MARTINGALE SELECTION THEOREM FOR A STOCHASTIC SEQUENCE WITH RELATIVELY OPEN CONVEX VALUES

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Abstract. For a set-valued stochastic sequence \((G_n)_{n=0}^N\) with relatively open convex values \(G_n(\omega)\) we give a criterion for the existence of an adapted sequence \((x_n)_{n=0}^N\) of selectors, admitting an equivalent martingale measure. Mentioned criterion is expressed in terms of supports of the regular conditional upper distributions of the elements \(G_n\). This result is a refinement of the main result of author’s previous paper (Teor. Veroyatnost. i Primen., 2005, 50:3, 480–500), where the sets \(G_n(\omega)\) were assumed to be open and where were asked if the openness condition can be relaxed.

Introduction

Let \((\Omega, \mathcal{F}, \mathbf{P})\) be a probability space endowed with the filtration \((\mathcal{F}_n)_{n=0}^N\). Consider a sequence of \(\mathcal{F}_n\)-measurable set-valued maps \(\Omega \mapsto G_n(\omega) \subset \mathbb{R}^d\), \(n = 0, \ldots, N\) with the nonempty relatively open convex values \(G_n(\omega)\). In this paper we give a criterion for the existence of a pair, consisting of an adapted single-valued stochastic process \(x = (x_n)_{n=1}^N\), \(x_n(\omega) \in G_n(\omega)\) and a probability measure \(\mathbf{Q}\) equivalent to \(\mathbf{P}\) such that \(x\) is a martingale under \(\mathbf{Q}\). Following [1], we say that the martingale selection problem (m.s.p.) is solvable if such a pair \((x, \mathbf{Q})\) exists.

This problem is motivated by some questions of arbitrage theory. In particular, if the mappings \(G_n\) are single-valued, then we obtain the well-known problem concerning the existence of an equivalent martingale measure for a given stochastic process \(G_n = x_n\). In this case the solvability of the m.s.p. is equivalent to the absence of arbitrage in the market, where the discounted asset price process is described by \(x\) [2–5]. It is shown in [4] that an equivalent martingale measure for \(x\) exists iff the convex hulls of the supports of \(x_n - x_{n-1}\) regular conditional distributions with respect to \(\mathcal{F}_{n-1}\) contain the origin as a point of relative interior [4, Theorem 3, condition (g)]. The aim of the present paper is to refine this result.

In the framework of market models with transaction costs [6–8] the role of equivalent martingale measures is played by strictly consistent price processes. This name is assigned to \(\mathbf{P}\)-martingales a.s. taking values in the relative interior of the random cones \(K^*\), conjugate to the solvency cones \(K\). Using the invariance of \(K\) under multiplication, it is easy to show (see [1, Introduction]) that the existence of a strictly consistent price process is

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equivalent to the solvability of the m.s.p. for the sequence \((\text{ri} K_n^*)_{n=0}^\infty\) of relative interiors of \(K_n^*\).

In the paper [1] there was obtained a criterion of the solvability of the m.s.p. under the assumption that the sets \(G_n(\omega)\) are open. This result is not completely satisfactory since, for instance, it does not include the case of single-valued \(G_n\) and it does not allow the cones \(K_n^*\) to have the empty interior. The last limitation means that the ”efficient friction” condition must be satisfied (according to the terminology of [6]).

In the present paper we refine the main result of [1] (see Theorem 1). Moreover, the proof given below, as compared to [1], is considerably simplified.

2. Preliminaries

Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a \(\sigma\)-algebra \(\mathcal{H} \subset \mathcal{F}\). In the sequel we assume that all \(\sigma\)-algebras are complete with respect to \(\mathbb{P}\) (i.e. they contain all the subsets of their \(\mathbb{P}\)-negligible sets). Denote by \(\text{cl} A\), \(\text{ri} A\), \(\text{conv} A\) the closure, the relative interior, and the convex hull of a subset \(A\) of a finite-dimensional space. Let \(\mathcal{B} = \mathcal{B}(\mathbb{R}^d)\) be the Borel \(\sigma\)-algebra of \(\mathbb{R}^d\).

