Effect of coherence of nonthermal reservoirs on heat transport in a microscopic collision model

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We investigate the heat transport between two nonthermal reservoirs based on a microscopic collision model. We consider a bipartite system consisting of two identical subsystems, and each subsystem interacts with its own local reservoir, which consists of a large collection of initially uncorrelated ancillas. Then a heat transport is formed between two reservoirs by a sequence of pairwise collisions (inter-subsystem and subsystem-local reservoir). In this paper we consider two kinds of reservoir’s initial states, the thermal state, and the state with coherence whose diagonal elements are the same as that of the thermal state and the off-diagonal elements are nonzero. In this way, we define the effective temperature of the reservoir with coherence according to its diagonal elements. We find that for two reservoirs having coherence the direction of the steady current of heat is different for different phase differences between the two initial states of two reservoirs, especially the heat can transfer from the “cold reservoir” to the “hot reservoir” in the steady regime for particular phase difference. And in the limit of the effective temperature difference between the two reservoirs $\Delta T \to 0$, for most of the phase differences, the steady heat current increases with the increase of effective temperature until to the high effective temperature limit; while for the thermal state or particular phase difference the steady heat current decreases with the increase of temperature at high temperatures, and in this case the conductance can be obtained.

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I. INTRODUCTION

The increasing abilities to control systems at smaller and smaller scales motivate us to understand how the laws of physics developed in the macroscopic domain are modified in the microscopic scale \cite{1}. From the fundamental perspective, a link between quantum dynamics and thermodynamic processes has been widely investigated \cite{2-4}. In particular, out-of-equilibrium thermodynamics of quantum systems represents one of the most active research areas in this field \cite{3-9}. And understanding how energy transport can be controlled and efficiently distributed has been identified as one of the crucial studies for the development of quantum thermodynamics \cite{10-12}. In common construction, heat flow through a quantum system is generated by coupling it to two thermal reservoirs with different temperatures \cite{13-14}, and heat transport problem has attracted considerable attention during the past decades \cite{15-19}. For example, Wichterich et al. have investigated heat transport in a spin-1/2 Heisenberg chain, locally coupled to independent thermal baths of different temperatures, and they have obtained a stationary energy current by quantum master equation method and provided the way for efficient numerical investigations of heat transport in larger systems \cite{15}: Werlang et al. have realized a heat transport between two pure-dephasing Markovian reservoirs connected through a chain of coupled sites, and quantum coherence between sites is generated in the steady regime and results in the underlying mechanism sustaining the effect of heat transport \cite{16}. Moreover, one of the conceptual pillars in energy transport, Fourier’s law of heat conduction, has become an important issue and has been investigated in classical \cite{20, 21} and quantum systems \cite{17, 22-24}. An important step in understanding how Fourier’s law emerges from the quantum domain has been made by Michel et al. \cite{22}. Subsequently Manzano et al. have analyzed the steady energy transfer in a chain of coupled two-level systems connecting two thermal reservoirs, and they have revealed that there is a distinct violation of Fourier’s law in the quantum transport scenario of their model \cite{17}.

As one of the representative model for studying open quantum system, collision model has been extensively studied during the past decades \cite{25-37}. A quantum collision model \cite{33, 38} is a microscopic framework to describe the open dynamics of a system interacting with a reservoir assumed to be consisted of a large collection of smaller constituents (ancillas). The system is assumed to interact with the environment via a sequence of “collisions” between the system and ancillas, and each collision being described by the same bipartite quantum map (usually a unitary one). The reduced system can be obtained in many cases without any approximations, hence the complete positivity (CP) of the dynamical map is guaranteed. The collision model has been
applied in quantum thermodynamics recently. For example, Lorenzo et al. have investigated the link between information and thermodynamics in the dynamics of a multipartite open quantum system, which is described in terms of a collision model with a finite temperature reservoir [31]; and they have also explored the relation between heat flux and quantum correlations of a bipartite system via a collision-model-based approach [32]. Moreover, Pezzutto et al. have studied the heat exchange between system and environment and the information-to-energy conversion through a collision-based model [33]. Actually a similar model of a system repeatedly in contact with a stream of independently prepared units being in any nonequilibrium state (acting as a reservoir), has been investigated from the perspective of quantum and information thermodynamics [39]. This kind of setups have been used in quantum optics, theoretically as well as experimentally, for example, a micromaser cavity interacts sequentially with a stream of flying atoms prepared in nonequilibrium states [40–42], and this setup has also been used to investigate the work extraction from nonequilibrium bath including the particle units crossing the cavity each time being single multi-level atom [43], single two-level atom [44], two- and three-atom clusters [45, 46].

As mentioned above, traditional thermodynamic setups consist of a system in contact with thermal reservoirs. In fact, quantum systems also open up the possibility for exploring more general reservoirs [43–46]. Especially, the effect of quantum nonequilibrium baths, for example reservoirs with quantum correlations [48], coherence [43, 49] and squeezing [49, 50], have been studied extensively in quantum thermodynamics. Recently, the exploration of the effect of a reservoir’s coherence in quantum thermodynamics has provoked great interest. Scully et al. firstly demonstrated that the improvement of the working efficiency can be realized by using the nonequilibrium state preparation, i.e., the bath with coherence, and the obtained efficiency is beyond the classical Carnot efficiency [43]. This observation has been used in different scenarios [52, 53]. However we have not seen any report about the effect of coherence of reservoir on heat transport between two reservoirs. Thus an interesting question concerns if, and how, the heat transfer between two reservoirs can be affected by intra-reservoir coherence.

In this paper, we consider a bipartite system which, interacts with its local reservoir consisting of a large collection of initially uncorrelated systems which we call ancillas, respectively (see Fig. 1). We consider two kinds of reservoir’s initial states, one is the thermal state and the other is the state with coherence. The diagonal elements of both states are identical, and compared with the thermal state, the off-diagonal elements of the state with coherence are nonzero. In this way we define the effective temperature of the reservoir with coherence according to its diagonal elements. Based on this, the heat transport is formed between two reservoirs of different effective temperature by a sequence of pairwise collisions (inter-subsystem and subsystem-local reservoir). If the ancillas of both reservoirs are in thermal states, the general form of heat transport appears, i.e., heat transfers from the hot reservoir to the cold reservoir in the steady regime. However in the case of ancillas’ states with coherence, the direction of steady heat current can be changed by manipulating the relative phase of ancillas. And in the limit of the effective temperature difference between the two reservoirs ΔT → 0, for most of phase differences the steady heat current increases with the increase of effective temperature until to the high effective temperature limit. While for the thermal state or particular phase difference (0 and π) the steady heat current decreases with the increase of temperature at high temperatures, and in this case the conductance can be obtained.

