COMMUTANTS OF TOEPLITZ OPERATORS WITH MONOMIAL SYMBOLS

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Abstract. In this note we describe the commutant of the multiplication operator by a monomial in the Toeplitz algebra of a complete strongly pseudoconvex Reinhardt domain.

Throughout given an $n$-tuple of nonnegative integers $\alpha = (i_1, \cdots, i_n)$ and $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we will denote monomial $z_1^{i_1} \cdots z_n^{i_n}$ by $z^\alpha$. Also, $\mathbb{Z}_+$ will denote the set of all nonnegative integers.

Recall that a bounded domain $\Omega \subset \mathbb{C}^n$ is said to be a complete Reinhardt domain if $(z_1, \cdots, z_n) \in \Omega$ implies $(a_1 z_1, \cdots, a_n z_n) \in \Omega$ for any complex numbers $a_i \in \bar{D}, 1 \leq i \leq n$, where $D = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disk.

As usual, the Bergman space of square integrable holomorphic functions on a bounded domain $\Omega \subset \mathbb{C}^n$ is denoted by $A^2(\Omega)$. Recall that given a bounded measurable function $f \in L^\infty(\Omega)$, one defines the corresponding Toeplitz operator $T_f : A^2(\Omega) \to A^2(\Omega)$ with symbol $f$ as $T_f(\phi) = P(f \phi), \phi \in A^2(\Omega)$, where $P : L^2(\Omega) \to A^2(\Omega)$ is the orthogonal projection. If the symbol $f$ is holomorphic, then $T_f$ is the multiplication operator on $A^2(\Omega)$ with symbol $f$. Let $\mathfrak{T}(\Omega)$ denote the $C^*$-algebra generated by $\{T_g : g \in L^\infty(\Omega)\}$.

We will refer to this algebra as the Toeplitz $C^*$-algebra of $\Omega$.

Given a Toplitz operator $T_f$, it is of great interest to study the commutant of $T_f$ in the Toeplitz $C^*$-algebra of $\Omega$.

To this end, in the case of the unit disk $\Omega = D$ in $\mathbb{C}$, Z. Cuckovic [Cu, Theorem 1.4] showed that the commutant of $T_{z^k}, k \in \mathbb{N}$ in the Toeplitz $C^*$-algebra of $D$ consists of the Toeplitz operators with bounded holomorphic symbols. Subsequently T. Le [Le, Theorem 1.1] has generalized Cuckovic’s result to the case of of the unit ball $\Omega = B_n \subset \mathbb{C}^n$ and a monomial $f = z_1^{m_1} \cdots z_n^{m_n}$ such that $m_i > 0$ for all $1 \leq i \leq n$.

In this note we extend Le’s result to the case of strongly pseudoconvex complete Reinhardt domains. (Theorem 0.2.) Moreover, our proof is simpler and computation free.

As in [Cu, Le], the following result plays the crucial role in the proof of Theorem 0.2.

**Theorem 0.1.** Let $\Omega \subset \mathbb{C}^n$ be a bounded complete Reinhardt domain, and let $f = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ be a monomial such that $m_i > 0$ for all $1 \leq i \leq n$. If $S : A^2(\Omega) \to A^2(\Omega)$ is a compact operator that commutes with $T_f$, then $S = 0$. 

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We will need the following trivial

**Lemma 0.1.** Let $\phi \in L^\infty(\Omega)$. Then there exists $\epsilon > 0$ such that $\int_{\Omega} |\phi|^{m+1} d\mu \geq \epsilon \int_{\Omega} |\phi|^m d\mu$, for all $m \in \mathbb{N}$.

**Proof.** Without loss of generality we may assume that $\mu(\phi^{-1}(0)) = 0$. Put $\Omega_{\epsilon} = |\phi|^{-1}((0, \epsilon))$. Choose $\epsilon > 0$ so that $\mu(\Omega_{\epsilon}) \leq \frac{1}{\epsilon} \mu(\Omega)$. Then for all $m \in \mathbb{N}$

$$\int_{\Omega} |\phi|^{m+1} d\mu \geq \int_{\Omega \setminus \Omega_{\epsilon}} |\phi|^{m+1} d\mu \geq \epsilon \int_{\Omega \setminus \Omega_{\epsilon}} |\phi|^m d\mu.$$ 

But

$$\int_{\Omega_{\epsilon}} |\phi|^m d\mu \leq \epsilon^m \mu(\Omega_{\epsilon}) \leq \int_{\Omega \setminus \Omega_{\epsilon}} |\phi|^m d\mu.$$ 

Therefore

$$\int_{\Omega} |\phi|^{m+1} d\mu \geq \frac{1}{\epsilon} \int_{\Omega} |\phi|^m d\mu.$$ 

