Axially symmetric stars in Einstein-Maxwell theory

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Abstract

In this work we show how to construct general relativistic star models with axial symmetry in the Einstein-Maxwell theory. Starting with a general axially symmetric interior fluid solution we show how to construct the corresponding electrically charged and also the corresponding magnetized solution of the Einstein-Maxwell-fluid equations. As an example, we provide an electrically charged version of the anisotropic Bower and Liang solution.

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1 Introduction

Since the pioneering work of Schwarzschild [1] and Tolman [2], in General Relativity (GR) compact objects are usually modeled using spherically symmetric perfect fluid solutions of the Einstein field equations. With the subsequent discovery of white dwarfs and neutron stars, in the last decades there has been a lot of interest in generating new exact solutions describing spherically symmetric relativistic stars. One should note that there are known by now various algorithms to generate new spherically symmetric exact solutions, which are sourced by perfect fluids [3], [4], [5]. However, as shown in [6] not all the perfect fluid interior solutions considered so far are physical. Moreover, in modeling realistic compact objects in GR one should consider more sophisticated models, involving deviations either from the spherical symmetry and/or the perfect fluid distribution of the source.

As is well-known, while preserving the requirement of spherical symmetry one can consider more general solutions that are sourced by anisotropic fluids (see for instance [7] - [17]). For such anisotropic fluids the radial pressure component, $p_r$, is not equal to the components in the transverse directions, $p_t$. There are strong theoretical reasons to believe that in realistic relativistically covariant stellar models in the high density regimes the pressures inside the star are anisotropic [18]. Such anisotropies in the fluid distributions can arise from various reasons: it can be due to a mixture of two fluid components [19], the existence of a superfluid phase, the presence of a magnetic field, etc. (for a review see [20] and references there). Other non-trivial examples of anisotropic fluid distributions are provided by the bosonic stars (see for instance [21] and the references within), by traversable wormholes [22], or the so-called gravastars [23], which are systems where anisotropic pressures occur naturally. The pressure anisotropy can have significant effects on the structure and the properties of the stellar models [24]. Anisotropic fluid models of neutron stars could be used to model the so-called magnetars [25], which denote a class of neutron stars whose emissions are powered by the decay of their huge magnetic field. For a magnetar the magnetic field strength can reach values as high as $10^{11}T$, while being even more intense inside the star. There are over 30 magnetars cataloged by now [26] (for recent reviews of their properties see [27], also [28]). This class of objects includes the soft gamma repeaters (SGRs) and the the anomalous X-ray pulsars (AXPs). Analytic non-perturbative solutions in GR describing anisotropic models of magnetars have been constructed in [29] and [30]. However, in presence of very strong magnetic fields, in order to construct more realistic magnetar models one should consider an axially symmetric treatment of the source [31]. In absence of rotation, the most general line element describing such systems has the form:

$$\text{ds}^2 = -A(r, \theta)^2 dt^2 + B(r, \theta)^2 dr^2 + C(r, \theta)^2 d\theta^2 + D(r, \theta)^2 d\varphi^2.$$  \hfill (1)

In general, this line element can be considered as a solution of Einstein equations$^1$ $G_{\mu\nu} = 8\pi T_{\mu\nu}^0$ sourced by an anisotropic fluid, which is described by a non-diagonal stress-energy tensor of the form:

$$T_{\mu\nu}^0 = \rho_0 u_\mu^0 u_\nu^0 + p_r^{0} \delta_{\mu\nu}^0 + p_{\theta}^{0} \delta_{\mu\varphi}^0 + p_{\varphi}^{0} \delta_{\mu\theta}^0 + 2 p_{r}^{0} \delta_{\mu\varphi}^0,$$ \hfill (2)

$^1$Note that we work using the natural units for which $G = c = 1.$
where $\rho^0$ is the fluid density, $p^0_r$ is the radial pressure, while $p^0_\varphi$, $p^0_\theta$ and $p^0_{r\theta}$ are transverse components of the fluid pressure. Also $u^0_\mu = (-A, 0, 0, 0)$ is the 4-velocity of the fluid, while $\lambda^0_\mu = (0, B, 0, 0)$, $\zeta^0_\mu = (0, 0, C, 0)$ and $\zeta^0_\mu = (0, 0, 0, D)$ are spacelike unit vectors in the radial and transverse directions.

