ZERO-CYCLES WITH MODULUS AND RELATIVE $K$-THEORY

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Abstract. We construct a cycle class map from the higher Chow groups of 0-cycles to the relative $K$-theory of a modulus pair. We show that this induces a pro-isomorphism between the additive higher Chow groups of relative 0-cycles and relative $K$-theory of truncated polynomial rings over a regular semi-local ring, essentially of finite type over a characteristic zero field.

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1. Introduction

The story of Chow groups with modulus began with the discovery of additive higher 0-cycles by Bloch and Esnault in [8] and [9]. Their hope was that these additive 0-cycle groups would serve as a guide in developing a theory of motivic cohomology with modulus which could describe the algebraic $K$-theory of non-reduced schemes. Recall that Bloch’s original higher Chow groups (equivalently, Voevodsky’s motivic cohomology) overlook the difference between non-reduced and reduced schemes.

Motivated by the work of Bloch and Esnault, a theory of motivic cohomology with modulus was proposed by Binda and Saito [5] in the name of ‘higher Chow groups with modulus’. The expectation was that one would be able to describe relative algebraic $K$-theory in terms of these Chow groups. The theory of Chow groups with modulus generalized the theory of additive higher Chow groups defined by Bloch-Esnault and further studied by Rülling [48], Krishna-Levine [32] and Park [46]. It also generalized the theory of 0-cycles with modulus of Kerz-Saito [27] and the higher Chow groups of Bloch [7].

Recall that one way to study the algebraic $K$-theory of a non-reduced (or any singular) scheme is to embed it as a closed subscheme of a smooth scheme and study the resulting
relative $K$-theory. Since there are motivic cohomology groups which can completely describe the algebraic $K$-theory of a smooth scheme, what one needs is a theory of motivic cohomology to describe the relative $K$-theory.

The expectation that the higher Chow groups with modulus should be the candidate for the motivic cohomology to describe the relative $K$-theory has generated a lot of interest in them in past several years. In a recent work [19], Iwasa and Kai constructed a theory of Chern classes from the relative $K$-theory to a variant of the higher Chow groups with modulus. In another work [20], they proved a Riemann-Roch type theorem showing that the relative group $K_0$-group of an affine modulus pair is rationally isomorphic to a direct sum of Chow groups with modulus. An integral version of this isomorphism for all modulus pairs in dimension up to two was earlier proven by Binda and Krishna [4]. They also constructed a cycle class map for relative $K_0$-group in all dimensions.

The above results suggest strong connection between cycles with modulus and relative $K$-theory. However, an explicit construction of cycle class maps in full generality or Atiyah-Hirzebruch type spectral sequences, which may directly connect Chow groups with modulus to relative algebraic $K$-theory, remains a challenging problem today.

1.1. Main results and consequences. The objective of this paper is to investigate the original question of Bloch and Esnault [9] in this subject. Namely, can 0-cycles with modulus explicitly describe relative $K$-theory in terms of algebraic cycles? We provide an answer to this question in this paper. We prove two results. The first is that there is indeed a direct connection between 0-cycles with modulus and relative $K$-theory in terms of an explicit cycle class map. The second is that in many cases of interest, these 0-cycles with modulus are strong enough to completely describe the relative algebraic $K$-theory. More precisely, we prove the following. The terms and notations used in the statements of these results are explained in the body of the text.

**Theorem 1.1.** Let $X$ be a non-singular quasi-projective variety of pure dimension $d \geq 1$ over a field $k$ and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map

$$\text{cyc}_{X|D}: \{\text{CH}^{n+d}(X|mD,n)\}_m \to \{K_n(X,mD)\}_m$$

between pro-abelian groups. This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero.

For a general divisor $D \subset X$, we do not expect that the cycle class map that we construct in Theorem 1.1 will exist without resorting to the setting of pro-abelian groups. However, if we use rational coefficients, then the usage of pro-abelian groups can indeed be avoided, as the following result shows. In this paper, we use this improved version in the proof of Theorem 1.3.

**Theorem 1.2.** Let $X$ be a non-singular quasi-projective variety of pure dimension $d \geq 1$ over a field $k$ and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map

$$\text{cyc}_{X|D}: \text{CH}^{n+d}(X|D,n)_\mathbb{Q} \to K_n(X,D)_\mathbb{Q}.$$

This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero. Furthermore, it coincides with the map (1.1) on the generators of $\text{CH}^{n+d}(X|D,n)$.

We now address as to why the cycle class maps of Theorems 1.1 and 1.2 should be non-trivial and what we expect about these maps. Recall that the relative $K$-theory $K_n(X,mD)$ has Adams operations (e.g., see [12] for their construction). From our construction, we expect the map (1.2) to be injective in the pro-setting, with image
\( \{K_n(X, mD)_{Q}^{(d+n)} \}_{m} \). Here, \( K_n(X, mD)_{Q}^{(d+n)} \) is \((d+n)\)-th eigen-space of the Adams operations. When \( D = \emptyset \), the cycle class map \( cyc_{X} := cyc_{X|\emptyset} \) is not new and it was constructed by Levine [11] by a different method. He also showed that in this special case, \( cyc_{X} \) is indeed injective with image \( K_n(X)^{(d+n)}_{Q} \).

When \( X = \text{Spec}(k) \) and \( D = \emptyset \), the cycle class map \( cyc_{X} \) coincides with Totaro’s map \( \text{CH}^{n}(k, n) \rightarrow K_{n}^{M}(k) \rightarrow K_{n}(k) \) [54]. Totaro showed that the map \( \text{CH}^{n}(k, n) \rightarrow K_{n}^{M}(k) \) is an isomorphism and one knows that the canonical map \( K_{n}^{M}(k)_{Q} \rightarrow K_{n}(k)_{Q}^{(n)} \) is an isomorphism. The remaining part of this paper is devoted to showing that \( cyc_{X|D} \) is in fact an isomorphism with integral coefficients for the modulus pair \( (A_{1}^{1}, \{0\}) \), where \( R \) is a regular semi-local ring essentially of finite type over a characteristic zero field.

We make some further remarks on the past works on the cycle class map for 0-cycles with modulus. Following Levine’s strategy, Binda [34] showed that there is a cycle class map to relative \( K \)-theory provided one makes the following changes: replace the higher Chow group with modulus by a variant of it (which imposes a stronger version of the modulus condition, originally introduced in [34]), assume that \( D_{\text{reg}} \) is a strict normal crossing divisor, and assume rational coefficients. Theorem 1.1 imposes none of these conditions. If \( D \subset X \) is a non-singular divisor, a cycle class map was defined in [40, Theorem 1.5] using the stable \( A^{1} \)-homotopy theory.

We now give details about the cycle class map of Theorem 1.1 for the modulus pair \( (A_{1}^{1}, \{0\}) \). Recall that in case of the higher \( K \)-theory of a smooth scheme \( X \), the cycle-class map \( \text{CH}^{n+d}(X, n) \rightarrow K_{n}(X) \) from the 0-cycle group can not be expected to describe all of \( K_{n}(X) \) (even with rational coefficients). However, we show in our next result that the cycle class map of Theorem 1.1 is indeed enough to describe all of the (integral) relative \( K \)-theory of nilpotent extensions of smooth schemes, if we work in the category of pro-abelian groups instead of the usual category of abelian groups. This demonstrates a remarkable feature of relative \( K \)-theory which is absent in the usual \( K \)-theory.

Before we state the precise result, recall that the additive higher Chow groups are special cases of higher Chow groups with modulus. More precisely, for an equi-dimensional scheme \( X \), the additive higher Chow group \( \text{TCH}^{n}(X, n+1; m) \) is the same as the higher Chow group with modulus \( \text{CH}^{n}(X \times A^{1}_{k}, X \times (m+1)\{0\}, n) \) for \( m, n, p \geq 0 \). To understand the reason for the shift in the value of \( n \), we need to recall that the additive higher Chow groups are supposed to compute the relative \( K \)-theory of truncated polynomial extensions and one knows that the connecting homomorphism \( \partial: K_{n+1}(X[t]/(t^{m+1}), \langle x \rangle) \rightarrow K_{n}(X \times A^{1}_{k}, X \times (m+1)\{0\}) \) is an isomorphism when \( X \) is non-singular. Under this dichotomy, we shall use the notation \( cyc_{X} \) for \( cyc_{A^{1}_{k}, \{0\}}(X(x)) \) whenever we use the language of additive higher Chow groups. In particular, for a ring \( R \), we shall write \( cyc_{R} \) for \( cyc_{A^{1}_{k}, \{0\}}(X) \) while using additive higher Chow groups.

Let \( R \) now be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Recall that there is a canonical map \( K_{*}^{M}(R) \rightarrow K_{*}(R) \) from the Milnor to the Quillen \( K \)-theory of \( R \). For \( n \geq 1 \), the group \( \text{TCH}^{n}(R, n; m) \) is not a 0-cycle group if \( \dim(R) \geq 1 \). Hence, Theorem 1.1 does not give us a cycle class map for this group. However, using this theorem for fields and various other deductions, we can in fact prove an improved version of Theorem 1.1. Namely, we can avoid the usage of pro-abelian groups for the existence of the cycle class map with integral coefficients.

**Theorem 1.3.** Let \( R \) be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Let \( m \geq 0 \) and \( n \geq 1 \) be two integers. Then the following hold.
(1) There exists a cycle class map
\[ \text{cyc}_R^M : \text{TCH}^n(R, n; m) \rightarrow K^M_n(R[x]/(x^{m+1}),(x)). \]

(2) The composite map
\[ \text{cyc}_R : \text{TCH}^n(R, n; m) \xrightarrow{\text{cyc}_R^M} K^M_n(R[x]/(x^{m+1}),(x)) \rightarrow K_n(R[x]/(x^{m+1}),(x)) \]
coincides with the map of Theorem 1.1 when \( R \) is a field.

(3) \( \text{cyc}_R^M \) and \( \text{cyc}_R \) are natural in \( R \).

(4) \( \text{cyc}_R^M \) is an isomorphism.

(5) The map
\[ \text{cyc}_R : \{ \text{TCH}^n(R, n; m) \}_m \rightarrow \{ K_n(R[x]/(x^{m+1}),(x)) \}_m \]
of pro-abelian groups is an isomorphism.

In other words, Theorem 1.3 (4) says that the relative \( K \)-theory of truncated polynomial rings can indeed be completely described by the relative 0-cycles over \( R \) (the cycles in \( \text{TCH}^n(R, n; m) \) have relative dimension zero over \( R \)). This shows that the additive Chow groups defined by Bloch-Esnault [9] and Rülling [48] are indeed the relative \( K \)-groups, at least in characteristic zero. This was perhaps the main target of the introduction of additive higher Chow groups by Bloch and Esnault.

By the works of several authors (see [11] and [25] for regular semi-local rings and [14] and [54] for fields), it is now well known that the motivic cohomology of a regular semi-local ring in the equal bi-degree (the Milnor range) coincides with its Milnor \( K \)-theory. Theorem 1.3 (3) says that this isomorphism also holds for truncated polynomial rings over such rings. This provides a concrete evidence that if one could extend Voevodsky’s theory of motives to the theory of ‘non-\( \mathbb{A}^1 \)-invariant’ motives over so-called fat points (infinitesimal extensions of spectra of fields), then the underlying motivic cohomology groups must be the additive higher Chow groups (see [34]).

It should be remarked that the objective of Theorem 1.3 is not to compute the relative \( K \)-groups. There are already known computations of these by many authors (e.g., see [14] and [17]). Instead, the above result addresses the question whether these relative (Milnor or Quillen) \( K \)-groups could be described by additive 0-cycles.

Theorem 1.3 has following consequences. The first corollary below is in fact part of our proof of Theorem 1.3.

**Corollary 1.4.** Let \( R \) be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Let \( n \geq 0 \) be an integer. Then the canonical map
\[ \{ K^M_n(R[x]/(x^m),(x)) \}_m \rightarrow \{ K_n(R[x]/(x^m),(x)) \}_m \]
of pro-abelian groups is an isomorphism. In particular, \( \{ K_n(R[x]/(x^m),(x)) \}^{(p)} \}_m = 0 \) for \( p \neq n \).

Let \( R \) be any regular semi-local ring containing \( \mathbb{Q} \). Then the Néron-Popescu desingularization theorem says that \( R \) is a direct limit of regular semi-local rings \( \{ R_i \} \), where each \( R_i \) is essentially of finite type over \( \mathbb{Q} \) (see [52, Theorem 1.1]). One knows from [48, Lemma 1.17] that if each \( \{ K^M_n(R_i[x]/(x^m),(x)) \}_{n,m \geq 1} \) is a restricted Witt complex over \( R_i \), then \( \{ K^M_n(R[x]/(x^m),(x)) \}_{n,m \geq 1} \) is a restricted Witt-complex over \( \lim_{i \rightarrow \mathbb{I}^1} R_i = R \) (see [48, Definition 1.14] for the definition of a restricted Witt-complex). On the other hand, it was shown in [35, Theorem 1.2] that each collection \( \{ \text{TCH}^n(R_i,n;m) \}_{n,m \geq 1} \) is
a restricted Witt-complex over \( R \). We therefore obtain our next consequence of Theorem 1.3.

**Corollary 1.5.** Let \( R \) be a regular semi-local ring containing \( \mathbb{Q} \). Then the relative Milnor \( K \)-theory \( \{K^M_n(R[x]/(x^m), (x))\}_{n,m \geq 1} \) is a restricted Witt-complex over \( R \).

In [6, Chapter II], Bloch had shown (without using the terminology of Witt-complex) that if \( R \) is a regular local ring containing a field of characteristic \( p > 2 \), then the subgroup of the relative Quillen \( K \)-theory of truncated polynomial rings over \( R \), generated by Milnor symbols (the symbolic \( K \)-theory in the language of Bloch), has the structure of a restricted Witt-complex. The above corollary extends the result of Bloch to characteristic zero.

The last consequence of Theorem 1.3 is the following. Park and Ünver [47] proposed a definition of motivic cohomology of truncated polynomial ring \( k[x]/(x^m) \) over a field. They showed that these motivic cohomology in the Milnor range coincide with the Milnor \( K \)-theory of \( k[x]/(x^m) \) when \( k \) is a characteristic zero field. Theorem 1.3 implies that the Milnor range (relative) motivic cohomology of Park-Ünver coincides with the additive higher Chow groups.

1.2. **Comments and questions.** We make a couple of remarks related to the above results.

1. Since Theorem 1.1 is characteristic-free, one would expect the same to be true for Theorem 1.3 and Corollary 1.4 as well. Our remark is that Theorem 1.3 and Corollary 1.4 are indeed true in all characteristics \( \neq 2 \). Since the techniques of our proofs in positive characteristics are different from the present paper, they are presented in [16].

2. Our second remark is actually a question. Recall that Chow groups with modulus are supposed to be the motivic cohomology to describe the relative \( K \)-theory, just as Bloch’s higher Chow groups describe \( K \)-theory. Analogous to Bloch’s Chow groups, the ones with modulus exist in all bi-degrees. However, as we explained earlier, Theorem 1.3 says that the 0-cycles groups with modulus are often enough to describe all of relative \( K \)-theory in the setting of pro-abelian groups. One can therefore ask the following.

**Question 1.6.** Let \( R \) be a regular semi-local ring essentially of finite type over a perfect field. Let \( n, p \geq 1 \) be two integers such that \( n \neq p \). Is \( \{TCH^p(R,n;m)\}_m = 0 \) ?

Note that this question is consistent with the second part of Corollary 1.4. Note also that it is already shown in [38] that the answer to this question is yes when \( p > n \). So the open case is when \( p < n \). We hope to address this question in a future work. Reader may recall that when \( p < n/2 \), the additive version of the deeper Beilinson-Soulé vanishing conjecture says that \( TCH^p(R,n;m) \) should vanish for every \( m \geq 1 \).

1.3. **An outline of the paper.** We end this section with a brief outline of the layout of this text. In sections 2 and 3 we set up our notations, recollect the main objects of study and prove some intermediate results. In §4 we define the cycle class map on the group of generators of 0-cycles. Our definition of the cycle class map is \textit{a priori} completely different from the one in [3] and [41]. The novelty of the new construction is that it is very explicit in nature and, therefore, it becomes possible to check that it factors through the rational equivalence. We also prove in this section that the cycle class map is natural for suitable proper and flat morphisms. One can check that this map does coincide with more abstractly defined maps of [3] and [41] on generators. But we do not discuss this in this paper (see however §4.4 for a sketch of this).

We break the proof of Theorem 1.1 into two steps. In §5 we prove it for a very specific type of curves using the results of §2. This is the technical part of the proof of
Theorem 1.1. It turns out that the general case can be reduced to the above type using the results of § 2.3. This is done in § 6. The idea that we have to increase the modulus for factoring the cycle class map through the rational equivalence is already evident in the technical results of § 8.2.

Sections 7 and 8 constitute the heart of the proof of Theorem 1.3. In § 7, we provide some strong relations between the additive 0-cycles, relative Milnor $K$-theory and the big de Rham-Witt complex. In particular, we show that it suffices to know the image of certain very specific 0-cycles under the cycle class map in order to show that it factors through the relative Milnor $K$-theory of a truncated polynomial ring (see Lemma 7.3). In § 8, we give an explicit description of the relative Milnor $K$-theory in terms of the module of Kähler differentials (see Lemma 8.4). This allows us to establish the isomorphism between the additive higher Chow groups of 0-cycles and the relative Milnor $K$-theory.

To pass to the Quillen $K$-theory, we prove a vanishing theorem (see Proposition 9.5) using some results of [30]. This allows us to show that the additive 0-cycle groups for fields are isomorphic to the relative $K$-theory in the setting of pro-abelian groups. In § 10, we extend the results of § 9 to regular semi-local rings using the main results of [39]. The last section is the appendix which contains some auxiliary results on the relation between Milnor and Quillen $K$-theory of fields. These results are used in the main proofs.

2. The relative $K$-theory and cycles with modulus

In this section, we fix our notations and prove some basic results in relative algebraic $K$-theory. We shall also recall the definition of the higher Chow groups with modulus.

2.1. Notations. We shall in general work with schemes over an arbitrary base field $k$. We shall specify further conditions on $k$ as and when it is required. We let $\textbf{Sch}_k$ denote the category of separated finite type schemes over $k$ and let $\textbf{Sm}_k$ denote the full subcategory of non-singular schemes over $k$. For $X,Y \in \textbf{Sch}_k$, we shall denote the product $X \times_k Y$ simply by $X \times Y$. For any point $x \in X$, we shall let $k(x)$ denote the residue field of $x$. For a reduced scheme $X \in \textbf{Sch}_k$, we shall let $X^N$ denote the normalization of $X$. For $p \geq 0$, we shall denote the set of codimension $p$ points of a scheme $X$ by $X^p$. For an affine scheme $X \in \textbf{Sch}_k$, we shall let $k[X]$ denote the coordinate ring of $X$.

We shall let $\overline{\mathbb{C}}$ denote the projective space $\mathbb{P}^n_k = \text{Proj}(k[y_0, y_1])$ and let $\sigma = \overline{\mathbb{C}} \setminus \{1\}$. We shall let $\mathbb{A}^n_k = \text{Spec}(k[y_1, \ldots, y_n])$ be the open subset of $\overline{\mathbb{C}}^n$, where $(y_1, \ldots, y_n)$ denotes the coordinate system of $\overline{\mathbb{C}}^n$ with $y_j = Y_j/Y_0$. Given a rational map $f: X \to \overline{\mathbb{C}}^n$ in $\textbf{Sch}_k$ and a point $x \in X$ lying in the domain of definition of $f$, we shall let $f_i(x) = (y_i \circ f)(x)$, where $y_i: \overline{\mathbb{C}}^n \to \mathbb{C}$ is the $i$-th projection. For any $1 \leq i \leq n$ and $t \in \mathbb{C}(k)$, we let $F^t_{n,i}$ denote the closed subscheme of $\overline{\mathbb{C}}^n$ given by $\{y_i = t\}$. We let $F^t_{n,i} = \sum_{i=1}^{n} F^t_{n,i}$.

By a closed pair $(X,D)$ in $\textbf{Sch}_k$, we shall mean a closed immersion $D \hookrightarrow X$ in $\textbf{Sch}_k$, where $X$ is reduced and $D$ is an effective Cartier divisor on $X$. We shall write $X \smallsetminus D$ as $X^O$. We shall say that $(X,D)$ is a modulus pair if $X^O \in \textbf{Sm}_k$. If $(X,D)$ is a closed pair, we shall let $mD \subset X$ be the closed subscheme defined by the sheaf of ideals $I_D^m$, where $D$ is defined by the sheaf of ideals $I_D$.

All rings in this text will be commutative and Noetherian. For such a ring $R$ and an integer $m \geq 0$, we shall let $R_m = R[t]/(t^{m+1})$ denote the truncated polynomial algebra over $R$. We shall write $\text{Spec}(R[t_1, \ldots, t_n])$ as $\mathbb{A}^n_R$. The tensor product $M \otimes_k N$ will be denoted simply as $M \otimes N$. Tensor products over other bases will be explicitly indicated.

2.2. The category of pro-objects. By a pro-object in a category $C$, we shall mean a sequence of objects $\{A_m\}_{m \geq 0}$ together with a map $\alpha^A_m: A_{m+1} \to A_m$ for each $m \geq 0$. We
shall write this object often as \( \{A_m\} \). We let \( \text{Pro}(\mathcal{C}) \) denote the category of pro-objects in \( \mathcal{C} \) with the morphism set given by

\[
\text{Hom}_{\text{pro}}(\{A_m\}, \{B_m\}) = \lim_{\rightarrow} \lim_{\leftarrow} \text{Hom}_{\mathcal{C}}(A_m, B_n).
\]

In particular, giving a morphism \( f \) as above is equivalent to finding a function \( \lambda : \mathbb{N} \rightarrow \mathbb{N} \), a map \( f_n : A_{\lambda(n)} \rightarrow B_n \) for each \( n \geq 0 \) such that for each \( n' \geq n \), there exists \( l \geq \lambda(n), \lambda(n') \) so that the diagram

\[
\begin{array}{ccc}
A_l & \xrightarrow{f_{n'}} & A_{\lambda(n')} \\
\downarrow & & \downarrow \\
A_{\lambda(n)} & \xrightarrow{f_n} & B_n
\end{array}
\]

is commutative, where the unmarked arrows are the structure maps of \( \{A_m\} \) and \( \{B_m\} \). We shall say that \( f \) is strict if \( \lambda \) is the identity function. If \( \mathcal{C} \) admits all sequential limits, we shall denote the limit of \( \{A_m\} \) by \( \lim \limits_{\rightarrow} A_m \in \mathcal{C} \). If \( \mathcal{C} \) is an abelian category, then so is \( \text{Pro}(\mathcal{C}) \). We refer the reader to [1, Appendix 4] for further details about \( \text{Pro}(\mathcal{C}) \).

