Study of thermal gradation on creep deformation of non-linear varying functionally graded rotating disc.

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Abstract. Rotating discs working at high temperature finds its application in many dynamic devices like turbo generators, turbojet engines, gas and steam turbine rotors, centrifugal compressors and components like flywheels, gears, etc. and in many such industrial applications. Thus, in the present work, mathematical modeling have been done to investigate the effect of parabolic temperature distribution on secondary stage deformation in non-linear functionally graded composite disc along both radial and tangential directions. Sherby’s law helps to calculate creep behavior of composite rotating disc. MATLAB software is used to model stress and strain rate distributions occurring in composite disc working at different temperature profiles.

1. Introduction
Analysis of creep is a major consideration as rotating discs are subjected to high temperatures for long duration. Thus, researchers find interest in the study of creep analysis of discs with various dimensional profile to make better designs for dynamic applications in turbo generators, turbojet engines, gas and steam turbine rotors, flywheels, gears, etc. The concept of functionally graded material (FGM) took birth in 1984 after extensive research by researchers in Sendai (Japan). FGM applications such as optical filter and fiber, machining tool, electric shaver blade, surface material for wristwatch, baseball shoes spikes, etc were proposed by Uemura [17]. The designing, processing and characteristics of FGM were stated by Chen and Hirai [11]. FGMs find use in high temperature impenetrable materials for dynamical components in vehicles used for space and air crafts working at high temperature. Bose et al. [4] analysed creep analysis of rotating disc made of functionally graded material at high temperatures. Bose and Rattan [3] modeled secondary stage deformation for linearly varying FGM rotating disc under thermal gradation. Creep distributions were calculated for discs operating at various temperature profiles. The study showed that disc operating at uniform temperature showed less strain as compared to disc under thermal gradation. Garg [9] studied effect of particle gradient and varying thickness of functionally graded disc on stresses and strains. He suggested disc with decreasing thickness showed lowest strains. Bhatnagar et al. [2] considered orthotropic rotating discs of variable thickness (constant, linearly varying and hyperbolically varying) for the study of steady state creep. Stresses and strain rates have been calculated in the discs using Norton’s creep law.
The computations to study five different types of anisotropy were employed. It was concluded that for better design of turbine discs suitable material anisotropy and profile of disk must be selected. Deepak et al. [7] investigated creep study of functionally graded rotating disc with linear thickness profile and linear particle gradient. They examined that disc with thickness and particle gradient and showed lower strain rates in discs with uniform distribution. Rattan et al. [15] modeled secondary stage of deformation in thermally graded isotropic rotating disc under the effect of residual stress. In the presence of thermal residual stress, deformation is significantly effected. Hoffman yield formula is used to carry out creep analysis. In the present work, mathematical modeling has been done to investigate creep deformation in non-linear functionally graded composite disc rotating at temperature profile along both radial and tangential directions.

2. Disc profile

Isotropic annular aluminium(Al) disc embedded with silicon carbide particles(SiCₚ) is considered as mentioned by Bose and Rattan [5]. Law of mixture expressed density in the composite as

$$\rho(r) = \rho_{Al} + (\rho_{SiC} - \rho_{Al}) \frac{vol(r)}{100}$$ \hspace{1cm} (1)

where $\rho_{SiC} = 3210 \text{kg/m}^3$ and $\rho_{Al} = 2713 \text{kg/m}^3$ are the respective densities ([6]). $Vol(r)$ is the volume content of the reinforcement at any radial distance $r$, where $a \leq r \leq b$, given by:

$$vol(r) = P + Q/r$$ \hspace{1cm} (2)

where

$$P = \frac{a.vol_{max} - b.vol_{min}}{a - b}$$ \hspace{1cm} (3)

and

$$Q = a.b.(vol_{max} - vol_{min}) \frac{b - a}{b - a}$$ \hspace{1cm} (4)

Here $vol_{max} = 30\text{vol}%$ at the inner radius $a = 0.03175m$ and $vol_{min} = 10\text{vol}%$ at the outer radius $b = 0.1524m$, respectively. From (1) and (2), we get

$$\rho(r) = \rho_{Al} + (\rho_{SiC} - \rho_{Al}) \frac{P + Q/r}{100}$$ \hspace{1cm} (5)

