NOTES ON THE JOYAL MODEL STRUCTURE

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Abstract. We give a new construction of the Joyal model structure on the category $\text{Set}_\Delta$ of simplicial sets, and we provide a simple characterization of the fibrations in it. We characterize the inner anodyne maps between simplicial sets in terms of categorical equivalences and use this characterization to establish the inner model structure on the category $\text{Set}_\Delta(O)$ of simplicial sets whose set of zero-simplices is equal to a fixed set $O$.

1. Introduction

Recall that a simplicial set $S$ is said to be a quasi-category if every map $\Lambda^n_i \to S$ with $0 < i < n$ can be extended along the inclusion $\Lambda^n_i \subseteq \Delta^n$ to give an $n$-simplex of $S$. Quasi-categories, or restricted Kan complexes, were introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [2].

It is a remarkable fact, and a great insight of Joyal, that most concepts and results from ordinary category theory can be extended to quasi-categories, see [5]. Lurie has vastly extended Joyal’s theory in the books [11] and [12]. We will follow Lurie and use the term $\infty$-category instead of quasi-category.

One of the key discoveries of Joyal was the existence of a primordial model structure on the category $\text{Set}_\Delta$ of simplicial sets in which the $\infty$-categories are the fibrant objects. This model structure is called the Joyal model structure and it plays an important role in the Joyal/Lurie theory of $\infty$-categories. For example, it makes precise the idea that every simplicial set $S$ presents an $\infty$-category by a generators and relations type construction. A choice of a fibrant replacement for $S$ in the Joyal model structure gives an $\infty$-category $\mathcal{C}$, well defined up to equivalence, which one may think of as an $\infty$-category generated by $S$.

While the Joyal model structure on $\text{Set}_\Delta$ is known to be cofibrantly generated, it is an open problem to describe an explicit set of generating acyclic cofibrations (compare with Remark 2.14 in [5]). Also, while the fibrations between fibrant objects in the Joyal model structure are well understood (see Corollary 2.4.6.5 of [11]), an explicit description of the fibrations in general is unknown. This is due to the fact that the existing treatments of the Joyal model structure (see [5] [11] and also the streamlined treatment in [3]) rely on set theoretic arguments which place a greater emphasis on the role of weak equivalences (compare with Proposition A.2.6.13 of [11]).

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In this paper we will attempt to address these questions with the following theorems.

**Theorem A.** A map \( p: X \to S \) in \( \text{Set}_\Delta \) is a fibration in the Joyal model structure if and only if it is an inner fibration and the induced functor \( h(p): h(X) \to h(S) \) on homotopy categories is an isofibration.

Here \( h: \text{Set}_\Delta \to \text{Cat} \) denotes the functor which sends a simplicial set to its homotopy category (see Section 1.2.3 of [11]).

**Theorem B.** A set of generating trivial cofibrations for the Joyal model structure on \( \text{Set}_\Delta \) is given by the following collection of morphisms:

- the inner horn inclusions \( \Lambda^n_i \subseteq \Delta^n \), \( 0 < i < n \) for \( n \geq 2 \);
- a set of representatives for the isomorphism classes of simplicial sets \( B \) with two vertices \( 0 \) and \( 1 \), with countably many non-degenerate simplices and such that the inclusion \( \{ 0 \} \to B \) is a categorical equivalence.

In order to prove these theorems we give a characterization of the inner anodyne maps in \( \text{Set}_\Delta \) which is of interest in its own right. Recall that the class of inner anodyne maps in \( \text{Set}_\Delta \) is the saturated class generated by the inner horn inclusions \( \Lambda^n_i \subseteq \Delta^n \) for \( 0 < i < n \).

For instance, if \( S \) is a simplicial set, then we can apply Quillen’s small object argument to the set of these inner horn inclusions to produce an inner anodyne map \( S \to C \), where \( C \) is an \( \infty \)-category. We may think of this \( \infty \)-category as being ‘freely generated’ by \( S \).

In Section 5 below we will prove the following theorem which answers a question left open by Joyal (see paragraph 2.10 of [9]).

**Theorem C.** A monomorphism in \( \text{Set}_\Delta \) is inner anodyne if and only if it is bijective on vertices and it is a categorical equivalence.

Note that this theorem has the following interesting consequence (see Proposition 5.9): the class of inner anodyne maps in \( \text{Set}_\Delta \) satisfies the 2-out-of-3 property in the class of monomorphisms in \( \text{Set}_\Delta \). In other words, if \( u: A \to B \) and \( v: B \to C \) are monomorphisms in \( \text{Set}_\Delta \) such that any two of the maps \( u \), \( v \) and \( vu \) are inner anodyne, then all three maps are inner anodyne. It is clear that the class of inner anodyne maps is closed under composition. The fact that the class of inner anodyne maps satisfies the so-called right cancellation property was established in [13]. It is not obvious that the analogous left cancellation property is also satisfied.

As an application of Theorem C we construct what we call the inner model structure on the category \( \text{Set}_\Delta(O) \), consisting of all simplicial sets whose set of 0-simplices is equal to a fixed set \( O \) and whose morphisms are the simplicial maps which induce the identity on \( O \).

More precisely, we prove the following result in Section 5.

**Theorem D.** There is the structure of a left proper, cofibrantly generated model category on \( \text{Set}_\Delta(O) \) for which

- the weak equivalences are the categorical equivalences in \( \text{Set}_\Delta(O) \);
- the cofibrations are the monomorphisms in \( \text{Set}_\Delta(O) \); and
• the fibrant objects are the $\infty$-categories whose set of objects is equal to $O$.

It turns out that the fibrations in this model structure are the inner fibrations in $\text{Set}_\Delta(O)$. The existence of this model structure explains the fact that inner anodyne maps satisfy the 2-out-of-3 property. We make use of it in our proof of Theorem 11.

To put this theorem into context, recall that in the paper [4], Dwyer and Kan constructed a model structure on the category $\text{Cat}_\Delta(O)$ of simplicial categories with a fixed object set $O$. This model structure plays a role in the construction of the Berger model structure on $\text{Cat}_\Delta$.

We summarize the contents of this paper. In Section 2 we review some basic properties of categorical equivalences — the weak equivalences in the Joyal model structure. Everything in this section is well-known and due in its original form to Joyal. We have included proofs of some of the statements here in an effort to make the construction of the Joyal model structure as self contained as possible (see Remark 6.7). In Section 3 we introduce the concept of what we have called (for lack of a better name) pre-fibrant simplicial sets. This is an ad-hoc device which we have introduced to counter the fact that the naive mapping spaces of a simplicial set $S$ do not in general have the homotopy type of the mapping spaces of the $\infty$-category generated by $S$ (see Remark 2.15). The key result that we prove here is Proposition 3.5 which shows that every pre-fibrant simplicial set $S$ freely generates an $\infty$-category whose mapping spaces are isomorphic to the naive mapping spaces of $S$.

In Section 4 we study a notion of descent for inner fibrations. Our main result is Proposition 4.12. This states that given an inner fibration $p: X \to S$, there exists an inner anodyne map $S \to T$, where $T$ is a pre-fibrant simplicial set, and an inner fibration $q: Y \to T$ forming part of a pullback diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow^p & & \downarrow^q \\
S & \longrightarrow & T
\end{array}
$$

This result is useful when one wants to prove that the inner fibration $p: X \to S$ is a left fibration or a trivial fibration, as one can then translate this problem into the problem of proving that the inner fibration $q: Y \to T$ is a left fibration or a trivial fibration. This problem is sometimes easier to solve since $T$ (and hence $Y$) are pre-fibrant simplicial sets. In Section 5 we apply these results to prove Theorem C and establish the inner model structure (Theorem D). In Section 6 we prove Theorem A and Theorem B and establish the Joyal model structure (Theorem 6.6). Finally, in Sections 7–9 we prove some technical results that are needed earlier in the paper.

Notation: for the most part we will adopt the notation and terminology from [11]. Thus $\text{Set}_\Delta$ will denote the category of simplicial sets for instance. Typically $\infty$-categories will be denoted with calligraphic letters, $\mathcal{C}, \mathcal{D}, \ldots$ etc, while simplicial sets will typically be denoted with capital Latin letters, $S, T, \ldots$ etc. We follow the standard convention of not distinguishing notationally between an ordinary category and its nerve.
2. Categorical equivalences

In this section we recall the notion of categorical equivalences between simplicial sets and give some examples. We prove that a map between $\infty$-categories is a categorical equivalence if and only if it is a Dwyer-Kan equivalence (Definition 2.16). Everything in this section is well-known and is due to Joyal and Lurie.

2.1. The homotopy category of a simplicial set. Recall (see Section 1.2.3 of [11]) the functor $h: \text{Set}_\Delta \to \text{Cat}$ which sends a simplicial set $S$ to its homotopy category $h(S)$. Recall also that an edge $e: x \to y$ in an $\infty$-category $C$ is said to be an equivalence if its image in the associated homotopy category $h(C)$ is an isomorphism. More generally we will say that an edge $e: x \to y$ in a simplicial set $S$ is an equivalence if its image in $h(S)$ is an isomorphism.

An edge $e: x \to y$ in an $\infty$-category $C$ is an equivalence if and only if the 1-simplex $e: \Delta^1 \to C$ classifying $e$ extends along the canonical map $\Delta^1 \to J$, where $J$ denotes the groupoid interval (i.e. the groupoid with two objects 0 and 1 and a unique isomorphism between them). Clearly the existence of such an extension is a sufficient condition; to see that it is also necessary observe that if $e: \Delta^1 \to C$ is an equivalence then $e$ factors through $C^\sim$, the maximal Kan complex contained in $C$. Since $\Delta^1 \to J$ is anodyne it follows that the map $\Delta^1 \to C^\sim$ extends along the inclusion $\Delta^1 \to J$.

2.2. Equivalences of $\infty$-categories. We recall the notion of an equivalence between $\infty$-categories and give some examples.

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories. A map $f: \mathcal{C} \to \mathcal{D}$ is said to be an equivalence if there exists a map $g: \mathcal{D} \to \mathcal{C}$ such that there are equivalences $gf \to \text{id}_\mathcal{C}$ and $fg \to \text{id}_\mathcal{D}$ in the $\infty$-categories $\text{Fun}(\mathcal{C}, \mathcal{C})$ and $\text{Fun}(\mathcal{D}, \mathcal{D})$ respectively.

Following Dugger and Spivak [3] we introduce the following notation.

Notation 2.2. We say that maps $f, g: S \to T$ in $\text{Set}_\Delta$ are $J$-homotopic if there exists a map $H: S \times J \to T$ such that $Hi_0 = f$ and $Hi_1 = g$, where $i_0, i_1: S \to S \times J$ are the obvious inclusions.

Remark 2.3. Thus $f: \mathcal{C} \to \mathcal{D}$ is an equivalence of $\infty$-categories if and only if it is a $J$-homotopy equivalence in the sense that there is a map $g: \mathcal{D} \to \mathcal{C}$ such that the composites $fg$ and $gf$ are $J$-homotopic to the respective identity maps.

Lemma 2.4. Let $f: K \to L$ be a homotopy equivalence, where $K$ and $L$ are Kan complexes. Then $f$ is an equivalence of $\infty$-categories.

Proof. This follows immediately from the fact that every edge in a Kan complex is an equivalence.

Example 2.5. The inclusion $\{0\} \subseteq J$ is an equivalence.
2.3. **Categorical equivalences.** We recall the definition and some properties of the class of categorical equivalences in $\text{Set}_\Delta$.

**Definition 2.6 (Joyal).** A map $f: S \to T$ of simplicial sets is said to be a *categorical equivalence* if for any $\infty$-category $\mathcal{C}$, the induced map

$$\text{Fun}(T, \mathcal{C}) \to \text{Fun}(S, \mathcal{C})$$

is an equivalence of $\infty$-categories.

**Remark 2.7.** Clearly the class of categorical equivalences in $\text{Set}_\Delta$ satisfies the 2-out-of-3 property.

The following lemma is proved by a straightforward adjointness argument.

**Lemma 2.8.** If $f: \mathcal{C} \to \mathcal{D}$ is an equivalence between $\infty$-categories then $f$ is a categorical equivalence.

**Lemma 2.9.** If $f: S \to T$ is inner anodyne then $f$ is a categorical equivalence.

*Proof.* This follows immediately from Corollary 2.3.2.5 of [11]. $\Box$

The following lemma is well-known and easy to prove.

**Lemma 2.10.** If $p: X \to Y$ is a trivial Kan fibration then $p$ is a categorical equivalence.

**Notation 2.11.** We recall the following notation from [10]: if $\mathcal{C}$ is an $\infty$-category and $K$ is a simplicial set then we write $\mathcal{C}(K)$ for the *full* simplicial subset (see Remark 2.12 below) of the $\infty$-category $\mathcal{C}^K$ whose vertices are the maps $K \to \mathcal{C}$ which factor through the maximal Kan complex $\mathcal{C}^\infty$.

**Remark 2.12.** Here we recall that a full simplicial subset $T \subseteq S$ of a simplicial set $S$ is determined by the subset $T_0 \subseteq S_0$ of 0-simplices, in the sense that there is a pullback diagram

$$\begin{array}{ccc}
T & \to & S \\
\downarrow & & \downarrow \\
\cosk_0 T_0 & \to & \cosk_0 S
\end{array}$$

Clearly every full simplicial subset $T \subseteq S$ is an inner fibration.

The following example is a special case of a more general result due to Joyal (see Theorem 5.10 of [8]).

**Example 2.13.** Let $\mathcal{C}$ be an $\infty$-category. Then the inclusion $\{0\} \subset \Delta^1$ induces a trivial Kan fibration $\mathcal{C}(\Delta^1) \to \mathcal{C}^{\{0\}}$. To see this it suffices to prove that the indicated diagonal filler exists in every commutative diagram of the form

$$\begin{array}{ccc}
\partial \Delta^n & \to & \mathcal{C}(\Delta^1) \\
\downarrow & & \downarrow \\
\Delta^n & \to & \mathcal{C}^{\{0\}}
\end{array}$$
for \( n \geq 1 \) (the existence of such fillers being clear when \( n = 0 \)). By adjointness, the existence of such a diagonal filler is equivalent to the existence of an extension of the induced map
\[
f : \partial \Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \to \mathcal{C}
\]
along the inclusion \( \partial \Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \subseteq \Delta^n \times \Delta^1 \). By assumption, for every vertex \( i \) of \( \Delta^n \), the restriction \( f : \{i\} \times \Delta^1 \to \mathcal{C} \) is an equivalence in the \( \infty \)-category \( \mathcal{C} \). It is a well known fact (see for instance the proof of Proposition 2.1.2.6 in [11]) that there exists a sequence of inclusions
\[
X(n + 1) \subseteq X(n) \subseteq \cdots \subseteq X(1) \subseteq X(0)
\]
with \( X(n + 1) = \partial \Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \) and \( X(0) = \Delta^n \times \Delta^1 \), and where each \( X(k) \) is obtained from \( X(k + 1) \) as a pushout of the form
\[
X(k) = X(k + 1) \cup_{\Delta_{k+1}^n} \Delta_{k+1}^n.
\]
Since the inclusion \( X(n + 1) \subseteq X(1) \) is inner anodyne we may choose an extension \( \tilde{f} : X(1) \to \mathcal{C} \) of \( f \). Since the edge \( f : \{0\} \times \Delta^1 \to \mathcal{C} \) is an equivalence in \( \mathcal{C} \) it follows from Theorem 1.3 of [7] that \( \tilde{f} \) extends along the inclusion \( X(1) \subseteq X(0) \).

**Remark 2.14.** The class of categorical equivalences in \( \text{Set}_\Delta \) is stable under filtered colimits.

### 2.4. Dwyer-Kan equivalences

If \( \mathcal{C} \) is an \( \infty \)-category and \( x, y \) are objects of \( \mathcal{C} \) then there is a mapping space \( \text{Map}_\mathcal{C}(x, y) \) of morphisms from \( x \) to \( y \). The space \( \text{Map}_\mathcal{C}(x, y) \) is well-defined as an object of \( \mathcal{H} \), the homotopy category of spaces. In [11] Lurie defines several models for the homotopy type \( [\text{Map}_\mathcal{C}(x, y)] \) in \( \mathcal{H} \). We shall make use of the model described in terms of left morphisms from \( x \) to \( y \) in \( \mathcal{C} \).

