INSCRIBABLE FANS I:
INSCRIBED CONES AND VIRTUAL POLYTOPES

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Abstract. We investigate polytopes inscribed into a sphere that are normally equivalent (or strongly isomorphic) to a given polytope $P$. We show that the associated space of polytopes, called the inscribed cone of $P$, is closed under Minkowski addition. Inscribed cones are interpreted as type cones of ideal hyperbolic polytopes and as deformation spaces of Delaunay subdivisions. In particular, testing if there is an inscribed polytope normally equivalent to $P$ is polynomial time solvable.

Normal equivalence is decided on the level of normal fans and we study the structure of inscribed cones for various classes of polytopes and fans, including simple, simplicial, and even. We classify (virtually) inscribable fans in dimension 2 as well as inscribable permutahedra and nestohedra.

A second goal of the paper is to introduce inscribed virtual polytopes. Polytopes with a fixed normal fan $\mathcal{N}$ form a monoid with respect to Minkowski addition and the associated Grothendieck group is called the type space of $\mathcal{N}$. Elements of the type space correspond to formal Minkowski differences and are naturally equipped with vertices and hence with a notion of inscribability. We show that inscribed virtual polytopes form a subgroup, which can be non-trivial even if $\mathcal{N}$ does not have actual inscribed polytopes.

We relate inscribed virtual polytopes to routed particle trajectories, that is, piecewise-linear trajectories of particles in a ball with restricted directions. The state spaces give rise to connected groupoids generated by reflections, called reflection groupoids. The endomorphism groups of reflection groupoids can be thought of as discrete holonomy groups of the trajectories and we determine when they are reflection groups.

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1. Introduction

Let $P \subseteq \mathbb{R}^d$ be a convex polytope. We call $P$ **inscribed** if its vertices $V(P)$ lie on a common sphere. Prominent examples include the regular polytopes and, more generally, vertex-transitive polytopes; cf. [9]. In 1832 Steiner (cf. [30]) asked if every 3-dimensional polytope is **inscribable**, that is, if for every 3-polytope $P$ there is an inscribed polytope $P'$ that is combinatorially equivalent to $P$. Almost a century later, Steinitz [31] gave a simple combinatorial criterion for non-inscribability and with it the first counterexample to Steiner’s question. Using Steinitz seminal result that combinatorial types of 3-polytopes are in one-to-one correspondence with 3-connected planar graphs (on $\geq 4$ nodes), Rivin [27] showed that testing inscribability can be done in polynomial time: to a given 3-connected planar graph, a system of linear equations and strict inequalities is associated and feasibility of said system is equivalent to an inscribed realization; see (10). This settled the inscribability problem in dimensions $\leq 3$; see [13, Ch.13.5] or [23] for more on the historic background.

By Corollary 4.16 of [1], the problem of finding an inscribed polytope combinatorially equivalent to a given (simplicial) polytope $P$ is polynomial-time equivalent to the **existential theory of the reals** (ETR) and hence NP-hard. In this paper, we focus on a more restrictive form of equivalence: two polytopes $P$ and $P'$ are **normally equivalent** (written $P \simeq P'$) if $P$ is combinatorially isomorphic to $P'$ and corresponding faces are parallel. Figure 1 shows three
normally-equivalent polytopes. Normal equivalence is a natural and important notion. It occurs, for example, in McMullen’s proof of the g-Theorem [20], in the study of torus invariant divisors on toric varieties [8, Ch. 6.3], and in parametric linear programming [2]. In particular, the deformation space of polytopes normally equivalent to \( P \) is (simply) connected, which prompts the following question:

\[ \text{Is there an inscribed polytope } P' \text{ normally equivalent to } P? \]

We define the \textit{inscribed cone}

\[ \mathcal{I}_+(P) := \{ P' \simeq P : P' \text{ inscribed} \} / \text{translations} \]

as the space of inscribed polytopes normally equivalent to \( P \) up to translation. A polytope \( P \) is said to be \textit{normally inscribable} if \( \mathcal{I}_+(P) \neq \emptyset \). Recall that the Minkowski sum of two polytopes \( Q, Q' \subset \mathbb{R}^d \) is the polytope \( Q + Q' := \{ q + q' : q \in Q, q' \in Q' \} \). It is easy to see that if \( Q \simeq Q' \simeq P \), then \( Q + Q' \simeq P \). Our first main result shows that \( \mathcal{I}_+(P) \) has an intriguing structure, which justifies the name.

\textbf{Theorem 1.1.} Let \( P \) be a convex \( d \)-dimensional polytope and \( Q, Q' \in \mathcal{I}_+(P) \). Then

\[ Q + Q' \in \mathcal{I}_+(P). \]

In particular, \( \mathcal{I}_+(P) \) has the structure of an open polyhedral cone of dimension \( \leq d \).

A direct application of Theorem 1.1 shows how symmetry is affected by inscribability.

\textbf{Corollary 1.2.} Let \( P \subset \mathbb{R}^d \) be a convex polytope invariant with respect to the action of a finite group of orthogonal transformations \( G \). If \( P \) is normally inscribable, then there is an inscribed polytope \( P' \in \mathcal{I}_+(P) \) that is invariant under \( G \).

We give a first proof of Theorem 1.1 in Section 2. In Section 5 we give an \textit{intrinsic} representation of \( \mathcal{I}_+(P) \) that makes no reference to the embedding of \( P \) (Theorem 5.20). We obtain an explicit description in terms of linear inequalities and equations, which then allows us to give a satisfactory algorithmic answer to the above question:

\textbf{Theorem 1.3.} Testing if a (rational) polytope \( P \) has a normally equivalent inscribed polytope can be done in polynomial time.

Moreover, if \( P \) is rational and \( \mathcal{I}_+(P) \neq \emptyset \), then there is a rational inscribed polytope \( P' \simeq P \).
Whereas testing if a given polytope $P$ is inscribable is tantamount to solving a system of polynomial equations and strict inequalities (see [23, Section 5]), the proof of Theorem 1.3 shows that testing if $P$ is normally inscribable can be phrased in terms of linear equations and strict inequalities. Moreover, the feasibility problem only makes use of the geometric graph of $P$ and is similar to that of Rivin [27].

Let us write $\mathcal{N}(P)$ for the normal fan of $P$. This is a complete and strongly connected polyhedral fan whose maximal cones are precisely the domains of linearity of the support function of $P$; see Section 2. It is straightforward to check that $P \simeq P'$ if and only if $\mathcal{N}(P) = \mathcal{N}(P')$. Hence, the class of polytopes normally equivalent to $P$ is represented by $\mathcal{N}(P)$. We call a fan $\mathcal{N}$ inscribable if there is an inscribed polytope $P$ with $\mathcal{N} = \mathcal{N}(P)$ and we denote by $\mathcal{I}_+(\mathcal{N})$ the inscribed cone of $\mathcal{N}$. Thus, $\mathcal{I}_+(P) = \mathcal{I}_+(\mathcal{N}(P))$. We study inscribed cones for certain classes of polytopes and fans in Section 4. In particular, we give a characterization of polytopes $P$ for which $\dim \mathcal{I}_+(P)$ has maximal dimension (Theorem 4.16). At the other extreme, we show that the $\dim \mathcal{I}_+(P) \leq 1$ whenever $P$ is a simplicial polytope (Corollary 4.8) and we show that this can even happen for simple polytopes (Corollary 4.5). Using a local characterization (Theorem 4.12) of inscribability for general polytopes, we show that a simple polytope $P$ is inscribed if and only if all $k$-faces are inscribed for some fixed $k > 1$ (Corollary 4.14). In the follow-up paper [17], we use that to show that inscribed zonotopes are determined by their 2-faces, extending the local characterization of zonotopes due to Bolker [5].

Many of these results rely on Section 3, in which we completely characterize (virtually) inscribable fans in the plane. We show that the set of (virtually) inscribable 2-dimensional fans with a fixed number of regions has the structure of a polytope (Proposition 3.1, Theorem 3.5). As an example, we investigate the fan determined by the braid arrangement $A_{d-1}$ in Section 2.3. We show that the polytopes in $\mathcal{I}_+(A_{d-1})$ are vertex-transitive with respect to the symmetric group and hence permutahedra. The example is continued in Section 4.3, where we determine the inscribed nestohedra, a class of simple, generalized permutahedra [25]. This characterizes all matroid polytopes (in the sense of [6]) among nestohedra.

In the upcoming paper [17], we study the inscribed cones for fans coming from general hyperplane arrangements and show strong ties to simplicial arrangements and reflection groups.

The following two subsections highlight two significant implications of Theorem 1.1.

1.1. Ideal hyperbolic type cones. The projective disk (or Beltrami–Klein) model identifies hyperbolic space $\mathbb{H}^d$ with the points in the open unit ball $D^d = \{ x \in \mathbb{R}^d : \| x \| < 1 \}$. Hyperplanes in $\mathbb{H}^d$ correspond to sets of the form $D^d \cap H$, where $H \subset \mathbb{R}^d$ is an ordinary hyperplane meeting $D^d$. The two halfspaces induced by $H$ are convex and finite intersections of halfspaces give rise to hyperbolic polyhedra. In particular, hyperbolic polytopes correspond to polytopes contained in the unit ball $B^d = \overline{D^d}$. A vertex $v$ of $P$ is called ideal if $v \in \partial B^d = S^{d-1}$ and $P \subseteq B^d$ is an ideal hyperbolic polyhedron if all vertices are ideal. For more information on hyperbolic geometry and hyperbolic polytopes we refer to [26, Ch. 6] and [32].

In this non-conformal model for hyperbolic space, hyperbolic polytopes are simply Euclidean polytopes and thus are equipped with the notion of normal equivalence and normal fans.
Steiner’s question and Rivin’s result pertain to the combinatorics of ideal hyperbolic polyhedra. Theorem 1.1 states that Minkowski sums extend to ideal hyperbolic polytopes with fixed normal fan. Let $Q, Q' \subset B^d$ be ideal hyperbolic polytopes. For a generic linear function $l(x)$, let $q$ and $q'$ be the unique vertices maximizing $l(x)$ over $Q$ and $Q'$, respectively. We define the angle between $Q$ and $Q'$ by $\theta(Q, Q') \in [0, \pi)$ as the angle between $q$ and $q'$.

**Corollary 1.4.** Let $Q, Q' \subset \mathbb{R}^d$ be normally equivalent ideal hyperbolic polytopes. The angle $\theta(Q, Q')$ is independent of the choice of linear function $l(x)$. Moreover,

$$\frac{1}{\sqrt{2 + 2 \cos \theta(Q, Q')}}(Q + Q')$$

is again an ideal hyperbolic polytope.

If $Q$ and $Q'$ are ideal hyperbolic polytopes which are not normally equivalent, then their Minkowski sum might still be ideal, provided $Q$ and $Q'$ are inscribed relative to $N$ with respect to the fan $\mathcal{N}(Q + Q')$; see Section 5.2 for details.

McMullen [18] introduced the type cone of a polytope $P$ as the space of polytopes normally equivalent to $P$ up to translation

$$\mathcal{T}_+(P) := \{P' : P \simeq P'\} / \text{translations}.$$ 

It follows from work of Shephard (cf. [13, Sect. 15.1]) that $\mathcal{T}_+(P)$ is an open polyhedral cone. In that sense, we may view $\mathcal{T}_+(P)$ as the ideal hyperbolic type cone of an ideal hyperbolic polytope $P$. In Section 5.2, we investigate the structure of the closure $\overline{\mathcal{T}}_+(P)$, which turns out to be more subtle than that of $\mathcal{T}_+(P)$.

We believe that further study of the relationship between $\mathcal{I}_+(P)$ and $\mathcal{T}_+(P)$ and their implications for ideal hyperbolic polytopes (such as ideal hyperbolic flips, for example) will be very exciting.

### 1.2. Deformations of Delaunay subdivisions.

A Delaunay subdivision of a full-dimensional polytope $P \subset \mathbb{R}^{d-1}$ is a subdivision $\mathcal{D} = \{P_1, \ldots, P_m\}$ into full-dimensional and inscribed polytopes that satisfy the following condition for all $i = 1, \ldots, m$: If $B$ is the unique ball to which $P_i$ is inscribed, then $B \cap V(P_j) \subseteq V(P_i)$ for all $j \neq i$. Delaunay subdivisions and in particular Delaunay triangulations play an important role in numerical computations; see [4] for more. The basic computational task is for a given $U \subset \mathbb{R}^{d-1}$ to construct a Delaunay subdivision $\mathcal{D}(U)$ of $P = \text{conv}(U)$ such that $V(P_i) \subseteq U$ for all $i = 1, \ldots, m$ and $U = \bigcup_i V(P_i)$.

Brown [7] observed a simple correspondence between inscribed polytopes and Delaunay subdivisions. Let $\eta : S^{d-1} \to \mathbb{R}^{d-1} \times \{0\}$ be the stereographic projection from the north-pole $e_d$ to the equatorial plane $\mathbb{R}^{d-1} \times \{0\}$. The polytope $\hat{P} := \text{conv}(\eta^{-1}(U))$ is inscribed into the unit sphere. The visibility complex of $\hat{P}$ with respect to some fixed $\xi \in S^{d-1}$ is the collection of faces $F \subset \hat{P}$ such that $\text{conv}(\xi \cup F)$ does not meet the interior of $\hat{P}$. Then the collection of facets of $\hat{P}$ not visible from $e_d$ stereographically projects to a Delaunay subdivision of $P = \text{conv}(U)$.

Thus every configuration $U$ has a Delaunay subdivision but slight perturbations of the points can result in drastic changes in the combinatorics of $\mathcal{D}(U)$. In Section 2.4, we distill a notion of normal equivalence for (labelled) Delaunay subdivisions, which allows us to interpret $\mathcal{I}_+(\hat{P})$
as a deformation space of the Delaunay subdivisions. Figure 2 shows two normally equivalent Delaunay subdivisions.

Corollary 1.5. Let $U \subset \mathbb{R}^{d-1}$ be an affinely-spanning point configuration. The space of Delaunay subdivisions $\mathcal{D}(U')$ normally equivalent to $\mathcal{D}(U)$ has the structure of a spherical polytope of dimension $\leq d - 1$.

If $\mathcal{D}(U)$ and $\mathcal{D}(U')$ are normally equivalent, then it is in generally not true that $\text{conv}(U)$ and $\text{conv}(U')$ are normally equivalent or even combinatorially equivalent. This stems from the fact that two normally equivalent polytopes $\hat{P}, \hat{P}'$ inscribed to the unit sphere can have quite different visibility complexes with respect to a fixed point $\xi \in S^{d-1}$. In Theorem 2.11 we determine equivalence relation on normally equivalent polytopes inscribed to the unit sphere with fixed visibility complex. The equivalence classes are nonconvex in general and can be disconnected. We give a simple necessary condition when a cell in this subdivision is convex (Corollary 2.12).

1.3. Incribed virtual polytopes. A second goal of this paper is to introduce and study inscribed virtual polytopes. A fan $\mathcal{N}$ is polytopal if there is a polytope $P$ with $\mathcal{N}(P) = \mathcal{N}$ and we define $\mathcal{T}_+(\mathcal{N}) := \mathcal{T}_+(P)$, the open polyhedral cone of polytopes with normal fan $\mathcal{N}$ modulo translations. If $P, Q, R \in \mathcal{T}_+(\mathcal{N})$ satisfy $P = Q + R$, then $R$ is called the Minkowski difference of $P$ and $Q$ and is denoted by $P - Q$. Minkowski differences exist for all pairs $P$ and $Q$ in the Grothendieck group $\mathcal{T}(\mathcal{N}) := (\mathcal{T}_+(\mathcal{N}) \times \mathcal{T}_+(\mathcal{N}))/\sim$ with $(Q + R, Q' + R) \sim (Q, Q')$ for $Q, Q', R \in \mathcal{T}_+(\mathcal{N})$ and $P - Q$ is called a virtual polytope if $P - Q \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{T}_+(\mathcal{N})$. Since $\mathcal{T}_+(\mathcal{N})$ is a convex cone, $\mathcal{T}(\mathcal{N})$ has the structure of an $\mathbb{R}$-vector space and will be called the type space of $\mathcal{N}$. We recall in Section 5.1 that $\mathcal{T}(\mathcal{N})$ can be defined in terms of piecewise-linear functions supported on $\mathcal{N}$ and is thus also defined for non-polytopal fans. Virtual polytopes are related to non-nef divisors in toric geometry [8, Ch. 6.1] and they embody reciprocity results for translation-invariant valuations by means of McMullen’s polytope algebra [19]. For more on virtual polytopes, see [24]. Important for us is that virtual polytopes are naturally equipped with vertices and hence a notion of inscribability (Section 5.3). Figure 3 depicts some examples of inscribed virtual polytopes with respect to a fixed fan.
We study inscribed virtual polytopes in Section 5.3 and show that virtual polytopes inscribed relative to \( \mathcal{N} \) form a vector subspace \( \mathcal{I}(\mathcal{N}) \subseteq \mathcal{T}(\mathcal{N}) \). Naturally, \( \mathcal{I}_+(\mathcal{N}) + (-\mathcal{I}_+(\mathcal{N})) \subseteq \mathcal{I}(\mathcal{N}) \) with equality if \( \mathcal{N} \) is inscribable. However, there are fans that only possess virtual inscribed polytopes. For example, we show that \( \mathcal{I}(\mathcal{N}) \) is always 1-dimensional for every 2-dimensional fan with an odd number of rays (Proposition 3.1).

### 1.4. Routed billiard trajectories and reflection groupoids.

The inscribed virtual polygons in Figure 3 are reminiscent of closed, piecewise-linear trajectories of particles inside the unit disc that bounce off the boundary in random directions. We make this analogy precise in Section 6, where we introduce routed particle trajectories. We model the state-space of a particle as a graph \( G = (V, E) \) together with a map \( \alpha : E \rightarrow \mathbb{P}^{d-1} \), which encodes the admissible directions. The pair \((G, \alpha)\) is called a routing scheme. A trajectory is then a map \( T : V \rightarrow S^{d-1} \) that records positions of a trajectory routed by \((G, \alpha)\). We show that the space of trajectories is isomorphic to a spherical subspace (Theorem 6.1) and we show that for routing schemes \((G, \alpha)\) derived from a fan, routed trajectories correspond to inscribed virtual polytopes (Theorem 6.3).

From the perspective of state spaces, routing schemes give rise to groupoids, whose morphisms are generated by reflections and are therefore called reflection groupoids. We discuss reflection groupoids in Section 6.2 and, in particular, study their associated endomorphism groups. These groups can be thought of as discrete holonomy groups and generalize Joswig’s groups of projectivities [15].

