On the Complexity of Intersection Non-emptiness for Star-Free Language Classes

Emmanuel Arrighi
University of Bergen, Norway

Henning Fernau
Universität Trier, Fachbereich IV, Informatikwissenschaften, Germany

Stefan Hoffmann
Universität Trier, Fachbereich IV, Informatikwissenschaften, Germany

Markus Holzer
Universität Giessen, Institut für Informatik, Germany

Ismaël Jecker
Institute of Science and Technology, Klosterneuburg, Austria

Mateus de Oliveira Oliveira
University of Bergen, Norway

Petra Wolf
Universität Trier, Fachbereich IV, Informatikwissenschaften, Germany

Abstract

In the Intersection Non-emptiness problem, we are given a list of finite automata $A_1, A_2, \ldots, A_m$ over a common alphabet $\Sigma$ as input, and the goal is to determine whether some string $w \in \Sigma^*$ lies in the intersection of the languages accepted by the automata in the list. We analyze the complexity of the Intersection Non-emptiness problem under the promise that all input automata accept a language in some level of the dot-depth hierarchy, or some level of the Straubing-Thérien hierarchy. Automata accepting languages from the lowest levels of these hierarchies arise naturally in the context of model checking. We identify a dichotomy in the dot-depth hierarchy by showing that the problem is already NP-complete when all input automata accept languages of the levels $B_0$ or $B_{1/2}$ and already PSPACE-hard when all automata accept a language from the level $B_1$. Conversely, we identify a tetrachotomy in the Straubing-Thérien hierarchy. More precisely, we show that the problem is in AC$^0$ when restricted to level $L_0$; complete for L or NL, depending on the input representation, when restricted to languages in the level $L_{1/2}$; NP-complete when the input is given as DFAs accepting a language in $L_1$ or $L_{3/2}$; and finally, PSPACE-complete when the input automata accept languages in level $L_2$ or higher. Moreover, we show that the proof technique used to show containment in NP for DFAs accepting languages in $L_1$ or $L_{3/2}$ does not generalize to the context of NFAs. To prove this, we identify a family of languages that provide an exponential separation between the state complexity of general NFAs and that of partially ordered NFAs. To the best of our knowledge, this is the first superpolynomial separation between these two models of computation.

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1 Introduction

The Intersection Non-emptiness problem for finite automata is one of the most fundamental and well studied problems in the interplay between algorithms, complexity theory, and automata theory [11,19,20,23,25,42,43,44]. Given a list $A_1, A_2, \ldots, A_m$ of finite automata over a common alphabet $\Sigma$, the goal is to determine whether there is a string $w \in \Sigma^*$ that is accepted by each of the automata in the list. This problem is PSPACE-complete when no restrictions are imposed [23], and becomes NP-complete when the input automata accept unary languages (implicitly contained already in [37]) or finite languages [33].

In this work, we analyze the complexity of the Intersection Non-emptiness problem under the assumption that the languages accepted by the input automata belong to a given level of the Straubing-Thérien hierarchy [32,38,39,41] or to some level of the Cohen-Brzozowski dot-depth hierarchy [5,10,32]. Somehow, these languages are severely restricted, in the sense that both hierarchies, which are infinite, are entirely contained in the class of star-free languages, a class of languages that can be represented by expressions that use union, concatenation, and complementation, but no Kleene star operation [5,7,32]. Yet, languages belonging to fixed levels of either hierarchy may already be very difficult to characterize, in the sense that the very problem of deciding whether the language accepted by a given finite automaton belongs to a given full level or half-level $k$ of either hierarchy is open, except for a few values of $k$ [2,14,15,32]. It is worth noting that while the problem of determining whether a given automaton accepts a language in a certain level of either the dot-depth or of the Straubing-Thérien hierarchy is computationally hard (Theorem 1), automata accepting languages in lower levels of these hierarchies arise naturally in a variety of applications such as model checking where the Intersection Non-emptiness problem is of fundamental relevance [1,3,4].

An interesting question to consider is how the complexity of the Intersection Non-emptiness problem changes as we move up in the levels of the Straubing-Thérien hierarchy or in the levels of the dot-depth hierarchy. In particular, does the complexity of this problem changes gradually, as we increase the complexity of the input languages? In this work, we show that this is actually not the case, and that the complexity landscape for the Intersection Non-emptiness problem is already determined by the very first levels of either hierarchy (see Figure 1). Our first main result states that the Intersection Non-emptiness problem for NFAs and DFAs accepting languages from the level $1/2$ of the Straubing-Thérien hierarchy are NL-complete and L-complete, respectively, under $\text{AC}^0$ reductions (Theorem 3). Additionally, this completeness result holds even in the case of unary languages. To prove hardness for NL and L, respectively, we will use a simple reduction from the reachability problem for DAGs and for directed trees, respectively. Nevertheless, the proof of containment in NL and in L, respectively, will require a new insight that may be of independent interest. More precisely, we will use a characterization of languages in the level $1/2$ of the Straubing-Thérien hierarchy as shuffle ideals to show that the Intersection Non-emptiness problem can be reduced to concatenation non-emptiness (Lemma 5). This allows us to decide Intersection Non-emptiness by analyzing each finite automaton given at the input individually. It is worth mentioning that this result is optimal in the sense that the problem becomes NP-hard even if we allow a single DFA to accept a language from $L_1$, and require all the others to
accept languages from $L_{1/2}$ (Theorem 8).

Subsequently, we analyze the complexity of INTERSECTION NON-EMPTINESS when all input automata are assumed to accept languages from one of the levels of $B_0$ or $B_{1/2}$ of the dot-depth hierarchy, or from the levels $L_1$ or $L_{3/2}$ of the Straubing-Thérien hierarchy. It is worth noting that NP-hardness follows straightforwardly from the fact that INTERSECTION NON-EMPTINESS for DFAs accepting finite languages is already NP-hard [33]. Containment in NP, on the other hand, is a more delicate issue, and here the representation of the input automaton plays an important role. A characterization of languages in $L_{3/2}$ in terms of languages accepted by partially ordered NFAs [36] is crucial for us, combined with the fact that INTERSECTION NON-EMPTINESS when the input is given by such automata is NP-complete [28]. Intuitively, the proof in [28] follows by showing that the minimum length of a word in the intersection of languages in the level $3/2$ of the Straubing-Thérien hierarchy is bounded by a polynomial on the sizes of the minimum partially ordered NFAs accepting these languages. To prove that INTERSECTION NON-EMPTINESS is in NP when the input automata are given as DFAs, we prove a new result establishing that the number of Myhill-Nerode equivalence classes in a language in the level $L_{3/2}$ is at least as large as the number of states in a minimum partially ordered automaton representing the same language (Lemma 12). Interestingly, we show that the proof technique used to prove this last result does not generalize to the context of NFAs. To prove this, we carefully design a sequence $(L_n)_{n \in \mathbb{N}_{\geq 1}}$ of languages over a binary alphabet such that for every $n \in \mathbb{N}_{\geq 1}$, the language $L_n$ can be accepted by an NFA of size $n$, but any partially ordered NFA accepting $L_n$ has size $2^{\Omega(\sqrt{n})}$. This lower bound is ensured by the fact that the syntactic monoid of $L_n$ has many $J$-factors. Our construction is inspired by a technique introduced by Klein and Zimmermann, in a completely different context, to prove lower bounds on the amount of look-ahead necessary to win infinite games with delay [21]. To the best of our knowledge, this is the first exponential separation between the state complexity of general NFAs and that of partially ordered NFAs. While this result does not exclude the possibility that INTERSECTION NON-EMPTINESS for languages in $L_{3/2}$ represented by general NFAs is in NP, it gives some indication that proving such a containment requires substantially new techniques.

Finally, we show that INTERSECTION NON-EMPTINESS for both DFAs and for NFAs is already PSPACE-complete if all accepting languages are from the level $B_1$ of the dot-depth hierarchy or from the level $L_2$ of the Straubing-Thérien hierarchy. We can adapt Kozen’s classical PSPACE-completeness proof by using the complement of languages introduced in [27] in the study of partially ordered automata. Since the languages in [27] belong to $L_{3/2}$, their complement belong to $L_2$ (and to $B_1$), and therefore, the proof follows.

2 Preliminaries

We let $\mathbb{N}_{\geq k}$ denote the set of natural numbers greater or equal than $k$.

We assume the reader to be familiar with the basics in computational complexity theory [30]. In particular, we recall the inclusion chain: $\text{AC}^0 \subset \text{NC}^1 \subset L \subset \text{NL} \subset P \subset \text{NP} \subset \text{PSPACE}$. Let $\text{AC}^0$ ($\text{NC}^1$, respectively) refer to the class of problems accepted by Turing machines with a bounded (unbounded, respectively) number of alternations in logarithmic time; alternatively one can define these classes by uniform Boolean circuits. Here, $L$ ($\text{NL}$, respectively) refers to the class of problems that are accepted by deterministic (nondeterministic, respectively) Turing machines with logarithmic space, $P$ ($\text{NP}$, respectively) denotes the class of problems solvable by deterministic (nondeterministic, respectively) Turing machines in polynomial time, and $\text{PSPACE}$ refers to the class of languages accepted by deterministic or
nondeterministic Turing machines in polynomial space \cite{34}. Completeness and hardness are always meant with respect to deterministic logspace many-one reductions unless otherwise stated. We will also consider the parameterized class \( \text{XP} \) of problems that can be solved in time \( n^f(k) \), where \( n \) is the size of the input, \( k \) is a parameter, and \( f \) is a computable function \cite{12}.

We mostly consider nondeterministic finite automata (NFAs). An NFA \( A \) is a tuple \( A = (Q, \Sigma, \delta, q_0, F) \), where \( Q \) is the finite state set with the start state \( q_0 \in Q \), the alphabet \( \Sigma \) is a finite set of input symbols, and \( F \subseteq Q \) is the final state set. The transition function \( \delta : Q \times \Sigma \rightarrow 2^Q \) extends to words from \( \Sigma^* \) as usual. Here, \( 2^Q \) denotes the powerset of \( Q \). By \( L(A) = \{ w \in \Sigma^* : \delta(q_0, w) \cap F \neq \emptyset \} \), we denote the language accepted by \( A \). The NFA \( A \) is a deterministic finite automaton (DFA) if \( \delta(q, a) = \{ q \} \) for every \( q \in Q \) and \( a \in \Sigma \). Then, we simply write \( \delta(q, a) = p \) instead of \( \delta(q, a) = \{ p \} \). If \( |\Sigma| = 1 \), we call \( A \) a unary automaton.

