CONTINUOUS QUASIPERIODIC SCHRÖDINGER OPERATORS WITH GORDON TYPE POTENTIALS

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Abstract. Let us concern the quasi-periodic Schrödinger operator in the continuous case,

\[(Hy)(x) = -y''(x) + V(x, \omega x)y(x),\]

where \(V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}\) is piecewisely \(\gamma\)-Hölder continuous with respect to the second variable. Let \(L(E)\) be the Lyapunov exponent of \(H_y = E_y\). Define \(\beta(\omega)\) as

\[\beta(\omega) = \limsup_{k \to \infty} -\frac{\ln ||k\omega||}{k}.\]

We prove that \(H\) admits no eigenvalue in regime \(\{E \in \mathbb{R} : L(E) < \gamma\beta(\omega)\}\).

1. Introduction

In this note, we study the continuous quasi-periodic Schrödinger operator, which is given by

\[(Hy)(x) = -y''(x) + V(x, \omega x)y(x),\]

where \(V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}\) is the potential and \(\omega \in \mathbb{R}\) is frequency.

We are interested in a particular class of functions \(V\). \(V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}\) is called piecewisely \(\gamma\)-Hölder continuous with respect to the second variable, denoted by \(A_\gamma\), if \(V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}\) is measurable and there exist \(a_1 < a_2 < \cdots < a_m\) (\(a_1 = 0\) and \(a_m = 1\)) such that

\[
\sup_{(x,y) \in (\mathbb{R}/\mathbb{Z})^2} |V(x,y)| + \sup_{x \in \mathbb{R}/\mathbb{Z}} \sup_{1 \leq i \leq m-1} \sup_{y_1, y_2 \in [a_i, a_{i+1}]} \frac{|V(x,y_1) - V(x,y_2)|}{|y_1 - y_2|^\gamma} < \infty.
\]

We emphasize that \(V \in A_\gamma\) implies \(V(x,y)\) is bounded and continuous with respect to \(y\). \(A_\gamma\) contains a special case \(V = V_1(x) + V_2(\omega x)\), where \(V_1 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) is measurable function and \(V_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) is piecewisely \(\gamma\)-Hölder continuous. We always assume that potentials \(V \in A_\gamma\) in this paper.

Recently, there has been a remarkable development of arithmetically spectral transition (singular spectrum and Anderson localization) for discrete quasiperiodic operator \(18\), in particular for explicit models: almost Mathieu operator \(1\) \(11\) \(13\) \(14\), Maryland model \(12\) and extended Harper model \(9\) (Jacobi operator). For the continuous, arithmetic phase transitions are currently very far from being established. The quantitative arguments for discrete case is to show the absence of eigenvalues if the frequency can be approximated by a rational number well (Gordon type potential in one dimension), which improved the previous result \(6, 19\), obtaining sharp thresholds for the smallness of small denominators in terms of the Lyapunov exponents. For various other recent developments on the Gordon-type potentials, see Damanik-Stolz \(4\), Damanik’s survey paper \(3\) and references therein, and Jitomirskaya-Zhang \(17\). The purpose of this note is to obtain a similar sharp result for the continuous

\[\text{We should mention that Gordon and Nemirovski also obtained the absence of eigenvalues for discrete Schrödinger operators with Gordon type potentials in higher dimensions} 6. \text{It is a interesting problem to make this arithmetic sharp.}\]
case. We should mention that the localization part for continuous case is only known for a full Lebesgue measure subset of Diophantine frequencies [2].

**Theorem 1.1.** Let $H$ be a quasiperiodic Schrödinger operator,
\[(Hy)(x) = -y''(x) + V(x, \omega x)y(x),\]
where $V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}$ is piecewisely Hölder continuous with respect to the second variable. Let $L(E)$ be the Lyapunov exponent of $Hy = Ey$. Define $\beta(\omega)$ as
\[
\beta(\omega) = \limsup_{k \to \infty} \frac{-\ln ||k\omega||}{k}.
\]
Then $H$ does not have any eigenvalue in the regime \(\{E \in \mathbb{R} : L(E) < \gamma \beta(\omega)\}\).

**Remark 1.2.**
- By the recent results of [1,12], the statement of Theorem 1.1 for the case $\gamma = 1$ is sharp for discrete Schrödinger operators with regular and singular potentials. We refer the reader to [10] for the potentials with finitely many singular points. We expect that Theorem 1.1 is sharp for $\gamma = 1$ and even for $0 < \gamma < 1$.
- The potential $V$ here is only Hölder continuous. Such Schrödinger operator with rough potential was studied in [8,15].

