On the modulus of continuity of solutions to complex Monge-Ampère equations

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Abstract

In this paper, we prove a uniform and sharp estimate for the modulus of continuity of solutions to complex Monge-Ampère equations, using the PDE-based approach developed by the first three authors in their approach to supremum estimates for fully non-linear equations in Kähler geometry. As an application, we derive a uniform diameter bound for Kähler metrics satisfying certain Monge-Ampère equations.

1 Introduction

The complex Monge-Ampère equation has been extensively studied ever since Yau’s seminal work on the solution of the Calabi conjecture [17]. Notably, assuming that the right hand side is in some Orlicz space, Kolodziej [11] showed using pluripotential theory that the solution must be in $L^\infty$ (see also [7] for a recent PDE-based proof), and in fact continuous. When the right hand side is in $L^q$ for some $q > 1$, it is known that the solution must be Hölder continuous [13, 4]. However, there are examples showing that Hölder continuity may not hold when the right hand side is not in $L^q$ for any $q > 1$. In general, when the solution is not Hölder continuous, its modulus of continuity is not known. For the complex Monge-Ampère equation, the modulus of continuity of the solution is especially important, as it is closely related to essential geometric properties of the corresponding Kähler metric such as its diameter. In particular, uniform bounds for the diameter of the metric are needed for Gromov-Hausdorff convergence, and this requires in turn uniform bounds for the modulus of continuity. This is the problem which we shall solve in the present paper.

Let $(X,\omega_0)$ be a compact Kähler manifold of complex dimension $n$. We consider the complex Monge-Ampère equation

$$(\omega_0 + i\partial\bar\partial u)^n = e^F \omega_0^n, \quad \omega := \omega_0 + i\partial\bar\partial u > 0 \text{ and } \inf_X u = 1,$$

where $F \in C^\infty(X,\mathbb{R})$ satisfies the compatibility condition $\int_X e^F \omega_0^n = \int_X \omega_0^n$. We shall use the following Orlicz norm for the right hand side $e^F$,

$$\|e^F\|_{L^1(\log L)^p} = \int_X e^F |F|^p \omega_0^n.$$

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Theorem 1  Fix $p > n$. There exists a constant $C > 0$ depending on $n, p, \omega_0, \|e^F\|_{L^1(\log L)^p}$ such that the following uniform estimate holds

$$|u(x) - u(y)| \leq \frac{C}{|\log d(x, y)|^\alpha}$$

(1.2)

for any $x, y \in X$. Here $d(x, y)$ denotes the geodesic distance of the two points $x, y$ in the Riemannian manifold $(X, \omega_0)$, and $\alpha = \min\{\frac{p-n}{n}, \frac{p}{n+1}\}$.

The case of Riemann surfaces (i.e. $n = 1$) shows that the solution $u$ to (1.1) may fail to be (uniformly) Hölder continuous if $e^F \not\in L^q$ for any $q > 1$. Theorem 1 says that the solution is still continuous with order $O(|\log d|^{-\alpha})$. This is a remarkable estimate in itself, as logarithmic moduli of continuity are rarely encountered, and this is the first example of it that we are aware of in the partial differential equations arising in Kähler geometry. The example at the end of Section 3 implies the exponent $\alpha > 0$ is sharp.

We remark that the estimate (1.2) continues to hold when the function $e^F$ is not smooth. This can be seen by a smoothing argument combined with the stability estimate of complex Monge-Ampère equations [12, 8] and Theorem 1. The continuity of $u$ when $e^F$ is not smooth had been obtained by Kolodziej [11] using pluripotential theory and an argument by contradiction. Theorem 1 sharpens the continuity estimate of $u$ and provides a uniform control.

In the case $e^F \in L^q(X, \omega_0^n)$ for some $q > 1$, it is known from [13, 4] that the solution $u$ is Hölder continuous, i.e. $|u(x) - u(y)| \leq C d(x, y)^\alpha$ for some $a \in (0, 1)$. Our proof of Theorem 1 can be easily modified to give a new and PDE-based proof of the Hölder continuity of $u$, in the spirit of the proof of $L^\infty$ estimates developed in [7]. We can readily show in this manner that, if $e^F \in L^q$ for some $q > 1$ and $q^* = \frac{q}{q-1}$ is the conjugate exponent of $q$, then $|u(x) - u(y)| \leq C d(x, y)^{\alpha_0}$ for $\alpha_0 = \frac{2}{1+(n+1)q^*}$, for some constant $C > 0$ depending only on $n, q, \omega_0$ and $\|e^F\|_{L^q}$. For the sake of interested readers, we provide a sketch of the proof in Section 4.

The complex Monge-Ampère equation (1.1) plays an important role in finding canonical Kähler metrics in complex geometry. It is natural to study the geometry of the Kähler metric $\omega_u = \omega_0 + i\partial\bar\partial u$ satisfying (1.1), for example, its Ricci curvature, volume growth of geodesic balls, and diameter bound. Under some general assumptions on $i\partial\bar\partial F$ and $e^F$, it is known by the work of Fu-Guo-Song [5] that these geometric quantities are indeed bounded in some sense. Without the assumption on $i\partial\bar\partial F$, Y. Li [14] proves a diameter bound of $(X, \omega_u)$ if the function $e^F \in L^q(X, \omega_0^n)$ for some $q > 1$. His proof requires the Hölder continuity of $u$ proved in [13, 4] and Morrey’s lemma. With the uniform modulus of continuity estimate in Theorem 1, we can generalize Li’s result with a weaker assumption on $e^F$. 

