Relation between optical Fresnel transformation and quantum tomography in two-mode entangled case

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Similar in spirit to the preceding work [9], where the relation between optical Fresnel transformation and quantum tomography is revealed, we explore this kind of relationship in a two-mode entangled case. We show that under the two-mode Fresnel transformation the entangled state density $|\psi\rangle$ becomes density operator $F_2|\psi\rangle = \int d^2y\rho(y)\delta(\xi - \sigma_1 - B\gamma_2) \Delta(\sigma, \gamma)$, which is just the Radon transform of the two-mode Wigner operator $\Delta(\sigma, \gamma)$ in entangled form, i.e.:

$$|\psi_{s,r}\rangle = \pi \int d^2y d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma_2) \delta(\eta_1 - D\sigma_1 - B\gamma_2) \Delta(\sigma, \gamma)$$

where $F_2$ is a two-mode Fresnel operator in quantum optics, and $s, r$ are the complex-value expressions of $(A, B, C, D)$. So the probability distribution $\rho_{s,r}(\eta) = \langle \psi | F_2^\dagger \rho F_2 | \psi \rangle$ is the tomography (Radon transform of the two-mode Wigner function), and $s, r(\eta) = \langle \psi | F_2^\dagger | \eta \rangle$ gives a tomogram. Similarly, we find

$$F_2|\xi\rangle = \int d^2y \delta(\xi - A\sigma_1 - C\gamma_2) \delta(\eta_2 - A\sigma_2 + C\gamma_2) \Delta(\sigma, \gamma)$$

where $|\xi\rangle$ is the conjugated state to $|\eta\rangle$.

Keywords: optical Fresnel transformation; quantum tomography; two-mode entangled state

1. Introduction

In [1] we proved that for optical Fresnel integration [2–4] transforming the input light field $f(x)$ to the output light field $g'(x')$ by

$$g'(x') = \frac{1}{\sqrt{2\pi iB}} \int_{-\infty}^{\infty} \exp\left[\frac{i}{2B} (Ax^2 - 2x'x + Dx^2)\right] f(x) dx$$

(1)

which is characteristic of ray transfer matrix elements $(A, B, C, D)$, $AD-BC=1$, there exists Fresnel operator $F_1(r, s)$ expressed in the Glauber–Klauder coherent state $\langle z |$ representation [5,6], i.e.

$$F_1(r, s) = \sqrt{\frac{1}{2\pi}} \int \frac{d^2z}{\pi} \exp[-|z|^2/2 + za^*]$$

(2)

where $z = \langle 0 | \exp[-|z|^2/2 + za^*]$, $[a, a^*] = 1$, and $(s, r)$ are complex numbers related to a classical ray transfer matrix $(A \ B \ C \ D)$ by

$$s = \frac{1}{2} [A + D - i(B - C)], r = -\frac{1}{2} [A - D + i(B + C)]$$

(3)

with $|s|^2 - |r|^2 = 1$ replacing the unimodularity condition $AD-BC=1$. We have shown that the coordinate matrix element of $F_1(r, s)$ is just the integral kernel in Equation (1). In fact, by using the technique of integration within an ordered product (IWOP) of operators we can derive the normally ordered form of $F_1$ [7],

$$F_1(r, s) = \frac{1}{\sqrt{2\pi iB}} \exp\left[\frac{-r a^2}{2s} \right] \exp\left[\frac{(1/s^2 - 1) a^* a}{2s} \right] \exp\left[\frac{r a^2}{2s} \right]$$

(4)

such that in the coordinate $\langle x |$ representation [1,8], the coordinate matrix element of $F_1(r, s)$ is given by

$$\langle x' | F_1(r, s) | x \rangle = \frac{1}{\sqrt{2\pi iB}} \exp\left[\frac{i}{2B} (Ax^2 - 2x'x + Dx^2)\right]$$

(5)

if we use Dirac's symbol to let $f(x) = \langle x | f \rangle$, then Equation (1) is expressed as

$$g'(x') = \int_{-\infty}^{\infty} \langle x' | F_1(r, s) | x \rangle f(x) dx = \langle x' | F_1(r, s) | f \rangle$$

(6)
which is just the quantum mechanical version of the Fresnel transformation.

