Self-Referential Definition of Orthogonality

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Abstract

There has for longer been an interest in finding equivalent conditions which define inner product spaces, and the respective literature is considerable, see for instance Amir, which lists 350 such results. Here, in this tradition, an alternative definition of orthogonality is presented which does not make use of any inner product. This definition, in the spirit of the recently developed non-wellfounded set theory, is self-referential, or circulatory.

0. Preliminaries

Let us note that a group \((G, \star)\) is defined by properties of its subgroups generated by no more than 3 elements. Indeed, the axioms of a group are

\[
(0.1) \quad \forall \ x, y \in G : x \star y \in G \\
(0.2) \quad \forall \ x, y, z \in G : x \star (y \star z) = (x \star y) \star z \\
(0.3) \quad \exists \ e \in G : \forall \ x \in G : x \star e = e \star x = x
\]
1. Linear Independence

Let $V$ be a vector space on $\mathbb{R}$, and $m \geq 2$, then we denote
(1.1) \( \text{Ind}^m V = \left\{ (a_1, \ldots, a_m) \mid a_1, \ldots, a_m \in V \text{ are linearly independent in } V \right\} \)

which is nonvoid, if and only if \( \text{dim} V \geq m \).

Further, for \((a_1, \ldots, a_m) \in \text{Ind}^m V\) we denote by

(1.2) \( \text{span}(a_1, \ldots, a_m) \)

the vector subspace in \( V \) generated by \( a_1, \ldots, a_m \). Then for every \( m \)-dimensional vector subspace \( E \) of \( V \), we obviously we have

\[
\forall \ x \in E : \\
\forall \ (a_1, \ldots, a_m) \in \text{Ind}^m E : \\
\exists ! \ \lambda_1(a_1, \ldots, a_m, x), \ldots, \lambda_m(a_1, \ldots, a_m, x) \in \mathbb{R} : \\
x = \sum_{1 \leq i \leq m} \lambda_i(a_1, \ldots, a_m, x) a_i
\]

2. Usual orthogonality based on inner product

Here for convenience and for setting the notation, we review the usual case of orthogonality, namely, that which is defined based on an inner product. Given any inner product \( < , > : V \times V \rightarrow \mathbb{R} \), then for \( m \geq 2 \), we denote

\[
\text{Ort}^m V (< , >) = \\
= \left\{ (a_1, \ldots, a_m) \in \text{Ind}^m V \mid a_1, \ldots, a_m \text{ are orthogonal with respect to } < , > \right\}
\]

Now, if \((a_1, \ldots, a_m) \in \text{Ort}^m V (< , >)\), then for any vector \( x \in \text{span}(a_1, \ldots, a_m) \), we have in a unique manner, see (1.3)

(2.2) \( x = \sum_{1 \leq i \leq m} \lambda_i(a_1, \ldots, a_m, x) a_i \)

where this time we have in addition the simple explicit formulae
(2.3) $\lambda_i(a_1, \ldots, a_m, x) = \langle a_i, x \rangle / \langle a_i, a_i \rangle, \quad 1 \leq i \leq m$

**Remark 1.**

An important point to note is that each $\lambda_i(a_1, \ldots, a_m, x)$ in (2.2), (2.3) can in fact only depend on $x$ and $a_i$, but *not* on $a_j$, with $1 \leq j \leq m$, $j \neq i$. Indeed, in view of (1.3), $x$ obviously does *not* depend on $a_1, \ldots, a_m$.

