On the $\epsilon = d - 2$ expansion of the Ising model

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Abstract: We study the Ising model in $d = 2 + \epsilon$ dimensions using the conformal bootstrap. As a minimal-model Conformal Field Theory (CFT), the critical Ising model is exactly solvable at $d = 2$. The deformation to $d = 2 + \epsilon$ with $\epsilon \ll 1$ furnishes a relatively simple system at strong coupling outside of even dimensions. At $d = 2 + \epsilon$, the scaling dimensions and correlation function coefficients receive $\epsilon$-dependent corrections. Using numerical and analytical conformal bootstrap methods in Lorentzian signature, we rule out the possibility that the leading corrections are of order $\epsilon^1$. The essential conflict comes from the $d$-dependence of conformal symmetry, which implies the presence of new states. A resolution is that there exist corrections of order $\epsilon^{1/k}$ where $k > 1$ is an integer. The linear independence of conformal blocks plays a central role in our analyses. Since our results are not derived from positivity constraints, this bootstrap approach can be extended to the rigorous studies of non-positive systems, such as non-unitary, defect/boundary and thermal CFTs.
1 Introduction

The $d$-dimensional Ising model is a fundamental model in statistical physics and condensed matter physics. Historically, it was proposed by Lenz to describe ferromagnetism and the case of $d=1$ was solved by Ising. This simple model displays rich physics and captures some main traits of phase transitions and many-body problems. At criticality, it belongs to one of the simplest universality classes, characterized by the global $Z_2$ symmetry.

The upper and lower critical dimensions of the Ising model are given by $d_{\text{upper}}=4$ and $d_{\text{lower}}=1$. For $d>4$, the critical behaviour of the Ising model is described by Landau’s mean-field theory [1], in which fluctuations are neglected due to the averaging effects of a large number of adjacent spins in square lattices. As $d$ decreases, fluctuations play a more significant role. According to Ginzburg’s criterion, the mean-field description is not sufficient for $d \leq 4$. For instance, the domain-wall fluctuations dominate at $d=1$ and the Ising model has no phase transition at finite temperature. For $1<d<4$, the Ising critical exponents are different from the mean field values and vary with $d$. \(^1\) As a continuation of Landau’s theory, Wilson and Fisher calculated the critical exponents in $d=4-\epsilon$ dimensions using the perturbative $\epsilon$ expansion [3]. The $\epsilon$ expansion has proved to be a valuable tool in the studies of critical phenomena [4, 5].

At $d=2$, it is well-known that the Ising model is solvable since Onsager’s groundbreaking results [6]. The critical behaviour is described by the fixed point of renormalization group flows. In particular, scale invariance of the fixed point is further promoted to conformal invariance. The critical correlation functions of the spin operator and energy operators \(^2\) are

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\(^1\)At $d=4$, there are logarithmic corrections to the mean field description [2].

\(^2\)They are the continuum counterparts of the lattice spin and interaction energy. More abstractly, they correspond to the lowest $Z_2$ odd and even operators in the spectrum.
known exactly from the $\mathcal{M}(4,3)$ minimal-model Conformal Field Theory (CFT). As a natural
continuation, it would be interesting to deform the 2d exact solution to $d = 2 + \epsilon$ dimensions.

The $\epsilon$ expansion usually concerns weakly coupled systems \(^3\), but the deformed Ising solution
around $d = 2$ remains strongly coupled \(^4\), so the strong coupling physics is more manifest
than the case around $d = 4$. More recently, the $\epsilon = d - 2$ expansion has also been used to
study deconfined quantum criticality \([11, 12]\). \(^5\) We will study the Ising model in $d = 2 + \epsilon$
dimensions using the conformal bootstrap methods.

The conformal bootstrap program aims to classify and solve CFTs by general principles and consistency conditions \([14, 15]\). For $d = 2$, conformal symmetry becomes infinite-dimensional and this program can be carried out rather successfully \([16, 17]\). The studies in $d > 2$ dimensions are more challenging because conformal symmetry is less constraining. However, considerable progress has been achieved due to the seminal work \([18]\), in which the unitarity assumption and the crossing equations are formulated as inequalities. This modern bootstrap approach has led to rigorous bounds on the space of unitary CFTs, such as the most precise determinations of the $d = 3$ Ising critical exponents \([19–22]\). We refer to \([23–27]\) for useful reviews and lecture notes.

The critical Ising model in various dimensions can be viewed as a continuous family of $\mathbb{Z}_2$-
covariant CFTs parametrized by $d$. The case of non-integer $d$ has been studied by the unitary
bootstrap methods in \([28–30]\). The bounds exhibit similar features as those at $d = 2, 3$ and the results near $d = 4$ are consistent with the $(4 - d)$ expansion. However, a subtle point is
that the Wilson-Fisher fixed point is non-unitary in non-integer dimensions, as the spectrum contains descendant states of complex scaling dimensions \([31]\). \(^6\) It will be useful to consider complementary approaches that are not based on unitarity, such as the flow method \([32]\) and the truncation method \([33]\). The truncated bootstrap approach has been applied to the study of non-positive problems \([34–45]\). In the original formulation \([33]\), the truncated problem is encoded in determinants. In \([46]\), we proposed some new ingredients, which we believe are important to a more systematic formulation. We emphasized the essential role of linear independence and introduced the concept of norm to the truncation approach. In this work, we will apply these notions to the numerical bootstrap study of the $d = 2 + \epsilon$ Ising CFT.

On the other hand, it was noticed in \([30]\) that the tentative spectrum from the unitary numerical approach exhibits a transition at $d = 2 + \epsilon$ with $\epsilon \sim 0.2$ small but finite. Such a transition is expected because $d = 2$ is special. At $d = 2$, the spectrum is organized into Virasoro multiplets and the corresponding Regge trajectories have constant twists $\tau = \Delta - \ell$ with integer spacing. At $d = 3$, the twist spectrum of the Ising CFT is additive and the Regge trajectories have more interesting dependence on spin. There are infinitely many high spin

\(^{3}\)The $\epsilon = d - 2$ expansion has been applied to the study of weakly coupled systems with $d_{\text{lower}} = 2$ \([7–10]\).