2.3. The relative algebraic \( K \)-theory. Given a closed pair \((X, D)\) in \( \text{Sch}_k \), we let \( K(X, D) \) be the homotopy fiber of the restriction map between the Thomason-Trobaugh non-connective algebraic \( K \)-theory spectra \( K(X) \rightarrow K(D) \). We shall let \( K_i(X) \) denote the homotopy groups of \( K(X) \) for \( i \in \mathbb{Z} \). We similarly define \( K_i(X, D) \). We shall let \( K^D(X) \) denote the homotopy fiber of the restriction map \( K(X) \rightarrow K(X \setminus D) \). Note that \( K^D(X) \) does not depend on the subscheme structure of \( D \) but \( K(X, D) \) does. Note also that if \( D' \subset X \) is another closed subscheme such that \( D \cap D' = \emptyset \), then \( K^D(X) \) is canonically homotopy equivalent to the homotopy fiber \( K^D(X, D') \) of the restriction map \( K(X, D') \rightarrow K(X \setminus D, D') \).

If \((X, D)\) is divisorial, we have the canonical restriction map \( K(X, (m+1)D) \rightarrow K(X, mD) \). In particular, this gives rise a pro-spectrum \( \{K(X, mD)\} \) and a level-wise homotopy fiber sequence of pro-spectra

\[
\{K(X, mD)\} \rightarrow K(X) \rightarrow \{K(mD)\}.
\]

Suppose that \( R \) is a regular semi-local ring. Let \( f(t) \in R[t] \) be a polynomial such that \( f(0) \in R^* \) and let \( Z = V((f(t))) \subset \mathbb{A}^1_R \) be the closed subscheme defined by \( f(t) \). Since \( Z \cap \{0\} = \emptyset \), the composite map \( K^Z_0(\mathbb{A}^1_R) \rightarrow K(\mathbb{A}^1_R) \rightarrow K(\{(m+1)\{0\}) \) is null-homotopic for all \( m \geq 0 \). Hence, there is a factorization \( K^Z_0(\mathbb{A}^1_R) \rightarrow K(\mathbb{A}^1_R, (m+1)\{0\}) \rightarrow K(\mathbb{A}^1_R) \). Let \( [\mathcal{O}_Z] \) denote the fundamental class of \( Z \) in \( K^Z_0(\mathbb{A}^1_R) \) (see [53, Exercise 5.7]). Note that \( Z \) may not be reduced or irreducible. Let \( \alpha_Z \) denote the image of \( [\mathcal{O}_Z] \) under the map \( K^Z_0(\mathbb{A}^1_R) \rightarrow K(\mathbb{A}^1_R, (m+1)\{0\}) \). Let \( \partial_n : K_n(R_m) \rightarrow K_{n-1}(\mathbb{A}^1_R, (m+1)\{0\}) \) denote the connecting homomorphism obtained by considering the long exact homotopy groups sequence associated to \((2.3)\). The homotopy invariance of \( K \)-theory on \( \text{Sm}_k \) shows that this map is an isomorphism. For \( g(t) \in R[t] \), let \( g(t) \) denote its image in \( R_m \).

**Lemma 2.1.** Given \( Z = V((f(t))) \) as above, we have

\[
\alpha_Z = \partial_1((f(0))^{-1}f(t)).
\]
Proof. Since \( f(0) \in R^x \), we note that \( Z = V((f(0))^{-1} f(t)) \). We let \( g(t) = (f(0))^{-1} f(t) \) so that \( g(0) = 1 \) and therefore \( g(t) \in \mathcal{K}_1(R_m) \). We let \( \Lambda = \{(a, b) \in R[t] \times R[t] | a - b \in (t^{m+1})\} \) be the double of \( R[t] \) along the ideal \((t^{m+1})\) as in [44, Chapter 2]. Let \( p_1 : \Lambda \rightarrow R[t] \) be the first projection. Then recall from [44, Chapter 6] that with the restriction \( K \) map factorization \( K \). We have two inclusions \( \iota_* : X \hookrightarrow S_X \) of the push-forward map. We shall denote the pull-back map \( \iota^*_u : K(S_X, X-) \rightarrow K(X, D) \).

We now let \( u : Z \hookrightarrow X \) be a closed immersion such that \( Z \cap D = \emptyset \). This gives rise to a push-forward map \( (X, D) \rightarrow (X, D) \) also by \( u_* \). It is clearly functorial in \((X, D)\) and \( Z \). Recall also that \( K(Z) \) and \( K(X, D) \) are module spectra over the ring spectrum \( K(X) \) (e.g., see [53, Chapter 3]). We shall need to follow the following result about the map \( u_* \) in the proof of Lemma 2.2.

**Lemma 2.2.** The push-forward map \( u_* : K_*(Z) \rightarrow K_*(X, D) \) is \( K_*(X) \)-linear.

**Proof.** Since \( Z \subset S_X \setminus X_- \), we also have the push-forward map \( v_* : K(Z) \rightarrow K(S_X, X_-) \), where we let \( v = \iota_* \circ u \). Suppose we know that \( u_* = \iota^*_u \circ v_* : K(Z) \rightarrow K(S_X, X_-) \rightarrow K(X, D) \) and the lemma holds for \( v_* \). Then for any \( \alpha \in K_*(X) \) and \( \beta \in K_*(Z) \), we get

\[
\begin{align*}
u_*(u^*(\alpha)\beta) &= \iota^*_u(v_*(v^*(\alpha)\beta)) = \iota^*_u(p^*(\alpha)v_*(\beta)) = (p \circ \iota_*)^*(\alpha)u_*(\beta) = \alpha u_*(\beta).
\end{align*}
\]

We thus have to show the following.

1. The lemma holds for the inclusion \( Z \hookrightarrow S_X \), and
2. \( u_* = \iota^*_u \circ v_* \).

To prove (1), we can use that the map \( K_*(S_X, X_-) \rightarrow K_*(S_X) \) is a split inclusion (as we saw above). Using this and the fact that \( K(S_X, X_-) \rightarrow K(S_X) \) is \( K(S_X) \)-linear, it suffices to prove (1) for the composite push-forward map \( v_* : K(Z) \rightarrow K(S_X) \). But we already saw above that \( K(Z) \) is a module spectrum over \( K(S_X) \).

We now prove (2). By the definition of the push-forward maps to the relative \( K \)-theory, we have factorizations

\[
\begin{array}{ccc}
K(Z) & \rightarrow & K^Z(S_X, X_-) \\
\downarrow & & \downarrow \iota^*_Z \\
K(Z) & \rightarrow & K^Z(X, D) \rightarrow K(X, D),
\end{array}
\]

The factorization of the push-forward map.
such that the square on the right is commutative and the top (resp. bottom) composite arrow is $v_*$ (resp. $u_*$). Hence, it suffices to show that the left square is commutative.

For showing this, we use the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
K(Z) & \xrightarrow{z} & K^Z(S_X, X_-) \\
\downarrow & & \downarrow \iota^*_z \\
K(Z) & \xrightarrow{\iota^*_z} & K^Z(X, D) \xrightarrow{u^*} K^Z(X \backslash D),
\end{array}
\end{array}
\]

where the horizontal arrows on the right are the restriction maps induced by the open immersions of modulus pairs. In particular, the square on the right is commutative. The right horizontal arrows are homotopy equivalences by the excision theorem. Hence, it suffices to show that the composite square in (2.5) commutes.

To see this, we note that the composite horizontal arrows in (2.5) have the factorizations:

\[
\begin{array}{c}
\begin{array}{ccc}
K(Z) & \xrightarrow{z} & G(Z) \xrightarrow{\iota^*_z} K^Z(S_X, X_-) \\
\downarrow & & \downarrow \\
K(Z) & \xrightarrow{\iota^*_z} & G(Z) \xrightarrow{\pi^*_z} K^Z(X, D),
\end{array}
\end{array}
\]

where $G(Z)$ is the $K$-theory of pseudo-coherent complexes on $Z$ ([53, Chapter 3]) and $K(Z) \rightarrow G(Z)$ is the canonical map. We are now done because the square on the right in (2.6) clearly commutes. □

2.5. The 0-cycles with modulus. Let $k$ be a field and let $(X, D)$ be an equi-dimensional divisorial closed pair in $\mathbf{Sch}_k$ of dimension $d \geq 1$. We recall the definition of the higher Chow groups with modulus from [5] or [36]. For any integers $n, p \geq 0$, we let $z^p(X|D, n)$ denote the free abelian group on the set of integral closed subschemes of $X \times \Delta^n$ of codimension $p$ satisfying the following.

(1) $Z$ intersects $X \times F$ properly for each face $F \subset \Delta^n$.

(2) If $\overline{Z}$ is the closure of $Z$ in $X \times \Delta^n$ and $\nu : \overline{Z}^N \rightarrow X \times \Delta^n$ is the canonical map from the normalization of $\overline{Z}$, then the inequality (called the modulus condition)

\[\nu^*(D \times \Delta^n) \leq \nu^*(X \times F^1_n)\]

holds in the set of Weil divisors on $\overline{Z}^N$.

An element of the group $z^p(X|D, n)$ will be called an admissible cycle. It is known that \{ $n \mapsto z^p(X|D, n)$ \} is a cubical abelian group (see [32, §1]). We denote this by $z^p(X|D, *)$. We let $z^p(X|D, *) = \sum_{\nu_0} z^p(X|D_\nu, *)$, where $z^p_{\nu_0}(X|D, *)$ is the degenerate part of the cubical abelian group $z^p(X|D, *)$. For $n \geq 0$, we let

\[\text{CH}_p(X|D, n) = H_n(z^p(X|D, *))\]

and call them the higher Chow groups with modulus of $(X, D)$. The direct sum

(2.7) \[\text{CH}_0(X|D, *) := \bigoplus_{n \geq 0} \text{CH}^{d+n}(X|D, n) = \bigoplus_{n \geq 0} \text{CH}_{-n}(X|D, n)\]

is called the higher Chow group of 0-cycles with modulus. The subject of this paper is to study the relation between $\text{CH}_0(X|D, *)$ and the relative $K$-theory $K_*(X, D)$.

We recall for the reader that the groups $\text{CH}^p(X|D, *)$ satisfy the flat pull-back and the proper push-forward properties under certain conditions. We refer the reader to [5] or [36] for these and other properties of the Chow groups with modulus.
3. The Milnor K-theory

Recall that for a semi-local ring $R$, the Milnor $K$-group $K_i^M(R)$ is defined to be the $i$-th graded piece of the graded Milnor $K$-theory $\mathbb{Z}$-algebra $K^M(R)$. The latter is defined to be the quotient of the tensor algebra $T_*(R^*)$ by the two-sided graded ideal generated by homogeneous elements $\{a \otimes (1-a) | a, 1-a \in R^*\}$. The image of an element $a_1 \otimes \cdots \otimes a_n \in T_n(R^*)$ in $K^M_n(R)$ is denoted by the Milnor symbol $a = \{a_1, \ldots, a_n\}$. If $I \subset R$ is an ideal, the relative Milnor $K$-theory $K_i^M(R, I)$ is defined to be the kernel of the natural surjection $K^M_n(R) \to K^M_n(R/I)$. It follows from [24, Lemma 1.3.1] that $K^M_n(R, I)$ is generated by Milnor symbols $\{a_1, \ldots, a_n\}$, where $a_i \in \text{Ker}(R^* \to (R/I)^*)$ for some $1 \leq i \leq n$, provided $R$ is a finite product of local rings.

The product structures on the Milnor and Quillen $K$-theories yield a natural graded ring homomorphism $\psi_R : K^M_*(R) \to K_*(R)$. If $I \subset R$ is an ideal, we have a natural isomorphism $K^M_1(R, I) \cong \bar{K}_1(R, I)$, where $\bar{K}_*(R, I)$ is the group $\text{Ker}(K_*(R) \to K_*(R/I))$. Using the module structure on $\bar{K}_*(R, I)$ over $K_*(R)$ and the ring homomorphism $K^M_*(R) \to K_*(R)$, we obtain a natural graded $K^M_*(R)$-linear map $\psi_{R, I} : K^M_*(R, I) \to \bar{K}_*(R, I)$. The cup product on Milnor $K$-theory yields maps $K^M_n(R) \otimes K^M_m(R, I) \to K^M_{n+m}(R, I)$. In the sequel, we shall loosely denote the image of this map also by $K^M_n(R)K^M_m(R, I)$ (e.g., see Lemma 3.3).

3.1. The improved Milnor $K$-theory. If $R$ is a semi-local ring whose residue fields are not infinite, then the Milnor $K$-theory $K^M_*(R)$ does not have good properties. For example, the Gersten conjecture does not hold even if $R$ is a regular local ring containing a field. If $R$ is a finite product of local rings containing a field, Kerz [26] defined an improved version of Milnor K-theory, which is denoted as $\bar{K}^M_*(R)$. This is a graded commutative ring and there is natural map of graded commutative rings $\eta^R : K^M_*(R) \to \bar{K}^M_*(R)$. For an ideal $I \subset R$, we let $\bar{K}^M_*(R, I) = \text{Ker}(\bar{K}^M_*(R) \to \bar{K}^M_*(R/I))$. We thus have a natural map $K^M_*(R, I) \to \bar{K}^M_*(R, I)$. We state some basic facts about $\bar{K}^M_*(R)$ in the following result and refer the reader to [26] for proofs.

**Proposition 3.1.** Let $R$ be a finite product of local rings containing a field. Then the map $\eta^R : K^M_*(R) \to \bar{K}^M_*(R)$ has following properties.

1. $\eta^R$ is surjective.
2. $\eta^R_n$ is an isomorphism for all $n \geq 0$ if $R$ is a field.
3. $\eta^R_n$ is an isomorphism for $n \leq 1$.
4. $\eta^R_n$ is an isomorphism for all residue fields of $R$ are infinite.
5. The natural map $K^M_n(R) \to K_n(R)$ factors through $\eta^R_n$.
6. The map $\bar{K}^M_2(R) \to K_2(R)$ is an isomorphism.
7. The Gersten conjecture holds for $\bar{K}^M_*(R)$.

If $R$ is a regular semi-local ring (not necessarily a product of local rings) containing a field, we let $\bar{K}^M_n(R)$ denote the kernel of the boundary map $\partial : K^M_n(F) \to \bigoplus_{\text{ht}(p)=1} K^M_{n-1}(k(p))$ of the Gersten sequence, where $F$ is the total quotient ring (a product of fields) of $R$. Note that the Gersten sequence

\[ K^M_n(F) \to \bigoplus_{\text{ht}(p)=1} K^M_{n-1}(k(p)) \to \cdots \to \bigoplus_{\text{ht}(p)=n-1} K^M_1(k(p)) \to \bigoplus_{\text{ht}(p)=n} K^M_0(k(p)) \]

is defined for $R$ and it is a complex by [23]. It follows from Proposition 3.1 and the Gersten resolution for Quillen $K$-theory that there are natural maps

\[ K^M_n(R) \to \bar{K}^M_n(R) \xrightarrow{\psi_R} K_n(R). \]
Furthermore, one can check that \( \widehat{K}_n^M(R) \) coincides with the group of global sections of the sheaf \( \widehat{K}_{n,X}^M \) on the scheme \( X = \text{Spec}(R) \), defined in [26].

Suppose that \( R \) is a regular semi-local integral domain of dimension one containing a field and \( I \subset R \) is an ideal of height one. Then \( R/I \) is a finite product of Artinian local rings. In particular, the improved Milnor \( K \)-theory \( \widehat{K}_n^M(R/I) \) is defined. We can write \( R/I = \prod_{i=1}^r R_{m_i}/IR_{m_i} \), where \( m_1, \ldots, m_r \) are the minimal primes of \( I \). We thus have the canonical maps

\[
\begin{align*}
\widehat{K}_n^M(R) & \to \prod_{i=1}^r \widehat{K}_n^M(R_{m_i}) \to \prod_{i=1}^r \widehat{K}_n^M(R_{m_i}/IR_{m_i}) \cong \widehat{K}_n^M(R/I),
\end{align*}
\]  

where the first arrow is induced from the definition of \( \widehat{K}_n^M(R) \) and the Gersten resolutions of the improved Milnor \( K \)-theory of the localizations of \( R \). We define the relative improved Milnor \( K \)-group \( \widehat{K}_n^M(R, I) \) as the kernel of the composite map. Note that this agrees with the relative improved Milnor \( K \)-groups defined earlier if \( R \) is a product of local rings.

Note that (3.3) also shows that the diagram

\[
\begin{array}{ccc}
K_n^M(R) & \to & \widehat{K}_n^M(R) \\
\downarrow & & \downarrow \\
K_n^M(R/I) & \to & \widehat{K}_n^M(R/I)
\end{array}
\]

commutes. We therefore get the canonical maps of relative \( K \)-theories

\[
(3.5) \quad K_n^M(R) \to \widehat{K}_n^M(R) \to K_n(R),
\]

where recall that \( K_n(R) = \text{Ker}(K_n(R) \to K_n(R/I)) \).

3.2. Some results on Milnor-\( K \)-theory. We shall need few results on the Milnor \( K \)-theory of discrete valuation rings. For a discrete valuation ring \( R \) with field of fractions \( F \), we shall let \( \text{ord} : F^* \to \mathbb{Z} \) denote the valuation map. We begin with the following elementary computation in Milnor \( K \)-theory. We shall use the additive notation for the group operation of the Milnor \( K \)-theory.

**Lemma 3.2.** Let \( R \) be a semi-local integral domain with field of fractions \( F \). Let \( a, b, s, t \) be non-zero elements of \( R \) such that \( 1 + as, 1 + bt \neq 0 \). Then we have the following identity in \( K_2^M(F) \).

\[
(3.6) \quad \{1 + as, 1 + bt\} = \begin{cases} 
-\{1 + \frac{ab}{1+as}st, -as(1 + bt)\} & \text{if } 1 + (1 + bt)as \neq 0 \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** Suppose first that \( 1 + (1 + bt)as = 0 \). Then we have

\[
\{1 + as, 1 + bt\} = \{1 + as, (-as)^{-1}\} = -\{1 + as, -as\} = 0.
\]

Otherwise, we write

\[
(3.7) \quad \{1 + \frac{ab}{1+as}st, -as(1 + bt)\} = \{1 + \frac{ab}{1+as}st, -as\} + \{1 + \frac{ab}{1+as}st, (1 + bt)\}
\]

\[
= \{1 + \frac{ab}{1+as}st, -as\} + \{1 + (1 + bt)as, 1 + bt\}
\]

\[
= -\{1 + as, 1 + bt\} - \{1 + as, -as\}
\]

\[
\quad + \{1 + (1 + bt)as, 1 + bt\} - \{1 + as, 1 + bt\}
\]

\[
= \{1 + (1 + bt)as, -(1 + bt)as\} - \{1 + as, -as\}
\]

\[
= -\{1 + as, 1 + bt\}
\]

\[
= \{1 - u, u\} - \{1 - v, v\} - \{1 + as, 1 + bt\}
\]

\[
= -\{1 + as, 1 + bt\}.
\]
where we let $u = -(1 + bt)as$ and $v = -as$. This proves the lemma.

\[ \square \]

**Lemma 3.3.** Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and field of fractions $F$. For $m, n \geq 1$, let $K^M_n(F, m)$ denote the subgroup of $K^M_n(F)$ generated by Milnor symbols $\{y_1, \ldots, y_n\} \in K^M_n(F)$ such that $\sum_{i=1}^n \text{ord}(y_i - 1) \geq m$. Then for any $n \geq 0$, we have

$$K^M_{n+1}(F, m) \subseteq (1 + m^n)K^M_n(F).$$

**Proof.** Note that for $n = 0$, we actually have $K^M_1(F, m) = 1 + m^m$ and this is obvious from the definition of $K^M_1(F, m)$. We shall prove $n \geq 1$ case by induction on $n$. We let $\pi$ denote a uniformizing parameter of $R$. We can write $y_i = 1 + u_i \pi^{m_i}$ for some $u_i \in R^\times$ and $m_i \in \mathbb{Z}$ for $1 \leq i \leq n$. We first observe that if $m_i \geq m$ for some $i \geq 1$, then $y_i = 1 + u_i \pi^{m_i} \in 1 + m^m$ and we are done.

We now assume that $n = 1$. In this case, if some $m_i \leq 0$, then we must have that some $m_j \geq m$ and we are done as above. We can therefore assume that $0 < m_1, m_2 < m$. In this case, Lemma 3.2 says that $\{y_1, y_2\}$ is either zero or it is $\{-1 + u_1 u_2 y_1^{-1} \pi^{m_1 + m_2}, -u_1 y_2 \pi^{m_1}\}$. Since $m_1 \geq 0$, we see that $y_1^{-1} \in R^\times$. In particular, $1 + u_1 u_2 y_1^{-1} \pi^{m_1 + m_2} \in 1 + m^m$. We therefore get $\{y_1, y_2\} \subseteq (1 + m^m)K^M_1(F)$. If $n \geq 2$, we have must have $m_i \geq 0$ for some $1 \leq i \leq n$ as $m \geq 0$. Since the permutation of coordinates of a Milnor symbol only changes its sign in the Milnor $K$-group, we can assume that $m_1 \geq 0$ so that $y_1 \in R^\times$. We can now write $\{y_1, \ldots, y_n\} = \{y_1, y_2\} \cdot \{y_3, \ldots, y_n\}$. We have seen before that the term $\{y_1, y_2\}$ is either zero or we have

$$\{y_1, \ldots, y_n\} = \{1 + u_1 \pi^{m_1}, 1 + u_2 \pi^{m_2}\} \cdot \{y_3, \ldots, y_n\} = \{-1 + u_1 u_2 y_1^{-1} \pi^{m_1 + m_2}, -u_1 \pi^{m_1} y_2\} \cdot \{y_3, \ldots, y_n\} = \{-u_1 y_2 \pi^{m_1}\} \cdot \{1 + u_1 u_2 y_1^{-1} \pi^{m_1 + m_2}, y_3, \ldots, y_n\}.$$ 

Since $y_1 \in R^\times$, it follows that $y_2' := 1 + u_1 u_2 y_1^{-1} \pi^{m_1 + m_2} \in 1 + m^{m_1 + m_2}$. In particular, we see that $\text{ord}(y_2' - 1) + \sum_{i=3}^n \text{ord}(y_i - 1) \geq \sum m_i \geq m$. Hence, the induction hypothesis implies that $\{y_2', y_3, \ldots, y_n\} \subseteq (1 + m^m)K^M_{n-1}(F)$. This implies that $\{y_1, \ldots, y_n\} = \{-u_1 y_2 \pi^{m_1}\} \cdot \{y_2, y_3, \ldots, y_n\} \subseteq (1 + m^m)K^M_n(F)$. This finishes the proof. \[ \square \]

**Lemma 3.4.** Let $R$ be a discrete valuation ring containing a field. Let $\mathfrak{m}$ and $F$ denote the maximal ideal and the field of fractions of $R$, respectively. Then the following hold for every integer $n \geq 0$.

1. $(1 + m)K^M_n(F) \subseteq \widehat{R}^M_{n+1}(R)$.
2. $(1 + m^{m+n})K^M_n(F) \subseteq (1 + m^n)\widehat{R}^M_{n+1}(R)$ for all $m \geq 1$.

