Multiplicity of solutions for the Dirichlet boundary value problem to a fractional quasilinear differential model with impulses

Xiaohui Shen¹,² and Tengfei Shen³*

¹Correspondence: sfcool@126.com
²School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, PR China
³School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, PR China
Full list of author information is available at the end of the article

Abstract

This paper aims to consider the multiplicity of solutions for a kind of boundary value problem to a fractional quasilinear differential model with impulsive effects. By establishing a new variational structure and overcoming the difficulties brought by the influence of impulsive effects, some new results are acquired via the symmetry mountain-pass theorem, which extend and enrich some previous results.

MSC: 26A33; 34G20; 34B15
Keywords: Fractional differential equation; Boundary value problem; Multiplicity; Impulsive effect

1 Introduction

In this paper, we are concerned with the following fractional quasilinear differential model with impulsive effects.

\[
\begin{align*}
\left\{ \\
\dot{D}^\alpha_I (\alpha D^\alpha_I u(t)) + b(t)u(t) + 2u(t)|\alpha D^\alpha_I u(t)|^2 + 2\varepsilon D^\alpha_T (|u(t)|^2 \alpha D^\alpha_I u(t)) \\
= f(t, u(t)), & \quad \text{a.e. } t \in J, \\
\Delta (I^{1-\alpha}_T (\alpha D^\alpha_I u(t_j))) = I_1 u(t_j), & \quad j = 1, 2, \ldots, m, \\
\Delta (I^{1-\alpha}_T (|u(t)|^2 \alpha D^\alpha_I u(t_j))) = I_2 u(t_j), & \quad j = 1, 2, \ldots, m, \\
u(0) = u(T) = 0,
\end{align*}
\]

where \( D^\alpha_I \) and \( \dot{D}^\alpha_I \) are the left and right Riemann–Liouville fractional derivatives, respectively, \( J^{1-\alpha}_T \) is the right Riemann–Liouville fractional integral, \( f(t, u) = g(t, u) + \xi h(t)|u(t)|^{v-2} u(t), \) \( g, h \in C([0, T] \times \mathbb{R}, \mathbb{R}), \) \( \alpha \in \left( \frac{1}{2}, 1 \right], \) \( b, h \in C([0, T], \mathbb{R}), \) \( t_0 = 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T, J = [0, T] \setminus \{t_1, t_2, \ldots, t_m\}, \) \( m \in \mathbb{N}, I_j \in C(\mathbb{R}, \mathbb{R}), \) \( \xi \in \mathbb{R}, v \in [1, 2), \)

\[
\begin{align*}
\Delta (I^{1-\alpha}_T (\alpha D^\alpha_I u(t_j))) &= I^{1-\alpha}_T (\alpha D^\alpha_I u(t_j)), \\
\Delta (I^{1-\alpha}_T (|u(t)|^2 \alpha D^\alpha_I u(t_j))) &= I^{1-\alpha}_T (|u(t)|^2 \alpha D^\alpha_I u(t_j)),
\end{align*}
\]

\( \alpha \in (\frac{1}{2}, 1], \) \( J = [0, T], \) \( t = t_j, \) \( u(t_j) \) is the right Riemann–Liouville fractional integral, \( f(t, u) = g(t, u) + \xi h(t)|u(t)|^{v-2} u(t), \) \( g, h \in C([0, T] \times \mathbb{R}, \mathbb{R}), \) \( \alpha \in \left( \frac{1}{2}, 1 \right], \) \( b, h \in C([0, T], \mathbb{R}), \) \( t_0 = 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T, J = [0, T] \setminus \{t_1, t_2, \ldots, t_m\}, \) \( m \in \mathbb{N}, I_j \in C(\mathbb{R}, \mathbb{R}), \) \( \xi \in \mathbb{R}, v \in [1, 2), \)

\[
\begin{align*}
\Delta (I^{1-\alpha}_T (\alpha D^\alpha_I u(t_j))) &= I^{1-\alpha}_T (\alpha D^\alpha_I u(t_j)), \\
\Delta (I^{1-\alpha}_T (|u(t)|^2 \alpha D^\alpha_I u(t_j))) &= I^{1-\alpha}_T (|u(t)|^2 \alpha D^\alpha_I u(t_j)),
\end{align*}
\]

© The Author(s) 2022. Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
In fact, the idea of a fractional quasilinear differential model comes from the standing-wave solutions \( \varphi(t, x) = e^{i\omega t}u(x), w \in \mathbb{R} \) of the following integer quasilinear Schrödinger equation.

\[
i\partial_t \varphi = -\partial_{xx} \varphi + V(x) \varphi - \partial_x (|\varphi|^2) \varphi - |\varphi|^{v-1} \varphi, \quad x \in \mathbb{R}, v > 1, \tag{1.2}
\]

which plays an important role in some research fields of physics (see [1, 2] and the references therein). An interesting question as to whether the existence or multiplicity of solutions to this fractional quasilinear differential model with suitable boundary conditions generated by impulsive effects can be obtained naturally comes to mind. It is well known that the impulsive differential models describe the discontinuous process and originate from some important research fields. In recent years, critical-point theory has been successfully applied to deal with the existence and multiplicity of solutions of boundary value problems (BVPs for short) to differential equations with impulsive effects. Based on some critical-point theorems, Nieto and O’Regan [3] considered the impulsive Dirichlet BVP

\[
\begin{aligned}
-\varphi''(t) + \lambda \varphi(t) &= f(t, u(t)), \quad \text{a.e. } t \in J, \\
\Delta (\varphi(t_j)) &= I_j(u(t_j)), \quad j = 1, 2, \ldots, m, \\
\varphi(0) &= \varphi(T) = 0
\end{aligned}
\tag{1.3}
\]

and obtained some existence results. Subsequently, more and more scholars have paid attention to this problem, such as Sun and Chen [4], Zhou and Li [5], Zhang and Yuan [6], etc. Moreover, for the case of impulsive BVPs with \( p \)-Laplacian operator, one can refer to [7, 8] and references therein.

