The Strong-Coupling Spectrum of the Seiberg-Witten Theory

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ABSTRACT

We carefully study the global structure of the solution of the $N = 2$ supersymmetric pure Yang-Mills theory with gauge group $SU(2)$ obtained by Seiberg and Witten. We exploit its $Z_2$-symmetry and describe the curve in moduli space where BPS states can become unstable, separating the strong-coupling from the weak-coupling region. This allows us to obtain the spectrum of stable BPS states in the strong-coupling region: we prove that only the two particles responsible for the singularities of the solution (the magnetic monopole and the dyon of unit electric charge) are present in this region. Our method also permits us to very easily obtain the weak-coupling spectrum, without using semi-classical methods. We discuss how the BPS states disintegrate when crossing the border from the weak to the strong-coupling region.

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1. Introduction

In an already classical paper, Seiberg and Witten [1] derived the exact low energy wilsonian effective action for the pure \( N = 2 \) supersymmetric Yang-Mills theory with gauge group \( SU(2) \). (Note that pure gauge \( SU(2) \), i.e. without extra matter, is actually equivalent to \( SO(3) \).) Since then, their work has been generalized to other gauge groups and to theories with matter (see, e.g. [2] and references therein). The main ingredient that allows for an exact solution of these strongly interacting theories is duality: one can perform certain duality transformations (the duality group being \( Sp(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z}) \) in the case of an \( SU(2) \) theory) which relate different descriptions of the same low energy theory in terms of different sets of elementary fields. The latter are mutually non local. In this sense, a duality transformation is like a change of variables, not a symmetry of the theory. However, in some particular models, like \( N = 4 \) super Yang-Mills theory [3] or \( N = 2 \) super \( SU(2) \) Yang-Mills theory with four flavors [4], these duality transformations are conjectured to be a symmetry of the theory (Montonen-Olive duality [5]). In particular, the spectrum of BPS saturated states should then be self-dual. Moreover, even in those theories in which Montonen-Olive duality does not hold in full, there may exist some class of duality transformations under which the spectrum is invariant. For instance, it was argued in [1] that the semiclassical (i.e. the weak-coupling) spectrum should be invariant under the transformation which corresponds to the monodromy around the point at infinity in the moduli space, \( (n_e, n_m) \rightarrow (-n_e + 2n_m, -n_m) \), where \( n_e \) and \( n_m \) are the electric and magnetic charges of the stable BPS states.

In the present paper, restricting our attention to the \( SU(2) \) case of [1], we exploit the existence of another duality (i.e. \( Sp(2, \mathbb{Z}) \)) transformation, related to the \( \mathbb{Z}_2 \)-symmetry of the moduli space, under which the spectrum should be invariant. This, together with a few other arguments insuring the physical consistency of the exact Seiberg-Witten solution, will allow us to unambiguously determine both the weak-coupling spectrum (which has already been investigated when \( n_m \leq 2 \) using semi-classical methods [6]), and, more important, also the strong-coupling spectrum. In the weak-coupling region the spectrum of BPS states contains all dyons \( \pm(n, 1) \) with unit magnetic charge and arbitrary integer electric charge \( n \), in addition to the perturbative W-bosons \( \pm(1, 0) \). When entering the strong-coupling region, almost all of these states decay and one is left with only two BPS states, namely the magnetic monopole \( \pm(0, 1) \) and the dyon of unit electric charge described either as \( \pm(1, 1) \) or as \( \pm(1, -1) \). This
dramatic change in the spectrum is possible precisely since the weak and strong-coupling regions in moduli space are separated by a curve $C$ where the otherwise stable BPS states can become unstable. Such a phenomenon was first considered in two dimensional theories in [7]. We use the above-mentioned $\mathbb{Z}_2$-symmetry to prove that, for the Seiberg-Witten solution, no other BPS states can be present in the strong-coupling region.

This paper is organized as follows: Section 2 is a brief review of the work of Seiberg and Witten [1] to fix our notation and insist on the exact $\mathbb{Z}_2$-symmetry of the theory. Particular attention is paid to the explicit form of the solution of [1] and to its analytic structure. In Section 3, we describe the curve $C$ on which the usually stable BPS states become unstable and which is the border between the strong and weak-coupling regions of the moduli space $\mathcal{M}$. Section 4 is devoted to the $\mathbb{Z}_2$-symmetry and its consequences. In Section 5, we then rigorously determine both the weak and strong-coupling spectra. In Section 6, we present a simple physical consistency check of our solution, illustrating how the BPS states decay* when crossing the curve $C$. Finally, in Section 7, we recapitulate our assumptions and conclusions.

2. Overview of Seiberg-Witten theory

For most of the material presented in this section, see [1] and references therein (see also [9] for a pedagogical introduction). The microscopic action $S_{\text{mic}}$ of $N = 2$ supersymmetric gauge theory without hypermultiplets (i.e. without extra matter) is expressed in terms of an $N = 2$ vector superfield $\Psi^a$ transforming in the adjoint representation of the gauge group, which for us will be $SU(2)$, or equivalently $SO(3)$. Among others, $\Psi$ contains a scalar field $\phi$ whose potential is $V(\phi) = \frac{1}{2} \text{tr}([\phi^\dagger, \phi])^2$. Thus, as long as $\phi$ and $\phi^\dagger$ commute in $su(2)$, the scalar potential remains zero even for a nonvanishing expectation value of $\phi$ which spontaneously breaks the $SU(2)$ gauge symmetry down to $U(1)$. This shows that, at least semiclassically, the theory has a continuum of gauge inequivalent vacua, called the moduli space, parametrized by the gauge invariant quantity $u = \langle \text{tr} \phi^2 \rangle$. Seiberg and Witten argued [1] that this picture is maintained quantum mechanically, $u$ being a good local coordinate on the quantum moduli space $\mathcal{M}$.

