CLIFFORD SYSTEMS IN OCTONIONIC GEOMETRY

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Dedicated to the memory of Sergio Console

Abstract. We give an inductive construction for irreducible Clifford systems on Euclidean vector spaces. We then discuss how this notion can be adapted to Riemannian manifolds, and outline some developments in octonionic geometry.

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1. Introduction

The notion of Clifford system, as formalized in 1981 by D. Ferus, H. Karcher and H. F. Münzner, has been used in the last decades both in the study of isometric hypersurfaces and of Riemannian foliations \([6, 18, 8]\). In particular, Clifford systems have been used by Sergio Console and Carlos Olmos \([4]\) to give an alternative proof of a Theorem of E. Cartan stating that a compact isoparametric hypersurface of a sphere with three distinct principal curvatures is a tube around the Veronese embedding of the projective planes \(\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2\) over the reals, complex numbers, quaternions and Cayley numbers, respectively.

In this statement, the Veronese embedding of the four projective planes goes into spheres \(S^4, S^7, S^{13}, S^{25}\) and these embeddings admit an analogy in complex projective geometry. Namely, the (so-called) projective planes

\[
\mathbb{C}P^2, (\mathbb{C} \otimes \mathbb{C})P^2, (\mathbb{C} \otimes \mathbb{H})P^2, (\mathbb{C} \otimes \mathbb{O})P^2
\]

over complex numbers and over the other three composition algebras of complex complex numbers, complex quaternions and complex octonions, admit an embedding into complex projective spaces \(\mathbb{C}P^5, \mathbb{C}P^8, \mathbb{C}P^{14}, \mathbb{C}P^{26}\). These latter embeddings are also named after Veronese and give rise to projective algebraic varieties of degrees 4, 6, 14, 78, respectively. Very interesting properties of the mentioned two series of Veronese embeddings have been pointed out in \([1]\).

The following Table A collects “projective planes” \((\mathbb{K} \otimes \mathbb{K'})P^2\) over composition algebras \(\mathbb{K} \otimes \mathbb{K'}\), where \(\mathbb{K}, \mathbb{K'} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}\). Here notations \(V_2^4, V_4^6, V_4^{14}, V_{16}^{78}\) (with lower and upper indices being the complex...
dimension and the degree, respectively) are for the so-called Severi varieties, smooth projective algebraic varieties with nice characterizations realizing the mentioned embeddings [20]. Table A will give a general reference for our discussion. In particular, the fourth Severi variety $E_6/\text{Spin}(10) \cdot U(1) \cong V_{16}^{78} \subset \mathbb{CP}^{26}$ has been recently studied both with respect to the structure given by its holonomy and in the representation of the differential forms that generate its cohomology [17].

**Table A. Projective planes**

| $\mathbb{K} = \mathbb{K}'$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|-----------------------------|--------------|--------------|--------------|--------------|
| $\mathbb{R}$                | $\mathbb{R}P^2$ | $\mathbb{C}P^2 \cong V_2^4$ | $\mathbb{H}P^2$ | $\mathbb{O}P^2 \cong F_4/\text{Spin}(9)$ |
| $\mathbb{C}$                | $\mathbb{C}P^2 \cong V_2^4$ | $\mathbb{C}P^2 \times \mathbb{C}P^2 \cong V_4^6$ | $G_2(\mathbb{C}^6) \cong V_8^{14}$ | $E_6/\text{Spin}(10) \cdot U(1) \cong V_{16}^{78}$ |
| $\mathbb{H}$                | $\mathbb{H}P^2$ | $G_2(\mathbb{C}^6) \cong V_8^{14}$ | $G_{14}^{or}(\mathbb{R}^{12})$ | $E_7/\text{Spin}(12) \cdot \text{Sp}(1)$ |
| $\mathbb{O}$                | $\mathbb{O}P^2 \cong F_4/\text{Spin}(9)$ | $E_6/\text{Spin}(10) \cdot U(1) \cong V_{16}^{78}$ | $E_7/\text{Spin}(12) \cdot \text{Sp}(1)$ | $E_8/\text{Spin}(16)$ |

Recall that a Clifford system on the Euclidean vector space $\mathbb{R}^N$ is the datum of an $(m + 1)$-ple $C_m = (P_0, \ldots, P_m)$ of symmetric transformations $P_\alpha$ such that:

$$P_\alpha^2 = \text{Id} \quad \text{for all } \alpha, \quad P_\alpha P_\beta = -P_\beta P_\alpha \quad \text{for all } \alpha \neq \beta.$$

A Clifford system on $\mathbb{R}^N$ is said to be irreducible if $\mathbb{R}^N$ is not direct sum of two positive dimensional subspaces that are invariant under all the $P_\alpha$.

From representation theory of Clifford algebras one recognizes (cf. [9] page 483, [10] page 163) that $\mathbb{R}^N$ admits an irreducible Clifford system $C = (P_0, \ldots, P_m)$ if and only if

$$N = 2\delta(m),$$

where $\delta(m)$ is given by the following

**Table B. Clifford systems**

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | $\ldots$ | 8 + $h$ |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|---|---|
| $\delta(m)$ | 1 | 2 | 3 | 4 | 4 | 8 | 8 | 8 | 8 | 16 | 32 | 64 | 64 | 128 | 128 | 128 | $\ldots$ | 16$\delta(h)$ |

One can discuss uniqueness as follows. Given on $\mathbb{R}^N$ two Clifford systems $C_m = (P_0, \ldots, P_m)$ and $C'_m = (P'_0, \ldots, P'_m)$, they are said to be equivalent if there exists $A \in O(N)$ such that $P'_\alpha = A^tP_\alpha A$ for all $\alpha$. Then for $m \equiv 0$ mod 4 there is a unique equivalence class of irreducible Clifford systems, and for $m \equiv 0$ mod 4 there are two, classified by the two possible values of $\text{tr}(P_0P_1 \ldots P_m) = \pm 2\delta(m)$.

In the approach by Sergio Console and Carlos Olmos to the mentioned E. Cartan theorem on isoparametric hypersurfaces with three distinct principal curvatures in spheres, the Clifford systems are related with the Weingarten operators of their focal manifolds, and the possible values of $m$ turn out to be here only 1, 2, 4 or 8, the multiplicities of the eigenvalues of the Weingarten operators.

In the present paper we outline an inductive construction for all Clifford systems on real Euclidean vector spaces $\mathbb{R}^N$, by pointing out how the four Clifford systems $C_1, C_2, C_4, C_8$ considered in [4] correspond to structures given by the groups $U(1), U(2), \text{Sp}(2) \cdot \text{Sp}(1), \text{Spin}(9)$. We also develop, following ideas contained in [13, 10, 14, 17], the intermediate cases as well as some further cases appearing in Table B. We finally discuss the corresponding notion on Riemannian manifolds and relate it with the notion
of even Clifford structure and with the octonionic geometry of some exceptional Riemannian symmetric spaces.

We just mentioned the even Clifford structures, a kind of unifying notion proposed by A. Moroianu and U. Semmelmann \cite{13}. It is the datum, on a Riemannian manifold $M$, of a real oriented Euclidean vector bundle $(E, h)$, together with an algebra bundle morphism $\varphi : Cl^0(E) \to \text{End}(TM)$ mapping $\Lambda^2 E$ into skew-symmetric endomorphisms. Indeed, a Clifford system gives rise to an even Clifford structure, but there are some even Clifford structures on manifolds that cannot be constructed, even locally, from Clifford systems. This will be illustrated by examples in Sections 6 and 8.

