Subsonic time-periodic solution to compressible
Euler equations with damping in a bounded
domain

Peng Qu\textsuperscript{*a}, Huimin Yu\textsuperscript{†b} and Xiaomin Zhang\textsuperscript{‡b}

\textsuperscript{a}School of Mathematical Sciences, Shanghai Key Laboratory for Contemporary Applied
Mathematics, Fudan University, Shanghai 200433 China
\textsuperscript{b}Department of mathematics, Shandong Normal University, Jinan 250014 China

Abstract: In this paper, we consider the one-dimensional isentropic compressible Euler
equations with linear damping $\beta(t, x)\rho u$ in a bounded domain, which can be used to describe
the process of compressible flows through a porous medium. And the model is imposed a
dissipative subsonic time-periodic boundary condition. Our main results reveal that the
time-periodic boundary can trigger a unique subsonic time-periodic smooth solution which
is stable under small perturbations on initial data. Moreover, the time-periodic solution
possesses higher regularity and stability provided a higher regular boundary condition.

Keywords: Isentropic compressible Euler equations, time-periodic boundary, source term,
global existence, stability, subsonic flow, time-periodic solutions

Mathematics Subject Classification 2010: 35B10, 35A01, 35Q31.

1 Introduction

In this paper, we consider the one-dimensional compressible Euler equations for
isentropic flow with linear damping in Eulerian coordinates

\begin{equation}
\begin{aligned}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p) &= \beta(t, x)\rho u,
\end{aligned}
\end{equation}

where temporal and spatial variables $(t, x) \in D = \{t, x\} | t \in \mathbb{R}_+, x \in [0, L]\}$. $\rho, u$
and $p(\rho) = A\rho^\gamma$ are the density, velocity and pressure respectively. Here the adia-
batic gas exponent $\gamma > 1$ and pressure coefficient $A$ is normalized to 1. Damping
coefficient $\beta = \beta(t, x)$ is a non-positive $C^2$ smooth function.

In recent years, much effort has been made on the time-periodic solutions to
the viscous fluids equations and the hyperbolic conservation laws, see for example [1, 2, 5, 9, 12, 13]. However, the time-periodic solutions mentioned above are usually caused by the time-periodic external forces. As far as we know, there is little work to consider the problem with time-periodical boundary. In 2019, Yuan [16] studied the existence and high-frequency limiting behavior of supersonic time-periodic solutions to the 1-D isentropic compressible Euler equations (i.e. \( \beta(t, x) \equiv 0 \)) with time-periodic inflow boundary conditions. Supersonic is an essential assumption in [16], which implies that all characteristic propagate forward in both space and time. Therefore, due to the finite propagation speed, Yuan obtained the existence of time-periodic solution after the start-up time \( T_0 > 0 \), which are needed for the boundary condition to effect the whole domain. From a physical point of view, the force produced by the wall of duct [11] or porous medium can be regarded as some source term added to the Euler equations. A natural question is: what kind of effects can the friction term do on the time-periodic solution? [14, 15] considered the existence and stability of supersonic time periodic flows for the compressible Euler equations with friction term. Due to wave interactions between families of different directions, it is well known that the case with subsonic boundary conditions is more complicated for compressible Euler equation. Motivated by [10], in which Qu considered the time-periodic solutions triggered by a kind of dissipative time-periodic boundary condition for the general quasilinear hyperbolic systems, we consider the subsonic time-periodical solutions of isentropic Euler equation with linear damping and the damping coefficient \( \beta(t,x) \) satisfy the following hypothesis:

\[
\begin{align*}
(H): \text{there exist two constants } \beta_* < 0 \text{ and } T_* > 0 \text{ such that } \\
\beta_* &\leq \beta(t, x) \leq 0, \\
|\partial_t \beta(t, x)|, |\partial_x \beta(t, x)| &\leq \epsilon, \\
\beta(t + T_*, x) &\equiv \beta(t, x),
\end{align*}
\]

where the small constant \( \epsilon \) will be determined later.

Obviously, (1.2)-(1.4) hold if \( \beta(t, x) \) is a negative constant.

It should be noted that: the third author Zhang [17] studied the same problem with \( \beta(t, x) = \beta(t) \). However, [17] needs the smallness of the friction coefficient \( \beta(t) \) and its integral needs to be zero in one cycle, i.e.

\[
\int_t^{t+T_*} \beta(s)ds = 0,
\]

which can not be satisfied for general constants. We remove these two restrictions and include the constant friction coefficient case in this paper. The main idea used in this paper is that we select a special linearized iteration scheme (3.3) and (3.5). The advantage of this modified iteration scheme is to make better use of the nonstrictly diagonally dominant structure brought by the linear damping. While the iteration scheme used in [17] can only utilize the smallness of the source term. [18] also studied this problem with nonlinear damping.

The rest of this paper is organized as follows. In Section 2 we first introduce the Riemann invariants of the homogeneous compressible Euler equations and give
the main results: Theorem 2.1–Theorem 2.4. In Section 3, we use the linearized iteration method to prove Theorem 2.1. In Section 4, we use the inductive method to prove Theorem 2.2. In Section 5 and Section 6, we consider the higher regularity and stability of the time-periodic solution.

2 Preliminaries and Main Results

In this section, we make some transformations on equations (1.1) and give the main results of this paper.

Firstly, it is easy to check that the two eigenvalues of system (1.1) are

\[ \lambda_1 = u - c, \quad \lambda_2 = u + c. \]

Then there holds

\[ \lambda_1(\rho, 0) < 0 < \lambda_2(\rho, 0) \]

and

\[ \lambda_1(\rho, u) < 0 < \lambda_2(\rho, u), \quad (\rho, u) \in \mathcal{I} \]  \hspace{1cm} (2.1)

for any positive constant \( \rho > 0 \) and a small neighborhood \( \mathcal{I} \) of \((\rho, 0)\).

Using the Riemann invariants \( m \) and \( n \) defined by

\[ m = \frac{1}{2}(u - \frac{2}{\gamma - 1}c), \quad n = \frac{1}{2}(u + \frac{2}{\gamma - 1}c), \]  \hspace{1cm} (2.2)

the system (1.1) becomes the following diagonal form

\[
\begin{align*}
 m_t + \lambda_1(m, n)m_x &= \frac{\beta}{2}(m + n), \\
 n_t + \lambda_2(m, n)n_x &= \frac{\beta}{2}(m + n),
\end{align*}
\]  \hspace{1cm} (2.3)

where

\[ c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\gamma \rho^{\frac{\gamma - 1}{2}}}, \]

\[ \lambda_1(m, n) = \frac{\gamma + 1}{2} m + \frac{3 - \gamma}{2} n, \quad \lambda_2(m, n) = \frac{3 - \gamma}{2} m + \frac{\gamma + 1}{2} n. \]

We impose the system (1.1) or (2.3) with the following initial data and boundary conditions

\[
\begin{align*}
 t = 0: & \quad m(0, x) = m_0(x), \quad n(0, x) = n_0(x), \quad (2.4) \\
 x = 0: & \quad n(t, 0) = n_b(t) + \kappa_2(m(t, 0) - m), \quad (2.5) \\
 x = L: & \quad m(t, L) = m_b(t) + \kappa_1(n(t, L) - n), \quad (2.6)
\end{align*}
\]
where the constants $|\kappa_1| < 1, |\kappa_2| < 1$ and $-\overline{m} = \overline{n} = \frac{\sqrt{\nu}}{2} \rho^{\frac{n-1}{2}}$. Moreover, in (2.5)-(2.4), we assume $m_b(t), n_b(t)$ are two periodic functions with the period $T_\ast > 0$, namely,

$$m_b(t + T_\ast) = m_b(t), \quad n_b(t + T_\ast) = n_b(t).$$

(2.5)-(2.6) is a kind of dissipative boundary condition in the sense of [3].

Denote the perturbation variable

$$\phi(t, x) = (\phi_1(t, x), \phi_2(t, x))^T \overset{df}{=} (m(t, x) - \overline{m}, n(t, x) - \overline{n})^T$$

and let $\Phi = (\overline{m}, \overline{n})^T$, then equations (2.3) can be rewritten as

$$\begin{cases}
\partial_t \phi_1 + \lambda_1(\phi + \Phi) \partial_x \phi_1 = \frac{\beta}{2} (\phi_1 + \phi_2), \\
\partial_t \phi_2 + \lambda_2(\phi + \Phi) \partial_x \phi_2 = \frac{\beta}{2} (\phi_1 + \phi_2),
\end{cases} \quad (2.7)$$

where $\lambda_1(\Phi) = -\ell = -\sqrt{\rho^{\frac{n-1}{2}}}, \lambda_2(\Phi) = \ell = \sqrt{\rho^{\frac{n-1}{2}}}$.

Correspondingly, the initial data and boundary conditions (2.4)-(2.6) become

$$t = 0 : \quad \phi(0, x) = \phi_0(x) = (\phi_{10}(x), \phi_{20}(x))^T = (m_0(x) - \overline{m}, n_0(x) - \overline{n})^T, \quad (2.8)$$

$$x = 0 : \quad \phi_2(t, 0) = \phi_{20}(t) + \kappa_2 \phi_1(t, 0), \quad t \geq 0, \quad (2.9)$$

$$x = L : \quad \phi_1(t, L) = \phi_{10}(t) + \kappa_1 \phi_2(t, L), \quad t \geq 0, \quad (2.10)$$

where $\phi_{10}(t) = m_b(t) - \overline{m}, \phi_{20}(t) = n_b(t) - \overline{n}$ satisfying

$$\phi_{ib}(t + T_\ast) = \phi_{ib}(t), \quad t > 0, i = 1, 2.$$ 

Obviously, we have the following two facts:

1: By (2.1), we get

$$
\lambda_1(\phi + \Phi) < 0 < \lambda_2(\phi + \Phi), \quad \forall \phi \in \Psi,
$$

where $\Psi$ is a small neighborhood of $O = (0, 0)^T$ corresponding to $\mathcal{A}$.

2: Define $\nu_i(\phi + \Phi) = \lambda_i^{-1}(\phi + \Phi)(i = 1, 2)$, then there exists a positive constant $A_0$, such that

$$
\max_{i=1, 2} \sup_{\phi \in \Psi} |\nu_i(\phi + \Phi)| \leq A_0. \quad (2.11)
$$

Next, we give the main results in the following theorems.

**Theorem 2.1.** *(Existence of time-periodic solutions)* There exists a small enough constant $\epsilon_1 > 0$ and a constant $C_\varepsilon > 0$ such that for any given $0 < \epsilon < \epsilon_1$, and any given $C^1$ smooth functions $\phi_{ib}(t)(i = 1, 2)$ satisfying

$$
\begin{align*}
\phi_{ib}(t + T_\ast) &= \phi_{ib}(t), \quad t > 0, i = 1, 2, \\
\|\phi_{ib}(t)\|_{C^1(\mathbb{R}_+)} &\leq \epsilon,
\end{align*} \quad (2.12) \quad (2.13)
$$

$$
through Theorem 2.1,
\[\phi_{\xi}\phi\]
where \(C\) has a unique global
\[\phi(t,x) = \phi(t,x), \quad \forall(t,x) \in D\]  \tag{2.15}
and
\[\|\phi(T_\ast)(t)\|_{C^1(D)} \leq C_2\epsilon\]  \tag{2.16}

**Theorem 2.2.** (Stability of time-periodic solutions) There exists a small constant \(\epsilon_1 \in (0, \epsilon_2)\) and any given \(\epsilon_1 > 0\), such that for any given \(\epsilon \in (0, \epsilon_2)\) and any given \(C^1\) smooth functions \(\phi_0 = \phi_0(x)\) and \(\phi_{i_1}(t)(i = 1, 2)\) satisfying \[\tag{2.14}\]

where \(\phi(T_\ast)\), depending on \(\phi_{i_1}(t)(i = 1, 2)\), is the time-periodic solution given through Theorem 2.1, \(\xi \in (0, 1)\) is a constant and \(T_0 = \max \sup_{i=1,2} \frac{1}{\lambda_i(T_\ast)}\).

