Closed Superstrings in a Uniform Magnetic Field
and Regularization Criterion

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We summarize exact solutions of closed superstrings in a constant magnetic field, from a viewpoint of the regularization criterion. Some models will be excluded according to this criterion. The spectrum-generating algebra is also constructed in these interacting models.

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I. INTRODUCTION

There is a long history of string models with electromagnetic interactions [1]-[4]. Recently a lot of interest has been drawn in exact solutions of closed strings placed in a uniform magnetic field [5]-[7]. We can see the Landau-like energy level in these solutions. In these models the electromagnetic field has so far been introduced as a Kaluza-Klein type or a gauge field with internal gauge group origin.

In the present paper we first summarize exact solutions of closed superstrings in a uniform magnetic field. We give the exact solutions in more generalized forms without taking any light-cone gauge.

The second aim of this paper is to construct the spectrum-generating algebra (SGA) in these interacting models. Physical states satisfying Virasoro conditions or equivalently the BRST charge condition are actually constructed from spectrum-generating operators.

As the third aim, we consider a regularization criterion. Some models will be excluded according to this criterion.

II. A CLOSED SUPERSTRING IN A CONSTANT MAGNETIC FIELD

The free closed bosonic Lagrangian is given by

\[ L^0 = -\frac{1}{2} \partial_\alpha x^\mu \partial^\alpha x^\mu = \partial_\alpha x^+ \partial^\alpha x^- - \frac{1}{2} \sum_{i=1}^{d-2} \partial_\alpha x_i \partial^\alpha x^i , \]

where \( x^\pm = (x^0 \pm x^{d-1})/\sqrt{2} \) are light-cone variables, \( x^{d-1} \) being regarded as one of the KK internal coordinates.

The electromagnetic field \( A_i \) is introduced as the KK type

\[ G_i - \partial_\alpha x^- \partial^\alpha x^i + B_i - \epsilon_{\alpha \beta} \partial^\alpha x^- \partial^\beta x^i \]

\[ = A_i (\eta_{\alpha \beta} + \epsilon_{\alpha \beta}) \partial^\alpha x^- \partial^\beta x^i , \]

where \( G_i = A_i \), and the anti-symmetric background field \( B_i = A_i \) is also defined as the same one. Here, two-dimensional matrices are given by

\[ \eta_{\alpha \beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \epsilon_{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \]

Introducing right-moving and left-moving variables

\[ s = \tau - \sigma , \quad \sigma = \frac{\partial}{\partial s} , \quad \bar{s} = \tau + \sigma , \quad \bar{\sigma} = \frac{\partial}{\partial \bar{s}} , \]

...
we have the total Lagrangian
\[ L = 2\left[ -\partial x^+ \partial x^- - \partial x^- \partial x^+ + \partial x_i \partial x^i - 2A_i \partial x^+ \partial x^i \right]. \] (II.5)

This is extended to the supersymmetric Lagrangian
\[ \hat{L} = 2\left[ -\hat{D}\hat{x}^+ \hat{D}\hat{x}^- - \hat{D}\hat{x}^- \hat{D}\hat{x}^+ + \hat{D}\hat{x}_i \hat{D}\hat{x}^i \right. \]
\[ \left. - 2A_i (\hat{x}) \hat{D}\hat{x}^- \hat{D}\hat{x}^i \right], \] (II.6)
where
\[ D = i\partial_\theta + \theta \partial, \quad \hat{D} = i\partial_\theta + \bar{\theta} \partial, \] (II.7)
\[ \hat{x}^\mu(s, \bar{s}, \theta, \bar{\theta}) = x^\mu(s, \bar{s}) + i\frac{1}{\sqrt{2}}\psi_\mu(s, \bar{s}) + i\frac{1}{\sqrt{2}}\bar{\psi}_\mu(s, \bar{s}) \]
\[ + i\theta\bar{\theta}C^\mu(s, \bar{s}). \] (II.8)

We choose the symmetric gauge
\[ A_i(\hat{x}) = -F_{ij}\hat{x}^j/2, \quad F_{ij} = \text{const.}, \] (II.9)
and concentrate on one of the 2 \times 2 blocks of $F_{ij}$ with $B$ real
\[ F_{ij} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \quad i, j = 1, 2, \] (II.10)
so that
\[ \hat{L} = 2\left[ -\hat{D}\hat{x}^+ \hat{D}\hat{x}^- - \hat{D}\hat{x}^- \hat{D}\hat{x}^+ \right. \]
\[ \left. + 2\left\{ \hat{D}\hat{x}_i \hat{D}\hat{x}^i + F_{ij}\hat{x}^j \hat{D}\hat{x}^- \hat{D}\hat{x}^i \right\} \right. \]
\[ \left. + \sum_{i=1}^{d-2} \hat{D}\hat{x}_i \hat{D}\hat{x}^i \right], \] (II.11)
from which free field equations for variables $\hat{X}^\pm, \hat{x}^- \hat{X}, \hat{y}^\pm$ and $\hat{x}^k$ are derived. When $F_{ij} = 0$, the Lagrangian (II.14) reduces to the free Lagrangian of (II.11).

However, in order to solve the field equations from the Lagrangian (II.21), it is necessary to find boundary conditions for variables at $\sigma = 0$ and $2\pi$. We use the complex variable notations, $\hat{X}^{(\pm)} = (\hat{X}^1 \pm i\hat{X}^2)/\sqrt{2}$. Then Eq. (II.18) turns out to be of the form
\[ \hat{X}^{(\pm)} = \exp[\pm i\hat{x}^- R(s)]\hat{B}\hat{x}^\pm. \] (II.25)

The Lagrangian (II.21) can be rewritten as
\[ L' = 2\left[ -\hat{D}\hat{y}^+ \hat{D}\hat{x}^- - \hat{D}\hat{x}^- \hat{D}\hat{y}^+ \right. \]
\[ \left. + 2\hat{D}\hat{x}^+ \hat{D}\hat{x}^- + \sum_{k=3}^{d-2} \hat{D}\hat{x}^+_k \hat{D}\hat{x}^-_k \right], \] (II.26)
whereas those of $\hat{x}^{(\pm)}$ and $\hat{x}^- \hat{X}$ are
\[ \hat{x}^{(\pm)} = x^{(\pm)} + i\frac{1}{\sqrt{2}}\theta\hat{C}^{(\pm)} + i\frac{1}{\sqrt{2}}\bar{\theta}\hat{C}^{(\pm)} \] (II.27)
Comparing both sides of Eq. (II.25), we have

\begin{align}
X^{(\pm)}(s) &= \exp(\pm i\omega B)X^{(\pm)}(s), \\
\chi^{(\pm)} &= \exp(\pm i\omega B)\psi^{(\pm)}, \\
\tilde{\chi}^{(\pm)} &= \exp(\pm i\omega B)(\tilde{\psi}^{(\pm)} \pm i\sqrt{2}\tilde{\psi}^{(\pm)}X^{(\pm)}), \\
C^X_X^{(\pm)} &= \exp(\pm i\omega B)(C^{(\pm)} \pm \frac{1}{\sqrt{2}}\tilde{\psi}^{(\pm)} \psi^{(\pm)}).
\end{align}

