Spacetime models, fundamental interactions and noncommutative geometry

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Abstract
We discuss the problem of determining the spacetime structure. We show that when we are using only topological methods the spacetime can be modelled as an \( \mathbb{R} \)- or \( \mathbb{Q} \)-compact space although the \( \mathbb{R} \)-compact spaces seem to be more appropriate. Demanding the existence of a differential structure substantially narrows the choice of possible models. The determination of the differential structure may be difficult if it is not unique. By using the noncommutative geometry construction of the standard model we show that fundamental interactions determine the spacetime in the class of \( \mathbb{R} \)-compact spaces. Fermions are essential for the process of determining the spacetime structure.

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1 Introduction

The outcomes of physical measurements are expressed in rational numbers. We believe that all possible values of physical variables constitute the set of real numbers $\mathbb{R}$. It is an idealized view since all measurements are performed with certain accuracy and it is hard to imagine how can they give irrational numbers. In this way the algebra of real continuous functions $C(M)$ on the spacetime manifold $M$ comes to play. This algebra play central rôle in classical and quantum physics, although this fact is not always perceived. Most of physical theories, including quantum gravity, make use of the notion of spacetime, at least approximately. Therefore one of the most important and fundamental open problems in theoretical physics is to explain the origin and structure of spacetime. Here we would like to discuss the problem of determining the spacetime structure. Put it another way, to analyse how faithful our theoretical models of the spacetime can be. We will try to be model independent and avoid unnecessary assumptions. Nevertheless, we will suppose that it is possible to determine the algebra $C(M)$ on the spacetime (assumed to be a topological space) with sufficient for our aim accuracy. This does not mean that we have to be able to find each element of $C(M)$ ”experimentally”: some inductive construction should be sufficient. By an abuse of language, we will call elements of $C(M)$ observables. Then we will
discuss to what extent the structure of the model $M$ of the spacetime is determined by $C(M)$, $M$ being a topological space. Further, we will analyse what happens if we admit of $M$ to have no topology or to be a differential manifold. We will also use the algebra of continuous $K$-valued functions $C(M, K)$, $K$ being a topological ring. Finally, we will show how $C(M, K)$ can be used to construct field theory via the A. Connes construction. We will also discuss to what extent the spacetime manifold is determined by electroweak interactions in the Connes’ noncommutative geometry formalism.

2 In quest of the topology of spacetime

A lot of properties of a topological space $M$ is encoded in the associated algebras $C(M, K)$ of continuous $K$-valued functions, $K$ being a topological ring, field, algebra etc. Even differential structures on a manifold $M$ can be equivalently defined by appropriate subalgebras $C^k(M, K)$ of real or complex differentiable functions on $M$. Suppose that our experimental technique is a priori powerful enough to reconstruct $C(M, \mathbb{R}) \equiv C(M)$ on our model of the spacetime $M$. What sort of information concerning $M$ can be extracted from these data? If $M$ is a set and $\mathcal{C}$ a family of real functions $M \to \mathbb{R}$ then $\mathcal{C}$ determines a (minimal) topology $\tau_{\mathcal{C}}$ on $M$ such that all function in $\mathcal{C}$ are continuous [1-2]. In general, there will be real continuous functions on $M$ that do not belong to $\mathcal{C}$ and more families of real functions on $M$ would
define the same topology on $M$. $(M, \tau_C)$ is a Hausdorff space if and only if for every pair of different points $p_1, p_2 \in M$ there is a function $f \in C$ such that $f(p_1) \neq f(p_2)$. Therefore it seems reasonable to assume that

$$f(x) = f(y) \quad \forall f \in C(M) \quad \Rightarrow \quad x = y. \quad (\star)$$

Physically this means that in order to be able to distinguish $x$ from $y$ in our model of spacetime we have to find such an observable $f \in C(M)$ that for $x, y \in M$ $f(x) \neq f(y)$. From the mathematical point of view, we have to identify all points that are not distinguished by $C(M)$, that is to demand $(\star)$. It is easy to show that such spaces are Hausdorff spaces. To proceed let us define [2-4].

**Definition 1.** Let $E$ be a topological space. A topological Hausdorff space $X$ is called $E$-compact ($E$-regular) if it is homeomorphic to a closed (arbitrary) subspace of some Tychonoff power of $E$, $E^Y$.

