Abstract—Quantum amplifier is an essential device in quantum information processing. As in the classical (non-quantum) case, its characteristic uncertainty needs to be suppressed by feedback, and in fact such a control theory for a single quantum amplifier has recently been developed. This paper extends this result to the case of cascaded quantum amplifier. In particular, we consider two types of structures: the case where controlled amplifiers are connected in series, and the case where a single feedback control is applied to the cascaded amplifier. Then, we prove that the latter is better in the sense of sensitivity to the uncertainty. A detailed numerical simulation is given to show actual performance of these two feedback schemes.

I. INTRODUCTION

Amplifiers are essential in modern technology [1]. Note that this device is not used in a stand-alone fashion, because its amplification gain cannot be exactly specified due to unavoidable characteristic uncertainty. Actually the amplified output signal produced from a bare amplifier can be largely distorted, leading to the low performance signal processing. Black discovered that feedback control resolves this issue [2], which has been further investigated in depth [3], [4]. This feedback amplification method, which is now known as one of the most successful examples of control theory, has made a significant contribution to the development of the today’s electronic technologies.

The idea of classical (non-quantum) feedback amplification is as follows. Figure 1 shows a system composed of a single amplifier with gain $G$ and another system (called the controller) with gain $K$. A simple calculation yields

$$y = G^b u, \quad G^b = \frac{G}{1 + GK}. \tag{1}$$

Therefore, in the limit $|G| \to \infty$, the closed-loop gain becomes $G^b \to 1/K$. This means that the robust amplification is realized by taking a passive and attenuating controller, such as a resistor, because the characteristic change in $K$ of those passive devices is in general quite small.

A single amplifier does not always provide sufficient gain and bandwidth due to the gain-bandwidth constraint, and thus cascaded amplifiers are often used in practice to satisfy the required performance [1]. Surely feedback stabilization is needed in this case as well, but it is not obvious how to construct a feedback configuration for such a multi-component network. In the classical control theory, as the most basic study, two types of feedback configurations depicted in Fig. 2 were first investigated. The type-a scheme shown in Fig. 2(a) is the cascade connection of the feedback-controlled amplifiers, and in the type-b scheme shown in (b), a single feedback loop is constructed for the cascaded amplifier. In Ref. [5], it was shown that the type-b scheme is more effective for improving the robustness than the type-a.

Turning our attention to the quantum regime, the (phase-preserving) quantum amplifier [6], [7], [8], [9], [10], [11] is expected to serve as a fundamental device in quantum information science, such as quantum sensing [12], [13], [14] and quantum communication [15], [16]. To make it practical, the quantum amplifier needs to be stabilized via feedback as in the classical case. In fact one of the authors has developed the theory of feedback stabilization for a single quantum amplifier [17]. It is thus important to extend the theory for the cascaded quantum amplifier [18], [19], [20], which has not been yet established; in particular, analyzing the quantum version of the above-described two classical feedback configurations should be an important basic study along this research direction. The contribution of this paper is to prove that a quantum version of the type-b scheme shown in Fig. 2(b) is better than a correspondence to the type-a, in the sense of the robustness to uncertainty. Note that this theorem is highly non-trivial, because the quantum amplifier is essentially a multi-input and multi-output (MIMO) device and eventually the analysis becomes quite involved compared to the classical case, as will be shown in the paper.

This paper is organized as follows. Section II is devoted to some preliminaries. In Sec. III we prove the main result. Section IV gives a detailed numerical simulation to show the robustness and stability of the controlled amplifiers.
II. PRELIMINARIES

A. Sensitivity function and classical cascaded amplifier

In Sec. I we have seen that the controlled amplifier is robust. To quantify the robustness, we define the sensitivity function. Suppose that a small characteristic change $\Delta G$ occurs in the gain as $G \rightarrow G + \Delta G$. Then the closed-loop gain also changes to $G^b + \Delta G^b$. The sensitivity function of $G^b$ with respect to $G$ is defined as

$$S = \frac{\Delta G^b / G^b}{\Delta G / G}. \quad (2)$$

Now the small deviation $\Delta G^b$ is calculated as

$$\Delta G^b = \frac{G + \Delta G}{1 + (G + \Delta G)K} - \frac{G}{1 + GK} \approx \frac{\Delta G}{(1 + GK)^2},$$

thus $S = 1/(1 + GK)$. Therefore, the open-loop gain $G K$ should be carefully designed so that $|S| < 1$ while retaining the stability of the closed-loop system.