Since \(x^{(\pm)}\) and \(C^{(\pm)}\) are periodic, \(\psi^{(\pm)}\) and \(\tilde{\psi}^{(\pm)}\) are anti-periodic (NS) or periodic (R), and \(x^{(\pm)}_L(s + 2\pi) = x^{(\pm)}_L(s) + 2\pi(\tilde{\alpha}_0 / \sqrt{2})\), we have boundary conditions for the component fields as

\begin{align}
X^{(\pm)}(\tau, \sigma + 2\pi) &= \exp(\pm 2\pi i\omega)X^{(\pm)}(\tau, \sigma), \\
\chi^{(\pm)}(\tau, \sigma + 2\pi) &= \mp \exp(\pm 2\pi i\omega)\chi^{(\pm)}(\tau, \sigma), \\
\tilde{\chi}^{(\pm)}(\tau, \sigma + 2\pi) &= \exp(\pm 2\pi i\omega)\tilde{\chi}^{(\pm)}(\tau, \sigma), \\
C^X_X^{(\pm)}(\tau, \sigma + 2\pi) &= \exp(\pm 2\pi i\omega)C^X_X^{(\pm)}(\tau, \sigma).
\end{align}

where - and + signs in front of the exponential function stand for NS sector and R sector, respectively, and \(\omega\) is defined by \(BA_0 / \sqrt{2} = N + \omega, N \in \mathbb{Z}, 0 < \omega < 1\), called the cyclotron frequency. Here, \(\tilde{\alpha}_0\) is the zero-mode operator of \(x^+_L\), related to its momentum \(\tilde{p}^-\) through \(\tilde{p}^- = \sqrt{2}\tilde{\alpha}_0\). We regard it as a c-number, concentrating on one of its eigenspace.

Integrating \(L'\) over \(\theta, \bar{\theta}\), we have

\begin{align}
L' &= 2\left[\partial^\mu X^{(\pm)}\partial x^{(-)} + \partial x^{(-)}\partial X^{(\pm)} + \frac{1}{2}(\chi^{(\pm)}\partial\tilde{\chi}^{(\pm)}) \\
&\quad + \chi^{(-)}\partial\chi^{(\pm)} + \chi^{(\pm)}\partial\chi^{(-)} + \chi^{(-)}\partial\chi^{(\pm)}\right] \\
&\quad + 2\sum_{\mu\neq 1,2} \left[\partial\chi^{(\pm)}\partial\psi^{(\pm)} + \frac{1}{2}(\psi^{(\pm)}\partial\psi^{(\pm)} + \tilde{\psi}^{(\pm)}\partial\tilde{\psi}^{(\pm)})\right].
\end{align}

(II.38)

Here, we have dropped the \(C^\mu\) field as usual. In spite of the quasi-periodicity of \(X^{(\pm)}\), \(\chi^{(\pm)}\) and \(\tilde{\chi}^{(\pm)}\), their periodic boundary conditions, which is necessary in the variational principle, is guaranteed, because the aperiodic phase factors \(\exp(\pm 2\pi i\omega)\) are always canceled out between the (+) and (−) components in the Lagrangian.

Equations of motion are all of free type, especially, important things are

\begin{align}
\partial\partial X^{(\pm)} &= 0, \\
\tilde{\chi}^{(\pm)} &= 0, \\
\partial\chi^{(\pm)} &= 0.
\end{align}

(II.39) (II.40)

Their solutions with boundary conditions (II.34), (II.35) and (II.36) are given by

\begin{align}
X^{(\pm)}(\tau, \sigma) &= X_R^{(\pm)}(s) + X_L^{(\pm)}(\bar{s}), \\
X_R^{(\pm)}(s) &= \frac{1}{\sqrt{2}}\sum_n \frac{1}{n + \omega} \exp[-i(n + \omega)\sigma]\alpha_n^{(\pm)}, \\
X_L^{(\pm)}(\bar{s}) &= \frac{1}{\sqrt{2}}\sum_n \frac{1}{n + \omega} \exp[-i(n + \omega)\bar{s}]\hat{\alpha}_n^{(\pm)},
\end{align}

(II.41)

\(\chi^{(\pm)}(\bar{s}) = \sum_{n \in \mathbb{Z}} d_n^{(\pm)}\exp[-i(n \pm \omega)\bar{s}],\) for Ramond sector,

and

\(\chi^{(\pm)}(s) = \sum_{n \in \mathbb{Z}} \tilde{d}_n^{(\pm)}\exp[-i(n \pm \omega)s],\) for NS sector.

The conjugate momenta to \(X^{(\pm)}(\tau, \sigma)\) are

\begin{align}
P^{(\pm)} &= \frac{\partial L'}{\partial \partial X^{(\pm)}} = \partial X^{(\mp)} + \tilde{\partial} X^{(\mp)} = \hat{X}^{(\mp)}.
\end{align}

(II.44)

The quantization is accomplished by setting the commutation rules

\[ [X^{(\pm)}(\tau, \sigma), P^{(\mp)}(\tau, \sigma') ] = 2i\pi\delta_{\pm\omega}(\sigma - \sigma'), \]

(II.45)

and other combinations are zero. Here \(\delta_{\pm\omega}(\sigma - \sigma')\) is the delta function with the same quasi-periodicity as \(X^{(\pm)}(\tau, \sigma)\) with respect to its argument. From these we have commutation relations:

\begin{align}
[\alpha_m^{(\pm)}, \alpha_n^{(\pm)}] &= (m + \omega)\delta_{m+n,0}, \quad 0 < \omega < 1 \quad (II.46) \\
[\hat{\alpha}_m^{(\pm)}, \hat{\alpha}_n^{(\pm)}] &= (m - \omega)\delta_{m+n,0}, \quad (II.47)
\end{align}

and other combinations are zero. Since \(0 < \omega < 1\), \(\alpha_m^{(\pm)}\), \(\hat{\alpha}_m^{(\pm)}\) are annihilation operators for \(m > 0\), and creation operators for \(m \leq 0\), while \(\alpha_m^{(\pm)}\), \(\hat{\alpha}_m^{(\pm)}\) are creation operators for \(m < 0\), and annihilation operators for \(m \geq 0\).

As for the fermionic parts, in the same way we get, for right-moving modes

\begin{align}
\{ b_r^{(\pm)}, b_s^{(\pm)} \} = \delta_{r+s,0}, \quad \text{others} = 0, \\
\{ d_m^{(\pm)}, d_n^{(\pm)} \} = \delta_{m+n,0}, \quad \text{others} = 0.
\end{align}

(II.48)

\(b_r^{(\pm)}\) are annihilation operators for \(r > 0\), and creation operators for \(r < 0\). The same is true for the left-moving modes. For the Ramond sector, we need a special care on the 0-modes, so the detail will be discussed in Sec. III and Appendix.

