Abstract

We consider a single Abelian Higgs vortex on a surface $\Sigma$ whose Gaussian curvature $K$ is small relative to the size of the vortex, and analyse vortex motion by using geodesics on the moduli space of static solutions. The moduli space is $\Sigma$ with a modified metric, and we propose that this metric has a universal expansion, in terms of $K$ and its derivatives, around the initial metric on $\Sigma$. Using an integral expression for the Kähler potential on the moduli space, we calculate the leading coefficients of this expansion numerically, and find some evidence for their universality. The expansion agrees to first order with the metric resulting from the Ricci flow starting from the initial metric on $\Sigma$, but differs at higher order. We compare the vortex motion with the motion of a point particle along geodesics of $\Sigma$. Relative to a particle geodesic, the vortex experiences an additional force, which to leading order is proportional to the gradient of $K$. This force is analogous to the self-force on bodies of finite size that occurs in gravitational motion.
1 Bogomolny Vortices

Solitons that satisfy Bogomolny equations experience no static forces, and soliton motion is known to be well approximated by a geodesic motion in moduli space \[^{[1][2]}\]. The metric on moduli space is induced from the field kinetic energy and this dominates the dynamics because the potential energy is constant. For one soliton moving on a curved base manifold, one may compare the geodesic motion in the 1-soliton moduli space with the geodesic motion of a test particle on the base manifold. They will differ because of the finite size of the soliton, and because of possible internal motion of the soliton, and it is interesting to study these effects. Here we calculate this difference for an Abelian Higgs vortex moving on a surface of small curvature. A vortex is a simple soliton, because it has a well defined location, and no internal motion. The 1-vortex moduli space, as a manifold, is therefore the same as the base manifold, and only the metrics differ.

In gravitational theory, it is one of the classic challenges to describe the motion of a massive body of finite size in a given gravitational background (see e.g. \[^{[3]}\]), or the detailed interaction between two finite-size bodies \[^{[4]}\]. Because a body’s self-gravity locally dominates its background, it is difficult to compare the trajectory of the body with the geodesic representing a test particle. The trajectory also seems sensitive to the internal structure of the body.

Vortex motion provides a conceptually simpler set-up to consider this issue. The mechanical properties of a vortex are determined by the Abelian Higgs field theory and the background surface geometry, but the background geometry is arbitrary and non-dynamical here, and not subject to an Einstein equation. A vortex has a precise centre, so its trajectory is unambiguous. Our work thus gives some understanding of how the motion through curved space of a body of small but finite size differs from the motion of a point-like test particle.

In the critically coupled Abelian Higgs theory, there is a moduli space \(M_N\) of static \(N\)-vortex solutions, which satisfy a coupled pair of Bogomolony equations. All these \(N\)-vortex solutions have the same potential energy, so there are no static forces. The moduli space acquires a metric from the kinetic energy of the theory, and its geodesics model \(N\)-vortex motion. Mathematically, the metric is the restriction to \(M_N\) of the natural \(L^2\) norm on the tangent space to the space of all field configurations, with gauge freedom quotiented out. Samols found a useful local expression for this metric on \(M_N\) \[^{[5]}\]. The accuracy of geodesic motion on \(M_N\), as an approximation to true \(N\)-vortex motion according to the field equations, was proved by Stuart \[^{[6]}\].

The above discussion applies not just to \(N\)-vortex motion in \(\mathbb{C}\), i.e. in the plane \(\mathbb{R}^2\), where the theory was originally defined, but to \(N\)-vortex motion on any Riemann surface \(\Sigma\) that satisfies the Bradlow area inequality, which we recall below. There is a modified Abelian Higgs theory, with Bogomolony equations and static \(N\)-vortex solutions, provided \(\Sigma\) has a metric compatible with the complex structure,

\[
g = \Omega(z, \bar{z}) dz d\bar{z},
\]

(1.1)

where \(z = x_1 + ix_2\) is a local holomorphic coordinate. The function \(\Omega : \Sigma \to \mathbb{R}^+\) is called the
conformal factor of the metric. The surface Σ should be metrically complete, and it may be compact and of finite area, or non-compact with boundaries at infinity.

The fields of the theory on Σ are a complex scalar field, the Higgs field φ, and an Abelian gauge potential $A_μ$. They are defined on the 2+1 dimensional product space-time $Σ \times ℝ$ with metric

$$ds^2 = dx_0^2 - g. \quad (1.2)$$

At the critical coupling the Lagrangian takes the form

$$L = \frac{i}{2} \int_Σ dz \wedge d\bar{z} \Omega(z, \bar{z}) \left[ -\frac{1}{4} F_{μν}F^{μν} + \frac{1}{2} D_μφD^μ\bar{φ} - \frac{1}{8} \left( \frac{1}{τ} - |φ|^2 \right)^2 \right], \quad (1.3)$$

where $F_{μν} = \partial_μA_ν - \partial_νA_μ$, and the Bradlow parameter τ is a positive constant. If Σ has a boundary, then $|φ|^2 = \frac{1}{τ}$ there.

The Lagrangian can be split into kinetic and potential terms $L = T - V$, where $T$ is the part of $L$ containing derivatives w.r.t. $x_0$. Completing the square in the potential energy $V$ one finds that, for a given magnetic flux, the total energy $E = T + V$ is minimal if the fields $(φ, A_μ)$ are static, with $A_0 = 0$, and satisfy the Bogomolny equations

$$D_\bar{z}\phi = 0, \quad F_{12} = \frac{Ω}{2} \left( \frac{1}{τ} - |φ|^2 \right). \quad (1.4)$$

Here $D_\bar{z} = \partial_\bar{z} - iA_\bar{z}$ is the anti-holomorphic part of the $U(1)$ covariant derivative. The vortex number $N$ is the number of zeros of $φ$, counted with multiplicity. The Bogomolny equations (with our choice of signs) only admit vortices of positive multiplicity, and solutions generically consist of $N$ single vortices at distinct locations. The potential energy of $N$ vortices is $Nπ/τ$, independently of their locations. Their total magnetic flux is $2πN$.

If the surface Σ is compact, its area is

$$A = \frac{i}{2} \int_Σ dz \wedge d\bar{z} \Omega(z, \bar{z}). \quad (1.5)$$

Bradlow [7] showed, by integrating the second Bogomolny equation, that $N$-vortex solutions exist on Σ only if $A > 4πNτ$. This can be interpreted as saying that the area of a vortex is $4πτ$ and the total area occupied by vortices must be less than the area of the surface. We shall mostly be concerned with compact surfaces of very large area, $A \gg 4πτ$, or non-compact surfaces of infinite area, on which solutions exist for all $N$. It is sometimes convenient to rescale $τ$ to unity. One must then, at the same time, rescale the size of the surface Σ to retain similar physics.

Using the parametrisation $|φ|^2 = \frac{1}{τ}e^h$, and eliminating the gauge potential from the first Bogomolny equation one finds that the second Bogomolny equation reduces to the gauge invariant Taubes equation [8]

$$∆h + \frac{1}{τ} \left( 1 - e^h \right) = \frac{4π}{Ω} \sum_{i=1}^N δ^2(z - Z_i) \quad (1.6)$$
where $Z_i$ are the vortex locations and $\Delta = 4\Omega^{-1}\partial_z\partial_{\bar{z}}$ is the covariant Laplacian on $\Sigma$. This is the fundamental tool for studying vortices on $\Sigma$ and their moduli space. The Bogomolny equations do not directly give the delta functions, but they occur because of the logarithmic singularity of $h$ wherever $|\phi| = 0$.

