SYMMETRIZATION AND SHARP SOBOLEV INEQUALITIES IN METRIC SPACES

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Abstract. We derive sharp Sobolev inequalities for Sobolev spaces on metric spaces. In particular, we obtain new sharp Sobolev embeddings and Faber-Krahn estimates for Hörmander vector fields.

1. Introduction

Recently, a rich theory of Sobolev spaces on metric spaces has been developed (cf. [7], [6], and the references therein). In particular, this has led to the unification of some aspects of the classical theory of Sobolev spaces with the theory of Sobolev spaces of vector fields satisfying Hörmander’s condition. At the root of these developments are suitable forms of Poincaré inequalities which, in fact, can be used to provide a natural method to define the notion of a gradient in the setting of metric spaces. In the theory of Hörmander vector fields the relevant Poincaré inequalities had been obtained much earlier by Jerison [10]:

\[(1.1) \left( \frac{1}{|B|} \int_B |f - f_B|^2 \, dx \right)^{1/2} \leq Cr(B) \left( \frac{1}{|B|} \int_B |Xf|^2 \, dx \right)^{1/2},\]

where \(X = (X_1, \ldots, X_m)\) is a family of \(C^\infty\) Hörmander vector fields, \(|Xf| = \left( \sum |X_i f|^2 \right)^{1/2}\), \(dx\) is Lebesgue measure, \(B\) is a ball of radius \(r(B)\), with respect to the Carnot-Carathéodory metric. For more on the connection between the theory of Sobolev spaces on metric spaces and Sobolev spaces on Carnot groups we refer to the Appendix below and [6].

The purpose of this paper is to prove sharp forms of the classical Sobolev inequalities in the context of metric spaces. In fact, we develop an approach to symmetrization in the metric setting which has applications to other problems as well. In particular, we will show some functional forms of the Faber-Krahn inequalities which are new even in the classical setting.

A well known, and very natural, approach to the Sobolev inequalities is through the use of the isoperimetric inequality and related rearrangement inequalities (for an account cf. [23]). For example, a good deal of the classical inequalities can be in fact derived from (cf. [2], [12], and also [16])

\[(1.2) \frac{1}{t} \int_0^t |f^*(s) - f^*(t)| \, ds \leq ct^{1/n} \left( \frac{1}{t} \int_0^t |\nabla f^*(s)| \, ds \right), t > 0, f \in C_0^\infty(\mathbb{R}^n).\]

For example, in [2] and [19] it is shown how, starting from (1.2), one can derive Sobolev inequalities which are sharp, including the borderline cases, within the class of Sobolev spaces based on rearrangement invariant spaces. Therefore, it seemed natural to us to try to extend (1.2) to the metric setting. At the outset
one obstacle is that the usual methods to prove (1.2) are not available for metric spaces (cf. [2], [16]). However, we noticed that, in the Euclidean setting, (1.2) is the rearranged version of a Poincaré inequality. More specifically, suppose that \( f \) and \( g \) are functions such that, for any cube \( Q \subset \mathbb{R}^n \) with sides parallel to the coordinate axes, we have

\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq c \frac{|Q|^{1/n}}{|Q|} \int_Q g(x) \, dx.
\]

Then, the following version of (1.2) holds,

\[
\frac{1}{t} \int_0^t [f^*(s) - f^*(t)] \, ds \leq ct^{1/n} \left( \frac{1}{t} \int_0^t g^*(s) \, ds \right).
\]

By Poincaré’s inequality, (1.3) holds with \( g = |\nabla f| \) and therefore the implication \( (1.3) \Rightarrow (1.4) \) provides us with a proof of (1.2). This is somewhat surprising since the usual proofs of (1.2) depend on a suitable representation of \( f \) in terms of \( \nabla f \). Since, in the context of metric spaces, the gradient is defined through the validity of (1.3), this is a crucial point for our development of the symmetrization method in this setting.

Since the mechanism involved in transforming (1.3) into (1.4) plays an important role in our approach, it is instructive to present it here in the somewhat simpler, but central, Euclidean case. The first step is to reformulate (1.3) as an inequality between maximal operators

\[
f_{1/n}^#(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1+1/n}} \int_Q |f(x) - f_Q| \, dx \leq c \sup_{Q \ni x} \frac{1}{|Q|} \int_Q g(x) \, dx = c M g(x),
\]

where \( M \) is the non-centered maximal operator of Hardy-Littlewood. At this point taking rearrangements on both sides of (1.5) leads to

\[
(f_{1/n}^#)^*(t) \leq c M g^*(t) \leq c g^{**}(t).
\]

Here the estimate for the maximal operator of Hardy-Littlewood is a well known, and easy, consequence of the fact that \( M \) is weak type \((1,1)\) and strong type \((\infty, \infty)\). Moreover, by a simple variant of an inequality of Bennett-DeVore-Sharples [3], we have

\[
(f^{**}(t) - f^*(t)) t^{-1/n} \leq c (f_{1/n}^#)^*(t).
\]

Combining (1.6) and (1.7), we see that if (1.3) holds then (1.4) holds.

