PERVERSE COHERENT T-STRUCTURES THROUGH TORSION THEORIES

JORGE VITÓRIA

Abstract. Bezrukavnikov (later together with Arinkin) recovered the work of Deligne defining perverse t-structures for the derived category of coherent sheaves on a projective variety. In this text we prove that these t-structures can be obtained through tilting torsion theories as in the work of Happel, Reiten and Smalø. This approach proves to be slightly more general as it allows us to define, in the quasi-coherent setting, similar perverse t-structures for certain noncommutative projective planes.

1. Introduction

A t-structure on a triangulated category $D$ is a pair of full subcategories of $D$,
\( (D^\leq, D^\geq) \), such that, for $D^\leq_n := D^\leq[-n]$ and $D^\geq_n := D^\geq[-n]$, $n \in \mathbb{Z}$ we have:

1. $\text{Hom}(X, Y) = 0$, $\forall X \in D^\leq$, $Y \in D^\geq$
2. $D^\leq \subseteq D^\leq_1$
3. For all $X \in D$, there is a distinguished triangle

\[
\begin{align*}
A & \rightarrow X & \rightarrow B & \rightarrow A[1]
\end{align*}
\]

such that $A \in D^\leq_0$ and $B \in D^\geq_1$.

$D^\leq_0 \cap D^\geq_0$ is called the heart of the t-structure and it is an abelian category (as proven in Beilinson, Bernstein and Deligne [7]). It is also well known (see for example [13]) that $D^\leq_0$, the aisle of the t-structure, determines the whole pair by setting $D^\geq_0 = (D^\leq_0)^\perp[1]$.

In Arinkin and Bezrukavnikov’s work ([3] and [2]) perverse coherent t-structures are constructed as follows. Let $X$ be a scheme and $X^{\text{top}}$ denote the set of generic points of all closed subschemes of $X$. A perversity is a map $p : X^{\text{top}} \rightarrow \mathbb{Z}$ satisfying the monotone and comonotone properties:

- **Monotone:** $y \in \bar{x} \Rightarrow p(y) \geq p(x)$
- **Comonotone:** $y \in \bar{x} \Rightarrow p(x) \geq p(y) - (\dim(x) - \dim(y))$.

Note that the image of a perversity on an $n$-dimensional scheme has at most $n + 1$ elements.

Now, given a perversity $p$, the perverse t-structure is defined by taking as aisle:

\[
D^p = \left\{ F^\bullet \in D^b(\text{coh}(X)) : \forall x \in X^{\text{top}}, Li_x^*(F^\bullet) \in D^\leq_{p(x)}(O_x - \text{mod}) \right\}
\]

where $i_x : \{x\} \rightarrow X$ is the inclusion map and $Li_x^*$ is the left derived functor of the pull-back functor by $i_x$. The proof that this is in fact an aisle can be seen in [3] and [2].

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A more algebraic treatment of these categories, in the projective setting, is provided by Serre in [23]. Let $X$ be a projective variety over an algebraically closed field $K$ of characteristic zero (which we shall fix throughout) and define $R = \Gamma(X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, O_X(n))$ where $\Gamma$ is the functor of global sections and $O_X$ is the structure sheaf of $X$. Serre proved that the category $\text{Qcoh}(X)$, of quasi-coherent sheaves over $X$, is equivalent to the quotient category $\text{Tails}(R) = \text{Gr}(R)/\text{Tors}(R)$ where $\text{Tors}(R)$ is the full subcategory of torsion modules of the category $\text{Gr}(R)$ of graded modules over $R$ for the following torsion theory: $x \in M$ is torsion if and only if there is $N \geq 0$ such that $xR_j = 0$, for all $j > N$, i.e., the submodule generated by $x$ is right bounded. When written in the lower case, $\text{tails}(R) = gr(R)/\text{tors}(R)$ will denote the subcategory of finitely generated objects in $\text{Tails}(R)$, thus getting a category equivalent to $\text{coh}(X)$.

Given a torsion theory in the heart of a $t$-structure we can construct, as in [12] and [9] a new $t$-structure. We shall use an iteration of this process to obtain the perverse coherent $t$-structure from the standard $t$-structure on $D^b(\text{coh}(X)) = D^b(\text{tails}(R))$ when $R$ is a commutative connected noetherian positively graded $K$-algebra generated in degree one.

This new method of getting perverse coherent $t$-structures can then be used in other contexts. In this text we exemplify this with noncommutative projective planes. The notion of noncommutative projective scheme associated to a noncommutative graded $K$-algebra was introduced in [6]. A noncommutative projective scheme can be thought of as an abstract space $\text{Proj}(R)$ whose category of quasi-coherent sheaves (respectively coherent sheaves) is the quotient category $\text{Tails}(R) = \text{Gr}(R)/\text{Tors}(R)$ (resp. $\text{tails}(R) = \text{grmod}(R)/\text{tors}(R)$) for a noncommutative $K$-algebra $R$. Artin and Schelter defined in [3] the class of algebras whose categories of tails play the role of coherent sheaves over noncommutative projective planes. These, called Artin-Schelter regular algebras of dimension 3, are algebras of global dimension 3, finite Gelfand-Kirillov dimension (in fact 3) and Gorenstein. Artin-Schelter (AS for short) regular algebras of dimension 3 have been classified (see [3] and [4]). They are quotients of the free algebra in $r$ generators by $r$ relations of degree $s$, where $(r, s) \in \{(2, 3), (3, 2)\}$. Furthermore, to each AS-regular algebra of dimension 3 we can associate a triple $(E, \sigma, L)$ where $E$ is a scheme, $\sigma \in \text{Aut}(E)$ and $L$ is an invertible sheaf on $E$. In the case where $r = 3$ the algebra is said to be elliptic if $E$ is a divisor of degree 3 in $\mathbb{P}^2$ and $L$ is the restriction of $O_{\mathbb{P}^2}(1)$. The only other case to consider is when $E = \mathbb{P}^2$ and then we say the algebra is linear. It can be proven that if $A$ is linear then $A \cong B$ where $B$ is a twisted coordinate ring of $\mathbb{P}^2$ and therefore $\text{tails}(A) \cong \text{coh}(\mathbb{P}^2)$. We will focus on the elliptic cases in which the automorphism $\sigma$ is of finite order to provide an example of a new construction of perverse quasi-coherent $t$-structures. In these cases the algebras obtained are finite over their centres (see [5]) and, therefore, they are fully bounded Noetherian.

Section 2 presents some basics on the theory of torsion theories for categories of graded modules. In section 3 we recall a result from [12] and shows how to obtain a $t$-structure by adequately iterating the use of torsion theories on a general Abelian category; section 4 show how torsion theories come into play when describing perverse coherent $t$-structures and section 5 applies section 3 to define perverse coherent $t$-structures on some noncommutative projective planes.
2. Torsion theories for graded modules

Let $R$ be a, not necessarily commutative, Noetherian graded ring. $Gr(R)$ is a Grothendieck category admitting injective envelopes which, for a graded module $M$, we will denote by $E^0(M)$. We shall use $\text{Hom}_{Gr(R)}(M, N)$ for homomorphisms in this category (i.e. $R$-linear, grading preserving) between graded modules $M$ and $N$. $h(M)$ shall denote the subset of homogeneous elements of $M$. It is clear that $M = \langle h(M) \rangle$. Also, for a prime ideal $P$, define $C^0(P) = C(P) \cap h(R)$, where $C(P)$ is the set of homogeneous regular elements mod $P$, i.e., the set of homogeneous elements $x$ of $R$ such that $x + P$ is neither left nor right zero divisor in $R/P$. If $R$ is commutative, then $C(P) = R \setminus P$. The following remark proves to be useful.