Next we discuss the cascaded amplifier. As explained in Sec. I we consider two types of configurations shown in Fig. 2 [5], which in both cases are composed of $N$ identical classical amplifiers. In the type-a scheme, the same feedback controller with gain $K_a$ is applied to each amplifier, and in the type-b scheme, the output of the terminal amplifier is fed back to the first one through the single controller with gain $K_b$. The overall gains are given by

$$G^b_a = (G^b)^N = \frac{G^N}{1 + GK_a^N}, \quad G^b_b = \frac{G^N}{1 + G^N K_b}.$$

Now suppose that the small change $G \rightarrow G + \Delta G$ occurs in one of the amplifiers, say, the j-th amplifier. Then the fluctuations of $G^b_a$ and $G^b_b$ are calculated as follows;

$$\Delta G^b_a = \frac{(G + \Delta G)G^{N-1}}{1 + (G + \Delta G)K_a(1 + GK_a)^{N-1} - (1 + GK_a)^N},$$

$$\Delta G^b_b = \frac{(G + \Delta G)G^{N-1}}{1 + (G + \Delta G)G^{N-1}K_b - (1 + G^N K_b)^2}.$$ 

From Eq. (2), the sensitivity functions are given by

$$S_a = 1/(1 + GK_a), \quad S_b = 1/(1 + G^N K_b).$$

Then, if the gains of both of the controlled systems are equal and these are smaller than the gain of the non-controlled cascaded amplifier, i.e., $|G^b_a| = |G^b_b| < |G|^N$, we have

$$\frac{|S_b|}{|S_a|} = \frac{1}{1 + GK_a}|N-1| < 1.$$

Thus the type-b feedback scheme has a better performance than the type-a scheme in the sense of sensitivity.

B. Quantum amplifier and feedback stabilization

Here we describe the phase-preserving linear quantum amplifier [6], [7], [8], [9], [10], [11], which we simply call the “amplifier” in what follows. Let $b(t)$ be the annihilation operator of the input quantum field which we call the signal; $b(t)$ has the meaning of a complex amplitude of the field and satisfies the canonical commutation relation (CCR) $b(i)b^\dagger(t') - b^\dagger(t')b(t) = \delta(t-t')$, where $b^\dagger(t)$ represents the Hermitian conjugate of $b(t)$. The amplifier transforms $b(t)$ to $\tilde{b}(t) = gb(t) + \sqrt{g^2 - 1}d(t)$, where $d(t)$ is the additional field operator called the idler, which is necessary to preserve the CCR of $b(t)$. Also $g \in \mathbb{R}$ is the amplification gain, which is assumed to be real for simplicity.

In quantum optics the non-degenerate parametric amplifier (NDPA) [21] shown in Fig. 3(a) is often used. This system is composed of an optical cavity with two input fields $b_1$ (signal) and $b_2$ (idler), and two internal cavity modes $a_1$ and $a_2$ couple with the pump beam through the nonlinear process. In the rotating-frame at the half of input laser frequency, the dynamics and input-output equations of the NDPA under ideal setup (i.e., zero-detuned and no-loss) are given by

$$\dot{a}_1 = -\frac{\kappa}{2}a_1 + \varepsilon a_2 - \sqrt{\kappa}b_1, \quad \dot{a}_2 = -\frac{\kappa}{2}a_2 + \varepsilon a_1 - \sqrt{\kappa}b_2,$$

$$\tilde{b}_1 = \sqrt{\kappa}a_1 + b_1, \quad \tilde{b}_2 = \sqrt{\kappa}a_2 + b_2,$$

where $\kappa$ is the cavity damping rate and $\varepsilon$ is the strength of nonlinearity. (The mirror $M_1$ is partially transmissive for $a_1$, but perfectly reflective for the other cavity mode.)