The method of proof outlined above can be developed in more general settings as long as suitable variants of the classical covering lemmas, which are needed to estimate the underlying maximal operators, are available. In the context of metric spaces the covering lemmas we need\(^1\) were obtained in [14]. Once the rearrangement inequalities are at hand we can use standard machinery to derive suitable Sobolev inequalities (see Section 3).  

\(^1\)In the PhD thesis of the first author [11], the required covering lemmas been obtained for domains with Lipschitz boundaries.
To give a more precise description of the contents of this paper we now recall the definition of a $p-q-$Poincaré inequality. In what follows $(X, \mu)$ is a homogenous metric space. Suppose that the operator we see that (1.2) implies

$$\|f\|_{B^p} \leq c_{P} r(B)^{1/q} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x)\right)^{1/p}$$

holds for every ball $B$ such that $\sigma B \subset \Omega$, where $f_B = (\mu(B))^{-1} \int_B f(x) d\mu(x)$. We may then refer to $f$ as a $(p-q-)$ Sobolev function and to $g$ as its gradient.

We can now state our main results. We start with the following extension of (1.2).

**Theorem 1.** (cf. Theorem 3 below) Let $B_0 \subset X$ be a ball, and suppose that $f$ and $g$ satisfy a $p-q-$Poincaré inequality on $4 \sigma B_0$. Then there exist constants $c_1 > 0, 0 < c_2 \leq 1$, independent of $B_0, f$ and $g$, such that

$$t^{-\frac{1}{p}} \left(\frac{1}{t} \int_{0}^{t} |f| \, ds\right)^{1/p} \leq c_1 (g^\ast(t))^{1/q}, 0 < t < c_2 \mu(B_0).$$

Following [2], given a r.i. space $Y$, we introduce the spaces $Y^p(\infty, s)$ (see Section 2 below) that contain all the functions for which the $Y$-norm of the expression on the left-hand side of (1.9) is finite. The following sharp Sobolev embedding theorem then follows immediately.

**Theorem 2.** (cf. Theorem 4 below) Let $B_0 \subset X$ be a ball, and let $Y(X)$ be an r.i. space. Suppose that the operator $P_{\max(p,q)}$ (cf. (2.3) below) is bounded on $Y(X)$. Then, if $f$ and $g$ satisfy a $p-q-$Poincaré inequality on $4 \sigma B_0$ with constant $c_1$, there exists a constant $c = c(B_0, c_1, p, q, Y) > 0$ such that

$$\|f \chi_{B_0}\|_{Y^p(\infty, s)} \leq c (\|g\|_{Y} + \|f\|_{Y}).$$

We also provide a new application of our rearrangement inequality (1.9) to the study of the so called functional forms of the Faber-Krahn inequalities in metric spaces (cf. Section 4 below). We now illustrate these ideas in the classical Euclidean case. For example, using $\int_{f^\ast(t)}^{\infty} \lambda_f(u) = t(f^{**}(t) - f^*(t))$, and Hölder’s inequality, we see that (1.2) implies

$$\int_{f^\ast(t)}^{\infty} \lambda_f(u) \leq t^{1/n} \int_0^t |\nabla f|^\ast(u) \, du \leq t^{1/n} \left(\int_0^t |\nabla f|^\ast(u)^p \, du\right)^{1/p}.$$  

Now let $t = \|f\|_1 = \int_{|f| > 0} dx$, and observe that then $f^*(t) = 0$, and $f_{f^\ast(t)} \lambda_f(u) = \|f\|_1$. We have thus obtained the following Faber-Krahn inequality

$$\|f\|_1 \leq \|f\|_0^{1/n + 1/p} \|\nabla f\|_p.$$  

\(^{2}\)See Definition 2 below.

\(^{3}\)Here and in what follows the symbol $\approx$ denotes equivalence modulo constants, and the symbol $\preceq$ denotes smaller or equal modulo constants.
More generally, if a $p - q$–Poincaré inequality holds then we can use \((1.9)\) and a similar argument to prove \((2.1)\).