The uniqueness of the time-periodic solution is a direct consequence from Theorem 2.2 by taking \(t \to +\infty\).

**Corollary 2.1.** (Uniqueness of the time-periodic solution) There exists a constant \(\epsilon_3 \in (0, \epsilon_2)\), such that for any given \(\epsilon \in (0, \epsilon_3)\) and any given \(C^1\) smooth functions \(\phi_{i_1}(t)(i = 1, 2)\) satisfying \[\tag{2.17}\]
and possess further \(W^{2,\infty}\) regularity with
\[\max_{i=1,2} \|m''_{i_1}(t)\|_{L^\infty} \leq M_2 < +\infty\]  \tag{2.18}
then there exist constants \(C_R > 0\) and \(\epsilon_4 \in (0, \epsilon_1)\), such that for any given \(\epsilon \in (0, \epsilon_4)\), the time-periodic solution \(\phi = \phi(T_\ast)(t,x)\) provided by Theorem 2.1 is also a \(W^{2,\infty}\) function with
\[\max\{\|\partial_t^2\phi(T_\ast)\|_{L^\infty}, \|\partial_t\partial_x\phi(T_\ast)\|_{L^\infty}, \|\partial_x^2\phi(T_\ast)\|_{L^\infty}\} \leq (1 + A_0)^2C_R < +\infty\]  \tag{2.19}
where \(A_0\) is defined in \[\tag{2.11}\]

**Theorem 2.4.** (Stabilization around the time-periodic solution) Assume that \[\tag{2.18}\] holds, then there exist constants \(C_{\ast} > 0\) and \(\epsilon_5 \in (0, \min\{\epsilon_2, \epsilon_4\})\), such that for any given \(\epsilon \in (0, \epsilon_5)\), we have not only the \(C^0\) convergence result
\[\max\{\|\partial_t\phi(t,\cdot) - \partial_t\phi(T_\ast)(t,\cdot)\|_{L^\infty}, \|\partial_x\phi(t,\cdot) - \partial_x\phi(T_\ast)(t,\cdot)\|_{L^\infty}\} \leq A_0C_{\ast}\epsilon\Xi[t/T_0], \quad \forall t \geq 0\]  \tag{2.20}
In this section, we give the proof of Theorem 2.1 by applying the linearized iteration method.

Firstly, multiplying \( \lambda_i^{-1} \) on both sides of the \( i \)-th equation of (2.7) for \( i = 1, 2 \) and swapping the positions of \( t \) and \( x \), we have

\[
\partial_x \phi_1 + \nu_1 (\phi + \Phi) \partial_t \phi_1 = \frac{\beta}{2} \nu_1 (\Phi) \phi_1 + \frac{\beta}{2} \left( \nu_1 (\phi + \Phi) - \nu_1 (\Phi) \right) \phi_1 \\
+ \frac{\beta}{2} \nu_1 (\phi + \Phi) \phi_2, 
\]

(3.1)

\[
\partial_x \phi_2 + \nu_2 (\phi + \Phi) \partial_t \phi_2 = \frac{\beta}{2} \nu_2 (\Phi) \phi_2 + \frac{\beta}{2} \left( \nu_2 (\phi + \Phi) - \nu_2 (\Phi) \right) \phi_2 \\
+ \frac{\beta}{2} \nu_2 (\phi + \Phi) \phi_1. 
\]

(3.2)

Then using (3.1)-(3.2) and (2.9)-(2.10), we establish the following "initial"-value problem of linearized system

\[
\partial_x \phi^{(l)}_1 + \nu_1 (\phi^{(l-1)} + \Phi) \partial_t \phi^{(l)}_1 = \frac{\beta}{2} \nu_1 (\Phi) \phi^{(l)}_1 + \frac{\beta}{2} \left( \nu_1 (\phi^{(l-1)} + \Phi) - \nu_1 (\Phi) \right) \phi^{(l-1)}_1 \\
+ \frac{\beta}{2} \nu_1 (\phi^{(l-1)} + \Phi) \phi^{(l-1)}_2, 
\]

(3.3)

\[
x = L : \quad \phi^{(l)}_1 (t, L) = \phi_{1r}(t) + \kappa_1 \phi^{(l-1)}_2 (t, L) 
\]

(3.4)

and

\[
\partial_x \phi^{(l)}_2 + \nu_2 (\phi^{(l-1)} + \Phi) \partial_t \phi^{(l)}_2 = \frac{\beta}{2} \nu_2 (\Phi) \phi^{(l)}_2 + \frac{\beta}{2} \left( \nu_2 (\phi^{(l-1)} + \Phi) - \nu_2 (\Phi) \right) \phi^{(l-1)}_2 \\
+ \frac{\beta}{2} \nu_2 (\phi^{(l-1)} + \Phi) \phi^{(l-1)}_1, 
\]

(3.5)

\[
x = 0 : \quad \phi^{(l)}_2 (t, 0) = \phi_{2r}(t) + \kappa_2 \phi^{(l-1)}_1 (t, 0), 
\]

(3.6)

where

\[
\phi_{1r}(t) = \begin{cases} 
\phi_{1b}(t), & t \geq 0, \\
\phi_{1l}(t), & t < 0,
\end{cases} \quad \phi_{2r}(t) = \begin{cases} 
\phi_{2b}(t), & t \geq 0, \\
\phi_{2l}(t), & t < 0,
\end{cases}
\]

are the time-periodic extensions of \( \phi_{ib}(t) (i = 1, 2) \).

Here we select a special linearized iteration scheme (3.3) and (3.5) to make better use of the nonstrictly diagonally dominant structure brought by damping.

For system (3.3)-(3.6), we start to iterate from

\[
\phi^{(0)}(t, x) = 0 
\]

(3.7)

and prove the following a priori estimates.
Proposition 3.1. There exists a small enough constant $\epsilon_1 > 0$ and constants $M_1 > 0$, $C_1 > 0$ and $\theta \in (0, 1)$, such that for any given $\epsilon \in (0, \epsilon_1)$, the following estimates hold:

\[
\phi^{(i)}(t + T_s, x) = \phi^{(i)}(t, x), \quad \forall (t, x) \in \mathbb{R} \times [0, L], l \in \mathbb{N}_+, \quad (3.8)
\]

\[
\|\phi^{(i)}\|_{C^1} \leq (C_1 + M_1)\epsilon, \quad \forall l \in \mathbb{N}_+, \quad (3.9)
\]

\[
\|\phi^{(i)} - \phi^{(l-1)}\|_{C^0} \leq M_1\epsilon^\theta, \quad \forall l \in \mathbb{N}_+, \quad (3.10)
\]

\[
\max_{i=1,2} \{\varpi(\delta|\partial_t\phi^{(i)}_1) + \varpi(\delta|\partial_x\phi^{(i)}_1)\} \leq \left(\frac{1}{3} + \frac{1}{2}A_0\right)\Omega(\delta), \quad \forall l \in \mathbb{N}_+, \quad (3.11)
\]

where

\[
\|\phi^{(i)}\|_{C^1(D)} \overset{def}{=} \max_{i=1,2} \left\{\|\phi^{(i)}_1\|_{C^0(D)}, \|\partial_t\phi^{(i)}_1\|_{C^0(D)}, \|\partial_x\phi^{(i)}_1\|_{C^0(D)}\right\},
\]

\[
\varpi(\delta|h) = \sup_{|t_1 - t_2| \leq \delta, |x_1 - x_2| \leq \delta} |h(t_1, x_1) - h(t_2, x_2)|,
\]

and $\Omega(\delta)$ is a continuous function of $\delta \in (0, 1)$ which is independent of $l$ and satisfies

\[
\lim_{\delta \to 0^+} \Omega(\delta) = 0.
\]

Proof. We prove the a priori estimates (3.8), (3.11) inductively, i.e., for each $l \in \mathbb{N}_+$, we show

\[
\phi^{(i)}(t + T_s, x) = \phi^{(i)}(t, x), \quad \forall (t, x) \in \mathbb{R} \times [0, L], \forall i = 1, 2, \quad (3.12)
\]

\[
\max_{i=1,2} \left\{\|\phi^{(i)}_1\|_{C^0}, \|\partial_t\phi^{(i)}_1\|_{C^0}\right\} \leq M_1\epsilon, \quad (3.13)
\]

\[
\max_{i=1,2} \left\{\|\partial_x\phi^{(i)}_1\|_{C^0}\right\} \leq C_1\epsilon, \quad (3.14)
\]

\[
\max_{i=1,2} \left\{\|\phi^{(i)}_1 - \phi^{(l-1)}_1\|_{C^0}\right\} \leq M_1\epsilon^\theta, \quad (3.15)
\]

\[
\max_{i=1,2} \varpi(\delta|\partial_t\phi^{(i)}_1(\cdot, x)) \leq \frac{1}{8A_0 + 1} \Omega(\delta), \quad \forall x \in [0, L] \quad (3.16)
\]

and

\[
\max_{i=1,2} \left\{\varpi(\delta|\partial_x\phi^{(i)}_1) + \varpi(\delta|\partial_x\phi^{(i)}_1)\right\} \leq \left(\frac{1}{3} + \frac{1}{2}A_0\right)\Omega(\delta) \quad (3.17)
\]

under the following hypothesis

\[
\phi^{(l-1)}_i(t + T_s, x) = \phi^{(l-1)}_i(t, x), \quad \forall (t, x) \in \mathbb{R} \times [0, L], \forall i = 1, 2, \quad (3.18)
\]

\[
\max_{i=1,2} \left\{\|\phi^{(l-1)}_i\|_{C^0}, \|\partial_t\phi^{(l-1)}_i\|_{C^0}\right\} \leq M_1\epsilon, \quad (3.19)
\]

\[
\max_{i=1,2} \left\{\|\partial_x\phi^{(l-1)}_i\|_{C^0}\right\} \leq C_1\epsilon, \quad (3.20)
\]

\[
\max_{i=1,2} \left\{\|\phi^{(l-1)}_i - \phi^{(l-2)}_i\|_{C^0}\right\} \leq M_1\epsilon^\theta, \quad \forall l \geq 2, \quad (3.21)
\]

\[
\max_{i=1,2} \varpi(\delta|\partial_t\phi^{(l-1)}_i(\cdot, x)) \leq \frac{1}{8A_0 + 1} \Omega(\delta), \quad \forall x \in [0, L] \quad (3.22)
\]
and

\[
\max_{i=1, 2} \{ \varpi(\delta|\partial_t \phi_i^{(l-1)}) + \varpi(\delta|\partial_x \phi_i^{(l-1)}) \} \leq \left( \frac{1}{3} + \frac{1}{2} |A_0| \right) \Omega(\delta), \tag{3.23}
\]

where \([A_0 + 1]\) represents the integer part of \(A_0 + 1\) and

\[
\varpi(\delta|h(\cdot, x)) = \max_{|t_1 - t_2| \leq \delta} |h(t_1, x) - h(t_2, x)|.
\]

In the remaining parts of this section, we will use several steps to check the above estimates (3.12)-(3.17) one by one.

