Associative realizations of $\kappa$-deformed extended Snyder model

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Abstract

Usually, the realizations of the noncommutative Snyder model lead to a nonassociative star product. However, it has been shown that this problem can be avoided by adding to the spacetime coordinates new tensorial degrees of freedom. The model so obtained, called extended Snyder model, can be subject to a $\kappa$-deformation, giving rise to a unification of the Snyder and the $\kappa$-Poincaré algebras in the formalism of extended spacetime.

In this paper we review this construction and consider the generic realizations of the $\kappa$-deformed extended Snyder model, calculating the associated star product, coproduct and twist in a perturbative setting. We also introduce a representation of the Lorentz algebra in the extended space and speculate on possible interpretations of the tensorial degrees of freedom.

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1. Introduction

Noncommutative geometries are an attempt to describe the structure of spacetime at scales of the order of the Planck length by postulating a noncommutative structure of the operators associated to the position of a point in spacetime [1,2]. A particularly successful approach to noncommutative geometry is that based on the formalism of Hopf algebras [3], that permits to describe the relativistic symmetries of the quantum spacetime, starting from the algebra of the position operators. This formalism has been extensively used in particular in the investigation of the Moyal [4] and the $\kappa$-Poincaré models [5].

However, not all noncommutative geometries fit easily in this framework. For example, the treatment of the Snyder model [6] presents some difficulties, since the algebra of the position operators does not close in this case, because the commutators are given by Lorentz generators: hence also the Lorentz generators should be included in the spacetime algebra from which the Hopf structure is derived. It is nevertheless possible to construct a sort of Hopf algebra even excluding the Lorentz generators, but in this way the resulting Hopf structure does not satisfy all the standard axioms, in particular it is not coassociative [6,7] and the related star product is not associative.

For this reason in [8], following a proposal of [7], a Hopf algebra was constructed adding to the position operators $\hat{x}_i$ of $N$-dimensional spacetime a set of antisymmetric rank-2 tensors $\hat{x}_{ij}$ as primary generators, in such a way that the algebra generated by the $\hat{x}_i$ and $\hat{x}_{ij}$ were isomorphic to the $N$-dimensional Snyder algebra. It was also shown that this algebra can be written more compactly as an $so(1,N)$ algebra with generators $\hat{x}_{\mu\nu}$, by identifying the $\hat{x}_i$ with the $\hat{x}_{iN}$ of $so(1,N)$.

To distinguish this structure from the usual realization of the Snyder model leading to nonassociativity, it was called extended Snyder spacetime. General realizations of the ensuing Hopf algebra in terms of an extended Heisenberg algebra were investigated and the related coproduct and twist were calculated.

Later in [9], using a formalism developed in [10], it was shown that this construction can be extended to include also a $\kappa$-deformation, by passing from an algebra $so(1,N)$ to an algebra $so(1,N;g)$ with generators $\hat{X}_{\mu\nu}$, in which the metric $\eta_{\mu\nu}$ is changed into a nondiagonal metric $g_{\mu\nu}$. In that paper, only the Weyl realization of the Hopf algebra was considered and the coproduct and twist were calculated only in that case. In particular, the determination of the twist is important also in view of applications, as for example the determination of dispersion relations [11]. In this context, the Heisenberg double construction was investigated in [12].

The aim of this paper is to review these models and extend the investigation of the Hopf algebra of the $\kappa$-deformed extended Snyder spacetime to generic realizations. We shall assume that the general realization of the $\kappa$-deformed algebra can be obtained from that of the extended Snyder algebra by changing the old variables to new ones defined by a transformation depending on the metric $g_{\mu\nu}$ defined above. In this way one can easily calculate the relevant quantities. We recall that a unification of Snyder and $\kappa$-Poincaré models was previously considered in the context of nonassociative realizations [13].

The paper is organized as follows: in sect. 2 we review the construction of the $\kappa$-deformed extended Snyder spacetime [9] and its relation with the extended Snyder spacetime [8] and define the action of the Lorentz algebra on it. In sect. 3 we discuss perturbatively, up to second order in an expansion parameter $\lambda$, the generic realizations of the extended Snyder algebra in terms of an extended Heisenberg algebra generated by the operators $x_i$ and $x_{ij}$ and their canonically conjugated momenta. In sect. 4 we construct realizations of $\hat{X}_i$ and $\hat{X}_{ij}$, the star product and the coproduct of the momenta for a generic realization of the $\kappa$-deformed extended Snyder model. In sect. 5 we obtain the twist up to first order in $\lambda$, while in sect. 6 the coproduct of the generators of the Lorentz transformations is computed. In sect. 7 we draw some conclusions, while in the Appendix we recall the construction of the twist in the Hopf algebroid approach.

2. $\kappa$-Minkowski Snyder spacetime and $so(1,N;g)$ algebra

2A. The algebras $so(1,N;g)$ and $so(1,N)$.

