Andrei Okounkov’s initial area of research is group representation theory, with particular emphasis on combinatorial and asymptotic aspects. He used this subject as a starting point to obtain spectacular results in many different areas of mathematics and mathematical physics, from complex and real algebraic geometry to statistical mechanics, dynamical systems, probability theory and topological string theory. The research of Okounkov has its roots in very basic notions such as partitions, which form a recurrent theme in his work. A partition $\lambda$ of a natural number $n$ is a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ adding up to $n$. Partitions are a basic combinatorial notion at the heart of the representation theory. Okounkov started his career in this field in Moscow where he worked with G. Olshanski, through whom he came in contact with A. Vershik and his school in St. Petersburg, in particular S. Kerov. The research programme of these mathematicians, to which Okounkov made substantial contributions, has at its core the idea that partitions and other notions of representation theory should be considered as random objects with respect to natural probability measures. This idea was further developed by Okounkov, who showed that, together with insights from geometry and ideas of high energy physics, it can be applied to the most diverse areas of mathematics.

This is an account of some of the highlights mostly of his recent research.

I am grateful to Enrico Arbarello for explanations and for providing me with very useful notes on Okounkov’s work in algebraic geometry and its context.

1. Gromov–Witten invariants

The context of several results of Okounkov and collaborators is the theory of Gromov–Witten (GW) invariants. This section is a short account of this theory. GW invariants originate from classical questions of enumerative geometry, such as: how many rational curves of degree $d$ in the plane go through $3d - 1$ points in general position? A completely new point of view on this kind of problems appeared at the end of the eighties, when string theorists, working on the idea that space-time is the product of four-dimensional Minkowski space with a Ricci-flat compact complex three-fold, came up with a prediction for the number of rational curves of given degree in the quintic $x_1^5 + \cdots + x_5^5 = 0$ in $\mathbb{C}P^4$. Roughly speaking, physics gives predictions for differential equations obeyed by generating functions of numbers of curves. Solving these equations in power series gives recursion relations for the numbers. In particular a recursion relation of Kontsevich gave a complete answer to the above question on rational curves in the plane.

In general, Gromov–Witten theory deals with intersection numbers on moduli spaces of maps from curves to complex manifolds. Let $V$ be a nonsingular projective variety over the complex numbers. Following Kontsevich, the compact moduli space $\overline{M}_{g,n}(V, \beta)$ (a Deligne–Mumford stack) of stable maps of class...
β ∈ H^2(V) is the space of isomorphism classes of data (C, p_1, ..., p_n, f) where C is a complex projective connected nodal curve of genus g with n marked smooth points p_1, ..., p_n and f : C → V is a stable map such that [f(C)] = β. Stable means that if f maps an irreducible component to a point then this component should have a finite automorphism group. For each j = 1, ..., n two natural sets of cohomology classes can be defined on these moduli space: 1) pull-backs ev_j^*α ∈ H^*(\overline{M}_{g,n}(V, \beta)) of cohomology classes α ∈ H^*(V) on the target V by the evaluation map ev_j : (C, p_1, ..., p_n, f) → f(p_j); 2) the powers of the first Chern class \psi_j = c_1(L_j) ∈ H^2(\overline{M}_{g,n}(V, \beta)) of the line bundle L_j whose fiber at (C, p_1, ..., p_n, f) is the cotangent space T^*_F, C to C at p_j. The Gromov–Witten invariants of V are the intersection numbers

\langle \tau_{k_1}(\alpha_1) \cdots \tau_{k_n}(\alpha_n) \rangle^{V}_{\beta,g} = \int_{\overline{M}_{g,n}(V, \beta)} \prod \psi_j^{k_j} ev_j^*\alpha_j.

If all k_i are zero and the \alpha_i are Poincaré duals of subvarieties, the Gromov–Witten invariants have the interpretation of counting the number of curves intersecting these subvarieties. As indicated by Kontsevich, to define the integral one needs to construct a virtual fundamental class, a homology class of degree equal to the “expected dimension”

(1) \text{vir dim } \overline{M}_{g,n}(V, \beta) = -\beta \cdot K_V + (g - 1)(3 - \dim V) + n,

where K_V is the canonical class of V. This class was constructed in works of Behrend–Fantechi and Li–Tian.

