Transition to chaos in magnetized, weakly coupled plasmas

Andrea Carati∗
Department of Mathematics,
Università degli Studi di Milano
Via Saldini 50, 20133 Milano, Italy

Francesco Benfenati†
Corso di Laurea in Fisica, Università degli Studi di Milano
Via Celoria 12, 20133 Milano, Italy

Alberto Maiocchi‡ and Luigi Galgani§
Department of Mathematics,
Università degli Studi di Milano
Via Saldini 50, 20133 Milano, Italy

Matteo Zuin¶
Consorzio RFX, Associazione EURATOM-ENEA sulla Fusione, Padova, Italy

(Dated: May 2, 2014)

We report the results of numerical simulations for a model of a one component plasma (a system of \(N\) point electrons with mutual Coulomb interactions) in a uniform stationary magnetic field. We take \(N\) up to 512, with periodic boundary conditions, and macroscopic parameters corresponding to the weak coupling regime, with a coupling parameter \(\Gamma = \frac{1}{64}\). We find that a transition from order to chaos takes place when the density is increased or the field decreased so that the ratio \(\omega_p/\omega_c\) between plasma and cyclotron frequencies becomes of order 1 (or equivalently the ratio \(r/L\) between Larmor radius and Debye length becomes of order 1). The result is in agreement with the theoretical prediction obtained in [1], on the basis of an old estimate of Iglesias, Lebowitz and MacGowan[2] for the intensity of the electric field acting on one electron and due to all the other ones. A comparison can be made with the threshold obtained from kinetic theory arguments, which corresponds to the condition \(\nu_{ee}/\omega_c = 1\), where \(\nu_{ee}\) is the electron collision frequency. The latter threshold has a completely different dependence on the physical parameters and, for \(\Gamma = \frac{1}{64}\), gives a critical value of \(\omega_p\) about 80 times larger.

PACS numbers: 05.45.-a, 52.55.-s

The microscopic foundations of plasma physics are usually formulated in terms of kinetic theory, in which a key role is played by the concept of “collision frequency” or by the related one of “mean free path”. It is usually stated that for the so–called weakly coupled plasmas, such as gaseous–discharge plasmas, fusion plasmas, or a plasma in the solar corona, the Coulomb coupling is so small that “their thermodynamic properties are analogous to those of an ideal gas” (see [3], page 10), i.e., the electrons behave essentially as if they were free.

A different approach was taken in paper [1], in which the microscopic Newton equations themselves were tackled directly, making use of the tools of ergodic theory and of a quite recent extension of Hamiltonian perturbation theory to the thermodynamic limit (see [4–6]). This allows one to obtain theoretical results for the microscopic model itself, with no need of passing through the approximation of the Boltzmann equation. In particular, for an infinite plasma immersed in a uniform stationary magnetic field \(B\) the electron motions were estimated to be ordered if the ratio \(\omega_p/\omega_c\) between electron plasma and cyclotron frequencies is below unity, chaotic in the opposite case. This means that the Coulomb interactions among the electrons are strong enough to produce a chaoticity threshold when \(\omega_p \simeq \omega_c\) (the definitions of these and of some other familiar quantities will be recalled in a moment). Here “ordered” has to be understood in the sense of ergodic theory, i.e., that there exists at least one dynamical variable the time–autocorrelation of which does not decay to zero, or decays in an extremely slow way. For completely chaotic motions, instead, the time–correlations of smooth dynamical variables are known to quickly decay to zero.

Notice that the estimate for the chaoticity threshold determined in [1] can be eventually expressed in an extremely intuitive way, namely, as the condition that the typical value of the perturbing force on any electron (the sum of the Coulomb forces due to all the other ones) just equals the typical value of the Lorentz force. So, denoting by \(E_j\) the modulus of the electric field acting on the
\( j \)-th electron and by \( v_j \) the modulus of the electron's velocity, the condition for the chaoticity threshold takes the simple form \( E_j \simeq Bv_j/c \) (in Gauss units). In turn, the electric field acting on one electron and due to all the other ones, looked at as a random variable, obviously has vanishing mean, so that its typical value is estimated by its standard deviation. The value of the latter, at density \( n_e \) and temperature \( T \), was estimated long ago by Iglesias, Lebowitz and MacGowan [2] to be given by \( \sqrt{4\pi n_ek_BT} \), where \( k_B \) is the Boltzmann constant. This leads for the threshold to the condition \( \omega_p/\omega_c \simeq 1 \), which in particular is independent of the coupling parameter \( \Gamma \).