**Remark 1.6.** All results regarding inscribed cones and inscribed spaces remain valid if we replace the unit sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : \langle x, x \rangle = 1 \} \) with a general, non-degenerate quadric \( Q = \{ x : \langle Ax, x \rangle = 1 \} \). See also [10] for work related to 3-polytopes inscribed in a general quadric.

**Remark 1.7.** A polyhedron \( Q \subseteq \mathbb{R}^d \) is inscribed if all vertices lie on a sphere \( S \) and all unbounded edges meet \( S \) only in a vertex. All our results can be adapted to inscribed polyhedra.

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2. Local reflections and inscribable fans

A non-empty collection $\mathcal{N}$ of polyhedral cones in some $\mathbb{R}^d$ is called a fan [34, Sect. 7] if

- (F1) if $C \in \mathcal{N}$ and $F \subseteq C$ is face, then $F \in \mathcal{N}$;
- (F2) if $C, C' \in \mathcal{N}$, then $C \cap C' \in \mathcal{N}$.

The dimension of $\mathcal{N}$ is $\dim \mathcal{N} := \max\{\dim C : C \in \mathcal{N}\}$. The inclusion-maximal cones of $\mathcal{N}$ are called regions and $\mathcal{N}$ is pure if all regions have the same dimension. The support of $\mathcal{N}$ is $|\mathcal{N}| = \bigcup_{C \in \mathcal{N}} C$ and $\mathcal{N}$ is complete if $|\mathcal{N}| = \mathbb{R}^d$. A convex cone $C$ is pointed, if its lineality space $\text{lineal}(C) := \{x \in C : -x \in C\}$ contains only the origin. All cones in a fan $\mathcal{N}$ share the same lineality space $\text{lineal}(\mathcal{N})$ and we will therefore call $\mathcal{N}$ pointed if $\text{lineal}(\mathcal{N}) = \{0\}$.

Let $P \subseteq \mathbb{R}^d$ be a non-empty convex polytope. For $c \in \mathbb{R}^d$, we write

$$P^c := \{x \in P : \langle c, x \rangle \geq \langle c, y \rangle \text{ for all } y \in P\}$$

for the (non-empty) face that maximizes the linear function $x \mapsto \langle c, x \rangle$.

The normal cone of $P$ at a face $F \subseteq P$ is the polyhedral cone

$$N_F P := \{c \in \mathbb{R}^d : F \subseteq P^c\}.$$ 

It is easy to verify that $\mathcal{N}(P) = \{N_F P : F \subseteq P \text{ face}\}$ is a complete fan, called the normal fan of $P$. The normal fan is pointed precisely when $P$ is full-dimensional. We call a fan $\mathcal{N}$ polytopal if it is the normal fan of a polytope.

Let $\text{aff}(P)$ be the affine hull of $P$. Recall that two polytopes $P_0, P_1 \subseteq \mathbb{R}^d$ are normally equivalent ($P_0 \simeq P_1$) if for every $c \in \mathbb{R}^d$ the affine spaces $\text{aff}(P_0^c)$ and $\text{aff}(P_1^c)$ differ by a translation. The upcoming characterization of normally equivalent polytopes follows from the definition of normal fans; see, for example, [34, Section 7.2].

**Proposition 2.1.** Let $P_0, P_1 \subseteq \mathbb{R}^d$ polytopes. Then

$$P_0 \simeq P_1 \quad \text{if and only if} \quad \mathcal{N}(P_0) = \mathcal{N}(P_1).$$

In particular, $(1 - \mu)P_0 + \mu P_1$ is normally equivalent to $P_0$ for all $0 \leq \mu \leq 1$.

The proposition shows that $\mathcal{T}_+(\mathcal{N})$ is a convex cone that depends only on $\mathcal{N}(P)$; see Section 5.1 for details.

**2.1. Local reflections.** The following is the key observation in the proof of Theorem 1.1.

**Lemma 2.2.** Let $P \subseteq \mathbb{R}^d$ be inscribed to a sphere centered at the origin with normal fan $\mathcal{N}(P) = \mathcal{N}$. Then $P$ is completely determined by $\mathcal{N}$ and a single vertex.

**Proof.** We may assume that $P$ is full-dimensional and hence $\mathcal{N}$ is pointed. Let $v \in V(P)$ be a vertex with normal cone $N_v P$. The polyhedral cone has an irredundant representation of the form

$$N_v P = \{c \in \mathbb{R}^d : \langle \alpha_i, c \rangle \leq 0 \text{ for } i = 1, \ldots, m\}$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^d \setminus \{0\}$. If $u \in V(P)$ is a neighbor of $v$, then $u = v + \lambda_i \alpha_i$ for some $i \in [m] := \{1, \ldots, m\}$ and $\lambda_i > 0$. Since $P$ is inscribed into a sphere centered at the origin, we have

$$\|v\|^2 = \|u\|^2 = \|v + \lambda_i \alpha_i\|^2. \quad (1)$$
This is a quadratic equation in \( \lambda_i \) with a unique solution \( \lambda_i > 0 \). Thus, knowing \( v \) and \( N_v P \), we can uniquely recover the neighbors of \( v \). As the graph of \( P \) is connected, we can recover all vertices of \( P \).

Let \( \mathcal{N} \) be a pure \( d \)-dimensional fan in \( \mathbb{R}^d \). The cones of dimension \( d - 1 \) in \( \mathcal{N} \) are called walls. Every wall \( W \in \mathcal{N} \) induces a hyperplane \( \text{lin}(W) = \{ x : \langle \alpha, x \rangle = 0 \} \) and we let \( s_W : \mathbb{R}^d \to \mathbb{R}^d \)
\[
s_W(x) := x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha
\]
be the corresponding reflection. By inspecting the proof of Lemma 2.2, we make the following observation.

**Corollary 2.3.** Let \( P \subset \mathbb{R}^d \) be a polytope inscribed to a sphere centered at the origin and \( v \in V(P) \). The neighbors of \( v \) are given by \( s_W(v) \) where \( W \) ranges over the walls of \( N_v P \).

**Proof.** The unique nonzero solution of (1) is given by \( \lambda_i = -2\frac{\langle \alpha_i, v \rangle}{\langle \alpha_i, \alpha_i \rangle} \) and hence
\[
u = v - 2\frac{\langle \alpha_i, v \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = s_W(v),
\]
where \( W = \{ c \in N_v P : \langle \alpha_i, c \rangle = 0 \} \) is the wall of \( N_v P \) corresponding to \( \alpha_i \). \( \square \)

A second observation drawn from the proof of Lemma 2.2 is the following relation between vertices and their normal cones.

**Corollary 2.4.** Let \( P \subset \mathbb{R}^d \) be a polytope inscribed to a sphere centered at the origin. Then \( v \in \text{int}(N_v P) \) holds for every vertex \( v \in V(P) \).

**Proof.** Observe that the nonzero solution \( \lambda_i \) to (1) is positive if and only if \( \langle \alpha_i, v \rangle < 0 \). Thus \( v \in \{ c : \langle \alpha_i, c \rangle < 0 \} \) for \( i = 1, \ldots, m \} = \text{int}(N_v P) \). \( \square \)

**Definition 2.5** (Inscribed cone of a fan). Let \( \mathcal{N} \) be a fan in \( \mathbb{R}^d \). The **inscribed cone** of \( \mathcal{N} \) is the set of all inscribed polytopes \( P \) with \( \mathcal{N}(P) = \mathcal{N} \) modulo translations. We call \( \mathcal{N} \) **inscribable** if \( \mathcal{I}_+(\mathcal{N}) \neq \emptyset \).

Calling \( \mathcal{I}_+(\mathcal{N}) \) a **cone** is justified as \( \lambda P \in \mathcal{I}_+(\mathcal{N}) \) for all \( P \in \mathcal{I}_+(\mathcal{N}) \) and \( \lambda > 0 \). Theorem 1.1 asserts that \( \mathcal{I}_+(\mathcal{N}) \) is in fact a convex cone.

For every translation class in \( \mathcal{I}_+(\mathcal{N}) \), a canonical representative can be obtained as follows: Let \( P \subset \mathbb{R}^d \) be a polytope inscribed into a sphere \( S \). If \( P \) is full-dimensional, then \( S \) is unique. If \( P \) is of lower dimension, then \( S \cap \text{aff}(P) \) is the unique inscribing sphere relative to its affine hull. We write \( c(P) \) for the **center** of \( S \cap \text{aff}(P) \). Since \( c(P + x) = c(P) + x \) for all \( x \in \mathbb{R}^d \), we can always assume that \( c(P) = 0 \) and we write \( \overline{P} := P - c(P) \).

The space \( \mathcal{I}_+(\mathcal{N}) \) is endowed with the Hausdorff metric
\[
d_H(P, Q) := \min\{ \mu \geq 0 : \overline{P} \subseteq Q + \mu B^d, \overline{Q} \subseteq \overline{P} + \mu B^d \},
\]
where \( B^d \) is the unit ball and \( P, Q \in \mathcal{I}_+(\mathcal{N}) \).

Let \( P \) be a polytope with normal fan \( \mathcal{N} \). For a fixed region \( R_0 \in \mathcal{N} \), we write \( v_{R_0}(P) \) for the unique vertex \( v \) of \( P \) with \( N_v P = R_0 \). Let us denote the set of possible \( v_{R_0}(P) \) for inscribed \( P \) by
\[
\mathcal{I}_+(\mathcal{N}, R_0) := \{ v_{R_0}(P) : \mathcal{N}(P) = \mathcal{N}, P \text{ inscribed}, c(P) = 0 \}.
\]
We call $\mathcal{I}_+ (\mathcal{N}, R_0)$ the **inscribed cone based at** $R_0$.

It follows from Lemma 2.2 and Corollaries 2.3 and 2.4 that the map $v_{R_0} : \mathcal{I}_+ (\mathcal{N}) \to \mathcal{I}_+ (\mathcal{N}, R_0)$ is a homeomorphism.

### 2.2. Virtually inscribable fans and the reflection game.

In order to find necessary conditions for a fan to be inscribed, we use the following **reflection game** for fans: Let $\mathcal{N}$ be a pure and full-dimensional fan. The **dual graph** of $\mathcal{N}$ is the simple undirected graph $G(\mathcal{N})$ with nodes given by the regions of $\mathcal{N}$. Two regions $R, R'$ are adjacent in $G(\mathcal{N})$ if $R \cap R'$ is a wall. We call $\mathcal{N}$ **strongly connected** if $G(\mathcal{N})$ is connected. For example, every complete fan is strongly connected. If $R, R'$ are two adjacent regions, then let $s_{RR'}$ be the reflection in the hyperplane $\text{lin}(R \cap R')$. Every walk $\mathcal{W} = R_0 R_1 \ldots R_k$ in $G(\mathcal{N})$ yields an orthogonal transformation

$$t_{\mathcal{W}} := s_{R_k R_{k-1}} \cdots s_{R_2 R_1} s_{R_1 R_0}.$$  

**Definition 2.6** (Virtually inscribable). Let $\mathcal{N}$ be a full-dimensional and strongly connected fan in $\mathbb{R}^d$ and let $R_0 \in \mathcal{N}$ be a region. The fan $\mathcal{N}$ is **virtually inscribable** if there is a point $v \in \mathbb{R}^d \setminus \text{lineal}(\mathcal{N})$ such that

$$t_{\mathcal{W}}(v) = v \quad (2)$$

for all closed walks $\mathcal{W}$ starting in $R_0$.

Note that we do not require that $\mathcal{N}$ is polytopal. Also note that every $t_{\mathcal{W}}$ fixes $\text{lineal}(\mathcal{N})$ pointwise. The linear subspace $\mathcal{I}(\mathcal{N}, R_0) \subset \mathbb{R}^d$ of all $v \in (\text{lineal}(\mathcal{N}))^\perp$ satisfying (2) for all closed walks $\mathcal{W}$ will be called the **based inscribed space** of $\mathcal{N}$. Thus, $\mathcal{N}$ is virtually inscribable if and only if $\mathcal{I}(\mathcal{N}) \neq \{0\}$. The actual choice of $R_0$ is immaterial: for a different base region $R_0'$, we have

$$\mathcal{I}(\mathcal{N}, R_0') = t_{\mathcal{W}} \mathcal{I}(\mathcal{N}, R_0)$$

for any walk $\mathcal{W}$ from $R_0$ to $R_0'$. Hence $\mathcal{I}(\mathcal{N}, R_0)$ is based at $R_0$. In Section 5.4, we will discuss inscribed spaces that do not require the choice of a base region. Clearly,

$$\mathcal{I}_+(\mathcal{N}, R_0) \subseteq \mathcal{I}(\mathcal{N}, R_0).$$

**Proposition 2.7.** Let $\mathcal{N}$ be a complete fan. Then $\mathcal{N}$ is inscribed if and only if $\mathcal{N}$ is virtually inscribed and there is $v_0 \in \mathcal{I}(\mathcal{N}, R_0)$ such that for every region $R$

$$t_{\mathcal{W}}(v_0) \in \text{int}(R)$$

for some path $\mathcal{W}$ from $R_0$ to $R$.

**Proof.** If $P \subset \mathbb{R}^d$ is a polytope inscribed into a sphere centered at the origin and $\mathcal{N}(P) = P$, then Lemma 2.2 and Corollary 2.3 show that $\mathcal{N}$ is virtually inscribable. The graph of $P$ is exactly $G(\mathcal{N})$ and $t_{\mathcal{W}}(v_0) = u$ is the vertex with region $R$. Corollary 2.4 now shows that $u \in \text{int}(R)$.

For the converse, let $\mathcal{N}$ be a virtually inscribed fan satisfying the given conditions. For every region $R$, let $v_R := t_{\mathcal{W}}(v_0)$, where $\mathcal{W}$ is a path connecting $R_0$ to $R$. Note that this implies $v_R \neq v_S$ for $R \neq S$. Since $\mathcal{N}$ is virtually inscribed, $v_R$ is independent of the chosen path. Define $P = \text{conv}(v_R : R \text{ region})$. Since $t_{\mathcal{W}}$ is an orthogonal transformation, all vertices lie on a sphere centered at the origin with radius $\|v_0\|$. In particular $P$ is an inscribed polytope with vertices $v_R$ for $R \in \mathcal{N}$ region.
We are left to show that $\mathcal{N}(P) = \mathcal{N}$. By construction $N_{v_R} \subseteq R$ for every region $R$. Indeed, for every region $S$ adjacent to $R_0$, $v_S - v_0$ is an outer normal for $R_0$ and

$$N_{v_{R_0}} = \{ c \in \mathbb{R}^d : \langle c, v_{R_0} \rangle \geq \langle c, v_R \rangle \text{ for } R \in \mathcal{N} \text{ region} \}.$$ 

As $P$ is independent of the choice of $R_0$, $N_{v_R} \subseteq R$ holds for all regions. Since $\mathcal{N}$ and $\mathcal{N}(P)$ are complete fans, this implies $N_{v_R} = R$ for all regions $R$ and completes the proof. \qed

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $Q, Q'$ be two polytopes inscribed into a sphere centered at the origin with $\mathcal{N}(Q) = \mathcal{N}(Q') = \mathcal{N}$. For every region $R \in \mathcal{N}$, let $q_R, q'_R \in R$ be the respective vertices, whose existence is vouched for by Corollary 2.4. Since $\mathcal{N}(Q + Q') = \mathcal{N}$, we have

$$Q + Q' = \text{conv}\{q_R + q'_R : R \in \mathcal{N} \text{ region} \}.$$ 

To show that $Q + Q'$ is inscribed, let $R_0 \in \mathcal{N}$ be a fixed region. By convexity, $q_{R_0} + q'_{R_0} \in \text{int}(R_0)$ and $q_R + q'_R = t_W(q_{R_0} + q'_{R_0}) \in \text{int}(R_0)$ for all walks $W$ from $R_0$ to any region $R \in \mathcal{N}$. Proposition 2.7 now shows that $Q + Q' \in \mathcal{I}_+(P)$.

It remains to see that $\mathcal{I}_+(\mathcal{N}) \cong \mathcal{I}_+(\mathcal{N}, R_0)$ is a relatively open polyhedral cone of dimension at most $d$. The collection of points $v_0 \in \mathbb{R}^d$ with $t_W(v_0) \in \text{int}(R)$ for all $W$ from $R_0$ to $R$ is an open polyhedral cone of dimension $d$. If $v_0 \in \mathcal{I}(\mathcal{N}, R_0)$, then $t_W(v_0)$ is independent of the choice of $W$. Hence, if we choose a path $W_R$ from $R_0$ to $R$ for every region $R$, then

$$\mathcal{I}_+(\mathcal{N}, R_0) = \mathcal{I}(\mathcal{N}, R_0) \cap \bigcap_R t^{-1}_{W_R}(\text{int}(R)).$$

(3)

The latter is the restriction of a linear subspace to the intersection of finitely many open polyhedral cones. \qed

Corollary 2.8. The map $v_{R_0} : \mathcal{I}_+(\mathcal{N}) \to \mathcal{I}_+(\mathcal{N}, R_0)$ is a linear homeomorphism:

$$\mathcal{I}_+(\mathcal{N}) \cong \mathcal{I}_+(\mathcal{N}, R_0).$$

(4)

We are now in a position to proof Corollary 1.2 and Corollary 1.4.

Proof of Corollary 1.2. Let $P$ be an normally inscribable polytope and let $G$ be a group of orthogonal transformations such that $gP = P$ for all $g \in G$. For $P_0 \in \mathcal{I}_+(P)$ consider

$$P' := \frac{1}{|G|} \sum_{g \in G} gP_0.$$ 

$P'$ is clearly invariant under $G$ and since $\mathcal{I}_+(\mathcal{N})$ is convex by Theorem 1.1, it follows that $P' \in \mathcal{I}_+(\mathcal{N})$. \qed

Proof of Corollary 1.4. Let $Q, Q' \subseteq \mathbb{R}^d$ be polytopes inscribed into the unit sphere with $\mathcal{N}(Q) = \mathcal{N}(Q') = \mathcal{N}$. Fix a region $R_0$ and let $q = v_{R_0}(Q)$ and $q' = v_{R_0}(Q')$. For any region $R \in \mathcal{N}$, we have that $v_R(Q) = t_W(q)$ and $v_R(Q') = t_W(q')$ for any path $W$ from $R_0$ to $R$. Since $t_W$ is a product of reflection, it follows that $\langle t_W(q), t_W(q') \rangle = \langle q, q' \rangle$ and hence $\theta(Q, Q')$ is independent of the choice of $R_0$ or, equivalently, the choice of a generic linear function $l(x)$.