In the Laplace domain, the amplified output signal $\tilde{b}_1$ is, together with the amplified idler $\tilde{b}_2$, represented as

$$\begin{bmatrix} \tilde{b}_1(s) \\ \tilde{b}_2(s) \end{bmatrix} = \begin{bmatrix} g_1(s) & g_2(s) \\ g_2(s) & g_1(s) \end{bmatrix} \begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix},$$

where $g_1(s)$ and $g_2(s)$ are the transfer functions given by

$$g_1(s) = (s^2 - \kappa^2/4 - \varepsilon^2)/D(s), \quad g_2(s) = -\kappa \varepsilon/D(s),$$

$$D(s) = s^2 + \kappa s + \kappa^2/4 - \varepsilon^2. \quad (3)$$

Note that $|g_1(\omega)|^2 - |g_2(\omega)|^2 = 1$, $\forall \omega$ holds to satisfy the CCR of the output, represented by $\tilde{b}(i\omega)\tilde{b}^\dagger(i\omega') - \tilde{b}^\dagger(i\omega')\tilde{b}(i\omega) = \delta(\omega - \omega')$ in the Fourier domain. Also the characteristic equation $D(s) = 0$ yields the stability condition $0 < x = 2\varepsilon/\kappa < 1$. The gain at the center frequency satisfies $|g_1(0)| = (1 + x^2)/|1 - x^2| \rightarrow \infty$ as $x \rightarrow 1 - 0$.

Here we review the general feedback method for the quantum amplifier [17]. The general quantum linear amplifier
is represented in the Laplace domain as
\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s)
\end{bmatrix} = G(s) \begin{bmatrix}
\tilde{b}_3(s) \\
\tilde{b}_4(s)
\end{bmatrix},
\]
where \(|G_{11}(i\omega)|^2 - |G_{12}(i\omega)|^2 = |G_{22}(i\omega)|^2 - |G_{21}(i\omega)|^2 = 1\) and \(G_{21}(i\omega)G_{11}^*(i\omega) - G_{22}(i\omega)G_{12}^*(i\omega) = 0\) hold for all \(\omega\). As for the controller, we take a passive and attenuating quantum system with the following input-output relation:
\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s)
\end{bmatrix} = K(s) \begin{bmatrix}
\tilde{b}_3(s) \\
\tilde{b}_4(s)
\end{bmatrix},
\]
where
\[
K^\dagger(i\omega)K(i\omega) = I, \quad \forall \omega
\] holds to satisfy the CCR in both \(\tilde{b}_3\) and \(\tilde{b}_4\). These two systems are connected through the feedback \(b_2 = \tilde{b}_4\) and \(b_3 = \tilde{b}_2\), as shown in Fig. 4b. The input-output relation of the closed-loop system is given by
\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_3(s)
\end{bmatrix} = \begin{bmatrix}
G_{11}^{fb}(s) & G_{12}^{fb}(s) \\
G_{21}^{fb}(s) & G_{22}^{fb}(s)
\end{bmatrix} \begin{bmatrix}
\tilde{b}_4(s) \\
\tilde{b}_2(s)
\end{bmatrix},
\]
(4) where
\[
G_{11}^{fb} = |G_{11} - K_2(G_{12}G_{22} - G_{12}G_{21})|/(1 - G_{22}K_2),
\]
\[
G_{12}^{fb} = G_{12}K_2/(1 - G_{22}K_2),
\]
\[
G_{21}^{fb} = G_{21}K_1/(1 - G_{22}K_2),
\]
\[
G_{22}^{fb} = |K_1 + G_{22}(K_1G_{22} - K_1K_2)|/(1 - G_{22}K_2).
\]
Then \(|G_{11}^{fb}(s)| \rightarrow 1/|K_2(s)| > 1\) holds in the high-gain limit \(|G_{11}| \rightarrow \infty\), meaning that the amplification process can be made robust by feedback as in the classical case.