The full Virasoro operators are almost the same as those of the free Virasoro operators except for the (±)
modes,

\[ L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \left( \alpha_{-m} \alpha_{n+m} + \alpha_{-m} \alpha_{n+m}^+ \right) : \quad (\text{II.49}) \]

\[ + \frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{\mu, \nu \neq 1, 2} : \eta_{\mu \nu} \alpha_{-m} \alpha_{n+m}^\nu : \]

\[ + \frac{1}{2} \sum_{r=-\infty}^{+\infty} \left( r + \frac{n}{2} \right) : b_{-r}^+ b_{n+r}^- : \]

\[ + \frac{1}{2} \sum_{r=-\infty}^{+\infty} \left( r + \frac{n}{2} \right) : b_{-r}^- b_{n+r}^+ : \]

\[ + \frac{1}{2} \sum_{r=-\infty}^{+\infty} \sum_{\mu, \nu \neq 1, 2} \left( r + \frac{n}{2} \right) : \eta_{\mu \nu} \alpha_{-m} b_{n+r}^\nu : \]

\[ G_r = \sum_{n=-\infty}^{+\infty} \left( \alpha_{-n} b_{r+n}^- + \alpha_{-n} b_{r+n}^+ \right) : \]

\[ + \sum_{n=-\infty}^{+\infty} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \alpha_{-n} b_{r+n}^\nu : \]

The left-moving Virasoro operators \( \hat{L}_n, \hat{G}_r \) are also the same as above, but the mode operators should be replaced by the tilded ones with \((+) \leftrightarrow (-)\), i.e., \( \alpha_n^{(+)}, b_r^{(+) \rightarrow b_r^{(-)}}, \)

\section{III. Calculation of Anomaly}

The Lagrangian \( \text{(III.3)} \) happens to appear as if it is a free type. However, the dynamical variables \( X^{(\pm)}(\tau, \sigma), \chi^{(\pm)}(\tau, \sigma) \) and \( \chi^{(\pm)}(\tau, \sigma) \) are subject to the quasi-periodicity \( \text{(III.3)} \) and \( \text{(III.3)} \), and this causes the inclusion of the cyclotron frequency in the commutators \( \text{(II.47)} \) and \( \text{(II.47)} \) for mode operators, which are different from those of the completely free case. Considering this fact, we should examine the validity of the super Virasoro algebra together with anomalies. Here we use the operator product expansion method for calculation of anomalies. Its detail is given in Appendix A. However, we give also other regularization methods in Appendices B, C and D, in order to emphasize the equivalence between the various regularizations.

The super Virasoro algebra for NS sector is shown to hold in a form

\[ [L_m, L_n] = (m-n) L_{m+n} + \delta_{m+n,0} A_m, \quad (\text{III.1}) \]

\[ [L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r}, \quad (\text{III.2}) \]

\[ \{ G_r, G_s \} = 2L_{r+s} + \delta_{r+s,0} B_r, \quad (\text{III.3}) \]

where the anomaly terms are given by

\[ A_m(\text{NS}) = \frac{d}{8} m(m^2 - 1) + m\omega, \quad (\text{III.4}) \]

\[ B_r(\text{NS}) = \frac{d}{2} (r^2 - \frac{1}{4}) + \omega, \quad (\text{III.5}) \]

\[ qB = N + \omega, \quad N \in \mathbb{Z}, \quad 0 < \omega < 1. \quad (\text{III.6}) \]

Anomalies for the left-moving part are the same as above, i.e., \( \hat{A}_m = A_m, \hat{B}_r = B_r \).

However, we find anomaly terms without the cyclotron frequency for Ramond sector

\[ A_m(\text{Ramond}) = \hat{A}_m(\text{Ramond}) = \frac{d}{8} m^3, \quad (\text{III.7}) \]

\[ B_m(\text{Ramond}) = \hat{B}_m(\text{Ramond}) = \frac{d}{2} m^2. \quad (\text{III.8}) \]

\section{IV. Spectrum-Generating Algebra}

The SGA for interacting dimensions \( \mu = 1, 2 \), or \((+), (-)\), is characterized by the cyclotron frequency \( \omega \). We summarize it for the right-moving NS sector:

\[ [A_m^{(+)}, A_n^{(-)}] = (m + \omega) \delta_{m+n,0}, \]

\[ \{ B_r^{(+)}, B_s^{(-)} \} = \delta_{r+s,0}, \]

\[ [A_m^{(+)}, B_r^{(-)}] = 0, \quad (\text{IV.1}) \]

\[ [A_m^{(+)}, A_n^{+}] = (m + \omega) A_{m+n}, \]

\[ [B_r^{(+)}, A_n^{+}] = \left( \frac{n}{2} + r + \omega \right) B_{r+n}, \]

\[ [A_m^{(+)}, B_r^{+}] = (m + \omega) B_{m+r}, \]

\[ \{ B_r^{+}, B_s^{+} \} = 2A_{r+s} + 4\tau^2 \delta_{r+s,0}. \]

The sub-algebra \( \text{(IV.2)} \) is completely the same as that in Ref. \( \text{[8]} \), so we omit explicit definitions of \( A_n^{(\pm)} \) and \( B_n^{(\pm)} \). They are composed of free operators with light-cone components. Any operator in Eqs. \( \text{(IV.1)} \) and \( \text{(IV.2)} \) is commutable with the super Virasoro operator \( G_r \).

The new operators \( A_n^{(\pm)} \) and \( B_n^{(\pm)} \) in Eq. \( \text{(IV.1)} \) are defined as

\[ A_n^{(\pm)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tau A_n^{(\pm)}(\tau) V^{n \pm \omega}(\tau), \quad (\text{IV.3}) \]

\[ A_n^{(\pm)}(\tau) = \rho^{(\pm)} - (n \pm \omega) \chi^{(\pm)} \psi^{(\pm)}, \]

\[ B_r^{(\pm)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tau B_r^{(\pm)}(\tau) V^{r \pm \omega}(\tau), \quad (\text{IV.4}) \]

\[ B_r^{(\pm)}(\tau) = \chi^{(\pm)}(1 - \frac{i}{2} \psi^{(\pm)} \partial_\tau \psi^{(\pm)} - \omega / 2) P^{1/2} \]

\[ - \psi^{(\pm)} P^{1/2}, \]
where
\[ V(\tau) = \exp[iX_-(\tau)] \, . \quad (IV.5) \]

Here, \(X_-, \ P_- \) and \(P^{(\pm)}\), as well as \(\chi^{(\pm)}, \psi_-\), are all right-moving operators defined by
\[ X_-(\tau) = \sqrt{2}x_R(\tau) = x_- + \tau p_- + \tau \sum_{n \neq 0} n^{-1} a_n e^{-i\tau} , \]
\[ P_\pm(\tau) = \sqrt{2} \partial_\tau x_R^\pm(\tau) = \sum_n \, e^{-i\tau} a_n^\pm \, , \quad (IV.6) \]
\[ P^{(\pm)}(\tau) = \sqrt{2} \partial_\tau X_R^{(\pm)}(\tau) = \sum_n e^{-i(\pm\omega)\tau} a_n^{(\pm)} \, . \]

The nude indices \(\pm\) denote the light-cone components defined as \(X_\pm = (X^0 \pm X^d^-)/\sqrt{2}\). The dressed indices \(\pm\) of \(X^{(\pm)} = (X^1 \pm iX^2)/\sqrt{2}\) should be distinguished from the light-cone indices \(\pm\). The new definitions for \(A_m^{(\pm)}\) and \(B_r^{(\pm)}\) reduce to the original ones proposed by Brower and Friedmann [5], if the cyclotron frequency \(\omega\) is set to be zero.

The proof of our SGA \((IV.1)\) is given by the same method as in Ref. [11].

V. ISOMORPHISMS

The algebra \((IV.2)\) is similar to the super Virasoro algebra for transverse operators
\[ [L_m^T, \, L_n^T] = (m - n)L_{m+n}^T + A^T(m)\delta_{m+n,0} \, , \]
\[ [L_m^T, \, G_r^T] = (\frac{m}{2} - r)G_{m+r}^T \, , \]
\[ \{ G_r^T, \, G_s^T \} = 2L_{r+s}^T + B^T(r)\delta_{r+s,0} \, , \quad (V.1) \]
where
\[ A^T(m) = \frac{d-2}{8}(m^2 - 2) + 2ma + m\omega \, , \quad (V.2) \]
\[ B^T(r) = \frac{d-2}{2}(r^2 - \frac{1}{4}) + 2a + \omega \, , \quad (V.3) \]
\[ qB = N + \omega, \quad N \in \mathbb{Z}, \quad 0 < \omega < 1 \, . \quad (V.4) \]

Here the superscript \(T\) means that the operators are constructed from \(L_m, G_r\), including only the oscillators with spacial components \(\mu = 1, 2, \cdots, d - 2\). The constant \(a\) is included in \(L_m^T\) as \(-a\delta_{m,0}\).

