INTEGRABILITY OF INVARIANT METRICS ON THE DIFFEOMORPHISM GROUP OF THE CIRCLE

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Abstract. Each $H^k$ Sobolev inner product ($k \geq 0$) defines a Hamiltonian vector field $X_k$ on the regular dual of the Lie algebra of the diffeomorphism group of the circle. We show that only $X_0$ and $X_1$ are bi-Hamiltonian relatively to a modified Lie-Poisson structure.

1. Introduction

Often motions of inertial mechanical systems are described in Lagrangian variables by paths on a configuration space $G$ that is a Lie group. The velocity phase space is the tangent bundle $TG$ and the kinetic energy

$$K = \frac{1}{2} < v, v >$$

for $v \in TG$. For example, in continuum mechanics the state of a system at time $t \geq 0$ can be specified by a diffeomorphism $x \mapsto \varphi(t, x)$ of the ambient space, giving the configuration of the particles with respect to their initial positions at time $t = 0$. Here $x$ is a label identifying a particle, taken to be the position of the particle at time $t = 0$ so that $\varphi(0, x) = x$. In this setting $G$ would be the group of diffeomorphisms. The material (Lagrangian) velocity field is given by $(t, x) \mapsto \varphi_t(t, x)$ while the spatial (Eulerian) velocity field is $u(t, y) = \varphi_t(t, x)$, where $y = \varphi(t, x)$, i.e. $u = \varphi_t \circ \varphi^{-1}$. Observe that for any fixed time-independent diffeomorphism $\eta$, the spatial velocity field $u = \varphi_t \circ \varphi^{-1}$ along the path $t \mapsto \varphi(t)$ remains unchanged if we replace this path by $t \mapsto \varphi(t) \circ \eta$. This right-invariance property suggests to extend the kinetic energy $K$ by right translation to a right-invariant Lagrangian $K : TG \rightarrow \mathbb{R}$, obtaining a Lagrangian system on $G$. The length of a path $\{\varphi(t)\}_{t \in [a,b]}$ in $G$ is defined as

$$l(\varphi) = \int_{a}^{b} < \varphi_t, \varphi_t >^{1/2} \, dt.$$ 

The Least Action Principle holds if the equation of motion is the geodesic equation. The set $\text{Diff}(S^1)$ of all smooth orientation-preserving diffeomorphisms of the circle represents the configuration space for the spatially periodic motion of inertial one-dimensional mechanical systems. $\text{Diff}(S^1)$ is an infinite dimensional Lie group, the group operation being composition [19] and its Lie algebra $\text{Vect}(S^1)$ being the space of all smooth vector fields on $S^1$.

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2. PRELIMINARIES

In this section, we review some fundamental aspects of finite dimensional smooth Poisson manifolds.

Definition 2.1. A symplectic manifold is a pair \((M, \omega)\), where \(M\) is a manifold and \(\omega\) is a closed nondegenerate 2-form on \(M\), that is \(d\omega = 0\) and for each \(m \in M\), \(\omega_m : T_mM \times T_mM \to \mathbb{R}\) is a continuous bilinear skew-symmetric map such that the induced linear map \(\tilde{\omega}_v : T_mM \to T^*_mM\) defined by \(\tilde{\omega}_v(w) = \omega(v, w)\) is an isomorphism for all \(v \in T_mM\).

Example 2.2. In the general study of variational problems, extensive use is made of the canonical symplectic structure on the cotangent bundle \(T^*M\) (representing the phase space) of the manifold \(M\) (representing the configuration space). This symplectic form is given in any local trivialization \((q, p) \in U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n\) of \(T^*M\) by
\[
\omega((Q, P), (\tilde{Q}, \tilde{P})) = \tilde{P} \cdot Q - P \cdot \tilde{Q}, \quad (Q, P), (\tilde{Q}, \tilde{P}) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Since a symplectic form \(\omega\) is nondegenerate, it induces an isomorphism (2.1)
\[b : TM \to T^*M, \quad X \mapsto X^\flat,\]
defined via \(X^\flat(Y) = \omega(X, Y)\). The symplectic gradient \(X_f\) of a function \(f\) is defined by the relation \(X_f^\flat = -df\). The inverse of the isomorphism \(b\) defines a skew-symmetric bilinear form \(W\) on the cotangent space of \(M\). This bilinear form \(W\) induces itself a bilinear mapping on \(C^\infty(M)\), the space of smooth functions \(f : M \to \mathbb{R}\), given by
\[
\{f, g\} = W(df, dg) = \omega(X_f, X_g), \quad f, g \in C^\infty(M),
\]
and called the Poisson bracket of the functions \(f\) and \(g\).

