LOCAL CLUB CONDENSATION AND L-LIKENESS

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Abstract. We present a forcing to obtain a localized version of Local Club Condensation, a generalized Condensation principle introduced by Sy Friedman and the first author in [3] and [5]. This forcing will have properties nicer than the forcings to obtain this localized version that could be derived from the forcings presented in either [3] or [5]. We also strongly simplify the related proofs provided in [3] and [5]. Moreover our forcing will be capable of introducing this localized principle at \( \kappa \) while simultaneously performing collapses to make \( \kappa \) become the successor of any given smaller regular cardinal. This will be particularly useful when \( \kappa \) has large cardinal properties in the ground model. We will apply this to measure how much L-likeness is implied by Local Club Condensation and related principles. We show that Local Club Condensation at \( \kappa^+ \) is consistent with \( \neg \Box_\kappa \) whenever \( \kappa \) is regular and uncountable, generalizing and improving a result of the third author in [14], and that if \( \kappa \geq \omega_2 \) is regular. \( \text{CC}(\kappa^+) \) - Chang’s Conjecture at \( \kappa^+ \) - is consistent with Local Club Condensation at \( \kappa^+ \), both under suitable large cardinal consistency assumptions.

§1. Condensation and L-likeness. Besides the presentation of the forcing announced in the abstract, the central theme of this paper is the relationship between generalized Condensation principles (i.e., generalizations of consequences of Gödel’s Condensation Lemma) and other L-like principles: we investigate the question of how close to Gödel’s constructible universe the universe of sets has to be given that it satisfies certain generalized Condensation principles. For definitions of generalized Condensation principles that will be relevant to this paper see Section 2.

In [3], Sy Friedman and the first author showed that Local Club Condensation allows for the existence of very large large cardinals, far beyond those compatible with \( V = L \)—namely they showed, by using the method of forcing, that Local Club Condensation is consistent with the existence of \( \omega \)-superstrong cardinals. This was further improved in [4] by showing that Local Club Condensation and Acceptability are simultaneously consistent with the existence of \( \omega \)-superstrong cardinals.

It is generally believed that the fine structural properties of \( L \) are necessary to prove that various square principles hold in \( L \). In [14], the third author showed that Strong Condensation for \( \omega_2 \) is consistent with \( \neg \Box_{\omega_1} \) from a stationary limit of measurable cardinals, thus giving additional support to this belief. One of the main aims of this paper is to generalize his result to cardinals beyond \( \omega_2 \), replacing Strong

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Condensation for $\omega_2$ by Local Club Condensation at $\kappa$ for $\kappa \geq \omega_2$\(^1\) and reducing the consistency assumption to a 2-Mahlo cardinal.\(^2\) We further investigate weaker square principles, Jónsson cardinals, and Chang’s Conjecture style principles, all in the context of generalized Condensation principles.

§2. Condensation Principles. The definitions of Strong Condensation and Local Club Condensation apply to models $\mathbf{M}$ of set theory with a hierarchy of levels of the form $\langle M_\alpha \mid \alpha \in \text{Ord} \rangle$ with the properties that $\mathbf{M} = \bigcup_{\alpha \in \text{Ord}} M_\alpha$, each $M_\alpha$ is transitive, $\text{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \subseteq M_\beta$ and if $\gamma$ is a limit ordinal, $M_\gamma = \bigcup_{\alpha < \gamma} M_\alpha$. We will also let $M_\alpha$ denote the structure $(M_\alpha, \in, \langle M_\beta \mid \beta < \alpha \rangle)$.\(^3\) where context will usually clarify the intended meaning. Moreover we denote $\langle M_\alpha \mid \alpha \in \text{Ord} \rangle$ or any of its restrictions by $\bar{M}$.

If $\mathcal{X}$ is a substructure of $\langle M_\alpha, \in, \bar{M} \rangle$ for some $\alpha \in \text{Ord}$, we say that $\mathcal{X}$ condenses or is a condensing substructure of $M_\alpha$ if $\mathcal{X}$ is isomorphic to $\langle M_\alpha, \in, \bar{M} \rangle$ for some $\bar{\alpha} \leq \alpha$. More generally, if $\mathcal{A}$ is a structure for a countable language of the form $\mathcal{A} = \langle M_\alpha, \in, \bar{M}, \ldots \rangle$ for some $\alpha \in \text{Ord}$ and $\mathcal{X}$ is a substructure of $\mathcal{A}$ with domain $X$, we say that $\mathcal{X}$ condenses or is a condensing substructure of $\mathcal{A}$ if $\langle X, \in, \bar{M} \mid X \rangle$ condenses.

Local Club Condensation is the statement that if $\alpha$ has uncountable cardinality $\kappa$ and $\mathcal{A}_\alpha = \langle M_\alpha, \in, \bar{M}, \ldots \rangle$ is a structure for a countable language, then there exists a continuous chain $\langle B_\gamma \mid \omega \leq \gamma < \kappa \rangle$ of condensing substructures of $\mathcal{A}_\alpha$ whose domains have union $M_\alpha$, where each $B_\gamma = \text{dom}(B_\gamma)$ is such that $\lvert B_\gamma \rvert = \lvert \gamma \rvert$ and $\gamma \subseteq B_\gamma$.

We will usually be in the situation that $\mathbf{M} = (\mathbf{L}[A], \in, A)$ for some $A \subseteq \text{Ord}$ and $\bar{M} = \langle L_\alpha[A] \mid \alpha \in \text{Ord} \rangle$. We say that $\mathbf{M}$ is of the form $\mathbf{L}[A]$ in that case.

If $\kappa = \lambda^+$ and $\lambda$ is uncountable, Local Club Condensation at $\kappa$ is the statement that $M_\kappa = H_\kappa$ and if $\alpha \in [\lambda, \kappa)$ and $\mathcal{A} = \langle M_\alpha, \in, \bar{M}, \ldots \rangle$ is a structure for a countable language, then there exists a continuous chain $\langle B_\gamma \mid \gamma < \lambda \rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union $M_\alpha$, where each $B_\gamma = \text{dom}(B_\gamma)$ is such that $\lvert B_\gamma \rvert < \lambda$ and $\gamma \subseteq B_\gamma$.\(^4\)

Note: The reason why one need not include the case $\alpha = \kappa$ in the above is that if $\mathcal{A} = \langle M_\kappa, \in, \bar{M}, \ldots \rangle$ is a structure for a countable language, then there is a continuous chain of transitive, elementary substructures of $\mathcal{A}$ of size $\lambda$ that each have domain some $M_{\bar{\alpha}}$, $\bar{\alpha} < \kappa$ and have union $M_\kappa$. Now we may apply Local Club Condensation at $\kappa$ to the least substructure of that chain to obtain a continuous chain $\langle B_\gamma \mid \gamma < \kappa \rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union $M_\kappa$, where each $B_\gamma = \text{dom}(B_\gamma)$ is such that $\lvert B_\gamma \rvert < \lambda$ for $\gamma < \lambda$, $\lvert B_\gamma \rvert = \lambda$ for $\gamma \in [\lambda, \kappa)$ and $\gamma \subseteq B_\gamma$ for every $\gamma < \kappa$.

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\(^1\)Local Club Condensation at $\omega_2$ implies Strong Condensation for $\omega_2$, see Theorem 2.1 below.

\(^2\)A cardinal $\kappa$ is 2-Mahlo if the set of Mahlo cardinals below $\kappa$ is a stationary subset of $\kappa$. In the first submitted version of this paper, our large cardinal assumption was a stationary limit of $\omega$-Erdős cardinals. The key hint on how to further reduce this consistency assumption was given to the authors in personal communication by Boban Veličković.

\(^3\)We assume some appropriate coding of the sequence $\langle M_\beta \mid \beta < \alpha \rangle$ here.

\(^4\)If $\lambda = \theta^+$ is a successor cardinal, we may equivalently demand that $\lvert B_\gamma \rvert = \theta$. 
In [3], the theorem below is shown assuming Local Club Condensation holds in $\mathbf{M}$, but all that is actually used is Local Club Condensation at $\kappa$, giving rise to the following (we will abbreviate the conclusion of the theorem by saying that *Transitive Condensation at $\kappa$* holds):

**Theorem 2.1** (Friedman, Holy, Wu [4, Theorem 88]). *If $(\mathbf{M}, \in, \vec{M})$ is a model of Local Club Condensation at $\kappa$, where $\kappa = (\tau^+)^{\mathbf{M}}$, $\tau$ is an $\mathbf{M}$-cardinal of uncountable cofinality, $F = \langle f_\alpha : \alpha \in [\tau, \kappa) \rangle$ where each $f_\alpha$ is a bijection from $\tau$ to $\alpha$ in $\mathbf{M}$,

$$X \prec (M_\kappa, \in, \vec{M}, F, S)$$

where $S$ is a set of Skolem functions for the above structure and $X$ is transitive below $\tau$, then $X$ condenses. In fact, $X$ need not be an element of $\mathbf{M}$ for the above to hold.*

**Strong Condensation** is the statement that for every ordinal $\alpha$, there is a structure $A_\alpha = (M_\alpha, \in, \vec{M}, \ldots)$ for a countable language such that each of its substructures condenses.\(^5\) Strong Condensation for $\alpha$ ($\alpha \in \text{Card}$) is the statement of Strong Condensation for a single cardinal $\alpha$ together with the assumption that $M_\alpha = H_\alpha$.

As was observed in [4], Local Club Condensation at $\omega_2$ implies (by Theorem 2.1) Transitive Condensation at $\omega_2$ which is easily seen to imply Strong Condensation. In Section 4, we will observe that this is not the case for $\kappa \geq \omega_3$, i.e., that Local Club Condensation at $\kappa$ does not imply Strong Condensation for $\kappa$.

We define one last version of Local Club Condensation that strengthens Local Club Condensation at $\kappa$ (we will observe that this is a proper strengthening in Section 4).

If $\omega \leq \lambda < \lambda^+ < \kappa$, *Local Club Condensation in $[\lambda, \kappa)$* is the statement that $M_\kappa = H_\kappa$ and if $\alpha \in [\lambda^+, \kappa)$ and $A_\alpha = (M_\alpha, \in, \vec{M}, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle B_\gamma : \lambda \leq \gamma < \kappa \rangle$ of condensing substructures of $A_\alpha$ whose domains have union $M_\alpha$, where each $B_\gamma = \text{dom}(B_\gamma)$ is such that $|B_\gamma| = |\gamma|$ and $\gamma \subseteq B_\gamma$. If $\lambda = \omega$ in the above, we call the resulting principle *Local Club Condensation up to $\kappa$*.