A set-valued map \(F\), assigning some set \(F(\omega) \subset \mathbb{R}^d\) to each \(\omega \in \Omega\), is called \(\mathcal{H}\)-measurable if \(\{\omega : F(\omega) \cap V \neq \emptyset\} \in \mathcal{H}\) for any open set \(V \subset \mathbb{R}^d\). The graph and the domain of \(F\) are defined by

\[
\text{gr} F = \{(\omega, x) : x \in F(\omega)\}, \quad \text{dom} F = \{\omega : F(\omega) \neq \emptyset\}.
\]

If \(\text{gr} F \in \mathcal{H} \otimes \mathcal{B}\), then the mapping \(F\) is \(\mathcal{H}\)-measurable [9, Corollary II.1.34].

The function \(f : \Omega \mapsto \mathbb{R}^d\) is called a selector of a set-valued map \(F\) if \(f(\omega) \in F(\omega)\) for all \(\omega \in \text{dom} F\). Denote by \(S(F, \mathcal{H})\) the set of \(\mathcal{H}\)-measurable selectors of \(F\). Note, that if the set-valued map \(F\) is \(\mathcal{H}\)-measurable, then the mapping

\[
F_* = FI_{\text{dom} F} + \mathbb{R}^dI_{\Omega \setminus \text{dom} F}
\]

is also \(\mathcal{H}\)-measurable and \(S(F, \mathcal{H}) = S(F_*, \mathcal{H})\). Here \(I_A(\omega) = 1, \omega \in A; I_A(\omega) = 0, \omega \not\in A\).

The countable family \(\{f_i\}_{i=1}^\infty\) of an \(\mathcal{H}\)-measurable selectors is called (an \(\mathcal{H}\)-measurable) Castaing representation for \(F\), if the sets \(\{f_i(\omega)\}_{i=1}^\infty\) are dense in \(F(\omega)\) for all \(\omega \in \text{dom} F\). The set-valued map \(F\) with nonempty closed values is \(\mathcal{H}\)-measurable iff it admits an \(\mathcal{H}\)-measurable Castaing representation [9, Proposition II.2.3].

An element \(f \in S(\text{conv} F, \mathcal{H})\) is said to have an \(\mathcal{H}\)-measurable Carathéodory representation, if there are some elements \(g_k \in S(F, \mathcal{H}), k = 1, \ldots, d+1\) and \(\mathcal{H}\)-measurable functions

\[
\alpha_k \geq 0, \quad k = 1, \ldots, d+1; \quad \sum_{k=1}^{d+1} \alpha_k = 1
\]

such that \(f = \sum_{k=1}^{d+1} \alpha_k g_k\) a.s. Under the assumption \(\text{gr} F \in \mathcal{H} \otimes \mathcal{B}\) any element \(f \in S(\text{conv} F, \mathcal{H})\) has an \(\mathcal{H}\)-measurable Carathéodory representation [10, Theorem 8.2(iii)].

Denote by \(\text{CL} = \text{CL}(\mathbb{R}^d)\) the family of nonempty closed subsets of \(\mathbb{R}^d\) and let \(\mathcal{E}(\text{CL})\) be the Effros \(\sigma\)-algebra, generated by the sets of the form

\[
A_V = \{D \in \text{CL} : D \cap V \neq \emptyset\},
\]
where $V$ is an open subset of $\mathbb{R}^d$.

Suppose $F$ is an $\mathcal{F}$-measurable set-valued map with nonempty closed values. It follows directly from the definitions that the corresponding single-valued map $F : (\Omega, \mathcal{F}) \mapsto (\text{CL}, \mathcal{E}(\text{CL}))$ is measurable. The measurable space $(\text{CL}, \mathcal{E}(\text{CL}))$ is a Borel space ([11, Theorem 3.3.10]). Consequently, the map $F$, considered as a random element taking values in $(\text{CL}, \mathcal{E}(\text{CL}))$, has the regular conditional distribution with respect to $\mathcal{H}$ [12, Chapter II, §7, Theorem 5].

