Preface

Recursive maps, nowadays called *primitive recursive maps*, PR maps, have been introduced by Gödel in his 1931 article for the arithmetisation, gödelisation, of metamathematics.

For construction of his *undecidable formula* he introduces a non-constructive, non-recursive predicate *beweisbar*, provable.

Staying within the area of categorical free-variables theory PR of primitive recursion or appropriate extensions opens the chance to avoid the two (original) Gödel’s incompleteness theorems: these are stated for *Principia Mathematica und verwandte Systeme*, “related
systems” such as in particular Zermelo-Fraenkel set theory $\text{ZF}$ and v. Neumann Gödel Bernays set theory $\text{NGB}$.

On the basis of primitive recursion we consider $\mu$-recursive maps as partial p. r. maps. Special terminating general recursive maps considered are complexity controlled iterations. Map code evaluation is then given in terms of such an iteration.

We discuss iterative p. r. map code evaluation versus termination conditioned soundness and based on this decidability of primitive recursive predicates. This leads to consistency provability and soundness for classical, quantified arithmetical and set theories as well as for the PR descent theory $\pi R$, with unexpected consequences:

We show inconsistency provability for the quantified theories as well as consistency provability and logical soundness for the theory $\pi R$ of primitive recursion, strengthened by an axiom scheme of non-infinite descent of complexity controlled iterations like (iterative) map-code evaluation.

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P.S. I am obviously not an English native speaker. As Joseph Helfer puts it, my mathematical thinking and speech is somewhat special, it is Germish.
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Introduction

We fix constructive foundations for arithmetic on a map theoretical, algorithmical level. In contrast to elementhood and quantification based traditional foundations such as Principia Mathematica PM or Zermelo-Fraenkel set theory ZF, our fundamental primitive recursive theory PR has as its “undefined” terms just terms for objects and
maps. On that language level it is *variable free*, and it is free from formal quantification on individuals like numbers or number pairs.

This theory $\text{PR}$ is a formal, *combinatorial category* with cartesian i.e. universal *product* and a natural numbers object (NNO) $\mathbb{N}$, a *PR cartesian category*, cf. Romàn 1989.

The NNO $\mathbb{N}$ admits *iteration of endo maps* and the *full scheme of primitive recursion*. Such NNO has been introduced in categorical terms by Freyd 1972, on the basis of the NNO of Lawvere 1964, and named later (e.g. by Maietti 2010) *parametrised NNO*.

We will remain on the purely *syntactical* level of this categorical theory, and later *extensions*: *no formal semantics* necessary into an outside, non-combinatorial world. Cf. Hilbert’s formalistic program.

We then introduce into our *variable-free* setting *free variables*, which are introduced by *interpretation* of these variables as *names* for *projections*. As a consequence, we have in the present context ‘*free variable*’ as a *defined* notion. We have object and map *constants* such as *terminal object*, NNO, zero etc. and use free metavariables for objects and for maps.

*Fundamental arithmetic* is further developed along Goodstein’s 1971 *free variables Arithmetic* whose *uniqueness rules* are derived as theorems of categorical theory $\text{PR}$, with its “eliminable” notion of a *free variable*. This gives the expected *structure theorem for the algebra and order* on NNO $\mathbb{N}$. “On the way”, via Goodstein’s *truncated subtraction*, and “his” *commutativity of maximum function*, we obtain the *Equality Definability theorem*: If predicative equality of two p.r. maps is derivably true, then map equality between these maps is derivable. It follows a section on the derivation of the Peano axioms
as theorems.

The subsequent chapter brings into the game an embedding theory extension of $\text{PR}$ by abstraction of predicates into “virtual” new objects. This enrichment makes emerging basic theory $\text{PRa} = \text{PR} + \text{(abstr)}$ more comfortable, in direction to set theories, with their sets and subsets.

Chapter 3 introduces the general concept of partial maps, proves a structure theorem on the theory $\hat{\text{PRa}}$ of these maps and shows that $\mu$-recursive maps and while-loop programs are just partial p.r. maps; in particular our evaluations will be such (formally) partial maps.

Categories of partial maps are introduced in the literature via idempotent monos taken as domains, see Robinson & Rosolini 1988, and Cockett & Lack 2002.

Partial maps are introduced here as map pairs consisting of a domain-of-definition enumeration (in general not mono) and of a rule to throw an enumeration index of a defined argument into the value of that argument. Equality of partial maps is by availability of extension maps between the enumeration domains of the two partial maps under consideration, in both directions.

These partial maps form a primitive recursive diagonal-monoidal half-cartesian theory $\hat{\text{PRa}}$ (cf. Budach & Hoehncke 1975) which contains theory $\text{PRa}$ embedded as theory of this type, composition being defined via composition of pullbacks: Structure theorem for partials. theory extension by partiality is a Closure operator: partial partial maps are just partial maps.

Chapter 4 then exhibits within theory $\text{PRa}$ a universal object, $X$, of all numerals and nested pairs of numerals, and constructs by means
of that object universe theories PRX and PRXa: theory PRX is good for a one-object map-code evaluation, PRXa contains PRa as a cartesian PR embedded theory with predicate extensions.

Chapter 5 on evaluation strengthens p.r. theory PRXa into descent theory πR, by an axiom of non-infinite iterative descent with order values in polynomial semiring N[ω] ordered lexicographically.

This theory is shown to derive the—free variable PR—consistency formula for p.r. theories PRXa (and PR). The proof relies on constructive, complexity controlled code evaluation, which is extended to evaluation of argumented deduction trees:

*theorem on p.r. soundness* within set theory as frame (chapter 6), and *termination conditioned soundness of PRa ⊂ PRXa within theory πR taken as frame* (chapter 7).

The consequence is decidability of p.r. predicates within both theories. Since consistency formulae Con of both theories can be expressed as (free variable) p.r. predicates, this leads to

1. *Inconsistency provability* of set theory by Gödel’s second incompleteness theorem, and to

2. *Consistency provability* and soundness of descent theory πR, under *assumption* of μ-consistency.

[The latter is a (set theoretically) equivalent variant of ω-consistency, expressible in PRa, πR.]

**Notes** to the literature are inserted which are based mainly on Remarks of the Referee to Pfender 2012.
1 Primitive Recursion

1.1 The fundamental theory PR of primitive recursion

We fix here terms and axioms for the fundamental categorical (formally variable-free) cartesian theory PR of primitive recursion. The fundamental objects of the theory PR are the natural numbers object (‘NNO’) $\mathbb{N}$ and the terminal object $1$.

Composed objects of PR come in as “cartesian” products $(A \times B)$ of objects already enumerated. Formally:

$\begin{align*}
\text{A, B objects} \\
\text{(Obj_{Cart})} \\
\text{(A \times B) object}
\end{align*}$

[Here outmost brackets may be dropped]

Maps: Basic maps (“map constants”) of the theory PR are

- the zero map $0 : 1 \to \mathbb{N}$, and
- the successor map $s : \mathbb{N} \to \mathbb{N}$

Structure of PR as a category:

- generation—enumeration—of identity maps

$\begin{align*}
\text{A an object} \\
\text{(id generation)} \\
\text{id}_A : A \to A \text{ map}
\end{align*}$
• Composition:

\[
f : A \to B, \ g : B \to C \text{ maps}
\]

(\circ)

\[
(g \circ f) : A \to C \text{ map, diagram:}
\]

\[
\begin{array}{c}
\text{A} \\
\downarrow f \\
\text{B} \\
\downarrow g \\
\text{C}
\end{array}
\]

Here are the axioms making PR into a category:

• **Associativity of composition:**

\[
f : A \to B, \ g : B \to C, \ h : C \to D \text{ maps}
\]

(\circ_{\text{ass}})

\[
h \circ (g \circ f) = (h \circ g) \circ f : A \to D
\]

• **Neutrality of identities**

\[
f : A \to B \text{ map}
\]

(neutr\text{id})

\[
(f \circ \text{id}_{A}) = f : A \to A \to B \quad \text{and} \quad (\text{id}_{B} \circ f) = f : A \to B \to B.
\]

Map equality \( f = g : A \to B \) satisfies the axioms of reflexivity, symmetry, and transitivity:
\[(\text{refl})\]
\[
f : A \to B \text{ map}
\]
\[
f = f : A \to B
\]

\[(\text{sym})\]
\[
f = g : A \to B \text{ map}
\]
\[
g = f : A \to B
\]

\[(\text{trans})\]
\[
f = g, \ g = h : A \to B \text{ maps}
\]
\[
f = h : A \to B
\]

Composition is compatible with equality:

\[(\circ_\equiv)\]
\[
f = f' : A \to B, \ g = g' : B \to C
\]
\[
(g \circ f) = (g' \circ f') : A \to B \to C
\]

Because of technical simplicity in later code evaluation, we split this \textbf{axiom} into the following two ones:

\[(\circ_\equiv \text{1st})\]
\[
f = f' : A \to B, \ g : B \to C
\]
\[
(g \circ f) = (g \circ f') : A \to B \to C
\]
\( f : A \to B, \ g = g' : B \to C \)

\((\circ = \text{2nd})\)

\((g \circ f) = (g' \circ f) : A \to B \to C\)

**Cartesian map structure:**

- enumeration of *terminal maps*
  
  \( A \) object
  
  \[ \Pi = \Pi_A : A \to 1 \text{ map} \]

  [In Eilenberg & Elgot's notation. Lawvere designates this projection \! : A \to 1.]

- uniqueness **axiom** for terminal map family:
  
  \( A \) object, \( f : A \to 1 \) map
  
  \((\Pi)\)

  \[ f = \Pi_A : A \to 1 \]

**\( \Pi \)-naturality Lemma:** \( \Pi = [\Pi : A \to 1]_A \) is natural, i.e.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\Pi_A} & = & \downarrow{\Pi_B} \\
\equiv & \equiv & \equiv \\
1 & \xrightarrow{id} & 1
\end{array}
\]
• generation of left and right projections:

\[ \ell = \ell_{A,B} : A \times B \to A \text{ left projection,} \]
\[ r = r_{A,B} : A \times B \to B \text{ right projection} \]

• generation of induced maps into products:

\[ (f,g) : C \to A \times B \text{ map,} \]
the map induced by \( f \) and \( g \)

• compatibility of induced map formation with equality:

\[ (f,g) = (f',g') : C \to A \times B \]

• characteristic (Godement) equations

\[ \ell \circ (f,g) = f : C \to A \]
as well as

\((\text{GODE}_r)\)

\[ f : C \to A, \ g : C \to B \]

\[ r \circ (f, g) = g : C \to B \]

in commutative diagram form:

\[
\begin{array}{ccc}
C & \xrightarrow{(f,g)} & A \times B \\
\downarrow{g} & & \downarrow{r} \\
B & \xrightarrow{\ell} & A
\end{array}
\]

- uniqueness of induced map (GODEMENT):

\[ f : C \to A, \ g : C \to B, \ h : C \to A \times B \text{ maps}, \]

\[ \ell \circ h = f : C \to A \text{ and } r \circ h = g : C \to B \]

\[(\text{ind!})\]

\[ h = (f, g) : C \to A \times B \]

**SP Lemma:** In presence of the other axioms, this *uniqueness of the induced map* is equivalent to the following equational **axiom** of Surjective Pairing, see Lambek-Scott 1986:

\[ h : C \to A \times B \]

\[(\text{SP})\]

\[(\ell \circ h, r \circ h) = h : C \to A \times B\]
**Proof** as an exercise: Use compatibility of forming the induced map with equality.

We will formally rely on this equation as an **axiom**. It replaces uniqueness of forming the induced map.

We eventually replace equivalently, given the other axioms, inferential axiom (ind=) by distributivity **equation**

\[(f, g) \circ h = (f \circ h, g \circ h) : D \to A \times B\]

taken from Lambek-Scott. Equivalence **proof** as an exercise, proof of uniqueness of the induced in op. cit. Draw the diagram.

**Definition:** we define, for a map \(g : B \to B'\), **cylindrification**

\[A \times g = \text{def} \quad \text{id}_{A \times g} = \text{def} \quad (\text{id}_A \circ \ell, g \circ r) : A \times B \to A \times B'.\]

Diagram:

This **ends** the list of axioms for the **cartesian structure** of the theory **PR**.

Axioms for the iteration of endo maps:
$f : A \to A$ (endo) map

$f^\$: $A \times \mathbb{N} \to A$ iterated of $f$, satisfies

$f^\$: $(id_A, 0) = id_A : A \to A$ \, $[0 := 0 \Pi]$ (anchor),

$f^\$: $(A \times s) = f \circ f^\$: $A \times \mathbb{N} \to A \to A$ (step).

“Pentagonal” diagram:

As a first example for an iterated endo map take addition $+$ : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, having properties

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As a first example for an iterated endo map take addition $+$ : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, having properties
i.e. satisfying the free-variables equations

\[ a + 0 = a : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \]
\[ a + s \ n = s(a + n) = (a + n) + 1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{s} \mathbb{N}, \]

where \( 1 =_{\text{def}} s \circ 0 : 1 \to \mathbb{N} \to \mathbb{N}. \)

[A formal introduction of free variables as projections see below.]

**uniqueness axiom** for the iterated:

\[ f : A \to A \text{ (endo map)} \]
\[ h : A \times \mathbb{N} \to A, \]
\[ h \circ (\text{id}_A, 0) = \text{id}_A \text{ and} \]
\[ h \circ (A \times s) = f \circ h \text{ “as well”} \]

\[ (\S!\S) \]
\[ h = f^\S : A \times \mathbb{N} \to A \]

By this uniqueness **axiom**, the iterated map is **characterised** by the commutative pentagonal diagram above.

**Theorem (compatibility of iteration with equality):** uniqueness **axiom** (\S!\S) infers

\[ f = g : A \to A \]

\[ (\S=\S) \]
\[ f^\S = g^\S : A \times \mathbb{N} \to A \]
Proof: Consider the diagram

$$\begin{array}{c}
\begin{array}{c}
A 
\downarrow ^{(id,0)} \\
A
\end{array}
\xymatrix{
A \times N 
\ar[r]^{A \times s}
\ar[d]_{f^\$}
& A \times N \\
A 
\ar[r]_g
\ar[d]_{f}
& A
\end{array}
\end{array}$$

Since $f^\$ is the unique commutative fill-in into this pentagonal diagram over endomorphism $f$, it is sufficient to show that $g^\$: $A \times N \rightarrow A$ equally is such a commutative fill in.

For the triangle (anchor) this is trivial: $g^\$(id, 0) = id : A → A by definition of the null-fold iterated.

For the square (step) we have

$$g^\$ \circ (A \times s) = g \circ g^\$ \ (definition \ of \ g^\$$

$$= f \circ g^\$: A \times N \rightarrow A,$$

by assumption $f = g$ and by compatibility of $\circ$ with $=$ in first composition factor, axiom ($\circ = \text{1st}$).

So $g^\$ turns out to be another iterated of endo $f$, whence in fact $g^\$ = $f^\$ by uniqueness of the iterated q.e.d.

These axioms give all objects and maps of theory PR.

Freyd’s uniqueness scheme which completes the axioms constituting theory PR, reads
\[ f : A \to B, \ g : B \to B, \ h : A \times \mathbb{N} \to B, \]
\[ h \circ (\text{id}_A, 0 \circ \Pi_A) = f : A \to B, \ (\text{init}) \]
\[ h \circ (A \times s) = g \circ h : A \times \mathbb{N} \to B, \ (\text{step}) \]
\[ h = g^\S \circ (f \times \mathbb{N}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B, \]

in form of Freyd’s pentagonal diagram:

Freyd’s uniqueness diagram (FR!)

**Remark:** This uniqueness of the *initialised iterated* obviously specialises to *axiom* \((\S!)\) of uniqueness of “simple” iterated \(f^\S : A \times \mathbb{N} \to A\) and so makes that uniqueness axiom redundant.

**Problem:** Is, conversely, stronger Freyd’s uniqueness *axiom* already covered by uniqueness \((\S!)\) of “simply” iterated \(f^\S : A \times \mathbb{N} \to A\) ? My guess is “no”.

Freyd’s existence and uniqueness of the *initialised iterated* is displayed as the following commutative diagram:
Freyd's uniqueness diagram (FR!)

Existence of \( g^\$ \) and commutativity of lower triangle and square follow directly from axiom (§). Upper right commutativity is splitting a cartesian product \( f \times s \) in the two ways into compositions of right and left cylindrified maps.

Remaining equation

\[
(id_B, 0 \circ \Pi_B) \circ f = (f \times \mathbb{N}) \circ (id_A, 0 \circ \Pi_A) : A \to B \times \mathbb{N}
\]

is given by uniqueness of the induced map into the cartesian product \( B \times \mathbb{N} \), in detail:

\[
\ell \circ (id_B, 0) \circ f = id_B \circ f = f \quad \text{and} \\
\ell \circ (f \times \mathbb{N}) \circ (id_A, 0) = f \circ \ell \circ (id_A, 0) = f \circ id_A = f, \\
r \circ (id_B, 0) \circ f = 0 \circ f = 0 \circ \Pi_A \quad \text{and} \\
r \circ (f \times \mathbb{N}) \circ (id_A, 0 \circ \Pi_A) = r \circ (id_A, 0) = 0 \circ \Pi_A. \\
\]

Together this shows constructive availability of wanted initialised iterated \( h : A \times \mathbb{N} \to B \).
uniqueness of $h$, namely

$$f : A \to B, \ g : B \to B, \ h : A \times N \to B$$

$$h \circ (\text{id}_A, 0) = f$$

$$h \circ (A \times s) = g \circ h$$

(FR!) \hspace{1cm} h = g^5 \circ (f \times N).

is just required as an axiom, final axiom of theory PR.

From (FR!) we get trivially, with data

$$A \xrightarrow{\text{id}_A} A \xrightarrow{f} A$$

specializing data $$A \xrightarrow{f} B \xrightarrow{g} B$$

uniqueness ($\S$1) of iterated map $f^\S : A \times N \to A$.

1.2 The full scheme of primitive recursion

Already for definition and characterisation of multiplication and moreover for proof of “the” laws of arithmetic, the following full scheme (pr) of primitive recursion is needed.\footnote{in pure categorical form see FREYD 1972, and (then) PFENDER, KRÖPLIN, and PAPE 1994, not to forget its uniqueness clause}

**Theorem (Full scheme of PR):** PR admits scheme
\( g : A \to B \) (init map)

\( h : (A \times \mathbb{N}) \times B \to B \) (step map)

\[ \text{(pr)} \]

\[ \text{pr}[g, h] : = f : A \times \mathbb{N} \to B \]

is given such that

\[ f(\text{id}_A, 0) = g : A \to B \] (init), and

\[ f(\text{id}_A \times s) = h(\text{id}_{A \times \mathbb{N}}, f) : \]

\( (A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B, \) (step)

as well as

\( \text{(pr!)} : f \) is unique with these properties.

**Proof:** construction of the map \( f = \text{pr}[g, h] : A \times \mathbb{N} \to B \) out of data \( g : A \to B \) (initialisation) and \( h : (A \times \mathbb{N}) \times B \to B \) (iteration step):

Wanted \( f : A \times \mathbb{N} \to B \) is to satisfy (init) und (step) given as the two commuting diagrams.
With $\hat{g} := ((\text{id}_A, 0), g)$ and $\hat{h} := ((A \times s) \circ l, h)$ we get by (FR!) a uniquely determined map

$$k = (k_L, k_R) : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$$

satisfying
\[ A \times N \xrightarrow{A \times s} A \times N \]

(i.e.)

\[ k \circ (\text{id}_A, 0) = \hat{g} \quad \text{and} \]

\[ k \circ (A \times s) = \hat{h} \circ k. \]

[It will turn out that \( k = (\text{id}_{A \times N}, f) \) for wanted map \( f : A \times N \to B \).]

For our unique \( k \), consider first its left component \( k_\ell = \ell \circ k : A \times N \to A \times N \), unique—by (FR!)—in

\[ (A \times N) \times B \xrightarrow{\hat{h}} (A \times N) \times B \]

We have

\[ \ell \circ k \circ (\text{id}_A, 0) = \ell \circ \hat{g} = (\text{id}_A, 0) \quad \text{and} \]

\[ \ell \circ k \circ (A \times s) = \ell \circ \hat{h} \circ k = (A \times s) \circ \ell \circ k \]
Since these two equations hold likewise for $\text{id}_{A \times N}$ instead of $\ell \circ k$, it follows by uniqueness (FR!) of such a map $\ell \circ k = \text{id}_{A \times N}$.

Taking now $f := r \circ k : A \times N \to B$, we have the following diagram for this (unique) right component of $k : A \times N \to (A \times N) \times B$:

$$
\begin{array}{c}
A \times N \\
A \times s \\
A \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(A \times N) \times B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(A \times N) \times B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B \\
B \\
\end{array}
$$

obtain

$$
k = (\ell \circ k, r \circ k) = (\text{id}_{A \times N}, f),
$$

$$
f \circ (\text{id}_A, 0) = r \circ k \circ (\text{id}_A, 0) = r \circ \hat{g} = g \quad \text{and}
$$

$$
f \circ (A \times s) = r \circ k \circ (A \times s) = r \circ \hat{h} \circ k
$$

$$
= h \circ k = h \circ (\text{id}_{A \times N}, f)
$$

So this map $f : A \times N \to B$ is available, to fulfill the requirements of $\text{pr} [g, h] : A \times N \to B$.

**uniqueness proof** for such map $f$: Let $f'$ be a map assumed likewise to satisfy equations (init) and (step).
Then take $k' := (id_{A \times N}, f') : A \times N \to (A \times N) \to B$ and calculate:

\[
\begin{align*}
k' \circ (id_A, 0) &= (id_{A \times N}, f') \circ (id_A, 0) \\
&= ((id_A, 0), f' \circ (id_A, 0)) \\
&= ((id_A, 0), g) = \hat{g} \quad \text{as well as}
\end{align*}
\]

\[
\begin{align*}
k' \circ (A \times s) &= (id_{A \times N}, f') \circ (A \times s) \\
&= ((A \times s), f' \circ (A \times s)) \\
&= ((A \times s), \hat{h} \circ k').
\end{align*}
\]

Since by (FR!), $k$ above is the unique map to satisfy the equations above, we have necessarily $k' = k$ and hence $f' = r \circ k' = r \circ k = f : A \times N \to B$. q.e.d.

\begin{center}
CLOETEENDE
\end{center}

1.3 Uniqueness of the NNO $N$

Strictly speaking, this subsection is not needed for the sequel.

1.4 A monoidal presentation of theory PR

straightforward categorically, not needed strictly.

1.5 Introduction of free variables

We start with a (“generic”) example of Elimination of free variables by their Interpretation into (possibly nested) projections:
a distributive law \( a \cdot (b + c) = a \cdot b + a \cdot c \) gets the map interpretation

\[
a \cdot (b + c) = (a \cdot b) + (a \cdot c) : \\
R^3 =_{\text{by def}} R^2 \times R =_{\text{by def}} (R \times R) \times R \rightarrow R,
\]

with systematic interpretation of variables:

\[
a := \ell \ll , \ b := r \ell , \ c := r : R^3 = (R \times R) \times R \rightarrow R,
\]

and infix writing of operations \( \text{op} : R \times R \rightarrow R \) prefix interpreted as

\[
\cdot \circ (a, + \circ (b, c)) = + \circ (\cdot \circ (a, b), \cdot \circ (a, c)) : R^3 \rightarrow R.
\]

In form of a commuting diagram:

An iterated \( f^8 : A \times \mathbb{N} \) may be written in free-variables notation as

\[
f^8 = f^8(a, n) = f^n(a) : A \times \mathbb{N} \rightarrow A
\]

with \( a := \ell : A \times \mathbb{N} \rightarrow A \), and \( n := r : A \times \mathbb{N} \rightarrow \mathbb{N} \).

**Systematic map Interpretation of free-variables Equations:**
1. extract the common codomain (domain of values), say $B$, of both sides of the equation (this codomain may be implicit);

2. “expand” operator priority into additional bracket pairs;

3. transform infix into prefix notation, on both sides of the equation;

4. order the (finitely many) variables appearing in the equation, e.g. lexically;

5. if these variables $a_1, a_2, \ldots, a_m$ range over the objects $A_1, A_2, \ldots, A_m$, then fix as common domain object (source of commuting diagram), the object

$$A = A_1 \times A_2 \times \ldots \times A_m = \text{def} \ ((A_1 \times A_2) \times \ldots) \times A_m;$$

6. interpret the variables as identities (possibly nested) projections, will say: replace, within the equation, all the occurrences of a resp. variable, by the corresponding—in general binary nested—projection;

7. replace each symbol “0” by “0 $\Pi_D$” where “$D$” is the (common) domain of (both sides) of the equation;

8. insert composition symbol $\circ$ between terms which are not bound together by an induced map operator as in $(f_1, f_2)$;

9. By the above, we have the following two-maps-cartesian-Product rule, forth and back: For
\[a := \ell_{A,B} : (A \times B) \to A, \ b := r_{A,B} : (A \times B) \to B, \text{ and } f : A \to A',\]
as well as \(g : B \to B',\) the following identity holds:

\[(f \times g)(a, b) = (f \times g) \circ (\ell_{A,B}, r_{A,B}) \]
\[= (f \times g) \circ \text{id}_{(A \times B)} = (f \times g) \]
\[= (f \circ \ell_{A,B}, g \circ r_{A,B}) \]
\[= (f \circ a, g \circ b) = (f(a), g(b)) : A \times B \to A' \times B';\]

10. for free variables \(a \in A, n \in \mathbb{N}\) interpret the term \(f^n(a)\) as the map \(f^b(a, n) : A \times \mathbb{N} \to A.\)

These 10 interpretation steps transform a (PR) free-variables equation into a variable-free, categorical equation of theory PR:

**Elimination of (free) variables** by Interpretation as projections, and vice versa: **Introduction of free variables** as names for projections. We allow for mixed notation too, all this, for the time being, only in the context of a cartesian (!) theory \(T.\)

All of our theories are free from classical, (axiomatic) formal quantification. Free variables equations are understood naively as universally quantified. But a free variable \((a \in A)\) occurring only in the premise of an implication takes (in suitable context, see below), the meaning

\[\text{for any given } a \in A : \text{premise } (a, \ldots) \implies \text{conclusion, i.e.}\]
\[\text{if exists } a \in A \text{ s.t. premise } (a, \ldots), \text{then conclusion.}\]

### 1.6 Goodstein FV arithmetic

In “Development of Mathematical Logic” (Logos Press 1971) R. L. Goodstein gives four basic uniqueness-rules for free-variable Arith-
metics. We show here these rules for theory PR, and that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction $a \div n$.

For our evaluation and consistency considerations below we need from present section equality predicate $[a \doteq b] : \mathbb{N} \times \mathbb{N} \rightarrow 2$, and that this predicate defines map equality, see equality definability scheme in the middle of section. This scheme is a consequence of commutativity $\max(a, b) = \text{def } a + (b \div a) = b + (a \div b) = \text{by def } \max(b, a)$ which is difficult to show and which you may take on faith.

Basic GA operations are addition ‘+’, predecessor ‘pre’, truncated subtraction ‘\(\div\)’, [in Goodstein predecessor written pre := (\(_\) \div 1)], as well as multiplication ‘\(\cdot\)’.

We include\(^2\) into Goodstein’s uniqueness rules a “passive parameter” $a$. These extended rules are derivable by use of Freyd’s uniqueness theorem (pr!), part of full scheme (pr) of primitive recursion which he deduces from his uniqueness (FR!) of the initialised iterated.

Freyd 1972 deduces the latter from availability of a natural numbers object $\mathbb{N}$ in Lawvere’s sense, axiomatic availability of higher order internal hom objects with, again axiomatic, evaluation map family for these objects, of form $\epsilon_{A,B} : B^A \times A \rightarrow B$ within the category considered.

**Goodstein’s rules with passive parameter:**

Let $f, g : A \times \mathbb{N} \rightarrow \mathbb{N}$ be maps, $s : \mathbb{N} \rightarrow \mathbb{N}$ the successor map

\(^2\)Sandra Andrasek and the author
$n \mapsto n + 1$ and $\text{pre} : \mathbb{N} \to \mathbb{N}$ the predecessor map, usually written as $n \mapsto n - 1$.

Then Goodstein’s rules read:

$$f(a, sn) = f(a, n) : A \times \mathbb{N} \to B$$

**U$_1$**

$$f(a, n) = f(a, 0) : A \times \mathbb{N} \to B$$

no change by application of successor

infers equality with value at zero for $f$

$$f(a, sn) = s \cdot f(a, n) : A \times \mathbb{N} \to \mathbb{N}$$

**U$_2$**

$$f(a, n) = f(a, 0) + n : A \times \mathbb{N} \to \mathbb{N}$$

accumulation of successors into $+n$

$$f(a, sn) = \text{pre} \cdot f(a, n) : A \times \mathbb{N} \to \mathbb{N}$$

**U$_3$**

$$f(a, n) = f(a, 0) \div n : A \times \mathbb{N} \to \mathbb{N}$$

accumulation of predecessors into $\div n$
\[ f(a, 0) = g(a, 0) : A \to \mathbb{N} \]
\[ f(a, sn) = g(a, sn) : A \times \mathbb{N} \to \mathbb{N} \]

\text{uniqueness of map definition by case-distinction}

Rule \( U_4 \) is nothing else than \textit{uniqueness} of the induced map out of the sum \( A \times \mathbb{N} \cong (A \times 1) + (A \times \mathbb{N}) \), this sum canonically realised via \textit{injections} \( \iota = (\text{id}_A, 0) : A \to A \times \mathbb{N} \) as well as—right injection—\( \kappa = \text{id}_A \times s : A \times \mathbb{N} \to A \times \mathbb{N} \).