**Theorem 3.** (cf. Theorem 4 below) Let $B_0 \subset X$ be a ball, and let $f$ be a function with $\text{supp}(f) \subset B_0$. Let $Z(X)$ be an r.i. space and let $\phi_{Z'}$ denote the fundamental function of its associate space $Z'(X)$.

(i) Let $f$ be a $p - q$–Sobolev function and let $g$ be a gradient of $f$. If $\|f\|_0 < c_2 \mu(B_0)$, where $c_2$ is the constant of Theorem 5 then

\[
\|f\|_{L^p(X)} \leq c \left[\|g\|_{Z(X)} \phi_{Z'}(\|f\|_0)\right]^{1/q} \|f\|_0^{\frac{p}{q} - \frac{1}{q}}.
\]

(ii) Let $f$ be a $1 - q$–Sobolev function, $q > s$, and let $g$ be a gradient of $f$. If $\|f\|_0 < c_2 \mu(B_0)$, where $c_2$ is the constant of Theorem 5 then

\[
\|f\|_{L^\infty(X)} \leq c \left[\|g\|_{Z(X)} \phi_{Z'}(\|f\|_0)\right]^{1/q} \|f\|_0^{\frac{1}{s} - \frac{1}{q}}.
\]

2. THE BASIC SYMMETRIZATION INEQUALITY IN METRIC SPACES

We start with a definition.

**Definition 2.** An homogeneous space consists of a metric space $X$ and a Borel measure $\mu$ on $X$, such that $0 < \mu(B(x, r)) < \infty$, for all $x \in X$, $r > 0$, and, moreover, the measure $\mu$ satisfies a doubling condition:

\[
\mu(B(x, 2r)) \leq c_d \mu(B(x, r)),
\]

for all $x \in X$ and $r > 0$. If $c_d$ is the smallest constant in (2.1) then the number $s = \log_2 c_d$ is called the doubling order or the dimension of $\mu$.

**Remark 1.** Note that if we fix a ball $\overline{B} \subset X$, then by iterating (2.1) (cf. Lemma 14.6 in [7]) we can find a positive constant $c$ (possibly depending on $\overline{B}$) such that for every ball $B \subset \overline{B}$ we have

\[
\frac{\mu(B)}{\mu(\overline{B})} \geq cr(B)^s.
\]

In what follows given a ball $B = B(x, r)$, $qB$ will denote the ball concentric with $B$, whose radius is $qr$.

A rearrangement invariant (r.i.) space $Y = Y(X)$ is a Banach function space of $\mu$–measurable functions on $X$ endowed with a norm $\|\cdot\|_Y$ such that if $f \in Y$ and $g^* = f^*$ then $g \in Y$ and $\|g\|_Y = \|f\|_Y$. The fundamental function $\phi_Y$ of $Y$ is defined for $t$ in the range of $\mu$ by

\[
\phi_Y(t) = \|\chi_E\|_Y,
\]

where $E$ is any subset of $X$ with $\mu(E) = t$. Recall that any rearrangement invariant (r.i.) space $Y$ has a representation as a function space $Y^-([0, \infty))$ such that (cf. [3] Theorem II.4.10)

\[
\|f\|_{Y([0, \infty))} = \|f^*\|_{Y^-([0, \infty))}.
\]

Since the measure space will be always clear from the context it is convenient to “drop the hat” and use the same letter $Y$ to indicate the different versions of the space $\hat{Y}$ that we use.

\(\text{For a different approach to (1.10) we refer to [15]. For a far reaching generalization of Theorem 4 using the beautiful ideas of Jawerth [5], see [9].}\)
Let $P$ denote the usual Hardy operator $P : f(t) \mapsto t^{-1} \int_0^t f^*(s)ds$. The operators $P_p, p \geq 1$, are defined by
\begin{equation}
(P_p f)(t) = [P((f^*)^p)(t)]^{1/p}.
\end{equation}

**Definition 3.** Let $Y$ be a r.i. space, and let $p \geq 1$ and $r > 0$. Let
\[ Y^p(\infty, r) = \{ f : \| f \|_{Y^p(\infty, r)} = \| t^{-1/r} \left( \frac{1}{t} \int_0^t |f^*(s) - f^*(t)|^p ds \right) \|_Y < \infty \}. \]

**Remark 2.** Under suitable assumptions the expression defining the “norm” of the $Y^p(\infty, r)$ spaces can be simplified. For example, suppose that $p$ and $r$ are such that
\begin{equation}
1 \leq p < \frac{pr}{pr - r},
\end{equation}
where $p_Y$ is the lower Boyd index of $Y$, and suppose, moreover (cf. [19]),
\begin{equation}
\int_1^\infty s^{1/r}dY \left( \frac{1}{s} \right) \frac{ds}{s} < \infty,
\end{equation}
where $d_Y(s)$ is the norm of the dilation operator $D_s : f(\cdot) \mapsto f(\cdot s)$. Then, for $f$ with $f^{**}(\infty) = 0$, we have
\[ \| f \|_{Y^p(\infty, r)} \approx \| f \|_{Y^1(\infty, r)}. \]

