COMODULES FOR SOME SIMPLE $\mathcal{O}$-FORMS OF $G_m$

N. E. CSIMA AND R. E. KOTTWITZ

Abstract. This paper gives a rather concrete description of the category Rep($G$) for certain flat commutative affine group schemes $G$ over a discrete valuation ring such that the general fiber of $G$ is the multiplicative group.

Tannakian theory allows one to understand an affine group scheme $G$ over a commutative base ring $A$ in terms of the category Rep($G$) of $G$-modules, by which is meant comodules for the Hopf algebra corresponding to $G$. The theory is especially well-developed [Sa] in the case that $A$ is a field, and some parts of the theory still work well over more general rings $A$, say discrete valuation rings (see [Sa, We]).

When $A$ is a field of characteristic zero and $G$ is connected reductive, the category Rep($G$) is very well understood. However, with the exception of groups as simple as the multiplicative and additive groups, little seems to be known about what Rep($G$) looks like concretely when $A$ is no longer assumed to be a field, even in the most favorable case in which $A$ is a discrete valuation ring and $G$ is a flat affine group scheme over $A$ with connected reductive general fiber.

The modest goal of this paper is to give a concrete description of Rep($G$) for certain flat group schemes $G$ over a discrete valuation ring $\mathcal{O}$ such that the general fiber of $G$ is $G_m$.

Choose a generator $\pi$ of the maximal ideal of $\mathcal{O}$ and write $F$ for the field of fractions of $\mathcal{O}$. For any non-negative integer $k$, the construction of 1.1, when applied to $f = \pi^k$, yields a commutative flat affine group scheme $G_k$ over $\mathcal{O}$ whose general fiber is $G_m$. The $\mathcal{O}$-points of $G_k$ are given by

$$G_k(\mathcal{O}) = \{ t \in \mathcal{O}^\times : t \equiv 1 \mod \pi^k \}.$$ 

These form a projective system

$$\cdots \to G_2 \to G_1 \to G_0 = G_m$$

in an obvious way, and we may form the projective limit $G_\infty := \text{proj lim} G_k$. The Hopf algebra $S_k$ corresponding to $G_k$ can be described explicitly (see 1.1 and 1.2). The Hopf algebra $S_\infty$ corresponding to $G_\infty$ is

$$\text{inj lim } S_k = \{ \sum_{i \in \mathbb{Z}} x_i T^i \in F[T, T^{-1}] : \sum_{i \in \mathbb{Z}} x_i \in \mathcal{O} \}.$$

The categories Rep($G_\infty$) and Rep($G_k$) can be described very concretely. Indeed, Rep($G_\infty$) consists of the category of $\mathcal{O}$-modules $M$ equipped with a $\mathbb{Z}$-grading on $F \otimes_{\mathcal{O}} M$ (see 2.2, where a much more general result is proved). As for Rep($G_k$), we proceed in two steps.

First, the full subcategory of Rep($G_k$) consisting of those $G_k$-modules that are flat as $\mathcal{O}$-modules is equivalent (see Theorem 1.3.1) to the category of pairs $(V, M)$.
consisting of a $\mathbb{Z}$-graded $F$-vector space $V$ and an \textit{admissible} $O$-submodule $M$ of $V$, where admissible means that the canonical map $F \otimes_O M \rightarrow V$ is an isomorphism and $C_n M \subset M$ for all $n \geq 0$, where $C_n : V \rightarrow V$ is the graded linear map given by multiplication by $\pi^n (t)$ on the $i$-th graded piece of $V$. The $G_k$-module corresponding to $(V, M)$ is $M$, equipped with the obvious comultiplication.

Second, any $G_k$-module (see [1.4]) is obtained as the cokernel of some injective homomorphism $M_1 \rightarrow M_0$ coming from a morphism $(V_1, M_1) \rightarrow (V_0, M_0)$ of pairs of the type just described.

When $O$ is a algebra, the situation is even simpler: $M$ is an admissible $O$-submodule of the graded vector space if and only if $C_1 M \subset M$ and $F \otimes_O M \cong V$.

Moreover, in case $O$ is the formal power series ring $\mathbb{C}[[\varepsilon]]$, there is an interesting connection with affine Springer fibers (see [1.5]).

