HOMOLOGICAL PROPERTIES OF PARAFREE LIE ALGEBRAS

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Abstract. In this paper, an explicit construction of a countable parafree Lie algebra over \( \mathbb{Z}/2 \) with nonzero second homology is given. It is also shown that the cohomological dimension of the pronilpotent completion of a free noncyclic finitely generated Lie algebra over \( \mathbb{Z} \) is greater than two. Moreover, it is proven that there exists a countable parafree group with nontrivial \( H_2 \).

1. Introduction

Historical remarks. In 1960-s G. Baumslag introduced the class of parafree groups [1], [2], [3], [4]. Recall that a group \( G \) is called parafree if \( G \) is residually nilpotent and there exists a free group \( F \) and a homomorphism \( F \to G \), which induces isomorphisms of the lower central quotients \( F/\gamma_i(F) \simeq G/\gamma_i(G) \), \( i \geq 1 \). The main motivation to introduce and study parafree groups was the problem how to characterize the class of groups of cohomological dimension one. G. Baumslag called the parafree groups as "just about free" and had a hope that some of them may give examples of non-free groups of cohomological dimension one. At the end of 1960-s the results of J. Stallings and R. Swan appeared [13], [14]. By Stallings-Swan theorem, the groups of cohomological dimension one are free, hence, the non-free parafree groups constructed in [1], [2] have cohomological dimension at least two. Despite this fact, there are series of properties which parafree groups share with free groups. G. Baumslag during many years studied these properties and stated a number of natural problems about parafree groups. The main conjecture about homological properties of parafree groups is known as Parafree Conjecture: for a finitely generated parafree group \( G \), \( H_2(G) = 0 \). There is an additional strong form of the conjecture: for a finitely generated parafree group \( G \), \( H_2(G) = 0 \) and the cohomological dimension of \( G \) is \( \leq 2 \). For the formulation of these conjectures we refer to [8], see also [9] for the discussion of topological applications of parafree groups. T. Cochran wrote the following: "Some (including Baumslag) believe that all finitely-generated parafree groups have cohomological dimension at most 2 and have trivial \( H_2 \)." In [8] the Parafree Conjecture is formulated for finitely generated groups only is due to the result of A.K. Bousfield [6]: for a non-cyclic finitely generated free group \( F \), and its pronilpotent completion \( \hat{F} := \varprojlim F/\gamma_i(F) \), \( H_2(\hat{F}) \) is uncountable. The pronilpotent completion \( \hat{F} \) is parafree, hence it gives an example of a non-free parafree group with \( H_2 \neq 0 \). Observe that, the initial interest of G. Baumslag was not in just finitely generated parafree groups, but in parafree groups in general, in particular, he constructed locally free non-free parafree groups in [4], and the problem of sharing the properties of parafree groups with free groups first was formulated in general, not only for finitely generated case.

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In 2010-2014 the third author worked with G. Baumslag on constructing examples of countable (non-finitely generated, in general) parafree groups with nonzero \( H_2 \) and (or) cohomological dimension greater than two. That project was not finished and, for the moment, we are not able to present a (probably uncountable) parafree group of cohomological dimension greater than two. The problem whether the cohomological length of \( \hat{F} \) is greater than two is still open. Next we will show that there exist countable parafree groups with \( H_2 \neq 0 \). However, an explicit construction of such groups seems problematic. In this paper we make a step in this direction, by constructing explicit examples of countable parafree Lie algebras with \( H_2 \neq 0 \) as well as an example of a parafree Lie algebra of cohomological length greater than two.

**Countable parafree groups with nonzero \( H_2 \).** If \( F \) is a finitely generated free group of rank at least 2, we say that \( G \) is a parafree subgroup of \( \hat{F} \), if \( F \subseteq G \subseteq \hat{F} \) and the embedding \( F \subseteq G \) induces isomorphisms \( F/\gamma_i(F) \cong G/\gamma_i(G) \), \( i \geq 1 \). It easy to prove that there exists a countable parafree group with non-trivial \( H_2 \) using the following proposition.