Example 2.3. In Example 2.2, the Poisson bracket is given by
\[
\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).
\]

The observation that a bracket like (2.3) could be introduced on \(C^\infty(M)\) for a smooth manifold \(M\), without the use of a symplectic form, leads to the general notion of a Poisson structure [26].
Definition 2.4. A Poisson structure on a $C^\infty$ manifold $M$ is a skew-symmetric bilinear mapping $(f,g) \mapsto \{f,g\}$ on the space $C^\infty(M)$, which satisfies the Jacobi identity
\begin{equation}
\{ \{f,g\}, h\} + \{ \{g,h\}, f\} + \{ \{h,f\}, g\} = 0,
\end{equation}
as well as the Leibnitz identity
\begin{equation}
\{f, gh\} = \{f, g\} h + g \{f, h\}.
\end{equation}

When the Poisson structure is induced by a symplectic structure $\omega$, the Leibnitz identity (2.4) is a direct consequence of (2.2), whereas the Jacobi identity (2.4) corresponds to the condition $d\omega = 0$ satisfied by the symplectic form $\omega$. In the general case, the fact that the mapping $g \mapsto \{f,g\}$ satisfies (2.5) means that it is a derivation of $C^\infty(M)$. Each derivation on $C^\infty(M)$ for a $C^\infty$ manifold (even in the infinite dimensional case cf. [1]) corresponds to a smooth vector field, that is, to each $f \in C^\infty(M)$ is associated a vector field $X_f : M \to TM$, called the Hamiltonian vector field of $f$, such that
\begin{equation}
\{f, g\} = X_f \cdot g = dg(X_f),
\end{equation}
where $dg(X_f) = L_{X_f}g$ is the Lie derivative of $g$ along $X_f$. Conversely, a vector field $X : M \to TM$ on a Poisson manifold $M$ is said to be Hamiltonian if there exists a function $f$ such that $X = X_f$.

Recall [29] that for a smooth vector field $X : M \to TM$, the Lie derivative operator $L_X : C^\infty(M) \to C^\infty(M)$ acts on smooth functions $g : M \to \mathbb{R}$ with differentials $dg : M \to T^*M$ by $(L_Xg)(m) = dg(m) \cdot X(m)$ for $m \in M$. The space $\text{Vect}(M)$ of smooth vector fields on $M$ and the space of operators $\{L_X : X \in \text{Vect}(M)\}$ are isomorphic as real vector spaces, the linear isomorphism between them being $X \mapsto L_X$ [1]. Therefore the elements of $\text{Vect}(M)$ can be regarded as operators on $C^\infty(M)$ via $X \cdot f = L_Xf$, forming a Lie algebra if endowed with the bracket $[X,Y] = L_X \circ L_Y - L_Y \circ L_X$. Notice that (2.4) yields
\begin{equation}
[X_f, X_g] = X_{\{f,g\}}.
\end{equation}
From (2.7) it follows (see [29]) that $g \in C^\infty(M)$ is a constant of motion for $X_f$ if and only if $\{f,g\} = 0$.

Jost [21] pointed out that, just like a derivation on $C^\infty(M)$ corresponds to a vector field, a bilinear bracket $\{f,g\}$ satisfying the Leibnitz rule (2.5) corresponds to a skew-symmetric bilinear form on $TM$. That is, there exists a $C^\infty$ tensor field $W \in \Gamma(\Lambda^2 TM)$, called the Poisson bivector of $(M,\{\cdot,\cdot\})$, such that
\begin{equation}
\{f,g\} = W(df, dg).
\end{equation}
Using the unique local extension of the Lie bracket of vector fields to skew-symmetric multivector fields, called the Schouten-Nijenhuis bracket [30], the condition (2.4) becomes
\begin{equation}
[W, W] = 0.
\end{equation}
Conversely, any $W \in \Gamma(\Lambda^2 TM)$ that satisfies (2.8) induces a Poisson structure on $M$ via (2.2). The only condition that must be satisfied by $W$ is (2.8) since (2.5) holds automatically. A Poisson structure on $M$ is therefore
equivalent to a bivector $W$ that satisfies (2.8). This induces a homomorphism

