D-Branes on Group Manifolds

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ABSTRACT

Possible Dirichlet boundary states for WZW models with untwisted affine super Kac-Moody symmetry are classified for all compact simple Lie groups. They are obtained by inner- and outer-automorphism of the group. D-brane world-volume turns out to be a group manifold of a symmetric subgroup, so that the moduli space of D-brane is an irreducible Riemannian symmetric space. It is also clarified how these D-branes are transformed to each other under abelian T-duality of WZW model. Our result implies, for example, there is no D-particle on the compact simple group manifold. When the D-brane world-volume contains $S^1$ factor, the D-brane moduli space becomes hermitian symmetric space and the open string world-sheet instantons are allowed.

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1 Introduction

It is now widely recognized that the D-branes are key ingredients of the non-perturbative physics of the string theories [1]. A conformal field theoretic (CFT) treatment of the interaction of the closed string with the background D-brane is described by the boundary state adapted to Dirichlet boundary condition. However, most of the CFT analysis so far are restricted to the flat D-brane, except a few attempts [2]. In order to get more insights on the D-brane dynamics and eventually to achieve a full quantum treatment of the D-branes themselves[3], it is valuable for us to experience various situations such as a curved D-brane in a curved space.

In the present paper, we will study Dirichlet boundary states which describe D-branes living in the group manifolds [4]. As is well known, strings on the group manifolds are described by the WZW models, which possess left and right affine (super or non-super) Kac-Moody algebras [5]. The condition imposed to the boundary state is given in terms of the Kac-Moody currents. For the purely Neumann boundary condition, the building blocks of the associated boundary state has been known as the Ishibashi [6] state. We will generalize it to accommodate the Dirichlet boundary condition. For the abelian currents, one can take an arbitrary number of Dirichlet directions. It is, however, not the case for the non-abelian currents; non-trivial commutation relations of the currents require consistency among the boundary conditions for diverse directions. We will classify possible Dirichlet boundary conditions with respect to super and non-super untwisted Kac-Moody currents for all compact simple Lie groups.

T-duality of the WZW model transforms one boundary state to another. We will show how the boundary states thus obtained are transformed to each other under the abelian T-duality.

It is known that under abelian T-duality non-trivial monodromy for world-sheet genus greater than one can be induced for the dual theory, when the level of the Kac-Moody algebra is greater than one [7]. That means twisted sectors may appear in the dual theory even if the original theory consists only of untwisted sector. Thus we need to be equipped with the boundary states for the twisted Kac-Moody currents. In the present paper, however, we restrict our study only to the untwisted sector. Our argument can be applied straightforwardly to the case where left and right states have the same twist, whereas the asymmetric twist

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3For D-particle, an effort toward this end is in ref.[3]
case should be considered separately. We will leave the latter case for a future publication.

The organization of the paper is the following. We give a criterion for the consistent Dirichlet boundary condition for the WZW models in the subsequent section. A generalized version of the Ishibashi state is also constructed there. In section 3, we make a classification of the Dirichlet boundary state for all compact simple Lie groups. Abelian T-duality transformation of these boundary states are discussed in section 4. The last section is devoted to discussions and comments. Some technical points are explained in two appendices.

2 Boundary states in WZW model

Let us begin by recalling the boundary states in flat space or abelian current case [8, 9]. Denoting \( \alpha_n \) and \( \tilde{\alpha}_n \) for the closed string left and right oscillators in a standard way, Neumann and Dirichlet boundary states are defined by the following conditions:

\[
\text{Neumann} : \quad (\alpha_n + \tilde{\alpha}_{-n})|N\rangle = 0, \\
\text{Dirichlet} : \quad (\alpha_n - \tilde{\alpha}_{-n})|D\rangle = 0.
\]

The solutions for these equations are well known as

\[
|N\rangle = \int dp \, f(p) \exp(-\sum_{n>0} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n})|p\rangle \otimes |\tilde{p}\rangle, \\
|D\rangle = \int dp \, f(p) \exp(+\sum_{n>0} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n})|p\rangle \otimes |\tilde{p}\rangle,
\]

where \(|p\rangle\) and \(|\tilde{p}\rangle\) are the eigenstates of \(\alpha_0\) and \(\tilde{\alpha}_0\), respectively, with eigenvalue \(p\) and \(f(p)\) is an arbitrary function.

Our aim in this section is to extend these boundary states to the case of the non-abelian affine super and non-super Kac-Moody algebras. We consider the algebras associated to the compact simple Lie group \(G\), for brevity. But the basic part of our argument can be easily extended to more general groups.