In the preceding paper [9] we also found that under the Fresnel transformation the pure position density $|x\rangle\langle x|$ becomes the tomographic density $|x\rangle_{s,rs}\langle x|$ which is just the Radon transform of the Wigner operator $\Delta(x,p)$, i.e.

$$F_1|x\rangle\langle x|F_1^* = |x\rangle_{s,rs}\langle x|$$

$$= \int_{-\infty}^{\infty} dp'dx'\delta[x-(Dx'-Bp')]|\Delta(x',p')|.$$  

(7)

So the probability distribution for the Fresnel quadrature phase is the tomography (Radon transform of the Wigner function [10–12]), and the tomogram of a state $|\psi\rangle$ is just the wave function of its Fresnel-transformed state $F_1|\psi\rangle$, i.e. $s,rs|x\rangle\langle x| = \langle x|F_1^*|\psi\rangle$, and $|x\rangle_{s,rs}$ is found as

$$|x\rangle_{s,rs} = \frac{\pi^{1/4}}{\sqrt{D+iB}}$$

$$\times \exp\left\{-\frac{A-ICx^2}{D+iB} + \frac{\sqrt{2x}}{D+iB} a^* - \frac{D-iBa^2}{D+iB} \right\}[0]$$

(8)

where $|0\rangle$ is the vacuum state. In this article we want to generalize the above conclusion to the two-mode entangled case. Firstly, we extend Equation (5) to the two-dimensional Fresnel transformation,

$$K_2(\eta^*,\eta) = \frac{1}{2\pi B} \exp \left\{ \frac{i}{2B} (A|\eta|^2-(\eta^*\eta^* + \eta^*\eta) + D|\eta|^2) \right\}$$

(9)

where $\eta$ is a complex number, then we construct the two-mode Fresnel operator $F_2(r,s)$ such that its transformation matrix element in the entangled state $|\eta\rangle$ representation (see below Equation (17)) is just the two-dimensional Fresnel transformation, i.e. $K_2(r,s)(\eta^*,\eta) = \frac{1}{2} (\eta^*|F_2(r,s)|\eta)$, then we shall prove

$$F_2|\eta\rangle\langle \eta|F_2^* = |\eta\rangle_{s,rs,rs}\langle \eta| = \pi \int d^2y d^2\sigma \delta(\eta_1 + \sigma_1 D - B\gamma_1)$$

$$\times \delta(\eta_2 - D\sigma_2 + B\gamma_2) \Delta(\sigma,\gamma)$$

where $\gamma = \gamma_1 + i\gamma_2$, $\eta = \eta_1 + i\eta_2$, i.e. we show that $|\eta\rangle_{s,rs,rs}\langle \eta|$ is just the Radon transform of the entangled Wigner operator $\Delta(\sigma,\gamma)$ (for its definition, see Equation (31)).

This paper is arranged as follows. In Section 2, after briefly reviewing the two-mode Fresnel operator $F_2(r,s)$ we derive the 2D Fresnel transformation in entangled state $|\eta\rangle$ representation, and we introduce a new representation $|\eta\rangle_{s,rs,rs}(= F_2(r,s)|\eta\rangle)$ in Section 3. Section 4 is devoted to proving Equation (10), i.e. $|\eta\rangle_{s,rs,rs}(\eta)$ appears as the Radon transform of entangled Wigner operator. Similar discussions are devoted to the Fresnel transformation in its ‘frequency domain’ in Section 5. In the final section, we derive the inverse Radon transformation of the entangled Wigner operator.

### 2. Two-mode Fresnel operator

Similar in spirit to the single-mode case, we introduce the two-mode Fresnel operator $F_2(r,s)$ through the following two-mode coherent state representation [1,13], i.e.

$$F_2(r,s) = s \int \frac{d^2z_1d^2z_2}{\pi^2} \exp \left\{-|z_1|^2 + |z_2|^2 + rs z_1 r^*_2 z_2^* \right\}$$

(11)

which indicates that $F_2(r,s)$ is a mapping of classical symplectic transform $(z_1,z_2) \rightarrow (z_1 + rz_2^*, rz_1^* + sz_2^*)$ in phase space, where $(z_1,z_2) = \exp[-\frac{1}{2} |z_1|^2 - \frac{1}{2} |z_2|^2 + z_1a_1^* + z_2a_2^*]|00\rangle$ is a usual two-mode coherent state. Concretely, the ket in Equation (11) is

$$|z_1 + rz_2^*, rz_1^* + sz_2^*\rangle = |z_1 + rz_2^*\rangle \otimes |rz_1^* + sz_2^*\rangle$$

