X structures in $B^+ \to J/\psi \phi K^+$ as one-loop and double-triangle threshold cusps

Satoshi X. Nakamura$^{1,2}$

$^1$University of Science and Technology of China, Hefei 230026, People’s Republic of China
$^2$State Key Laboratory of Particle Detection and Electronics (IHEP-USTC), Hefei 230026, People’s Republic of China

The LHCb data on $B^+ \to J/\psi \phi K^+$ show four peaks and three dips in the $J/\psi \phi$ invariant mass distribution, and the peaks are interpreted as $X(4140)$, $X(4274)$, $X(4500)$ and $X(4685)$. The LHCb Collaboration conducted a first six-dimensional amplitude analysis and claimed four $X$ states along with their spin-parity $(J^P)$: $X(4140)$ and $X(4274)$ with $J^P = 1^+$; $X(4500)$ and $X(4700)$ with $J^P = 0^+$. Recent higher statistics data confirmed these $X$ states, and added $1^+X(4685)$, $2^-X(4150)$, and $1^-X(4630)$ $^{12}$. Moreover, structures in $M_{J/\psi K^+}$ distribution were interpreted as charmonium $(\chi_{cJ})$, $^{13}2^{++}$, hybrid $^{10}$, and tetraquark $(cs\bar{s}s)$ $^{23}4^{+}$. Hadron molecule models were developed for $X(4140)$. $^{35}$

It is recognized $^{53}54$ that the $X(4274)$ and $X(4500)$ peak positions are virtually at the $D_{s0}^{*}(2317)\bar{D}_s$ and $D_{s1}^{*}(2536)\bar{D}_s$ thresholds, respectively, and $X(4700)$ and $X(4685)$ are at the $\psi \phi$ threshold; see Fig. 4. The $X(4140)$ structure is close to the $D_s^* \bar{D}_s$ threshold. Furthermore, three dip structures have their lowest points at the $D_s^* \bar{D}_s$ and $D_{s0}^{*}(2317)\bar{D}_s$ thresholds. This suggests that the $X$ and dip structures are associated with openings of the $D_s^* \bar{D}_s^{(*)}$ and $D_{s0}^{*}(2317)\bar{D}_s^{(*)}$ channels through kinematical effects such as threshold cusps and triangle singularities $^{55}$.

Indeed, the LHCb confirmed that the $X(4140)$ structure can be described with a $D_s^* \bar{D}_s$ threshold cusp, albeit using a rather small cutoff in form factors $^{11}68$. Similarly, Liu studied triangle diagrams that cause $D_s^* \bar{D}_s$ and $\psi \phi$ threshold cusps, and found $X(4140)$- and $X(4700)$-like enhancements, respectively $^{55}$. Dong et al. also suggested that $X(4140)$ could be caused by a $D_s^0 \bar{D}_s^0$ virtual state and the associated threshold cusp $^{54}$. $X(4140)$ as the kinematical effect may be supported by a lattice QCD that found no $J^{PC} = 1^{++}$ $cs\bar{s}s$ state below 4.2 GeV $^{57}$. On the other hand, $X(4274)$ and $X(4500)$ as an s-wave $D_{s0}^{*}(2317)\bar{D}_s$ threshold cusp has $J^P$ that conflicts with the experimentally determined ones $^{11}12$. Non s-wave threshold cusps from one-loop diagrams are unlikely either, since they should be suppressed $^{53}$. Thus, until the present work, there exists no explanation of $X(4274)$ and $X(4500)$ based on kinematical effects.

Now let us assume negligibly small $D_{s0}^{*}(2317)\bar{D}_s^{(*)}$.

---

1. The charge conjugate decays are implied throughout.
2. We follow the hadron naming scheme of Ref. $^{1}$. For simplicity, however, $J/\psi$ and $\psi(2S)$ are often denoted by $\psi$ and $\psi'$, respectively. We generically denote $D_{s0}^{*}(2317)$ and $D_{s1}^{*}(2536)$ by $D_{sJ}^{*}$. Charge indices are often suppressed.

3. This LHCb’s finding should be viewed with a caution since a small cutoff makes a cusp significantly sharper by suppressing the high momentum contribution.
4. Our present analysis assumes that $J^P$ of the $X$ structures determined by the LHCb $^{11}12$ are correct. It is noted, however, that the LHCb’s $J^P$ determination is based on fitting the $X$ structures with Breit-Wigner models and thus is not model-independent. If the $X$ and dip structures are described with more complicated mechanisms that might involve kinematical effects, it is unclear whether $J^P$ of the $X$ structures remain unchanged.
$D_{s1}(2536)\bar{D}_s^{(*)} \rightarrow J/\psi\phi$ transition strengths caused by short-range (e.g., quark-exchange) interactions. This assumption may seem reasonable for the $D_{s1}^{(*)}(2317)$ cases, because previous theoretical studies \cite{58,61} indicated a dominant $DK$-molecule component in $D_{s0}^{(*)}(2317)$. Under this assumption, double triangle (DT) mechanisms of Fig. 1(a) should be the most important among those including $D_{sJ}^{(*)}\bar{D}_s^{(*)}$. The DT mechanisms are worthwhile studying to understand the $X$ and dip structures and their locations. The DT mechanisms cause threshold cusps that are significantly sharper than ordinary one-loop ones with the same $J^P$. This is because the DT is close to causing the leading kinematical singularity. Thus, the DT can generate $X$-like and dip structures at the $D_{sJ}^{(*)}\bar{D}_s^{(*)}$ thresholds.

In this paper, we develop a $B^+ \rightarrow J/\psi\phi K^+$ decay model. The model includes the DT mechanisms that cause enhanced threshold cusps at the $D_{sJ}^{(*)}\bar{D}_s^{(*)}$ thresholds. $D_s^{(*)}\bar{D}_s^{(*)}$ and $\psi\phi$ threshold cusps are also generated by one-loop mechanisms. We first examine singular behaviors of the DT amplitudes. We then analyze the $M_{J/\psi\phi}$ distribution from the LHCb. Since one-dimensional analysis would not reliably extract partial wave amplitudes or determine spin-parity of resonances, this is not our intention. The present one-dimensional analysis keeps $J^P$ of $X$ from the LHCb analysis. Under this constraint, we demonstrate that all the $X$ and dip structures can be well described with the threshold cusps. The purpose of this work is to propose a novel interpretation of the $X$ and dip structures in this way.

II. MODEL

In describing $B^+ \rightarrow J/\psi\phi K^+$, we explicitly consider mechanisms that generate the structures in the $M_{J/\psi\phi}$ distribution through kinematical effects or resonance excitations; others are subsumed in contact mechanisms. Thus we consider diagrams shown in Fig. 1. To derive the corresponding amplitudes, we write down effective Lagrangians of relevant hadrons and their matrix elements, and combine them following the time-ordered perturbation theory. We consider the DT diagrams [Fig. 1(a)] that include $p$-wave pairs of

$$D_{s0}^{*}(2317)\bar{D}_s(1^+), \quad D_{s0}^{*}(2317)\bar{D}_s(0^+, 1^+),$$
$$D_{s1}(2536)\bar{D}_s(0^+, 1^+), \quad D_{s1}(2536)\bar{D}_s(0^+, 1^+), \quad (1)$$

where $J^P$ of a pair is indicated in the parenthesis. In principle, more quantum numbers are possible such as $J^P$ from $s$-wave pairs and $J^P = 2^+$ from $p$-wave pairs which the LHCb did not find relevant to the $X$ structures. While the kinematical effects can generate structures in lineshapes almost model-independently, it is the dynamics that determines the strength of the kinematical effects. Since the relevant dynamical information is scarce, we need to rely on the LHCb analysis to select the quantum numbers to take into account at the model. Most phenomenological models share this limitation of predicting quantum numbers relevant to the process. We assume that contributions from the other quantum numbers are relatively minor and can be absorbed by mechanisms included in the model. We also do not consider $D_{s1}(2460)\bar{D}_s^{(*)}$ pairs since their thresholds are either not clear in the data or replaceable by a $D_{s0}^{*}(2317)\bar{D}_s^{*}$ threshold cusp. The one-loop diagram [Fig. 1(b)] includes $s$-wave pairs of $D_s^{*}\bar{D}_s^{*}(1^+)$, $D_s^{*}\bar{D}_s^{*}(0^+)$, and $\psi\phi(0^+, 1^+)$. $D_s^{*}\bar{D}_s^{*}(1^+)$ is not included since $D_s^{*}\bar{D}_s^{*}(1^+)$ is forbidden by the $C$-parity conservation. We denote the DT and one-loop amplitudes by $A^{DT}_{D_s^{*}\bar{D}_s^{*}(J^P)}$ and $A^{UL}_{D_s^{*}\bar{D}_s^{*}(J^P)}$ respectively.

