Fluctuations of particle motion in granular avalanches
– from the microscopic to the macroscopic scales

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In this study, we have investigated the fluctuations of particle motion, i.e. the non-affine motion, during the avalanche process, discovering a rich dynamics from the microscopic to the macroscopic scales. We find that there is strong correlation between the magnitude of the velocity fluctuation and the velocity magnitude in the spatial and temporal domains. In addition the velocity magnitude of the system and the stress fluctuations of the system are strongly correlated temporally. Our finding will pose challenges to the development of more rigorous theories to describe the avalanche dynamics based on the microscopic approach. Moreover, our finding presents a plausible mechanism of the particle entrainment in a simple system.

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I. INTRODUCTION

Granular avalanches are ubiquitous in nature. A simple demonstration will be a continuous tilting of a pile of granular materials, eventually causing an avalanche with a massive rearrangement of particles to self-organize the system to a new stable configuration at a lower energy state [1,2]. The study of such processes has important applications in geophysics, in mechanical and civil engineering, in pharmaceutical applications, and in studying the natural disasters such as the landslide and mud-slide, et al. [3–6]. Historically, the study of the avalanche of sand pile has played an important role in the development of important theoretical concepts such as the Coulomb’s laws of friction [7] and the self-organized criticality [8–10], et al.

In recent years, granular avalanches have become a field of intense research activities. These include: (1) the experimental measurement of the critical behavior at the onset of the avalanche [11,12]; (2) the numerical analysis of the bimodal contact network contributions to the instability of the avalanche [13]; (3) the continuum theory description of the surface flow profile of the avalanche [14,15] and the extended theory [18] and the combined theoretical/experimental analysis for particle segregation and for surface wave instability analysis [19,22]; (4) the novel dynamics caused by particle shapes [23], particle cohesion [24], and the variation of volume fractions [25]; (5) the bifurcation of flow dynamics and instabilities due to the change of the external driving [26,28]; (6) the self-similarity of the surface mean velocity profile [29] and correlations between the starting and the ending angles [30]. Nevertheless, there is still a lack of a coherent physical picture which connects the macroscopic/mesoscopic dynamics of a granular system to the microscopic dynamics at the particle scale in the granular avalanche process [31].

In this study, we try to fill the missing link between the microscopic and the macroscopic/mesoscopic dynamics by focusing much of the attention on analyzing the non-affine particle motion from the particle scale to the system scale during granular avalanches in a quasi two dimensional (2D) system – a rotating drum partially filled with a 2D layer of granular disks to create avalanches. Here we show that the magnitude of velocity \( v \) and the non-affine velocity \( \delta v \) are strongly correlated in the temporal domain from the microscopic particle scale to the macroscopic system scale. In addition, we also find strong correlations of these two fields in the spatial domain. This is surprising since normally one would expect the fluctuations of particle motion become weaker as the scale increases. The strong spatial correlation between \( \delta v \) and \( v \) can be partially understood from the numerical model of a stochastic steady shear flow. The fluctuations of particle velocity can have profound influence on the macroscopic stress changes of the system. We have discovered a connection between the velocity dynamics of particles and the fluctuation of the total stress of the system from the perspective of momentum transfer. Our finding may pose a challenge to the development of the theoretical description of avalanche dynamics based on microscopic dynamics of particles. Our finding could be important to some geological processes such as particle entrainment and bed erosion in snow avalanches and landslides.

This paper is organized as follows. In Sec. II, we first describe the related experimental techniques. In Sec. III, we present experimental results along with the discussion, which include the spatial and temporal correlations of velocity and velocity fluctuations, the numerical simulation, and the stress fluctuations of the system. Finally, we will draw conclusions in Sec. IV.

II. EXPERIMENTAL METHODS

The experimental setup mainly consists of a thin rotating drum of a diameter of 80 cm and a width of 1 cm, as shown in
The drum is driven by a stepping motor to rotate at a slow speed of $1/15$ revolutions per minute. The construction of the drum is composed of two Plexiglas plates with a gap of $8 \text{mm}$ and with the inner surfaces coated to eliminate the accumulation of electrostatic charges. A thin layer of 736 photo-elastic disks is sandwiched in between the two plates. These quarter-inch-thick disks are bi-disperse with a large size of $1.4 \text{ cm}$ in diameter and a small size of $1.2 \text{ cm}$ in diameter and with a size ratio of $1 : 1$. They are machined from PSM4 materials from Vishay and only fill less than half of the space of the drum. In front of the drum, two high-speed cameras (2048 $\times$ 2048 pixels and with a frame rate up to 180/sec) are mounted to continuously record images during the avalanche. In front of the lens of one camera, a piece of polarizer is placed in order to capture the force-chain network simultaneously.