As a direct corollary, we obtain

**Corollary 1.3.** Let $H$ be a quasiperiodic Schrödinger operator,
\[(Hy)(x) = -y''(x) + V(x, \omega x)y(x),\]
where $V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}$ is a Hölder continuous with respect to the second variable. Suppose the frequency $\omega$ satisfies
\[
\limsup_{k \to \infty} \frac{-\ln ||k\omega||}{k} = \infty.
\]
Then $H$ does not have any eigenvalue.

If the analytical potential $V$ is small or the energy $E$ is large, the Lyapunov exponent is zero in the spectrum [10]. Thus we have the following result.

**Corollary 1.4.** Let $H_\lambda$ be a quasiperiodic Schrödinger operator,
\[(H_\lambda y)(x) = -y''(x) + \lambda V(x, \omega x)y(x),\]
where $V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}$ is analytic. Suppose the frequency $\omega$ satisfies
\[
\limsup_{k \to \infty} \frac{-\ln ||k\omega||}{k} > 0.
\]
Then there exists $\lambda_0 > 0$ such that $H_\lambda$ does not have any eigenvalue for $|\lambda| \leq \lambda_0$.

**Corollary 1.5.** Let $H$ be a quasiperiodic Schrödinger operator,
\[(Hy)(x) = -y''(x) + V(x, \omega x)y(x),\]
where $V : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}$ is analytic. Suppose the frequency $\omega$ satisfies
\[
\limsup_{k \to \infty} \frac{-\ln ||k\omega||}{k} > 0.
\]
Then there exists $E_0 > 0$ such that $H$ does not have any eigenvalue in regime $[E_0, \infty)$.
2. Proof of Theorem 1.1

Let $u$ be a solution of $Hu = Eu$. Define the transfer matrix $T(E, y, x)$ as

$$T(E, y, x) \begin{bmatrix} u'(y) \\ u(y) \end{bmatrix} = \begin{bmatrix} u'(x) \\ u(x) \end{bmatrix}.$$ (3)

**Lemma 2.1**. [14] Let $B \in SL(2, \mathbb{R})$ and $\varphi$ be a unit vector in $\mathbb{R}^2$, then

$$\max \{ ||B^2\varphi||, ||B\varphi||, ||B^{-1}\varphi|| \} \geq \frac{1}{4}.$$ (7)

Let $\frac{q_n}{n}$ be the continued fraction expansion to $\omega$. By (2), one has

$$\limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n} = \beta(\omega).$$ (4)

Below, $\varepsilon > 0$ is arbitrarily small, and $c(C)$ is small (large) constant depending on $E$, $\omega$ and the potential $V$. Since $V \in A_{\gamma}$, there exist $a_1, a_2, \ldots, a_m$ ($a_1 = 0$ and $a_m = 1$) such that (1) holds.

Write down the eigen-equation $Hy = Ey$ in first order,

$$\begin{bmatrix} y'(x) \\ y(x) \end{bmatrix}' = \begin{bmatrix} 0 & V(x, \omega x) - E \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y'(x) \\ y(x) \end{bmatrix}. $$ (5)

By the definition of transfer matrix and Lyapunov exponent (see [3]), we have

$$||T(E, x, y)|| \leq Ce^{(L(E) + \varepsilon)|x-y|}. $$ (6)

For simplicity, sometimes we ignore the dependence of $E$ in $T(E, x, y)$ and $L(E)$. By (4), there exists a sequence $\{ q_{n_k} \}$ such that

$$q_{n_k+1} \geq e^{(\beta(\omega)-\varepsilon)q_{n_k}}.$$ (7)

By the property of continued fraction expansion, one has

$$||q_{n_k} \omega|| \leq e^{-(\beta(\omega)-\varepsilon)q_{n_k}},$$

where $||x|| = \text{dist}(x, \mathbb{Z})$.

We start the proof with some basic Lemmas.