Remark 2.1. Given a connected positively graded ring $R$ generated in degree one and a homogeneous prime ideal $P \neq R_+ := \bigoplus_{i \geq 1} R_i$, we have $P_n \neq R_n$ for all $n > 1$. In fact, suppose there is $n_0 > 1$ such that $P_{n_0} = R_{n_0}$. Then, since the ring is generated in degree one, we have $P_n = R_n$ for all $n > n_0$. Now, if $x_1 \in R_1 \setminus P_1$ then, since $P$ is prime, there is $r_1 \in R$ such that $x_2 = x_1 r_1 x_1 \notin P$. Now, deg$(x_2) \geq 2$ since $R$ is positively graded. Thus we can inductively construct a sequence of elements $(x_n)_{n \in \mathbb{N}}$ none of them lying in $P$ and such that deg$(x_n) >$ deg$(x_{n-1})$, thus yielding a contradiction.

We recall the definition of torsion theory and then we define the torsion theories we shall be concerned with in this section.

Definition 2.2. Let $C$ be an Abelian category and $(T,F)$ a pair of full subcategories. $(T,F)$ is said to be a torsion theory if:

1. $\text{Hom}(T,F) = 0$, for all $T \in T$ and $F \in F$
2. For all $M \in C$ there is an exact sequence

$$0 \longrightarrow \tau(M) \longrightarrow M \longrightarrow M/\tau M \longrightarrow 0$$

where $\tau(M) \in T$ and $M/\tau(M) \in F$.

Definition 2.3. To an injective object $E$ in $Gr(R)$ we can associate a natural torsion theory in $Gr(R)$, for which a module $M$ is torsion if $\text{Hom}_{Gr(R)}(M, E) = 0$. This torsion theory is said to be cogenerated by $E$ in $Gr(R)$.

Given that $R$ is Noetherian, $gr(R)$ is closed under taking subobjects. Observe then that the torsion theories above defined in $Gr(R)$ restrict to torsion theories in $gr(R)$. In fact, given $M$ finitely generated graded $R$-module and $\tau$ the torsion radical functor induced by a torsion theory in $Gr(R)$, $\tau(M)$ and $M/\tau(M)$ are finitely generated. Therefore axiom (2) is satisfied by considering in $gr(R)$ the same exact sequence as in $Gr(R)$ (axiom (1) is, in its turn, automatic).

The following lemma proves a useful criterion for graded modules to be torsion for the torsion theory associated with an injective object. The arguments of the proof mimic the ungraded case proved in [10].

Lemma 2.4. Given graded modules $T$ and $F$ over a graded ring $R$, the following conditions are equivalent:

1. $\text{Hom}_{Gr(R)}(T,E^0(F)) = 0$;
2. $\forall t \in h(T), \forall f \in h(F) \setminus 0, \text{deg}(f) = \text{deg}(t), \exists r \in h(R): tr = 0 \land fr \neq 0$.

Proof. Suppose $\text{Hom}_{Gr(R)}(T,E^0(F)) \neq 0$. Let $\alpha$ be one of its nonzero elements. Choose $u \in h(T)$ such that $\alpha(u) \neq 0$. $F$ is a graded essential submodule of $E^0(F)$,
i.e., given any nontrivial graded submodule of $E^g(F)$, its intersection with $F$ is nontrivial. Hence there is $s \in h(R)$ such that $0 \neq \alpha(u)s = \alpha(us) \in F$. If we choose $t = us$ and $f = \alpha(us)$ they are homogeneous of the same degree and clearly, given $r \in R$, if $tr = 0$ then $fr = 0$.

Suppose now that (2) is false, i.e., there are $t \in T$ and $f \in F \setminus \{0\}$ homogeneous of the same degree such that for all $r \in h(R)$, if $tr = 0$ then $fr = 0$. Then, there is a well defined nonzero graded homomorphism

$$tR \rightarrow F, \ tr \mapsto fr$$

since $\langle h(R) \rangle = R$. Since $E^g(F)$ is an injective object in the category of graded modules, we can find a nonzero graded homomorphism from $T$ to $E^g(F)$.

The following corollary shows that, even though in general it does not make sense to talk about the degree zero of localisation (simply because localisation may not exist), we can reformulate such statement it in terms of torsion.

**Corollary 2.5.** Let $R$ be a commutative graded ring, $P$ a homogeneous prime ideal in $R$ and $S = h(R \setminus P)$. Given $M$ a graded $R$-module then $(S^{-1}M)_0 = 0$ if and only if $\text{Hom}_{Gr(R)}(M, E^g(R/P)) = 0$.

**Proof.** This follows from the fact $(S^{-1}M)_0 = 0$ is equivalent, by definition of graded localisation, to condition (2) of the above lemma. □

We consider now the so called rigid torsion theories (19). We shall consider the following subset of $\text{Hom}_R(M, N)$: $\text{Hom}(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{Gr(R)}(M, N(i))$.

**Definition 2.6.** We say that a torsion theory in $Gr(R)$ is rigid if the class of torsion modules (equivalently, the class of torsion-free modules) is closed under shifts of the grading. The rigid torsion theory associated to an injective object $E$ in $Gr(R)$ is defined such that a module $M$ is torsion if $\text{Hom}(M, E) = 0$.

We easily get a lemma similar to 2.4.

**Lemma 2.7.** Given graded modules $T$ and $F$ over a graded ring $R$, the following conditions are equivalent:

1. $\text{Hom}(T, E^g(F)) = 0$;
2. $\forall t \in h(T), \ \forall f \in h(F) \setminus 0, \ \exists r \in h(R)$ such that $tr = 0$ and $fr \neq 0$.

**Proof.** The argument is the same as before. Observe that not having a relation between the degrees of $t$ and $f$ we can get homomorphisms of any degree from $tR$ to $F$. Since $\text{Hom}(T, E^g(F)) = \bigoplus_i \text{Hom}_{Gr(R)}(T, E^g(F)(i))$, the result follows. □

As a consequence, one gets a result similar to 2.4.

**Corollary 2.8.** Let $R$ be a commutative graded ring, $P$ a homogeneous prime ideal in $R$ and $S = R \setminus P$. Given $M$ a graded $R$-module then $S^{-1}M = 0$ if and only if $\text{Hom}(M, E^g(R/P)) = 0$.

**Proof.** This follows from the fact $S^{-1}M = 0$ is equivalent, by definition of graded localisation, to condition (2) of the above lemma. □

One can, in fact, get a more general statement, including some noncommutative rings. This can be done by comparing this rigid torsion theory with the torsion
theory associated to a multiplicative set. For a right homogeneous ideal \( J \) of a graded ring \( R \) we use the notation \( J \triangleleft R \) and, given \( r \in R \) we define a right ideal
\[
r^{-1}J := \{a \in R : ra \in J\}.
\]

Recall the following result [19].

**Proposition 2.9.** Let \( R \) be a graded ring, \( S \) a multiplicative subset contained in \( h(R) \). Then the class of modules \( M \) such that there is \( J \in L_S \) with \( MJ = 0 \), where
\[
L_S = \{ J \triangleleft R : r^{-1}J \cap S \neq \emptyset, \forall r \in h(R) \},
\]
is a torsion class for a rigid torsion theory in \( \text{Gr} R \).

\( L_S \) as above is said to be a graded Gabriel filter for that torsion theory. If \( S = C^g(P) \) for some homogeneous prime ideal \( P \), then we denote the filter by \( L_P \).

The rigid torsion theory associated to an injective graded module \( E \) also has an associated graded Gabriel filter given by:
\[
L'_E = \{ J \triangleleft R : \text{Hom}_R(R/J, E) = 0 \}.
\]

In fact hereditary rigid torsion theories are in bijection with graded Gabriel filters [19]. Thus the graded Gabriel filter determines the torsion theory and vice-versa.

**Theorem 2.10.** Let \( P \) be a homogeneous prime ideal of a graded ring \( R \) and \( R/P \) right Noetherian. Let \( M \) be a graded right \( R \)-module. Then \( M \) is torsion with respect to \( C^g(P) \) if and only if \( M \) is torsion with respect to the rigid torsion theory associated to \( E^g(R/P) \).