$$\# : T^*M \to TM, \quad \alpha \mapsto \alpha^\#,$$

such that $\beta(\alpha^\#) = W(\beta, \alpha)$ for every $\beta \in T^*M$. Notice that for $f \in C^\infty(M)$ we have $(df)^\# = X_f$. If the homomorphism (2.9) is an isomorphism we call the Poisson structure nondegenerate. A nondegenerate Poisson structure on $M$ is equivalent to a symplectic structure where the symplectic form $\omega$ is just $\#W$, the closedness condition corresponding to the Jacobi identity [30].

Remark 2.5. The notion of a Poisson manifold is more general than that of a symplectic manifold. For example, in the symplectic case the Poisson bracket satisfies the additional property that $\{f, g\} = 0$ for all $g \in C^\infty(M)$ only if $f \in C^\infty(M)$ is constant, whereas for Poisson manifolds such non-constant functions $f$ might exist, in which case they are called Casimir functions. To highlight this, notice that by Darboux’ theorem [29] a finite dimensional symplectic manifold $M$ has to be even dimensional and locally there are coordinates $\{q_1, ..., q_n, p_1, ..., p_n\}$ such that $\{f, g\}$ is given by (2.3). On the other hand, on $M = \mathbb{R}^{2n+1}$ with coordinates $\{q_1, ..., q_n, p_1, ..., p_n, \zeta\}$ we determine a Poisson structure defining the Poisson bracket of $f, g \in C^\infty(\mathbb{R}^{2n+1})$ by the same formula (2.3). Notice that any $f \in C^\infty(\mathbb{R}^{2n+1})$ which depends only on $\zeta$ is a Casimir function.

Two Poisson bivectors $W_1$ and $W_2$ define compatible Poisson structures if

$$[W_1, W_2] = 0.$$

This is equivalent to say that for any $\lambda, \mu \in \mathbb{R}$,

$$\{f, g\}_{\lambda, \mu} = \lambda \{f, g\}_1 + \mu \{f, g\}_2$$

is also a Poisson bracket. On a manifold $M$ equipped with two compatible Poisson structures, a vector field $X$ is said to be (formally) integrable or bi-Hamiltonian if it is Hamiltonian for both structures.

On a symplectic manifold $(M, \omega)$, a necessary condition for a vector field $X$ to be Hamiltonian is that $L_X \omega = 0$ [29]. A similar criterion exists for a Poisson manifold $(M, W)$. It is instructive for later considerations to present a short proof of this known result.

**Proposition 2.6.** On a Poisson manifold $(M, W)$ a necessary condition for a vector field $X$ to be Hamiltonian is

$$L_X W = 0.$$

**Proof.** If $X$ is Hamiltonian, there is a function $h \in C^\infty(M)$ such that $X = X_h$. Let $f$ and $g$ be arbitrary smooth functions on $M$. We have

$$L_X W(df, dg) = L_X (W(df, dg)) - W(L_X df, dg) - W(df, L_X dg).$$

But $L_{X_h} f = \{h, f\}$ and $L_{X_h} df = dL_{X_h} f = d\{h, f\}$. Therefore

$$L_X W(df, dg) = L_X \{f, g\} - W(df, \{h, f\}, dg) - W(df, d\{h, g\})
= \{h, \{f, g\}\} - \{h, f\}, g\} - \{f, \{h, g\}\}.$$

This last expression equals zero because of the Jacobi identity. □
The fundamental example of a non-symplectic Poisson structure is the Lie-Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$.

**Definition 2.7.** On the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group $G$, there is a Poisson structure defined by

\[
\{ f, g \}(m) = m([d_mf, d_mg])
\]

for $m \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$, called the canonical Lie-Poisson structure \(^1\).

**Remark 2.8.** The canonical Lie-Poisson structure has the remarkable property to be linear. A Poisson bracket on a vector space is said to be linear if the bracket of two linear functionals is itself a linear functional.