**Step 1:** Transformations of the principal equations. Define the characteristic curve \(t = t_i^l(x; t_0, x_0) (i = 1, 2)\) as the following form:

\[
\begin{align*}
\frac{dt_i^l(x; t_0, x_0)}{dx} &= \nu_i(\phi^{(l-1)} + \Phi)(t_i^l(x; t_0, x_0), x), \\
t_i^l(x; t_0, x_0) &= t_0.
\end{align*}
\tag{3.24}
\]

Denote

\[F_1(t, x) = e^{\int_0^t \frac{\partial t_i^l}{\partial x} \nu_i(\Phi) ds}, \quad F_2(t, x) = e^{-\int_0^t \frac{\partial t_i^l}{\partial x} \nu_i(\Phi) ds}.\]

Noticing \(\beta(t, x) \leq 0, \nu_1(\Phi) < 0, \nu_2(\Phi) > 0\) and \(x \in [0, L]\), we have

\[
\begin{align*}
F_1(t, x), F_2(t, x) &\geq 1, \\
\frac{\partial F_1(t, x)}{\partial x} &= -\frac{\beta(t, x)}{2} \nu_1(\Phi) F_1(t, x) < 0, \\
\frac{\partial F_1(t, x)}{\partial t} &= \int_x^L \frac{\partial \beta(t, s)}{\partial x} \nu_1(\Phi) ds F_1(t, x), \\
\frac{\partial F_2(t, x)}{\partial x} &= -\frac{\beta(t, x)}{2} \nu_2(\Phi) F_2(t, x) > 0, \\
\frac{\partial F_2(t, x)}{\partial t} &= -\int_0^x \frac{\partial \beta(t, s)}{\partial x} \nu_2(\Phi) ds F_2(t, x).
\end{align*}
\]

Furthermore, by (1.2) and (2.11), we get

\[
1 \leq F_1(t, x) \leq e^{-\frac{\beta}{|\kappa|} A_0 L} \overset{\text{def}}{=} M_0 > 1, \tag{3.28}
\]

\[
1 \leq F_2(t, x) \leq M_0. \tag{3.29}
\]

Let \(M_1 = \frac{100}{1-\kappa}\) with \(\kappa = \max\{|\kappa_1|, |\kappa_2|\} < 1\), then

\[
M_1 \geq |\kappa_1| M_1 + 100, \quad M_1 \geq |\kappa_2| M_1 + 100. \tag{3.30}
\]

Now, we turn problem (3.3)-(3.6) into the system of \(F_i(t, x)\phi_i^{(l)}\) as follows:

\[
\begin{align*}
\partial_x \left( F_1 \phi_1^{(l)} \right) + \nu_1(\phi^{(l-1)} + \Phi) \partial_t \left( F_1 \phi_1^{(l)} \right) &= \frac{\beta}{2} F_1 \left( \nu_1(\phi^{(l-1)} + \Phi - \nu_1(\Phi)) \phi_1^{(l-1)} + \phi_2^{(l-1)} \right) \\
&\quad + \frac{\beta}{2} F_1 \nu_1(\Phi) \phi_2^{(l-1)} + \nu_1(\phi^{(l-1)} + \Phi) F_1 \phi_1^{(l)} \int_x^L \frac{\partial \beta(t, s)}{\partial x} \nu_1(\Phi) ds, \tag{3.31}
\end{align*}
\]

8
\[ x = L : \quad F_1(t, L) \phi_1^{(l)}(t, L) = \phi_1^{(l)}(t, L) = \phi_{1\nu}(t) + \kappa_1 \phi_2^{(l-1)}(t, L), \quad (3.32) \]

\[ \partial_x \left( F_2 \phi_2^{(l)} \right) + \nu_2 (\phi^{(l-1)} + \Phi) \partial_t \left( F_2 \phi_2^{(l)} \right) = \frac{\beta}{2} F_2 \left( \nu_2 (\phi^{(l-1)} + \Phi) - \nu_2 (\Phi) \right) (\phi_1^{(l-1)} + \phi_2^{(l-1)}) \]
\[ + \frac{\beta}{2} F_2 \nu_2 (\Phi) (\phi_1^{(l-1)} + \nu_2 (\phi^{(l-1)} + \Phi) F_2 \phi_2^{(l)} \int_0^x \frac{\partial_1 \beta(t, s)}{2} \nu_2 (\Phi) ds, \quad (3.33) \]

\[ x = 0 : \quad F_2(t, 0) \phi_2^{(l)}(t, 0) = \phi_2^{(l)}(t, 0) = \phi_{2\nu}(t) + \kappa_2 \phi_1^{(l-1)}(t, 0). \quad (3.34) \]

By (1.4), (2.12) and (3.18), it is easy to know that if \( F_i(t, x) \phi_i^{(l)}(t, x) (i = 1, 2) \) solves problem (3.31)-(3.34), so does \( F_i(t + T_x, x) \phi_i^{(l)}(t + T_x, x) (i = 1, 2) \). Then we get \( F_i(t + T_x, x) \phi_i^{(l)}(t + T_x, x) = F_i(t, x) \phi_i^{(l)}(t, x) \) by the uniqueness of this linear system. Thus (3.12) is proved.

**Step 2:** The proof of (3.13) and (3.14).

Next, we prove the \( C^0 \) estimate for \( \phi_i^{(l)} (i = 1, 2) \). By the aid of conditions (2.13), (3.4), (3.6), (3.30), we get

\[ \| \phi_1^{(l)} (\cdot, L) \|_{C^0} \leq \epsilon + |\kappa_1| \epsilon \leq M_1 \epsilon - 99 \epsilon, \quad (3.35) \]
\[ \| \phi_2^{(l)} (\cdot, 0) \|_{C^0} \leq M_1 \epsilon - 99 \epsilon. \quad (3.36) \]

Then we integrate (3.31) along the 1st characteristic curve \( t = \tau^{(l)}(y; t, x) \) from \( L \) to \( x \) to get

\[ F_1(t, x) \phi_1^{(l)}(t, x) - F_1(t, L) \phi_1^{(l)}(t_1^{(l)}(y; t, x), 0) \]
\[ = \int_L^x \left( \frac{\beta(\tau, y)}{2} F_1(\tau, y) \left( \nu_1 (\phi^{(l-1)} + \Phi) - \nu_1 (\Phi) \right) (\phi_1^{(l-1)} + \phi_2^{(l-1)}) \right. \]
\[ + \nu_1 (\phi^{(l-1)} + \Phi) F_1(\tau, y) \phi_1^{(l)} \left| \int_y^L \frac{\partial_1 \beta(s, t)}{2} \nu_2 (\Phi) ds \right|_{\tau = \tau^{(l)}(y; t, x)} dy \]
\[ + \int_L^x - \frac{\partial}{\partial y} F_1(\tau^{(l)}(y; t, x), y) \phi_2^{(l-1)} \left( \tau^{(l)}(y; t, x), y \right) dy, \]

where \( F_1(t, L) = 1. \)

Using (1.2)-(1.3), (2.11), (3.19) and (3.35), we get

\[ \| \phi_i^{(l)} \|_{C^0} \leq \frac{M_1 \epsilon - 99 \epsilon}{F_1(t, x)} + C \epsilon^2 + \frac{F_1(t, x) - 1}{F_1(t, x)} M_1 \epsilon \]
\[ \leq \frac{M_1 \epsilon - 99 \epsilon}{F_1(t, x)} + \frac{\epsilon}{F_1(t, x)} + \frac{F_1(t, x) - 1}{F_1(t, x)} M_1 \epsilon \]
\[ = M_1 \epsilon - \frac{98 \epsilon}{F_1(t, x)} < M_1 \epsilon, \quad (3.37) \]
Differentiating equations (3.3), (3.5) with respect to $t$

Correspondingly, at the boundary $x$ and $t$

and unless specified, in this paper $C$ denotes a generic constant.

Similarly, we can get

$$
\|\phi_2^{(l)}\|_{C^0} < M_1 \epsilon.
$$

(3.38)

Now, we consider the temporal derivative estimates.

Let

$$z_i^{(l)} = \partial_t \phi_i^{(l)}, \quad i = 1, 2, \forall l \in \mathbb{N}.
$$

Differentiating equations (3.3), (3.5) with respect to $t$, we get

$$
\begin{align*}
\partial_t z_1^{(l)} &+ \nu_1 (\phi^{(l-1)} + \Phi) \partial_t z_1^{(l)} \\
&= \frac{\beta}{2} \nu_1 (\Phi) z_1^{(l)} + \frac{\partial_\beta}{2} \nu_1 (\Phi) \phi_1^{(l)} + \frac{\beta}{2} \nu_1 (\Phi) z_2^{(l-1)} + \frac{\partial_\beta}{2} \nu_1 (\Phi) \phi_2^{(l-1)} \\
&+ \frac{\beta}{2} \left( \nu_1 (\phi^{(l-1)} + \Phi) - \nu_1 (\Phi) \right) (z_1^{(l-1)} + z_2^{(l-1)}) \\
&+ \frac{\partial_\beta}{2} \left( \nu_1 (\phi^{(l-1)} + \Phi) - \nu_1 (\Phi) \right) (\phi_1^{(l-1)} + \phi_2^{(l-1)}) \\
&+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_1 (\phi^{(l-1)} + \Phi)}{\partial \phi_j} (z_j^{(l-1)} \phi_1^{(l-1)} + z_j^{(l-1)} \phi_2^{(l-1)}) \\
&- \sum_{j=1}^2 \frac{\partial \nu_1 (\phi^{(l-1)} + \Phi)}{\partial \phi_j} z_j^{(l-1)} z_1^{(l)}.
\end{align*}
$$

(3.39)

and

$$
\begin{align*}
\partial_t z_2^{(l)} &+ \nu_2 (\phi^{(l-1)} + \Phi) \partial_t z_2^{(l)} \\
&= \frac{\beta}{2} \nu_2 (\Phi) z_2^{(l)} + \frac{\partial_\beta}{2} \nu_2 (\Phi) \phi_2^{(l)} + \frac{\beta}{2} \nu_2 (\Phi) z_1^{(l-1)} + \frac{\partial_\beta}{2} \nu_2 (\Phi) \phi_1^{(l-1)} \\
&+ \frac{\beta}{2} \left( \nu_2 (\phi^{(l-1)} + \Phi) - \nu_2 (\Phi) \right) (z_1^{(l-1)} + z_2^{(l-1)}) \\
&+ \frac{\partial_\beta}{2} \left( \nu_2 (\phi^{(l-1)} + \Phi) - \nu_2 (\Phi) \right) (\phi_1^{(l-1)} + \phi_2^{(l-1)}) \\
&+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_2 (\phi^{(l-1)} + \Phi)}{\partial \phi_j} (z_j^{(l-1)} \phi_1^{(l-1)} + z_j^{(l-1)} \phi_2^{(l-1)}) \\
&- \sum_{j=1}^2 \frac{\partial \nu_2 (\phi^{(l-1)} + \Phi)}{\partial \phi_j} z_j^{(l-1)} z_2^{(l)}.
\end{align*}
$$