II. MODEL

We consider a bipartite system $S$, consisting of two identical subsystem: qubits $S_A$ and $S_B$, with logical states $\{|0\}, |1\rangle$ for each qubit. Subsystem $S_A$ is in contact with a reservoir $R^h$ consisted of a collection of $N$ identical noninteracting ancillas $\{R^h_1, R^h_2, ..., R^h_N\}$, and subsystem $S_B$ is also in contact with a reservoir $R^c$ consisted of a collection of $N$ identical noninteracting ancillas $\{R^c_1, R^c_2, ..., R^c_N\}$. Here we focus on the ancillas of the reservoir $R^h (R^c)$ consisting of simple two-level systems (qubits) whose logical states are $\{|0\}, |1\rangle_j$ ($j=1, ..., N$). The two reservoirs are, therefore, in the product state $\eta_{tot}^h = \otimes_{j=1}^N \eta_j^h$ and $\eta_{tot}^c = \otimes_{j=1}^N \eta_j^c$ respectively. The evolution of the whole system including the two subsystems and their reservoirs is: $S_A$ and $S_B$ interact first, subsequently $S_A$ and $S_B$ collide with the individual ancillas of

FIG. 1: (Color online) Sketch of the protocol: A bipartite system $S$ consists of subsystems $S_A$ and $S_B$; after inter-subsystem interaction, $S_A$ interacts with $n$th ancilla (prepared in $\eta_j^n$), and $S_B$ interacts with $n$th ancilla (prepared in $\eta_j^n$); then this process is repeated and is directed to the $(n+1)$th ancilla. Thus the heat transport is established between the two reservoirs by a sequence of this process.
their local reservoirs, respectively; and then this process is repeated. The general scheme is illustrated in Fig. 1. The assumption of two big reservoirs ($R^h$ and $R^c$) implies that the subsystem never interacts twice with the same ancilla, i.e., at each collision the state of the ancilla is refreshed.

In our model, we consider a coherent interaction between the subsystem and the $j$th ancilla of the corresponding reservoir, i.e., a mechanism that can be described by a Hamiltonian model of some form, specifically in this paper we suppose that the interaction Hamiltonian is

$$\hat{H}_{\text{int}} = g(\hat{s} \hat{A}(\theta) \hat{\sigma}_x^A + \hat{s} \hat{A}(\theta) \hat{\sigma}_y^A + \hat{s} \hat{A}(\theta) \hat{\sigma}_z^A),$$

where $\hat{s}_i^A(\theta)$ and $\hat{s}_i^A(\theta) = (i = x, y, z)$ are the Pauli matrices, and $g$ is a coupling constant; and each collision is described by a unitary operator $\hat{U}_{S(A,B)} = e^{-i\hat{H}_{\text{int}} \tau}$, where $\tau$ is the collision time and we set $\hbar = 1$ throughout this paper.Using the following result [25]

$$e^{\frac{i}{\hbar}(\hat{d}_x \hat{d}_x + \hat{d}_y \hat{d}_y + \hat{d}_z \hat{d}_z)} = e^{-i\frac{\hbar}{2}(\cos \phi \hat{n} + i \sin \phi \hat{S}_{\text{sw}})},$$

where $\hat{n}$ is the identity operator, and $\hat{S}_{\text{sw}}$ is the two-particle swap operator, i.e., it is the unitary operation whose action is $|\psi_1\rangle \otimes |\psi_2\rangle \rightarrow |\psi_2\rangle \otimes |\psi_1\rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2$. We can now write the unitary time-evolution operator

$$\hat{U}_{S(A,B)} = (\cos \gamma) \hat{I}_{S(A,B), \hat{H}_{\text{int}}} + i(\sin \gamma) \hat{S}_{\text{sw}}\hat{S}_{S(A,B), \hat{H}_{\text{int}}},$$

where $\gamma = 2\gamma \tau$ is a dimensionless interaction strength. And we have assumed above that each ancilla of the two reservoirs have two energy levels, and in the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \hat{S}_{\text{sw}}\hat{S}_{S(A,B), \hat{H}_{\text{int}}}$ in Eq. (3) reads $[54]

$$\hat{S}_{\text{sw}}\hat{S}_{S(A,B), \hat{H}_{\text{int}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\] $$

Similar to Eq. (3), the interaction between two subsystems $S_A$ and $S_B$ is implemented through the unitary evolution

$$\hat{V}_{S_A,S_B} = (\cos \delta) \hat{I}_{S_A,S_B} + i(\sin \delta) \hat{S}_{\text{sw}}\hat{S}_{S_A,S_B},$$

with $\delta \neq \gamma$, in general, and the analog of the operations introduced above applies to $\hat{I}_{S_A,S_B}, \hat{S}_{\text{sw}}\hat{S}_{S_A,S_B}$ (swap gate between two subsystems). Such interactions give rise to the dynamical maps

$$\hat{\Phi}_{S(A,B), \hat{H}_{\text{int}}}[\rho] = \hat{U}_{S(A,B), \hat{H}_{\text{int}}} \rho \hat{U}_{S(A,B), \hat{H}_{\text{int}}}^\dagger,$$

$$\hat{\Psi}_{S_A,S_B}[\rho] = \hat{V}_{S_A,S_B} \rho \hat{V}_{S_A,S_B}^\dagger.$$

As mentioned above the dynamics of system $S$ consists of sequential inter-system interaction interspersed with subsystem-reservoir collisions. Therefore, each collision is treated in the following process, specifically $S_A$ and $S_B$ interact first, subsequently $S_A$ and $S_B$ collide with one of the ancillas in $R^h$ and $R^c$ respectively. Thus the system is brought from step $n$ to step $n + 1$ through the process

$$\rho^S_n \otimes |\eta^h_{n+1} \rangle \langle \eta^h_{n+1}| \otimes \rho^c_{n+1} \rightarrow \rho^S_{n+1} = \hat{U}(\rho^S_n \otimes |\eta^h_{n+1} \rangle \langle \eta^h_{n+1}|) \hat{U}^\dagger,$$

with $\hat{U}$ the overall unitary evolution experienced by the SE system (system plus the $(n+1)$th ancillas $R^h_{n+1}, R^c_{n+1})$ being generated by the composition of the set of unitary gates introduced above, i.e., $\hat{U} = \hat{U}_{R^h_{n+1}, R^c_{n+1}}(\gamma) \hat{U}_{S_A,S_B}(\gamma) \hat{V}_{S_A,S_B}(\delta)$. Hence after the $(n+1)$th collision, the state of system is $\rho^S_{n+1} = \hat{V}_{R^h_{n+1}, R^c_{n+1}}(\gamma) \hat{V}_{S_A,S_B}(\delta)$, and the state of the $(n+1)$th ancilla of reservoir $R^h$ ($R^c$) is

