Elastodynamic modeling and parameter sensitivity analysis of a parallel manipulator with articulated traveling plate

Binbin Lian • Lihui Wang • Xi Vincent Wang

Received: 3 May 2018 / Accepted: 26 December 2018
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Abstract
This paper deals with the elastodynamic modeling and parameter sensitivity analysis of a parallel manipulator with articulated traveling plate (PM-ATP) for assembling large components in aviation and aerospace. In the elastodynamic modeling, the PM-ATP is divided into four levels, i.e., element, part, substructure, and the whole mechanism. Herein, three substructures, including translation, bar, and ATP, are categorized according to the composition of the PM-ATP. Based on the kineto-elastodynamic (KED) method, differential motion equations of lower levels are formulated and assembled to build the elastodynamic model of the upper level. Degrees of freedom (DoFs) at connecting nodes of parts and deformation compatibility conditions of substructures are considered in the assembling. The proposed layer-by-layer method makes the modeling process more explicit, especially for the ATP having complex structures and multiple joints. Simulations by finite element software and experiments by dynamic testing system are carried out to verify the natural frequencies of the PM-ATP, which show consistency with the results from the analytical model. In the parameter sensitivity analysis, response surface method (RSM) is applied to formulate the surrogate model between the elastic dynamic performances and parameters. On this basis, differentiation of performance reliability to the parameter mean value and standard variance are adopted as the sensitivity indices, from which the main parameters that greatly affect the elastic dynamic performances can be selected as the design variables. The present works are necessary preparations for future optimal design. They can also provide reference for the analysis and evaluation of other PM-ATPs.

Keywords Parallel manipulator • Articulated traveling plate • Elastodynamic modeling • Parameter sensitivity

1 Introduction
Parallel manipulator with articulated traveling plate (PM-ATP) is one of the most well-recognized mechanisms in the research community of parallel manipulators [1, 2]. The common parallel manipulator is composed of one fixed base, one moving platform, and several kinematic chains linking to them. The moving platform is usually a single plate. Alternatively, the articulated traveling plate (ATP) is a special type of moving platform consisting of two or more in-parts and one out-part [3]. Besides the mobility provided by the kinematic chains, PM-ATP gains extra motions from the ATP. Therefore, PM-ATPs are more flexible in terms of motion capability. One typical example of PM-ATPs is the parallel manipulator with Schönflies motion (three translations and one rotation around the vertical axis, i.e., 3T1R) whose rotation is provided by the relative translation of the two in-parts. The well-known Par4 [4], I4 [5], and the four degree of freedom (DoF) parallel robot [6] belong to this group of PM-ATPs. The extra rotation from the in-parts is up to 720 degree, making the PM-ATP attractive to posture changing of disorder products. In practice, these PM-ATPs have been successfully applied for the high-speed pick-and-place in food packaging, medicine, and semiconductor manufacturing.

Inspired by the successful applications of parallel manipulators with Schönflies motion, more and more attention has been drawn to the investigation of PM-ATPs. Referring to the industrialization of conventional parallel manipulators in the sequence of topology innovation [7, 8], optimal design [9–13], calibration, and control [14–18], the developments of PM-ATPs start from the topology synthesis. The aim is to invent new PM-ATPs for wider industrial scenarios [19, 20].
In this regard, Sun [3] discussed the kinematic constraints within the ATP and proposed a group of PM-ATPs that can be applied as tracking mechanism, docking equipment, or machine tools. The parameterized topological models were further analyzed [21–23]. By filling in the gap between finite and instantaneous screw theory, Sun [24, 25] succeeded in connecting topology analysis to the following performance analysis and even optimal design, which is a milestone in the topology synthesis of parallel manipulators.

A PM-ATP (details shown in Section 2) is then selected from the topology synthesis and used as a pose-adjusting mechanism for assembling large components in aviation and aerospace. The next problem for the development of the PM-ATP is the optimal design that builds an actual mechanism from certain topological structure. The concerned performances would be optimized by adjusting the structural parameters which are regarded as design variables. Therefore, the two essential elements for the optimal design are the performance and the structural parameters.

The commonly concerned performance indices of parallel mechanisms include workspace [26], singularity avoidance [27, 28], stiffness [29, 30], and dynamic [31–33]. Since the studied PM-ATP is targeted for assembling large components, elastodynamic performance [34, 35] catering on large load-carrying, lightweight structure and small deflections is of importance. Performance indices such as natural frequency or elastic deformations can be adopted as objectives in the optimal design, which require for the mapping model between the elastodynamic performances and the structural parameters.

The existing elastodynamic modeling methods are mainly for the high-speed pick-and-place PM-ATPs whose links are made of light weight material. The elastic deformations of the links are coupled with the pick-and-place motions, resulting in an ongoing trend of applying flexible multibody dynamic methods to model the elastodynamic performances [36–38]. Although the obtained elastodynamic models are with high accuracy, the modeling process is computationally expensive due to the nonlinear couplings between link deformations and rigid motions. For the rigid structures moving in a low speed, however, elastic deformations of parts are much smaller than the rigid body motions. The deformations are assumed not to affect the mechanism motions and the kineto-elastodynamic (KED) method can be applied [39, 40]. In the KED framework, the motions of PM-ATP are firstly analyzed by the rigid body kinematics, and then the elastic deformations computed by the structural dynamics at each instantaneous moment are added. The modeling procedure is greatly simplified while the effectiveness in describing the elastodynamic performance of the whole mechanism can be kept.

The KED method is applicable under the assumption that the parts are relatively rigid and the mechanism moves slowly. Since the pose-adjusting motions of the studied PM-ATP are relatively slow and the parts are designed towards high stiffness, the KED method can be adopted. Two challenges need to be addressed in the elastodynamic modeling of the studied PM-ATP by the KED method. (1) The parts are usually with irregular shapes, increasing difficulty in analytically computing the dynamic behavior. (2) The ATP contains complex structures and multiple joints, complicating the whole system.

In order to address the modeling difficulties, the studied PM-ATP is divided into four levels, i.e., element, part, substructure, and the whole mechanism. By applying the element as basic unit, the elastic deformation of irregular parts can be captured. The elastodynamic model of each level is built by the KED method and the model of upper level is assembled by the model of lower level. The obtained elastodynamic model of the PM-ATP will be verified by simulations in finite element software and elastic dynamic experiments in the following sections. Herein, the relation of structural parameters and elastodynamic performance are the major concern in the modeling. Noises [41, 42] that impose influence on the dynamic performance in practical application are not included. They are regarded as system disturbances and solved by in the controller development after the optimal design of the PM-ATP.

However, the optimal design is still challenging if the elastodynamic model is directly applied. Large amounts of structural parameters are involved because of the irregular parts, the complex structures, and the compositions of the PM-ATP. Parameter sensitivity analysis is usually implemented to exclude the trivial parameters and simplify the model [43]. By analyzing the change of the output performances when varying the input parameters, parameter sensitivity identifies the effects of parameters to the performances. Parameters with high sensitivity impose more influence to the performance and should be chosen as design variables while the parameters with low sensitivity can be eliminated in the optimal design.

