Affleck-Kennedy-Lieb-Tasaki state on a honeycomb lattice is a universal quantum computational resource

Tzu-Chieh Wei, Ian Affleck, and Robert Raussendorf

Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia V6T 1Z1, Canada

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Universal quantum computation can be achieved by simply performing single-qubit measurements on a highly entangled resource state, such as cluster states. The family of Affleck-Kennedy-Lieb-Tasaki (AKLT) states has recently been intensively explored and shown to provide restricted computation. Here, we show that the two-dimensional AKLT state on a honeycomb lattice is a universal resource for measurement-based quantum computation.

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Introduction. Quantum computation promises exponential speedup over classical computation by exploiting the quantum mechanical nature of physical processes [1]. In addition to the standard circuit model based on unitary evolution, surprisingly, local measurement alone provides the same power of computation, given only a prior sufficiently entangled state [2,3]. For this model of measurement-based quantum computation (MBQC), universal resource states are known to be very rare [4], but examples do exist [5–8]. The 2D cluster state on the square lattice is a universal resource state [2,5]. Cluster states can be created by the Ising interaction from unentangled states [5,9], but they do not arise from unentangled states [5–8]. The 2D cluster state on

FIG. 1: Illustrations of the AKLT state and the 2D cluster state. (a) AKLT state. Spin singlets of two virtual spins 1/2 (i.e., qubits) are located on the edges of the honeycomb lattice. A projection $P_{S,v}$ at each lattice site $v$ onto the symmetric subspace of three virtual spins defines the AKLT state. In one hexagon, sites are labeled by $A$ or $B$ to show the bipartite (or bi-colorable) property of the honeycomb lattice. (b) 2D Cluster state. One qubit resides at each lattice site and one stabilizer generator is shown.

state $\ket{G(A)}$ hinges solely on the connectivity of $G(A)$, and is thus a percolation problem. Third, we demonstrate via Monte Carlo simulation that the typical graphs $G(A)$ are indeed deep in the supercritical phase.

The AKLT state [13] on the honeycomb lattice $L$ has one spin-3/2 per site of $L$. The state space of each spin 3/2 can be viewed as the symmetric subspace of three virtual spin-1/2’s, i.e., qubits. In terms of these virtual qubits, the AKLT state on $L$ is (see Fig. 1a)

$$|\Phi_{\text{AKLT}}\rangle \equiv \bigotimes_{v \in V(L)} P_{S,v} \bigotimes_{e \in E(L)} |\phi\rangle_e,$$  \hspace{1cm} (1)

where $V(L)$ and $E(L)$ denote the set of vertices and edges of $L$, respectively. $P_{S,v}$ is the projection onto the symmetric (equivalently, spin 3/2) subspace at site $v$ of $L$ [20]. For an edge $e = (v,w)$, $|\phi\rangle_e$ denotes a singlet state, with one spin 1/2 at vertex $v$ and the other at $w$.

A graph state $\ket{G}$ is a stabilizer state [21] with one qubit per vertex of the graph $G$ and is the unique eigenstate of a set of commuting operators [5], usually called the stabilizer generators [22].

$$X_v \bigotimes_{u \in \text{nb}(v)} Z_u \ket{G} = \ket{G}, \hspace{0.5cm} \forall v \in V(G),$$  \hspace{1cm} (2)
where \( \text{nb}(v) \) denotes the neighbors of vertex \( v \), and \( X \equiv \sigma_x, Y \equiv \sigma_y \) and \( Z \equiv \sigma_z \) are the Pauli matrices. A cluster state is a special case of graph states, with the underlying graph being a regular lattice (see Fig. 1b). Any 2D cluster state is a universal resource for measurement-based quantum computation [22].