We consider $Z_{cs}$ excitations [Fig. 1(c)] since the data \cite{62} shows their effects on the $M_{J/\psi\phi}$ distribution. In particular, $Z_{cs}(4000)$ seems to enhance the $X(4274)$ peak through an interference. The LHCb presented the $Z_{cs}(4000)$ and $Z_{cs}(4220)$ properties. Meanwhile, coupled-channel analyses \cite{63,64} found virtual states below the $D_s^{(*)}\bar{D}^{(*)}$ thresholds that may be identified with $Z_{cs}(4000)$ and $Z_{cs}(4220)$. The $D_s^{(*)}\bar{D}^{(*)}$ threshold cusps enhanced by the virtual states can fit the $M_{J/\psi K^+}$ distribution. We thus considered the above two options. We use a Breit-Wigner form without addressing the $Z_{cs}$ internal structures. To simulate the $D_s^{(*)}\bar{D}^{(*)}$ threshold cusps, two $Z_{cs}$ masses are 3975 MeV and 4119 MeV from the $D_s^{(*)}\bar{D}^{(*)}$ thresholds; $Z_{cs}$ widths are set to be 100 MeV (constant width values); see Eqs. (A39) and (A40) for formulas. For each

---

**FIG. 1.** $B^+ \rightarrow J/\psi\phi K^+$ mechanisms: (a) double triangle; (b) one-loop; (c) $Z_{cs}$ excitation; (d) direct decay.
$Z_{cs}$, we use a $p$-wave $B^+ \to Z_{cs} \phi$ decay vertex which contributes to the $1^+ J/\psi \phi$ final state. Our fits visibly favored the threshold-cusp-based $Z_{cs}$; we thus use them hereafter.

All the other mechanisms such as non-resonant and $K^*_1(1270)$ excitations are simulated by two independent direct decay mechanisms [Fig. 1(d)] creating $J/\psi\phi(0^+,1^+)$. We consider $J/\psi\phi(0^+,1^+)$ partial waves. Although the LHCb amplitude analysis found resonances in $1^+$ and $2^-$ partial waves, their contributions are rather small in the $M_{J/\psi \phi}$ spectrum. We confirmed that the $1^+$ and $2^-$ resonance contributions only marginally improved our fits; we thus do not consider them.

The DT and one-loop diagrams are respectively initiated by $B^+ \to D_{s}^{(+)} D_{s}^{(+)} K^+$ and $B^+ \to D_{s}^{0} D_{s}^{0} K^+$ which may be dominated by color-favored quark mechanisms. Although charge analogous $B^+ \to D_{s}^{(*)} D_{s}^{(*)} K^+$ and $B^+ \to D_{s}^{(*)} D_{s}^{(*)} K^+$ generally have independent decay strengths, the corresponding DT and one-loop amplitudes have the same singular behaviors as the original ones. Thus we do not explicitly consider the charge analogous processes, but their effects and projections onto positive $C$-parity states are understood to be taken into account in coupling strengths of the considered processes.

We present amplitude formulas for representative cases; see Appendix A for complete formulas. We use the particle mass and width values from Ref. [65] unless otherwise specified, and denote the energy, momentum, and polarization vector of a particle $x$ by $E_x$, $p_x$, and $\epsilon_x$, respectively. A DT diagram [Fig. 1(a)] that includes $D_{s}^{(*)}(2317) D_{s}^{(*)}(1^+)$ consists of four vertices such as $B^+ \to D_{s}^{(*)} D_{s}^{(*)} K^+$, $D_{s}^{(*)} \to D K$, $D K D_s \to J/\psi K$, and $K \bar{K} \to \phi$, given as

$$ c_{D_{s}^{(*)} D_{s}^{(*)}} P_{D_{s}^{(*)} D_{s}^{(*)}} \cdot p_{K} F_{D_{s}^{(*)} D_{s}^{(*)} K}^{11} ;$$
$$ c_{D K D_{s}^{(*)}} P_{D K D_{s}^{(*)}}^{0} F_{D K D_{s}^{(*)}}^{1} ;$$
$$ c_{\psi \phi K D_{s}^{(*)}}^{1} i\left(p_{K} \times \epsilon_{K}ight) \cdot p_{D} D_{s}^{(*)} J_{D_{s}^{(*)} D_{s}^{(*)}}^{1} ;$$
$$ c_{K \bar{K} \phi}^{1} \epsilon_{K} \cdot \epsilon_{\phi} F_{K \bar{K} \phi}^{1} ;$$

respectively; $p_{ab} \equiv p_{a} - p_{b}$. We have introduced dipole form factors $F_{ij}^{k}$, $f_{ij}^{k}$, and $f_{ij}^{k}$, including a cutoff $\Lambda$. We use a common cutoff value $\Lambda = 1$ GeV in all form factors unless otherwise stated. We used a $p$-wave $D D_{s}^{(*)} \to J/\psi K$ interaction; $s$-wave is forbidden by the spin-parity conservation [63]. The coupling $c_{K \bar{K} \phi}$ can be determined by the $\phi \to K \bar{K}$ decay width. Experimental information for the other couplings ($c_{D_{s}^{(*)} D_{s}^{(*)}}$, $c_{D K D_{s}^{(*)}}$, $c_{\psi \phi K D_{s}^{(*)}}$) are unavailable. Thus we determine their product, which is generally a complex value, by fitting the data. The DT amplitude from the above ingredients is

$$ A_{D_{s}^{(*)} D_{s}^{(*)} K}^{DT} = c_{K K \phi}^{1} \epsilon_{K}^{0} c_{\psi K D_{s}^{(*)}}^{1} c_{D K D_{s}^{(*)}} c_{D_{s}^{(*)} D_{s}^{(*)} D_{s}^{(*)} K}^{0} ;$$

$$ \times \int d^{3} p_{D_{s}^{(*)}} \frac{\hat{p}_{K} \cdot \epsilon_{\phi}}{W - E_{K} - E_{\bar{K}} - E_{\phi} + i\epsilon} ;$$

$$ \times \frac{\hat{i}(p_{K} \times \epsilon_{K}) \cdot p_{D_{s}^{(*)}}}{W - E_{K} - E_{D_{s}^{(*)}} + i\epsilon} \cdot \frac{\hat{p}_{K} \cdot \epsilon_{\phi}}{W - E_{D_{s}^{(*)}} - E_{\bar{K}} + 4 \Gamma_{D_{s}^{(*)}} K} ;$$

where the summation over $D^+ K^0 \bar{K}^0$ and $D^0 K^+ K^-$ intermediates states with the charge dependent particle masses is implicit; $K^+$ in the final state is denoted by $K_f$, and $W$ is related to the total energy $E$ by $W = E - E_{K_f}$.

The $D_{s}^{(*)}$ width ($\Gamma_{D_{s}^{(*)}}$) should be small because the dominant $D_{s}^{(*)} \to D_{s} \pi$ decay is isospin-violating. Experimentally, only an upper limit has been set: $\Gamma_{D_{s}^{(*)}} < 3.8$ MeV [65]. Theoretically, $\Gamma_{D_{s}^{(*)}} \sim 0.1$ MeV (0.01 MeV) has been given by a hadron molecule model [65] ($c_8$ models [67]). We use $\Gamma_{D_{s}^{(*)}} = 0.1$ MeV; our results do not significantly change for $\Gamma_{D_{s}^{(*)}} < 1$ MeV.

Similarly, we consider other $p$-wave $D_{s}^{(*)} D_{s}^{(*)}$ pairs of Eq. (11) in DT diagrams. The $D_{s} D_{s}$ cusp needs to be $0^+$ to be consistent with the LHCb result for $X(4500)$. Interestingly, the DT amplitudes of the $s$-wave $D_{s}^{(*)} D_{s}^{(*)}$ and $p$-wave $D_{s}^{(*)} D_{s}^{(*)}$ share the same DT amplitudes of the $D_{s}^{(*)} D_{s}^{(*)} K^+$ interaction of Eq. (A16), while the $p$-wave $D_{s}^{(*)} D_{s}^{(*)}$ is weaker than that of Eq. (A15). The dips in the $M_{J/\psi \phi}$ distribution, and the LHCb did not assign any spin-parity to these structures; we can thus choose their spin-parity to obtain a good fit.

The $D_{s}^{(*)} D_{s}^{(*)}$ one-loop amplitude includes $B^+ \to D_{s}^{(*)} D_{s}^{(*)} K^+$ and $B^+ \to D_{s}^{(*)} D_{s}^{(*)} K^+$ vertices given by

$$ c_{D_{s}^{(*)} D_{s}^{(*)}} P_{D_{s}^{(*)} D_{s}^{(*)}}^{0} F_{D_{s}^{(*)} D_{s}^{(*)} K}^{1} ;$$

$$ i\left(p_{D_{s}^{(*)}} \times \epsilon_{D_{s}^{(*)}} \right) \cdot \epsilon_{\phi} F_{\psi \phi D_{s}^{(*)} D_{s}^{(*)}}^{0} ;$$

respectively, from which the one-loop amplitude is

$$ A_{D_{s}^{(*)} D_{s}^{(*)} K}^{1L} = c_{\psi \phi D_{s}^{(*)} D_{s}^{(*)}}^{1} i\left(p_{\phi} \times \epsilon_{\phi} \right) \cdot p_{K} ;$$

$$ \times \int d^{3} p_{D_{s}^{(*)}} \frac{\hat{p}_{\phi} \cdot \epsilon_{\phi}}{W - E_{D_{s}^{(*)}} - E_{\phi} + i\epsilon} \cdot \frac{\hat{p}_{D_{s}^{(*)}} \cdot \epsilon_{\phi}}{W - E_{D_{s}^{(*)}} - E_{\phi} + i\epsilon} ;$$

The $D_{s}^{(*)}$ width is expected to be tiny (~0.1 keV [68]) and thus neglected.