Current parameter sensitivity methods mainly compute performance changes by randomly changing the values of one parameter at a time, in which the rest of the parameters remain unchanged [44, 45]. The coupling effects of multiple parameters are ignored in these methods. In order to efficiently and effectively analyze parameter sensitivity, more comprehensive parameter sensitivity indices are required. They should (1) consider the possible coupling effects among parameters, and (2) evaluate the change of each parameter via statistic technique, for instance mean value and standard variance.

In summary, this paper focuses on the preparations for the optimal design of a PM-ATP, i.e., the elastodynamic modeling and the parameter sensitivity analysis. The difficulties of this work are resulted from the complicated composition and the substantial parameters. To illustrate the adopted methods, this paper is organized as follows. Section 2 briefly introduces the PM-ATP and carries out the inverse kinematics analysis. Based on the KED method, elastodynamic modeling of the
PM-ATP is implemented in Section 3. Translation, bar, and ATP substructures are assigned to assemble the dynamic model of the whole mechanism. In Section 4, simulation and experiment are conducted to verify the elastodynamic model. Section 5 is dedicated to the parameter sensitivity analysis, from which the main parameters are identified and selected as design variables in future optimal design. Conclusions are drawn in Section 6.

2 Mechanism description and inverse kinematics

The studied PM-ATP is named as PaQuad PM (Fig. 1a). The PaQuad PM is composed of a fixed base, an ATP, and four identical PRS limbs. Herein, P, R, and S denote actuated prismatic, revolute, and spherical joint. The PRS limbs connect to the fixed base and the ATP by P joint and S joint, respectively. The ATP consists of in-part 1, in-part 2, and out-part. The in-part 1 links to the out-part through helical (H) joint, whereas the in-part 2 joins to out-part by R joint. The axes of the H joint and R joint are collinear. According to the mobility analysis, the ATP has one translational and two rotational capabilities provided by the PRS limbs. Additionally, the relative translation of in-part 1 and in-part 2 results in the rotation of H joint, which adds an extra rotation to the out-part. Hence, the PaQuad PM has one translation and three rotations.

In order to formulate the kinematic model of the PaQuad PM, some denotations and coordinate frames are defined as shown in Fig. 1b. Point Ai is assigned to the connecting point of the ith (i = 1, 2, ⋯, 4) PRS limb and the fixed base. The fixed base is defined by a circle whose center is point O and radius is a. Points Bi, Ci, and Di denote centers of P joint, R joint, and S joint, respectively. The lengths of in-part 1 (DiDi) and in-part 2 (DiDi) are both 2b and the vertical distance between them is e. The traveling distance of P joint and the length of bar are represented by qi and l. A fixed reference frame O-xyz is assigned to point O. The x-axis is collinear with OB2 and the z-axis is vertical to the fixed base. A moving reference frame O-uvw is attached to the point O on the out-part. Its w-axis points to the same direction as H joint and the u-axis is parallel to DA2 at home position.

The rotation matrix of frame O-uvw with respect to frame O-xyz can be computed by

\[
R = R_{\alpha,\beta,\gamma} = [u\ v\ w]=\begin{bmatrix}
\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\
\sin\gamma & \cos\beta\cos\gamma & -\sin\beta \\
\cos\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha
\end{bmatrix}
\]

where α, β, and γ are the three Euler angles. c and s denote cosine and sine, respectively.

Considering the projection of point Di on the O-uvw plane, the position vector of point Di in frame O-uvw can be expressed as

\[
d_{i\alpha'} = (-\sin\gamma \ -b\cos\beta \ -d_0 \ -e)^T,
\]

\[
d_{i\beta'} = (b\cos(\gamma_1 + \gamma) \ -b\sin(\gamma_1 + \gamma) \ -d_0)^T
\]

\[
d_{i\gamma'} = (b\sin(\gamma_1 + \gamma) \ -b\cos(\gamma_1 + \gamma) \ -d_0)^T,
\]

\[
d_{i\beta'} = (-b\cos(\gamma_1 + \gamma) \ b\sin(\gamma_1 + \gamma) \ -d_0)^T
\]

where γ1 denotes additional rotation angle produced by Euler angles α and β.

Point Di can also be described in frame O-xyz as

\[
d_i = Rd_{i\alpha'} + r_{i\alpha'}, \quad i = 1, 2, \cdots, 4
\]

where \(r_{i\alpha'} = (x_{i\alpha'} y_{i\alpha'} z_{i\alpha'})^T\).

The point Di moves within the plane spanned by s1i and s3i due to the limitation of the R joint, which can be mathematically described by

\[
d_i^T s_{2i} = 0, \quad i = 1, 2, \cdots, 4
\]

where

\[
s_{21} = (1 \ 0 \ 0)^T, \quad s_{22} = (0 \ 1 \ 0)^T, \quad s_{23} = (-1 \ 0 \ 0)^T,
\]

\[
s_{24} = (0 \ -1 \ 0)^T
\]

Substituting Eq. (1) and Eq. (2) into Eq. (3) yields

\[
\begin{align*}
x_{i\alpha'} &= (d_0 + e)\sin\beta \\
y_{i\alpha'} &= -d_0 \sin\alpha\cos\beta \\
\gamma_1 &= \arctan(\tan\alpha\sin\beta)
\end{align*}
\]

where \(e = \frac{P_\theta}{2\pi} + e_0\). \(P_\theta\) denotes screw pitch, \(e = e_0\) when PaQuad PM is at home position. \(d_0\) represents distance between point O and D2D4.
The closed-loop equation can be formulated as
\[ \mathbf{r}_{ji} + \mathbf{d}_j = a_i + qs_{1,j} + ls_{3,j}, \quad i = 1, 2, \cdots, 4 \] (5)
where
\[ \mathbf{d}_j = \mathbf{R} \mathbf{d}_{i'}, \quad a_i = \begin{pmatrix} \sin \phi_i & -\cos \phi_i & 0 \end{pmatrix}^T, \quad \phi_i = (i-1)\pi/2, \]
\[ s_{1,j} = (0 \quad 0 \quad 1)^T. \]

For the inverse kinematics of the PaQuad PM, the \( z \) value of point \( O' \) and the three Euler angles \((\alpha, \beta, \gamma)\) are the known parameters. By solving Eq. (1) to Eq. (5), the traveling distance of the \( P \) joint is obtained as follows
\[ q_i = z_{O'} - \sqrt{R^2 - M^2 - N^2}, \quad i = 1, 2, \cdots, 4 \] (6)
where \( M = x_{0'} + d_{x'} - \sin \phi_i, N = y_{0'} + d_{y'} + \cos \phi_i, \)
\( R, G, \) and \( L \) are made as follows. (1) The rigid body motions and the elastic deformations of the substructures are concerned. (2) The Euler-Bernoulli beam is applied to formulate the elastic model of the whole mechanism. (3) The first-order differentiation of \( U \) and \( \theta_1 \) are the first-order differentiation of \( U \) and \( \theta_1 \) and \( V' \) and \( W' \) are the second-order differentiation of \( V \) and \( W \).

