
k\text{-Simpl}(\mathcal{L}\text{Top}) := \iota^* \mathcal{Z} (\mathcal{L}\text{Top} \times (\Delta^k)^{op}) \to \mathcal{L}\text{Top},
\]

whose fibre over any \( \infty \)-topos \( \mathcal{X} \) is equivalent to \( \text{Fun}((\Delta^k)^{op}, \mathcal{X}) \) and which associates to any geometric morphism of \( \infty \)-topoi \( f^* : \mathcal{X} \to \mathcal{Y} : f_* \) the adjunction given by composition with \( f^* \) (resp. \( f_* \))

\[
(f^*)_! : \text{Fun}((\Delta^k)^{op}, \mathcal{X}) \to \text{Fun}((\Delta^k)^{op}, \mathcal{Y}) : (f_*)!. 
\]

Suppose that \( X_{\bullet, \ldots, \bullet} \in k\text{-Simpl}(\mathcal{L}\text{Top}) \) is a vertex lying over \( \mathcal{X} = p(X_{\bullet, \ldots, \bullet}) \). We say that \( X_{\bullet, \ldots, \bullet} \) is a complete Segal object if it lies in the essential image of \( CSS_k(\mathcal{X}) \hookrightarrow p^{-1}(\mathcal{X}) \). We define \( \int CSS_k \) to be the full subcategory of \( k\text{-Simpl}(\mathcal{L}\text{Top}) \) spanned by the complete Segal objects.

**Lemma 3.2.** \( \int CSS_k \to \mathcal{L}\text{Top} \) is a presentable fibration.

**Proof.** We begin by showing that \( \int CSS_k \) is a Cartesian fibration. It suffices to show that for any complete Segal object \( Y_{\bullet, \ldots, \bullet} \in k\text{-Simpl}(\mathcal{L}\text{Top}) \) and any \( p \)-Cartesian morphism \( \hat{f} : X_{\bullet, \ldots, \bullet} \to Y_{\bullet, \ldots, \bullet} \), the vertex \( X_{\bullet, \ldots, \bullet} \in k\text{-Simpl}(\mathcal{L}\text{Top}) \) is also a complete Segal object.

\(^{11}\)Recall that a fibration is presentable if it is both a Cartesian and a coCartesian fibration each of whose fibres are presentable \( \infty \)-categories.
Let $f^* : \mathcal{X} \to \mathcal{Y}$ denote the image of $\tilde{f}$ under $p$, and let $\mathcal{X} \leftarrow \mathcal{Y} : f_*$ denote a right adjoint to $f^*$. Then $X_{\bullet, \ldots, \bullet} \cong (f_*)!(Y_{\bullet, \ldots, \bullet})$ by the construction of $p$, and the latter is a complete Segal object by [8, Proposition 2.20].

Indeed, this shows that $\int C S S_k \to \mathcal{L} T o p$ is a Cartesian fibration classified by a functor

$$\chi : \mathcal{L} T o p^{op} \to \hat{C a t}_\infty$$

such that

- for every $\infty$-topos $\mathcal{X} \in \mathcal{L} T o p$, the image $\chi(\mathcal{X})$ is equivalent to the presentable $\infty$-category $C S S_k(\mathcal{X})$, and
- for every geometric morphism $f^* : \mathcal{X} \rightleftarrows \mathcal{Y} : f_*$, the functor

$$C S S_k(\mathcal{X}) \leftarrow C S S_k(\mathcal{Y}) : \chi(f^*)$$

is equivalent to

$$C S S_k(\mathcal{X}) \leftarrow C S S_k(\mathcal{Y}) : (f_*)_!,$$

which has a left adjoint (cf. [8, Proposition 2.20]).

It follows that $\int C S S_k \to \mathcal{L} T o p$ is a presentable fibration (cf. [13, Proposition 5.5.3.3]).

Remark 3.1. By construction, the objects $X_{\bullet, \ldots, \bullet}$ in $\int C S S_k$ over an $\infty$-topos $\mathcal{X}$ can be identified with complete Segal objects in $\mathcal{X}$, and morphisms $X_{\bullet, \ldots, \bullet} \to Y_{\bullet, \ldots, \bullet}$ in $\int C S S_k$ over a geometric morphism $f^* : \mathcal{X} \rightleftarrows \mathcal{Y} : f_*$ of $\infty$-topoi can be identified with either

(a) morphisms $X_{\bullet, \ldots, \bullet} \to (f_*)!(Y_{\bullet, \ldots, \bullet})$ in $C S S_k(\mathcal{X})$, or
(b) morphisms $L_k, Y(f^*)!(X_{\bullet, \ldots, \bullet}) \to Y_{\bullet, \ldots, \bullet}$ in $C S S_k(\mathcal{Y})$,

where $L_k, Y : S e g_k(\mathcal{Y}) \to C S S_k(\mathcal{Y})$ is the localization functor which sends a $k$-fold Segal object in $\mathcal{Y}$ to its completion. The equivalence between morphisms of types (a) and (b) is given by the adjunction

$$L_k, Y(f^*)! : C S S_k(\mathcal{X}) \rightleftarrows C S S_k(\mathcal{Y}) : (f_*)_!$$

of [8, Proposition 2.20].

Recall that a geometric morphism $f^* : \mathcal{X} \rightleftarrows \mathcal{Y} : f_*$ is said to be étale if it admits a factorization

$$f^* : \mathcal{X} \rightleftarrows \mathcal{X}/U \cong \mathcal{Y} : f_*$$

for some object $U \in \mathcal{X}$. We let $\mathcal{L} T o p_{\acute{e}t} \subset \mathcal{L} T o p$ denote the subcategory spanned by the étale geometric morphisms, and we define $\int^{\acute{e}t} C S S_k \to \mathcal{L} T o p_{\acute{e}t}$ to be the presentable fibration fitting into the pullback square:

$$
\begin{array}{ccc}
\int^{\acute{e}t} C S S_k & \longrightarrow & \int C S S_k \\
\downarrow & & \downarrow \\
\mathcal{L} T o p_{\acute{e}t} & \longrightarrow & \mathcal{L} T o p
\end{array}
$$
Remark 3.2. Since $\int CSS_k \to \mathcal{L}\text{Top}$ and $\int^{\text{et}} CSS_k \to \mathcal{L}\text{Top}_{\text{et}}$ are both presentable fibrations [13 Corollary 4.3.1.11] implies that they both admit all small relative limits and colimits. Since $\mathcal{L}\text{Top}$ admits all small limits and colimits (cf. [13 § 6.3]), it follows that $\int CSS_k$ admits all small limits and colimits, and that the functor $\int CSS_k \to \mathcal{L}\text{Top}$ preserves those limits and colimits (cf. [7 Lemma 9.8]).

Finally, [13 Theorem 6.3.5.13] implies that $\mathcal{L}\text{Top}_{\text{et}} \subset \mathcal{L}\text{Top}$ is closed under small limits, which implies that $\int^{\text{et}} CSS_k \subset \int CSS_k$ is also closed under small limits.