For the second statement, we simply note that $\|q + q'\|^2 = 2 + 2\langle q, q' \rangle = 2 + 2\cos\theta(Q, Q')$. \qed
2.3. Inscribed Permutahedra. The braid arrangement $A_{d-1}$ is the arrangement of linear hyperplanes
\[ H_{ij} = \{ x \in \mathbb{R}^d : x_i = x_j \} \quad \text{for } 1 \leq i < j \leq d. \]
Every connected component of $\mathbb{R}^d \setminus \bigcup_{H \in A_{d-1}} H$ is an open cone, and the closures of these cones form a fan which we will also denote by $A_{d-1}$. Note that $A_{d-1}$ is not pointed. Its lineality space is the line spanned by $(1, 1, \ldots, 1)$. It is straightforward to verify that the $d!$ regions of $A_{d-1}$ are given by
\[ R_{\sigma} := \{ z \in \mathbb{R}^d : z_{\sigma(1)} \leq z_{\sigma(2)} \leq \cdots \leq z_{\sigma(d)} \}, \]
where $\sigma \in \mathfrak{S}_d$ is a permutation. Two regions $R_{\sigma}$ and $R_{\tau}$ are adjacent if they differ by an adjacent transposition, that is, $\sigma^{-1} = (i, i+1)$ for some $1 \leq i < d$.

Let $R_0 = \{ x_1 \leq x_2 \leq \cdots \leq x_d \}$ be the region for the identity permutation. If $\mathcal{W}$ is a path from $R_0$ to $R_{\sigma}$, then $t_{\mathcal{W}}(x_1, \ldots, x_d) = (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ is the permutation of coordinates by $\sigma$. Thus $t_{\mathcal{W}}^{-1}(R_{\sigma}) = R_0$. This shows that $\mathcal{I}(A_{d-1}, R_0) = \mathbb{R}^d$ and we can conclude
\[ \mathcal{I}_+(A_{d-1}, R_0) = R_0 = \{ z : z_1 \leq z_2 \leq \cdots \leq z_d \} \]
and for $z \in R_0$, the corresponding polytope is
\[ P(z) = \text{conv}\{(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) : \sigma \in \mathfrak{S}_d\} \]
a permutahedron or weight polytope of type $A_{d-1}$; cf. [3].

The rays of the closure $\mathcal{I}_+(A_{d-1}) \cong R_0$ are of the form $(0, \ldots, 0, 1, \ldots, 1)$ and the associated polytopes are precisely the hypersimplices $\Delta(d, k)$ for $0 < k < d$; see next section.

Corollary 2.9. Let $A_{d-1}$ be the fan of the braid arrangement. Then every $P \in \mathcal{I}_+(A_{d-1})$ is of the form
\[ P = P(z) = z_1 \Delta(d, 1) + (z_2 - z_1) \Delta(d, 2) + \cdots + (z_d - z_{d-1}) \Delta(d, d-1) \]
for $z = (z_1 \leq z_2 \leq \cdots \leq z_d)$. In particular, every $P \in \mathcal{I}_+(A_{d-1})$ is symmetric with respect to $\mathfrak{S}_d$.

2.4. Delaunay subdivisions and visibility complexes. Let $[n] = \{1, \ldots, n\}$. A labelled point configuration is an injective map $U : [n] \rightarrow \mathbb{R}^{d-1}$. We will mostly identify $U$ with its underlying set $U([n])$. Let $P = \text{conv}(U)$ be the corresponding convex hull and let $\hat{P} = \text{conv}(\eta^{-1}(V))$, where $\eta : S^{d-1} \rightarrow \mathbb{R}^{d-1} \times \{0\}$ is the stereographic projection from $e_d$. The projection of the faces of $\hat{P}$ not visible from $e_d$ is the Delaunay subdivision $D(U)$ of $P$. The distinguishing feature of the Delaunay subdivision is that it is the coarsest subdivision such that every face $F$ of some cell $P_i \in D(U)$ is inscribed to some $(d-1)$-dimensional ball $B$ such that $U \cap B = V(F)$.

A pair of distinct points $u_1, u_2 \in U$ is a hidden edge if the segment $[u_1, u_2]$ is inscribed to some ball $B$ containing all points $U \setminus \{u_1, u_2\}$ in its interior. It follows that if $u_1 u_2$ is a hidden edge, then $u_1, u_2$ are contained in the boundary of $P$. We write $G(U)$ for the graph with nodes $[n]$ and $i, j \in [n]$ form an edge if $U(i)U(j)$ is an edge or a hidden edge of $D(U)$. It is not hard to see that $G(U)$ is exactly the edge graph of $\hat{P}$.

Let $I \subset \mathbb{R}^{d-1}$ be a segment. There is a unique sphere $S = S(I)$ such that $I$ is invariant under inversion in $S$ and $S$ meets $S^{d-2}$ in a great-sphere. We call two segments $I, I'$ co-circular if $S(I) = S(I')$. If $I$ and $I'$ are oriented, then they are positively co-circular if
the positive endpoints of $I$ and $I'$ are not separated by $S(I)$. Figure 4 shows three positively co-circular segments. Let $U$ and $U'$ be labelled point configurations. We call $D(U)$ and $D(U')$ normally equivalent if $G(U) = G(U')$ as labelled graphs and for every edge $e$ of $G(U)$, the corresponding segments of $U$ and $U'$ are positively co-circular.

With respect to stereographic projection, we get the following.

**Proposition 2.10.** Let $U, U'$ be labelled configurations. Then $D(U)$ and $D(U')$ are normally equivalent if and only if $\hat{P}(U) \cong \hat{P}(U')$.

**Proof of Corollary 1.5.** Normally equivalent Delaunay subdivisions are represented by a polyhedral fan $N$ in $\mathbb{R}^d$. Recalling that for a fixed region $R_0$, the cone $I_+(N, R_0)$ represents all polytopes $P \subset \mathbb{R}^d$ with $N(P) = N$ and inscribing sphere centered at the origin, we obtain that the spherical polytope $S(N, R_0) := I_+(N, R_0) \cap S^{d-1}$ parametrizes all normally equivalent Delaunay subdivisions in $\mathbb{R}^{d-1}$ represented by $N$. □

Choose $\xi \in S^{d-1}$ and let $P \subset \mathbb{R}^d$ be a full-dimensional polytope inscribed to the unit sphere. Recall from the introduction that the visibility complex $\text{Vis}_\xi(P)$ is the collection of faces $F \subseteq P$ such that $\text{conv}(\xi \cup F)$ does not meet the interior of $P$. We call such $F$ visible from $\xi$. The visibility complex is a full-dimensional pure polyhedral complex and hence is determined by the collection of facets of $P$ visible from $\xi$. For a ray $r \in \mathcal{N}$, let us denote by $\tau := r \cap S^{d-1}$ the unit vector in $r$. Every ray $r$ determines the oriented hyperplane $H_r := \{x \in \mathbb{R}^d : \langle \tau, x \rangle = \langle \tau, \xi \rangle\}$

It follows that the facet $P^\tau$ is visible from $\xi$ if and only if $P^\tau$ is contained in the open halfspace $H_r^- = \{x \in \mathbb{R}^d : \langle \tau, x \rangle < \langle \tau, \xi \rangle\}$. In fact, if $v$ is a vertex of $P$ contained in $P^\tau$, then this is equivalent to $v \in H_r^-$. We denote the opposite closed halfspace by $H_r^+$.

Let $\mathcal{N}$ be a polyhedral fan with base region $R_0$. For every ray $r \in \mathcal{N}$, choose a path $\mathcal{W}_r$ to a region $R$ with $r \in R$. We define the arrangement of hyperplanes $\mathcal{H}(\mathcal{N}, R_0, \xi) := \{t_{\mathcal{W}_r}(H_r) : r \in \mathcal{N} \text{ ray}\}$.

The arrangement determines an equivalence relation on $\mathbb{R}^d$ by setting $x \sim y$ if and only if $|\{x, y\} \cap H^-| \neq 1$ for all $H \in \mathcal{H}(\mathcal{N}, R_0, \xi)$. If $P$ is a polytope with normal fan $\mathcal{N}$ inscribed to the unit sphere, then $t_{\mathcal{W}_r}(v_{R_0}(P)) \in P^\tau$ for all rays $r$. This proves the following.

![Figure 4. Three positively co-circular segments in the plane.](image-url)
Theorem 2.11. Let $\mathcal{N}$ be an inscribable fan with base region $R_0$ and $\xi \in S^{d-1}$. Then $P, Q \in \mathcal{I}^+(\mathcal{N}, R_0)$ have the same visibility complex if and only if $v_{R_0}(P)$ and $v_{R_0}(Q)$ are equivalent with respect to $H(\mathcal{N}, R_0, \xi)$.

Note that the equivalence classes on $S^{d-1}$ with respect to $\sim$ are not necessarily convex or even connected. This is due to the fact that the hyperplanes $H_r$ do not pass through the origin and therefore do not induce great-spheres. We close this section with a simple criterion when the set of points in $S(\mathcal{N}, R_0)$ with a fixed visibility complex is convex. If $H = \{x : \langle c, x \rangle = \delta\}$ is an affine hyperplane with normal vector $c$, then $H^{-} \cap S^{d-1}$ is a convex subset if and only if $\delta < 0$.

Corollary 2.12. Let $M$ be a collection of rays of $\mathcal{N}$. Then the polytopes $P \in S(\mathcal{N}, R_0)$ with visibility complex induced by $P_r$ for $r \in M$ form a (spherically) convex subset if $\langle r, \xi \rangle < 0$ for $r \in M$ and $\langle r, \xi \rangle \geq 0$ for all rays $r \notin M$.

3. Inscribable fans in dimension 2

In this section we study inscribable fans in the plane. The situation in the plane is simple enough to give a complete classification of inscribable fans. However, since faces of inscribable polytopes are inscribable, the results obtained in this section give simple necessary conditions for inscribable fans in higher dimensions.

Throughout this section, let $\mathcal{N}$ be a complete and pointed fan in $\mathbb{R}^2$. We order the $n \geq 3$ regions of $\mathcal{N}$ counterclockwise and denote them by $R_0, \ldots, R_{n-1}$. For $i = 0, \ldots, n-1$, let $\beta_i$ the angle of $R_i$ (cf. Figure 5). We call $\beta(\mathcal{N}) = (\beta_0, \ldots, \beta_{n-1})$ the profile of $\mathcal{N}$. The profile $\beta(\mathcal{N})$ determines $\mathcal{N}$ up to rotation.

It is clear that the set of all profiles of complete fans with $n$ regions is

$$B_n := \left\{ \beta \in \mathbb{R}^n : 0 < \beta_i < \pi \text{ for } i = 0, \ldots, n-1, \beta_0 + \cdots + \beta_{n-1} = 2\pi \right\}.$$
Recall that the \((n,k)\)-hypersimplex [34, Ex. 0.11] is the polytope given by the convex hull of points \(v \in \{0,1\}^n\) with exactly \(k\) entries equal to 1. The \((n,1)\)-hypersimplex is the standard simplex \(\Delta_{n-1} = \{x \in \mathbb{R}^n : x \geq 0, x_1 + \cdots + x_n = 1\} = \text{conv}(e_1, \ldots, e_n)\), where \(e_i\) are the standard basis vectors.

It follows that \(\mathcal{B}_n\) is the relative interior of \(\pi \cdot \Delta(n,2)\). This description also highlights the fact that cyclic shifts of \(\beta\) correspond to cyclic relabellings of \(\mathcal{N}\). To ease notation, we decree

\[
\beta_{n+i} := \beta_i \quad \text{for} \quad 0 \leq i < n.
\]

Our first result determines the locus of virtually inscribable fans.

**Proposition 3.1.** Let \(\mathcal{N}\) be a 2-dimensional fan with \(n\) regions and \(\beta(\mathcal{N}) = (\beta_0, \ldots, \beta_{n-1})\). If \(n\) is odd, then \(\mathcal{N}\) is virtually inscribable and \(\dim \mathcal{I}(\mathcal{N}) = 1\). If \(n\) is even, then \(\mathcal{N}\) is virtually inscribable if and only if

\[
\beta_2 + \beta_4 + \cdots + \beta_{n-2} = \pi.
\]

In this case \(\dim \mathcal{I}(\mathcal{N}) = 2\).

**Proof.** The only relevant closed walk is \(\mathcal{W} = R_0 R_1 \ldots R_{n-1} R_0\) and the corresponding transformation

\[
t_{\mathcal{W}} := s_{R_0 R_{n-1}} \cdots s_{R_2 R_1} s_{R_1 R_0}
\]

is a product of \(n\) reflections in \(\mathbb{R}^2\). If \(n\) is odd, then \(t_{\mathcal{W}}\) is a reflection and hence there is a unique \(v \in \mathbb{R}^2\) up to scaling such that \(t_{\mathcal{W}}(v) = v\). If \(n\) is even, then \(t_{\mathcal{W}}\) is a rotation by \(2(\beta_2 + \beta_4 + \cdots + \beta_{n})\) and \(t_{\mathcal{W}}\) has a fixpoint if and only if \(\beta_2 + \beta_4 + \cdots + \beta_{n} = \pi\). In this case \(t_{\mathcal{W}} = \text{id}\) and \(\mathcal{I}(\mathcal{N}) = \mathbb{R}^2\).

**Corollary 3.2.** Let \(\mathcal{B}_n^{\text{vin}} \subseteq \mathcal{B}_n\) be the profiles of virtually inscribable fans. If \(n\) is odd, then \(\mathcal{B}_n^{\text{vin}} = \mathcal{B}_n\). If \(n = 2k\) is even, then \(\frac{1}{2^k} \mathcal{B}_n^{\text{vin}}\) is the relative interior of \(\Delta_{k-1} \times \Delta_{k-1}\), the product of two standard simplices.

**Example 3.3.** Let \(\mathcal{N}\) be a two-dimensional fan with 4 regions. Then Proposition 3.1 shows that \(\mathcal{N}\) is virtually inscribed if and only if its profile \(\beta(\mathcal{N}) = (\beta_0, \beta_1, \beta_2, \beta_3)\) satisfies:

\[
\beta_0 + \beta_2 = \beta_1 + \beta_3 = \pi.
\]

This is precisely the condition that \(\mathcal{N}\) is the fan of a cyclic quadrangle of Euclidean geometry and therefore any realization of \(\mathcal{N}\) will be inscribed, if \(\mathcal{N}\) satisfies this condition. Thus, any rhombus \((\beta_0 = \beta_2, \beta_1 = \beta_3)\) which is not a rectangle can not be (virtually) inscribed, while any isosceles trapezoid \((\beta_0 = \beta_1, \beta_2 = \beta_3)\) is virtually inscribed (see Figure 6).

**Example 3.4.** Figure 7 shows a two-dimensional fan with profile \(\beta = (2\pi/3, \pi/3, \pi/3, \pi/3)\). It is virtually inscribable, but there does not exist a inscribed polygon with this normal fan, since all trajectories starting from the based inscribed space degenerate to triangles.

We next determine the subset \(\mathcal{B}_n^{\text{vin}} \subseteq \mathcal{B}_n^{\text{vin}}\) of profiles of inscribable fans.

**Theorem 3.5.** Let \(\beta \in \mathcal{B}_n^{\text{vin}}\) be a virtually inscribable profile. If \(n\) is odd, then \(\beta \in \mathcal{B}_n^{\text{vin}}\) if and only if

\[
\beta_{j+0} - \beta_{j+1} + \cdots - \beta_{j+n-2} + \beta_{j+n-1} > 0
\]

for all \(0 \leq j < n\). If \(n = 2m\) is even, then \(\beta \in \mathcal{B}_n^{\text{vin}}\) if and only if

\[
\sum_{i=1}^{h} \beta_{2i+j} + \sum_{i=h+1}^{m-1} \beta_{2i+1+j} < \pi
\]
**Figure 6.** (A): Normal fan of rhombus which can not be virtually inscribed. (B) - (D): Multiple inscribed realizations of the normal fan of an isosceles trapezoid. (E) - (F): Virtually inscribed realizations. While the trajectory of (E) closes to a polygon, its normal fan differs. The trajectory of (F) overlaps with itself.

**Figure 7.** (A) Virtually inscribable fan $\mathcal{N}$, which is not inscribable. Base region $R_0$ in highlighted. (B) and (C) two virtually inscribed realizations, where multiple vertices coincide. The 1-dimensional inscribed space in red.
for all $0 \leq h < m$ and $0 \leq j < n$.

**Proof.** Consider $n$ distinct points on the unit circle which are labelled counterclockwise by $v_0, v_1, \ldots, v_{n-1}$. Then $P = \text{conv}(v_0, \ldots, v_{n-1})$ is inscribed and clearly every inscribable fan is obtained this way. In particular if we label the regions of $\mathcal{N}$ by $R_i := N_i P$, then every such choice of $n$ points yields a unique inscribable profile $\beta \in \mathcal{B}_n$ up to rotation.

For $n$ ordered points $v_0, \ldots, v_{n-1}$, define $\alpha_i$ to be the angle between $\overline{0v_i}$ and $\overline{0v_{i+1}}$, where we set $v_n := v_0$; see Figure 5. Then $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ determines $v_0, \ldots, v_{n-1}$ up to rotation. In particular $\alpha$ arises from a point configuration if and only if

$$\alpha_0 + \cdots + \alpha_{n-1} = 2\pi \quad \text{and} \quad \alpha_i > 0 \quad \text{for} \quad 0 \leq i < n.$$

Thus, the set of admissible $\alpha$ is $\text{int}(2\pi \cdot \Delta_{n-1})$. By Corollary 2.3, each angle $\alpha_i$ is bisected by the ray of $\mathcal{N}(P)$ corresponding to $[v_i, v_{i+1}]$ and therefore $2\beta_i = \alpha_{i-1} + \alpha_i$, which gives a linear map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Psi(\text{int}(2\pi \cdot \Delta_{n-1})) = \mathcal{B}_n$.

If $n$ is odd, then $\Psi$ is bijective and $\alpha = \Psi^{-1}(\beta)$ is given by

$$\alpha_i = \frac{1}{2} \sum_{j=0}^{n-1} (-1)^j \beta_{i+j}$$

for $0 \leq i < n$.