Let \( S \) be a simplicial set and let \( x \) and \( y \) be vertices of \( S \). Recall (see Section 1.2.2 of [11]) that the simplicial set \( \text{Hom}^L_S(x, y) \) of *left morphisms* from \( x \) to \( y \) in \( S \), is the simplicial set whose set of \( n \)-simplices is the set of all maps \( u : \Delta^{n+1} \to S \) such that \( u(0) = x \) and \( u|\Delta^{1,...,n} \) is equal to the constant \( n \)-simplex on the vertex \( y \). The face and degeneracy maps are induced from those of \( S \) in the obvious way. In other words,
\[
\text{Hom}^L_S(x, y) = S_{x/} \times_{S} \{ y \}.
\]
It is an important fact that when \( \mathcal{C} \) is an \( \infty \)-category, the simplicial set \( \text{Hom}^L_S(x, y) \) is a Kan complex which presents the mapping space \( \text{Map}_\mathcal{C}(x, y) \) of morphisms in \( \mathcal{C} \) from \( x \) to \( y \).

**Remark 2.15.** One should beware that if \( S \) is a simplicial set which presents an \( \infty \)-category \( \mathcal{C} \) in the sense that there is a categorical equivalence \( S \to \mathcal{C} \), then the naive mapping space \( \text{Hom}^L_S(x, y) \) need not have the homotopy type of \( \text{Map}_\mathcal{C}(x, y) \).

**Definition 2.16.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories. A map \( f : \mathcal{C} \to \mathcal{D} \) is said to be

(i) *essentially surjective* if the functor \( h(f) : h(\mathcal{C}) \to h(\mathcal{D}) \) is essentially surjective.

(ii) *fully faithful* if the induced map
\[
\text{Hom}^L_S(C, C') \to \text{Hom}^L_D(f(C), f(C'))
\]
is a homotopy equivalence for every pair of vertices \( C, C' \in \mathcal{C} \).
If \( f : \mathcal{C} \to \mathcal{D} \) is fully faithful and essentially surjective then we will say that \( f \) is a Dwyer-Kan equivalence.

We will show in Proposition 2.24 below that a map between \( \infty \)-categories is an equivalence if and only if it is a Dwyer-Kan equivalence. We begin by proving that every equivalence between \( \infty \)-categories is fully faithful (it is clear that every such equivalence is essentially surjective).

**Lemma 2.17.** Let \( f : \mathcal{C} \to \mathcal{D} \) be an equivalence between \( \infty \)-categories \( \mathcal{C} \) and \( \mathcal{D} \). Then \( f \) is fully faithful.

**Proof.** Suppose first that \( f : \mathcal{C} \to \mathcal{D} \) is a trivial Kan fibration. Then for any vertex \( C \in \mathcal{C} \) the induced map
\[
\mathcal{C}_{C/} \to \mathcal{D}_{f(C)/} \times_{\mathcal{D}} \mathcal{C}
\]
is a trivial Kan fibration. Therefore, for any vertex \( C' \in \mathcal{C} \), the map
\[
\text{Hom}_{\mathcal{C}}^L(C, C') \to \text{Hom}_{\mathcal{D}}^L(f(C), f(C'))
\]
is also a trivial Kan fibration. Hence \( f \) is fully faithful in this case.

Suppose now that \( f : \mathcal{C} \to \mathcal{D} \) is an arbitrary equivalence between \( \infty \)-categories \( \mathcal{C} \) and \( \mathcal{D} \). Let \( g : \mathcal{D} \to \mathcal{C} \) be an inverse equivalence for \( f \) as in Definition 2.1. Let \( h : \mathcal{C} \to \mathcal{C}^J \) be a homotopy witnessing the relation \( gf = \text{id}_C \) in \( \text{h}(\text{Fun}(\mathcal{C}, \mathcal{C})) \); by an abuse of notation write \( h \) also for the composite with the canonical projection \( \mathcal{C}^J \to \mathcal{C}(\Delta^1) \). We have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{id_C} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{h} & \mathcal{C}(\Delta^1) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{gf} & \mathcal{C} \\
\end{array}
\]
in which the vertical arrows are trivial Kan fibrations by Example 2.13. It follows from the discussion above that \( gf \) is fully faithful. By symmetry, \( fg \) is fully faithful.

Let \( C, C' \in \mathcal{C} \) be vertices. We have an induced diagram
\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}^L(C, C') & \xrightarrow{f} & \text{Hom}_{\mathcal{D}}^L(f(C), f(C')) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{C}}^L(gf(C), gf(C')) & \xrightarrow{f} & \text{Hom}_{\mathcal{D}}^L(fgf(C), fgf(C'))
\end{array}
\]
The vertical arrows are homotopy equivalences since \( gf \) and \( fg \) are fully faithful. Therefore, by the 2-out-of-6 property for homotopy equivalences, it follows that the map
\[
\text{Hom}_{\mathcal{C}}^L(C, C') \to \text{Hom}_{\mathcal{D}}^L(f(C), f(C'))
\]
is a homotopy equivalence. Hence \( f \) is fully faithful. \( \square \)
Remark 2.18. More generally we can make sense of what it means for a map of simplicial sets to be essentially surjective or fully faithful (the above definitions are invariant under equivalences of ∞-categories).

We would now like to prove that every Dwyer-Kan equivalence \( f : C \to D \) is a categorical equivalence. For this we shall need to isolate a particular class of inner fibrations between ∞-categories. With the benefit of hindsight (see Corollary 2.4.6.5 of [11]) we make the following definition.

Definition 2.19. We will say that an inner fibration \( p : C \to D \) between ∞-categories is a categorical fibration if the functor \( h(p) : h(C) \to h(D) \) is an isofibration of categories.

Remark 2.20. An inner fibration \( p : C \to D \) between ∞-categories is a categorical fibration if and only if it has the right lifting property with respect to the inclusion \( \{0\} \subseteq J \), if and only if the canonical map \( C(\Delta^1) \to C \times_D D(\Delta^1) \) induced by the inclusion \( \Delta\{0\} \subseteq \Delta^1 \) is surjective.

Lemma 2.21 (Joyal). Suppose \( p : C \to D \) is a categorical fibration between ∞-categories. If \( p \) is fully faithful and essentially surjective then \( p \) is a trivial Kan fibration.

Proof. Suppose \( p \) is fully faithful and essentially surjective. Since \( p \) has the right lifting property against the inclusion \( \{0\} \subseteq J \) it follows that \( p \) is surjective.

We prove that for every \( n \geq 1 \) the indicated diagonal filler exists in every commutative diagram of the form

\[
\begin{array}{cccc}
\partial \Delta^n & \xrightarrow{u} & C \\
\downarrow & & \downarrow p \\
\Delta^n & \xrightarrow{v} & D.
\end{array}
\]

By an adjointness argument this is equivalent to proving that the indicated diagonal filler exists in the induced diagram

\[
\begin{array}{cccc}
\partial \Delta\{1,\ldots,n\} & \xrightarrow{\sim} & C_{u(0)}/ \\
\downarrow & & \downarrow \\
\Delta\{1,\ldots,n\} & \xrightarrow{\sim} & D_{v(0)}/ \times_D C.
\end{array}
\]

The induced map

\[
C_{u(0)}/ \to D_{v(0)}/ \times_D C
\]

is a left fibration (Proposition 2.1.2.1 of [11]). Since \( p \) is fully faithful it follows that the fibers of \( \delta_1 \) are contractible. Hence \( \delta_1 \) is a trivial Kan fibration (Lemma 2.1.3.4 of [11]). Hence the indicated lift exists in the diagram \( \delta_1 \). It follows that \( p \) is a trivial Kan fibration. \( \Box \)

The following example appears as Proposition 5.16 in [8]. The laborious account that we give of it here is an unfortunate consequence of our efforts to keep the paper self-contained.
Example 2.22. Let \( C \) be an \( \infty \)-category. The canonical map \( C^{(\Delta^1)} \to C \times C \) induced by the inclusion \( \{0,1\} \subseteq \Delta^1 \) is an inner fibration by Corollary 2.3.2.5 of [11]. In fact this canonical map is also a categorical fibration. To see this, suppose given a commutative diagram of the form

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{u} & C^{(\Delta^1)} \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{(f,g)} & C \times C
\end{array}
\]

in which the edges \( f \) and \( g \) are equivalences in the \( \infty \)-category \( C \). We must prove that the indicated diagonal filler exists. The existence of such a diagonal filler is equivalent to the existence of a map \( \phi: \Delta^1 \times \Delta^1 \to C \) such that the restriction \( \phi|_{\Delta^1 \times \Delta^1} \) is equal to \( u \), the restriction \( \phi|_{\Delta^1 \times \{0\}} \) is equal to \( f \), and the restriction \( \phi|_{\Delta^1 \times \{1\}} \) is equal to \( g \). The existence of such a map \( \phi \) is easily proven using Proposition 1.2.4.3 of [11] (notice the restriction \( \phi|_{\Delta^1 \times \Delta^1} \) is necessarily an equivalence in \( C \), so that \( \phi \) corresponds to a map \( \Delta^1 \to C^{(\Delta^1)} \)).

Example 2.23. (Mapping path space construction). Given a map \( f: C \to D \) between \( \infty \)-categories \( C \) and \( D \) we define the \( \infty \)-category \( Q(f) \) by the pullback diagram

\[
\begin{array}{ccc}
Q(f) & \to & C \times D \\
\downarrow & & \downarrow f \times \text{id}_D \\
D^{(\Delta^1)} & \longrightarrow & D \times D
\end{array}
\]

It follows from the discussion in the paragraph above that the composite map \( Q(f) \to C \times D \to D \) is a categorical fibration. Denote this composite map by \( \pi: Q(f) \to D \). The diagonal map \( D \to D^{(\Delta^1)} \) induces a map \( i: C \to Q(f) \) which is right inverse to the trivial Kan fibration given by the composite map \( Q(f) \to C \times D \to C \). It follows that \( i \) is a categorical equivalence. The factorization \( f = \pi i \) is called the mapping path space factorization of \( f \).

Proposition 2.24 (Joyal). Let \( C \) and \( D \) be \( \infty \)-categories. A map \( f: C \to D \) is an equivalence if and only if it is a Dwyer-Kan equivalence.

Proof. We have seen above that every equivalence between \( \infty \)-categories is fully faithful and essentially surjective. We need to prove the converse. Suppose that \( f: C \to D \) is fully faithful and essentially surjective, where \( C \) and \( D \) are \( \infty \)-categories. Example 2.23 gives a factorization of the map \( f: C \to D \) as

\[
C \overset{i}{\longrightarrow} Q(f) \overset{p}{\longrightarrow} D
\]

where \( i \) is right inverse to a trivial Kan fibration and where \( p \) is a categorical fibration. It follows that \( i \) is fully faithful and essentially surjective. Therefore we may suppose without loss of generality that the map \( f: C \to D \) is a categorical fibration. Lemma 2.21 then shows that \( f \) is a trivial Kan fibration. □
Lemma 2.25. For any pushout diagram

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow i & & \downarrow j \\
B & \longrightarrow & D
\end{array}
\]

of simplicial sets in which \( i \) is a monic categorical equivalence then so is \( j \).

Proof. Since trivial Kan fibrations are stable under base change it suffices to prove that for any \( \infty \)-category \( C \), the map \( C^B \to C^A \) induced by \( i \) is a trivial Kan fibration. The induced map is a categorical equivalence by hypothesis, and is an inner fibration by Corollary 2.3.2.5 of [11]. Therefore it suffices by Lemma [2.21] to prove that this inner fibration is a categorical fibration. Suppose given a commutative diagram

\[
\begin{array}{ccc}
\Delta \{0\} & \longrightarrow & C^B \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & C^A
\end{array}
\]

in which the edge \( f: \Delta^1 \to C^A \) is an equivalence. We need to prove that the indicated diagonal filler exists. Choose an extension \( g: J \to C^A \) of \( f \) along the map \( \Delta^1 \to J \). Observe that a diagonal filler exists in the diagram above if and only if the indicated diagonal filler exists in the diagram

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & C^B \\
\downarrow & & \downarrow \\
J & \longrightarrow & C^A
\end{array}
\]

Denote by \( \overline{f}: A \to C^{\Delta^1} \) and \( \overline{g}: A \to C^J \) the maps conjugate to \( f \) and \( g \) respectively. Since \( f \) is equal to the composite \( \Delta^1 \to J \overset{g}{\longrightarrow} C^A \), the map \( \overline{f} \) is equal to the composite \( A \overset{\overline{g}}{\longrightarrow} C^J \to C^{\Delta^1} \). In particular it follows that \( \overline{f} \) factors through \( C^{(\Delta^1)} \) via a unique map \( k: A \to C^{(\Delta^1)} \). Since the composite map \( C^{(\Delta^1)} \to C^{\Delta^1} \to C^{\{0\}} \) is a trivial Kan fibration (Example 2.13), it follows that there is a map \( B \to C^{(\Delta^1)} \) whose composite with \( i: A \to B \) is equal to the map \( k \). It follows that the indicated diagonal filler in the diagram

\[
\begin{array}{ccc}
A & \overline{f} & \longrightarrow & C^{\Delta^1} \\
\downarrow & & \downarrow & \\
B & \longrightarrow & C^{\{0\}}
\end{array}
\]
exists. Therefore, by adjointness, we can find a diagonal filler \( \hat{f} \) for the diagram

\[
\begin{array}{ccc}
\{ 0 \} & \longrightarrow & \mathcal{C}^B \\
\downarrow & \swarrow \hat{f} & \downarrow \\
\Delta^1 & \longrightarrow & \mathcal{C}^A
\end{array}
\]

Therefore, we have proven that for every equivalence \( f \) in \( h(\mathcal{C}^A) \) and every vertex \( u \) in \( h(\mathcal{C}^B) \) such that \( ui = f(0) \), there is an edge \( \hat{f} \) in \( h(\mathcal{C}^B) \) which projects to \( f \). Since \( h(\mathcal{C}^B) \to h(\mathcal{C}^A) \) is fully faithful, it follows that \( h(\mathcal{C}^B) \to h(\mathcal{C}^A) \) is an isofibration. \( \square \)

**Remark 2.26.** Of course the above Lemma is an immediate consequence of the existence of the Joyal model structure on \( \text{Set}_\Delta \). One of our aims is to give an independent proof of the existence of this model structure; in order to avoid a circular argument (see the proof of Proposition 4.11) we will need to use this lemma.

### 3. Pre-fibrant simplicial sets

We have observed above that if \( S \) is a simplicial set and \( x, y \) are vertices of \( S \) then the naive mapping space \( \text{Hom}_{\mathcal{C}}^L(x, y) \) may not have the homotopy type of the mapping space \( \text{Map}_{\mathcal{C}}(x, y) \) for an \( \infty \)-category \( \mathcal{C} \) generated by \( S \). In this section we give a condition on a simplicial set to ensure that the naive mapping spaces have the correct homotopy types.

#### 3.1. Pre-fibrant simplicial sets

We introduce the following definition.

**Definition 3.1.** We will say that simplicial set \( S \) is **pre-fibrant** if it satisfies the following conditions:

(i) every map \( \Lambda^n_1 \to S \) extends along the inclusion \( \Lambda^n_1 \subseteq \Delta^n \).

(ii) for every \( 0 < i < n \), if \( f: \Lambda^n_i \to S \) is a map such that \( d_0 f \) is a constant \((n - 1)\)-simplex, then \( f \) extends along the inclusion \( \Lambda^n_i \subseteq \Delta^n \).

Here we will say that an \( n \)-simplex \( x \in S \) is **constant** if \( x = s^n_0(v) \) for some vertex \( v \) of \( S \).

**Example 3.2.** Every \( \infty \)-category is a pre-fibrant simplicial set.

**Remark 3.3.** If \( S \) is a pre-fibrant simplicial set then for every pair of vertices \( x, y \in S \), the naive mapping space \( \text{Hom}_{\mathcal{C}}^L(x, y) \) is a Kan complex.

**Remark 3.4.** Note that if \( p: X \to S \) is an inner fibration, and \( S \) is pre-fibrant, then \( X \) is also pre-fibrant. In particular if \( S \) is pre-fibrant and \( T \subseteq S \) is a full simplicial subset (see Remark 2.12) then \( T \) is pre-fibrant.

#### 3.2. Pre-fibrant simplicial sets and \( \infty \)-categories

In this section we prove that the naive mapping spaces for a pre-fibrant simplicial set \( S \) compute the mapping spaces of an \( \infty \)-category generated by \( S \). More precisely, we prove the following result.