The purpose of this paper is to show that starting from any solution (1) sourced by the stress-energy tensor (2) one can easily generate the corresponding solutions in Einstein-Maxwell theory that correspond either to an electrically charged metric or to a magnetized solution. In GR the electromagnetic field is described using the anti-symmetric Faraday tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, where $A_\mu$ is the vector potential of the electromagnetic field. The Maxwell equations are then written as:

$$\nabla_\nu(\star F)^{\mu\nu} = 0, \quad \nabla_\nu F^{\mu\nu} = 4\pi j_\mu,$$

where $\star F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\gamma\delta} F^{\gamma\delta}$ is the Hodge-dual tensor, while $\epsilon_{\mu\nu\gamma\delta}$ is the Levi-Civita tensor. Also, $j_\mu$ is the 4-current that sources the electromagnetic field. The electromagnetic stress-energy tensor, which enters the Einstein field equations is defined as:

$$T_{em}^{\mu\nu} = \frac{1}{4\pi} \left( F_{\nu\gamma} F^{\gamma\nu} - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g_{\mu\nu} \right).$$

For an electrically charged solution, in absence of a magnetic field, the only non-zero component of the electromagnetic potential is $A_t$. On the other hand, the magnetic field of a magnetized star can have both toroidal and poloidal components. However, when one takes into account the toroidal components of the magnetic field then the spacetime geometry (1) has to be modified and it will include other non-vanishing metric components [32]. Therefore, in our work we shall assume that the toroidal components are zero and the magnetic field is purely poloidal.

The structure of this paper is as follows: in the next section we present the general electrically charged version of the metric in (1). As an example we show how to obtain the charged Bowers and Liang solution [33]. In section 3 we construct the general magnetized version of the metric (1). The final section contains a summary of our work and avenues for further work.

## 2 The electrically charged model

Starting with the metric (1) solution of the Einstein equations sourced by an anisotropic fluid described by (2) one can construct the following metric:

$$ds^2 = -\frac{A(r, \theta)^2}{\Lambda^2} dt^2 + \Lambda^2 \left[ B(r, \theta)^2 dr^2 + C(r, \theta)^2 d\theta^2 + D(r, \theta)^2 d\varphi^2 \right],$$

where we defined $\Lambda = 1 - E_0^2 A(r, \theta)^2$, with $E_0$ being a constant. This will be a solution of the Einstein-Maxwell-fluid equations:

$$G_{\mu\nu} = 8\pi T_{em}^{\mu\nu} + 8\pi T_{fluid}^{\mu\nu}.$$
together with the Maxwell equations (3) if the electromagnetic 4-vector potential is \( A_\mu = (A_t, 0, 0, 0) \) with
\[
A_t = \frac{E_0 A(r, \theta) ^2}{\Lambda},
\]
while the fluid stress-energy tensor has the form:
\[
T^\text{fluid}_{\mu\nu} = (\rho + \sigma_e) u_\mu u_\nu + p_r \chi_\mu \chi_\nu + p_\theta \xi_\mu \xi_\nu + p_\varphi \zeta_\mu \zeta_\nu + 2 p_{r\theta} \chi(\mu \xi_\nu).
\]
Here we defined
\[
\rho = \frac{\rho^0}{\Lambda^2}, \quad p_r = \frac{p^0_r}{\Lambda^2}, \quad p_\theta = \frac{p^0_\theta}{\Lambda^2}, \quad p_\varphi = \frac{p^0_\varphi}{\Lambda^2}, \quad p_{r\theta} = \frac{p^0_{r\theta}}{\Lambda^2}.
\]
Note that \( u_\mu = \left( -\frac{A_\Lambda}{\Lambda}, 0, 0, 0 \right) \) is the 4-velocity of the fluid, while \( \chi_\mu = (0, B \Lambda, 0, 0) \), \( \xi_\mu = (0, 0, C \Lambda, 0) \) and \( \zeta_\mu = (0, 0, 0, D \Lambda) \) are respectively spacelike unit vectors in the radial and the transverse angular directions. Finally, the charge density is:
\[
\sigma_e = 2(\rho + p_r + p_\theta + p_\varphi) \frac{E_0^2 A(r, \theta) ^2}{\Lambda}.
\]
and the electric current is \( j_\mu = (j_t, 0, 0, 0) \) where:
\[
j_t = -2(\rho + p_r + p_\theta + p_\varphi) \frac{E_0 A(r, \theta) ^2}{\Lambda^2}.
\]
We explicitly checked using Maple [34] that the fields given in (5), (7), (8) are an exact solution of the coupled Einstein-Maxwell-fluid system if (1) and (2) is an exact solution of the Einstein-fluid equations of motion.