$$\hat{\eta}^h_{n+1} = \hat{V}_{R^h_{n+1}, R^c_{n+1}}(\gamma) \hat{V}_{S_A,S_B}(\delta), \] $$

where $\hat{V}_{R^h_{n+1}, R^c_{n+1}}(\gamma) \hat{V}_{S_A,S_B}(\delta)$ means the trace of $S$ and $\hat{R}^h_{n+1}$ degrees of freedom. And note that in Appendix A, we have demonstrated that the total change of energy of reservoir $R^h_{n+1}$ after the $(n+1)$th collision, is equivalent to the sum of energy change of the $j$th ancilla (in reservoir $R^h_{n+1}$) during the $j$th collision. Therefore from Eq. (9) we can obtain the heat exchange between reservoir $R^h_{n+1}$ and system during the $(n+1)$th collision

$$\Delta Q^R^h_{n+1} = \text{Tr}[\hat{H}_{n+1}^h(\eta^h_{n+1} - \eta^h_{n+1})],$$

where $\hat{H}_{n+1}^h = \frac{i}{\hbar} \hat{\omega} \hat{\sigma}_z$ (in this paper we assume that $\hat{H}_{n+1}^h = \frac{i}{\hbar} \hat{\omega} \hat{\sigma}_z$ is the local Hamiltonian of each ancilla of two reservoirs). And we consider energy-conserving $S(A,B) - \hat{R}^h_j$ interactions, i.e., $[\hat{H}_{\text{int}}, \hat{H}_{S(A,B)} + \hat{H}_{\text{int}}^h_j] = 0$ (this can be realized in the case of resonance interaction between $S(A,B)$ and the ancilla). Hence, $\Delta Q^R^h_{n+1} = -\Delta Q_{n+1}^{R^h_{n+1}}$ in the steady regime. Therefore, Eq. (10) can also be defined as the heat current

$$J^h_{n+1} = \Delta Q^R^h_{n+1},$$

i.e., the heat flows into (or out of) reservoir $R^h_{n+1}$ during the $(n+1)$th collision.

In order to investigate the effect of coherence of reservoirs on the heat current, we consider the initial state of each ancilla of two reservoirs as

$$\rho^{\text{coh}} = p |\psi\rangle \langle \psi| + (1 - p) \rho_{\beta},$$

where $p \in [0, 1], |\psi\rangle = \frac{1}{\sqrt{2}} (e^{-i \omega \beta} |0\rangle + e^{i \omega \beta} |1\rangle)$ with a relative phase $\phi$, and $\rho_{\beta}$ is the thermal state assumed to be of canonical equilibrium form, i.e., $\rho_{\beta} = \hat{Z}^{-\beta}$ $e^{-\beta \hat{H}^h_{n+1}}$. Here $\beta = 1/T_h(n+1)$ and $Z = \text{Tr} [e^{-\beta \hat{H}^h_{n+1}}]$ are the corresponding inverse temperature and the partition function.
Note that the diagonal elements of states $\rho_{coh}$ and $\rho_\beta$ are identical, and compared with the thermal state, the off-diagonal elements of state $\rho_{coh}$ are nonzero if $p \neq 0$. Therefore, Eq. (12) can also be written as

$$\rho_{coh} = \rho_\beta + pp_{non},$$

where $p_{non}$ is the non-diagonal part of state $|\psi\rangle\langle\psi|$, i.e., the off-diagonal elements of $p_{non}$ are the same as that of state $|\psi\rangle\langle\psi|$ and the diagonal elements are zero. As the diagonal elements of state $\rho_{coh}$ are the same as that of the thermal state, the effective temperature of the reservoir is defined by its diagonal elements. In this paper, we assume that the ancillas of reservoir $R^h$ and $R^c$ are in state (13) with different effective temperature $T_h$ and $T_c$ respectively, and $T_h > T_c$, which composes the “hot reservoir” $R^h$ and the “cold reservoir” $R^c$. Based on this, an interesting question concerns if, and how, the heat current can be affected by the non-zero off-diagonal elements of state $\rho_{coh}$, i.e., reservoir-coherence.

III. HEAT CURRENT

We first consider the heat flow from the reservoir $R^h$, i.e., $J_h$. In Fig. 2, we plot the heat current $J_h$ against the number of collisions $n$. We suppose that the two reservoirs are in the state (13) and the system is initially in the ground state $|\psi\rangle_s = |11\rangle$. And we let $T_h = 2$ and $T_c = 1$ in Eq. (13) for reservoirs $R^h$ and $R^c$ respectively. Based on this we consider two initial states of the two reservoirs, one is $p = 0.8$ with different phase difference between the two reservoirs $\phi_h - \phi_c = \{0, \pi/4, \pi/2, \pi, 5\pi/4, 3\pi/2\}$ (the two reservoirs with coherence); and the other is $p = 0$, i.e., the thermal states $\rho_{\beta}$ of two reservoirs. It can be seen from Fig. 2 that a unidirectional heat current (heat flows out of hot reservoir $R^h$) is formed during the whole dynamics for the two reservoirs with thermal state (dashed black line), which is in contrast to the case of two reservoirs with coherence. For the two reservoirs with coherence, from numerical calculations we find that the heat current $J_h$ is independent of their respective phases of two reservoirs, and is dependent only on the phase difference between the two reservoirs, i.e., $\phi_h - \phi_c$. And in this case, it can be seen from Fig. 2 that the heat currents are oscillating between positive and negative values as $n$ increases, which reveals that the energy flows into (positive $J_h$) and out of (negative $J_h$) reservoir $R^h$ in the early stages of dynamics. And then at the later stage the heat currents reach their stationary values respectively. And from numerical calculations we find that the steady heat current of reservoir $R^h$ ($J_h^{steady}$) and the steady heat current of reservoir $R^c$ ($J_c^{steady}$) satisfy the relation $J_h^{steady} = -J_c^{steady}$ as expected. When $\phi_h - \phi_c \in [0, \pi]$, $J_h^{steady}$ are negative, which corresponds to heat flowing from $R^h$ to $R^c$ in the steady regime. However an interesting thing appears: When $\phi_h - \phi_c \in (\pi, 2\pi)$, $J_h^{steady}$ are positive, which corresponds to heat flowing from the “cold reservoir” ($R^c$) to the “hot reservoir” ($R^h$), i.e., the “cold reservoir” is ‘cooled’. In other words, we can realize a countercurrent of heat from the “cold reservoir” to the “hot one” for particular phase differences. This could appear to be counterintuitive at first, as heat always transfers from hot reservoir to cold reservoir in general. This can be understood as follows. First from Eqs. (8)-(11), after some calculations we can obtain the steady heat current (for simplicity we choose $\gamma = \pi/2$, i.e., a complete swap between the subsystem and the corresponding ancilla) as