On the other hand, recently, Jiao and Zhou [9] proved that under the Dirichlet boundary condition \( u(0) = u(T) = 0 \), the operator \( \mathcal{L}_T^\alpha \) has a variational structure. Also, by the mountain-pass theorem, the existence of solutions to the following systems was obtained under the Ambrosetti–Rabinowitz condition:

\[
\begin{aligned}
\mathcal{L}_T^\alpha \varphi(t) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\
u(0) &= u(T) = 0
\end{aligned}
\tag{1.4}
\]

where \( \alpha \in (\frac{1}{2}, 1) \). After that, Bonanno, Rodríguez-López and Tersian [10] discussed the existence of three solutions to the following problem with impulsive effects and parameters:

\[
\begin{aligned}
\mathcal{L}_T^\alpha \varphi(t) + a(t)u(t) &= \lambda f(t, u(t)), \quad \text{a.e. } t \in J, \\
\Delta (\mathcal{L}_T^\alpha \varphi(t_j)) &= \mu I_j(u(t_j)), \quad j = 1, 2, \ldots, m, \\
u(0) &= u(T) = 0
\end{aligned}
\tag{1.5}
\]
where \( \alpha \in \left( \frac{1}{2}, 1 \right) \). Nyamoradi and Rodríguez-López [11] extended the scalar model of (1.5) to the case of Hamiltonian systems and obtained the multiplicity of solutions by the variant Fountain theorems. Moreover, by the gene property and the mountain-pass theorem, Ledesma and Nyamoradi [12] investigated the eigenvalue problem \( ID_t^\alpha \phi_p(0D_t^\alpha u) = \lambda \phi_p(u) \) with the Dirichlet boundary conditions \( u(0) = u(T) = 0 \) and obtained the existence of solutions to the Dirichlet boundary problem of a fractional \( p \)-Laplacian equation with impulsive effects. Liu, Wang and Shen [13] extended the results of [12] to the case of combined nonlinearity. Furthermore, for the Dirichlet BVPs and other BVPs of fractional differential equations with or without impulsive effects, one can refer to [14–21] and references therein.

Motivated by the works mentioned above, we are concerned with the multiplicity of solutions to the fractional quasilinear differential model with impulsive effects (1.1). Let us present our paper’s contribution: To begin with, the variational structure of (1.1) is established, which makes the critical-point theory applicable to discuss the existence and multiplicity of solutions to this problem. Moreover, the impulsive effects produced by the quasilinear term \( u_0D_t^\alpha u^2 + ID_t^\alpha (|u|^2D_t^\alpha u) \) are more complex than the case of \( ID_t^\alpha (D_t^\alpha u) \), which make this problem challenging. Furthermore, there are few papers considering this problem.

In order to describe our main conclusion, the following assumptions are presented:

(1) For any \( t \in \mathbb{R} \), \( I_{ij}(t) \) and \( I_{ij}(t) \) are odd on \( t \) and

\[
\int_0^t \left( I_{ij}(s) + I_{ij}(s) \right) ds \geq 0.
\]

(2) There exist constants \( a_{ij}, a_{ij}, d_{ij}, d_{ij}, d_{ij}, d_{ij}, d_{ij} > 0 \) such that

\[
\left| I_{ij}(t) \right| \leq a_{ij} + d_{ij}|t|^{\gamma_{ij}} \quad \text{for any } t \in \mathbb{R}, \gamma_{ij} \in [0, 1),
\]

\[
\left| I_{ij}(t) \right| \leq a_{ij} + d_{ij}|t|^{\gamma_{ij}} \quad \text{for any } t \in \mathbb{R}, \gamma_{ij} \in [2, 3).
\]

(3) For any \( t \in \mathbb{R} \), \( I_{ij}(t) \) and \( I_{ij}(t) \) satisfy

\[
\theta \int_0^t \left( I_{ij}(s) + I_{ij}(s) \right) ds - \left( I_{ij}(t) + I_{ij}(t) \right) t \geq 0,
\]

where \( \theta \geq 4 \) is a constant.

(G1) \( \lim_{|u| \to +\infty} \frac{G(t,u)}{|u|^2} = +\infty \) uniformly for \( t \in [0, T] \).

(G2) There exist constants \( M_1 > 0, L_1 > 0 \) such that for \( t \in [0, T], |u| \geq L_1 \),

\[
ug(t, u) - \theta G(t, u) \geq -M_1|u|^2.
\]

(G3) There exist constants \( M_2 > 0, L_2 > 0, \mu > \theta \) such that for \( t \in [0, T], |u| \geq L_2 \),

\[
G(t, u) \leq M_2|u|^\mu.
\]

(G4) \( g(t, u) = o(|u|) \) as \( |u| \to 0 \) uniformly for \( t \in [0, T] \).

(G5) \( g(t, u) \) is odd on \( u \).

Now, we state our main results.
Theorem 1.1  Assuming that the conditions (I1)–(I3) and (G1)–(G5) are satisfied, there exists a constant $\zeta^* > 0$ such that the problem (1.1) has infinitely many nontrivial weak solutions, provided that $\zeta \in [0, \zeta^*)$.