**Note:** Whenever $\lambda_0 < \lambda_1 < \lambda_1^+ < \kappa$, Local Club Condensation in $[\lambda_0, \kappa)$ is stronger than Local Club Condensation in $[\lambda_1, \kappa)$ which in turn is stronger than Local Club Condensation at $\kappa$. If $\kappa = \lambda^{++}$, Local Club Condensation in $[\lambda, \kappa)$ is the same as Local Club Condensation at $\kappa$.

### §3. Easy observations regarding L-likeness.

Local Club Condensation implies the GCH (see [3, Lemma 1]). If $\kappa = \lambda^+$ and $\lambda$ is uncountable, Local Club Condensation at $\kappa$ implies $2^\lambda = \kappa$ and $2^{<\lambda} = \lambda$, in fact it is easily seen to imply that $H_\lambda = M_\lambda$ has cardinality $\lambda$. We will see in Section 4 that it does not impose any further restrictions on the values of the continuum function. Strong Condensation for $\kappa$ implies that the GCH holds below $\kappa$, in fact it implies that $M_\lambda = H_\lambda$ has cardinality $\lambda$ for every uncountable $\lambda \leq \kappa$. We provide a proof of this last statement.

**Lemma 3.1.** *If $\kappa$ is an uncountable cardinal, Strong Condensation for $\kappa$ implies that for every uncountable cardinal $\lambda \leq \kappa$, $M_\lambda = H_\lambda$.*

\(^5\)Strong Condensation was originally introduced by Hugh Woodin in [13].
Proof. This is part of the definition of Strong Condensation for $\kappa$ if $\lambda = \kappa$. Thus we may assume that $\lambda < \kappa$.

Assume $x \in H_{\lambda}$. Let $N$ be a condensing elementary substructure of $M_\kappa$ that contains the transitive closure of $x$ as a subset and has size less than $\lambda$. It follows that $x \in M_\lambda$, i.e., $H_{\lambda} \subseteq M_\lambda$.

Let $\mathcal{A}$ be a Skolemized structure on $H_\kappa$ witnessing Strong Condensation for $\kappa$ holds. For $\alpha < \lambda$, let $N_\alpha$ be the Skolem hull of $\alpha$ in $\mathcal{A}$. Each $N_\alpha$ condenses to some $M_{f(\alpha)}$, where $f(\alpha) < \lambda$ and $\{ f(\alpha) \mid \alpha < \lambda \}$ is cofinal in $\lambda$. It follows that $M_\alpha$ has size less than $\lambda$ whenever $\alpha < \lambda$. Since each $M_\alpha$ is transitive, it thus follows that $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha \subseteq H_\lambda$ has cardinality $\lambda$.

We mention some further facts about Local Club Condensation, localizations of which may also be easily obtained.

**Fact 3.2 ([4, Lemma 95]).** Local Club Condensation implies

- $\diamondsuit_\kappa (E)$ whenever $\kappa$ is regular and $E \subseteq \kappa$ is stationary.
- $\diamondsuit_\kappa^+$ for all successor cardinals $\kappa$.

**Fact 3.3 ([3, Theorem 37]).** Local Club Condensation implies that whenever $\kappa$ is regular, there is a $\Delta_1$-definable wellorder of $H_{\kappa^+}$.

Strong Condensation has some even more striking consequences:

**Fact 3.4 ([3, Theorem 4]).** Strong Condensation implies that there is no $\omega_1$-Erdős cardinal.

**Fact 3.5 ([9, Corollary 1.13]).** Strong Condensation for $\omega_3$ implies that there is no precipitous ideal on $\omega_1$.

§4. The Forcing Construction. In this section, we present our main forcing construction. This is a (strongly simplified and improved) variation of forcing constructions (and the corresponding proofs of their properties) from [3] and [5], that allows us to obtain Local Club Condensation at $\kappa$ for a given regular cardinal $\kappa$ while collapsing $\kappa$ to become the successor of any given smaller regular uncountable cardinal $\lambda$.\(^6\) When $\kappa = \lambda^+ = \omega_2$, the construction below is a significant simplification to obtain the main technical result of [14], namely a small forcing to obtain Strong Condensation for $\omega_2$.

Assume $\kappa$ is regular. We want to extend a given model $V$ of set theory to a model of Local Club Condensation at $\kappa$ while, for some regular cardinal $\lambda < \kappa$, collapsing all cardinals in $(\lambda, \kappa)$ so that $\kappa = \lambda^+$ in the generic extension. We assume that $V$ satisfies $2^{< \lambda} = \lambda$, $2^\lambda \leq \kappa$, and $\theta^{< \lambda} < \kappa$ for every $\theta < \kappa$, define a forcing iteration $P = P(\lambda, \kappa)$ and show that $P$-generic extensions of the universe satisfy Local Club Condensation at $\kappa$, model $\kappa = \lambda^+$ and $2^\lambda = \kappa$ and that forcing with $P$ preserves $\lambda$ and both the continuum function and all cardinals outside of the interval $[\lambda, \kappa)$. $P$ will be $< \lambda$-directed closed and $\kappa$-cc. We define $P$ inductively. $P_{< \lambda}$, the forcing up to $\lambda$, is just the product of length $\lambda$ with $< \lambda$-sized support of the lottery of $\{0, 1\}$.\(^7\)

\(^6\)This is supposed to include the case where $\kappa = \lambda^+$ initially, i.e., when no actual collapses are performed by the forcing.

\(^7\)So $P_{< \lambda}$ is just the forcing to add a Cohen subset of $\lambda$, in disguise.
If $\alpha \geq \lambda$, a condition at $\alpha$ is a pair $(f_\alpha, c_\alpha)$ which is either trivial, i.e., $(f_\alpha, c_\alpha) = (\emptyset, \emptyset)$, or there is $\gamma_\alpha < \lambda$ such that

- $c_\alpha : \gamma_\alpha \to 2$ is such that $C_\alpha = \{ \delta < \gamma_\alpha \mid c_\alpha(\delta) = 1 \}$ is closed in $\lambda$.
- $f_\alpha : \max(C_\alpha) \to \alpha$ is an injection and
- $f_\alpha[\max(C_\alpha)] \supseteq \max(C_\alpha)$.

If $p^0 = (f^0, c^0)$ and $p^1 = (f^1, c^1)$ are conditions at $\alpha$, we let $p^1 \leq p^0$ iff $p^0$ is trivial or

1. $f^1 \supseteq f^0$ and
2. $c^1 \supseteq c^0$, i.e., $c^1|\dom(c^0) = c^0$.

Note that if we force with the poset consisting of conditions at $\alpha$, ordered as above, this will generically add a bijection from $\lambda$ to $\alpha$.

While defining $P_{<\alpha}$ for $\alpha \in (\lambda, \kappa]$ we also define a function $A$ with domain $[\lambda, \kappa)$ such that for every $\alpha$, $A(\alpha)$ is a $P_{<\alpha}$-name for either 0 or 1. We fix a wellorder $\mathcal{W}$ of $H_\alpha$ of order-type $\kappa$. Let $\beta \in [\lambda, \kappa)$ and assume $A|\beta$ and $P_{<\beta}$ have been defined. Let $A(\beta)$ be the canonical $P_{<\alpha}$-name for either 0 or 1 such that for any $P_{<\beta}$-generic $G_{<\beta}$, $A(\beta)|G_{<\beta} = 1$ iff $\beta = <\gamma, <\delta, e>, 8, \hat{x}$ is the $\gamma^\text{th}$ (in the sense of $\mathcal{W}$) $P_{<\beta}$-nice name for a subset of $\lambda$, $e < \lambda$ and $e \in \hat{x}|G_{<\beta}$.

Now assume $\alpha \in (\lambda, \kappa]$ and we defined $\langle P_{<\beta} \mid \beta < \alpha \rangle$ and $A \upharpoonright \alpha$. Then $P_{<\alpha}$ is the set of all $\alpha$-sequences $p$ with $<\lambda$-support such that

- $p, p|\beta \in P_{<\beta}$ for every $\beta < \alpha$ and if $\alpha = \beta + 1$, the following hold:
  - $p(\beta) = (f_\beta, c_\beta)$ is a condition at $\beta$,
  - if $C_\beta \neq \emptyset$, then $p|\beta$ decides $A(\beta) = a_\beta \in \{0, 1\}$,
  - $\forall \delta \in C_\beta \ p(\ot f_\beta[\delta]) = a_\beta$.
- * We let $C\supp(p) = \{ \beta \mid C_\beta \neq \emptyset \}$. The following need to hold for $p$ as well:10
  - $\gamma_p := \supp(p) \cap \lambda = \gamma_\beta = \dom(c_\beta)$ for any $\beta \in C\supp(p)$,
  - $\exists \delta^p \forall \beta \in C\supp(p) \max(C_\beta) = \delta^p$,
  - $\forall \beta_0 < \beta_1$ both in $C\supp(p)$,

$f_{\beta_0}[\delta^p]$ is an initial segment of $f_{\beta_1}[\delta^p]$ and

$f_{\beta_1}[\delta^p] \setminus \beta_0 \neq \emptyset$.

For $p$ and $q$ in $P_{<\alpha}$, we let $q \leq p$ iff $q|\lambda \leq p|\lambda$ and for every $\beta \in [\lambda, \alpha)$, $q(\beta) \leq p(\beta)$. We let $P = P_{<\kappa}$. Note that if $\beta < \alpha$ then $P_{<\beta} \subseteq P_{<\alpha}$.

**Claim 4.1.** If $p \in P_{<\alpha}$, $\beta \in [\lambda, \alpha)$, and $\delta < \lambda$, then there is $q \leq p$ with $\beta \in C\supp(q)$ and $\delta^q > \delta$.