(3.40)

Correspondingly, at the boundary $x = 0$ and $x = L$, we have

$$
\begin{align*}
x = L : \quad & z_1^{(l)} (t, L) = \phi_{1_{\alpha}}^{(l)} (t) + \kappa_1 z_2^{(l-1)} (t, L), \quad t \in \mathbb{R},
\end{align*}
$$

(3.41)

and

$$
\begin{align*}
x = 0 : \quad & z_2^{(l)} (t, 0) = \phi_{2_{\alpha}}^{(l)} (t) + \kappa_2 z_1^{(l-1)} (t, 0), \quad t \in \mathbb{R}.
\end{align*}
$$

(3.42)
By (2.13) and (3.19), we get

\[ \| z_1^{(l)}(\cdot, L) \|_{C^0} \leq \varepsilon + |\kappa_1|M_1 \varepsilon \leq M_1 \varepsilon - 99 \varepsilon, \]  
(3.43)

\[ \| z_2^{(l)}(\cdot, 0) \|_{C^0} \leq \varepsilon + |\kappa_2|M_1 \varepsilon \leq M_1 \varepsilon - 99 \varepsilon. \]  
(3.44)

In order to overcome the complexity of linearized equations caused by the quasi-linear nature of the original system, we introduce the method of "estimation by twice integration". We first get a rough estimate of \( z_1^{(l)} \) as follows.

Multiplying \( \text{sgn}(z_1^{(l)}) \) on both sides of (3.39), we get

\[
\frac{\partial_x}{\partial x}|z_1^{(l)}| + \nu_1(\phi^{(l-1)} + \Phi) \partial_t |z_1^{(l)}| \\
= \left( \frac{\beta}{2} \nu_1(\Phi) - \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j^{(l-1)} \right) |z_1^{(l)}| + \frac{\partial_t}{\partial t} \text{sgn}(z_1^{(l)}) \nu_1(\Phi) \phi_1^{(l)} \\
+ \frac{\beta}{2} \text{sgn}(z_1^{(l)}) \nu_1(\Phi) z_2^{(l-1)} + \frac{\partial_t}{\partial t} \text{sgn}(z_1^{(l)}) \nu_1(\Phi) \phi_2^{(l-1)} \\
+ \frac{\beta}{2} \text{sgn}(z_1^{(l)}) \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) (z_1^{(l-1)} + z_2^{(l-1)}) \\
+ \frac{\partial_t}{\partial t} \text{sgn}(z_1^{(l)}) \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) (\phi_1^{(l-1)} + \phi_2^{(l-1)}) \\
+ \frac{\beta}{2} \text{sgn}(z_1^{(l)}) \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j^{(l-1)} \phi_1^{(l-1)} + z_j^{(l-1)} \phi_2^{(l-1)}.
\]

Using the sign of \( \beta(t, x) \) in (1.2) and noticing (2.11), (3.19), it is easy to see

\[
\frac{\beta}{2} \nu_1(\Phi) - \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j^{(l-1)} > 0.
\]

Then integrating the result along the 1st characteristic curve \( t = t_1^{(l)}(y; t, x) \) from \( L \) to \( x \) and using (1.2)-(1.3), (2.11), (3.19) and (3.43), one has

\[
|z_1^{(l)}(t, x)| \leq |z_1^{(l)}(t_1^{(l)}(L; t, x), L)| + \left( \frac{\beta}{2} |\nu_1(\Phi)| + |\partial_t \beta||\nu_1(\Phi)| \right) \int_{L}^{x} M_1 \varepsilon dy | + C \varepsilon^2 \\
\leq C_0 \varepsilon, \quad \forall (t, x) \in \mathbb{R} \times [0, L],
\]  
(3.45)

where \( C_0 > 0 \) is a constant independent of \( l \).
Next we transform equation (3.39) into the following equation of $F_1z_1^{(l)}$

$$\partial_t(F_1z_1^{(l)}) + \nu_1(\phi^{(l-1)} + \Phi)\partial_t(F_1z_1^{(l)})$$

$$= \frac{\partial \beta}{2} F_1 \nu_1(\Phi)\partial_t z_1^{(l)} + \frac{\beta}{2} F_1 \nu_1(\Phi)z_2^{(l-1)} + \frac{\partial \beta}{2} F_1 \nu_1(\Phi)\phi_2^{(l-1)}$$

$$+ \frac{\beta}{2} F_1 \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) \left( z_1^{(l-1)} + z_2^{(l-1)} \right)$$

$$+ \frac{\partial \beta}{2} F_1 \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) \left( \phi_1^{(l-1)} + \phi_2^{(l-1)} \right)$$

$$+ \frac{\beta}{2} F_1 \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} \left( z_j^{(l-1)} \phi_1^{(l-1)} + z_j^{(l-1)} \phi_2^{(l-1)} \right) - F_1 \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j^{(l-1)} z_1^{(l)}$$

$$+ \nu_1(\phi^{(l-1)} + \Phi) F_1z_1^{(l)} \int_x^L \frac{\partial \beta(t, s)}{\nu_1(\Phi)} ds,$$  \hspace{1cm} (3.46)

then integrate it along the 1st characteristic curve $t = t_1^{(l)}(y; t, x)$ from $L$ to $x$ to get

$$F_1(t, x)z_1^{(l)}(t, x) - F_1(t_1^{(l)}(L; t, x), L) z_1^{(l)}(t_1^{(l)}(L; t, x), L)$$

$$= \int_L^x \left( \frac{\partial}{\partial y} F_1(t_1^{(l)}(y; t, x), y) \right) z_2^{(l-1)}(t_1^{(l)}(y; t, x), y) dy$$

$$+ \int_L^x \left( \frac{\partial \beta}{\nu_1(\Phi)} + \frac{\partial \beta}{2} F_1 \nu_1(\Phi) \right) \phi_1^{(l-1)} + \frac{\beta}{2} F_1 \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) \left( z_1^{(l-1)} + z_2^{(l-1)} \right)$$

$$+ \frac{\partial \beta}{2} F_1 \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) \left( \phi_1^{(l-1)} + \phi_2^{(l-1)} \right)$$

$$+ \frac{\beta}{2} F_1 \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} \left( z_j^{(l-1)} \phi_1^{(l-1)} + z_j^{(l-1)} \phi_2^{(l-1)} \right) - F_1 \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j^{(l-1)} z_1^{(l)}$$

$$+ \nu_1(\phi^{(l-1)} + \Phi) F_1z_1^{(l)} \int_y^L \frac{\partial \beta(t, s)}{\nu_1(\Phi)} ds \right) (t_1^{(l)}(y; t, x), y) dy.$$  \hspace{1cm} (3.47)

By (1.2), (1.3), (2.11), (3.19), (3.43) and the rough estimate (3.45), we get

$$|z_1^{(l)}(t, x)| \leq \frac{1}{F_1(t, x)} (M_1 \epsilon - 99 \epsilon) + \frac{F_1(t, x) - 1}{M_1 \epsilon} M_1 \epsilon + C \epsilon^2$$

$$\leq M_1 \epsilon - \frac{99}{F_1(t, x) \epsilon} + C \epsilon^2$$

$$< M_1 \epsilon.$$  \hspace{1cm} (3.47)

Similarly, we have

$$|z_2^{(l)}(t, x)| < M_1 \epsilon.$$  \hspace{1cm} (3.48)
By applying the equations (3.3), (3.5) and noting (1.2), (2.11), (3.19), (3.37)-(3.38) and (3.47)-(3.48), we gain

$$
\| \partial_x \phi_l^{(i)} \|_{C^0} \leq A_0 M_1 \epsilon + \beta_4 A_0 M_1 \epsilon + C \epsilon^2 \\
\leq C_1 \epsilon,
$$

(3.49)

where we choose the constant $C_1 > A_0 M_1 + \beta_4 A_0 M_1$, which is independent of $l$.

**Step 3:** $\{ \phi_l^{(i)} \} (i = 1, 2)$ is a Cauchy sequence in $C^0$.

By (3.7), we get (3.15) for $l = 1$ from (3.37)-(3.38) directly. Next we prove (3.15) for $l \geq 2$. Select $\theta < 1$ satisfying

$$
\theta > |\kappa_1|, \quad \theta > |\kappa_2|.
$$

At the boundary $x = 0$ and $x = L$, by (3.21), we have

$$
\| \phi_l^{(i)} (t, L) - \phi_l^{(i-1)} (t, L) \|_{C^0} \leq |\kappa_1| \| \phi_l^{(i-1)} (t, L) - \phi_l^{(i-2)} (t, L) \|_{C^0} \\
\leq |\kappa_1| M_1 \epsilon \theta^{l-1},
$$

(3.50)

$$
\| \phi_l^{(i)} (t, 0) - \phi_l^{(i-1)} (t, 0) \|_{C^0} \leq |\kappa_2| \| \phi_l^{(i-1)} (t, 0) - \phi_l^{(i-2)} (t, 0) \|_{C^0} \\
\leq |\kappa_2| M_1 \epsilon \theta^{l-1}.
$$

(3.51)

