Leray–Hopf solutions to a viscoelastoplastic fluid model with nonsmooth stress-strain relation

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We consider a fluid model including viscoelastic and viscoplastic effects. The state is given by the fluid velocity and an internal stress tensor that is transported along the flow with the Zaremba–Jaumann derivative. Moreover, the stress tensor obeys a nonlinear and nonsmooth dissipation law as well as stress diffusion. We prove the existence of global-in-time weak solutions satisfying an energy inequality under general Dirichlet conditions for the velocity field and Neumann conditions for the stress tensor.

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1 Introduction

In this article we investigate the equations of motion that describe the flow of a viscoelastoplastic fluid with stress diffusion modeled in the following way. On a time interval \((0, T)\) and a bounded domain \(\Omega \subset \mathbb{R}^3\), we consider the system of equations

\[
\begin{align*}
\rho \partial_t V - \text{div} \left( \eta_1 S + 2\mu D(V) - PI \right) &= F & \text{in } \Omega \times (0, T), \\
\text{div} V &= 0 & \text{in } \Omega \times (0, T), \\
\dot{S} + \partial P(S) - \gamma \Delta S &\ni \eta_2 D(V) & \text{in } \Omega \times (0, T).
\end{align*}
\]

(1.1)

Here the first two equations describe the flow of an incompressible fluid with Eulerian velocity field \(V: \Omega \times (0, T) \to \mathbb{R}^3\) and pressure field \(P: \Omega \times (0, T) \to \mathbb{R}\) affected by a prescribed external force \(F: \Omega \times (0, T) \to \mathbb{R}^3\). The relevant Cauchy stress tensor

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\[ T = \eta_1 S + 2\mu D(V) - PI \] consists of the classical term \( 2\mu D(V) - PI \) for Newtonian fluids and an extra stress tensor

\[ S : \Omega \times (0, T) \to \mathbb{R}^{3 \times 3} := \{ M \in \mathbb{R}^{3 \times 3} \mid M = M^T, \ Tr M = 0 \}, \]

which satisfies the additional evolution equation (1.1) and is thus subject to a special transport encoded in \( S \) along the velocity field \( V \), a nonlinear dissipation law via \( \partial \mathcal{P}(S) \), and another diffusion process. Here \( \rho, \eta_1, \eta_2, \mu \) and \( \gamma \) denote positive constants, and \( D(V) := \frac{1}{2}(\nabla V + \nabla V^\top) \) denotes the symmetric rate-of-strain tensor. Following [MDM02, GeY07, HGv17, PH\textsuperscript{*}19], we choose \( S \in \mathbb{R}^{3 \times 3} \) to be the deviatoric stress tensor (i.e. \( \text{Tr } S = 0 \)), which corresponds to the incompressibility of the fluid encoded in (1.1), such that the pressure \( P \) contains the full spherical part of the Cauchy stress tensor. For other modeling choices we refer to Section 3.4.

System (1.1) is complemented by boundary and initial conditions. The former are given by

\[ V \cdot n = 0, \quad V - (V \cdot n)n = g, \quad n \cdot \nabla S = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.2) \]

which means that there is no boundary flux, that the tangential part of the fluid velocity at the boundary coincides with some prescribed function \( g : \partial \Omega \times (0, T) \to \mathbb{R}^3 \) and that \( S \) has vanishing normal derivative. The first two conditions can also be summarized as \( V = g \) on \( \partial \Omega \times (0, T) \) for some \( g \) with \( g \cdot n = 0 \). Note that, from a physical point of view, only the case \( g \cdot n = 0 \) seems to fit to the Neumann boundary condition \( n \cdot \nabla S = 0 \). If we would allow for \( g \cdot n \neq 0 \), more complicated boundary conditions for \( S \) would be needed. The initial conditions are

\[ V(\cdot, 0) = V_0, \quad S(\cdot, 0) = S_0 \quad \text{in } \Omega. \quad (1.3) \]

For a fluid velocity \( V \), the material derivative is given by

\[ D_t A := \partial_t A + V \cdot \nabla A, \]

and as an objective derivative of a tensor \( S \) we use the Zaremba–Jaumann derivative

\[ \overset{\overset{\circ}{\cdot}}{S} := D_t S + SW(V) - W(V) S = \partial_t S + V \cdot \nabla S + SW(V) - W(V) S, \]

also called co-rotational derivative, where \( W(V) := \frac{1}{2}(\nabla V - \nabla V^\top) \). Note that this choice of the objective derivative is not canonical and there are different ways to define objective derivatives for tensors. However, the choice made here is commonly used in geodynamics (cf. [MDM02, GeY07, HGv17, PH\textsuperscript{*}19]) and comes along with special features that are very useful for the mathematical analysis, see below.

The mathematical study of viscoelastic fluids with different choices of the objective derivatives (including the upper and lower convected Maxwell derivatives) started in the middle 1980s, see e.g. [JRS85, ReR86, RHN87, CoS91, Ren00]. Because of the strong nonlinearities arising from the objective derivatives, a first global existence result was only established years later in [LiM00] based on the Zaremba–Jaumann derivative and a linear dissipation law \( \partial \mathcal{P}(S) = aS \) with \( a > 0 \). More recently, the more difficult case of a Maxwell fluid with \( \mu = 0 \) (and without stress diffusion, i.e. \( \gamma = 0 \)) has also been considered, see [CL\textsuperscript{*}19] and references therein.

For more general nonlinear situations there is a series of works involving implicitly defined stress-strain relations of the type \( G(S, D(V)) = 0 \), see [BG\textsuperscript{*}12] and the references
in the recent survey [BMR20]. Viscoelastic fluids have a constitutive relation of rate-type, i.e. they involve suitable convective derivatives of the strain tensor $D(V)$ or of the stress tensor, as in our equation (1.1)$_3$. The treatment of such nonlinearities is possible by using the recently introduced regularization of stress diffusion, i.e. $\gamma > 0$, as first illustrated in [BM*18] for a simplified model replacing the tensor evolution by a scalar problem. We refer to [MP*18] for a careful thermodynamical modeling of such viscoelastic fluids and to [BBM21], where a large data global existence result for weak solutions was obtained for a one-parameter family of convected tensor derivatives including the (simpler) case of the Zaremba–Jaumann rate.

Our work is in a similar spirit as the latter one, but it generalizes the conventional linear or quadratic stress-strain relation by allowing in (1.1)$_3$ for subdifferentials

$$S \mapsto \partial \mathcal{P}(S) = \left\{ A \in L^2(\Omega; \mathbb{R}^{3 \times 3}) \mid \mathcal{P}(\tilde{S}) \geq \mathcal{P}(S) + \int_\Omega A : (\tilde{S} - S) \, dx \right\}$$

of a general dissipation potential $\mathcal{P} : L^2(\Omega; \mathbb{R}^{3 \times 3}) \to [0, \infty]$ that is convex, lower semicontinuous and satisfies $\mathcal{P}(0) = 0$. The space $L^2(\Omega; \mathbb{R}^{3 \times 3})$ is a natural choice in virtue of the formal energy-dissipation balance (1.6) below. While smooth stress-strain relations of polynomial type seem to be sufficient for the modeling of polymeric fluids (cf. [Ren00, BM*18, MP*18, BBM21]), such nonsmooth dissipation potentials are important for viscoelastoplastic fluid models that are used in geodynamics for the deformation of rocks in lithospheric plates, namely

$$\mathcal{P}(S) = \int_\Omega \varphi(S(x)) \, dx \quad \text{with} \quad \varphi(S) = \begin{cases} \frac{\sigma}{2} |S|^2 & \text{for} \ |S| \leq \sigma_{\text{yield}}, \\
\infty & \text{for} \ |S| > \sigma_{\text{yield}}, \end{cases} \quad (1.4)$$

where the yield stress $\sigma_{\text{yield}} > 0$ determines the onset of plastic flow behavior, see [MDM02, GeY07] and Section 3.4. Observe that the potential defined in (1.4) is indeed convex, lower semicontinuous in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ and satisfies $\mathcal{P}(0) = 0$.

In the context of geodynamics, it is also crucial to allow for nontrivial boundary data $g \neq 0$ in (1.2), because often the prescribed drifts of tectonic plates act as boundary data for the specific region of interest.

The basic features of the model include, of course, all the difficulties of the three-dimensional Navier–Stokes equations such that we cannot expect better solutions than Leray–Hopf solutions for the velocity component $V$. For $\gamma > 0$, the equation (1.1)$_3$ for the stress tensor $S$ is a (semilinear) parabolic equation with linear source term $D(V)$, but, crucially, also coupled nonlinearly to $V$ via the Zaremba–Jaumann derivative $\nabla S$.

In our analysis we essentially exploit the fact that for sufficiently smooth functions $V$ and $S$ satisfying $n \cdot V = 0$ on $\partial \Omega$ we have the identity

$$\frac{d}{dt} \int_\Omega \frac{1}{2} |S(t, x)|^2 \, dx = - \int_\Omega \partial_t S : S \, dx = \int_\Omega D_t S : S \, dx = \int_\Omega \nabla S : S \, dx. \quad (1.5)$$

Exploiting this identity and assuming for the moment that $V = 0$ on $\partial \Omega$, one can show
that smooth solutions satisfy the energy-dissipation balance

$$
\int_\Omega \left( \frac{\rho}{2} |V(t)|^2 + \frac{\eta_1}{2\eta_2} |S(t)|^2 \right) \, dx \\
+ \int_0^t \int_\Omega \left( 2\mu |D(V)|^2 + \frac{\eta_1}{\eta_2} S : \partial \mathcal{P}(S) + \frac{\eta_2}{\eta_1} |\nabla S|^2 \right) \, dx \, d\tau
\right)
$$

(1.6)

for all $t \in (0, T)$. For the more general case with non-trivial boundary data, we refer to (3.11). We clearly see how the quadratic energy consisting of the kinetic energy and an elastic energy associated with $S$ can be changed by the external force $F$ and is dissipated by three mechanisms: (i) a direct fluid viscosity given by $\mu > 0$, (ii) a stress dissipation encoded in the dissipation potential $\mathcal{P}$, and (iii) the stress diffusion associated with $\gamma > 0$.

To simplify the notation we will fix two constants and choose $\rho = 1$ and $\eta_1 = \eta_2 = \eta$ subsequently. With this choice the quadratic energy is simply given as one half of the $L^2$ norm of $(V, S)$. We should note that this choice of parameters rules out the (non-trivial) case $\eta_1 \neq 0$ and $\eta_2 = 0$, in which the system is still fully coupled. In this case, we need to replace the weight $\frac{\eta_1}{2\eta_2}$ in the energy density appearing in (1.6) by some positive constant, leading to an extra term of the form $-\eta_1 \int_0^t \int_\Omega S : \nabla V \, dx \, d\tau$ on the right-hand side of (1.6). However, as will become clear from the proofs, our existence result can be extended to this case.

**Brief description of the main results** In this article, we perform a large-data global-in-time existence analysis for system (1.1)–(1.3) that is valid for the general class of convex potentials $\mathcal{P}$ specified above. See Theorem 3.4 for the main result. A key novelty with respect to existing literature is the ability to deal with $\mathcal{P}$ nonsmooth, in which case the differential inclusion (1.1) cannot be replaced by an equation. Determining an appropriate notion of solution that allows for a reasonable existence theory is part of our results (see Sec. 3.1). The inhomogeneous time-dependent boundary data for the velocity field introduce additional technicalities. Here, we adapt a construction for the incompressible Navier–Stokes equations that allows us to deduce a global-in-time energy control provided that, in some averaged sense, the data decay to zero as $t \to \infty$, see Corollary 3.5. The proof of the existence result for nonsmooth potentials relies on approximation by a suitable family of $C^{1,1}$-smooth potentials with the property that $S \mapsto \partial \mathcal{P}(S)$ is monotone and globally Lipschitz continuous. As long as the potential $\mathcal{P}$ is smooth, somewhat better results can be obtained, see Theorem 3.7.

**Outline** In Section 2 we specify the notations used in the subsequent analysis. The statements of our main results including core definitions and hypotheses can be found in Section 3, which also provides some details on the strategy of the proofs and some further remarks on the modeling. The proof of Theorem 3.7 concerning smooth potentials is provided in Section 4. Section 5 contains the proof of our results concerning nonsmooth potentials, including the proof of our main existence result.
2 Notations

Here, we specify general notations, definitions and conventions required for the subsequent analysis.

**General notations** For two vectors \(a, b \in \mathbb{R}^3\) we denote their inner product by \(a \cdot b = a_j b_j\) and their tensor product by \(a \otimes b\) with \((a \otimes b)_{jk} = a_j b_k\). Here and in the following we use Einstein’s summation convention and implicitly sum over repeated indices from 1 to 3. The inner product of two tensors \(A, B \in \mathbb{R}^{3 \times 3}\) is denoted by \(A : B = A_{jk} B_{jk}\). Moreover, \(A^\top\) and \(\text{Tr} A\) denote the transpose and the trace of \(A\). We further set \(a \otimes b : A = (a \otimes b) : A = a_j A_{jk} b_k\) and, if \(C \in \mathbb{R}^{3 \times 3}\) is a third tensor, \(AB : C = (AB) : C = A_{jk} B_{k\ell} C_{\ell j}\).

Usually, \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain and \(T \in (0, \infty]\). Points \((x, t)\) in the space-time cylinder \(\Omega \times (0, T)\), consist of a spatial variable \(x \in \Omega\) and a time variable \(t \in (0, T)\). For a sufficiently regular function \(u\), we denote its partial derivatives in time and space by \(\partial_t u\) and \(\partial_j u\), \(j = 1, 2, 3\), respectively. The symbols \(\nabla\) and \(\Delta\) denote (spatial) gradient and Laplace operator. If \(v\) is a vector-valued function, we let \(\text{div} v = \partial_j v_j\) denote its divergence and set \(v \cdot \nabla u = \partial_j v_j \partial_j u\). Symmetric and antisymmetric parts of \(v \cdot \nabla = (\partial_j v_j)\) are given by

\[
D(v) := \frac{1}{2}(\nabla v + \nabla v^\top), \quad W(v) := \frac{1}{2}(\nabla v - \nabla v^\top),
\]

respectively. If \(S\) is a tensor-valued function, its divergence \(\text{div} S\) is given by \((\text{div} S)_j = \partial_k S_{jk}\). If \(T\) is another tensor-valued function, we define \(\nabla T : \nabla S = \partial_k T_{jk} \partial_k S_{jk}\) and \(v \cdot \nabla T : S = v_j (\partial_j T_{k\ell}) S_{k\ell}\).

**Function spaces** Let \(k \in \mathbb{N}_0 \cup \{\infty\}\) and \(A \in \{\Omega, \overline{\Omega}\}\). Then the class \(C^k(A)\) consists of all \(k\)-times continuously differentiable (real-valued) functions on \(A\), and \(C^0_0(A)\) contains all compactly supported functions in \(C^k(A)\). By \(L^q(\Omega)\) with \(q \in [1, \infty]\) we denote the classical Lebesgue spaces with corresponding norm \(\|\cdot\|_q\), and \(H^k(\Omega)\) with \(k \in \mathbb{N}\) denotes the \(L^2\)-based Sobolev space of order \(k\), equipped with the norm \(\|\cdot\|_{k, 2}\). Moreover, \(H^0_0(\Omega)\) contains all elements of \(H^1(\Omega)\) with vanishing boundary trace, and \(H^{1/2}(\partial \Omega)\) denotes the class of boundary traces of functions from \(H^1(\Omega)\). By \(H^{-1}(\Omega)\) and \(H^{-1/2}(\partial \Omega)\) we denote the dual spaces of \(H^0_0(\Omega)\) and \(H^{1/2}(\partial \Omega)\), respectively, where we use the distributional duality pairing.

The norm of a Banach space \(X\) is denoted by \(\|\cdot\|_X\), and the same symbol is used for the norms of \(X^3\) and \(X^{3 \times 3}\). When the dimension is clear from the context, we simply write \(X\) instead of \(X^3\) or \(X^{3 \times 3}\). Moreover, \(X'\) denotes the dual space of \(X\), and \(C^{1, 1}(X)\) is the set of all continuously Fréchet differentiable functions \(X \to \mathbb{R}\) with globally Lipschitz continuous derivative.