**Lemma 2.2**. The following estimate holds,

$$\int_0^{q_{n_k}} |V(t, \omega t) - V(t, \omega(t + q_{n_k}))| dt \leq Ce^{-(\gamma \beta - \varepsilon)q_{n_k}}.$$ (9)

**Proof.** Let

$I = \{ t \in [0, q_{n_k}] : \text{dist}(\omega t, a_i + \ell) \geq 2e^{-(\beta(\omega)-\varepsilon)q_{n_k}} \text{ for all } i = 1, 2, \ldots, m-1 \text{ and } \ell = 0, 1, 2, \ldots, q_{n_k} - 1 \}$. (10)

By (7) and the assumption on $V$, one has

$$|V(t, \omega t) - V(t, \omega(t + q_{n_k}))| dt \leq Ce^{-(\gamma \beta - \varepsilon)q_{n_k}} \text{ for } t \in I.$$ (11)

We also have

$$|I'| \leq Cq_{n_k} e^{-(\beta(\omega)-\varepsilon)q_{n_k}} \leq Ce^{-(\beta(\omega)-\varepsilon)q_{n_k}},$$ (12)

where $I' = [0, q_{n_k}] \setminus I$. Thus we obtain

$$\int_0^{q_{n_k}} |V(t, \omega t) - V(t, \omega(t + q_{n_k}))| dt = \int_I |V(t, \omega t) - V(t, \omega(t + q_{n_k}))| dt + \int_{I'} |V(t, \omega t) - V(t, \omega(t + q_{n_k}))| dt \leq Ce^{-(\gamma \beta - \varepsilon)q_{n_k}}.$$ (13)

$\square$
Lemma 2.3. The following estimates hold

\[ \|T(E, 0, q_{n_k}) - T(E, q_{n_k}, 2q_{n_k})\| \leq Ce^{(L(E) - \gamma \beta) \epsilon q_{n_k}}, \]

and

\[ \|T(E, 0, -q_{n_k}) - T(E, 0, q_{n_k})^{-1}\| \leq Ce^{(L(E) - \gamma \beta) \epsilon q_{n_k}}. \]

Proof. Consider the two differential equations \( y_1, y_2 \) on \([0, q_{n_k}]\) with the same initial condition at 0,

\[ \begin{bmatrix} y_1'(x) \\ y_1(x) \end{bmatrix} = \begin{bmatrix} 0 & V(x, \omega x) - E \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_1(x) \end{bmatrix}. \]

and

\[ \begin{bmatrix} y_2'(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0 & V(x + q_{n_k}, \omega(x + q_{n_k})) - E \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_2(x) \\ y_2(x) \end{bmatrix}. \]

Let \( Y(x) = \begin{bmatrix} y_1(x) - y_2(x) \\ y_1(x) - y_2(x) \end{bmatrix} \). Denote by

\[ F(t) = \begin{bmatrix} 0 & V(t, \omega t) - V(t, \omega(t + q_{n_k})) \\ 0 & 0 \end{bmatrix} T(E, 0, t) \begin{bmatrix} y_2'(0) \\ y_2(0) \end{bmatrix}. \]

Thus

\[ Y'(x) = F(x) + \begin{bmatrix} 0 & V(x, \omega x) - E \\ 1 & 0 \end{bmatrix} Y(x), \]

and \( Y(0) = 0 \), since \( V(x + q_{n_k}, \omega(x + q_{n_k})) = V(x, \omega(x + q_{n_k})) \). By the constant variation method, we obtain that

\[ Y(x) = T(x) \int_0^x T^{-1}(t)F(t)dt \]

(10)

\[ = \int_0^x T(x)T^{-1}(t)F(t)dt. \]

(11)

We give the proof of (11) in the Appendix.

By Lemma 2.2, we have for \( 0 < x \leq q_{n_k} \),

\[ \int_0^x \|F(t)\|dt \leq \left\| \begin{bmatrix} y_2'(t) \\ y_2(t) \end{bmatrix} \right\| \int_0^x \left\| \begin{bmatrix} 0 & V(t, \omega t) - V(t, \omega(t + q_{n_k})) \\ 0 & 0 \end{bmatrix} \right\|dt \]

\[ \leq e^{-(\gamma \beta - \epsilon)q_{n_k}}\|T(E, 0, t)\| \left\| \begin{bmatrix} y_2'(0) \\ y_2(0) \end{bmatrix} \right\| \]

(12)

\[ \leq Ce^{-(\gamma \beta - \epsilon)q_{n_k}e^{L}t} \left\| \begin{bmatrix} y_2'(0) \\ y_2(0) \end{bmatrix} \right\|, \]

where the second inequality holds by (9). For \( x > t \), by (8) again, one has

\[ \|T(x)T^{-1}(t)\| = \|T(E, t, x)\| \leq Ce^{(L+\epsilon)|x-t|}. \]

(13)
By (11), (12) and (13), we obtain
\[ ||Y(q_{n_k})|| \leq C \epsilon^{(L-\gamma\beta+c)q_{n_k}} \left[ \begin{array}{c} y'_2(0) \\ y_2(0) \end{array} \right] ||, \]
which implies (8) by the arbitrary choice of \[ \left[ \begin{array}{c} y'_2(0) \\ y_2(0) \end{array} \right]. \]

By the fact that \[ T(E, 0, q_{n_k})^{-1} = T(E, q_{n_k}, 0) \] and following the similar arguments of proof of (8), we can prove (9).

