COVERING MAPS FOR LOCALLY PATH-CONNECTED SPACES

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Abstract. We define Peano covering maps and prove basic properties analogous to classical covers. Their domain is always locally path-connected but the range may be an arbitrary topological space. One of characterizations of Peano covering maps is via the uniqueness of homotopy lifting property for all locally path-connected spaces.

Regular Peano covering maps over path-connected spaces are shown to be identical with generalized regular covering maps introduced by Fischer and Zastrow [15]. If \( X \) is path-connected, then every Peano covering map is equivalent to the projection \( \tilde{X}/H \to X \), where \( H \) is a subgroup of the fundamental group of \( X \) and \( \tilde{X} \) equipped with the topology used in [2], [15] and introduced in [23, p.82]. The projection \( \tilde{X}/H \to X \) is a Peano covering map if and only if it has the unique path lifting property. We define a new topology on \( \tilde{X} \) for which one has a characterization of \( \tilde{X}/H \to X \) having the unique path lifting property if \( H \) is a normal subgroup of \( \pi_1(X) \). Namely, \( H \) must be closed in \( \pi_1(X) \). Such groups include \( \pi(U, x_0) \) (\( U \) being an open cover of \( X \)) and the kernel of the natural homomorphism \( \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0) \).

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1. Introduction

As locally complicated spaces naturally appear in mathematics (examples: boundaries of groups, limits under Gromov-Hausdorff convergence) there is an effort to extend homotopy-theoretical concepts to such spaces. This paper is devoted to...
a theory of coverings by locally path-connected spaces. Zeeman’s example [16] 6.6.14 on p.258 demonstrates difficulty in constructing a theory of coverings by non-locally path-connected spaces (that example amounts to two non-equivalent classical coverings with the same image of the fundamental groups). For coverings in the uniform category see [1] and [3].

To simplify exposition let us introduce the following concepts:

**Definition 1.1.** A topological space $X$ is an **lpc-space** if it is locally path-connected. $X$ is a **Peano space** if it is locally path-connected and connected.

Fischer and Zastrow [15] defined generalized regular coverings of $X$ as functions $p: \tilde{X} \rightarrow X$ satisfying the following conditions for some normal subgroup $H$ of $\pi_1(X)$:

R1. $\tilde{X}$ is a Peano space.

R2. The map $p: \tilde{X} \rightarrow X$ is a continuous surjection and $\pi_1(p): \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is a monomorphism onto $H$.

R3. For every Peano space $Y$, for every continuous function $f: (Y, y) \rightarrow (X, x_0)$ with $f_*(\pi_1(Y, y)) \subset H$, and for every $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x_0$, there is a unique continuous $g: (Y, y) \rightarrow (\tilde{X}, \tilde{x})$ with $p \circ g = f$.

Our view of the above concept is that of being universal in a certain class of maps and we propose a different way of defining covering maps between Peano spaces in Section 6.

Our first observation is that each path-connected space $X$ has its universal Peano space $P(X)$, the set $X$ equipped with new topology, such that the identity function $P(X) \rightarrow X$ corresponds to a generalized regular covering for $H = \pi_1(X)$. That way quite a few results in the literature can be formally deduced from earlier results for Peano spaces.

The way the projection $P(X) \rightarrow X$ is characterized in 2.2 generalizes to the concept of **Peano maps** in Section 6 and our **Peano covering maps** combine Peano maps with two classical concepts: Serre fibrations and unique path lifting property. Peano covering maps possess several properties analogous to the classical covering maps [18] (example: local Peano covering maps are Peano covering maps). One of them is that they are all quotients $\tilde{X}_H$ of the universal path space $\tilde{X}$ equipped with the topology defined in the proof of Theorem 13 on p.82 in [23] and used successfully by Bogley-Sieradski [2] and Fischer-Zastrow [15]. It turns out the endpoint projection $\tilde{X}_H \rightarrow X$ is a Peano covering map if and only if it has the uniqueness of path lifts property (see 6.4). In an effort to unify Peano covering maps with uniform covering maps of [1] and [3] (we will explain the connection in [4]) we were led to a new topology on $\tilde{X}_H$ (see Section 3). Its main advantage is that there is a necessary and sufficient condition for $\tilde{X}_H \rightarrow X$ to have the unique path lifting property in case $H$ is a normal subgroup of $\pi_1(X)$. It is $H$ being closed in $\pi_1(X)$. That explains Theorem 6.9 of [15] as the basic groups there turn out to be closed in $\pi_1(X)$. As an application of our approach we show existence of a universal Peano covering map over a given path-connected space.

We thank Sasha Dranishnikov for bringing the work of Fischer-Zastrow [15] to our attention. We thank Greg Conner, Katsuya Eda, Aleš Vavpetič, and Ziga Virk for helpful comments.
2. Constructing Peano spaces

The purpose of this section is to discuss various ways of constructing new Peano spaces.

2.1. Universal Peano space. In analogy to the universal covering spaces we introduce the following notion:

Definition 2.1. Given a topological space $X$ its universal lpc-space $\text{lpc}(X)$ is an lpc-space together with a continuous map (called the universal Peano map) $\pi: \text{lpc}(X) \to X$ satisfying the following universality condition:

For any map $f: Y \to X$ from an lpc-space $Y$ there is a unique continuous lift $g: Y \to \text{lpc}(X)$ of $f$ (that means $\pi \circ g = f$).

Theorem 2.2. Every space $X$ has a universal lpc-space. It is homeomorphic to the set $X$ equipped with a new topology, the one generated by all path-components of all open subsets of the existing topology of $X$.

Proof. Let $U$ be an open set in $X$ containing the point $x$ and $c(x, U)$ be the path component of $x$ in $U$. Since $z \in c(x, U) \cap c(y, V)$ implies $c(z, U \cap V) \subset c(x, U) \cap c(y, V)$, the family $\{c(x, U)\}$, where $U$ ranges over all open subsets of $X$ and $x$ ranges over all elements of $U$, forms a basis.

Given a map $f: Y \to X$ and given an open set $U$ of $X$ containing $f(y)$ one has $f(c(y, f^{-1}(U))) \subset c(f(y), U)$. That proves $f: Y \to \text{lpc}(X)$ is continuous if $Y$ is an lpc-space. It also proves $\text{lpc}(X)$ is locally path-connected as any path in $X$ induces a path in $\text{lpc}(X)$. \qed

Remark 2.3. The topology above was mentioned in Remark 4.17 of [15]. After the first version of this paper was written we were informed by Greg Conner of his unpublished preprint [7] with David Fearnley, where that topology is discussed and its properties (compactness, metrizability) are investigated.

If $X$ is path-connected, then $\text{lpc}(X)$ is a universal Peano space $P(X)$ in the following sense: given a map $f: Z \to X$ from a Peano space $Z$ to $X$ there is a unique lift $g: Z \to P(X)$ of $f$.

In the remainder of this section we give sufficient conditions for a function on an lpc-space to be continuous. Those conditions are in terms of maps from basic Peano spaces: the arc in the first-countable case and hedgehogs (see Definition 2.8) in the arbitrary case.

Proposition 2.4. Suppose $f: Y \to X$ is a function from a first-countable lpc-space $Y$. $f$ is continuous if $f \circ g$ is continuous for every path $g: I \to Y$ in $Y$.

Proof. Suppose $U$ is open in $X$. It suffices to show that for each $y \in f^{-1}(U)$ there is an open set $V$ in $Y$ containing $y$ such that the path component of $y$ in $V$ is contained in $f^{-1}(U)$. Pick a basis of neighborhoods $\{V_n\}_{n \geq 1}$ of $y$ in $Y$ and assume for each $n \geq 1$ there is a path $\alpha_n$ in $V_n$ joining $y$ to a point $y_n \notin f^{-1}(U)$. Those paths can be spliced to one path $\alpha$ from $y$ to $y_1$ and going through all points $y_n$, $n \geq 2$. $f \circ \alpha$ starts from $f(y)$ and goes through all points $f(y_n)$, $n \geq 1$. However, as $U$ is open, it must contain almost all of them, a contradiction. \qed

The construction of the topology on $\text{lpc}(X)$ in 2.2 can be done in the spirit of the finest topology on $X$ that retains the same continuous maps from a class of spaces.
Proposition 2.5. Suppose $X$ is a path-connected topological space and $\mathcal{P}$ is a class of Peano spaces. The family $T$ of subsets of $X$ such that $f^{-1}(U)$ is open in $Z \in \mathcal{P}$ for any map $f: Z \to X$ in the original topology, is a topology and $\mathcal{P}(X):=(X,T)$ is a Peano space.

Proof. Since $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, $T$ is a topology on $X$. Suppose $U \in T$ and $C$ is a path component of $U$ in the new-topology. Suppose $f: Z \to X$ is a map and $f(z_0) \in C$. As $f^{-1}(U)$ is open, there is a connected neighborhood $V$ of $z_0$ in $Z$ satisfying $f(V) \subset U$. As $f(V)$ is path-connected, $f(V) \subset C$ and $C \in T$.

In case of first-countable spaces $X$ we have a very simple characterization of the universal Peano map of $X$:

Corollary 2.6. If $X$ is a first-countable path-connected topological space, then a map $f: Y \to X$ is a universal Peano map if and only if $Y$ is a Peano space, $f$ is a bijection, and $f$ has the path lifting property.

Proof. Consider $\mathcal{A}(X)$ as in 2.4, where $\mathcal{A}$ consists of the unit interval. Notice the identity function $P(X) \to \mathcal{A}(X)$ is continuous as $P(X)$ is first-countable (use 2.4). Since the topology on $\mathcal{A}(X)$ is finer than that on $P(X)$, $P(X) = \mathcal{A}(X)$. Since $f$ induces a homeomorphism from $\mathcal{A}(Y)$ to $\mathcal{A}(X)$ (due to the uniqueness of path lifting property of $f$), the composition $\mathcal{A}(Y) \to \mathcal{A}(X) \to P(X)$ is a homeomorphism and $f: Y \to P(X)$ must be a homeomorphism (its inverse is $P(X) \to \mathcal{A}(Y) \to Y$).

The construction in 2.5 can be used to create counter-examples to 2.6 in case $X$ is not first-countable.

Example 2.7. Let $X$ be the cone over an uncountable discrete set $B$. Subsets of $X$ that miss the vertex $v$ are declared open if and only if they are open in the CW topology on $X$. A subset $U$ of $X$ that contains $v$ is declared open if and only if $U$ contains all but countably many edges of the cone and $U \setminus \{v\}$ is open in the CW topology on $X$ (that means $X$ is a hedgehog if $B$ is of cardinality $\omega_1$ - see 2.8). Notice $\mathcal{A}(X)$ is $X$ equipped with the CW topology, the identity function $\mathcal{A}(X) \to X$ has the path lifting property but is not a homeomorphism.

Proof. Notice every subset of $X \setminus \{v\}$ that meets each edge in at most one point is discrete. Hence a path in $X$ has to be contained in the union of finitely many edges. That means $\mathcal{A}(X)$ is $X$ with the CW topology.

We generalize 2.7 as follows:

Definition 2.8. A generalized Hawaiian Earring is the wedge 
\[ (Z, z_0) = \bigvee_{s \in S} (Z_s, z_s) \] 
of pointed Peano spaces indexed by a directed set $S$ and equipped with the following topology (all wedges in this paper are considered with that particular topology):

1. $U \subset Z \setminus \{z_0\}$ is open if and only if $U \cap Z_s$ is open for each $s \in S$.
2. $U$ is an open neighborhood of $z_0$ if and only if there is $t \in S$ such that $Z_t \subset U$ for all $s > t$ and $U \cap Z_s$ is open for each $s \in S$.

A hedgehog is a generalized Hawaiian Earring $(Z, z_0) = \bigvee_{s \in S} (Z_s, z_s)$ such that each $(Z_s, z_s)$ is homeomorphic to $(I, 0)$. 
Our definition of generalized Hawaiian Earrings is different from the definition of Cannon and Conner [5]. Also, the preferred terminology in [5] is that of a **big Hawaiian Earring**.

Observe each generalized Hawaiian Earring is a Peano space.