The theory of Gromov–Witten invariants is already non-trivial and deep in the case where V is a point. In this case \overline{M}_{g,n} = \overline{M}_{g,n}(\{\text{pt}\}) is the Deligne–Mumford moduli space of stable curves of genus g with n marked points. Witten conjectured and Kontsevich proved that the generating function

\[ F(t_0, t_1, \ldots) = \sum_{n=0}^{\infty} \frac{1}{h!} \sum_{k_1 + \cdots + k_n = 3g - 3 + n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle^{pt}_g \prod_{j=1}^{n} t_{k_j}, \]

involving simultaneously all genera and numbers of marked points, obeys an infinite set of partial differential equations (it is a tau-function of the Korteweg-de Vries integrable hierarchy obeying the “string equation”) which are sufficient to compute all the intersection numbers explicitly. One way to write the equations is as Virasoro conditions

\[ L_k(\varepsilon^F) = 0, \quad k = -1, 0, 1, 2, \ldots, \]

for certain differential operators L_k of order at most 2 obeying the commutation relations [L_j, L_k] = (j - k)L_{j+k} of the Lie algebra of polynomial vector fields.

Before Okounkov few results were available for general projective varieties V and they were mostly restricted to genus g = 0 Gromov–Witten invariants (quantum cohomology). For our purpose the conjecture of Eguchi, Hori and Xiong is relevant here. Again, Gromov–Witten invariants of V can be encoded into a generating function \hat{F}_V depending on variables t_{j,a} where a labels a basis of the cohomology of V. Eguchi, Hori and Xiong extended Witten’s definition of the differential operators L_k and conjectured that \hat{F}_V obeys the Virasoro conditions L_k(\varepsilon^{\hat{F}_V}) = 0 with these operators.
2. Gromov–Witten invariants of curves

In a remarkable series of papers [10, 12], Okounkov and Pandharipande give an exhaustive description of the Gromov–Witten invariants of curves. They prove the Eguchi–Hori–Xiong conjecture for general projective curves $V$, give explicit descriptions in the case of genus 0 and 1, show that the generating function for $V = \mathbb{P}^1$ is a tau-function of the Toda hierarchy and consider also in this case the $\mathbb{C}^\times$-equivariant theory, which is shown to be governed by the 2D-Toda hierarchy. They also show that GW invariants of $V = \mathbb{P}^1$ are unexpectedly simple and more basic than the GW invariants of a point, in the sense that the latter can be obtained as a limit, giving thus a more transparent proof of Kontsevich’s theorem.

A key ingredient is the Gromov–Witten/Hurwitz correspondence relating GW invariants of a curve $V$ to Hurwitz numbers, the numbers of branched covering of $V$ with given ramification type at given points. A basic beautiful formula of Okounkov and Pandharipande is the formula for the stationary with given ramification type at given points. A basic beautiful formula of Okounkov and Pandharipande is the formula for the stationary GW invariants of a curve $V$ of genus $g(V)$, namely those for the Poincaré dual $\omega$ of a point:

$$\tau_k(\omega) \cdot \tau_{\lambda}(\omega) \cdot \cdots \tau_{\lambda_n}(\omega) \cdot \left[\frac{\dim \lambda}{d!}\right]^{g(V)} \prod_{i=1}^n \frac{p_{k_i+1}(\lambda_i)}{(k_i+1)!}.$$  

The (finite!) summation is over all partitions $\lambda$ of the degree $d$ and $\dim \lambda$ is the dimension of the corresponding irreducible representation of $S_d$. The genus $g$ of the domain is fixed by the condition that the cohomological degree of the integrand is equal to the dimension of the virtual fundamental class. It is convenient here to include also stable maps with possibly disconnected domains and this is indicated by the bullet. The functions $p_k(\lambda)$ on partitions are described below.