Thus, there exists a function $\mathbf{P}^* : \Omega \times \mathcal{E}(\text{CL}) \mapsto [0, 1]$ with the following properties:

(i) for every $\omega$ the function $C \mapsto \mathbf{P}^*(\omega, C)$ is a probability measure on $\mathcal{E}(\text{CL})$;

(ii) for every $C \in \mathcal{E}(\text{CL})$ the function $\omega \mapsto \mathbf{P}^*(\omega, C)$ a.s. coincides with the conditional probability $\mathbf{P}({\{F \in C}\} | \mathcal{H})(\omega)$.

Following [1], we define the regular conditional upper distribution of the mapping $F$ with respect to $\mathcal{H}$ by the formula $\mu_{F, \mathcal{H}}(\omega, V) = \mathbf{P}^*(\omega, A_V)$ for any open subset $V \subset \mathbb{R}^d$. The set

$$\mathcal{K}(F, \mathcal{H}; \omega) = \{ y \in \mathbb{R}^d : \mu_{F, \mathcal{H}}(\omega, \{ y' : |y' - y| < \varepsilon \}) > 0 \text{ for all } \varepsilon > 0 \}$$

is called the support of $\mu_{F, \mathcal{H}}(\omega, \cdot)$ [1]. Note, that if $F$ is a single-valued map, then $\mu_{F, \mathcal{H}}$ is its regular conditional distribution with respect to $\mathcal{H}$ and $\mathcal{K}(F, \mathcal{H})$ is the support of the measure $\mu_{F, \mathcal{H}}$.

The set-valued map $\omega \mapsto \mathcal{K}(F, \mathcal{H}; \omega)$ has nonempty closed values and is $\mathcal{H}$-measurable [1, Proposition 4(a)]. Let $\{f_i\}_{i=1}^{\infty}$ be an $\mathcal{F}$-measurable Castaing representation for $F$. Then the following equality holds true (see [1, Lemma 1]):

$$\mathcal{K}(F, \mathcal{H}) = \text{cl} \left( \bigcup_{i=1}^{\infty} \mathcal{K}(f_i, \mathcal{H}) \right) \text{ a.s.} \quad (3)$$

If the values of $F$ are empty on a $\mathbf{P}$-null set, then we put $\mathcal{K}(F, \mathcal{H}) = \mathcal{K}(F_*, \mathcal{H})$, where $F_*$ is defined by (1). Evidently, equality (3) still holds true in this case.

Provided $F(\omega) = \emptyset$ on a set of positive measure, we put $\mathcal{K}(F, \mathcal{H}) = \emptyset$ for all $\omega$.

3. MAIN RESULT

Suppose $\Omega \mapsto G_n(\omega) \subset \mathbb{R}^d$, $n = 0, 1, \ldots, N$ is a sequence of $\mathcal{F}_n$-measurable set-valued maps with nonempty relatively open convex values $G_n(\omega)$. Define the sequence $(W_n)_{n=0}^{N}$ of set-valued maps recursively by

$$W_N = \text{cl} G_N,$$

$$W_{n-1} = \text{cl}(G_{n-1} \cap \text{ri} Y_{n-1}), \quad Y_{n-1} = \text{conv} \mathcal{K}(W_n, \mathcal{F}_{n-1}), \quad 1 \leq n \leq N.$$  

This sequence is well-defined and is adapted to the filtration. Indeed, suppose the map $W_n$ is $\mathcal{F}_n$-measurable. If $W_n \neq \emptyset$ a.s., then the map $\text{conv} \mathcal{K}(W_n, \mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$-measurable (see [1, Proposition 4(a)] and [9, Proposition II.2.26]). Furthermore, the graphs of the maps $G_{n-1}, \text{ri} Y_{n-1}$ are measurable with respect to the $\sigma$-algebra $\mathcal{F}_{n-1} \otimes \mathcal{B}$ [13, Lemma 1(c)]. Consequently, the map $G_{n-1} \cap \text{ri} Y_{n-1}$ is $\mathcal{F}_{n-1}$-measurable [9, Corollary II.1.34]. Its closure $W_{n-1}$ has the same property [9, Proposition II.1.8].