Denoting left and right (non-super for the moment) Kac-Moody currents by \(J^a\) and \(\tilde{J}^a\) \((a = 1, 2, \cdots, d = \dim(G))\) respectively, a generalization of the above boundary condition is

\[
\text{Neumann} : \quad (J^a_n + \tilde{J}^a_{-n})|B\rangle = 0, \\
\text{Dirichlet} : \quad (J^a_n - \tilde{J}^a_{-n})|B\rangle = 0.
\]
It is, however, not possible to impose arbitrarily Neumann or Dirichlet condition. Since the currents satisfy non-trivial algebra

\[
\begin{align*}
[ J^a_n, J^b_m ] &= i f^{abc} J^c_{n+m} + \frac{k}{2} n \delta^{ab} \delta_{n+m,0} , \\
[ \bar{J}^a_n, \bar{J}^b_m ] &= i f^{abc} \bar{J}^c_{n+m} + \frac{k}{2} n \delta^{ab} \delta_{n+m,0} , \\
[ J^a_n, \bar{J}^b_m ] &= 0 ,
\end{align*}
\]

the boundary condition has to be consistent with the algebra. For example, if we impose Dirichlet condition in two directions, say \(a\) and \(b\), then the relation

\[
[ J^a_n - \bar{J}^a_{-n} , J^b_m - \bar{J}^b_{-m} ] |B\rangle = i f^{abc} ( J^c_{n+m} + \bar{J}^c_{-n-m} ) |B\rangle
\]

forces us to impose Neumann conditions to the direction \(c\) of non-vanishing \(f^{abc}\). It is clear, in particular, that we cannot put Dirichlet conditions in all the directions. That means there is no D-particle on the compact simple group manifold.

We shall look for consistent sets of boundary conditions in the following form:

\[
\left( J^a_n + \tau(\bar{J}^a_{-n}) \right) |B\rangle = 0 ,
\]

where \(\tau\) is a certain map of the current. It is easily seen that the consistency of this condition requires a relation

\[
[ \tau(\bar{J}^a_n) , \tau(\bar{J}^b_m) ] = \tau([\bar{J}^a_n , \bar{J}^b_m ]) ,
\]

so that the \(\tau\) should be an automorphism of the algebra. Since we are concerned with Dirichlet and Neumann condition, we only consider automorphism of the type

\[
\tau(\bar{J}^a_n) = U^{ab} \bar{J}^b_n ,
\]

with a orthogonal matrix \(U^{ab}\). The orthogonality of \(U\) guarantees the invariance of the \(\text{tr}(T^a T^b)\), where \(T^a\)'s are the generators of the corresponding non-affine Lie algebra. The condition that eq.(13) gives automorphism is

\[
U^{ad} U^{be} f^{def} = f^{abc} U^{cf} .
\]

This is equivalent to the invariance of \(\text{tr}([T^a, T^b] T^c)\). One can easily see \(\tau(\bar{L}_n) = \bar{L}_n\) for the Virasoro generator given in the Sugawara form in terms of \(\bar{J}^a_n\). Hence our boundary

\[\text{Lattice translation in the affine Weyl group, which mixes the generators on the different level, cannot be written in this form, but does not give a simple Dirichlet or Neumann condition.}\]
condition is automatically consistent with the conformal invariance

\[(L_n - \bar{L}_{-n})|B\rangle = 0.\]  \hspace{1cm} (15)

None of the above argument is changed in the case of super Kac-Moody algebra:

\[ [J^a_n, j^b_m] = i f^{abc} J^c_{n+m} + \frac{k}{2} n \delta^{ab} \delta_{n+m,0}, \] \hspace{1cm} (16)

\[ [J^a_n, j^b_m] = i f^{abc} j^c_{n+m}, \] \hspace{1cm} (17)

\[ \{ j^a_n, j^b_m \} = \frac{k}{2} \delta^{ab} \delta_{n+m,0}, \] \hspace{1cm} (18)

where \( j^a_n \) is a fermionic partner of the current \( J^a_n \), and the same algebra is satisfied by the right moving current \( \tilde{J}^a_n \) and its fermionic partner \( \tilde{j}^a_n \). We take the boundary condition as

\[ (J^a_n + \tau(\tilde{J}^a_n)) |B\rangle = 0, \] \hspace{1cm} (19)

\[ (j^a_n \pm i\tau(\tilde{j}^a_n)) |B\rangle = 0. \] \hspace{1cm} (20)

If we take identity map for \( \tau \), then this reduces to the usual Neumann boundary condition. As in the non-super case, \( \tau \) should be an automorphism of the super algebra, and we consider the one of the type given by

\[ \tau(\tilde{J}^a_n) = U^{ab} \tilde{j}^b_n, \] \hspace{1cm} (21)

\[ \tau(j^a_n) = U^{ab} j^b_n. \] \hspace{1cm} (22)

Here the orthogonal matrix \( U \) is common to \( \tilde{J} \) and \( j \) because of supersymmetry, and the condition of automorphism is given by the same eq.(14) as before. Since \( \tau(\tilde{L}_n) = \tilde{L}_n \) and \( \tau(\tilde{G}_n) = \tilde{G}_n \) for super-Virasoro generators \( \tilde{L}_n \) and \( \tilde{G}_n \), they are consistent with the super-conformal invariance

\[ (L_n - \bar{L}_{-n})|B\rangle = 0, \] \hspace{1cm} (23)

\[ (G_n \mp i\tilde{G}_{-n})|B\rangle = 0. \] \hspace{1cm} (24)

Thus our problem for both super and non-super algebra is to find possible orthogonal matrices \( U \) satisfying eq.(14).