For the case $n = 2m$, we consider the closure $Q = \frac{1}{\pi} \text{cl}(\mathcal{B}_n) = 2 \cdot \Psi(\Delta_{n-1})$. The polytope $Q$ is the convex hull of the $n$ cyclic shifts of $(1, 1, 0, \ldots, 0)$. This polytope is known as the *edge polytope* of the $n$-cycle; cf. Ohsugi–Hibi [22] and Villarreal [33]. Consider the undirected cycle $C_n$ with nodes $0, \ldots, n - 1$ and edges $(i - 1, i)$ for $1 \leq i < n$ and $(0, n - 1)$. The vertices of $Q$ are naturally in bijection to the edges of $C_n$. Since $n$ is even, we may color the edges alternatingly with two colors. The facets, and hence the defining linear inequalities, are given by omitting an edge of each color. Deleting the two edges from $C_n$ leaves a disjoint union of two paths of length $2r$ and $2s$, respectively. Summing the coordinates of the independent sets in each of the two paths of size $r$ and $s$ yields the given inequalities. \qed

**Corollary 3.6.** If $n$ is odd, then the closure of $\mathcal{B}_n$ is an $(n - 1)$-dimensional simplex. If $n = 2k$ is even, then the closure of $\mathcal{B}_n$ is a free sum of two $(k - 1)$-simplices. In both cases, the vertices are given by the cyclic shifts of $(\pi, \pi, 0, \ldots, 0)$.

**Example 3.7.** We can reprove Corollary 2.9 for the braid arrangement $\mathcal{A}_2$ using Theorem 3.5. $\mathcal{A}_2$ becomes pointed when dividing by its lineality space. The resulting fan $\mathcal{N}$ is the normal fan of a regular hexagon and has the profile

$$\beta := \beta(\mathcal{N}) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$$

and a two-dimensional inscribed cone. With $\Psi : \mathbb{R}^6 \to \mathbb{R}^6$ as in the proof of Theorem 3.5, we see that:

$$\Psi^{-1}(\beta) = \{\alpha \in \mathbb{R}^6 : \alpha_0 = \alpha_2 = \alpha_4, \alpha_1 = \alpha_3 = \alpha_5, \alpha_0 + \alpha_1 = \frac{2\pi}{3}\}.$$

The extreme rays of $\overline{\mathcal{I}}_+(\mathcal{N})$ are given by extreme rays of $\Psi^{-1}(\beta) \cap \mathbb{R}_{\geq 0}^6$, which correspond to two triangles $\Delta = \Delta(3, 1), \nabla = \Delta(3, 2)$, see Figure 8. Therefore:

$$\overline{\mathcal{I}}_+(\mathcal{N}) = \{\mu_1 \Delta + \mu_2 \nabla : \mu_1, \mu_2 \in \mathbb{R}_{\geq 0}\}.$$
Let us close with a different parametrization of $B_{\infty}^n$. Again, let $v_0, \ldots, v_{n-1}, v_n = v_0$ be the cyclically labelled vertices of a polygon $P$ inscribed to a circle centered at the origin. Then $P$ is determined up to rotation by the edge length $\ell_i = \|v_{i-1} - v_i\|$ for $0 \leq i < n$. Set $\ell(P) = (\ell_0, \ldots, \ell_{n-1})$. Up to positive scaling, $\ell(P)$ uniquely determines $N(P)$.

**Proposition 3.8.** For $n \geq 3$, we have

$$B_{\infty}^n \cong \mathrm{int}(\Delta(n, 2)).$$

**Proof.** Let $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$ for all $i$. It should be clear that $\ell$ can be realized as the length vector of a polygon if and only if $\ell$ can be realized as the length vector of an inscribed polygon. Hence, we only need to determine admissible $\ell$.

As scaling $\ell$ does not change the fan, we can normalize $\ell_1 + \cdots + \ell_n = 2$. It follows from the triangle inequality that $\ell$ is admissible if and only if $\ell_j < \sum_{i \neq j} \ell_i$, which is equivalent to $\ell_j < 1$. Thus

$$B_{\infty}^n \cong \left\{ \ell \in \mathbb{R}^n : 0 < \ell_i < 1 \text{ for } i = 1, \ldots, n, \ell_1 + \cdots + \ell_n = 2 \right\} = \mathrm{int}(\Delta(n, 2)). \quad \Box$$

Note that the isomorphism in the proposition maps $\ell(P)$ to $\beta(N)$, which is a highly nonlinear map.

4. Inscribable fans in general dimensions

Using the results of the previous section, we can give some conditions for inscribability of fans in higher dimensions as well as for particular classes of fans and polytopes.

4.1. Restrictions from faces. We spell out the trivial observation that makes the connection to the previous section transparent.

**Proposition 4.1.** If $P \subset \mathbb{R}^d$ is an inscribed polytope and $F \subseteq P$ is a face, then $F$ is inscribed.

Let $N$ be a fan in $\mathbb{R}^d$ and a $C \in N$ a cone. For two cones $C, D \subset \mathbb{R}^d$ we write $D - C$ for the convex cone $\{d - c : d \in D, c \in C\}$. The localization of $N$ at $C$, is the fan

$$N_C := \{D - C : D \in N, C \subseteq D\}.$$

This is a fan with lineality space $\text{lineal}(N) = C - C$.

**Proposition 4.2.** Let $N$ be a (virtually) inscribable fan and $C \in N$ a cone. Then the localization $N_C$ is (virtually) inscribable.
Proof. If $\mathcal{N}$ is inscribable and hence polytopal, then the proposition easily follows from Proposition 4.1: Let $P \in \mathcal{I}_+ (\mathcal{N})$. There is a face $F \subseteq P$ such that $C = N_F P$ and it is easy to see that $\mathcal{N}_C$ is the normal fan of $F$. By Proposition 4.1, it follows that $\mathcal{N}_C$ is inscribable.

For the general case, pick a region $R \in \mathcal{N}$ with $C \subseteq R$. If $W \in \mathcal{N}$ is a wall with $C \subseteq W$, then $(W - C) - (W - C) = W - W$ and hence $s_W = s_{W-C}$. Moreover, $G(\mathcal{N}_C)$ is a vertex-induced subgraph of $G(\mathcal{N})$. This shows that $\mathcal{I}(\mathcal{N}, R) \subseteq \mathcal{I}(\mathcal{N}_C, R - C)$. □

Note that the lineality space of $\mathcal{N}_C$ contains $C - C$. Hence, we may consider the intersections $\{ (D - C) \cap C^\perp : D - C \in \mathcal{N}_C \}$. This corresponds to the projection of $\mathcal{N}_C$ to $C^\perp$, which is again a (virtually) inscribable fan. If $C \in \mathcal{N}$ is a cone of codimension 2, then we can identify $\mathcal{N}_C$ with a 2-dimensional fan.

Corollary 4.3. Let $\mathcal{N}$ be a virtually inscribable fan. Then for all codimension-2 cones $C \in \mathcal{N}$ one has $\beta(\mathcal{N}_C) \in \mathcal{B}_n^{\text{vin}}$, where $n$ is the number of regions containing $C$. If $\mathcal{N}$ is inscribed, then $\beta(\mathcal{N}_C) \in \mathcal{B}_n^{\text{ins}}$.

It is non-trivial to compute the profiles of 2-dimensional localizations of $\mathcal{N}$ and hence the previous result is of limited applicability. It would be very interesting if realizations of 3-dimensional fans with a fixed combinatorics can be parametrized by their profiles around rays. To be precise, consider a planar 3-connected graph $G$. By Steinitz result, this is graph of a 3-polytope. Consider the collection of fans $\mathcal{N}$ with $G(\mathcal{N}) = G$. Let $r_1, \ldots, r_m$ be the rays of $\mathcal{N}$ and $\mathcal{N}_i := \mathcal{N}_{r_i}$ the corresponding 2-dimensional localizations.

Question 1. What can be said about the image $\{ (\beta(\mathcal{N}_i))_{i=1}^{m} : \mathcal{N} 3$-dimensional fan with $G(\mathcal{N}) = G \}$ ?

Let $\mathcal{N}$ be a fixed pointed and polytopal fan in $\mathbb{R}^d$ and $R \in \mathcal{N}$ a region. We may use the realization of $\mathcal{I}_+ (\mathcal{N})$ as a subcone of $R$ as given in (4). It follows that

$$\mathcal{I}_+ (\mathcal{N}, R) \subseteq \mathcal{I}_+ (\mathcal{N}_C, R - C) + \text{lineal}(\mathcal{N}_C)$$

and the inclusion is typically strict, as can be seen for a pyramid over a quadrilateral; cf. Example 4.7 below.

We record the following simple observation.

Corollary 4.4. Let $\mathcal{N}$ be a full-dimensional and strongly connected fan in $\mathbb{R}^d$ and $R$ a region. For fixed $1 \leq k \leq d$

$$\mathcal{I}_+ (\mathcal{N}, R) \subseteq \bigcap_C \mathcal{I}_+ (\mathcal{N}_C, R - C) + \text{lineal}(\mathcal{N}_C)$$

where the intersection is over all $k$-cones $C \subseteq R$.

Recall that a 2-dimensional polytope is even/odd if it has an even/odd number of vertices.

Corollary 4.5. Let $P$ be a normally inscribed 3-polytope. If $P$ has an odd 2-face, then $\dim \mathcal{I}_+ (P) \leq 2$. If $P$ has two adjacent odd 2-faces, then $\dim \mathcal{I}_+ (P) = 1$, that is, up to homothety $P$ is the unique inscribed polytope with normal fan $\mathcal{N}(P)$. 

Proof. Let $F_1 \subseteq P$ be an odd 2-face and choose a vertex $v \in F_1$. We set $\mathcal{N} = N(P)$, $R = N_v P$, and $C_1 := N_{F_1} P$ the 1-dimensional normal cone of $F_1$. It follows from Proposition 3.1 that $\mathcal{I}_+(N_{C_1}, R - C_1)$ is 2-dimensional and the first claim follows from Corollary 4.4.

Now, let $F_2$ be an odd 2-face sharing an edge with $F_1$ and set $C_2 = N_{F_2} P$. We may also assume that $v \notin F_2$ and by the same argument as above $\mathcal{I}_+(N_{C_2}, R - C_2)$ is 2-dimensional. In fact $\mathcal{I}_+(N_{C_1}, R - C_1) = \mathbb{R}_{\geq 0} v + \mathbb{R} C_1$. Since $F_1 \cap F_2$ is an edge, it follows immediately that $\mathcal{I}_+(N_{C_1}, R - C_1) \cap \mathcal{I}_+(N_{C_2}, R - C_2)$ is 1-dimensional. Appealing to Corollary 4.4 again yields the claim. \hfill \Box

Example 4.6 (Dodecahedra). Let $P_{12}$ be a 3-polytope combinatorially equivalent to the dodecahedron. There is at most one inscribed normally equivalent realization of $P_{12}$ up to scaling.

Example 4.7 (Pyramids). Every 3-dimensional pyramid over an inscribable polygon has a unique inscribed realization with fixed normal fan, up to scaling.

Corollary 4.8. Let $P$ be a simplicial polytope of dimension $d \geq 3$. Then $\dim \mathcal{I}_+(P) \leq 1$.

Proof. We can reuse the argument of the proof of Corollary 4.5: If $F_1, F_2 \subset P$ are adjacent facets with $\dim \mathcal{I}_+(F_i) \leq 1$, then Corollary 4.4 implies the desired result. It is straightforward (for example, by induction), to show that every simplex $S$ satisfies $\dim \mathcal{I}_+(S) = 1$. \hfill \Box

4.2. Simple and even polytopes. We now turn to the other extremal class of simple polytopes. For a set $X \subseteq \mathbb{R}^d$, denote by $M(X)$ the affine subspace of centers of spheres containing $X$, that is,

$$M(X) := \{c \in \mathbb{R}^d : \|x - c\| = \|y - c\| \text{ for all } x, y \in X\}.$$ 

We call $M(X)$ the generalized bisector of $X$, since for two points $x, y \in \mathbb{R}^2$, $M(\{x, y\})$ is the well-known bisector of $x$ and $y$ of Euclidean geometry. Clearly $M(X) \subseteq M(Y)$ for $Y \subseteq X \subseteq \mathbb{R}^d$. Stronger even:

Lemma 4.9. Let $X, Y \subseteq \mathbb{R}^d$. Then $M(X \cup Y) \subseteq M(X) \cap M(Y)$. If $X \cap Y \neq \emptyset$, then $M(X \cup Y) = M(X) \cap M(Y)$.

Proof. The first claim follows directly from the definition. Now let $c \in M(X) \cap M(Y)$, $x \in X$, $y \in Y$ and $z \in X \cap Y$. Then $\|c - x\| = \|c - z\| = \|c - y\|$ so $c \in M(X \cup Y)$. \hfill \Box

For two affine subspaces $L, L' \subseteq \mathbb{R}^d$, write $L \parallel L'$ if $L = L' + t$ for some $t \in \mathbb{R}^d$.

Lemma 4.10. Let $X \subseteq \mathbb{R}^d$. Either $M(X) = \emptyset$ or $M(X) \parallel \text{aff}(X)$. \hfill \Box

Proof. Let $\dim \text{aff}(X) = k$ and let $B$ be an affine basis of $X$, $|B| = k + 1$. Then there exists a unique sphere centered in a point $c_B \in \text{aff}(X)$ that contains $B$ and $M(B) = c_B + \text{aff}(X)$. If $c_B \neq c_{B'}$ for two affine bases $B, B'$ of $X$, then $M(X) = \emptyset$, otherwise $M(X) = c_B + \text{aff}(X)$, which is parallel to $\text{aff}(X)$. \hfill \Box

For a polytope $P \subseteq \mathbb{R}^d$, $P$ is inscribed if and only if $M(V(P)) \neq \emptyset$. For $F \subseteq P$ a face, let

$$\mathcal{F}_k(F, P) := \{G \subseteq P \text{ face} : F \subseteq G, \dim G = \dim F + k\}$$
and set
\[ K(F, P) := \bigcup_{G \in \mathcal{F}_1(F, P)} V(G). \]
By projecting along \( F \), one can convince oneself that \( \text{aff}(K(F, P)) = \text{aff}(P) \).

**Corollary 4.11.** Let \( F \) be a proper, non-empty face of an inscribed polytope \( P \subseteq \mathbb{R}^d \). Then
\[ M(K(F, P)) = M(V(P)). \]

**Proof.** Since \( \text{aff}(K(F, P)) = \text{aff}(P) \), we have \( M(K(F, P)) \parallel \text{aff}(K(F, P)) = P^\perp \). Moreover, \( K(F, P) \subseteq V(P) \), so \( M(V(P)) \subseteq M(K(F, P)) \). But \( M(V(P)) \parallel P^\perp \parallel M(K(F, P)) \), so \( M(K(F, P)) = M(V(P)) \). □

The following result gives a local criterion on inscribability.

**Theorem 4.12.** Let \( P \subseteq \mathbb{R}^d \) be a polytope and \( 0 \leq j < j + 2 \leq k \leq d \). Then \( P \) is inscribed if and only if the following two conditions hold: every \( k \)-face is inscribed and \( M(K(F, P)) \neq \emptyset \) for all \( j \)-faces \( F \).

The necessity of both conditions is exemplified by simplicial and simple polytopes respectively, that is, every \( k \)-face of a simplicial polytope is a simplex and thus inscribed, while \( M(K(F, P)) \neq \emptyset \) for every simple polytope. Thus, it is necessary to combine both.

For \( j = 0 \), we obtain the following remarkable characterization:

**Corollary 4.13.** Let \( P \subseteq \mathbb{R}^d \) be a d-polytope and \( 2 \leq k \leq d \). If every vertex \( v \) together with its neighbors lie on some sphere and if all \( k \)-faces are inscribed to some sphere, then \( P \) is inscribed.

**Proof of Theorem 4.12.** If \( P \) is inscribed, then every face of \( P \) is inscribed. Moreover, since \( K(F, P) \subseteq V(P) \) we have \( \emptyset \neq M(V(P)) \subseteq M(K(F, P)) \). For the converse, we only need to consider the case \( k = j + 2 \). Let \( F \) be a \( j \)-face and \( G \supseteq F \) a \((j + 1)\)-face of \( P \). We want to show that \( M(K(F, P)) = M(K(G, P)) \). By varying \( F \) and \( G \) over all \( j \)- and \((j + 1)\)-faces respectively, we can then conclude that \( M(K(F, P)) = M(K(F', P)) \) for any two \( j \)-faces \( F, F' \) of \( P \), so that \( M(V(P)) = K(F, P) \neq \emptyset \), i.e. \( P \) is inscribed.

Thus, we are left to show that \( M(K(F, P)) = M(K(G, P)) \). We have
\[
M(K(F, P)) \overset{(4.9)}{=} \bigcap_{G \in \mathcal{F}_1(F, P)} M(V(G)) = \bigcap_{H \in \mathcal{F}_2(F, P)} M(K(F, H)) \overset{(4.11)}{=} \bigcap_{H \in \mathcal{F}_2(F, P)} M(V(H)) \subseteq \bigcap_{H \in \mathcal{F}_2(G, P)} M(V(H)) \overset{(4.9)}{=} M(K(G, P)).
\]
Since \( \emptyset \neq M(K(F, P)) \subseteq M(K(G, P)) \), we obtain \( M(K(G, P)) \parallel P^\perp \parallel M(K(F, P)) \) by Lemma 4.10. We conclude that \( M(K(F, P)) = M(K(G, P)) \). □

We stated above that the condition \( M(K(F, P)) \neq \emptyset \) in Theorem 4.12 is not necessary for simple polytopes. Stronger even, a \( d \)-polytope is called \textbf{k-simple} if every \((d - k - 1)\)-face is contained in exactly \( k + 1 \) facets [13, Ch. 4.5]. Every polytope is at least 1-simple and a \((d - 1)\)-simple polytope is simply a simple polytope.
Corollary 4.14. Let $P$ be a $k$-simple $d$-polytope, $1 \leq k \leq d - 1$. Then $P$ is inscribed, if and only if all its $(d - k + 1)$-faces are inscribed. In particular a simple polytope is inscribed if and only if all its $k$-faces are inscribed.

