Bogomolny Yang-Mills-Higgs Solutions in (2+1) anti-de Sitter Space

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Abstract
This paper investigates an integrable system which is related to hyperbolic monopoles; ie the Bogomolny Yang-Mills-Higgs equations in (2+1) anti-de Sitter space which are integrable and whose solutions can be obtained using analytical methods. In particular, families of soliton solutions have been constructed explicitly and their dynamics has been investigated in some detail.

I. Introduction

Static BPS monopoles are solutions of a nonlinear elliptic partial differential equation on some three-dimensional Riemannian manifold. Most work on monopoles has dealt with the case when this manifold is Euclidean space $\mathbb{R}^3$ since the equations are integrable and geometrical techniques can be applied. [The introduction of time dependence destroys the integrability]. In addition, the monopole equations on hyperbolic space $\mathbb{H}^3$ are also integrable [1] and often hyperbolic monopoles turn out to be easier to study than the Euclidean (see, for example, [2]). Moreover, recently, it has been rigorously established [3] that in the limit as the curvature of hyperbolic space tends to zero then Euclidean monopoles are recovered. In this paper, we consider an integrable system [4] which is related to hyperbolic monopoles and follows from replacing the positive definite space $\mathbb{H}^3$ by a Lorentzian version, ie the anti-de Sitter space. In recent years, the $n$-dimensional anti-de Sitter spacetime has been of continuing interest since it is the base of M-theory and a source of simple examples studying methods and spacetime concepts both on classical and quantum level. It also arises as the natural ground state of gauged supergravity theories when quantized [5].

The Bogomolny version of Yang-Mills-Higgs equations for Yang-Mills-Higgs fields on a
three-dimensional Riemannian manifold \( (\mathcal{M}) \) with gauge group \( SU(2) \) have the form

\[
D_i \Phi = \frac{1}{2 \sqrt{|g|}} g_{ij} \epsilon^{jkl} F_{kl}.
\] (1)

Here \( A_k \), for \( k = 0, 1, 2 \), is the \( su(2) \)-valued gauge potential, with field strength \( F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \) and \( \Phi = \Phi(x^\mu) \) is the \( su(2) \)-valued Higgs field; while \( x^\mu = (x^0, x^1, x^2) \) represent the local coordinates on \( M \). The action of the covariant derivative \( D_i = \partial_i + A_i \) on \( \Phi \) is:

\[
D_i \Phi = \partial_i \Phi + [A_i, \Phi].
\]

Equation (1) is integrable in the sense that a Lax pair exists for constant curvature. In particular, the solutions of (1) correspond to Euclidean or hyperbolic BPS monopoles when \( (\mathcal{M}, g) \) is Euclidean \( \mathbb{R}^3 \) or hyperbolic \( \mathbb{H}^3 \) space.

There are two curved spacetimes with constant curvature: (i) the de Sitter space with positive scalar curvature and (ii) the anti-de Sitter space with negative curvature. By definition the \( (2+1) \)-dimensional anti-de Sitter space is the universal covering space of the hyperboloid \( \mathcal{H} \) satisfied by the equation

\[
U^2 + V^2 - X^2 - Y^2 = 1
\] (2)

with metric given by

\[
ds^2 = -dU^2 - dV^2 + dX^2 + dY^2.
\] (3)

By parametrizing the hyperboloid \( \mathcal{H} \) by

\[
U = \sec \rho \cos \theta \\
V = \sec \rho \sin \theta \\
X = \tan \rho \cos \phi \\
Y = \tan \rho \sin \phi
\] (4)

for \( \rho \in [0, \pi/2) \), the corresponding metric takes the form

\[
ds^2 = \sec^2 \rho \left( -d\theta^2 + d\rho^2 + \sin^2 \rho \, d\phi^2 \right).
\] (5)

The spacetime contains closed timelike curves, due to the periodicity of \( \theta \) (for more details, see Ref. [6]). In fact, anti-de Sitter space (as a manifold) is the product of an open spatial disc with \( \theta \) and constant curvature equal to minus six; where \((\rho, \phi)\) correspond to polar coordinates and \( \theta \in \mathbb{R} \) being the time. Null spacelike infinity \( \mathcal{I} \) consists of the timelike cylinder \( \rho = \pi/2 \) and this surface is never reached by timelike geodesics.