**Proof.** We obtain an $\alpha$-sequence $r = \langle (f^\xi_\zeta, c^\xi_\zeta) \mid \zeta < \alpha \rangle$ from $p$ by extending $p|\beta$ to $r|\beta \in P_{<\beta}$ such that $r|\beta$ decides $A(\beta) = a_\beta \in \{0, 1\}$ and setting

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8Thus if $\gamma_\alpha$ is a limit ordinal, $C_\alpha$ is bounded in $\gamma_\alpha$. To avoid this case (which we don’t), one could simply demand that $\gamma_\alpha$ is a successor ordinal.

9$<\cdot, \cdot>$ denotes the Gödel pairing function.

10The remaining clauses will help ensure both that our forcing is sufficiently closed and that the following proofs go through easily. None of these clauses (nor the last clause for a condition at $\alpha$) were used in the forcing constructions presented in either [3] or [5]. While this made the presentation of the forcing constructions itself somewhat easier, it made the corresponding proofs much more difficult (and also the forcings provided there were not closed).
Choose \( \xi > \gamma \) such that \( \gamma^\gamma + \xi = \xi \). We want to find a componentwise end-extension \( q \) of \( r \) with \( q \in P_{\alpha}, \delta^\gamma = \xi \), and \( \beta \in \text{C-sup}(q) \). It then follows that \( q \leq p \), i.e., \( q \) is as desired.

Let \( \text{C-sup}(r) = \{ \xi : C_\xi^r \neq \emptyset \} \). For every \( \xi \in \text{C-sup}(r) \cup \{ \beta \} \), we choose \( c_\xi^r : (\xi + 1) \rightarrow 2 \) such that \( c_\xi^0 \mid \text{dom}(c_\xi^r) = c_\xi^r \), and \( c_\xi^0(\xi) = 0 \) and \( c_\xi^r(\xi) = 1 \). By our assumptions on \( \xi \), we may extend \( f_\xi^r \) to \( f_\xi^q \) with domain \( \xi \) such that \( f_\xi^q[\xi] \supseteq \xi \) and such that whenever \( \xi_0 < \xi_1 \) are both in \( \text{C-sup}(r) \cup \{ \beta \} \), then \( f_\xi^q[\xi] \) is an initial segment of \( f_{\xi_1}^q[\xi] \) and \( f_{\xi_1}^q[\xi_0] \neq \emptyset \), using that \( \xi \) was chosen sufficiently large. This now allows us to choose \( q \) of \( f_\xi^q[\xi] = a_\xi \) for \( \xi \in \text{C-sup}(r) \cup \{ \beta \} \), where if \( \xi \neq \beta \), \( a_\xi \in \{ 0, 1 \} \) is such that either \( p[\xi] (\xi > \beta) \) or \( r[\xi] (\text{if } \xi < \beta) \) decides \( A(\xi) = a_\xi \).

The following useful fact can easily be extracted from the proof of Claim 4.1:

**Fact 4.2.** If \( p \in P_{\alpha} \), \( \beta \in [\lambda, \alpha) \), \( q \in P_{\beta} \) and \( q \leq p \mid \beta \), then there is \( r \in P_{\alpha} \) stronger than both \( p \) and \( q \).

**Claim 4.3.** If \( \beta < \alpha \), then \( P_{\beta} \) is a complete subforcing of \( P_{\alpha} \).

**Proof.** Let \( X \) be a maximal antichain of \( P_{\beta} \) and let \( p \) be a condition in \( P_{\alpha} \). Then \( p \mid \beta \in P_{\beta} \) is compatible with some element of \( X \) as witnessed by \( q \in P_{\beta} \) which is stronger than both. By Fact 4.2, \( p \) and \( q \) are compatible in \( P_{\alpha} \).

**Notation:** Given a decreasing sequence of conditions \( \langle p^i \mid i < \delta \rangle \) in \( P_{\alpha} \), we say that \( r = \langle r(\beta) : \beta < \lambda \rangle \) is the componentwise union of \( \langle p^i \mid i < \delta \rangle \) if for every \( \beta < \lambda \),

\[
r(\beta) = \bigcup_{i<\delta} p^i(\beta),
\]

and for \( \beta \geq \lambda \), \( c_\beta^r = \bigcup_{i<\delta} c_\beta^{p^i} \) and \( f_\beta^r = \bigcup_{i<\delta} f_\beta^{p^i} \). \( r \) is usually not a condition in \( P_{\alpha} \) as the \( c_\beta^r \) are not necessarily closed. We let \( \text{C-sup}(r) \) denote \( \{ \beta : C_\beta^r \neq \emptyset \} = \bigcup_{i<\delta} \text{C-sup}(p^i) \).

**Claim 4.4.** \( P_{\alpha} \) is \( \lambda \)-closed.

**Proof.** Let \( \delta < \lambda \) be a limit ordinal and let \( \langle p^i \mid i < \delta \rangle \) be a decreasing sequence of conditions in \( P_{\alpha} \). Let \( r \) be their componentwise union. Let \( \gamma^\gamma = \text{sup}(r) \cap \lambda = \bigcup_{i<\delta} \gamma^\gamma \). Let \( \delta^\gamma = \bigcup_{i<\delta} \delta^\gamma \). If \( \delta^\gamma < \gamma^\gamma \), then \( r \) is a condition in \( P_{\alpha} \) and a lower bound for \( \langle p^i \mid i < \delta \rangle \). Thus assume that \( \delta^\gamma = \gamma^\gamma \) is a limit ordinal from now on. We want to form \( q \) out of \( r \) (by componentwise end-extension) such that \( q \leq p^i \) for every \( i < \delta \). We have to set \( C_\beta^q = C_\beta^r \cup \{ \delta^\gamma \} \) for every \( \beta \in \text{C-sup}(r) \).

As \( \gamma \), \( f_\beta^q \mid [\delta^\gamma] \supseteq \delta^\gamma \), we obtain that \( f_\beta^q[\delta^\gamma] = \bigcup_{i<\delta} f_\beta^{p^i}[\delta^\gamma] \supseteq \delta^\gamma \) for every \( \beta \in \text{C-sup}(r) \). Similarly, if \( \beta_0 < \beta_1 \) are both in \( \text{C-sup}(r) \), we obtain that \( f_\beta^q[\delta^\gamma] \) is an initial segment of \( f_{\beta_1}^q[\delta^\gamma] \) and \( \beta_0 \) and \( \beta_1 \) are both in \( \text{C-sup}(p^i) \) for some \( i < \delta \) and hence \( f_\beta^q[\delta^\gamma] \setminus \beta_0 \neq \emptyset \), implying that \( f_{\beta_1}^q[\delta^\gamma] \setminus \beta_0 \neq \emptyset \). This now allows us to choose \( q \) of \( \langle f_\beta^q[\delta^\gamma] \rangle = a_\beta \) for every \( \beta \in \text{C-sup}(r) \). Where \( a_\delta \) is such that some \( p^i \) decides \( A(\beta) = a_\beta \in \{ 0, 1 \} \), and moreover set \( c_\beta^q = c_\beta^r \cup \{ (\delta, \gamma^\gamma) \cup \{ (\gamma, 0) : \gamma \in [\delta, \text{sup}(r) + 1 \mid \beta \in \text{C-sup}(r) \} \} \) for every \( \beta \in \text{C-sup}(r) \), to obtain a condition \( q \) as desired.

If \( \delta < \lambda \) is a limit ordinal, \( \langle p^i \mid i < \delta \rangle \) is a decreasing sequence of conditions in \( P_{\alpha} \) and \( q \) is the lower bound of \( \langle p^i \mid i < \delta \rangle \) as obtained in the proof of Claim 4.4, then we write \( q = \bigcup_{i<\delta} p^i \).

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11 As our forcing is no standard iteration, it is not necessarily the case here that \( r \in P_{\alpha} \), as for example possibly \( \gamma^\gamma \beta > \gamma^\gamma \).
**Definition 4.5.** If $Q$ is a poset, $D \subseteq Q$ is directed if any two elements of $D$ have a lower bound in $D$. We say that $Q$ is $<\lambda$-directed closed if for any directed $D \subseteq Q$ of size less than $\lambda$, there is a condition in $Q$ below all elements of $D$.

**Corollary 4.6.** The proof of Claim 4.4 in fact shows that $P_{<\alpha}$ is $<\lambda$-directed closed.

**Claim 4.7.** $P_{<\alpha}$ is $\kappa$-cc.

**Proof.** Assume for a contradiction that $X$ is an antichain of $P_{<\alpha}$ of size $\kappa$. By a $\Delta$-System argument using $2^{<\lambda} = \lambda$, there is $r$ of size $<\lambda$ and $Y \subseteq X$ of size $\kappa$ such that for any $p_0, p_1$ in $Y$, $C\text{-supp}(p_0) \cap C\text{-supp}(p_1) = r$. $2^{<\lambda} = \lambda$ now implies that $\kappa$-many conditions in $Y$ are compatible in $P_{<\alpha}$, contradicting our assumption. \hfill \Box

**Claim 4.8.** For every $\alpha < \kappa$, $P_{<\alpha}$ has size less than $\kappa$. $P \subseteq H_\kappa$ has size $\kappa$, forces that $\kappa = \lambda^+$ and $2^\lambda = \kappa$ and preserves $\lambda$ and both cardinals and the continuum function below $\lambda$ and at and above $\kappa$.

**Proof.** The first and second statements follow since $\theta^{<\kappa} < \kappa$ for every $\theta < \kappa$. If $G$ is $P$-generic over $V$ and $f^G_\alpha$ denotes $\bigcup_{p \in G} f^p_\alpha$, then $f^G_\alpha$ is a bijection from $\lambda$ to $\alpha$ for every $\alpha \in [\lambda, \kappa)$ by an easy density argument. That $P$ forces $2^\lambda = \kappa$ follows from the first statement of the claim and Claim 4.7. The rest of the claim is immediate by Claim 4.4, Claim 4.7 and the fact that our assumptions imply that $\kappa^{<\kappa} = \kappa$. \hfill \Box

We will use the following easy fact, a proof of which may be found in [3].

**Fact 4.9.** Assume $\beta$ has regular cardinality $\gamma$ and for every $\gamma \leq \beta$, $f_\gamma$ is a bijection from card $\gamma$ to $\gamma$. Then there is a club of $\delta < \nu$ such that $f_\alpha[\delta] = f_\beta[\delta] \cap \alpha$ for all $\alpha \in f_\beta[\delta] \setminus \nu$.