In the domain $D$, from (3.3), we get

$$
\partial_x (\phi_l^{(i)} - \phi_l^{(i-1)}) + \nu_1 (\phi_l^{(i-1)} + \Phi) \partial_t (\phi_l^{(i)} - \phi_l^{(i-1)}) \\
= \frac{\beta}{2} \nu_1 (\Phi) (\phi_l^{(i)} - \phi_l^{(i-1)}) - \left( \nu_1 (\phi_l^{(i-1)} + \Phi) - \nu_1 (\phi_l^{(i-2)} + \Phi) \right) \partial_t \phi_l^{(i-1)} \\
+ \frac{\beta}{2} \left( \nu_1 (\phi_l^{(i-1)} + \Phi) - \nu_1 (\phi_l^{(i-2)} + \Phi) \right) (\phi_l^{(i-1)} - \phi_l^{(i-2)}) \\
+ \frac{\beta}{2} \nu_1 (\phi_l^{(i-1)} + \Phi) (\phi_l^{(i-1)} - \phi_l^{(i-2)}) \\
+ \frac{\beta}{2} \left( \nu_1 (\phi_l^{(i-1)} + \Phi) - \nu_1 (\phi_l^{(i-2)} + \Phi) \right) \phi_l^{(i-2)}.
$$
then we multiply $F_1(t, x)$ on both sides of the above equality to gain
\[
\begin{align*}
\partial_t \left(F_1(\phi_1^{(l)} - \phi_1^{(l-1)})\right) + \nu_1(\phi^{(l-1)} + \Phi)\partial_t \left(F_1(\phi_1^{(l)} - \phi_1^{(l-1)})\right) \\
= - F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)\partial_t \phi_1^{(l-1)} \\
+ \nu_1(\phi^{(l-1)} + \Phi)F_1(\phi_1^{(l)} - \phi_1^{(l-1)}) \int_x^L \frac{\partial t \beta(t, s)}{2} \nu_1(\Phi)ds \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)(\phi_1^{(l-1)} - \phi_1^{(l-2)}) \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)(\phi_2^{(l-1)} - \phi_2^{(l-2)}) \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)\phi_2^{(l-2)} \\
\end{align*}
\]
and integrate it along the 1st characteristic curve $t = t_1^{(l)}(y; t, x)$ from $L$ to $x$ to get
\[
F_1(t, x)(\phi_1^{(l)}(t, x) - \phi_1^{(l-1)}(t, x)) \\
= F_1(t, L)(\phi_1^{(l)}(t_1^{(l)}(L; t, x), L) - \phi_1^{(l-1)}(t_1^{(l)}(L; t, x), L)) \\
+ \int_L^x \left( - F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)\partial_t \phi_1^{(l-1)} \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)(\phi_1^{(l-1)} - \phi_1^{(l-2)}) \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)(\phi_2^{(l-1)} - \phi_2^{(l-2)}) \\
+ \frac{\beta}{2} F_1 \left(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\phi^{(l-2)} + \Phi)\right)\phi_2^{(l-2)} \\
\right) \\
+ \nu_1(\phi^{(l-1)} + \Phi)F_1(\phi_1^{(l)} - \phi_1^{(l-1)}) \int_y^L \frac{\partial t \beta(\cdot, s)}{2} \nu_1(\Phi)ds \left(t_1^{(l)}(y; t, x), y\right)dy \\
+ \int_L^x -\left(\frac{\partial t \beta}{\partial y}(\cdot, y)\right)(\phi_2^{(l-1)} - \phi_2^{(l-2)})(t_1^{(l)}(y; t, x), y)dy.
\]
By (3.19), (3.21) and (3.50), we have
\[
\|\phi_1^{(l)} - \phi_1^{(l-1)}\|_{C^0} \leq \frac{|\kappa|}{F_1(t, x)} + \frac{F_1(t, x) - 1}{F_1(t, x)} M_1 \epsilon^{\theta - 1} + Ce M_1 \epsilon^{\theta - 1} \\
\leq M_1 \epsilon^{\theta - 1} - \frac{1 - |\kappa|}{F_1(t, x)} M_1 \epsilon^{\theta - 1} + Ce M_1 \epsilon^{\theta - 1} \\
\leq M_1 \epsilon^\theta.
\]
Thus, we have
\[ \| (\phi_2^{(l)} - \phi_2^{(l-1)}) \|_{C^0} \leq M_1 \epsilon \theta^l. \]  
(3.53)

**Step 4:** The proof of \( (3.16) \) and \( (3.17) \).

Now, we show the modulus of continuity for \( z_i^{(l)}(i = 1, 2) \) on the temporal direction \( (3.16) \), which is very important to prove \( (3.17) \).

For \( \delta \in (0, 1) \), we choose
\[ \Omega(\delta) = \frac{24}{1 - \kappa} [A_0 + 1] (\sqrt{\kappa} \delta + \omega(\delta|\phi_{1x}^l) + \omega(\delta|\phi_{2x}^l) + \omega(\delta|\partial_t^l)). \]  
(3.54)

Since \( \omega(\delta|\phi_{ix}^l)(i = 1, 2) \) and \( \omega(\delta|\partial_t^l) \) are monotonically increasing, bounded and continuous concave functions of \( \delta \) and \( \lim_{\delta \to 0^+} \omega(\delta|\phi_{ix}^l) = 0 \), then \( \Omega(\delta) \) is also such a function and
\[ \lim_{\delta \to 0^+} \Omega(\delta) = 0. \]

At the boundary \( x = L \), for any given \( t_1, t_2 \in \mathbb{R} \) with \( |t_1 - t_2| \leq \delta \ll 1 \), one has
\[ |z_1^{(l)}(t_1, L) - z_1^{(l)}(t_2, L)| \leq |\phi_{1x}^l(t_1) - \phi_{1x}^l(t_2)| + |\kappa_1||z_2^{(l-1)}(t_1, L) - z_2^{(l-1)}(t_2, L)|. \]

Thus, by \( (3.22) \) and \( (3.54) \), we get
\[ \omega(\delta|z_1^{(l)}(\cdot, L)) \leq \omega(\delta|\phi_{1x}^l) + \kappa \omega(\delta|z_2^{(l-1)}(\cdot, L)) \]
\[ \leq \frac{1 - \kappa}{24[A_0 + 1]} \Omega(\delta) + \frac{\kappa}{8[A_0 + 1]} \Omega(\delta) \]
\[ = \left( \frac{1}{24[A_0 + 1]} + \frac{\kappa}{12[A_0 + 1]} \right) \Omega(\delta) \]
\[ < \frac{1}{8[A_0 + 1]} \Omega(\delta). \]  
(3.55)

In the domain \( D \), for any \( x \in [0, L] \) and \( t_1, t_2 \in \mathbb{R} \) with \( |t_1 - t_2| \leq \delta \), by the definition of the characteristic curve, one has
\[ \int_{t_1}^{t_2} \nu_1(\phi^{(l-1)}(t_1^{(l)}(\tilde{y}; t_*, x), \tilde{y}) + \Phi)d\tilde{y} + t_* . \]

Thus,
\[ |t_1^{(l)}(y; t_1, x) - t_1^{(l)}(y; t_2, x)| \]
\[ \leq |t_1 - t_2| + \int_{t_1}^{t_2} |\nu_1(\phi^{(l-1)}(t_1^{(l)}(\tilde{y}; t_1, x), \tilde{y}) + \Phi) - \nu_1(\phi^{(l-1)}(t_1^{(l)}(\tilde{y}; t_2, x), \tilde{y}) + \Phi)|d\tilde{y} \]
\[ \leq |t_1 - t_2| + \int_{t_1}^{t_2} \sum_{j=1}^2 \left| \frac{\partial \nu_1}{\partial \phi_j} \right| C_j^{(l-1)} \| C_\theta \| |t_1^{(l)}(\tilde{y}; t_1, x) - t_1^{(l)}(\tilde{y}; t_2, x)|d\tilde{y}, \]

then by the Gronwall’s inequality and \( (3.14) \), we have
\[ |t_1^{(l)}(y; t_1, x) - t_1^{(l)}(y; t_2, x)| \leq e^{C\epsilon}|t_1 - t_2| \leq (1 + \sqrt{\epsilon})|t_1 - t_2|. \]  
(3.56)
Noticing the concavity of $\Omega(\delta)$, we have
\[
\frac{1}{1+\sqrt{\varepsilon}}\Omega((1+\sqrt{\varepsilon})\delta) + \frac{\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}\Omega(0) \leq \Omega(\delta),
\]
then
\[
\Omega((1+\sqrt{\varepsilon})\delta) \leq (1+\sqrt{\varepsilon})\Omega(\delta).
\]
Thus, by (3.22) we get
\[
\frac{1}{8[A_0+1]}\Omega((1+\sqrt{\varepsilon})\delta) \leq \frac{1}{8[A_0+1](1+\sqrt{\varepsilon})\Omega(\delta)}.
\]
Integrating (3.39) along $t = t_1^{(l)}(y; t_1, x)$ and $t = t_1^{(l)}(y; t_2, x)$ respectively and then subtracting the two results, we get
\[
z_1^{(l)}(t_1, x) - z_1^{(l)}(t_2, x) = z_1^{(l)}(t_1^{(l)}(L; t_1, x), L) - z_1^{(l)}(t_1^{(l)}(L; t_2, x), L)
\]
\[\quad + \int_L^\infty \frac{\beta}{2} \nu_1(\Phi)z_1^{(l)} \left|\left| (t_1^{(l)}(y; t_1, x), y) - z_1^{(l)}(t_1^{(l)}(y; t_2, x), y) \right|\right| dy + \int_L^\infty \frac{\beta}{2} \nu_1(\Phi)z_2^{(l-1)} \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad + \int_L^\infty \frac{\partial_1 \beta(t_1^{(l)}(y; t_1, x), y) - \partial_1 \beta(t_1^{(l)}(y; t_2, x), y)}{2} \nu_1(\Phi)\phi_1^{(l)} \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad + \int_L^\infty \frac{\partial_1 \beta(t_1^{(l)}(y; t_2, x), y) - \partial_1 \beta(t_1^{(l)}(y; t_2, x), y)}{2} \nu_1(\Phi)\phi_1^{(l)} \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad + \nu_1(\Phi)\left(\frac{\phi_1^{(l-1)} + \phi_2^{(l-1)}}{2}\right) \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad + \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_1, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad - \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_1, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad + \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_1, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad - \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_1, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_1, x), y) \right|\right| dy\]
\[\quad + \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad - \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad + \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad - \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad + \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
\[\quad - \nu_1(\phi^{(l-1)}(t_1^{(l)}(y; t_2, x), y) + \Phi) \left|\left| (t_1^{(l)}(y; t_2, x), y) \right|\right| dy\]
By (1.2)-(1.3), (2.11), (3.19), (3.55)-(3.57) and Gronwall’s inequality, we have

\[ |z^{(l)}_1(t_1, x) - z^{(l)}_1(t_2, x)| \leq \frac{1}{8|A_0 + 1|} C_2 \Omega(\delta), \quad (3.58) \]

where \( C_2 > 0 \) is a constant independent of \( l \).

Using the integral expression (3.10) of \( F_1(t, x)z^{(l)}_1(t, x) \) and then by (1.2)-(1.3),
we have
\[
|z_1^{(l)}(t_1, x) - z_1^{(l)}(t_2, x)| \leq \frac{1}{F_1(t, x)} \left( \frac{1}{24|A_0 + 1|} + \frac{1}{12|A_0 + 1|} \right) (1 + \sqrt{c}) \Omega(\delta)
\]
\[
+ \frac{F_1(t_1, x) - 1}{F_1(t, x)} \frac{1}{8|A_0 + 1|} (1 + \sqrt{c}) \Omega(\delta) + C \varepsilon^2 (1 + \sqrt{c}) |t_1 - t_2|
\]
\[
+ C \varepsilon \left( \frac{1 + \sqrt{c}}{8|A_0 + 1|} \Omega(\delta) + \frac{1}{8|A_0 + 1|} \right) C \epsilon \Omega(\delta)
\]
\[
\leq \frac{1}{8|A_0 + 1|} \Omega(\delta) - \frac{1 - \kappa}{12|A_0 + 1|} - \sqrt{c} \Omega(\delta)
\]
\[
< \frac{1}{8|A_0 + 1|} \Omega(\delta).
\]

Finally, we prove (3.17). We first consider the special case that two given points \((t_1, x_1)\) and \((t_2, x_2)\) with \(|t_1 - t_2| \leq \delta, |x_1 - x_2| \leq \delta\) locate on the same characteristic curve \(t = t_1^{(l)}(x; t_0, x_0)\), namely, \(t_2 = t_1^{(l)}(x; t_1, x_1)\). Using the similar method of (3.17), we can get
\[
|z_1^{(l)}(t_1, x_1) - z_1^{(l)}(t_2, x_2)| \leq C \varepsilon \delta \leq \frac{1}{12} \Omega(\delta).
\]

Then, for general two points \((t_1, x_1)\) and \((t_2, x_2)\) with \(|t_1 - t_2| \leq \delta, |x_1 - x_2| \leq \delta\), we can choose a point \((t_3, x_1)\) locating on the 1st characteristic curve passing through \((t_2, x_2)\), namely, \(t_3 = t^{(l)}_1(x_1; t_2, x_2)\).