For the $D_{s}^{(*)} D_{s}^{(*)}$ transition in the one-loop diagram, we consider a single-channel $D_{s}^{(*)} D_{s}^{(*)}$ scattering followed by a perturbative $D_{s}^{(*)} D_{s}^{(*)}$ to $J/\psi \phi$ transition. Details are given in Sec. 2 of the Supplemental Material in Ref. [65]. Since attractive $D_{s}^{(*)} D_{s}^{(*)}$ interactions are preferred in fitting the LHCb data, we fix the $D_{s}^{(*)} D_{s}^{(*)}$ interaction strengths so that the scattering length $(a)$ is

---

5 $s$-wave $DD_{s}^{(*)}, D^+ D_{s}^{(*)} \to J/\psi K$ interactions are allowed in DT mechanisms including other $D_{s}^{(*)} D_{s}^{(*)}$ pairs. However, such DT amplitudes are suppressed due to their tensor structures. See a discussion above Eq. (A14) in Appendix A.
FIG. 2. Double triangle amplitudes; (a) real, and (b) imaginary parts. The red solid curves are from Fig. 1(a) with $D^{(*)}_{sJ}D^{(*)}_{sJ}(J^{P}) = D^{(s0)}_{s}(2317)^{+}D^{(1+)}_{s}$ and $D^{(*)}_{s}K\bar{K} = D^{+}K^{0}\bar{K}^{0} + D^{0}K^{+}\bar{K}^{-}$. The green dotted curves have been arbitrary scaled to have the same magnitude at the threshold indicated by the dotted vertical lines. An overall constant phase factor has been multiplied to the double triangle amplitude to compare well with the one-loop amplitude. The two amplitudes are obtained from those in (a) [(b)] by replacing $D^{(s0)}_{s}(2317)^{+}D^{(1+)}_{s}$ and $D$ with $D^{(s1)}_{s}(2536)^{+}D_{s}(0^{+})$ and $D^{+}$, respectively.

FIG. 3. Ratio of double triangle ($A_{DT}$) and one-loop ($A_{1L}$) amplitudes. The red dotted and blue solid curves show Re[$A_{DT}/A_{1L}$] and Im[$A_{DT}/A_{1L}$], respectively. The ratios in the panels (a) and (b) are obtained using the amplitudes shown in Fig. 2(a,b) and Fig. 2(c,d), respectively.

III. RESULTS

A. Singular behaviors of double triangle amplitudes

A DT amplitude ($A_{DT}$) such as Fig. 1(a) can cause a kinematical singularity (anomalous threshold) $^{72}$. The DT singularity can generate a resonance-like structure in a decay spectrum, as first demonstrated in Refs. $^{70,73}$. According to the Coleman-Norton theorem $^{74}$, $A_{DT}$ has the leading singularity if the whole DT process is kinematically allowed at the classical level: the energy and momentum are always conserved; in Fig. 1(a), all internal momenta are collinear in the $D^{(*)}_{sJ}D^{(*)}_{sJ}$ center-of-mass frame; $D^{(*)}_{sJ}$ and $D^{(*)}_{sJ}$ ($\bar{K}$ and $K$) are moving to the same direction and the former is faster than the latter. Whether a given diagram has a singularity is solely determined by the participating particles’ masses.

The DT amplitudes included in our $B^{+} \rightarrow J/\psi\phi K^{+}$ model do not cause the leading singularity. Yet, the leading singularity is close to being caused $^{74}$, and its effect is expected to be visible as an enhancement of the threshold cusp. We thus study the singular behavior of $A_{DT}$ numerically $^{8}$. In Fig. 2(a,b), we show $A_{DT}$ of Eq. 6 by the red solid curve, and a one-loop amplitude $A_{1L}$ by the green dotted curve. The $p$-wave pair of $D^{(*)}_{sJ}(2317)D^{(1+)}_{s}$ is included in $A_{DT}$ and $A_{1L}$. While both amplitudes have threshold cusps at the $D^{(*)}_{sJ}(2317)D_{s}$ threshold, $A_{DT}$ is sharper. This can be seen more clearly by taking a ratio $A_{DT}/A_{1L}$ as shown in Fig. 3(a). The ratio is still singu-

---

$^{6}$ The scattering length ($a$) is related to the phase shift ($\delta$) by $p\cot \delta = 1/a + O(p^2)$.

$^{7}$ See Appendix A for a discussion on how closely $A_{DT}$ satisfies the kinematical condition for the leading singularity.

$^{8}$ In principle, the singular behavior of $A_{DT}$ can also be studied more analytically by examining the corresponding Landau equation $^{74,75}$. 
lar; the derivative of the imaginary part with respect to \(M_{J/\psi}\) seems divergent at the threshold. The ratio may also serve to isolate from \(A_{\text{PT}}\) the kinematical singularity effect other than the ordinary threshold cusp. Similarly, \(A_{\text{PT}}\) and \(A_{\text{1L}}\) including \(D_{1}(2536)\) and \(D_{0}(0^{+})\) p-wave pairs and their ratio are shown in Fig. 2(c,d) and Fig. 2(b), respectively. At the threshold, \(A_{\text{PT}}\) is even sharper and the imaginary part of the ratio is singular. The quantitative difference in the singular behavior between \(A_{\text{PT}}D_{0}^{*}D_{0}(1^{+})\) and \(A_{\text{PT}}D_{0}D_{0}(0^{+})\) is from the fact that \(D_{1}(2536)\) \(\rightarrow D^{*}K\) is allowed at on-shell while \(D_{0}^{*}(2317)\) \(\rightarrow DK\) is not.

B. Analysis of the LHCb data

To analyze the \(B^{+} \rightarrow J/\psi K^{+}\) data, we have seven DT diagrams, four one-loop diagrams, two \(Z_{cs}\)-excitation diagrams, and two direct decay diagrams. Each of the diagrams has a complex overall factor that comes from the product of unknown coupling constants. The overall normalization and phases of the \(0^{+}\) and \(1^{+}\) full amplitudes are arbitrary. We totally have 20 fitting parameters from the coupling constants after removing relatively unimportant parameters; see Tables II and III in Appendix A for coupling parameters and fit fractions. Also, we use cut-offs different from the common value for the direct decay diagrams so that their \(M_{J/\psi}\) distributions are similar to the phase-space shape. We note that no parameter can adjust the DT and one-loop threshold cusp positions where the experimental peaks are located. Fitting the \(M_{J/\psi}\)-distribution lineshape requires the adjustable parameters. In contrast, the quark and hadron-molecule models need adjustable parameters to get pole positions at the experimental peak positions; parameters for fitting the lineshape are needed additionally. It is therefore likely that our model can fit the \(M_{J/\psi}\)-distribution with fewer parameters.

We compare in Fig. 4 our calculation with the \(M_{J/\psi}\) distribution data. Theoretical curves are smeared with the experimental bin width. The data are well fitted by the full model (red solid curve). In particular, the resonancelike four peaks and three dips are well described by threshold cusps from the DT and one-loop amplitudes. We used common cutoff values over \(\Lambda = 0.8 - 1.5\) GeV and confirmed the stability of the fit quality. This is understandable since the structures in the spectrum are generated by the threshold cusps that are insensitive to a particular choice of the form factors.

In the same figure, we plot the \(1^{+}\) partial wave contribution without \(Z_{cs}\) [blue dashed curve]. There are two clear resonancelike cusps from \(A_{\text{1L}}D_{0}(1^{+})\) and \(A_{\psi\phi}(1^{+})\) at \(M_{J/\psi} \sim 4.14\) and \(4.7\) GeV, respectively. These threshold cusps would play a role similar to those of \(X(4140)\) and \(X(4685)\) found in the LHCb analysis. The dip at \(M_{J/\psi} \sim 4.65\) GeV is caused by the \(A_{\text{PT}}D_{0}^{*}D_{0}(1^{+})\) cusp.