The following facts justify our assumption $(\star)$. For a topological space $X$, not necessarily a Hausdorff one, we can construct an $E$-regular space $\tau_E X$ and its $E$-compact extension $\nu_E X$ so that we have [3-4]

$$C(X, E) \cong C(\tau_E X, E) \cong C(\nu_E X, E) \cong C(\nu_E \tau_E X, E),$$
where \( \cong \) denotes isomorphism. The spaces \( \tau_E X \) and \( \nu_E \tau_E X \) have the nice property (*)). Now, it is obvious that, in general, our theoretical model of the spacetime may not be uniquely determined. This is an important result that says we can always model our spacetime as a subset of some Tychonoff power of \( \mathbb{R} \) provided \( C(M) \) is known! But it also says that we can model it as a subset of a Tychonoff power of a different topological space e.g. the rational numbers \( \mathbb{Q} \) (cf the discussion at the beginning). So its our choice! The topological number fields \( \mathbb{R} \) and \( \mathbb{Q} \) have the additional nice property of determining uniquely (up to a homeomorphism) \( \mathbb{R} \)- and \( \mathbb{Q} \)-compact sets, respectively:

\[
C(X, E) \cong C(Y, E) \iff X \text{ is homeomorphic to } Y, \ E = \mathbb{R} \text{ or } \mathbb{Q}. \quad (**)
\]

Other topological rings can also have this property. But this does not mean that the spacetime modeled by \( C(M, E) \) is homeomorphic to the one modeled by \( C(M, E') \). Hewitt have shown that \( \mathbb{R} \)-compact spaces are determined up to a homeomorphism by \( C(X, E) \), where \( E = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), the topological fields of complex numbers and quaternions, respectively [5]. This means that if we are interested in modeling spacetime as an \( \mathbb{R} \)-compact (\( \mathbb{Q} \)-compact) space then we can use \( C(M, \mathbb{R}), \ C(M, \mathbb{C}) \) or \( C(M, \mathbb{H}) \) (\( C(M, \mathbb{Q}) \)) to determine it. Another problem we will face is to decide if we are dealing with
the algebra $C(X, E)$ or only with the algebra of all continuous bounded $E$-valued functions on $X$, $C^*(X, E)$ [2-4]. For a compact space $X$ we have $C(X, E) = C^*(X, E)$, but in general, they are distinct. Spaces on which all continuous real functions are bounded are called pseudocompact. An $\mathbb{R}$-compact pseudocompact space is compact. We might get hints that some observables may in fact be unbounded but we are unlikely to be able to "measure infinities". An unbounded observable is necessary to show that the spacetime is a noncompact topological space. If we suppose that we can only recover $C^*(M, \mathbb{R}) \equiv C^*(M)$, then we can as well suppose that $M$ is compact (for an $\mathbb{R}$-compact $M$). In general, there will be more spaces with $C^*(M)$ as the algebra of real bounded continuous functions on them (they may not be compact or even $\mathbb{R}$-compact). Compactness (or paracompactness) of the space is a welcome property. For example pseudodifferential operators have discrete spectrum on compact spaces. Physicists often compactify configuration spaces by adding extra points or imposing appropriate boundary conditions. Demanding that all physical fields vanish at infinity is usually equivalent to the one point compactification of the spacetime and requiring that all fields vanish at the added "infinity point". In general, a topological space $X$ has more then one compactification. In some sense the one point compactification is minimal and the Stone-Čech compactification is maximal [2]. We will probably have to make nontopological assumptions
to choose one among the possible compactifications although they can be distinguished by regular subrings of $C(M)$ if they contain constant functions [3-4].

**Definition 2.** We shall say that a subspace $X$ of $M$ is $C$-embedded in $M$ if every function in $C(X, E)$ can be extended to a function in $C(M, E)$. Likewise, we shall say that $X$ is $C^*$-embedded in $M$ if every function in $C^*(X, E)$ can be extended to a function in $C^*(M, E)$.