**Proof.** We will prove that the Gabriel filters of both torsion theories are the same. Let \( E = E^g(R/P) \) and \( J \in L'_E \), i.e., \( J \triangleleft R : \text{Hom}_R(R/J, E) = 0 \). By lemma 2.7 for any choice of \( a \in h(R) \) and \( b \in h(R) \setminus P \), there is a choice of \( c \in h(R) \) such that \( ac \in J \) and \( bc \notin P \). Now, since \( P \) is a two-sided ideal, \( c \notin P \). Thus we conclude that for all \( a \in h(R) \), \( a^{-1}J \) is not contained in \( P \). This means that \( (a^{-1}J + P)/P \triangleleft R/P \) is nontrivial and since \( P \) is prime, it is essential. By Goldie’s theorem for graded rings (see [10]) we have that \( (a^{-1}J + P)/P \) has a homogeneous regular element and thus, for all \( a \in h(R) \), \( a^{-1}J \cap C^g(P) \neq \emptyset \) which means \( J \in L_P \).

Conversely, suppose \( J \in L_P \) and let \( a, b \in h(R) \), \( b \notin P \). By hypothesis, \( a^{-1}J \cap C^g(P) \neq \emptyset \). Let \( z \) be one of its elements. Then, clearly, \( az \in J \) and \( bz \notin P \). Again, by lemma 2.7 the result follows. \( \square \)

In the commutative case, however, the torsion theories discussed above coincide.

**Proposition 2.11.** Let \( R \) be a commutative Noetherian positively graded connected \( \mathbb{K} \)-algebra generated in degree 1, \( P \) a homogeneous prime ideal in \( R \) not equal to the irrelevant ideal and \( M \) a graded \( R \)-module. Then, for \( S = h(R \setminus P) \), \( S^{-1}M = 0 \) if and only if \( (S^{-1}M)_0 = 0 \).

**Proof.** One direction is clear. For the converse, suppose that \( (S^{-1}M)_0 = 0 \). Let \( m \in M \) such that \( \deg(m) > 0 \). Then from remark 2.4 we conclude that there is \( s \in S \) such that \( \deg(m) = \deg(s) \) and, therefore, since \( (S^{-1}M)_0 = 0 \), we get \( \frac{m}{r} = 0 \), i.e., there is \( r \in S \) such that \( mr = 0 \). Thus, for any \( s' \in S \) we have \( \frac{m}{r} = 0 \) and hence \( S^{-1}M_{\geq 0} = 0 \).
Note that since $R$ is positively graded, $M_{\geq 0}$ is a submodule of $M$. Consider the following short exact sequence:

$$0 \rightarrow M_{\geq 0} \rightarrow M \rightarrow M/M_{\geq 0} \rightarrow 0$$

and apply to it the exact localisation functor, thus obtaining

$$0 \rightarrow S^{-1}M_{\geq 0} \rightarrow S^{-1}M \rightarrow S^{-1}(M/M_{\geq 0}) \rightarrow 0$$

Let $x \in M/M_{\geq 0}$. Using again remark 2.13 there is $r \in S$ such that $\deg(r) = -\deg(x)$. This means that $xr = 0$ and thus $\frac{x}{r} = 0$ for all $s \in S$. This shows that $S^{-1}(M/M_{\geq 0}) = 0$ and since we already had $S^{-1}M_{\geq 0} = 0$, it shows that $S^{-1}M = 0$, finishing the proof. □

This shows that under the conditions of the proposition above, the torsion theory cogenerated by $E^g(R/P)$ is rigid. This statement, however, can be reproduced by dropping the commutativity assumption.

**Lemma 2.12.** Let $R$ be a Noetherian positively graded connected $\mathbb{K}$-algebra generated in degree 1, $P$ a homogeneous prime ideal in $R$ not equal to the irrelevant ideal and $M$ a right graded $R$-module. Then, $\text{Hom}_{\text{Gr}(R)}(M,E^g(R/P)) = 0$ if and only if $\overline{\text{Hom}}(M,E^g(R/P)) = 0$.

**Proof.** One direction is clear. Suppose $\overline{\text{Hom}}(M,E^g(R/P)) \neq 0$. Then by lemma 2.7 there is $m \in h(M)$ such that $\text{Ann}(m) \cap C^g_i(P) = \emptyset$, where $\text{Ann}(m)$ stands for right annihilator of $m$ and $C^g_i(P)$ stands for homogeneous left regular elements mod $P$.

We want to prove $\text{Hom}_{\text{Gr}(R)}(M,E^g(R/P)) = 0$ which, by lemma 2.7 and remark 2.1 is equivalent to the existence of $\tilde{m} \in h(M_{\geq 0})$ such that $\text{Ann}(\tilde{m}) \cap C^g_i(P) = \emptyset$.

Note that the irrelevant ideal, $R_+$, is a homogeneous maximal ideal containing $P$. So, by graded Goldie’s theorem (see [10]), $R_+/P$ is an essential ideal in the graded prime Goldie ring $R/P$, thus containing a regular element. This means that there is a homogeneous regular element of positive degree in $R/P$ and thus $C^g_i(P)_{\geq k} \neq \emptyset$ for all $k \in \mathbb{N}$. Choose $s \in C^g_i(P)$ such that $\deg(ms) \geq 0$. Note that if there is $a \in \text{Ann}(ms) \cap C^g_i(P)$, then $sa \in \text{Ann}(m) \cap C^g_i(P)$ yielding a contradiction. Therefore, take $\tilde{m} = ms$ and we are done. □

**Remark 2.13.** We summarise the results of this section. If $R$ is a noetherian positively graded connected $\mathbb{K}$-algebra generated in degree 1, $P$ a homogeneous prime ideal not equal to the irrelevant ideal $R_+$ and $M$ a graded $R$-module, then the following are equivalent:

1. $\text{Hom}_{\text{Gr}(R)}(M,E^g(R/P)) = 0$;
2. $\overline{\text{Hom}}(M,E^g(R/P)) = 0$;
3. $M$ is torsion with respect to $C^g_i(P)$.

If, furthermore, $R$ is commutative and $S = h(R \setminus P)$, then (1), (2) and (3) are equivalent to $S^{-1}M = 0$ and to $(S^{-1}M)_0 = 0$.

3. **T-structures via torsion theories**

Let $\mathcal{A}$ be a complete and cocomplete Abelian category. Fix $a \in \mathbb{Z}$, $n \in \mathbb{N}$ and an ordered set (indexed by a string of integers of length $n$ starting at $a$) of hereditary torsion classes $S = \{T_a, T_{a+1}, \ldots, T_{a+n-1}\}$ such that $\bigcap_{i=a}^{a+n-1} T_i = 0$. 


Our target is to prove that the following subcategory is the aisle of a $t$-structure:

$$D^{S, \leq 0} = \{ X^* \in D(A) : H^i(X^*) \in T_j, \ \forall i > j \}.$$  

Remark 3.1. Clearly such a category is a subcategory of $D^{S_0 + n - 1}$, a shift of the aisle of the standard $t$-structure. This is because the intersection of all torsion classes is zero.

We will start by proving a simple but useful lemma regarding how truncations between two $t$-structures are related when their aisles are subject to a similar relation of the remark above.

Lemma 3.2. Suppose $(D^{\leq 0}_A, D^{\geq 0}_A)$ and $(D^{\leq 0}_B, D^{\geq 0}_B)$ are two $t$-structures in a triangulated category $D$ with truncation functors $t^{\leq 0}_A$ and $t^{\leq 0}_B$, respectively. If $D^{\leq 0}_A \subset D^{\leq 0}_B$, then for all $X \in D$ there is a triangle:

$$t^{\leq 0}_A(X) \rightarrow t^{\leq 0}_B(X) \rightarrow Y \rightarrow t^{\leq 0}_A(X)[1]$$

such that $Y \in D^{\geq 1}_A \cap D^{\leq 0}_B$.

