Algorithm that Solves 3-SAT in Polynomial Time

JASON W. STEINMETZ
(No institution, self-funded)

The question of whether the complexity class P is equal to the complexity class NP has been a seemingly intractable problem for over 4 decades. It has been clear that if an algorithm existed that would solve the problems in the NP class in polynomial time then P would equal NP. However, no one has yet been able to create that algorithm or to successfully prove that such an algorithm cannot exist. The algorithm that will be presented in this paper runs in polynomial time and solves the 3-satisfiability or 3-SAT problem, which has been proven to be NP-complete, thus indicating that P = NP.

Key Words and Phrases: Complexity classes, P=NP, SAT, NP-complete

1. INTRODUCTION

Whether P does or does not equal NP primarily affects the creation of effective algorithms for fairly common but often very difficult problems, which directly affects the efficiency and at the extreme, efficacy of computing in general. This question also has potentially large implications for cryptography because cryptography relies on P not being equal to NP. Cryptography doesn't rely on the idea that the encrypted code absolutely cannot be cracked, because all codes that can be deciphered can also be cracked. Instead, cryptography relies on the encrypted code being too complex to crack in any feasible amount of time.

Polynomial time is related to the idea of a feasible amount of time. Technically, polynomial time refers to an algorithm with a maximum number of steps that can be expressed as a fixed polynomial based on the length of the input to the algorithm, with the length of the input generally expressed as n. If an algorithm has a polynomial run time, which may be expressed as O(n^2) then the algorithm will theoretically run in a feasible amount of time. If an algorithm has an exponential run time, which may be expressed as O(2^n) then the algorithm will theoretically not run in a feasible amount of time.

To put the current problem into simple terms, if you had a bunch of "stuff" and 3 drawers to store it in, how could you know if all the stuff would fit in the 3 drawers? To answer this question you would attempt to rearrange and fit all the stuff into the 3 drawers. If the stuff doesn't all fit after trying every possible combination to make the stuff fit then the problem is not solvable. The problem of how or if the stuff will fit in the 3 drawers represents an NP problem. The process of going through and trying every possible combination to make the stuff fit is equivalent to an algorithm that has an exponential run time. The algorithms that currently exist to solve the problems in the NP class have exponential run times, or similarly it could be stated that no algorithm currently exists to solve the problems in the NP class in a feasible amount of time.

The complexity class P is a class of problems that can be solved in polynomial time on a deterministic Turing Machine, which is to say that the problems can be solved in polynomial time and once solved, produce a solution that can be verified in polynomial time. The complexity class NP is a class of problems that can be solved in polynomial time on a non-deterministic Turing Machine, which is to say that once solved the problems have a solution that can be verified in polynomial time, however, coming up with the solution in the first place currently takes a [potentially] exponential amount of time. To refer to the previous example, once all the stuff has been successfully packed into the 3 drawers it is easy to explain how it was done, although actually doing it in the first place may not have been easy or simple at all.

Thus, it is clear that if the problems in the NP class could be solved in polynomial time on a deterministic Turing Machine then P would equal NP. The problems in both complexity classes could then be solved and the solution verified in polynomial time. The algorithm that will be presented in this paper solves an NP-complete problem in polynomial time, which implies that all NP problems can now be solved in polynomial time.
2. THE ALGORITHM

The algorithm runs in polynomial time and solves the 3-SAT problem, which is a restricted version of Boolean Satisfiability or SAT. 3-SAT was proven to be NP-complete in 1971 by Stephen Cook [1] and was then listed as one of 21 proven NP-complete problems in 1972 by Richard Karp [2] in his paper, "Reducibility Among Combinatorial Problems".

Algorithm: Algorithm that solves 3-SAT in polynomial time

Input: A SAT formula in conjunctive normal form (CNF) with a list of the variables in the formula.
Output: TRUE/FALSE indicating whether the formula is satisfiable.

/*Variables represents a list or array of all the distinct variables in the formula*/

Let @Result = 0
Let @Unsolvable = (2 ^ (2 ^ Variables.Count)) - 1

For each @Clause in the Formula
    Let @Base = 2
    Let @ClauseResult = 1
    For each @Variable in Variables
        Let @Exists = FALSE
        For each @Literal in @Clause
            If @Variable = @Literal.Variable /*if the variable exists in the clause...*/
                @Exists = TRUE
            If @Literal.Negated = TRUE /*... and it is negated*/
                @ClauseResult = @ClauseResult * @Base
            End if
            Exit for
        End if
        @Base = @Base ^ 2
    End if
    @Result = @Result | @ClauseResult
End if