As explained in [13 Remark 6.3.5.10] for any $\infty$-topos $\mathcal{X}$, the Cartesian fibration

$$\text{Fun}(\Delta^1, \mathcal{X}) \to \text{Fun}(\{1\}, \mathcal{X}) \cong \mathcal{X}$$

is classified by a functor

$$\mathcal{X}^{\text{op}} \xrightarrow{f : U \to V} (\mathcal{X}/_U \rightleftarrows \mathcal{X}/_V : f^*) \to \mathcal{L}\text{Top}_{\text{et}},$$

which factors as

$$\mathcal{X}^{\text{op}} \xrightarrow{\cong} (\mathcal{L}\text{Top}_{\text{et}})_/ \mathcal{X} \to \mathcal{L}\text{Top}_{\text{et}},$$

where the first functor is an equivalence of categories.

Definition 3.1 ([13 Notation 6.3.5.19]). Given a functor $F : \mathcal{L}\text{Top} \to \mathcal{C}$, let $F_\mathcal{X} : \mathcal{X}^{\text{op}} \to \mathcal{C}$ denote the composite

$$\mathcal{X}^{\text{op}} \to \mathcal{L}\text{Top}_{\text{et}} \subset \mathcal{L}\text{Top} \xrightarrow{F} \mathcal{C}.$$ 

We say that $F$ is a sheaf if for every $\infty$-topos $\mathcal{X}$, the composite functor $F_\mathcal{X}$ preserves small limits.

Theorem 3.1. The functor

$$CSS_k : \mathcal{L}\text{Top}_{\text{et}} \xrightarrow{\mathcal{X} \mapsto CSS_k(\mathcal{X})} \hat{\mathcal{C}}_{\text{at}}$$

classifying $\int^{\text{et}} CSS_k \to \mathcal{L}\text{Top}_{\text{et}}$ preserves small limits. In particular,

$$CSS_k : \mathcal{L}\text{Top} \xrightarrow{\mathcal{X} \mapsto CSS_k(\mathcal{X})} \hat{\mathcal{C}}_{\text{at}}$$

is a sheaf.

Proof. By definition, for any étale geometric morphism $f^* : \mathcal{X} \rightleftarrows \mathcal{X}/_U : f_*$, the canonical projection $f_1 : \mathcal{X}/_U \to \mathcal{X}$ forms part of an adjoint triple,

$$(f_1 \dashv f^* \dashv f_*) : \mathcal{X}/_U \xrightarrow{f^*} \mathcal{X}.$$ 

Moreover, the forgetful functor from the over category $f_1 : \mathcal{X}/_U \to \mathcal{X}$ preserves pullbacks, so that $f_1 : \mathcal{X}/_U \rightleftarrows \mathcal{X} : f^*$ is a pseudo-geometric morphism (cf. [8]). In particular, $(f^*)_! : \text{Fun}((\Delta^k)^{\text{op}}, \mathcal{X}) \to \text{Fun}((\Delta^k)^{\text{op}}, \mathcal{X}/_U)$ preserves complete $k$-fold Segal objects.
(cf. [8, Proposition 2.20]); and consequently the inclusion

$$
\int^\text{ét} \mathcal{C}SS_k \longrightarrow \text{k-Simpl}(\mathcal{L}\text{Top}_\text{ét})
$$

preserves both Cartesian and coCartesian edges. Therefore,

- \( \mathcal{C}SS_k : \mathcal{L}\text{Top}_\text{ét} \rightarrow \widehat{\text{Cat}}_\infty \) is a (fully faithful) subfunctor of the composite functor \( \text{k-Simpl}' : \mathcal{L}\text{Top}_\text{ét} \hookrightarrow \widehat{\text{Cat}}_\infty \xrightarrow{\mathcal{C}}} \text{Fun((} \text{k} \text{)}^{\text{op}}, \mathcal{C}) \rightarrow \widehat{\text{Cat}}_\infty \),

- this latter functor is continuous (cf. [13, Proposition 6.3.2.3, Theorem 6.3.5.13]).

We will leverage these facts to show that \( \mathcal{C}SS_k|_{\mathcal{L}\text{Top}_\text{ét}} \) is continuous. For simplicity of exposition, we restrict to the case that \( k = 1 \).

Suppose that \( q : I \rightarrow \mathcal{L}\text{Top}_\text{ét} \) is a diagram. Then we may identify the limit of \( 1\text{-Simpl}' \circ q \) with the \( \infty \)-category

$$
\lim (1\text{-Simpl}' \circ q) \subset \text{Fun}_I \left(I, q^* (1\text{-Simpl}(\mathcal{L}\text{Top}_\text{ét}))\right)
$$

of coCartesian sections of the pulled-back presentable fibration \( q^* (1\text{-Simpl}(\mathcal{L}\text{Top}_\text{ét})) \rightarrow I \).

Similarly, we may identify the limit of \( \mathcal{C}SS_1 \circ q \) with the \( \infty \)-category

$$
\lim (\mathcal{C}SS_1 \circ q) \subset \text{Fun}_I \left(I, q^* \left( \int^\text{ét} \mathcal{C}SS_1 \right)\right)
$$

of coCartesian sections of the pulled-back presentable fibration \( q^* (\int^\text{ét} \mathcal{C}SS_1) \rightarrow I \).

Now let \( X \cong \lim q \in \mathcal{L}\text{Top}_\text{ét} \) be the limit of \( q \). Then \( \mathcal{C}SS_1(X) \) is the accessible localization of \( \text{Fun}((\Delta)^{\text{op}}, X) \cong \lim (1\text{-Simpl}' \circ q) \) spanned by those objects which satisfy

1. the Segal conditions (3.1) which specify the category objects;
2. the completeness conditions; namely (in the case that \( k = 1 \)) that \( \text{Gp}_X : \Delta^{\text{op}} \rightarrow X \) is equivalent to the constant functor.

So we have full and faithful inclusions of both \( \mathcal{C}SS_1(X) \) and \( \lim (\mathcal{C}SS_1 \circ q) \) into the \( \infty \)-category, \( \text{Fun}((\Delta)^{\text{op}}, X) \), of co-Cartesian sections of \( q^* (1\text{-Simpl}(\mathcal{L}\text{Top}_\text{ét})) \rightarrow I \). Using the universal property for the limit yields a diagram of full and faithful inclusions:

$$
\mathcal{C}SS_1(X) \hookrightarrow \lim (\mathcal{C}SS_1 \circ q) \hookrightarrow \text{Fun}((\Delta)^{\text{op}}, X).
$$

Thus, it suffices to show that any coCartesian section of \( q^* (\int^\text{ét} \mathcal{C}SS_1) \rightarrow I \) lies in the essential image of the leftmost functor - i.e. satisfies conditions (1) and (2). As a first step, notice that \( q^* (\int^\text{ét} \mathcal{C}SS_1) \rightarrow I \) satisfies conditions (1) and (2) fibrewise.

\[\text{when } k > 1, \text{ one also has constancy conditions } \text{(3.3)}, \text{ which are likewise given as limits.}\]
For every \(i \in I\), let \(\pi^*_i : \mathcal{X} \xrightarrow{\sim} \mathcal{X}(i) : \pi_{i*}\) denote the étale geometric morphism fitting into the limit cone. Recall that left adjoints of étale geometric morphisms \(f^* : \mathcal{Y} \to \mathcal{Z}\) are continuous. Now, since the conditions for a simplicial object \(X_\bullet \in \text{Fun}((\Delta)^{op}, \mathcal{X})\) to be a category object are given in terms of limits, Theorem 2.1 implies that \(X_\bullet\) is a category object if and only if each of the simplicial objects \((\pi^*_i)_!(X_\bullet)\) are category objects.