Proof. Let $F$ be a $(d - k - 1)$-face of $P$. Let $B_F \subseteq V(F)$ be an affine basis off $\text{aff}(F)$, and for each $G \in \mathcal{F}_1(F, P)$, let $v_G \in V(G) \setminus V(F)$. Set $B := B_F \cup \{v_G : G \in \mathcal{F}_1(F, P)\}$. Then $|B_F| = d + 1$ and $\text{aff}(B_F) = \text{aff}(K(F, P)) = P^\perp$, so $M(B_F) \neq \emptyset$. Let $c \in M(B_F)$. Since $F$ is a facet of all $G \in \mathcal{F}_1(F, P)$ and $G$ is inscribed, we have $c \in M(V(G))$ and therefore $c \in M(K(F, P))$. We can now apply Theorem 4.12. \hfill $\Box$

Remark 4.15. There is also a dual notion of an generalized angular bisector. For this, take a circumscribed polytope $P$, that is, a polytope all of whose facets are tangent to a sphere. Assume that the sphere is centered at the origin. Then its polar $P^\Delta$ is inscribed. For $F \subseteq P$ a face, let $F^o \subseteq P^\Delta$ denote its associated dual face. Let $\hat{M}(F)$ be the linear subspace of all points which are the center of a sphere tangent to all affine hulls of facets in $P$ containing $F$. We call $\hat{M}(F)$ the generalized angular bisector at $F$. Then $\hat{M}(F^o) = M(V(F))$. This generalizes a well-known theorem of Euclidean geometry: The angular bisectors of a triangle coincide with the perpendicular bisectors of the Gergonne triangle (the contact triangle of the inscribed circle).

Let us call a polytope $P$ even if all 2-faces are even. Equivalently, $P$ is even if the graph of $P$ is bipartite.

Theorem 4.16. Let $P \subset \mathbb{R}^d$ be a normally inscribed and full-dimensional polytope. Then the following are equivalent:

(i) $P$ is even;
(ii) $\dim \mathcal{I}_+(P) = d$;
(iii) $\mathcal{I}(P, R_0) = \mathbb{R}^d$ for any region $R_0$ of $\mathcal{N}(P)$.

Proof. Let $N = \mathcal{N}(P)$ and choose a region $R_0 \in N$. The representation (3) of $\mathcal{I}_+(N)$ given in the proof of Theorem 1.1 is the intersection of open $d$-dimensional cones with the inscribed space $\mathcal{I}(N, R_0)$. Thus it suffices to show that $\mathcal{I}(N, R_0) = \mathbb{R}^d$ if and only if $P$ is even.

For a 2-face $F \subseteq P$ and $C = N_F P$, let us write $W_C$ for the unique cycle in $\mathcal{N}_C$. Now $\mathcal{I}(N, R_0)$ is the intersection of $\ker(t_{W_C} W^{-1} - \text{id})$, where $C$ ranges over all codimension-2 cones and $W$ is a walk to a region of $\mathcal{N}_C$ and $W^{-1}$ is the reversed walk. It follows from Proposition 3.1 that $\mathcal{I}(N_C, R_0 - C) = C^\perp$ precisely if $F$ is even. Hence $\mathcal{I}(N, R_0) = \mathbb{R}^d$ if and only if $P$ is even. \hfill $\Box$

We call a complete fan $N$ even if all its codimension-2 cones are incident to an even number of regions. Using essentially the same argument we arrive at:

Corollary 4.17. Let $N$ be a full-dimensional, strongly connected and even fan in $\mathbb{R}^d$. Then $\mathcal{I}(N, R_0) = \{0\}$ or $= \mathbb{R}^d$ for any region $R_0$ of $N$. Moreover, $N$ is virtually inscribable if and only if $N_C$ is virtually inscribable for all $C \in N$ of codimension 2.

Remark 4.18. If $P$ is simple and even, then its combinatorial dual polytope is balanced; see [15]. Let us call a 3-connected planar graph $G$ even if all faces are even. Interestingly, Dillencourt–Smith [11] showed that every trivalent and even polytopal graph can be realized as
the graph of an inscribed 3-polytope. This translates to our setting as follows: Every trivalent and even polyhedral graph is the dual graph of an inscribable 3-dimensional fan.

A last observation on even fans stems from Corollary 4.4. Given a fan $\mathcal{N}$, can we extend every inscribed realization of a localization $\mathcal{N}_C$ ($C$ being some cone of $\mathcal{N}$) to an inscribed realization of $\mathcal{N}$? Together with Theorem 4.16, we can answer this in the affirmative for virtual inscribability of even fans. Let $\pi_C : \mathbb{R}^d \to C^\perp$ be the orthogonal projection.

**Corollary 4.19.** Let $\mathcal{N}$ be a full-dimensional and strongly connected fan, $C \in \mathcal{N}$ a cone and $C \subseteq R \in \mathcal{N}$ a region. Then $\pi_C(\mathcal{I}(\mathcal{N}, R)) \subseteq \mathcal{I}(\mathcal{N}_C, R - C)$. Equality holds when $\mathcal{N}$ is inscribable and even.

Clearly, $\pi_C(\mathcal{I}_+(\mathcal{N}, R)) \subseteq \mathcal{I}_+(\mathcal{N}_C, R - C)$ but the inclusion is strict in general. The next example shows that equality is not attained for simplicial and even fans.

**Example 4.20.** Let $P := \text{conv}\{ (x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = 61 \}$. This is an inscribed 3-dimensional polytope with 72 vertices shown in Figure 9. It can be checked that $P$ is even and simple. Let $F \subseteq P$ be the hexagonal 2-face that maximizes the linear function $c = (1, 1, 1)$. This is a permutahedron for the point $v_0 = (3, 4, 6)$. Let $R_0 := N_{v_0} P$ and $R'_0 := N_{v_0} F \cap c^\perp$. The based inscribed cone of $F$ coincides with $R'_0$; cf. Section 2.3. Figure 9 shows the projection $\pi_C(\mathcal{I}_+(P, R_0))$ as a proper subcone of $R'_0 = \mathcal{I}_+(F, R'_0)$.

This behavior is similar to that of simplicial fans and their polytopal (virtual) realizations, see [21, Lem. 8.5].

4.3. Inscrribable Nestohedra. In this section, we use Corollary 4.14 to give a characterization of inscribed nestohedra, an important subclass of generalized permutahedra.

A polytope $P \subseteq \mathbb{R}^d$ is a **generalized permutahedron** if for any two adjacent vertices $u, v \in V(P)$, there are $i \neq j$ such that $u - v = \mu(e_i - e_j)$ for some $\mu \in \mathbb{R}$. Generalized permutahedra were introduced by Postnikov [25], where it was also shown that the above
condition is equivalent to the existence of a polytope $Q$ such that $P + Q$ is a permutohedron; cf. Section 2.3. Generalized permutohedra constitute an important class in the field of algebraic and geometric combinatorics and many well-known combinatorial polytopes can be realized as generalized permutohedra; cf. [25, Sect. 8]. In [6], a type-A matroid polytope is defined as a polytope whose edges are along directions $e_i - e_j$ and whose whose vertices are equidistant from some point. That is, type-A matroid polytopes are precisely inscribed generalized permutohedra.

Postnikov described a large class of simple generalized permutohedra, the so-called nestohedra. A collection $\mathcal{B}$ of subsets of $[d]$ is called a building set if $I \cap J \neq \emptyset$ implies $I \cup J \in \mathcal{B}$ for any $I, J \in \mathcal{B}$. For $I \subseteq [d]$, write $\Delta_I := \text{conv}(e_i : i \in I)$. The generalized permutohedron associated to a building set $\mathcal{B}$

$$\Delta_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I$$

is called a nestohedron. The name derives from the fact that $\Delta_{\mathcal{B}}$ is a simple polytope whose face lattice is anti-isomorphic to the complex of nested sets of $\mathcal{B}$. A nested set is a subset $N \subseteq B$ such that $I \cap J \neq \emptyset$ implies $I \subseteq J$ or $J \subseteq I$ for any $I, J \in N$ and if $J_1, \ldots, J_k \in N$ are disjoint, then $\bigcup_i J_i \not\subseteq B$. The nested complex is the collection of nested sets of $\mathcal{B}$ ordered by inclusion.

We mentioned two important examples of nestohedra: For $d \geq 2$, let

$$\mathcal{B}_{\text{ass}} := \{\{i, i + 1, \ldots, j\} : 1 \leq i < j \leq d\}.$$ 

then $\Delta_{\mathcal{B}_{\text{ass}}}$ is combinatorially isomorphic to the associahedron or Stasheff polytope [28]. The collection of all cyclic intervals of $[d]$

$$\mathcal{B}_{\text{cyc}} := \mathcal{B}_{\text{ass}} \cup \{[d] \setminus \{i, \ldots, j\} : 1 \leq i < j < d\}$$

gives rise to the cyclohedron [28].

For $i, j, k \in [d]$, let us write $N_{i,j}^k(\mathcal{B})$ for the set of $I \in \mathcal{B}$ with $i, j \in I$ and $k \not\in I$ and $n_{i,j}^k(\mathcal{B}) := |N_{i,j}^k(\mathcal{B})|$ for its size. Furthermore, for $J \subseteq [d]$, define the restriction of $\mathcal{B}$ be the building set

$$\mathcal{B}|_J := \{I \in \mathcal{B} : I \subseteq J\}.$$ 

**Theorem 4.21.** Let $\mathcal{B}$ be a building set. The nestohedron $\Delta_{\mathcal{B}}$ is inscribed if and only if for all $J \subseteq [d]$ the following condition holds: If for $i, j, k \in J$ both $n_{i,j}^k(\mathcal{B}|_J) > 0$ and $n_{j,k}^i(\mathcal{B}|_J) > 0$ holds, then

$$n_{i,j}^k(\mathcal{B}|_J) = n_{j,k}^i(\mathcal{B}|_J) = n_{i,k}^j(\mathcal{B}|_J).$$

**Proof.** Since $\Delta_{\mathcal{B}}$ is simple, Corollary 4.14 yields that it suffices to check that all 2-faces of $\Delta_{\mathcal{B}}$ are inscribed. For $c \in \mathbb{R}^d$, we have

$$\Delta_{\mathcal{B}}^c = \sum_{I \in \mathcal{B}} \Delta_I^c.$$ 

Hence, for every face $F \subseteq \Delta_{\mathcal{B}}$ of dimension 2, there is a $c$ with $d - 2$ distinct coordinates such that $\Delta_{\mathcal{B}}^c = F$. If $c_i = c_j < c_k = c_l$ for some $i, j, k, l$, then $F$ is the Cartesian product of two segments and hence inscribable. The remaining case is if $c_i = c_j = c_k$ for some distinct $i, j, k$. 

Let \( J := \{ i \in [d] : c_i \leq c_i \} \). If \( I \in \mathcal{B} \setminus \mathcal{B}|J \), then \( \Delta_{i,j}^r \) is a vertex. For this reason, the face \( F \) is a translate of

\[
n_{i,j}^k (\mathcal{B}|J) \Delta_{i,j} + n_{j,k}^i (\mathcal{B}|J) \Delta_{j,k} + n_{i,k}^j (\mathcal{B}|J) \Delta_{i,k} + m \Delta_{i,j,k}
\]
for some \( m \geq 0 \). The normal fan of \( F \) is determined by how many numbers of \( n_{i,j}^k (\mathcal{B}|J) \), \( n_{j,k}^i (\mathcal{B}|J) \), \( n_{i,k}^j (\mathcal{B}|J) \) are greater than zero. If zero of them are, then \( F \) is a translate of \( m \Delta_{i,j,k} \) and thus inscribable. If there is only one, then \( F \) is a translate of an isosceles triangle and inscribable again. If there are two, then \( F \) is a rhombus or a pentagon with normal fan as in Example 3.4 and not inscribable. Finally, if there are three, then \( F \) is a hexagon with normal fan as in Example 3.7, which is inscribable if and only if all three numbers are equal. \( \square \)

**Example 4.22** (Pitman–Stanley polytopes). The \((d - 1)\)-dimensional Pitman–Stanley polytope is the nestohedron of the building set

\[
\mathcal{B}_{flag} := \{ \{1, \ldots, k\} : 1 \leq k \leq d \},
\]
see [25, Sect. 8.5]. It is combinatorially but not linearly isomorphic to the \((d - 1)\)-dimensional cube. By Theorem 4.21, it is an inscribed nestohedron.

A particularly nice subclass of nestohedra is given by the graph associahedra. Let \( G = (V, E) \) be a simple graph on nodes \( V = [d] \). The **graphical building set** \( \mathcal{B}(G) \) consists of all non-empty \( I \subseteq [d] \) such that the vertex-induced subgraph \( G[I] \) is connected. It is easy to see that \( \mathcal{B}(G) \) is indeed a building set. If \( G \) is a path or a cycle, then \( \Delta_{\mathcal{B}(G)} \) is the associahedron and the cyclohedron, respectively. If \( G \) is the complete graph, then \( \Delta_{\mathcal{B}(G)} \) is the permutahedron.

**Corollary 4.23.** Let \( G \) be a graph. Then \( \Delta_{\mathcal{B}(G)} \) is inscribed if and only if \( G \) is a disjoint union of complete graphs.

**Proof.** If \( G \) is disconnected with connected components \( G_1, \ldots, G_k \), then

\[
\Delta_{\mathcal{B}(G)} = \Delta_{\mathcal{B}(G_1)} \times \Delta_{\mathcal{B}(G_2)} \times \cdots \times \Delta_{\mathcal{B}(G_k)}.
\]

So it suffices to restrict to connected \( G \) and \( |V| = d \geq 2 \).

Assume that \( G \) is not complete. We can find nodes \( i, j, k \) such that \( ij, jk \in E \) and \( ik \notin E \). Let \( J := \{ i, j, k \} \). Then \( n_{i,j}^k (\mathcal{B}|J) = 1, n_{i,j}^k (\mathcal{B}|J) = 1, \) but \( n_{i,k}^j (\mathcal{B}|J) = 0 \). It follows from Theorem 4.21 that \( \Delta_{\mathcal{B}(G)} \) is not inscribed. \( \square \)

The corollary implies that neither the associahedron nor the cyclohedron (in their realization as a generalized permutahedron) is inscribed.

For \( I, J \subseteq [d] \), let \( I \Delta J := (I \setminus J) \cup (J \setminus I) \) be their **symmetric difference**. A building set is called \( \Delta \)-**closed**, if for all \( I, J \in \mathcal{B} \) with \( I \not\subseteq J \) and \( J \not\subseteq I \)

\[
I \cap J \neq \emptyset \implies I \Delta J \in \mathcal{B}.
\]

For example, both the building sets of Pitman–Stanley polytopes, as well as the graphical building sets of complete graphs are \( \Delta \)-closed.

**Proposition 4.24.** If \( \mathcal{B} \) is a \( \Delta \)-closed building set, then \( \Delta_{\mathcal{B}} \) is inscribed.

**Proof.** We want to check the condition of Theorem 4.21. Clearly, if \( \mathcal{B} \) is triangle closed, then also \( \mathcal{B}|J \) for any \( J \subseteq [d] \), so we only need to show the condition for \( J = [d] \). Let \( i, j, k \in \mathcal{B} \) and
both \( n_{i,j}^k(\mathcal{B}) > 0 \) and \( n_{j,k}^i(\mathcal{B}) > 0 \). Furthermore, let \( K \in N_{i,j}^k(\mathcal{B}) \). Then for all \( J \in N_{i,j}^k(\mathcal{B}) \), we see that \( J \cap K \not\in \{\emptyset, J, K\} \) and thus \( J \triangle K \in N_{j,k}^i(\mathcal{B}) \). Therefore, the map

\[
N_{i,j}^k(\mathcal{B}) \to N_{j,k}^i(\mathcal{B}), \quad J \mapsto J \triangle K,
\]
is a bijection. By a symmetric argument, using some \( I \in N_{j,k}^i(\mathcal{B}) \), the map \( J \mapsto J \triangle I \) gives a bijection between \( N_{j,k}^i(\mathcal{B}) \) and \( N_{i,j}^k(\mathcal{B}) \), so all three set have equal size. \(\square\)

While this gives a quite general class of inscribed nestohedra, not all examples arise this way.

**Example 4.25.** Consider the following building set for \( d = 4 \):

\[
\mathcal{B} := \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.
\]

Using Theorem 4.21 one can check that \( \Delta_{\mathcal{B}} \) is inscribed, but \( \mathcal{B} \) is not \( \triangle \)-closed.

5. **Type cones and inscribed virtual polytopes**

In this section, we develop a different perspective on the inscribed cone \( I_+ (\mathcal{N}) \) that is invariant under orthogonal transformations of \( \mathcal{N} \) and that allows us to give a polynomial time algorithm to check if a rational polytope \( P \) is normally inscribable (Theorem 1.3).

The collection \( T_+ (P) \) of polytopes normally equivalent to \( P \) modulo translation is an open polyhedral cone \([18]\) and we realize \( I_+ (\mathcal{N}) \) as a subcone. We recall in Section 5.1 that polytopes normally equivalent to \( P \) can be seen as the cone of strictly-convex piecewise-linear functions on \( \mathcal{N} \) modulo linear functions. We will realize the inscribed space \( I(P) \) as a subspace of \( \mathrm{PL}(\mathcal{N})/(\mathbb{R}^d)^* \). This gives an interpretation of \( I(\mathcal{N}, R_0) \setminus \mathcal{I}_+ (\mathcal{N}, R_0) \) as inscribed virtual polytopes (Section 5.3).

Interestingly, a fan \( \mathcal{N} \) may have virtual inscribed polytopes but no actual inscribed polytopes. This parallels the situation of fans not admitting a strictly convex PL-functions. Explicit descriptions in terms of inequalities and equations are given in Section 5.4.

5.1. **Type cones and piecewise-linear functions.** The type cone \( T_+ (P) \) of a polytope \( P \subset \mathbb{R}^d \) is the collection of polytopes \( P' \subset \mathbb{R}^d \) normally equivalent to \( P \), up to translation. In this section we recall the well-known result that \( T_+ (P) \) is an open polyhedral cone. In order to understand the polytopes in the closure \( \overline{T}_+ (P) \), let us call a polytope \( Q \subset \mathbb{R}^d \) a **weak Minkowski summand** of \( P \) if there are \( \mu > 0 \) and a polytope \( R \) such that

\[
\mu P = Q + R.
\]

In particular, \( Q \) is normally equivalent to \( P \) if, in addition, \( P \) is a weak Minkowski summand of \( Q \). We shall see that \( \overline{T}_+ (P) \) is precisely the collection of weak Minkowski summands of \( P \), up to translation.

To understand the boundary structure of \( \overline{T}_+ (P) \), let \( \mathcal{N}, \mathcal{N}' \) be fans with the same support. We say that \( \mathcal{N}' \) **coarsens** \( \mathcal{N} \) (or that \( \mathcal{N} \) **refines** \( \mathcal{N}' \)) if for every region \( C \in \mathcal{N} \) there is \( C' \in \mathcal{N}' \) with \( C \subset C' \). ‘Coarsening’ defines a partial order on complete fans in \( \mathbb{R}^d \) with minimum \( \{\mathbb{R}^d\} \).