$$J_h^{steady} = J_\beta + J_{coh}$$

$$= \frac{\sin^2 \delta (e^{i\beta} - e^{-i\beta})}{(1 + e^{i\beta})(1 + e^{-i\beta})}$$

$$- \frac{p^2 \sin(2\delta) e^{i\phi_h} (e^{i\phi_c})^4}{(1 + e^{i\phi_h})(1 + e^{-i\phi_c})} \sin(\phi_h - \phi_c),$$

where $\delta \in (0, \frac{\pi}{2})$. Obviously, the steady heat current can be divided into two parts, the first term in Eq. (14), $J_\beta = \frac{\sin^2 \delta (e^{i\beta} - e^{-i\beta})}{(1 + e^{i\beta})(1 + e^{-i\beta})}$, is the contribution of the diagonal elements in Eq. (13) and is independent of $p$. In other words, $J_\beta$ is the steady current of heat for the states of two reservoir without coherence, i.e., the thermal states. And from the expression of $J_\beta$ it can be seen that the direction of heat current depends on the temperature difference between the two reservoirs. As in our paper $T_h > T_c$, so $J_\beta < 0$ is always satisfied. The second term in Eq. (14), $J_{coh} = -\frac{p^2 \sin(2\delta) e^{i\phi_h} (e^{i\phi_c})^4}{(1 + e^{i\phi_h})(1 + e^{-i\phi_c})} \sin(\phi_h - \phi_c)$ is the contribution of non-diagonal elements in Eq. (13), i.e., the contribution of coherence, and depends on the parameter $p$. Note that Eq. (14) is the steady heat current in the case of complete swap, and we analyze how the phase difference influence it in this case in the following. From Eq. (14), if $\phi_h - \phi_c \in (0, \pi)$, $J_{coh} < 0$ and $J_h^{steady} < 0$, which corresponds to heat flowing from the “hot reservoir” to the “cold reservoir” ($R^h \rightarrow R^c$) in the steady regime, and now $J_h^{steady}$ is larger than that of the initial thermal state $\rho_{\beta}$. If $\phi_h - \phi_c \in (\pi, 2\pi)$, $J_{coh} > 0$, the sign of $J_h^{steady}$ depends on the absolute values of $J_\beta$ and $J_{coh}$. In other words, in this case the direction of steady flow of heat depends upon different contribution of the effective temperature difference and coherence of reservoirs. For example when $\phi_h - \phi_c = 5\pi/4, 3\pi/2, J_\beta < 0$, $J_{coh} > 0$, and $|J_\beta| < |J_{coh}|$, which leads to $J_h^{steady} > 0$ and a heat current from the “cold reservoir” into the “hot reservoir” ($R^c \rightarrow R^h$) appears. And, if $\phi_h - \phi_c = 0, \pi$, $J_{coh} = 0$ and $J_h^{steady} = J_\beta$ which returns to the cases of initial thermal state introduced above.

Above, we have discussed the case of completely swap ($\gamma = \pi/2$), and now we analyze how the interaction strength $\gamma$ influence the steady heat current $J_h^{steady}$. In Fig. 3, we plot $J_h^{steady}$ as a function of $\phi_h - \phi_c$ for initial state (13) with $p = 0.8$ and $\delta = \pi/4$, and different $\gamma$, $\gamma = \{\pi/2, \pi/4, \pi/12, \pi/32\}$. It can be seen from
when the interaction strength coupling is weak in general, therefore we mainly consideratively enhanced. However, in practice the system-bath difference between the two regions corresponding hence, it reveals that the effect of effective tempera-
ture difference (i.e., \( J_{\text{eff}} \)). For all plots we choose the ground state of system |\( \psi \rangle_S = |11\rangle, \gamma = \pi/32, \delta = \pi/4, T_h = 2T_c = 2 \) (i.e., \( \beta_h = \frac{1}{2} \beta_c = \frac{1}{2} \)), and \( \omega = 1 \).

Fig. 2 that \( J_{\text{eff}} \) moves down overall with the increase of \( \gamma \), which leads to the region of \( J_{\text{eff}} < 0 \) increasing and that of \( J_{\text{eff}} > 0 \) decreasing, and the maximal difference between the two regions corresponding to \( J_{\text{eff}} > 0 \) and \( J_{\text{eff}} < 0 \) respectively is reached when \( \gamma = \pi/2 \) (Fig. 3 (a)). It is noted that \( J_\beta < 0 \), therefore, it reveals that the effect of effective temperature difference (i.e., \( J_\beta \)) on \( J_{\text{eff}} \) is weakened with the decrease of \( \gamma \). In other words, with the decrease of \( \gamma \), the contribution of reservoir-coherence on \( J_{\text{eff}} \) is relatively enhanced. However, in practice the system-bath coupling is weak in general, therefore we mainly consider the interaction strength \( \gamma = \pi/32 \) in this paper. In the case of \( \gamma = \pi/32 \), for \( \phi_h - \phi_c = 0, \pi \) the steady heat current depends not only on the effective temperatures of two reservoirs but also on the reservoir-coherence, i.e., \( J_{\text{eff}} \neq J_\beta \) (in contrast to \( J_{\text{eff}} = J_\beta \) in the case of completely swap). This can be seen from the inset of Fig. 2: For \( \phi_h - \phi_c = 0, \pi \) though the direction of the steady heat current is the same as that of the initial thermal state, the magnitudes are slightly different, which satisfy the relation \( J_{\text{eff}}(\phi_h - \phi_c = 0) < J_{\text{eff}}(\phi_h - \phi_c = \pi) < J_{\text{eff}}(\beta) \). Especially the heat currents in the early stages of dynamics (before reaching the stationary state) are greatly different. When \( \phi_h - \phi_c = 0, \pi \) the heat flows out of “hot reservoir” \( \mathcal{R}^h \) and flows back into \( \mathcal{R}^h \) alternatively before reaching the stationary state, and the backflow of heat into \( \mathcal{R}^h \) is especially obvious for \( \phi_h - \phi_c = 0, \pi \).

which is in contrast to the thermal state as mentioned above (a unidirectional heat current during the whole dynamics). In other words, even if there is no phase difference between the two reservoirs, the heat current is also different from that of the thermal state. Though the phase difference of two reservoirs is zero, the two reservoirs with coherence can also influence the heat transfer between the two reservoirs. And from the discussion above, the heat transport strongly depends on the phase difference between two reservoirs resulted from the quantum interference effects of the two reservoirs.

