A K-THEORETIC FULTON CLASS

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1. Summary

Fix a quasi-projective scheme $M$ over the complex numbers, and pick a global embedding in a smooth ambient variety $A$. Let $I \subset O_A$ denote the ideal sheaf of $M$. We get the cone on the embedding $M \hookrightarrow A$,

$$C_M A := \text{Spec } \bigoplus_{i \geq 0} I^i / I^{i+1}.$$  

Then Fulton’s total Chern class of $M$ [Fu, Example 4.2.6] is defined to be

$$c_F(M) := c(T_A|_M) \cap s(C_M A) \in A_* (M),$$

where $c$ is the total Chern class and $s$ denotes the Segre class. The result is independent of the choice of embedding. When $M$ is smooth, $c_F(M)$ is just the total Chern class $c(T_M) \cap [M] = \sum_{i \geq 0} c_i(M) \cap [M]$ of $M$.

We define a K-theoretic analogue. For notation see Section 2; in particular $t$ denotes the class of the weight one irreducible representation of $\mathbb{C}^*$.  

**Definition/Theorem.** Let $\mathbb{C}^*$ act trivially on $M$, with weight 1 on $\Omega_A|_M$, and with weight $i$ on $I^i / I^{i+1}$. The K-theoretic Fulton class

$$\Lambda_M := \Lambda^* \Omega_A|_M \otimes \left( \bigoplus_{i \geq 0} I^i / I^{i+1} \right) \in K_0(M)[t],$$

is independent of the smooth ambient space $A \supset M$ and is polynomial in $t$,

$$\Lambda_M = \sum_{i=0}^d (-1)^i \Lambda^i_M t^i \in K_0(M)[t],$$

where $d$ is the embedding dimension of $M$. When $M$ is smooth, $\Lambda^i_M = \Omega^i_M$.

In Proposition 3.3 we describe $\Lambda_M$ in de Rham terms for $M$ lci, but for more general $M$ we have not seen these classes in the literature.

Suppose $M$ has a perfect obstruction theory $E^* \rightarrow L_M$ [BF] of virtual dimension $\text{vd} := \text{rank}(E^*)$. Then we get a virtual cycle [BF] for which Siebert [Sie] gave the following formula

$$[M]^{\text{vir}} = [s((E^*)^\vee) \cap c_F(M)]_{\text{vd}} \in A_{\text{vd}}(M).$$

The K-theoretic analogue is the following.

**Theorem.** Given a perfect obstruction theory $E^* \rightarrow L_M$ the virtual structure sheaf can be calculated in terms of the K-theoretic Fulton class $\Lambda_M$ as

$$\mathcal{O}^{\text{vir}}_M = \left[ \frac{\Lambda_M}{\Lambda^* (E^*)} \right]_{t=1} = \left[ \frac{\Lambda_M}{\Lambda^* L^{\text{vir}}_M} \right]_{t=1}.$$
Siebert’s formula (1.2) showed that \([M]^\text{vir}\) depends only on the scheme structure of \(M\) and the K-theory class of the virtual cotangent bundle \(\mathbb{L}_M^\text{vir} := E^\bullet\) (and not the specific map in the perfect obstruction theory). Similarly the above Theorem implies that \(\mathcal{O}_M^\text{vir}\) also depends only on \(M\) and \(E^\bullet\). The analogy we refer to is that — with some work — (1.2) follows from (1.3) by the virtual Riemann-Roch theorem of [CFK, FG].

To understand why \(\Lambda_M\) might be polynomial in \(t\), consider what happens in the case that \(M\) is a zero dimensional Artinian scheme. (The general case is a family version of this.) By [AM, Section 11] the Hilbert series of the graded \(\mathcal{O}_M\)-module \(\bigoplus_{i \geq 0} I^i/I^{i+1}\) is \(p(t)/(1 - t)^d\), where \(d = \dim A\) and \(p\) is polynomial in \(t\). But \(\Omega_A|_M \cong \mathcal{O}_M^{\oplus d} \otimes t\) so tensoring by \(\Lambda^\bullet \Omega_A|_M = (1 - t)^d \mathcal{O}_M\) gives the result.

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2. K-theoretic analogue of Fulton’s Chern class

Throughout this paper we fix a quasi-projective scheme \(M\) over \(\mathbb{C}\), endow it with the trivial \(\mathbb{C}^*\) action, and work with the \(\mathbb{C}^*\)-equivariant K-theory of coherent sheaves on \(M\). In fact we use only the subgroup generated by coherent sheaves with \(\mathbb{C}^*\) actions with nonnegative weights, and its completion \(K^\mathbb{C}^*(M)_{\geq 0} \otimes \mathbb{Z}[t] = K_0(M)[[t]]\).