To improve the statistics, the experiment has been repeated for ten times following the same protocol. Results of different runs are similar. In this paper, we will present experimental results in two groups: the first group includes results from a randomly selected experimental run in order to show details of the avalanche process; the second group consists results of all ten runs in order to present the statistics of the granular avalanche.

### TABLE I

The statistics of the spatial correlations $C_{\delta v, v}$ and $C_{\bar{v}, v}$ in ten different runs. Here $\delta v$ is the magnitude of the nonaffine velocity of a particle, $v$ is the magnitude of the particle velocity and $\bar{v}$ is the magnitude of the local mean velocity at the center of the particle.

| Avalanche number $m_0$ | $m_0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------------------------|------|---|---|---|---|---|---|---|---|---|----|
| $C_{\delta v, v}$       | mean | 0.73 | 0.89 | 0.87 | 0.90 | 0.84 | 0.84 | 0.63 | 0.92 | 0.70 | 0.84 |
|                         | std  | 0.14 | 0.047 | 0.14 | 0.054 | 0.10 | 0.20 | 0.083 | 0.057 | 0.10 | 0.10 |
| $C_{\bar{v}, v}$        | mean | 0.85 | 0.91 | 0.89 | 0.92 | 0.77 | 0.74 | 0.86 | 0.67 | 0.88 | 0.80 |
|                         | std  | 0.070 | 0.034 | 0.067 | 0.058 | 0.095 | 0.10 | 0.046 | 0.090 | 0.042 | 0.072 |

### III. RESULTS AND DISCUSSION

#### A. Velocity fluctuations

We first analyze the evolution of the local and the global velocity fluctuations. In Fig. 3(a), we plot $\delta v$ (red) and $v$ (blue) versus time respectively for an arbitrary particle, which shows that the curves of $\delta v$ and $v$ have a similar trend and thus may be correlated. The average temporal correlation between $\delta v$ and $v$ of individual particles is around 0.79 of all runs, which
is reasonably strong. Here the velocity fluctuation of a given particle $\delta v = v - \bar{v}$, as illustrated in Fig. 2(b). Note that in this paper we use bold fonts for vectors, e.g. $\delta v$ and $v$, and regular fonts for moduli, e.g. $\delta v$ and $v$. We define $\theta$ as the deviation angle between $\bar{v}$ and $v$. The local mean velocity $\bar{v}$ is measured by averaging local velocities of particles centered around the given particle using a Gaussian coarse-graining function with a standard deviation of $2D$ ($D$ is the average diameter of a particle) [32,33]. A sample image of $v$ and the corresponding sample image of $\delta v$ are shown in Fig. 2(c-d). The dynamics of the spatial correlations between $\delta v$ and $v$ are analyzed as shown in Fig. 2(e), where the correlation function $C_{\delta v,v}$ reveals a strong spatial correlation of the two quantities during the whole avalanche process. There are random fluctuations around the average, close to 0.89, as indicated using the red horizontal line in this figure. A strong correlation $C_{\bar{v},v}$ between the local mean velocity $\bar{v}$ and $v$ is also shown in Fig. 2(f). Table I summarizes the statistics of the average and the standard deviation of $C_{\delta v,v}$ and $C_{\bar{v},v}$ from ten different runs, which are consistent for all runs.

We define the global velocity fluctuation $\delta V$ and global velocity $V$ as $\delta V = \sqrt{\sum \delta v^2}$, $V = \sqrt{\sum v^2}$, and plot them in Fig. 3(a). Clearly, their time evolutions are similar and the scatter plot of the two variables is shown in (b). The strong correlation between $\delta V$ and $V$ is unexpected since normally the velocity fluctuations will become much weaker at the system scale. This means that the ordered directional movement of all particles can hardly occur in granular avalanches. Note that the above observations are quite different from the thermodynamical systems under phase transitions, e.g. melting of a crystal where fluctuations increase as the system becomes more disordered but directed motion seldom occurs [34,35]. In continuum theory description of granular flows of avalanche, fluctuations are usually ignored [15–17,31]. However, our findings suggest that ignoring such fluctuations might be problematic especially for applications such as parti-

FIG. 4: (a) $\theta(t)$ of a moving particle during one avalanche. Red line is the average of $\theta(t)$, which equals 0.69. (b) The spatial distribution of $\theta$ at $t = 1.25$ in one avalanche. (c) The distribution of $\theta$ of all moving particles at different time instants for all ten different runs in the logarithm coordinate. Here the solid line is the total distribution of all runs (main plain). The spatial average of $\theta$ over all the moving particles as a function of time in one avalanche (inset). (d) The distributions of $\frac{\delta v}{v}$ of all moving particles in different runs. The solid line is the total distribution of all runs. Here the positive $\alpha$ is clockwise with respect to the inclination of the surface (main plain). A schematic of the definition of $\alpha$, which is the deviation angle of $v$ with respect to the inclination of the surface (inset).