Next, \([3, \text{Proposition 2.20}]\) implies that left adjoints of étale geometric morphisms \(f^* : \mathcal{Y} \to \mathcal{Z}\) commute with the underlying groupoid functors, i.e.

\[
\begin{array}{ccc}
\text{Seg}(\mathcal{Y}) & \xrightarrow{(f^*)_!} & \text{Seg}(\mathcal{Z}) \\
\downarrow \text{Gp} & & \downarrow \text{Gp} \\
\text{Gpd}(\mathcal{Y}) & \xrightarrow{(f^*)_!} & \text{Gpd}(\mathcal{Z})
\end{array}
\]

commutes. Suppose now that \(X_\bullet \in \text{Seg}(\mathcal{X})\) is a category object, which we may identify with a coCartesian section \(X'_\bullet : I \to q^*(1\text{-Simpl}(\mathcal{L}\text{Top}_{\text{et}}))\). Applying Theorem 2.2 we see that the underlying groupoid \(\text{Gp} X_\bullet\) can be identified with the coCartesian section \(\text{Gp} \circ X'_\bullet : I \to q^*(1\text{-Simpl}(\mathcal{L}\text{Top}_{\text{et}}))\) obtained by applying the underlying groupoid functor fibrewise.\(^{13}\) Consequently, for every \(i \in I\) we have

\[(\pi^*_i)_! \text{Gp} X_\bullet \cong \text{Gp} ((\pi^*_i)_! X_\bullet).\]

Therefore

\[\text{Gp} X_\bullet : \Delta^{op} \to \mathcal{X}\]

is essentially constant if and only if each

\[\text{Gp} ((\pi^*_i)_! X_\bullet) : \Delta^{op} \to q(i)\]

is essentially constant.

Thus, we have shown \(CSS_1(\mathcal{X}) \xrightarrow{\lim} (CSS_1 \circ q)\) is an equivalence, which proves that \(3.8a\) preserves small limits.

Now for any \(\infty\)-topos \(\mathcal{X}\), the functor \(\mathcal{X}^{op} \to \mathcal{L}\text{Top}_{\text{et}}\) given by \(3.7a\) factors as an equivalence followed by the forgetful functor from an undercategory \(3.7b\); hence it preserves small limits (cf. \([13, \text{Proposition 1.2.13.8}]\)). It follows that the composite

\[\mathcal{X}^{op} \to \mathcal{L}\text{Top}_{\text{et}} \xrightarrow{CSS_k} \hat{\text{Cat}}_{\infty}\]

also preserves small limits, so \(3.8b\) is a sheaf.

\(^{13}\)See \([12, \text{Proposition 7.3.2.6}]\) to confirm that this right adjoint can be applied fibrewise in a coherent manner.
4. ∞-CATEGORIES OF SPANS.

Let \( \mathcal{C} \) be an \( \infty \)-category with pullbacks. In [4], Barwick introduces the \( \infty \)-category \( \text{Span}(\mathcal{C}) \), which has the same space of objects as \( \mathcal{C} \), but whose morphisms between two objects \( c_0, c_1 \in \mathcal{C} \) is the space of diagrams in \( \mathcal{C} \) of the form

\[
\begin{array}{c}
\scriptstyle x \\
\scriptstyle c_1 \downarrow \\
\scriptstyle c_0
\end{array}
\]

That is, spans \( c_0 \to c_1 \) in \( \mathcal{C} \). Composition of two such morphisms is given by taking the fibred product. Haugseng [8] extends this construction, introducing an \((\infty, k)\)-category \( \text{Span}_k(\mathcal{C}) \) of iterated spans in \( \mathcal{C} \), whose 2-morphisms are spans between spans, and so forth. In this section, we show that the functor \( \mathcal{C} \to \text{Span}_k(\mathcal{C}) \) depends continuously on \( \mathcal{C}. \)

4.1. Continuity of the formation of \( \infty \)-categories of iterated spans. We now briefly recall Haugseng’s construction. Let \( \Sigma^n \) denote the partially ordered set whose objects are pairs of numbers \((i, j)\) such that \( 0 \leq i \leq j \leq n \), and \((i, j) \leq (i', j')\) if \( i \leq i' \) and \( j' \leq j \). We may picture the poset \( \Sigma^n \) (using Barwick’s notation \( \bar{p} = n - p \)) as follows:

For any map of totally ordered sets \( \phi : [n] \to [m] \), the map \((i, j) \mapsto (\phi(i), \phi(j))\) induces a monotone map \( \Sigma^n \to \Sigma^m \); and thus we have a functor \( \Sigma^* : \Delta \to \text{Cat}_\infty \). Similarly, taking \( k \)-fold product, \( \Sigma^{n_1, \ldots, n_k} := \Sigma^{n_1} \times \cdots \times \Sigma^{n_k} \) defines a functor

\[
\Sigma^{*, \ldots, *}_k : \Delta^k \to \text{Cat}_\infty.
\]

Suppose that \( \mathcal{C} \) is an \( \infty \)-category with finite limits. We will be interested in functors \( f : \Sigma^{n_1, \ldots, n_k} \to \mathcal{C} \). We let \( \Delta^k \subseteq \Sigma^k \) denote the full subcategory

\[
\begin{array}{c}
\scriptstyle 00 \\
\scriptstyle 11 \\
\scriptstyle 22 \\
\scriptstyle 00
\end{array}
\]

\[
\begin{array}{c}
\scriptstyle 01 \\
\scriptstyle 12 \\
\scriptstyle 22 \\
\scriptstyle 01
\end{array}
\]

\[
\begin{array}{c}
\scriptstyle 02 \\
\scriptstyle 13 \\
\scriptstyle 20 \\
\scriptstyle 02
\end{array}
\]

\[
\begin{array}{c}
\scriptstyle 10 \\
\scriptstyle 20 \\
\scriptstyle 11 \\
\scriptstyle 10
\end{array}
\]

\[
\begin{array}{c}
\scriptstyle 31 \\
\scriptstyle 21 \\
\scriptstyle 11 \\
\scriptstyle 00
\end{array}
\]

\[
\begin{array}{c}
\scriptstyle 00 \\
\scriptstyle 11 \\
\scriptstyle 22 \\
\scriptstyle 00
\end{array}
\]
spanned by those pairs \((i, j)\) with \(j - i \leq 1\). Similarly, we define \(\Lambda^{n_1 \ldots n_k} := \Lambda^{n_1} \times \ldots \times \Lambda^{n_k}\), and let \(\iota_{n_1 \ldots n_k} : \Lambda^{n_1 \ldots n_k} \to \Sigma^{n_1 \ldots n_k}\) denote the inclusion.