In this section we review the construction of the $\kappa$-Minkowski-Snyder spacetime introduced in [9]. We define the algebra $so(1,N;g)$ with dimensionless generators $\hat{X}_{\mu\nu}$ as

$$[\hat{X}_{\mu\nu},\hat{X}_{\rho\sigma}] = i\lambda(g_{\mu\rho}\hat{X}_{\nu\sigma} - g_{\mu\sigma}\hat{X}_{\nu\rho} - g_{\nu\rho}\hat{X}_{\mu\sigma} + g_{\nu\sigma}\hat{X}_{\mu\rho}),$$

(2.1)

1 In this paper, Greek indices run from 0 to $N$, Latin indices from 0 to $N - 1$. 

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where $\lambda$ is a real parameter and $g_{\mu\nu}$ is a symmetric matrix that plays the role of a metric. The parameter $\lambda$ is dimensionless in natural units and interpolates between an abelian algebra and a $SO(1, N)$ algebra; its actual value is arbitrary, depending on the units chosen, so that we can assume $\lambda \ll 1$ and use it for series expansions.

In the following, we shall be interested in metrics of the form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & \ldots & 0 & g_0 \\ 0 & 1 & \ldots & 0 & g_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & g_{N-1} \\ g_0 & g_1 & \ldots & g_{N-1} & g_N \end{pmatrix},$$  \hspace{1cm} (2.2)

with $g^T = g$ and $\det g = -g_0^2 + \sum_{i=1}^{N-1} g_i^2 - g_N \neq 0$.

If we introduce noncommutative coordinates $\hat{X}_i = \frac{1}{\kappa} \hat{X}_{iN}$, where $\kappa$ is a parameter with dimension of mass, and define $a_i = \frac{1}{\kappa} g_i$, $\beta = \frac{1}{\kappa^2} g_N$, the relations (2.1) split into

$$[\hat{X}_i, \hat{X}_j] = i\lambda (a_i \hat{X}_j - a_j \hat{X}_i + \beta \hat{X}_{ij}),$$

$$[\hat{X}_{ij}, \hat{X}_k] = i\lambda (\eta_{jk} \hat{X}_i - \eta_{ik} \hat{X}_j + a_i \hat{X}_{jk} + a_j \hat{X}_{ik}),$$

$$[\hat{X}_{ij}, \hat{X}_{kl}] = i\lambda (\eta_{lk} \hat{X}_{ij} - \eta_{ij} \hat{X}_{lk} - \eta_{jk} \hat{X}_{il} + \eta_{jl} \hat{X}_{ik}).$$  \hspace{1cm} (2.3)

In this way we obtain a unification of extended Snyder and $\kappa$-Minkowski spacetimes using the formalism of extended coordinates $\hat{X}_{\mu\nu}$. The $\hat{X}_i$ are interpreted as spacetime coordinates, while the $\hat{X}_{ij}$ are new tensorial degrees of freedom, whose physical interpretation we leave open. The algebra (2.1) generated by the $\hat{X}_{\mu\nu}$ reduces to the $\kappa$-Poincaré algebra if $g_N = 0$, and to the Snyder algebra if $g_0 = \ldots = g_{N-1} = 0$.

In the latter case, the algebra (2.1) reduces to the standard $so(1, N)$ algebra with generators $\hat{x}_{\mu\nu}$, considered in [8] and defined by

$$[\hat{x}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda (\eta_{\mu\rho} \hat{x}_{\nu\sigma} - \eta_{\mu\sigma} \hat{x}_{\nu\rho} - \eta_{\nu\rho} \hat{x}_{\mu\sigma} + \eta_{\nu\sigma} \hat{x}_{\mu\rho}),$$  \hspace{1cm} (2.4)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, and we have set $g_N = 1$.

Clearly, if one defines, in analogy with the previous case, $\hat{x}_i = \frac{1}{\kappa} \hat{X}_{iN}$, and identifies $\frac{1}{\kappa^2}$ with $\beta$ (i.e. sets $g_N = 1$), the commutation relations (2.4) take the form of the extended Snyder algebra since

$$[\hat{x}_i, \hat{x}_j] = i\frac{\lambda}{\kappa^2} \hat{x}_{ij},$$

$$[\hat{x}_{ij}, \hat{x}_k] = i\lambda (\eta_{jk} \hat{x}_i - \eta_{ik} \hat{x}_j),$$

$$[\hat{x}_{ij}, \hat{x}_{kl}] = i\lambda (\eta_{lk} \hat{x}_{ij} - \eta_{ij} \hat{x}_{lk} - \eta_{jk} \hat{x}_{il} + \eta_{jl} \hat{x}_{ik}).$$  \hspace{1cm} (2.5)

One can write the relation between the $\hat{X}_{\mu\nu}$ and $\hat{x}_{\mu\nu}$ and between the metrics $g_{\mu\nu}$ and $\eta_{\mu\nu}$ by means of a matrix $O$ such that

$$\hat{X}_{\mu\nu} = (O \hat{x}O^T)_{\mu\nu}, \hspace{1cm} g_{\mu\nu} = (O\eta O^T)_{\mu\nu},$$  \hspace{1cm} (2.6)

with

$$O_{\mu\nu} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -g_0 & g_1 & \ldots & g_{N-1} & \rho \end{pmatrix},$$  \hspace{1cm} (2.7)

where $\rho = \sqrt{g_N - g_kg_k} = \sqrt{-\det g}$. The matrix $O$ is defined up to $SO(1, N)$ transformations.