Each element $\gamma \in \bigwedge^2 \mathfrak{g}^*$ can be viewed as a Poisson bivector. Indeed, $[\gamma, \gamma] = 0$ since $\gamma$ is a constant tensor field. As such, $\gamma$ defines a Poisson structure on $\mathfrak{g}^*$. The condition of compatibility with the canonical Lie-Poisson structure, $[W_0, \gamma] = 0$, can be written as (see [30], Chapter 3)

\[
\gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v) = 0, \quad u, v, w \in \mathfrak{g}.
\]

On a Lie group $G$, a right-invariant $k$-form $\omega$ is completely defined by its value at the unit element $e$, and hence by an element of $\bigwedge^k \mathfrak{g}^*$. In other words, there is a natural isomorphism between the space of right-invariant $k$-forms on $G$ and $\bigwedge^k \mathfrak{g}^*$. Moreover, since the exterior differential $d$ commutes with right translations, it induces a linear operator $\partial : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$ that satisfies $\partial \circ \partial = 0$ and

1. $\partial \gamma = 0$ for $\gamma \in \bigwedge^0 \mathfrak{g}^* = \mathbb{R}$;
2. $\partial \gamma (u, v) = -\gamma([u, v])$ for $\gamma \in \bigwedge^1 \mathfrak{g}^* = \mathfrak{g}^*$;
3. $\partial \gamma (u, v, w) = \gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v)$ for $\gamma \in \bigwedge^2 \mathfrak{g}^*$, where $u, v, w \in \mathfrak{g}$, as one can check by direct computation (see [18], Chapter 24). The kernel $Z^n(\mathfrak{g})$ of $\partial : \bigwedge^n(\mathfrak{g}^*) \to \bigwedge^{n+1}(\mathfrak{g}^*)$ is the space of $n$-cocycles and the range $B^n(\mathfrak{g})$ of $\partial : \bigwedge^{n-1}(\mathfrak{g}^*) \to \bigwedge^n(\mathfrak{g}^*)$ is the spaces of $n$-coboundaries. Notice that $B^n(\mathfrak{g}) \subset Z^n(\mathfrak{g})$. The quotient space $H^n_{CE}(\mathfrak{g}) = Z^n(\mathfrak{g})/B^n(\mathfrak{g})$ is the $n$-th Lie algebra cohomology or Chevaley-Eilenberg cohomology group of $\mathfrak{g}$. Notice that in general the Lie algebra cohomology is different from the de Rham cohomology $H^n_{DR}$. For example, $H^1_{DR}(\mathbb{R}) = \mathbb{R}$ but $H^2_{CE}(\mathbb{R}) = 0$.

Each 2-cocycle $\gamma$ defines a Poisson structure on $\mathfrak{g}^*$ compatible with the canonical one. Indeed (2.13) can be recast as $\partial \gamma = 0$. Notice that the Hamiltonian vector field $X_f$ of a function $f \in C^\infty(\mathfrak{g}^*)$ computed with respect to the Poisson structure defined by the 2-cocycle $\gamma$ is

\[
X_f(m) = \gamma(d_mf, \cdot).
\]

**Definition 2.9.** A modified Lie-Poisson structure is a Poisson structure on $\mathfrak{g}^*$ whose Poisson bivector is given by $W_\gamma = W_0 + \gamma$, where $W_0$ is the canonical Poisson bivector and $\gamma$ is a 2-cocycle.

\(^1\)Here, $d_mf$, the differential of a function $f \in C^\infty(\mathfrak{g}^*)$ at $m \in \mathfrak{g}^*$ is to be understood as an element of the Lie algebra $\mathfrak{g}$.
Example 2.10. A special case of modified Lie-Poisson structure is given by a 2-cocycle $\gamma$ which is a coboundary. If $\gamma = \partial m_0$ for some $m_0 \in g^*$, the expression

$$\{ f, g \}_\gamma (m) = m_0([d_m f, d_m g])$$

looks like if the Lie-Poisson bracket had been "frozen" at a point $m_0 \in g^*$ and for this reason some authors call it a "freezing" structure.

