1. Introduction

With more and more people using social media, it has become increasingly a source of news and information for many. This is recognized by marketing firms who employ brand "influencers" to promote their products on Facebook and Instagram, and expect that those endorsements will quickly spread among influencer's network. Now, what if we consider multiple influencers all posting an endorsement the same product, but at different times? How long would it take that post to propagate through the entire social network?

In \cite{3}, Bonato, Janssen and Roshanbin introduce a simplified, deterministic version of this problem called graph burning. Given a finite connected graph, the process of burning a graph begins with all vertices being unburned. At time step 1, a single vertex is chosen to be burned. In each subsequent time step, two things occur: (1) the fire spreads to all neighbours of a previously burned vertex, and those vertices become burned, and (2) another vertex is selected to be burned. Note that once a vertex is burned, it cannot be unburned. The process is complete when all vertices of the graph have been burned. The objective is to burn the entire graph in the minimum number of time steps.

Given a graph $G$, suppose vertex $x_i$ is selected in time step $i$ of the burning process for each $i = 1, \ldots, k$. We call each $x_i$ a source of fire. If all vertices of the graph are burned after these $k$ times steps, this sequence $(x_1, x_2, \ldots, x_k)$ is referred to as a burning sequence. We note that once we choose $x_i$ as a source of fire at time step $i$, the fire propagates to all vertices of $K_n$ in time step $2$, making the choice of $x_2$ redundant. However, we still choose a vertex $x_2$ so that the length of the burning sequence is equal to the number of time steps used in the burning process.
The burning numbers of various classes of graphs have been determined; this includes paths and cycles [3], and complete bipartite graphs [7]. Upper and lower bound on the burning number of the Cartesian products and Lexicographic products of graphs have also been determined [7]. In this paper, we examine the burning number of circulant graphs. Due to properties such as symmetry, scalability, and small average node distance, circulant graphs serve as good models for local area networks and parallel computer architectures [8]. (For a survey of circulant graph see [6].)

The circulant graph on $n$ vertices with distance set $S$ has vertex set $\mathbb{Z}_n$ and edge set $\{xy | x - y \in S\}$, where $S \subseteq \mathbb{Z}_n$ and $x \in S$ implies $-x \in S$, with addition done modulo $n$. We denote this circulant graph $C(n, S)$. Circulant graphs are regular and vertex transitive, and are a subset of the more general family of Cayley graphs.

Since $S$ can be written as $S = \{s_1, -s_1, s_2, -s_2, \ldots, s_t, -s_t\}$ where $0 < s_1 < s_2 < \cdots < s_t \leq n/2$, we also use the notation $C(n; s_1, s_2, \ldots, s_t)$ to represent $C(n, S)$. The notation $C(n; s_1, s_2, \ldots, s_t)$ is used in the majority of the paper, with the notation $C(n, S)$ mainly appearing in the final section regarding lexicographic products. Also note that we limit our discussion to connected circulant graphs. Therefore, we assume $\gcd(n, s_1, s_2, \ldots, s_t) = 1$, since $S$ much generate $\mathbb{Z}_n$.

In this paper, we begin by finding the burning number of $C(n; 1, \frac{n}{2})$ for any even integer $n$ such that $n \geq 6$. We then find upper and lower bounds for the class of 4-regular circulant graphs $C(n; 1, m)$ where $1 < m < \frac{n}{2}$. Next, we give the burning numbers for $C(n; 1, m)$ for the specific cases $m = 2$ and $m = 3$. Finally, we examine the burning numbers of circulant graphs of higher degree, including those found by taking the lexicographic product of $C(n; 1, m)$ with another circulant graph.

2. Burning Numbers of 3-Regular Circulant Graphs

We now consider the graph $(n; 1, m)$ where $n$ is even and $m = n/2$. This is the class of all 3-regular circulant graphs.

**Lemma 1.** Suppose $G \cong (n; 1, n/2)$, where $n$ is even and $n \geq 4$. For any $x \in V(G)$ and any $\ell$ such that $1 \leq \ell \leq n/4$, $N_\ell[x] = [x - \ell, x + \ell] \cup [x - \ell + \frac{n}{2} + 1, x + \ell + \frac{n}{2} - 1]$.

**Proof.** Consider vertex $i$ in $G$. It can be easily verified that $N_1[i] = \{i - 1, i, i + 1, i + m\}$. Therefore, $N_1[i] = [i - 1, i + 1] \cup [i + \frac{n}{2}, i + \frac{n}{2}]$. We now proceed by induction.
Assume that $N_k[x] = [x - \ell, x + \ell] \cup [x + \frac{n}{2} - \ell + 1, x + \frac{n}{2} + \ell - 1]$, where $0 \leq \ell \leq \frac{n}{4} - 1$.

We now find the $N_{\ell+1}[0]$, noting that $N_k[0] = [-\ell, \ell] \cup \left[\frac{n}{2} - \ell + 1, \frac{n}{2} + \ell - 1\right]$: 

$$N_{\ell+1}[0] = \bigcup_{i \in N_k[0]} N_1[i] = \bigcup_{i \in N_k[0]} \left\{-1 + i, i, i + 1, i + \frac{n}{2}\right\} = [-\ell - 1, \ell + 1] \cup \left[\frac{n}{2} - \ell, \frac{n}{2} + \ell\right]$$

By symmetry, it can be similarly shown that $N_{\ell+1}[x] = [x - \ell - 1, x + \ell + 1] \cup [i + \frac{n}{2} - \ell, i + \frac{n}{2} + \ell]$. The result follows by induction.

**Theorem 2.** Suppose $G \cong C(n; 1, n/2)$, where $n$ is even and $n \geq 4$. Then $b(G) = \left\lfloor \frac{1 + \sqrt{2n+1}}{2} \right\rfloor$.