As a manifold, the $N$-vortex moduli space $\mathcal{M}_N$ is the symmetrised $N$-th power of $\Sigma$,

$$\mathcal{M}_N = \Sigma^N/S_N,$$

(1.7)

where $S_N$ is the permutation group. This is because an $N$-vortex solution is completely determined by the $N$ unordered zeros of $\phi$, whose locations $\{Z_i : i = 1, 2, \ldots, N\}$ are anywhere on $\Sigma$ (and can coincide). $\mathcal{M}_N$ is a smooth complex manifold, whose natural coordinates are the symmetric polynomials in $\{Z_i\}$.

The metric on $\mathcal{M}_N$ is not explicitly known, in general. However, using the Taubes equation, and its linearisation when the vortex locations are infinitesimally varied, Samols showed that the metric could be expressed in a localised form using $\{Z_i\}$ as holomorphic coordinates [5]. The metric coefficients depend on local data obtained from $h$ in the neighbourhoods of these vortex centres. From Samols’ formula it can be deduced that the metric on $\mathcal{M}_N$ is Kähler. It is also smooth, even where vortex centres coincide. The cohomology class of the Kähler 2-form on $\mathcal{M}_N$ can be determined [9], so if $\Sigma$ is compact and $A$ is finite, the volume of $\mathcal{M}_N$ can be found. It depends on $N, A$ and the genus $g$ of $\Sigma$.

Subsequent to Samols’ work, Chen and Manton [10] found an expression for the Kähler potential on $\mathcal{M}_N$ in the case of $N$ vortices on $\mathbb{C}$, with its flat metric. This Kähler potential involves an integral over $\mathbb{C}$ of an elementary function of $h$. However, the integral has logarithmic divergences at each point $Z_i$, and these need to be regularised, again using local data from the neighbourhoods of these points. Remarkably, the regularised integral can be interpreted as the action whose Euler-Lagrange equation reproduces the Taubes equation. It is the on-shell action, by which we mean the action evaluated on a solution of the Taubes equation, that is the main contribution to the Kähler potential. This on-shell action is not the standard potential energy of vortices in the Abelian Higgs theory, and is a non-trivial function of the vortex locations. In the Appendix we will re-derive the Kähler potential on $\mathcal{M}_1$, obtaining an expression that is valid for any curved surface $\Sigma$.

2 The 1-Vortex Moduli Space

The simplest example of moduli space geometry is for one vortex on the flat plane, $\mathbb{C}$. The moduli space $\mathcal{M}_1$ is $\mathbb{C}$, with the same flat metric. (This means a factor $\pi/\tau$, representing the mass of the vortex, which occurs in the kinetic energy, has been scaled out.) Geodesic motion is straight line motion at constant speed. This accurately describes the non-relativistic limit of the exact solution where a static vortex is Lorentz boosted.

In the case that $\Sigma$ is curved, even the 1-vortex moduli space $\mathcal{M}_1$ is geometrically interesting [11]. A 1-vortex has a single Higgs zero at an arbitrary location $z = Z$, and is the
unique solution of the Taubes equation

$$\Delta h + \frac{1}{\tau} (1 - e^h) = \frac{4\pi}{\Omega} \delta^2(z - Z). \quad (2.1)$$

The moduli space is therefore $M_1 = \Sigma$, with $Z$ as coordinate, and Samols’ analysis implies that the metric on $M_1$ has the form

$$\tilde{g} = \tilde{\Omega}(Z, \bar{Z}; \tau) dZ d\bar{Z}. \quad (2.2)$$

The change of notation, from $z$ to $Z$, is slight, and it is convenient to think of the metric on $M_1$ as a modified version of the original metric $g$ on $\Sigma$. Notice that the complex structure is unchanged in going from $\Sigma$ to $M_1$; just the conformal factor changes, from $\Omega$ to $\tilde{\Omega}$. In this 1-vortex case, Samols’ formula simplifies to

$$\tilde{\Omega}(Z, \bar{Z}; \tau) = \Omega(Z, \bar{Z}) + 2\tau \frac{\partial b}{\partial Z}, \quad (2.3)$$

where $\frac{1}{2}b(Z, \bar{Z})$ is the coefficient of $\bar{z} - \bar{Z}$ in the expansion of $h$ around the vortex location $Z$,

$$h(z, \bar{z}) = 2 \log |z - Z| + a(Z, \bar{Z}) + \frac{1}{2}b(Z, \bar{Z})(z - Z) + \frac{1}{2}b(Z, \bar{Z})(\bar{z} - \bar{Z})$$

$$+ c(Z, \bar{Z})(z - Z)^2 + d(Z, \bar{Z})(z - Z)(\bar{z} - \bar{Z}) + \cdots. \quad (2.4)$$

Apart from the leading logarithmic term, this expansion is a Taylor series in $z - Z$ and its conjugate. The Taubes equation (2.1) requires that $d(Z, \bar{Z}) = -\Omega(Z, \bar{Z})/4\tau$, but the other coefficients shown here are not determined purely locally, but only from the complete 1-vortex solution. They also depend on $\tau$.

The function $b(Z, \bar{Z})$ is not explicitly known, except on some especially symmetric surfaces, e.g. a round sphere, so $\tilde{\Omega}$ is not known either. Despite lacking explicit knowledge of the function $b(Z, \bar{Z})$, one can show that if $\Sigma$ is compact, and of genus $g$, then the total area of $M_1$ is $A_1 = A - 4\pi\tau(1 - g)$ [9]. This result is a consequence of being able to integrate $\frac{\partial b}{\partial Z}$. In detail, it relies on $h$ being a globally defined function of $z$ and $Z$ (and their conjugates), with a singularity of type $2 \log |z - Z|$, which implies a particular type of transformation for $b$ under holomorphic changes of the local coordinate $z$.

The principal aim of this paper is to gain an understanding of the conformal factor $\tilde{\Omega}$ in the case that $\Sigma$ has small curvature. We expect $\tilde{\Omega}$ to differ rather little from $\Omega$.

To be precise about what is small in our approach, consider a metric $g_0 = \Omega_0(z, \bar{z}) dz d\bar{z}$ on $\Sigma$ that has Gaussian curvature $K_0$ and derivatives of $K_0$ all of order 1, and now assume that $\Omega = L^2\Omega_0$, where $L$ is a large constant scale factor. Lengths get rescaled by $L$ and areas by $L^2$. The Gaussian curvature $K$ of $\Sigma$ is

$$K = -\frac{1}{2} \Delta (\log \Omega) \quad (2.5)$$

$$= -\frac{1}{2L^2} \Delta_0 (\log \Omega_0 + 2 \log L) = \frac{1}{L^2} K_0. \quad (2.6)$$
So $K$ is small, of order $1/L^2$. Similarly $K^2$ and $\Delta K$ are of order $1/L^4$. Each factor of $K$ and each application of $\Delta$ brings in a further factor of $1/L^2$.

Now, a vortex is a smooth localised solution of the Bogomolny equations, centred at $Z$, whose characteristic area is of order $\tau$. It is sensitive only to the local aspects of the background metric and curvature near $Z$ if the surface is large and the curvature small. Long-range effects due to the curvature, including topological effects, are exponentially small, and we neglect them. These remarks also apply to the conformal factor of the moduli space, $\tilde{\Omega}$.

We propose that $\tilde{\Omega}$ has an asymptotic expansion in $\tau/L^2$ if expressed in terms of the conformal factor $L^2\Omega_0$ and its curvature. If we work directly with $\Omega$ and its curvature $K$, then $\tau$ occurs explicitly, but the inverse powers of $L$ occur implicitly.

The expansion we propose (provisionally) is

$$\tilde{\Omega}(Z, \bar{Z}; \tau) = \Omega(Z, \bar{Z})(1 + \alpha_0 \tau K + \beta_0 \tau^2 K^2 + \gamma_0 \tau^2 \Delta K + \cdots),$$

(2.7)

where $\alpha_0, \beta_0, \gamma_0, \ldots$ are universal constants of order 1, independent of the function $\Omega$. $K$ and its derivatives are all evaluated at $Z$. Pulling out the overall factor $\Omega$ is dimensionally right, and ensures that all the terms inside the bracket are invariant under holomorphic coordinate transformations. $\tau K$ is also dimensionless, and $\Delta$ has the same dimension as $K$. We have explicitly shown all the terms up to order $\tau^2 L^4$ that can occur. This expansion is a refinement of what was proposed and studied in [11].