\(^{4}\)It is well known that the 2d Ising model can be mapped to a free fermion theory. It would be interesting to extend the fermionization to $2 + \epsilon$ dimensions.

\(^{5}\)See \([13]\) for a related study based on the numerical bootstrap.

\(^{6}\)At first order in $\epsilon$, some evanescent states have scaling dimensions $\Delta = 23 + \Delta^{(1)} \epsilon + O(\epsilon^2)$, where $\Delta^{(1)}$ is complex \([31]\).
operators whose twists are asymptotic to the sum of two lower twists [47, 48]. For example, the Regge trajectories $[\sigma \sigma]_{n}$ are associated with the lowest $\mathbb{Z}_2$-odd scalar $\sigma$ and they have twist accumulation points at $2\Delta_{\sigma} + 2n$. In this work, we will also discuss the location of the transition to the double-twist spectrum using analytic bootstrap techniques.

In the standard case of $d = 4 - \epsilon$, the corrections to the CFT data can be computed order by order in $\epsilon$, and the results are known to be given by asymptotic series [49]. It has been argued that they should take the form of integer power series based on the Renormalization Group (RG) analysis in the minimal subtraction scheme [50, 51]. Since we are interested in the Ising model in $2 + \epsilon$ dimensions, we encounter a basic question:

Is the $\epsilon = d - 2$ expansion of the Ising data given by integer power series?

Naively, one might think that this should be the case based on the $(4 - d)$ expansion results. For instance, the scaling dimension of the lowest $\mathbb{Z}_2$-even operator was assumed to receive integer power corrections in the study of disorder effects in $2 + \epsilon$ dimensions [52]. However, the $(4 - d)$ expansion is around a Gaussian theory and the weakly coupled arguments do not easily extend to the strong coupling situation.

In this work, we will assume that this is the case and analyze the consequences. We will study the 4-point correlation function of the lowest $\mathbb{Z}_2$-odd scalar operator $\sigma$ and examine if this is a consistent scenario for the leading corrections to the $2d$ data. To be more specific, we will make the following assumptions:

1. The critical Ising model is conformally invariant in $d = 2 + \epsilon$ dimensions with $|\epsilon| \ll 1$.

2. The leading corrections to the $2d$ data are linear in $\epsilon$:

\[
\Delta_{\sigma} = \Delta_{\sigma}^{(0)} + \epsilon \Delta_{\sigma}^{(1)} + \ldots, \quad \Delta_{i} = \Delta_{i}^{(0)} + \epsilon \Delta_{i}^{(1)} + \ldots, \quad \lambda_{i} = \lambda_{i}^{(0)} + \epsilon \lambda_{i}^{(1)} + \ldots \tag{1.1}
\]

where $\Delta_{i} = \Delta_{O_{i}}$ and $\lambda_{i} = \lambda_{\sigma O_{i}}$ are the scaling dimension and OPE coefficient of $O_{i}$. The zeroth order values can be derived from the exact solution in $d = 2$ dimensions.

For the convergence of the conformal block summation, we will further assume that $\Delta_{i}^{(1)}, \lambda_{i}^{(1)}$ do not grow too rapidly with $\Delta_{i}^{(0)}$, so the $\epsilon$ expansion of the correlator to first order is given by a finite function of $z, \bar{z}$ in the regime $0 \leq z, \bar{z} < 1$. 8

In the first assumption, we assume that scale invariance of the Ising fixed point is enhanced to conformal invariance. There is ample evidence for conformal invariance in $d = 2, 3, 4 - \epsilon$ dimensions, so we expect that this property extends to $d = 2 + \epsilon$ dimensions. 9 In the second assumption, the leading corrections cannot be more singular because they have positive integer powers in $\epsilon$. In addition, they cannot start from second or higher orders because the $d$-dependence of conformal blocks lead to first-order terms in the crossing equation.

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7 Note that a divergent series is not analytic in $\epsilon$ even if all the powers are non-negative integers.

8 I thank Slava Rychkov for emphasizing this point.

9 For conformal invariance, a crucial question to address is whether a virial current is present [53]. The case of $d = 4 - \epsilon$ dimensions has been revisited from various CFT perspectives [54–63].
We will focus on the 4-point function of $\sigma$
\[
\langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle = \frac{G(z, \bar{z})}{x_{12}^{2\Delta_{\sigma}} x_{34}^{2\Delta_{\sigma}}},
\]
where $z, \bar{z}$ are related to the conformally invariant cross-ratios by
\[
\frac{z \bar{z}}{x_{12}^2 x_{34}^2} = (1 - z)(1 - \bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},
\]
The crossing equation for $G(z, \bar{z})$ reads:
\[
\left[ (1 - z)(1 - \bar{z}) \right]^{\Delta_{\sigma}} G(z, \bar{z}) = (z \bar{z})^{\Delta_{\sigma}} G(1 - \bar{z}, 1 - z).
\]
In the $\epsilon = d - 2$ expansion, we have
\[
G(z, \bar{z}) = G^{(0)}(z, \bar{z}) + \epsilon G^{(1)}(z, \bar{z}) + \ldots,
\]
where the 2d solution reads
\[
G^{(0)}(z, \bar{z}) = \left( \frac{1 + \sqrt{z \bar{z}} + \sqrt{(1 - z)(1 - \bar{z})}}{\sqrt{2} (1 - z)^{1/4}(1 - \bar{z})^{1/4}} \right)^{1/2},
\]
and $G^{(1)}(z, \bar{z})$ can be written as a convergent power series in $z, \bar{z}$ in the regime $0 \leq z, \bar{z} < 1$. After the conformal block decomposition, the crossing equation becomes
\[
\sum_i \lambda_i^2 F_i(z, \bar{z}) = 0,
\]
with
\[
F_i^{(d)}(z, \bar{z}) = ((1 - z)(1 - \bar{z}))^{\Delta_{\sigma}} G^{(d)}_{\Delta_i, \ell_i}(z, \bar{z}) - (z \leftrightarrow 1 - \bar{z}).
\]
Note that $G^{(d)}_{\Delta_i, \ell_i}$ is the global conformal block for the conformal multiplet labelled by the primary operator $O_i$. To first order in $\epsilon$, the crossing equation reads
\[
\sum_i \lambda_i^{(0)} \left( \lambda_i^{(0)} \Delta^{(1)}_{\sigma} \partial_{\Delta_{\sigma}} + \lambda_i^{(0)} \Delta^{(1)}_i \partial_{\Delta_i} + 2 \lambda_i^{(1)} \right) F_i^{(d)}(z, \bar{z}) = (-1) \sum_i \lambda_i^2 \partial_d F_i^{(d)}(z, \bar{z}),
\]
which will be written in a more compact form in (1.15). Note that the derivative $\partial_d$ extracts the $d$-dependence of $G^{(d)}_{\Delta, \ell}$. After taking the derivatives, the leading equation can be obtained by setting
\[
d \rightarrow 2, \quad \Delta_i \rightarrow \Delta_i^{(0)}, \quad \lambda_i \rightarrow \lambda_i^{(0)}.
\]
We do not assume that the free parameters $\{\Delta^{(1)}_{\sigma}, \Delta^{(1)}_i, \lambda_i^{(1)}\}$ have definite signs.
Although the left hand side of (1.9) involves an infinite number of free parameters, the building blocks are given by the simple 2d conformal blocks^{10}