Then

$$\hat{X}_i = \rho \hat{x}_i + a_j \hat{x}_{ij}, \hspace{1cm} \hat{X}_{ij} = \hat{x}_{ij},$$  \hspace{1cm} (2.8)

with inverse

$$\hat{x}_i = \frac{1}{\rho} (\hat{X}_i - a_j \hat{X}_{ij}), \hspace{1cm} \hat{x}_{ij} = \hat{X}_{ij}.$$  \hspace{1cm} (2.9)
An important limit is obtained for \( \lambda = 0 \): in this limit, one is left with an abelian algebra, i.e. a commutative extended spacetime, from which it is possible to construct an extended Heisenberg algebra. In fact, calling \( X_{\mu \nu} \) and \( x_{\mu \nu} \) the commuting operators corresponding to \( \hat{X}_{\mu \nu} \) and \( \hat{x}_{\mu \nu} \) in this limit, we can define their canonically conjugate momenta \( P^{\mu \nu} \) and \( p^{\mu \nu} \), respectively. These are also related by the matrix \( O \) as

\[
P^{\mu \nu} = (O^T \rho O^{-1})^{\mu \nu},
\]

where

\[
O^T \equiv (O^{-1})^T = \frac{1}{\rho} \begin{pmatrix}
\rho & 0 & \ldots & 0 & g_0 \\
0 & \rho & \ldots & 0 & -g_1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \rho & -g_{N-1} \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix},
\] (2.11)

or, more explicitly\(^2\) defining \( P_i = \kappa P^{iN} \), and \( p_i = \kappa p^{iN} \),

\[
P_i = \frac{1}{\rho} p_i, \quad P_{ij} = p_{ij} + \frac{1}{\rho} (a_i p_j - a_j p_i).
\] (2.12)

The inverse relations are

\[
p_i = \rho P_i, \quad p_{ij} = P_{ij} - a_i P_j + a_j P_i.
\] (2.13)

Together with the respective position variables, the momenta satisfy the extended Heisenberg algebras

\[
[X_{\mu \nu}, X_{\rho \sigma}] = [P^{\mu \nu}, P^{\rho \sigma}] = 0, \quad [X_{\mu \nu}, P^{\rho \sigma}] = i(\delta_{\rho}^{\mu} \delta_{\nu}^{\sigma} - \delta_{\rho}^{\sigma} \delta_{\nu}^{\mu}),
\] (2.14)

and

\[
x_{\mu \nu}, x_{\rho \sigma}] = [p^{\mu \nu}, p^{\rho \sigma}] = 0, \quad [x_{\mu \nu}, p^{\rho \sigma}] = i(\delta_{\rho}^{\mu} \delta_{\nu}^{\sigma} - \delta_{\rho}^{\sigma} \delta_{\nu}^{\mu}),
\] (2.15)

respectively. Moreover, analogs of the relations (2.6) and (2.8) are valid for \( X_{\mu \nu} \) and \( x_{\mu \nu} \).

\[
X_{\mu \nu} = (OxO^T)_{\mu \nu}, \quad X_{i} = \rho x_{i} + a_{j} x_{ij}, \quad X_{ij} = x_{ij},
\] (2.16)

with \( X_i = \frac{1}{\kappa} X_{iN} \) and \( x_i = \frac{1}{\kappa} x_{iN} \), and inverse

\[
x_{i} = \frac{1}{\rho} (X_{i} - a_{j} X_{ij}), \quad x_{ij} = X_{ij}.
\] (2.17)

The relevance of this formalism is that it permits to pass from the standard results relative to the extended Snyder model to those of its \( \kappa \)-deformed extension by simply transforming the \( X, P \) variables to the \( x, p \) ones with (2.8), (2.12) and use the formulas valid in this case, and then go back to the \( \kappa \)-deformed variables by means of (2.9), (2.13), obtaining the corresponding deformed relations.

2.B. The Lorentz algebra acting on the extended Heisenberg algebra.

One can also define an \( N \)-dimensional Lorentz algebra \( so(1,N-1) \) acting on the coordinates \( x_{\mu \nu} \) in such a way that the \( x_i \) transform as \( N \)-vectors and the \( x_{ij} \) as antisymmetric tensors. The generators of the Lorentz algebra are given by the operators \( M_{ij} \) defined as

\[
M_{ij} = x_i p_j - x_j p_i + x_{ik} p_{jk} - x_{jk} p_{ik},
\] (2.18)

and satisfy

\[
[M_{ij}, M_{kl}] = i(\eta_{ik} M_{jl} - \eta_{il} M_{jk} - \eta_{jk} M_{il} + \eta_{jl} M_{ik}),
\] (2.19)

\(^2\) We adopt the convention that Latin indices are lowered and raised by the flat metric, thus in the following we shall always write them in lower position.
together with
\[
[M_{ij}, x_k] = i(\eta_{ik} x_j - \eta_{jk} x_i), \quad [M_{ij}, x_{kl}] = i(\eta_{ik} x_j - \eta_{jk} x_i - \eta_{il} x_{kj} + \eta_{jl} x_{ki}),
\]
\[
[M_{ij}, p_k] = i(\eta_{ik} p_j - \eta_{jk} p_i), \quad [M_{ij}, p_{kl}] = i(\eta_{ik} p_j - \eta_{jk} p_i - \eta_{il} p_{jk} + \eta_{jl} p_{ki}).
\] (2.20)