It follows from (5) that \( \mathcal{N}(Q) \) coarsens \( \mathcal{N}(P) \) whenever \( Q \) is a weak Minkowski summand of \( P \). In the introduction, we called a fan \( \mathcal{N} \) **polytopal** if \( \mathcal{N} = \mathcal{N}(P) \) for some polytope \( P \) and we write \( \mathcal{I}_+(\mathcal{N}) := \mathcal{I}_+(P) \). Conversely, Theorem 15.1.2 of [13] implies that every polytopal
coarsening of $\mathcal{N}(P)$ is the normal fan of a weak Minkowski summand of $P$. The following captures the most important structural properties of type cones.

**Theorem 5.1** ([18, Theorem 7]). Let $\mathcal{N}$ be a complete fan. Then $\mathcal{T}_+(\mathcal{N})$ is a polyhedral cone. The set of non-empty faces of $\mathcal{T}_+(P)$ ordered by inclusion is isomorphic to the poset of polytopal coarsenings of $\mathcal{N}$. If $F \subseteq \mathcal{T}_+(\mathcal{N})$ is a non-empty face corresponding to $\mathcal{N}'$, then $F = \mathcal{T}_+(\mathcal{N}')$.

A polytope $P$ is **indecomposable** if $\mathcal{T}_+(P)$ is 1-dimensional, that is, if every weak Minkowski summand of $P$ is homothetic to $P$. Theorem 5.1 implies that the rays of $\mathcal{T}_+(P)$ correspond to the indecomposable weak Minkowsi summands of $P$.

It is advantageous to view Theorem 5.1 from the perspective of piecewise-linear functions. In fact, the results leading to Corollary 5.6 essentially give a complete proof of Theorem 5.1. Let $\mathcal{N}$ be a full-dimensional and strongly connected fan. Theorem 5.1 (cf. [18, Theorem 7]) captures the most important structural properties of type cones.

(ii) For all $x \in \mathbb{R}^d$, let $h_P : \mathbb{R}^d \to \mathbb{R}$

$$h_P(c) := \max \{ \langle c, x \rangle : x \in P \} = \max \{ \langle c, v \rangle : v \in V(P) \}.$$ 

This is a convex PL-function supported on $\mathcal{N}(P)$ that uniquely determines $P$. A convex PL-function $\ell$ is strictly convex with respect to $\mathcal{N}$ if

$$\ell(x + y) < \ell(x) + \ell(y)$$

for any points $x, y \in \mathbb{R}^d$ not contained in the same region.

**Proposition 5.2.** Let $\mathcal{N}$ be a polytopal fan and $\ell$ a PL-function supported on $\mathcal{N}$. The following are equivalent:

(i) $\ell$ is strictly convex;

(ii) For all $x \in \mathbb{R}^d$ it holds that $\ell(x) = \max \{ \ell_R(x) : R \in \mathcal{N} \}$ region;

(iii) $\ell = h_P$ for some polytope $P$ with $\mathcal{N}(P) = \mathcal{N}$.

**Proof.** (i) $\Rightarrow$ (ii): Let $R \in \mathcal{N}$ be a region and $r \in \text{int}(R)$.

For any other region $S \in \mathcal{N}$, let $s \in S$ such that $r + s \in \text{int}(S)$.

Then

$$\ell_S(r) + \ell_S(s) = \ell_S(r + s) = \ell(r + s) < \ell(r) + \ell(s) = \ell_R(r) + \ell_S(s).$$

This shows $\ell_R(r) \geq \ell_S(r)$ for all $r \in R$ and all regions $S$ and hence (ii) holds.

(ii) $\Rightarrow$ (iii): For $R \in \mathcal{N}$, let $v_R \in \mathbb{R}^d$ such that $\ell_R(x) = \langle v_R, x \rangle$ for all $x \in R$ and define $P = \text{conv}(v_R : R \in \mathcal{N})$. Convexity now forces $h_P(x) = \max \{ \langle v_R, x \rangle : R \}$ for all $x$.

(iii) $\Rightarrow$ (i): Let $r \in \text{int}(R)$ and $s \in \text{int}(S)$ for two distinct regions $R, S \in \mathcal{N} = \mathcal{N}(P)$. Let $v_R = P^r$ and $v_S = P^s$ be the corresponding vertices. That is, $h_P(r) = \langle r, v_R \rangle > \langle r, x \rangle$ for all $x \in P \setminus \{ v_R \}$ and likewise for $s$ and $v_S$. For $r + s$ there is a point $x \in P$ such that

$$h_P(r + s) = \langle r + s, x \rangle = \langle r, x \rangle + \langle s, x \rangle < \langle r, v_R \rangle + \langle s, v_S \rangle = h_P(r) + h_P(s).$$

The domains of linearity of a convex PL-function $\ell \in \text{PL}(\mathcal{N})$ are convex and yield a coarsening of $\mathcal{N}$. The proof now implies that any convex PL-function $\ell$ determines a polytope...
Recall that the dual graph of $\mathcal{N}$ is the undirected graph $G(\mathcal{N}) = (V, E)$ whose nodes $V(\mathcal{N})$ are the regions of $\mathcal{N}$ and regions $R, S$ satisfy $RS \in E(\mathcal{N})$ if $\dim R \cap S = d - 1$. If $R, S$ are adjacent, then let $\alpha_{RS}$ be the unit vector such that $\lin(R \cap S) = \alpha_{RS}^\perp$ and $\langle \alpha_{RS}, x \rangle \leq 0$ for all $x \in R$. That is, $\alpha_{RS}$ is an outer normal vector to $R$ and note that $\alpha_{SR} = -\alpha_{RS}$. Let $\ell_R : \mathbb{R}^n \to \mathbb{R}$ be a linear function for each region $R \in \mathcal{N}$. The collection $(\ell_R)_R$ determines a PL function supported on $\mathcal{N}$ if $\ell_R(x) = \ell_S(x)$ for all $x \in R \cap S$. If $R$ and $S$ are adjacent, then this is equivalent to

$$\ell_S(x) - \ell_R(x) = \lambda_{RS} \langle \alpha_{RS}, x \rangle \quad (6)$$

for some $\lambda_{RS} \in \mathbb{R}$. Clearly, $\lambda_{SR} = \lambda_{RS}$ and thus we can regard $\lambda$ as a function $E(\mathcal{N}) \to \mathbb{R}$.

Let $R_0$ be a fixed base region with $\ell_0 := \ell_{R_0}$. For any region $R$ let $R_0 R_1 \ldots R_k = R$ be a walk in $G(\mathcal{N})$. Then

$$\ell_R(x) = \ell_0(x) + \sum_{i=1}^k \lambda_{R_{i-1} R_i} \langle \alpha_{R_{i-1} R_i}, x \rangle. \quad (7)$$

In particular, this means that for any closed walk $(R_k = R)$, we have

$$\sum_{i=1}^k \lambda_{R_{i-1} R_i} \alpha_{R_{i-1} R_i} = 0. \quad (8)$$

A map $\lambda : E(\mathcal{N}) \to \mathbb{R}$ satisfying (8) for all closed walks is called a 1-weight on $\mathcal{N}$; cf. [21]. As we do not consider higher weights in this paper, we simply call $\lambda$ a weight. The argument above now shows the following.

**Proposition 5.3.** Let $\mathcal{N}$ be a full-dimensional and strongly connected fan in $\mathbb{R}^d$. For a fixed region $R_0$ we have that $\text{PL}(\mathcal{N})$ is isomorphic to the collection of pairs $(\ell_0, \lambda) \in (\mathbb{R}^d)^* \times \mathbb{R}^{E(\mathcal{N})}$, where $\ell_0$ is a linear function and $\lambda = (\lambda_{RS})_{RS \in E(\mathcal{N})}$ is a weight. Changing the base region only affects $\ell_0$ and leaves $\lambda$ invariant.

For a PL-function $\ell \in \text{PL}(\mathcal{N})$ we will write $\lambda(\ell)$ for the corresponding element in $\mathbb{R}^{E(\mathcal{N})}$ in this isomorphism. We next give a well-known description of strictly convex PL-functions in terms of the weights $\lambda \in \mathbb{R}^{E(\mathcal{N})}$.

**Theorem 5.4.** Let $\mathcal{N}$ be a complete fan in $\mathbb{R}^d$ and let $\ell \in \text{PL}(\mathcal{N})$. Then $\ell$ is strictly convex if and only if $\lambda(\ell)_{RS} > 0$ for all adjacent regions $R, S \in \mathcal{N}$.

**Proof.** Let $\ell$ be strictly convex and $R, S$ adjacent regions. Let $x \in \text{int}(R)$ and $y \in \text{int}(S)$ such that $x + y \in R$. It follows from (6) that

$$\ell(x + y) = \ell_R(x + y) < \ell_R(x) + \ell_S(y) = \ell_R(x) + \ell_R(y) + \lambda_{RS} \langle \alpha_{RS}, y \rangle.$$

Since $S$ and $R$ are separated by $\alpha_{RS}^\perp$ and $y \notin R$, we have $\langle \alpha_{RS}, y \rangle > 0$ and hence $\lambda_{RS} > 0$.

For the converse, note that by Proposition 5.2(ii) it is sufficient to show that for $x \in \text{int}(R_0)$, $\ell_{R_0}(x) > \ell_S(x)$ for all regions $S \neq R_0$. As $R_0$ is arbitrary, this shows it for all regions. Let $x \in \text{int}(R_0)$ be generic, that is, $\langle \alpha_{RS}, x \rangle \neq 0$ for all $RS \in E(\mathcal{N})$. Any such point induces an acyclic orientation on $G(\mathcal{N})$ as follows: For two adjacent regions $R, S$, we orient the edge
from $R$ to $S$ if $\langle \alpha_{RS}, x \rangle < 0$. Note that $R_0$ is the unique source and for every region $R$ there is a directed path $R_0R_1 \ldots R_k = R$. We compute
\[
\ell_{R_0}(x) - \ell_R(x) = - \sum_{j=1}^{k} \lambda_{R_{j-1}R_j} \alpha_{R_{j-1}R_j} > 0.
\]

The result implies that for a convex PL-function $\ell$ supported on $\mathcal{N}$, we can read off the induced coarsening of $\mathcal{N}$.

**Corollary 5.5.** Let $\ell$ be a convex PL-function supported on $\mathcal{N}$ and let $\lambda = \lambda(\ell)$. Denote the connected components of the graph $(V(\mathcal{N}), \{RS \in E(\mathcal{N}) : \lambda_{RS} = 0\})$ by $V_1, \ldots, V_s \subseteq V(\mathcal{N})$. The coarsening $\mathcal{N}'$ of $\mathcal{N}$ induced by $\ell$ has regions $\bigcup V_i$ for $i = 1, \ldots, s$.

As we are only interested in polytopes $P$ with normal fan $\mathcal{N}$ up to translation, we get the following.

**Corollary 5.6.** Let $\mathcal{N}$ be a complete fan with dual graph $G(\mathcal{N}) = (V, E)$. Then
\[
\mathcal{T}_+(\mathcal{N}) \cong \{ \lambda \in \mathbb{R}_\geq 0 : \lambda \text{ satisfies (8) for all closed walks} \}.
\]

Note that (8) has to hold for only finitely many closed walks. We give two particular choices for these finitely many walks. Let $T$ be a spanning tree of $G(\mathcal{N})$. Every edge $RS \in E(\mathcal{N}) \setminus E(T)$ closes a cycle $C_{RS}$ in $T \cup \{RS\}$, called the fundamental cycle with respect to $T$. The following is standard and follows from the fact that both collections of cycles give a basis for the cycle space of $G(\mathcal{N})$.

**Proposition 5.7.** Let $\mathcal{N}$ be a full-dimensional and strongly connected fan and $\lambda \in \mathbb{R}^E$. Then $\lambda$ satisfies (8) for all closed walks if $\lambda$ satisfies (8) for
\begin{enumerate}[(i)]
  \item all fundamental cycles with respect to a fixed spanning tree $T$, or
  \item the cycles of links of codimension-2 cones not in the boundary of $|\mathcal{N}|$.
\end{enumerate}

5.2. **Inscribed Minkowski summands.** Let $\mathcal{N}$ be a polytopal fan. Theorem 5.1 states that the facial structure of the polyhedral cone $\mathcal{T}_+(\mathcal{N})$ is given in terms of polytopal coarsenings of $\mathcal{N}$, partially ordered by refinement. Let $\mathcal{I}_+(\mathcal{N}) \subseteq \mathcal{T}_+(\mathcal{N})$ be the closure of $\mathcal{I}_+(\mathcal{N})$ in the Hausdorff metric. The following example illustrates that the geometry and the combinatorics of $\mathcal{I}_+(\mathcal{N})$ is more subtle and deserves further study.

**Example 5.8.** Let $\mathcal{N}$ be the normal fan of the regular hexagon $P$. We already computed the inscribed cone of $\mathcal{N}$ using its profile in Example 3.7, but we now want to illustrate how the inscribed cone is embedded into the type cone. Corollary 5.6 allows us to embed $\mathcal{T}_+(\mathcal{N})$ as a 4-dimensional subcone of $\mathbb{R}^6$. The hyperplane $\lambda_1 + \lambda_2 + \cdots + \lambda_6 = 1$ meets all rays of $\mathcal{T}_+(\mathcal{N})$ and the intersection yields a 3-dimensional polytope $T$, depicted in Figure 10. The faces of $T$ are in one-to-one correspondence with the polytopal coarsenings of $\mathcal{N}$. The vertices of $T$ correspond to the indecomposable summands of $P$. North- and south pole correspond to the triangles $\triangle$ and $\nabla$. The three vertices on the equator of $T$ are segments along the three distinct edge directions of $P$.

All indecomposable summands are inscribed to some sphere but, as we have seen in Example 3.7, the inscribed cone $\mathcal{I}_+(\mathcal{N})$ is spanned by two rays corresponding to $\triangle$ and $\nabla$. In $T$, $\mathcal{I}_+(\mathcal{N})$ is thus given by the segment connecting the two polytopes; see Figure 10 left. The
other decomposable summands of $P$ are rhombi, isosceles trapezoids, and certain pentagons. We already worked through all weak Minkowski summands of $P$ in the Examples 3.3 and 3.4 and saw that only the isosceles trapezoids are normally inscribable. The extraneous inscribed examples are marked in red in the right of Figure 10.

The example demonstrates that not every inscribable coarsening $N'$ of $N$ occurs in the boundary of $\bar{I}_+(N)$. We will call a polytope $Q$ inscribed \textbf{relatively to} $N$ if $Q \in \bar{I}_+(N)$. Consequently, a coarsening $N'$ of $N$ is inscribable \textbf{relative to} $N$, if $N' = N(Q)$ for some $Q \in I_+(N)$.

**Theorem 5.9.** Let $N$ be an inscribable fan and $Q \in \bar{T}_+(N)$. Then $Q \in I_+(N)$ if and only if $Q$ is inscribed and $(V(Q) - c(Q)) \cap R \neq \emptyset$ for all regions $R \in N$.

**Proof.** Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of polytopes in $I_+(N)$ which converges to $Q$. We may assume that $c(P_n) = 0$ for all $n \geq 0$ and by Lemma 2.2 and Corollary 2.4, $P_n$ is uniquely determined by $\{p_n\} = V(P_n) \cap \text{int}(R)$ for some arbitrary but fixed region $R \in N$. It follows that $\lim_{n \to \infty} p_n = p \in R$. We infer from Corollary 2.3 that $Q$ is inscribed with $c(Q) = 0$ and thus $p$ is a vertex of $Q$. This shows that $V(Q) \cap R \neq \emptyset$ for all regions $R \in N$.

For the only if part, assume that $c(Q) = 0$. Since $Q \in \bar{T}_+(N)$, the normal fan $\mathcal{N}(Q)$ is a coarsening of $N$ and there is a region $C \in \mathcal{N}(Q)$ with $R \subseteq C$. Now $Q$ is inscribed and Corollary 2.4 gives that $\text{int}C$ contains exactly one vertex of $Q$ and that $\partial C$ does not contain any vertex of $Q$. Since $V(Q) \cap R \neq \emptyset$, we see that $|V(Q) \cap R| = 1$. We write $q_R$ for the unique vertex of $Q$ contained in $R$.

Let $R, R' \in N$ be two regions such that $R \cap R'$ is a wall and associated reflection $s_{RR'}$. Let $C, C' \in \mathcal{N}(Q)$ be the unique regions such that $R \subseteq C$ and $R' \subseteq C'$. If $C = C'$, then $q_R = q_{R'} \in R \cap R'$ and therefore $s_{RR'}(q_R) = q_{R'}$. If $C \neq C'$, then $S_{CC'} = s_{RR'}$ and therefore $s_{RR'}(q_R) = s_{CC'}(q_C) = q_{CC'} = q_{R'}$. Let $P \in I_+(N)$ with $c(P) = 0$. It follows that $Q + \varepsilon P \in I_+(N)$ for every $\varepsilon > 0$ and hence $Q \in \bar{I}_+(N)$.

\[ \square \]
Corollary 5.10. Let $R_0 \in \mathcal{N}$ be a region. The linear map $v_{R_0}$ extends to a linear homeomorphism

$$\mathcal{I}_+(\mathcal{N}) \cong \mathcal{I}_+(\mathcal{N}, R_0).$$

This extension is given by $\{v_{R_0}(Q)\} = (V(Q) - c(Q)) \cap R$.

A second characterization of inscribability relative to a fan is given by the following. Let $C \in \mathcal{N}$ be a cone and $P \in \mathcal{T}_+(\mathcal{N})$. The face $P^C := P^c$ does not depend on $c \in \text{relint} C$.

Theorem 5.11. Let $\mathcal{N}$ be a polytopal fan and $Q \in \mathcal{T}_+(\mathcal{N})$ be an inscribed polytope with $c(Q) = 0$. The following are equivalent:

(i) $Q \in \mathcal{I}_+(\mathcal{N})$;
(ii) $c(Q^C) \in \text{lin} C$ for all walls $C$ of $\mathcal{N}$.

Proof. (i) $\Rightarrow$ (ii): Let $C \in \mathcal{N}$ be a cone of codimension 1 and let $R, R' \in \mathcal{N}$ be the regions such that $C = R \cap R'$. If $v_R = v_{R'}$, then $P^C = v_R$ and $v_R \in R \cap R' = C$ by Theorem 5.9. Otherwise, $v_R \neq v_{R'}$ are the endpoints of the edge $P^C$ whose center is in $C$ by Corollary 2.3.