* Such kinematics of possible decay reactions were also considered in [8], indicating already the possibility of a strong-coupling spectrum consisting only of the monopole and the dyon.
Among other symmetries, the action $S_{\text{mic}}$ has a global $U(1)_R$ $R$-symmetry under which $\phi$ has charge 2. This $U(1)_R$ symmetry is reduced by an anomaly down to $\mathbb{Z}_8$. This can be seen from the form of the instanton contributions in the low energy effective action [1,10]. Since $u = \langle \text{tr} \phi^2 \rangle$ has charge 4 under this symmetry, a given vacuum with a non-vanishing value of $u$ furthermore breaks $\mathbb{Z}_8$ to $\mathbb{Z}_4$. Nevertheless, let us stress that $u$ and $-u$ correspond to physically equivalent vacua related by the $\mathbb{Z}_8$-symmetry of the quantum theory. This is the $\mathbb{Z}_2$-symmetry on the moduli space which we will extensively use in the following.

As already mentioned, at a generic point $u \in \mathcal{M}$ the gauge symmetry is broken down to $U(1)$ by the vacuum expectation value of $\phi$, $\langle \phi \rangle = \frac{1}{2}a(u)\sigma_3$. The low energy wilsonian effective lagrangian $\mathcal{L}$ then is expressed in terms of the light fields of the microscopic theory. By $N = 2$ supersymmetry, the most general form for $\mathcal{L}$ is, in terms of the $N = 1$ abelian vector ($W$) and chiral ($A$) superfields,

$$\mathcal{L} = \frac{1}{8\pi} \Im m \left[ 2 \int d^2\theta d^2\bar{\theta} A_D \bar{A} + \int d^2\theta \mathcal{F}''(A)W^2 \right]. \quad (2.1)$$

where $\mathcal{F}$ is a holomorphic function and $A_D \equiv \mathcal{F}'(A)$ is the dual superfield of $A$. We also note $a_D = \mathcal{F}'(a)$. An $Sp(2,\mathbb{Z})$ transformation on $\Omega = (a_D(u), a(u))$ is simply a duality transformation under which $\mathcal{L}$ is invariant. Then $\Omega$ is naturally interpreted as a section of a holomorphic $Sp(2,\mathbb{Z})$ vector bundle $E$ over the moduli space $\mathcal{M}$, with fiber $\mathbb{C}^2$. One can then define a symplectic product $\eta$ of two sections $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ by $\eta(\alpha, \beta) = \alpha_2\beta_1 - \alpha_1\beta_2$. In this notation, the Kähler potential on $\mathcal{M}$ is $K = \frac{i}{2}\eta(\Omega, \bar{\Omega})$ from which the Kähler metric is derived as $ds^2 = \Im m (da_D \bar{a})$. This is a positive definite metric as a consequence of unitarity.

In this language, a BPS state will be represented by a locally constant section $p = (n_e, n_m)$ over $\mathcal{M}$. The mass of such a state is :

$$m = \sqrt{2} |Z| , \quad Z = \eta(\Omega, p) = an_e - a_D n_m , \quad (2.2)$$

where $Z$ is the central charge of the supersymmetry algebra. The mass $m$, being given by the symplectic product of $\Omega$ and $p$, is obviously an $Sp(2,\mathbb{Z})$-invariant. This remarkable formula shows that once we know the section $\Omega$ and the set of sections $p$ representing the BPS particles, we also have the mass spectrum of the theory. Seiberg and Witten completely determined $\Omega$. This will be the starting point of our analysis.
Let us carefully examine the explicit form of $\Omega$. $a_D$ and $a$ can be expressed in terms of hypergeometric functions as [9]:

\begin{align*}
a_D(u) &= i \frac{u-1}{2} F \left( \frac{1}{2}, \frac{1}{2}; 2, \frac{1-\pi}{2} \right) \\
a(u) &= \sqrt{2}(u+1)^{\frac{3}{2}} F \left( -\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1} \right) .
\end{align*}

Here the square-root is defined with the argument of a complex number always running from $-\pi$ to $\pi$. Recall that $F(a, b, c; z)$ has a cut on the positive real axis from $z = 1$ to $z = +\infty$. Hence $a_D(u)$ has a cut on the real line from $-\infty$ to $-1$, while $a(u)$ has two cuts, both on the real line, one from $-\infty$ to $-1$ and another from $-1$ to $1$, see Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The branch cuts of $a_D(u)$ (left) and of $a(u)$ (right).}
\end{figure}

Near the singular points which are the branch points $1$ and $-1$ and the point at infinity, the asymptotic behaviour of $\Omega$ is:

\begin{align*}
a_D(u) &\approx i \frac{u-1}{2} \log u + 3 \log 2 - 2 \quad \text{as } u \to \infty \\
a(u) &\approx \sqrt{2} u \\
\end{align*}

\begin{align*}
a_D(u) &\approx i \frac{u-1}{2} \\
a(u) &\approx 4 \pi - \frac{1}{\pi} u \log \frac{u}{2} + \frac{1}{\pi} u - \frac{1}{\pi} (1 + 4 \log 2) \quad \text{as } u \to 1
\end{align*}

\begin{align*}
a_D(u) &\approx i \left( -\frac{1}{2} \log \frac{u+1}{2} + \frac{u+1}{2} (1 + 4 \log 2) - 4 \right) \quad \text{as } u \to -1, \\
a(u) &\approx i \left( \frac{u+1}{2} \log \frac{u+1}{2} + \frac{u+1}{2} (-i \pi - \epsilon (1 + 4 \log 2)) + 4 \epsilon \right)
\end{align*}

where $\epsilon$ is the sign of $\Im u$. From these formula one can recover the monodromies associated with the analytic continuations of $\Omega$ around the three singular points. Around $\infty$ and $1$ they
are:

\[
M_{\infty} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.
\] (2.5)

Around \(-1\), the monodromy depends explicitly on the base point \(u_P\) chosen to define the monodromy group. This is due to the appearance of \(\epsilon\) in eq. (2.4). One obtains the matrix \(M_{-1}\) if \(\Im u_P < 0\) and \(M'_{-1}\) if \(\Im u_P > 0\):

\[
M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M'_{-1} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}.
\] (2.6)

We have \(M_1M_{-1} = M_{\infty} = M'_{-1}M_1\). Note that going round a singular point one is bound to cross a cut and thus change from the principal branch to another branch of the multivalued function \((a_D(u), a(u))\). The asymptotics (2.4) and monodromy matrices (2.5) and (2.6) are valid for the principal branch. If then one goes around a second singularity, one obtains a monodromy matrix corresponding to the new branch and which may differ from the ones quoted above by conjugation by the first monodromy matrix. This invalidates simple contour composition arguments suggested in [1] for determining \(M_{-1}\) for instance. Though such subtleties were completely irrelevant in the analysis of [1], they are important here.