A priori, after determining $C(M, E)$ or $C^*(M, E)$ we may find out that some space in which $M$ is $C$- or $C^*$-embedded is as good a model of the spacetime as $M$ is, and vice versa. Fortunately, for most topological spaces $X$ (completely $\mathbf{R}$-regular ones [2-5]) there is a unique compact space $\beta X$ (the Stone-Čech compactification) in which $X$ is dense and $C^*$-embedded and a unique $\mathbf{R}$-compact space $\nu_\mathbf{R} X$ in which $X$ is dense and $C$-embedded [2]. It can be proven that $\nu_\mathbf{R} X$ can be embedded in $\beta X$ and that $\nu_\mathbf{R} X$ is the smallest $\mathbf{R}$-compact space between $X$ and $\beta X$ [2-4]. The spaces $\beta X$ and $\nu_\mathbf{R} X$ are in some sense (see below) upper and lower limits on the spaces we are looking for. As $C(X)$ distinguishes among $\mathbf{R}$-compact spaces [2], we have to find a physical phenomenon that is not describable in terms of $C(X)$ to prove the assumption that the spacetime is an $\mathbf{R}$-compact space to be wrong. In
general one can say that $C(X, K)$ is more sensitive than $C^*(X, K)$ (e.g. it can distinguish between $X$ and $\beta X$). The following theorems give us some sense of the limitations of the determination of the spacetime modeled by $C(M, R)$ [2].

**Theorem 1.** If $X$ is dense in $T$ then the following statements are equivalent.

i. Every continuous mapping from $X$ into any $R$-compact space $Y$ has an extension to a continuous mapping from $T$ to $Y$.

ii. $X \subset T \subset v_RX$.

iii. $v_T = v_RX$.

iii. $X$ is $C$-embedded in $T$.

**Theorem 2.** $v_R Y$ contains a $C$-embedded copy of $X$ if and only if $C(X, R)$ is a homeomorphic image of $C(Y, R)$.

**Theorem 3.** If $X$ is dense in $T$ then the following statements are equivalent.

i. $X$ is $C^*$-embedded in $T$.

ii. $X \subset T \subset \beta X$.

iii. $\beta T = \beta X$.

**Theorem 4.** $\beta Y$ contains a $C^*$-embedded copy of $X$ if and only if
$C^*(X, \mathbb{R})$ is a homeomorphic image of $C^*(Y, \mathbb{R})$.

One can try to estimate the cardinality of the difference between various spaces in question. Theorems 5 and 6 [2] say that it can be essential.

**Theorem 5.** If $X$ is locally compact and $\mathbb{R}$-compact then the cardinal of a closed infinite set in $\beta X - X$ is at least $2^c$.

**Theorem 6.** The cardinal of a nondiscrete, closed set in $\beta X - \nu_{\mathbb{R}}X$ is at least $2^c$.

Physicists frequently raise questions concerning the potential discreteness of spacetime. One can formulate conditions of finiteness in terms of $C(X, K)$ [2-4]. It is unlikely that the spacetime forms a finite set (although various finite approximation have been put forward [6]). The answer to the question if discreteness can be defined in terms of $C(X, K)$ depends on the axioms of set theory! If one assume the existence of measurable cardinals, then conditions of discreteness of $X$ cannot be formulated in terms of $C(X, K)$ or $C^*(X, K)$, [3-4]. Nevertheless, the following theorem can be proven [2]:

**Theorem 7.** A discrete space is $\mathbb{R}$-compact if and only if its cardinal
is nonmeasurable.

The existence of measurable cardinals cannot be proven in the standard axioms of set theory. Even if they do exist they must be so huge that it is unlikely that the spacetime is so "potent". Therefore if the spacetime is discrete we certainly will be able to model it as an $\mathbb{R}$-compact space and discover this fact "on inspection" of $C(M,E)$, $E = \mathbb{R}, \mathcal{C}, \mathcal{H}$. Cardinality of such space can also be enormous (e.g. $c$, $2^c$, $2^{2^c}$, ...).

The problems of cardinality, dimension, density and tightness of the spacetime can also be addressed in terms of rings of real continuous functions with various topologies although experimental verification of these features (except dimension) is unlikely. The reader is referred to [7] for details. Here, we would like to mention only the following two facts. $\mathbb{R}$-compact spaces $X$ are precisely those with countable Hewitt numbers, $q(X) \leq \aleph_0$ [7]. For an arbitrary topological space and cardinal $\tau$ there is a subspace $\nu_\tau X$ of $\beta X$ so that every continuous function $f : X \to \mathbb{R}$ can be extended to a continuous real function on $\nu_\tau X$ and $q(\nu_\tau X) \leq \tau$ [7].