We study Intersection Non-emptiness problems and their complexity. For finite automata, this problem is defined as follows:

- Input: Finite automata \( A_i = (Q_i, \Sigma, \delta_i, q_{0,i}, F_i) \), for \( 1 \leq i \leq m \).
- Question: Is there a word \( w \) that is accepted by all \( A_i \), i.e., \( \bigcap_{i=1}^{m} L(A_i) \neq \emptyset \)?

Observe that the automata have a common input alphabet. Note that the complexity of the non-emptiness problem for finite automata of a certain type is a lower bound for the Intersection Non-emptiness for this particular type of automata. Throughout the paper we are mostly interested in the complexity of the Intersection Non-emptiness problem for finite state devices whose languages are contained in a particular language class.

We study the computational complexity of the intersection non-emptiness for languages from the classes of the Straubing-Thérien \cite{38,41} and Cohen-Brzozowski’s dot-depth hierarchy \cite{10}. Both hierarchies are concatenation hierarchies that are defined by alternating the use of polynomial and Boolean closures. Let’s be more specific. Let \( \Sigma \) be a finite alphabet. A language \( L \subseteq \Sigma^* \) is a marked product of the languages \( L_0, L_1, \ldots, L_k \), if \( L = L_0 a_1 L_1 \cdots a_k L_k \), where the \( a_i \)’s are letters. For a class of languages \( \mathcal{M} \), the polynomial closure of \( \mathcal{M} \) is the set of languages that are finite unions of marked product of languages from \( \mathcal{M} \).

The concatenation hierarchy of basis \( \mathcal{M} \) (a class of languages) is defined as follows (also refer to \cite{31}):

1. \( \mathcal{M}_{n+1/2} \), that is, level \( n + 1/2 \), is the polynomial closure of level \( n \) and
2. \( \mathcal{M}_{n+1/2} \), that is, level \( n + 1/2 \), is the Boolean closure of level \( n + 1/2 \).

The basis of the dot-depth hierarchy is the class of all finite and co-finite languages\footnote{The dot-depth hierarchy, apart level \( \mathcal{B}_0 \), coincides with the concatenation hierarchy starting with the language class \( \{\emptyset, \{\lambda\}, \Sigma^+, \Sigma^*\} \).} and their classes are referred to as \( \mathcal{B}_n \) (\( \mathcal{B}_{n+1/2} \), respectively), while the basis of the Straubing-Thérien hierarchy is the class of languages that contains only the empty set and \( \Sigma^* \) and their classes.
are denoted by $\mathcal{L}_n$ ($\mathcal{L}_{n+1/2}$, respectively). Their inclusion relation is given by

$$B_{n+1/2} \subseteq B_{n+1} \subseteq B_{n+3/2} \quad \text{and} \quad L_{n+1/2} \subseteq L_{n+1} \subseteq L_{n+3/2},$$

for $n \geq 0$, and

$$L_{n-1/2} \subseteq B_{n-1/2} \subseteq L_{n+1/2} \quad \text{and} \quad L_n \subseteq B_n \subseteq L_{n+1},$$

for $n \geq 1$. In particular, $L_0 \subseteq B_0$, $B_0 \subseteq B_{1/2}$, and $L_0 \subseteq L_{1/2}$. Both hierarchies are infinite for alphabets of at least two letters and completely exhaust the class of star-free languages, which can be described by expressions that use union, concatenation, and complementation, but no Kleene star operation. For singleton letter alphabets, both hierarchies collapse to $B_0$ and $L_1$, respectively. Next, we describe the first few levels of each of these hierarchies:

**Straubing-Thérien hierarchy:** A language of $\Sigma^*$ is of level 0 if and only if it is empty or equal to $\Sigma^*$. The languages of level 1/2 are exactly those languages that are a finite (possibly empty) union of languages of the form $\Sigma^* a_1 \Sigma^* a_2 \cdots a_k \Sigma^*$, where the $a_i$’s are letters from $\Sigma$. The languages of level 1 are finite Boolean combinations of languages of the form $\Sigma^* a_1 \Sigma^* a_2 \cdots a_k \Sigma^*$, where the $a_i$’s are letters. These languages are also called piecewise testable languages. In particular, all finite and co-finite languages are of level 1.

Finally, the languages of level 3/2 of $\Sigma^*$ are the finite unions of languages of the form $\Sigma_0 a_1 \Sigma_1 a_2 \cdots a_k \Sigma_k$, where the $a_i$’s are letters from $\Sigma$ and the $\Sigma_i$’s are subsets of $\Sigma$.

**Dot-depth hierarchy:** A language of $\Sigma^*$ is of dot-depth (level) 0 if and only if it is finite or co-finite. The languages of dot-depth 1/2 are exactly those languages that are a finite union of languages of the form $u_0 \Sigma^* u_1 \Sigma^* u_2 \cdots u_{k-1} \Sigma^* u_k$, where $k \geq 0$ and the $u_i$’s are words from $\Sigma^*$. The languages of dot-depth 1 are finite Boolean combinations of languages of the form $u_0 \Sigma^* u_1 \Sigma^* u_2 \cdots u_{k-1} \Sigma^* u_k$, where $k \geq 0$ and the $u_i$’s are words from $\Sigma^*$.

It is worth mentioning that in [36], it was shown that partially ordered NFAs (with multiple initial states) characterize the class $L_{3/2}$, while partially ordered DFAs characterize the class of $R$-trivial languages [6], a class that is strictly in between $L_1$ and $L_{3/2}$. For an automaton $A$ with input alphabet $\Sigma$, a state $q$ is reachable from a state $p$, written $p \leq q$, if there is a word $w \in \Sigma^*$ such that $q \in \delta(p, w)$. An automaton is partially ordered if $\leq$ is a partial order. Partially ordered automata are sometimes also called acyclic or weakly acyclic automata.

We refer to a partially ordered NFA (DFA, respectively) as poNFA (poDFA, respectively). with input alphabet

**Theorem 1.** For each level $\mathcal{L}$ of the Straubing-Thérien or the dot-depth hierarchies, the $\mathcal{L}$-MEMBERSHIP problem for NFAs is PSPACE-hard, even when restricted to binary alphabets.

**Proof.** For the PSPACE-hardness, note that each of the classes contains $\{0, 1\}^*$ and is closed under quotients, since each class is a positive variety. As NON-UNIVERSALITY is PSPACE-hard for NFAs, we can apply Theorem 3.1.1 of [18], first reducing regular expressions to NFAs.

For some of the lower levels of the hierarchies, we also have containment in PSPACE, but in general, this is unknown, as it connects to the famous open problem if, for instance, $\mathcal{L}$-MEMBERSHIP is decidable for $\mathcal{L} = L_5$; see [26,32] for an overview on the decidability status of these questions. Checking for $L_0$ up to $L_2$ and $B_0$ up to $B_1$ containment for DFAs can be done in NL and is also complete for this class by ideas similar to the ones used in [8].
A language of $\Sigma^*$ belongs to level 0 of the Straubing-Thérien hierarchy if and only if it is empty or $\Sigma^*$. The Intersection Non-emptiness problem for language from this language family is not entirely trivial, because we have to check for emptiness. Since by our problem definition the property of a language being a member of level 0 is a promise, we can do the emptiness check within $\text{AC}^0$, since we only have to verify whether the empty word belongs to the language $L$ specified by the automaton. In case $\varepsilon \in L$, then $L = \Sigma^*$; otherwise $L = \emptyset$. Since in the definition of finite state devices we do not allow for $\varepsilon$-transitions, we thus only have to check whether the initial state is also an accepting one. Therefore, we obtain:

**Theorem 2.** The Intersection Non-emptiness problem for DFAs or NFAs accepting languages from $L_0$ belongs to $\text{AC}^0$.

For the languages of level $L_{1/2}$ we find the following completeness result.

**Theorem 3.** The Intersection Non-emptiness problem for NFAs accepting languages from $L_{1/2}$ is $\text{NL}$-complete. Moreover, the problem remains $\text{NL}$-hard even if we restrict the input to NFAs over a unary alphabet. If the input instance contains only DFAs, the problem becomes $L$-complete (under weak reductions).

Hardness is shown by standard reductions from variants of graph accessibility [16,40].

**Lemma 4.** The Intersection Non-emptiness problem for NFAs accepting languages from $L_{1/2}$ is $\text{NL}$-hard. If the input instance contains only DFAs, the problem becomes $L$-hard under weak reductions.

**Proof.** The $\text{NL}$-complete graph accessibility problem $2$-$\text{GAP}$ [10] is defined as follows: given a directed graph $G = (V,E)$ with outdegree (at most) two and two vertices $s$ and $t$. Is there a path linking $s$ and $t$ in $G$? The problem remains $\text{NL}$-complete if the outdegree of every vertex of $G$ is exactly two and if the graph is ordered, that is, if $(i,j) \in E$, then $i < j$ must be satisfied. The complexity of the reachability problem drops to $L$-completeness, if one considers the restriction that the outdegree is at most one. In this case the problem is referred to as $1$-$\text{GAP}$ [16].

First we consider the Intersection Non-emptiness problem for NFAs. The $\text{NL}$-hardness is seen as follows: let $G = (V,E)$ and $s, t \in V$ be an ordered 2-GAP instance. Without loss of generality, we assume that $V = \{1, 2, \ldots, n\}$, the source vertex $s = 1$, and the target vertex $t = n$. From $G$ we construct a unary NFA $A = (V, \{a\}, \delta, 1, n)$, where $\delta(i, a) = \{ j \mid (i,j) \in E \} \cup \{ i \}$. The 2-GAP instance has a solution if and only if the language accepted by $A$ is non-empty. Moreover, by construction the automaton accepts a language of level $1/2$, because (i) the NFA without $a$-self-loops is acyclic, since $G$ is ordered, and thus does not contain any large cycles and (ii) all states do have self-loops. This proves the hardness and moreover the $\text{NL}$-completeness.