1. A description of $\text{Rep}(G)_f$ for certain group schemes $G$

Throughout this section we consider a commutative ring $A$ and a nonzerodivisor $f \in A$. Thus the canonical homomorphism $A \rightarrow A_f$ is injective, where $A_f$ denotes the localization of $A$ with respect to the multiplicative subset $\{f^n : n \geq 0\}$. For the rest of this section we denote $A_f$ by $B$ and use the canonical injection $A \rightarrow B$ to identify $A$ with a subring of $B$.

1.1. The group scheme $G$ over $A$. We are now going to define a commutative affine group scheme $G$, flat and finitely presented over $A$. There will be a canonical homomorphism $G \rightarrow \mathbb{G}_m$ that becomes an isomorphism after extending scalars from $A$ to $B$.

We begin by specifying the functor of points for $G$. For any commutative $A$-algebra $R$ we put

$$G(R) := \{ (t, x) \in R^\times \times R : t - 1 = fx \}$$

$$= \{ x \in R : 1 + fx \in R^\times \}.$$

Then $G$ is represented by the $A$-algebra

\begin{equation}
S := A[T, T^{-1}, X]/(T - 1 - fX)
= A[X]_{1+fX},
\end{equation}

which is clearly flat and finitely presented.

The multiplication on $G(R)$ is defined as $(t, x)(t', x') = (tt', x + x' + fxx')$. The identity element is $(1, 0)$ and the inverse of $(t, x)$ is $(t^{-1}, -t^{-1}x)$.

There is a canonical homomorphism $\lambda : G \rightarrow \mathbb{G}_m$, given by $(t, x) \mapsto t$. When $f$ is a nonzerodivisor in $R$, the homomorphism $\lambda : G(R) \rightarrow R^\times$ identifies $G(R)$ with $\ker[R^\times \rightarrow (R/fR)^\times]$, and when $f$ is a unit in $R$, then $G(R) = R^\times$, showing that the homomorphism $\lambda : G \rightarrow \mathbb{G}_m$ becomes an isomorphism after extending scalars from $A$ to $B$. Thus there is a canonical isomorphism $B \otimes_A S \cong B[T, T^{-1}]$.

\begin{lemma}
Let $M$ be an $A$-module on which $f$ is a nonzerodivisor. Let $F$ be any flat $A$-module. Then $f$ is also a nonzerodivisor on $F \otimes_A M$.
\end{lemma}

\begin{proof}
Tensor the injection $M \rightarrow M$ over $A$ with $F$.
\end{proof}

\begin{corollary}
The canonical homomorphism $S \rightarrow B \otimes_A S = B[T, T^{-1}]$ is injective, so that we may identify $S$ with a subring of $B[T, T^{-1}]$.
\end{corollary}
Proof. Just note that $S$ is flat over $A$ and $f$ is a nonzerodivisor on $A$. Therefore $f$ is a nonzerodivisor on $S \otimes_A A = S$, and this means that $S \rightarrow B \otimes_A S$ is injective. □

1.2. Description of $S$ as a subring of $B[T,T^{-1}]$. We have just identified $S$ with a subring of $B[T,T^{-1}]$. It is obvious from (1.1.1) that $S$ is the $A$-subalgebra of $B[T,T^{-1}]$ generated by $T, T^{-1}, \frac{T}{f}$. However there is a more useful description of $S$ in terms of $B$-module maps

$$L_n : B[T, T^{-1}] \rightarrow B,$$

one for each non-negative integer $n$, defined by the formula

$$L_n \left( \sum_{i \in \mathbb{Z}} b_i T^i \right) = \sum_{i \in \mathbb{Z}} \binom{i}{n} b_i.$$

Here $\binom{i}{n}$ is the binomial coefficient $i(i-1) \ldots (i-n+1)/n!$, defined for all $i \in \mathbb{Z}$. When $n = 0$, we have $\binom{i}{0} = 1$ for all $i \in \mathbb{Z}$.