**Claim 4.10.** $P$ forces Local Club Condensation at $\kappa = \lambda^+$.

**Proof.** Let $G$ be $P$-generic. Let $B$ be the generic predicate obtained by letting $B[\lambda] = A_{<\lambda}$, where $A_{<\lambda}$ denotes the generic Cohen subset of $\lambda$ added by $G$ restricted to $P_{<\lambda}$, and for $\alpha \in [\lambda, \kappa)$, $B(\alpha) = a_\alpha$, where $a_\alpha \in \{0, 1\}$ is such that some $p \in G$ decides $A(\alpha) = a_\alpha$. Note that $H_\kappa^{|G|} = L_\kappa[B]$, as Claim 4.8 and Claim 4.7 show that every subset of $\lambda$ in $V[G]$ has a $P_{<\alpha}$-nice name in $H_\kappa$ for some $\alpha < \kappa$. We claim that $\langle M_\alpha \mid \alpha < \kappa \rangle$ witnesses Local Club Condensation at $\kappa$ in $V[G]$ with $M_\alpha = L_\alpha[B]$.

Assume $\alpha \in [\lambda, \kappa)$ and let $A_\alpha = \langle M_\alpha, \mathbb{E}, M_\beta \mid \beta < \alpha, \ldots \rangle$ be a structure for a countable language. We may assume that $A_\alpha$ is Skolemized. Note that for $\beta \in \alpha \setminus \lambda$ we have $B(\beta) = B(\otimes f^\beta[\delta])$ for all $\delta$ in the club $\bigcup_{p \in G} c^p_\beta \subseteq \lambda$. It follows that for a club $C$ of $\delta < \lambda$, $B(\beta) = B(\otimes f^\beta[\delta])$ and moreover $f^\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \lambda$: this is seen using Fact 4.9. For any $X \subseteq \alpha$ let $A_\alpha(X)$ be the least substructure of $A_\alpha$ containing $X$ as a subset of its domain $A_\alpha(X)$. Consider the continuous chain $\langle A_\alpha(f_\alpha[\delta]) \mid \delta \in D \rangle$, where $D$ consists of all elements $\delta$ of $C$ such that $\delta = f_\alpha[\delta] \cap \lambda$ and $f_\alpha[\delta] = A_\alpha(f_\alpha[\delta]) \cap \text{Ord}$. Then $A_\alpha(f_\alpha[\delta])$ condenses for each $\delta \in D$. \hfill \Box

**Note:** If $\kappa = \lambda^+$ and $\lambda$ is regular and uncountable, the above provides a cofinality-preserving forcing to obtain Local Club Condensation at $\kappa$, generalizing [14] and in the case of $\kappa = \omega_2$ providing a strongly simplified version of the proofs given in [14], [3], and [5]. Note moreover that if $\kappa \geq \omega_3$, we may perform the above forcing
over a model of $2^{\aleph_0} = \aleph_2$, to obtain a model of Local Club Condensation at $\kappa$ in which CH fails. This contrasts the situation with both Strong Condensation for $\kappa$ and Local Club Condensation up to $\kappa$, as they both imply CH to hold.

§5. Local Club Condensation and the negation of Square. In [14], the third author obtained Strong Condensation for $\omega_2$ (which is implied by Local Club Condensation at $\omega_2$) and $-\square_{\omega_1}$ starting from a stationary limit of measurable cardinals. In this section, building on the methods introduced in [14] and on the forcing construction of Section 4, we generalize his result to cardinals larger than $\omega_2$ and also reduce the large cardinal hypothesis to a 2-Mahlo cardinal. For convenience, we assume GCH throughout.

**Definition 5.1.** If $\kappa$ is regular and greater than $\omega_1$, $\square(\kappa)$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that the following hold:

1. Whenever $\alpha$ is a limit ordinal, $C_\alpha$ is a closed unbounded subset of $\alpha$.
2. If $\beta$ is a limit point of $C_\alpha$ then $C_\beta = C_\alpha \cap \beta$.
3. There is no club $C \subseteq \kappa$ such that for every limit point $\alpha$ of $C$, $C_\alpha = C \cap \alpha$.

**Definition 5.2.** If $\kappa = \lambda^+ > \omega_1$, $\square_\lambda$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that (1) and (2) from above hold together with the following:

(3*) For every $\alpha$, $\text{ot}(C_\alpha) \leq \lambda$.

**Lemma 5.3.** If $\eta$ is Mahlo, $\theta \geq \eta$ is regular, $\lambda < \eta$ is regular and $\mathcal{A}$ is a structure for a countable language with domain $H_\theta$, then there is a pair of models $M_0^\mathcal{A}$ and $M_1^\mathcal{A}$ such that

1. $M_0^\mathcal{A}$ and $M_1^\mathcal{A}$ are both substructures of $\mathcal{A}$.
2. $M_0^\mathcal{A}$ and $M_1^\mathcal{A}$ both have size $\lambda$.
3. $\lambda \subseteq M_0^\mathcal{A} \cap M_1^\mathcal{A}$.
4. Let $\delta = \sup(\eta \cap M_0^\mathcal{A} \cap M_1^\mathcal{A})$. Then $P_{\omega_1}(M_0^\mathcal{A} \cap V_\delta) \subseteq M_1^\mathcal{A}$.
5. $\min(M_0^\mathcal{A} \setminus \delta)$ has cofinality $\geq \lambda$.
6. $\delta < \sup(M_0^\mathcal{A} \cap \eta) = \sup(M_1^\mathcal{A} \cap \eta)$ and the latter have cofinality $\omega$.

**Proof.** Large parts of this proof are based on the proof of [12, Lemma 3.5] and much of the adaptation below was essentially suggested to us by Boban Velčković.

Let $A = H_\theta$ denote the domain of $\mathcal{A}$ and let $F : [A]^{<\omega} \to A$ be such that $X \prec A$ whenever $X$ is closed under $F$. Consider the following two player game with perfect information of length $\omega$. I starts by playing $\rho \in [\lambda, \eta]$ with $\text{cof}(\rho) > \omega$. Then II plays $\delta_0 \in (\rho, \eta)$ and $A_0 \in [V_\rho]^\lambda$. If II has played $\delta_i$ and $A_i$ for some $i < \omega$, I responds by playing $\alpha_i$ and $\beta_i$ such that $\delta_i < \alpha_i < \beta_i < \eta$. Moreover I has to choose $\alpha_0$ such that $\text{cof}(\alpha_0) \geq \lambda$. If I has played $\alpha_i$ and $\beta_i$ for some $i < \omega$, II responds by playing $\delta_{i+1}$ and $A_{i+1}$ such that $\beta_i < \delta_{i+1} < \eta$ and $A_i \in [V_\rho]^\lambda$.

| I | $\rho$ | $\alpha_0, \beta_0$ | $\alpha_1, \beta_1$ | ... |
|---|---|---|---|---|
| II | $\delta_0, A_0$ | $\delta_1, A_1$ |

Let $X$ be the closure under $F$ of $\lambda \cup \bigcup_{i<\omega} A_i \cup \{ \alpha_i \mid i < \omega \}$. I wins the game if $\text{Ord}(X) \subseteq \rho \cup \bigcup_{i<\omega}[\alpha_i, \beta_i]$. 
If II wins, he knows so by a finite stage, i.e., the above is an open game for II. It follows by [6] that the game is determined, i.e., one of the players has a winning strategy. We will now show that if I has a winning strategy, then we can construct $M_0^*$ and $M_i^*$ as desired, and then show that II does not have a winning strategy.

Thus assume now that I has a winning strategy in the above game. We will play the game on two boards simultaneously, denoting moves on the first board as above and adding a * for moves on the second board in our notation. On the first and second board, let I start by playing $\rho = \rho^*$ according to her strategy. On the first board, let II respond with $\delta_0 = \rho + 1$ and $A_0 = \emptyset$. Assume $\delta_i$ and $A_i$ are played on the first board. Let I respond with $\alpha_i$ and $\beta_i$ according to her strategy. On the second board, let II respond with $\delta_i^* = \beta_i$ and $A_i^* = [V_{\rho} \cap \text{cl}_F(\lambda \cup \{\alpha_i \mid j \leq i\})]^\omega_\omega$ and let I respond with $\alpha_i^*$ and $\beta_i^*$ according to her strategy. Now on the first board let II respond with $\delta_{i+1} = \beta_i^*$ and $A_{i+1} = \emptyset$.

After playing as above for $\omega$-many stages, let $M_0^* = \text{cl}_F(\lambda \cup \{\alpha_i \mid i < \omega\})$ and let $M_i^* = \text{cl}_F(\lambda \cup \bigcup_{i<\omega} A_i \cup \{\alpha_i^* \mid i < \omega\})$. We claim that $M_0^*$ and $M_i^*$ are as desired. (1), (2), and (3) are obvious. If $\bar{\delta} = \sup(\eta \cap M_0^* \cap M_i^*)$, it follows that $\bar{\delta} \leq \rho$, thus $P_\omega(M_0^* \cap V_{\bar{\delta}}) \subseteq M_i^*$ by our choice of the $A_i^*$, i.e., (4) holds. (5) and (6) are again obvious from our construction.

