The Energy of Conjugate Graph of a Dihedral Group

M. S. Mahmud a*, Pokalas P. Tal a, M. Z. Idris b and A. A. Malle c

a Department of Mathematics and Statistics, Federal University of Kashere, Gombe, Nigeria.
b Department of Mathematics, University of Jos, Jos, Nigeria.
c Department of Mathematical Sciences, Bauchi State University Gadau, Nigeria.

Authors’ contributions
This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information
DOI: 10.9734/JAMCS/2021/v36i1230424

Abstract

Let $\Gamma^{C}_{D_{2n}}$ and $E(\Gamma)$ denote the conjugate graph of a dihedral group of order $2n$ ($n \in \mathbb{N}$) and the energy of a graph respectively. The sum of the absolute values of the eigenvalues of an adjacency matrix's eigenvalues is the energy of a graph. In this paper, we use group representation of a dihedral group of order $2n$ with its conjugacy classes to explicitly design admissible conjugate graphs. We further introduced the general formula for the energy of conjugate graphs of dihedral groups in various circumstances. Also, we deduced the general formula for the conjugate graph of generalized dihedral groups of order $2n$ depending on the nature of $n$.

Keywords: Dihedral group; adjacency matrix; conjugacy class; conjugate graph; energy of a graph.

1 Introduction

The application of group theory to graph theory has been the subject of numerous studies over the years. The aspect of conjugate graph was first introduced by Erfanian and Tolue [1] and researches in this aspect commenced since then. These researches ultimately extended to calculating the various energies of the
conjugate graphs. It was remarked in Woods [2] that Gutman was the first to interpret the energy of a generic simple graphs in 1978 after getting inspired by Hückel’s Molecular Orbital Theory introduced in 1930. This discovery by Hückel has found application in Chemistry in the aspect of calculating the energy associated with π-electron orbital in conjugated hydrocarbons. A further analysis on Hückel’s work was undertaken in Gunthard and Primas [3] where they discovered that Hückel’s technique is actually based on the first-degree polynomial of a graph matrix.

In one development, Pirzada and Gutman [4] displayed that a graph’s energy cannot be the square root of an odd integer. Further, Mahmoud et al. [5] came up with a formula for the energy of a non-commuting graph of a dihedral group. Moreover, Mahmoud et al. [6] computed the energy of the conjugate graph of some finite Metabelian groups of order less than 24. Furthermore, Kinkar et al. [7] present some lower and upper bounds on the energy of a chain graph G, as well as demonstrating that the star provides the minimal energy of connected chain graphs of order n. And recently, Yinzhen et al. [8] obtain some upper and lower bounds on the energy and Laplacian energy of chain graphs by comparing algebraic connectivity, where they achieve the maximal Laplacian energy among all connected bicyclic chain graphs.

In this paper, we focus our attention on the dihedral group of order 2n.

This paper is subdivided into three (3) sections. In the second section we present existing definitions and results which will ultimately give light for understanding the results of this work. We present in the third section the findings of this work.

2 Preliminaries

We present in this section preliminary definitions and results. We shall find these definitions and results instrumental in the next section. Some of these definitions can be found in any textbook on Algebra.

2.1 Definition

Let G be a finite group and H be its normal subgroup. The factor group of G modulo H is a group defined by 
\[ G/H = \{gH: g \in G \}. \]

2.2 Definition

The group of all symmetries of a regular polygon with n vertices is known as the dihedral group, \( D_{2n} \). The group is of order \( 2n (n \in \mathbb{N}) \). \( D_{2n} \) is defined in the following way:

\[ D_{2n} \cong \langle a, b : a^3 = b^2 = 1, bab = a^{-1} \rangle \]

Its elements is presented below:

\[ \{1, a, a^2, ..., a^{n-1}, b, ab, a^2b, ..., a^{n-1}b \} \]

2.3 Definition

Let G be a group and \( a, b \in G \). The commutator of a and b denoted by \([a, b]\) is the element

\[ [a, b] = aba^{-1}b^{-1} \in G. \]

2.4 Definition

Let G be a finite group and let \( x_1, x_2 \) be elements in G. The elements \( x_1, x_2 \) are said to be conjugate if \( x_2 = h x_1 h^{-1} \) for some h in G. The set of all conjugate elements of \( x_1 \) is called the conjugacy class of \( x_1 \).