**Proposition 1.** \( \hat{F} \) is a filtered union of its countable parafree subgroups.

This proposition implies that \( \hat{F} = \varinjlim G \), where \( G \) runs over the directed set of countable parafree subgroups of \( \hat{F} \). Since \( H_2 \) commutes with filtered colimits, we obtain \( \varinjlim H_2(G) = H_2(\hat{F}) \neq 0 \). Therefore, there exists \( G \) such that \( H_2(G) \neq 0 \). Moreover, using that \( H_2(\hat{F}) \) is uncountable, we obtain the following.

**Corollary.** There exists an uncountable set of countable parafree subgroups \( G \subset \hat{F} \) such that \( H_2(G) \neq 0 \).

**Parafree Lie algebras.** The concept of parafreeness can be naturally extended from groups to other algebraic categories, such as Lie algebras or augmented associative algebras. The following question rises naturally: what kind of properties do free and parafree objects have in common? Can one construct a finitely generated but not finitely presented parafree object? What can one say about homology and cohomological dimension of parafree objects?

Parafree Lie algebras are considered in [5]. Suppose that \( R \) is a commutative ring. For a Lie algebra \( A \) over \( R \) we denote by \( A^{(i)} \) the lower central series of \( A \), so that \( A^{(1)} = A \) and inductively \( A^{(i+1)} = [A^{(i)}, A] \). We also denote by \( A^{(\omega)} \) the intersection of all \( A^{(n)} \):

\[
A^{(\omega)} = \bigcap_{i \geq 1} A^{(i)}.
\]

A Lie algebra \( A \) is called parafree if \( A^{(\omega)} = 0 \) and there exists a free Lie algebra \( E \) with a homomorphism \( E \rightarrow A \) which induces isomorphisms \( E/E^{(i)} \cong A/A^{(i)} \) for all \( i = 1, 2, \ldots \).

For a Lie algebra \( A \), the completion \( \hat{A} \) is the inverse limit \( \lim \lim A/A^{(i)} \). It follows immediately from definition that, for a parafree Lie algebra, the completion is the same as of a free one.

The main results of this paper are Theorems A and B, formulated below. Let us denote by \( B \) a Lie algebra over \( \mathbb{Z}/2 \) generated by elements \( a, b, \{x_i\}, \{y_i\} \) for \( i \geq 1 \).
with the following relations\footnote{For elements of Lie algebras we will use the left-normalized notation \([a_1, \ldots, a_i] := [[a_1, \ldots, a_{i-1}], a_i]\) and the following notation for Engel commutators 
\([a, 0 b] := a, \ldots, [a, i+1 b] = [[a, i b], b]\) for \(i \geq 0\).}

\[ x_1 = [a, b, b] + [x_2, b, b], \quad y_1 = [a, b, a] + [y_2, b, b], \]
\[
\vdots
\]
\[ x_i = [a, b, b] + [x_{i+1}, 2b, b], \quad y_i = [a, b, a] + [y_{i+1}, 2b, b], \text{ for all } i \geq 1.\]

Theorem A. The Lie algebra \(B/B^{(\omega)}\) is a parafree and \(H_2(B/B^{(\omega)}) \neq 0\).

Theorem B. Let \(L\) be the free Lie algebra over \(\mathbb{Z}\) of rank two. The homology group \(H_2(\hat{L})\) contains a 2-divisible element. Hence, the cohomological dimension of \(\hat{L}\) is greater than two.

The proofs are based on the method used in the solution of Bousfield’s problem \cite{bousfield}. All results in \cite{bousfield} are for groups, here we prove their analogs for Lie algebras, in particular, we introduce the Lie-analog \(L\) of the lamplighter group and show that certain elements of the second homology are nonzero by projecting them onto the elements of \(H_2(\hat{L})\). The main tool for showing that an element in \(H_2(\hat{L})\) is nonzero is given by Corollary \ref{cor:non-triviality} non-triviality of an element in homology follows from the non-rationality of certain function.

Observe that, the existence of a countable parafree Lie algebra with \(H_2 \neq 0\) immediately implies the existence of a countable parafree associative augmented algebra (i.e. an augmented algebra with trivial intersection of the augmentation powers and the same augmentation quotients as a free augmented algebra) with \(H_2 \neq 0\). For that, one can take, for example, the universal enveloping algebra of \(B/B^{(\omega)}\) constructed above.