*Proof.* Let $k = b(G)$. There is an optimal burning sequence $(x_1, x_2, \ldots, x_k)$, where $n = |N_0(x_k) \cup N_1(x_{k-1}) \cup \cdots \cup N_{k-1}(x_1)|$. From Lemma 1, we find that $|N_\ell(x)| = (2\ell + 1) + (2\ell - 1) = 4\ell$ for any $x \in V(G)$ and $1 \leq \ell \leq \frac{n}{4}$, and $|N_0(v)| = 1$. It is straightforward to verify that $1 \leq k - 1 \leq \frac{n}{4}$ for $n \geq 4$. Therefore, $\sum_{\ell=0}^{k-1} |N_\ell[x_k-i]| = 1 + \sum_{\ell=1}^{k-1} 4\ell = 1 + 2k^2 - 2k$. It follows that $n \leq 1 + 2k^2 - 2k$. However, since $n$ is even, we have $n \leq 2k^2 - 2k$. It follows that $k \geq \frac{1 + \sqrt{2n+1}}{2}$, and $b(G) \geq \left\lceil \frac{1 + \sqrt{2n+1}}{2} \right\rceil$.

Now let $k = \left\lceil \frac{1 + \sqrt{2n+1}}{2} \right\rceil$, and consider the vertex sequence $(x_1, x_2, \ldots, x_k)$ where

$$x_j = \begin{cases} \frac{n}{2} - 2kj - 2k + j^2 - 1 & : j \text{ is even} \\ -2kj - 2k + j^2 - 1 & : j \text{ is odd and } j \neq k - 1 \\ \frac{n}{2} + k & : j \text{ is odd and } j = k - 1 \end{cases}$$

Since $N_k[x] = [x - \ell, x + \ell] \cup [x + \frac{n}{2} - \ell + 1, x + \frac{n}{2} + \ell - 1]$, it follows that when $j$ is odd

$$N_{k-j}[x_j] = \left[-2kj + j^2 + j + k - 1, -2kj + j^2 + j - j + 3k - 1\right] \cup \left[\frac{n}{2} - 2jk + j^2 + j + k, \frac{n}{2} - 2jk + j^2 - j + 3k - 2\right]$$

and when $j$ is even

$$N_{k-j}[x_j] = \left[\frac{n}{2} - 2kj + j^2 + j + k - 1, \frac{n}{2} - 2kj + j^2 - j + 3k - 1\right] \cup \left[-2jk + j^2 + j + k, -2jk + j^2 - j + 3k - 2\right]$$
Let $f(j) = -2jk + j^2 + j + k$ and $g(j) = \frac{n}{2} - 2jk + j^2 + j + k - 1$. We claim that for any even $j$ such that $2 \leq j \leq k - 1$,

$$
\bigcup_{i=1}^{j} N_{k-i}[x_i] = [f(j), k-1] \cup \left[ g(j), \frac{n}{2} + k - 2 \right]
$$

First, we verify the result for $j = 2$. We have

$$
N_{k-1}[x_1] \cup N_{k-2}[x_2] = [-k + 1, k-1] \cup \left[ \frac{n}{2} - k + 2, \frac{n}{2} + k - 2 \right]
$$

$$
\cup \left[ \frac{n}{2} - 3k + 5, \frac{n}{2} - k + 1 \right] \cup [-3k + 6, -k]
$$

$$
= [-3k + 6, k-1] \cup \left[ \frac{n}{2} - 3k + 5, \frac{n}{2} + k - 2 \right]
$$

Assume that the result holds for some even $j$ such that $1 \leq j \leq k - 3$.

$$
\bigcup_{i=1}^{j+2} N_{k-i}[x_i] = [f(j), k-1] \cup \left[ g(j), \frac{n}{2} + k - 2 \right]
$$

$$
\cup N_{k-(j+1)}[x_{j+1}] \cup N_{k-(j+2)}[x_{j+2}]
$$

Since $j$ is even, we have

$$
N_{k-(j+1)}[x_{j+1}] \cup N_{k-(j+2)}[x_{j+2}]
$$

$$
= [-2k + j^2 + 3j - k + 1, -2k + j^2 + j + k - 1] \cup \left[ \frac{n}{2} - 2k + j^2 + 3j - k + 2, \frac{n}{2} - 2k + j^2 + j + k - 2 \right]
$$

$$
\cup \left[ \frac{n}{2} - 2k + j^2 + 5j - 3k + 5, \frac{n}{2} - 2k + j^2 + 3j - k + 1 \right] \cup [-2k + j^2 + 5j - 3k + 6, -2k + j^2 + 3j - k]
$$

$$
= [-2k + j^2 + 5j - 3k + 6, -2k + j^2 + j + k - 1] \cup \left[ \frac{n}{2} - 2k + j^2 + 5j - 3k + 5, \frac{n}{2} - 2k + j^2 + j + k - 2 \right]
$$

$$
= [f(j + 2), f(j) - 1] \cup [g(j + 2), g(j) - 1]
$$

Therefore, $\bigcup_{i=1}^{j+2} N_{k-i}[x_i] = [f(j + 2), k-1] \cup \left[ g(j + 2), \frac{n}{2} + k - 2 \right]$.

**Case 1: k is even.** By induction, when $k$ is even, $k - 2$ is even and $\bigcup_{i=1}^{k-2} N_{k-i}[x_i] = [f(k-2), k-1] \cup \left[ g(k-2), \frac{n}{2} + k - 2 \right] = [-k^2 + 2k + 2, k-1] \cup \left[ \frac{n}{2} - k^2 + 2k + 1, \frac{n}{2} + k - 2 \right]$. 