For an interval \(I \subset \mathbb{R}\), the class \(C^0(I; X)\) consists of all continuous \(X\)-valued functions, and \(C_w(I; X)\) consists of all weakly continuous \(X\)-valued functions. The Bochner–Lebesgue spaces of \(X\)-valued functions are denoted by \(L^q(I; X)\) for \(q \in [1, \infty]\), and \(L^q_{loc}(I; X)\) denotes the class of all functions that belong to \(L^q(J; X)\) for all compact subintervals \(J \subset I\). When \(I = (0, T)\), we set \(C^0(0, T; X) = C^0(I; X)\) and \(L^q(0, T; X) = L^q(I; X)\). For functions \(A\) on \(\Omega \times I\) we use the shorthand \(A(t) := A(\cdot, t)\).

We further need spaces of solenoidal vector fields and of symmetric deviatoric tensor fields. The corresponding classes of smooth functions on \(\Omega\) are given by

\[
C^\infty_0(\Omega) := \{\varphi \in C^\infty_0(\Omega)^3 \mid \text{div} \, \varphi = 0\},
\]

\[
C^\infty_0(\Omega) := \{\psi \in C^\infty(\Omega)^{3 \times 3} \mid \psi = \psi^\top, \text{Tr} \, \psi = 0\}.
\]
We further set
\[
C^\infty_{0,\sigma}(\Omega \times I) := \{ \Phi \in C^\infty_0(\Omega \times I)^3 \mid \text{div} \, \Phi = 0 \},
\]
\[
C^\infty_{0,\delta}(\Omega \times I) := \{ \Psi \in C^\infty_0(\overline{\Omega} \times I)^{3 \times 3} \mid \Psi = \Psi^T, \, \text{Tr} \, \Psi = 0 \},
\]
and we set \( C^\infty_{\delta}(\Omega \times I) := C^\infty_{0,\delta}(\overline{\Omega} \times I) \) if \( I \) is a compact interval. We define the associated \( L^2 \) spaces on \( \Omega \) by
\[
L^2_\sigma(\Omega) := \{ v \in L^2(\Omega)^3 \mid \text{div} \, v = 0, \, v|_{\partial \Omega} \cdot n = 0 \} = C^\infty_{0,\sigma}(\Omega)^{\|\cdot\|_2},
\]
\[
L^2_\delta(\Omega) := \{ S \in L^2(\Omega)^{3 \times 3} \mid S = S^T, \, \text{Tr} \, S = 0 \} = C^\infty_{0,\delta}(\Omega)^{\|\cdot\|_2}.
\]

Here the conditions \( \text{div} \, v = 0 \) and \( v|_{\partial \Omega} \cdot n = 0 \) in the definition of \( L^2_\sigma(\Omega) \) have to be understood in a weak sense; see [Gal11, Theorem III.2.3] for example. We further introduce the corresponding Sobolev spaces
\[
H^{1,\delta}(\Omega) := \{ v \in H^1_0(\Omega)^3 \mid \text{div} \, v = 0 \} = \overline{C^\infty_{0,\delta}(\Omega)}^{\|\cdot\|_{H^1_0(\Omega)}}.
\]

We can now define the solution spaces
\[
LH_T := L^\infty(0,T;L^2_\delta(\Omega)) \cap L^2(0,T;H^1(\Omega)^3)
\]
for the fluid velocity and
\[
X_T := L^\infty(0,T;L^2_\delta(\Omega)) \cap L^2(0,T;H^1(\Omega)^{3 \times 3})
\]
for the stress tensor. Observe that \( LH_T \) is the classical Leray–Hopf class for weak solutions to the Navier–Stokes equations, and \( X_T \) is the analog for semilinear parabolic equations taking values in deviatoric tensor fields. Moreover, for the introduction of the variational formulation of (1.1) below, the space
\[
Z_T := H^1(0,T;L^2_\delta(\Omega)) \cap L^2(0,T;H^1(\Omega)^{3 \times 3}) \cap L^5(0,T;L^5(\Omega)^{3 \times 3})
\]
will serve as the class of test functions.

**Convex subdifferential** Let \( \mathcal{P} : L^2_\delta(\Omega) \to [0,\infty) \) be convex with \( \mathcal{P}(0) = 0 \). We denote by \( \partial \mathcal{P} \) the convex subdifferential of \( \mathcal{P} \), i.e., for \( S \in L^2_\delta(\Omega) \) we let
\[
\partial \mathcal{P}(S) := \{ \tau \in L^2_\delta(\Omega) \mid (\tau,\tilde{S} - S)_{L^2_\delta(\Omega)} + \mathcal{P}(S) \leq \mathcal{P}(\tilde{S}) \quad \text{for all} \quad \tilde{S} \in L^2_\delta(\Omega) \}.
\]

Observe that, by definition, \( \partial \mathcal{P}(S) = \emptyset \) if \( \mathcal{P}(S) = +\infty \). If \( \partial \mathcal{P}(S) = \{ \tau \} \) for some \( \tau \in L^2_\delta(\Omega) \), we identify the set \( \partial \mathcal{P}(S) \) with its unique element \( \tau \). In this case, we call \( \tau \) the (Gâteaux) differential of \( \mathcal{P} \) at \( S \).

### 3 Main results

The main contribution of our analysis to the large-data existence theory for rate-type viscoelastoplastic fluid models lies in its ability to deal with *nonsmooth* dissipation potentials \( \mathcal{P} \) under the mild hypothesis that
\[
\mathcal{P} : L^2_\delta(\Omega) \to [0,\infty] \text{ is convex and lower semicontinuous with } \mathcal{P}(0) = 0.
\]
Throughout this paper, we let \( T \in (0, \infty) \) and impose the following conditions on the initial data and external forcing
\[
V_0 \in L^2_0(\Omega), \quad S_0 \in L^2_0(\Omega), \quad F = F_0 + \text{div} \, F_1, \\
F_0 \in L^1_{\text{loc}}([0, T); L^2(\Omega)^3), \quad F_1 \in L^2_{\text{loc}}([0, T); L^2(\Omega)^3 \times 3).
\] (3.2)
In our main results we focus on boundary data \( g \) that satisfy
\[
g \in L^\infty(0, T; H^{1/2}(\partial \Omega)^3), \quad \partial_t g \in L^\infty(0, T; H^{-1/2}(\partial \Omega)^3)
\] (3.3)
and \( g \cdot n = 0 \) on \( \partial \Omega \times (0, T) \).

Section 3.4 will be devoted to the motivation for allowing for nonsmooth and set-valued stress-strain relations \( S \mapsto \partial\mathcal{P}(S) \) and non-trivial boundary data \( g \) by discussing applications in the geodynamics of lithospheric plate motion.

### 3.1 Generalized solution concept

For nonsmooth potentials \( \mathcal{P} \) the subdifferential \( \partial\mathcal{P} \) may be multi-valued, and hence, line \((1.1)_3\) cannot be replaced with an equality but rather has to be understood as a suitable inclusion. Our notion of solution for problem \((1.1)_3\) adapts the weak solution concept in [Rou13, Chapter 10] for semilinear parabolic equations with nonsmooth potentials, which involves an evolutionary variational inequality that avoids the multi-valued function \( \partial\mathcal{P}(S) \) by exploiting the inequality
\[
\int_\Omega \partial\mathcal{P}(S) : (\tilde{S} - S) \, dx \leq \mathcal{P}(\tilde{S}) - \mathcal{P}(S)
\] (3.4)
for suitably regular test function \( \tilde{S} \). (If \( \partial\mathcal{P}(S) \) is multi-valued, the left-hand side of the last inequality should be understood elementwise, replacing \( \partial\mathcal{P}(S) \) by \( \beta \in \partial\mathcal{P}(S) \).)

In comparison to [Rou13, Chapter 10], the analysis of the present problem is, however, greatly complicated by the presence of the geometric nonlinearities in \((1.1)_3\) (coming from the objective derivative \( \tilde{S} \)) as well as the coupling term \( \eta D(V) \).

To motivate the evolutionary variational inequality we are going to propose for \((1.1)_3\), let us assume for the moment smoothness of the functions involved and suppose that \((1.1)_3\) holds as an equality. Multiplying this equation by \( (\tilde{S} - S) \) and integrating over \( \Omega \) then allows us to use ineq. (3.4) and avoid \( \partial\mathcal{P}(S) \) at the expense of an inequality. The integral involving the Zaremba–Jaumann derivative can be modified via the identity (1.5),
\[
\int_\Omega \tilde{S} : (\tilde{S} - S) \, dx = \int_\Omega \left( \partial_t S : (\tilde{S} - S) + (\tilde{S} - \partial_t S) : \tilde{S} \right) \, dx \\
= \int_\Omega \left( \partial_t \tilde{S} : (\tilde{S} - S) + (\tilde{S} - \partial_t S) : \tilde{S} \right) \, dx - \frac{d}{dt} \int_\Omega \frac{1}{2} |\tilde{S} - S|^2 \, dx,
\] (3.5)
where \( \tilde{S} - \partial_t S = V \cdot \nabla S + SW - WS \) and where we have used the fact that \( \int_\Omega (\tilde{S} - \partial_t S) : S \, dx = 0 \) for \( (V, S) \) is sufficiently regular. Upon integration in time from \( t = 0 \) to \( t = T' < T \), we then arrive at
\[
\int_0^{T'} \int_\Omega \partial_t \tilde{S} : (\tilde{S} - S) + \gamma \nabla S : \nabla (\tilde{S} - S) \, dx \, dt + \int_0^{T'} \left( \mathcal{P}(\tilde{S}) - \mathcal{P}(S) \right) \, dt \\
+ \int_0^{T'} \int_\Omega V \cdot \nabla S : \tilde{S} \left( SW(V) - W(V)S \right) : \tilde{S} - \eta D(V) : (\tilde{S} - S) \, dx \, dt \\
\geq \frac{1}{2} \|\tilde{S}(T') - S(T')\|^2_2 - \frac{1}{2} \|\tilde{S}(0) - S_0\|^2_2.
\]
To avoid issues when passing to the limit along approximate solutions, we follow [Rou13, Chapter 10] and drop the positive term $\frac{1}{2} ||S(T') - S(T')||^2_2$ on the right-hand side in our ultimate variational inequality for (1.1) (cf. eq. (3.7) below).

**Definition 3.1 (Generalized solution).** Let $P$ satisfy (3.1). We call a couple $(V, S)$ a *generalized solution to* (1.1)–(1.3) if for all $T' \in (0, T)$ the following holds: $(V, S) \in LH_{T'} \times X_{T'}$, satisfies $V|_{\partial \Omega \times (0, T)} = g$ as well as

$$
\int_0^T \int_{\Omega} \left[ - V \cdot \partial_t \Phi - V \otimes V : \nabla \Phi + \eta S : \nabla \Phi + \mu \nabla V : \nabla \Phi \right] \, dx \, dt
\quad = \quad \int_0^T \int_{\Omega} F_0 \cdot \Phi \, dx \, dt - \int_0^T \int_{\Omega} F_1 : \nabla \Phi \, dx \, dt + \int_{\Omega} V_0 \cdot \Phi (\cdot, 0) \, dx,
$$

for all $\Phi \in C^\infty_0 (\Omega \times [0, T])$ and

$$
\int_0^T \int_{\Omega} \partial_t \tilde{S} : (\tilde{S} - S) + \gamma \nabla S : \nabla (\tilde{S} - S) \, dx \, dt + \int_0^T \left( P(\tilde{S}) - P(S) \right) \, dt + \int_0^T \int_{\Omega} V \cdot \nabla S : \tilde{S} + (SW(V) - W(V)S) : \tilde{S} - \eta D(V) : (\tilde{S} - S) \, dx \, dt
\quad \geq \quad -\frac{1}{2} \||S(0) - S_0||^2_2
$$

for all $\tilde{S} \in Z_{T'}$.

Observe that (3.6) is obtained by multiplying (1.1) by the respective test functions and formally integrating by parts. In particular, (3.6) is in accordance with the notion of weak solutions for the classical Navier–Stokes problem, since we take divergence-free test functions and omit the pressure term. Moreover, it is easy to see that the terms in (3.7) are well-defined. For the integrals involving the nonlinear terms of the Zaremba–Jaumann rate, this follows from the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{6}(\Omega)$, the interpolation

$$
L^\infty (0, T'; L^2(\Omega)) \cap L^2 (0, T'; L^6(\Omega)) \hookrightarrow L^{10/3}(0, T'; L^{10/3}(\Omega)),
$$

and the generalized Hölder inequality with inverse exponents $\frac{3}{10} + \frac{1}{2} + \frac{1}{5} = 1$.

**Remark 3.2.** In the formulation (3.7) it is crucial that in the integral involving the convective part, only the term $V \cdot \nabla S : \tilde{S}$ occurs, and not $V \cdot \nabla S : (\tilde{S} - S)$, since under the natural regularity hypotheses of $S$ in Def. 3.1, integrability of the term $V \cdot \nabla S : S$ is not ensured.

It is worth noting that, despite the absence of the term $\frac{1}{2} ||S(T') - S(T')||^2_2$ in ineq. (3.7), generalized solutions in the sense of Def. 3.1 obey the natural partial energy dissipation inequality for $S$.

**Proposition 3.3 (Partial energy inequality).** *Any generalized solution* $(V, S)$ *in the sense of Definition 3.1 satisfies the partial energy dissipation inequality*

$$
\frac{1}{2} ||S(T')||^2_2 + \gamma \| \nabla S \|^2_{L^2(\Omega \times (0, T'))} + \int_0^{T'} \mathcal{P}(S) \, dt
\quad \leq \quad \frac{1}{2} ||S_0||^2_2 + \int_0^T \int_{\Omega} \eta D(V) : S \, dx \, dt
$$

*for almost all* $T' \in (0, T)$.

Observe that this proposition applies to any solution conforming to Def. 3.1 and is independent of the approximation scheme chosen in our construction. Its proof will therefore be postponed to Section 5.4.
3.2 Main results

The main achievement of this article is the following result, which shows global-in-time existence of generalized solutions in the sense of Definition 3.1.

**Theorem 3.4** (Existence of generalized solutions to (1.1)–(1.3)). Let \( T \in (0, \infty) \), let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with \( C^{1,1} \)-boundary, and let \( \mathcal{P} \) satisfy (3.1). Let \( V_0, S_0 \) and \( F \) be as in (3.2), and let \( g \) satisfy (3.3) and \( g \cdot n = 0 \) on \( \partial \Omega \times (0, T) \). Then there exists a generalized solution \((V,S)\) to (1.1)–(1.3) in the sense of Definition 3.1.

Moreover, there exists an extension \( w \) of the boundary data \( g \) such that \( v = V - w \) satisfies the following energy-dissipation inequality: For a.a. \( t \in (0, T) \) we have

\[
\int_{\Omega} \frac{1}{2} |v(t)|^2 + \int_0^t \int_{\Omega} \mu |\nabla v|^2 \, dx \, d\tau \leq \int_{\Omega} \frac{1}{2} |V_0 - w(0)|^2 \, dx + \int_0^t \int_{\Omega} \left[ F_0 \cdot v - (F_1 - \tilde{F}_1) : \nabla v + w \otimes (v+w) : \nabla v - \eta S : \nabla v \right] \, dx \, d\tau,
\]

where \( \tilde{F}_1 \) is determined by \( \text{div} \tilde{F}_1 = \partial_t w - \mu \Delta w \). In particular, for a.a. \( t \in (0, T) \) we have the total energy-dissipation inequality

\[
\int_{\Omega} \frac{1}{2} |v(t)|^2 + \frac{1}{2} |S(t)|^2 \, dx + \int_0^t \int_{\Omega} \left[ \mu |\nabla v|^2 + \gamma |\nabla S|^2 \right] \, dx \, d\tau + \int_0^t \mathcal{P}(S) \, d\tau \\
\leq \int_{\Omega} \frac{1}{2} |V_0 - w(0)|^2 + \frac{1}{2} |S_0|^2 \, dx + \int_0^t \int_{\Omega} \left[ F_0 \cdot v - (F_1 - \tilde{F}_1) : \nabla v + w \otimes (v+w) : \nabla v + \eta D(w) : S \right] \, dx \, d\tau.
\]

The function \( w \) can be chosen such that \( w \in L^{\infty}(0,T; H^1(\Omega)^3), \partial_t w \in L^{\infty}(0,T; H^{-1}(\Omega)^3) \), and such that \( v \) satisfies the estimates

\[
\int_0^t \int_{\Omega} w \otimes v : \nabla v \, dx \, d\tau \leq \frac{\mu}{2} \|\nabla v\|_{L^2(0,T;H^1(\Omega))}^2,
\]

and

\[
\|w\|_{L^q(0,T';H^0(\Omega))} \leq C \|g\|_{L^q(0,T';H^{1/2}(\partial\Omega))},
\|
\partial_t w\|_{L^q(0,T';H^{-1}(\Omega))} \leq C \|\partial_t g\|_{L^q(0,T';H^{-1/2}(\partial\Omega))},
\]

for all \( T' \in (0,T) \) and \( q \in [1, \infty) \) and a suitable constant \( C = C(q,\Omega,\mu) > 0 \).

