NEUTRINOS, ELECTRONS AND MUONS IN ELECTROMAGNETIC FIELDS AND MATTER: THE METHOD OF EXACT SOLUTIONS

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We present a powerful method for exploring various processes in the presence of strong external fields and matter. The method implies utilization of the exact solutions of the modified Dirac equations which contain the effective potentials accounting for the influences of external electromagnetic fields and matter on particles. We briefly discuss the basics of the method and its applications to studies of different processes, including a recently proposed new mechanism of radiation by neutrinos and electrons moving in matter (the spin light of the neutrino and electron). In view of a recent "prediction" of an order-of-magnitude change of the muon lifetime under the influence of an electromagnetic field of a CO₂ laser, we revisit the issue and show that such claims are nonrealistic.

1 Introduction

The problem of particles’ interactions under the influence of external electromagnetic fields and matter is one of the important topics in particle physics. Besides the possibility for better visualization of fundamental properties of particles and their interactions when they are influenced by external conditions, the interest to this problem is also stimulated by astrophysical and cosmological applications, where strong electromagnetic fields and dense matter may play important roles. There are well established methods for such kind of investigations that have a long-standing history.

In particular, the method of exact solutions of quantum equations, which is based on a Furry representation of QED, is widely used in studies of particles’ interactions in external electromagnetic fields. In this technique, the evolution operator $U_F(t_1,t_2)$, which determines the matrix element of the process, is presented in the usual form

$$U_F(t_1,t_2) = T \exp \left[ -i \int_{t_1}^{t_2} j_\mu(x) A_\mu^\text{cl} dx \right],$$

where $A_\mu(x)$ is the quantized part of the potential corresponding to the radiation field, which is accounted for within the perturbation-series techniques. At the same time, the electron (a charged particle) current is presented as

$$j_\mu(x) = \frac{e}{2} \left[ \Psi_e \gamma_\mu, \Psi_e \right],$$

where $\Psi_e$ are the exact solutions of the Dirac equation for an electron in the presence of an external electromagnetic field given by the classical non-quantized potential $A_\mu^\text{ext}(x)$:

$$\left\{ \gamma^\mu \left( i \partial_\mu - e A_\mu^\text{cl}(x) \right) - m_e \right\} \Psi_e(x) = 0.$$
Dirac equations for each of the particles can be written in the following form: electron neutrinos and electrons with matter composed of neutrons, the corresponding modified, and an electron, a quasiclassical treatment. Neutrino 7 and an electron 8 quantum processes that can take part in the presence of matter. In particular, we have elaborated the quantum theory of the new type of electromagnetic radiation that can be emitted by a neutrino 7 and an electron 8, and an electron, SLν, while these particles move in dense matter (spin light of a neutrino, SLν, and an electron, SLν). Note that SLν in matter was first considered in 11 on the basis of a quasiclassical treatment.

As it was shown in 7,8,9, in the case of the standard model interactions of the electron neutrinos and electrons with matter composed of neutrons, the corresponding modified Dirac equations for each of the particles can be written in the following form:

\[ \left\{ i\gamma_\mu \partial^\mu - \frac{1}{2} \gamma_\mu (c_\ell + \gamma_5) \vec{f}^\mu - m_\ell \right\} \Psi^{(l)}(x) = 0, \]

where for a neutrino \( m_\ell = m_\nu \) and \( c_\ell = c_\nu = 1 \), whereas for an electron \( m_\ell = m_e \) and \( c_\ell = c_e = 1 - 4 \sin^2 \theta_W \). For unpolarized matter \( \vec{f}^\mu = \frac{G_F}{\sqrt{2}} (n_n, n_n \nu) \), \( n_n \) and \( \nu \) are, respectively, the neutron number density and average speed. The solutions of these equations are