In addition, gravity potential energy is given by
\[ E_{p2} = \frac{1}{2} \rho g T r_{p2} \] (9)
where \( \rho, g, \) and \( r_{p2} \) are density, gravity acceleration, and vector from point \( O \) to point \( E_1 \) in frame \( E_1-xyz \).

Moreover, kinetic energy can be expressed as
\[ E_k = \frac{1}{2} \rho u_{0} \cdot \mathbf{G} \cdot \mathbf{u}_{0} \cdot \mathbf{d} \mathbf{r} \]
\[ \mathbf{u}_{0} = \mathbf{N} (\mathbf{u} + \mathbf{u}) = \mathbf{N} \mathbf{u} \]
Here, \( \mathbf{u}_{0} \) is the absolute velocity of point \( E_0 \). \( \mathbf{u} \) and \( \mathbf{u} \) denote velocities of rigid body motion and elastic deformation.

Hence, Lagrange equation is formulated as
\[ \frac{d}{dt} \left( \frac{\partial E_k}{\partial \dot{x}} \right) - \frac{\partial E_k}{\partial x} + \frac{\partial E_p}{\partial \dot{x}} = \mathbf{f} \]
where \( E_p = E_{p1} + E_{p2}, \) \( \mathbf{f} \) is the vector of external forces.

Differential motion equation of the beam can be formulated by substituting Eq. (8) to Eq. (10) into Eq. (11) as follows.
\[ \mathbf{M}^c \dot{\mathbf{U}}^c + \mathbf{K}^c \mathbf{U}^c = \mathbf{F}^c + \mathbf{Q}^c \]
(12)
where \( \mathbf{U}^c \) is the generalized coordinates of nodes. \( \mathbf{M}^c, \mathbf{K}^c, \mathbf{F}^c, \) and \( \mathbf{Q}^c \) are the mass matrix, stiffness matrix, and vector of external and internal forces in frame \( O-\tau \).

\[ \mathbf{M}^c = \mathbf{T}^T \mathbf{m} \mathbf{T}, \quad \mathbf{K}^c = \mathbf{T}^T \mathbf{k} \mathbf{T}, \quad \mathbf{F}^c = \mathbf{T}^T \mathbf{f}_1, \]
\[ \mathbf{Q}^e = \mathbf{T}^T \mathbf{f}_2 - \frac{1}{2} \rho A \mathbf{T}^T \mathbf{N}^T \mathbf{g} \mathbf{d} \mathbf{r} - \mathbf{M}^e \dot{\mathbf{U}}^e \]
\[ \mathbf{T} = \text{diag}(\mathbf{R}^c, \mathbf{R}^c, \mathbf{R}^c, \mathbf{R}^c) \]

herein \( \mathbf{T} \) is the transformation matrix of frame \( E_1-xyz \) with respect to frame \( O-\tau \xyz. \)

\[ \frac{\partial \mathbf{U}^c}{\partial x} \]
\[ \frac{\partial \mathbf{U}^c}{\partial y} \]
\[ \frac{\partial \mathbf{U}^c}{\partial z} \]
3.2 Differential motion equation of substructures

The differential motion equation of the beam is applied to assemble the elastic dynamic of the parts by considering DoFs of the connecting points. Then, the parts will be used to construct the substructure. The PaQuad PM is divided into three substructures, i.e., translation, bar, and ATP. The first two substructures are from the PRS limb, where the former is the main body of the P joint and the latter is the connecting structure of the R and S joints. The effects of P, R, and S joints will be taken into account in the mechanism level. Unlike the other two substructures, ATP contains internal joints (H and R joints) whose influences need to be addressed in the substructure level.

3.2.1 Translation substructure

The P joint is composed of screw pair and guide slider, as is shown in Fig. 3. Ball screw and slider are the major components to deform; thus, they are assumed to be elastic and represented by two and three beam elements, respectively. The elements are named from iE1 to iE5 and the seven nodes are denoted by iN1 to iN7. Element coordinate frames are firstly established for establishing the differential motion equation of elements. Frame iN1 - xiyiz is assigned to node iN1. The xi-axis is collinear with the rotational axis of ball screw, the yi-axis is parallel to the x-axis in frame O - xyz, and zi-axis satisfies right hand rule. Frames of element iE2, iE3, and iE4 are parallel to frame iN1 - xiyiz. Frame iN7 - xiyiz is established at node iN7, whose xi-axis points to the axis of element iE5 and zi-axis is in the same direction as z-axis of O - xyz.

Based on Eq. (12), the differential motion equation of ball screw can be computed as

\[
M_{iP1} \mathbf{U}_{iP1} + K_{iP1} \mathbf{U}_{iP1} = F_{iP1} + Q_{iP1}
\]  

(13)

where \( \mathbf{U}_{iP1} = [U_{iN1} \ldots U_{iN18}]^T \) is the vector of generalized coordinates of ball screw. \( M_{iP1}, K_{iP1}, F_{iP1}, \) and \( Q_{iP1} \) are...
Thus, the five elastic deformations of node motor. The other end is linked to the fixed base by bearings. Restricted because one end of the ball screw is fixed to the servo motor. As referred to Appendix.

Next, the boundary conditions and relations of connecting points are analyzed. The elastic deformations of node N1 are restricted because one end of the ball screw is fixed to the servo motor. The other end is linked to the fixed base by bearings. Thus, the five elastic deformations of node N3 are zeros except for the rotation about the screw axis. Node N2 of the ball screw and node N5 of the slider are connected by H joint, hence

\[
U_{N25} = U_{N7} + \frac{p_h}{2\pi} (U_{N28} - U_{N10})
\]

\[
(U_{N8} \quad U_{N9} \quad U_{N11} \quad U_{N12}^T) = (U_{N26} \quad U_{N27} \quad U_{N29} \quad U_{N30})^T
\]

where \(p_h\) is the pitch of ball screw.

Finally, the differential motion equation of translation substructure is obtained by Eq. (13) to Eq. (16) as

\[
M_i p \ddot{U}_i + K_i U_i = F_i + Q_i, \quad i = 1, \cdots, 4
\]

where \(U_i\) is generalized coordinates of translation substructure. \(U_i, M_i, K_i, F_i,\) and \(Q_i\) are shown under Fig. 3, in which \(B_{ip}\) is referred to Appendix.

### 3.2.2 Bar substructure

According to the features of the bar, ten elements with nine nodes are assigned to the bar, as is shown in Fig. 4. The element coordinate frames \(x_i, y_i, z_i (i = 5, \cdots, 8; j = 1, \cdots, 10)\) are defined, where the \(x_i\)-axis is along the length of the elements. Differential motion equation of the bar is expressed as

\[
M_i p \ddot{U}_i + K_i U_i = F_i + Q_i, \quad i = 1, \cdots, 4
\]
are assigned to point $D_i (i = 1, 2, \cdots, 4)$. The $x_{R_i}$-axis is collinear with direction of element length and $z_{R_i}$-axis is perpendicular to the plane of in-parts. For the elements R5, R6, and R8, frame $O - x_{R_i}y_{R_i}z_{R_i} (i = 5, 6)$ is defined at point $O'$. The $x_{R_i}$-axis is along the axis of screw, and $z_{R_i}$-axis and $z_{R_6}$-axis are parallel to $x_{R_5}$-axis and $x_{R_1}$-axis.