The \(1^{+}\) contribution also has a relatively small cusp at \(M_{J/\psi} \sim 4.29\) GeV caused by \(A_{\text{PT}}D_{0}^{*}D_{0}(1^{+})\). This cusp interferes with \(Z_{cs}(4000)\) accompanied by \(p\)-wave \(\phi\) to create the prominent \(X(4274)\) structure as seen in the brown dash-two-dotted curve. Similarly, threshold cusps play a major role to form resonancelike and dip structures in the \(0^{+}\) contribution [green dotted curve]. The cusp from \(A_{\text{PT}}D_{0}D_{0}(0^{+})\) develops the \(X(4500)\) structure. This structure is made even sharper by the neighboring two dips at \(M_{J/\psi} \sim 4.46\) and \(4.66\) GeV due to the \(A_{\text{PT}}D_{0}^{*}D_{0}(0^{+})\) and \(A_{\text{PT}}D_{0}D_{0}(0^{+})\) cusps, respectively. There is another peak at \(M_{J/\psi} \sim 4.75\) GeV, near the \(X(4700)\) peak, that is caused by the dip at \(M_{J/\psi} \sim 4.66\) GeV and the rapidly shrinking phase-space near the kinematical endpoint. Another dip is created at \(M_{J/\psi} \sim 4.23\) GeV by \(A_{\text{1L}}D_{0}^{*}D_{0}(0^{+})\). The contributions from the lighter and heavier \(Z_{cs}\) are shown by the magenta dash-dotted [gray two-dash-two-dotted] curve. The dotted vertical lines indicate thresholds for, from left to right, \(D_{0}^{*}D_{0}, D_{0}^{*}D_{0}^{*}, D_{0}(2317)D_{0}, D_{0}^{*}(2317)D_{0}, D_{0}(2536)D_{0}^{*}, D_{0}(2536)D_{0}^{*},\) and \(\psi\phi\), respectively. Data are from Ref. 12.
molecule models. We need more experimental and lattice QCD inputs to judge the different interpretations.

The LHCb presented \( M_{K^+K^-} \) distribution for \( B^+ \to J/\psi K^+ K^- K^+ \). In Fig. 5 the data show the \( \phi \) peak and a small backgroundlike contribution. The data actually put a constraint on the contribution from the DT diagrams of Fig. 1(a). This is because when the DT diagrams followed by \( \phi \to K^+K^- \) contribute to \( B^+ \to J/\psi K^+ K^- K^+ \), there must be a contribution from the diagrams of Fig. 1(a) with the last \( K\bar{K} \to \phi \) vertex removed. This single triangle contribution has to be smaller than the backgroundlike data. Thus, in Fig. 5 we plot the \( M_{K^+K^-} \) distribution from our model with (red solid curve) and without (blue dashed curve) the single triangle contribution. The single triangle contribution does not significantly change the \( \phi \) peak and slightly enhances the \( \phi \)-tail region well within the experimental constraint. The unexplained part of the backgroundlike data should be from non-\( \phi \) mechanisms not considered here.

The key assumption in our model is that short-range \( D_{sJ} \bar{D}_s^{(*)} \to J/\psi \phi \) transition strengths are weak. The assumption naturally leads to the DT mechanisms that cause enhanced threshold cusps consistent with the LHCb data. If the assumption is wrong, the initial \( B^+ \to D_{sJ}^{(*)} \bar{D}_s^{(*)} K^+ \) decays would be followed by \( D_{sJ}^{(*)} \bar{D}_s^{(*)} \) one-loop like Fig. 1(b), and ordinary threshold cusps are expected. However, s-wave cusps are disfavored by the LHCb data, and p-wave cusps are too suppressed to fit the data as shown in Ref. [11]. Thus the LHCb’s result seems to be in favor of the assumption.

For \( D_{sJ}^{(*)} = D_{s0}^{(*)}(2317) \), the assumption is also partly supported by previous theoretical works that analyzed lattice QCD energy spectrum and found \( D_{s0}^{(*)}(2317) \) to be mainly a \( DK \) molecule [55, 56]. On the other hand, \( D_{sJ}^{(*)}(2536) \) have been mostly considered to be a p-wave \( cs \) [61, 76] and, thus, the assumption is not intuitively understandable. Yet, \( D_{sJ}^{(*)}(2536) \) is known to have a strong coupling to \( D^*K \), which has been utilized in our model.

**ACKNOWLEDGMENTS**

I thank F.-K. Guo for useful comments on the manuscript. This work is in part supported by National Natural Science Foundation of China (NSFC) under contracts U2032103 and 11625523, and also by National Key Research and Development Program of China under Contracts 2020YFA0406400.

**Appendix A: \( B^+ \to J/\psi \phi K^+ \) amplitudes**

For double triangle (DT) diagrams of Fig. 1(a), we consider those including \( F_{s0J}^D D_{s0J}^{(*)} K^+ \) -wave pairs. Each of the DT diagrams includes four vertices. The initial \( B^+ \to F_{s0J}^D D_{s0J}^{(*)} K^+ \) vertices are given by

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} P_{D_{s0J}^D} \cdot \mathbf{P}_{K} F_{s0J}^{11} D_{s0J}^D D_{s0J}^K K_B, \tag{A1}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} P_{D_{s0J}^D} \cdot \epsilon_{D_{s0J}^D} F_{s0J}^{10} \bar{D}_{s0J}^D D_{s0J}^K K_B, \tag{A2}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} \mathbf{i} (\mathbf{P}_{D_{s0J}^D} \times \epsilon_{D_{s0J}^D}) \cdot \mathbf{P}_{K} F_{s0J}^{11} D_{s0J}^D D_{s0J}^K K_B, \tag{A3}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} \epsilon_{D_{s0J}^D} F_{s0J}^{10} \bar{D}_{s0J}^D D_{s0J}^K K_B, \tag{A4}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} \mathbf{i} (\mathbf{P}_{D_{s0J}^D} \times \epsilon_{D_{s0J}^D}) \cdot \mathbf{P}_{K} F_{s0J}^{11} D_{s0J}^D D_{s0J}^K K_B, \tag{A5}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} \epsilon_{D_{s0J}^D} F_{s0J}^{10} \bar{D}_{s0J}^D D_{s0J}^K K_B, \tag{A6}
\]

\[
c_{D_{s0J}^D} \bar{D}_{s0J}^{(*)} \epsilon_{D_{s0J}^D} \mathbf{i} (\mathbf{P}_{D_{s0J}^D} \times \epsilon_{D_{s0J}^D}) \cdot \mathbf{P}_{K} F_{s0J}^{11} D_{s0J}^D D_{s0J}^K K_B, \tag{A7}
\]

respectively, with complex coupling constants \( c_{D_{s0J}^{(*)} D_{s0J}^{(*)}}(J') \). Here and in what follows, the initial vertices contributing to the final \( 0^+ \) and \( 1^+ \) \( J/\psi \phi \) partial waves are parity-conserving and parity-violating, respectively. We use dipole form factors \( F_{i,j,k,l}^{LL'} \) defined by

\[
F_{i,j,k,l}^{LL'} = \frac{1}{E_i E_j E_k E_l} \left( -\frac{\Lambda^2}{\Lambda^2 + q_{ij}^2} \right)^{2+\frac{L}{2}} \left( -\frac{\Lambda'^2}{\Lambda'^2 + p_k^2} \right)^{2+\frac{L'}{2}}, \tag{A8}
\]

where \( q_{ij} \) is the momentum of \( i \) (\( k \)) in the \( ij \) (total) center-of-mass frame. The second vertices \( D_{s0J}^{(*)} \to DK \) and \( D_{s1} \to DK \) are given by

\[
c_{DK,D_{s0J}^D} F_{DK,D_{s0J}^D}^{00}, \tag{A9}
\]

\[
c_{D_{s0J}^D,D_{s1}^D} \epsilon_{D_{s0J}^D} \epsilon_{D_{s1}^D} F_{D_{s0J}^D,D_{s1}^D}^{10}, \tag{A10}
\]
with form factors \( f_{ij,k}^D \equiv f_i^D / \sqrt{E_k} \) and

\[
f_{ij}^D = \frac{1}{\sqrt{E_i E_j}} \left( \frac{\Lambda^2}{\Lambda^2 + q_{ij}^2} \right)^{2+(L/2)}.
\]

The third vertices \( D^{(+)\bar{D}^{(*)}} \rightarrow J/\psi \bar{K} \) are \( p \)-wave interactions between the pairs with the same spin-parity \( j^P = 0^- \) or \( 1^- \), and are given with coupling constants \( c_{D^{(*)}\bar{D}^{(*)}J} \) as

\[
c_{\psi K, D\bar{D}}^{1-} i(p_{\bar{K}} \times \epsilon_\psi) \cdot p_{D\bar{D}} f_{\psi K} f_{D\bar{D}}^D f_{\bar{D}D}, \quad \quad \quad (12)
\]

\[
c_{\psi K, D\bar{D}^*}^{1-} \epsilon_\psi \cdot p_{D\bar{D}^*} f_{\psi K} f_{D\bar{D}^*}^D f_{\bar{D}D}, \quad \quad \quad (13)
\]

\[
c_{\psi K, D\bar{D}^*}^{1-} (p_{D\bar{D}^*} \times \epsilon_\psi) \cdot (p_{\bar{K}} \times \epsilon_\psi) f_{\psi K} f_{D\bar{D}^*}^D f_{\bar{D}D}, \quad \quad \quad (14)
\]

\[
A_{D_{a_{10}}, D_{a_{10}}^{(+)\bar{D}^{(*)}}}^{DT} \equiv c_{\psi K, \bar{D}D}^{1-} i(p_{\bar{K}} \times \epsilon_\psi) \cdot p_{DD} f_{\psi K} f_{D\bar{D}}^D f_{\bar{D}D}.
\]

respectively, where a notation of \( p_{ab} \equiv p_a - p_b \) has been used. The fourth vertex \( K\bar{K} \rightarrow \phi \) is common for all the DT diagram, and is given as

\[
c_{K\bar{K}, \phi} p_{\bar{K}K} \cdot \epsilon_\phi f_{\phi K}^{f_1} f_{D\bar{D}^*} \quad (19)
\]