2
Theorem 2 Let $\omega_u$ be the solution to (1.1). Suppose $p > 3n$, in other words, $\alpha > 2$, then there exists a constant $C > 0$ depending only on $n, p, \omega_0, \|e^F\|_{L^1(\log L)^p}$ such that
\[ \text{diam}(X, \omega_u) \leq C. \]

The diameter bound of Kähler metrics satisfying certain complex Monge-Ampère equations or Kähler-Einstein type equations is a necessary ingredient in the study of degenerations of these metric spaces in the Gromov-Hausdorff sense [15]. Theorem 2 provides a uniform diameter bound under a mild assumption on $e^F$, and we expect it to be useful in studying the geometry of complex Monge-Ampère equations.

Convention: we say a constant $C > 0$ is uniform if it depends only on $n, p > n, \omega_0$ and $\|e^F\|_{L^1(\log L)^p}$. The constants $C$’s in different lines may not be the same, but are all uniform unless otherwise stated.

2 Preliminaries

We collect some necessary background materials from [4]. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a smoothing kernel which is supported in $[0, 1]$ and normalized to satisfy $\int_{\mathbb{R}_+} \rho(t) dt = 1$ and $\rho(t) = \text{const}$ for $t \in [0, 3/4]$. Given a function $u \in L^1(X)$ and $\delta \in (0, 1)$, we define its $\delta$-regularization to be
\[ \rho_{\delta}u(z) = \frac{1}{\delta^{2n}} \int_{\zeta \in T_zX} u(\exp z(\zeta)) \rho(\delta^{-1} |\zeta|_{\omega_0}^2) dV_{\omega_0}(\zeta), \]
where $\exp_z : T_zX \to X$ is the exponential map of the Riemannian manifold $(X, \omega_0)$. $\rho_{\delta}u$ is a suitable weighted average of $u$ over the geodesic ball $B_{\omega_0}(z, \delta)$, so if $0 \leq u \in PSH(X, \omega_0)$, by mean value inequality $\rho_{\delta}u(z)$ control the maximum of $u$ over $B_{\omega_0}(z, \delta/2)$. The following lemma is proved in [4, 1].

Lemma 1 Let $u$ be an $L^1$ function in $PSH(X, \omega_0)$. Then
\begin{enumerate}
\item ([1]) There exists a constant $K > 0$ depending on the curvature of $(X, \omega_0)$ such that $t \mapsto \rho_t u(z) + Kt^2$ is monotone increasing for any $z \in X$.
\item ([4]) There exists a constant $C > 0$ depending on only $n, \omega_0$ such that
\[ \int_X |\rho_{\delta}u - u|_{\omega_0^n} \leq C\delta^2, \quad \forall \delta \in (0, 1). \]
\end{enumerate}

For a given small $c > 0$, we define the Kiselman-Legendre transformation of $u$ as
\[ u_{c,\delta}(z) = \inf_{t \in (0, \delta]} \{ \rho_t u(z) + K t^2 - c \log \frac{t}{\delta} - K \delta^2 \} \]
where $K > 0$ is the constant in Lemma 1. By applying Kiselman’s minimum principle it can be shown that (see [3, 1]) for $u \in PSH(X, \omega_0)$
\[ \omega_0 + i\partial\bar{\partial} u_{c,\delta} \geq -(A c + K \delta^2) \omega_0 \]
where $-A$ is a lower bound of the bisectional curvature of the fixed Kähler metric $\omega_0$. 

3
3 Proof of Theorem 1

Let $u \in PSH(X, \omega_0)$ be the solution to the equation (1.1), where we normalize $u$ so that $u \geq 1$. We assume $p > n$ in this section. First recall the $L^\infty$ estimate of $u$ in [11, 7].

Lemma 2 There exists a constant $C_0 > 0$ depending on $n, p, \|e^F\|_{L^1((\log L)^p), \omega_0}$ such that

$$1 \leq u \leq C_0, \quad \text{on } X.$$

We fix a small $\delta > 0$. Let $c = \frac{1}{|\log \delta|^{\alpha}}$ for some $\alpha = \min(\frac{p-n}{n}, \frac{p}{n+1}) > 0$, and $U_\delta = u_{c,\delta}$ be the Kiselman-Legendre transformation of $u$ at the level $c$ as in (2.3). From (2.4) we get

$$\omega_0 + i\partial \bar{\partial} U_\delta \geq - (A c + K \delta^2) \omega_0 \geq - A' c \omega_0$$

for some $A' = A'(n, \omega_0) > 0$. Hence the function $u_\delta = U_\delta^{1+ A' c}$ belongs to $PSH(X, \omega_0)$, i.e. $\omega_0 + i\partial \bar{\partial} u_\delta \geq 0$. We note from (2.3) and normalization of $u$ that $u_\delta$ is positive.

For any $s \geq 0$ we denote the set

$$E_s = \{ u \leq -2 \delta + (1-r)u_\delta - s \} ,$$

where $r = |\log \delta|^{-\frac{p}{n+1}} > 0$ is a small constant.