\[ \Psi_{\varepsilon, \mathbf{p}, s}^{(l)}(\mathbf{r}, t) = \frac{e^{-i(E^{(l)}_{\varepsilon} t - \mathbf{p} \cdot \mathbf{r})}}{2L^2} \left( \frac{s \sqrt{1 + \frac{m_l}{E^{(l)}_{\varepsilon} - c_\alpha m_l}}}{s \sqrt{1 + \frac{m_l}{E^{(l)}_{\varepsilon} - c_\alpha m_l}}} \right) \left( \frac{1 + s \frac{p_x}{p} e^{i\delta}}{1 + s \frac{p_x}{p} e^{i\delta}} \right). \]

where the energy spectra are

\[ E^{(l)}_{\varepsilon} = \varepsilon \eta \sqrt{p^2 \left(1 - s \alpha_n m_l \right)^2 / p^2 + m^2 + c_\ell c_\alpha m_l}, \quad \alpha_n = \pm \frac{1}{2 \sqrt{2}} G_F \frac{n_n}{m_l}. \]

Here \( p, s, \) and \( \varepsilon \) are the particles’ momenta, helicities and signs of energy, \( \pm \) corresponds to \( e \) and \( \nu_e \). The value \( \eta = \text{sign}(1 - s \alpha_n m_l / p) \) is introduced to provide a proper behavior of the neutrino wave function in a hypothetical massless case.

The developed approach to description of the matter effect on neutrinos and electrons, driven by (electro)weak forces, is valid as long as interactions of particles with the background is coherent. This condition is satisfied when a macroscopic amount of the background particles are confined within the scale of a neutrino or electron de Broglie wave length. For relativistic

\[ 2 \text{ Neutrino and electron quantum states in matter} \]

Recently we have applied the “method of exact solutions” for treating different interactions of neutrinos and electrons in the presence of matter 7,8,9 (see also 10). The developed method is based on the use of the exact solutions of the modified Dirac equations that include effective matter potentials. It has been demonstrated how this method works in application to different quantum processes that can take part in the presence of matter. In particular, we have elaborated the quantum theory of the new type of electromagnetic radiation that can be emitted by a neutrino 7 and an electron 8, and an electron, SLν, while these particles move in dense matter (spin light of a neutrino, SLν, and an electron, SLν). Note that SLν in matter was first considered in 11 on the basis of a quasiclassical treatment.

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neutrinos or electrons the following condition should be satisfied \( n/(\gamma l m^3) \gg 1 \), where \( n \) is the number density of matter, \( \gamma_l = E_l/m_l \) and \( l = \nu \) or \( e \). In a case of varying density of the background matter, there is an additional condition for applicability of the developed approach (see, for instance, [12]). The characteristic length of matter density variations should be much larger than the de Broglie wavelength, \( |\nabla n| \ll 1 \).

Using the exact solutions of the above Dirac equations for a neutrino and an electron we have performed detailed investigations of \( SL\nu \) and \( SLe \) in matter. In particular, in the case of ultra-relativistic neutrinos \( (p \gg m) \) and a wide range of the matter density parameter \( \alpha \) for the total rate of \( SL\nu \) we obtained

\[
\Gamma_{SL\nu} = 4 \mu^2 \alpha^2 m^2 \nu p, \quad m_\nu/p \ll \alpha \ll p/m_\nu. \tag{7}
\]

Performing the detailed study of the \( SLe \) in neutron matter [9], we have found for the total rate

\[
\Gamma_{SLe} = e^2 m^2 \nu/(2p) \left[ \ln (4\alpha_n p/m_e) - 3/2 \right], \quad m_e/p \ll \alpha_n \ll p/m_e, \tag{8}
\]

where it is supposed that \( \ln(4\alpha_n p/m_e) \gg 1 \). It was also found that for relativistic electrons the emitted photon energy can reach the range of gamma-rays. Furthermore, the electron can lose almost the whole initial energy due to the \( SLe \) mechanism.