Before going into the classification of automorphisms, let us touch on the construction of the boundary state. For purely Neumann case, the building block of the boundary state was constructed by Ishibashi [6]:

\[ |w\rangle = \sum_n |w, n\rangle \otimes \Theta |\bar{w}, n\rangle, \] \hspace{1cm} (25)
where $|w,n\rangle$ and $|\tilde{w},n\rangle$ are complete orthogonal basis of a representation $w$ for left and right current algebra, and $\Theta$ is an anti-unitary operator which satisfies $\Theta \tilde{J}_n^a \Theta^{-1} = -\tilde{J}_n^a$ and $\Theta j_n^a \Theta^{-1} = \mp i j_n^a$. For our general boundary condition, it should be slightly modified:

$$|w\rangle = \sum_n |w,n\rangle \otimes T \Theta |\tilde{w},n\rangle,$$

(26)

where $T$ is an unitary operator which generates an automorphism $T \tilde{J}_n^a T^{-1} = \tau(\tilde{J}_n^a)$ and $T \tilde{j}_n^a T^{-1} = \tau(\tilde{j}_n^a)$. We can prove that the state (26) satisfies the condition (19) as almost same way as Ishibashi state does Neumann condition. Let us take an arbitrary bra-state $\langle i | \otimes \langle \tilde{j} |$ in the representation space of $w$. Then we can show

$$\langle i | \otimes \langle \tilde{j} | (J_n^a + \tau(\tilde{J}_n^a)) |w\rangle = 0.$$  

(27)

Since the state $\langle i | \otimes \langle \tilde{j} |$ is arbitrary, $(J_n^a + \tau(\tilde{J}_n^a)) |w\rangle = 0$ holds. Similarly, we can show $(j_n^a \mp i \tau(\tilde{j}_n^a)) |w\rangle = 0$. Thus a general boundary state is given by their linear combination $|B\rangle = \sum_w c(w) |w\rangle$.

### 3 Classification of Dirichlet boundary states

Now we are to classify possible automorphism which gives consistent boundary condition. From the results in the previous section, it is sufficient to consider the automorphism $\tau(T^a) = U^{ab} T^b$ of non-affine version of the algebra

$$[T^a, T^b] = i f^{abc} T^c.$$  

(28)

Since we are interested in the Dirichlet or Neumann condition, not its mixture in one direction, our orthogonal matrix $U^{ab}$ should satisfy $U^2 = 1$. If we use an automorphism other than $Z_2$, corresponding boundary condition we get is a mixed Dirichlet-Neumann condition which can be interpreted as the boundary coupled with open string background fields.

\footnote{If we use an automorphism other than $Z_2$, corresponding boundary condition we get is a mixed Dirichlet-Neumann condition which can be interpreted as the boundary coupled with open string background fields.}
More explicitly, if we take an appropriate hermitian basis, \( U \) becomes a diagonal matrix whose entries are ±1. Then in this basis we can divide all the generators into two sets; \( S = \{ T^a | \tau(T^a) = T^a \} \) and \( A = \{ T^a | \tau(T^a) = -T^a \} \). They satisfy

\[
\begin{align*}
[S, S] &\subset S, \quad [S, A] \subset A, \quad [A, A] \subset S.
\end{align*}
\] (29)

Thus \( S \) forms subalgebra which is known as symmetric subalgebra. So our task is now reduced to the classification of the symmetric subalgebras. Though the answer is already known \[10\], we want not just a list of the subalgebra but their mutual relations under T-duality. Therefore, we go into a fine structure of the automorphism which will be relevant for the argument in section 4.

There are two classes of the automorphism for the compact simple Lie algebra: inner-automorphism and outer-automorphism. In order to study these, it is convenient to use Cartan-Weyl basis instead of hermitian basis. Let us denote Cartan generators by \( H^i \) (\( i = 1, 2, \cdots, r = \text{rank}(G) \)) and a generator associated to a root \( \alpha \) by \( E^\alpha \).