III. CASCaded QUANTUM FEEDBACK AMPLIFIER

In this section we show the quantum version of the classical cascade amplification theory given in Sec. 11-A. First note that, because the quantum amplifier is an MIMO system and hence it essentially differs from the classical one, specifying the feedback configuration composed of quantum amplifiers and controllers, which corresponds to the classical one shown in Fig. 2, is a non-trivial problem. Here we particularly consider the case where the idler mode of the quantum amplifier can be used, in addition to the signal mode, to construct the feedback network; actually in the standard experiments of quantum optics and superconductivity [22], the idler mode is artificially implemented and is thus accessible. In this formulation, the reasonable quantum versions of the classical feedback configurations are illustrated in Fig. 3. That is, the type-A and type-B schemes correspond to the classical type-a and type-b schemes, respectively. In both cases, the signal-out and the idler-out are connected to the signal-in and the idler-in, respectively, and eventually the whole system has only one idler input from outside. Note that, if the idler modes are not accessible and only the signal modes can be connected, then in both configurations the whole closed-loop system has multiple idler inputs and eventually it is subjected to a large noise coming from those idler input channels.

Now the problem is to compare the above two schemes in terms of the sensitivity. We tackle this problem under the following setting. First, we focus on the gain at the center frequency \(\omega = 0\). Then we consider the quantum amplifier whose transfer function matrix at \(\omega = 0\) is of the form
\[
G(0) = \begin{bmatrix}
G_1 & G_2 \\
G_2 & G_1
\end{bmatrix}, \quad G_1^2 - G_2^2 = 1, \quad G_i \in \mathbb{R}.
\]
(5)
Note that the ideal NDPA with transfer functions \(\{3\}\) indeed fulfills this condition. Moreover we suppose that both feedback networks are composed of \(N\) identical quantum amplifiers characterized by Eq. (5), and that the gain of the \(j\)-th amplifier changes as \(G_j \rightarrow G_j + \Delta G_j\) and \(G_j \rightarrow G_j + \Delta G_{2j}\). Lastly, without loss of generality, the transfer function matrix of the controller at \(\omega = 0\) can be set to:
\[
K_*(0) = \begin{bmatrix}
K_{1*} & K_{2*} \\
-K_{2*} & K_{1*}
\end{bmatrix}, \quad K_{1*}^2 + K_{2*}^2 = 1, \quad K_* \in \mathbb{R},
\]
where \(\bullet = A, B\); i.e., \(K_A(0)\) and \(K_B(0)\) are applied to the type-A and the type-B schemes, respectively.

First, we derive the overall gain for the type-A scheme. From Eq. (4), each feedback-controlled amplifier has the following transfer function matrix:
\[
G^{fb}(0) = \begin{bmatrix}
G_{1A}^{fb} & G_{1B}^{fb} \\
G_{2A}^{fb} & G_{1A}^{fb}
\end{bmatrix} = \frac{1}{1 + G_1 K_{2A}} \begin{bmatrix}
G_1 + K_{2A} & G_{2A}K_{1A} \\
G_2 K_{1A} & G_1 + K_{2A}
\end{bmatrix}.
\]
(6)
This matrix can be diagonalized using the orthogonal matrix \(P = 1/\sqrt{2}[1, 1; 1, -1]\) as follows;
\[
P^{-1}G^{fb}(0)P = \text{diag}\{\lambda_{1A}^{fb}, \lambda_{2A}^{fb}\} = \begin{bmatrix}
\lambda_{1A}^{fb} & 0 \\
0 & \lambda_{2A}^{fb}
\end{bmatrix},
\]
where \(\lambda_{1A}^{fb} = (G_1 + K_{2A} \pm G_{2A} K_{1A})/(1 + G_1 K_{2A})\). The overall transfer function matrix is the \(N\) product of \(G^{fb}(0)\):
\[
G_A^{fb} = \begin{bmatrix}
G_{1A}^{fb} & G_{2A}^{fb} \\
G_{2A}^{fb} & G_{1A}^{fb}
\end{bmatrix} = [G^{fb}(0)]^N
= \frac{1}{2} \left[ \lambda_{1A}^{fb}N + (\lambda_{1A}^{fb})^N (\lambda_{2A}^{fb})N - (\lambda_{1A}^{fb})N + (\lambda_{2A}^{fb})N \right].
\]
The gain of interest is the (1,1) element of \(G_A^{fb}\), i.e., \(G_{1A}^{fb}\). Using \(G_2 \Delta G_2 = G_1 \Delta G_1\), we find that the fluctuation of \(G_{1A}^{fb}\)
is calculated as
\[
\Delta G_{1A}^{fb} = \frac{1}{2} \left[ \frac{G_1 + \Delta G_1 + K_{2A} + (G_2 + \Delta G_2)K_{1A}}{1 + (G_1 + \Delta G_1)K_{2A}} - \frac{G_1 + K_{2A} + G_2K_{1A}}{1 + G_1K_{2A}} \right] \left( \lambda^{fb} \right)^{N-1} - \frac{1}{2} \left[ \frac{G_1 + \Delta G_1 + K_{2A} - (G_2 + \Delta G_2)K_{1A}}{1 + (G_1 + \Delta G_1)K_{2A}} - \frac{G_1 + K_{2A} - G_2K_{1A}}{1 + G_1K_{2A}} \right] \left( \lambda^{fb} \right)^{N-1} \approx \frac{K_{1A} \Delta G_1}{2G_2(1 + G_1K_{2A})} \left( \lambda^{fb} \right)^N \left( \lambda^{fb} \right)^{-N} \right].
\]