Note that (3.11) directly follows from summation of (3.10) and (3.9), which holds by Proposition 3.3, and using \( S = S : D(v) \). One readily verifies that all terms in (3.11) are well defined when \( w \) has the stated regularity. Moreover, for \( g = 0 \) we have \( w = 0 \), in which case the right-hand side of the energy-dissipation inequality (3.11) simplifies significantly. Another consequence is the following result, which gives a class of data such that the total energy stays bounded as \( t \to \infty \).

**Corollary 3.5.** In the situation of Theorem 3.4, let \( T = \infty \) and assume

\[
F_0 \in L^1(0,\infty;L^2(\Omega)), \quad F_1 \in L^2(0,\infty;L^2(\Omega)),
\]

\[
g \in L^{\infty}(0,\infty;H^{1/2}(\partial\Omega)) \cap L^1(0,\infty;H^{1/2}(\partial\Omega)),
\]

\[
\partial_t g \in L^{\infty}(0,\infty;H^{-1/2}(\partial\Omega)) \cap L^2(0,\infty;H^{-1/2}(\partial\Omega)).
\]

Then \((V,S) \in LH_T \times X_T \) for \( T = \infty \), so that the total energy remains bounded as \( t \to \infty \).
The proof of both Theorem 3.4 and Corollary 3.5 will be given in Subsection 5.3.

3.3 Strategy of the proof and existence for smooth potentials

Moreau envelope. The proof of Theorem 3.4 is based on approximating the nonsmooth potential $P$ by its Moreau envelope

$$
P_\varepsilon(S) = \inf_{S' \in L^2_2(\Omega)} \left( \frac{1}{2\varepsilon} \|S - S'\|_{L^2_2(\Omega)}^2 + P(S') \right), \quad \varepsilon \in (0, 1].
$$

(3.14)

This regularization preserves the basic properties (3.1) imposed on our potentials, that is, for each $\varepsilon \in (0, 1]$ the regularized potential $P_\varepsilon : L^2_2(\Omega) \to [0, \infty)$ is convex and lower semicontinuous with $P_\varepsilon(0) = 0$. Furthermore, each $P_\varepsilon$ is Fréchet differentiable and its differential $\partial P_\varepsilon$ is globally Lipschitz continuous with Lipschitz constant $1/\varepsilon$ (see [BaC17, Section 12.4] for example). Since $S = 0$ is a minimum of $P_\varepsilon$ and $P_\varepsilon(0) = 0$, the Lipschitz continuity of $\partial P_\varepsilon$ implies that

$$
\|\partial P_\varepsilon(S)\|_{L^2_2(\Omega)} = \|\partial P_\varepsilon(S) - \partial P_\varepsilon(0)\|_{L^2_2(\Omega)} \leq \varepsilon^{-1}\|S\|_{L^2_2(\Omega)}.
$$

Weak solutions for smooth potentials. In a first step of our analysis, we provide an existence result to system (1.1)–(1.3) for smooth potentials $P \in C^{1,1}(L^2_2(\Omega))$ allowing us to use a standard concept of weak solutions.

Definition 3.6 (Weak solutions). Let $P \in C^{1,1}(L^2_2(\Omega))$. We call a couple $(V, S)$ a weak solution to (1.1)–(1.3) if $(V, S) \in LH_{T'} \times X_{T'}$ for all $0 < T' < T$, if $V|_{\partial \Omega \times (0, T')} = g$, and if the identities (3.6) and

$$
\int_0^T \int_\Omega \left[ -S : \partial_t \Psi + V \cdot \nabla S : \Psi + SW(V) : \Psi - W(V)S : \Psi \\
+ \partial P(S) : \Psi + \gamma \nabla S : \nabla \Psi - \eta \nabla V : \Psi \right] \, dx \, dt = \int_\Omega S_0 : \Psi(\cdot, 0) \, dx
$$

(3.15)

hold for all $\Phi \in C^\infty_{0,\alpha}(\Omega \times [0, T'))$ and $\Psi \in C^\infty_{0,\delta}(\overline{\Omega} \times [0, T'))$.

Our existence result for $C^{1,1}(L^2_2(\Omega))$-smooth potentials states as follows.

Theorem 3.7. In addition to the hypotheses of Theorem 3.4, assume $P \in C^{1,1}(L^2_2(\Omega))$. Then there exists a weak solution $(V, S)$ to (1.1)–(1.3) in the sense of Definition 3.6. This solution is weakly continuous in $L^2(\Omega)$ in the sense that

$$
V \in C_w(0, T; L^2_2(\Omega)), \quad S \in C_w(0, T; L^2_2(\Omega)),
$$

(3.16)

with $(V(0), S(0)) = (V_0, S_0)$. Moreover, $V$ can be decomposed as $V = v + w$, where $w$ is the same extension of $g$ as in Theorem 3.4, such that for all $t \in (0, T)$ we have the partial energy-dissipation inequalities (3.10) and

$$
\frac{1}{2} \|S(t)\|_2^2 + \gamma \|\nabla S\|_{L^2_2(\Omega \times (0, t))}^2 + \int_0^t \int_\Omega \partial P(S) : S \, dx \, d\tau \\
\leq \frac{1}{2} \|S_0\|_2^2 + \int_0^t \int_\Omega \eta D(V) : S \, dx \, d\tau.
$$

(3.17)
Observe that the stress tensor $S$ obtained in Theorem 3.7 enjoys better regularity properties than that in Theorem 3.4. In particular, we note that the fact that $(V, S)$ satisfies the tensor evolution problem in the weak sense implies an estimate on the time derivative $\partial_t S$ in $L^{8/7}(0, T'; (H^1(\Omega))')$ (cf. Remark 4.11) as well as the weak continuity of $t \mapsto S(t)$ in $L^2_2(\Omega)$. Further note that

$$
\int_{\Omega} \partial \mathcal{P}(S) : S \, dx = \int_{\Omega} \partial \mathcal{P}(S) : (S - 0) \, dx \geq \mathcal{P}(S) - \mathcal{P}(0) = \mathcal{P}(S). \tag{3.18}
$$

Hence, the partial energy-dissipation inequality (3.9), satisfied by any generalized solution due to Proposition 3.3, may in general be weaker than the corresponding inequality (3.17), which the weak solution constructed in Theorem 3.7 conform to.

Our construction of solutions for smooth $\mathcal{P}$ is based on a Galerkin approximation that is manufactured in such a way that the global energy control in Corollary 3.5 holds true for data with suitable time decay. Exploiting the standard energy estimates and relying on compactness arguments of Aubin–Lions type for $V$ and $S$ allows us to pass to the weak limit even in the nonlinear terms $V \cdot \nabla S$ and $SW(V)$. Of course, in the limit the energy-dissipation balance (1.6) will only survive as an energy-dissipation inequality.

When approaching the nonsmooth dissipation potentials $\mathcal{P}$ by their Moreau envelope $\mathcal{P}_\varepsilon$, we lose the compactness for $S^\varepsilon$ because $\partial \mathcal{P}_\varepsilon(S^\varepsilon)$ cannot be controlled. Nevertheless, we are able to pass to the limit in the critical terms of the form $S^\varepsilon \nabla V^\varepsilon$ by integration by parts and relying on the boundary conditions, see Lemma 5.4.

### 3.4 Some remarks concerning the modeling

Our work is motivated by the modeling of geophysical flows, where rock is considered as a fluid flowing on very long time scales with speeds of millimeters per year, see [MDM02, GeY07, HGv17, PH*19]. The aim of those works is to understand the deformations of tectonic plates, the formation and development of faults, and the simulation of aseismic slipping processes. The modeling often concerns a smaller fault area of weaker material in the domain $\Omega$ that is driven by the more rigid tectonic plates around the weaker material. This situation is modeled in our case by prescribing the velocity $g(t, \cdot)$ along the boundary $\partial \Omega$.

The important common feature of these geodynamical models is the plasticity threshold for the shear stress, which is defined in terms of the norm of the deviatoric stress tensor, namely $|S| \leq \sigma_{\text{yield}}$, because of our assumption $\text{Tr}(S) = 0$. This condition is mathematically formulated via dissipation potentials $\mathcal{P}$ of the form $\mathcal{P}(S) = \int_{\Omega} \mathfrak{P}(S(x)) \, dx$ with $\mathfrak{P}$ satisfying $\mathfrak{P}(S) = \infty$ for $|S| > \sigma_{\text{yield}}$.

Indeed the evolution equation (1.1) for $S$ has to be invariant under time-dependent changes of the observer (cf. [Ant98]) which means that $\mathfrak{P}$ has to satisfy

$$
\mathfrak{P}(QSQ^T) = \mathfrak{P}(S) \quad \text{for all } S \in \mathbb{R}^{3 \times 3} \text{ and } Q \in \text{SO}(\mathbb{R}^3).
$$

Thus, $\mathfrak{P}$ can only depend on the three invariants of $S \in \mathbb{R}^{3 \times 3}$, namely $\text{Tr} S$, $\text{Tr}(S^2) - \text{Tr}(S)^2$, and $\det S$. Since we have restricted our analysis to the case $\text{Tr} S = 0$ (implying that $\text{Tr}(S^2) - \text{Tr}(S)^2 = |S|^2$), a very typical choice is given in the form

$$
\mathfrak{P}(S) = p(|S|) \quad \Rightarrow \quad \partial \mathfrak{P}(S) = \partial p(|S|) \frac{1}{|S|} S,
$$
where \( p : \mathbb{R} \to [0, \infty] \) is a lower semicontinuous, convex function with \( p(\sigma) = 0 \) for \( \sigma \leq 0 \). The choice \( p(\sigma) = \frac{\sigma}{2} \sigma^2 \) for \( \sigma \in [0, \sigma_{\text{yield}}] \) and \( p(\sigma) = \infty \) for \( \sigma > \sigma_{\text{yield}} \) gives the case in (1.4).

More general dissipation potentials may involve \( \det S \) as well. Using that \( S \mapsto |S|^6 + b(\det S)^2 \) is convex for \( |b| \leq b_* \) (e.g. \( b_* = 4 \) suffices) and that \( \frac{\partial \det S}{\partial s_{ij}} = \text{cof}(S)_{ij} \) (with \( \text{cof}(A) = \det(A)A^{-T} \) for invertible matrices) we may consider, for \( a_2, a_4, a_6 \geq 0 \),

\[
\mathcal{P}(S) = \frac{a_2}{2} |S|^2 + \frac{a_4}{4} |S|^4 + \frac{a_6}{6} (|S|^6 + b(\det S)^2)
\]

\[
\implies \partial \mathcal{P}(S) = (a_2 + a_4 |S|^2 + \frac{1}{3} \text{cof}(S) + \frac{1}{6} |S|^2 I),
\]

where the last term was added to ensure \( \partial \mathcal{P}(S) \in \mathbb{R}^{3 \times 3}_{\text{sym}} \), and we used that \( \text{Tr} S = 0 \) implies \( \text{Tr} (\text{cof}(S)) = -|S|^2/2 \).

In fact, without changing much in our analysis, we could have allowed the internal stresses \( S \) to have a spherical part, i.e. we may have considered \( \tilde{S} = S + \hat{p}I \in \mathbb{R}^{3 \times 3}_{\text{sym}} \) with \( S \in \mathbb{R}^{3 \times 3}_{\text{sym}} \) and \( \hat{p} \in \mathbb{R} \). The analysis remains unchanged as long as we keep the constraint \( \text{div} V = 0 \) in (1.1)\(_2\). This would allow us to consider more general dissipation potentials allowing for a coupling of \( S \) and \( \hat{p} \), e.g. in the form

\[
\mathcal{P}(\tilde{S}) = \begin{cases} 
\frac{a_2}{2} |S|^2 + \frac{b}{2} \hat{p}^2 & \text{for } \sigma \leq \sigma_{\text{yield}} + c\hat{p}, \\
\infty & \text{for } \sigma > \sigma_{\text{yield}} + c\hat{p},
\end{cases}
\]

where \( a, b, c \) are positive material parameters.

However, \( \hat{p} \) takes the position of an evolving additional pressure that should be modeled together with the true pressure taking compressibility of the fluid into account. Following [MDM02, GeY07, HGv17] the modeling assumption is then to identify \( \hat{p} \) and the pressure \( P \) in (1.1)\(_1\). Moreover, \( P \) is then treated as an independent variable solving an evolution equation like \( \frac{1}{\xi} \text{D}_t P + \frac{1}{\xi} P = -\text{div} V \) for positive constants \( K \) and \( \xi \), which replaces (1.1)\(_2\). The arising model is quite different from ours and will be studied in subsequent work.

### 4 Existence for smooth potentials

In this section we focus on the case of a smooth potential \( \mathcal{P} \in C^{1,1}(L^2_0(\Omega)) \) and show existence of a weak solution to the system (1.1)–(1.3) as stated in Theorem 3.7. To this end, we decompose the velocity field \( V \) into a suitable extension of the boundary data \( g \) and a remainder \( v \) with homogeneous boundary conditions such that \( (v, S) \) satisfies a perturbed system. While the existence of \( w \) will follow from well-known results, we show existence of \( (v, S) \) by a Galerkin method. In the end, \( (V, S) = (v + w, S) \) will be a weak solution with the properties asserted in Theorem 3.7.

#### 4.1 Decomposition of the velocity field

To show existence of a weak solution to the system (1.1)–(1.3), we decompose the velocity and pressure fields into two parts, \( V = v + w \) and \( P = p + q \), where \( (w, q) \) is a solution
to the Stokes initial-value problem with boundary data $w = g$

$$
\begin{align*}
\partial_t w - \text{div} \left( 2\mu D(w) - qI \right) &= \tilde{F} \quad \text{in } \Omega \times (0,T), \\
\text{div} w &= 0 \quad \text{in } \Omega \times (0,T), \\
w &= g \quad \text{on } \partial \Omega \times (0,T), \\
w(\cdot,0) &= w_0 \quad \text{in } \Omega
\end{align*}
$$

(4.1)

for some functions $\tilde{F}$ and $w_0$, and $v$ satisfies the remaining system with homogeneous boundary data $v = 0$

$$
\begin{align*}
\partial_t v + (v+w) \cdot \nabla (v+w) - \text{div} \left( \eta S + 2\mu D(v) - pI \right) &= f \quad \text{in } \Omega \times (0,T), \\
\text{div} v &= 0 \quad \text{in } \Omega \times (0,T), \\
\partial_t S + (v+w) \cdot \nabla S + SW(v+w) - W(v+w)S \\
+ \partial \mathcal{P}(S) - \gamma \Delta S - \eta D(v+w) &= 0 \quad \text{in } \Omega \times (0,T), \\
v &= 0, \quad n \cdot \nabla S = 0 \quad \text{on } \partial \Omega \times (0,T), \\
v(\cdot,0) &= v_0, \quad S(\cdot,0) = S_0 \quad \text{in } \Omega
\end{align*}
$$

(4.2)

with $f = F - \tilde{F}$ and $v_0 = V_0 - w_0$. In this way, we have decomposed the question of existence of solutions to (1.1)–(1.3) into two separate problems.

This decomposition method is a common way to treat inhomogeneous boundary data $g \neq 0$, and was successfully used to show existence of weak solutions to the classical Navier–Stokes initial-value problem in different configurations, see [FGS06, FKS10, FKS11a, FKS11b] for example. Observe that this decomposition is by no means unique since the functions $\tilde{F}$ and $w_0$ are not determined by the original problem (1.1)–(1.3). This freedom allows us to construct $w$ as a divergence-free extension of the boundary data $g$ satisfying (3.12) and (3.13) and to specify $\tilde{F}, w_0$ afterwards.

### 4.2 Weak solutions to the modified system

In this section we prescribe a class of vector fields $w$ such that the modified system (4.2) has a solution, see Assumption 4.1 below. In particular, for the moment we do not assume that the function $w$ in (4.2) is related to some given boundary function $g$.

For showing the existence of weak solutions to the modified system (4.2) we make the following assumptions. Let $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the dissipation potential $\mathcal{P}$ satisfy (3.1) and $\mathcal{P} \in C^{1,1}(L^2_{\delta}(\Omega))$. Similarly to (3.2), for the data we assume

$$
\begin{align*}
v_0 &\in L^{2}_{\delta}(\Omega), \quad S_0 \in L^{2}_{\delta}(\Omega), \quad f = f_0 + \text{div} f_1, \\
f_0 &\in L^{2}_{\text{loc}}([0,T);L^{2}(\Omega)^3), \quad f_1 \in L^{2}_{\text{loc}}([0,T);L^{2}(\Omega)^{3\times3}).
\end{align*}
$$

(4.3)

Moreover, the function $w$ is assumed to have the following properties.