Finally, we concentrate on the $L$-hardness of the Intersection Non-emptiness problem for DFAs. Here we use the 1-GAP variant to prove our result. Let $G = (V,E)$ and $s, t \in V$ be a 1-GAP instance, where we can assume that $V = \{1, 2, \ldots, n\}$, $s = 1$, and $t = n$. From $G$ we construct a unary DFA $A = (V, \{a\}, \delta, 1, n)$ with $\delta(i, a) = j$, for $(i,j) \in E$ and $1 \leq i < n$, and $\delta(n, a) = n$. By construction the DFA $A$ accepts either the empty language or a unary

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2 Some form of $\text{AC}^0$ reducibility can be employed.
language where all words are at least of a certain length. In both cases \( L(A) \) is a language from level 1/2 of the Straubing-Thérien hierarchy. Moreover, it is easy to see that there is a path in \( G \) linking \( s \) and \( t \) if and only if \( L(A) \neq \emptyset \). Hence, this proves \( L \)-hardness and moreover \( L \)-completeness for an intersection non-emptiness instance of DFAs.

It remains to show containment in logspace. To this end, we utilize an alternative characterization of the languages of level 1/2 of the Straubing-Thérien hierarchy as exactly those languages that are shuffle ideals. A language \( L \) is a shuffle ideal if, for every word \( w \in L \) and \( v \in \Sigma^* \), the set \( w \uplus v \) is contained in \( L \), where \( w \uplus v := \{ w_0v_0w_1v_1 \ldots w_kv_k \mid w = w_0w_1 \ldots w_k \text{ and } v = v_0v_1 \ldots v_k \text{ with } w_i, v_i \in \Sigma^* \text{, for } 0 \leq i \leq k \} \). The operation \( \uplus \) naturally generalizes to sets. For the level \( L_{1/2} \), we find the following situation.

\( \blacktriangleright \) **Lemma 5.** Let \( m \geq 1 \) and languages \( L_i \subseteq \Sigma^* \), for \( 1 \leq i \leq m \), be shuffle ideals, i.e., they belong to \( L_{1/2} \). Then, \( \bigcap_{i=1}^m L_i \neq \emptyset \) iff the shuffle ideal \( L_1L_2 \cdots L_m \neq \emptyset \) for every \( i \) with \( 1 \leq i \leq m \). Finally, \( L_i \neq \emptyset \), for \( 1 \leq i \leq m \), iff \( (a_1a_2 \ldots a_k)^{\ell_i} \in L_i \), where \( \Sigma = \{ a_1, a_2, \ldots, a_k \} \) and the shortest word in \( L_i \) is of length \( \ell_i \).

**Proof.** The implication from left to right holds, because if \( \bigcap_{i=1}^m L_i \neq \emptyset \), then there is a word \( w \) that belongs to all \( L_i \), and hence the concatenation \( L_1L_2 \cdots L_m \) is nonempty, too. Since this argument has not used the prerequisite that the \( L_i \)'s belong to the first half level of the Straubing-Thérien hierarchy, this implication does hold in general.

For the converse implication, recall that a language \( L \) of the first half level is a finite (possibly empty) union of languages of the form \( \Sigma^*a_1\Sigma^*a_2 \cdots \Sigma^*a_k \Sigma^* \), where the \( a_i \)'s are letters. Hence, whenever word \( w \) belongs to \( L \), any word of the form \( u w v \) with \( u, v \in \Sigma^* \) is a member of \( L \), too. Now assume that \( L_1L_2 \cdots L_m \neq \emptyset \), which can be witnessed by words \( w_i \in L_i \), for \( 1 \leq i \leq m \). But then the word \( w_1w_2 \cdots w_m \) belongs to every \( L_i \), by setting \( u = w_1w_2 \ldots w_{i-1} \) and \( v = w_i+1w_{i+2} \ldots w_m \) and using the argument above. Therefore, the intersection of all \( L_i \), i.e., the set \( \bigcap_{i=1}^m L_i \), is nonempty, because of the word \( w_1w_2 \ldots w_m \).

The statement that \( L_1L_2 \cdots L_m \) is an ideal and that \( L_1L_2 \cdots L_m \neq \emptyset \) if and only if \( L_i \neq \emptyset \), for every \( i \) with \( 1 \leq i \leq m \), is obvious.

For the last statement, assume \( \Sigma = \{ a_1, a_2, \ldots, a_k \} \). The implication from right to left is immediate, because if \( (a_1a_2 \ldots a_k)^{\ell_i} \in L_i \), for \( \ell_i \) as specified above, then \( L_i \) is non-empty. Conversely, if \( L_i \) is non-empty, then there is a shortest word \( w \) of length \( \ell_i \) that is contained in \( L_i \). But then \( (a_1a_2 \ldots a_k)^{\ell_i} \) belongs to \( w \uplus \Sigma^* \), which by assumption is a subset of the language \( L_i \), since \( L_i \) is an ideal. Therefore, \( L_i \neq \emptyset \) implies \( (a_1a_2 \ldots a_k)^{\ell_i} \in L_i \), which proves the stated claim.

\( \blacktriangleright \)

**Lemma 6.** The Intersection Non-emptiness problem for NFAs accepting languages from \( L_{1/2} \) belongs to NL. If the input instance contains only DFAs, the problem is solvable in L.

**Proof.** In order to solve the Intersection Non-emptiness problem for given finite automata \( A_1, A_2, \ldots, A_m \) with a common input alphabet \( \Sigma \), regardless of whether they are deterministic or nondeterministic, it suffices to check non-emptiness for all languages \( L(A_i) \), for \( 1 \leq i \leq m \), in sequence, because of Lemma 5. To this end, membership of the words \( (a_1a_2 \ldots a_k)^{\ell_i} \) in \( L_i \) has to be tested, where \( \ell_i \) is the length of the shortest word in \( L_i \). Obviously, all \( \ell_i \) are linearly bounded in the number of states of the appropriate finite automaton that accepts \( L_i \). Hence, for NFAs as input instance, the test can be done on a nondeterministic logspace-bounded Turing machine, guessing the computations in the individual NFAs on
the input word \((a_1a_2\ldots a_k)^e\). For DFAs as input instance, nondeterminism is not needed, so that the procedure can be implemented on a deterministic Turing machine.

4 NP-Completeness

In contrast to the Straubing-Thérien hierarchy, the Intersection Non-emptiness problem for languages from the dot-depth hierarchy is already NP-hard in the lowest level \(B_0\). More precisely, Intersection Non-emptiness for finite languages is NP-hard [33 Theorem 1] and \(B_0\) already contains all finite languages. Hence, the Intersection Non-emptiness problem for languages from the Straubing-Thérien hierarchy of level \(L_1\) and above is NP-hard, too. For the levels \(B_0, B_{1/2}, L_1,\) or \(L_{3/2}\), we give matching complexity upper bounds if the input are DFAs, yielding the first main result of this section proven in Subsection 4.1.

\textbf{Theorem 7.} The Intersection Non-emptiness problem for DFAs accepting languages from either \(B_0, B_{1/2}, L_1,\) or \(L_{3/2}\) is NP-complete. The same holds for poNFAs instead of DFAs. The results hold even for a binary alphabet.

For the level \(L_1\) of the Straubing-Thérien hierarchy, we obtain with the next main theorem a stronger result. Recall that if all input DFAs accept languages from \(L_{1/2}\), the Intersection Non-emptiness problem is L-complete due to Lemmata 4 and 6.

\textbf{Theorem 8.} The Intersection Non-emptiness problem for DFAs is NP-complete even if only one DFA accepts a language from \(L_1\) and all other DFAs accept languages from \(L_{1/2}\) and the alphabet is binary.

The proof of this theorem will be given in Subsection 4.2.

For the level \(B_0\), we obtain a complete picture of the complexity of the Intersection Non-emptiness problem, independent of structural properties of the input finite automata, i.e., we show that here the problem is NP-complete for general NFAs.

For the level \(L_{3/2}\), if the input NFA are from the class of poNFA, which characterize level \(L_{3/2}\), then the Intersection Non-emptiness problem is known to be NP-complete [27]. Recall that \(L_{3/2}\) contains the levels \(B_{1/2}\), and \(L_1\) and hence also languages from these classes can be represented by poNFAs. But if the input automata are given as NFAs without any structural property, then the precise complexity of Intersection Non-emptiness for \(B_{1/2}, L_1,\) and \(L_{3/2}\) is an open problem and narrowed by NP-hardness and membership in PSPACE. We present a “No-Go-Theorem” by proving that for an NFA accepting a co-finite language, the smallest equivalent poNFA is exponentially larger in Subsection 4.3.

\textbf{Theorem 9.} For every \(n \in \mathbb{N}_{\geq 1}\), there exists a language \(L_n \in B_0\) on a binary alphabet such that \(L_n\) is recognized by an NFA of size \(O(n^2)\), but the minimal poNFA recognizing \(L_n\) has more than \(2^{n-1}\) states.

While for NFAs the precise complexity for Intersection Non-emptiness of languages from \(L_1\) remains open, we can tackle this gap by narrowing the considered language class to commutative languages in level \(L_1\); recall that a language \(L \subseteq \Sigma^*\) is commutative if, for any \(a, b \in \Sigma\) and words \(u, v \in \Sigma^*\), we have that \(uabv \in L\) implies \(ubav \in L\). We show that for DFAs, this restricted Intersection Non-emptiness problem remains NP-hard, in case the alphabet is unbounded. Concerning membership in NP, we show that even for NFAs, the Intersection Non-emptiness problem for commutative languages is contained in NP in general and in particular for commutative languages on each level. This generalizes the case of unary NFAs. Note that for commutative languages, the Straubing-Thérien hierarchy collapses at level \(L_{3/2}\). See Subsection 4.4 for the proofs.
Theorem 10. The Intersection Non-emptiness problem is NP-hard for DFAs accepting commutative languages in $L_1$, but is contained in NP for NFAs accepting commutative languages that might not be star-free.

The proof of NP-hardness for commutative star-free languages in $L_1$ requires an arbitrary alphabet. However, we show that Intersection Non-emptiness is contained in XP for specific forms of NFAs such as poNFAs or DFAs accepting commutative languages, with the size of the alphabet as the parameter, i.e., for fixed input alphabets, our problem is solvable in polynomial time.

4.1 NP-Membership

Next, we focus on the NP-membership part of Theorem 7 and begin by proving that for $B_0$, regardless of whether the input automata are NFAs or DFAs, the Intersection Non-emptiness problem is contained in NP and therefore NP-complete in combination with [33].

Lemma 11. The Intersection Non-emptiness problem for DFAs or NFAs all accepting languages from $B_0$ is contained in NP.