The following remarks may help in understanding the maps $L_n$. For any non-negative integer $n$, we have the divided-power differential operator

$$D^{[n]} : B[T, T^{-1}] \rightarrow B[T, T^{-1}]$$

defined by

$$(1.2.1) \quad D^{[n]} \left( \sum_{i \in \mathbb{Z}} b_i T^i \right) = \sum_{i \in \mathbb{Z}} \binom{i}{n} b_i T^{i-n}.$$

The Leibniz formula says that

$$(1.2.2) \quad D^{[n]}(gh) = \sum_{r=0}^{n} D^{[r]}(g)D^{[n-r]}(h).$$

For any $g \in B[T] \subset B[T,T^{-1}]$ the Taylor expansion of $g$ at $T = 1$ reads

$$(1.2.3) \quad g = \sum_{n=0}^{\infty} (D^{[n]}g)(1) \cdot (T-1)^n,$$

the sum having only finitely many non-zero terms.

For any $g \in B[T, T^{-1}]$ we have $L_n(g) = f^n(D^{[n]}g)(1)$. It follows from (1.2.2) that for all $g, h \in B[T, T^{-1}]$

$$(1.2.4) \quad L_n(gh) = \sum_{r=0}^{n} L_r(g)L_{n-r}(h),$$

and for all $h \in B[T] \subset B[T,T^{-1}]$ it follows from (1.2.3) that

$$(1.2.5) \quad h = \sum_{n=0}^{\infty} L_n(h) \left( \frac{T-1}{f} \right)^n.$$

Now we are in a position to prove

**Proposition 1.2.1.** The subring $S$ of $B[T, T^{-1}]$ is equal to

$$\{ g \in B[T, T^{-1}] : \forall n \geq 0 \quad L_n(g) \in A \}. $$
Proof. Write $S'$ for $\{g \in B[T, T^{-1}] : \forall n \geq 0 \ L_n(g) \in A\}$. Obviously $S'$ is an $A$-submodule of $B[T, T^{-1}]$, and it follows from (1.2.3) that $S'$ is a subring of $B[T, T^{-1}]$. A simple calculation shows that $T, T^{-1}, (T-1)/f$ lie in $S'$, and as these three elements generate $S$ as $A$-algebra, we conclude that $S \subset S'$.

Now let $g \in S'$. There exists an integer $n$ large enough that $h := T^{-m}g$ lies in the subring $B[T]$. Note that $h \in S'$. Equation (1.2.5) shows that $h \in S$, since $(T-1)/f \in S$ and $L_n(h) \in A$. Therefore $g = T^{-m}h \in S$. \hfill $\Box$

Now let $M$ be an $A$-module on which $f$ is a nonzerodivisor, so that we may use the canonical $A$-module map $M \to B \otimes_A M$ (sending $m$ to $1 \otimes m$) to identify $M$ with an $A$-submodule of $N := B \otimes_A M$.

It follows from Lemma 1.1.1 that the canonical $A$-module map

$$S \otimes_A M \to B \otimes_A (S \otimes_A M) = B[T, T^{-1}] \otimes_B N$$

identifies $S \otimes_A M$ with an $A$-submodule of $B[T, T^{-1}] \otimes_B N$. We will now derive from the previous proposition a description of $S \otimes_A M$ inside $B[T, T^{-1}] \otimes_B N$. For this we will need the $B$-module maps $L_n : B[T, T^{-1}] \otimes_B N \to N$ defined by

$$L_n \left( \sum_{i \in \mathbb{Z}} T^i \otimes x_i \right) = \sum_{i \in \mathbb{Z}} f^n \left( \begin{pmatrix} i \\ n \end{pmatrix} \right) x_i.$$ 

Here $x_i \in N$, all but finitely many being 0.

Lemma 1.2.2. The $A$-submodule $S \otimes_A M$ of $B[T, T^{-1}] \otimes_B N$ is equal to

$$\{x \in B[T, T^{-1}] \otimes_B N : \forall n \geq 0 \ L_n(x) \in M\}.$$ 

Proof. From Proposition 1.2.1 we see that there is an exact sequence

$$0 \to S \to B[T, T^{-1}] \xrightarrow{L} \prod_{n \geq 0} B/A,$$

the $n$-th component of the map $L$ being the composition

$$B[T, T^{-1}] \xrightarrow{L_n} B \to B/A.$$ 

In fact the map $L$ takes values in $\oplus_{n \geq 0} B/A$. Indeed, for any $g \in B[T, T^{-1}]$ there exists an integer $m$ large enough that $f^m g \in A[T, T^{-1}]$, and then $L_n(g) \in A$ for all $n \geq m$. Moreover $L$ maps $B[T, T^{-1}]$ onto $\oplus_{n \geq 0} B/A$. Indeed, a simple calculation shows that for $b \in B$ and $m \geq 0$

$$L_n(bf^{-m}(T-1)^m) = \begin{cases} b & \text{if } m = n \\ 0 & \text{otherwise}. \end{cases}$$