We hope that the results of this paper will help to attack the homological problems for groups. In particular, the proof of \(cd(\hat{L}) > 2\) gives an approach for the proof of \(cd(\hat{F}) > 2\), however, the group case is more complicated. Generally speaking, the theories of parafree Lie algebras and parafree groups are very similar, but the theory of parafree associative augmented algebras is different from groups or Lie algebras. The prounilpotent completion of the free augmented associative algebra has cohomological dimension one but contains subobjects of cohomological dimension \(\geq 2\), what is not possible in the category of groups. In view of this difference, it is not surprising that, a finitely presented parafree augmented associative algebra of cohomological dimension \(> 2\) can be constructed. The authors hope to give such kind of examples in the forthcoming papers.

2. Proofs

During the work we denote by \(L = L_R(a, b)\) a free Lie algebra over \(R\) with two generators \(a, b\). Then \(L = \oplus_{n \geq 1} L_n\) is a sum of its homogeneous components \(L_n\). Note that an element \(x\) from \(L\) can be regarded as an infinite series \(x = \sum x_n\) with \(x_n \in L_n\).
2.1. **Free Lie algebra completion homology.** It was shown in [8] and later in [10] that $H_2(\hat{\mathbb{F}}_2,\mathbb{Z})$ is uncountable, where $\hat{\mathbb{F}}_2$ is a $\mathbb{Z}$-completion of free group on two generators. Here we provide an analogue of that result for a free Lie algebra $L$ using methods from [10].

**Definition 2.1.** Suppose that $R$ is a commutative ring. By $\mathcal{L} = \mathcal{L}_R$ we will denote a semidirect sum of abelian lie algebras $R[x] \rtimes R$ with the following action of the generator $t \in R$ on a polynomial $p \in R[x]$:

$$[p, t] = p \cdot x.$$ 

For a commutative ring $R$ we denote the ring of formal power series over $R$ by $R[[x]]$. We consider it as an abelian Lie algebra.

**Lemma 2.2.** The completion $\hat{\mathcal{L}}$ is a semidirect sum $R[[x]] \rtimes R$ with the action

$$[f, t] = f \cdot x$$

for $f \in R[[x]]$ and the generator $t \in R$.

**Proof.** Since $\mathcal{L}^{(n)} = x^n \cdot R[x]$ and $\mathcal{L}/\mathcal{L}^{(n)} = R[x]/x^n \rtimes R$ with similar action, the assertion follows. $\square$

**Lemma 2.3.** There is an isomorphism

$$(R[[x]] \wedge R[[x]])_R \cong H_2(\hat{\mathcal{L}}_R, R)$$

induced by the inclusion $R[[x]] \to \hat{\mathcal{L}}$, where the action $R$ on $R[[x]] \wedge R[[x]]$ is given by

$$(f \wedge g) \cdot t = fx \wedge g + f \wedge gx.$$ 

**Proof.** Consider the short exact sequence $R[[x]] \hookrightarrow \hat{\mathcal{L}}_R \twoheadrightarrow R$ and associated spectral sequence $E$. Since $R$ is one-dimensional, we have $H_n(R, -) = 0$ for $n \geq 2$. It follows that $E^{2}_{i,j} = 0$ for $i \geq 2$, and hence, there is a short exact sequence

$$0 \to E^{2}_{0,2} \to H_2(\hat{\mathcal{L}}_R, R) \to E^{2}_{1,1} \to 0.$$ 

The action of $R$ on $R[[x]]$ has no invariants, so $E^{2}_{1,1} = H_1(R, R[[x]]) = 0$, and the inclusion

$$H_2(R[[x]], R) = E^{2}_{0,2} \hookrightarrow H_2(\hat{\mathcal{L}}_R, R)$$

is an isomorphism. $\square$

**Lemma 2.4.** Suppose that $\sigma : R[[x]] \to R[[x]]$ is the involution such that

$$\sigma(\sum_{i \geq 0} q_i x^i) = \sum_{i \geq 0} (-1)^i q_i x^i.$$ 

Then there is an isomorphism

$$(R[[x]] \wedge R[[x]])_R \cong R[[x]] \wedge R[[x]]$$

given by $f \wedge g \mapsto f \wedge \sigma(g)$.