Lemma 3 There is a constant $C_1 > 0$ depending on $n, p, \|e^F\|_{L^1((\log L)^p), \omega_0}$ such that

$$\int_{E_0} e^F \omega_0^n \leq \frac{C_1}{|\log \delta|^p}.$$ 

Proof. We observe the following elementary inclusions of sets

$$E_0 = \{ 2 \delta \leq (1-r)u_\delta - u \} \subset \{ 2 \delta \leq u_\delta - u \} \subset \{ 2 \delta \leq U_\delta - u \} \subset \{ 2 \delta \leq \rho_\delta u - u \} = : \Omega,$$

where the last inclusion follows from the fact that $U_\delta \leq \rho_\delta u$. Thus it suffices to prove the lemma for the domain $\Omega$.

We define $v = \log \frac{\rho_\delta u - u}{\delta^{3/2}}$ as a function on $\Omega$. It is clear that $v \geq \log \frac{2}{\delta^{3/2}} > 0$ on $\Omega$. Take a weight function $\eta(x) = (\log (1+x))^p$ on $\mathbb{R}_+$, and we apply the generalized Young's inequality with this weight. It follows that at any point $z \in \Omega$

$$v^p e^F \leq \int_0^v \eta(x)dx + \int_0^{v^p} \eta^{-1}(y)dy \leq e^F (1 + |F|)^p + v^p e^v.$$

Integrating the above over $\Omega$, we get

$$\int_{\Omega} v^p e^F \omega_0^n \leq C + \int_{\Omega} C|\log \delta|^{p} \frac{\rho_\delta u - u}{\delta^{3/2}} \omega_0^n \leq C.$$
where in the last inequality we use (2) in Lemma 1. Since on Ω, \( v \geq \log 2 + \frac{1}{2} |\delta| \geq \frac{1}{2} |\log \delta| \), we conclude that

\[
\frac{1}{2p} |\log \delta|^p \int_\Omega e^{F_n} \omega^n_0 \leq \int_\Omega v^p e^{F_n} \omega^n_0 \leq C.
\]

Then the lemma follows easily.

We consider the auxiliary equations

\[
(\omega_0 + i\partial\bar{\partial}\psi_{s,k})^n = \frac{\tau_k (1-u)(1-r)u_\delta - 2\delta - s)}{A_{s,k}} e^{F_n} \omega^n_0, \quad \sup_x \psi_{s,k} = 0,
\]

where \( \tau_k(x) : \mathbb{R} \to \mathbb{R}_+ \) is a sequence of positive smooth functions converging decreasingly and pointwise to \( x \cdot \chi_{\mathbb{R}_+}(x) \) on \( \mathbb{R} \), and

\[
A_{s,k} = \int_X \tau_k (1-u)(1-r)u_\delta - 2\delta - s) e^{F_n} \omega^n_0 \to \int_{E_\delta} (1-u)(1-r)u_\delta - 2\delta - s) e^{F_n} \omega^n_0 := A_s.
\]

as \( k \to \infty \). The equation (3.2) admits a unique smooth solution by Yau’s theorem [17]. As in [7, 8, 9], we aim to compare \( \psi_{s,k} \) with the solution \( u \) to (1.1). Consider the following test function

\[
\Psi = -\varepsilon (-\psi_{s,k} + \Lambda)^{\frac{n}{n+1}} + [-u + (1-r)u_\delta - 2\delta - s],
\]

where

\[
\varepsilon = (\frac{n+1}{n})^{\frac{n+1}{n+1}} \Lambda = \frac{n}{n+1} \frac{A_{s,k}}{A_{s,k}}.
\]

We claim that \( \Psi \leq 0 \) on \( X \).

If the maximum point \( x_{\text{max}} \) of \( \Psi \) lies in \( X \setminus E_s^0 \), by definition of \( E_s \), it is clear that \( \Psi(x_{\text{max}}) < 0 \). If \( x_{\text{max}} \in E_s^0 \), then by maximum principle, at \( x_{\text{max}} \)

\[
0 \geq \Delta_\omega \Psi \geq \frac{n\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{\frac{n}{n+1}} \Delta_\omega \psi_{s,k} + (1-r)\Delta_\omega u_\delta - \Delta_\omega u
\]

\[
\geq \frac{n\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}} \Delta_\omega \psi_{s,k} + (1-r)\Delta_\omega u_\delta - n + (r - \frac{n\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}) \Delta_\omega \omega_0
\]

\[
\geq \frac{n^2\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}} (\omega_{s,k}^n)^{1/n} - n
\]

\[
= \frac{n^2\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}} (\tau_k (1-u)(1-r)u_\delta - 2\delta - s)^{1/n} - n
\]

\[
\geq \frac{n^2\varepsilon}{n+1} (-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}} (\frac{(-u + (1-r)u_\delta - 2\delta - s)}{A_{s,k}})^{1/n} - n
\]

where we denote \( \omega_\varphi = \omega_0 + i\partial\bar{\partial}\varphi \) for a function \( \varphi \in PSH(X, \omega_0) \), in the third line we used the arithmetic-geometric inequality, the fact that \( u_\delta \in PSH(X, \omega_0) \) and the choices of \( \Lambda \) and \( \varepsilon \) in (3.4). It follows easily that \( \Psi(x_{\text{max}}) \leq 0 \) in this case. Hence we have \( \Psi \leq 0 \). From the definition of \( \Psi \) we obtain that on \( E_s \)

\[
\frac{(-u + (1-r)u_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \leq C(n) (-\psi_{s,k} + A_{s,k}^{1/(n+1)}).
\]
By the Hörmander estimate [16, 10] we can find a small \( \beta_0 = \beta_0(n, \omega_0) > 0 \) such that

\[
\int E_s \exp \left( \beta_0 \frac{(-u + (1 - r)u_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \right) \omega_0^n \leq \int E \exp \left( -C(n)\beta_0 \psi_{s,k} + C(n)\beta_0 \frac{A_{s,k}}{r^{n+1}} \right) \\
\leq C \exp \left( C \frac{A_{s,k}}{r^{n+1}} \right).
\]