The isomorphisms
\[ A^+_m \sim L_m^T, \quad B^+_r \sim G_r^T \, , \quad (V.5) \]
are completed, if there hold equations
\[ A^T(m) = \frac{d-2}{8}(m^3 - m) + 2ma + m\omega = m^3 \, , \quad (V.6) \]
\[ B^T(r) = \frac{d-2}{2}(r^2 - \frac{1}{4}) + 2a + \omega = 4r^2 \, . \]

These two equations are consistent to give the solution,
\[ d = 10 \, , \quad (V.7) \]
\[ a = \frac{1}{2}(1 - \omega) \, . \]

As for the Ramond sector, we have \(d^R = 10\) and \(a^R = 0\). The isomorphisms \((V.3)\) are also extended to other components interacting with the magnetic field. The algebra \((IV.1)\) is similar to
\[ [\alpha^{(\pm)}_m, \, \alpha^{(-)}_n] = (m + \omega)\delta_{m+n,0} \, , \]
\[ \{ b^{(+)}_r, \, b^{(-)}_s \} = \delta_{r+s,0} \, , \]
\[ [\alpha^{(\pm)}_m, \, b^{(\pm)}_r] = 0 \, , \]
\[ [\alpha^{(\pm)}_m, \, L_n^T] = (m + \omega)\alpha^{(\pm)}_{m+n} \, , \quad (V.8) \]
\[ [b^{(\pm)}_r, \, L_n^T] = (\frac{n}{2} + r + \omega)b^{(\pm)}_{r+n} \, , \]
\[ [\alpha^{(\pm)}_m, \, G_r^T] = (m + \omega)\alpha^{(\pm)}_{m+r} \, , \]
\[ \{ b^{(\pm)}_r, \, G_s^T \} = \alpha^{(\pm)}_{r+s} \, . \]

The isomorphisms are now completed by
\[ A_m^{(\pm)} \sim \alpha_m^{(\pm)}, \quad B_r^{(\pm)} \sim b_r^{(\pm)} \, , \quad (V.9) \]
The same conclusion is obtained for the left-moving part.

Any physical state should satisfy the BRST condition \(Q_{BRST} \mid \text{phys.} \rangle = 0\), or equivalently the super Virasoro conditions,
\[ G_{r>0} \mid \text{phys.} \rangle = 0 \, , \quad (L_{n>0} - \delta_{n,0}a) \mid \text{phys.} \rangle = 0 \, , \quad (V.10) \]
\[ \tilde{G}_{r>0} \mid \text{phys.} \rangle = 0 \, , \quad (\tilde{L}_{n>0} - \delta_{n,0}a) \mid \text{phys.} \rangle = 0 \, , \quad (V.11) \]
for the NS sector with \(a = (1 - \omega)/2\), and
\[ F_{n>0} \mid \text{phys.} \rangle = 0 \, , \quad L_{n>0} \mid \text{phys.} \rangle = 0 \, , \quad (V.12) \]
\[ \tilde{F}_{n>0} \mid \text{phys.} \rangle = 0 \, , \quad \tilde{L}_{n>0} \mid \text{phys.} \rangle = 0 \, , \quad (V.13) \]
for the Ramond sector. It is well known that such physical states can be constructed by using spectrum-generating operators.

VI. ANOMALIES IN RELATED SOLVABLE MODELS

A. A closed bosonic string

If the fermionic field \(\psi^a(\tau, \sigma)\) is neglected in our model, we have a closed bosonic string in the constant magnetic field. In this case the Virasoro constraint constant is given by \(a = 1 - (\omega - \omega^2)/2\), and the space-time dimension is \(d = 20\). The \((\omega - \omega^2)/2\) anomaly comes from the
operator product expansion

\[ T^B(z)T^B(z') = \frac{1}{(z-z')^4} + \frac{\omega - \omega^2}{z' - z'} \]

\[ + \frac{2T^B(z')}{(z-z')^2} + \frac{\partial^2 T^B(z')}{z - z'} , \]  

(VI.1)

for the bosonic energy-momentum tensor, as is seen by Eq. (A.12) in Appendix A.

**B. The heterotic string**

As an interesting possibility of exactly solvable models, we consider the heterotic string [9][10] in the constant magnetic field. This heterotic model is obtained from our model by replacing the left-moving fermion with the 32 Lorentz singlet Majorana-Weyl fermion. This heterotic model is obtained from the anomaly in Eqs. (VI.4) comes from the anomaly term in Eq. (VI.1).

Let us now consider the difference

\[ (L_0 - \tilde{L}_0 - a + \tilde{a}) \mid \text{phys. } \]

\[ = (N - \tilde{N} + \frac{1}{2} + \frac{1}{2} \omega^2) \mid \text{phys. } = 0 . \]  

(VI.9)

The last equation fails to be valid, because the \( \omega^2 \) term cannot be canceled by any eigenvalue of the number operator difference, \( N - \tilde{N} \), which gives at most the first order of \( \omega \). Therefore, there is no possibility of the solvable heterotic model with Lagrangian (VI.2).

**VII. UNIQUENESS OF REGULARIZATION**

In calculation of the normal ordering constant \( a \) of the Virasoro operator \( L_0 \), some authors [12] have proposed a new kind of regularization based on the formula

\[ \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} n^{-c}(n + \omega) = -\frac{1}{12} - \frac{\omega}{2} . \]  

(VII.1)

This differs by \(-\omega^2/2\) from the usual regularization based on the generalized zeta function of Riemann defined as

\[ \zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} , \quad 0 < a \leq 1 , \]  

(VII.2)

from which we have (see Appendix D)

\[ \zeta(-1, \omega) - \omega = \lim_{s \to -1} \sum_{n=1}^{\infty} (n + \omega)^{-s} = -\frac{1}{12} - \frac{\omega^2}{2} \cdot \frac{\omega}{2} . \]  

(VII.3)

Their proposal is based on their observation that the regularization of divergent sum is ambiguous. They have
proposed that the correct way to regularize it, which leads to a modular invariant partition function, is to use the first prescription \textbf{(VII.1)}.

If the regularization \textbf{(VII.1)} is used in the heterotic model, the normal ordering constants are given as

\[
\begin{align*}
a &= \frac{1}{2} - \frac{\omega}{2}, & \text{for the right-moving NS sector ,} & \quad \text{(VII.4)} \\
\tilde{a} &= 1 - \frac{\omega}{2}, & \text{for the left-moving sector .} & \quad \text{(VII.5)}
\end{align*}
\]

Since there is no $\omega^2$ anomaly here, there is no inconsistency in the Virasoro constraint equation,

\[
(L_0 - \tilde{L}_0 - a + \tilde{a}) | \text{phys. } \rangle = (N - \tilde{N} + \frac{1}{2}) | \text{phys. } \rangle = 0 .
\]

(VII.6)

On the other hand, the usual regularization \textbf{(VII.3)} gives

\[
\begin{align*}
a &= \frac{1}{2} - \frac{\omega}{2}, & \text{for the right-moving NS sector ,} & \quad \text{(VII.7)} \\
\tilde{a} &= 1 - \frac{\omega}{2} + \frac{\omega^2}{2}, & \text{for the left-moving sector ,} & \quad \text{(VII.8)}
\end{align*}
\]

which lead to the same inconsistency as Eq.\textbf{(VI.9)}.