Acknowledgements. We thank F. Reese Harvey for his interest in the present work and for taking the reference \cite{5} to our attention, cf. Remark 4.5. We also thank the referee for pointing out how representation theory of Clifford algebras allows to give the basic Proposition 3.3 on essential even Clifford structures. Also proofs of Theorems 6.2 and 8.1 have been simplified through this approach.

2. From $\mathbb{R}$ to $\mathbb{C}$ and to $\mathbb{H}$: the Clifford systems $C_1, C_2, C_3, C_4$

We examine here the first four columns of Table B, describing with some details irreducible Clifford systems $C_1, C_2, C_3, C_4$, acting on $\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$, respectively.

A Clifford system $C_1$ on $\mathbb{R}^2$ (here $m = 1$ and $\delta(m) = 1$) is given by matrices

$$N_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

representing in $\mathbb{C} \cong \mathbb{R}^2$ the involutions $z \in \mathbb{C} \to i\overline{z}$ and $z \in \mathbb{C} \to \overline{z}$, whose composition

$$N_{01} = N_0 N_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the complex structure on $\mathbb{C} \cong \mathbb{R}^2$.

Going to the next case, the Clifford system $C_2$ (now $m = 2$ and $\delta(m) = 2$) is the prototype example of the Pauli matrices:

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that we will need in their real representation:

$$P_0 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & -N_{01} \\ N_{01} & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}.$$  

The compositions $P_{\alpha\beta} = P_\alpha P_\beta$, for $\alpha < \beta$, yield as complex structures on $\mathbb{R}^4$ the multiplication on the right $R^H_i, R^H_j, R^H_k$ by unit quaternions $i, j, k$:

$$P_{01} = R^H_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad P_{02} = R^H_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_{12} = R^H_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$
Multiplication $L^H_i$ on the left by $i$ coincides with

$$P_{012} = P_0 P_1 P_2 = L^H_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and to complete $R^H_i, R^H_j, R^H_j, L^H_i$ to a basis of the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ one has to add the two further left multiplications

$$L^H_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L^H_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$ 

Thus:

**Proposition 2.1.** Orthogonal linear transformations in $\mathbb{R}^4$ preserving the individual $P_0, P_1, P_2$ are the ones in $U(1) = SO(2) \Delta \subset SO(4)$, those preserving the vector space $E^3 = \langle P_0, P_1, P_2 \rangle$ are the ones in $U(2) = Sp(1) \cdot U(1)$.

The next Clifford systems $C_3$ and $C_4$ act on $\mathbb{R}^8$. They can be defined by the following $4 \times 4$ block matrices

$$C_3 : \quad Q'_0 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad Q'_1 = \begin{pmatrix} 0 & -P_{01} \\ P_{01} & 0 \end{pmatrix},$$

$$Q'_2 = \begin{pmatrix} 0 & -P_{02} \\ P_{02} & 0 \end{pmatrix}, \quad Q'_3 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix},$$

and

$$C_4 : \quad Q_0 = Q'_0, \quad Q_1 = Q'_1, \quad Q_2 = Q'_2, \quad Q_3 = \begin{pmatrix} 0 & -P_{12} \\ P_{12} & 0 \end{pmatrix}, \quad Q_4 = Q'_3.$$

The following characterizations of the structures on $\mathbb{R}^8$ associated with $C_3$ and $C_4$ are easily seen.

**Proposition 2.2.** The structure defined in $\mathbb{R}^8$ by the datum of the vector space $E^4 = \langle Q'_0, Q'_1, Q'_2, Q'_3 \rangle \subset \text{End}(\mathbb{R}^8)$ can be described as follows. Matrices

$$B = \begin{pmatrix} B' & B'' \\ B''' & B'''' \end{pmatrix}$$

commuting with the single endomorphisms $Q'_0, Q'_1, Q'_2, Q'_3$ are characterized by the conditions $B' = B'''' \in \text{Sp}(1) \subset SO(4)$ and $B'' = B''' = 0$. Accordingly, matrices of SO(8) that preserve the vector space $E^4$ belong to a subgroup Sp(1) \cdot Sp(1) \cdot Sp(1) \subset SO(8).

To prepare the next Clifford systems, namely $C_5, C_6, C_7, C_8$ on $\mathbb{R}^{16}$, we need to look at the complex structures $Q_{0\alpha} = Q_0 Q_{\alpha}$ on $\mathbb{R}^8$. They indeed coincide with $R_i, R_j, R_k, R_e$, the right multiplication on $\mathbb{R}^8 \cong \mathbb{C}$ by octonions $i, j, k, e$, respectively:
\[ Q_{01} = R_i = \begin{pmatrix} R_i^H & 0 \\ 0 & -R_i^H \end{pmatrix}, \quad Q_{02} = R_j = \begin{pmatrix} R_j^H & 0 \\ 0 & -R_j^H \end{pmatrix}, \]
\[ Q_{03} = R_k = \begin{pmatrix} R_k^H & 0 \\ 0 & -R_k^H \end{pmatrix}, \quad Q_{04} = R_\epsilon = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}. \]

Associated with the Clifford system \( C_4 = (Q_0, Q_1, Q_2, Q_3, Q_4) \), we have ten complex structures \( Q_{\alpha\beta} = Q_{\alpha}Q_{\beta} \) with \( \alpha < \beta \), a basis of the Lie algebra \( \mathfrak{sp}(2) \subset \mathfrak{so}(8) \).

Their Kähler forms \( \theta_{\alpha\beta} \), written in the coordinates of \( \mathbb{R}^8 \), and using short notations like \( 12 = dx_1 \wedge dx_2 \), read:
\[
\begin{align*}
\theta_{01} &= -12 + 34 + 56 - 78, \quad \theta_{02} = -13 + 24 + 57 + 68, \quad \theta_{03} = -14 + 23 + 58 - 67, \\
\theta_{12} &= -14 + 23 - 58 + 67, \quad \theta_{13} = 13 + 24 + 57 + 68, \quad \theta_{23} = -12 + 34 - 56 + 78, \\
\theta_{04} &= -15 - 26 - 37 - 48, \quad \theta_{14} = -16 + 25 + 38 - 47, \\
\theta_{24} &= -17 - 28 + 35 + 46, \quad \theta_{34} = -18 + 27 - 36 + 45.
\end{align*}
\]

Thus, the second coefficient of the characteristic polynomial of their skew-symmetric matrix \( \theta = (\theta_{\alpha\beta}) \) turns out to be:
\[
\tau_2(\theta) = \sum_{\alpha<\beta} \theta_{\alpha\beta}^2 = -121234 - 41256 - 41357 + 41368 - 41278 - 41467 - 41458 + \star = -2\Omega_L,
\]
where \( \star \) denotes the Hodge star of what appears before it, and where
\[
\Omega_L = \omega_{L_i}^2 + \omega_{L_j}^2 + \omega_{L_k}^2
\]
is the left quaternionic 4-form.