(12)

where $s$ and $r$ are complex and satisfy the unimodularity condition $|s|^2-|r|^2=1$. Using the IWOP technique [14,15] and the normal ordering of the vacuum projector $|00\rangle\langle 00| = \exp(-a_1^*a_1 - a_2^*a_2)$, we perform the integral in Equation (11) and obtain

$$F_2(r,s) = s \int \frac{1}{\pi^2} d^2z_1d^2z_2 \exp \left\{-|z_1|^2 + |z_2|^2 \right\}$$

$$- r^* z_1 a_2 + r^* z_2^* a_1^* + (z_1 + rz_2^*) a_1^* + (rz_1^* + sz_2^*) a_2^* + z_1a_1^* + z_2a_2^*$$

$$+ z_1a_2^* - a_1^* a_2 - r a_1^* a_2$$

$$= \frac{1}{s^*} \exp\left(\frac{s^*}{s} a_1^* a_2^*\right) \exp\left[\frac{1}{s^*} - 1\right] \left(a_1^* a_1 + a_2^* a_2\right)$$

$$: \exp\left(-\frac{r^* a_1 a_2}{s^*}\right)$$

$$= \exp\left(\frac{s^*}{s} a_1^* a_2^*\right) \exp\left[\left(a_1^* a_1 + a_2^* a_2 + 1\right) \ln(s^*)^{-1}\right]$$

$$\times \exp\left(-\frac{r^* a_1 a_2}{s^*}\right)$$

(13)

Thus, $F_2(r,s)$ induces the transforms

$$F_2(r,s)a_1 F_2^{-1}(r,s) = s^* a_1 - r a_2^*$$

$$F_2(r,s)a_2 F_2^{-1}(r,s) = s^* a_2 - r a_1^*$$

(14)

and $F_2$ is actually a generalized 2-mode squeezing operator [16–18].
It is remarkable that \( F_2(r,s) \) abides by the group multiplication rule. Using the IWOP technique and Equation (11) we have

\[
F_2(r,s)F_2(r',s') = \frac{d^2z_1d^2z_2d^2z_1'd^2z_2'}{\pi^4} \exp \left\{ -|s|^2(|z_1|^2 + |z_2|^2) - r^*s z_1z_2 
\right.
\]

\[
- rs^*z_1z_2 - \frac{1}{2}\left[ |z_1|^2 + |z_2|^2 + |s'|z_1' + r^*z_2' + |s|^2z_1 + r^*z_2 \right]
\]

\[
+ (sz_1 + rz_2^*)a_1^* + (rz_1^* + sz_2)a_2^* + z_1^*a_1 + z_2^*a_2
\]

\[
+ z_1^*(s'z_1' + r^*z_2') + z_2^*(r^*z_1' + s'z_2') - a_1^*a_1 - a_2^*a_2 \right\}
\]

\[
= \frac{1}{s^*}\exp \left( \frac{r^*}{2s^*}a_1^*a_2 \right) \exp \left( \frac{1}{s^*} \left( a_1^*a_1 + a_2^*a_2 \right) \right)
\]

\[
\times \exp \left( \frac{r^*}{2s^*}a_1^*a_2 \right) = F_2(r',s')
\]  (15)

where \((r^*, s^*)\) are given by

\[
\begin{pmatrix}
  s & -r^* \\
  -r & s^*
\end{pmatrix} = \begin{pmatrix}
  s & -r \\
  -r^* & s^*
\end{pmatrix} = \begin{pmatrix}
  s' & -r^* \\
  -r & s
\end{pmatrix}.
\]  (16)

Therefore, Equation (15) is a loyal representation of the multiplication rule for ray transfer matrices in Matrix Optics. Here we should mention that the group property of the operators \( F_2(r,s) \) in (13) can be seen clearly, in principle, from property of Lie algebra of the operators in the exponents.

3. Two-mode Fresnel transformation in entangled state representations

By introducing the bipartite entangled state \(|\eta\rangle\)  \[19,20\]

\[
|\eta\rangle = \exp \left\{ -\frac{1}{2} |\eta|^2 + \eta a_1^* - \eta^* a_1 \right\}
\]  (00)

\[
|\eta\rangle = |\eta_1 + i\eta_2\rangle
\]

is the common eigenstate of relative coordinate \(Q_1 - Q_2\) and the total momentum \(P_1 + P_2\), i.e.