Firstly we study inner-automorphism, which is given by an adjoint action

\[
\tau(T^a) = g T^a g^{-1}, \quad g \in G.
\] (30)

It is general enough to take \( g \) as being made only of the Cartan generators \[12\]: \( g = \exp(i\eta \cdot H) \). Then each generator is transformed under this automorphism as

\[
\begin{align*}
\tau(H^i) &= H^i, \\
\tau(E^\alpha) &= \exp(i\eta \cdot \alpha) E^\alpha.
\end{align*}
\] (31)

(32)

Since every positive (resp. negative) root is expressed by a linear sum of simple roots \( \alpha_i \) (\( i = 1, 2, \cdots, \text{rank}(G) \)) with non-negative (non-positive) integer coefficients, the phase appeared in eq.(32) for a positive (negative) root is written as a product of the phases for simple roots with non-negative (non-positive) integer power. So the independent freedom of our choice is in a set of \( \{ \epsilon_i = \exp(i\eta \cdot \alpha_i) \} \). Possible values of \( \epsilon_i \) are necessarily ±1 from the property \( U^2 = 1 \). For a generic root \( \alpha = \sum_i m_i \alpha_i \), the phase factor becomes a product \( \prod_i \epsilon_i^{m_i} \).

It is useful to define a quantity

\[
T_G(\{\epsilon_i\}) = \sum_{\alpha \in \Delta_+} \exp(i\eta \cdot \alpha) = \sum_{\sum_i m_i \alpha_i \in \Delta_+} \prod_{i=1}^{\text{rank}(G)} \epsilon_i^{m_i},
\] (33)
where $\Delta_+$ is a set of all positive roots. If we denote $N_\pm$ as the number of positive root generators $E^\alpha$ whose phase factor $\exp(i\eta \cdot \alpha)$ is $\pm 1$ respectively, then $T_G$ gives $N_+ - N_-$ under a given $\{\epsilon_i\}$. Thereby the number of Dirichlet directions $N_D$, i.e. $2N_-$, is given by

$$N_D = \frac{1}{2}(d - r) - T_G, \quad (34)$$

where $d = \dim(G)$ and $r = \text{rank}(G)$.

On the other hand, the number of Neumann direction is the dimension of the D-brane world-volume embedded in the group manifold $G$. It is clear from the above argument that the D-brane world-volume itself is a group manifold associated to the symmetric subgroup which we denote $H$.

We examined all possible choices of $\{\epsilon_i\}$ and determined $N_D$ and $H$ for all compact simple Lie groups. The result is shown in Table 1. We omitted in the table the identity automorphism which gives $N_D = 0$ and $H = G$, i.e. pure Neumann boundary condition. Some technical statements are in the Appendix A.

We note here that, for the inner-automorphism case, the symmetric subgroup is a maximal regular subgroup. Smaller (non-maximal) subgroup cannot be obtained by $\mathbb{Z}_2$ (i.e. $U^2 = 1$) automorphism. This is followed by that if we consider multiple D-branes they are likely to intersect.

Next we turn to the study of outer-automorphism. Outer-automorphism is achieved by the automorphism of Dynkin diagram. Therefore only $A_n$, $D_n$ and $E_6$ are relevant. Under the automorphism each generator is transformed as

$$\tau(\alpha \cdot H) = \tau(\alpha) \cdot H, \quad (35)$$

$$\tau(E^\alpha) = \varepsilon_\alpha E^{\tau(\alpha)}, \quad (36)$$

where $\varepsilon_\alpha = \varepsilon_{-\alpha}$ is either $+1$ or $-1$ consistently with the algebra. Note that if $\alpha \in \Delta_+$ then $\tau(\alpha) \in \Delta_+$. The phase factor $\varepsilon_\alpha$ satisfies the following properties (proof is given in Appendix B):

**Lemma 1**

1. $\varepsilon_\alpha \varepsilon_{\tau(\alpha)} = 1$.

2. If $\alpha \neq \tau(\alpha)$ and $\alpha, \alpha + \tau(\alpha) \in \Delta_+$, then $\varepsilon_{\alpha + \tau(\alpha)} = -1$.

3. If $\alpha \neq \tau(\alpha)$, $\beta = \tau(\beta)$ and $\alpha, \beta, \alpha + \beta, \tau(\alpha) + \beta, \alpha + \beta + \tau(\alpha) \in \Delta_+$ but $\alpha + \tau(\alpha) \notin \Delta_+$, then $\varepsilon_{\alpha + \beta + \tau(\alpha)} = \varepsilon_\beta$. 

7
\[
A_n = SU(n+1) \quad d = n(n+2)
\]

\[
B_n = SO(2n+1) \quad d = n(2n+1)
\]

\[
C_n = Sp(n) \quad d = n(2n+1)
\]

\[
D_n = SO(2n) \quad d = n(2n+1)
\]

\[
E_6 \quad d = 78
\]

\[
E_7 \quad d = 133
\]

\[
E_8 \quad d = 248
\]

\[
F_4 \quad d = 52
\]

\[
G_2 \quad d = 14
\]