It is also easy to check that
\[
[M_{ij}, \hat{x}_k] = i(\eta_{ik} \hat{x}_j - \eta_{jk} \hat{x}_i), \quad [M_{ij}, \hat{x}_{kl}] = i(\eta_{ik} \hat{x}_j - \eta_{jk} \hat{x}_i - \eta_{il} \hat{x}_{jk} + \eta_{jl} \hat{x}_{ki}).
\] (2.21)

3. Generic realization and star product for the extended Snyder model

3A. Generic realization of the extended Snyder model

We now recall how the extended Snyder model (2.5) can be realized in terms of the extended Heisenberg algebra (2.15) [8]. A realization of the model is defined as
\[
\hat{x}_i = x_k \phi_1(\lambda p)_{k,i} + x_{kl} \phi_2(\lambda p)_{kl,i}, \quad \hat{x}_{ij} = x_k \phi_3(\lambda p)_{k,ij} + x_{kl} \phi_4(\lambda p)_{kl,ij},
\] (3.1)

where the matrix functions \(\phi(\lambda p)\) satisfy the differential equations that can be obtained from (2.5) with boundary conditions \(\hat{x}_i = x_i, \hat{x}_{ij} = x_{ij}\) for \(\lambda = 0\).

In [8] the most general realization of these commutation relations in terms of elements of the extended Heisenberg algebra (2.15) up to second order in \(\lambda\) was shown to depend on 14 parameters, namely
\[
\hat{x}_i = x_i + \lambda \left[ \beta c_0 x_k p_k + c_1 x_k p_{ki} \right] + \lambda^2 \left[ \beta c_2 x_k p_k + c_3 x_k p_{ki} + c_4 x_k p_{ki} + c_5 x_k p_{ki} \right] + \lambda^3 \left[ \beta c_6 x_k p_k + c_7 x_k p_{ki} \right],
\]
\[
\hat{x}_{ij} = x_{ij} + \lambda \left[ d_0 x_k p_{ki} + d_1 x_{ki} - (i \leftrightarrow j) \right] + \lambda^2 \left[ d_2 x_k p_{ki} + d_3 x_k p_{ki} + d_4 x_k p_{ki} + d_5 x_k p_{ki} \right] + \lambda^3 \left[ d_6 x_k p_{ki} \right].
\] (3.2)

where the coefficients of the first-order terms must satisfy
\[
c_0 = -\frac{1}{2}, \quad d_0 = \frac{1}{2}, \quad c_1 + d_1 = 1,
\] (3.3)

while those of the second-order terms satisfy six further independent relations,
\[
\frac{c_1}{2} - 2c_3 + c_3 = d_1, \quad \frac{c_1}{2} + c_4 + c_5 = \frac{1}{2}, \quad d_3 - 2d_4 = -\frac{1}{4},
\]
\[
c_5 - d_2 = \frac{1}{4}, \quad \frac{c_1}{2} + c_6 - d_6 = 0, \quad \frac{c_1}{2} - c_1 d_1 + c_6 + d_5 = 0.
\] (3.4)

Therefore, to first order in \(\lambda\) a single free parameter, \(c_1\), is left, while up to second order one has five free parameters. Note that setting \(\beta = 0\) in (3.2) one obtains realizations of the Poincaré algebra. In particular, for \(\beta = 0, c_1 = c_6 = 0\), it follows \(\hat{x}_i = x_i\), describing ordinary Minkowski space.

3B. The star product for the extended Snyder model

To define the star product for a generic realization of the kind \(\hat{x}_\alpha = x_\beta \phi_{\beta \alpha}(p)\), where \(\alpha, \beta\) are arbitrary indices, using the action \(\triangleright\) we calculate [13]
\[
e^{ik\hat{x}} \triangleright 1 = e^{iK(k)x}.
\] (3.5)

Then
\[
e^{ikx} \triangleright e^{iqx} = e^{iK^{-1}(k)\hat{x}} \triangleright 1, \quad e^{ikx} \ast e^{iqx} = e^{iD(k,q)x} \equiv e^{iD(k,q)x}.
\] (3.6)