The average particle content in the FGM disc with $t$ as constant thickness is given as

$$\int_a^b 2\pi r t vol(r)dr = vol_{avg} \pi t (b^2 - a^2)$$ \hspace{1cm} (6)

(2) and (6) gives

$$vol_{avg} = P - \frac{2Q}{b + a}$$ \hspace{1cm} (7)

3. Thermal analysis

In steady-state conduction, 1st thermodynamics law for energy equation is given as:

$$\frac{1}{r} \frac{d}{dr} \left( r.E_\Theta(r) \frac{d}{dr} \Theta(r) \right) = 0$$ \hspace{1cm} (8)
The lower as well as upper surfaces of disc are insulated and there is negligible heat dissipation is assumed. Thermal conductivity varies radially on the basis of law of mixtures ([8], [12] and [13]) as

$$E_\Theta(r) = E_\Theta(Al) + (E_\Theta(SiC) - E_\Theta(Al)) \frac{vol(r)}{100}$$  \hspace{1cm} (9)

where $E_\Theta(Al) = 120 \text{ Wm}^{-1}\text{C}^{-1}$ and $E_\Theta(SiC) = 237 \text{ Wm}^{-1}\text{C}^{-1}$ are the respective thermal conductivity.

Combining (9) with heat conduction equation (8) we get,

$$\frac{d^2 \Theta}{dr^2} + \left ( \frac{1}{r} + \frac{1}{E_\Theta(r)} \frac{d}{dr} E_\Theta(r) \right ) \frac{d\Theta}{dr} = 0$$  \hspace{1cm} (10)

Figure 1 shows that solution is evaluated by splitting radial zone with thickness $t^i$ by semi analytical approach [12]. For each division, second-order ordinary differential equation (ODE) (10) with variable coefficients is transformed into same order differential equation with constant coefficients.

![Figure 1. Splitting of radial zone into finite sub zones](image)

The coefficients of (10) are calculated at $r_i$, mean radius of $i^{th}$ zone and the ODE with constant coefficients are well grounded only in the $i^{th}$ sub zone given as:

$$c_k = \frac{1}{r_i} + \frac{1}{E_\Theta(r_i)} \frac{d}{dr} E_\Theta(r) \bigg|_{r=r_i} ; i = 1, 2, ..., m$$  \hspace{1cm} (11)

The exact solution of these ODEs are

$$\Theta(r_i) = iX_1 + iX_2 \exp(-r_ic_i)$$  \hspace{1cm} (12)

where $iX_1$, $iX_2$ are unknowns for $i^{th}$ zone and are calculated by application of boundary conditions in between two adjoining sub zones. The continuity conditions imposed at the interfaces are

$$\Theta(r) \bigg|_{r=r_i + \frac{\mu}{2}} = \Theta(r) \bigg|_{r=r_i+1-\frac{\mu+1}{2}}$$

$$\frac{d\Theta(r)}{dr} \bigg|_{r=r_i + \frac{\mu}{2}} = \frac{d\Theta(r)}{dr} \bigg|_{r=r_i+1-\frac{\mu+1}{2}}$$  \hspace{1cm} (13)

Boundary conditions imposed at internal and external parts of rotating disc globally are

$$\Theta(r) = \Theta_a \hspace{1cm} at \hspace{0.5cm} r = a$$

$$\Theta(r) = \Theta_b \hspace{1cm} at \hspace{0.5cm} r = b$$  \hspace{1cm} (14)

Here, $\Theta_a = 673K$ and $\Theta_b = 573K$ are the imposed temperatures respectively.

The global boundary conditions (14) along with continuity conditions (13) yield a system of linear algebraic equations in $iX_1$ and $iX_2$. Solving these equations for $iX_1$ and $iX_2$ in each sub-region, temperature $\Theta(r)$ can be evaluated. Temperature distribution in the disc is done in MATLAB using curve fitting given by

$$\Theta(r) = 1254.7r^2 - 1108.9r + 710.4$$  \hspace{1cm} (15)

Discs with following temperature profiles are taken in this work.
1. $D_1$ disc working at constant temperature ($\Theta = 623K$) throughout the radius.
2. $D_2$ disc working according to temperature field $\Theta(r)$ (15) along radius.