**Proof.** Since $fx \wedge g + f \wedge gx \to fx \wedge \sigma(g) + f \wedge \sigma(gx)$, $\sigma(gx) = -x \sigma(g)$ and $fx \wedge \sigma(g) = f \wedge x \sigma(g)$, the map is well defined. It is sufficient to show that the inverse map is also well defined, i.e. the inverse map is $R[x]$-bilinear. Since $fx \wedge \sigma(g) = f \wedge \sigma(xg)$ in $(R[[x]] \wedge R[[x]])_R$, we have

$$fx^k \wedge \sigma(g) = f \wedge \sigma(x^k g)$$

for all $k \geq 0$, the assertion follows. $\square$
Corollary 2.5. Suppose that $R$ is a countable or finite field, then the kernel $K$ of the map $R[[x]] \to H_2(\hat{L}_R, R)$ given by $p \mapsto p \wedge 1$ is countable.

Proof. Let us denote by $R(x)$ the field of fractions of $R[x]$ and by $R((x))$ the field of Laurent power series $R((x))$. Since the kernel $K$ contains in the kernel of the composition $R[[x]] \to R((x)) \wedge R(x)$, $K$ contains in $R(x)$ and $R(x)$ is countable. □

Corollary 2.6. Suppose that $R$ is a countable or finite field and that

$$f = \sum_{i=0}^{\infty} q_i x^i \in R[[x]]$$

is not rational function, then $f \wedge 1$ is nontrivial element of $H_2(\hat{L}_R, R)$.

Theorem 2.7. The image $\overline{\varphi} : H_2(\hat{L}, R) \to H_2(\hat{L}, R)$ is uncountable, where $\varphi$ induced by the map $\varphi : L(a, b) \to L$ given by $a \mapsto 1$ and $b \mapsto t$.

Proof. It was shown in [10, Lemma 4.1] that the following relations hold in any Lie algebra $A$ for $n \geq 1$:

$$[[a, 2n], a] = \left[ \sum_{i=0}^{n-1} (-1)^i [[a, 2n-1-i], [a, i]], b \right].$$

Consider the following elements in a free algebra $L(a, b)$

$$r_{2n} = [a, 2n] \wedge a + \left( \sum_{i=1}^{n} (-1)^i [[a, 2n-i], [a, i-1]], b \right) \wedge b \in L \wedge L.$$

Then for $q \in \{0, 1\}^N$ we have

$$\overline{\varphi} \left( \sum q_n r_{2n} \right) = \sum q_n x^{2n} \wedge 1$$

and the set of such images is uncountable by corollary 2.5. □

2.2. Proof of Theorem A.

Lemma 2.8. Suppose that $A$ is a Lie algebra over $\mathbb{Z}/2\mathbb{Z}$ and $a, b, c \in A$. Then the following identities hold

$$[[a, 2n], [c, 2n]] = [[a, 2n+1], b] + [a, 2n, b, c, 2n]$$

for $n \geq 0$. Hence for any $a, b \in A$

$$[a, 2n+1, b, a] = [a, 2n, b, a, 2n], \quad [a, 2n+1, b, a] = [a, b, a, 2n+1].$$

Proof. The case $n = 0$ follows from the Jacobi identity. Let us denote

$$a_1 = [a, 2n], \quad c_1 = [c, 2n], \quad a_2 = [a, 2n+1].$$

By induction we have

$$[[a, 2n+1], [c, 2n+1]] = [a_1, 2n+1, c_1] + [a_1, 2n, c_1, 2n],$$

$$[a_1, 2n+1, c_1] = [a_2, 2n+1, b, c_1] + [a_2, 2n, b, c_2],$$

$$[a_1, 2n, c_1, 2n] = [a_1, 2n+1, b, c_2] + [a_1, 2n, b, c_2],$$

and the assertion follows. □

Next we consider the Lie algebra $B$ defined in introduction.
Lemma 2.9. The second homology group $H_2(B, \mathbb{Z}) = 0$.

Proof. Let $L_0$ be a free algebra over $\mathbb{Z}$ on variables $a, b, \{x_n\}, \{y_n\}$ and $R$ be an ideal generated by relations. Since the set relations are linearly independent relative to $[L_0, L_0]$, $R \cap [L_0, L_0] = [R, L_0]$ and the statement follows from the Hopf’s formula. □

The next lemma comes immediately from the analogue of the Stallings theorem for Lie algebras.