One can check that matrices \( B = \begin{pmatrix} B' & B'' \\ B''^\prime & B''' \end{pmatrix} \) \( \in \text{SO}(8) \) commuting with each of the \( Q_\alpha \) are again the ones satisfying \( B'' = B''' = 0 \) and \( B' = B''' \in \text{Sp}(1) \subset \text{SO}(4) \). Hence the stabilizer of all individual \( Q_\alpha \) is the diagonal \( \text{Sp}(1) \Delta \subset \text{SO}(8) \), and the stabilizer of their spanned vector space \( E^5 \) is the quaternionic group \( \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8) \). Thus:

**Proposition 2.3.** The vector space \( E^5 \) spanned by the Clifford system \( C_4 = (Q_0, Q_1, Q_2, Q_3, Q_4) \) gives rise to the quaternion Hermitian structure of \( \mathbb{R}^8 \), and it is therefore equivalent to the datum either of the reduction to \( \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8) \) of its structure group, or to the quaternionic 4-form \( \Omega_L \). The complex structures \( Q_{\alpha\beta} = Q_{\alpha}Q_{\beta} \) are for \( \alpha < \beta \) a basis of the Lie subalgebra \( \mathfrak{sp}(2) \subset \mathfrak{so}(8) \).

**Remark 2.4.** As mentioned in the Introduction, when \( m \equiv 0 \mod 4 \), there are two equivalence classes of Clifford systems. It is clear from the construction of \( C_4 \) that a representative of the other class is just \( \tilde{C}_4 = (Q_0, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, Q_4) \), where:
\[
\begin{align*}
\tilde{Q}_1 &= \begin{pmatrix} 0 & -L_i^H \\ L_i^H & 0 \end{pmatrix}, \quad \tilde{Q}_2 = \begin{pmatrix} 0 & -L_j^H \\ L_j^H & 0 \end{pmatrix}, \\
\tilde{Q}_3 &= \begin{pmatrix} 0 & -L_k^H \\ L_k^H & 0 \end{pmatrix}.
\end{align*}
\]

3. **Statement on how to write new Clifford systems and representation theory**

The Clifford systems \( C_3 \) and \( C_4 \) have been obtained from \( C_2 \) through the following procedure. Similarly for the step \( C_1 \to C_2 \).

**Theorem 3.1.** [Procedure to write new Clifford systems from old] Let \( C_m = (P_0, P_1, \ldots, P_m) \) be the last (or unique) Clifford system in \( \mathbb{R}^N \). Then the first (or unique) Clifford system
\[ C_{m+1} = (Q_0, \tilde{Q}_1, \ldots, Q_m, Q_{m+1}) \]
in $\mathbb{R}^{2N}$ has as first and as last endomorphisms respectively

\[ Q_0 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad Q_{m+1} = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}, \]

where the blocks are $N \times N$. The remaining matrices are

\[ Q_\alpha = \begin{pmatrix} 0 & -P_\alpha \\ P_\alpha & 0 \end{pmatrix} \quad \alpha = 1, \ldots, m. \]

Here $P_\alpha$ are the complex structures given by compositions $P_0 P_\alpha$ in the Clifford system $C_m$. When the complex structures $P_\alpha$ can be viewed as (possibly block-wise) right multiplications by some of the unit quaternions $i, j, k$ or unit octonions $i, j, k, e, f, g, h$, and if the dimension permits, similarly defined further endomorphisms $Q_\beta$ can be added by using some others among $i, j, k$ or $i, j, k, e, f, g, h$.

**Proof.** Since $P_0 P_\alpha = -\text{Id}$, it is straightforward that $C_{m+1}$ is a Clifford system. As for the statement concerning the further $Q_\beta$, its proof follows as in the cases of $C_3$ (already seen), the further cases of $C_6, C_7, C_8$ in the next Section, and of $C_{12}, C_{14}, C_{15}, C_{16}$ in Section 7.

We now discuss some aspects of Clifford systems and of even Clifford structures (defined in the Introduction) related with representation theory of Clifford algebras. As pointed out in in [6, pages 482–483], any irreducible Clifford system $C_m = (P_0, \ldots, P_m)$ in $\mathbb{R}^N$, $N = 2\delta(m)$, gives rise to an irreducible representation of the Clifford algebra $\text{Cl}_{0,m-1}$ in $\mathbb{R}^{|\delta(m)|}$. This latter is given by skew-symmetric matrices

\[ E_1, \ldots, E_{m-1} \in \mathfrak{so}(\delta(m)) \]

satisfying

\[ E_\alpha E_\beta + E_\beta E_\alpha = -2\delta_\alpha \beta I. \]

To get such matrices $E_\alpha$ from $C_m$ consider the $\delta(m)$-dimensional subspace $E_+$ of $\mathbb{R}^N$ defined as the $(+1)$-eigenspace of $P_0$, and observe that $E_+$ is also invariant under the compositions $P_1 P_{\alpha+1}$. Then define the skew-symmetric endomorphisms on $\mathbb{R}^{\delta(m)}$

\[ E_\alpha = P_\alpha P_{\alpha+1} E_+ \quad (\alpha = 1, \ldots, m-1). \]

This gives the system of $E_1, \ldots, E_{m-1} \in \mathfrak{so}(\delta(m))$, thus a representation of the Clifford algebra $\text{Cl}_{0,m-1}$ in $\mathbb{R}^{|\delta(m)|}$. Conversely, given the anti-commuting $E_1, \ldots, E_{m-1} \in \mathfrak{so}(\delta(m))$, define on $\mathbb{R}^N$ the Clifford system $C_m$ given by the symmetric involutions

\[ P_0(u, v) = (v, u), \ldots, P_m(u, v) = (-E_\alpha v, E_\alpha u), \ldots, P_{m}(u, v) = (u, -v). \]

As a consequence, the procedure given by Theorem 3.1 can be seen as rephrasing the way to get irreducible representations of Clifford algebras. For these latter one can see [12, pages 30–40], and more explicitly the construction of Clifford algebra representations in [19, pages 18–20].

**Remark 3.2.** An alternative aspect of Clifford systems is to look at $C_m$ in $\mathbb{R}^N$, $N = 2\delta(m)$, as a representation of the Clifford algebra $\text{Cl}_{0,m+1}$ in $\mathbb{R}^N$ such that any vector of the pseudo-euclidean $\mathbb{R}^{0,m+1} \subset \text{Cl}_{0,m+1}$ acts as a symmetric endomorphism in $\mathbb{R}^N$.

Recall now from the structure of Clifford algebras the following periodicity relations

\[ \text{Cl}_{m+8} \cong \text{Cl}_m \otimes \mathbb{R}(16), \quad \text{Cl}_{0,m+8} \cong \text{Cl}_{0,m} \otimes \mathbb{R}(16), \]

where $\mathbb{R}(16)$ denotes the algebras of all real matrices of order 16.

Look now at the even Clifford structures, mentioned in the Introduction. First observe that a natural notion of irreducibility can be given for them, by requiring the Euclidean vector bundle $(E, h)$ not to be a direct sum. Then, by definition an irreducible even Clifford structure of rank $m + 1$ is equivalent to an irreducible representation of the even Clifford algebra $\text{Cl}_{m+1}^0 \cong \text{Cl}_m$ in $\mathbb{R}^N$, $N = 2\delta(m)$, mapping $\Lambda^2 \mathbb{R}^{m+1}$ into skew-symmetric endomorphisms of $\mathbb{R}^N$. 