Hurwitz numbers can be computed combinatorially and are given in terms of representation theory of the symmetric group by an explicit formula of Burnside. If the covering map at the $i$th point looks like $z \rightarrow z^{k_i+1}$, i.e., if the monodromy at the $i$th point is a cycle of length $k_i + 1$, the formula is

$$H^\vee_d(k_1 + 1, \ldots, k_n + 1) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g(V)} \prod_{i=1}^n f_{k_i+1}(\lambda).$$

Thus in this case the GW/Hurwitz correspondence is given by the substitution rule $f_{k+1}(\lambda) \rightarrow p_{k+1}(\lambda)/(k+1)!$. The functions $f_k$ and $p_k$ are basic examples of shifted symmetric functions, a theory initiated by Kerov and Olshanski, and the results of Okounkov and Pandharipande offer a geometric realization of this theory. A shifted symmetric polynomial of $n$ variables $\lambda_1, \ldots, \lambda_n$ is a polynomial invariant under the action of the symmetric group given by permuting $\lambda_j - j$. A shifted symmetric function is a function of infinitely many variables $\lambda_1, \lambda_2, \ldots$, restricting for each $n$ to a shifted symmetric polynomial of $n$ variables if all but the first $n$ variables are set to zero. Shifted symmetric functions form an algebra $\Lambda^* = \mathbb{Q}[p_1, p_2, \ldots]$ freely generated by the regularized shifted power sums, appearing in the GW invariants:

$$p_k(\lambda) = \sum_j \left(\left(\lambda_j - j + \frac{1}{2}\right)^k - \left(-j + \frac{1}{2}\right)^k\right) + (1 - 2^{-k})\zeta(-k).$$

The second term and the Riemann zeta value “cancel out” in the spirit of Ramanujan’s second letter to Hardy: $1 + 2 + 3 + \cdots = -\frac{1}{12}$. The shifted symmetric functions $f_k(\lambda)$ appearing in the Hurwitz numbers are central characters of the symmetric
groups \( S_n \): \( f_1 = |\lambda| = \sum \lambda_i \) and the sum of the elements of the conjugacy class of a cycle of length \( k \geq 2 \) in the symmetric group \( S_n \) is a central element acting as \( f_k(\lambda) \) times the identity in the irreducible representation corresponding to \( \lambda \). The functions \( p_k \) and \( f_k \) are two natural shifted versions of Newton power sums.

In the case of genus \( g(V) = 0,1 \) Okounkov and Pandharipande reformulate \([4]\) in terms of expectation values and traces in fermionic Fock spaces and get more explicit descriptions and recursion relations. In particular if \( Z \) functions inspired by ideas of string theory \([14]\) states that suitably normalized generating information:

\[
p(\mathbf{V}, \beta, \chi; E_i) = \prod q^d(\tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega))^{\bullet E}
\]

belongs to the ring \( \mathbb{Q}[E_2, E_4, E_6] \) of quasimodular forms.

A shown by Eskin and Okounkov \([2]\), one can use quasimodularity to compute the asymptotics as \( d \to \infty \) of the number of connected ramified degree \( d \) coverings of a torus with given monodromy at the ramification points. By a theorem of Kontsevich–Zorich and Eskin–Mazur, this asymptotics gives the volume of the moduli space of holomorphic differentials on a curve with given orders of zeros, which is in turn related to the dynamics of billiards in rational polygons. Eskin and Okounkov give explicit formulae for these volumes and prove in particular the Kontsevich–Zorich conjecture that they belong to \( \pi^{-2g} \mathbb{Q} \) for curves of genus \( g \).