We close this section by pointing out that, according to the previous remarks, the explicit solution (2.3) is not the only one compatible with the physical constraints used to derive it. One may use the analytic continuations of \(a_D\) and \(a\) obtained by going around infinity \(p\) times. This amounts to conjugating the representation of the monodromy group by replacing the monodromy matrices \(M\) by \(M^p_{\infty}MM_{\infty}^{-p}\). This does not change the asymptotics of \(a_D\) and \(a\) as \(u \to \infty\) which was the physical input. We will call this the democracy transformation because it is at the origin of the “democracy” between dyons as noted in [1].
3. The curve $\Im m (a_D/a) = 0$

As already noted, a BPS state is represented by a locally constant section $p = (n_e, n_m)$ where $n_e$ and $n_m$ are relatively prime integers. Recall that its mass is proportional to the euclidean length of the complex vector $a(u)n_e - a_D(u)n_m$ (eq. (2.2)). Recall also that if $a_D/a$ is not real, the set of these vectors forms a lattice in the complex plane. If $n_e$ and $n_m$ are not relatively prime, i.e. if $(n_e, n_m) = q(n, m)$ for $n, m, q \in \mathbb{Z}$, $q \neq \pm 1$, then the BPS state is unstable against decay into $q$ BPS states $(n, m)$ since this reaction conserves the total electric and magnetic charges as well as the total mass. On the contrary, if $n_e$ and $n_m$ are relatively prime it follows from the conservation laws that this state cannot decay and hence is stable as long as $a_D/a \notin \mathbb{R}$. On the other hand, if $a_D/a$ is real, it becomes much easier to satisfy charge and mass conservation, and otherwise stable BPS states can decay (see e.g. [1]). We will study examples of such decays in Section 6.

We define the curve $C$ on the moduli space $\mathcal{M}$ as $C = \{ u \in \mathcal{M} \mid \Im m (a_D/a) = 0 \}$. Note that the solution (2.3) is such that $a(u)$ never vanishes. In fact, for all $u \in \mathcal{M}$ one has $|a(u)| \geq |a(0)| \simeq 0.76$. This means that the singular point $a = 0$ of the classical moduli space where the full $SU(2)$ gauge symmetry is restored does no longer exist in the quantum moduli space $\mathcal{M}$. The curve $C$ is of utmost importance if one wants to study the spectrum of the theory. As long as two points $u$ and $u'$ in $\mathcal{M}$ are not separated by the curve $C$, i.e. if they can be joined by a continuous path in $\mathcal{M}$ that does not cross $C$, one can deform the theory at $u$ into the theory at $u'$ without changing the spectrum. By spectrum, we mean the set of locally constant sections representing BPS particles. Of course the mass spectrum will change as $u$ varies.

To try to determine the curve $C$ analytically one observes [9] that $a_D$ and $a$ are both solutions of the same second-order differential equation

$$\left[ -\frac{d^2}{du^2} + V(u) \right] \left( \frac{a_D(u)}{a(u)} \right) = 0 \ , \quad V(u) = -\frac{1}{4(u^2 - 1)} \ . \quad (3.1)$$

It is then well-known [11] that the ratio of two solutions of (3.1) satisfies the Schwarz equation

$$w(u) = \frac{a_D(u)}{a(u)} \Rightarrow \{ w, u \} = -2\ V(u) \ , \quad (3.2)$$

where $\{ w, u \}$ denotes the Schwarzian derivative of $w$ with respect to $u$: $\{ w, u \} = \frac{w'''}{w'} - 3 \left( \frac{w''}{w'} \right)^2$. 


The curve $\mathcal{C}$ precisely is the set of points where $w(u)$ is real. Hence a parametrisation of $\mathcal{C}$ would be given by the inverse function $u(w)$ with $w$ a parameter in an appropriate real interval. Using this line of reasoning, a parametric form of $\mathcal{C}$ was obtained in [12]. The shape of $\mathcal{C}$ was also discussed in [8,13] (see Fig. 2). Actually the precise determination of $\mathcal{C}$ is completely irrelevant for our purposes. The general features which we need are easy to check numerically and are summarized below.

To a very good approximation (about $10^{-2}$) $\mathcal{C}$ looks like an ellipse (although it is not exactly an ellipse), centered at the origin of the complex plane and with semimajor and semiminor axis equal respectively to 1 and $\simeq 0.86$. In particular the points $u = \pm 1$ are on the curve $\mathcal{C}$. $\mathcal{C}$ does not contain any other disconnected component elsewhere in the complex $u$-plane. The curve $\mathcal{C}$ being closed, it separates a strong-coupling region $\mathcal{R}_S$ containing $u = 0$ from a weak-coupling region $\mathcal{R}_W$ containing $u = \infty$. The physical spectrum of BPS states may change from one region to the other, but it is necessarily the same at any two points of $\mathcal{M}$ in the same region. We will call $\mathcal{S}_S$ and $\mathcal{S}_W$ the two a priori different spectra.

