Construction of Gaussian Integer Periodic Complementary Sequence Set with Zero Correlation Zone

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Abstract. Based on aperiodic complementary sequence (ACS) sets and orthogonal matrices, the construction of Gaussian integer (GI) zero correlation zone (ZCZ) periodic complementary sequence (ZPCS) sets is proposed. The constructed GI ZPCS sets can achieve the theoretical bound, and the length of ZCZ can be chosen flexibly. The GI ZPCS sets obtained by this paper can be used in multi-carrier code division multiple access system to obtain higher spectrum efficiency and eliminate interference.

1. Introduction

Since Golay proposed the complementary pairs in the study of infrared spectrometry in 1961 [1], this kind of optimal code characterized by the property of ideal autocorrelation sum has been widely concerned and studied for decades. It has been shown that complementary sequence (CS) sets derived from Golay complementary pairs (GCPs) have been applied in spread spectrum communication systems, such as multiple input multiple output (MIMO) code division multiple access (CDMA) systems and multi-carrier CDMA (MC-CDMA) [2],[3]. CS sets help to suppress multipath interference (MPI) and multiple access interference (MAI) because of ideal autocorrelation and cross correlation performance [4]. Nevertheless, as we know, the key drawback of CS sets is that the set size is bounded by the number of the flock size [5]. Consequently, CS sets cannot support a large number of users in multi-carrier communication circumstance. In order to increase the user capacity, in other words, to enlarge the set size of CS sets, Fan introduced the idea of zero correlation zone (ZCZ) into CSs and constructed binary ZPCS sets [5]. The theoretical bound [6] shows that ZCZ CS sets have much larger set size than conventional CS sets. ZCZ CS sets break the dilemma that the amount of terminal access in a block is limited by the number of channel carriers in practical applications.

Gaussian integer (GI) sequence is a class of complex sequences whose elements of real and imaginary parts are both integers. The quaternary sequences and quadrature amplitude modulation (QAM) sequences widely used in communications are special forms of GI sequences. Due to high transmission efficiency and spectrum utilization, sequence design over GI alphabet set has attracted widespread attention. GI perfect sequences (PS) [7],[8] with ideal autocorrelation characteristics and GI ZCZ sequences [9] suitable for quasi-synchronous communication systems have been well studied successively. Unfortunately, they are not competent to multi-carrier systems. Accordingly, the study of GI ZCZ CS sets has been carried out in recent years. Some constructions of GI ZCZ periodic CS (ZPCS) sets have been presented [10]-[15]. In [10] and [11], based on the binary ZPCS sets and
inverse Gray mapping, quaternary ZPCS sets were constructed. Subsequently, on the basis of quaternary ZPCS sets, Zeng investigated the construction of 16-QAM ZPCS sets by aid of mapping methods [12]. In [13], the 8-QAM+ ZPCS sets are constructed based on the ternary PS and the binary orthogonal matrix. Besides, the ZPCS over general GI alphabet set were developed. In [14], two constructions of GI ZPCS sets were presented, one of which completely depends on original ZPCS sets, so the parameter is same as that of the original sets, while the other provides the flexible choices of parameters by use of the PS and the orthogonal matrix. In [15], two constructions of GI ZPCS sets were proposed by shifting the periodic CS sets and using GI orthogonal matrix and PS respectively. For the existing GI ZPCS sets, the length of ZCZ always be limited to a certain extent, shown in Table 1. Therefore, we propose a class of GI ZPCS sets with new parameters based on aperiodic complementary sequence (ACS) sets and orthogonal matrices in this paper, the length of ZCZ in which are flexible. Our construction can produce more GI ZPCS sets and provide a large number of address choices for high-speed communication.