(First check that $D^{[m]}((T-1)^m) = \binom{m}{n}(T-1)^{m-n}$, say by induction on $m$; note that this formula is valid even if $n > m$, since $\binom{m}{n} = 0$ when $0 \leq m < n$.)

We now have a short exact sequence

$$0 \to S \to B[T, T^{-1}] \xrightarrow{L} \bigoplus_{n \geq 0} B/A \to 0$$

of $A$-modules. Tensoring with the $A$-module $M$, we obtain an exact sequence

$$S \otimes_A M \to B[T, T^{-1}] \otimes_A M \xrightarrow{L \otimes \text{id}_M} \bigoplus_{n \geq 0} B/A \otimes_A M \to 0.$$
Now
\[ B[T, T^{-1}] \otimes_A M = B[T, T^{-1}] \otimes_B B \otimes_A M = B[T, T^{-1}] \otimes_B N \]
and
\[ \left( \bigoplus_{n \geq 0} B/A \right) \otimes_A M = \bigoplus_{n \geq 0} N/M. \]

With these identifications (and recalling that \( S \otimes_A M \to B[T, T^{-1}] \otimes_B N \) is injective), we see that (1.2.3) describes \( S \otimes_A M \) as the subset of \( B[T, T^{-1}] \otimes_B N \) consisting of elements \( x \) such that \( L_n(x) \in M \) for all \( n \geq 0 \), and this completes the proof. \( \square \)

1.3. **Comodules for** \( S \). Since \( G \) is an affine group scheme over \( A \), the \( A \)-algebra \( S \) is actually a commutative Hopf algebra, and we can consider \( \text{Rep}(G) \), the category of \( S \)-comodules. We denote by \( \text{Rep}(G)_f \) the full subcategory of \( \text{Rep}(G) \) consisting of \( S \)-comodules \( M \) such that \( f \) is a nonzerodivisor on the \( A \)-module underlying \( M \). Our next goal is to give a concrete description of \( \text{Rep}(G)_f \).

In order to do so, we need one more construction. Let \( N = \oplus_{i \in \mathbb{Z}} N_i \) be a \( \mathbb{Z} \)-graded \( B \)-module. For each non-negative integer \( n \) we define an endomorphism \( C_n : N \to N \) of the graded \( B \)-module \( N \) by requiring that \( C_n \) be given by multiplication by \( f^n(i) \) on \( N_i \). Thus

\[ C_n \left( \sum_{i \in \mathbb{Z}} x_i \right) = \sum_{i \in \mathbb{Z}} f^n(i) x_i. \]

Here \( x_i \in N_i \), all but finitely many being \( 0 \).

Let \( C \) be the category whose objects are pairs \((N, M)\), \( N \) being a \( \mathbb{Z} \)-graded \( B \)-module, and \( M \) being an \( A \)-submodule of \( N \) such that the natural map \( B \otimes_A M \to N \) is an isomorphism and such that \( C_n M \subset M \) for all \( n \geq 0 \). A morphism \((N, M) \to (N', M')\) is a homomorphism \( \phi : N \to N' \) of graded \( B \)-modules such that \( \phi M \subset M' \).

We now define a functor \( F : \text{Rep}(G)_f \to C \). Let \( M \) be an object of \( \text{Rep}(G)_f \). Then \( N := B \otimes_A M \) is a comodule for \( B \otimes_A S = B[T, T^{-1}] \). It is known (see [SGA3, Exposé 1]) that the category of \( B[T, T^{-1}] \)-comodules is equivalent to the category of \( \mathbb{Z} \)-graded \( B \)-modules. Thus \( N \) has a \( \mathbb{Z} \)-grading \( N = \oplus_{i \in \mathbb{Z}} N_i \), and the comultiplication \( \Delta_N : N \to B[T, T^{-1}] \otimes_B N \) is given by \( \sum_{i \in \mathbb{Z}} x_i \mapsto \sum_{i \in \mathbb{Z}} T^i \otimes x_i \).