**Proposition 3.5.** Suppose that \( S \) is a pre-fibrant simplicial set. Then there exists an inner anodyne map \( f: S \to T \) with the following properties:
(1) \( T \) is an \( \infty \)-category;
(2) the induced map
\[ S_{xy} \times S \{ y \} \to T_{xy} \times T \{ y \} \]
is an isomorphism for every pair of vertices \( x, y \in S \).

Proof. We construct a simplicial set \( T \) which contains \( S \) as a subcomplex as follows. Define \( \text{sk}_3 T \) by the pushout diagram
\[
\bigsqcup_{a \in A_3} 
\Delta^3_{ia} 
\to 
\text{sk}_3 S

\downarrow

\downarrow

\bigsqcup_{a \in A_3} 
\Delta^3 
\to 
\text{sk}_3 T
\]
where \( A_3 \) denotes the set of maps
\[ a: \Lambda^3_i \to \text{sk}_3 S \]
with \( 0 < i < 3 \) and where \( d_0(a) \) is non-constant. Clearly the canonical map \( \text{sk}_3 S \to \text{sk}_3 T \) is inner anodyne. Note the following facts: (i) if \( \sigma: \Delta^3 \to \text{sk}_3 T \) does not belong to \( \text{sk}_3 S \), then \( d_0(\sigma) \) is non-constant, and (ii) \( \text{sk}_1 S = \text{sk}_1 T \).

Assume inductively that \( \text{sk}_n T \) has been constructed for \( n \geq 3 \), containing \( \text{sk}_n S \) as a subcomplex and with the following properties:

(P-1) the inclusion \( \text{sk}_n S \subseteq \text{sk}_n T \) is inner anodyne;
(P-2) if \( \sigma: \Delta^n \to \text{sk}_n T \) is an \( n \)-simplex which does not belong to \( \text{sk}_n S \) then \( d_0(\sigma) \) is non-constant.

Define \( \text{sk}_{n+1} T \) by the pushout diagram
\[
\bigsqcup_{a \in A_{n+1}} 
\Lambda^{n+1}_{ia} 
\to 
\text{sk}_{n+1} S \cup \text{sk}_n T

\downarrow

\downarrow

\bigsqcup_{a \in A_{n+1}} 
\Delta^{n+1} 
\to 
\text{sk}_{n+1} T
\]
where \( A_{n+1} \) denotes the set of maps
\[ a: \Lambda^{n+1}_i \to \text{sk}_{n+1} S \cup \text{sk}_n T \]
with \( 0 < i < n + 1 \) and where \( d_0(a) \) is non-constant.

To close the inductive loop we need to prove that \( \text{sk}_{n+1} T \) satisfies the corresponding properties (P-1) and (P-2). For (P-1), we need to prove that the canonical map \( \text{sk}_{n+1} S \to \text{sk}_{n+1} T \) is inner anodyne. For this, note that the map \( \text{sk}_{n+1} S \cup \text{sk}_n T \to \text{sk}_{n+1} T \) in the diagram (3) above is inner anodyne. Since the map
\[ \text{sk}_{n+1} S \to \text{sk}_{n+1} S \cup \text{sk}_n T \]
is a pushout of the inner anodyne map \( \text{sk}_n S \to \text{sk}_n T \), it follows that the composite map \( \text{sk}_{n+1} S \to \text{sk}_{n+1} T \) is inner anodyne. For (P-2), we need to prove that if \( \sigma: \Delta^{n+1} \to \text{sk}_{n+1} T \) is an \( (n + 1) \)-simplex which does not belong to \( \text{sk}_{n+1} S \) then \( d_0(\sigma) \) is non-constant. If \( \sigma \)
is non-degenerate then $d_0(\sigma)$ is non-constant by construction. Suppose then that $\sigma$ is degenerate — suppose $\sigma = s_i(x)$ where $x \in sk_n T$. Suppose for a contradiction that $d_0(\sigma)$ is constant — suppose $d_0(\sigma) = s_0^n(v)$ for some vertex $v$ of $S$. If $i = 0$ then $x = s_0^n(v)$; if $i \geq 1$ then

$$
\begin{align*}
    d_0(x) &= d_0 d_i s_i(x) \\
    &= d_0 d_i (\sigma) \\
    &= d_{i-1} s_0^n(v) \\
    &= s_0^{n-1}(v).
\end{align*}
$$

In either case we see that $d_0(x)$ is constant. This is a contradiction to the inductive hypothesis, since $\sigma$ does not belong to $sk_{n+1} S$ and hence $x$ does not belong to $sk_n S$. Therefore $d_0(\sigma)$ is not constant.

This completes the inductive step and hence the construction of the simplicial set $T$ containing $S$ as a subcomplex. Since each inclusion $sk_n S \subseteq sk_n T$, $n \geq 3$, is inner anodyne, it follows from Lemma [7.8](#) that the inclusion $S \subseteq T$ is inner anodyne.

It remains to prove that

1. the induced map

$$
S_x/ \times S \{ y \} \to T_{x/} \times_T \{ y \}
$$

is an isomorphism for every pair of vertices $x, y \in S$;

2. $T$ is an $\infty$-category.

We prove the first statement. Clearly we have an injective map

$$
S_{x/} \times_S \{ y \} \to T_{x/} \times_T \{ y \}
$$

since $S \to T$ is a monomorphism. Let $\sigma: \Delta^n \to T_{x/} \times_T \{ y \}$ be an $n$-simplex. Then $\sigma$ corresponds to an $(n + 1)$-simplex $\tilde{\sigma}: \Delta^{n+1} \to T$ such that $d_0(\tilde{\sigma})$ is the constant $n$-simplex at $y$ and $\tilde{\sigma}|\Delta^{\{0\}} = x$. Since $d_0(\tilde{\sigma})$ is constant, $\tilde{\sigma}$ must belong to $sk_{n+1} S$ by construction. Hence $\tilde{\sigma}$ factors through $S$ and hence the $n$-simplex $\sigma: \Delta^n \to T_{x/} \times_T \{ y \}$ factors through $S_{x/} \times_S \{ y \}$. Hence we have an isomorphism

$$
S_{x/} \times_S \{ y \} \simeq T_{x/} \times_T \{ y \}.
$$

We prove the second statement, i.e. we prove that $T$ is an $\infty$-category. Suppose given a map

$$
f: \Lambda^n_i \to T
$$

with $0 < i < n$ and where $n \geq 3$. Then $f$ factors through $sk_{n-1} T$. If $d_0 f$ is non-constant, then the composite map $\Lambda^n_i \xrightarrow{f} sk_{n-1} T \to sk_n T$ extends along the inclusion $\Lambda^n_i \subseteq \Delta^n$.

Suppose now that $d_0 f$ is constant. Then for every face $\partial_j \Delta^{n-1}$ with $j \neq i$, the induced map $d_j f := f|\partial_j \Delta^{n-1}$ is an $(n - 1)$-simplex $d_j f: \Delta^{n-1} \to sk_{n-1} T$ with $d_0(d_j f)$ constant. Hence each map $d_j f$ factors through $sk_{n-1} S$. Hence $f$ factors through $sk_{n-1} S$.

By hypothesis, the composite map $\Lambda^n_i \xrightarrow{f} sk_{n-1} S \to sk_n S$ extends along the inclusion $\Lambda^n_i \subseteq \Delta^n$.
Finally, since \( \text{sk}_1 S = \text{sk}_1 T \) and \( S \) has the right lifting property against the inclusion \( \Lambda^2_1 \subseteq \Delta^2 \), it follows that \( T \) is an \( \infty \)-category.

**Remark 3.6.** Suppose that \( S \) is a pre-fibrant simplicial set. If \( u: S \to T \) is an inner anodyne map where \( T \) is an \( \infty \)-category then the induced map

\[
\tilde{u}: S_{s/} \times_S \{ s' \} \to T_{u(s)/} \times_T \{ u(s') \}
\]

is a homotopy equivalence between Kan complexes for any pair of vertices \( s, s' \in S \). To see this, choose an inner anodyne map \( v: S \to T' \) with the properties described in Proposition 3.5. Then there exists a map \( w: T \to T' \) such that \( v = wu \). The statement follows by the 2-out-of-3 property for weak homotopy equivalences.

**Remark 3.7.** It follows from Proposition 3.5 that a map \( f: S \to T \) of pre-fibrant simplicial sets is fully faithful if and only if the induced map \( \text{Hom}_S(x, y) \to \text{Hom}_T(f(x), f(y)) \) on the naive mapping spaces is a homotopy equivalence between Kan complexes for all objects \( x \) and \( y \) of \( S \).

### 3.3. The homotopy category of a pre-fibrant simplicial set.
In this section we make some observations about the homotopy category of pre-fibrant simplicial sets that we shall put to use later.

**Lemma 3.8.** Let \( S \) be a pre-fibrant simplicial set. Then for any pair of vertices \( s, s' \in S \), there is an isomorphism

\[
\pi_0(S_{s/} \times_S \{ s' \}) \simeq h(S)(s, s').
\]

**Proof.** Let \( u: S \to T \) be an inner anodyne map as in Proposition 3.5. The functor \( h(S) \to h(T) \) induced by \( u \) is an isomorphism. For any pair of vertices \( s, s' \in S \) we have an isomorphism

\[
\pi_0(S_{s/} \times_S \{ s' \}) \simeq \pi_0(T_{u(s)/} \times_T \{ u(s') \}).
\]

But \( \pi_0(T_{u(s)/} \times_T \{ u(s') \}) = \pi(T)(u(s), u(s')) \) and we have an isomorphism \( h(T) \simeq \pi(T) \) (here \( \pi(T) \) denotes the category constructed in Section 1.2.3 of [11]).

**Lemma 3.9.** Let \( p: X \to S \) be an inner fibration between pre-fibrant simplicial sets. Then \( h(p): h(X) \to h(S) \) is an isofibration if and only if for every equivalence \( f: \Delta^1 \to S \) and vertex \( x \in X \) such that \( p(x) = f(0) \), there exists an equivalence \( u: \Delta^1 \to X \) such that \( p(u) = f \).

**Proof.** Suppose that \( h(p): h(X) \to h(S) \) is an isofibration. Choose an inner anodyne map \( S \to \mathcal{D} \) with the properties of Proposition 3.5 so that \( \mathcal{D} \) is an \( \infty \)-category. The composite map \( X \to S \to \mathcal{D} \) factorizes as \( X \to \mathcal{C} \to \mathcal{D} \) where \( X \to \mathcal{C} \) is inner anodyne and \( \mathcal{C} \to \mathcal{D} \) is an inner fibration. Thus we have a commutative diagram of the form

\[
\begin{array}{ccc}
X & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{D}
\end{array}
\]
Since \( h(p) : h(X) \to h(S) \) is an isofibration it follows that the induced map \( h(\mathcal{C}) \to h(\mathcal{D}) \) is an isofibration. Hence there exists an equivalence \( v : \Delta^1 \to \mathcal{C} \) such that \( v(0) = x \) and \( p(v) = f \). Let \( y = v(1) \) and \( t = f(1) \). We have an induced map

\[
\begin{array}{ccc}
X_{x/} \times X \{ y \} & \xrightarrow{\phi} & \mathcal{C}_{x/} \times \mathcal{C} \{ y \} \\
\downarrow & & \downarrow \\
S_{s/} \times S \{ t \} & \xrightarrow{\psi} & \mathcal{D}_{s/} \times \mathcal{D} \{ t \}
\end{array}
\]

where the lower horizontal map is an isomorphism (Proposition 3.5), the vertical maps are Kan fibrations between Kan complexes, and the upper horizontal map is a homotopy equivalence (Remark 3.6). It follows that we can choose a vertex \( u \) in the fiber of \( X_{x/} \times X \{ y \} \to S_{s/} \times S \{ t \} \) over \( f \) whose image in \( \mathcal{C}_{x/} \times \mathcal{C} \{ y \} \) is homotopic to \( v \). Thus \( u \) represents a morphism in \( h(X)(x, y) \) whose image in \( h(\mathcal{C})(x, y) \) is equal to \([v]\). Since \( h(X) \to h(\mathcal{C}) \) is an isomorphism it follows that \([u]\) is an isomorphism in \( h(\mathcal{C}) \). The proof of the converse statement is straightforward and is left to the reader. \( \square \)

**Remark 3.10.** If \( S \) is a pre-fibrant simplicial set and \( f : x \to y \) is an equivalence in \( S \) (Section 2.1) then it is not necessarily true that the map \( f : \Delta^1 \to S \) classifying \( f \) extends along the canonical map \( \Delta^1 \to J \). However, it follows from Lemma 3.8 that there exists an edge \( g : y \to x \) such that \([g][f] = \text{id}_x \) and \([f][g] = \text{id}_y \) in \( h(S)(x, x) \) and \( h(S)(y, y) \) respectively. Hence there exist 2-simplices \( \sigma, \sigma' : \Delta^2 \to S \) corresponding to diagrams

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{\phi} & & \downarrow{\psi} \\
x & \xrightarrow{g} & x
\end{array}
\quad
\text{and}
\quad
\begin{array}{ccc}
x & \xrightarrow{g} & x \\
\downarrow{\phi} & & \downarrow{\psi} \\
x & \xrightarrow{f} & y
\end{array}
\]

respectively. Moreover \([\phi] = \text{id}_x \) in \( h(S)(x, x) \) and \([\psi] = \text{id}_y \) in \( h(S)(y, y) \). Therefore, using Lemma 3.8 again, we see that, since \( S_{x/} \times S \{ x \} \) and \( S_{y/} \times S \{ y \} \) are Kan complexes, there exist 2-simplices \( \tau, \tau' : \Delta^2 \to S \) corresponding to diagrams

\[
\begin{array}{ccc}
x & \xrightarrow{\text{id}_x} & x \\
\downarrow{\phi} & & \downarrow{\psi} \\
x & \xrightarrow{\text{id}_y} & y
\end{array}
\quad
\text{and}
\quad
\begin{array}{ccc}
x & \xrightarrow{\text{id}_x} & x \\
\downarrow{\phi} & & \downarrow{\psi} \\
x & \xrightarrow{\text{id}_y} & y
\end{array}
\]

respectively. Notice that if \( S' \subseteq S \) denotes the subcomplex generated by the 2-simplices \( \sigma, \sigma', \tau \) and \( \tau' \) then \( h(S') = J \).

**Lemma 3.11.** Suppose given a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{p} & & \downarrow{q} \\
S & \xrightarrow{v} & T
\end{array}
\]
of pre-fibrant simplicial sets. If \( q: Y \to T \) is an inner fibration, then the induced diagram

\[
\begin{array}{ccc}
\text{h}(X) & \longrightarrow & \text{h}(Y) \\
\downarrow & & \downarrow \\
\text{h}(S) & \longrightarrow & \text{h}(T)
\end{array}
\]

is a pullback diagram in \( \text{Cat} \).

Proof. It suffices to prove that for every pair of vertices \( x, x' \in X \), the induced diagram

\[
\begin{array}{ccc}
\text{h}(X)(x, x') & \longrightarrow & \text{h}(Y)(u(x), u(x')) \\
\downarrow & & \downarrow \\
\text{h}(S)(s, s') & \longrightarrow & \text{h}(T)(v(s), v(s'))
\end{array}
\]

is a pullback, where \( s = p(x) \) and \( s' = p(x') \). The diagram

\[
\begin{array}{ccc}
X_{x/} \times_X \{ x' \} & \longrightarrow & Y_{u(x)/} \times_Y \{ u(x') \} \\
\downarrow^p & & \downarrow^q \\
S_{s/} \times_S \{ s' \} & \longrightarrow & T_{v(s)/} \times_T \{ v(s') \}
\end{array}
\]

is a homotopy pullback. The result follows by applying \( \pi_0 \).

\[ \Box \]

4. Descent for inner fibrations

In this section we introduce a notion of descent for inner fibrations. The case that will be of most interest to us is the case of descent along an inner anodyne map constructed via the small object argument. Our main result is Proposition 4.12.

**Definition 4.1.** Suppose that \( p: X \to S \) is an inner fibration between simplicial sets and that \( f: S \to T \) is a map in \( \text{Set}_\Delta \). We will say that \( p: X \to S \) satisfies descent relative to \( f: S \to T \) if there exists an inner fibration \( q: Y \to T \) forming part of a pullback diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow^p & & \downarrow^q \\
S & \longrightarrow & T
\end{array}
\]

If \( f: S \to T \) is inner anodyne then we will say that \( p: X \to S \) satisfies inner anodyne descent with respect to \( f \) if the map \( g \) in the diagram above is inner anodyne.