If the Poincaré coordinates \((r, x, t)\) for \( r > 0 \) are defined as

\[
r = \frac{1}{U + X} \\
x = \frac{Y}{U + X} \\
t = \frac{-V}{U + X}
\] (6)
Figure 1: The Penrose diagram of anti-de Sitter space. The boundary of anti-de Sitter is the boundary of the cylinder.

the metric simplifies to the following form

\[ ds^2 = r^{-2}(-dt^2 + dr^2 + dx^2). \]  

(7)

Note that, the Poincaré coordinates cover a small part of anti-de Sitter space, i.e., that corresponding to half of the hyperboloid \( \mathcal{H} \) for \( U + X > 0 \); which is the shaded region in FIG. [4]. The surface \( r = 0 \) is part of infinity \( \mathcal{I} \).

Hitchin [7] show that the minitwistor space corresponding to Poincaré space (7) is \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) and can be visualized as a quadric \( Q \) in \( \mathbb{C}P^3 \); while the points of spacetime correspond to certain plane sections (conics) of \( Q \) with space \( \mathbb{C}P^3 \). The relevant conics which have to be real and nondegenerate, are given by the expression [4]

\[ \omega = v - r^2 (\mu - u)^{-1} \]  

(8)

where \((\omega, \mu)\) are standard coordinates on the two \( \mathbb{C}P^1 \) factor of \( Q \), while \( u = x + t \) and \( v = x - t \). Note that the Poincaré coordinates \((r, x, t)\) cover all of the space of these conics (which is the top half of \( \mathbb{R}P^3 \)) except for a set of measure zero. In order to see the correspondence between spacetime and twistor space \( Q \) one needs to substitute (6) into (8).

Consider the set of linear equations

\[
\begin{align*}
[rD_r - 2(\lambda - u)D_u - \Phi] \Psi &= 0 \\
[2D_v + \frac{\lambda - u}{r}D_r - \frac{\lambda - u}{r^2} \Phi] \Psi &= 0.
\end{align*}
\]  

(9)

Here \( \lambda \in \mathbb{C} \) and \((r, u, v)\) are the Poincaré coordinates which cover, only, the shaded region of FIG. [4]. The gauge fields \((\Phi, A_r, A_u, A_v)\) are \( 2 \times 2 \) trace-free matrices depending only on \((r, u, v)\) and \( \Psi(\lambda, r, u, t) \) is a unimodular \( 2 \times 2 \) matrix function satisfying the reality condition \( \Psi(\lambda)\Psi(\bar{\lambda})^\dagger = I \) (where \( \dagger \) denotes the complex conjugate transpose). The system
(1) is overdetermined and in order for a solution $\Psi$ to exist the following integrability conditions need to be satisfied

$$
D_u \Phi = r F_{ur},
$$
$$
D_v \Phi = -r F_{vr},
$$
$$
D_r \Phi = -2r F_{uv}.
$$

(10)
The above equations are consistent with the ones obtained from (1) using the Poincaré coordinates.

The gauge and Higgs fields in terms of the function $\Psi$ can be obtained from the Lax pair (9). Note that, as $\lambda \to \infty$ the function $\Psi$ goes to the identity matrix which implies that

$$
A_u = 0, \quad A_r = \frac{1}{r} \Phi.
$$

(11)

On the other hand, for $\lambda = 0$ and using (11) the rest of the gauge fields are defined as

$$
\Phi = \frac{r}{2} J_r J^{-1} - u J_u J^{-1}
$$
$$
A_v = \frac{u}{2r} J_r J^{-1} - J_v J^{-1}
$$

(12)

where $J(r, u, v) \doteq \Psi(\lambda = 0, r, u, v)$. Note that, in this case, the first equation of the system (10) is automatically satisfied (due to the specific gauge choice).