**Assumption 4.1.** The function $w$ satisfies

$$
w \in L^{4}_{\text{loc}}([0,T);L^{4}(\Omega)^3), \quad \nabla w \in L^{2}_{\text{loc}}([0,T);L^{2}(\Omega)^{3\times3}),
$$

and one of the following three properties:

(a) $w \in L^{r}_{\text{loc}}([0,T);L^{s}(\Omega)^3)$ for some $r \in (3,\infty)$, $s \in (2,\infty)$ with $\frac{2}{r} + \frac{3}{s} = 1$, 

13
(b) \( w \in C^0(0, T; L^3(\Omega)^3) \) with \( \|w\|_{L^\infty(0, T; L^3(\Omega))} \leq \alpha \) for \( \alpha > 0 \) sufficiently small,

(c) \( w \in C^0(0, T; L^3(\Omega)^3) \) and for all \( t \in (0, T) \) we have

\[
\forall v \in H^1_{0, \sigma}(\Omega) : \quad \int_\Omega w(t) \otimes v : \nabla v \, dx \leq \frac{\mu}{2} \|\nabla v\|_2^2, \tag{4.4}
\]

\[
\forall S \in H^1_{\delta}(\Omega) : \quad \int_\Omega w(t) : \nabla S : S \, dx \leq \frac{\gamma}{2} \|S\|_{1,2}^2, \tag{4.5}
\]

where \( \mu \) and \( \gamma \) are the constants appearing in (1.1).

Observe that in our main results in Theorem 3.4 and Theorem 3.7 we consider boundary data \( g \) satisfying (3.3). As we will see in Subsection 4.3, this suffices to construct an extension \( w \) of \( g \) satisfying Assumption 4.1(c). However, since there is not much difference in the proof, we shall establish solutions to (4.2) in all cases described in Assumption 4.1.

**Remark 4.2.** Note that condition (a) cannot directly be generalized to the case \( r = 3 \), \( s = \infty \), which is treated in (b) and (c). Moreover, the smallness of the extension \( w \) required in (b) naturally transfers to a smallness assumption on the associated boundary data \( g \). In contrast, although condition (c) is a direct consequence of (b), it does not require such a condition. As we shall see in Subsection 4.3, we can find an extension \( w \) satisfying (c) without imposing a smallness assumption on \( g \).

Below, we will show existence to problem (4.2) in the following sense.

**Definition 4.3.** We call a couple \((v, S)\) a weak solution to (4.2) if \((v, S) \in LH_T \times X_T\) for all \( 0 < T' < T \), if \( v|_{\partial\Omega \times (0, T')} = 0 \) and if the identities

\[
\int_0^T \int_\Omega \left[ -v \cdot \partial_t \Phi - (v+w) \otimes (v+w) : \nabla \Phi + \eta S : \nabla \Phi + \mu \nabla v : \nabla \Phi \right] \, dx \, dt = \int_0^T \int_\Omega [f_0 \cdot \Phi - f_1 : \nabla \Phi] \, dx \, dt + \int_\Omega v_0 \cdot \Phi(\cdot, 0) \, dx, \tag{4.6}
\]

\[
\int_0^T \int_\Omega \left[ -S \cdot \partial_t \Psi + (v+w) \cdot \nabla S : \Psi + (SW(v+w) - W(v+w)S) : \Psi \right. \\
\left. + \partial \mathcal{P}(S) : \Psi + \gamma \nabla S : \nabla \Psi - \eta D(v+w) : \Psi \right] \, dx \, dt = \int_\Omega S_0 : \Psi(\cdot, 0) \, dx \tag{4.7}
\]

hold for all \( \Phi \in C^\infty_{0, \sigma}(\Omega \times [0, T)) \) and \( \Psi \in C^\infty_{0, \delta}(\overline{\Omega} \times [0, T)) \).

We shall provide a proof of the following existence result.

**Theorem 4.4.** Let \( v_0, S_0 \) and \( f \) be as in (4.3), let \( \mathcal{P} \in C^{1,1}(L^2_\mathcal{P}(\Omega)) \) satisfy (3.1), and let \( w \) be as in Assumption 4.1. Then there exists a weak solution \((v, S)\) to (4.2) in the sense of Definition 4.3. Additionally, this solution is weakly continuous in \( L^3(\Omega) \), that is,

\[
v \in C_w(0, T; L^3(\Omega)^3), \quad S \in C_w(0, T; L^2(\Omega)^{3 \times 3}), \tag{4.8}
\]
with \((v(0), S(0)) = (v_0, S_0)\), and it satisfies the energy inequalities

\[
\frac{1}{2} \|v(t)\|_2^2 + \mu \|\nabla v\|^2_{L^2(\Omega \times (0,t))} \\
\leq \frac{1}{2} \|v_0\|_2^2 + \int_0^t \int_{\Omega} [f_0 \cdot v - f_1 : \nabla v + w \otimes (v+w) : \nabla v - \eta S : \nabla v] \, dx \, d\tau,
\]

\[
\frac{1}{2} \|S(t)\|_2^2 + \gamma \|\nabla S\|^2_{L^2(\Omega \times (0,t))} + \int_0^t \int_{\Omega} \partial \mathcal{P}(S) : S \, dx \, d\tau \\
\leq \frac{1}{2} \|S_0\|_2^2 + \int_0^t \int_{\Omega} [-w \cdot \nabla S : S + \eta \nabla (v+w) : S] \, dx \, d\tau
\]

for all \(t \in [0,T]\). In particular, we conclude the total energy inequality

\[
\frac{1}{2} \|(v(t))_{L^2(\Omega)} + \|S(t)\|_{L^2(\Omega)}\| + \mu \|\nabla v\|^2_{L^2(0,T; L^2(\Omega))} + \gamma \|\nabla S\|^2_{L^2(0,T; L^2(\Omega))}
\]

\[
+ \int_0^T \int_{\Omega} \partial \mathcal{P}(S) : S \, dx \, d\tau \leq \frac{1}{2} \|(v_0)_{L^2(\Omega)} + \|S_0\|_{L^2(\Omega)}\|
\]

\[
+ \int_0^T \int_{\Omega} [f_0 \cdot v - f_1 : \nabla v + w \otimes (v+w) : \nabla v - \eta \nabla w : S] \, dx \, d\tau
\]

for all \(t \in [0,T]\).

### 4.2.1 Approximate solutions

First, we construct a sequence of approximate solutions to (4.2). To this end, we introduce suitable basis functions.

**Lemma 4.5.** There exists a sequence \((\varphi_k) \subset C_0^\infty(\Omega)\), which is an orthonormal basis of \(L^2(\Omega)\), such that for all \(\Phi \in C_0^\infty(\Omega \times [0,T])\) and all \(\varepsilon > 0\) there exist \(N \in \mathbb{N}\) and \(\gamma_1, \ldots, \gamma_N \in C^1([0,T])\) such that

\[
\max_{t \in [0,T]} \left\| \sum_{k=1}^N \gamma_k(t) \varphi_k - \Phi \right\|_{C^2(\Omega)} + \max_{t \in [0,T]} \left\| \sum_{k=1}^N \partial_t \gamma_k(t) \varphi_k - \partial_t \Phi \right\|_{C^1(\Omega)} < \varepsilon. \tag{4.12}
\]

**Proof.** See [Gal00, Lemma 2.3].

**Lemma 4.6.** There exists a sequence \((\psi_k) \subset C_0^\infty(\Omega)\), which is an orthonormal basis of \(L^2(\Omega)\), such that for all \(\Psi \in C_0^\infty(\Omega \times [0,T])\) and all \(\varepsilon > 0\) there exist \(N \in \mathbb{N}\) and \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N \in C^1([0,T])\) such that

\[
\max_{t \in [0,T]} \left\| \sum_{k=1}^N \tilde{\gamma}_k(t) \psi_k - \Psi \right\|_{C^2(\Omega)} + \max_{t \in [0,T]} \left\| \sum_{k=1}^N \partial_t \tilde{\gamma}_k(t) \psi_k - \partial_t \Psi \right\|_{C^1(\Omega)} < \varepsilon. \tag{4.13}
\]

**Proof.** One can follow the proof of [Gal00, Lemma 2.3].

**Remark 4.7.** Observe that \((\varphi_k)\) is a basis of \(H^1_0(\Omega)\), and \((\psi_k)\) is a basis of \(H^1_\delta(\Omega)\). To see this, consider \(\varphi \in C_0^\infty(\Omega)\) and let \(\Phi \in C_0^\infty(\Omega \times [0,T])\) such that \(\Phi(\cdot,0) = \varphi\). Let \(\varepsilon > 0\) and \(\gamma_1, \ldots, \gamma_N\) as in Lemma 4.5. Then

\[
\left\| \sum_{k=1}^N \gamma_k(0) \varphi_k - \varphi \right\|_{C^2(\Omega)} \leq \max_{t \in [0,T]} \left\| \sum_{k=1}^N \gamma_k(t) \varphi_k - \Phi(\cdot,t) \right\|_{C^2(\Omega)} < \varepsilon.
\]
Since $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $H_{0,\sigma}^{1}(\Omega)$ and $\Omega$ is bounded, this shows the claim for $(\varphi_k)$. Taking $\psi \in C_{0,\sigma}^{\infty}(\Omega)$ and $\Psi \in C_{0,\sigma}^{\infty}(\Omega \times [0, T))$ with $\Psi(\cdot, 0) = \psi$ instead, we can use Lemma 4.6 to conclude the statement for $(\psi_k)$ in the same way.

With these bases at hand, we now construct a sequence of approximate solutions in the following way. For $k \in \mathbb{N}$ we call $(v, S)$ an approximate solution (of order $k$) if there exist $\alpha_r, \beta_r \in C^1(0, T) \cap C^0([0, T))$, $r = 1, \ldots, k$, such that

\begin{equation}
    v(x, t) = v_k(x, t) = \sum_{r=1}^{k} \alpha_r(t) \varphi_r(x), \quad S(x, t) = S_k(x, t) = \sum_{r=1}^{k} \beta_r(t) \psi_r(x), \quad (4.14)
\end{equation}

and for all $\ell \in \{1, \ldots, k\}$ the pair $(v, S)$ satisfies

\begin{equation}
\begin{aligned}
    \int_{\Omega} \left[ \partial_t v \cdot \varphi_\ell - (v+w) \otimes (v+w) : \nabla \varphi_\ell + \eta S : \nabla \varphi_\ell + \mu \nabla v : \nabla \varphi_\ell \right] \, dx \\
    = \int_{\Omega} f_0 \cdot \varphi_\ell \, dx - \int_{\Omega} f_1 : \nabla \varphi_\ell \, dx,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
    \int_{\Omega} \left[ \partial_t S \cdot \psi_\ell + (v+w) \cdot \nabla S : \psi_\ell + SW(v+w) : \psi_\ell - W(v+w)S : \psi_\ell \right. \\
    \left. + \partial \mathcal{P}(S) : \psi_\ell + \gamma \nabla S : \nabla \psi_\ell - \eta \nabla (v+w) : \psi_\ell \right] \, dx = 0
\end{aligned}
\end{equation}

in $(0, T)$ and

\begin{equation}
\begin{aligned}
    \int_{\Omega} v(0) \cdot \varphi_\ell \, dx = \int_{\Omega} v_0 \cdot \varphi_\ell \, dx, \quad \int_{\Omega} S(0) : \psi_\ell \, dx = \int_{\Omega} S_0 : \psi_\ell \, dx.
\end{aligned}
\end{equation}

The existence of approximate solutions is guaranteed by the following result.

**Lemma 4.8.** For all $k \in \mathbb{N}$ there exists an approximate solution $(v, S) = (v_k, S_k)$, which satisfies the partial energy-dissipation equalities

\begin{equation}
\begin{aligned}
    &\frac{1}{2} \|v(t)\|_2^2 + \mu \|\nabla v\|_{L^2(\Omega \times (0, t))}^2 \\
    &= \frac{1}{2} \|v(0)\|_2^2 + \int_0^t \int_{\Omega} \left[ f_0 \cdot v - f_1 : \nabla v + w \otimes (v+w) : \nabla v - \eta S : \nabla v \right] \, dx \, d\tau,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
    &\frac{1}{2} \|S(t)\|_2^2 + \gamma \|\nabla S\|_{L^2(\Omega \times (0, t))}^2 + \int_0^t \int_{\Omega} \partial \mathcal{P}(S) : S \, dx \, d\tau \\
    &= \frac{1}{2} \|S(0)\|_2^2 + \int_0^t \int_{\Omega} \left[ - w \cdot \nabla S + \eta \nabla (v+w) : S \right] \, dx \, d\tau
\end{aligned}
\end{equation}

for all $t \in (0, T)$. Moreover, for $0 < T' < T$, there exists a constant $M_{T'} > 0$, only depending on the data and $T'$ but independent of $k$ and $\mathcal{P}$, such that

\begin{equation}
\begin{aligned}
    \sup_{t \in (0, T')} \left( \|v(t)\|_2^2 + \|S(t)\|_2^2 \right) + \int_0^{T'} \left( \|v(t)\|_{1,2}^2 + \|S(t)\|_{1,2}^2 + \mathcal{P}(S) \right) \, dt \leq M_{T'},
\end{aligned}
\end{equation}

so that the sequence $(v_k, S_k)_{k \in \mathbb{N}}$ is bounded in $LH_{T'} \times X_{T'}$. 

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Remark 4.9. The energy balances of the kinetic energy and of the stored elastic energy are expressed in (4.18) in two separate equations. By summation we obtain the total energy equality

\[ \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|S(t)\|^2 + \mu \|\nabla v\|^2_{L^2(\Omega \times (0,t))} + \gamma \|\nabla S\|^2_{L^2(\Omega \times (0,t))} + \int_0^t \int_{\Omega} \partial P(S) : S \, dx \, d\tau = \frac{1}{2} \|v(0)\|^2 + \frac{1}{2} \|S(0)\|^2 \]  

(4.20)

Proof. We reduce the equations (4.15)–(4.17) to an initial-value problem for the coefficient function \((\alpha, \beta) = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)\). Let \((\cdot)’ = \frac{d}{dt}\) denote the time derivative. Due to orthogonality properties of the two bases \((\varphi_k)\) and \((\psi_k)\), we then obtain

\[
\alpha’(t) = F^1(\alpha(t), \beta(t), t), \quad \beta’(t) = F^2(\alpha(t), \beta(t)),
\]

\[
\alpha_\ell(0) = \int_\Omega v_0 \cdot \varphi_\ell \, dx, \quad \beta_\ell(0) = \int_\Omega S : \psi_\ell \, dx \quad (\ell = 1, \ldots, k),
\]

where \(F^j = (F^j_1, \ldots, F^j_k)\), \(j = 1, 2\), with

\[
F^1_\ell(\alpha, \beta, t) = \sum_{r,s=1}^n \alpha_r \alpha_s \int_\Omega \varphi_r \otimes \varphi_s : \nabla \varphi_\ell \, dx
\]

\[+ \sum_{r=1}^k \alpha_r \int_\Omega (w \otimes \varphi_r + \varphi_r \otimes w) : \nabla \varphi_\ell \, dx + \int_\Omega w \cdot \nabla w \cdot \varphi_\ell \, dx
\]

\[ - \eta \sum_{r=1}^k \beta_r \int_\Omega \psi_r : \nabla \varphi_\ell \, dx - \mu \sum_{r=1}^k \alpha_r \int_\Omega \nabla \varphi_r : \nabla \varphi_\ell \, dx
\]

\[+ \int_\Omega f_0(\cdot, t) \cdot \varphi_\ell - f_1(\cdot, t) \cdot \nabla \varphi_\ell \, dx,
\]

and

\[
F^2_\ell(\alpha, \beta) = - \sum_{r,s=1}^k \alpha_r \beta_s \int_\Omega \varphi_r \cdot \nabla \psi_s : \psi_\ell \, dx - \sum_{r=1}^k \beta_r \int_\Omega w \cdot \nabla \psi_s : \psi_\ell \, dx
\]

\[ - \sum_{r,s=1}^k \beta_r \alpha_s \int_\Omega \left[ \psi_r W(\varphi_r) - W(\varphi_r)\psi_s \right] : \psi_\ell \, dx
\]

\[ - \int_\Omega \left[ \psi_r W(w) - W(w)\psi_s \right] : \psi_\ell \, dx
\]

\[ - \int_\Omega \partial P(\sum_{r=1}^k \alpha_r \psi_r) : \psi_\ell \, dx - \gamma \sum_{r=1}^k \beta_r \int_\Omega \nabla \psi_r : \nabla \psi_\ell \, dx
\]

\[+ \eta \sum_{r=1}^k \alpha_r \int_\Omega \nabla \varphi_r : \psi_\ell \, dx + \eta \int_\Omega \nabla w : \psi_\ell \, dx.
\]

for \(\ell = 1, \ldots, k\). In particular, since \(\partial P\) is Lipschitz continuous by assumption, (4.21) is an initial-value problem with right-hand side \(F = (F^1, F^2)\) that satisfies a local Lipschitz condition. By the Picard–Lindelöf theorem we thus obtain a unique local solution \((\alpha, \beta)\)
to (4.21). Let \((0, T_k) \subset (0, T)\) denote its maximal existence interval. Then \((v, S)\) defined by (4.14) satisfy (4.15)–(4.16) in \((0, T_k)\) and (4.17).