| \(G\) | \(H\) | \(N_D\) |
|---|---|---|
| \(A_n = SU(n+1)\) | \(A_{n-k} \otimes A_{k-1} \otimes U(1)\) | \(2k(n+1-k), \quad k = 1, 2, \ldots, \left[\frac{n+1}{2}\right]\) |
| \(B_n = SO(2n+1)\) | \(B_{n-k} \otimes D_k\) | \(2k(2n+1-2k), \quad k = 1, 2, \ldots, n-1\) |
| \(C_n = Sp(n)\) | \(A_{n-1} \otimes U(1)\) | \(n(n+1)\) |
| \(D_n = SO(2n)\) | \(A_{n-1} \otimes U(1)\) | \(n(n-1)\) |
| \(E_6\) | \(D_5 \otimes U(1)\) | 32 |
| \(E_7\) | \(E_6 \otimes U(1)\) | 54 |
| \(E_8\) | \(D_6 \otimes A_1\) | 64 |
| \(F_4\) | \(B_4\) | 16 |
| \(G_2\) | \(A_1 \otimes A_1\) | 8 |

Table 1: List of symmetric subgroup \(H\) other than \(G\) itself obtained by inner-automorphism. \(N_D\) is the number of Dirichlet directions.
Let us denote a simple root with property $\tau(\alpha_i) = \alpha_i$ by $\alpha_{i\parallel}$ and one with $\tau(\alpha_i) \neq \alpha_i$ by $\alpha_{i\perp}$. Then apparently $\left(\alpha_{i\perp} - \tau(\alpha_{i\perp})\right) \cdot H$ belongs to the set $A$, and the rest of Cartan generators do to symmetric subalgebra $S$. As for the root generator $E^\alpha$ with $\alpha \neq \tau(\alpha)$,

$$E^\alpha + \varepsilon_\alpha E^{\tau(\alpha)} \in S,$$
$$E^\alpha - \varepsilon_\alpha E^{\tau(\alpha)} \in A,$$

while $E^\alpha$ with $\alpha = \tau(\alpha)$ depends on its phase $\varepsilon_\alpha$. For the latter, invariance of the algebra restricts arbitrariness. Since $\alpha \in \Delta_+$ can be written as

$$\alpha = \sum m_{i\parallel} \alpha_{i\parallel} + \sum m_{i\perp} \left(\alpha_{i\perp} + \tau(\alpha_{i\perp})\right) \quad \text{for} \quad \alpha = \tau(\alpha),$$

the above lemma lead us to the relation:

$$\varepsilon_\alpha = \begin{cases} 
-1 & \text{if all } m_{i\parallel} = 0 \\
\prod_{i\parallel} \varepsilon_{\alpha_{i\parallel}} m_{i\parallel} & \text{otherwise}
\end{cases} \quad \text{for} \quad \alpha = \tau(\alpha).$$

This looks like the relation of the phases encountered in the inner-automorphism case. Actually, if we set $\varepsilon_{\alpha_{i\parallel}} = 1$ for all invariant simple roots as a representative, then each of the other choices is expressed as a product of this representative automorphism and an inner-automorphism with $\epsilon_{i\parallel} = \varepsilon_{\alpha_{i\parallel}}$ and $\epsilon_{i\perp} = 1$ which commutes with the representative. We show in Table 2 the list of symmetric subgroup thus obtained. For each $G$, the first $H$ is the representative. Note that the symmetric subgroup obtained by outer-automorphism is a maximal special subgroup [10].

Combining the Table 1 and 2 we have obtained all the symmetric subgroup $H$ of compact simple Lie group $G$. To make a distinction between inner- and outer-automorphism is important for the study of abelian T-duality transformation of these boundary states which is the subject of the next section.

## 4 Abelian T-duality transformations

An abelian T-duality transformation in the WZW model is nothing but an automorphism on the left and/or right Kac-Moody currents [12, 13]:

$$J_n^a \rightarrow L^{ab} J_n^b,$$
$$\tilde{J}_n^a \rightarrow R^{ab} \tilde{J}_n^b,$$
| $G$ | $H$    | $N_D$ |
|-----|-------|-------|
| $A_{2l}$ | $B_l$ | $2l^2 + 3l$ |
| $A_{2l-1}$ | $C_l$ | $2l^2 - l - 1$ |
|       | $D_l$ | $2l^2 + l - 1$ |
| $D_n$ | $B_{n-1}$ | $2n - 1$ |
|       | $B_{n-k-1} \otimes B_k$ | $2n - 1 + 4k(n - k - 1), \quad k = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$ |
| $E_6$ | $F_4$ | 26 |
| $C_4$ | 42 |

Table 2: List of symmetric subgroup $H$ obtained by outer-automorphism. $N_D$ is the number of Dirichlet directions.

where the matrices $L$ and $R$ are orthogonal matrices which satisfy the same relation of eq.(14) as $U$. For the super Kac-Moody case, they are supplemented by the transformation on the partners with the same matrices

\[
\tilde{j}_a^n \rightarrow L^{ab}_n \tilde{j}_b^n , \quad (43)
\]
\[
\tilde{j}^a_n \rightarrow R^{ab}_n \tilde{j}^b_n . \quad (44)
\]