Reduction to a graph state. To show that the 2D AKLT state of four-level spin-3/2 particles can be converted to a graph state of two-level qubits, we need to preserve the local two-dimensional structure at each site. This is achieved by a local generalized measurement [1], also called positive-operator-value measure (POVM), on every site \( v \) on \( \mathcal{L} \). The POVM consists of three rank-two elements

\[
F_{v,z} = \sqrt{\frac{1}{3}}(|000\rangle \langle 000| + |111\rangle \langle 111|) \tag{3a}
\]

\[
F_{v,x} = \sqrt{\frac{1}{3}}(|+++angle \langle +++| + |----\rangle \langle ----|) \tag{3b}
\]

\[
F_{v,y} = \sqrt{\frac{1}{3}}(|i,i,i\rangle \langle i,i,i| + |−i,−i,−i\rangle \langle −i,−i,−i|) \tag{3c}
\]

where \( |0/1\rangle, |\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2} \) and \( |\pm i\rangle \equiv (|0\rangle \pm |i\rangle)/\sqrt{2} \) are eigenstates of \( Z, X \) and \( Y \), respectively, and \( |0\rangle \equiv |\uparrow\rangle, |1\rangle \equiv |\downarrow\rangle \). Physically, \( F_{v,a} \) is proportional to a projector onto the two-dimensional subspace spanned by the \( S_a = \pm 3/2 \) states. The above POVM elements obey the relation \( \sum_{v \in \{x,y,z\}} F_{v,u} F_{v,v} = P_{S_a} \), i.e., project onto the symmetric subspace, as required. The outcome \( a_{uv} \) of the POVM at site \( v \) is random, which can be \( x, y \) or \( z \), and it is correlated with the outcomes at other sites due to the entanglement in the AKLT state [13, 24]. As demonstrated below, the resulting quantum state, dependent on the random POVM outcomes \( \mathcal{A} = \{a_{uv}, v \in V(\mathcal{L})\} \), is equivalent under local unitary transformations to an encoded graph state \( \tilde{G}(\mathcal{A}) \). We show that the corresponding graph \( \tilde{G}(\mathcal{A}) \) is constructed from graph \( \mathcal{L} \) by applying the following two rules:

| POVM outcome | \( z \) | \( x \) | \( y \) |
|--------------|------|------|------|
| \( X \)      | \( \lambda_1 \lambda_2 Z_j \) | \( \lambda_1 \lambda_2 X_j \) | \( \lambda_1 \lambda_2 Y_j \) |
| \( Z \)      | \( \sum_{j=1}^{\mathcal{L}} Z_j \) | \( \sum_{j=1}^{\mathcal{L}} X_j \) | \( \sum_{j=1}^{\mathcal{L}} Y_j \) |

TABLE I: The dependence of stabilizers and encodings on the local POVM outcome. \(|c|\) denotes the total number of sites contained in a domain and \( i,j = 1..3|c| \). The honeycomb lattice \( \mathcal{L} \) is bi-partite and all sites can be divided into either \( A \) or \( B \) sublattice, \( V(\mathcal{L}) = A \cup B \); see Fig. 1b. One choice of the sign is \( \lambda_1 = 1 \) if the virtual qubit \( i \in v \in A \) and \( \lambda_1 = -1 \) if \( i \in v' \in B \); this is due to the negative sign in the stabilizer generator for a singlet, i.e., \( -\sigma_\mu,\sigma_\mu \rangle \langle \sigma_\mu,\sigma_\mu \rangle \)

\( \equiv \rangle \langle \rangle \)

FIG. 2: Graphical rules for transformation of the lattice \( \mathcal{L} \), depending on the POVM outcomes \( \mathcal{A} \).

(a) Rule R1. (b) Rule R2. (c) An example to illustrate the qubit encoding of a domain. (d) An example for demonstrating the stabilizer generator.

R1 (Edge contraction): Contract all edges \( e \in E(\mathcal{L}) \) that connect sites with the same POVM outcome.

R2 (Mod-2 edge deletion): In the resultant multi graph, delete all edges of even multiplicity and convert all edges of odd multiplicity into conventional edges of multiplicity 1.