We used here the definitions \( \omega_c \defeq eB/me \), \( \omega_p \defeq \sqrt{e^2n_e/m} \), \( \Gamma \defeq e^2/ak_BT \), where \( m \) and \( e \) are the electron mass and charge, \( c \) the speed of light, and \( \alpha \defeq n_e^{-1/3} \) the mean interparticle distance. A relevant related quantity is the Debye length \( \lambda_D \defeq \sqrt{k_BT/n_ee^2} \).

The theoretical estimate for the chaoticity threshold at \( \omega_p \simeq \omega_c \), found in [1] was quite unexpected because kinetic arguments apparently suggest that, in the weakly coupled regime \( \Gamma \ll 1 \), a transition might occur at \( \nu_{ee} \simeq \omega_c \), where \( \nu_{ee} \) is the electron-phonon collision frequency (see for example [7]). On the other hand one has (see for example [3], page 35) \( \nu_{ee} \simeq \Gamma^{3/2} \left| \log \Gamma \right| \omega_p \), and this gives a threshold at \( \omega_p \simeq \omega_c \left( \Gamma^{3/2} \left| \log \Gamma \right| \right) \), i.e., at \( \omega_p \gg \omega_c \) (for \( \Gamma < 1 \)).

In this letter we report the results of numerical simulations at \( \Gamma = 1/64 \). A transition to chaotic motion is seen to occur for \( \omega_p/\omega_c \) in the interval between 0.25 and 2, in agreement with the prediction given in [1].

Let us recall that a one component plasma model is just a system of \( N \) point electrons with mutual Coulomb interactions. We denote their position vectors by \( x_j, j = 1, \ldots, N \), and take them to lie in a box of side \( L \) (so that the electron density is given by \( n_e = N/L^3 \)). They are subject to the Lorentz force \( e/c \mathbf{E} + \mathbf{B} \times \mathbf{x}_j \) due to a constant homogeneous magnetic field \( \mathbf{B} \) (which we take directed along the \( z \) axes), and to their mutual Coulomb forces. The electric force on the \( j \)-th electron, which depends on the positions \( x_1, \ldots, x_N \) of all electrons, may be simply denoted by \( e^2 \mathbf{E}(x_j) \), where \( \mathbf{E} \) is the electric field acting on that electron and due unit charges located in the positions of the other electrons. As we are using periodic boundary conditions (so that we are actually dealing with a system of infinitely many electrons), the latter field can be computed [8] by the Ewald summation of the field due to an infinite cubic lattice of charges of the form \( x_j + mL \). Here, \( n \) is a vector with integer coordinates, i.e., \( n \defeq (nx_x + ny_y + nz_z) \), with \( nx, ny \) and \( nz \in \mathbb{Z} \), while \( e_x, \ldots \) are the unit vectors along the axes.

Rescaling time by the electron cyclotron frequency \( \omega_c \) and position vectors by the mean interparticle distance \( a \), i.e., introducing \( \tau \defeq \omega_c t \) and \( y_j \defeq x_j/a \), the equations of motion take the form

\[
\dot{y}_j = e_z \wedge \dot{y}_j + \left( \frac{\omega_p}{\omega_c} \right)^2 E(y_j)
\]

(the dots denoting now derivatives with respect to \( \tau \), and so contain only one (dimensionless) parameter, namely \( \omega_p/\omega_c \), while the rescaled density is obviously equal to 1. In the simulations, the explicit form of the Ewald resummed field \( \mathbf{E} \) acting on the \( j \)-th particle is given by

\[
\mathbf{E}(y_j) = \sum_n \sum_l \frac{r_{l,n}}{|r_{l,n}|^3} \left[ \text{erfc}(\alpha_{l,n}) + \frac{\alpha_{l,n}}{\sqrt{\pi}} \exp(-\alpha_{l,n}^2) \right] + 4\pi \sum_{k \neq 0} \frac{\mathbf{k}}{|\mathbf{k}|} \exp(-\frac{\mathbf{k}^2}{4\alpha}) \sin(\mathbf{k} \cdot \mathbf{y}_j).
\]

Here \( r_{l,n} \defeq y_j - y_l \), while \( r_{l,n} \defeq y_j - y_l + Ln/a \); the function \( \text{erfc}(x) \) is the usual error function, and \( \alpha \) is the Ewald convergence parameter which we chose as \( \alpha \defeq \pi^{1/2}N^{1/6}L^{-1} \). In the first sum the term corresponding to the self-force on the \( j \)-th particle should be excluded.

These are the equations of motion that were actually integrated numerically, using a symplectic splitting method. The conservation of energy in every run was better than a part over 10^4. The integration time was chosen proportional to \( \omega_c \), in order that all different cases be integrated for the same physical time. In any case, the time was always some hundreds cyclotron periods.