2.5 Definition

Let G be a finite group. The centre of G denoted \( Z(G) \) is defined by:

\[ Z(G) = \{ a \in G : ax = xa \forall x \in G \}. \]
The centre of $G$ is a subset of elements in $G$ that commute with every element of $G$.

**2.6 Definition**

Suppose $G$ is a group, two elements $a$ and $b$ of $G$ are conjugate if there exists an element $g$ in $G$ with $gag^{-1} = b$.

**2.7 Definition**

Let $G$ be a finite group. Then the conjugacy class of the element $a$ in $G$ is given as:

$$\text{cl}(a) = \{ g \in G : xax^{-1} = g, x \in G \}.$$  

**2.8 Definition**

A pair consisting of a set $V$ of vertices and a set $E$ of edges labeled as $\Gamma = (V, E)$ is called a graph. The elements of $E$ are the lines that combine two elements of $V$.

**2.9 Definition**

A complete graph is a simple graph in which every pairs of distinct vertices are adjacent. The complete graph with $n$ vertices is denoted as $K_n$.

**2.10 Definition**

Let $G$ be a finite non-abelian group with centre $Z(G)$. The conjugate graph is a graph whose vertices are non-central elements of $G$ in which two vertices are adjacent if they are conjugate.

**2.11 Preposition**

Let $G$ be finite group and let $a$ and $b$ be elements of $G$. $a$ and $b$ are said to be conjugate if they belong to precisely one conjugacy class. That is $\text{Cl}(a)$ and $\text{Cl}(b)$ are equal.

**2.12 Theorem [9]**

Depending on the parity of $n$, the conjugacy classes in the dihedral group $D_{2n}$ are as follows:

1. $\{1\}, \{a, a^{-1}\}, ..., \{a^{\frac{n-1}{2}}, a^{-\left(\frac{n-1}{2}\right)}\}, \{a^ib, 0 \leq i \leq n-1\}$, if $n$ is odd;
2. $\{1\}, \{a^2\}, \{a, a^{-1}\}, \{a^2, a^{-1}\}, ..., \{a^{\frac{n-2}{2}}, a^{-\left(\frac{n-2}{2}\right)}\}, \{a^{2i}b, 0 \leq i \leq \frac{n-2}{2}\}, \{a^{2i+1}b, 0 \leq i \leq \frac{n-2}{2}\}$, if $n$ is even.

**3. Main Result**

We present in this section the findings of this paper. In the result that follows, we characterize the conjugate graph of a dihedral group.

**3.1 Theorem**

Let $G$ be a dihedral group of order $2n$ where $n \geq 3, n \in \mathbb{N}$. Then, the conjugate graph of $D_{2n}$ is

$$\Gamma_{D_{2n}}^\text{c} = \begin{cases} 
\{K_i\}_{i=1}^{\frac{n-1}{2}} \cup \text{Knifn is odd} \\
\{K_i\}_{i=1}^{\frac{n-2}{2}} \cup \{K_{2i}\}_{i=1}^\text{if n is even} 
\end{cases}$$
3.1.1 Proof

Suppose $D_{2n}$ is a dihedral group of order $2n$ and $\Gamma^C_{D_{2n}}$ its conjugate graph. Now,

**Case 1:** if $n$ odd,

By Theorem 2.2, the set of vertex of the conjugate graph is $V(\Gamma^C_{D_{2n}}) = 2n - 1$, by the vertex adjacency of conjugate graph and by proposition 2.1, the elements are conjugate if they belong to one conjugate class i.e. $\text{cl}(a) = \text{cl}(b)$. Thus, $\Gamma^C_{D_{2n}} \cong \{K_i^2\}_{i=1}^{\frac{n-1}{2}} \cup Kn$.