\[ G^{(d=2)}_{\Delta, \ell}(z, \bar{z}) = k_{\Delta+\ell}(z) k_{\Delta-\ell}(\bar{z}) + (1 - \delta_{\ell,0}) k_{\Delta+\ell}(\bar{z}) k_{\Delta-\ell}(z), \]  

where we have used the \( SL(2, \mathbb{R}) \) block

\[ k_\beta(x) = x^{\beta/2} F_1(\beta/2, \beta/2, \beta, x). \]  

(1.12)

Since each term is multiplied by \( \lambda(0) \), the intermediate states are the same as those in 2d with twist

\[ \{ \tau^{(0)} \} = \{ 4n, 4n + 1 \}, \]  

(1.13)

where \( n = 0, 1, 2, \ldots \) but \( \tau^{(0)} \neq 5 \). Note that the twist-5 trajectory and the twist-1 spin-2 state are absent in the 2d intermediate spectrum of the \( \sigma \times \sigma \) OPE. New intermediate states do not contribute to the OPE at order \( \epsilon^1 \) because their squared OPE coefficients are at least of order \( \epsilon^2 \). The absence of new intermediate states applies to both primary and descendant states.\(^{11}\) Since the 2d Ising model is unitary, \( \sum_{N=1}^{N} \lambda_{i,k}^{(0)} \lambda_{i,k}^{(1)} = 0 \) implies \( \lambda_{i,1}^{(0)} = \cdots = \lambda_{i,N}^{(0)} = 0 \) and we do not need to worry about \( \lambda_{i,k}^{(0)} \neq 0 \) due to the mixing problem. To be more precise, the first-order parameters are the average corrections

\[ \lambda_i^{(1)} = \frac{\sum_k \lambda_{i,k}^{(0)} \lambda_{i,k}^{(1)}}{\sqrt{\sum_k (\lambda_{i,k}^{(0)})^2}}, \quad \Delta_i^{(1)} = \frac{\sum_k (\lambda_{i,k}^{(0)})^2 \Delta_{i,k}^{(1)}}{\sum_k (\lambda_{i,k}^{(0)})^2}, \quad \lambda_i^{(0)} = \sqrt{\sum_k (\lambda_{i,k}^{(0)})^2}, \]  

(1.14)

where \( k \) labels the degenerate states with identical \( \Delta \) and \( \ell \) at \( d = 2 \).

On the other hand, the right hand side of (1.9) has no free parameter. We use the general \( d \) formula of conformal blocks to compute the sum over the intermediate states based on the 2d operator product expansion. Then we take the \( d \) derivative and set \( d \rightarrow 2 \).

In terms of the bra-ket notation, the crossing equation (1.9) reads

\[ \Delta_{\sigma}^{(1)}|\sigma\rangle + \sum_i (\Delta_i^{(1)}|\Delta_i\rangle + \lambda_i^{(1)}|\lambda_i\rangle) = -|d\rangle, \]  

(1.15)

where \( |a\rangle \) denotes the contribution associated with the change in the parameter \( a \). We are interested in the following question:

**Do \( |\Delta_{\sigma}\rangle, |\lambda_i\rangle, |\Delta_i\rangle \) form a complete set of basis for \( |d\rangle \)?**

\(^{10}\)The study of conformal blocks dates back to the 1970’s [64–67]. In \( d = 2, 4, 6 \) dimensions, the closed-form expressions of 4-point scalar conformal blocks with general \( \ell \) were found in [68, 69].

\(^{11}\)For \( d = 4 - \epsilon \), the absence of new states is an assumption in the conformal multiplet recombination in [54]. Then conformal invariance leads to constraints on the deformed data because the states should be organized into conformal multiplets. One can argue that no new states appear at first order by assuming integer power series in \( \epsilon \).
These vectors $|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle, |d\rangle$ can be interpreted as wave functions in $(z, \bar{z})$. If a solution exists \(^{12}\), then the coefficients characterize the overlap between a basis wave function and the target wave function $|d\rangle$. It turns out that the answer is negative. In other words, $|d\rangle$ does not belong to the vector space spanned by \{ $|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle$ \}. \(^{13}\)

Before studying the crossing equation (1.15), let us discuss the explicit expressions of the building blocks $|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle, |d\rangle$. The first one can be derived from the 2d solution (1.6):

$$|\Delta_\sigma\rangle = \frac{1}{\sqrt{2}} \left( 1 + \sqrt{z \bar{z}} + \sqrt{(1-z)(1-\bar{z})} \right)^{1/2} \log[(1-z)(1-\bar{z})] - (z \leftrightarrow 1-\bar{z}). \quad (1.17)$$