Under these transformation, our general boundary condition is transformed to

\[
\left( L^{ab}_n j^b_n + U^{ab} R^{bc}_n \tilde{j}^c_{-n} \right) |B\rangle = 0 , 
\]
\[
\left( L^{ab}_{-n} j^b_{-n} \pm iU^{ab} R^{bc}_n \tilde{j}^c_{-n} \right) |B\rangle = 0 . 
\]

In use of an appropriate basis change, this is rewritten as

\[
\left( J^a_n + (URL^T)^{ab}_n \tilde{j}^b_{-n} \right) |B\rangle = 0 , 
\]
\[
\left( j^a_n \pm i(UR\tilde{L}^T)^{ab}_n \tilde{j}^b_{-n} \right) |B\rangle = 0 . 
\]

Hence the boundary condition of a matrix $U$ is transformed to that of $URL^T$. Especially, pure Neumann condition is transformed to the boundary condition of $RL^T$. If we require this to be one of our Dirichlet conditions, then the matrix $V = RL^T$ should satisfy $V^2 = 1$, which we assume in the following. The matrix $V$ satisfies all conditions that $U$ satisfies: orthogonality, eq.(14) and $V^2 = 1$. Therefore $V$ falls into our classification of $U$. 
Now we investigate how the boundary conditions of $U$ are related to each other by the transformation $V$. Since $(UV)^2 = 1$ is satisfied if and only if $[U, V] = 0$, $U$ should be diagonal in the basis in which $V$ is diagonal. Let us fix the basis in this way. Obviously, there are two classes, one with and without outer-automorphism. We shall call them inner class and outer class respectively.

Firstly we consider the inner class, which we denote $I_G$. This is a whole set of possible $U$ via inner-automorphism for a given $G$, i.e. the listed in Table 1 plus identity, all diagonalized in the same basis. Each element in $I_G$ is uniquely labeled by $\{\epsilon_i\}$ and a product of two elements are defined by

$$U_{\{\epsilon_i^{(1)}\}} U_{\{\epsilon_i^{(2)}\}} = U_{\{\epsilon_i^{(1)} \epsilon_i^{(2)}\}}.$$  

The corresponding symmetric subgroup is identified by $N_D$ using the formula (34). Hence, if a transformation $V$ belongs to $I_G$, then a boundary condition $U \in I_G$ is transformed to $UV \in I_G$ according to the product rule (49).

Next we turn to the outer class, which we denote $O_G$. This is a whole set of possible $U$ via outer-automorphism for a given $G$ (Table 2) and all $U \in I_G$ which commute with the matrices of outer-automorphism. The latter forms a closed subset (we denote it $\hat{I}_G$) whose element is uniquely labeled by $\{\varepsilon_{\alpha i}\}$. The product rule in $\hat{I}_G$ is similar to the rule (49):

$$U_{\{\varepsilon_{\alpha i}^{(1)}\}} U_{\{\varepsilon_{\alpha i}^{(2)}\}} = U_{\{\varepsilon_{\alpha i}^{(1)} \varepsilon_{\alpha i}^{(2)}\}}.$$  

If a matrix $\hat{U}$ corresponds to the representative of outer-automorphism, then $O_G$ consists of $U_{\{\varepsilon_{\alpha i}\}}$ and $\hat{U}_{\{\varepsilon_{\alpha i}\}} = \hat{U} U_{\{\varepsilon_{\alpha i}\}}$. The product rule in $O_G$ is given by the eq.(50) combined with

$$\hat{U}_{\{\varepsilon_{\alpha i}^{(1)}\}} U_{\{\varepsilon_{\alpha i}^{(2)}\}} = \hat{U}_{\{\varepsilon_{\alpha i}^{(1)} \varepsilon_{\alpha i}^{(2)}\}},$$  

$$U_{\{\varepsilon_{\alpha i}^{(1)}\}} \hat{U}_{\{\varepsilon_{\alpha i}^{(2)}\}} = U_{\{\varepsilon_{\alpha i}^{(1)} \varepsilon_{\alpha i}^{(2)}\}}.$$  

As is for the inner class, a boundary condition $U \in O_G$ is transformed to $UV \in O_G$ by a transformation $V \in O_G$ according to the product rule (51)~(52).

Thus we have shown mutual relationship of the general boundary conditions under the abelian T-duality of the WZW model which satisfies $(RLT)^2 = 1$. If we loosen this condition, we have to include boundary conditions with $U$ such that $U^2 \neq 1$. That means we need to argue the boundary coupled to the open string background, which is out of scope of the present paper.
5 Discussions

We have shown that the general Dirichlet boundary states of the WZW model can be constructed by the $\mathbb{Z}_2$ automorphism of the group so that the D-brane world-volume is the group manifold of the symmetric subgroup. The coordinates in the directions perpendicular to the D-brane world-volume corresponds to the moduli of the D-brane. Therefore the moduli space of the D-brane on group manifold is a symmetric space $G/H$ given by the target group manifold $G$ divided by the world-volume $H$. Indeed Table 1 and 2 correspond essentially to the Cartan’s classification of the irreducible Riemannian symmetric spaces [14].