We will be interested in finding conditions on the inner fibration \( p \) which ensure that \( p \) satisfies descent relative to certain maps \( f: S \to T \). The following lemma gives a convenient sufficient condition for an inner fibration to satisfy descent.
Lemma 4.2. Suppose that \( f: T \to S \) and \( p: X \to S \) are maps of simplicial sets and that \( f \) is surjective. If there is a pullback diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow^{q} & & \downarrow^{p} \\
T & \longrightarrow & S
\end{array}
\]

in which \( q \) is an inner fibration then \( p \) is also an inner fibration.

Proof. We prove that \( p \) is an inner fibration. For this, we note that, since \( f \) is surjective, for any commutative diagram of the form

\[
\begin{array}{ccc}
\Lambda^{n}_{k} & \longrightarrow & X \\
\downarrow & & \downarrow^{p} \\
\Delta^{n} & \longrightarrow & S
\end{array}
\]

the map \( u \) factors as \( fv \) for some map \( v: \Delta^{n} \to T \). It follows that the map \( \Lambda^{n}_{k} \to X \) factors through \( Y \) so that we have a commutative diagram

\[
\begin{array}{ccc}
\Lambda^{n}_{k} & \longrightarrow & Y \\
\downarrow & & \downarrow^{q} \\
\Delta^{n} & \longrightarrow & T
\end{array}
\]

Since \( q \) is an inner fibration we may find a diagonal filler in this diagram and hence in the diagram (4). Therefore \( p \) is an inner fibration.

\[\square\]

4.1. Colimits and descent. In this section we give some examples of descent for coprojection maps into cocones on certain colimit diagrams.

Lemma 4.3. Suppose given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X_{0} & \longrightarrow & X_{1} & \longrightarrow & X_{2} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots \\
S_{0} & \longrightarrow & S_{1} & \longrightarrow & S_{2} & \longrightarrow & \cdots
\end{array}
\]

in which each square is a pullback, all horizontal maps are monomorphisms, and all vertical maps are inner fibrations. Then the inner fibration \( X_{m} \to S_{m} \) satisfies descent with respect to the canonical map \( S_{m} \to \lim_{\longrightarrow} S_{m} \) for every \( m \geq 0 \).

Proof. Let \( Y = \lim_{\longrightarrow} X_{m} \) and let \( T = \lim_{\longrightarrow} S_{m} \). Under the given hypotheses, the squares

\[
\begin{array}{ccc}
X_{m} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S_{m} & \longrightarrow & T
\end{array}
\]
are pullbacks for every \( m \). It follows that any commutative diagram of the form
\[
\begin{array}{ccc}
\Lambda^n & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & T
\end{array}
\]
(5)
can be extended to a commutative diagram of the form
\[
\begin{array}{ccc}
\Lambda^n & \longrightarrow & X_m & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^n & \longrightarrow & S_m & \longrightarrow & T
\end{array}
\]
(6)
for some \( m \geq 0 \). A choice of diagonal filler for the left hand square in (6) determines a diagonal filler for the square (5).

**Remark 4.4.** One outcome of the proof of Lemma 4.3 is that under the stated hypotheses the canonical map
\[
\lim X_m \rightarrow \lim S_m
\]
is an inner fibration.

**Lemma 4.5.** Suppose given a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
X_1 & \longleftrightarrow & X_0 & \longrightarrow & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
S_1 & \longleftrightarrow & S_0 & \longrightarrow & S_2
\end{array}
\]
in which both squares are pullbacks, the arrow \( S_0 \rightarrow S_2 \) is a monomorphism, and all vertical maps are inner fibrations. Then for \( i = 1 \) and \( i = 2 \) the inner fibration \( X_i \rightarrow S_i \) satisfies descent with respect to the canonical map \( S_i \rightarrow S \), where \( S = S_1 \cup S_0 S_2 \).

**Proof.** Let \( X = X_1 \cup_{S_0} X_2 \) and let \( S = S_1 \cup_{S_0} S_2 \). Under the given hypotheses the squares
\[
\begin{array}{ccc}
X_i & \longrightarrow & X \\
\downarrow & & \downarrow \\
S_i & \longrightarrow & S
\end{array}
\]
are pullbacks for \( i = 1 \) and \( i = 2 \). The result then follows from Lemma 4.2 using the fact that the canonical map \( S_1 \cup S_2 \rightarrow S \) is surjective.

**Remark 4.6.** If follows from the proof of Lemma 4.5 that the canonical map
\[
X_1 \cup_{X_0} X_2 \rightarrow S_1 \cup_{S_0} S_2
\]
is an inner fibration. Note also that if \( S_0 \rightarrow S_1 \) and \( X_0 \rightarrow X_1 \) are inner anodyne then the canonical maps \( S_2 \rightarrow S \) and \( X_2 \rightarrow X \) are also inner anodyne.
4.2. **Descent and the small object argument.** Suppose $\Sigma$ is a set of inner anodyne maps in $\text{Set}_\Delta$ with small domains. Let $S$ be a simplicial set. Construct an inner anodyne map $S \rightarrow T$ by the following mild variation of the small object argument. Define a sequence of inner anodyne maps

$$S = S(0) \rightarrow S(1) \rightarrow S(2) \rightarrow \cdots$$

inductively as follows. Assuming that $S(m)$ has been defined, suppose given a set $A_m$ of maps $\alpha: I_\alpha \rightarrow S(m)$, where $I_\alpha$ is the domain of $\alpha$ in $\Sigma$ for each $\alpha \in A_m$. Define $S(m+1)$ by the pushout diagram

$$\bigsqcup_{\alpha \in A_m} I_\alpha \quad \phi \quad S(m)$$

$$\downarrow {\omega f_\alpha} \quad \downarrow$$

$$\bigsqcup_{\alpha \in A_m} J_\alpha \quad \rightarrow \quad S(m+1)$$

where $\phi$ is equal to the map $\alpha: I_\alpha \rightarrow S(m)$ on the summand labelled by $\alpha$, and where $f_\alpha: I_\alpha \rightarrow J_\alpha$ is a map in $\Sigma$ for all $\alpha \in A_m$. Let $T = \lim_{\rightarrow} S(m)$ so that we have a canonical map $S \rightarrow T$.

**Proposition 4.7.** Suppose that every map $\alpha: I_\alpha \rightarrow S(m)$ in $A_m$ has the following property: if $Z \rightarrow S(m)$ is an inner fibration, then the induced inner fibration $\alpha^* Z \rightarrow I_\alpha$ satisfies inner anodyne descent with respect to the map $f_\alpha: I_\alpha \rightarrow J_\alpha$. Then the following is true: every inner fibration $p: X \rightarrow S$ satisfies inner anodyne descent with respect to the map $S \rightarrow T$.

**Proof.** Define $X(0) = X$. Under the above hypothesis it follows from Lemma 4.5 and Remark 4.6 that we can construct inductively a sequence of inner fibrations $p(m): X(m) \rightarrow S(m)$ such that for every $m \geq 0$ we have a pullback diagram

$$\begin{array}{ccc}
X(m) & \longrightarrow & X(m+1) \\
\downarrow p(m) & & \downarrow p(m+1) \\
S(m) & \longrightarrow & S(m+1)
\end{array}$$

in which the horizontal maps are inner anodyne. Define $Y = \lim_{\rightarrow} X(m)$. Then the canonical map $q: Y \rightarrow T$ is an inner fibration and the diagram

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow p & & \downarrow q \\
S & \longrightarrow & T
\end{array}$$

is a pullback by Lemma 4.3. It is straightforward to check that the horizontal maps in this diagram are inner anodyne. \qed

4.3. **Descent for inner anodyne maps.** Every inner fibration $X \rightarrow \Lambda^2_1$ satisfies inner anodyne descent with respect to the inclusion $\Lambda^2_1 \subseteq \Delta^2$. More precisely we have the following lemma.
Lemma 4.8. Suppose that \( p: X \to \Lambda^2_1 \) is an inner fibration of simplicial sets where \( n \geq 2 \). Then there is a pullback diagram

\[
\begin{array}{ccc}
X & \rightarrow & C \\
\downarrow p & & \downarrow q \\
\Lambda^2_1 & \rightarrow & \Delta^2
\end{array}
\]

in which the map \( v \) is inner anodyne and \( q \) is an inner fibration.

Proof. We use a modification of the small object argument: we consider diagrams

\[
(D) \quad \Lambda^m_i \rightarrow X \\
\downarrow \downarrow \\
\Delta^m \rightarrow \Delta^2
\]

in which \( \beta \) does not factor through \( \Lambda^2_1 \) and where \( 0 < m < i \). We define a simplicial set \( C(1) \) by the pushout

\[
\bigsqcup_D \Lambda^m_i \rightarrow X \\
\downarrow \downarrow \\
\bigsqcup_D \Delta^m \rightarrow C(1)
\]

in which the coproducts are taken over all diagrams \( D \) as above. The key point is that the canonical map \( X \to \Lambda^2_1 \times_\Delta^2 C(1) \) is an isomorphism. To see this it suffices to prove that for each diagram \( D \) as above, the map \( \beta \) induces an isomorphism

\[
(\Lambda^2_1 \times_\Delta^2 \Lambda^m_i)_p = (\Lambda^2_1 \times_\Delta^2 \Delta^m)_p
\]

on \( p \)-simplices for all \( p \geq 0 \). If \( p \leq m - 2 \) this is clear, since \((\Lambda^m_i)_p = (\Delta^m)_p\) in this case. Suppose that \( \beta(\partial_i \Delta^{m-1}) \) is the image of an \((m-1)\)-simplex of \( \Lambda^2_1 \) in \( \Delta^2 \). This can only happen if \( \beta \) restricts to a map \( \partial_i \Delta^{m-1} \to \Delta^k \) for some \( k \geq 0 \). In particular \( \beta(0), \beta(m) \in \{k, k+1\} \). Since \( \beta(0) \leq \beta(i) \leq \beta(m) \) it follows that \( \beta \) factors through \( \Lambda^2_1 \), contradicting the hypothesis that it does not. It follows that \((7)\) holds when \( p = m - 1 \). In a similar fashion one proves that \((7)\) holds for all \( p \geq m \).

Returning to the modification of the small object argument, we continue in this manner to obtain the usual sequence

\[
X \rightarrow C(1) \rightarrow C(2) \rightarrow \cdots \rightarrow C(n) \rightarrow \cdots
\]

where each map is inner anodyne. The argument above shows that we have isomorphisms \( X \to \Lambda^2_1 \times_\Delta^2 C(n) \) for each \( n \geq 1 \). Let \( C \) denote the colimit of the \( C(n) \)'s. The canonical map \( X \to C \) is inner anodyne and induces an isomorphism \( X \to \Lambda^2_1 \times_\Delta^2 C \). The canonical
map $\mathcal{C} \to \Delta^2$ from the colimit is still an inner fibration, for given a commutative diagram

$$
\begin{align*}
\Lambda^m_i & \xrightarrow{\alpha} \mathcal{C} \\
\Delta^m & \xrightarrow{\beta} \Delta^2
\end{align*}
$$

in which $0 < i < m$, if the map $\beta$ does not factor through $\Lambda^2_j$ then a diagonal filler exists by construction. Otherwise, if $\beta$ does factor through $\Lambda^2_j$ then it must factor through some $\Delta^{\{i,i+1\}}$. In this case the map $\alpha$ must factor through $X|\Delta^{\{i,i+1\}}$ and hence the required diagonal filler exists since $X|\Delta^{\{i,i+1\}}$ is an $\infty$-category. □

**Remark 4.9.** In fact, it is easy to extend the above argument to show that every inner fibration $X \to I_n$ satisfies descent with respect to the spine inclusion $I_n \subseteq \Delta^n$ (see Example 5.1).

The story for inner horn inclusions $\Lambda^n_i \subseteq \Delta^n$ with $n \geq 3$ is more complicated. In general it is not true that every inner fibration $X \to \Lambda^n_i$ satisfies descent with respect to the inner horn inclusion $\Lambda^n_i \subseteq \Delta^n$.

**Example 4.10.** The inner fibration $\Delta^{\{0,2\}} \cup \Delta^{\{2,3\}} \to \Lambda^3_1$ does not satisfy descent with respect to the inner horn inclusion $\Lambda^3_1 \subseteq \Delta^3$.

The following proposition gives a sufficient condition for an inner fibration over $\Lambda^n_i$ to satisfy descent with respect to the inclusion $\Lambda^n_i \subseteq \Delta^n$.

**Proposition 4.11.** Let $p: \mathcal{C} \to \Lambda^n_i$ be an inner fibration where $n \geq 3$. Let $q$ denote the projection

$$q: \Delta^{\{1,...,n\}} \times_{\Lambda^n_i} \mathcal{C} \to \Delta^{\{1,...,n\}}.$$

Suppose that the following conditions are satisfied:

(i) for every vertex $x \in \mathcal{C}_j$, $1 \leq j \leq n-2$, there exists a $q$-cocartesian morphism $f: x \to y$ where $y \in \mathcal{C}_{j+1}$;

(ii) for every vertex $y \in \mathcal{C}_j$, $2 \leq j \leq n-1$ there exists a $q$-cartesian morphism $f: x \to y$ where $x \in \mathcal{C}_{j-1}$.

Then $p: \mathcal{C} \to \Lambda^n_i$ satisfies inner anodyne descent with respect to the inclusion $\Lambda^n_i \subseteq \Delta^n$.

The proof of Proposition 4.11 is lengthy and is postponed until Section 8. We can now state and prove the main result of this section.

**Proposition 4.12.** Let $S$ be a simplicial set. Then there exists an inner anodyne map $S \to T$, where $T$ is a pre-fibrant simplicial set, with the property that every inner fibration $X \to S$ satisfies inner anodyne descent with respect to $S \to T$.

**Proof.** We use the modification of the small object argument described in Section 4.2 in which $\Sigma$ is the set of inner horn inclusions $\Lambda^n_i \subseteq \Delta^n$, $0 < i < n$, in $\text{Set}_\Delta$, and where $A_m$ is the set of maps $\alpha: \Lambda^n_i \to S(m)$ which satisfy $d_0(\alpha)$ is constant if $n > 2$. This produces
an inner anodyne map $S \to T$, where $T$ is a pre-fibrant simplicial set. By Proposition 4.7 it suffices to prove that for any map $\alpha \in A_m$, and for any inner fibration $Z \to S(m)$, the induced inner fibration $\alpha^*Z \to \Lambda^n_i$ satisfies inner anodyne descent with respect to the inclusion $\Lambda^n_i \subseteq \Delta^n$. If $n = 2$ this follows from Lemma 4.8. Therefore it suffices to verify that for any map $\alpha \in A_m$ and for any inner fibration $Z \to S(m)$, the hypotheses of Proposition 4.11 are satisfied by the inner fibration $\alpha^*Z \to \Lambda^n_i$. Suppose then that $\alpha: \Lambda^n_i \to S(m)$ satisfies $d_0(\alpha)$ is constant. Let $C \to \Lambda^n_i$ denote the induced inner fibration $\alpha^*Z \to \Lambda^n_i$. The projection $$ q: \Delta^{\{1,\ldots,n\}} \times_{\Lambda^n_i} C \to \Delta^{\{1,\ldots,n\}} $$ is the induced inner fibration $$ d_0(\alpha)^*Z \to \Delta^{\{1,\ldots,n\}} $$ Since $d_0(\alpha)$ is constant this induced inner fibration is isomorphic to one of the form $$ \Delta^{\{1,\ldots,n\}} \times Z_v \to \Delta^{\{1,\ldots,n\}} $$ where $Z_v$ denotes the fiber of $Z$ over some vertex $v \in S(m)$; this inner fibration clearly satisfies the hypotheses of Proposition 4.11. □

5. INNER ANODYNE MAPS AND THE INNER MODEL STRUCTURE

In this section we study inner anodyne maps in a little more detail. We give some characterizations of the class of inner anodyne maps and prove Theorem C. We introduce the inner model structure and prove Theorem D.

5.1. Examples and basic properties. Recall that the class of inner anodyne maps in $\text{Set}_\Delta$ is the saturated class of monomorphisms generated by the inner horn inclusions $\Lambda^n_i \subseteq \Delta^n$, $0 < i < n$. It follows immediately that every inner anodyne map is bijective on vertices, since the inner horn inclusions have this property.