Note that a global conformal block takes a factorized form at $d=2$, given in (1.11). \(^{14}\)

According to the $z$ dependence, we can rewrite the sum of $|\Delta_i\rangle, |\lambda_i\rangle$ as

$$\sum_i \left( \Delta_i^{(1)} |\Delta_i\rangle + \lambda_i^{(1)} |\lambda_i\rangle \right) = \sum_{\beta=(\tau^{(0)})} \left( A_{\beta}(\bar{z}) k_{\beta}(z) + B_{\beta}(\bar{z}) \partial_{\beta} k_{\beta}(z) \right) (1-z)^{1/8} \left(1-\bar{z}\right)^{1/8} - (z \leftrightarrow 1-\bar{z}), \quad (1.18)$$

where $\{\tau^{(0)}\}$ is defined in (1.13) and $A_{\beta}(\bar{z}), B_{\beta}(\bar{z})$ encode the dependence on $\bar{z}$. On the right hand side of (1.15), $|d\rangle$ is given by the sums of $\partial_d G_{\Delta,\ell}^{(d)}$ with 2d OPE coefficients. \(^{15}\)

Numerically, we can use the general $d$ expression of conformal blocks in [71] to compute $|d\rangle$ order by order in $z$ at any $\bar{z}$ in $[0,1)$. Near the lightcone $\bar{z}=1$, a uniform precision evaluation requires summing the spin $\ell$ from 0 to $C (1-\bar{z})^{-1/2}$, where $C \gg 1$ determines the precision.

Analytically, we can divide the sum over spin into a low spin part and a high spin part. We expand individual conformal blocks in small $1-\bar{z}$ and perform the low spin sum of the regular terms exactly. To compute the high spin part, we first derive the large spin expansion of the 2d OPE coefficients, so the singular terms can be derived exactly to high order, such as $(1-\bar{z})^{40-1/8}$. Then we use the $SL(2,\mathbb{R})$ resummation identities in [72] to evaluate the regular terms to high precision. Together with the low spin part, we can compute the full coefficients of the regular terms to high precision, such as $10^{-50}$.

2 Numerical conformal bootstrap

In this section, we will consider a numerical approach. If $|d\rangle$ is linearly independent of \{ $|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle$ \}, then the latter is not a complete set of basis. We can detect the linear

\(^{12}\)There can be more than one solution if $|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle$ are not linearly independent. Then the set of basis is overcomplete.

\(^{13}\)We can illustrate the essential obstruction by a finite-dimensional example:

$$a_0 + a_2 t^2 = b_0 + b_1 t^4 + b_2 t^2, \quad (1.16)$$

where $t$ is the time coordinate. The basis vector associated with $t^4$ is linearly independent of all the basis vectors on the left hand side, so (1.16) has no solution that is valid at generic $t$ if $b_1 \neq 0$.

\(^{14}\)A general $d$ formula for the factorized lightcone expansion of global conformal blocks can be found in [70].

\(^{15}\)For each $\partial_d G_{\Delta,\ell}^{(d)}$, we first set $\ell$ to an integer and then take the limit $d \to 2$. 

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independence by a norm:

\[ \eta = \| d + \Delta_\sigma^{(1)} | \Delta_\sigma \rangle + \sum_i \Delta_i^{(1)} | \Delta_i \rangle + \sum_i \lambda_i^{(1)} | \lambda_i \rangle \|, \]

(2.1)

which is the distance between the target point determined by \( | d \rangle \) and a point in the space spanned by \( | \Delta_\sigma \rangle, | \lambda_i \rangle, | \Delta_i \rangle \). If there exists at least one crossing solution, then we should find \( \eta_{\text{min}} = 0 \). For the numerical implementation, we define the norm in terms of a set of sampling points

\[ \| H(z, \bar{z}) \| = \sqrt{\langle H | H \rangle} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \mu(z_i, \bar{z}_i) | H(z_i, \bar{z}_i) |^2}, \]

(2.2)

where the measure \( \mu(z, \bar{z}) \) will be specified later. The general inner product \( \langle H_1 | H_2 \rangle \) is defined as a weighted sum of the product \( H_1^* H_2 \). We consider the Lorentzian regime in which \( z, \bar{z} \) are independent, real variables. Furthermore, we focus on the region near the double-lightcone limit with \( 0 < z \ll 1 \) and \( 0 \ll \bar{z} < 1 \), which can also be studied by the analytic methods in Sec. 3. We use sampling rather than derivative equations to define the norm because it is easier to assign a proper measure \( \mu(z, \bar{z}) \).

2.1 Numerical implementation of \( \eta \) minimization

In practice, we need to truncate the conformal block summation to a finite sum in the numerical studies. This is sometimes called OPE truncation. \(^{17}\) Then we need to know if a finite \( \eta_{\text{min}} \) is due to the OPE truncation or absence of crossing solution. Since we are sampling in a subregion, the prefactor of \( \eta_{\text{min}} \) is scheme-dependent and the finite \( \eta_{\text{min}} \) becomes smaller as we increase the truncation cutoff. \(^{18}\) To distinguish between the two origins of \( \eta_{\text{min}} > 0 \), we will examine the dependence of \( \eta_{\text{min}} \) on \( z_0 \), which labels the local sampling regions. If \( \eta_{\text{min}} > 0 \) comes from the OPE truncation, then the functional form of \( \eta_{\text{min}}(z_0) \) will change significantly according to the leading contributions that are not taken into account. If \( \eta_{\text{min}} > 0 \) is due to absence of crossing solution, then \( \eta_{\text{min}}(z_0) \) will only get a smaller prefactor as more intermediate states are introduced. In the near lightcone region, we can readily distinguish between the two origins of \( \eta_{\text{min}}(z_0) > 0 \) based on the scaling behaviour.

