A Note on Cosmic \((p, q, r)\) Strings

Mark G. Jackson
Particle Astrophysics Center, Fermi National Accelerator Laboratory, Batavia IL 60510

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There has been recent interest in the cosmic rehabilitation of fundamental strings [1]. This has led to the exciting possibility that the properties of a cosmic network made of fundamental strings may have observationally distinct signatures from more mundane solitonic objects.

There are two smoking guns. The first is the probability of reconnection which scales as $P \sim g_s^2 \mu^2$ for fundamental strings [2], while it is essentially unity for abelian vortices [3, 4]. Non-abelian vortices also typically reconnect with $P = 1$, although it is sometimes possible to get probability $P < 1$ with the velocity dependence substantially different from the fundamental string case [3].

The second smoking gun [5, 6] is the existence of both F- and D-strings in warped IIB compactifications which form a distinctive spectrum of bound states with tension

$$\mu_{(p,q)} = \sqrt{p^2 \mu_F^2 + q^2 \mu_D^2}. \quad (1)$$

A simple consequence of these bound states is the existence of 3-string junctions, with angles dictated by charge conservation and the tension formula [11]. Aspects of network formation and gravitational lensing of such junctions were studied in [9, 10].

It is rather simple to construct field theories which admit bound states of vortices and the corresponding 3-string junctions. Examples include vortices charged under discrete symmetries [11] and multiple abelian gauge groups [12]. However, none of the field theoretic models studied so far reproduce the stringy spectrum [11]. The purpose of this short note is to show that the general Bogomol’nyi bound in gauge theories with multiple $U(1)$ gauge groups includes the stringy spectrum [11]. In fact, we shall see that there is a maximum of three different types of supersymmetric vortices, with the tensions bounded by

$$\mu = \sqrt{k_1 \mu_1^2 + k_2 \mu_2^2 + k_3 \mu_3^2} \quad (2)$$

where $k_i$ are integer gauge winding charges. If we choose a field theory without the third type of vortex, then this mimics the IIB string theory spectrum of cosmic strings. Although we find that no nontrivial Bogomol’nyi-Prasad-Sommerfield (BPS) solutions exist which have this square-root spectrum, it is possible that non-BPS solutions exist which could closely resemble it.

THE BOGOMOLNYI BOUND

In theories with $\mathcal{N} = 2$ supersymmetry, the real D-term and complex F-term are unified into a triplet, transforming under an $SU(2)_R$ R-symmetry. This existence of this triplet is responsible for the three different tensions appearing in (2). Recall that matter lives in a hypermultiplet, consisting of two complex scalars $\phi$ and $\dot{\phi}$ transforming in conjugate representations of the gauge group. For a single scalar charged under a single $U(1)$ gauge group, the D- and F-terms in the scalar potential [16] are fixed by $\mathcal{N} = 2$ supersymmetry to be,

$$V = \frac{e^2}{2} (|\phi|^2 - |\dot{\phi}|^2 - r_3)^2 + \frac{e^2}{2} |2\bar{\phi}\phi - r_1 - ir_2|^2. \quad (3)$$

Here $e^2$ is the gauge coupling constant. There are three vacuum expectation values $r_1, r_2$ and $r_3$ allowed by supersymmetry (often referred to as a Fayet-Iliopoulos parameters). The $SU(2)_R$ symmetry of this potential can be made manifest by defining the doublet $\omega^T = (\phi, \dot{\phi})$ and writing

$$V = \frac{e^2}{2} (\bar{\omega} \sigma \omega - \bar{r} r)^2 \quad (4)$$

where $\bar{r} = (r_1, r_2, r_3)$ and $\sigma$ are the triplet of Pauli matrices.

Consider now a $U(1)^N$ gauge theory with gauge coupling $e_a^2, a = 1, \ldots, N$. We couple $N$ hypermultiplets $\omega_i$ with integer charges $Q'_a$ under the $a^{th}$ gauge group. The covariant derivatives are given by $D\omega_i = \partial \omega_i - i(\sum_{a=1}^N Q'_a A_a) \omega_i$. The energy functional for static
\((\partial_0 = A_0 = 0)\) configurations is

\[
\mathcal{E} = \sum_{i=1}^{N} |D_1 \omega_i|^2 + \sum_{a=1}^{N} \frac{1}{2e_a} B_a^2 + \frac{e_a^2}{2} \left( \sum_{i=1}^{N} Q^i_a \omega_i \right)^2.
\]

with \(B_a\) the magnetic field for the \(a\)th gauge group. We choose \(\det Q \neq 0\) to ensure that in the ground state, defined by \(\sum_{i=1}^{N} Q^i_a \omega_i \equiv \vec{r}_a\), the \(U(1)^N\) gauge group is fully broken and the theory exhibits a mass gap.

