Systematic Ionospheric Residual Errors in GNSS Radio Occultation: Theory for Spherically Stratified Media

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Abstract The standard ionospheric correction of bending angles in Global Navigation Satellite System (GNSS) radio occultation (RO) measurements removes most of the influence from the ionosphere, but leaves small systematic residual errors in the corrected data. The main reasons for residual errors at stratospheric and mesospheric altitudes are the neglect of higher-order terms in the expansion of the ionospheric refractive index, and the fact that the two GNSS signals follow slightly different paths through the ionosphere. Another reason is the neglect of the local electron density at the receiver in orbit. These residual errors depend on the geomagnetic field and the spatial distribution of the ionospheric electron density and its gradients. In this work we derive this dependency to high accuracy for a spherically stratified ionosphere, including the “bi-local” situation where the ionosphere is different (though still spherically stratified) on the inbound and outbound side of the RO event tangent points. As part of the derivations, we find a small residual error term not previously noted, which can become appreciable for elliptical satellite orbits. The results are verified by ray tracing through simple ionospheric and geomagnetic field models. The accuracy of a higher-order bi-local ionospheric residual error correction based on these results would be limited by the uncertainty in knowledge of the electron density and by horizontal electron density gradients along the ray paths.

1. Introduction

Global Navigation Satellite System (GNSS) radio occultation (RO) data are influenced by the ionosphere. To use RO data for numerical weather prediction or climate monitoring, ionospheric correction is therefore necessary. Ionospheric correction of RO bending angles was first described by Vorob'ev and Krasil'nikova (1994). The formula is similar to the well-known dual-frequency linear combination of phase data at equal time samples (Spilker, 1978), and to the dual-frequency linear combination of derived Doppler shifts (Ladreiter & Kirchengast, 1996). However, because the correction of RO bending angles is done at a common impact parameter instead of at a common time, systematic residual errors are fundamentally different and smaller in size (Gorbunov et al., 1996; Ladreiter & Kirchengast, 1996). The theoretical relation between the two approaches was studied by Syndergaard (2000), showing that a major systematic residual remaining after the linear combination of GPS L1 and L2 phase paths at a common time, is inherently corrected for in the linear combination of L1 and L2 bending angles at a common impact parameter. Thus, ionospheric correction of bending angles at a common impact parameter has become the standard approach. Still, small systematic residual errors remain. Although GNSS RO measurements without further correction have already contributed to climate monitoring and the improvement of numerical weather prediction, these small residual errors may be dominant in the upper stratosphere and mesosphere, and limit the upper altitude to which RO data are useful for climate applications.

Gorbunov et al. (1996) studied the accuracy of the bending angle correction method and suggested the additional correction of systematic residual errors by numerical simulations given a priori knowledge of the ionospheric electron density. Over the past decade, an increased number of studies have assessed the systematic residual errors in bending angle either by simulation experiments using ray tracing (Coleman & Forte, 2017; Danzer et al., 2013, 2015; Li et al., 2020; Liu et al., 2013, 2015, 2018; Mannucci et al., 2011; Qu et al., 2015) or by theoretical considerations and empirical analyses using the relation between refractivity and bending angle under the assumption of local spherical symmetry (Angling et al., 2018; Danzer et al., 2020, 2021; Fan et al., 2017; Healy & Culverwell, 2015; Liu et al., 2020). On the latter account, Healy and Culverwell (2015) derived and suggested a simple addition to the standard correction using the already measured L1 and L2 bending angles.
thereby reducing the sensitivity of residual ionospheric correction to a priori knowledge, as well as reducing algorithm complexity. This addition is now commonly known as the kappa-correction (Danzer et al., 2020, 2021).

The kappa-correction is based on the residual originally derived by Vorob’ev and Krasiǎlikova (1994). It mainly takes into account the ray path splitting, that is, the fact that the L1 and L2 ray paths, for a common impact parameter, are slightly different in the ionosphere due to dispersion. The size of this residual depends on the squared electron density, and ray path splitting therefore dominates the total systematic error in the standard correction when the electron density is large. In worst case scenarios the order of magnitude is 0.1 μrad, and in a spherically symmetric ionosphere the residual is negative (thus, the kappa-correction adds a small positive term to the bending angle after the standard correction). However, in simulations using ray tracing, partly corroborated by observations, Li et al. (2020) show that the residual can be either positive or negative when horizontal gradients are taken into account.

Another contribution to the residual error in the standard correction is due to the geomagnetic field. The size of this residual depends on the product between the electron density and the geomagnetic field strength, and on the direction between the ray paths and the field lines. Also this residual can be either positive or negative; because of the orientation of the geomagnetic field, it is more or less positive when the azimuth of the propagation direction relative to the north is between 90° and 270°, and negative otherwise (Qu et al., 2015). The magnitude, even in worst case scenarios, is somewhat less than 0.1 μrad (Liu et al., 2020).

Both of the residuals mentioned above depend on the altitude of the low earth orbit (LEO) satellite. Using simulations, Mannucci et al. (2011) assessed the residual error in retrieved refractivity profiles for a satellite in a relatively low orbit (400 km), where the electron density at the receiver might be appreciable, and found a strong sensitivity to the assumption in the retrieval of setting the refractive index to unity at the receiver. Thus, to assess the residuals accurately, it is important to take into account the orbit altitude and the non-zero electron density at the receiver.

There are a number of practical aspects of ionospheric correction of RO data that we do not address in this paper. Those include ionospheric correction in combination with statistical optimization of bending angles (Gobiet & Kirchengast, 2004; Gorbunov, 2002), optimal filtering or smoothing of the L1 and L2 bending angle difference (Hajj et al., 2002; Kuo et al., 2004; Schwarz et al., 2018; Sokolovskiy et al., 2009; Steiner et al., 1999), reduction of fluctuations in the corrected bending angle via comparative discrimination or back propagation (Sokolovskiy et al., 2014), and ionospheric correction in the troposphere via extrapolation when one of the two GNSS signals is unavailable due to tracking limitations (Rocken et al., 1997; Zeng et al., 2016). Nor do we address residual errors in derived refractivity or temperature using the Abel transform and hydrostatic integration (Danzer et al., 2020, 2021; Kursinski et al., 1997; Rieder & Kirchengast, 2001; Schwarz et al., 2017). The residual errors in these variables are linked to the error in bending angle, but the details depend on the particular approach of statistical optimization and upper altitude initialization, and is therefore more difficult to describe in a general sense. Thus, we limit ourselves to the standard correction of bending angles given by

\[ \alpha_c(a) = \frac{f_1^2 \alpha_1(a) - f_2^2 \alpha_2(a)}{f_1^2 - f_2^2}, \]

(1)

where \( \alpha_1(a) \) and \( \alpha_2(a) \) are the L1 and L2 bending angles, respectively, at a given impact parameter, \( a \), \( \alpha_4(a) \) is the standard corrected bending angle at that impact parameter, and \( f_1 \) and \( f_2 \) are the L1 and L2 signal frequencies, respectively.

In this paper we establish a rigorous theory of the systematic ionospheric residual error for spherically stratified media when using the standard correction of bending angles given by Equation 1. The theory takes into account all residuals that potentially could be larger than 0.01 μrad using the GPS L1 and L2 signal frequencies. It builds on the work by Syndergaard (2000), and extends that work by taking into account the residual error due to the geomagnetic field, as well as by taking into account the receiver’s finite orbit altitude and the electron density at the LEO receiver, including a subtle effect for non-circular orbits. The mathematical derivations are given in Section 2. The final result in Section 2.6 is related to previous theoretical works in Section 3. Although it is a very comprehensive theory, taking into account all known residual errors, it is not an ultimate theory; it is only strictly valid under the assumption of a stratified atmospheric and ionospheric environment, including a simplified contribution from the geomagnetic field. The development of an all-encompassing theory for non-stratified
media is much more challenging than what we present here, and may be the subject of future efforts. However, as a start, in Section 4, we apply the theory for stratified media to the situation where the ionosphere is different (though still spherically stratified) on the inbound (from transmitter) and outbound (toward receiver) side of the RO event tangent points. As opposed to local spherical symmetry, we refer to this as bi-local spherical symmetry. In Section 5, we show that even the theory with bi-local spherical symmetry can give inaccurate estimates of the residual errors in the more general situations with horizontal gradients along the ray paths. Section 6 contains the conclusions.

2. Derivations of the Ionospheric Residual Error in Stratified Media

The mathematical derivations throughout this section were complemented and supported by forward model simulations (ray tracing), and subsequent simulated retrievals. These simulations were performed as described in the first two subsections below (Sections 2.1 and 2.2). The profiles of ionospheric electron density and the neutral atmospheric refractivity used in the ray tracing simulations are shown in Figure 1.

In the context of the theory, the true bending angle is defined as the angle between the outgoing and incoming directions of a ray going from the GNSS transmitter to the LEO receiver. It is important to distinguish this from the retrieved bending angle defined as the one that can be calculated from the measurements without knowledge of the ionospheric environment. The (hypothetical) ionosphere-free bending angle is defined as the one that would come about in the absence of the ionosphere, for example, from a perfect simulation without the ionosphere. The ionosphere-free bending angle is thus the one that we strive to obtain by ionospheric correction.

The derivations are divided into four subsections (Sections 2.3–2.6), which clarify a four-step approach: (a) derivation of the systematic residual error accounting for non-circular orbits and non-zero electron density at the satellites; (b) higher-order derivation of the systematic error in the retrieved L1 and L2 bending angles at a common impact parameter; (c) higher-order derivation of the difference, at a common impact parameter, between the true L1 and L2 bending angles and the ionosphere-free bending angle; (d) higher-order derivation of the residual error in the standard ionospheric correction. The first step is a primer to the higher-order derivation in the second step. The entire approach is schematically illustrated in Figure 2.