3. Donaldson–Thomas invariants

As is clear from the dimension formula \([1] \) the case of three-dimensional varieties \( V \) plays a very special role. In this case, which in the Calabi-Yau case \( K_V = 0 \) is the original context studied in string theory, it is possible to define invariants counting curves by describing curves by equations rather than in parametric form. Curves in \( V \) of genus \( g \) and class \( \beta \in H_2(V) \) given by equations are parametrized by Grothendieck’s Hilbert schemes \( \text{Hilb}(V; \beta, \chi) \) of subschemes of \( V \) with given Hilbert polynomial of degree 1. The invariants \( \beta, \chi = 2 - g \) are encoded in the coefficients of the Hilbert polynomial. R. Thomas constructed a virtual fundamental class of \( \text{Hilb}(V; \beta, \chi) \) for three-folds \( V \) of dimension \( -\beta \cdot K_V \), the same as the dimension of \( \overline{M}_{g,0}(V, \beta) \). Thus one can define Donaldson–Thomas (DT) invariants as intersection numbers on this Hilbert scheme. There is no direct geometric relation between \( \text{Hilb}(V; \beta, \chi) \) and \( \overline{M}_{g,0}(V, \beta) \), and indeed the (conjectural) relation between Gromov–Witten invariants and Donaldson–Thomas invariants is quite subtle. In its simplest form, it relates the GW invariants \( \int_{\overline{M}_{g,n}(V; \beta, \chi)} \prod \text{ev}_i^* \gamma_i \) to the DT invariants \( \int_{\text{Hilb}(V; \beta, \chi)} \prod c_2(\gamma_i) \). The class \( c_2(\gamma) \) is the coefficient of \( \gamma \in H^*(V) \) in the Künneth decomposition of the second Chern class of the ideal sheaf of the universal family \( V \subset \text{Hilb}(V; \beta, \chi) \times V \).

The conjecture of Maulik, Nekrasov, Okounkov and Pandharipande \([6,7]\), inspired by ideas of string theory \([14]\) states that suitably normalized generating functions \( Z'_{\text{GW}}(\gamma; u)_\beta, Z'_{\text{DT}}(\gamma; q)_\beta \) are essentially related by a coordinate transformation:

\[
(-iu)^{-d} Z'_{\text{GW}}(\gamma_1, \ldots, \gamma_n; u)_\beta = q^{-d/2} Z'_{\text{DT}}(\gamma_1, \ldots, \gamma_n; q)_\beta, \quad \text{if } q = -e^{iu}, \beta \neq 0.
\]
Here \( d = -\beta \cdot K_V \) is the virtual dimension. Moreover these authors conjecture that there \( Z_{DT}(\gamma; q)_\beta \) is a rational function of \( q \). This has the important consequence that all (infinitely many) GW invariants are determined in principle by finitely many DT invariants. Versions of these conjectures are proven for local curves and the total space of the canonical bundle of a toric surface. The GW/DT correspondence can be viewed as a far-reaching generalization of formula (2), to which it reduces in the case where \( V \) is the product of a curve with \( \mathbb{C}^2 \).

4. Other uses of partitions

Here is a short account of other results of Okounkov based on the occurrence of partitions.

One early result of Okounkov [9] is his first proof of the Baik–Deift–Johansson conjecture (two further different proofs followed, one by Borodin, Okounkov and Olshanski and one by Johansson). This conjecture states that, as \( n \to \infty \), the joint distribution of the first few rows of a random partition of \( n \) with the Plancherel measure \( P(\lambda) = (\dim \lambda)^2/|\lambda|! \), natural from representation theory, is the same, after proper shift and rescaling, as the distribution of the first few eigenvalues of a Gaussian random hermitian matrix of size \( n \). The proof involves comparing random surfaces given by Feynman diagrams and by ramified coverings and contains many ideas that anticipate Okounkov’s later work on Gromov–Witten invariants.

Random partitions also play a key role in the work [8] of Nekrasov and Okounkov on \( N = 2 \) supersymmetric gauge theory in four dimensions. Seiberg and Witten gave a formula for the effective “prepotential”, postulating a duality with a theory of monopoles. The Seiberg–Witten formula is given in terms of periods on a family of algebraic curves, closely connected with classical integrable systems. Nekrasov showed how to rigorously define the prepotential of the gauge theory as a regularized instanton sum given by a localization integral on the moduli space of antiselfdual connections on \( \mathbb{R}^4 \). Nekrasov and Okounkov show that this localization integral can be written in terms of a measure on partitions with periodic potential and identify the Seiberg–Witten prepotential with the surface tension of the limit shape.