From Eq. (13), we know that the amount of coherence of reservoir is dependent on the value of \( p \), and the amount of coherence increase with the increase of \( p \). Therefore we will investigate how the amount of reservoir-coherence influence the heat transfer next. In the limit \( p \to 0 \) the effect of reservoir-coherence on \( J_{\text{eff}} \) is negligible, i.e., \( J_{\text{eff}} \sim J_\beta \), thus the reversed steady current \( (\mathcal{R}^c \to \mathcal{R}^h) \) disappears. In the case of complete swap \( (\gamma = \pi/2) \): When \( J_{\text{eff}} = 0 \) in Eq. (14) with \( \phi_h - \phi_c \in (\pi, 2\pi) \), we can obtain

\[
p_c = \sqrt{\frac{\tan \delta(e^{i\phi_h} - e^{i\phi_c})}{2e^{\delta\phi_c} \sin(\phi_h - \phi_c)}},
\]

which is the critical value of \( p \) appearing reversed steady current, i.e., we cannot obtain the reversed steady current if \( p < p_c \). When \( \gamma \neq \pi/2 \) (partial swap) it is difficult to obtain the analytical expression of the steady heat current, and with the same parameters in Fig. 2, \( p \approx 0.155 \) is the critical value appearing the reversed steady current now. In other words, in this case we can realize the steady heat transfer from the “cold reservoir” \( (T_c = 1) \).
into the “hot reservoir” \((T_h = 2)\) by manipulating the relative phase only when \(p \gtrsim 0.155\).

IV. ENTROPY CHANGE AND HEAT CURRENT

As mentioned above, the evolution of the total system (system+two reservoirs) is unitary, therefore the entropy of the total system is conserved in this process. Based on this, and according to Ref. \[53\], for the initial separable state of the total system, \(\rho_{0}^{tot} = \rho_{0}^{S} \otimes \eta_{0}^{R} (\eta_{0}^{R} = \eta_{0}^{hc} \otimes \eta_{0}^{tot})\), the change of the von Neuman entropy of system \(S\) after the \((n+1)\)th collision, can be expressed as

\[
\Delta S = S(\rho_{n+1}^{S}) - S(\rho_{0}^{S}) = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{S} \otimes \eta_{n+1}^{hc}],
\]

(16)

where \(\rho_{n+1}^{tot}\) is the total state of system \(S\) and the \((n+1)\) ancillas in each reservoir which have interacted with the system, \(\eta_{n+1}^{hc} = \eta_{n}^{hc} \otimes \eta_{n}^{hc}\) is the initial state of the \(j\)th ancilla of the two reservoirs totally, and \(\eta_{n+1}^{hc}\) is the marginal state of \(\rho_{n+1}^{tot}\), the change of energy of the two reservoirs during the \((j+1)\)th collision. Since the relative entropy is positive, Eq. (20) indicates the positivity of the entropy production, i.e., \(\Delta S_{ir}^{tot} \geq 0\) (equal to zero only when \(\rho_{n+1}^{tot}\) and \(\rho_{n+1}^{S} \otimes \eta_{n+1}^{hc}\) are identical). And in Ref. \[56–58\] it has been claimed that in this case (the environment is in the thermal equilibrium) the second law is fulfilled. However, for the two reservoirs with coherence, i.e., Eq. (13) \((p \neq 0)\), the entropy change for the second term in Eq. (16), cannot be associated with the heat flow.

In order to study the relation of entropy change and heat exchange with single reservoir \((R^{h}\) or \(R^{c}\)), firstly we take the system \(S\) and reservoir \(R^{c}\) as a composite system \(SR^{c}\), and the change in the von Neuman entropy of \(SR^{c}\), after the \((n+1)\)th collision, can be expressed as

\[
\Delta S_{1} = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{SR^{c}} \otimes \eta_{n+1}^{c}],
\]

(21)

where \(\rho_{n+1}^{SR^{c}} = Tr_{R}(\rho_{n+1}^{tot})\) means the trace of reservoir \(R^{c}\) degrees of freedom. And we name the second term in Eq. (21),

\[
\Delta S_{1}^{re} = \sum_{j=1}^{n+1} Tr_{j}(\eta_{n+1}^{hc} \otimes \eta_{n+1}^{hc}) \ln \eta_{n+1}^{hc},
\]

(22)

can be written in the standard thermodynamics form

\[
\Delta S_{1}^{re} = \sum_{j=1}^{n+1} \beta Q_{1}^{j},
\]

(23)

where \(\Delta Q_{1}^{j} = Tr_{j}[\hat{H}_{n+1}^{j} - \hat{H}_{n+1}^{j}(\eta_{n+1}^{hc} \otimes \eta_{n+1}^{hc})],\) is the change of energy of the two reservoirs during the \(j\)th collision, hence \((\hat{H}_{n+1}^{j} + \hat{H}_{n+1}^{j})\) represents the total Hamiltonian of \(SR^{c} + R^{c}\). As we focus on energy-conserving system-reservoir interactions, hence \(\Delta Q_{1}^{j} = -\Delta Q_{j}^{S}\) (\(\Delta Q_{j}^{S}\) is the change of energy of the system \(S\)). Therefore Eq. (18) can also be written as

\[
\Delta S_{1}^{re} = \sum_{j=1}^{n+1} \beta \Delta Q_{j}^{S},
\]

(24)

and Eq. (19) is the total entropy flow after the \((n+1)\)th collision, and accordingly the first term in Eq. (16),

\[
\Delta S_{ir}^{tot} = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{S} \otimes \eta_{n+1}^{hc}],
\]

(20)

is the total entropy production after the \((n+1)\)th collision. Note that in Ref. \[56\], it has been shown that if the initial state of the composite system \(SR^{h}\) is separable (\(p = 0\)), the second term in Eq. (16), cannot be associated with the heat flow.