Provided $W_n = \emptyset$ on a set of positive measure, we have $W_{n-1} = \emptyset$ by the definition.
Theorem 1. The following conditions are equivalent:
(a) there exist an adapted to the filtration \((\mathcal{F}_n)_{n=0}^N\) stochastic process \(x = (x_n)_{n=0}^N\) and an equivalent to \(\mathbb{P}\) probability measure \(Q\) such that \(x_n \in \mathcal{S}(G_n, \mathcal{F}_n)\), \(n \geq 0\) and \(x\) is a \(Q\)-martingale;
(b) \(W_n \neq \emptyset\) a.s., \(n = 0, \ldots, N - 1\).

Denote by \(E(f|\mathcal{H})\) the generalized conditional expectation of the \(\mathcal{F}\)-measurable random variable \(f\) with respect to \(\mathcal{H}\) (under the measure \(\mathbb{P}\)) \([12,\text{p.229}], [5,\text{p.117}]\). The proof of Theorem 1 is based on the following result.

Lemma 1. Let \(F\) be an \(\mathcal{F}\)-measurable set-valued mapping with nonempty closed convex values. For any \(\mathcal{H}\)-measurable selector \(\xi\) of the map \(\text{ri}(\text{conv}\mathcal{K}(F, \mathcal{H}))\) there exist an element \(\eta \in \mathcal{S}(\text{ri} F, \mathcal{F})\) and an \(\mathcal{F}\)-measurable random variable \(\gamma > 0\) such that
\[
\xi = E(\gamma \eta|\mathcal{H}), \quad E(\gamma|\mathcal{H}) = 1 \text{ a.s.} \tag{4}
\]

Proof. Let \(\{f_i\}_{i=1}^\infty\), \(f_i \in \mathcal{S}(\text{ri} F, \mathcal{F})\) be a Castaing representation for \(\text{ri} F\). Since \(\text{ri} F \in \mathcal{F} \otimes \mathcal{B}\) \([13,\text{Lemma 1(c)}]\), such a representation exists (see \([9,\text{Proposition II.2.17}]\)).

Evidently, \(\{f_i\}_{i=1}^\infty\) is also a Castaing representation for \(F\). Applying (3) we get
\[
\xi \in \text{ri}(\text{conv} \mathcal{K}(F, \mathcal{H})) = \text{ri} \left(\text{conv} \left(\text{cl} \left(\bigcup_{i=1}^\infty \mathcal{K}(f_i, \mathcal{H})\right)\right)\right) \text{ a.s.}
\]

Note that for any collection of sets \(\{A_i\}_{i=1}^\infty\), \(A_i \subset \mathbb{R}^d\) the following inclusion holds true
\[
B_1 = \text{ri} \left(\text{conv} \left(\text{cl} \left(\bigcup_{i=1}^\infty A_i\right)\right)\right) \subset \text{conv} \left(\bigcup_{i=1}^\infty \text{ri}(\text{conv} A_i)\right) = B_2.
\]

Indeed, suppose \(x \not\in B_2\). Then by the separation theorem there exist \(p \in \mathbb{R}^d\), \(j \in \mathbb{N}\), \(\overline{y} \in A_j\) such that
\[
\langle p, x \rangle \geq \langle p, y \rangle, \quad y \in A_i, \ i \in \mathbb{N};
\]
\[
\langle p, x \rangle > \langle p, \overline{y} \rangle.
\]

Here \(\langle \cdot, \cdot \rangle\) is the usual scalar product in \(\mathbb{R}^d\). Obviously, \(\langle p, x \rangle \geq \langle p, z \rangle\) for all \(z \in \text{cl} B_1\). Since \(\overline{y} \in \text{cl} B_1\), it follows that \(\{x\}\) and \(\text{cl} B_1\) are properly separated. Therefore, \(x \not\in \text{ri}(\text{cl} B_1) = B_1\).