We will thus finish the proof of Lemma 5.3 by the following:

**Claim 5.4.** II does not have a winning strategy in the above game.

**Proof.** Assume for a contradiction that II has a winning strategy in the above game. Let $\theta^* > \theta$ be sufficiently large and regular. Let $\mathcal{H}$ be a structure for a finite language on $H_{\theta^*}$ that contains everything relevant, in particular $A$ and the winning strategy for II, as a constant, together with a wellorder of $H_{\theta^*}$. Let $\{M_i \mid i < \eta\}$ be a continuous increasing $\in$-chain of elementary substructures of $\mathcal{H}$ which are transitive below $\eta$, such that there is $\delta < \eta$ which is inaccessible and $M_\delta \cap \eta = \delta$, using that $\eta$ is Mahlo. For $i < \eta$, let $\xi_i = M_i \cap \eta$ and note that $\xi_\delta = \bar{\delta}$. Choose a singular strong limit cardinal $\rho < \delta$ such that if $N = H^\mathcal{H}(V_{\rho} \cup \{\xi_i \mid 1 \leq i < \omega\})$, we get $N \cap \delta = \rho$. Since $|V_{\rho}| = \rho$, $N$ is bounded in $\xi_i$ for each $i \geq 1$. We claim that I wins by playing (independent of II’s moves) $\rho$, $\alpha_i = \xi_{\delta \cdot (i+1)}$ and $\beta_i = \sup(N \cap \xi_{\delta \cdot (i+2)})$. This is because of elementarity of $M_i$ for $i < \eta$. II is forced to play $\delta_i \in M_{\delta \cdot (i+1)}$ and thus $\delta_i < \alpha_i$. We obtain

$$\text{cl}_F(\lambda \cup \bigcup_{i<\omega} A_i \cup \{\alpha_i \mid i < \omega\}) \subseteq H^\mathcal{H}(V_{\rho} \cup \{\alpha_i \mid i < \omega\}).$$

But now I has won the run of the game, for the intersection of the latter set with the ordinals was arranged to be contained in $\rho \cup \bigcup_{i<\omega}[\alpha_i, \beta_i)$. This gives the desired contradiction.

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**Theorem 5.5.** Given $\kappa$ 2-Mahlo and $\lambda < \kappa$ regular, $P(\lambda, \kappa)$ forces Local Club Condensation at $\kappa = \lambda^+$ and $\neg \square_\lambda$.

**Proof.** Let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa \rangle$ is a $\square_\lambda$-sequence in a $P$-generic extension. As $q$ plays no role in the proof, we assume that $q = 1$.

If $\eta < \kappa$, $\dot{C}_\eta$ is a name for an object that consists of $<\kappa$-many ordinals and thus using the chain condition of the forcing, there is $\xi < \kappa$ such that $\dot{C}|\eta$ is in fact a
$P_{< \eta}$-name. For a club of $\eta < \kappa$ we thus have a $P_{< \eta}$-name $\dot{C} | \eta$. By the large cardinal properties of $\kappa$, we may choose such an $\eta < \kappa$ that is Mahlo. Assume some condition forces that $\dot{C}_\eta$ has a $P_{< \eta}$-name. Then $\dot{C}_\eta$ has order-type $\eta > \lambda$, as $\eta$ is regular in any $P_{< \eta}$-generic extension, contradicting that $\dot{C}$ is a name for a $\square_\eta$-sequence. Thus $\dot{C}_\eta$ doesn’t have a $P_{< \eta}$-name.

By the above, there are $t_0 \perp t_1$ in $P$ with $t_0 | \eta = t_1 | \eta$ and some $\xi < \eta$ such that $t_0$ and $t_1$ disagree about whether $\xi \in \dot{C}_\eta$. Let $M_0^\ast$ and $M_1^\ast$ be elementary substructures of $(H_\theta, \in, \eta, \lambda, \xi, t_0, t_1, \dot{C}_\eta, \ldots)$ for some large, regular $\theta$ as provided by Lemma 5.3. Let $\delta$ denote $\sup(M_0^\ast \cap \eta) = \sup(M_1^\ast \cap \eta)$, let $\bar{\delta} = \sup(M_0^\ast \cap M_1^\ast \cap \eta)$. Let $M_0 \prec M_0^\ast$ be countable with $\sup(M_0 \cap \eta) = \delta$ and $t_0 \in M_0$, let $s_0 \subseteq t_0$ be $(M_0, P)$-complete, so that $s_0 = \bigcup_{i < \omega} p_i$ for some decreasing sequence of conditions $\langle p_i | i < \omega \rangle \subseteq M_0$ with $p_0 = t_0$ and such that whenever $D \in M_0$ is a dense subset of $P$ there is $i < \omega$ such that $p_i \in D$.

**Claim 5.6.** $s_0[\bar{\delta}] \in M_1^\ast$.

**Proof.** $s_0[\bar{\delta}] \in H_\delta$ and is thus an element of $M_1^\ast$ by Clause 3 of Lemma 5.3. For every $i < \omega$, $p_i[\langle \bar{\delta}, \bar{\delta} \rangle] = p_i[\langle \bar{\delta}, \min(M_0^\ast \setminus \bar{\delta}) \rangle] \in M_0^\ast$; the equation holds since $\sup(p_i) \subseteq M_0^\ast$. But by Clause 5 of Lemma 5.3, $p_i[\langle \bar{\delta}, \bar{\delta} \rangle] \in V_{\delta_1}$ holds as well, for $M_0^\ast$ thinks that $p_i[\langle \bar{\delta}, \bar{\delta} \rangle] \in V_{\min(M_0^\ast \setminus \bar{\delta})}$. By Clause 4 of Lemma 5.3, $\langle p_i[\langle \bar{\delta}, \bar{\delta} \rangle] | i < \omega \rangle \in M_1^\ast$ and thus so is $s_0[\langle \bar{\delta}, \bar{\delta} \rangle]$ for it is easily definable from that sequence.

Let $M_1$ be a countable elementary submodel of $M_1^\ast$ such that $s_0[\bar{\delta}] \in M_1$ and $\sup(M_1 \cap \eta) = \delta$. Note that by Fact 4.2, $s_0[\bar{\delta}]$ and $t_1$ are compatible. Let $s_1$ be stronger than both and $(M_1, P)$-complete. By the properties of $M_0$ and $M_1$, both $s_0$ and $s_1$ force that $\delta \in \text{Lim}(\dot{C}_\eta)$. Thus both $s_0$ and $s_1$ force that $\xi \in \dot{C}_\eta \iff \xi \in \dot{C}_\delta$.

**Claim 5.7.** $s_0[\eta]$ and $s_1[\eta]$ are compatible.

**Proof.** $s_1[\bar{\delta}] \leq s_0[\bar{\delta}]$ and $\sup(s_0 \cap \bar{\delta}) \subseteq M_0^\ast$ and $\sup(s_1 \cap \bar{\delta}, \eta) \subseteq M_1^\ast$, and hence these supports are disjoint by the disjointness properties of $M_0^\ast$ and $M_1^\ast$. \(\neg\) Theorem 5.5

We can now strengthen $s_0[\eta]$ and $s_1[\eta]$ to conditions agreeing about whether $\xi \in \dot{C}_\delta$, which clearly gives a contradiction to our choice of $t_0$ and $t_1$.

**Note:** One could replace Local Club Condensation at $\lambda^+$ by Local Club Condensation up to $\lambda^+$ (or, with a little more work, by Local Club Condensation) in the statement of Theorem 5.5. However this would require providing a $\sigma$-closed forcing construction to obtain this Condensation principle (and, in the case of Local Club Condensation, working around the $\kappa$-cc by standard reduction arguments). This is straightforward to do (but with a significant increase in complexity of notation) by combining the construction from Section 4 with some of the ideas from [5]. To keep things more easily readable and because the most obvious interaction between $\square_\eta$ and Local Club Condensation (and thus the most interesting aspect of their independence) should naturally occur within the interval $[\lambda, \kappa]$, we decided not to present such a construction.

§6. Variations of Square. We first improve Theorem 5.5 by showing that in fact a whole hierarchy of weaker square principles is forced to fail by $P(\lambda, \kappa)$. We again assume GCH throughout.
**Definition 6.1.** $\square_\lambda^n$ is the statement that there exists a sequence $\langle C_\alpha : \alpha < \lambda^+ \rangle$ such that the following hold:

1. Each $C_\alpha$ has at most $n$-many elements.
2. Whenever $\alpha$ is a limit ordinal, elements of $C_\alpha$ are closed unbounded subsets of $\alpha$ of order-type at most $\lambda$.
3. If $\beta$ is a limit point of $X \in C_\alpha$ then there is $Y \in C_\beta$ such that $Y = X \cap \beta$.

**Theorem 6.2.** Given $\kappa$ 2-Mahlo, $\lambda < \kappa$ regular and $2 \leq n < \omega$, $P(\lambda, \kappa)$ forces Local Club Condensation at $\kappa = \lambda^+$ and $\neg \square_\lambda^n$.

We need the following minor generalization of Lemma 5.3.

**Lemma 6.3.** If $\eta$ is Mahlo and $\theta \geq \eta$ is regular, $c \in H_\theta$ and $\lambda < \eta$ is regular, then there is a sequence of models $\langle M^*_i \mid i < \omega \rangle$ such that

1. $M^*_i < (H_\theta, e, c, \eta, \lambda)$ for every $i < \omega$.
2. Each $M^*_i$ has size $\lambda$.
3. $\lambda \subseteq M^*_i$ for every $i < \omega$.
4. Let $\bar{\delta}_i = \sup(\eta \cap M^*_i \cap M^*_{i+1})$. Then $P_{\omega n}(M^*_i \cap V_\theta) \subseteq M^*_{i+1}$ for every $i < \omega$.
5. $\langle \bar{\delta}_i \mid i < \omega \rangle$ is increasing.
6. $\min(M^*_i \setminus \bar{\delta}_i)$ has cofinality $\geq \lambda$.
7. $\delta_i < \sup(M^*_i \cap \eta) = \sup(M^*_j \cap \eta)$ and the latter have cofinality $\omega$ for any $i, j < \omega$.

**Proof.** We consider the game described in Lemma 5.3, for which player I has a winning strategy. Using this, we can simultaneously play on $\omega$-many boards to produce the desired models, in a similar way as we produced two models in the proof of Lemma 5.3: Use a disjoint partition $\langle S_i \mid i < \omega \rangle$ of $\omega$ into infinite sets and at stage $j$ play on board $i$ if $j \in S_i$. We leave the (easy) details to the reader.

**Proof of Theorem 6.2.** Fix $n < \omega$ and let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\hat{C} = \langle \hat{C}_\eta \mid \eta < \kappa \rangle$ is a $\square_\lambda^n$-sequence in a $P$-generic extension. As $q$ plays no role in the proof, we assume that $q = 1$.