**Definition 4.1** (\([13]\)). We say that a functor \(f : \Sigma^{n_1 \ldots n_k} \to C\) is *Cartesian* if it is a right Kan extension of \(f \circ \iota_{n_1 \ldots n_k}\), and we let \(\text{Fun}^{-\text{Cart}}(\Sigma^{n_1 \ldots n_k}, C) \subseteq \text{Fun}(\Sigma^{n_1 \ldots n_k}, C)\) denote the full subcategory spanned by the Cartesian functors. We let

\[ \text{Map}^{-\text{Cart}}(\Sigma^{n_1 \ldots n_k}, C) := \iota_! \text{Fun}^{-\text{Cart}}(\Sigma^{n_1 \ldots n_k}, C) \subseteq \text{Fun}^{-\text{Cart}}(\Sigma^{n_1 \ldots n_k}, C) \]

denote the classifying space of Cartesian functors.

For example, when \(k = 1\), a Cartesian functor \(f : \Sigma^{n_1} \to C\) is a diagram of the form

\[
\begin{array}{ccccccc}
  c_{00} & \downarrow & c_{01} & \downarrow & & & \downarrow & c_{10} \\
  \downarrow & & c_{02} & \downarrow & c_{12} & \downarrow & & c_{20} \\
  & & c_{11} & \downarrow & c_{22} & \downarrow & c_{21} & \downarrow & c_{00}
\end{array}
\]

where each square is a pullback in \(C\). Such a diagram is to be understood as a composable sequence of spans

\[ c_{00} \to c_{01} \to c_{11} \to c_{22} \to \cdots \to c_{nn} \]

where for \(i < j < k\) each \(c_{ik}\) is the composite (fibre product) of \(c_{ii} \to c_{ij} \to c_{jk} \to c_{kk}\).

Recall that \(\text{Cat}_\infty\) is a Cartesian closed \(\infty\)-category, in particular, there is an internal mapping object bi-functor (cf. \([12]\) Remark 4.2.1.31)),

\[
\text{Cat}_\infty^{op} \times \text{Cat}_\infty \xrightarrow{(D,C)\mapsto \text{Fun}(D,C)} \text{Cat}_\infty,
\]

which is separately continuous in either variable. Composing with \(\Sigma^{* \cdots * : \Delta^k \to \text{Cat}_\infty}\) yields a functor \((\Delta^k)^{op} \times \text{Cat}_\infty \to \text{Cat}_\infty\), or equivalently, a functor

\[
\overline{\text{SPAN}}^+_k : \text{Cat}_\infty \to \text{Fun}((\Delta^k)^{op}, \text{Cat}_\infty),
\]

\[
\mathcal{C} \to \left[ (n_1, \ldots, n_k) \to \text{Fun}(\Sigma^{n_1 \ldots n_k}, C) \right]
\]

which is continuous (by \([13]\) Corollary 5.1.2.3 and the continuity of \((\Delta^k)^{op}\) in the second variable).
Let $\mathcal{C}_{\text{at}}^{\text{lex}} \subset \mathcal{C}_{\infty}$ consist of those $\infty$-categories with finite limits and functors preserving finite limits. Suppose that $\mathcal{C} \in \mathcal{C}_{\text{at}}^{\text{lex}}$ has finite limits, and $f : \Sigma^{n_1, \ldots, n_k} \to \mathcal{C}$ is Cartesian (in the sense of Definition 4.1); then for any finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$, the composite $F \circ f : \Sigma^{n_1, \ldots, n_k} \to \mathcal{D}$ is also Cartesian. Therefore, following [8], we may define

$$\text{SPAN}^+_k : \mathcal{C}_{\text{at}}^{\text{lex}} \to \text{Fun}((\Delta^k)^{op}, \mathcal{C}_{\infty}),$$

(4.3b)

to be the subfunctor of $\text{SPAN}^+_k |_{\mathcal{C}_{\text{at}}^{\text{lex}}}$ which assigns to each $\mathcal{C} \in \mathcal{C}_{\text{at}}^{\text{lex}}$ and each $(n_1, \ldots, n_k) \in \Delta^k$ the full subcategory spanned by the Cartesian functors $\Sigma^{n_1, \ldots, n_k} \to \mathcal{C}$. As explained in [8] the functor (4.3b) takes values in $k$-uple category objects (see also [4]).

**Lemma 4.1.** The functor

$$\text{SPAN}^+_k : \mathcal{C}_{\text{at}}^{\text{lex}} \to \text{Cat}^k(\mathcal{C}_{\infty})$$

is continuous.

**Proof.** Following Lemma 3.1, we need only show that the composite

$$\text{Cat}_{\text{at}}^{\text{lex}} \xrightarrow{\text{SPAN}^+_{\text{lex}}} \text{Cat}^k(\mathcal{C}_{\infty}) \xrightarrow{i^*} \text{Fun}((\text{Mor} \Delta^k)^{op}, \mathcal{C}_{\infty})$$

(4.4a)

is continuous, where $i : \text{Mor} \Delta^k \to \Delta^k$ is as in Lemma 3.1. But (4.4a) is equivalent to the composite

$$\text{Cat}_{\text{at}}^{\text{lex}} \xrightarrow{\text{SPAN}^+_{\text{at}}} \text{Fun}((\Delta^k)^{op}, \mathcal{C}_{\infty}) \xrightarrow{i^*} \text{Fun}((\text{Mor} \Delta^k)^{op}, \mathcal{C}_{\infty}).$$

(4.4b)

The first arrow in (4.4b) is continuous by [17] (or Theorem 2.1, the second arrow is continuous since $\text{Cat}_{\text{at}}^{\text{lex}}$ is Cartesian closed (as explained above), and the final arrow is continuous by [18] Corollary 5.1.2.3].

Next, let $\iota : \mathcal{C}_{\text{at}}^{\text{lex}} \to \mathcal{S}$ denote the right adjoint to the inclusion, which sends an $\infty$-category $\mathcal{C}$ to its classifying space of objects, the largest Kan complex contained in $\mathcal{C}$. Then as in [8], we define

$$\text{SPAN}_k : \mathcal{C}_{\text{at}}^{\text{lex}} \to \text{Cat}^k(\mathcal{S}),$$

(4.5)

to be the composite $\iota \circ \text{SPAN}^+_k$.

Finally let $U_{\text{seg}} : \text{Cat}^k(\mathcal{S}) \to \text{Seg}^k(\mathcal{S})$ denote a right adjoint to $\text{Seg}^k(\mathcal{S}) \to \text{Cat}^k(\mathcal{S})$. Then $\text{Span}_k := U_{\text{seg}} \circ \text{SPAN}_k$ takes values in complete Segal spaces [8 Corollary 3.18].