Putting \(A_i = \mathcal{K}(f_i, \mathcal{H})\), we conclude that
\[
\xi \in \text{conv} \left(\bigcup_{i=1}^\infty \text{ri}(\text{conv} \mathcal{K}(f_i, \mathcal{H}))\right) \text{ a.s.}
\]

The results of the theory of measurable set-valued maps mentioned above, readily imply that \(\xi\) has an \(\mathcal{H}\)-measurable Caratheodory representation:
\[
\xi = \sum_{k=1}^{d+1} \alpha_k \xi_k \text{ a.s.},\quad \xi_k \in \mathcal{S} \left(\bigcup_{i=1}^\infty \text{ri}(\text{conv} \mathcal{K}(f_i, \mathcal{H})), \mathcal{H}\right),
\]
where \( \mathcal{H} \)-measurable functions \( \alpha_k \) satisfy conditions (2).

Put \( A_i^k = \{ \omega : \xi_k \in \text{ri}(\text{conv} \mathcal{K}(f_i, \mathcal{H})) \} \) and consider the covering of \( \Omega \), consisting of the sets \( A_1^1 \cap \cdots \cap A_{d+1} \), where the upper indexes run through all natural numbers. It is easy to show (see [1, Lemma 2]) that there exists an \( \mathcal{H} \)-measurable partition \( \{ D_j \}_{j \in J}, J \subset \mathbb{N} \) of \( \Omega \) such that

\[
\emptyset \neq D_j \subset A_1^1 \cap \cdots \cap A_{d+1}^i, \ j \in J,
\]

where the set \((i_1, \ldots, i_{d+1})\) depends on \( j \).

For almost all \( \omega \in D_j \) we have

\[
\xi_k \in \text{ri}(\text{conv} \mathcal{K}(f_i(j), \mathcal{H})), \ k = 1, \ldots, d + 1,
\]

or, in other words, \( 0 \in \text{ri}(\mathcal{K}(\zeta_{k,j}, \mathcal{H})) \) a.s., where \( \zeta_{k,j} = I_{D_j}(f_i(j) - \xi_k) \).

According to [4, Theorem 3] it follows that for any \( k \in \{1, \ldots, d + 1\} \) and \( j \in J \) there exists an equivalent to \( \mathbb{P} \) probability measure \( Q_{k,j} \) with a.s. bounded density \( 0 < \gamma_{k,j} = dQ_{k,j} / d\mathbb{P} \) such that

\[
I_{D_j} \xi_k = I_{D_j} E_{Q_{k,j}}(f_i(j) | \mathcal{H}) = \frac{I_{D_j}}{E(\gamma_{k,j} | \mathcal{H})} E(\gamma_{k,j} f_i(j) | \mathcal{H}) \text{ a.s.}
\]

In the last equality the generalized Bayes formula [5, Ch. V, §3a] is used.

Hence, we get the representation

\[
\xi = \sum_{k=1}^{d+1} \alpha_k \xi_k = E \left( \sum_{k=1}^{d+1} \frac{\alpha_k \gamma_{k,j}}{E(\gamma_{k,j} | \mathcal{H})} f_i(j) | \mathcal{H} \right) \text{ a.s. on } D_j.
\]

Here we take into account that the equality \( E(gh | \mathcal{H}) = hE(g | \mathcal{H}) \) holds true if the function \( g \) is \( F \)-measurable and \( \mathbb{P} \)-integrable, and the function \( h \) is \( \mathcal{H} \)-measurable (see the remark in [12, p. 236]).