If $\eta < \kappa$, $\hat{C} \upharpoonright \eta$ is a name for an object that consists of $<\kappa$-many ordinals and thus using the club condition of the forcing, there is $\xi < \kappa$ such that $\hat{C} \upharpoonright \eta$ is in fact a $P_{\eta\kappa}$-name. For a club of $\eta < \kappa$ we thus have a $P_{\eta\kappa}$-name $\hat{C} \upharpoonright \eta$. By the large cardinal properties of $\kappa$, we may choose such an $\eta < \kappa$ that is Mahlo. Choose $X$ to be a name for an element of $\hat{C}_\eta$. Assume some condition forces that $\hat{X}$ has a $P_{\eta\kappa}$-name. Then $\hat{X}$ has order-type $\eta > \lambda$, as $\eta$ is regular in any $P_{\eta\kappa}$-generic extension, contradicting that $\hat{C}$ is a name for a $\square_\lambda^n$-sequence. Thus $\hat{X}$ doesn’t have a $P_{\eta\kappa}$-name.

By the above, there are countably many incompatible conditions $t_i$ in $P$ with equal restrictions to $\eta$ and some $\xi_i < \eta$ for $i < \omega$ such that for every $i < \omega$, $t_i$ and $t_{i+1}$ agree about whether $\xi_j \in \hat{X}$ for $j < i$ but disagree about whether $\xi_i \in \hat{X}$. Let $\langle M^*_i \mid i < \omega \rangle$ be an $\omega$-sequence of elementary substructures of $(H_\theta, e, \eta, \lambda, \langle \xi_i \mid i < \omega \rangle, \langle t_i \mid i < \omega \rangle, \hat{X}, \ldots)$ for some large, regular $\theta$ as provided by Lemma 6.3. Let $\bar{\delta}$ denote $\sup(M^*_i \cap \eta) = \sup(M^*_i \cap \eta)$ for any $i < \omega$, let $\delta_i = \sup(M^*_i \cap M^*_{i+1} \cap \eta)$.

Let $M_0 < M^*_0$ be countable with $\sup(M_0 \cap \eta) = \delta$ and $t_0 \in M_0$, let $s_0 < t_0$ be $(M_0, P)$-complete. Exactly as in the proof of Claim 5.6, we can now show that $s_0|\bar{\delta}_0 \in M^*_1$. Now given $s_i$, let $M_{i+1}$ be a countable elementary submodel of $M^*_{i+1}$.
such that $s_i | \delta_i \in M_{i+1}$ and $\sup(M_{i+1} \cap \eta) = \delta$. Note that $s_i | \delta_i$ and $t_{i+1}$ are compatible and let $s_{i+1}$ be stronger than both and $(M_{i+1}, P)$-complete. Using that $s_i | \delta_i \in M_{i+1}$ analogous to above, we may perform this construction for every $i < \omega$.

By the properties of the $M_i$, each $s_i$ forces that $\delta \in \check{X}$. Since $\check{C_\delta}$ has size at most $n$, we can pick a $P_\eta$-name $\check{Y}$ for an element of $\check{C_\delta}$ and indices $i < j$ such that both $s_i$ and $s_j$ force that $\check{X} \cap \delta = \check{Y}$. Now both $s_i$ and $s_j$ force that $\check{\xi_i} \in \check{X} \iff \check{\xi_j} \in \check{Y}$.

CLAIM 6.4. $s_i | \eta$ and $s_j | \eta$ are compatible.

Proof. $s_j | \delta_i \leq s_i | \delta_i$ (this uses Clause 5 of Lemma 6.3). $\sup(s_i) \cap [\delta_i, \eta) \subseteq M_i^*$ and $\sup(s_j) \cap [\delta_i, \eta) \subseteq M_j^*$ - hence those supports are disjoint by the disjointness properties of $M_i^*$ and $M_j^*$ (Clause 6 of Lemma 6.3).

We can now strengthen $s_i | \eta$ and $s_j | \eta$ to conditions agreeing about whether $\check{\xi_i} \in \check{Y}$ and hence about whether $\check{\xi_j} \in \check{X}$, which clearly gives a contradiction to our choice of $t_i$ and $t_j$.

Now we consider $\square(\kappa)$. Note that our consistency strength assumption in Theorem 6.5 is optimal, essentially because weakly compact cardinals are 2-Mahlo.

THEOREM 6.5. Given $\kappa$ which is weakly compact and $\lambda < \kappa$ regular, $P(\lambda, \kappa)$ forces Local Club Condensation at $\lambda^+$ and $\neg \square(\lambda^+)$. 

Proof. Let $P = P(\lambda, \kappa)$. Assume for a contradiction that $q \in P$ forces that $\check{C} = \langle \check{C_\eta} | \eta < \kappa \rangle$ is a $\square(\kappa)$-sequence in a $P$-generic extension. As $q$ plays no role in the proof, we assume that $q = 1$. Using the $\kappa$-cc of $P$, we may assume that $\check{C} \subseteq V_\kappa$.

$$(V_\kappa, \in, P, \Vdash_P, \check{C}) \models \Vdash_P \check{C} \text{ is a } \square(\kappa)-\text{sequence}.$$  

As $\kappa$ being Mahlo is a $\Pi^1_1$-property of $V_\kappa$, we may invoke $\Pi^1_1$-indescribability of $\kappa$ to find a Mahlo cardinal $\eta < \kappa$ such that 

$$P_{\eta \kappa} \models \check{C} | \eta \text{ is a } \square(\eta)-\text{sequence}.$$  

It follows that $\check{C_\eta}$ cannot have a $P_{\eta \kappa}$-name, as this would contradict that $\check{C} | \eta$ is forced to be a $\square(\eta)$-sequence in any $P_{\eta \kappa}$-generic extension. But this now allows us to finish exactly as in the proof of Theorem 5.5.

Note: Similar to the remark at the end of the previous section, one could replace Local Club Condensation at $\lambda^+$ by Local Club Condensation up to $\lambda^+$ in the statement of Theorem 6.5. However, we do not know whether it could also be replaced by Local Club Condensation in this case, for the proof of Theorem 6.5 heavily uses that $P \subseteq V_\kappa$. Moreover just like the proof for $\neg \square_2$ was improved to $\neg \square_n$ for $n < \omega$, one could improve the above from $\neg \square(\lambda^+)$ to $\neg \square(\lambda^+, n)$ for $n < \omega$, where the latter is defined correspondingly.

§7. Condensation and Jónsson cardinals. In [3], it was shown that Strong Condensation refutes the existence of an $\omega_1$-Erdős cardinal. We slightly improve this result by showing that it refutes the existence of an $\omega_1$-Jónsson cardinal.

DEFINITION 7.1. If $\omega < \delta < \kappa$ are cardinals, $\kappa$ is $\delta$-Jónsson if for every first order structure $A$ for a countable language with universe $\kappa$, there is $A' \prec A$ with universe $A'$ such that $\ot(A') = \delta$.


\( \kappa \) is \( \kappa \)-Jónsson or Jónsson if every structure \( A \) as above has a proper substructure \( A' \) with universe \( A' \) of size \( \kappa \).

It is easy to see ([10]) that every \( \delta \)-Erdős cardinal is \( \delta \)-Jónsson. Whether the reverse implication holds is not known. The proof of the next result closely follows a proof by Keisler and Rowbottom (announced in [8]) which shows that if there is a Jónsson cardinal, then \( V \neq L \). Their proof can be found in [7].

**Theorem 7.2.** Assume \( \kappa \) is an uncountable cardinal. Strong Condensation for \( \kappa \) implies that \( \kappa \) is not Jónsson and that for no \( \delta < \kappa \), \( \kappa \) is \( \delta \)-Jónsson.

**Proof.** First assume for a contradiction that Strong Condensation for \( \kappa \) holds and \( \kappa \) is Jónsson. Let \( A = (M_\kappa, \in, \vec{M}, \ldots) \) be a structure for a countable language witnessing Strong Condensation for \( \kappa \). As \( \kappa \) is Jónsson, \( A \) has a proper substructure \( B_0 \) of cardinality \( \kappa \). By our choice of \( A \), \( (B_0, \in, \vec{M}) \) is isomorphic to \( (M_\kappa, \in, \vec{M}) \); let \( \pi \) be the inverse of the collapsing isomorphism of \( (B_0, \in) \), let \( \delta > \omega \) be the critical point of \( \pi \). Define \( U \) by

\[
X \in U \iff X \subseteq \delta \land \delta \in \pi(X).
\]

Since \( M_\kappa = H_\kappa \), \( U \) is easily seen to be a \( \delta \)-complete ultrafilter and hence \( \delta \) is a measurable cardinal, contradicting the above-mentioned result of [3], for measurable cardinals are in particular \( \omega_1 \)-Erdős.

Now assume Strong Condensation for \( \kappa \) holds (note that this implies the GCH below \( \kappa \)), \( \delta < \kappa \) and \( \kappa \) is \( \delta \)-Jónsson. Assume that \( A \) is as above, but also includes \( \delta \) as a constant. Let \( \rho \) be an isomorphism between \( \kappa \) and \( H_\kappa \) such that for every cardinal \( \lambda < \kappa \), \( \rho|\lambda \) is an isomorphism between \( \lambda \) and \( H_\lambda \) (this uses the GCH below \( \kappa \)). This means that \( A \) is isomorphic (via \( \rho^{-1} \)) to a structure \( B \) on \( \kappa \) that has a substructure \( B' \) with underlying set \( B' \) of order-type \( \delta \). \( \rho''B' \) induces a substructure \( A' \) of \( A \) with underlying set \( A' \). By Strong Condensation, \( A' \) condenses, say to \( A'' \supseteq H_\delta \).

Let \( \pi \) denote the elementary embedding from \( A'' \) to \( A \). If \( \text{crit}(\pi) = \theta < \delta \), then \( \mathcal{P}(\theta) \subseteq \text{dom} \pi \) and \( \theta \) is seen to be measurable, leading to a contradiction as above. If \( \text{crit} \pi \geq \delta \), this means that \( H_\delta \cup \{\pi(\delta)\} \subseteq A' \) and hence \( \delta \cup \{\rho^{-1}(\pi(\delta))\} \subseteq B' \) by our choice of \( \rho \). But this contradicts elementarity of \( \pi \), as \( \delta \notin H_\delta \).

In contrast to this, Local Club Condensation is clearly consistent with the existence of Jónsson cardinals, both inaccessible and accessible: Start with countably many measurable cardinals and force Local Club Condensation preserving those measurable by the techniques of [3] (preservation of measurables is a standard argument that is not carried out in that paper). In the extension, Local Club Condensation holds and the supremum of the measurable cardinals is accessible and Jónsson (see [11]). Preservation of a measurable while forcing Local Club Condensation clearly yields the consistency of Local Club Condensation with inaccessible Jónsson cardinals, as measurable cardinals are Jónsson.