Lowest energy vortex states may be found by the usual Bogomolnyi method. We search for straight strings, extended in the \(x^3\) direction, by setting \(\partial_3 = A_3 = 0\) and writing

\[
\mathcal{E} = \sum_{i=1}^{N} |D_1 \omega_i - i(\vec{m} \cdot \vec{\sigma}) D_2 \omega_i|^2 + \sum_{a=1}^{N} \frac{1}{2e_a} \left( \vec{m} B_a - e_a^2 \sum_{i=1}^{N} Q^i_a \omega_i \vec{r}_a \right)^2 - B_a \vec{m} \cdot \vec{r}_a.
\]

The above decomposition holds for any unit vector \(\vec{m}\). The last term yields a topological charge when integrated over the plane transverse to the vortex string: \(\int d^2 x B_a = -2\pi k_a\). Noting that the first two terms are squares, we derive the bound on the tension

\[
\mu = \int d^2 x \mathcal{E} \geq 2\pi \sum_a k_a \vec{m} \cdot \vec{r}_a.
\]

This is maximized by choosing \(\vec{m}\) parallel to \(\sum_a k_a \vec{r}_a\). In IIB string theory the tension-squared for a string with integer charge vector \(k_a = (p, q)\) is expressed as

\[
\mu^2 = \sum_{a,b=1,2} (M^{-1})^{ab} k_a k_b
\]

where \(M_{ab}\) is the metric on the IIB auxiliary torus of modular parameter \(\tau\). We obtain the same spectrum by defining \((M^{-1})^{ab} = 2\pi \vec{r}_a \cdot \vec{r}_b\), where now \(a, b = 1, \ldots, N\).

In the special case where \((M^{-1})^{ab} = \mu_a^2 \delta_{ab}\), this takes the form of \([2]\).

Note that since the Fayet-Iliopoulos (FI) parameters \(r_a\) contain only three linearly independent directions, there are only three linearly independent effective string tensions, and we may henceforth assume that \(N = 3\) and so \(k_a = (p, q, r)\).

The bound is saturated by solutions to the equations

\[
\vec{m} B_a = e_a^2 \sum_{i=1}^{N} Q^i_a \omega_i \vec{r}_a
\]

and

\[
D_1 \omega_i = i(\vec{m} \cdot \vec{\sigma}) D_2 \omega_i
\]

where, as explained above, \(\vec{m}\) is the unit vector parallel to \(\sum_a k_a \vec{r}_a\). When all \(\vec{r}_a\) lie parallel, for example \(\vec{r}_a = (0, 0, r_a)\), these reduce to the usual coupled vortex equations studied in \([13]\). They have solutions only when the winding \(n_i\) of all scalar fields with non-zero expectation value, defined by \(n_i = \sum_a Q^i_a k_a\) is non-negative. (This is simply the statement that there is no holomorphic vector bundle of negative degree). In this case there is no attractive force between vortices. In contrast, when the \(\vec{r}_a\) do not lie parallel and the vortices in different gauge groups are coupled through the scalars \(\omega_i\), one may expect bound states to form.

In both the field theoretic and string theoretic contexts, the Bogomolnyi bound is expected to receive corrections at the scale at which the protectorate supersymmetry is broken. In warped IIB compactifications, supersymmetry is broken from 16 supercharges (in the orientifold background) to 4 at the compactification scale, with subsequent low-energy breaking at the TeV scale.

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**NON-EXISTENCE OF BPS SOLUTIONS**

Now decompose each field \(\omega_i\) into eigenvectors of \(\vec{m} \cdot \vec{\sigma}\), writing

\[
\omega_i = \psi_i |\vec{m}_+\rangle + \tilde{\psi}_i |\vec{m}_-\rangle.
\]

Then the covariant derivatives become

\[
D_2 \psi_i \equiv (\partial_1 - i\partial_2) \psi_i - i \sum_a Q^i_a (A^1_a - iA^2_a) \psi_i = 0,
\]

\[
D_2 \tilde{\psi}_i \equiv (\partial_1 + i\partial_2) \tilde{\psi}_i - i \sum_a Q^i_a (A^1_a + iA^2_a) \tilde{\psi}_i = 0.
\]