Proof. First note that, since $D^{\leq 0}_A \subset D^{\leq 0}_B$, we have $D^{\geq 1}_B \subset D^{\geq 1}_A$. The triangle

$$t^{\leq 0}_B(X) \rightarrow X \rightarrow t^{\geq 1}_B(X) \rightarrow t^{\leq 0}_B(X)[1]$$

then shows that the natural map $t^{\leq 0}_A(X) \rightarrow X$ must factor through $t^{\leq 0}_B(X)$ (since $\text{Hom}(t^{\leq 0}_A(X), t^{\geq 1}_B(X)) = 0$). Let $Y$ be defined by the following triangle

$$t^{\leq 0}_A(X) \rightarrow t^{\leq 0}_B(X) \rightarrow Y \rightarrow t^{\leq 0}_A(X)[1].$$

Since aisles are closed under taking cones and $t^{\leq 0}_A(X) \in D^{\leq 0}_B$, we have that $Y \in D^{\leq 0}_B$. We want to prove $Y \in D^{\geq 1}_A$. Consider the diagram

$$
\begin{array}{ccc}
{t^{\leq 0}_A(X)} & \rightarrow & {t^{\leq 0}_B(X)} \\
| & | & | \\
{t^{\leq 0}_A(X)} & \rightarrow & X \\
| & | & | \\
{t^{\leq 0}_B(X)} & \rightarrow & X \\
\end{array}
\rightarrow
\begin{array}{ccc}
Y & \rightarrow & t^{\leq 0}_A(X)[1] \\
| & | & | \\
{t^{\leq 0}_A(X)} & \rightarrow & X \\
| & | & | \\
{t^{\leq 0}_B(X)} & \rightarrow & X \\
\end{array}
\rightarrow
\begin{array}{ccc}
Y & \rightarrow & t^{\leq 0}_A(X)[1] \\
| & | & | \\
{t^{\leq 0}_A(X)} & \rightarrow & X \\
\end{array}
\rightarrow
\begin{array}{ccc}
Y & \rightarrow & t^{\leq 0}_A(X) \\
| & | & | \\
{t^{\leq 0}_B(X)} & \rightarrow & X \\
\end{array}
$$

where rows are triangles and the squares commute by the observation above. Then, the octahedral axiom gives us a new triangle

$$Y \rightarrow t^{\geq 1}_A(X) \rightarrow t^{\geq 1}_B(X) \rightarrow Y[1].$$

Since $t^{\geq 1}_B(X) \in D^{\geq 1}_A$, so is $t^{\geq 1}_B(X)[-1]$. By the long exact sequence of cohomology induced from this triangle, it is easy to see that this shows that $Y \in D^{\geq 1}_A$. \qed

Our proof relies on a suitable iteration of the following well-known theorem. Although the original result is due to Happel, Reiten and Smalø (122) we present here a slightly different version of that result presented by Bridgeland (9).
Theorem 3.3 (Happel, Reiten, Smalø, Bridgeland, [22, 9]). Let $A$ be the heart of a bounded t-structure in $D$, a triangulated category. Suppose that $(T, F)$ is a torsion theory on $A$ and that $H^i$ denotes the $i$-th cohomology functor with respect to $A$. Then $(D^\leq 0, D^\geq 0)$ is a t-structure on $D$, where

$$D^\leq 0 = \{ E \in D : H^i(E) = 0, \forall i > 0, H^0(E) \in T \}$$

and

$$D^\geq 0 = \{ E \in D : H^i(E) = 0, \forall i < -1, H^{-1}(E) \in F \}.$$ 

Theorem 3.4. For $n \geq 2$, $D^{S, \leq 0}$ is the aisle of a t-structure in $D(A)$.

Proof. We shall use induction on $n$. Without loss of generality we shall assume $a = -n + 1$.

Suppose $n = 2$ and $S = \{ T_{-1}, T_0 \}$. Thus we have

$$D^{S, \leq 0} = \{ X^* \in D(A) : H^0(X^*) \in T_{-1}, H^1(X^*) = 0, \forall i > 0 \}.$$ 

To see this note that for $X \in D^{S, \leq 0}$, we have $H^i(X^*) = 0$ for all $i > 0$ since, for such $i$, $H^i(X^*) \in T_{-1} \cap T_0 = 0$. This is the aisle of a t-structure: it is obtained via tilting with respect to $T_{-1}$, by theorem 3.3.

Suppose the result is valid for any ordered set of $n$ hereditary torsion classes with zero intersection. Let $S$ be such a set with $n + 1$ elements, i.e., $S = \{ T_{-n}, T_{-n+1}, \ldots, T_0 \}$. We want to prove that $D^{S, \leq 0}$ defines a t-structure on $D(A)$.

First let us consider $\bar{S} = \{ \bar{T}_{-n+1}, \bar{T}_{-n+2}, \ldots, \bar{T}_0 \}$ where $\bar{T}_i = T_{i-1}$ for $i < 0$ and $\bar{T}_0 = T_{-1} \cap T_0$. Clearly, by assumption on $S$, this is also an ordered set of torsion classes with zero intersection in $A$. We fall into the case of $n$ torsion classes and by the induction hypothesis we have an associated t-structure whose heart will be denoted by $B$. The corresponding cohomological functor will be denoted by $H^0_\bar{S} := t^{\leq 0}_\bar{S} t^{\leq 0}_\bar{S}$, where the $t^{\leq 0}_\bar{S}$'s are the associated truncation functors.

Consider now the following subcategory of $B$:

$$W = \{ X^* \in B : H^0(X^*) \in T_{-1}, H^1(X^*) = 0, \forall i < 0 \}.$$ 

$W$ can be seen as the stalk subcategory $0 \to 0 \to \bigcap_{i = -n}^{i = -1} T_i \to 0$ (since it is a subcategory of $B$ and therefore all positive cohomologies vanish). Since $A$ is complete and cocomplete, to see that $W$ is a torsion class inside $B$ we only need to check that it is closed under epimorphic images, coproducts and extensions (see [22] for details). Indeed, since homomorphisms in $W$ can be seen as homomorphisms in $A$ and $W$ is a torsion class in $A$, $W$ is closed under epimorphic images and coproducts. It is also closed under extensions since exact sequences in $B$ are precisely the distinguished triangles of $D(A)$ that lie in $B$ and the result follows from the long exact sequence of cohomology of a distinguished triangle.

Now the crucial observation is the following lemma:

Lemma 3.5. $X^* \in D^{S, \leq 0}$ if and only if $H^0_\bar{S}(X^*) \in W$ and $H^i_\bar{S}(X^*) = 0$ for all $i > 0$.

Proof. Note first that $H^i_\bar{S}(X^*) = 0$, for all $i > 0$, is equivalent to $X^* \in D^{S, \leq 0}$.

Suppose $X^* \in D^{S, \leq 0}$. It is clear from the definition of the perversity $\bar{S}$ that $D^{S, \leq 0} \subset D^{S, \leq 0}$, thus proving the vanishing of positive $\bar{S}$-cohomologies.

Now, we can fit $\bar{S}$-cohomology in the following distinguished triangle:

$$t^{\leq -1}_\bar{S}(X^*) \to t^{\leq 0}_\bar{S}(X^*) \to H^0_\bar{S}(X^*) \to t^{\leq -1}_\bar{S}(X^*)(1)$$
which, again due to the fact that $D^{S,\leq 0} \subset D^{\bar{S},\leq 0}$, amounts to the distinguished triangle

$$t^{\leq -1}(X^\bullet) \to X^\bullet \to H^0_S(X^\bullet) \to t^{\leq -1}(X^\bullet)[1].$$

Now, lemma 3.2 applied to $D^{\bar{S},\leq -1} \subset D^{\bar{S},\leq 1}$ (see remark 3.1) shows that

$$t^{\leq -1}(X^\bullet) \to t^{\leq -1}(X^\bullet) \to Y \to t^{\leq 0}(X^\bullet)[1]$$

where $Y \in D^{\bar{S},\leq 0} \cap D^{\leq -1}$. Since $X^\bullet \in D^{S,\leq 0}$, we have that $t^{\leq -1}(X^\bullet) \in D^{S,\leq -1}$ and thus $Y = 0$. Therefore, $t^{\leq -1}(X^\bullet) \cong t^{\leq -1}(X^\bullet)$. Since in any distinguished triangle two of the vertices determine the third one up to isomorphism, we have

$$H^0_S(X^\bullet) = H^0(X^\bullet)$$

which, by definition of $D^{S,\leq 0}$, tells us that $H^0_S(X^\bullet) \in \mathcal{W}$.