Here \(t\) is the class of the weight one \(\mathbb{C}^*\) irreducible representation. For \(E\) locally free we use \(\Lambda^\bullet E\) to refer to the K-theory class \(\sum_{i=0}^{\text{rank } E} (-1)^i \Lambda^i E\). When \(E\) has only strictly positive weights then \(\Lambda^\bullet E = 1/\text{Sym}^\bullet E\) in completed K-theory.

Remark 2.1. The results here commute with operations such as localisation with respect to a nontrivial \(\mathbb{C}^*\) action on \(M\) (as used in [F]), for instance) since any such \(\mathbb{C}^*\) action commutes with the trivial \(\mathbb{C}^*\) action used here.

Let \(\mathbb{C}^*\) act on \(\Omega_A|_M\) with weight 1 and on \(I^i/I^{i+1}\) with weight \(i\). Consider

\[
\Lambda_M := \Lambda^\bullet \Omega_A|_M \otimes \left( \bigoplus_{i \geq 0} I^i/I^{i+1}\right) \in K_0(M)[[t]].
\]

Theorem 2.3. This \(\Lambda_M\) is independent of the smooth ambient space \(A \supseteq M\) and is polynomial in \(t\), defining the K-theoretic Fulton classes

\[
\Lambda_M = \sum_{i=0}^d (-1)^i \Lambda^i_M t^i \in K_0(M)[[t]].
\]

Here \(d\) is the embedding dimension of \(M\). When \(M\) is smooth, \(\Lambda^i_M = \Omega^i_M\).

Proof. Fix two embeddings \(M \subset A_i, \ i = 1, 2\) with ideal sheaves \(I_i\) and cones \(C_i := C_M A_i\). We get the induced diagonal inclusion

\[
M \subset A_1 \times A_2
\]
with ideal \( I_{i_{12}} \) and cone \( C_{12} := C_M(A_1 \times A_2) \). This gives the exact sequence of cones [Fu, Example 4.2.6],

\[
(2.4) \quad 0 \rightarrow T_{A_2}|_M \rightarrow C_{12} \rightarrow C_1 \rightarrow 0.
\]

That is, \( I_{i_{12}}/I_{i_{12}}^{i+1} \) has an increasing filtration beginning with \( I_{i_{12}}/I_{i_{12}}^{i+1} \) and with graded pieces \( \text{Sym}^j \Omega_{A_2}|_M \otimes I_{1}^{-j}/I_{1}^{i-j+1} \). Therefore, in completed equivariant K-theory,

\[
\Lambda^\bullet \Omega_{A_1 \times A_2}|_M \otimes \left( \bigoplus_{i_{12} \geq 0} I_{i_{12}'}/I_{i_{12}}^{i-1} \right) \\
= \Lambda^\bullet \Omega_{A_1 \times A_2}|_M \otimes \left( \bigoplus_{i,j \geq 0} \text{Sym}^j \Omega_{A_2}|_M \otimes I_{1}^{-j}/I_{1}^{i-j+1} \right) \\
= \Lambda^\bullet \Omega_{A_1}|_M \otimes \Lambda^\bullet \Omega_{A_2}|_M \otimes \bigoplus_{i \geq 0} I_{i}/I_{1}^{i+1} \\
= \Lambda^\bullet \Omega_{A_1}|_M \otimes \bigoplus_{i_{12} \geq 0} I_{i_{12}}/I_{i_{12}}^{i+1}.
\]

This gives the independence from \( A \). We now show (2.2) is polynomial in \( t \) of degree \( \leq d := \dim A \). Taking \( A \) to have minimal dimension (the embedding dimension of \( M \) — the maximum of the dimensions of the Zariski tangent spaces at closed points of \( M \)) will complete the proof of the Theorem.

Writing (2.2) as \( \pi_* \pi^* \Lambda^\bullet \Omega_A|_M \), where \( \pi : C_M A \rightarrow M \) denotes the projection, the power series in \( t \) comes from the fact that \( \pi \) is not proper. So we only need show that \( \pi^* \Lambda^\bullet \Omega_A|_M \) is equivalent in \( K_{C}^\bullet (C_A M) \) to a class pushed forward from \( M \) with \( \mathbb{C}^* \) weights in \([0, d]\).