Lemma 2.10. The Lie algebra $A = B/B^{(\omega)}$ is parafree.

Proof. Suppose that $L = L_\mathbb{Z}(a, b)$ and consider the map $\varphi : L \to B$ that is identical on $a$ and $b$. Since $\varphi$ induces an isomorphism on $H_1(L, \mathbb{Z}) \to H_1(B, \mathbb{Z})$ and a surjective map on second homologies by Lemma 2.9 it induces isomorphisms $L/L^{(n)} \to B/B^{(n)} = A/A^{(n)}$ by the Stallings theorem, and so on completions. □

The next lemma is following by induction.

Lemma 2.11. There are the following identities in $A/A^{(2^n)}$

$$x_k \equiv [a, b, b] + \sum_{i=k+1}^{n} [a, 2i-2^k+2b] \pmod{A^{(2^n)}},$$

$$y_k \equiv [a, b, a] + \sum_{i=k+1}^{n} [a, b, a, 2i-2^k b] \pmod{A^{(2^n)}}.$$

Lemmas 2.8 and 2.11 imply that, in $A$,

$$[x_1, a] + [y_1, b] = 0.$$

Hence, the element $x_1 \wedge a + y_1 \wedge b$ is a cycle in the Chevalley-Eilenberg complex

$$A \wedge A \wedge A \to A \wedge A \xrightarrow{[\cdot, \cdot]} A.$$

Now we are ready to show that $H_2(A, \mathbb{Z}) \neq 0$, finishing the proof of Theorem A. Consider the map $\psi : H_2(A, \mathbb{Z}) \to H_2(\hat{L}/2\hat{L}, \mathbb{Z}/2\mathbb{Z})$ induced by the composition of embedding $A \to \hat{L}$ and $\varphi : \hat{L} \to \hat{L}$. Since $\sum x_i^2$ is not rational function, the image

$$\tilde{\psi}(x_1 \wedge a + y_1 \wedge b) = \left( \sum_{i=1}^{\infty} x_i^2 \right) \wedge 1$$

is not trivial by corollary 2.6. Hence, the cycle $x_1 \wedge a + y_1 \wedge b$ defines a nonzero element of homology $H_2(A)$.

2.3. Proof of Theorem B. Analogously with the proof of Theorem 2.7 consider the following elements

$$r_{2^n} = [a, 2^n b] \wedge a + \sum_{i=1}^{2^n-1} (-1)^i [a, 2^n b], [a, b] \wedge b \in L \wedge L$$

for $n = 1, 2, \ldots$. Then the sum $\sum_n 2^n r_{2^n}$ defines a cycle in

$$\hat{L} \wedge \hat{L} \to \hat{L} \wedge \hat{L} \xrightarrow{[\cdot, \cdot]} \hat{L}.$$
We claim that the power series $\sum_n 2^n x^{2^n}$ is not rational. Indeed, if $p = b_0 + b_1 x + \cdots + b_N x^N$ is a polynomial of degree $N$, then for any $k$ such that $2^k > N$ the coefficient of $x^{2^k + N}$ in the product $(\sum_n 2^n x^{2^n}) \cdot p$ is $b_N 2^k \neq 0$, an hence it is not a polynomial. Then the image of the sum $\sum 2^n r_{2^n}$

$$\psi(\sum_n 2^n r_{2^n}) = \left( \sum_{i=1}^{\infty} 2^n x^{2^n} \right) \land 1$$

is not trivial by corollary 2.6. Hence, $\sum_n 2^n r_{2^n}$ defines a nonzero element in $H_2(\hat{L})$. Now observe that $\sum_n 2^n r_{2^n}$ is 2-divisible. Indeed, the finite sum $\sum_{n=1}^k 2^n r_{2^n}$ defines a cycle in the Chevalley-Eilenberg complex $L \land L \land L \to L \land L$ and since the second homology of a free Lie algebra are trivial, $\sum_{n=1}^k 2^n r_{2^n} \in L \land L$ lies in the image of $L \land L \land L$. Presenting the cycle $\sum_{n=1}^k 2^n r_{2^n} = \sum_{n=1}^k 2^n r_{2^n} + \sum_{n\geq k+1} 2^{n-k} r_{2^n}$, we see that the final part in this presentation can be omitted in homology, i.e. $\sum_{n=1}^k 2^n r_{2^n} + \sum_{n\geq k+1} 2^{n-k} r_{2^n}$ define the same homology class in $H_2(L)$. Since the element $\sum_{n\geq k+1} 2^{n-k} r_{2^n}$ also defines a nonzero element in homology, we conclude that the homology class $\sum_{n} 2^n r_{2^n}$ is $2^k$-divisible for every $k$. That is, we constructed a 2-divisible element in $H_2(\hat{L})$.