Proposition 3.3. Irreducible even Clifford structures of rank \( r \) can be used. This gives the following: see whether an irreducible even Clifford structure is essential, Table C and the mentioned periodicity

\[ \text{essential} \]

\[ \text{not essential} \]

Thus, to way, and not only by dimensional reasons, as we will see on manifolds, cf. Section 6. We call essential a Clifford system that is not induced by an irreducible Clifford system. Thus, to see whether an irreducible even Clifford structure is essential, Table C and the mentioned periodicity relation can be used. This gives the following:

**Proposition 3.3.** Irreducible even Clifford structures of rank \( m + 1 \) on \( \mathbb{R}^{2\delta(m)} \) are essential when \( m \equiv 3, 5, 6, 7 \mod 8 \), and non essential when \( m = 0, 4 \mod 8 \). For \( m = 1, 2 \mod 8 \) both possibilities are open.

We will can back to this point on manifolds, see the last Sections.

4. From \( \mathbb{H} \) to \( \mathbb{O} \): the Clifford systems \( C_5, C_6, C_7, C_8 \)

According to Table B and to Theorem 3.1, the next Clifford system to consider is

\[ C_5 = (S'_0, S'_1, S'_2, S'_3, S'_4, S'_5) \]

in \( \mathbb{R}^{16} \), where:

\[ S'_0 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad S'_1 = \begin{pmatrix} 0 & -Q_{01} \\ Q_{01} & 0 \end{pmatrix}, \quad S'_2 = \begin{pmatrix} 0 & -Q_{02} \\ Q_{02} & 0 \end{pmatrix}, \]

\[ S'_3 = \begin{pmatrix} 0 & -Q_{03} \\ Q_{03} & 0 \end{pmatrix}, \quad S'_4 = \begin{pmatrix} 0 & -Q_{04} \\ Q_{04} & 0 \end{pmatrix}, \quad S'_5 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}. \]

A computation shows that:

**Proposition 4.1.** The orthogonal transformations in \( \mathbb{R}^{16} \) commuting with the individual \( S'_0, \ldots, S'_5 \) are the ones in the diagonal \( \text{Sp}(1)_\Delta \subset \text{SO}(16) \). The orthogonal transformations preserving the vector subspace \( E^6 = \langle C_5 \rangle \subset \text{End}(\mathbb{R}^{16}) \) are the ones in the subgroup \( \text{SU}(4) \cdot \text{Sp}(1) \subset \text{SO}(16) \). The complex structures \( S'_\alpha = S'_\alpha S'_\beta \) are for \( \alpha < \beta \) a basis of a Lie subalgebra \( \mathfrak{su}(4) \subset \mathfrak{so}(16) \).

By reminding that

\[ Q_{01} = R_i, \quad Q_{02} = R_j, \quad Q_{03} = R_k, \quad Q_{04} = R_e, \]

are the right multiplications on \( \mathbb{O} \) by \( i, j, k, e \), one completes \( C_5 \) to the Clifford system

\[ C_8 = (S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) \]

with

\[ S_0 = S'_0, \quad S_1 = S'_1, \quad S_2 = S'_2, \quad S_3 = S'_3, \quad S_4 = S'_4, \quad S_5 = S'_5 \]
and

\[
S_5 = \begin{pmatrix}
0 & -R_f \\
R_f & 0
\end{pmatrix}, \\
S_6 = \begin{pmatrix}
0 & -R_g \\
R_g & 0
\end{pmatrix}, \\
S_7 = \begin{pmatrix}
0 & -R_h \\
R_h & 0
\end{pmatrix}.
\]

It is now natural to compare the Clifford system \(C_8\) with the following notion, that was proposed by Th. Friedrich in [7].

**Definition 4.2.** A Spin(9)-structure on a 16-dimensional Riemannian manifold \((M, g)\) is a rank 9 real vector bundle

\[E^9 \subset \text{End}(TM) \to M,\]

locally spanned by self-dual anti-commuting involutions \(S_\alpha : TM \to TM:\)

\[g(S_\alpha X, Y) = g(X, S_\alpha Y), \quad \alpha = 1, \ldots, 9,
\]

\[S_\alpha S_\beta = -S_\beta S_\alpha, \quad \alpha \neq \beta,
\]

\[S_\alpha^2 = \text{Id}.
\]

From this datum one gets on \(M\) local almost complex structures \(S_{\alpha \beta} = S_\alpha S_\beta\), and the 9 \times 9 skew-symmetric matrix of their Kähler 2-forms

\[
\psi = (\psi_{\alpha \beta})
\]

is naturally associated. The differential forms \(\psi_{\alpha \beta}, \alpha < \beta\), are thus a local system of Kähler 2-forms on the Spin(9)-manifold \((M^{16}, E^9)\).

On the model space \(\mathbb{R}^{16}\), the standard Spin(9)-structure is defined by the generators \(S_1, \ldots, S_9\) of the Clifford algebra \(\text{Cl}_9\), the endomorphisms’ algebra of its 16-dimensional real representation \(\Delta_9 = \mathbb{R}^{16} = \mathbb{O}^2\). Accordingly, unit vectors in \(\mathbb{R}^9\) can be viewed as self-dual endomorphisms

\[w : \Delta_9 = \mathbb{O}^2 \to \Delta_9 = \mathbb{O}^2,
\]

and the action of \(w = u + r \in S^8\) (\(u \in \mathbb{O}, r \in \mathbb{R}, u\bar{u} + r^2 = 1\)), on pairs \((x, x') \in \mathbb{O}^2\) is given by

\[
\begin{pmatrix}
x \\
x'
\end{pmatrix} \mapsto \begin{pmatrix}
r \\
R_\pi \\
R_u \\
-R_r
\end{pmatrix} \begin{pmatrix}
x \\
x'
\end{pmatrix},
\]

where \(R_u, R_\pi\) denote the right multiplications by \(u, \pi\), respectively (cf. [9] page 288]). The choices

\[w = (0, 1), (0, i), (0, j), (0, k), (0, e), (0, f), (0, g), (0, h)\]

and \((1, 0) \in S^8 \subset \mathbb{O} \times \mathbb{R} = \mathbb{R}^9\),

define the symmetric endomorphisms:

\[S_0, S_1, \ldots, S_8\]

that constitute our Clifford system \(C_8\).

The subgroup Spin(9) \(\subset \text{SO}(16)\) is characterized as preserving the vector subspace

\[
E^9 = \langle S_0, \ldots, S_8 \rangle \subset \text{End}(\mathbb{R}^{16})
\]

whereas it is easy to check that the only matrices in \(\text{SO}(16)\) that preserve all the individual involutions \(S_0, \ldots, S_8\) are just \(\pm \text{Id}\).

It is useful to have explicitly the following right multiplications on \(\mathbb{O}\):
Theorem 4.3. (i) The families of complex structures

\[ S^A = \{ S_{\alpha\beta} \}_{1 \leq \alpha < \beta \leq 7}, \quad S^B = \{ S_{\alpha\beta} \}_{0 \leq \alpha < \beta \leq 7}, \quad \text{and} \quad S^C = \{ S_{\alpha\beta} \}_{0 \leq \alpha < \beta \leq 8}, \]

provide bases of Lie subalgebras spin_{\Delta}(7), spin(8), and spin(9) \subset so(16), respectively.