Since \( f \) is a nonzerodivisor on \( M \), the canonical map \( M \to B \otimes_A M = N \) identifies \( M \) with an \( A \)-submodule of \( N \).

We define our functor \( F : \text{Rep}(G)_f \to C \). For this to make sense we must check that \( C_n M \subset M \) for all \( n \geq 0 \). Let \( m \in M \), and write \( m = \sum_{i \in \mathbb{Z}} x_i \) in \( \oplus_{i \in \mathbb{Z}} N_i = N \). Since the comodule \( N \) was obtained from \( M \) by extension of scalars, the element \( x = \Delta_N m = \sum_{i \in \mathbb{Z}} T^i \otimes x_i \in B[T, T^{-1}] \otimes_B N \) lies in the image of \( S \otimes_A M \to B[T, T^{-1}] \otimes_B N \). Lemma [1.2.2] then implies that \( L_n(x) = \sum_{i \in \mathbb{Z}} f^n(i) x_i = C_n(m) \) lies in \( M \), as desired.

**Theorem 1.3.1.** The functor \( F : \text{Rep}(G)_f \to C \) is an equivalence of categories.

**Proof.** Let us first show that \( F \) is essentially surjective. Let \((N, M)\) be an object in \( C \). We are going to use the comultiplication \( \Delta_N : N \to B[T, T^{-1}] \otimes_B N \) to turn \( M \) into an \( S \)-comodule.

Since \( M \) is an \( A \)-submodule of \( N \), it is clear that \( f \) is a nonzerodivisor on \( M \). As we have seen before, it follows that \( f \) is a nonzerodivisor on \( S \otimes_A M \) and hence that the natural map \( S \otimes_A M \to B \otimes_A (S \otimes_A M) = B[T, T^{-1}] \otimes_B N \) identifies \( S \otimes_A M \) with an \( A \)-submodule of \( B[T, T^{-1}] \otimes_B N \).
Using Lemma 1.2.2, we see that our assumption that \( C_n M \subset M \) for all \( n \geq 0 \) is simply the statement that \( \Delta_N M \subset S \otimes_A M \). In other words there exists a unique \( A \)-module map \( \Delta_M : M \to S \otimes_A M \) such that \( \Delta_M \) yields \( \Delta_N \) after extending scalars from \( A \) to \( B \).

We claim that \( \Delta_M \) makes \( M \) into an \( S \)-comodule. For this we must check the commutativity of two diagrams, and this follows from the commutativity of these diagrams after extending scalars from \( A \) to \( B \), once one notes that for any two \( A \)-modules \( M_1, M_2 \) on which \( f \) is a nonzerodivisor
\[
\text{Hom}_A(M_1, M_2) = \{ \phi \in \text{Hom}_B(B \otimes_A M_1, B \otimes_A M_2) : \phi(M_1) \subset M_2 \}.
\]
Here of course we are identifying \( M_1, M_2 \) with \( A \)-submodules of \( B \otimes_A M_1, B \otimes_A M_2 \) respectively. (At one point we need that \( f \) is a nonzerodivisor on \( S \otimes_A S \otimes_A M \), which is true since \( S \otimes_A S \) is flat over \( A \).)

As \( F \) takes \( M \) to \( (N, M) \), we are done with essential surjectivity. It is easy to see that \( F \) is fully faithful; this too uses (1.3.1). \( \square \)

1.4. **Principal ideal domains** \( A \). One defect of the theorem we just proved is that it only describes those \( G \)-modules on which \( f \) is a nonzerodivisor. When \( A \) is a principal ideal domain, as we assume for the rest of this subsection, we can do better. Now \( f \) is simply any non-zero element of \( A \). As a consequence of our theorem we obtain an equivalence of categories between the category \( \text{Rep}(G)_{\text{flat}} \) of \( G \)-modules \( M \) such that \( M \) is flat as \( A \)-module and the full subcategory of \( \mathcal{C} \) consisting of pairs \((N, M)\) for which \( M \) is a flat \( A \)-module (in which case \( N \cong B \otimes_A M \) is necessarily a flat \( B \)-module).