Near the lightcone \( z = 0 \), we can truncate the sum (1.18) to low \( \beta \). But we will not truncate the spin sum, so \( A_\beta(\bar{z}), B_\beta(\bar{z}) \) are arbitrary. \(^{19}\) They will be evaluated near the

\(^{16}\)The sampling approach was first proposed in [73] as an alternative way of discretizing the crossing equation.

\(^{17}\)For unitary CFTs, both the direct- and cross-channel conformal block decompositions converge rapidly near the crossing symmetric point \( z = \bar{z} = 1/2 \) \([74]\). Although the case of non-unitary CFTs has not been rigorously established, it is reasonable to assume that they share some similarities based on the 2d examples. See [75, 76] for recent discussions of the OPE convergence in the distributional sense and in Minkowski space.

\(^{18}\)Introducing more conformal blocks can cancel out some part of the small difference in a subregion.

\(^{19}\)In fact, \( A_\beta(\bar{z}), B_\beta(\bar{z}) \) should satisfy some constraints, as they are sums of \( k_\beta(\bar{z}), \partial_\beta k_\beta(\bar{z}) \) with \( \beta = \{ \tau^{(0)} \} \). We will not impose these constraints, but the enlarged functional space has no solution to the crossing equation as well.
other lightcone $\bar{z} = 1$. In the $\eta$ minimization, $A_\beta(\bar{z})$ and $B_\beta(\bar{z})$ are approximated by truncated Taylor series about the center of the sampling region. We use high order Taylor polynomials so that this approximation has negligible errors.

We can also view $A_\beta(\bar{z})$, $B_\beta(\bar{z})$ at different $\bar{z}$ as independent free parameters. We have checked some concrete cases that the corresponding $\eta_{\text{min}}$ match that from Taylor polynomials. As we increase the $\beta$ cutoff, the number of free parameters, i.e. $A_\beta(\bar{z})$, $B_\beta(\bar{z})$ at different $\bar{z}$, becomes greater than that of equations, but $\eta_{\text{min}}$ remains finite in the cases under examination.

**Simple measure**

A simple measure is given by

$$\mu(z, \bar{z}) = 1.$$ (2.3)

In Fig. 1, we show the dependence of $\eta_{\text{min}}$ on the sampling region labelled by $z_0$. The sampling points are given by $z, 1 - \bar{z} = z_0 \times 10^{-k/10}$ with $k = 0, 1, 2, \ldots, 10$ and $z \neq 1 - \bar{z}$. One can notice the scaling behaviour

$$\eta_{\text{min}}(z_0) \propto z_0^\alpha,$$ (2.4)

which becomes more precise at small $z_0$. The exponent $\alpha$ is about $1.13(1)$ in the regime $10^{-6} < z_0 < 10^{-3}$, in which $|d|$ can be computed directly from the numerical summation over spin. For $z_0 < 10^{-6}$, we can use the analytic expression of $|d|$ in Sec. 3 to obtain a more precise value $\alpha \approx 1.125$. The exact exponent is expected to be $9/8$. Furthermore, the leading scaling behaviour is not sensitive to the $\beta$ truncation. This implies that $|d|$ contains a vector that scales as $\lambda^{9/8}$ under the transformation $z \rightarrow \lambda z, 1 - \bar{z} \rightarrow \lambda (1 - \bar{z})$. In particular, it does not belong to the space spanned by $\{|\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle\}$, otherwise the $z_0$ dependence should change with the cutoff $\beta_\ast$. Therefore, no crossing solution can be found. The analytic discussion of the linear independence will be given in Sec. 3.

**Refined measure**

We can consider a refined norm with a cutoff dependent measure. Near the lightcone, the lowest $\beta$ contribution dominates the OPE truncation error. We can define the refined measure as

$$\mu_\ast(z, \bar{z}) = |z^{\beta_\ast/2} - (1 - \bar{z})^{\beta_\ast/2}|^{-2},$$ (2.5)

where $\beta_\ast$ is the cutoff for the $\beta$ summation in (1.18). If a crossing solution exists, the exponents should always be positive because the OPE truncation errors are of higher order in $z, 1 - \bar{z}$ than $\mu_\ast^{-1/2}$. In Fig. 2, we compare the results of different $\beta$ truncations. If the $\beta$ truncation is the main source of $\eta_{\text{min}}$, then the exponent $\alpha$ should always be positive. One can see that

\[20\] We should enhance the numerical precision as $\beta_\ast$ increases because the higher $\beta$ contributions are smaller.
the exponent $\alpha$ decreases with the cutoff $\beta_*$ and becomes negative, implying that the OPE truncation is not the essential origin of $\eta_{\text{min}}$. The negative exponents are consistent with the results from the simple measure $\mu = 1$, i.e. $\alpha_{\text{refined}} \approx \alpha_{\text{simple}} - \beta_*/2$. A negative exponent also implies a divergent $\eta_{\text{min}}$ in the double lightcone limit $z, 1 - \bar{z} \to 0$, providing a clear signature for absence of crossing solution.

### 2.2 Geometric interpretation of $\eta$ minimization

In Fig. 3, we provide a geometric picture of the $\eta$ minimization. The $\eta$ minimization induces a special vector $|N\rangle$ orthogonal to the basis vectors:

$$\langle \Delta_\sigma |N\rangle = \langle \Delta_i |N\rangle = \langle \lambda_i |N\rangle = 0,$$

where $|N\rangle$ is defined as

$$|N\rangle = \left(|d\rangle + \Delta_\sigma^{(1)} |\Delta_\sigma\rangle + \sum_i \Delta_i^{(1)} |\Delta_i\rangle + \sum_i \lambda_i^{(1)} |\lambda_i\rangle \right)_{\text{arg min } \eta}.$$  

The minimal distance $\eta_{\text{min}}$ is precisely the inner product of $|N\rangle$ and $|d\rangle$:

$$\eta_{\text{min}}^2 = \langle N|N\rangle = \langle d|N\rangle > 0.$$  

There is no crossing solution when $\langle N|N\rangle > 0$ because there is a finite distance between the target point and the space spanned by the basis vectors.
Figure 2. Log-log plot of $\eta_{\text{min}}(z_0)$ with a refined measure. Here the measure is given by (2.5) and depends on the cutoff $\beta_* = 1, 4, 8$. The sampling points are at $z, 1 - z = z_0 \times 10^{-k}/10$ with $z \neq 1 - z$ and $k = 0, 1, 2, \ldots, 10$. The scaling exponents increase with the $\beta$ cutoff and become negative, so the $\beta$ truncation is not the main source for $\eta_{\text{min}} > 0$. The origin for finite $\eta_{\text{min}}$ is the absence of crossing solution. The scaling exponents are $0.63(1), -0.87(1), -2.87(1)$. The negative exponents imply that the $\beta$ truncation is essential to the finiteness of $\eta_{\text{min}}$.