Conversely, suppose $X^\bullet \in D^{S,\leq 0}$ and $H^0_S(X^\bullet) \in \mathcal{W}$. As before, we have an exact triangle

$$t^{\leq -1}(X^\bullet) \to X^\bullet \to H^0_S(X^\bullet) \to t^{\leq -1}(X^\bullet)[1]$$

whose long exact sequence of cohomology (for the standard cohomology functor) tells us that $H^i(t^{\leq -1}(X^\bullet)) \cong H^i(X^\bullet)$ for all $i < 0$ (since negative cohomologies vanish for $H^0_S(X^\bullet)$) and that $H^0(X^\bullet) \cong H^0(H^0_S(X^\bullet)) \in \mathcal{W}$. Note that we will have $H^i(t^{\leq -1}(X^\bullet)) \in T_{i-1}$ and thus $H^i(X^\bullet) \in T_{i-1}$ for all $-n + 1 \leq i < 0$. This is because $D^{S,\leq -1} = D^{\bar{S},\leq 0}[1]$. On the other hand $H^0(X^\bullet) \cong H^0(H^0_S(X^\bullet)) \in \mathcal{W}$ proving that $H^0(X^\bullet) \in T_{-1}$. This is precisely the additional conditions that an element in $D^{S,\leq 0}$ needs to satisfy to be in $D^{S,\leq 0}$, thus finishing the proof.

$D^{S,\leq 0}$ can then be obtained by tilting the heart $\mathcal{B}$, defined earlier in this proof, with respect to the torsion theory whose torsion class is $\mathcal{W}$. Therefore it is the aisle of a $t$-structure.

\[\Box\]

Remark 3.6. Note that the assumption that $\mathcal{A}$ is complete and cocomplete was important to prove that $\mathcal{W}$ is a torsion class in $\mathcal{B}$. This fact would also hold if $\mathcal{B}$ were noetherian (since in this case it would also be enough to check that $\mathcal{W}$ is closed under coproducts, epimorphic images and extensions). However, it is well known that $\mathcal{B}$ might not be noetherian even when $\mathcal{A}$ is (see [21] for a discussion of this topic), hence our assumption.

Recall that a $t$-structure is said to be nondegenerate if

$$\bigcap_{n \in \mathbb{Z}} \text{Ob } D^{\leq n} = 0 \text{ and } \bigcap_{n \in \mathbb{Z}} \text{Ob } D^{\geq n} = 0.$$  

Clearly, the standard $t$-structure is nondegenerate.

Lemma 3.7. The $t$-structure associated to an ordered set of hereditary torsion classes $S$ with zero intersection, as defined above, is nondegenerate.

Proof. Suppose without loss of generality that the maximal index in $S$ is zero. Then, as before, $D^{S,\leq 0} \subseteq D^{\leq 0}$. The standard $t$-structure is nondegenerate and thus $\bigcap_{n \in \mathbb{Z}} D^{S,\leq n} = 0$.

On the other hand

$$\bigcap_{n \in \mathbb{Z}} D^{S,\geq n} = \bigcap_{n \in \mathbb{Z}} (D^{S,\leq n-1})^\perp$$

where $D^{S,\leq n-1}$ is the subcategory of $D^{S,\leq n}$ determined by the torsion classes in $S$. Since $D^{S,\leq n}$ is the collection of objects that are zero in $D^{S,\leq n}$, it follows that $\bigcap_{n \in \mathbb{Z}} D^{S,\geq n} = 0$.

\[\Box\]
Then \( \text{Hom}_{\text{Gr}} \) is injective, \( B \) is torsion-free and injective. Since \( D^{\leq n-1} \subseteq D^{\leq n-1+k} \) for any \( k \geq 0 \) and \( \bigcup_{n \in \mathbb{Z}} D^{\leq n} = D \) we also have \( \bigcup_{n \in \mathbb{Z}} D^{S_i, \leq n-1} = D \) and thus \( \bigcup_{n \in \mathbb{Z}} D^{S_i, \geq n} = D. \)

\[ \text{Remark 3.8. A nondegenerate t-structure restricts well to the bounded derived category (ES). Thus, the construction above can be restricted to bounded derived categories.} \]

4. **Perverse coherent t-structures through torsion theories**

We are now going to prove the main theorem of this text but let us fix some notation beforehand. Let \( X \) be a smooth projective scheme such that its homogeneous coordinate ring \( R = \Gamma(X) \) is an algebro-geometric positively graded \( \mathbb{K} \)-algebra generated in degree 1, where \( \mathbb{K} \) is, as before, algebraically closed of characteristic zero. \( \pi \) shall denote the projection functor from \( \text{Gr}(R) \) to its quotient \( \text{Tails}(R) \) (and the corresponding restriction to \( \text{gr}(R) \)) and note that \( \Gamma(\pi M) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Tails}(R)}(\pi R, \pi M(i)) \) is a left adjoint of \( \pi \). For more details on the formalism of these quotient categories check [3], for instance.

Let \( p : X^{\text{top}} \to \mathbb{Z} \) be a perversity as in the introduction. Suppose that the perversity has \( n \) values and that, without loss of generality, the maximal value of the perversity is zero. Set \( E_i = \prod_{x \in X, p(x) = i} E^q(R/I_x) \) for \( i \in \text{Im}(p) \), where \( I_x \) is the homogeneous ideal of functions vanishing at \( x \).

**Lemma 4.1.** Let \( A \) and \( B \) be graded modules over \( R \), \( B \) torsion-free and injective. Then \( \text{Hom}_{\text{Gr}(R)}(A, B) = 0 \) if and only if \( \text{Hom}_{\text{Tails}(R)}(\pi A, \pi B) = 0 \)

**Proof.** Suppose \( f \in \text{Hom}_{\text{Tails}(R)}(\pi A, \pi B) \neq 0. \) \( B \) is torsion-free and so we have

\[ \text{Hom}_{\text{Tails}(R)}(\pi A, \pi B) = \lim_{A' \leq A: A' \text{ is torsion}} \text{Hom}_{\text{Gr}(R)}(A', B). \]

Let \( A' \) be such that there is \( \tilde{f} \in \text{Hom}_{\text{Gr}(R)}(A', B) \) such that \( \pi \tilde{f} = f \). Then, since \( B \) is injective, \( \tilde{f} \) can be extended to \( A \), proving that \( \text{Hom}_{\text{Gr}(R)}(A, B) \neq 0 \).

Conversely, suppose \( f \in \text{Hom}_{\text{Gr}(R)}(A, B) \neq 0. \) Since \( B \) is torsion-free, we have that for all \( A' \leq A \) such that \( A/A' \) is torsion, \( f|_{A'} \neq 0 \) (since otherwise we would have a nonzero map from \( A/A' \) to \( B \), which is not allowed by definition of torsion). Thus \( \pi f \neq 0 \) in \( \text{Hom}_{\text{Tails}(R)}(\pi A, \pi B). \)

Recall that a set of injective objects \( \{I_1, \ldots, I_n\} \) in an abelian category \( \mathcal{A} \) is a cogenerating set for \( \mathcal{A} \) if, for any \( X \in \mathcal{A}, \text{Hom}(X, I_j) = 0 \) for all \( j \) implies \( X = 0. \)

Clearly, if \( T_i \) is the torsion class in \( \mathcal{A} \) associated to \( I_i \) (i.e., the set of objects \( X \) such that \( \text{Hom}_A(X, I_i) = 0) \) then \( \{I_1, \ldots, I_n\} \) is a cogenerating set if and only if \( \bigcap_i T_i = 0. \)

**Remark 4.2.** Note that given \( R \) positively graded noetherian connected \( \mathbb{K} \)-algebra, \( R/P \) is torsion-free for any homogeneous prime ideal \( P \) not equal to the irrelevant ideal. Indeed if, for \( x \notin P, xR_{\geq n} = 0 \) then \( (Rx)(R_{\geq n}) \subset P \) and hence, by [2.1] \( RxR \subset P \) which yields a contradiction.
Corollary 4.3. The objects $\pi E_i$ cogerenerate $\text{Tails}(R)$, where $R = \Gamma(X)$ is as above.