2. Preliminaries

Definition 1: Given two complex sequences \( a = (a(0), a(1), \ldots, a(N-1)) \) and \( b = (b(0), b(1), \ldots, b(N-1)) \). The aperiodic and periodic cross-correlation functions (CCFs) of \( a \) and \( b \) are defined by

\[
C_{a,b}(\tau) = \begin{cases} 
\sum_{i=0}^{N-1} a(i)b^*(i+\tau), & 0 \leq \tau \leq N-1 \\
\sum_{i=0}^{N-1} a(i)b^*(1-i), & 1-N \leq \tau < 0 \\
0, & |\tau| \geq N.
\end{cases}
\]

(1)

\[
R_{a,b}(\tau) = \sum_{i=0}^{N-1} a(i)b^*(i+\tau)
\]

(2)

where * denotes the complex conjugate, \( t + \tau = t + \tau (\mod N) \). If \( a = b \), \( C_{a,a}(\tau) \) and \( R_{a,a}(\tau) \) are the aperiodic and periodic autocorrelation functions (ACFs) and simply denoted by \( C_a(\tau) \) and \( R_a(\tau) \). Moreover, the correlation functions of \( a \) and \( b \) satisfy

\[
R_{a,b}(\tau) = C_{a,b}(\tau) + C_{a,b}(\tau-N)
\]

(3)

Definition 2: Let \( a = (a(0), a(1), \ldots, a(N-1)) \) be a complex sequence, then \( a \) is a PS if

\[
R_a(\tau) = \begin{cases} 
E, & \tau = 0 (\mod N) \\
0, & \tau \neq 0 (\mod N),
\end{cases}
\]

(4)

where \( E \) is the energy of sequence \( a \) and \( E = \sum_{i=0}^{N-1} |a(i)|^2 \).

Definition 3: Given a set composed of \( M \) sequence sets \( \mathbb{A} = \{\mathbb{A}^m, 0 \leq m < M \} \), where each sequence set contains \( P \) sequences of length \( N \), \( \mathbb{A}^m = \{a^m_p, 0 \leq p < P \} \). Suppose \( \mathbb{A}^m, \mathbb{A}^n \in \mathbb{A} \), where \( 0 \leq m_1, m_2 < M \), then \( \mathbb{A} \) is called an aperiodic ZCZ CS set or a periodic ZCZ CS set and denoted by \( (M, Z)ACS_P^N \) or \( (M, Z)PCS_P^N \) if
Given an orthogonal matrix $A$, let $Q$ and $P$ be a ZPCS set, and $N$ a PCS set, if $P \subseteq Q$, then we call the ZPCS set optimal.

Lemma 1[6]: For a $(M, Z)PCS^p$ ZPCS set, the theoretical bound is

$$M \leq P \cdot \left\lfloor \frac{N}{Z} \right\rfloor.$$

Let $M = P \cdot \left\lfloor \frac{N}{Z} \right\rfloor$, if $M = M_o$, then we call the ZPCS set optimal.

Definition 4: Given two sequences $a = (a(0), a(1), \ldots, a(N - 1))$ and $b = (b(0), b(1), \ldots, b(N - 1))$. The sequence $u = (u(0), u(1), \ldots, u(N - 1))$ is obtained by filtering the sequence $a$ by the sequence $b$ as follows

$$u(n) = a(t) \cdot b(t + n) = R_{ab}(n), \quad 0 \leq t, n < N$$

Lemma 2[16]: The correlation property of the resultant $u$ is the same as the base sequence $a$ if the filtering sequence $b$ is a perfect sequence.

3. Construction of ZPCS Sets

In this section, a GI ZPCS set is constructed based on an ACS set and an orthogonal matrix.

Step 1: Let $A = \{A^p, 0 \leq p < P\}$ be an ACS set with parameters $(P, Q, N)ACS$, where $A^p = \{a^p_q, 0 \leq q < Q\}$, $a^p_q = (a^p_q(0), a^p_q(1), \ldots, a^p_q(N - 1))$. By adding $J$ zeros to $a^p_q$, the sequence $a^p_q = (a^p_q(0), a^p_q(1), \ldots, a^p_q(N - 1), 0, \ldots, 0)$ can be obtained, where $L = N + J$, $J \geq 0$. Given an orthogonal matrix $H = [h_{i,j}]_{m \times m}$ and $\gcd(L, M) = 1$.