Figure 3. Geometric interpretation of $\eta$ minimization. The red vector represents the target vector $|d\rangle$. The grey plane denotes the space spanned by $\{ |\Delta_\sigma\rangle, |\lambda_i\rangle, |\Delta_i\rangle \}$ and we do not impose positivity constraints. The $\eta$ minimization searches for the minimal distance between the target point and the grey plane. The result is given by the purple vector, i.e. the projection of the target vector on the normal vector. There is no solution to the crossing equation (1.15) if the purple vector has a finite length.
3 Analytical conformal bootstrap

Let us study the analytic structure of the crossing equation (1.15). This requires the analytic expression of $|d\rangle$, which is defined as

$$
|d\rangle = \left. \sum_i (\lambda^{(0)}_{r_i,\ell_i})^2 \partial_d G^{(d)}_{r_i,\ell_i}(z,\bar{z}) \right|_{d\to 2} - (z \leftrightarrow 1 - \bar{z}).
$$

(3.1)

Using the large spin expansion and $SL(2, \mathbb{R})$ identities, we can compute the contribution of a twist trajectory order by order in $z, 1 - \bar{z}$

$$
\sum_{\ell} (\lambda^{(0)}_{r,\ell})^2 \partial_d G^{(d)}_{r,\ell}(z,\bar{z}) \bigg|_{d\to 2} = \sum_{n=1}^{\infty} z^{n/2-n} \left( D^{(s)}_{r,n}(1 - \bar{z}) + D^{(r)}_{r,n}(1 - \bar{z}) \right),
$$

(3.2)

which encodes the contributions with identical twist. Here $D^{(s)}$ and $D^{(r)}$ denote singular and regular terms, which have finite and vanishing double discontinuities, respectively. The $n$ summation starts from $n = 1$ because the $n = 0$ part is independent of $d$.

Below we will present the explicit results that are relevant to our analyses near the double lightcone limit. The leading terms of the singular part are

$$
D^{(s)}_{0,1}(x) = \left( \frac{1}{4} + \frac{1}{8} x^2 - \frac{23}{224} x + \frac{59}{704} x^2 - \frac{37}{512} x^2 + \ldots \right) \frac{x^{-\frac{3}{8}}}{\sqrt{2}},
$$

(3.3)

$$
D^{(s)}_{0,2}(x) = \left( \frac{1}{8} + \frac{1}{16} x^2 - \frac{39}{448} x^2 + \frac{107}{1408} x^2 - \frac{1291}{15360} x^2 + \ldots \right) \frac{x^{-\frac{3}{8}}}{\sqrt{2}},
$$

(3.4)

$$
D^{(s)}_{1/2,1}(x) = \left( \frac{1}{8} - \frac{1}{16} x - \frac{95}{3136} x^2 - \frac{905}{15488} x^2 + \frac{177881}{3763200} x^2 + \ldots \right) \frac{x^{-\frac{3}{8}}}{\sqrt{2}}.
$$

(3.5)

We only give the exact results to order $x^2$, but they have been computed to high order, such as order $x^{20}$. To order $z^2$, the above results are sufficient, but we have also computed the explicit expressions with higher $\tau$ and $n$, which are of higher order than $z^2$. The regular terms can also be computed order by order to high precision

$$
D^{(r)}_{0,1}(x) = -\frac{3}{16 G},
$$

(3.6)

$$
D^{(r)}_{0,2}(x) \approx -0.10471666 + \frac{1}{128} \log(1 - x) + \frac{1}{256} \log x,
$$

(3.7)

$$
D^{(r)}_{1/2,1}(x) \approx 0.06743304 + 0.02179289 x + 0.00814652 x^2 + \ldots
+ 0.00986926 \left( 1 - \frac{1}{4} x - \frac{7}{64} x^2 + \ldots \right) \log x
$$

(3.8)

21The double discontinuity is defined by analytical continuation around the branch point $\bar{z} = 1$ [77].
where \( G = \Gamma(1/4)^2(2\pi)^{-3/2} \) is Gauss’s constant. We only write the numerical coefficients to precision \( 10^{-8} \), but they have been evaluated at much higher precision. The exact coefficients were guessed from the high-precision numerical values and we have checked that they are consistent with the direct summation over spin in Sec. 2. For example, we first computed \( D_{0,1}^{(r)} \) to precision \( 10^{-50} \) using the large spin expansion and \( SL(2, \mathbb{R}) \) resummation identities and guessed the analytic expression. Then we verified that the direct summation from \( \ell = 0 \) to \( \ell = 40476 \approx 2^7 \times 10^5 / 2 \) matches \( D_{0,1}^{(r)}(x) \) at \( x = 10^{-5} \) with precision \( 10^{-100} \).