It is known that, when the symmetric subgroup $H$ contains $U(1)$ or $SO(2)$ factor, the resulting $G/H$ becomes hermitian symmetric space. We do have such cases as is seen from Table 1. In this case, we can construct the $N=2$ superconformal generators from the defining super Kac-Moody currents [15]. Since they are related to the geometrical information of the D-brane moduli space, some interesting implications for the D-brane dynamics may be expected, though we have not fully clarified yet.

Related to this, we should also note that the open string world-sheet instantons are allowed for the above case. Let us consider a disk, for instance. Its boundary, which is $S^1$, should be attached to the D-brane world-volume $H$. Then the non-trivial homotopy $\pi_1(H) = \pi_1(U(1)) = \mathbb{Z}$ implies the existence of the world-sheet instantons, which is smooth everywhere on the disk if $\pi_1(G) = 0$. For simple $G$, instantons and anti-instantons contribute to the disk amplitude with equal weight.

We have not dealt with twisted sectors in the present paper. As we mentioned in the introduction, our argument can be directly applicable to the case where the left and right sectors have same twist. Because in that case the left and right currents have same moding and our automorphism is independent from moding. When the left and right sectors have different twists, boundary state itself carries “twist”. The boundary condition for this case can be written similarly as the same-twist case, if we expressed it in terms of currents themselves, not of the modes of them. However, the boundary state has to be constructed like a spin-field for the string on orbifold.

The final comment goes to the case of non-compact groups. The theory of symmetric space tells us that each pair $(G, H)$ accompanies a non-compact sibling $(G^*, H)$ in such a way that $H$ is a maximal compact subgroup of a non-compact simple group $G^*$. Our argument

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\[6\] This is also called “duality” [18].
can be straightforwardly extended to the non-compact group $G^*$ of this type. D-brane world-volume remains compact subgroup manifold for the pair $(G^*, H)$, while T-duality transforms it to a non-compact one in general.

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**Appendix A**

Here we collect some technical details concerning the quantity $T_G(\{\epsilon_i\})$.

Let us consider the simplest case of $A_n$. We number each simple root in the Dynkin diagram of $A_n$ orderly, say from left to right. The important properties of $T_{A_n}(\{\epsilon_i\})$ are

$$T_{A_n}(\{1, \cdots, 1\}) = \frac{1}{2} n(n + 1),$$

$$T_{A_n}(\{(-1)^{\delta_{i,k}} \epsilon_i\}) = -T_{A_n}(\{\epsilon_i\}) + 2T_{A_{n-1}}(\{\epsilon_{i=1,\cdots,k-1}\}) + 2T_{A_{n-k}}(\{\epsilon_{i=k+1,\cdots,n}\}).$$

These are enough to determine the value $T_{A_n}(\{\epsilon_i\})$. The following corollaries are also useful:

$$T_{A_n}(\{\epsilon_i\}) = T_{A_n}(\{\epsilon_{n+1-i}\}),$$

$$T_{A_n}(\{(-1)^{\delta_{i,k}} \epsilon_i\}) = T_{A_n}(\{(-1)^{\delta_{i,n+1-k}} \epsilon_i\}) = \frac{1}{2} n(n + 1) - 2k(n + 1 - k),$$

$$T_{A_n}(\{(-1)^{\delta_{i,k}+\delta_{i,l}} \epsilon_i\}) = T_{A_n}(\{(-1)^{\delta_{i,\min(k,n+1-k)+\min(l,n+1-l)}} \epsilon_i\}).$$

For the other groups, each $T_G$ can be given in terms of $T_{A_n}$'s:

$$T_{B_n}(\{\epsilon_i\}) = T_{A_n}(\{\epsilon_i\}) + T_{A_{n-1}}(\{\epsilon_{i=1,\cdots,n-1}\}),$$

$$T_{C_n}(\{\epsilon_i\}) = T_{A_n}(\{\epsilon_i\}) + \epsilon_n \left( T_{A_{n-2}}(\{\epsilon_{i=1,\cdots,n-2}\}) + n - 1 \right),$$

$$T_{D_n}(\{\epsilon_i\}) = (1 + \epsilon_{n-1} \epsilon_n) T_{A_{n-1}}(\{\epsilon_{i=1,\cdots,n-1}\}),$$

$$T_{G_2}(\{\epsilon_i\}) = 2T_{A_2}(\{\epsilon_1\}),$$

$$T_{F_4}(\{\epsilon_i\}) = (T_{A_2}(\{\epsilon_{i=1,2}\}) + 1) \left( T_{A_2}(\{\epsilon_{i=3,4}\}) + 4 \right) - 4,$$