**Case 2:** if $n$ even,

By Theorem 2.2, the set of vertex of conjugate graph is $V(\Gamma^C_{D_{2n}}) = 2(n - 1)$ by the vertex adjacency of conjugate graph and by proposition 2.1, the element are conjugate if they belong to precisely one conjugate class i.e. $\text{cl}(a) = \text{cl}(b)$. Thus, $\Gamma^C_{D_{2n}} \cong \{K_i^2\}_{i=1}^{\frac{n-2}{2}} \cup \{K_i^n\}_{i=1}^{\frac{n}{2}}$.

3.2 Example

Let $D_6$ be a dihedral group of order 6, $D_6 = \langle a, b; a^3 = b^2 = 1, bab = a^{-1} \rangle$ and let $\Gamma^G_{D_6}$ be a conjugate graph of $D_6$. Then $\Gamma^G_{D_6} = K_2 \cup K_3$.

3.2.1 Solution

Following from theorem 2.2, there are five non-central elements in $D_6$ and thus $|V(\Gamma^C_{D_6})| = 5$. By vertices adjacency of conjugate graph and proposition 2.1, the element are conjugate if they belong to one conjugacy class $\{\text{cl}(a), \text{cl}(a^2), \text{cl}(b), \text{cl}(ab), \text{cl}(a^2b)\}$, the related vertices are joined by an edge and form a single graph $K_2$. In the same way, the other three points make up a full graph of $K_3$. Hence $\Gamma^C_{D_6} = K_2 \cup K_3$. The conjugate graph of $\Gamma^G_{D_6}$ is shown below:

![Graph](image)

**Fig. 1.** The complete graph of $K_2 \cup K_3$

3.3 Example

Let $G$ be a dihedral group of order 8, $D_8 = \langle a, b; a^4 = b^2 = 1, bab = a^{-1} \rangle$ and let $\Gamma^G_{D_8}$ be a conjugate graph of $D_8$. Then $\Gamma^G_{D_8} = \{K_i^2\}_{i=1}^{3}$.

3.3.1 Solution

Considering theorem 2.2, there will be five non-central elements in $D_8$, thus $|V(\Gamma^C_{D_8})| = 6$. By vertices adjacency of conjugate graph and proposition 1, the element are conjugate if they belong to one conjugacy class
{cl(a), cl(a^3), [cl(b), cl(a^2b)], [cl(ab), cl(a^3b)]}. The related vertices are joined by an edge and formed three complete graph of $K_2$. Hence $\Gamma_{D_22}^C = \{K_i \}_{i=1}^3$ the conjugate graph of $\Gamma_{D_22}^C$ is shown below:

Fig. 2. The complete graph of $\{K_i \}_{i=1}^3$

3.4 Example

Let $G$ be a dihedral group of order $22, D_{22} = \langle a, b : a^2 = b^2 = 1, bab = a^{-1} \rangle$ and let $\Gamma_{D_{22}}^C$ be a conjugate graph of $D_{22}$. Then $\Gamma_{D_{22}}^C = \{K_i \}_{i=1}^5 \cup K_{11}$

3.4.1 Solution

It follows from theorem 2.2 that there are five non-central elements in $D_{22}$, thus $|V(\Gamma_{D_{22}}^C)| = 21$. And by vertices adjacency of conjugate graph and proposition 1, the element are conjugate if they belong to one conjugacy class $\{cl(a), cl(a^{10})\}, \{cl(a^3), cl(a^9)\}, \{cl(a^4), cl(a^8)\}, \{cl(a^5), cl(a^5)\}, \{cl(b), cl(ab)\}$, the related vertices are joined by an edge and form five single graph of $K_2$. In the same way, the other eleven points make up a full graph of $K_{11}$. Hence $\Gamma_{D_{22}}^C = \{K_i \}_{i=1}^5 \cup K_{11}$. The conjugate graph of $\Gamma_{D_{22}}^C$ is shown below:

Fig. 3. The complete graph of $\{K_i \}_{i=1}^5 \cup K_{11}$
3.4.2 Remark

We remark here that it is not hard to see from the examples above that as we display the conjugate graph of the dihedral groups of order $2n$, the size of the graph gets bigger in proportion to the order.