From this, we see that both \(\psi_i\) and \(\tilde{\psi}_i\) transform with charge \(Q^i_a\) under the \((U(1))_g\) gauge group. Taking the complex conjugate of the second equation, this ensures that both \(\psi_i\) and \(\tilde{\psi}_i\) are covariantly holomorphically constant, i.e.

\[
D_2 \psi_i = D_2 \tilde{\psi}_i = 0
\]

where now \(\psi_i\) has charge \(Q^i_a\) while \(\tilde{\psi}_i\) has charge \(-Q^i_a\). In other words, as the notation suggests, these are the rotated form of \(\phi_i\) and \(\tilde{\phi}_i\). We can now look at the first Bogomolnyi equation. Dotting with the unit vector \(\vec{m}\) tells us

\[
B_a = e_a^2 \left( \sum_{i=1}^{N} Q^i_a |\psi_i|^2 - Q^i_a |\tilde{\psi}_i|^2 - \vec{m} \cdot \vec{r}_a \right).
\]

Equations \([9]\) and \([10]\) are now in the form of the usual coupled vortex equations described, for example, in Morrison and Plesser \([13]\). The criterion for the existence of solutions is that for each scalar field \(\psi_i\) we can define the winding \(n_i = \sum_a Q^i_a k_a\), while for each \(\tilde{\psi}_i\) we have \(\tilde{n}_i = -\sum_a Q^i_a k_a\). Clearly \(n_i = -\tilde{n}_i\). Solutions to \([9]\) and \([10]\) exist if \(n_i\) is non-negative for each \(\psi_i\) that gains an expectation value. (If \(\psi_i\) has no expectation value for some \(i\) then it may remain zero throughout the solution).

Similarly, each \(\tilde{n}_i\) must be non-negative for each \(\tilde{\psi}_i\) which is non-zero. Clearly, since \(n_i = -\tilde{n}_i\), either \(\psi_i\) or \(\psi_i\) is allowed an expectation value, but not both.
There are two further real equations that come from dotting the first equation in (7) with \( \vec{I}_a \), where \( \vec{I}_a \cdot \vec{m} = 0 \), for \( \alpha = 1, 2 \). We write \( \vec{I} = \vec{I}_1 + i\vec{I}_2 \). There is an ambiguous phase to the vector \( \vec{I} \), associated to rotating the basis \( \vec{I}_1 \) and \( \vec{I}_2 \), but we can always pick a basis so that the remaining two real equations combine into the complex equation

\[
\sum_{i=1}^{N} Q_i^a \tilde{\psi}_i \psi_i = \vec{I} \cdot \vec{r}_a. \tag{11}
\]

Thus we see that in order for either \( \psi_i \) or \( \tilde{\psi}_i \) to be zero, the vector \( \sum_a (Q^{-1})_a \vec{r}_a \) must be perpendicular to \( \vec{I} \), making it proportional to \( \vec{m} \). Since \( \vec{m} \propto \sum_a k_a \vec{r}_a \), this requires \( k_a \) to be proportional to \( (Q^{-1})_a \) (and of course \( k_a \) must be integer-valued). That is, in order for only \( n_i \) to be nonzero we select \( k_a \) to be the \( i \)th entry of \( Q^{-1} \).

There is also the trivial solution when all \( \vec{r}_a \) lie parallel, so that each \( \vec{r}_a \) is proportional to \( \vec{m} \) and thus perpendicular to \( \vec{I} \), but this does not produce a square-root spectrum.

While these are the necessary conditions for BPS solutions, we find they are not sufficient. Consider the radially symmetric field ansatz (a non-radially symmetric solution would necessarily have higher energy and thus could not be BPS):

\[
\omega_i = \begin{pmatrix} \psi_i \\ \tilde{\psi}_i \end{pmatrix} = (1 - q_i(\rho) \vec{m} \cdot \vec{\sigma}) \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} e^{in_i \theta} \tag{12}
\]

where we use polar coordinates \((\rho, \theta)\) in the \((x^1, x^2)\)-plane. The asymptotic boundary condition is

\[ q_i \rightarrow 0 \quad \text{as} \quad \rho \rightarrow \infty \]

making the vacuum selection at infinity

\[
\sum_i Q_i^a (s_i^* \tilde{s}_i^*) \vec{\sigma} \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} = \vec{r}_a. \tag{13}
\]

The ansatz for the gauge potential is

\[
A^a_{\mu} = \frac{e^\mu x^\nu}{\rho^2} A_a(\rho), \quad A_a = -k_a + \rho f_a(\rho) \tag{14}
\]

where we similarly require \( f_a \rightarrow 0 \) as \( \rho \rightarrow \infty \). The field strength is given by the simple expression

\[
B^a = \partial_\mu A^a_\mu - \partial_\mu A^a_\mu = \frac{1}{\rho} \frac{\partial A_a}{\partial \rho}. \tag{15}
\]