\[
(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \quad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle
\]  (18)

where \(Q_i = (a_i + a_i^\dagger)/\sqrt{2}\), \(P_i = (a_i - a_i^\dagger)/(i\sqrt{2})\), \((i = 1, 2)\), are two ordinary quadrature operators, respectively. \(|\eta\rangle\) compose a complete set \(\int \frac{d^2a}{\pi^2} |\eta\rangle\langle\eta| = 1\), then using the overlap

\[
|\langle z_1, z_2|\eta\rangle| = \exp \left\{ -\frac{1}{2} |z_1|^2 + |z_2|^2 + |\eta|^2 \right. 
\]

\[
\left. + \eta z_1^* - \eta^* z_1 + z_1^* z_2 \right\}
\]  (19)

and

\[
|\langle z_1, z_2|F_2(r,s)|z_1, z_2\rangle| = \frac{1}{s^*} \exp \left\{ -\frac{1}{2} \left( |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2 \right) 
\right.
\]

\[
+ \frac{r}{s^*} z_1^* z_2^* - \frac{r^*}{s^*} z_1 z_2 + \frac{1}{s^*} \left( z_1^* z_1 + z_2^* z_2 \right)
\]  (20)

as well as the over-completeness relation of the coherent state we can calculate the integral kernel

\[
K^{(2)}_2(q', \eta) = \frac{1}{\pi^2} \left( \langle \eta| F_2(r,s) |\eta\rangle \right) = \frac{d^2z_1d^2z_2d^2z_1'd^2z_2'}{\pi^5} \left( \langle \eta| F_2(r,s) |\eta\rangle \right)
\]

\[
\times \exp \left\{ -\left( |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2 \right) - \frac{1}{2} (|\eta|^2 + |\eta|^2) \right\}
\]

\[
\times \exp \left( \frac{r^*}{2s^*}a_1^*a_2 \right) = \frac{1}{\pi^2} \exp \left\{ -\frac{1}{2} |\eta|^2 - (\eta\eta^* + \eta^*\eta) + D |\eta|^2 \right\}
\]

\[
\times \exp \left( \frac{r^*}{2s^*}a_1^*a_2 \right) = \frac{1}{\pi^2} \exp \left\{ -\frac{1}{2} |\eta|^2 - (\eta\eta^* + \eta^*\eta) + D |\eta|^2 \right\}
\]  (21)

Using the relation between \(s, r\) and \((A, B, C, D)\) in Equation (3) we see that Equation (21) becomes

\[
K^{(2)}_2(q', \eta) = \frac{1}{2IB\pi^2} \exp \left\{ \frac{i}{2B} \left( A|\eta|^2 - (\eta\eta^* + \eta^*\eta) + D |\eta|^2 \right) \right\}
\]

\[
= K^{(2)}_M(q', \eta)
\]  (22)

where the superscript \(M\) only means the parameters of \(K^{(2)}_M\) are \([A, B, C, D]\), and the subscript 2 implies the two-dimensional kernel.

Operating \(F_2(r,s)\) on \(|\eta\rangle\) and using Equations (13) and (19) yields

\[
F_2(r,s)|\eta\rangle
\]

\[
= \frac{1}{s^*} \int \frac{d^2z_1d^2z_2}{\pi^2} \exp \left( \frac{r^*}{s^*}a_1^*a_2 + \frac{1}{s^*} a_1^*z_1 + a_2^*z_2 \right)
\]

\[
- \frac{r^*}{s^*} \int \frac{d^2z_1d^2z_2}{\pi^2} \left( z_1, z_2 \right) \langle z_1, z_2|\eta\rangle
\]

\[
= \frac{1}{s^*} \int \frac{d^2z_1d^2z_2}{\pi^2} \exp \left( -|z_1|^2 + \frac{1}{s^*} \left( a_1^* - r^* z_2 \right) z_1 
\right.
\]

\[
+ \left( \eta + z_2^* \right) z_1 \right]
\]

\[
\times \exp \left( -\frac{1}{2} |\eta|^2 - |z_2|^2 + \frac{1}{s^*} z_2a_1^* + a_2^*z_2 + \frac{r}{s^*} a_1^*a_2 \right) \right\}
\]  (00)
where we have used the integration formula
\[
\int \frac{d^2z}{\pi} e^{iz^2 + tz + sz^*} = -\frac{1}{\xi} e^{-\xi}, \quad \text{Re}(\xi) < 0.
\] (25)