The basic idea of the proof is the following. Suppose we could pick a \( \mathbb{C}^* \)-invariant section of \( \pi^* T_A|_M \) which — on the complement \( C_M A \setminus M \) of \( M \) — has vanishing locus of dimension 0. By \( \mathbb{C}^* \) invariance, the vanishing locus is then just \( M \) (one which the section must vanish since \( T_A|_M \) has weight \(-1\)). Therefore \( \pi^* \Lambda^\bullet \Omega_A|_M \) becomes the Koszul resolution of a class which is zero on \( C_M A \setminus M \). This is the best way to understand the proof below, using only the \( i = 0 \) parts with no \( O_D \) terms in (2.5). In general, however, there only exist such sections with poles so we have to twist by the divisor \( D \) below, leading to all the \( i > 0 \) terms in (2.5).

So pick a line bundle \( L \gg 0 \) on \( M \) and \( D \in |L| \). We work with

\[
E := T_A|_M \otimes L
\]

since it has sections which we can use to cut down to lower dimensions. We give \( L \) the trivial \( \mathbb{C}^* \) action of weight 0 so that \( E \) has weight \(-1\). We have

\[
\Lambda^k \Omega_A|_M = \Lambda^k E^* \otimes L^k = \Lambda^k E^* \otimes (O - O_D)^{-k}.
\]
Using $\otimes$ for the derived tensor product, in K-theory we deduce

\[
\Lambda^* \Omega_A |_M = \sum_{k=0}^{d} (-1)^k \Lambda^k E^* \otimes \sum_{i=0}^{d} \left( \frac{k+i-1}{i} \right) \mathcal{O}_D^{\otimes i}
\]

(2.5)

\[
= \sum_{i=0}^{d} \mathcal{O}_D^{\otimes i} \otimes \sum_{k=0}^{d} (-1)^k \Lambda^k E^* \otimes \text{Sym}^{k-1} \mathcal{O}^{i+1}.
\]

Here we have used $[\mathcal{O}_D^{\otimes i}] = 0$ for $i > d \geq \dim M$. For $L \gg 0$ basepoint free this follows from taking divisors $D_1, \ldots, D_{d+1} \in |L|$ with empty intersection.

More generally, fix any $i \geq 0$. Since $L$ is basepoint free, we may pick generic divisors

\[
D^1, \ldots, D^i \in |L| \quad \text{with intersection} \quad Z_i := D^1 \cap \cdots \cap D^i \subset M
\]

with the following transversality property. On $M$, the locus where $I/I^2$ has rank $r$ is of dimension $d-r$. We choose the divisors such that the intersection of $Z_i$ with this locus has dimension $d - r - i$ (for each $r$), and such that the derived pull back $\pi^*$ of the $i$th term of (2.5) is

\[
\sum_{k=0}^{d} (-1)^k \Lambda^k (\pi^* E^*) |_{\pi^* Z_i} \otimes \text{Sym}^{k-1} \mathcal{O}^{i+1}_{\pi^* Z_i}.
\]

(2.6)

So we are left with showing that (2.6) is equal, in equivariant K-theory, to a class pushed forward from $Z_i \subset \pi^* Z_i$ with $\mathbb{C}^*$ weights in $[0, d]$.

To do this we use a variant of the Koszul resolution. (It is precisely the Koszul resolution for a section of $E$ when $i = 0$.) Let $p$ denote any of the projections down $\mathbb{P}^i$ such as

\[
p: \mathbb{P}^i \times C_M A \rightarrow C_M A,
\]

or the same with $C_M A$ replaced by $M$, $Z_i$ or $\pi^* Z_i$. Since $E$ has weight $-1$, the $\mathbb{C}^*$-invariant (i.e. weight 0) sections of $p^* \pi^* E(1)$ are

\[
\Gamma_{M \times \mathbb{P}^i}(p^* E(1) \otimes I/I^2) \subset \Gamma_{M \times \mathbb{P}^i}(p^* E(1) \otimes \pi_* \mathcal{O}_{C_M A})
\]

\[
= \Gamma_{C_M A \times \mathbb{P}^i}(p^* \pi^* E(1)),
\]

i.e. those which are linear on the fibres of $\pi: C_M A \rightarrow M$.