Remark 1.2 It should be pointed out that if $\zeta = 0$, the condition (G3) can be removed. Moreover, the conditions (G1) and (G2) are weaker than the following classical Ambrosetti–Rabinowitz condition:

$$0 < \theta G(t, u) \leq uG(t, u), \quad \theta > 4, u \in \mathbb{R} \setminus \{0\}.$$

Remark 1.3 If $\alpha = 1$, the quasilinear term $2u|D_t^\alpha u|^2 + 2rD_t^\alpha (|u|^2 D_t^\alpha u)$ is equal to $-(|u|^2)'u$. Moreover, $I_2(u) = |u|^2 I_1(u)$.

Corollary 1.4 If the assumptions (I2) and (G1) in Theorem 1.1 are replaced by (I2)* there exist constants $a_{ij}, a_{2j}, d_{1j}, d_{2j} > 0$ such that

$$|I_1(t)| \leq a_{ij} + d_{1j}|t|^{\gamma_{ij}} \text{ for any } t \in \mathbb{R}, \gamma_{ij} \in [0, \theta - 3),$$

$$|I_2(t)| \leq a_{2j} + d_{2j}|t|^{\gamma_{2j}} \text{ for any } t \in \mathbb{R}, \gamma_{2j} \in [2, \theta - 1).$$

(G1)* $\lim_{|u| \to +\infty} \frac{F(t, u)}{|u|^\theta} = +\infty$ uniformly for $t \in [0, T]$. Then, the conclusion of Theorem 1.1 is also true.

Remark 1.5 It should be pointed out that the impulsive nonlinearity $I_{ij}$ could be superlinear growth when $\theta > 4$.

2 Preliminaries

Set $C := C([0, T], \mathbb{R})$ with norm $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$ and $L^p := L^p([0, T], \mathbb{R})$ with norm $\|u\|_L^p = (\int_0^T |u(t)|^p dt)^{1/p}$. For the definitions of fractional integrals and derivatives relating to the well-known left and right Riemann–Liouville and Caputo, one can refer to references [22, 23]. Next, some of the necessary results and properties will be presented. Define the Sobolev space

$$E^\alpha_0 = \{ u : [0, T] \to \mathbb{R} \mid u, 0D_t^\alpha u \in L^2, u(0) = u(T) = 0 \}$$

by $C_0^\infty([0, T], \mathbb{R}) \cap 1_\alpha$, where $\| \cdot \|_\alpha = (\langle \cdot, \cdot \rangle_\alpha)^{1/2}$,

$$\langle u, v \rangle = \int_0^T ((u(t)v(t) + 0D_t^\alpha u(t)0D_t^\alpha v(t)) dt.$$

Let

$$P(u) = \frac{1}{2} \|u\|_a^2 - \frac{1}{2} \int_0^T (1 - b(t))u^2(t) dt.$$

By the method of [24], the space $E^\alpha_0$ can be decomposed as follows. In fact, based on the Riesz representation theorem, we can find a linear self-adjoint operator $Q : E^\alpha_0 \to E^\alpha_0$ such
Noting that the embedding $E_0^q \hookrightarrow C$ is compact (see [9]), it implies that $Q$ is compact. In view of the well-known compact operator’s spectral theory, for the operator $I - Q$, we can decompose the Sobolev space $E_0^q$ into the orthogonal sum of invariant subspaces as

$$E_0^q = E^- \oplus E^0 \oplus E^+,$$

where $E^-$ and $E^+$ are negative and positive spectral subspaces corresponding to the operator $I - Q$, $E^0 = N(I - Q)$. Moreover, letting $\Omega = \{1, 2, \ldots, i\}$ with $i \in \mathbb{N}$, $Q$ possesses only finitely many eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ satisfying $\lambda_i > 1$ because $Q$ is compact on $E_0^q$, which implies that the dimension of subspace $E^-$ is finite. By the classical self-adjoint operator theory, for $I - Q$ that can be viewed as a compact perturbation relating to the self-adjoint operator $I$, it is clear that 0 is excluded in the essential spectrum of $I - Q$. Thus, the dimension of subspace $E^0$ is also finite. Furthermore, there exists a positive constant $\kappa$ such that

$$\pm P(u) \geq \kappa \|u\|_2^2, \quad u \in E^\pm. \quad (2.1)$$

**Lemma 2.1** ([22, 23]) Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If $u$ is a function defined on $[a, b]$ for which the Caputo fractional derivatives $\overset{\circ}{D}_t^\alpha u(t)$ and $\overset{\circ}{D}_b^\alpha u(t)$ of order $\alpha$ exist together with the Riemann–Liouville fractional derivatives $\overset{\circ}{D}_t^\alpha u(t)$ and $\overset{\circ}{D}_b^\alpha u(t)$, then

$$\overset{\circ}{D}_t^\alpha u(t) = \overset{\circ}{D}_t^\alpha u(t) - \frac{\alpha(a)}{\Gamma(\alpha + 1)}(t - a)^{\alpha-1}, \quad t \in [a, b], \quad (2.2)$$

$$\overset{\circ}{D}_b^\alpha u(t) = \overset{\circ}{D}_b^\alpha u(t) - \frac{\alpha(b)}{\Gamma(\alpha + 1)}(b - t)^{\alpha-1}, \quad t \in [a, b]. \quad (2.3)$$

**Remark 2.2** From (2.2) and (2.3), one has $\overset{\circ}{D}_t^\alpha u(t) = \overset{\circ}{D}_b^\alpha u(t), \overset{\circ}{D}_b^\alpha u(t) = \overset{\circ}{D}_b^\alpha u(t), \quad t \in [0, T]$ by $u(0) = u(T) = 0$.