### 2.1 The electrically charged version of the Bowers-Liang solution

As an example of this solution-generating technique, let us consider the charged version of the anisotropic Bowers-Liang solution. This solution, which was found by Bowers and Liang [33] corresponds to a anisotropic fluid with an homogeneous density distribution \( \rho = \rho^0 = \text{constant} \). In their work they considered a spherically symmetric relativistic matter distribution and studied the behavior of such systems by incorporating the pressure anisotropy effects in the equation of the hydrostatic equilibrium. Their solution is given by (1) where:

\[
A(r, \theta) ^2 = \left[ \frac{3 \left( 1 - \frac{2 M}{R} \right)^{\frac{b}{2}} - \left( 1 - \frac{2 m(r)}{r} \right)^{\frac{b}{2}}}{2} \right]^{\frac{2}{b}}, \quad B(r, \theta)^2 = \frac{1}{1 - \frac{2 m(r)}{r}}, \quad C(r, \theta) = r.
\]

\[
D(r, \theta) = r \sin \theta, \quad p^0 = \frac{3 M}{4 \pi R^3}, \quad p^0_r = \rho^0 \left( 1 - \frac{2 m(r)}{r} \right)^{\frac{b}{2}} - \left( 1 - \frac{2 M}{R} \right)^{\frac{b}{2}}, \quad p^0_{r\theta} = \frac{3 \left( 1 - \frac{2 m(r)}{r} \right)^{\frac{b}{2}} - \left( 1 - \frac{2 M}{R} \right)^{\frac{b}{2}}}{\left( 1 - \frac{2 m(r)}{r} \right)^{\frac{b}{2}}},
\]

\[
\Delta^0 = p^0_t - p^0_r = \frac{4 \pi}{3} C r^2 \frac{(\rho^0 + p^0_r)(\rho^0 + 3 p^0_r)}{1 - \frac{2 m(r)}{r}}.
\]
where \( h = 1 - 2C \), \( m(r) = \frac{4\pi}{3}r^3\rho^0 \) and \( C \) is the anisotropy parameter. Note that for this solution \( p_\theta^0 = p_\varphi^0 = p_t^0 \), while \( p_{r\theta}^0 = 0 \) in (2).

Then, using the results from section 2 the electrically charged Bowers-Liang solution will simply be given by (5) supplemented by (7) and (8). The final geometry is still spherically symmetric, while the anisotropic fluid source has the pressures different in radial and transverse directions. Note also that for \( C = 0 \) one obtains the electrically charged interior Schwarzschild solution discussed in [35] in a slightly different form, in absence of the dilaton field.

Since in origin \( r = 0 \)

\[
\Lambda_0 = 1 - E_0^2 \left( \frac{3 \left( 1 - \frac{2M}{R} \right)^{\frac{3}{2}}}{2} - 1 \right),
\]

then in the electrically charged Bowers-Liang solution the radial pressure in origin becomes:

\[
p_r(0) = \frac{p_\theta^0 \Lambda_0^2}{3 \left( 1 - \frac{2M}{R} \right)^{\frac{3}{2}} - 1},
\]

and the critical value of the quantity \( \frac{2M}{R} \) for which the central pressure becomes infinite is\(^2\):

\[
\frac{2M}{R}|_{cr} = 1 - \left( \frac{1}{3} \right)^{\frac{3}{2}}.
\]

The critical value of the ratio \( \frac{2M}{R} \) is the same as the critical value of the original Bowers and Liang solution.