and the star product of two plane waves is defined as
\[
e^{ikx} \ast e^{iqx} = e^{iK^{-1}(k)\hat{x}} \triangleright e^{iqx} \equiv e^{iD(k,q)x}.
\] (3.7)
Applying this definition to the realizations (3.2) we obtain
\[ e^{ik_1x_1+\frac{1}{2}k_1x_{ij}} \ast e^{iq_{k}x_k+\frac{1}{2}q_kx_{kl}} = e^{iD_i x_i+\frac{1}{2}D_{ij} x_{ij}}, \] (3.8)
where [8]
\[
D_i(k, q) = k_i + q_i + \lambda \left[ -c_1 k_j q_{ij} + d_1 k_j q_j \right] + \frac{\lambda^2}{2} \left[ \beta(c_0 c_1 + c_3) k_j^2 q_{ij} + \beta(-c_0 c_1 + 2c_2 + c_3) k_j k_j q_{ij} \
+ (c_1^2 - c_1 d_0 - c_1 d_1 + c_0 + d_0) k_j q_{jk} + (c_1 d_0 + c_1 d_1 + c_1 d_2) k_j q_{jk} \
+ (d_1^2 - d_2 d_0 - d_2 d_1 + d_1) k_j q_{jk} + 2\beta c_2 k_j q_{ij}^2 + 2\beta c_3 k_j q_{ij} + 2 \phi c_0 k_j q_{jk} q_{ik} + 2 d_3 k_j q_{jk} q_{ik} \
+ 2d_0 k_j q_{jk} q_{ik} \right] + o(\lambda^3),
\]
\[
D_{ij}(k, q) = k_{ij} + q_{ij} + \lambda \left[ -d_0 k_{ik} q_{jk} + \beta c_0 k_i q_j - (i \leftrightarrow j) \right] + \frac{\lambda^2}{2} \left[ \beta(-c_0 c_1 + c_4 - c_5) k_i k_k q_j \
+ (-d_0^2 + d_3 + 2d_4) k_{ik} k_{jl} q_{kl} + d_0^2 + d_3) k_{ik} k_{jl} q_{kl} + \beta(c_0 d_0 + c_5 + d_2) k_i k_k q_j \
+ \beta(c_1 d_0 + c_1 d_1 + c_4 - d_2) k_i k_j k_{jk} q_k + 2\beta d_2 k_{ik} q_{jk} q_{ij} + 2d_3 k_{ik} q_{kl} q_{jl} + 2d_4 k_{ik} q_{ijkl} \
+ 2\beta c_3 k_{ik} q_{jk} q_{ij} + 2\beta c_5 k_{ik} q_{jk} q_{ij} - (i \leftrightarrow j) \right] + o(\lambda^3). (3.9)
\]

The star product so defined is associative.

4. Generic realization and star product for the κ-deformation

4.1. Realization of \( \hat{X}_i \) and \( \hat{X}_{ij} \)

In this section we extend the previous results to the κ-deformation of the model, employing the formalism introduced in sect. 2. For simplicity, we shall consider only expansions up to order \( \lambda \).

We assume that the generators \( \hat{X}_i \) and \( \hat{X}_{ij} \) of the κ-deformed extended Snyder model in a generic realization can be obtained in terms of the \( x \) and \( p \) by substituting (2.8) and (2.12) in (3.2). Taking into account the relations (3.3), at first order in \( \lambda \) we get

\[
\hat{X}_i = \rho \hat{x}_i + \hat{x}_{ij} a_j = \rho x_i + x_{ij} a_j + \frac{\lambda}{2} \left[ -\beta \rho x_{ij} p_j + 2 c_1 \rho x_j p_{ij} + (x_{ik} p_{jk} - x_{jk} p_{ik}) a_j \
+ 2(1 - c_1)(x_{ik} p_k a_k - x_k p_{ik} a_k) \right],
\]
\[
\hat{X}_{ij} = \hat{x}_{ij} = x_{ij} + \frac{\lambda}{2} \left[ x_{ik} p_{jk} + 2(1 - c_1) x_{ij} p_j - (i \leftrightarrow j) \right]. (4.1)
\]

Using (2.9) and (2.13), the general realization can then be written in terms of \( X_i, X_{ij}, P_i, \) and \( P_{ij}, \)

\[
\hat{X}_i = X_i + \frac{\lambda}{2} \left[ -\beta X_{ij} P_j + 2 c_1 (X_j P_{ij} - X_{ij} P_j) a_i \right] + 2(1 - c_1) X_{ij} P_j a_j + X_{ik} P_{jk} a_j \
+ (2c_1 - 1) (2X_j P_{ij} a_k - X_{ik} P_j a_k + X_{ik} P_{jk} a_k) \right],
\]
\[
\hat{X}_{ij} = X_{ij} + \frac{\lambda}{2} \left[ X_{ik} P_{jk} + 2(1 - c_1) X_{ij} P_j - X_{ik} P_{jk} a_j + (2c_1 - 1) X_{ij} P_j a_k - (i \leftrightarrow j) \right]. (4.2)
\]

Note that in the limit \( \kappa \to \infty \), i.e. \( a_i = \beta = 0 \), in (4.2) one obtains realizations of the Poincaré algebra. In particular if also \( c_1 = 0 \), it follows that \( \hat{X}_i = X_i \) spans the ordinary Minkowski spacetime, which decouples from the tensorial coordinates.

The previous limit implies \( g_0 = \ldots = g_{N-1} = g_N = 0 \). In this case the matrices \( g \) and \( O \) are singular and do not have an inverse and (2.8), (2.9), as well as (4.1) do not make sense. If instead \( g_N = 0, g_2 = 0, \) but \( g_1 \neq 0 \), it follows that \( \rho = 0, \det O = \det g = 0 \) and \( \hat{X}_i = a_i \hat{x}_{ij} \), corresponding to a realization of lightlike κ-Minkowski spacetime [14]. Finally, if \( g_N = 0, g_1 \neq 0, g_2^2 < 0 \) it follows that \( \rho \neq 0, \det O \neq 0, \det g \neq 0 \) and the corresponding algebra is isomorphic to the timelike κ-Poincaré algebra.
4.B. The star product and coproduct for the $\kappa$-deformation