Proof. Suppose that $M$ is not torsion, i.e., that there is an element $m \in h(M)$ such that $\text{Ann}(m) \neq R_{\geq n}$ for any $n > 1$. We prove that $\text{Ann}(m)$ is contained in a homogeneous prime ideal. It is clear that, since $m$ is not torsion, the radical of $\text{Ann}(m)$, which we shall denote by $\sqrt{\text{Ann}(m)}$, is not the augmentation ideal $R_+$. Thus we can choose $f \in R_1$ such that $f \notin \sqrt{\text{Ann}(m)}$. Applying Zorn’s lemma to the set $S = \{ f \supset \text{Ann}(m) \text{ homogeneous} : f \notin \sqrt{J} \}$ (which is nonempty since $\text{Ann}(m) \in S$) we get a maximal element - call it $P$. We prove that $P$ is prime. In fact, for $a, b \in h(R)$, if $ab \in P$ and $a \notin P$, then there is an integer $l$ such that $f^l \in aR + P$ (since $P$ is maximal in $S$). If there is an integer $s$ such that $f^s \in bR + P$, then $f^{l+s} \in (aR + P)(bR + P) \subset P$, a contradiction. Hence $b \in P$. This proves that $P$ is a homogeneous gr-prime ideal.

To complete the proof we need the lemma below. Recall that in noncommutative ring theory primality of an ideal $P$ is defined in terms of products of ideals, i.e., if $IJ \subset P$ for some ideals $I$ and $J$, then $I \subset P$ or $J \subset P$. If this property holds at the level of elements (i.e., if $ab \in P$ for some elements $a, b$ of the ring, then $a \in P$ or $b \in P$) then we say $P$ is strongly prime. There are obvious graded counterparts of these notions and the following property holds.

Lemma 4.4 (Nastasescu, Van Oystaeyen, [20]). For a $\mathbb{Z}$-graded ring, a homogeneous ideal is gr-strongly prime if and only if it is strongly prime.

Since $R$ is commutative, the notions of prime and strongly prime are the same and hence $P$ is prime.

Note now that there is a graded isomorphism from $R/\text{Ann}(m)(\text{deg}(m))$ to $mR$ and thus a graded injection from $R/\text{Ann}(m)(\text{deg}(m))$ to $M$. Since $\text{Ann}(m)$ is contained in a homogeneous prime ideal $P$, $R/\text{Ann}(m)(\text{deg}(m))$ maps nontrivially to $E^2(R/P)(\text{deg}(m))$ and thus so does $M$. Since $R$ satisfies the hypothesis of lemma 2.12, one has that $M$ maps nontrivially to $E^2(R/P)$ and thus, by the previous lemma, $\text{Hom}_{\text{Tails}}(\pi M, \pi E^2(R/P)) \neq 0$. □

Before stating the main theorem, we need to prove the following useful lemma.

Lemma 4.5. Suppose $R$ is a commutative local ring with maximal ideal $m$. Given $X^\bullet$ a bounded complex of finitely generated free $R$-modules, define $Y^\bullet$ to be the complex $R/m \otimes_R X^\bullet$. If, for some fixed integer $\alpha$, $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$, then $H^j(X^\bullet) = 0$ for all $j \geq \alpha$.

Proof. Suppose $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$. Suppose $X^k = 0$ for all $k \geq p$ ($X^\bullet$ is bounded). If $\alpha > p$ then it is done. If $\alpha \leq p$, consider the following exact sequence:

$$X^{p-1} \longrightarrow X^p \longrightarrow \text{coker}(d_{X}^{p-1}) \longrightarrow 0$$

and apply to it the functor $F := R/m \otimes_R -$ thus getting another exact sequence

$$Y^{p-1} \longrightarrow Y^p \longrightarrow R/m \otimes_R \text{coker}(d_{X}^{p-1}) \longrightarrow 0,$$
since F is right exact. By definition of $Y^\bullet$, the first map of the sequence is $d^p_{Y^0}$. Since $\alpha \leq p$, $H^p(Y^\bullet) = 0$, thus proving that $(d^p_{Y^0})$ is surjective ($Y^{p+1} = 0$ by definition). Therefore $R/m \otimes_R \text{coker}(d^p_{X^0}) = 0$ which, by Nakayama’s lemma (since $R$ is local and $\text{coker}(d^p_{X^0})$ is a finitely generated $R$-module), implies that $\text{coker}(d^p_{X^0}) = 0$. Hence $d^p_{X^0}$ is surjective, thus proving that $H^p(X^\bullet) = 0$.

If $\alpha = p$ it is done. Otherwise assume $H^{p-1}(Y^\bullet) = 0$ and we prove that $H^{p-1}(X^\bullet) = 0$ as well. Note that then the result follows by iterating this process a finite number of times (the difference between $\alpha$ and $p$). First, since $X^p$ is free, the short exact sequence

$$0 \rightarrow \text{Ker}(d^p_{X^0}) \rightarrow X^{p-1} \rightarrow X^p \rightarrow 0$$

splits and thus $\text{Ker}(d^p_{X^0})$ is a summand of the free module $X^{p-1}$, i.e., a projective module (it is exact because $H^p(X^\bullet) = 0$ and $X^{p+1} = 0$). However it is well-known (Kaplansky’s theorem) that projective modules over local rings are free.

Now we observe that $\text{Ker}(d^p_{X^0}) \cap mX^{p-1} = m\text{Ker}(d^p_{X^0})$. In fact let $z_1, ..., z_n$ be a basis for $X^{p-1}$ such that the first $t$ elements form a basis for $\text{Ker}(d^p_{X^0})$. Given $x \in \text{Ker}(d^p_{X^0}) \cap mX^{p-1}$ we have, on one hand $x = \sum_{i=1}^t a_i z_i$ with $a_i \in R$ and on the other hand $x = \sum_{i=1}^n b_i z_i$, with $b_i \in m$. Linear independence of the elements of the basis assure $b_i = 0$ for $i > t$ and $a_i = b_i$ for $i \leq t$, thus proving that $x \in m\text{Ker}(d^p_{X^0})$. The converse inclusion is trivial. This allows us to see, by considering the natural map from $\text{Ker}(d^p_{X^0})$ to $(\text{Ker}(d^p_{X^0}) + mX^{p-1})/mX^{p-1}$, that

$$\text{Ker}(d^p_{X^0}) = \frac{\text{Ker}(d^p_{X^0}) + mX^{p-1}}{mX^{p-1}} \cong \frac{\text{Ker}(d^p_{X^0})}{m\text{Ker}(d^p_{X^0})} = \frac{R}{m} \otimes_R \text{Ker}(d^p_{X^0}).$$

This means that, by definition of truncation, $\hat{Y}^\bullet := t_{\leq p-1}(Y^\bullet) = R/m \otimes_R \hat{X}^\bullet$ where $\hat{X}^\bullet = t_{\leq p-1}(X^\bullet)$. Note that, since $H^{p-1}(Y^\bullet) = 0$, $\text{Ker}(d^p_{Y^0}) = \text{Im}(d^{p-1}_{Y^0})$ and thus

$$\hat{Y}^{p-2} = Y^{p-2} \rightarrow \text{Ker}(d^p_{Y^0}) = \hat{Y}^{p-1}$$

is surjective, allowing us to conclude, by repeating first argument of the proof, that $H^{p-1}(\hat{X}^\bullet) = 0$, which concludes the proof since $\hat{X}^\bullet$ is quasi-isomorphic to $X^\bullet$, thus having the same cohomology.

\[\square\]

**Corollary 4.6.** Suppose $R$ is a commutative local $\mathbb{K}$-algebra. Given $X^\bullet$ a bounded complex of finitely generated free $R$-modules and $Z^\bullet = \mathbb{K} \otimes_R X^\bullet$, if $H^j(Z^\bullet) = 0$ for all $j \geq \alpha$, then $H^j(X^\bullet) = 0$ for all $j \geq \alpha$, for some fixed integer $\alpha$.

**Proof.** Let $m$ denote the unique maximal ideal of $R$. Note that

$$Y^\bullet = R/m \otimes_R X^\bullet = (R/m \otimes_\mathbb{K} \mathbb{K}) \otimes_R X^\bullet = R/m \otimes_\mathbb{K} Z^\bullet.$$ 

Since tensoring over $\mathbb{K}$ is exact, we have that $H^j(Z^\bullet) = 0$ for all $j \geq \alpha$ implies $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$ and hence, by the previous lemma, we have the result. \[\square\]

Finally we prove the main theorem of the chapter.