If one takes \( h = 0 \) in the electrically charged Bowers-Liang solution one obtains the charged version of the so-called Florides solution [36]. It corresponds to an anisotropic object with zero radial pressure \( p_r = 0 \), which is sustained only by tangential stresses.

### 3 The magnetized solution

Similarly to the electrically charged case, given a general solution (1) - (2) of the Einstein-fluid field equations, one can write down directly the corresponding magnetized solution in the following form:

\[
d s^2 = \Lambda^2 [ - A(r, \theta)^2 d t^2 + B(r, \theta)^2 d r^2 + C(r, \theta)^2 d \theta^2 ] + \frac{D(r, \theta)^2}{\Lambda^2} d \varphi^2,
\]

\[
A_\varphi = \frac{B_0 D(r, \theta)^2}{\Lambda}, \quad \Lambda = 1 + B_0^2 D(r, \theta)^2,
\]

where the stress-energy tensor of the anisotropic fluid is given by:

\[
T_{\mu\nu}^{\text{fluid}} = \rho u_\mu u_\nu + p_r \chi_\mu \chi_\nu + p_\theta \xi_\mu \xi_\nu + (p_\varphi + \sigma_m) \zeta_\mu \zeta_\nu + 2 p_r \chi (\mu \xi_\nu),
\]

\(^2\)For this value \( \Lambda_0 \rightarrow 1 \), there is no physical critical value of \( \frac{2M}{R} \) for which \( \Lambda_0 = 0 \).
with
\[ \rho = \frac{\rho_0}{\Lambda^2}, \quad p_r = \frac{p_r^0}{\Lambda^2}, \quad p_\theta = \frac{p_\theta^0}{\Lambda^2}, \quad p_\phi = \frac{p_\phi^0}{\Lambda^2}, \quad p_{r\theta} = \frac{p_{r\theta}^0}{\Lambda^2}, \] (18)

while:
\[ \sigma_m = -2(\rho - p_r - p_\theta + p_\phi) \frac{B_0^2 D(r, \theta)^2}{\Lambda} \] (19)

and the only non-vanishing component of the 4-current \( j_\mu \) is:
\[ j_\phi = 2(\rho - p_r - p_\theta + p_\phi) \frac{B_0 D(r, \theta)^2}{\Lambda^2}. \] (20)

Finally, \( u_\mu = (-A\Lambda, 0, 0, 0) \) is the 4-velocity of the fluid, while \( \chi_\mu = (0, B\Lambda, 0, 0) \), \( \xi_\mu = (0, 0, C\Lambda, 0) \) and \( \zeta_\mu = (0, 0, 0, \frac{D}{\Lambda}) \) are respectively spacelike unit vectors in the radial and the transverse angular directions.

This solution is a direct generalization of the magnetized solutions considered in [30]. One should note that we explicitly checked using Maple [34] that (16) and (17) represent a full exact solution of the Einstein-Maxwell-fluid equations (3) - (6).

### 4 Conclusions

In this work we presented a simple solution-generating technique that enabled us to construct the electrically charged or the magnetized solution for every axially-symmetric geometry (1), sourced by an anisotropic fluid described by a non-diagonal anisotropic stress-energy tensor (2). As an example, we showed how to derive the charged Bowers and Liang solution. Note that using our method one should be able to construct the charged/magnetized version of every spherically symmetric fluid solution. However, our solution-generating technique can be successfully applied to more general interior solutions with axial symmetry as found for instance in [37], [38].

As avenues for further work, the magnetized solution presented in our paper should be suitable to construct more realistic models of magnetars, by adding the slow-rotation in a perturbative way, along the lines of [39], [40]. Another interesting extension of the present work would involve a study of the stars anisotropy effect on the propagation of various fields in this background, on the lines of the study presented in [41], [42]. Work on these matters is in progress and it will be presented elsewhere.

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