**Proof.** We shall prove the lemma by induction on $n$. As the base case $n = 0$ trivially follows, we shall assume that $n \geq 1$. Suppose we show that

$$1. (1 + m)K^M_1(F) \subseteq (1 + m)\widehat{R}^M_2(R) \subseteq \widehat{R}^M_2(R).$$

We will then have

$$(1 + m)K^M_n(F) \subseteq (1 + m)\widehat{R}^M_{n+1}(R)K^M_{n-1}(F) = \widehat{R}^M_n(R)(1 + m)K^M_{n-1}(F)$$

$$\subseteq (1 + m)\widehat{R}^M_{n+1}(R) \subseteq \widehat{R}^M_{n+1}(R),$$

where $\subseteq$ holds by induction on $n$. This will prove (1).

Similarly, suppose we show for all $m \geq 1$ that

$$2. (1 + m^{m+1})K^M_1(F) \subseteq (1 + m^m)\widehat{R}^M_1(R) \subseteq \widehat{R}^M_2(R).$$

Then for any $m \geq 1$ and $n \geq 2$, we will have

$$(1 + m^{m+n})K^M_n(F) \subseteq (1 + m^{m+n-1})\widehat{R}^M_{n+1}(R)K^M_{n-1}(F) = \widehat{R}^M_n(R)(1 + m^{m+n-1})K^M_{n-1}(F)$$
\[ \subseteq^1 \widehat{K}^M_1(R)(1 + m^n) \subseteq (1 + m^n) \widehat{K}^M_n(R), \]

where \( \subseteq^1 \) holds by induction on \( n \). This will prove (2). We are therefore left with showing (3.8) and (3.9) in order to prove the lemma.

To prove (3.8), we let \( \pi \) be a uniformizing parameter of \( R \). For \( j \in \mathbb{Z} \) and \( u, v \in R^\ast \), we then have

\[
\{1 + u\pi, v\pi^j\} = \{1 + u\pi, v\} + \{1 + u\pi, \pi^j\} \\
= \{1 + u\pi, v\} + j\{1 + u\pi, \pi\} \\
= \{1 + u\pi, v\} - j\{1 + u\pi, -u\},
\]

where the last equality holds because \( \{1 + u\pi, -u\pi\} = 0 \). It follows that \( \{1 + u\pi, v\pi^j\} \in (1 + m)\widehat{K}^M_1(R) \). If \( i \geq 2 \), then \( \{1 + u\pi, v\pi^j\} \in (1 + m^{i-1})\widehat{K}^M_1(R) \subseteq (1 + m)\widehat{K}^M_1(R) \) by (3.9). It remains therefore to prove (3.9).

We now fix \( m \geq 1, j \in \mathbb{Z}, a \in R \) and \( u \in R^\ast \). We consider the element \( \{1 + a\pi^{m+1}, u\pi^j\} \in (1 + m^{m+1})\widehat{K}^M_1(F) \). We set

\[ t = -a\pi^m, \quad v' = (1 + t(-1 - \pi))^{-1} \quad \text{and} \quad v'' = v'' = -1 - \pi. \]

With these notations, it is clear that \( 1 + v't, 1 + v''t \in (1 + m^m) \) and \( (1 + v't)(1 + v''t) = 1 - \pi t \).

In \( K^M_2(F) \), we now compute

\[
\{1 + a\pi^{m+1}, u\pi^j\} = \{1 + a\pi^{m+1}, u\} + j\{1 + a\pi^{m+1}, \pi\} \\
= \{1 + a\pi^{m+1}, u\} + j\{1 + (a\pi^m)\pi, \pi\} \\
= \{1 + a\pi^{m+1}, u\} - j\{1 + (a\pi^m)\pi, -(a\pi^m)\} \\
= \{1 + a\pi^{m+1}, u\} - j\{1 - \pi, t\} \\
= \{1 + a\pi^{m+1}, u\} - j\{(1 + v't)(1 + v''t), t\} \\
= \{1 + a\pi^{m+1}, u\} - j\{1 + v't, t\} - j\{1 + v''t, t\} \\
= \{1 + a\pi^{m+1}, u\} + j\{1 + v't, -v'\} + j\{1 + v''t, -v''\} \\
\in (1 + m^m)\widehat{K}^M_1(R).
\]

This proves (3.9) and completes the proof of the lemma.

Let us now assume that \( R \) is a regular semi-local integral domain of dimension one containing a field. Let \( \{m_1, \ldots, m_r\} \) be the set of maximal ideals of \( R \) and let \( F \) be the field of fractions of \( R \). If \( \mathbf{m} = (m_1, \ldots, m_r) \) is an \( r \)-tuple of positive integers, we shall write the relative improved Milnor K-theory \( \widehat{K}^M_*(R, \mathbf{m}^{m_1}, \ldots, m^{m_r}) \) (see (3.3)) as \( \widehat{K}^M_*(R, \mathbf{m}) \). For \( n \geq 1 \), we let \( K^M_n(F, \mathbf{m}_{\sum}) = \bigcap_{i=1}^{r} (1 + m^{m_i} R_{m_i}) \widehat{K}^M_{n-1}(F) \). We let \( \mathbf{m}_{\sum} = (m_1 + n, \ldots, m_r + n) \).

**Lemma 3.5.** For every integer \( n \geq 1 \), we have \( K^M_n(F, \mathbf{m}_{\sum}) \subseteq \widehat{K}^M_n(R, \mathbf{m}). \)

**Proof.** If \( R \) is local, then Lemma 3.4 says that \( K^M_n(F, \mathbf{m}_{\sum}) \subseteq (1 + m^n)\widehat{K}^M_{n-1}(R) \subseteq \widehat{K}^M_n(R, \mathbf{m}) \). In particular, the lemma holds if \( R \) is local.

In general, let \( y \in K^M_n(F, \mathbf{m}_{\sum}) \). Then it is clear that \( y \in (1 + m^n R_{m_i}) \widehat{K}^M_{n-1}(R_{m_i}) \) for each \( 1 \leq i \leq r \). It follows by Lemma 3.4 that \( y \in (1 + m^n R_{m_i}) \widehat{K}^M_{n-1}(R_{m_i}) \subseteq \widehat{K}^M_n(R_{m_i}) \) for each \( i \). We conclude from (3.1) and Proposition 3.1 that \( y \in \widehat{K}^M_n(R) \).
We now consider the diagram

\[
\begin{array}{c}
\xymatrix{
\widehat{R}_n^M(R) \ar[r]^\pi & K_n^M(F) \\
\Delta_R & \widehat{R}_n^M(R/m_1^{n_1} \cdots m_r^{n_r}) \ar[l]_\Delta \\
\bigoplus_{i=1}^r \widehat{R}_n^M(R_{m_i}) \ar[r]^\phi & \bigoplus_{i=1}^r K_n^M(F) \\
\bigoplus_{i=1}^r \widehat{R}_n^M(R_{m_i}/m_i^{m_i}) \ar[u]_\Theta_1 \ar[r] & \bigoplus_{i=1}^r K_n^M(F) \ar[u]_\Phi_1 \\
}\end{array}
\]

where \(\phi\) is the last in the sequence of arrows in (3.3).

This diagram is clearly commutative. It follows from the case of local rings shown above that \((\Theta_1, \Phi_1) \circ \Delta_R(y) = 0\). Since \(\phi\) is an isomorphism, we conclude that \(\pi(y) = 0\), which is what we wanted to show. This finishes the proof. \(\square\)

4. The cycle class map

In this section, we shall define the cycle class map on the group of 0-cycles with modulus and prove a very special case of Theorem 1.1. The final proof of this theorem will be done by the end of the next section. We fix an arbitrary field \(k\).

4.1. The map \(\text{cyc}_{X|D}\) on generators. Let \((X, D)\) be a modulus pair in \(\text{Sch}_k\) of dimension \(d \geq 1\). Let \(n \geq 0\) be an integer. We begin by defining the cycle class map \(\text{cyc}_{X|D}\) on the group \(z^{d+n}(X|D, n)\). Let \(Z \in X \times \square^n\) be an admissible closed point. Since \(Z\) is a closed point, we have that \(Z = \text{Spec}(k(Z))\). We let \(p_X: X \times \square^n \to \square^n\) and \(p_X: X \times \square^n \to X\) denote the projection maps. We let \(f: Z \to X\) denote the projection map. It is clear that \(f\) is a finite map and its image is a closed point \(x \in X\) which does not lie in \(D\). We thus have a factorization \(Z \to \text{Spec}(k(x)) \to X^o \to X\) of \(f\). The latter is actually a map of modulus pairs \(f: (Z, \emptyset) \to (X, D)\). Hence, it induces the proper push-forward \(f_*: \text{CH}_q(Z, *) \to \text{CH}_q(X|D, *)\), where \(\text{CH}_q(Z, *)\) are Bloch’s higher Chow groups of \(Z\) \cite{7}.

Now, the closed point \(Z \in X \times \square^n\) defines a unique \(k(Z)\)-rational point (which we also denote by \(Z\)) in \(\square^n\) such that the composite projection map \(Z \to \square^n\) is identity. Furthermore, \([Z] \in z^{d+n}(X|D, n)\) is the image of \([Z] \in z^n(Z, n)\) under the push-forward map \(f_*\). Since \(Z \to \square^n\) does not meet any face of \(\square^n\), it follows that \(y_i(Z) \in k(Z)^*\) for every \(1 \leq i \leq n\), where \(y_i: \square^n \to \square_i\) is the projection to the \(i\)-th factor. In particular, \((y_1(Z), \ldots, y_n(Z))\) is a well-defined element of \(K^M_n(k(Z))\). We let

\[
\text{cyc}_Z^M([Z]) = \{y_1(Z), \ldots, y_n(Z)\} \in K^M_n(k(Z)) \quad \text{and}
\]

\[
\text{cyc}_Z([Z]) = \psi_Z \circ \text{cyc}_Z^M([Z]) \in K_n(Z),
\]

where recall that \(\psi_Z: K^M_n(k(Z)) \to K_n(k(Z)) = K_n(Z)\) is the canonical map from the Milnor to the Quillen \(K\)-theory.

We next recall from \(\S\) 2.3 that as \(x = f(Z) \in X^o\) (which is regular), the finite map \(f\) defines a map of spectra \(f_*: K(Z) \to K(X, D)\) such that the composite map \(K(Z) \to \)
$K(X,D) \to K(X)$ is the usual push-forward map. The same holds for the inclusion $i^*: \text{Spec}(k(x)) \to X$. We let

$\text{cyc}_{X|D}([Z]) = f_* \circ \text{cyc}_Z([Z]) \in K_n(X,D)$.

Extending this linearly, we obtain our cycle class map

$\text{cyc}_{X|D}: z^{d+n}(X|D,n) \to K_n(X,D)$.

If $Z$ is an admissible closed point as above and $x = f(Z)$, then we have a commutative diagram

$\begin{array}{ccc}
K^M_n(k(Z)) & \xrightarrow{\psi_x} & K_n(k(Z)) \\
N_{Z/\mathbf{k}} \downarrow & & \downarrow f_* \\
K^M_n(k(x)) & \xrightarrow{\psi_x} & K_n(k(x)) \xrightarrow{i^*} K_n(X,D),
\end{array}$

where $N_{Z/\mathbf{k}}$ is the Norm map between the Milnor $K$-theory of fields [2] (see also [25]) and the right vertical arrow is the transfer (push-forward) map between the Quillen $K$-theory of fields. The square on the left commutes by Lemma 11.3. We can therefore write

$\text{cyc}_{X|D}([Z]) = f_* \circ \text{cyc}_Z([Z]) = f_* \circ \psi_Z \circ \text{cyc}_Z^D([Z]) = f_* \circ \psi_Z([y_1(Z), \ldots, y_n(Z)]) = i^* \circ \psi_x \circ N_{Z/\mathbf{k}}([y_1(Z), \ldots, y_n(Z)])$.

It is clear from the definition that for any integer $m \geq 1$, there is a commutative diagram

$\begin{array}{ccc}
z^{d+n}(X|m+1D,n) & \xrightarrow{\text{cyc}_{X|D}} & K_n(X,(m+1)D) \\
\downarrow & & \downarrow \\
z^{d+n}(X|mD,n) & \xrightarrow{\text{cyc}_{X|D}} & K_n(X,mD),
\end{array}$

where the vertical arrows are the canonical restriction maps. We therefore have a strict map of pro-abelian groups

$\text{cyc}_{X|D}: \{z^{d+n}(X|mD,n)\}_m \to \{K_n(X,mD)\}_m$.

We next prove that the map $\text{cyc}_{X|D}$ is covariant with respect to proper morphisms of modulus pairs and contravariant for the flat morphisms of modulus pairs which are of relative dimension 0. Note that these are the only general cases where the functoriality of the cycle class map makes sense.

**4.2. Naturality for flat morphisms.** Let $(Y,E)$ and $(X,D)$ be modulus pairs. Let $h: Y \to X$ be a flat morphisms of relative dimension 0 such that $E = h^*(D)$. Recall from [36, Proposition 2.12] that we have a pull-back map $h^*: z^{d+n}(X|D,n) \to z^{d+n}(Y|E,n)$ such that $h^*([Z]) = [W]$, where $W = (h \times \text{id}_{\mathbb{P}^n})^{-1}(Z)$ and $Z$ a closed point in $X \times D \times \mathbb{P}^n$.

**Lemma 4.1.** With notations as above, the following diagram commutes:

$\begin{array}{ccc}
z^{d+n}(X|D,n) & \xrightarrow{\text{cyc}_{X|D}} & K_n(X,D) \\
\downarrow h^* & & \downarrow h^* \\
z^{d+n}(Y|E,n) & \xrightarrow{\text{cyc}_{Y|E}} & K_n(Y,E).
\end{array}$
Proof. Let $Z$ be a closed point in $z^{d+n}(X,D,n)$ and let $[W] = \sum_{i=1}^{r} m_i [W_i] \in z^{d+n}(Y,E,n)$, where $W_i$ are irreducible components of the inverse image scheme $W$ with multiplicities $m_i$. Let $f^Z : Z \to X$, $f^W : W \to Y$ and $f^W_i : W_i \to Y$ denote the respective projections. We then have to show that

\begin{equation}
(4.9) \quad h^* \circ f^Z_* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i ( f^W_{*i} \circ \psi_{W_i}(y(W_i))).
\end{equation}

Consider the following diagram:

\begin{equation}
(4.10) \quad K(Z) \to G(Z) \to K^Z(X \setminus D) \xrightarrow{\psi} K^Z(X,D) \to K(X,D)
\end{equation}

Since $h$ is flat, it follows that all the squares in (4.10) commute. Indeed, since the canonical map $K(-) \to G(-)$ respects flat pull-back, the left-most square in (4.10) commutes. The middle left square commutes by [53, Proposition 3.18] and the middle right square commutes because each map is a pull-back map. Lastly, the right-most square in (4.10) commutes by the definition of the left arrow in the square. As discussed in §2.4 the composition of the top horizontal arrows is the push forward map $f^Z_*$ and the composition of the top horizontal arrows is the push forward map $f^W_*$. It then suffices to show that

\begin{equation}
(4.11) \quad f^W_* \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i ( f^W_{*i} \circ \psi_{W_i}(y(W_i))).
\end{equation}

Since $W = (h \times \text{id}_D)^{-1}(Z)$, we have $y(W_i) = y(Z)$ for each $1 \leq i \leq r$ under the injective map $k(Z)^x \hookrightarrow k(W_i)^x$. It then follows that $\psi_{W_i}(y(W_i)) = h^*_i \psi_Z(y(Z)) \in K_n(W_i)$, where $h_i : W_i \to Z$ is the induced map. Note that $h_i = h \circ g_i$, where $g_i : W_i \to W$ denotes the inclusion of the irreducible component $W_i$ into $W$. We are therefore reduced to show that

\begin{equation}
(4.12) \quad f^W_* \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i ( f^W_{*i} \circ g^*_i \circ h^* \circ \psi_Z(y(Z))).
\end{equation}

We shall actually show that for all $a \in K_n(W)$, we have

\begin{equation}
(4.13) \quad f^W_*(a) = \sum_{i=1}^{r} m_i f^W_{*i} \circ g^*_i(a).
\end{equation}

Observe that the equality (4.12) follows from (4.13) with $a = h^* \circ \psi_Z(y(Z))$. To show (4.13), consider the diagram:

\begin{equation}
(4.14) \quad \bigsqcup_{i=1}^{r} G(W_i) \xrightarrow{\psi} G(W_{\text{red}}) \xrightarrow{\psi} K^W(Y \setminus E) \xrightarrow{\psi} K^W(Y,E) \to K(Y,E)
\end{equation}

Since $W_{\text{red}} = \bigsqcup_i W_i$, it follows that the push-forward map $\prod_{i=1}^{r} G(W_i) \to G(W_{\text{red}})$ is an isomorphism and the left triangle in (4.14) commutes. Observe that the left-most square commutes because all arrow in the square are (compatible) push-forward maps. As before, the composition of the top horizontal arrows on $G(W_i)$ is the push-forward map $f^W_{*i}$. Let $b \in G_n(W)$ be the image of $a \in K_n(W)$ under the map $K_n(W) \to G_n(W)$
induced by the bottom left arrow in \((4.14)\). The equality \((4.13)\) then follows if we show that
\[
(4.15) \quad b = g_*(\sum_{i=1}^{r} m_i \cdot g_i^*(b)) \in G_n(W).
\]

The equality \((4.15)\) however follows from the following calculation:
\[
g_*(\sum_{i=1}^{r} m_i \cdot g_i^*(b)) = \sum_{i=1}^{r} m_i \cdot g_i \circ g_i^*(b)
= 1 \sum_{i=1}^{r} m_i \cdot g_i (g_i^*(b)[\mathcal{O}_{W_i}])
= 2 b \left( \sum_{i=1}^{r} m_i \cdot g_i ([\mathcal{O}_{W_i}]) \right)
= 3 b[\mathcal{O}_W] = b,
\]
where \(=1\) follows because for each \(i\), we have \([\mathcal{O}_{W_i}] = 1 \in G_0(W_i)\), \(=2\) follows from the projection formula \([53, \text{Proposition 3.17}]\) for \(G\)-theory because \(a \in K_n(W)\) and \(=3\) follows as \(1 = [\mathcal{O}_W] = \sum_{i=1}^{r} m_i \cdot g_i ([\mathcal{O}_{W_i}]) \in G_0(W)\). This completes the proof of the lemma. \(\square\)

4.3. Naturality for proper morphisms. Let \((Y, E)\) and \((X, D)\) be modulus pairs such that \(X\) and \(Y\) are non-singular schemes over \(k\) of pure dimension \(d_X\) and \(d_Y\). Let \(h : Y \to X\) be a proper morphisms such that \(E = h^*(D)\). By \([56, \text{Proposition 2.10}]\), we have a proper push-forward map \(h_* : z^{d_Y+n}(Y[E,n]) \to z^{d_X+n}(X[D,n])\) such that \(h_*([z]) = ([w])\), where \(w = (h \times \text{id}_Y)(z)\) and \(z\) is a closed point in \(Y \setminus E \times \square^n\). The existence of the push-forward map \(h_* : K(Y, E) \to K(X, D)\) follows from the following lemma.

**Lemma 4.2.** The map \(h\) induces a proper push-forward map \(h_* : K(Y, E) \to K(X, D)\) of relative \(K\)-theory spectra.

**Proof.** Since \(Y\) is non-singular, we can assume with out loss of generality that \(Y\) is integral. Since \(X\) is non-singular, the map \(h\) has finite tor-dimension. By \([53, 3.16.4]\), we have a push-forward map \(f_* : K(Y) \to K(X)\). Observe that if \(E = \emptyset\), then we have a push-forward map \(f_* : K(Y) \to K(X, D)\). We can therefore assume that \(E \neq \emptyset\). To prove the lemma, it suffices to show that \(Y\) and \(D\) are tor-independent over \(X\). This will in particular imply that \(E \to D\) also has finite tor-dimension. For tor-independence, we note that \(D\) is an effective Cartier divisor. Hence, the only possible non-trivial tor term can be \(\text{Tor}^1_{\mathcal{O}_Y} (\mathcal{O}_Y, \mathcal{O}_D)\). But this is same as the \(\mathcal{I}_D\)-torsion subsheaf of \(\mathcal{O}_Y\). Since \(Y\) is integral, this torsion subsheaf is non-zero if and only if the ideal \(\mathcal{I}_E\) is zero. But this can not happen as \(E\) is a proper divisor on \(Y\). This finishes the proof. \(\square\)

**Remark 4.3.** Observe that Lemma 4.2 is true for a general integral scheme (may not be non-singular) \(Y\) over a non-singular scheme \(X\).

We now prove that the map \(\text{cyc}_{X[D]}\) in \((4.4)\) commutes with the push-forward map.

**Lemma 4.4.** For a cycle \(\alpha \in z^{d_Y+n}(Y[E,n])\), we have \(\text{cyc}_{X[D]} \circ h_* (\alpha) = h_* \circ \text{cyc}_{Y[E]} (\alpha)\).

**Proof.** We can assume \(\alpha\) is represented by a closed point \(z \in z^{d_Y+n}(Y[E,n])\). We set \(w = (h \times \text{id}_Y)(z) \in X \times \square^n\) and \(x = p_X(w)\). The compatibility between norm maps in the Milnor \(K\)-theory of fields and push-forward maps in the Quillen \(K\)-theory (see
Lemma \ref{lem:main} yields a commutative diagram

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\xymatrix{
K_n^M(k(z)) \ar[r]^{N_{z/w}} & K_n^M(k(w)) \\
K_n(k(z)) \ar[u]^{\psi_z} \ar[d]^{p_*} \ar[r]^{t_{z,*}} & K_n(k(w)) \ar[u]^{\psi_w} \ar[d]^{p_*} \\
K_n(Y, E) \ar[r]^{h_*} & K_n(X, D).
}
\end{array}
\end{array}
\end{equation}

Using this commutative diagram, we get

\[cyc_{X|D} \circ h_*(\{z\}) = cyc_{X|D}(\{k(z) : k(w)\}[[w]])\]
\[= t_{w,*}(\{y_1(w), \cdots, y_n(w)\})^{[k(z):k(w)]}\]
\[= t_{w,*} \circ t_{z,w,*} \circ t_{z,w,*}^{*}(\{y_1(z), \cdots, y_n(z)\})\]
\[= h_* \circ t_{z,*}(\{y_1(z), \cdots, y_n(z)\})\]
\[= h_* \circ cyc_{Y|E}(\{z\}).\]

In this set of equalities, recall our notation (preceding Lemma \ref{lem:main}) that \(y_i(z)\) is the image of \(z\) under the \(i\)-th projection \(\text{Spec}(k(z)) \to \square_k(z)\). In particular, \(y_i(z) \in k(z)^x\) for all \(1 \leq i \leq n\). The coordinates \(y_i(w) \in k(w)^x\) have similar meaning. The map \(t_{z,w} : \text{Spec}(k(z)) \to \text{Spec}(k(w))\) is the projection. The equality \(=^1\) is a consequence of the projection formula for the Milnor K-theory associated to the resulting inclusion \(k(w) \to k(z)\). The equality \(=^2\) follows from the fact that \(y_i(z) = y_i(w) = t_{z,w,*}^{*}(y_i(w))\) via the inclusion \(k(w) \to k(z)\) for all \(1 \leq i \leq n\). This proves the lemma. \(\square\)

We end this section with some comments below on \(cyc_{X|D}\) in the non-modulus case.