\textbf{Proof} of these four rules is straight forward for theory \textbf{PR}, using \textsc{Freyd}’s uniqueness (\textsc{FR!}) and uniqueness clause (\textsc{pr!}) of the \textit{full scheme of primitive recursion} respectively, as follows:

For scheme \( U_1 \) consider, with free variable \( a := \ell : A \times \mathbb{N} \to A \),

\( (\text{FR!}) \)

\[ f(a, n) = f = f(a, 0). \]

\textbf{Proof} of \( U_2 \) of “\textit{summing up successors}”: 31
\[ f(a, n) = f(a, 0) + n \]

**Proof** of \( U_3 \) is exactly analogous to the above. Replace in statement of \( U_2 \) and its proof stepwise augmentation \( f(a, sn) = sf(a,n) \) by stepwise descent

\[ f(a, sn) = f(a,n) - 1 = \text{by def} \quad \text{pre} f(a,n). \]

On right hand side replace successor \( s: \mathbb{N} \to \mathbb{N} \) by predecessor \( \text{pre}: \mathbb{N} \to \mathbb{N} \) which in turn is defined by the full scheme (pr) of primitive recursion. In postcedent replace iterated successor \( a + n: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by iterated predecessor \( a - n: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \).

[In Goodstein's original, \( \text{pre}(n) = n - 1 : \mathbb{N} \to \mathbb{N} \) is a basic, “undefined” map constant]

We give a Direct Proof of \( U_4 \):

We tailor first this scheme for convenient use of “full” uniqueness
scheme (pr!), as follows:

\[
\begin{align*}
  f &= f(a, n), \quad f' = f'(a, n) : A \times \mathbb{N} \to B, \\
  f(a, 0) &= f'(a, 0) : A \to B, \\
  f(a, s n) &= f'(a, s n) : A \times \mathbb{N} \to A \times \mathbb{N} \to B \\
  f &= f' : A \times \mathbb{N} \to B.
\end{align*}
\]

Choose the anchor map

\[
  g = g(a) := f(a, 0) = f'(a, 0) : A \to A \times \mathbb{N} \to B
\]

and the step map

\[
  h = h((a, n), b) := f(a, s n) = f'(a, s n) : (A \times \mathbb{N}) \times B \rightarrow A \times \mathbb{N} \to B.
\]

We obtain, via the full scheme (pr!) of PR:

\[
\begin{align*}
  f(a, 0) &= g(a) = f'(a, 0), \quad \text{(anchor hypothesis)} \\
  f(a, s n) &= h((a, n), f(a, n)) = f'(a, s n) \quad \text{(step hypothesis)}
\end{align*}
\]

\[
  f = \text{pr}[g, h] = f' : A \times \mathbb{N} \to B \quad \text{q.e.d.}
\]

Together with reflexivity, symmetry, and transitivity of equality \( f = g : A \to B \): between maps as well as with the defining equations for the fundamental operations and \( U_1, \ldots, U_4 \) above, we define categorical Goodstein’s free-variables Arithmetic which we name Goodstein Arithmetic, GA.
We now quote, with passive parameters made visible, GOODSTEIN’s arithmetical equations together with his proofs.

The first equation is (Goodstein’s statement numbers)

**Lemma:**

\[(a \div n) \div 1 =^{\mathbb{GA}} (a \div 1) \div n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \]  
\[a \in \mathbb{N} \text{ free, “passive”, } a := \ell : A \times \mathbb{N} \to A, \]
\[n \in \mathbb{N} \text{ free, recursive, } n := r : A \times \mathbb{N} \to \mathbb{N}. \]

**Proof:**

\[
\begin{align*}
(a \div s n) \div 1 &= \text{by def} \ ( (a \div n) \div 1 ) \div 1 \\
&= \text{by def} \ (a \div 1) \div n : \mathbb{N}^2 \to \mathbb{N} \quad \text{q.e.d.}
\end{align*}
\]

Next equation is

**stepwise simplification rule** for truncated subtraction:

\[s a \div s b = a \div b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (1.1)\]

**Proof:**

\[
\begin{align*}
s a \div s s b &= \text{by def} \ (s a \div s b) \div 1 \\
&= \text{by def} \ a \div b : \mathbb{N}^2 \to \mathbb{N}, \quad \text{q.e.d.}
\end{align*}
\]
Lemma: \( a \div a = 0 : \mathbb{N} \to \mathbb{N} \). \hspace{1cm} (1.2)

Proof:
\[
\begin{align*}
\quad & s a \div s a = a \div a \\
\text{U}_1 & (\text{by stepwise simplification 1.1 above}) \\
\hline
\quad & a \div a = 0 \div 0 =_{\text{by def}} 0 \quad \text{q.e.d.}
\end{align*}
\]

Lemma: \( 0 \div a = 0 : \mathbb{N} \to \mathbb{N} \). \hspace{1cm} (1.3)

Proof:
\[
\begin{align*}
\quad & 0 \div s a =_{\text{by def}} (0 \div a) \div 1 \\
\quad & = (0 \div 1) \div a \quad (\text{by (1.) above}) \\
\text{U}_1 & = 0 \div a : \mathbb{N} \to \mathbb{N} \\
\hline
\quad & 0 \div a = 0 \div 0 = 0 : \mathbb{N} \to \mathbb{N} \quad \text{q.e.d.}
\end{align*}
\]

Proposition:
\[
a \div (b + c) = (a \div b) \div c : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}. \quad (1.31)
\]

Proof:
\[ a := \ell_{N,N} \circ \ell_{N \times N,N} : (N \times N) \times N \rightarrow N \times N \rightarrow N, \]
\[ b := r \circ \ell : (N \times N) \times N \rightarrow N \times N \rightarrow N, \]
\[ (a, b) = \ell_{N \times N,N} : A \times N = N^2 \times N \rightarrow A = N^2, \]
\[ c := r : A \times N = N^2 \times N \rightarrow N. \]

\[ a \div (b + sc) = \text{by def} \ a \div s(b + c) \quad \text{(definition of + )}, \]
\[ = \text{by def} \ (a \div (b + c)) \div 1 \quad \text{(definition of \div )} \]

\[ a \div (b + c) = (a \div (b + 0)) \div c = \text{by def} \ (a \div b) \div c. \quad \text{q.e.d.} \]

**Full Simplification:**

\[ (a + n) \div (b + n) = a \div b : N^2 \times N \rightarrow N. \quad (1.4) \]

**Proof:**

\[ (a + sn) \div (b + sn) \]
\[ = \text{by def} \ s(a + n) \div s(b + n) = (a + n) \div (b + n), \]
by substitution—realised essentially as composition
—of \((a + n)\) into \(a\), and \((a + n)\) into \(b\) within

\[ \text{stepwise simplification equation 1.1 above} \]
\[ (U_1) \]

\[ (a + n) \div (b + n) = (a + 0) \div (b + 0) = \text{by def} \ a \div b. \]
Lemma: $0 + n = n [ = \text{by def} \ n + 0 ] : \mathbb{N} \rightarrow \mathbb{N}$, \hspace{1cm} (2)

Proof:

$$
\begin{array}{c}
\text{id}_{\mathbb{N}} \ s \ a = s \ a \\
\text{U}_2 \hfill \\
\text{id}_{\mathbb{N}}(a) = \text{id}_{\mathbb{N}}(0) + a,
\end{array}
$$

and hence

$$a = \text{id}_{\mathbb{N}}(a) = \text{id}_{\mathbb{N}}(0) + a = 0 + a : \mathbb{N} \rightarrow \mathbb{N} \hspace{1cm} \text{q.e.d.}
$$

Lemma: $a + s \ b = s \ a + b : \mathbb{N} \times \mathbb{N} \rightarrow B$. \hspace{1cm} (2.1)

Proof by $\text{U}_2$ as follows, with free variable $b := r : \mathbb{N}^2 \rightarrow \mathbb{N}$ as recursion variable:

For $f = f(a,b) = \text{def} \ a + s \ b : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$:

$$
\begin{array}{c}
f(a,s \ b) = \text{by def} \ a + s \ s \ b = s(a + s \ b) = s \ f(a,b) : \mathbb{N}^2 \rightarrow \mathbb{N} \\
\text{U}_2 \hfill \\
\hfill
\end{array}
$$

$$f(a,b) = a + s \ b = f(a,0) + b$$

$$= \text{by def} \ (a + s \ 0) + b = \text{by def} \ s \ a + b \hspace{1cm} \text{q.e.d.}
$$

Theorem:

$$a + b = b + a : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, \hspace{1cm} (2.2),$$

$$a := \ell : \mathbb{N}^2 \rightarrow \mathbb{N},$$

$$b := r : \mathbb{N}^2 \rightarrow \mathbb{N}.$$
Proof:

\[ a + 0 =_{\text{by def}} a = 0 + a \text{ by (2) above}, \]

\[ a + sb = sa + b \text{ by (2.1) above (and symmetry of equality)} \]

\[ U_4 \]

\[ a + b =_{\text{by def}} f(a,b) = g(a,b) \]

\[ =_{\text{by def}} sa + b : N^2 \rightarrow N \quad \text{q.e.d.} \]

This gives also sort of permutability for truncated subtraction:

\[ (a \div b) \div c = (a \div c) \div b : (N \times N) \times N \rightarrow N. \]

Proof:

\[ (a \div b) \div c = a \div (b + c) \text{ by (1.31) above} \]

\[ = a \div (c + b) \text{ by commutativity of addition above} \]

\[ = (a \div c) \div b \text{ again by (1.31) q.e.d.} \]

Lemma:

\[ (a + n) \div n = (a + n) \div (0 + n) = a : N \times N \rightarrow N \quad \text{q.e.d.} \quad (2.3) \]

Associativity of Addition

\[ (a + b) + c = a + (b + c) : (N \times N) \times N \rightarrow N, \]

with free variables

\[ a := \ell \circ \ell : (N \times N) \times N \rightarrow N \times N \rightarrow N, \]

\[ b := r \circ \ell : (N \times N) \times N \rightarrow N \times N \rightarrow N, \]

\[ c := r : (N \times N) \times N. \]
Proof: for $f((a, b), c) = \text{def } a + (b + c) : \mathbb{N}^2 \times \mathbb{N}$:

$$f((a, b), s c) = a + (b + s c) = a + s(b + c)$$

$$= s(a + (b + c)) = s f((a, b), c)$$

\[ U_2 \]

$$a + (b + c) = f((a, b), c) = f((a, b), 0) + c$$

$$= \text{by def } (a + (b + 0)) + c = (a + b) + c : \mathbb{N}^2 \times \mathbb{N} \to \mathbb{N} \quad \text{q.e.d.}$$

Recall p. r. Definition of Multiplication:

$$a \cdot 0 = 0 : \mathbb{N} \to \mathbb{N},$$

$$a \cdot (n + 1) = (a \cdot n) + a.$$  

For this operation, we have not only annihilation by zero from the right, but also

Left zero-Annihilation $0 \cdot n = 0 : \mathbb{N} \to \mathbb{N}$.

Proof:

$$0 \cdot s n = (0 \cdot n) + 0 = 0 \cdot n$$

\[ U_1 \]

$$0 \cdot n = 0 \cdot 0 = 0 \quad \text{q.e.d.}$$

For proving the other equational laws making the natural numbers object $\mathbb{N}$ into a unitary commutative semiring with in addition truncated subtraction introduced above, GOODSTEIN’s derived scheme $V_4$ below is helpfull.

For proof of that scheme, we rely on
Commutativity of maximum operation:

\[ \max(a, b) = \text{def} \quad a + (b - a) = b + (a - b) = \text{by def} \quad \max(b, a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \]

**Proof:** As a first step, we show

**Diagonal Reduction Lemma for maximum:**

\[
\max(a, b) = \max(a \div 1, b \div 1) + \text{sign}(a + b)
\]

\[= \text{by def} \quad \max(a \div 1, b \div 1) + (1 \div (1 \div (a + b))) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \]

\[
\max(a, s \ b) = \max(a \div 1, s \ b \div 1) + \text{sign}(a + s \ b), \quad (1)
\]

(where \(\text{sign}(0) = 0, \ \text{sign}(s \ n) = 1\)), as follows:

\[
\max(0, s \ b) = s \ b = \max(0, b) + 1 : \mathbb{N} \to \mathbb{N}, \quad (2)
\]

\[
\max(s \ a, s \ b) = s \ \max(a, b) = \max(a, b) + 1
\]

\[= \max(s \ a \div 1, s \ b \div 1) + \text{sign}(s \ a + s \ b) \quad (3)
\]

From (2) and (3) follows (1) by uniqueness U₄.

Furthermore

\[
\max(a, 0) = a = (a \div 1) + \text{sign}(a)
\]

\[= \max(a \div 1, 0 \div 1) + \text{sign}(a + 0). \quad (4)
\]

Together with (1) above, this gives, again by U₄, the **Diagonal Reduction Lemma**.

From this we get immediately by substitution

\[3\text{in GOODSTEIN 1957 this is taken as an axiom} \]
Opposite Diagonal Reduction Lemma for maximum:

\[
\max(b, a) = \max(b \div 1, a \div 1) + \operatorname{sign}(b + a)
\]

\[
= \max(b \div 1, a \div 1) + \operatorname{sign}(a + b) \quad \text{q.e.d.}
\]

Now let

\[
\phi = \phi(n, (a, b)) : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}
\]

by

\[
\phi(0, (a, b)) = 0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}
\]

and

\[
\phi(s n, (a, b)) = \phi(n, (a, b)) + \operatorname{sign}((a \div n) + (b \div n))
\]

\[
\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}
\]

We show for this increment map \( \phi \)

\[
\max(a \div n, b \div n) + \phi(n, (a, b))
\]

as well as

\[
\max(b \div n, a \div n) + \phi(n, (a, b))
\]

(5)

\[
\max(a \div s n, b \div s n) + \phi(s n, (a, b))
\]

(6)

(same increment).

First we show equation (5): Substitution of \((a \div n)\) for \(a\) and \((b \div n)\) for \(b\) within Reduction Lemma above gives

\[
\max(a \div n, b \div n)
\]

\[
= \max((a \div n) \div 1, (b \div n) \div 1) + \operatorname{sign}((a \div n) + (b \div n))
\]
Adding $\phi(n, (a, b))$ to both sides of this equation gives

$$\max(a \div n, b \div n) + \phi(n, (a + b))$$
$$= \max((a \div n) \div 1, (b \div n) \div 1)$$
$$+ \text{sign}((a \div n) + (b \div n)) + \phi(n, (a + b))$$
$$= \text{by def } \max(a \div s n, b \div s n) + \phi(s n, (a, b)),$$

i.e. equation (5).

We show equation (6): By substitution of $(b \div n)$ for $b$ and $(a \div n)$ for $a$ in Opposite Reduction Lemma and addition of $\phi(n, (a, b))$ on both sides, we get

$$\max(b \div n, a \div n) + \phi(n, (a, b))$$
$$= \max((b \div n) \div 1, (a \div n) \div 1)$$
$$+ \text{sign}((b \div n) + (a \div n)) + \phi(n, (a, b))$$
$$= \max((b \div n) \div 1, (a \div n) \div 1)$$
$$+ \text{sign}((a \div n) + (b \div n)) + \phi(n, (a, b))$$
$$= \text{by def } \max((b \div n) \div 1, (a \div n) \div 1) + \phi(s n, (a, b))$$
$$= \max(b \div s n, a \div s n) + \phi(s n, (a, b)),$$

i.e. equation (6).

From the two Lemmata, we get by uniqueness $U_1$

$$\max(a \div n, b \div n) + \phi(n, (a, b))$$
$$= \max(a \div 0, b \div 0) + \phi(0, (a, b)) = \max(a, b) + 0 = \max(a, b)$$
as well as

$$\max(b \div n, a \div n) + \phi(n, (a, b))$$
$$= \max(b \div 0, a \div 0) + \phi(0, (a, b)) = \max(b, a) + 0 = \max(b, a)$$

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and hence
\[
\max(a, b) = \max(a \div n, b \div n) + \phi(n, (a, b)) \text{ as well as }
\]
\[
\max(b, a) = \max(b \div n, a \div n) + \phi(n, (a, b)),
\]
and so, by substitution of \( b \) into \( n \):
\[
\max(a, b) = \max(a \div b, b \div b) + \phi(b, a, b)
\]
\[
= (a \div b) + \phi(b, (a, b))
\]
\[
= \max(b \div b, a \div b) + \phi(b, (a, b))
\]
\[
= \max(b, a) : N \times N \rightarrow N
\]
q.e.d.

This given, we can now show, for \( GA \) (and hence for \( PR \)), scheme

\[
f, g, h : A \times N \rightarrow N
\]
\[
f(a, 0) = g(a, 0) : A \rightarrow N
\]
\[
f(a, sn) = f(a, n) + h(a, n) : A \times N \rightarrow N
\]
\[
g(a, sn) = g(a, n) + h(a, n) : A \times N \rightarrow N
\]
\[
V_4
\]
\[
f(a, n) = g(a, n).
\]

Rule \( V_4 \) can be derived, by applying rule \( U_1 \) to the distance map
\[
d(a, n) = |f(a, n), g(a, n)| = |f(a, n) - g(a, n)|
\]
\[
= \text{by def} \quad (f(a, n) \div g(a, n)) + (g(a, n) \div f(a, n)) : A \times N \rightarrow N^2 \rightarrow N :
\[
\begin{align*}
d(a,0) &= (f(a,0) \div g(a,0)) + (g(a,0) \div f(a,0)) = 0 \\
d(a,sn) &= (f(a,sn) \div g(a,sn)) + (g(a,sn) \div f(a,sn)) \\
&= (f(a,n) + h(a,n)) \div (g(a,n) + h(a,n)) \\
&\quad + (g(a,n) + h(a,n)) \div (f(a,n) + h(a,n)) \\
&= (f(a,n) \div g(a,n)) + (g(a,n) \div f(a,n)) \\
&= d(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{align*}
\]

whence, by U\(_1\):

\[
\begin{align*}
d(a,n) &= d(a,0) = 0, \ i.e. \\
(f(a,n) \div g(a,n)) + (g(a,n) \div f(a,n)) &= 0, \ \text{whence} \\
f(a,n) \div g(a,n) &= 0 = g(a,n) \div f(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{align*}
\]

and hence

\[
\begin{align*}
f(a,n) &= f(a,n) + (g(a,n) \div f(a,n)) \\
&= \max(f(a,n), g(a,n)) \\
&= \max(g(a,n), f(a,n)) \\
&= g(a,n) + (f(a,n) \div g(a,n)) \\
&= g(a,n) \quad \text{q.e.d.}
\end{align*}
\]

**individual equality**, equality *predicate*

\[
[m \doteq n] : \mathbb{N}^2 \to 2
\]

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is defined via weak order as follows:

\[ [m \leq n] = \text{def} \; \neg [m \div n] : \mathbb{N}^2 \to \mathbb{N}, \text{ where} \]

\[ \neg n = \text{def} \; 1 \div n, \text{ directly p.r. defined by} \]

\[ \neg 0 = \text{def} \; 1 \equiv \text{true} : 1 \to \mathbb{N}, \]

\[ \neg s n = \text{def} \; 0 \equiv \text{false} : 1 \to \mathbb{N}. \]

This order on \( \mathbb{N} \) is reflexive and transitive.

**Individual equality**—first on \( \mathbb{N} \)—then is easily defined by

\[ [m \div n] = \text{def} \; [m \leq n \land n \leq m] : \mathbb{N}^2 \to \mathbb{N}. \]

Almost by definition, the triple \( \{ \leq, \div, \geq \} : \mathbb{N}^2 \to \mathbb{N} \) fullfills the law of trichotomy, and max\((a, b) : \mathbb{N}^2 \to \mathbb{N} \) above is characterised as the maximum map with respect to the order \([a \leq b] : \mathbb{N}^2 \to \mathbb{N} \) just introduced, a posteriori.

We now have at our disposition all ingredients for the

**Equality definability theorem:**

\[ f = f(a) : A \to B, \; g = g(a) : A \to B \text{ in PR}, \]

\[ \text{PR} \vdash \text{true}_A = [f(a) \div_B g(a)] : \]

\[ A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{\div_B} 2 \]

(EqDef)

\[ \text{PR} \vdash f = g : A \to B, \text{ i.e. } f = \text{PR} g : A \to B. \]

**Proof:**
We begin with the special case $B = \mathbb{N}$: Let $f, g : A \rightarrow \mathbb{N}$ PR-maps satisfying the antecedent of (EqDef). Then

\[
\text{PR} \vdash f(a) = f(a) + 0 = f(a) + (g(a) \div f(a)) \\
= \max(f(a), g(a)) \\
= \max(g(a), f(a)) \\
= g(a) : A \rightarrow B.
\]

The general case for codomain object $B$ follows, since individual equality on (binary) cartesian Products is canonically defined componentwise, and $B$ is a cartesian product of $\mathbb{N}$’s q.e.d.

These fundamentals given, we can continue with properties of the algebraic structure on $\mathbb{N}$.

**Algebra, Order and Logic** on $\mathbb{N}$:

- $\mathbb{N}$ admits the structure

  ![Diagram](image)

  of a **unary, commutative semiring with zero**—properties of $\div$, sign : $\mathbb{N} \rightarrow \mathbb{N}$ (“positiveness”), order, and equality $\equiv$ see below.

- $\mathbb{N}$ admits a foundational important additional algebraic structure, namely **truncated subtraction** $m \div n : \mathbb{N}^2 \rightarrow \mathbb{N}$, with
its simplification properties, such that multiplication distributes over this kind of subtraction.

This distributivity will further entail that of multiplication over “full”, not truncated subtraction within

\[ \mathbb{Z} =_{\text{def}} (\mathbb{N} \times \mathbb{N}) / \equiv_{\mathbb{Z}} , \]

with defining equality predicate

\[ [(p, q) \equiv_{\mathbb{Z}} (p', q')] =_{\text{def}} [p + q' \equiv q + p'] : \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N} \times \mathbb{N} \overset{\equiv}{\to} \mathbb{N} . \]

- \( \mathbb{N} \) admits linear order \([ m \leq n ] : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \subset \mathbb{N} \) as a weak reflexive and transitive predicate—this order is p.r. decidable.

- As basic logical structures, \( \mathbb{N} \) admits negation

  \( \neg = \neg n : \mathbb{N} \to \mathbb{N} , \) as well as

  \( \text{sign} = \text{sign } n = \neg \neg n : \mathbb{N} \to \mathbb{N} , \)

  \( \text{sign}(n) \) is directly p.r. defined by

  \( \text{sign } 0 =_{\text{def}} 0 \equiv \text{false} , \text{sign } s n =_{\text{def}} 1 \equiv s 0 : \)

  \( \text{sign } n = [ n > 0 ] : \mathbb{N} \to \mathbb{N} \) PR decides on positiveness.

Furthermore, we have a fundamental equality predicate

\[ [ m \equiv n ] =_{\text{by def}} [ m \leq n ] \land [ m \geq n ] : \mathbb{N} \times \mathbb{N} \to \mathbb{N} , \]

\[ [ a \land b =_{\text{def}} \text{sign}(a \cdot b) \text{ logical ‘and’} ] , \]
which is an equivalence predicate, and which makes up a trichotomy with strict order

\[ [m < n] = \text{def} \ sign(n - m) \]
\[ = [m \leq n] \land \neg [m = n] : \mathbb{N}^2 \to \mathbb{N}, \]

\textbf{Proof} of the latter equation is left as an Exercise.

- object \( \mathbb{N} \) admits definition of (Boolean) “logical functions” by truth tables, as does set \( 2 \) classically—and below in theory \( \text{PRa} = \text{PR} + (\text{abstr}) \) of primitive recursion with predicate abstraction: draw the commuting diagrams.

- Algebra Combined with Order: As expected, addition is strongly monotonic in both arguments, multiplication is strongly monotonic for both arguments strictly greater than zero, and truncated subtraction is weakly monotonic in its first argument and weakly antitonic in its second.

\textbf{Theorem:} In free–variables arithmetics the \textbf{commutative law} for multiplication: \( n \cdot m = m \cdot n \), holds.

\textbf{Proof:} We need the following

\textbf{Lemma:}

(i) \( 0 \cdot n = 0 \)

(ii) \( sa \cdot n = a \cdot n + n \)

\textbf{Proof:}
(i) $0 \cdot 0 = 0$ and
\[ 0 \cdot sn = 0 \cdot (n + 1) = 0 \cdot n + 0 = 0 \cdot n = 0 \cdot 0 = 0. \]

(ii) We show $f(a, n) := sa \cdot n = g(a, n) := a \cdot n + n$ using $V_4$:
\[ f(a, 0) = g(a, 0) \] because for $n = 0$ we get $(sa) \cdot 0 = 0$ as well as $a \cdot 0 + 0 = a \cdot 0 = 0$.

\[
\begin{align*}
    f(a, sn) &= (sa) \cdot (sn) = (a + 1) \cdot (n + 1) \\
               &= (a + 1) \cdot n + (a + 1) = (sa) \cdot n + sa \\
               &= f(a, n) + h(a, n), \quad \text{with } h(a, n) := sa \\
    g(a, sn) &= a \cdot (sn) + sn = a \cdot (n + 1) + (n + 1) \\
               &= a \cdot n + a + n + 1 = a \cdot n + n + a + 1 \\
               &= a \cdot n + n + sa \\
               &= g(a, n) + h(a, n).
\end{align*}
\]

So $V_4$ gives $f(a, n) = g(a, n)$ i.e. $sa \cdot n = a \cdot n + n$.

q.e.d.

We continue with the proof of $a \cdot n = n \cdot a$:

From $a \cdot 0 = 0 = 0 \cdot a$ and $a \cdot sn = a \cdot n + n = sn \cdot a$ by the Lemma, we conclude $a \cdot n = n \cdot a$ by $V_4^4$.

q.e.d.

\[4\text{ corrected by S. Lee may 21, 2013}\]
Theorem: In free–variable arithmetics multiplication distributes over addition: \( a \cdot (m + n) = a \cdot m + a \cdot n. \)

Proof:

Case \( n = 0 \) is trivial by definition of + and \( \cdot \).

From the hypothesis \( a \cdot (m + n) = a \cdot m + a \cdot n \) we infer the next step \( a \cdot (m + sn) = a \cdot m + a \cdot sn \) by rule \( V_4 \) above—with passive parameter \((a, m)\)—as follows:

with \( f((a, m), n) := a \cdot (m + n), \)
\( g((a, m), n) := a \cdot m + a \cdot n \) and \( h((a, m), n) := a \)

we have

\[
\begin{align*}
 f((a, m), sn) &= a \cdot (m + sn) = a \cdot (m + (n + 1)) \\
 &= a \cdot ((m + n) + 1) = a \cdot (m + n) + a \\
 &= f((a, m), n) + h((a, m), n) \\
 g((a, m), sn) &= a \cdot m + a \cdot sn = a \cdot m + a \cdot (n + 1) \\
 &= a \cdot m + a \cdot n + a \\
 &= g((a, m), n) + h((a, m), n).
\end{align*}
\]

So by \( V_4 \) we get \( f((a, m), n) = g((a, m), n) \), i.e. \( a \cdot (m + n) = a \cdot m + a \cdot n. \)

q.e.d.

Theorem: In free–variable arithmetics the associative law holds, i.e. \( a \cdot (m \cdot n) = (a \cdot m) \cdot n. \)
**Proof:** We prove the law applying rule $V_4$ with “active” parameter $n$ and passive parameter $(a,m)$ to

\[
\begin{align*}
  f((a,m),n) &:= a \cdot (m \cdot n), \\
  g((a,m),n) &:= (a \cdot m) \cdot n \quad \text{and} \\
  h((a,m),n) &:= a \cdot m.
\end{align*}
\]

For $n = 0$ we have: $a \cdot (m \cdot n) = a \cdot 0 = 0$, and on the other hand: $(a \cdot m) \cdot 0 = 0$.

For $V_4$–step we have:

\[
\begin{align*}
  f((a,m),sn) &= a \cdot (m \cdot sn) = a \cdot (m \cdot (n + 1)) \\
  &= a \cdot (m \cdot n + m) = a \cdot (m \cdot n) + a \cdot m \\
  &= f((a,m),n) + h((a,m),n) \\
  g((a,m),sn) &= (a \cdot m) \cdot (n + 1) = (a \cdot m) \cdot n + a \cdot m \\
  &= g((a,m),n) + h((a,m),n).
\end{align*}
\]

By $V_4$ we get $f((a,m),n) = g((a,m),n)$, i.e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

q.e.d.