A set of sites in \( \mathcal{L} \) that are contracted into a single vertex of \( G(\mathcal{A}) \) by rule R1 is called a domain. Each domain supports a single encoded qubit. The stabilizer generators and the encoded operators for the resulting codes are summarized in Table I

Rule R1 derives intuitively from the antiferromagnetic property of the AKLT state: neighbor spin-3/2 particles must not have the same \( S_a = \pm 3/2 \) (or \( -3/2 \)) configuration [23]. Hence, after the projection onto \( S_a = \pm 3/2 \) subspace by the POVM, the configurations for all sites inside a domain can only be \( |3/2,−3/2,\ldots\rangle \) or \(|−3/2,3/2,\ldots\rangle \), and these form the basis of a single qubit. This can also be understood in terms of the stabilizer. Consider the case where two neighboring POVMs yield the same outcome, say \( z \); see Fig. 2c. Due to the projections \( F_{v,z} \) and \( F_{u,z} \) (with \( u \in \{1,2,3\} \) and \( v \in \{4,5,6\} \) each containing three virtual qubits), the operators \( Z_1 Z_2, Z_2 Z_3, \) and \( Z_4 Z_5, Z_5 Z_6 \) become stabilizer generators of \( |\Psi(\mathcal{A})\rangle \). Moreover, the stabilizer \( −Z_3 Z_4 \) of \( |\phi\rangle_3 \) commutes with \( F_{u,z} \otimes F_{u,z} \), and thus remains a stabilizer element for \( |\Psi(\mathcal{A})\rangle \) [24]. In brief, the stabilizer generators \( Z_1 Z_2, Z_2 Z_3, −Z_3 Z_4, Z_4 Z_5, Z_5 Z_6 \) lead to a single encoded qubit

\[
\alpha(|000\rangle_u |111\rangle_v) + \beta(|111\rangle_u |000\rangle_v),
\]

where \( \alpha, \beta \) are complex numbers.
supported by the two sites $u$ and $v$ jointly. Note the antiferromagnetic ordering \cite{13} among groups of three virtual qubits. To reduce the support of this logical qubit to an individual site, a measurement in the basis \{$(000)_u \pm (111)_{v, w}$\} is performed. The resulting state is $\alpha [(000)_u] + \beta [(111)_w]$, with the sign “$\pm$” known from the measurement outcome. This is the proper encoding for a domain consisting of a single site. Domains of more than two sites are thereby reduced to single sites.

To see that $|\Psi(A)\rangle$ is indeed equivalent under local unitary transformations to an encoded graph state $|\tilde{G}(A)\rangle$, we consider the example of four domains $c, u, v, w$, each consisting of a single site of $\mathcal{L}$, where the POVM outcome is $z$ on the central domain $c$ and $x$ on all lateral domains $u$, $v$ and $w$; see Fig. \ref{fig:domain}b. By similar arguments as above \cite{20}, the operator $O = -X_1X_1'X_2X_2'X_3X_3'$ is in the stabilizer of $|\Psi(A)\rangle$. Using the encoding in Table \ref{table:encoding} i.e., with the encoded Pauli operators $\Xi_c = Z_1Z_2Z_3$, $\Xi_u = \pm X_1$, $\Xi_v = \pm X_2$, and $\Xi_w = \pm X_3$, we find that $O = \pm X_c \Xi_u \Xi_v \Xi_w$, which is (up to a possible sign) one stabilizer generator defining the graph state; see Eq. \ref{eq:stabilizer}.

By the above construction, if two domains $u, v$ are connected by an edge of multiplicity $m$, the inferred graph state stabilizer generators will contain factors of $\Xi_u \Xi_v$ or $\Xi_u \Xi_v^m$. Rule 2 thus follows the observation that $Z^2 = I$. Generalizing these ideas, one can rigorously prove that for any $A$ of POVM outcomes, the state $|\Psi(A)\rangle$ is local-unitarily equivalent to an encoded graph state $|\tilde{G}(A)\rangle$ \cite{25}. We shall denote by $|\tilde{G}(A)\rangle$ the same graph state but with domains of single sites.

Random graph states and percolation. Whether or not typical graph states $|\tilde{G}(A)\rangle$ are universal resources hinges solely on connectivity properties of $G(A)$, and is thus a percolation problem \cite{26}. Specifically, for a large initial $\mathcal{L}$ the random graph state $|\tilde{G}(A)\rangle$ can, with close to unit probability, be efficiently reduced to a large two-dimensional cluster state if the following properties hold:

C1 The distribution of the number of sites in a domain (i.e. domain size) is microscopic, i.e., the largest domain size can at most scale logarithmically with the total number of sites $|V(\mathcal{L})|$ in the large $\mathcal{L}$ limit.