To conclude the energy equalities (4.18a) and (4.18b) for all \(t \in (0, T_k)\), we multiply (4.15) by \(\alpha_\ell\) and (4.16) by \(\beta_\ell\), sum over \(\ell = 1, \ldots, k\) and integrate over the time interval \((0, t)\). This leads to the two equalities (4.18) by employing the identities

\[
\int_\Omega v \otimes (v + w) : \nabla v \, dx = \frac{1}{2} \int_\Omega \nabla |v|^2 \, dx - \frac{1}{2} \int_{\partial \Omega} (v + w) \cdot n |v|^2 \, d\sigma = 0,
\]

\[
\int_\Omega v \cdot \nabla S : S \, dx = \frac{1}{2} \int_\Omega v \cdot \nabla |S|^2 \, dx - \frac{1}{2} \int_{\partial \Omega} v \cdot n |S|^2 \, d\sigma = 0,
\]

which hold due to \(v = 0\) on \(\partial \Omega \times (0, T_k)\).

Next we show that \(T_k = T\). To this end, we add (4.18a) and (4.18b) to obtain (4.20) and further estimate the right-hand side of (4.20) for \(t \in (0, T_k)\). The initial terms can be estimated with Bessel’s inequality as

\[
\|v(0)\|_2^2 \leq \|v_0\|_2^2, \quad \|S(0)\|_2^2 \leq \|S_0\|_2^2.
\]  \hspace{1cm} (4.22)

For the linear terms, we employ a combination of H"older’s and Young’s inequalities to obtain

\[
\int_0^t \int_\Omega f_0 \cdot v \, dx \, d\tau \leq c_0(\varepsilon) \|f_0\|_{L^1(0, t; L^2(\Omega))}^2 + \varepsilon \|v\|_{L^\infty(0, t; L^2(\Omega))}^2,  \hspace{1cm} (4.23)
\]

\[
\int_0^t \int_\Omega f_1 : \nabla v \, dx \, d\tau \leq c_1(\varepsilon) \|f_1\|_{L^2(\Omega \times (0, t))}^2 + \varepsilon \|\nabla v\|_{L^2(\Omega \times (0, t))}^2,  \hspace{1cm} (4.24)
\]

\[
\int_0^t \int_\Omega \nabla w : S \, dx \, d\tau \leq c_2(\varepsilon) \|\nabla w\|_{L^1(0, t; L^2(\Omega))}^2 + \varepsilon \|S\|_{L^\infty(0, t; L^2(\Omega))}^2,  \hspace{1cm} (4.25)
\]

\[
\int_0^t \int_\Omega (w \otimes w) : \nabla v \, dx \, d\tau \leq c_3(\varepsilon) \|w\|_{L^4(\Omega \times (0, t))}^4 + \varepsilon \|\nabla v\|_{L^2(\Omega \times (0, t))}^2,  \hspace{1cm} (4.26)
\]

for any \(\varepsilon > 0\). Next we address the nonlinear terms, where we need to discuss the different cases in Assumption 4.1.

\textbf{Case }s < \infty\textbf{ : We use part (a) of Assumption 4.1 with }r > 3\textbf{ and define }p \in (2, 6)\textbf{ via }1/p = 1/2 \- 1/r.\textbf{ With }\theta = 3/2 \- 3/2 = 3/r = 1 \- 2/s \in (0, 1)\textbf{ the Gagliardo–Nirenberg inequality gives}

\[
\|v(t)\|_p \leq c_4 \|v(t)\|_2^{1-\theta} \|\nabla v(t)\|_2^\theta,  \quad \|S(t)\|_p \leq c_5 \|S(t)\|_2^{1-\theta} \|\nabla S(t)\|_2^\theta + c_6 \|S(t)\|_2
\]
for all \( t \in [0, T_k] \), so that Hölder’s and Young’s inequalities lead to

\[
\int_0^t \int_\Omega w \otimes v : \nabla v \, dx \, d\tau \leq \int_0^t \|w\|_r \|v\|_p \|\nabla v\|_2 \, d\tau \\
\leq c_7 \int_0^t \|w\|_r \|v\|_2^{1-\theta} \|\nabla v\|_2^{1+\theta} \, d\tau \\
\leq \varepsilon \|\nabla v\|_{L^2(\Omega \times (0,t))}^2 + c_8(\varepsilon) \int_0^t \|w\|_r^p \|v\|_2^p \, d\tau ,
\]

(4.27)

\[
\int_0^t \int_\Omega w \cdot \nabla S : S \, dx \, d\tau \leq \int_0^t \|w\|_r \|\nabla S\|_2 \|S\|_2 \, d\tau \\
\leq c_9 \int_0^t \|w\|_r \left( \|\nabla S\|_2^{1+\theta} \|S\|_2^{1-\theta} + \|\nabla S\|_2 \|S\|_2 \right) \, d\tau \\
\leq \varepsilon \|\nabla S\|_{L^2(\Omega \times (0,t))}^2 + c_{10}(\varepsilon) \int_0^t \left( \|w\|_r^p + \|w\|_r^2 \right) \|S\|_2^2 \, d\tau .
\]

(4.28)

Let now \( t' \in (0, T_k) \) be arbitrary. Using (4.22)–(4.28) and choosing \( \varepsilon = \frac{1}{6} \), for \( t \in [0, t'] \) we may then estimate the right-hand side of (4.20) as

\[
\|v(t)\|_{L^2(\Omega)}^2 + \|S(t)\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega \times (0,t))}^2 + \|\nabla S\|_{L^2(\Omega \times (0,t))}^2 \\
+ \int_0^t \int_\Omega \partial \mathcal{P}(S) : S \, dx \, d\tau \\
\leq c_{11} \left( \|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|_{L^1(0,t,L^2(\Omega))}^2 + \|f_1\|_{L^2(\Omega \times (0,t))}^2 + \|\nabla w\|_{L^4(\Omega \times (0,t))}^4 \right) \\
+ \int_0^t \left( \|w\|_r^p + \|w\|_r^2 \right) \left( \|v\|_2^2 + \|S\|_2^2 \right) \, d\tau \\
+ \frac{1}{2} \left( \|v\|_{L^\infty(0,t';L^2(\Omega))}^2 + \|S\|_{L^\infty(0,t';L^2(\Omega))}^2 + \|\nabla v\|_{L^2(\Omega \times (0,t'))}^2 + \|\nabla S\|_{L^2(\Omega \times (0,t'))}^2 \right) .
\]

We now take the supremum over \( t \in [0, t'] \) and make use of the elementary inequality \( \sum_{i=1}^M \sup_{t} N_i(t) \leq M \sup_{t} \left( \sum_{i=1}^M N_i(t) \right) \) for \( M \in \mathbb{N} \) and functions \( N_i \geq 0 \). After rearranging terms, relabelling \( t' \) by \( t \), and adding the squared norm of \( v \) and \( S \) in \( L^2(\Omega \times (0,t)) \), this yields the inequality

\[
\|v\|_{L^\infty(0,t;L^2(\Omega))}^2 + \|S\|_{L^\infty(0,t;L^2(\Omega))}^2 + \|v\|_{L^2(0,t;H^1(\Omega))}^2 + \|S\|_{L^2(0,t;H^1(\Omega))}^2 \\
+ \int_0^t \int_\Omega \partial \mathcal{P}(S) : S \, dx \, d\tau \\
\leq c_{12} \left( \|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|_{L^1(0,t,L^2(\Omega))}^2 + \|f_1\|_{L^2(\Omega \times (0,t))}^2 + \|\nabla w\|_{L^2(0,t;L^2(\Omega))}^4 \right) \\
+ \int_0^t \left( \|w\|_r^p + \|w\|_r^2 \right) \left( \|v\|_2^2 + \|S\|_2^2 \right) \, d\tau .
\]

Since the last term in the last line is bounded above by

\[
\int_0^t (\|w\|_r^p + \|w\|_r^2 + 1) \left( \|v\|_{L^\infty(0,t;L^2(\Omega))}^2 + \|S\|_{L^\infty(0,t;L^2(\Omega))}^2 \right) \, d\tau ,
\]
an application of Gronwall’s inequality eventually leads to the bound
\[
\|v\|_{L^\infty(0,t;L^2(\Omega))} + \|S\|_{L^\infty(0,t;L^2(\Omega))} + \|v\|_{L^2(0,t;H^1(\Omega))} + \|S\|_{L^2(0,t;H^1(\Omega))} \\
+ \int_0^t \mathcal{P}(S) \, d\tau \leq c_{13} (\|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|_{L^1(0,t;L^2(\Omega))}^2 + \|f_1\|_{L^2(\Omega \times (0,t))}^2) \\
+ \|\nabla w\|_{L^2(0,t;L^2(\Omega))} + \|w\|_{L^4(\Omega \times (0,t))}^4 \exp \left( c_{14} \int_0^t (\|w\|_r^r + \|w\|_r^r + 1) \, d\tau \right)
\]
for \(t \in (0, T_k)\) in the case \(s < \infty\), where we also used (3.18).

Case \(s = \infty\): We can apply (4.4) and (4.5) to obtain
\[
\int_0^t \int_\Omega w \otimes v : \nabla v \, dx \, d\tau \leq \frac{\mu}{2} \|\nabla v\|_{L^2(\Omega \times (0,t))}^2, \tag{4.30}
\]
\[
\int_0^t \int_\Omega w \cdot \nabla S : S \, dx \, d\tau \leq \frac{\gamma}{2} (\|S\|_{L^2(\Omega \times (0,t))}^2 + \|\nabla S\|_{L^2(\Omega \times (0,t))}^2). \tag{4.31}
\]
Using (4.30) and (4.31) instead of (4.27) and (4.28), the argument from above leads to
\[
\|v\|_{L^\infty(0,t;L^2(\Omega))} + \|S\|_{L^\infty(0,t;L^2(\Omega))} + \|v\|_{L^2(0,t;H^1(\Omega))} + \|S\|_{L^2(0,t;H^1(\Omega))} \\
+ \int_0^t \mathcal{P}(S) \, d\tau \leq c_{15} (\|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|_{L^1(0,t;L^2(\Omega))}^2 + \|f_1\|_{L^2(\Omega \times (0,t))}^2) \\
+ \|\nabla w\|_{L^1(0,t;L^2(\Omega))} + \|w\|_{L^4(\Omega \times (0,t))}^4) e^{\gamma_0 t}
\]
in the case \(s = \infty\).

Now consider general \(s \in (2, \infty)\) again. For any \(T' \in (0, T_k)\), the right-hand side of (4.29) and (4.32) can be bounded uniformly in \(t \in (0, T')\) by a constant \(M_{T'} > 0\) that only depends on the data and \(T'\). In particular, \(M_{T'}\) is independent of \(k \in \mathbb{N}\) and \(\mathcal{P}\), and we conclude (4.19) in both cases. By Parseval’s identity, this shows
\[
\sum_{r=1}^k |\alpha_r(t)|^2 + |\beta_r(t)|^2 = \|v(t)\|_2^2 + \|S(t)\|_2^2 \leq M_{T'}
\]
for all \(t \in (0, T')\). Hence the solution \((\alpha, \beta)\) does not blow-up at \(t = T'\), and we conclude \(T_k = T\) together with (4.19). \(\square\)

**Lemma 4.10.** For any \(0 < T' < T\), the sequence \((\partial_t v_k, \partial_t S_k)_{k \in \mathbb{N}}\) is bounded in
\[
L^1(0, T'; (H^1(\Omega))') \times L^{8/7}(0, T'; (H^1(\Omega))')
\]
with
\[
\|\partial_t v_k\|_{L^1(0, T'; (H^1(\Omega))')} \leq M_{T'}, \\
\|\partial_t S_k\|_{L^{8/7}(0, T'; (H^1(\Omega))')} \leq M_{T',\mathcal{P}},
\]
where \(M_{T'}\) is independent of \(\mathcal{P}\), but \(M_{T',\mathcal{P}}\) depends on \(\mathcal{P}\).

**Proof.** These bounds follow as for the classical Navier–Stokes equations, and we only give a brief sketch here, where we focus on the estimate of \((\partial_t S_k)\). The bound for \((\partial_t v_k)\) follows similarly. Let \(S = S_k\) and recall the interpolation inequality
\[
\|S(t)\|_4 \leq c_0 \|S(t)\|_2^{1/4} \|S(t)\|_1^{3/4}
\]
for a.a. $t \in (0, T)$. Since $S$ is a finite linear combination of elements of the basis $(\psi_k)$ of $H^1_0(\Omega)$, one deduces from (4.16) that $\partial_t S(t) \in (H^1(\Omega))'$ for a.a. $t \in (0, T)$ and

\[
\|\partial_t S\|_{(H^1(\Omega))'} \leq c_1 \left( \|v\|_{L^2}^{1/4} \|v\|_{L^1}^{3/4} \|\nabla S\|_{L^2} + \|w\|_{L^1} \|\nabla S\|_{L^2} \right)
\]

\[
+ \|S\|_{L^2}^{1/4} \|S\|_{L^1}^{3/4} \|\nabla(v+w)\|_{L^2} + \|S\|_{L^2} \|\nabla(v+w)\|_{L^2},
\]

where we used the estimate $\|\mathcal{P}(S)\|_{L^2} \leq c_2 \|S\|_{L^2}$ for some $\mathcal{P}$-dependent constant $c_2 > 0$, which follows from the Lipschitz continuity of $\partial \mathcal{P}$ and $\partial \mathcal{P}(0) = 0$. Using Hölder’s inequality and (4.19), one concludes the asserted uniform bound for $(\partial_t S_k)$.

\[\square\]

4.2.2 Existence of weak solutions to (4.2)

Based on the previous preparations we establish the existence of weak solutions to (4.2) as stated in Theorem 4.4.