Proof. Let $A_1, A_2, \ldots, A_m$ be NFAs accepting languages from $B_0$. If all NFAs accept co-finite languages, which can be verified in deterministic polynomial time, the intersection $\bigcap_{i=1}^m L(A_i)$ is non-empty. Otherwise, there is at least one NFA accepting a finite language, where the longest word is bounded by the number of states of this device. Hence, if $\bigcap_{i=1}^m L(A_i) \neq \emptyset$, there is a word $w$ of polynomial length in the length of the input that witnesses this fact. Such a $w$ can be nondeterministically guessed by a Turing machine checking membership of $w$ in $L(A_i)$, for all NFAs $A_i$, in sequence. This shows containment in NP as desired.

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Notice that Masopust and Krötzsch have shown in [27] that Intersection Non-emptiness for poDFAs and for poNFAs is NP-complete. Also the unary case is discussed there, which can be solved in polynomial time. We cannot directly make use of these results, as we consider arbitrary NFAs or DFAs as inputs, only with the promise that they accept languages from a certain level of the studied hierarchies. In order to prove that for the levels $B_0$, $B_{1/2}$, $L_1$, and $L_{3/2}$, the Intersection Non-emptiness problem for DFAs is contained in NP, it is sufficient to prove the claim for $L_{3/2}$ as all other stated levels are contained in $L_{3/2}$. We prove the latter statement by obtaining a bound, polynomial in the size of the largest DFA, on the length of a shortest word accepted by all DFAs. Therefore, we show that for a minimal poNFA $A$, the size of an equivalent DFA is lower-bounded by the size of $A$ and use a result of [27] for poNFAs. They have shown that given poNFAs $A_1, A_2, \ldots, A_m$, if the intersection of these automata is non-empty, then there exists a word of size at most $\sum_{i=1}^m d_i$, where $d_i$ is the depth of $A_i$ [27]. Here, the depth of $A_i$ is the length of the longest path (without self-loops) in the state graph of $A_i$. This result implies that the Intersection Non-emptiness problem for poNFAs accepting languages from $L_{3/2}$ is contained in NP. We will further use this result to show that the Intersection Non-emptiness problem for DFAs accepting languages from $L_{3/2}$ is NP-complete. First, we show that the number of states in a minimal poNFA is at most the number of classes in the Myhill-Nerode equivalence relation.

Lemma 12. Let $A = (Q, \Sigma, \delta, q_0, F)$ be a minimal poNFA. Then, $L(q_1, A) \neq L(q_2, A)$ for all states $q_1, q_2 \in Q$, where $qA$ is defined as $(Q, \Sigma, \delta, q, F)$.

Proof. Let $A = (Q, \Sigma, \delta, q_0, F)$ be a minimal poNFA and $q_1, q_2 \in Q$ be two states. Suppose that $L(q_1, A) = L(q_2, A)$. We have two cases.
By Lemma 12, we have that the number of states in a minimal poNFA is at most the number of classes of the Myhill-Nerode equivalence relation. Hence, given a DFA accepting a language \( L \in \mathcal{L}_{3/2} \), there exists a smaller poNFA that recognizes \( L \). By [27, Theorem 3.3], if the intersection is not empty, then there is a certificate of polynomial size.
4.2 NP-Hardness

Recall that by \cite{33} Theorem 1] INTERSECTION NON-EMPTINESS for finite languages accepted by DFAs is already NP-complete. As the level $B_0$ of the dot-depth hierarchy contains all finite language, the NP-hardness part of Theorem 7 follows directly from inclusion of language classes. Combining Lemma 13 and \cite{27} Theorem 3.3 with the inclusion between levels in the Straubing-Thérien and the dot-depth hierarchy, we conclude the proof of Theorem 7.

\vspace{0.5cm}

\textbf{Remark 14.} Recall that the dot-depth hierarchy, apart form $B_0$, coincides with the concatenation hierarchy starting with the language class $\{\emptyset, \{\lambda\}, \Sigma^+, \Sigma^*\}$. The INTERSECTION NON-EMPTINESS problem for DFAs or NFAs accepting only languages from $\{\emptyset, \{\lambda\}, \Sigma^+, \Sigma^*\}$ belongs to AC$^0$, by similar arguments as in the proof of Theorem 2.

We showed in Section 3 that INTERSECTION NON-EMPTINESS for DFAs, all accepting languages from $L_{1/2}$, belongs to L. If we allow only one DFA to accept a language from $L_1$, the problem becomes NP-hard. The statement also holds if the common alphabet is binary.

\vspace{0.5cm}

\textbf{Theorem 8.} The INTERSECTION NON-EMPTINESS problem for DFAs is NP-complete even if only one DFA accepts a language from $L_1$ and all other DFAs accept languages from $L_{1/2}$ and the alphabet is binary.

\vspace{0.5cm}

\textbf{Proof idea.} The reduction is from VERTEX COVER. Let $k \in \mathbb{N}_{\geq 0}$ and let $G = (V, E)$ be a graph with vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{e_0, e_1, \ldots, e_{m-1}\}$. The only words $w = a_0a_1\ldots a_k$ accepted by all DFAs will be of length exactly $n = \ell + 1$ and encode a vertex cover by: $v_j$ is in the vertex cover if and only if $a_j = 1$. Therefore, we construct for each edge $e_i = \{v_{i_1}, v_{i_2}\} \in E$, with $i_1 < i_2$, a DFA $A_{e_i}$, as depicted in Figure 2 that accepts the language $L(A_{e_i}) = \Sigma^i \cdot 1 \cdot \Sigma^{n-i-1} \cup \Sigma^i \cdot 1 \cdot \Sigma^{2n-i-1} \cup \Sigma^{2n+1}$. We show that $L(A_{e_i})$ is from $L_{1/2}$, as it also accepts all words of length at least $n + 1$. We further construct a DFA $A_{\leq k}$ that accepts all words of length exactly $n$ that contain at most $k$ letters 1. The finite language $L(A_{\leq k})$ is the only language from $L_1$ in the instance.

\vspace{0.5cm}

\textbf{Proof.} The NP-membership follows from Lemma 13 by inclusion of language classes. For the hardness, we give a reduction from the VERTEX COVER problem: given an undirected graph $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq V \times V$ and integer $k$. Is there a subset $S \subseteq V$ with $|S| \leq k$ and for all $e \in E, S \cap e \neq \emptyset$? If yes, we call $S$ a vertex cover of $G$ of size at most $k$.

Let $k \in \mathbb{N}_{\geq 0}$ and let $G = (V, E)$ be an undirected graph with vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{e_0, e_1, \ldots, e_{m-1}\}$. From $(G, k)$ we construct $m + 1$ DFAs over the common alphabet $\Sigma = \{0, 1\}$. The input word for these automata will encode which vertices are in the vertex cover. Therefore, we assume a linear order on $V$ indicated by the indices of the vertices. More precisely, a word accepted by all automata will have a 1 at position $j$ if and only if the vertex $v_j$ will be contained in the vertex cover $S$. For a word $w = a_0a_1\ldots a_k$ with $a_j \in \Sigma$ for $0 \leq j < \ell$ we denote $w[j] = a_j$. We may call a word $w$ of length $n$ a vertex cover and say that the vertex cover covers an edge $e = \{v_{j_1}, v_{j_2}\}$ if $w[j_1] = 1$ or $w[j_2] = 1$.

For every edge $e_i = \{v_{i_1}, v_{i_2}\} \in E$ with $i_1 < i_2$, we construct a DFA $A_{e_i}$ as depicted in Figure 2 consisting of two chains, one of length $n + 1$ and one of length $n - (i_1 + 1)$ (The length of a chain is the number of transitions in the chain). The DFA is defined as $A_{e_i} = (Q, \Sigma, \delta, q_0, F)$ with state set $Q = \{q^j | 0 \leq j \leq n + 1\} \cup \{q^j | i_1 + 1 \leq j \leq n\}$ and final states $F = \{q^{n+1}, q^n\}$. We first focus on the states $\{q^j | 0 \leq j \leq n + 1\}$. The idea is that there, the first $n + 1$ states correspond to the sequence of vertices and reading a 1 at position $j$ for which $v_j \in e_i$ will cause the automaton to switch to the chain consisting of states $\{q^j | i_1 + 1 \leq j \leq n\}$. There, only one state is accepting namely the state that
we reach after reading a vertex cover of length exactly \(n\) that satisfies the edge \(e_i\). Note that the paths from \(q_0\) to \(q^n\) are one transition shorter than the path from \(q_0\) to \(q^{n+1}\). To be more formal, we define \(\delta(q^1,1) = q^{n+1}\) and \(\delta(q^2,1) = q^{n+2}\). All other transitions are leading to the next state in the corresponding chain. Formally, we define \(\delta(q^i,0) = q^{i+1}\) and \(\delta(q^i,0) = q^{i+1}\) and for all \(0 \leq j \leq n\) with \(j \not\in \{i_1, i_2\}\), we define \(\delta(q^j, \sigma) = q^{j+1}\), for both \(\sigma \in \Sigma\), and for all \(i+1 \leq j \leq n-1\), we define \(\delta(q^j, \sigma) = q^{j+1}\). We conclude the definition of \(\delta\) by defining self-loops for the two accepting states, i.e., we define \(\delta(q^{n+1}, \sigma) = q^{n+1}\) and \(\delta(q^n, \sigma) = q^n\) for both \(\sigma \in \Sigma\). Clearly, \(A_{e_i}\) is deterministic and of size \(O(n)\).

Note that the only words of length exactly \(n\) that are accepted by \(A_{e_i}\) contain a 1 at position \(i_1\) or position \(i_2\) and therefore cover the edge \(e_i\). All other words accepted by \(A_{e_i}\) are of length at least \(n+1\). More precisely \(A_{e_i}\) accepts all words which are of size at least \(n+1\). Hence, we can describe the language accepted by \(A_{e_i}\) as

\[
L(A_{e_i}) = \Sigma_{i_1} \cdot 1 \cdot \Sigma_{i_2} \cdot 1 \cdot \Sigma \cdot \Sigma^* \cdot \Sigma \cdot \Sigma^* \cdot 1 \cdot \Sigma \cdot \Sigma^*.
\]

Consider a word \(w \in L(A_{e_i})\) of length \(n\). W.l.o.g., assume \(w[i_1] = 1\). If we insert into \(w\) one letter somewhere before or after position \(i_1\), then the size of \(w\) increases by 1 and hence \(w\) falls into the subset \(\Sigma^{\geq n+1}\) of \(L(A_{e_i})\). Hence, we can rewrite the language \(L(A_{e_i})\) by the following equivalent expression.

\[
L(A_{e_i}) = \Sigma_{i_1} \cdot 1 \cdot \Sigma_{i_2} \cdot 1 \cdot \Sigma_{i_1} \cdot \Sigma_{i_2} \cdot 1 \cdot \Sigma \cdot \Sigma \cdot 1 \cdot \Sigma \cdot \Sigma^* \cdot \Sigma \cdot \Sigma^*.
\]

As we can rewrite a language of the form \(\Sigma^e \Sigma^*\) equivalently as a union of languages of the form \(\Sigma^e w_i \Sigma^* w_2 \ldots w_i \Sigma^*\) for \(w_i \in \Sigma, \forall 1 \leq i \leq \ell\), it is clear that \(L(A_{e_i})\) is a language of level \(L_{1/2}\).