**Theorem 4.1.** The functor

$$\text{Span}_k : \mathcal{C}_{\text{at}}^{\text{lex}} \to \text{CSS}_k(\mathcal{S})$$

is continuous.
Proof. \( \text{Span}_k \) is the composite \( U_{\text{Seg}} \circ \iota \circ \text{SPAN}_k^+ \), the first two functors are continuous (since they are right adjoints), and the last functor is continuous by Lemma 4.1. \( \square \)

Remark 4.1. Let \( \mathcal{K} \) be the subcategory inclusions \( i_{n_1,...,n_k} : \Delta^{n_1,...,n_k} \rightarrow \Sigma^{n_1,...,n_k} \) used in the definition of a Cartesian functor (cf. Definition 4.1). Let \( \mathcal{Cat}_{\infty}^K \subset \mathcal{Cat}_{\infty} \) denote the subcategory consisting of \( \infty \)-categories which admit all right Kan extensions along any \( i_{n_1,...,n_k} : \Delta^{n_1,...,n_k} \rightarrow \Sigma^{n_1,...,n_k} \) and of functors which preserve those right Kan extensions. Then \( \mathcal{Cat}_{\infty}^K \) is the maximal subcategory of \( \mathcal{Cat}_{\infty} \) on which the functor \( \text{SPAN}_k^+ \) may be defined. As a consequence of Corollary 2.1, each of the functions

\[
\text{SPAN}_k^+ : \mathcal{Cat}_{\infty}^K \rightarrow \text{Cat}(\mathcal{Cat}_{\infty}) \\
\text{SPAN}_k : \mathcal{Cat}_{\infty}^K \rightarrow \text{Cat}(\mathcal{S}) \\
\text{Span}_k : \mathcal{Cat}_{\infty}^K \rightarrow \text{CSS}_k(\mathcal{S})
\]

are continuous.

4.2. The sheaf of iterated spans with local systems. Suppose that \( \mathcal{X} \) is an \( \infty \)-topos, and \( X_{\bullet,...,\bullet} \in \text{CSS}_k(\mathcal{X}) \) is a complete \( k \)-fold Segal object in \( \mathcal{X} \). In [8], Haugseng gave an elegant construction of the \((\infty, k)\)-category \( \text{Span}_k(\mathcal{X}, X_{\bullet,...,\bullet}) \) of iterated \( k \)-fold spans in \( \mathcal{X} \) with local systems valued in \( X_{\bullet,...,\bullet} \).

- whose objects are objects in \( \mathcal{X} \) equipped with a map to the objects of the local system, \( X_{\bullet,...,\bullet} \).
- whose morphisms are spans in \( \mathcal{X} \) equipped with compatible maps to the space of morphisms of the local system, \( X_{\bullet,...,\bullet} \).
- \( \ldots \)
- and whose \( i \)-morphisms are \( i \)-fold spans in \( \mathcal{X} \) equipped with compatible maps to the space of \( i \)-morphisms of the local system, \( X_{\bullet,...,\bullet} \).

In this section, we show that for any continuous functor \( \sigma : \mathcal{X}^{\text{op}} \rightarrow \int \text{CSS}_k \) over

\[
\mathcal{X}^{\text{op}} \xrightarrow{U \mapsto \mathcal{X} \cup U} \mathcal{L}\text{Top},
\]

the functor

\[
\mathcal{X}^{\text{op}} \xrightarrow{U \mapsto \text{Span}_k(\mathcal{X} \cup U, \sigma(U))} \text{CSS}_k(\mathcal{S})
\]

forms an \((\infty, k)\)-stack over \( \mathcal{X} \).

We begin by describing the functor

\[
\int \text{CSS}_k \xrightarrow{(\mathcal{X}, X_{\bullet,...,\bullet}) \mapsto \text{Span}_k(\mathcal{X}, X_{\bullet,...,\bullet})} \text{CSS}_k(\mathcal{S})
\]

in more detail.

As in [8], we let \( \mathcal{S} \xrightarrow{q} \Delta^{\text{op}} \) denote the Grothendieck fibration classified by the functor \( \Sigma^* : \Delta \rightarrow \text{Cat} \), whose objects are pairs \( ([n], (i, j)) \) with \( 0 \leq i \leq j \leq n \), and whose
are pairs of morphisms $\phi: [m] \to [n]$ in $\Delta$ and $(i, j) \to (\phi(i'), \phi(j'))$ in $\Sigma^n$. Let $i^* \mathcal{Z} \to \mathcal{L}\text{Top}$ denote the (canonical) presentable fibration classified by the inclusion $i: \mathcal{L}\text{Top} \hookrightarrow \mathcal{C}at_{\infty}$. The functor $\text{SPAN}^+_k: \mathcal{L}\text{Top} \times (\Delta^k)^{op} \to \mathcal{C}at_{\infty}$ classifies the coCartesian fibration (cf. [13, Corollary 3.2.2.13])

$$(i^* \mathcal{Z} \times (\Delta^k)^{op})_{\mathcal{L}\text{Top} \times ^\ast k} \to \mathcal{L}\text{Top} \times (\Delta^k)^{op},$$

whose fibre over any $(\mathcal{X}; (n_1, \ldots, n_k)) \in \mathcal{L}\text{Top} \times (\Delta^k)^{op}$ is equivalent to

$$\text{Fun}(\Sigma^{n_1, \ldots, n_k}, \mathcal{X}).$$

Similarly, $\text{SPAN}^+_k: \mathcal{L}\text{Top} \times (\Delta^k)^{op} \to \mathcal{C}at_{\infty}$ classifies the coCartesian fibration defined as the full subcategory

$$\int \text{SPAN}^+_k \subset (i^* \mathcal{Z} \times (\Delta^k)^{op})_{\mathcal{L}\text{Top} \times ^\ast k}$$

spanned (over $(\mathcal{X}; (n_1, \ldots, n_k)) \in \mathcal{L}\text{Top} \times (\Delta^k)^{op}$) by the Cartesian functors $\Sigma^{n_1, \ldots, n_k} \to \mathcal{X}$ (cf. Definition 4.1).

There is a second functor

$$\Pi: \hat{\Sigma} \xrightarrow{[n].(i,j)\mapsto[j-i]} \Delta^{op}$$

which sends the map (4.6) to

$$j-i \xrightarrow{k \mapsto \phi(k+i'-i)} [j' - i'].$$

The corresponding morphism of Cartesian fibrations

$$\Delta^{op} \times \Delta^{op} \xrightarrow{\Pi \times q} \hat{\Sigma}$$

induces a morphism of coCartesian fibrations

$$(i^* \mathcal{Z} \times (\Delta^k)^{op})_{\mathcal{L}\text{Top} \times (k)^{op} \times (k)^{op}} \to (i^* \mathcal{Z} \times (\Delta^k)^{op})_{\mathcal{L}\text{Top} \times ^\ast k}$$

$$(\mathcal{L}\text{Top} \times (\Delta^k)^{op})$$
By [8, Lemma 4.3] this restricts to a morphism:

\[(\int CSS_k) \times (\Delta^k)^{op} \to \int \text{SPAN}_k^+ \]

\[s_0 \]

\[\mathcal{L}\text{Top} \times (\Delta^k)^{op} \]

(4.7)

which sends any \((\mathcal{X}; (n_1, \ldots, n_k)) \in \mathcal{L}\text{Top} \times (\Delta^k)^{op}\) to

\[X_{\bullet, \ldots, \bullet} \circ \Pi_{n_1, \ldots, n_k} \in \text{Fun} \rightarrow \text{Cart}(\Sigma^{n_1, \ldots, n_k}, \mathcal{X}),\]

where \(\Pi_{n_1, \ldots, n_k} := \Pi|_{n_1, \ldots, n_k} \).