Step 2: Construct a sequence set $U = \{U^m, 0 \leq m < MP\}$, $u^m = (u^m(0), u^m(1), \ldots, u^m(LM - 1))$ and

$$u^m_n(t) = a^m_{n \text{mod}P} \cdot h_{\frac{n}{P} \text{mod}M}, \quad 0 \leq t < LM$$

Step 3: Let $v = (v(0), v(1), \ldots, v(LM - 1))$ be a PS of length $LM$. The sequence set $S = \{S^m, 0 \leq m < MP\}$ is obtained by the filtering operation, where $S^m = (s^m_n, 0 \leq p < P)$, $s^m_n = (s^m_n(0), s^m_n(1), \ldots, s^m_n(LM - 1))$, and

$$s^m_n(q) = u^m_n(t) \cdot v(t + q) = R_{s^m_n}, \quad 0 \leq q < LM.$$

Theorem 1: The sequence set $S$ generated from above steps is a GI ZPCS set with parameters $(MP, LM)PCS^m_{Q^m}$.

Proof: Let $U^m, U^m_n \in U$ and $0 \leq m_1, m_2 < MP$, $t = M \cdot t_1 + t_2$, the periodic CCF is calculated as
Obviously, we have $PP < 1 \mod 2 \mod tt$.

Furthermore, the filtering is a PCS set according to $tt LM = \tau \tau \tau \tau$.

The sequence set $U$ is a ZPCS set with parameters $(MP, L)PCS_{\varphi}^{LM}$.

In Step 3, since the filtering sequence $v$ is a PS, the correlation property of $S$ is same as that of $U$. Therefore, $S$ is also a $(MP, L)PCS_{\varphi}^{LM}$ ZPCS set according to Lemma 2. Furthermore, the filtering
operations between \( u^n \) and \( v \) cover addition, subtraction and multiplication operations on the GI set, so all the items of the resultant sequence \( s^n \) also belong to the GI set.

From the above, the sequence set \( S \) is a GI ZPCS set with parameters \((MP, L)PCS^{LM}_0\).

**Example 1:** Given an ACS set \( A = \{ A^0, A^1 \} \) with parameters \((2,2,4)ACS\), where \( A^0 = \{(1,1,1,1), (1,-1,1,1)\}, A^1 = \{(1,1,1,1), (1,-1,1,-1)\} \). Let the zero-padded parameter \( J \) equal to 3, then ACS \( A' \) with parameters \((2,2,7)ACS\) is obtained, where \( A'^0 = \{(1,1,1,-1,0,0,0,1,-1,1,1,0,0,0)\}, A'^1 = \{(1,1,1,1,0,0,0,1,-1,1,1,0,0,0)\} \). \( A'^0 = \{(1,1,1,1,0,0,0,1,-1,1,1,0,0,0)\}, A'^1 = \{(1,1,1,1,0,0,0,1,-1,1,1,0,0,0)\} \). Given an orthogonal matrix \( H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), the sequence set \( U = \{ U^0, U^1 \} \) is obtained as \( U^0 = \{(1,1,1,1,0,0,0,1,1,1,0,0,0)\}, (1,-1,1,1,0,0,0,1,-1,1,1,0,0,0)\}, U^1 = \{(1,1,1,1,0,0,0,1,-1,1,1,0,0,0)\}, (1,-1,1,1,0,0,0,1,-1,1,1,0,0,0)\}\). Hence, \( U \) is a ternary ZPCS set with parameters \((4,7)PCS^1_2\). Next, take a GI PS \( v = \{7, \ldots, 2, 3 + j, 2, 3 + j, -1 + 3, -5 + 7, \ldots\} \) as the filtering sequence, the set \( S \) is constructed as follows.

\[
S^0 = \{(4 + 20j, 4 - 2j, 4 + 2j, 4 - 2j, 4 - 2j, 4 + 2j, 4 - 2j, 4 - 2j, 4 - 2j, 4 - 2j, 4 - 18j, 4 - 2j, 4 - 2j, 4 - 8j)\}, \\
S^1 = \{(4 + 8j, 4 + 10j, 4 - 2j, 4 + 14j, 4 - 2j, 4 + 14j, 4 - 2j, 4 + 14j, 4 - 2j, 4 + 14j, 4 - 2j, 4 - 18j, 4 - 8j)\}
\]