We in fact know more about the nature of this expansion, because of our precise knowledge of the area of $\mathcal{M}_1$ when $\Sigma$ is compact [9]. Because $A_1 = A - 4\pi \tau (1 - g)$, independently of $L$, we know that $\alpha_0 = -1$. This follows from the Gauss–Bonnet formula

$$\frac{i}{2} \int_\Sigma dZ \wedge d\bar{Z} \Omega K = 4\pi (1 - g).$$

(2.8)

Moreover, all subsequent terms in the expansion, involving higher powers of $K$ and the operator $\Delta$, must integrate to zero. For this to occur universally, the only terms allowed inside the bracket must be of the form of $\Delta$ applied to some further globally defined scalar expressions (as $\Omega \Delta = 4\partial_Z \partial_{\bar{Z}}$ is a total derivative, and a vortex is a scalar soliton, with no internal degrees of freedom). Therefore $\beta_0 = 0$, but $\gamma_0$ can be non-zero. We may therefore rewrite the expansion (2.7) in its final form (and with new coefficients) as

$$\tilde{\Omega}(Z, \bar{Z}; \tau) = \Omega(Z, \bar{Z})(1 - \tau K + \alpha \tau^2 \Delta K + \beta \tau^3 \Delta^2 K + \cdots),$$

(2.9)

where we have written out all terms up to order $\tau^3/L^6$. This holds for both compact and non-compact $\Sigma$.

When $\Sigma$ is non-compact, simply-connected and asymptotically planar, a slight generalization of the Gauss–Bonnet formula yields

$$\frac{i}{2} \int_\Sigma dZ \wedge d\bar{Z} \Omega K = 0.$$  

(2.10)

This equation combined with (2.9) tells us that, whether the area deficit between $\Sigma$ and the flat plane, obtained by integrating $\Omega - \Omega_{\text{flat}}$, is finite or infinite, the deficit area between $\mathcal{M}_1$
and $\Sigma$ is always zero, as
\[ \frac{i}{2} \int_{\Sigma} dZ \wedge d\bar{Z} (\tilde{\Omega} - \Omega) = \frac{i}{2} \int_{\Sigma} dZ \wedge d\bar{Z} [-\tau \Omega K + \partial Z \partial \bar{Z}(\cdots)] = 0. \tag{2.11} \]

Ideally, we would now calculate the coefficients $\alpha, \beta, \gamma, \ldots$ using a general argument. However this would require constructing a 1-vortex solution in a completely general background of small curvature. Such a solution is close to the solution in flat space, and it may be possible to construct it iteratively using a Green’s function. However, neither the flat space solution nor the relevant Green’s function are known in closed form, so we have not been able to pursue this approach.

Instead, we have exploited the proposed universality of the coefficients, and have calculated the first few of them – $\alpha, \beta, \gamma$ – using a few carefully selected examples of conformal factors $\Omega$. These conformal factors have more than just three parameters, and for these we have verified the universality of the coefficients. We have set $\tau = 1$ but have included $L$ as a parameter and taken the limit $L \to \infty$ to extract the coefficients. In this way we avoid contamination by the neglected higher order terms.

Our method involves solving Taubes’ equation numerically. The background metric $g = \Omega dz d\bar{z}$ is axially symmetric around the origin $z = 0$ in all cases, and is asymptotic to the flat planar metric of $\mathbb{C}$ as $|z| \to \infty$. So the curvature is concentrated in a neighbourhood of the origin. If the vortex is also located at the origin, then Taubes’ equation reduces to an ODE, which is straightforward to solve. From the solution we can extract $a$ and $b$, the leading coefficients occurring in the expansion (2.4) of $h$. However, this method now runs into difficulty. The conformal factor $\tilde{\Omega}$ involves not $b$ (which actually vanishes at the origin if there is axial symmetry), but $\partial b / \partial Z$, and to find this derivative we would need to relocate the vortex away from the origin. Accurately solving Taubes’ equation, with an off-centre delta function, would not be numerically simple.

We have found a way round this. Rather than trying to find $\tilde{\Omega}$, we instead calculate its Kähler potential $\tilde{K}$. The general relation between $\tilde{K}$ and the conformal factor $\tilde{\Omega}$ is
\[ \tilde{\Omega} = \partial Z \partial \bar{Z} \tilde{K}. \tag{2.12} \]

Let $K$ be the Kähler potential of the background $\Omega$, so $\Omega = \partial Z \partial \bar{Z} K$. The expansion (2.9) can be integrated to give
\[ \tilde{K}(Z, \bar{Z}; \tau) = K(Z, \bar{Z}) + 2\tau \log \Omega + 4\alpha \tau^2 K + 4\beta \tau^3 K^2 + 4\gamma \tau^3 \Delta K + \cdots \tag{2.13} \]
where we have used (2.5) to integrate the term $\Omega K$. There is some ambiguity in a Kähler potential, but if we insist that $\tilde{K}$ and $K$ both have the asymptotic form $Z \bar{Z}$, and their difference vanishes asymptotically, then the ambiguity is resolved. For our selected metrics we can calculate the quantities $\log \Omega$, $K$ and $\Delta K$ at the origin, by elementary differentiation, and hence estimate $\tilde{K} - K$ using (2.13), as a function of $\alpha, \beta, \gamma$. As we explain in the Appendix, we can also independently calculate $\tilde{K} - K$ in terms of the on-shell action $S$, using the solution of Taubes’ equation with the vortex at the origin. The integral expression for $S$ is given in
equation (A12). By comparing these we gain information about the coefficients \( \alpha, \beta, \gamma \), and by varying the parameters of our metrics, we determine the coefficients precisely. We have found

\[
\alpha = -0.325 , \quad \beta = -0.01 , \quad \gamma = -0.08 .
\]

(2.14)

The calculations are presented in section 3.

In [11] it was conjectured that \( \tilde{\Omega} \) is determined from \( \Omega \) by a Ricci flow, \( \Omega(\tau) \), starting at \( \tau = 0 \) with \( \Omega \). (The Bradlow parameter \( \tau \) is twice the usual ‘time’ encountered in Ricci flow.) The Ricci flow equation on a surface, in terms of the Gaussian curvature, is

\[
\frac{\partial}{\partial \tau} \Omega(\tau) = -K(\tau)\Omega(\tau) .
\]

(2.15)

One can formally integrate this equation from \( \tau = 0 \), obtaining an expansion of the type (2.9). This approach gives the correct result for \( \tilde{\Omega} \) if \( \Omega \) has constant curvature, but one example in [11] showed that the conjecture could not be correct in general.

The conjecture is not entirely wrong. The expansion (2.9) is structurally similar to what one gets from Ricci flow. The leading terms \( \Omega(1 - \tau K) \) are identical, but the subsequent coefficients are different. Interestingly, they are not very different. The comparison is discussed in section 4.

The conformal factor (2.9) rather precisely characterises the 1-vortex moduli space metric \( \tilde{g} \) when the background curvature is small. Using this we can compare the geodesic motion on the moduli space to the geodesic motion using \( g \). Recall that the latter represents the motion of a point particle on \( \Sigma \), whereas geodesics using \( \tilde{g} \) represent the motion of a vortex on \( \Sigma \). A vortex has a finite size of order \( \tau \), so we expect its motion to sample the background metric over a larger region than a point particle. We find that, relative to the geodesic of a point particle, the vortex experiences a force proportional (to leading order in \( \tau \)) to the gradient of the curvature \( K \). The additional force is also proportional to the velocity squared, as one expects for geodesic motion on the moduli space. As mentioned in section 1, the additional force is analogous to the self-force experienced by a body of finite size in general relativity. These issues are explored in more detail in section 5.

