CURVATURE ESTIMATE ON THE FINITE GRAPH WITH LARGE GIRTH

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ABSTRACT. The CD inequalities and CDE inequalities are useful in the estimate of curvature on graphs. This article is based on the infinite graph with large girth, and finally concludes some curvature estimate in CD and CDE.

1. Introduction

Ricci curvature on graphs is an important aspect in the study of graphs and geometry. Some related work has been done in [6][7]. Recently, there is some work has been done about graphs with large girth in [2]. So we want to do some research about the curvature estimate under the special conditions—when the girth of a finite graph is large than 5. Maybe the conclusion is beautiful.

Moreover, the graph referred is unweighted, the corresponding Laplacian is called unweighted normalized (i.e., \( \mu = 1 \) on \( E \) and \( m = \mu \) on \( V \)) [1][3][4]. Let \((V,E)\) be a undirected graph with the set of vertices \( V \) and the set of edges \( E \); i.e. two-elements subsets in \( V \). The graph is called simple if there is no self-loops and multiple edges. The graph is called locally finite, if the combinatorial degree \( d_x < \infty \) for any \( x \in V \). We say a vertex \( x \) is a pending vertex if \( d_x = 1 \). For any subsets \( A, B \subset V \), we denote by \( E(A,B) := x, y \in E : x \in A, y \in B \) the set of edges between \( A \) and \( B \). For vertices \( x \) and \( y \), a walk from \( x \) to \( y \) is a sequence of vertices \( x_i \) such that

\[ x = x_0 \sim x_1 \sim ... \sim x_k = y, \]

where \( k \) is called the length of the walk. A graph is said to be connected if for any \( x, y \in V \) there is a walk from \( x \) to \( y \). In this paper, we only consider undirected, connected, locally finite simple graphs.

This paper gives an estimate of the curvature on finite graph with girth large than 5.

The paper is organized into four parts:

Chapter 1 is the introduction of the finite graph, the girth, the Laplacians, CD inequalities and CDE inequalities on it.

Chapter 2 introduces some easy conclusions about the calculation of the operator which is helpful for chapter 3.

Chapter 3 is the main conclusion of this thesis which includes curvature estimate about CD inequality and CDE inequality.
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2. GRAPHS, GIRTH, LAPLACIANS CD INEQUALITIES AND CDE INEQUALITIES

Given a graph $G = (V, E)$, for an $x \in V$, if there exists another $y \in V$ that satisfies $(x, y) \in E$, we call them are neighbors, and written as $x \sim y$. If there exists an $x \in V$ satisfying $(x, x) \in E$, we call it a self-loop.

Now we will introduce some basic definitions and theorems before we get the main results.

**Definition 2.1.** (locally finite graph) We call a graph $G$ is a locally finite graph if for any $x \in V$, it satisfies $\#\{y \in V | y \sim x\} < \infty$. Moreover, it is called connected if there exists a sequence $\{x_i\}_{i=0}^n$ satisfying: $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$.

**Definition 2.2.** (Laplacians on locally finite graphs) On a locally finite graph $G = (V, E, \mu, m)$ the Laplacian has a form as follows:

$$\triangle f(x) = \frac{1}{m(x)} \sum_{y \in V} \mu_{xy}(f(y) - f(x)), \quad \forall f \in C_0(V).$$

**Definition 2.3.** (gradient operator $\Gamma$) The operator $\Gamma$ is defined as follows:

$$\Gamma(f, g)(x) = \frac{1}{2}(\triangle(fg) - f\triangle g - g\triangle f)(x)$$

Always we write $\Gamma(f, f)$ as $\Gamma(f)$.

**Definition 2.4.** (gradient operator $\Gamma_i$) The operator $\Gamma_i$ is defined as follows:

$$\Gamma_0(f, g) = fg$$

$$\Gamma_{i+1}(f, g) = \frac{1}{2}(\triangle(\Gamma_i(f, g)) - \Gamma_i(f, \triangle g) - \Gamma_i(\triangle f, g))$$

Also we have $\Gamma_2(f) = \Gamma_2(f, f) = \frac{1}{2}\triangle\Gamma(f) - \Gamma(f, \triangle f)$.

**Definition 2.5.** (the girth of a graph) The girth of a verth $x$ in $(V, E)$, denoted by $Gir(x)$, is defined to the minimal length of cycles passing through $x$. (If there is no cycle passing through $x$, define $Gir(x) = \infty$.) The girth of a graph is defined as $\inf_{x \in V} Gir(x)$.