For $L \gg 0$ we claim that on restriction to $Z_i \times \mathbb{P}^i$, the generic such section $p^* E^*(-1)|_{Z_i \times \mathbb{P}^i} \rightarrow p^* I/I^2|_{Z_i \times \mathbb{P}^i}$ is onto. We already saw that on $Z_i \times \mathbb{P}^i$ the locus where $p^* I/I^2$ has rank $r$ is of dimension $(d - r - i) + i = d - r$. Therefore, after tensoring by the very ample line bundle $L(1)$, the sheaf $p^* I/I^2$ is surjected onto by $r + (d - r) = d$ generic sections. Equivalently $p^* I/I^2$ is surjected onto by $d$ copies of $L^{-1}(-1)$, or by a very negative rank $d$ bundle like $p^* E^*(-1)$.

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1When $k = 0 = i$ we have to work with the standard negative binomial convention that \( \binom{k+i-1}{i} = \binom{-1}{0} = 1 \). Therefore we also set $\text{Sym}^{-1} \mathcal{O}^{i+1}$ to be $\mathcal{O}$ for $i = 0$ and 0 for $i > 0$. 
This surjectivity means that the section cuts out (scheme-theoretically) the zero section $Z_i \times \mathbb{P}^i \subset \pi^*Z_i \times \mathbb{P}^i$. Therefore the corresponding Koszul resolution on $\pi^*Z_i \times \mathbb{P}^i$,

(2.7) $\Lambda^d(p^*\pi^*E^*)(-d) \to \cdots \to \Lambda^2(p^*\pi^*E^*)(-2) \to p^*\pi^*E^*(-1) \to \mathcal{O}$

has zeroth cohomology $h^0 = \mathcal{O}_{Z_i \times \mathbb{P}^i}$. It follows that all its other cohomology sheaves are also supported, scheme-theoretically, on $Z_i \times \mathbb{P}^i$ (thinking of (2.7) as a differential graded sheaf of algebras, its cohomology sheaves are then modules over its $h^0$ sheaf of algebras).

But $Z_i \times \mathbb{P}^i$ is $\mathbb{C}^*$-fixed, and $\mathbb{C}^*$ acts on the $\Lambda^kE^*$ term of (2.7) with weight $k$ for $k = 0, 1, \ldots, d$. Therefore the cohomology sheaves of (2.7) have $\mathbb{C}^*$ weights in $[0, d]$.

Finally then, the push down of (2.7) to $Z_i$ by $R\rho_i(\cdot \otimes \mathcal{O}(-i))$ has weights in $[0, d]$. Since $R\rho_i\mathcal{O}(-j) \cong \text{Sym}^{j-i+1}\mathcal{O}^{i+1}[-i]$ this pushdown is

(2.8) $\Lambda^d(\pi^*E^*)|_{\pi^*Z_i} \otimes \text{Sym}^{d-1}\mathcal{O}_{\pi^*Z_i} \to \Lambda^{d-1}(\pi^*E^*)|_{\pi^*Z_i} \otimes \text{Sym}^{d-2}\mathcal{O}_{\pi^*Z_i} \to \cdots \to \Lambda^2(\pi^*E^*)|_{\pi^*Z_i} \otimes \mathcal{O}_{\pi^*Z_i} \to \pi^*E^*|_{\pi^*Z_i}$.

(For $i = 0$ we get an extra term $\mathcal{O}$ at the right hand end. Cf. footnote [1]) But its K-theory class is precisely (2.6).

If we pick a locally free resolution $F^* \xrightarrow{s} I$ on $A$ (i.e. a vector bundle $F \to A$ with a section $s$ cutting out $s^{-1}(0) = M \subset A$) then we can express $\Lambda_M$ differently as follows. Give $F$ the $\mathbb{C}^*$ action of weight $-1$, so the embedding

(2.8) $C_M A \hookrightarrow F|_M$

induced by $s$ is equivariant. Let $\iota: M \hookrightarrow F|_M$ and $\pi: F|_M \to M$ denote the zero section and projection respectively.

**Lemma 2.9.** The K-theoretic Fulton class equals

$$\Lambda_M = \text{Lt}^*\mathcal{O}_{C_M A} \otimes \frac{\Lambda^*\Omega_A}{\Lambda^*F^*}|_M.$$

**Proof.** Applying $\pi_*\iota_* = \text{id}$ to the right hand side gives

$$\pi_*\left(\iota_*\mathcal{O}_M \otimes \mathcal{O}_{C_M A}\right) \otimes \frac{\Lambda^*\Omega_A}{\Lambda^*F^*}|_M.$$

Then $\iota_*\mathcal{O}_M = \pi^*\Lambda^*F^*|_M$ by the Koszul resolution of the zero section $M \hookrightarrow F|_M$, so by (1.1) we get

$$\Lambda_M = \left(\bigoplus_{i \geq 0} I^i/I^{i+1}\right) \otimes \frac{\Lambda^*\Omega_A}{\Lambda^*F^*}|_M.$$

3. de Rham cohomology

Consider the pushforward of $\Lambda_M$ to the formal completion $\hat{A}$ of $A$ along $M$. Its K-theory class looks remarkably similar to that of Hartshorne’s algebraic de Rham complex $[\Pi]$.