3. Modified Lie-Poisson structures on $\text{Vect}(S^1)$

The group $\text{Diff}(S^1)$ of smooth orientation-preserving diffeomorphisms of the circle $S^1$ is endowed with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$. The composition and the inverse are both smooth maps $\text{Diff}(S^1) \times \text{Diff}(S^1) \to \text{Diff}(S^1)$, respectively $\text{Diff}(S^1) \to \text{Diff}(S^1)$, so that $\text{Diff}(S^1)$ is a Lie group [19]. Its Lie algebra $\text{Vect}(S^1)$ is the space of smooth vector fields on $S^1$, which is isomorphic to the space $C^\infty(S^1)$ of periodic functions. The Lie bracket on $\text{Vect}(S^1)$ is given by

$$[u, v] = uv - u_xv.$$

Since the topological dual of the Fréchet space $\text{Vect}(S^1)$ is too big, being isomorphic to the space of distributions on the circle, we restrict our attention in the following to the regular dual $\text{Vect}^*(S^1)$, the subspace of distributions defined by linear functionals of the form

$$u \mapsto \int_{S^1} mu \, dx$$

for some function $m \in C^\infty(S^1)$. The regular dual $\text{Vect}^*(S^1)$ is therefore isomorphic to $C^\infty(S^1)$ by means of the $L^2$ inner product 2

$$< u, v > = \int_{S^1} uv \, dx.$$

Let $f$ be a smooth real valued function on $C^\infty(S^1)$. Its Fréchet derivative at $m$, $df(m)$ is a linear functional on $C^\infty(S^1)$. We say that $f$ is a regular function if there exists a smooth map $\delta f : C^\infty(S^1) \to C^\infty(S^1)$ such that

$$df(m) M = \int_{S^1} M \cdot \delta f(m) \, dx, \quad m, M \in C^\infty(S^1).$$

That is, the Fréchet derivative $df(m)$ belongs to the regular dual $\text{Vect}^*(S^1)$ and the mapping $m \mapsto \delta f(m)$ is smooth. The map $\delta f$ is a vector field on $C^\infty(S^1)$, called the gradient of $f$ for the $L^2$-metric. In other words, a regular function is a smooth function on $C^\infty(S^1)$ which has a smooth gradient.

Example 3.1. Typical examples of regular functions are nonlinear functionals over the space $C^\infty(S^1)$, like

$$f(m) = \int_{S^1} \left( m^2 + mm^2 \right) \, dx \quad \text{with} \quad \delta f(m) = 2m - m_x^2 - 2mm_{xx},$$

as well as linear functionals

$$f(m) = \int_{S^1} um \, dx \quad \text{with} \quad \delta f(m) = u \in C^\infty(S^1).$$

2In the sequel, we use the notation $u, v, \ldots$ for elements of $\text{Vect}(S^1)$ and $m, n, \ldots$ for elements of $\text{Vect}^*(S^1)$ to distinguish them, although they all belong to $C^\infty(S^1)$. 
Notice that the smooth function \( f_\theta : C^\infty(S^1) \to \mathbb{R} \) defined by \( f_\theta(m) = m(\theta) \) for some fixed \( \theta \in S^1 \) is not regular as \( \delta f_\theta \) is the Dirac measure at \( \theta \).

Conversely, a smooth vector field \( X \) on \( \text{Vect}^*(S^1) \) is called a gradient if there exists a regular function \( f \) on \( \text{Vect}^*(S^1) \) such that \( X(m) = \delta f(m) \) for all \( m \in \text{Vect}^*(S^1) \). Observe that if \( f \) is a smooth real valued function on \( C^\infty(S^1) \) then its second Fréchet derivative is symmetric \([19]\), that is,
\[
d^2 f(m)(M, N) = d^2 f(m)(N, M), \quad m, M, N \in C^\infty(S^1).
\]

For a regular function, this property can be written as
\[
(3.1) \quad \int_{S^1} \left( d \delta f(m) M \right) N dx = \int_{S^1} \left( d \delta f(m) N \right) M dx,
\]
for all \( m, M, N \in C^\infty(S^1) \). Hence the linear operator \( d \delta f(m) \) is symmetric for the \( L^2 \)-inner product on \( C^\infty(S^1) \) for each \( m \in C^\infty(S^1) \). We will resume this fact in the following lemma.

