SOLUTIONS IN SOME BORDERLINE CASES OF ELLIPTIC EQUATIONS WITH DEGENERATE COERCIVITY

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Abstract. We study a degenerate elliptic equation, proving existence results of distributional solutions in some borderline cases.

Bernardo, come vide li occhi miei ...
(Dante, Paradiso XXXI)

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1. Introduction

The Sobolev space $W^{1,2}_0(\Omega)$ is the natural functional framework (see [10], [12]) to find weak solutions of nonlinear elliptic problems of the following type

\begin{align}
-\text{div} \left( \frac{a(x) \nabla u}{(1+|u|)^\theta} \right) &= f, \quad \text{in} \ \Omega; \\
u &= 0, \quad \text{on} \ \partial \Omega,
\end{align}

where the function $f$ belongs to the dual space of $W^{1,2}_0(\Omega)$, $\Omega$ is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $\theta$ is a real number such that $0 \leq \theta \leq 1$,

and $a : \Omega \to \mathbb{R}$ is a measurable function satisfying the following conditions:

$$
\alpha \leq a(x) \leq \beta,
$$

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for almost every $x \in \Omega$, where $\alpha$ and $\beta$ are positive constants. The main difficulty to use the general results of [10], [12] is the fact that

$$A(v) = -\text{div} \left( \frac{a(x)\nabla v}{(1 + |v|)^\theta} \right),$$

is not coercive. Papers [7], [4] and [3] deal with the existence and summability of solutions to problem (1) if $f \in L^m(\Omega)$, for some $m \geq 1$.

Despite the lack of coercivity of the differential operator $A(v)$ appearing in problem (1), in the papers [7], [4] and [1], the authors prove the following existence results of solutions of problem (1), under assumption (3):

A) a weak solution $u \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$, if $m > \frac{N}{2}$ and (2) holds true;

B) a weak solution $u \in W^{1,2}_0(\Omega) \cap L^{m^*(1-\theta)}(\Omega)$, where $m^* = \frac{mN}{N-2m}$; if

$$0 < \theta < 1, \quad \frac{2N}{N + 2 - \theta(N - 2)} \leq m < \frac{N}{2};$$

C) a distributional solution $u$ in $W^{1,q}_0(\Omega)$, if $q = \frac{Nm(1-\theta)}{N - m(1+\theta)} < 2$, if

$$\frac{1}{N - 1} \leq \theta < 1, \quad \frac{N}{N + 1 - \theta(N - 1)} < m < \frac{2N}{N + 2 - \theta(N - 2)}.$$

D) an entropy solution $u \in M^{m^*(1-\theta)}$, with $|\nabla u| \in M(q)(\Omega)$, for

$$1 \leq m \leq \max \left\{ 1, \frac{N}{N+1-\theta(N-1)} \right\}.$$

The borderline case $\theta = 1$ was studied in [3], proving the existence of a solution $u \in W^{1,2}_0(\Omega) \cap L^p(\Omega)$ for every $p < \infty$. The case where the source is $\frac{A}{|x|^2}$ was analyzed too.

About the different notions of solutions mentioned above, we recall that the notion of entropy solution was introduced in [2]. Let

$$T_k(s) = \begin{cases} 
    s & \text{if } |s| \leq k, \\
    k \frac{s}{|s|} & \text{if } |s| > k.
\end{cases}$$

Then $u$ is an entropy solution to problem (1) if $T_k(u) \in W^{1,2}_0(\Omega)$ for every $k > 0$ and

$$\int_{\Omega} \frac{a(x)\nabla u}{(1 + |u|)^\theta} \cdot \nabla T_k(u - \varphi) \leq \int_{\Omega} f T_k(u - \varphi), \quad \forall \varphi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega).$$
Moreover, we say that $u$ is a distributional solution of (1) if

$$
\int_\Omega \frac{a(x)\nabla u}{(1 + |u|)^\theta} \cdot \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in C^\infty_0(\Omega).
$$

The figure below can help to summarize the previous results, where the name of a given region corresponds to the results that we have just cited.

- **Theorem 1.1.** Let $f$ be a function in $L^m(\Omega)$. Assume (3) and

$$
(6) \quad m = \frac{N}{N + 1 - \theta(N - 1)} , \quad \frac{1}{N - 1} < \theta < 1
$$

Then there exists a $W^{1,1}_0(\Omega)$ distributional solution to problem (1).

Observe that this case corresponds to the curve between the regions $C$ and $D$ of the figure. Note that $m > 1$ if and only if $\theta > \frac{1}{N-1}$.

In the following result too, we will prove the existence of a $W^{1,1}_0(\Omega)$ solution.

- **Theorem 1.2.** Let $f$ be a function in $L^m(\Omega)$. Assume (3), $f \log(1 + |f|) \in L^1(\Omega)$ and $\theta = \frac{1}{N-1}$. Then there exists a $W^{1,1}_0(\Omega)$ distributional solution of (1).

