Orbital and field angular momentum in the nucleon

D. Singleton *

Dept. of Physics, CSU Fresno, 2345 East San Ramon Ave. M/S 37, Fresno, CA 93740-8031

V. Dzhunushaliev †
Theor. Physics Dept., Kyrgyz State National University, 720024, Bishkek, Kyrgyzstan, and
Universität Potsdam, Institut für Mathematik, 14469, Potsdam, Germany

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Abstract

The nucleon spin problem raises experimental and theoretical questions regarding the contribution of the orbital angular momentum of the quarks to the total spin of the nucleon. In this article we examine the commutation relationships of various operators that contribute to the total angular momentum of the nucleon. We find that the sum of the orbital plus gluon field angular momenta should satisfy the angular momentum commutators, at least up to the one-loop level. This requirement on the sum of these operators imposes a non-trivial restriction on the form of the color electric and magnetic fields. This is similar to the magnetic monopole/electric charge system where it is only the sum of the orbital plus field angular momentum that satisfies the correct commutation relationships.
1. INTRODUCTION

Since the European Muon Collaboration (EMC) experiment at CERN where muons were scattered off polarized protons it has been realized that, contrary to the simple quark model, the spin of the nucleon comes not only from the spin of the valence quarks, but also has contributions from the quark orbital angular momentum and from the gluons. Mathematically this is written as

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma(q^2) + L_q(q^2) + J_g(q^2) \]  

where \( q^2 \) is the scale at which the operators are determined. The spin \( 1/2 \) of the nucleon is broken into contributions from the quark spin \( (\frac{1}{2} \Delta \Sigma(q^2)) \), the quark orbital angular momentum \( (L_q(q^2)) \), and the total gluon contribution \( (J_g(q^2)) \). (Technically one should also include the contribution of the photons, \( J_\gamma \) and weak gauge bosons \( J_{WZ} \)). In Ref. [3] an explicitly gauge-invariant form of the total QCD angular momentum was given

\[ \vec{J}_{QCD} = \int d^3x \left[ \frac{1}{2} \bar{\psi} \gamma_5 \psi + \psi^\dagger \left( -i \vec{D} \right) \psi + \vec{x} \times (\vec{E} \times \vec{B}) \right] \]  

where \( D_i = \partial_i - igA^a_i T^a \) is the covariant derivative with \( T^a \) the \( SU(3) \) generators. The first term in Eq. (2) is associated with the spin of the quarks, the second term with the orbital angular momentum of the quarks, and the last term is the total gluonic contribution. Eq. (2) can be cast in four-vector notation by defining

\[ J^{\mu\nu} = \int d^4x M^{0\mu\nu}(\vec{x}) \]  

where \( M^{\alpha\mu\nu} \) is a rank 3 tensor which can be defined in terms of the energy-momentum tensor, \( T^{\mu\nu} \)

\[ M^{\alpha\mu\nu} = T^{\alpha\nu} x^\mu - T^{\alpha\mu} x^\nu \]  

The QCD energy momentum tensor is given by

\[ T^{\mu\nu} = \frac{1}{4} \bar{\psi} \gamma^{\mu i} \vec{D} \gamma^{\nu j} \psi + \left( \frac{1}{4} g^{\mu\nu} F^{\alpha\beta a} F_{\alpha\beta} - F^{\mu\alpha a} F_{\alpha}^{\nu a} \right) \]  

where \( \gamma^{\mu i} \vec{D} \gamma^{\nu j} \) means that the indices are symmetrized, and \( a \) is a group index. With these definitions the angular momentum components of Eq. (2) can be expressed in terms
of Eq. (3) as \( J_{QCD}^i = \frac{1}{2} \epsilon_{ijk} J^{jk} \). In examining the commutators of the various terms in Eq. (4) we will deal with the angular momentum density operators. Also for the first two terms in Eq. (2) we will not always write out the \( \psi \)'s.