Letting \( k \to \infty \) in (3.5) we obtain a Trudinger-type estimate

\[
\int E_s \exp \left( \beta_0 \frac{(-u + (1 - r)u_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \right) \omega_0^n \leq C \exp \left( C \frac{A_s}{r^{n+1}} \right).
\]

Lemma 4 We have a uniform constant \( C > 0 \) independent of \( \delta \) and \( s \) such that

\[
A_s \leq C.
\]

Proof. By the uniform \( L^\infty \) bound of \( u \) and \( u_\delta \), we have

\[
A_s = \int E_s (-u + (1 - r)u_\delta - 2\delta - s)e^F \omega_0^n \leq C \int E_s e^F \omega_0^n \leq C \int E_0 e^F \omega_0^n \leq \frac{C}{|\log \delta|^p}.
\]

The lemma follows from this and the choice of \( r = |\log \delta|^{-p/(n+1)} \).

Combined with the lemma above, (3.6) implies

\[
\int E_s \exp \left( \beta_0 \frac{(-u + (1 - r)u_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \right) \omega_0^n \leq C,
\]

for some uniform constant \( C > 0 \). Given (3.7), we can apply the generalized Young’s inequality as in [7] to conclude that

\[
\int E_s [-u + (1 - r)u_\delta - 2\delta - s]^{(n+1)p/n} e^F \omega_0^n \leq C A_s^{p/n},
\]

where \( C > 0 \) is independent of \( s \) and \( \delta \). By Hölder inequality we have

\[
A_s \leq \left( \int E_s [-u + (1 - r)u_\delta - 2\delta - s]^{(n+1)p/n} e^F \omega_0^n \right)^{n/(p(n+1))} \left( \int E_s e^F \omega_0^n \right)^{1/q} \\
\leq CA_s^{1/(n+1)} \left( \int E_s e^F \omega_0^n \right)^{1/q},
\]

where \( q = \frac{p(n+1)}{n(n+1)-n} \) is the conjugate exponent of \( p(n+1)/n \). So we get

\[
A_s \leq C \left( \int E_s e^F \omega_0^n \right)^{(1+n)/qn} = C \left( \int E_s e^F \omega_0^n \right)^{1+a_0}
\]
where $a_0 = \frac{p-n}{pn} > 0$. If we denote $\phi(s) = \int_{E_s} (-u + (1-r)u_\delta - 2\delta - s) e^{F} \omega_0^n$, then it follows easily that
\[
s'\phi(s + s') \leq C_3 \phi(s)^{1+a_0}, \quad \forall s \geq 0, s' \geq 0,
\]
for some uniform constant $C_3 > 0$ independent of $\delta \in (0, 1/2]$. Next we apply a De Giorgi-type iteration argument (c.f. [11, 7, 8]). Choose a $\delta_0 > 0$ small which depends only on $n, p, \omega_0, \|e^F\|_{L^1(\log L)^p}$ such that for all $\delta \in (0, \delta_0]$, it follows from Lemma 3
\[
C_3 \phi(0)^{a_0} = C_3 \left( \int_{E_0} e^F \omega_0^n \right)^{a_0} \leq \frac{C_3 C_1^{a_0}}{\|\log \delta\|_{pa_0}} < \frac{1}{2}.
\]
And $\phi(0)^{a_0} \leq C |\log \delta|^{-pa_0}$. With this choice of $\delta_0$, by a simple iteration argument (e.g. [7]), we get the set $E_s = \emptyset$ for all $s > S_\infty$, where
\[
S_\infty \leq \frac{2C_3}{1 - 2^{-a_0}} \phi(0)^{a_0} \leq \frac{C}{|\log \delta|^{pa_0}}.
\]
We thus conclude that
\[
u_\delta - u \leq 2\delta + ru_\delta + \frac{C}{|\log \delta|^{pa_0}}, \quad \text{on } X.
\]
From the definition of $u_\delta$ and $U_\delta$, we get that on $X$
\[
U_\delta - u \leq 2\delta + ru_\delta + A'cu + \frac{C}{|\log \delta|^{pa_0}}.
\]
At each point $z \in X$, there exists a $t_z \in (0, \delta]$ realizing the infimum of $U_\delta = u_{c, \delta}$ in the definition (2.3). From (3.10) it holds that
\[
\rho_{t_z} u + K t_z^2 - u - c \log \frac{t_z}{\delta} - K \delta^2 \leq 2\delta + ru_\delta + A'cu + \frac{C}{|\log \delta|^{pa_0}}.
\]
Also (1) in Lemma 1 shows that $\rho_{t_z} u + K t_z^2 - u \geq 0$. So
\[
\log \frac{t_z}{\delta} \geq \frac{-K \delta^2 - 2\delta}{c} - \frac{r}{c} u_\delta - A'u - \frac{C}{|\log \delta|^{pa_0}}.
\]
by the choice of $c = \frac{1}{|\log \delta|^\alpha}$ with $\alpha = \min(pa_0, \frac{p}{n+1})$, it follows that there exists a uniform constant $C > 0$ such that $\log \frac{t_z}{\delta} \geq -C$, thus $t_z \geq \theta \delta$ for some uniform $\theta \in (0, 1)$. Again by (1) in Lemma 1 we have at $z \in X$
\[
\rho_{\theta \delta} u + K \theta^2 \delta^2 - u \leq \rho_{t_z} u + K t_z^2 - u
\]
\[
\leq K \delta^2 + c \log \frac{t_z}{\delta} + 2\delta + ru_\delta + A'cu + \frac{C}{|\log \delta|^{pa_0}} \leq \frac{C}{|\log \delta|^\alpha}.
\]
This yields that for any $z \in X$ and $\delta \in (0, \delta_0]$, $\rho_\delta u(z) - u(z) \leq \frac{C}{|\log \delta|^\alpha}$; or equivalently for any $\delta \in (0, \theta\delta_0]$