Similarly, the change in the von Neumann entropy of the composite system \(SR^{h}\), after the \((n+1)\)th collision, can be expressed as

\[
\Delta S_{2} = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{SR^{h}} \otimes \eta_{n+1}^{h}],
\]

(25)

where Eq. (19) is the total entropy flow after the \((n+1)\)th collision, and accordingly the first term in Eq. (16),

\[
\Delta S_{ir}^{tot} = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{S} \otimes \eta_{n+1}^{hc}],
\]

(20)

is the total entropy production after the \((n+1)\)th collision. Since the relative entropy is positive, Eq. (20) indicates the positivity of the entropy production, i.e., \(\Delta S_{ir}^{tot} \geq 0\) (equal to zero only when \(\rho_{n+1}^{tot}\) and \(\rho_{n+1}^{S} \otimes \eta_{n+1}^{hc}\) are identical). And in Ref. \[56–58\] it has been claimed that in this case (the environment is in the thermal equilibrium) the second law is fulfilled. However, for the two reservoirs with coherence, i.e., Eq. (13) \((p \neq 0)\), the entropy change for the second term in Eq. (16), cannot be associated with the heat flow.

In order to study the relation of entropy change and heat exchange with single reservoir \((R^{h}\) or \(R^{c}\)), firstly we take the system \(S\) and reservoir \(R^{c}\) as a composite system \(SR^{c}\), and the change in the von Neuman entropy of \(SR^{c}\), after the \((n+1)\)th collision, can be expressed as

\[
\Delta S_{1} = D[\rho_{n+1}^{tot} \| \rho_{n+1}^{SR^{c}} \otimes \eta_{n+1}^{c}],
\]

(21)

where \(\rho_{n+1}^{SR^{c}} = Tr_{R}(\rho_{n+1}^{tot})\) means the trace of reservoir \(R^{c}\) degrees of freedom. And we name the second term in Eq. (21),

\[
\Delta S_{1}^{re} = \sum_{j=1}^{n+1} \beta Q_{1}^{j},
\]

(22)

can be written in the standard thermodynamics form

\[
\Delta S_{1}^{re} = \sum_{j=1}^{n+1} \beta h \Delta Q_{1}^{j},
\]

(23)

where \(\Delta Q_{1}^{j} = \Delta Q_{j}^{S}\) is the change of energy of the system \(S\) during the \(j\)th collision, i.e., the heat flow from reservoir \(R^{h}\). However, for the state of reservoir \(R^{h}\) with coherence, i.e., Eq. (13) \((p \neq 0)\), it can be seen from Eq. (22) that Eq. (23) is not valid. In other words, for the reservoir with coherence, there is no longer a direct connection between the entropy change of the system and the heat flow from its environment. And Nejad et.al. have pointed out that for the general quantum setting there may be not a direct association between heat flux and entropy change \[58\].
where \( \rho_{n+1}^{\text{tot}} = \text{Tr}_{R^c}(\rho_{n+1}^{\text{tot}}) \) means the trace of reservoir \( R^c \) degrees of freedom. And the second term in Eq. (24), i.e., the entropy exchanged with the reservoir \( R^c \),

\[
\Delta S^{\text{re}}_2 = \sum_{j=1}^{n+1} \text{Tr}_j (\eta_j^c - \eta_j^R) \ln \eta_j^R,
\]

(25)
can also be written in the standard thermodynamic form for the thermal equilibrium state of reservoir \( R^c \)

\[
\Delta S^{\text{re}}_2 = - \sum_{j=1}^{n+1} \beta_j \Delta Q^R_j,
\]

(26)
where the heat flowing from reservoir \( R^c \) during the \( j \)th collision is \( \Delta Q^R_j \). Of course, for the state of reservoir \( R^c \) with coherence Eq. (26) is also not valid. From Eqs. (17), (22) and (25), we find that the relation \( \Delta S^{\text{re}} = \Delta S^{\text{eq}}_1 + \Delta S^{\text{eq}}_2 \) is always satisfied for the two reservoirs being in state (13), which is independent of the value of \( p \) and the phase difference of two reservoirs. And clearly, for the two reservoirs are in the thermal equilibrium with the same temperature \( (\beta_h = \beta_c = \beta) \), this relation can be written as

\[
\sum_{j=1}^{n+1} \Delta Q^R_j = \sum_{j=1}^{n+1} (\Delta Q^h_j + \Delta Q^c_j),
\]

(27)
which can be interpreted as that, after the \( (n+1) \)th collision, the total heat flow (energy change) of the two reservoirs equals to the sum of the heat flow of the two reservoirs respectively.

V. THERMAL CONDUCTANCE

Thus far, we have only concerned the heat transport with finite effective temperature difference between the two reservoirs \( (T_h = 2T_c = 2) \). Now we consider the case of small effective temperature difference between the two reservoirs, and we begin this study with Fourier’s law of heat conduction. Fourier’s law of heat conduction states that the heat current through a classical macroscopic object is proportional to the applied temperature gradient

\[
J = -\kappa \nabla T,
\]

(28)
where \( \kappa \) is the conductance. In our case we assume that \( T_h = T + \Delta T/2 \) and \( T_c = T - \Delta T/2 \), and write

\[
J_h = -\kappa \Delta T,
\]

(29)
then the conductance is obtained from \( \kappa = -J_h/\Delta T \) by taking the limit \( \Delta T \to 0 \).

Thermal state. For thermal initial state of the two reservoirs, in Appendix B we provide an analytic expression of steady heat current, and we show that the steady heat current can be written in the form of \( J_h^{\text{steady}} = -\kappa \Delta T \), therefore the conductance \( \kappa \) is a constant for fixed \( T, \delta \) and \( \gamma \). And from Eq. (B1) in Appendix B, we find that \( \kappa \) increases with the increase of \( T \) at low temperatures \( T \lesssim 0.45 \); and \( \kappa \) decreases with the increase of \( T \) at high temperatures \( T \gtrsim 0.45 \). This indicates that a high conductance can be obtained at low temperature of reservoir. Note that a similar result has been obtained in Ref. [24] that the conductance firstly increases (low temperatures) and then decreases (high temperatures) with the increase of temperature, and at high temperatures the conductance is proportional to the inverse of temperature.