Noticing the completeness relation and the orthogonality of \(|\eta\rangle\) we immediately derive
\[
\int \frac{d^2\eta}{\pi} |\eta\rangle_{s,r} \langle \eta| = 1, \quad \text{for } |\eta\rangle_{s,r} = \pi \delta(\eta - \eta') \delta(\eta^* - \eta^{*'})
\] (26)
a generalized entangled state representation \(|\eta\rangle_{s,r}\) with the completeness relation (26). From Equation (24) we can see that
\[
a_1|\eta\rangle_{s,r} = \left( \frac{\eta}{D + iB} + \frac{D - iB a_2}{D + iB a_2^*} \right) |\eta\rangle_{s,r}
\] (27)
\[
a_2|\eta\rangle_{s,r} = \left( -\frac{\eta^*}{D + iB} + \frac{D - iB a_1^*}{D + iB a_1^*} \right) |\eta\rangle_{s,r}
\] (28)
so we have the eigen-equations for \(|\eta\rangle_{s,r}\) as follows
\[
[D(Q_1 - Q_2) - B(P_1 - P_2)]|\eta\rangle_{s,r} = \sqrt{2}|\eta\rangle_{s,r}
\] (29)
\[
[B(Q_1 + Q_2) + D(P_1 + P_2)]|\eta\rangle_{s,r} = \sqrt{2}|\eta\rangle_{s,r}
\] (30)
We can also check Equations (27)–(30) in another way (see the appendix).

**4. |\eta\rangle_{s,r,s,r} as the Radon transform of entangled Wigner operator**

For a two-mode correlated system, it is convenient to express the Wigner operator in the \(|\eta\rangle\) representation in another way:
\[
\Delta(\sigma, \gamma) = \int \frac{d^2\eta}{\pi^2} |\sigma - \eta\rangle \langle \sigma + \eta| e^{\eta r^* - \eta^* r}. (31)
\]
When \(\sigma = \alpha - \beta^*, \gamma = \alpha + \beta^*,\) one can be sure that Equation (31) is just equal to the direct product of two single-mode Wigner operators [26], i.e., \(\Delta(\sigma, \gamma) = \Delta(\alpha, \alpha^*) \otimes \Delta(\beta, \beta^*).\) Then according to the Weyl correspondence rule [27]
\[
H(a_1, a_2; a_1, a_2) = \int d^2\gamma d^2\sigma h(\sigma, \gamma)\Delta(\sigma, \gamma)\] (32)

where \(h(\sigma, \gamma)\) is the Weyl correspondence of \(H(a_1, a_2; a_1, a_2),\) and
\[
h(\sigma, \gamma) = 4\pi^2 \text{Tr} \left[ H(a_1, a_2; a_1, a_2) \Delta(\sigma, \gamma) \right]
\] (33)
the classical Weyl correspondence of the projection operator \(|\eta\rangle_{s,r,s,r} \langle \eta|\) can be calculated as,
\[
4\pi^2 \text{Tr} \left[ |\eta\rangle_{s,r,s,r} \langle \eta| \Delta(\sigma, \gamma) \right]
\] = \[4\pi^2 \int \frac{d^2\eta}{\pi^3} \langle \eta|_{s,r,s,r} \langle \sigma - \eta| \langle \sigma + \eta|_{s,r,s,r} \exp(\eta^* r - \eta r^*)\]
\[= 4\pi^2 \int \frac{d^2\eta}{\pi^3} \langle \eta|_{s,r,s,r} \langle \sigma - \eta| \langle \sigma + \eta|_{s,r,s,r} \exp(\eta^* r - \eta r^*).\]
(34)

Then using Equation (21) we have
\[
4\pi^2 \text{Tr} \left[ |\eta\rangle_{s,r,s,r} \langle \eta| \Delta(\sigma, \gamma) \right]
\] = \[\pi \delta(\eta_2 - D\sigma_2 + B\gamma) \delta(\eta_1 - D\sigma_1 - B\gamma)\]
\[= \pi \delta(\eta_2 - D\sigma_2 + B\gamma) \delta(\eta_1 - D\sigma_1)\]
\[\times \delta(\eta_1 - D\sigma_1 - B\gamma) \Delta(\sigma, \gamma)\]
(35)
which means the following Weyl correspondence
\[
|\eta\rangle_{s,r,s,r} \langle \eta| = \pi \int d^2\gamma d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma)
\] \[\times \delta(\eta_1 - D\sigma_1 - B\gamma) \Delta(\sigma, \gamma)\]
(36)
so the projector operator \(|\eta\rangle_{s,r,s,r} \langle \eta|\) is just the Radon transformation of \(\Delta(\sigma, \gamma),\) \(D\) and \(B\) are the Radon transformation parameter. Combining Equations (23)–(36) we complete the proof in Equation (10). Therefore, the quantum tomography in a two-mode entangled case is expressed as
\[
|\psi\rangle_{s,r} \langle \psi| = |\eta| F^\dagger \langle \psi| = \pi \int d^2\gamma d^2\sigma \delta(\eta_2 - D\sigma_2 + B\gamma)
\] \[\times \delta(\eta_1 - D\sigma_1 - B\gamma) \psi \Delta(\sigma, \gamma) \psi\]
(37)
where \(\psi \Delta(\sigma, \gamma) \psi\) is the Wigner function. So the probability distribution for the Fresnel quadrature
phase (see the line below Equation (62) in the appendix) is the tomography (Radon transform of the two-mode Wigner function). This is the main result of the present paper. This new relation between quantum tomography and optical Fresnel transform may enable experimentalists to figure out a new approach for generating tomography.