The following lemma is well known for Schrödinger operator. We give the proof for completeness.

**Lemma 2.4.** Suppose \( u \) is an eigensolution, that is \( Hu = Eu \) and \( u \in L^2(\mathbb{R}) \) for some \( E \), then
\[ \limsup_{x \to \infty} || \left[ \begin{array}{c} u'(x) \\ u(x) \end{array} \right] || = 0. \]

**Proof.** Suppose
\[ || \left[ \begin{array}{c} u'(x_0) \\ u(x_0) \end{array} \right] || \geq \epsilon. \]
By equation (5), one has
\[ \int_{x_0-1}^{x_0+1} (|u'(x)| + |u(x)|)dx \geq c\epsilon. \]

By a result of [20, Lemma 3.1], we have
\[ \int_{x_0-2}^{x_0+2} |u(x)|dx \geq c\epsilon. \]

Thus
\[ \int_{x_0-2}^{x_0+2} |u(x)|dx \geq c\epsilon, \]
which leads to (14) by the fact that \( u \in L^2(\mathbb{R}). \)

**Proof of Theorem 1.1**

**Proof.** Let \( E \) be such that \( L(E) < \gamma\beta(\omega) \). Suppose \( Hy = Ey \) and \( y \in L^2(\mathbb{R}) \). Setting \( B = T(E, 0, q_{n_k}) \) and applying Lemma 2.1, we have
\[ \max \{ ||B^2\varphi||, ||B\varphi||, ||B^{-1}\varphi|| \} \geq \frac{1}{4}, \]
where \( \varphi = \left[ \begin{array}{c} y'(0) \\ y(0) \end{array} \right] \) is unit. We claim that
\[ \max \{ ||y'(-q_{n_k})/y(-q_{n_k})||, ||y'(q_{n_k})/y(q_{n_k})||, ||y'(2q_{n_k})/y(2q_{n_k})|| \} \geq \frac{1}{8}. \]

If \[ || \left[ \begin{array}{c} y'(q_{n_k}) \\ y(q_{n_k}) \end{array} \right] || = ||B\varphi|| \geq \frac{1}{4}, \] there is nothing to prove. If \[ ||B^{-1}\varphi|| \geq \frac{1}{4}, \] by (9),
\[ || \left[ \begin{array}{c} y'(-q_{n_k}) \\ y(-q_{n_k}) \end{array} \right] || = ||T(E, 0, -q_{n_k})\varphi|| \geq \frac{1}{8}. \]
If \( \|B^2 \varphi\| \geq \frac{1}{4} \) and \( \|B \varphi\| < \frac{1}{4} \), by (8),
\[
\begin{align*}
\left\| \begin{bmatrix} y'(2q_{n_k}) \\ y(2q_{n_k}) \end{bmatrix} - B^2 \varphi \right\| &= \left\| T(E, q_{n_k}, 2q_{n_k}) B \varphi - B^2 \varphi \right\| \\
&\leq \left\| T(E, q_{n_k}, 2q_{n_k}) - B \right\| \left\| B \varphi \right\| \\
&\leq e^{(L(E)-\gamma \beta(\omega,\epsilon)q_{n_k})} \leq \frac{1}{8}.
\end{align*}
\]
This implies
\[
\left\| \begin{bmatrix} y'(2q_{n_k}) \\ y(2q_{n_k}) \end{bmatrix} \right\| \geq \frac{1}{8}.
\]
We finish the proof of the claim. By Lemma 2.4 this is impossible. □

3. Appendix. Proof of (11)

Proof. Let \( \tilde{Y} \) be
\[
\tilde{Y}(x) = T(x) \int_0^x T^{-1}(t) F(t) dt
\]
(15)
Obviously,
\[
\tilde{Y}(0) = 0.
\]
By the definition of \( T(x) \) and (3), we have
\[
T'(x) = \begin{bmatrix} 0 & V(x, \omega x) - E \\ 1 & 0 \end{bmatrix} T(x).
\]
(16)
By (15) and (16), we have
\[
\tilde{Y}'(x) = F(x) + \begin{bmatrix} 0 & V(x, \omega x) - E \\ 1 & 0 \end{bmatrix} \tilde{Y}(x).
\]
Thus \( Y \) and \( \tilde{Y} \) satisfy the same differential equation and initial condition. This implies \( Y \equiv \tilde{Y} \). □

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