As a result, the sensitivity function is represented as
\[
S_A = \frac{\Delta G_{1A}^{fb}/G_{1A}^{fb}}{\Delta G_{1A}/G_{1A}} = \frac{K_{1A}G_1}{G_2(1 + G_1K_{2A})} \frac{G_{2A}^{fb}}{G_{1A}^{fb}}.
\]

Next we consider the type-B scheme, where the single feedback loop is applied to the cascade connection of the quantum amplifier with transfer function matrix \(G(0)\) given in Eq. (5). Noting that \(G(0)\) is diagonalized in terms of the orthogonal matrix \(P\) as \(P^{-1}G(0)P = \text{diag}(\lambda_s, \lambda_B)\) with \(\lambda_\pm = G_1 \pm G_2\), we have
\[
[G(0)]^N = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} = \frac{1}{2} \left[ \lambda_+^N + \lambda_-^N - \lambda_+^N \lambda_-^N - \lambda_+^N \lambda_-^N \right].
\]

From Eq. (4), the transfer function matrix of the whole closed-loop system is then given by
\[
G_B^{fb} = \begin{bmatrix} G_{1B}^{fb} & G_{2B}^{fb} \\ G_{2B}^{fb} & G_{1B}^{fb} \end{bmatrix} = \frac{1}{1 + M_1K_{2B}} \begin{bmatrix} M_1 + K_{2B} & M_2K_{1B} \\ M_2K_{1B} & M_1 + K_{2B} \end{bmatrix}.
\]

The characteristic change in \(G_1\) and \(G_2\) induces a small fluctuation in the overall gain, \(G_{1B}^{fb}\), as follows:
\[
\Delta G_{1B}^{fb} = \frac{M_1 + \Delta M_1 + K_{2B}}{1 + (M_1 + \Delta M_1)K_{2B}} - \frac{M_1 + K_{2B}}{1 + M_1K_{2B}} = \frac{K_{1B}^2 \Delta M_1}{(1 + M_1K_{2B})(1 + (M_1 + \Delta M_1)K_{2B})} = \frac{K_{2B}^2 M_2 \Delta G_1}{(1 + M_1K_{2B})(G_2 + (M_1G_2 + M_2G_1)K_{2B})} \approx \frac{K_{1B}G_{2B}^{fb}\Delta G_1}{G_2(1 + M_1K_{2B})},
\]

where the following equality is used:
\[
G_2M_1 = M_2G_1.
\]

The proof of this equation is given in Appendix. Therefore we arrive at the sensitivity function:
\[
S_B = \frac{\Delta G_{1B}^{fb}/G_{1B}^{fb}}{\Delta G_{1A}/G_{1A}} = \frac{K_{1B}G_1}{G_2(1 + M_1K_{2B})} \frac{G_{2B}^{fb}}{G_{1B}^{fb}}.
\]

Now we show the main result of this paper; if the gains of both of the controlled systems are equal and these are smaller than the gain of the non-controlled cascaded amplifier, i.e., \(|G_{1A}^{fb}| = |G_{1B}^{fb}| < |M_1|\), we prove that
\[
|S_B| < |S_A|.
\]

The proof is given in Appendix. Therefore, the type-B feedback scheme is better than the type-A scheme in terms of the sensitivity to the characteristic uncertainty \(\Delta G_1\).