**Proof.** From (2.4) it follows that, for $f^{**}(\infty) = 0$, we have (cf. [19, Lemma 2.6]),
\begin{equation}
\| t^{-1/r}(f^{**}(t) - f^*(t)) \|_Y \approx \| t^{-1/r}f^{**}(t) \|_Y.
\end{equation}

On the other hand, since $\| [D_a f(t)t^{-1/r}] \|_Y = a^{-1/r}[\| D_a f(t) \|_t^{-1/r}]_Y$, we have
\[ \| D_a[f(t)t^{-1/r}] \|_Y \leq c a^{-1/p}\| f(t)t^{-1/r} \|_Y \]
if and only if
\[ \| D_a[f(t)]t^{-1/r} \|_Y \leq ca^{-1/p+1/r}\| f(t)t^{-1/r} \|_Y. \]

Consequently, if we let $Y(t^{-1/r})$ be the space defined by the norm $\| f(t)t^{-1/r} \|_Y$ then the lower Boyd index of $Y(t^{-1/r})$ is equal to $p_Y r/(pr - r)$, where $p_Y$ is the lower Boyd index of $Y$. Now, in view of (2.3), with $r = p$, it follows from [20 Theorem 2 (i)], that the operator $P_p$ is continuous on $Y(t^{-1/r})$. Thus,
\[ \| \left( \frac{1}{t} \int_0^t |f^*(s)|^p ds \right)^{1/p} t^{-1/r} \|_Y \leq c \| f^*(t)t^{-1/r} \|_Y \leq \| f^{**}(t)t^{-1/r} \|_Y. \]

Combining the last inequality with (2.6) we obtain,
\[ \| f \|_{Y^p(\infty, r)} \lesssim \| f \|_{Y^1(\infty, r)}. \]

The reverse inequality follows readily from Hölder’s inequality.

In [9] sets similar to $Y^p(\infty, r)$ are considered and conditions are given for these sets to be equivalent to normed spaces.
For the proof of our symmetrization inequality we need the following version of a covering lemma from [14].

**Lemma 1.** There exist positive constants $c, \lambda$, with $\lambda < 1$, such that for any ball $B$, and any open set $E \subset B$ with $\mu(E) \leq \lambda \mu(B)$, there exists a countable family of balls $\{B_i\}_{i=1}^\infty$ such that

1. $B_i \subset 4B$ for $i = 1, \ldots$
2. $E \subset \bigcup_{i=1}^\infty B_i$
3. $\sum_{i} \mu(B_i) \leq c \mu(E)$
4. $\mu(B_i \cap E) \leq \frac{1}{2} \mu(B_i \cap B)$ for $i = 1, \ldots$

**Proof.** Follows readily from the proof of [14, Lemma 3.1]. $\square$

**Theorem 4.** There exist positive constants $c_1, c_2$, such that, for any ball $B_0 \subset X$, $p, q \geq 1$, and for all $f \in L^1_{\text{loc}}$, $0 < t < c_2 \mu(B_0)$, we have

\[
\int_0^t \left( [f \chi_{B_0}]^*(s) - [f \chi_{B_0}]^*(t) \right)^p ds \leq c_1 (f^{\#}_{4B_0, p, q})^*(t).
\]

**Proof.** Follows along the lines of the corresponding proof in [4, Theorem V.7.3].

It suffices to establish (2.7) for nonnegative functions. Let $\lambda$ be as in Lemma 1 and fix $0 < t < \frac{3}{4} \mu(B_0)$. Let

\[
E = \{ x \in B_0 : f(x) > [f \chi_{B_0}]^*(t) \}
\]

and

\[
F = \{ x \in B_0 : f^{\#}_{4B_0, p, q}(x) > [f^{\#}_{4B_0, p, q} \chi_{B_0}]^*(t) \}.
\]