Next, we define a \(n\)-DFA \(A_{=n,\leq k}\) which accepts the finite language of all binary words of length \(n\) which contain at most \(k\) appearances of the letter 1. We define \(A_{=n,\leq k} = \{\{q^i \mid 0 \leq i \leq n+1, 0 \leq j \leq k\} \} \cup \{\delta(q^i, \sigma) \mid q^i \in A_{=n,\leq k}\} \}. The state graph of \(A_{=n,\leq k}\) is a \((n,k)\)-grid graph, where each letter increases the \(x\) dimension represented by the subscript \(i\) up to the value \(n+1\), and each letter that is a 1 increases the \(y\) dimension represented by the superscript \(j\) up to the value \(k+1\). More formally, we define \(\delta(q^i, \sigma) = q^{i+1}\), and \(\delta(q^i,1) = q^{i+1}\) for \(0 \leq i \leq n\) and \(0 \leq j \leq k\); and \(\delta(q^i, \sigma) = q^i\) for \(i = n+1\) or \(j = k+1\). The size of \(A_{=n,\leq k}\) is bounded by \(O(nk)\). For readability, we defined \(A_{=n,\leq k}\) as a non-minimal DFA. As \(L(A_{=n,\leq k})\) is finite, it is of level \(B_0 \subseteq L_1\).

By the arguments discussed above, the set of words accepted by all of the automata \((A_{e_i})_{e_i \in E}\) and \(A_{=n,\leq k}\) are of size exactly \(n\) and encode a vertex cover for \(G\) of size at most \(k\).

### 4.3 Large Partially Ordered NFAs

The results obtained in the last subsection left the precise complexity membership of \textsc{Intersection Non-emptiness} in the case of input automata being NFAs without any structural properties for the levels \(B_{1/2}, L_1\), and \(L_{3/2}\) open. We devote this subsection to the proof of Theorem 9 showing that already for languages of \(B_0\) being accepted by an NFA, the size of an equivalent minimal poNFA can be exponential in the size of the NFA.

\begin{theorem}
For every \(n \in \mathbb{N}_{\geq 1}\), there exists a language \(L_n \in B_0\) on a binary alphabet such that \(L_n\) is recognized by an NFA of size \(O(n^2)\), but the minimal poNFA recognizing \(L_n\) has more than \(2^{n-1}\) states.
\end{theorem}
We then define $M'_n$ and $M''_n$ over the alphabet $\{1, 2, \ldots, n\}$ as follows. The language $M'_n$ contains all the words of odd length, and $M''_n$ contains all the words in which there are two occurrences of some letter $i \in \{1, 2, \ldots, n\}$ with only letters smaller than $i$ appearing in between. Formally,

$$M'_n = \{ x \in \{1, 2, \ldots, n\}^* \mid |x| \text{ is odd} \},$$
$$M''_n = \{ xiyz \in \{1, 2, \ldots, n\}^* \mid i \in \{1, 2, \ldots, n\}, y \in \{1, 2, \ldots, i-1\}^* \}. $$

We then define $M_n$ by encoding $M'_n$, $M''_n$ and the language $L_n$ by encoding $M_n$ with the binary alphabet $\{a, b\}$. Let us consider the function $\phi_n : \{1, 2, \ldots, n\}^* \rightarrow \{a, b\}^*$ defined by $\phi(i_1 i_2 \ldots i_n) = a^{i_1} b^{i_2} a^{i_3} b^{i_4} \ldots a^{i_{n-1}} b^{i_n}$. We set $L_n \subseteq \{a, b\}^*$ as the union of $\phi_n(M_n)$ with the language $\{a, b\}^* \setminus \phi(\{1, 2, \ldots, n\}^*)$ containing all the words that are not a proper encoding of some word in $\{1, 2, \ldots, n\}^*$.

The statement of the theorem immediately follows from the following claim

\begin{itemize}
\item[1.] The languages $M_n$ and $L_n$ are cofinite, thus they are in $B_0$.
\item[2.] The languages $M_n$ and $L_n$ are recognized by NFAs of size $n + 4$, resp. $O(n^2)$.
\item[3.] Every pNFA recognizing either $M_n$ or $L_n$ has a size greater than $2^n - 1$.
\end{itemize}

**Proof of Item 1** We begin by proving that $M_n$ is cofinite. Note that, by itself, the language $M'_n$ is not in $B_0$, as it is not even star-free. We show that $M''_n$ is cofinite, which directly implies that $M_n = M'_n \cup M''_n$ is also cofinite. This follows from the fact that every word $u \in \{1, 2, \ldots, n\}^*$ satisfying $|u| \geq 2^n$ is in $M''_n$. This is easily proved by induction on $n$:

If $n = 1$, we immediately get that $1^j \in M'_n$ for every $j \geq 2 = 2^1$: such a word contains two adjacent occurrences of 1. Now suppose that $n > 1$, and that the property holds for $n - 1$.

Every word $u \in \{1, 2, \ldots, n\}^*$ satisfying $|u| \geq 2^n$ can be split into two parts $u_0, u_1$ such that $|u_0|, |u_1| \geq 2^{n-1}$. We consider two possible cases, and prove that $u \in M''_n$ in both of them.

1. If either $u_0$ or $u_1$ contains no occurrence of the letter $n$, then by the induction hypothesis, either $u_0 \in M''_{n-1}$ or $u_1 \in M''_{n-1}$, which directly implies that $u \in M''_n$.

2. If both $u_0$ and $u_1$ contain (at least) one occurrence of the letter $n$, then $u \in M''_n$ since it contains two occurrences of the letter $n$ with only letters smaller than $n$ appearing in between (the latter part trivially holds, as $n$ is the largest letter).

Finally, we also get that $L_n$ is cofinite: for all $u \in \{a, b\}^*$ satisfying $|u| \geq 2^n \cdot n$, either $u$ is not a proper encoding of a word of $\{1, 2, \ldots, n\}^*$, thus $u \not\in L_n$, or $u$ encodes a word $v \in \{1, 2, \ldots, n\}^*$ satisfying $|v| \geq 2^n$, hence $v \not\in M_n$, which again implies that $u \not\in L_n$.

**Proof of Item 2** We first construct an NFA $A$ of size $n + 4$ recognizing $M_n = M'_n \cup M''_n$ as the disjoint union of an NFA $A'$ (Figure 3) of size 2 recognizing $M'_n$ and an NFA $A''$ (Figure 4) of size $n + 2$ recognizing $M''_n$. The language $M'_n$ of words of odd length is trivially recognized by an NFA of size 2, thus we only need to build an NFA $A'' = (Q, \{1, 2, \ldots, n\}, \delta, q_f, \{q_F\})$ of size $n + 2$ that recognizes $M''_n$. The state space $Q$ is composed of the start state $q_1$, the single final state $q_F$, and $n$ intermediate states $\{q_1, q_2, \ldots, q_n\}$. The NFA $A''$ behaves in three phases:

---

3 The languages $(M'_n)_{n \in \mathbb{N}^+}$ were previously studied in [21] with a game-theoretic background. We also refer to [29] for similar “fractal languages.”
1. First, $A''$ loops over its start state until it non-deterministically guesses that it will read two copies of some $i \in \Sigma$ with smaller letters in between: $\delta(q_1, i) = \{q_1, q_i\}$ for all $i \in \Sigma$.
2. To check its guess, $A''$ loops in $q_i$ while reading letters smaller than $i$ until it reads a second $i$: $\delta(q_i, j) = \{q_i\}$ for all $j \in \{1, 2, \ldots, i-1\}$ and $\delta(q_i, i) = \{q_F\}$.
3. The final state $q_F$ is an accepting sink: $\delta(q_F, j) = \{q_F\}$ for all $j \in \Sigma$.

This definition guarantees that $A''$ accepts the language $M''_n$.

Finally, we build an NFA $B$ of size $O(n^2)$ that recognizes $L_n$ by following similar ideas. Once again, $B$ is defined as the disjoint union of two NFAs $B'$ and $B''$: The NFA $B'$ uses $4n$ states to check that either the input is not a proper encoding, or the input encodes a word $u \in \{1, 2, \ldots, n\}^*$ of odd length. Then, the NFA $B''$ with $O(n^2)$ states is obtained by adapting the NFA $A''$ to the encoding of the letters $\{1, 2, \ldots, n\}$: we split each of the $2n$ intermediate transitions of $A''$ into $n$ parts by adding $n-1$ states, and we add $2(n-1)$ states to each self-loop of $A''$ in order to check that the encoding of an adequate letter is read.

Proof of Item 3. It is sufficient to prove the result for $M_n$, as we can transform each poNFA $A = (Q, \{a,b\}, \delta_A, q_I, F)$ recognizing $L_n$ into a poNFA $B = (Q, \{1, 2, \ldots, n\}, \delta_B, q_I, F)$ recognizing $M_n$ with the same set of states by setting $\delta_B(q, i) = \delta_A(q, a^ib^{n-i})$.

Note that, by itself, the language $M''_n$ is recognized by the poNFA $A$ of size $n + 2$ defined in the proof of Item 2. Let $A'$ be a poNFA recognizing $M_n$. To show that $A'$ has more than $2^{n-1}$ states, we study its behavior on the Zimin words, defined as follows:

Let $u_1 = 1$ and $u_j = u_{j-1}ju_{j-1}$ for all $1 < j \leq n$.