**§8. Condensation and variants of Chang’s Conjecture.**

**Definition 8.1.** For infinite cardinals \( \alpha, \beta, \gamma, \delta \) with \( \alpha > \beta > \delta \) and \( \alpha \geq \gamma > \delta \),

\[
(\alpha, \beta) \rightarrow^* (\gamma, \delta)
\]
is the statement that for every countable language $L$ with a unary predicate $A \in L$ and every $L$-structure $M = (M, A^M, \ldots)$ with $card M = \alpha$ and $card A^M = \beta$, there exists $N = (N, A^N, \ldots)$ such that

1. $N$ is a substructure of $M$ and
2. $card N = \gamma$ and $card(A^N) = \delta$.

**Theorem 8.2.** *Strong Condensation for $\alpha$ refutes $(\alpha, \beta) \rightarrow (\gamma, \delta)$.*

**Proof.** Assume for a contradiction that Strong Condensation for $\alpha$ holds and $(\alpha, \beta) \rightarrow (\gamma, \delta)$. Thus $(M_\alpha, \in, \bar{M}, \beta)$ has an elementary substructure $N$ with $card N = \gamma$ and $card(N \cap \beta) = \delta$, which condenses to some $M_{\bar{\alpha}}$: let $\pi$ denote the collapsing map. By taking the Skolem Hull of $N \cup (\delta + 1)$ in that structure w.r.t. some wellordering of $M_\alpha$, we may as well assume that $\delta + 1 \subseteq N$. Hence $\pi(\delta) = \delta$. As $card(N \cap \beta) = \delta$, $\pi(\beta) < \delta^+$. Since, using Lemma 3.1, $H_{\bar{\alpha}} = M_{\delta^+} \subseteq M_{\bar{\alpha}}$, it follows that $M_{\bar{\alpha}} \models \pi(\beta)$ is not a cardinal, contradicting elementarity of $N$. \hfill $\square$

**Theorem 8.3.** *Local Club Condensation at $\kappa^{++}$ refutes $(\kappa^{++}, \kappa^+) \rightarrow (\kappa^+, \kappa)$ for any infinite cardinal $\kappa$.*

**Proof.** Let $F$ be as in Theorem 2.1. Assume for a contradiction Local Club Condensation at $\kappa^{++}$ holds and $(\kappa^{++}, \kappa^+) \rightarrow (\kappa^+, \kappa)$. Thus $(M_{\kappa^{++}}, \in, \bar{M}, \kappa^+, F)$ has an elementary substructure $N$ with $card N = \kappa^+$ and $card N \cap \kappa^+ = \kappa$. We may assume that $(\kappa + 1) \subseteq N$ (and hence $N$ is transitive below $\kappa^+$) as in the proof of Theorem 8.2 and thus by Theorem 2.1, $N$ condenses to some $M_\alpha \supseteq M_{\kappa^+} = H_{\kappa^+}$ (the final equality is mentioned in Section 3). Let $\kappa^+$ denote the $M_\alpha$-version of $\kappa^+$. $\bar{\kappa^+} \in (\kappa, \kappa^+)$, contradicting $M_\alpha \supseteq H_{\kappa^+}$. \hfill $\square$

Are any nontrivial instances of Chang’s Conjecture consistent with Local Club Condensation? Under sufficient large cardinal assumptions, the following fact answers this positively.

**Fact 8.4.** *Assume $\delta^+ < \kappa$ and $\kappa$ is $\delta^+$-Jónsson. Then $(\kappa, \delta^+) \rightarrow (\delta^+, \delta)$.***

**Proof.** Let $A$ be a Skolemized structure on $\kappa$ using $\delta^+$ as a predicate. Then by $\delta^+$-Jónssonness of $\kappa$, there is an $M \prec A$ of order-type $\delta^+, \delta^{++} \in M$ by elementarity and thus $M$ is bounded in $\delta^+$ (as otherwise $ot(M) > \delta^+$). But this means that $M \cap \delta^+$ has cardinality at most $\delta$. We may enlarge $M$ to contain $\delta$ as a subset (by taking the Skolem Hull of $M \cup \delta$ in $A$), so that $M \cap \delta^+$ has cardinality $\delta$. Since any structure of size $\kappa$ with a unary predicate $T$ of size $\delta^+$ has an extension which is isomorphic to some extension of $A$ and identifies $T$ and $\delta^+$, this shows that $(\kappa, \delta^+) \rightarrow (\delta^+, \delta)$. \hfill $\square$

We next present a positive result for small cardinals, strongly based on a proof of James Baumgartner in [1]. Let $CC(\kappa)$ denote the statement that for every $\lambda < \kappa$, $(\kappa, \lambda) \rightarrow (\omega_1, \omega)$. We will show that, assuming the existence of an $\omega_1$-Erdős cardinal $\kappa$, we may collapse $\kappa$ to become the successor of any regular cardinal $\lambda \geq \omega_2$ and obtain $CC(\kappa)$ together with Local Club Condensation at $\kappa$. By Theorem 8.3, this cannot work for $\lambda = \omega_1$.

**Definition 8.5.** A cardinal $\kappa$ is $\alpha$-Erdős if for any Skolemized structure $A$ for a countable language with universe $\kappa$ and for any closed unbounded $C \subseteq \kappa$, there is $I \subseteq C$ of order-type $\alpha$ such that $I$ is a set of indiscernibles for $A$ that is remarkable.
Lemma 8.7. [1, Theorem 5.2], which however refers to what are recalled Easton-like is

\[ \tau^A(\alpha_0, \ldots, \alpha_n) = \tau^A(\alpha_0, \ldots, \alpha_{i-1}, \beta_i, \ldots, \beta_n). \]

Definition 8.6. [1] Suppose \( A \) is a Skolemized structure for a countable language with universe \( \kappa \). For each \( k < \omega \) let \( f_k \) be given so that \( f_k : \kappa^n \to \kappa \) for some \( n < \omega \) that may depend on \( k \). We say \( I \subseteq \kappa \) is remarkable for \( A \) and the \( f_k \) if for any \( \alpha \in H^A(I) \) there is \( k < \omega \) and increasing \( \alpha_0, \ldots, \alpha_{n-1} \) from \( I \) so that \( \alpha = f_k(\alpha_0, \ldots, \alpha_{n-1}) \) and if \( \alpha_{i-1} \leq \alpha < \alpha_i \) then for any increasing \( \beta_i, \ldots, \beta_{n-1} \) from \( I \) with \( \alpha_{i-1} < \beta_i \) we have \( \alpha = f_k(\alpha_0, \ldots, \alpha_{i-1}, \beta_i, \ldots, \beta_{n-1}) \) and \( \alpha < \beta_i \).

We say that a cardinal \( \kappa \) is \( \alpha \)-remarkable if for any \( A \) as above there exist \( f_k \) as above so that for any closed unbounded set \( C \subseteq \kappa \) there is \( I \subseteq C \) of order-type \( \alpha \) such that \( I \) contains none of its limit points and is remarkable for \( A \) and the \( f_k \).

Lemma 8.7 ([1, Proposition 5.1]). If \( \kappa \) is \( \omega_1 \)-remarkable, then \( CC(\kappa) \) holds.

Definition 8.8. We say that \( \langle P_{<\alpha} \mid \alpha \leq \kappa \rangle \) is an Easton bounded iteration of length \( \kappa \) if whenever \( \alpha_0 < \alpha_1 < \kappa \), \( P_{<\alpha_0} \) is a complete subforcing of \( P_{<\alpha_1} \), and \( P_{<\alpha} \) is the direct limit of \( \langle P_{<\beta} \mid \beta < \alpha \rangle \) if \( \alpha \) is inaccessible.

The proof of Theorem 8.9 is almost an exact copy of one of the two cases of [1, Theorem 5.2], which however refers to what are called Easton-like partial orderings (those are particular kinds of products with Easton support). All that we do below is to essentially observe that the same proof can be carried out for Easton bounded iterations of length \( \kappa \). For the benefit of the reader, we provide the basic framework of the proof (and omit the proofs of several auxiliary lemmas, which can all be found in [1]).

Theorem 8.9. Let \( \kappa \) be \( \omega_1 \)-Erdős and let \( \langle P_{<\alpha} \mid \alpha \leq \kappa \rangle \) be an Easton bounded iteration of length \( \kappa \) such that \( P_{<\alpha} \) has size \( < \kappa \) for \( \alpha < \kappa \), and \( P = P_{<\kappa} \) is \( <\omega_2 \)-directed closed. Then \( \Vdash_P \kappa \) is \( \omega_1 \)-remarkable and hence \( CC(\kappa) \) holds by Lemma 8.7.

Proof. We may assume that \( P_{<\alpha} \in V_\kappa \) for \( \alpha < \kappa \), and so \( P \subseteq V_\kappa \). Let \( \hat{A} \) be a term for a Skolemized structure with universe \( \kappa \). Let

\[ \mathcal{B} = (V_\kappa, \in, <, P, \Vdash_{\phi}), \]

where \( < \) is a well-ordering of \( V_\kappa \), \( \phi \) ranges over all formulas of the language of \( \hat{A} \) and \( \Vdash_{\phi} \) is the relation

\[ \{ (p, \alpha_0, \ldots, \alpha_{n-1}) \mid p \Vdash \hat{A} \models \phi(\alpha_0, \ldots, \alpha_{n-1}) \}. \]

Let \( \{ f_k \mid k < \omega \} \) enumerate all functions of the form \( f : \kappa^n \to \kappa \) definable over \( \mathcal{B} \). By the \( \kappa \)-cc of \( P \), every club subset of \( \kappa \) in \( V^P \) contains a club subset of \( \kappa \) in \( V \). Let \( C \subseteq \kappa \) be club. Let \( I \subseteq C \) be a cofinal remarkable set of indiscernibles for \( \mathcal{B} \) with order-type \( \omega_1 \) (we say that \( I \) is cofinal if whenever \( \alpha_0, \ldots, \alpha_n \) is an increasing sequence of elements of \( I \) and \( \tau \) is a term in the language of \( \mathcal{B} \), then \( \tau^\mathcal{B}(\alpha_0, \ldots, \alpha_{n-1}) < \alpha_n \); this property can be ensured by shrinking \( C \) before applying the large cardinal properties of \( \kappa \) to obtain \( I \)). By standard arguments we may assume that \( I \) consists only of Mahlo cardinals and \( \forall \alpha < \beta \, P_\alpha \in V_\beta \) for every
\[ \beta \in I. \] In particular, \( I \) contains none of its limit points. We will show that in \( V^P \), \( I \) contains a set of order-type \( \omega_1 \) remarkable for \( \mathcal{A} \) and the \( f_k \).