For the sake of clarity, we recall here that by inner product on $V$ we mean a function $\langle , \rangle: V \times V \rightarrow \mathbb{R}$, such that

(2.4) $\langle , \rangle$ is symmetric

(2.5) $\langle x, . \rangle$ is linear on $V$

(2.6) $\langle x, x \rangle \geq 0$

(2.7) $\langle x, x \rangle = 0 \iff x = 0$

3. Orthogonality without inner product

In view of the above it may appear that one could give the following definition of *orthogonality without inner product*, namely

**Definition 1.**

If $m \geq 2$ and $(a_1, \ldots, a_m) \in \text{Ind}^m V$, then by definition
\( a_1, \ldots, a_m \) are orthogonal \( \iff \)

\[
(3.1) \quad \iff \quad \begin{aligned}
\forall x \in \text{span}\,(a_1, \ldots, a_m) : \\
\forall \ 1 \leq i \leq m : \\
\lambda_i(a_1, \ldots, a_m, x) \text{ in (1.3) does not depend on } a_j, \text{ with } 1 \leq j \leq m, j \neq i
\end{aligned}
\]

Consequently, we denote

\[
(3.2) \quad \text{Orth}_m V = \left\{ (a_1, \ldots, a_m) \in \text{Ind}_m V \mid a_1, \ldots, a_m \text{ are orthogonal in the sense of (3.1)} \right\}
\]

Remark 2.

One should note above that the meaning of "does not depend on" in (3.1) needs further clarification. Indeed, when we are given an explicit expression, such as for instance in (2.3), then it is quite clear what "does not depend on" means, since the entities on which the respective expressions are supposed not to depend simply do not appear in the respective expressions.

However, in (3.1) we are actually dealing with (1.3). And then, the meaning of

" \lambda_i(a_1, \ldots, a_m, x) \text{ in (1.3) does not depend on } a_j, \\
\text{with } 1 \leq j \leq m, j \neq i "

is not so immediately obvious.

4. The two dimensional case

Orthogonality, in its simplest nontrivial case, involves only two nonzero vectors, hence it is a property which can already be formulated in 2-dimensional vector spaces. Let therefore \( E \) be any 2-dimensional
vector space on $\mathbb{R}$. Then the property in (1.3) becomes

$$\exists! \lambda, \mu : \text{Ind}^2_E \times E \to \mathbb{R} :$$

$$(4.1) \quad \forall (a, b) \in \text{Ind}^2_E, x \in E :$$

$$x = \lambda(a, b, x)a + \mu(a, b, x)b$$

Therefore, in view of (3.1) in the above Definition 1 which does not use inner product for defining orthogonality, we have

Proposition 1.

Given $(a, b) \in \text{Ind}^2_E$, then

$$(4.2) \quad \left( a, b \text{ are orthogonal in the sense of (3.1)} \right) \iff$$

$$\left\{ \begin{array}{l}
(4.2.1) \lambda(a, b, x) \text{ in (4.1)} \\
\text{does not depend on } b \\
(4.2.2) \mu(a, b, x) \text{ in (4.1)} \\
\text{does not depend on } a
\end{array} \right.$$ 

Clearly, if we want to consider the concept of orthogonality in its natural minimal nontrivial context of 2-dimensionality, then we can take the above property (4.2) as a definition of orthogonality, without the use of inner product, instead of referring to the more general condition (3.1) in Definition 1.

Remark 3.
It is useful to note the following. For every two vectors \((a, b) \in \text{Ind}^2_E\), there exists an inner product \(< , >_{a,b}\) on \(E\), such that \(< a, b >_{a,b} = 0\), that is, \(a\) and \(b\) are orthogonal with respect to \(< , >_{a,b}\), or equivalently, \((a, b) \in \text{Ort}^2_E(< , >_{a,b})\). This, however, does not contradict (4.2), since in such a case we still have, see (4.1)

\[
\forall \ x \in E : \\
x = \lambda(a, b, x) a + \mu(a, b, x) b
\]

with, see (2.3), (4.2)

\[
\lambda(a, b, x) = \frac{< a, x >_{a,b}}{< a, a >_{a,b}} \\
\mu(a, b, x) = \frac{< b, x >_{a,b}}{< b, b >_{a,b}}
\]

5. One possible meaning of "Does not depend on"

Here we shall specify within a rather general context one possible meaning of the above property "does not depend on", which was used in (3.1) and (4.2).