Therefore, the burning sequence \( \cup_{i=1}^{k} N_{k-i}[x_i] \supseteq \{\frac{k}{2} + 2k - 1, \frac{n}{2} + k + 1\} \) and \([\frac{k}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2] \). Therefore, \( N_{k-i}[x_i] \cup \{\frac{k}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2\} \). Therefore, \( N_{k-i}[x_i] \cup \{\frac{k}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2\} \). Therefore, \( [\frac{k}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2] \). Therefore, \( [\frac{k}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2] \).

We know \( \frac{1+\sqrt{2n+1}}{2} \leq k < \frac{1+\sqrt{2n+1}}{2} + 1 \). Therefore, \( -k^2 + k - 1 < -\frac{1+\sqrt{2n+1}+2n+1}{2} + \frac{1+\sqrt{2n+1}}{2} = -\frac{n}{2} \). It follows that, \( -k^2 + 2k - 1 \leq -\frac{n}{2} + k - 1 \) and \( \frac{n}{2} - k^2 + 2k - 1 \leq \frac{n}{2} - \frac{n}{2} + k - 1 \). Therefore, \( [\frac{n}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2] \). Therefore, \( [\frac{n}{2} - k^2 + 2k - 1, \frac{n}{2} + k - 2] \) and \( \frac{n}{2} + k - 1 \). Therefore, \( [\frac{n}{2} + k - 1, \frac{n}{2} + k - 2] \) and \( \cup_{i=1}^{k} N_{k-i}[x_i] \). Since addition is done modulo \( n \), \( -\frac{n}{2} = \frac{n}{2} \). Therefore, the burning sequence \( (x_1, \ldots, x_k) \) burns all the vertices in \( G \).

Therefore, \( b(G) \leq \left\lfloor \frac{1+\sqrt{2n+1}}{2} \right\rfloor \), and the result follows. \( \square \)

3. Bounds on the Burning Numbers of 4-regular Circulant Graphs

**Lemma 3.** Let \( G \cong (n; 1, m) \) for some \( m < n/2 \). For any \( \ell \geq 0 \), \( N_{\ell}[x] = \cup_{j=0}^{\ell} \left( [x + (j - \ell)m - j, x + (j - \ell)m + j] \cup [x + (\ell - j)m - j, x + (\ell - j)m + j] \right) \).

**Proof.** It is straightforward to verify that the result holds when \( \ell = 0 \) and \( \ell = 1 \). We proceed by induction. Assume that \( N_\ell[x] = \cup_{j=0}^{\ell} \left( [x + (j - k)m - j, x + (j - k)m + j] \cup [x + (k - j)m - j, x + (k - j)m + j] \right) \) for some \( k \geq 1 \). Now consider \( N_{k+1}[0] = \cup_{i \in N_k[0]} N_1[i] \). It follows that, when increasing the to the \((k + 1)\)st closed neighbourhood, each interval in \( N_k[0] \) will be have its lower bound decreased by 1 and its upper bound increased by 1. For example, the interval \([k-j)m - j, (k-j)m + j]\) in \( N_k[0] \) will expand to \([k-j)m - j - 1, (k-j)m + j + 1]\) due to the distance 1 constraint being applied to the interval \([k-j)m - j, (k-j)m + j]\). The upper bound of \([k-j)m - j, (k-j)m + j]\) also increases due to the distance \( m \) constraint.
Proof. Without loss of generality, consider $N_\ell[0] = \bigcup_{j=0}^\ell ([j+(\ell-j)m-j, (j+1\ell)m+j])$ where $\ell \geq 0$. We note that $[(j-\ell)m-j, (j-\ell)m+j]$ corresponds to $j=\ell$. Furthermore, when $j \neq \ell$, $[(j-\ell)m-j, (j-\ell)m+j] = [(\ell-j)m-j, (\ell-j)m+j]$ when $j=\ell$. It follows that $|N_\ell[x]| = |N_\ell[0]| \leq (\sum_{j=0}^{\ell-1} 2(2j+1)) + (2\ell + 1) = 2\ell^2 + 2\ell + 1$. \hfill \Box

We note that $|N_{\ell}[x]| = 2\ell^2 + 2\ell + 1$ is only achieved when all the intervals in the union given in Lemma 3 are disjoint. When $\ell > \frac{m}{2}$, this will not be the case, and the upper bound on $|N_{\ell}[x]|$ given in Corollary 4 can be improved.

**Lemma 5.** Let $G \cong (n; 1, m)$ where $1 < m < n/2$. For any $\ell$ such that $\ell > m/2$,

$$N_\ell[x] = \left( \bigcup_{j=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor-1} (j+(\ell-j)m-j, (j+1\ell)m+j) \cup [x+(\ell-j)m-j, x+(\ell-j)m+j] \right)$$

$$\cup \left[ x-(\ell-\left\lfloor \frac{m}{2} \right\rfloor), x+(\ell-\left\lfloor \frac{m}{2} \right\rfloor) \right]$$

**Proof.** Without loss of generality, consider $N_{\ell}[0]$ for some $\ell > \frac{m}{2}$. From Lemma 3, it suffices to show that $\bigcup_{j=\left\lfloor \frac{\ell}{2} \right\rfloor}^\ell ([j+(\ell-j)m-j, (j+\ell)m+j] \cup [(\ell-j)m-j, (\ell-j)m+j]) = \left[ -(\ell-\left\lfloor \frac{m}{2} \right\rfloor), m + \left\lfloor \frac{m}{2} \right\rfloor \right]$ when $\ell \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$.