The next lemma is a variant of [Sc Prop. 3].

**Lemma 1.4.1.** Let \( A \) be a principal ideal domain, let \( C \) be a flat \( A \)-coalgebra, and let \( E \) be a \( C \)-comodule. Then there exists a short exact sequence of \( C \)-comodules
\[
0 \to F_1 \to F_0 \to E \to 0
\]
in which \( F_0, F_1 \) are flat as \( A \)-modules.

**Proof.** We imitate Serre’s proof. Recall (see 1.2 in loc. cit.) that for any \( A \)-module \( M \) the map \( \Delta \otimes id_M : C \otimes_A M \to C \otimes_A C \otimes_A M \) (\( \Delta \) being the comultiplication for \( C \)) gives \( C \otimes_A M \) the structure of \( C \)-comodule, and that (see 1.4 in loc. cit.) the comultiplication map \( \Delta_E : E \to C \otimes_A E \) is an injective comodule map when \( C \otimes_A E \) is given the comodule structure just described. We use \( \Delta_E \) to identify \( E \) with a subcomodule of \( C \otimes_A E \).

Now choose a surjective \( A \)-linear map \( p : F \to E \), where \( F \) is a free \( A \)-module. Let \( F_0 \) denote the preimage of \( E \) under the surjective comodule map \( id \otimes p : C \otimes_A F \to C \otimes_A E \). Since \( F_0 \) is the kernel of
\[
C \otimes_A F \to C \otimes_A E \to (C \otimes_A E)/E,
\]
it is a subcomodule of \( C \otimes_A F \), and \( id \otimes p \) restricts to a surjective comodule map \( F_0 \to E \), whose kernel we denote by \( F_1 \). Since \( C \) and \( F \) are flat, so too are \( C \otimes_A F \), \( F_0 \), and \( F_1 \), and we are done. We used that for principal ideal domains, a module is flat if and only if it is torsion-free, and being torsion-free is a property that is inherited by submodules. \( \square \)

Returning to our Hopf algebra \( S \), we see that any \( G \)-module \( E \) has a resolution \( 0 \to F_1 \to F_0 \to E \to 0 \) in which \( F_1, F_0 \) are objects of \( \text{Rep}(G)_{\text{flat}} \) and hence are described by our theorem. We conclude that \( E \) has the following form. There exist an
injective homomorphism $\phi : N \to N'$ of graded $B$-modules and flat $A$-submodules $M, M'$ of $N, N'$ respectively such that $\phi M \subset M'$ and $(N, M), (N', M') \in \mathcal{C}$, having the property that $E$ is isomorphic to $M'/\phi M$ as $G$-module.

1.5. A special case. When $A$ is a $\mathbb{Q}$-algebra, the category $\mathcal{C}$ is very simple. Indeed, there is a polynomial $P_n \in \mathbb{Q}[U]$ of degree $n$ such that $\binom{n}{i} = P_n(i)$, and therefore

$$C_n = Q_n(C),$$

where $C = C_1$ and $Q_n := f^n P_n(f^{-1}U) \in A[U]$. Therefore $\mathcal{C}$ is the category of pairs $(N, M)$ consisting of a $\mathbb{Z}$-graded $B$-module $N$ and an $A$-submodule $M$ of $N$ such that the natural map $B \otimes_A M \to N$ is an isomorphism and such that $CM \subset M$, where $C$ is the endomorphism of the graded module $N = \oplus_{i\in\mathbb{Z}} N_i$ given by multiplication by $f^i$ on $N_i$.