Conventionally a quantum angular momentum operator should satisfy the standard angular momentum commutation relationship

\[
[J_i, J_j] = i \epsilon_{ijk} J_k
\]

(6)

For example the first term \( (\frac{1}{2} \gamma_0 \vec{\gamma}_5) \) in Eq. (2) can be written as

\[
\frac{1}{2} \begin{pmatrix}
\vec{\sigma} & 0 \\
0 & \vec{\sigma}
\end{pmatrix}
\]

(7)

Since the \( \vec{\sigma} \) matrices satisfy \( [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \) one finds that the first term in Eq. (2) satisfies Eq. (3). At higher orders the various operators in Eq. (2) can be scale dependent and mix with one another, which complicates matters. However, in Ref. [4] it was shown that at the one-loop level the contribution, \( \frac{1}{2} \Delta \Sigma \), associated with the first term in Eq. (2), was scale independent so that, at least up to this order, the first operator in Eq. (2) should satisfy Eq. (3). The quantity, \( \vec{J}_{QCD} \), should always satisfy Eq. (3) regardless of \( q^2 \), since it is the total, conserved angular momentum of the nucleon. If we rearrange Eq. (2) by placing the spin term on the left hand side (so that \( \vec{J}_{QCD} - \vec{S} = \vec{L}_{QCD} + \vec{G}_{QCD} \)) then at least to the one loop level \( \vec{J}_{QCD} - \vec{S} \), and therefore \( \vec{L}_{QCD} + \vec{G}_{QCD} \), should satisfy Eq. (3). We will find that this requirement places restrictions on the color electric/magnetic fields.

2. QUARK ORBITAL ANGULAR MOMENTUM

The quark orbital angular momentum part of Eq. (2) can be written in component form as \( L_{QCD}^i = -i \epsilon^{ilm} x^l D^m \) with \( D^m = \vec{\sigma}^m - ig A^m T^a \). Inserting this into Eq. (3) gives after some commutator algebra

\[
[L_{QCD}^i, L_{QCD}^j] = -\epsilon^{ilm} \epsilon^{ipq} \left( x^l x^p [D^m, D^q] + x^l [D^m, x^p] D^q + x^p [x^l, D^q] D^m \right)
\]

(8)

Using \( [D^m, x^p] = \delta^{mp} \) and \( [D^m, D^q] = ig G^{mq} = ig e^{mqk} B^k \) (where the matrix \( B^k = B^{ak} T^a \) is the color magnetic field) we can transform Eq. (8) to
\[ [L^i_{QCD}, L^j_{QCD}] = -\epsilon^{ilm} \left( \epsilon^{mq} x^l D^q - \epsilon^{pl} x^p D^m + ig x^l x^p \epsilon^{mq} \epsilon^{pk} B^k \right) \]  

(9)

Using the standard formula \( \epsilon^{ilm} \epsilon^{jqm} = \delta^{ij} \delta^l_q - \delta^{iq} \delta^j_l \) several times and eliminating terms like \( x^l x^m \epsilon^{ilm} \) by symmetry we find

\[ [L^i_{QCD}, L^j_{QCD}] = \left( x^i D^j - x^j D^i + ig \epsilon^{ijl} x^l B^k \right) \]  

(10)

The first two terms can be written as \( x^i D^j - x^j D^i = i \epsilon^{ijk} \epsilon^{klm} x^l (-i D^m) = i \epsilon^{ijk} L^k_{QCD} \). Using this our final result is

\[ [L^i_{QCD}, L^j_{QCD}] = i \epsilon^{ijk} L^k_{QCD} - i \epsilon^{ijk} (g x^k x^l B^l) \]  

(11)

where in the last term we have renamed some of the dummy indices. The second term in Eq. (11) prevents \( \vec{L}_{QCD} \) from satisfying Eq. (9). This has also been remarked upon in Ref. [5]. One could force \( \vec{L}_{QCD} \) to satisfy Eq. (9) by requiring that color magnetic fields obey the condition \( x^l B^l = 0 \). This would be a strong restriction on the form of the color magnetic fields. This option is probably not viable, since some successful phenomenological models of baryons [6] (or see Ref. [7] for a general overview) explain the mass differences between baryons of similar quark content but different spins in terms of a color magnetic dipole-dipole interaction, in analogy with electromagnetism. For such a color magnetic dipole field \( x^l B^l \neq 0 \). In addition there is no real reason to try and force \( \vec{L}_{QCD} \) to satisfy Eq. (9), since from the discussion at the end of the previous section it is only the sum, \( \vec{L}_{QCD} + \vec{G}_{QCD} = \vec{J}_{QCD} - \vec{S} \), which was required to satisfy Eq. (9), at least up to the one-loop level.