It may be too optimistic to assume that we are able to determine $C(M,\mathbb{R})$ with the required precision. Suppose that our experimental technique allows
only for sort of yes or no answer to questions concerning spacetime structure [8]. In this case we have to consider determination of a topological space $X$ by the ring $C(X, D)$ of continuous functions into $D = \{0, 1\}$ with various topological and/or algebraic structures. In general, $C(X, D)$ does not determine the space $X$ although $C(X, \mathbb{Z}_2)$ fulfills (**) with $E = \mathbb{Z}_2$. One can also consider other discrete fields e.g. $\mathbb{Z}_3$ [3-4]. In such case we can only try to determine the space in the class of $E$-compact spaces for some discrete $E$. Topological subfields of $\mathbb{R}$ can also be used for that purpose because they fulfil (**) [2,3,9].

Most of physical models of spacetime require that it is metrizable. Metrizable spaces with nonmeasurable cardinals are $\mathbb{R}$-compact [2]. This means that "practically all" models of spacetime are $\mathbb{R}$-compact (cf the discussion of discreteness).

Up to now we have considered the arbitrariness of our mathematical model $X$ of the spacetime as determined by $C(X, \mathbb{R})$. But one can also ask if any algebra that we identify as an algebra of physical observables on the spacetime always defines a topological space. The answer is negative: a commutative algebra must fulfil various sets of conditions to be a $C(X, \mathbb{R})$ of some topological space $X$. If we suppose that our model of the spacetime
is not a topological space we can deal with $\mathbb{R}^X$, the algebra of all real functions on $X$. But to have some ”deterministic power” we have to demand the existence of some additional structure on $X$, that is to distinguish a family of subsets of $X$ and/or an algebraic structure on the class of functions we are dealing with [10]. For example, if $(X, \tau)$ is a pair consisting of a set $X$ and a family $\tau$ of its subsets then we can define ”continuity” and ”homeomorphisms” by replacing topology by the family $\tau$. In this case one can prove [3-4].

**Theorem 8.** Let $X$ and $Y$ be sets and $\tau$ and $\sigma$ families of their subsets containing the empty set, closed with respect to finite intersections and summing up to $X$ and $Y$, respectively. Then $X$ and $Y$ are ”homeomorphic” if and only if there is an isomorphism of the semigroups $D^X$ and $D^Y$ such that ”$C(X, D)$” is mapped onto ”$C(Y, D)$”.

Such generalized space are more difficult to deal with than ordinary topological spaces therefore we think that spacetime should be modelled in the class of topological spaces.

One may also wonder if the knowledge of some symmetries might be of any help. In general, a topological space $X$ is not determined by its symmetries (homeomorphisms $X \to X$) [12-13] but sometimes can provide us with
useful information, e. g. if we know that some group $G$ acts transitively on $X$ then the cardinality of $X$ is not greater than the cardinality of $G$ [14]. For example, if we are pretty sure that the Lorentz group acts transitively on the spacetime we have got an upper bound on the cardinality of the spacetime.

Let us sum up the above consideration. We can model the spacetime as a topological $\mathbb{R}$-compact or $\mathbb{Q}$-compact space although the $\mathbb{R}$-compact spaces seem to be more appropriate. This two spaces are not necessarily homeomorphic. We might have serious problems with identification of some of the topological properties of the spacetime. This is because more then one space will have the same algebra of $C(M, E)$. If we decide to model the spacetime as a (completely regular) $\mathbb{R}$-compact space $M$ then we are able to reconstruct $M$ from $C(M)$ or $C^*(M)$ in the following sense [2,11]. $C(M)$ or $C^*(M)$ determine its Stone-Čech compactification $\beta M$ with $M$ as a dense $\mathbb{R}$-compact subspace. All fixed ideals in $C(M)$ correspond to points in $M$ [2-4,11]. Such spaces are Hausdorff. In order to distinguish two spacetime points we need an observable that takes different values at these points. If we fail to do this we have to identify these points and this may result in a discrete or even finite model that would also be an $\mathbb{R}$-compact space and can be reconstructed from $C(M)$. Of course, spacetime points may have "reach structure" that is beyond our experimental scope. This corresponds to determining only some
subalgebra of $C(M)$. We have to find a phenomenon that is not describable in terms of $C(M)$ to reject the assumptions of $\mathbb{R}$-compactness. We do not know if the physical world can be described by using only topological methods. The most spectacular example is the existence of the Whitehead spaces. These are three-dimensional topological manifolds that are not homeomorphic to $\mathbb{R}^3$ but their products with $\mathbb{R}$ are homeomorphic to $\mathbb{R}^4$. In other words when an $\mathbb{R}^1$ is factored out in $\mathbb{R}^4$ the result will not necessary be $\mathbb{R}^4$. One have to demand differentiability for this to be case. More sophisticated formalism would involve further assumptions about the spacetime structure but it may not be easy to find out if these assumptions are necessary or just convenient tools. We will discuss it in the following sections.