Another interesting property of the curve $\mathcal{C}$ is that massless BPS states can only exist on this curve. Indeed, since $m = \sqrt{2}|an_e - a_Dn_m|$, for a massless state one must have $\frac{a_D}{a} = \frac{n_e}{n_m}$.
which is rational, and \textit{a fortiori} real. In this respect it is interesting to know which (real) values \(\frac{a_D}{a}\) takes on the curve \(C\). Let us call \(C^+\) the part of \(C\) in the upper half \(u\)-plane and \(C^-\) the part in the lower half-plane. (We include the two common end-points \(u = \pm 1\) into both, \(C^+\) and \(C^-\).) It is easy to see from the explicit expressions (2.3) that on \(C^+\) one has \(\frac{a_D}{a} \leq 0\) and \(\frac{a_D}{a}(u) \to -1\) as \(u \to -1\), while on \(C^-\) one has \(\frac{a_D}{a} \geq 0\) and \(\frac{a_D}{a}(u) \to +1\) as \(u \to -1\). Furthermore, it is clear that \(\frac{a_D}{a}(1) = 0\). This just expresses the well-known result [1] that at \(u = 1\) the magnetic monopole is massless. At \(u = -1\), the massless state is a dyon, but it is described as \((n_e, n_m) = \pm (1, -1)\) if one approaches \(u = -1\) from the upper half-plane, and as \((n_e, n_m) = \pm (1, 1)\) if the same point is approached from the lower half-plane. The fact that the same BPS state has two different descriptions is important for us and will be discussed in more detail below. As one goes along \(C^+\) from \(u = -1\) to \(u = 1\), \(\frac{a_D}{a}\) increases continuously and monotonically from \(-1\) to \(0\). Going back from \(u = 1\) to \(u = -1\) on \(C^-\), \(\frac{a_D}{a}\) continues to increase continuously and monotonically from \(0\) to \(+1\). Again, one can numerically determine \(\frac{a_D}{a}(u)\) on the curve \(C\). However, the precise form will not be important for our analysis below. What is important is that \(\frac{a_D}{a}(u)\) can take \textit{any} value in the real interval \([-1, 1]\) for \(u \in C\), with \textit{any} value in \([-1, 0]\) for \(u \in C^+\) and \textit{any} value in \([0, 1]\) for \(u \in C^-\).

It follows from the above remarks that a BPS state \((n_e, n_m)\) of the weak-coupling spectrum \(S_W\) will become massless somewhere on the curve \(C\) if \(\frac{n_e}{n_m} \in [-1, 1]\). Now, when a charged particle becomes massless, their should be a singularity in the effective gauge coupling and thus in the low energy wilsonian action. We know that there are precisely two such singularities at \(u = 1\) and \(u = -1\), where \(a_D/a = \pm 1\) or \(0\). From this we conclude that the monopole \(\pm (0, 1)\) as well as the dyon described either as \(\pm (1, 1)\) or \(\pm (1, -1)\) do exist in both \(R_W\) and \(R_S\) since these are the only stable states able to yield these singularities. We also conclude from the absence of other singularities that there are no other states in \(R_W\) or \(R_S\) that become massless on \(C\). Finally, note that any state \(p\) which becomes massless somewhere on \(C\) must exist both in \(R_W\) and \(R_S\). Indeed, first, the singularity produced by such a state can be seen from the two sides of the curve. Second, one can cross \(C\) precisely at the point where \(p\) becomes massless and it is then stable under decay since it is the only massless charged particle at that point. This is the case for the monopole and the dyon.
4. The $\mathbb{Z}_2$-symmetry

4.1. Global symmetries on the moduli space

We are now going to examine the realization of the $\mathbb{Z}_2$-symmetry $u \rightarrow u' = -u$ on the moduli space. As we recalled above, this is a global symmetry. A global symmetry relating two points $u$ and $u'$ implies that the two corresponding quantum theories are equivalent and must have the same physical content. In particular, the mass spectrum with its degeneracies must be the same at $u$ and $u'$. Since $m = \sqrt{2} |a_n e - a_D n_m|$, this implies that for each BPS state $p = (n_e, n_m)$ at $u$ there exists a BPS state $p' = (n'_e, n'_m)$ at $u'$ such that

$$|\eta(\Omega(u'), p')| \equiv |a(u') n'_e - a_D(u') n'_m| = |a(u) n_e - a_D(u) n_m| \equiv |\eta(\Omega(u), p)|.$$  \hfill (4.1)

This implies that there exists a matrix $G \in Sp(2, \mathbb{Z})$ and a phase $e^{i\omega}$ such that

$$\left( \begin{array}{c} a_D \\ a \end{array} \right)(u') = e^{i\omega} G \left( \begin{array}{c} a_D \\ a \end{array} \right)(u), \quad \left( \begin{array}{c} n'_e \\ n'_m \end{array} \right) = G \left( \begin{array}{c} n_e \\ n_m \end{array} \right),$$  \hfill (4.2)

since then $|\eta(\Omega(u'), p')| = |\eta(G \Omega(u), Gp)| = |\eta(\Omega(u), p)|$. Thus, if the BPS state $(n_e, n_m)$ exists at $u$, the BPS state $^*(n'_e, n'_m) = \pm G(n_e, n_m)$ must also exist at $u'$ with the same mass. The sign has no importance since $-(n_e, n_m)$ is the antiparticle of $(n_e, n_m)$ and is always present with $(n_e, n_m)$.

The phase in (4.2) may be surprising. It shows that the relation between $\Omega(u)$ and $\Omega(u')$ is not in general a duality transformation. Nevertheless, this sort of new $U(1)$ clearly is a symmetry of the lagrangian $\mathcal{L}$ and of the metric $ds^2$ which are invariant under the change $A \rightarrow e^{i\omega} A$ and $A_D \rightarrow e^{i\omega} A_D$. This amounts to performing the transformation $\mathcal{F}(a) \rightarrow e^{2i\omega} \mathcal{F}(e^{-i\omega} a)$. We will see that such a phase does indeed arise for the $\mathbb{Z}_2$-symmetry. Before doing so, however, we need to clarify the mathematical description of the BPS states.

* Of course, by $G(n_e, n_m)$ we mean $G \left( \begin{array}{c} n_e \\ n_m \end{array} \right)$, but it is typographically more convenient to write $G(n_e, n_m)$.

We will adopt this convention in the following.
4.2. The mathematical description of BPS states

As we have discussed in the previous section, the curve $\mathcal{C}$ separates a weak-coupling region $\mathcal{R}_W$ (outside $\mathcal{C}$) from a strong-coupling region $\mathcal{R}_S$ (inside $\mathcal{C}$). We already mentioned that the physical spectrum (by which we mean the set of BPS states, not the mass spectrum) does not depend on the point $u$ inside a given region $\mathcal{R}_W$ or $\mathcal{R}_S$. This means that if a locally constant section $p$ representing a BPS state exists at $u \in \mathcal{R}_S$ ($u \in \mathcal{R}_W$), it will exist at any other point $u' \in \mathcal{R}_S$ ($u' \in \mathcal{R}_W$). However, in the strong-coupling region $\mathcal{R}_S$, the section cannot be represented by a unique couple of integer numbers $(n_e, n_m)$ through all $\mathcal{R}_S$.