Prove that cohomological dimension of $\hat{L}$ is at least 3. Assume the contrary, that $\text{cd}(\hat{L}) \leq 2$. Consider a projective resolution $P_\bullet \to \mathbb{Z}$ over the enveloping algebra $U L$. Set $\Omega^n = \text{Coker}(P_{n+1} \to P_n)$ for $n \geq 0$. Then the long sequences of the short exact sequence $\Omega^{n+1} \to P_n \to \Omega^n$ imply that

$$\text{Ext}^m(\Omega^{n+1}, \cdot) = \text{Ext}^{m+1}(\Omega^n, \cdot), \quad H_m(\hat{L}, \Omega^{n+1}) = H_{m+1}(\hat{L}, \Omega^n)$$

for $m \geq 1$ and there is a monomorphism

$$H_1(\hat{L}, \Omega^n) \to H_0(\hat{L}, \Omega^{n+1}).$$

Therefore $\text{Ext}^1(\Omega^2, \cdot) = \text{Ext}^3(\mathbb{Z}, \cdot) = H^3(\hat{L}, \cdot) = 0$. It follows that $\Omega^2$ is projective. On the other hand we have a monomorphism $H_2(\hat{L}, \mathbb{Z}) = H_1(\hat{L}, \Omega^1) \to H_0(\hat{L}, \Omega^2)$. Since $\Omega^2$ is projective, $H_0(\hat{L}, \Omega^2)$ is a free abelian group, and hence $H_2(\hat{L})$ is a free abelian group. This contradicts to the fact that $H_2(\hat{L})$ has a nontrivial 2-divisible element.

2.4. Proof of Proposition 1. In order to prove this proposition we need to recall some statements from the theory of $H\mathbb{Z}$-localization that can be found in [11] and [6].

Let $G$ be a group, $X$ be a set that we call the set of variables and $F = F(X)$ be the free group generated by $X$. An element $w$ of the free product $G * F$ is called monomial with coefficients in $G$. A monomial $w$ is called acyclic, if its image in $F_{ab}$ is trivial. Let $S = (w_x)_{x \in X}$ be a family of acyclic monomials indexed by $X$. A $\Gamma$-system of equations defined by $S$ is the family of equations $(x = w_x)_{x \in X}$. A solution of such a system is a map $X \to G$ such that $x w_x^{-1}$ is in the kernel of the induced map $G * F \to G$.

A homomorphism $f : G \to G'$ is called 2-connected if it induces an isomorphisms $H_1(G) \cong H_1(G')$ and an epimorphism $H_2(G) \to H_2(G')$. 
Lemma 2.12. Let \( S = (w_x)_{x \in X} \) be an \( X \)-indexed family of acyclic monomials in \( G * F \). Consider the group \( G_2 = (G * F)/R \), where \( R \) is the normal subgroup generated by the elements \( xw_i^{-1} \). Then the map
\[
G \map G_2
\]
is 2-connected.

Proof. The fact that the induced map \( H_1(G) \map H_1(G_2) \) is an isomorphism, is obvious.