**Proof of Theorem 4.4.** Let $(v_k, S_k)_{k \in \mathbb{N}}$ be the sequence of approximate solutions in $(0, T)$ from Lemma 4.8. Due to the uniform bounds from (4.19) for $T' \in (0, T)$, combined with a classical diagonalization argument, we obtain the existence of a subsequence, which we also denote by $(v_k, S_k)$, and a pair $(v, S)$ with $(v, S) \in LH_{T'} \times X_{T'}$ for each $T' \in (0, T)$ such that

\[
v_k \to v \quad \text{in } L^2(0, T'; H^1(\Omega)^3),
\]

\[
S_k \to S \quad \text{in } L^2(0, T'; H^1(\Omega)^{3 \times 3}),
\]

\[
v_k \rightharpoonup v \quad \text{in } L^\infty(0, T'; L^2(\Omega)),
\]

\[
S_k \rightharpoonup S \quad \text{in } L^\infty(0, T'; L^2(\Omega)),
\]

as $k \to \infty$. Moreover, since by Lemma 4.10 the sequences $(\partial_t v_k)$ and $(\partial_t S_k)$ are bounded in $L^1(0, T'; (H^1_0(\Omega))^3)'$ and $L^{8/7}(0, T'; (H^1(\Omega)^{3 \times 3})')$, respectively, and since the Sobolev embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the Aubin–Lions lemma further implies the strong convergence

\[
v_k \to v \quad \text{in } L^2(0, T'; L^2(\Omega)^3),
\]

\[
S_k \to S \quad \text{in } L^2(0, T'; L^2(\Omega)^{3 \times 3})
\]

as $k \to \infty$. Let us show that $(v, S)$ is a weak solution to (4.2), that is, that (4.6) and (4.7) are satisfied. To this end, let $\chi \in C^\infty_0([0, T])$ and let $T' \in (0, T)$ such that supp $\chi \subset [0, T')$. Fix $\ell \in \mathbb{N}$. Multiply (4.15) and (4.16) by $\chi(t)$, integrate over $(0, T)$ and pass to the limit $k \to \infty$ exploiting the above convergence properties. For example, in view of (4.17), for $k \geq \ell$ we have

\[
\int_0^T \int_\Omega \partial_t v_k \cdot \varphi \chi dx \, dt = - \int_0^T \int_\Omega v_k \cdot \varphi \partial_t \chi dx \, dt - \int_\Omega v_0 \cdot \varphi \chi(0) dx
\]

\[
- \int_0^T \int_\Omega v \cdot \varphi \partial_t \chi dx \, dt - \int_\Omega v_0 \cdot \varphi \chi(0) dx.
\]

Employing the strong convergence of $(v_k)$ in $L^2(\Omega \times (0, T'))$, we further conclude

\[
\int_0^T \int_\Omega v_k \otimes v_k : \nabla \varphi \chi dx \, dt \to \int_0^T \int_\Omega v \otimes v : \nabla \varphi \chi dx \, dt.
\]

Similarly, we can derive

\[
\int_0^T \int_\Omega \partial \mathcal{P}(S_k) : \psi \chi dx \, dt \to \int_0^T \int_\Omega \partial \mathcal{P}(S) : \psi \chi dx \, dt
\]
from the Lipschitz continuity of $\partial P$ and the strong convergence of $(S_k)$ in $L^2(\Omega \times (0, T'))$. Convergence of the remaining terms can be shown in a similar fashion, and we conclude (4.6) and (4.7) for all $\Phi$ and $\Psi$ of the form $\Phi(x, t) = \varphi(x) \chi(t)$ and $\Psi(x, t) = \psi(x) \chi(t)$ with $\ell \in \mathbb{N}$. Finally, an approximation argument based on Lemma 4.5 and Lemma 4.6 allows us to pass to general $\Phi \in C^{0, \sigma}_0(\Omega \times [0, T])$ and $\Psi \in C^{0, \delta}_0(\overline{\Omega} \times [0, T])$, respectively. Consequently, $(v, S)$ is a weak solution to (4.2).

Now let us show the energy inequalities (4.9) and (4.10). Similarly to [Gal00, Proof of Theorem 3.1], one can show that $(v_k(t), S_k(t)) \to (v(t), S(t))$ in $L^2(\Omega)$ as $k \to \infty$ for all $t \in (0, T)$ after possibly modifying the solution $(v, S)$ on a set of measure zero in $(0, T)$. This property and the weak convergence in $L^2(0, T'; H^1(\Omega))$ imply

$$\frac{1}{2} \|v(t)\|^2_2 + \mu \|\nabla v\|^2_{L^2(\Omega)} \leq \liminf_{k \to \infty} \left( \frac{1}{2} \|v_k(t)\|^2_2 + \mu \|\nabla v_k\|^2_{L^2(\Omega)} \right),$$

$$\frac{1}{2} \|S(t)\|^2_2 + \gamma \|\nabla S\|^2_{L^2(\Omega)} \leq \liminf_{k \to \infty} \left( \frac{1}{2} \|S_k(t)\|^2_2 + \gamma \|\nabla S_k\|^2_{L^2(\Omega)} \right).$$

(4.33)

Moreover, the strong convergence $S_k \to S$ in $L^2(\Omega \times (0, T'))$ and the Lipschitz continuity of $\partial P$ lead to

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} \partial P(S_k) : S_k \, dx \, dt = \int_0^t \int_{\Omega} \partial P(S) : S \, dx \, dt.$$

(4.35)

By construction we further have $\|v(0)\|_2 \leq \|v_0\|_2$ and $\|S(0)\|_2 \leq \|S_0\|_2$ due to Bessel’s inequality, and we can directly conclude

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} [f_0 \cdot v_k - f_1 \cdot \nabla v] \, dx \, dt = \int_0^t \int_{\Omega} [f_0 \cdot v - f_1 \cdot \nabla v] \, dx \, dt,$$

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} w \otimes w : \nabla v \, dx \, dt = \int_0^t \int_{\Omega} w \otimes w : \nabla v \, dx \, dt,$$

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} \eta \nabla w : S_k \, dx \, dt = \int_0^t \int_{\Omega} \eta \nabla w : S \, dx \, dt$$

since $f, w \otimes w, \nabla w \in L^2(\Omega \times (0, T'))$. Moreover, the strong convergence of $(S_k)$ in $L^2(\Omega \times (0, T'))$ and the weak convergence of $(\nabla v_k)$ in $L^2(\Omega \times (0, T'))$ imply

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} S_k : \nabla v \, dx \, dt = \int_0^t \int_{\Omega} S : \nabla v \, dx \, dt.$$

(4.39)

For the remaining terms, assume for the moment that $w \in C^0_0(\Omega \times (0, T))$. Then $(w \otimes v_k)$ converges to $w \otimes v$ strongly in $L^2(\Omega \times (0, T))$. Hence we obtain

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} w \otimes v_k : \nabla v \, dx \, dt = \int_0^t \int_{\Omega} w \otimes v : \nabla v \, dx \, dt.$$

(4.40)

For general $w \in L^q(0, T; L^r(\Omega))$ we obtain (4.40) by approximating $w$ by elements from $C^0_0(\Omega \times (0, T))$ and exploiting that $LH_p \hookrightarrow L^q(0, T; L^p(\Omega))$ with $1/p = 1/2 - 1/r$, $1/q = 1/2 - 1/s$, so that $2/q + 3/p = 3/2$. Observe that here we use $w \in C^0(0, T; L^3(\Omega))$ if $s = \infty$. An analogous argument shows

$$\lim_{k \to \infty} \int_0^t \int_{\Omega} w \cdot \nabla S_k : S_k \, dx \, dt = \int_0^t \int_{\Omega} w \cdot \nabla S : S \, dx \, dt.$$
Finally, we combine (4.33)–(4.41) with the energy equalities (4.18a) and (4.18b) to conclude the energy inequalities (4.9) and (4.10). Moreover, in the same way as for the classical Navier–Stokes initial-value problem (see [Gal00, Lemma 2.2] for example), the weak solution can be redefined on a set of measure zero such that it is weakly continuous in the sense of (4.8). This finishes the proof of Theorem 4.4.

**Remark 4.11.** From the proof of Theorem 4.4 we directly obtain

\[
\|v\|^2_{L^\infty(0,T',L^2(\Omega))} + \|S\|^2_{L^\infty(0,T';L^2(\Omega))} + \|v\|^2_{L^2(0,T';H^1(\Omega))} + \|S\|^2_{L^2(0,T';H^1(\Omega))} + \int_0^{T'} \mathcal{P}(S) \, dt \leq M_{T'},
\]

\[
\|\partial_t v\|_{L^1(0,T';(H^1_0(\Omega))')} \leq M_{T'},
\]

\[
\|\partial_t S\|_{L^3(0,T';(H^1(\Omega))')} \leq M_{T',\mathcal{P}},
\]

for each \(0 < T' < T\), where \(M_{T'}, M_{T'}\), and \(M_{T',\mathcal{P}}\) are given in Lemma 4.8 and Lemma 4.10.

### 4.3 Existence of a suitable extension

Since we have shown existence of a weak solution to (4.2) in Theorem 4.4, it remains to address the existence of solutions \(w\) to the Stokes initial-value problem (4.1). As mentioned above, the forcing term \(\bar{F}\) and the initial value \(w_0\) in (4.1) are not prescribed by the original problem, whence we have some freedom in their choice. For example, we can simply consider data \(\bar{F} = w_0 = 0\) and use existing theory for the Stokes initial-value problem with inhomogeneous Dirichlet data (see [Gru01, FGH02, Ray07] for example) to obtain a suitable extension \(w\) satisfying Assumption 4.1 in case (a). Proceeding like this for the cases (b) and (c) would require smallness of \(g\). In the following we focus on case (c), which is more general than (b), and show that smallness of \(g\) is not necessary if we exploit the freedom we have in the choice of \(\bar{F}\) and \(w_0\). For this purpose, we use the following well-known lemma.

**Lemma 4.12.** Let \(\Omega\) be a bounded domain with connected \(C^{1,1}\)-boundary, \(T \in (0,\infty]\) and let \(g\) satisfy

\[
g \in L^\infty(0, T; H^{1/2}(\partial\Omega)^3), \quad \partial_t g \in L^\infty(0, T; H^{-1/2}(\partial\Omega)^3), \quad (4.42a)
\]

\[
\int_{\partial\Omega} g(t) \cdot n = 0 \quad \text{for a.a. } t \in (0, T). \quad (4.42b)
\]

Then, for each \(\alpha > 0\) there exists a function

\[
w_\alpha \in L^\infty(0, T; H^1(\Omega)^3), \quad \partial_t w_\alpha \in L^\infty(0, T; H^{-1}(\Omega)^3) \quad (4.43)
\]

with \(w_\alpha = g\) on \(\partial\Omega \times (0, T)\) and \(\text{div} \, w_\alpha = 0\) in \(\Omega \times (0, T)\) such that

\[
\forall v_1, v_2 \in H^1_0(\Omega) : \left| \int_\Omega w_\alpha(t) \otimes v_1 : \nabla v_2 \, dx \right| \leq \alpha \|\nabla v_1\|_2 \|\nabla v_2\|_2
\]

and for all \(q \in [1, \infty]\) there exists a constant \(C_1 = C_1(\Omega, \alpha, q) > 0\) such that for all \(T' \in (0, T)\) it holds

\[
\|w_\alpha\|_{L^q(0, T'; H^1(\Omega))} \leq C_1 \|g\|_{L^q(0, T'; H^{1/2}(\partial\Omega))},
\]

\[
\|\partial_t w_\alpha\|_{L^q(0, T'; H^{-1}(\Omega))} \leq C_1 \|\partial_t g\|_{L^q(0, T'; H^{-1/2}(\partial\Omega))}. \quad (4.44)
\]
Proof. In [FKS11a, Proposition 5.4] the statement was shown with \( \alpha = 1/4 \). An adaptation of the proof for arbitrary \( \alpha > 0 \) is straightforward.

In order to ensure (4.5), it is sufficient to assume smallness of \( g \cdot n \) in a suitable norm.

**Lemma 4.13.** In the situation of Lemma 4.12 it holds

\[
\forall S \in H^1(\Omega)^{3 \times 3} : \left| \int_\Omega w_\alpha(t) \cdot \nabla S : S \, dx \right| \leq C_2 \| g \cdot n \|_{L^\infty(0,T;L^2(\partial \Omega))} \| S \|^2_{L^2(\Omega)}
\]

for some constant \( C_2 = C_2(\Omega) > 0 \).

Proof. We have

\[
\int_\Omega w_\alpha(t) \cdot \nabla S : S \, dx = \frac{1}{2} \int_\Omega w_\alpha(t) \cdot \nabla |S|^2 \, dx = \frac{1}{2} \int_{\partial \Omega} g(t) \cdot n |S|^2 \, d\sigma.
\]

Now Hölder’s inequality and Sobolev embeddings imply

\[
\left| \int_\Omega w_\alpha(t) \cdot \nabla S : S \, dx \right| \leq c_0 \| g(t) \cdot n \|_2 \| S \|^2_{L^4(\Omega)} \leq c_1 \| g(t) \cdot n \|_2 \| S \|^2_{H^{1/2}(\partial \Omega)}.
\]

The statement now follows from a standard trace inequality.

In particular, Lemma 4.13 implies that condition (4.5) is satisfied automatically when \( g \cdot n = 0 \), which is the case we are interested in.

### 4.4 Existence of a weak solution

Combining now Theorem 4.4, Lemma 4.12 and Lemma 4.13, we show existence of a weak solution to the original problem (1.1)–(1.3) in the sense of Definition 3.6.

**Proof of Theorem 3.7.** Let \( w = w_\alpha \) from Lemma 4.12 with \( \alpha = \mu/2 \). Then the Aubin–Lions lemma implies \( w \in C^0(0,T;L^2(\Omega)) \), and in virtue of \( g \cdot n = 0 \) and Lemma 4.13, we see that Assumption 4.1(c) is satisfied. From Hölder’s inequality and Sobolev embeddings we further conclude the remaining properties of Assumption 4.1. Since \( w \in C^0(0,T;L^2(\Omega)) \), we can define \( w_0 := w(\cdot,0) \in L^2(\Omega) \). Moreover, we set \( \tilde{F} := \partial_t w - \mu \Delta w \). Since every element of \( H^{-1}(\Omega)^n \) can be represented as the divergence of a tensor field from \( L^2(\Omega)^{n \times n} \) (see [Soh01, Lemma 1.6.1] for example), we obtain \( \tilde{F} = \text{div} \tilde{F}_1 \) for some \( \tilde{F}_1 \in L^\infty(0,T;L^2(\Omega)^{n \times n}) \). Now we set \( f_0 := F_0 \), \( f_1 := F_1 - \tilde{F}_1 \), \( v_0 := V_0 - w_0 \), and let \((v,S)\) be the weak solution to (4.2) from Theorem 4.4. Since for all \( \Phi \in C^\infty_0(\Omega \times [0,T])^3 \) we have

\[
\int_0^T \int_\Omega \left[ - w \cdot \partial_t \Phi + \mu \nabla w : \nabla \Phi \right] \, dx \, dt = - \int_0^T \int_\Omega \tilde{F}_1 : \nabla \Phi \, dx \, dt + \int_\Omega w_0 \cdot \Phi(\cdot,0) \, dx,
\]

the pair \((V,S) := (v+w,S)\) is a weak solution to (1.1)–(1.3) in the sense of Definition 3.6. Moreover, (3.16) follows from (4.8) and \( w \in C^0(0,T;L^2(\Omega)) \), and (3.10) is a direct consequence of (4.9). Similarly to the proof of Lemma 4.13, we further derive

\[
\int_\Omega w(t) \cdot \nabla S : S \, dx = \frac{1}{2} \int_{\partial \Omega} g(t) \cdot n |S|^2 \, d\sigma = 0
\]

since \( g \cdot n = 0 \). With this identity, (3.17) directly follows from (4.10).
We can further derive the following estimate, which will be needed for passing from smooth Moreau envelopes $P_\varepsilon$ to nonsmooth potentials $P$.

**Lemma 4.14.** For all $0 < T' < T$ there exists a constant $M_{T'} > 0$, which is independent of $P$, such that

\[
\|V\|_{L^\infty(0,T';L^2(\Omega))} + \|V\|_{L^2(0,T';H^1(\Omega))} + \|\partial_t V\|_{L^1(0,T';(H^1_0,\sigma(\Omega)))'} + \|S\|_{L^\infty(0,T';L^2(\Omega))} + \|S\|_{L^2(0,T';H^1(\Omega))} + \int_0^{T'} P(S) \, dt \leq M_{T'}.
\]  

**Proof.** By (4.44) we have

\[
\|w\|_{L^\infty(0,T';L^2(\Omega))} + \|w\|_{L^2(0,T';H^1(\Omega))} + \|\partial_t w\|_{L^1(0,T';(H^1_0,\sigma(\Omega)))'} \\
\leq c_0(T') (\|w\|_{L^\infty(0,T';H^1(\Omega))} + \|\partial_t w\|_{L^\infty(0,T';H^{-1}(\Omega))}) \\
\leq c_1(T') (\|g\|_{L^\infty(0,T';H^{1/2}(\partial\Omega))} + \|\partial_t g\|_{L^\infty(0,T';H^{-1/2}(\partial\Omega))}).
\]

In view of Remark 4.11 and the identity $V = v + w$, this shows (4.45). \qed

## 5 Generalized solutions for nonsmooth potentials

The primary purpose of this section is to prove our main result, Theorem 3.4, on the existence of generalized solutions in case of a nonsmooth potential $P$, see Section 5.3. Recall that throughout this manuscript $P: L^2_0(\Omega) \to [0,\infty]$ is assumed to be convex and lower semicontinuous with $P(0) = 0$. As outlined in Section 3.3, we proceed by regularization, approximating $P$ by its Moreau envelope $\{P_\varepsilon\}$, which allows us to take advantage of Theorem 3.7 providing existence for smooth potentials. The key to passing to the limit $\varepsilon \to 0$ in the tensorial evolutionary variational inequality for the approximate solutions is the convergence result in Lemma 5.4.