We have already encountered the example of the dyon which becomes massless at $u = -1$ and which is represented as $(1, -1)$ or $(1, 1)$ depending on whether one approaches $u = -1$ from the upper or lower half-plane. This is a consequence of the presence of the singularities and branch cuts (see Fig. 1) which prevent the bundle $E$ from being trivial. To see this, pick a section $p$ represented by $(n_e, n_m)$ at $u \in \mathcal{R}_S \cap H_+ = \mathcal{R}_{S,+}$ where $H_+$ is the upper half-plane ($\mathcal{R}_{S,-}$ is defined similarly). The mass of the BPS state associated with $p$ will be $m_p(u) = \sqrt{2}|a(u)n_e - a_D(u)n_m|$. Now transport this section through the cut $(-1, 1)$ to a point $u'$ in $\mathcal{R}_{S,-}$ (see Fig. 3 where the case $u' = -u$ is depicted). Of course the mass $m_p(u)$ will vary continuously in this process, as physically nothing happens on the cut. But once

Fig. 3: Taking $u$ to $u' = -u$ inside the strong-coupling region $\mathcal{R}_S$ one has to cross the cut on $[-1, 1]$. 
one passes through the cut, \( m_p \) will no longer be expressed in terms of \( a_D \) and \( a \) but in terms of their analytic continuations: 
\[
m_p(u') = \sqrt{2}|\hat{a}(u')n_e - \hat{a}_D(u')n_m|,
\]
where \((\hat{a}_D, \hat{a})(u') = M_1(a_D, a)(u')\). One has then 
\[
m_p(u') = \sqrt{2}|a(u')\tilde{n}_e - a_D(u')\tilde{n}_m| \text{ with } \left(\tilde{n}_e, \tilde{n}_m\right) = M_1^{-1}(n_e, n_m).
\]
Hence, the section \( p \) will be represented in \( \mathcal{R}_{S,-} \) by 
\[
(M_1^{-1}(n_e, n_m) = (n_e, 2n_e + n_m).
\]
This transformation insures the continuity of the mass of the state. Note that the different descriptions of the same state in terms of different couples of integers is consistent with the notion of stability. Indeed, if \( n_e \) and \( n_m \) are relatively prime, then it follows from Bézout’s theorem that any \( n'_e \) and \( n'_m \), obtained through an \( Sp(2, \mathbb{Z}) \) transformation from \( n_e \) and \( n_m \), are also relatively prime.

We have learned that, though there is a unique spectrum \( S_S \) valid through all the region \( \mathcal{R}_S \), we must introduce two different sets of couples \((n_e, n_m)\) to represent it. We will denote these two sets by \( S_{S,+} \) and \( S_{S,-} \). We have:
\[
S_{S,-} = M_1^{-1}(S_{S,+}) , \quad S_{S,+} = M_1(S_{S,-}) . \tag{4.3}
\]

In the weak-coupling region \( \mathcal{R}_W \) the situation is simpler. Since any two points \( u, u' \in \mathcal{R}_W \) can be joined by a path not crossing a cut, for such a path \( a_D \) and \( a \) at \( u' \) are always given by the same branch as the one at \( u \) (the principal branch). Hence, a section \( p \) can be represented by the same couple of integers through all of \( \mathcal{R}_W \). However, if one wants to compare two sections just below and above the cut \((-\infty, -1]\), one again needs to compare different representations, this time related by \( M_\infty \). In particular, for the dyon which becomes massless at \( u = -1 \) one has \((1, 1) = M_\infty^{-1}(1, -1)\), in analogy with the first relation (4.3).

### 4.3. The \( \mathbb{Z}_2 \)-symmetry

Consider now the \( \mathbb{Z}_2 \)-symmetry \( u \to u' = -u \). To start with, we take \( u \) in the upper half-plane and outside the curve \( C \), \textit{i.e.} in the weak-coupling region \( \mathcal{R}_W \). Then \( a(-u) \) and \( a_D(-u) \) are obtained by analytical continuation along the path in \( \mathcal{R}_W \) shown in Fig. 4 which does not cross any of the cuts on \((-\infty, 1]\). Using Kummer’s relations [11] between hypergeometric functions, namely
\[
(1 - z)^{c-b-1} F\left(1-a, b + 1 - c, 2 - c; \frac{z}{z - 1}\right) = F\left(a + 1 - c, b + 1 - c, 2 - c; z\right) \tag{4.4}
\]
with \( a = \frac{1}{2}, \, b = -\frac{1}{2}, \, c = 1, \) and
\[
e^{i\pi(a+1-c)\tilde{\epsilon}} \frac{\Gamma(a+1-c)\Gamma(b)}{\Gamma(a+b+1-c)} F(a, b, a+b+1-c; 1-z) \\
= \frac{\Gamma(a+1-c)\Gamma(1-a)}{\Gamma(2-c)} z^{1-c} (1-z)^{c-a-b} F(1-a, 1-b, 2-c; z) + e^{i\pi(a+1-c)\tilde{\epsilon}} \frac{\Gamma(b)\Gamma(1-a)}{\Gamma(b+c-a, b+1-a; \frac{1}{1-z})} F(b, c-a, b+1-a; \frac{1}{1-z})
\]
with \( a = b = \frac{1}{2}, \, c = 0, \) and \( \tilde{\epsilon} = \text{sgn}(\Im m u), \) it is easy to show that
\[
\begin{pmatrix} a_D \\ a \end{pmatrix} (-u) = -i\epsilon \, G_{W,\epsilon} \begin{pmatrix} a_D \\ a \end{pmatrix} (u), \quad G_{W,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \quad (4.6)
\]
where \( \epsilon \) is the sign of \( \Im m u. \) Note that \( G_{W,+} = G_{W,-1}^{-1} \), as required by consistency.