$$\rho_\delta u(z) - u(z) \leq \frac{C_4}{|\log \delta|^\alpha}, \quad (3.11)$$

for some uniform constant $C_4 > 0$.

We are now ready to finish the proof of Theorem 1. For any $\delta > 0$, we denote

$$\bar{u}_\delta(z) = \max_{x \in B(z,\delta)} u(x),$$

where $B(z, \delta)$ denotes the geodesic ball with center $z$ and radius $\delta$ in the Riemannian manifold $(X, \omega_0)$. We claim that there exists a uniform constant $C > 0$ such that $\bar{u}_\delta(z) - u(z) \leq \frac{C}{|\log \delta|^\alpha}$ for any $z \in X$ and $\delta \in (0, \theta\delta_0/\beta_n]$ for some $\beta_n > 0$ sufficiently large depending only on $n$ and $\alpha$. This would be sufficient to prove the theorem. We follow the arguments in [6] closely. In the paragraphs below we shall assume $\delta_0 > 0$ is chosen to be smaller than the injectivity radius of $(X, \omega_0)$, so the exponential maps considered are diffeomorphisms on relevant domains.

We denote $\Omega(\delta) = \sup_{z \in X} (\bar{u}_\delta(z) - u(z))$. Define $A > 0$ to be (3.17) which depends only on $n, p, \omega_0$ and $C_4 > 0$, then we claim that $\Omega(\delta) \leq \frac{A}{|\log \delta|^\alpha}$ for any $\delta \in (0, \theta\delta_0/\beta_n]$. Suppose not, then there exists some $0 < \delta' < \theta\delta_0/\beta_n$ such that $\Omega(\delta') > \frac{A}{|\log \delta'|^\alpha}$. We define

$$\delta := \inf\{0 < t < \frac{\theta\delta_0}{\beta_n} : \Omega(s) \leq \frac{A}{|\log s|^\alpha} \text{ for all } s \in [t, \frac{\theta\delta_0}{\beta_n}]\}, \quad (3.12)$$

The existence of $\delta'$ implies that $\delta \geq \delta' > 0$. Since $u$ is continuous and $X$ is compact, there exists $z_0 \in X$ such that $\Omega(\delta) = \bar{u}_\delta(z_0) - u(z_0) = u(w_0) - u(z_0)$ for some $w_0 \in \overline{B}(z_0, \delta)$. From the definition of $\delta$ in (3.12), it follows that

$$\Omega(\delta) = \frac{A}{|\log \delta'|^\alpha}, \quad \text{and} \quad \Omega(s) \leq \frac{A}{|\log s|^\alpha} \text{ for all } s \in [\delta, \frac{\theta\delta_0}{\beta_n}], \quad (3.13)$$

We fix a constant $b > 1$ but close to 1, and observe that $d(x, w_0) \geq b\delta$ for any $x \in B(z_0, 3b\delta) \setminus B(w_0, b\delta)$, hence by (3.12)

$$u(w_0) - u(x) \leq \Omega(d(x, w_0)) \leq \frac{A}{|\log d(x, w_0)|^\alpha} = \frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha} \Omega(\delta) \leq \hat{C} \Omega(\delta), \quad (3.14)$$

where $\hat{C} > 0$ is an upper bound of $\frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha}$ for all $\delta \in (0, \theta\delta_0/\beta_n]$, which is uniform. By taking $\beta_n$ large enough (depending only on $n$ and $p$), we can choose $\hat{C}$ arbitrarily close to 1, since $\lim_{\delta \to 0} \frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha} = 1$. Thus we can assume that $\hat{C} < 1 + \frac{1}{100^\alpha}$, say. (3.14) yields that