The derivations rely heavily on Taylor-series expansions, and thus everything is in essence approximations. However, for the sake of simplicity, we use the equal sign (=) without asymptotic notation (e.g., big O notation) in expansions that include all terms that could potentially lead to residuals larger than 0.01 μrad. In a few places where we deliberately do not take enough terms into account in a series expansion, or where we make other approximations, we use the approximately equal sign (≈).

Because the simulations supporting the derivations were based on the GPS L1 and L2 frequencies, the number of terms included in the series expansions depends to some degree on the choice of these frequencies. The L1 and L2 frequencies could be replaced with other frequencies in the L-band without altering the sizes of terms too much, but for frequencies lower than ∼1 GHz there could be additional important terms to consider, and generally the highest order terms would be of relatively larger importance than in the derivations here. For higher frequencies,
all terms naturally becomes smaller, and for very high frequencies such as those anticipated in future LEO-LEO microwave occultations (Liu et al., 2017), the highest order terms are of very little importance. Also, if more than two signal frequencies would be available, of which at least one should be somewhat larger than the frequencies in the L-band, higher-order terms could be eliminated by multi-frequency combinations (Syndergaard, 2013).

2.1. Forward Model Simulations of Excess Phases

Forward model simulations were carried out using the Radio Occultation Simulation for Atmospheric Profiling (ROSAP) ray tracing software (Høeg et al., 1995). The ray tracing through analytical models of the ionospheric electron density and neutral atmospheric refractivity results in very accurate synthetic excess phases, besides providing the true and ionosphere-free bending angles directly. The orbits used in the simulations were coplanar with the GNSS in a 26,570 km circular orbit, and the LEO satellite in a 6,820 km shifted circular orbit; it was shifted 100 km toward the GNSS satellite to simulate in a simple way the effects of a non-circular orbit. Although a shifted circular orbit is still circular, we refer to it as non-circular throughout the paper, because the origin is shifted and the relatively short arc of the orbit during an occultation event is such that the velocity vector is not exactly perpendicular to the radius vector from the center of the Earth.

The ionospheric model was a 1D profile of the electron density of a double Chapman layer equivalent to the one used in the study by Syndergaard (2000) (see also Text S1 in Supporting Information S1 for its formulation). Differently from that study, though, was that we here included higher-order ionospheric contributions to the refractive index according to a series expansion of the Appleton-Lassen formula. This also requires a model of the geomagnetic field. For simplicity, the geomagnetic field parallel to the propagation direction, as a function of radius, $r$, was taken to be spherically symmetric and given by $B_\parallel = B_0 (R_{\text{earth}}/r)^3$, with $B_0 = 30 \, \mu T$, and $R_{\text{earth}} = 6,370 \, \text{km}$. This is not a realistic field model, but suffices the purpose of giving a near worst case estimate of the influence from the geomagnetic field. The anisotropy of the ionospheric refractive index was ignored.

The neutral atmospheric model was a 1D profile of refractivity given by $N(z) = N_0 \exp(-z/H_0)$, with $N_0 = 278.39 \, \text{N-units}$, $H = 7.5 \, \text{km}$, and $z$ being the altitude above Earth’s surface.

The Earth was considered to be a sphere and the signals simulated (at a sampling of 50 Hz) were L1 at 1.57542 GHz and L2 at 1.22760 GHz, both with and without the ionospheric model included.

2.2. Bending Angle Profile Retrievals

The simulated L1 and L2 excess phases were processed to bending angles as a function of impact parameter via the calculation of the Doppler shift (derivative of the phase) using Equations 2–5 below. Because we are
interested in very small residual errors, and because these depend on basically the second derivative of the excess phases, it was necessary to first filter out high-frequency numerical noise from the ray tracing (which has a standard deviation of about $2 \times 10^{-3}$ mm); this was done using regularization, minimizing third derivatives (Syndergaard, 1999). Thus, the 50 Hz simulated excess phases were smoothed to a resolution of about 0.1 s, with a resulting reduction of the numerical error to less than $10^{-3}$ mm.

Given the definition of symbols in Figure 3, the equation relating the measured Doppler shift, or excess Doppler, $\dot{A} \Delta \dot{L}$, to impact parameter, $a$, is given by Melbourne et al. (1994); Syndergaard (1999):

$$\dot{L} + \dot{R}_{LG} = \left( |\dot{R}_L| \cos \varphi(a) - |\dot{R}_G| \cos \chi(a) \right) = 0$$

(2)

with

$$\varphi(a) = \zeta - \arcsin \left( \frac{a}{|\dot{R}_L|} \right),$$

(3)

$$\chi(a) = (\pi - \eta) - \arcsin \left( \frac{a}{|\dot{R}_G|} \right).$$

(4)

The bending angle, $\alpha$, for that $a$, is then given by

$$\alpha = \Theta - \arccos \left( \frac{a}{|\dot{R}_L|} \right) - \arccos \left( \frac{a}{|\dot{R}_G|} \right).$$

(5)

This set of equations is valid in the geometrical optics approximation and when the satellites are well outside the ionosphere where the electron density can be neglected.

### 2.3. Accounting for Non-Circular Orbits and Non-Zero Electron Density at the Satellites

In the following, we take into account the electron density at the satellites and we simplify the notation slightly defining $L = \Delta L + \dot{R}_{LG}$, $v_L = |\dot{R}_L|$, $v_G = |\dot{R}_G|$, $r_L = |\dot{R}_L|$, and $r_G = |\dot{R}_G|$. Then we have what we shall refer to as the general Doppler equation:

$$\dot{L} = n_L v_L \cos \varphi(a) - n_G v_G \cos \chi(a)$$

(6)

with

$$\varphi(a) = \zeta - \arcsin \left( \frac{a}{n_L r_L} \right).$$

(7)
\[ \chi(a) = (\pi - \eta) - \arcsin \left( \frac{a}{n_G r_G} \right), \]  

(8)

and

\[ a = \Theta - \arccos \left( \frac{a}{n_L r_L} \right) - \arccos \left( \frac{a}{n_G r_G} \right), \]  

(9)

where \( n_L \) and \( n_G \) are the refractive indices at the LEO and GNSS, respectively.

The equations above are consistent with those of Melbourne et al. (1994). In a later work by Hajj and Romans (1998), however, the refractive indices in the general Doppler equation are omitted, and the estimate of the size of the error in bending angle of neglecting the electron density at the satellites in the appendix of that work is therefore incorrect; we found it to be a factor of about 20 too small. Nevertheless, parts of the derivations below are inspired by the work of Hajj and Romans (1998). Note also that the general Doppler equation has this particular form, and not the one given in many other works (e.g., Gorbunov & Kornblueh, 2001; Vorob'ev & Krasil'nikova, 1994; Zeng et al., 2016), because the GNSS velocity (or rather its projection into the occultation plane) is an apparent velocity; it is the derivative of the GNSS position at the time of transmission with respect to the time of reception (Syndergaard, 2012).

For circular satellite orbits (\( \zeta = \eta = 90^\circ \)), we find that \( \cos \varphi(a) = a(n_L r_L)^{-1} \) and \( \cos \chi(a) = a(n_G r_G)^{-1} \), and thus from Equation 6 that

\[ a = \frac{L}{v_L r_L^{-1} - v_G r_G^{-1}}. \]  

(10)

Note that even for out-of-plane circular orbits, where the projection in the occultation plane is an ellipse, the angle between the radius vector and the velocity vector in the occultation plane is exactly 90°. This is because the radius vector, as the occultation plane is redefined for each epoch, is always along the major axis of the ellipse.

Equation 10 shows the already known result (e.g., Schreiner et al., 1999) that for circular satellite orbits the retrieved impact parameter only depends on the measured Doppler shift and the satellite orbits; it is independent of the assumption of the electron density at the satellites. The corresponding bending angle (Equation 9) does depend on the assumed electron density at the satellites, but to lowest order the dependency becomes proportional to the inverse of the squared signal frequency, and thus cancel to a large degree when forming the ionospheric correction of bending angles at a common impact parameter. Normally the assumption is that \( n_L = n_G = 1 \), that is, ignoring the electron density at the satellites.

However, satellite orbits are generally not circular, and we should have in mind that the orbit coordinates in a retrieval would be with respect to a shifted origin (not the center of Earth) because of the oblateness correction (Syndergaard, 1998). The oblateness correction typically shifts the origin by several tens of kilometers, and the angle \( \zeta \) can become comparable to that of an orbit with an eccentricity similar to that of the Ørsted satellite (Larsen et al., 2005), for which the altitude difference between the apogee and the perigee was about 200 km. As we shall show shortly, non-circular orbits give rise to errors in the retrieved impact parameters when ignoring the electron density at the satellites, in particular that at the receiver, \( N(r_L) \). This in turn results in a residual error in the standard retrieval of the neutral atmospheric bending angle. It is a small error, though of sufficient size to be seen in simulations with highly non-circular orbits, and it is thus necessary to include it in a complete description of the ionospheric residual errors.