Such a situation, where it is only the sum of the field plus orbital angular momentum which satisfies Eq. (9), is encountered in the electromagnetic system of an electric charge and magnetic monopole [8]. Also if one considers an electric charge/point magnetic dipole system one again finds [8] that it is only the combination of orbital plus field angular momentum that satisfies the angular momentum commutators. In the next section we make some assumptions about the form of the color electric and color magnetic fields of the nucleon, and show that the angular momentum commutators impose a non-trivial restriction on the form of these fields. In the electromagnetic examples cited the fields are known explicitly and one must only check that the resulting particle plus field angular momentum satisfies...
Eq. (6). In the QCD case, however, the explicit form of the color fields is not known, so this restriction coming from the angular momentum commutation relationships might give some extra insight into the structure of these fields.

3. FIELD ANGULAR MOMENTUM

At the one-loop level the combination $\vec{J}^{QCD} - \vec{S}$ obeys the angular momentum commutation rules, since each term separately obeys them ($\vec{J}^{QCD}$ satisfies the angular momentum commutation rules to all orders). Thus at the same one-loop level $\vec{L}^{QCD} + \vec{G}^{QCD} = \vec{J}^{QCD} - \vec{S}$ is also required to satisfy the angular momentum commutation relationships. Applying this requirement to the combination of orbital plus gluon field angular momentum we will find that certain restrictions are placed on the form of the fields. The commutator of the orbital plus field angular momentum operators is

$$[(L^i_{QCD} + G^i_{QCD}), (L^j_{QCD} + G^j_{QCD})] = [L^i_{QCD}, L^j_{QCD}] + [G^i_{QCD}, G^j_{QCD}]$$

$$+ [G^i_{QCD}, L^j_{QCD}] + [L^i_{QCD}, G^j_{QCD}]$$

(12)

The first term on the right hand side was calculated in the last section. In order to calculate the last three terms we need to express the field angular momentum in index notation as $G^i_{QCD} = \epsilon^{ijk} x^j \epsilon^{klm} E^a_k B^a_m$. Next, the matrix chromoelectric and chromomagnetic fields are usually defined as $E^a_p(x) = E^a_p(x)T^a$ and $B^a_p(x) = B^a_p(x)T^a$. Using the standard normalization for the $T^a$'s ($Tr[T^aT^b] = \delta^{ab}/2$ where $Tr$ is the trace) gives

$$E^a_p(x) = 2Tr[T^aE^a_p(x)]$$

$$B^a_p(x) = 2Tr[T^aB^a_p(x)]$$

(13)

An example of specific forms for the chromoelectric and chromomagnetic fields in a generic quark model can be found in Ref. [10]. With Eq. (13) the second commutator in Eq. (12) becomes

$$[G^i_{QCD}, G^j_{QCD}] = [\epsilon^{ilm} \epsilon^{mpq} x^l E^a_p B^a_q, \epsilon^{irs} \epsilon^{stu} x^r E^b_t B^b_u]$$

$$= 16\epsilon^{ilm} \epsilon^{mpq} \epsilon^{irs} \epsilon^{stu} x^l x^r [Tr(T^aE^a_p)Tr(T^aB^a_q), Tr(T^bE^b_t)Tr(T^bB^b_u)]$$

(14)