**Definition 2.6.** ($CD(K, n)$ condition) We call a graph satisfies $CD(K, n)$ condition if for any $x \in V$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\triangle f)^2(x) + K\Gamma(f)(x). \quad K \in \mathbb{R}.$$
Definition 2.7. \((CDE(K,n)\) condition) Let \(f: V \rightarrow \mathbb{R}^+\) satisfy \(f(x) > 0\), \(\Delta f(x) < 0\). We call a graph satisfies \(CDE(x,K,n)\) condition if for any \(x \in V\), we have
\[
\Gamma_2(f)(x) - \Gamma \left( f, \frac{\Gamma(f)}{f} \right)(x) \geq \frac{1}{n}(\Delta f)(x)^2 + K\Gamma(f)(x). \quad K \in \mathbb{R}.
\]

3. MAIN RESULTS

Remark 3.1. In this section, we use the definition in part 2. They are the work from [7].

Lemma 3.2.
\[
\Gamma(f)(x) = \frac{1}{2} \mu_x \sum_{y \in N_x} f(x,y)^2.
\]

Here we define \(N_x = y \in V : xy \in E\) and \(d_x = |N_x|\). For notational simplicity we work with \(\mu_x = \frac{1}{d_x}\).

Proof We have
\[
\Gamma(f)(x) = \frac{1}{2} \Delta (f^2)(x) - f(x)(\Delta f)(x)
\]
\[
= \frac{1}{2} \mu_x \sum_{y \in N_x} (f^2)(x,y) - f(x)\mu_x \sum_{y \in N_x} f(x,y)
\]
\[
= \frac{1}{2} \mu_x \sum_{y \in N_x} (f(x,y)(f(y) + f(x)) - 2f(x,y)f(x))
\]
\[
= \frac{1}{2} \mu_x \sum_{y \in N_x} f(x,y)^2.
\]

Lemma 3.3.
\[
\Gamma_2(f)(x) = \frac{1}{2} (\Delta f(x)^2 + \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y,z)^2 - \frac{1}{2}f(x,z)^2)).
\]

Proof We have
\[
\Delta(\Gamma(f))(x) = \mu_x \sum_{y \in N_x} \Gamma(f)(x,y) = \mu_x \sum_{y \in N_x} \frac{1}{2} \mu_y \sum_{z \in N_y} (f(y,z)^2 - f(x,y)^2)
\]
and

$$\Gamma(f, \Delta f)(x) = \frac{1}{2}(\Delta (f \cdot \Delta f)(x) - f(x) \cdot (\Delta^2 f)(x) - (\Delta f)^2(x))$$

$$= -\frac{1}{2}(\Delta f)^2(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} (f(y, z)^2 - f(x, y)^2) - \frac{1}{2} f(x, y) f(y, z)$$

$$= -\frac{1}{2}(\Delta f)^2(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} f(x, y) \Gamma(y, z)$$

thus

$$\Gamma_2(f)(x) = \frac{1}{2}(\Delta f)^2(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - f(x, y)^2) - \frac{1}{2} f(x, y) f(y, z)$$

$$= \frac{1}{2}(\Delta f)^2 + \frac{1}{2} \mu_y \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - f(x, z)^2)$$

4. BASIC CONCLUSION

Remark 4.1. In this section, we will give the curvature estimate about finite graphs with girth larger than 5 in the CD inequality.

Theorem 4.2. We have a finite graph with girth larger than 5 and we concern the fixed point $x$, assume the neighborhood of $x$ are $y_1, y_2, ..., y_n$, so just from the definition above, we have $d_x = n$. Also, the neighborhood of $y_1$ is $z_1, z_2, ..., z_{k_1-1}$, the neighborhood of $y_2$ is $z_1, z_2, ..., z_{k_2-1}$, the neighborhood of $y_n$ is $z_1, z_2, ..., z_{k_n-1}$. So $d_{y_i} = k_i, i = 1, 2, ..., n$. Let $k = \min\left\{ \frac{2-k_1}{k_1}, \frac{2-k_2}{k_2}, ..., \frac{2-k_n}{k_n} \right\}$. Then we have the conclusion that the graph satisfies $CD(k, 2, x)$.