To derive this error, we first seek the error, \( \Delta a \), in the retrieved impact parameter that we get when ignoring the electron density at the satellites. Let \( \hat{a} = a + \Delta a \) denote the retrieved impact parameter for either the L1 or the L2 signal, while \( a \) in Equations 6–9 is the true solution. Thus, we write Equations 2–5 as

\[ L = v_L \cos \varphi(\hat{a}) - v_G \cos \chi(\hat{a}), \]  

(11)

with

\[ \varphi(\hat{a}) = \zeta - \arcsin \left( \frac{\hat{a}}{r_L} \right), \]  

(12)
\[ \chi(\tilde{a}) = (\pi - \eta) - \arcsin \left( \frac{\tilde{a}}{r_G} \right), \]

and

\[ \tilde{a} = \Theta - \arccos \left( \frac{a}{r_L} \right) - \arccos \left( \frac{\tilde{a}}{r_G} \right). \]

From Equations 6 and 11 we have

\[ \nu_L [n_L \cos \varphi(a) - \cos \varphi(\tilde{a})] = \nu_G \left[ n_G \cos \chi(a) - \cos \chi(\tilde{a}) \right], \]

which, with Equations 7, 8, 12, and 13, leads to

\[ \frac{\nu_L}{r_L} \left( \left( \frac{r_L^2}{a^2} - \tilde{a}^2 \right) - a^2 \right) \cos \zeta = \Delta a \sin \zeta \]

\[ \frac{\nu_G}{r_G} \left( \left( \frac{r_G^2}{a^2} - \tilde{a}^2 \right) - a^2 \right) \cos \eta = \Delta a \sin \eta. \]

Using series expansions to lowest order in the three small quantities given by \( \Delta a \), we end up with

\[ \Delta a = \frac{\nu_L \cos \zeta}{r_L^2} \left[ a \left( r_L^2 - a^2 \right)^{-1/2} \cos \zeta - \sin \zeta \right] + \frac{\nu_G \cos \eta}{r_G^2} \left[ a \left( r_G^2 - a^2 \right)^{-1/2} \cos \eta + \sin \eta \right]. \]

The left panel of Figure 4 shows the differences between retrieved impact parameters \( \tilde{a} \) and the corresponding ones obtained directly from the ray tracing \( a \) for the L1 and L2 signals (red and green, respectively). The results using Equation 17 are plotted on top of that (blue and magenta, respectively). This shows the very high accuracy of the analytical expression, and that the possible size of the error in the retrieved impact parameter because of non-circular orbits can be several meters.

Equation 17 can be simplified if we neglect \( \epsilon_G \) (assuming zero electron density at the GNSS satellite) and introduce

\[ H_L = \frac{\nu_L \cos \zeta}{r_L^2} \left[ a \left( r_L^2 - a^2 \right)^{-1/2} \cos \zeta - \sin \zeta \right] + \frac{\nu_G \cos \eta}{r_G^2} \left[ a \left( r_G^2 - a^2 \right)^{-1/2} \cos \eta + \sin \eta \right]. \]

With \( \epsilon_L \approx C N_L(r_L) f^{-2} \), being the signal frequency and \( C = 40.3082 \text{ m}^3 \text{ s}^{-2} \), we can then write

\[ \Delta a = \frac{C}{f^2} \frac{r_L n_L N_L(r_L)}{\sqrt{r_L^2 - a^2}}. \]

Thus, the error is inversely proportional to the squared signal frequency, and if the electron density at the GNSS satellite is small enough, the error can be considered proportional to the electron density at the LEO satellite.

Given the retrieved impact parameter, which in the geometrical optics approximation belongs to a given time, the error in the bending angle at this time, \( \Delta a = \tilde{a} - a \), can be derived by subtracting Equation 14 from Equation 9. As before, using series expansion to lowest order in the small quantities given by \( \Delta a \), we end up with

\[ \Delta a \approx \frac{\Delta a}{D} = \frac{ae_L}{\sqrt{r_L^2 - a^2}} - \frac{ae_G}{\sqrt{r_G^2 - a^2}}, \]

where \( D = \left( \frac{r_L^2}{a^2} \right)^{-1/2} + \left( \frac{r_G^2}{a^2} \right)^{-1/2} \). As described by Syndergaard (1999), the relation between time and impact parameter gives rise to an additional term, such that the resulting error in the bending angle, \( \delta a \), when regarded as a function of impact parameter, is given by

\[ \delta a = \frac{\partial a}{\partial a} \Delta a. \]
Whereas $\Delta \alpha$ is mainly proportional to $f^{-2}$, the additional term is mainly proportional to $f^{-4}$ in the mesosphere and above because both $\alpha$ (at high altitudes) and $\Delta a$ are mainly proportional to $f^{-2}$. We have previously, in the series expansions, only included terms proportional to $f^{-2}$, but because $\frac{d\alpha}{da}$ in practice can become large, we cannot generally ignore this additional term. In the simulations here, it results in an error in the ionospheric corrected bending angle at very high altitudes (due to the E-layer gradients) that is about one order of magnitude larger than the error we make by ignoring the $f^{-3}$ dependency on $\Delta \alpha$ (via $\Delta \alpha$, $e_L$, and $e_G$). Since the $f^{-2}$ dependency on $\Delta \alpha$ cancel in the standard ionospheric correction, the additional term in Equation 21 ends up being the one that gives rise to most of the error in the corrected bending angle at very high altitudes when not taking into account the electron density at the LEO satellite.

Thus, again neglecting $e_G$, the standard ionospheric correction applied to the L1 and L2 bending angle errors according to Equation 21 gives

$$\frac{f_1^2 \delta \alpha_1 - f_2^2 \delta \alpha_2}{f_1^2 - f_2^2} = \frac{C}{(f_1^2 - f_2^2)} \frac{d(a_2 - a_1)}{da} \frac{r_L N_e(r_L)}{\sqrt{r_L^2 - a^2}}, \tag{22}$$

where $\alpha_1$ and $\alpha_2$ are the true L1 and L2 bending angles, respectively. We shall refer to this as the bending gradient term because it depends on the derivative of the bending angles with respect to the impact parameter.

The right panel of Figure 4 shows the shape and size of the bending gradient term given the simulations described above. Since the ionosphere can be quite variable in time and space, the shape and size shown here is only an example that should not be generalized. On the other hand, such an example gives an indication of the possible size, and helps us understand how the shape depends on the electron density vertical gradients. Later, we will compare the bending gradient term to other residual terms.

To derive all the residual terms, we need to take into account higher-order contributions in Equation 20, and derive the residual error when applying the standard ionospheric correction to retrieved L1 and L2 bending angles. In line with the work by Syndergaard (2000), the recognition of which terms to keep and which to ignore was partly based on simulations with models representing a near worst case scenario (electron density, geomagnetic field, and orbits as described above). As also noted by Syndergaard (2000), it is important to emphasize that since the models were smooth, the estimates are only valid on the large scale, and consequently, the results cannot be used to correct small-scale features. In the following, we shall include all contributions up to order $f^{-4}$ that results in residual terms of size $\sim 0.01 \ \mu$rad, or larger. We start by revisiting the derivations leading to Equation 20, which leads us to a higher-order expression for the residual error in the retrieved bending angle. After that we shall derive a higher-order expression for the difference between the true bending angle (L1 or L2) and the ionosphere-free bending angle that we are interested in. The combination of the residual errors in the retrieved bending angles (compared to the true bending angles), and the difference between the true bending angles and the ionosphere-free bending angle, allows the derivation of the residual error using the standard ionospheric correction (see Figure 2).
2.4. Higher-Order Expression for the Error in Retrieved Bending Angles

In Equation 20, \( \Delta a/D \) is much smaller than the term including \( \varepsilon_L \), and we therefore do not have to consider a higher-order expression for \( \Delta a \). Thus, to first order in \( \Delta a/a \), but second order in \( \varepsilon_L \), and disregarding \( \varepsilon_\text{CE} \), we get

\[
\Delta a = \frac{\Delta a}{D} - \frac{a\varepsilon_L}{\sqrt{r_L^2 - a^2}} - \frac{1}{2} \frac{a\varepsilon_L^2}{\sqrt{r_L^2 - a^2}} \left( 2 + \frac{a^2}{r_L^2 - a^2} \right),
\]

(23)

with

\[
\varepsilon_L = \frac{C}{f^2} N_e(r_L) + \frac{K}{f^3} B_{\parallel}(r_L) N_e(r_L) + \frac{1}{2} \frac{C^2}{f^2} N_e^2(r_L),
\]

(24)

where \( B_{\parallel} = |\vec{B}| \cos \theta \) is the component of the geomagnetic field along the wave normal, \( \theta \) being the angle between the wave normal and geomagnetic field vector, \( \vec{B} \), and \( K = 1.1283 \cdot 10^{12} \text{m}^3 \text{T}^{-1} \text{s}^{-3} \).

Equation 24 comes from the high-frequency series expansion of the Appleton-Lassen formula (Budden, 1985). A complete high-frequency series expansion to order \( f^{-2} \), with indication of the approximate order of magnitude of different terms for GNSS frequencies, was given by Høeg et al. (1995). In our derivations here we retain the three largest terms. As noted by Høeg et al. (1995), the lowest order contribution from positive ions would be of a similar size as the last term in Equation 24, and should in principle be included, but since such a term is proportional to \( f^{-2} \), it cancel in the standard ionospheric correction, and we will not consider it. We write the term containing \( B_{\parallel} \) following the considerations by Petrie et al. (2010): The GPS and the Galileo signals are mainly right-hand circularly polarized, therefore giving rise to mainly ordinary waves when \( \cos \theta \) is negative and mainly extraordinary waves when \( \cos \theta \) is positive. Thus, the wave mode and the sign of the contribution from the term containing \( B_{\parallel} \) changes as \( \theta \) goes through \( \pi/2 \). In principle, Equation 24 is not a correct series expansion when \( \theta \approx \pi/2 \) (within ~0.1° of 90°), but the error we make is probably insignificant. The theory when \( \theta \) goes through \( \pi/2 \) for high-frequency signals, such as the ones in the GPS, is rather complex and includes coupling of the ordinary and the extraordinary wave modes (Cohen, 1960). Combining Equation 23 and Equation 24 to consistent order, gives

\[
\Delta a = \frac{\Delta a}{D} - \frac{C}{f^2} \frac{aN_e(r_L)}{\sqrt{r_L^2 - a^2}} - \frac{K}{f^3} \frac{AB_{\parallel}(r_L)N_e(r_L)}{\sqrt{r_L^2 - a^2}} - \frac{1}{2} \frac{C}{f^2} \frac{aN_e^2(r_L)}{\sqrt{r_L^2 - a^2}} \left( 3 + \frac{a^2}{r_L^2 - a^2} \right).
\]

(25)

Equation 25 is a higher-order expression for the error in the retrieved bending angle (L1 or L2) at a given time, and the second term on the right-hand side is by far the largest term. Below we shall refer to this as the main term; in our simulations it amounts to about 30% of the true bending angle. To get a higher-order expression for the error as a function of impact parameter we apply Equation 21. There is, however, a subtlety that needs to be considered when we later want to use the resulting expression in the derivation of the residual error using the standard ionospheric correction at a common impact parameter: At a common impact parameter, the L1 and L2 bending angles belong to different times, and for a non-circular orbit, \( r_i \) is slightly different for the L1 and the L2 signals, since the two signals did not arrive at the receiver at the same time. The difference is small, and need only be taken into account in the main term. We note that, in practice, the bending angles at a common impact parameter are obtained by interpolation, and the signals and times discussed here are not necessarily the discretely measured signals and times.