Now for any matrices $A, B, C, D$

$$[Tr(A)Tr(B), Tr(C)Tr(D)] = 0$$

(15)
Thus the right hand side of Eq. \((14)\) equals zero, and \([G^i_{QCD}, G^j_{QCD}] = 0\) with respect to the color group structure of \(G^i_{QCD}\). However, \([G^i_{QCD}, G^j_{QCD}]\) could have a possible non-zero contribution arising from the canonical commutation relationships of the non-Abelian gauge potentials, \(A^a_\mu\). First, using standard commutator techniques the complex commutator of Eq. \((14)\) can be simplified into a sum of two term commutators

\[
[E^a_p B^a_q, E^b_t B^b_a] = [E^a_p, E^b_t] B^b_a B^a_q + E^a_p [E^b_t B^a_q, B^b_a] + E^b_t [E^a_p, B^a_q] B^b_a + E^a_p [B^a_q, E^b_t] B^b_a \tag{16}
\]

In order to evaluate these commutators we expand the chromoelectric and chromomagnetic fields in terms of the gauge potentials. The general expressions are

\[
E^a_p = F^a_{0p} = \partial_0 A^a_p - \partial_p A^a_0 + g f^{abc} A^b_0 A^c_p
\]
\[
B^a_p = F^a_{mn} = \partial_m A^a_n - \partial_n A^a_m + g f^{abc} A^b_m A^c_n \tag{17}
\]

where \(f^{abc}\) are the group structure constants. In the canonical formalism the gauge potentials satisfy the following commutation rules \([11]\)

\[
[\partial_0 A^a_\mu(x, t), A^b_\nu(x', t)] = ig_{\mu\nu}\delta^{ab}\delta^3(x - x')
\]
\[
[A^a_\mu(x, t), A^b_\nu(x', t)] = [\partial_0 A^a_\mu(x, t), \partial_0 A^b_\nu(x', t)] = 0 \tag{18}
\]

These are similar to the Abelian field commutators except for the Kronecker delta, \(\delta^{ab}\), in the group indices. These commutation relationships along with the expansions given in Eq. \((17)\) give rise to the possibility that some of the commutators in Eq. \((16)\) may be non-zero. The expansion of the chromomagnetic field in Eq. \((17)\) involves only \(A^a_n\) and spatial derivatives of \(A^a_n\), so the second commutator of Eq. \((18)\) implies that \([B^a_q, B^b_u] = 0\) in Eq. \((16)\). The other commutators of Eq. \((16)\) involve the chromoelectric field, which in the most general case has \(\partial_0 A^a_p\) terms. Thus because of the non-zero first commutator in Eq. \((18)\) the remaining three terms from Eq. \((16)\) could in general be non-zero. For example, in the Abelian case a similar development using the time-dependent electric and magnetic fields of a photon results in the commutators between the electric and magnetic fields \(i.e. [E_i, B_j]\) being non-zero. These non-zero commutators between \(E_i\) and \(B_j\) ensure that the photon angular momentum operator satisfies Eq. \((1)\) \([12]\). Also in the general non-Abelian case one would expect some or all of the commutators from Eq. \((16)\) to be non-zero. However, since
the nucleon is a bound state we will assume that the gauge fields are static – \( \partial_0 A_\mu = 0 \) – so that the chromoelectric field for this specific case of the nucleon simplifies to

\[
E_\mu^a = -\partial_\mu A_0^a + g f^{abc} A_0^b A_c^a
\]  

(19)

Under this assumption the remaining commutators of Eq. (16) involve only the non-Abelian gauge potentials, \( A_\mu^a \), or their spatial derivatives. As with the pure chromomagnetic commutator, \([B_q^a, B_u^b]\), the second commutator from Eq. (18) now also implies that the remaining commutators from Eq. (16) are zero. This in turn gives \([G_{QCD}^i, G_{QCD}^j] = 0\). One subtle point in the above considerations is that rigorously one should impose the time-independent condition not on the gauge field operators, \( A_\mu^a \), via \( \partial_0 A_\mu^a |p\rangle = 0 \), but rather one should impose this as a condition on the nucleon bound states. Taking \(|p\rangle \) as the state vector for the nucleon bound state of a certain momentum and helicity, the time-independent condition becomes \( \partial_0 A_\mu^a |p\rangle = 0 \). Using this one can show that each term on the right hand side of Eq. (16) vanishes. For example, taking the third term and inserting \( 1 = \sum |p\rangle \langle p| \) or \( 1 = \sum |p\rangle \langle p| \) appropriately gives

\[
\sum E_b^b |p\rangle \langle p| [\partial_0 A_\mu^a, B_u^b] |p\rangle \langle p| B_q^a
\]  