**Lemma 2.9.** Let $S$ be a basis of neighborhoods of $x_0$ in $X$ ordered by inclusion (i.e., $U \subseteq V$ means $V \cap U \subseteq U$). If, for each $U \in S$, $\alpha_U : I \rightarrow U$ is a path in $U$ starting from $x_0$, then their wedge

$$
\bigvee_{U \in S} \alpha_U : \bigvee_{U \in S} (I_U, 0_U) \rightarrow (X, x_0)
$$

is continuous, where $(I_U, 0_U) = (I, 0)$ for each $U \in S$.

**Proof.** Only the continuity of $g = \bigvee_{U \in S} \alpha_U$ at the base-point of the hedgehog wedge $(I_U, 0_U)$ is not totally obvious. However, if $V$ is a neighborhood of $x_0$ in $X$, then $g^{-1}(V)$ contains all $I_U$ if $U \subseteq V$ and $g^{-1}(V) \cap I_W$ is open in $I_W$ for all $W \in S$.

**Proposition 2.10.** Suppose $f : Y \rightarrow X$ is a function from an lpc-space $Y$. $f$ is continuous if $f \circ g$ is continuous for every map $g : Z \rightarrow Y$ from a hedgehog $Z$ to $Y$.

**Proof.** Assume $U$ is open in $X$ and $x_0 = f(y_0) \in U$. Suppose for each path-connected neighborhood $V$ of $y_0$ in $Y$ there is a path $\alpha_V : (I, 0) \rightarrow (V, y_0)$ such that $\alpha_V(1) \notin f^{-1}(U)$. By Lemma 2.9 the wedge $g = \bigvee_{V \in S} \alpha_V$ is a map $g$ from a hedgehog to $Y$ (here $S$ is the family of all path-connected neighborhoods of $y_0$ in $Y$). Hence $h = f \circ g$ is continuous and there is $V \in S$ so that $I_V \subseteq h^{-1}(U)$. That means $f(\alpha_V(I)) \subseteq U$, a contradiction.

**2.2. Basic topology on $\tilde{X}$.** The philosophical meaning of this section is that many results can be reduced to those dealing with Peano spaces via the universal Peano space construction. Let us illustrate this point of view by discussing a topology on $\tilde{X}$.

Suppose $(X, x_0)$ is a pointed topological space. Consider the space $\tilde{X}$ of homotopy classes of paths in $X$ originating at $x_0$. It has an interesting topology (see the proof of Theorem 13 on p.82 in [23]) that has been put to use in [2] and [15]. Its basis consists of sets $B([\alpha], U)$ ($U$ is open in $X$, $\alpha$ joins $x_0$ and $\alpha(1) \in U$) defined as follows: $[\beta] \in B([\alpha], U)$ if and only if there is a path $\gamma$ in $X$ from $\alpha(1)$ to $\beta(1)$ such that $\beta$ is homotopic rel. endpoints to the concatenation $\alpha * \gamma$.

$\tilde{X}$ equipped with the above topology will be denoted by $\tilde{X}$ as in [2].

Both [2] and [15] consider quotient spaces $\tilde{X}/H$, where $H$ is a subgroup of $\pi_1(X, x_0)$. We find it more convenient to follow [23] pp.82-3:

**Definition 2.11.** Suppose $H$ is a subgroup of $\pi_1(X, x_0)$. Define $\tilde{X}_H$ as the set of equivalence classes of paths in $X$ under the relation $\alpha \sim_\tilde{H} \beta$ defined via $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1)$ and $[\alpha * \beta^{-1}] \in H$ (the equivalence class of $\alpha$ under the relation $\sim_\tilde{H}$ will be denoted by $[\alpha]_H$).

To introduce a topology on $\tilde{X}_H$ we define sets $B_H([\alpha]_H, U)$ (denoted by $< \alpha, U >$ on p.82 in [23]), where $U$ is open in $X$, $\alpha$ joins $x_0$ and $\alpha(1) \in U$, as follows:
[β]_H ∈ B_H([α]_H, U) if and only if there is a path γ in U from α(1) to β(1) such that [β + (α ∗ γ)−1] ∈ H (equivalently, β ∼_H α ∗ γ).

\(X_H\) equipped with the topology (which we call the basic topology on \(X_H\)) whose basis consists of \(B_H([α]_H, U)\), where U is open in \(X\), α joins \(x_0\) and \(α(1) \in U\), is denoted by \(X_H\) in analogy to the notation \(\hat{X}\) in [2] that corresponds to \(H\) being trivial.

Given a path α in X and a path β in X from \(x_0\) to \(α(0)\) one can define a standard lift \(\hat{α}\) of it to \(\hat{X}_H\) originating at \([β]_H\) by the formula \(\hat{α}(t) = [β + α_t]_H\), where \(α_t(s) = α(s \cdot t)\) for \(s, t \in I\) [see 10 Proposition 6.6.3].

Let us extract the essence of the proof of [23] Theorem 13 on pp.82–83:

**Lemma 2.12.** Suppose X is a path-connected space and H is a subgroup of \(π_1(X, x_0)\). An open set \(U \subset X\) is evenly covered by \(p_H: \hat{X}_H \to X\) if and only if U is locally path-connected and the image of \(h_α: π_1(U, x_1) \to π_1(X, x_0)\) is contained in H for any path α in X from \(x_0\) to any \(x_1 \in U\).

**Proof.** Recall that U is evenly covered by \(p_H\) (see [23] p.62) if \(p_H^{-1}(U)\) is the disjoint union of open subsets \(\{U_s\}_{s \in S}\) of \(\hat{X}_H\) each of which is mapped homeomorphically onto U by \(p_H\). Also, recall \(h_α: π_1(U, x_1) \to π_1(X, x_0)\) is given by \(h_α(γ) = [α ∗ γ ∗ α_t]_H\).

Suppose U is evenly covered, γ is a loop in \((U, x_1)\), and α is a path from \(x_0\) to \(x_1\). If \([α]_H ≠ [α ∗ γ]_H\), then they belong to two different sets \(U_α\) and \(U_v, u, v \in S\). However, there is a path from \([α]_H\) to \([α ∗ γ]_H\) in \(p_H^{-1}(U)\) given by the standard lift of γ, a contradiction. Thus \([α]_H = [α ∗ γ]_H\) and \([α ∗ γ ∗ α_t]_H \in H\).

To show that U is locally path-connected, take a point \(x_1 \in U\), pick a path α from \(x_0\) to \(x_1\) and select the unique \(s \in S\) so that \([α]_H \in U_s\). There is an open subset V of U satisfying \(B_H([α]_H, V) \subset U_s\). As \(p_H(\{U_s\})\) maps \(U_s\) homeomorphically onto U, \(p_H(B_H([α]_H, V))\) is an open neighborhood of \(x_1\) in U and it is path-connected.

Suppose U is locally path-connected and the image of \(h_α: π_1(U, x_1) \to π_1(X, x_0)\) is contained in H for any path α in X from \(x_0\) to any \(x_1 \in U\). Pick a path component V of U and notice sets \(B_H([β]_H, V)\), \(β\) ranging over paths from \(x_0\) to points of V, are either identical or disjoint. Observe \(p_H(B_H([β]_H, V))\) maps \(B_H([β]_H, V)\) homeomorphically onto V. Thus each V is evenly covered and that is sufficient to conclude U is evenly covered.

As in [23] p.81, given an open cover \(U\) of X, π(\(U, x_0)\) is the subgroup of \(π_1(X, x_0)\) generated by elements of the form \([α ∗ γ ∗ α_t]_H\), where γ is a loop in some \(U \in U\) and α is a path from \(x_0\) to \(γ(0)\).

Here is our improvement of [23] Theorem 13 on p.82 and [15] Theorem 6.1:

**Theorem 2.13.** If X is a path-connected space and H is a subgroup of \(π_1(X, x_0)\), then the endpoint projection \(p_H: \hat{X}_H \to X\) is a classical covering map if and only if X is a Peano space and there is an open covering \(U\) of X so that \(π(\{U\}, x_0) \subset H\).

**Proof.** Apply 2.12.

**Proposition 2.14.** \(\hat{P}(X)_H\) is naturally homeomorphic to \(\hat{X}_H\) if X is path-connected.

**Proof.** Since continuity of \(f: (Z, z_0) \to (P(X), x_0)\), for any Peano space Z, is equivalent to the continuity of \(f: (Z, z_0) \to (X, x_0)\), paths in \(P(X, x_0)\) correspond to paths in \((X, x_0)\). Also, \(π_1(P(X), x_0) \to π_1(X, x_0)\) is an isomorphism so H is a subgroup of both \(π_1(P(X), x_0)\) and \(π_1(X, x_0)\), and the equivalence classes of
relations $\sim_H$ are identical in both spaces $\tilde{P}(X)$ and $\tilde{X}$. Notice that basis open sets are identical in $\tilde{P}(X)_H$ and $\tilde{X}_H$. □

**Remark 2.15.** In view of 2.14 some results in [15] dealing with maps $f: Y \to X$, where $Y$ is Peano, can be derived formally from corresponding results for $f: Y \to P(X)$. A good example is Lemma 2.8 in [15].

$p: \tilde{X} \to X$ has the unique path lifting property if and only if $\tilde{X}$ is simply connected.

It follows formally from Corollary 4.7 in [2]:

The universal endpoint projection $p: \tilde{Z} \to Z$ for a connected and locally path-connected space $Z$ has the unique path lifting property if and only if $\tilde{Z}$ is simply connected.

When working in the pointed topological category the space $\tilde{X}_H$ is equipped with the base-point $\tilde{x}_0$ equal to the equivalence class of the constant path at $x_0$.

Let us illustrate $\tilde{X}_H$ in the case of $H = \pi_1(X, x_0)$.

**Proposition 2.16.** If $H = \pi_1(X, x_0)$, then

a. The endpoint projection $p_H: (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ is an injection and $p_H(B(\alpha|_H, U))$ is the path component of $\alpha(1)$ in $U$,

b. $\tilde{X}_H$ is a Peano space,

c. Given a map $g: (Z, z_0) \to (X, x_0)$ from a pointed Peano space to $(X, x_0)$, there is a unique lift $h: (Z, z_0) \to (\tilde{X}_H, \tilde{x}_0)$ of $g \circ h = g$.

**Proof.** a). Clearly, $p_H(B_H([\alpha]_H, U))$ equals path component of $\alpha(1)$ in $U$. If $[\beta_1]_H$ and $[\beta_2]_H$ map to the same point $x_1$, then $\beta_1(1) = \beta_2(1)$ and $\gamma = \beta_1 \beta_2^{-1}$ is a loop. Hence $[\gamma] \in H$ and $[\beta_2]_H = [\gamma \beta_2]_H = [\beta_1]_H$ proving $p_H$ is an injection.

b) is well-established in both [2] and [15]. Notice it follows from a).

c). For each $z \in Z$ pick a path $\alpha_z$ from $z_0$ to $z$ in $Z$. Define $h(z)$ as $[\alpha_z]_H$ and notice $h$ is continuous as $h^{-1}(B_H([\alpha_z]_H, U))$ equals the path component of $g^{-1}(U)$ containing $z$ (use Part a)). As $p_H$ is injective, there is at most one lift of $g$. □

In view of [2.16] we have a convenient definition of a universal Peano space in the pointed category:

**Definition 2.17.** By the universal Peano space $P(X, x_0)$ of $(X, x_0)$ we mean the pointed space $(\tilde{X}_H, \tilde{x}_0)$, $H = \pi_1(X, x_0)$, and the universal Peano map of $(X, x_0)$ is the endpoint projection $P(X, x_0) \to (X, x_0)$. Equivalently, $P(X, x_0)$ is $(P(C), x_0)$, where $C$ is the path component of $x_0$ in $X$.

Due to standard lifts the endpoint projection $p_H: \tilde{X}_H \to (X, x_0)$ always has the path lifting property. Thus the issue of interest is the uniqueness of path lifting property of $p_H$.