We denote the DT amplitudes including \( D^{(*) \bar{D}^{(*)}} (J^P) \) by \( A_{D_{a_{10}}^{(+)} D_{a_{10}}^{(*)}}^{DT} \). The DT amplitudes are constructed with the above ingredients, and are given by

\[
A_{D_{a_{10}}, D_{a_{10}}^{(+)\bar{D}^{(*)}}}^{DT} = c_{\psi K, \bar{D}D}^{1-} i(p_{\bar{K}} \times \epsilon_\psi) \cdot p_{DD} f_{\psi K} f_{D\bar{D}}^D f_{\bar{D}D}. 
\]

\[
A_{D_{a_{10}}, D_{a_{10}}^{(+)\bar{D}^{(*)}}}^{DT} = c_{\psi K, \bar{D}D^*}^{1-} \epsilon_\psi \cdot p_{D\bar{D}^*} f_{\psi K} f_{D\bar{D}^*} f_{\bar{D}D}.
\]

\[
A_{D_{a_{10}}, D_{a_{10}}^{(+)\bar{D}^{(*)}}}^{DT} = c_{\psi K, \bar{D}D^*}^{1-} (p_{D\bar{D}^*} \times \epsilon_\psi) \cdot (p_{\bar{K}} \times \epsilon_\psi) f_{\psi K} f_{D\bar{D}^*} f_{\bar{D}D}.
\]

where, in each amplitude, the summation over \( D^{(+)} K^0 \bar{K}^0 \) and \( D^{(*)0} K^+ \bar{K}^- \) intermediates states with
the charge dependent particle masses is implicit; $K^+$ in the final state is denoted by $K_f$, and $W \equiv E - E_{K_f}$.

Regarding the $D_{sJ}^{(*)}$ widths in the third energy denominators, while $\Gamma_{D_{sJ}}$ is well determined experimentally, $\Gamma_{D_{sJ}^{(*)}}^\pi$ is given only an upper limit [6]. We use $\Gamma_{D_{sJ}^{(*)}}^{\pi} = 0.1$ MeV; the result does not significantly change for $\Gamma_{D_{sJ}^{(*)}}^{\pi}$, $\Gamma_{D_{sJ}^{(*)}} < 1$ MeV. We neglect $\Gamma_{D_s}$ and $\Gamma_{D_s^*}$ which are expected to be very small ($\Gamma_{D_s} \sim 55$ keV [7], $\Gamma_{D_s^*} \sim 0.1$ keV [6, 9]). From Eqs. (A20)-(A26), we remove terms including $p_\gamma, \epsilon_\gamma$ and $p_\gamma \cdot \epsilon_\gamma$, to maintain a consistency with the Lorentz condition, $p_\gamma \cdot \epsilon_\gamma = p_\gamma \cdot \epsilon_\gamma = 0$, of a relativistic formulation.

For a given DT amplitude, we can analytically integrate the angular part of the loop-integrals by ignoring smaller angle dependences from denominators and form factors. If the DT amplitude is (non-)vanishing after this angular integral, the DT integrand has a suppressed (favorable) tensor structure. The p-wave $D^{(s)}_{sJ} D^{(*)}_{sJ} \rightarrow J/\psi K$ interactions of Eqs. (A12)-(A18) are chosen so that the DT integrands of Eqs. (A20)-(A26) have favored tensor structures. We did not use s-wave $D^{(s)}_{sJ} D^{(*)}_{sJ} \rightarrow J/\psi K$ interactions because the resultant DT integrands have suppressed tensor structures. We numerically confirmed the suppression.

Next we present formulas for one-loop amplitudes of Fig. 1(b) including s-wave pairs of $D_{sJ}^* D_{sJ}(1^{+})$, $D_{sJ}^* D_{sJ}^*(0^{+})$, and $\psi(2S)^0, (1S)^0$ in the loop. The one-loop processes are initiated by $B^+ \rightarrow D_{sJ}^* D_{sJ}^{(*)} K^+$ and $B^+ \rightarrow \psi(2S)^0 K^+$ vertices given as

$$c_{D_s^* D_s^*(1^+)} P K \cdot \epsilon_{D_s^*} F_{D_s^* D_s^* K, B}^{00} \quad (A27)$$

$$c_{D_s^* D_s^*(0^+)} e_{D_s^*} \cdot \epsilon_{D_s^*} F_{D_s^* D_s^* K, B}^{00} \quad (A28)$$

$$c_{\psi(2S)^0 K, B}^{00} \quad (A29)$$

$$c_{\psi(1S)^0} \cdot \epsilon_\psi \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A30)$$

The subsequent $D_{sJ}^* D_{sJ}^{(*)}, \psi(2S)^0 \rightarrow J/\psi K$ interactions in $J/\psi$ partial waves are given, with coupling constants $c_{J/\psi, D_{sJ}^{(*)}}$ and $c_{J/\psi, \psi(2S)^0}$, as

$$c_{J/\psi, D_{sJ}^{(*)}}^{1+} \cdot \epsilon_\psi \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A31)$$

$$c_{J/\psi, D_{sJ}^{(*)}}^{0+} e_{D_s^*} \cdot \epsilon_{D_s^*} F_{D_s^* D_s^* K, B}^{00} \quad (A32)$$

$$c_{J/\psi, \psi(2S)^0}^{0+} e_{\psi} \cdot \epsilon_{\psi} \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A33)$$

$$c_{J/\psi, \psi(2S)^0}^{1+} \cdot \epsilon_\psi \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A34)$$

We denote the one-loop amplitudes including $D_{sJ}^* D_{sJ}^{(*)}(J/\psi)$ and $\psi(2S)^0 J/\psi$ by $A_{J/\psi, D_{sJ}^{(*)}}^{1+}(J/\psi)$ and $A_{J/\psi, \psi(2S)^0}^{1+}(J/\psi)$ respectively. The amplitudes are given with the above ingredients as

$$A_{J/\psi, D_{sJ}^{(*)}}^{1+}(J/\psi) = c_{J/\psi, D_{sJ}^{(*)}}^{1+} \cdot \epsilon_\psi \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A35)$$

$$A_{J/\psi, D_{sJ}^{(*)}}^{0+}(J/\psi) = c_{J/\psi, D_{sJ}^{(*)}}^{0+} e_{\psi} \cdot \epsilon_{\psi} \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A36)$$

$$A_{J/\psi, \psi(2S)^0}(J/\psi) = c_{J/\psi, \psi(2S)^0}^{1+} \cdot \epsilon_\psi \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A37)$$

$$A_{J/\psi, \psi(2S)^0}^{0+}(J/\psi) = c_{J/\psi, \psi(2S)^0}^{0+} e_{\psi} \cdot \epsilon_{\psi} \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A38)$$

where $\Gamma_{\psi(2S)^0}$ has been neglected since $\Gamma_{\psi(2S)^0} \ll \Gamma_{\psi}$. Regarding the $Z_{cs}^0$ excitation mechanisms [Fig. 1(c)], we consider lighter and heavier ones, respectively denoted by $Z_{cs}$ and $Z_{cs}^0$, that could be identified with $Z_{cs}(4000)$ and $Z_{cs}(4220)$ from the LHCb analysis [12]. Our $Z_{cs}^0$ would simulate the $D_{sJ}^{(*)} + D_{sJ}^{(*)}$ threshold cusps enhanced by virtual states found in coupled-channel analyses [62-64]. We consider s- and p-wave $B^+ \rightarrow Z_{cs}^{(1S)^0, (0S)^0}$ decays followed by $Z_{cs}^{(1S)^0, (0S)^0} \rightarrow J/\psi K^+$. The s- and p-wave decays are parity-conserving and parity-violating, respectively, and contribute to the $0^+$ and $1^+$ $J/\psi K$ final states, respectively. The corresponding amplitudes $A_{Z_{cs}^0}^{J/\psi}$ are given by

$$A_{Z_{cs}^0}^{J/\psi} = c_{Z_{cs}^0}^{J/\psi} \cdot \epsilon_{\psi} \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A39)$$

$$A_{Z_{cs}^0}^{J/\psi} = c_{Z_{cs}^0}^{J/\psi} e_{\psi} \cdot \epsilon_{\psi} \cdot F_{\psi(2S)^0 K, B}^{00} \quad (A40)$$

where the $Z_{cs}$ and $Z_{cs}^0$ masses are 3975 MeV and 4119 MeV from the $D_{sJ}^{(*)} + D_{sJ}^{(*)}$ thresholds, respectively; their widths are set to be 100 MeV (constants).