Next, differential motion equation of each component can be formulated by the elements as

$$ M_{R_i} \ddot{U}_{R_i} + K_{R_i} U_{R_i} = F_{R_i} + Q_{R_i}, \quad i = 1, \cdots, 4 \quad (19) $$

$$ M_{R_7} \ddot{U}_{R_7} + K_{R_7} U_{R_7} = F_{R_7} + Q_{R_7} \quad (20) $$

where $U_{R_i}$ is the generalized coordinates of components; $M_{R_i}$, $K_{R_i}$, $F_{R_i}$, and $Q_{R_i}$ are the mass matrix, stiffness matrix, and vector of external and internal forces. Differential motion equation of R7 and R8 is formulated in frame $O' - x_{R_5}y_{R_6}z_{R_6}$.

The relations at connecting nodes of each part are then assessed by the assembling conditions as shown in Fig. 5. Node R8 and node R7N3 have the same generalized coordinates since the out-part and the screw is fixed. Node R8 and node R5 are linked by R joint; thus, the five coordinates are the same except for the angular deformation about rotational axis. Cylindrical joint is formed between node R5 and node R6, and the coordinates about/along the joint axis are different. Node R6 and node R7 is connected by H joint. Node R6 is attached to node R1 and R3 rigidly, while node R5 is fixed to node R2 and R4. These assembling conditions can be mathematically expressed as

$$ [U_{R_{5N1}} \cdots U_{R_{5N6}}]^T = [U_{R_{7N1}} \cdots U_{R_{7N18}}]^T \quad (22) $$

$$ \begin{bmatrix} R_{56} & 0 \\ 0 & R_{56} \end{bmatrix} U_{R_5} = U_{R_6}, \quad U_{R_{5N1}} = U_{R_{5N4}} = U_{R_{6N1}} = U_{R_{6N4}} = 0 \quad (24) $$

$$ U_{R_{7N7}} = U_{R_{6N1}} + \frac{P h c}{2\pi} (U_{R_{7N18}} - U_{R_{7N10}}) \quad (25) $$

$$ \begin{bmatrix} U_{R_{1N7}} \\ U_{R_{1N12}} \end{bmatrix} = \begin{bmatrix} R_{61} & -R_{61} S(a_1) \\ 0 & R_{61} \end{bmatrix} U_{R_6} \quad (26) $$

$$ \begin{bmatrix} U_{R_{3N7}} \\ U_{R_{3N12}} \end{bmatrix} = \begin{bmatrix} R_{63} & -R_{63} S(a_3) \\ 0 & R_{63} \end{bmatrix} U_{R_6} \quad (27) $$

$$ \begin{bmatrix} U_{R_{2N7}} \\ U_{R_{2N12}} \end{bmatrix} = \begin{bmatrix} R_{52} & -R_{52} S(a_2) \\ 0 & R_{52} \end{bmatrix} U_{R_5} \quad (28) $$
\[
\begin{bmatrix}
U_{RAN7} \\
\vdots \\
U_{RAN12}
\end{bmatrix} = 
\begin{bmatrix}
R_{s4} & -R_{s4}S(a_4) \\
0 & R_{s4}
\end{bmatrix}
U_{R5}
\]  
(29)

where \( R_{s6} \) is the rotation matrix of frame \( O \) with respect to frame \( O_1 \), and \( R_{s1} \) and \( R_{s3} \) are rotation matrices of frame \( O \) with respect to frame \( D_1 \) and frame \( D_3 \), respectively. \( R_{s2} \) and \( R_{s4} \) are the rotation matrices of frame \( O \) with respect to frame \( D_2 \). The skew matrices of node \( R_6 \) to node \( R_1 \) and node \( R_3 \) to node \( R_5 \) are the skew matrices of node \( R_6 \) to node \( R_1 \) and node \( R_3 \) to node \( R_5 \). Finally, the differential motion equation of ATP can be formulated from the equations of parts and relations of connecting nodes as follows.

\[
M_{sp} \ddot{U}_{qp} + K_{sp} U_{qp} = F_{qp} + Q_{qp}
\]  
(30)

where \( U_{qp} \) is a 40 \times 1 vector representing generalized coordinates.

\[
\begin{align*}
M_{sp} &= \sum_{j=1}^{8} B_{spj}^T M_{spj} B_{spj}, \\
K_{sp} &= \sum_{j=1}^{8} B_{spj}^T K_{spj} B_{spj}, \\
F_{qp} &= \sum_{j=1}^{8} B_{tpj}^T F_{tpj}, \\
Q_{qp} &= \sum_{j=1}^{8} B_{tpj}^T Q_{tpj}
\end{align*}
\]

herein \( B_{tpj} \) is referred to Appendix.

### 3.3 Dynamic model of PaQuad PM

With the differential motion equations of substructures available at hand, the elasto-dynamic model of the whole mechanism is assembled by the deformation compatibility conditions among substructures. Node \( n7 \) of translation substructure connects to node \( n8 \) and \( n9 \) of bar substructure by \( R \) joints. Hence, the other five generalized coordinates of these nodes are the same except for the rotational deformations about the axis of \( R \) joint.

\[
\begin{align*}
U_{(k=5)p}(43:45) &= U_{(k=5)p}(49:51) = U_{(k=1)p}(37:39), \\
U_{(k=5)p}(47:48) &= U_{(k=5)p}(53:54) = U_{(k=1)p}(41:42), \\
U_{(k=6)p}(43:46) &= U_{(k=6)p}(49:52) = U_{(k=2)p}(37:40), \\
U_{(k=6)p}(48) &= U_{(k=6)p}(54) = U_{(k=2)p}(42).
\end{align*}
\]  
(31)

where \( k = 0, 2 \).

Node \( n2 \) (5 \( \cdots \) 8) links to the in-parts through \( S \) joints. Considering the stiffness of the \( S \) joints, the generalized coordinates of connecting nodes between bar structure and ATP are expressed as

\[
K_{1S} \begin{bmatrix}
U_{sp}(7:9) \\
U_{sp}(1:3)
\end{bmatrix} = Q_{1S}, \\
K_{2S} \begin{bmatrix}
U_{tp}(7:9) \\
U_{tp}(7:9)
\end{bmatrix} = Q_{2S}
\]  
(32)

\[
K_{S} = \begin{bmatrix}
-k_{s1} & -k_{s1} \\
-k_{s1} & k_{s1}
\end{bmatrix} (i = 1, 2, \cdots, 4) \text{ is stiffness matrix of } S \text{ joint. } K_{S} = \text{diag}(k_{s1x}, k_{s1y}, k_{s1z}). Q_{S} \text{ is the external forces of } 4 \text{th } PRS \text{ limb.}
\]

Therefore, the elasto-dynamic model of the PaQuad PM is obtained by assembling differential motion equations of substructures as follows

\[
M \ddot{U} + KU = F
\]  
(34)

where \( M, K, \) and \( F \) are the mass matrix, stiffness matrix, and external forces of PaQuad PM.

\[
M = \sum_{i=1}^{9} D_{i}^T M_{ip} D_{i}, \quad F = \sum_{i=1}^{9} D_{i}^T F_{ip} + \sum_{i=1}^{4} A_{i}^T Q_{is}, \quad K = \sum_{i=1}^{9} D_{i}^T M_{ip} A_{i} + \sum_{i=1}^{4} A_{i}^T K_{is} A_{i}
\]

herein \( D_{i} \) and \( A_{i} \) are listed in Appendix.