4.4. Agreement with Levine’s map. We had mentioned in \S 4.1 that for Bloch’s higher Chow groups of 0-cycles, a cycle class map to the ordinary K-theory of a non-singular variety was constructed by Levine \cite{Levine} with rational coefficients. Binda \cite{Binda} constructed such a map in the modulus setting and his map is identical to that of Levine by definition. It is not hard to check that the map \(cyc_{X|D}\) coincides with Levine’s map when \(D = \emptyset\) (note that \(z^{d+n}(X|D,n) \subseteq z^{d+n}(X,n))\). In particular, it turns out that Levine’s cycle class map for the ordinary 0-cycles exists with integral coefficients. We give a sketch of this agreement and leave the details for the reader.

Let \(n \geq 0\) be an integer. Since \(X\) is non-singular, the homotopy invariance implies that the multi-relative K-theory exact sequence (see \cite{Levine} \S 1) yields an isomorphism

\begin{equation}
\theta_n : K_n(X) \xrightarrow{\cong} K_0(X \times \square^n, \partial \square^n),
\end{equation}

where \(K(X \times \square^n, \partial \square^n)\) is the iterated multi-relative K-theory of \(X \times \square^n\) relative to all codimension one faces.

Let \(Z \subset X \times \square^n\) be a closed point. Then Levine’s cycle class map \(cyc_X^L([Z])\) is the image of \(1 \in K_0(Z)\) under the composition

\begin{equation}
K_0(Z) \cong K_0(Z \times \square^n, \partial \square^n) \to K_0(X \times \square^n, \partial \square^n) \xrightarrow{\theta_n} K_n(X).
\end{equation}

Let \(f : Z \to X\) denote the projection map (which is finite). We then have a finite map of multi-closed pairs \(f : (Z \times \square^n, \partial \square^n) \to (X \times \square^n, \partial \square^n)\). Since the relative K-theory fiber sequence commutes with finite push-forward for non-singular schemes (see Lemma \ref{lem:finite}), and since \(cyc_X^L([Z])\) is the image of \([Z] \in K_0(Z \times \square^n, \partial \square^n)\) under the push-forward map \(f_*\), we can assume that \(X = \text{Spec}(k)\) and \(f : Z \to \text{Spec}(k)\) is the identity map.
When \( n = 0 \), the agreement of \( \text{cyc}_k([Z]) \) and \( \text{cyc}_k^L([Z]) \) is immediate. When \( n \geq 1 \) and if we follow our notation of \((\ref{eq:cycle}),\), then we see that \( y_i(Z) = a_i \in k^\times \) for each \( 1 \leq i \leq n \). In particular, we have \( \text{cyc}_k([Z]) = \{a_1, \ldots, a_n\} \in K_n(k) \). One therefore has to show that if \( z = (a_1, \ldots, a_n) \in (k^n)^n \) is a \( k \)-rational point in \( \mathbb{A}^n \), then the class \([k(z)] \in K_0(\mathbb{A}^n, \partial \mathbb{A}^n)\) coincides with \( \theta_n(\{a_1, \ldots, a_n\}) \). But this is an elementary exercise in \( K \)-theory using repeated application of relative \( K \)-theory exact sequence. For \( n = 1 \), it already follows from a straightforward generalization of Lemma \( \ref{lem:cycle} \) (where we replace \((t^{m+1})\) by any ideal of \( R[t] \)) with identical proof. We leave it to the reader to check the details for \( n \geq 2 \).

5. The case of non-singular curves

The goal now is to show that \( \text{cyc}_{X|D} \) kills the rational equivalence if we allow the modulus to vary along \( \{mD\}_{m \geq 1} \). In this section, we shall prove a very special case of this. The proof of Theorem \( \ref{thm:main} \) will be reduced to this case in the next section. We consider the following situation. Let \( n \geq 0 \) be an integer. We let \( X \) be a non-singular connected curve over \( k \) and let \( D \subset X \) be an effective Cartier divisor. Let \( \overline{W} \subset X \times \mathbb{A}^{n+1} \) be a closed subscheme such that the following hold.

1. \( \text{The composite map } \overline{W}^X \to X \times \mathbb{A}^{n+1} \xrightarrow{p_X} X \text{ is an isomorphism.} \)
2. \( \mathcal{W} = \overline{W} \cap (X \times \mathbb{A}^{n+1}) \) is an admissible cycle on \( X \times \mathbb{A}^{n+1} \) with modulus \((n+1)D\).

That is, \([\mathcal{W}] \in z^{n+1}(X|(n+1)D, n+1)\).

Let \( \partial = \sum_{i=1}^{n+1} (-1)^i (\partial_i^\infty - \partial_i^0) : z^{n+1}(X|E, n+1) \to z^{n+1}(X|E, n) \) be the boundary map in the cycle complex with modulus for an effective divisor \( E \subset X \). We want to prove the following result in this subsection.

**Proposition 5.1.** The class \( \text{cyc}_{X|D}(\partial \mathcal{W}) \) dies in \( K_n(X, D) \) under the cycle class map

\[
\text{cyc}_{X|D}(\partial \mathcal{W}) : z^{n+1}(X|D, n) \xrightarrow{\text{cyc}_{X|D}} K_n(X, D).
\]

We shall prove this proposition in several steps. We begin with the following description of the cycle class map on various boundaries of \( \overline{W} \). Let \( F \) denote the function field of \( X \). Let \( g \) denote the Milnor symbol \( \{g_1, \ldots, g_{n+1}\} \in K_n^M(F) \), where \( g_i : \overline{W} \to \mathbb{A}^1 \) is the \( i \)-th projection for \( 1 \leq i \leq n + 1 \). Note that this symbol is well defined because no \( g_i \) can be identically zero by the admissibility of \( W \). For any closed point \( z \in \overline{W} \), let \( \text{ord}_z : F^\times \to \mathbb{Z} \) denote the valuation associated to the discrete valuation ring \( \mathcal{O}_{\overline{W}, z} \). We let \( \partial_z^M : K^M_{i+1}(F) \to K^M_i(k(z)) \) denote a boundary map in the Gersten complex \((\ref{eq:gersten})\).

We let \( m_z \) denote the maximal ideal of the local ring \( \mathcal{O}_{\overline{W}, z} \). A symbol \( \{a_1, \ldots, a_i, \ldots, a_n\} \) will mean the one obtained from \( \{a_1, \ldots, a_n\} \) by omitting \( a_i \). For any point \( z \in X \times \mathbb{A}^{n+1} \), let \( y_i : \mathbb{A}^{n+1}_k \to \mathbb{A}^1_{k(z)} \) denote the projection map to \( i \)-th factor. Let \( f^z : \text{Spec}(k(z)) \to X \) denote the projection to \( X \).

**Lemma 5.2.** For \( 1 \leq i \leq n + 1 \), we have

\[
\text{cyc}_{X|D}(\partial_i^0 \mathcal{W}) = \sum_{z \in \partial_i^0 \mathcal{W}} \text{ord}_z(g_i) f^z \circ \psi_z(\{y_1(z), \ldots, y_i(z), \ldots, y_{n+1}(z)\}),
\]

\[
\text{cyc}_{X|D}(\partial_i^\infty \mathcal{W}) = \sum_{z \in \partial_i^\infty \mathcal{W}} \text{ord}_z(1/g_i) f^z \circ \psi_z(\{y_1(z), \ldots, y_i(z), \ldots, y_{n+1}(z)\}).
\]

**Proof.** We should first observe that the admissibility of \( W \) implies that if \( z \in \partial_i^t \mathcal{W} \) for \( t \in \{0, \infty\} \), then we must have \( y_j(z) \neq 0 \) for all \( j \neq i \). In particular, the element
{y_1(z), \ldots, y_i(z), \ldots, y_{n+1}(z)} \in K^M_n(k(z)) is well defined. By the definition of $cyc_{X \mid D}$, it suffices to show that for $1 \leq i \leq n + 1$, we have

$$(5.1) \quad \partial_i^0 W = \sum_{z \in \partial_i^0 W} \ord_z(g_i)[z] \quad \text{and} \quad \partial_i^\infty W = \sum_{z \in \partial_i^\infty W} \ord_z(1/g_i)[z].$$

But this is an immediate consequence of the definition of the intersection product of an integral cycle with the faces of $X \times \mathbb{A}^n$. \hfill \square

**Lemma 5.3.** Suppose that $n \geq 1$ and $\partial_X^M(g) \in \bigoplus_{z \in X^n} K^M_n(k(z))$ under the Milnor boundary map $\partial_X^M : K^M_{n+1}(F) \to \bigoplus_{z \in X^{(1)}} K^M_n(k(z))$. Then

$$(5.2) \quad \sum_{z \in X^n} f^1_z \circ \psi_z \circ \partial_X^M(g) = 0.$$

**Proof.** Suppose first that $D = \emptyset$. In this case, we have a diagram

$$(5.3) \quad \begin{array}{ccc}
K^M_{n+1}(F) & \xrightarrow{\partial_X^M} & \bigoplus_{z \in X^{(1)}} K^M_n(k(z)) \\
\psi_F \downarrow & & \downarrow (\psi)_z \\
K_{n+1}(F) & \xrightarrow{\partial_X^O} & \bigoplus_{z \in X^{(1)}} K_n(k(z))^{(f^1_z)_z} K_n(X).
\end{array}$$

The Gersten complex for Milnor $K$-theory canonically maps to the Gersten complex for the Quillen $K$-theory by Lemma [11.1]. In particular, the square in the above diagram is commutative. The bottom row is exact by Quillen’s localization sequence and a limit argument. The lemma follows immediately from this diagram.

We now let $D = m_1 x_1 + \cdots + m_r x_r$, where $x_1, \ldots, x_r$ are distinct closed points of $X$ and $m_1, \ldots, m_r$ are positive integers. Let $A = \mathcal{O}_{X,D}$ be the semi-local ring of $X$ at $D$ and let $I$ denote the ideal of $D$ inside $\text{Spec}(A)$. The localization and relativization sequences give us the commutative diagram of homotopy fiber sequences

$$(5.4) \quad \begin{array}{ccc}
\bigoplus_{z \in X^n} K(k(z)) & = & \bigoplus_{z \in X^n} K(k(z)) \\
\downarrow & & \downarrow \\
K(X,D) & \longrightarrow & K(X) \longrightarrow K(D) \\
\downarrow & & \downarrow \\
K(A,I) & \longrightarrow & K(A) \longrightarrow K(D).
\end{array}$$

The associated homotopy groups long exact sequences yield the commutative diagram

$$(5.5) \quad \begin{array}{ccc}
K_{n+1}(A,I) & \xrightarrow{\partial_A^O} & \bigoplus_{z \in X^n} K_n(k(z))^{(f^1_z)_z} K_n(X,D) \\
\downarrow & & \downarrow \\
K_{n+1}(A) & \xrightarrow{\partial_A^O} & \bigoplus_{z \in X^n} K_n(k(z))^{(f^1_z)_z} K_n(X) \\
u_i & & \downarrow \\
K_{n+1}(D),
\end{array}$$
where \( w : D \hookrightarrow \text{Spec}(A) \) is the inclusion map. It follows from this diagram that there is an exact sequence

\[
\tilde{K}_{n+1}(A, I) \xrightarrow{\partial^Q_A} \oplus K_n(k(z)) \xrightarrow{(f^*_z)} K_n(X, D).
\]

In order to compare this with the Milnor \( K \)-theory, we consider the diagram

\[
\tilde{K}_{n+1}^M(A) \xrightarrow{\partial^M_A} \oplus K_n^M(k(z)) \xrightarrow{\partial^M_X} \oplus K_n^M(k(z))
\]

The map \( \psi_A \) comes from (3.2). By the definition of \( \tilde{K}_{n+1}^M(A) \), we know that the composite map \( \tilde{K}_{n+1}^M(A) \to K_{n+1}^M(F) \xrightarrow{\partial^M_z} K_n^M(k(z)) \) is zero for all closed points \( z \in D \). Hence, the composite \( \tilde{K}_{n+1}^M(A) \to K_{n+1}^M(F) \xrightarrow{\partial^M_A} \oplus K_n^M(k(z)) \) factors through the map denoted by \( \partial^A \) in the above diagram. In particular, the top face of (5.7) commutes. Exactly the same reason shows that the bottom face also commutes. Furthermore, \( \partial^Q_A \) is same as the boundary map in the bottom row of (5.5). The left and the right faces clearly commute and so does the front face by Lemma 11.1 A diagram chase shows that the back face of (5.7) commutes too.

In order to show (5.2), we consider our final diagram

\[
\tilde{K}_{n+1}^M(A) \xrightarrow{\partial^M_A} \oplus K_n^M(k(z)) \xrightarrow{\partial^Q_A} \oplus K_n(k(z)) \xrightarrow{(f^*_z)} K_n(X, D).
\]

The left face of this diagram commutes by (3.5) and we just showed above that the right face commutes. In particular, the bottom face also commutes. Furthermore, the bottom row is same as the exact sequence (5.6). Our assertion will therefore follow if we can show that \( g \in \tilde{K}_{n+1}^M(A, I) \) provided \( W \in z^{n+1}(X|(n+1)D, n+1) \).

Suppose now that \( W \in z^{n+1}(X|(n+1)D, n+1) \). By the definition of the modulus condition (see § 2.5), it means that \( \sum_{j=1}^{n+1} \text{ord}_{r_i}(g_j - 1) \geq (n+1)m_i \) for each \( 1 \leq i \leq r \).
Since $n, m_i \geq 1$, we must have $(n + 1)m_i \geq m_i + n$ for each $1 \leq i \leq r$. This implies that $g = \{g_1, \ldots, g_{n+1}\} \in K_{n+1}^M(F, m_i + n)$ for every $1 \leq i \leq r$ in the notations of Lemma 5.2. If we now apply Lemma 5.3 with $R = A_{m_i}$, it follows that $g \in (1 + m_i^{m_i + n})K_{n+1}^M(F)$ for every $1 \leq i \leq r$. In other words, $g$ lies in the intersection $\bigcap_{i=1}^r (1 + m_i^{m_i + n})K_{n+1}^M(F)$ as an element of $K_{n+1}^M(F)$. It follows from Lemma 5.3 that $g \in \hat{K}_{n+1}(A, m) = \hat{K}_{n+1}(A, I)$. This finishes the proof of the lemma.

**Proof of Proposition 5.4**. We assume first that $n = 0$. In this case, we will show the stronger assertion that $\text{cy}_X^{|D|}(\partial W)$ dies in $K_0(X, D)$ if $W \in z^{n+1}(X|D, n + 1)$. Let $A$ be the semi-local ring and $I \subset A$ the ideal as in the proof of Lemma 5.3. By (5.4), we have an exact sequence

$$K_1(A, I) \xrightarrow{\partial_A^1} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_z^1)} K_0(X, D) \to 0. \tag{5.9}$$

Comparing this with the exact sequence

$$K_1(A) \xrightarrow{\partial_A^1} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_z^1)} K_0(X),$$

we see that we can replace $K_1(A, I)$ by $\hat{K}_1(A, I)$ in (5.9). But then, it is same as the exact sequence

$$(1 + I)^X \xrightarrow{\partial_A^1} \bigwedge^1(X|D, 0) \xrightarrow{\text{cy}_X^{|D|}} K_0(X, D) \to 0.$$

Moreover, one knows that $\text{Coker}(\partial_A^1) \cong CH^1(X|D, 0)$ (e.g., see [31 § 2]). We therefore showed that $\text{cy}_X^{|D|} : \text{CH}^1(X|D, 0) \to K_0(X, D)$ is actually an isomorphism.

We now assume for the remaining part of the proof that $n \geq 1$. As before, let $g_i : W \to \overline{\mathbb{F}}$ denote the projections and let $f = p_X \circ \nu : W \to X$ be the projection to $X$. We also recall the element $g = \{g_1, \ldots, g_{n+1}\} \in K_{n+1}(F)$. Our task is to show that

$$\text{cy}_X^{|D|}(\partial W) = \sum_{i=1}^{n+1} (-1)^i \text{cy}_X^{|D|}(\partial_i \partial W - \partial_i^0 W) = 0 \quad \text{in} \quad K_{n}(X, D) \tag{5.10}$$

if $W$ satisfies the modulus $(n + 1)D$.

Our idea is to compute the cycle class of $\partial W$ in terms of the cycle class of the Milnor boundary $\partial_A^1(g)$. In order to do this, we consider in general a closed point $z \in W$. If $z \in \overline{W} \cap F_1$, then we must have $g_i(z) = 1$ for some $1 \leq i \leq n + 1$. This means that $g_i - 1 \in m_z$. Lemma 5.4 then implies that $\partial_z^M(g) = 0$. If $z \in W \setminus \partial_i^{0, \infty}W$ for all $1 \leq i \leq n + 1$, then we must have $g_i \in \mathcal{O}_{W, z}$ for all $1 \leq i \leq n + 1$ and hence $\partial_z^M(g) = 0$. If $z \in \partial_i^0 W$ for some $1 \leq i \leq n + 1$, then $z \notin \partial_j W$ unless $(t, j) = (0, i)$. This implies that $g_j(z) \in \mathcal{O}_{W, z}$ for all $j \neq i$. Furthermore, the image of $g_j(z)$ under the map $\mathcal{O}_{W, z}^\times \to k(z)$ is simply $y_j(z)$. By the definition of the boundary map in the Gersten complex for the Milnor $K$-theory (e.g., see [2]), we therefore have

$$\partial_z^M(g) = (-1)^i \text{ord}_z(g_i)\{y_i(z), \ldots, \overline{y_i(z)}, \ldots, y_{n+1}(z)\} \in K_{n+1}^M(k(z)).$$

We have the same expression for $\partial_z^M(g)$ if $z \in \partial_i^\infty W$ for some $1 \leq i \leq n + 1$.

If we identify $W$ with $X$ via $f$ so that $W \subseteq X^o = X \setminus D$, it follows from the above computation of $\{\partial_z^M(g)z \in W^{(1)}\}$ and the comparison of (5.10) and Lemma 5.2 that the two things hold. Namely,

1. The image of $g$ under the Milnor boundary $K_{n+1}^M(F) \xrightarrow{\partial_A^1} \bigoplus_{z \in X^o} K_{n+1}^M(k(z))$ lies in the subgroup $\bigoplus_{z \in \partial W} K_{n+1}^M(k(z)) \subset \bigoplus_{z \in X^o} K_{n+1}^M(k(z))$. 


The element $\partial_x^M(g)$ maps to $\text{cyc}_{X|D}(\partial W)$ under the composition of maps
\[
\bigoplus_{z \in X^o} K_n^M(k(z)) \xrightarrow{(\psi_z)^*} \bigoplus_{z \in X^o} K_n(k(z)) \xrightarrow{(f_z^*)} K_n(X,D).
\]

The proposition is therefore reduced to showing that $\sum_{z \in X^o} f_z^* \circ \psi_z \circ \partial_z^M(g) = 0$ if $W$ lies in $z^{n+1}(X|(n+1)D,n+1)$. But this follows at once from Lemma 5.3.

6. PROOF OF THEOREM 1.1

In this section, we shall prove Theorem 1.1 using the case of non-singular curves. So let $k$ be any field. Let $X$ be a non-singular quasi-projective variety of pure dimension $d \geq 1$ over $k$ and let $D \subset X$ be an effective Cartier divisor. We fix an integer $n \geq 0$. In § 4.1 we constructed the cycle class map
\[
\text{cyc}_{X|D}: z^{d+n}(X|D,n) \to K_n(X,D).
\]

The naturality statements in Theorem 1.1 follow from Lemma 4.4 and Lemma 4.1. To prove Theorem 1.1 it therefore suffices to show the following.

**Proposition 6.1.** Let $W \subset z^{d+n}(X|(n+1)D,n+1)$ be an integral cycle. Then the image of $W$ under the composition
\[
z^{d+n}(X|(n+1)D,n+1) \xrightarrow{\partial} z^{d+n}(X|D,n+1) \xrightarrow{\text{cyc}_{X|D}} K_n(X,D)
\]
is zero.

We shall prove this proposition in several steps. Let $\overline{W} \subset X \times \overline{m}$ be the closure of $W$ and let $\nu: \overline{W}^N \to X \times \overline{m}$ be the map induced on the normalization of $\overline{W}$. We begin with a direct proof of one easy case of the proposition as a motivating step.

**Lemma 6.2.** Suppose that $W$ lies over a closed point of $X$. Then the assertion of Proposition 6.1 holds.

*Proof.* In this case, the modulus condition implies that such a closed point must lie in $X^o$. In other words, there is a closed point $x \in X^o$ such that $W \subset z^n(\text{Spec}(k(x)),n+1) \subset z^{d+n}(X|D,n+1)$. Using the commutative diagram (see the construction of $\text{cyc}_{X|D}$ in § 4.1)
\[
z^n(\text{Spec}(k(x)),n+1) \xrightarrow{\partial} z^n(\text{Spec}(k(x)),n) \xrightarrow{\text{cyc}_x} K_n(k(x))
\]
\[
z^{d+n}(X|D,n+1) \xrightarrow{\partial} z^{d+n}(X|D,n) \xrightarrow{\text{cyc}_{X|D}} K_n(X,D),
\]
it suffices to show that $\text{cyc}_x(\partial W) = 0$ in $K_n(k(x))$. We can thus assume that $X = \text{Spec}(k)$ and $D = \emptyset$.

We let $F$ denote the function field of $W$ and let $g = \{g_1, \ldots, g_{n+1}\} \subset K_{n+1}^M(F)$ denote the Milnor symbol given by the projection maps $g_i: \overline{W} \to \overline{m}$. Following the proof of Proposition 5.1 our assertion is equivalent to showing that the composition
\[
K_{n+1}^M(F) \xrightarrow{\partial_M} \bigoplus_{z \in W^{(1)}} K_n^M(k(z)) \xrightarrow{\bigoplus_{z \in W^{(1)}} (f_z^*)} K_n(k(z)) \xrightarrow{(f_z^*)} K_n(k)
\]
kills $g$. Arguing as in (5.3), it suffices to show that the composite map
\[
K_{n+1}(F) \xrightarrow{\partial_M} \bigoplus_{z \in W^{(1)}} K_n(k(z)) \xrightarrow{(f_z^*)} K_n(k)
\]
killing. Since this map is same as the composite map
\[ K_{n+1}(F) \xrightarrow{\partial^Q_{W^N}} \bigoplus_{z \in (W^N)^{(1)}} K_n(k(z)) \xrightarrow{(f^*_z)} K_n(k), \]
we need to show that \( g \) dies under this map. But this is the well known Weil reciprocity theorem in algebraic \( K \)-theory (e.g., see [36, Chapter IV, Theorem 6.12.1]). \( \square \)

We now proceed to the proof of the general case of Proposition 6.1. In view of Lemma 6.2, we can assume that the projection map \( f : W^N \to X \) is finite.