(3.1) $\Lambda^*\Omega_{\hat{A}}$. 
If we discard the de Rham differential in (3.1) and filter by order of vanishing along $M$ (with $n$th filtered piece $I^n \otimes \Lambda^n \Omega_A$) then the associated graded is

$$(3.2) \quad \Lambda^n \Omega_A|_M \otimes \left( \bigoplus_{i \geq 0} I^i/I^{i+1} \right).$$

This is just (the push forward from $M$ to $\hat{A}$ of) $\Lambda_M$ with its $C^*$ action forgotten. However convergence issues stop us from equating (3.2) with (3.1) in K-theory. Putting the $C^*$ action back into (3.2) we get convergence to $\Lambda_M$ in completed equivariant K-theory, but in general there is no $C^*$ action on (3.1).

The algebraic de Rham complex (3.1) — with the de Rham differential — has been shown by Illusie [Ill, Corollary VIII.2.2.8] (and more generally Bhatt [Bh]) to be quasi-isomorphic to the pushforward of the derived de Rham complex $\Lambda^* \mathbb{L}_M$ of $M$. (Here $\Lambda^*$ denotes the alternating sum of derived exterior powers.) And, as kindly suggested to us by Ben Antieau, we can prove that the K-theory class of $\Lambda^* \mathbb{L}_M$ (again without its de Rham differential) can be identified with $\Lambda_M$ when $M$ is a local complete intersection.

**Proposition 3.3.** Let $C^*$ act on $\mathbb{L}_M$ with weight 1, and suppose $M$ is lci. Then $\Lambda_M = \Lambda^* \mathbb{L}_M$ in $K_0(M)[[t]]$.

**Proof.** For $M$ lci we have $\mathbb{L}_{M/A} = I/I^2[1]$ so the exact triangle $\mathbb{L}_A|_M \to \mathbb{L}_M \to \mathbb{L}_{M/A}$ gives, in K-theory,

$$\mathbb{L}_M = \Omega_A|_M - I/I^2.$$  

Using the weight one $C^*$ action on $\mathbb{L}_M$ this gives

$$\Lambda^* \mathbb{L}_M = \Lambda^* \Omega_A|_M \otimes \text{Sym}^* I/I^2 \in K_0(M)[[t]].$$

Furthermore $I/I^2$ is locally free so

$$\text{Sym}^i I/I^2 \to I^i/I^{i+1}$$

is a surjection from a locally free sheaf to a sheaf of the same rank. It is therefore an isomorphism and we have

$$\Lambda^* \mathbb{L}_M = \Lambda^* \Omega_A|_M \otimes \left( \bigoplus_{i \geq 0} I^i/I^{i+1} \right).$$

By Theorem 2.3 this means $\Lambda^* \mathbb{L}_M$ is in fact polynomial in $t$, so we can set $t = 1$ to get a class in non-equivariant K-theory.

However it does not follow (and indeed is not in general true) that the push forward of $\Lambda_M$ can be equated with the algebraic de Rham complex (3.1) when $M$ is lci. Firstly, the de Rham differential does not preserve the $C^*$ action we have used, so we cannot lift Illusie’s theorem to equivariant K-theory. Secondly, this therefore gives us convergence issues; Illusie and Bhatt use the “Hodge completion” of the derived de Rham complex to get their quasi-isomorphism, and this differs from our completion.
4. A FORMULA FOR THE VIRTUAL STRUCTURE SHEAF

The foundations of cohomological virtual cycles are laid down in [BF, LT]; we use the notation from [BF]. The foundations for K-theoretic virtual cycles (or “virtual structure sheaves”) are laid down in [CFK, FG]; we use the notation from [FG].

Again let $M$ be a quasi-projective scheme over $\mathbb{C}$. A perfect obstruction theory $E^\bullet \to \mathbb{L}_M$ is a 2-term complex of vector bundles $E^\bullet = \{E^{-1} \to E^0\}$ with a map in $D(Coh M)$ to the cotangent complex $\mathbb{L}_M$ which is an isomorphism on $h^0$ and a surjection on $h^{-1}$.