$$T_{E_6}(\{\epsilon_i\}) = (1 + \epsilon_6) T_{A_5}(\{\epsilon_{i=1,\cdots,5}\}) + \epsilon_6 \left( T_{A_2}(\{\epsilon_{i=1,2}\}) - 1 \right) \left( T_{A_2}(\{\epsilon_{i=4,5}\}) - 1 \right) + \epsilon_1 \epsilon_3 \epsilon_5 (1 + \epsilon_6),$$

$$T_{E_7}(\{\epsilon_i\}) = (1 + \epsilon_7) \left[ T_{A_6}(\{\epsilon_{i=1,\cdots,6}\}) + \epsilon_1 \epsilon_3 \left( T_{A_2}(\{\epsilon_{i=5,6}\}) + \epsilon_4 \epsilon_6 \right) \right]$$
\[ T_{E\delta}(\{\epsilon_i\}) = (1 + \epsilon_8)T_{\Delta\tau}(\{\epsilon_i=1,\ldots,7\}) + \epsilon_8 (T_{\Delta\tau}(\{\epsilon_i=1,\ldots,7\}) - 1) (T_{\Delta\alpha}(\{\epsilon_i=4,\ldots,6\}) - 1) + \epsilon_1 \epsilon_3 \epsilon_5 (1 + \epsilon_8) + [\epsilon_1 \epsilon_3 \epsilon_5 (1 + \epsilon_8)] (1 + \epsilon_4 + \epsilon_5) + \epsilon_4 \epsilon_6 T_{\Delta\alpha}(\{\epsilon_i=1,\ldots,7\}) (1 + \epsilon_7) + \epsilon_7 [\epsilon_3 + \epsilon_3 \epsilon_4 + \epsilon_3 \epsilon_4 \epsilon_5] \epsilon_1 (1 + \epsilon_8) + T_{\Delta\alpha}(\{\epsilon_i=1,\ldots,7\}) T_{\Delta\tau}(\{\epsilon_i=4,\ldots,6\}) + \epsilon_2 \epsilon_3 \epsilon_5 (1 + \epsilon_1) (1 + \epsilon_8) + \epsilon_3 \epsilon_5 (1 + \epsilon_8) + T_{\Delta\alpha}(\{\epsilon_i=4,\ldots,6\}) \epsilon_8 + \epsilon_4 \epsilon_6 \epsilon_8 + \epsilon_4 \epsilon_6 \epsilon_8 , \]

where we have used the same numbering convention of simple roots as ref. [10].

In general, different sets of \( \{\epsilon_i\} \) may give a same value of \( T_G \). In this case corresponding subgroups are same as a group, although the directions are different. To get all independent values (i.e., all different \( H \)) of \( T_G \) it is sufficient to examine \( \epsilon_i = (-1)^{\delta_{ik}} \), even for which degeneracy still exists.

**Appendix B**

Here we prove Lemma 1.

1. It immediately follows from \( \tau(\tau(E^\alpha)) = E^\alpha \).

2. For \( \alpha, \beta, \alpha + \beta \in \Delta_+ \), the relation \([E^\alpha, E^\beta] = N_{\alpha,\beta} E^{\alpha+\beta} \) is transformed by \( \tau \) to

\[
[\varepsilon_\alpha E^{\tau(\alpha)}, \varepsilon_\beta E^{\tau(\beta)}] = N_{\alpha,\beta} \varepsilon_\alpha \varepsilon_\beta E^{\tau(\alpha)+\tau(\beta)}
\]

which should be compared with \([E^{\tau(\alpha)}, E^{\tau(\beta)}] = N_{\tau(\alpha),\tau(\beta)} E^{\tau(\alpha)+\tau(\beta)} \). Then we have

\[
\varepsilon_{\alpha+\beta} = \varepsilon_\alpha \varepsilon_\beta \frac{N_{\tau(\alpha),\tau(\beta)}}{N_{\alpha,\beta}} .
\]

By taking \( \beta = \tau(\alpha) \) the relation \( N_{\alpha,\beta} = -N_{\beta,\alpha} \) leads us to the desired result.

3. Jacobi identity for \( E^\alpha, E^\beta \) and \( E^{\tau(\alpha)} \) leads to \( N_{\alpha,\beta} N_{\alpha+\beta,\tau(\alpha)} + N_{\beta,\tau(\alpha)} N_{\beta+\tau(\alpha),\alpha} = 0 \), then

\[
\frac{N_{\tau(\alpha),\tau(\beta)}}{N_{\alpha,\beta}} \frac{N_{\tau(\alpha+\beta),\tau(\alpha)}}{N_{\alpha+\beta,\tau(\alpha)}} = 1 .
\] Using the relation (67) this gives the desired result.
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