This gives rise to a Cartesian square
\[
\begin{array}{ccc}
W^N & \xrightarrow{\phi} & X \\
\downarrow{id} & & \downarrow{p_X} \\
W^N \times \mathbb{Q}^{n+1} & \xrightarrow{p'} & X \times \mathbb{Q}^{n+1} \\
\end{array}
\]

(6.2)

Note that \( \phi \) is a closed immersion. Using the finiteness of \( f \) and admissibility of \( W \), it is evident that that \( W^N = \phi(W^N) \cap (W^N \times \mathbb{Q}^{n+1}) \) is an admissible cycle on \( W^N \times \mathbb{Q}^{n+1} \). In other words, it intersects the faces of \( W^N \times \mathbb{Q}^{n+1} \) properly, and satisfies the modulus \((n+1)E\), if we let \( E = f^*(D) \nsubseteq W^N \). Notice that a consequence of the modulus condition for \( W \) is that \( E \) is a proper Cartier divisor on \( W^N \). Since \( f \) is finite and \( E = f^*(D) \), we have a push-forward map \( f_* : z^{n+1}(W^N, (n+1)E, \ast) \to z^{n+d}(X, (n+1)D, \ast) \) (see [36, Proposition 2.10]). Since \( (f \times id_{\mathbb{Q}^n}) \) takes \( W^N \) to \( W \) and since \( W^N \to W \) is the normalization map, we see that \( f_*([W^N]) = [W] \in z^{n+d}(X, (n+1)D, n+1) \). In particular, we get
\[
f_*\left( \partial W^N \right) = \partial(f_*([W^N])) = \partial W.
\]

**Proof of Proposition 6.1:** In view of Lemma 6.2 we can assume that \( f : W^N \to X \) is finite. Using (6.3) and Lemma 4.4, we have that
\[
cyc_{X|D}(\partial W) = cyc_{X|D} \circ f_*\left( \partial W^N \right) = f_* \circ cyc_{W^N|E}(\alpha).
\]

We can therefore assume that \( X \) is non-singular curve and \( W \in z^{n+1}(X, (n+1)D, n+1) \) is an integral cycle such that the map \( p_X : W \to X \) is an isomorphism. We can now apply Proposition 6.1 to finish the proof. \( \square \)

**Remark 6.3.** We remark that throughout the proof of Theorem 1.1 it is only in Lemma 4.2 where we need to assume that \( X \) is non-singular everywhere (see Remark 4.3). One would like to believe that for a proper map of modulus pairs \( f : (Y, f^*(D)) \to (X, D) \), there exists a push-forward map \( f_* : K_*(Y, f^*(D)) \to K_*(X, D) \). But we do not know how to prove it.

6.1. **The cycle class map with rational coefficients.** If we work with rational coefficients, we can prove the following improved version of Theorem 1.1. This may not be very useful in positive characteristic. However, one expects it to have many consequences in characteristic zero. The reason for this is that the relative algebraic \( K \)-groups of nilpotent ideals are known be \( \mathbb{Q} \)-vector spaces in characteristic zero. The proofs of Theorem 1.3 and its corollaries in this paper are crucially based on this improved version.
Theorem 6.4. Let $X$ be a non-singular quasi-projective variety of pure dimension $d \geq 1$ over a field $k$ and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map

$$(6.4) \quad \text{cyc}_{X|D}: \text{CH}^{n+d}(X|D,n)_\mathbb{Q} \rightarrow K_n(X,D)_\mathbb{Q}.$$

Proof. We shall only indicate where do we use rational coefficients in the proof of Theorem 1.1 to achieve this improvement as rest of the proof is just a repetition. Since we work with rational coefficients, we shall ignore the subscript $A_\mathbb{Q}$ in an abelian group $A$ in this proof and treat $A$ as a $\mathbb{Q}$-vector space.

As we did before, we need to prove Proposition 6.1 with $W \in z^{d+n}(X,D,n+1)$. We can again reduce the proof of this proposition to the case when $X$ is a non-singular curve and $W \in z^{n+1}(X,D,n+1)$ is an integral cycle such that the map $p_X : \overline{W} \rightarrow X$ is an isomorphism. We thus have to prove Proposition 5.1 with $W \in z^{n+1}(X,D,n+1)$. In turn, this is reduced to proving Lemma 5.3 when $W \in z^{n+1}(X,D,n+1)$. If we go through the proof of this lemma, we see that we finally need to prove an improved version of Lemma 5.5 where we have to replace $K^M_n(F,m+n,\text{sum})$ by $K^M_n(F,m,\text{sum})$.

Now, the reduction from the semi-local ring $R$ to it being a local ring (dvr) goes through exactly as in the proof of Lemma 5.5 without any change. So the whole proof of the theorem is eventually reduced to showing the following improved version of Lemma 5.4 (2) for every pair of integers $n \geq 0$ and $m \geq 1$:

$$(6.5) \quad (1+m^n)K^M_n(F) \subseteq (1+m^n)\overline{R}^M_n(R).$$

This inclusion is obvious for $n = 0$. To prove this for $n \geq 1$, an easy induction (see (3.9) in the proof of Lemma 3.4) reduces to the case $n = 1$. We now let $\pi$ be a uniformizing parameter of $R$ and let $\{1+a\pi^m,u\pi^j\} \in (1+m^n)K^M_1(F)$, where $u \in R^\times$, $a \in R$ and $j \in \mathbb{Z}$. If $a = 0$, there is nothing to show and so we can write $a = u_0\pi^i$ with $u_0 \in R^\times$ and $i \geq 0$. We then get the following in $K^M_2(F)$.

$$\{1+a\pi^m,u\pi^j\} = \left\{1+u_0\pi^{i+m},u\pi^j\right\}$$
$$= \{1+u_0\pi^{i+m},u\} + \left\{1+u_0\pi^{i+m},\pi^j\right\}$$
$$= \{1+u_0\pi^{i+m},u\} + \pi^j\left\{1+u_0\pi^{i+m},\pi^{-i-m}\right\}$$
$$= \{1+u_0\pi^{i+m},u\} - \pi^j\left\{1+u_0\pi^{i+m},-u_0\right\}.$$ 

Since the last term clearly belongs to $(1+m^n)\overline{R}^M_1(R)$, we conclude the proof of (6.5) and hence the theorem.

6.2. Chow groups and $K$-theory with compact support. Let $X$ be a quasi-projective scheme of pure dimension $d$ over a field $k$ and let $\overline{X}$ be a proper compactification of $X$ such that $\overline{X} \setminus X$ is supported on an effective Cartier divisor $D$. Recall from [5, Lemma 2.9] that $\text{CH}^p(X,n)_c := \lim_{m \to 1} \text{CH}^p(\overline{X}|mD,n)$ is independent of the choice of $\overline{X}$ and is called the higher Chow group of $X$ with compact support. One can similarly define the algebraic $K$-theory with compact support by $K_n(X)_c := \lim_{m \to 1} K_n(\overline{X},mD)$. It follows from [28, Theorem A] that this is also independent of the choice of $\overline{X}$. As a consequence of Theorem 1.1 we get

Corollary 6.5. Let $X$ be a non-singular quasi-projective scheme of pure dimension $d$ over a field admitting resolution of singularities. Then there exists a cycle class map

$$(6.6) \quad \text{cyc}_{X}: \text{CH}^{n+d}(X,n)_c \rightarrow K_n(X)_c.$$
7. Milnor $K$-theory, 0-cycles and de Rham-Witt complex

Our next goal is to show that the cycle class map of Theorem 7.1 completely describes the relative $K$-theory of truncated polynomial rings in terms of additive 0-cycles in characteristic zero. We shall give a precise formulation of our main result for fields in §19 and for semi-local rings in §10. In this section, we prove some results on the connection between $a$ priori three different objects: the additive 0-cycles, the relative Milnor $K$-theory and the de Rham-Witt forms. These results will form one of the two key steps in showing that the cycle class map $cyc_k$ (see (7.2)) factors through the Milnor $K$-theory in characteristic zero.

7.1. The additive 0-cycles. To set up the notations, let $k$ be a field of any characteristic. Let $m \geq 0$ be an integer. Recall that the additive higher Chow groups $TCH^p(X, *; m)$ of $X \in Sch_k$ with modulus $m$ are defined so that there are canonical isomorphisms $T\zeta^p(X, n + 1; m) \cong \zeta^p(X \times A^1_k, X \times (m + 1)\{0\}, n)$ and

\[TCH^p(X, n + 1; m) \cong CH^p(X \times A^1_k, X \times (m + 1)\{0\}, n),\]

where the term on the right are the Chow groups with modulus defined in §2.5. Using a similar isomorphism between the relative $K$-groups, Theorem 7.1 provides a commutative diagram of pro-abelian groups

\[\{TCH^{d+n+1}(X, n + 1; m)\}_m \xrightarrow{cycX} \{K_{n+1}(X[t]/(t^{m+1}), (t))\}_m \cong \{CH^{d+n+1}(X \times A^1_k, X \times (m + 1)\{0\}, n)\}_m \xrightarrow{cycX}_{X, n} \{K_n(X \times A^1_k, X \times (m + 1)\{0\})\}_m\]

for an equi-dimensional non-singular scheme $X$ of dimension $d$ and integer $n \geq 0$.

7.2. Connection with de Rham-Witt complex. Let $k$ be a field with $\text{char}(k) \neq 2$. Let $R$ be a regular semi-local ring which is essentially of finite type over $k$. Let $m, n \geq 1$ be two integers. Let $\mathbb{W}_m\Omega^r_R$ be the big de Rham-Witt complex of Hesselholt and Madsen (see [48, §1]). We shall let $\underline{a} = (a_1, \ldots, a_m)$ denote a general element of $\mathbb{W}_m(R)$. Recall from [48, Appendix] that there is an isomorphism of abelian groups $\gamma: \mathbb{W}(R) \cong (1 + tR[[t]])^\times$ (with respect to addition in $\mathbb{W}(R)$ and multiplication in $R[[t]]$) such that $\gamma(a) = \gamma((a_1, \ldots)) = \prod_{i=1}^\infty (1 - a_i t^r)$. This map sends $\text{Ker}(\mathbb{W}(R) \rightarrow \mathbb{W}_m(R))$ isomorphically onto the subgroup $(1 + t^m k[[t]])^\times$ and hence there is a Canonical isomorphism of abelian groups

\[\gamma_m: \mathbb{W}_m(R) \cong (1 + tR[[t]])^\times / (1 + t^mR[[t]])^\times.\]

Under this isomorphism, the Verschiebung map $V_r: \mathbb{W}_m(R) \rightarrow \mathbb{W}_{m+r-1}(R)$ corresponds to the map on the unit groups induced by the $R$-algebra homomorphism $R[[t]]/(t^{m+1}) \rightarrow R[[t]]/(t^{m+r})$ under which $t \mapsto t^r$. Recall also that there is a restriction ring homomorphism $\xi_0^R: \mathbb{W}_{m+1}(R) \rightarrow \mathbb{W}_m(R)$ as part of the Witt-complex structure. We shall often use the notation $V_r$ also for the composition $\mathbb{W}_m(R) \xrightarrow{V_r} \mathbb{W}_{m+r-1}(R) \xrightarrow{(\xi_0^R)^{-1}} \mathbb{W}_{mr}(R)$. With this interpretation of the Verschiebung map, every element $\underline{a} \in \mathbb{W}_m(R)$ has a unique presentation

\[\underline{a} = \sum_{i=1}^m V_i([a_i]_{m/i}),\]
where for a real number $x \in \mathbb{R}_{\geq 0}$, one writes $[x]$ for the greatest integer not bigger than $x$ and $\lceil x \rceil: R \to \mathbb{W}_{[x]}(R)$ for the Teichmüller map $[a]_{[x]} = (a, 0, \ldots, 0)$.

It was shown in [35] Theorem 7.10 that $\{\text{TCH}^*(R, *, m)\}_{m \geq 1}$ is a pro-differential graded algebra which has the structure of a restricted Witt-complex over $R$ in the sense of [48] Definition 1.14. Using the universal property of $\{\mathbb{W}_m \Omega^*_{R,n} \}_{m \geq 1}$ as the universal restricted Witt-complex over $R$, one gets a functorial morphism of restricted Witt-complexes

\[ \tau^R_{n,m}: \mathbb{W}_m \Omega^{n-1}_{R} \to \text{TCH}^n(R, n; m). \]

It was shown in [39] Theorem 1.0.2 that this map is an isomorphism. When $R$ is a field, this isomorphism was shown earlier by Rülling [48]. We shall use this isomorphism throughout the remaining part of this paper and consequently, will usually make no distinction between the source and target of this map.

### 7.3. Connection with Milnor $K$-theory

Continuing with the above notations, we have another set of maps

\[ \text{TCH}^1(R, 1; m) \otimes \text{CH}^{n-1}(R, n-1) \xrightarrow{\tau^R_{1,m} \otimes \nu^R_{n-1}} \mathbb{W}_m(R) \otimes K^{M}_{n-1}(R) \xrightarrow{\psi^R_{n,m}} \text{TCH}^n(R, n; m). \]

Here, $\nu^R_{n,m}: K^{M}_{n}(R) \to \text{CH}^{n}(R, n)$ is the semi-local ring analog of the Milnor-Chow homomorphism of Totaro [54]. It takes a Milnor symbol $\{b_1, \ldots, b_{n-1}\}$ to the graph of the function $\{b_1, \ldots, b_{n-1}\}: \text{Spec}(R) \to \square^{n-1}$. A combination of the main results [11] and [25] implies that this map is an isomorphism. The map $\psi^R_{n,m}$ is given by the action of higher Chow groups on the additive higher Chow groups, shown in [32]. It takes cycles $\alpha \in \text{TCH}^j(R, n; m)$ and $\beta \in \text{TCH}^j(R, n')$ to $\Delta^j_R(\alpha \times \beta) \in \text{TCH}^{j+1}(R, n+n'; m)$, where $\Delta^j_R: \text{Spec}(R) \times \mathbb{A}^j_k \times \square^{n+n'-1} \to \text{Spec}(R) \times \text{Spec}(R) \times \mathbb{A}^j_k \times \square^{n+n'-1}$ is the diagonal on $\text{Spec}(R)$ and identity on $\mathbb{A}^j_k \times \square^{n+n'-1}$.

**Lemma 7.1.** For $a = (a_1, \ldots, a_m) \in \mathbb{W}_m(R)$ and $b = (b_1, \ldots, b_{n-1}) \in K^{M}_{n-1}(R)$, one has $\psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1})(a \otimes b) = [Z]$, where $Z \subset \text{Spec}(R) \times \mathbb{A}^1_k \times \square^{n-1} \cong \mathbb{A}^1_R \times \square^{n-1}$ is the closed subscheme given by

\[ Z = \{ (t, y_1, \ldots, y_n) | \prod_{i=1}^m (1 - a_i t^i) = y_1 = b_1 = \ldots = y_{n-1} - b_{n-1} = 0 \}. \]

**Proof.** We let $f(t) = \prod_{i=1}^m (1-a_i t^i)$. In view of the description of $\psi^R_{n,m}$, we only have to show that $\tau^R_{1,m}(a) = V(f(t))$. But this is a part of the definition of the restricted Witt-complex structure on $\{\text{TCH}^*(R, *, m)\}_{m \geq 1}$ over $R$ (see [35] Proposition 7.6).

Let $\phi^R_{n,m}: \mathbb{W}_m(R) \otimes K^{M}_{n-1}(R) \to \mathbb{W}_m \Omega^{n-1}_{R}$ be the unique map such that $\tau^R_{n,m} \circ \phi^R_{n,m} = \psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1})$. The following lemma describes the map $\phi^R_{n,m}$.

**Lemma 7.2.** For any $a \in \mathbb{W}_m(R)$ and $b = (b_1, \ldots, b_{n-1}) \in K^{M}_{n-1}(R)$, we have

\[ \phi^R_{n,m}(a \otimes b) = a dlog([b_1]) \wedge \ldots \wedge dlog([b_{n-1}]). \]

**Proof.** For an ideal $I = (f_1, \ldots, f_r) \subset \mathbb{R}(t, y_1, \ldots, y_{n-1})$, we let $Z(f_1, \ldots, f_r)$ denote the closed subscheme of $\text{Spec}(R[t, y_1, \ldots, y_{n-1}])$ defined by $I$. We let $a \in \mathbb{W}_m(R)$ and $b = (b_1, \ldots, b_{n-1}) \in K^{M}_{n-1}(R)$. We write $b = b_1 \cdots b_{n-1} \in R^*$. Then we have by (7.4),

\[ \tau^R_{n,m}(a dlog([b_1]) \wedge \ldots \wedge dlog([b_{n-1}])) = \]

...
\[
\sum_{i=1}^{m} \tau_{n,m}^{R}(V_{i}([a_{i}]_{m/i}) d\log([b_{1}]) \wedge \cdots \wedge d\log([b_{n-1}])).
\]

We now recall that \(\tau_{n,m}^{R}\) is a part of the morphism of restricted Witt-complexes. In particular, we have for each \(1 \leq i \leq m\)

\[
(7.9) \quad \tau_{n,m}^{R}(V_{i}([a_{i}]_{m/i}) d\log([b_{1}]) \wedge \cdots \wedge d\log([b_{n-1}])) =
\]

\[
V_{i}(Z(1-a_{i}t)) \left(\prod_{j=1}^{n-1} Z(1-b_{i}^{-1}t)\right) (d(Z(1-b_{1}t)) \wedge \cdots \wedge d(Z(1-b_{n-1}t)))
\]

\[
= Z(1-a_{i}t) \left(\prod_{j=1}^{n-1} Z(1-b_{i}^{-1}t)\right) (d(Z(1-b_{1}t)) \wedge \cdots \wedge d(Z(1-b_{n-1}t))),
\]

\[
= Z(1-a_{i}t) Z(1-b^{-1}t)(d(Z(1-b_{1}t)) \wedge \cdots \wedge d(Z(1-b_{n-1}t))),
\]

where \(=^{1}\) follows from the fact the Verschiebung map on the additive higher Chow groups is induced by the pull-back through the power map \(\pi_{*} \colon A_{R}^{1} \to A_{R}^{1}\), given by \(\pi_{*}(t) = t^{r}\) (see [35, § 6]). The equality \(=^{2}\) follows from the fact that the product in \(\text{TCH}^{1}(R; 1; m)\) is induced by the multiplication map \(\mu^{*} \colon A_{R}^{1} \times R A_{R}^{1} \to A_{R}^{1}\) (see [35, § 6]).

Since the differential of the additive higher Chow groups is induced by the anti-diagonal map \((t, y) \mapsto (t, t^{-1}, y)\), we see that \(d(Z(1-b_{1}t)) = Z(1-b_{1}t, y_{1}, \ldots, y_{n-1}, -b_{n-1})\). In particular, we get

\[
d(Z(1-b_{1}t)) \wedge \cdots \wedge d(Z(1-b_{n-1}t)) = Z(1-bt, y_{1}-b_{1}, \ldots, y_{n-1}-b_{n-1}).
\]

As \(Z(1-b^{-1}t) \cdot Z(1-bt) = Z(1-t) = \tau_{1,m}^{R}(1)\) is the identity element for the differential graded algebra structure on \(\text{TCH}^{*}(R; *; m)\), we therefore get

\[
\tau_{n,m}^{R}(V_{i}([a_{i}]_{m/i}) d\log([b_{1}]) \wedge \cdots \wedge d\log([b_{n-1}])) = Z(1-a_{i}t^{i}, y_{1}-b_{1}, \ldots, y_{n-1}-b_{n-1}).
\]

Combining this with (7.8), we get

\[
(7.10) \quad \tau_{n,m}^{R}(d\log([b_{1}]) \wedge \cdots \wedge d\log([b_{n-1}])) =
\]

\[
Z(\prod_{i=1}^{m} (1-a_{i}t^{i}), y_{1}-b_{1}, \ldots, y_{n-1}-b_{n-1}).
\]

Since \(\tau_{n,m}^{R}\) is an isomorphism, we now conclude the proof by applying Lemma [7.1] \(\square\)

7.4. **Additive 0-cycles in characteristic zero.** We shall assume in this subsection that the base field \(k\) has characteristic zero. As above, we let \(R\) be a regular semi-local ring which is essentially of finite type over \(k\) and \(m,n \geq 1\) two integers. For any \(\mathfrak{a} = \{b_{1}, \ldots, b_{n-1}\} \in K_{n-1}^{M}(R)\), we let \(b = b_{1} \cdots b_{n-1} \in R^{*}\). Under our assumption on \(\text{char}(k)\), we can prove the following result which is the first key step for showing that the cycle class map \(\text{cycc}_{k}\) (see (7.2)) factors through the Milnor \(K\)-theory in characteristic zero.

**Lemma 7.3.** The map

\[
\psi_{n,m}^{R} \circ (\tau_{1,m}^{R} \otimes \nu_{n-1}^{R}) : \mathbb{W}_{m}(R) \otimes K_{n-1}^{M}(R) \to \text{TCH}^{n}(R, n; m)
\]

is surjective. In particular, \(\psi_{n,m}^{R}\) is surjective.

**Proof.** Let \(\phi_{n,m}^{R} : \mathbb{W}_{m}(R) \otimes K_{n-1}^{M}(R) \to \mathbb{W}_{m} \Omega^{n}_{R}\) be the unique map such that \(\tau_{n,m}^{R} \circ \phi_{n,m}^{R} = \psi_{n,m}^{R} \circ (\tau_{1,m}^{R} \otimes \nu_{n-1}^{R})\). The lemma is then equivalent to showing that \(\phi_{n,m}^{R}\) is surjective.

For \(\mathfrak{a} \in \mathbb{W}_{m}(R)\) and \(\mathfrak{b} = \{b_{1}, \ldots, b_{n-1}\} \in K_{n-1}^{M}(R)\), it follows from Lemma [7.2] that \(\phi_{n,m}^{R}(\mathfrak{a} \otimes \mathfrak{b}) = d\log([b_{1}]) \wedge \cdots \wedge d\log([b_{n-1}])\). We shall now use that the ground field has
characteristic zero. In this case, it follows from [48, Theorem 1.11, Remark 1.12] that there is a canonical decomposition

\[
(7.11) \quad \zeta_{r,m}^R: \mathcal{W}_m \Omega^r_R \to \prod_{i=1}^m \Omega^r_R
\]

such that the following diagram commutes:

\[
(7.12) \quad \mathcal{W}_m(R) \otimes K^M_{n-1}(R) \xrightarrow{\phi_{n,m}^R} \mathcal{W}_m \Omega^{n-1}_R \\
\downarrow \quad \downarrow \\
\prod_{i=1}^m R \otimes K^M_{n-1}(R) \to \prod_{i=1}^m \Omega^{n-1}_R
\]

where the bottom arrow is defined component-wise so that \((a_i) \otimes \{b_1, \ldots, b_{n-1}\}\) maps to \(\left(\frac{1}{m+1} a_i \log(b_j) \wedge \cdots \wedge \log(b_{n-1})\right)_i\). Since we are working with characteristic zero field, it then suffices to show that the map \(R \otimes K^M_{n-1}(R) \to \Omega^{n-1}_R\), given by \(a \otimes b \mapsto a \log(b_1) \wedge \cdots \wedge \log(b_{n-1})\), is surjective. By an iterative procedure, it suffices to prove this surjectivity when \(n = 2\).