**Lemma 3.2.** A necessary condition for a vector field \( X \) on \( C^\infty(S^1) \) to be a gradient is that its Fréchet derivative \( dX(m) \) is a symmetric linear operator.

To define a Poisson bracket on the space of regular functions on \( \text{Vect}^*(S^1) \), we consider a one-parameter family of linear operators \( J(m) \) and set
\[
(3.2) \quad \{ f, g \}(m) = \int_{S^1} \delta f(m) J(m) \delta g(m) dx.
\]
The operators \( J(m) \) must satisfy certain conditions in order for \( (3.2) \) to be a valid Poisson structure on \( \text{Vect}^*(S^1) \).

**Definition 3.3.** A family of linear operators \( J(m) \) on \( \text{Vect}^*(S^1) \) defines a Poisson structure on \( \text{Vect}^*(S^1) \) if \( (3.2) \) satisfies

1. \( \{ f, g \} \) is regular if \( f \) and \( g \) are regular,
2. \( \{ g, f \} = - \{ f, g \} \),
3. \( \{ \{ f, g \}, h \} + \{ \{ g, h \}, f \} + \{ \{ h, f \}, g \} = 0 \).

Notice that the second condition above simply means that \( J(m) \) is a skew-symmetric operator for each \( m \).

**Example 3.4.** The canonical Lie-Poisson structure on \( \text{Vect}^*(S^1) \) given by
\[
\{ f, g \}(m) = m([\delta f, \delta g]) = \int_{S^1} \delta f(m)(mD + Dm) \delta g(m) dx
\]
is represented by the one-parameter family of skew-symmetric operators
\[
(3.3) \quad J(m) = mD + Dm
\]
where \( D = \partial_x \). It can be checked that all the three required properties are satisfied. In particular, we have
\[
\delta \{ f, g \} = d \delta f(J \delta g) - d \delta g(J \delta f) + \delta f \delta g_x - \delta g \delta f_x.
\]

**Definition 3.5.** The Hamiltonian of a regular function \( f \), for a Poisson structure defined by \( J \) is defined as the vector field
\[
X_f(m) = J(m) \delta f(m).
\]
Proposition 3.6. A necessary condition for a smooth vector field $X$ on $\text{Vect}^*(S^1)$ to be Hamiltonian with respect to the Poisson structure defined by a constant linear operator $K$ is the symmetry of the operator $dX(m) \circ K$ for each $m \in \text{Vect}^*(S^1)$.

Proof. If $X$ is Hamiltonian, we can find a regular function $f$ such that

$$X(m) = K \delta f(m).$$

Moreover, since $K$ is a constant linear operator, we have

$$d(K \delta f)(m) M = K \circ (d\delta f(m)) M.$$

Therefore, we get

$$< dX(m) \circ K M, N > = < K \circ d\delta f(m) \circ K M, N >$$

$$= < M, K \circ d\delta f(m) \circ K N >$$

$$= < M, dX(m) \circ K N >,$$

since $K$ is skew-symmetric and $d\delta f(m)$ is symmetric. □

A 2-cocycle on $\text{Vect}(S^1)$ is a bilinear functional $\gamma$ represented by a skew-symmetric operator $K : C^\infty(S^1) \to C^\infty(S^1)$ such that

$$\gamma(u, v) = < u, Kv > = \int_{S^1} u K v \, dx,$$

and satisfying the Jacobi identity

$$< [u, v], Kw > + < [v, w], Ku > + < [w, u], Kv > = 0.$$

If $K$ is a differential operator we call $\gamma$ a differential cocycle. Gelfand and Fuks [16] observed that all differential 2-cocycles of $\text{Vect}(S^1)$ belong to the one-dimensional cohomology class generated by $[D^3]$. Moreover, each regular 2-coboundary is represented by the skew-symmetric operator

$$m_0 D + D m_0,$$

for some $m_0 \in C^\infty(S^1)$. Therefore, each differential 2-cocycle of $\text{Vect}(S^1)$ is represented by an operator of the form

$$K = m_0 D + D m_0 + \beta D^3$$

(3.4)

where $m_0 \in C^\infty(S^1)$ and $\beta \in \mathbb{R}$ (see also [17]).