Recently, Ward [4] has shown that holomorphic vector bundles $V$ over $Q$ determine multi-soliton solutions of (10) in anti-de Sitter space via the usual Penrose transform. This way a five-parameter family of soliton solutions can be obtained, in a similar way as for flat spacetime [8]. Later, more solutions of equations (10) were obtained by Zhou [9, 10] using Darboux transformations with constant and variable spectral parameters. In what follows, we use the Riemann problem with zeros to construct families of soliton solutions and observe the occurrence of different types of scattering behaviour. More precisely, we present families of multi-soliton solutions with trivial and nontrivial scattering.

II. Construction of Solitons

The integrable nature of (1) means that there is a variety of methods for constructing solutions. Here, we indicate a general method for constructing soliton solutions of (1) which is a variant of that in Ref. [8]. Using the standard method of Riemann problem with zeros in order to construct the multi-soliton solution, we assume that the function $\Psi$ has the simple form in $\lambda$, i.e.

$$
\Psi = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k}
$$

(13)

where $M_k$ are $2 \times 2$ matrices independent of $\lambda$ and $n$ is the soliton number. The components of the matrix $M_k$ are given in terms of a rational function $f_k(\omega_k) = a_k \omega_k + c_k$ of the complex variable: $\omega_k = v - r^2 (\mu_k - u)^{-1}$. Here $a_k, c_k$ and $\mu_k$ are complex constants which determine
the size, position and velocity of the $k$-th solitons. Remark: The rational dependence of
the solutions $\Psi$ follows (directly) when the inverse spectral theory is considered. In [11]
(for the flat spacetime), it was shown by solving the Cauchy problem that the spectral
data is a function of a parameter similar to (8).

The matrix $M_k$ has the form

$$M_k = \sum_{l=1}^{n} (\Gamma^{-1})^{kl} \bar{m}_a^l m_b^k$$

with $\Gamma^{-1}$ the inverse of

$$\Gamma^{kl} = \sum_{a=1}^{2} (\bar{\mu}_k - \mu_l)^{-1} \bar{m}_a^k m_a^l$$

and $m_a^k$ holomorphic functions of $\omega_k$, of the form $m_a^k = (m_1^k, m_2^k) = (1, f_k)$. The Yang-
Mills-Higgs fields ($\Phi, A_r, A_v, A_u$) can then be read off from (11-12) and they automatically
satisfy (10). The corresponding solitons are spatially localized since $\Phi \to 0$ at spatial
infinity (ie at $r = 0$).

By way of example, let us look at the special case where $\mu_1 = i, \mu_2 = 2i, a_1 = 2, a_2 = 1,$
c_1 = 5 and $c_2 = -10$. FIG. 2 represents a snapshot of the positive definite gauge quantity
$(-\text{tr}\Phi^2)$ at time $t = 8$. The corresponding solution consists of two solitons which travel
towards $r = 0$ and bounce back while their sizes change as they move.

III. Scattering Solutions

The Riemann problem with zeros approach assumes that the parameters $\mu_k$ are distinct
and also $\bar{\mu}_k \neq \mu_l$ for all $(k, l)$. However, examples of generalizations of these constructions
can be obtained either involving higher order poles in $\mu_k$ or when $\bar{\mu}_k = \mu_l$. When this pro-
cedure has been applied in flat spacetime the corresponding solitons scatter in a nontrivial
way. In particular, as it has been shown in [12 13], in head-on collisions of $N$ indistin-
guishable solitons the scattering angle of the emerging solitons relative to the incoming
ones is $\pi/N$. As a result, it would be of great interest to see the scattering behaviour of the corresponding solitons in the anti-de Sitter spacetime. Note that, it is not clear what to expect as a nontrivial scattering (for example, $90^\circ$ scattering) in the shaded region of FIG. 1. Another interesting point to be considered is the extension of the corresponding solutions to the whole anti-de Sitter space, i.e., the plot of the corresponding configuration in terms of the coordinates $(\rho, \theta, \phi)$. This issue will be addressed towards the end of the paper.