**Example 5.1.** In addition to being bijective on vertices, every inner anodyne map is both left and right anodyne. These three properties do not suffice to characterize inner anodyne maps, as the following example shows. Regard $\Delta^1 \times \Delta^1$ as a subcomplex of $\Delta^3$ by identifying $\Delta^1 \times \Delta^1$ with $\Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}}$. Then the inclusion $\Delta^1 \times \Delta^1 \subseteq \Delta^3$ is left and right anodyne, and is also bijective on vertices. But it is not inner anodyne.

An important example of inner anodyne maps are the spine inclusions.

**Example 5.2.** For any $n \geq 2$ the inclusion $I_n \subseteq \Delta^n$ is inner anodyne, where $$ I_n = \Delta^{\{0,1\}} \cup \ldots \cup \Delta^{\{n-1,n\}} $$ denotes the spine of $\Delta^n$ (for a proof that $I_n \subseteq \Delta^n$ is inner anodyne, see Proposition 2.13 of [5]).

**Definition 5.3.** Recall that a class $\mathcal{A}$ of monomorphisms in $\text{Set}_\Delta$ is said to satisfy the right cancellation property if the following condition is satisfied: if $u: A \to B$ and $v: B \to C$ are monomorphisms in $\text{Set}_\Delta$ such that $vu, u \in \mathcal{A}$, then $v \in \mathcal{A}$.
It is well-known that the classes of left anodyne and right anodyne maps in \( \mathbf{Set}_\Delta \) have the right cancellation property. The same is also true for the class of inner anodyne maps.

**Lemma 5.4** ([13]). The class of inner anodyne morphisms in \( \mathbf{Set}_\Delta \) satisfies the right cancellation property.

We shall make extensive use of Lemma 5.4 in Section 8 but for now let us note that it quickly gives the following characterization of the class of inner anodyne maps in \( \mathbf{Set}_\Delta \).

**Proposition 5.5.** The class of inner anodyne maps in \( \mathbf{Set}_\Delta \) is the smallest saturated class of monomorphisms in \( \mathbf{Set}_\Delta \) which contains the spine inclusions \( I_n \subseteq \Delta^n \) for every \( n \geq 2 \) and which has the right cancellation property.

**Proof.** Let \( A \) be a saturated class of monomorphisms which contains the spine inclusions \( I_n \subseteq \Delta^n \) for all \( n \geq 2 \) and which has the right cancellation property. Then \( A \) contains the class of inner anodyne maps by Lemma 3.5 of [10]. Conversely, the class of inner anodyne maps is saturated, has the right cancellation property, and contains the spine inclusions. The result follows. \( \square \)

One should beware that the weakly saturated class generated by the set of spine inclusions \( I_n \subseteq \Delta^n, n \geq 2 \), is not equal to the class of inner anodyne maps, as the following example shows.

**Example 5.6.** Let \( S \) be the simplicial set defined by the pushout diagram

\[
\begin{array}{ccc}
I_3 & \rightarrow & \partial \Delta^3 \\
\downarrow & & \downarrow \\
\Delta^3 & \rightarrow & S
\end{array}
\]

Then for every \( n \geq 2 \), any map \( I_n \rightarrow S \) extends along the inclusion \( I_n \subseteq \Delta^n \). But the composite map \( \Lambda^3_1 \rightarrow \partial \Delta^3 \rightarrow S \) does not extend along the inclusion \( \Lambda^3_1 \subseteq \Delta^3 \).

### 5.2. Further properties of inner anodyne maps

Our aim in this section is to prove Theorem C from the introduction. We begin by proving a special case of it, Lemma 5.7 below. We then give some different, equivalent reformulations of Theorem C. We prove one of these reformulations, Theorem 5.10 below.

The following lemma gives a characterization of the inner anodyne maps with codomain an \( \infty \)-category \( \mathcal{C} \).

**Lemma 5.7.** Suppose that \( u: A \rightarrow \mathcal{C} \) is a monic categorical equivalence which is a bijection on objects. If \( \mathcal{C} \) is an \( \infty \)-category then \( u \) is inner anodyne.

**Proof.** Factor \( u \) as \( u = pi \) where \( i: A \rightarrow B \) is inner anodyne and \( p: B \rightarrow \mathcal{C} \) is an inner fibration. Since the functors \( h(u) \) and \( h(i) \) are isomorphisms it follows that \( h(p) \) is an isomorphism. Since \( \mathcal{C} \) is an \( \infty \)-category it follows quickly that \( p \) is a categorical fibration. Since \( p \) is a categorical equivalence between \( \infty \)-categories it follows that \( p \) is a trivial Kan fibration (Lemma 2.21 and Proposition 2.24). Choosing a section of \( p \) exhibits \( u \) as a codomain retract of the inner anodyne map \( i \). Hence \( u \) is inner anodyne. \( \square \)
Example 5.8. In particular, the canonical inclusion $\Delta^n \times \{0\} \cup \partial \Delta^n \times J \subseteq \Delta^n \times J$ is inner anodyne for every $n \geq 1$.

This next proposition lists some reformulations of Theorem C. We shall see below that the third statement in this list can be proven by a descent argument using pre-fibrant simplicial sets.

**Proposition 5.9.** The following statements are equivalent:

1. The class of inner anodyne maps satisfies the 2-out-of-3 property in the class of monomorphisms in $\text{Set}_\Delta$;
2. Every monic categorical equivalence which is a bijection on objects is inner anodyne;
3. if $p: X \to S$ is an inner fibration between simplicial sets which is a bijection on objects and a categorical equivalence, then $p$ is a trivial Kan fibration.

We clarify the meaning of statement (1) above: if $u$ and $v$ are composable monomorphisms in $\text{Set}_\Delta$ such that any two of the maps $u$, $v$ and $vu$ are inner anodyne, then all three of the maps are inner anodyne.

**Proof.** We prove that $(1) \implies (2)$. Suppose that the class of inner anodyne maps satisfies the 2-out-of-3 property. Let $u: A \to B$ be a monic categorical equivalence which is a bijection on objects. Choose an inner anodyne map $B \to B'$ where $B'$ is an $\infty$-category. Then the composite map $A \to B \to B'$ is a monic categorical equivalence and a bijection on objects with codomain an $\infty$-category. Hence it is inner anodyne (Lemma 5.7). Therefore, under the assumption (1), it follows that $u: A \to B$ is inner anodyne.

We prove that $(2) \implies (1)$. Suppose that (2) is true. Let $u: A \to B$ and $v: B \to C$ be monomorphisms of simplicial sets such that two of the maps $u$, $v$ and $vu$ are inner anodyne. Under this assumption the 2-out-of-3 property of categorical equivalences shows that all of the maps $u$, $v$ and $vu$ are categorical equivalences. Similarly all of the maps $u$, $v$ and $vu$ must be bijections on objects. It follows, under the assumption of (2), that all of the maps $u$, $v$ and $vu$ are inner anodyne.

We prove that $(2) \implies (3)$. Let $p: X \to S$ be an inner fibration between simplicial sets which is a categorical equivalence and a bijection on objects. Choose an inner anodyne map $S \to S'$ where $S'$ is an $\infty$-category. We may factor the composite map $X \to S \to S'$ as $X \to X' \to S'$ where $X \to X'$ is inner anodyne and $p': X' \to S'$ is an inner fibration, so that we have a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow^p & & \downarrow^{p'} \\
S & \longrightarrow & S'
\end{array}
$$

The induced map $h(X') \to h(S')$ is an isomorphism of categories, from which it follows easily that $p': X' \to S'$ is a categorical fibration. Since $p'$ is a categorical equivalence between $\infty$-categories it follows (Lemma 2.21) that $p'$ is a trivial Kan fibration. The induced map $X \to X' \times_{S'} S$ is a monic categorical equivalence which is a bijection on
objects. Hence it is inner anodyne, under the assumption that (2) holds. Therefore, using the fact that \( p \) is an inner fibration, we can find the indicated diagonal filler in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow & & \downarrow^p \\
X' \times^S S & \longrightarrow & S
\end{array}
\]

It follows that \( p: X \to S \) is a retract of the trivial Kan fibration \( X \times^S S \to S \), and hence must itself be a trivial Kan fibration.

Finally, to complete the proof it suffices to prove that (3) \( \implies \) (2). Under the assumption of (3), the proof of Lemma 5.7 can be easily adapted to show that (2) is true. \( \square \)

**Theorem 5.10.** Let \( p: X \to S \) be an inner fibration between simplicial sets. If \( p \) is a categorical equivalence and if the map \( p_0: X_0 \to S_0 \) on objects is a bijection then \( p \) is a trivial Kan fibration.

**Proof.** By Proposition 4.12 we may find a pullback diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow^p & & \downarrow^{p'} \\
S & \longrightarrow & S'
\end{array}
\]

in which the horizontal maps are inner anodyne, \( p': X' \to S' \) is an inner fibration and \( S' \) (and hence \( X' \)) is a pre-fibrant simplicial set.

Clearly \( p': X' \to S' \) is a categorical equivalence and also a bijection on objects. It follows that without loss of generality we may suppose at the outset that \( S \) (and hence \( X \) — see Remark 3.4) is a pre-fibrant simplicial set.

Our aim is to prove that \( p: X \to S \) has the right lifting property against the boundary inclusions \( \partial \Delta^n \subseteq \Delta^n \) for \( n \geq 0 \). The case \( n = 0 \) is clear, since \( p \) is a bijection on objects.

Suppose that \( n \geq 1 \) and suppose given a commutative diagram

\[
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{u} & X \\
\downarrow & & \downarrow^p \\
\Delta^n & \longrightarrow & S
\end{array}
\]

When \( n \geq 2 \) we may write

\[
\partial \Delta^n = \Delta^{(0)} \star \partial \Delta^{\{1,2,\ldots,n\}} \cup \Delta^{\{1,2,\ldots,n\}},
\]

from which it follows that a diagonal filler exists in (8) if and only if the indicated diagonal filler exists in the diagram

\[
\begin{array}{ccc}
\partial \Delta^{\{1,2,\ldots,n\}} & \longrightarrow & X_x/ \\
\downarrow & & \downarrow^p \\
\Delta^{\{1,2,\ldots,n\}} & \longrightarrow & S_x/ \times_S X
\end{array}
\]
where \( x = u(0) \) and where the right hand vertical map is the induced map. Since \( p: X \to S \) is an inner fibration the induced map \( X_{x/} \to S_{x/} \times_S X \) is a left fibration. Therefore it suffices to prove that the induced map has contractible fibers (Lemma 2.1.3.4 of [11]). Let \( y \) be a vertex of \( X \). It suffices to prove that the left fibration

\[
X_{x/} \times_X \{y\} \to S_{x/} \times_S \{y\}
\]

has contractible fibers. But this latter map is a trivial Kan fibration by the hypothesis that \( p: X \to S \) is a categorical equivalence, using Proposition 3.5 and the fact that \( X \) and \( S \) are pre-fibrant simplicial sets (see Remark 3.7).

Finally, we need to prove that there is a diagonal filler in (8) in the case when \( n = 1 \). In this case, write \( x = u(0) \) again and write \( y = u(1) \). Then the canonical map

\[
X_{x/} \times_X \{y\} \to S_{x/} \times_S \{y\}
\]

is a trivial Kan fibration and in particular is surjective on vertices. It follows that the edge \( \Delta^1 \to S \) can be lifted to an edge \( \Delta^1 \to X \), as required. \( \square \)

We can now give the proof of Theorem C from the introduction.

**Proof of Theorem C.** We have observed earlier that every inner anodyne map is a bijection on objects and a categorical equivalence. The converse statement follows from Theorem 5.10 and Proposition 5.9. \( \square \)

5.3. The inner model structure. Let \( \text{Set}_\Delta(O) \) denote the subcategory of \( \text{Set}_\Delta \) whose objects consist of the simplicial sets whose set of 0-simplices is equal to a fixed set \( O \). The morphisms in \( \text{Set}_\Delta(O) \) are the simplicial maps which induce the identity on the set \( O \).

Since limits and colimits are computed pointwise in \( \text{Set}_\Delta \), it follows that the inclusion \( \text{Set}_\Delta(O) \subseteq \text{Set}_\Delta \) creates all limits and colimits, and hence that \( \text{Set}_\Delta(O) \) is complete and cocomplete.

We will say that a map \( X \to Y \) in \( \text{Set}_\Delta(O) \) is an inner fibration, or inner anodyne, or a categorical equivalence, or a trivial Kan fibration, just in case the underlying map of simplicial sets is such a map.

We make the following observation.

**Lemma 5.11.** A map \( u: A \to B \) in \( \text{Set}_\Delta(O) \) is a monomorphism if and only it has the left lifting property with respect to all trivial Kan fibrations in \( \text{Set}_\Delta(O) \).

**Proof.** Observe that \( u: A \to B \) in \( \text{Set}_\Delta(O) \) is a monomorphism if and only if its image in \( \text{Set}_\Delta \) is a monomorphism, if and only if its image in \( \text{Set}_\Delta \) has the left lifting property with respect to all trivial fibrations in \( \text{Set}_\Delta \). Therefore, if \( u: A \to B \) is a monomorphism in \( \text{Set}_\Delta(O) \) then it has the left lifting property with respect to all trivial Kan fibrations in \( \text{Set}_\Delta(O) \). We need to prove that the converse statement is true.

Suppose given a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow^u & & \downarrow^p \\
B & \longrightarrow & S
\end{array}
\]

(9)
where \( p: X \to S \) is a trivial Kan fibration in \( \text{Set}_\Delta \) and where \( u: A \to B \) is a monomorphism in \( \text{Set}_\Delta(O) \). Define an object \( S_O \) of \( \text{Set}_\Delta(O) \) by the pullback diagram

\[
\begin{array}{c}
S_O \\
\downarrow \\
\cosk_0 O \\
\end{array} \quad \begin{array}{c}
\to \\
\downarrow \\
\cosk_0 S \\
\end{array}
\]

Define similarly an object \( X_O \) of \( \text{Set}_\Delta(O) \) and observe that \( p \) induces a map \( p_O: X_O \to S_O \) which is a trivial Kan fibration. We have a commutative diagram

\[
\begin{array}{ccc}
A & \to & X_O & \to & X \\
\downarrow u & & \downarrow p_O & & \downarrow p \\
B & \to & S_O & \to & S
\end{array}
\]

A choice of the indicated diagonal filler in (10) induces a diagonal filler in (9). The result follows.

In particular it follows from the lemma that a map in \( \text{Set}_\Delta(O) \) is a trivial Kan fibration if and only if it has the right lifting property against the set of maps in \( \text{Set}_\Delta(O) \) of the form

\[
O \cup \{0,\ldots,n\} \partial \Delta^n \to O \cup \{0,\ldots,n\} \Delta^n
\]

for some map \( \{0,\ldots,n\} \to O \), where \( n \geq 1 \).

A similar argument shows that the class of inner anodyne maps in \( \text{Set}_\Delta(O) \) is precisely the class of maps which belong to the weakly saturated class in \( \text{Set}_\Delta(O) \) generated by the maps in \( \text{Set}_\Delta(O) \) of the form

\[
O \cup \{0,\ldots,n\} \Lambda_i^n \to O \cup \{0,\ldots,n\} \Delta^n
\]

for some map \( \{0,\ldots,n\} \to O \), where \( 0 < i < n \).

We can now establish the inner model structure on \( \text{Set}_\Delta(O) \), proving Theorem D from Section I. We have the following theorem.

**Theorem 5.12.** There is the structure of a model category on \( \text{Set}_\Delta(O) \) in which

- the cofibrations are the monomorphisms in \( \text{Set}_\Delta(O) \);
- the weak equivalences are the categorical equivalences in \( \text{Set}_\Delta(O) \); and
- the fibrations are the inner fibrations in \( \text{Set}_\Delta(O) \).

The model structure is cofibrantly generated and left proper.

**Proof.** Let \( \mathcal{W} \) denote the class of categorical equivalences in \( \text{Set}_\Delta(O) \). Clearly \( \mathcal{W} \) satisfies the 2-out-of-3 property. Let \( \mathcal{C} \) denote the class of monomorphisms in \( \text{Set}_\Delta(O) \) and let \( \mathcal{F}_W \) denote the class of trivial Kan fibrations in \( \text{Set}_\Delta(O) \). Note that \( \mathcal{F}_W \subseteq \mathcal{F} \cap \mathcal{W} \). By the small object argument, the pair \( (\mathcal{C}, \mathcal{F}_W) \) forms a weak factorization system on \( \text{Set}_\Delta(O) \).