(ii) Let

\[ \psi^A = (\psi_{\alpha\beta})_{1 \leq \alpha, \beta \leq 7}, \quad \psi^B = (\psi_{\alpha\beta})_{0 \leq \alpha, \beta \leq 7}, \quad \psi^C = (\psi_{\alpha\beta})_{0 \leq \alpha, \beta \leq 8} \]

be the skew-symmetric matrices of the Kähler 2-forms associated with the mentioned families of complex structures \( S_{\alpha\beta} \). If \( \tau_2 \) and \( \tau_4 \) are the second and fourth coefficient of the characteristic polynomial, then:

\[ \frac{1}{6} \tau_2(\psi^A) = \Phi_{Spin_{\Delta}(7)}, \quad \frac{1}{4} \tau_2(\psi^B) = \Phi_{Spin(8)}, \quad \tau_2(\psi^C) = 0, \quad \frac{1}{360} \tau_4(\psi^C) = \Phi_{Spin(9)}. \]

where \( \Phi_{Spin_{\Delta}(7)} \in \Lambda^4(\mathbb{R}^16) \) restricts on any summand of \( \mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8 \) to the usual Spin(7) 4-form, and where \( \Phi_{Spin(9)} \in \Lambda^8(\mathbb{R}^16) \) is the canonical form associated with the standard Spin(9)-structure in \( \mathbb{R}^{16} \).

The 8-form \( \Phi_{Spin(9)} \) was originally defined by M. Berger in 1972, cf. [2].

Proof. (i) The three families refer to Lie subalgebras of spin(9). Now, the family \( S^C = \{ S_{\alpha\beta} \}_{0 \leq \alpha < \beta \leq 8} \) is known to be a basis of \( \text{spin}(9) \), cf. [2, 15]. Look at the construction of the \( S_{\alpha\beta} = S_{\alpha \beta} \), following the approach to Spin(9) as generated by transformations of type \( (4.1) \) (cf. [3] pages 278–279). In this
construction, matrices in the subalgebra $\mathfrak{spin}(8) \subset \mathfrak{spin}(9)$ are characterized through the infinitesimal triality principle as:

$$
\begin{pmatrix}
    a_+ & 0 \\
    0 & a_-
\end{pmatrix},
$$

where $a_+, a_- \in \mathfrak{so}(8)$ are *triality companions*, i.e. for each $u \in \mathfrak{g}$ there exists $v = a_0(u)$ such that $R_v = a_+ R_a a_-$. It is easily checked that all matrices $S_{\alpha\beta}$ with $0 \leq \alpha < \beta \leq 7$ satisfy this condition. Moreover, matrices in $\mathfrak{spin}_7(7) \subset \mathfrak{spin}(8)$ are characterized as those with $a_+ = a_-$ (thus $a_0 = 1$) ([15] pages 278–279, 285)). Thus only the $S_{\alpha\beta}$ with $1 \leq \alpha < \beta \leq 7$ are in $\mathfrak{spin}_7(7)$.

(ii) Here one can write explicit expressions of the $\psi_{\alpha\beta}$ in the coordinates of $\mathbb{R}^{16}$ (cf. [15] pages 334–335). These formulas allow to compute the $\tau_2$ and the $\tau_4$ appearing in the statements. It is convenient to begin with the matrix $\psi^B$, by adding up squares of the 2-forms $\psi_{\alpha\beta}$ with $0 \leq \alpha < \beta \leq 7$:

$$
\frac{1}{4} \tau_2(\psi^B) = \frac{1}{4} \sum_{0 \leq \alpha < \beta \leq 7} \psi_{\alpha\beta}^2 = 121'2' + 123'4' + 125'6' - 127'8' + 341'2' + 343'4' - 345'6'
$$

and this can be defined as $\Phi_{\mathfrak{spin}(8)}$. By computing the sum of squares of the $\psi_{\alpha\beta}$ with $1 \leq \alpha < \beta \leq 7$ one gets instead:

$$
\tau_2(\psi^A) = \sum_{1 \leq \alpha < \beta \leq 7} \psi_{\alpha\beta}^2 = \frac{6}{4} \tau_2(\psi^B) + 6[1234 + 5678 + 1'2'3'4' + 5'6'7'8']
$$

Thus, by defining $\Phi_{\mathfrak{spin}_7(7)} = \frac{1}{6} \tau_2(\psi^A)$ one has that its restriction to any of the two summands $\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$ is the usual $\mathfrak{spin}(7)$ form (cf. [15] pages 332–333]). Moreover, computations carried out in [15] pages 336 and 343] show that:

$$
\tau_2(\psi^C) = \sum_{1 \leq \alpha < \beta \leq 9} \psi_{\alpha\beta}^2 = 0, \quad \Phi_{\mathfrak{spin}(9)} = \frac{1}{360} \tau_4(\psi^C).
$$

The coefficients in the above equalities are chosen in such a way that, when reading

$$
\Phi_{\mathfrak{spin}_7(7)}, \quad \Phi_{\mathfrak{spin}(8)} \in \Lambda^4, \quad \Phi_{\mathfrak{spin}(9)} \in \Lambda^8
$$

in the coordinates of $\mathbb{R}^{16}$, the g.c.d. of coefficients be 1.
Theorem 4.3 suggests to consider, besides the Clifford systems \( C_5 \) and \( C_8 \) on \( \mathbb{R}^{16} \), the following intermediate Clifford systems:

\[
C_6 = (S_1, \ldots, S_7), \quad C_7 = (S_0, \ldots, S_7),
\]
or, in accordance with Theorem 3.1, the equivalent Clifford systems

\[
C'_6 = (S_0, S_1, \ldots, S_5, S_8), \quad C'_7 = (S_0, S_1, \ldots, S_6, S_8).
\]

Remark 4.4. Similarly to what observed in Remark 2.4, a representative of the other equivalence class of Clifford systems with \( m = 8 \) can be constructed as \( \tilde{C}_8 = (S_0, \tilde{S}_1, \ldots, \tilde{S}_7, S_8) \), where \( S_1, \ldots, S_7 \) are defined like \( S_1, \ldots, S_7 \) but using the left octonion multiplications \( L_i, \ldots, L_h \) instead of the right ones \( R_i, \ldots, R_h \).

Remark 4.5. A family of calibrations \( \phi_{4k}(\lambda) \in \Lambda^{4k}(\mathbb{R}^{16}) \) has been constructed by J. Dadok and F. R. Harvey for any \( \lambda \) in the unit sphere \( S^7 \subset \mathbb{R}^8 \) by squaring positive spinors \( S(\lambda) \in \mathbb{S}^+(16) \), through the following procedure, cf. [5]. Write \( \mathbb{R}^{16} \) as \( \mathbb{O}^+ \oplus \mathbb{O}^- \), and let \( (s_1 = i, \ldots, s_8 = h) \) be the standard basis of the octonions \( \mathbb{O}^+ \mathbb{O}^- \). Then one looks at the model for the Clifford algebra

\[
\Lambda(\mathbb{R}^{16}) \cong \text{Cl}_{16} \cong \text{Cl}_8 \otimes \text{Cl}_8 \cong \text{End}(\mathbb{O}^+ \otimes \mathbb{O}^-) \otimes \text{End}(\mathbb{O}^+ \otimes \mathbb{O}^-) \cong \text{End}(\mathbb{O}^+ \otimes \mathbb{O}^+) \oplus \text{End}(\mathbb{O}^- \otimes \mathbb{O}^-) \oplus \text{End}(\mathbb{O}^+ \otimes \mathbb{O}^-) \oplus \text{End}(\mathbb{O}^- \otimes \mathbb{O}^+),
\]