State with coherence. From our study, we find that in the case of \( \Delta T \to 0 \), generally the steady heat current can be expressed as

\[
J_h^{\text{steady}} = -\lambda \Delta T + c,
\]

(30)
where \( \lambda \) and \( c \) are constant for fixed parameters \( (p, \gamma, \delta \) and \( T) \). When the phase difference between the two reservoirs is 0 or \( \pi \) with fixed parameters \( (p = 0.8, \gamma = \pi/32, \delta = \pi/4, \omega = 1, T_h = T + \Delta T/2 \) and \( T_c = T - \Delta T/2 \),

\[
J_h^{\text{steady}} = \begin{cases} 
-6.1063 \times 10^{-4} \Delta T, & (\phi_h - \phi_c = 0) \\
-7.3643 \times 10^{-4} \Delta T, & (\phi_h - \phi_c = \pi)
\end{cases}
\]

(31)
and clearly for \( \phi_h - \phi_c = 0, \pi \) the constant \( c \) in Eq. (30) is zero, i.e., we can also obtain the conductance now similar to the case of thermal state. Obviously, from Eq. (31), the conductance is different for the phase difference \( \phi_h - \phi_c = 0, \pi \). And in order to compare the conductance for the phase difference \( \phi_h - \phi_c = 0, \pi \) and thermal state, in Fig. 4, we plot the conductance as a function of \( T \) for \( \phi_h - \phi_c = 0, \pi \) and thermal state \( \rho_\beta \). It can be seen from Fig. 4 that the relation
\( \kappa(\phi_h - \phi_c = 0) < \kappa(\phi_h - \phi_c = \pi) < \kappa(\rho_{ij}) \) is always true for arbitrary \( T \). And they have the similar behaviors that \( \kappa \) firstly increases (at low temperatures \( T \lesssim 0.45 \)) and then decreases (at high temperatures \( T \gtrsim 0.45 \)) with the increase of temperatures, and as mentioned above this result is consistent with Ref. [24]. Physically, this can be easily understood as following. At high temperature, increasing \( T \) would weaken the influence of temperature difference \( \Delta T \) on two reservoirs and lead to the decrease of heat flows.

However, if the phase difference \( \phi_h - \phi_c \neq 0, \pi \), the constant \( c \neq 0 \) in Eq. (30), for example when \( \phi_h - \phi_c = \pi/4, 5\pi/4 \) with the fixed parameters \((p = 0.8, \gamma = \pi/32, \delta = \pi/4 \) and \( T = 1 \)), the steady heat currents are

\[
J_{\text{steady}}^h = \begin{cases} 
-6.3147 \times 10^{-4} \Delta T - 8.8067 \times 10^{-3}, & (\phi_h - \phi_c = \frac{\pi}{4}) \\
-7.3239 \times 10^{-4} \Delta T + 1.1089 \times 10^{-2}. & (\phi_h - \phi_c = \frac{5\pi}{4})
\end{cases}
\]  

(32)

From Eq. (32), clearly for \( \Delta T \to 0 \), \( J_{\text{steady}}^h < 0 \) for \( \phi_h - \phi_c = \pi/4 \) and \( J_{\text{steady}}^h > 0 \) for \( \phi_h - \phi_c = 5\pi/4 \). In other words, the steady heat current flows from the reservoir \( R^h \) to \( R^c \) for \( \phi_h - \phi_c = \pi/4 \), while for \( \phi_h - \phi_c = 5\pi/4 \) the direction of the steady heat current is opposite, i.e., from \( R^c \) to \( R^h \). In Fig. 5, we show the steady heat currents as a function of \( \Delta T \) and \( T \) for initial state (13) with \( p = 0.8; \phi_h - \phi_c = \pi/4 \) and \( \phi_h - \phi_c = 5\pi/4 \) in Fig. 5 (a) and Fig. 5 (b), respectively. It can be seen from Fig. 5 that the steady heat currents increase with the increase of \( T \), which is in contrast to the cases of thermal state and the state with coherence \( \phi_h - \phi_c = 0, \pi \) (Fig. 4) that \( J_{\text{steady}}^h \) decreases with the increase of \( T \) at high effective temperatures. This could appear to be counterintuitive at first, as one always expects that high temperature would weaken the effect of temperature difference \( \Delta T \) on heat transfer and lead to the decrease of heat flows. However, as mentioned above for the reservoirs with coherence the steady heat current depends not only on the reservoir effective temperature but also on the coherence (non-diagonal part of initial state (13)). And physically this can be understood as following. From Eq. (13), as the effective temperature \( T \) increases, the values of the off-diagonal elements of Eq. (13) increases, i.e., the coherence increases. And in the high effective temperature limit the coherence of two reservoirs reaches the maximum, and now the values of the diagonal elements of Eq. (13) are almost the same (both of them equal to 1/2 approximately). In a word the higher the effective temperature, the more the coherence of reservoir and the more the contribution of the reservoir-coherence on the steady heat current.

We can also understand the result above assisted by the complete swap case \((\gamma = \pi/2)\) as follows. We expand Eq. (14) in series up to the first order of \( \Delta T \) and obtain

\[
J_{\text{steady}}^h = -\lambda_1 \Delta T + c_1,
\]

(33)

where \( \lambda_1 = \frac{1}{4p^2} \text{sech}^2(\frac{1}{2T}) \sin^2 \delta \) and \( c_1 = -\frac{1}{4p^2} \text{sech}^2(\frac{1}{2T}) \sin(\phi_h - \phi_c) \). Clearly, \( J_{\text{steady}}^h \) can be divided into two parts, the first term in Eq. (33) is the contribution of diagonal elements of state (13) (thermal state) only. And note that we can also obtain the same expression of the first term in Eq. (33) from Eq. (B1) in Appendix B. In other words, for the two reservoirs in thermal states, the conductance \( \kappa = \lambda_1 \). And the second term in Eq. (33) is the contribution of all the non-diagonal elements of state (13), i.e., the total coherence. Now we discuss the effect of phase difference on the steady heat current in two cases below. (i) \( \phi_h - \phi_c = 0, \pi \). From Eq. (33), when \( \phi_h - \phi_c = 0, \pi \), \( J_{\text{steady}}^h \) returns to the case of thermal state. However for partial swap \((\gamma \neq \pi/2)\), the first term in Eq. (30) is determined by the diagonal and non-diagonal elements of state (13) jointly, which leads to the difference of the conductance for \( \phi_h - \phi_c = 0, \pi \) and thermal state discussed above. (ii) \( \phi_h - \phi_c \neq 0, \pi \). When \( \phi_h - \phi_c \neq 0, \pi \), for very small \( \Delta T \) \((\Delta T \to 0)\), the first term in Eq. (33) is negligible, and Eq. (33) can be written approximatively as

\[
J_{\text{steady}}^h \sim c_1,
\]

(34)
and obviously for fixed and nonzero parameters $p$, $\delta$ and $\sin(\phi_h - \phi_c)$, Eq. (34) can be re-written as
\[
J_h^{\text{steady}} \sim c_1 = \xi \text{sech}^2 \left( \frac{1}{2T} \right),
\] (35)

where $\xi$ is a constant. Because $T > 0$ and from Eq. (35), clearly $J_h^{\text{steady}}$ increases with the increase of effective temperature $T$, and $J_h^{\text{steady}}$ is approaching a constant $\xi (\xi = -\frac{1}{2}p^2 \sin(2\delta) \sin(\phi_h - \phi_c))$ in the high effective temperature limit.