However, we would like to stress that the first regularization prescription \textbf{(VII.1)} is inconsistent with usual regularization prescriptions, such as the operator product expansion in Appendix A, the contraction method in Appendix B and the damping factor method in Appendix C. On the other hand, the second regularization prescription \textbf{(VII.2)} based on the generalized zeta function of Riemann is consistent with those of three regularizations A, B and C. The uniqueness of regularization can be specially seen in the generalized damping factor method of Appendix C.

**VIII. CONCLUDING REMARKS**

The heterotic string in a constant magnetic field can be solved exactly for the KK type without taking any light-cone gauge. However, we pointed out that they include inconsistency coming from anomalies, which was explicitly explained in Eq.\textbf{(VII.9)}. The bosonic string in the left-moving sector carries the anomaly, $(\omega^2 - \omega)/2$, whereas the superstring (NS or R) in the right-moving sector carries the anomaly, $(-\omega/2)$ or 0). The $\omega^2$ factor in Eq.\textbf{(VII.9)} cannot be canceled by any eigenvalue of the number operator difference, $N - \tilde{N}$, which gives at most the first order of $\omega$.

From this observation, we conclude that the exact solution in NSR superstring with the constant magnetic field exists in each of combined sectors, (NS-NS), (NS-R) and (R-R), where there is no $\omega^2$ anomaly. Of course, (bosonic-bosonic) combination is allowed to have the exact solution in a constant magnetic field.

We have also given the spectrum-generating algebra for our interacting system, which is necessary to construct actually physical states satisfying the Virasoro conditions.

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**Appendix A: Calculation of anomalies based on the operator product expansion method**

1. The Neveu-Schwarz sector

It is enough to consider only the right-moving part. Let us define current operators for interacting parts by

\[
J(\pm)(z) = i \partial_z X_R^{(\pm)}(z) = \frac{1}{\sqrt{2}} z^{\mp \omega} \sum_n z^{-n-1} a_n^{(\pm)}
\]

\[
= \frac{1}{\sqrt{2}} z^{\mp \omega} J_0^{(\pm)}(z),
\]

(A.1)

z = \exp(is),

with

\[
J_0^{(\pm)}(z) = \sum_n z^{-n-1} a_n^{(\pm)}. \quad \text{(A.2)}
\]

In the following we use the notation $\partial_z$ for the derivative $\partial_z = \partial/\partial z$ by omitting the index $z$.

The operator product expansions for them are given by

\[
J_0^{(+)}(z)J_0^{(-)}(z') = \frac{1}{(z - z')^2} + \frac{\omega}{z'}(z - z'), \quad \text{(A.3)}
\]

\[
J_0^{(-)}(z)J_0^{(+)}(z') = \frac{1}{(z - z')^2} - \frac{\omega}{z}(z - z'), \quad \text{(A.4)}
\]

Here, we have used the following contractions:

\[
\langle \alpha_m^{(+)} \alpha_n^{(-)} \rangle = \delta_{m+n,0} \theta_{m \geq 0}(m + \omega),
\]

\[
\langle \alpha_m^{(-)} \alpha_n^{(+)} \rangle = \delta_{m+n,0} \theta_{m \geq 0}(m - \omega), \quad \text{(A.5)}
\]

\[
0 \leq \omega < 1, \quad \theta_{1} = \begin{cases} 1, & \text{if } \Gamma \text{ is true} \\ 0, & \text{if } \Gamma \text{ is false} \end{cases}
\]

For the fermionic fields we confine ourselves to the NS sector,

\[
\chi^{(\pm)}(z) = z^{\mp \omega} \sum_r z^{-r-1/2} b_r^{(\pm)} = z^{\mp \omega} \chi_0^{(\pm)}(z), \quad \text{(A.6)}
\]

\[
\chi_0^{(\pm)}(z)\chi_0^{(\mp)}(z') = \frac{1}{z - z'}, \quad \text{(A.7)}
\]
with contractions \( \langle b_{i}^{(+)} b_{j}^{(-)} \rangle = \delta_{i \pm} \theta_{r \mp} \). The exponent \(-1/2\) on \( z \) in Eq. (A.13) is only for convenience.

Define the super current operator for interacting parts by

\[
G(z) = \sqrt{2} \sum_{\mu=1}^{2} \chi_{\mu}(z) J^{\mu}(z) = \chi_{0}^{(+)}(z) J_{0}^{(-)}(z) + \chi_{0}^{(-)}(z) J_{0}^{(+)}(z) .
\]

Then we calculate the operator product \( G(z)G(z') \) to yield the conformal operator \( T(z) \), i.e.,

\[
G(z)G(z') = \frac{2}{(z-z')^{3}} + \frac{\omega}{zz'(z-z')} + \frac{2T(z')}{z-z'} , \tag{A.9}
\]

where

\[
T(z) = T^{B}(z) + T^{F}(z) \tag{A.10}
\]

\[
= \frac{1}{2} \sum_{\mu=1}^{2} \left[ : J_{0\mu} J_{0\mu}^{\dagger} : + : \partial \chi_{\mu} \chi_{\mu}^{\dagger} : \right] .
\]

Here we have used the formula for the fermionic part

\[
\partial \chi_{\mu} \chi_{\mu}^{\dagger} := \partial \chi_{\mu}(z) \chi_{\mu}^{\dagger}(z) + \partial \chi_{\mu}^{\dagger}(z) \chi_{\mu}(z) ; \tag{A.11}
\]

\[
= \partial (z^{-1} \chi_{0}^{(+)}(z) \chi_{0}^{(i)}(z) + \chi_{0}^{(i)}(z) \chi_{0}^{(+)}(z) ;
\]

\[
= - \frac{2\omega}{z} \chi_{0}^{(+)}(z) \chi_{0}^{(i)}(z) + \partial \chi_{0}^{(+)}(z) \chi_{0}^{(i)}(z) + \partial \chi_{0}^{(i)}(z) (z) ;
\]

In the same way we get

\[
T^{B}(z)T^{B}(z') = \frac{1}{(z-z')^{4}} + \frac{\omega - \omega}{zz'(z-z')} + \frac{2T^{B}(z')}{z-z'}
\]

\[
= \frac{\partial T^{B}(z')}{z-z'} , \tag{A.12}
\]

for the bosonic part \( T^{B} = (1/2) \sum_{\mu=1}^{2} : J_{0\mu} J_{0\mu}^{\dagger} : \), and

\[
T^{F}(z)T^{F}(z') = \frac{1/2}{(z-z')^{4}} + \frac{\omega}{zz'(z-z')} + \frac{2T^{F}(z')}{z-z'}
\]

\[
= \frac{\partial T^{F}(z')}{z-z'} , \tag{A.13}
\]

for the fermionic part \( T^{F} = (1/2) : \partial \chi \cdot \chi : \). Totally, it follows that

\[
T(z)T(z') = \frac{3/2}{(z-z')^{4}} + \frac{\omega}{zz'(z-z')} + \frac{2T(z')}{z-z'}
\]

\[
= \frac{\partial T(z')}{z-z'} . \tag{A.14}
\]