Here is a necessary and sufficient condition for $p_H$ to have the unique path lifting property (compare it to [2] Theorem 4.5) for Peano spaces):

**Proposition 2.18.** If $X$ is a path-connected space and $x_0 \in X$, then the following conditions are equivalent:

a. $p_H: (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ has the unique path lifting property,

b. The image of $\pi_1(p_H): \pi_1(\tilde{X}_H, \tilde{x}_0) \to \pi_1(X, x_0)$ is contained in $H$. 
Proposition 3.2.  If $X$ is a path-connected space and $H$ is a subgroup of $\pi_1(X, x_0)$, then the following conditions are equivalent:

a) A fiber of the endpoint projection $p_H : \tilde{X}_H \to X$ has an isolated point,

b) The endpoint projection $p_H : \tilde{X}_H \to X$ has discrete fibers,

c) There is an open covering $\mathcal{U}$ of $X$ so that $\pi(\mathcal{U}, x_0) \subset H$,

d) $\tilde{X}_H$ is a Peano space and $p_H : \tilde{X}_H \to P(X)$ is a classical covering map.
Proof. a) \(\implies\) c). Suppose \([\alpha]_H \in p_H^{-1}(x_1)\) is isolated. There is an open covering \(\mathcal{U}\) of \(X\) and \(V \in \mathcal{U}\) containing \(x_1\) such that \(B_H([\alpha]_H, \mathcal{U}, V) \cap p_H^{-1}(x_1) = \{[\alpha]_H\}\). Given \(\gamma \in \pi(\mathcal{U}, x_0)\), the homotopy class \([\alpha^{-1} \cdot \gamma \cdot \alpha]_H\) belongs to \(\pi(\mathcal{U}, x_1)\), so \([\alpha \cdot \alpha^{-1} \cdot \gamma \cdot \alpha]_H = [\gamma \cdot \alpha]_H\) belongs to \(B_H([\alpha]_H, \mathcal{U}, V) \cap p_H^{-1}(x_1)\). Hence \([\gamma \cdot \alpha]_H = [\alpha]_H\) and \([\gamma] \in H\).

c) \(\implies\) d). Suppose there is an open covering \(\mathcal{U}\) of \(X\) so that \(\pi(\mathcal{U}, x_0) \subseteq H\) and \(W\) is a path component of \(U \in \mathcal{U}\). Notice \(B_H([\alpha]_H, \mathcal{U}, U)\) is mapped by \(p_H\) bijectively onto \(W\) and that is sufficient for d).

d) \(\implies\) b) and b) \(\implies\) a) are obvious.

Applying 3.2 to \(H\) being trivial one gets the following (see [12] for analogous result in case of a different topology on the fundamental group):

**Corollary 3.3.** If \(X\) is a path-connected space, then \(\pi_1(X, x_0)\) is discrete if and only if \(X\) is semilocally simply connected.

**Proposition 3.4.** If \(\pi(\mathcal{V}, x_0) \subseteq H\) for some open cover \(\mathcal{V}\) of \(X\), then the identity function \(\tilde{X}_H \to \tilde{X}_H\) is a homeomorphism.

**Proof.** Let us show \(B_H([\alpha]_H, \mathcal{U}, W) = B_H([\alpha]_H, W)\) if \(\mathcal{U}\) is an open cover of \(X\) refining \(\mathcal{V}\) and \(W\) is an element of \(\mathcal{U}\) containing \(\alpha(1)\). Clearly, \(B_H([\alpha]_H, W) \subseteq B_H([\alpha]_H, \mathcal{U}, W)\), so assume \([\beta]_H \in B_H([\alpha]_H, \mathcal{U}, W)\). There are \(h \in H, \lambda \in \pi(\mathcal{U}, \alpha(1))\), and a path \(\gamma\) in \(W\) such that \([\beta] = [h * \alpha * \lambda * \gamma]\). Choose \(h_1 \in H\) so that \([h_1 * \alpha] = [\alpha * \lambda]\), \([h_1 = [\alpha * \lambda * \alpha^{-1}] \in \pi(\mathcal{U}, x_0) \subseteq H]\). Now \([\beta] = [h * \alpha * \lambda * \gamma] = [h * h_1 * \alpha * \gamma] \in B_H([\alpha]_H, W)\).

Now we can show the identity function \(\tilde{X}_H \to \tilde{X}_H\) is open: given an open cover \(\mathcal{W}\) of \(X\) and given a path \(\alpha\) from \(x_0\) to \(x_1\) pick an element \(W\) of \(\mathcal{U} = \mathcal{W} \cap \mathcal{V}\) containing \(x_1\) and notice \(B_H([\alpha]_H, \mathcal{U}, W) \subseteq B_H([\alpha]_H, W)\). \(\square\)

**Lemma 3.5.** If \(G \subseteq H\) are subgroups of \(\pi_1(X, x_0)\), then the projection \(p: \tilde{X}_G \to \tilde{X}_H\) is open.

**Proof.** It suffices to show \(p(B_G([\alpha]_G, \mathcal{U}, V)) = B_H([\alpha]_H, \mathcal{U}, V)\). Clearly, \(p(B_G([\alpha]_G, \mathcal{U}, V)) \subseteq B_H([\alpha]_H, \mathcal{U}, V)\), so suppose \([\beta]_H \in B_H([\alpha]_H, \mathcal{U}, V)\) and \([\beta] = [h * \alpha * \lambda * \gamma]\), where \([\lambda] \in \pi(\mathcal{U}, \alpha(1))\) and \(\gamma\) is a path in \(V\) originating at \(\beta(1)\). Observe \([\beta]_H = [\alpha * \lambda * \gamma]_H = p([\alpha * \lambda * \gamma]_G)\). \(\square\)

We arrived at the fundamental result for the new topology on \(\tilde{X}_H:\)

**Theorem 3.6.** Suppose \(G \subseteq H\) are subgroups of \(\pi_1(X, x_0)\). If \(G\) is normal in \(\pi_1(X, x_0)\), then \(H/G\), identified with the fiber \(p^{-1}(\tilde{x}_0 H)\) of the projection \(p: \tilde{X}_G \to \tilde{X}_H\), is a topological group and acts continuously on \(\tilde{X}_G\) so that

a) The natural map \((H/G) \times \tilde{X}_G \to \tilde{X}_G \times \tilde{X}_G\) defined by \(([\alpha]_G, [\beta]_G) \mapsto ([\alpha * \beta]_G, [\beta]_G)\) is an embedding.

b) The quotient map from \(\tilde{X}_G\) to the orbit space corresponds to the projection \(p: \tilde{X}_G \to \tilde{X}_H\).

**Proof.** The fiber \(F\) of the projection \(p: \tilde{X}_G \to \tilde{X}_H\) is the set of classes \([\alpha]_G\) such that \([\alpha] \in H\), so it corresponds to \(H/G\). Define \(\mu: F \times \tilde{X}_G \to \tilde{X}_G\) as follows: given \([\alpha]_G \in F\) and given \([\beta]_G \in \tilde{X}_G\) put \(\mu([\alpha]_G, [\beta]_G) = [\alpha * \beta]_G\). To see \(\mu\) is well defined assume \([\gamma_1], [\gamma_2] \in G\). Now \([\gamma_1 * \alpha * \gamma_2 * \beta]_G([\alpha * \gamma_2 * \alpha^{-1}] * (\alpha * \beta)]_G = [\alpha * \beta]_G\) as \([\alpha * \gamma_2 * \alpha^{-1}] \in G\) due to normality of \(G\) in \(H\).

Suppose \(\mathcal{U}\) is an open cover of \(X, V, V_1 \in \mathcal{U}\), and
(1) \([\alpha]_G \in F, [\beta]_G \in \tilde{X}_G,\)
(2) \([\alpha]_G \in B_G([\alpha]_G, U, V_1) \cap F, \) and \([\beta]_G \in B_G([\beta]_G, U, V).\)

Thus \([\alpha]_1 = [g_1 \ast \alpha \ast \lambda_1]\) for some \([\lambda_1] \in \pi(U, x_0)\) and \([g_1] \in G.\) Similarly, \([\beta]_1 = [g_2 \ast \beta \ast \lambda_2 \ast \gamma],\) where \([g_2] \in G, [\lambda_2] \in \pi(U, \beta(1)),\) and \(\gamma\) is a path in \(V.\) Now,
\[
[\alpha_1^{-1} \ast \beta_1]_G = [\lambda_1^{-1} \ast \alpha^{-1} \ast g_1 \ast \beta \ast \lambda_2 \ast \gamma]_G =
\]
\[
[\lambda_1^{-1} \ast \alpha^{-1} \ast g_1 \ast \alpha \ast \lambda_1] \ast \lambda_1^{-1} \ast \alpha^{-1} \ast \beta \ast \lambda_2 \ast \gamma]_G =
\]
\[
[\lambda_1^{-1} \ast \alpha^{-1} \ast g_1^{-1} \ast g_2 \ast \alpha \ast \lambda_1] \ast \lambda_1^{-1} \ast \alpha^{-1} \ast \beta \ast \lambda_2 \ast \gamma]_G =
\]
\[
[\lambda_1^{-1} \ast \alpha^{-1} \ast \beta \ast \lambda_2 \ast \gamma]_G = [(\alpha^{-1} \ast \beta) \ast \beta^{-1} \ast \alpha \ast \lambda_1^{-1} \ast \alpha^{-1} \ast \beta]_G \in B_G([\alpha^{-1} \ast \beta]_G, U, V)
\]
as \([\lambda_1^{-1} \ast \alpha^{-1} \ast g_1^{-1} \ast g_2 \ast \alpha \ast \lambda_1] \in G\) and \([\beta^{-1} \ast \alpha \ast \lambda_1^{-1} \ast \alpha^{-1} \ast \beta] \in \pi(U, (\alpha^{-1} \ast \beta)(1)).\)

The above calculations amount to
\[\rho((F \cap B_G(x, U, V_1)) \times B_G(y, U, V)) \subset B_G(\rho(x, y), U, V),\]
where \(\rho(x, y) := \mu(x^{-1}, y),\) which implies the following
(1) \(F\) is a topological group,
(2) \(\mu\) is continuous,
(3) \((x, y) \rightarrow (\mu(x^{-1}, y), y)\) from \(F \times \tilde{X}_G\) onto its image is open.

As the map in (3) is injective, it is an embedding. Hence \((x, y) \rightarrow (\mu(x, y), y)\) is an embedding.

To see b) use 3.3 or check it directly. \(\square\)

4. Path Lifting

**Definition 4.1.** A pointed map \(f: (X, x_0) \rightarrow (Y, y_0)\) has the **path lifting property** if any path \(\alpha: (I, 0) \rightarrow (Y, y_0)\) has a lift \(\beta: (I, 0) \rightarrow (X, x_0)\).

A surjective map \(f: X \rightarrow Y\) has the **path lifting property** if for any path \(\alpha: I \rightarrow Y\) and any \(y_0 \in f^{-1}(\alpha(0))\) there is a lift \(\beta: I \rightarrow X\) of \(\alpha\) such that \(\beta(0) = y_0.\)

**Definition 4.2.** A pointed map \(f: (X, x_0) \rightarrow (Y, y_0)\) has the **uniqueness of path lifts property** if any two paths \(\alpha, \beta: (I, 0) \rightarrow (X, x_0)\) are equal if \(f \circ \alpha = f \circ \beta.\)

A pointed map \(f: (X, x_0) \rightarrow (Y, y_0)\) has the **unique path lifting property** if it has both the path lifting property and the uniqueness of path lifts property.

A map \(f: X \rightarrow Y\) has the **uniqueness of path lifts property** if any two paths \(\alpha, \beta: I \rightarrow X\) are equal if \(f \circ \alpha = f \circ \beta\) and \(\alpha(0) = \beta(0).\)

A surjective map \(f: X \rightarrow Y\) has the **unique path lifting property** if it has both the path lifting property and the uniqueness of path lifts property.