With that in mind, let \( t_i \) be the time of arrival of the L1 signal at the LEO, and \( t_e \) be the time of arrival of a hypothetical signal that is not affected by the ionosphere (one can think of a signal at a much higher frequency), both signals having the same impact parameter. For the L1 signal, with carrier frequency \( f_c \), we can then write

\[
\left. \frac{C}{f^2} \right|_{r(t_1)} \frac{aN_e[r(r(t_1))]}{\sqrt{r_i^2(r(t_1)) - a^2}} = \left. \frac{C}{f^2} \right|_{r(t_0)} \frac{aN_e[r(r(t_0))]}{\sqrt{r_i^2(r(t_0)) - a^2}} + \left. \frac{C}{f^2} \right|_{r(t_0)} \frac{a\Delta r_i}{\sqrt{r_i^2(r(t_0)) - a^2}} \frac{dN_e}{dr} \bigg|_{r=r_i(t_0)},
\]

(26)

where \( \Delta r_i = r_i(t_i) - r_i(t_0) \), and where we have included only the most important term in a series expansion, all other terms being less than 0.01 μrad. A similar expression holds for the L2 signal, just replacing the index 1 with...
is the L1 bending angle retrieved via Equation 25. It is mainly proportional to the standard ionospheric correction, whereas the radial gradient residual, although it is mainly proportional to the geomagnetic residual and the squared density residual give rise to residual errors in the standard ionospheric correction. Thus, we choose to use an identical term that appears in the derivation (in the following subsection) of the difference between the true bending angle and the ionosphere-free bending angle.

2. It is not important whether we use \( r_1(t_e) \), \( r_1(t_0) \), or \( r_1(t_{10}) \) in the other terms in Equation 25, or in the expression for \( \Delta \alpha \), but it makes a difference which one we use in the main term, and we want to use the same for the L1 and the L2 signals, such that the main term cancel in the derivation of the residual error in the standard ionospheric correction. Thus, we choose to use \( r_1(t_0) \) for both signals, and in all terms, which ensures that the following derivations for the L1 signal are applicable also for the L2 signal, just replacing the index 1 with 2. To simplify the notation, from now on we shall use \( r_L \) to denote \( r_1(t_0) \), unless otherwise specified.

It can be shown (see Appendix A) that \( \Delta r_L \), to lowest order, can be written as

\[
\Delta r_L = -H_L (\tilde{\alpha}_1 - \alpha_0),
\]

where \( \tilde{\alpha}_1 \) is the L1 bending angle retrieved via Equation 14, and \( \alpha_0 \) is the bending angle of the hypothetical ionosphere-free path. Since \( \tilde{\alpha}_1 - \alpha_0 \) is mainly proportional to \( f_i^{-1} \), the second term on the right-hand side of Equation 25 is mainly proportional to \( f_i^{-4} \). The quantity \( H_L \) can be either positive or negative (or zero). In our simulations, \( H_L \) is about 80 km (\( \cos \zeta \) is about –0.014), resulting in a negative \( \Delta r_L \) (and \( \Delta r_2 \)) with size of a few tens of meters. Finally, combining Equations 19, 21, 25, 26, and 27, we get

\[
\delta \alpha_1 = -\frac{C}{f_i^2} \frac{a N_e(r_L)}{\sqrt{r_L^2 - a^2}} \left( 1 - \frac{r_i H_L}{a D} \right) - \frac{C}{f_i^2} \frac{r_L H_L N_e(r_L) \, d\alpha}{\sqrt{r_L^2 - a^2} \, da}
\]

\[
= -\frac{K}{f_i^2} \frac{a B_0(r_L) N_e(r_L)}{\sqrt{r_L^2 - a^2}} - \frac{1}{2} \frac{C}{f_i^2} \frac{a N_e^2(r_L)}{\sqrt{r_L^2 - a^2}} \left( 3 + \frac{a^2}{r_L^2 - a^2} \right)
\]

\[
+ \frac{C}{f_i^2} \frac{a H_L (\tilde{\alpha}_1 - \alpha_0) \, dN_e}{\sqrt{r_L^2 - a^2} \, dr} \bigg|_{r_L},
\]

and similarly for \( \delta \alpha_2 \).

The first term (the combination of the main term and \( \Delta \alpha fD \)) on the right-hand side of Equation 28 cancel in the standard ionospheric correction, and the second term gives rise to the bending gradient term that was already discussed in the previous section. The remaining three terms are small residuals which we refer to as the geomagnetic residual, the squared density residual, and the radial gradient residual, in order of appearance in Equation 28. In our simulations, these three terms are smaller than 0.1 \( \mu \)rad, but still significant if we want to include contributions of size 0.01 \( \mu \)rad and larger. Their variation with impact height are shown for both L1 and L2 in Figure 5. The geomagnetic residual and the squared density residual give rise to residual errors in the standard correction, whereas the radial gradient residual, although it is mainly proportional to \( f_i^{-4} \), turns out to cancel with an identical term that appears in the derivation (in the following subsection) of the difference between the true bending angle and the ionosphere-free bending angle.
2.5. Higher-Order Expression for the Difference Between the True Bending Angle and the Ionosphere-Free Bending Angle

Similarly to the derivations by Syndergaard (2000), we shall write integral expressions for the true bending angle and for the hypothetical ionosphere-free bending angle, and transform the integrals into the same integration domain by appropriate changes of variables. Assuming local spherical symmetry, we start by writing the ionosphere-free bending angle at a given impact parameter as

\[ \alpha_0(a) = \alpha_0(a) = -a \int_{r_0}^{r} \frac{(d \ln n_0/dr) dr}{\sqrt{n_0^2 r^2 - a^2}} - a \int_{r_1}^{r} \frac{(d \ln n_0/dr) dr}{\sqrt{n_0^2 r^2 - a^2}}, \]

where \( r_0 \) is the perigee radius for the free path and \( n_0 = n_0(r) \) is the refractive index in the neutral atmosphere. To limit the length of equations later on, we denote the two integrals in Equation 29 by \( \alpha_0(a) \) and \( \alpha_0(a) \), in order of appearance. Likewise, for the true L1 bending angle, we write

\[ \alpha_1(a) = \alpha_0(a) + \alpha_1(a) = -a \int_{r_1}^{r} \frac{(d \ln n_1/dr) dr}{\sqrt{n_0^2 r^2 - a^2}} - a \int_{r_1}^{r} \frac{(d \ln (1-\varepsilon_1)/dr) dr}{\sqrt{(1-\varepsilon_1)^2 r^2 - a^2}}. \]

where \( r_1 \) is the perigee radius for the L1 path, and \( n_1 = n_0 - \varepsilon_1 \) with

\[ \varepsilon_1(r) = \frac{C}{f_1} N_e(r) + \frac{K}{f_1} B(r) N_e(r) + \frac{1}{2} \frac{C^2}{f_1^2} N_e^2(r). \]

We note that the upper limit in the second integral in Equation 30 is \( r_1(t_1) \), whereas a similar specification is not necessary in the first integral, because we assume that \( \ln n_1 = \ln n_2 = 0 \) at and near the altitude of the GNSS satellite. Since \( \ln n_2 = 0 \) anywhere in the ionosphere (above ~90 km), and \( \varepsilon_1 = 0 \) in the neutral atmosphere, and there is no region where both are appreciable (cf. Figure 1), we can split up the integrals in Equation 30, and write

\[ \alpha_{10}(a) = -a \int_{r_0}^{r_1} \frac{(d \ln n_0/dr) dr}{\sqrt{n_0^2 r^2 - a^2}} - a \int_{r_1}^{r_1} \frac{(d \ln (1-\varepsilon_1)/dr) dr}{\sqrt{(1-\varepsilon_1)^2 r^2 - a^2}}. \]

and

\[ \alpha_{11}(a) = -a \int_{r_0}^{r_1} \frac{(d \ln n_0/dr) dr}{\sqrt{n_0^2 r^2 - a^2}} - a \int_{r_1}^{r_1} \frac{(d \ln (1-\varepsilon_1)/dr) dr}{\sqrt{(1-\varepsilon_1)^2 r^2 - a^2}}. \]

where \( r_1 = a(1-\varepsilon_1)^{-1} \). It now follows that

\[ \alpha_1(a) - \alpha_0(a) = -a \int_{x_1}^{x_2} \frac{(d \ln (1-\varepsilon_1)/dx) dx}{\sqrt{x^2 - a^2}} - a \int_{x_1}^{x_2} \frac{(d \ln (1-\varepsilon_1)/dx) dx}{\sqrt{x^2 - a^2}}. \]