Partitions of \( n \) also label \((\mathbb{C}^*)^2\)-invariant ideals of codimension \( n \) in \( \mathbb{C}[x, y] \) and thus appear in localization integrals on the Hilbert scheme of points in the plane. Okounkov and Pandharipande [13] describe the ring structure of the equivariant quantum cohomology (genus zero GW invariants) of this Hilbert scheme in terms of a time-dependent version of the Calogero–Moser operator from integrable systems.

5. Dimers

Dimers are a much studied classical subject in statistical mechanics and graph combinatorics. Recent spectacular progress in this subject is due to the discovery by Okounkov and collaborators of a close connection of planar dimer models with real algebraic geometry.

A dimer configuration (or perfect matching) on a bipartite graph \( G \) is a subset of the set of edges of \( G \) meeting every vertex exactly once. For example if \( G \) is a square grid we may visualize a dimer configuration as a tiling of a checkerboard by dominoes. In statistical mechanics one assigns positive weights (Boltzmann weights) to edges of \( G \) and defines the weight of a dimer configuration as the product of the weights of its edges. The basic tool is the Kasteleyn matrix of \( G \), which is up to certain signs the weighted adjacency matrix of \( G \). For finite \( G \) Kasteleyn proved
that the partition function (i.e., the sum of the weights of all dimer configurations) is the absolute value of the determinant of the Kasteleyn matrix.

Kenyon, Okounkov and Sheffield consider a doubly periodic bipartite graph $G$ embedded in the plane with doubly periodic weights. For each natural number $n$ one then has a probability measure on dimer configurations on $G_n = G/n\mathbb{Z}^2$ and statistical mechanics of dimers is essentially the study of the asymptotics of these probability measures in the thermodynamic limit $n \to \infty$. One key observation is the Kasteleyn matrix on $G_1$ can be twisted by a character $(z, w) \in (\mathbb{C}^\times)^2$ of $\mathbb{Z}^2$ and thus one defines the spectral curve as the zero set $P(z, w) = 0$ of the determinant of the twisted Kasteleyn matrix $P(z, w) = \det K(z, w)$. This determinant is a polynomial in $z^{\pm1}, w^{\pm1}$ with real coefficients and thus defines a real plane curve.

The main observation of Kenyon, Okounkov and Sheffield is that the spectral curve belongs to the very special class of (simple) Harnack curves, which were studied in the 19th century and have reappeared recently in real algebraic geometry. Kenyon, Okounkov and Sheffield show that in the thermodynamic limit, three different phases (called gaseous, liquid and frozen) arise. These phases are characterized by qualitatively different long-distance behaviour of pair correlation functions. One can see these phases by varying two real parameters $(B_1, B_2)$ (the “magnetic field”) in the weights, so that the spectral curve varies by rescaling the variables. The regions in the $(B_1, B_2)$-plane corresponding to different phases are described in terms of the amoeba of the spectral curve, namely the image of the curve by the map $\text{Log}: (z, w) \mapsto (\log |z|, \log |w|)$. The amoeba of a curve is a closed subset of the plane which looks a bit like the microorganism with the same name. The amoeba itself corresponds to the liquid phase, the bounded components of its complement to the gaseous phase and the unbounded components to the frozen phase. This insight has a lot of consequences for the statistics of dimer models and lead Okounkov and collaborators to beautiful results on interfaces with various boundary conditions.

Such a precise and complete description of phase diagrams and shapes of interfaces is unprecedented in statistical mechanics.