Recently we apply our method to a particular case where a neutrino is propagating in a rotating medium of constant density [13] (see also [10, 14]). Suppose that a neutrino is propagating perpendicular to the uniformly rotating matter composed of neutrons. This can be considered for modelling of the neutrino propagation inside a rotating neutron star. The corresponding modified Dirac equation for the neutrino wave function is given by [11] with the matter potential accounting for rotation,

\[
\hat{f}^\mu = -G(n, n\nu), \quad \nu = (\omega y, 0, 0), \tag{9}
\]

where \( G = G_F/\sqrt{2} \). Here \( \omega \) is the angular frequency of the matter rotation around the \( z \) axis, here also is accounted that all radial directions orthogonal to the \( z \) axis are physically equal. For the energy of the active left-handed neutrino we get

\[
p_0 = \sqrt{p_3^2 + 2pN - Gn}, \quad N = 0, 1, 2, \ldots. \tag{10}
\]

The energy depends on the neutrino momentum component \( p_3 \) along the rotation axis of matter and the quantum number \( N \) that determines the magnitude of the neutrino momentum in the orthogonal plane. For description of antineutrinos one has to consider the “negative sign” energy eigenvalues (for details see, for instance, [10]). The energy of an electron antineutrino in the rotating matter composed of neutrons is given by

\[
\tilde{p}_0 = \sqrt{p_3^2 + 2pN + Gn}, \quad N = 0, 1, 2, \ldots. \tag{11}
\]

Thus, the transversal motion of the active neutrino and antineutrino is quantized in moving matter very much alike an electron energy is quantized in a constant magnetic field that corresponds to the relativistic form of the Landau energy levels (see, for instance, [2]).

In conclusion of this section, we note that the developed new approach establishes a basis for investigation of different phenomena which can emerge when neutrinos and electrons move in dense media, including those peculiar to astrophysical and cosmological environments.

### 3 Muon decay \( \mu \rightarrow e\nu\bar{\nu} \) in electromagnetic field

In this section we inspect, using the method of exact solutions, some aspects of the muon decay process in which the muon is embedded in a field of a linearly polarized electromagnetic wave.
with the wave vector \( k = (\omega, \mathbf{k}) \), where \( \omega \) is the frequency and \( |\mathbf{k}| = \omega \). The field is thus described by the vector potential \( A(x) = a \cos(k \cdot x) \) satisfying the Lorenz condition, where \( a = (0, \mathbf{a}) \) is a constant four-vector such that \( a \cdot k = 0 \).

The theoretical framework for the considered process was developed in the basic papers of Ritus\(^{16,13}\). Its key ingredients are the standard theory of the weak interaction and description of the muon and electron states by the Volkov functions\(^{15}\). The decay rate is thus given by\(^{16}\)

\[
W = \frac{G^2}{48\pi^4 q_0} \sum_{s > 0} \int \frac{d^3 q}{q_0} \left\{ \left[ \frac{1}{2} (m_\mu^2 - m_e^2)^2 + \frac{1}{2} (m_\mu^2 + m_e^2) Q^2 - Q^4 \right] B_0^2 + \right.
\]
\[
+ \left[ \frac{(2Q^2 + m_\mu^2 + m_e^2)(k \cdot Q)^2}{(k \cdot q)(k \cdot q')} + 2Q^2 \right] e^2 a^2 (B_1^2 - B_0 B_2) \right\}, \tag{12}
\]

where \( G \) is the weak interaction constant, \( q = p - k(e^2 a^2/[4(k \cdot p)]) \) and \( q' = p' - k(e^2 a^2/[4(k \cdot p')] \) are the muon and electron four-quasimomenta (\( p \) and \( p' \) are the field-free four-momenta, respectively), \( m_\mu \) and \( m_e \) are the muon and electron masses, \( s \) is the number of photons absorbed from the wave (emitted into the wave if \( s < 0 \)), \( s_0 = (m_\mu^2 - m_e^2)/(2k \cdot q) \), \( Q = sk + q - q' \). The functions \( B_i \equiv B_i(s; \alpha, \beta) \) \((i = 0, 1, 2)\), with