$$u(x) \geq u(w_0) - \hat{C} \Omega(\delta), \quad \forall x \in B(z_0, 3b\delta) \setminus B(w_0, b\delta). \quad (3.15)$$
By the definition of $\rho_{3b\delta}u$ in (2.1), we have
\[
\rho_{3b\delta}u(z_0) = \frac{1}{(3b\delta)^{2n}} \int_{\zeta \in T_{z_0}X} u(\exp_{z_0}(\zeta)) \rho\left(\frac{|\zeta|_{\omega}^2}{(3b\delta)^2}\right) dV_{\omega}(\zeta)
\]
\[
\geq \frac{1}{(3b\delta)^{2n}} \int_{F} u(\exp_{z_0}(\zeta)) \rho\left(\frac{|\zeta|_{\omega}^2}{(3b\delta)^2}\right) dV_{\omega}(\zeta) + (1 - \varepsilon_0)(u(w_0) - \hat{C}\Omega(\delta)),
\]
where $F \subset T_{z_0}X$ is the inverse image of $B(w_0, b\delta)$ under $\exp_{z_0}: T_{z_0} \rightarrow X$, and
\[
\varepsilon_0 = \frac{1}{(3b\delta)^{2n}} \int_{F} \rho\left(\frac{|\zeta|_{\omega}^2}{(3b\delta)^2}\right) dV_{\omega}(\zeta) \in [0, 1].
\]
Note that we can choose $\varepsilon_0 \geq \frac{1}{3^{2n}}$. By Gauss’ Lemma, $|\zeta|_{\omega}^2 \leq (b + 1)^2\delta^2$ for any $\zeta \in F$. By the choice of the kernel function $\rho$, we have $\rho\left(\frac{|\zeta|_{\omega}^2}{(3b\delta)^2}\right) = \text{const}$ for such $\zeta$. So
\[
\frac{1}{(3b\delta)^{2n}} \int_{F} u(\exp_{z_0}(\zeta)) \rho\left(\frac{|\zeta|_{\omega}^2}{(3b\delta)^2}\right) dV_{\omega}(\zeta) = \varepsilon_0 \frac{1}{\mu(B(w_0, b\delta))} \int_{B(w_0, b\delta)} u(z) d\mu(z),
\]
where $d\mu = (\exp_{z_0})_{\ast}dV_{\omega(z_0)}$ is the pushforward of the “Euclidean measure” in $T_{z_0}X$ to $X$ under the exponential map $\exp_{z_0}$. We observe that $B(w_0, b\delta)$ can be viewed as a domain in the normal coordinates chart at $z_0$, and under this coordinates system, the measure $\mu$ differs from the Euclidean one by $C\delta$ (for some uniform $C = C(\omega_0)$). Moreover, $u + \varphi_{z_0}$ is pluri-subharmonic for some local potential $\varphi_{z_0}$ of $\omega_0$ which satisfies $|\varphi_{z_0}| \leq C\delta$ (e.g. consider $\varphi_{z_0} - \varphi_{z_0}(w_0)$ if necessary). Then by the standard mean-value inequality for subharmonic functions in Euclidean space, we get
\[
\varepsilon_0 \frac{1}{\mu(B(w_0, b\delta))} \int_{B(w_0, b\delta)} u(z) d\mu(z) \geq \varepsilon_0 u(w_0) - C_5\delta,
\]
where $C_5 > 0$ is a constant depending only on $n, \omega_0$. Combining the above we get
\[
\rho_{3b\delta}u(z_0) \geq u(w_0) - C_5\delta - (1 - \varepsilon_0)\hat{C}\Omega(\delta) = u(z_0) - C_5\delta + (1 - (1 - \varepsilon_0)\hat{C})\Omega(\delta) \geq u(z_0) - C_5\delta + \frac{1}{100^n}\Omega(\delta),
\]
where in the last inequality, we use the choices of $\varepsilon_0 \geq 4^{-2n}$ and $\hat{C} \leq 1 + \frac{1}{100^n}$. Combined with (3.11), this yields that
\[
\frac{C_4}{|\log 3b\delta|^{\alpha}} + C_5\delta \geq 10^{-2n}\Omega(\delta) = 10^{-2n} \frac{A}{|\log \delta|^{\alpha}}, \quad (3.16)
\]
If at the beginning we choose $A > 0$ to be
\[
A = 1 + \left|\log \frac{\theta_{\delta_0}}{\beta_n}\right|^{\alpha} \Omega\left(\frac{\theta_{\delta_0}}{\beta_n}\right) + \sup_{\delta \in (0, b\delta_0/b_n]} 10^{2n}|\log \delta|^{\alpha} \left(\frac{C_4}{|\log 3b\delta|^{\alpha}} + C_5\delta\right) \quad (3.17)
\]
9
The proof of Theorem 1 is complete.

**Example 1.** Let \( D \subset \mathbb{C} \subset \mathbb{C}P^1 \) be the disk with radius 1/2. Consider the function \( \varphi(z) = (-\log |z|^2)^{-a} \) for some \( a > 0 \) defined on \( D \) where \( z \) is the standard coordinate on \( \mathbb{C} \). Straightforward calculations show that on \( D \setminus \{0\} \)

\[
i\partial \bar{\partial} \varphi = a(a + 1)\frac{idz \wedge d\bar{z}}{|z|^2(-\log |z|^2)^{a+2}} = e^F.
\]

It is easy to see that \( e^F \in L^1(\log L)^p(D) \) for any \( p < a + 1 \). The exponent \( a \) in Theorem 1 is \( \alpha = p - 1 \) in this case. This example shows that the exponent \( \alpha \) is sharp. Moreover, \( \varphi \) is not Hölder continuous for any exponent.

Though \( \varphi \) is singular in this example, we can consider a regularization of \( \varphi \), for example, \( \varphi_\epsilon(z) = (-\log (\epsilon + |z|^2))^{-a} \) for \( \epsilon \to 0^+ \) to get a smooth example. We can also glue \( \varphi \) or \( \varphi_\epsilon \) to the whole space \( \mathbb{C}P^1 \) to get an example on a compact Kähler manifold.