There exists an open set $\Omega \supset E \cup F$, with measure at most $3t \leq \lambda \mu(B_0)$. Consequently, we can apply Lemma 1 to obtain a family of balls $\{B_j\}_j$, such that all the conditions this Lemma are verified. Define disjoint sets by letting $M_1 = B_1$ and $M_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i$ for $k = 2, \ldots$. We have

\[
\int_0^t \left( [f \chi_{B_0}]^*(s) - [f \chi_{B_0}]^*(t) \right)^p ds = \int_E \{ f(x) - [f \chi_{B_0}]^*(t) \}^p d\mu(x)
\]

\[
= \sum_{j=1}^\infty \int_{E \cap M_j} \{ f(x) - [f \chi_{B_0}]^*(t) \}^p d\mu(x)
\]

\[
\leq c \sum_{j=1}^\infty \int_{B_j} |f - f_{B_j}|^p d\mu(x) +
\]

\[
c \sum_{j=1}^\infty \mu(E \cap M_j) \{ f_{B_j} - [f \chi_{B_0}]^*(t) \}^p
\]

\[
= c(\alpha + \beta), \text{ say.}
\]
Now,

\[ \beta \leq \sum_{\{j : f_{B_j} > [f \chi_{B_0}]^*(t)\}} \mu(E \cap B_j) [f_{B_j} - [f \chi_{B_0}]]^p \]

\[ \leq \sum_{\{j : f_{B_j} > [f \chi_{B_0}]^*(t)\}} \mu(B_0 \cap B_j \setminus E) [f_{B_j} - [f \chi_{B_0}]]^p \]

\[ \leq \sum_{\{j : f_{B_j} > [f \chi_{B_0}]^*(t)\}} \int_{B_0 \cap B_j \setminus E} [f_{B_j} - f(x)]^p d\mu(x) \]

\[ \leq \sum_{j=1}^{\infty} \int_{B_j} |f - f_{B_j}|^p d\mu(x) = \alpha. \]

Combining the previous estimates we obtain

\[ \int_0^t ([f \chi_{B_0}]^*(s) - [f \chi_{B_0}]^*(t))^p ds \leq 2c_0 = 2c \sum_{j=1}^{\infty} \frac{\mu(B_j)^q}{\mu(B_j)^p} \int_{B_j} |f - f_{B_j}|^p d\mu(x). \]

By (4) of Lemma 1, the set \( B_0 \cap B_j \setminus F \) is nonempty and, therefore, we can find a point \( x_j \in B_0 \cap B_j \setminus F \). It follows that

\[ \int_0^t ([f \chi_{B_0}]^*(s) - [f \chi_{B_0}]^*(t))^p ds \leq 2c \sum_{j=1}^{\infty} \mu(B_j)^q [f_{B_j}^\#(y_{B_0}(x_j))]^p \]

\[ \leq c t^q [[f_{B_j}^\#(y_{B_0}(x_j))]^p]. \]

\[ \square \]

**Corollary 1.** (cf. [22 page 228]) Suppose that \( \mu(X) = \infty \), and let \( c_1 \) be the constant of Theorem 4. Then

\[ t^{-q/p} \left( \int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 (f_{X,p}^\#)^*(t), \]

for all \( f \in L_{1,loc}^1(X), t > 0 \).

**Proof.** Let \( t > 0 \), and let \( c_2 \) be as in Theorem 4 Fix an arbitrary \( x_0 \in X \), since \( \mu(X) = \infty \), we can find a positive integer \( n_0 \) such that, for \( n \geq n_0 \), and \( B_n := B(x_0, n) \), we have \( t < c_2 \mu(B_n) \). Therefore, by (2.7),

\[ t^{-q/p} \left( \int_0^t ([f \chi_{B_n}]^*(s) - [f \chi_{B_n}]^*(t))^p ds \right)^{1/p} \leq c_1 (f_{B_n}^\#)^*(t), \]

and consequently

\[ t^{-q/p} \left( \int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 (f_{X,p}^\#)^*(t). \]

Letting \( n \rightarrow \infty \), and using Fatou’s lemma, we see that

\[ t^{-q/p} \left( \int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 (f_{X,p}^\#)^*(t). \]

\[ \square \]

Once the inequality (2.7) is available then it can be combined with the Poincaré inequality, as described in the introduction, to obtain the symmetrization inequality.
Theorem 5. Let $B_0 \subset X$ be a ball, and suppose that $f$ and $g$ satisfy a $p - q$-Poincaré inequality on $4B_0$ (with constant $c_P$). Then there exist positive constants $c_1 = c_1(B_0, c_P)$ and $1 \geq c_2 = c_2(X)$, such that, for $0 < t < c_2 \mu(B_0)$,

$$t^{-\frac{1}{p}} \left( \frac{1}{t} \int_0^t |(f \chi_{B_0})^*(s) - [f \chi_{B_0}]^*(t)|^p ds \right)^{1/p} \leq c_1 [(g^q)^*(t)]^{1/q}. \tag{2.10}$$

Proof. From the underlying Poincaré inequality and (2.2) (with $\tilde{B} = 4B_0$) we get

$$\left( \frac{1}{(\mu(B))^{1+p/s}} \int_B |f - f_B|^p d\mu \right)^{1/p} \leq c \left( \frac{1}{\mu(\sigma B)} \int_{\sigma B} g^q d\mu \right)^{1/q},$$

for every ball $B$ with $B \subset 4B_0$.