Firstly, let us look at an example in which the function $\Psi$ has a double pole in $\lambda$ and no others. In this case, $\Psi$ has the form

$$\Psi = I + \sum_{k=1}^{2} \frac{R_k}{(\lambda - \mu)^k}$$

(where $R_k$ are $2 \times 2$ matrices independent of $\lambda$). Then, as in flat spacetime [12], $\Psi$ corresponds to a solution of (9) if and only if it factorizes as

$$\Psi(\lambda) = \left(1 - \frac{\bar{\mu} - \mu}{(\lambda - \mu)} \frac{q^\dagger \otimes q}{|q|^2}\right) \left(1 - \frac{\bar{\mu} - \mu}{(\lambda - \mu)} \frac{p^\dagger \otimes p}{|p|^2}\right)$$

for some two vectors $q$ and $p$. One way to obtain the form of these vectors is by taking the formula (13) for $n = 2$ and setting $\mu_1 = \mu + \epsilon$, $\mu_2 = \mu - \epsilon$, $f_1(\omega_1) = f(\omega_1) + \epsilon h(\omega_1)$, $f_2(\omega_2) = f(\omega_2) - \epsilon h(\omega_2)$, with $f$ and $h$ being rational function of one variable. In the limit $\epsilon \to 0$ the two vectors $q$ and $p$ can be obtained and are of the form:

$$q = (1 + |f|^2)(1, f) + (\bar{\mu} - \mu) \left(\frac{r^2 f'}{(\mu - u)^2} + h\right)(\bar{f}, -1)$$

$$p = (1, f).$$

(18)

In this case, the constraint $f_2(\omega_2) - f_1(\omega_1) \to 0$ as $\epsilon \to 0$ has to be imposed in order for the resulting solution $\Psi$ to be smooth for all $(r, u, v)$, which is true due to (8). Note that the solution depends on the parameter $\mu$ and on the two arbitrary functions $f$ and $h$.

Another way to obtain the aforementioned solutions is by using the Uhlenbeck construction [14]; i.e., by assuming that the function $\Psi$ is a product of projectors which satisfy first-order partial differential equations and can easily be solved [15].

In order to illustrate the above family of solutions, two simple cases are going to be examined, by giving specific values to the parameters $\mu$, $f(\omega)$ and $h(\omega)$.

(i) Let us study the simple case, where $\mu = i$, $f(\omega) = \omega$ and $h(\omega) = 0$. Then, the quantity $-\text{tr}\Phi^2$ simplifies to

$$-\text{tr}\Phi^2 = 32r^2 \frac{(r^2 + x^2 - t^2 + 1)^2 + 4t^2][r^2 + x^2 - t^2 - 1)^2 + 4x^2]}{((r^2 + x^2 - t^2)^2 + 1 + 2t^2 + 2x^2)^2 + 4r^4)^2},$$

which is time reversible. The basic characteristics of the time-evolution of the above solution are given qualitatively by FIG. 3. The time-dependent solution is a traveling
soliton configuration which for negative $t$, goes towards spatial infinity ($r = 0$); approaches it at $t = 0$ and then bounces back at positive $t$. During this period the soliton configuration deforms.

(ii) Next, we investigate the solution which corresponds to a nontrivial scattering, at least in the flat spacetime. FIG. 4 represent the solution given by (17-18) for $\mu = i$, $f(\omega) = \omega$ and $h(\omega) = \omega^4$. The picture consists of two solitons with nontrivial scattering since, for large (negative) $t$, the $-\text{tr} \Phi^2$ is peaked at two points which changes to a lump at $t = 0$ and then two solitons emerge, for large (positive) $t$, with the small one been shifted to the left.

This method can be extended to derive solutions which correspond to the case where the function $\Psi$ has higher order pole in $\lambda$ (and no others). Then, $\Psi$ can be written as a product of three (or more) factors with three (or more) arbitrary vectors (for more details, see [13]).

Secondly, let us construct a large family of solutions which correspond to the case where $\bar{\mu}_k = \mu_i$. One way of proceeding is to take the solution (13) with $n = 2$, put...
\[ t = -3 \quad t = -1 \]

\[ t = 0 \quad t = 3 \]

Figure 5: A soliton configuration at different times.

