Tight Beltrami fields with symmetry.

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Abstract

Let $M$ be a compact orientable Seifered fibered 3-manifold without a boundary, and $\alpha$ an $S^1$-invariant contact form on $M$. In a suitable adapted Riemannian metric to $\alpha$, we provide a bound for the volume $\text{Vol}(M)$ and the curvature, which implies the universal tightness of the contact structure $\xi = \ker \alpha$.

keywords: contact structures, Beltrami fields, curl eigenfields, adapted metrics, nodal sets, characteristic hypersurface, dividing sets.

1 Introduction.

Recall that a contact structure $\xi$ on a 3-manifold $M$ is a fully “nonintegrable” subbundle of the tangent bundle of $M$. If $\xi$ is defined by a global 1-form $\alpha$, namely $\xi = \ker \alpha$, the nonintegrability condition for $\xi$ can be conveniently expressed as

$$\alpha \wedge d\alpha \neq 0.$$  

The contact structure $\xi$ is called overtwisted, if there exists an embedded disk $D \subset M$, such that $T_pD = \xi_p$ along $\partial D$, $\xi$ is called tight if it is not overtwisted. If all covers of a contact structure are tight, we call it universally tight. A contact structure which is not universally tight is either overtwisted or virtually overtwisted (i.e. its lift to a covering space is overtwisted).

As shown in [11], tight and overtwisted structures constitute two different types of isotopy classes among all contact structures in dimension 3. Recall that two contact structures $\xi_0$ and $\xi_1$ are isotopic if and only if there exists a homotopy $\xi_t$, $0 \leq t \leq 1$, such that each $\xi_t$ is a contact structure. Clearly, the equivalence up to isotopy is stronger than the equivalence up to homotopy of plane fields. Overtwisted structures are rather “flexible”, and in each homotopy class of plane fields there exists an overtwisted representative (c.f. [11]). On the other hand, tight contact structures are “rare”, for instance, on manifolds $S^3$, $\mathbb{R}P^3$.

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there exists the unique, up to isotopy, tight contact structure. In addition, there exist 3-manifolds which admit no tight contact structures (c.f. [16]).

A more geometric perspective on the contact structures in dimension 3 has been initiated by Chern and Hamilton [9]. They showed that contact forms can be equipped with an adapted Riemannian metric $g_\alpha$ (see Section 2). Their main theorem states that an arbitrary adapted metric can be conformally deformed to a metric of constant Webster curvature [9]. However, the questions of relations between the Riemannian geometry and the tight/overtwisted dichotomy, in dimension 3, have not received much attention in the literature [4]. One may indicate work in [3, 18], on normal CR-structures, where tightness of Sasakian manifolds is concluded, and also [13, 14, 15, 20], where the hydrodynamical perspectives on contact geometry, related to the Riemannian geometry, have been studied. The principal motivation behind this paper can be formulated as follows.

Find conditions on geometric parameters, such as volume, curvature and eigenvalues, for a Riemannian metric adapted to a contact structure $\xi$, that imply tightness of $\xi$.

We achieve this geometric tightness for a certain class of $S^1$-invariant contact structures on Seifered fibered 3-manifolds (theorems in Section 6). This result may be viewed as a translation of Giroux’s classification of $S^1$-invariant contact structures on $S^1$-bundles, and their quotients, into a condition on the volume, curvature and eigenvalues of $M$. In a nutshell, these theorems describe lower bounds for the volume of $M$ in terms of geometric parameters of $M$ and the magnitude $\|\alpha\|$ of a contact form $\alpha$, which defines overtwisted or virtually overtwisted contact structure $\xi$ on $M$.

In Section 7 we present concluding remarks and a slightly different perspective on the problem of geometric tightness, which is motivated by the work of Etnyre and Ghrist in [13, 14, 15]. We also indicate geometric conditions that imply tightness of certain contact structures on various products and circle bundles, and obtain the well known result [28] about universal tightness of $\xi_n = \ker\{\cos(nz)dx + \sin(nz)dy\}$, $n \in \mathbb{Z}$, on $T^3$.

2 Contact structures and adapted metrics.

In this section we show that, given an $S^1$-invariant contact form $\alpha$, we may adapt a suitable Riemannian metric to $\alpha$ with a Killing vector field tangent to the $S^1$-fibers preserving $\alpha$. First, we recall basic results about adapted metrics.

**Definition 2.1.** We say that the Riemannian metric $g_\alpha$ is adapted to $\alpha$, provided

$$\ast d\alpha = \mu \alpha, \quad \mu \neq 0, \quad \mu \in C^\infty(M),$$

where $\ast$ is the Hodge star operator in $g_\alpha$ (it is shown [7] that one may additionally prescribe $\|\alpha\| = 1$, and $\mu = 2$).

Here and throughout the paper, we work in the smooth category of closed 3-dimensional Riemannian manifolds. We denote by $DT$ the covariant derivative of a tensor field $T$, in the
Levi-Civita connection of $g_\alpha$, and $\nabla f$ the gradient vector field of a scalar function $f$. We often use the notation $\langle \cdot, \cdot \rangle$ for the inner product $g_\alpha(\cdot, \cdot)$. By $\xi$ we denote an orientable contact structure defined by a global 1-form $\alpha$, i.e. the contact form on a Riemannian 3-dimensional manifold $(M, g_\alpha)$. Every such $\alpha$ admits the unique, transverse to $\xi$, vector field $X_\alpha$ called the Reeb-field of $\alpha$ satisfying,

$$\alpha(X_\alpha) = 1, \quad \iota(X_\alpha) d\alpha = 0, \quad (3)$$

where $\iota(X_\alpha)$ is the contraction of $d\alpha$ by $X_\alpha$. Notice that $X_\alpha$ defines a projection $\pi_\alpha : TM \mapsto \xi$ on $\xi = \ker \alpha$ via the formula

$$\pi_\alpha(X) = X - \alpha(X) X_\alpha. \quad (4)$$

If $\|\alpha\| \neq 0$, and $\alpha$ satisfies (2) one quickly verifies that $\xi = \ker \alpha$ defines a contact structure. Indeed, the nonintegrability condition (1) holds

$$\alpha \wedge d\alpha = \mu \alpha \wedge \ast \alpha = \mu \|\alpha\|^2 \ast 1 \neq 0. \quad (5)$$

When we allow $\|X_\alpha\|$ to be non constant, one expects more “flexibility” in metrics adapted to $\alpha$. Following [6, Example 3.7 on p. 93] we may argue that in the class of analytic metrics Equation (2) can always be locally solved for a nonvanishing 1-form $\alpha$, and an arbitrary choice of a constant $\mu$. As a result, the hyperbolic metric can be locally adapted to a contact structure (recall that all contact structures are locally equivalent up to a diffeomorphism [19]). In contrast, if the condition $\|X_\alpha\| = 1$ is imposed, the hyperbolic metric cannot be an adapted metric (c.f. [4]).

**Remark 2.2.** On a manifold equipped with an adapted Riemannian metric $g_\alpha$, the dual vector field $v$ to a contact form $\alpha$ satisfies the Euler equations for the inviscid incompressible fluid flow (see [14]). Such solutions of the Euler equations are known as Beltrami fields. Clearly, if $\mu$ is constant, Beltrami fields are just the eigenfields of the curl operator $\ast d$ (c.f. [20]).

The following lemma provides a useful characterization of adapted metrics.

**Lemma 2.3.** Given a contact form $\alpha$, a local choice of a metric $g_\alpha$ adapted to $\alpha$ is equivalent to a choice of a local orthonormal frame $\{e_1, e_2, e_3\}$ satisfying

(i) $e_1 = v X_\alpha$, where $X_\alpha$ is a Reeb field of $\alpha$ and $v$ a positive function,

(ii) $\xi = \text{span}\{e_2, e_3\}$.

(one may also define the associated almost complex structure $J : \xi \mapsto \xi$ on $\xi$ in terms of the frame as follows: $Je_2 = -e_3$, $Je_3 = e_2$.)

**Proof.** Given an adapted metric $g_\alpha$, we have the unique dual vector field $X$, such that $\alpha(\cdot, \cdot) = g_\alpha(X, \cdot)$. We define $e_1 = X/\|X\|$, and choose an arbitrary frame on $\xi$ satisfying (ii). For the dual coframe $\{\eta_i\}$ to $\{e_i\}$, Equation (2) implies

$$\iota(X) d\alpha = \iota(X) \mu \ast \alpha = \iota(e_1) \mu \|X\| \ast \eta_1 \quad = \iota(e_1) \mu \|X\| \eta_2 \wedge \eta_3 = 0.$$
By (3) we conclude that $e_1 = v X_\alpha$, for some function $v \neq 0$.