When determining the corresponding matrices in the strong-coupling region \( \mathcal{R}_S \) one has to be careful about picking the correct analytic continuation; since then \( u \) is inside the curve \( \mathcal{C} \) (see Fig. 3), when going from \( u \) to \(-u\) one either has to go through a cut or cross \( \mathcal{C} \) twice. Since the BPS spectrum changes when crossing the curve \( \mathcal{C} \) we have to go through the cut instead (see Section 5.2). We will then use the analytic continuation of \( a_D \) and \( a \) through the
cut \([-1, 1]\), that is \(M_1(-iG_{W,+})\Omega(u)\) if \(\Im u > 0\) and \((M_1)^{-1}(+iG_{W,-})\Omega(u)\) if \(\Im u < 0\), so that in the strong-coupling region \(R_S\) one has

\[
G_{S,\epsilon} = (M_1)^\epsilon G_{W,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ -2\epsilon & -1 \end{pmatrix}.
\]

(4.7)

Note that

\[
(G_{S,\epsilon})^2 = -1,
\]

(4.8)
as well as \(M_1^{-1}G_{S,+}M_1 = -G_{S,-}\), so that \(G_{S,-} = -G_{W,+}M_1\) and \(G_{S,+} = -G_{W,-}M_1^{-1}\).

One could ask if some new \(G\) matrices occurred when going from \(u\) to \(-u\) in the region \(R_W\) crossing the cut \((-\infty, -1]\). The answer is no, as the matrices we obtain in this case are \(G'_{W,\epsilon} = (M_\infty)^\epsilon G_{W,\epsilon} = -G_{W,-}\). Finally, let us mention that the democracy transformation amounts to conjugating the \(G\) matrices by \(M_\infty^p\), exactly as for the monodromy matrices.

5. The spectrum of BPS states

5.1. The weak-coupling spectrum

Here we will prove that \(S_W\) is composed of the massive gauge bosons \(\pm(1, 0)\), usually called \(W^\pm\), and the dyons \(\pm(n, 1), n \in \mathbb{Z}\). We insist on the fact that our method of proof is completely different from the usual semiclassical approach. It relies on the knowledge of the low energy wilsonian action only, \(i.e.\) the solution (2.3) with its \(\mathbb{Z}_2\)-symmetry, and involves very simple arguments.

The states \(\pm(1, 0)\) are in the perturbative spectrum of the theory and belong trivially to \(S_W\). Note that they are invariant under the \(\mathbb{Z}_2\)-symmetry since \(G_{W,\epsilon}(1, 0) = (1, 0)\). Moreover, as seen above, the monopole and antimonopole \(\pm(0, 1)\) also are in \(S_W\). At a point \(u \in R_W\), this monopole has a mass \(m(u) = \sqrt{2|a_D(u)|} = \sqrt{2|\eta((0, 1), \Omega(u))|}\). As discussed above, the same section must also exist at \(-u\), since it can be transported along the path of Fig. 4, but with

* To be precise, there is also the everywhere massless abelian \(N = 2\) vector multiplet with \(n_e = n_m = 0\), describing the photon, etc., which is the only one appearing in the low-energy effective action. This multiplet is present on all of \(\mathcal{M}\), but it has a status very different from all other BPS states since it is a vector and not a hyper-multiplet.
a mass \( m(-u) = \sqrt{2} |\eta((0, 1), \Omega(-u))| = \sqrt{2} |\eta((0, 1), G_W\Omega(u))| = \sqrt{2} |\eta(G_W^{-1}(0, 1), \Omega(u))|. \)

By the \( \mathbb{Z}_2 \)-symmetry, at \( u \), there must exist a state \( G_W^{-1}(0, 1) = (-\epsilon, 1) \) which has the same mass (at \( u \)) as \((0, 1)\) has at \(-u\). (Recall that \( \epsilon = \pm \) is always the sign of \( \Im m u \)). This proves the existence of the dyon \((-\epsilon, 1)\) at \( u \). Repeating this reasoning, the \( \mathbb{Z}_2 \)-symmetry and the existence of the dyon \((-\epsilon, 1)\) at \( u \) implies the existence of a state \( G_W^{-1}(\epsilon, 1) = (-2\epsilon, 1) \) at \( u \), and hence everywhere in \( \mathcal{R}_W \). Since \( \epsilon = \pm 1 \), depending on where one started in \( \mathcal{R}_W \), it follows by induction that all dyons \((n, 1), n \in \mathbb{Z} \) exist in \( \mathcal{R}_W \). In other words, \( \mathcal{S}_W \) is invariant under the transformations generated by the matrices \( G_W^\epsilon \):

\[
\mathcal{S}_W = G_W^\epsilon(\mathcal{S}_W),
\]

and all the dyons \( \pm(n, 1), n \in \mathbb{Z} \) indeed belong to \( \mathcal{S}_W \).

Next, let us show that there are no other states in \( \mathcal{S}_W \). Suppose \((n_e, n_m)\) is in \( \mathcal{S}_W \). We exclude the case \( n_m = 0 \) since this is either the \( W^\pm \)-boson \( \pm(1, 0) \) which we know is part of \( \mathcal{S}_W \) or, if \( n_e \neq \pm 1 \), an unstable state. Then, as before, the \( \mathbb{Z}_2 \)-symmetry implies that all the states generated by \( G_W^\pm \) from \((n_e, n_m)\), i.e. all the states of the form \((n_e + kn_m, n_m)\), \( k \in \mathbb{Z} \), are also in \( \mathcal{S}_W \). Of course, there always exists a \( k_0 \in \mathbb{Z} \) such that \((n_e + k_0n_m)/n_m = n_e/n_m + k_0 \in [-1, 1] \). Thus \((n_e + k_0n_m, n_m)\) will become massless at the point \( u^* \) on \( \mathcal{C} \) where \((a_D/a)(u^*) = n_e/n_m + k_0 \) and hence must equal \( \pm(0, 1), \pm(1, 1) \) or \( \pm(-1, 1) \). In all cases this implies that \( n_m = \pm 1 \) and thus \((n_e, n_m)\) is one of the states \( \pm(n, 1) \).