FIG. 5: The distributions of $\alpha$ of different runs. Here the solid line is the total distribution of all runs. Here the positive $\alpha$ is clockwise with respect to the inclination of the surface (main plain). A schematic of the definition of $\alpha$, which is the deviation angle of $v$ with respect to the inclination of the surface (inset).

FIG. 6: (a) A portion of an artificially created flow field within a total of $200 \times 10$ grid size when $\alpha$ is assumed to obey an exponential distribution, i.e. $P_1(\alpha) = 0.75e^{-1.5\alpha}$. Here the flow field is drawn on a square lattice with blue arrows representing for the velocity $v$ on each grid point and red arrows for the local coarse-grained mean velocity $\bar{v}$. (b) Similar with (a) but here $\alpha$ is assumed to obey a Gaussian distribution, i.e. $P_2(\alpha) = \frac{1}{\sqrt{2\pi}e^{-2\alpha^2}}$. (c) Distribution of $\theta$ with separate results from $P_1(\alpha)$ (green squares) and $P_2(\alpha)$ (blue circles). (d) Distribution of $\frac{\delta v}{v}$ with separate results from $P_1(\alpha)$ (green squares) and $P_2(\alpha)$ (blue circles).
cle entrainment or bed erosion \([4,6]\) where mesoscopic particle dynamics is critical. This may pose a theoretical challenge for developing more rigorous description of granular flows of avalanches \([32]\).

To better characterize the fluctuations of the moving particles, in addition to \(\delta v\), we also track the deviation angle \(\theta\) between the mean velocity \(\bar{v}\) and \(v\), as sketched in Fig. 2(b). In Fig. 3(a), we present the evolution of \(\theta\) of one particle, which fluctuates irregularly with the mean value of 0.69 rad, showing no discernible temporal correlation with either \(v\) or \(\delta v\). We show the spatial distribution of \(\theta\) at a given instant in Fig. 3(b) and its distribution of different runs in Fig. 3(c) on a semi-logarithmic plot, exhibiting an exponential-like form. Here the random distribution of \(\theta\) suggests an intrinsic disordered characteristics of the avalanche. In the inset of Fig. 3(c), \(\bar{\theta}(t)\) is computed by averaging \(\theta\) over all moving particles at the time instant \(t\). It fluctuates randomly with a time average around 0.62, as indicated using the red line in the figure. Note that at
the limit when $|v - \bar{v}|/v \ll 1$, we can have $\delta v \approx v\theta$ if the fluctuations of $\theta$ is small. At this limit, the correlation between $\delta v$ and $v$ can thus be expected. However, the distribution of $(v - \bar{v})/v$ is broad as shown in Fig. 4(d) where the histograms of at different time are plotted, showing a Gaussian-like distribution.

B. A numerical model

To get some understanding of the correlations shown above, we have constructed a numerical model where we simplify the particle motion using a 2D steady shear flow with stochastic orientations of local velocity compared to the mean. This shear flow is constructed by assigning a velocity vector field on a $200 \times 10$ square lattice according to $v \propto z\alpha$ where $z$ represents the vertical coordinate and $\alpha$ is the unit vector of the horizontal direction. The stochastic motion of the particle velocity is then introduced by setting the orientation angle $\alpha$ of each particle’s velocity vector with respect to $\hat{z}$ on every grid point according to a given distribution. Two distributions of $\alpha$ have been used: (1) an exponential distribution $P_1(\alpha) = 0.5 \lambda e^{-\lambda |\alpha|}$; (2) a Gaussian distribution $P_2(\alpha) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{\alpha^2}{2\sigma^2}}$. Here the parameter $\lambda$ is set to equal 1.5, and the parameter $\sigma$ is set to 0.5 as estimated from Fig. 3 where the experimentally measured distribution of $\alpha$ is neither an exponential nor a Gaussian distribution and the distribution is nonsymmetric in the positive and negative regimes. Hence the above two types of distribution used in the simulation are only an approximation.

Small portions of the velocity field $v$ are shown in Fig. 6(a-b) in blue color in panel (a) generated using $P_1(\alpha)$ and in green color in panel (b) generated using $P_2(\alpha)$. Along with the $v$ fields, at each grid point the local mean velocity fields $\bar{v}$ can be computed following the exactly same Coarse-Graining method in analyzing the experimental results. The $v$ fields are drawn in red color in Fig. 6(a-b).