We end our introduction just mentioning a uniqueness result of solutions to problem (1) can be found in [13]. Moreover, in [4, 5, 11, 6] it was showed that the presence of a lower order term has a regularizing effect on the existence and regularity of the solutions.
To prove our results, we will work by approximation, using the following sequence of problems:

\[
\begin{aligned}
- \text{div} \left( \frac{a(x) \nabla u_n}{(1 + |u_n|)^\theta} \right) &= T_n(f), \quad \text{in } \Omega; \\
 u_n &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]  

The existence of weak solutions \( u_n \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) to problem (7) is due to [7].

2. \( W^{1,1}_0(\Omega) \) solutions

In the sequel \( C \) will denote a constant depending on \( \alpha, N, \text{meas}(\Omega), \theta \) and the \( L^m(\Omega) \) norm of the source \( f \).

We are going to prove Theorem 1.1, that is, the existence of a solution to problem (1) in the case where \( m = \frac{N}{N+1-\theta(N-1)} \) and \( \frac{1}{N-1} < \theta < 1 \).

Note that \( m > 1 \) if and only if \( \theta > \frac{1}{N-1} \).

Proof. (of Theorem 1.1) We consider \( T_k(u_n) \) as a test function in (7):

\[
\int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{(1 + k)^\theta \|f\|_{L^1(\Omega)}}{\alpha}
\]

by assumption (3) on \( a \).

Choosing \( [(1 + |u_n|)^p - 1] \text{sign}(u_n) \), for \( p = \theta - \frac{1}{N-1} \), as a test function in (7) we have, by Hölder’s inequality on the right hand side and assumption (3) on the left one

\[
\alpha p \int_{\Omega} \left( \frac{\nabla u_n}{(1 + |u_n|)^{\frac{N}{N-1}}} \right)^2 \leq \int_{\Omega} |f|[((1 + |u_n|)^p - 1] \leq \|f\|_{L^m(\Omega)} \left[ \int_{\Omega} [(1 + |u_n|)^p - 1]^{m'} \right]^\frac{1}{m'}. \]

The Sobolev embedding used on the left hand side implies

\[
\left[ \int_{\Omega} \left\{ (1 + |u_n|)^{\frac{N-2}{2(N-1)}} - 1 \right\}^{\frac{2N}{N-2}} \right]^\frac{2}{N} \leq C \left[ \int_{\Omega} [(1 + |u_n|)^p - 1]^{m'} \right]^\frac{1}{m'}. \]

We observe that \( \frac{N-2}{2(N-1)} \frac{2N}{N-2} = pm' \); moreover \( \frac{2}{p'} > \frac{1}{m'} \), since \( m < \frac{N}{2} \).

Therefore the above inequality implies that

\[
\int_{\Omega} |u_n|^\frac{N}{N-1} \leq C.
\]
One deduces that
\begin{equation}
\int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C
\end{equation}
from (10) and (9). Let \( v_n = \frac{2(N-1)}{N-2} (1 + |u_n|)^{\frac{N-2}{N-1}} \text{sign}(u_n) \). Estimate (11) is equivalent to say that \( \{v_n\} \) is a bounded sequence in \( W^{1,2}_0(\Omega) \); therefore, up to a subsequence, there exists \( v \in W^{1,2}_0(\Omega) \) such that \( v_n \rightharpoonup v \) weakly in \( W^{1,2}_0(\Omega) \) and a.e. in \( \Omega \). If we define the function
\[ u = \left( \frac{2(N-1)}{N-2} |v| \right)^{\frac{N-2}{N-1}} - 1 \text{sign}(v), \]
the weak convergence of \( \nabla v_n \rightharpoonup \nabla v \) means that
\begin{equation}
\int_{\Omega} \frac{\nabla u_n}{(1 + |u_n|)^{\frac{N}{N-1}}} \rightharpoonup \frac{\nabla u}{(1 + |u|)^{\frac{N}{N-1}}} \text{ weakly in } L^2(\Omega).
\end{equation}

Moreover, the Sobolev embedding for \( v_n \) implies that \( u_n \to u \) in \( L^s(\Omega) \), for every \( 1 \leq s < \frac{N}{N-1} \).