Conversely, let $\{e_i\}$ be an adapted frame to $\alpha$ satisfying (i) and (ii), and $\{\eta_i\}$ the coframe. We must show that the metric $g_\alpha = \sum_i \eta_i^2$ is adapted to $\alpha$. By (ii), $e_1 \perp \xi$ thus $\alpha(\cdot) = g(X, \cdot)$ for $X = h e_1 = h v X_\alpha$ and $\eta_1 = w \alpha$, where $v, h, w$ are positive functions. Relations among $v, h, w$ follow from the identities:

$$
\alpha(X_\alpha) = g(X, X_\alpha) = h v \|X_\alpha\|^2 = 1, \\
e_1 = v X_\alpha = \frac{X_\alpha}{\|X_\alpha\|}, \\
\eta_1(\cdot) = g(v X_\alpha, \cdot) = \frac{1}{h} g(X, \cdot) = \frac{1}{h} \alpha(\cdot).
$$

Thus,

$$
v = w = h = \frac{1}{\|X_\alpha\|}.
$$

Let $a, b, c$ be the coefficients of $d\alpha$ in $\{\eta_i\}$, because

$$
\iota(e_1)d\alpha = v \iota(X_\alpha)d\alpha = 0
$$

we obtain

$$
\iota(e_1)d\alpha = \iota(e_1) [a\eta_1 \wedge \eta_2 + b\eta_1 \wedge \eta_3 + c\eta_2 \wedge \eta_3] = a\eta_2 + b\eta_3 = 0.
$$

Thus $a = b = 0$, and

$$
d\alpha = c \eta_2 \wedge \eta_3 = c \iota \eta_1 = c v \iota \alpha.
$$

Equation (2) follows by defining $\mu = c v$. Because $\alpha \wedge d\alpha \neq 0$, we conclude that $\mu \neq 0$. \( \square \)

**Lemma 2.4.** Let $\{e_i\}$ be the frame defined locally as in Lemma 2.3 and $\{\eta_i\}$ the coframe. We have the following formula for the adapted metric $g_\alpha$:

$$
g_\alpha(X, Y) = \sum_i \eta_i^2(X, Y) = \frac{1}{v^2} \alpha(X)\alpha(Y) + \frac{2v}{\mu} d\alpha(X, J\pi_\alpha Y), \quad (6)
$$

for any $X, Y$, where

$$
\mu = v d\alpha(e_2, e_3) = v \alpha([e_2, e_3]), \text{ and } v = \|X_\alpha\|.
$$
Proof.

\[ d\alpha(\cdot, J\cdot) = \frac{\mu}{v} \eta_2 \wedge \eta_3(\cdot, J\cdot) \]

\[ = \frac{\mu}{2v} [\eta_2(\cdot) \otimes \eta_3(J\cdot) - \eta_3(\cdot) \otimes \eta_2(J\cdot)] \]

\[ = \frac{\mu}{2v} (\eta_2^2(\cdot, \cdot) + \eta_3^2(\cdot, \cdot)) \]

(the last equality follows from (iii) in Lemma \[23\]). But \( \eta_1 = \frac{1}{v} \alpha \), and

\[ g(\cdot, \cdot) = \sum_i \eta_i^2(\cdot, \cdot) \]

\[ = \frac{1}{v^2} \alpha^2(\cdot, \cdot) + \eta_2^2(\cdot, \cdot) + \eta_3^2(\cdot, \cdot) \]

\[ = \frac{1}{v^2} \alpha^2(\cdot, \cdot) + \frac{2}{\mu} \alpha(\cdot, J\pi_\alpha \cdot) \].

As a corollary we conclude a global existence of adapted metrics \[9\].

**Corollary 2.5.** Given a contact form \( \alpha \), one may always adapt the Riemannian metric \( g_\alpha \) to \( \alpha \), such that Equation (2) is satisfied on \( (M, g_\alpha) \).

**Proof.** Indeed, by Formula (6) for \( g_\alpha \), it suffices to choose a global almost complex structure \( J : \xi \mapsto \xi \), \( \xi = \ker \alpha \), and a vector field \( e_1 = vX_\alpha \), where \( v \) is a positive function. The only issue is to define \( J \) globally, but this may be achieved via an arbitrary choice of a metric \( g_\xi \) on \( \xi \), and defining \( J \) by the \( \pi \)-rotation in \((\xi, g_\xi)\).

These results lead to the following,

**Proposition 2.6.** Suppose that \( M \) is a Seifert fibered 3-manifold, and \( \alpha \) an invariant contact 1-form on \( M \) (i.e. invariant under the action of a nonsingular vector field \( X \) tangent to \( S^1 \)-fibers of \( M \)). Then,

(iv) There exists an adapted Riemannian metric \( g_\alpha \) to \( \alpha \), such that \( X \) is a Killing vector field in \( g_\alpha \).

(v) Moreover, there is an adapted metric \( g'_\alpha \), conformal to \( g_\alpha \), such that \( X \) is a unit Killing vector field in \( g'_\alpha \).

**Proof.** Assume that \( \alpha \) defines a positive contact structure i.e. \( \alpha \wedge d\alpha > 0 \) (if \( \alpha \) defines a negative contact structure the proof is analogous). Recall the observation from \[32\, Proposition 1.3 on p. 336\] stating existence of a Riemannian metric \( g \) on \( M \), such that \( X \) is the unit Killing vector field for \( g \). Because \( X \) preserves \( \alpha \) and \( \xi \), the flow \( \varphi^t_X \) of \( X \) maps \( \xi_p \)
isometrically to $\xi_{\phi^t(p)}$. Consequently, $\mathcal{L}_X g_\xi = 0$, where $g_\xi$ denotes the restriction of $g$ to $\xi$. By positivity of $\alpha$ and $\mathcal{L}_X d\alpha = 0$, we have a positive function $\nu$ such that

$$2\nu d\alpha(\pi_\alpha \cdot, J\pi_\alpha \cdot) = g_\xi(\pi_\alpha \cdot, \pi_\alpha \cdot),$$

where $J$ is a rotation by $\frac{\pi}{2}$ in $g_\xi$, and $\pi_\alpha$ is the projection defined in (4). Notice that $\mathcal{L}_X \nu = 0$, therefore we may define $g_\alpha$ by Formula (6). The conclusion (iv) now follows from the tensor product formula for the Lie derivative.

In order to prove (v), consider $h^2 = g_\alpha(X, X)$. If $h = 1$, we are done, if $h \neq 1$ define $g'_\alpha = \frac{1}{h^2} g_\alpha$. We verify that $g'_\alpha$ is adapted by plugging into (6):

$$g'_\alpha(X, Y) = \frac{1}{h^2} g_\alpha(X, Y)$$

$$= \frac{1}{h^2} \nu(\alpha(X)\alpha(Y) + \frac{2}{h^3} \frac{\nu}{\mu} d\alpha(X, J\pi_\alpha Y).$$

This calculation confirms that $g'_\alpha$ is adapted with $\mu' = h^3 \mu$. Moreover, $X$ is a unit vector field in $g'_\alpha$ and, because $\mathcal{L}_X h = 0$, $X$ has to be a Killing vector field in $g'_\alpha$.

**Question 2.7.** Can we find $g_\alpha$, which admits both a unit Killing vector field $X$ tangent to the fibers of $M$ and $\mu$ as a constant function?

### 3 Characteristic hypersurface as a nodal set.

Among known techniques of contact topology, which allow us to detect a contact isotopy type of a contact structure $\xi$, is the technique of convex surfaces and dividing curves, originally introduced by Giroux in [23, 24]. We adapt Giroux’s technique, which is crucial in our further investigation. First, we briefly review its basic notions.

**Definition 3.1.** Recall that a vector field $X$ on $M$ is called the contact vector field for $\xi$ if and only if its flow preserves the plane distribution $\xi$. The set of tangencies $\Gamma_X = \{p \in M : X_p \in \xi_p\}$ of $X$ and $\xi$ is called the characteristic surface of $X$ and is denoted by $\Gamma_X$. An embedded surface $\Sigma$ in $M$ is called the convex surface if and only if there exists a transverse contact vector field $X$ to $\Sigma$. The set of curves $\Gamma = \Gamma_X \cap \Sigma$ is called the dividing set on $\Sigma$.

Another way to express the condition for $X$ to be a contact vector field is the following equation for the contact form $\alpha$:

$$\mathcal{L}_X \alpha = h \alpha,$$

for some $h \in C^\infty(M)$.

The special case occurs when $h = 0$ and the contact field $X$ also preserves the contact form $\alpha$, we consider this case in our further investigation. Also, notice that the characteristic surface $\Gamma_X$ is a zero set of the function $f = \alpha(X)$, i.e. $\Gamma_X = f^{-1}(0)$. This function is commonly known as the contact hamiltonian (c.f. [19]).
Classification of contact structures on $S^1$-bundles over a surface, has been partially achieved by Giroux in [24], and completed in full generality by Honda [25, 26]. As it is presented in the following theorem, $S^1$-invariant contact structures on $S^1$-bundles are fully characterized by the topology of the dividing set $\Gamma$ on the base (i.e. projected $S^1$-invariant characteristic surface $\Gamma_{S^1}$) and the Euler number of the bundle.

**Theorem 3.2 ([24])**. Let $\xi$ be an $S^1$-invariant contact structure on the principal $S^1$-bundle $\pi : P \to \Sigma$, where $\Sigma$ is an orientable surface. Let $\Gamma = \pi(\Gamma_{S^1})$ be a projection of the characteristics surface $\Gamma_{S^1}$ onto $\Sigma$. Denote by $e(P)$ the Euler number of $P$.

(a) If $\xi$ is tight and one of the connected components of $\Sigma/\Gamma$ bounds a disc, then $\Gamma$ has to be a single circle and $e(P)$ must satisfy

$$
\begin{cases}
  e(P) > 0, & \text{if } \Sigma \neq S^2 \\
  e(P) \geq 0, & \text{if } \Sigma = S^2.
\end{cases}
$$

(b) For $\xi$ to be universally tight it is necessary and sufficient that one of the following holds

- (b.1) for $\Sigma \neq S^2$, none of the connected components of $\Sigma/\Gamma$ is a disc,
- (b.2) for $\Sigma = S^2$, $e(P) < 0$ and $\Gamma = \emptyset$,
- (b.3) for $\Sigma = S^2$, $e(P) \geq 0$ and $\Gamma$ is connected.