We then compute the statistics of the relevant physical quantities – $\theta$, $\frac{\|v\|}{v}$, $\bar{\theta}$, $C_{\bar{\theta}v}$, and $C_{\bar{\theta}v}$ in order to compare the results with the experiment, as shown in Fig. 6 and Fig. 7. In Fig. 6(c), the distribution of $\theta$ decays slightly faster compared to an exponential distribution, similar with Fig. 4(c). In Fig. 6(d) the distributions of $\frac{\|v\|}{v}$ are Gaussian-like with peak positions and widths close to the values seen in Fig. 4(d). Note that the PDF of $\frac{\|v\|}{v}$ shows a second minor peak if $\alpha$ obeys the Gaussian distribution $P_2(\alpha)$, whose cause is unknown. Compared to the experimental results in Fig. 4(d), it is hard to see whether there is a definite minor peak at the tail due to the noise of the data.

In Fig. 7(a), $\bar{\theta}$ is defined as the spatial average of $\theta$ over all grid points in each simulation, and the statistics are collected over 1000 simulations. The distribution shows an average value around 0.65, in good agreement with the experiment. if $\alpha$ assumes the distribution of $P_1(\alpha)$; the average of $\bar{\theta}$ is around 0.5, slightly smaller than the experimental value if $\alpha$ assumes the distribution of $P_2(\alpha)$. In Fig. 7(b), the average of $C_{\bar{\theta}v}$ is around 0.60~0.65, which indicates a reasonably strong correlation but is smaller than the experimentally measured values listed in Table I. In addition, in Fig. 7(c) the average of $C_{\bar{\theta}v}$ is around 0.91~0.95, which is larger than the experimental measurement in Table I. In Fig. 6 and Fig. 7 suggest that our numerical model agrees qualitatively and to some extent quantitatively with the experimental measurement. It suggests that to the first approximation, the mean velocity profile resembles a steady shear flow. However, this model is oversimplified by ignoring the dynamics, which is complex as discussed in detail in our previous paper [30] and can also be seen from the quantitative difference between the results in Fig. 7(b-c) and experimental measurements. In addition there is no steady state in our system. The above model suggests that the random fluctuations of velocity is an intrinsic characteristic of the system, independent of scales. The confirmed correlations between $\delta v$ and $v$ can enhance the fluidization of the stable granular configurations under the surface particle flow, which might be important to some geological processes, such as particle entrainment and bed erosion during avalanches [4–6].

### C. Stress fluctuations

Finally, we try to make a connection between the velocity $v$ of individual particles and the change of stress for the whole system – a reflection of the energy dissipation and transformation. Here we measure the change of internal stress $\delta F$ within the system by the means of $G^2$ (see ref [37]), which computes the square of the gradient of the differential image between two stress images in successive time frames, as shown in Fig. 8 $\delta F$ versus time is plotted in Fig. 9(a). In ref [37], a simple model is proposed to describe an impact process where the intruder transfers momentum to the granular materials through a sequence of random collisions with the force network. For our system, during an avalanche there are random collisions between the flowing particles at the surface layers and massive stable particle configurations at the bottom where the kinetic energy is dissipated and the momentum is transferred to excite new force-chain networks. We first analyse the collision between a single particle and the bottom as shown in the inset of Fig. 5. The change of $\delta F$ is connected with the momentum transfer perpendicular to the surface layers by the collision, which imparts momentum $\Delta p = (1 + e) \frac{mM}{m + M} \sin \alpha$, where $\alpha$ is the angle between $v$ and the inclination of the surface, $e$ is the restitution coefficient, and $m$ and $M$ are the mass.