In turn, (4.7) defines a section of the left hand arrow in the pullback square

\[
\begin{array}{ccc}
\mathcal{Q} & \rightarrow & \int \text{SPAN}_k^+ \\
\downarrow s & & \downarrow \\
\left(\int CSS_k \right) \times (\Delta^k)^{op} & \to & \mathcal{L}\text{Top} \times (\Delta^k)^{op}
\end{array}
\]

For brevity, we denote \(\mathcal{D} = (\int CSS_k) \times (\Delta^k)^{op}\), and we define \(Q/s \to \mathcal{D} \cong (\int CSS_k) \times (\Delta^k)^{op}\) to be the simplicial set satisfying the universal property that for any morphism of simplicial sets \(Y \to \mathcal{D}\), commutative diagrams of the form

\[
\begin{array}{ccc}
\mathcal{D} & \rightarrow & Q \\
\downarrow Y \circ \mathcal{D} & & \downarrow \\
\mathcal{D} & \rightarrow & \mathcal{D}
\end{array}
\]

(where \(Y \circ \mathcal{D} = Y \times \Delta^1 \prod_{Y \times \{1\}} \mathcal{D}\)) correspond to diagrams of the form

\[
\begin{array}{ccc}
\mathcal{D} & \rightarrow & Q/s \\
\downarrow & & \\
\mathcal{D} & \rightarrow & 
\end{array}
\]

By [13, Proposition 4.2.2.4.] \(Q/s \to (\int CSS_k) \times (\Delta^k)^{op}\) is a coCartesian fibration whose fibre over \((\mathcal{X}, X_{\bullet, \ldots, \bullet}; (n_1, \ldots, n_k)) \in (\int CSS_k) \times (\Delta^k)^{op}\) is equivalent to the overcategory

\[
\text{Fun} \rightarrow \text{Cart}(\Sigma^{n_1, \ldots, n_k}, \mathcal{X})/X_{\bullet, \ldots, \bullet} \circ \Pi_{n_1, \ldots, n_k}
\]

Let

\[
(\text{SPAN}_k^+)^{op} : CSS_k \to \text{Fun} ((\Delta^k)^{op}, \text{Cat}_\infty)
\]
be a functor classifying $Q/s$. By [8, Proposition 4.5], $(\text{SPAN}_k^+)/s$ takes values in $k$-uple category objects in $\hat{\text{Cat}}_\infty$. We define

$$\text{Span}_k := U_{\text{Seg}} \circ i \circ (\text{SPAN}_k^+)/s : \int CSS_k \to \text{Seg}_k(\hat{\mathcal{S}}).$$

It follows from [8, Proposition 4.8] that $\text{Span}_k$ takes values in complete $k$-fold Segal spaces, i.e. we have a functor

$$(4.8) \quad \text{Span}_k : \int CSS_k \to CSS_k(\hat{\mathcal{S}})$$

which sends a complete $k$-fold Segal object $X \in CSS_k(\mathcal{X})$ to the complete $k$-fold Segal space

$$\text{Span}_k(\mathcal{X}, X \in CSS_k(\mathcal{X}))$$

of iterated spans in $\mathcal{X}$ with local systems valued in $X \in CSS_k(\mathcal{X})$ (cf. [8, § 4]).

4.2.1. **Continuity of** $(\mathcal{X}, X \in CSS_k(\mathcal{X})) \to \text{Span}_k(\mathcal{X}, X \in CSS_k(\mathcal{X}))$.

**Lemma 4.2.** The functor $(4.8)$ preserves small products.

**Proof.** Since $\int^{\text{ét}} CSS_k \subseteq \int CSS_k$ is a continuous inclusion of a wide subcategory (i.e. it contains all the objects), it suffices to show that the restriction of $(4.8)$ to $\int^{\text{ét}} CSS_k$ preserves small products.

Now suppose $\{X^j \in CSS_k(\mathcal{X}^j)\}_{j \in J}$ is a set of complete Segal objects indexed by a small set $J$. Since $p : \int^{\text{ét}} CSS_k \to \mathcal{L}\text{Top}_{\text{ét}}$ is a presentable fibration and $\mathcal{L}\text{Top}_{\text{ét}}$ has small products, we may compute the product

$$\prod_{j \in J}(\mathcal{X}^j, X^j \in CSS_k(\mathcal{X}^j)) \in \int CSS_k$$

by first computing the product $\prod_{j \in J} \mathcal{X}^j$ in $\mathcal{L}\text{Top}_{\text{ét}}$, and then computing the $p$-relative product of $\{X^j \in CSS_k(\mathcal{X}^j)\}_{j \in J}$ over $\prod_{j \in J} \mathcal{X}^j$.

Note that [13, Proposition 6.3.2.3 and Theorem 6.3.5.13] imply that $\prod_{j \in J} \mathcal{X}^j$ is just the product of the $\infty$-categories $\mathcal{X}^j$ (i.e. we can take this product in $\hat{\text{Cat}}_\infty$ rather than $\mathcal{L}\text{Top}_{\text{ét}}$). Next, Theorem [32] implies that the fibre of $\int^{\text{ét}} CSS_k$ over $\prod_{j \in J} \mathcal{X}^j$ is just

$$CSS_k(\prod_{j \in J} \mathcal{X}^j) \cong \prod_{j \in J} CSS_k(\mathcal{X}^j),$$

where the right hand product is taken in $\hat{\text{Cat}}_\infty$. Consequently, the $p$-relative product of $\{X^j \in CSS_k(\mathcal{X}^j)\}_{j \in J}$ over $\prod_{j \in J} \mathcal{X}^j$ is

$$\prod_{j \in J} X^j \in CSS_k(\mathcal{X}^j).$$
Next we argue that the restriction of \((\text{SPAN}^+_k)^/s\) to \(\int^\text{et} CSS_k\) preserves small products. In view of Lemma 3.1, we need only show that for any \(0 \leq i_1, \ldots, i_k \leq 1\), the functor
\[
\int^\text{et} CSS_k \xymatrix{ (X,X_\bullet,\ldots,\bullet) \ar[r] & \text{Fun}(\Sigma^{i_1,\ldots,i_k} X_\bullet,\ldots,\bullet,\Pi_{i_1,\ldots,i_k}) & \hat{\text{Cat}}_\infty. }
\]
preserves small products.

Notice that the continuous functor
\[
\prod_{j \in J} CSS_k(\mathcal{X}^j) \cong CSS_k(\prod_{j \in J} \mathcal{X}^j) \xymatrix{ \ar[r] & \text{Fun}(\Delta_k^{op}, \prod_{j \in J} \mathcal{X}^j) \ar[r]^-{\Pi_{i_1,\ldots,i_k}} & \text{Fun}(\Sigma^{i_1,\ldots,i_k}, \prod_{j \in J} \mathcal{X}^j)}
\]
takes \(\prod_{j \in J} X_j^i, \ldots, \bullet\) to
\[
(\prod_{j \in J} X_j^i, \ldots, \bullet) \circ \Pi_{i_1,\ldots,i_k} \cong \prod_{j \in J} (X_j^i, \ldots, \bullet) \circ \Pi_{i_1,\ldots,i_k}).
\]
So
\[
\text{Fun}(\Sigma^{i_1,\ldots,i_k}, \prod_{j \in J} \mathcal{X}^j) / (\Pi_{j \in J} X_j^i, \ldots, \bullet) \circ \Pi_{i_1,\ldots,i_k}) \cong \prod_{j \in J} \text{Fun}(\Sigma^{i_1,\ldots,i_k}, \mathcal{X}^j) / (X_j^i, \ldots, \bullet) \circ \Pi_{i_1,\ldots,i_k}),
\]
which implies that (4.9) preserves small products.