**Corollary 4.3.** Suppose \(G \subset H\) are subgroups of \(\pi_1(X, x_0).\) If \(G\) is normal in \(\pi_1(X, x_0),\) then the following conditions are equivalent:

a) The natural map \(\tilde{X}_G \rightarrow \tilde{X}_H\) has the uniqueness of path lifts property,

b) \(\pi_0(H/G) = H/G, i.e. H/G has trivial path components.\)

**Proof.** a) \(\implies\) b). If \(H/G\) has a non-trivial path component, then there is a non-trivial lift of the constant path at the base-point of \(\tilde{X}_H.\)

b) \(\implies\) a). Suppose \(\alpha\) and \(\beta\) are two lifts of the same path \(\gamma\) in \(\tilde{X}_H\) and \(\alpha(0) = \beta(0).\) By 3.3 there is a path \(\lambda\) in \(H/G\) with the property \(\lambda(t) \cdot \alpha(t) = \beta(t)\) for each \(t \in I.\) As \(\lambda(0) = 1 \in H/G\) and \(H/G\) has trivial path components, \(\lambda(t) = 1 \in H/G\) for all \(t \in I\) and \(\alpha = \beta.\) \(\square\)
Proposition 4.4. Suppose $G \subseteq H$ are subgroups of $\pi_1(X, x_0)$. If $G$ is normal in $\pi_1(X, x_0)$, then the following conditions are equivalent:

- $H/G$ is a $T_0$-space,
- $H/G$ is Hausdorff,
- Fibers of the projection $p: \tilde{X}_G \to \tilde{X}_H$ are $T_0$,
- Fibers of the projection $p: \tilde{X}_G \to \tilde{X}_H$ are Hausdorff,
- For each $h \in H - G$ there is a cover $U$ such that $(G \cdot h) \cap \pi(U, x_0) = \emptyset$,
- $G$ is closed in $H$.

Proof. In view of 3.6 (a)⇒(c) and (b)⇒(d).

(a)⇒(e). Assume $H/G$ is $T_0$ and $h \in H - G$. Since $[\beta]_G \in s_B([\alpha]_G, U, V)$ is equivalent to $[\alpha]_G \in B([\beta]_G, U, V)$, there is an open cover $U$ and $V \in U$ containing $x_0$ such that $G \cdot h \notin B(G \cdot 1, U, V)$. That means precisely there is no $\lambda \in \pi(U, x_0)$ such that $G \cdot h = G \cdot \lambda$, hence $(G \cdot h) \cap \pi(U, x_0) = \emptyset$.

(b)⇒(d) and (a)⇒(c) follow from 3.6.

e)⇒(d). Suppose $\alpha, \beta$ are two paths in $(X, x_0)$ so that $[\alpha]_H = [\beta]_H$ but $[\alpha]_G \neq [\beta]_G$. Choose $h \in H - G$ satisfying $[h \cdot \alpha] = [\beta]$. Pick an open cover $U$ of $X$ satisfying $G \cdot h \cap \pi(U, x_0) = \emptyset$ and let $V \in U$ contain $\alpha(1)$. Suppose $[\gamma]_G \in B([\alpha]_G, U, V) \cap B([\beta]_G, U, V)$ and $[\gamma]_H = [\alpha]_H$. Let $h_0 \in H$ satisfy $[h_0 \cdot \alpha] = [\gamma]$. Choose $\lambda_1, \lambda_2 \in \pi(U, \alpha(1))$ such that $G \cdot [h_0 \cdot \alpha] = G \cdot \alpha \cdot \lambda_1$ and $G \cdot [h_0 \cdot \alpha] = G \cdot [h \cdot \alpha] \cdot \lambda_2$. As $G$ is normal in $H$, $G \cdot h = G \cdot G \cdot (\alpha \cdot \lambda_1 \cdot \lambda_2^{-1} \cdot \alpha^{-1})$, a contradiction as $\alpha \cdot \lambda_1 \cdot \lambda_2^{-1} \cdot \alpha^{-1} \in \pi(U, x_0)$.

Corollary 4.5. Suppose $G \subseteq H$ are subgroups of $\pi_1(X, x_0)$. If $G$ is a normal subgroup of $\pi_1(X, x_0)$, then the following conditions are equivalent:

- $H/G$ has trivial components,
- $H/G$ has trivial path components,
- $G$ is closed in $H$.

Proof. b)⇒c). Suppose $H/G$ has trivial path components. In view of 4.4 it suffices to show $H/G$ is $T_0$ to deduce $G$ is closed in $H$. If there are two points $u$ and $v$ of $H/G$ such that any open subset of $H/G$ either contains both of them or contains none of them, then any function $I \to \{u, v\} \subseteq H/G$ is continuous. Hence $u = v$ as $H/G$ does not contain non-trivial paths.

c)⇒a).

Claim. If $h_1, h_2 \in H$ and $G \cdot f \in B_H(G \cdot h_1, U, V) \cap B_H(G \cdot h_2, U, V) \cap (H/G)$ for some open cover $U$ of $X$ and some $V \in U$ containing $x_0$, then $G \cdot h_1^{-1} \cdot h_2 \in G \pi(U, x_0)$.

Proof of Claim: $G \cdot f = G \cdot h_1 \cdot \lambda_1$ and $G \cdot f = G \cdot h_2 \cdot \lambda_2$ for some $\lambda_1, \lambda_2 \in \pi(U, x_0)$. Now $h_1 \cdot G = h_2 \cdot G \cdot (\lambda_1 \cdot \lambda_1^{-1})$ and $(h_1^{-1} \cdot h_2) \cdot G = G \cdot (\lambda_1 \cdot \lambda_2^{-1}) \subseteq G \pi(U, x_0)$. □

Suppose $G$ is closed in $H$ and $h \in H - G$. By 4.4 there is a cover $U$ such that $(G \cdot h) \cap \pi(U, x_0) = \emptyset$. If there is a connected subset $C$ of $H/G$ containing $G \cdot h_1$ and $G \cdot h_1$ for some $h_1 \in H$, we consider the open cover $\{C \cap B_G(z, U, V)\}_{z \in C}$ of $C$ and define the equivalence relation on $C$ determined by that cover ($z \sim z'$ if there is a finite chain $z = z_1, \ldots, z_k = z'$ in $C$ such that $B_G(z_i, U, V) \cap B_G(z_{i+1}, U, V) \cap C \neq \emptyset$ for all $i < k$). Equivalence classes of that relation are open, hence closed and
must equal $C$. Thus there is a finite chain $h_1, \ldots, h_k = h_1 \cdot h$ in $H$ such that $B_G([h_1], U, V) \cap B_G([h_1], U, V) \cap (H/G) \neq \emptyset$ for all $i < k$. By Claim there are elements $g_i \in G$ $(i < k)$ so that $g_i \cdot h_i \cdot h_{i+1} \in \pi(U, x)$. By normality of $G$ in $H$ there is $g \in G$ satisfying $g \cdot \prod_{i=1}^{k-1} h_i^{-1} \cdot h_{i+1} = g \cdot h \in \pi(U, x)$, a contradiction. □

**Theorem 4.6.** If $G$ is a normal subgroup of $\pi_1(X, x_0)$, then the following conditions are equivalent:

a. The endpoint projection $p_G: (\tilde{X}_G, \tilde{x}_0) \to (X, x_0)$ has the unique path lifting property,

b. $G$ is closed in $\pi_1(X, x_0)$,

c. $\pi_1(p_G): \pi_1(\tilde{X}_G, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism and its image equals $G$.

**Proof.** Put $H = \pi_1(X, x_0)$ and observe $\tilde{X}_H$ is the Peanianization of $(X, x_0)$ by Proposition 4.7. Suppose $(X, x_0)$ is a pointed topological space and $H$ is a subgroup of $\pi_1(X, x_0)$. The closure of $H$ in $\pi_1(X, x_0)$ consists of all elements $g \in \pi_1(X, x_0)$ such that for each open cover $U$ of $X$ there is $h \in H$ and $\lambda \in \pi(U, x_0)$ satisfying $g = h \cdot \lambda$. If $H$ is a normal subgroup of $\pi_1(X, x_0)$, then so is its closure.

**Proposition 4.7.** Suppose $(X, x_0)$ is a pointed topological space and $H$ is a subgroup of $\pi_1(X, x_0)$. The closure of $H$ in $\pi_1(X, x_0)$ consists of all elements $g \in \pi_1(X, x_0)$ such that for each open cover $U$ of $X$ there is $h \in H$ and $\lambda \in \pi(U, x_0)$ satisfying $g = h \cdot \lambda$. If $H$ is a normal subgroup of $\pi_1(X, x_0)$, then so is its closure.

**Proof.** Suppose $g \in \pi_1(X, x_0)$ and for each open cover $U$ of $X$ there is $h \in H$ and $\lambda \in \pi(U, x_0)$ satisfying $g = h \cdot \lambda$. Notice $B(\mathcal{U}, U)$ contains $h$, so $g$ belongs to the closure of $H$. If $H$ is normal, then $k \cdot g \cdot k^{-1} = (k \cdot h \cdot k^{-1}) \cdot (k \cdot \lambda \cdot k^{-1})$ also belongs to the closure of $H$.

**Corollary 4.8.** The closure of the trivial subgroup of $\pi_1(X, x_0)$ in $\pi_1(X, x_0)$ equals $
\bigcap_{\mathcal{U} \in \mathcal{COV}} \pi(\mathcal{U}, x_0), \text{ where } \mathcal{COV} \text{ stands for the family of all open covers of } X.$

**Example 4.9.** The Harmonic Archipelago $HA$ of Bogley and Sieradski [2] is a Peano space such that $\pi_1(X, x_0)$ equals $\bigcap_{\mathcal{U} \in \mathcal{COV}} \pi(\mathcal{U}, x_0)$. Hence $\pi_1(X, x_0)$ is the only closed subgroup of $\pi_1(X, x_0)$. $HA$ is built by stretching disks $B(2^{-n}, 2^{-n-2})$ to form cones over its boundary with the vertices at height 1 in the 3-space.

**Corollary 4.10.** Suppose $(X, x_0)$ is a pointed topological space. The following subgroups of $\pi_1(X, x_0)$ are closed:

a) Subgroups $H$ containing $\pi(\mathcal{U}, x_0)$ for some open cover $\mathcal{U}$ of $X$,

b) $\bigcap_{\mathcal{U} \in S} \pi(\mathcal{U}, x_0)$ for any family $S$ of open covers of $X$, where COV stands for the family of all open covers of X.
c) The kernel of \( \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) for any map \( f : (X, x_0) \to (Y, y_0) \) to a pointed semilocally simply connected space.

d) The kernel of the natural homomorphism \( \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0) \) from the fundamental group to the Čech fundamental group.

**Proof.** a) Any subgroup containing \( \pi(U, x_0) \) is open. Any open subgroup of a topological group is closed.

b) easily follows from a).

c) follows from \([3, 3]\) and \([5, 1]\) as \( \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) is continuous and \( \pi_1(Y, y_0) \) is discrete.

d) follows from c). Indeed \( \tilde{\pi}_1(X, x_0) \) is defined (see \([9]\) or \([19]\)) as the inverse limit of an inverse system \( \{ \pi_1(K_s, k_s) \}_{s \in S} \), where each \( K_s \) is a simplicial complex and there are maps \( f_s : (X, x_0) \to (K_s, k_s) \) so that for \( t > s \) the map \( f_s \) is homotopic to the composition of \( f_t \) and the bonding map \( (K_t, k_t) \to (K_s, k_s) \). That means the kernel of the natural homomorphism \( \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0) \) is the intersection of kernels of all \( \pi_1(f_s), s \in S \).

The concept of a space \( X \) being **homotopically Hausdorff** was introduced by Conner and Lamoreaux \([8, Definition 1.1]\) to mean that for any point \( x_0 \) in \( X \) and for any non-homotopically trivial loop \( \gamma \) at \( x_0 \) there is a neighborhood \( U \) of \( x_0 \) in \( X \) with the property that no loop in \( U \) is homotopic to \( \gamma \) rel. \( x_0 \) in \( X \). Subsequently, Fischer and Zastrow \([15]\) defined a space \( X \) to be **homotopically Hausdorff relative to a subgroup** \( H \) of \( \pi_1(X, x_0) \) if for any \( g \notin H \) and for any path \( \alpha \) originating at \( x_0 \) there is an open neighborhood \( U \) of \( \alpha(1) \) in \( X \) such that none of the elements of \( G \cdot g \) can be expressed as \( [\alpha * \gamma * \alpha^{-1}] \) for some loop \( \gamma \) in \((U, \alpha(1))\).