The star product of exponentials in terms of coordinates $X_i$, $X_{ij}$ and momenta $K_i$, $K_{ij}$ and $Q_i$, $Q_{ij}$ is defined as

$$e^{iK_iX_i+i\frac{1}{2}K_{ij}X_{ij}} * e^{iQ_iX_i+i\frac{1}{2}Q_{ij}X_{ij}} = e^{iD_i(K,Q)X_i+i\frac{1}{2}D_{ij}(K,Q)X_{ij}},$$

(4.3)

Using the relations (2.16) and (2.12) it follows that

$$K_iX_i + \frac{1}{2}K_{ij}X_{ij} = k_i x_i + \frac{1}{2}k_{ij}x_{ij}, \quad Q_iX_i + \frac{1}{2}Q_{ij}X_{ij} = q_i x_i + \frac{1}{2}q_{ij}x_{ij},$$

(4.4)

and therefore

$$e^{iK_iX_i+i\frac{1}{2}K_{ij}X_{ij}} * e^{iQ_kX_k+i\frac{1}{2}Q_{kl}X_{kl}} = e^{iD_i(k,q)x_i+i\frac{1}{2}D_{ij}(k,q)x_{ij}},$$

(4.5)

Using then the inverse relations (2.17), we obtain

$$D_i(K,Q) = \frac{1}{\rho}D_i(k,q), \quad D_{ij}(K,Q) = D_{ij}(k,q) + \frac{1}{\rho}(a_iD_j(k,q) - a_jD_i(k,q)).$$

(4.6)

Finally, from (3.9) and the relations (2.13) for $k_i$, $k_{ij}$ and $q_i$, $q_{ij}$, we get

$$D_i(K,Q) = K_i + Q_i + \lambda \left[ -c_1(K_jQ_{ij} + K_jQ_{iaj}) + (1 - c_1)(K_iQ_j + K_iP_ja_j) + (2c_1 - 1)K_jQ_ja_i \right],$$

$$D_{ij}(K,Q) = K_{ij} + Q_{ij} + \frac{\lambda}{2} \left[ \beta K_jQ_j - (K_{ik}Q_{jk} + K_{ik}Q_{jka_k} + K_{jk}Q_{ika_k}) - (2c_1 - 1)(K_kQ_ja_i + K_jQ_ka_i - (i \leftrightarrow j) \right].$$

(4.7)

The coproduct is then obtained as (see Appendix)

$$\Delta P_i = D_i(P \otimes 1 + 1 \otimes P) \quad \Delta P_{ij} = D_{ij}(P \otimes 1 + 1 \otimes P).$$

(4.8)

At first order in $\lambda$ we have therefore

$$\Delta P_i = \Delta_0 P_i + \lambda \left[ -c_1(P_j \otimes P_{ij} + P_j \otimes P_{iaj}) + (1 - c_1)(P_{ij} \otimes P_j + P_i \otimes P_ja_j) + (2c_1 - 1)P_j \otimes P_ja_i \right],$$

$$\Delta P_{ij} = \Delta_0 P_{ij} + \frac{\lambda}{2} \left[ -\beta P_i \otimes P_j - (P_{ik} \otimes P_{jk} + P_i \otimes P_{jka_k} + P_{jk} \otimes P_{ia_k}) - (2c_1 - 1)(P_k \otimes P_{jka_k} + P_k \otimes P_{ka_k}a_i + P_{jka_k}a_i + P_j \otimes P_{ka_k}a_i - (i \leftrightarrow j) \right].$$

(4.9)

where

$$\Delta_0 P_i = P_i \otimes 1 + 1 \otimes P_i, \quad \Delta_0 P_{ij} = P_{ij} \otimes 1 + 1 \otimes P_{ij}.$$ \hfill (4.10)

This coproduct is coassociative,

$$\Delta \otimes 1 \Delta = (1 \otimes \Delta)\Delta.$$ \hfill (4.11)

Finally, note that $\Delta P_i = \frac{1}{\rho}\Delta p_i$ and $\Delta P_{ij} = \Delta \left( p_{ij} + \frac{1}{\rho}(a_i p_j - a_j p_i) \right)$, and after inserting (2.12) in these relations, we get the expressions (4.9) for $\Delta P_i$ and $\Delta P_{ij}$. This is consistent with the results (4.7) for $D_i(K,Q)$ and $D_{ij}(K,Q)$.

5. The twist

In the Hopf algebroid approach, the twist is defined as a bilinear operator such that $\Delta h = \mathcal{F} \Delta_0 h \mathcal{F}^{-1}$ for each $h$ in the algebroid, and it has been discussed in several papers [15,16], see Appendix A. It is given by

$$\mathcal{F}^{-1} = e^{\frac{\mu}{2}(1 \otimes X_{\mu\nu}) (\Delta - \Delta_0) X_{\mu\nu}},$$

(5.1)
where $: \cdot :$ denotes normal ordering [16]. Note that this definition of twist differs from that of Drinfeld twist in the Hopf algebra sense, since in the latter case the coordinates $X$ do not appear.