For \( X \subseteq I \), we say \( G \) is \( P \)-generic over \( H^B(X) \) if \( G \subseteq P \cap H^B(X) \) is a filter meeting every dense subset of \( P \) lying in \( H^B(X) \), the Skolem Hull of \( X \) in \( B \).

The following lemmas can be proven exactly as in [1] by using our requirements on \( P \), so we will just provide their statements and refer to the corresponding lemmas in [1].

**Lemma 8.10 ([1, Lemma 5.3]).** Let \( G \) be \( P \)-generic over \( V \). In \( V[G] \) suppose \( J \subseteq I \) is uncountable and \( G \cap H^B(J) \) is \( P \)-generic over \( H^B(J) \). Then \( J \) is remarkable for \( \mathcal{A} \) and the \( f_k \).

We thus want to find \( J \subseteq I \) as in the hypothesis of the above lemma. We work in \( V \). Let \( F = \{ X \subseteq I \mid X \) is countable and has no last element \}. For \( X, Y \in F \) put \( X < Y \) if \( X \) is a proper initial segment of \( Y \).

**Lemma 8.11 ([1, Lemma 5.4]).** Suppose \( X, Y \in F \), \( X < Y \) and \( \beta = \text{min}(Y \setminus X) \). If \( G \) is \( P \)-generic over \( H^B(X) \) then \( G \) is \( P_\beta \)-generic over \( H^B(Y) \).

**Lemma 8.12 ([1, Lemma 5.5]).** Suppose \( X, Y \in F \), \( X < Y \) and \( G \) is \( P \)-generic over \( H^B(X) \). Then there is \( H \supseteq G \) such that \( H \) is \( P \)-generic over \( H^B(Y) \).

**Lemma 8.13 ([1, Lemma 5.6]).** Suppose that for \( n \in \omega \), \( X_n \in F \), \( G_n \) is \( P \)-generic over \( H^B(X_n) \), \( X_n < X_{n+1} \) and \( G_n \subseteq G_{n+1} \). Then \( \bigcup_{n \in \omega} G_n \) is \( P \)-generic over \( H^B(\bigcup_{n \in \omega} X_n) \).

Using the above lemmas, it is straightforward to find \( G \subseteq P \cap H^B(I) \) such that \( G \) is \( P \)-generic over \( H^B(I) \). Since by Corollary 4.6 \( P \) is \( <\omega_2 \)-directed closed, there is \( p \in P \) stronger than any condition in \( G \). Hence if \( G \) is the canonical name for the \( P \)-generic,

\[ p \Vdash G \cap H^B(I) \text{ is } P \text{-generic over } H^B(I) \]

and we are thus finished proving Theorem 8.9.

**Corollary 8.14.** Given the consistency of an \( \omega_1 \)-Erdős cardinal and \( 2 < n < \omega \), Local Club Condensation at \( \omega_n \) is consistent with \( \text{CC}(\omega_n) \). More generally, assuming the GCH, if \( \kappa \) is \( \omega_1 \)-Erdős and \( \omega_2 \leq \lambda < \kappa \) is regular, we may obtain a forcing extension in which \( \kappa \) becomes \( \lambda^+ \), all cardinals up to \( \lambda \) and \( \geq \kappa \) the GCH are preserved, and \( \text{CC}(\kappa) \) and Local Club Condensation at \( \kappa \) hold. In fact, a similar statement can be obtained using only weaker assumptions on the continuum function, namely those made in Section 4.

**Proof.** If \( \kappa \) and \( \lambda \) are as in the second statement above, force with \( P = P(\lambda, \kappa) \).

**Note:** Similar to the remark at the end of Section 5, one could replace Local Club Condensation at \( \kappa \) by Local Club Condensation in \([\omega_1, \kappa) \) in the statement of Corollary 8.14. This would require a forcing construction to obtain the latter principle by \( <\omega_2 \)-directed closed forcing which (as explained in Section 5) we omitted for the sake of simplicity. It is worth noting however that forcing Local Club Condensation up to \( \kappa \) seems to require us to add new subsets of \( \omega_1 \) in general and hence any forcing to obtain this principle should not be \( <\omega_2 \)-directed closed. As the proof of Theorem 8.9 seems to crucially require this closure property, we do not know
whether Local Club Condensation up to $\kappa$ is consistent with $\text{CC}(\kappa)$ for any $\kappa > \omega_2$ (they are inconsistent for $\kappa = \omega_2$ by Theorem 8.3 above).

We close this section by considering weak Chang’s Conjecture.

**DEFINITION 8.15.** [2] Assume $\kappa$ is a successor cardinal. The weak Chang Conjecture for $\kappa$, $\text{wCC}(\kappa)$, is the following assertion: Whenever $\mathcal{A}$ is a first order structure for a countable language, with universe $A$ and $\kappa^+ \subseteq A$, then there is $\alpha < \kappa$ such that for all $\beta < \kappa$ there is $X \triangleleft \mathcal{A}$ with $X \cap \kappa \subseteq \alpha$ and $\text{ot}(X \cap \kappa^+) > \beta$.\(^{12}\)

We present a well-known auxiliary lemma.

**LEMMA 8.16 (folklore).** Assume $N \triangleleft \mathcal{A}$ for some Skolemized structure $\mathcal{A}$ on $H_\kappa$. Assume $\lambda_0 < \lambda_1$ are regular cardinals below $\kappa$ and $\lambda_0, \lambda_1 \in N$. There is $M \triangleleft \mathcal{A}$ such that $\sup(M \cap \lambda_1) = \sup(N \cap \lambda_1), M$ is transitive below $\lambda_0$ and $M \cap \lambda_0 = \sup(N \cap \lambda_0)$.

**PROOF.** Let $M = H^\mathcal{A}(\sup(N \cap \lambda_0) \cup N)$. Assume that $\xi \in M \cap \sup(N \cap \lambda_1, \lambda_1)$. Then for some $\delta_0, \ldots, \delta_m \in N, \gamma_0, \ldots, \gamma_n < \sup(N \cap \lambda_0)$ and some formula $\varphi$ in the language of $\mathcal{A}, M \models \varphi$ is the unique $x$ with $\varphi(x, \gamma_0, \ldots, \gamma_n, \delta_0, \ldots, \delta_m)$. Let $v$ be such that $M$ thinks that $v$ is the supremum of all $x < \lambda_1$ which are uniquely defined by some formula $\varphi$ in the language of $\mathcal{A}$ of the form $\varphi(x, \xi_0, \ldots, \xi_n, \delta_0, \ldots, \delta_m)$ for some $\xi_0, \ldots, \xi_n < \lambda_0$. By regularity of $\lambda_1$, $v < \lambda_1$. By elementarity, $v \in N$. As $\xi < v$, this implies that $\sup(M \cap \lambda_1) = \sup(N \cap \lambda_1)$. The final statement of our claim follows similarly, noting that if $\sup(N \cap \lambda_0)$ is not a cardinal, we have $M = H^\mathcal{A}(\text{card}(\sup(N \cap \lambda_0)) \cup N)$.

**THEOREM 8.17.** Local Club Condensation at $\kappa^+$ refutes $\text{wCC}(\kappa)$ whenever $\kappa = \lambda^+$ is a successor cardinal.

**PROOF.** Assume that both Local Club Condensation at $\kappa^+$ and $\text{wCC}(\kappa)$ hold with $\kappa$ regular. Assume $\mathcal{A} = (M_{\kappa^+}, \in, M, F, \ldots)$ is a Skolemized structure for a countable language with $F$ as in Theorem 2.1 and assume $\alpha$ is for $\mathcal{A}$ as in Definition 8.15. For each $\beta < \kappa$, let $X_\beta$ be a witnessing structure, i.e., $X_\beta \triangleleft \mathcal{A}$ with $X_\beta \cap \kappa \subseteq \alpha$ and $\text{ot}(X_\beta \cap \kappa^+) > \beta$. We may assume that each $X_\beta$ is transitive below $\kappa$ by Lemma 8.16 and hence condenses by Theorem 2.1. Let $\pi_\beta$ denote the collapsing map of $X_\beta$. $\tilde{X}_\beta$ its transitive collapse. Then $\lambda + 1 \subseteq X_\beta$ and therefore $\pi_\beta(\kappa) \in (\lambda, \alpha]$. But $\tilde{X}_\beta \supseteq M_\beta$, hence for sufficiently large $\beta < \kappa$, $\tilde{X}_\beta \models \alpha \cong \lambda$, contradicting elementarity of $X_\beta$. \(\neg\)

§9. Open Questions.

**QUESTION 9.1.** Does Lemma 5.3 hold true if $\eta$ is assumed only to be inaccessible?

If the above has a positive answer, we would obtain the result of Theorem 5.5 from the optimal consistency assumption, namely a Mahlo cardinal, i.e., a positive answer to the following question.

**QUESTION 9.2.** Does Theorem 5.5 hold true if $\kappa$ is assumed only to be Mahlo?

**QUESTION 9.3.** Assume $\lambda$ is regular and GCH holds. Under sufficient large cardinal hypothesis, can one force to obtain a model of Local Club Condensation at $\lambda^+$ and $\neg \Box^\omega_{\lambda^+}$ while preserving all cofinalities $\leq \lambda^+?^{13}\)

\(^{12}\)The same definition would make perfect sense if $\kappa$ were inaccessible. The resulting principle though is easily seen to be inconsistent with ZFC.

\(^{13}\)If $\Box^\omega_{\lambda^+}$ is the same as $\Box^n_{\lambda^+}$ for $n < \omega$ except that one allows for the $C_\alpha$ to be at most countable.
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