The initial data were chosen in the following way: the electron positions \( y_j \) were taken uniformly distributed in the box of side \( N^{1/3} \), while the velocities were extracted from a Maxwellian with a given temperature \( T \). This introduces in the model (in addition to \( \omega_p/\omega_c \) the further parameter \( T \), or equivalently the dimensionless Coulomb coupling parameter \( \Gamma \), to be used in the Maxwell distribution for the velocities.

For what concerns the number \( N \) of electrons in the box, our computational power allows us to go up to \( N = 512 \). This induces a lower bound on \( \Gamma \), namely, \( \Gamma \geq N^{-2/3} \). Indeed, in order to correctly simulate the Coulomb cumulative force acting on an electron in a plasma, the side of the box has to be at least equal to the Debye length, which, in our rescaled units, takes the value \( \lambda_D = \Gamma^{-1/2} \). We took \( \Gamma = N^{-2/3} \). Computations were performed both for \( N = 128 \) and \( N = 512 \), which correspond to \( \Gamma = 128^{-2/3} \simeq 0.04 \) and \( \Gamma = 512^{-2/3} = 1/64 \simeq 0.016 \) respectively.

We now come to the main issue, i.e., whether the motions are ordered or chaotic. Obviously what plays the role of the unperturbed system with completely ordered motions is the limit case with \( \omega_p/\omega_c = 0 \), for which the Coulomb interaction disappears and one has pure Larmor gyrations. The problem then is to determine whether a threshold for chaotic motions takes place as the parameter \( \omega_p/\omega_c \) is increased and \( \Gamma \) is varied.

To this end we considered the magnetization of a box, \( \mathcal{M} \defeq (e/2mc) \sum x_j \wedge x_j \), looking at its autocorrelation function (normalized by \( Nk_BT \))

\[
\mathcal{C}_\mathcal{M}(t) \defeq \frac{\langle \mathcal{M}(0)\mathcal{M}(t) \rangle}{Nk_BT},
\]
and at its Fourier transform $\hat{C}_M(\omega)$. The latter is a physically very relevant quantity because, according to linear response theory (see [9, 10], or Appendix B of [11]), $i\omega \hat{C}_M(\omega)$ gives the susceptibility $\chi(\omega)$ at frequency $\omega$. In the formula for the time–autocorrelation $C_M(t)$, the average $<\cdot>$ is meant as a phase–average with respect to Gibbs measure; in our computations, however, we estimated it by the time–average along an orbit (with initial data extracted as previously explained), as often done in numerical works. We did not investigate the relations between the two averages. Moreover, the Fourier transform $\hat{C}_M(\omega)$ was estimated by the amplitude of the discrete Fourier transform of $C_M(t)$, which will be simply called the spectrum. So we report figures of the time–autocorrelation $C_M$ versus $t$, and of the corresponding spectrum versus angular frequency $\omega/\omega_c$. Having fixed $\Gamma = 1/64$, by increasing $\omega_p/\omega_c$ we found that a threshold occurs for $\omega_p/\omega_c$ between 0.25 and 2.
This is exhibited in Fig. 1, where the results are reported for such two values of $\omega_p/\omega_c$, 0.25 on the left and 2 on the right. The autocorrelations are reported in the upper part of the figure, and the spectra in the lower part.

For $\omega_p/\omega_c = 0.25$ the autocorrelation is seen to display regular oscillations with a decreasing amplitude: we were unable to follow this relaxation process up to the end. The oscillations are apparently peaked about the cyclotron frequency and its low harmonics (as should be, due to the nonlinearities in the equations of motions). This is clearly exhibited by the spectrum, with its large peak at $\omega/\omega_c = 1$, and the smaller ones about the low harmonics $\omega/\omega_c = 2,3,\ldots$. Of special relevance is the peak at $\omega = 0$, which corresponds to the existence of a nonvanishing static susceptibility, i.e., to the existence of diamagnetism. There also appears a continuous component, which accounts for the extremely slow drift towards equilibrium. This case clearly corresponds to prevalently diamagnetic plasmas, through numerical computations of the chaoticity threshold given in [1]. The main point is that the perturbation due to the Coulomb interactions is not yet sufficiently large to produce prevalent chaotic motions. The passage to chaos, however, already occurred at $\omega_p/\omega_c = 2$. Indeed in this case the autocorrelation is seen to go to zero in an extremely short lapse of time (even shorter than one cyclotron period $2\pi/\omega_c$), so that the peaks disappear from the spectrum and one only remains with the continuous part. This means that for $\Gamma = 1/64$ the threshold in $\omega_p/\omega_c$ lies between 0.25 and 2. For $\Gamma = 128^{-2/3}$ the corresponding figures at those same values of $\omega_p/\omega_c$ are qualitatively similar to the above ones, and are not reported here.