3 Calculating the moduli space metric

In this section the Bradlow parameter \( \tau \) is set to 1. We work with the difference between the moduli space Kähler potential \( \tilde{K} \) and the original Kähler potential \( K \), both defined on \( \Sigma \), and using the common coordinate \( Z \). As shown in the Appendix, this has the form

\[
\tilde{K} - K = S - S_{flat}
\]

(3.1)

where \( S \) is the regularised on-shell action for Taubes’ equation, which simplifies to

\[
S(Z, \bar{Z}) = -\frac{i}{4\pi} \int_{\Sigma} dz \wedge d\bar{z} \Omega h(1 + e^{h}) + 2a - 4 .
\]

(3.2)
Here, \( h \) is the solution of the Taubes equation, and \( a \) is the constant term in the expansion (2.4) which, like the function \( h \), depends on the vortex location \( Z \). The constant \( S_{\text{flat}} \) is the on-shell action \( S \) evaluated on the solution \( h_{\text{flat}} \) of the Taubes equation on the flat background, \( \Omega = 1 \).

From (2.13) we expect that \( S - S_{\text{flat}} \) has the small curvature expansion, for \( \tau = 1 \),

\[
S - S_{\text{flat}} = 2 \log \Omega + 4\alpha K + 4\beta K^2 + 4\gamma \Delta K + \cdots. \tag{3.3}
\]

Our aim is to find the coefficients \( \alpha, \beta, \gamma \) and check their universality. For this we only need to consider a selection of axially symmetric surfaces which are asymptotically planar, and calculate \( S \) for a vortex at the origin, \( Z = 0 \). To do the calculation we numerically determine the solution \( h \) of Taubes’ equation, and from its behaviour near the origin extract the coefficient \( a \). Then we compute the regularised integral \( S \) and subtract the constant \( S_{\text{flat}} \).

When \( \Sigma \) is axially symmetric, the metric (1.1) becomes

\[
g = \Omega(r)(dr^2 + r^2 d\theta^2) \tag{3.4}
\]

where \( z = re^{i\theta} \). For a vortex located at the origin, Taubes’ equation (2.1) simplifies to the ODE

\[
\frac{1}{\Omega} \left( \frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} \right) + 1 - e^h = \frac{4\pi}{\Omega} \delta^2(z). \tag{3.5}
\]

The expansion of the solution \( h(r) \) around \( r = 0 \) is the simplified version of (2.4),

\[
h(r) \sim 2 \log r + a - \frac{\Omega(0)}{4} r^2 + O(r^4), \tag{3.6}
\]

where coefficients of odd powers of \( r \) vanish. The asymptotic form of \( h \) for large \( r \), which is needed for the numerical calculations, can be found by noticing that \( h \) vanishes as \( r \to \infty \), and (3.5) reduces to a Bessel equation, with leading asymptotic solution

\[
h(r) \sim \frac{\lambda}{\sqrt{r}} e^{-\sqrt{\Omega_{\text{as}}} r}, \tag{3.7}
\]

where \( \Omega_{\text{as}} = \lim_{r \to \infty} \Omega(r) \). Usually \( \Omega_{\text{as}} = 1 \).

Taubes’ equation uniquely determines the two asymptotic constants \( a, \lambda \), but since an explicit solution is not known, they generally have to be computed numerically. In the flat case it is possible [12] to relate the two asymptotic expansions and effectively reduce the problem of solving Taubes’ equation to a system of transcendental algebraic equations relating \( a \) and \( \lambda \). A striking, implicitly integrable case is when the metric is related to the vortex profile function by \( \Omega = e^{-h/2} \). In this case Taubes’ equation reduces to a \( \sinh \)-Gordon equation [13]. Also in this case an explicit solution is not known, but the two asymptotic expansions of \( h \) can be related using connection formulae for a particular case of the Painlevé III ODE (a radial reduction of the \( \sinh \)-Gordon equation) and the constant \( a \) is uniquely determined, fixing in this way the solution.
For the class of metrics of small curvature we are interested in, no such methods exist and one has to compute the solution and asymptotic constants numerically.

Let us first consider the flat metric with $\Omega = 1$. To analyse equation (3.5) numerically, we consider a finite interval $r \in [\delta, R]$ and then examine the limits $\delta \to 0$ and $R \to \infty$. Instead of working with the function $h$ we remove the logarithmic singularity and consider

$$u(r) = h(r) - 2 \log(r/R).$$

The delta function disappears, and $u$ satisfies

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + 1 - \frac{r^2}{R^2} e^u = 0.\quad (3.9)$$

The boundary conditions are $u(\delta) \sim a + 2 \log R - \frac{1}{4} \delta^2 + O(\delta^4)$, while for $r = R$ the logarithmic term that we added vanishes (but not its derivative) so $h(R) = u(R) \sim \frac{\lambda}{\sqrt{R}} e^{-R}$. To obtain the solution for $u$ we implemented both a shooting and a cooling method, and the two solutions coincide within numerical errors. We set $\delta = 10^{-4}$ and $R = 30$ so $u(R)$ effectively vanishes. The values of the constant $a$ and the on-shell action $S$ are found to be

$$a_{\text{flat}} = -1.011, \quad S_{\text{flat}} = -1.598.\quad (3.10)$$

These values can also be obtained after some manipulations from the asymptotic analysis of $[12]$ and they agree within numerical errors.

We now consider the family of metrics with conformal factors

$$\Omega(r) = \frac{A L^4 + B L^2 r^2 + r^4}{C L^4 + D L^2 r^2 + r^4}\quad (3.11)$$

with $A, B, C, D$ chosen such that the metric has no zeros or singularities for $r \in \mathbb{R}^+$. These metrics are asymptotically planar, with $\Omega_{as} = 1$, while at the origin $\Omega(0) = A/C$. Thanks to the axial symmetry of $\Omega$ the Gaussian curvature can be easily obtained using equation (2.5) and takes the form

$$K = \frac{r \Omega'(r)^2 - \Omega(r) (\Omega'(r) + r \Omega''(r))}{2 r \Omega(r)^3},\quad (3.12)$$

and further differentiation gives $\Delta K$. Since $\Omega$ is quadratic around the origin with no linear term, $K(0) = -\Omega''(0)/\Omega(0)^2$. In Figure 1 we plot $\Omega$ and $K$ as a function of $r$ for the particular representative of our family with $A = 2, B = 3, C = 1, D = 4$ and $L = 1$. 

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Figure 1. The conformal factor $\Omega$ in blue and the Gaussian curvature $K$ in red when $A = 2, B = 3, C = 1, D = 4$ and $L = 1$.

Recall that the covariant Laplacian $\Delta$ carries a factor $1/L^2$, so $K \sim 1/L^2$ while $K^2, \Delta K \sim 1/L^4$ and all higher terms in the expansion (3.3) come with higher powers of $1/L^2$. For the class of conformal factors $Ω(0) = 1$, and for $Z = 0$, equation (3.3) takes the form:

$$S - S_{\text{flat}} = 2 \log \frac{A}{C} + 8\alpha \frac{AD - BC}{A^2 L^2} + 16\beta \frac{(AD - BC)^2}{A^4 L^4} - 32\gamma \frac{4AC^2 - 4A^2C + 2ABCD + A^2D^2 - 3B^2C^2}{A^4 L^4} + O\left(\frac{1}{L^6}\right).$$ (3.13)

We now fix $L = 10$ to suppress all higher order terms. We have also increased the value of $L$ to check that the parameters $\alpha, \beta, \gamma$ do not significantly change.