| Table II: The statistics of the degree of linear temporal correlations between macroscopic variables in ten different runs. Here ‘ccf’ means the correlation coefficient of the linear fitting. |
|---|
| avalanche number $n_0$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | mean | std |
| $c_{\delta F - v}$ | 0.93 | 0.93 | 0.93 | 0.92 | 0.93 | 0.94 | 0.95 | 0.95 | 0.95 | 0.94 | 0.94 | 0.95 |
| $c_{\delta F - \bar{v}}$ | 0.93 | 0.92 | 0.91 | 0.93 | 0.92 | 0.91 | 0.93 | 0.94 | 0.93 | 0.94 | 0.93 | 0.94 | 0.92 |
| $c_{\delta F - \bar{\theta}}$ | 0.93 | 0.95 | 0.93 | 0.91 | 0.92 | 0.91 | 0.93 | 0.95 | 0.94 | 0.93 | 0.94 | 0.95 | 0.92 |
| $c_{\delta F - C_{\bar{\theta}v}}$ | 0.93 | 0.95 | 0.93 | 0.91 | 0.92 | 0.91 | 0.93 | 0.95 | 0.94 | 0.93 | 0.94 | 0.95 | 0.92 |
| $c_{\delta F - C_{\bar{\theta}v}}$ | 0.93 | 0.95 | 0.93 | 0.91 | 0.92 | 0.91 | 0.93 | 0.95 | 0.94 | 0.93 | 0.94 | 0.95 | 0.92 |
of a single particle and the bottom configuration, respectively. The typical collision time can be approximated as $\Delta t = \frac{\sqrt{m}}{\gamma D}$, where $D$ is the diameter of a particle and $\gamma$ is a constant. Thus, the average force acting on the bottom perpendicular to the surface is

$$f = \frac{\Delta p}{\Delta t} = \frac{(1+\epsilon)v^2 \sin^2 \alpha}{\gamma D} \frac{mM}{m+M}$$

To obtain the total stress change of the system, we need to sum the forces over all moving particles, i.e.

$$\delta F = \sum_i f_i = \frac{(1+\epsilon)mM}{\gamma D(m+M)} \sum_i v_i^2 \sin^2 \alpha_i$$

The histograms of $\alpha$ of all runs are shown in Fig. 5. We see that $\alpha$ follows a broad nonsymmetric distribution with zero mean. We also analyze its spatial distribution at each instant (figures are not plotted here) and find that it is randomly distributed and has no correlation with $v$. Due to the independence of $\alpha$ and $v$, the average of $\delta F$ can be obtained by

$$\langle \delta F \rangle = C \sum_i \langle v_i^2 \rangle \langle \sin^2 \alpha_i \rangle = C \langle \sin^2 \alpha \rangle \sum_i \langle v_i^2 \rangle$$

where $C = \frac{(1+\epsilon)mM}{\gamma D(m+M)}$ and $\langle \sin^2 \alpha \rangle$ are constant for each particle, for example, $\langle \sin^2 \alpha \rangle = 0.197$ for a Gaussian distribution with $\mu = 0, \sigma = 0.5$. So $\delta F$ will strongly correlate with the summation of $v^2$ over each particle in this model. We note that if the collision time is a constant then $\langle \delta F \rangle$ is proportional to $\Sigma_i \langle v_i \rangle$ instead. To verify this, we plot the summation of $v$ (inset) and $v^2$ (main plain) versus time in Fig. 9(b). After calculation, the correlation between $\delta F$ and $\Sigma v$ is 0.942, and between $\delta F$ and $\Sigma v^2$ is 0.925, which agrees with the experiment, as displayed in Table II.

The combination of Fig. 3 and Fig. 9 points to a coherent physical picture of a plausible mechanism of particle entrainment. When the kinetic energy of the flowing particles increases, the velocity fluctuation increases; as a consequence, the stress fluctuation in the particle configurations under the surface flowing particles increases. Therefore more particles in the stable particle configuration will be fluidized, a phenomenon of particle entrainment [11, 12]. In the inset of Fig. 9, the dynamics of the total number of particle $N$ indeed confirms the above scenario. Note that in more realistic geological processes, the saturated water content in the stable particle configuration can make significant contributions to the particle entrainment [6]. In addition, our system consists of circular shaped particles without cohesive forces. It is known from the literature that the change of the particle roughness or the inclusion of the cohesion force may change the dynamics of the avalanches [23, 24]. It remains an open question on how the main results in Fig. 3 and Fig. 9 would be changed if more complex particle shapes and interactions are considered.

IV. CONCLUSIONS

To conclude, we have analyzed the nonaffine particle motion in the granular avalanches and we find that there are strong correlations between velocity fluctuations and velocity during granular avalanche: at the particle scale these two quantities correlate in both the spatial and the temporal domains; at the macroscopic scales, the nonaffine motion increases as the total kinetic energy increases. By simulating a steady shear flow with random orientations of velocity vector relative to the mean flow, we can reproduce strong correlations between $\delta v$ and $v$, and $\delta v$ and $v$, which agrees qualitatively with the experimental measurements. In addition, by adapting an analytical model from impact dynamics, we find that there is a strong correlation between the total velocity magnitude or kinetic energy of the system and the stress fluctuations of the system. Our findings might be important for the development of more rigorous theories to describe the dynamics of the granular avalanche. Our findings also provides a coherent physical picture of a plausible mechanism of particle entrainment in a simple granular system.

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