We claim that for $t$ such that $1 \leq t \leq \ell - \left\lfloor \frac{m}{2} \right\rfloor$

$$\bigcup_{j=\ell-t}^\ell [(j-\ell)m-j, (\ell-j)m+j] \cup [(\ell-j)m-j, (\ell-j)m+j] = \left[ -(tm-(\ell-t)), tm+(\ell-t) \right]$$

When $t = 1$, we see that

$$\bigcup_{j=\ell-1}^\ell [(j-\ell)m-j, (\ell-j)m+j] \cup [(\ell-j)m-j, (\ell-j)m+j]$$

$$= \left[ -m-\ell+1, m+\ell-1 \right] \cup [m-\ell+1, m+\ell-1] \cup [\ell, \ell]$$

$$= \left[ -m-\ell+1, m+\ell-1 \right]$$
since \( \ell \leq \left\lfloor \frac{m}{2} \right\rfloor + 1 \) and, as a result \( \ell \geq m - \ell + 1 \).

Assuming that the result holds for \( t = w \) where \( 1 \leq w \leq \ell - \left\lfloor \frac{m}{2} \right\rfloor - 1 \), it follows that when \( t = w + 1 \)

\[
\bigcup_{j=\ell-w-1}^{\ell} (\{(j-\ell)m-j,(j-\ell)m+j\} \cup \{(\ell-j)m-j, (\ell-j)m+j\})
\]

\[
= [-(w+1)m-\ell+w+),(w+1)m+\ell-w-1]
\]

\[
\bigcup \{(w+1)m-\ell+w+1,(w+1)m+\ell-w-1\}
\]

\[
\bigcup \{-wm-(\ell-w),wm+(\ell-w)\}
\]

\[
= [-(w+1)m-\ell+w+1,(w+1)m+\ell-w-1]
\]

Since \( wm + (\ell - w) \geq (w+1)m - \ell + w + 1 \). Therefore, by induction, the claim holds for \( t = \ell - \left\lfloor \frac{m}{2} \right\rfloor \). The result follows. \( \square \)

**Corollary 6.** Let \( G \cong (n; 1, m) \) for some even \( m \) such that \( m < n/2 \). For any \( \ell \) such that \( \ell > m/2 \), \( |N_\ell[i]| \leq 2 (\left\lfloor \frac{m}{2} \right\rfloor) + 2\ell m - 2 \left\lfloor \frac{m}{2} \right\rfloor + 2 + 1 \).

**Proof.**

\[
|N_\ell[i]| \leq \left( \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} 2(2j+1) \right) + 2\ell m - 2 \left\lfloor \frac{m}{2} \right\rfloor m + 2 \left\lfloor \frac{m}{2} \right\rfloor + 1
\]

\[
= 2 \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + 2\ell m - 2 \left\lfloor \frac{m}{2} \right\rfloor m + 2 \left\lfloor \frac{m}{2} \right\rfloor + 1
\]

\( \square \)

**Theorem 7.** For any circulant graph \( C(n; 1, m) \) such that \( 2 \leq m < \frac{n}{2} \)

\[
b(C(n; 1, m)) \geq \left\lfloor \frac{\left(162n + 6\sqrt{729n^2 + 6}\right)^{2/3} - 6}{\left(162n + 6\sqrt{729n^2 + 6}\right)^{1/3}} \right\rfloor
\]

Furthermore, when \( n > 1 + \frac{m^3}{12} + \frac{m^2}{2} + \frac{7m}{6} \),

\[
b(C(n; 1, m)) \geq \begin{cases} 
\left\lfloor \frac{3m^2 + \sqrt{-3m^4 + 12m^2 + 144m^2 + 36} - 6}{12m} \right\rfloor & : m \text{ is even} \\
\left\lfloor \frac{3m^2 + \sqrt{-3m^4 - 6m^2 + 144m^2 + 9} - 3}{12m} \right\rfloor & : m \text{ is odd}
\end{cases}
\]
Proof. Suppose \((x_1, x_2, \ldots, x_k)\) is an optimal burning sequence in \(G\), where \(G \cong C(n; 1, m)\) such that \(m \geq 2\) and \(n > \frac{1}{12}m^3 + \frac{1}{2}m^2 + \frac{7}{6}m + 1\).

By Corollary 4 it follows that

\[
n \leq \sum_{i=0}^{k-1} |N_i[x_k-i]| \leq \sum_{i=0}^{k-1} (2i^2 + 2i + 1) = \frac{2}{3}k^3 + \frac{1}{3}k
\]

Solving this inequality for \(k\), we obtain

\[
k \geq \frac{\left(162n + 6\sqrt{729n^2 + 6}\right)^{2/3} - 6}{\left(162n + 6\sqrt{729n^2 + 6}\right)^{1/3}}
\]

Now, given that \(n \leq \frac{2}{3}k^3 + \frac{1}{3}k\), if we assume \(k-1 \leq \frac{m}{2}\), we obtain \(n \leq \frac{2}{3}(\frac{m}{2}+1)^3 + \frac{1}{3}(\frac{m}{2}+1) = \frac{1}{12}m^3 + \frac{1}{2}m^2 + \frac{7}{6}m + 1\). Therefore, if we assume \(n > \frac{1}{12}m^3 + \frac{1}{2}m^2 + \frac{7}{6}m + 1\), and use Corollaries 4 and 6 we obtain an improved upper bound on \(n\):

\[
n \leq \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} |N_i[x_k-i]| + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor + 1}^{k-1} |N_i[x_k-i]|
\]

\[
\leq \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (2i^2 + 2i + 1) + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor + 1}^{k-1} \left(2\left\lfloor \frac{m}{2} \right\rfloor \right)^2 + 2\ell m - 2 \left\lfloor \frac{m}{2} \right\rfloor m + 2 \left\lfloor \frac{m}{2} \right\rfloor + 1
\]

We now consider two cases based on the parity of \(m\).