With a change of variable, \( x = r(1-\varepsilon_1) \), the lower integration limits and the denominators in the integrands become independent of frequency:

\[ \alpha_1(a) - \alpha_0(a) = -a \int_{x_1}^{x_2} \frac{(d \ln (1-\varepsilon_1)/dx) dx}{\sqrt{x^2 - a^2}} - a \int_{x_1}^{x_2} \frac{(d \ln (1-\varepsilon_1)/dx) dx}{\sqrt{x^2 - a^2}}. \]

where \( x_1(t_1) = r_1(t_1) (1-\varepsilon_1 r_1(t_1)) \). As in the work by Syndergaard (2000), we keep in mind that \( \varepsilon_1 \) in the numerators are functions of \( x \) via \( r(x) \). We have

\[ r(x) = \frac{x}{1-\varepsilon_1 r(x)} \]

which is an implicit equation with the sensitivity of the right-hand side to \( r(x) \) through \( \varepsilon_1 \) being rather weak. We can therefore, in general, obtain an explicit equation by iteratively inserting Equation 36 into the right-hand side, and expand in powers of \( \varepsilon_1 \) and its derivatives. To first order, we simply get
Here \( \varepsilon_1(x) \) is the expression in Equation 31, but with the argument \( r \) replaced by \( x \). With this, we can now use Taylor expansion to first order and write
\[
\varepsilon_1[r(x)] = \varepsilon_1(x) + x \varepsilon_1'(x) \frac{dx_1}{dx},
\]
and furthermore, to concordant order,
\[
\ln(1 - \varepsilon_1[r(x)]) = -\varepsilon_1(x) - x \varepsilon_1(x) \frac{dx_1}{dx} - \frac{1}{2} \varepsilon_1''(x).
\]
Finally, for the numerators in Equation 35, we arrive at
\[
\frac{d \ln(1 - \varepsilon_1[r(x)])}{dx} = -\frac{dx_1}{dx} - \frac{1}{2} x \frac{dx^2_1}{dx^2} - \frac{dx^2_1}{dx^2}.
\]
On the GNSS side, subtracting \( a_{t_0} \) from \( a_{t_1} \), using Equations 31 and 40, then gives
\[
a_{t_1}(a) - a_{t_0}(a) = \int_{r_0}^{r_1} \left[ C \frac{dN_e}{dx} + K \frac{dN_i}{dx} + 1 \frac{C^2}{2} \frac{dN_e^2}{dx} + 1 \frac{C^2}{2} \frac{dN_i^2}{dx} + 1 \frac{C^2}{2} \frac{dN_e^2}{dx} + x \frac{dN_i^2}{dx} \right] \frac{adx}{\sqrt{x^2 - a^2}}.
\]
In Equation 41, the first term in the square brackets cancel in the standard ionospheric correction. The remaining higher-order terms give rise to residual errors. On the LEO side, we get something very similar because \( x_1(t_1) \) can be replaced by \( r_1 \) when integrating the higher-order terms, but we need to retain \( x_2(t_1) \) as the upper limit when integrating the first term. Thus, we get
\[
a_{t_1}(a) - a_{t_0}(a) = \int_{x_1(t_1)}^{r_1} \frac{C}{f_1^1} \frac{dN_e}{dx} \frac{adx}{\sqrt{x^2 - a^2}}.
\]
Noting that, to sufficient accuracy, \( r_1 = x_1(t_1) = r_1 C f_1^1 N_e(r_1) - \Delta r_1 \), the last integral in Equation 42 results in
\[
\int_{x_1(t_1)}^{r_1} \frac{C}{f_1^1} \frac{dN_e}{dx} \frac{adx}{\sqrt{x^2 - a^2}} = \frac{C^2}{2 f_1^1} \frac{ar_1}{\sqrt{r_1^2 - a^2}} \frac{dN_e}{dr} \bigg|_{r_1} + \frac{C}{f_1^1} \frac{aH_i(a_1 - a_0)}{\sqrt{r_1^2 - a^2}} \frac{dN_e}{dr} \bigg|_{r_1},
\]
where we recognize the second term on the right-hand side as the radial gradient residual. The first term on the right-hand side of Equation 43 also appears in the following exact equation, for which the proof can be found in Appendix B.

\[
\frac{1}{2a} \int_{f_1^1} \frac{dN_e}{dx} \frac{adx}{\sqrt{x^2 - a^2}} - \frac{a^2 N_e^2(r_1)}{\sqrt{r_1^2 - a^2}} = \int_{a}^{r_1} \frac{1}{2} \frac{C^2}{2 f_1^1} \frac{dN_e^2}{dx} + x \frac{dN_i^2}{dx} \frac{adx}{\sqrt{x^2 - a^2}} - \frac{1}{2} \frac{C^2}{2 f_1^1} \frac{aN_i^2}{\sqrt{r_1^2 - a^2}} \frac{dN_e}{dr} \bigg|_{r_1}.
\]

On the GNSS side, we have just
\[
\frac{1}{2a} \int_{f_1^1} \frac{dN_e}{dx} \frac{adx}{\sqrt{x^2 - a^2}} = \int_{a}^{r_1} \frac{1}{2} \frac{C^2}{2 f_1^1} \frac{dN_e^2}{dx} + x \frac{dN_i^2}{dx} \frac{adx}{\sqrt{x^2 - a^2}}.
\]
Combining Equations 41–45, we end up with
\[ \alpha_1(a) - \alpha_0(a) = \int_a^\infty \left( \frac{C \, dN_e}{f_1^2} + \frac{K \, dB_1 N_e}{f_1} \right) \frac{a \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \left( \frac{C \, dN_e}{f_1^2} + \frac{K \, dB_1 N_e}{f_1} \right) \frac{a \, da}{\sqrt{x^2 - a^2}} \]

\[ + \frac{1}{2} \frac{C^2}{f_1^2} \int_a^{\infty} \frac{a^2 \, da}{\sqrt{x^2 - a^2}} + \frac{a^2}{\sqrt{r_L^2 - a^2}} \left[ \alpha \right] \right] \]

and similarly for \( \alpha_2(a) - \alpha_0(a) \).

### 2.6. Higher-Order Expression for the Residual Error in the Standard Ionospheric Correction of Bending Angles

Adding the results from Sections 2.4 and 2.5 we have the difference between the retrieved bending angles as functions of impact parameter and the ionosphere-free bending angle. For the L1 frequency, adding Equation 28 to Equation 46 we have

\[ \tilde{\alpha}_1(a) - \alpha_0(a) = \frac{C}{f_1^2} \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} - \frac{a N_e(r_1)}{\sqrt{r_L^2 - a^2}} \]

\[ + \frac{K}{f_1^2} \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} - \frac{a B_1 N_e(r_1)}{\sqrt{r_L^2 - a^2}} \]

\[ + \frac{1}{2} \frac{C^2}{f_1^2} \int_a^{\infty} \frac{a^2 \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{a^2 \, da}{\sqrt{x^2 - a^2}} - \frac{a^2 N_e^2(r_1)}{\sqrt{r_L^2 - a^2}} \]

\[ + \frac{C}{f_1^2} \int_a^{\infty} \frac{r_1 H_l N_e(r_1)}{\sqrt{r_L^2 - a^2}} \frac{d \alpha_1}{d a} \right], \]

and similarly for \( \tilde{\alpha}_2(a) - \alpha_0(a) \). Applying Equation 1 to these differences, and defining \( \delta \alpha_e = \alpha_e(a) - \alpha_0(a) \), we finally find the residual error in the standard ionospheric correction of retrieved bending angles:

\[ \delta \alpha_e = \frac{f_1^2 [\tilde{\alpha}_1(a) - \alpha_0(a)] - f_2^2 [\tilde{\alpha}_2(a) - \alpha_0(a)]}{f_1^2 - f_2^2} \]

\[ = \frac{K}{f_1^2} \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{dN_e}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} - \frac{a B_1 N_e(r_1)}{\sqrt{r_L^2 - a^2}} \]

\[ - \frac{1}{2} \frac{C^2}{f_1^2} \int_a^{\infty} \frac{a^2 \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{a^2 \, da}{\sqrt{x^2 - a^2}} - \frac{a^2 N_e^2(r_1)}{\sqrt{r_L^2 - a^2}} \]

\[ = \frac{C}{f_1^2} \int_a^{\infty} \frac{r_1 H_l N_e(r_1) d \alpha_1}{\sqrt{r_L^2 - a^2}} \]

If we introduce the functional \( F \), working on a function \( X \), such that

\[ F(X) = \int_a^{\infty} \frac{dX}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} + \int_a^{\infty} \frac{dX}{dx} \frac{a \, da}{\sqrt{x^2 - a^2}} - \frac{a X(r_1)}{\sqrt{r_L^2 - a^2}}. \]

then the result in Equation 48 can be written in a more condensed form as
The left panel of Figure 6 shows the residual bending angle error after applying the standard ionospheric correction of bending angles to simulated L1 and L2 signals (red). The theoretical result using Equation 48 is plotted on top of that (green). The same is shown in the right panel, except that the theoretical results are with $B_\parallel = 0$ (black), $H_L = 0$ (cyan), and $N_e(r_L) = 0$ (blue). Together, these results illustrate the very high accuracy of the derived expression, and that ignoring the geomagnetic field, the ellipticity of the LEO, or the electron density at the LEO satellite, in principle result in a wrong estimate of the residual error.