Fix an arbitrary point $x \in B_0$. Taking a supremum over all balls containing $x$ on the right hand side, and over all balls $B \subset 4B_0$ containing $x$ on the left hand side, we arrive at

$$f^#_{4B_0, p, 1+p/s}(x) \leq c (Mg^q(x))^{1/q},$$

where $M$ is the maximal operator of Hardy-Littlewood. After passing to rearrangements, and using (recall that the underlying measure is doubling)

$$(Mh)^*(t) \leq ch^{**}(t),$$

combined with Theorem 4 we obtain positive constants $c_1, c_2$ such that

$$t^{-\frac{1}{p}} \left( \frac{1}{t} \int_0^t |(f \chi_{B_0})^*(s) - [f \chi_{B_0}]^*(t)|^p ds \right)^{1/p} \leq c_1 [(g^q)^*(t)]^{1/q}, \quad 0 < t < c_2 \mu(B_0).$$

\[\square\]

Remark 3. It may not be possible to extend the inequality (2.10) to all $0 < t < \mu(B_0)$. This can be seen from the following counterexample for $p = q = 1$.

Let $f_k(x) = 1$ if $x \in [-1,1]^2 \setminus B_{1/k}(0)$, $f_k(x) = k|x|$ for $x \in B_{1/k}(0)$. Now, $|\nabla f_k| = k \chi_{[0,1/k]}(t)$ and $|\nabla f_k|^{**} = k \chi_{[0,1/k]}(t) + \frac{1}{k^2} \chi_{(1/k, \infty)}(t)$. If (2.10) were true for $0 < t < 4$, then taking the limit as $t \to 4$ of (2.10) would give us

$$|f_k|_{L^1(Q)} \leq c \frac{\pi}{4} \frac{1}{k}.$$

But whereas the right-hand side $\to 0$ as $k \to \infty$ the left-hand side $\approx 4$. One possible way to overcome this problem is to consider functions with zero average, by means of replacing $f$ by $f - f_Q$ (cf. [16]).

Remark 4. Suppose that the following global growth condition holds for every ball $B \subset X$,

$$\mu(B) \geq cr(B)^* \tag{2.11}$$

Then using the proof of Theorem 5 together with (2.8) yields

$$t^{-\frac{1}{p}} \left( \frac{1}{t} \int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 [(g^q)^*(t)]^{1/q}, \quad t > 0.$$
3. Applications

First we consider the Sobolev embedding theorem for metric spaces.

**Theorem 6.** Let $B_0 \subset X$ be a ball and let $Y = Y(X)$ be an r.i. space. Suppose that the operator $P_{\max\{p,q\}}$ is bounded on $Y$, and let $f$ and $g$ satisfy a $p-q$–Poincaré inequality on $4\sigma B_0$. Then there exists a constant $c > 0$, independent of $f$ and $g$, such that

$$\|f\chi_{B_0}\|_{Y^{p(\infty, \infty)}} \leq c(\|g\|_Y + \|f\chi_{B_0}\|_Y).$$

**Proof.** By Theorem 5 there are constants $c_1, c_2$ such that, for $0 < t < c_2 \mu(B_0)$,

$$t^{-\frac{1}{q}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^s(s) - [f\chi_{B_0}]^s(t))^p ds\right)^{1/p} \leq c_1 ([g^q]^s(t))^{1/q}.$$

If $t \geq c_2 \mu(B_0)$, we have

$$t^{-\frac{1}{q}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^s(s) - [f\chi_{B_0}]^s(t))^p ds\right)^{1/p} \leq (c_2 \mu(B_0))^{-\frac{1}{q}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^s(s))^p ds\right)^{1/p}.$$

Therefore,

$$t^{-\frac{1}{q}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^s(s) - [f\chi_{B_0}]^s(t))^p ds\right)^{1/p} \leq c_1 ([g^q]^s(t))^{1/q}.$$

for all $t > 0$. In view of our assumption on $P_{\max\{p,q\}}$ it follows, upon applying the $Y'$ norm to both sides of the previous inequality, that

$$\|t^{-\frac{1}{q}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^s(s) - [f\chi_{B_0}]^s(t))^p ds\right)^{1/p} \|_{Y} \leq c \|g\|_Y + \|f\chi_{B_0}\|_Y,$$

as we wished to show. 