We now let \(a, b \in R\). By Lemma [7.4] below, we can write \(b = b_1 + b_2\), where \(b_1, b_2 \in R^x\). We then get \(adb = abd_1 + adb_2 = ab_1 \log(b_1) + ab_2 \log(b_2)\). Since \(\Omega^1_R\) is generated by the universal derivations of the elements of \(R\) as an \(R\)-module, we are done.

\[\square\]

**Lemma 7.4.** Let \(R\) be a semi-local ring which contains an infinite field \(k\). Then every element \(a \in R\) can be written as \(a = u_1 + u_2\), where \(u_1 \in k^x\) and \(u_2 \in R^x\).

**Proof.** Let \(M = \{m_1, \ldots, m_r\}\) denote the set of all maximal ideals of \(R\). Fix \(a \in R\). Suppose that there exists \(u \in k^x \subset R^x\) such that \(a + u \in R^x\). Then we are done. Otherwise, every element \(u \in k^x\) has the property that \(a + u \in m_i\) for some \(i\). Since \(k\) is infinite and \(M\) is finite, there are two distinct elements \(u_1, u_2 \in k^x\) such that \(a + u_1\) and \(a + u_2\) both belong to a maximal ideal \(m_j\). Then we get \(u_1 - u_2 \in m_j\). But \(u_1 \neq u_2\) in \(k\) implies that \(u_1 - u_2 \in k^x\) and this forces \(m_j = R\), a contradiction. We conclude that there must exist \(u \in k^x\) such that \(a + u \in R^x\).

\[\square\]

8. The relative Milnor \(K\)-theory

In this section, we shall prove our second key step (see Lemma [8.4]) to show the factorization of the cycle class map through the relative Milnor \(K\)-theory and prove the isomorphism of the resulting map.

8.1. Recollection of relative Hochschild and cyclic homology. In the next two sections, we shall use Hochschild, André-Quillen and cyclic homology of commutative rings as our tools. We refer to [43] for their definitions and some properties that we shall use. While using a specific result from [43], we shall mention the exact reference.

Let \(R\) be a commutative ring. For an integer \(m \geq 0\), recall from §[2.1] that the truncated polynomial algebra \(R[t]/(t^{m+1})\) is denoted by \(R_m\). Throughout our discussion of truncated polynomial algebras, we shall make no distinction between the variable \(t \in R[t]\) and its image in \(R_m\).

If \(k \subset R\) is a subring, we shall use the notation \(HH_*(R|k)\) for the Hochschild homology of \(R\) over \(k\). Similarly, \(D_*(q)(R|k)\) for \(q \geq 0\) and \(HC_*(R|k)\) will denote the André-Quillen and cyclic homology of \(R\) over \(k\), respectively. When \(k = \mathbb{Z}\), we shall write \(HH_*(R|k)\) simply as \(HH_*(R)\). Similar notations will be used for \(D_*(q)(R|\mathbb{Z})\) and \(HC_*(R|\mathbb{Z})\). Note
that $HH_*(R) \cong HH_*(R/\mathbb{Q})$, $D^{(q)}_*(R) \cong D^{(q)}_*(R/\mathbb{Q})$ and $HC_*(R) \cong HC_*(R/\mathbb{Q})$ if $R$ contains $\mathbb{Q}$. We also have $\Omega^q_R := \Omega^q_{R/\mathbb{Z}} \cong \Omega^q_{R/\mathbb{Q}}$ for $q \geq 0$.

Recall that for an ideal $I \subset R$, the relative Hochschild homology $HH_*((R, I)|k)$ is defined as the homology of the complex $\text{Ker}(HH(R) \rightarrow HH(R/I|k))$, where $k \subset R$ is a subring and $HH(R)$ is the Hochschild complex of $R$ over $k$. The relative cyclic homology $HC_*((R, I)|k)$ is defined to be the homology of the complex $\text{Ker}(CC(R) \rightarrow CC(R/I|k))$, where $CC(R|k)$ is the total cyclic complex of $R$ over $k$. We refer to [33, 1.1.16, 2.1.15] for these definitions. One defines $D^{(q)}_*((R, I)|k)$ similarly. If $R$ is a commutative ring, we shall write $HH_{\lambda}((R_m, (t))|\mathbb{Z})$, $D^{(q)}_{\lambda}((R_m, (t))|\mathbb{Z})$ and $HC_{\lambda}((R_m, (t))|\mathbb{Z})$ simply as $\overline{HH}_{\lambda}(R_m)$, $\overline{D}^{(q)}_{\lambda}(R_m)$ and $\overline{HC}_{\lambda}(R_m)$, respectively. We let $\overline{\Omega}^n_{R_m} = \text{Ker}(\Omega^n_{R_m} \rightarrow \Omega^n_R)$. Suppose now that $R$ contains $\mathbb{Q}$. For $x \in tR_m$, we shall write $\exp(x) = \sum x^i/i!$ and $\log(1 + x) = \sum (-1)^{i-1}x^i/i$. Note that these are finite sums and define homomorphisms

$$tR_m \xrightarrow{\exp} \overline{K}_1(R_m) \xrightarrow{\log} tR_m$$

which are inverses to each other.

8.2. Relative Milnor $K$-theory of truncated polynomial rings. Let $R$ be a semi-local ring and let $m \geq 0, n \geq 1$ be two integers. Recall from §2.3 that $\overline{K}_*(R_m)$ denotes the relative $K$-theory $K_*(R_m, (t))$. We shall write the relative Milnor $K$-groups (see §3) $K^M_*(R_m)$ as $\overline{K}^M_*(R_m)$. Since $\overline{K}_*(R_m)$ is same as the kernel of the augmentation map $K_*(R_m) \rightarrow K_*(R)$, we have the canonical map $\psi_{R_m}: \overline{K}^M_*(R_m) \rightarrow \overline{K}_*(R_m)$.

Let us now assume that $R$ is a semi-local ring containing $\mathbb{Q}$. Recall that there is a Dennis trace map $\text{tr}^R_{m,n}: \overline{K}_n(R_m) \rightarrow HH_n(R_m)$ which restricts to the dlog map on the Milnor $K$-theory (e.g., see [33 Example 2.1]). Equivalently, there is a commutative diagram

$$K^M_n(R_m) \xrightarrow{d\log} \Omega^n_{R_m} \xrightarrow{\psi_{R_m,n}} K_n(R_m) \xrightarrow{\text{tr}^R_{m,n}} HH_n(R_m),$$

where $\epsilon_n$ is the canonical anti-symmetrization map from Kähler differentials to Hochschild homology (see [33 §1.3.4]).

A very well known result of Goodwillie [14] says that the relativization of the Dennis trace map with respect to a nilpotent ideal factors through a trace map $\overline{K}_n(R_m) \rightarrow \overline{HC}_{n-1}(R_m)$ such that its composition with the canonical Connes’ periodicity map $B: \overline{HC}_{n-1}(R_m) \rightarrow \overline{HH}_n(R_m)$ is the relative Dennis trace map. This factorization is easily seen on $\overline{K}_1(R_m)$ via the chain of maps

$$\overline{K}_1(R_m) \xrightarrow{\log} tR_m \cong \overline{HC}_0(R_m) \xrightarrow{d} \overline{HH}_1(R_m) \cong \overline{\Omega}^1_{R_m}.$$  

Recall that Connes’ periodicity map $B$ on $HC_0(R_m)$ coincides with the differential $d: R_m \rightarrow \Omega^1_{R_m}$ under the isomorphisms $R_m \cong HC_0(R_m)$ and $\Omega^1_{R_m} \cong HH_1(R_m)$. Goodwillie showed that his factorization $\overline{K}_n(R_m) \rightarrow \overline{HC}_{n-1}(R_m)$ is an isomorphism of $\mathbb{Q}$-vector spaces. We shall denote this Goodwillie’s isomorphism also by $\text{tr}^R_{m,n}$.

Going further, Cathelineau showed (see [10] Theorem 1]) that the $K$-group $\overline{K}_n(R_m)$ and the relative cyclic homology group $\overline{HC}_{n-1}(R_m)$ are $\lambda$-rings. Furthermore, Goodwillie’s map is an isomorphism of $\lambda$-rings. In particular, it induces an isomorphism
between the Adams graded pieces \( \text{tr}^R_{m,n}: \widetilde{K}_n^{(q)}(R_m) \xrightarrow{\cong} \overline{H}_n^{(q-1)}(R_m) \) for every \( 1 \leq q \leq n \).

As a corollary of this isomorphism and [33, Theorem 4.6.8], we get the following.

**Lemma 8.1.** The Dennis trace map induces an isomorphism of \( \mathbb{Q} \)-vector spaces

\[
\text{tr}^R_{m,n}: \widetilde{K}_n^{(n)}(R_m) \xrightarrow{\cong} \overline{\Omega}_{R_m}^{n-1} / d\Omega_{R_m}^{n-2}.
\]

In order to relate these groups with the Milnor \( K \)-theory, we first observe that as the map \( K_1^M(R_m) \rightarrow K_1(R_m) \) is an isomorphism, it follows from the properties of \( \gamma \)-filtration associated to the \( \lambda \)-ring structure on \( K \)-theory that the canonical map \( K_n^M(R_m) \rightarrow K_n(R_m) \) factors through \( K_n^M(R_m) \rightarrow F^n_\gamma R_m(R_n) \), where \( F^n_\gamma R_n(R_n) \) denotes the \( \gamma \)-filtration. If we consider the induced map on the relative \( K \)-groups, it follows that the canonical map \( \overline{K}_n^M(R_m) \rightarrow \widetilde{K}_n(R_m) \) factors as

\[
\psi_{R_m,n}: \overline{K}_n^M(R_m) \rightarrow \widetilde{K}_n^{(n)}(R_m) = F^n_\gamma \widetilde{K}_n(R_m) \rightarrow \widetilde{K}_n(R_m).
\]

It follows from [51, Theorem 12.3] that \( \text{Ker}(\psi_{R_m,n}) \) is a torsion group. On the other hand, it follows from [15, Proposition 5.4] that \( \overline{K}_n^M(R_m) \) is a \( \mathbb{Q} \)-vector space. If we now apply Soulé’s computation of \( F^n_\gamma \widetilde{K}_n(R_m) \) in [50, Théorème 2], we conclude that the map \( \psi_{R_m,n} \) is in fact an isomorphism. We have thus shown the following.

**Lemma 8.2.** The maps

\[
\overline{K}_n^M(R_m) \xrightarrow{\psi_{R_m,n}} \widetilde{K}_n^{(n)}(R_m) \xrightarrow{\text{tr}^R_{m,n}} \overline{\Omega}_{R_m}^{n-1} / d\Omega_{R_m}^{n-2}
\]

are all isomorphisms of \( \mathbb{Q} \)-vector spaces.

One knows from [12, 3.4.4] that the map \( d = B: \overline{\Omega}_{R_m}^{n-1} \rightarrow \overline{\Omega}_{R_m}^{n} \) is injective. Using the fact that \( \text{tr}^R_{m,n}: \overline{K}_n^M(R_m) \rightarrow \overline{\Omega}_{R_m}^{n} \) is multiplicative (e.g., see [29, Property 1.3] or [43, 8.4.12]) and it is the usual logarithm on \( \overline{K}_1^M(R_m) \) (see [8.3]), it follows from [8.2] that modulo \( d\Omega_{R_m}^{n-2} \), the composite map \( \text{tr}^R_{m,n} \circ \psi_{R_m,n} \) is the dlog map:

\[
(8.4) \quad d\log((1-tf(t), b_1, \ldots, b_{n-1})) = \log(1-tf(t))d\log(b_1) \land \cdots \land d\log(b_{n-1}).
\]

8.3. **More refined structure on \( \overline{K}_n^M(R_m) \).** We shall further simplify the presentation of \( \overline{K}_n^M(R_m) \) in the next result. This will be our second key step in factoring the cycle class map through the relative Milnor \( K \)-theory and showing that the resulting map is an isomorphism. For this, we assume \( m \geq 1 \) and look at the diagram

\[
(8.5) \quad K_n^M(R_m) \otimes tR_m \rightarrow \overline{K}_n^M(R_m)
\]

where the top horizontal arrow is the product \( \{b_1, \ldots, b_{n-1}\} \otimes a \rightarrow \{\exp(a), b_1, \ldots, b_{n-1}\} \).

The map \( \theta_{R_m,n}^{-1} \) in (8.5) is the composition of the canonical map \( \Omega_{R_m}^{n-1} \otimes tR_m \rightarrow \overline{\Omega}_{R_m}^{n-1} \) (sending \( \omega \otimes t^i \) to \( t^i \omega \)) with the surjection \( \overline{\Omega}_{R_m}^{n-1} \rightarrow \overline{\Omega}_{R_m}^{n-1} / d\Omega_{R_m}^{n-2} \). Using (8.4), it is easy to check that (8.5) is commutative. It follows from Lemma 8.2 that \( \theta_{R_m,n}^{-1} \) factors through a unique
map $\tilde{\theta}^n_{R_m} : \Omega^{n-1}_R \otimes_R tR_m \rightarrow \tilde{K}^M_n(R_m)$. We want to show that this map is an isomorphism. Equivalently, $\theta^{n-1}_{R_m}$ an isomorphism. Since the proof of this is a bit long, we prove that it is surjective and injective in separate lemmas.

**Lemma 8.3.** The map $\theta^n_{R_m}$ is surjective for all $n \geq 0$.

**Proof.** This is obvious for $n = 0$ and so we assume $n \geq 1$. We now consider the exact sequence

$$\Omega^1_R \otimes_R R_m \rightarrow \Omega^1_{R_m} \rightarrow \Omega^1_{R_m/R} \rightarrow 0.$$  

We claim that the first arrow in this sequence is split injective. For this, we consider the map $d' : R_m \rightarrow \Omega^1_R \otimes_R R_m$, given by $d' (\sum_{i=0}^m a_i t^i) = \sum_{i=0}^m d(a_i) \otimes t^i$. The computation

$$d' ((\sum_{i,j} a_i b_j t^{i+j})) = \sum_{i,j} d(a_i b_j) \otimes t^{i+j} = \sum_{i,j} b_j d(a_i) \otimes t^{i+j} + \sum_{i,j} a_i d(b_j) \otimes t^{i+j} = (\sum_{i} a_i t^i') \otimes (\sum_{j} b_j t^j')$$

shows that $d'$ is a $Q$-linear derivation on the $R_m$-module $\Omega^1_R \otimes_R R_m$. Hence, it induces an $R_m$-linear map $\nu : \Omega^1_{R_m} \rightarrow \Omega^1_{R_m} \otimes_R R_m$. Moreover, it is clear that the composite $\Omega^1_R \otimes_R R_m \rightarrow \Omega^1_{R_m} \rightarrow \Omega^1_{R_m/R}$ is identity. This proves the claim. We thus get a direct sum decomposition of $R_m$-modules $\Omega^1_{R_m} = (\Omega^1_R \otimes_R R_m) \oplus \Omega^1_{R_m/R}$. As $\Omega^1_{R_m/R} = 0$ for $n \geq 2$, we get $\Omega^n_{R_m} = (\Omega^n_R \otimes_R R_m) \oplus (\Omega^{n-1}_R \otimes_R \Omega^1_{R_m/R})$ for any $n \geq 1$. This implies that

$$\tilde{\Omega}^n_{R_m} = (\Omega^n_R \otimes tR_m) \oplus (\Omega^{n-1}_R \otimes \Omega^1_{R_m/R}).$$

The other thing we need to observe is that the exact sequence

$$(t^{m+1})/(t^{2m+2}) \rightarrow \Omega^1_{R[t]/R} \rightarrow \Omega^1_{R[t]/R} \rightarrow 0$$

implies that $\Omega^1_{R[t]/R} \cong \Omega^1_{R[t]/R}$. In particular, $\Omega^{n-1}_R \otimes_R \Omega^1_{R_m/R}$ is generated as an $R$-module by elements of the form $\omega \otimes (\sum_{i=1}^{m-1} a_i t^i dt)$, where $a_i \in R$.

We now let $\omega \in \Omega^{n-1}_R$. We then get

$$d(\omega \otimes \sum_{i=1}^{m-1} a_i t^i dt) = d\omega \otimes \sum_{i=1}^{m-1} \frac{a_i}{i+1} t^{i+1} + \omega \otimes \sum_{i=0}^{m-1} \frac{da_i}{i+1} t^i \otimes t^i - \omega \otimes \sum_{i=0}^{m-1} a_i t^i dt$$

$$= d\omega \otimes \sum_{i=1}^{m-1} \frac{a_i}{i+1} t^{i+1} + \sum_{i=0}^{m-1} \frac{\omega \otimes da_i}{i+1} \otimes t^i - \omega \otimes \sum_{i=0}^{m-1} a_i t^i dt$$

$$= \sum_{i=0}^{m-1} \frac{a_i d\omega + \omega \otimes da_i}{i+1} \otimes t^i - \omega \otimes \sum_{i=0}^{m-1} a_i t^i dt.$$  

It follows from this that the composite map $\Omega^n_R \otimes_R tR_m \rightarrow \tilde{\Omega}^n_{R_m} \rightarrow \tilde{\Omega}^n_{R_m}$ is surjective. \qed

**Lemma 8.4.** For $m, n \geq 1$, the map

$$\tilde{\theta}^n_{R_m} : \Omega^n_{R_m} \otimes_R tR_m \rightarrow \tilde{K}^M_n(R_m);$$

$$\tilde{\theta}^n_{R_m} (\log(b_1) \wedge \ldots \wedge \log(b_n-1)) \otimes a = \{ \exp(a), b_1, \ldots, b_{n-1} \},$$

is an isomorphism.

**Proof.** In view of Lemmas 8.2 and 8.3, the assertion is equivalent to showing that the map $\theta^n_{R_m}$ is injective for all $n \geq 0$ and $m \geq 1$. We can again assume that $n \geq 1$. We shall prove this by induction on $m \geq 1$. Assume first that $m = 1$. In this case, we want to show that the map $\Omega^n_R \otimes_R tR \rightarrow \tilde{\Omega}^n_{R_1}$ is injective. To show this, we should first observe that
every element of $\Omega^{n-1}_R \otimes_R R t$ must be of the form $\omega \otimes t$ and every element of $\Omega^{n-2}_R \otimes_R R dt$ must be of the form $\omega' \otimes dt$. In this case, we get

$$d(\omega \otimes t - \omega' \otimes dt) = d\omega \wedge t + dt \wedge \omega - dt \wedge d\omega'$$

(8.8)

$$= d\omega \wedge t - \omega \wedge dt + d\omega' \wedge dt$$

$$= d\omega \wedge t + (d\omega' - \omega) \otimes dt.$$

If $d(\omega \otimes t - \omega' \otimes dt)$ has to lie in $\Omega^n_R \otimes_R R t$, then we must have $(d\omega' - \omega) \otimes dt = 0$. But this implies that $\omega = d\omega'$. Putting this in (8.8), we get $d\omega \otimes t = 0$. This shows that $d\Omega^{n-1}_R \cap (\Omega^n_R \otimes_R R t) = 0$. But this is equivalent to the desired injectivity.

To prove the $m \geq 2$ case by induction, we shall need the following

**Claim:** The restriction map $\text{Ker}(\Omega^{n-1}_{R+1} \to \Omega^n_{R+1}) \to \text{Ker}(\Omega^{n-1}_R \to \Omega^n_R)$ is surjective for every $m \geq 1$.

To prove the claim, recall from (8.6) that

$$\Omega^{n-1}_{R_m} \cong (\Omega^{n-1}_R \otimes_R t R_m) \oplus (\Omega^{n-1}_R \otimes_R R_m dt t/ (t^m) dt) \cong \left( \oplus_{i=0}^{m-1} \Omega^{n-1}_R \otimes_R R^{i+1} t \oplus \left( \oplus_{i=0}^{m-1} \Omega^{n-2}_R \otimes_R R^i dt \right) \right).$$

(8.9)

Suppose now that $\omega \in \Omega^{n-1}_{R_m}$ is such that $d\omega = 0$. It follows from (8.9) that we can write

$$\omega = \sum_{i=0}^{m-1} \omega_i t^{i+1} + \sum_{i=0}^{m-1} \omega_i^t dt.$$

Hence, we have

$$d\omega = \sum_{i=0}^{m-1} (i+1) \omega_i t^{i+1} - \sum_{i=0}^{m-1} \omega_i^t dt \wedge \omega_i + \sum_{i=0}^{m-1} \omega_i^t dt \wedge d\omega_i'$$

(8.10)

$$= \sum_{i=0}^{m-1} (i+1) (-1) \omega_i^t dt + \sum_{i=0}^{m-1} (-1) \omega_i^t dt$$

$$= \sum_{i=0}^{m-1} (i+1) (-1) \omega_i^t dt + \sum_{i=0}^{m-1} (\omega_i - d\omega_i') t^i dt.$$

Since the left hand term is zero, it implies from the decomposition (8.9) (for $\Omega^{n-1}_{R_m}$) that $d\omega_i = 0 = d\omega_i' - \omega_i$ for $0 \leq i \leq m - 1$. If we now let $\omega' = \sum_{i=0}^{m-1} (i+1) \omega_i^t t^{i+1}$, we get

$$d\omega' = \sum_{i=0}^{m-1} (i+1) (-1) \omega_i^t dt + \sum_{i=0}^{m-1} \omega_i^t dt = \omega.$$

(8.11)

The claim now follows from the surjectivity of the map $\Omega^{n-2}_{R_{m+1}} \to \Omega^{n-2}_{R_m}$. Indeed, this implies that some $\omega' \in \Omega^{n-2}_{R_{m+1}}$ maps onto $\omega$ and therefore $d\omega' \in \Omega^{n-1}_{R_{m+1}}$ maps onto $\omega$. Since $d\omega'$ is clearly a closed form, we are done.