**Case 1: \(m\) is even.** The previous inequality can be simplified as follows:

\[
n \leq \sum_{i=0}^{\frac{m}{2}} (2i^2 + 2i + 1) + \sum_{i=m/2+1}^{k-1} (2im - \frac{m^2}{2} + m + 1)
\]

\[
\leq mk^2 + \left(1 - \frac{m^2}{2}\right)k + \frac{m^3}{12} - \frac{m}{3}
\]
We now solve this inequality for \( k \) and obtain the following:

\[
k \geq \frac{3m^2 + \sqrt{-3m^4 + 144mn + 36} - 6}{12m}
\]

**Case 2: \( m \) is odd.** Again, we simplify the previous inequality relating \( n \) and \( k \) to obtain

\[
n \leq m \left( k^2 + \frac{1}{2} \right) + \frac{m^3}{12} - \frac{m}{12}
\]

As in the previous case, we solve the inequality for \( k \) and obtain the following:

\[
k \geq \frac{3m^2 + \sqrt{-3m^4 - 6m^2 + 144mn + 9} - 3}{12m}
\]

Note that, since \( n > \frac{1}{12}m^3 \), each bound on \( k \) is a real value. \( \square \)

**Theorem 8.** Let \( G \cong C(n; 1, m) \) such that \( n > 2m \) and \( m \geq 4 \). Then \( b(G) < \sqrt{\frac{m}{2}} + \frac{m}{2} + 1 \).

**Proof.** Suppose \( n = qm + r \) for some \( q \geq 2 \) and \( 0 \leq r \leq m - 1 \). For each \( i = 0, -1, 1, -2, 2, \ldots, -\left\lceil \frac{m}{2} \right\rceil, \left\lfloor \frac{m}{2} \right\rfloor \), let \( H_i \) be the subgraph of \( G \) induced on the vertices in \( \{i, i + m, \ldots, i + (q - 1)m\} \). Note that if \( r = 0 \), each \( H_i \) is a cycle on \( q \) vertices. Otherwise, each \( H_i \) is a path on \( q \) vertices. Suppose \( S = (x_1, x_2, \ldots, x_t) \) is an optimal burning sequence in \( H_0 \). It follows that, \( t = \left\lceil \sqrt{q} \right\rceil \).

We note that for each vertex \( x \in V(H_0) \), both \( x + i \) and \( x - i \) in \( N_i[x] \). It follows that, for each \( i \) after \( t + i \) time steps, all the vertices in \( H_i \) and \( H_{-i} \) are burned. Therefore, for any burning sequence \( S' \) of \( G \) of length \( \left\lceil \frac{m}{2} \right\rceil \) that has \( x_1, x_2, \ldots, x_t \) as its first \( t \) vertices, the vertices \( \cup_{i=1}^{\left\lceil \frac{m}{2} \right\rceil} (V(H_i) \cup V(H_{-i})) \) are burned. Therefore, all vertices in \( [- (q - 1)m - \left\lceil \frac{m}{2} \right\rceil, (q - 1)m + \left\lceil \frac{m}{2} \right\rceil] \). As a result, after \( t + \left\lceil \frac{m}{2} \right\rceil \) time steps all except \( r \) vertices are burned as a result of the initial \( t \) vertices in the burning sequence \( S' \).

If \( r \geq 1 \), then the only vertices in \( G \) that are unburned by \( S \) after \( t + \left\lceil \frac{m}{2} \right\rceil \) time steps are those in the interval \( [(q - 1)m + \left\lceil \frac{m}{2} \right\rceil + 1, (q - 1)m + \left\lceil \frac{m}{2} \right\rceil + r] \). Let \( (x_{t+1}, \ldots, x_{t+\left\lceil \sqrt{r} \right\rceil}) \) be an optimal burning sequence on the path induced by this interval. Since \( r \leq m - 1 \) and \( m \geq 4 \), it follows that \( \sqrt{r} \leq \left\lceil \frac{m}{2} \right\rceil \).

Therefore, for any \( r \) such that \( 0 \leq r \leq m - 1 \), the burning sequence \( (x_1, x_2, \ldots, x_{t+\left\lceil \sqrt{r} \right\rceil}) \) will burn all of the vertices of \( G \) after \( t + \left\lceil \frac{m}{2} \right\rceil \) time steps. Hence, any sequence of vertices
of length $t + \left\lfloor \frac{m}{2} \right\rfloor$ that has $x_1, x_2, \ldots, x_{t + \left\lfloor \sqrt{q} \right\rfloor}$ as the initial sequence of vertices is a burning sequence in $G$.

It follows that $b(G) \leq \left\lfloor \sqrt{q} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor = \left\lfloor \sqrt{\frac{m}{q}} \right\rfloor + \sqrt{\frac{m}{q}} + \frac{m}{2} + 1$. \hfill \square

**Corollary 9.** For any $q > 2$ and $m \geq 2$, $\left\lfloor \sqrt{q} \right\rfloor \leq b(C mq; 1, m) \leq \left\lceil \sqrt{q} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor$.

**Proof.** Let $G \cong b(C mq; 1, m)$ for some $q > 2$ and $m \geq 2$, the subgraph of $G$ induced on the set $\{0, m, 2m, \ldots, (q - 1)m\}$ is a cycle of length $q$. Furthermore, this cycle is an isometric subgraph of $G$. Call this subgraph $H$. It follows from CITE RESULT that $b(H) \leq b(G)$. Since $H$ is a cycle on $q$ vertices, $b(H) = \left\lceil \sqrt{q} \right\rceil$. This, together with Theorem 8, gives the desired result. \hfill \square