In [9], the twist for the $\kappa$-deformed extended Snyder model was calculated perturbatively for a Weyl realization. Here we compute it for a generic realization: defining $F^{-1} = \epsilon^F$, one gets at leading order

$$ F = i(1 \otimes X_i)(\Delta - \Delta_0)P_i + \frac{i}{2}(1 \otimes X_{ij})(\Delta - \Delta_0)P_{ij} $$

$$ = iP_i \otimes (\hat{X}_i - X_i) + \frac{i}{2}P_{ij} \otimes (\hat{X}_{ij} - X_{ij}), $$

which in the Weyl realization yields [9]

$$ F = \frac{i\lambda}{2} \left[ P_i \otimes \left( -\beta X_{ij} P_j + X_j P_{ij} - X_j P_j a_i + X_i P_j a_j + X_{ij} P_{k j} a_k \right) + P_{ij} \otimes (X_{ik} P_{jk} + X_j P_{ij} - X_{ik} P_{k j} a_j) \right]. $$

In the general case, using either (4.2) or (4.9), we obtain

$$ F = \frac{i\lambda}{2} \left[ P_i \otimes \left( -\beta X_{ij} P_j + 2c_1(X_j P_{ij} - X_j P_j a_i) + 2(1 - c_1)X_j P_j a_j + X_{ik} P_{jk} a_j \
+ (2c_1 - 1)(2X_j P_{a j} - X_{jk} P_{ai} a_k + X_{ik} P_{aj} a_k) \right) + P_{ij} \otimes (X_{ik} P_{jk} + 2(1 - c_1)X_j P_j a_j + (2c_1 - 1)X_{ik} P_{a k} a_j) \right]. $$

It is easy to check that

$$ \Delta P^i = F \Delta_0 P^i F^{-1}, \quad \Delta P^{ij} = F \Delta_0 P^{ij} F^{-1}. $$

It also holds

$$ \hat{X}_i = m F^{-1}(\beta \otimes 1)(X_i \otimes 1), \quad \hat{X}_{ij} = m F^{-1}(\beta \otimes 1)(X_{ij} \otimes 1). $$

6. The Lorentz algebra

The Lorentz algebra so(1, $N - 1$) with generators $M_{ij}$ is defined in (2.19). From (2.8) and (2.12) one obtains

$$ [M_{ij}, X_k] = i(\eta_{ik} X_j - \eta_{jk} X_i + a_j X_{ik} - a_i X_{jk}), $$

$$ [M_{ij}, X_{kl}] = i(\eta_{ik} X_{jl} - \eta_{jk} X_{il} - \eta_{il} X_{jk} + \eta_{jl} X_{ik}), $$

where now

$$ M_{ij} = X_i P_j + X_{ik} P_{jk} - a_j X_{ik} P_k - (i \leftrightarrow j). $$

The definition of $M_{ij}$ is obtained by the previous recipe of substituting (2.17), (2.13) in (2.18) In the same way we obtain

$$ [M_{ij}, P_k] = i(\eta_{ik} P_j - \eta_{jk} P_i), $$

$$ [M_{ij}, P_{kl}] = i \left[ \eta_{ik}(P_{jl} - a_j P_l) - \eta_{il}(P_{jk} - a_j P_k) - (i \leftrightarrow j) \right]. $$

It is easy to see that the relations (6.1) also hold for $\hat{X}_i, \hat{X}_{ij}$ in a generic realization,

$$ [M_{ij}, \hat{X}_k] = i(\eta_{ik} \hat{X}_j - \eta_{jk} \hat{X}_i + a_j \hat{X}_{ik} - a_i \hat{X}_{jk}), $$

$$ [M_{ij}, \hat{X}_{kl}] = i(\eta_{ik} \hat{X}_{jl} - \eta_{jk} \hat{X}_{il} - \eta_{il} \hat{X}_{jk} + \eta_{jl} \hat{X}_{ik}). $$

The commutation relations of the $\hat{X}_i, \hat{X}_{ij}$ and $M_{ij}$ satisfy all the relevant Jacobi identities and generate a Lie algebra. We have defined the action of these operators on the unity as

$$ \hat{X}_i \triangleright 1 = X_i, \quad \hat{X}_{ij} \triangleright 1 = X_{ij}, \quad M_{ij} \triangleright 1 = 0, $$

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Interesting special cases are $g_N = 0$, that corresponds to $\kappa$-Minkowski spacetime with extra tensorial coordinates and momenta, and $g_i = 0$, that corresponds to the extended Snyder model.

Using the twist (5.4) we can calculate

$$\Delta M_{ij} = \Delta_0 M_{ij} + o(\lambda^2), \quad (6.6)$$

where $\Delta_0 M_{ij} = 1 \otimes M_{ij} + M_{ij} \otimes 1$, since order-$\lambda$ corrections vanish.

We can improve this result, noting that if $a_i = 0$, the twist (5.1) can be written as

$$\mathcal{F} = \exp \left( \sum_{k,l=1}^{\infty} \lambda^{k+l-1} f_{k,l} \right), \quad (6.7)$$

where $f_{k,l} \sim p^k \otimes x^l$, with all indices contracted. Then, $[f_{k,l}, M_{ij} \otimes 1 + 1 \otimes M_{ij}] = 0$ for any $k, l$ and hence $\Delta M_{ij} = \Delta_0 M_{ij}$. Moreover, our construction implies that also for $a_i \neq 0$, $\Delta M_{ij} = \Delta_0 M_{ij}$.