**Proposition 2.3** ([23]) The following property of fractional integration

$$\int_a^b \left[ \overset{\circ}{D}_t^\alpha f(t) \right] g(t) \, dt = \int_a^b \left[ \overset{\circ}{D}_t^\alpha g(t) \right] f(t) \, dt, \quad \alpha > 0$$

holds, provided that $u \in L^p([a, b], \mathbb{R}^N), g \in L^q([a, b], \mathbb{R}^N)$ and $p \geq 1, q \geq 1, 1/p + 1/q \leq 1 + \alpha$ or $p \neq 1, q \neq 1, 1/p + 1/q = 1 + \alpha$, where $\overset{\circ}{D}_t^\alpha$ and $\overset{\circ}{D}_b^\alpha$ are the left and right Riemann–Liouville fractional integrals, respectively.
Lemma 2.4 ([9]) Let $0 < \alpha \leq 1$. If $u \in E_0^\alpha$, one has
\[
\|u\|_{L^2} \leq S_\alpha \|\mathcal{D}^\alpha u\|_{L^2},
\] (2.4)
where $S_\alpha = \frac{T^\alpha}{\Gamma(\alpha+1)}$. Moreover, if $\alpha > \frac{1}{2}$, then
\[
\|u\|_\infty \leq S_\infty \|\mathcal{D}^\alpha u\|_{L^2},
\] (2.5)
where $S_\infty = \frac{T^{\alpha-1/2}}{\Gamma(\alpha+1/4)}$. Based on (2.4), clearly, the norm of $E_0^\alpha$ is equivalent to $\|\mathcal{D}^\alpha u\|_{L^2}$.

Proposition 2.5 ([9]) Let $0 < \alpha \leq 1$. Assume that $\alpha > \frac{1}{2}$ and the sequence $u_n$ converges weakly to $u$ in $E_0^\alpha$, i.e., $u_n \rightharpoonup u$. Then, $u_n \to u$ in $C$, i.e., $\|u_n - u\|_\infty \to 0$, $n \to +\infty$.

If $u \in E_0^\alpha$ is a solution of the problem (1.1), for $\nu \in E_0^\alpha$, based on Lemma 2.1 and Proposition 2.3, it implies that
\[
\begin{align*}
\int_0^T \mathcal{D}^\alpha (|u(t)|^2 \mathcal{D}^\alpha u(t)) \nu(t) \, dt \\
= -\sum_{j=0}^m \int_{I_j} \nu(t) d\left[ I_{1/\alpha} \left( |u(t)|^2 \mathcal{D}^\alpha u(t) \right) \right] \\
+ \sum_{j=0}^m \int_{I_j} |u(t)|^2 \mathcal{D}^\alpha u(t) (I_{1/\alpha} \left( |u(t)|^2 \mathcal{D}^\alpha u(t) \right)) \nu(t) \, dt \\
= \sum_{j=0}^m \int_{I_j} |u(t)|^2 \mathcal{D}^\alpha u(t) (I_{1/\alpha} \left( |u(t)|^2 \mathcal{D}^\alpha u(t) \right)) \nu(t) \, dt.
\end{align*}
\]
Similarly, one has
\[
\begin{align*}
\int_0^T \mathcal{D}^\alpha (\mathcal{D}^\alpha u(t)) \nu(t) \, dt = \sum_{j=0}^m I_{j/\alpha} (u(t_j)) \nu(t_j) + \int_0^T |u(t)|^2 \mathcal{D}^\alpha u(t) \mathcal{D}^\alpha \nu(t) \, dt.
\end{align*}
\]
As a conclusion, the definition of a weak solution is shown as follows.

Definition 2.6 A function $u \in E_0^\alpha$ is a weak solution of problem (1.1) if
\[
\begin{align*}
\int_0^T \mathcal{D}^\alpha u(t) \mathcal{D}^\alpha v(t) \, dt + \int_0^T \mathcal{D}^\alpha \mathcal{D}^\alpha u(t) \mathcal{D}^\alpha v(t) \, dt \\
+ \int_0^T b(t) u(t) v(t) \, dt + \int_0^T 2|\mathcal{D}^\alpha u(t)|^2 u(t) v(t) \, dt \\
+ \int_0^T 2 |u(t)|^2 \mathcal{D}^\alpha u(t) \mathcal{D}^\alpha v(t) \, dt + \sum_{j=1}^m \left( I_{j/\alpha} (u(t_j)) + I_{j/\alpha} (u(t_j)) \right) v(t_j) \\
= \int_0^T f(t, u(t)) v(t) \, dt
\end{align*}
\]
holds for any $\nu \in E_0^\alpha$. 
Define the functional $\Phi : E^0_0 \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} \int_0^T \left| D_x^2 u(t) \right|^2 \, dt + \frac{1}{2} \int_0^T b(t) |u(t)|^2 \, dt + \sum_{j=1}^m \int_0^{\nu_j} (I_{ij}(t) + I_{2j}(t)) \, dt$$

$$+ \int_0^T \left| D_x^2 u(t) \right|^2 |u(t)|^2 \, dt - \int_0^T G(t,u(t)) \, dt - \frac{\xi}{\nu} \int_0^T h(t)|u(t)|^\nu \, dt,$$

where $G(t,u) = \int_0^T g(t,s) \, ds$. Since $g, I_{ij}$, and $I_{2j}$ are continuous, by the standard arguments, one can obtain that $\Phi(u) \in C^1(E^0_0, \mathbb{R})$. Moreover, it is clear that the critical points of $\Phi(u)$ are weak solutions of the problem (1.1).

**Lemma 2.7** ([25]) Let $E$ be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even with $\Phi(0) = 0$. Assume that $E = V \oplus X$, where $V$ is finite-dimensional. Moreover, $\Phi$ satisfies the (PS)-condition and the following conditions.

(i) There exist constants $\rho, \sigma > 0$ such that $\Phi |_{\partial B_\rho \cap X} \geq \sigma$.