and the spinors \( S(\lambda) \) are in the diagonal \( D \subset \mathbb{S}^+(16) = (\mathbb{O}^+ \otimes \mathbb{O}^+) \oplus (\mathbb{O}^- \otimes \mathbb{O}^-) \). Then it is proved in [5] that by squaring such spinors one gets inhomogeneous exterior forms in \( \mathbb{R}^{16} \) as

\[
256 \, S(\lambda) \circ S(\lambda) = 1 + \phi_1 + \phi_8 + \phi_{12} + \text{vol},
\]

where the \( \phi_{4k} \) are calibrations. In particular, calibrations corresponding to \( \text{Spin}(7) \) and \( \text{Spin}(8) \) geometries are determined and discussed in [5]. This construction can be related with the present point of view in terms of Clifford systems, as we plan to show in a forthcoming work.

5. Clifford systems \( C_m \) on Riemannian manifolds

The definition 4.2 of a \( \text{Spin}(9) \) structure on a Riemannian manifold \( M^{16} \), using locally defined Clifford systems \( C_8 \) on its tangent bundle, and yielding a rank 9 vector subbundle of the endomorphism bundle, suggests to give the following more general definition.

Definition 5.1. A Clifford system \( C_m \) on a Riemannian manifold \( (M^N, g) \) is the datum of a rank \( m + 1 \) vector subbundle \( E^{m+1} \subset \text{End}(TM) \) locally generated by Clifford systems that are related in the intersections of trivializing open sets by matrices in \( \text{SO}(m + 1) \).

Some of the former statements, like Propositions 2.1, 2.2, 2.3, 4.1, and similar properties discussed for \( C_6, C_7, C_8 \) in Section 4, can be interpreted on Riemannian manifolds. One can then recognize that the datum of a Clifford system \( C_m \) on a Riemannian manifold \( M^N \), \( N = 2\delta(m) \), is equivalent to the reduction of its structure group to the group \( G \) according to the following Table D.

| \( m \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( N \) | 16 | 16 | 16 | 16 | 16 | 32 | 64 | 128 | 128 |
| \( G \) | \( \text{U}(1) \) | \( \text{U}(2) \) | \( \text{Sp}(1)^3 \) | \( \text{Sp}(2)\text{Sp}(1) \) | \( \text{SU}(4)\text{Sp}(1) \) | \( \text{Spin}(7)\text{U}(1) \) | \( \text{Spin}(8) \) | \( \text{Spin}(9) \) | \( \text{Spin}(10) \) | \( \text{Spin}(11) \) | \( \text{Spin}(12) \) | \( \text{Spin}(13) \) |

Remark 5.2. Although \( \text{Spin}(7) \) structures on 8-dimensional Riemannian manifolds cannot be described through a rank 7 vector bundle of symmetric endomorphisms of the tangent bundle (cf. [15] Corollary 9), this can definitely be done for a \( \text{Spin}_\Delta(7) \) structure in dimension 16. The above discussion shows in fact that the 7 symmetric endomorphisms \( S_1, \ldots, S_7 \) allow to deal with a \( \text{Spin}_\Delta(7) \) structure as a...
Clifford system on a 16-dimensional Riemannian manifold. Indeed, most of the known examples of Spin(9) manifolds carry the subordinated structure \( \text{Spin}_\Delta(7) \), cf. [7, 14].

On the other hand, a Spin(7) structure on an 8-dimensional Riemannian manifold is an example of even Clifford structure, as defined in the Introduction. Here the defining vector bundle \( E \) has rank 7 and one can choose \( E \subset \text{End}(TM)^- \) locally spanned as
\[
\langle I, J, K, E, F, G, H \rangle,
\]
i.e. multiplying by elements in the canonical basis of octonions. As already mentioned, in situations like this, we call essential the even Clifford structure.

6. The Clifford system \( C_9 \) and the essential Clifford structure on \( E_{III} \)

Our recipe for producing Clifford systems, according to Theorem 3.1, gives on \( \mathbb{R}^{32} \) the following Clifford system \( C_9 = (T_0, T_1, \ldots, T_9) \):

\[
T_0 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \ldots, T_\alpha = \begin{pmatrix} 0 & -S_{\alpha} \\ S_{\alpha} & 0 \end{pmatrix}, \ldots, T_9 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}.
\]

Here \( \alpha = 1, \ldots, 8 \) and any block in the matrices is now of order 16.

In [17] we showed that the group of orthogonal transformations preserving the vector subspace \( E^{10} = C_9 \subset \text{End}(\mathbb{R}^{32}) \) is the image of Spin(10) under a real representation in SO(32). Indeed, one can also look at the half-Spin representations of Spin(10) into SU(16) \( \subset \text{SO}(32) \), that are related with the notion of even Clifford structure, as defined in the Introduction.

Note that, according to the definitions, any Clifford system \( C_m \) on a Riemannian manifold \( \mathcal{M} \) gives rise to an even Clifford structure of rank \( m + 1 \): this is for example the case of \( C_4 \) on 8-dimensional quaternion Hermitian manifolds, or of \( C_8 \) on 16-dimensional Spin(9) manifolds. Indeed, one has also a notion of parallel even Clifford structure, requiring the existence of a metric connection \( \nabla \) on \( (E, h) \) such that \( \varphi \) is connection preserving. Thus for example parallel even Clifford structures with \( m = 4, 8 \) correspond to a quaternion Kähler structure in dimension 8 and to holonomy Spin(9) in dimension 16. In [13, page 955], a classification is given of complete simply connected Riemannian manifolds with a parallel non-flat even Clifford structure.

This classification statement includes one single example for each value of the rank \( m + 1 = 9, 10, 12, 16 \) (and no other examples when \( m + 1 > 8 \)). These examples are the ones in the last row (or column) of Table A, namely the projective planes over the four algebras \( \mathbb{O}, \mathbb{C} \otimes \mathbb{O}, \mathbb{H} \otimes \mathbb{O}, \mathbb{O} \otimes \mathbb{O} \).

In the Cartan labelling they are the symmetric spaces:

\[
\text{FII} = \mathbb{F}_4/\text{Spin}(9), \quad \text{EIII} = \mathbb{E}_6/\text{Spin}(10) \cdot \text{U}(1), \\
\text{EVI} = \mathbb{E}_7/\text{Spin}(12) \cdot \text{Sp}(1), \quad \text{EVIII} = \mathbb{E}_8/\text{Spin}(16)^+.
\]

In this respect we propose the following

**Definition 6.1.** Let \( \mathcal{M} \) be a Riemannian manifold. An even Clifford structure \( (E, h) \), with rank \( m + 1 \) and defining map \( \varphi : \text{Cl}_0(E) \to \text{End}(TM) \), is said to be essential if it is not a Clifford system, i.e. if it is not locally spanned by anti-commuting self-dual involutions.