Throughout the derivations we have shown results using the parameters described in Section 2.1 with the electron density as shown in Figure 1. In the following we refer to this as the basis. In Figure 7 we additionally show three variations to illustrate the sensitivity to the orbit altitude, the shift of the circular orbit toward the GNSS satellite, and the overall electron density, respectively. Regarding the orbit altitude, it is about 420 km for the basis (green; 6,820 km circular orbit shifted 100 km), about 710 km for the first variation (magenta; 7,120 km circular orbit shifted 100 km), about 435 km for the second variation (orange; 6,820 km circular orbit shifted 50 km), and about 420 km for the third variation (gray; same orbit as the basis). We note that the electron density at the receiver is about one order of magnitude smaller in the first variation than in the basis (cf. Figure 1), but that the full residual in the left panel of Figure 7 is almost unaffected below 80 km. Thus, the electron density at the receiver has only little influence on the full residual in the standard ionospheric correction, at least for a stratified ionosphere. A similar conclusion was reached by Liu et al. (2020) who used the theory derived here (more precisely the extension of the theory presented in Section 4 below) and estimated the ionospheric residual error in ensembles for RO missions in both low and high altitude orbits and at different solar activity levels. That study also provides an alternative formulation of the last two terms in Equation 49 to better understand why the full residual is only slightly affected by the orbit altitude and thereby the electron density at the receiver. The bending gradient term in the right panel of Figure 7 is relatively highly affected above 80 km, and quite small in the first variation, although below 80 km the effect of the orbit altitude is small in absolute terms.

In the second variation, the orbit shift toward the GNSS satellite was 50 km instead of 100 km, but also this has little influence on the full residual in Figure 7, while reducing the bending gradient term by a factor of about two ($H_L$ was about 40 km in this case). In the third variation the electron density was reduced by a factor of two at all altitudes, which results in a significant reduction in a relative sense in both the full residual and the bending gradient term. The reduction is almost a factor of four because the electron density in the simulations is large enough so that the second term on the right-hand side of Equation 50, which depends on the electron density squared, dominates. For smaller electron density, the first term on the right-hand side of Equation 50 will dominate over.

\[
\delta a_c = -\frac{KF(B_\parallel N_e)}{f_1 f_2 (f_1 + f_2)} \left( \frac{C^2}{2 f_1^2 f_2^3} \frac{d}{da} \left[ a^2 P(N_e^2) \right] \right) - \frac{C}{(f_1^2 - f_2^2)} \frac{r_1 N_e(r_1) d(a_1 - a_2)}{\sqrt{r_1^2 - a^2}}. \tag{50}
\]
the second term, but overall the full residual will of course be smaller. The reason for the oscillations above 80 km in the bending gradient term (and in the full residual) is the influence of the E-layer in the simulations and the fact that this term depends on the derivative of the L1 and L2 bending angle difference, which is highly affected by the E-layer gradients. In all cases the bending gradient term is minuscule below 60 km.

As discussed above, the ray tracing results are affected by numerical noise. In Figure 6 it is seen that the numerical noise in the residual bending angle increases in the troposphere and blurs the fact that the size of the theoretical estimate in the left panel at low altitudes is slightly underestimated. Additional analysis, with much larger electron density, revealed more clearly a small systematic difference between the curves at low altitudes that increases downward in the troposphere. The analysis also showed that the difference is mainly due to a higher-order term proportional to the second derivative of the bending angle, which was omitted in Equation 21. The size of the difference here (a little more than 0.01 μrad near the surface) is unimportant since the bending angle in the troposphere is huge in comparison. However, it should be kept in mind, that we have used smooth models without any sharp gradients in the troposphere, and therefore the derivatives of the bending angle are small. Also, and likely more important, our simulations are without ionospheric horizontal gradients, and the errors in the retrieved impact parameters (and thereby the last term in Equation 21) exist only for non-circular orbits. In our simulations these errors are small (on the order of 10 m). In simulations with ionospheric horizontal gradients, in combination with a strong inversion layer at the top of the planetary boundary layer, Zeng et al. (2016) found large vertical shifts in L1 and L2 impact heights (on the order of 100 m). Although the theory here cannot be applied directly to such situations, it is possible that the last term in Equation 21, or a corresponding term in a theory including ionospheric horizontal gradients, could play a role in a theoretical understanding of such large vertical shifts and the accompanying bending angle errors.

Related to the remark above, we note that it is the true bending angles that are used in the derivative of the bending angles in the last term in Equation 48, which stems from Equation 21. It is shown in Appendix C, that if we instead use the retrieved bending angles, the main effect is that the residual in the troposphere becomes slightly overestimated by the same term (but with opposite sign) as the one responsible for the underestimation in Figure 6. Thus, in principle we may use the retrieved bending angles in Equation 48 without introducing additional terms in the derivations.

3. Relation to Previous Theoretical Formulations

With \( B_1 = 0 \) and \( N_e(r_L) = 0 \), Equation 48 reduces to

\[
\delta d_C^{[N_e=0,N_e(r_L)=0]} = -\frac{1}{2} \int \int \frac{dN_e^2}{dx} \frac{a^2}{\sqrt{x^2 - a^2}} + \int \frac{dN_e^2}{dx} \frac{a^2}{\sqrt{x^2 - a^2}}.
\]

Figure 7. Variations of residual bending angle errors in the standard dual-frequency ionospheric correction as a function of impact height. The basis is the result in Figure 6, with the parameters in the simulation described earlier, and each variation is relative to the basis as indicated in the legends. Left: According to the full expression in Equation 48. Right: The bending gradient term according to Equation 22 (last term in Equation 48).
Following the derivations in Appendix B (cf. Equation B3), and extending the upper integration limits to infinity in both integrals, we can write

$$\delta a_C^{[B]=0,N_r(\alpha)=0]} = - \frac{C^2}{f_1^2 f_2^2} \lim_{h \to 0} \left[ a \int_b^m \frac{dN_e^2}{dx} \left( 3x^2 - 2a^2 \right) dx - \frac{a^2}{\sqrt{b^2 - a^2}} \frac{dN_e^2}{dr} \right].$$

(52)

where $b = a + h$. For a non-zero vertical gradient of the electron density at the lower integration limit, each of the two terms in the brackets on the right-hand side of Equation 52 become infinite as $h \to 0$, but their difference converge to a finite value. Under the assumption that $h$ corresponds to an altitude well below the ionosphere, we can disregard taking the limit $h \to 0$, and the second term vanishes. We thereby get an expression for the ionospheric residual error equivalent to the one given by Vorob’ev and Krasil’nikova (1994):

$$\delta a_C^{[B]=0,N_r(\alpha)=0]} = - \frac{C^2}{f_1^2 f_2^2} a \int_b^m \frac{dN_e^2}{dx} \left( 3x^2 - 2a^2 \right) dx - \frac{a^2}{\sqrt{b^2 - a^2}} \frac{dN_e^2}{dr}.$$ 

(53)

If we are a little less restrictive, and take into account a non-zero $N_r(r_1)$, but set $H_\ell = 0$, then from Equation 50, we can write

$$\delta a_C^{[B]=0,H_\ell=0]} = \frac{1}{2} \frac{d}{da} \int F N_e^2 ds.$$ 

(54)

Similarly to some of the derivations in Appendix B, it can be shown that when $dr_1/da = 0$ (which follows from $H_\ell = 0$) we have

$$P(N_e^2) = \frac{d}{da} \int F N_e^2 ds,$$ 

(55)

where

$$\int F N_e^2 ds = \int_a^\infty N_e^2(x) \frac{dx}{\sqrt{x^2 - a^2}} + \int_a^\infty N_e^2(x) \frac{dx}{\sqrt{x^2 - a^2}}$$ 

(56)

is the integral of the square of the electron density between the GNSS and the LEO; the index $F$ indicates that the integral is along the hypothetical ionosphere-free path, and

$$ds = \frac{n_{ord} dr}{\sqrt{n_{ord}^2 - a^2}}$$ 

(57)

is an element of length along the path. Equation 56 can be deduced from the work by Syndergaard (2000). Here we note that the hypothetical ionosphere-free path in the ionosphere is basically a straight line (i.e., $dr/da = 1$ with $x = n_{ord}$), whereas in the part of the neutral atmosphere where the ionosphere-free path deviates appreciably from a straight line, the electron density is negligible. Inserting Equation 55 into Equation 54, we find

$$\delta a_C^{[B]=0,H_\ell=0]} = \frac{1}{2} \frac{C^2}{f_1^2 f_2^2} \frac{d^2}{da^2} \int F N_e^2 ds - \frac{C^2}{f_1^2 f_2^2} \frac{d}{da} \int F N_e^2 ds.$$ 

(58)

The first term on the right-hand side of Equation 58 is identical to the expression for the ionospheric residual error given by Syndergaard (2000). The second term is generally much smaller and was ignored by Syndergaard (2000); in the simulations above it is about 0.01 μrad.

One half of the last term in Equation 58 can be traced back to the last term in Equation 31. Without the last term in Equation 31, the factor in the numerator in Equation 53 becomes $(2x^2 - a^2)$ instead of $(3x^2 - 2a^2)$, which explains the basic difference between Equation 53 and the expression for the ionospheric residual error found by Healy and Culverwell (2015). Making assumptions about the vertical distribution of the electron density, Healy and Culverwell (2015) were able to show that

$$\delta a_C^{[B]=0,N_r(\alpha)=0]} \approx -\kappa(a)[a_1(a) - a_2(a)]^2,$$ 

(59)
where $\kappa(a)$ is a slowly varying function of the impact parameter. Following the approach of Healy and Culverwell (2015), but starting from Equation 53, assuming that the vertical distribution of the electron density is that of a Chapman layer with scale height $H$, and with the peak at $r_m$ we find

$$\kappa(a) = \frac{3}{8\pi} \frac{f_1^2 f_2^2}{(f_1^2 - f_2^2)^2} \frac{r_m \sqrt{r_m^2 - a^2}}{aH}.$$  \hspace{1cm} (60)

The difference to the result by Healy and Culverwell (2015) is minor (2%–3% for relevant values of $r_m$ and $a$), but the expression in Equation 60 is based on consistent inclusion of the last terms in Equations 31 and 40.