C2 The probability of the existence of a path through $G(A)$ from the left to the right (or top to bottom) approaches unity in the limit of large $\mathcal{L}$.

Condition C1 ensures that the graph $G(A)$ remains macroscopic if the initial $\mathcal{L}$ was, and Condition C2 ensures that the system is in the supercritical phase with a macroscopic spanning cluster.

Together with planarity, which holds for all graphs $G(A)$ by construction, the conditions C1 and C2 are sufficient for the reduction of the random graph state to a standard universal cluster state. The proof \cite{25} extends a similar result already established for site percolation on a square lattice \cite{27}. The physical intuition comes from percolation theory. In the supercritical phase (where there exists a macroscopic spanning cluster and connects one boundary to the other), the spanning cluster contains a subgraph which is topologically equivalent to a coarse-grained two-dimensional lattice structure. This subgraph can be carved out and subsequently cleaned off all imperfections by local Pauli measurements, leading to a perfect two-dimensional percolation.

**Numerical results.** We used Monte Carlo simulations to sample typical random graphs resulting from the POVM and compute their properties. The simulations utilize a generalized Hoshen-Kopelman algorithm \cite{28} to identify domains. Due to the entanglement in the AKLT state, the local POVM outcomes are correlated which is sufficiently taken into account in our simulations. In particular, to sample typical POVM outcomes $A$ correctly, we use a Metropolis method to update configurations. For each site we attempt to flip the type (either $x$, $y$ or $z$) to one of the other two equally and accept the flip with a probability $p_{\text{acc}} = \min \{1, 2^{|\mathcal{V}| - |\mathcal{E}|} / (2^{|\mathcal{V}| + |\mathcal{E}|}\}$, where $|\mathcal{V}|$ and $|\mathcal{E}|$
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With an appropriate choice of boundary conditions, the AKLT state is the unique ground state; see also Ref. cite-sup. In contrast to 1D, the 2D AKLT state on the honeycomb lattice was only shown to possess exponentially decaying correlation functions [13], nevertheless suggesting a spectral gap.

Concluding remarks. We investigated MBQC on the AKLT states and established one crucial missing ingredient in this area: the two-dimensional spin-3/2 AKLT state on a honeycomb lattice is indeed a universal resource. The approach described in this work also applies to other trivalent lattices, such as the Archimedean lattices: (3,125), (4,6,12) and (4,82), which have higher percolation thresholds than the honeycomb lattice.

After the completion of our work, we learned of a similar result by Miyake with a different approach [30].

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Appendix A: Projection onto the symmetric subspace of three qubits

The addition of angular momenta for three qubits (spin-1/2) gives rise to three subspaces: \( \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \otimes \frac{3}{2} \). The four basis states of the \( S = 3/2 \) subspaces are \(|3/2, 3/2\rangle, |3/2, -3/2\rangle, |3/2, 1/2\rangle \) and \(|3/2, -1/2\rangle \) (with the quantization axis being assumed to be \( z \)), and they can be expressed in terms of the three-qubit basis states (with \(|0/1\rangle \equiv |\uparrow / \downarrow \rangle = |1/2, \pm 1/2\rangle\))

\[
|3/2, 3/2\rangle = |000\rangle, \quad |3/2, -3/2\rangle = |111\rangle, \quad |3/2, 1/2\rangle = |010\rangle + |100\rangle, \quad |3/2, -1/2\rangle = |110\rangle + |101\rangle + |011\rangle.
\]

(A1a) (A1b) (A1c)

As can be clearly seen, states in this subspace are symmetric under permutation of the three qubits and the corresponding projection operator is

\[
P_{S,v} = |000\rangle\langle 000| + |W\rangle\langle W| + |\overline{W}\rangle\langle \overline{W}| + |111\rangle\langle 111|.
\]

(A2)