**Remark 5.** Arguing as in Remark 4 we conclude that, if the global growth condition holds for all balls $B \subset X$, then, for all $f$ and $g$ as in Theorem 4, we have

$$\|f\|_{Y^{p(\infty, \infty)}} \leq c \|g\|_Y.$$

We now apply our symmetrization inequality to derive functional forms of Faber-Krahn inequalities (see [1] for a brief introduction to inequalities of this type). In the following we denote

$$\|f\|_0 := \mu(\text{supp}(f)).$$

We will also assume that $\mu$ is nonatomic.

**Theorem 7.** Let $B_0 \subset X$ be a ball and let $f$ be a function with $\text{supp}(f) \subset B_0$. Let $Z(X)$ be an r.i. space and let $\varphi_{Z'}$ denote the fundamental function of its associate space $Z'(X)$.

(i) Let $f$ be a $p-q$–Sobolev function and let $g$ be a gradient of $f$. If $\|f\|_0 < c_2 \mu(B_0)$, where $c_2$ is the constant of Theorem 5, then

$$\|f\|_{L^r(X)} \leq c \left[\|g^q\|_{Z(X)} \varphi_{Z'}((\|f\|_0)^{1/r})\right]^{1/q} \|f\|_0^{1/r - \frac{1}{q}}.$$

(ii) Let $f$ be a $1-q$–Sobolev function, $q > s$, and let $g$ be a gradient of $f$. If $\|f\|_0 < c_2 \mu(B_0)$, where $c_2$ is the constant of Theorem 5, then

$$\|f\|_{L^\infty(X)} \leq c \left[\|g^q\|_{Z(X)} \varphi_{Z'}((\|f\|_0)^{1/q})\right]^{1/q} \|f\|_0^{1/s - 1/q}.$$
Proof. (i) By Theorem 5 we have, for $0 < t < c_2 \mu(B_0)$,\n\begin{equation}
\tag{3.3}
t^{-\frac{c_2 p}{p'}} \left( \int_0^t \left[ f^*(s) - f^*(t) \right]^p ds \right)^{1/p} \leq c_1 \left[ (g^q)^{**}(t) \right]^{1/q}.
\end{equation}
Now, since $\|f\|_0 < c_2 \mu(B_0)$, using right-continuity of the decreasing rearrangement we can substitute $t = \|f\|_0$ in (3.3). Thus,\n\[ \|f\|_0^{-\frac{c_2 p}{p'}} \|f\|_{L^p} \leq c_1 \left[ \frac{1}{\|f\|_0} \int_0^{\|f\|_0} (g^q)^*(s) ds \right]^{1/q}. \]
Applying Hölder’s inequality we finally obtain\n\[ \|f\|_{L^p} \leq c_1 \left[ \|g^q\|_{Z\phi Z^*} (\|f\|_0) \right]^{1/q} \|f\|_0^{-\frac{c_2 p}{p'}}. \]
(ii) We first observe that $-\frac{d}{dt} f^{**}(t) = \left[ f^{**}(t) - f^*(t) \right]/t$. Thus, by Theorem 5 we have\n\[ -\frac{d}{dt} f^{**}(t) \leq c_1 t^{1/s-1} \left( \frac{1}{t} \int_0^t (g^q)^*(s) ds \right)^{1/q}. \]
Integrating over $(0, \|f\|_0)$ yields\n\[ \|f\|_{L^\infty(X)} - f^{**}(\|f\|_0) \leq c_1 \int_0^{\|f\|_0} t^{1/s-1/4} \left( \int_0^t (g^q)^*(s) ds \right)^{1/q} dt. \]
Thus, estimating the inner integral using Hölder’s inequality,\n\[ \|f\|_{L^\infty(X)} \leq c_1 \left[ \|g^q\|_{Z\phi Z^*} (\|f\|_0) \right]^{1/q} \int_0^{\|f\|_0} t^{1/s-1/4} dt + f^{**}(\|f\|_0) \]
\[ = c_1 \left[ \|g^q\|_{Z\phi Z^*} (\|f\|_0) \right]^{1/q} \frac{1}{1/s - 1/4} \|f\|_0^{1/s-1/q} + \|f\|_{L^1(X)} \]
\[ \leq c \left[ \|g^q\|_{Z\phi Z^*} (\|f\|_0) \right]^{1/q} \|f\|_0^{1/s-1/q}, \]
where in the last line we used the result obtained in the first half of the theorem.\n
Finally for a different connection between Poincaré inequalities and symmetrization with other interesting applications we refer to [17] (see also [13] for the relevant family of Poincaré inequalities). It would be of interest to extend the results of these papers to the metric setting.