For instance, $u_4 = 121312141213121$. It is known that $|u_j| = 2^j - 1$ and $u_j \notin M''_n$ for every $1 \leq j \leq n$ [21]. These two properties are easily proved by induction on $j$: Trivially, $u_1$ is
Moreover, the hierarchy collapses at level one. \[ |a| = |b| = 2 \cdot |a| + 1 = 2 \cdot (2^{j-1} - 1) + 1 = 2^j - 1; \]

To prove the induction step for the second property, we suppose, towards building a contradiction, that \( u_j \in M''_n \). Then \( u_j \) contains two occurrences of some letter \( i \in \{1, 2, \ldots, n\} \) with only letters smaller than \( i \) appearing in between. Since \( u_j \) contains only one occurrence of the letter \( j \) and no letter is greater than \( j \), \( i \) is strictly smaller than \( j \). Moreover, as only letters smaller than \( i \) (thus no \( j \)) can appear between these two occurrences, they both need to appear in one of the copies of \( u_{j-1} \). Therefore \( u_{j-1} \) is also in \( M''_n \), which contradicts the induction hypothesis.

To conclude, remark that the word \( u_n \) is not in \( M''_n \), but since \( |u_n| = 2^n - 1 \) is odd, it is in \( M_n = L(A') \). Consider a sequence \( \rho \in Q^* \) of states leading \( A' \) from its start state to a final state over the input \( u_n \), Observe that the word \( u_n \) contains \( 2^n \) occurrences of the letter \( 1 \), and deleting (any) one of these occurrences results in a word of even length that is still not in \( M''_n \), thus it is also not in \( M_n = L(A') \). This proves that the sequence \( \rho \) cannot loop over any of the \( 1 \)'s in \( u_n \). Moreover, as \( A' \) is partially ordered by assumption, once it leaves a state, it can never return to it. Therefore, \( \rho \) contains at least \( 2^n - 1 + 1 \) distinct states while processing the \( 2^n - 1 \) occurrences of \( 1 \) in \( u_n \), which shows that the automaton \( A' \) has more than \( 2^n - 1 \) many states.

This concludes the proof.

### 4.4 Commutative Star-Free Languages

In the case of commutative languages, we have a complete picture of the complexities for both hierarchies, even for arbitrary input NFAs. Observe, that commutative languages generalize unary languages, where it is known that for unary star-free languages both hierarchies collapse. For commutative star-free languages, a similar result holds, employing [17, Prop. 30].

**Theorem 16.** For commutative star-free languages the levels \( L_n \) of the Straubing-Thérien and \( B_n \) of the dot-depth hierarchy coincide for all full and half levels, except for \( L_0 \) and \( B_0 \). Moreover, the hierarchy collapses at level one.

**Proof.** The strict inclusion \( L_0 \subset B_0 \) even in the commutative case is obvious. Since \( L_{1/2} \subset B_{1/2} \) we only need to show the converse inclusion in the case of commutative languages. For the sake of notational simplicity, we shall give the proof only in a special case. Observe that, by commutativity, if \( \Sigma^* \alpha \beta \Sigma^* \subseteq L \), then \( \Sigma^* \alpha \Sigma^* \beta \Sigma^* \subseteq L \); moreover, \( \Sigma^* \alpha \beta \Sigma^* \subseteq \Sigma^* \alpha \Sigma^* \beta \Sigma^* \). Using this idea repeatedly for marked products as they describe languages from \( B_{1/2} \), we can write them as equivalent polynomials used for defining languages from \( L_{1/2} \).

It remains to show that every commutative star-free language is contained in \( L_1 \). As shown in [17, Prop. 30], every star-free commutative language can be written as a finite union of languages of the form \( L = \text{perm}(u) \cup \Gamma^* \) for some \( u \in \Sigma^* \) and \( \Gamma \subseteq \Sigma \). Here \( \text{perm}(u) = \{ w \in \Sigma^* \mid |w|_a = |w|_{\alpha} \text{ for every } a \in \Sigma \} \), where \( |w|_a \) is equal to the number of occurrences of \( a \) in \( w \). Since \( \text{perm}(u) \) is a finite language, clearly, language \( L \) is equal to the finite union of all \( v \cup \Gamma^* \) for \( v \in \text{perm}(u) \), and thus belongs to \( L_{1/2} \), since \( \Gamma \subseteq \Sigma \).
Now, note that \( v \cup \Sigma^* = \Sigma^* v_1 \Sigma^* \cdots \Sigma^* v_{|v|} \Sigma^* \), where \( v = v_1 \cdots v_{|v|} \) with \( v_i \in \Sigma \), is in level one of the hierarchy. Further,

\[
v \cup \Gamma^* = (v \cup \Sigma^*) \cap \bigcup_{a \in \Sigma \setminus \Gamma} \text{perm}(va) \cup \Sigma^*.
\]

Hence, we can conclude containment in \( \mathcal{L}_1 \).

Next we will give the results, summarized in Theorem 10, for the case of the commutative (star-free) languages. The \( \text{NP} \)-hardness follows by a reduction from \( 3\text{-CNF-SAT} \).

\begin{lemma}

The Intersection Non-emptiness problem is \( \text{NP} \)-hard for DFAs accepting commutative languages in \( \mathcal{L}_1 \).

\end{lemma}

\begin{proof}
The \( \text{NP} \)-complete \( 3\text{-CNF-SAT} \) problem is defined as follows: given a Boolean formula \( \varphi \) as a set of clauses \( C = \{c_1, c_2, \ldots, c_m\} \) over a set of variables \( V = \{x_1, x_2, \ldots, x_n\} \) such that \( |c_i| \leq 3 \) for \( i \leq m \). Is there a variable assignment \( \beta : V \to \{0, 1\} \) such that \( \varphi \) evaluates to true under \( \beta \)?

Let \( \varphi \) be a Boolean formula in \( 3\text{-CNF} \) with clause set \( C = \{c_1, c_2, \ldots, c_m\} \) and variable set \( V = \{x_1, x_2, \ldots, x_n\} \). Let \( \Sigma = \{x_1, x_2, \ldots, x_n, \overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\} \). It is straightforward to construct polynomial-size DFAs for the following languages from \( \mathcal{B}_1 \):

\[
L_{c_i} = \bigcup_{x \in c_i} \Sigma^* x \Sigma^* \quad \text{and} \quad L_{x_j} = \Sigma^* \setminus (\Sigma^* x_j \Sigma^* \overline{x_j} \Sigma^* \cup \Sigma^* \overline{x_j} \Sigma^* x_j \Sigma^*),
\]

where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Then, the intersection of all \( L_{c_i} \) and all \( L_{x_j} \) is non-empty if and only if the \( 3\text{-CNF-SAT} \) instance \( \varphi \) is satisfiable.

The upper bound shown next also holds for arbitrary commutative languages.

\begin{theorem}
The Intersection Non-emptiness problem for NFAs accepting arbitrary, i.e., not necessarily star-free, commutative languages is in \( \text{NP} \).

\end{theorem}

\begin{proof}
It was shown in [37] that Intersection Non-emptiness is \( \text{NP} \)-complete for unary NFAs as input. Fix some order \( \Sigma = \{a_1, a_2, \ldots, a_r\} \) of the input alphabet. Let \( A_1, A_2, \ldots, A_m \) be the NFAs accepting commutative languages with \( A_i = (Q_i, \Sigma, \delta_i, q_0, i, F_i) \) for \( 1 \leq i \leq m \). Without loss of generality, we may assume that every \( F_i \) is a singleton set, namely \( F_i = \{q_{f,i}\} \).

For each \( 1 \leq i \leq m \) and \( 1 \leq j \leq r \), let \( B_{i,j} \) be the automaton over the unary alphabet \( \{a_j\} \) obtained from \( A_i \) by deleting all transitions labeled with letters different from \( a_j \) and only retaining those labeled with \( a_j \). Each \( B_{i,j} \) will have one initial and one final state. Let \( \tilde{q}_0 = (q_{0,1}, q_{0,2}, \ldots, q_{0,m}) \) be the tuple of initial states of the NFAs; they are the initial states of \( B_{1,1}, B_{2,1}, \ldots, B_{m,1} \), respectively. Then, nondeterministically guess further tuples \( \tilde{q}_j \) from \( Q_1 \times Q_2 \times \ldots \times Q_m \) for \( 1 \leq j \leq r - 1 \). The \( j \)th tuple is considered as collecting the final states of the \( B_{i,j} \) but also as the start states for the \( B_{i,j+1} \). Finally, let \( \tilde{q}_f = (q_{f,1}, q_{f,2}, \ldots, q_{f,m}) \) and consider this as the final states of \( B_{1,r}, B_{2,r}, \ldots, B_{m,r} \). Then, for each \( 1 \leq j \leq r \) solve Intersection Non-emptiness for the unary automata \( B_{1,j}, B_{2,j}, \ldots, B_{m,j} \). If there exist words \( w_j \) in the intersection of \( L(B_{1,j}), L(B_{2,j}), \ldots, L(B_{m,j}) \), for each \( 1 \leq j \leq r \), then, by commutativity, there exists one in \( a_1^*a_2^* \cdots a_r^* \), namely, \( w_1w_2 \cdots w_m \), and so the above procedure finds it. Conversely, if the above procedure finds a word, this is contained in the intersection of the languages induced by the \( A_i \)’s.

For fixed alphabets, we have a polynomial-time algorithm, showing that the problem is in \( \text{XP} \) for alphabet size as a parameter, for a class of NFAs generalizing, among others,
poNFA s and DFAs (accepting star-free languages). This is in contrast to the other results on the INTERSECTION NON-EMPTINESS problem in this paper. We say that an NFA $A = (Q, \Sigma, \delta, q_0, F)$ is totally star-free, if the language accepted by $q_A = (Q, \Sigma, \delta, q, \{ p \})$ is star-free for any states $q, p \in Q$. For instance, partially ordered NFAs are totally star-free.

An example of a non-totally star-free NFA accepting a star-free language is given next. Consider the following NFA $A = (\{ q_0, q_1, q_2, q_3 \}, \delta, q_0, \{ q_0, q_2 \})$ with $\delta(q_0, a) = \{ q_1, q_2 \}$, $\delta(q_1, a) = \{ q_0 \}$, $\delta(q_2, a) = \{ q_3 \}$, and $\delta(q_3, a) = \{ q_2 \}$ that accepts the language $\{ a \}^*$. The automaton is depicted in Figure 5. Yet, neither $L(q_0, A_{q_0}) = \{ a \}^*$ nor $L(q_0, A_{q_2}) = \{ a \} \{ a a \}^* \cup \{ \varepsilon \}$ are star-free.

The proof of the following theorem uses classical results of Chrobak and Schützenberger [9, 35].

> **Theorem 19.** The INTERSECTION NON-EMPTINESS problem for totally star-free NFAs accepting star-free commutative languages, i.e., commutative languages in $L(3/2)$, is contained in XP (with the size of the alphabet as the parameter).