(ii) For each finite-dimensional subspace $\tilde{X} \subset E$, there exists an $l = l(\tilde{X})$ such that $\Phi \leq 0$ on $\tilde{X} \setminus B_l$.

Then, $\Phi$ has an unbounded sequence of critical values.

### 3 Main Results

In order to prove our main conclusions, we need the following lemmas. First, in $E^0_0$, let $V = E^- \oplus E^0$ and $X = E^+$, then the dimension of subspace $V$ is finite and $E^0_0 = V \oplus X$.

**Lemma 3.1** Assuming that the conditions (I1), (G3), and (G4) are satisfied, we can find constants $\rho, \sigma, \xi^* > 0$ such that $\Phi |_{\partial B_\rho \cap X} \geq \sigma$, provided that $\xi \in [0, \xi^*)$.

**Proof** Based on (G3) and (G4), for any $\varepsilon > 0$, we can find a constant $c_\varepsilon$ such that for $t \in [0, T]$,

$$G(t,u) \leq \varepsilon |u|^2 + \varepsilon_c |u|^\mu,$$

where $\Phi(u) \geq \kappa \left| D_x^2 \right|^2 + \sum_{j=1}^m \int_0^{\nu_j} (I_{ij}(t) + I_{2j}(t)) \, dt + \int_0^T |D_x^2 u(t)|^2 |u(t)|^2 \, dt$

$$- \int_0^T G(t,u(t)) \, dt - \frac{\xi}{\nu} \int_0^T h(t)|u(t)|^\nu \, dt$$

$$\geq \kappa \left| D_x^2 \right|^2 - \int_0^T G(t,u(t)) \, dt - \frac{\xi}{\nu} \int_0^T h(t)|u(t)|^\nu \, dt$$
\[ \geq \|u\|_\alpha^2 \left( \|u\|_\alpha^{2-v} - c_\varepsilon TS_{\alpha}^{\mu} \|u\|_\alpha^{\mu-v} - \frac{\xi}{\nu} TS_{\alpha}^{\nu} \|h\|_{L^1} \right) \]

Letting \( \varepsilon = \frac{\xi}{2TS_{\alpha}^{\nu}} \), leads to

\[ \Phi(u) \geq \Phi_1(u) = \left( \|u\|_\alpha^{2-v} - c_\varepsilon TS_{\alpha}^{\mu} \|u\|_\alpha^{\mu-v} - \frac{\xi}{\nu} TS_{\alpha}^{\nu} \|h\|_{L^1} \right). \]

Set

\[ y(t) = \frac{\kappa}{2} t^{2-v} - c_\varepsilon TS_{\alpha}^{\mu} t^{\mu-v}, \quad t \geq 0. \]

Clearly, there exists a \( \rho = \left[ \frac{\kappa(2-v)}{2c_\varepsilon TS_{\alpha}^{\mu}(\mu-v)} \right]^{\frac{1}{2-v}} \) such that

\[ y(\rho) = \max_{t \geq 0} y(t) = \left( \frac{\kappa(2-v)}{2c_\varepsilon TS_{\alpha}^{\mu}(\mu-v)} \right)^{\frac{1}{2-v}}. \]

Therefore, we can find

\[ \zeta^* = \frac{v\kappa(\mu-2)}{TS_{\alpha}^{\mu}(\mu-v)\|h\|_{L^1}} \left( \frac{\kappa(2-v)}{2c_\varepsilon TS_{\alpha}^{\mu}(\mu-v)} \right)^{\frac{1}{2-v}}. \]

If \( \zeta \in [0, \zeta^*) \), there exists a constant \( \sigma > 0 \) such that \( \Phi|_{X \cap S_{\rho}} \geq \sigma \). \( \square \)

**Lemma 3.2** If the conditions (I2) and (G1) are satisfied, there exists a constant \( l > 0 \) such that for each finite-dimensional subspace \( \tilde{X} \subset E_\alpha \), \( \Phi(u) \leq 0 \), \( \forall u \in \tilde{X} \setminus B_l \), provided that \( \zeta \in [0, +\infty) \).

**Proof** Actually, for \( \zeta \in [0, +\infty) \), the key point is to prove that \( \Phi(u) \) is anticoercive, i.e.,

\[ \Phi(u) \to -\infty \quad \text{as} \quad \|u\|_\alpha \to +\infty \quad \text{for} \quad u \in \tilde{X}. \quad (3.2) \]

If not, let the sequence \( \{u_n\} \subset \tilde{X} \) and \( \tau \in \mathbb{R} \) such that

\[ \Phi(u_n) \geq \tau \quad \text{when} \quad \|u_n\|_\alpha \to +\infty \quad \text{as} \quad n \to +\infty. \quad (3.3) \]

Setting \( \omega_n = \frac{\mu}{1+\bar{\|u_n\|}_\alpha} \), then \( \|\omega_n\|_\alpha = 1 \). Since \( \dim \tilde{X} < \infty \), we can find a subsequence of \( \{\omega_n\} \) (named again \( \{\omega_n\} \)) such that \( \omega_n \to \omega \) in \( E_\alpha^0 \), which implies \( \|\omega\|_\alpha = 1 \). From \( \omega \neq 0 \), one has \( |\omega_n(t)| \to +\infty \) as \( n \to +\infty \). Define

\[ W(t, u) = G(t, u) + \frac{\xi}{\nu} h(t) |u|^{\nu} - \frac{1}{2} b(t) |u|^2. \]

In view of (G1), it follows that for any \( t \in [0, T] \),

\[ \lim_{|u| \to +\infty} \frac{W(t, u)}{|u|^2} = +\infty. \quad (3.4) \]
Moreover, by a standard measure estimation on a finite-dimensional space (see [4]), it follows that there exists a positive constant $\epsilon > 0$ such that

$$\text{meas}\{t \in [0, T] : |u(t)| \geq \epsilon \|u\|_a \} \geq \epsilon \quad \text{for } u \in \overline{X} \setminus \{0\}. \quad (3.5)$$