5. Radon transform of entangled Wigner operator in the ‘frequency’ domain

Next we turn to the ‘frequency’ domain, that is to say, we shall prove that the \((A, C)\) related Radon transform of entangled Wigner operator \(\Delta(\sigma, \gamma)\) is just the pure state density operator \(\langle \xi | s, r \rangle \langle \xi | \rangle\), i.e.

\[
F_s |\xi\rangle \langle F_s^+ |\xi\rangle = \int (\xi - A \sigma_1 - C \gamma_2) \delta(\xi - A \sigma_2 + C \gamma_1) \Delta(\sigma, \gamma) d^2 \sigma d^2 \gamma
\]

where

\[
|\xi\rangle = \exp \left[ -\frac{1}{2} |\xi|^2 + \xi^* a_1^+ + \xi a_1^+ \right] |00\rangle
\]

is an entangled state conjugate to \(|\eta\rangle\). By analogy with the above procedures, we obtain the two-dimensional Fresnel transformation in its ‘frequency domain’, i.e.

\[
\mathcal{K}_2^N(\xi', \xi) = \frac{1}{\pi} \int |\xi'| |F_s(r, s)| |\xi\rangle
\]

\[
= \int d^2 \eta d^2 \sigma \frac{1}{\pi^2} (\xi'^* \eta'^* + \xi'^* \eta^* + \xi^* \eta^* - \xi^* \eta^*) K_{2, 0}^{(r, s)}(\sigma, \eta)
\]

\[
= \frac{1}{8iB\pi} \int d^2 \sigma d^2 \eta \exp \left( i \frac{\xi^* \eta'^* - \xi^* \eta^* + \xi^* \eta^* - \xi^* \eta^*}{2} \right) K_{2, 0}^{(r, s)}(\sigma, \eta)
\]

\[
= \frac{1}{2\Delta(-C)} \int \exp \left[ i \left( D|\xi|^2 + A|\xi|^2 - \xi^* \eta - \xi \eta^* \right) \right]
\]

where the superscript \(N\) means that this transform kernel corresponds to the parameter matrix \(N = \{D, -C, -B, A\}\). Thus, the 2D Fresnel transformation in its ‘frequency domain’ is given by

\[
\Psi(\xi') = \int \mathcal{K}_2^N(\xi', \xi) \Phi(\xi) d^2 \xi.
\]

Operating \(F_2(r, s)\) on \(|\xi\rangle\) we have (also see the appendix)

\[
|\xi|_{s, r} = \frac{1}{A - iC} \exp \left\{ -\frac{D + iB}{2(A - iC)} |\eta|^2 + \frac{\xi^* a_1^+}{A - iC} \right\}
\]

\[
\xi^* a_2^+ \frac{A + iC}{A - iC} \left\{ 00 \right\}
\]

or

\[
|\xi|_{s, r} = \frac{1}{s^* - r^*} \exp \left\{ -\frac{s^* + r^*}{2(s^* - r^*)} |\eta|^2 + \frac{\xi^* a_1^+}{s^* - r^*} \right\}
\]

\[
-\frac{s - r}{s^* - r^*} a_1^+ a_2^+ \left\{ 00 \right\}.
\]