### IV. STABILITY AND SENSITIVITY ANALYSIS

The superiority of the type-B scheme over the type-A, given in Eq. (6), holds only at the center frequency \(\omega = 0\). Hence, in this section, for a specific amplifier and controller, we numerically investigate the frequency dependence of the control performance of those two schemes, particularly paying attention to the robustness and stability properties.

The amplifier is the ideal NDPA discussed in Sec. II-B. The controller is a partially transmissive mirror called the beam-splitter (BS), which is a 2-inputs and 2-outputs passive static system with the following transfer function matrix:
\[
K_\bullet(s) = \begin{bmatrix} \alpha_\bullet & -\beta_\bullet \\ \beta_\bullet & \alpha_\bullet \end{bmatrix}, \quad \alpha_\bullet^2 + \beta_\bullet^2 = 1,
\]

where \(\bullet = A, B\). The real parameters \(\alpha_\bullet, \beta_\bullet \in \mathbb{R}\) represent the transmissivity and reflectivity of the mirror, respectively. Note that, from Eq. (4), the single NDPA with this controller has the amplification gain \(1/|\beta_\bullet|\) in the limit \(s \rightarrow 1 - 0\).

Specifically, we consider the four cases summarized in Table I; the number of amplifiers is \(N = 2\) (Cases 1 and 2) or \(N = 5\) (Cases 3 and 4); the gain of the (1,1) element of \(G(0)^N\), i.e., the non-controlled cascaded NDPA at \(\omega = 0\), is \(M_1 = 45\) dB (Cases 1 and 3) or \(M_1 = 30\) dB (Cases 2 and 4). In each case the cavity decay rate of the NDPA is fixed to \(\kappa = 1.8 \times 10^7\) Hz [23], while \(x = 2\varepsilon/\kappa\) is chosen so that \(M_1\) equals to 45 dB or 30 dB. The reflectivity \(\beta_\bullet\) was determined as follows; first we fix the parameters of the type-A controlled system, \(x\) and \(\beta_\bullet\), and then \(\beta_\bullet\) is determined so that the gains at \(\omega = 0\) of both of the feedback schemes are the same, i.e., \(|G_{1A}^{fb}| = |G_{1B}^{fb}|\).

First, let us see the stability of the feedback-controlled system. For the type-A system, it is enough to analyze the stability of the single feedback-controlled NDPA; its characteristic equation is given by
\[
s^2 + \frac{\kappa}{1 - \beta_\bullet s} + \frac{1 + \beta_\bullet}{1 - \beta_\bullet} = 0.
\]
that is, the nominal parameter \( n \) generated from the uniform distribution over 

\[ (1 + \beta_A)/(1 - \beta_B). \]

This condition is always satisfied if the NDPA is stable (\( x < 1 \)) and \( 0 \leq \beta_B < 1 \).

To analyze the stability property of the type-B system, we use the Nyquist plot, which is now directly applicable because all the parameters \((\kappa, \epsilon, \alpha, \beta)\) are real. The Nyquist plot is the vector plot of the open-loop transfer function \( L(s) \), i.e., the trajectory of \((\text{Re}\{L(i\omega)\}, \text{Im}\{L(i\omega)\})\) with \( \omega \in (-\infty, +\infty) \); note that \( L(s) = G(s)K(s) \) for the classical system [1]. The feedback-controlled system is stable if and only if there is no encirclement of the point \((-1, 0)\), provided that \( L(s) \) has no unstable poles. Now, from Eq. (4) the type-B system has the open-loop transfer function \( L(s) = -[G^N]_{22}(s)K_{21}(s) \), where \([G^N]_{22}(s)\) is the \((2, 2)\) element of \( G(s) \), and \( K_{21}(s) \) is the \((2, 1)\) element of \( K(s) \). The Nyquist plots are shown in Fig. 5; hence, from the above stability criterion, the type-B system is stable in all Cases.