We generalize this definition as follows:

**Definition 4.11.** Suppose \( G \subset H \) are subgroups of \( \pi_1(X, x_0) \). \( X \) is \((H, G)\)-**homotopically Hausdorff** if for any \( h \in H \setminus G \) and any path \( \alpha \) originating at \( x_0 \) there is an open neighborhood \( U \) of \( \alpha(1) \) in \( X \) such that none of the elements of \( G \cdot h \) can be expressed as \( [\alpha * \gamma * \alpha^{-1}] \) for any loop \( \gamma \) in \((U, \alpha(1))\).

Notice \( X \) being homotopically Hausdorff relative to \( H \) corresponds to \( X \) being \((\pi_1(X, x_0), H)\)-homotopically Hausdorff.

Let us characterize the concept of being \((H, G)\)-homotopically Hausdorff in terms of the basic topology on the fundamental group.

**Proposition 4.12.** If \( G \subset H \) are subgroups of \( \pi_1(X, x_0) \), then \( X \) is \((H, G)\)-**homotopically Hausdorff** if and only if for every path \( \alpha \) in \( X \) that terminates at \( x_0 \) the group \( h_\alpha(G) \) is closed in \( h_\alpha(H) \) in the basic topology.

**Proof.** \( h_\alpha(G) \) being closed in \( h_\alpha(H) \) means existence, for each \( h \in H \setminus G \), of a neighborhood \( U \) of \( x_1 = \alpha(0) \) such that \( B([\alpha*h*\alpha^{-1}], U) \cap ([\alpha] : G : [\alpha^{-1}]) = \emptyset \). Thus, for every loop \( \gamma \) in \( U \) at \( x_1 \), there is no \( g \in G \) satisfying \( [\alpha*h*\alpha^{-1}*\gamma^{-1}] = [\alpha*g*\alpha^{-1}] \).

The last equality is equivalent to \( [g*h] = [\alpha^{-1}*\gamma*\alpha] \) which completes the proof.

**Example 4.13.** Proposition 4.12 allows for an easy construction of subgroups \( H \) of \( \pi_1(X, x_0) \) such that \( X \) is not homotopically Hausdorff relative to \( H \). Namely, \( X = S^1 \times S^1 \times \ldots \) and \( H = \bigoplus Z \subset \prod Z = \pi_1(X) \).

Let us show \( G \) being closed in \( H \) (in the new topology) is a stronger condition than \( X \) being \((H, G)\)-homotopically Hausdorff.
Lemma 4.14. Suppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$. If $G$ is closed in $H$, then $X$ is $(H, G)$-homotopically Hausdorff.

Proof. Given $h \in H \setminus G$ pick an open cover $U$ and $W \in U$ containing $x_0$ so that $B(h, U, W)$ does not intersect $G$. Given a path $\alpha$ in $X$ from $x_0$ to $x_1$ choose $V \in U$ containing $x_1$. Suppose there is a loop $\gamma$ in $(V, x_1)$ so that $[\alpha \ast \gamma \ast \alpha^{-1}] = g \ast h$ for some $g \in G$. Now $[\alpha \ast \gamma^{-1} \ast \alpha^{-1}] \in \pi(U, x_0)$ and $g^{-1} = h \ast [\alpha \ast \gamma^{-1} \ast \alpha^{-1}] \in G \cap B(h, U, W)$, a contradiction.

Remark 4.15. The proof of Lemma 4.14 suggests that the trivial subgroup of $\pi_1(X, x_0)$ being closed is philosophically related to the concept of $X$ being strongly homotopically Hausdorff (see [22]). Recall a metric space $X$ is strongly homotopically Hausdorff if for any non-nil-homotopic loop $\alpha$ in $X$ there is an $\epsilon > 0$ such that $\alpha$ is not freely homotopic to a loop of diameter less than $\epsilon$.

Lemma 4.16. Given subgroups $G \subset H$ of $\pi_1(X, x_0)$ the following conditions are equivalent:

a) The fibers of the natural projection $p: \hat{X}_G \to \hat{X}_H$ are $T_0$,

b) The fibers of the natural projection $p: \hat{X}_G \to \hat{X}_H$ are Hausdorff,

c) $X$ is $(H, G)$-homotopically Hausdorff.

Proof. a) $\implies$ c). Suppose $h \in H \setminus G$ and $\alpha$ is a path in $X$ from $x_0$ to $x_1$. As $[h \ast \alpha]_G \neq [\alpha]_G$ belong to the same fiber of $p$, there is a neighborhood $U$ of $x_1$ so that $[h \ast \alpha]_G \notin B_G([\alpha]_G, U)$ or $[\alpha]_G \notin B_G([h \ast \alpha]_G, U)$. Notice $[h \ast \alpha]_G \notin B_G([\alpha]_G, U)$ is equivalent to $[\alpha]_G \notin B_G([h \ast \alpha]_G, U)$. Suppose there is a loop $\gamma$ in $(U, x_1)$ so that $g \ast h = [\alpha \ast \gamma \ast \alpha^{-1}]$ for some $g \in G$. Now $[h \ast \alpha]_G = [g \ast h \ast \alpha]_G = [\alpha \ast \gamma]_G \in B_G([\alpha]_G, U)$, a contradiction.

c) $\implies$ b). Any two different elements of the same fiber of $p$ can be represented as $[h \ast \alpha]_G \neq [\alpha]_G$ for some path $\alpha$ in $X$ from $x_0$ to $x_1$ and some $h \in H \setminus G$. Choose a neighborhood $U$ of $x_1$ with the property that none of the elements of $G \cdot h$ can be expressed as $[\alpha \ast \gamma \ast \alpha^{-1}]$ for any loop $\gamma$ in $(U, x_1)$. Suppose $[\beta]_G \in (H/G) \cap B_G([\alpha]_G, U) \cap B_G([h \ast \alpha]_G, U)$. That means existence of loops $\gamma_1, \gamma_2$ in $(U, x_1)$ so that $[\beta]_G = [h \ast \alpha \ast \gamma_1]_G = [\alpha \ast \gamma_2]_G$. Hence $[h]_G = [\alpha \ast (\gamma_2 \ast \gamma_1^{-1}) \ast \alpha^{-1}]_G$, a contradiction.

Lemma 4.17. Suppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$, $G$ is normal in $\pi_1(X, x_0)$, and $X$ is $(H, G)$-homotopically Hausdorff. If $\alpha, \beta: (I, 0) \to (\hat{X}_G, \hat{x}_0)$ are two continuous lifts of the same path $\gamma: (I, 0) \to (\hat{X}_H, \hat{x}_0)$, then for every $h \in H$ the set

$$S = \{t \in I | \alpha(t) = h \cdot \beta(t)\}$$

is closed.

Proof. Choose paths $u_t, v_t$ in $(X, x_0)$ so that $\alpha(t) = [u_t]_G$ and $\beta(t) = [v_t]_G$ for all $t \in I$. Assume $[u_t]_G \neq [v_t]_G$ for some $t \in I$. Pick a neighborhood $U$ of $x_1 = u_t(1)$ so that $[v_t \ast u^{-1}_t]_G \neq [v_t \ast \gamma \ast v^{-1}_t]_G$ for any loop $\gamma$ in $(U, x_1)$. There is a neighborhood $V$ of $t$ in $I$ so that $[u_s]_G \in B_G([u_t]_G, U)$ and $[v_s]_G \in B_G([v_t]_G, U)$ for all $s \in V$. That means $[u_s]_G = [g_1 \ast u_t \ast \gamma_1]_G$ and $[v_s]_G = [g_2 \ast v_t \ast \gamma_2]_G$ for some $g_1, g_2 \in G$ and some paths $\gamma_1, \gamma_2$ in $U$ joining $x_1$ and $u_1(1) = v_1(1)$. Put $\gamma = \gamma_1 \ast \gamma_2^{-1}$ and notice $[u_s \ast v^{-1}_s]_G = [g_1 \ast u_t \ast v^{-1}_t \ast (v_t \ast \gamma \ast v^{-1}_t) \ast g_2^{-1}]$. As $G$ is normal in $\pi_1(X, x_0)$, there is $g_3 \in G$ satisfying $[g_1 \ast u_t \ast v^{-1}_t \ast (v_t \ast \gamma \ast v^{-1}_t) \ast g_2^{-1}] = [g_3 \ast u_t \ast v^{-1}_t \ast (v_t \ast \gamma \ast v^{-1}_t)]$ and that element cannot belong to $G \cdot h$ by the choice of $U$. \(\Box\)
Corollary 4.18. Suppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$. If $H/G$ is countable, $G$ is normal in $\pi_1(X, x_0)$, and $X$ is $(H, G)$-homotopically Hausdorff, then the natural map $\tilde{X}_G \to \tilde{X}_H$ has the uniqueness of path lifts property.

Proof. Pick representatives $h_i \in H$, $i \geq 1$, of all right cosets of $H/G$ so that $h_1 = 1$. If $\alpha$ and $\beta$ are two continuous lifts in $\tilde{X}_G$ of the same path in $\tilde{X}_H$, then each set $S_i = \{ t \in I | \alpha(t) = h_i \cdot \beta(t) \}$ is closed, they are disjoint, and their union is the whole interval $I$. Hence only one of them is non-empty and it must be $S_1$. Thus $\alpha = \beta$. \(\square\)

5. Peano maps

This section is about one of the main ingredients of our theory of covering maps for lpc-spaces. It amounts to the following generalization of Peano spaces:

Definition 5.1. A map $f : X \to Y$ is a Peano map if the family of path components of $f^{-1}(U)$, $U$ open in $Y$, forms a basis of neighborhoods of $X$.

Notice $X$ is an lpc-space if $f : X \to Y$ is a Peano map. One may reword the above definition as follows: $X$ is an lpc-space and lifts of short paths in $Y$ are short in $X$. Indeed, given a neighborhood $U$ of $x_0 \in X$ there is a neighborhood $V$ of $f(x_0)$ in $Y$ such that any path $\alpha$ in $(f^{-1}(V), x_0)$ (i.e. $f \circ \alpha$ is contained in $V$, hence short) must be contained in $U$.

Proposition 5.2. Any product of Peano maps is a Peano map.

Proof. Suppose $f_s : X_s \to Y_s$, $s \in S$, are Peano maps. Observe $X = \prod_{s \in S} X_s$ is an lpc-space. Given a neighborhood $U$ of $x = \{x_s\}_{s \in S} \in X$, we find a finite subset $T$ of $S$ and neighborhoods $U_s$ of $x_s$ in $X_s$ such that $\prod_{s \in S \setminus T} U_s \subset U$ and $U_s = X_s$ for $s \notin T$. Choose neighborhoods $V_s$ of $f_s(x_s)$ in $Y_s$, $s \in T$, so that the path-component of $x_s$ in $f_s^{-1}(V_s)$ is contained in $U_s$. Put $V_s = X_s$ for $s \notin T$ and observe the path component of $x$ in $f^{-1}(V)$, $f = \prod_{s \in S} f_s$ and $V = \prod_{s \in S} V_s$, is contained in $U$. \(\square\)

Here is our basic class of Peano maps:

Proposition 5.3. If $H$ is a subgroup of $\pi_1(X, x_0)$, then the endpoint projection $p_H : \tilde{X}_H \to X$ is a Peano map.