When $A$ is the formal power series ring $\mathcal{O} := \mathbb{C}[\![\varepsilon]\!])$, and $f = \varepsilon^k$ (for some non-negative integer $k$) our constructions yield a group scheme $G$ over $\mathcal{O}$ such that $G(\mathcal{O}) = \{t \in \mathcal{O}\times : t \equiv 1 \mod \varepsilon^k\}$, and the category of representations of $G$ on free $\mathcal{O}$-modules of finite rank is equivalent to the category of pairs $(V, M)$, where $V$ is a finite dimensional graded vector space over $\mathcal{F} := \mathbb{C}((\varepsilon))$ and $M$ is an $\mathcal{O}$-lattice in $V$ such that $CM \subset M$, where $C$ is given by multiplication by $i\varepsilon^k$ on the $i$-th graded piece of $V$. It is amusing to note that for fixed $V$, the space of all $M$ satisfying $CM \subset M$ is an affine Springer fiber, which, when all the non-zero graded pieces of $V$ are one-dimensional, is actually one of the affine Springer fibers studied at some length in [GKM], where it was shown to be paved by affine spaces. Finally, since $\mathcal{O}$ is a principal ideal domain, the results in [14] give a concrete description of all $G$-modules.

2. Certain Hopf algebras and their comodules

Throughout this section $A$ is a commutative ring and $B$ is a commutative algebra such that the canonical homomorphism $B \otimes_A B \to B$ (given by $b_1 \otimes b_2 \mapsto b_1 b_2$) is an isomorphism. For example $B$ might be of the form $S^{-1}A/I$ for some multiplicative subset $S$ of $A$ and some ideal $I$ in $S^{-1}A$.

Let $N$ be a $B$-module. Then the canonical $B$-module map $B \otimes_A N \to N$ (given by $b \otimes n \mapsto b n$) is an isomorphism. It follows that the canonical $A$-module homomorphism $N \to B \otimes_A N$ (given by $n \mapsto 1 \otimes n$) is actually an isomorphism of $B$-modules (since $N \to B \otimes_A N \to N$ is the identity).

Moreover, for any two $B$-modules $N_1, N_2$, we have isomorphisms

\begin{equation}
\text{Hom}_B(N_1, N_2) \cong \text{Hom}_A(N_1, N_2)
\end{equation}

and

\begin{equation}
N_1 \otimes_A N_2 \cong N_1 \otimes_B N_2.
\end{equation}

2.1. General remarks on Hopf algebras and their comodules. Let $S$ be a Hopf algebra over $A$. The composition $A \to S \to A$ of the unit and counit is the identity, and therefore there is a direct sum decomposition $S = A \oplus S_0$ of $A$-modules, where $S_0$ is by definition the kernel of the counit $S \to A$. In this subsection all tensor products will be taken over $A$ and the subscript $A$ will be omitted.

We denote by $\Delta : S \to S \otimes S$ the comultiplication for $S$. The counit axioms imply that $\Delta$ takes the form $\Delta(a + s_0) = a + s_0 \otimes 1 + 1 \otimes s_0 + \Delta(s_0)$, when we identify $S$ with $A \oplus S_0$ and $S \otimes S$ with $A \oplus (S_0 \otimes A) \oplus (A \otimes S_0) \oplus (S_0 \otimes S_0)$. Here $\Delta$ is a uniquely determined $A$-module map $S_0 \to S_0 \otimes S_0$. 

For any $S$-comodule $M$ with comultiplication $\Delta_M : M \to S \otimes M$ the counit axiom for $M$ implies that $\Delta_M(m) = 1 \otimes m + \tilde{\Delta}_M(m)$ for a uniquely determined $A$-module map
$$\tilde{\Delta}_M : M \to S_0 \otimes M.$$In this way we obtain an equivalence of categories between $S$-comodules and $A$-modules $M$ equipped with an $A$-linear map $\tilde{\Delta}_M : M \to S_0 \otimes M$ such that the diagram
$$(2.1.1)$$
\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{\Delta}_M} & S_0 \otimes M \\
\downarrow{\Delta_M} & & \downarrow{\tilde{\Delta} \otimes id} \\
S_0 \otimes M & \xrightarrow{id \otimes \tilde{\Delta}_M} & S_0 \otimes S_0 \otimes M
\end{array}
\]
commutes.

2.2. Hopf algebras for $B$ give Hopf algebras for $A$. Let $S$ be a Hopf algebra over $B$. As in the previous subsection we decompose $S$ as $B \oplus S_0$. It is easy to see that there is a unique Hopf algebra structure on $R := A \otimes S_0$ such that the unit and counit for $R$ are the obvious maps $A \to R$ and $R \to A$ and such that the Hopf algebra structure on $\bigotimes B R$ agrees with the given one on $S$ under the natural $B$-module isomorphism $B \otimes_A R \cong S$. What makes this work is $(2.0.2)$, a consequence of our assumption that $B \otimes_A B \to B$ is an isomorphism, so that, for example, $S_0 \otimes_B S_0 \cong S_0 \otimes_A S_0$.