**Theorem 10.** Taking $m$ to be constant, $b(C(n; 1, m)$ is on the order of $\sqrt{\frac{m}{n}}$.

**Proof.** Let $G = (n; 1, m)$ where $n > 1 + \frac{m^3}{12} + \frac{m^2}{2} + \frac{7m}{6}$. Let

$$f(n) = \begin{cases} \left\lfloor \frac{3m^2 + \sqrt{-3m^4 + 12m^2 + 114mn + 36 - 6}}{12m} \right\rfloor & : m \text{ is even} \\ \left\lfloor \frac{3m^2 + \sqrt{-3m^4 - 6m^2 + 114mn + 9 - 3}}{12m} \right\rfloor & : m \text{ is odd} \end{cases}$$

and $g(n) = \frac{\sqrt{m}}{n} + \frac{m}{2} + 1$. By Theorems 7 and 8, $f(n)$ and $g(n)$ are lower and upper bounds on $b(G)$, respectively. It follows that

$$\lim_{n \to \infty} \frac{g(n)}{\sqrt{\frac{m}{n}}} \leq \lim_{n \to \infty} \frac{f(n)}{\sqrt{\frac{m}{n}}} \leq \lim_{n \to \infty} \frac{b(C(n; 1, m)}{\sqrt{\frac{m}{n}}}$$

and

$$1 \leq \lim_{n \to \infty} \frac{b(G)}{\sqrt{\frac{m}{n}}} \leq 1.$$

Therefore, $b(C(n; 1, m)$ and $\sqrt{\frac{m}{n}}$ are asymptotically equal, and the result follows. \hfill \square

4. Burning Numbers of $C(n; 1, 2)$ and $C(n; 1, 3)$

We now use the lower bounds presented in Theorem 7 to find the burning numbers for circulant graphs $C(n; 1, 2)$ and $C(n; 1, 3)$.

**Theorem 11.** For any $n \geq 5$, $b(C(n; 1, 2)) = \left\lfloor \frac{1 + \sqrt{1 + 8n}}{4} \right\rfloor$.

**Proof.** Let $G \cong C(n; 1, 2)$, where $n \geq 5$, and let $k = \left\lfloor \frac{1 + \sqrt{1 + 8n}}{4} \right\rfloor$.

By Theorem 7, $b(C(n; 1, 2) \geq \left\lfloor \frac{1 + \sqrt{1 + 8n}}{4} \right\rfloor$. It therefore suffices to show that there is sequence of vertices $(x_1, x_2, \ldots, x_k)$ that serves as a burning sequence for $G$. 


We now define \(x_{k-i}\) for each \(i \in \{0, \ldots, k-1\}\). First, when \(0 \leq i \leq k-1\), we let \(x_{k-i} = 2i^2 + i\). We note that for any vertex \(x\) in \(G\), \(N_i[x] = [x - 2i, x + 2i]\). Since \(x_{k-i} = 2i^2 + i\) for \(0 \leq i \leq k-2\), it follows that \(N_i[x_{k-i}] = [2i^2 - i, 2i^2 + 3i]\) for \(0 \leq i \leq k-2\).

We claim that for all \(m = 0, 2, \ldots, k-1\),
\[
[0, 2m^2 + 3m] \subseteq N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}]
\]

We now proceed by induction. Note that \(N_0[x_k] = N_0[0] = \{0\} = [0, 2(0)^2 + 3(0)]\), and assume \(N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}] \supseteq [0, 2m^2 + 3m]\) for some \(m\) such that \(0 \leq m \leq k-2\).

Since \(N_{m+1}[x_{k-(m+1)}] = [2(m+1)^2 - (m+1), 2(m+1)^2 + 3(m+1)] = [2m^2 + 3m + 1, 2(m+1)^2 + 3(m+1)]\), it follows that \(N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}] \cup N_{m+1}[x_{k-(m+1)}] \supseteq [0, 2(m+1)^2 + 3(m+1)]\). Therefore, \(N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_{k-1}[x_1] \supseteq [0, 2(k-1)^2 + 3(k-1)] = [0, (2k+1)(k-1)].\)

Since \(k \geq \frac{1+\sqrt{1+8n}}{4}\),
\[
(2k+1)(k-1) \geq \left(\frac{1 + \sqrt{1 + 8n}}{2} + 1\right)\left(\frac{1 + \sqrt{1 + 8n}}{4} - 1\right) = n - 1
\]

Therefore, \(N_{k-1}[x_1] \supseteq [2k^2 - 5k + 3, n - 1]\), and \(N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_{k-1}[x_1] = V(G)\).

Since \(b(C(n; 1, 2)) \geq \left\lceil \frac{1+\sqrt{1+8n}}{4} \right\rceil\), it follows that \((x_1, x_2, \ldots, x_k)\) is an optimal burning sequence and \(b(C(n; 1, 2)) = \left\lceil 1 + \frac{\sqrt{1 + 8n}}{4} \right\rceil\).

**Theorem 12.** For a circulant graph \(C(n; 1, 3)\) where \(n \geq 7\), we have that \(b(C(n; 1, 3)) = \left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil + 1\).

**Proof.** Let \(G\) be the circulant graph \(C(n; 1, 3)\), where \(n \geq 7\). We begin by proving that \(b(C(n; 1, 3)) \geq \left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil + 1\). Using the first lower bound given in Theorem 7, we obtain \(b(C(n; 1, m)) \geq 3\) when \(7 \leq n \leq 11\). When \(n \geq 12\), we use the odd case for the second lower bound given in Theorem 7. This gives \(b(C(n; 1, 3)) \geq \left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil\). This generalizes to \(b(C(n; 1, 3)) \geq \left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil\) for all \(n \geq 7\). Note that whenever \(\frac{2+\sqrt{3n-2}}{3}\) is non-integer, \(\left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil = \left\lceil \frac{2+\sqrt{3n-2}}{3} \right\rceil + 1\). Therefore, it suffices to show that \(b(C(n; 1, 3)) > \frac{2+\sqrt{3n-2}}{3}\) for all \(n \geq 7\).