Proof. From the theorem above, we know that

$$\Gamma_2(f)(x) = \frac{1}{2}(\Delta f)^2(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2} f(x, z)^2).$$

So with the definition of $CD(k, 2, x)$ inequality, we need to prove the following inequality

$$\frac{1}{2}(\Delta f)^2 + \frac{1}{2} \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2} f(x, z)^2) \geq \frac{1}{2}(\Delta f)^2 + k \Gamma(f).$$

Which is the same to:

$$\frac{1}{2} \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2} f(x, z)^2) \geq k \cdot \frac{1}{2} \mu_x \sum_{y \in N_x} f(x, y)^2.$$
Here $\mu_x = \frac{1}{n}$, $\mu_y = \frac{1}{k_i} (i = 1, 2, ..., n)$, and without loss of generality we can assume $f(x) = 0$.

So the inequality is equal to the following:

$$\sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(z) - f(y))^2 - \frac{1}{2} f(z)^2 \geq k \cdot \sum_{y \in N_x} f(y)^2.$$ 

Firstly, we concern the $y_1$ part in the left inequality. The sum is:

$$\frac{1}{k_1}((f(z_{11}) - f(y_1))^2 - \frac{1}{2} f(z_{11})^2 + (f(z_{12}) - f(y_1))^2 + (f(z_{21}) - f(y_1))^2 - \frac{1}{2} f(z_{12})^2 + ...)$$

$$= \frac{1}{k_1}(f(z_{11})^2 + f(y_1)^2 - 2f(z_{11})f(y_1) - \frac{1}{2} f(z_{11})^2 + ... f(y_1)^2)$$

We use the knowledge of quadratic function, take $f(z_{1i}) = 2f(y_1)$, for $i = 1, 2, ..., k - 1$, so we can get the minimal sum of the function. The sum about $y_1$ is equal to:

$$\frac{1}{k_1}(-f(y_1)^2 \cdot (k - 1) + f(y_1)^2) = \frac{1}{k_1} f(y_1)^2 (2 - k_1) = -\frac{k_1 - 2}{k_1} f(y_1)^2.$$ 

And the situation is all the same to $y_2, y_3, ..., y_n$. At last we have the following inequality:

$$\frac{2 - k_1}{k_1} f(y_1)^2 + \frac{2 - k_2}{k_2} f(y_2)^2 + ... + \frac{2 - k_n}{k_n} f(y_n)^2 \geq k f(y_1)^2 + k f(y_2)^2 + ... k f(y_n)^2.$$

From the assumption of the theorem, we know that $\frac{2 - k_i}{k_i} \geq k$ for $i = 1, 2, ..., n$. Also $f(y_i)^2 \geq 0$. So the above inequality is correct. We end the proof.

**Theorem 4.3.** Suppose $G$ is a graph with girth larger than 5 and let $f : V \to \mathbb{R}^+$ satisfy $f(x) > 0$, $\Delta f(x) < 0$. Then we have the conclusion that the graph satisfy CDE$(2, -\frac{n}{2} - 1)$

**Proof.** Because the CDE condition satisfies that:

$$\Gamma_2(f) - \Gamma(f, \frac{\Gamma(f)}{f}) \geq \frac{1}{m} (\Delta f)^2 + k \cdot \Gamma(f).$$

From the above proposition, we have that:

$$\Gamma_2(f)(x) = \frac{1}{2} ((\Delta f)^2(x) + \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} ((f(y, z)^2 - \frac{1}{2} f(x, z)^2)).$$
So the most important work for us is to simplify the $\Gamma(f, \frac{\Gamma(f)}{f})$.

$$\Gamma(f, \frac{\Gamma(f)}{f})(x) = \frac{1}{2}(\Delta(\Gamma(f))(x) - \Delta(\frac{\Gamma(f)}{f}) - \Delta f \cdot \Gamma(f))$$

$$= \frac{1}{2} \Delta(\Gamma(f))(x) - \frac{1}{2} \Delta(\frac{\Gamma(f)}{f}) - \frac{1}{2} \Delta f \cdot \Gamma(f)$$

$$= I_1 - I_2 - I_3$$

Firstly:

$$I_1 = \frac{1}{2} \Delta(\Gamma(f))(x) = \frac{1}{2} \mu_x \sum_{y \in N_x} \Gamma(f)(x, y)$$

$$= \frac{1}{2} \mu_x \sum_{y \in N_x} \frac{1}{2} \mu_y \sum_{z \in N_y} (f(y, z)^2 - f(x, y)^2)$$