6. Random surfaces

One useful interpretation of dimers is as models for random surfaces in three-dimensional space. In the simplest case one considers a model for a melting or dissolving cubic crystal in which at a corner some atoms are missing (Fig. 1).
Viewing the corner from the (1,1,1) direction one sees a tiling of the plane by 60° rhombi, which is the same as a dimer configuration on a honeycomb lattice (each tile covers one dimer of the dimer configuration). In this simple model one gives the same probability for every configuration with given missing volume. If one lets the size of the cubes go to zero keeping the missing volume fixed, the probability measure concentrates on an a surface, the limit shape. More generally, every planar dimer model can be rephrased as a random surface model and limit shapes for more general crystal corner geometries can be defined. Kenyon, Okounkov and Sheffield show that the limit shape is given by the graph of (minus) the Ronkin function

\[ R(x,y) = (2\pi i)^{-2} \int_{|z|=|w|=1} \log(P(e^{x}z,e^{y}w))dzdw/zw \]

of the spectral curve (in the case of the honeycomb lattice with equal weights, \( P(z,w) = z + w + 1 \)). This function is affine on the complement of the amoeba and strictly convex on the amoeba. So the connected components of the complement of the amoeba are the projections of the facets of the melting crystal.

In addition to this surprising connection with real algebraic geometry, random surfaces of this type are essential in the GW/DT correspondence, see Section 3, as they arise in localization integrals for DT invariants of toric varieties.

7. The moduli space of Harnack curves

The notions used by Okounkov and collaborators in their study of dimer models arose in an independent recent development in real algebraic geometry. Their result brings a new probabilistic point of view in this classical subject.

In real algebraic geometry, unsurmountable difficulties already appear when one consider curves. The basic open question is the first part of Hilbert’s 16th problem: what are the possible topological types of a smooth curve in the plane given by a polynomial equation \( P(z,w) = 0 \) of degree \( d \)? Topological types up to degree 7 are known but very few general results are available. In a recent development in real algebraic geometry in the context of toric varieties the class of Harnack curves plays an important role and can be characterized in many equivalent way. In one definition, due to Mikhalkin, a Harnack curve is a curve such that the map to its amoeba is 2:1 over the interior, except at possible nodal points; equivalently, by a theorem of Mikhalkin and Rullgård, a Harnack curve is a curve whose amoeba has area equal to the area of the Newton polygon of the polynomial \( P \). These equivalent properties determine the topological type completely.

Kenyon and Okounkov prove [4] that every Harnack curve is the spectral curve of some dimer model. They obtain an explicit parametrization of the moduli space of Harnack curves with fixed Newton polygon by weights of dimer models, and deduce in particular that the moduli spaces are connected.

8. Concluding remarks

Andrei Okounkov is a highly creative mathematician with both an exceptional breadth and a sense of unity of mathematics, allowing him to use and develop, with perfect ease, techniques and ideas from all branches of mathematics to reach his research objectives. His results not only settle important questions and open new avenues of research in several fields of mathematics, but they have the distinctive feature of mathematics of the very best quality: they give simple complete answers to important natural questions, they reveal hidden structures and new connections
between mathematical objects and they involve new ideas and techniques with wide applicability.

Moreover, in addition to obtaining several results of this quality representing significant progress in different fields, Okounkov is able to create the ground, made of visions, intuitive ideas and techniques, where new mathematics appears. A striking example for this concerns the relation to physics: many important developments in mathematics of the last few decades have been inspired by high energy physics, whose intuition is based on notions often inaccessible to mathematics. Okounkov’s way of proceeding is to develop a mathematical intuition alternative to the intuition of high energy physics, allowing him and his collaborators to go beyond the mere verification of predictions of physicists. Thus, for example, in approaching the topological vertex of string theory, instead of stacks of D-branes and low energy effective actions we find mathematically more familiar notions such as localization and asymptotics of probability measures. As a consequence, the scope of Okounkov’s research programme goes beyond the context suggested by physics: for example the Maulik–Nekrasov–Okounkov–Pandharipande conjecture is formulated (and proved in many cases) in a setting which is much more general than the Calabi–Yau case arising in string theory.

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