Suppose this is not the case. Suppose \(b(C(n; 1, 3)) = \frac{2+\sqrt{3n-2}}{3}\) for some \(n \geq 7\). It follows from the proof of Theorem 7 that there must be an optimum burning sequence \((x_1, x_2, \ldots, x_k)\) such that the elements in \(\{N_{k-i}[x_i] | 1 \leq i \leq k\}\) are pair-wise disjoint.
Without loss of generality, assume that $x_k = 0$. It follows that for some $x$ in the optimum burning sequence, where $x = x_{k-\ell}$ and $\ell \geq 1$, $1 \in N_\ell[x]$, but $0 \notin N_\ell[x]$. By Lemma 5, $N_\ell[x] = \{x - 3\ell, x + 3\ell\} \cup \{x - 3\ell + 2, x + 3\ell - 2\}$. Hence, either $1 = x - 3\ell$, $1 = x + 3\ell$, or $1 = x - 3\ell + 2$. In other words, $x = 1 + 3\ell$, $x = 1 - 3\ell$, or $x = -1 + 3\ell$. Without loss of generality, we consider the two cases $x = 1 + 4\ell$ and $x = 1 - 3\ell$.

**Case 1:** $x = 1 + 3\ell$. Then $N_\ell[x] = \{1, 1 + 6\ell\} \cup \{3, 6\ell - 1\}$. It follows that $2 \notin N_\ell[s]$. Otherwise, we would have $1 + 6\ell = 2$ and $6\ell - 1 = 0$, which contradicts the fact that $0 \notin N_\ell[x]$. It follows that there must be some vertex $y$, where $y = x_{k-\ell'}$, in the optimum burning sequence such that $2 \in N_\ell'[y]$. It follows that $0, 1 \notin N_\ell'[y]$. As with $x$, this gives us three possible values for $y$: $y = 2 + 3\ell'$, $y = 2 - 3\ell'$, and $y = 3\ell'$. Since $0 \notin N_\ell'[y]$, it follows that $y = 2 + 3\ell'$. However, this means $N_\ell'[y] = \{2\} \cup \{4, 6\ell\}$. This implies 4 is in both $N_\ell[x]$ and $N_\ell'[y]$, which is a contradiction.

**Case 2:** $x = 1 - 3\ell$. Then $N_\ell[x] = \{1 - 6\ell, 1\} \cup \{3 - 6\ell, -1\}$. Now, we again consider a vertex $y$, where $y = x_{k-\ell'}$, in the optimum burning sequence such that $2 \in N_\ell'[y]$. Of the three possible values for $y$ only $y = 2 + 3\ell'$ gives $N_\ell'[y] \cap N_\ell[x] = \emptyset$. This means $N_\ell'[y] = \{2, 2 + 6\ell'\} \cup \{4, 6\ell'\}$. It follows that 3 is in neither $N_\ell[x]$ nor $N_\ell'[y]$, and there must be some vertex $z$ and $\ell'' \geq 1$ such that $3 \in N_\ell''[z]$. However, it is impossible to choose such a vertex $z$ and integer $\ell''$ such that $N_\ell''[z]$ is disjoint from both $N_\ell[x]$ and $N_\ell'[y]$.

It follows that there is no optimal burning sequence that results in a set of disjoint neighbourhoods. Therefore, $b(C(n; 1, 3)) > \frac{2 + \sqrt{3n-2}}{3}$, and $b(C(n; 1, 3)) \geq \left\lfloor \frac{2 + \sqrt{3n-2}}{3} \right\rfloor + 1$.

Next, we assume $k = \left\lfloor \frac{2 + \sqrt{3n-2}}{3} \right\rfloor + 1$. Then for $0 \leq i \leq k - 1$, we choose $x_{k-i} = 3i^2 - i$.

Let us prove that this burning sequence covers the vertex set of $G$. We need to show that

$$N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_{k-1}[x_1] = [0, n - 1]$$

Since $N_i[x_{k-i}] = [x_{k-i} - 3i + 2, x_{k-i} + 3i - 2] \cup \{x_{k-i} - 3i\} \cup \{x_{k-i} + 3i\}$ and $x_{k-i} = 3i^2 - i$, we have that

$$N_i[x_{k-i}] = [3i^2 - 4i + 2, 3i^2 + 2i - 2] \cup \{3i^2 - 4i, 3i^2 + 2i\}$$

We claim for all $m = 1, 2, \ldots, k - 1$ that

$$N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}] = [-1, 3m^2 + 2m - 2] \cup \{3m^2 + 2m\}.$$ 

This will proved via induction on $m$, with the $m = 1$ case verified below.

Since $N_0[x_k] = \{0\}$ and $N_1[x_{k-1}] = \{1, 3\} \cup \{-1, 5\}$, it follows that

$$N_0[x_k] \cup N_1[x_{k-1}] = [-1, 3] \cup \{5\} = [-1, 3(1)^2 + 2(1) - 2] \cup \{3(1)^2 + 2(1)\}.$$
Now assume \( N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}] \supseteq [-1, 3m^2 + 2m - 2] \cup \{3m^2 + 2m\} \) for some \( m \) such that \( 1 \leq m \leq k - 2 \). It follows that

\[
N_0[x_k] \cup N_1[x_{k-1}] \cup \cdots \cup N_m[x_{k-m}] \cup N_{m+1}[x_{k-(m+1)}]
= \left[ -1, 3m^2 + 2m - 2 \right] \cup \{3m^2 + 2m\} \cup \left[ 3(m+1)^2 - 4(m+1) + 2, 3(m+1)^2 + 2(m+1) - 2 \right]
\cup \{3m^2 + 2m + 1, 3(m+1)^2 + 2(m+1)\}
= \left[ -1, 3m^2 + 2m + 1 \right] \cup \{3m^2 + 2m + 1\}
\]