The POVM and the post-POVM state. To show that the 2D AKLT state of four-level spin-3/2 particles can be converted to a graph state of two-level qubits, we need to preserve a local two-dimensional structure at each site. This is achieved by a local generalized measurement \( \Pi \), also called positive-operator-value measure (POVM), on every site \( v \) on \( L \). The POVM consists of three rank-two elements \( F_v^t, a F_v, a ^t \) with \( a = x, y, \) or \( z \), and

\[
F_{v,x} = \frac{\sqrt{2}}{3}(|000\rangle\langle 000| + |111\rangle\langle 111|)
\]

(A3a)

\[
F_{v,y} = \frac{\sqrt{2}}{3}(|++\rangle\langle ++| + |--\rangle\langle --|)
\]

(A3b)

\[
F_{v,z} = \frac{\sqrt{2}}{3}(|i,i,i\rangle\langle i,i,i| + |i,-i,-i\rangle\langle i,-i,-i|)
\]

(A3c)

where \(|0/1\rangle, |\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2} \) and \(| \pm i\rangle \equiv (|0\rangle \pm i|1\rangle)/\sqrt{2} \) are eigenstates of the three Pauli operators \( Z, X \) and \( Y \), respectively. Physically, \( F_v, a \) is proportional to a projector onto the two-dimensional subspace spanned by the \( S_a = \pm 3/2 \) states, i.e.,

\[
F_v, a = \sqrt{2}/3(|3/2, +3/2\rangle a \langle 3/2, +3/2| + |3/2, -3/2\rangle a \langle 3/2, -3/2|).
\]

(A4)

with \( \hat{S}_a |3/2, \pm 3/2\rangle a = \pm 3/2|3/2, \pm 3/2\rangle a \) and \( a \) indicating the quantization axis.

The above POVM elements obey the relation \( \sum_{v \in \{x,y,z\}} F_v, a ^t F_v, a = P_{S,v} = 1_{S=3/2} \), i.e., project onto the symmetric subspace, as required. The outcome \( a_v \) of the POVM at site \( v \) is random, which can be \( x, y \) or \( z \). The resulting quantum state, dependent on the random POVM outcomes \( \mathcal{A} = \{a_v, v \in V(L)\} \),

\[
|\Psi(\mathcal{A})\rangle = \bigotimes_{v \in V(L)} F_v, a_v |\Phi_{\text{AKLT}}\rangle = \bigotimes_{v \in V(L)} F_v, a_v \bigotimes_{e \in E(L)} |\phi_e\rangle,
\]

(A5)

where \(|\phi_e\rangle \) denotes a singlet state \(|\langle 01 | - |10\rangle\rangle/\sqrt{2} \) on the edge \( e \).

Example 1: qubit encoding. Consider the case where two neighboring POVMs yield the same outcome, say \( z \); see Fig. 2c in the main text. Due to the projections \( F_u, z \) and \( F_v, z \) (with \( u = \{1, 2, 3\} \) and \( v = \{4, 5, 6\} \) each containing three virtual qubits), the operators \( Z_1 Z_2, Z_2 Z_3, \) and \( Z_4 Z_5, Z_5 Z_6 \) become stabilizer generators of \( |\Psi(\mathcal{A})\rangle \), as, e.g.,

\[
Z_1 Z_2 F_{u,z} = F_{u,z}
\]

(A6)

and hence

\[
Z_1 Z_2 |\Psi(\mathcal{A})\rangle = Z_1 Z_2 \bigotimes_{v \in V(L)} F_v, a_v \bigotimes_{e \in E(L)} |\phi_e\rangle = |\Psi(\mathcal{A})\rangle.
\]

(A7)
Moreover, the stabilizer \(-Z_3Z_4\) of |\(\phi\rangle_{34}\) (i.e., \(-Z_3Z_4|\phi\rangle_{34} = |\phi\rangle_{34}\)) commutes with \(F_{u,z} \otimes F_{v,z}\), and thus remains a stabilizer element for |\(\Psi(A)\rangle\),

\[
-Z_3Z_4|\Psi(A)\rangle = \bigotimes_{v \in V(\mathcal{L})} F_{v,a_v}(-Z_3Z_4) \bigotimes_{v \in E(\mathcal{L})} |\phi\rangle_v = |\Psi(A)\rangle.
\] (A8)

Therefore, regarding sites \(u\) and \(v\), \(|(000)_u(111)_v\rangle\) and \(|(111)_u(000)_v\rangle\) are the only two basis states that are stabilized (i.e., common eigenstates with eigenvalue equal to unity) by the above five operators, \(Z_1Z_2, Z_2Z_3, -Z_3Z_4, Z_4Z_5, Z_5Z_6\). This leads to a single encoded qubit

\[
|\alpha(000)_u(111)_v\rangle + |\beta(111)_u(000)_v\rangle,
\] (A9)

supported by the two sites \(u\) and \(v\) jointly. If more neighboring sites share the same POVM outcome, the encoded qubit can be easily extended.