**Theorem 4.7.** Given a perversity $p$, denote by $T_i$ the torsion theory cogenerated by $\pi E_i = \pi \prod_{\{x \in X_{p(\pi)}\}} E^q(R/I_x)$ in $\text{Tails}(R)$. Let $S := \{T_i : i \in \text{im}(p)\}$. Then

$$D^p_{\leq 0} = D^S_{\leq 0} \cap D^b(\text{coh}(X)).$$
Proof. Let us denote by \( T_i \) the torsion theory cogenerated by \( E_i \) in \( Gr(R) \). We start by rewriting the conditions defining the aisle \( D^{S, \leq 0} \). By definition, we have

\[
D^{S, \leq 0} = \{ F^* \in D^b(\text{Tails}(R)) : H^j(F^*) \in T_k, \forall j > k \}
\]

and given that the objects \( E_k \) are torsion-free injective objects, by lemma \( \text{[1.1]} \) we have

\[
D^{S, \leq 0} = \{ F^* \in D^b(\text{Tails}(R)) : \Gamma(H^j(F^*)) \in T_k, \forall j > k \} = \{ F^* \in D^b(\text{Qcoh}(X)) : \forall x \in X^{\text{top}}, \text{Hom}_{Gr(R)}(\Gamma(H^j(F^*)), E^q(R/I_x)) = 0, \forall j > p(x) \}.
\]

Now we intersect with \( D^b(\text{coh}(X)) \) (i.e., pass from the quasi-coherent setting to the coherent one). For simplicity, define \( D^{p, \leq 0} := D^{S, \leq 0} \cap D^b(\text{coh}(X)) \). Using corollary \( \text{[2.5]} \) we get

\[
D^{p, \leq 0} = \{ F^* \in D^b(\text{coh}(X)) : \forall x \in X^{\text{top}}, \text{Hom}_{Gr(R)}(\Gamma(H^j(F^*)), E^q(R/I_x)) = 0, \forall j > p(x) \}.
\]

Recall that

\[
D^{p, \leq 0} = \{ F^* \in D^b(\text{coh}(X)) : \forall x \in X^{\text{top}}, Li_x^*(F^*) \in D^{\leq p(x)}(O_x - \text{mod}) \}
\]

which is clearly the same as

\[
\{ F^* \in D^b(\text{coh}(X)) : \forall x \in X^{\text{top}}, H^j(Li_x^*(F^*)) = 0, \forall j > p(x) \}.
\]

Hence, it suffices to prove that \( H^j(F^*_x) = 0 \) for all \( j > p(x) \) is equivalent to \( Li_x^*(F^*) = 0, \forall j > p(x) \).

Suppose \( F^* \) such that \( H^j(F^*_x) = 0 \) for all \( j > p(x) \). By definition of the pullback functor \( \psi_x^*(V) = V_x \otimes_{O_{X,x}} \mathbb{K} \) for any coherent sheaf \( V \), there is a spectral sequence of Grothendieck type of the following form:

\[
E_2^{a,b} = \text{Tor}^a_{\mathcal{O}_{X,x}}(\mathbb{K}_x, H^b(F^*)) \Rightarrow L_{a+b}^i(F^*),
\]

where \( \mathbb{K}_x \) is the skyscraper sheaf over \( x \). Our hypothesis assures that \( E_2^{a,b} = 0 \) for all \( a < 0 \) or \( b > p(x) \) and thus \( E_2^{a,b} = 0 \) for all \( a < 0 \) or \( b > p(x) \). Let \( F^i \) denote the \( i \)-th part of the decreasing filtration assumed to exist (by definition of convergent spectral sequence) on the limit object \( \Omega^{a+b} := L_{a+b}^i(F^*) \). Then, for \( q > p(x) \) we get

\[
\_ = \mathbb{F}^{-2} \Omega^{-2+(q+2)} = \mathbb{F}^{-1} \Omega^{-1+(q+1)} = \mathbb{F}^0 \Omega^q = \mathbb{F}^1 \Omega^q
\]

and thus they are all equal to zero, proving that \( \Omega^q = L_{q}^i(F^*) = 0 \) for all \( q > p(x) \).

Conversely, suppose we have \( F^* \) such that \( L_{j}^i(F^*) = 0 \) for all \( j > p(x) \). Since \( X \) is smooth, let \( G^* \) be a complex of locally free sheaves such that \( G^* \) is quasi-isomorphic to \( F^* \) (thus isomorphic in the derived category) - [1.3]. Then \( L_{j}^i(F^*) = 0 \) means that \( H^j((i^*_x G)^*) \), where \( (i^*_x G)^* \) denotes the complex resulting from applying \( i^*_x \) componentwise to \( G^* \). Take now \( X^* = G^* \) and \( Y^* = (i^*_x G)^* \) and recall that \( G^* \) is a complex of free modules over the local ring \( O_{X,x} \). This leaves us in the context of corollary 5.4.5, thus proving that \( H^j(G^*_x) = H^j(G^*)_x = 0 \) for all \( j > p(x) \). Finally we have \( H^j(F^*_x) = H^j(F^*)_x = H^j(G^*)_x = 0 \) for all \( j > p(x) \), hence finishing the proof. \( \square \)
Remark 4.8. Note that as a consequence we get that, for $S$ defined as above, the t-structure constructed in section 3 restricts well to the derived category of finitely generated objects (coherent sheaves).

5. PERVERSE QUASI-COHERENT T-STRUCTURES FOR NONCOMMUTATIVE PROJECTIVE PLANES

The aim of this section is to use the construction of section 3 to create an analogue of perverse coherent t-structures in the derived categories of certain noncommutative projective planes. This entails finding a cogenerating set of injective objects in $\text{Tails}(R)$ for a suitable class of $\mathbb{K}$-algebras $R$ and set up a definition of perversity that generalises the commutative one.

Remark 5.1. $\text{Tails}(R)$ is not complete nor cocomplete and therefore, in this section, we can only do the construction of section 3 in $\text{Tails}(R)$ (hence the word quasi-coherent rather than coherent in the title). However, taking into account theorem 4.7, we conjecture that indeed the constructions in this section restrict well to $D^b(\text{Tails}(R))$.

We shall focus on the case where $R$ is a graded elliptic 3-dimensional Artin-Schelter regular algebra which is finite over its centre. These algebras are interesting for our purposes since they are fully bounded noetherian (more than that, they are PI - [5]). Also, a graded noetherian algebra which is fully bounded is graded fully bounded ([25]). This is important for the following result that allows us to parametrise a useful collection of injective objects via prime ideals. In this sense, although these examples are noncommutative, we are still very close to the commutative setting ([17]).

Recall that there is a map from the set of indecomposable injective graded modules to the set of homogeneous prime ideals given by assigning to an injective $E$ its homogeneous assassinator ideal, $\text{Ass}(E)$. The assassinator ideal of an indecomposable object is the only prime ideal associated to $E$, i.e., the only prime ideal which is maximal among the annihilators of nonzero submodules of $E$ (and there is a natural graded version of this concept - see [19] and [25]).

Proposition 5.2 (Natalescu, Van Oystaeyen, [19]). Let $R$ be a positively graded noetherian ring. Then $R$ is graded fully bounded if and only if the map that assigns the corresponding assassinator ideal to an indecomposable injective graded module induces a bijection between indecomposable injective modules in $\text{Gr}(R)$ (up to isomorphism and graded shift) and homogeneous prime ideals of $R$.

Remark 5.3. In the context of the proposition, the indecomposable injective associated with a homogeneous prime $P$ is the unique (up to isomorphism and shifts) direct summand of $E^0(R/P)$ ([19]), thus establishing an inverse map.