3 Differential structure.

Differential calculus have proven to be a powerful tool in the hands of physicists. But is it indispensable? Not every topological space or even topological manifold can support differential structures and demanding the existence of a differential structure on the spacetime can severely restrict our choice of spaces for modeling the spacetime. A differential structure on a topological manifold $M$ can be defined by specifying a subalgebra of differentiable functions $C^k(M, \mathbb{R})$ of the algebra $C(M)$. The algebra $C^\infty(M)$ of smooth real functions on $M$ determines $M$ up to a diffeomorphism [16] (the points of
$M$ are in one-to-one correspondence with maximal ideals in $C^\infty (M)$). The algebra of continuous function on $M$ is larger than $C^k (M, \mathbb{R})$ and may correspond to more topological spaces than $M$ but if two manifolds have at some points $p$ and $q$ isomorphic rings of germs of continuous functions then the points $p$ and $q$ have homeomorphic neighbourhoods (local dimensions are the same) [17]. If the laws of physics are "smooth" the spacetime should be modeled on a smooth manifold. If this is the case then $C^\infty (M, \mathbb{R})$ is sufficient to determine $M$ and describe all physical phenomena. Geometrical quantization is one of the most popular efforts in this direction. One can even try to "quantize" differential equations in terms of $C^k (M, \mathbb{R})$ [18]. It should be noted here that $C(M)$ or even $C^\infty (M)$ are far to big as potential algebras of observables to be used for constructing a quantum theory that agrees with experiment. Additional information (assumptions) is necessary to deal with this problem [19]. Any manifold can be embedded in $\mathbb{R}^n$ for some $n$ and therefore is $\mathbb{R}$-regular. The most popular models of spacetime are riemannian or pseudoriemannian manifolds. Such spaces are metrizable and as such $\mathbb{R}$-compact (cf the discussion in the previous section). This means that these manifolds are as topological spaces determined by $C(X, E)$, where $E = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ but additional knowledge of the algebra of differentiable functions is needed to determine the differential structure [16]. But even in the smooth case we face a new nonuniqueness problem because some manifolds can sup-
port many nonequivalent differential structures [20-27]. Such "additional" differential structures are usually referred to as fake or exotic ones. They are specially abundant in the four-dimensional case (it is sufficient to remove one point from a given manifold to get a manifold with exotic structures [24]).

More astonishing is the fact that the topologically trivial fourdimensional Euclidean space $\mathbb{R}^4$ can be given uncountably many exotic structures (in fact a two-parameter family of them) [24]. We have to interpret these mathematical results in physical language [25-27]. This is not an easy task. Although one can put forward many arguments that exotic smoothness might have physical sense [26,30-31], the lack of any explicit (pseudo-) riemannian structure hinders physical predictions. Nevertheless some problems can be discussed.