Next we discuss the sensitivity. To see this explicitly, suppose that the characteristic change of the amplifier, \( \Delta G_1 \), stems from the fluctuation of the parameter \( \epsilon \). We model this uncertainty as \( \epsilon' = (1 + 0.05r)\epsilon \), where \( r \) is the random number generated from the uniform distribution over \([-1, 1]\); that is, the nominal parameter \( x = 2\epsilon/\kappa \) given in Table I experiences up to 5% deviation. The gain plots are shown in Fig. 6, where the red and blue lines represent the gains of the type-A and the type-B controlled systems, respectively. Also the black lines are the gain plots of the cascaded amplifier without feedback. In each scheme (color), 100 sample paths are produced from the above-mentioned probability distribution. The figure shows that, in all Cases, the gain fluctuation of the controlled systems at \( \omega = 0 \) are smaller than that of the uncontrolled system; that is, the feedback control always works well to suppress the gain fluctuation of the amplifier, at the price of decreasing the gain. Moreover, the fluctuation of the gain at \( \omega = 0 \) of the type-B controlled system is always smaller than that of the type-A, i.e., \( |S_B| < |S_A| \), as proven in Sec. III. However, importantly, this fact does not hold over all frequencies; in particular in Cases 1 and 3, the type-A scheme is better than the type-B, at the frequency \( \omega \sim \kappa/10 \) where there is a peaking in the gain.

Finally we discuss the control performance by focusing on both stability and sensitivity. The Nyquist plot can be used to quantify how much the system is stable, in terms of the gain margin \( g_m = 1/[L(i\omega_{pc})] \) with \( \omega_{pc} \) the phase crossover frequency satisfying \( \angle L(i\omega_{pc}) = -180^\circ \). Now, as shown in Table I, the gain margin \( g_m \) in cases 1 and 3 are smaller than that in Cases 2 and 4. Hence the systems in Cases 1 and 3 are less stable than those in Cases 2 and 4; actually a peaking appears in Figs. (a) and (c), but not in (b) and (d). However, as implied by Fig. 6, it is harder to reduce the sensitivity in Cases 2 and 4, compared to Cases 1 and 3. That is, there is a tradeoff between the stability and robustness. Note also that the controlled system with less number of amplifiers has the better sensitivity; in fact the controlled system composed of \( N = 5 \) amplifiers, which yet has the same level of sensitivity as that of the system with \( N = 2 \), is often unstable.

V. CONCLUDING REMARK

The long-term goal of this work is to develop the design theory for feedback-controlled quantum networks containing amplifiers, corresponding to the classical theory [1], [2], [3], [4], [5]. Toward this goal, as an important first step, in this paper we have proven that, to construct a robust high-gain quantum amplifier from some low-gain amplifiers, it is always better to stabilize the cascaded amplifier via a single feedback controller, than to take a cascade connection of feedback-controlled amplifiers. Recall that, although this is the same conclusion as the classical one, the proof of this fact is highly non-trivial as demonstrated in this paper. Furthermore, we have learned through the investigation that, although feedback control problems for quantum amplifiers seem to be generally involved due to their MIMO nature, analytic studies can still be possible by making a reasonable assumption. That is, Eq. 5, which corresponds to the ideal
op-amp assumption taken in the classical network theory, will enable us to pursue the long-term goal mentioned above.

APPENDIX

First we prove Eq. (7). If the gain of the $j$-th amplifier changes as $G_1 \rightarrow G_1 + \Delta G_1$ and $G_2 \rightarrow G_2 + \Delta G_2$, then $M_1 = (\lambda^N + \lambda^N)/2$ changes as follows;

$$\Delta M_1 = \frac{1}{2} \left[ (G_1 + G_1 + G_2 + G_2) \lambda^N - ((G_1 + G_1 - G_2 - G_2) \lambda^N) - M_1 \right]$$

$$= \frac{1}{2} \left[ (G_1 + G_2) \lambda^N + (G_1 - G_2) \lambda^{-N} \right]$$

$$= \frac{\Delta G_1}{2G_2} \left[ \lambda^N - \lambda^{-N} \right] = \frac{M_2}{G_2}.$$ 