\[
\alpha = e \left( \frac{a \cdot p}{k \cdot p} - \frac{a \cdot p'}{k \cdot p'} \right), \quad \beta = \frac{e^2 a^2}{8} \left( \frac{1}{k \cdot p'} - \frac{1}{k \cdot p} \right),
\]

are defined as follows:

\[
B_0(s; \alpha, \beta) = \sum_{-\infty}^{\infty} J_{s+2}(\alpha) J_1(\beta),
\]

\[
B_1(s; \alpha, \beta) = \frac{1}{2} \left[ B_0(s - 1; \alpha, \beta) + B_0(s + 1; \alpha, \beta) \right],
\]

\[
B_2(s; \alpha, \beta) = \frac{1}{4} \left[ B_0(s - 2; \alpha, \beta) + 2B_0(s; \alpha, \beta) + B_0(s + 2; \alpha, \beta) \right], \tag{13}
\]

where \( J_s \) is a Bessel function of order \( s \). Note that practical calculations of Eq.\(^{12}\) are hindered by the sums and integrations involving rapidly oscillating functions.

We have reanalyzed the muon decay process in the electromagnetic field because recently Liu et al\(^{17}\) made a rather unexpected theoretical conclusion, based on their numerical calculations, that the muon lifetime can be changed dramatically in an intense laser field achievable with present-day laser sources. For example, they predicted an order-of-magnitude reduction of the muon lifetime in the case of a CO\(_2\) laser (\( \omega = 0.117 \) eV) with the electric field amplitude \( \mathcal{E}_0 = \omega |\mathbf{a}| = 10^6 \) V/cm. It should be remarked that Liu et al\(^{17}\) employed instead of Eq.\(^{12}\) an approximate model which neglects the laser influence on the muon and drops in the electron’s Volkov state the dependence on terms quadratic in the vector potential. Such an approach as well as a resultant surprising prediction met a serious criticism by Narozhny and Fedotov\(^{18}\) who classified the obtained result as “fallacious” and attributed it either to the mistakes in analytics or to the erroneous numerical calculation (see also the reply of Liu et al\(^{19}\)).

Let us examine the muon decay to the erroneous numerical calculation. We note that, due to the properties of Bessel functions, the functions \( B_1^2 \) and \( B_1^2 - B_0 B_2 \) decrease exponentially with \( s \) if \(|s| > \alpha\), and therefore we can replace the sum in Eq.\(^{12}\) with \( \sum_{s=\tilde{s}}^\infty \), where \( \tilde{s} \sim ea/\omega \). For a CO\(_2\) laser with \( \mathcal{E}_0 = 10^6 \) V/cm we have \( \tilde{s} \sim 2 \times 10^3 \) \((|s_0| \sim 5 \times 10^8)\).

Further, since all the items in the integrand of Eq.\(^{12}\) except \( B_0^2 \) and \( B_1^2 - B_0 B_2 \) practically do not vary with \( s \leq \tilde{s} \), we can following Ritus\(^3\) perform the summation over \( s \) setting approximately \( \tilde{s} = \infty \) and using the formulas \( \sum_{-\infty}^\infty B_0^2 = 1 \) and \( \sum_{-\infty}^\infty (B_1^2 - B_0 B_2) = 0 \). The remaining
integrations can be performed analytically and they yield the well known vacuum result, i.e. \( W \approx W_0 = G^2 m_\mu^5 / (192 \pi^3) \). We can also estimate the relative change in the decay rate as

\[
\delta = \frac{W - W_0}{W_0} \approx \frac{37 \epsilon^2 a^2}{48 m_\mu^2} \sim -10^{-12}
\]

which is due to the difference between the muon and electron four-quasimomenta, \( q \) and \( q' \), and their field-free analogs, \( p \) and \( p' \). This estimate clearly shows the nonphysical character of the prediction made by Liu \textit{et al.}\(^\text{17}\).

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