### 5.1 Weak versus generalized solutions

Here, we provide a basic consistency analysis concerning our new notion of solution for rate-type viscoelastoplastic models involving a nonsmooth potential in the tensorial evolution. In particular, we will show that the weak solutions from Theorem 3.7 obtained in the case of a smooth potential satisfy a variational inequality (for $S$) and hence are generalized solutions in the sense of Definition 3.1. This result will further serve as a basic ingredient in the proof of Theorem 3.4. We will also provide sufficient regularity conditions for $(V, S)$ that allow us to conclude that generalized solutions are already weak solutions in the sense of Definition 3.6. For this purpose, we need an approximation property for the induced potential $\mathcal{P}$ acting on Bochner functions

\[
\mathcal{P}(\tilde{S}) := \int_0^{T'} \mathcal{P}(\tilde{S}(t)) \, dt, \quad \tilde{S} \in L^2(0,T';L^2_0(\Omega)).
\]  

The approximation condition states:

\[
\forall \tilde{S} \in L^2(0,T';L^2_0(\Omega)) \quad \exists (\tilde{S}_n)_{n \in \mathbb{N}} \subset Z_{T'} : \ \\
\tilde{S}_n \to \tilde{S} \text{ in } L^2(0,T';L^2(\Omega)) \text{ and } \mathcal{P}(\tilde{S}_n) \to \mathcal{P}(\tilde{S}).
\]  

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This property is certainly satisfied if \( \mathcal{P}(S) = \int_0^T \int_\Omega \Psi(S(x,t)) \, dx \, dt \) for a continuous convex function \( \Psi: \mathbb{R}_{\delta}^{3 \times 3} \to [0, \infty) \) with \( \Psi(0) = 0 \) and a quadratic upper bound. In this case it suffices to choose an arbitrary sequence \((\tilde{S}_n)\) converging to \( \tilde{S} \) strongly in \( L^2(0,T';L^2(\Omega)) \), because \( \mathcal{P} \) is norm continuous. If \( \Psi \) is the sum of such a function and an indicator function

\[
\iota_A(S) := \begin{cases} 
0 & \text{for } S \in A, \\
\infty & \text{for } S \notin A 
\end{cases}
\]

for a closed and convex subset \( A \subset \mathbb{R}_{\delta}^{3 \times 3} \), a suitable sequence \((\tilde{S}_n)\) can be constructed by means of mollification taking into account the convexity and closedness of the set \( A \) as well as the fact that \( \Psi(0) = 0 \) (which implies \( 0 \in A \)). In particular, (5.2) holds for the plasticity potential \( \mathcal{P} \) defined in (1.4).

**Lemma 5.1** (Weak versus generalized solutions).

(A) Assume that \( \mathcal{P} \in C^{1,1}(L^2_\delta(\Omega)) \). If \( (V,S) \) is a weak solution in the sense of Definition 3.6 that additionally satisfies the partial energy dissipation inequality (3.17), then it is also a generalized solution in the sense of Definition 3.1.

(B) If \( (V,S) \) is a generalized solution in the sense of Definition 3.1 with the additional regularity

\[
S \in H^1(0,T';L^2(\Omega)) \cap L^2(0,T';H^2(\Omega)) \cap L^\infty(0,T';L^\infty(\Omega))
\]

(5.3) for all \( T' < T \), and if \( \mathcal{P} \) satisfies the approximation property (5.2), then it is also a weak solution in the sense of Definition 3.6, where (3.15) is replaced by

\[
\int_0^T \int_\Omega \left[ -S : \partial_t \Psi + V : \nabla S : \Psi + (SW(V) - W(V)S) : \Psi + \beta : \Psi + \gamma \nabla S : \nabla \Psi - \eta D(V) : \Psi \right] \, dx \, dt = \int_\Omega S_0 : \Psi(0) \, dx
\]

(5.4)

for all \( \Psi \in C^{\infty}_{0,\delta}(\overline{\Omega} \times [0,T)) \), where \( \beta \in L^2(0,T;L^2_\delta(\Omega)) \) with \( \beta(t) \in \partial \mathcal{P}(S(t)) \) for a.a. \( t \in (0,T) \).

**Remark 5.2.** By Proposition 3.3, generalized solutions with the extra regularity assumed in Lemma 5.1 (B) not only satisfy the weak formulation (5.4) as asserted in the above lemma, but further fulfill the partial energy inequality (3.9), which is genuinely encoded in the evolutionary variational inequality.

**Proof.** Part (A): Let \( (V,S) \) be a weak solution. We have to derive the evolutionary variational inequality (3.7). For this purpose, we first show that the weak equation (3.15) for \( S \) implies a modified version that holds true for test functions \( \tilde{S} \in C^{\infty}_{\delta}(\overline{\Omega} \times [0,T']) \), \( T' < T \), not necessarily compactly supported in time. More precisely, we assert that for all \( T' < T \) and all \( \tilde{S} \in C^{\infty}_{\delta}(\overline{\Omega} \times [0,T']) \)

\[
\int_0^{T'} \int_\Omega \left[ -S : \partial_t \tilde{S} + V : \nabla S : \tilde{S} + (SW(V) - W(V)S) : \tilde{S} + \partial \mathcal{P}(S) : \tilde{S} + \gamma \nabla S : \nabla \tilde{S} - \eta D(V) : \tilde{S} \right] \, dx \, dt
\]

(5.5)

\[
= \int_\Omega S_0 : \tilde{S}(0) \, dx - \int_\Omega S(T') : \tilde{S}(T') \, dx.
\]

The proof of this assertion follows from a standard extension and approximation argument applied to the test functions \( \tilde{S} \): given \( \tilde{\tilde{S}} \in C^{\infty}_{\delta}(\overline{\Omega} \times [0,T']) \), we can find an extension
\( \hat{S} \in C^\infty_0(\overline{\Omega} \times [0, T]) \) of \( \hat{S} \). We then choose \( \theta \in C^\infty(\mathbb{R}) \), \( \theta' \geq 0 \), \( \theta = 0 \) on \( (-\infty, -1] \), \( \theta = 1 \) on \( [0, \infty) \), let \( \theta_{T', \delta}(t) := \theta(T' - t) \) and define for \( 0 < \delta \ll T - T' \) the function \( \Psi_\delta(t, x) := \theta_{T', \delta}(t) \hat{S}(t, x) \in C^\infty_0(\overline{\Omega} \times [0, T]) \), which satisfies \( \Psi_\delta(t, \cdot) = \hat{S}(t, \cdot) \) for all \( t \in [0, T'] \) and \( \Psi_\delta(t, \cdot) = 0 \) for all \( t \in [T' + \delta, T) \). Inserting \( \Psi_\delta \) as a test function in the weak formulation \( (3.15) \), performing standard manipulations and sending \( \delta \to 0 \), we arrive at \( (5.5) \). (For more details on the limit \( \delta \to 0 \), we refer the proof of Prop. 3.3 in Section 5.4, where related arguments are carried out in a setting of lower regularity.)

Subtracting inequality \( (3.17) \) (at time \( t = T' \)) we find, upon rearranging terms,

\[
\int_0^{T'} \int_{\Omega} \left[ -S : \partial_t \hat{S} + \partial \mathcal{P}(S) : (\hat{S} - S) + \gamma \nabla S : \nabla (\hat{S} - S) + V \cdot \nabla S : \hat{S} \right. \\
+ \left. (SW(V) - W(V)S) : \hat{S} - \eta D(V) : (\hat{S} - S) \right] \, dx \, dt \\
\geq -\frac{1}{2} \| S_0 \|^2_2 + \int_{\Omega} S_0 : \hat{S}(0) \, dx + \frac{1}{2} \| S(T') \|^2_2 - \int_{\Omega} S(T') : \hat{S}(T') \, dx.
\]

By the density of \( C^\infty_0(\overline{\Omega} \times [0, T']) \) in \( Z_T \), the last inequality continues to hold for all \( \hat{S} \in Z_T \). The assertion is now obtained by adding \( \int_0^{T'} \int_{\Omega} \partial_t \hat{S} : S_0 \, dx \, dt = \frac{1}{2} \| \hat{S}(T') \|^2_2 - \frac{1}{2} \| \hat{S}(0) \|^2_2 \), and using the fact that \( \frac{1}{2} \| \hat{S}(T') - S(T') \|^2_2 \geq 0 \) as well as the inequality \( \mathcal{P}(\hat{S}(t)) - \mathcal{P}(S(t)) \geq \int_{\Omega} \partial \mathcal{P}(S(t)) : (\hat{S}(t) - S(t)) \, dx \).

**Part (B):** For a generalized solution \((V, S)\) with the smoothness as in \( (5.3) \) we have

\[
\beta := -\hat{S} + \gamma \Delta S + \eta D(V) \in L^2([0, T']; L^2_2(\Omega)).
\]

As a consequence of \( (3.7) \), we further note that \( \mathcal{P}(S) = \int_0^{T'} \mathcal{P}(S) \, dt < \infty \).

Using the Zaremba–Jaumann identity \( (1.5) \) and the definition of \( \beta \), we find

\[
\int_0^{T'} \int_{\Omega} \left( V \cdot \nabla S : \hat{S} + (SW(V) - W(V)S) : \hat{S} \right) \, dx \, dt \\
= \int_0^{T'} \int_{\Omega} \left( \hat{S} - \partial_t S \right) : \hat{S} \, dx \, dt = \int_0^{T'} \int_{\Omega} \left( \hat{S} - \partial_t S \right) : (\hat{S} - S) \, dx \, dt \\
= \int_0^{T'} \int_{\Omega} \left( -\partial_t S - \beta + \gamma \Delta S + \eta D(V) \right) : (\hat{S} - S) \, dx \, dt.
\]

Inserting this identity into \( (3.7) \), we are left with the variational inequality

\[
\int_0^{T'} \int_{\Omega} \left( (\partial_t \hat{S} - \partial_t S) : (\hat{S} - S) - \beta : (\hat{S} - S) \right) \, dx \, dt + \mathcal{P}(\hat{S}) - \mathcal{P}(S) \\
\geq -\frac{1}{2} \| \hat{S}(0) - S_0 \|^2_2
\]

for all \( \hat{S} \in Z_T \), where we recall the definition of \( \mathcal{P} \) in \( (5.1) \). Given \( \hat{R} \in Z_T \) with \( \hat{R}(0) = 0 = \hat{R}(T') \), we choose in \( (5.7) \) the test function \( \hat{S} = S + \hat{R} \), which by \( (5.3) \) lies in \( Z_T \) and moreover satisfies \( \hat{S}(0) = S_0 \), to infer

\[
\mathcal{P}(S + \hat{R}) - \mathcal{P}(S) \geq \int_0^{T'} \int_{\Omega} \beta : \hat{R} \, dx \, dt
\]

for all such \( \hat{R} \).
We assert that by means of an approximation argument, ineq. (5.8) can be extended to general \( \tilde{R} \in Z_{T'} \), not necessarily vanishing at the boundary of \((0, T')\). To see this, we pick a sequence \( \{\theta_j\} \subset C^\infty_c((0, T')) \) with \( 0 \leq \theta_j \leq \theta_{j+1} \leq 1 \) for all \( j \in \mathbb{N} \) and such that \( \lim_{j \to \infty} \theta_j(t) = 1 \) for all \( t \in (0, T') \). We then infer from ineq. (5.8) for general \( \tilde{R} \in Z_{T'} \)

\[
\int_0^{T'} (\mathcal{P}(S + \theta_j \tilde{R}) - \mathcal{P}(S)) \, dt \geq \int_0^{T'} \theta_j(t) \int_\Omega \beta : \tilde{R} \, dx \, dt. \tag{5.9}
\]

Since \( \mathcal{P} \) is convex, we have for every \( \theta = \theta_j(t) \in [0, 1] \)

\[
\mathcal{P}(S + \theta \tilde{R}) - \mathcal{P}(S) = \mathcal{P}(\theta(S + \tilde{R}) + (1 - \theta)S) - \mathcal{P}(S) \leq \theta \mathcal{P}(S + \tilde{R}) - \theta \mathcal{P}(S).
\]

Inserting this inequality into (5.9) gives

\[
\int_0^{T'} \theta_j(t) \mathcal{P}(S + \tilde{R}) \, dt - \int_0^{T'} \theta_j(t) \mathcal{P}(S) \, dt \geq \int_0^{T'} \theta_j(t) \int_\Omega \beta : \tilde{R} \, dx \, dt.
\]

Invoking the monotone convergence theorem for the first term in the last line and using dominated convergence for the remaining two time integrals, we can take the limit \( j \to \infty \) in the last inequality and arrive at (5.8) for general \( \tilde{R} \in Z_{T'} \).

Due to the approximation property (5.2) of \( \mathcal{P} \), we can further extend (5.8) to general \( R \in L^2(0, T'; L^2_\delta(\Omega)) \). Indeed, letting \( \tilde{S} := S + R \in L^2(0, T'; L^2_\delta(\Omega)) \), property (5.2) provides us with a sequence \( (\tilde{S}_n) \subset Z_{T'} \) such that \( \tilde{S}_n \to S + R \) in \( L^2(0, T'; L^2_\delta(\Omega)) \) and \( \mathcal{P}(\tilde{S}_n) \to \mathcal{P}(S + R) \). Hence, inserting \( \tilde{R} = \tilde{R}_n := \tilde{S}_n - S \) in (5.8) and passing to the limit \( n \to \infty \) we obtain

\[
\mathcal{P}(S + R) \geq \mathcal{P}(S) + \int_0^{T'} \int_\Omega \beta : R \, dx \, dt \quad \text{for all } R \in L^2(0, T'; L^2_\delta(\Omega)).
\]

But this is exactly the definition of \( \beta \in \partial \mathcal{P}(S) \), and the special definition of \( \mathcal{P} \) in terms of \( \mathcal{P} \) (cf. (5.1)) implies \( \beta(t) \in \partial \mathcal{P}(S(t)) \) a.e. on \((0, T')\).

The definition of \( \beta \) in (5.6) implies the desired weak equation (5.4). \qed

### 5.2 Properties of the Moreau envelope

Recall the definition of the Moreau envelope \( \{\mathcal{P}_\varepsilon\} \) in (3.14). As an immediate consequence of (3.14), we find that \( \mathcal{P}_\varepsilon(S) \leq \mathcal{P}(S) \) and hence

\[
\limsup_{\varepsilon \to 0} \int_0^T \mathcal{P}_\varepsilon(S(t)) \, dt \leq \int_0^T \mathcal{P}(S(t)) \, dt \quad \text{for all } S \in L^2(0, T; L^2_\delta(\Omega)). \tag{5.10}
\]

The proof of Theorem 3.4 further makes use of the following version of the classical approximation properties of the Moreau envelope [BaC17, Rou13].

**Lemma 5.3.** Let \( \mathcal{P} \) be as in (3.1). The Moreau envelope \( \{\mathcal{P}_\varepsilon\} \) of \( \mathcal{P} \), as defined in (3.14), satisfies the inequality

\[
\liminf_{\varepsilon \to 0} \int_0^T \mathcal{P}_\varepsilon(S_\varepsilon(t)) \, dt \geq \int_0^T \mathcal{P}(S(t)) \, dt \tag{5.11}
\]

whenever \( S_\varepsilon \rightharpoonup S \) in \( L^2(0, T; L^2_\delta(\Omega)) \).
Proof. Let $\delta > \varepsilon > 0$. Then, by definition, $\mathcal{P}_\varepsilon \geq \mathcal{P}_\delta$. Hence
\[
\liminf_{\varepsilon \to 0} \int_0^T \mathcal{P}_\varepsilon(S_\varepsilon(t)) \, dt \geq \liminf_{\varepsilon \to 0} \int_0^T \mathcal{P}_\delta(S_\varepsilon(t)) \, dt \geq \int_0^T \mathcal{P}_\delta(S(t)) \, dt. \tag{5.12}
\]
The second step follows from the fact that for $\delta > 0$ the functional
\[
F_\delta : L^2(0, T; L^2_\delta(\Omega)) \ni S \mapsto \int_0^T \mathcal{P}_\delta(S(t)) \, dt
\]
is convex and continuous, and thus weakly lower semicontinuous. The convexity of $F_\delta$ is inherited from the convexity of $\mathcal{P}_\delta$, while continuity follows from standard theory on Nemytskii operators (see e.g. [Rou13, Theorem 1.43]) together with the growth condition
\[
0 \leq \mathcal{P}_\delta(S) \leq \|S\|^2_{L^2(\Omega)}/(2\delta),
\]
which is a consequence of the definition of the Moreau envelope.

To show the assertion, it now remains to prove that
\[
\lim_{\delta \to 0} \int_0^T \mathcal{P}_\delta(S(t)) \, dt = \int_0^T \mathcal{P}(S(t)) \, dt. \tag{5.13}
\]

By [BaCl17, Proposition 12.33 (ii)], $\mathcal{P}_\delta(S(t)) \to \mathcal{P}(S(t))$ for a.e. $t \in (0, T)$. The nonnegativity of $\mathcal{P}_\delta$ and Beppo Levi’s monotone convergence imply the identity (5.13). Together with (5.12) the desired assertion (5.11) follows. \hfill \Box

5.3 Proof of the main result (Theorem 3.4)

In this subsection, we will prove our main theorem. Let us start by highlighting the following elementary result, which presents the crucial idea for passing to the limit, along approximate solutions, in the tensorial evolutionary variational equality and in particular in the nonlinear terms arising from the Zaremba–Jaumann derivative.