4. **Creep law**

Sherby’s law [16] defines secondary deformation of Al-SiC$_p$ composite as:

$$\dot{\epsilon} = [M(\bar{\sigma} - \sigma_0)]^8$$

(16)

where $\dot{\epsilon}$ is effective strain rate, $\bar{\sigma}$ is effective stress under biaxial stress and $\sigma_0$ is threshold stress.

Values of $M$ and $\sigma_0$ (16) are procured from Pandey et al. [14] results for Al-SiC$_p$ composite under uniaxial loading.

In this study, $p = 1.7 \mu m$ and Data Fit software is used to develop regression equations taking $M$, $\sigma_0$ as functions of $p$, $V(r)$ and $\Theta(r)$ given as:

$$M(r) = p^{0.2112} \Theta(r)^{4.89} V(r)^{-0.591} e^{-34.91}$$

(17)

$$\sigma_0(r) = p^{-0.02050} \Theta(r)^{0.0378} V(r)^{1.033} e^{-4.9695}$$

(18)

5. **Mathematical formulation**

Following assumptions are made for modeling taking principal stresses in $r$, $\theta$ and $z$ directions.

(i) Steady state condition of stress prevails.
(ii) Elastic deformations are neglected being small in magnitude.
(iii) Existence of biaxial stress at any radius of disc.

For secondary stage deformation, generalized constitutive equations are:

$$\dot{\epsilon}_r = \frac{\dot{\epsilon}}{2\bar{\sigma}} \left[2\sigma_r - (\sigma_\theta + \sigma_z)\right]$$

(19)

$$\dot{\epsilon}_\theta = \frac{\dot{\epsilon}}{2\bar{\sigma}} \left[2\sigma_\theta - (\sigma_z + \sigma_r)\right]$$

(20)

$$\dot{\epsilon}_z = \frac{\dot{\epsilon}}{2\bar{\sigma}} \left[2\sigma_z - (\sigma_r + \sigma_\theta)\right]$$

(21)

where,

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \left[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_r - \sigma_z)^2\right]^{1/2}$$

(22)

and $\dot{\epsilon}_r$, $\dot{\epsilon}_\theta$, $\dot{\epsilon}_z$ are the strain rates and $\sigma_r$, $\sigma_\theta$, $\sigma_z$ are stresses. For $\sigma_z = 0$ (biaxial stress), constitutive equations are,

$$\dot{\epsilon}_r = \frac{d\dot{u}_r}{dr} = \frac{\dot{\epsilon}(2y - 1)}{2(1 - y + y^2)^{1/2}}$$

(23)

$$\dot{\epsilon}_\theta = \frac{\dot{u}_r}{r} = \frac{\dot{\epsilon}(2 - y)}{2(1 - y + y^2)^{1/2}}$$

(24)

$$\dot{\epsilon}_z = - (\dot{\epsilon}_r + \dot{\epsilon}_\theta)$$

(25)
where \( y \) is the ratio of \( \sigma_r(r) \) and \( \sigma_\theta(r) \) and \( u_r \) is the radial displacement.

For rotating disc, equation of motion is given as

\[
\frac{d}{dr} \left[ r \sigma_r(r) \right] - \sigma_\theta(r) + \rho(r) \omega^2 r^2 = 0
\] (26)

where \( \omega = 15000 \text{ rpm} \) is taken as the angular velocity of the disc.

Equations (23) and (24) are solved to get \( \sigma_\theta(r) \) as:

\[
\sigma_\theta(r) = \frac{(\dot{u}_a)^{1/8}}{M(r)} \psi_1(r) + \psi_2(r)
\] (27)

where

\[
\zeta_1(r) = \frac{\zeta(r)}{(1 - y + y^2)^{1/2}},
\] (28)

\[
\zeta_2(r) = \frac{\sigma_0(r)}{(1 - y + y^2)^{1/2}},
\] (29)

\[
[\zeta(r)]^8 = \frac{2(1 - y + y^2)^{1/2}}{r(2 - y)} \exp \int_a^r \frac{\Psi(r)}{r} \, dr,
\] (30)

and

\[
\Psi(r) = \frac{(2y - 1)}{(2 - y)}.
\] (31)