It follows by induction that

\[
\bigcup_{i=0}^{k-1} N_i[x_{k-i}] = \left[ -1, 3(k-1)^2 + 2(k-1) - 2 \right] \cup \{3(k-1)^2 + 2(k-1)\}
= \left[ -1, 3k^2 - 4k - 1 \right] \cup \{3k^2 - 4k + 1\}
\]

Since \( k = \left\lfloor \frac{2 + \sqrt{3n-2}}{3} \right\rfloor + 1 \), it follows that \( k > \frac{2 + \sqrt{3n-2}}{3} \). Therefore,

\[
3k^2 - 4k - 1
= (3k - 1)(k - 1) - 2
> \left(\sqrt{3n-2} + 1\right)\left(\frac{\sqrt{3n-2} - 1}{3}\right) - 2
= n - 3
\]

Therefore, \( 3k^2 - 4k - 1 \geq n - 2 \) and \( [-1, 3k^2 - 4k - 1] = [-1, n - 2] = [0, n - 1] \). It follows that \( \bigcup_{i=0}^{k-1} N_i[x_{k-i}] = [0, n - 1] \), and \( b(C(n; 1, 3)) \leq \left\lfloor \frac{2 + \sqrt{3n-2}}{3} \right\rfloor + 1 \). Hence,

\[
b(C(n; 1, 3)) = \left\lfloor \frac{2 + \sqrt{3n-2}}{3} \right\rfloor + 1.
\]

\( \square \)

5. Burning Numbers of Circulant Graphs of Higher Degree

**Theorem 13.** For any \( m \geq 2 \) and \( n > 2m \),

\[
b(C(n; 1, 2, \ldots, m)) = \left\lfloor \frac{(m - 1) + \sqrt{4mn + (m - 1)^2}}{2m} \right\rfloor.
\]

**Proof.** Let \( G = b(C(n; 1, 2, \ldots, m)) \). We note that for any \( x \in V(G) \) and \( \ell \leq 0 \), \( N_\ell[x] = [x - \ell m, x + \ell m] \). Furthermore, \( |N_\ell[x]| = 2\ell + 1 \) for any \( \ell \leq \frac{n-1}{2m} \); otherwise,
Theorem 14. Given by Roshanbin [7], as stated the following theorem.

\[ b(C(n; 1, 2, \ldots, m)) \geq \left\lceil \frac{(m-1)+\sqrt{4mn+(m-1)^2}}{2m} \right\rceil. \]

Next, we select the burning sequence \((x_1, x_2, \ldots, x_k)\) where \(k = \left\lceil \frac{(m-1)+\sqrt{4mn+(m-1)^2}}{2m} \right\rceil\) and \(x_{k-i} = i^2m + i\) for each \(i = 0, \ldots, k - 1\). Again, using a proof similar to that for Theorem 11 it can be shown that this sequence burns all the vertices of \(G\). The result follows.

The lexicographic product (or wreath product) of graphs \(G\) and \(H\) is denoted \(G \cdot H\), where \(V(G \cdot H) = V(G) \times V(H)\), and \(E(G \cdot H) = \{(x_1, y_1)(x_2, y_2) | x_1x_2 \in E(G), \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(H)\}\). In other words, \(G \cdot H\) is obtained by (1) replacing each vertex of \(G\) with a copy of \(H\), and (2) adding all possible edges between two copies of \(H\), whenever they have replaced adjacent vertices of \(G\). It is well known that the lexicographic product of two circulant graphs is itself a circulant graph. Specifically, if \(G = C(n_1; S)\) and \(H = C(n_2, T)\), then \(G \cdot H \cong C(n_1n_2; n_1T \cup (\bigcup_{s \in S} (n_1Z_{n_2} + s))\) [5].

Lower and upper bounds of the burning number of a lexicographic product \(G \cdot H\) were given by Roshanbin [7], as stated the following theorem.

**Theorem 14.** [7] For any two graphs \(G\) and \(H\), we have
\[ b(G) \leq b(G \cdot H) \leq b(G) + 2. \]

Therefore, from Theorems, [2] [11] [12], [13] and [14] we obtain the following.

**Corollary 15.** For any graph \(H\),
\[
(1) \quad \left\lceil \frac{1+\sqrt{2n+1}}{2} \right\rceil \leq b\left( C(n; 1, \frac{n}{2}) \cdot H \right) \leq \left\lceil \frac{1+\sqrt{2n+1}}{2} \right\rceil + 2.
\]
\[
(2) \quad \left\lceil \frac{1+\sqrt{1+8n}}{4} \right\rceil \leq b(C(n; 1, 2) \cdot H) \leq \left\lceil \frac{1+\sqrt{1+8n}}{4} \right\rceil + 2.
\]
\[
(3) \quad \left\lceil \frac{2+\sqrt{4n-2}}{3} \right\rceil + 1 \leq b(C(n; 1, 3) \cdot H) \leq \left\lceil \frac{2+\sqrt{4n-2}}{3} \right\rceil + 3.
\]
\[
(4) \quad \left\lceil \frac{(m-1)+\sqrt{4mn+(m-1)^2}}{2m} \right\rceil \leq b(C(n; 1, 2, \ldots, m) \cdot H) \leq \left\lceil \frac{(m-1)+\sqrt{4mn+(m-1)^2}}{2m} \right\rceil + 2.
\]

Finally from Theorem 10 we have the following, more general result.

**Corollary 16.** For any graph \(H\), \(b(C(n; 1, m) \cdot H)\) is on the order of \(\sqrt{\frac{n}{m}}\).

While these results hold for any graph \(H\), when \(H = C(n'; T)\) for \(n \geq 4\) and an appropriate choice of set \(T\), they establish bounds on the burning numbers of infinite families of circulant graphs.
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