Noticing that the entangled Wigner operator in \(|\xi\rangle\) representation is expressed as

\[
\Delta(\sigma, \gamma) = \int d^2 \frac{\xi}{\pi^2} |\gamma + \xi\rangle \langle \gamma - \xi| \exp(\xi^* \sigma - \sigma^* \xi),
\]

and using the classical correspondence of \(|\xi|_{s, r} \langle \xi|\rangle\) which is calculated by

\[
h(\sigma, \gamma) = 4\pi^2 \int \langle \xi|_{s, r} \langle \xi| \Delta(\sigma, \gamma)
\]

\[
= 4 \int d^2 \frac{\xi}{\pi^2} |\gamma - \xi| |F_2(\xi)| |\xi|^2 |\gamma + \xi| \exp(\xi^* \sigma - \sigma^* \xi)
\]

\[
= 4\pi \delta(\xi - A \sigma_1 - C \gamma_2) |\delta(\xi - A \sigma_2 + C \gamma_1)|
\]

we obtain

\[
|\xi|_{s, r} \langle \xi| = \int \delta(\xi - A \sigma_1 - C \gamma_2) \delta(\xi - A \sigma_2 + C \gamma_1)
\]

\[
\times \Delta(\sigma, \gamma) d^2 \sigma d^2 \gamma
\]

so the projector operator \(|\xi|_{s, r} \langle \xi|\rangle\) is another Radon transformation of the two-mode Wigner operator, with \(A\) and \(C\) being the Radon transformation parameter (‘frequency’ domain). Therefore, the quantum tomography in \(s, r\) \(|\xi|\) representation is expressed as the Radon transformation of the Wigner function

\[
|\xi| |\psi\rangle|^2 = \langle \xi| |\psi\rangle|^2 = \pi \int d^2 \gamma d^2 \delta(\xi - A \sigma_1 - C \gamma_2)
\]

\[
\times \delta(\xi - A \sigma_2 + C \gamma_1) |\psi| |\Delta(\sigma, \gamma)| |\psi\rangle
\]

and \(s, r\langle \xi|\rangle\) is \(|\xi|^\dagger\).

6. Inverse Radon transformation

Now we consider the inverse Radon transformation. For instance, using Equation (36) we see the Fourier transformation of \(|\eta|_{s, r} \langle \eta|\rangle\) is

\[
\left[ d^2 \eta |\eta|_{s, r} \langle \eta| \right] \exp(-i\xi_1 \eta_1 - i\xi_2 \eta_2)
\]

\[
= \pi \int d^2 \gamma d^2 \delta(\sigma, \gamma) \exp\left\{ -i\xi_1 (D \sigma_1 + B \gamma_2)
\]

\[
- i\xi_2 (D \sigma_2 - B \gamma_1) \right\}
\]
the right-hand side of Equation (48) can be regarded as a special Fourier transformation of $\Delta(\sigma, \gamma)$, so by making its inverse Fourier transformation, we get

$$
\Delta(\sigma, \gamma) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dr_1 |r_1|^2 \int_{-\infty}^{\infty} dr_2 |r_2|^2 \int_0^\pi d\theta_1 d\theta_2 \times \int_{-\infty}^{\infty} d^2 \eta |s, r, \eta \rangle |K(r_1, r_2, \theta_1, \theta_2)\rangle
$$

where $\cos \theta_1 = \cos \theta_2 = \frac{\rho}{\sqrt{\rho^2 + \sigma^2}}$, $r_1 = \xi \sqrt{\rho^2 + \sigma^2}$, $r_2 = \zeta \sqrt{\rho^2 + \sigma^2}$ and

$$
K(r_1, r_2, \theta_1, \theta_2) \equiv \exp \left[ -ir_1 \left( \frac{\eta_1}{\sqrt{\rho^2 + \sigma^2}} - \sigma_1 \cos \theta_1 - \gamma_1 \sin \theta_1 \right) \right] \times \exp \left[ -ir_2 \left( \frac{\eta_2}{\sqrt{\rho^2 + \sigma^2}} - \sigma_2 \cos \theta_2 + \gamma_1 \sin \theta_2 \right) \right].
$$

Equation (49) is just the inverse Radon transformation of entangled Wigner operator in the entangled state representation. This is different from the direct product of two independent Radon transformations of two independent single-mode Wigner operators, because in Equation (24) the $|\eta\rangle_s$ is an entangled state. Therefore, the Wigner function of quantum state $|\psi\rangle$ can be reconstructed from the tomographic inversion of a set of measured probability distributions $|s, r, \eta \rangle |\psi\rangle|^2$, i.e.

$$
W_\psi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2 \eta |s, r, \eta \rangle |\psi\rangle|^2 \int_{-\infty}^{\infty} d^2 \eta |s, r, \eta \rangle |K(r_1, r_2, \theta_1, \theta_2)\rangle
$$

In summary, based on the preceding paper [9], we have further extended the relation connecting optical Fresnel transformation with quantum tomography to the entangled case. The tomography representation $s, r, \eta \rangle |\psi\rangle^2$ is set up, thus the tomogram of quantum state $|\psi\rangle$ is just the squared modulus of the wave function $s, r, \eta \rangle |\psi\rangle$, i.e. the probability distribution for the Fresnel quadrature phase is the tomogram (Radon transform of the Wigner function). For more discussions about the relation between quantum optics transformation and classical optics transformation, we refer to [28,29].