(20)

Only the time derivative part of the chromoelectric field from inside the commutator has been written out, since by Eq. (18) this is the only term which could give a non-zero result in the commutator. Expanding the commutator part of Eq. (20) and again inserting \( 1 = \sum |p\rangle \langle p| \) appropriately gives

\[
\sum \left( \langle p\rangle [\partial_0 A_\mu^a, B_u^b] |p\rangle \langle p| B_q^a \right) = 0
\]

(21)

Each term above separately equals zero by the condition \( \partial_0 A_\mu^a |p\rangle = \partial_0 A_\mu^a |p\rangle = ... = 0 \). In the same way all the other terms from Eq. (18) which contain the possibly non-trivial \( \partial_0 A_\mu^a \) factors in the commutator can be shown to vanish by appropriately inserting \( 1 = \sum |p\rangle \langle p| \) and imposing the condition \( \partial_0 A_\mu^a |p\rangle = \partial_0 A_\mu^a |p\rangle = ... = 0 \).

The result that \([G_{QCD}^i, G_{QCD}^j] = 0\) may seem strange, but a similar result arises in the electromagnetic system of a charge and monopole at rest with respect to one another. The commutator of the components of the electromagnetic field angular momentum \((G_{EM}^i)\) among themselves could be calculated by expanding the electric and magnetic fields in
terms of the Abelian gauge potential and using equations Eqs. (14) (18) with the color indices dropped. The only non-trivial terms would come from commutators between $\partial^j A^i$ and $A^j$ or $\partial^j A^k$. However, since the electric and magnetic charges are at rest, the gauge fields are time-independent so that $\partial^0 A^i = 0$, and one again finds that the commutator for the components of the electromagnetic field angular momentum with itself are zero. In this electromagnetic case the calculation is actually much easier if one first calculates the explicit form for the field angular momentum, namely $G_{EM}^i = e g x^i / r$ where $e, g, x^i$ are the electric charge, magnetic charge and displacement between the two charges respectively. The commutator of the components of $G_{EM}^i$ among themselves is zero since $x^i / r$ commutes with itself. In the color field case this latter procedure was not an option since, unlike the Abelian charge/monopole case, one does not have explicit forms for the chromoelectric and chromomagnetic field, so that one can not obtain an explicit form for $G_{QCD}^i$.

In the present context the fact that $\vec{G}_{QCD}$ does not satisfy Eq. (6) is not crucial. If on the other hand one considered pure gluon states (glueballs) - so that $\vec{J}_{QCD} = \vec{G}_{QCD}$ - then the gluon field angular momentum would be the total, conserved angular momentum. In this case $\vec{G}_{QCD}$ must satisfy Eq. (6). The fact that we get $[G_{QCD}^i, G_{QCD}^j] = 0$ could have several possible resolutions: First, the form that we used from the color fields in Eq. (13) may not be valid for pure gluon states. One might have to consider functions $(\mathcal{E}_p(x), \mathcal{B}_p(x))$ which have a non-trivial commutation with one another beyond those already coming form the group factors and/or the annihilation and creation operator representation of the gauge fields. Also one could question the ability to split the spatial and group factors as in Eq. (13). Such a change in the form of the non-Abelian electric and magnetic fields would require a similar change in the analogous expression for the matrix gauge potentials, $A_\mu = T^a A^a_\mu$. Since this form of the potential is central in arriving at various fundamental relationships of Yang-Mills theory (e.g. $[D^\mu, D^\nu] = ig F^{\mu\nu}$) it is hard to see how one could change Eq. (13) without significantly affecting the structure of Yang-Mills theory. A second possible resolution to having $[G_{QCD}^i, G_{QCD}^j] = 0$ would be if pure gluon systems always had zero angular momentum (i.e. if glueballs were restricted to being spin zero objects). One might be tempted to immediately discard this conclusion since one expects by simple angular momentum addition that a glueball with two valence gluons should have angular
momenta of $0, 2, \ldots$, and for a glueball with three valence gluons one could have in addition odd angular momenta $(1, 3, \ldots)$. However, such considerations are based on the same type of simple arguments that had the spin of the nucleon coming from the spin of the valence quarks. Since the EMC experiments have shown that such a simple picture is not correct for quark bound states like the nucleon it is not unreasonable to raise questions about applying the same procedure to pure gluon bound states.