Next we prove Eq. (8). To make a fair comparison, we assume that both the controlled systems have the same amplification gain at $\omega = 0$, i.e., $|G\beta_{1A}| = |G\beta_{1B}|$, which leads to $|G\beta_{2A}| = |G\beta_{2B}|$. Then we have

$$\frac{|S_B|}{|S_A|} = \frac{K_{1B} + G_1 K_{2A}}{K_{1A} + M_1 K_{2B}} \left[ 1 + \frac{G_1 K_{2A}}{K_{1A}} \right] \left( \frac{\lambda^N - \lambda^{-N}}{\lambda^N - \lambda^{-N}} \right).$$

Here, from the relations $a^n - b^n = (a - b) \sum_{k=1}^{n} a^{n-k}b^k$ and $\lambda^N - \lambda^{-N} = \lambda^{N-1} \lambda^{N-2} - \lambda^{-N-1} \lambda^{-N-2}$, we have

$$\left( \lambda^{(b)} \right)^N - \left( \lambda^{(b)} \right)^N = \left( \lambda^{(b)} - \lambda^{(b)} \right) \left[ \left( \lambda^{(b)} \right)^{N-1} + \left( \lambda^{(b)} \right)^{N-3} + \cdots + \left( \lambda^{(b)} \right)^{-(N-1)} \right]$$

$$= 2G_2K_{1A} \sum_{k=1}^{N} \lambda^{N-2k+1}.$$ 

and likewise $\lambda^{N} - \lambda^{-N} = \lambda^{N} - \lambda^{-N}$. Hence, Eq. (9) is rewritten as

$$\frac{|S_B|}{|S_A|} = \sum_{k=1}^{N} \left( \lambda^{(b)} \right)^{N-2k+1}.$$ 

In addition to the condition $|G\beta_{1A}| = |G\beta_{1B}|$, we assume that the gains of both of the type-A and type-B controlled systems are smaller than the gain of the non-controlled cascaded amplifier: $|G\beta_{1A}| = |G\beta_{1B}| < |M_1|$, which is represented as

$$\frac{|G\beta_{1A}|}{|M_1|} = \frac{\lambda^{(b)} + \lambda^{(b)}}{\lambda^N + \lambda^{-N}} < 1, \quad \forall k = 1, \ldots, N.$$ 

Then, if $N$ is odd, Eq. (10) leads to

$$\frac{|S_B|}{|S_A|} = \frac{1 + \sum_{k=1}^{(N-1)/2} \left( \lambda^{(b)} \right)^{N-2k} + \left( \lambda^{(b)} \right)^{-2k}}{1 + \sum_{k=1}^{(N-1)/2} \left( \lambda^{(b)} \right)^{2k} + \left( \lambda^{(b)} \right)^{-2k}} < 1.$$ 

Also, if $N$ is even, particularly $N = 4l - 2$ ($l = 1, 2, \ldots$),

$$\frac{|S_B|}{|S_A|} = \frac{\lambda^{(b)} + \left( \lambda^{(b)} \right)^{-1}}{\lambda^N + \lambda^{-N}} \frac{1 + \sum_{k=1}^{l-1} \left( \lambda^{(b)} \right)^{4k} + \left( \lambda^{(b)} \right)^{-4k}}{1 + \sum_{k=1}^{l-1} \left( \lambda^{(b)} \right)^{2k} + \left( \lambda^{(b)} \right)^{-2k}},$$ 

and if $N = 4l$ ($l = 1, 2, \ldots$),

$$\frac{|S_B|}{|S_A|} = \frac{\lambda^{(b)} + \left( \lambda^{(b)} \right)^{-1}}{\lambda^N + \lambda^{-N}} \frac{\sum_{k=1}^{l} \left( \lambda^{(b)} \right)^{4k-2} + \left( \lambda^{(b)} \right)^{-4k-2}}{\sum_{k=1}^{l} \left( \lambda^{(b)} \right)^{2k-2} + \left( \lambda^{(b)} \right)^{-2k-2}},$$

which are both less than 1. This completes the proof.

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