We may simplify further the class of conformal factors (3.11) by setting $A = C = 1$. Now $\Omega(0) = 1$, and the curvature $K$ at the origin is proportional to $B - D$. We consider two cases: $B > -2, D = 0$ so

$$S - S_{\text{flat}} = -8\alpha \frac{B}{L^2} + 16(\beta + 6\gamma)\frac{B^2}{L^4} + O(1/L^6),$$ (3.14)

and $B = 0, D > -2$ so

$$S - S_{\text{flat}} = 8\alpha \frac{D}{L^2} + 16(\beta - 2\gamma)\frac{D^2}{L^4} + O(1/L^6).$$ (3.15)

For $D = 0$ and $B \in [0, 1]$, for each value of $B$ we compute the solution to Taubes’ equation, extract the constant term $a$ near the origin and compute the regularised on-shell action $S$. By fitting $S - S_{\text{flat}}$ with a quadratic polynomial in $B$ we extract the values $\alpha = -0.32528$ and $\beta + 6\gamma = -0.5125$. Repeating the analysis for $B = 0$ and $D \in [0, 1]$, a quadratic polynomial fit in $D$ for $S - S_{\text{flat}}$ gives the values $\alpha = -0.32529$ and $\beta - 2\gamma = 0.156$. The value of $\alpha$
in the two cases coincides within numerical error and is the first check of the universality of the expansion; the other coefficients are determined to be

$$\beta = -0.011, \quad \gamma = -0.084. \quad (3.16)$$

In Figure 2 we show the numerical data for $S - S_{flat}$ and the quadratic fits. The two match perfectly for small values of $B, D$ but when $B, D \sim 10$ the neglected terms in the expansion (3.13) become of order 1 and our quadratic fits should be replaced by higher order ones.

![Figure 2](image_url)

**Figure 2.** Numerical data in blue and quadratic fits in red for $S - S_{flat}$, for the family of metrics with $A = C = 1$ and (left) $B \in [0, 10], D = 0$ and (right) $B = 0, D \in [0, 10]$.

To specifically check the universality of the coefficient $\gamma$ we reinstate the parameters $A$ and $C$ in (3.11) and set $B = D = 0$. The curvature $K$ now vanishes at the origin but $\Delta K$ does not. We consider the cases: $A > 0, C = 1$ so

$$S - S_{flat} = 2 \log A + 128 \gamma (A - 1)/(A^3 L^4) + O(1/L^6), \quad (3.17)$$

and $A = 1, C > 0$ so

$$S - S_{flat} = -2 \log C - 128 \gamma C(C - 1)/L^4 + O(1/L^6), \quad (3.18)$$

and proceed similarly as before. We first set $C = 1$ and vary $A$ in the interval $[1, 2]$; for each value of $A$ we numerically solve Taubes’ equation and calculate the on-shell action $S$. Fitting $S - S_{flat}$ with the sum of $2 \log A$ and a cubic polynomial in $1/A$ gives a consistent value $\gamma = -0.086$. Then with $A = 1$ and $C \in [1, 2]$ we fit with a sum of $-2 \log C$ and a quadratic polynomial in $C$ and find $\gamma = -0.085$. Figure 3 shows the fits in the two cases, compared with data over larger ranges of $A$ and $C$. 

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To further check the universality of the coefficients within the family of conformal factors \([3.11]\) we also performed a random sampling of the multi-dimensional space of parameters \(A, C \in [0.5, 2]^2, B, D \in [-1, 1]^2\). For 2000 points in this space we computed \(h\) and evaluated \(S - S_{\text{flat}}\). We then performed a multi-dimensional nonlinear fit to obtain \(\alpha, \beta\) and \(\gamma\), finding results consistent with the values given above.

Having shown that our coefficients \(\alpha, \beta\) and \(\gamma\) are universal within the chosen family \([3.11]\) we have also changed the structure of the conformal factors, for example to a family with exponential behaviour, sharing the same fundamental features: axial symmetry, asymptotic planarity, and hierarchical dependence on \(L\) for the curvature \(K\) and its derivatives. We investigated these families as above and always obtained similar values for the parameters \(\alpha, \beta\) and \(\gamma\), thus confirming the universality of our expansion.

In summary, our final estimated values of the coefficients, taking into account the uncertainty in \(\gamma\), are:

\[
\alpha = -0.325, \quad \beta = -0.01, \quad \gamma = -0.08. \tag{3.19}
\]

4 Comparison with the Ricci flow

Our underlying idea is that the Taubes equation gives rise to a \(\tau\)-dependent functor mapping a background metric \(g\) on \(\Sigma\) to a moduli space metric \(\tilde{g}(\tau)\) on the surface \(\mathcal{M}_1\), which is identified with \(\Sigma\) as we discussed in section 2. This vortex functor has the asymptotic expansion \([2.9]\) in the Bradlow parameter \(\tau\) (or equivalently in the scaling factor \(1/L^2\)), and in particular, \(g(0) = g\). The terms in this expansion include the curvature of the background metric, and other higher order, local scalar invariants constructed by acting with powers of the covariant Laplacian on powers of the curvature.

The Ricci flow \([14]\) on \(\Sigma\) also provides a \(\tau\)-dependent functor on the space of metrics. Here, \(\tau\) becomes identified with a multiple of the time-parameter of the flow. This functor is definitely local, and in \([11]\) some evidence was given to suggest that the vortex functor and the Ricci flow functor are closely related, perhaps after reparametrisation of \(\tau\).
are clearly similarities between the two: the Ricci flow expands negatively curved regions and shrinks positively curved regions, and the same is true for the moduli space metric as a function of $\tau$. In this section we shall demonstrate that the functors agree to lowest order in $\tau$, but differ at higher order in a way which is parametrisation independent.

Let $g = g_{ab}(t)$ be a one-parameter family of Riemannian metrics on a surface. The Ricci flow equation in two dimensions is

$$\frac{\partial}{\partial t} g_{ab} = - R g_{ab}, \quad (4.1)$$

where the Ricci scalar $R = 2K$ is twice the Gaussian curvature. The Ricci flow preserves the conformal class of $g$, so (4.1) can be regarded as a scalar equation for the conformal factor $\Omega$. Moreover, to compare the series solution of (4.1) with the expansion (2.9) of the conformal factor $\tilde{\Omega}(\tau)$ we shall set $t = \tau/2$, so that (4.1) becomes

$$\frac{\partial}{\partial \tau} \Omega = - K \Omega. \quad (4.2)$$

This is the fundamental equation analysed in this section. It implies that

$$\frac{\partial}{\partial \tau} K = \frac{1}{2} \Delta K + K^2, \quad \left[ \frac{\partial}{\partial \tau}, \Delta \right] = K \Delta, \quad (4.3)$$

where $\Delta = g^{ab} \nabla_a \nabla_b = 4\Omega^{-1} \partial_z \partial_{\bar{z}}$. We claim that formally

$$\Omega(\tau) = \left( 1 + \tau \gamma_1 + \frac{1}{2!} \tau^2 \gamma_2 + \ldots \right) \Omega(0), \quad (4.4)$$

where the $\tau$-independent scalars $\gamma_k$ are homogeneous polynomials of degree $k$ in $K$ and the Laplacian operator $\Delta$, both evaluated at $\tau = 0$. Using (4.3) we find

$$\gamma_1 = - K, \quad \gamma_2 = - \frac{1}{2} \Delta K, \quad \gamma_3 = - \Delta \left( \frac{1}{4} \Delta K + \frac{1}{2} K^2 \right), \quad (4.5)$$

$$\gamma_4 = - \Delta \left( \frac{1}{8} \Delta^2 K + \frac{1}{4} \Delta K^2 + \frac{3}{4} K \Delta K + K^3 \right), \quad \ldots$$

Moreover, for all $k \geq 2$, $\gamma_k$ is of the form of $\Delta$ acting on a homogeneous polynomial in $(K, \Delta)$ of degree $k - 1$. This can be seen by an inductive argument: if

$$\frac{\partial^k}{\partial \tau^k} \Omega = - \Delta(\rho_k) \Omega \quad (4.6)$$

is the $(k - 1)$st derivative of (4.2), then (4.2) and (4.3) imply that

$$\frac{\partial^{(k+1)}}{\partial \tau^{(k+1)}} \Omega = \left( -K\Delta(\rho_k) - \Delta \left( \frac{\partial \rho_k}{\partial \tau} \right) + \Delta(\rho_k) K \right) \Omega$$

$$= - \Delta \left( \frac{\partial \rho_k}{\partial \tau} \right) \Omega. \quad (4.7)$$
The coefficients (4.5) are most easily obtained by using this formula, and setting $\tau = 0$.