\[ \mu_1 = \mu + \epsilon, \mu_2 = \bar{\mu} - \epsilon \] and take the limit \( \epsilon \to 0 \). In order for the resulting \( \Psi \) to be smooth it is necessary to take \( f_1(\omega_1) = f(\omega_1), f_2(\omega_2) = -1/f(\omega_2) - \epsilon h(\omega_2) \), where \( f \) and \( h \) are rational functions of one variable. On taking the limit we obtain a solution \( \Psi \) of the form

\[
\Psi = I + \frac{n^1 \otimes m^1}{\lambda - \mu} + \frac{n^2 \otimes m^2}{\lambda - \bar{\mu}} \tag{20}
\]

where \( n^k, m^k \) for \( k = 1, 2 \) are complex valued two vector functions of the form

\[
m^1 = (1, f), \quad m^2 = (-\bar{f}, 1)
\]

\[
\begin{pmatrix}
n^1 \\
n^2
\end{pmatrix} = \frac{2(\mu - \bar{\mu})}{4(1 + |f|^2)^2 - (\mu - \bar{\mu})^2 |w|^2} \begin{pmatrix}
2(1 + |f|^2) & -(\mu - \bar{\mu})\bar{w} \\
(\mu - \bar{\mu})w & -2(1 + |f|^2)
\end{pmatrix} \begin{pmatrix}
m_1^1 \\
m_2^1
\end{pmatrix} \tag{21}
\]

with

\[
w \equiv \frac{2r^2}{(\mu - u)^2} f' + \bar{h} f^2. \tag{22}
\]

So we generate a solution which depends on the parameter \( \mu \) and the two arbitrary rational functions \( f = f(\omega) \) and \( h = h(\bar{\omega}) \).

In FIG. 3 we represent snapshots of the solution (20) for \( \mu = i, f = \omega, h = \bar{\omega} \). The configuration consists of two solitons with nontrivial scattering behaviour. Again, the quantity \( -\text{tr}\Phi^2 \) is peaked at two points, for (negative) \( t \), which are still distinct at \( t = 0 \) and then two shifted (compared to the initial ones at \( t = -3 \)) solitons emerge, for (positive) \( t \). Throughout the time-evolution their sizes change.

Note that, the scattering solutions belong to a large family since \( f \) and \( h \) can be taken to be any rational functions of \( \omega \).
IV. Conclusions

Currently a great deal of attention has been focused on anti-de Sitter spacetimes since they arise naturally in black holes and p-branes. For the case of Yang-Mills theory with \( \mathcal{N} = 4 \) supersymmetries and a large number of colours it has been conjectured that gauge strings are the same as the fundamental strings but moving in a particular curved space: the product of five-dimensional anti-de Sitter space and a five sphere \([16]\). Then, using Poincaré coordinates the anti-de Sitter solutions play the role of classical sources for the boundary field correlators, as shown in \([17]\); while extensions of the corresponding statements can be applied to gravity theories, like the black holes which arise in anti-de Sitter backgrounds.

In this paper, we illustrate the construction of time-dependent solutions related to hyperbolic monopoles. In particular, families of solutions of the Bogomolny Yang-Mills-Higgs equations in the (2+1)-dimensional anti-de Sitter space have been constructed and their dynamics has been in studied in some detail. As a result, it would be interesting to understand the role of higher poles in algebraic-geometry approach like twistor theory (for example, the function \( \Psi \) \([L7]\) correspond to \( n = 2 \) bundles), and also to investigate the construction of the corresponding solutions and their dynamics in de Sitter space. Finally, it would be interesting to extend our construction in higher dimensional gauged theories and investigate the scattering behaviour of the corresponding classical solutions and, also, consider and study its noncommutative version (see, for example, Ref. \([L8]\)).

Remark: The extension of the obtained classical solutions in the whole anti-de Sitter space, ie using the coordinates \((\rho, \theta, \phi)\) is unambiguous. For example, the simplest solution which corresponds to the one soliton (first derived in \([4]\)) given by (13) for \( n = 1, \mu_1 = i \) and \( f_1 = \omega_1 \) implies that

\[
- \text{tr} \Phi^2 = \frac{8r^4}{(r^2 + x^2 - t^2)^2 + 2x^2 + 2t^2 + 1)^2} = \frac{2 \cos^4 \rho}{(\cos^2 \rho - 2)^2}
\]  

(23)
which means that the positive definite quantity $-\text{tr} \Phi^2$ is independent of the variables $(\theta, \phi)$ as shown in FIG. 6.

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