It is remarkable that the \( \omega^{2} \) anomaly in each of the bosonic term in Eq. (A.12) and the fermionic term in Eq (A.13) is canceled out with each other in the total equation in (A.14). The algebra is closed by

\[
T(z)G(z') = \frac{3/2}{(z-z')^{2}} G(z') + \frac{1}{z-z'} \partial G(z') . \tag{A.15}
\]

We have so far considered only the \((1,2)\) plane, where the constant magnetic field is placed. The other \((d-2)\) space-time components of fields are all free, and their Virasoro algebras are well known. Collecting all of them, we get

\[
T(z)T(z') = \frac{3d/4}{(z-z')^{4}} + \frac{\omega}{zz'(z-z')} + \frac{2T(z')}{z-z'}
\]

\[
= \frac{\partial T(z')}{z-z'} , \tag{A.16}
\]

\[
T(z)G(z') = \frac{3/2}{(z-z')^{2}} G(z') + \frac{1}{z-z'} \partial G(z') , \tag{A.17}
\]

\[
G(z)G(z') = \frac{d}{(z-z')^{3}} + \frac{\omega}{zz'(z-z')} + \frac{2T(z')}{z-z'} . \tag{A.18}
\]

These are equivalent to the super Virasoro algebra

\[ [L_{m}, L_{n}] = (m-n)L_{m+n} + \delta_{m+n,0}A_{m} , \tag{A.19} \]

\[ [L_{m}, G_{r}] = \left( \frac{m}{2} + r \right) G_{m+r} , \tag{A.19} \]

\[ \{ G_{r}, G_{s} \} = 2L_{r+s} + \delta_{r+s,0}B_{r} , \]

where the anomaly terms are given by

\[ A_{m} = \frac{d}{3} m (m^{2} - 1) + m \omega , \tag{A.20} \]

\[ B_{r} = \frac{d}{2} (r^{2} - \frac{1}{4}) + \omega , \tag{A.21} \]

\[ qB = N + \omega , \quad N \in Z , \quad 0 < \omega < 1 . \tag{A.22} \]

Anomalies for the left-moving part are the same as above, i.e., \( A_{m} = A_{m}, B_{r} = B_{r} \).

2. The Ramond sector

As for the Ramond sector, we should be careful for the mode expansions of fermionic fields are given by

\[
\chi_{R}(z) = z^{\tau} \omega \sum_{n} \chi_{n}^{(e)}(z) = z^{\tau} \omega \chi_{R0}^{(e)}(z) , \tag{A.23}
\]

where \( n \) runs over the integral-numbers. The mode operators obey the commutation relation,

\[
\{ d_{n}^{(+)} , d_{n}^{(-)} \} = \delta_{m+n,0} . \tag{A.24}
\]

Usually the 0-mode \( d_{0}^{\mu} \) is regarded as the Dirac \( \gamma \)-matrix. However, in the presence of the magnetic field, it is not the case. The reason is as follows: Note that the super Virasoro operator \( F_{0} \) contains factors, \( \gamma_{0}^{(+)} d_{0}^{(-)} + \gamma_{0}^{(-)} d_{0}^{(+)}. \) Since \( \gamma_{0}^{(-)} \) is the creation operator, the second term contradicts with the Virasoro condition \( F_{0} \mid \mathrm{ground \ state} \rangle = 0, \) if \( d_{0}^{(+)} \) is regarded as the Dirac \( \gamma \) matrix. In the sector of the presence of magnetic fields, therefore, \( d_{0}^{(+)} \) should be regarded as the annihilation operator, whereas other components \( d_{0}^{\mu} \) without magnetic
fields behave as $\gamma$ matrices.

From this reason $d_0^{(+)}$ is regarded as annihilation operator for $m \geq 0$, and creation operator for $m < 0$, while $d_0^{(-)}$ is annihilation operator for $m > 0$, and creation operator for $m \leq 0$. The contractions are, therefore, defined as

$$\langle d_m^{(+)} d_n^{(-)} \rangle = \begin{cases} \delta_{m+n,0}, & (m \geq 0) \\ 0, & (m < 0) \end{cases}$$
$$\langle d_m^{(-)} d_n^{(+)} \rangle = \begin{cases} \delta_{m+n,0}, & (m > 0) \\ 0, & (m \leq 0) \end{cases}$$
(\text{A.25})

The operator product expansions for fermionic fields are, then, given by

$$\chi_{R0}^{(+)}(z)\chi_{R0}^{(-)}(z') = \frac{z}{z - z'}, \quad \chi_{R0}^{(+)}(z)\chi_{R0}^{(-)}(z') = \frac{z'}{z - z'}.$$  (\text{A.26})

$$\chi_{R0}^{(-)}(z)\chi_{R0}^{(+)}(z') = \frac{z'}{z - z'}.$$  (\text{A.27})

For the super operator, $F(z) = \chi_{R0}^{(+)}(z)J_0^{(+)}(z) + \chi_{R0}^{(-)}(z)J_0^{(-)}(z)$, we have

$$\text{Anomaly terms of } F(z)F(z') = \frac{z + z'}{(z - z')^2}.$$  (\text{A.28})

From the formula

$$\{ F_m, F_n \} = \int dz dz' z^m z^n F(z)F(z'),$$
(\text{A.29})

it follows that

$$\{ F_m, F_n \} = 2\delta_{m+n,0}B_m^{\text{(Ramond)}},$$
(\text{A.30})

For the fermionic part $T^F = (1/2) : \partial \chi_R \chi_R :$ with

$$2T^F =: \partial \chi_R^{(+)} \chi_R^{(-)} + \partial \chi_R^{(-)} \chi_R^{(+)} :$$
$$= -\frac{2\omega}{z} \chi_{R0}^{(+)} \chi_{R0}^{(-)} + \partial \chi_{R0}^{(+)} \chi_{R0}^{(-)} + \partial \chi_{R0}^{(-)} \chi_{R0}^{(+)} : ,$$

we have

$$\text{Anomaly parts of } T^F(z)T^F(z') = \frac{1}{(z - z')^2} \frac{z^2 + z'^2}{4} + \frac{\omega^2 - \omega}{(z - z')^2}.$$  (\text{A.31})

For the bosonic part $T^B =: J_0^{(+)} J_0^{(-)} :$, we already had the product $T^B(z)T^B(z')$ before as

$$\text{Anomaly parts of } T^B(z)T^B(z') = \frac{zz'}{(z - z')^4} + \frac{\omega - \omega^2}{(z - z')^4}.$$  (\text{A.32})

Here the equation has been multiplied by the factor $zz'$, in order to make it of the same power as the fermionic one. Then the total sum of the anomaly is given by

$$\text{Anomaly of } \left( T^B(z) + T^F(z) \right) \left[ T^B(z') + T^F(z') \right]$$
$$= \frac{1}{(z - z')^4} \left( zz' + \omega^2 + \omega^2 \right),$$  (\text{A.33})

where $\omega$, $\omega^2$ terms are cancelled out from Eqs. (\text{A.31}) and (\text{A.32}). This gives the anomaly term without the cyclotron frequency

$$A_m^{\text{(Ramond)}} = \frac{d}{8} m^3,$$  (\text{A.34})

$$B_m^{\text{(Ramond)}} = \frac{d}{2} m^2.$$  (\text{A.35})

The same is true for the left-moving part.