(ii) $\Rightarrow$ (i): Let $R \in \mathcal{N}$ be a region and $v_R$ be the corresponding vertex of $Q$. Let $C \subset R$ be a wall with supporting hyperplane $H := \text{lin} C$. Now $v_R \in Q^C$ and $c(Q^C) \in H$ implies that $v_R$ is not separated from $R$ by $H$. This holds for all walls $C$ and shows $v_R \in R$. Theorem 5.9 then implies $Q \in \mathcal{I}_+(\mathcal{N})$. $\square$

We close this section with a description of the facets of $\mathcal{I}_+(\mathcal{N})$ when $\mathcal{N}$ is an even inscribable fan. The coarsenings of $\mathcal{N}$ are encoded by certain contractions of $G(\mathcal{N})$. Recall that the contraction of an edge $e = uv$ in a simple graph $G = (V,E)$ is the graph $G/e$ with nodes $V \setminus e$ and edges $\{e \in E : v \notin e\} \cup \{uw : vw \in E\}$. Our definition of contractions does not produce parallel edges and hence stays in the category of simple graphs. If $\mathcal{N}'$ coarsens $\mathcal{N}$, then let

$$\mathcal{E}(\mathcal{N}, \mathcal{N}') := \{RS \in E(\mathcal{N}) : R, S \subseteq C \text{ for some region } C \in \mathcal{N}'\}.$$ 

Then $G(\mathcal{N}')$ is isomorphic to $G(\mathcal{N})/\mathcal{E}(\mathcal{N}, \mathcal{N}')$. On the level of weights $\lambda(\ell)$ for PL-functions $\ell$ supported on $\mathcal{N}$, coarsenings correspond to $\lambda : E(\mathcal{N}) \to \mathbb{R}$ with $\lambda(e) = 0$ for $e \in \mathcal{E}(\mathcal{N}, \mathcal{N}')$.

Proposition 5.12. Let $\mathcal{N}$ be an inscribable and even fan. Let $\mathcal{N}'$ be inscribable relative to $\mathcal{N}$ corresponding to a facet of $\mathcal{I}_+(\mathcal{N})$. Then $\mathcal{E}(\mathcal{N}, \mathcal{N}') \subseteq E(\mathcal{N})$ is a matching.

Proof. Let $Q \in \partial \mathcal{I}_+(\mathcal{N})$ be an inscribed polytope with $c(Q) = 0$ and $\mathcal{N}'(Q) = \mathcal{N}'$ and assume that $\mathcal{E}(\mathcal{N}, \mathcal{N}')$ is not a matching. Thus, there exists a region $R_0$ and two edges $R_0S, R_0S' \in \mathcal{E}(\mathcal{N}, \mathcal{N}')$, i.e. $R_0$, $S$ and $S'$ are coarsened to a common region $C$ in $\mathcal{N}'(Q)$. By Theorem 5.9 we see that $v_{C}(Q) \in R_0 \cap S \cap S'$. But, since $\mathcal{N}$ is even and hence $\mathcal{I}_+(\mathcal{N}, R_0) \subseteq R_0$ is a full-dimensional subcone, $v_{C}(Q)$ does not lie in the interior of a facet of $\mathcal{I}_+(\mathcal{N}, R_0)$, so neither does $Q$ lie in the interior of a facet of $\mathcal{I}_+(\mathcal{N})$ by Corollary 5.10. $\square$

5.3. Virtual polytopes and the inscribed space. Recall from Section 1.3 that the type space $\mathcal{T}(\mathcal{N})$ is the Grothendieck group of the monoid $(\mathcal{T}_+(\mathcal{N}), +)$. Since $\mathcal{T}_+(\mathcal{N})$ is a convex cone, we can identify $\mathcal{T}(\mathcal{N})$ with $\mathcal{T}_+(\mathcal{N}) + (-\mathcal{T}_+(\mathcal{N}))$. The following well-known result gives $\mathcal{T}(\mathcal{N})$ a simple interpretation in terms of piecewise-linear functions.
Proposition 5.13. Let $\mathcal{N}$ be a polytopal fan in $\mathbb{R}^d$. For all $\ell \in \text{PL}(\mathcal{N})$ there exist polytopes $P, Q \subseteq \mathbb{R}^d$ with $\mathcal{N} = \mathcal{N}(P) = \mathcal{N}(Q)$ such that $\ell = h_P - h_Q$. Moreover, if $\ell = h_{P'} - h_{Q'}$ for polytopes $P', Q'$, then $P + Q' = P' + Q$.

Proof. Let $P$ be a polytope with normal fan $\mathcal{N}$. By linearity, for any $\mu > 0$, we have $\lambda(\ell + \mu h_P) = \lambda(\ell) + \mu \lambda(h_P)$. Since $h_P$ is strictly convex, there exists a $\mu > 0$ such that $\lambda(\ell) + \mu \lambda(h_P) > 0$. That is, the function $\ell + \mu h_P$ is strictly convex and hence the support function of a polytope $Q \in T_+(\mathcal{N})$, by Proposition 5.2. This implies $\ell = h_Q - h_P$. For the final statement, observe that $h_{P+Q'} = h_P + h_{Q'}$. \hfill \Box

Hence $T(\mathcal{N})$ can be identified with piecewise-linear functions supported on $\mathcal{N}$ up to translation and we write $P - Q$ for $h_P - h_Q$. If $h_P - h_Q$ is convex, then it is the support function of a polytope $R$ with $P = Q + R$ and hence $P - Q = R$ is a polytope. Otherwise, we call $P - Q$ a virtual polytope. Virtual polytopes arise naturally in a number of settings; cf. [24]. Notably, $T(\mathcal{N})$ is the first graded piece in McMullen’s polytope algebra [19] and its dual space is 1-homogeneous translation-invariant valuations on polytopes with normal fan $\mathcal{N}$.

Let $\ell \in \text{PL}(\mathcal{N})$. For every region $R \in \mathcal{N}$, let $v_R(\ell) \in \mathbb{R}^d$ such that $\langle v_R(\ell), x \rangle \equiv \ell_R(x)$. We define the vertex set $V(\ell) := \{v_R(\ell) : R \in \mathcal{N} \text{ region}\}$. If $\ell = h_P$, then $V(h_P)$ is precisely the vertex set of $P$. In general $V(\ell)$ is not in bijection to the regions of $\mathcal{N}$. We call a PL function $\ell$ inscribed if $V(\ell)$ lies on a sphere. In this case, there exists a unique sphere containing $V(\ell)$ whose center, denoted $c(\ell)$, is in aff $V(\ell)$. For a different geometric perspective, let us associate an

$$H(\ell) := \{\ker(\ell_P(x) - t) \subset \mathbb{R}^d \times \mathbb{R} : R \in \mathcal{N} \text{ region}\}.$$ 

This is the smallest arrangement of linear hyperplanes containing the graph of $\ell$.

It follows that $\ell$ is inscribed if and only if there is a sphere in $\mathbb{R}^{d+1}$ that is tangent to all planes in $H(\ell)$. If $\ell$ is strictly convex, then the sphere can be chosen to be contained in the epigraph of $\ell$ and shows the duality between inscribed and circumscribed polytopes.

If $C \in \mathcal{N}$ is a cone, we define the localization $\ell_C$ of $\ell$ at $C$ to be the PL function supported on the localization $\mathcal{N}_C$, where $\ell_C$ coincides with $\ell_R$ on the cone $C - R \in \mathcal{N}_C$. If $F$ is a face of a polytope $P$, then $(h_P)_C = h_F$, where $C := N_F P$ is the normal cone of $F$, so localizations are the natural generalization of faces to PL functions. Clearly, $V(\ell_C) \subseteq V(\ell)$ and therefore an inscribed PL function has inscribed localizations.

Example 5.8 implies that the collection $\{P - Q \in T(\mathcal{N}) : h_P - h_Q \text{ inscribed}\}$ is not a linear subspace of $T(\mathcal{N})$. In light of Theorem 5.11, we define an inscribed PL function $\ell \in \text{PL}(\mathcal{N})$ to be inscribed relative to $\mathcal{N}$, if $c(\ell_C) - c(\ell) \in \text{lin} C$ for all cones $C \in F$ of codimension 1. Note that this definition retains Corollary 2.3 in that “neighboring vertices” should be obtained by reflecting at the hyperplanes of walls. We define

$$\mathcal{I}(\mathcal{N}) := \{P - Q \in T(\mathcal{N}) : h_P - h_Q \text{ inscribed relative to } \mathcal{N}\}.$$ 

to be the inscribed space of $\mathcal{N}$. Clearly, $P \in \mathcal{I}(\mathcal{N})$ if and only if $h_P \in \mathcal{I}(\mathcal{N})$, thus:

Proposition 5.14. Let $\mathcal{N}$ be a polytopal fan. Then $\mathcal{I}(\mathcal{N}) = \mathcal{I}(\mathcal{N}) \cap T_+(\mathcal{N})$. If $\mathcal{N}$ is inscribable, then $\mathcal{I}+(\mathcal{N}) = \mathcal{I}(\mathcal{N}) \cap T_+(\mathcal{N})$.

As a first consequence, we can now easily describe the facial structure of $\mathcal{I}(\mathcal{N})$:
Corollary 5.15. Let $\mathcal{N}$ be an inscribable fan. The faces of $\overline{\mathcal{I}}_+(\mathcal{N})$ are in correspondence to relatively inscribable coarsenings of $\mathcal{N}$.

Proof. Since by assumption $\mathcal{I}_+(\mathcal{N}) \neq \emptyset$, we see from Proposition 5.14 and Theorem 5.1 that the faces if $\overline{\mathcal{I}}_+(\mathcal{N})$ stem from relatively inscribable coarsenings of $\mathcal{N}$. \qed

We will now show that the based inscribed space $\mathcal{I}(\mathcal{N}, R_0)$ of Section 2 is linearly isomorphic to $\mathcal{I}(\mathcal{N})$. Recall that for a complete fan $\mathcal{N}$, the based inscribed space $\mathcal{I}(\mathcal{N}, R_0)$ is given by all $v \in \mathbb{R}^d$ such that

$$v = t_W(v) = s_{R_k R_{k-1}} \cdots s_{R_2 R_1} s_{R_1 R_0}(v)$$

for every closed walk $W = R_0 R_1 \ldots R_k$ in $G(\mathcal{N})$ starting in $R_0$.

Let $v_0 \in \mathcal{I}(\mathcal{N}, R_0)$ and $R \in \mathcal{N}$ a region. Choose a walk $W$ from $R_0$ to $R$ and define $v_R := t_W(v_0)$. We claim that the collection $(\ell_R)_R$ given by

$$\ell_R(x) := \langle v_R, x \rangle \quad \text{for all } x \in R,$$

is a piecewise-linear function supported on $\mathcal{N}$. If $S$ and $R$ are adjacent regions, then choose an appropriate walk $W$ from $R_0$ to $R$ which extends to a walk $W' = W S$. Then for $v_R := t_W(v_0)$ and $v_S := t_W(v_0)$

$$\ell_S(x) - \ell_R(x) = \langle v_S, x \rangle - \langle v_R, x \rangle = \langle s_{R S}, v_R, x \rangle - \langle v_R, x \rangle = -2 \langle \alpha_{RS}, v_R \rangle \cdot \langle \alpha_{RS}, x \rangle,$$

so it satisfies equation (6), from which we can deduce that $(\ell_R)_R$ is in fact a PL function on $\mathcal{N}$. We denote this PL-function by $\ell_{\mathcal{N}, R_0, v} \in \text{PL}(\mathcal{N})$.

Proposition 5.16. Let $\mathcal{N}$ be a full-dimensional and strongly connected fan in $\mathbb{R}^d$. The map

$$\Xi_{\mathcal{N}, R_0} : \mathcal{I}(\mathcal{N}, R_0) \to \text{PL}(\mathcal{N}) \quad \Xi_{\mathcal{N}, R_0}(v) := \ell_{\mathcal{N}, R_0, v}$$

is a linear embedding of $\mathcal{I}(\mathcal{N}, R_0)$ into $\text{PL}(\mathcal{N})$. The image $\Xi_{\mathcal{N}, R_0}(\mathcal{I}(\mathcal{N}, R_0))$ does not depend on $R_0$.

Proof. Let us note that the map $\Xi_{\mathcal{N}, R_0}$ is indeed linear. We will first prove that $\Xi_{\mathcal{N}, R_0}$ defines a linear map of $\mathcal{I}(\mathcal{N}, R_0)$ into $\text{PL}(\mathcal{N})$.

We will prove the claim using Theorem 5.4. Recall that for $RS \in E(\mathcal{N})$, $\alpha_{RS}$ is the unit vector satisfying $\langle \alpha_{RS}, x \rangle \leq 0$ for all $x \in R$. If $W = R_0 R_1 \ldots R_k$ is a path in $G(\mathcal{N})$ and $v \in \mathcal{I}(\mathcal{N}, R_0)$, then define $v_0 := v$ and $v_i := s_{R_{i-1} R_i}(v_{i-1})$ for $1 \leq i \leq k$. Since $s_{R_{i-1} R_i}$ is a reflection in $\alpha_{R_{i-1} R_i}^t$, we get

$$v_i = s_{R_{i-1} R_i}(v_{i-1}) = v_{i-1} - 2 \langle \alpha_{R_{i-1} R_i}, v_{i-1} \rangle \alpha_{R_{i-1} R_i}.$$

If we set $\lambda_{R_{i-1} R_i} := -2 \langle \alpha_{R_{i-1} R_i}, v_{i-1} \rangle$, then

$$v_i = v_0 + \sum_{i=1}^k \lambda_{R_{i-1} R_i} \alpha_{R_{i-1} R_i}.$$

To see that this is well defined, let $W' := R_0' R_1' \ldots R_{i-1} R_i' R_l'$ be a different walk with $R_0' = R_0$ and $R_{l-1} R_l = R_{k-1} R_k$. Then $R_0 \ldots R_{k-1} R_l' \ldots R_1' R_0$ is a closed walk and the definition of $\mathcal{I}(\mathcal{N}, R_0)$ implies that hence $v_k = s_{R_{k-1} R_l}(v_{l-1})$. 

If \( \mathcal{W} \) is a closed walk in \( G(\mathcal{N}) \), then this implies \( v_k = v_0 \) and hence \( \sum_{i=1}^{k} \lambda R_{i-1} R_i \alpha R_{i-1} R_i = 0 \). This means that \( \lambda \) satisfies (8) and hence \( \Xi_{\mathcal{N}, R_0}(v) \) is a piecewise-linear function supported on \( \mathcal{N} \) with data \( (\langle v, \cdot \rangle, \lambda) \).

If \( R_0 \) is a different base region and \( \mathcal{W}' \) is a walk from \( R_0 \) to \( R_0' \), then \( I(\mathcal{N}, R_0') = t_{\mathcal{W}'}(\mathcal{N}, R_0) \) and \( \ell_{\mathcal{N}, R_0, v} = \ell_{\mathcal{N}, R_0', v'} \) for \( v' = t_{\mathcal{W}'}(v) \in I(\mathcal{N}, R_0') \), thus \( \Xi_{\mathcal{N}, R_0}(I(\mathcal{N}, R_0)) = \Xi_{\mathcal{N}, R_0'}(I(\mathcal{N}, R_0')) \).

To show that \( \Xi_{\mathcal{N}, R_0} \) yields an embedding of \( I(\mathcal{N}, R_0) \) into \( \text{PL}(\mathcal{N}) \), we argue that the point \( v \) can be recovered from \( \lambda \). For this observe that \( R_0 \) is a full-dimensional cone and let \( l := \dim \text{lineal}(\mathcal{N}) \). There are regions \( S_1, \ldots, S_d \in \mathcal{N} \) adjacent to \( R_0 \) such that the vectors \( \alpha R_{0} S_i \) for \( i = 1, \ldots, d \) are linearly independent and span \( \text{lineal}(\mathcal{N})^\perp \). Thus \( v \) is the unique point with \( -2(\alpha R_{0} S_i, v) = \lambda R_{0} S_i \) and therefore can be reconstructed from \( \lambda \). \( \square \)

Notice that \( \Xi_{\mathcal{N}, R_0}(v) \) is inscribed relatively to \( \mathcal{N} \) for all \( v \in I(\mathcal{N}, R_0) \), since \( v_R(\ell_{\mathcal{N}, R_0, v}) = v_R \). Conversely, if \( \ell \) is inscribed relative to \( \mathcal{N} \), then for some linear function \( l(x) \), we have that \( V(\ell+l) \) is contained in a sphere centered at the origin. It is now easy to see that \( \ell+l = \Xi_{\mathcal{N}, R_0}(v) \) for \( v := v_{R_0}(\ell+l) \). This shows the isomorphism between \( I(\mathcal{N}, R_0) \) and \( I(\mathcal{N}) \).

**Corollary 5.17.** \( I(\mathcal{N}, R_0) \cong I(\mathcal{N}) \) for every complete fan \( \mathcal{N} \) and base region \( R_0 \in \mathcal{N} \).

Using Proposition 5.14, we also see, that:

**Corollary 5.18.** Let \( \mathcal{N} \) be an inscribable fan. Then \( I_+(\mathcal{N}) + (-I_+(\mathcal{N})) = I(\mathcal{N}) \).

Clearly, this result is false, if \( \mathcal{N} \) is not inscribable but virtually inscribable, i.e. \( I_+(\mathcal{N}) = \emptyset \), but \( I(\mathcal{N}) \neq \emptyset \). For example, Proposition 3.1 implies that 2-dimensional complete fans with an odd number of rays always have an inscribed virtual polytope. Figure 3 shows an even 2-dimensional fan all whose inscribed polytopes are virtual.

According to Steinitz, combinatorial types of 3-dimensional polytopes are in bijection with planar and 3-connected graphs; cf. [34]. While there are complete 3-dimensional fans \( \mathcal{N} \) that are not polytopal [34, Example 7.5], Steinitz result shows that \( G(\mathcal{N}) \) is always the dual graph of a polytopal fan. It was shown by Steinitz that there are planar and 3-connected graphs \( G \) that cannot be realized as the graph of an inscribed 3-polytope.

**Question 2.** Is there a planar and 3-connected graph \( G \) that cannot be realized as the graph of an inscribed polytope but as the the graph of an inscribed virtual polytope?

Let \( G \) be a planar and 3-connected graph and \( G' = (V, E) \) be its planar dual. Rivin [27] showed that there is an inscribed 3-polytope \( P \) with graph \( G \) if and only if there is \( \omega : E \to (0, \pi) \) such that for every cycle in \( C \subseteq E \)

\[
\sum_{e \in C} \omega(e) = 2\pi \quad \text{if } C \text{ is non-separating (i.e. bounding a 2-face of } G')
\]

\[
\sum_{e \in C} \omega(e) > 2\pi \quad \text{if } C \text{ is separating.} \quad (10)
\]

Let us call a planar and 3-connected graph \( G \) virtually inscribable if there is a virtually inscribable polytopal fan \( \mathcal{N} \) with \( G(\mathcal{N}) = G \).