For $k \geq 0$ and $u, v \in \text{Vect}(S^1) \equiv C^\infty(S^1)$, let us now define the $H^k$ (Sobolev) inner product by

$$< u, v >_k = \int_{S^1} \sum_{i=0}^{k} (\partial_x^i u) (\partial_x^i v) \, dx = \int_{S^1} A_k(u) v \, dx,$$

where

$$A_k = 1 - \frac{d^2}{dx^2} + \ldots + (-1)^k \frac{d^{2k}}{dx^{2k}}$$

(3.5)

is a continuous linear isomorphism of $C^\infty(S^1)$. Note that $A_k$ is a symmetric operator for the $L^2$ inner product since

$$\int_{S^1} A_k(u) v \, dx = \int_{S^1} u A_k(v) \, dx.$$
The operator $A_k$ gives rise to a Hamiltonian function on $\text{Vect}^*(S^1)$ given by

$$h_k(m) = \int_{S^1} \frac{1}{2} m (A_k^{-1} m) \, dx.$$ 

The corresponding Hamiltonian vector field $X_k$ is given by

$$X_k(m) = (mD + Dm)(A_k^{-1} m) = 2mu_x + um_x,$$

if we let $m = A_ku$.

**Theorem 3.7.** The Hamiltonian vector field $X_k$ is bi-Hamiltonian relatively to a modified Lie-Poisson structure if and only if $k \in \{0, 1\}$.

**Proof.** It is well known (see [28]) that $X_0$ is bi-Hamiltonian with respect to the operator $D$ which represents a coboundary. It is also known that $X_1$ is a bi-Hamiltonian vector field with respect to the cocycle represented by the operator $D(1 - D^2)$ cf. [2, 11, 14]. Notice that this cocycle is not a coboundary.

We will now show that there is no differential cocycle $K = m_0D + Dm_0 + \beta D^3$ for which $X_k$ could be Hamiltonian unless $k \in \{0, 1\}$. We have

$$dX_k(m) = 2u_xI + uD + 2mA_{k}^{-1} + m_xA_{k}^{-1},$$

and in particular, for $m = 1$,

$$dX_k(1) = D + 2DA_{k}^{-1}.$$ 

Letting

$$P(m) = dX_k(m) \circ K,$$

we obtain that

$$P(1) = (D + 2DA_{k}^{-1}) \circ (m_0D + Dm_0) + \beta D^4(1 + 2A_{k}^{-1}),$$

whereas

$$P(1)^* = (m_0D + Dm_0) \circ (D + 2DA_{k}^{-1}) + \beta D^4(1 + 2A_{k}^{-1}).$$

Therefore, denoting $m_0' = \partial_x m_0$, we have

$$P(1) - P(1)^* = (m_0' D + Dm_0') + 2(A_{k}^{-1} Dm_0 D - Dm_0 D A_{k}^{-1}) + 2(A_{k}^{-1} D^2 m_0 - m_0 D^2 A_{k}^{-1}).$$

If this operator is zero, we must have in particular the relation

$$A_k(P(1) - P(1)^*) A_k(e^{irx}) = 0,$$

for all $r \in \mathbb{Z}$. But, for $r \neq \pm 1$,

$$A_k(e^{irx}) = f_k(r) e^{irx} \quad \text{with} \quad f_k(r) = \frac{r^{2k+2} - 1}{r^2 - 1},$$

and

$$A_k(P(1) - P(1)^*) A_k(e^{irx})$$

is of the form $e^{irx}$ times a polynomial expression in $r$ with highest order term $2m_0'(x) r^{ik+1}$. Therefore, a necessary condition for $X_k$ to be Hamiltonian relatively to the Poisson operator $K$ defined by (3.4) is that $m_0$ is a constant.
Let $\alpha = 2m_0 \in \mathbb{R}$. Then

\[
P(m) = dX_k(m) \circ K = \alpha \{ 2u_x D + u D^2 + 2m D^2 A_k^{-1} + m_x D A_k^{-1} \} + \beta \{ 2u_x D^3 + u D^4 + 2m D^4 A_k^{-1} + m_x D^3 A_k^{-1} \},
\]

because $D$ and $A_k$ commute. By virtue of Proposition 3.6, a necessary condition for $X_k$ to be Hamiltonian with respect to the cocycle represented by $K$ is that $P(m)$ is symmetric. That is