If @Result = @Unsolvable /*the formula cannot be satisfied*/
    Return FALSE
End if

Return TRUE /*the formula is satisfiable*/

The algorithm takes as an input a SAT formula in conjunctive normal form (CNF), or a CNF-SAT formula, with a list of all the distinct variables in the formula. The algorithm is not limited to the 3 literals per clause that the 3-SAT problem is limited to. The essential requirements are that the formula is a conjunction of clauses and a clause is a disjunction of literals, with a literal representing a variable or its negation, and also that none of the clauses can contain a variable and its negation. The algorithm’s output is then either TRUE indicating that the formula is satisfiable or FALSE indicating that it is not. The formula is satisfiable if there exists some assignment to the Boolean variables in the formula that would cause the formula to evaluate to TRUE.
The run time of the algorithm is \( c \times v \times l \), where \( c \) represents the number of clauses in the formula, \( l \) the number of literals per clause, and \( v \) the number of distinct variables in the formula. If the length of the input, expressed as \( n \) is equal to the total number of literals in the formula, or the number of clauses multiplied by the number of literals per clause, then \( n = c \times l \). Subsequently, since the number of distinct variables is bounded by the total number of literals in the formula, then the number of distinct variables is also equal to the length of the input, \( n = v \). If \( n = c \times l \) and \( n = v \) then the run time \( (c \times l \times v) \) can be expressed as \((n \times n) \) or \( n^2 \). Thus, the algorithm’s maximum run time is \( O(n^2) \), also expressed as \( T(n) = O(n^2) \).

3. WHY THE ALGORITHM WORKS

The reason why the present problem has seemed to be intractable is because the problem has been approached from the wrong direction. When approaching an NP problem from the beginning, when trying to figure out the “correct combination” to solve the problem, the only solution available is to try every possible combination, which is equivalent to an algorithm with an exponential run time, as mentioned in the introduction, section 1. Thus, as the algorithm presented in this paper demonstrates, the only way to know in a feasible amount of time if a solution to the problem exists is to approach the problem from the end instead of from the beginning, by focusing on the conditions that would make the problem unsolvable, or what I am calling the halt conditions. If all the halt conditions for an NP problem could be defined and then the problem checked for any of them, it could then be determined whether the problem is solvable or not. If a halt condition exists then the problem is not solvable.

Now I would like to present a table. The clauses are represented with \( A, B, C \) as opposed to \( X_1, X_2, X_3 \) and \( | \) is used as opposed to \( \land \) for formatting and clarity.

| \( A, B, C \) | 1,1,1 | 1,1,0 | 1,0,1 | 1,0,0 | 0,1,1 | 0,1,0 | 0,0,1 | 0,0,0 |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( A \)       | 1     | 1     | 1     | 1     | 0     | 0     | 0     | 0     |
| \( \neg A \)  | 0     | 0     | 0     | 0     | 1     | 1     | 1     | 1     |
| \( B \)       | 1     | 1     | 0     | 0     | 1     | 1     | 0     | 0     |
| \( \neg B \)  | 0     | 0     | 1     | 1     | 0     | 1     | 1     | 1     |
| \( C \)       | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 0     |
| \( \neg C \)  | 0     | 1     | 0     | 1     | 0     | 1     | 0     | 1     |
| \( A \lor B \)| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| \( A \lor \neg B \)| 1     | 1     | 0     | 0     | 1     | 1     | 1     | 1     |
| \( \neg A \lor B \)| 1     | 1     | 1     | 1     | 0     | 1     | 1     | 1     |
| \( \neg A \lor \neg B \)| 0     | 0     | 0     | 0     | 1     | 1     | 0     | 0     |
| \( A \lor C \)| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| \( A \lor \neg C \)| 1     | 1     | 1     | 1     | 0     | 1     | 1     | 1     |
| \( \neg A \lor C \)| 1     | 1     | 0     | 1     | 1     | 1     | 1     | 1     |
| \( \neg A \lor \neg C \)| 0     | 0     | 1     | 1     | 1     | 1     | 1     | 1     |

In the leftmost column are all the reduced, nontrivial clauses that could result from a 3 variable CNF-SAT formula. In the topmost row are all the possible 1/0 (TRUE/FALSE) values of those 3 variables. When solving a SAT formula in conjunctive normal form, because the formula is a conjunction of clauses, if at any time any clause in the formula evaluates to zero then the entire formula will evaluate to zero. Thus, the halt conditions for a CNF-SAT formula with 3 variables could be defined as any combination of clauses in Table 1 that when combined would contain at least one 0 in every column. This would indicate that for every possible set of values for the 3
variables the formula would contain at least one clause that evaluates to zero, which would thus indicate that the formula is not satisfiable. For example, referencing the first 2 rows, literals $A$ and $\neg A$, which when combined contain a 0 in every column, it is clear that no matter how many clauses a formula contains if a formula contains those two clauses then the formula is not satisfiable.