The expansion (4.4) can also be derived using Picard iterations of conformal rescalings. Recall that the Gaussian curvature of a conformally rescaled metric $\tilde{g} = \omega g$ is

$$\tilde{K} = \omega^{-1} \left( K - \frac{1}{2} \Delta (\log \omega) \right).$$

(4.8)

Let $\Omega_0 = \Omega$ be a conformal factor on $\Sigma$ which does not depend on $\tau$. Define a sequence $\Omega_0, \Omega_1, \ldots$ of $\tau$-dependent conformal factors by

$$\Omega_{n+1}(\tau) = \Omega_0 - \int_0^\tau K_n(s) \Omega_n(s) ds,$$

where $K_n$ is the Gaussian curvature of $\Omega_n$. The limit of this sequence is the solution to the Ricci flow equation (4.2), by Picard’s theorem. At this stage we do not assume that $\tau$ is small. At each step of the iteration we find $\Omega_n(\tau) = \omega_n(\tau) \Omega_0$, where $\omega_n$ is some conformal factor of the form

$$\omega_n = 1 + \tau \gamma_1 + \cdots + \frac{\tau^n}{n!} \gamma_n.$$

(4.10)

We can therefore find $K_n$ using formula (4.8). The Picard expansion yields

$$\omega_{n+1}(\tau) = 1 - \tau K + \frac{1}{2} \Delta \int_0^\tau \log (\omega_n(s)) ds,$$

(4.11)

where $\Delta$ is the Laplacian associated with $\Omega_0$. If we now assume that $\tau$ is small, and expand (in $s$ which is also small) the logarithms in the integrand above keeping all terms up to $s^n$, then the successive iterations agree with (4.4), in the sense that the $n$th iteration reproduces the first $n+1$ terms in (4.4). Moreover $\omega_{n+1}(\tau)$ differs from $\omega_n(\tau)$ by a monomial of the form $\tau^{n+1} \gamma_{n+1}/n!$, and so the $n$th Picard iteration preserves the first $n$ terms in the expansion.

To compare the Ricci flow expansion (4.4) with the expansion (2.9) of the conformal factor $\tilde{\Omega}(\tau)$ on the moduli space, write out the first four terms in (4.4) as

$$\tilde{\Omega}(\tau) = \left( 1 - \tau K - \frac{1}{4} \tau^2 \Delta K - \frac{1}{24} \tau^3 (\Delta^2 K + 2 \Delta K^2) + \ldots \right) \Omega(0),$$

(4.12)

and set $\Omega(0) = \Omega$. This expansion is of the form of (2.9) with

$$\alpha = -\frac{1}{4}, \quad \beta = -\frac{1}{12}, \quad \gamma = -\frac{1}{24}.$$

(4.13)

These coefficients differ from those in (2.14), obtained by calculating the Kähler potential on the moduli space. Therefore the Ricci flow does not integrate to the functor giving the moduli space conformal factor $\tilde{\Omega}(\tau)$, although it does not differ from it greatly for small values of $\tau$.

One may ask if $\tilde{\Omega}(\tau)$ can be regarded as the solution of a different flow equation in $\tau$, which we refer to as vortex flow in contrast to Ricci flow. Let us assume that this is a local geometric flow on $\Sigma$. It has to be of first order in $\partial/\partial \tau$ because the background metric $g$ on
\[\Sigma\] specifies the moduli space metric uniquely, so there is no room for more initial data. As the first term in the expansion \((2.9)\) agrees with that of the Ricci flow, the RHS of \((4.2)\) has to be modified in a non-autonomous way. To the next order in \(\tau\), the vortex flow is
\[
\frac{\partial}{\partial \tau} \tilde{\Omega} = (-K + c\tau \Delta K)\tilde{\Omega},
\]
for some constant \(c\). Using the Picard method to iterate the resulting integral equation, we find this vortex flow reproduces \((2.9)\) up to quadratic terms in \(\tau\), provided one chooses \(c = 2\alpha + \frac{1}{2}\). The cubic terms now need to be corrected by a further modification of the RHS of \((4.14)\) and so forth. The problem essentially reduces to finding an operator \(f(\tau\Delta)\) with \(f(0) = -Id\) such that the vortex flow is
\[
\frac{\partial}{\partial \tau} \tilde{\Omega} = [f(\tau\Delta)(K)]\tilde{\Omega}.
\]
The iterative analytical procedure for constructing the operator \(f\) should involve a Green’s function for the linearised Taubes equation, but this function is not known in closed form.

On the basis of the dimensional analysis carried out in section 2, the vortex flow expansion \((2.9)\) could in principle contain terms involving the norm of the gradient of the Gaussian curvature. These terms can not arise at orders lower then four, but already the coefficient of \(\tau^4\) might involve a constant multiple of \(\Delta |\nabla K|^2\). On the other hand we have established that the Ricci flow expansion contains no such terms, and it may be possible to prove their absence in the vortex flow.

5 Particle geodesics and vortex paths

On a simply-connected surface of constant curvature \(K\) the 1-vortex moduli space metric is known to be \(\tilde{g} = (1 - \tau K)g\), just a constant multiple of \(g\). It can be constructed exactly without using the expansion \((2.9)\), but instead relying on symmetry arguments \([15, 5]\). Thus a vortex moves along a geodesic of the original surface, and its path coincides with that of a point particle, although a vortex and a point particle with the same initial position and kinetic energy will usually reach the same destination at different times. The vortex and particle have different inertial masses, so their velocities differ even if they have the same kinetic energy. On a sphere the vortex has smaller mass – measured by the conformal factor – than the particle. On a hyperbolic plane the vortex has larger mass.

This picture changes if the curvature of the background metric is not constant, as a vortex of finite size is then affected by the background metric in a larger region than a point particle. Thus we expect the vortex and particle paths with the same initial conditions to be different. The acceleration of the vortex will differ from that of the particle – an effect which can be attributed to an additional force. In this section we shall investigate this effect.

To first order in the Bradlow parameter \(\tau\), the moduli space metric is
\[
\tilde{g}(\tau) = \left(1 - \tau K + O(\tau^2)\right)g,
\]
\[\text{16}\]
where $K$ is the Gaussian curvature of the background metric $g$. The point particle follows the geodesics of $g$, and in the geodesic approximation the vortex moves along the geodesics of $\tilde{g}$. If $x^a = (x, y)$ are local real coordinates on $\Sigma$, then the affinely parametrised geodesics $x^a(s)$ of $\tilde{g}$ are integral curves of the system of ODEs

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = -\dot{x}^b \Upsilon_b^a + \frac{1}{2} g_{bc} \dot{x}^b \dot{x}^c \Upsilon^a, \quad a = 1, 2 \quad(5.2)$$

where overdots denote derivatives w.r.t. $s$ and $\Gamma^a_{bc}$ is the Levi–Civita connection of the metric $g$. The quantity $\Upsilon_a$ on the RHS is

$$\Upsilon_a = \nabla_a \log (1 - \tau K + O(\tau^2)) \cong -\tau \nabla_a K + O(\tau^2), \quad(5.3)$$

which vanishes when $\tau = 0$ or if $K$ is constant. Using the Kähler coordinates on $\Sigma$ such that the metric $g$ is given by (1.1), the vortex geodesics are approximated by integral curves of the equation

$$\ddot{z} + (\Omega^{-1} \partial_z \Omega) \dot{z}^2 = \tau (\partial_z K) \dot{z}^2 + O(\tau^2) \quad(5.4)$$

and the complex conjugate of this. The equation of motion for a vortex differs from the equation for a point particle by the terms on the RHS of (5.2) or (5.4). Therefore, to first order in $\tau$, the vortex experiences a force proportional to the gradient of the curvature, and also proportional to the velocity squared.