7. Conclusions

In this paper we have discussed the $\kappa$-deformation of the extended Snyder model, by introducing a deformation of the flat metric in the definition of the extended Snyder algebra [9]. This formalism permits to have an associative star product and a coassociative coproduct, contrary to the standard implementations of the $\kappa$-deformed Snyder model [13]. Using this formalism, we were able to calculate in a straightforward way several properties of the associated Hopf algebra. We also introduced a suitable definition of the Lorentz algebra acting on the extended space.

Before concluding, we explicitly relate the parameters of our model to the standard ones in $\kappa$-Minkowski and Snyder: in the present paper we have introduced the dimensionless parameters $\lambda$, and $g_i, g_N$ coming from the metric $g_{\mu\nu}$, together with a mass parameter $\kappa$. The timelike $\kappa$-Minkowski spacetime is described by

$$[\hat{x}_0, \hat{x}_i] = -\frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad i = 1, \ldots, N - 1, \quad (7.1)$$

while the original Snyder model is described by

$$[\hat{x}_i, \hat{x}_j] = \frac{i}{m^2} M_{ij}, \quad (7.2)$$

where $M_{ij}$ are Lorentz generators and $m$ is a constant of the order of the Planck mass.

We assume that the matrix $O$ is real. Consequently, $\rho^2 = g_N - g_k g_k > 0$. Let us define

$$a_i = \frac{g_i}{\kappa} = \frac{|g| u_i}{\kappa}, \quad (7.3)$$

where $|u_i| \leq 1$, $|u_k u_k| = 1$ and $|g| = \sqrt{g_k g_k} \neq 0$. Then, comparing (2.3) with (7.1) we obtain

$$\frac{1}{\kappa} = \frac{\lambda |g|}{\kappa}, \quad (7.4)$$

while comparing (2.3) with (7.2),

$$\frac{1}{m^2} = \frac{\lambda g_N}{\kappa^2}. \quad (7.5)$$

Hence,

$$\frac{\tilde{\kappa}}{m} = \sqrt{\frac{g_N}{\lambda |g|^2}} \quad (7.6)$$

If $m$ is of the order of the Planck mass, and $\tilde{\kappa} < m$, $g_N < \lambda g_k^2$. Note that in the limit $|g| = 0$ it follows $\tilde{\kappa} = 0$ and we are left with the extended Snyder model. If instead $g_N = 0$ with $a^2 < 0$ the corresponding algebra is isomorphic to the timelike $\kappa$-Poincaré algebra.
From a physical point of view, the major problem of our formalism is the interpretation of the tensorial degrees of freedom corresponding to the coordinates $\hat{X}_{ij}$. When $N = 4$, one may interpret them as parametrizing six extra dimensions. In the limit $\kappa \to \infty$, the four-dimensional spacetime and the internal space are completely independent. However, at Planck-scale distances or energies these two spaces mix, giving rise to possible physical effects.

This is even more evident if one assigns the $\hat{X}_{ij}$ the dimension of a length, like the $\hat{X}_i$, in contrast with the previous discussion where we have implicitly assumed that the coordinates $\hat{X}_{ij}$ have dimension 0 in natural units, as in the original Snyder model. However, no compelling reason constrains the choice of the physical dimension of the tensorial degrees of freedom, which depends on their physical interpretation. With this choice, $\lambda$ becomes dimensional and we may identify it with the Planck length. One may then interpret the $X_{ij}$ as position coordinates, that parametrize a disconnected universe, that interacts with ours very weakly only at the Planck scale. However, its gravitational effects are still effective and may contribute to the dark matter content of our universe.

At the present level of development of the theory the previous considerations are only conjectural. Of course, a more elaborate model, taking into account also the dynamics would be necessary to discuss quantitatively this hypothesis.

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Appendix: Twist in the Hopf algebroid approach

The twist in the Hopf algebroid approach [16] is constructed so that the following relation holds:

$$e^{ikx} \star e^{iqx} = mF^{-1}(\triangleright \otimes \triangleright)(e^{ikx} \otimes e^{iqx}) = e^{iD(k,q)x}.$$  \hspace{1cm} (A.1)

The result for the twist is, using normal ordering,

$$F^{-1} = e^{(1 \otimes x)(\Delta - \Delta_0)p};,$$  \hspace{1cm} (A.2)

with action

$$\Delta p(\triangleright \otimes \triangleright)(e^{ikx} \otimes e^{iqx}) = D(k,q)(e^{ikx} \otimes e^{iqx}).$$  \hspace{1cm} (A.3)

Hence,

$$\Delta p_\mu = D_\mu(p \otimes 1, 1 \otimes p),$$  \hspace{1cm} (A.4)

where $\Delta p_\mu$ is the coproduct of the momentum $p_\mu$ since

$$\Delta p_\mu = F\Delta_0 p_\mu F^{-1}, \quad \text{with} \quad \Delta_0 p_\mu = (p_\mu \otimes 1, 1 \otimes p_\mu).$$  \hspace{1cm} (A.5)

Since the star product is associative, the coproduct so defined is coassociative.

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