Prove that \( H_2(G) \map H_2(G_2) \) is an epimorphism. It is easy to see that \( H_2(G) = H_2(G * F) \). Hence we need to prove that \( H_2(G * F) \map H_2(G_2) \) is an epimorphism. Recall that for any group \( A \) and any its normal subgroup \( U \) the cokernel of \( H_2(A) \map H_2(A/U) \) is isomorphic to \((U \cap [A, A])/[U, A]\). Therefore we need to prove that \((R \cap [G * F, G * F])/[R, G * F] = 0\). Let us write elements of \( R/G * F = (R_{ab}G * F) \) and \( F_{ab} \) in the additive notation. Then any element of \( R/[R, G * F] \) can be presented as a linear combination \( \theta = \sum \alpha_x(xw_i^{-1}) \). The image of \( \theta \) in \( F_{ab} \) is \( \sum \alpha_x x \). So \( \theta \) in \((R \cap [G * F, G * F])/[R, G * F] \) only if \( \sum \alpha_x x = 0 \), and hence \( \alpha_x = 0 \) for any \( x \). Then \((R \cap [G * F, G * F])/[R, G * F] = 0\). \( \square \)

Lemma 2.13. Let \( G \) be a finitely generated group. Then any element of \( \hat{G} \) is an element of a solution of a countable \( \Gamma \)-system of equations with coefficients in \( \Im(G \map \hat{G}) \).

Proof. It is proved in [11] that a group is \( HZ \)-local if and only if any \( \Gamma \) system of equations has a unique solution. Moreover, they prove that it is enough to consider countable \( \Gamma \)-systems of equations: a group \( L \) is \( HZ \)-local if and only any countable \( \Gamma \)-system of equations has a unique solution.

If \( G \subseteq H \), then the set of all solutions of \( \Gamma \)-systems of equations with constants in \( G \) is called \( \Gamma \)-closure of \( G \) in \( H \). It is proved in [11] that \( \Gamma \)-closure of \( G \) equals to Bousfiled’s \( HZ \)-closure of \( G \) in \( H \). Again, one can check that it is enough to consider countable \( \Gamma \)-systems of equations. In particular, this means that any element of \( HZ \)-localization \( LG \) is an element of a solution of a countable \( \Gamma \)-system of equations with coefficients in the image of \( G \). Then the assertion follows from the fact \( \hat{G} = LG/\gamma_\omega(LG) \). \( \square \)

Proposition 2.14. Let \( G \) be a finitely generated group. Then for any countable subset \( A \subseteq \hat{G} \) there exists a subgroup \( H \subseteq \hat{G} \) such that \( \Im(G \map \hat{G}) \cup A \subseteq H \) and the induced maps \( G/\gamma_n(G) \map H/\gamma_n(H) \) are isomorphisms.

Proof. For simplicity we set \( A/\gamma_n = A/\gamma_n(A) \) for any group \( A \). Lemma 2.13 implies that any element \( a \in A \) is an element of a solution of a countable \( \Gamma \)-system of equations. A countable union of countable \( \Gamma \)-systems of equations is a \( \Gamma \)-system of equations. Therefore there exists a countable family of acyclic monomials \( S = (w_i)_{i=1}^{\infty} \) from \( G * F(x_1, x_2, \ldots) \) such that \( A \) lies in the image of of the solution \( \{x_1, x_2, \ldots\} \map \hat{G} \) of the \( \Gamma \)-system of equations \( (x_i = w_i) \). Consider the group
\[
G_S = (G * F(x_1, x_2, \ldots))/R,
\]
where \( R \) is the normal subgroup generated by the elements \( x_iw_i^{-1} \). Since the map \( G \map G_S \) is 2-connected, then by Stallings’ theorem we have an isomorphism \( G/\gamma_n \cong G_S/\gamma_n \). In particular \( \hat{G} \cong \hat{G}_S \). Therefore we obtain a map \( f : G_S \map \hat{G} \), whose
kernel is \( \gamma_\omega(G_S) \). Denote by \( H \) the image of \( f \). Then \( H \cong G_S/\gamma_\omega \) and \( H/\gamma_n \cong G_S/\gamma_n \cong G/\gamma_n \). The restriction \( f|_{\{x_1,x_2,\ldots\}} \) is a solution of the \( \Gamma \)-system of equations \((x_i = w_i)\). Since \( \hat{G} \) is \( HZ \)-local, any \( \Gamma \)-system of equations has a unique solution. Therefore \( A \subseteq H \).

**Corollary 2.15.** Let \( F \) be a finitely generated free group. Then for any countable subset \( A \subseteq \hat{F} \) there exists a countable parafree subgroup \( G \subseteq \hat{F} \) such that \( A \subseteq G \).

The Proposition 1 follows from this corollary.

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