Let \( \mathcal{F} \) denote the class of inner fibrations in \( \text{Set}_\Delta(O) \), and let \( \mathcal{C}_W \) denote the class of inner anodyne maps in \( \text{Set}_\Delta(O) \). Note that \( \mathcal{C}_W \subseteq \mathcal{C} \cap \mathcal{W} \). Again, by the small object argument the pair \( (\mathcal{C}_W, \mathcal{F}) \) forms a weak factorization system on \( \text{Set}_\Delta(O) \). It suffices (see Proposition
E.11 of [8]) to prove the reverse inclusion $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W$. This follows immediately from Theorem 5.10. □

6. The Joyal model structure

In this section we prove Theorem [A] and Theorem [B] and prove the existence of the Joyal model structure (see Theorem 6.6 below).

6.1. Generating acyclic cofibrations. In this section we will define a set of generating acyclic cofibrations for the Joyal model structure on $\text{Set}_\Delta$. This set will turn out to be analogous to the set of generating acyclic cofibrations for the Bergner model structure on $\text{Cat}_\Delta$.

**Notation 6.1.** Denote by $\mathcal{J}$ a chosen set of representatives for the isomorphism classes of simplicial sets $B$ with two vertices 0 and 1, with countably many non-degenerate simplices and such that the inclusion $\{0\} \hookrightarrow B$ is a categorical equivalences.

In the next section we will prove that a set of generating acyclic cofibrations for the Joyal model structure is given by the set $\mathcal{A}$, which is equal to the union of the set $\mathcal{J}$ and the set of inner anodyne inclusions $\Lambda_i^n \subseteq \Delta^n$, $0 < i < n$.

In this section, we characterize the maps in $\text{Set}_\Delta$ which have the right lifting property with respect to the maps in $\mathcal{A}$. More precisely, we will prove the following proposition.

**Proposition 6.2.** A map $p: X \rightarrow S$ in $\text{Set}_\Delta$ has the right lifting property with respect to the maps in the set $\mathcal{A}$ if and only if it is an inner fibration such that the induced functor $h(p): h(X) \rightarrow h(S)$ is an isofibration.

Proposition 6.2 will follow immediately from Propositions 6.4 and 6.5 below, which are the direct analogs for $\text{Set}_\Delta$ of Proposition 2.3 and Proposition 2.5 from [1]. We shall see that, with some work, the arguments used in [1] to establish these propositions can be adapted to the setting of simplicial sets.

We shall need the following analog of Lemma 2.4 from [1]. It is closely related to the bounded cofibration property introduced in [6].

**Lemma 6.3.** Let $S$ be a pre-fibrant simplicial set with two vertices $x$ and $y$ and suppose that $f: x \rightarrow y$ is an equivalence in $S$. Then the map $\Delta^1 \rightarrow S$ classifying the edge $f$ factors as $\Delta^1 \rightarrow B \rightarrow S$ in such a way that the composite $\{0\} \rightarrow B$ belongs to the set $\mathcal{J}$.

The proof of Lemma 6.3 is somewhat complicated and its proof is deferred until Section 9.

To begin with, we prove the following version of Proposition 2.3 from [1], adapting the proof of that proposition from op. cit. to the present context.

**Proposition 6.4.** Suppose that $p: X \rightarrow S$ is an inner fibration of simplicial sets which has the right lifting property against the set of maps $\{0\} \rightarrow B$ in $\mathcal{J}$. Then $h(f): h(X) \rightarrow h(S)$ is an isofibration.
Proof. By Proposition 4.12 there exists a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{p} & & \downarrow{q} \\
S & \xrightarrow{j} & T
\end{array}
\]

in which the horizontal arrows are inner anodyne maps, \( q \) is an inner fibration, and where \( T \) is a pre-fibrant simplicial set. It follows that without loss of generality we may suppose that \( S \) and \( X \) are pre-fibrant simplicial sets.

Suppose that \( x \in X \) and that \( f: s \to t \) is an equivalence in \( S \), where \( s = p(x) \). We will show that there is an equivalence \( u: x \to y \) in \( X \) such that \( p(u) = f \). Suppose to begin with that \( s \neq t \). Let \( \Delta^1 \to S \) classify the edge \( f \). Write \( S' \) for the full simplicial subset of \( S \) spanned by \( s \) and \( t \). We claim that \( f \) restricts to an equivalence in \( S' \). To see this, it suffices to prove that \( h(S') \subseteq h(S) \) is a full subcategory. To see this, observe that by Lemma 3.11 the pullback diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & \cosk_0 \{ s, t \} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \cosk_0 S
\end{array}
\]

of pre-fibrant simplicial sets shows that the induced diagram

\[
\begin{array}{ccc}
h(S') & \longrightarrow & h(\cosk_0 \{ s, t \}) \\
\downarrow & & \downarrow \\
h(S) & \longrightarrow & h(\cosk_0 S)
\end{array}
\]

is also a pullback, which immediately implies the claim. Using Lemma 6.3 we may factor the induced map \( \Delta^1 \to S' \) as \( \Delta^1 \to B \to S' \) in such a way that the map \( \{ 0 \} \to B \) belongs to \( J \).

By hypothesis we can find the indicated diagonal filler in the diagram

\[
\begin{array}{ccc}
\{ 0 \} & \longrightarrow & X \\
\downarrow & & \downarrow{p} \\
B & \longrightarrow & S'
\end{array}
\]

Pre-composing the map \( B \to X \) with the map \( \Delta^1 \to B \) defines an edge \( u: \Delta^1 \to X \) which lifts \( f \) and is an equivalence.

Suppose now that \( s = t \). Let \( C \) denote the category with two objects \( s \) and \( s' \) and in which every set of morphisms in \( C \) is equal to the set \( h(S)(s, s) \). Composition in \( C \) is given by composition in \( h(S) \). There is a canonical functor \( C \to h(S) \) which maps every object
to $s$. Define a simplicial set $S'$ by the pullback diagram

$$
\begin{array}{ccc}
S' & \longrightarrow & S \\
\downarrow & & \downarrow \\
C & \longrightarrow & h(S)
\end{array}
$$

Since $S$ is pre-fibrant and $C \to h(S)$ is an inner fibration it follows that $S'$ is pre-fibrant. Moreover $h(S') = C$ by Lemma 3.11. The edge $f: \Delta^1 \to S$ induces an edge $\tilde{f}: \Delta^1 \to S'$ which is clearly an equivalence (its image in $C$ is an isomorphism). The argument then proceeds as in the previous case. □

Next, we prove the following version of Proposition 2.5 from [1]. Again, the proof is an adaptation of the arguments used in [1] to the present context.

**Proposition 6.5.** Suppose that $p: X \to S$ is an inner fibration between simplicial sets such that the induced functor $h(X) \to h(S)$ is an isofibration. Then $p$ has the right lifting property with respect to the maps in the set $\mathcal{J}$.

**Proof.** We need to show that the indicated diagonal filler exists in any diagram of the form

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & X \\
\downarrow & & \downarrow p \\
B & \longrightarrow & S
\end{array}
$$

where $\{0\} \to B$ is a map in $\mathcal{J}$. Proposition 4.12 shows that without loss of generality we may suppose $S$, and hence $X$, are pre-fibrant simplicial sets.

Using Remark 3.4 and the small object argument, we see that we may choose an inner anodyne map $B \to B'$, where $B'$ is a pre-fibrant simplicial set, such that the map $B \to S$ factors through $B'$. Note that $\{0\} \to B'$ is a categorical equivalence and that $B'$ can be chosen to have countably many non-degenerate simplices. It follows that without loss of generality we may assume at the outset that $B$ is a pre-fibrant simplicial set.

Let $g: 0 \to 1$ be an equivalence in $B$ and write $g: \Delta^1 \to B$ for the simplex classifying $g$. Under the map $B \to S$, $g$ is mapped to an equivalence in $S$. Since the functor $h(X) \to h(S)$ is an isofibration, and $X$ and $S$ are pre-fibrant simplicial sets, we can use Lemma 3.9 to find the indicated map making the following diagram commute:
We need to show that we can extend \(f: \Delta^1 \to X\) to a map \(B \to X\) making the diagram \((\text{11})\) commute. Form a pullback diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow^p \\
B & \longrightarrow & S
\end{array}
\]

and let \(\tilde{f}: \Delta^1 \to Y\) denote the map induced by \(f: \Delta^1 \to X\). It follows from Lemma 3.11 that \(\tilde{f}\) is an equivalence in \(Y\).

Let \(Z\) denote the full simplicial subset of \(Y\) spanned by the objects \(\tilde{f}(0)\) and \(\tilde{f}(1)\); clearly \(\tilde{f}(0) \neq \tilde{f}(1)\). We regard \(Z\) as an object of \(\text{Set}_\Delta(\{0,1\})\). Since \(\tilde{f}\) is an equivalence in \(Y\), and \(Z \subseteq Y\) is a full simplicial subset, it follows that \(\tilde{f}\) is an equivalence in \(Z\).

Observe that \(Z \to B\) is an inner fibration in \(\text{Set}_\Delta(\{0,1\})\), since it is the composite of the inner fibrations \(Z \to Y\) and \(Y \to B\). In particular it follows that \(Z\) is a pre-fibrant simplicial set. Lemma 6.3 implies that the map \(\tilde{f}: \Delta^1 \to Z\) factorizes as \(\Delta^1 \to B' \xrightarrow{\alpha} Z\), in such a way that the map \(\{0\} \to B'\) belongs to \(\beta\). Since both maps \(\{0\} \to B\) and \(\{0\} \to B'\) are categorical equivalences, a 2-out-of-3 argument shows that the composite map \(\Delta^1 \to Z\) is a categorical equivalence.

We have a commutative diagram

\[
\begin{array}{ccc}
\Delta^1 & \longrightarrow & \Delta^1 \\
\downarrow & \quad & \downarrow \\
\{0\} & \longrightarrow & \text{Set}_\Delta(\{0,1\})
\end{array}
\]

in \(\text{Set}_\Delta(\{0,1\})\). Lemma 2.6 from \([1]\), applied to the inner model structure on \(\text{Set}_\Delta(\{0,1\})\) (Theorem 5.12), shows that the indicated diagonal filler \(\beta: B \to Z\) exists in the diagram \((\text{12})\). Composing \(\beta\) with the canonical map \(Z \to Y\) defines a diagonal filler for the diagram \((\text{11})\). \(\square\)

6.2. The Joyal model structure. In this section we give a proof of the existence of the Joyal model structure on \(\text{Set}_\Delta\). As immediate outcomes of our proof we obtain Theorem A and Theorem B.

Recall that the Joyal model structure is uniquely determined by the fact that its fibrant objects are the \(\infty\)-categories and its cofibrations are the monomorphisms (see \([3]\)). Recall also that the fibrations in the Joyal model structure are called categorical fibrations in \([11]\).

**Theorem 6.6** (Joyal/Lurie). There is the structure of a model category on \(\text{Set}_\Delta\) for which

- the cofibrations are the monomorphisms,
- the fibrant objects are the \(\infty\)-categories,
- the weak equivalences are the categorical equivalences.

The model structure is cofibrantly generated and left proper.

**Proof.** The class \(W\) of categorical equivalences in \(\text{Set}_\Delta\) satisfies the 2-out-of-3 property. Let \(\mathcal{C}_W\) denote the class of monomorphisms generated by the set of maps \(\mathcal{A}\) and let \(\mathcal{F}\) denote...
the class of maps which have the right lifting property against the maps in \( A \). The small object argument gives a weak factorization system \(( \mathcal{C}_W, \mathcal{F} \))

Let \( \mathcal{F}_W \) denote the class of trivial Kan fibrations and let \( \mathcal{C} \) denote the class of monomorphisms in \( \text{Set}_\Delta \). The small object argument gives a weak factorization system \(( \mathcal{C}, \mathcal{F}_W \))

Clearly we have \( \mathcal{C}_W \subseteq \mathcal{C} \cap \mathcal{W} \) and \( \mathcal{F}_W \subseteq \mathcal{F} \cap \mathcal{W} \). To prove that the classes \( \mathcal{C}, \mathcal{F} \) and \( \mathcal{W} \) determine a model structure it suffices (see Proposition E.1.11 of [8]) to prove that \( \mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W \).

Suppose that \( p: X \to S \) belongs to the class \( \mathcal{F} \cap \mathcal{W} \). Then \( p \) is an inner fibration such that \( h(p): h(X) \to h(S) \) is an isofibration (Proposition 6.2), and \( p \) is a categorical equivalence. There exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{p} & & \downarrow{q} \\
S & \xrightarrow{v} & T
\end{array}
\]

in which \( u \) and \( v \) are inner anodyne, \( q: Y \to T \) is an inner fibration and \( T \) is an \( \infty \)-category. It follows that \( q \) is a categorical fibration. Since \( q \) is also a categorical equivalence it follows that \( q \) is a trivial Kan fibration (Lemma 2.21). Therefore the horizontal map

\[
\begin{array}{ccc}
X & \to & S \times_T Y \\
\downarrow{p} & & \leftarrow \downarrow{q} \\
S & &
\end{array}
\]

in the induced diagram is a categorical equivalence. Since it is also a monomorphism which is bijective on objects it must be inner anodyne (Theorem C). Therefore, since \( p: X \to S \) is an inner fibration, \( p \) is a retract of the trivial Kan fibration \( S \times_T Y \to S \) and hence is itself a trivial Kan fibration.

To complete the proof we need to show that the fibrant objects for this model structure are the \( \infty \)-categories. It is clear that every fibrant object is an \( \infty \)-category. We must prove that if \( \mathcal{C} \) is an \( \infty \)-category then the canonical map \( \mathcal{C} \to \Delta^0 \) has the right lifting property with respect to every map \( \{0\} \to B \) in the set \( \mathcal{J} \). This is clear however, since we can just retract \( B \) onto \( \{0\} \).

As immediate corollaries of our proof of Theorem 6.6 we obtain Theorem A and Theorem B.

**Proof of Theorem A** From the proof of Theorem 6.6 we see that a map is a categorical fibration if and only if it has the right lifting property with respect to the maps belonging to the set \( A \). The result then follows from Proposition 6.2.

**Proof of Theorem B**. This follows immediately from the fact established in the proof above that the class of acyclic cofibrations in the Joyal model structure on \( \text{Set}_\Delta \) is equal to the class \( \mathcal{C}_W \).

**Remark 6.7**. In addition to the results from this paper, the construction of the Joyal model structure that we have given here depends on the results proven in the first 13 pages
of [7] (the key result being Theorem 1.3 of that paper), Corollary 2.3.6.5 of [11], and the fact that a left fibration with contractible fibers is a trivial Kan fibration (Lemma 2.1.3.4 of [11]).

7. Technical lemmas on inner anodyne maps

In this section we collect together some technical lemmas on inner anodyne maps that we use at various points in the paper. A key ingredient in the proof of these lemmas is the right cancellation property for inner anodyne maps (Lemma 5.4).

The following lemma is straightforward. We shall need it for the proof of Proposition 4.11.

**Lemma 7.1.** If $u: A \to B$ is an inner anodyne map between simplicial sets, then the induced map $u \star \text{id}_C: A \star C \to B \star C$ is inner anodyne for any simplicial set $C$.

**Proof.** In the pushout diagram

$$
\begin{array}{ccc}
A & \to & A \star C \\
\downarrow^u & & \downarrow \\
B & \to & A \star C \cup B
\end{array}
$$

both vertical maps are inner anodyne. The canonical map $A \star C \cup B \to B \star C$ is inner anodyne by Lemma 2.1.2.3 of [11]. Therefore the composite map

$$A \star C \to A \star C \cup B \to B \star C$$

is inner anodyne. □

The next lemma from [13] is a straightforward application of the right cancellation property.

**Lemma 7.2 ([13]).** Suppose given a commutative diagram of simplicial sets of the form

$$
\begin{array}{ccc}
A_1 & \leftarrow & A_0 & \to & A_2 \\
\downarrow & & \downarrow & & \downarrow \\
B_1 & \leftarrow & B_0 & \to & B_2
\end{array}
$$

in which at least one of the squares is a pullback. If all of the vertical arrows are inner anodyne then the induced map

$$A_1 \cup_{A_0} A_2 \to B_1 \cup_{B_0} B_2$$

is also inner anodyne.

We have following refinement of Lemma 5.4.5.10 from [11].