In what follows, we determine the eigenvalues of the conjugate graph of the dihedral group of order $2n$ as well as the general formulas for its energy. We consider this in cases depending on the parity of $n$.

3.5 Theorem

Let $G$ be a dihedral group of order $2n$, where $n$ is an odd integer such that $n \geq 3, n \in \mathbb{Z}^+$ and let $\Gamma_{D_{2n}}^C$ be its conjugate graph as usual. Then the energy of the graph $\Gamma_{D_{2n}}^C$ is $\varepsilon(\Gamma_{D_{2n}}^C) = \frac{5n-3+2i}{2}$ for $i = 0, 1, 2, 3,...$

3.5.1 Proof

Suppose $G$ is a dihedral group of order $2n$ and $\Gamma_{D_{2n}}^C$ be its conjugate graph, then from theorem 3.1, case 1, the conjugate graph of the group $G$ can be expressed as a complete graphs given by $\Gamma_{D_{2n}}^C = \left(\sum_{i=1}^{n-1} K_{2i}\right) \cup K_n$. Hence, the eigenvalues of the complete graph are:

$\lambda_1 = n - 1, \lambda_2 = 1$ (with $n-1$ repeats) and $\lambda_3 = -1$ (with $(n + i)$ repeats) where $i = 0, 1, 2, ...$ and so the energy is $\varepsilon(\Gamma_{D_{2n}}^C) = \frac{5n-3+2i}{2}$.

3.6 Example

Let $G = D_6$. Then, the energy of the conjugate graph of $G$, $\varepsilon(D_6) = 6$.

Then, the adjacency matrix $A$ for the conjugate graph $\Gamma_{D_6}^C = K_2 \cup K_3$ is given in the following:

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}$$

Thus, the characteristic polynomial of $A$ is given as in the following:

$$2 + \lambda^5 - 4\lambda^3 - 2\lambda^2 + 3\lambda$$

Hence, It is discovered that the eigenvalues are $\lambda = 2, \lambda = 1$ and $\lambda = -1$ with 3 repeats. Therefore, the energy of the conjugate graph for $D_6$ is $\varepsilon(\Gamma_{D_6}) = \sum_{i=1}^{n} |\lambda_i| = 6$.

Meanwhile by theorem 3.2, $\varepsilon(\Gamma_{D_{10}}^C) = \frac{5n-3+2i}{2} = 6$

3.7 Example

$G = D_{10}$. Then, the energy of the conjugate graph of $G$, $\varepsilon(D_{10}) = 12$.

The adjacency matrix $A$ for the conjugate graph $\Gamma_{D_{10}}^C = \left(\sum_{i=1}^{5} K_{2i}\right) \cup K_5$ is given in the following:
Thus, the characteristic polynomial of $A$ is given as in the following:

$$-4 + \lambda^9 - 12\lambda^7 - 20\lambda^6 - 6\lambda^5 + 36\lambda^4 + 20\lambda^3 - 12\lambda^2 - 15\lambda$$

Hence, It is discovered that the eigenvalues are $\lambda = 4$, $\lambda = 1$ with 2 repeats and $\lambda = -1$ with 6 repeats. Therefore, the energy of the conjugate graph for $D_{10}$ is $\varepsilon(G_{D_{10}}) = \sum_{i=1}^{n}\lambda_i = 12$.

Meanwhile by using theorem 3.1 $\varepsilon(G_{D_{14}}) = \frac{5n-34zi}{2} = 12$

### 3.8 Example

Let $G = D_{14}$. Then, the energy of the conjugate graph of $G$, $\varepsilon(G_{D_{14}}) = 18$.

The adjacency matrix $A$ for the conjugate graph $G_{D_{14}} = \{K_{i_2}\}_{i=1}^{3} \cup K_{5}$ is given in the following:

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}$$

Thus, the characteristic polynomial of $A$ is given as in the following:

$$(-1 + \lambda^2)^3(-6 + \lambda^7 - 21\lambda^5 - 70\lambda^4 - 105\lambda^3 - 84\lambda^2 - 35\lambda)$$

Hence, It is discovered that the eigenvalues are $\lambda = 6$, $\lambda = 1$ with 3 repeats and $\lambda = -1$ with 9 repeats. Therefore, the energy of the conjugate graph for $D_{14}$ is $\varepsilon(G_{D_{14}}) = \sum_{i=1}^{n}\lambda_i = 18$.