Values of \( \sigma_r(r) \) is obtained after tangential stress distribution \( \sigma_\theta(r) \) is calculated from 26:

\[
\sigma_r(r) = \frac{1}{r} \int_a^r \sigma_\theta(r) \, dr - \frac{\omega^2}{r} \left[ \frac{(r^3 - a^3)}{3} \left( \rho_m + (\rho_d - \rho_m) \frac{P}{100} \right) - \frac{(\rho_d - \rho_m)}{100} \frac{Q(r^2 - a^2)}{2} \right].
\] (32)

\( \sigma_\theta \) and \( \sigma_r \) are calculated by (27) and (28) at any radius of rotating disc. Strain rates \( \dot{\epsilon}_r \), \( \dot{\epsilon}_\theta \) and \( \dot{\epsilon}_z \) are determined using (19), (20) and (21) respectively.

5.1. Numerical analysis

Creep deformation is evaluated for the above analysis with the help of given algorithm. The iteration yields values of stresses at various radial points and converges until the boundary conditions, \( \sigma_r(a) = 0 \) and \( \sigma_r(b) = 0 \) are attained (Garg et al.[10] and Arnold et al. [1]).

5.1.1. Algorithm

Step A: Begin.
Step B: Compute volume, \( V(r) \) at various radial points.
Step C: Compute \( \sigma_{\theta_{avg}} \).
Step D: At various radial points, evaluate \( M \) and \( \sigma_0 \).
Step E: Initialize \( \sigma_\theta = \sigma_{\theta_{avg}} \).
Step F: Iteration=1
Step G: Compute \( \Psi(r) \), \( \zeta(r) \), \( \zeta_1(r) \), \( \zeta_2(r) \), \( \dot{u}_a \).
Step H: Compute \( \sigma_r(r) \).
Step I: Compute \( \sigma_\theta(r) \).
Step J: If Iteration > 1 then move to Step K else Step L.
Step K: If \( \text{error} = \frac{[\sigma_r(r)]_{\text{iteration}} - [\sigma_r(r)]_{\text{iteration}-1}}{[\sigma_r(r)]_{\text{iteration}-1}} \leq \text{Threshold error value} (=0.01) \) then move to Step M otherwise Step L.
Step L: Iteration = Iteration + 1.
Step M: If Iteration < Maximum no. of iteration then move to Step L otherwise Step N.
Step N: Compute strain rates in \( r, \theta \) and \( z \) directions.
Step O: End.

6. Results and interpretation

6.1. Validation of MATLAB program

Numerical calculations established on the basis of mathematical formulation exhibited in the algorithm aids to investigate steady-state creep of the composite disc. For accomplishing this task, in the current analysis, the stresses and strain rates for a rotating steel disc obtained by experimental results of [18] are compared for disc operating under similar conditions, the outcomes for which are graphically expressed in Figure 2.

As component spends major part of life in secondary stage so the current work studies effect of thermal gradation on secondary deformation of rotating FGM disc with non-linear reinforcement. The analysis of stress and strain rate distribution have been discussed for the discs operating at uniform and parabolic temperature gradient.

In Figure 3(a) as radius increases from 0.0318m to 0.1524m radial stress traces a parabolic path for both discs D1 and D2. For disc D1, radial stress attains the peak at 0.0800m with \( \sigma_r = 37.17 \text{MPa} \) and D2 with \( \sigma_r = 37.617 \text{MPa} \). Thus, disc operating at parabolic gradient shows more stress value than disc operating at constant temperature. In Figure 3(b), circumferential stresses increases to 85.62\text{MPa} at radius 0.0559m and then gradually decreases at the outer surface of disc for D1. However, D2 shows an increase in circumferential stress values throughout the radial length of the disc.

Figure 4(a) shows a sharp decline in radial strain rate values as radius increases from 0.0318m to 0.1524m for disc D1. For disc D2, radial strain rate initially decreases and then increases to \(-4.153 \times 10^{-5} \text{s}^{-1}\) towards middle of the disc and then further decreases towards the outer surface.
Figure 3. Comparison of (a) radial stress($\sigma_r$) (b) tangential stress($\sigma_{\theta\theta}$) along radius in disc $D_1$ working at constant temperature and $D_2$ operating at parabolic gradient.