### 3.1 Inconsistency from regular terms

Near the double lightcone limit, the target vector can be well approximated by

\[
|d| = \frac{1}{4 \sqrt{2}} z - \frac{3}{16 G} z (1 - \bar{z})^{\frac{1}{2}} + \frac{1}{8 \sqrt{2}} z (1 - \bar{z})^{\frac{3}{2}} + \ldots
\]

\[ -(z \leftrightarrow 1 - \bar{z}), \quad (3.9) \]

where \( \ldots \) indicates higher order terms. In the numerical discussion, we show that \( \eta_{\text{min}}(z_0) \) scales as \( z_0^{-1.125} \). This is associated with the leading regular term in (3.9):

\[
z (1 - z)^{\frac{1}{2}} - z^{\frac{1}{2}} (1 - \bar{z}), \quad (3.10)
\]

whose coefficient is given in (3.9). In the lightcone limit \( z \to 0 \), the exponents of \( z \) are associated with the half twists of primary and descendant states in the direct-channel OPE. In the case of cross-channel, we need to interchange \( z \) and \( 1 - \bar{z} \). Since the double-twist trajectories \([\sigma \sigma]_n\) with \( \tau = 2 \Delta_\sigma + 2n = 1/4 + 2n \) are absent, the first term \( z (1 - \bar{z})^{1/8} \) can only come from the direct-channel contribution. From \( k_\beta(z) = \sum_{n=0}^{\infty} c_n(\beta) z^{\beta/2 + n} \), we know that \( z (1 - \bar{z})^{1/8} \) should be associated with \( \beta = 0 \) in (1.18) and other \( \beta \) in (1.13) will not contain such a term. However, the absence of \( z^0 (1 - z)^{1/8} \) in \( |d| \) is inconsistent with \( (1 - z)^{1/8} k_0(z) \). The \( \beta \) derivative term \( \partial_\beta k_\beta(z) \) will introduce unwanted log \( z \) terms and cannot resolve the problem. Therefore, the crossing equation (1.15) has no solution.

The \( d = 2 \) solution is very special. All the regular terms have vanishing coefficients, so crossing solutions do not require the presence of double-twist trajectories with \( \tau_{[\sigma \sigma]}_n = \)

\[ 2^2 \]
\[ 2\Delta_\sigma + 2n = 1/4 + 2n. \]

At first order in \( \epsilon = d - 2 \), the \( d \)-dependence of conformal symmetry requires the presence of double twist states in the \( \sigma \times \sigma \) OPE. From this analytic perspective, the spectrum transition takes place at \( d = 2 + 0^+ \), which may be difficult to see clearly in the numerical studies. Furthermore, we expect the existence of other double/multi-twist states due to the generic finiteness of regular terms. The more complicated states should be suppressed by higher powers of \( \epsilon \).

### 3.2 Inconsistency from bi-singular terms

The presence of double-twist trajectories is not sufficient. Another obstruction to solving the crossing equation is the large spacing of the twist spectrum \((1.13)\). Let us focus on the bi-singular terms in \(|d\rangle_{\text{b.s.}}\)

\[ |d\rangle_{\text{b.s.}} = \frac{1}{4\sqrt{2}} (z - (1 - \bar{z})) + \frac{1}{8\sqrt{2}} (\sqrt{z} - \sqrt{1 - \bar{z}})(\sqrt{z} + \sqrt{1 - \bar{z}})^2 \]

\[ + \frac{1}{32\sqrt{2}} (\sqrt{z} - \sqrt{1 - \bar{z}})(\sqrt{z} + \sqrt{1 - \bar{z}})(3z - 2\sqrt{z}\sqrt{1 - \bar{z}} + 3(1 - \bar{z})) \]

\[ + \frac{1}{256\sqrt{2}} (\sqrt{z} - \sqrt{1 - \bar{z}})(23z^2 + 35z^{3/2}(1 - \bar{z})^{1/2} + \frac{11481}{1078}z(1 - \bar{z} + (z \leftrightarrow 1 - \bar{z})) \]

\[ + \ldots, \quad (3.13) \]

which are determined by the asymptotic behaviour of the 2d OPE coefficients at large spin. To match the power laws in \(|d\rangle_{\text{b.s.}}\), the functions \( A_\beta(z) \), \( B_\beta(z) \) in \((1.18)\) should take the following form

\[ [A_\beta(z)]_{\text{b.s.}} = \sum_{k=0}^{\infty} \left( a_{0,k} + a_{1,k} \log(1 - \bar{z}) \right) (1 - \bar{z})^{k/2 - 1/8}, \quad (3.14) \]

\[ [B_\beta(z)]_{\text{b.s.}} = \sum_{k=0}^{\infty} \left( b_{0,k} + b_{1,k} \log(1 - \bar{z}) \right) (1 - \bar{z})^{k/2 - 1/8}. \quad (3.15) \]

We introduce \( \log(1 - \bar{z}) \) because \( \partial_\beta k_{\beta}(z) \) involves \( \log z \). The exponents take the expected values and there is no double-twist exponents, so naively we may try to solve the crossing equation order by order. Here we count the total power of \( z, 1 - \bar{z} \). For example, the first line of \((3.13)\) contains terms of order 1 and 3/2, while the second and third lines give terms of order 2 and 5/2. After solving the crossing equation to order 2, we substitute the solutions of \( a_{n,k}, b_{n,k} \) into the 5/2 order equation. It turns out that one term has a fixed coefficient

\[ \left( \Delta^{(1)}_\sigma |\Delta_\sigma\rangle + \sum_i \Delta^{(1)}_i |\Delta_i\rangle + \sum_i \lambda^{(1)}_i |\lambda_i\rangle + |d\rangle \right)_{\text{b.s.}} \text{solution to order 2} \]

\[ = -\frac{6\sqrt{2}}{539} \left( z(1 - \bar{z})^{3/2} - z^{3/2}(1 - \bar{z}) \right) + \ldots, \quad (3.16) \]

\(^{23}\)We verify that the 2d regular terms vanish to high numerical precision using the large spin expansion and \( SL(2, \mathbb{R}) \) resummation identities, as a test of our computational method.
so the crossing equation has no solution beyond order 2. To construct a crossing solution, we can reduce the spacing of the twist spectrum from 4 to 2, as in the standard case of generalized free theory. With the additional free parameters, one can solve the bi-singular part of the crossing equation order by order.