### 4 Natural frequencies

#### 4.1 Case study

The virtual and physical prototypes of the PaQuad PM were built, which can be used to verify the elasto-dynamic model formulated in Section 3. The natural frequency of the PaQuad PM can be computed by the elasto-dynamic model shown in Eq. (34) as

\[
\det(-\omega^2 M + K) = 0
\]  
(35)

where \( \omega_n \) (i.e., \( i = 1, 2, \cdots, n \)) is the natural frequencies.

Dimensional parameters of the PaQuad PM are shown in Table 1. The major material for most components is 45# steel, whose density is \( 7.8 \times 10^3 \text{ kg/m}^3 \), Young’s modulus is \( 2.2 \times 10^{11} \text{ Pa} \) and shear modulus is \( 7.938 \times 10^{10} \text{ Pa} \). The mass of in-part \( 1 \) is \( 13.13 \text{ kg} \). Its moment of inertia about \( x-, y-, \) and \( z- \) axis is \( 0.0598, 0.0598, \) and \( 0.0572 \text{ kg} \cdot \text{m}^2 \), respectively. The mass of in-part \( 2 \) is \( 20.46 \text{ kg} \), whose moment of inertia is \( 0.146, 0.146, \) and \( 0.278 \text{ kg} \cdot \text{m}^2 \). The mass of out-part is \( 34.27 \text{ kg} \), the moment of inertia is \( 0.348, 0.348, \) and \( 0.681 \text{ kg} \cdot \text{m}^2 \).

By applying the parameter value to the elasto-dynamic model, the distribution of natural frequencies within workspace is shown in Fig. 6. The first frequency is symmetrical to the plane \( \beta = 0^\circ \) and the maximum value is at \( \beta = 0^\circ \), then it decreases with the increasing of \( \beta \). With the increment of \( \alpha \), the first frequency drops when \( \beta \) is fixed. The second frequency is symmetrical to the axis \( \alpha = \beta = 0^\circ \). It decreases with the changes of \( \alpha \) and \( \beta \). The peak value is at the symmetrical axis.
The third frequency monotonously climbs up as $\alpha$ increases and obtains the maximum value at $\alpha = 10^\circ$ when $\beta$ keeps the same. However, the third frequency decreases if $\alpha$ increases to $\alpha = 30^\circ$, then it increases again. The fourth frequency is plane symmetrical. Maximum and minimum values are mainly on the boundary of the workspace. The former are at $\alpha = 0^\circ$, $\beta = \pm 40^\circ$, and $\alpha = \pm 40^\circ$, $\beta = 0^\circ$, while the latter are at $\alpha = \pm 40^\circ$ and $\beta = \pm 40^\circ$. The fifth frequency is axial symmetrical to $\alpha = \beta = 0^\circ$, where it gets the minimum value. Distribution of the sixth frequency is similar to the first frequency but the change is sharper.

In summary, the natural frequencies change versus configurations and they show plane-symmetrical features due to the plane-symmetrical structure of the PaQuad PM.

4.2 Simulation and experiment

Simulations on the virtual prototype by FEA software are applied to verify the elastodynamic model of PaQuad PM. SAMCEF from SEMTECH Inc. is chosen to analyze the natural frequencies of eight typical configurations within workspace. The simulation is implemented as follows.

1. Compute actuations of the first configuration through the inverse kinematics. Drive the $P$ joints according to the calculated input value. Save the corresponding 3D model under the first configuration as name.x-p file.
2. Choose Structural analysis and Modal in the Solver Driver Setting of SEMCEF. In the Modeler, insert the file from step (1).
3. Define material property of all components. They are the same as theoretical model in Section 4.1. Assign assembling conditions to the adjacent parts and add boundary condition to the fixed base in Analysis Data.
4. Select the type of finite elements and mesh the elastic components in Mesh.
5. Calculate the natural frequencies in Solver and analyze in Result.
6. Choose the second configuration and proceed to step (1) to step (5). Repeat until PaQuad PM under all eight configurations is simulated.

The simulation results are shown in Table 2. The changing tendencies of the first to sixth frequencies are similar for both theoretical analysis and simulations. Generally, the simulated frequencies are smaller because non-standard features, such as...
shape and dimension, are included in the simulation while they are approximately represented by standard beam element.

Elastic dynamic experiments are also carried out as shown in Fig. 7. Measuring points are firstly assigned to different parts of PaQuad PM, on which acceleration sensors are attached. Hammer is then applied to excite the PaQuad PM at the end reference point. Through collecting exciting forces and response signals from the sensors, the modes are fitted and analyzed by LMS dynamic testing system (including SCSASA III data collecting hardware and LMS Test. Lab software from SIEMENS). Finally, the frequencies of PaQuad PM are obtained. The experimental procedure can be summarized as follows.

(1) Geometrical modeling. In the Geometry interface, the overall and local coordinate frames (fixed and element frames) are defined according to Section 3. Measuring points are assigned in the local frames based on the structures. By connecting the measuring points in the overall frame, the geometrical model of PaQuad PM is obtained. There are 68 measuring points in total.

(2) Sensor setting. In the Channel Setup interface, channel 1 is assigned to measure force. The actual sensitivity is 0.2838 mv/N. Channels 2 to 4 are chosen to collect signals from acceleration sensors in three directions. Their sensitivity are 9.822, 9.99, and 9.864 mv/(m/s²), respectively.

![Fig. 7 Experiment setup for natural frequencies of the PaQuad PM](image-url)
(3) Excitation setting. In the Impact Scope interface, the excitation of hammer is set as free form mode with 200 Hz bandwidth. The pre-excitation time and signal setups are also defined.

(4) Data collection. In the Measure interface, excitations are applied to the end reference point by hammer. These excitations are exerted along x-, y-, and z-axis and repeated five times at each direction. Measurement data is collected from the measuring points.

(5) Data analysis. In the Validate interface, the data is checked. Modal parameter is then analyzed in the Modal Data Selection.

The experimental results are shown in Table 2 and Fig. 8. Comparing with the analytical models and simulations, experimental frequencies are the smallest due to (1) detail features such as chamfer and groove are included in the physical prototype while they are ignored in the other two cases; (2) contacts between parts, especially within joints are not ideal in experiments; and (3) there are external influences like measurement noise, non-rigid fixed base, or low bearing preload.

The modals of the PaQuad PM are further analyzed to check the consistency of the analytical, simulation, and experiment results. Taking pose I in Table 2 as an example, the first and second modals are the relative vibrations of opposite PRS limbs along x- and y-axis. The third and fourth modals are the torsions of opposite PRS limbs and the connected in-parts. The fifth and sixth modals are the vibrations along z-axis. The first to sixth modals have the same rules.