Hölder’s inequality with exponent 2 applied to
\begin{equation}
\int_{\Omega} |\nabla u_n| = \int_{\Omega} \frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2(N-2)}}(1 + |u_n|)^{\frac{N}{2(N-2)}}}
\end{equation}
gives
\begin{equation}
\int_{\Omega} |\nabla u_n| \leq C,
\end{equation}
due to (10) and (11). We are now going to estimate \( \int_{\{k \leq |u_n|\}} |\nabla u_n| \). By using \( \frac{1}{(1 + |u_n|)^{\frac{N}{N-1}}} - (1 + k)^{\frac{N}{N-1}} \text{sign}(u_n) \) as a test function in (7), we have
\begin{equation}
\int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C \left[ \int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{m}} \left[ \int_{\Omega} (1 + |u_n|)^{\frac{N}{N-1}} \right]^{\frac{1}{m}},
\end{equation}
which implies, by (10),
\begin{equation}
\int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C \int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{m}}.
\end{equation}

Hölder’s inequality, estimates (10) and (14) on
\begin{equation}
\int_{\{k \leq |u_n|\}} |\nabla u_n| = \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2(N-2)}}(1 + |u_n|)^{\frac{N}{2(N-2)}}} \leq C \int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{m}},
\end{equation}
give
\begin{equation}
\int_{\{k \leq |u_n|\}} \frac{\nabla u_n}{(1 + |u_n|)^{\frac{N}{2} - 1}} \leq C \left[ \int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{2m}}.
\end{equation}

Thus, for every measurable subset \( E \), due to (8) and (15), we have
\begin{equation}
\int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n| \leq \text{meas}(E) \frac{1}{2} \left[ (1 + k)^{\frac{\theta}{\alpha}} \|f\|_{L^1(\Omega)} \right]^{\frac{1}{2}} + C \left[ \int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{2m}}.
\end{equation}

Now we are going to prove that \( u_n \) weakly converges to \( u \) in \( W^{1,1}_0(\Omega) \) following [5]. Estimates (16) and (10) imply that the sequence \( \{ \frac{\partial u_n}{\partial x_i} \} \) is equiintegrable. By Dunford-Pettis theorem, and up to subsequences, there exists \( Y_i \) in \( L^1(\Omega) \) such that \( \frac{\partial u_n}{\partial x_i} \) weakly converges to \( Y_i \) in \( L^1(\Omega) \).

Since \( \frac{\partial u_n}{\partial x_i} \) is the distributional partial derivative of \( u_n \), we have, for every \( n \) in \( \mathbb{N} \),
\begin{equation*}
\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi = -\int_\Omega u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).
\end{equation*}

We now pass to the limit in the above identities, using that \( \frac{\partial u_n}{\partial x_i} \) weakly converges to \( Y_i \) in \( L^1(\Omega) \), and that \( u_n \) strongly converges to \( u \) in \( L^1(\Omega) \): we obtain
\begin{equation*}
\int_\Omega Y_i \varphi = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).
\end{equation*}

This implies that \( Y_i = \frac{\partial u}{\partial x_i} \), and this result is true for every \( i \). Since \( Y_i \) belongs to \( L^1(\Omega) \) for every \( i \), \( u \) belongs to \( W^{1,1}_0(\Omega) \).

We are now going to pass to the limit in problems (7). For the limit of the left hand side, it is sufficient to observe that \( \frac{\nabla u_n}{(1 + |u_n|)^{\frac{N}{2} - 1}} \) weakly in \( L^2(\Omega) \) due to (12) and that \( |a(x)\nabla \varphi| \) is bounded.

We prove Theorem 1.2, that is, the existence of a \( W^{1,1}_0(\Omega) \) solution in the case where \( \theta = \frac{N}{N-1} \) and \( f \log(1 + |f|) \in L^1(\Omega) \).

**Proof.** (of Theorem 1.2) Let \( k \geq 0 \) and take \( [\log(1 + |u_n|) - \log(1 + k)]^+ \text{sign}(u_n) \), as a test function in problems (7). By assumption (3) on
We now use the following inequality on the left hand side:

\[ a \log(1 + b) \leq \frac{a}{\rho} \log \left( 1 + \frac{a}{\rho} \right) + (1 + b)^\rho \]

where \( a, b \) are positive real numbers and \( 0 < \rho < \frac{N-2}{N-1} \). This gives, for any \( k \geq 0 \)

\[ \alpha \int \{ k \leq |u_n| \} |\nabla u_n|^2 \leq \int \{ k \leq |u_n| \} |f| \log \left( 1 + \frac{|f|}{\rho} \right) + \int \{ k \leq |u_n| \} (1 + |u_n|)^\rho. \]  

In particular, for \( k \geq 1 \) we have

\[ \frac{\alpha}{2^{q+1}} \int \{ k \leq |u_n| \} \frac{|\nabla u_n|^2}{|u_n|^{q+1}} \leq \int \{ k \leq |u_n| \} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) + 2^\rho \int \{ k \leq |u_n| \} |u_n|^\rho. \]