**Appendix B: Calculation of anomalies based on contractions**

The most simple method to obtain anomalies for relevant parts is to calculate contractions of $[L_m, L_n]$, or $\{G_r, G_s\}$. For the bosonic case we have

$$\{ [L_m, L_n] \} = \sum_{k,l} \langle \alpha_k^{(+)} \alpha_{m-k}^{(-)} \alpha_l^{(+)} \alpha_{n-l}^{(-)} :$$
$$\langle \alpha_k^{(+)} \alpha_{m-k}^{(-)} \alpha_l^{(+)} \alpha_{n-l}^{(-)} : angle \rangle,$$  (\text{B.1})

where contractions are defined as

$$\langle \alpha_m^{(+)} \alpha_n^{(-)} : = \delta_{m+n,0};$$
$$\langle \alpha_m^{(-)} \alpha_n^{(+)} : = \delta_{m+n,0} ;$$

$$0 < \omega < 1, \quad \theta_F = \begin{cases} 1, & \text{if } \Gamma \text{ is true} \\ 0, & \text{if } \Gamma \text{ is false} \end{cases}$$

These equations give a finite sum so that we have the unique anomaly $A_m^B$ with $\omega$ term.

$$\langle [L_m, L_n] \rangle = \delta_{m+n,0} A_m^B ,$$  (\text{B.3})

$$A_m^B = \sum_{k=0}^{m-1} (m - k - \omega)(k + \omega)$$
$$= \frac{1}{6} m(m^2 - 1) + m \omega(1 - \omega).$$

In the same way, for the superstring case we have

$$\{ G_r, G_s \} = \sum_{m,n} \langle b_{r-m}^{(+)} \alpha_{m}^{(-)} b_{s-n}^{(-)} \alpha_{n}^{(+)} + b_{r-m}^{(+)} \alpha_{m}^{(-)} b_{s-n}^{(-)} \alpha_{n}^{(+)} \rangle$$
$$= \sum_{m,n} \langle (b_{r-m}^{(+)} \alpha_{m}^{(-)} \alpha_{n}^{(+)} + b_{r-m}^{(+)} \alpha_{m}^{(-)} \alpha_{n}^{(+)} \rangle$$
$$+ \langle b_{r-m}^{(-)} \alpha_{m}^{(+)} \alpha_{n}^{(-)} \rangle$$,  (\text{B.4})
Here contractions for fermionic operators have been defined as where contractions are defined as

\[ \langle b_{r\rightarrow m}^{(+)} b_{s\rightarrow n}^{(-)} \rangle = \langle b_{r\rightarrow m}^{(-)} b_{s\rightarrow n}^{(+)} \rangle = \delta_{l+s,m+n} \theta_{r-m>0} , \]

(B.5)

so that we get a finite sum for each contraction to yield

\[ \langle G_r G_s \rangle = \delta_{r+s,0} B_r , \]

(B.6)

\[ B_r = r^2 - \frac{1}{4} + \omega . \]

The relevant two dimensional Virasoro operator is given by

\[ \text{The result is} \]

\[ A_m = \frac{1}{4} m (m^2 - 1) + m \omega , \]

(B.9)

without \( \omega^2 \) term.

**Appendix C: Uniqueness of anomalies based on the damping factor method**

Commutation relations for bosonic string are given by

\[ \left[ \alpha_m^{(+)}, \alpha_n^{(-)} \right] = (m + \omega) \delta_{m+n,0} g_m , \]

(C.1)

\[ \left[ \alpha_m^{(-)}, \alpha_n^{(+)} \right] = (m - \omega) \delta_{m+n,0} g_{-m} , \]

where a damping factor \( g_m \) is inserted. Its detailed functional form is irrelevant, but it should be set as \( g_m = 1 \) when series is a finite sum.

The relevant two dimensional Virasoro operator is given by

\[ \left[ L_m, L_{r+s} \right] = \frac{1}{2} \left[ L_m, \{ G_r, G_s \} \right] , \]

(B.7)

and

\[ \left[ L_m, G_r \right] = \frac{1}{2} m (m - r) G_{m+r} . \]

(B.8)

Then we have

\[ \left[ L_m, L_{n-r} \right] = \sum_{k,l} \left[ \alpha_k^{(+)} \alpha_{m-k}^{(-)} + \alpha_l^{(-)} \alpha_{n-l}^{(+)} \right] \]

(B.3)

\[ = \sum_{k,l} \left[ \alpha_k^{(+)} \alpha_{m-k}^{(-)} \right] \delta_{k,n} \alpha_{l,m}^{(+)} \]

\[ + \delta_{k,n} \alpha_{m-k}^{(+)} \alpha_{l,m}^{(-)} \]

\[ = \sum_k \left\{ (m-k-\omega) g_{m+k} \alpha_k^{(+)}, \alpha_{n-k}^{(-)} \right\} \]

\[ + \delta_{k+n-1,0} (k+\omega) g_k \alpha_{n+k}^{(+)} \alpha_{m-k}^{(-)} \]

\[ = \left( m-n \right) L_{m+n} + \delta_{m+n,0} A_m^B , \]

\[ A_m^B = \sum_{k \geq 0} (m-k-\omega)(k+\omega) g_{m+k}^k \]

\[ + \sum_{k \geq m} (k+\omega)(k-m+\omega) g_k g_{k-m} . \]

When \( m > 0 \), it follows that

\[ A_m^B = \sum_{0 \leq k < m} (m-k-\omega)(k+\omega) g_k g_{k-m} \]

\[ = \sum_{k=0}^{m-1} (m-k-\omega)(k+\omega) \]

\[ = \frac{1}{6} \left( m^2 - 1 \right) + m \omega (1 - \omega) . \]

Here the damping factors, \( g_k, g_{k-m} \), have been set as unity, since the series is finite. The same is true when \( m \leq 0 \). The result agrees with Eq. (B.3) with \( \omega^2 \) term.

For the NS string case, commutation relations of fermionic mode operators are given by

\[ \{ b_r^{(+)}, b_s^{(-)} \} = \delta_{r+s,0} \gamma_r , \]

\[ \{ b_r^{(-)}, b_s^{(+)} \} = \delta_{r+s,0} \gamma_{-r} . \]

(C.5)
where $\gamma_r$ is the damping factor. By using Eqs. (3.1) and (3.3), let us calculate the anti-commutator

\[
\{ G_r, G_s \} = \sum_{m,n} \left\{ b_{r-m}^{(+)\alpha_m^{(-)}} \right\} + b_{s-n}^{(-)} \alpha_n^{(+)} + b_{s+n}^{(+)} \alpha_n^{(-)} + \delta_{m+n,0} \gamma_r \alpha_m^{(+)} \alpha_n^{(-)} + \delta_{m+n,0} \gamma_r \alpha_m^{(-)} \alpha_n^{(+)} + \gamma_{r+m} \alpha_m^{(+)} \alpha_n^{(-)} + \gamma_{r-m} \alpha_m^{(-)} \alpha_n^{(+)} + \frac{\omega}{2} \alpha_m^{(+)} \alpha_n^{(-)} + \frac{\omega}{2} \alpha_m^{(-)} \alpha_n^{(+)} .
\]

Accordingly we get

\[
\{ G_r, G_s \} = 2L_{r+s} + \delta_{r+s,0} \bar{B}_r , \quad (C.7)
\]

\[
B_r = \sum_{m>0} \gamma_{r-m} g_m (m - \omega) - \sum_{m>0} \gamma_{r-m} g_m (m + \omega) + \gamma_{r+m} g_m (m + \omega) - \sum_{m>0} \gamma_{r-m} g_m (m + \omega) .
\]

When $r > 0$, it follows that

\[
B_r = \sum_{m=1}^{r-1/2} \gamma_{r-m} g_m (m - \omega) + \sum_{m=0}^{r-1/2} \gamma_{r+m} g_m (m + \omega) .
\]

Since this is a finite sum, one can set as $\gamma_r = g_m = 1$ to yield

\[
B_r = 2 \sum_{m=1}^{r-1/2} m + \omega = r^2 - \frac{1}{4} + \omega .
\]

The same is true even when $r < 0$.