The proof of Theorem 19 is based on a combinatorial number theoretical result that might be of independent interest.

> **Lemma 20.** Let $n \geq 1$ and $t_i, p_i \in \mathbb{N}_{\geq 0}$ for $1 \leq i \leq n$. Set $X = \bigcup_{i=1}^n (t_i + \mathbb{N}_{\geq 0} \cdot p_i)$, where $\mathbb{N}_{\geq 0} \cdot p_i = \{ x \cdot p_i \mid x \in \mathbb{N}_{\geq 0} \}$. If there exists a threshold $T \geq 0$ such that $\{ x \in \mathbb{N}_{\geq 0} \mid x \geq T \} \subseteq X$, then already for $T_{\text{max}} = \max \{ t_i \mid 1 \leq i \leq n \}$, we find $\{ x \in \mathbb{N}_{\geq 0} \mid x \geq T_{\text{max}} \} \subseteq X$.

**Proof.** The assumption basically says that every integer $y$ greater than $T - 1$ is congruent to $t_\ell$ modulo to $t_\ell$ for some $1 \leq \ell \leq n$. More specifically, if $x$ is an arbitrary number with $x \geq T_{\text{max}}$, then $y = x + \text{lcm}(p_1, p_2, \ldots, p_n)$ is congruent to $t_\ell$ modulo $p_\ell$ for some $1 \leq \ell \leq n$. But this implies that $x$ itself is congruent to $t_\ell$ modulo $p_\ell$, and so, as $x \geq t_\ell$, we can write $x = t_\ell + k_\ell \cdot p_\ell$ for some $k_\ell \geq 0$, i.e., $x \in X$. □

This number-theoretic result can be used to prove a polynomial bound for star-free unary languages on an equivalence resembling Schützenberger’s characterization of star-freeness [35].

> **Lemma 21.** Let $L$ be a unary star-free language specified by an NFA $A$ with $n$ states. Then, there is a number $N$ of order $O(n^2)$ such that $a^N \in L$ if and only if for all $k \in \mathbb{N}_{\geq 0}$, $a^{N+k} \in L$.

**Proof.** By a classical result of Chrobak [9], the given NFA $A$ on $n$ states can be transformed into a normal form where we have an initial tail with length at most $O(n^2)$ that branches at a common endpoint into several cycles, where every cycle is of size at most $n$, see [9, Lemma 4.3]. Moreover, this transformation can be performed in polynomial time [13]. Note that a unary star-free language is either finite or co-finite [9]. If $L$ is finite, then there are no final states on the cycles and we can set $N$ to be equal to the length of the tail, plus one. Otherwise, if $L \subseteq \{ a \}^*$ is co-finite, then it can be expressed as a union of a finite language corresponding to the final states on the tail and finitely many languages of the form $\{ a^\ell \mid \ell \in \{ t + \mathbb{N}_{\geq 0} \cdot p \} \}$, where the numbers $t$ and $p$ are induced by the Chrobak normal form. Then we can apply...
Lemma 20, where the set $X$ is built from the $t$’s and $p$’s, and where the $t$’s are bounded by $T_{\text{max}}$, the sum of the longest tail and the largest cycle, plus one. Note that $T_{\text{max}}$ is in $O(n^3)$ and that the threshold from Lemma 20 guarantees that every word $a^\ell$ with $\ell \geq T_{\text{max}}$ is a member of $L$, as desired.

\begin{theorem}

The Intersection Non-emptiness problem for totally star-free NFAs accepting star-free commutative languages, i.e., commutative languages in $\mathcal{L}_{3/2}$, is contained in $\text{XP}$ (with the size of the alphabet as the parameter).

\end{theorem}

\begin{proof}

Let $A_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$, for $i \in \{1, 2, \ldots, m\}$, be totally star-free NFAs accepting commutative languages. Let $n_i = |Q_i|$ be the number of states of $A_i$. Fix some order $\Sigma = \{a_1, a_2, \ldots, a_r\}$.

For $1 \leq i \leq m$ and $1 \leq j \leq r$, as well as $q, p \in Q_i$, let the automaton $B_{i,j,q,p} = (Q_i, \{a_j\}, \delta_i, q, \{p\})$ be obtained from $A_i$ by deleting all transitions not labeled with the letter $a_j$ and only retaining those labeled with $a_j$. Further, let $A_{i,q,p}$ be obtained from $A_i$ by taking $q$ as (new) initial state and $p$ as the new (and only) final state. As $A_i$ is totally star-free, $L(A_i,q,p)$ is also star-free. By Schützenberger’s Theorem characterizing star-freeness [35], it is immediate that $L(A_{i,q,p}) \cap \Gamma^*$ is also star-free for each $\Gamma \subseteq \Sigma$. In particular, $L(B_{i,j,q,p}) = L(A_{i,q,p}) \cap \{a_j\}^*$ is star-free and commutative.

Recall that $\text{perm}(u) = \{ w \in \Sigma^* \mid |w|_a = |w|_a \text{ for every } a \in \Sigma \}$, where $|w|_a$ is equal to the number of letters $a$ in $w$. Moreover, $\text{perm}(L) = \bigcup_{v \in L} \text{perm}(v)$. By commutativity, the following property is clear:

$$L(A_i) = \text{perm} \left( \bigcup_{p_1, p_2, \ldots, p_{r-1} \in Q_i} \bigcup_{p_r \in F_i} L(B_{i,1,q_1,p_1}) \cdot L(B_{i,2,p_1,p_2}) \cdots L(B_{i,r,p_{r-1},p_r}) \right).$$

As $A_i$ accepts a commutative language, by ordering the letters, we find that $w \in L(A_i)$ if and only if $a_1^{\ell_1} a_2^{\ell_2} \cdots a_r^{\ell_r} \in L(A_i)$, for $\ell_j$ being the number of occurrences of $a_j$ in $w$, with $1 \leq j \leq r$. Furthermore, the word $a_1^{\ell_1} a_2^{\ell_2} \cdots a_r^{\ell_r}$ is in $L(A_i)$ if and only if for all $j$ with $1 \leq j \leq r$, there is a state $p_j \in Q_i$ such that $a_j^{\ell_j} \in L(B_{i,j,p_j-1,p_j})$, where $p_0 = q_i$ and $p_r \in F_i$. We can apply Lemma 21 to get constants $N_{i,1}, N_{i,2}, \ldots, N_{i,r} \in \mathcal{O}(n^2)$ such that checking membership of $a_j^{\ell_j}$ in $L(B_{i,j,p_j-1,p_j})$ can be restricted to checking membership for a word of length at most $N_j$. Now, we describe a polynomial-time procedure to solve Intersection Non-emptiness for fixed alphabets. Set $N_i = \max\{N_{i,1}, N_{i,2}, \ldots, N_{i,r}\}$ with the numbers $N_{i,j}$ from above. Then, we know that a word $a_1^{\ell_1} a_2^{\ell_2} \cdots a_r^{\ell_r}$ is accepted by an input automaton $A_i$ if and only if the word $a_1^{\min\{\ell_1, N_i\}} a_2^{\min\{\ell_2, N_i\}} \cdots a_r^{\min\{\ell_r, N_i\}}$ is accepted by it. If we let $N = \max\{N_1, N_2, \ldots, N_m\}$ and $n = \max\{n_1, n_2, \ldots, n_m\}$, we only need to test the $(N + 1)^r$ many words $a_1^{\ell_1} a_2^{\ell_2} \cdots a_r^{\ell_r}$ with $0 \leq \ell_j \leq N$ and $1 \leq j \leq r$ if we can find a word among them that is accepted by all automata $A_i$ for $1 \leq i \leq m$. Altogether, ignoring polynomial factors, this leads to a running time of the form $(N + 1)^r$.

\begin{remark}

Note that Theorem 19 does not hold for arbitrary commutative languages concerning a fixed alphabet, but only for star-free commutative languages, since in the general case, the problem is NP-complete even for languages over a common unary alphabet [37].

\end{remark}

\section{PSPACE-Completeness}

Here, we prove that even when restricted to languages from $\mathcal{B}_1$ or $\mathcal{L}_2$, Intersection Non-emptiness is PSPACE-complete, as it is for unrestricted DFAs or NFAs. We will profit from
the close relations of Intersection Non-emptiness to the Non-universality problem for NFAs: Given an NFA $A$ with input alphabet $\Sigma$, decide if $L(A) \neq \Sigma^*$. Conversely, we can also observe that Non-universality for NFAs is PSPACE-complete for languages from $B_1$.

**Theorem 23.** The Intersection Non-emptiness problem for DFAs or NFAs accepting languages from $B_1$ or $L_2$ is PSPACE-complete, even for binary input alphabets.

As $B_1 \subseteq L_2$, it is sufficient to show that the problem is PSPACE-hard for $B_1$. While without paying attention to the size of the input alphabet, this result can be readily obtained by re-analyzing Kozen’s original proof in [23], the restriction to binary input alphabets needs some more care. We modify the proof of Theorem 3 in [24] that showed PSPACE-completeness for Non-universality for poNFAs (that characterize the level $3/2$ of the Straubing-Thérien hierarchy). Also, it can be observed that the languages involved in the intersection are actually locally testable languages. Without giving details of definitions, we can therefore formulate:

**Corollary 24.** The Intersection Non-emptiness problem for DFAs or NFAs accepting locally testable languages is PSPACE-complete, even for binary input alphabets.

**Proof.** To see our claims, we re-analyze the proof of Theorem 3 in [24] that shows PSPACE-completeness for the closely related Non-universality problem for NFAs. Similar to Kozen’s original proof, this gives a reduction from the general word problem of deterministic polynomial-space bounded Turing Machines. In the proof of Theorem 3 in [24] that showed PSPACE-completeness for Non-universality for poNFAs (that characterize the level $3/2$ of the Straubing-Thérien hierarchy), a polynomial number of binary languages $L_i$ was constructed such that $\bigcup_i L_i \neq \{0,1\}^*$ if and only if the $p$-space-bounded Turing machine $M$, where $p$ is some polynomial, accepts a word $x \in \{0,1\}^*$ using space $p(|x|)$. Observe that each of the languages $L_i$ is a polynomial union of languages of the forms $E\{0,1\}^*$, $\{0,1\}^*E$, $\{0,1\}^*E\{0,1\}^*$, or $E$ for finite binary languages $E$. This means that each $L_i$ belongs to $B_{1/2}$. Now, observe that $\bigcup_i L_i \neq \{0,1\}^*$ if and only if $\bigcap_i L_i \neq \emptyset$. As $L_i \in B_1$ and each $L_i$ (and hence its complement $\overline{L_i}$) can be described by a polynomial size DFA, the claims follow. ▷

By the proof of Theorem 3 in [24], also $\bigcup L_i$ belongs to $B_1$, so that we can conclude:

**Corollary 25.** The Non-universality problem for NFAs accepting languages from $B_1$ is PSPACE-complete, even for binary input alphabets.