Writing the above inequality for \( k = 1 \) and using the Sobolev inequality on the left hand side, one has

\[ \left[ \int \left( |u_n|^\frac{1-\theta}{2} - 1 \right)^2 \right]^\frac{1}{2^\theta} \leq C \int \left\{ 1 \leq |u_n| \right\} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) + C \int \left\{ 1 \leq |u_n| \right\} |u_n|^\rho, \]

which implies that

\[ \left[ \int \left( |u_n|^{\frac{(1-\theta)2^\rho}{2}} \right) \right]^\frac{1}{2^\rho} \leq C + C \left[ \int \left\{ 1 \leq |u_n| \right\} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) + C \int \Omega |u_n|^\rho \right]. \]

By using Hölder’s inequality with exponent \( \frac{(1-\theta)2^\rho}{2\rho} \) on the last term of the right hand side, we get

\[ \left[ \int \left( |u_n|^{\frac{(1-\theta)2^\rho}{2}} \right) \right]^\frac{1}{2^\rho} \leq C + C \left[ \int \left\{ 1 \leq |u_n| \right\} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) + C \left[ \int \Omega |u_n|^{\frac{(1-\theta)2^\rho}{2}} \right] \right]^{\frac{\rho}{(1-\theta)2^\rho}}. \]

By the choice of \( \rho \), this inequality implies that

\[ \int \Omega |u_n|^{\frac{(1-\theta)2^\rho}{2}} \leq C. \]

Inequalities (20) and (18) written for \( k = 0 \) imply that \( \left\{ v_n \right\} = \left\{ \frac{2}{1-\theta} (1 + |u_n|^{\frac{1-\theta}{2}} \text{sign}(u_n)) \right\} \) is a bounded sequence in \( W^{1,2}_0(\Omega) \), as in the proof.
of Theorem 1.1. Therefore, up to a subsequence there exists \( v \in W^{1,2}_0(\Omega) \) such that \( v_n \rightharpoonup v \) weakly in \( W^{1,2}_0(\Omega) \) and a.e. in \( \Omega \). Let \( u = \left\{ \left[ \frac{1-\theta}{2} v \right]^{\frac{2+\theta}{2}} - 1 \right\} \text{sign}(v) \); the weak convergence of \( \nabla v_n \rightharpoonup \nabla v \) means that

\[
\frac{\nabla u_n}{(1 + |u_n|)^{\frac{2+\theta}{2}}} \rightharpoonup \frac{\nabla u}{(1 + |u|)^{\frac{2+\theta}{2}}} \quad \text{weakly in } L^2(\Omega).
\]

Moreover, the Sobolev embedding for \( v_n \) implies that \( u_n \to u \) in \( L^s(\Omega), s < \frac{N}{N-1} \).

By (8) one has

\[
\int_{\Omega} |\nabla u_n| = \int_{\Omega} |\nabla T_k(u_n)| + \int_{\{1 \leq |u_n|\}} |\nabla u_n| \leq C + \int_{\{1 \leq |u_n|\}} \frac{|\nabla u_n|}{|u_n|^\theta} |u_n|^\frac{\theta+1}{2}.
\]

Hölder’s inequality on the right hand side, and estimates (19) written with \( k = 1 \) and (20) imply that the sequence \( \{u_n\} \) is bounded in \( W^{1,1}_0(\Omega) \).

Moreover, due to (19)

\[
\int_{\{k \leq |u_n|\}} |\nabla u_n| = \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{|u_n|^\theta} |u_n|^\frac{\theta+1}{2} \leq C \left[ \int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) \right] \int_{\{k \leq |u_n|\}} |u_n|^\rho.
\]

For every measurable subset \( E \), the previous inequality and (8) imply

\[
\int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n| \leq C \text{meas}(E)^{\frac{1}{2}} (1 + k)^{\frac{\theta}{2}} + C \left[ \int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) \right] \int_{\{k \leq |u_n|\}} |u_n|^\rho.
\]

Since \( \rho < \frac{(1-\theta)^2}{2} \), by using Hölder’s inequality on the last term and estimate (20), one has

\[
\int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq C \text{meas}(E)^{\frac{1}{2}} (1 + k)^{\frac{\theta}{2}} + C \left[ \int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left( 1 + \frac{|f|}{\rho} \right) \right] \int_{\{k \leq |u_n|\}} |u_n|^\rho \leq \text{meas}(\{|u_n| \geq k\})^{1-(\frac{1-\theta}{(1-\theta)^2})}.\]
One can argue as in the proof of Theorem 1.1 to deduce that $u_n \to u$ weakly in $W^{1,1}_0(\Omega)$.

To pass to the limit in problems (7), as in the proof of Theorem 1.1, it is sufficient to observe that

$$
\frac{\nabla u_n}{(1+|u_n|)^{\frac{N}{2(N-1)}}} \rightharpoonup \frac{\nabla u}{(1+|u|)^{\frac{N}{2(N-1)}}}
$$

weakly in $L^2(\Omega)$, due to (21).

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