And so,$\varepsilon(G_{D_{14}}) = \frac{5n-34zi}{2} = 18$

### 3.9 Theorem

Let $G$ be a dihedral group of order $2n$ where $n$ is even, $n \geq 6 \in N$ and let $G_{D_{2n}}$ be its conjugate graph. Then the energy of the graph $G_{D_{2n}}$ is $\varepsilon(G_{D_{2n}}) = 3(n - 2)$.

#### 3.9.1 Proof

Suppose $G$ is a dihedral group of order $2n$ and $G_{D_{2n}}$ be its conjugate graph from theorem 3.1, case 2, the conjugate graph of the group $G$ is complete given by $G_{D_{2n}} = \{K_{i_2}\}_{i=1}^{2} \cup \{K_{i_3}\}_{i=1}^{2}$. Then the eigenvalues of the
complete graph are \( \lambda_1 = n - 2 \), \( \lambda_2 = n - 4 - i \) and \( \lambda_3 = -1 \) with multiplicity \( n + i \) where \( i = 0, 1, 2, 3, 4, 5, 6, \ldots \). Hence, the energy is \( \varepsilon(G_{D_{2n}}) = 3(n - 2) \).

### 3.10 Example

Let \( G = D_{12} \). Then, the energy of the conjugate graph of \( G \), \( \varepsilon(G_{D_{12}}) = 12 \).

Then, the adjacency matrix \( A \) for the conjugate graph \( G_{D_{12}} \) is given in the following:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus, the characteristic polynomial of \( A \) is given as in the following:

\[
4 + \lambda^{10} - 8\lambda^6 - 4\lambda^7 + 22\lambda^5 + 20\lambda^6 - 20\lambda^4 - 28\lambda^3 + \lambda^2 + 12\lambda
\]

Hence, It is discovered that the eigenvalues are \( \lambda = 2 \) with 2 repeats, \( \lambda = 1 \) with 2 repeats and \( \lambda = -1 \) with 6 repeats. Therefore, the energy of the conjugate graph for \( D_{12} \) is

\[
\varepsilon(G_{D_{12}}) = \sum_{i=1}^{n} |\lambda_i| = 12.
\]

Meanwhile by using theorem 3.2. \( \varepsilon(G_{D_{2n}}) = 3(n - 2) = 12 \)

### 3.11 Example

Let \( G = D_{16} \). Then, the energy of the conjugate graph of \( G \), \( \varepsilon(G_{D_{16}}) = 18 \).

Then, the adjacency matrix \( A \) for the conjugate graph \( G_{D_{16}} \) is given in the following:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Thus, the characteristic polynomial of $A$ is given as in the following:

\[-1 + \lambda^2)^3 (-3 + \lambda^4 - 36 \lambda^2 - 6 \lambda^2 - 8 \lambda)^2\]

Hence, it is discovered that the eigenvalues are $\lambda = 3$ with 2 repeats, $\lambda = 1$ with 4 repeats and $\lambda = -1$ with 9 repeats. Therefore, the energy of the conjugate graph for $D_{16}$ is

\[\varepsilon(\Gamma_{D_{16}}) = \sum_{i=1}^{n} |\lambda_i| = 18\]

Meanwhile by using theorem 3.3, $\varepsilon(\Gamma_{D_{20}}^c) = 3(n - 2) = 18$

3.12 Example

Let $G = D_{20}$. Then, the energy of the conjugate graph of $G$, $\varepsilon(\Gamma_{D_{20}}^c) = 24$.