To compare vortex paths with the paths of point particles we only require unparametrised geodesics. Eliminating the affine parameter $s$ between the two equations (5.2) leads, for each $\tau$, to a single second order ODE for $y$ as a function of $x$,

$$\frac{d^2 y}{dx^2} = \Gamma^1_{22}(\tau) \left( \frac{dy}{dx} \right)^3 + (2 \Gamma^1_{12}(\tau) - \Gamma^2_{22}(\tau)) \left( \frac{dy}{dx} \right)^2 + (\Gamma^1_{11}(\tau) - 2 \Gamma^2_{12}(\tau)) \left( \frac{dy}{dx} \right) - \Gamma^2_{11}(\tau), \quad(5.5)$$

where $\Gamma^a_{bc}(\tau) = \Gamma^a_{bc} - \tau (\delta^a_b \nabla_c K + \delta^a_c \nabla_b K - g_{bc} \nabla^a K)/2 + O(\tau^2)$.

We shall solve this equation numerically, for $(\Sigma, g)$ a surface of revolution with metric (3.4) and Gaussian curvature $K$ given by (3.12). The ODE (5.5) becomes, to first order in $\tau$,

$$\frac{d^2 r}{d\theta^2} = r + \frac{r^2}{2} \frac{\partial_r \Omega}{\Omega} - \frac{\tau}{2} r^2 \partial_r K + \left( \frac{2}{r} + \frac{1}{2} \frac{\partial_r \Omega}{\Omega} - \frac{\tau}{2} \partial_r K \right) \left( \frac{dr}{d\theta} \right)^2. \quad(5.6)$$

As an example we consider a conformal factor of the form (3.11), with $L = 1$, and choose the constants $A, B, C, D$ so that

$$\Omega = \frac{2 + 7r^2 + r^4}{1 + r^2 + r^4}. \quad(5.7)$$

This metric is asymptotically planar, and the radial plots of the Gaussian curvatures of the background metric $g$ and the modified metric $\tilde{g} = (1 - K)g$ as functions of $r$ are given in Figure 4. Note that both curvatures integrate to zero as anticipated in equation (2.10).
Figure 4. Gaussian curvatures of the asymptotically flat surface of revolution with conformal factor (5.7) (thin red curve) and the modified vortex metric (thicker blue curve).

We have set $\tau = 1$ here, but still work to first order in $\tau$. The resulting curvature is not small in the sense explained in section 2, and thus the deviation effects are approximate and exaggerated on the Figures below. However, the nature of these effects would not change for small $\tau$ or equivalently small curvature, but the differences in the Figures would be less visible.

For the chosen conformal factor the radial force $\partial_r K$ in the ODE (5.6) is attractive for $r \in (0, r_1)$ and $r \in (r_2, \infty)$ and repulsive for $r \in (r_1, r_2)$, where $r_1 \approx 0.76, r_2 \approx 2.89$ (Figure 5).

Figure 5. The radial force $\partial_r K$ as a function of $r$.

Figure 6 shows both the point particle geodesic ($\tau = 0$) and the vortex geodesic ($\tau = 1$) approaching from the same point in the asymptotically flat region and in the same initial direction. The paths coincide initially, but the vortex path moves apart from the particle path in the region where the curvature gradient is large. The paths then approach straight lines again in the asymptotically flat region, but in different directions. The vortex path can
either extend further out in the radial direction, or bend towards the origin. This depends on the sign of the radial force $\partial_r K$ in the region where its absolute value becomes large. This in turn depends on the initial conditions.

Figure 6. Particle geodesics (thin, red), and vortex geodesics (thicker, blue) with the same initial conditions: \(y(-5) = 0, y'(-5) = 1\) (left) and \(y(-1.5) = 0, y'(-1.5) = 1\) (right).

Figure 7 shows the geodesics of a point particle and a vortex going through the same initial and final points. These points are in the region of small negative curvature of both the background metric $g$ and the approximate moduli space metric $\tilde{g}$. The vortex path extends further out in $r$ than the point particle path.

Figure 7. Geodesic of a point particle (thin, red), and a vortex (thicker, blue) passing through the same initial and final points $(-5,5)$ and $(5,5)$.

Let us finally consider an example of vortex motion on a compact surface. We assume that our analysis leading to the numerical values (2.14) of the coefficients in the expansion of the conformal factor of the moduli space, (2.9), applies in the compact case, although we
have not established this. The first term in the expansion is certainly universal. To see the curvature effects, we choose to work with the ellipsoid of revolution

\[ x^2 + y^2 + \frac{\zeta^2}{b^2} = 1, \quad (5.8) \]

whose Riemannian metric induced from \( \mathbb{R}^3 \) by eliminating \( \zeta \) and setting \( x + iy = re^{i\theta} \) is

\[ g = \frac{1 - (1 - b^2)r^2}{1 - r^2}dr^2 + r^2 d\theta^2. \quad (5.9) \]

As before, the vortex path deviates from a point particle path in the region of large curvature gradient, which for an ellipsoid with \( b < 1 \) is close to the equator. This is illustrated in Figure 8.

![Figure 8](image)

**Figure 8.** Point particle geodesic (thin, red) and vortex geodesic (thicker, blue) on the ellipsoid of revolution \( x^2 + y^2 + 4\zeta^2 = 1 \).

### 6 Conclusions

We have studied the moduli space metric for one Abelian Higgs vortex on a surface of small curvature, or equivalently, for a vortex of small size on a fixed surface. This work complements a previous study of the moduli space metric close to the Bradlow limit where the vortex is of comparable size to the whole surface, and is close to dissolving [16]. The moduli space metric has a universal expansion in the Bradlow parameter (controlling the size of the vortex) involving the background metric, its curvature and the Laplacian operator acting on powers of the curvature. We are not able to calculate the coefficients in this expansion analytically, but have calculated the first few of them numerically using a family of asymptotically planar, circularly symmetric surfaces, and have checked their universality.

The moduli space metric as a function of the Bradlow parameter has some similarity to the solution of the Ricci flow as a function of time with the underlying metric as initial
data. Their Taylor expansions agree to first order but not to higher order. We have found a modification of the Ricci flow which reproduces correctly the expansion in the Bradlow parameter to second order and further improvements are possible.

We have investigated vortex motion by calculating geodesics on the moduli space and have compared these to geodesics on the original surface, which model the motion of point particles. They differ because of the finite vortex size. The leading effect is an additional force acting on a vortex, proportional to the gradient of the curvature of the surface. This additional force has some analogy with the self-force experienced by a finite-size body moving in a gravitational background.

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**Appendix. The regularised Taubes action**

Here we construct an integral formula for the Kähler potential on the 1-vortex moduli space, for a vortex moving on the Riemann surface Σ with conformal factor Ω(z, ̄z). We assume that Σ is asymptotically planar with Ω = 1 at infinity.