**Distributivity theorem:** In free–variable arithmetics multiplication distributes over truncated subtraction:

\[
a \cdot (m \div n) = a \cdot m \div a \cdot n.
\]

**Proof** by equality definability, namely

\[
[f = g \quad \text{iff} \quad [f \div g] = true],
\]

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it is sufficient to show
\[ f((a, m), n) := a \cdot (m \div n) \doteq a \cdot m \div a \cdot n =: g((a, m), n)] = true. \]

**Proof** of this law becomes comparatively easy with *diagonal induction* out of Pfender, Kröplin, Pape 1994:

**Anchoring** \((m = 0 \text{ resp. } n = 0)\):
\[
a \cdot (0 \div n) = a \cdot 0 = 0 = 0 \div a \cdot n = a \cdot 0 \div a \cdot n, \quad \text{as well as}
a \cdot (m \div 0) = a \cdot m = a \cdot m \div 0 = a \cdot m \div a \cdot 0.
\]

**Diagonal induction step:**
\[
f(a, m, n) := a \cdot (m \div n) \doteq a \cdot m \div a \cdot n =: g(a, m, n)
\implies f(a, sm, sn) = a \cdot (sm \div sn) \doteq a \cdot sm \div a \cdot sn = g(a, sm, sn),
\]
since
\[
f(a, sm, sn) = a \cdot (sm \div sn) = a \cdot (m \div n)
= f(a, m, n),
g(a, sm, sn) = a \cdot sm \div a \cdot sn = a \cdot (m + 1) \div a \cdot (n + 1)
= (a \cdot m + a) \div (a \cdot n + a)
= a \cdot m \div a \cdot n \quad \text{by absorption law for } \div
= a \cdot (m \div n)
= g(a, m, n).
\]

q.e.d.
**Proposition:** Addition and multiplication in free-variable arithmetics are weakly monotonous, i.e.

\[ m \leq n \implies m \div n = 0 \]
\[ \implies (a + m) \div (a + n) = 0 \quad \text{by absorption law for } \div \]
\[ \implies a + m \leq a + n \]
\[ m \leq n \implies m \div n = 0 \]
\[ \implies (a \cdot m) \div (a \cdot n) = a \cdot (m \div n) = 0 \]
\[ \implies a \cdot m \leq a \cdot n \]

q.e.d.

**Boolean Structure on \( \mathbb{N} \)**

In present framework GA of **Goodstein Arithmetic** we introduce on NNO \( \mathbb{N} \) the following *proto Boolean* structure:

\[
\begin{array}{ccc}
\text{false} & \equiv & 0 \\
\text{true} & \equiv & 1 \\
\vee & \equiv & + \\
\wedge & \equiv & \cdot \\
\Rightarrow & \equiv & \leq \\
\Leftrightarrow & \equiv & = \\
\end{array}
\]

[Successors are all viewed logically to represent truth value true.]

1.7 Sum objects and definition by distinction of cases

“Hilbert’s infinite hotel” \( \mathbb{N} \cong \mathbb{N} \):
Consider the sum diagram

\[
\begin{array}{c}
A \times 1 \xrightarrow{\sim} A \\
\downarrow_{a \times 0} \quad (a,0) \downarrow \qquad (f|g) \\
A \times N \xrightarrow{\sim} B \\
\downarrow_{a \times s} \quad g \\
A \times N
\end{array}
\]

where

\[(f|g) \overset{\text{def}}{=} \text{pr}[f : A \to B, \ g \circ \ell : (A \times N) \times B \to A \times N \to B]\]

is the unique commutative fill-in into this sum diagram: full scheme (pr) of primitive recursion. Symbolically:

\[A \times N = A + (A \times N) \cong (A \times 1) + (A \times N).\]

An important consequence is the following scheme of map defini-
tion by case distinction:

$$\chi = \text{sign} \circ \chi : A \to \mathbb{N} \text{ p.r. predicate},$$

$$g, h : A \to B \text{ p.r. maps}$$

(\text{IF})

$$f = \text{if}[\chi, (g|h)] \text{ "if } \chi \text{ then } g \text{ else } h"$$

$$=_{\text{def}} (h|g \circ \ell) \circ (\text{id}_A, \chi) :$$

$$A \to A \times \mathbb{N} \to B,$$

$$\chi(a) \implies \text{if}[\chi, (g|h)] = g(a),$$

$$\neg \chi(a) \implies \text{if}[\chi, (g|h)] = h(a).$$

\textbf{Proof:} Commuting diagram:

\[ \begin{array}{ccc}
A & \xrightarrow{(\text{id},\chi)} & A \\
\downarrow (\text{id},0) & & \downarrow (h|g\ell) \\
A \times \mathbb{N} & \xrightarrow{(h|g\ell)} & B \\
\downarrow f & & \downarrow h \\
A \times \mathbb{N} & \xrightarrow{\ell} & A \\
\end{array} \]

with \((h|g\ell) : A \times \mathbb{N} = A + (A \times \mathbb{N}) \to B\) the induced map out of the sum ("coproduct"), coproduct \textit{injections} \((\text{id},0), A \times s\).
free-variable notation:

\[
f = f(a) = \text{if}[\chi, (g|h)](a)
\]

\[
= \begin{cases} 
  g(a) & \text{if } \chi(a) \\
  h(a) & \text{if } \neg \chi(a) \ (\text{otherwise}).
\end{cases}
\]

This terminates presentation (and discussion) of terms and equational axioms presenting fundamental categorical free variables theory \textbf{PR} of primitive recursion.

Note:

In Pfender, Kröplin, Pape 1994 section 4, D. Pape has adapted the classical concept of primitive recursion out of Yashuhara 1971 to the (free-variables) categorical setting, and shown equivalence with fundamental theory \textbf{PR} above.

1.8 Substitutivity and Peano induction

Leibniz substitutivity theorem for predicative equality:

\[
f : A \to B \ 	ext{PR-map}
\]

\[
a \equiv a' \implies f(a) \equiv f(a') : A \times A \to \mathbb{N}.
\]

Proof by structural induction on \( f \):

\bullet f = 0 : \mathbb{1} \to \mathbb{N} : \text{clear since } 0 \equiv 0 : \mathbb{1} \to \mathbb{N} \times \mathbb{N} \overset{\text{Eq}}{\to} \mathbb{N}.

• \( f = s : \mathbb{N} \to \mathbb{N} \): Use \([sm \div sn] = [m \div n]\) and 
  \([a \div b] = [a \leq b] \land [b \leq a] = \neg[a \div b] \land \neg[b \div a].\)

• \( f = \Pi : A \to 1 \): trivial since \( \hat{} \equiv \text{true}_{1 \times 1}.\)

• \( f = \ell : A \times B \to A \):
  \((a, b) \equiv (a', b') \iff [a \equiv a'] \land [b \equiv b']\)
  \(\implies [a \equiv a'] \iff [\ell(a, b) \equiv \ell(a', b')]:\)
  \((A \times B) \times (A \times B) \to \mathbb{N}.\)

• \( f = r : A \times B \to B \): analogous.

Further recursively:

• for a composition \( g \circ f : A \to B \to C \):
  \( a \equiv a' \implies f a \equiv f a' \) (hypothesis)
  \(\implies g(f a) \equiv g(f a') \) (hypothesis)
  \(\iff (g \circ f)(a) \equiv (g \circ f)(a') : A \times A \to \mathbb{N}.\)

• for an induced \((f, g) : C \to A \times B\):
  \( c \equiv c' \implies f(c) \equiv f(c') \land g(c) \equiv g(c') \) (hypothesis)
  \(\iff (f(c), g(c)) \equiv (f(c'), g(c'))\)
  \(\iff (f, g)(c) \equiv (f, g)(c') : C \times C \to \mathbb{N}.\)

• for an iterated map \( f^\#: A \times \mathbb{N} \to A \) to show:
  \((a, n) \equiv (a', n') \implies f^\#(a, n) \equiv f^\#(a', n) : (A \times \mathbb{N})^2 \to A.\)
Diagonal induction on \((n, n') \in \mathbb{N} \times \mathbb{N}\):

\((a, 0) \not\bowtie (a', 0) \implies f^\$\!(a, 0) \not\bowtie a \bowtie a' \bowtie f^\$\!(a', 0)\);

**left axis:** \((a, 0) \neq (a, s\, \text{pre}(n'))\), premise fails;

**right axis:** \((0, a') \neq (s\, \text{pre}(n), a')\), premise fails;

diagonal induction step:

\[(a, s\, n) \not\bowtie (a', s\, n') \implies a \bowtie a' \land s\, n \bowtie s\, n'\]

\[\implies a \bowtie a' \land n \bowtie n' \ (\text{injectivity of } s)\]

\[\implies (a, n) \not\bowtie (a', n') \implies f^\$\!(a, n) \not\bowtie f^\$\!(a', n')\]

(\text{induction hypothesis})

\[\implies f^\$\!(a, s\, n) \not\bowtie f(f^\$\!(a, n)) \bowtie f(f^\$\!(a', n')) \bowtie f^\$\!(a', s\, n')\]

(\text{structural recursion hypothesis on } f)

q.e.d.

Peano’s **axioms** read in categorical free-variables form:

**Peano theorem:**

- **P1:** **zero is a natural number:**

  \[0 : 1 \to \mathbb{N}\] is a map constant of \(\mathbb{N}\), a **natural number** as such.

  [Other natural numbers are free variables on \(\mathbb{N}\)]

- **P2:** **to any natural number (free variable) \(n\)** is assigned a successor:

  This **assignment** is realised categorically by successor **map**

---

5 see Reiter 1982 as well as Pfender, Kröplin & Pape
Such successor $s(n)$ is unique:

This is given categorically by Leibniz's substitutivity for the successor map:

$$\text{PR} \vdash m \models n \implies s(m) \models s(n) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$ 

- **P3: 0 is not a successor:**
  This follows from $sn > 0$, whence $sn \neq 0$, by definition of $m \models n$ via $m < n$ via $m \div n$.

- **P4: equality $s(m) \models s(n)$ implies $m \models n$:**
  This is derived injectivity of successor map $s : \mathbb{N} \to \mathbb{N}$ which reads in free variables:

  $$s m \equiv s(m) \models s(n) \equiv s n$$

  $$\implies m \models \text{pre } s m \models \text{pre } s n \models n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$ 

- **P5: Peano-induction, derived from uniqueness part (pr!) of full scheme (pr) of primitive recursion (Freyd):**

  \[
  \begin{align*}
  \phi &= \phi(a, n) : A \times \mathbb{N} \to \mathbb{N} \quad \text{predicate} \\
  \phi(a, 0) &= \text{true}_A(a) \quad \text{(anchor)} \\
  [\phi(a, n) \implies \phi(a, s n)] &= \text{true}_{A \times \mathbb{N}} \quad \text{(induction step)} \\
  \phi(a, n) &= \text{true}_{A \times \mathbb{N}} \quad \text{(conclusio).}
  \end{align*}
  \]
**Proof** of Peano induction principle (P5) from *full scheme* (pr) of primitive recursion\(^6\)

For scheme (pr!) choose as anchor map

\[
g = g(a) = \varphi(a, 0) = \text{true}(a) : A \to \mathbb{N}, \text{and as step map}\]

\[
h = h((a, n), b) = b \lor \varphi(a, s\, n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}\]

By (pr) we get a unique \(f = f(a, n) : A \times \mathbb{N} \to \mathbb{N}\) which satisfies

\[
f(a, 0) = \varphi(a, 0) = \text{true}(a) \quad \text{and} \quad f(a, s\, n) = h((a, n), f(a, n)) = f(a, n) \lor \varphi(a, s\, n).
\]

This works for \(f = \text{true} : A \times \mathbb{N} \to \mathbb{N}\) as well as for \(f = \varphi\), the latter since

\[
\varphi(a, n) \lor \varphi(a, s\, n)
\]

\[
= (\varphi(a, n) \lor \varphi(a, s\, n)) \land (\varphi(a, n) \Rightarrow \varphi(a, s\, n))
\]

by 2nd hypothesis

\[
= \varphi(a, s\, n) \quad \text{by boolean tautology}
\]

\[
(\alpha \lor \beta) \land (\alpha \Rightarrow \beta) = \beta:
\]

*test with* \(\beta = 0 \equiv \text{false} \text{ and } \beta = 1 \equiv \text{true}.*

q.e.d.

By replacing predicate \(\varphi\) with

\[
\psi(a, n) := \bigwedge_{i \leq n} \varphi(a, i) : A \times \mathbb{N} \to \mathbb{N}
\]

in this **proof** we get

---

\(^6\) **Reiter** 1982 and **Pfender, Kröplin, Pape** 1994
Course of values induction principle:

\[
\varphi = \varphi(a, n) : A \times \mathbb{N} \rightarrow \mathbb{N} \quad \text{predicate}
\]

\[
\varphi(a, 0) = \text{true}_A(a) \quad \text{(anchor)}
\]

\[
[ \land_{i \leq n} \varphi(a, i) \implies \varphi(a, s \, n) ] = \text{true}_{A \times \mathbb{N}} \quad \text{(induction step)}
\]

(P5)

\[
\varphi(a, n) = \text{true}_{A \times \mathbb{N}} \quad \text{(conclusio).}
\]

Here predicate \( \land_{i \leq n} \varphi(a, i) : A \times \mathbb{N} \rightarrow \mathbb{N} \) is p. r. defined by

\[
\land_{i \leq 0} \varphi(a, i) = \varphi(a, 0) : A \rightarrow \mathbb{N},
\]

\[
\land_{i \leq s \, n} \varphi(a, i) = \land_{i \leq n} \varphi(a, i) \land \varphi(a, s \, n) : A \times \mathbb{N} \rightarrow \mathbb{N}.
\]

1.9 Integer division and related

Integer division with remainder (Euclidean)

\[
(a \div b, a \, \text{rem} \, b) : \mathbb{N} \times \mathbb{N}_\succ \rightarrow \mathbb{N} \times \mathbb{N}
\]

is characterised by

\[
a \div b = \max\{c \leq a \mid b \cdot c \leq a\} : \mathbb{N} \times \mathbb{N}_\succ \rightarrow \mathbb{N},
\]

\[
a \, \text{rem} \, b = a \div (a \div b) \cdot b : \mathbb{N} \times \mathbb{N}_\succ \rightarrow \mathbb{N}.
\]

[for \( \mathbb{N}_\succ = \{n \in \mathbb{N} \mid n > 0\} \) and objects defined by p. r. predicate abstraction in general see next chapter.]

Explicitly, we define

\[
\div = a \div b : \mathbb{N} \times \mathbb{N}_\succ \rightarrow \mathbb{N}
\]
via initialised iteration \( h = h((a, b), n) \) of

\[
g = g((a, b), c) = \begin{cases} 
((a, b), c) & \text{if } a < b, \\
((a \div b, b), c + 1) & \text{if } a \geq b
\end{cases}
\]

in

\[
\begin{array}{c}
\xymatrix{
\mathbb{N} \times \mathbb{N}_{>0} & \ar[l] \mathbb{N} \ar[r]^{(N \times N_{>0}) \times s} \ar[d]_{(id,0)} & \mathbb{N} \times \mathbb{N}_{>0} \ar[d]_{h} & \ar[l] \mathbb{N} \times \mathbb{N}_{>0} \ar[r]^{g} \ar[d]_{(id,0)} & \mathbb{N} \times \mathbb{N}_{>0} \ar[d]_{h} \\

& & & & \\
\end{array}
\]

\[
a \div b =_{\text{def}} r h((a, b), a) : \mathbb{N} \times \mathbb{N}_{>0} \rightarrow (\mathbb{N} \times \mathbb{N}_{>0})\mathbb{N} \rightarrow \mathbb{N},
\]

\[
a \text{rem } b =_{\text{def}} \ell \ell h((a, b), a) = a \div b \cdot (a \div b) : \mathbb{N} \times \mathbb{N}_{>0} \rightarrow \mathbb{N}.
\]

The predicate \( a | b : \mathbb{N}_{>0} \times \mathbb{N} \rightarrow \mathbb{N} \) is a divisor of \( b \), \( a \) divides \( b \) is defined by

\[
a | b = [b \text{ rem } a \div 0].
\]

**Exercise:** Construct the Gaussian algorithm for determination of the \( \gcd \) of \( a, b \in \mathbb{N}_{>0} \) defined as

\[
\gcd(a, b) = \max\{c \leq \min(a, b) \mid c|a \wedge c|b\} : \mathbb{N}_{>0} \times \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}
\]

by iteration of mutual rem.
Primes

Define the predicate *is a prime* by

\[ \mathbb{P}(p) = \prod_{m=1}^{p} [m|p \Rightarrow m \doteq 1 \lor m \doteq p] : \mathbb{N} \rightarrow 2 : \]

Only 1 and \( p \) divide \( p \).

Write \( \mathbb{P} \) for \( \{n \in \mathbb{N}|\mathbb{P}(n)\} \subset \mathbb{N} \) too.

The (euclidean) count \( p_n : \mathbb{N} \mapsto \mathbb{N} \) of all primes is given by

\[
p_0 = 2, \quad p_{n+1} = \min\{p \in \mathbb{N}|\mathbb{P}(p), p_n < p \leq \prod_{q} [q \leq p_n \land \mathbb{P}(q)]\} + 1 \]

\[
= \min\{p \in \mathbb{N}|\mathbb{P}(p), p < 2p_n\} : \quad \mathbb{P} \mapsto \mathbb{P},
\]

iterated binary product and iterated binary minimum.

The latter presentation is given by Bertrand’s theorem.

Notes

(a) An NNO, within a cartesian Closed category of sets, was first studied by Lawvere 1964.

(b) Eilenberg-Elgot 1970 iteration, here special case of one-successor iteration theory \( \text{PR} \), is, because of Freyd’s uniqueness scheme (FR!), a priori stronger than classical free-variables *primitive recursive arithmetic* \( \text{PRA} \) in the sense of Smorynski 1977. If viewed as a subsystem of \( \text{PM}, \text{ZF} \) or \( \text{NGB} \), that \( \text{PRA} \) is stronger than our \( \text{PR} \).
(c) Within Topoi (with their cartesian closed structure), Freyd 1970 characterised Lawvere’s NNO by unique initialised iteration. Such Freyd’s NNO has been called later, e.g. in Maietti 2010??, parametrised NNO.

(d) Lambek-Scott 1986 consider in parallel a weak NNO: uniqueness of Lawvere’s sequences \( a : \mathbb{N} \rightarrow A \) not required. We need here uniqueness (of the initialised iterated) for proof of Goodstein’s 1971 uniqueness rules basic for his development of p.r. arithmetic. Without the latter uniqueness requirement, the definition of parametrised (weak) NNO is equational.

(e) For uniqueness of the set of natural numbers (out of the Peano-axioms), classical set theory needs higher order. This corresponds in category theory to the use of free meta-variables on maps.

In first order classical, elementhood based Peano-arithmetic there are other models of the natural numbers, even uncountable ones. Others than the “standard” (e.g. von Neumann) model.

2 Predicate Abstraction

We extend the fundamental theory \( \text{PR} \) of primitive recursion definitionally by predicate abstraction objects \( \{ A \mid \chi \} = \{ a \in A \mid \chi(a) \} \). We get an (embedding) extension \( \text{PR} \sqsupseteq \text{PRA} \) having all of the expected properties.

---

\[ 7 \text{ This was brought to my attention 2013 in a seminar talk of J. Busse and A. Schlote who quote Barwise ed. 1977 as well as Ebbinghaus et al. 1996 and 2008.} \]
2.1 Extension by predicate abstraction

We discuss a p.r. abstraction scheme as a definitional enrichment of PR, into theory PRa of PR decidable objects and PR maps in between, decidable subobjects of the objects of PR. The objects of PR are, up to isomorphism,

\[ \mathbb{I}, \mathbb{N}^1 = \text{def} \mathbb{N}, \mathbb{N}^{m+1} = \text{def} (\mathbb{N}^m \times \mathbb{N}). \]

\[ [m] \text{ is a free metavariable, over the NNO constants } 0, 1 = s0, 2 = ss0, \ldots \in \mathbb{N}. ]\]

The extension PRa is given by adding schemes (ExtObj), (ExtMap), and (Ext=) below. Together they correspond to the scheme of abstraction in set theory, and they are referred below as schemes of PR abstraction.

Our first predicate-into-object abstraction scheme is

\[ \chi : A \rightarrow \mathbb{N} \text{ a PR-predicate:} \]

\[ \text{sign} \circ \chi = \chi : A \rightarrow \mathbb{N} \rightarrow \mathbb{N}, \]

\[ Au \xrightarrow{\chi} \mathbb{N} \xrightarrow{\text{sign}} \mathbb{N} \]

\( (\text{Ext}_{\text{Obj}}) \)

\( \{ A \mid \chi \} \text{ object (of emerging theory PRa)} \)

Subobject \( \{ A \mid \chi \} \subseteq A \cong \mathbb{N}^a \) may be written alternatively, with bound variable \( a \), as

\[ \{ A \mid \chi \} = \{ a \in A \mid \chi(a) \}. \]
\{A \mid \chi\} is just another name for the (external) code \(\chi \in \text{PR} \subset \mathbb{N}\), a NNO constant out of \(\mathbb{N}\), the external set of natural number constants
\[0, 1 \equiv s\ 0,\ 2 \equiv ss\ 0\ \text{etc.}\ n \equiv s\ldots s\ 0 \equiv \text{num}(n) \in \mathbb{N}\ \text{etc.} \]

The maps of \(\text{PRA} = \text{PR} + (\text{abstr})\) come in by

\[
\{A \mid \chi\},\ \{B \mid \varphi\} \ \text{PRA-objects,}
\]
\[
f : A \rightarrow B \ \text{a PR-map,}
\]
\[
\text{PR} \vdash \chi(a) \implies \varphi f(a), \ \text{i.e.}
\]
\[
(\text{ExtMap}) \quad \frac{[\chi \implies \varphi \circ f] =^\text{PR} \text{true}_A : A \overset{\Pi} \rightarrow \overset{1} \rightarrow \mathbb{N}}{f \ \text{is a PRA-map} f : \{A \mid \chi\} \rightarrow \{B \mid \varphi\}}
\]

In particular, if for predicates \(\chi',\ \chi'' : A \rightarrow \mathbb{N}\)

\[
\text{PR} \vdash \chi'(a) \implies \chi''(a) : A \rightarrow \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N},
\]
then \(\text{id}_A : \{A \mid \chi\} \rightarrow \{A \mid \chi''\}\) in \(\text{PRA}\) is called an inclusion, and written \(\subseteq : A' = \{A \mid \chi\} \rightarrow A'' = \{A \mid \chi''\}\) or \(A' \subseteq A''\).

Nota bene: For predicate (terms!) \(\chi, \varphi : A \rightarrow \mathbb{N}\) such that \(\text{PR} \vdash \chi = \varphi : A \rightarrow \mathbb{N}\) (logically: such that \(\text{PR} \vdash [\chi \iff \varphi]\)) we have

\[
\{A \mid \chi\} \subseteq \{A \mid \varphi\} \ \text{and} \ \{A \mid \varphi\} \subseteq \{A \mid \chi\},
\]
but—in general—not equality of objects. We only get in this case

\[
\text{id}_A : \{A \mid \chi\} \overset{\simeq} \rightarrow \{A \mid \varphi\}
\]
as an PRa isomorphism.

A posteriori, we introduce as Reiter does, the formal truth Algebra $2$ as

$$2 \overset{\text{def}}{=} \{ n \in \mathbb{N} \mid \chi(n) \}, \text{ where } \chi(n) = [n \leq 1] : \mathbb{N} \to \mathbb{N},$$

with proto Boolean operations on $\mathbb{N}$ restricting—in codomain and domain—to boolean operations on $2$ resp.

$$2 \times 2 \overset{\text{def}}{=} \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m, n \leq s 0\},$$

by definition below of cartesian Product of objects within PRa.

PRa-maps with common PRa domain and codomain are considered equal, if their values are equal on their defining domain predicate. This is expressed by the scheme

$$f, g : \{A \mid \chi\} \to \{B \mid \varphi\} \text{ PRa-maps,}$$

$$\text{PR} \vdash \chi(a) \implies f(a) \equiv_B g(a)$$

(Ext$_=$)

$$f = g : \{A \mid \chi\} \to \{B \mid \varphi\},$$

explicitly:

$$f =^\text{PRa} g : \{A \mid \chi\} \to \{B \mid \varphi\}, \text{ also noted}$$

$$\text{PRa} \vdash f = g : \{A \mid \chi\} \to \{B \mid \varphi\}.$$
\textbf{PRa} is a cartesian p.r. theory. The theory \textit{PR} is cartesian p.r. embedded. The theory \textbf{PRa} has universal extensions of all of its predicates and a boolean truth object as codomain of these predicates, as well as map definition by case distinction. In detail:

(i) \textbf{PRa} inherits associative \textbf{map composition} and identities from \textit{PR}.

(ii) \textbf{PRa} has \textit{PR} fully \textbf{embedded} by

\[
\langle f : A \to B \rangle \mapsto \langle f : \{ A \mid \text{true}_A \} \to \{ B \mid \text{true}_B \} \rangle
\]

(iii) \textbf{PRa} has \textbf{cartesian product}

\[
\{ A \mid \chi \} \times \{ B \mid \varphi \} = \text{def} \ \{ A \times B \mid \chi \land \varphi : A \times B \to \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \},
\]

with \textit{projections} and universal property inherited from \textit{PR}. We abbreviate \{ A \mid \text{true}_A \} by \textit{A}.

(iv) object \textbf{2} comes as a \textit{sum} \ \begin{array}{c}
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\begin{array}{c}
\begin{array}{c}
\text{true}
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\text{false}
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\end{array}\end{array} \ \text{over which cartesian product } A \times \_ \ \text{distributes.}

This allows in fact for the usual \textbf{truth-table definitions} of all \textit{boolean operations} on object \textbf{2} and for \textit{PR} map \textbf{definition} by \textbf{case distinction}.

(v) The embedding \( \sqsubseteq : \textit{PR} \longrightarrow \textbf{PRa} \) is a \textbf{cartesian functor}: it preserves Products and their \textit{cartesian} universal property with respect to the \textit{projections} inherited from \textit{PR}.
(vi) **PRa** has **extensions** of its *predicates*, namely

\[
\text{Ext} [ \varphi : \{A \mid \chi \} \to 2] =_{\text{def}} \{A \mid \chi \land \varphi \} \subseteq \{A \mid \chi \},
\]

characterised as (**PRa**)-**equalisers**

\[
\text{Equ} (\chi \land \varphi, \text{true}_A) : \{A \mid \chi \} \to 2
\]

[mutatis mutandis: within theory **PRa**, we identify predicates \(\chi = \text{sign} \circ \chi : A \to \mathbb{N} \to \mathbb{N}\) with maps \(\chi : A \to 2\).]

(vii) **PRa** has **all equalisers**, namely equalisers

\[
\text{Equ}[f, g] =_{\text{def}} \{a \in A \mid \chi(a) \land f(a) =_B g(a)\}
\]

\[
= \text{Ext}[\sim_B \circ (f, g) : A' \to B' \times B' \xrightarrow{\sim} 2],
\]

of arbitrary **PRa** map pairs \(f, g : A' = \{A \mid \chi\} \to B' = \{B \mid \varphi\}\), and hence all finite projective **limits**, in particular **pullbacks**, which we will rely on later.

A **pullback**, of a map \(f : A \to C\) along a map \(g : B \to C\), also of \(g\) along \(f\), is a square in

\[
\begin{align*}
D \xrightarrow{(h, k)} A & \\
\downarrow h \quad & \\
\downarrow g' & \\
\downarrow f' & \\
B \xrightarrow{g} C & \\
\downarrow f & \\
\end{align*}
\]
[I prefer this “set theoretical” way to construct extension sets out of the cartesian category structure of fundamental theory PR, and then I construct equalisers and the other finite limits on this basis. Another possibility—ROMÀN(?)—is to add equalisers as *undefined notion* and to construct directly from these and cartesian product. The relation between (vi) and (vii) is best understood set theoretically: use free variable argument chase, and recall set theoretical definition of an equaliser.]

The embedding preserves such limits as far as available already in PR. Equality predicate extends to cartesian Products componentwise as

\[(a, b) \triangleq_{A \times B} (a', b') = \text{def} \ [a \triangleq_A a'] \land [b \triangleq_B b'] : (A \times B)^2 \to 2,\]

and to (predicative) subobjects \(\{A \mid \chi\}\) by restriction.

(viii) arithmetical structure extends from PR to PRa, i.e. PRa admits the *iteration* scheme as well as Freyd’s *uniqueness* scheme: the iterated

\[f^\#: \{A \mid \chi\} \times \{N \mid \text{true}_N\} \to \{A \mid \chi\}\]

is just the *restricted* PR-map \(f^\# : A \times N \to A\), the uniqueness schemes follow from definition of \(=^{\text{PRa}}\) via PRa’s scheme (Ext) above.

(ix) In particular, our equality predicate \(\triangleq_A : A^2 \to N\), restricted to subobjects \(A' = \{A \mid \chi\} \subseteq A\), inherits all of the properties of equality on \(N\) and the other *fundamental objects.*
(x) **Countability:** Each fundamental object $A$ i.e. $A$ a finite power of $N \equiv \{ N \mid \text{true}_N \}$, admits, by CANTOR’s isomorphism

$$ct = ct_{N \times N}(n) : N \xrightarrow{\cong} N \times N,$$

a retractive count $ct_A(n) : N \to A$.

**Problem:** For which predicates $\chi : A \to 2$ ($A$ fundamental) does theory PRa admit a retractive count

$$ct = ct_{\{ A \mid \chi \}}(n) : N \to \{ A \mid \chi \}?$$

The difficulty is seen already in case $\emptyset_A = \text{by def} \{ A \mid \text{false}_A \}$. A sufficient condition is $\{ A \mid \chi \}$ to come with a point, $a_0 : 1 \to \{ A \mid \chi \}$. But there may be non-empty objects without points in suitable theories.

**Remarks:**

- a PRa-map $f : \{ A \mid \chi \} \to \{ B \mid \varphi \}$ can be viewed as a defined partial PR map from $A$ to $B$ with values in $\varphi :$ the object of defined arguments, namely $\{ a \in A \mid \chi(a) \}$ is p.r. decidable. By definition of PRa’s equality, PR-map $f : A \to B$ “doesn’t care” about arguments $a$ in the complement $\{ a \in A \mid \neg \chi(a) \}$.