**Example 2.** Let \( (X, \omega_0) \) be a compact Kähler manifold and \( L \to X \) be a holomorphic line bundle over \( X \). Suppose \( s \in \mathcal{H}^0(X, \mathcal{O}_X(L)) \) is a nonzero holomorphic section of \( \mathcal{O}_X(L) \). Consider the following complex Monge-Ampère equations (for \( \epsilon \in (0, 1) \))

\[
(\omega_0 + i\partial \bar{\partial} u_\epsilon)^n = \frac{C_\epsilon}{(\epsilon + |s|^2_h)(-\log (\epsilon + |s|^2_h))^{a+1}} \omega_0^n,
\]

where \( h \) is a Hermitian metric on \( L \) such that \( |s|^2_h < 1, C_\epsilon > 0 \) is a normalizing constant so that the equation is solvable, and \( a > 0 \) is a constant. Note that the function on the RHS belongs to \( L^1(\log L)^p \) for any \( p < a + 1 \). So for any fixed \( a > n - 1 \), Theorem 1 implies a uniform estimate on the modulus of continuity of \( u_\epsilon \). Note that this estimate still holds for \( \epsilon = 0 \), if we interpret the complex Monge-Ampère equation in the Bedford-Taylor sense.

## 4 Hölder estimates

We provide now a sketch of the proof of the Hölder continuity stated in the Introduction. Since the proof is analogous to that of Theorem 1, we shall only point out the major differences. We use the same notations and definitions as before.

We assume \( e^F \in L^q(X, \omega_0^n) \) for some \( q > 1 \). Denote \( \alpha_0 = \frac{2}{1+(n+1)q} \) where \( q^* = \frac{q}{q-1} \).

The sets \( E_\delta \) in (3.1) is replaced by \( E_\delta = \{ u \leq -2\delta^{\alpha_0} + (1-r)u_\delta - s \} \) for \( s \geq 0 \), and here \( r = \delta^{2-\alpha_0}/(n+1)q^* \). With these choices, Lemma 3 holds as \( \int_{E_\delta} e^F u_0^n \leq C\delta^{(2-\alpha_0)/q^*} \) by Hölder inequality. Lemma 4 continues to hold by the choice of \( r \). We can proceed exactly as before to conclude (3.9) with any \( \alpha_0 < 1 \), since in this case we can take \( p \) as large as we like. The same iteration argument gives that \( u_\delta - u \leq C\delta^{\alpha_0} \). In the choice of \( c \) in \( U_{c,\delta} \) we can take \( c = \delta^{\alpha_0} \). This will give \( \rho_{\theta\delta} u - u \leq C\delta^{\alpha_0} \) for some uniform \( \theta \in (0, 1) \) and any \( \delta \in (0, \delta_0] \) for some uniform \( \delta_0 \in (0, 1] \). Then it suffices to finish the proof of Hölder continuity of \( u \) by invoking the estimates in [6] or the direct arguments in our proof of Theorem 1.
5 Geometric applications

In this section, we apply a trick from [14] to see that the uniform continuity of the solution to (1.1) leads to the diameter bound of the Kähler metric \( \omega_u = \omega_0 + i\partial\bar{\partial}u \), where \( u \) satisfies (1.1).

Recall a function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is called Dini continuous, if \( \int_0^1 \frac{f(r)}{r} dr < \infty \). As before, we denote \( \Omega(r) = \sup_{d(x,y) \leq r} |u(x) - u(y)| \) to be the modulus of continuity of \( u \), which is the oscillation of \( u \) over geodesic balls of radius \( r \).

Lemma 5 On the Kähler manifold \( (X, \omega_0) \), let \( u \) be a smooth and strictly \( \omega_0 \)-PSH function. If \( \sqrt{\Omega} \) is Dini continuous, then the diameter of the Kähler manifold \( (X, \omega_u) \) is bounded by a constant depending on \( \omega_0 \) and \( \int_0^1 \frac{\sqrt{\Omega(r)}}{r} dr \).

Proof. Since \( (X, \omega_0) \) is compact, we can take a finite open cover \( \{U_a\}_{a=1}^N \), where each \( U_a \) is a bounded domain in \( \mathbb{C}^n \), and without loss of generality we assume each \( U_a \) is biholomorphic to the Euclidean ball \( B_{\mathbb{C}^n}(0,2) \) and \( \{\frac{1}{2}U_a\} \) also covers \( X \). It is clear that \( \omega_0|_{U_a} \) is equivalent to \( \omega_{\mathbb{C}^n}|_{U_a} \). For notational convenience, in the proof of this lemma we write \( B_r(z) = B_{\mathbb{C}^n}(z,r) \) and \( \omega_E = \omega_{\mathbb{C}^n} \).