According to Theorem 1, \( S \) is a GI ZPCS set with parameters \((4,7)PCS^1_2\). The periodic ACFs and CCFs of \( S \) are listed by

\[
R_s^s(\tau) = R_s^s(\tau) = (3712, 0, \ldots, 0, 3712, 0, \ldots, 0), \quad R_s^s(\tau) = R_s^s(\tau) = (3712, 0, \ldots, 0, -3712, 0, \ldots, 0), \quad R_s^s(\tau) = R_s^s(\tau) = (0_{14}), \quad \text{where } (0_{14}) \text{ denotes } (0, \ldots, 0).
\]

4. **Discussion on the Parameters of the ZPCS Set**

In this section, we discuss the theoretical bound of the ZPCS set obtained by Theorem 1 and compare with the known ZPCS sets.

According to Lemma 1, the set size of the ZPCS set with parameters \((MP, L)PCS^LM_0\) obtained by Theorem 1 is bounded by

\[
M_s = Q \cdot \frac{LM}{L} = QM
\]

Accordingly, when \( P=Q \), the ZPCS set constructed from Theorem 1 is optimal with respect to the theoretical bound.

The ACS sets as base sequences in Theorem 1 can be provided by a large number of existing references [17][18], so that the constructed GI ZPCS sets are abundant. The length of ZCZ of constructed ZPCS sets can be chosen flexibly by adding the number of zero-padding. Table 1 shows the comparison of known ZPCS sets.
Table 1. Comparison of several constructions of ZPCS sets.

| Constructions | Initial sequences | Parameters | Optimal or not | Length of ZCZ | Length of sequence |
|---------------|-------------------|------------|----------------|---------------|-------------------|
| [10]          | ZPCS sets         | (T, Z)PCS\(_M^N\) | when \(T = M \left\lceil N/Z \right\rceil\), \(M \leq N\) | not flexible | not flexible |
|               |                   | \((2MG, \min(L, L_n))\)PCS\(_p^N\) | \(MK = \left\lfloor \frac{PN}{2\min(L, L_n)} \right\rfloor\), \(M \leq N\) | not flexible | not flexible |
| [11]          | Quaternary ZPCS   | (T, Z)PCS\(_M^N\) or (T, Z)PCS\(_{2M}^N\) when \(T = M \left\lceil N/Z \right\rceil\) or \(T = 2M \left\lceil N/Z \right\rceil\), \(M \leq N\) | not flexible | not flexible |
| [12]          | Ternary PS of length \(N\), binary orthogonal matrix \(H_{M \times M}\) \(\left(\mathbf{MK}, L\right)\)PCS\(_p^N\) when \(L = qL + r\), where \(0 \leq r < L\), \(q\) is even, \(M \leq N\) | optimal | flexible | not flexible |
| Construction 1 in [14] | ZPCS set | \((M, L)\)PCS\(_p^N\) when \(M = \left\lfloor N/L \right\rfloor\) | optimal | flexible | not flexible |
| Construction 2 in [14] | PS of length \(N\), orthogonal matrix \(H_{M \times M}\) \(\left(M, K\right)\)PCS\(_p^N\) when \(MK = \left\lfloor N/L \right\rfloor\) | flexible | not flexible | |
| Construction 1 in [15] | PCS set | \((MM', L)\)PCS\(_p^N\) when \(MM' = \left\lfloor N/L \right\rfloor\) | flexible | not flexible | |
| Construction 2 in [15] | PS of length \(NL\), orthogonal matrix \(H_{MN \times MNL}\) \(\left(MN, L\right)\)PCS\(_{MN}^N\) | optimal | flexible | |
| Theorem 1      | ACS set \(\left(P, Q, N\right)\)ACS, orthogonal matrix \(H_{M \times M}\) \(\left(MP, L\right)\)PCS\(_{QJM}^N\) when \(P = Q\) | optimal | flexible | flexible |

5. Conclusion
By zero-padding and filtering operation, this paper proposes a new class of GI ZPCS sets. By aid of abundant ACS sets and orthogonal matrices, the construction method can produce a large number of GI ZPCS sets with flexible ZCZ length. The parameters of the resulting set can achieve the theoretical bound if the original ACS set is optimal. This study is suitable for high-speed multi-carrier communication and provides rich address choices.

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