Secondly:

$$I_2 = \frac{1}{2} \Delta(\frac{\Gamma(f)}{f}) = \frac{1}{2n} \sum_{y \in N_x} \frac{\Gamma(f)(x)}{f(y)} = \frac{1}{2n} \sum_{y \in N_x} \frac{\Gamma(f)(x)}{f(x)}$$

$$= \frac{1}{2n} \sum_{y \in N_x} \frac{1}{2f(y)} \mu_y \sum_{z \in N_y} f(y, z)^2 - \frac{1}{2n} \sum_{y \in N_x} f(x, y)^2$$

$$= \frac{1}{4n} \sum_{y \in N_x} \mu_y \sum_{z \in N_y} \frac{f(y, z)^2}{f(y)} - \frac{1}{4n} \sum_{y \in N_x} f(x, y)^2$$

Thirdly:

$$I_3 = \frac{1}{2} \Delta f(x) \Gamma(f)(x) = \frac{1}{2n} \sum_{y \in N_x} f(x, y) \frac{1}{2n} \sum_{y \in N_x} f(x, y)^2.$$ 

To simplify the question, we assume that $f(x) = 1; f(x, y_i) = v_i, i = 1, 2, \ldots, n$.

So combine these together, we have:

$$k \geq \frac{1}{2n} \sum_{y \in N_x} v_i^2 \left( \frac{1}{2} (\Delta f)^2 + \frac{1}{2n} \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2} f(x, z)^2) \right) +$$

$$\frac{1}{4n} \sum_{y \in N_x} \mu_y \sum_{z \in N_y} \frac{f(y, z)^2}{f(y)} - \frac{1}{4n} \sum_{y \in N_x} f(x, y)^2 + \frac{1}{2n} \sum_{y \in N_x} f(x, y) \frac{1}{2n} f(x, y)^2 - \frac{1}{2n^2} \left( \sum_{y \in N_x} f(x, y) \right)^2$$

$$= \frac{1}{2n} \sum_{y \in N_x} \mu_y \left( \frac{1}{2n} \sum_{z \in N_y} \left( (f(y, z)^2 - \frac{1}{2} f(x, z)^2 - \frac{1}{2} f(y, z)^2 \right) \right)$$
\begin{align*}
&\frac{1}{2}f(x, y)^2 + \frac{1}{2}f(y, z)^2 + \frac{1}{2n} \sum_{y \in N_x} v_i \left( \frac{1}{2n} \sum_{y \in N_x} v_i \right) \\
&- \frac{1}{4n} \sum_{y \in N_x} v_i^2 \\
\text{Absolute we have:} \\
&f(y, z)^2 - \frac{1}{2}f(x, z)^2 - \frac{1}{2}f(y, z)^2 + \frac{1}{2}f(x, y)^2 + \frac{f(y, z)^2}{2f(y)} \\
&= \frac{f(z)^2}{f(y)} - f(y)f(z) + f(y)^2 - \frac{1}{2}f(y) \\
\text{They are minimized when } f(z) = f(y)^2, \text{whence the sum is} \\
&-\frac{1}{2}f(y)^3 + f(y)^2 - \frac{1}{2}f(y) = -\frac{1}{2} \sum v_i^3 - \frac{1}{2} \sum v_i^2. \\
\text{So} \\
k \geq -1 + \frac{1}{\sum v_i^2} \left( \frac{1}{2n} \sum v_i^2 \sum v_i - \frac{1}{2} \sum v_i^3 \right) \\
\text{Because} \\
f(y_i) > 0, \\
\text{so} \\
v_i = f(y_i) - f(x) = f(y_i) - 1 > -1. \\
\text{Also according to } \Delta f(x) = \frac{1}{n} \sum_{y \in N_x} f(x, y_i) < 0, \text{we have } \sum v_i < 0. \\
\text{we can assume } v_1 < v_2 < ... < v_n, \text{so } v_n < n - 1. \\
\text{At last we need only to prove:} \\
\frac{\sum v_i \sum v_i^2 - n \sum v_i^3}{2n \sum v_i^2} > -\frac{n}{2}. \\
\sum v_i > -n \rightarrow \frac{\sum v_i \sum v_i^2}{2n \sum v_i^2} > -\frac{n}{2n} = -\frac{1}{2}. \\
\sum v_i^3 = \sum v_i^2 v_i < \sum v_i^2 v_n < \sum v_i^2 (n-1) \rightarrow \frac{\sum v_i^3}{2n \sum v_i^2} > -\frac{n(n-1)}{2n} = -\frac{n}{2} + \frac{1}{2}. \\
\text{we end the proof.}
\end{align*}

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