We now present all proof details, because the construction is somewhat subtle.

The proof is based on simulating a $p$-space-bounded Turing machine $M$. We are interested in simulating a run of $M$ on a string $x$. Its configurations are encoded as words over an alphabet $\Delta$, so that with the help of the enhanced alphabet $\Delta_\# = \Delta \cup \{\#\}$, runs of $M$ can be encoded, with $\#$ serving as a separator between configurations. More precisely, if $\Sigma_M$ is the input alphabet of $M$, $\Gamma_M$ (containing a special blank symbol $\omega$) is the tape alphabet, and $Q_M$ is the state alphabet, then transitions take the form $f_M : Q_M \times \Gamma_M \to Q_M \times \Gamma_M \times \{L, R\}$, where $L, R$ indicate the movements of the head. For simplicity, define $\Delta = \Gamma_M \times (Q_M \cup \{\$\})$. A configuration $\gamma \in \Delta^+$ has then the specific properties that it contains exactly one symbol from $\Gamma_M \times Q_M$ and that it has length $p(|x|)$ always, i.e., we are possibly filling up a string that is too short by the blank symbol $\omega$. Configuration sequences of $M$, or runs for short, can be encoded by words from $\#(\Delta^+ \#)^*$, or more precisely, from $L_{\text{simple-run}} = \#(\Gamma_M \times \{\$\})^*(\Gamma_M \times Q_M)(\Gamma_M \times \{\$\})^+ \#)^*$. The latter language can be encoded by a 3-state DFA. However, we will not make use of this language in the following, as it does not fit in the level of the dot-depth hierarchy that we are aiming at.
Let $\Sigma = \{0, 1\}$ be the binary target alphabet. A letter $a \in \Delta_\#$ is first encoded by a binary word $\hat{a}$ of length $K = \lceil \log_2(|\Delta_\#|) \rceil$, but this is only an auxiliary encoding, used to define the block-encoding

$$enc(a) = 001\hat{a}[1]1\hat{a}[2]1 \cdots \hat{a}[K]1$$

of length $L = 2K + 3$. This block-encoding is extended to words and sets of words as usual. In order to avoid some case distinctions, we assume that $|\Delta_\#|$ is a power of two, so that $enc(\Delta_\#) = 001\Sigma \Sigma 1 \cdots \Sigma 1$. Hence, $enc(\Delta_\#) = 00\Sigma^{L-2} \setminus enc(\Delta_\#) = 00\{a_1b_1a_2b_2 \cdots a_Kb_K \mid a_1a_2 \cdots a_K \in \Sigma^K \land b_1b_2 \cdots b_K \in 1^*\Sigma^*\}$. Clearly, there are DFAs with $O(L)$ many states accepting $enc(\Delta_\#)$ and $\Sigma^* enc(\Delta_\#) \{1\}$. In this proof, we will call DFAs with $O(L \cdot p(|x|))$ many states small.

Any encoded word $enc(w)$, with $w \in \Delta_\#^+$, contains the factor 00 only at positions (minus one) that are multiples of $L$, more precisely: $enc(w)[i] = enc(w)[i + 1] = 0$ if and only if $i - 1$ is divisable by $L$. This observation allows us to construct small DFAs for $\Sigma^* enc(\Delta_\#)\Sigma^*$ (for $e \in \{1, 00\}$) and for $\Sigma^* \Sigma^* enc(\Delta_\#) \Sigma^*$, based on (1). As shown in the proof of Theorem 3 in [24], the language of words that are not encodings over $\Delta_\#$ at all is the union of the following languages:

1. $(1 \cup 01)\Sigma^*$,
2. $\Sigma^* \Sigma^* enc(\Delta_\#) \Sigma^*$,
3. $\Sigma^* enc(\Delta_\#)(1 \cup 01)\Sigma^*$, and
4. $\Sigma^* 00(\bigcup_{i=1}^{L-3} \Sigma^i) = \{w \in \Sigma^* \mid$ The factor 00 is in the last $L - 1$ positions\}.

Each of these languages can be accepted by small DFAs $A_1, A_2, A_3, A_4$.

Then, we have to take care of the binary words that cannot be encodings of configuration sequences, because the first configuration is not initial. By our construction, the (unique) initial configuration $\gamma$ is encoded by a binary string $enc(\gamma)$ of length $L \cdot p(|x|)$, i.e., we consider a language $L'$ which is the complement of $enc(#\gamma\#)\Sigma^*$, the language of all binary strings that do not start with the encoding of the initial configuration. Let $\#\gamma\# = a_1a_2 \cdots a_{p(|x|)+2}$. As we already described non-encodings by automata $A_1$ through $A_4$, instead of $L'$, we describe $\bigcup_{j=0}^{p(|x|)+2} L_j'$, where $L_0' = \bigcup_{j=0}^{p(|x|)+2} L_j'$ is a finite language (of strings that are too short), $L_j' = \Sigma^{(j-1)L \Sigma^*} \Sigma^*$ for $j = 1$ to $p(|x|) + 2$, describing a violation at symbol $a_j$ of the initial configuration $\gamma$. Moreover, there are small DFAs $A_5, A_6, \ldots, A_{p(|x|)+7}$ that accept $L_0', L_1', \ldots, L_{p(|x|)+2}'$.

Assuming a unique final state and also assuming that $M$ cleans up the tape after processing, there is a unique final configuration $\gamma_f$ that should be reached. Then, invalidity of a computation with respect to the final configuration can be checked as for the initial configuration, giving us small DFAs $A_{p(|x|)+8}, A_{p(|x|)+9}, \ldots, A_{2p(|x|)+10}$.

Finally, we want to check the (complement of the) following property of a valid configuration sequence $\rho \in \#(\Delta_\#)^*$: any sequence of three letters $a, b, c$ in $\rho$ determines the letter $f(a, b, c)$ that should be present at a distance of $p(|x|) - 1$ to the right. More precisely, we are interested in any factor $abcdef(a, b, c)e$ of $\rho$ where $|ed| = p(|x|) - 1$. Different scenarios can occur; we only describe three typical situations in the following.

- If $a = (a', \$), b = (b', \$), c = (c', \$), then $f(a, b, c) = b$. For $d$, we know $d \in \{a'\} \times (Q \cup \{\$\})$ and similarly for $e$, we know $e \in \{c'\} \times (Q \cup \{\$\})$.

- If $a = (a', \#), b = \#, c = (c', \$), then $f(a, b, c) = \#$. For $d$, we know $d \in \{a'\} \times (Q \cup \{\$\})$ and similarly for $e$, we know $e \in \{c'\} \times (Q \cup \{\$\})$.

- If $a = (a', \$), b = (b', q), c = (c', \$)$ and if $f_M(q, b') = (p, \hat{b}', L)$, then $d = (a', p)$, $f(a, b, c) = (\hat{b}, \$)$, and $e = e$. 

We refrain from describing all such situations in detail. Yet with some more sloppiness, we write $\text{enc}(d(f(a, b, c)e)$ for all situations that do not obey the rules for $df(a, b, c)e$ as tentatively formulated before. Now, for each triple $a, b, c \in \Delta_{\#}$, consider the binary language $L_{a,b,c} = \Sigma^* \cdot \text{enc}(abc) \cdot \Sigma^L(p(|x|) - 1) \cdot \text{enc}(d(f(a, b, c)e) \cdot \Sigma^*$. This language can be accepted by a small DFA $A_{a,b,c}$.

Altogether, we described $2p(|x|) + 10 + (|\Delta_{\#}|)^3$ many languages from $B_1$ such that their union does not yield $\Sigma^*$ if and only if $M$ accepts $x$ using $p(|x|)$ space. Moreover, for each of the languages, we can build small DFAs.

6 Conclusion and Open Problems

We have investigated how the increase in complexity within the dot-depth and the Straubing-Thérien hierarchies is reflected in the complexity of the \textsc{Intersection Non-emptiness} problem. We have shown the complexity of this problem is already completely determined by the very first levels of either hierarchy.

Our work leaves open some very interesting questions and directions of research. First, we were not able to prove containment in \textsf{NP} for the \textsc{Intersection Non-emptiness} problem when the input automata are allowed to be NFAs accepting a language in the level $3/2$ or in the level $1$ of the Straubing-Thérien hierarchy. Interestingly, we have shown that such containment holds in the case of DFAs, but have shown that the technique we have used to prove this containment does not carry over to the context of NFAs. In particular, to show this we have provided the first exponential separation between the state complexity of general NFAs and partially ordered NFAs. The most immediate open question is if \textsc{Intersection Non-emptiness} for NFAs accepting languages in $B_{1/2}$, $L_1$, or $L_{3/2}$ is complete for some level higher up in the polynomial-time hierarchy (\textsf{PH}), or if this case is already \textsf{PSPACE}-complete. Another tantalizing open question is whether one can capture the levels of \textsf{PH} in terms of the \textsc{Intersection Non-emptiness} problem when the input automata are assumed to accept languages belonging to levels of a sub-hierarchy of $L_2$. Such sub-hierarchies have been considered for instance in [22].

It would also be interesting to have a systematic study of these two well-known sub-regular hierarchies for related problems like \textsc{Non-universality} for NFAs or \textsc{Union Non-universality} for DFAs. Notice the technicality that \textsc{Union Non-universality} (similar to \textsc{Intersection Non-emptiness}) has an implicit Boolean operation (now union instead of intersection) within the problem statement, while \textsc{Non-universality} lacks this implicit Boolean operation. This might lead to a small “shift” in the discussions of the hierarchy levels that involve Boolean closure. Another interesting hierarchy is the group hierarchy [31], where we start with the group languages, i.e., languages acceptable by automata in which every letter induces a permutation of the state set, at level $0$. Note that for group languages, \textsc{Intersection Non-emptiness} is \textsf{NP}-complete even for a unary alphabet [37]. As $\Sigma^*$ is a group language, the Straubing-Thérien hierarchy is contained in the corresponding levels of the group hierarchy, and hence, we get \textsf{PSPACE}-hardness for level $2$ and above in this hierarchy. However, we do not know what happens in the levels in between.

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