By definition (3.24) and (2.11), we have
\[
|t_3 - t_2| \leq |t_3 - t_1|, |x_1 - x_2| \leq A_0 \delta,
\]
and thus
\[
|t_3 - t_1| \leq |t_3 - t_2| + |t_2 - t_1| \leq (A_0 + 1) \delta.
\]

Now we combine estimates (3.59)-(3.60) to get
\[
|z_1^{(l)}(t_1, x_1) - z_1^{(l)}(t_2, x_2)| \leq \frac{[A_0 + 1] + 1}{8|A_0 + 1|} \Omega(\delta) + \frac{1}{12} \Omega(\delta)
\]
\[
\leq \frac{1}{3} \Omega(\delta).
\]
The combination of (3.60) and (3.61) leads to
\[ \varpi(\delta|z_1^{(l)}) \leq \frac{1}{3} \Omega(\delta). \] (3.62)
In a similar way, we obtain
\[ \varpi(\delta|z_2^{(l)}) \leq \frac{1}{3} \Omega(\delta). \] (3.63)
By the aid of equations (3.3), (3.5) and by (1.2), (2.11), (3.19), (3.23), (3.54) and (3.62)-(3.63), we have
\[ \varpi(\delta|\partial_\tau \phi_i^{(l)}) \leq A_0 \frac{1}{3} \Omega(\delta) + C\epsilon^2 \delta + C\epsilon \delta \]
\[ \leq A_0 \frac{1}{2} \Omega(\delta), \quad i = 1, 2. \] (3.64)
Thus, (3.62)-(3.64) indicate (3.17).

With the help of Proposition 3.1 and the similar arguments as in [10], the proof of Theorem 2.1 could be presented, here we omit the details.

4 Stability of the Time-periodic Solution

In this section, we give the proof of Theorem 2.2 to consider the stability of the time-periodic solution obtained in Theorem 2.1. For the sake of proving the existence of the classical solutions \( \phi = \phi(t, x) \), we only need to prove the following Lemma 4.1 on the basis of the existence and uniqueness of local \( C^1 \) solution for the mixed initial-boundary value problem for quasilinear hyperbolic system (cf. Chapter 4 in [1]). Using the method in [3], we can give the proof of Lemma 4.1.

Here we omit the details.

**Lemma 4.1.** There exists a small constant \( \epsilon_0 > 0 \), for any given \( \epsilon \in (0, \epsilon_0) \), there exists \( \sigma = \sigma(\epsilon) > 0 \) such that if
\[ \|\phi_i\|_{C^1(\mathbb{R}_+)} \leq \sigma, \quad i = 1, 2, \]
\[ \|\phi_0\|_{C^1[0, L]} \leq \sigma, \]
then the \( C^1 \) solution \( \phi = \phi(t, x) \) to the initial-boundary value problem (2.7)-(2.10) satisfies
\[ \|\phi\|_{C^1(D)} \leq \epsilon. \] (4.1)

Now, we prove (2.17) inductively. For any \( t_* > 0 \) and \( N \in \mathbb{N} \), we prove
\[ \max_{i=1, 2} \|\phi_i(t, \cdot) - \phi_i(T_0)(t, \cdot)\|_{C^0} \leq C_N \epsilon \xi^{N+1}, \quad \forall t \in [t_* + T_0, t_* + 2T_0], \] (4.2)
under the hypothesis
\[ \max_{i=1,2} \| \phi_i(t, \cdot) - \phi_i^{(T_\tau)}(t, \cdot) \|_{C^0} \leq C_S \epsilon \xi^N, \quad \forall t \in [t_*, t_* + T_0], \quad (4.3) \]
where \( \xi \in (0, 1) \) is a constant to be determined later and \( \phi_i^{(T_\tau)}(t, x), i = 1, 2 \) is the time-periodic solution obtained in Theorem 2.1. Let
\[ \theta(t) = \max_{1 \leq i \leq 2} \sup_{x \in [0, L]} | \phi_i(t, x) - \phi_i^{(T_\tau)}(t, x) |, \]
Obviously, \( \theta(t) \) is continuous and
\[ \theta(t_* + T_0) \leq C_S \epsilon \xi^N \]
follows (4.3). Then it’s just necessary to prove
\[ \theta(t) \leq C_S \epsilon \xi^{N+1}, \quad \forall t \in [t_* + T_0, \tau] \quad (4.4) \]
under the assumption
\[ \theta(t) \leq C_S \epsilon \xi^N, \quad \forall t \in [t_*, \tau] \quad (4.5) \]
for any \( \tau \in [t_* + T_0, t_* + 2T_0] \).

At the boundary \( x = L \), one has
\[ \phi_1(t, L) - \phi_1^{(T_\tau)}(t, L) = \kappa_1(\phi_2(t, L) - \phi_2^{(T_\tau)}(t, L)), \]
then from (4.5), we have
\[ | \phi_1(t, L) - \phi_1^{(T_\tau)}(t, L) | \leq | \kappa_1 | C_S \epsilon \xi^N. \quad (4.6) \]
Similarly, at the boundary \( x = 0 \), we get
\[ | \phi_2(t, 0) - \phi_2^{(T_\tau)}(t, 0) | \leq | \kappa_2 | C_S \epsilon \xi^N. \quad (4.7) \]
As for the interior estimates, we have
\[ \partial_x \phi_1^{(T_\tau)} + \nu_1(\phi^{(T_\tau)} + \Phi) \partial_t \phi_1^{(T_\tau)} = \frac{\beta(t, x)}{2} \nu_1(\Phi) \phi_1^{(T_\tau)} + \frac{\beta(t, x)}{2} (\nu_1(\phi^{(T_\tau)} + \Phi) - \nu_1(\Phi) \phi_1^{(T_\tau)} + \nu_1(\phi^{(T_\tau)} + \Phi) \phi_2^{(T_\tau)}). \quad (4.8) \]
Then by (4.1) and (4.8), we get
\[ \partial_x (\phi_1 - \phi_1^{(T_\tau)}) + \nu_1(\phi + \Phi) \partial_t (\phi_1 - \phi_1^{(T_\tau)}) = - \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_\tau)} + \Phi) \right) \partial_t \phi_1^{(T_\tau)} + \frac{\beta(t, x)}{2} \nu_1(\Phi) (\phi_1 - \phi_1^{(T_\tau)}) + \frac{\beta(t, x)}{2} (\nu_1(\phi + \Phi) - \nu_1(\Phi)) (\phi_1^{(T_\tau)} + \phi_2^{(T_\tau)}), \quad (4.9) \]
Multiplying $F_1(t, x)$ on both sides of (4.9), we gain
\[
\partial_x \left( F_1(\phi_1 - \phi_1^{(T_i)}) \right) + \nu_1(\phi + \Phi) \partial_t \left( F_1(\phi_1 - \phi_1^{(T_i)}) \right) \\
= -F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\phi(T_i) + \Phi) \right) \partial_t \phi_1^{(T_i)} \\
+ \nu_1(\phi + \Phi) F_1(\phi_1 - \phi_1^{(T_i)}) \int_x^T \frac{\partial_t \beta(t, s)}{2} \nu_1(\Phi) ds \\
+ \frac{\beta(t, x)}{2} F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\Phi) \right) \left( (\phi_1 + \phi_2) - (\phi_1^{(T_i)} + \phi_2^{(T_i)}) \right) \\
+ \frac{\beta(t, x)}{2} F_1 \nu_1(\Phi)(\phi_2 - \phi_2^{(T_i)}),
\]
then integrate the result along the 1st characteristic curve $t = t_1(x; \hat{t}, \hat{x})$ defined by
\[
\begin{align*}
\frac{dt_1}{dx}(x; \hat{t}, \hat{x}) &= \nu_1(\phi(t_1(x; \hat{t}, \hat{x}), x) + \Phi), \\
t_1(\hat{t}; \hat{t}, \hat{x}) &= \hat{t}.
\end{align*}
\]
Noting $T_0 = L \max_{i=1,2} \sup_{\phi \in \Psi} |\nu_i(\phi + \Phi)|$, for each points $(\hat{t}, \hat{x}) \in [t_*, T_0 \times [0, L]$, the backward curve $t = t_1(x; \hat{t}, \hat{x})$ will intersect the boundary in a time interval shorter than $T_0$, namely,
\[
t_1(L; \hat{t}, \hat{x}) \in [\hat{t} - T_0, \hat{t}] \subseteq [t_*, \tau], \quad \forall (\hat{t}, \hat{x}) \in [t_*, T_0 \times [0, L],
\]
and thus we can use estimates (4.6)–(4.7) on the boundary.

Using (1.2), (2.10) and (4.5)–(4.6), we get
\[
|\phi_1(\hat{t}, \hat{x}) - \phi_1^{(T_i)}(\hat{t}, \hat{x})| \\
\leq \frac{|\kappa_1|}{F_1(\hat{t}, \hat{x})} C S \epsilon \xi_N + C \epsilon C S \epsilon \xi_N + \frac{F_1(\hat{t}, \hat{x}) - 1}{F_1(\hat{t}, \hat{x})} C S \epsilon \xi_N \\
\leq C S \epsilon \xi^{N+1}.
\]
Here we choose a constant $0 < \xi < 1$ satisfying
\[
\frac{|\kappa_1|}{F_1(\hat{t}, \hat{x})} + C \epsilon + \frac{F_1(\hat{t}, \hat{x}) - 1}{F_1(\hat{t}, \hat{x})} \leq \xi.
\]
Similarly, we get
\[
|\phi_2(\hat{t}, \hat{x}) - \phi_2^{(T_i)}(\hat{t}, \hat{x})| \leq C S \epsilon \xi^{N+1}.
\]
Thus, we have
\[
\theta(\hat{t}) \leq C S \epsilon \xi^{N+1}.
\]
Since $\hat{t} \in [t_*, T_0, \tau]$ is arbitrary, we get (4.4).
5 Regularity of the Time-periodic Solution

In this section, we will prove higher regularity of the time-periodic solutions, provided that all boundary functions \( m_i(t)(i = 1, 2) \) possess higher regularity.

In order to get the regularity of \( \phi^{(T)} \), we use the iteration scheme (3.3)-(3.6) introduced in Section 3 and prove the following proposition.