Suppose that the spacetime manifold is topologically $\mathbb{R}^4$. If we require that all physical observables vanish at infinity then our model is equivalent to the one on $S^4$ with the "boundary condition" that all observables vanish at one point. Then if the smooth Poincare hypothesis (there is only one differential structure on $S^4$) [28] is correct we are left with only one (standard) differential structure. This may be a solution to the nonuniqueness problem but certainly is not a satisfactory explanation of the fact! If we suppose that all observables have compact supports then we are not able to eliminate exotic structures [27]. Bizaca and Etnyre proved that for any compact 3-manifold $M$ the open manifold $M \times \mathbb{R}$ has infinitely many different smooth structures.
(this is true for a wider class of $M$'s) [29]. Spaces of these form are often explored by physicists. Below we give some examples. Bag models are popular models of hadrons and astrophysical objects [32-33]. Such models may have their exotic versions because we do not know if the choice of metric tensor and boundary conditions is sufficient to eliminate the possible exotic structures, especially in astrophysical considerations. We are accustomed to the coordinate representation of quantum mechanics. Most of the configuration spaces for quantum problems are of the form $M \times \mathbf{R}$. If the Schrödinger operator is of the form $-\Delta + V$, where $\Delta$ is the Laplace operator and $V$ the potential then some $\Delta$ may not be consistent with all smooth structures (that is the metric tensor may not be smooth). We think that if one chooses the metric tensor and boundary condition for the above problems then the additional differential structures are physically unimportant. This means that we are only interested in the spectral problem of the operators in question and suppose that physically interesting isospectral homeomorphic manifolds are diffeomorphic, cf [30-31]. But in general relativity metric tensor is one of the variables and the question is what determines differential structure in this case? Once more we dare to conjecture that the spectral problem for physical operators should give an answer to this nonuniqueness problem. Unfortunately, our present knowledge is too poor to give a definite answer. H. Brans has conjectured that "localized" exoticness can act as a source for
some externally regular gravitational field, just as matter or a wormhole can [25]. In this context one can also ask if there is an analogue of the Bohm-Aharonov effect. That is suppose that some points are "excluded" from the spacetime. Such singularities allows for exotic structures. The "standard" metric tensor defined by matter might not be smooth with respect to some exotic differential structures. Can such effect be detected, say in gravitational measurements? This would mean that there is "additional" curvature required by consistency of differential structures. The existence of exotic differential structures is certainly a challenge to physicists [26]. We will return to this problem in the following section.

4 Noncommutative differential geometry and physical models.

We have seen that differential geometry can be formulated in terms of the commutative algebra of real smooth functions on the manifold in question. A. Connes managed to generalize it for much larger class of algebras, not necessarily commutative [34-35]. His noncommutative geometry have found profound physical applications. The basic ingredients are a $C^*$-algebra $\mathcal{A}$ represented in some Hilbert space $H$ and an operator $\mathcal{D}$ acting in $H$. The differential $da$ of an $a \in \mathcal{A}$ is defined by $[\mathcal{D}, a]$ and the integral is replaced by the Diximier trace, $Tr_\omega$, with an appropriate inverse n-th power of $|\mathcal{D}|$.
instead of the volume element $d^n x$. The Dixmier trace of an operator $O$ is roughly speaking the logarithmic divergence of the ordinary trace:

$$Tr_{\omega} O = \lim_{n \to \infty} \frac{\lambda_1 + \ldots + \lambda_n}{\log n},$$

where $\lambda_i$ is the i-th proper value of $O$. See [34-40] for details. One can generalize the notions of covariant derivative ($\nabla$), connection ($A$) and curvature ($F$) forms so that "standard" properties are conserved:

$$\nabla = d + A, \quad F = \nabla^2 = dA + A^2,$$

where $A \in \Omega^1_D$ is the algebra of one forms defined with respect to $d$. Fiber bundles became projective modules on $A$ in this language. The $n$-dimensional Yang-Mills fermionic action is given by the formula [35-38]

$$L(A, \psi, D) = Tr_{\omega} \left( F^2 | D |^{-n} \right) + <\psi | D + A | \psi >,$$

where $<|>$ denotes the inner product in the Hilbert space. For $A = \mathcal{C}^\infty(M)$ and $D$ being the Dirac operator we recover the ordinary riemannian geometry of the spin manifold $M$. Physicists have learned from the noncommutative geometry that one can describe fundamental interactions by specifying the Hilbert space of fermionic states and a representation of an $C^*$ algebra in
this Hilbert space. If one takes

\[ \mathcal{A} = C^\infty(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H}) \oplus M_{3 \times 3}(C^\infty(M, \mathbb{C})) , \quad (****) \]