4. Bi-Local Spherical Symmetry

The derivations in Section 2 were based on the assumption of local spherical symmetry. With that assumption, Bouguer’s law for a ray path in an isotropic spherically stratified medium can be applied (Budden, 1985). If $\psi$ is the angle between the radius vector and the ray path at any given point along the path, Bouguer’s law states that

$$a = n(r) r \sin \psi = \text{constant},$$  \hspace{1cm} (61)

where $n(r)$ is the refractive index at radial distance $r$. Thus, we can say that the impact parameter is invariant along a given ray path, or that the impact parameter belongs to a given ray path.

A ray having the tangent point in the neutral atmosphere will travel first through the ionosphere on the inbound side, then through the neutral atmosphere, and finally through the ionosphere on the outbound side. If we assume that the stratification of the ionosphere on the inbound side is different from the stratification on the outbound side, the impact parameter for ray paths with tangent points well below the ionosphere is still invariant along the path, because there are only radial gradients in the vicinity of the path. Such a situation is illustrated in Figure 8. We shall refer to this as bi-local spherical symmetry.

All previous derivations are valid in the case of bi-local spherical symmetry for impact parameters with tangent points below the ionosphere. We just need to emphasize the possibility that the ionosphere on the GNSS side can be different from the ionosphere on the LEO side. We do this by modifying Equation 49 to

$$F'_b(X) = \int_a^b \frac{dX}{dx} \left| \frac{a}{\sqrt{x^2 - a^2}} \right| = \int_a^b \frac{a}{\sqrt{x^2 - a^2}} + \int_a^b \frac{a}{\sqrt{x^2 - a^2}} - \frac{aX(r_L)}{\sqrt{r_L^2 - a^2}},$$  \hspace{1cm} (62)

where the symbol $b$ is used as subscript to emphasize the bi-local spherical symmetry and the difference to Equation 49.

Figure 9 illustrates the ionospheric horizontal change and the residual bending angle errors in two cases of bi-local spherical symmetry that is referred to as “night-day” and “day-night.” The electron density in the simulations was constructed such that there was a very sharp horizontal gradient near the tangent points (which by design in the simulation setup were near zero degree longitude in the equator plane), but otherwise each side was spherically stratified. On the day-side the vertical distribution was identical to that of the double Chapman layer used above, whereas on the night-side, the E-layer was removed and the F-layer was reduced to 20% of the one on the day-side (analytical description can be found in Text S2 Supporting Information S1). The geomagnetic field and the refractivity in the neutral atmosphere were unchanged. In both cases, the ray tracing was such that rays propagated from west (negative longitudes) to east (positive longitudes) in the equator plane. The right panels show how the theoretical results matches the ray tracing and retrieval simulations as long as the tangent point is below the ionosphere (~90 km), but that the theory generally breaks down when the tangent point is situated within the ionosphere. Fortunately, we are only interested in the residual errors below about 90 km in atmospheric and climate applications of RO.

With the considerations above, and given vertical profiles of $N_e(r)$ and $B_z(r)$ on both sides of the tangent points, the standard ionospheric correction can be augmented with a bi-local residual error correction to yield
\[ \alpha_i(a) = \frac{f_1^2 \tilde{\alpha}_1(a) - f_2^2 \tilde{\alpha}_2(a)}{f_1^2 - f_2^2} + \delta \alpha_i(a), \]  

(63)

where \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) are the retrieved L1 and L2 bending angles, and

\[ \delta \alpha_i(a) = \frac{K \Psi_i(B_0 N_e)}{f_1 f_2 (f_1 + f_2)} + \frac{C}{2 f_1^2 f_2^2} \frac{d}{da} \left[ a^2 \Psi_i(N_e^2) \right] + \frac{C}{(f_1^2 - f_2^2)} \frac{r_i H_i N_i(r_i) d(\tilde{\alpha}_1 - \tilde{\alpha}_2)}{\sqrt{r_i^2 - a^2}} \]  

(64)

is the bi-local residual correction.

5. Impact of Ionospheric Horizontal Gradients

To assess the validity of the theory in conditions where there is not perfect bi-local spherical symmetry, but where the ionospheric horizontal gradients are more gradual along the ray paths, we made simulations similar to the ones in Section 4, but with less sharp horizontal electron density gradients near the tangent points (analytical description can be found in Text S2 Supporting Information S1). The integrals in Equation 49 were solved numerically using equalities similar to the one in Equation 56, with \( ds \) given by Equation 57, thus integrating along the ionosphere-free path that is known from the simulations. More specifically, in
Figure 10. The ionospheric horizontal change of electron density along a line with tangent point altitude of 60 km (left, indicating also the peak values of the E- and F-layers as a function of longitude) and the residual bending angle errors in the standard ionospheric correction as a function of impact height (right, co-showing also an approximate kappa-correction term with $\kappa(\alpha) = 14$) in three cases of horizontal gradients referred to by local time (LT) labels. Top: 15:30 LT transition case. Middle: 16:00 LT transition case. Bottom: 16:30 LT transition case. In all cases, the receiver was located at about 18° longitude (to the right in the left panels).
the numerical integrations (also in the previous sections) we used the following form of the functional in Equation 49:

\[ F(X) = \int_{F} \frac{a \, dX}{d \xi} = \frac{aX(\xi)}{\sqrt{\xi^{2} - a^{2}}} \]  

(65)

Figure 11. Same as Figure 10, but for three additional cases. Top: 17:00 local time (LT) transition case. Middle: 17:30 LT transition case. Bottom: 18:00 LT transition case.
In this way, we are able to evaluate $dX/dx$ without ambiguity when there are horizontal gradients along the paths, and easily include the subtle complexity when $r_L$ depends on $a$ for a non-circular LEO. In practice, as done in the study by Liu et al. (2020), when the electron density and exact paths are unknown, it may be more convenient to decide on fixed profiles of $dX/dx_{\kappa}$ and $dV/dx_{\kappa}$ (e.g., based on a model, taking representative profiles of electron density at fixed locations on each side of the tangent points) and use Equation 62 with a constant $r_L$.

Figures 10 and 11 show a series of cases with the horizontal gradients illustrated on the left, and the corresponding residuals plotted on the right. The cases are labeled by local times (LT) used in the simulations; the times are only given to distinguish the different cases and are not otherwise important. What is important is that there are ionospheric horizontal gradients along the ray paths. These could as well represent horizontal gradients across the equator, turning the geometry of the setup by $90^\circ$. Besides the theoretical results based on the equations in Section 2.6 (green) and the ray tracing and retrieval simulations (red), we also plot the residuals given by Equation 59 with $\kappa(a) = 14$, and where $\alpha_1(a)$ and $\alpha_2(a)$ are the retrieved bending angles (blue).

The series of cases illustrates the limitation of the theoretical results. It shows that the theoretically derived residual errors, whether based on Equation 50 or Equation 59, can be significantly off when there are horizontal gradients along the ray paths. This seems to be true in particular if there are large gradients at the location of the receiver (at 16:30LT and 17:00LT in these simulations). The receiver in the simulations was located at approximately $18^\circ$ longitude for a ray path with tangent point altitude at 60 km. However, we should note here that the part of the residual due to $B_\|$, responsible for a good part of the difference between the blue and the green curves below 80 km, can be either negative or positive (it was negative in our simulations). Horizontal gradients can be in either direction as well, and in general be much more disorderly than simulated here.

The ray tracing and retrieval simulations here and in the previous sections are qualitatively consistent with those of Li et al. (2020). Using 3D ray tracing and an ionospheric model with large-scale electron density horizontal gradients, though without a geomagnetic field model, Li et al. (2020) showed that the residual can be either positive or negative, whereas if there is spherical symmetry it is always negative below 90 km. It is also negative when taking the mean over many cases. With data from the Constellation Observing System for Meteorology, Ionosphere and Climate (COSMIC), they furthermore showed that the sign of the residual in the 60–80 km range at daytime varies with latitude across the geomagnetic equator due to horizontal gradients in the meridional direction.

6. Conclusions

For spherically stratified media, taking into account all known sources of systematic ionospheric residual errors, the total residual error in the standard correction of GNSS radio occultation (RO) bending angles can be very accurately written as a sum of three terms as given in the final Equation 50 of Section 2. The residual error depends on the vertical distribution of the ionospheric electron density, the geomagnetic field, and the satellite positions and velocities. The third term in Equation 50 is only appreciable for satellites in highly non-circular orbits. These terms were obtained by first deriving a higher-order expression for the error in the retrieved bending angles from dual-frequency RO excess phases in the geometrical optics approximation, second by deriving a higher-order expression for the difference between the true bending angle (“true” in the context of the theory) and the hypothetical ionosphere-free bending angle, and finally by deriving a higher-order expression for the residual error in the standard ionospheric correction of retrieved bending angles using the previously derived results. Along the road, the derivations were supported and verified by high-accuracy ray tracing simulations and corresponding bending angle retrievals, ensuring that all relevant contributions were included. It was pointed out (in Section 3) how the theory relates to less accurate previous theoretical works and how these can be derived from the end result using different approximations.