**Example 2: graph-state stabilizer generator.** To see that |\(\Psi(A)\rangle\) is indeed equivalent under local unitary transformations to an encoded graph state \(|\tilde{G}(A)\rangle\), we consider the example of four domains \(c, u, v, w\), each consisting of a single site of \(\mathcal{L}\), where the POVM outcome is \(z\) on the central domain \(c\) and \(x\) on all lateral domains \(u, v\) and \(w\); see Fig. 2d of the main text. By direct computation we have that

\[
[-X_1X'_1, F_{u,z}] = [-X_2X'_2, F_{v,z}] = [-X_3X'_3, F_{w,z}] = 0.
\] (A10)

Although \(-X_iX'_i\) individually does not commute with \(F_{c,z}\), the operator \(O = X_1X'_1X_2X'_2X_3X'_3\) does, as by direction computation,

\[
X_1X_2X_3(|000\rangle\langle000| + |111\rangle\langle111|) = |111\rangle\langle000| + |000\rangle\langle111| = (|000\rangle\langle000| + |111\rangle\langle111|)X_1X_2X_3,
\] (A11)

we have

\[
[X_1X_2X_3, F_{c,z}] = 0.
\] (A12)

This shows that \(O\) is in the stabilizer of |\(\Psi(A)\rangle\), i.e., \(|O\Psi(A)\rangle = |\Psi(A)\rangle\). Using the encoding in Table 1 of the main text, i.e., with the encoded Pauli operators \(\hat{X}_c = Z_1Z_2Z_3, \hat{Z}_u = X'_1, \hat{Z}_v = X'_2,\) and \(\hat{Z}_w = X'_3\), we find that \(O = \hat{X}_c\hat{Z}_u\hat{Z}_v\hat{Z}_w\) which is (up to a possible sign) one stabilizer generator defining the graph state.

**The Hamiltonian and boundary conditions.** The construction of the AKLT state by the valence-bond picture gives rise to a state that is the ground state of the following Hamiltonian

\[
H = \sum_{\text{edge } (i,j)} \tilde{P}_{i,j}^{(S=3)} = \sum_{\text{edge } (i,j)} \left[ \hat{S}_i \cdot \hat{S}_j + \frac{116}{243} (\hat{S}_i \cdot \hat{S}_j)^2 + \frac{16}{243} (\hat{S}_i \cdot \hat{S}_j)^3 + \frac{55}{108} \right],
\] (A13)

where \(\tilde{P}_{i,j}^{(S=3)}\) is a projector of the neighboring sites \(i\) and \(j\) onto a total \(S = 3\) subspace. In the case of the periodic boundary condition, the AKLT is the unique ground state. In the case of the open boundary condition, one can terminate every boundary spin-3/2 by a spin-1/2, and add a corresponding term in the Hamiltonain,

\[
h_{i,i'} = P_{i,i'}^{(S=2)} = \frac{1}{2} \hat{S}_i \cdot \hat{S}_{i'} + \frac{5}{4},
\] (A14)

where \(\hat{S}_i\) is the spin-3/2 operator at the boundary and \(\hat{S}_{i'}\) is the associated spin-1/2 operator. These additional Hamiltonian terms for all boundary pairs make the AKLT state a unique ground state.

In the case of the open boundary condition, the planarity of the graph is preserved even after the POVMs. However, in the case of the periodic boundary condition, the underlying topology is that of a torus. To make the graph of the corresponding graph state after the POVMs be planar, one simply measures the logical \(Z\) on sites along the two independent cycles and this will cut the torus into a plane.