**Theorem 3.3 ([24])**. Let $\Sigma$ be a convex surface of nonzero genus in the contact manifold $(M, \xi)$. Let $X$ be the contact vector field transverse to $\Sigma$, and $\Gamma_{\Sigma} = \Gamma_X \cap \Sigma$ the dividing set of $\Sigma$. Then $\xi$ is tight, in a tubular neighborhood of $\Sigma$, if and only if none of the components of $\Sigma/\Gamma$ is a disc.

These results indicate that the topology of characteristic surfaces is an indicator of tightness/overtwistedness both on a local and global level. In the remainder of this section the goal is to interpret $\Gamma_X$ in the Riemannian geometric setting of adapted metrics. The following result provides such a characterization (compare to [30, Lemma 2.7]).

**Theorem 3.4.** Assume that $X$ is a global contact vector field on the Riemannian manifold $(M, g_0)$ which preserves a contact form $\alpha$ satisfying (2). Let $f = \alpha(X)$, denote by $\{e_1 = \frac{X}{\|X\|}, e_2, e_3\}$ a local adapted orthonormal frame and by $\{\eta_1, \eta_2, \eta_3\}$ the dual coframe. Then, the coefficients of $\alpha = a_k \eta_k = \frac{4}{v}\eta_1 + a_2\eta_2 + a_3\eta_3$ locally satisfy the first order system

$$
\begin{cases}
  D_1 f = 0, \\
  D_2 f = -\mu v a_3, \\
  D_3 f = \mu v a_2,
\end{cases}
$$

where $v = \|X\|$. Furthermore, $f$ satisfies globally the subelliptic equation:

$$
\Delta_E f - (\nabla \ln h, \nabla f) + \mu(\mathcal{E} - \mu)f = 0,
$$
where \( E = (\ast d\eta_1)(e_1), \) \( h = 1/\mu v, \) and \( \Delta_E \) is the Laplacian on the subbundle \( E = \ker \eta_1. \) One may express Equation (8) in terms of the global Laplacian \( \Delta_M, \) on \( M, \) as follows

\[
\Delta_M f + \frac{1}{v^2} \nabla^2 f(X, X) - \langle \nabla \ln \left( \frac{1}{\mu v} \right), \nabla f \rangle + \mu (E - \mu) f = 0. \tag{9}
\]

Proof. Recall formulas [27] for the Hessian and the Laplacian in a frame \( \{e_i\}:
\[
\nabla^2 f = \nabla^2 f(e_i, e_j) = D_i D_j f + \sum_k D_k f \omega^j_{ik}, \tag{10}
\]
and the following definitions (summation is assumed over the repeating indices)

\[
D_i e_j = \omega^k_{ij} e_k, \quad \omega^k_i = D_i \eta_k = -\omega^k_{ij} \eta_j, \quad \omega^j_{ik} = -\omega^j_{ki},
\]

\[
d\alpha = \eta_i \wedge D_i \alpha,
\]

\[
D\alpha = da_k \otimes \eta_k + a_k D\eta_k = da_k \otimes \eta_k - a_k \omega^k_j \otimes \eta_j,
\]

\[
\Delta_M = -D_i D_i + \omega^j_{ii} D_j,
\]

where \( D_i \equiv D_{e_i}. \) The proof of Theorem 3.4 is a calculation in the adapted coframe \( \{\eta_i\}. \) Using Cartan’s formula and Equation (2) we obtain (for \( v = \|X\|:\)

\[
0 = \mathcal{L}_X \alpha = \iota(X)d\alpha + df
= \mu \iota(X) \ast \alpha + D_i f \eta_i,
\]

and

\[
-D_i f \eta_i = \mu v \iota(X_1) \ast \alpha
- D_1 f \eta_1 - D_2 f \eta_2 - D_3 f \eta_3 = \mu v (a_2 \eta_3 + a_3 \eta_2).
\]

These expressions lead to the following equations

\[
\begin{cases}
D_1 f = 0, \\
D_2 f = -\mu v a_3, \\
D_3 f = \mu v a_2,
\end{cases} \tag{11}
\]

and

\[
d\alpha = \sum_{i < j} a_{ij} \eta_i \wedge \eta_j = \eta_i \wedge D_i \alpha
= \eta_i \wedge (D_1 a_k \eta_k - a_k \omega^k_{ij} \eta_j)
= D_i a_k \eta_i \wedge \eta_k - a_k \omega^k_{ij} \eta_i \wedge \eta_j.
\]
Collecting terms in front of $\eta_1 = \eta_2 \wedge \eta_3$, we have

$$a_{23} = D_2 a_3 - D_3 a_2 + a_k (\omega^k_{32} - \omega^k_{23})$$
$$= D_2 a_3 - D_3 a_2 + a_1 (\omega^1_{32} - \omega^1_{23}) - a_2 \omega^2_{23} + a_3 \omega^3_{32}.$$ 

Applying Equations (2) and (11):

$$a_{23} = \frac{\mu}{v} f = -D_2 (\frac{1}{\mu v} D_2 f) - D_3 (\frac{1}{\mu v} D_3 f) + \frac{f}{v} (\omega^1_{32} - \omega^1_{23}) - \frac{1}{\mu v} D_3 f \omega^2_{32} - \frac{1}{\mu v} D_2 f \omega^3_{32},$$

and distributing terms we obtain (for $h = 1/(\mu v)$)

$$\mu^2 h f = h (-D_2 D_2 f - D_3 D_3 f + \omega^2_{32} D_3 f + \omega^2_{33} D_2 f)$$

$$+ \mu h f (\omega^1_{32} - \omega^1_{23}) - D_2 h D_2 f - D_3 h D_3 f.$$ 

Dividing the above equation by $h$ yields

$$(\Delta_E + L + \nu) f = 0, \quad (12)$$

where

$$\Delta_E = -D_2 D_2 - D_3 D_3 + \omega^2_{32} D_3 + \omega^2_{33} D_2,$$

$$L = -\frac{1}{h} (D_2 h D_2 + D_3 h D_3) = -\langle \nabla \ln h, \nabla \cdot \rangle$$

$$\nu = \mu (\omega^1_{32} - \omega^1_{23} - \mu) = \mu (\mathcal{E} - \mu),$$

$$\mathcal{E} = i(e_1) \ast d\eta_1.$$ 

Applying $D_1 f = 0$ and $D_1 e_1 = \omega^k_{11} e_k$, in Equation (11), we express (12) in terms of the Laplacian $\Delta_M$ on $M$:

$$\Delta_E f = \Delta_M f - \langle \nabla f, D_1 e_1 \rangle$$

$$= \Delta_M f + \nabla^2 f (e_1, e_1) = \Delta_M f + \frac{1}{v^2} \nabla^2 f (X, X),$$

where in the second equation we noticed that

$$\langle \nabla f, D_1 e_1 \rangle + \langle D_1 \nabla f, e_1 \rangle = D_1 \langle \nabla f, e_1 \rangle = 0,$$

and

$$\nabla^2 f (e_1, e_1) = -\langle \nabla f, D_1 e_1 \rangle.$$ 



\[ \square \]

**Corollary 3.5.** If the contact field $X$ is a unit vector field in the metric and $\mu \equiv \text{const}$, Equation (8) becomes

$$\Delta_E f + \mu (\mathcal{E} - \mu) f = 0. \quad (13)$$

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In the remaining part of this section we work under the assumptions of Theorem 3.4.

**Theorem 3.6.** A characteristic surface $\Gamma_X = f^{-1}(0)$ is the zero set (also known as the nodal set) of the solution $f$ to the subelliptic equation (9), and consists of a finite disjoint union of smooth 2-tori: $\Gamma_X \cong \bigsqcup_i T^2_i$, $T^2_i \cong S^1 \times S^1$.

**Proof.** Since $f$ is invariant, under a nonsingular vector field $X$, regular level sets of $f$ must be 2-dimensional tori. Clearly, $\Gamma_X$ cannot contain a singular point $p$, since it would imply $\alpha(p) = 0$ which contradicts the contact condition (1). Now, the claim follows from Theorem 3.4. 

**Remark 3.7.** Equation (8) can be modified to become an elliptic equation by adding the term $D_1 D_1 f$. It follows, from the methodology in [2], that given any solution $f$ to (8), or (13), the nodal set $N = f^{-1}(0)$ is a union $N = N_{\text{sing}} \cup N_{\text{reg}}$ of the singular part $N_{\text{sing}}$ of codimension at least 2 and the regular part $N_{\text{reg}}$ which is a codimension 1 submanifold. We remark that, in general, the nodal set of a solution to an elliptic partial differential equation can be very irregular. For example, any closed subset $A \subset \mathbb{R}^n$ is a nodal set of a solution to some elliptic equation, [2].

The following proposition is a standard result from the elliptic theory [21] and Aronszajn’s Unique Continuation Principle (see [1, p. 235]):

**Proposition 3.8.** Equation (9) admits nontrivial solutions, if and only if

$$\mu(\mu - \mathcal{E}) \geq 0 \quad \text{on } M. \quad (14)$$

Moreover, if the above inequality is strict then $f$ cannot be a locally constant function.