the known fermionic states to span the Hilbert space and the generalized Dirac operator with the Kobayashi-Maskawa mass matrix as \( \mathcal{D} \) one gets the standard model lagrangian [35-36]. The structure of the ”world algebra” [36] (***) and the analysis given in the previous sections allow us to conclude that the spacetime structure is uniquely determined in the class of \( \mathbb{R} \)-compact spaces by fundamental interactions of fermions (gravitation is hidden in the metric tensor that ”enters” the Dirac operator [35,40]) as the result of the properties of \( C^\infty(M, \mathbb{C}) \) and \( C^\infty(M, \mathbb{H}) \). The knowledge of \( C^\infty(M) \) is sufficient for the construction of the manifold \( M \) but the Higgs mechanism to be at work requires that \( M \) be multiplied by some discrete space [34-40]. This means that we may not know the structure of the spacetime with satisfactory precision but nevertheless fundamental interactions determine it in a quite unique way. It should be noted here that if other rings would appear in (*** then this conclusion may not be true (for example, grand unified models can be less determinative than the ”low energy approximation” [40]). Of course, it is still possible that the \( C^* \) algebra \( \mathcal{A} \) that describes correctly fundamental interactions do not correspond to any topological space. This would mean that spacetime can only approximately
be described as a topological space, say, defined by some subalgebra of $\mathcal{A}$ or that fundamental interactions does not determine it uniquely. It should be stressed here that matter fields (fermions) and their interactions are essential in the process determining the spacetime structure. The pure gauge sector is insufficient because two $E$-compact spaces $X$ and $Y$ are homeomorphic if and only if the categories of all modules over $C(X, E)$ and $C(Y, E)$ are equivalent. The noncommutative geometry formalism even suggest that fermions define the spacetime via the Dirac operator at least on the theoretical level.

Let us now return to the smooth case. By using the heat kernel method [41] we can express the Yang-Mills action in the form [36, 40]:

$$\mathcal{L}_{YM}(F) \sim \lim_{t \to 0} \frac{\text{tr} (F^2 \exp (-tD^2))}{\text{tr} (\exp (-tD^2))}.$$ 

Suppose that we have a one parameter ($z$) family of differential structures and the corresponding family of Dirac operators $D(z)$. The Duhamel’s formula [40]

$$\partial_z (e^{-tD^2(z)}) = \int_0^t e^{-(t-s)\triangle(z)} \partial_z (D^2(z)) e^{-s\triangle(z)} ds ,$$

where $\triangle$ is the scalar Laplacian, can be used to calculate the possible variation of $\mathcal{L}_{YM}(F)$ with respect to $z$. Unfortunately our present knowledge of exoticness is too poor for performing such calculations. For an operator $K$
with a smooth kernel we have the following asymptotic formula [40]:

\[ tr \left( Ke^{-tD^2} \right) \sim tr \left( K \right) + \sum_{i=1}^{\infty} t^i a_i. \]

So if \( F^2 \) is smooth with respect to all differential structures (e.g. has compact support [27]) then the possible effects of exoticness are negligible. This means that we are unlikely to discover exoticness by performing "local" experiments involving gauge interactions. (Brans proved that exoticness can be localized in arbitrary small spatial region but they should cause extremely strong gravitational effects to be detectable.) If we consider only matter (fermions) coupled to gravity then the action can be expressed in terms of the coefficients of the heat kernel expansion of the Dirac Laplacian, \( D^2 \) [40]. In this case we may be able to determine the differential structure only if the Dirac operator specifies it uniquely [26, 31]. In general case the possible physical effect of exotic smoothness is still an open problem.

5 Conclusions

We have analysed the problem of determining the spacetime structure. We should be able to determine the spacetime in the class of \( \mathbb{R} \)-compact spaces. We have to find a phenomenon that cannot be described in terms of the algebra \( C(M) \) to reject the assumption of \( \mathbb{R} \)-compactness. If we are using only topological methods we will not be able to construct the topological
model $M$ of the spacetime uniquely. An unbounded observable is necessary to prove noncompactness of spacetime. In the general case, we will be able to construct only the Stone-Čech compactification of the space in question. The existence of a differential structure on $M$ allows for the identification of $M$ with the set of maximal ideals of $C^\infty(M)$, although we anticipate that the determination of the differential structure may be problematic. Connes’ construction of the standard model lagrangian imply that fundamental interactions determine the model of spacetime in the class $\mathbb{R}$-compact spaces although more general models may not. Matter fields are essential for defining and determining the spacetime properties. If we are not able to determine $C(M, \mathbb{R})$ or $C(M, \mathbb{Q})$ then our knowledge of the spacetime structure is substantially limited. If this is the case we have a bigger class of spaces ”at our disposal” and we are more free in making assumptions about the spacetime.

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