The comultiplications $\Delta_R, \Delta_S$ on $R,S$ respectively are given by
$$(2.2.1)$$
\[
\Delta_R(a + s_0) = a + s_0 \otimes 1 + 1 \otimes s_0 + \tilde{\Delta}(s_0)
\]
$$(2.2.2)$$
\[
\Delta_S(b + s_0) = b + s_0 \otimes 1 + 1 \otimes s_0 + \tilde{\Delta}(s_0)
\]
and similar considerations apply to the multiplication maps $R \otimes_B R \to R, S \otimes_B S \to S$ and the antipodes $R \to R, S \to S$.

**Proposition 2.2.1.** The category of $R$-comodules is equivalent to the category of $A$-modules $M$ equipped with an $S$-comodule structure on $N := B \otimes_A M$.

**Proof.** We have already observed that giving an $R$-comodule is the same as giving an $A$-module $M$ equipped with an $A$-module map $\tilde{\Delta}_M : M \to S_0 \otimes_A M$ such that $\ref{2.1.1}$ commutes. Since $S_0$ is a $B$-module and $B \otimes_A B \cong B$, giving $\tilde{\Delta}_M$ such that $\ref{2.1.1}$ commutes is the same as giving a $B$-module map $\tilde{\Delta}_N : N \to S_0 \otimes_B N$ such that
$$(2.1.1)$$
\[
\begin{array}{ccc}
N & \xrightarrow{\tilde{\Delta}_N} & S_0 \otimes_B N \\
\downarrow{\Delta_N} & & \downarrow{\tilde{\Delta} \otimes id} \\
S_0 \otimes_B N & \xrightarrow{id \otimes \tilde{\Delta}_N} & S_0 \otimes_B S_0 \otimes_B N
\end{array}
\]
commutes, or, in other words, giving an $S$-comodule structure on $N$. \qed

2.3. Special case. Let $\mathcal{O}$ be a valuation ring and $F$ its field of fractions. Let $G$ be an affine group scheme over $F$ and let $S$ be the corresponding commutative Hopf algebra over $F$. Decompose $S$ as $F \oplus S_0$ and define a commutative Hopf algebra $R$ over $\mathcal{O}$ by $R := \mathcal{O} \oplus S_0$. Corresponding to $R$ is an affine group scheme $\tilde{G}$ over $\mathcal{O}$, and giving a representation of $\tilde{G}$ (that is, an $R$-comodule) is the same as giving an $\mathcal{O}$-module $M$ together with an $S$-comodule structure on $F \otimes_{\mathcal{O}} M$. 

For example, when $G$ is the multiplicative group $\mathbb{G}_m$, the Hopf algebra $R$ is
\[ \{ \sum_{i \in \mathbb{Z}} a_i T^i \in F[T, T^{-1}] : \sum_{i \in \mathbb{Z}} a_i \in \mathcal{O} \} \],
which is easily seen to be the union of the Hopf subalgebras $S_k$ discussed in the introduction.

References

[GKM] M. Goresky, R. Kottwitz and R. MacPherson, *Purity of equivalued affine Springer fibers*, Represent. Theory 10 (2006), 130–146.

[Sa] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics 265, Springer-Verlag, Berlin, 1972.

[Se] J-P. Serre, *Groupes de Grothendieck des schémas en groupes réductifs déployés*, Publ. Math. IHES 34 (1968), 37–52.

[SGA3] M. Demazure and A. Grothendieck, *SGA3, Schémas en Groupes, Tome I*, Lecture Notes in Mathematics 151, Springer-Verlag, Heidelberg, 1970.

[We] T. Wedhorn, *On Tannakian duality over valuation rings*, J. Algebra 282 (2004), 575–609.

N. E. Csima, Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637

E-mail address: ecsima@math.uchicago.edu

Robert E. Kottwitz, Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637

E-mail address: kottwitz@math.uchicago.edu