Proof. It suffices to show that for any $U$ open in $X$ the path component of any $[\alpha]_H$ in $p_H^{-1}(U)$ is precisely $B_H([\alpha]_H, U)$. It’s straightforward that $B_H([\alpha]_H, U)$ is path-connected so suppose $\beta$ is a path in $p_H^{-1}(U)$ starting at $[\alpha]_H$. We wish to show that $\beta([0, 1]) \subset B_H([\alpha]_H, U)$. Let $T = \{ t : \beta(t) \in B_H([\alpha]_H, U) \}$. Now $T$ is nonempty since $\beta(0) = [\alpha]_H$ and open as the inverse image of an open set. It suffices to prove $[0, t) \subset T$ implies $[0, t] \subset T$. Set $\beta(t) = [b]_H$. Now $p_H \beta([0, 1]) \subset U$ so in particular $p_H([b]_H) \in U$. Consider $B_H([b]_H, U)$. There is an $\epsilon > 0$ such that $\beta(t - \epsilon, t) \subset B_H([b]_H, U)$. Pick $s \in (t - \epsilon, t)$. Then $\beta(s) = [c_s]_H$ and $[b]_H = [b_1]_H$ such that $c_1 \simeq b_1 * \gamma_1$ for some $\gamma_1$ with $\gamma_1 [0, 1] \subset U$. But $\beta(s) \in B_H([\alpha]_H, U)$ so $\beta(s) = [c_2]_H$ and $[\alpha]_H = [a_1]_H$ such that $c_2 \simeq a_1 * \gamma_2$ for some $\gamma_2$ with $\gamma_2 [0, 1] \subset U$. Then $b \simeq_H b_1 \simeq c_1 * \gamma_1 \simeq_H c_2 * \gamma_1 \simeq a_1 * \gamma_2 * \gamma_1 \simeq_H a * \gamma_2 * \gamma_1$ and $(\gamma_2 * \gamma_1)([0, 1]) \subset U$ so $[b]_H \in B_H([\alpha]_H, U)$ and $t \in T$. Therefore $T = [0, 1]$. \(\square\)

In analogy to path lifting and unique path lifting properties (see 4.1 and 4.2 one can introduce the corresponding concepts for hedgehogs:
Definition 5.4. A surjective map \( f: X \to Y \) has the **hedgehog lifting property** if for any map \( \alpha: \bigsqcup_{s \in S} I_s \to Y \) from a hedgehog and any \( y_0 \in f^{-1}(\alpha(0)) \) there is a continuous lift \( \beta: \bigsqcup_{s \in S} I_s \to X \) of \( \alpha \) such that \( \beta(0) = y_0 \).

Definition 5.5. \( f: X \to Y \) has the **unique hedgehog lifting property** if it has both the hedgehog lifting property and the uniqueness of path lifts property.

Theorem 5.6. If \( f: X \to Y \) has the unique hedgehog lifting property, then \( f: \text{lp}(X) \to Y \) is a Peano map.

**Proof.** Assume \( U \) is open in \( X \) and \( x_0 \in U \). Suppose for each neighborhood \( V \) of \( f(x_0) \) in \( X \) there is a path \( \alpha_V: (I, 0) \to (f^{-1}(V), x_0) \) such that \( \alpha_V(1) \notin U \). By \ref{2.9} the wedge \( \bigvee_{V \in S} f \circ \alpha_V \) is a map \( g \) from a hedgehog to \( Y \) (here \( S \) is the family of all neighborhoods of \( f(x_0) \) in \( Y \)). Its lift must be the wedge \( h = \bigvee_{V \in S} \alpha_V \). However \( h^{-1}(U) \) is not open in \( \text{lp}(X) \), a contradiction. \( \square \)

Definition 5.7. Given a map \( f: X \to Y \) of topological spaces its **Peano map** \( P(f): P_f(X) \to Y \) is \( f \) on \( X \) equipped with the topology generated by path components of sets \( f^{-1}(U) \), \( U \) open in \( Y \).

Notice that in the case of \( f = \text{id}_X \) the range \( P_{\text{id}_X}(X) \) of \( P(\text{id}_X) \), where \( \text{id}_X: X \to X \) is the identity map, is identical to \( \text{lp}(X) \) as defined in \ref{2.2}.

Recall \( f: X \to Y \) is a **Hurewicz fibration** if every commutative diagram

\[
\begin{array}{ccc}
K \times \{0\} & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow f \\
K \times I & \xrightarrow{H} & Y
\end{array}
\]

has a filler \( G: K \times I \to X \) (that means \( f \circ G = H \) and \( G \) extends \( \alpha \)). If the above condition is satisfied for \( K \) being any \( n \)-cell \( I^n \), \( n \geq 0 \) (equivalently, for any finite polyhedron \( K \)), then \( f \) is called a **Serre fibration**. Notice for \( K \) being a point this is the classical **path lifting property**.

If the above condition is satisfied for \( K \) being any hedgehog, then \( f \) is called a **hedgehog fibration**. If the above condition is satisfied for \( K \) being any Peano space, then \( f \) is called a **Peano fibration**.

We will modify those concepts for maps between pointed spaces as follows:

Definition 5.8. A map \( f: (X, x_0) \to (Y, y_0) \) is a **Serre 1-fibration** if any commutative diagram

\[
\begin{array}{ccc}
(I \times \{0\}, \{\frac{1}{2}, 0\}) & \xrightarrow{(X, x_0)} & (I \times I, \{\frac{1}{2}, 0\}) \\
\downarrow & & \downarrow f \\
(I \times I, \{\frac{1}{2}, 0\}) & \xrightarrow{H} & (Y, y_0)
\end{array}
\]

has a filler \( G: (I \times I, \{\frac{1}{2}, 0\}) \to (X, x_0) \) (that means \( f \circ G = H \) and \( G \) extends \( \alpha \)).

Observe Serre 1-fibrations have the path lifting property in the sense that any path in \( Y \) starting at \( y_0 \) lifts to a path in \( X \) originating at \( x_0 \).
Theorem 5.9. Suppose

\[(T, z_0) \xrightarrow{g_1} (X, x_0)\]

is a commutative diagram in the topological category such that \((Z, z_0)\) is a Peano space and \(i\) is the inclusion from a path-connected subspace \(T\) of \(Z\). If \(f\) is a Serre 1-fibration, then there is a continuous lift \(h: (Z, z_0) \rightarrow (P_f(X), x_0)\) of \(g\) extending \(g_1\) if the image of \(\pi_1(g): \pi_1(Z, z_0) \rightarrow \pi_1(Y, y_0)\) is contained in the image of \(\pi_1(f): \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)\).

**Proof.** For each point \(z \in Z\) pick a path \(\alpha_z\) in \(Z\) from \(z_0\) to \(z\) and let \(\beta_z\) be a lift of \(g: \alpha_z \rightarrow Y\). In case of \(z = z_0\) we pick the constant paths \(\alpha_z\) and \(\beta_z\). In case \(z \in T\) the path \(\alpha_z\) is contained in \(T\) and \(\beta_z = g_1 \circ \alpha_z\). Define \(h: (Z, z_0) \rightarrow (P_f(X), x_0)\) by \(h(z) = \beta_z(1)\). Given a neighborhood \(U\) of \(g(z)\) in \(Y\), let \(V\) be the path component of \(h(z)\) in \(f^{-1}(U)\) and let \(W\) be the path component of \(g^{-1}(U)\) containing \(z\). Our goal is to show \(h(W) \subset V\) as that is sufficient for \(h: (Z, z_0) \rightarrow (P_f(X), x_0)\) to be continuous. For any \(t \in W\) choose a path \(\mu_t\) in \(W\) from \(z\) to \(t\). Let \(\gamma\) be a loop in \(X\) at \(x_0\) so that \(f(\gamma)\) is homotopic to \(g(\alpha_z \ast \mu_t \ast \alpha_t^{-1})\). Notice \(f(\beta_z)\) is homotopic to \(f(\gamma \ast \beta_t)\) via a homotopy \(H\) so that \(H(\{1\} \times I) \subset U\). By lifting that homotopy to \(X\) we get a path in \(f^{-1}(U)\) from \(h(z)\) to \(h(t)\), i.e., \(h(t) \in V\). □

Corollary 5.10. A Peano map \(f: X \rightarrow Y\) is a Peano fibration if and only if it is a Serre 1-fibration.

**Proof.** Assume \(f: X \rightarrow Y\) is a Peano map and a Serre 1-fibration (in the other direction [5,10] is left as an exercise), \(g: Z \times \{0\} \rightarrow X\) is a map from a Peano space, and \(H: Z \times I \rightarrow Y\) is a homotopy starting from \(f \circ g\). Pick \(z_0 \in Z\) and put \(x_0 = g(z_0, 0), y_0 = f(x_0)\). Notice the image of \(\pi_1(g): \pi_1(Z \times \{0\}, (z_0, 0)) \rightarrow \pi_1(Y, y_0)\) is contained in the image of \(\pi_1(f)\). Use 5.9 to produce an extension \(G: Z \times I \rightarrow X\) of \(g\) that is a lift of \(H\). □

6. Peano covering maps

[5.9] suggests the following concept:

**Definition 6.1.** A map \(f: X \rightarrow Y\) is called a **Peano covering map** if the following conditions are satisfied:

1. \(f\) is a Peano map,
2. \(f\) is a Serre fibration,
3. The fibers of \(f\) have trivial path components.

Notice 3) above can be replaced by \(f\) having the unique path lifting property (see [5.20]). Also notice that, in case fibers of a Peano map \(f: X \rightarrow Y\) are \(T_0\) spaces, path-components of fibers are trivial. Indeed, two points in a path-component of a fiber are always in any open set that contains one of them.

**Proposition 6.2.** Any product of Peano covering maps is a Peano covering map.

**Proof.** Suppose \(f_s: X_s \rightarrow Y_S, s \in S\), are Peano covering maps. Put \(f = \prod_{s \in S} f_s\), \(X = \prod_{s \in S} X_s\), and \(Y = \prod_{s \in S} Y_s\). By 5.2, \(f\) is a Peano map. It is obvious \(f\) is a Serre fibration and has the uniqueness of path lifting property. □
Corollary 6.3. Suppose \( f : (X, x_0) \to (Y, y_0) \) is a Peano covering map. If \( (Z, z_0) \) is a Peano space, then any map \( g : (Z, z_0) \to (Y, y_0) \) has a unique continuous lift \( h : (Z, z_0) \to (X, x_0) \) if the image of \( \pi_1(g) \) is contained in the image of \( \pi_1(f) \).

**Proof.** By 5.6 a lift \( h \) exists and is unique by the uniqueness of path lifting property. □

Our basic example of Peano covering maps is related to the basic topology:

Theorem 6.4. If \( X \) is a path-connected space and \( x_0 \in X \), then the following conditions are equivalent:

a. \( p_H : (\hat{X}_H, \hat{x}_0) \to (X, x_0) \) has the unique path lifting property,

b. \( p_H : \hat{X}_H \to X \) is a Peano covering map.

**Proof.** a) \( \implies \) b). In view of 5.3 and 8.4 it suffices to show \( p_H : (\hat{X}_H, \hat{x}_0) \to (X, x_0) \) is a Serre fibration. Suppose \( f : (Z, z_0) \to (X, x_0) \) is a map from a simply connected Peano space \( Z \) (the case of \( Z = I^n \) is of interest here). There is a standard lift \( g : (Z, z_0) \to \hat{X}_H \) of \( f \) defined as \( g(z) = [\alpha_z]_H \), where \( \alpha_z \) is a path in \( Z \) from \( z_0 \) to \( z \). If \( T \) is a path-connected subspace of \( Z \) containing \( z_0 \) and \( h : (T, z_0) \to (\hat{X}_H, \hat{x}_0) \) is any continuous lift of \( f|T \), then \( h = g|T \) due to the uniqueness of the path lifting property of \( p_H \). That proves \( p_H \) is a Serre fibration in view of 8.4.

b) \( \implies \) a) is obvious. □

Theorem 6.5. If \( f : X \to Y \) is a map and \( X \) is an lpc-space, then the following conditions are equivalent:

a) \( f \) is a Peano covering map,

b) \( f \) is a Peano fibration and has the uniqueness of path lifting property,

b) \( f \) is a hedgehog fibration and has the uniqueness of path lifting property,

d) For any \( x_0 \in X \) and any map \( g : (Z, z_0) \to (Y, f(x_0)) \) from a simply-connected Peano space there is a lift \( h : (Z, z_0) \to (X, x_0) \) of \( g \) and that lift is unique.

**Proof.** a) \( \implies \) b). Suppose \( H : Z \times I \to Y \) is a homotopy, \( Z \) is a Peano space, and \( G : Z \times \{0\} \to X \) is a lift of \( H|Z \times \{0\} \). Pick \( z_0 \in Z \), put \( x_0 = G(z_0, 0) \) and \( y_0 = f(x_0) \), and notice \( im(\pi_1(Z \times I, (z_0, 0))) \subset im(\pi_1(f)) \). Using 5.9 there is a lift of \( H \) and that lift is unique, hence it agrees with \( G \) on \( Z \times \{0\} \).

b) \( \implies \) c) is obvious.

d) \( \implies \) c) is obvious.

a) \( \implies \) b) follows from 5.9

c) \( \implies \) a). Notice \( f \) has the unique hedgehog lifting property and is a Serre 1-fibration. By 5.6 \( f \) is a Peano map. □

Corollary 6.6. Suppose \( f : X \to Y \) and \( g : Y \to Z \) are maps of path-connected spaces and \( Y \) is a Peano space. If any two of \( f, g, h = g \circ f \) are Peano covering maps, then so is the third provided its domain is an lpc-space.