So wouldn’t it be easier to realise this view to defined partial maps just by throwing the undefined arguments into a waste basket $\{ \bot \}$?

But where to place this waste basket, this for each codomain object $B$? The fundamental objects have a zero-vector as a candidate. For example we could interprete truncated subtraction as a defined partial map

$$a - b : \{(m, n) \in N \times N \mid m \geq n \} \to N,$$
and throw the complement \( \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n \} \) into waste basket \( \{0\} \subset \mathbb{N} \). But this is not a good interpretation of truncated (!) subtraction: Value 0 is not waste, it has an important meaning as zero.

“The” waste basket \( \{\bot\} \) should be an entity with a natural extra representation, and we should have only one such entity in a later theory of defined partial p.r. maps to come. This theory, to be called \( \text{PRx}a \), will be constructed with the help of a universal object \( \mathbb{X} \) which is to contain all numerals (codes of numbers) and all nested pairs of numerals. It then has place for \( \LaTeX \) codes of all symbols, in particular for the code \( \bot \) of undefined value symbol \( \bot \), in a “Hilbert’s hotel”.

- a \( \text{PR}-\)map \( f : A \to B \) such that \( f \) is a \( \text{PR}a\)-map

\[
f : \{A \mid \chi \vee \chi' : A \to 2\} \to \{B \mid \varphi\}
\]
also works as a \( \text{PR}a\)-map

\[
f : \{A \mid \chi\} \to \{B \mid \varphi\}, \text{ and a } \text{PR}a\text{-map}
\]
\[
g : \{A \mid \chi\} \to \{B \mid \varphi \land \varphi'\}
\]
also works as a \( \text{PR}a\)-map

\[
g : \{A \mid \chi\} \to \{B \mid \varphi\}.
\]

Since map-properties of injectivity, epi-property of \( \text{PR}\)-maps viewed as \( \text{PR}a\)-maps depend on choice of hosting \( \text{PR}a\) objects—examples above—specification of a \( \text{PR}a\) map \( f : \{A \mid \chi\} \to \{B \mid \varphi\} \) must contain, besides \( \text{PR}\)-map \( f : A \to B \), domain and codomain objects \( \chi : A \to 2 \) and \( \varphi : B \to 2 \) as well.
This way the members of map set family $\text{PRa}(A, B) : A, B$ $\text{PRa}$-objects, become mutually disjoint. Inclusions $i : A' \subseteq A''$ are realised in $\text{PRa}$ as restricted $\text{PR}$-identities

$id_A : \{A \mid \chi'\} \overset{\subseteq}{\rightarrow} \{A \mid \chi''\}, \chi' \Longrightarrow \chi''$.

### 2.2 Predicate calculus

**Free Variables Predicate Calculus**

In the framework $\text{GA} \subseteq \text{PR} \sqsubset \text{PRa}$ of **Goodstein Arithmetic** we have introduced on NNO $\mathbb{N}$ the following proto Boolean structure:

This structure is turned, within $\text{PRa}$, into a two-valued Boolean algebra on object

\[
\begin{align*}
2 & = \text{by def} \quad \{0, 1\} \\
& = \text{def} \quad \{n \in \mathbb{N} \mid n \equiv 0 \lor n \equiv 1\} \\
& = \text{by def} \quad \{n \in \mathbb{N} \mid n \leq 1\} : \\
\end{align*}
\]
A **PR** predicate on an object \( A \) of **PR** has been a **PR** map \( \chi : A \to \mathbb{N} \) with

\[
\text{sign} \circ \chi = \chi,
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\chi} & \mathbb{N} \\
\downarrow & & \downarrow \text{sign} \\
\chi & & \mathbb{N}
\end{array}
\]

A **PRa** predicate on an object \( \{ A | \chi \} \) is a **PRa** map \( \varphi = \varphi(a) : \{ A | \chi \} \to 2 = \{0, 1\} \).

Using the Boolean operations on \( 2 \) above, a **free-Variables boolean predicate calculus** is easily **defined**, making the set of **PR** predicates on (any) object \( A \) of **PRa** into a boolean algebra:

- **overall negation:**
  
  \[ \neg \varphi(a) = \neg \circ \varphi : A \to 2 \to 2, \]

- **conjunction:**
  
  \[ \chi(a) \land \varphi(a) = \land \circ (\chi, \varphi) : A \to 2^2 \to 2, \]

- **disjunction:**
  
  \[ \chi(a) \lor \varphi(a) = \lor \circ (\chi, \varphi) : A \to 2^2 \to 2, \]

- **implication:**
  
  \[ [\chi(a) \Rightarrow \varphi(a)] = \Rightarrow \circ (\chi, \varphi) : A \to 2^2 \to 2, \]

- **equivalence:**
  
  \[ [\chi(a) \Leftrightarrow \varphi(a)] = \Leftrightarrow \circ (\chi, \varphi) : A \to 2^2 \to 2, \]

Verification of the logical properties of such free-variables predicates and their interrelationships by the **truth table method** inherited from the Boolean algebra \( 2 \).
Axiomatic Images and Quantification

As a step aside, we discuss here classical quantification, introduced axiomatically via image predicates. These correspond to topos theoretic characteristic functions of non-necessarily monic (injective) maps. Quantification + cartesian PR allows for the original version of Gödel’s theorems. It seems to be necessary for that original theorems and proof, since existential quantification plays a prominent rôle in statement and proof. Nevertheless, Incompleteness can be shown in a different way for weaker theories, cf. GOODSTEIN 1957. We do not exclude that PR, PRa turn out to be incomplete in Goodstein’s sense.

**Definition:** A (total) predicate $\chi : B \to 2$ is a (the) *image predicate* of a map $f = f(a) : A \to B$, if

- $\chi \circ f = \text{true}_A : A \to B \to 2$ and
- $\chi : B \to 2$ minimal in this regard i.e.

\[
\varphi \circ f = \text{true}_A : A \to B \to 2
\]

\[
[\chi(b) \Rightarrow \varphi(b)] = \text{true}_B
\]

If available, such $\chi$, noted $\text{im}[f] = \text{im}[f] : B \to 2$, is unique, this by minimality and Equality Definability.

In case of $f : A \to B$ monic, such $\chi$ is just the characteristic map of $f$ in the sense of Elementary Topos theory ETT, with respect to $2 = \{0, 1\} \subset \mathbb{N}$ taken as its *subobject classifier, truth object.*
If available, image

\[ \text{im}[\{A \times B | \varphi\}] \xrightarrow{\subseteq} A \times B \rightarrow A] : A \rightarrow 2 \]

works as right existential quantification

\[ (\exists b \in B) \varphi(a, b) = (\exists_r \varphi)(a) : A \rightarrow 2, \]

with the categorical properties of this quantification known from (ETT and categorical) set theory.

If available, define right universal quantification

\[ (\forall b \in B) \varphi(a, b) =_{\text{def}} \neg (\exists b \in B) \neg \varphi(a, b) : A \rightarrow 2. \]

Our (weak, categorical) set theories \( T \) will here always be Extensions of quantified p.r. theory \( \text{PRa} \exists = \text{PRa} + (\exists) \), defined to be theory \( \text{PRa} \) closed under formation of images and hence closed under (two-valued) quantification \( \exists, \forall \).

**Comment:** These semi-classical theories will be taken as background for Consistency questions: we will show differences in internal consistency between these classical set theories \( T \), in particular between Osius’ categorical pendants of the different stages of Zermelo-Fraenkel set theory \( \text{ZF} \) on one hand, and the categorical theories here: \( \text{PR}, \text{PRa} \) above, and \( \text{PRX}, \text{PRxa}, \pi R \) to come. For fixing ideas, you may always read set theory \( T \) as \( T := \text{PRa}\exists \) : Gödel’s Incompleteness theorems apply to \( \text{PRa}\exists \), not to descent p.r. theory \( \pi R \) to come.
Notes

(a) we have equalisers, products distributing over sums, sums certainly stable under pullbacks, quotients by equivalence predicates (not yet quotients by equivalence relations).

(b) in comparison with doctrines: Kock-Reyes 1977, and in comparison with pretopoi: Maietti 2010??, (axiomatic) quantification is lacking for “our” strengthenings $S$ of PRa.

3 Partial Maps

We introduce general recursive maps as partial p. r. maps, coming as a p. r. enumeration of defined arguments together with a p. r. rule mapping the enumeration index of a defined argument into the value of that argument. This covers $\mu$-recursive maps and content driven loops as in particular while-loops. Code evaluation will be definable as such a while-loop.

3.1 Theory of partial maps

Definition: A partial map $f : A \rightarrow B$ is a pair

$$f = \langle d_f : D_f \rightarrow A, \hat{f} : D_f \rightarrow B \rangle : A \rightarrow B,$$

\[ D_f \]
\[ \hat{f} \]
\[ d_f \]
\[ A \]
\[ f \]
\[ B \]
The pair \( f = \langle d_f, \hat{f} \rangle \) is to fullfill the right-uniqueness condition
\[
d_f(\hat{a}) \simeq_A d_f(\hat{a}') \implies \hat{f}(\hat{a}) \simeq_B \hat{f}(\hat{a}') :\]

We now define the theory \( \hat{S} \) of partial S-maps \( f : A \to B \).

Objects of \( \hat{S} \) are those of \( S \), i.e. of \( PRa \). The morphisms of \( \hat{S} \) are the partial S-maps \( f : A \to B \).

**Definition:** Given \( f', f : A \to B \) in \( \hat{S} \), we say that \( f \) extends \( f' \) or that \( f' \) is a restriction of \( f \), written \( f' \subset f \), if there is given a map \( i : D_{f'} \to D_f \) in \( S \) such that

\[
\begin{array}{c}
\xymatrix{D_{f'} \ar[dr]^{d_{f'}} & D_f \ar[dl]_{d_f} \ar[rr]^s & & \ar[dl]_{f'} \hat{f} \\
& A \ar[rr]_f & & B
}\end{array}
\]

The partial maps \( f \) and \( f' \) are equal in \( \hat{S} \), if \( f \) extends \( f' \) and \( f' \) extends \( f \):

\[
( \overset{\subset}{=} S ) \quad \text{(equivalent)} \quad f' \subset f, \ f \subset f' : A \to B
\]

**Notation:** From now on, \( f = g : A \to B \) will always denote equality between maps within theory \( S \) chosen as basic, cartesian p.r.

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theory. Equality between partial S-maps, \( \hat{\mathcal{S}} \)-morphisms \( f, g : A \to B \) is denoted \( f \hat{=} g : A \to B \), see the above. Pointed equality \( \hat{=} : \mathbb{N}^2 \to 2 \) resp. \( \hat{=}_A : A^2 \to 2 \) is reserved for equality predicates (special maps), on \( \mathbb{N} \) resp. on objects \( A \) of \( \mathcal{S} \).

**Definition:** Composition \( h = g \hat{\circ} f : A \to B \to C \) of \( \hat{\mathcal{S}} \) maps

\[
f = \langle (d_f, \hat{f}) : D_f \to A \times B \rangle : A \to B \text{ and } g = \langle (d_g, \hat{g}) : D_g \to B \times C \rangle : B \to C
\]

is defined by the diagram

![Composition diagram for \( \hat{\mathcal{S}} \)](attachment:image)

Composition diagram for \( \hat{\mathcal{S}} \)

[The idea is from BRINKMANN-PUPPE 1969: They construct composition of relations this way via pullback]

**Remark:** The standard form of the pullback \( D_h \) is

\[
D_h = \{(\hat{a}, \hat{b}) \in D_f \times D_g \mid \hat{f}(\hat{a}) \hat{=}_B d_g(\hat{b})\},
\]
with pullback-projections

\[ \ell = \pi_\ell = \ell \circ \subseteq : D_h \to D_f \times D_g \to D_f \text{ and} \]

\[ r = \pi_r = r \circ \subseteq : D_h \to D_f \times D_g \to D_g. \]

[ We may abbreviate such restricted projections—pullback "projections"—\( \pi_\ell \) and \( \pi_r \) respectively, by \( \ell, r \)—as suggested above ]

In a sense, the pullback \( D_h \) represents the inverse image \( D_h = f^{-1}[D_g] \), more precisely: \( [D_h \xrightarrow{\ell} D_f] = f^{-1}[D_g \xrightarrow{d_g} B] \). But the definability domains \( d_f, d_g, d_h \) need not be monic (injective).

Composition \( h = g \hat{\circ} f : A \to B \to C \) gives a well-defined partial map \( h \), since for \((\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h \) free:

\[
\begin{align*}
d_h(\hat{a}, \hat{b}) \mathbin{\vdash}_A d_h(\hat{a}', \hat{b}') & \iff d_f(\hat{a}) \mathbin{\vdash}_A d_f(\hat{a}') \\
\implies \hat{f}(\hat{a}) \mathbin{\vdash}_B \hat{f}(\hat{a}') \text{ (f well-defined)}, \\
\iff \hat{f} \ell(\hat{a}, \hat{b}) \mathbin{\vdash} \hat{f} \ell(\hat{a}', \hat{b}') \\
\implies d_g(r(\hat{a}, \hat{b})) \mathbin{\vdash}_B d_g(r(\hat{a}', \hat{b}')) \\
& \quad \quad \text{ (}(\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h, \text{ p.b. commutes}) \\
\iff d_g(\hat{b}) \mathbin{\vdash}_B d_g(\hat{b}') \implies \hat{g}(\hat{b}) \mathbin{\vdash}_C \hat{g}(\hat{b}') \\
\implies \hat{h}(\hat{a}, \hat{b}) = \hat{g}(\hat{b}) \mathbin{\vdash}_C \hat{g}(\hat{b}') = \hat{h}(\hat{a}', \hat{b}') : D_h \times D_h \to 2.
\end{align*}
\]

Obviously, \( \hat{S} \)-map \( \text{id}_A^{\hat{S}} = \text{def} \langle (\text{id}_A, \text{id}_A) : A \to A^2 \rangle : A \to A \) works as identity for object \( A \) with respect to composition \( \hat{\circ} \) for \( \hat{S} \).

If one of two \( \hat{S} \) maps to be composed, is an \( S \) map, \( \hat{S} \) composition becomes simpler:

Mixed Composition Lemma:
(i) For \( f : A \rightarrow B \) in \( \hat{S} \), and \( g : B \rightarrow C \) in \( S \):

\[
g \circ \hat{f} = \langle (d_f, g \circ \hat{f}) : D_f \rightarrow A \times C \rangle : A \rightarrow C,
\]

in diagram form:

![Diagram](image)

(ii) For \( f : A \rightarrow B \) in \( S \), \( g : B \leftarrow C \) in \( \hat{S} \):

\[
g \circ \hat{f} = \langle (\bar{f} [d_g], \hat{g} \circ \bar{f}) : \bar{f} [D_g] \rightarrow A \times C \rangle : A \rightarrow C,
\]

as diagram:

![Diagram](image)

**Proof:** Left as an exercise.
3.2 Structure theorem for $\hat{P}\hat{R}a$:

(i) $\hat{S}$ carries a canonical structure of a diagonal symmetric monoidal category, with composition $\hat{\circ}$ and identities introduced above, monoidal product $\times$ extending $\times$ of $S$, association $\text{ass} : (A \times B) \times C \xrightarrow{\simeq} A \times (B \times C)$, symmetry $\Theta : A \times B \xrightarrow{\simeq} B \times A$, and diagonal $\Delta : A \to A \times A$ inherited from $S$.

(ii) The defining diagram for a $\hat{S}$-map—namely

\[
\begin{array}{c}
D_f \\
\downarrow d_f \\
A \\
\end{array} \xrightarrow{f} \begin{array}{c}
B \\
\downarrow f \\
\end{array}
\]

Partial Map DIAGRAM

is a commuting $\hat{S}$ diagram.

Conversely the minimised opposite $\hat{S}$ map $d_{\hat{f}}^- : A \to D_f$ to $S$ map $d_f : D_f \to A$ fullfills

\[
\begin{array}{c}
D_f \\
\downarrow d_{\hat{f}}^- \\
A \\
\end{array} \xrightarrow{\hat{f}} \begin{array}{c}
B \\
\downarrow f \\
\end{array}
\]

Put together:
(iii) “section lemma:” The first factor $f : A \to B$ in an $\hat{S}$ composition

$$h = g \hat{\circ} f : A \to B \to C,$$

when giving an (embedded) $S$ map $h : A \to C$, is itself an (embedded) $S$ map:

_a first composition factor of a total map is total._

So each section (“coretraction”) of theory $\hat{S}$ is an $S$ map, in particular an $\hat{S}$ section of an $S$ map belongs to $S$.

[We will rely on this lemma below.]

### 3.3 Equality definability for partials

Not needed for the Gödel discussion.

### 3.4 Partial-map extension as closure

Not needed for the discussion of the Gödel theorems.
3.5 \( \mu \)-recursion without quantifiers

We define \( \mu \)-recursion within the free-variables framework of partial p. r. maps as follows:

Given a PR predicate \( \varphi = \varphi(a, n) : A \times \mathbb{N} \to 2 \), the \( \hat{S} \) morphism

\[
\mu \varphi = \langle (d_{\mu \varphi}, \hat{\mu} \varphi) : D_{\mu \varphi} \to A \times \mathbb{N} : A \rightarrow \mathbb{N} \rangle
\]

is to have (S) components

\[
\begin{align*}
D_{\mu \varphi} &= \text{def} \{ A \times \mathbb{N} \mid \varphi \} \subseteq A \times \mathbb{N}, \\
d_{\mu \varphi} = d_{\mu \varphi}(a, n) &= \text{def} \ a = \ell \circ \subseteq : \\
\{ A \times \mathbb{N} \mid \varphi \} &\subseteq A \times \mathbb{N} \xrightarrow{\ell} A, \text{ and} \\
\hat{\mu} \varphi = \hat{\mu} \varphi(a, n) &= \text{def} \ \min\{ m \leq n \mid \varphi(a, m) \} : \\
\{ A \times \mathbb{N} \mid \varphi \} &\subseteq A \times \mathbb{N} \rightarrow \mathbb{N}.
\end{align*}
\]

Comment: This definition of \( \mu \varphi : A \rightarrow \mathbb{N} \) is a static one, by enumeration \((\ell, \hat{\mu} \varphi) : \{ A \times \mathbb{N} \mid \varphi \} \rightarrow A \times \mathbb{N} \) of its graph, as is the case in general here for partial p. r. maps: We start with given pairs in enumeration domain \( \{ A \times \mathbb{N} \mid \varphi \} \), and get defined arguments a “only” as \( d_{\mu \varphi} \)-enumerated “elements” (dependent variable) \( a = d_{\mu \varphi}(\langle a, n \rangle) = d_{\mu \varphi}(a, n), \langle a, n \rangle = (a, n) \) “already known” to lie in \( D_{\mu \varphi} = \{ A \times \mathbb{N} \mid \varphi \} : \) No need—and in general no “direct” possibility—to decide, for a given \( a \in A \), if \( a \) is of form \( a = d_{\mu \varphi}(a, n) \) with \( (a, n) \in D_{\mu \varphi} \), i.e. if \( \text{Exists} \ n \in \mathbb{N} \text{ such that } \varphi(a, n) \). In particular, if \( D_{\mu \varphi} = \{ A \times \mathbb{N} \mid \varphi \} = \emptyset_{A \times \mathbb{N}} \), then \( d_{\mu \varphi} \) as well as \( \hat{\mu} \varphi \) are empty maps.

\( \mu \)-Lemma: \( \hat{S} \) admits the following (free-variables) scheme \((\mu)\) combined with \((\mu!)\)—uniqueness—as a characterisation of the \( \mu \)-operator \( \langle \varphi : A \times \mathbb{N} \rightarrow 2 \rangle \mapsto \langle \mu \varphi : A \rightarrow \mathbb{N} \rangle \) above:
\[
\varphi = \varphi(a, n) : A \times \mathbb{N} \to 2 \quad S - \text{map} \text{ ("predicate"),}
\]

\[
\mu \varphi = \langle (\mu \varphi, \hat{\mu} \varphi) : D_{\mu \varphi} \to A \times \mathbb{N} \rangle : A \to \mathbb{N}
\]

is an \(\widehat{S}\)-map such that

\[
S \vdash \varphi(d_{\mu \varphi}(\hat{a}), \hat{\mu} \varphi(\hat{a})) = \text{true}_{D_{\mu \varphi}} : D_{\mu \varphi} \to 2,
\]

+ "argumentwise" \textbf{minimality:}

\[
S \vdash [\varphi(d_{\mu \varphi}(\hat{a}), n) \Rightarrow \hat{\mu} \varphi(\hat{a}) \leq n] : D_{\mu \varphi} \times \mathbb{N} \to 2
\]

as well as \textbf{uniqueness}—by maximal extension:

\[
f = f(a) : A \to \mathbb{N} \text{ in } \widehat{S} \text{ such that}
\]

\[
S \vdash \varphi(d_{f}(\hat{a}), \hat{f}(\hat{a})) = \text{true}_{D_{f}} : D_{f} \to 2,
\]

\[
S \vdash \varphi(d_{f}(\hat{a}), n) \Rightarrow \hat{f}(\hat{a}) \leq n : D_{f} \times \mathbb{N} \to 2
\]

\[
\mu! \quad S \vdash f \subseteq \mu \varphi : A \to \mathbb{N} \quad \text{(inclusion of graphs)}
\]

[Requiring this maximality of \(\mu \varphi\) is \textbf{necessary}, since—for example—\(\mu\) alone is fulfilled already by the \textit{empty} partial function \(\emptyset : A \to \mathbb{N}\)]

\section{3.6 Content driven loops}

By a \textit{content driven} loop we mean an \textit{iteration} of a given \textit{step endo map}, whose number of performed steps is not known at \textit{entry time} into the \textit{loop}—as is the case for a PR iteration \(f^8(a, n) : A \times \mathbb{N} \to A\)
with iteration number \( n \in \mathbb{N} \)—, but whose (re) entry into a “new” endo step \( f : A \to A \) depends on content \( a \in A \) reached so far:

This (re) entry or exit from the loop is now controlled by a (control) predicate \( \chi = \chi(a) : A \to 2 \).

First example: a while loop \( \text{wh}[\chi|f] : A \to A \), for given p.r. control predicate \( \chi = \chi(a) : A \to 2 \), and (looping) step endo \( f : A \to A \), both in \( S \), both \( S \)-maps for the time being, \( S \) as always in our present context an extension of \( \text{PRa} \), admitting the scheme of (predicate) abstraction. Examples for the moment: \( \text{PRa} = \text{PR} + (\text{abstr}) \) itself, Universe theory \( \text{PRx}a \) as well as \( \text{PA} \restriction \text{PR} \), restriction of \( \text{PA} \) to its p.r. terms, with inheritance of all \( \text{PA} \)-equations for this term-restriction.

Classically, with variables, such \( \text{wh} = \text{wh}[\chi|f] \) would be “defined”—in pseudocode—by

\[
\text{wh}(a) : = \begin{cases} 
a' : = a; 
\text{while } \chi(a') \text{ do } a' : = f(a') \text{ od}; 
\text{wh}(a) : = a'.
\end{cases}
\]

The formal version of this—within a classical, element based setting—, is the following partial-(\text{Peano})-map characterisation:

\[
\text{wh}(a) = \text{wh}[\chi|f](a) = \begin{cases} 
a & \text{if } \neg \chi(a) \\
\text{wh}(f(a)) & \text{if } \chi(a)
\end{cases} : A \to A.
\]

But can this dynamical, bottom up “definition” be converted into a p.r. enumeration of a suitable graph “of all argument-value pairs” in terms of an \( \hat{S} \)-morphism

\[
\text{wh} = \text{wh}[\chi|f] = \langle (d_{\text{wh}}, \hat{\text{wh}}) : D_{\text{wh}} \to A \times A \rangle : A \to A?
\]
In fact, we can give such suitable, static Definition of \( \text{wh} = \text{wh}[\chi | f] : A \rightarrow A \)—within \( \hat{S} \sqsupset S \)—as follows:

\[
\begin{align*}
\text{wh} &= \text{def} \quad f^\hat{g} \circ (\text{id}_A, \mu \varphi[\chi | f]) \\
&= \text{by def} \quad f^\hat{g} \circ (A \times \mu \varphi[\chi | f]) \circ \Delta_A : \\
A \rightarrow A \times A &\rightarrow A \times \mathbb{N} \rightarrow A, \text{where} \\
\varphi = \varphi[\chi | f](a, n) &= \text{def} \quad \neg \chi f^\hat{g}(a, n) : A \times \mathbb{N} \rightarrow A \rightarrow 2 \rightarrow 2.
\end{align*}
\]

Within a quantified arithmetical theory like \( \text{PA} \), this \( \hat{S} \)-Definition of \( \text{wh}[\chi | f] : A \rightarrow A \) fulfills the classical characterisation quoted above, as is readily shown by Peano-Induction “on” \( n := \mu \varphi[\chi | f](a) : A \rightarrow \mathbb{N} \), at least within \( \text{PA} \) and its extensions.

[Classically, partial definedness of this—dependent—induction parameter \( n \) causes no problem: use a case distinction on definedness of \( \mu \varphi_{\chi, f}(a) \in \mathbb{N} \). Even in our quantifier-free context such dependent induction on a partial dependent induction parameter will be available, see below]

In this generalised sense, we have—within theories \( \hat{S} \sqsupset S \)—all while loops, for the time being at least those with control \( \chi : A \rightarrow 2 \) and step \( \text{endo} f : A \rightarrow A \) within \( S \).

It is obvious that such \( \text{wh}[\chi | f] : A \times A \) is in general “only” partial—as is trivially exemplified by integer division by divisor \( 0 \), which would be endlessly subtracted from the dividend, although in this case control and step are both PR.
4 Universal Sets and Universe Theories

4.1 Strings as polynomials

Strings $a_0 a_1 \ldots a_n$ of natural numbers (in set $\mathbb{N}^+ = \mathbb{N}^* \setminus \{\square\}$ of non-empty strings) are coded as prime power products

$$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_0 \subset \mathbb{N}, \ p_j \text{ the } j \text{ th prime number.}$$

Formally: euclidean prime power factorisation gives rise to a p. r. projection family

$$\pi = \pi_j(a) : \mathbb{N} \times \mathbb{N}_0 \rightarrow \mathbb{N}, \ a = p_0^{\pi_0(a)} \cdot p_1^{\pi_1(a)} \cdot \ldots \cdot p_n^{\pi_n(a)},$$

unique $\pi_j(a)$, $\pi_j(a) = 0$ for $j > n$, $n = n(a) : \mathbb{N}_0 \rightarrow \mathbb{N}$ suitable p. r.

Strings $a_0 a_1 \ldots a_n \equiv p_0^{a_0} \cdot \ldots \cdot p_n^{a_n}$ are identified with (the coefficient lists of) “their” polynomials

$$p(X) = a_0 + a_1 X^1 + \ldots + a_n X^n \text{ as well as}$$

$$p(\omega) = a_0 + a_1 \omega^1 + \ldots + a_n \omega^n,$$

in indeterminate $X$ resp. $\omega$.

Componentwise addition (and truncated subtraction), as well as

$$p(\omega) \cdot \omega = \sum_{j=0}^{n} a_j \omega^{j+1} = \prod_{j=0}^{n} p_j^{a_j},$$

special case of Cauchy product of polynomials.

Lexicographical Order of NNO strings and polynomials has—intuitively, and formally within sets—only finite descending chains.

This applies in particular to descending complexities of CCI’s: Complexity Controlled Iterations below, with complexity values in $\mathbb{N}[\omega]$; p. r. map code evaluation will be resolved into such a CCI.
4.2 Universal object $X$ of numerals and nested pairs

We begin the construction of Universal object by internal *numeralisation* of all objective natural numbers, of objective numerals

\[
\text{num}(0) \equiv 0 : 1 \to \mathbb{N},
\]

\[
\text{num}(1) \equiv 1 = \text{def} (s(0)) : 1 \to \mathbb{N} \to \mathbb{N},
\]

\[
\text{num}(2) \equiv 2 = \text{def} (s(s(0))) : 1 \to \mathbb{N},
\]

\[
\text{num}(n + 1) \equiv n + 1 = \text{def} (s(n)) : 1 \to \mathbb{N},
\]

\[n \in \mathbb{N}\] meta-variable.

Internal numerals, *numeralisation*

\[
\nu = \nu(n) : \mathbb{N} \to \mathbb{N}^+ \equiv \mathbb{N}^* \setminus \{0\} \equiv \mathbb{N}_> \subset \mathbb{N} :
\]

\[
\nu(0) = \text{def} \langle \Gamma 0 \rangle : 1 \to \mathbb{N} \text{ code (goedel number) of 0},
\]

\[
\nu(1) = \text{def} \langle \langle s \rangle \circ \nu(0) \rangle = \text{by def} \langle \Gamma s \Gamma \circ \Gamma 0 \rangle : 1 \to \mathbb{N},
\]

abbreviation for (string) goedelisation, here in particular for $\LaTeX$ source code

\[
\Gamma (\neg \Gamma s \neg \circ \nu(0) \Gamma) \neg \equiv p_0 \quad p_1 \quad p_2 \quad p_3 \quad p_4
\]

\[
\equiv 2^{40} 3^{115} 5^{48} 11^{41} : 1 \to \mathbb{N},
\]

an element of $\mathbb{N}$, a *constant* of $\mathbb{N},$

\[
\nu(2) = \text{def} \langle \langle s \rangle \circ \nu(1) \rangle = \langle \langle s \rangle \circ \langle \langle s \rangle \circ \nu(0) \rangle \rangle \text{ etc. PR:}
\]

\[
\nu(n + 1) = \text{def} \langle \langle s \rangle \circ \nu(n) \rangle \in \mathbb{N}.
\]

\[
\nu(n) \text{ has } n \text{ closing brackets (at end).}
\]
This internal numeralisation distributes the “elements”, numbers of
the NNO \( \mathbb{N} \), with suitable gaps over \( \mathbb{N} \): the gaps then will receive in
particular codes of any other symbols of object Languages \( \text{PR} \) and
\( \text{PRa} \) as well as of Universe Languages \( \text{PRX} \) and \( \text{PRXa} \) to come.