□

**Proof.** (Theorem 0.1.) Let us write $f = z^\tau, \tau = (m_1, \ldots, m_n)$. Since $\Omega$ is a complete Reinhardt domain, it is well-known that monomials $\{z^\gamma, \gamma \in \mathbb{Z}_+^n\}$ form an orthogonal basis of $A^2(\Omega)$. Assume that there exits a nonzero compact operator $S : A^2(\Omega) \to A^2(\Omega)$ that commutes with $T_z$. Thus $S(gz^{\mu\tau}) = S(g)z^{\mu\tau}$, for all $g \in A^2(\Omega), m \in \mathbb{N}$. Let $\alpha, \beta \in \mathbb{Z}_+^n$ be such that $\langle S(z^\alpha), z^\beta \rangle_{A^2(\Omega)} \neq 0$. Write $S(z_\alpha) = \sum_\gamma c_\gamma z^\gamma, c_\gamma \in \mathbb{C}$. Thus $c_\beta \neq 0$. It follows that for any $m \in \mathbb{N}$

$$\|S(z^{\alpha + m\tau})\|_{A^2(\Omega)} = \|z^{m\tau}S(z^\alpha)\|_{A^2(\Omega)} \geq |c_\beta| \|z^{\beta + m\tau}\|_{A^2(\Omega)}.$$ 

Let $\beta' \in \mathbb{N}^n, k \in \mathbb{N}$ be such that $z^\beta z^{\beta'} = z^{k\tau}$. Such $\beta', k$ exist because $m_i > 0$, for all $1 \leq i \leq m$. Let $\epsilon' > 0$ be such that $\epsilon'\|gz^{\beta'}\|_{A^2(\Omega)} \leq \|g\|_{A^2(\Omega)}$ for all $g \in A^2(\Omega)$. Then for $\epsilon = |c_\beta|\epsilon' > 0$, we have

$$\|S(z^{\alpha + m\tau})\|_{A^2(\Omega)} \geq \epsilon \|z^{(m+k)\tau}\|_{A^2(\Omega)}, m \geq 0.$$ 

By Lemma 0.1 there exists $\delta' > 0$ such that for all $m \geq 0$

$$\|z^{(m+k)\tau}\|_{A^2(\Omega)} \geq \delta' \|z^{m\tau}\|_{A^2(\Omega)}.$$ 

Put $\delta = \epsilon\delta'$. Combining the above inequalities we get that for all $m \geq 0$

$$\|S(z^{\alpha + m\tau})\|_{A^2(\Omega)} \geq \delta \|z^{m\tau}\|_{A^2(\Omega)}.$$ 

On the other hand since

$$\|z^{\alpha + m\tau}\|_{A^2(\Omega)} \leq \|z^\alpha\|_{L^\infty(\Omega)} \|z^{m\tau}\|_{A^2(\Omega)},$$

we get that

$$\frac{\|S(z^{\alpha + m\tau})\|_{A^2(\Omega)}}{\|z^{\alpha + m\tau}\|_{A^2(\Omega)}} \geq \frac{\delta}{\|z^\alpha\|_{L^\infty(\Omega)}}, m \in \mathbb{N}.$$
However, the sequence \( \left\{ \frac{z^{\alpha + m\tau}}{\|z^{\alpha + m\tau}\|_{A^2(\Omega)}} \right\}, m \in \mathbb{N} \) converges to 0 weakly. Thus compactness of \( S \) implies that

\[
\lim_{m \to \infty} \frac{\|S(z^{\alpha + m\tau})\|_{A^2(\Omega)}}{\|z^{\alpha + m\tau}\|_{A^2(\Omega)}} = 0,
\]
a contradiction.

In the proof of Theorem 0.2, we will use the Hankel operators. Recall that given a function \( \phi \in L^\infty(\Omega) \), the Hankel operator \( H_\phi : A^2(\Omega) \to L^2(\Omega) \) with symbol \( \phi \) is defined by \( H_\phi(g) = \phi g - P(\phi g), g \in A^2(\Omega) \).

The proof of the following uses Theorem 0.1 and is essentially the same as in [Cu, page 282].

**Theorem 0.2.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded smooth strongly pseudoconvex complete Reinhardt domain, and let \( f = z_1^{m_1} \cdots z_n^{m_n}, m_i > 0 \) be a monomial. If \( S \) is an element of the Toeplitz \( C^* \)-algebra of \( \Omega \) which commutes with \( T_f \), then \( S \) is a multiplication operator by a bounded holomorphic function on \( \Omega \).

**Proof.** Recall that for any \( g \in L^\infty(\Omega) \) and a holomorphic \( \psi \in A^\infty(\Omega) \) we have \( [T_g, T_\psi] = H_{\psi}^* H_g \). On the other hand \( H_{z_i} \) is a compact operator for all \( 1 \leq i \leq n \), as easily follows from [Pe]. Thus, \( [T_g, T_{z_i}] \) is a compact operator for all \( 1 \leq i \leq n, g \in L^\infty(\Omega) \). This implies that the commutator \( [s, T_{z_i}] \) is compact for any \( s \in \mathcal{S}(\Omega), 1 \leq i \leq n \). If \( S \in \mathcal{S}(\Omega) \) commutes with \( T_f \), then so do compact operators \( [S, T_{z_i}], 1 \leq i \leq n \). Therefore, by Theorem 0.2 we have \( [S, T_{z_i}] = 0, 1 \leq i \leq n \). This implies that \( S = T_g \) for some bounded holomorphic \( g \) by [SSU, proof of Theorem 1.4].

\[ \square \]

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