In order to use the above claim, we let $F^m_{m+1} = (\Omega^n_R \otimes_R R t^m) \oplus (\Omega^{n-1}_R \otimes_R R t^{m-1} dt)$. We then have an exact sequence of $R_{m+1}$-modules

$$0 \to F^m_{m+1} \to \Omega^n_{R_{m+1}} \to \Omega^n_{R_m} \to 0.$$

(8.12)

Letting $m \geq 1$ and taking the quotient of this short exact sequence by the similar exact sequence for $n-1$ via the differential map, we obtain a commutative diagram

$$0 \to \Omega^n_R \otimes_R R t^{m+1} \to \Omega^{n-1}_R \otimes_R R t^{m+1} \to \Omega^{n-2}_R \otimes_R R t^m \to 0.$$

(8.13)
The top row is clearly exact and the above claim precisely says that the bottom sequence is exact. The right vertical arrow is injective by induction. It suffices therefore to show that the map $Ω^n_R ⊗_R R t^{m+1} → F_{m+1}^n/dF_{m+1}^{n-1}$ is injective. The proof of this is almost identical to that of $m = 1$ case. Indeed, it is easy to check that every element of $F_{m+1}^{n-1}$ must be of the form $ω ⊗ t^{m+1} − ω' ⊗ t^m dt$. We therefore get

\[
\begin{align*}
\text{d}(ω ⊗ t^{m+1} − ω' ⊗ t^m dt) &= dω ∧ t^{m+1} − (m+1)t^m dt ∧ ω + t^m dt ∧ dω' \\
&= dω ∧ t^{m+1} + ((m+1)ω − dω') ∧ t^m dt \\
&= dω ⊗ t^{m+1} + ((m+1)ω − dω') ⊗ t^m dt.
\end{align*}
\]

(8.14)

If $d(ω ⊗ t^{m+1} − ω' ⊗ t^m dt)$ lies in $Ω^n_R ⊗_R R t^{m+1}$, then we must have $(dω' − (m+1)ω) ⊗ t^m dt = 0$. But this implies that $ω = d((m+1)^{-1}ω')$. Putting this in (8.14), we get $dω ⊗ t^{m+1} = 0$. This shows that $dF_{m+1}^n ∩ (Ω^n_R ⊗_R R t^{m+1}) = 0$. Equivalently, the left vertical arrow in (8.10) is injective. This proves that $θ^n_{R_m}$ is injective for all $m ≥ 1$ and completes the proof of the lemma.

We shall also need the following related result later on.

**Lemma 8.5.** Let $R$ be as above. Let $n ≥ 0$ and $m ≥ 1$ be integers. Then the map $d: Ω^n_R ⊗_R R tR_m → Ω^{n+1}_R R_m$ is injective.

**Proof.** Suppose $ω = \sum_{i=0}^{m-1} ω_i ⊗ t^{i+1}$ and $d(ω) = 0$. That is, $\sum_{i=0}^{m-1} dω_i ⊗ t^{i+1} − \sum_{i=0}^{m-1} (i+1)ω_i ⊗ t^i dt = 0$. But this implies by (8.6) that $\sum_{i=0}^{m-1} (i+1)ω_i ⊗ t^i dt = 0$. Since $Ω^n_R ⊗_R Ω^1_{R_m} ⊆ Ω^n_R dt ⊗ ⋯ ⊗ Ω_R^n t^{m-1} dt ⊆ (Ω^n_R)^m$ as an $R$-module, we must have $ω_i = 0$ for each $i$. In particular, we have $ω = 0$.

9. The cycle class map in characteristic zero

In this section, we shall show that the cycle class map for the additive 0-cycles completely describes the relative $K$-theory of the truncated polynomial rings over a characteristic zero field in terms of additive 0-cycles. This was perhaps the main target for the introduction of the additive higher Chow groups by Bloch and Esnault [9]. We formulate our precise result as follows.

Let $k$ be a characteristic zero field. Let $R$ be a regular semi-local ring which is essentially of finite type over $k$. Let $m ≥ 0, n ≥ 1$ be two integers. Recall from § 8.2 that the canonical map from the Milnor to the Quillen $K$-theory induces a map $ψ_{R_m, n}: K^M_n(R_m) → K_n(R_m)$. Since this map is clearly compatible with change in $m ≥ 0$, we have a strict map of pro-abelian groups

\[
ψ_{R_m, n}: \{K^M_n(R_m)\}_m → \{K_n(R_m)\}_m.
\]

(9.1)

In this section, we shall restrict our attention to the case when $R$ is the base field $k$ itself. We shall prove a general result for regular semi-local rings in the next section. In the case of the field $k$, every integer $n ≥ 1$ has associated to it a diagram of pro-abelian groups:

\[
\begin{array}{c}
\{K^M_n(k_m)\}_m \\
\psi_k
\end{array}
\]

\[
\{TCII^n(k, n; m)\}_m \xrightarrow{\text{cycϕ}} \{K_n(R_m)\}_m.
\]

(9.2)

The following is our main result.

**Theorem 9.1.** If $k$ is a field of characteristic zero, then all maps in (9.2) are isomorphisms.
The proof of this theorem will be done by combining the results of §4 and §5 with a series of new steps.

9.1. **Factorization of cyck into Milnor K-theory.** We follow two step strategy for proving Theorem 9.1. We shall first show that cyck factors through the Milnor K-theory and the resulting map is an isomorphism. The second step will be to compare the Milnor and Quillen K-groups in the pro-setting. Apart from showing factorization through the Milnor K-theory, the following result also improves Theorem 1.4 in that it tells us that for additive higher Chow groups of 0-cycles, the cycle class map is a strict morphism of pro-abelian groups (see §2.2).

**Lemma 9.2.** Let \( m \geq 0, n \geq 1 \) be two integers. Then the following hold.

1. The map \( \text{cyck}_{\text{TCH}}: T^n(k; n; m) \to \tilde{K}_n(k_m) \) descends to a group homomorphism
   \[
   \text{cyck}_{\text{TCH}}: T_{\text{CH}}(k; n; m) \to \tilde{K}_n(k_m).
   \]

2. The map \( \text{cyck} \) has a factorization
   \[
   T_{\text{CH}}(k; n; m) \xrightarrow{\text{cyck}} \tilde{K}_n^M(k_m) \xrightarrow{\psi_{k,m,n}} \tilde{K}_n(k_m).
   \]

**Proof.** Since \( \text{char}(k) = 0 \), we know by the main result of [48] (see also [35, Theorem 1.2]) that each \( T^n(k; n; m) \) is a \( k \)-vector space. Similarly, \( \tilde{K}_n(k_m) \) is a \( \mathbb{Q} \)-vector space because it is isomorphic to \( \tilde{H}_{\text{C}}_{n-1}(k_m) \) by [14]. The first part of the lemma therefore follows directly from Theorem 6.4.

We shall now prove the second part. Since \( \psi_{k,m,n} \) is injective for each \( m \geq 0, n \geq 1 \) by Lemma 8.2, we only need to show that \( \text{cyck} \) takes a set of generators of the group \( T^n(k; n; m) \) to \( \tilde{K}_n^M(k_m) \).

We first assume that \( n = 1 \). Let \( z \in \mathbb{A}^1_k \) be a closed point. We can write \( z = \text{Spec}(k[t]/(f(t))) \), where \( f(t) \) is an irreducible polynomial. The modulo condition for \( z \) implies that \( f(0) \in k^s \). If we let \( g(t) = (f(0))^{-1}f(t) \) and let \( \overline{g(t)} \) denote the image \( g(t) \) in \( k_m \), then we see that \( g(t) \in \tilde{K}_1^M(k_m) \). By the definition of the cycle class map in [4.3], we have that \( \text{cyck}(z) = [k(z)] \in \tilde{K}_0(\mathbb{A}_k^1(m + 1)\{0\}) \). But Lemma 2.1 says that \( [k(z)] = \partial(\overline{g(t)}) \) under the isomorphism \( \tilde{K}_1^M(k_m) = \tilde{K}_1(k_m) \xrightarrow{\partial} K_0(k[t],(t^{m+1})) \). So we are done.

Suppose now that \( n \geq 2 \). In this case, Lemma 7.3 says that \( T_{\text{CH}}(k; n; m) \) is generated by closed points \( z \in \mathbb{A}^{n-1}_k \times \mathbb{A}^1 \) which lie in \( \mathbb{A}^{n-1}_k \times \mathbb{G}_{m-1}^n \subset \mathbb{A}^n_k \). Furthermore, \( z \in \mathbb{A}^n_k \) is defined by an ideal \( I \subset k[t, y_1, \ldots, y_{n-1}] \) of the type \( I = (f(t), y_1-b_1, \ldots, y_{n-1}-b_{n-1}) \), where \( b_i \in k^s \) for each \( 1 \leq i \leq n-1 \). Since \( z \) is a closed point, \( f(t) \) must be an irreducible polynomial in \( k[t] \). Moreover, \( f(t) \) defines an element of \( \mathbb{W}_m(k) \subset \tilde{K}_1^M(k_m) \). In particular, \( f(0) \in k^s \).

We next note that the push-forward map \( K_*(k(z)) \to K_*(k(z')) \) is \( K_*(k) \)-linear, where \( z' = \text{Spec}(k[t]/(f(t))) \) (see [54] Chapter 3). The map \( K_*(k(z')) \to K_*(k[t],(t^{m+1})) \) is \( K_*(k) \)-linear by Lemma 2.2. It follows that the composition \( K_*(k(z)) \to K_*(k[t],(t^{m+1})) \) is \( K_*(k) \)-linear.

It follows therefore from the definition of the cycle class map in (4.3) that under the map \( \text{cyck}_{\text{TCH}}: T^n(k; n; m) \to K_{n-1}(k[t],(t^{m+1})) \), we have

\[
\text{cyck}([z]) = \{b_1, \ldots, b_{n-1}\} \cdot [k(z')] \in K_{n-1}^M(k) \cdot K_0(k[t],(t^{m+1})) \subset K_{n-1}(k[t],(t^{m+1})),
\]

where \( \{b_1, \ldots, b_{n-1}\} \in K_{n-1}^M(k) \). We let \( g(t) = (f(0))^{-1}f(t) \) and let \( \overline{g(t)} \) be the image of \( g(t) \) in \( k_m \) via the surjection \( k[t] \to k_m \).

Since \( \partial: \tilde{K}_n(k_m) \xrightarrow{\partial} K_{n-1}(k[t],(t^{m+1})) \) is \( K_*(k) \)-linear, we see that

\[
\{b_1, \ldots, b_{n-1}\} \cdot [k(z')] = \{b_1, \ldots, b_{n-1}\} \cdot \partial(\overline{g(t)}) = \partial(\{b_1, \ldots, b_{n-1}\} \cdot g(t)).
\]
We are now done because \((b_1, ..., b_{n-1}) \cdot g(t) \in K^M_{n-1}(k) \cdot \tilde{K}_1^M(k_m) \subseteq \tilde{K}_n^M(k_m)\). We have thus shown that \(cyc_k([z])\) lies in the image of \(\tilde{K}_n^M(k_m)\) under the map \(\partial\). This finishes the proof.

9.2. The main result for \(cyc_k^M\). Let \(k\) be a characteristic zero field as before. We shall now show that the cycle class map \(cyc_k^M\) that we obtained in Lemma 9.2 is an isomorphism. Since the map \(\tau_{n,m}^k: \mathbb{W}_m \Omega_{k}^{n-1} \to \text{TCH}^n(k, n; m)\) is an isomorphism, we shall make no distinction between \(\mathbb{W}_m \Omega_{k}^{n-1}\) and \(\text{TCH}^n(k, n; m)\) throughout our discussion of the proof of Theorem 9.1. Furthermore, we shall denote \(cyc_k \circ \tau_{n,m}^k\) also by \(cyc_k\) in what follows.

**Theorem 9.3.** For every pair of integers \(m \geq 0, n \geq 1\), the map

\[
cyc_k^M : \text{TCH}^n(k, n; m) \to \tilde{K}_n^M(k_m)
\]

is an isomorphism.

**Proof.** When \(m = 0\), the group on the right of \(cyc_k\) is zero by definition and the one on the left is zero by [37, Theorem 6.3]. We can therefore assume that \(m \geq 1\).

We can replace \(\text{TCH}^n(k, n; m)\) by \(\mathbb{W}_m \Omega_{k}^{n-1}\). Accordingly, we can identify \(cyc_k^M\) with \(cyc_k^M \circ \tau_{n,m}^k\). Let \(\eta_m^k : \Omega_{k}^{n-1} \to t^m \Omega_{k}^{m-1} \otimes \Omega_{k}^{n-1}\) denote the \((k\text{-linear})\) map \(\eta_m^k(a \omega) = -at^m \otimes \omega\). This is clearly an isomorphism of \(k\)-vector spaces.

We shall prove the theorem by induction on \(m \geq 1\). Suppose first that \(m = 1\). In this case, it follows from Lemmas 7.1 and 7.2 that

\[
cyc_k^M(ad \log(b_1) \wedge ... \wedge \log(b_{n-1})) = cyc_k^M(Z(1 - at, y_1 - b_1, ..., y_{n-1} - b_{n-1}))
\]

\[
= \{1 - at, b_1, ..., b_{n-1}\}
\]

\[
= \theta_k^M((-at) \otimes \log(b_1) \wedge ... \wedge \log(b_{n-1}))
\]

\[
= \theta_k^M \circ \eta_1^M(ad \log(b_1) \wedge ... \wedge \log(b_{n-1})).
\]

It follows from this that \(cyc_k^M = \theta_k^M \circ \eta_1^M\). We now apply Lemma 8.4 to conclude that \(cyc_k^M\) is an isomorphism.

Suppose now that \(m \geq 2\). Let \(F_m^n\) denote the kernel of the restriction map \(\tilde{K}_n^M(k_m) \to \tilde{K}_n^M(k_{m-1})\). It is easy to see that the isomorphism \(\theta_k^M\) of Lemma 8.4 commutes with the restriction map \(k_m \to k_{m-1}\) for all \(m \geq 1\). It follows therefore from Lemma 8.4 and the snake lemma that \(\theta_k^M\) restricts to an isomorphism

\[
\begin{equation}
\theta_k^M : \Omega_{k}^{n-1} \otimes_k t^{m+1} \Omega_{k}^{m-1} \rightarrow F_m^n;
\end{equation}
\]

\[
\begin{equation}
\theta_k^M : ((d \log(b_1) \wedge ... \wedge d \log(b_{n-1})) \otimes at^{m+1}) = \{1 + at^{m+1}, b_1, ..., b_{n-1}\}.
\end{equation}
\]

We now consider the diagram

\[
\begin{equation}
0 \rightarrow \Omega_{k}^{n-1} V_{m+1} \mathbb{W}_m \Omega_{k}^{n-1} \rightarrow \mathbb{W}_m \Omega_{k}^{n-1} \rightarrow 0
\end{equation}
\]

\[
\begin{equation}
\theta_k^M : \Omega_{k}^{n-1} \otimes_k t^{m+1} \Omega_{k}^{m-1} \rightarrow F_m^n;
\end{equation}
\]

\[
\begin{equation}
\theta_k^M : (d \log(b_1) \wedge ... \wedge d \log(b_{n-1})) \otimes at^{m+1} = \{1 + at^{m+1}, b_1, ..., b_{n-1}\}.
\end{equation}
\]

where the horizontal arrows on the right in both rows are the restriction maps. In particular, the square on the right is commutative.

We next note that \(V_{m+1}(ad \log(b_1) \wedge ... \wedge d \log(b_{n-1})) = V_{m+1}(a \log(b_1) \wedge ... \wedge d \log(b_{n-1}))\) (see [49, Proposition 4.4]). Indeed, by iteration, it is enough to check this for \(n = 2\). Furthermore, we can check it in \(\text{TCH}^2(k, 2; m)\). Now, we have

\[
\begin{equation}
V_{m+1}(ad \log(b_1)) = \begin{cases} V_{m+1}(Z(1 - at, y_1 - b_1)) & \text{if } k \text{ is algebraically closed} \\ Z(1 - at, y_1 - b_1) & \text{otherwise} \end{cases}
\end{equation}
\]
where the last two equalities follow from the definitions of differential and Verschiebung on the additive higher Chow groups (see the proof of Lemma 7.2). On the other hand, we have

\[
V_{m+1}(a) d\log(b_1) = V_{m+1}(a) b_1^{-1} db_1 = (V_{m+1}(Z(1-at))Z(1-b_1^{-1}t)d(Z(1-b_1t))
\]

\[
= Z(1-at^{m+1})Z(1-b_1^{-1}t)d(Z(1-b_1t)) = Z(1-at^{m+1})Z(1-b_1t)Z(1-b_1t, y_1-b_1)
\]

\[
= Z(1-at^{m+1})Z(1-t, y_1-b_1) = Z(1-at^{m+1}, y_1-b_1),
\]

(9.6)

where the equalities again follow from various definitions (see the proof of Lemma 7.2). A combination of (9.5) and (9.6) proves what we had claimed. Since the map \(k \otimes K^{n-1}_{m}(k) \rightarrow \Omega_k^{n-1}\) (given by \(a \otimes b \mapsto a \log(b) \wedge \cdots \wedge d\log(b_{n-1})\)) is surjective, it follows from (7.10) and (9.3) that the left square in (9.4) is also commutative.

The top row of (9.4) is well known to be exact in characteristic zero (e.g., see [49, Remark 4.2]). The bottom row is exact by the definition of (9.4) that the left square in (9.4) is also commutative. We shall now prove our final step in the proof of Theorem 9.1. Namely, we shall show that the reduced Milnor and Quillen relative cyclic homology isomorphism is based on the following general result about pro-vanishing of fields. In more detail, the isomorphism \(\eta_{n, \mathcal{R}}\) (given by \(\eta_{n} = \bar{\Omega}_{R}^{n}/\bar{\Omega}_{R}^{n-1}\)) makes sense. Lemmas 7.1 and 7.2 as well as (9.3) are all valid for \(R\). The final step (9.4) also holds for \(R\). This observation will be used in the next section in extending Theorem 9.1 to regular semi-local rings.

9.3. The final step. We shall now prove our final step in the proof of Theorem 9.1. Namely, we shall show that the reduced Milnor and Quillen \(K\)-theories of the truncated polynomial rings are pro-isomorphic. This will finish the proof of Theorem 9.1. The desired pro-isomorphism is based on the following general result about pro-vanishing of relative cyclic homology.

Let \(R\) be a ring containing \(\mathbb{Q}\). Recall from [43, § 4.5, 4.6] that for integers \(m \geq 0\) and \(n \geq 1\), there are functorial \(\lambda\)-decompositions \(\overline{HC}_{n}(R_{m}) = \bigoplus_{i=1}^{n} \overline{HH}_{n}^{(i)}(R_{m})\) and \(\overline{HC}_{n}(R_{m}) = \bigoplus_{i=1}^{n} \overline{HC}_{n}^{(i)}(R_{m})\). Moreover, there are isomorphisms

\[
\overline{HH}_{n}^{(n)}(R_{m}) \cong \bar{\Omega}_{R_{m}}^{n} \text{ and } \overline{HC}_{n}^{(n)}(R_{m}) \cong \frac{\bar{\Omega}_{R_{m}}^{n}}{d\bar{\Omega}_{R_{m}}^{n-1}}\]

such that the canonical map \(\overline{HH}_{n}^{(n)}(R_{m}) \rightarrow \overline{HC}_{n}^{(n)}(R_{m})\) is the quotient map \(\bar{\Omega}_{R_{m}}^{n} \rightarrow \frac{\bar{\Omega}_{R_{m}}^{n}}{d\bar{\Omega}_{R_{m}}^{n-1}}\). We also have the functorial isomorphisms

\[
\overline{HH}_{n}^{(i)}(R_{m}) \cong \bar{D}_{n-1}^{(i)}(R_{m}).
\]

(9.7)

(9.8)

The key result is the following.

**Proposition 9.5.** Let \(R\) be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Let \(0 < i < n\) be two integers. Then \(\{\overline{HC}_{n}^{(i)}(R_{m})\}_{m} = 0\).

**Proof.** First we show that \(\{\overline{HH}_{n}^{(i)}(R_{m})\}_{m} = 0\). By (9.8), this is equivalent to showing that \(\{\bar{D}_{n-1}^{(i)}(R_{m})\}_{m} = 0\). To show this latter vanishing, we let \(A = R[t^2, t^3] \subset R[t]\) be the
monomial subalgebra generated by \( \{ t^2, t^3 \} \). We then know that the inclusion \( A \hookrightarrow R[t] \) is the normalization homomorphism whose conductor ideal is \( I = (t^2, t^3) \subset A \) such that \( IR[t] = (t^3) \). Since \( R[t] \) is regular, we know that \( D_{n-i}^{(i)}(R[t]) = 0 \) for \( 0 < i < n \) by [43 Theorem 3.5.6]. We now apply part (ii) of [43 Proposition 5.2] with \( A = R[t^2, t^3] \) and \( B = R[t] \) to conclude that \( \{ D_{n-i}^{(i)}(R_{2m-1}) \}_{m} = \{ D_{n-i}^{(i)}(R[t])/(t^2m) \}_{m} = 0 \). But this is same as saying that \( \{ D_{n-i}^{(i)}(R_m) \}_{m} = 0 \). In particular, we get \( \{ D_{n-i}^{(i)}(R_m) \}_{m} = 0 \).

To prove the result for cyclic homology, we first assume \( n \geq 2 \) and \( i = n - 1 \). Connes’ periodicity exact sequence (see [43 theorem 4.6.9]) gives an exact sequence

\[
\overline{H}(R_m) \xrightarrow{d} \overline{H}(R_{m-1}) \xrightarrow{s} \overline{H}(R_{m-2}) \xrightarrow{\partial} \overline{H}(R_m) \xrightarrow{d} \overline{H}(R_{m-1}) \xrightarrow{s} \overline{H}(R_{m-2}).
\]

We have shown that the first term in this exact sequence vanishes. It suffices therefore to show that the map \( B \xrightarrow{d} \overline{H}(R_{m-2}) \xrightarrow{\partial} \overline{H}(R_m) \) is injective. But this follows from Lemma 8.5.

We prove the general case by induction on \( n \geq 2 \). The case \( n = 2 \) is just proven above. When \( n \geq 3 \), we can assume that \( 1 \leq i < n - 2 \) by what we have shown above. We now again use the periodicity exact sequence

\[
\overline{H}(R_m) \xrightarrow{d} \overline{H}(R_{m-1}) \xrightarrow{s} \overline{H}(R_{m-2}) \xrightarrow{\partial} \overline{H}(R_m) \xrightarrow{d} \overline{H}(R_{m-1}) \xrightarrow{s} \overline{H}(R_{m-2}).
\]

Since \( i \leq n - 2 \), we get \( i - 1 \leq n - 3 \). Hence, the induction hypothesis implies that the last term of this exact sequence is zero. We have shown above that the first term is zero. We conclude that the middle term is zero as well. This finishes the proof.

**Corollary 9.6.** Let \( R \) be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Let \( n \geq 0 \) be an integer. Then the canonical map

\[
\psi_{R,n}: \{ \overline{K}^M_n(R_m) \}_{m} \rightarrow \{ \overline{K}_n(R_m) \}_{m}
\]

is an isomorphism of pro-abelian groups.