**Corollary 3.9.** If $\mu(\mu - \mathcal{E}) > 0$ then $M$ is fillable almost everywhere by 2-tori.

### 4 Geometry of the dividing set.

Our strategy for this part is to investigate further how topology of dividing sets is “controlled” by the geometry of the underlying manifold. These considerations are essential for the proof of the main theorem, where Equation (8) “projects” onto an orientable surface $\Sigma$ and reduces to the eigenequation

$$\Delta_\Sigma f = \lambda f, \quad f \in C^\infty(\Sigma), \quad \lambda \in \mathbb{R}_+. \quad (15)$$

We seek conditions on $g_\Sigma$ which imply that $f^{-1}(0)$ is a homotopically essential collection of curves. In context of Theorem 3.2 and 3.3 these conditions determine, under appropriate assumptions, tightness of the underlying contact structure. The next result is inspired by [33, Lemma 11].

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Proposition 4.1. Let $f$ be a solution to (15) and $\Omega$ a domain in $\Sigma \setminus f^{-1}(0)$. Denote by $K^\pm$ the positive (negative) part of the scalar curvature $K$ of $(\Sigma, g_\Sigma)$. If $\Omega$ is diffeomorphic to a 2-disc $D^2$ with smooth boundary then

$$4\pi - 2 \int_{\Omega} K^+ \leq \lambda \text{Vol}(\Omega).$$

Proof. The Euler characteristic $\chi(\Sigma)$ of a closed orientable surface $\Sigma$ can be computed via the celebrated Gauss-Bonnet formula

$$2\pi \chi(\Sigma) = \int_{\Sigma} K + \int_{\partial \Sigma} \kappa_\nu,$$  \hspace{1cm} (16)

where $K$ is the scalar curvature, and $\nu$ the unit outward normal along $\partial \Sigma$. Given a smooth function $f$, on $\Sigma$, every regular level set $N = f^{-1}(c)$ is a codimension 1 submanifold in $\Sigma$. In [33] the following formula for the mean curvature of $N$ has been obtained

$$H_\nu = \frac{\Delta_{\Sigma} f}{\|\nabla f\|^2} + \frac{1}{2} \langle \nabla \ln \|\nabla f\|^2, \nu \rangle,$$  \hspace{1cm} (17)

where $\nu = \frac{\nabla f}{\|\nabla f\|}$ is pointing towards $\{f > c\}$, $\Delta_{\Sigma}$ is the scalar Laplacian on $N$, and $H_\nu$ the mean curvature in the $\nu$ direction (c.f. [8]).

By Equation (15) and (17), we obtain a formula for the geodesic curvature of $\partial \Omega$:

$$\kappa_\nu = H_\nu = \frac{\Delta_{\Sigma} f}{\|\nabla f\|^2} + \frac{1}{2} \langle \nabla \ln \|\nabla f\|^2, \nu \rangle = \frac{1}{2} \langle \nabla \ln \|\nabla f\|^2, \nu \rangle,$$

where $\nu = \frac{\nabla f}{\|\nabla f\|}$ points towards $\{f > 0\}$, and because $f \mid_{\partial \Omega} = 0$, the last equality is a consequence of (15). Assume that $f > 0$ on $\Omega$, so that $-\nu = \nu_{\text{out}}$ points outwards (it can be done without loss of generality since both $f$ and $-f$ satisfy (15)). Define

$$q = (\|\nabla f\|^2 + \frac{\lambda}{2} f^2).$$

Clearly, $q \mid_{\partial \Omega} = \|\nabla f\|^2$ and as a result we have $\kappa_\nu = \frac{1}{2} \langle \nabla \ln q, \nu \rangle$. Dong’s theorem [10] implies the following estimate for the function $q$:

$$\Delta \ln q \leq \lambda - 2 K^-, \quad K^- = \min(K, 0).$$  \hspace{1cm} (18)

By Green’s formula (see e.g. [8] p. 7), we obtain

$$\frac{1}{2} \int_{\Omega} \text{div} \nabla \ln q = \int_{\partial \Omega} \frac{1}{2} \langle \nabla \ln q, \nu_{\text{out}} \rangle = - \int_{\partial \Omega} \frac{1}{2} \langle \nabla \ln q, \nu \rangle,$$

$$\frac{1}{2} \int_{\Omega} \Delta \ln q = \int_{\partial \Omega} \kappa_\nu.$$
(Note that $\Delta = -\text{div} \circ \nabla$, and the orientation $e_1$ of $\partial \Omega$ is chosen, so that $\{\nu_{\text{out}}, e_1\}$ agrees with the orientation of $\Omega$.) Applying estimates (18) and (16), we derive

$$
\int_{\partial \Omega} \kappa \nu \leq \frac{\lambda}{2} \Vol(\Omega) - \int_{\Omega} K^-
$$

$$
2\pi \chi(\Omega) - (\int_{\Omega} K^+ + \int_{\Omega} K^-) \leq \frac{\lambda}{2} \Vol(\Omega) - \int_{\Omega} K^-
$$

$$
2\pi \chi(\Omega) - \int_{\Omega} K^+ \leq \frac{\lambda}{2} \Vol(\Omega).
$$

Since $\chi(D^2) = 1$, the claim follows from (16).

**Corollary 4.2.** Let $f$ satisfy Equation (15), if $\Sigma$ is a nonpositively curved surface (i.e. $K \leq 0$) of area $\Vol(\Sigma)$, then the following is a necessary condition for one of the domains in $\Sigma \setminus f^{-1}(0)$ to be a disc with smooth boundary,

$$
\frac{4\pi}{\Vol(\Sigma)} \leq \lambda.
$$

In the remaining part of this section, we focus on the case of a convex surface $\Sigma$ embedded in $(M, \xi, g_\alpha)$, $\xi = \ker \alpha$. These results are interesting in their own right, and later provide an essential ingredient in the proof of the main theorem.

**Proposition 4.3.** Assume the setup of Theorem 3.4, let $\Sigma$ be a convex surface embedded in $(M, \xi)$, $\xi = \ker \alpha$. If a contact field $X$ is orthogonal to $\Sigma$ and $K \leq 0$, the sufficient condition for tight tubular neighborhood of $\Sigma$ reads

$$
\max_{\Sigma}(\Delta \Sigma \ln \|\alpha\|) < \frac{2\pi}{\Vol(\Sigma)}.
$$

**Proof.** Since $X \perp \Sigma$, Equation (8) simplifies as

$$
\Delta_{\Sigma} f + \left(\frac{\nabla \mu v, \nabla f}{\mu v}\right) - \mu^2 f = 0.
$$

In order to show (20), we must prove: $\Delta_{E} = \Delta_{\Sigma}$ in the frame $\{e_1 = \frac{X}{\|X\|}, e_2, e_3\}$, where $\{e_2, e_3\}$ span $T\Sigma$. Equation (12) yields ($E = T\Sigma$)

$$
\Delta_{E} = -D_2 D_2 - D_3 D_3 + \omega_{22}^2 D_3 + \omega_{33}^2 D_2.
$$

Local vector fields $\{e_2, e_3\}$ are tangent to $\Sigma$ and thus the bracket $[e_2, e_3]$ satisfies: $[e_2, e_3] \in T\Sigma$. By the general formula for Christoffel symbols in the frame [27]:

$$
\omega_{ij}^k = \frac{1}{2} \{\{e_i, e_j\}, e_k\} - \{\{e_j, e_k\}, e_i\} + \{\{e_k, e_i\}, e_j\},
$$

(21)
we conclude that the formula $\Delta_{\Sigma} = -D_i D_i + \omega_{ij}^2 D_j$ implies $\Delta_{E} = \Delta_{\Sigma}$ on $\Sigma$. In addition,

$$\langle [e_2, e_3], e_1 \rangle = \eta_1([e_2, e_3]) = 0,$$
$$d\eta_1(e_2, e_3) = 0,$$
$$\mathcal{E} = (d\eta_1)(e_1) = 0.$$

Secondly, we express the middle term in (8) as follows ($h = 1/(\mu v)$):

$$-\langle \nabla \ln h, \nabla f \rangle = \langle -\frac{\mu v}{\mu v} \nabla \left( \frac{f}{\mu v} \right), \nabla f \rangle = \frac{1}{\mu v} \langle \nabla (\mu v), \nabla f \rangle,$$

which leads to Equation (20).