The argument is not modified after a democracy transformation. The \( G \) matrices relevant here are not changed since \( M^p_\infty G_W^\epsilon M_\infty^{-p} = G_W^\epsilon \). After a democracy transformation \( a_D/a \) will run from \(-1 - 2p\) to \( 1 - 2p \) on the curve \( \mathcal{C} \), so that the states that become massless are \( \pm(-2p, 1) \) and \( \pm(1 - 2p, 1) \equiv \pm(-1 - 2p, 1) \). It is clear that exactly the same spectrum \( \mathcal{S}_W \) is generated from \(-2p, 1\) and \( 1, 0 \).

5.2. The strong-coupling spectrum

Now take a section \( p \) in the strong-coupling spectrum, which at a point \( u \in \mathcal{R}_{S,+} \) is represented by \((n_e, n_m) \in \mathcal{S}_{S,+} \). It should now be clear that, by the \( \mathbb{Z}_2 \)-symmetry, the state \( G_{S,+}(n_e, n_m) \) then also is in \( \mathcal{S}_{S,+} \). However, since the argument in the strong-coupling region, with its distinction between \( \mathcal{S}_{S,+} \) and \( \mathcal{S}_{S,-} \), is potentially more confusing, we will give the argument in detail again.

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Since one can go from \( u \) to \(-u\) without crossing the curve \( C \), the same section \( p \) must also exist at \(-u\), but is represented by \((\tilde{n}_e, \tilde{n}_m) = M_1^{-1}(n_e, n_m)\) according to our discussion in Section 4.2. At \(-u\) this state then has a mass

\[
m_p(-u) = \sqrt{2} |\tilde{n}_e a(-u) - \tilde{n}_m a_D(-u)| = \sqrt{2} |\eta(M_1^{-1}(n_e, n_m), \Omega(-u))| = \sqrt{2} |\eta((n_e, n_m), M_1 \Omega(-u))| \tag{5.2}
\]

which by eq. (4.7) equals

\[
\sqrt{2} |\eta((n_e, n_m), G_{S,+} \Omega(u))| = \sqrt{2} |\eta((G_{S,+})^{-1}(n_e, n_m), \Omega(u))| . \tag{5.3}
\]

By the \( \mathbb{Z}_2 \)-symmetry there must be a section \( p' \), represented by \((n'_e, n'_m)\) in \( R_{S,+} \), which at \( u \) has the same mass as \( p \) has at \(-u\), i.e.

\[
p' = G_{S,+}^{-1} p = -G_{S,+} p . \tag{5.4}
\]

We conclude that if \( p \in S_{S,+} \) then also \( G_{S,+} p \in S_{S,+} \). The same applies for \( S_{S,-} \). Hence

\[
S_{S,\pm} = G_{S,\pm} (S_{S,\pm}) . \tag{5.5}
\]

Now we are in a position to determine the strong-coupling spectrum \( S_S \). We know that the magnetic monopole, becoming massless on the curve \( C \) at \( u = 1 \), must exist in \( R_S \) and hence be in \( S_S \). Since \( M_1(0,1) = (0,1) \) it is described by the same couple of integers in \( R_{S,+} \) and \( R_{S,-} \). Let us determine \( S_{S,+} \) first. Take \( p \) in eq. (5.4) to be the monopole. Then \( p' = G_{S,+} p \) is the dyon \((1, -1)\). Applying \( G_{S,+} \) again yields \(- (0,1)\), and hence gives back the monopole. This is very different from the weak-coupling spectrum where all dyons (with unit magnetic charge) are generated from the monopole. Actually, since \((G_{S,\pm})^2 = -1\), all BPS states in \( S_{S,+} \) come in \( \mathbb{Z}_2 \)-pairs. For a general stable BPS state described by \((n_e, n_m) \in S_{S,+} \) the \( \mathbb{Z}_2 \)-pairs are

\[
(n_e, n_m) \in S_{S,+} \iff G_{S,+} (n_e, n_m) = (n_e + n_m, -2n_e - n_m) \in S_{S,+} . \tag{5.6}
\]

We will now show that for each of the \( \mathbb{Z}_2 \)-pairs one or the other member becomes massless somewhere on \( C^+ \). Since we know that \( \pm (0,1) \) and \( \pm (1,-1) \) are the only states in the physical spectrum that become massless, we then conclude that this is the only pair in \( S_{S,+} \).
Recall that \( C^+ \) is the part of \( C \) in the upper half-plane, which is the only part seen from \( R_{S,+} \), and that \( \frac{a_D}{a}(u) \) takes all real values in \([-1, 0]\) on \( C^+ \). First consider the case \( n_m = 0 \). Then, for stability, one has \( n_e = 1 \) (up to an irrelevant sign). By (5.6), the \( \mathbb{Z}_2 \)-transformed state is \((1, -2)\). At the point \( u \in C^+ \) such that \( \frac{a_D}{a}(u) = -\frac{1}{2} \), this state becomes massless. Hence \((1, 0)\) and \((1, -2)\) cannot be in \( S_{S,+} \). The nonexistence of the W-bosons \((1, 0)\) in the strong-coupling region has been suggested before in [14] using completely different arguments.

Next, let \( n_m \neq 0 \). Then \((n_e, n_m)\) will become massless on \( C^+ \) if there is a point \( u \in C^+ \) where \( a_D(u) = \frac{n_e n_m}{2 n_e + n_m} \equiv r \), i.e. if \( r \in [-1, 0] \). The \( \mathbb{Z}_2 \)-partner will become massless on \( C^+ \) if there is a point \( u' \in C^+ \) where \( a_D(u') = -(n_e + n_m)/(2n_e + n_m) = -(r + 1)/(2r + 1) \equiv \varphi(r) \), i.e. if \( \varphi(r) \in [-1, 0] \). It is easy to see from the properties of the function \( \varphi(r) \) that one or the other case is always realised, i.e. either \( r \in [-1, 0] \) (and then \( \varphi(r) \notin (-1, 0) \)) or \( \varphi(r) \in (-1, 0) \) (for \( r \notin [-1, 0] \)). So one or the other \( \mathbb{Z}_2 \)-partner always becomes massless on \( C^+ \), and we conclude that

\[
S_{S,+} = \{(\pm 0, 1), \ (\pm 1, -1)\} .
\]  

(5.7)

Exactly the same reasoning applies to \( S_{S,-} \) with \( G_{S,-} \) replacing \( G_{S,+} \). But \( S_{S,-} \) is most easily determined by using eq. (4.3), and we obtain

\[
S_{S,-} = \{(0, 1), \ (1, 1)\} .
\]  

(5.8)

Recall that \((1, -1) \in S_{S,+} \) and \((1, 1) \in S_{S,-} \) are the two different descriptions of the same section \( p \) corresponding to one and the same dyon. So the strong-coupling spectrum contains exactly two BPS states.