In summary, frequencies from the analytical model, simulation, and experiment are close, and the changing tendencies are similar. In addition, the modal of the eight typical poses are the same. The elastodynamic model is validated. The parametric elastodynamic model can be applied to the re-design of the PaQuad PM under specific requirements from different application scenarios.

Table 3 Parameters of PaQuad PM for sensitivity analysis (unit for mean value is m, standard variance ×10^{-5})

| Parameters | Mean value | Standard variance | Parameters | Mean value | Standard variance |
|------------|------------|-------------------|------------|------------|-------------------|
| \(d_{sc}\) Screw diameter | 0.025 | 0.2091 | \(d_{sj}\) S joint diameter | 0.09 | 0.677 |
| \(l_{a1}\) Horizontal length of slider | 0.285 | 27.17 | \(l_{sj}\) S joint length | 0.13 | 1.413 |
| \(h_{a1}\) Horizontal height of slider | 0.085 | 2.417 | \(l_{ip1}\) Length of in-part 1 | 0.63 | 33.19 |
| \(b_{a1}\) Horizontal width of slider | 0.278 | 25.85 | \(h_{ip1}\) Height of in-part 1 | 0.04 | 0.535 |
| \(l_{a2}\) Vertical length of slider | 0.096 | 3.083 | \(b_{ip1}\) Width of in-part 1 | 0.23 | 4.42 |
| \(h_{a2}\) Vertical height of slider | 0.115 | 4.424 | \(l_{ip2}\) Length of in-part 2 | 0.63 | 33.19 |
| \(b_{a2}\) Vertical width of slider | 0.12 | 4.817 | \(h_{ip2}\) Height of in-part 2 | 0.37 | 45.80 |
| \(l_{b}\) Bar length | 0.205 | 3.515 | \(b_{ip2}\) Width of in-part 2 | 0.04 | 0.535 |
| \(h_{b}\) Bar height | 0.16 | 2.141 | \(d_{hj}\) H joint diameter | 0.02 | 0.134 |
| \(b_{h}\) Bar width | 0.12 | 1.204 | \(l_{hj}\) H joint length | 0.25 | 5.227 |
5 Parameter sensitivity

The complicated composition and structures of the PaQuad PM lead to large amounts of parameters. In the re-design or the optimization process, however, it is not necessary to optimize all the parameters since some parameters impose little effects to the elastodynamic performance. Besides, the substantial parameters increase the difficulties of the optimization. Based on the engineering experience, the parameters are scaled down as shown in Table 3. Parameter sensitivity analysis is then implemented to categorize the main or subordinate parameters according to their effects to the natural frequencies. In our previous work, we proposed a parameter sensitivity analysis method by response surface method (RSM) based model, parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44].

(2) Comprehensive evaluation is achieved by considering the parameter sensitivity. The merits of this method are variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44]. The RSM method is to build the surrogate model based on the explicit mapping model, the parameter mean value, and variance-based indices [44].

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Based on design of experiment (DoE), RSM employs a set of experimental sets, is decided by the DoE strategy. Due to the amount of parameters and uncertain order of polynomial functions, the Latin hypercube design (LHD) strategy [47] is chosen since it is capable of dealing with difficult problems with limited number of experiments. After setting up by LHD strategy and implementing the experiments by the elastodynamic model, the linear, quadratic, cubic, and quartic RSM functions can be formulated as

\[
 f_{\text{linear}}(\mathbf{x}) = a_0 + \sum_{i=1}^{20} b_i x_i \tag{36}
\]

\[
 f_{\text{quadratic}}(\mathbf{x}) = a_0 + \sum_{i=1}^{20} b_i x_i + \sum_{i=1}^{20} c_i x_i^2 + \sum_{i=1}^{20} \sum_{i<j} d_{ij} x_i x_j \tag{37}
\]

\[
 f_{\text{cubic}}(\mathbf{x}) = a_0 + \sum_{i=1}^{20} b_i x_i + \sum_{i=1}^{20} c_i x_i^2 + \sum_{i=1}^{20} \sum_{i<j} d_{ij} x_i x_j + \sum_{i=1}^{20} e_i x_i^3 \tag{38}
\]

\[
 f_{\text{quartic}}(\mathbf{x}) = a_0 + \sum_{i=1}^{20} b_i x_i + \sum_{i=1}^{20} c_i x_i^2 + \sum_{i=1}^{20} \sum_{i<j} d_{ij} x_i x_j + \sum_{i=1}^{20} e_i x_i^3 + \sum_{i=1}^{20} f_i x_i^4 \tag{39}
\]

where \( \mathbf{x} = (x_1, x_2, \ldots, x_{20})^T \) is the vector of parameters. \( a_0, b_i, c_i, d_{ij}, e_i, f_i \) are the estimated coefficients obtained from the least square regression. \( x_i, x_j \) is the interaction of any two parameters. \( x_i^2, x_i^3, x_i^4 \) are the second, third, and fourth nonlinearity of \( \mathbf{x} \).

Accuracy assessment is then carried out to verify the RSM models. Additional parameter sets are randomly generated. The errors between the actual responses and the results from RSM models are compared via four metrics [48], i.e., R square (RS), relative average absolute error (RAAE), relative maximum absolute error (RMAE), and root mean square error (RMSE), as follows

\[
 \text{RAAE} = \frac{\sum_{i=1}^{m} |y_i - \hat{y}_i|}{\sum_{i=1}^{m} |y_i|}, \quad \text{R}^2 = 1 - \frac{\sum_{i=1}^{m} (y_i - \hat{y}_i)^2}{\sum_{i=1}^{m} (y_i - \bar{y})^2} \tag{40}
\]

\[
 \text{RMAE} = \max\{ |y_1 - \hat{y}_1|, \ldots, |y_m - \hat{y}_m| \} \tag{41}
\]

\[
 \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{m} (y_i - \hat{y}_i)^2}{m}} \tag{42}
\]

where \( y_i \) is the natural frequency obtained from analytical model, \( \hat{y}_i \) is the value computed by RSM model, and \( \bar{y} \) is the mean value of \( y_i \).

In these accuracy metrics, RS, RAAE, and RMSE evaluate the overall accuracy of RSM models within the parameter ranges while RMAE shows the maximum error. Through the simultaneous consideration of global and worst accuracy, RSM model that has smaller RS, RAAE, and RMSE and larger RS are selected as the surrogate model for parameter sensitivity analysis.