We consider the function \( \rho(z) = d_{\omega_u}(z,0) \), which is a Lipschitz function. We fix a cut-off function \( \chi : \mathbb{R}_+ \to [0,1] \) such that \( \chi(x) = 1 \) for \( x \in [0,1] \) and vanishes on \( [2, \infty) \). Following [14], we look at the integral of \( |\nabla \rho|^2 \omega_{\omega_0} \). For any fixed \( r < 1 \) and any \( p \in \frac{1}{2}U_a \cong B_1(0) \), we have

\[
\int_{B_r(p)} |\nabla \rho|^2 \omega_{\omega_E} \leq \int_{B_r(p)} |\nabla \rho|^2 \omega_u (\text{tr}_{\omega_u} \omega_u) \omega_{\omega_E} = \int_{B_r(p)} (n + \Delta_{\omega_E} u) \omega_{\omega_E}^n \\
\leq C r^{2n} + \int_{B_{2r}(p)} \Delta_{\omega_E} \chi \left( \frac{d_{\omega_E}(z,p)}{r} \right) \cdot (u(z) - u(p)) \omega_{\omega_E}^n \\
\leq C r^{2n} + C r^{2n-2} \Omega(2r),
\]

where in the second line we apply the integration by parts. By Poincaré inequality it follows that

\[
\int_{B_r(p)} (\rho - \rho_{r,p})^2 \omega_{\omega_E}^n \leq r^2 \int_{B_r(p)} |\nabla \rho|^2 \omega_{\omega_E}^n \leq C r^2 + C \Omega(r), \tag{5.1}
\]

where \( \int_{B_r(p)} f \) denotes the average of \( f \) over the ball \( B_r(p) \), \( \rho_{r,p} = \frac{1}{|B_r(p)|} \rho \omega_{\omega_E}^n \), and in the last inequality we have applied \( \Omega(2r) \leq 2\Omega(r) \) which follows from the triangle inequality.

We now follow closely the proof of the classical Morrey’s lemma in PDE theory. By Hölder inequality and (5.1)

\[
|\rho_{r,p} - \rho_{r/2,p}| \leq \int_{B_{r/2}(p)} |u(z) - \rho_{r,p}| \omega_{\omega_E}^n \leq C r + C \sqrt{\Omega(r)}. \tag{5.2}
\]
We apply (5.2) with \( r = 2^{-j} \) for \( j = 1, 2, 3, \ldots \). Then
\[
|\rho_{2^{-j},p} - \rho_{2^{-j-1},p}| \leq C 2^{-j} + C \Omega(2^{-j})^{1/2}. \tag{5.3}
\]
Under the assumption that \( \sqrt{\Omega(r)} \) is Dini continuous, we see that the series \( \hat{\rho} = \sum_{j=1}^{\infty} (\rho_{2^{-j},p} - \rho_{2^{-j-1},p}) \) converges absolutely, and \(|\hat{\rho}|\) is uniformly bounded, since \( \sum_j 2^{-j} \) converges and \( \sum_j \Omega(2^{-j})^{1/2} \leq 2 \int_0^1 \frac{\sqrt{\Omega(t)}}{t} \, dt < \infty \). By Lebesgue differentiation theorem it is clear that
\[
d_{\omega_u}(p,0) = \rho(p) = \lim_{j \to \infty} \rho_{2^{-j},p} = \rho_{1/2,p} - \hat{\rho}. \tag{5.4}
\]
To get the desired bound on \( d_{\omega_u}(p,0) \) it suffices to estimate \( \rho_{1/2,p} \). To this end, we observe that the inequalities above are uniform for any \( p \in B_1(0) \). In particular we can apply (5.2) and (5.3) with \( r = 3 \cdot 2^{-j} \) for \( j = 1, 2, \ldots \) and \( p = 0 \) to conclude that
\[
d_{\omega_u}(0,0) = 0 = \rho_{3/2,0} - O(1)
\]
where \( O(1) \) denotes a uniformly bounded constant. This gives the bound on \( \rho_{3/2,0} = \int_{B_{3/2}} \rho \). Finally for any \( p \in B_1(0) \), we have \( B_{1/2}(p) \subset B_{3/2}(0) \) by triangle inequality, hence
\[
\rho_{1/2,p} = \int_{B_{1/2}(p)} \rho \omega^n_E \leq C \int_{B_{3/2}(0)} \rho \omega^n_E = C \rho_{3/2,0}
\]
is uniformly bounded, as desired. Combined with (5.4), this gives the expected bound on \( d_{\omega_u}(p,0) \) for any \( p \in B_1(0) \). Since finitely many these balls cover \( (X, \omega_0) \), we get the diameter bound of \( (X, \omega_u) \). The proof of the lemma is complete.

**Proof of Theorem 2.** Let \( u \) be the solution to (1.1). Suppose \( p > 3n \), then
\[
\alpha = \min\left\{ \frac{p-n}{n}, \frac{p}{1+n} \right\} > 1.
\]

Theorem 1 implies that \( |u(x) - u(y)| \leq \frac{C}{|\log d(x,y)|^{\alpha}} \) for any \( x, y \in X \). So we have for the modulus of continuity of \( u \), \( \Omega(r) \leq \frac{C}{|\log r|^{\alpha}} \). It is now elementary to see that
\[
\int_0^{1/2} \frac{\sqrt{\Omega(r)}}{r} \, dr \leq \int_{\log 2}^{\infty} \frac{C}{t^{\alpha}} \, dt < \infty.
\]
We can now apply Lemma 5 to conclude the uniform diameter bound of \( (X, \omega_u) \).

**Example.** In **Example 2** at the end of Section 3, if \( a > 3n - 1 \), Theorem 2 implies a uniform diameter bound of the Kähler metrics \( \omega_\epsilon = \omega_0 + i \partial \bar{\partial} u_\epsilon \), which is independent of \( \epsilon \in (0,1] \).

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