Finally, we have \(\text{Span}_k = U_{\text{Seg}} \circ \iota \circ (\text{SPAN}^+_k)^/s\), and since \(U_{\text{Seg}}\) and \(\iota\) are both right adjoints, they are continuous, which implies the statement we wished to prove.

\[\square\]

**Lemma 4.3.** The functor (4.8) preserves pullbacks.

**Proof.** We begin by arguing that \((\text{SPAN}^+_k)^/s : \int CSS_k \to \hat{\text{Cat}}_\infty\) preserves pullbacks. In view of Lemma 3.1, we need only show that for any \(0 \leq i_1, \ldots, i_k \leq 1\), the functor
\[
\int CSS_k \xymatrix{ (X,X_\bullet,\ldots,\bullet) \ar[r] & \text{Fun}(\Sigma^{i_1,\ldots,i_k} X_\bullet,\ldots,\bullet,\Pi_{i_1,\ldots,i_k}) & \hat{\text{Cat}}_\infty. }
\]
preserves pullbacks.

Suppose we have a diagram
\[
(\mathcal{X}, X_\bullet,\ldots,\bullet) \to (\mathcal{Z}, Z_\bullet,\ldots,\bullet) \leftarrow (\mathcal{Y}, Y_\bullet,\ldots,\bullet)
\]
in \(\int CSS_k\) (here \(X_\bullet,\ldots,\bullet \in CSS_k(\mathcal{X})\), \(Y_\bullet,\ldots,\bullet \in CSS_k(\mathcal{Y})\), and \(Z_\bullet,\ldots,\bullet \in CSS_k(\mathcal{Z})\)). To compute the pullback of (4.11a) we use [13 Corollary 4.3.1.11]. That is, we first compute
the pullback

\[
\begin{array}{ccc}
W & \xrightarrow{b^*} & Y \\
\downarrow a^* & & \downarrow c^* \\
X & \xrightarrow{f^*} & Z
\end{array}
\]

(4.11b)

in \(\mathcal{L}Top\) and then take the relative pullback

\[
\begin{array}{ccc}
W & \xrightarrow{(a_*,X \times_{(c_*)} Z (b_*))_*} & Y \\
\downarrow (a_*,X \times_{(c_*)} Z (b_*))_* & & \downarrow (b_*, Y \times_{(c_*)} Z (b_*))_* \\
X & \xrightarrow{f^*} & Z
\end{array}
\]

(4.11c)

in the fibre \(CSS_k(W)\) over \(W\) (here we have dropped the abstract multi-indices on \(X_{\bullet,...,\bullet}\), \(Y_{\bullet,...,\bullet}\), and \(Z_{\bullet,...,\bullet}\)).

For any \(\infty\)-category \(\mathcal{C}\), let

\[
(\mathcal{C}_{\infty}/\mathcal{C}) \xhookrightarrow{\mathcal{P}(\mathcal{C})}
\]

(4.12a)

denote the functor which sends a presheaf over \(\mathcal{C}\) to the corresponding right fibration over \(\mathcal{C}\). Then \([13, Corollary\ 2.1.2.10]\) implies that (4.12a) is equivalent to a reflective left localization of \((\mathcal{C}_{\infty}/\mathcal{C})\); in particular (4.12a) is continuous (see also \([7, Theorem\ 4.5]\)). Since the Yoneda embedding is continuous, and the forgetful functor \((\mathcal{C}_{\infty}/\mathcal{C}) \rightarrow \mathcal{C}_{\infty}\) preserves pullbacks, it follows that the composite

\[
\mathcal{C} \xrightarrow{\mathcal{P}(\mathcal{C})} (\mathcal{C}_{\infty}/\mathcal{C}) \rightarrow \mathcal{C}_{\infty}
\]

(4.12b)

does as well.

Applying (4.10) to \(W_{\bullet,...,\bullet}\) (the top left corner of (4.11c)) and using the continuity of (4.12b) yields a pullback diagram

\[
\begin{array}{ccc}
\text{Fun}(\Sigma^{i_1,...,i_k}, W)_{/W_{\bullet,...,\bullet} \circ \Pi_{i_1,...,i_k}} & \rightarrow & \text{Fun}(\Sigma^{i_1,...,i_k}, W)_{/(b_*), Y_{\bullet,...,\bullet} \circ \Pi_{i_1,...,i_k}} \\
\downarrow & & \downarrow \\
\text{Fun}(\Sigma^{i_1,...,i_k}, W)_{/(a_*), X_{\bullet,...,\bullet} \circ \Pi_{i_1,...,i_k}} & \rightarrow & \text{Fun}(\Sigma^{i_1,...,i_k}, W)_{/(c_*), Z_{\bullet,...,\bullet} \circ \Pi_{i_1,...,i_k}}
\end{array}
\]

(4.13)
Now, since 
\[(a^*)_! : \text{Fun}(\Sigma i_1, \ldots, i_k, W) \backslash\!/ (a^*)_! \xleftarrow{\sim} \text{Fun}(\Sigma i_1, \ldots, i_k, X) : (a^*)_! \]
is an adjunction, we have a pullback square

\[
\begin{array}{ccc}
\text{Fun}(\Sigma^{i_1, \ldots, i_k}, W)/ (a^*)_! X \circ \Pi_{i_1, \ldots, i_k} & \xrightarrow{f^*} & \text{Fun}(\Sigma^{i_1, \ldots, i_k}, X) / X \circ \Pi_{i_1, \ldots, i_k} \\
\text{Fun}(\Sigma^{i_1, \ldots, i_k}, W) & \xrightarrow{(a^*)_!} & \text{Fun}(\Sigma^{i_1, \ldots, i_k}, X) \\
\end{array}
\]

Similarly, the right hand terms of (4.13) fit into analogous pullback squares. It follows that

\[
\text{Fun}(\Sigma^{i_1, \ldots, i_k}, W) / W \circ \Pi_{i_1, \ldots, i_k}
\]
fits into a limit diagram

\[
\begin{array}{ccc}
\text{Fun}(\Sigma^I, W) / W \circ \Pi_I & \xrightarrow{f^*_I} & \text{Fun}(\Sigma^I, X) / X \circ \Pi_I \\
\text{Fun}(\Sigma^I, X) / X \circ \Pi_I & \xrightarrow{g^*_I} & \text{Fun}(\Sigma^I, Y) / Y \circ \Pi_I \\
\text{Fun}(\Sigma^I, W) & \xrightarrow{b^*_I} & \text{Fun}(\Sigma^I, Y) \\
\end{array}
\]

(4.14)

where we have abbreviated the multi-index \(i_1, \ldots, i_k = I\) and dropped the abstract multi-indices on \(W \circ \cdot, X \circ \cdot, Y \circ \cdot, Z \circ \cdot\).