The direct decay amplitudes [Fig. 1(d)] can be projected onto the $J/\psi(2S)^0$ partial waves. Thus we employ
the product of coupling constants to fit the data, and its value and unit are given in the fourth and fifth columns, respectively.

**TABLE II. Fit fractions and parameter values.** The common cut off value Λ = 1 GeV is used. The first column lists each mechanism considered in our model, and the second column is its fit fraction (%) defined in Eq. (A33). The third column lists the product of coupling constants to fit the data, and its value and unit are given in the fourth and fifth columns, respectively. Amplitude formulas are given in the equations in the last column.

| Mechanism | Fit Fraction (%) | Value | Unit |
|-----------|------------------|-------|------|
| κKφ,ωD*Δ− | 15 | −158 | GeV−3 |
| κKφ,ωD*Δ+ | 50 | 44.6 | GeV−2 |
| κKφ,ωD+Δ− | 25 | −32.0 | GeV−3 |
| κKφ,ωD+Δ+ | 10 | −45.2 | GeV−2 |

**TABLE III. Parameter values for the full model (Λ = 1 GeV) not fitted to the LHCb data.** The last two parameters are elastic D*+D−(s) interaction strengths defined by Eq. (43) in the Supplemental Material of Ref. [73].

| Parameter | Value | Source |
|-----------|-------|--------|
| Λ (MeV) | 1000 | | |
| A0+ | 850 | Eq. (A41) |
| A1+ | 630 | Eq. (A42) |
| hD*Δ− | 2 | |
| hD*Δ+ | 2 | |

A form as follows:

\[
A_{0+}^{\text{dir}} = c_{0+}^{\text{dir}} \cdot \epsilon_{0+} F_{0+}^{\psi K J,B} , \tag{A41}
\]

\[
A_{1+}^{\text{dir}} = c_{1+}^{\text{dir}} (\epsilon_{1+} \times \epsilon_{0+}) \cdot P_{K J,B} F_{01}^{\psi K J,B} , \tag{A42}
\]

where \(c_{0+}^{\text{dir}}\) is a coupling constant for the \(J/\psi\phi(J^P)\) partial wave amplitude.

We basically use a common cutoff value (1 GeV unless otherwise stated) in the form factors for all the interaction vertices discussed above. One exception applies to Eqs. (A41) and (A42) where we adjust \(\Lambda'\) of Eq. (A30) so that the \(M_{J/\psi\phi}\) distribution from the direct decay amplitude is similar to the phase-space shape.

In numerical calculations, for convenience, the above amplitudes are evaluated in the \(J/\psi\phi\) center-of-mass frame. An exception is the \(Z_{\text{dir}}(s)\) amplitudes that are evaluated in the total center-of-mass frame. With the relevant kinematical factors multiplied to the amplitudes, the invariant amplitudes are obtained and plugged into the Dalitz plot distribution formula; see Appendix B of Ref. [78] for details.

Parameter values obtained from and not from the fit are listed in Tables II and III, respectively. In Table II we also list each mechanism’s fit fraction defined by

\[
FF = \frac{\Gamma_{A_x}}{\Gamma_{\text{full}}} \times 100 \, \text{(} \%	ext{)} , \tag{A43}
\]

where \(\Gamma_{\text{full}}\) and \(\Gamma_{A_x}\) are \(B^+ \to J/\psi\phi K^+\) decay rates calculated with the full model and with an amplitude \(A_x\) only, respectively. In Table II \(A_{1+}^{\text{dir}}\) seems to have a rather large fit fraction of \(\sim 36\%\). This mechanism causes a threshold cusp at \(M_{J/\psi\pi} \sim 4.7\) GeV, and its height (without interference) is about 80\% of the data. The mechanism also has a long tail toward the lower \(M_{J/\psi\pi}\) region, which makes its fit fraction rather large.

The amplitudes \(A_{0+}^{\text{dir}}\) and \(A_{1+}^{\text{dir}}\) also have large fit fractions of \(\sim 67\%\) and \(\sim 45\%\), respectively. This may be because \(K_{0+}\)-excitation mechanisms have been subsumed in this mechanism. The LHCb analysis [12] found large fit frac-
tions of the $K_J^{(+)}$-excitation mechanisms.

**Appendix B: Double triangle amplitudes and closeness to the leading singularity**

The Coleman-Norton theorem [74] states that a DT amplitude like Fig. 1(a) has the leading singularity if the whole DT process is kinematically allowed at the classical level: the energy and momentum are always conserved; in Fig. 1(a), all internal momenta are collinear in the $D_J^{(+)} \bar{D}_J^{(-)}$ center-of-mass frame; $D^{(+)}$ and $\bar{D}_J^{(+)}$ ($\bar{K}$ and $K$) are moving to the same direction and the former is faster than the latter.

The DT amplitudes presented in Eqs. (A20)-(A26) do not exactly satisfy the above kinematical condition, and thus do not have the leading singularity. Yet, their threshold cusps are significantly enhanced compared with an ordinary one-loop threshold cusp. This is because the DT amplitudes are fairly close to satisfying the kinematical condition of the leading singularity, and here we examine how close.

Let us study $A_{D_J^{(0+)},1}^{DT}$ of Eq. (A23) that generates an $X(4500)$-like threshold cusp. Apart from the coupling constants and the dependence on the external $K^+$, we can express Eq. (A23) as

$$A_{D_J^{(0+)},1}^{DT} = \int dp_K G(p_K) H(p_K), \quad (B1)$$

with

$$G(p_K) = \int d\Omega_{p_K} \frac{p_K^2 p_{K\bar{K}} \epsilon_{\bar{K}} p_{\bar{K}J/\psi} \epsilon_{\bar{J}/\psi}}{W - E_K - E_{\bar{K}} - E_{\bar{J}/\psi} + i\epsilon} f^1_{D_J^{(0+)}} f^1_{D_J^{(0+)}} (B2)$$

$$H(p_K) = \int d^3 p_{\bar{D}_J} \frac{p_{D_J^{(0+)}}}{W - E_{\bar{D}_J} - E_{\bar{D}_J} + i\epsilon} \times \frac{f^1_{D_J^{(0+)}} f^1_{D_J^{(0+)}} f^1_{D_J^{(0+)}}}{W - E_{D_J^{(0+)}} - E_{D_J^{(0+)}} + i\epsilon}, \quad (B3)$$

where $G(p_K)$ and $H(p_K)$ have been implicitly projected onto $0^+$ of the $J/\psi$ pair. Here, we suppose that $G(p_K)$ and $H(p_K)$ include only $K = K^+$, $\bar{K} = K^-$, $D^* = D^{*0}$ in the two-loop, although their isospin partners are also included in Eq. (A23). Now we plot in Fig. 6 $G(p_K)$ and $H(p_K)$ for $W = m_{D_J^{(0+)}} + m_{D_J^{(0+)}} + 1.4$ MeV $\sim 4505$ MeV where the DT amplitude is close to cause the leading singularity. Both $G(p_K)$ and $H(p_K)$ show singular behaviors. The real part of $H(p_K)$ [red solid curve] shows a peak at $p_K \sim 170$ MeV (peak A). The peak A is due to a triangle singularity from the $D^{*0}_{J} \bar{D}_J^{(-)} D^{*0}$ triangle loop; see Fig. 1(a). Meanwhile, the angular integral of the $KKJ/\psi$ energy denominator in $G(p_K)$ causes a logarithmic end-point singularity. Thus the real part of $G(p_K)$ [blue dashed curve] shows two peaks, and the one at $p_K \sim 240$ MeV (peak B) is relevant to the leading singularity. The other peak does not create a singular behavior in the amplitude of Eq. (B1). If the peaks A and B occurred at the same $p_K$, the DT leading singularity (pinch singularity) would have occurred. Yet, Fig. 6 indicates a substantial overlap between the peaks A and B, and this is the cause of the enhancement of the DT threshold cusps. The proximity of $A_{D_J^{(0+)},1}^{DT}$ to the leading singularity condition is due to the fact that: (i) each vertex is kinematically allowed to occur at on-shell; (ii) the mass deficit in $D^* \bar{D}_J^{(-)} \rightarrow \bar{K} J/\psi$ enables the relatively light $\bar{K}$ to chase $K$ with a velocity faster than $K$. In fact, if the exchanged $K^-$ mass were in the range of $445 \lesssim m_{K^+} \lesssim 455$ MeV, $A_{D_J^{(0+)},1}^{DT}$ would have hit the leading singularity.