Figure 4. Comparison of (a) radial strain rate($\dot{\varepsilon}_r$) (b) tangential strain rate($\dot{\varepsilon}_\theta$) along radius in disc $D_1$ working at constant temperature and $D_2$ operating at parabolic gradient.

of disc. Figure 4(b) shows that initially FGM disc at constant temperature shows less tangential strain rate as compared to FGM disc subjected to parabolic thermal gradation, however this phenomenon reverses at the outer radius.

7. Conclusion
It is clearly observed that thermal gradient effects deformation in non-linear varying FGM disc significantly. On application of thermal gradient, radial strain rates initially decreases, then increases to the middle of the disc and then further decreases towards the outer radius. However, tangential strain rates show the opposite behavior. Hence, creep deformation plays a major role in the safe designs of rotating discs and thus cannot be ignored.

References
[1] Arnold S M, Saleeb A F and Al-Zoubi N R 2001 Deformation and life analysis of composite flywheel disk and multi-disk systems NASA/TM-2001-210578
[2] Bhatnagar N S, Kulkarni P S and Arya V K, 1986 Steady state creep of orthotropic rotating disks of variable thickness Nuclear Engineering Design 91 121-41
[3] Bose T and Rattan M 2016 Modeling creep behavior of thermally graded rotating disc of functionally graded material Differential Equations and Dynamical Systems DOI:10.1007/s12591-017-0350-1
[4] Bose T, Rattan M and Chamoli N 2017 Steady state creep of Isotropic Rotating composite Disc under Thermal Gradation International Journal of Applied Mechanics 9(6) 1750077
[5] Bose T and Rattan M 2018 Effect of thermal gradation on steady state creep of functionally graded rotating disc European Journal of Mechanics A/Solids 67 169-76

[6] Clyne T W and Withers P J An Introduction to Metal Matrix Composites Cambridge University Press 479

[7] Deepak D, Gupta V K and Dham A K 2010 Creep modeling in functionally graded rotating disc of variable thickness Journal of Mechanical Science and Technology 24(11) 2221-32

[8] Fesharaki J J, Loghman A, Yazdipoor M and Golabi S 2014 Semi-analytical solution of time-dependent thermomechanical creep behavior of FGM hollow spheres Mechanics of Time-Dependent Materials 18 41-53

[9] Garg M 2017 Stress analysis of variable rotating FG disc International Journal of Pure and Applied Physics 13(1) 158-61

[10] Garg M, Salaria B S and Gupta V K 2013 Effect of thermal gradient on steady state creep in a rotating disc of variable thickness Procedia Engineering 55 542-47

[11] Hirai T and Chen L 1999 Recent and prospective development of functionally graded materials in Japan Material Science Forum 308-311 509-14

[12] Kordkheili S A H and Naghdabadi R 2007 Thermo-elastic analysis of a functionally graded rotating disk Composite Structures 79 508-16

[13] Loghman A, Arani A G, Shajari A R and Amir S 2011 Time-dependent thermoelastic creep analysis of rotating disk made of Al-SiC composite Archive of Applied Mechanics 81(12) 1853-64

[14] Pandey A B, Mishra R S and Mahajan Y R 1992 Steady State Creep Behavior Of Silicon Carbide Particulate Reinforced Aluminum Composites Acta Metall. Mater., 40(8) 2045-52

[15] Rattan M, Kaushik A, Chamoli N and Bose T 2016 Steady state creep behavior of thermally graded isotropic rotating disc of composite taking into account the thermal residual stress European Journal of Mechanics A/Solids 60 315-26

[16] Sherby O D, Klundt R H and Miller A K 1977 Flow stress, subgrain size and subgrain stability at elevated temperature Metall. Trans., A, 8 843-50

[17] Uemura S 2003 The activities of FGM on new application Material Science Forum 423-425 1-10

[18] Wahl A M, Sankey G O, Manjoine M J and Shoemaker E 1954 Creep tests of rotating disks at elevated temperature and comparison with theory Journal Applied Mechanics 76 225-35