We sometimes call $E^\bullet$ the virtual cotangent bundle $\mathbb{L}_M^\text{vir}$ of $M$. Its rank is the virtual dimension $vd := \text{rank } E^0 - \text{rank } E^{-1}$.

By [BF] this data defines a cone $C \subset E_1 := (E^{-1})^*$ from which we may define $M$’s virtual cycle

$$[M]\text{vir} := \iota ! [C] \in A_{vd}(M),$$

where $\iota: M \to E_1$ is the zero section. Siebert [Sie] proved the alternative formula

$$[M]\text{vir} = \left[ s((E^\bullet)^*) \cap c_F(M) \right]_{vd}.$$

The K-theoretic analogue of $[M]\text{vir}$ is the virtual structure sheaf [FG]

$$\mathcal{O}_M^\text{vir} := \left[ L\iota^* \mathcal{O}_C \right] \in K_0(M),$$

where $L\iota^* \mathcal{O}_C$ is a bounded complex because $\iota$ is a regular embedding.

The construction of Section 2 allows us to give a K-theoretic analogue.

**Theorem 4.2.** The virtual structure sheaf (4.1) can be calculated in terms of the K-theoretic Fulton class $\Lambda_M$ (2.2) as

$$\mathcal{O}_M^\text{vir} = \left[ \Lambda_M \Lambda^\bullet (E^\bullet) \right]_{t=1} = \left[ \Lambda_M \Lambda^\bullet \mathcal{L}_M^\text{vir} \right]_{t=1}.$$

In particular, when $M$ is smooth $\Lambda_M = \Lambda^\bullet \Omega_M$ and $[E^\bullet] = \Omega_M - \text{ob}^*_M$, so (4.3) recovers $\mathcal{O}^\text{vir} = \Lambda^\bullet \text{ob}^*_M$.

**Proof.** By [BF], the perfect obstruction theory $E^\bullet \to \mathbb{L}_M$ induces a cone $C \subset E_1$ which Siebert [Sie] proof of Proposition 4.4] shows sits inside an exact sequence of cones

$$0 \to T_A|_M \to C_M A \oplus E_0 \to C \to 0.$$

Here $A$ is any smooth ambient space containing $M$ with ideal $I$, so that $C_M A = \text{Spec } \bigoplus_{i \geq 0} I^i/I^{i+1}$.

As before we give the $E_i$ and $T_A|_M$ weight $-1$ (so $E^i$ and $\Omega_A|_M$ have weight 1) and let $\iota: M \to E_1$ and $\pi: E_1 \to M$ denote the zero section and projection respectively. Then

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2 Or its $\tau^{[-1,0]}$ truncation.

3 After possibly replacing $E^\bullet$ by a quasi-isomorphic 2-term complex of vector bundles.
\[ O_{\text{vir}} = L_L^*O_C = \pi_*\iota_\ast L_L^*O_C = \pi_*(O_C \otimes \iota_\ast O_M) = \pi_*(O_C \otimes \Lambda^\bullet E^{-1}) \]
evaluated at \( t = 1 \), by (4.1) and the Koszul resolution of \( \iota_\ast O_M \). By (4.4),
\[ \pi_*O_C = \text{Sym}^\bullet E^0 \otimes \left( \bigoplus_{i \geq 0} \mathcal{I}^i/\mathcal{I}^{i+1} \right) \otimes \Lambda^\bullet \Omega^1_M, \]
so by (2.2),
\[ O_{\text{vir}} = \left[ \Lambda_M \otimes \text{Sym}^\bullet E^0 \otimes \Lambda^\bullet E^{-1} \right]_{t=1} = \left[ \Lambda_M/\Lambda^\bullet(E^\bullet) \right]_{t=1}. \]

□

**Corollary 4.5.** \( O_{\text{vir}}^M \) depends only on \( M \) and the K-theory class of \( E^\bullet \).

Of more interest in enumerative K-theory is the twisted virtual structure sheaf
\[ \hat{O}_{\text{vir}}^M = O_{\text{vir}}^M \otimes \text{det}(E^\bullet)^{1/2} \]
of Nekrasov-Okounkov [NO]. Here we twist by a choice of square root of the virtual canonical bundle \( \text{det}(E^\bullet) = \text{det}E^0 \otimes (\text{det}E^{-1})^* \). The above shows it depends only on \( M \), the K-theory class of \( E^\bullet \), and the choice of square root (“orientation data”).

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