\[
(3.6) \quad < P(m) M, N > = < M, P(m) N >,
\]

for all $m, M, N \in C^\infty(S^1)$. Since this last expression is tri-linear in the variables $m, M, N$, the equality can be checked for complex periodic functions $m, M, N$ where the $L^2$ inner product is extended naturally into a complex bilinear functional. That is, the extension is not a hermitian product, we just allow homogeneity with respect to the complex scalar field in both components. Let $m = A_k u$, $u = \exp(iax)$, $M = \exp(ibx)$ and $N = \exp(icx)$ with $a, b, c \in \mathbb{Z}$. We have

\[
< P(m) M, N > = \left[ (2ab^3 + b^4)\beta - (2ab + b^2)\alpha + \left( (ab^3 + 2b^4)\beta - (ab + 2b^2)\alpha \right) \frac{f_k(a)}{f_k(b)} \right] \int_{S^1} e^{i(a+b+c)x} dx,
\]

whereas

\[
< M, P(m) N > = \left[ (2ac^3 + c^4)\beta - (2ac + c^2)\alpha + \left( (ac^3 + 2c^4)\beta - (ac + 2c^2)\alpha \right) \frac{f_k(a)}{f_k(c)} \right] \int_{S^1} e^{i(a+b+c)x} dx.
\]

For $a = n$, $b = -2n$ and $c = n$, we obtain

\[
(3.7) \quad < P(m) M, N > = (24n^4 \beta - 6n^2 \alpha) \frac{f_k(n)}{f_k(2n)}, \quad < M, P(m) N > = 6n^4 \beta - 6n^2 \alpha.
\]

The equality of the two expressions in (3.7) for all $n \in \mathbb{N}$ is ensured by means of (3.6). For $k = 1$ this leads to the condition $\alpha + \beta = 0$ and we recover the second Poisson structure given by $K = D - D^3$ for which $X_1$ is known to be Hamiltonian with Hamiltonian function

\[
\tilde{h}_1(m) = \frac{1}{2} \int_{S^1} \left( (A_1^{-1} m)^3 + (A_1^{-1} m) [(A_1^{-1} m)_x]^2 \right) dx.
\]

In the general case, if $\beta \neq 0$, the leading term with respect to $n$ in the first expression in (3.7) is $(-48 \beta 2^{-2k})$, whereas in the second it is $(-12 \beta)$. Thus unless $\beta = 0$ we must have $k = 1$. On the other hand, if $\beta = 0$, from (3.6)-(3.7) we infer that $\alpha f_k(n) = \alpha f_k(2n)$ for all $n \in \mathbb{N}$. Thus $\alpha = 0$ unless $k = 0$. For $k = 0$ we recover the Poisson structure given by $K = D$ for which $X_0$ is Hamiltonian with Hamiltonian function

\[
\tilde{h}_0(m) = \frac{1}{2} \int_{S^1} m^3 dx.
\]

This completes the proof. \qed
We showed that among all $H^k$ Sobolev inner products on $C^\infty(S^1)$, only for $k \in \{0, 1\}$ is the associated vector field bi-Hamiltonian relatively to a modified Lie-Poisson structure. Endowing $\text{Diff}(S^1)$ with the $H^1$ right-invariant metric, the associated geodesic equation turns out to be the Camassa-Holm equation \[ u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2} u_x^2) = 0, \]
a model for shallow water waves (see [2] and the alternative derivations in [5, 13, 15, 20]) that accommodates waves that exist indefinitely in time [3, 7] as well as breaking waves [6, 8]. The bi-Hamiltonian structure is reflected in the existence of infinitely many conserved integrals for the equation \[ u_t + uu_x = 0. \]
This model of gas dynamics has been thoroughly studied (see [9] and references therein). In contrast to the case of the $H^1$ right-invariant metric [10], the Riemannian exponential map is not a $C^1$ local diffeomorphism in the case of the $L^2$ right-invariant metric [9]. This means that of the two bi-Hamiltonian vector fields $X_0$ and $X_1$, the second generates a flow on $\text{Diff}(S^1)$ with properties that parallel those of geodesic flows on finite-dimensional Lie groups.

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