Let $\Pi = \{t \in [0, T] : |u(t)| \geq \epsilon \|u\|_a \}$. Based on (3.4), it means that for $\frac{2\epsilon^2}{\epsilon^2} > 0$, there exists $\eta > 0$ such that

$$W(t, u) \geq \frac{2\epsilon^2}{\epsilon^4} |u|^4 \quad \text{for } |u| \geq \eta. \quad (3.6)$$

Hence, for $u \in \overline{X}$ with $\|u\|_a \geq \frac{\eta}{\epsilon}$, we can obtain that

$$W(t, u) \geq 2\epsilon^2 \|u\|_a^4 \quad \text{for } t \in \Pi. \quad (3.7)$$

Let $\|u_n\|_a \geq \frac{\eta}{\epsilon}$ for $n$ large enough. From (I2), one has

$$\Phi(u) = \frac{1}{2} \int_0^T \left| \int_0^T D^\alpha_t u_n(t) \right|^2 dt + \sum_{j=1}^m \int_0^T (I_{1j}(t) + I_{2j}(t)) dt$$

$$+ \int_0^T \left| \int_0^T D^\alpha_t u_n(t) \right|^2 |u_n(t)|^2 dt$$

$$- \int_0^T W(t, u_n(t)) dt$$

$$\leq \frac{1}{2} \left[ \sum_{j=1}^m a_j S_{\infty} \|u_n\|_a + \sum_{j=1}^m a_{2j} S_{\infty} \|u_n\|_a + \sum_{j=1}^m d_{1j} S_{\infty} \|u_n\|_a \right]$$

$$+ \sum_{j=1}^m d_{2j} S_{\infty}^2 \|u_n\|_a^2 + \sum_{j=1}^m \left( \sum_{j=1}^m a_{2j} S_{\infty} \right) \frac{1}{\|u_n\|_a}$$

$$\leq \frac{1}{2} \left[ \sum_{j=1}^m a_j S_{\infty} \|u_n\|_a + \sum_{j=1}^m a_{2j} S_{\infty} \|u_n\|_a + \sum_{j=1}^m d_{1j} S_{\infty} \|u_n\|_a \right]$$

$$+ \sum_{j=1}^m d_{2j} S_{\infty}^2 \|u_n\|_a^2 + \sum_{j=1}^m \left( \sum_{j=1}^m a_{2j} S_{\infty} \right) \frac{1}{\|u_n\|_a}$$

$$\rightarrow -\infty \quad \text{if } \|u_n\|_a \rightarrow +\infty \text{ as } n \rightarrow +\infty,$$
which is in contradiction to (3.3). Hence, $\Phi(u)$ is anticoercive. Therefore, there exists a constant $l > 0$ such that $\Phi(u) \leq 0$, $\forall u \in \tilde{X} \backslash B_l$ for $\zeta \in [0, +\infty)$.

Lemma 3.3 If the assumptions (I2), (I3), (G1), (G2), and (G4) are satisfied, $\Phi(u)$ meets the (PS)-condition, provided that $\zeta \in [0, +\infty)$. 

Proof Let $\{u_n\} \subset E_0^\alpha$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ as $n \to +\infty$, which implies that there exists a constant $\beta > 0$ such that

$$\left|\Phi(u_n)\right| \leq \beta, \quad \left\|\Phi'(u_n)\right\|_{E_0^{\alpha}} \leq \beta.$$

We claim that the sequence $\{u_n\}$ is bounded. If not, let $\|u_n\| \to +\infty$ as $n \to +\infty$. Setting $\omega_n = \frac{u_n}{\|u_n\|}$, it follows that $\omega_n$ is bounded in $E_0^\alpha$. Noting that $E_0^\alpha$ is a reflexive Banach space, it implies that $\{\omega_n\}$ has a convergent subsequence (named again $\{\omega_n\}$) such that $\omega_n \rightharpoonup \omega$ in $E_0^\alpha$ and $\omega_n \to \omega$ uniformly in $C$.

In view of (I2), one has

$$\int_0^T W(t, u_n(t)) \, dt = \frac{1}{2} \int_0^T \left|\partial_t^\alpha u_n(t)\right|^2 \, dt + \sum_{j=1}^m \int_0^T (I_{1j}(t) + I_{2j}(t)) \, dt$$

$$+ \int_0^T \left|\partial_t^\alpha u_n(t)\right|^2 |u_n(t)|^2 \, dt - \Phi(u_n)$$

$$\leq \frac{1}{2} \|u_n\|^2_\alpha + \sum_{j=1}^m a_{1j} S_\infty \|u_n\|_\alpha + \sum_{j=1}^m a_{2j} S_\infty \|u_n\|_\alpha$$

$$+ \sum_{j=1}^m d_{1j} S_\infty^2 \|u_n\|_\alpha^2 + d_{2j} S_\infty^2 \|u_n\|_\alpha^4 + \beta,$$

which shows that for $n$ large enough,

$$\int_0^T \frac{W(t, u_n)}{\|u_n\|^2_\alpha} \, dt \leq S_\infty^2 + o(1). \quad (3.8)$$

Based on the continuity of $g$, we can find a constant $\vartheta_1 > 0$ such that

$$|ug(t, u) - \vartheta G(t, u)| \leq \vartheta_1 \quad \text{for} \quad |u| \leq L_1, t \in [0, T],$$

which together with (G2) yields

$$ug(t, u) - \vartheta G(t, u) \geq -M_1 |u|^2 - \vartheta_1 \quad \text{for} \quad |u| \in \mathbb{R}, t \in [0, T]. \quad (3.9)$$