**Lemma 7.3.** Suppose that $A$, $B$ and $C$ are simplicial sets and that $B$ is weakly contractible. Then the canonical map

$$A \star B \cup B \star C \subseteq A \star B \star C$$

is inner anodyne.
Proof. One can check that the steps (1) – (7) of Lurie’s proof of Lemma 5.4.5.10 go through with monic categorical equivalences replaced by inner anodyne maps using Lemma 7.2 and the right cancellation property for inner anodyne maps (Lemma 5.4). □

We shall make use of the following generalization of Lemma 7.3 in the proof of Proposition 8.

Lemma 7.4. Let $A_0, \ldots, A_n$ be simplicial sets where $n \geq 2$. Suppose that $0 < i < n$ and that each of the simplicial sets $A_1, \ldots, A_{n-1}$ is weakly contractible. Then the inclusion

\[ \bigcup_{0 \leq j \leq n, j \neq i} A_0 \cdots \hat{A}_j \cdots A_n \to A_0 \cdots A_n \]

is inner anodyne, where the hat indicates omission.

Proof. The proof is by induction on $n$. Lemma 7.3 gives the statement when $n = 2$. Let $0 < i < n$ where $n \geq 3$. Without loss of generality we may suppose by a duality argument that $i < n - 1$. We have a pushout diagram of the form

\[
\begin{array}{ccc}
\bigcup_{0 \leq j \leq n-1, j \neq i} A_0 \cdots \hat{A}_j \cdots A_{n-1} & \to & \bigcup_{0 \leq j \leq n-1, j \neq i} A_0 \cdots \hat{A}_j \cdots A_n \\
\downarrow & & \downarrow \\
A_0 \cdots A_{n-1} & \to & \bigcup_{0 \leq j \leq n, j \neq i} A_0 \cdots \hat{A}_j \cdots A_n
\end{array}
\]

The left hand vertical map is inner anodyne by the inductive hypothesis and hence so is the right hand vertical map. The map

\[ \bigcup_{0 \leq j \leq n-1, j \neq i} A_0 \cdots \hat{A}_j \cdots A_n \to (A_0 \cdots A_{n-1}) \ast A_n \]

is inner anodyne by Lemma 7.1. The result then follows by the right cancellation property of inner anodyne maps (Lemma 5.4). □

Remark 7.5. Observe that if one of the simplicial sets $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n-1}$ in the statement of Lemma 7.4 is empty then the inclusion (13) is an equality. Therefore the hypotheses of Lemma 7.4 can be relaxed to cover the cases where each of the simplicial sets $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n-1}$ are either empty or weakly contractible and $A_i$ is weakly contractible.

Corollary 7.6. Let $A_0, \ldots, A_n$ be simplicial sets where $n \geq 5$. If each of $A_1, \ldots, A_{n-1}$ is weakly contractible then for any $i + 1 < j < n$ the inclusion

\[ \bigcup_{k=0}^{n} A_0 \cdots \hat{A}_k \cdots A_n \subseteq A_0 \cdots A_n \]

is inner anodyne.
Proof. This follows easily from Lemma 7.4 and Lemma 2.1.2.3 of \[11\].

Remark 7.7. If one of the simplicial sets \(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n-1}\) in the statement of Corollary 7.6 is empty then the inclusion (14) is an isomorphism. Therefore the hypotheses of Corollary 7.6 can be relaxed to cover the cases where each of the simplicial sets \(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n-1}\) are either empty or weakly contractible and \(A_i\) is weakly contractible.

Finally, we have the following lemma which we have made use of above in the proof of Proposition 3.5.

Lemma 7.8. Suppose given a commutative diagram in \(\text{Set}_\Delta\) of the form

\[
\begin{array}{ccc}
A_0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & \cdots
\end{array}
\]

in which the horizontal arrows are inclusions. If the map \(A_n \rightarrow B_n\) is inner anodyne for all \(n \geq 0\), then the induced map

\[
\lim_n A_n \rightarrow \lim_n B_n
\]

is inner anodyne.

Proof. A standard argument shows that it suffices to prove that the canonical map

\[
A_{n+1} \cup B_n \rightarrow B_{n+1}
\]

induced from the commutative diagram

\[
\begin{array}{ccc}
A_n & \rightarrow & A_{n+1} \\
\downarrow & & \downarrow \\
B_n & \rightarrow & B_{n+1}
\end{array}
\]

is inner anodyne. But this is a straightforward consequence of the right cancellation property (Lemma 5.4) of the class of inner anodyne maps, using the fact that the vertical maps in the diagram above are inner anodyne.

8. Proof of Proposition 4.11

In this section we prove Proposition 4.11. For the proof we will need the following straightforward lemma (in fact, for the application we have in mind — see the proof of Proposition 4.12 — we could use an ad-hoc argument which does not refer to cocartesian edges).
Lemma 8.1. Let $p: X \to S$ be an inner fibration between simplicial sets. Suppose given a commutative diagram of the form

$$
\begin{array}{ccc}
\Delta \{0,2,\ldots,n\} \cup \Delta \{0,1\} & \xrightarrow{u} & X \\
\downarrow & & \downarrow p \\
\Delta \{0,\ldots,n\} & \xrightarrow{p} & S
\end{array}
$$

If the edge $u|\Delta \{0,1\}$ is $p$-cocartesian then the indicated diagonal filler exists.

Proof. If $u|\Delta \{0,1\}$ is $p$-cocartesian, then we can find the indicated diagonal filler in the diagram

$$
\begin{array}{ccc}
\Delta \{0,2,\ldots,n\} \cup \Delta \{0,1\} & \xrightarrow{u} & X \\
\downarrow & & \downarrow p \\
\Delta \{0,2,\ldots,n\} \cup \Delta \{0,1,2\} & \xrightarrow{p} & S
\end{array}
$$

A straightforward argument shows that the map $\Delta \{0,2,\ldots,n\} \cup \Delta \{0,1,2\} \to \Delta \{0,\ldots,n\}$ is inner anodyne. This suffices to complete the proof. \qed

We now prove Proposition 4.11. As noted earlier, this proposition is a generalization of Lemma 3.1.2.4 from [12]. We recall the statement.

Proposition. Let $p: \mathcal{C} \to \Lambda^n_0$ be an inner fibration where $n \geq 3$. Let $q$ denote the projection $q: \Delta \{1,\ldots,n\} \times \Lambda^n_0 \to \Delta \{1,\ldots,n\}$. Suppose that the following conditions are satisfied:

(i) for every vertex $x \in \mathcal{C}_j$, $1 \leq j \leq n-2$, there exists a $q$-cocartesian morphism $f: x \to y$ where $y \in \mathcal{C}_{j+1}$;

(ii) for every vertex $y \in \mathcal{C}_j$, $2 \leq j \leq n-1$ there exists a $q$-cartesian morphism $f: x \to y$ where $x \in \mathcal{C}_{j-1}$.

Then $p: \mathcal{C} \to \Lambda^n_0$ satisfies inner anodyne descent with respect to the inclusion $\Lambda^n_0 \subseteq \Delta^n$.

Proof. The proof that we give is an adaptation of Lurie’s proof of Lemma 3.1.2.4 of [12]. The objective is to construct a sequence

$$
\mathcal{C} = \mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \mathcal{C}(2) \subseteq \cdots
$$

in $(\text{Set}_\Delta)/\Delta^n$ where each inclusion $\mathcal{C}(m) \subseteq \mathcal{C}(m+1)$ is inner anodyne and where the following conditions are satisfied:

(i) if $\Lambda^n_0 \to \mathcal{C}(m)$ is a map in $(\text{Set}_\Delta)/\Delta^n$ with $0 < k < r$, then the composite map $\Lambda^n_0 \to \mathcal{C}(m) \to \mathcal{C}(m+1)$ factors through the inclusion $\Lambda^n_k \subseteq \Delta^n$ in $(\text{Set}_\Delta)/\Delta^n$;
(ii) for each \( m \geq 0 \) the diagram
\[
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}(m) \\
\downarrow p & & \downarrow \\
\Lambda_i^n & \longrightarrow & \Delta^n
\end{array}
\]
is cartesian.

The construction proceeds by induction on \( m \geq 0 \). Assuming that \( \mathcal{C}(m) \) has been defined, let \( A \) denote the set of maps \( \alpha: \Lambda_k^r \to \mathcal{C}(m) \) in \((\text{Set}_\Delta)_{/\Delta^n}\) with \( 0 < k < r \). It suffices to construct, for each map \( \alpha \) in \( A \), a simplicial set \( \mathcal{C}(m, \alpha) \) in \((\text{Set}_\Delta)_{/\Delta^n}\) with the following properties:

(iii) the inclusion \( \mathcal{C}(m) \subseteq \mathcal{C}(m, \alpha) \) is inner anodyne;

(iv) the composite map \( \Lambda_k^r \overset{\alpha}{\to} \mathcal{C}(m) \subseteq \mathcal{C}(m, \alpha) \) can be extended to an \( r \)-simplex of \( \mathcal{C}(m, \alpha) \);

(v) the diagram
\[
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}(m, \alpha) \\
\downarrow p & & \downarrow \\
\Lambda_i^n & \longrightarrow & \Delta^n
\end{array}
\]
is cartesian.

Assuming that the simplicial sets \( \mathcal{C}(m, \alpha) \) have been constructed satisfying the properties (iii) – (v) above, we set \( \mathcal{C}(m+1) \) to be the coproduct \( \sqcup_{\alpha \in A} \mathcal{C}(m, \alpha) \) in \((\text{Set}_\Delta)_{/\Delta^n}\). It is not hard to show that the canonical map \( \mathcal{C}(m) \to \mathcal{C}(m+1) \) is inner anodyne and that the properties (i) and (ii) above are satisfied.

Suppose given a map \( \alpha: \Lambda_k^r \to \mathcal{C}(m) \) in \( A \); let \( \alpha_0: \Lambda_k^r \to \Delta^n \) denote the composite map. Without loss of generality we may suppose that the image of \( \alpha_0 \) contains \( \partial_i \Delta^n = \Delta\{0,\ldots,i-1,i+1,\ldots,n\} \subseteq \Delta^n \). Note that \( \alpha_0(0) = 0 \) and \( \alpha_0(r) = n \). Let \( \tilde{\alpha}_0: \Delta^r \to \Delta^n \) denote the unique extension of \( \alpha_0 \). Without loss of generality we may suppose that the canonical map
\[
\Lambda_k^r \times_{\Delta^r} \Lambda_i^n \hookrightarrow \Delta^r \times_{\Delta^r} \Lambda_i^n
\]
is not surjective (if the canonical map is surjective then we may take \( \mathcal{C}(m, \alpha) = \mathcal{C}(m) \cup \Lambda_k^r \Delta^r \) and check that the properties (i) and (ii) above are satisfied).

Therefore, the simplex \( \partial_k \Delta^r = \Delta\{0,\ldots,k-1,k+1,\ldots,r\} \) is mapped into \( \Lambda_i^n \) by \( \tilde{\alpha}_0 \). Since \( \tilde{\alpha}_0(0) = 0 \) and \( \tilde{\alpha}_0(r) = n \), the simplex \( \partial_k \Delta^r \) must be mapped into a simplex \( \partial_j \Delta^n = \Delta\{0,\ldots,j-1,j+1,\ldots,n\} \) of \( \Lambda_i^n \) by \( \tilde{\alpha}_0 \). Note that \( 0 < j < n \) and \( j \neq i \). Under the given hypotheses we may suppose, by a duality argument, that without loss of generality \( j \leq n-2 \). Note also that we must have \( \tilde{\alpha}_0^{-1}\{j\} = \{k\} \), since otherwise the vertex \( j \) belongs to the image of \( \partial_k \Delta^r \) under \( \tilde{\alpha}_0 \) (recall that the image of \( \tilde{\alpha}_0 \) contains \( \partial_i \Delta^n \)).

Write \( A_l = \tilde{\alpha}_0^{-1}\{l\} \) for each \( l = 0, \ldots, n \). Note that \( A_j = \{x\} \), where \( x \) denotes the \( k \)-th vertex of \( \Delta^r \). It follows that we have a canonical isomorphism
\[
\Delta^r = A_0 \star \cdots \star A_n.
\]
Note that $A_0$ and $A_n$ are non-empty and that each $A_i$ (if non-empty) is isomorphic to a simplex. It is convenient to introduce the following shorthand notation: if $\{i_1, \ldots, i_p\}$ is a subset of $\{0, \ldots, n\}$ then we write

$$A_{i_1, \ldots, i_p} = A_{i_1} \ast \cdots \ast A_{i_p}.$$  

Let $X$ denote the image of the vertex $x$ under the map $\alpha$, and choose a $q$-cocartesian arrow $f: X \to Y$ in $C$, where $Y \in C_{j+1}$. Since $f$ is $q$-cocartesian, Lemma 8.1 shows that we can find the indicated diagonal filler $\beta$ making the diagram

$$\begin{array}{ccc}
\{x\} \ast \{y\} & \rightarrow & C \\
\downarrow & \searrow \beta & \downarrow p \\
\{x\} \ast \{y\} \ast A_{j+1, \ldots, n} & \rightarrow & \Lambda^n_{j+1}
\end{array}$$

commute, where the upper horizontal map is induced by $f$ and $\alpha$.

Let $K_3 = A_0 \ast \cdots \ast A_j \ast \{y\} \ast A_{j+1} \ast \cdots \ast A_n$ so that $K_3$ is isomorphic to $\Delta^{r+1}$. By an abuse of notation let us denote by $\overline{\alpha}_0: K_3 \to \Delta^n$ the canonical extension of $\overline{\alpha}_0: \Delta^r \to \Delta^n$ with $\overline{\alpha}_0(y) = j + 1$.

Our aim is to construct (following Lurie’s proof of Lemma 3.1.2.4 in [12]) a sequence of simplicial sets

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3$$

where each inclusion is a categorical equivalence, together with maps $K_i \to C(m)$ for $i = 0, 1, 2$, compatible with the inclusions and with the maps $\alpha$ and $\beta$. We will then set

$$C(m, \alpha) = C(m) \cup_{K_3} K_3$$

and show that $C(m, \alpha)$ has the desired properties.

To begin with, we define the subcomplex $K_0$ of $K_3$ to be equal to the inverse image of $\Lambda^n_0 \cap \Lambda^n_1 \cap \Lambda^n_{j+1}$ in $K_3$ under the map $\overline{\alpha}_0$. In other words

$$K_0 = \bigcup_{l=0, l \neq i}^{j-1} A_{0, \ldots, \hat{l}, \ldots, j} \ast \{y\} \ast A_{j+1, \ldots, n} \cup \bigcup_{l=j+2, l \neq i}^n A_{0, \ldots, j} \ast \{y\} \ast A_{j+1, \ldots, \hat{l}, \ldots, n}$$

where the hats indicate omission. Next, we show that there is a map $\phi: K_0 \to C$ which is compatible with $\alpha$ and $\beta$. Define a sequence of inclusions

$$K_0^{(0)} \subseteq K_0^{(1)} \subseteq \cdots \subseteq K_0^{(j-1)} = K_0^{(j)} = K_0^{(j+1)} \subseteq \cdots \subseteq K_0^{(n)} = K_0$$

by setting

$$K_0^{(t)} = \bigcup_{l=0, l \neq i}^{t} A_{0, \ldots, \hat{l}, \ldots, j} \ast \{y\} \ast A_{j+1, \ldots, n}$$

for $0 \leq t \leq j - 1$ and

$$K_0^{(t)} = \bigcup_{l=j+2, l \neq i}^{t} A_{0, \ldots, j} \ast \{y\} \ast A_{j+1, \ldots, \hat{l}, \ldots, n}$$

for $0 \leq t \leq j$.
for $j + 2 \leq t \leq n$.

For each $t = 0, \ldots, n$ we will construct a compatible sequence of maps $\phi_t: K_0^{(t)} \to \mathcal{C}(m)$, each of which is compatible with $\alpha$ and $\beta$. The construction is by induction on $t$.

For the case $t = 0$, we observe that the inclusion $A_j \star \{ y \} \star A_{j+1, \ldots, n} \subseteq A_{1, \ldots, j} \star \{ y \} \star A_{j+1, \ldots, n} = K_0^{(0)}$ is inner anodyne, since the inclusion $A_j \subseteq A_{1, \ldots, j}$ is right anodyne (Lemma 2.1.2.3 of [11]). Therefore, since $p: \mathcal{C} \to \Lambda^i_n$ is an inner fibration, we can choose the indicated diagonal filler $\phi_0$ making the diagram

$$
\begin{array}{ccc}
A_j \star \{ y \} \star A_{j+1, \ldots, n} & \subseteq & A_{1, \ldots, j} \star \{ y \} \star A_{j+1, \ldots, n} \\
\downarrow & & \downarrow \\
K_0^{(0)} & \xrightarrow{\phi_0} & \Lambda^i_n
\end{array}
$$

commute, where the upper horizontal map is induced by $\alpha$ and $\beta$.