\( \nu \)-Predicate lemma: Enumeration \( \nu : \mathbb{N} \rightarrow \mathbb{N} \) defines a charac-
teristic predicate \( \text{im}[\nu] = \chi_\nu : \mathbb{N} \rightarrow 2 \), and by this object
\[
\nu \mathbb{N} = \{ \mathbb{N} | \chi_\nu \} \subset \mathbb{N}^+
\]
of internal numerals \( \nu \mathbb{N} \cong \mathbb{N} \).

Proof: Use finite \( \exists \)—iterative ‘\( \lor \)’—for definition of \( \text{im}[\nu] \), as fol-
lows:
\[
\chi_\nu(c) = \text{def} \ \lor_{n \leq c}[c \doteq \nu(n)]
\]
\[
= [c \doteq \nu(0) \lor c \doteq \nu(1) \lor \ldots \lor c \doteq \nu(c)] : \mathbb{N} \rightarrow 2 \quad \text{q.e.d.}
\]
\( \nu : \mathbb{N} \rightarrow \mathbb{N}^+ \subset \mathbb{N} \) has codomain restriction
\[
\nu : \mathbb{N} \rightarrow \nu \mathbb{N} = \text{def} \ \{ \mathbb{N} | \chi_\nu \}
\]
and is then an iso with p. r. inverse
\[
\nu^{-1} = \nu^{-1}(c) = \text{def} \ \min_{n \leq c} [\nu(n) \doteq c] : \nu \mathbb{N} \cong \mathbb{N}.
\]
For a \( \text{PR} \)-map \( f : \mathbb{N} \rightarrow \mathbb{N} \) define its numeral twin
\[
\hat{f} = \text{def} \ \nu \circ f \circ \nu^{-1} : \nu \mathbb{N} \xrightarrow{\nu^{-1}} \mathbb{N} \xrightarrow{f} \mathbb{N} \xrightarrow{\nu} \nu \mathbb{N},
\]
giving trivially (local) naturality
**Extension** of numeral sets and numeralisation to all **objects** of PR (and of PRa):

- \( \nu \mathbb{1} = \{ \nu 0 \} = \{ 0 \downarrow \} \subset \nu \mathbb{N} \subset \mathbb{N} \),
  \( \nu_\mathbb{1}(0) = \nu(0) : 1 \xrightarrow{\sim} \nu \mathbb{1} \subseteq \nu \mathbb{N} \).

- recursive extension to products:
  \[ \nu(A \times B) = \langle \nu A \times \nu B \rangle = \text{def} \{ \langle \nu A(a); \nu B(b) \rangle \mid a \in A, b \in B \} \]
  predicatively
  \[ = \{ \langle c; d \rangle \in \mathbb{N} \mid \chi_{\nu A}(c) \wedge \chi_{\nu B}(d) \} \].

- Extension to (predicative) subsets:
  \[ \chi = \chi(a) : A \rightarrow \mathbb{N} \text{ predicate} \]
  \[ \nu\{A|\chi\} = \text{def} \{ \nu(a) \mid a \in \{A|\chi\} \} \subseteq \nu A \]

- **remark:** \( \chi, \nu \chi \subset \mathbb{N}, \nu \chi \cong \chi \), but \( \nu \chi \nsubseteq \chi \), parallel to \( \nu \mathbb{N} \nsubseteq \mathbb{N} \).

- \( \nu \) isomorphy (and **naturality**) extend to \( A, B \) in PR and in PRa.
Universal objects $X, X_\bot$ of numerals and (nested) pairs of numerals:

As code for waste symbol we take
$$\bot \overset{\text{def}}{=} \bot \triangleq \bot \\bot \text{ : } 1 \to \mathbb{N}.$$ Define sets

$$X, X_\bot = \{ N \mid X, X_\bot : N \to 2 \} \subset \mathbb{N}$$

of all (codes of)

- undefined value $\bot$,
- numerals $\nu(n) \in \nu\mathbb{N}$, and
- (possibly nested) pairs

$$\langle x; y \rangle = \overset{\text{by def}}{= \bot x \\bot y \\bot} \text{ of numerals}$$
as follows:

- $\nu\mathbb{N} \subset X \subset \mathbb{N}$, numerals proper; further recursively enumerated:
- $\langle X \times X \rangle = \overset{\text{def}}{=} \{ \langle x; y \rangle \mid x, y \in X \} \subset X$,
  set of (nested) pairs of numerals, general numerals, in particular

$$\langle X \times \nu\mathbb{N} \rangle = \{ \langle x; vn \rangle \mid x \in X, n \in \mathbb{N} \} \subset X;$$

- $X_\bot = \overset{\text{def}}{=} X \cup \{ \bot \} \subset \mathbb{N}^+.$

**X-Predicative Lemma:** $X$ has predicative form

$$X = \{ N \mid \chi_X \}, \text{ and } X_\bot = \{ N \mid \chi_X \vee \{ \bot \} \}. $$

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**Proof** as (technically advanced) **Exercise.**

This terminates recursive **definition** of (“minimal”) predicative *Universal objects* \(X\) and \(X_{\bot}\), of *nested pairs of numerals*, both

\[
X, \ X_{\bot} \subset \mathbb{N}^+ \equiv \mathbb{N}_> =_{by \ def} \ N_{>0} \subset \mathbb{N} \equiv \mathbb{N}^*.
\]

**Remark:** A *superUniversal object* \(U \supset X, \ U \subset \mathbb{N}\) of lists (bracketed strings) of numerals can be **defined** p.r. by

- \(\nu \mathbb{N} \subseteq U\),
- \(x \in U, y \in U \implies x; y \in U\),
- \(x \in U \implies \langle x \rangle \in U\).

(Predicative) set \(U \subset \mathbb{N}\) can be interpreted as set of (numeralised) coefficient lists \(\mathbb{N}[X_1, X_2, \ldots, X_m, \ldots]\) of polynomials in *several indeterminates* \(X_1, X_2, \ldots\) with (numeralised) coefficients out of \(\nu \mathbb{N}\), written in form \(\cup_m \mathbb{N}[X_1][X_2] \ldots [X_m]\).

### 4.3 Universe monoid \(PRX\)

The endomorphism set \(PR(\mathbb{N}, \mathbb{N}) \subset PR\) is itself a **monoid**, a categorical theory with just one object.

*Embedded “cartesian p.r. Monoid” \(PRX\):*

- the basic, “super” object of \(PRX\) is
  \[
  X_{\bot} = X \cup \{\bot\} = X \cup \{\bot \bot\} \subset \mathbb{N},
  \]
  \[
  X : \mathbb{N} \to \mathbb{N} \text{ in } PR(\mathbb{N}, \mathbb{N}) \text{ predicate/set of (internal) numerals and nested pairs of numerals.}
  \]
the rôle of the NNO will be taken by the above predicative subset
\[ \nu \mathbb{N} = \{ c \in \mathbb{N} | \chi_\nu(c) \} \subset X \subset X \bot \subset \mathbb{N} \]
of the internal numerals.

the basic “universe” map constants of \text{PR}_X, \ ba \in \text{bas set of those maps}, are

- “identity” \text{id} = \text{id}_X : \mathbb{N} \supset X \bot \supset X \rightarrow X \subset X \bot,

\[ X \ni x \mapsto x \in X, \]
\[ \mathbb{N} \setminus X \ni z \mapsto \bot \] (trash),

\text{PR} map code set “from” \mathbb{N} “to” \mathbb{N}, same for all codes below.

- “zero” (redefined for \text{PR}_X) \hat{0} : X \rightarrow X \bot,

\[ X \ni \nu 0 \mapsto \nu 0 \in \nu \mathbb{N} \subset X, \]
\[ \mathbb{N} \setminus \{ \nu 0 \} \ni z \mapsto \bot, \]

- “successor” \hat{s} : X \bot \rightarrow X \bot :

\[ \nu n \mapsto \nu (s n) = \text{by def} \ \langle \ begin{array}{c} \nu (s\downarrow) \circ \nu (n) \end{array} \rangle, \]
\[ \mathbb{N} \setminus \nu \mathbb{N} \ni z \mapsto \bot. \]

- “terminal map”: \hat{1} : X \rightarrow \nu \bot \subset X,

\[ X \ni x \mapsto \nu 0 \in \nu \bot = \{ \nu 0 \} \subset X, \]
\[ \mathbb{N} \setminus X \ni z \mapsto \bot. \]

- “left projection”:

\[ \hat{\ell} : \mathbb{N} \supset X \supset (X \times X) \rightarrow X \bot, \]
\[ \langle x; y \rangle \mapsto x \in X, \nu \mathbb{N} \ni \nu n \mapsto \bot, \bot \mapsto \bot. \]
– “right projection” $\hat{r} \in \text{bas analogous.}$

• close Monoid $\text{PR}_X$ under composition of theory $\text{PR}$:

\[
\begin{align*}
\text{f, g in } \text{PR}_X & \subseteq \text{PR}(\mathbb{N}, \mathbb{N}) \\
\circ & \\
\text{(g \circ f) in } \text{PR}_X, \\
\text{trash propagation clear.}
\end{align*}
\]

• “induced map”:

\[
\begin{align*}
\text{f, g in } \text{PR}_X & \\
\text{(ind)} & \\
\langle f, g \rangle & \text{ in } \text{PR}_X, \text{ defined by} \\
\mathbb{X} \ni x & \mapsto \langle f(x; g x) \rangle \in \mathbb{X}.
\end{align*}
\]

• “product map”:

\[
\begin{align*}
\text{f, g in } \text{PR}_X & \\
\hat{\times} & \\
\langle f \hat{\times} g \rangle & \text{ in } \text{PR}_X, \text{ defined by} \\
\mathbb{X} \ni \langle x; y \rangle & \mapsto \langle f(x; g y) \rangle \in \mathbb{X}, \\
\mathbb{N} \setminus \langle \mathbb{X} \hat{\times} \mathbb{X} \rangle & \ni z \mapsto \bot.
\end{align*}
\]
• “iterated” (formally interesting, see last lines):

\[ f : X \rightarrow X {\text{ PRX map, in particular }} \bot \mapsto \bot \]

\[ f^\delta : X \supset (X \times \nu N) \rightarrow X \text{ in PRX, } \]

\[ \langle x; \dot{n} \rangle \mapsto f^n(x) \in X, \]

\[ n = \nu^{-1}(\dot{n}), \dot{n} \in \dot{N} = \nu N =_{\text{by def}} \{N|\chi_{\nu}\} \text{ free, } \]

\[ N \ni z \mapsto \bot \text{ for } z \text{ not of form } \langle x; \dot{n} \rangle. \]

[Predicates \( \nu N \) and \( (X \times \nu N) : N \rightarrow N \) work as auxiliary objects, subobjects of \( X : N \rightarrow N \).]

• Notion of map equality for theory PRX is inherited(!) from PR(N,N) i.e. from theory PR.

**PRX Structure theorem:** With emerging (predicative) objects \( X, \nu \mathbb{1}, \nu N \),

\[ A, B \text{ objects } \]

\[ \langle A \times B \rangle \text{ object, } \]

constants, maps, composition above,

• \( \nu \mathbb{1} = \{\nu 0\} \) taken as “terminal object”,

• \( \check{I} : X \rightarrow \nu \mathbb{1} \) taken as “terminal map,”
• “Product” taken

\[
\langle \ell : \langle A \times B \rangle \to A, \langle x; y \rangle \to x, \\
\hat{r} : \langle A \times B \rangle \to B, \langle x; y \rangle \to y \rangle,
\]

• \(\langle f \cdot g \rangle : C \to \langle A \times B \rangle, x \mapsto \langle f x; g x \rangle\), taken as “induced map,”

• \(\langle f \times g \rangle : \langle A \times B \rangle \to \langle A' \times B' \rangle, \langle x; y \rangle \mapsto \langle f x; g y \rangle\), taken as “map product,”

• \(\langle \nu \mathbb{N} \to \nu \mathbb{N} \rangle\) taken as NNO,

• and \(f^\hat{s} : \langle \mathbb{X} \times \nu \mathbb{N} \rangle \to \mathbb{X}\) as iterated of

\[\text{PRX endomap } f : \mathbb{X} \to \mathbb{X}, \langle x; \nu n \rangle \mapsto f^n(x) = f^\hat{s}(x, n),\]

\text{PRX becomes a cartesian p.r. category with universal object.}

• Fundamental theory \text{PR} is naturally embedded into theory \text{PRX}, by faithful functor \(I\) say.

4.4 Typed universe theory \text{PRXa}

Let emerge within universe monoid/universe cartesian p.r. theory all \text{PRa} objects \(\{A | \chi\}\) as additional objects \(\nu\{A | \chi\}\) and get this way a p.r. cartesian theory \text{PRXa} with extensions of predicates, finite limits, finite sums, coequalisers of equivalence predicates, as well as with (formal, “including”) universal object \(\mathbb{X}\), of numerals and (nested) pairs of numerals.

\textbf{Universal embedding theorem:}
(i) \( I : \mathbf{PR} \rightarrow \mathbf{PR}_X \subset \mathbf{PR}(\mathbb{N}, \mathbb{N}) \) above is a faithful functor.

(ii) theory \( \mathbf{PR}_X a \) “inherits” from category \( \mathbf{PR}_a \) all of its (categorically described) structure: cartesian p.r. category structure, equality predicates on all objects, scheme of predicate abstraction, equalisers, and—trivially—the whole algebraic, logic and order structure on \( \text{NNO} \cap \mathbb{N} \) and truth object \( \nu 2 \).

(iii) \( \mathbf{PR} \) map embedding \( I \) “canonically” extends into a cartesian p.r. functorial embedding (!)

\[
I : \mathbf{PR}_a \rightarrow \mathbf{PR}_X a \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})
\]

of theory \( \mathbf{PR}_a = \mathbf{PR} + (\text{abstr}) \) into emerging universe theory \( \mathbf{PR}_X a \) with predicate abstraction.

(iv) Embedding \( I \) defines a p.r. isomorphism of categories

\[
I : \mathbf{PR}_a \xrightarrow{\cong} I[\mathbf{PR}_a] \sqsubseteq \mathbf{PR}_X a.
\]

(v) (internal) code set is

\[
[X, X] = \text{by def} \quad [X, X]_{\mathbf{PR}_X a} = [X, X]_{\mathbf{PR}_X} = \mathbf{PR}_X.
\]

Internal notion \( \hat{=} \) of equality is in both cases inherited from internal notion of equality of theories \( \mathbf{PR}, \mathbf{PR}(\mathbb{N}, \mathbb{N}) \), given as enumeration of internally equal pairs

\[
\hat{=} = \hat{=}_k : \mathbb{N} \rightarrow \mathbf{PR}_X \times \mathbf{PR}_X \subset \mathbb{N} \times \mathbb{N},
\]

as well as predicatively as

\[
\hat{=} = u \hat{=}_k v : \mathbb{N} \times (\mathbf{PR} \times \mathbf{PR}) \rightarrow 2 :
\]

\( k \)th internal equality instance equals pair \((u, v)\) of internal maps.
(vi) put things together into the following diagram:

\[
\begin{array}{c}
\{A \mid \chi\} \xrightarrow{f} \{B \mid \varphi\} \\
\downarrow \nu\{A \mid \chi\} \cong = \downarrow \nu\{B \mid \varphi\} \\
\nu\{A \mid \chi\} \xrightarrow{\text{IF}} \{A \mid \chi\} \xrightarrow{\subset} \{B \mid \varphi\} \xrightarrow{c} \{B \mid \varphi\} \cup \{\bot\}
\end{array}
\]

\[
\begin{array}{c}
\downarrow c \\
\downarrow c = \downarrow f = \text{by def} \ I_{\text{PR}} f \\
\downarrow c \\
\downarrow c
\end{array}
\]

\[
\begin{array}{c}
\text{PRa embedding DIAGRAM for I}\ f \ q.e.d.
\end{array}
\]

5 Evaluation of p. r. map codes

5.1 Complexity controlled iteration

The data of such a CCI are an endomap \( p = p(a) : A \to A \) (predecessor), and a complexity map \( c = c(a) : A \to \mathbb{N}[\omega] \) on \( p \)'s domain. Complexity values are taken in lexicographically ordered polynomial object \( \mathbb{N}[\omega] \equiv \mathbb{N}^+ \equiv \mathbb{N}^* \setminus \{\square\} \equiv \mathbb{N}_* \).

**Definition:** \([c : A \to \mathbb{N}[\omega], p : A \to A]\) constitute the data of a *Complexity Controlled Iteration* \( \text{CCI} = \text{CCI}[c, p] \), if

- \((a \in A)[c(a) > 0 \implies c p(a) < c(a)] \) (descent)
- as well as, for commodity,
- \((a \in A)[c(a) = 0 \implies p(a) = a] \) (stationarity).
Such data define a while loop

\[
\text{wh}[c > 0, p] : A \rightarrow A, \text{ more explicetly written }
\]

\[
\text{while } c(a) > 0 \text{ do } a := p(a) \text{ od.}
\]

We rely on scheme of non-infinite iterative descent

\[
\text{CCI}[c = c(a) : A \rightarrow \mathbb{N}[^\omega], \; p = p(a) : A \rightarrow A] : \\
c, p \text{ make up a complexity controlled iteration, }
\]

\[
\psi = \psi(a) : A \rightarrow 2 \; \text{“negative” test predicate:}
\]

\[
(a \in A)(n \in \mathbb{N})[\psi(a) \implies cp^n(a) > 0]
\]

\[
\text{ (“all } n \text{”, to be excluded)}
\]

\[
\psi(a) = \text{false}_A(a) : A \rightarrow 2.
\]

A predicate \( \psi \) which implies a CCI to infinitely descend must be (overall) false.

By contraposition this can be turned into

\[
c, p \text{ define a CCI,}
\]

\[
\varphi = \varphi(a) : A \rightarrow 2 \; \text{“positive” test predicate:}
\]

\[
[cp^n(a) \not\equiv 0 \implies \varphi(a)] : A \times \mathbb{N} \rightarrow 2
\]

\[
\text{ (“exists } n \text{”, to be asserted)}
\]

\[
\varphi(a) = \text{true}_A(a) : A \rightarrow 2.
\]
A predicate which holds under the premise of termination of a CCI must be true by itself. This is to express that a CCI must terminate anyway. It says that the defined arguments enumeration of a CCI considered as a while loop is a p.r. epimorphism (not a retraction in general.) Technically, we will rely on the (negative) form \((\pi)\) of the axiom.

- **central example:** *general recursive, Ackermann type PR-code evaluation* \(ev\) to be resolved into such a CCI.

- **scheme** \((\pi)\) is a theorem for set theory \(T\) with its quantifiers \(\exists\) and \(\forall\), and with its having \(N[\omega] \equiv \omega^\omega\) as a (countable) ordinal: existential guarantee of finiteness of descending chains within \(\omega^\omega\).

- without quantification, namely for theories like \(PRa, PR\alpha\), we are led to this inference-of-equations scheme guaranteeing (intuitively) termination of CCIs, in particular termination of iterative p.r. code evaluation.

**Comment:** The point is that \((\pi)\) expresses an axiom which “we all” believe in (and which is a theorem in set theory): Nobody has pointed to—will be able (?) to point to—any infinitely descending chain in \(N[\omega] =_{by\,def} N^+ \subset N^*\) (provided with its lexicographical order), a fortiori not to an iterative such, to an infinitely descending CCI.

**Definition:** Call \(PR\) descent theory universe theory \(\piR =_{def} PR\alpha + (\pi)\) strengthened by axiom scheme \((\pi)\) above of non-infinite descent.
5.2 PR code set

The map code set—set of gödel numbers—we want to evaluate is $\text{PRX} = \{X, X\} \subset \mathbb{N}$. It is p.r. defined as follows:

- $\forall ba \in \text{PRX}$—formal categorically:
  $\text{PRX} \circ \langle ba \rangle = \text{true}$—this for basic map constant $ba \in \text{bas} = \{0, \hat{s}, \text{id}, \hat{\Pi}, \hat{\Delta}, \hat{\ell}, \hat{r}\}$: zero, successor, identity, terminal map, diagonal, left and right projection. All of these interpreted into endo map Monoid $\text{PRX} \subset \text{PR}(\mathbb{N}, \mathbb{N})$ of fundamental cartesian p.r. theory $\text{PR}$.

- for $u, v$ in $\text{PRX}$ in general add
  - internally composed: $\langle v \circ u \rangle = \langle \langle v \circ \rangle \circ \langle u \rangle \rangle$:
    $\text{PRX} \times \text{PRX} \to \text{PRX}, u, v \in \text{PRX}$ both free,
    in particular $\langle \langle g \circ f \rangle \rangle = \langle \langle g \rangle \circ \langle f \rangle \rangle \in \text{PRX}$
    for $f, g : X \to X$ in $\text{PRX}$;
  - internally induced: $\langle u ; v \rangle = \langle \langle u \rangle ; \langle v \rangle \rangle \in \text{PRX},$
    in particular $\langle \langle f, g \rangle \rangle = \langle \langle f \rangle . \langle g \rangle \rangle \in \text{PRX};$
  - internal cartesian product: $\langle u \# v \rangle \in \text{PRX},$
    $u, v \in \text{PRX}$ free, in particular
    $\langle \langle f \times g \rangle \rangle = \langle \langle f \rangle \# \langle g \rangle \rangle \in \text{PRX};$
  - internally iterated: $u^\$ = $u^\hat{\$} \in \text{PRX}, u \in \text{PRX},$
    in particular $\langle \langle f \rangle ^\$ \rangle = \langle \langle f \rangle ^\$ \rangle \in \text{PRX}.$
5.3 Iterative evaluation

For Definition of evaluation ev we first introduce evaluation step of form

\[ e(u, x) = (e_{\text{map}}(u, x), e_{\text{arg}}(u, x)) : \text{PR} \times X_{\perp} \to \text{PR} \times X_{\perp}, \]

by primitive recursion. This within “outer” theory PRXa which already has PR predicates \( X, X_{\perp} = \text{by def} \ X \cup \{\perp\} = X \cup \{\top, \perp\} \), and \( \langle X \times \nu N \rangle : N \to N \) as objects.

Comment: \( e_{\text{arg}}(u, x) \in X_{\perp} \) means here one-step \( u \)-evaluated argument, and \( e_{\text{map}}(u, x) \) denotes the remaining part of map code \( u \) still to be evaluated after that evaluation step.

PR Definition of step \( e \), p. r. on depth(\( u \)) \( \in \mathbb{N} \), now runs as follows:

- depth(\( u \)) = 0, i. e. \( u \) of form \( \top \text{ba} \),

\[ \text{ba} \in \text{bas} = \text{by def} \ \{\text{id}, \hat{0}, \hat{s}, \hat{\Pi}, \hat{\Delta}, \hat{\ell}, \hat{r}\} \]

one of the basic map constants of theory PRX \( \subset \text{PR} \) :

\[ e_{\text{arg}}(\top \text{ba}, x) = \text{def} \ \text{ba}(x) \in X_{\perp}, \]
\[ e_{\text{map}}(\top \text{ba}, x) = \text{def} \ \top \text{id} \in \text{PR}. \]

- cases of internal composition:

\[ e \left( \langle v \odot \top \text{ba} \rangle, x \right) = \text{def} \ \langle v, \text{ba}(x) \rangle \in \text{PR} \times X_{\perp} \]

and for \( u \notin \{\top \text{ba} \mid \text{ba} \in \text{bas}\} : \)

\[ e \left( \langle v \odot u \rangle, x \right) = \text{def} \ \langle \langle v \odot e_{\text{map}}(u, x) \rangle, e_{\text{arg}}(u, x) \rangle : \]

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step-evaluate first map code $u$, on argument $x$, and preserve remainder of $u$ followed by $v$ as map code to be step-evaluated on intermediate argument $e_{\text{arg}}(u, x)$.

- cartesian cases:

$$e \left( \langle \text{id} \# \text{id} \rangle, \langle y; z \rangle \right) = \text{def} \left( \langle \text{id} \# \langle y; z \rangle \rangle, \langle \text{id}, \langle y; z \rangle \rangle \right) \in \mathbb{P} \mathbb{R} \times \mathbb{X},$$

a terminating case.

For $⟨u\#v⟩ \neq ⟨\text{id} \# \text{id}⟩$:

$$e \left( ⟨u\#v⟩, ⟨y; z⟩ \right) = \text{def} \left( ⟨e_{\text{map}}(u, y)\#e_{\text{map}}(v, z)⟩, ⟨e_{\text{arg}}(u, y); e_{\text{arg}}(v, z)⟩ \right),$$

evaluate $u$ and $v$ in parallel.

Here free variable $x$ on $\mathbb{X}$ legitimately runs only on $⟨\mathbb{X} \times \mathbb{X}⟩ \subset \mathbb{X}$, takes there the pair form $⟨y; z⟩$. $x \in \mathbb{X} \setminus ⟨\mathbb{X} \times \mathbb{X}⟩$ results in present evaluation case into $\perp$.

- Cases of an induced (redundant via $\text{id} \# \text{id}$ and $\circ$):

$$e \left( \langle \text{id} \# \text{id} \rangle, z \right) = \text{def} \left( \langle \text{id} \# \langle z; z \rangle \rangle, \langle \text{id}, \langle z; z \rangle \rangle \right),$$

a terminating case.

For $⟨u; v⟩ \neq ⟨\text{id} \# \text{id}⟩$:

$$e \left( ⟨u; v⟩, z \right) = \text{def} \left( ⟨e_{\text{map}}(u, z); e_{\text{map}}(v, z)⟩, ⟨e_{\text{arg}}(u, z); e_{\text{arg}}(v, z)⟩ \right),$$

evaluate both components $u$ and $v$. 
• iteration case, with $\$: $\}$ designating internal iteration:

\[ e(u^\$, \langle y; \nu n \rangle) = (u^{[n]}, y) : \]

\[ \text{PRX} \times \text{X} \ni \text{PRX} \times \langle \text{X} \times \nu \text{N} \rangle \rightarrow \text{PRX} \times \text{X}. \]

Here $\nu n \in \nu \text{N}$ free, $n := \nu^{-1}(\nu n) \in \text{N}$, and $u^{[n]}$ is given by code expansion as

\[ u^{[0]} = \text{def} \: \text{id}^\$, \: u^{[n+1]} = \text{def} \: \langle u \odot u^{[n]} \rangle. \]

• trash case $e(u, x) = (\: \text{id}^\$, , \bot) \in \text{PRX} \times \text{X}_\bot$ if $(u, x)$ in none of the above—regular—cases.

For to convince ourselves on termination of iteration of step $e : \: \text{PRX} \times \text{X}_\bot \rightarrow \text{PRX} \times \text{X}_\bot$—on a pair of form $(\: \text{id}^\$, , x)$—we introduce:

\[ (\text{Descending}) \: \text{complexity} \]

\[ c_{ev}(u, x) = c(u) : \text{PRX} \times \text{X} \xrightarrow{\ell} \text{PRX} \rightarrow \text{N}[\omega] \]

defined p. r. as

\[ c(\: \text{id}^\$) = \text{def} \: 0 = 0 \cdot \omega \in \text{N}[\omega], \]

\[ c(\: \text{ba'}^\$) = \text{def} \: 1 \in \text{N}[\omega] \]

for $\text{ba'}$ one of the other basic map constants in $\text{bas}$,

\[ c(\langle v \odot u \rangle) = \text{def} \: c(u) + c(v) + 1 = c(u) + c(v) + 1 \cdot \omega^0 \in \text{N}[\omega], \]

\[ c(\langle u \# v \rangle) = \text{def} \: c(u) + c(v) + 1, \]

\[ c(\langle u; v \rangle) = \text{def} \: c(u) + c(v) + 1, \]

\[ c(u^\$) = \text{def} \: (c(u) + 1) \cdot \omega^1 \in \text{N}[\omega]. \]
[\omega^1 \text{ is to account for unknown } \text{iteration count } n \text{ in argument } \langle x; n \rangle \text{ before code expansion.}]

**Example:** Complexity of *addition* \( \_ + _{=_{by \text{ def}}} s^\delta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} :\)

\[
c \mapsto + = c \mapsto s^\delta \mapsto c \mapsto (s^\delta \mapsto) \\
=(c \mapsto s^\delta + 1) \cdot \omega^1 = 2 \cdot \omega \in \mathbb{N}[\omega] \quad [\equiv 0; 2 \in \mathbb{N}^+]\]

**Motivation** for the above definition—in particular for this latter iteration case—will become clear with the corresponding case in **proof** of descent **Lemma** below for *evaluation*

\[ev = ev(u,v) =_{=_{by \text{ def}}} \text{r} \hat{\circ} \text{wh} [c_{ev} > 0 , e] : \text{PR} \times X_\perp \rightarrow \text{PR} \times X_\perp \xrightarrow{r} X_\perp\]

defined by a **while** loop which reads

\[
\text{while } c_{ev}(u) > 0 \text{ do } (u,x) : = e(u,x) \text{ od.}
\]

Evaluation *step* and *complexity* above are in fact the right ones to give

**Basic descent lemma:** For formally *partially defined* and “nevertheless” *epi-terminating* evaluation map: the defined-arguments p. r. enumeration of partial map is epi—this by axiom scheme \((\pi)\)—,

\[ev = ev(u,x) =_{=_{by \text{ def}}} \text{r} \hat{\circ} \text{wh} [c_{ev} > 0 , e] : \text{PR} \times X_\perp \rightarrow \text{PR} \times X_\perp \xrightarrow{r} X_\perp\]

(epi-terminating within theory \(\pi R = \text{PRA} + (\pi))\)

i. e. for step \(e = e(u,x) = (e_{\text{map}}, e_{\text{arg}}) : \text{PR} \times X_\perp \rightarrow \text{PR} \times X_\perp\) and complexity \(c_{ev} = c_{ev}(u,x) =_{=_{by \text{ def}}} c(u) : \text{PR} \rightarrow \mathbb{N}[\omega], \text{we have descent}\]
above $0 \in \mathbb{N}[\omega]$, and Stationarity at complexity 0:

\[
\begin{align*}
\text{PRX} & \vdash c_{ev}(u, x) > 0 \implies c_{ev}(u, x) < c_{ev}(u, x) : \\
\text{PRX} \times \mathbb{X}_\bot & \to \mathbb{N}[\omega] \times \mathbb{N}[\omega] \to 2 \text { i.e. } \\
\text{PRX} & \vdash c(u) > 0 \implies e_{\text{map}}(u, x) < c(u) \quad \text{(Desc)} \\
\text{as well as } \\
\text{PRX} & \vdash c(u) \doteq 0 \quad [ \iff u \equiv \langle \text{id} \rangle ] \\
& \implies c_{ev}(u, x) \doteq 0 \land e(u, x) \doteq (u, x) \quad \text{(Sta)}
\end{align*}
\]

This with respect to the canonical, lexicographic, and—intuitively—finite-descent order of polynomial semiring $\mathbb{N}[\omega]$.