Conceptually, if Table 1 above is viewed as a set of numbers with each row being a number expressed in binary code (read from right to left), the algorithm effectively flips the 0's and 1's in the table and uses the corresponding values (see Table 2 below) to create a bit field that then catches the halt condition and evaluates the formula. If all the bits are set in the bit field then a halt condition exists, proving that the formula is not satisfiable and doing so in polynomial time.

| Table 2 The values used to create the bit field |
|-----------------------------------------------|
| $\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline A & \sim A & B & \sim B & C & \sim C & A \mid B & \sim A \mid \sim B & A \mid \sim A \mid B & \sim A \mid \sim B \mid C \\ \hline 1,1,1 & 1,1,0 & 1,0,0 & 1,0,1 & 1,0,0 & 0,0,0 & 1,0,1 & 1,0,0 & 1,0,0 & 1,0,0 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline \end{array}$

All the values in Table 1 were flipped, from 0 to 1 and vice versa, and then a column added to display the base 10 number created from its binary, or base 2 representation in the row. It should now be quite clear to see why the algorithm calculates the value for each clause the way that it does (for the $\text{@ClauseResult}$ variable). Starting from clause $(A \mid B \mid C)$, whose value is 1 and could be considered the baseline; if the variable $C$ is negated, clause $(A \mid B \mid \neg C)$, then the value is 2 ($\text{@ClauseResult} \ast \text{@Base}$), and if $C$ does not exist in the clause, clause $(A \mid B)$, then the value is 3 ($\text{@ClauseResult} \ast (\text{@Base} + 1)$). Similarly, if $A$, the 3rd variable is negated, clause $(\neg A \mid B \mid C)$, then the value is 16, and if $A$ does not exist in the clause, clause $(B \mid C)$, then the value is 17.

In addition, although it is not required to solve the present problem, it is clear that if a formula is satisfiable then the bit field in the algorithm (the $\text{@Result}$ variable) could be used to determine which sets of values could be assigned to the variables to satisfy the formula. If a list were created of all the possible values for the variables and the list sorted in a simple binary order beginning with all the variables being assigned to 0 or FALSE, then each bit in the bit field that was not set would correspond to a set of values in the list that would satisfy the formula. If a formula contained 3 variables then the topmost row in Table 1 above could be used to determine the sets of values that would satisfy the formula. If the first bit in the bit field was not set then 0,0,0 would satisfy the formula, if the second bit was not set then 0,0,1 would satisfy the formula, etc.
4. CONCLUSION, DOES P = NP?

The impetus for me to work on this problem was the Clay Mathematics Institute posing this question as one of their Millennium problems [4]. Thus, I will address their fundamental question specifically and directly, which is, does P = NP? The Cook-Levin theorem [1,3], also known as Cook’s theorem proves that any NP problem can be reduced in polynomial time to an NP-complete problem. This indicates that if an algorithm were created that would solve any NP-complete problem in polynomial time then every NP problem could be solved in polynomial time. Subsequently, because the algorithm presented in this paper solves an NP-complete problem in polynomial time this indicates that every NP problem can now be solved in polynomial time proving that NP ⊆ P, and thus also proving that P = NP.

ELECTRONIC APPENDIX

A windows application written in VB.Net to implement the algorithm along with the source code for the application is available.

ACKNOWLEDGMENTS

The author would like to thank Alex Krivitsky for being invaluable in helping work out the details and with composing this paper.

REFERENCES

[1] Stephen A. Cook. The complexity of theorem-proving procedures, Proceedings of the third annual ACM symposium on Theory of computing, May 03-05, 1971, Shaker Heights, Ohio, United States
[2] R.M. Karp, "Reducibility among combinatorial problems", "Complexity of Computer Computations", R.E. Miller and J.W. Thatcher, eds. Plenum Press, New York, 1972.
[3] Levin, L. 1973. Universal search problems (in Russian). Problemy Peredachi Informatsii 9, 3, 265-266. (English translation in Trakhtenbrot, B. A.: A survey of Russian approaches to Perebor (brute-force search) algorithms. Ann. Hist. Comput. 6 (1984), 384-400.)
[4] http://www.claymath.org/millennium/P_vs_NP/

© Copyright 2011 Jason Steinmetz, all rights reserved.