Put \( \beta_{k,j} = \gamma_{k,j} / E(\gamma_{k,j} | \mathcal{H}) \) and introduce the functions

\[
\gamma_j = \sum_{k=1}^{d+1} \alpha_k \beta_{k,j}, \quad \eta_j = \sum_{k=1}^{d+1} \frac{\alpha_k \beta_{k,j}}{\gamma_j} f_i(j).
\]

We have

\[
\xi = E(\gamma_j \eta_j | \mathcal{H}) \text{ a.s. on } D_j.
\]

It remains to note that \( \gamma_j > 0, E(\gamma_j | \mathcal{H}) = 1, \)

\[
\eta_j \in \text{conv} \{ f_1(j), \ldots, f_{d+1}(j) \} \subset \text{ri} \mathcal{F} \text{ a.s. on } D_j,
\]

and the functions

\[
\gamma = \sum_{j \in J} I_{D_j} \gamma_j, \quad \eta = \sum_{j \in J} I_{D_j} \eta_j
\]

satisfy conditions (4). The proof of Lemma 1 is complete.
**Proof of Theorem 1.** Assume that condition (b) is satisfied. Starting from an arbitrary selector \(x_0 \in S(\text{ri} W_0, \mathcal{F}_0)\) let us construct adapted sequences \(x_n \in \text{ri} W_n, \gamma_n > 0\), meeting the conditions

\[
x_{n-1} = \mathbb{E}(\gamma_n x_n | \mathcal{F}_n), \quad \mathbb{E}(\gamma_n | \mathcal{F}_{n-1}) = 1 \text{ a.s., } n = 1, \ldots, N.
\]

The existence of the selector \(x_0\) is implied by already mentioned results [13, Lemma 1(c)], [9, Proposition II.2.17]. The existence of the above sequences follows from Lemma 1, since \(x_{n-1} \in S(\text{ri} W_{n-1}, \mathcal{F}_{n-1})\) imply that \(x_{n-1} \in S(\text{ri} (\text{conv} \ K(x_n, \mathcal{F}_{n-1})), \mathcal{F}_{n-1})\).

Consider the positive \(P\)-martingale

\[
(z_n)_{n=0}^N, \quad z_0 = 1, \quad z_n = \prod_{k=1}^{n} \gamma_k, \ n \geq 1
\]

and the equivalent to \(P\) probability measure \(Q'\) with the density \(dQ'/dP = z_N\). According to the generalized Bayes formula we have

\[
x_{n-1} = \frac{1}{z_{n-1}} \mathbb{E}(x_n z_n | \mathcal{F}_{n-1}) = \mathbb{E}Q'(x_n | \mathcal{F}_{n-1}) \text{ a.s.}
\]

Thus, the process \(x\) is a generalized (or, equivalently, a local) \(Q'\)-martingale and it admits an equivalent martingale measure \(Q\) ([4, Theorem 3]).

As long as, moreover, \(x_n \in S(\text{ri} W_n, \mathcal{F}_n) \subset S(G_n, \mathcal{F}_n)\), condition (a) is verified.

Now assume that condition (a) is satisfied. Note that \(x_N \in G_N \subset W_N\). Suppose the relations \(x_j \in W_j, \ j \geq n\) are already established. Since

\[
0 \in \text{ri} (\text{conv} \ K(x_n - x_{n-1}, \mathcal{F}_{n-1})) \text{ a.s., } n \geq 1
\]

(see [4, Theorem 3]) and \(K(x_n, \mathcal{F}_{n-1}) \subset K(W_n, \mathcal{F}_{n-1})\), it follows that

\[
x_{n-1} \in G_{n-1} \cap \text{ri} (\text{conv} \ K(x_n, \mathcal{F}_{n-1})) \subset G_{n-1} \cap \text{ri} Y_{n-1} \subset W_{n-1} \text{ a.s.}
\]

Particularly, \(W_n \neq \emptyset \) a.s. for all \(n\). The proof is complete.

In the paper [1] Theorem 1 was proved under one of the following additional assumptions: (i) the sets \(G_n(\omega)\) are open; (ii) the set \(\Omega\) is finite.

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