VI. CONCLUSION

In this paper, we have investigated the heat transport between two nonthermal reservoirs by collision-model-based approach, and we have studied the effect of reservoir’s coherence on the heat current. Specifically, we have considered a bipartite system consisting of two identical subsystems, and each subsystem interacts with its local reservoir, which consists of a large collection of initially uncorrelated ancillas in a state with coherence. We have realized a heat transport between two reservoirs by a sequence of pairwise collisions (inter-subsystem and subsystem-local reservoir). We have found that the direction of heat current depends on the relative phases (phase difference between the two reservoirs) strongly. For example, we have realized heat transfer from the “cold reservoir” to the “hot reservoir” in the steady regime. This result appears to be counterintuitive at first, as heat always transfer from hot reservoir to cold reservoir in general. However it is due to the contribution of reservoir-coherence on the heat current.

Then we have explored the relation of heat current and entropy exchanged with the reservoir in our model. We have shown that there is a linear relation between the heat current and entropy flux for the reservoir in thermal state, and there is not a direct connection between them for the reservoir with coherence. Finally, we have studied the steady current of heat in the limit of the effective temperature difference between the two reservoirs $\Delta T \to 0$. For most of phase differences of two reservoirs, the steady heat current increases with the increase of effective temperature until to the high effective temperature limit, and this is in contrast to the thermal states of reservoirs (heat current decreases with the increase of temperature at high temperatures). In a word, in the presence of reservoir’s coherence we can observe the effect of reservoir-interference on the heat transport.

It is noted that in this paper we have used the collision model to investigate the effect of coherence of reservoirs on the heat flow. The reason to consider this simple model is that exact solutions can be obtained for a general class of initial states of reservoirs with coherence. We expect that some features of the heat flow in this simple model might be similar to those in more involved but less tractable heat transfer models so we can gain some insight into the general feature of effect of reservoirs with coherence on heat flow.

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APPENDIX A

After the $(n+1)$th collision, the total state of $\mathcal{S}$ plus all the ancillas of the two reservoirs which have been interacted with the system is
\[
\rho_{n+1}^{\text{tot}} = \hat{U}_{n+1} \cdots \hat{U}_1 \rho_{n+1}^{\text{tot}} (0) \hat{U}_1^\dagger \cdots \hat{U}_{n+1}^\dagger,
\] (A1)

where $\rho_{n+1}^{\text{tot}} (0) = \rho_0^{\text{tot}} \otimes \prod_{j=1}^{n+1} \mathbb{I}^{\mathcal{R}_j}$, is the initial state of $\mathcal{S}$ plus the $(n+1)$ ancillas of the two reservoirs respectively; and $\hat{U}_j = \hat{U}_{\mathcal{S}_A, \mathcal{R}_j}(\gamma) \hat{U}_{\mathcal{S}_A, R_j}(\gamma) \hat{V}_{\mathcal{S}_A, \mathcal{S}_B}(\delta)$ is the unitary operator of the $j$th collision. Hence, the total energy of the $(n+1)$ ancillas interacted with $\mathcal{S}$ in reservoir $\mathcal{R}_c^{(c)}$ after the $(n+1)$th collision can be written as
\[
E_{n+1}^{(c)} = \frac{1}{n+1} \sum_{i=1}^{n+1} \text{Tr} [\hat{H}_i^{(c)} \rho_{n+1}^{\text{tot}}],
\]

\[
= \sum_{i=1}^{n+1} \text{Tr} [\hat{H}_i^{(c)} (\hat{U}_{n+1} \cdots \hat{U}_1 \rho_{n+1}^{\text{tot}} (0) \hat{U}_1^\dagger \cdots \hat{U}_{n+1}^\dagger)].
\] (A2)

Because $[\hat{H}_i^{(c)}, \hat{U}_j] = 0$ for $j > i$, therefore Eq. (A2) can be written as
\[
E_{n+1}^{(c)} = \sum_{i=1}^{n+1} \text{Tr} [\hat{H}_i^{(c)} (\hat{U}_i \cdots \hat{U}_1 \rho_{n+1}^{\text{tot}} (0) \hat{U}_1^\dagger \cdots \hat{U}_i^\dagger)]
\]

\[
= \sum_{i=1}^{n+1} \text{Tr} [\hat{H}_i^{(c)} \tilde{\eta}_i^{(c)}],
\] (A3)

where $\tilde{\eta}_i^{(c)} = \text{Tr}_{\bar{\mathcal{I}}} [\hat{H}_i^{(c)} \rho_{n+1}^{\text{tot}}]$ means the trace of all except the $i$th ancilla of the reservoir $\mathcal{R}_c^{(c)}$ degrees of freedom. Clearly, Eq. (A3) is equivalent to the sum of energy change of the $i$th ancilla (in reservoir $\mathcal{R}_c^{(c)}$) during the $i$th collision.

APPENDIX B

The expression of steady heat current for thermal state $\rho_\beta$ in the limit of small temperature difference $\Delta T$ be-
between the two reservoirs can be obtained,

\[
J_{\text{steady}}^{h} = \frac{-4e^{\frac{2\gamma}{T}} [5 + \cos(4\gamma) + 4 \cos(2\gamma) \cosh^{2}\left(\frac{1}{2T}\right) + 6 \cosh\left(\frac{1}{T}\right)] \sin^{2}\gamma \sin^{2}\delta}{(1 + e^{\frac{2\gamma}{T}})^{2} T^{2} \{a (1 + e^{\frac{2\gamma}{T}}) + e^{\frac{2\gamma}{T}} [20 + 7 \cos(2\gamma) + 4 \cos(4\gamma) + \cos(6\gamma) - 32 \cos^{4}\gamma \cos(2\delta)]\} \Delta T}, \tag{B1}
\]

where \(a = 11 + 4 \cos(2\gamma) + \cos(4\gamma) - 16 \cos^{2}\gamma \cos(2\delta)\). Therefore, from Eq. (B1), for thermal state the conductance \(\kappa\) can be obtained by the expression \(\kappa = -J_{h}^{\text{steady}} / \Delta T\). And it is worth mentioning that in the high temperature limit the conductance reduces to

\[
\kappa = \frac{(11 + 4 \cos(2\gamma) + \cos(4\gamma)) \sin^{2}\gamma \sin^{2}\delta}{2(3 + \cos(2\gamma))(7 + \cos(4\gamma) - 8 \cos^{2}\gamma \cos(2\delta)) T^{2}}. \tag{B2}
\]
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