To determine the long-distance behavior of the fields, we insert the ansatz into the equations (7) and expand to linear order in \( q_i \) and \( f_a \) to obtain

\[
\begin{align*}
\left( L^a_{\mu} + f^a_\mu \right) &= -2e_a^\mu \sum_i Q_i^a (|s_i|^2 + |\tilde{s}_i|^2) q_i, \\
\left( \vec{m} \cdot \vec{\sigma} \partial_\rho \right) - \sum_a Q_i^a f_a (1 - q_i(\rho) \vec{m} \cdot \vec{\sigma}) \begin{pmatrix} s_i \\ \tilde{s}_i \end{pmatrix} &= 0.
\end{align*} \tag{16}
\]

The second equation can be solved (to linear order!) to give

\[
q_i = -\sum_a Q_i^a f_a. \tag{17}
\]

Differentiation of these first-order equations then produces the modified Bessel equations

\[
\begin{align*}
Q''_a + \frac{1}{\rho} Q'_a - \frac{1}{\rho^2} Q_a - \sum_b L_{ab} f_b &= 0, \\
q''_i + \frac{1}{\rho} q'_i - \sum_j M_{ij} q_j &= 0
\end{align*}
\]

where the mass-squared matrices are given by

\[
\begin{align*}
L_{ab} &= 2\epsilon^2_a \sum_i (|s_i|^2 + |\tilde{s}_i|^2) Q_i^a Q^b_i, \\
M_{ij} &= 2(|s_i|^2 + |\tilde{s}_i|^2) \sum_a \epsilon^2_a Q_i^a Q^j_a.
\end{align*}
\]

As should be expected from a BPS solution, the gauge and matter mass-squared matrices \( L_{ab} \) and \( M_{ij} \) have identical eigenvalues, which can be seen by acting with \( Q^a \) as a similarity transformation. Denoting the mass eigenvalues as \( \lambda_A \) (so that the mass-squared eigenvalues are \( \lambda_A^2 \)), the solution to (16) and (17) is then given by

\[
\begin{align*}
f_a &= \sum_A S_{aA} C_A \lambda_A K_1(\lambda_A \rho), \\
q_i &= \sum_{a,A} Q^i_a S_{aA} C_A K_0(\lambda_A \rho) \tag{18}
\end{align*}
\]

where \( S_{aA} \) is the diagonalization matrix for \( L_{ab} \) and the coefficients \( C_A \) cannot be determined in the linear approximation and would have to be fixed from numerical comparison to the non-linear solution. We would expect that only gauge fields charged under the given \( k_a \) and the matter field \( \omega_i \) with nonzero winding \( n_i \) should attain a profile, but from (18) we see that the \( Q^a \) mix the matter and gauge fields into a basis such that non-charged fields are excited. To prevent this it must be that the \( L_{ab} \) and \( M_{ij} \) are diagonal, making the \( C_A \) proportional to \( k_a \).

This gives us a total of 6 constraints (3 each from setting the off-diagonal components of a symmetric matrix to zero). These are precisely enough constraints to set the off-diagonal components of \( Q^a \) to zero, which now makes the interaction trivial. The acceptable BPS windings are then simply of the form \( k_a = (\rho, 0, 0), (0, q, 0) \) or \( (0, 0, r) \), which will not display any distinctive square-root behavior. The lack of BPS solutions for such \( N = 2 \) theories has been noted previously in the literature \[14\] \[15\] but not in the context of cosmic \((p, q, r)\) strings.

The situation is very reminiscent of supersymmetric quantum mechanics, with the doublet \( \omega_i = (\psi_i, \tilde{\psi}_i) \) taking the place of the bosonic and fermionic components of the wavefunction. The ground state (BPS) solution
(when it exists) is given by first-order solutions which allow one component or the other, but not both. It is still likely that non-BPS solutions exist which will contain both components, but it will have energy greater than the BPS bound.

**CONCLUSION**

We have shown the BPS spectrum for supersymmetric vortices exhibits the same square-root cosmic string spectrum as superstring theory, including not just two but three types of vortices. Unfortunately no BPS solutions exist which actually exhibit this square-root spectrum. It is still likely that non-BPS solutions exist which would have an energy higher than the BPS bound, but which might approximate the square-root BPS spectrum for a certain choice of parameters. It would be interesting to make a full analysis of these solutions.

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