Since the bottom square in (4.14) is already a pullback square, it follows that the top square is also a pullback square, which proves that (4.10) preserves pullbacks.

Finally, we have \(\text{Span}_k = U_{\text{Seg}} \circ \iota \circ (\text{SPAN}_k^+)/\#,\) and since \(U_{\text{Seg}}\) and \(\iota\) are both right adjoints, they are continuous, which implies the statement we wished to prove.

\[\square\]

**Theorem 4.2.** The functor (4.8),

\[
\int CSS_k \xrightarrow{(X, X, \ldots)} \text{Span}_k (X, X, \ldots) \xrightarrow{CSS_k(\hat{S})} CSS_k(\hat{S}),
\]

preserves small limits.
Remark 4.2 (Warning!). The inclusion of a fibre $CSS_k(\mathcal{X}) \looparrowright \int CSS_k$ doesn’t preserve products or terminal objects (though it does preserve small limits with connected diagrams). So the functor

$$CSS_k(\mathcal{X}) \xrightarrow{X_\bullet,\ldots \mapsto \text{Span}_k(X_\bullet,\ldots)} \int CSS_k(\hat{S})$$

is not continuous: while it does preserve small limits with connected diagrams, it generally fails to preserve products or terminal objects.

Proof. According to [13, Proposition 4.4.2.7], it suffices to prove this result for pullbacks and small products; thus the result follows from Lemmas [4.2 and 4.3]. □

Theorem 4.3. Suppose that $\mathcal{X}$ is an $\infty$-topos and $\sigma : \mathcal{X}^{\text{op}} \to \int CSS_k$ is a continuous functor fitting into the diagram

$$\xymatrix{ \mathcal{X}^{\text{op}} \ar[r]^-{\sigma} \ar[dr] & \int CSS_k \ar[dl] \\
\mathcal{L}\text{Top} & }$$

where the left diagonal arrow $\mathcal{X}^{\text{op}} \xrightarrow{U\mapsto X_i/U} \mathcal{L}\text{Top}$ is [3.7]. Then

$$(4.15) \quad \text{Span}_k \circ \sigma : \mathcal{X}^{\text{op}} \xrightarrow{U\mapsto \text{Span}_k(X_i/U,\sigma(U))} \int CSS_k(\hat{S})$$

forms an $(\infty,k)$-stack over $\mathcal{X}$.

In particular, given any complete $k$-fold Segal space $X_\bullet,\ldots \in CSS_k(\mathcal{X})$, iterated spans in $\mathcal{X}$ with local systems valued in $X_\bullet,\ldots$ form an $(\infty,k)$-stack

$$(4.16) \quad \mathcal{X}^{\text{op}} \xrightarrow{U\mapsto \text{Span}_k(X_i/U,\mathcal{X}_U,\ldots)} \int CSS_k(\hat{S})$$

over $\mathcal{X}$.

Proof. The first statement is equivalent to the continuity of (4.15), which follows directly from Theorem 4.2.

Let $F : \mathcal{X}^{\text{op}} \to \mathcal{L}\text{Top}_{\text{et}}$ be defined by [3.7]. Of course, we have $\mathcal{X}^{\text{op}} \cong (\mathcal{L}\text{Top}_{\text{et}})_{X/}$, so $F^*(\int^{\text{et}} CSS_k)$ is equivalent to the pullback

$$\xymatrix{ F^*(\int^{\text{et}} CSS_k) \ar[r] \ar[d] & \int^{\text{et}} CSS_k \ar[d] \\
(\mathcal{L}\text{Top}_{\text{et}})_{X/} \ar[r]^-F & \mathcal{L}\text{Top}_{\text{et}}}$$

Since $\mathcal{X} \in (\mathcal{L}\text{Top}_{\text{et}})_{X/}$ is an initial object, [13, Proposition 3.3.3.1] and Theorem 3.1 imply that the $\infty$-category of coCartesian sections of $F^*(\int^{\text{et}} CSS_k) \to (\mathcal{L}\text{Top}_{\text{et}})_{X/} \cong \mathcal{X}^{\text{op}}$
is equivalent to $CSS_k(\mathcal{A})$. In particular, any complete $k$-fold complete Segal object $X_{•,\ldots,•} \in CSS_k(\mathcal{A})$ determines a coCartesian section $\sigma : \mathcal{A}^{\text{op}} \to F^*(\int^{\text{ét}}CSS_k)$. By [13, Lemma 6.3.3.5], $\sigma$ is continuous, which implies that (4.16) is an $(\infty,k)$-stack. □

REFERENCES

[1] Dimitri Ara. Higher quasi-categories vs higher Rezk spaces. Journal of K-Theory, 14(3):701, 2014.
[2] Clark Barwick. ($\infty,n$)-Cat as a closed model category. PhD thesis, 2005.
[3] Clark Barwick. From operator categories to topological operads. February 2013. Preprint available at http://arxiv.org/abs/1302.5756v2
[4] Clark Barwick. On the Q construction for exact quasicategories. January 2013. Preprint available at http://arxiv.org/abs/1301.4725v2
[5] Clark Barwick and Christopher Schommer-Pries. On the Unicity of the Homotopy Theory of Higher Categories. November 2011. Preprint available at http://arxiv.org/abs/1112.0040v4
[6] Damien Calaque. Lagrangian structures on mapping stacks and semi-classical TFTs. 2013. Preprint available at http://arxiv.org/abs/1306.3235
[7] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in $\infty$-categories. pages 1–33, January 2015. Preprint available at http://arxiv.org/abs/1501.02161
[8] Rune Haugseng. Iterated spans and “classical” topological field theories. September 2014. Preprint available at http://arxiv.org/abs/1409.0837
[9] André Joyal. The theory of quasi-categories and its applications. 2008. Preprint.
[10] David Li-Bland and Alan Weinstein. Selective Categories and Linear Canonical Relations. SIGMA. Symmetry, Integrability and Geometry. Methods and Applications, 10:1–31, October 2014. 46 pages.
[11] Jacob Lurie. Derived Algebraic Geometry X: Formal Moduli Problems. Preprint available at http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf
[12] Jacob Lurie. Higher Algebra. Preprint available at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf
[13] Jacob Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.
[14] Jacob Lurie. $(\infty,2)$-Categories and the Goodwillie Calculus I. pages 1–183, May 2009. Preprint available at http://www.math.harvard.edu/~lurie/papers/GoodwillieI.pdf
[15] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. Institut des Hautes Études Scientifiques. Publications Mathématiques, 117:271–328, 2013.
[16] Charles Rezk. A model for the homotopy theory of homotopy theory. Transactions of the American Mathematical Society, 353(3):973–1007 (electronic), 2001.
[17] Emily Riehl and Dominic Verity. Completeness results for quasi-categories of algebras, homotopy limits, and related general constructions. Homology, Homotopy and Applications, 17:1–33, 2015.
[18] Alan Weinstein. The symplectic “category”. Differential geometric methods in mathematical physics (Clausthal, 1980). Springer Lecture Notes in Mathematics, 905:45–51.
[19] Alan Weinstein. Lectures on symplectic manifolds, volume 29 of CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, R.I., 1979. Corrected reprint.