**Proof:** The only non-trivial case $(v, b) \in \text{PRX} \times \mathbb{X}$ for descent $c_{ev}(v, b) < c_{ev}(v, b)$ is iteration case $(v, b) = (u^\$, \langle x; n \rangle)$. In this “acute” iteration case we have

\[
c(u^{[n]}) = c(\langle u \odot (u \ldots \odot u) \ldots \rangle) \\
= n \cdot c(u) + (n \div 1) < \omega \cdot (c(u) + 1) = c(u^\$),
\]

proved in detail by induction on $n$ q.e.d.

### 5.4 Evaluation characterisation

Dominated characterisation theorem for evaluation:

\[
ev = ev(u, a) : \text{PRX} \times \mathbb{X} \to \mathbb{X} \text{ is characterised by}
\]

- \[
\begin{align*}
\text{PRXa} & \vdash [ ev(\langle ba\rangle, x) \doteq ba(x) ] \\
as well as, again within \text{PRXa}, \pi R \text{ and strengthenings, by:}
\end{align*}
\]
• \([ m \text{ def } ev (v \odot u, x)] \implies ev(⟨v \odot u⟩, x) = ev(v, ev(u, x))\);

this reads: if \(m\) defines the left hand iteration \(ev\), i.e. if iteration \(ev\) of step \(e\) terminates on the left hand argument after at most \(m\) steps, then \(ev\) terminates in at most \(m\) steps on right hand side as well, and the two evaluations have equal results.

• \([ m \text{ def } ev (⟨u#v⟩, ⟨x; y⟩)] \implies ev(⟨u#v⟩, ⟨x; y⟩) = ⟨ev(u, x); ev(v, y)⟩\),

• \([ m \text{ def } ev (⟨u; v⟩, z)] \implies ev(⟨u; v⟩, z) = ⟨ev(u, z); ev(v, z)⟩\).

• \(ev(u$, $⟨x; ^*0\rangle) = x,$

\([ m \text{ def } ev (u$, $⟨x; ν(sn)⟩)] \implies :\)

\([ m \text{ def } all \ ev \ below] \land ev(u$, $⟨x; ν(sn)⟩) = ev(u, ev(u$, $⟨x; νs⟩))\).

• it terminates, with all properties above, when situated in a set theory \(T\), since there complexity receiving ordinal \(\mathbb{N}[\omega]\) has (only) finite descent, in terms of existential quantification.

Corollary: within \(T\), we have the double recursive equations

• \(ev(⌜ba⌝, x) = ba(x),\)

• \(ev(⟨v \odot u⟩, x) = ev(v, ev(u, x)),\)

• \(ev(⟨u#v⟩, ⟨x; y⟩) = ⟨ev(u, x); ev(v, y)⟩,\)

\(ev(⟨u; v⟩, z) = ⟨ev(u, z); ev(v, z)⟩,\)
• $ev(u^$, $\langle x; \nu 0^n \rangle) \doteq x$, and

$ev(u^$, $\langle x; \nu(s n) \rangle) \doteq ev(u, ev(u^$, $\langle x; \nu n \rangle))$.

Within $T$—as well as within partial p.r. theories $\mathcal{PRXa, \pi R}$—these equations can be taken as definition for $\mathcal{PRX}$ code evaluation $ev$. Within $T$, they define evaluation as a total map.

Proof of theorem by primitive recursion (Peano Induction) on $m \in \mathbb{N}$ free, via case distinction on codes $w$, and arguments $z \in X$ appearing in the different cases of the asserted conjunction (case $w$ one of the basic map constants being trivial). All of the following—induction step—is situated in $\mathcal{PRXa}$, read: $\mathcal{PRXa} \vdash \ldots$ etc. If you are interested first in the negative results for set theories $T$, you can read it “$T \vdash \ldots$” but $T$ still deriving properties just of $\mathcal{PRX}$ map codes.

• case $(w, z) = (\langle v \diamond u \rangle, x)$ of an (internally) composed, subcase $u = \langle id \rangle$: obvious.

Non-trivial subcase $(w, z) = (\langle v \diamond u \rangle, x), u \neq \langle id \rangle$:

$m + 1 \\text{def} \Rightarrow ev(\langle v \diamond u \rangle, x) \Rightarrow :$

$ev(\langle v \diamond u \rangle, x) \doteq e^8((\langle v \diamond e_{map}(u, x) \rangle, e_{arg}(u, x), m))$

by iterative definition of $ev$ in this case

$\doteq ev(v, ev(e_{map}(u, x), e_{arg}(u, x)))$

by induction hypothesis on $m$

$\Rightarrow :$

$m + 1 \\text{def} \Rightarrow ev(v, ev(e_{map}(u, x), e_{arg}(u, x)))$

$\land ev(v, ev(e_{map}(u, x), e_{arg}(u, x))) \doteq ev(v, ev(u, x)) :$
The latter implication “holds” same way back, by the same induction hypothesis on \( m \) (map code \( v \) unchanged.)

- case \((w, z) = (\langle u\#v \rangle, \langle x; y \rangle)\) of an (internal) cartesian product: Obvious by definition of \( ev \) on a cartesian product map codes. Pay attention to arguments out of \( X \setminus \langle X \times X \rangle \) evaluated into \( \bot \) in this case (and in similar cases). In more detail:

\[
ev(w, z) :=
\]
\[
ev(\langle u\#v \rangle, \langle x; y \rangle)
\]
\[
= \text{by def } ev(\langle e_{\text{map}}(u, x)\#e_{\text{map}}(v, y) \rangle, \langle e_{\text{arg}}(u, x), e_{\text{arg}}(v, y) \rangle)
\]
\[
= \langle ev(e_{\text{map}}(u, x), e_{\text{arg}}(u, x)), ev(e_{\text{map}}(v, y), e_{\text{arg}}(v, y)) \rangle
\]
\[
\in \langle X \times X \rangle
\]

- alternatively (or both): case \((w, z) = (\langle u; v \rangle, z)\) of an internal induced:

\[
ev(w, z) = \langle ev(u, z), ev(v, z) \rangle \in \langle X \times X \rangle.
\]

- case \((w, z) = (u^\$, \langle x; \bot^0 \rangle)\) of a null-fold (internally) iterated: again obvious.
• case \((w, z) = (u^\$, \langle x; \nu (s n) \rangle)\) of a genuine (internally) iterated:

\[
m + 1 \text{ def } ev (u^\$, \langle x; \nu (s n) \rangle) \implies
m + 1 \text{ def } \text{all instances of } ev \text{ below, and:}
\]

\[
ev (u^\$, \langle x; \nu (s n) \rangle)
\]

\[
\doteq ev (e_{\text{map}} (u^\$, \langle x; \nu (s n) \rangle), e_{\text{arg}} (u^\$, \langle x; \nu (s n) \rangle))
\]

\[
\doteq ev (u^{[n+1]}, x) \doteq ev (\langle u \odot u^{[n]} \rangle, x) \doteq ev (u, ev (u^{[n]}, x))
\]

the latter by induction hypothesis on \(m\),

case of internal composed

\[
\doteq ev (u, \langle ev (u^\$, x); \nu n \rangle) : \text{same way back.}
\]

This shows the (remaining) predicative iteration equations “anchor” and “step” for an (internally) iterated \(u^\$\), and so proves fulfillment of the above double recursive system of equations for \(ev : PRXa \times X \rightarrow X\) subordinated to global evaluation \(ev : PRX \times X \rightarrow X\) q.e.d.

**Characterisation corollary:** Evaluation—\(PRXa\) map—

\[
ev = ev (u, x) : PRX \times X \rightarrow X
\]

defined as complexity controlled iteration—CCI—with complexity values in ordinal \(\mathbb{N}[\omega]\), epi-terminates in theory \(\hat{\pi}R\) : has epimorphic defined arguments enumeration. This by definition of this theory strengthening \(PRXa\). And it satisfies there the characteristic double-recursive equations above for evaluation \(ev\).

**Objectivity theorem:** Evaluation \(ev\) is objective, i.e. for each
single, (meta free) \( f : A \rightarrow B \) in theory PR\(X_a \) itself, we have

\[
\text{PR}\(X_a \), \pi R \vdash [m \text{ deff } ev(\tau f^\land, a)] \implies ev(\tau f^\land, a) = f(a), \ \text{symbolically:} \\
\pi R \vdash ev(\tau f^\land, \_ ) = f : A \rightarrow B.
\]

For frame a set theory T, there is no need for explicit domination \( m \text{ deff etc.} \)

**Proof** by substitution of codes of PR\(X_a \) maps into code variables \( u, v, w \in \text{PR}X \subset \mathbb{N} \) in Evaluation Characterisation above, in particular:

- \( [m \text{ deff } ev(\tau g \circ f^\land, a)] \implies ev(\tau g \circ f^\land, a) = ev(\tau g^\land, ev(\tau f^\land, a)), \)
  \( \overset{\text{def}}{=} g(f(a)) = (g \circ f)(a) \) recursively (on \( m \)) and

- \( [m \text{ deff } ev(\tau f^{\land \&}, \langle a; \nu(s n) \rangle)] \implies : \\
  [m \text{ deff all ev below}] \land \\
  ev(\tau f^{\land \&}, \langle a; \nu(s n) \rangle) = ev(\tau f^\land, ev(\tau f^{\land \&}, \langle a; \nu n \rangle)) \)
  \( \overset{\text{def}}{=} f(f^{\&}(a, \nu n)) = f^{\&}(a, \nu(s n)) \) recursively on \( m \).

- it terminates, with this objectivity, within set theory T.

### 6 PR Decidability by Set Theory

We embed evaluation \( \varepsilon(u, x) : \text{PR}X \times X \rightarrow X \) of PR map codes into set theory, theory T.
Notion $f =_{\text{PR}} g$ of p.r. maps is externally p.r. enumerated, by complexity of (binary) deduction trees.

Internalising—*formalising*—gives internal notion of PR equality (not: stronger $T$-equality)

$$u \overset{\text{PT}}{=} k v \in \text{PRX} \times \text{PRX}$$

coming by internal *deduction tree* $\text{dtree}_{k}$, which can be canonically provided with arguments in $X$—top down from (suitable) argument $x$ given to the *root* equation $u \overset{\text{PT}}{=} k v$ of $\text{dtree}_{k}$.

We denote internal deduction tree argumented this way by $\text{dtree}_{k}/x$, *root* of $\text{dtree}_{k}/x$ then is $u/x \overset{\text{PT}}{=} k v/x$.

### 6.1 PR soundness framed by set theory

**PR Evaluation soundness theorem Framed by set theory $T$**:

For p.r. theory $\text{PR}$ with its internal notion of equality ‘$\overset{\text{PT}}{=}$’ we have:

(i) **PRX to $T$ evaluation soundness:***

$$T \vdash u \overset{\text{PT}}{=} k v \Longrightarrow \text{ev}(u, x) = \text{ev}(v, x)$$

Substituting in the above “concrete” $\text{PRXa}$ codes into $u$ resp. $v$, we get, by *objectivity* of evaluation $\varepsilon$:

(ii) **$T$-Framed Objective soundness of $\text{PR}$:**

For $\text{PRXa}$ maps $f, g : X \supset A \rightarrow B \subset X$:

$$T \vdash \langle \langle f \rangle \rangle \overset{\text{PT}}{=} \langle \langle g \rangle \rangle \Longrightarrow f(a) = g(a).$$
(iii) Specialising to case \( u := \neg \chi \), \( \chi : X \rightarrow 2 \) a p.r. predicate, and to \( v := \text{true} \), we get

\[ T \vdash \exists k \, \text{Prov}_{PR}(k, \neg \chi) \implies \forall x \chi(x) : \]

**If** a p.r. predicate is—within \( T \)—\( PR \)-internally provable, **then it holds in** \( T \) **for all of its arguments.**

**Proof** of logically central assertion (\( \bullet \)) by primitive recursion on \( k \), \( \text{dtree}_k \) the \( k \)th deduction tree of the theory. These (argument-free) deduction trees are counted in lexicographical order.

**Remark:** A detailed **proof** is given for frame theory \( PR_X a \) and termination-conditioned evaluations in next section. This proof logically includes present case of frame theory a set theory \( T \) : within such \( T \) as frame, both evaluations, \( ev \) as well as **deduction tree evaluation** \( ev_d \), terminate on all of their arguments.

**Super Case** of **equational** internal **axioms:**

- associativity of (internal) composition:
  \[ \langle \langle w \circ v \rangle \circ u \rangle \approx_k \langle w \circ \langle v \circ u \rangle \rangle \implies \]
  \[ ev(\langle \langle w \circ v \rangle \circ u \rangle, x) = ev(\langle w \circ v \rangle, ev(u, x)) \]
  \[ = ev(w, ev(\langle v \circ u \rangle, ev(u, x))) \]
  \[ = ev(w, ev(\langle v \circ u \rangle, x)) = ev(w \circ \langle v \circ u \rangle, x). \]

This **proves** assertion (\( \bullet \)) in present **associativity-of-composition** case.
• Analogous proof for the other flat, equational cases, namely *reflexivity of equality, left and right neutrality* of $id =_{\text{by def}} id_{\mathfrak{X}}$, all substitution equations for the map constants, Godement’s equations for the induced map as well as surjective pairing and *distributivity equation for composition with an induced*.

• proof of (●) for the last equational case, the

*Iteration step*, case of genuine iteration equation

$$\text{dtree}_k = \langle u^s \circ \langle \text{id} \# \text{id} \rangle \hat{=} k u \circ u^s \rangle :$$

$$T \vdash ev (u^s \circ \langle \text{id} \# \text{id} \rangle, \langle y; \nu(n) \rangle)$$  \hspace{1cm} (1) 

$$= ev (u^s, ev (\langle \text{id} \# \text{id} \rangle, \langle y; \nu(n) \rangle))$$ 

$$= ev (u^s, \langle y; \nu(s) \rangle)$$ 

$$= ev (u, ev (u^s, \langle y; \nu(n) \rangle))$$ 

$$= ev (u \circ u^s, \langle y; \nu(n) \rangle).$$  \hspace{1cm} (2)

**Proof** of termination-conditioned inner soundness for the remaining *deep*—genuine HORN cases—for dtree$_k$, HORN type deduction of root:

**Transitivity-of-equality** case: with map code variables $u, v, w$ we start here with argument-free deduction tree

$$\text{dtree}_k = \begin{array}{c} u \hat{=} _k w \\ \uparrow \hline \end{array}$$

$$u \hat{=} _i v \land v \hat{=} _j w$$

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Evaluate at argument $x$ and get in fact

\[ T \vdash u \overset{\sim}{=} w \]

\[ \implies ev(u, x) = ev(v, x) \land ev(v, x) = ev(w, x) \]

(by hypothesis on $i, j < k$)

\[ \implies ev(u, x) = ev(w, x) : \]

transitivity export q.e.d. in this case.

Case of symmetry axiom scheme for equality is now obvious.

Compatibility case of composition with equality

\[ \langle v \odot u \rangle \overset{\sim}{=}_{k} \langle v \odot u' \rangle \]

\[ \text{dedu}_k = \uparrow \]

\[ u \overset{\sim}{=}_{i} u' \]

By induction hypothesis on $i < k$ we have

\[ \langle v \odot u \rangle \overset{\sim}{=}_{k} \langle v \odot u' \rangle \implies : \]

\[ [ev(u, x) = ev(u', x) \implies \]
\[ ev(v \odot u, x) = ev(v, ev(u, x)) = ev(v, ev(u', x)) \]
\[ = ev(v \odot u', x)] \]

by hypothesis on $u \overset{\sim}{=}_{i} u'$ and by Leibniz' substitutivity, q.e.d. in this 1st compatibility case.

Case of composition with equality in second composition factor:

\[ \text{dedu}_k = \uparrow \]

\[ v \overset{\sim}{=}_{i} v' \]

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Here $\text{mtree}_i$ is not (yet) provided with all of its arguments, it is completely argumented during top down tree evaluation.

$$\langle v \odot u \rangle \cong_k \langle v' \odot u \rangle \implies :$$

$$ev(\langle v \odot u \rangle, x) = ev(v, ev(u, x)) = ev(v', ev(u, x)) \quad (*)$$

$$(*)$$ holds by $v \cong_i v'$, induction hypothesis on $i < k$, and Leibniz’ substitutivity: same argument into equal maps.

This proves soundness assertion ($\bullet$) in this 2nd compatibility case.

(Redundant) Case of compatibility of forming the induced map, with equality is analogous to compatibilities above, even easier, since the two map codes concerned are independent from each other.

(Final) Case of Freyd’s (internal) uniqueness of the initialised iterated, is case

$$\text{dedu}_k/\langle y; \nu(n) \rangle$$

$$w/\langle y; \nu(n) \rangle \cong_k \langle w^\# \odot \langle \text{id}^\top \rangle / y \rangle / \langle y; \nu(n) \rangle$$

$$= \frac{\text{root} (t_i) \quad \text{root} (t_j)}{}$$

where

$$\text{root} (t_i)$$

$$= \langle w \odot \langle \text{id}^\top ; 0^\top \odot \Pi^\top \rangle / y \rangle \cong_i u / y \rangle,$$

$$\text{root} (t_j)$$

$$= \langle w \odot \langle \text{id}^\top \# s^\top \rangle / \langle y; \nu(n) \rangle \rangle \cong_j \langle v \odot w \rangle / \langle y; \nu(n) \rangle \rangle.$$
Comment: \( w \) is here an internal *comparison candidate* fullfilling the same internal p. r. equations as \( \langle v^\$ \odot \langle u^\# \Gamma \text{id}^\\top \rangle \rangle \). It should be—**is**: *soundness*—evaluated equal to the latter, on \( \langle X \times \nu \mathbb{N} \rangle \subset X \).

Soundness **assertion** (●) for the present Freyd’s *uniqueness case* recurs on \( \overset{\cong_i}{} \), \( \overset{\cong_j}{} \) turned into predicative equations ‘\( = \)’, these being already deduced, by hypothesis on \( i, j < k \). Further ingredients are transitivity of ‘\( = \)’ and established properties of basic evaluation \( ev \) of map terms.

So here is the remaining—inductive—**proof**, prepared by

\[
T \vdash ev(w, \langle y; \nu(0) \rangle) = ev(u; y) \tag{0}
\]

as well as

\[
ev(w, \langle y; \nu(s n) \rangle) = ev(w, \langle y; \Gamma s^\top \odot \nu(n) \rangle)
= ev(w \odot \langle \Gamma \text{id}^\top \# \Gamma s^\top \rangle, \langle y; \nu(n) \rangle)
= ev(v \odot w, \langle y; \nu(n) \rangle), \tag{s}
\]

the same being true for \( w' := v^\$ \odot \langle u^\# \Gamma \text{id}^\\top \rangle \) in place of \( w \), once more by (characteristic) double recursive equations for \( ev \), this time with respect to the *initialised internal iterated* itself.

(0) and (s) put together for both then show, by induction on *iteration count* \( n \in \mathbb{N} \)—all other free variables \( k, u, v, w, y \) together form the *passive parameter* for this induction—*truncated soundness assertion* (●) for this *Freyd’s uniqueness case*, namely

\[
T \vdash ev(w, \langle y; \nu(n) \rangle) = ev(v^\$ \odot \langle u^\# \Gamma \text{id}^\\top \rangle, \langle y; \nu(n) \rangle).
\]

**Induction** runs as follows:

**Anchor** \( n = 0 \):

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\[ ev(w, \langle y; \nu(0) \rangle) = ev(u, y) = ev(w', \langle y; \nu(0) \rangle), \]

**step:**

\[ ev(w, \langle y; \nu(n) \rangle) = ev(w', \langle y; \nu(n) \rangle) \implies : \]

\[ ev(w, \langle y; \nu(sn) \rangle) = ev(v, ev(w, \langle y; \nu(n) \rangle)) = ev(v, ev(w', \langle y; \nu(n) \rangle)) = ev(w', \langle y; \nu(sn) \rangle), \]

the latter since evaluation \( ev \) preserves predicative equality ‘\( = \)’ (Leibniz) q.e.d.

**Comment:** Already for stating the evaluations, we needed the—categorical, free-variables theories \( \text{PR}, \text{PRa}, \text{PRX}, \text{PRXa} \) of primitive recursion. Since this type of soundness is a corner stone in our approach, the above complicated categorical combinatorics seem to be necessary, even for the negative results on classical foundations.

### 6.2 PR-predicate decision by set theory

We consider here \( \text{PRXa} \) predicates for decidability by set theorie(s) \( T \). Basic tool is \( T \)-framed soundness of \( \text{PRXa} \) just above, namely

\[
\chi = \chi(a) : A \to 2 \ \text{PRXa predicate}
\]

\[
T \vdash \exists k \ \text{Prov}_{\text{PRXa}}(k, \Gamma \chi^\top) \implies \forall a \chi(a).
\]

Within \( T \) define for \( \chi : A \to 2 \) out of \( \text{PRXa} \) a partially defined (alleged, individual) \( \mu \)-recursive **decision** \( \nabla\chi = \nabla^{\text{PR}}\chi : \mathbb{I} \to 2 \) by first fixing **decision domain**

\[
D = D\chi := \{ k \in \mathbb{N} | \neg \chi(\text{ct}_A(k)) \lor \text{Prov}_{\text{PRXa}}(k, \Gamma \chi^\top) \},
\]

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\( \text{ct}_A : \mathbb{N} \to A \) (retractive) Cantor count of \( A \); and then, with (partial) recursive \( \mu D : 1 \to D \subseteq \mathbb{N} \) within \( T \):

\[
\nabla \chi \ = \ \text{def} \begin{cases} 
\text{false if } \neg \chi(\text{ct}_A(\mu D)) \\
\text{(counterexample),}
\text{true if } \text{Prov}_{\mathbb{P}R}A(\mu D, \lnot \chi) \\
\text{(internal proof),}
\perp \text{ (undefined) otherwise, i.e.}
\text{if } \forall a \chi(a) \land \forall k \neg \text{Prov}_{\mathbb{P}R}A(k, \lnot \chi).
\end{cases}
\]

[This (alleged) decision is apparently (\( \mu \)-)recursive within \( T \), even if apriori only partially defined.]

There is a first consistency problem with this definition: are the defined cases disjoint?

Yes, within frame theory \( T \) which soundly frames theory \( \mathbb{P}R \)

\[
T \vdash (\exists k \in \mathbb{N}) \text{Prov}_{\mathbb{P}R}A(k, \lnot \chi) \implies \forall a \chi(a).
\]

T-framed \( \mathbb{P}R \)-soundness leads to

**Complete T derivation alternative** for \( \mathbb{P}R \) predicate \( \chi \):

(a) \( T \vdash \nabla \chi = \text{false} \) iff \( T \vdash \exists a \neg \chi(a) \),

(b) \( T \vdash \nabla \chi = \text{true} \) iff \( T \vdash \exists k \text{Prov}_{\mathbb{P}R}A(k, \lnot \chi) \)

\iff \( T \vdash \exists k \text{Prov}_{\mathbb{P}R}A(k, \lnot \chi) \land \forall a \chi(a) \),

the latter iff by \( T \)-framed soundness of \( \mathbb{P}R \).

(c) \( T \vdash \nabla \chi = \perp \) iff \( T \vdash \forall a \chi(a) \land \forall k \neg \text{Prov}_{\mathbb{P}R}A(k, \lnot \chi) \).
Remark:

- within quantified arithmetic $T$ we have the right to replace
  $\chi(ct_A(\mu D))$ by $\exists a (\chi(a))$ in the above, and
  $\text{Prov}_{\text{PR}_A}(\mu D, \Gamma \chi \top)$ by $\exists k \text{Prov}_{\text{PR}_A}(k, \Gamma \chi \top)$.

- for consistent $T$, $\chi$ an arbitrary $T$-formula, and $\text{Proof} \ T$ in place of $\text{Prov}_{\text{PR}_A}$: soundness—and therefore disjointness of (termination) cases (a) and (b) above—does not work anymore:
take for $\chi$ Gödel’s undecidable formula $\varphi$ with its “characteristic” property

$$ T \vdash \neg \varphi \iff \exists k \text{Prov}_T(k, \Gamma \varphi \bot). $$

Merging now the (right hand sides) of the latter two cases gives
the following complete alternative,

Decidability of primitive recursive free-variable predicates by quantified extension $T$ (via $\mu$-recursive decision algorithm $\nabla \chi : \mathbb{1} \to \mathbb{2}$):

For (arbitrary) $\text{PR}_A$ predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$ T \vdash \forall a \chi(a) \quad \text{or} \quad T \vdash \exists a \neg \chi(a). $$

"Theorem or derivable existence of a counterexample" q.e.d.

Decision Remark: this does not mean a priori that decision algorithm $\nabla \chi$ terminates for all such predicates $\chi$. The theorem says only that $\chi$ is decidable "by", within theory $T$, that it is not independent from $T$. 
For free-variable PRXa (!) predicate $\chi := \neg \text{Prov}_T(k, \lceil \text{false} \rceil)$ : 
$\mathbb{N} \to 2$ the above entails the alternative

\[ T \vdash \forall k \neg \text{Prov}_T(k, \lceil \text{false} \rceil) \text{ or } T \vdash \exists k \text{Prov}_T(k, \lceil \text{false} \rceil), \]

will say the alternative

\[ T \vdash \text{Con}_T \text{ or } T \vdash \neg \text{Con}_T, \]

i.e. consistency decidability for set theory T.

First assertion of Gödel’s 2nd incompleteness theorem says: 
$T \nvDash \text{Con}_T$, if T consistent, 
whence we get 2nd alternative above:

\[ T \vdash \neg \text{Con}_T : \]

set theory T derives/proves its own inconsistency (formula).

Proof of first assertion of 2nd incompleteness theorem in Smorynski 1977, adapted to categorical language in next section.

This concerns set theories as PM, ZF, and NGB as well as “already” Peano arithmetic PA.

6.3 Gödel’s incompleteness theorems

We visit §2. Gödel’s theorems, in Smorynski 1977.

First incompleteness theorem. Let T be a formal theory containing arithmetic. Then there is a sentence $\varphi$ which asserts its own unprovability and such that:
(i) If \( T \) is consistent, \( T \not\models \varphi \).

(ii) If \( T \) is \( \omega \)-consistent, \( T \not\models \neg \varphi \).

In §3.2.6 Smorynski discusses possible choices of arithmetic (theory) \( S \), namely

(a) \( \text{PRA} \) = (classical, free-variables) primitive recursive arithmetic, 
    S. Feferman: “my \( \text{PRA} \)”, in contrast to \( \text{PRa} \) above.

(b) \( \text{PA} \) = Peano’s arithmetic.

Conjecture: \( \text{PA} \cong \text{PR} \cap \text{PRa} \).

(c) \( \text{ZF} \) = Zermelo-Fraenkel set theory. “This is both a good and a 
    bad example. It is bad because the whole encoding problem is 
    more easily solved in a set theory than in an arithmetical theory. 
    By the same token, it is a good example.”

Conjecture: \( \text{PRA} \) can categorically be viewed as cartesian theory with weak NNO in Lambek’s sense.

We take \( S := \text{PRA} \), embedding extension of categorical theory \( \text{PR} \), formally stronger than \( \text{PRA} \) because of uniqueness of maps 
defined by the full schema of primitive recursion, and weaker than 
\( \text{PA} \cong \text{PR} \).

By construction of arithmetic \( \text{PRa} \), “one can adequately encode syntax in this \( S = \text{PRa} \),” since Smorynski’s conditions (i)-(iii) for the 
representation of p. r. functions are fulfilled.