The derivation of the theory, in combination with further geometrical considerations, led to the suggestion of a so-called bi-local residual error correction in Section 4. The bi-local correction allows for the situation where the ionosphere is different (though still spherically stratified) on the inbound (from transmitter) and outbound (toward receiver) sides of the RO event tangent points. In principle, the bi-local residual correction is accurate to the 0.01 μrad level, even in worst case scenarios with very high electron density concentrations and highly non-circular orbits. However, in more realistic situations including horizontal gradients, uncertainty in the knowledge about the true ionospheric environment, and violation of the assumption of bi-local spherical symmetry, will limit its accuracy.
Appendix A: Derivation of Equation 27

The bending angle of a hypothetical signal that is not affected by the ionosphere (e.g., in a simulation without the ionosphere) can be derived using Equations 6–9 with \( n_L = n_G = 1 \). Thus

\[
\alpha_0 = \Theta - \arccos \left( \frac{a}{r_L} \right) - \arccos \left( \frac{a}{r_G} \right).
\]

(A1)

Let us say that this signal arrived at the receiver at time \( t = t_0 \). The retrieved bending angle of the L1 signal, arriving at the receiver at a slightly different time, \( t_1 \), but having the same impact parameter, can be derived using Equations 11–14 with \( \tilde{a}(t_1) = a \), and written as

\[
\tilde{\alpha}_1(t_1) = \Theta(t_1) - \arccos \left( \frac{a}{r_{L}(t_1)} \right) - \arccos \left( \frac{a}{r_{G}(t_1)} \right).
\]

(A2)

To simplify the notation, we shall not explicitly note the time dependence when \( t = t_0 \). Using Taylor expansion to first order in \( \Delta t = t_1 - t_0 \), we therefore write

\[
\tilde{\alpha}_1(t_1) = \Theta + \frac{d\Theta}{dt} \Delta t - \arccos \left( \frac{a}{r_L} \right) - \frac{1}{\sqrt{r_L^2 - a^2}} \frac{a}{r_L} \frac{dr_L}{dt} \Delta t - \arccos \left( \frac{a}{r_G} \right) - \frac{1}{\sqrt{r_G^2 - a^2}} \frac{a}{r_G} \frac{dr_G}{dt} \Delta t.
\]

(A3)

Noting (from Figure 3) that

\[
\frac{d\Theta}{dt} = \frac{\nu_L}{r_L} \sin \zeta - \frac{\nu_G}{r_G} \sin \eta,
\]

(A4)

\[
\frac{dr_L}{dt} = v_L \cos \zeta,
\]

(A5)

\[
\frac{dr_G}{dt} = v_G \cos \eta,
\]

(A6)

and subtracting Equation A1 from Equation A3 we get
\[ \bar{a}_1(t_1) - a_0 = \left[ \frac{\nu_L}{r_L} \left( \frac{a \cos \zeta}{\sqrt{r_L^2 - a^2}} \sin \zeta \right) + \frac{\nu_G}{r_G} \left( \frac{a \cos \eta}{\sqrt{r_G^2 - a^2}} + \sin \eta \right) \right] \Delta t. \] (A7)

Noting furthermore that
\[ \Delta r_1 = \frac{dr_1}{dt} \Delta t, \] (A8)
and combining this with Equation A5 and Equation A7 leads to Equation 27 with \( H_L \) given by Equation 18.

### Appendix B: Derivation of Equation 44

In this appendix we show that

\[ \frac{d}{da} \left( \int_a^{r_h} \frac{a^2 dx}{\sqrt{x^2 - a^2}} - a^2 N_e^2(r_h) \right) = \int_a^{r_h} \left( 3 \frac{dN_e^2}{dx} + \frac{x^2 N_e^2}{\sqrt{x^2 - a^2}} \right) \frac{a^2 dx}{\sqrt{x^2 - a^2}} \]

\[ - \frac{a^2 N_e^2(r_h)}{\sqrt{r_h^2 - a^2}} \left( 3 + \frac{a^2}{r_h^2 - a^2} \right) - \frac{a^2 r_h}{\sqrt{r_h^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{r_h}, \] (B1)

from which Equation 44 readily follows by multiplication with \( C^2(2a \sigma_f^4)^{-1} \).

To handle the singularity in the lower limit of the integral on the left-hand side of Equation B1 we introduce \( b = a + h \), where \( h \) is small and positive, and independent of \( a \). We write

\[ \frac{d}{da} \int_a^{r_h} \frac{a^2 dx}{\sqrt{x^2 - a^2}} = \lim_{b \to a} \frac{d}{da} \int_b^{r_h} \frac{a^2 dx}{\sqrt{x^2 - a^2}}. \] (B2)

Using Leibniz's rule for differentiation of integrals we have

\[ \frac{d}{da} \int_b^{r_h} \frac{a^2 dx}{\sqrt{x^2 - a^2}} = \int_b^{r_h} 3 \frac{dN_e^2}{dx} \frac{a^2 dx}{\sqrt{x^2 - a^2}} + \int_b^{r_h} \frac{a^4 dx}{(x^2 - a^2)^{3/2}} \]

\[ + \frac{a^3}{\sqrt{r_h^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{r_h} - \frac{a^3}{\sqrt{b^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{b} \] (B3)

where we note that \( dr/da \neq 0 \) in the general case of non-circular orbits, and that \( db/da = 1 \) because \( h \) has been defined to be independent of \( a \). Furthermore, using integration by parts we find that

\[ \int_b^{r_h} \frac{a^4 dx}{(x^2 - a^2)^{3/2}} \]

\[ = \int_b^{r_h} x^2 N_e^2 \frac{a^2 dx}{\sqrt{x^2 - a^2}} + \frac{a^2 r_h}{\sqrt{r_h^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{r_h} + \frac{a^2 b}{\sqrt{b^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{b}. \] (B4)

Substituting Equation B4 into the second integral on the right-hand side of Equation B3, and taking the limit \( h \to 0 \), for which we note that \( a^2(b - a)(b^2 - a^2)^{-1/2} \to 0 \), we get

\[ \frac{d}{da} \int_a^{r_h} \frac{a^2 dx}{\sqrt{x^2 - a^2}} = \int_a^{r_h} \left( 3 \frac{dN_e^2}{dx} + x \frac{d^2 N_e^2}{dx^2} \right) \frac{a^2 dx}{\sqrt{x^2 - a^2}} + \frac{a^2}{\sqrt{r_h^2 - a^2}} \frac{dN_e^2}{dr} \bigg|_{r_h} \left( a \frac{dr_h}{da} - r_h \right). \] (B5)
For the second term on the left-hand side of Equation B1 we have
\[
\frac{d}{da} \left( \frac{a^2 N_2^2(r_L)}{\sqrt{r_L^2 - a^2}} \right) = \frac{a^2 N_2^2(r_L)}{\sqrt{r_L^2 - a^2}} \left( 3 + \frac{a^2}{r_L^2 - a^2} \right) + \frac{a^2}{\sqrt{r_L^2 - a^2}} \left( \frac{d N_2^2}{dr} \right) \left|_{r_L} \right| \frac{dr}{da}.
\] (B6)

Subtracting Equation B6 from Equation B5 gives Equation B1.

**Appendix C: Expansions Related to the Last Term in Equation 48**

The difference in using $\alpha$ or $\tilde{\alpha}$ in the derivative of the bending angles in the last term in Equation 48 is quite subtle. To show this, we recall that the error in the retrieved bending angle at a given time is
\[
\Delta \alpha = \tilde{\alpha}(\tilde{a}) - \alpha(a),
\] (C1)

where we refer to $\alpha$ as the true bending angle, $\tilde{\alpha}$ as the retrieved bending angle, and where $\tilde{a} = a + \Delta a$ is the retrieved impact parameter in the general case of non-circular orbits (with $\Delta a$ given by Equation 17). We seek a higher-order expression for the difference $\delta \alpha(a) = \tilde{\alpha}(a) - \alpha(a)$, that is, the error in the retrieved bending angle as a function of impact parameter. Using Taylor expansion to second order in $\Delta a$, we write
\[
\tilde{\alpha}(\tilde{a}) = \tilde{\alpha}(a) + \frac{d \tilde{\alpha}}{da} \Delta a + \frac{1}{2} \frac{d^2 \tilde{\alpha}}{da^2} (\Delta a)^2,
\] (C2)

which inserted into Equation C1, and rearranging, gives
\[
\delta \alpha(a) = \Delta a - \frac{d \tilde{\alpha}}{da} \Delta a - \frac{1}{2} \frac{d^2 \tilde{\alpha}}{da^2} (\Delta a)^2.
\] (C3)

To get a corresponding higher-order expression using $da/da$ instead of $d\tilde{a}/da$, we take the derivative of Equation C3, while retaining only the derivative of the first two terms on the right-hand side, and multiply by $\Delta a$. This leads to
\[
\frac{d \tilde{\alpha}}{da} \Delta a - \frac{da}{da} \Delta a = \frac{d \Delta a}{da} \Delta a - \frac{1}{2} \frac{d^2 \tilde{\alpha}}{da^2} (\Delta a)^2 - \frac{d \Delta a}{da} \frac{d \tilde{\alpha}}{da} \Delta a.
\] (C4)

The last term in Equation C4 can safely be ignored because $d \Delta a/da \ll 1$. The first term on the right-hand side was less than 0.01 μrad in our simulations, and about an order of magnitude smaller than the second term in the lower troposphere. Thus, to sufficient accuracy we have
\[
\frac{d \tilde{\alpha}}{da} \Delta a = \frac{da}{da} \Delta a - \frac{1}{2} \frac{d^2 \tilde{\alpha}}{da^2} (\Delta a)^2,
\] (C5)

which inserted into Equation C3 gives
\[
\delta \alpha(a) = \Delta a - \frac{da}{da} \Delta a + \frac{1}{2} \frac{d^2 \tilde{\alpha}}{da^2} (\Delta a)^2.
\] (C6)

Comparing Equation C3 and Equation C6, it is seen that to first order there is no difference between the two, other than the use of $d\tilde{a}/da$ in Equation C3 versus the use of $da/da$ in Equation C6. However, the error that we make in a first order expression using $d\tilde{a}/da$ is equal in size but with opposite sign of the error that we make when using $da/da$. The size of the error is only appreciable in the lower troposphere where the bending angle and its vertical variability can become large. In Equation 21 we included only the first two terms on the right-hand side of Equation C6, resulting in a slightly underestimated size (in the lower troposphere) of the total ionospheric residual error given in Equation 48.
Data Availability Statement

Data were not used, nor created for this research.

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