The $d = 2$ solution is possible because of the special structure of 2d conformal blocks. As 2d global conformal blocks are invariant under $\ell \to -\ell$, the spectrum is symmetric in twist $\Delta - \ell$ and conformal spin $\Delta + \ell$. This explains the large spacing in intermediate twist spectrum, which is dual to that of $2\ell$. 24 The fact that only even spin states appear in the $\sigma \times \sigma$ OPE implies that the spacing of the 2d twist spectrum is 4. 25 This large spacing is inconsistent with the general $d$ structure of conformal blocks.

4 Discussion

We have investigated the critical Ising model in $d = 2 + \epsilon$ dimensions using the conformal bootstrap methods. At order $\epsilon^1$, the crossing equation (1.15) is not consistent with the naive assumption that the leading corrections to the 2d OPE data are linear in $\epsilon$. The $d$-dependence of global conformal symmetry implies the existence of new intermediate states, such as double twist trajectories, but the spectrum is almost the same as the $d = 2$ case at order $\epsilon$. No solution to the crossing equation can be found due to the linear independence of conformal blocks. Since the obstructions are related to the speciality of $d = 2$, we expect them also appear in other minimal-model CFTs.

As a result, the leading corrections to the 2d Ising data should be more singular than $\epsilon^1$. The presence of non-integer power corrections $\epsilon^a$ with $0 < a < 1$ will imply strong non-unitarity of the Ising CFT below $d = 2$. 26 This is consistent with the observation of two kinks in the $d < 2$ unitary bootstrap bounds in [80]. If the leading corrections are of order $\epsilon^{1/k}$ and $k > 1$ is an integer, then there can be new contributions to the crossing equation at order $\epsilon^1$, as required by the $d$-dependence of conformal blocks. There exist more exotic scenarios that can generate new terms at order $\epsilon^1$, as it only requires that the sum of exponents gives 1.

Another consequence is that different Regge trajectories are connected by the analyticity in the $SL(2, \mathbb{R})$ conformal spin in the negative $\ell$ regime. In this way, they can form a more rigid structure, i.e. Virasoro blocks. It would be interesting to generalize this “negative spin” continuation to $d > 2$ dimensions. Previously, the closed-form expression of conformal blocks in the crossed lightcone limit in [79] was inspired by the pole structure of the Lorentzian inversion in the negative spin regime.

The spacing of the twist spectrum of the 2d $\sigma \times \sigma$ OPE is 4, but that of the full spectrum is 2 at higher twist. According to the character decomposition in [30], we checked that there exist even-spin states of twist $4n + 2$ and $4n + 3$ with $n \geq 3$.

We mentioned in Sec. 1 that the unitary numerical bootstrap results are compatible with the $(4 - d)$ expansion, in spite of the non-unitary issue at non-integer $d$. At low orders in $\epsilon$, the non-unitary states do not appear in the $\sigma \times \sigma$ OPE. They may be present at higher orders in $\epsilon$ or as non-perturbative contributions, but their effects must be small near $d = 4$. In this sense, the unitarity violation is weak below $d = 4$, as oppose to the strong case below $d = 2$.
The simplest resolution could be that the leading corrections are of order $\epsilon^{1/2}$. For $d < 2$, the scaling dimensions can be complex conjugate pairs and the OPE coefficients of new states can be imaginary numbers. In the context of perturbative RG fixed points, the $\epsilon^{1/2}$ behavior has been found in the cases with two marginal operators, such as the random Ising model in $d = 4 - \epsilon$ dimensions [81]. In general, the square root behaviour can appear around a bifurcation point at which two fixed points collide [82–85].

Furthermore, we can rule out the possibility that only scaling dimensions receive $\epsilon^{1/k}$ corrections by adding higher $\beta$ derivatives of $k_{\beta}(z)$ to (1.18). Then there must be infinitely many new intermediate states. It is easy to see that they will not resolve the problem associated with regular terms (3.10). We also examine the bi-singular part of the crossing equation with $k = 2, 3, 4, 5$ by adding higher order $\beta$-derivatives of $k_{\beta}(z)$ and higher powers of $\log(1 - \bar{z})$. We find the same terms with constant coefficients given explicitly in (3.16), so the bi-singular crossing equation has no solution and we conjecture that this is a general result. Previously, it has been shown in [31] that there are non-unitary evanescent states in $d = 4 - \epsilon$ dimensions, which are absent in integer dimensions.

We have shown that infinitely many of operators receive corrections more singular than $\epsilon^1$, but it is still unclear if the leading corrections to the low-lying data are linear in $\epsilon$. It would be important to use analytic methods [54–63, 77, 86–90] to solve the exact series at least to order $\epsilon^1$. A crucial question is address is what are the defining properties of the Ising solution. One of them should be related to the phenomena of operator decoupling observed in $d = 2, 3$. It would be fascinating to study other strongly coupled CFTs in $2 + \epsilon$ dimensions.

For many statistical physics problems, unitarity is violated due to absence of reflection positivity. Similarly, the boundary/defect bootstrap [91, 92] and thermal bootstrap [93–95] problems do not obey positivity constraints. In the usual numerical bootstrap studies, the positivity constraints from unitarity play a crucial role in deriving rigorous bounds on the CFT space. Here we show that one can rule out the inconsistent theory space without using positivity. We plan to revisit these non-positive bootstrap problems from the new perspective.

In this work, the intermediate spectrum is severely constrained by our assumptions. It would be important to extend this bootstrap approach and apply it to more complicated bootstrap problems without resorting to positivity constraints. The $d = 3$ Ising CFT has a more complex spectrum due to additivity of the twist spectrum. A natural extension of the spectrum assumption is that there exists a hierarchical structure in operator product expansion [96]. For example, the resolution of the subleading trajectories should only lead to subleading corrections to the dominant double-twist behaviour. The crossing equation can be solved to high precision using a few Regge trajectories and low-spin states. There would be no solution if we assume an unrealistic tolerance. The minimization of tolerance then extracts interesting solutions. One may generalize the concept of minimal models to $d > 2$ dimensions based on the complexity of bootstrap solutions.

\[27\] A more ambitious approach is to deduce the absence of bootstrap solutions from a candidate low-lying spectrum and linear independence.
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