We take for formal extension \( T \) of \( S \) one of the categorical pendants to suitable set theories (subsystems of \( \text{ZF} \), see Osius 1974), or
the (first order) elementary theory of two-valued Topoi with NNO, cf. Freyd 1972, or, minimal choice, \( T := \text{PRa} \sqsupset \text{PA} \).

**Derivability theorem:** Our \( S \) encoding, extended from \( \text{PRa} \) to \( T \), meets the following (quantifier free categorically expressed) **Derivability Conditions** in §2.1 of Smorynski:

\[
\begin{align*}
\text{D1} & \quad T \vdash \varphi \quad \text{infers} \quad S \vdash \text{Prov}_T(\text{num}(k), \text{⌜\varphi⌝}). \\
\text{D2} & \quad S \vdash \text{Prov}_T(k, \text{⌜\varphi⌝}) \implies \text{Prov}_T(j_2(k), \text{Prov}_T(k, \text{⌜\varphi⌝})), \\
& \quad j_2 = j_2(k) : \mathbb{N} \to \mathbb{N} \text{ suitable.} \\
\text{D3} & \quad S \vdash \text{Prov}_T(k, \text{⌜\varphi⌝}) \land \text{Prov}_T(k', \text{⌜\varphi \implies \psi⌝}) \\
& \quad \implies \text{Prov}_T(j_3(k, k'), \text{⌜\psi⌝}), \\
& \quad j_3 = j_3(k, k') : \mathbb{N}^2 \to \mathbb{N} \text{ suitable.}
\end{align*}
\]

Smorynski’s **proof** gives the **First Gödel’s incompleteness theorem**, and from that the

**Second incompleteness theorem:** Let \( T \) be one of the extensions above of \( \text{PR} \exists \), and \( T \) consistent. Then

\[
T \not\vdash \text{Con}_T,
\]

where \( \text{Con}_T = \forall k \neg \text{Prov}_T(k, \text{⌜false⌝}) \) is the sentence asserting the consistency of \( T \).

From this Gödel’s theorem and our **PR Decidability theorem** for quantified arithmetic \( \text{PRa} \exists, T \) we get

**Inconsistency provability theorem** for quantified arithmetical (set) theories \( T \):

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If $T$ is consistent, then

$$T \vdash \neg \text{Con}_T.$$ 

[If not, then it derives everything, in particular $\neg \text{Con}_T$. We will see that p.r. arithmetic, under a mild termination condition for external evaluation, yields inconsistency of $T$.]

### 7 Consistency Decision within $\pi R$

#### 7.1 Termination conditioned evaluation soundness

**ES**\(^9\) Theorem on termination-conditioned soundness:

For p.r. theory $\text{PRXa}^{10}$ and internal notion of equality $\doteq = \doteq_k : \mathbb{N} \to \text{PRX} \times \text{PRX}$, $\text{dtree}_k$ the $k$th deduction tree of universe theory $\text{PRX} \subset \text{PR}(\mathbb{N}, \mathbb{N})$, we have:

(i) **Termination-Conditioned Inner soundness:**

With $r = r(u, x) = x : \text{PRX} \times \text{X} \to \text{X}$ right projection:

$$\text{PRXa} \vdash \langle u \doteq_k v \rangle \doteq \text{root} (\text{dtree}_k)$$

$$\wedge \ m \ \text{deff} \ ev_d (\text{dtree}_k / x)$$

$$\implies ev(u, x) \doteq ev(v, x). \quad (\bullet)$$

---

\(^9\)Evaluation soundness

\(^{10}\)presumably not directly for $\pi R$ with respect to its own internal equality, without assumption of “$\pi$-consistency,” in this regard RCF 2 contains an error
explicitly:

\[ \text{PR} a \vdash u \equiv_k v \land c_d e_d^m (\text{dtree}_k/x) \vdash 0 \]

\[ \Rightarrow ev(u,x) \equiv e^m(u,x) \equiv e^m(v,x) \]

\[ \equiv ev(v,x), \quad (\bullet) \]

free map-code variables \( u, v \), variable \( x \) free in universal set \( \mathbb{X} \).

[ Argumentation \( \text{dtree}_k/x \) of \( \text{dtree}_k \) and definition of argumented tree evaluation \( ev_d \) based on its evaluation step \( e_d \) and complexity \( c_d \) is by merged recursion on depth(\( \text{dtree}_k \)), within proof below]

In words, this “\( m \)-Truncated”, “\( m \)-Dominated” Inner soundness says that theory \( \text{PR} a \) derives:

If for an internal \( \text{PR} X \) equation \( u \equiv_k v \) argumented deduction tree \( \text{dtree}_k/x \) for \( u \equiv_k v \), argumented with \( x \in \mathbb{X} \), admits complete argumented-tree evaluation, i.e.

if tree-evaluation becomes completed after a finite number \( m \) of evaluation steps,

then both sides of this internal (!) equation are completely evaluated on \( x \) by (at most) \( m \) steps \( e \) of basic evaluation \( ev \), into equal values.

Substituting in the above “concrete” codes into \( u \) resp. \( v \), we get, by objectivity of evaluation \( ev \), formally “mutatis mutandis”:

(ii) Termination-Conditioned Objective soundness for Map Equality:
For PR\(\alpha\) maps \(f, g : A \rightarrow B:\)

\[
\text{PR}\alpha \vdash [ \Gamma \vdash_k \Gamma \vdash \land m \text{def} ev_d(\text{dtree}_k/a)] \\
\implies f(a) \vdash_B r e^m (\Gamma \vdash, a) \vdash_B g(a), a \in A \text{ free}:
\]

If an internal PR deduction-tree for (internal) equality of \(\Gamma f\) and \(\Gamma g\) is available, and if on this tree—top down argumented with \(a\) in \(A\)—tree evaluation terminates, then equality \(f(a) \vdash_B g(a)\) of \(f\) and \(g\) at this argument is the consequence.

(iii) Specialising this to case of \(f := \chi : A \rightarrow 2\) a p.r. predicate and to \(g := \text{true}_A : A \rightarrow 2\) we eventually get

Termination-Conditioned Objective Logical soundness:

\[
\text{PR}\alpha \vdash \text{Prov}_{\text{PR}\alpha}(k, \Gamma \chi) \land m \text{def} ev_d(\text{dtree}_k/a) \implies \chi(a):
\]

If tree-evaluation of an internal deduction tree for a free variable p.r. predicate \(\chi : A \rightarrow 2\) — the tree argumented with \(a \in A\) — terminates after a finite number \(m\) of evaluation steps, then \(\chi(a) \vdash \text{true}\) is the consequence, within \(\text{PR}\alpha\) as well as within its extensions \(\pi R\) — and set theory \(T\).

Remark to proof below: in present case of frame theory \(\text{PR}\alpha\) (and stronger theory \(\pi R\)) we have to control all evaluation step iterations, and we do that by control of iterative evaluation \(ev_d\) of whole argumented deduction trees, whose recursive definition will be—merged—part of this proof.

Proof of—basic—termination-conditioned inner soundness, i.e. of implication \((\bullet)\) in \(ES\) theorem is by induction on deduction tree
counting index $k \in \mathbb{N}$ counting family $\text{dtree}_k : \mathbb{N} \rightarrow \text{Bintree}$, starting with (flat) $\text{dtree}_0 = \langle \text{id} \rangle \xrightarrow{0} \langle \text{id} \rangle$. $m \in \mathbb{N}$ is to dominate argumented-deduction-tree evaluation $ev_d$ to be recursively defined below: condition

$$m \text{ deff } ev_d(\text{dtree}_k/x), \text{ step } e_d, \text{ complexity } c_d.$$ 

We argue by recursive case distinction on the form of the top up-to-two layers—top (implicational) deduction—$\text{dedu}_k/x$ of argumented deduction tree $\text{dtree}_k/x$ at hand.

**Flat super case** $\text{depth}(\text{dtree}_k) = 0$, i.e. super case of unconditioned, axiomatic (internal) equation $u \xrightarrow{k} v$:

The first involved of these cases is associativity of (internal) composition:

$$\text{dtree}_k = \langle \langle w \odot v \rangle \odot u \rangle \equiv_k \langle w \odot \langle v \odot u \rangle \rangle$$

In this case—no need of a recursion on $k$—

$$\text{PRXa} \vdash m \text{ deff } ev_d(\text{dtree}_k/x) \implies [m \text{ deff } ev(\langle w \odot v \rangle \odot u, x)]$$

$$\wedge [m \text{ deff } ev(\langle w \odot v \rangle, ev(u, x))]$$

$$\wedge [m \text{ deff } ev(w, ev(v, ev(u, x)))]$$

$$\wedge [m \text{ deff } ev(w, ev(\langle v \odot u \rangle, x))]$$

$$\wedge [m \text{ deff } ev(\langle w \odot \langle v \odot u \rangle \rangle, x)] \wedge \langle \langle w \odot v \rangle \odot u, x \rangle \equiv ev(\langle w \odot v \rangle, ev(u, x))$$

$$\equiv ev(w, ev(v, ev(u, x)))$$

$$\equiv ev(w, ev(\langle v \odot u \rangle, x)) \equiv ev(w \odot \langle v \odot u \rangle, x).$$

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This proves assertion (●) in present *associativity-of-composition* case. [New in comparison to previous *Inconsistency* chapter is here only the "preamble" *m deff* etc.]

Analogous **proof** for the other **flat**, equational cases, namely *reflexivity of equality, left and right neutrality* of $\text{id} = \text{by def} \; \text{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement’s equations for the induced map as well as surjective pairing and distributivity of composition over forming the induced map.

Godement’s equations $\ell \circ (f, g) = f$, $r \circ (f, g) = g$:

\[
\begin{align*}
\text{m deff ev etc. } \Rightarrow & \\
\text{ev}(\langle \ell \downarrow \circ \langle u; v \rangle, z \rangle) & \vdash r \text{e}^m(\langle \ell \downarrow \circ \langle u; v \rangle, z \rangle) \\
& \vdash \ell(\langle \text{ev}(u, z); \text{ev}(v, z) \rangle) \vdash \text{ev}(u, z),
\end{align*}
\]

analogously for composition with right projection.

Fourman’s equation $(\ell \circ h, r \circ h) = h$:

\[
\begin{align*}
\text{m deff ev etc. } \Rightarrow & \\
\text{ev}(\langle \ell \downarrow \circ w; \ell \downarrow \circ w \rangle, z) & \vdash \ell(\langle \text{ev}(w, z); \text{ev}(w, z) \rangle) \\
& \vdash \ell(\text{ev}(w, z)); \ell(\text{ev}(w, z)) \vdash \text{ev}(w, z)
\end{align*}
\]

by SP equation on objective level.

Now here are the **proofs**—with preambles—of (●), for the last equational case, the

**Iteration step**, case of genuine iteration equation
\[
dtree_k = \langle u^s \circ \langle \Gamma \text{id} \# \Gamma s \rangle \vdash_k u \circ u^s \rangle:
\]

\[
\text{PRa} \vdash m \text{ deff ev}_d(\text{dtree}_k/(y; \nu(n))) \implies
\]
\[
m \text{ deff all instances of ev below, and:}
\]
\[
ev (u^s \circ \langle \Gamma \text{id} \# \Gamma s \rangle, \langle y; \nu(n) \rangle)
\]
\[
\vdash ev (u^s, ev(\langle \Gamma \text{id} \# \Gamma s \rangle, \langle y; \nu(n) \rangle))
\]
\[
\vdash ev (u^s, \langle y; \nu(s n) \rangle)
\]
\[
\vdash ev (u^{[s n]}, y) \quad \text{(by definition of ev step e)}
\]
\[
\vdash ev (u \circ u^{[n]}, y)
\]
\[
\vdash ev (u, ev(u^s; \langle y; \nu(n) \rangle))
\]
\[
\vdash ev (u \circ u^s; \langle y; \nu(n) \rangle).
\]

**Proof** of termination-conditioned inner soundness for the remaining *deep*—genuine HORN cases— for \(\text{dtree}_k\), HORN type (at least) at deduction of root:

**Transitivity-of-equality** case: with map code variables \(u, v, w\) we start here with argument-free deduction tree

\[
dtree_k = \begin{array}{c}
    u \vdash_k w \\
    \hline
    u \vdash_i v & v \vdash_j w \\
    \hline
    \text{dtree}_{ii} & \text{dtree}_{ji} & \text{dtree}_{ij} & \text{dtree}_{jj}
\end{array}
\]

It is argumented with argument \(x\) say, recursively spread down:
\[ \text{dtree}_k/x = \frac{u/x \quad w/x}{u/x \quad v/x \quad v/x \quad w/x} \]

\[ \text{dtree}_{ii}/x_{ii} \quad \text{dtree}_{ji}/x_{ji} \quad \text{dtree}_{ij}/x_{ij} \quad \text{dtree}_{jj}/x_{jj} \]

Spreading down arguments from upper level down to 2nd level must/is given explicitly, further arguments spread down is then recursive by the type of deduction (sub)trees \( \text{dtree}_i, \text{dtree}_j, i, j < k \).

Now by induction hypothesis on \( i, j \) we have for tree evaluation \( ev_d \):

\[
\begin{align*}
 u &\equiv_k w \land m \ \text{deff} \ ev_d(\text{dtree}_k/x) \\
 &\implies m \ \text{deff} \ ev_d(\text{dtree}_i/x), ev_d(\text{dtree}_j/x) \land \\
 &ev_d(\text{dtree}_i/x) \equiv \langle \text{id} \downarrow / ev(u, x) \equiv \text{id} \downarrow / ev(v, x) \rangle \\
 &\land ev_d(\text{dtree}_j/x) \equiv \langle \text{id} \downarrow / ev(v, x) \equiv \text{id} \downarrow / ev(w, x) \rangle \\
 &\implies ev(u, x) \equiv ev(v, x) \land ev(v, x) \equiv ev(w, x) \\
 &\implies ev(u, x) \equiv ev(w, x).
\end{align*}
\]

and this is what we wanted to show in present transitivity of equality case.

[Transitivity axiom for equality is a main reason for necessity to consider (argumented) deduction trees: intermediate map code equalities ‘\( \equiv \)’ in a transitivity chain must be each evaluated, and pertaining deduction trees may be of arbitrary high evaluation complexity]

Case of symmetry axiom scheme for equality is now obvious.
Compatibility Case of composition with equality

\[
\text{foreach } k \in \mathbb{N} \text{ do }
\]
\[
dtree_k/x = \langle v \odot u \rangle/x \equiv_k \langle v \odot u' \rangle/x
\]
\[
\text{foreach } j \in \mathbb{N} \text{ do }
\]
\[
u/x \equiv_j u'/x
\]
\[
dtree_{ij}/x \quad dtree_{jj}/x
\]

By induction hypothesis on \( j < k \)

\[
m \text{ deff } ev_d(dtree_k/x) \implies
\]
\[
m \text{ deff } ev_d(dtree_j/x) \implies
\]
\[
ev(u, x) \equiv ev(u', x) \implies
\]
\[
ev(v \odot u, x) \equiv ev(v, ev(u, x)) \equiv ev(v, ev(u', x)) \equiv ev(v \odot u', x)
\]

by dominated characteristic equations for \( ev \) and Leibniz’ substitutivity, q.e.d. in this 1st compatibility case.

Spread down arguments is more involved in

Case of composition with equality in second composition factor: argument spread down merged with tree evaluation \( ev_d \) and proof of result.

\footnote{this simplified version has been suggested by Joseph}
\[
\begin{align*}
\text{dtree}_k / x &= \langle v \odot u \rangle / x \quad \langle v' \odot u \rangle / x \\
&= v \overset{=}{\sim} v' \\
&= \text{dtree}_{ii} \quad \text{dtree}_{ji}
\end{align*}
\]

[Here \( \text{dtree}_i \) is not (yet) provided with argument, it is argumented during top down tree evaluation below]

\[
m \text{ deff } \text{ev}_d(\text{dtree}_k / x) \implies \\
m \text{ deff } \text{all instances of } \text{ev} \text{ below, and:} \\
\text{ev}(\langle v \odot u \rangle, x) \overset{=}{\sim} \text{ev}(v, \text{ev}(u, x)) \overset{=}{\sim} \text{ev}(v', \text{ev}(u, x)) \quad (*)
\]
\[
\overset{=}{\sim} \text{ev}(\langle v' \odot u \rangle, x).
\]

(\*) holds by Leibniz’ substitutivity and

\[
m \text{ deff } \text{ev}_d(\text{dtree}_k / x) \implies \\
m \text{ deff } \text{ev}_d(\text{dtree}_i / \text{ev}(u, x))
\]

[ argumentation of \text{dtree}_i \text{ with } \\
\text{ev}(u, x)—calculated \text{ en cours de route,} \\
\text{extra definition of } e_d ]

\[
\implies \\
m \text{ deff } \text{ev}(v, \text{ev}(u, x)) \overset{=}{\sim} \text{ev}(v', \text{ev}(u, x)),
\]

by induction hypothesis on \( i < k \): The hypothesis is independent of substituted argument, provided—and this is here the case—that \text{dtree}_i is evaluated on that argument, in \( m' < m \) steps, \( m' \) suitable (minimal).
This proves assertion (●) in this 2nd compatibility case.

(Redundant) case of compatibility of forming the induced map with map equality is analogous to compatibilities above, even easier, because of almost independence of any two inducing map codes from each other.

(Final) case of Freyd’s (internal) uniqueness of the initialised iterated, is case

\[
\text{dedu}_k / \langle y; \nu(n) \rangle = \frac{w / \langle y; \nu(n) \rangle \cong_k \langle v^\$ \circ \langle u^\# \circ \langle \text{id} \rangle \rangle / \langle y; \nu(n) \rangle}{\text{root} (t_i) \quad \text{root} (t_j)}
\]

where

\[
\text{root} (t_i) = \langle w \circ \langle \text{id} \rangle ; \text{0} ; \text{1} \rangle / \langle y \rangle \cong_i u / y,
\]

\[
\text{root} (t_j) = \langle w \circ \langle \text{id} \rangle \# \text{0} \rangle / \langle y; \nu(n) \rangle \cong_j \langle v \circ w \rangle / \langle y; \nu(n) \rangle
\]

Comment: \( w \) is here an internal comparison candidate fullfilling the same internal PR equations as \( \langle v^\$ \circ \langle u^\# \circ \langle \text{id} \rangle \rangle \)\. It should be—is: soundness—evaluated equal to the latter, on \( \langle X \times \nu \mathbb{N} \rangle \subset \mathbb{X} \).

soundness assertion (●) for the present Freyd’s uniqueness case recurs on \( \cong_i, \cong_j \) turned into predicative equations ‘\( \cong \)’, these being already deduced, by hypothesis on \( i, j < k \). Further ingredients are transitivity of ‘\( \cong \)’ and established properties of basic evaluation \( ev \) of map terms.
So here is the remaining—inductive—\textbf{proof}, prepared by

\[
\mathbf{T} \vdash m \text{ deff } \text{dtree}_k / \langle y; \nu(n) \rangle \implies m \text{ deff all of the following ev-terms and } \\
ev(w, \langle y; \nu(0) \rangle) \doteq ev(u; y) \\
\text{as well as } \\
m \text{ deff both of the following ev-terms, and } \\
ev(w, \langle y; \nu(s\, n) \rangle) \doteq ev(w, \langle y; \overline{s} \circ \nu(n) \rangle) \\
\doteq ev(w \odot \langle \overline{s} \odot {\nu(n) \rangle,} \\
\doteq ev(w \odot \langle \overline{s} \odot \nu(n) \rangle), \\
\text{the same being true for } w' : = v^s \odot \langle u \# \nu(s) \rangle \text{ in place of } w, \text{ once more by (characteristic) double recursive equations for ev, this time with respect to the \textit{initialised internal iterated} itself. }
\]

(\bar{0}) \text{ and } (\bar{s}) \text{ put together for both then show, by induction on iteration count } n \in \mathbb{N}—\text{all other free variables } k, u, v, w, y \text{ together form the passive parameter for this induction—truncated soundness assertion } (\bullet) \text{ for this Freyd’s uniqueness case, namely }

\[
\mathbf{T} \vdash m \text{ deff } \text{dtree}_k / \langle y; \nu(n) \rangle \implies m \text{ deff all of the ev-terms concerned above, and } \\
ev(w, \langle y; \nu(n) \rangle) \doteq ev(v^s \odot \langle u \# \nu(s) \rangle, \langle y; \nu(n) \rangle). \\
\textbf{Induction} \text{ runs as follows: }
\]

\textbf{Anchor } n = 0 : \\
ev(w, \langle y; \nu(0) \rangle) \doteq ev(u, y) \doteq ev\left(w', \langle y; \nu(0) \rangle\right),
Step: \( m \text{ def } \) etc. \( \Rightarrow \)

\[
ev(w, \langle y; \nu(n) \rangle) \equiv ev(w', \langle y; \nu(n) \rangle) \quad \Rightarrow \\
ev(w, \langle y; \nu(s n) \rangle) \equiv ev(v, ev(w, \langle y; \nu(n) \rangle)) \\
\equiv ev(v, ev(w', \langle y; \nu(n) \rangle)) \equiv ev(w', \langle y; \nu(s n) \rangle),
\]

the latter since evaluation \( \ev \) preserves predicative equality ‘\( \equiv \)’ (Leibniz) \textbf{q.e.d.} Termination Conditioned PR soundness theorem.

Comment: Already for stating the evaluations, we needed the—categorical, free-variables theories \( \text{PR, PRa, PRX, PRXa} \) of primitive recursion, as well as—for termination, even in classical frame \( T \)—PR complexities within \( \mathbb{N}[\omega] \). Since this type of soundness is a cornerstone in our approach, the above complicated categorical combinatorics seem to be necessary, even for the negative results on classical Foundations.

### 7.2 Framed consistency

From termination-conditioned soundness—resp. from \( T \)-framed PR soundness—we get

\textbf{\( \pi \text{R-framed internal PR consistency corollary:} \)} For descent theory \( \pi \text{R} = \text{PRXa} + (\pi) \), axiom (\( \pi \)) stating non-infinite iterative descent in ordinal \( \mathbb{N}[\omega] \), we have

\( \pi \text{R} \vdash \text{Con}_{\text{PRX}}, \) i.e. “necessarily” in free-variables form:

\( \pi \text{R} \vdash \neg \text{Prov}_{\text{PRX}}(k, \lceil \text{false} \rceil) : \mathbb{N} \to 2, \ k \in \mathbb{N} \) free,

\( T \vdash \text{Con}_{\text{PRX}} : \)

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theory $\pi R$—as well as set theories $T$ as an extension of $\pi R$—derive that no $k \in \mathbb{N}$ is the internal $\text{PRX}$-Proof for \textit{false}.

\textbf{Proof} for this corollary from termination-conditioned soundness: By assertion (iii) of that theorem, with $\chi = \chi(a) := \text{false}(a) = \text{false} : 1 \to 2$, we get:

\textit{Evaluation-effective internal inconsistency} of $\text{PRX}$—i.e. availability of an \textit{evaluation-terminating} internal deduction tree of \textit{false}—implies \textit{false}:

$$\text{PRX} a, \pi R \vdash \text{Prov}_{\text{PRX}}(k, \text{false}^-) \land c_d e_d^m(\text{dtree}_k/\langle 0 \rangle) \not\equiv 0 \implies \text{false}.$$  

Contraposition to this, still with $k, m \in \mathbb{N}$ free:

$$\pi R \vdash \text{true} \implies \neg \text{Prov}_{\text{PRX}}(k, \text{false}^-) \lor c_d e_d^m(\text{dtree}_k/\langle 0 \rangle) > 0,$$

i.e. by free-variables (boolean) tautology:

$$\pi R \vdash \text{Prov}_{\text{PRX}}(k, \text{false}^-) \implies c_d e_d^m(\text{dtree}_k/\langle 0 \rangle) > 0 : \mathbb{N}^2 \to 2.$$  

For $k$ “fixed”, the conclusion of this implication—$m$ free—means infinite descent in $\mathbb{N}[\omega]$ of iterative argumented deduction-tree evaluation $\text{ev}_d$ on $\text{dtree}_k/0$, which is excluded intuitively. Formally it is excluded within our theory $\pi R$ taken as frame:

We apply non-infinite-descent scheme ($\pi$) to $\text{ev}_d$, which is given by \textit{step} $e_d$ and complexity $c_d$—the latter descends (this is \textit{argumented-tree evaluation descent}) with each application of $e_d$, as long as complexity $0 \in \mathbb{N}[\omega]$ is not (“yet”) reached. We combine this with—choice of—overall “negative” condition

$$\psi = \psi(k) : = \text{Prov}_{\text{PRX}}(k, \text{false}^-) : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free}$$
and get—by that scheme (π)—overall negation of this (overall) excluded predicate ψ, namely

\[ \pi R \vdash \neg \text{Prov}_{PRX}(k, \neg \text{false}) : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free, i.e.} \]

\[ \pi R \vdash \text{Con}_{PRX} \quad \text{q.e.d.} \]

So “slightly” strengthened theory \( \pi R = PRXa + (\pi) \) derives free variables Consistency Formula for theory PRX of primitive recursion.

Scheme (\( \pi \)) holds in set theory, since there \( O := \mathbb{N}[\omega] \) is an ordinal, not quite to identify with set theoretical ordinal \( \omega^\omega \), because classical ordinal addition on that ordinal \( \omega^\omega \) does not commute, e.g. classically \( \omega + 1 \neq 1 + \omega = \omega \). As linear orders (with non-infinite descent) the two are identical.

As is well known, consistency provability and soundness of a theory are strongly tied together. We get in fact even

**Theorem on \( \pi R \)-framed objective soundness of theory \( PRXa \):**

- for a \( PRXa \) predicate \( \chi = \chi(a) : A \to 2 \) we have

  \[ \pi R \vdash \text{Prov}_{PRX}(k, \neg \chi) \implies \chi(a) : \mathbb{N} \times A \to 2. \]

- more general, for \( PRXa \)-maps \( f, g : A \to B \) we have

  \[ \pi R \vdash \neg f \sim_k \neg g \implies f(a) \equiv g(a). \]

[Same for set theory \( T \) taken as frame]

**Proof** of first assertion is a slight generalisation of proof of framed Internal Consistency above as follows—take predicate \( \chi \) instead of false:

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Use termination-conditioned soundness, assertion (iii) directly:

Evaluation-effective internal provability of $\Gamma \chi$ within PRXa—i.e. availability of an evaluation-terminating internal deduction tree of $\Gamma \chi$—implies $\chi(a), a \in A$ free:

$$\text{PRXa, } \pi R \vdash \text{Prov}_{\text{PRX}}(k, \Gamma \chi) \land c_d e_d^m(\text{d}t\text{ree}_k/\langle 0 \rangle) \leq 0 \implies \chi(a) : \mathbb{N}^2 \times A \to 2.$$ 

Boolean free-variables calculus, tautology

$$[\alpha \land \beta \Rightarrow \gamma] = [\neg [\alpha \Rightarrow \gamma] \Rightarrow \neg \beta]$$

(test with $\beta = 0$ as well as with $\beta = 1$),

gives from this, still with $k, m, a$ free:

$$\pi R \vdash \neg [\text{Prov}_{\text{PRX}}(k, \Gamma \chi) \Rightarrow \chi(a)] \implies c_d e_d^m(\text{d}t\text{ree}_k/\langle 0 \rangle) > 0 : (A \times \mathbb{N}) \times \mathbb{N} \to 2.$$ 

As before, we apply non-infinit scheme ($\pi$) to $ev_d$, in combination with—choice of—overall “negative” condition

$$\psi = \psi(k, a) := \neg [\text{Prov}_{\text{PRX}}(k, \Gamma \chi) \Rightarrow \chi(a)] : \mathbb{N} \times A \to 2,$$

and get—scheme ($\pi$)—overall negation of this (overall) excluded predicate $\psi$, namely

$$\pi R \vdash \text{Prov}_{\text{PRX}}(k, \Gamma \chi) \implies \chi(a) : \mathbb{N} \times A \to 2.$$ 

q.e.d. for first assertion.

For proof of second assertion, take in the above

$$\chi = \chi(a) := [f(a) \triangleq g(a)] : A \to B^2 \to 2$$
and get
\[
\pi R \vdash \chi \frac{\forall f \forall g}{f \equiv_k g} \\
\Rightarrow \mathsf{Prov}_{\mathsf{PR}_\chi}(j(k), f \equiv g) \\
\text{(substitutivity into } \equiv) \\
\Rightarrow [f(a) \equiv g(a)]: \mathbb{N} \times A \to 2 \quad \text{q.e.d.}
\]

7.3 $\pi R$ decision

As the kernel of decision for p. r. predicate $\chi = \chi(a): A \to 2$ by theory $\pi R$ we introduce a (partially defined) $\mu$-recursive decision algorithm $\nabla \chi = \nabla^{\mathsf{PR}_\chi} : 1 \to 2$ for (individual) $\chi$. This decision algorithm is viewed as a map of theory $\pi \hat{R}$, of partial $\pi R$ maps.

As a partial p. r. map it is given—see chapter 2—by three (PR) data:

- its index domain $D = D_{\nabla \chi}$, typically (and here): $D \subseteq \mathbb{N}$,
- its enumeration $d = d_{\nabla \chi}: D \to 1$ of its defined arguments, as well as
- its rule $\hat{\nabla} = \hat{\nabla}_\chi: D \to 2$ mapping indices $k, k'$ in $D$ pointing to the same argument $d(k) \equiv d(k')$ in domain $1$, to the same value $\hat{\nabla}(k) \equiv \hat{\nabla}(k')$.