**Proof.** In view of 0.5 it amounts to verifying that the map has uniqueness of lifts of simply-connected Peano spaces, an easy exercise. □

Proposition 6.7. Suppose \( f : X \to Y \) is a map.

a. If \( f : X \to Y \) is a Peano covering map, then \( f : f^{-1}(U) \to U \) is a Peano covering map for every open subset \( U \) of \( Y \).
b. If every point \( y \in Y \) has a neighborhood \( U \) such that \( f: f^{-1}(U) \to U \) is a Peano covering map, then \( f \) is a Peano covering map.

**Proof.** a). \( f: f^{-1}(U) \to U \) is clearly a Peano map, is a fibration, and has the unique path lifting property.

b). \( f \) is a Serre 1-fibration and path components of fibers are trivial. If \( V \) is an open subset of \( Y \) containing \( y \) we pick an open subset \( U \) of \( X \) containing \( f(y) \) such that \( f: f^{-1}(U) \to U \) is a Peano covering map. There is an open neighborhood \( W \) of \( f(y) \) in \( U \) so that the path component of \( y \) in \( f^{-1}(W) \) is open and is contained in \( V \cap f^{-1}(U) \). That proves \( f: Y \to X \) is a Peano map.

In analogy to regular classical covering maps let us introduce regular Peano covering maps:

**Definition 6.8.** A Peano covering map \( f: X \to Y \) is regular if lifts of loops in \( Y \) are always loops of are always non-loops.

**Corollary 6.9.** Given a map \( f: X \to Y \) the following conditions are equivalent if \( X \) is path-connected:

a) \( f \) is a regular Peano covering map,

b) \( f \) is a Peano covering map and the image of \( \pi_1(f) \) is a normal subgroup of \( \pi_1(Y, f(x_0)) \) for all \( x_0 \in X \),

c) \( f: X \to Y \) is a generalized covering map in the sense of Fischer-Zastrow.

**Proof.** a) \( \implies \) b). If the image of \( \pi_1(f) \) is not a normal subgroup of \( \pi_1(Y, f(x_0)) \) for some \( x_0 \in X \), then there is a loop \( \alpha \) in \( Y \) at \( y_0 = f(x_0) \) that lifts to a loop in \( X \) at \( x_0 \) and there is a loop \( \beta \) in \( Y \) at \( y_0 \) such that \( \beta \ast \alpha \ast \beta^{-1} \) does not lift to a loop in \( X \) at \( x_0 \). Let \( \gamma \) be a lift of \( \alpha \) originating at \( x_0 \). Let \( x_1 = \beta(1) \). Notice the lift of \( \alpha \) originating at \( x_1 \) cannot be a loop, a contradiction.

b) \( \implies \) c). As \( \text{im}(\pi_1(f)) \) is a normal subgroup \( H \) of \( \pi_1(Y, y_0) \), it does not depend on the choice of the base-point of \( X \) in \( f^{-1}(y_0) \). Using \( \text{Proposition 6.10} \) one gets \( f \) is a generalized covering map.

c) \( \implies \) a). Since each hedgehog is contractible, \( f \) has the unique hedgehog lifting property and is a Peano map by \( \text{Proposition 6.10} \). It is also a Serre fibration, hence a Peano covering map. Also, as \( \text{im}(\pi_1(f)) \) is a normal subgroup \( H \) of \( \pi_1(Y, y_0) \), it does not depend on the choice of the base-point of \( X \) in \( f^{-1}(y_0) \). Hence a loop in \( Y \) lifts to a loop in \( X \) if and only if it represents an element of \( H \). Thus \( f \) is a regular Peano covering map.

In the remainder of this section we will discuss the relation of Peano covering maps to classical covering maps.

**Proposition 6.10.** If \( f: Y \to X \) is a Peano covering map and \( U \) is an open subset of \( X \) such that every loop in \( U \) is null-homotopic in \( X \), then \( f^{-1}(V) \to P(V) \) is a a trivial discrete bundle for every path component \( V \) of \( U \).

**Proof.** Consider a path component \( W \) of \( f^{-1}(U) \) intersecting \( f^{-1}(V) \). \( f \) maps \( W \) bijectively onto \( V \) and it is easy to see \( f|W: W \to V \) is equivalent to \( P(V) \to V \).

**Corollary 6.11.** If \( X \) is a semilocally simply connected Peano space, then \( f: Y \to X \) is a Peano covering map if and only if it is a classical covering map and \( Y \) is connected.
Proof. If \( f \) is a classical covering map and \( Y \) is connected, then \( Y \) is locally path-connected, \( f \) has unique path lifting property and is a Serre 1-fibration. Thus it is a Peano covering map.

Suppose \( f \) is a Peano covering map and \( x \in X \). Choose a path-connected neighborhood \( U \) of \( x \) in \( X \) such that any loop in \( U \) is null-homotopic in \( X \). By Proposition 6.10 \( U \) is evenly covered by \( f \).

Corollary 6.12. If \( f: Y \to P(X) \) is a classical covering map, then \( f: Y \to X \) is a Peano covering map.

Proof. By Proposition 6.11 \( f: Y \to P(X) \) is a Peano covering map. As the identity function induces a Peano covering map \( P(X) \to X \), \( f: Y \to X \) is a Peano covering map by Proposition 6.12.

Proposition 6.13. If \( f: Y \to X \) is a Peano covering map and \( X \) is path-connected, then all fibers of \( f \) have the same cardinality.

Proof. Given two points \( x_1, x_2 \in X \) fix a path \( \alpha \) from \( x_1 \) to \( x_2 \) and notice lifts of \( \alpha \) establish bijectivity of fibers \( f^{-1}(x_1) \) and \( f^{-1}(x_2) \).

The following result has its origins in Lemma 2.3 of [8] and Proposition 6.6 of [15].

Proposition 6.14. Suppose \( f: Y \to X \) is a regular Peano covering map. If \( f^{-1}(x_0) \) is countable and \( x_0 \) has a countable basis of neighborhoods in \( X \), then there is a neighborhood \( U \) of \( x_0 \) in \( X \) such that \( f^{-1}(U) \to P(V) \) is a classical covering map, where \( V \) is the path component of \( x_0 \) in \( U \).

Proof. Switch to \( X \) being Peano by considering \( f: Y \to P(X) \). Notice \( x_0 \) has a countable basis of neighborhoods and \( f \) is open. Suppose there is no open subset \( U \) of \( X \) containing \( x_0 \) such that \( U \) is evenly covered. That means path components of \( f^{-1}(U) \) are not mapped bijectively onto their images.

First, we plan to show there is a neighborhood \( U \) of \( x_0 \) in \( X \) such that the image of \( \pi_1(U, x_0) \to \pi_1(X, x_0) \) is contained in the image of \( \pi_1(f): \pi_1(Y, y_0) \to \pi_1(X, x_0) \).

In particular, there is a lift of \( P(U, x_0) \to (Y, y_0) \) of the inclusion induced map \( P(U, x_0) \to (X, x_0) \).

Suppose no such \( U \) exists. By induction we will find a basis of neighborhoods \( \{U_i\} \) of \( x_0 \) in \( X \) and elements \( [\alpha_i] \in \pi_1(U_i, x_0) \) that are not contained in the image of \( \pi_1(U_i+1, x_0) \to \pi_1(X, x_0) \) and whose lifts are not loops and end at points \( y_i \) such that \( y_i \neq y_j \) if \( i \neq j \). Given a neighborhood \( U_i \) pick a loop \( \alpha_i \) in \( (U_i, x_0) \) whose lift (as a path) in \( (Y, y_0) \) is not a loop and ends at \( y_i \neq y_0 \). There is a neighborhood \( U_{i+1} \) of \( x_0 \) in \( U_i \) such that the no path components of \( f^{-1}(U_{i+1}) \) contains both \( y_0 \) and some \( y_j, j \leq i \). Pick a loop \( \alpha_{i+1} \) in \( (U_{i+1}, x_0) \) whose lift is not a loop.

As in [21] one can create infinite concatenations \( \alpha_{i(1)} \ast \ldots \ast \alpha_{i(k)} \ast \ldots \), for any increasing sequence \( \{i(k)\}_{k \geq 1} \). By looking at lifts of those infinite concatenations, there are two different infinite concatenations \( \alpha_{i(1)} \ast \ldots \ast \alpha_{i(k)} \ast \ldots \) and \( \alpha_{j(1)} \ast \ldots \ast \alpha_{j(k)} \ast \ldots \) whose lifts end at the same point \( y \in f^{-1}(x_0) \). Pick the smallest \( k_0 \) so that \( i(k_0) \neq j(k_0) \). We may assume \( i(k_0) < j(k_0) \) and conclude there are loops \( \beta \) in \( (U_{k_0+1}, x_0) \) and \( \gamma \) in \( (Y, y_0) \) such that \( \alpha_{i(k_0)} = f(\gamma) \ast \beta \) which case the lift of \( \alpha_{i(k_0)} \) in \( (Y, y_0) \) ends in the path component of \( f^{-1}(U_{i(k_0)+1}) \) containing \( y_0 \), a contradiction.

As \( f \) is a regular Peano covering map, we can find lifts \( (U, x_0) \to (Y, y) \) of the inclusion map \( (U, x_0) \to (X, x_0) \) for any \( y \in f^{-1}(x_0) \).
7. Peano subgroups

**Definition 7.1.** Suppose \((X, x_0)\) is a pointed path-connected space. A subgroup \(H\) of \(\pi_1(X, x_0)\) is a **Peano subgroup** of \(\pi_1(X, x_0)\) if there is a Peano covering map \(f: Y \to X\) such that \(H\) is the image of \(\pi_1(f): \pi_1(Y, y_0) \to \pi_1(X, x_0)\) for some \(y_0 \in f^{-1}(x_0)\).

**Proposition 7.2.** If \(H\) is a Peano subgroup of \(\pi_1(X, x_0)\), then \(X\) is homotopically Hausdorff relative to \(H\). In particular, \(H\) is closed in \(\pi_1(X, x_0)\) equipped with the basic topology.

**Proof.** Choose a Peano covering map \(f: Y \to X\) so that \(\text{im}(\pi_1(f)) = H\) for some \(y_0 \in f^{-1}(x_0)\). If \(g \in \pi_1(X, x_0)\setminus H\) and \(\alpha\) is a path in \(X\) from \(x_0\) to \(x_1\), then lifts of \(\alpha\) and \(g \cdot \alpha\) end in two different points \(y_1\) and \(y_2\) of the fiber \(f^{-1}(x_1)\) and there is a neighborhood \(U\) of \(x_1\) in \(X\) such that no path component of \(f^{-1}(U)\) contains both \(y_1\) and \(y_2\). Suppose there is a loop \(\gamma\) in \((U, x_1)\) with the property \([\alpha \cdot \gamma \cdot \alpha^{-1}] \in H \cdot g\). In that case the lifts of both \(\alpha \cdot \gamma\) and \(g \cdot \alpha\) end at \(y_2\). Since the lift of \(\alpha\) ends in the same path component of \(f^{-1}(U)\) as the lift of \(\alpha \cdot \gamma\), both \(y_1\) and \(y_2\) belong to the same component of \(f^{-1}(U)\), a contradiction.

Use [4.12](#4.12) to conclude \(H\) is closed in \(\pi_1(X, x_0)\) equipped with the basic topology. \(\square\)

**Remark 7.3.** In case of \(H\) being the trivial subgroup, Lemma 2.10 of [15](#15) seems to imply that \(X\) is homotopically Hausdorff but the proof of it is valid only in a special case.

**Proposition 7.4.** If \(H\) is a Peano subgroup of \(\pi_1(X, x_0)\), then any conjugate of \(H\) is a Peano subgroup of \(\pi_1(X, x_0)\).