So, the numerical results obtained for $\Gamma = 128^{-2/3} \approx 0.04$ and $\Gamma = 1/64 \approx 0.016$ are in rather good agreement with the theoretical prediction found in [1], namely: at $\omega_p/\omega_c = 1$ the interactions become strong enough as to make the motions chaotic. On the other hand this is apparently in contrast with kinetic theory arguments, according to which Coulomb interactions should be negligible up to values of $\omega_p$ larger by a factor 20 and 80 respectively. The discrepancy would become enormous in physically relevant cases, as gaseous–discharge plasmas, fusion plasmas, or a plasma in the solar corona, for which $\Gamma$ takes the typical values $10^{-3}, 10^{-5}, 10^{-7}$ respectively. Indeed, in terms of densities, for fusion plasmas the coupling should be negligible up to densities about thirteen orders of magnitudes larger than according to the law $\omega_p/\omega_c = 1$. Notice by the way that, as shown in [1], the latter threshold appears to fit pretty well, at least as orders of magnitude are concerned, the empirical data for disruptions in fusion machines.

In conclusion, the present numerical work confirms, for weakly coupled plasmas, the theoretical predictions for the chaoticity threshold given in [1]. The main point is the possibility that the numbers 128 and 2. For $\Gamma = 128^{-2/3}$ the corresponding figures at those same values of $\omega_p/\omega_c$ are qualitatively similar to the above ones, and are not reported here.

So, the numerical results obtained for $\Gamma = 128^{-2/3} \approx 0.04$ and $\Gamma = 1/64 \approx 0.016$ are in rather good agreement with the theoretical prediction found in [1], namely: at $\omega_p/\omega_c = 1$ the interactions become strong enough as to make the motions chaotic. On the other hand this is apparently in contrast with kinetic theory arguments, according to which Coulomb interactions should be negligible up to values of $\omega_p$ larger by a factor 20 and 80 respectively. The discrepancy would become enormous in physically relevant cases, as gaseous–discharge plasmas, fusion plasmas, or a plasma in the solar corona, for which $\Gamma$ takes the typical values $10^{-3}, 10^{-5}, 10^{-7}$ respectively. Indeed, in terms of densities, for fusion plasmas the coupling should be negligible up to densities about thirteen orders of magnitudes larger than according to the law $\omega_p/\omega_c = 1$. Notice by the way that, as shown in [1], the latter threshold appears to fit pretty well, at least as orders of magnitude are concerned, the empirical data for disruptions in fusion machines.

Such an acquaintance might perhaps help elucidating also the situation met in the problem of anomalous transport, where it occurs that “... measured energy transport rates typically exceed those calculated for binary collisions ...” [13], or even “... greatly exceed” them [14]. We hope to come back to this point in the future. For a study on anomalous diffusion in a strongly coupled one component plasma, through numerical computations of the same type as those performed here, see [15].

Acknowledgments. The present paper is dedicated to Francesco Guerra (La Sapienza University at Rome) on the occasion of his seventieth birthday.

[1] A. Carati, M. Zuin, A. Maiocchi, M. Marino, E. Martinez L. Galgani, Chaos 22, 033124 (2012).
[2] C.A. Iglésias, J.L. Lebowitz, D. MacGowan, Phys. Rev. A 28, 1667 (1983).
[3] S. Ichimaru, Plasma Physics: an Introduction to Statistical Physics of Charged Particles, Benjamin (Menla Park, 1955).
[4] A. Carati, J. Stat. Phys. 128, 1057 (2007).
[5] A. Maiocchi, A. Carati, Commun. Math. Phys. 297, 427 (2010).
[6] A. Carati, A. Maiocchi, Commun. Math. Phys. 314, 129 (2012).
[7] R.D. Hazeltine, J.D. Meiss, Plasma Confinement, Addison–Wesley (Redwood City, 1991).
[8] P. Gibbon, G. Sutmann, in Quantum Simulation of Complex Many–Body Systems: from Theory to Algorithms, J. Grotendorst, D. Marx, A. Muramatsu eds., NIC Series 10, 467–506 (2002).
[9] R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).
[10] Yu.L. Klimontovich, Statistical Physics, Harwood Academic (Chur, 1982).
[11] F. Benfenati, A. Carati, L. Galgani, Chaos 21, 023134 (2011).
[12] T.J.M. Boyd, J.J. Sanderson, The Physics of Plasmas, Cambridge U.P. (Cambridge, 2003).
[13] E.J. Doyle et al., Nucl. Fusion 47, S18 (2007).
[14] J.W. Connor, H.R. Wilson, Plasma Phys. Control. Fusion 36, 719 (1994).
[15] T. Ott, M. Bonitz, Phys. Rev. Lett. 107, 135003 (2011).