Given a function \(f : \Delta \subseteq X \times Y \rightarrow Z\), where \(X, Y\) and \(Z\) are arbitrary sets, and further given \(\Gamma \subseteq \Delta\), we say that, on \(\Gamma\), the function \(f\) does not depend on \(x \in X\), if and only if there exists a function \(g : pr_Y(\Gamma) \subseteq Y \rightarrow Z\), such that

\[
f|_{\Gamma} = g \circ pr_Y|_{\Gamma}
\]

or equivalently, we have the commutative diagram

\[
\begin{align*}
\Delta & \quad \xrightarrow{f} \quad Z \\
\subseteq & \quad \xrightarrow{\Gamma} \quad \xrightarrow{g} \quad pr_Y(\Gamma) \\
\Gamma & \quad \xrightarrow{pr_Y|_{\Gamma}} \quad \xrightarrow{f|_{\Gamma}} \quad Z \\
\end{align*}
\]
where \( \text{pr}_Y : X \times Y \ni (x, y) \mapsto y \in Y \) is the usual projection mapping.

The above concept can easily be adapted to the case of \( m \geq 2 \) functions

\[
f_1, \ldots, f_m : \Delta \subseteq X_1 \times \ldots \times X_m \times Y \rightarrow Z
\]

where \( X_1, \ldots, X_m, Y \) and \( Z \) are arbitrary sets. Indeed, given \( \Gamma \subseteq \Delta \), we say that, on \( \Gamma \), each function \( f_i \), with \( 1 \leq i \leq m \), does not depend on \( x_j \in X_j \), with \( 1 \leq j \leq m, j \neq i \), if and only if there exist functions

\[
g_i : \text{pr}_{X_i \times Y}(\Gamma) \subseteq X_i \times Y \rightarrow Z, \quad 1 \leq i \leq m
\]

such that

\[
f_i|_{\Gamma} = g_i \circ \text{pr}_{X_i \times Y}|_{\Gamma}, \quad 1 \leq i \leq m
\]

Returning to the Definition in section 2, we note that there we have

\[
X_1 = \ldots = X_m = Y = V, \quad Z = \mathbb{R}
\]

\[
\Delta = \bigcup_{(a_1, \ldots, a_m) \in \text{Ind}^m V} \{ (a_1, \ldots, a_m) \} \times \text{span}(a_1, \ldots, a_m)
\]

\[
\Gamma = \bigcup_{(a_1, \ldots, a_m) \in \text{Ort}^m V} \{ (a_1, \ldots, a_m) \} \times \text{span}(a_1, \ldots, a_m)
\]

\[
f_i(a_1, \ldots, a_m, x) = \lambda_i(a_1, \ldots, a_m, x),
\]

for \( 1 \leq i \leq m, (a_1, \ldots, a_m, x) \in \Delta \)

\[
g_i(a_i, x) = \langle a_i, x \rangle / \langle a_i, a_i \rangle,
\]

for \( 1 \leq i \leq m, a_i \in X_i, a_i \neq 0, x \in Y \)

It follows that

\[
\text{Ort}^m V = \text{pr}_{X_1 \times \ldots \times X_m} \Gamma
\]

And in the particular case when

\[
dim V = m
\]
we obtain the simpler forms
\[ \Delta = Ind^m V \times V \]
(5.9)
\[ \Gamma = Orth^m V \times V \]
\[ Orth^m V = pr_{V^m} \Gamma \]
Furthermore, through any vector space isomorphism between \( V \) and \( \mathbb{R}^m \), we obtain
\[ Ind^m V \text{ is open in } V^m, \quad \Delta \text{ is open in } V^{m+1} \]
(5.10)
Obviously, for \( m = 2 \), the relations (5.9) and (5.10) apply as well to the situation in (4.1) and (4.2).