**Proposition 5.1.** For the iteration scheme (3.3)-(3.6), assuming that (2.18) holds, then exists a large enough constant \( C_R > 0 \), such that for any given \( l \in \mathbb{N}_+ \), we have

\[
\| \partial^2_t \phi^{(l)}_i \|_{L^\infty} \leq C_R, \tag{5.1}
\]
\[
\| \partial_t \partial_x \phi^{(l)}_i \|_{L^\infty} \leq A_0 C_R, \tag{5.2}
\]
\[
\| \partial^2_x \phi^{(l)}_i \|_{L^\infty} \leq A_0^2 C_R \tag{5.3}
\]

under the hypothesis

\[
\| \partial^2_t \phi^{(l-1)}_i \|_{L^\infty} \leq C_R < +\infty, \tag{5.4}
\]
\[
\| \partial_t \partial_x \phi^{(l-1)}_i \|_{L^\infty} \leq A_0 C_R, \tag{5.5}
\]
\[
\| \partial^2_x \phi^{(l-1)}_i \|_{L^\infty} \leq A_0^2 C_R. \tag{5.6}
\]

**Proof.** Since we use actually the same sequence constructed in Section 3. By Proposition 3.1, we already have (3.13)-(3.17) for each \( l \) and especially,

\[
\| \phi^{(l)} \|_{C^1} \leq (C_1 + M_1) \epsilon, \quad \| \phi^{(l-1)} \|_{C^1} \leq (C_1 + M_1) \epsilon. \tag{5.7}
\]

Let

\[
\phi^{(l)}_i = \partial_t z^{(l)}_i = \partial^2_t \phi^{(l)}_i, \quad i = 1, 2; l \in \mathbb{N}_+.
\]

First, by the aid of boundary conditions (3.41) and (3.42) and using (2.18) and (5.1), we have

\[
|\phi^{(l)}_1(t, L)| \leq M_2 + |\kappa_1| C_R, \tag{5.8}
\]
\[
|\phi^{(l)}_2(t, 0)| \leq M_2 + |\kappa_2| C_R. \tag{5.9}
\]

Then we use a method similar to (3.47) to get

\[
\| \phi^{(l)}(t, x) \|_{L^\infty} \leq h_1 C_R, \tag{5.10}
\]

where constant \( 0 < h_1 < 1 \) is independent of \( l \).

Similarly, we have

\[
\| \phi^{(l)}_2(t, x) \|_{L^\infty} \leq h_1 C_R. \tag{5.11}
\]

Next, using (3.39) and (3.40) and by (1.2), (1.3), (2.11), (3.13) and (5.10)-(5.11), we get

\[
\| \partial_x \partial_t \phi^{(l)}_i \|_{L^\infty} \leq A_0 h_1 C_R + C \epsilon + C \epsilon^2 \leq A_0 h_2 C_R, \tag{5.12}
\]

22
where \( h_2 \in (h_1, 1) \) is a constant independent of \( l \).

Then taking the spatial derivative to \( t \), we have

\[
\partial_x^2 \phi^{(l)}_1 = -\nu_1(\phi^{(l-1)} + \Phi)\partial_x \partial_t \phi^{(l)}_1 - \sum_{j=1}^{2} \frac{\partial \nu_1(\phi^{(l-1)} + \Phi)}{\partial \phi_j} \partial_x \phi^{(l-1)} z^{(l)}_1 \\
+ \frac{\beta}{2} \nu_1(\Phi) \partial_x \phi^{(l-1)}_1 + \frac{\beta}{2} \sum_{j=1}^{2} \frac{\partial \nu_1(\phi^{(l-1)} + \Phi)}{\partial \phi_j} \partial_x \phi^{(l-1)}_1 \phi^{(l-1)}_1 \\
+ \frac{\partial_x \beta}{2} \nu_1(\Phi) \phi^{(l-1)}_1 + \frac{\beta}{2}(\nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi)) \phi^{(l-1)}_1 \phi^{(l-1)}_1 \\
+ \frac{\partial_x \beta}{2} \left( \nu_1(\phi^{(l-1)} + \Phi) - \nu_1(\Phi) \right) \phi^{(l-1)}_1 + \frac{\beta}{2} \nu_1(\phi^{(l-1)} + \Phi) \partial_x \phi^{(l-1)}_2 \\
+ \frac{\beta}{2} \sum_{j=1}^{2} \frac{\partial \nu_1(\phi^{(l-1)} + \Phi)}{\partial \phi_j} \partial_x \phi^{(l-1)}_1 \phi^{(l-1)}_2 + \frac{\partial_x \beta}{2} \nu_1(\phi^{(l-1)} + \Phi) \phi^{(l-1)}_2. 
\]

Thus, using \( 1.2-1.3, 2.11, 3.13-3.14, 5.7 \) and \( 6.12 \), we get

\[
\| \partial_x^2 \phi^{(l)}_1 \|_{L^\infty} \leq A_0^2 h_2 C_R + C \epsilon \\
\leq A_0^2 C_R. 
\]

In a similar way, we also get

\[
\| \partial_x^2 \phi^{(l)}_2 \|_{L^\infty} \leq A_0^2 C_R. 
\]

**The Proof of Theorem 2.3.** By \( 5.1-5.3 \), we know that \( \{ \phi^{(l)} \}_{l=1}^\infty \) is uniformly \( W^{2,\infty} \) bounded and then \( weak^* \) convergent. Moreover, noting that \( \{ \phi^{(l)} \}_{l=1}^\infty \) converges strongly to \( \phi^{(T^\ast)} \) in \( C^1 \), so we get the \( W^{2,\infty} \) regularity of \( \phi^{(T^\ast)} \).

### 6 Boundary Stabilization around the Time-periodic Solution

In this section, we will give the proof of Theorem 2.4.

Noting that we have got the \( C^0 \) exponential convergence in Theorem 2.2 as follows:

\[
\| \phi(t, \cdot) - \phi^{(T^\ast)}_l(t, \cdot) \|_{C^0} \leq C_S \epsilon \xi^N, \quad \forall t \in [NT_0, (N + 1)T_0), \forall l \in \mathbb{N}_+, 
\]

which also shows that

\[
\| \phi(t, \cdot) - \phi^{(T^\ast)}_l(t, \cdot) \|_{C^0} \leq C_S \epsilon \xi^{N+1}, \quad \forall t \in [(N + 1)T_0, (N + 2)T_0), \forall l \in \mathbb{N}_+. 
\]

Moreover, by Theorem 2.1, Theorem 2.3 and Lemma 4.1, we have

\[
\| \phi \|_{C^1} \leq C_S \epsilon, \quad \| \phi^{(T^\ast)} \|_{C^1} \leq C_E \epsilon, \quad \| \phi^{(T^\ast)} \|_{W^{2,\infty}} \leq (1 + A_0)^2 C_R. 
\]

23
By the continuity, we will inductively get the estimates for the convergence of the first derivatives, namely, for each \( N \in \mathbb{N}, \) and \( \tau \in [(N+1)T_0, (N+2)T_0], \) we will prove
\[
\| \partial \phi_i(t, \cdot) - \partial \phi_i^{(T_r)}(t, \cdot) \|_{C^0} \leq C_S^* \epsilon x^{N+1}, \quad \forall t \in [(N+1)T_0, \tau], \forall l \in \mathbb{N}_+, \quad (6.4)
\]
\[
\| \partial_x \phi_i(t, \cdot) - \partial_x \phi_i^{(T_r)}(t, \cdot) \|_{C^0} \leq A_0 C_S^* \epsilon x^{N+1}, \quad \forall t \in [(N+1)T_0, \tau], \forall l \in \mathbb{N}_+. \quad (6.5)
\]
under the assumption
\[
\| \partial_t \phi_i(t, \cdot) - \partial_t \phi_i^{(T_r)}(t, \cdot) \|_{C^0} \leq C_S^* \epsilon x^N, \quad \forall t \in [NT_0, \tau], \forall l \in \mathbb{N}_+, \quad (6.6)
\]
\[
\| \partial_x \phi_i(t, \cdot) - \partial_x \phi_i^{(T_r)}(t, \cdot) \|_{C^0} \leq A_0 C_S^* \epsilon x^N, \quad \forall t \in [NT_0, \tau], \forall l \in \mathbb{N}_+. \quad (6.7)
\]

Let
\[
z_i = \partial_t \phi_i, \quad w_i = \partial_x \phi_i
\]
and
\[
z_i^{(T_r)} = \partial_t \phi_i^{(T_r)}, \quad w_i^{(T_r)} = \partial_x \phi_i^{(T_r)}.
\]

Taking the temporal derivative on boundary conditions \((6.9)\) and \((6.10)\), we get
\[
\begin{align*}
    z_2(t, 0) &= \phi_2'(t) + \kappa_2 z_1(t, 0), \quad t > 0, \\
    z_1(t, L) &= \phi_1'(t) + \kappa_1 z_2(t, L), \quad t > 0
\end{align*}
\]
and
\[
\begin{align*}
    z_2^{(T_r)}(t, 0) &= \phi_2'(t) + \kappa_2 z_1^{(T_r)}(t, 0), \quad t > 0, \\
    z_1^{(T_r)}(t, L) &= \phi_1'(t) + \kappa_1 z_2^{(T_r)}(t, L), \quad t > 0.
\end{align*}
\]

Thus, on the boundary \( x = 0, \) we have
\[
\sup_{t \in [NT_0, \tau]} |z_2(t, 0) - z_2^{(T_r)}(t, 0)| \leq |\kappa_2||z_1(t, 0) - z_1^{(T_r)}(t, 0)|
\]
\[
\leq |\kappa_2|C_S^* \epsilon x^N. \quad (6.8)
\]

Similarly, on \( x = L, \) we have
\[
\sup_{t \in [NT_0, \tau]} |z_1(t, L) - z_1^{(T_r)}(t, L)| \leq |\kappa_1|C_S^* \epsilon x^N. \quad (6.9)
\]

In the domain, we take the temporal derivative of \((3.1)\) and \((4.8)\) to get
\[
\begin{align*}
\partial_x z_1 + \nu_1(\phi + \Phi) \partial_t z_1 \\
+ \frac{\beta}{2} \nu_1(\phi + \Phi) z_1 + \frac{\beta}{2} \nu_1(\Phi) \phi_1 + \frac{\beta}{2} \nu_1(\Phi) z_2 + \frac{\beta}{2} \nu_1(\Phi) \phi_2 \\
+ \frac{\beta}{2} \left( \nu_1(\phi + \Phi) - \nu_1(\Phi) \right) (z_1 + z_2) \\
+ \frac{\beta}{2} \left( \nu_1(\phi + \Phi) - \nu_1(\Phi) \right) (\phi_1 + \phi_2) \\
+ \frac{\beta}{2} \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} (z_j \phi_1 + z_j \phi_2) - \frac{\nu_1}{2} \sum_{j=1}^{2} \frac{\partial \nu_1}{\partial \phi_j} z_j z_1,
\end{align*}
\]
\[
(6.10)
\]
\[
24
\]
Furthermore, by (6.10)-(6.11), we get

\[
\partial_t z_1^{(T_1)} + \nu_1(\phi^{(T_1)} + \Phi)\partial_t z_1^{(T_1)} \\
= \frac{\beta}{2} \nu_1(\Phi)z_1^{(T_1)} + \frac{\partial \beta}{2} \nu_1(\Phi)\phi_1^{(T_1)} + \frac{\beta}{2} \nu_1(\Phi)\phi_2^{(T_1)} + \frac{\partial \beta}{2} \nu_1(\Phi)\phi_2^{(T_1)} \\
+ \frac{\beta}{2} \left( \nu_1(\phi^{(T_1)} + \Phi) - \nu_1(\Phi) \right) \left( z_1^{(T_1)} + z_2^{(T_1)} \right) \\
+ \frac{\partial \beta}{2} \left( \nu_1(\phi^{(T_1)} + \Phi) - \nu_1(\Phi) \right) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right) \\
+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (z_j^{(T_1)} \phi_1^{(T_1)} + z_j^{(T_1)} \phi_2^{(T_1)}) - \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (z_j^{(T_1)} z_1^{(T_1)}). \quad (6.11)
\]