Then, the adjacency matrix $A$ for the conjugate graph $\Gamma_{D_{20}}^c = \{K_{ij}\}_{i=1}^{4} \cup \{K_{ij}\}_{i=1}^{2}$ is given in the following:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus, the characteristic polynomial of $A$ is given as in the following:

\[-1 + \lambda^2)^4(-4 + \lambda^2 - 10\lambda^2 - 20\lambda^2 - 15\lambda)^2\]

Hence, it is discovered that the eigenvalues are $\lambda = 4$ with 2 repeats, $\lambda = 1$ with 4 repeats and $\lambda = -1$ with 12 repeats. Therefore, the energy of the conjugate graph for $D_{20}$ is

\[\varepsilon(\Gamma_{G}) = \sum_{i=1}^{n} |\lambda_i| = 24\]

Meanwhile by using theorem 3.3, $\varepsilon(\Gamma_{D_{2n}}^c) = 3(n - 2) = 24$

4. Conclusion

In this paper, we deduced the general formulas for the energy of a conjugate graph of dihedral groups. This formula was found to be $\varepsilon(\Gamma_{D_{2n}}^c) = \frac{5n^3-3+2i}{2}$ for an odd integer $n$ and $\varepsilon(\Gamma_{D_{2n}}^c) = 3(n - 2)$ for an even integer $n$.

In the case of the conjugate graph of the dihedral groups of order $2n$, we discovered that as the size of the graph increases in proportion to the order, so does the adjacency matrix $A$, which has a significant impact on the energy of the aforementioned group.

Competing Interests

Authors have declared that no competing interests exist.
References

[1] Erfanian A, Tolue B. Conjugate graph of finite groups; 2012. [Online]
Available: https://www.researchgate.net/publication/267186181_Conjugate_graphs_of_finite_groups [Accessed 1 Jan. 2022].

[2] Woods C. My favorite application using graph eigenvalues: Graph energy; 2013. [Online]
Available: http://www.math.ucsd.edu/~fan/teach/262/13/262notes/Woods_Midterm.pdf [Accessed 22 Jan. 2022].

[3] Günthard H, Primas H. Zusammenhang von Graphentheorie und MO-Theorie von Molekeln mit Systemen konjugierter Bindungen. Helvetica Chimica Acta. 1956;39(6):1645–1653.

[4] Pirzada S, Gutman I. Energy of a graph is never the square root of an odd integer. Applicable Analysis and Discrete Mathematics. 2008;2(1):118-121.

[5] Mahmoud R, Sarmin NH, Erfanian A. On the energy of non-commuting graph of dihedral groups; 2017. [online]
Available: https://doi.org/10.1063/1.4980960 [Accessed 21 Jan. 2022].

[6] Mahmud MS, Malle AA, Chibuisi C, Idris MZ. The energy of conjugate graph of some finite metabelian groups of order less than 24. Journal of Advances in Mathematics and Computer Science. 2021;36(7):45–51.
Available: https://www.journaljamcs.com/index.php/JAMCS/article/view/30383 [Accessed 21 Jan. 2022].

[7] Das K, Alazemi A, Andelic M. On energy and Laplacian energy of chain graphs. Discrete Applied Mathematics. 2020;284:391-400.

[8] Mei Y, Guo C, Liu M. The bounds of the energy and Laplacian energy of chaingraphs. AIMS Mathematics, 2021;6(5):4847-4859.

[9] Samaila D, Abba BI, Pur MP. On the Conjugacy Classes, Centers and Representation of the Groups $S_n$ and D; 2013 [online].
Available:https://web.p.ebscohost.com/abstract?direct=true&profile=ehost&scope=site&authtype=crawler&rjnl=22296107&AN=87991943&h=H6IpY%2feTogurUuFkJKG2mQaMjbc0MA%2fCdZk6L1fgrAzgovtMCE72UJz%2bYewhGlcqtray4Mth9pYtmA%3d%3d&crl=f&resultNs=AdminWebAuthor&resultLocal=ErrCtrlNotAuth&crllhashurl=login.aspx%3fdirect%3dtrue%26profile%3dhost%26scope%3dsite%26authtype%3dcrwaler%26jnl%3d22296107%26AN%3d87991943 [Accessed 21 Jan. 2022].

© 2021 Mahmud et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.