The third term in Eq. (12) can be written as

$$[G_{QCD}^i, L_{QCD}^j] = [\epsilon^{ilm} \epsilon^{mpq} x^l E^{pa} B^{qa}, \epsilon^{jrs} x^r (-i D^s)]$$

$$= -i \epsilon^{ilm} \epsilon^{mpq} \epsilon^{jrs} \left( x^l x^r [E^{pa} B^{qa}, D^s] + x^r [x^l, D^s] E^{pa} B^{qa} \right)$$

(22)

The first commutator can be broken into two parts as $[E^{pa} B^{qa}, \partial^s] - ig[E^{pa} B^{qa}, A^{ab} T^b]$. The last part can be reduced to $-ig[4Tr(T^a E^p) Tr(T^a B^q), T^b A^{ab}]$ (the $T^b A^{ab}$ matrix term should be bracketed by $\psi \dagger$ and $\psi$). This commutator could be non-trivial from either the group factor or from representing $A^{ab}$ (and therefore $E^p, B^q$) in terms of creation/annihilation operators. The group factors do not give a non-zero commutator since the first term is a trace over the group factors. Representing the gauge fields as operators also does not give a non-zero contribution to the commutator for the same reason as for the pure gluon commutator: only $[\partial_0 A^a_\mu(x, t), A^b_\nu(x', t)]$ is non-zero, and in the present case we are assuming static fields so $\partial_0 A^a_\mu(x, t) = 0$ (or more rigorously $\partial_0 A^a_\mu(x, t)|_{p_\perp^2} = 0$).

The first part becomes $[E^{pa} B^{qa}, \partial^s] = -\partial^s (E^{pa} B^{qa})$. The second commutator in Eq. (22) is just $[x^l, D^s] = -\delta^{ls}$. Combining these results and using $\epsilon^{ilm} \epsilon^{jrs} = \delta^{ij} \delta^{mr} - \delta^{ir} \delta^{mj}$ we find

$$[G_{QCD}^i, L_{QCD}^j] = i \left( x^l \epsilon^{ipq} E^{pa} B^{qa} \right)$$

$$+ i \epsilon^{jrs} x^l x^r \partial^s \left( E^{ia} B^{la} - E^{ia} B^{la} \right)$$

(23)

The other cross term from Eq. (12) can be obtained from Eq. (23) by multiplying the result by a minus sign to account for the reversed order of the terms, and by interchanging the indices $i \leftrightarrow j$. Combining these two cross terms of Eq. (12) gives

$$[G_{QCD}^i, L_{QCD}^j] + [L_{QCD}^i, G_{QCD}^j] = i \left( x^l \epsilon^{ipq} E^{pa} B^{qa} - x^l \epsilon^{ipq} E^{pa} B^{qa} \right)$$