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Appendix

Checking Equations (27)–(30)

From Equation (14) we have

$$\begin{align*}
F_2Q_1F_2^* &= \frac{1}{2}((A+D)Q_1 - (B-C)P_1 + (A-D)Q_2 + (B+C)P_2)
\end{align*}$$

(52)
\[ F_2 Q_2 F^e_2 = \frac{1}{2}((A+D)Q_2-(B-C)P_2+(A-D)Q_1+(B+C)P_1) \]  
\[ F_2 P_1 F^e_2 = \frac{1}{2}((A+D)P_1+(B-C)Q_1-(A-D)P_2+(B+C)Q_2) \]  
\[ F_2 P_2 F^e_2 = \frac{1}{2}((A+D)P_2+(B-C)Q_2-(A-D)P_1+(B+C)Q_1) \]

and

\[ F_2 (Q_1 - Q_2) F^e_2 = D(Q_1 - Q_2) - B(P_1 - P_2) \]  
\[ F_2 (P_1 + P_2) F^e_2 = B(Q_1 + Q_2) + D(P_1 + P_2) \]  
\[ F_2 (P_1 - P_2) F^e_2 = A(P_1 - P_2) - C(Q_1 - Q_2). \]

Noticing that \( [F_2 (Q_1 - Q_2) F^e_2, F_2 (P_1 + P_2) F^e_2] = 0 \) and (18) thus the eigenequation of communicative operators \( D(Q_1 - Q_2) - B(P_1 - P_2) \) and \( B(Q_1 + Q_2) + D(P_1 + P_2) \) is

\[ [D(Q_1 - Q_2) - B(P_1 - P_2)]|\eta\rangle_{s,r} = F_2 (Q_1 - Q_2) F^e_2 |\eta\rangle_{s,r} = \sqrt{2}\eta_s |\eta\rangle_{s,r} \]  
\[ = F_2 (P_1 + P_2) F^e_2 |\eta\rangle_{s,r} = \sqrt{2}\eta_s |\eta\rangle_{s,r} \]  
\[ = F_2 (P_1 + P_2) F^e_2 |\eta\rangle_{s,r} = \sqrt{2}\eta_s |\eta\rangle_{s,r} \]

\[ |\eta\rangle_{s,r} = F_2 |\eta\rangle = \text{Equation (23)} \]

and we name \( D(Q_1 - Q_2) - B(P_1 - P_2) \) or \( B(Q_1 + Q_2) + D(P_1 + P_2) \) the Fresnel quadrature phase.

On the other hand, due to the communicative relation

\[ [F_2 (Q_1 + Q_2) F^e_2, F_2 (P_1 - P_2) F^e_2] = 0, \]  
and \( |\xi\rangle \) (the conjugate state to \( |\eta\rangle \)) is the common eigen-equation of \( (Q_1 + Q_2) \) and \( (P_1 - P_2) \), i.e.

\[ (Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_s |\eta\rangle, (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_p |\eta\rangle \]

so the common eigenvector of \( F_2 (Q_1 + Q_2) F^e_2 \) and \( F_2 (P_1 - P_2) F^e_2 \) is given by

\[ |\xi\rangle_{s,r} = F_2 |\xi\rangle = F_2 \left( \frac{d^2\eta}{2\pi} |\eta\rangle |\eta\rangle |\eta\rangle \right) \]

\[ = \left( \frac{d^2\eta}{2\pi} \exp \left( \frac{\xi_s - \xi_p \eta}{2} \right) \right) |\eta\rangle_{s,r} \]

\[ = \text{Equation (42)} = \text{Equation (43)} \]

where we have used the overlap relation of \( |\eta\rangle |\xi\rangle = \frac{1}{2} \exp \left( \frac{\xi_s - \xi_p \eta}{2} \right) \). The corresponding eigen-equations of \( |\xi\rangle_{s,r} \) are

\[ [A(Q_1 + Q_2) + C(P_1 + P_2)]|\xi\rangle_{s,r} = \sqrt{2}\xi_s |\xi\rangle_{s,r} \]  
\[ [A(P_1 - P_2) - C(Q_1 - Q_2)]|\xi\rangle_{s,r} = \sqrt{2}\xi_p |\xi\rangle_{s,r}. \]