![Graph](image-url)

**FIG. 6.** $G$ and $H$ from the DT amplitude including the $p$-wave $D_J^{(0+)}, 1$ pair, as defined in Eqs. (B1) and (B3): $W = m_{D_J^{(0+)}} + m_{D_J^{(0+)}} + 1.4$ MeV $\sim 4505$ MeV. The real and imaginary parts of $G$ [red] and $H$ [blue] show a logarithmic singularity near $p_K \sim 240$ MeV (peak B). The real part of $H$ [green] shows two peaks, and the one at $p_K \sim 170$ MeV (peak A). The peak A is due to a triangle singularity from the $D^{*0}_{J} \bar{D}_J^{(-)} D^{*0}$ triangle loop; see Fig. 1(a). Meanwhile, the angular integral of the $KKJ/\psi$ energy denominator in $G(p_K)$ causes a logarithmic end-point singularity. Thus the real part of $G(p_K)$ [blue dashed curve] shows two peaks, and the one at $p_K \sim 240$ MeV (peak B) is relevant to the leading singularity. The other peak does not create a singular behavior in the amplitude of Eq. (B1). If the peaks A and B occurred at the same $p_K$, the DT leading singularity (pinch singularity) would have occurred. Yet, Fig. 6 indicates a substantial overlap between the peaks A and B, and this is the cause of the enhancement of the DT threshold cusps. The proximity of $A_{D_J^{(0+)},1}^{DT}$ to the leading singularity condition is due to the fact that: (i) each vertex is kinematically allowed to occur at on-shell; (ii) the mass deficit in $D^* \bar{D}_J^{(-)} \rightarrow \bar{K} J/\psi$ enables the relatively light $\bar{K}$ to chase $K$ with a velocity faster than $K$. In fact, if the exchanged $K^-$ mass were in the range of $445 \lesssim m_{K^+} \lesssim 455$ MeV, $A_{D_J^{(0+)},1}^{DT}$ would have hit the leading singularity.

[1] P.A. Zyla et al. (Particle Data Group), The Review of Particle Physics, Prog. Theor. Exp. Phys. 2020, 083C01 (2020).
[2] T. Aaltonen et al. (CDF Collaboration), Evidence for a Narrow Near-Threshold Structure in the $J/\psi \phi$ Mass Spectrum in $B^+ \rightarrow J/\psi \phi K^+$ Decays, Phys. Rev. Lett. 102, 242002 (2009).
[3] J. Brodzicka, Heavy flavour spectroscopy, Conference Proceedings C0908171, 299 (2009).
[4] T. Aaltonen et al. (CDF Collaboration), Observation of
the $Y(4140)$ structure in the $J/\psi$ mass spectrum in $B^\pm \to J/\psi K^\pm$ decays, Mod. Phys. Lett. A 32, 1750139 (2017).

[5] R. Aaij et al. (LHCb Collaboration), Search for the $X(4140)$ state in $B^+ \to J/\psi K^{*+}$ decays, Phys. Rev. D 85, 091103(R) (2012).

[6] S. Chatrchyan et al. (CMS Collaboration), Observation of a peaking structure in the $J/\psi$ mass spectrum from $B^\pm \to J/\psi K^\pm$ decays, Phys. Lett. B 734, 261 (2014).

[7] V.M. Abazov et al. (D0 Collaboration), Search for the $X(4140)$ state in $B^+ \to J/\psi K^{*+}$ decays with the D0 detector, Phys. Rev. D 89, 012004 (2014).

[8] J.P. Lees et al. (BABAR Collaboration), Study of $B_{s}^{\pm,0} \to J/\psi K^{+}K^{-}K_{s}^{0}$ and search for $B_{s}^{0} \to J/\psi$ at BABAR, Phys. Rev. D 91, 012003 (2015).

[9] V.M. Abazov et al. (D0 Collaboration), Inclusive Production of the $X(4140)$ State in $p\bar{p}$ Collisions at D0, Phys. Rev. Lett. 115, 232001 (2015).

[10] R. Aaij et al. (LHCb Collaboration), Observation of $J/\psi$ Structures Consistent with Exotic States from Amplitude Analysis of $B_{s}^{\pm} \to J/\psi K^{(*)+}$ Decays, Phys. Rev. Lett. 118, 022003 (2017).

[11] R. Aaij et al. (LHCb Collaboration), Amplitude analysis of $B_{s}^{*} \to J/\psi K^{(*)+}$ decays, Phys. Rev. D 95, 012002 (2017).

[12] R. Aaij et al. (LHCb Collaboration), Observation of new resonances decaying to $J/\psi K^{*}$ and $J/\psi$, Phys. Rev. Lett. 127, 082001 (2021).

[13] P.G. Ortega, J. Segovia, D.R. Entem, and F. Fernández, Canonical description of the new LHCb resonances, Phys. Rev. D 94, 114018 (2016).

[14] D.-Y. Chen, Where are $\chi_{cJ}(3P)$ ?, Eur. Phys. J. C 76, 671 (2016).

[15] W.-J. Deng, H. Liu, L.-Ch. Gui, and X.-H. Zhong, Charmonium spectrum and their electromagnetic transitions with higher multipole contributions, Phys. Rev. D 95, 034026 (2017).

[16] R. Oncala and J. Soto, Heavy Quarkonium Hybrids: Spectrum, Decay and Mixing, Phys. Rev. D 96, 014004 (2017).

[17] D. Molina and M. De Sanctis, and C. Fernandez-Ramirez, Charmonium spectrum with a Dirac potential model in the momentum space, Phys. Rev. D 95, 094021 (2017).

[18] L.-C. Gui, L.-S. Lu, Q.-F. Lü, X.-H. Zhong, and Q. Zhao, Strong decays of higher charmonium states into open-charm meson pairs, Phys. Rev. D 98, 016010 (2018).

[19] A.M. Badalian and B.L.G. Bakker, Radial and orbital Regge trajectories in heavy quarkonia, Phys. Rev. D 100, 054036 (2019).

[20] J. Ferretti, E. Santopinto, M.N. Anwar, and Y. Lu, Quark structure of the $\chi_{c}(3P)$ and $X(4274)$ resonances and their strong and radiative decays, Eur. Phys. J. C 80, 464 (2020).

[21] J. Ferretti and E. Santopinto, Quark Structure of the $X(4500), X(4700)$ and $\chi_{c}(4P,5P)$ States, Front.in Phys. 9, 76 (2021).

[22] M.-X. Duan and X. Liu, Where are $3P$ and higher $P$-wave states in the charmonium family?, arXiv:2107.14435 [hep-ph].

[23] Q.-F. Lü and Y.-B. Dong, X(4140), X(4274), X(4500), and X(4700) in the relativized quark model, Phys. Rev. D 94, 074007 (2016).

[24] Fl. Stancu, Can $Y(4140)$ be a $c\bar{c}s\bar{s}$ tetraquark?, J. Phys.G 37, 075017 (2010).

[25] L. Maiani, A.D. Polosa, and V. Riquer, Interpretation of Axial Resonances in $J/\psi$ at LHCb, Phys. Rev. D 94, 054026 (2016).

[26] R. Zhu, Hidden charm octet tetraquarks from a diquark-antidiquark model, Phys. Rev. D 94, 054009 (2016).

[27] J. Wu, Y.-R. Liu, K. Chen, X. Liu, and S.-L. Zhu, $X(4140), X(4270), X(4500)$ and $X(4700)$ and their $c\bar{c}s\bar{s}$ tetraquark partners, Phys. Rev. D 94, 094031 (2016).

[28] S.S. Agaev, K. Azizi, and H. Sundu, Exploring the resonances $X(4140)$ and $X(4274)$ through their decay channels, Phys. Rev. D 95, 114003 (2017).

[29] H.-X. Chen, E.-L. Cui, W. Chen, X. Liu, and S.-L. Zhu, Understanding the internal structures of the $X(4140)$, $X(4274)$, $X(4500)$ and $X(4700)$, Eur. Phys. J. C 77, 160 (2017).

[30] Z.-G. Wang, Reanalysis of the $X(3915), X(4500)$ and $X(4700)$ with QCD sum rules, Eur. Phys. J. A 53, 19 (2017).

[31] Z.-G. Wang, Scalar tetraquark state candidates: $X(3915), X(4500)$ and $X(4700)$, Eur. Phys. J. C 77, 78 (2017).

[32] Z.-G. Wang, Analysis of the mass and width of the $Y(4274)$ as axialvector molecule-like state, Eur. Phys. J. C 77, 174 (2017).

[33] C. Deng, J. Ping, H. Huang, and F. Wang, Hidden charmed states and multibody color flux-tube dynamics, Phys. Rev. D 98, 014026 (2018).

[34] M.N. Anwar, J. Ferretti, and E. Santopinto, Spectroscopy of the hidden-charm $[q\bar{c}][\bar{q}\bar{c}]$ and $[s\bar{c}][s\bar{c}]$ tetraquarks in the relativized diquark model, Phys. Rev. D 98, 094015 (2018).

[35] A. Türkan and H. Dag, Exploratory study of $\chi_{c1}(4140)$ and like states in QCD sum rules, Nucl. Phys. A 985, 38 (2019).