This result brings us closer to the desired cogenerating set. Its significance in our context comes from the work of Matlis on the decomposition of injective modules over noetherian rings. Matlis proved that $R$ is (right) noetherian if and only if every injective (right) module is the direct sum of indecomposable injective (right) modules ([17]). This shows in particular that the set of indecomposable injective objects cogenerates the category of modules over a noetherian ring. One may hope a similar phenomena for graded rings and indeed one has the following ([22]).
Proof. Let $\hat{Y}$ denote the set of all such classes where $E$ runs over indecomposable injective objects, up to isomorphism and graded shift, such that its assasinator ideal is not the irrelevant ideal, i.e., $Ass(E) \neq R_+$. Analogously define $Y$ to be the set of the torsion classes of the form $T_{\pi E}$ in Tails$(R)$. 

**Corollary 5.5.** Let $R$ be a positively graded fully bounded connected noetherian $\mathbb{K}$-algebra generated in degree 1. Then the intersection (in Tails$(R)$) of the torsion classes in $Y$ is zero.

**Proof.** Suppose $\pi M$ lies in the intersection of the torsion classes of $Y$. Then by lemma 4.1, $M$ lies in the intersection of the torsion classes in $Y$. By proposition 5.4 the indecomposable injective objects cogenerate $Gr(R)$ and thus $E^0(M)$ must be a finite direct sum of the indecomposable injective associated with $R_+$. This indecomposable is a direct summand of $E^0(R/R_+)$ (see remark 5.3), whose projection in Tails$(R)$ is therefore zero. Thus $\pi E^0(M) = 0$ and so is $\pi M$. 

We proceed now to do the desired construction. Let $R$ be a positively graded Artin-Schelter regular algebra of dimension 3 generated in degree one which is finitely generated over its centre. As discussed before, it is graded fully bounded noetherian. We need to define a perversity in $Y$, where $Y$ is as above.

**Definition 5.6.** A perversity is a map $p : Y \rightarrow \mathbb{Z}$ such that, given $T_{\pi E_1}, T_{\pi E_2}$ in $S$, if there is a nonzero homomorphism from $\pi E_2$ to $\pi E_1$ then $p(T_{\pi E_1}) - (GKdim(R/Ass(E_2)) - GKdim(R/Ass(E_1))) \leq p(T_{\pi E_2}) - p(T_{\pi E_1})$.

We now prove that this definition of perversity coincides, when the algebra is commutative, with the definition of perversity of the introduction. We start by a supporting lemma.

**Lemma 5.7.** Let $R$ be a positively graded commutative noetherian $\mathbb{K}$-algebra and $X = \text{Proj}(R)$. The following are equivalent.

1. For $x_1, x_2 \in X^{\text{top}}$, $x_1 \in \overline{x_2}$;
2. $P_2 := \text{Ann}(x_2) \subset \text{Ann}(x_1) =: P_1$, where $\text{Ann}(x_1)$ denotes the homogeneous ideal of functions vanishing in $x_1$;
3. There is a nonzero homomorphism from $R/P_2$ to $R/P_1$;
4. There is a nonzero homomorphism from $E^0(R/P_2)$ to $E^0(R/P_1)$.

**Proof.** It is clear that (1) $\iff$ (2) $\implies$ (3) $\implies$ (4). We only need to prove (4) $\implies$ (2). Let $f$ be a homomorphism from $E^0(R/P_2)$ to $E^0(R/P_1)$ and $a \in h(P_2) \setminus h(P_1)$. Clearly $N := R/P_1 \cap \text{im}(f) \neq 0$ since $R/P_1$ is a graded essential submodule of $E^0(R/P_1)$. Now, $N \cap (a + P_1)R \neq 0$ since any homogeneous ideal of a commutative graded domain is graded essential (the product of two nonzero ideals is nonzero and
it is contained in the intersection. Hence, \( 0 \neq (a + P_1)R \cap N \subset (a + P_1)R \cap \text{im}(f) \).

Let then \( b \) be a nonzero element in \((a + P_1)R \cap \text{im}(f)\) and \( y \in E^g(R/P_2)\) such that \( b = ar + P_1 = f(y) \). Note that \( ya \in P_2E^g(R/P_2) \) and \( P_2 \) annihilates \( E^g(R/P_2) \), thus \( 0 = f(ya) = a^2r + P_1 \) and \( r \in P_1 \) since \( P_1 \) is prime. Hence \( b = 0 \) in \( R/P_1 \), reaching a contradiction and proving the result. \( \Box \)

**Proposition 5.8.** If \( R \) is a positively graded noetherian connected commutative \( \mathbb{K} \)-algebra generated in degree 1, the definition of perversity above is equivalent to the commutative definition of perversity in the introduction.

**Proof.** Note that points \( x \in X^{\text{top}} \) are in bijection with homogeneous prime ideals not equal to the irrelevant ideal of \( R \) and these are in bijection with graded torsion-free indecomposable injectives in \( \text{Gr}(R) \) (up to isomorphisms and shifts). Suppose \( x_1, x_2 \in X^{\text{top}}, P_1, P_2 \) the associated homogeneous prime ideals and \( E_1, E_2 \) the corresponding injectives. The condition \( x_1 \preceq x_2 \) translates to the existence of a nonzero map from \( E_2 \) to \( E_1 \) by the lemma above (note that, in this case, \( E_i = E^g(R/P_i) \) since \( R/P \) is indecomposable in \( \text{Gr}(R) \) and hence so is its injective envelope) and by lemma 4.14 this is equivalent to the existence of a map from \( \pi E_2 \) to \( \pi E_1 \).

Since \( R \) is finitely generated over \( \mathbb{K} \) (as it is noetherian), and hence are all its quotients, it is known the Krull dimension of \( R/P \) (which is the same as \( \dim(x_i) \)) in the geometric definition of perversity - see introduction) coincides with the Gelfand-Kirillov dimension of \( R/P \) (15). The result then follows by making the adequate substitutions in the geometric definition of perversity (see introduction). \( \Box \)

Recall that 3-dimensional Artin-Schelter regular algebras are noetherian domains - in particular, they are prime rings (3, 5). This allows us to prove the following useful lemma.

**Lemma 5.9.** Let \( R \) be a positively graded connected 3-dimensional Artin-Schelter regular algebra generated in degree 1 which is finitely generated over its centre. Then the image of a perversity \( p \) as defined above is finite.

**Proof.** Since \( R \) is prime, \((0)\) is a prime ideal not equal to the irrelevant ideal. Thus it corresponds to an indecomposable injective object which we shall denote by \( E_0 \). Furthermore, as a consequence of remark 6.3, \( E^g(R) \) is a finite direct sum of copies of \( E_0 \). Similarly, \( E^g(R/P) \) is a finite direct sum of copies of \( E_P \), the indecomposable injective object associated to the homogeneous prime ideal \( P \). We observe that for any such \( P \), there is a map from \( E^g(R) \) to \( E^g(R/P) \) induced by the canonical projection from \( R \) to \( R/P \). Therefore, there is a nontrivial map from \( E_0 \) to \( E_P \). The perversity condition then assures that:

\[
p(T_{\pi E_P}) - (GK \dim(R) - GK \dim(R/P)) \leq p(T_{\pi E_0}) - p(T_{\pi E_P}) .
\]

Since, by definition, the Gelfand-Kirillov dimension of \( R \) is finite (and so is the dimension of any of its quotients - 15) we have that, for a fixed value of \( p(T_{\pi E_0}) \), \( p(T_{\pi E_P}) \) is an integer that differs at most \( GK \dim(R) \) from it. Hence the image of \( p \) is finite. \( \Box \)

Thus, for \( R \) Artin-Schelter regular algebra of dimension 3 and finite over its centre, we can form a finite set of hereditary torsion classes

\[
S := \left\{ T_i := \bigcap_{T:p(T)=i} T, \min(p) \leq i \leq \max(p) \right\} .
\]
By corollary 5.5 the intersection of all its elements is zero. Finally, 3 provides a way of building a perverse quasi-coherent t-structure with respect to $p$ by defining its aisle to be $D^{S \leq 0}$ in $D^b(Tails(R))$. As mentioned in remark 5.1 in light of section 4 we conjecture that these t-structures restrict to $D^b(tails(R))$.

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Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
email: j.n.s.vitoria@warwick.ac.uk