$$+ i \epsilon^{jrs} x^l x^r \partial^s \left( E^{ia} B^{la} - E^{ia} B^{la} \right)$$
The first term on the right hand side becomes
\[ i\epsilon^{ijk} (\epsilon^{klm} x^l \epsilon^{mpq} E^{pa} B^{qa}) = i\epsilon^{ijk} G^{k}_{QCD}. \]
Also by making the definitions
\[ A^{ij} \equiv \epsilon^{irs} x^l x^r \partial^s (E^{ja} B^{la}) \]
and
\[ C^{ij} \equiv \epsilon^{irs} x^l x^r \partial^s (E^{la} B^{ja}) \]
we can write the right hand side of Eq. (24) as
\[ i\epsilon^{ijk} G^{k}_{QCD} + i(A^{ji} - A^{ij}) + i(C^{ij} - C^{ji}) = i\epsilon^{ijk} G^{k}_{QCD} + i\epsilon^{ijk} \left( \epsilon^{kmn} C^{mn} - \epsilon^{kmn} A^{mn} \right) \]
Using \( \epsilon^{kmn} \epsilon^{mrs} = -\delta^{kr} \delta^{ns} + \delta^{ks} \delta^{nr} \) and renaming some summed over dummy indices Eq. (25) becomes
\[ i\epsilon^{ijk} G^{k}_{QCD} + i\epsilon^{ijk} x^l x^k \partial^s \left( E^{sa} B^{la} - E^{la} B^{sa} \right) \]
Combining the result of Eq. (26) with the results from Eqs. (11) and (14) we find that the commutator for the combined field plus orbital angular momentum de nsity is
\[ [L^{i}_{QCD} + G^{i}_{QCD}, L^{i}_{QCD} + G^{i}_{QCD}] = i\epsilon^{ijk} \left( L^{k}_{QCD} + G^{k}_{QCD} \right) + i\epsilon^{ijk} x^l x^k \left( -gB^{la} T^{a} + \partial^s \left( E^{sa} B^{la} - E^{la} B^{sa} \right) \right) \]
where the \( L^{k}_{QCD} \) and \( gB^{la} T^{a} \) terms should be bracketed by \( \psi^{\dagger} \) and \( \psi \). The first line on the right hand side is the correct term to close the angular momentum algebra of the combination \( L^{i}_{QCD} + G^{i}_{QCD} \), however the second line prevents this. Now the combination \( L^{i}_{QCD} + G^{i}_{QCD} \) would satisfy Eq. (3) if the color electric and magnetic fields, and/or the spinors, \( \psi \), obeyed certain restrictions. For example, if
\[ -g x^{l} \bar{\psi} B^{la} T^{a} \psi + x^{l} \partial^s \left( E^{sa} B^{la} - E^{la} B^{sa} \right) = 0 \]
or in terms of the nucleon state vector
\[ \left< \frac{1}{2} \int d^3 x \left( -g x^{l} \bar{\psi} B^{la} T^{a} \psi + x^{l} \partial^s \left( E^{sa} B^{la} - E^{la} B^{sa} \right) \right) x^{k} \right| \frac{1}{2} \right> = 0 \]
then \( L^{i}_{QCD} + G^{i}_{QCD} \) would be a proper angular momentum density. Although these restrictions may seem ad hoc it should be noted that the commutation relationships of \( G^{i}_{QCD} \) with itself and with other parts of the QCD angular momentum operator will in general depend on the specific form of \( E^{ia} \) and \( B^{ia} \). Therefore it would be more surprising if no restrictions
were placed on these fields by the commutation relationships, or if the commutation relationships worked for any form of the fields. As an example one can consider the fields of a color charge located at $\vec{R}$ and a color magnetic dipole, $\vec{m}$, located at the origin. In the one-gluon exchange approximation the color fields will take the following electromagnetic-like form

$$E^{ia} = \frac{g x'^i}{r^3} T^a, \quad B^{ia} = \left( \frac{3 x'^i x'^l m^l}{r^5} - \frac{m^i}{r^3} \right) T^a$$  \hspace{1cm} (30)

where $\vec{x}' = \vec{x} - \vec{R}$, the color dipole has been oriented along the positive z-axis, and the right hand sides should be bracketed by $\psi^\dagger$ and $\psi$. From Refs. [2] [9] this particular field configuration results in $\vec{L}_{QCD} + \vec{G}_{QCD}$ satisfying the angular momentum commutator (i.e. the second term in Eq. (27) does not contribute). The condition in Eq. (28) does not uniquely determine the color fields, since $E^{la} = B^{la}$ and $x'^l B^{la} = 0$ also satisfy Eq. (28).

As another example one could postulate that the color fields are split into a perturbative plus non-perturbative part → Perturbative + Non-perturbative. The perturbative part could be of some form similar to the one-gluon exchange form given in Eq. (30), while the non-perturbative part could be the purely chromoelectric flux tubes that are thought to form between quarks in the standard picture of confinement. As already mentioned a perturbative part of the form given in Eq. (30) would make the commutation relationships come out correctly, and a purely chromoelectric (i.e. $B^{la} = 0$), non-perturbative part would also make the commutation relationships come out correctly so that Eq. (28) would be satisfied.