[36] J. Wu, X. Liu, Y.-R. Liu, S.-L. Zhu, Systematic studies of charmonium-, bottomonium-, and $B_{c}$-like tetraquark states, Phys. Rev. D 99, 014037 (2019).

[37] Z.-G. Wang and Z.-Y. Di, Analysis of the mass and width of the $X(4140)$ as axialvector tetraquark state, Eur. Phys. J. C 79, 72 (2019).

[38] Y. Yang and J. Ping, Investigation of $c\bar{c}s\bar{s}$ tetraquark in the chiral quark model, Phys. Rev. D 99, 094032 (2019).

[39] Z.-G. Wang, Revisit the $X(4274)$ as the Axialvector Tetraquark State, Acta Phys.Polon.B 51, 435 (2020).

[40] C. Deng, H. Chen, and J. Ping, Can the state $Y(4626)$ be a $P$-wave tetraquark state $[s\bar{c}][s\bar{c}]$ ?, Phys. Rev. D 101, 054039 (2020).

[41] Z. Ghalenovi and M.M. Sorkhi, Spectroscopy of hidden-charm tetraquarks in diquark model, Eur. Phys. J. Plus 135, 399 (2020).

[42] P.-P. Shi, F. Huang, and W.-L. Wang, Hidden charm tetraquark states in a diquark model, Phys. Rev. D 103, 094038 (2021).

[43] Z.-G. Wang, Assignments of the $X(4140), X(4500), X(4630)$ and $X(4685)$ based on the QCD sum rules, arXiv:2103.04236 [hep-ph].

[44] X. Liu, H. Huang, J. Ping, D. Chen, and X. Zhu, The explanation of some exotic states in the $c\bar{c}s\bar{s}$ tetraquark system, arXiv:2103.12425 [hep-ph].

[45] Z.-M. Ding, H.-Y. Jiang, D. Song, and J. He, Hidden and doubly heavy molecular states from interactions $D_{c}^{(*)}D_{s}^{(*)}/B_{c}^{(*)}B_{s}^{(*)}$ and $D_{c}^{(*)}D_{s}^{(*)}/B_{c}^{(*)}B_{s}^{(*)}$, Eur. Phys. J. C 81, 732 (2021).
[46] Y.A. Simonov, Recoupling Mechanism for exotic mesons and baryons, JHEP 04, 051 (2021).
[47] J. He, Understanding spin parity of $P_c(4450)$ and $Y(4274)$ in a hadronic molecular state picture, Phys. Rev. D 95, 074004 (2017).
[48] S.L. Olsen, T. Skwarnicki, and D. Zieminska, Nonstandard heavy mesons and baryons: Experimental evidence, Rev. Mod. Phys. 90, 015003 (2018).
[49] A. Esposito, A. Pilloni, and A.D. Polosa, Multiquark Resonances, Phys. Rept. 668, 1 (2017).
[50] M. Karliner, J.L. Rosner, and T. Skwarnicki, Multiquark States, Ann. Rev. Nucl. Part. Sci. 68, 17 (2018).
[51] R.M. Albuquerque, J.M. Dias, K.P. Khemchandani, A. Martínez Torres, F.S. Navarra, M. Nielsen, and C.M. Zanetti, QCD sum rules approach to the $X$, $Y$ and $Z$ states, J. Phys. G 46, 093002 (2019).
[52] S. Agaev, K. Azizi, and H. Sundu, Four-quark exotic mesons, Turk. J. Phys. 44, 95 (2020).
[53] X.-H. Liu, How to understand the underlying structures of $X(4140)$, $X(4274)$, $X(4500)$ and $X(4700)$, Phys. Lett. B 766, 117 (2017).
[54] X.-K. Dong, F.-K. Guo, and B.-S. Zou, A survey of heavy-antihadron molecules, Progr. Phys. 41, 65 (2021).
[55] F.-K. Guo, X.-H. Liu, and S. Sakai, Threshold cusp and triangle singularities in hadronic reactions, Prog. Part. Nucl. Phys. 112, 103757 (2020).
[56] E.S. Swanson, Cusps and Exotic Charmonia, Int. J. Mod. Phys. E 25, 1642010 (2016).
[57] M. Padmanath, C. B. Lang, and S. Prelovsek, $X(3872)$ and $Y(4140)$ using diquark-antidiquark operators with lattice QCD, Phys. Rev. D 92, 034501 (2015).
[58] L. Liu, K. Orginos, F.-K. Guo, C. Hanhart, and U.-G. Meißner, Interactions of charmed mesons with light pseudoscalar mesons from lattice QCD and implications on the nature of the $D_{s0}^*(2317)$, Phys. Rev. D 87, 014508 (2013).
[59] A. Martínez Torres, E. Oset, S. Prelovsek, and A. Ramos, Reanalysis of lattice QCD spectra leading to the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$, JHEP 05, 153 (2015).
[60] G.K.C. Cheung, C.E. Thomas, D.J. Wilson, G. Moir, M. Peardon, and S.M. Ryan (Hadron Spectrum Collaboration), $DK = 0$, $DK = 0$, and $DK = 0$ scattering and the nature of the $D_{s0}^*(2317)$ from lattice QCD, JHEP 02, 100 (2021).
[61] Z. Yang, G.-J. Wang, J.-J. Wu, M. Oka, and S.-L. Zhu, Novel coupled channel framework connecting quark model and lattice QCD: an investigation on near-threshold $D_s$ states, arXiv:2107.04860.
[62] P.G. Ortega, D.R. Entem, and F. Fernández, The strange partner of the $Z_c$ structures in a coupled-channels model, Phys. Lett. B 818, 136382 (2021).
[63] V. Baru, E. Epelbaum, A.A. Filin, C. Hanhart, and A.V. Nefediev, Is $Z_{c0}(3982)$ a molecular partner of $Z_c(3900)$ and $Z_c(4020)$ states?, Phys. Rev. D 105, 034014 (2022).
[64] M.-L. Du, M. Albaladejo, F.-K. Guo, and J. Nieves, A combined analysis of the $Z_c(3900)$ and the $Z_c(3985)$ exotic states, arXiv:2201.08253.
[65] B. Aubert et al. (BABAR Collaboration), Study of the $D_{sJ}^*(2317)^+$ and $D_{sJ}^*(2460)^+$ mesons in inclusive $cc$ production near $\sqrt{s} = 10.6$ GeV, Phys. Rev. D 74, 032007 (2006).
[66] S. Godfrey, Testing the nature of the $D_{sJ}^*(2317)^+$ and $D_{sJ}^*(2463)^+$ states using radiative transitions, Phys. Lett. B 568, 254 (2003).
[67] P. Colangelo and F. De Fazio, Understanding $D_{sJ}^*(2317)$, Phys. Lett. B 570, 180 (2003).
[68] B. Aubert et al. (BABAR Collaboration), Measurement of the branching ratios $\Gamma(D_{sJ}^{*+} \rightarrow D_s^* \pi^0)/\Gamma(D_{sJ}^{*+} \rightarrow D_s^* \pi^0)$ and $\Gamma(D_{sJ}^{*0} \rightarrow D_s^0 \pi^0)/\Gamma(D_{sJ}^{*0} \rightarrow D_s^0 \pi^0)$, Phys. Rev. D 72, 091101(R) (2005).
[69] B. Yang, B. Wang, L. Meng, and S.-L. Zhu, Isospin violating decay $D_{sJ}^* \rightarrow D_s \pi^0$ in chiral perturbation theory, Phys. Rev. D 101, 054019 (2020).
[70] S.X. Nakamura, $P_c(4312)^+$, $P_c(4380)^+$, and $P_c(4457)^+$ as double triangle cusp, Phys. Rev. D 103, L111503 (2021).
[71] X.-K. Dong, F.-K. Guo, and B.-S. Zou, Explaining the Many Threshold Structures in the Heavy-Quark Hadron Spectrum, Phys. Rev. Lett. 126, 152001 (2021).
[72] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The Analytic S-Matrix, (Cambridge University Press, Cambridge, England, 1966).
[73] S.X. Nakamura, Novel description of $P_c(4312)^+$, $P_c(4380)^+$, and $P_c(4457)^+$ with double triangle cusp, PoS CHARM2020, 029 (2021).
[74] S. Coleman and R.E. Norton, Singularities in the physical region , Nuovo Cimento 38, 438 (1965).
[75] L.D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. 13, 181 (1959).
[76] P.G. Ortega, J. Segovia, D.R. Entem, and F. Fernández, Molecular components in $P$-wave charmed-strange mesons, Phys. Rev. D 94, 074037 (2016).
[77] S.X. Nakamura, Triangle singularity appearing as an $X(3872)$-like peak in $B \to (J/\psi \pi^+ \pi^-)K\pi$, Phys. Rev. D 102, 074004 (2020).
[78] H. Kamano, S.X. Nakamura, T.-S.H. Lee, and T. Sato, Unitary coupled-channels model for three-mesons decays of heavy mesons, Phys. Rev. D 84, 114019 (2011).