6. Maximality

With the notation in (5.3) - (5.5), let \( \Gamma \subseteq \Delta \), and let us suppose that, on \( \Gamma \), each function \( f_i \), with \( 1 \leq i \leq m \), does not depend on \( x_j \in X_j \), with \( 1 \leq j \leq m \), \( j \neq i \). Then obviously, the same holds on every subset \( \Gamma' \subseteq \Gamma \).

Let now \( \Gamma_\alpha \subseteq \Delta \), with \( \alpha \in A \), be a family of subsets which is totally ordered by inclusion. Further, let us suppose that on each \( \Gamma_\alpha \), each of the functions \( f_i \), with \( 1 \leq i \leq m \), does not depend on \( x_j \in X_j \), with \( 1 \leq j \leq m \), \( j \neq i \). The according to (5.4), (5.5), there exist functions
\[ g_{\alpha, i} : pr_{X_i \times Y}(\Gamma_\alpha) \subseteq X_i \times Y \longrightarrow Z, \quad \alpha \in A, \ 1 \leq i \leq m \]
(6.1)
such that
\[ f_i|_{\Gamma_\alpha} = g_{\alpha, i} \circ pr_{X_i \times Y}|_{\Gamma_\alpha}, \quad \alpha \in A, \ 1 \leq i \leq m \]
(6.2)
If we consider now the subset
\[ \Gamma = \bigcup_{\alpha \in A} \Gamma_\alpha \subseteq \Delta \]
(6.3)
then for every \( 1 \leq i \leq m \), we obviously have

\[
(6.4) \quad pr_{X_i \times Y}(\Gamma) = \bigcup_{\alpha \in A} pr_{X_i \times Y}(\Gamma_\alpha)
\]

hence in view of (6.1), (6.2), there exists a function

\[
(6.5) \quad g_i : pr_{X_i \times Y}(\Gamma) \subseteq X_i \times Y \rightarrow Z
\]
such that

\[
(6.6) \quad f_i|_\Gamma = g_i \circ pr_{X_i \times Y}|_\Gamma
\]

In view of the above and the Zorn lemma, we obtain

**Lemma**

Given

\[
(6.7) \quad f_1, \ldots, f_m : \Delta \subseteq X_1 \times \ldots \times X_m \times Y \rightarrow Z
\]

and a nonvoid subset \( \Gamma \subseteq \Delta \), such that, on \( \Gamma \), each function \( f_i \), with \( 1 \leq i \leq m \), does not depend on \( x_j \in X_j \), with \( 1 \leq j \leq m \), \( j \neq i \).

Then there exists

\[
(6.8) \quad \text{a maximal subset } \bar{\Gamma} \subseteq \Delta, \text{ with } \bar{\Gamma} \supseteq \Gamma
\]
such that, on \( \bar{\Gamma} \), each function \( f_i \), with \( 1 \leq i \leq m \), does not depend on \( x_j \in X_j \), with \( 1 \leq j \leq m \), \( j \neq i \).

**Remark 4.**

A similar Lemma obviously holds for the particular situation in (5.1), (5.2).

**Theorem**

\( \Gamma \) as defined in (5.6) is maximal.
Proof

It follows easily.

7. The self-referentiality or circularity of orthogonality

In section 5 one possible precise meaning of ”does not depend on”, a concept used in (3.1) and (4.2), was presented. However, in view of the expressions of $\Gamma$ and $\mathcal{O}_{\text{rt}}.m_v$ in (5.6), (5.7), or for that matter, in (5.9), it is obvious that a self-referentiality or circularity appears with respect to the concept of orthogonality when one defines it - as in (3.1), (3.2) or (4.2) - without any inner product.

In this regard, and in view of recent developments in what is called non-wellfounded set theory, see Barwise & Moss, Forti & Honsell, Aczél, and the literature cited there, such self-referential or circulatory definitions are acceptable.

A detailed application of non-wellfounded set theory to the above problem of orthogonality will be presented in a number of subsequent papers.

Reference

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