Furthermore, by (6.10), (6.11), we get

\[
\partial_t (z_1 - z_1^{(T_1)}) + \nu_1(\phi + \Phi)\partial_t (z_1 - z_1^{(T_1)}) \\
= - \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \partial_t z_1^{(T_1)} + \frac{\beta}{2} \nu_1(\Phi)(z_1 - z_1^{(T_1)}) \\
+ \frac{\partial \beta}{2} \nu_1(\Phi)(1 - \phi_1^{(T_1)}) + \frac{\beta}{2} \nu_1(\Phi)(z_2 - z_2^{(T_1)}) + \frac{\partial \beta}{2} \nu_1(\Phi)(\phi_2^{(T_1)}) \\
+ \frac{\beta}{2} \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \left( z_1 + z_2 \right) - \left( z_1^{(T_1)} + z_2^{(T_1)} \right) \\
+ \frac{\beta}{2} \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right) \\
+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) z_j \left( (1 - \phi_2) - (\phi_1^{(T_1)} + \phi_2^{(T_1)}) \right) \\
+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) (z_j - z_j^{(T_1)}) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right) \\
+ \frac{\beta}{2} \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) \left( \phi_2^{(T_1)} \right) \left( z_j^{(T_1)} \phi_j^{(T_1)} + \phi_2^{(T_1)} \right) \\
- \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) z_j (z_1 - z_1^{(T_1)}) - \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) (z_j - z_j^{(T_1)}) z_1^{(T_1)} \\
- \sum_{j=1}^2 \left( \frac{\partial \nu_1}{\partial \phi_j} (\phi + \Phi) - \frac{\partial \nu_1}{\partial \phi_j} (\phi^{(T_1)} + \Phi) \right) z_j^{(T_1)} z_1^{(T_1)}. \quad (6.12)
\]
And we multiply $F_1(t, x)$ on both sides of (6.12) to gain

$$
\partial_x \left( F_1(z_1 - z_1^{(T_1)}) \right) + \nu_1(\phi + \Phi) \partial_t \left( F_1(z_1 - z_1^{(T_1)}) \right)
$$

$$
= - F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \partial_t z_1^{(T_1)}
$$

$$
+ \nu_1(\phi + \Phi) F_1(z_1 - z_1^{(T_1)}) \int_x^L \frac{\partial \beta(t, s)}{2} \nu_1(\Phi) ds
$$

$$
+ \frac{\partial \beta}{2} F_1 \nu_1(\Phi)(\phi_1 - \phi_1^{(T_1)}) + \frac{\partial F_1}{\partial x}(z_2 - z_2^{(T_1)})
$$

$$
+ \frac{\partial \beta}{2} F_1 \nu_1(\Phi)(\phi_2 - \phi_2^{(T_1)})
$$

$$
+ \frac{\beta}{2} F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\Phi) \right) \left( (z_1 + z_2) - (z_1^{(T_1)} + z_2^{(T_1)}) \right)
$$

$$
+ \frac{\beta}{2} F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \left( z_1^{(T_1)} + z_2^{(T_1)} \right)
$$

$$
+ \frac{\partial \beta}{2} F_1 \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T_1)} + \Phi) \right) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right)
$$

$$
+ \frac{\beta}{2} F_1 \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi) z_j \left( (\phi_1 + \phi_2) - (\phi_1^{(T_1)} + \phi_2^{(T_1)}) \right)
$$

$$
+ \frac{\beta}{2} F_1 \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi)(z_j - z_j^{(T_1)}) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right)
$$

$$
+ \frac{\beta}{2} F_1 \sum_{j=1}^2 \left( \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi) - \frac{\partial \nu_1}{\partial \phi_j}(\phi^{(T_1)} + \Phi) \right) \left( z_j^{(T_1)} \right) \left( \phi_1^{(T_1)} + \phi_2^{(T_1)} \right)
$$

$$
- F_1 \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi) z_j \left( z_1 - z_1^{(T_1)} \right)
$$

$$
- F_1 \sum_{j=1}^2 \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi)(z_j - z_j^{(T_1)}) z_1^{(T_1)}
$$

$$
- F_1 \sum_{j=1}^2 \left( \frac{\partial \nu_1}{\partial \phi_j}(\phi + \Phi) - \frac{\partial \nu_1}{\partial \phi_j}(\phi^{(T_1)} + \Phi) \right) z_j^{(T_1)} z_1^{(T_1)}.
$$

Then we integrate it along the corresponding backward characteristic curve $t = t_1(x; \hat{t}, \hat{x})$ defined by (1.9) and by (1.2)-(1.3), (2.11), (6.1)-(6.3), (6.6)-(6.7).
and (6.9), we have

\begin{align*}
|z_1(\hat{t}, \hat{x}) - z_1^{(T)}(\hat{t}, \hat{x})| &
\leq \frac{|\kappa | C_S^* \xi^N}{F_1(\hat{t}, \hat{x})} + C\epsilon C_S^* \xi^N + C\epsilon C_S^* \xi^N \\
&
+ \frac{F_1(\hat{t}, \hat{x}) - 1}{F_1(\hat{t}, \hat{x})} C_S^* \xi^N + F_1(\hat{t}, \hat{x}) \sup_{\phi \in \Psi} |\nabla \nu_1| C_R C_S \epsilon \xi^N.
\end{align*}

Here we choose constant

\[ C_S^* > 30 \max_{1 \leq i \leq 2} \sup_{\phi \in \Psi} |\nabla \nu_1| C_R C_S + 30 \beta^* A_0 C_S + C_S, \]

then we can get

\[ |z_1(\hat{t}, \hat{x}) - z_1^{(T)}(\hat{t}, \hat{x})| \leq h_3 C_S^* \xi^{N+1}, \quad \forall (\hat{t}, \hat{x}) \in [(N + 1)T_0, \tau] \times [0, L], \]

where constant \(0 < h_3 < 1\).

Similarly, we get

\[ |z_2(\hat{t}, \hat{x}) - z_2^{(T)}(\hat{t}, \hat{x})| \leq h_3 C_S^* \xi^{N+1}, \quad \forall (\hat{t}, \hat{x}) \in [(N + 1)T_0, \tau] \times [0, L]. \]

Thus, by the arbitrariness of \( \hat{t} \in [(N + 1)T_0, \tau] \), we have

\[ \|z_i(t, \cdot) - z_i^{(T)}(\cdot, \cdot)\|_{C^\alpha} \leq h_3 C_S^* \xi^{N+1}, \quad \forall t \in [(N + 1)T_0, \tau]. \quad (6.13) \]

Using (6.1) and (4.8), we get

\begin{align*}
w_1(t, \cdot) - w_1^{(T)}(\cdot, \cdot) &
= \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T)} + \Phi) \right) z_1 - \nu_1(\phi^{(T)} + \Phi) (z_1 - z_1^{(T)}) \\
&
+ \frac{\beta}{2} \nu_1(\phi - \phi \epsilon^{(T)}) + \frac{\beta}{2} \left( \nu_1(\phi + \Phi) - \nu_1(\phi^{(T)} + \Phi) \right) (\phi_1 + \phi_2) \\
&
+ \frac{\beta}{2} \nu_1(\phi^{(T)} + \Phi) \left( \phi_1 + \phi_2 + (\phi_1^{(T)} + \phi_2^{(T)}) \right) \\
&
+ \frac{\beta}{2} \nu_1(\Phi)(\phi_2 - \phi_2^{(T)}),
\end{align*}

then by (1.2), (2.11), (6.1)-(6.3) and (6.13), we have

\[ \|w_1(t, \cdot) - w_1^{(T)}(\cdot, \cdot)\| \]

\[ \leq A_0 h_3 C_S^* \epsilon \xi^{N+1} + C\epsilon C_S^* \xi^N + 2\beta^* A_0 C_S \epsilon \xi^N \]

\[ \leq A_0 C_S^* \epsilon \xi^{N+1}. \quad (6.14) \]

This indicates that we complete the proof of Theorem 2.4.
References

[1] H. Cai, Z. Tan, Time periodic solutions to the compressible Navier-Stokes-Poisson system with damping, *Commun. Math. Sci.*, **15**(3)(2017), 789-812.

[2] J. Greenberg, M. Rascle, Time-periodic solutions to systems of conservation laws, *Arch. Rational Mech. Anal.*, **115**(4)(1991), 395-407.

[3] T. Li, Global classical solutions for quasilinear hyperbolic systems, *Research in Appl. Math.*, vol.34, Wiley/Masson, New York/Paris, 1994.

[4] T. Li, W. Yu, Boundary value problem for quasilinear hyperbolic systems, *Duke University Math. Series*, vol.5, 1985.

[5] T. Luo, Bounded solutions and periodic solutions of viscous polytropic gas equations, *Chin. Ann. Math. Ser. B*, **18**(1)(1997), 99-112.

[6] H. Ma, S. Ukai, T. Yang, Time periodic solutions of compressible Navier-Stokes equations, *J. Differ. Equ.*, **248**(9)(2010), 2275-2293.

[7] A. Matsumura, T. Nishida, Periodic solutions of a viscous gas equation, *North-Holland Math. Stud.*, **160**(1989), 49-82.

[8] N. Tsuge, Existence of a time periodic solution for the compressible Euler equations with a time periodic outer force, *Nonlinear Anal. Real World Appl.*, **53**(2020), 103080, 22 pp.

[9] M. Ohnawa, M. Suzuki, Time-periodic solutions of symmetric hyperbolic systems, *J. Hyperbolic Differ. Equ.*, **17**(4)(2020), 707-726.

[10] P. Qu, Time-periodic solutions to quasilinear hyperbolic systems with time-periodic boundary conditions, *J. Math. Pures Appl.*, **139**(2020), 356-382.

[11] A. Shapiro, The Dynamics and Thermodynamics of Compressible Fluid Flow, Vol.1. New York: Wiley, 1953.

[12] S. Takeno, Time-periodic solutions for a scalar conservation law, *Nonlinear Anal.*, **45**(8)(2001), 1039-1060.

[13] B. Temple, R. Young, A Nash-Moser framework for finding periodic solutions of the compressible Euler equations, *J. Sci. Comput.*, **64**(3)(2015), 761-772.

[14] H. Yu, X. Zhang, J. Sun, Time-periodic solution to nonhomogeneous isentropic compressible Euler equations with time-periodic boundary conditions, *arXiv preprint arXiv:2207.09116* (2022).

[15] H. Yu, X. Zhang, J. Sun, Global existence and stability of time-periodic solution to isentropic compressible Euler equations with source term, *arXiv preprint arXiv:2204.01939* (2022).
[16] H. Yuan, Time-periodic isentropic supersonic Euler flows in one-dimensional ducts driving by periodic boundary conditions, *Acta. Math. Sci. Ser. B (Engl. Ed.)*, **39**(2)(2019), 403-412.

[17] X. Zhang, Global existence and stability of time-periodic solution to 1-D isentropic compressible Euler equations with source term, *J. Appl. Math. Comput.*, **6**(1)(2022), 41-52.

[18] X. Zhang, H. Yu, J. Sun, Global existence and stability of subsonic time-periodic solution to the damped compressible Euler equations in a bounded domain, *arXiv preprint arXiv:2205.05858* (2022).