In the next step, we calculate the geodesic curvature of $\partial \Omega$, where $\Omega$ is a domain in $\Sigma \setminus f^{-1}(0)$, and $f = \alpha(X)$ is a solution to Equation (20). By (17)

$$\kappa_\nu = \frac{\Delta_{\Sigma} f}{\|\nabla f\|} + \frac{1}{2} \langle \nabla \ln \|\nabla f\|^2, \nu \rangle.$$

Equations (20) and (17) yield

$$\kappa_\nu = -\frac{\langle \nabla (\mu v), \nabla f \rangle}{\mu v \|\nabla f\|} + \frac{\mu^2 f}{\|\nabla f\|} + \frac{1}{\|\nabla f\|} \langle \nabla \|\nabla f\|, \nu \rangle. \quad (22)$$

Let $\alpha = a_i \eta_i$, (7) implies that $D_1 f = 0$ and

$$\|\alpha\|^2 = \sum_i a_i^2 = \left( \frac{f}{v} \right)^2 + \left( \frac{D_2 f}{\mu v} \right)^2 + \left( \frac{D_3 f}{\mu v} \right)^2,$$

$$\mu v \|\alpha\|^2 = (\mu f)^2 + \|\nabla f\|^2,$$

$$v^2 = \frac{f^2}{\|\alpha\|^2} + \frac{\|\nabla f\|^2}{(\mu \|\alpha\|)^2}.$$

Because $f \mid_{\partial \Omega} = 0$, we derive

$$\mu v \mid_{\partial \Omega} = \frac{\|\nabla f\|}{\|\alpha\|},$$

$$\nabla (\mu v) \mid_{\partial \Omega} = -\frac{1}{\|\alpha\|^2} (\nabla \|\alpha\|) \|\nabla f\| + \frac{1}{\|\alpha\|} \nabla \|\alpha\| \nabla f|,$$

and for $\nu = \frac{\nabla f}{\|\nabla f\|}$:

$$\frac{\langle \nabla (\mu v), \nabla f \rangle}{\mu v \|\nabla f\|} \mid_{\partial \Omega} = \frac{1}{\mu v} \langle \nabla \mu v, \nu \rangle = -\frac{1}{\|\alpha\|} \langle \nabla \|\alpha\|, \nu \rangle + \frac{\langle \nabla \|\nabla f\|, \nu \rangle}{\|\nabla f\|}.$$
Substituting in (22) yields
\[
\kappa_{\nu} = \frac{1}{\|\alpha\|} \langle \nabla \|\alpha\|, \nu \rangle - \frac{\langle \nabla \|\nabla f\|, \nu \rangle}{\|\nabla f\|} + \frac{\mu^2 f}{\|\nabla f\|} + \frac{\langle \nabla \|\nabla f\|, \nu \rangle}{\|\nabla f\|}
\]
\[
= \langle \nabla \ln \|\alpha\|, \nu \rangle.
\]
As before, we apply (16) and the Green’s formula to obtain
\[
2\pi \chi(\Omega) = \int_{\Omega} K + \int_{\Omega} \Delta_{\Sigma} \ln \|\alpha\|.
\]
Clearly, if \( K \leq 0 \) the expression (19), provides a sufficient condition for \( \Sigma \) to have a tight tubular neighborhood.

Therefore, if we have an orthogonal contact vector field \( X \) to an embedded convex surface \( \Sigma \), Equation (19) provides a condition for a tight tubular neighborhood of \( \Sigma \). Contrary to the method of general convex surfaces, convex surfaces admitting an orthogonal contact field, as described in Proposition 4.3, are special (see Remark 4.5). In such circumstances, \( X \) is tangent to \( \xi \) along the dividing set \( \Gamma \chi \), and the orthogonality assumption \( X \perp \Sigma \) forces \( X_{\alpha} \) to be tangent to \( \Sigma \), because \( X_{\alpha} \perp \xi \). Based on Equations (7), we conclude that the Reeb field \( X_{\alpha} \) is tangent to the dividing set \( \Gamma_{\Sigma} \), and \( \Gamma_{\Sigma} \) is a set of periodic orbits of \( X_{\alpha} \). We have proved,

**Proposition 4.4.** For an embedded surface \( \Sigma \), in the contact manifold \((M, \xi)\), satisfying assumptions of Proposition 4.3 the dividing set \( \Gamma_{\Sigma} \) is a set of periodic orbits of the Reeb field \( X_{\alpha} \).

**Remark 4.5.** Example in [19, p. 327] demonstrates that the dynamics of \( X_{\alpha} \) may change drastically depending on a choice of a contact form \( \alpha \) defining \( \xi \). Consider the following family of contact forms on \( S^3 \subset \mathbb{R}^4 \), for \( t \geq 0 \):
\[
\alpha_t = (x_1 dy_1 - y_1 dx_1) + (1 + t)(x_2 dy_2 - y_2 dx_2),
\]
\[
X_{\alpha_t} = (x_1 \partial y_1 - y_1 \partial x_1) + \frac{1}{1 + t}(x_2 \partial y_2 - y_2 \partial x_2),
\]
where we consider \( S^3 \) as a unit sphere in the standard coordinates \((x_1, x_2, x_3, x_4)\) in \( \mathbb{R}^4 \).
If \( t = 0 \), \( X_{\alpha_0} \) defines a Hopf fibration on \( S^3 \), in particular, all orbits of \( X_{\alpha_0} \) are closed. For \( t \in \mathbb{R} \setminus \mathbb{Q}^+ \), \( X_{\alpha_t} \) defines an irrational flow on tori of the Hopf fibration and has just two periodic orbits (at \( x_1 = y_1 = 0 \), and \( x_2 = y_2 = 0 \)). It demonstrates that in the irrational case any embedded surface away from the periodic orbits cannot admit the contact vector required in Proposition 4.4. (It also demonstrates that contact forms are not stable, i.e. in the above example there exists no family of diffeomorphisms \( \psi_t \) such that \( \psi_t \circ \alpha_t = \alpha_0 \), otherwise the flows of \( X_{\alpha_t} \) would have to be conjugate).
5 Riemannian submersions and the horizontal Laplacian.

Proposition 4.3 describes situations where the horizontal Laplacian $\Delta_E$, in Equation (12), becomes the Laplacian on a surface. We begin by proving a similar statement in the setting of a Riemannian submersion on a principal $S^1$-bundle. The reader may consult [22] for the general treatment of related questions for the Hodge Laplacian on forms.

A submersion $\pi : M \to N$ is Riemannian if and only if

$$ \pi^* : T_pM \supset \ker(\pi^*)^\perp_p \to T_{\pi(p)}N, $$

determines a linear isometry, for all $p \in M$. In other words, for $V,W \in TM$ which are perpendicular to the kernel of $\pi^*$, we have $g_M(V,W) = g_N(\pi^*V, \pi^*W)$. Every Riemannian submersion determines an orthogonal decomposition $TM = V \oplus H$ of the tangent bundle into a vertical subbundle $V = \ker(\pi^*)$ and a horizontal subbundle $H = V^\perp$. The main feature of $\pi$ is a possibility of lifting orthogonal frames on $N$ to horizontal vectors on $M$, which stay mutually orthogonal. Consequently, we may complete a lifted frame to an orthogonal frame on $M$.

We summarize useful, for us, properties of Riemannian submersions through a series of lemmas, where vectors on the base $N$ are denoted with capital letters $E,F$ and lifted vectors on $M$ by small letters $e,f$. We summarize properties of the horizontal lift operation, $\mathcal{H} : T_{\pi(p)}N \to T_pM$, in the following (c.f. [22]).

**Lemma 5.1** (22). Let $\pi : M \to N$ be a Riemannian submersion then

(a) Lifted $f_p = \mathcal{H}(F_{\pi(p)})$ is horizontal i.e. $f_p \in H_p$.

(b) For any point $p \in M$ and a vector $F_{\pi(p)} \in T_{\pi(p)}N$, $\pi^* \mathcal{H}(F_{\pi(p)}) = F_{\pi(p)}$.

(c) Let $f_i = \mathcal{H}(F_i)$, then $\pi^*([f_1, f_2]) = [F_1, F_2]$.

(d) Let $D^M_i e_j = \omega^k_{ij} e_k$, and $D^N_a E_b = \Omega^c_{ab} e_c$. Christoffel symbols satisfy

$$ \omega^a_{cb} = \Omega^a_{cb} \circ \pi. \quad (23) $$

**Lemma 5.2.** Suppose $\pi : P \to \Sigma$ is a projection of an $S^1$-bundle $P$, equipped with a Riemannian metric $g_P$, which admits a vertical unit Killing vector field $X$. We have the following.

(e) $\pi$ defines the Riemannian submersion with an appropriate choice of the metric on $\Sigma$.

(f) In a local orthogonal frame of vector fields $\{e_1, e_2, e_3\}$, where $e_1 = X$ and $\{e_2 = \mathcal{H}(E_2), e_3 = \mathcal{H}(E_3)\}$ is the horizontal lift of a frame $\{E_2, E_3\}$ from $\Sigma$:

$$ [e_1, e_k] = 0, \quad k = 1, 2, 3, \quad (24) $$

$$ \pi \circ \Delta_E = \Delta_{\Sigma} \circ \pi, \quad (25) $$

$\Delta_\Sigma$ denotes the Laplacian on $\Sigma$, and $\Delta_E$ is defined in (12), where $E = \text{span}\{e_2, e_3\}$. 

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**Proof.** Since \( X \) is a unit Killing vector field, its flow \( \phi^t \) is a flow of isometries on \( P \). Therefore, in a local trivialization: \((t,x) \in V \cong S^1 \times U, \ x \in U \subset P \) where \( X = \partial_t \), the flow \( \phi^t \) acts by translations in the \( t \)-direction. Thus, we may choose a \( \partial_t \)-invariant frame \( \{e_1, e_2, e_3\} \), where \( e_1 = \partial_t = X \) on \( V \) satisfying:

\[
[e_1, e_k] = [\partial_t, e_k] = 0.
\]

Any local vector field \( f \) on \( U \) lifts, in a natural fashion, to the vector field \( F \) on \( V \cong S^1 \times U \), so that the equation \( \pi_*(F) = f \) holds, and we may define a metric \( g_{\Sigma} \) on \( U \subset \Sigma \) by

\[
g_{\Sigma}(f, f') = g_P(F, F').
\]

This turns \( \pi \) into a Riemannian submersion on \( V \), and defines \( g_{\Sigma} \) pointwise on the whole \( \Sigma \). In the next step we obtain (25) as a corollary of Lemma 5.1. Since the Christoffel symbols project under Riemannian submersions (see (d) in Lemma 5.1), for \( u \in C^2(\Sigma) \) in a local frame \( \{E_2, E_3\} \) on \( \Sigma \) we derive

\[
(\Delta_{\Sigma} u) \circ \pi = -D_{E_2}D_{E_2}u - D_{E_3}D_{E_3}u + \Omega_{22}^3 D_{E_2}u + \Omega_{33}^2 D_{E_2}u \circ \pi
= -D_{e_2}D_{e_2}(u \circ \pi) - D_{e_3}D_{e_3}(u \circ \pi) + \omega_{22}^3 D_{e_2}(u \circ \pi) + \omega_{33}^2 D_{e_2}(u \circ \pi)
= \Delta_E(u \circ \pi),
\]

where \( e_2 = f_2(E_2) \) and \( e_3 = f_3(E_3) \). \( \square \)

The following lemma (see [5], p. 148, Lemma 2.4.22) is an important ingredient in the proof of the main theorem, thus we provide a proof for more complete exposition.