4. Appendix: Vector fields satisfying Hörmander’s condition

We present a concrete example of our embedding theorem. But first let us briefly review some relevant definitions.

Let $X_1, \ldots, X_m$, be a collection of $C^\infty$ vector fields defined in a neighborhood $\Omega$ of the closure of $B(0, 1)$, the unit ball in $\mathbb{R}^n$. For a multiindex $\alpha = (i_1, \ldots, i_k)$ denote by $X_\alpha$ the commutator $[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}], \ldots]]$ of length $|\alpha| = k$. We shall assume that $X_1, \ldots, X_m$, satisfy Hörmander’s condition: there exists an integer $d$ such that the family of commutators, up to order $d$, $\{X_\alpha\}_{|\alpha| \leq d}$, spans the tangent space $\mathbb{R}^n$ at each point of $\Omega$. An admissible path $\gamma$ is a Lipschitz curve, $\gamma : [a, b] \to \Omega$, such that there exist functions $c_i(t)$, $a \leq t \leq b$, satisfying $\sum_{i=1}^m c_i^2(t) \leq 1$, and\n\[ \gamma'(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t)), \]
for a.e. $t \in [a, b]$. A natural metric (the so-called Carnot-Carathéodory metric) on $\Omega$ associated to $X_1, \ldots, X_m$, is defined by

$$\rho(\xi, \nu) = \min\{b \geq 0 : \text{there is an admissible path } \gamma : [0, b] \to \Omega \text{ such that } \gamma(0) = \xi \text{ and } \gamma(b) = \nu\}.$$  

Jerison [10] proved the following general theorem.

**Theorem 8.** For every $1 \leq p < \infty$, there exist a constant $c > 0$, and a radius $r_0$, such that, for every $\xi \in \{x \in \mathbb{R}^n : |x| < 1\}$, and every $r$, $0 < r < r_0$, for which $B(\xi, 2r) = \{\eta : \rho(\xi, \eta) < 2r\} \subset \Omega$, we have

$$\left(\int_{B(\xi, r)} |f(x) - f_{B(\xi, r)}|^p dx\right)^{1/p} \leq cr \left(\int_{B(\xi, r)} \left[\sum_{i=1}^m |X_i f(x)|\right]^p dx\right)^{1/p},$$

for all $f \in \mathcal{C}^\infty(B(\xi, r))$, where the integration is with respect to Lebesgue measure.

Moreover, by [21, §3 and Theorem 4], for an arbitrary compact set $K \subset \Omega$, there exist $c > 0$ and $r_0 > 0$ such that for all $\xi \in K$ and $\rho < r_0$ we have

$$|B(\xi, 2\rho)| = c|B(\xi, \rho)|,$$

where $|\cdot|$ indicates Lebesgue measure. Therefore, if $\xi \in \Omega$ and $r > 0$ such that $B(\xi, 4r) \subset \Omega$ and (1.1) hold for every $\rho < 4r$, then $B(\xi, 4r)$ with Lebesgue measure, is a homogeneous space. Thus, combining Theorem 8 and the theory of this paper implies the following Sobolev inequality for vector fields satisfying Hörmander’s condition.

**Theorem 9.** There exists a positive constant $c$ such that

$$\|f \chi_{B(\xi, r)}\|_{Y^p(\infty, s)} \leq c \left(\|\sum_{i=1}^m |X_i f|\|_Y + \|f\|_Y\right)$$

for all $f \in \mathcal{C}^\infty(B(\xi, 4r))$.

A slightly modified version of Theorem 9 sharpens known results. For example, starting with $1 - 1$–Poincaré inequality for vector fields satisfying Hörmander’s condition, we can proceed as in the proof of [16, Theorem 2] and together with Theorem 5 we obtain the inequality

$$\|f \chi_B - f_B\|_{Y^1(\infty, s)} \leq c\|\sum_{i=1}^m |X_i f|\|_Y,$$

where $B = B(\xi, r)$. According to [2, Theorem 3.1] the space $Y^1(\infty, s)$ is contained in the Hansson-Brézis-Wainger space

$$H_s(B) := \left\{f : \int_0^{||B||} \left(\frac{f^{**}(t)}{1 + \log \frac{||B||}{t}}\right)^s dt \frac{||B||}{t} < \infty\right\}$$

which, in turn, is known to be contained in the space described in [7, Theorem 6.1].

For other examples of metric spaces complying with the Poincaré inequality condition see [7].
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