Lemma 5.4. Let $V_\varepsilon = (V_\varepsilon^i)$ and $S_\varepsilon = (S_\varepsilon^{i,j,k})$ satisfy the conditions
\[
V_\varepsilon \to V \text{ in } L^2(\Omega \times (0, T')), \quad V_\varepsilon \to V \text{ and } S_\varepsilon \to S \text{ in } L^2(0, T'; H^1(\Omega)),
\]
let $V_\varepsilon|_{\partial \Omega} = g \in L^2(0, T'; L^2(\partial \Omega))$ be fixed, and assume that $\|S_\varepsilon\|_{L^5(\Omega \times (0, T'))} \leq C$.

Then, for all $i, j, k, l \in \{1, 2, 3\}$ we have
\[
\lim_{\varepsilon \to 0} \int_0^{T'} \int_\Omega S_\varepsilon^{i,j} \partial_k V_\varepsilon^i \psi \, dx \, dt = \int_0^{T'} \int_\Omega S_{ij} \partial_k V_1 \psi \, dx \, dt \tag{5.14}
\]
for all $\psi \in L^5(\Omega \times (0, T'))$.

Proof. We first show that (5.14) holds for $\psi \in C^1(\overline{\Omega} \times [0, T'])$. In this case, integration by parts with respect to the spatial variable gives
\[
\int_0^{T'} \int_\Omega S_\varepsilon^{i,j} \partial_k V_\varepsilon^i \psi \, dx \, dt = \int_0^{T'} \int_{\partial \Omega} S_{ij}^\varepsilon g \sigma_k \psi \, d\sigma \, dt - \int_0^{T'} \int_\Omega \partial_k (S_\varepsilon^{i,j} \psi) V_\varepsilon^i \, dx \, dt,
\]
where we have already exploited the boundary conditions $V_\varepsilon = g$ on $\partial \Omega$.

Because of the continuity of the trace operator from $H^1(\Omega)$ to $L^2(\partial \Omega)$, we have $S_\varepsilon \to S$ in $L^2(0, T'; L^2(\partial \Omega))$ and can pass to the limit in the first term on the right-hand side.
For the last term we use the weak convergence of $S_\varepsilon$ to $S$ in $L^2(0,T';H^1(\Omega))$ as well as the strong convergence of $V_\varepsilon$ to $V$ in $L^2(\Omega \times (0,T'))$. Undoing the spatial integration by parts, we obtain the desired result for $\psi \in C^1(\overline{\Omega} \times [0,T'])$.

The validity of (5.14) for general $\psi \in L^5(\Omega \times (0,T'))$ is now a consequence of the density of $C^1(\overline{\Omega} \times [0,T'])$ in $L^5(\Omega \times (0,T'))$ and the fact that, by Hölder’s inequality (with $\frac{3}{10} + \frac{1}{2} = \frac{4}{5}$), the sequence $\{S_{ij}^\varepsilon \partial_k V^\varepsilon\}_\varepsilon$ is $\varepsilon$-uniformly bounded in $L^\frac{4}{5}(\Omega \times (0,T'))$. \hfill $\Box$

We are now in a position to complete the proof of Theorem 3.4 and show existence of generalized solutions to (1.1).

**Proof of Theorem 3.4.** For $\varepsilon \in (0,1]$ consider the Moreau envelope $P_\varepsilon$ for $P$ as introduced in Section 2, and denote by $(V_\varepsilon,S_\varepsilon)$ the weak solution constructed in Theorem 3.7 with $P$ replaced by $P_\varepsilon$. By estimate (4.45), we have the $\varepsilon$-uniform bound

$$
\|V_\varepsilon\|_{L^\infty(0,T';L^2(\Omega))} + \|V_\varepsilon\|_{L^2(0,T';H^1(\Omega))} + \|\partial_t V_\varepsilon\|_{L^1(0,T';(H^1_{\text{div}}(\Omega)))'}
+ \|S_\varepsilon\|_{L^\infty(0,T';L^2(\Omega))} + \|S_\varepsilon\|_{L^2(0,T';H^1(\Omega))} + \int_0^T \int_\Omega P_\varepsilon(S_\varepsilon) \, dx \, dt 
\leq M_1,
$$

(5.15)

for all $T' < T$. Hence, there exists a sequence $\varepsilon \to 0$ (not relabeled) and a pair $(V,S)$ such that for all $T' < T$ one has $(V,S) \in \mathcal{L}H_{T'} \times X_{T'}$ and

$$
V_\varepsilon \rightharpoonup V \quad \text{in} \quad L^2(0,T';H^1(\Omega))^3,
S_\varepsilon \rightharpoonup S \quad \text{in} \quad L^2(0,T';H^1(\Omega))^{3 \times 3},
V_\varepsilon \to V \quad \text{in} \quad L^\infty(0,T';L^2(\Omega)),
S_\varepsilon \to S \quad \text{in} \quad L^\infty(0,T';L^2(\Omega))^3,
V_\varepsilon \to V \quad \text{in} \quad L^2(0,T';L^2(\Omega))^3,
$$

where, as in the proof of Theorem 4.4, the strong convergence of $(V_\varepsilon)$ is obtained from an Aubin–Lions compactness result.

The passage to the limit $\varepsilon \to 0$ in the weak form (3.6) of the equation for the velocity field $V_\varepsilon$ follows from standard arguments based on the above convergence properties. As a result, the limiting vector field $V$ satisfies eq. (3.6). Moreover, the fact that $V_\varepsilon|_{\partial \Omega \times (0,T)} = g$ combined with the above convergence properties easily yields $V|_{\partial \Omega \times (0,T)} = g$. Thus, it remains to show that $S$ satisfies inequality (3.7) for all $\tilde{S} \in Z_{T'}$.

By Lemma 5.1 (A), $S_\varepsilon$ satisfies the variational inequality

$$
\int_0^T \int_\Omega \partial_t \tilde{S} : (\tilde{S} - S_\varepsilon) + \gamma \nabla S_\varepsilon : \nabla (\tilde{S} - S_\varepsilon) \, dx \, dt + \int_0^T \int_\Omega \mathcal{P}_\varepsilon(\tilde{S}) - \mathcal{P}_\varepsilon(S_\varepsilon) \, dt
\geq -\frac{\gamma}{2} \|\tilde{S}(0) - S_0\|_2^2.
$$

(5.16)

We will deduce ineq. (3.7) by estimating the $\limsup_{\varepsilon \to 0}$ of the left-hand side.

First, the weak convergence $S_\varepsilon \to S$ in $L^2(0,T';H^1(\Omega))^{3 \times 3}$ implies that

$$
\int_0^T \int_\Omega \partial_t \tilde{S} : (\tilde{S} - S) + \gamma \nabla S : \nabla (\tilde{S} - S) \, dx \, dt
\geq \limsup_{\varepsilon \to 0} \int_0^T \int_\Omega \partial_t \tilde{S} : (\tilde{S} - S_\varepsilon) + \gamma \nabla S_\varepsilon : \nabla (\tilde{S} - S_\varepsilon) \, dx \, dt,
$$

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where we used weak upper semicontinuity of the concave quadratic term.

For the second term on the left-hand side of (5.16), we use the bound
\[
\limsup_{\varepsilon \to 0} \int_0^{T'} (\mathcal{P}_\varepsilon(\bar{S}) - \mathcal{P}_\varepsilon(S_\varepsilon)) \, dt \leq \int_0^{T'} (\mathcal{P}(\bar{S}) - \mathcal{P}(S)) \, dt
\]
which is consequence of Lemma 5.3 and inequality (5.10).

Further note that \( V_\varepsilon \to V \) in \( L^2(0,T';L^2(\Omega)) \) and \( S_\varepsilon \to S \) in \( L^2(0,T';H^1(\Omega)) \) imply
\[
\lim_{\varepsilon \to 0} \int_0^{T'} \int_\Omega V_\varepsilon \cdot \nabla S_\varepsilon : \bar{S} \, dx \, dt = \int_0^{T'} \int_\Omega V \cdot \nabla S : \bar{S} \, dx \, dt.
\]

The term \( (S_\varepsilon W(V_\varepsilon) - W(V_\varepsilon)S_\varepsilon) : \bar{S} + \eta D(V_\varepsilon) : S_\varepsilon \) consists of a finite linear combination of terms handled in Lemma 5.4. This lemma can be applied thanks to the convergence properties of \( (V_\varepsilon, S_\varepsilon) \) and the interpolation (3.8) ensuring the boundedness of \( (S_\varepsilon) \) in \( L^{4/3}(\Omega \times (0,T')) \). Hence we can pass to the limit with all remaining parts in the left-hand side.

The above observations allow us to estimate the \( \limsup_{\varepsilon \to 0} \) of the left-hand side of (5.16) above by the left-hand side of inequality (3.7). Hence \((V,S)\) is a generalized solution to (1.1)–(1.3) in the sense of Definition 3.1.

Since (3.11) directly follows from (3.10) and (3.9), and the latter follows from Proposition 3.3, which is proved below, it remains to establish the partial energy-dissipation inequality (3.10). But this is a simple consequence of the previously established partial energy-dissipation inequality. In particular, we note that the boundary extension \( w \) constructed for Theorem 3.7 depends only on \( g \) and, hence, is independent of the regularization parameter \( \varepsilon \). Thus, we can use the partial energy-dissipation inequality (3.10) for \( v_\varepsilon = V_\varepsilon - w \) and \( S_\varepsilon \). With the given weak and strong convergences, we can pass to the limit \( \varepsilon \to 0 \) and obtain the corresponding inequality for the limits \( v = V - w \) and \( S \).  \( \square \)

Finally, we show that, under suitable decay assumptions on the data, the total energy of the solution remains bounded.

**Proof of Corollary 3.5.** The pair \((v,S)\) satisfies the energy inequality (3.11), and \( w \) is constructed in such a way that Assumption 4.1 (c) is satisfied. Proceeding as in the proof of Theorem 4.4, we can then estimate the right-hand side of (3.11) and use an absorption argument to conclude
\[
\|v\|_{L^\infty(0,T';L^2(\Omega))}^2 + \|\nabla v\|_{L^2(\Omega \times (0,T'))}^2 + \|S\|_{L^2(\Omega \times (0,T'))}^2 + \int_0^{T'} \mathcal{P}(S) \, dt \leq C \left( \|V_0 - w(0)\|_2^2 + \|S_0\|_2^2 + \|F_0\|_{L^2(\Omega \times (0,T'))}^2 + \|F_1\|_{L^2(\Omega \times (0,T'))}^2 \right)
\]
for any \( T' \in (0,\infty) \). To further estimate the right-hand side, recall from the proof of Theorem 3.7 in Subsection 4.4 that \( w \in C^0(0,T';L^2(\Omega)) \) and \( \text{div} \bar{F}_1 = \partial_t w - \mu \Delta w \in L^2(0,T';H^{-1}(\Omega)) \). We further obtain
\[
\|w(0)\|_2 \leq \|w\|_{L^\infty(0,T';L^2(\Omega))} \leq C \|g\|_{L^\infty(0,T';H^{1/2}(\partial\Omega))}
\]
and
\[ \| \bar{F}_1 \|_{L^2(\Omega \times (0,T'))} \leq C \left( \| \partial_t w \|_{L^2(0,T';H^{-1}(\Omega))} + \| w \|_{L^2(0,T';H^1(\Omega))} \right) \]
\[ \leq C \left( \| \partial_t g \|_{L^2(0,T';H^{-1/2}(\Omega))} + \| g \|_{L^2(0,T';H^{1/2}(\Omega))} \right), \]
where we used (3.13) in the respective last estimate. By means of (3.13) we can also estimate the last two terms in (5.17) in terms of \( g \). Then a standard interpolation argument shows that the right-hand side of (5.17) is bounded by a constant independent of \( T' \in (0, \infty) \). This shows \( (v, S) \in L^2(0, T') \times X_T \) for \( T = \infty \). Since we have \( V = v + w \), and \( w \in L^2_T \) for \( T = \infty \) by (3.13), this completes the proof.

### 5.4 Partial energy inequality

**Proof of Proposition 3.3.** Observe that choosing \( \tilde{S} \equiv 0 \) in (3.7) shows that \( \int_0^{T'} P(S) \, d\tau < \infty \). Let us further note that since \( S \in L^\infty(0, T'; L^2(\Omega)) \), almost every \( T' \in (0, T) \) is a left Lebesgue point of \( t \mapsto S(t) \in L^2(\Omega) \).

Extend now \( S \) by zero for \( t < 0 \) and consider for \( \kappa > 0 \)
\[ S_\kappa(t) = \kappa^{-1} \int_{t-\kappa}^t S(\tau) \, d\tau. \]

Further let \( \eta \in C^\infty(\mathbb{R}; [0, 1]) \) be nondecreasing, \( \eta(t) = 0 \) for \( t \leq -1 \) and \( \eta(t) = 1 \) for \( t \geq 0 \), and define \( \eta_\delta(t) := \eta_\delta(T')(t) := \eta((t - T')/\delta) \). We then choose in (3.7) the test function \( \hat{S} := \hat{S}_{\kappa, \delta} := \eta_\delta S_\kappa \in Z_{T'}, \) where \( \delta \in (0, \delta_*] \) and \( \kappa \in (0, \kappa_*] \) are chosen sufficiently small and, in particular, such that \( \hat{S}_{\kappa, \delta}(0) = 0 \).

Since \( P \) is convex with \( P(0) = 0 \) and \( 0 \leq \eta_\delta \leq 1 \), we can estimate using Jensen’s inequality
\[ \int_0^{T'} P(\hat{S}_{\kappa, \delta}) \, d\tau \leq \kappa^{-1} \int_{T' - \delta}^{T'} \eta_\delta(t) \int_{t-\kappa}^t P(S(\tau)) \, d\tau \, dt \leq \kappa^{-1} \int_0^{T'} P(S) \, d\tau \cdot \delta, \]
where we also used the nonnegativity of \( P \). Since \( \int_0^{T'} P(S) \, d\tau < \infty \), the last line implies that \( \lim_{\delta \to 0} \int_0^{T'} P(\hat{S}_{\kappa, \delta}) \, d\tau = 0 \) for any \( \kappa \in (0, \kappa_*] \).

Let us next turn to the integral involving the time derivative. Using the fact that \( \eta_\delta(T') = 1 \), we find
\[ \int_0^{T'} \int_\Omega \partial_t \hat{S} : (\hat{S} - S) \, dx \, dt = \frac{1}{2} \| S_\kappa(T') \|_2^2 - \int_0^{T'} \eta_\delta'(t) \int_\Omega S_\kappa : S \, dx \, dt \]
\[ - \int_{T' - \delta}^{T'} \eta_\delta(t) \int_\Omega \partial_t S_\kappa : S \, dx \, dt. \]

Since \( S \in L^\infty(0, T'; L^2(\Omega)) \), we easily see that the term in the last line vanishes as \( \delta \to 0 \). Furthermore, we have the following convergence results, valid for almost all \( T' \in (0, T) \):
\[ \lim_{\delta \to 0} \int_0^{T'} \eta_\delta(t) \int_\Omega S_\kappa : S \, dx \, dt = \int_\Omega S_\kappa(T') : S(T') \, dx, \]
\[ \lim_{\kappa \to 0} \int_\Omega S_\kappa(T') : S(T') \, dx = \| S(T') \|_2^2, \]
\[ \lim_{\kappa \to 0} \frac{1}{2} \| S_\kappa(T') \|_2^2 = \frac{1}{2} \| S(T') \|_2^2. \]
Thus, for almost all $T'$ we obtain

$$
\lim_{\kappa \to 0} \lim_{\delta \to 0} \int_0^T \int_\Omega \partial_t \tilde{S}_{\kappa,\delta} : (\tilde{S}_{\kappa,\delta} - S) \, dx \, dt = -\frac{1}{2} \|S(T')\|^2_2.
$$

All remaining integrals in (3.7) involving $\tilde{S}$ converge to zero as $\delta \to 0$, as long as $\kappa$ is positive. Thus, sending first $\delta \to 0$ in (3.7) (with $\tilde{S} = \tilde{S}_{\kappa,\delta}$), and taking subsequently the limit $\kappa \to 0$, we arrive at (3.9).

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