**Lemma 5.3 ([5]).** Every closed, compact orientable Seifert fibered 3-manifold \( M \), with the base which is a “good” orbifold \( \Sigma \), is covered by a total space of a circle bundle \( P \). We have the following diagram:

\[
P \xrightarrow{p} M \\
\downarrow \pi \quad \downarrow \pi
\]

\[
\Sigma \xrightarrow{r} \Sigma
\]

where \( p \) is the covering map, \( r \) is the orbifold covering, and the maps \( \pi, \Pi \) are fibrations.

**Proof.** Any good 2-orbifold \( \Sigma \) is a quotient of one of the model spaces \( S = S^2, \mathbb{R}^2 \) or \( \mathbb{H}^2 \), by a discrete group of isometries, i.e. \( \Sigma = S/G \). Since \( \Sigma \) is good then any finitely generated discrete subgroup \( G \) of isometries of \( S \), with compact quotient space, has a torsion free subgroup \( G' \) of finite index, [34]. Clearly, such a subgroup is isomorphic to the fundamental group of a closed surface \( \Sigma \). Define \( \Sigma = S/G' \), and \( r : \Sigma \rightarrow \Sigma \) to be a quotient map, notice that \( r \) is generally not a cover in the usual sense, [34]. Let \( h \in \pi_1(M) \) represent a regular fiber of \( M \). The subgroup \( \langle h \rangle \) of \( \pi_1(M) \) generated by \( h \) is infinite cyclic, and \( \pi_1(M)/\langle h \rangle = \pi_1(M)/\langle \pi_1(M) \rangle = G \). Denote by \( K \) the inverse image in \( \pi_1(M) \) of a torsion free subgroup \( G' \), under the induced group homomorphism \( \pi_* \). Let \( P \) be the covering space of \( M \) corresponding to \( K \). If \( \tilde{h} \) is represented in \( \pi_1(P) = K \subset \pi_1(M) \) by a regular fiber then \( \langle \tilde{h} \rangle = \langle h \rangle \). Because \( K/\langle h \rangle = \pi_1(P)/\langle h \rangle = G' \) is torsion free, \( P \) has no singular fibers, and has to be an \( S^1 \)-bundle over \( \tilde{\Sigma} \). Diagram (26) follows accordingly. \( \square \)
6 Proof of the Main Theorem.

We now prove our main results. In a nutshell, these theorems describe lower bounds for the volume of \( M \) in terms of geometric parameters of \( M \) and the magnitude \( \| \alpha \| \) of a contact form \( \alpha \), which defines overtwisted or virtually overtwisted contact structure \( \xi \) on \( M \). Clearly, opposite inequalities provide sufficient conditions for the universal tightness of \( \xi \). These results require existence of an \( S^1 \)-action by a contact vector field which is also Killing in the adapted metric. Proposition 2.6 demonstrates that, given an \( S^1 \)-invariant contact form \( \alpha \), we may always adapt a suitable Riemannian metric satisfying these requirements. Topologically, \( M \) is a Seifert fibered manifold covered by a principal \( S^1 \)-bundle \( P \). Therefore, as Theorem 3.2 assures, the universal tightness of \( \xi \) is completely characterized by the topology of the dividing set on the base of \( P \), and techniques developed in Section 4 can be applied.

**Theorem 6.1** (Main Theorem (version 1)).

(A) Let \((M, g_M)\) be a compact closed orientable Riemannian 3-manifold, equipped with a contact structure \( \xi \) defined by \( \alpha \) and satisfying (2). Assume that \( \alpha \) admits a contact vector field \( X \) (i.e. \( L_X \alpha = 0 \)) with circular orbits, which is a unit Killing vector field for \( g_M \).

(B) Additionally, let the sectional curvature \( \kappa_E \) of planes \( E \), orthogonal to the fibers, obey

\[
\kappa_E \leq -\frac{3}{4} \ell^2.
\]

If \( \xi \) is an overtwisted, or virtually overtwisted contact structure on \( M \), then we have the following lower bound for the volume of \( M \):

(C)

\[
\text{Vol}(M) \geq \frac{2\pi l_{\text{min}}}{m_{\alpha} k}, \quad m_{\alpha} = \max_M (0, \Delta_M \ln \|\alpha\|),
\]

where \( l_{\text{min}} \) is a lower bound for lengths of orbits of \( X \), and \( k \) depends on the Seifert fibration of \( M \) induced by \( X \).

**Proof.** Since \( X \) has circular orbits, the result of Epstein \([12]\) implies that \( M \) is a Seifert fibered manifold and the lengths of orbits of \( X \) are bounded. Consequently, \( X \) induces an \( S^1 \)-action by isometries on \( M \), and we obtain an orbifold bundle: \( \pi : M \mapsto M/S^1 \cong \Sigma \). In the first part of the proof, we show how (A) and (B) imply that the base \( \Sigma \) of the Seifert fibration \( M \) is a negatively curved orbifold. Such orbifolds are good thus, \( M \) is covered by an \( S^1 \)-bundle \( P \), as concluded in Lemma 5.3. The constant \( k \) is the degree of a cover. This allows us, in the second part of the proof, to lift the structure from \( M \) to the covering space \( P \), using Diagram (26), and perform the analysis on \( P \).
Let $C = \{x_1, \ldots, x_k\}$ be the cone points of $\Sigma$, and $S = \pi^{-1}(C)$ the set of singular fibers in $M$. Since $M \setminus S \cong S^1 \times (\Sigma \setminus C)$, by Lemma 5.2 we may define a metric $g_\Sigma$ on $M \setminus C$ so that $\pi: M \setminus S \mapsto \Sigma \setminus C$ is a Riemannian submersion. The metric $g_\Sigma$ is smooth and extends continuously to $\Sigma$. In the first step, we prove that the scalar curvature of $(\Sigma \setminus C, g_\Sigma)$ is nonpositive, implying that $\Sigma$ is a good orbifold (c.f. [34]).

Let us fix a local frame of vector fields $\{e_1 = X, e_2, e_3\}$, and the dual coframe $\{\eta = \eta_1, \eta_2, \eta_3\}$. Since $X$ is the Killing vector field (i.e. $\mathcal{L}_X g_M = 0$), for any pair of vector fields $V, W$:

$$\langle D_V X, W \rangle = -\langle V, D_W X \rangle.$$  

Consequently, we obtain the following identities for the Christoffel symbols in the frame $\{e_i\}$:

$$\omega^2_{11} = \omega^3_{11} = \omega^2_{21} = \omega^3_{31} = 0, \quad \omega^2_{ij} = -\omega^j_{ik}, \quad -\omega^2_{31} = \omega^3_{21} = \frac{\mathcal{E}}{2}, \quad (29)$$

where $D_i e_j = \omega^k_{ij} e_k$. Cartan's structure equations imply that the 1-form $\eta = g_M(X, \cdot)$ satisfies

$$* d \eta = \mathcal{E} \eta. \quad (30)$$

Since $d \mathcal{E} \ast \eta = 0$, we have $d \mathcal{E}(X) = X \mathcal{E} = 0$, thus $\mathcal{E}$ is $S^1$-invariant. Using (29), we compute the sectional curvature $\kappa_E$ as follows (c.f. [32, p. 8])

$$\begin{align*}
D_2 D_3 e_3 &= D_2 (\omega^k_{33} e_k) = (D_2 \omega^2_{33}) e_2 + \omega^2_{33} D_2 e_2 \\
&= (D_2 \omega^2_{33}) e_2 + \omega^2_{33} \omega^3_{22} e_3, \\
D_3 D_2 e_3 &= D_3 (\omega^k_{22} e_k) = (-\frac{1}{2} D_3 \mathcal{E}) e_1 - \frac{1}{2} \mathcal{E} D_3 e_1 + (D_3 \omega^2_{23}) e_2 + \omega^2_{23} D_3 e_2 \\
&= -\frac{1}{2} D_3 \mathcal{E} e_1 + \frac{\mathcal{E}^2}{4} e_2 + (D_3 \omega^2_{23}) e_2 + \frac{\mathcal{E}}{2} \omega^3_{22} e_1 + \omega^3_{22} \omega^2_{33} e_3,
\end{align*}$$