Even if the above examples are not entirely realistic or do not give a unique restriction of the fields, the point is that even more realistic field configurations, determined dynamically from the QCD field equations via lattice gauge theory or some other method, may have kinematical restrictions coming from the angular momentum commutators.

In contrast, for electromagnetic systems such as the electric charge/magnetic charge or electric charge/magnetic dipole, one knows the form of the fields from the outset. In these cases one just needs to check that the fields lead to the correct angular momentum commutation relationship for the sum of the orbital plus field angular momentum. However, if the form of the electric and magnetic fields of these systems were not given analytically, then also in these cases one would find some restrictions on the form of the fields coming from the angular momentum algebra. Actually the Dirac condition on magnetic charges –
\( eg = 1/2 \) – can be viewed as a restriction on the electric charge/magnetic charge system that arises from the angular momentum algebra \[8\].

4. DISCUSSION AND CONCLUSION

Using the gauge-invariant form of the QCD angular momentum operator given in Ref. \[3\] we have investigated the commutators of the various terms in Eq. (2). There were two crucial assumptions made in arriving at these results. First, we only considered up to one-loop corrections to the various parts of the angular momentum operators. This greatly simplified the analysis since the quark spin operator (the first term in Eq. (2)) only begins to experience renormalization scaling effects at the two-loop level. This implies that the quark spin operator still satisfies the angular momentum commutation relationships up to one-loop. This in turn implies that \( J^i_{QCD} - S^i \), and therefore \( L^i_{QCD} + G^i_{QCD} \), should satisfy the angular momentum commutation relationships up to one-loop. This requirement was the starting point in the calculations leading to Eqs. (28) (29). Second, in order to obtain the commutator of some of the angular momentum parts (e.g. the field part, \( G^i_{QCD} \)) we assumed that since the nucleon is a bound state that the gauge potentials were time-independent so that we could set \( \partial_0 A^a_\mu = 0 \) (or more rigorously taking the time-independent condition to apply to the state vector as \( \partial_0 A^a_\mu|p_{1/2} = 0 \)). From Eq. (18) the only non-trivial field commutators involved \( \partial_0 A^a_\mu \) so for the nucleon the chromoelectric and chromomagnetic field operators commuted with one another, which in turn implied that \( [G^i_{QCD}, G^j_{QCD}] = 0 \).

Without this assumption both \( [G^i_{QCD}, G^j_{QCD}] \) and the commutators like \( [G^i_{QCD}, L^j_{QCD}] \) would have been much more complicated, and our final results – Eqs. (28) (29) – would be substantially altered.

In analogy with certain electromagnetic systems, such as the charge/monopole system \[8\] or the charge/magnetic dipole system \[9\], where it is only the combination of orbital plus field angular momentum which satisfies the correct commutation relationships, we examined the commutator for the combination \( L^i_{QCD} + G^i_{QCD} \). Although for \( L^i_{QCD} \) we did not make any assumptions about the specific form of the color fields, we did have to assume some general form - Eq. (13) - for the color fields in order to calculate the commutators containing \( G^i_{QCD} \). The calculation of the commutator of \( L^i_{QCD} + G^i_{QCD} \), given in Eq. (27), shows that in
addition to the expected $i\varepsilon^{ijk}(L_{QCD}^k + G_{QCD}^k)$ there were additional terms (the second line on the right hand side of Eq. (27)) which ruined the closure of the commutation relationship. We took this to indicate one of the following two possibilities:

1. The assumptions about the particular form of the color fields (i.e. Eq. (13) or that the color fields are static) were not valid. However, given the close connection of this form of the fields with the basic structure of Yang-Mills theory it is not apparent how one could alter this without altering the structure of Yang-Mills theory.

2. The form of the color fields are restricted in some way (e.g. Eqs. (28) or (29)) so that $L_{QCD}^i + G_{QCD}^i$ does satisfy the angular momentum commutation rules.

Both of these possibilities indicate that some restrictions are placed on the form of the color electric and magnetic fields by the commutation relationships.

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