We first note [10] that Taubes’ equation for a vortex located at

\[ S = \lim_{\varepsilon \to 0} \left\{ \frac{i}{2\pi} \int_{\Sigma_{\varepsilon}} dz \wedge d\bar{z} \left[ 2\partial_z h \partial_{\bar{z}} h + \frac{\Omega}{\tau} \left( e^h - 1 - h \right) \right] + 4a + 4 \log \varepsilon \right\}, \tag{A1} \]

where \( \Sigma_{\varepsilon} = \Sigma \setminus D_{\varepsilon}(Z) \) and \( D_{\varepsilon}(Z) \) is a disc of radius \( \varepsilon \) centred around \( Z \). It is assumed that \( h \) has an expansion of the form (2.4) around \( Z \), with the log-term fixed but the coefficients \( a, b, c, d \) etc. free to vary. The last two terms in (A1) can be seen as a contribution coming from a boundary action and are necessary to obtain a well defined variational principle, given that \( h \) has the logarithmic singularity \( 2 \log |z - Z| \). The \( 4 \log \varepsilon \) term makes \( S \) finite, and the variation of \( 4a \) cancels the boundary term coming from the integration by parts. By requiring \( S \) to be stationary under any variation \( h \to h + \delta h \) with \( \delta h \) vanishing at infinity and finite at the origin, one obtains Taubes’ equation (2.1). The delta function is absent since we have removed its support from the bulk action, but it is recovered by taking account of the logarithmic singularity.

We now define the on-shell action to be the action \( S \) evaluated for a solution of Taubes’ equation. This depends on the vortex location \( Z \), and the background geometry. In section 2 we pointed out that the metric on the moduli space can be written in terms of the coefficient \( b \) in (2.4). Here we show that, on-shell, \( b \) is related to a derivative of \( S \), and hence \( S \) is part of the Kähler potential.
The derivative of $S$ with respect to $Z$ is

$$
\frac{\partial S}{\partial Z} = \lim_{\varepsilon \to 0} \left\{ \frac{i}{2\pi} \int_{\Sigma_\varepsilon} dz \wedge d\bar{z} \left[ 2\partial_z h \partial_{\bar{z}} h + \frac{\Omega}{\tau} (e^{h} - 1 - h) \right] - \frac{i}{2\pi} \oint_{\gamma_\varepsilon} d\bar{z} \left[ 2\partial_z h \partial_{\bar{z}} h + \frac{\Omega}{\tau} (e^{h} - 1 - h) \right] \right\} + 4 \frac{\partial a}{\partial Z},
$$

(A2)

where the second integral comes from the variation with respect to $Z$ of the domain of integration $\Sigma_\varepsilon$, giving an integral over its boundary $\gamma_\varepsilon = \partial \Sigma_\varepsilon$. On-shell, we can make use of Taubes’ equation to put the first integral in the form

$$
\lim_{\varepsilon \to 0} \left\{ \frac{i}{2\pi} \int_{\Sigma_\varepsilon} dz \wedge d\bar{z} \left[ 2\partial_z h \partial_{\bar{z}} h + \frac{\Omega}{\tau} (e^{h} - 1 - h) \right] = \lim_{\varepsilon \to 0} \left\{ \frac{i}{\pi} \left( - \oint_{\gamma_\varepsilon} d\bar{z} \partial h \partial_{\bar{z}} h \partial_{\bar{z}} h + \oint_{\gamma_\varepsilon} dz \partial h \partial_{\bar{z}} h \partial_{\bar{z}} h \right) \right\}
$$

$$
= -4 \frac{\partial a}{\partial Z} + 3\bar{b},
$$

(A3)

where we have made use of the behaviour of $h$ near $Z$ and used a variant of Cauchy’s residue theorem. For the second integral, in the limit $\varepsilon \to 0$, the only term contributing is $\bar{b}/(\bar{z} - \bar{Z})$ from $2\partial_z h \partial_{\bar{z}} h$, so the integral is $-\bar{b}$, again using the residue theorem. Combining these integrals and the $4 \frac{\partial a}{\partial Z}$ term, we find that \(\frac{\partial S}{\partial Z} = 2\bar{b}\), and as $S$ is real,

$$
\frac{\partial S}{\partial Z} = 2\bar{b}.
$$

(A4)

Using this result we can rewrite the conformal factor of the Samols metric (2.3) as

$$
\tilde{\Omega}(Z, \bar{Z}) = \Omega(Z, \bar{Z}) + \tau \frac{\partial^2 S}{\partial Z \partial \bar{Z}}.
$$

(A5)

This means that the Kähler potential on the 1-vortex moduli space is given by

$$
\tilde{K}(Z, \bar{Z}) = K(Z, \bar{Z}) + \tau S(Z, \bar{Z}) + \text{const.}
$$

(A6)

where $K$ is the Kähler potential of $\Sigma$.

If we insist that $K$ and $\tilde{K}$ have the same asymptotic form $Z\bar{Z}$, and their difference tends to zero asymptotically, then $\tau S + \text{const.}$ must tend to zero as the vortex location tends to infinity. For a vortex on an asymptotically planar surface, located far from the region where the Gaussian curvature $K$ and its derivatives differ significantly from zero, the profile function $h$ will be almost identical to that in the flat case; only the exponential tail of $h$ will experience the region with significant $K$. Therefore

$$
\lim_{\bar{Z} \to \infty} S(Z, \bar{Z}) = S_{\text{flat}},
$$

(A7)

where $S_{\text{flat}}$ is the on-shell action $S$ evaluated on the solution $h_{\text{flat}}$ of the Taubes equation on the flat background. The constant term in (A6) must therefore be $-\tau S_{\text{flat}}$. 22
To facilitate the numerical studies it is better to simplify the on-shell action by making use of Taubes’ equation again to rewrite
\[
2\partial_z h \partial_{\bar{z}} h = -2h \partial_z \partial_{\bar{z}} h + 2\partial_z (h \partial_{\bar{z}} h) = -\frac{\Omega}{2\tau} h \left(e^h - 1\right) + 2\partial_z (h \partial_{\bar{z}} h) .
\] (A8)

This yields
\[
S = \lim_{\varepsilon \to 0} \left\{ \frac{i}{2\pi} \int_{\Sigma_{\varepsilon}} dz \wedge d\bar{z} \frac{\Omega}{\tau} \left[ \left(1 - \frac{h}{2}\right) e^h - \frac{h}{2} - 1\right] + \frac{i}{\pi} \oint_{\gamma_{\varepsilon}} dz \left(h \partial_z h\right) + 4a + 4 \log \varepsilon \right\} .
\] (A9)

As \( h \sim 2 \log \varepsilon + a \) and \( \partial_z h \sim \frac{1}{Z} \) for \( \varepsilon \) small, the residue theorem implies that
\[
\frac{i}{\pi} \oint_{\gamma_{\varepsilon}} dz \left(h \partial_z h\right) = -4 \log \varepsilon - 2a .
\] (A10)

The \( \log \varepsilon \) terms now cancel and the remaining integral part of \( S \) becomes well defined on \( \Sigma \) (the integrand still has a logarithmic singularity at the vortex location but that is integrable) so that the limit \( \varepsilon \to 0 \) can easily be taken. We can simplify \( S \) even further by noticing that
\[
\frac{i}{2} \int_{\Sigma} dz \wedge d\bar{z} \frac{\Omega}{\tau} (1 - e^h) = 4\pi ,
\] (A11)

since this integral is twice the magnetic flux. The on-shell action can therefore be written in the form
\[
S = -\frac{i}{4\pi} \int_{\Sigma} dz \wedge d\bar{z} \frac{\Omega}{\tau} h \left(1 + e^h\right) + 2a - 4 ,
\] (A12)

which can be easily computed numerically once the solution to Taubes’ equation is known. Equation (A6) takes the final form
\[
\tilde{K} = K + \tau (S - S_{\text{flat}}) ,
\] (A13)

with \( S \) given by (A12).

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