We have already seen that Spin(7) structures in dimension 8 are examples of essential even Clifford structures, cf. Remark 4.4. As mentioned, both quaternion Hermitian structures in dimension 8 and Spin(9) structures in dimension 16 are instead non-essential. For example, on the Cayley plane FII, local Clifford systems on its three coordinate open affine planes \( \mathbb{O}^2 \) fit together to define the Spin(9) structure and hence the even Clifford structure. This property has no analogue for the other three projective planes EIII, EVI, EVIII. As a matter of fact it has been proved in [11] that the projective plane EIII over complex octonions cannot be covered by three coordinate open affine planes \( \mathbb{C} \otimes \mathbb{O}^2 \). We have:
Theorem 6.2. The parallel even Clifford structure on $\text{EIII}$ is essential.

Proof. Note first that the statement cannot follow from Proposition 3.3. However, as observed in Table D, the structure group of a 32-dimensional manifold carrying a Clifford system $C_9$ reduces to $\text{Spin}(10) \subset \text{SU}(16)$. This would be the case of the holonomy group, assuming that such a Clifford system induces the parallel even Clifford structure of $\text{EIII}$. Thus, $\text{EIII}$ would have a trivial canonical bundle, in contradiction with the positive Ricci curvature property of Hermitian symmetric spaces of compact type. □

As showed in [17], the vector bundle defining the even Clifford structure is the $\text{E}^{10}$ locally spanned as $<I> \oplus <S_0, \ldots, S_8>$. Here $I$ is the global complex structure of the Hermitian symmetric space $\text{EIII}$, and $S_0, \ldots, S_8$ (matrices in $\text{SO}(16) \subset \text{SU}(16)$) define, together with $I$, the $\text{Spin}(10) \cdot \text{U}(1) \subset \text{U}(16)$ structure given by its holonomy. The construction of the even Clifford structure follows of course the alternating composition, so that $I \wedge S_\alpha = -S_\alpha \wedge I$, allowing to get a skew symmetric matrix $\psi^D = (\psi^D_{\alpha\beta})$ of Kähler forms associated with all compositions of two generators (cf. Theorem 4.3).

Remark 6.3. As proved in [17], the cohomology classes of the Kähler form and of the global differential form $\tau_4(\psi^D)$ generate the cohomology of $\text{EIII}$.

7. Clifford systems $C_{10}$ up to $C_{16}$

For the next step, we need now the following order 32 matrices:

$$
T_{01} = \begin{pmatrix}
R_i & 0 & 0 & 0 \\
0 & -R_i & 0 & 0 \\
0 & 0 & -R_i & 0 \\
0 & 0 & 0 & R_i
\end{pmatrix}, \ldots, T_{07} = \begin{pmatrix}
R_h & 0 & 0 & 0 \\
0 & -R_h & 0 & 0 \\
0 & 0 & -R_h & 0 \\
0 & 0 & 0 & R_h
\end{pmatrix},
\nonumber
$$

$$
T_{08} = \begin{pmatrix}
0 & -\text{Id} & 0 & 0 \\
\text{Id} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{Id} \\
0 & 0 & -\text{Id} & 0
\end{pmatrix}, \quad T_{09} = \begin{pmatrix}
0 & 0 & -\text{Id} & 0 \\
0 & 0 & 0 & -\text{Id} \\
\text{Id} & 0 & 0 & 0 \\
0 & \text{Id} & 0 & 0
\end{pmatrix},
\nonumber
$$

and now we can write on $\mathbb{R}^{64}$ the following matrices of the Clifford system $C_{10}$:

$$
U_0 = \begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix}, \ldots, U_\alpha = \begin{pmatrix}
0 & -T_{0\alpha} \\
T_{0\alpha} & 0
\end{pmatrix}, \ldots, U_{10} = \begin{pmatrix}
\text{Id} & 0 \\
0 & -\text{Id}
\end{pmatrix}.
\nonumber
$$

Here $\alpha = 1, \ldots, 9$ and any block in the matrices is of order 32.

The subgroup of $\text{SO}(64)$ preserving the subbundle $E^{11} = <C_{10}> \subset \text{End}(\mathbb{R}^{64})$ is now $\text{Spin}(11)$, a subgroup of $\text{Sp}(16) \subset \text{SO}(64)$ under its Spin representation.

Still another step, through

$$
U_{01} = \begin{pmatrix}
T_{01} & 0 \\
0 & -T_{01}
\end{pmatrix}, \ldots, U_{09} = \begin{pmatrix}
T_{09} & 0 \\
0 & -T_{09}
\end{pmatrix}, U_{0,10} = \begin{pmatrix}
0 & -\text{Id} \\
\text{Id} & 0
\end{pmatrix},
\nonumber
$$

we go to the first Clifford system $C_{11}$ in $\mathbb{R}^{128}$. Its matrices are:

$$
V'_0 = \begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix}, \ldots, V'_\alpha = \begin{pmatrix}
0 & -U_{0\alpha} \\
U_{0\alpha} & 0
\end{pmatrix}, \ldots, V'_{11} = \begin{pmatrix}
\text{Id} & 0 \\
0 & -\text{Id}
\end{pmatrix},
\nonumber
$$

and now we can write on $\mathbb{R}^{128}$ the following matrices of the Clifford system $C_{11}$:

$$
V'_0 = \begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix}, \ldots, V'_\alpha = \begin{pmatrix}
0 & -U_{0\alpha} \\
U_{0\alpha} & 0
\end{pmatrix}, \ldots, V'_{11} = \begin{pmatrix}
\text{Id} & 0 \\
0 & -\text{Id}
\end{pmatrix}.
\nonumber
$$
now with $\alpha = 1, \ldots, 10$ and any block of order 64.

To recognize the next Clifford system, $C_{12}$ and again in $\mathbb{R}^{128}$, introduce the following matrices, of order 32:

\[
\text{Block}_{R_i} = \begin{pmatrix} 0 & -\text{Id} & 0 & 0 \\ \text{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id} \\ 0 & 0 & -\text{Id} & 0 \end{pmatrix}, \quad \text{Block}_{R_j} = \begin{pmatrix} 0 & 0 & -\text{Id} & 0 \\ 0 & 0 & 0 & -\text{Id} \\ \text{Id} & 0 & 0 & 0 \\ 0 & \text{Id} & 0 & 0 \end{pmatrix}, \quad \text{Block}_{L_k} = \begin{pmatrix} 0 & 0 & 0 & -\text{Id} \\ 0 & 0 & -\text{Id} & 0 \\ 0 & \text{Id} & 0 & 0 \\ \text{Id} & 0 & 0 & 0 \end{pmatrix},
\]

block-wise extensions of matrices $R_i^H, R_j^H, L_k^H$ considered in Section 2. We need also the further matrices, of order 64, block-wise extension of $R_i, R_j, R_e, R_h$:

\[
\text{Block}_{R_i} = \begin{pmatrix} \text{Block}_{R_i} & 0 \\ 0 & -\text{Block}_{R_i} \end{pmatrix}, \quad \text{Block}_{R_j} = \begin{pmatrix} \text{Block}_{R_j} & 0 \\ 0 & -\text{Block}_{R_j} \end{pmatrix}, \quad \text{Block}_{R_e} = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad \text{Block}_{R_h} = \begin{pmatrix} 0 & \text{Block}_{L_k} \\ \text{Block}_{L_k} & 0 \end{pmatrix}.
\]

Then one easily writes the last matrices in the Clifford system $C_{12}$ as:

\[
V_8' = \begin{pmatrix} 0 & -\text{Block}_{R_i} \\ \text{Block}_{R_i} & 0 \end{pmatrix}, \quad V_9' = \begin{pmatrix} 0 & -\text{Block}_{R_j} \\ \text{Block}_{R_j} & 0 \end{pmatrix}, \quad V_{10}' = \begin{pmatrix} 0 & -\text{Block}_{R_e} \\ \text{Block}_{R_e} & 0 \end{pmatrix}.
\]

One gets:

**Proposition 7.1.** The matrices

\[
V_0 = V_0', \ldots, V_{10} = V_{10}', V_{11} = \begin{pmatrix} 0 & -\text{Block}_{R_h} \\ \text{Block}_{R_h} & 0 \end{pmatrix}, V_{12} = V_{11}'
\]

give rise to the Clifford system $C_{12}$ in $\mathbb{R}^{128}$.