**Proof.** Choose a Peano covering map \(f: Y \to X\) so that \(\text{im}(\pi_1(f)) = H\) for some \(y_0 \in f^{-1}(x_0)\). Suppose \(G = g \cdot H \cdot g^{-1}\) and choose a loop \(\alpha\) in \((X, x_0)\) representing \(g^{-1}\). Let \(\beta\) be a path in \((Y, y_0)\) that is the lift of \(\alpha\). Put \(y_1 = \beta(1)\) and notice the image of \(\pi_1(f): \pi_1(Y, y_0) \to \pi_1(X, x_0)\) is \(G\). \(\square\)

**Proposition 7.5.** Suppose \((X, x_0)\) is a pointed path-connected topological space. If \(f: (Y, y_0) \to (X, x_0)\) is a Peano covering map with image of \(\pi_1(f)\) equal \(H\), then \(f\) is equivalent to the projection \(p_H: \tilde{X}_H \to X\).

**Proof.** Define \(h: (\tilde{X}_H, \tilde{x}_0) \to (Y, y_0)\) by choosing a lift \(\tilde{\alpha}\) of every path \(\alpha\) in \(X\) starting at \(x_0\) and declaring \(h([\alpha]_H) = \tilde{\alpha}(1)\). Note \(h\) is a bijection. Given \(y_1 = \tilde{\alpha}(1)\) and given a neighborhood \(U\) of \(y_1\) in \(Y\) choose a neighborhood \(V\) of \(f(y_1) = \alpha(1)\) in \(X\) so that the path component of \(f^{-1}(V)\) containing \(y_1\) is a subset of \(U\). Observe \(B_H([\alpha]_H, V) \subset h^{-1}(U)\) which proves \(h\) is continuous.

Conversely, given a neighborhood \(W\) of \(\alpha(1)\) in \(X\) the image \(h(B_H([\alpha]_H, W))\) of \(B_H([\alpha]_H, W)\) equals the path component of \(\tilde{\alpha}(1)\) in \(f^{-1}(W)\) and is open in \(Y\). \(\square\)

**Theorem 7.6.** If \(X\) is a path-connected space, \(x_0 \in X\), and \(H\) is a subgroup of \(\pi_1(X, x_0)\), then the following conditions are equivalent:

a. \(H\) is a Peano subgroup of \(\pi_1(X, x_0)\),

b. The endpoint projection \(p_H: (\tilde{X}_H, \tilde{x}_0) \to X\) is a Peano covering map,

c. The image of \(\pi_1(p_H): \pi_1(\tilde{X}_H, \tilde{x}_0) \to \pi_1(X, x_0)\) is contained in \(H\),

d. \(p_H: (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)\) has the unique path lifting property.
Proof. c)⇒d) is done in 2.18 b)⇒d) is contained in 6.4
a)⇒b) follows from 7.5
b)⇒a) holds as c) implies the image of \( \pi_1(p_H) \) is \( H \). □

Let us state a straightforward consequence of 7.6:

**Corollary 7.7.** If \( X \) is a path-connected space and \( x_0 \in X \), then the following conditions are equivalent:

a. The endpoint projection \( p: \hat{X} \to X \) is a Peano covering map,
b. \( \pi_1(p): \pi_1(\hat{X}, \hat{x}_0) \to \pi_1(X, x_0) \) is trivial,
c. \( \hat{X} \) is simply connected,
d. \( p: (\hat{X}, \hat{x}_0) \to (X, x_0) \) has the unique path lifting property.

**Corollary 7.8.** Closed and normal subgroups of \( \pi_1(X, x_0) \) are Peano subgroups of \( \pi_1(X, x_0) \).

Proof. By 4.6 the endpoint projection \( p_H: (\hat{X}_H, \hat{x}_0) \to X \) has unique path lifting property. Since \( p_H: (\hat{X}_H, \hat{x}_0) \to X \) has path lifting property, this implies \( p_H: (\hat{X}_H, \hat{x}_0) \to X \) has the unique path lifting property. □

**Corollary 7.9.** If \( H(s) \) is a Peano subgroup of \( \pi_1(X, x_0) \) for each \( s \in S \), then \( G = \bigcap_{s \in S} H(s) \) is a Peano subgroup of \( \pi_1(X, x_0) \).

Proof. The projection \( p_G: (\hat{X}_G, \hat{x}_0) \to (X, x_0) \) factors through \( p_{H(s)}: (\hat{X}_{H(s)}, \hat{x}_0) \to (X, x_0) \) for each \( s \in S \). Therefore \( im(p_G) \subset \bigcap_{s \in S} H(s) = G \) and 6.4 (in conjunction with 2.18) says \( G \) is a Peano subgroup of \( \pi_1(X, x_0) \). □

**Corollary 7.10.** For each path-connected space \( X \) there is a universal Peano covering map \( p: Y \to X \). Thus, for each Peano covering map \( q: Z \to X \) and any points \( z_0 \in Z \) and \( y_0 \in Y \) satisfying \( q(z_0) = p(y_0) \), there is a Peano covering map \( r: Y \to Z \) so that \( r(y_0) = z_0 \). Moreover, the image of \( \pi_1(Y) \) is normal in \( \pi_1(X) \).

Proof. Let \( H \) be the intersection of all Peano subgroups of \( \pi_1(X, x_0) \) by 7.8 and 7.3 it is a normal Peano subgroup of \( \pi_1(X, x_0) \). Put \( Y = \hat{X}_H \) and use 6.3. □

It would be of interest to characterize path-connected spaces \( X \) admitting a universal Peano covering that is simply connected (that amounts to \( \hat{X} \) being simply connected). Here is an equivalent problem:

**Problem 7.11.** Characterize path-connected spaces \( X \) so that the trivial group is a Peano subgroup of \( \pi_1(X, x_0) \).

So far the following classes of spaces belong to that category:

1. Any product of spaces admitting simply connected Peano cover (see 6.2).
2. Subsets of closed surfaces: it is proved in 14 that if \( X \) is any subset of a closed surface, then \( \pi_1(X, x_0) \to \hat{\pi}_1(X, x_0) \) is injective.
3. 1-dimensional, compact and Hausdorff, or 1-dimensional, separable and metrizable: \( \pi_1(X, x_0) \to \hat{\pi}_1(X, x_0) \) is injective by 11 Corollary 1.2 and Final Remark. It is shown in 10 (see proof of Theorem 1.4) that the projection \( \hat{X} \to X \) has the uniqueness of path-lifting property if \( X \) is 1-dimensional and metrizable. See 6 for results on the fundamental group of 1-dimensional spaces.
(4) Trees of manifolds: If $X$ is the limit of an inverse system of closed PL-manifolds of some fixed dimension, whose consecutive terms are obtained by connect summing with closed PL-manifolds, which in turn are trivialized by the bonding maps, then $X$ is called a tree of manifolds. Every tree of manifolds is path-connected and locally path-connected, but it need not be semilocally simplyconnected at any one of its points. Trees of manifolds arise as boundaries of certain Coxeter groups and as boundaries of certain negatively curved geodesic spaces [13]. It is shown in [13] that if $X$ is a tree of manifolds (with a certain denseness of the attachments in the case of surfaces), then $\pi_1(X, x_0) \to \hat{\pi}_1(X, x_0)$ is injective.

Notice Example 2.7 in [15] gives $X$ so that $p: \tilde{X} \to X$ does not have the unique path lifting property (one can construct a simpler example with $X$ being the Harmonic Archipelago). However, $X$ is not homotopically Hausdorff.

**Problem 7.12.** Is there a homotopically Hausdorff space $X$ such that $p: \tilde{X} \to X$ does not have the uniqueness of path lifting property?

**Corollary 7.13.** Suppose $H$ is a normal subgroup of $\pi_1(X, x_0)$. If there is a Peano subgroup $G$ of $\pi_1(X, x_0)$ containing $H$ such that $G/H$ is countable, then $H$ is a Peano subgroup of $\pi_1(X, x_0)$ if and only if $X$ is homotopically Hausdorff relative to $H$.

**Proof.** By [7.2] $X$ is homotopically Hausdorff relative to $H$ if $H$ is a Peano subgroup of $\pi_1(X, x_0)$.

Suppose $X$ is homotopically Hausdorff relative to $H$. Given two lifts in $\tilde{X}_H$ of the same path in $X$, their composition with $\tilde{X}_H \to \tilde{X}_G$ are the same by [7.6]. By 4.18 the two lifts are identical and 7.6 says $H$ is a Peano subgroup of $\pi_1(X, x_0)$.

**Corollary 7.14.** Suppose $H$ is a normal subgroup of $\pi_1(X, x_0)$. If $\pi_1(X, x_0)/H$ is countable, then $H$ is a Peano subgroup of $\pi_1(X, x_0)$ if and only if $X$ is homotopically Hausdorff relative to $H$.

8. Appendix: Pointed versus unpointed

In this section we discuss relations between pointed and unpointed lifting properties.

**Lemma 8.1.** If $f: (X, x_0) \to (Y, y_0)$ has the uniqueness of path lifts property and $X$ is path-connected, then $f: X \to Y$ has the uniqueness of path lifts property.

**Proof.** Given two paths $\alpha$ and $\beta$ in $X$ originating at the same point and satisfying $f \circ \alpha = f \circ \beta$, choose a path $\gamma$ in $X$ from $x_0$ to $\alpha(0)$. Now $f \circ (\gamma \ast \alpha) = f \circ (\gamma \ast \beta)$, so $\gamma \ast \alpha = \gamma \ast \beta$ and $\alpha = \beta$.

**Lemma 8.2.** If $f: (X, x_0) \to (Y, y_0)$ has the unique path lifting property and $X$ is path-connected, then $f: X \to Y$ has the unique path lifting property.

**Proof.** In view of [8.2] it suffices to show $f: X \to Y$ is surjective and has the path lifting property. If $y_1 \in Y$, we pick a path $\alpha$ from $y_0$ to $y_1$ and lift it to $(X, x_0)$. The endpoint of the lift maps to $y_1$, hence $f$ is surjective. Suppose $\alpha$ is a path in $Y$ and $f(x_1) = \alpha(0)$. Choose a path $\beta$ in $X$ from $x_0$ to $x_1$ and lift $(f \circ \beta) \ast \alpha$ to a path $\gamma$ in $(X, x_0)$. Due to the uniqueness of path lifts property of $f: (X, x_0) \to (Y, y_0)$
one has $\gamma(t) = \beta(2t)$ for $t \leq \frac{1}{2}$. Hence $\gamma(\frac{1}{2}) = x_1$ and $\lambda$ defined as $\lambda(t) = \gamma(\frac{1}{2} + \frac{t}{2})$ for $t \in I$ is a lift of $\alpha$ originating from $x_1$. \hfill \Box

**Lemma 8.3** (Lemma 15.1 in [17]). If $f: X \to Y$ is a Serre 1-fibration, then $f$ has the unique path lifting property if and only if path components of fibers of $f$ are trivial.

**Proof.** Suppose the fibers of $f$ have trivial path components and $\alpha, \beta$ are two lifts of the same path in $Y$ that originate at $x_1 \in X$. Let $H: I \times I \to Y$ be the standard homotopy from $f \circ (\alpha^{-1} * \beta)$ to the constant path at $f(x_1)$. There is a lift $G: I \times I \to X$ of $H$ starting from $\alpha^{-1} * \beta$. As path components of $f$ are trivial, $\alpha = \beta$ due to the way the standard homotopy $H$ is defined. \hfill \Box

**Lemma 8.4.** Suppose $n \geq 1$. If $f: (X, x_0) \to (Y, y_0)$ is a Serre $n$-fibration, both $X$ and $Y$ are path-connected, and $f$ has the uniqueness of path lifts property, then $f: X \to Y$ is a Serre $n$-fibration.

**Proof.** Suppose $H: I^n \times I \to Y$ is a homotopy and $G: I^n \times \{0\} \to X$ is its partial lift. Choose a path $\alpha$ in $X$ from $x_0$ to $G(b, 0)$, where $b$ is the center of $I^n$. We can extend $G$ to a homotopy $G: I^n \times [-1, 0] \to X$ starting from the constant map to $x_0$. By splicing $f \circ G$ with original $H$, we can extend $H$ to $H: I^n \times [-1, 1] \to Y$. That $H$ can be lifted to $X$ and the lift must agree with $G$ on $I^n \times [-1, 0]$ due to the uniqueness of path lifts property of $f$. \hfill \Box

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