$$\begin{align*}
[e_2, e_3] &= D_2 e_3 - D_3 e_2 = (\omega^k_{23} - \omega^k_{32}) e_k = -\mathcal{E} e_1 + \omega^2_{23} e_2 - \omega^3_{32} e_3, \\
D[e_2, e_3] e_3 &= -\mathcal{E} D_1 e_3 + \omega^2_{23} D_2 e_3 - \omega^3_{32} D_3 e_3 \\
&= -\frac{\mathcal{E}^2}{2} \varphi e_2 + \frac{1}{2} \omega^2_{23} \mathcal{E} e_1 + ((\omega^3_{22})^2 + (\omega^3_{32})^2) e_2, \\
\kappa_E &= \langle R(e_2, e_3) e_3, e_2 \rangle \\
&= D_2 \omega^2_{33} + D_3 \omega^2_{23} - \frac{\mathcal{E}^2}{4} + \mathcal{E} \varphi - ((\omega^3_{22})^2 + (\omega^3_{32})^2) \\
&= \sigma - \frac{\mathcal{E}^2}{4} + \mathcal{E} \varphi,
\end{align*}$$

where $\sigma = D_2 \omega^2_{33} + D_3 \omega^2_{23} - ((\omega^3_{22})^2 + (\omega^3_{32})^2)$, and $\varphi = \omega^2_{13}$. Notice that

$$D_1 e_2 = \omega^3_{12} e_3 = -\varphi e_3, \quad D_1 e_3 = \omega^2_{13} e_2 = \varphi e_2.$$
Therefore, \( \varphi \) measures a rotation of the frame in \( E \), when parallel transported along orbits of \( X \). By Lemma 5.1 the Christoffel symbols project under \( \pi : (M \setminus S, g_M) \mapsto (\Sigma \setminus C, g_\Sigma) \), and the scalar curvature \( K \) of \( \Sigma \) obeys:

\[
K \circ \pi(x) = \sigma(x), \quad \text{for } x \in M,
\]

where \( \sigma \) is defined in (31). Assuming that \( \{e_2, e_3\} \) are horizontal lifts of a frame from \( \Sigma \), Equation (24) yields

\[
0 = [e_2, e_1] = D_1 e_2 - D_2 e_1,
\]

thus

\[
\varphi = -\frac{\mathcal{E}}{2}.
\]

Because \( \kappa_E = \sigma - \frac{\mathcal{E}^2}{4} + \mathcal{E} \varphi = \sigma - \frac{3}{4} \mathcal{E}^2 \), by the assumption (B):

\[
K \circ \pi = \kappa_E + \frac{3}{4} \mathcal{E}^2 \leq 0.
\]  

(32)

By the Gauss-Bonnet theorem for orbifolds [34]: \( \chi_{\text{orb}}(\Sigma) \leq 0 \), and \( \Sigma \) must be covered by a closed surface \( \tilde{\Sigma} \) of nonzero genus (denote the covering projection by \( r : \tilde{\Sigma} \mapsto \Sigma \)). Now, Lemma 5.3 tells us how to choose a principal bundle \( \Pi : P \mapsto \tilde{\Sigma} \), such that the total space \( P \) is a covering space for \( M \). This is done as follows, recall that Diagram (26) commutes, and \( p : P \mapsto M \) is a fiber preserving covering map. Define a metric \( g_P \) on \( P \) by pulling back the metric \( g_M \) from \( M \) via \( p \), this makes \( p : (P, g_P) \mapsto (M, g_M) \) into a local isometry, and \( \Pi : P \mapsto \tilde{\Sigma} \) into a Riemannian submersion. Let \( \tilde{X} \) be the unique lift of \( X \), because \( p \) respects the fibers, which are orbits of the flow \( \phi_X \) of \( X \), we have

\[
\phi_X(t, \cdot) \circ p = p \circ \phi_{\tilde{X}}(t, \cdot).
\]

Clearly, the lift \( \tilde{X} \) of \( X \) must also be a Killing vector field on \( P \) with circular orbits.

In the second part of the proof, we show the bound in (C), under the assumption that \( \xi \) is overtwisted or virtually overwisted. Notice that it suffices to work with the \( S^1 \)-invariant contact structure \( \tilde{\xi} \) on \( P \), obtained by lifting \( \xi \) to \( P \) (i.e. \( \tilde{\xi} = \ker \tilde{\alpha}, \tilde{\alpha} = p_* \alpha \)), since \( \tilde{\xi} \) cannot be universally tight either. But \( \tilde{\Sigma} \neq S^2 \) thus, by the necessary and sufficient condition (b.1) in Theorem 3.2 \( \tilde{\xi} \) satisfies

\[
(\star) \quad \text{The dividing set } \Gamma_{\tilde{\Sigma}} \text{ on } \tilde{\Sigma}, \text{which is a projection of the characteristic surface } \Gamma_X \text{ under } \Pi, \text{contains a contractible closed curve.}
\]

Because \( \mathcal{L}_{\tilde{X}} \tilde{\alpha} = 0 \) and \( *d\tilde{\alpha} = \mu \tilde{\alpha} \), Theorem 3.4 and Theorem 3.6 imply that \( \Gamma_{\tilde{X}} = f^{-1}(0) \), and \( f = \alpha(X) \circ p \) is an \( S^1 \)-invariant solution to Equation (8). By Lemma 5.2 the following equation for \( f \) holds on \( \tilde{\Sigma} \):

\[
\Delta_{\tilde{\Sigma}} f + \tilde{\mu}(\tilde{\mathcal{E}} - \tilde{\mu}) f = 0,
\]  

(33)

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where $\tilde{E} = \mathcal{E} \circ p$, and $\tilde{\mu} = \mu \circ p$. The function $f$ cannot be a trivial solution, for the following topological reason: $f \equiv 0$ implies that $\tilde{\xi}$ is tangent everywhere to the $S^1$-fibers of $P$. But for $S^1$-invariant $\tilde{\alpha}$, it would violate the nonintegrability condition (1). By Proposition 3.8 we must have $\tilde{\mu}(\tilde{\mu} - \tilde{E}) \geq 0$ on $M$, and unless $\tilde{\mu} = \tilde{E}$ on $M$, $f$ cannot be a constant function. (When $f$ is constant then $X$ is equal to the Reeb field of $X_\alpha$ and we arrive at Corollary 7.1.)

If $f \neq \text{const}$, $f$ must change sign on $\tilde{\Sigma}$, and the dividing set $\tilde{\Gamma} = \Pi(\tilde{\Gamma}_X)$ is nonempty. (Notice that Theorem 3.6 implies that curves $\Gamma_X$ cannot have self-intersections.) By condition (*) one of the domains $\tilde{\Omega} \in \tilde{\Sigma} \setminus \tilde{\Gamma}_X$ is a disc $\tilde{\Omega} \cong D^2$. Notice that the function $\|\tilde{\alpha}\|$ is $\tilde{X}$-invariant (where $\tilde{\alpha} = p_* \alpha$), applying the technique of Proposition 4.3 we obtain

$$2\pi = 2\pi \chi(\tilde{\Omega}) = \int_{\tilde{\Omega}} K + \int_{\tilde{\Omega}} \Delta_{\tilde{\Sigma}} \ln \|\tilde{\alpha}\|$$

$$\leq \operatorname{Vol}(\tilde{\Omega}) \max_{\tilde{\Sigma}} \left(0, \Delta_{\tilde{\Sigma}} \ln \|\tilde{\alpha}\|\right),$$

because $K \leq 0$ (by (32)). In the next step, we bound the area of $\tilde{\Omega}$. Since $r : \tilde{\Sigma} \setminus r^{-1}(C) \mapsto \Sigma \setminus C$ is a $k$-sheeted cover and $\Sigma$ is a quotient of $\tilde{\Sigma}$ by a discrete subgroup of isometries, we obtain

$$\operatorname{Vol}(\tilde{\Omega}) \leq \operatorname{Vol}(\tilde{\Sigma}) = k \operatorname{Vol}(\Sigma),$$

and

$$\operatorname{Vol}(M) = \int_M \eta \wedge \pi^* \omega = \int_{\Sigma} \int_{S^1} \eta(X) \pi^* \omega$$

$$= \int_{\Sigma} l(x) \omega \geq l_{\min} \operatorname{Vol}(\Sigma),$$

where $l_{\min} = \min_{x \in \Sigma} l(x)$ for the “length of the fiber function” $l : \Sigma \mapsto \mathbb{R}$, and $\omega$ is the volume form on $\Sigma$. Bounds in (34) and (35) yield

$$2\pi \leq \frac{k}{l_{\min}} \operatorname{Vol}(M) \max_{M} \left(0, \Lambda_M \ln \|\alpha\|\right).$$

For $m_\alpha$ defined in (C) we obtain Inequality (28). \hfill \Box

**Corollary 6.2.** If $m_\alpha = 0$, then $\xi$ is universally tight.

**Corollary 6.3.** When $\mathcal{E}$ and $\mu$ are constant functions and $X$ has constant length $l$ orbits, Inequality (C) simplifies to

$$(D) \quad \operatorname{Vol}(M) \geq \frac{4\pi l}{\mu^2 k} + \frac{2\pi e(M) l}{\mu},$$

where $e(M)$ is the Euler number of the Seifert fibration of $M$ induced by $X$. 20
Proof. The 1-form $\tilde{\eta} = \langle \tilde{X}, \cdot \rangle$ can be regarded as a connection form on $P$ (since $\mathcal{L}_{\tilde{X}} \tilde{\eta} = 0$, and $\ker \tilde{\eta}$ is orthogonal to the $S^1$-fibers). Therefore, we obtain the following relation between the function $E$ and the Euler number of $P$ (c.f. [31, p. 75]):

$$e(P) = \frac{1}{2\pi} \int_{\Sigma} \tilde{E} = \frac{k}{2\pi} \int_{\Sigma} E,$$

thus,

$$E = \frac{2\pi e(P)}{\Vol(\Sigma) k} = \frac{2\pi e(M)}{\Vol(\Sigma)}. \quad (37)$$

By (32), $\tilde{\Sigma}$ has nonpositive curvature. Because $E = \tilde{E}$ and $\mu = \tilde{\mu}$ are constant, Equation (33) is an eigenequation and Proposition 4.1 together with derivation (35) yield

$$4\pi \leq \mu(\mu - E)\Vol(\tilde{\Omega}) = \frac{k}{l}\mu(\mu - E)\Vol(M),$$

substituting (37) for $E$ proves the claim. \hfill \Box

Remark 6.4. In Theorem 6.1, it may be possible to drop the assumption of circular orbits of $X$. By the compactness of the group of isometries of $(M, g_M)$, one easily shows that there exists a regular Killing vector field $X_\varepsilon$ arbitrarily close to $X$ (c.f. [32]). One may expect that Equation (8) will hold for $f_\varepsilon = \alpha(X_\varepsilon)$ with possibly an error term. Consequently, one could imagine an approximation argument, with $\varepsilon \to 0$, showing that the limit function $f_\varepsilon \to f$, is a solution to (8).