Suppose that a map $\phi_t: K_0^{(t)} \to \mathcal{C}$ has been constructed, whose restriction to $K_0^{(r)}$ is equal to $\phi_r$ for all $r \leq t$. We construct a map $\phi_{t+1}: K_0^{(t+1)} \to \mathcal{C}$ whose restriction to $K_0^{(t)}$ is equal to $\phi_t$.

If $t < j$ then the inclusion $K_0^{(t)} \cup A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star A_{j+1, \ldots, n} \subseteq K_0^{(t+1)}$ is obtained as the pushout

$$
\begin{array}{ccc}
K_0^{(t)} \cup A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star A_{j+1, \ldots, n} & \subseteq & K_0^{(t+1)} \\
\downarrow & & \\
A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star \{ y \} \star A_{j+1, \ldots, n} & \subseteq & K_0^{(t+1)}
\end{array}
$$

An application of Lemma 7.4 combined with Lemma 7.1 shows that the left hand vertical map is inner anodyne, and hence that the inclusion $K_0^{(t)} \cup A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star A_{j+1, \ldots, n} \subseteq K_0^{(t+1)}$ is inner anodyne. Therefore, the indicated diagonal filler $\phi_{t+1}$ exists in the diagram

$$
\begin{array}{ccc}
K_0^{(t)} \cup A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star A_{j+1, \ldots, n} & \subseteq & \mathcal{C} \\
\downarrow & & \downarrow \\
K_0^{(t+1)} & \xrightarrow{\phi_{t+1}} & \Lambda^i_n
\end{array}
$$

where the upper horizontal map is induced by $\phi_t$ and $\alpha$. If $t > j + 1$ then the argument is analogous, except that either Lemma 7.4 or Corollary 7.6 is combined with Lemma 7.1 to show that the inclusion $K_0^{(t)} \cup A_{0, \ldots, \overleftarrow{t+1}, \ldots, j} \star A_{j+1, \ldots, \overrightarrow{t+1}, \ldots, n} \subseteq K_0^{(t+1)}$ is inner anodyne. This completes the inductive step, and hence the construction of the map $\phi: K_0 \to \mathcal{C}$.

We now construct a subcomplex $K_1 \subseteq K_3$ by

$$K_1 = K_0 \cup \Lambda^i_k.$$
Hence $K_1$ forms part of a pushout diagram

$$
\begin{array}{c}
K_0 \cap \Lambda^*_k \\
\downarrow \\
\Lambda^*_k
\end{array}
\begin{array}{c}
a \hookrightarrow K_0 \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
K_1 \cap \Lambda^*_k \\
\downarrow \\
\Lambda^*_k
\end{array}
$$

The inclusion $K_0 \cap \Lambda^*_k \subseteq \Delta^r$ is equal to the map

$$
\bigcup_{i=0}^n A_{0,\ldots,i,\ldots,n} \subseteq \Delta^r.
$$

Since $A_j \star A_{j+1}$ is weakly contractible it follows from Lemma 7.4 that $K_0 \cap \Lambda^*_k \subseteq \Delta^r$ is inner anodyne. Therefore, $K_0 \cap \Lambda^*_k \subseteq \Lambda^*_k$ is a categorical equivalence, by the 2-out-of-3 property of categorical equivalences (Remark 2.7). It follows (Lemma 2.25) that the inclusion $K_0 \subseteq K_1$ is a categorical equivalence. Let $\psi: K_1 \to C(m)$ denote the unique map extending $\alpha: \Lambda^*_k \to C(m)$ and $\phi: K_0 \to C$.

Finally, we construct the subcomplex $K_2 \subseteq K_3$ by setting

$$
K_2 = K_1 \cup A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n}.
$$

The inclusion $K_1 \subseteq K_2$ is a pushout of the inclusion

$$
K_0 \cap A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n} \subseteq A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n}
$$

since $A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n} \cap \Lambda^*_k = \emptyset$ (recall that $\Lambda^*_k$ is the union of the faces of $\Delta^r$ which contain the vertex $x$). An argument analogous to the argument used above to show that $K_0 \cap \Lambda^*_k \subseteq \Delta^r$ is inner anodyne shows that the inclusion $K_0 \cap A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n} \subseteq A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n}$ is inner anodyne. Since $p: C \to \Lambda^n_i$ is an inner fibration, the restriction of $\phi$ extends along the inclusion $K_0 \cap A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n} \subseteq A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n}$ to define a map

$$
\chi: A_{0,\ldots,j-1} \star \{y\} \star A_{j+1,\ldots,n} \to C.
$$

There is a unique map $K_2 \to C(m)$ extending the map $\psi$ and the map $\chi$. We define

$$
C(m, \alpha) = C(m) \cup_{K_2} K_3
$$

An easy 2-out-of-3 argument shows that the inclusion $K_2 \subseteq K_3$ is a categorical equivalence; since $K_3$ is an $\infty$-category it follows by Lemma 5.7 that $K_2 \subseteq K_3$ is inner anodyne. Hence the inclusion $C(m) \subseteq C(m, \alpha)$ is inner anodyne.

To complete the construction we need to show that the induced map

$$
C \subseteq C(m, \alpha) \times_{\Delta^n} \Lambda^n_i
$$

is an isomorphism. It suffices to prove that the induced map

$$
K_2 \times_{\Delta^n} \Lambda^n_i \subseteq K_3 \times_{\Delta^n} \Lambda^n_i
$$

is an isomorphism, which is clear. □
9. Proof of Lemma 6.3

Our objective in this section is to prove Lemma 6.3. Recall that this is the analog for simplicial sets of Lemma 2.4 in [1]. We will need some preliminary lemmas.

Firstly, we shall need the following result asserting that inner anodyne maps pullback along Kan fibrations to inner anodyne maps (this could be deduced from Proposition 3.3.1.3 of [11] however the proof of the latter result is much more difficult).

Lemma 9.1. Suppose that \( p : X \to S \) is a Kan fibration between simplicial sets. Then the induced map

\[ A \times_S X \to B \times_S X \]

is inner anodyne for any inner anodyne map \( A \to B \) in \( (\text{Set}_\Delta)/S \).

Proof. Let \( A \) be the class of all monomorphisms \( u : A \to B \) in \( (\text{Set}_\Delta)/S \) with the property that the induced map

\[ A \times_S X \to B \times_S X \]

is inner anodyne. We will prove that \( A \) contains the class of all inner anodyne maps in \( (\text{Set}_\Delta)/S \). The functor \( (\text{Set}_\Delta)/S \to (\text{Set}_\Delta)/X \) defined by pullback along the map \( p : X \to S \) is a left adjoint and hence sends saturated classes to saturated classes. Therefore it suffices to prove the statement in the special case that \( u : A \to B \) is an inner horn inclusion \( \Lambda^n_i \subseteq \Delta^n \) in \( (\text{Set}_\Delta)/S \).

Suppose given then a Kan fibration \( p : X \to \Delta^n \). We need to prove that the induced map \( \Lambda^n_i \times_{\Delta^n} X \to X \) is inner anodyne. Since the class of inner anodyne maps is closed under retracts we may assume without loss of generality that the Kan fibration \( X \to \Delta^n \) is minimal. But then a classical theorem asserts that \( X \to \Delta^n \) is a trivial bundle and hence there is an isomorphism \( X \simeq \Delta^n \times K \) for some Kan complex \( K \). But then the induced map \( \Lambda^n_i \times_{\Delta^n} X \to X \) is isomorphic to the map \( \Lambda^n_i \times K \to \Delta^n \times K \) which is clearly inner anodyne. \( \Box \)

Most of the work involved in proving Lemma 6.3 is contained in proving the following result.

Lemma 9.2. Suppose that \( B \) is a simplicial set with two vertices 0 and 1 and suppose that \( \{0\} \to B \) is a categorical equivalence. There is a countable subcomplex \( D \subseteq B \) satisfying the conditions that (i) \( \{0\} \to D \) is a categorical equivalence and (ii) the vertices 0 and 1 both belong to \( D \).

The proof of Lemma 9.2 will require some preparation. Let \( \mathcal{B} \) denote the subcategory of \( (\text{Set}_\Delta)_{sk_2 B//B} \) whose objects are the subcomplexes \( B' \) of \( B \) such that \( sk_2(B') = sk_2(B) \), and whose morphisms are the inclusions between them.

Construction 9.3. We construct a functor

\[ X : \mathcal{B} \to (\text{Set}_\Delta)_{\{0\}//B} \]

from \( \mathcal{B} \) to the arrow category of \( (\text{Set}_\Delta)_{\{0\}//B} \) which sends a subcomplex \( B' \in \mathcal{B} \) to a diagram

\[ \{0\} \to X(B') \to B' \]
in which \( \{0\} \to X(B') \) is a categorical equivalence and \( X(B') \to B' \) is a Kan fibration which is natural in \( B' \in \mathcal{B} \). Moreover the functor \( X \) is compatible with filtered colimits in \( \mathcal{B} \).

The construction proceeds as follows. It is directly inspired by the proof of Lemma X.2.8 from [5]. The small object argument for the set of inner horn inclusions \( \Lambda^n_i \subseteq \Delta^n \) with \( 0 < i < n \) gives rise to a functor

\[
R: \text{Set}_\Delta \to \text{Set}_\Delta
\]

together with a natural transformation \( \text{id}_{\text{Set}_\Delta} \to R \) such that \( R(S) \) is an \( \infty \)-category for every simplicial set \( S \) and the component \( S \to R(S) \) of the natural transformation is an inner anodyne map for every simplicial set \( S \). Moreover the functor \( R \) is compatible with filtered colimits.

The functor \( R \) induces a functor \( R: \mathcal{B} \to (\text{Set}_\Delta)_{/R(B)} \) such that for every subcomplex \( B' \in \mathcal{B} \) there is a commutative diagram, natural in \( B' \), of the form

\[
\begin{array}{ccc}
B' & \longrightarrow & R(B') \\
\downarrow & & \downarrow \\
B & \longrightarrow & R(B)
\end{array}
\]

in which the horizontal maps are inner anodyne. Note that since \( \text{sk}_2 B' = \text{sk}_2 B \) for every subcomplex \( B' \) in \( \mathcal{B} \), and \( h(B) \) is a (contractible) groupoid, it follows that \( R(B') \) is a Kan complex for every \( B' \in \mathcal{B} \) (Corollary 1.4 of [7]). In particular \( R(B) \) is a Kan complex.

Using the small object argument for the set of horn inclusions \( \Lambda^n_i \subseteq \Delta^n \) with \( 0 \leq i \leq n \), we see that for every \( B' \in \mathcal{B} \), there exists a commutative diagram, natural in \( B' \), of the form

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & T_{B'}(\{0\}) \\
\downarrow & & \downarrow \\
B' & \longrightarrow & R(B')
\end{array}
\]

in which \( \{0\} \to T_{B'}(\{0\}) \) is a homotopy equivalence and \( T_{B'}(\{0\}) \to R(B') \) is a Kan fibration. Since both \( \{0\} \) and \( T_{B'}(\{0\}) \) are Kan complexes, it follows that the map \( \{0\} \to T_{B'}(\{0\}) \) is a categorical equivalence (Lemma 2.4).

Since the natural transformation \( B' \to R(B') \) is compatible with filtered colimits in \( B' \) it follows that the diagram (15) is compatible with filtered colimits in \( B' \).

Since each map \( B' \to R(B') \) is inner anodyne, it follows (Lemma 9.1) that the induced maps \( B' \times_{R(B')} T_{B'}(\{0\}) \to T_{B'}(\{0\}) \) is inner anodyne, and hence that the map \( \{0\} \to B' \times_{R(B')} T_{B'}(\{0\}) \) is a categorical equivalence by the 2-out-of-3 property for categorical equivalences.

Define \( X(B') := B' \times_{R(B')} T_{B'}(\{0\}) \). Then the following statements are true for every \( B' \in \mathcal{B} \):

- The induced map \( X(B') \to B' \) is a Kan fibration which is natural in \( B \in \mathcal{B} \);
- The map \( \{0\} \to X(B') \) is a categorical equivalence;
• The map $\{0\} \to B'$ is a categorical equivalence if and only if the map $X(B') \to B'$ is a trivial Kan fibration.

Moreover, since $R(B')$ and $T_{B'}(\{0\})$ are natural in $B'$ and compatible with filtered colimits in $\mathcal{B}$, the construction $X(B')$ extends to define a functor

$$X: \mathcal{B} \to (\mathcal{S}et)^{[1]}_{\{0\} // B}$$

which is compatible with filtered colimits in $\mathcal{B}$.

Observe that if $B' \in \mathcal{B}$ has only countably many non-degenerate simplices then $R(B')$ has only countably many non-degenerate simplices. Therefore the same is true for $T_{B'}(\{0\})$ and hence for $X(B')$.

We can now prove Lemma 9.2.

**Proof of Lemma 9.2.** Let $D_0 = \text{sk}_2 B$. Consider all lifting problems for diagrams of the form

$$\begin{array}{ccc}
\partial \Delta^n & \to & X(D_0) \to X(B) \\
\downarrow & & \downarrow \\
\Delta^n & \to & D_0 \to B
\end{array}$$

(16)

Since $X(B) \to B$ is a trivial Kan fibration each one of these diagrams has a solution. The simplicial set $B$ is a filtered colimit of its countable subcomplexes and hence $X(B)$ is a filtered colimit of subcomplexes $X(B')$ with countably many non-degenerate simplices. Since there are only countably many such diagrams of the form (16) and countably many such diagonal fillers, it follows that there is a countable subcomplex $D_1 \subseteq B$ containing $D_0$ such that each of the lifting problems above has a solution in $X(D_1)$. Note that $\text{sk}_2 D_1 = \text{sk}_2 B$ since we have the inclusions $\text{sk}_2 B \subseteq \text{sk}_2 D_1 \subseteq \text{sk}_2 B$.

Repeat this construction countably many times to obtain a sequence $D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$ of $B$ such that all lifting problems of the above form over $D_i$ can be solved over $D_{i+1}$. Let $D = \bigcup_{i \geq 0} D_i$. Then $D$ is a countable subcomplex belonging to $\mathcal{B}$ which contains the vertices 0 and 1 and is such that $\{0\} \to D$ is a categorical equivalence. This last statement follows from the fact that the map $X(D) \to D$ is a trivial Kan fibration since each lifting problem

$$\begin{array}{ccc}
\partial \Delta^n & \to & X(D) \\
\downarrow & & \downarrow \\
\Delta^n & \to & D
\end{array}$$

factors through $X(D_i)$ for some $i$ and hence can be solved over $D_{i+1}$.

□

**Proof of Lemma 6.3.** Since $S$ is a pre-fibrant simplicial set and $f: x \to y$ is an equivalence in $S$, it follows by Remark 3.10 that there exists a subcomplex $S' \subseteq S$ containing $f$ and such that $h(S') = J$.

Let $\Delta^1 \to S'$ classify the edge $f$. Choose an inner anodyne map $S' \to K$ where $K$ is an $\infty$-category. Since $h(K) = J$ is a groupoid it follows that $K$ is a Kan complex (Corollary
1.4 of [7]. The composite map \( \Delta^1 \to S' \to K \) factors through \( J \), so that we have a commutative diagram

\[
\begin{array}{ccc}
\Delta^1 & \longrightarrow & J \\
\downarrow & & \downarrow \\
S' & \longrightarrow & K
\end{array}
\]

Factor the map \( J \to K \) as \( J \to Z \to K \) where \( J \to Z \) is anodyne and \( Z \to K \) is a Kan fibration. Let \( G \) be defined by the pullback diagram

\[
\begin{array}{ccc}
G & \longrightarrow & Z \\
\downarrow & & \downarrow \\
S' & \longrightarrow & K
\end{array}
\]

so that \( G \to S' \) is a Kan fibration and \( G \to Z \) is inner anodyne by Lemma 9.1.

The map \( J \to Z \) is a homotopy equivalence between Kan complexes and hence is a categorical equivalence (Lemma 2.4). It follows that the composite map \( \{0\} \to J \to Z \) is a categorical equivalence. Therefore the composite map \( \{0\} \to \Delta^1 \to G \) is a categorical equivalence. The desired result then follows from Lemma 9.2. \( \square \)

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NOTES ON THE JOYAL MODEL STRUCTURE

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