In view of (I3) and (3.9), we have

$$\vartheta \beta + \beta \|u_n\|_\alpha \geq \vartheta \Phi(u_n) - \left\langle \Phi'(u_n), u_n \right\rangle.$$
which means that there exists a positive constant \( \vartheta_2 \) such that

\[
\lim_{n \to +\infty} \| \omega_n \|_\infty = \lim_{n \to +\infty} \frac{\| u_n \|_\infty}{\| u_n \|_\alpha} \geq \vartheta_2 > 0.
\]

Therefore, we can obtain \( \omega \neq 0 \). Define

\[
\Xi_1 = \{ t \in [0, T] : \omega \neq 0 \}, \quad \Xi_2 = [0, T] \setminus \Xi_1.
\]

In view of (G1), there exists a constant \( \vartheta_3 > 0 \) such that \( G(t, u) \geq 0 \), for \( t \in [0, T] \), \( |u| \geq \vartheta_3 \), which together with (G4) yields that there exist constants \( \vartheta_4, \vartheta_5 > 0 \) such that

\[
G(t, u) \geq -\vartheta_4 u^2 - \vartheta_5 \quad \text{for } t \in [0, T], u \in \mathbb{R}.
\]

Based on Fatou's lemma, it follows that

\[
\liminf_{n \to +\infty} \int_{\Xi_2} \frac{G(t, u_n)}{\| u_n \|_\alpha^2} \, dt > -\infty.
\]

By (G1), for \( t \in [0, T] \), we can obtain that

\[
\liminf_{n \to +\infty} \int_0^T \frac{G(t, u_n)}{\| u_n \|_\alpha^2} \, dt = \liminf_{n \to +\infty} \left( \int_{\Xi_1} \frac{G(t, u_n)}{|u_n|^4} |\omega_n|^4 \, dt + \int_{\Xi_2} \frac{G(t, u_n)}{|u_n|^4} |\omega_n|^4 \, dt \right) \to +\infty,
\]

which is in contradiction to (3.8). Thus, \( \{ u_n \} \) is bounded, which implies that \( \{ u_n \} \) possesses a convergent subsequence (named again \( \{ u_n \} \)) such that \( u_n = u_n^+ + u_n^- + u_n^0 \xrightarrow{\text{weak}} u = u^+ + u^- + u^0 \) and \( u_n^0 \xrightarrow{\text{weak}} u^0 \) in \( E_0^\alpha \). Moreover, \( u_n \to u \) and \( u_n^+ \to u^+ \) uniformly in \( C \). It should be
mentioned that the dimensions of subspaces $E^-$ and $E^0$ are finite. Hence, $u_n^- \to u^-$ and $u_n^0 \to u^0$ in $E_n^\alpha$. Furthermore, if $n \to +\infty$, one has
\[
\langle \Phi'(u_n) - \Phi'(u), u_n^\alpha - u^\alpha \rangle \to 0,
\]
\[
\int_0^T b(t)(u_n(t) - u(t))(u_n^\alpha(t) - u^\alpha(t)) \, dt \to 0,
\]
\[
\sum_{j=1}^m (I_{1j}(u_n(t_j)) - I_{1j}(u(t_j)))(u_n^\alpha(t_j) - u^\alpha(t_j)) \to 0,
\]
\[
\sum_{j=1}^m (I_{2j}(u_n(t_j)) - I_{2j}(u(t_j)))(u_n^\alpha(t_j) - u^\alpha(t_j)) \to 0,
\]
\[
\int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n^\alpha(t) - u^\alpha(t)) \, dt \to 0,
\]
\[
\int_0^T \left( |\partial D^\alpha_1 u_n(t)|^2 u_n(t) - |\partial D^\alpha_1 u(t)|^2 u(t) \right) (u_n^\alpha(t) - u^\alpha(t)) \, dt \to 0
\]
and
\[
\int_0^T \left( |\partial D^\alpha_1 u_n(t)|^2 - |u(t)|^2 \right) \partial D^\alpha_1 u_n(t) - |u(t)|^2 \partial D^\alpha_1 u(t) \right) (u_n^\alpha(t) - u^\alpha(t)) \, dt
\]
\[
= \int_0^T \left( |u(t)|^2 \partial D^\alpha_1 u_n(t) - |u(t)|^2 \partial D^\alpha_1 u(t) \right) (u_n^\alpha(t) - u^\alpha(t)) \, dt
\]
\[
= \int_0^T |u(t)|^2 |\partial D^\alpha_1 u_n(t) - \partial D^\alpha_1 u^\alpha(t)||^2 \, dt + o(1),
\]
which implies that
\[
\langle \Phi'(u_n) - \Phi'(u), u_n^\alpha - u^\alpha \rangle \to 0
\]
\[
= \int_0^T |\partial D^\alpha_1 u_n(t) - \partial D^\alpha_1 u^\alpha(t)||^2 \, dt + \int_0^T |u(t)|^2 |\partial D^\alpha_1 u_n(t) - \partial D^\alpha_1 u^\alpha(t)||^2 \, dt + o(1).
\]
Since the norm of $E_n^\alpha$ is equivalent to $\|\partial D^\alpha_1 u\|_{L^2}$, it is clear that $u_n^\alpha \to u^\alpha$ in $E_n^\alpha$. Thus, $u_n \to u$ in $E_n^\alpha$. Therefore, $\Phi(u)$ satisfies the (PS)-condition. 

Proof of Theorem 1.1 From Lemma 3.1, Lemma 3.2, and Lemma 3.3, Theorem 1.1 can be proven immediately by Lemma 2.7. 

Proof of Corollary 1.4 The proof is similar to Theorem 1.1. Therefore, we omit the detail. 