The curvature assumption (B), in Theorem 6.1, is necessary to carry out the argument on the covering space $P$ of $M$, and also simplifies the bound in (36). By assuming that the base of the Seifert fibration $M$ is a good orbifold, we obtain

Theorem 6.5 (Main Theorem (version 2)).

(E) Let $M$, $\alpha$, $X$, $g_M$ satisfy the assumption (A), and let $M/S^1 \simeq \Sigma$ be a good orbifold covered by a smooth surface of nonzero genus.

Then,

(F) The form $\alpha$ defines a universally tight contact structure on $M$, provided that the volume of $M$ obeys

$$\Vol(M) < \frac{2\pi l_{\min}}{m_\alpha k}, \quad m_\alpha = \max_M (0, (\Delta_M \ln \|\alpha\| + \kappa_E + \frac{3}{4} E^2)),$$

where $k$ is the degree of a cover (as in Theorem 6.1).
Proof. We may apply analogous reasoning as in the second part of the proof of Theorem 6.1. The only difference is that $K = \kappa_E + \frac{3}{4} \mathcal{E}^2$ may be positive on $\tilde{\Sigma}$ so we must adjust the estimate in (34) as follows

$$2\pi = 2\pi \chi(\tilde{\Omega}) = \int_{\tilde{\Omega}} K + \int_{\tilde{\Omega}} \Delta_{\tilde{\xi}} \ln \| \tilde{\alpha} \|$$

$$\leq \text{Vol}(\tilde{\Omega}) \max_{\tilde{\Sigma}} \left( 0, \Delta_{\tilde{\xi}} \ln \| \tilde{\alpha} \| + \kappa_E + \frac{3}{4} \mathcal{E}^2 \right).$$

Now, the condition (F) may be derived analogously, as an opposite inequality.

The results presented do not fully address the case when $\Sigma = M/S^1$ is homeomorphic to $S^2$, and is either a bad orbifold or all covering spaces are $S^1$-bundles over $S^2$. In these cases, universal tightness is determined by conditions (b.2) and (b.3) in Theorem 3.2. Corollary 7.1 provides only a partial answer here, and a different geometric condition is needed. Also, a drawback of lifting structures to the covering space is that one must know the degree $k$ of a cover, or its upper bound. Such upper bounds may be hard to obtain in full generality, but may be known for particular types of orbifolds (c.f. [34]). The author will address these issues in the future work.

7 Conclusions

In concluding remarks, we want to point out several examples which demonstrate effectiveness of proposed theorems.

The case when the contact field $X$ is the Reeb field $X_\alpha$, and thus $\xi$ is transverse to $S^1$-fibers, is captured by the following

**Corollary 7.1.** If $\mu = \mathcal{E}$, under the assumption (E), $\xi$ is universally tight. Thus, every regular Sasakian 3-manifold is universally tight (see [3, p. 150]). The standard tight contact structure on $S^3$ is universally tight.

But, we also obtain more general result

**Corollary 7.2.** If a contact form $\alpha$ is of constant length ($\|\alpha\| = \text{const}$), then under assumptions (A) and (B), $\alpha$ defines a universally tight contact structure.

As a subsequent corollary, we obtain the well known result [28] concerning tight contact structures on a 3-torus $T^3 \simeq S^1 \times S^1 \times S^1$.

**Corollary 7.3.** Every contact form $\alpha_n = \cos(nz)dx + \sin(nz)dy$, $n \in \mathbb{Z}$, in standard coordinates on the 3-torus $T^3$, defines a universally tight contact structure on $T^3$.

**Proof.** The flat metric is adapted to $\alpha_n$ for all $n$ (c.f. [14]) and the coordinate vector fields $\partial_x$, $\partial_y$ are unit Killing vector fields with circular orbits. In addition, $\mathcal{E} = 0$, $\|\alpha_n\| = 1$ and $\kappa_E = 0$, thus $m_{\alpha_n} = 0$ in Theorem 6.5.

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To further demonstrate usefulness of proposed results, we look at the problem of geometric tightness from a slightly different perspective. Rather than adapting a metric to a contact structure, we consider an arbitrary closed Riemannian 3-manifold \((M, g_M)\). The standard elliptic theory implies that the eigenproblem \((\mu = \text{const})\):

\[ * d \alpha = \mu \alpha, \quad \mu \neq 0, \tag{38} \]

admits infinitely many solutions, in particular we may diagonalize the operator \(*d\) in the orthonormal basis of curl eigenfields: \(\{\alpha_i\}\). If a solution \(\alpha\) is a nonvanishing 1-form, i.e. \(\|\alpha\| \neq 0\), the derivation in \((5)\) shows that the distribution \(\xi = \ker \alpha\) defines a contact structure on \(M\). Clearly, the Riemannian metric \(g_M\) is adapted to every nonvanishing curl eigenfield.

**Question 7.4.** When are these contact structures tight/overtwisted?

If \((M, g_M)\) admits nonsingular Killing fields it also admits an \(S^1\)-action by the group of isometries (see Remark \(6.4\)). As shown in \(20\), \(*d\) has the \(S^1\)-invariant portion of the spectrum, thus \((38)\) admits \(S^1\)-invariant solutions. In the case of a unit Killing field, Theorem \(6.1\) and \(6.5\) address the above question. To further demonstrate, let us consider a product of \(S^1\) and a closed surface \(\Sigma\): \(M = S^1 \times \Sigma\). Assume that \(M\) has a product metric \(g_{S^1 \times \Sigma}\) and \(S^1\)-fibers are of constant length \(l\). In such setting we have a vertical vector field \(X\) which is unit Killing in the metric \(g_{S^1 \times \Sigma}\). Since \(X\) is orthogonal to \(\Sigma\), the dual 1-form \(\eta\) is closed and we obtain \(\mathcal{E} = 0\), by \((30)\). If \(\Sigma\) has nonpositive curvature, i.e. \(K = \kappa_E \leq 0\), conditions \((A)\) and \((B)\) of Theorem \(6.1\) are satisfied. Because \(\mu\) and \(\mathcal{E}\) are constant, every \(S^1\)-invariant solution to \((38)\) is universally tight provided

\[ \text{Vol}(M) < \frac{4\pi l}{\mu^2}, \tag{39} \]

(compare to Corollary \(4.2\)). It can be easily shown (see Lemma 4.2, \(20\), p. 46) that eigenvalues \(\mu^2\) of \(S^1\)-invariant curl eigenfields are equal to eigenvalues \(\lambda^2\) of the surface (in \((15)\)). Therefore, Inequality \((39)\) will hold if \(\Sigma\) has small eigenvalues with respect to the area of \(\Sigma\).

Examples of Buser \(7\) show that hyperbolic surfaces with small eigenvalues are not uncommon. Specifically, Buser constructs a family of hyperbolic genus \(g(\Sigma)\) surfaces (so called Löbell surfaces) with \(\text{Vol}(\Sigma) = 4(g(\Sigma) - 1)\pi\), where the eigenvalues satisfy

\[ \lambda_i \leq \varepsilon, \quad \text{for } i = 1, \ldots, 2g(\Sigma) - 3, \]

(for arbitrary small \(\varepsilon > 0\)). As a consequence, Inequality \((39)\) holds and the corresponding curl eigenfields are universally tight. In a nutshell, \(\lambda_i\)-eigenfunctions, for \(i = 1, \ldots, 2g(\Sigma) - 3\) cannot have a nodal domain which bounds a disc, and the corresponding curl eigenfields cannot have contractible dividing curves on \(\Sigma\), which implies universal tightness. Consult \(29\), p. 43 for detailed considerations and a specific condition on Löbell surfaces in Theorem
2.6.17. Techniques introduced in [29, 20] show that one may produce analogous examples in
the setting of arbitrary $S^1$-bundles.

These considerations confirm that knowledge of eigenvalues and the geometry of a man-
ifold, in certain symmetric situations, is sufficient to determine tightness of curl eigenfields
without an explicit knowledge of solutions to the problem (38). In general, this is not the
case that for low eigenvalues nonsingular curl eigenfields define tight contact structures,
[20]. However, one may still hope that this claim [13, p. 17] is true in a certain large class
of Riemannian manifolds.

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