Let us remark that a democracy transformation does not affect these conclusions, except that, for \( S_{S,+} \) for example, the sections corresponding to \((0, 1)\) and \((-1, 1)\) are now described by \((-2p, 1)\) and \((-1 - 2p, 1)\), and that \( \frac{a_D}{a}(u) \in [-1 - 2p, -2p] \) on \( C^+ \), and \( \varphi(r) \to \varphi_p(r) = -((1 + 4p)r + 8p^2 + 4p + 1)/(2r + 4p + 1) \) with either \( r \in [-1 - 2p, -2p] \), or if \( r \notin [-1 - 2p, -2p] \) then \( \varphi_p(r) \in (-1 - 2p, -2p) \). In any case, the strong-coupling spectrum precisely consists of the two sections describing the two BPS states that become massless at \( u = 1 \) or \( u = -1 \).
6. Crossing the curve $C$

We have seen that the strong-coupling spectrum only contains $\pm (0, 1)$ and $\pm (1, -1)$ in $R_{S, +}$ and only $\pm (0, 1)$ and $\pm (1, 1)$ in $R_{S, -}$. A nice physical picture of this is that any BPS state in the weak-coupling spectrum, $\pm (1, 0)$ or $\pm (n, 1)$, has to decay into these two states when crossing the curve $C$. We will now illustrate how this goes.

Suppose one crosses the curve $C$ at the point $u^*$ in the upper half-plane, i.e. on $C^+$, where $\frac{a_D}{a}(u^*) = r \in [-1, 0]$. Start with a dyon $(n, 1)$ with $n > 0$. By conservation of the electric and magnetic charges the decay reaction must be

$$ (n, 1) \rightarrow n \times (1, -1) + (n + 1) \times (0, 1) . \quad (6.1) $$

The masses of $(n, 1)$, $(1, -1)$ and $(0, 1)$ at $u^*$ are $\sqrt{2} |na(u^*) - a_D(u^*)| = \sqrt{2} |a(u^*)| |n - r|$, $\sqrt{2} |a(u^*)| |1 + r|$ and $\sqrt{2} |a(u^*)| |r|$, and the decay is possible (and does take place) since one has the conservation of total mass:

$$ |n - r| = n + |r| = n \times (1 - |r|) + (n + 1) \times |r| = n \times |1 + r| + (n + 1) \times |r| . \quad (6.2) $$

For $n < 0$, the decay reaction is $(n, 1) \rightarrow |n| \times (-1, 1) + (|n| - 1) \times (0, -1)$ ; (where $(-1, 1)$ and $(0, -1)$ are the anti-dyon and anti-monopole) with the masses working out similarly. The W-bosons $(\pm 1, 0)$ decay as $(\pm 1, 0) \rightarrow (\pm 1, \mp 1) + (0, \pm 1)$, with the mass balance given by $1 = |1 + r| + |r|$ which is satisfied since $-1 \leq r \leq 0$.

When one crosses $C^-$ instead of $C^+$, $r \in [0, 1]$ is positive instead, and $(\pm 1, \mp 1)$ is replaced by $\pm (1, 1)$, so that everything works out exactly the same way. Also the decay of anti-dyons $-(n, 1)$ is exactly the mirror of the decay of the dyons $(n, 1)$.

An alternative way of studying the strong-coupling spectrum may be to compute all the possible decays of the states belonging to the weak coupling spectrum into arbitrary states $(n_e, n_m)$, at any point on $C$. Only those states which can be produced by such a process at all the points on $C$ can eventually be present in $S_S$. This seems to be a strong constraint, and the monopole and the dyon may well be the unique states having this property. However, a

\* Of course, the kinematic possibility of the decay reactions mentioned in this section is well-known, see e.g. [8].
proof of this fact does not seem to exist yet. Moreover, it is impossible with this method to prove that a state \textit{a priori} of marginal stability actually does decay. The main ingredient we used to overcome this difficulty is the global $\mathbb{Z}_2$-quantum symmetry on the moduli space.

7. Conclusions and outlook

Let us recapitulate our assumptions:

1. $a(u)$ and $a_D(u)$ are given by the Seiberg-Witten solution (2.3).

2. A charged massless BPS state at a point $u \in \mathcal{M}$ leads to a singularity in the low-energy effective action and hence in $a(u)$ or $a_D(u)$, and thus there are no other charged massless states than those associated with the two singularities at $u = \pm 1$.

3. The $\mathbb{Z}_2$-symmetry is a true quantum symmetry acting on the moduli space as $u \to -u$.

4. The mass of a BPS state is given by $m = \sqrt{2} |n_e a - n_m a_D|$.

Of course, as physicists we believe that all of these assumptions are true. In any case, assuming them to be valid, we did show that

1. the weak-coupling spectrum is the well-known one composed of the dyons $\pm (n, 1)$ and the W-bosons $\pm (1, 0)$, and

2. the strong-coupling spectrum contains only the two BPS particles that can become massless and are responsible for the singularities of the Seiberg-Witten solution: the monopole $(0, 1)$ and the dyon, described either as $(1, 1)$ or as $(1, -1)$ (as well as their antiparticles, of course).

One may speculate on what happens for the generalisations to gauge groups other than $SU(2)$ and to theories including extra matter. We are tempted to conjecture that there, too, the strong-coupling spectra consist of those BPS states that become massless at the singularities in moduli space. However, the structure of the moduli space is much more complex than the one studied here, and a detailed investigation clearly is necessary.

Note Added

By now, we have confirmed this conjecture for the $N = 2$ $SU(2)$ theories with one, two or three massless quark hypermultiplets [15].
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