**Proof.** The only point to check is that $V_{11}$ anti-commutes with all the other matrices. This is a straightforward computation. □

The orthogonal transformations preserving $C_{12}$ correspond to a real representation of Spin(12) in SO(128).

As a further step, we construct $C_{13}$, the first Clifford system in $\mathbb{R}^{256}$, whose involutions are:

\[
W_0' = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \ldots, W_\alpha' = \begin{pmatrix} 0 & -V_0 \alpha \\ V_0 \alpha & 0 \end{pmatrix}, \ldots, W_{13}' = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix},
\]

now with $\alpha = 1, \ldots, 12$. In particular

\[
W_{12}' = \begin{pmatrix} 0 & 0 & 0 & \text{Id} \\ 0 & 0 & -\text{Id} & 0 \\ 0 & -\text{Id} & 0 & 0 \\ \text{Id} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\text{Block}_{R_e}^{128} \\ \text{Block}_{R_e}^{128} & 0 \end{pmatrix},
\]
when now the block matrix is of order 128. Then one recognizes that one can add three similar block matrices with $\text{Block}_{R_R}^{128}$, $\text{Block}_{R_L}^{128}$, $\text{Block}_{L_R}^{128}$, extending $C_{13}$ up to $C_{16}$, still on $\mathbb{R}^{256}$, and with intermediate Clifford systems $C_{14}$ and $C_{15}$.

8. The symmetric spaces EVI and EVIII

As a consequence of Proposition 3.3 we have:

**Theorem 8.1.** The parallel even Clifford structures on EVI and on EVIII are essential.

These even Clifford structures can in fact be defined by vector subbundles $E^{12}$ and $E^{16}$ of the endomorphism bundle, locally generated as follows:

\[(8.1) \quad E^{12} : < I, J, K > \oplus < S_0, \ldots, S_9 > \rightarrow \text{EVI},\]

\[(8.2) \quad E^{16} : < I, J, K, E, F, G, H > \oplus < S_0, \ldots, S_9 > \rightarrow \text{EVIII},\]

where $S_0, \ldots, S_9$ are the involutions in $\text{SO}(16)$ defining the $\text{Spin}(9)$ structures.

Note that the quaternionic structure of EVI, one of the quaternion Kähler Wolf spaces, appears as part of its even Clifford structure.

As already mentioned concerning EIII, also on EVI and EVIII the compositions of generators of the even Clifford structure follows the alternating property e.g. $I \wedge S_\alpha = -S_\alpha \wedge I$. In this way one still has skew-symmetric matrices of Kähler forms associated with compositions of two generators (cf. Theorem 4.3 and Theorem 6.2). We denote these skew-symmetric matrices by $\psi^E$ for EVI and $\psi^F$ for EVIII. One can look at the following sequence of the matrices we introduced:

$$\psi^A \subset \psi^B \subset \psi^C \subset \psi^D \subset \psi^E \subset \psi^F,$$

all producing, via invariant polynomials, global differential forms associated with the associated structure groups

$$\text{Spin}(7) \Delta, \text{Spin}(8), \text{Spin}(9), \text{Spin}(10) \cdot \text{U}(1), \text{Spin}(12) \cdot \text{Sp}(1), \text{Spin}(16)^+.$$

(cf. also Remark 6.3). We can mention here that the (rational) cohomology of EVI is generated, besides by the class of the quaternion Kähler 4-form, by a 8-dimensional class and by a 12-dimensional class. It is thus tempting to represent these classes by $\tau_4(\psi^E)$ and by $\tau_6(\psi^F)$. As for EVIII, it is known that its rational cohomology is generated by classes of dimension 8, 12, 16, 20. One can also observe, in this last situation of EVIII, that the local Kähler forms $\psi_{\alpha \beta}$ associated with the group $\text{Spin}(16)^+$ can be seen for $\alpha < \beta$ in correspondence with a basis of its Lie algebra $\text{so}(16)$. As such, they exhaust both families of $36 + 84 = 120$ exterior 2-forms appearing in decomposition 4.2.

We conclude with two remarks relating the discussed subjects with some of our previous work.

**Remark 8.2.** In [16] we described a procedure to construct maximal orthonormal systems of tangent vector fields on spheres. For that we essentially used, besides multiplication in $\mathbb{C}, \mathbb{H}, \mathbb{O}$, the Spin(9) structure of $\mathbb{R}^{16}$, applied also block-wise in higher dimension. Remind that the maximal number $\sigma(N)$ of linear independent vector field on an odd-dimensional sphere $S^{N-1}$, with $N = (2k + 1)2^p16^q$ and $0 \leq p \leq 3$, is given by the Hurwitz-Radon formula

$$\sigma(N) = 2^p + 8q - 1.$$ 

Thus, it does not appear easy to read this number out of Table B, even considering also reducible Clifford systems.

On the other hand, one can recognize from the construction of [16] that there is instead a simple relation with even Clifford structures, and that for example the construction of a maximal system of tangent vector fields on spheres $S^{31}, S^{63}, S^{127}$ can be rephrased using the essential even Clifford structures of rank 10, 12, 16 on $\mathbb{R}^{32}, \mathbb{R}^{64}, \mathbb{R}^{128}$. Such even Clifford structures exist and are parallel non-flat on the symmetric spaces EIII, EVI, EVIII (cf. proof of Theorem 6.2 and equations (8.1), (8.2)). Following [16], this point of view can be suitably applied to spheres of any odd dimension.
Remark 8.3. In [14] we studied the structure of compact locally conformally parallel Spin(9) manifolds. They are of course examples, together with their Kähler, quaternion Kähler, and Spin(7) counterparts, of manifolds equipped with a locally conformally parallel even Clifford structure. We can here observe that the following Hopf manifolds

\[ S^{31} \times S^1, \ S^{63} \times S^1, \ S^{127} \times S^1 \]

are further examples of them, with the locally conformally flat metric coming from their universal covering. One can also describe some finite subgroups of Spin(10), Spin(12), Spin(16) acting freely on \( S^{31}, S^{63}, S^{127} \), respectively, and the list of groups \( K \) mentioned in Example 6.6 of [14] certainly applies to these three cases. Accordingly, finite quotients like \( (S^{N-1}/K) \times S^1 \), with \( N = 32, 64, 128 \), still carry a locally conformally parallel even Clifford structure. Note however that the structure Theorem C proved in [14] cannot be reproduced for these higher rank locally conformally parallel even Clifford structures.

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