For the PaQuad PM, the parameter values shown in Table 1 are regarded as the baseline and the range of parameters is set as ±10%. Since the number of involving parameters is 20, the

| Error (accepted level) | Linear | Quadratic | Cubic | Quartic |
|-----------------------|--------|-----------|-------|---------|
| RS (> 0.9)            | 0.98898| 0.99985   | 0.99985| 0.99984 |
| RAAE (< 0.2)          | 0.01807| 0.00157   | 0.00175| 0.0022  |
| RMAE (< 0.3)          | 0.05898| 0.00789   | 0.00676| 0.0107  |
| RMSE (< 0.2)          | 0.02401| 0.00201   | 0.00232| 0.00284 |
number of parameter sets required by LHD is 42, 462, 502, and 542 for linear, quadratic, cubic, and quartic RSM functions. The formulation of RSM models shown in Eq. (32) to Eq. (34) can be implemented with the aid of MATLAB and Isight software. Additional 21, 231, 251, and 271 sets of parameters are randomly generated to assess the accuracy of the RSM models (see Table 4). The errors of quadratic and cubic RSM models are smaller than the linear and quartic models. RS values of quadratic and cubic RSM models are the same. The RAAE and RMSE values of quartic RSM model are lower than cubic model, but the RMAE value is higher. From the comparison, the quadratic RSM model is finally chosen.

### 5.2 Parameter sensitivity indices

Based on the explicit RSM model, parameter sensitivity indices are defined by the differentiation of performance reliability to the parameter mean value and variance. Performance reliability describes the probability of the studied PM achieving target performance with given range of parameters. It is expressed as

\[ R_p = P\{g(x) > 0\} = 1 - P\{Y > -\beta_j\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta_j} e^{-\frac{1}{2} t^2} dt \tag{43} \]

where \( P\{g(x) > 0\} \) denotes the probability of \( g(x) > 0 \). And \( g(x) \) is the subtraction of RSM model and the allowable values. \( Y = [g(x) - \mu_j]/\sigma_j \) is a random variable determined by \( \beta_j = \mu_j/\sigma_j \). Herein, \( Y \in [0, 1] \) is subjected to normal distribution. \( \mu_j \) and \( \sigma_j \) are the mean value and variance of \( g(x) > 0 \).

Taking differentiation of Eq. (43) to the mean value and covariance of parameters yields

\[
\frac{\partial R_p}{\partial \mu_j} = \frac{\partial R_p}{\partial \beta_j} \frac{\partial \beta_j}{\partial \mu_j} = \frac{e^{-\frac{\beta_j^2}{2}}}{\sigma_j \sqrt{2\pi}} \left[ \frac{\partial g(\mu)}{\partial \mu_1}, \ldots, \frac{\partial g(\mu)}{\partial \mu_{20}} \right]^T \quad \text{and} \quad \frac{\partial R_p}{\partial \text{cov}(x)} = \frac{\partial R_p}{\partial \beta_j} \frac{\partial \beta_j}{\partial \text{cov}(x)}
\]

where \( \mu_j \) and \( \sigma_j (j = 1, 2, \cdots, 20) \) are mean value and standard variance of \( j \)th parameter (see Table 3).

Considering both parameter mean value and variance, a global sensitivity index is defined as

\[ \varepsilon_j = \sqrt{\left( \frac{\kappa_j}{\kappa_{\text{max}}} \right)^2 + \left( \frac{\sigma_j}{\sigma_{\text{max}}} \right)^2}, \quad j = 1, 2, \cdots, 20 \tag{46} \]

where \( \kappa_j = \frac{\partial R_p}{\partial \mu_j}, \sigma_j = \frac{\partial R_p}{\partial \text{cov}(x)} \). \( \kappa_{\text{max}} \) and \( \sigma_{\text{max}} \) are the maximum sensitivity of mean value and variance among all the parameters.

Based on Eq. (37) to Eq. (39), the parameter sensitivity of PaQuad PM is computed and shown in Table 5 and Fig. 9. In summary, ATP imposes great effects to the elastodynamic

---

**Table 5** Parameter sensitivity of first-order natural frequency

| \( \kappa_j \) | \( \sigma_j \times 10^{-3} \) | \( \varepsilon_j \) | \( \kappa_j \) | \( \sigma_j \times 10^{-3} \) | \( \varepsilon_j \) |
|-------------|----------------|-------|-------------|----------------|-------|
| \( d_{\text{c}} \) | 0.0024 | 0.1057 | 0.0017 | \( d_{\text{c}} \) | 0.0684 | -1.0789 | 0.0483 |
| \( h_{\text{a1}} \) | -0.0005 | -0.003 | 0.0004 | \( h_{\text{a1}} \) | 0.0111 | -1.1146 | 0.0008 |
| \( h_{\text{a2}} \) | 0.0006 | 0.1566 | 0.0004 | \( h_{\text{a2}} \) | 0.0018 | -0.0772 | 0.0013 |
| \( b_{\text{a1}} \) | -0.0006 | 0.0005 | 0.005 | \( b_{\text{a1}} \) | 0.0107 | 4.8707 | 0.0076 |
| \( b_{\text{a2}} \) | 0.0082 | -0.0909 | 0.0058 | \( b_{\text{a2}} \) | 0.0123 | -0.2166 | 0.0087 |
| \( l_{\text{a1}} \) | 0.0041 | -0.0892 | 0.0029 | \( l_{\text{a1}} \) | 0.0884 | -0.006 | 0.0265 |
| \( l_{\text{a2}} \) | 0.0148 | -0.0737 | 0.0105 | \( l_{\text{a2}} \) | 1.4142 | -1.0022 | 1.0 |
| \( h \) | 0.0019 | 0.1608 | 0.0013 | \( h \) | 0.1063 | -0.0094 | 0.0752 |
| \( b_h \) | 0.0073 | -0.0519 | 0.0052 | \( b_h \) | 0.3499 | -0.5964 | 0.2474 |
| \( b_h \) | 0.0449 | -0.3417 | 0.0318 | \( b_h \) | 0.0007 | 0.2139 | 0.0005 |

---

**Fig. 9** Global sensitivity and parameter sensitivity to mean value and variance.
performance, especially central screw and in-part 2 that directly
link to the end-effector. Comparatively, PRS limbs are more rigid
and hence have little influence to the natural frequency. For
the sensitivity to mean values, diameter of central screw in H joint
\((d_{h})\), parameters of in-part 2 \((l_{ip2}, h_{ip2}, \text{ and } b_{ip2})\), diameter of S
joint \((d_{s})\), and width of fixed bar \((b_{h})\) have larger values than the
rest of the parameters. Herein, \(h_{ad2}\) is the maximum, indicating it
has the most significant influence to the natural frequency.

For the sensitivity to parameter variance, the top parameters
are \(h_{ip1}, h_{ip2}, d_{g}, l_{ip}, \text{ and } b_{h}\). Considering both parameter
mean value and variance, global indices show that \(b_{h}, d_{g}, l_{ip2}, h_{ip2}, b_{ip2},\)
and \(d_{s}\) are the major parameters that would greatly affect the natural frequencies. By increasing the values
of these parameters, the resulted natural frequency is expected
to increase. Therefore, they can be chosen as the design pa-
rameters in the future optimization.

6 Conclusions

Aiming at the optimal design of a PM-ATP named as PaQuad
PM, this paper carries out the elastodynamic modeling by KED
method and parameter sensitivity analysis by RSM method and
reliability sensitivity indices. Conclusions are drawn as follows.

(1) The PaQuad PM is divided into four levels, i.e., beam ele-
ment, parts, substructure, and the whole mechanism.
Differential motion equation of lower level is assembled to
formulate the model of upper level. Through this layer-
layer modeling method, multiple parts and joints within
ATP can be considered explicitly. The proposed elastodynamic modeling method can also be applied to oth-
er types of PMs.