**Proof.** The case \( n \leq 2 \) is well known (e.g., see [28 Proposition 2]) and in fact an isomorphism at every level \( m \geq 0 \). We can therefore assume that \( n \geq 3 \). We have seen in § 8.2 that Goodwillie provides an isomorphism of \( \mathbb{Q} \)-vector spaces

\[
\text{tr}_{m,n}^R \overline{K}_n(R_m) \cong \overline{HC}_{n-1}(R_m) \cong \bigoplus_{i=1}^{n-1} \overline{HC}_{n-1}^{(i)}(R_m).
\]

Moreover, Lemmas 8.1 and 8.2 say that the map \( \psi_{R,n}: \overline{K}^M_n(R_m) \rightarrow \overline{K}_n(R_m) \) is injective and \( \text{tr}_{m,n}^R \) maps \( \overline{K}^M_n(R_m) \) isomorphically onto \( \overline{HC}_{n-1}^{(i)}(R_m) \). We therefore have to show that \( \{ \overline{HC}_{n-1}^{(i)}(R_m) \}_{m} = 0 \) for \( 1 \leq i \leq n - 2 \). But this follows from Proposition 9.5.

**Proof of Theorem 9.1:** The proof is a combination of Lemma 9.2, Theorem 9.3 and Corollary 9.6.

10. **The cycle class map for semi-local rings**

In this section, we shall define the cycle class map for relative 0-cycles over regular semi-local rings and prove an extension of Theorem 9.1 for such rings. Let \( k \) be a characteristic zero field and let \( R \) be a regular semi-local ring which is essentially of finite type over \( k \). We shall let \( F \) denote the total quotient ring of \( R \). Note that \( R \) is a product of regular semi-local integral domains and \( F \) is the product of their fields of fractions. Since all our proofs for regular semi-local integral domains directly generalize
to finite products of such rings, we shall assume throughout that $R$ is an integral domain. We shall let $\pi : \text{Spec}(F) \to \text{Spec}(R)$ denote the inclusion of generic point. We shall often write $X = \text{Spec}(R)$ and $\eta = \text{Spec}(F)$. We shall let $\Sigma$ denote the set of all maximal ideals of $R$.

10.1. The sfs cycles. Let $m \geq 0$ and $n \geq 1$ be two integers. Recall from §2.4 and §7.1 that $\text{TCH}^n(R,n;m)$ is the defined as the middle homology of the complex $Tz^n(R,n+1;m) \xrightarrow{\partial} Tz^n(R,n;m) \xrightarrow{\partial} Tz^{n-1}(R,n;m)$. A cycle in $Tz^n(R,n;m)$ has relative dimension zero over $R$. For this reason, $\text{TCH}^n(R,n;m)$ is often called the additive higher Chow group of relative $0$-cycles on $R$. When $R$ is a field, it coincides with the one used in the statement of Theorem 9.1.

Since $\text{TCH}^n(R,n;m)$ does not consist of 0-cycles if dim$(R) \geq 1$, we can not directly apply Theorem 10.1 to define a cycle class map for $\text{TCH}^n(R,n;m)$. We have to use a different approach for constructing the cycle class map. We shall use the main results of §3 and the case of fields to construct a cycle class map in this case. We shall show later in this section that this map is an isomorphism. We shall say that an extension of regular semi-local rings $R_1 \subset R_2$ is simple if there is an irreducible monic polynomial $f \in R_1[t]$ such that $R_2 = R_1[t]/(f(t))$.

Let $Z \subset X \times \mathbb{A}^1_k \times □^{n-1}$ be an irreducible admissible relative 0-cycle. Recall from §2.5.2, Proposition 2.5.3 that $Z$ is called an sfs-cycle if the following hold.

1. $Z$ intersects $\Sigma \times \mathbb{A}^1_k \times F$ properly for all faces $F \subset □^{n-1}$.
2. The projection $Z \to X$ is finite and surjective.
3. $Z$ meets no face of $X \times \mathbb{A}^1_k \times □^{n-1}$.
4. $Z$ is closed in $X \times \mathbb{A}^1_k \times □^{n-1} = \text{Spec}(R[t,y_1,\ldots,y_{n-1}])$ (by (2) above) and there is a sequence of simple extensions of regular semi-local rings $R = R_1 \subset R_0 \subset \cdots \subset R_{n-1} = k[Z]$ such that $R_0 = R[t]/(f_0(t))$ and $R_i = R_{i-1}[y_i]/(f_i(y_i))$ for $1 \leq i \leq n-1$.

Note that an sfs-cycle has no boundary by (3) above. We let $Tz^n_{sfs}(R,n;m) \subset Tz^n(R,n;m)$ be the free abelian group on integral sfs-cycles and define

$$\text{TCH}^n_{sfs}(R,n;m) = \frac{Tz^n_{sfs}(R,n;m)}{\partial(Tz^n(R,n+1;m)) \cap Tz^n_{sfs}(R,n;m)}.$$ (10.1)

We shall use the following result from §3 Theorem 1.1.

Proposition 10.1. The canonical map $\text{TCH}^n_{sfs}(R,n;m) \to \text{TCH}^n(R,n;m)$ is an isomorphism.

10.2. The cycle class map. By Proposition 10.1 it suffices to define the cycle class map on $\text{TCH}^n_{sfs}(R,n;m)$. We can now repeat the construction of §4 word by word to get our map. So let $Z \subset X \times \mathbb{A}^1_k \times □^{n-1}$ be an irreducible sfs-cycle and let $R' = k[Z]$. Let $f:Z \to X \times \mathbb{A}^1_k$ be the projection map. Let $g_i:Z \to □$ denote the $i$-th projection. Then the sfs property implies that each $g_i$ defines an element of $R'^{\times \infty}$, and this in turn gives a unique element $\text{cyc}_{R'}^n([Z]) = \{g_1,\ldots,g_{n-1}\} \in K_{n-1}^M(R')$. We let $\text{cyc}_R([Z])$ be its image in $K_{n-1}(R')$ under the map $K_{n-1}^M(R') \to K_{n-1}(R')$. Since $Z$ does not meet $X \times \{0\}$, we see that the finite map $f$ defines a push-forward map of spectra $f_*:K(R') \to K(R[t],(t^{m+1}))$. We let $\text{cyc}_{R'}([Z]) = f_*(\text{cyc}_R([Z])) \in K_{n-1}(R[t],(t^{m+1}))$. We extend this definition linearly to get a cycle map $\text{cyc}_R:Tz^n_{sfs}(R,n;m) \to K_{n-1}(R[t],(t^{m+1}))$. We can now prove our first result of this section.

Theorem 10.2. The assignment $[Z] \mapsto \text{cyc}_R([Z])$ defines a cycle class map

$$\text{cyc}_R: \text{TCH}^n_{sfs}(R,n;m) \to K_{n-1}(R[t],(t^{m+1}))$$
which is functorial in $R$.

**Proof.** Let $F$ be the fraction field of $R$. We consider the diagram

$$
\begin{array}{ccc}
\partial^{-1}(T_{sfs}^n(R, n; m)) & \xrightarrow{\partial} & T_{sfs}^n(R, n; m) \xrightarrow{\text{cyc}} K_{n-1}(R[t], (l^{m+1})) \\
\pi^* & & \pi^* \\
T_{sfs}^n(F, n + 1; m) & \xrightarrow{\partial} & T_{sfs}^n(F, n; m) \xrightarrow{\text{cyc}} K_{n-1}(F[t], (l^{m+1})).
\end{array}
$$

Assume first that this diagram is commutative. Then Theorem [9.3] says that $\text{cyc} \circ \partial \circ \pi^* = 0$. Equivalently, $\pi^* \circ \text{cyc} \circ \partial = 0$. We will be therefore done if we know that the right vertical map $\pi^*$ is injective. To show this, we can replace these relative $K$-groups by the relative cyclic homology groups by [14]. These relative cyclic homology groups in turn can be replaced by the Hochschild homology $\text{HH}_*(R)$ and $\text{HH}_*(F)$ by [18 Proposition 8.1]. Since $R$ is regular, we can go further and replace $\text{HH}_*(R)$ and $\text{HH}_*(F)$ by $\Omega_R$ and $\Omega_F$, respectively, by the famous Hochschild-Kostant-Rosenberg theorem. We therefore have to show that the map $\Omega_R \to \Omega_F$ is injective. But this follows from Lemma [10.3].

To show that $\text{cyc}_R$ is natural for homomorphisms of regular semi-local rings $R \to R'$, we first observe that (10.2) shows that $\partial$ is injective. But this follows from Lemma 10.3. The right-most vertical arrow in (10.2) is injective, we can replace $R$ and $R'$ by their fraction fields to check the naturality of $\text{cyc}_R$ in general. In this case, the naturality of $\text{cyc}_R$ follows from Theorem [11.1]. It remains now to show that (10.2) is commutative.

The left square is known to be commutative by the flat pull-back property of additive cycle complex. To show that the right square commutes, let $Z \subset \mathbb{A}^1_R \times_R \mathbb{A}^1_R$ be an irreducible sfs-cycle and let $R_{n-1} = k[Z]$. Let $R_0$ be the coordinate ring of the image of $Z$ in $\mathbb{A}^1_R$ as in the definition of the sfs-cycles. By definition of sfs-cycles, we have the commutative diagram

$$
\begin{array}{ccc}
R & R_0 & R_{n-1} \\
\pi & & \pi_{n-1} \\
F & F_0 & F_{n-1},
\end{array}
$$

where each term in the bottom row is the quotient field of the corresponding term on the top row. Note also that all horizontal arrows are finite maps of regular semi-local integral domains. In particular, we have $F_0 = R_0 \otimes_R F$ and $F_{n-1} = R_{n-1} \otimes_R F = R_{n-1} \otimes_{R_0} F_0$.

We let $f: Z \to \mathbb{A}^1_R$ be the projection and let $p: \text{Spec}(R_0) \to \mathbb{A}^1_R$ be the inclusion. We denote the projection $\text{Spec}(F_{n-1}) \to \mathbb{A}^1_F$ and inclusion $\text{Spec}(F_0) \to \mathbb{A}^1_F$ also by $f$ and $p$, respectively. Note that $f$ is a finite map which has factorization $Z \to \mathbb{A}^1_R \setminus \{0\} \subset \mathbb{A}^1_R$.

We let $\alpha = \text{cyc}_{F_{n-1}}^M([Z]) = (g_1, \ldots, g_{n-1}) \in K_{n-1}^M(R_{n-1})$. We then have, by definition, $\text{cyc}_R([Z]) = f_* \circ \psi_{R_{n-1}}(\alpha)$ and $\text{cyc}_F(\pi^*([Z])) = f_* \circ \psi_{F_{n-1}} \circ \pi_{n-1}^*(\alpha)$. Using (10.3), we can write these as $\text{cyc}_R([Z]) = p_* \circ f_0 \circ \psi_{R_{n-1}}(\alpha)$ and $\text{cyc}_F(\pi^*([Z])) = p_* \circ f_0 \circ \psi_{F_{n-1}} \circ \pi_{n-1}^*(\alpha)$. Since $\pi_{n-1}^* \circ \psi_{R_{n-1}} = \psi_{F_{n-1}} \circ \pi_{n-1}^*$, we only have to show that the diagram

$$
\begin{array}{ccc}
K(R_{n-1}) & K(R_0) & K(R[t], (l^{m+1})) \\
\pi_{n-1}^* & & \pi^* \\
K(F_{n-1}) & K(F_0) & K(F[t], (l^{m+1})).
\end{array}
$$

commutes.

The left square commutes by [53 Proposition 3.18] since $F_{n-1} = R_{n-1} \otimes_{R_0} F_0$ and $f_0$ is finite. To see that the right square commutes, note that we can replace $K(R_0)$ by
We can do this because in (10.2) are commutative. This also shows that the homotopy fibers of the two rows of (10.5). We have now shown that both squares in (10.5) are commutative. This also shows that cycR is compatible with the inclusion \( R \to F \). The proof of the theorem is complete.

Throughout the remaining part of our discussion, we shall identify \( \text{TCH}^n(R, n; m) \) with \( \mathbb{W}_m \Omega^n_R \) (by (7.3)) and \( K_{n-1}(R[t], (t^{m+1})) \) with \( \tilde{K}_n(R_m) \) (via the connecting homomorphism).

10.3. **Factorization through Milnor \( K \)-theory.** We shall now show show that cycR factors through the relative Milnor \( K \)-theory. The proof is identical to the case of fields and we shall only sketch it. We shall reduce the proof to the case of fields using the following result.

**Lemma 10.3.** For \( n \geq 0 \) and \( m \geq 1 \), the map \( \pi^*: \mathbb{W}_m \Omega^n_R \to \mathbb{W}_m \Omega^n_F \) is injective. In particular, the map \( \pi^*: \tilde{K}^M_n(R_m) \to \tilde{K}^M_n(F_m) \) is injective for all \( m \geq 0 \).

**Proof.** Since \( \mathbb{W}_m \Omega^n_R \cong (\Omega^n_R)^m \) (and also for \( F \)), we need to show that \( \Omega^n_R \to \Omega^n_F \) is injective to prove the first assertion of the lemma. Since \( \Omega^n_R \cong \Omega^n_R \otimes_R F \), it suffices to show that \( \Omega^n_R \) is a free \( R \)-module. Since \( R \) is regular, we have \( D_1(R[k]) = 0 \) and \( \Omega^n_{R/k} \) is a free \( R \)-module. The Jacobi-Zariski exact sequence (see [43, 3.5.5]) therefore tells us that \( \Omega^n_R \cong (\Omega^n_R \otimes_k R) \otimes \Omega^n_{R/k} \). This proves the first part.

For the second part, there is nothing to prove when \( m = 0 \). When \( m \geq 1 \), Lemma 8.3.4 reduces to showing that the map \( \Omega^n_R \otimes_R tR_m \to \Omega^n_F \otimes_F tF_m \) is injective for all \( n \geq 0 \). Since \( \Omega^n_R \otimes_F tF_m \cong \Omega^n_R \otimes_R tR_m \), the problem is reduced to showing that \( \Omega^n_R \to \Omega^n_F \) is injective. We can now use the first part of the lemma.

Our second main result of this section is the following. This generalizes the main results of [11], [45] and [54] to truncated polynomial rings.

**Theorem 10.4.** Let \( R \) and \( m \geq 0, n \geq 1 \) be as above. Then the cycle class map cycR has a factorization

\[
\text{TCH}^n(R, n; m) \xrightarrow{\text{cyc}_R^M} \tilde{K}^M_n(R_m) \xrightarrow{\psi_{R^{m+1}, m}} \tilde{K}_n(R_m).
\]

Furthermore, cycR is natural in \( R \) and is an isomorphism.

**Proof.** We shall use Proposition 10.1 which allows us to repeat the proof of the field case (Lemma 9.2) word by word. When \( n = 1 \), any sfs irreducible cycle \( Z \subset \mathbb{A}^1_R \) is of the form \( Z = V(f(t)) \), where \( f(t) \) is an irreducible polynomial such that \( f(0) \in \mathbb{F}_p^* \). We now repeat the argument of the field case and use Lemma 2.1 to finish the proof. The \( n \geq 2 \) case follows from Lemma 7.3 and the proof is identical to the case of fields. To prove that cycR is an isomorphism, we again repeat the case of fields and use Remark 9.4. The naturality of cycR follows from Theorem 10.2 since \( \psi_{R^{m+1}, n} \) is injective.

Finally, we are now ready to prove Theorem 1.1. We restate it again for reader’s convenience.
Theorem 10.5. Let \( R \) be a regular semi-local ring which is essentially of finite type over a characteristic zero field. Let \( n \geq 1 \) be an integer. Then the cycle class map
\[
cyc_R : \{\text{TCP}^n(R,n;m)\}_m \to \{\overline{K}_n(R_m)\}_m
\]
is an isomorphism of the pro-abelian groups.

Proof. Combine Theorem 10.4 and Corollary 9.6. \( \square \)

11. Appendix: Milnor vs Quillen \( K \)-theory

In this section, we collect some results on the compatibility of various maps between Milnor and Quillen \( K \)-theories of fields. They are used in the proofs of the main results of this paper. We expect these results to be known to experts but could not find their written proofs in the literature.

Let \( k \) be a field and let \( X \) be a non-singular scheme which is essentially of finite type over \( k \). Let \( x, y \in X \) be two points in \( X \) of codimensions \( p \) and \( p - 1 \), respectively, such that \( x \in \{y\} \). Let \( Y = \{y\}, \ F = k(y), \ A = \mathcal{O}_{Y,x}, \) and \( l = k(x) \).

Lemma 11.1. For any \( n \geq 1 \), the diagram
\[
\begin{array}{ccc}
K_n^M(F) & \xrightarrow{\partial^M} & K_{n-1}(l) \\
\downarrow & & \downarrow \\
K_n(F) & \xrightarrow{\partial^Q} & K_{n-1}(l)
\end{array}
\]
is commutative.

Proof. Let \( B \) denote the normalization of \( A \) and let \( S \) denote the set of maximal ideals of \( B \). Note that \( B \) is semi-local so that \( S \) is finite. Since the localization sequence for Quillen \( K \)-theory of coherent sheaves is functorial for proper push-forward, we have a commutative diagram
\[
\begin{array}{ccc}
G_n(B) & \xrightarrow{\partial^M} & K_n(F) \oplus K_{n-1}(k(z)) \\
\downarrow & & \downarrow \\
G_n(A) & \xrightarrow{\partial^Q} & K_{n-1}(l),
\end{array}
\]
where \( T_{k(z)/l} \) is our notation for the finite push-forward \( K_*(k(z)) \to K_*(l) \) and \( G_*(-) \) is Quillen \( K \)-theory of coherent sheaves functor (for proper morphisms). Note that \( K_*(B) \xrightarrow{\partial} G_*(B) \).

On the other hand, the boundary map in Milnor \( K \)-theory also has the property that the diagram
\[
\begin{array}{ccc}
K_n^M(F) & \xrightarrow{\partial^M} & K_n^M(k(z)) \\
\downarrow & & \downarrow \\
K_n^M(F) & \xrightarrow{\partial^Q} & K_{n-1}(l)
\end{array}
\]
commutes, where the right vertical arrow is the sum of the Norm maps in Milnor \( K \)-theory of fields (see [2] and [22]). The lemma therefore follows if we prove Lemmas 11.2 and 11.3 below. \( \square \)
Lemma 11.2. Let \( z \in S \) be a closed point as above and let \( R \) denote the discrete valuation ring of \( F \) associated to \( z \). Then the diagram

\[
\begin{array}{ccc}
K_n^M(F) & \xrightarrow{\partial^M} & K_{n-1}^M(k(z)) \\
\downarrow & & \downarrow \\
K_n(F) & \xrightarrow{\partial^Q} & K_{n-1}(k(z))
\end{array}
\]

is commutative for every \( n \geq 1 \).

Proof. It is well known and elementary to see (using the Steinberg relations) that \( K^M_n(F) \) is generated by \( K^M_1(F) \) as an \( K^*_R \)-module. Furthermore, \( \partial^M \) is \( K^*_R \)-linear (see [2] § 4, Proposition 4.5). Since the localization sequence such as the one on the top of (11.2) (with \( B \) replaced by \( R \)) is \( K^*_R \)-linear, it follows that all arrows in (11.4) are \( K^*_R \)-linear. It therefore suffices to prove the lemma for \( n = 1 \). But in this case, both \( \partial^M \) and \( \partial^Q \) are simply the valuation map of \( F \) corresponding to \( z \). \( \square \)

Lemma 11.3. Let \( k \hookrightarrow k' \) be a finite extension of fields and \( n \geq 0 \) an integer. Then we have a commutative diagram

\[
\begin{array}{ccc}
K_n^M(k') & \xrightarrow{N_{k'/k}} & K_n^M(k) \\
\downarrow & & \downarrow \\
K_n(k') & \xrightarrow{T_{k'/k}} & K_n(k)
\end{array}
\]

Proof. Assume first that \( k \hookrightarrow k' \) is a simple extension so that \( k' = k[t]/m \) for some maximal ideal \( m \subset k[t] \). Let \( v_\infty \) be the valuation of \( k(t) \) associated to the point \( \infty \in \mathbb{P}^1_k \). Its valuation ring \( R_\infty \subset k(t) \) has uniformizing parameter \( t^{-1} \). In this case, we have the following diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{K_{n+1}^M(k)} & K_{n+1}^M(k(t)) \xrightarrow{\partial^M_{v_\infty}} K_n^M(k(v)) \xrightarrow{\oplus} 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{K_{n+1}(k)} & K_{n+1}(k(t)) \xrightarrow{\partial^Q_{v_\infty}} K_n(k(v)) \xrightarrow{\oplus} 0.
\end{array}
\]

The horizontal arrows on the left in both rows are induced by the inclusion \( k \subset k(t) \). The top row is Milnor’s exact sequence (see [55, Chapter III, Theorem 7.4]). The bottom row is the localization sequence in Quillen \( K \)-theory (using the isomorphism \( K^*_k(k) \cong K_*^e(k[t]) \)) and is known to be exact (see [55, Chapter V, Corollary 6.7.1]). The right square commutes by Lemma 11.2.
On the other hand, we have another diagram

\[
\begin{array}{c}
\K^M_n(k') \\
\K^M_{n+1}(k(t)) \xrightarrow{\partial^M_{\infty}(\partial_v)} \bigoplus_{v \in \pi_{\infty}} \K^M_n(k(v)) \xrightarrow{\sum_{v} N_{k' k}(v)} \K^M_n(k) \\
\K^Q_{n+1}(k(t)) \xrightarrow{\partial^Q_{\infty}(\partial_v)} \bigoplus_{v \in \pi_{\infty}} \K^Q_n(k(v)) \xrightarrow{\sum_{v} T_{k' k}(v)} \K^Q_n(k) \\
\K^Q_n(k').
\end{array}
\]

By the definition of the norm \( N_{k' k} \) in Milnor \( K \)-theory, the composition of the horizontal arrows on the top is the map \((-1)\partial^M_{\infty}: K^M_{n+1}(k(t)) \to K^M_n(k)\) (see [55, Chapter III, Definition 7.5]). Similarly, the composite of the horizontal arrows on the bottom is the map \((-1)\partial^Q_{\infty}: K^Q_{n+1}(k(t)) \to K^Q_n(k)\) (see [55, Chapter V, 6.12.1]). Note that both of these assertions are another way of stating the Weil reciprocity formulas for the Milnor and Quillen \( K \)-theories.

Since the left horizontal arrows in both rows are surjective, we are reduced to showing therefore that the diagram

\[
\begin{array}{c}
\K^M_{n+1}(k(t)) \xrightarrow{\partial^M_{\infty}} \K^M_n(k) \\
\K^Q_{n+1}(k(t)) \xrightarrow{\partial^Q_{\infty}} \K^Q_n(k)
\end{array}
\]

commutes. But this follows from Lemma [11.2]. This proves the lemma for simple extensions.

In general, we can write \( k' = k(x_1, \ldots, x_r) \). Since the norm maps in Milnor \( K \)-theory and the push-forward maps in Quillen \( K \)-theory satisfy the transitivity property, and since \( k \to k' \) is a composite of simple extensions, the proof of the lemma follows.

\[\Box\]

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