Now define alleged decision algorithm by fixing its graph
\[
\nabla \chi = \langle (d, \hat{\nabla}): D \to 1 \times 2 \rangle : 1 \to 2
\]

as follows:
Enumeration domain for defined arguments is to be

\[ D = D_{\nabla \chi} = \{ k \mid \neg \chi \text{ct}_A(k) \lor \text{Prov}_{\text{PR}_X}(k, \upuparrows \chi) \} \subset \mathbb{N}, \]

with \( \text{ct}_A : \mathbb{N} \to A \) (retractive) Cantor count, \( A \) assumed pointed. Defined arguments enumeration is here “simply”

\[ d = \text{def} \Pi : D \subseteq \mathbb{N} \to 1 \]

—not a priori a retraction or empty—, and rule is taken

\[ \hat{\nabla}(k) = \hat{\nabla}_X(k) = \text{def} \begin{cases} \text{false if } \neg \chi \text{ct}_A(k), \\ \text{true if } \text{Prov}_{\text{PR}_X}(k, \upuparrows \chi), \\ \text{undefined otherwise.} \end{cases} : D \to 2. \]

\( \hat{\nabla} : D \to 2 \) is in fact a well defined rule for enumeration \( d : D \to \mathbb{N} \to 1 \) of defined argument(s) since by (earlier) framed logical soundness theorem

\[ \pi_R \vdash \text{Prov}_{\text{PR}_X}(k, \upuparrows \chi) \implies \chi(a) : \mathbb{N} \times A \to 2, \]

whence disjointness of the alternative within \( D = D_{\nabla \chi} \).

This taken together means intuitively within \( \pi_R \)—and formally within set theory \( T \):

\[ \nabla(k) = \nabla_X(k) = \begin{cases} \text{false if } \neg \chi \text{ct}_A(k), \\ \text{true if } \text{Prov}_{\text{PR}_X}(k, \upuparrows \chi), \\ \text{undefined otherwise.} \end{cases} \]

We have the following complete—metamathematical—case distinction on \( D \subset \mathbb{N} \):
• 1st case, termination: $D$ has at least one ("total") PR point $1 \to D \subseteq \mathbb{N}$, and hence

$$t = t_{\nabla \chi} = \text{by def } \mu D = \min D : 1 \to D$$

is a (total) p.r. point.

Subcases:

- 1.1st, negative (total) subcase:
  $$\neg \chi \text{ct}_A(t) = \text{true}.$$  
  [Then $\pi R \vdash \nabla \chi = \text{false}.]

- 1.2nd, positive (total) subcase:
  $$\text{Prov}_{PRX}(t, \neg \chi) = \text{true}.$$  
  [Then $\pi R \vdash \nabla \chi = \text{true},$
  by $\pi R$-framed objective soundness of $PRX$.]

These two subcases are disjoint, disjoint here by $\pi R$ framed soundness of theory $PRX$ which reads

$$\pi R \vdash \text{Prov}_{PRX}(k, \neg \chi) \Rightarrow \chi(a) : \mathbb{N} \times A \to 2, \ k \in \mathbb{N} \text{ free, and } a \in A \text{ free},$$

here in particular—substitute $t : 1 \to \mathbb{N}$ into $k$ free:

$$\pi R \vdash \text{Prov}_{PRX}(t, \neg \chi) \Rightarrow \chi(a) : A \to 2, \ a \text{ free}.$$

So furthermore, by this framed soundness, in present subcase:

$$\pi R \vdash \chi(a) \land \text{Prov}_{PRX}(t, \neg \chi) : A \to 2.$$

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• 2nd case, derived non-termination:
  \( \pi R \vdash D = \emptyset_N \equiv \{ N \mid false_N \} \subset N \)
  [then in particular \( \pi R \vdash \neg \chi = false_A : A \to 2 \),
  so \( \pi R \vdash \chi \) in this case],
  and
  \( \pi R \vdash \neg \text{Prov}_{PR}(k, \Gamma^\chi) : N \to 2, k \text{ free}; \)

• 3rd, remaining, ill case is:
  \( D \) (metamathematically) has no (total) points \( 1 \to D \), but is nevertheless not empty.

Take in the above the (disjoint) union of 2nd subcase of 1st case and of 2nd case, last assertion. And formalise last, remaining case frame \( \pi R \). Arrive at the following

Quasi-Decidability Theorem: p.r. predicates \( \chi : A \to 2 \) give rise within theory \( \pi R \) to the following complete (metamathematical) case distinction:

(a) \( \pi R \vdash \chi : A \to 2 \) or else

(b) \( \pi R \vdash \neg \chi \text{ ct}_A t : 1 \to D_{\neg \chi} \to 2 \)
   (defined counterexample), or else

(c) \( D = D_{\neg \chi} \) non-empty, pointless, formally: in this case we would have within \( \pi R \):

\[
[D \circ \mu D \equiv \text{true} : 1 \to N \to 2]
\]
  and “nevertheless” for each p.r. point \( p : 1 \to N \)
  \( \neg D \circ p = \text{true} : 1 \to N \to 2. \)
We rule out the latter—general—possibility of a non-empty, pointless predicate, for quantified arithmetical frame theory $T$ by gödelian assumption of $\omega$-consistency which rules out above instance of $\omega$-inconsistency.

For frame $\pi R$ we rule it out by (corresponding) metamathematical assumption of “$\mu$-consistency,” as follows:

**Intermission on two variants of $\omega$-consistency:**

Gödelian assumption of $\omega$-consistency—non-$\omega$-inconsistency—for a quantified arithmetical theory $T$ reads:

For no p. r. predicate $\varphi : \mathbb{N} \rightarrow 2$

\[
T \vdash (\exists n \in \mathbb{N}) \varphi(n)
\]

and (nevertheless)

\[
T \vdash \neg \varphi(0), \neg \varphi(1), \neg \varphi(2), \ldots
\]

Adaptation to (categorical) recursive theory $\pi R$ is the following assumption of $\mu$-consistency, non-$\mu$-inconsistency for $\pi R$:

For no p. r. predicate $\varphi : \mathbb{N} \rightarrow 2$

\[
\pi R \vdash \varphi(\mu \varphi) \overset{\text{by def}}{=} \varphi \circ \mu \varphi \overset{\text{true}}{=} \text{true} : \mathbb{N} \rightarrow 2
\]

and

\[
\pi R \vdash \neg \varphi(0), \neg \varphi(1), \ldots, \neg \varphi(\text{num}(n)), \ldots
\]

For quantified $T$ first line reads: $T \vdash \exists n \varphi(n)$, and hence $\mu$-consistency is equivalent to gödelian $\omega$-consistency for such $T$.

**Alternative to $\mu$-consistency:** $\pi$-consistency.
By assertion (iii) of **Structure theorem** in chapter 2—*section lemma*—for theories $\tilde{S}$ of partial p.r. maps, first factor $\mu \varphi : \mathbb{1} \to \mathbb{N}$ of (total) p.r. map $\text{true} : \mathbb{1} \to \mathbb{2}$ above is necessarily itself a—*totally defined*—PR map: Intuitively, a first factor of a total map cannot have undefined arguments, since these would be undefined for the composition.

Now consider—here available—(external) point evaluation into numerals\(^{12}\), externalisation of objective evaluation

$$
\text{ev} : \left[ 1, \mathbb{N} \right] \xrightarrow{\sim} \left[ 1, \mathbb{N} \right] \times 1 \xrightarrow{\text{ev}} \mathbb{N} \xrightarrow{\sim} \nu \mathbb{N} \subseteq \left[ 1, \mathbb{N} \right]
$$

of point codes into (internal) numerals, $\text{ev}(u) \overset{\sim}{=} u \in \left[ 1, \mathbb{N} \right]$. This externalised evaluation $\text{ev}$ is **assumed**—meta-axiom of $\pi$-consistency—to (correctly) terminate:

$$
\pi \mathbb{R}(1, \mathbb{N}) \supset \text{num} \mathbb{N} \ni \text{ev}(p) =^{\pi} p \in \pi \mathbb{R}(1, \mathbb{N}).
$$

**Comment**: $\pi$-consistency means **Semantical Completeness** of descent axiom ($\pi$), this axiom is modeled into the external world of p.r. Metamathematic. But $\pi$-consistency is somewhat stronger: it assumes termination of $\text{ev}$ instead of non-infinite descent.

**Non-$\mu$-inconsistency** (of $\pi \mathbb{R}$) is then a consequence of $\pi$-consistency of theory $\pi \mathbb{R}$ above:

$$
\pi \mathbb{R} \vdash \text{true} = \varphi(\mu \varphi) = \varphi \circ \mu \varphi = \varphi \circ \mu \varphi : 1 \to \mathbb{N} \to 2
$$

entails $\pi \mathbb{R} \vdash \neg \left( \neg \varphi(\text{num}(n_0)) \right)$, with $\text{ev}(\mu \varphi) = \text{num}(n_0)$.

**End of Intermission.**

\(^{12}\text{Lassmann 1981}\)
First consequence: Theory $\pi R$ admits no non-empty predicative subset $\{n \in \mathbb{N} \mid \varphi(n)\} \subseteq \mathbb{N}$ such that for each numeral $\text{num}(n): 1 \to \mathbb{N}$

$$\pi R \vdash \neg \varphi \circ \text{num}(n) : 1 \to \mathbb{N} \to 2.$$ 

This rules out—in quasi-decidability above—possibility (c) for decision domain $D = D_{\chi} \subseteq \mathbb{N}$ of decision operator $\nabla_{\chi}$ for predicate $\chi : A \to 2$, and we get two unexpected results:

Decidability theorem: Each free-variable p.r. predicate $\chi : A \to 2$ gives rise to the following complete case distinction within, by $\pi R$:

- Under **assumption** of $\mu$-consistency or $\pi$-consistency for $\pi R$:
  - $\pi R \vdash \chi(a) : A \to 2$ (theorem) or
  - $\pi R \vdash \neg \chi \, \text{ct}_A \, \mu D : 1 \to D_{\chi} \to 2$
    (defined counterexample.)

- Under **assumption** of $\omega$-consistency for set theory $T$:
  - $T \vdash \chi(a) : A \to 2$ (theorem) or
  - $T \vdash \neg \chi \, \text{ct}_A \, \mu D : 1 \to D_{\chi} \to 2$, i.e.
    $T \vdash (\exists a \in A) \neg \chi(a)$.

Take here, in case of set theory $T$, for predicate $\chi$, $T$’s own free-variable consistency formula $\text{Con}_T = \neg \text{Prov}_T(\langle \text{false}\rangle) : \mathbb{N} \to 2$, and get, under **assumption** of $\omega$-consistency for $T$, consistency decidability for $T$. 

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This contradiction to (the postcedent) of Gödel’s 2nd Incompleteness theorem shows that the assumption of $\omega$-Consistency for set theories $T$ must fail.

Now take in the theorem for $\chi$ $\pi R$’s own free variable PR consistency formula

\[
\text{Con}_{\pi R} = \neg \text{Prov}_{\pi R}(k, \lceil \text{false} \rceil) : \mathbb{N} \rightarrow 2 \text{ and get}
\]

**Consistency Decidability** for descent theory $\pi R$:

- $\pi R \vdash \text{Con}_{\pi R} : 1 \rightarrow 2$ or else
- $\pi R \vdash \neg \text{Con}_{\pi R}$, will say

  $\pi R \vdash \text{Prov}_{\pi R}(\mu \text{Prov}_{\pi R}(k, \lceil \text{false} \rceil), \lceil \text{false} \rceil) = \text{true} \quad \text{q.e.d.}$

**Consistency provability theorem:** $\pi R \vdash \text{Con}_{\pi R}$, under assumption of $\pi$-consistency of theory $\pi R$.

**Proof:** Suppose we have 2nd alternative in consistency decidability above,

\[
\pi R \vdash \text{Prov}_{\pi R}(t, \lceil \text{false} \rceil),
\]

$t =_{\text{def}} \mu \text{Prov}_{\pi R}(k, \lceil \text{false} \rceil) : 1 \rightarrow \mathbb{N}$, necessarily (”total”) PR. Meta p.r. point evaluation $\text{ev}$ would turn—$\pi$-consistency—$t$ into a numeral $\text{num}(k_0) : 1 \rightarrow \mathbb{N}$, $k_0 \in \mathbb{N}$, $\text{num}(k_0) =^\pi t$, hence

\[
\pi R \vdash \text{Prov}_{\pi R}(\text{num}(k_0), \lceil \text{false} \rceil).
\]

But by derivation-into-proof internalisation we have

\[
\pi R \vdash \text{Prov}_{\pi R}(\text{num}(k), \lceil \chi \rceil) \text{ (only) iff } \pi R \vdash_{k_0} \chi, \text{ whence we would get inconsistency } \pi R \vdash_{k_0} \text{false, (and an inconsistent theory derives everything.)}
\]
This rules out in fact 2nd alternative in consistency decidability and so proves the theorem, here our main goal.

For proof of soundness of $\pi R$ below we need $\nu$-Lemma for theory $\pi R$:

(i) family $\nu_A : A \to [\mathbb{1}, A]_\pi = [\mathbb{1}, A] / \sim^\pi$ is a natural transformation, will say

$$(\nu_B \circ f)(a) = \nu_B(f(a))$$

\[ \overset{\sim^\pi}{\overset{k(a)}{\downarrow}} f \downarrow \circ \nu_A(a) \]

$= [\mathbb{1}, f]_\pi(\nu_A(a)),$

$k(a) : A \to \mathbb{N}$ suitable PR.

As a commuting diagram:

(ii) $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$ is injective, i.e.

$$\nu(m) \sim^\pi \nu(n) \implies m \doteq n.$$
(iii) same for all objects $A$ of $\pi R : \nu_A = \nu_A(a) : A \to [1,A]_\pi$ is injective.

**Proof:** We show assertion (i) by structural recursion on $f : A \to B$.

- anchor cases $f = \text{id}_A$ as well as $f = 0 : 1 \to \mathbb{N}$ are obvious.
- anchor case $f = s : \mathbb{N} \to \mathbb{N}$:
  - $\nu(s(a)) = \text{by def} \ s \downarrow \circ \nu(a) = [1,s] (\nu(a))$.

Map composition $g \circ f : A \to B \to C$ : combine the two commuting squares for $f$ and for $g$ into commuting rectangle for $g \circ f$.

- cartesian Structure: use

  - $\nu_{(A \times B)} = \text{by def} \ \text{ind} \circ (\nu_A \times \nu_B) : A \times B \to [1,A] \times [1,B] \xrightarrow{\cong} [1,A \times B] \to [1,A \times B]$, componentwise definition of (any) equality on cartesian product, as well as the universal properties of the cartesian product $A \times B$ and $[1,A \times B] \cong [1,A] \times [1,B]$, projections $[1,\ell],[1,r]$.

  - Iterated $f^\natural(a,n) : A \times \mathbb{N} \to A$ of (already tested) endo $f : A \to A$ :

    Straight forward by recursion on $n$, since iteration is repeated composition.
Assertion (ii) on injectivity of $\nu = \nu(n) : \mathbb{N} \to [1, \mathbb{N}]_{\pi}$:

$$\nu(m) \approx^{\pi} \nu(n) \implies \top \approx \top \circ (\nu(m) \times \nu(n)) \approx^{\pi} \top$$

by internal substitutivity into predicative equality $\approx$

$$\iff [1, \top] \circ (\nu \times \nu)(m, n) \approx^{\pi} \top$$

$$\implies \nu_2[m \approx n] \approx^{\pi} \nu_2(\top)$$

by naturality of transformation $\nu$

$$\implies m \approx n, \text{ by self-consistency (!) of theory } \pi \mathcal{R}.$$  

General $\nu$ injectivity assertion (iii) now follows from that special just above, from componentwise definition of $\nu$—and componentwise definition of injectivity—on cartesian products (and restriction of both to predicative subobjects), via naturality of transformation $[\nu_A : A \to [1, A]_{\pi}]_{A \in \pi \mathcal{R}}$ q.e.d.

This is to give self-consistency $\pi \mathcal{R} \vdash \text{Con}_{\pi \mathcal{R}}$ to be equivalent to

**Objective soundness theorem for descent theory $\pi \mathcal{R}$:**

- for $\pi \mathcal{R}$-maps $f, g : A \to B$:

  $$\pi \mathcal{R} \vdash [ \top f \vdash^k \top g \top ] \implies f(a) \vdash_B g(a) : \mathbb{N} \times A \to 2. $$

- this gives in particular logical soundness of theory $\pi \mathcal{R}$:

  For a predicate $\chi = \chi(a) : A \to 2$ we have

  $$\pi \mathcal{R} \vdash \text{Prov}_{\pi \mathcal{R}}(k, \top \chi \top) \implies \chi(a) : \mathbb{N} \times A \to 2,$$

  $a \in A$ free, meaning here $\forall a$, and $k \in \mathbb{N}$ free, meaning here $\exists k.$
**Proof:** Granted self-consistency of theory $\pi_R$ means just injectivity of numeralisation

$$\nu_2 : 2 \to [1, 2]_{\pi} = [1, 2]/\equiv_{\pi}.$$ 

The **Lemma** deduces that this injectivity carries over first to numeralisation $\nu_N = \nu : N \to [1, N]_{\pi}$, and then to all numeralisations

$$\nu_B : B \to [1, B]_{\pi}, \text{ } B \text{ a } \pi_R \text{ object.}$$

Now compatibility of internal composition with internal equality as well as—**Lemma** again—naturality of transformation $\nu_A : A \to [1, A]_{\pi}$ give

$$\pi_R \vdash [\Gamma f \equiv_{k}^R g \Gamma]$$

$$\implies \Gamma f \equiv_{k}^R \nu_A(a) \equiv_{\pi}^\gamma g \equiv_{\pi} \nu_A(a)$$

$$\implies \nu_B(f(a)) \equiv_{\pi} \nu_B(g(a))$$

$$\implies f(a) \equiv g(a),$$

the latter implication following from injectivity of $\nu_B : B \to [1, B]_{\pi}$

**q.e.d.**

**$\omega$-completeness theorem** for theory $\pi_R$: theory $\pi_R$ admits the following scheme of test by all internal numerals:

$$\chi = \chi(a) : A \to 2 \text{ predicate,}$$

$$k = k(a) : A \to N \text{ such that}$$

$$\pi_R \vdash \text{Prov}_{\pi_R}(k(a), \Gamma \chi \equiv_{\pi} \odot \nu_A(a)) : A \to 2$$

$$(\omega\text{-Comp}) \vdash \pi_R \vdash \chi : A \to 2.$$
**Proof:** By $\nu$ naturality—within $\pi R$—the antecedent gives

$$\pi R \vdash \text{Prov}_{\pi R}(k'(a), \nu_2 \circ \chi(a)) : A \to 2,$$

and from this, by $\pi R$ self-consistency: injectivity of $\nu_2$ within $\pi R$,

$$\pi R \vdash \chi(a) : A \to 2 \quad \text{q.e.d.}$$

**Interpretation:** The $\nu_A(a), a \in A$ are jointly epic, $\nu A$ lies dense in $[\mathbb{1}, A]_\pi$. theory $\pi R$ is in particular internally $\mu$-consistent, object $\mathbb{1}$ is an internal separator, all of this with respect to $\pi R$ maps (on object language level). Would it work for (free variable) internal map codes either?

**Question:** Can we then have/assume this test to work on the external level too? can we have/assume at least object $\mathbb{1}$ to be/to become a separator for category $\pi R$?

**Attempt to an answer:** logic/arithmetic externalisation of axioms and theorems, as opposite to—successfull—internalisation/arithmeticisation seems me to be legitimate/consistent: both internalisation and externalisation can be seen/formalised as preserving/reflecting logical invariants. A theory $T$ for which this is not always possible—Consistency/consistency provability—has a defect in this regard, it is not *sound* in the technical sense, see SMORYNSKI 1977.

**Conclusion:** descent theory $\pi R$—in the role of metamathematic—derives its own consistency (formula) as well as—see below—the inconsistency (formulae) for set theories $T$, the latter including Peano-arithmetic $\text{PA}^+$ with order of $\mathbb{N}[\omega]$ to satisfy finite descent.
All of this under assumption, meta-axiom, that theory $\pi R$ is $\pi$-consistent, that it externalises its axiom ($\pi$) into (correct) termination of (external) evaluation $ev$.

The $\pi R$ (in part) internal version of $\mu$-consistency, consequence of $\pi$-consistency, is $\omega$-completeness above.

**Question:** Are quantified arithmetical theories $T$, in particular theory $PA^+$, even inconsistent?

By Gödel’s 2nd Incompleteness theorem, first assertion, $T \not\vdash Con_T$ if $T$ consistent, hence $\pi R \not\vdash Con_T$ if $T$ consistent: this since $T$ is an extension of $\pi R$. But then, by Decidability theorem above, for $\pi R$ and p.r. free-variable predicate $Con_T = \neg Prov_T(k, \lceil false \rceil) : \mathbb{N} \to 2$,

$$\pi R \vdash \neg Con_T, [a\text{ fortiori } T \vdash \neg Con_T.]$$

Now if we take as metamathematic the external version $PR$ of fundamental theory $PR$, then the consistency questions are open.

But if we take as metamathematic an external version $\pi R$ of descent theory $\pi R$, then we get in fact consistency of p.r. theories $PR, PRa, PRXa$—and of descent theory $\pi R$—as well as inconsistency of set theories $T$.

**Problems:**

1. Is axiom scheme ($\pi$) redundant, $\pi R \cong PRXa$? Certainly not, since isotonic maps from lexicographically ordered $\mathbb{N} \times \mathbb{N}, \ldots, \mathbb{N}^+ \equiv \mathbb{N}[\omega] \equiv \omega^\omega$ to $\mathbb{N}$ are not available.

2. Can we get internal soundness for theory $\pi R$ itself? Up to now we have only Objective soundness: this is the one considered by
8 Discussion (tentative)

The claim for our set theories is that $T$ proves $\neg \text{Con}_T$ which formally denies Gödel’s second incompleteness theorem:

Its second postcedent and hence the assumption of $\omega$-consistency for $PM$ and $ZF$. Gödel himself was said to be not completely convinced of this assumption.

All of our theories, in particular $PA \cong PR \exists$, are standard recursively axiomatized extensions of primitive recursive arithmetic $PR$. Everybody then expects for these set theories $T\,\omega$-consistency. But this is only an assumption. Remains the possibility that this text contains a formal irreparable error. If so, where?

Axiomatisation and predicate $\text{Prov}_T$ of “being a proof for”, are constructed in categorical parallel to Smorynski (and to Gödels predicate 45. $x B y$, $x$ ist ein Beweis für die Formel $y$, not to Rosser’s $\text{Prov}_T^R$),

no room for “informally motivated” formal proof predicates.

References

[1] J. Barwise ed. 1977: Handbook of Mathematical Logic. North Holland.
[2] H.-B. Brinkmann, D. Puppe 1969: *Abelsche und exakte Kategorien, Korrespondenzen*. Lecture Notes in Math. 96.

[3] L. Budach, H.-J. Hoehnke 1975: *Automaten und Funktoren*. Akademie-Verlag Berlin.

[4] C. Chevalley 1956: *Fundamental Concepts of Algebra*. Academic Press.

[5] H. Ehrig, W. Kühnel, M. Pfender 1975: Diagram Characterization of Recursion. LN in Comp. Sc. 25, 137-143.

[6] H. Ehrig, M. Pfender und Studenten 1972: *Kategorien und Automaten*. De Gruyter.

[7] S. Eilenberg, C. C. Elgot 1970: *Recursiveness*. Academic Press.

[8] S. Eilenberg, G. M. Kelly 1966: Closed Categories. *Proc. Conf. on Categorical Algebra*, La Jolla 1965, pp. 421-562. Springer.

[9] S. Eilenberg, S. Mac Lane 1945: General Theory of natural Equivalences. *Trans. AMS* 58, 231-294.

[10] U. Felgner 2012: Das Induktions-Prinzip. *Jahresber. DMV* 114, 23-45.

[11] M. P. Fourman 1977: The Logic of Topoi. Part D.6 in BARWISE ed. 1977. *Handbook of Mathematical Logic*. North Holland.
[12] G. Frege 1879: Begriffsschrift. Reprint in “Begriffsschrift und andere Aufsätze”, 2te Auflage 1971, I. Angelelli editor. Georg Olms Verlag Hildesheim, New York.

[13] P. J. Freyd 1972: Aspects of Topoi. Bull. Australian Math. Soc. 7, 1-76.

[14] K. Gödel 1931: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. Monatsh. der Mathematik und Physik 38, 173-198.

[15] R. L. Goodstein 1957/64: Recursive Number Theory. A Development of Recursive Arithmetic in a Logic-Free Equation Calculus. North-Holland.

[16] R. L. Goodstein 1971: Development of Mathematical Logic, ch. 7: Free-Variable Arithmetics. Logos Press.

[17] P. T. Johnstone 1977: Topos Theory. Academic Press

[18] A. Joyal 1973: Arithmetical Universes. Talk at Oberwolfach.

[19] W. Kühnel, M. Pfender, J. Meseguer, I. Sols 1977: Primitive recursive algebraic theories and program schemes. Bull. Austral. Math. Soc. 17, 207-233.

[20] J. Lambek, P. J. Scott 1986: Introduction to higher order categorical logic. Cambridge University Press.

[21] M. Lassmann 1981: Gödel’s Nichtableitbarkeitstheoreme und Arithmetische Universen. Diploma Thesis. Techn. Univ. Berlin.
[22] F. W. Lawvere 1964: An Elementary Theory of the category of Sets. *Proc. Nat. Acad. Sc. USA* 51, 1506-1510.

[23] F. W. Lawvere 1970: Quantifiers and Sheaves. *Actes du Congrès International des Mathématiciens, Nice*, Tome I, 329-334.

[24] F. W. Lawvere, S. H. Schanuel 1997, 2000: *Conceptual Mathematics*. Cambridge University Press.

[25] S. Mac Lane 1972: *Categories for the working mathematician*. Springer.

[26] M. E. Maietti 2010: Joyal’s arithmetic universe as list-arithmetic pretopos. *Theory and Applications of Categories* 24(3), 39-83.

[27] G. Osius 1974: Categorical set theory: a characterisation of the category of sets. *J. Pure and Appl. Alg.* 4, 79-119.

[28] B. Pareigis 1969: *Kategorien und Funktoren*. Teubner.

[29] B. Pareigis 2004: *Category Theory*. pdf Script, author’s Home page LMU München.

[30] R. Péter 1967: *Recursive Functions*. Academic Press.

[31] M. Pfender 1974: Universal Algebra in S-Monoidal Categories. *Algebra-Berichte Nr. 20*, Mathematisches Institut der Universität München. Verlag Uni-Druck München.
[32] M. Pfender 2008b: RCF 2: Evaluation and Consistency. arXiv:0809.3881v2 [math.CT]. Has a gap.

[33] M. Pfender 2008c: RCF 3: Map-Code Interpretation via Closure. arXiv:0809.4970v1 [math.CT]. Has a gap.

[34] M. Pfender 2012: α version of present text. http://www3.math.tu-berlin.de/preprints/files/Preprint-38-2012.pdf

[35] M. Pfender 2013a: RCF 3: Inconsistency Provability for Set Theory. Preliminary submission to TAC.

[36] M. Pfender, M. Kröplin, D. Pape 1994: Primitive Recursion, Equality, and a Universal Set. Math. Struct. in Comp. Sc. 4, 295-313.

[37] M. Pfender, R. Reiter, M. Sartorius 1982: Constructive Arithmetics. Lecture Notes in Math. 962, 228-236.

[38] B. Poonen 2008: Undecidability in Number Theory. Notices of the AMS 55, 344-350.

[39] W. Rautenberg 1995/2006: A Concise Introduction to Mathematical Logic. Universitext Springer 2006.

[40] R. Reiter 1980: Mengentheoretische Konstruktionen in arithmetischen Universen. Diploma Thesis. Techn. Univ. Berlin.

[41] R. Reiter 1982: Ein algebraisch-konstruktiver Abbildungskalkül zur Fundierung der elementaren Arithmetik. Dissertation, rejected by Math. dpt. of TU Berlin.
[42] E. P. ROBINSON, G. ROSOLINI 1988: Categories of partial maps. Inform. Comp. 79, 94-130.

[43] L. ROMÀN 1989: Cartesian categories with natural numbers object. J. Pure and Appl. Alg. 58, 267-278.

[44] M. SARTORIUS 1981: Kategorielle Arithmetik. Diploma Thesis. Techn. Univ. Berlin.

[45] D. SCOTT 1975: Lambda calculus and recursion theory. Proc. 3rd Scandinavian Logic Symposium (Univ. Uppsala 1973), 154-193. Stud. Logic Found. Math. 82, North Holland, Amsterdam.

[46] J. R. SHOENFIELD 1967: Mathematical Logic. Addison-Wesley.

[47] Th. Skolem 1970: Selected Works in Logic. Universitetsforlaget Oslo - Bergen - Tromsö.

[48] C. Smorynski 1977: The Incompleteness Theorems. Part D.1 in Barwise ed. 1977. Handbook of Mathematical Logic. North Holland.

[49] A. Tarski, S. Givant 1987: A formalization of set theory without variables. AMS Coll. Publ. vol. 41.

[50] M. Tierney 1972: Sheaf theory and the continuum hypothesis. Toposes, algebraic geometry and logic. LN in Math. 274, 13-42.

[51] A. Yashuhara 1971: Recursive function theory and logic. Academic Press.
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