In this calculation we do not need any symmetry of $\gamma_r$, $g_m$, with respect to $\pm r$ and $\pm m$.

In conclusion this regularization never depends on any functional form of the damping factor. Therefore, there is no ambiguity in this method, giving a unique result.

\begin{align*}
\text{Appendix D: Regularization by means of the generalized zeta function of Riemann} \\
\zeta(s, a) &= \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} , \quad 0 < a \leq 1 . \quad (D.1) \\
\zeta(-1, \omega) - \omega &= \sum_{n=1}^{\infty} (a + \omega) = -\frac{1}{12} \frac{\omega^2}{2} - \frac{\omega}{2} , \quad (D.2) \\
\sum_{n=1}^{\infty} (n - \omega) &= -\frac{1}{12} \frac{\omega^2}{2} + \frac{\omega}{2} , \quad (D.3)
\end{align*}

Especially we have

\begin{align*}
\zeta(-1, \omega) - \omega &= \sum_{n=1}^{\infty} (a + \omega) = -\frac{1}{12} \frac{\omega^2}{2} - \frac{\omega}{2} , \quad (D.2) \\
\sum_{n=1}^{\infty} (n - \omega) &= -\frac{1}{12} \frac{\omega^2}{2} + \frac{\omega}{2} , \quad (D.3)
\end{align*}

in the region $0 < \omega < 1$.

By using these formulae let us calculate normal ordering constants for NS sector. The un-normal ordered Virasoro 0-th operator is given by

\[
L_0 = \frac{1}{2} \sum_n \left( \alpha_n^{(+)\alpha_n^{(-)}} + \alpha_n^{(-)} \alpha_n^{(+)\gamma} \right) - \frac{1}{2} \sum_{\mu,\nu \neq 1,2} \eta_{\mu\nu} \alpha_n^{(\mu)} \alpha_n^{(\nu)}\gamma + \frac{1}{2} \sum_{r} \frac{\gamma_{r+n} \alpha_n^{(r)} \gamma - \omega^2}{2} b_r^{(+)\gamma} b_r^{(-)} - \frac{1}{2} \sum_{r} \frac{\gamma_{r-m} \alpha_n^{(r)} \gamma + \omega}{2} b_r^{(+)\gamma} b_r^{(-)} ,
\]

with

\[
\left[ \alpha_n^{(+)\alpha_n^{(-)}} , \alpha_{n'}^{(-)} \right] = (n \pm \omega) , \quad (D.5) \\
\{ b_r^{(+)\gamma} , b_r^{(-)} \} = \delta_{r+s,0} . \quad (D.6)
\]
1. Bosonic sector

The bosonic sector becomes

\[
L_B^0 = \frac{1}{2} \sum_{n \geq 1} (a_n^{(+)} a_n^{(-)} + a_n^{(-)} a_n^{(+)}) \quad \text{(D.7)}
\]

\[
+ \frac{1}{2} \sum_{n \geq 1} (a_n^{(+)} a_{-n}^{(-)} + a_n^{(-)} a_{-n}^{(+)})
\]

\[
+ \frac{1}{2} \sum_{n \geq 1} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} (a_{\mu n}^{+} a_{\nu n}^{+} + a_{\mu n}^{-} a_{\nu n}^{-})
\]

\[
+ \frac{1}{2} \left(a_0^{(+)} a_0^{(-)} + a_0^{(-)} a_0^{(+)} \right) + \frac{1}{2} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \eta_{\mu \nu} n
\]

\[
+ \frac{1}{2} \sum_{n \geq 0} a_n^{(+)} a_n^{(-)} + \frac{1}{2} \omega + \frac{1}{2} p^2 .
\]

From Eqs. (D.2) and (D.3) we have

\[
\frac{1}{2} \sum_{n \geq 1} (n + \omega) + \frac{1}{2} \sum_{n \geq 1} (n - \omega) = -\frac{1}{12} - \frac{1}{2} \omega^2 .
\]

Also

\[
\frac{1}{2} \sum_{n \geq 1} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \eta_{\mu \nu} n = -\frac{1}{4} ,
\]

(only for transverse sectors).

Then, totally we get

\[
-a_B = -\frac{1}{3} - \frac{1}{2} \omega^2 + \frac{1}{2} \omega .
\]

It may be remarkable that the $\omega^2$ anomaly appears in the bosonic sector.

2. Fermionic sector

The fermionic sector becomes

\[
L_0^F = \frac{1}{2} \sum_{r \geq 1/2} (r - \omega) b_{r_r}^{(+)} b_{r_r}^{(-)} \quad \text{(D.9)}
\]

\[
+ \frac{1}{2} \sum_{r \geq 1/2} (-r - \omega) b_{r_r}^{(+)} b_{r_r}^{(-)}
\]

\[
+ \frac{1}{2} \sum_{r \geq 1/2} (r + \omega) b_{r_r}^{(-)} b_{r_r}^{(+)}
\]

\[
+ \frac{1}{2} \sum_{r \geq 1/2} (-r + \omega) b_{r_r}^{(-)} b_{r_r}^{(+)}
\]

\[
+ \frac{1}{2} \sum_{r \geq 1/2} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \eta_{\mu \nu} (b_{r_r}^{+} b_{r_r}^{+} b_{r_r}^{+} - b_{r_r}^{+} b_{r_r}^{+})
\]

\[
= \sum_{r \geq 1/2} (r - \omega) b_{r_r}^{(+)} b_{r_r}^{(-)} + \frac{1}{2} \sum_{r \geq 1/2} (-r - \omega)
\]

\[
+ \sum_{r \geq 1/2} (r + \omega) b_{r_r}^{(-)} b_{r_r}^{(+)} + \frac{1}{2} \sum_{r \geq 1/2} (-r + \omega)
\]

\[
+ \sum_{r \geq 1/2} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \eta_{\mu \nu} (b_{r_r}^{+} b_{r_r}^{+} b_{r_r}^{+} - b_{r_r}^{+} b_{r_r}^{+})
\]

By using the formulae

\[
\frac{1}{2} \sum_{r \geq 1/2} (-r - \omega) = -\frac{1}{48} + \frac{\omega^2}{4} ,
\]

\[
\frac{1}{2} \sum_{r \geq 1/2} (-r + \omega) = -\frac{1}{48} + \frac{\omega^2}{4} ,
\]

and

\[
-\frac{1}{2} \sum_{r \geq 1/2} \sum_{\mu, \nu \neq 1, 2} \eta_{\mu \nu} \eta_{\mu \nu} r = -\frac{1}{8}
\]

(only for transverse sectors),

we get

\[
-a_F = -\frac{1}{6} + \frac{\omega^2}{2} .
\]

The total sum is given by

\[
a = a_B + a_F = 1 - \frac{\omega}{2} .
\]

It is remarkable that the $\omega^2$ anomaly is cancelled out in $a$.

For the heterotic string model we have

\[
a = \frac{1 - \omega}{2} ,
\]

for the right-moving NS sector,

\[
\tilde{a} = 1 - \frac{\omega - \omega^2}{2} ,
\]

for the left-moving sector.
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Other refs. are therein.