4 Conclusions
By establishing a new variational structure and overcoming the difficulties brought by the influence of impulsive effects, the multiplicity of solutions for a kind of boundary value problem to a fractional quasilinear differential model with impulsive effects is obtained, which extend and enrich some previous results. Moreover, the impulsive effects
produced by the quasilinear term $u|_{(0}D_{t}^{\alpha}u|^2 + \lambda D_{t}^{\alpha}(u|_{(0}D_{t}^{\alpha}u)$ are more complex than the case of $\lambda D_{t}^{\alpha}(u|_{(0}D_{t}^{\alpha}u)$, which makes this problem challenging. Furthermore, there are few papers that consider this problem.

Acknowledgements
The authors really appreciate the reviewers’ valuable comments, which helped improve the former version of this paper.

Funding
X. Shen is supported by the National Natural Science Foundation of China (No. 12101532), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 21KJB110020) and the Research Initiation Foundation of Xuzhou Medical University (No. D2019003). T. Shen is supported by the Natural Science Foundation of Jiangsu Province (No. BK20190620) and the Fundamental Research Funds for the Central Universities (No. 2019QNA05).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The authors contributed equally to this paper. All authors reviewed the manuscript.

Author details
1Department of Health Statistics, Xuzhou Medical University, Xuzhou, Jiangsu, 221004, PR. China. 2Center for Medical Statistics and Data Analysis, Xuzhou Medical University, Xuzhou, Jiangsu, 221004, PR. China. 3School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, PR. China.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 July 2022 Accepted: 22 August 2022 Published online: 31 August 2022

References
1. Shen, T., Liu, W.: Infinitely many solutions for second-order quasilinear periodic boundary value problems with impulse effects. Mediterranean Journal of Mathematics 14, 1–12 (2017)
2. Shen, Z., Han, Z.: Existence of solutions to quasilinear Schrödinger equations with indefinite potential. Electronic Journal of Differential Equations 2015, 91 (2015)
3. Nieto, J., O’Regan, D.: Variational approach to impulsive differential equations. Nonlinear Analysis 10, 680–690 (2009)
4. Sun, J., Chen, H.: Multiplicity of solutions for a class of impulsive differential equations with Dirichlet boundary conditions via variant fountain theorems. Nonlinear Analysis, Real World Applications 11, 4062–4071 (2010)
5. Zhou, J., Li, Y.: Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects. Nonlinear Analysis 71, 2856–2865 (2009)
6. Zhang, Z., Yuan, R.: An application of variational methods to Dirichlet boundary value problem with impulses. Nonlinear Analysis 11, 155–162 (2010)
7. Tian, Y., Ge, W.: Applications of variational methods to boundary value problem for impulsive differential equations. Proceedings of the Edinburgh Mathematical Society 51, 509–527 (2008)
8. Xu, J., Wei, Z., Ding, Y.: Existence of weak solutions for $p$-Laplacian problem with impulsive effects. Taiwanese Journal of Mathematics 17, 501–515 (2013)
9. Jiao, F., Zhou, Y.: Existence results for fractional boundary value problem via critical point theory. International Journal of Bifurcation and Chaos 4, 1–17 (2012)
10. Bonanno, G., Rodríguez-López, R., Tersian, S.: Existence of solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. 17, 717–744 (2014)
11. Nyamoradi, N., Rodríguez-López, R.: Multiplicity of solutions to fractional Hamiltonian systems with impulsive effects. Chaos Solitons Fractals 102, 254–263 (2017)
12. Ledesma, C., Nyamoradi, N.: Impulsive fractional boundary value problem with $p$-Laplace operator. Journal of Applied Mathematics and Computation 55, 257–278 (2017)
13. Liu, W., Wang, M., Shen, T.: Analysis of a class of nonlinear fractional differential models generated by impulsive effects. Boundary Value Problems 175, 1–18 (2017)
14. Torres, C.: Mountain pass solution for fractional boundary value problem. Fract. Calc. Appl. Anal. 1, 1–10 (2014)
15. Chen, T., Liu, W.: Solvability of fractional boundary value problem with $p$-Laplacian via critical point theory. Boundary Value Problems 175, 1–12 (2016)
16. Ahmad, B., Alsaedi, A., Ntouyas, S.K., Tariboon, J., Alzahrani, F.: Nonlocal boundary value problems for impulsive fractional $q_k$-$q_k$-difference equations. Advances in Difference Equations 124, 1–16 (2016)
17. Tariboon, J., Ntouyas, S.K.: Oscillation of impulsive conformable fractional differential equations. Open Mathematics 14, 497–508 (2016)
18. Yukunthorn, W., Ahmad, B., Ntouyas, S.K., Tariboon, J.: On Caputo–Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions. Nonlinear Anal. Hybrid Syst. 19, 77–92 (2016)
19. Candito, P., Gasinski, L., Livrea, R., Santos Júnior, J.R.: Multiplicity of positive solutions for a degenerate nonlocal problem with p-Laplacian. Adv. Nonlinear Anal. 11, 357–368 (2022)
20. Diblik, J.: Bounded solutions to systems of fractional discrete equations. Adv. Nonlinear Anal. 11, 1614–1630 (2022)
21. Guo, G., Yang, Z.: On the singularly perturbation fractional Kirchhoff equations: critical case. Adv. Nonlinear Anal. 11, 1097–1116 (2022)
22. Podlubny, I.: Fractional Differential Equation. Academic Press, San Diego (1999)
23. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
24. Li, X., Wu, X., Wu, K.: On a class of damped vibration problems with super-quadratic potentials. Nonlinear Anal. 72, 135–142 (2010)
25. Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 65. Am. Math. Soc., Washington (1986)