(2) Natural frequencies are regarded as the dynamic perfor-
ance indices. Simulation by finite element software and experiments by LMS dynamic test system are imple-
mented to verify the elastodynamic model of PaQuad
PM. Although the results from the three methods are slightly different, the changing tendency and modal are the same. The analytical model is confirmed.

(3) RSM models are investigated to get explicit polynomial
functions between natural frequency and parameters of
PaQuad PM. The coupling effects among parameters can
be evaluated. Parameter sensitivity indices are defined
based on the differentiation of performance reliability
to the parameter mean value and standard variance. The
statistic features of parameters are considered; hence, the
obtained parameter sensitivity is comprehensive.

Elastodynamic modeling is the basis of formulating objec-
tive functions while the parameter sensitivity analysis deter-
mines the design variables. They are both necessary prepara-
tions for the optimal design which is the future work for the
development of the PaQuad PM.

Appendix

The transformation matrices of the element frames in translation
substructure with respect to fixed frame \(O - xyz\) are computed as
\[
\mathbf{J}_{E1} = \begin{bmatrix} E_{12\times12} & 0_{12\times6} \\ 0_{6\times12} & E_{6\times6} \end{bmatrix}, \quad \mathbf{J}_{E2} = \begin{bmatrix} 0_{12\times6} & E_{12\times12} \\ 0_{6\times12} & 0_{6\times6} \end{bmatrix}
\]
\[
\mathbf{J}_{E3} = \begin{bmatrix} 0_{6\times6} & E_{6\times6} \\ 0_{6\times30} & E_{6\times30} \end{bmatrix}, \quad \mathbf{J}_{E4} = \begin{bmatrix} 0_{6\times12} & E_{6\times6} \\ 0_{6\times24} & 0_{6\times24} \end{bmatrix}
\]
\[
\mathbf{J}_{E5} = \begin{bmatrix} 0_{12\times18} & E_{12\times12} & 0_{12\times24} \end{bmatrix}, \quad \mathbf{J}_{E6} = \begin{bmatrix} 0_{6\times18} & E_{6\times6} & 0_{6\times30} \\ 0_{6\times30} & E_{6\times30} & 0_{6\times18} \end{bmatrix}
\]
\[
\mathbf{J}_{E7} = \begin{bmatrix} 0_{12\times36} \\ 0_{12\times12} \\ 0_{12\times12} \end{bmatrix}, \quad \mathbf{J}_{E8} = \begin{bmatrix} 0_{6\times30} & E_{6\times6} & 0_{6\times18} \\ 0_{6\times42} & E_{6\times6} & 0_{6\times6} \end{bmatrix}
\]
\[
\mathbf{J}_{E9} = \begin{bmatrix} 0_{6\times36} & E_{6\times6} & 0_{6\times12} \\ 0_{6\times48} & E_{6\times6} \end{bmatrix}, \quad \mathbf{J}_{E10} = \begin{bmatrix} 0_{6\times24} & E_{6\times6} & 0_{6\times24} \\ 0_{6\times36} & E_{6\times6} & 0_{6\times12} \end{bmatrix}
\]

The transformation matrices of the element frames in bar
substructure are expressed as
The connecting relations of nodes within ATP can be calculated as

\[ \mathbf{B}_{\text{sp1}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 4} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp2}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp3}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp4}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp5}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp6}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ \mathbf{B}_{\text{sp7}} = \begin{bmatrix} \mathbf{E}_{6 \times 6} & \mathbf{0}_{6 \times 28} \\ \mathbf{0}_{6 \times 25} & \mathbf{0}_{6 \times 25} \\ \mathbf{0}_{6 \times 28} & \mathbf{0}_{6 \times 28} \end{bmatrix} \]

\[ (A - 4) \]
where $l_1, l_2, l_3,$ and $l_4$ are the length of elements in R1, R2, R3, and R4. $p_{hc}$ is the pitch of central screw.

The converting matrices of PaQuad PM are as follows.

\[
A_1 = \begin{bmatrix} 0_{3 \times 50} & E_{3 \times 3} & 0_{3 \times 203} \\ 0_{3 \times 216} & E_{3 \times 3} & 0_{3 \times 37} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0_{3 \times 93} & E_{3 \times 3} & 0_{3 \times 160} \\ 0_{3 \times 222} & E_{3 \times 3} & 0_{3 \times 31} \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0_{3 \times 136} & E_{3 \times 3} & 0_{3 \times 177} \\ 0_{3 \times 228} & E_{3 \times 3} & 0_{3 \times 25} \end{bmatrix}
\]

\[
A_4 = \begin{bmatrix} 0_{5 \times 179} & E_{3 \times 3} & 0_{5 \times 74} \\ 0_{5 \times 234} & E_{3 \times 3} & 0_{5 \times 19} \end{bmatrix}, \quad D_1 = \begin{bmatrix} E_{11 \times 11} & 0_{11 \times 245} \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0_{11 \times 11} & E_{11 \times 11} \end{bmatrix}, \quad D_9 = \begin{bmatrix} 0_{40 \times 216} & E_{40 \times 40} \end{bmatrix}
\]

\[
D_3 = \begin{bmatrix} 0_{11 \times 22} & E_{11 \times 11} & 0_{11 \times 223} \end{bmatrix}, \quad D_4 = \begin{bmatrix} 0_{11 \times 33} & E_{11 \times 11} & 0_{11 \times 212} \end{bmatrix}, \quad D_5 = \begin{bmatrix} 0_{42 \times 44} & E_{42 \times 42} & 0_{42 \times 170} \\ 0_{2 \times 5} & E_{2 \times 2} & 0_{2 \times 248} \\ 0_{1 \times 86} & E_{1 \times 1} & 0_{1 \times 169} \end{bmatrix}, \quad D_6 = \begin{bmatrix} 0_{42 \times 87} & E_{42 \times 42} & 0_{42 \times 127} \\ 0_{4 \times 16} & E_{4 \times 4} & 0_{4 \times 236} \\ 0_{3 \times 129} & E_{3 \times 3} & 0_{3 \times 234} \end{bmatrix},
\]

\[
D_7 = \begin{bmatrix} 0_{42 \times 130} & E_{42 \times 42} & 0_{42 \times 184} \\ 0_{1 \times 27} & E_{1 \times 1} & 0_{1 \times 226} \\ 0_{1 \times 172} & E_{1 \times 1} & 0_{1 \times 83} \\ 0_{3 \times 231} & E_{3 \times 3} & 0_{3 \times 223} \end{bmatrix}, \quad D_8 = \begin{bmatrix} 0_{42 \times 173} & E_{42 \times 42} & 0_{42 \times 41} \\ 0_{4 \times 38} & E_{4 \times 4} & 0_{4 \times 214} \\ 0_{1 \times 215} & E_{1 \times 1} & 0_{1 \times 40} \\ 0_{3 \times 43} & E_{3 \times 3} & 0_{3 \times 212} \end{bmatrix}
\]

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