**Question 3.** Can Rivin’s characterization (10) be extended to decide if a graph is virtually inscribable?
5.4. Computing the inscribed cone. The reflection game of Section 2 gives a mean to compute \( \mathcal{I}(\mathcal{N}, R_0) \) via (3). However, the process is computationally quite involved. In the remainder of this section, we give a simpler and more elegant description of \( \mathcal{I}(\mathcal{N}) \) utilizing Corollary 5.6.

In order to describe \( \lambda(\mathcal{I}(\mathcal{N})) \), we first consider a single cycle in \( G(\mathcal{N}) \). More generally, let \( A = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{d \times n} \) be an ordered collection of unit vectors. We write \( s_i \) for the reflection in \( \alpha_i^\perp \). We are interested in the question when there is a point \( q_0 \in \mathbb{R}^d \) such that \( s_n s_{n-1} \ldots s_1(q_0) = q_0 \). Let \( q_i := s_i s_{i-1} \ldots s_1(q_0) \), that is,

\[
q_i = s_i(q_{i-1}) = q_{i-1} - 2(\alpha_i, q_{i-1}) \alpha_i.
\]

Furthermore, set \( \lambda_i := -2(\alpha_i, q_{i-1}) \). Then

\[
q_i = q_0 + \sum_{j=1}^{i} \lambda_i \alpha_i
\]

and, in particular,

\[
\lambda_i = -2(\alpha_i, q_0) - 2 \sum_{j=1}^{i-1} (\alpha_i, \alpha_j) \lambda_j.
\]

Note that \( \lambda : \mathbb{R}^d \to \mathbb{R}^n \) with \( q_0 \mapsto (\lambda_i)_{i=1,\ldots,n} \) is a linear map and there is \( q_0 \) with \( q_n = q_0 \) if and only if \( A \lambda(q_0) = 0 \).

Let \( G \) be a Gale transform of \( A \), that is, a matrix \( G \) such that \( \ker G = \text{im} A^\top \). We define the skew Gram matrix of \( \alpha_1, \ldots, \alpha_n \) as the skew-symmetric matrix \( R = R(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n \times n} \) with \( R_{ij} = -R_{ji} = (\alpha_i, \alpha_j) \) for \( i < j \) and \( R_{ii} = 0 \).

**Lemma 5.19.** Let \( A = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{d \times n} \) be an ordered collection of unit vectors. Let \( G \) be a Gale transform for \( A \) and \( R = R(\alpha_1, \ldots, \alpha_n) \) the skew Gram matrix. There is \( q_0 \notin \text{im}(A)^\perp \) with \( s_n \ldots s_1(q_0) = q_0 \) if and only if there is \( \lambda \in \mathbb{R}^n \setminus \{0\} \) with

\[
A \lambda = 0 \quad \text{and} \quad GR\lambda = 0.
\]

**Proof.** Let \( \gamma = A^\top q_0 = (\langle \alpha_i, q_0 \rangle)_{i=1,\ldots,n} \) and denote by \( \delta_{i,j} \) the Kronecker delta. Then \( \lambda = S\gamma \), where \( S = S_0 S_{n-1} \ldots S_1 \) and

\[
(S_k)_{ij} = \begin{cases} 
-2(\alpha_k, \alpha_j) & \text{if } j \leq i = k, \\
\delta_{i,j} & \text{otherwise}.
\end{cases}
\]

In particular, \( \lambda = \lambda(q_0) \) for some \( q_0 \) if and only if \( S^{-1} \lambda \in \text{im} A^\top \) if and only if \( GS^{-1} \lambda = 0 \).

Note that \( S_k \) is a lower triangular matrix with inverse given by

\[
(S_k)^{-1}_{ij} = \begin{cases} 
-\frac{1}{2} & \text{if } j = i = k, \\
-(\alpha_k, \alpha_j) & \text{if } j < i = k, \\
\delta_{i,j} & \text{otherwise}.
\end{cases}
\]

In particular, \( S^{-1} = S_1^{-1} \ldots S_n^{-1} \) is given by

\[
S_{ij}^{-1} = \begin{cases} 
-(\alpha_i, \alpha_j) & \text{if } j < i, \\
-\frac{1}{2} \delta_{i,j} & \text{otherwise}.
\end{cases}
\]
If we consider the canonical decomposition of $S^{-1}$ as a sum of a symmetric and a skew-symmetric matrix, we see
\[ S^{-1} + (S^{-1})^t = -A^tA \quad \text{and} \quad S^{-1} - (S^{-1})^t = R. \]
Since $G$ is a Gale-transform of $A^t$, we have $GS^{-1} = \frac{1}{2}GR$ and this yields the claim. \hfill \Box

We can combine Corollary 5.6 with Proposition 5.7(i) and Lemma 5.19 to give a description of $\mathcal{I}(N)$ and of $\mathcal{I}_+(N)$ as a subcone of $\mathcal{T}_+(N)$ that is effectively computable.

For the purpose of computation, we view $G(N) = (V, E)$ as a directed symmetric graph. More precisely, $G(N)$ is a directed graph with nodes $V$ corresponding to the regions of $N$ and directed edges $E \subseteq V \times V$ corresponding to the walls of $N$, with the property that $RS \in E$ implies $SR \in E$. If $RS \in E$, then $\alpha_{RS}$ is the unit outer normal to $R$. This yields a map $\alpha : E \to S^{d-1}$ into the unit sphere $S^{d-1} \subset \mathbb{R}^d$ with $\alpha_{SR} = -\alpha_{RS}$. The pair $(G(N), \alpha)$ gives an encoding of $N$ that is the input to our algorithm.

If $N = N(P)$ for a convex polytope $P \subset \mathbb{R}^d$, then $G(N)$ can be computed from $P$ by first determining the edge graph of $P$. The regions $R$ are in bijection to the vertices $v_R$ of $P$ and $\alpha_{RS}$ is the unit vector $\frac{v_S - v_R}{\|v_S - v_R\|}$.

Let $\text{Cyc}(N)$ be the directed cycles around codimension-2 cones of $N$. That is, cycles corresponding to 2-faces of $P$. For any such cycle $C = R_1 \ldots R_n$, let $A_C = (\alpha_{R_1, R_2}, \ldots, \alpha_{R_{n-1}, R_n})$.

Further, let $G_C$ and $R_C$ be a Gale transform and skew Gram matrix for $A_C$. Moreover, for every region $R \in V$, let $N(R) = \{S : RS \in E\}$ be the neighbors of $R$ and $G_R$ a Gale transform of $A_R = (\alpha_{RS} : S \in N(R))$.

**Theorem 5.20.** Let $N$ be a complete fan represented by $(G(N), \alpha)$. Then $\mathcal{I}(N)$ is linearly isomorphic to the collection of $\lambda \in \mathbb{R}^{E(N)}$ such that
\begin{align}
A_C \lambda|_C &= 0 \quad \text{for all } C \in \text{Cyc}(N) \quad \text{(11)} \\
G_C R_C \lambda|_C &= 0 \quad \text{for all } C \in \text{Cyc}(N) \quad \text{(12)} \\
G_R \lambda|_{N(R)} &= 0 \quad \text{for all } R \in N \quad \text{(13)}
\end{align}

The inscribed cone $\mathcal{I}_+(N)$ is given by those $\lambda \in \mathbb{R}^{E(N)}$ with $\lambda_{RS} > 0$ for all $RS \in E(N)$.

**Proof.** We work with the based inscribed space $\mathcal{I}(N, R_0)$ for some fixed region $R_0 \in N$. Every element of $\mathcal{I}(N)$ is thus represented by some $v_{R_0} \in \mathbb{R}^d$ and all other $v_R$ are determined by (9) for walks $R_0$ to $R$. This yields a $\lambda : E(N) \to \mathbb{R}$ that satisfies (11) and (13). We also note that $-2A_R^t v_R = \lambda|_{N(R)}$ for all $R$, which shows that (13) is satisfied.

Now let $\lambda$ satisfy the given conditions. For any $R \in N$, $G_R$ is a Gale transform for $A_R$ and $A_R$ has full rank, so there is a unique $v_R$ with $-2\langle \alpha_{RS}, v_R \rangle = \lambda_{RS}$ for all $S \in N(R)$. Fix $RS \in E$ and let $v = s_{RS}(v_R)$. We claim that $v_S = v$. Note that $\langle \alpha_{RS}, v_S \rangle = \langle \alpha_{SR}, v \rangle = -\frac{1}{2} \lambda_{RS}$ by construction.

Let $C \in \text{Cyc}(N)$ be a cycle containing $QRST$ for regions $Q, T \in N$. Let $s_C$ be the composition of reflections along $C$ starting at $R$. By condition (11) and (12), Lemma 5.19 assures us that there is point $u$ such that $u = s_C(u)$. The vectors $A_C$ span a subspace of dimension 2 and $\alpha_{QR}$ and $\alpha_{RS}$ are linearly independent. Therefore, $s_C(u) = u$ if and only if $-2\langle \alpha_{QR}, u \rangle = \lambda_{QR}$ and $-2\langle \alpha_{RS}, u \rangle = \lambda_{RS}$. In particular $s_C(v_R) = v_R$. This implies $\langle \alpha_{ST}, v \rangle = \langle \alpha_{ST}, v_S \rangle = -\frac{1}{2} \lambda_{ST}$.

Every codimension-1 cone is contained in at least $d - 1$ codimension-2 cones. Thus there are...
at least \( d - 1 \) such cycles \( C \) through \( RS \) for which the vectors \( \alpha_{ST} \) are linearly independent. This means that \( v \) is the unique solution to \(-2A_S^Tv = \lambda|_{N(S)}\). \( \square \)

**Remark 5.21.** The result that precedes can also be interpreted in the context of Corollary 4.13: For an ordinary (that is non-virtual) polytope \( P \), conditions (11) and (12) are equivalent to all 2-faces being inscribed, while (11) and (13) are equivalent to the existence of a sphere containing every vertex together with its neighbors.

This, finally, allows us to prove Theorem 1.3.

**Proof of Theorem 1.3.** If \( \mathcal{N} \) is rational, that is, if \( \alpha_{RS} \in \mathbb{Q}^d \) for all \( RS \in E(\mathcal{N}) \), then \( \mathcal{I}_+(\mathcal{N}) \subset \mathcal{T}_+(-\mathcal{N}) \) is a rational cone and hence contains rational points in the relative interior, provided it is non-empty. This shows the second statement.

For the first statement, let \( \mathcal{N} \) be a (polytopal) fan in \( \mathbb{R}^d \) with \( n \) regions. The linear equations of type (13) is of order \( d \) times the number of walls of \( \mathcal{N} \) and is of order \( dn^2 \). If \( C \in \text{Cyc}(\mathcal{N}) \) is a cycle of length \( m < n \), then the linear equations of type (11) and (12) are of order \( m^2 \). There are at most \( \binom{n}{3} \) cycles. Computing a point in \( \mathcal{I}_+(\mathcal{N}) \) is therefore equivalent to finding a solution to a system of polynomially many linear equations and strict inequalities. The bit complexity of the system is polynomial in \( (G(\mathcal{N}), \alpha) \) and the ellipsoid method determines in polynomial time if a solution exists; cf. [12]. This completes the proof of Theorem 1.3. \( \square \)

### 5.5. Non-inscribable fans and inscribable coarsenings.

If \( \mathcal{N} \) is not polytopal, that is, not the normal fan of a polytope, then there is a unique finest coarsening \( \mathcal{N}_{pc} \) of \( \mathcal{N} \) such that \( \mathcal{N}_{pc} \) is polytopal. A representative is given by taking the Minkowski sum of polytopes for all possible polytopal coarsenings of \( \mathcal{N} \). A different way to see is, is to note that by Corollary 5.6,

\[
\mathcal{T}_+(\mathcal{N}_{pc}) \cong \mathbb{R}_{\geq 0}^{E(\mathcal{N})} \cap \{ \lambda \in \mathbb{R}^{E(\mathcal{N})} : \lambda \text{ satisfies (8) for all closed walks} \},
\]

where the last set is isomorphic to \( \mathcal{T}(\mathcal{N}) \). This embedding of the type space \( \mathcal{T}(\mathcal{N}) \hookrightarrow \mathbb{R}^{E(\mathcal{N})} \) meets the cone \( \mathbb{R}_{\geq 0}^{E(\mathcal{N})} \) in the relative interior of some inclusion-maximal face and the intersection is precisely the closed type cone of \( \mathcal{N}_{pc} \).

Analogously, there is a **canonical inscribable coarsening** (cic) of \( \mathcal{N} \) given by the fan \( \mathcal{N}_{cic} \) corresponding to the largest face of \( \mathcal{T}_+(\mathcal{N}_{pc}) \) meeting the inscribed space \( \mathcal{I}(\mathcal{N}) \). If \( \mathcal{N} \) is inscribed, then \( \mathcal{N}_{cic} = \mathcal{N}_{pc} = \mathcal{N} \). It follows from Example 5.8 that \( \mathcal{N}_{cic} \) is not the common refinement of all inscribable coarsenings of \( \mathcal{N} \), but the common refinement of all relatively inscribable coarsenings of \( \mathcal{N} \). In particular, \( \mathcal{N}_{cic} = \{ \mathbb{R}^d \} \) is possible even if \( \mathcal{N} \) has inscribable coarsenings. It is clear that

\[
\mathcal{T}_+(\mathcal{N}_{pc}) \cap \mathcal{I}(\mathcal{N}) \subseteq \mathcal{I}_+(\mathcal{N}_{cic}),
\]

but the following example shows that the inclusion can be strict in general.

**Example 5.22.** Let \( \mathcal{N} \) be a 2-dimensional fan with 5 rays. If no rays are antipodal, then \( \mathcal{T}_+(\mathcal{N}) \) is a 3-dimensional cone with 5 rays, i.e., the cone over a (different) pentagon. The precise combinatorics, i.e., which cones can be merged in a coarsening, depend on the geometry of \( \mathcal{N} \).

For example, let \( \mathcal{N} \) be a 2-dimensional fan with profile \( \beta(\mathcal{N}) = (\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}) \). A cross section of \( \mathcal{T}_+(\mathcal{N}) \) is shown in Figure 11. The vertices of the cross section are labelled by
the corresponding indecomposable coarsenings of \( \mathcal{N} \). All of these indecomposable coarsenings correspond to triangles and are therefore inscribable.

By Proposition 3.1 and Theorem 3.5, \( \mathcal{I}(\mathcal{N}) \) is a 1-dimensional subspace that does not meet the interior of \( \overline{\mathcal{T}}_+(\mathcal{N}) \). In fact, \( \mathcal{I}(\mathcal{N}) \) meets the relative interior of a 2-dimensional face \( F \subseteq \overline{\mathcal{T}}_+(\mathcal{N}) \). The corresponding fan \( \mathcal{N}_{\text{cic}} \) has 4 rays and it follows from Example 3.3 that \( \mathcal{I}_+(\mathcal{N}_{\text{cic}}) = F \). Thus, \( \mathcal{I}(\mathcal{N}) \cap \overline{\mathcal{T}}_+(\mathcal{N}) \subseteq \mathcal{I}_+(\mathcal{N}_{\text{cic}}) \). This situation is depicted in Figure 11.

6. Routed trajectories and reflection groupoids

The constructions in the Sections 2 and 5 prompt two related generalizations, namely routed trajectories and reflection groupoids, which may provide a broader perspective on normally inscribed polytopes.

6.1. Routed trajectories. Let \( B^d \subset \mathbb{R}^d \) be the \( d \)-dimensional unit ball and consider a particle that starts from a point \( t_0 \in \partial B^d = S^{d-1} \) along a straight line trajectory. Upon collision with the boundary at the point \( t_1 \in S^{d-1} \), the particle again takes off in a random direction and continues to produce a trajectory \( T = (t_0, t_1, \ldots, t_k) \). We will assume that the trajectory is closed and hence \( t_k = t_0 \). We define the route of the trajectory \( T \) as \( \alpha(T) = (\alpha_0, \ldots, \alpha_{k-1}) \), where \( \alpha_i = \mathbb{R}(t_{i+1} - t_i) \) is the line along which the particle moves. We will be interested in the space of all trajectories with the same route.

More generally, let \( G = (V, E) \) be a simple connected graph and let \( \alpha : E \rightarrow \mathbb{P}^{d-1} \), where \( \mathbb{P}^{d-1} \) is the space of lines through the origin in \( \mathbb{R}^d \). We call \( (G, \alpha) \) a routing scheme. A trajectory for \( (G, \alpha) \) is a map \( T : V \rightarrow S^{d-1} \) such that if \( C = v_0, v_1, \ldots, v_k \) is a cycle in \( G \), then \( T|_C = (T(v_0), \ldots, T(v_k)) \) is a trajectory with route \( \alpha|_C = (\alpha(v_0v_1), \ldots, \alpha(v_{k-1}v_k)) \).

Thus, we may interpret \( G \) as the state space of a particle with \( \alpha \) restricting the admissible directions for each state. The trajectory space is then

\[
S(G, \alpha) := \{ T : V \rightarrow S^{d-1} : T \text{ is a trajectory for } (G, \alpha) \}.
\]
Theorem 6.1. Let \((G, \alpha)\) be a routing scheme. Then \(S(G, \alpha)\) is homeomorphic to a subsphere \(S^{d-1} \cap U\), where \(U\) is a linear subspace.

The crucial observation is again that if \(uv \in E\), then \(T(v)\) is the reflection of \(T(u)\) in the hyperplane \(\alpha(uv)^{-1}\). Hence \(T\) is determined by \(T(v_0)\) for an arbitrary but fixed \(v_0 \in V\). Moreover, let us write \(s_{uv}\) for the reflection in \(\alpha(uv)^{-1}\). Then \(t_0 := T(v_0)\) has to satisfy
\[
s_{v_0 v_{k-1} \cdots s_{v_1 v_0}}(t_0) = t_0.
\]
for all closed walks \(W = v_0 v_1 \ldots v_k\) starting at \(v_0\). As in Section 2, we note that the collection of points \(t_0 \in \mathbb{R}^d\) satisfying these conditions yield a linear subspace \(U \subset \mathbb{R}^d\) and hence \(S(G, \alpha)\) is homeomorphic to \(S^{d-1} \cap U\).

Every full-dimensional and strongly connected fan \(\mathcal{N}\) naturally determines a routing scheme \((G, \alpha)\). The results of Sections 2 and 5 can now be interpreted as follows.

Corollary 6.2. Let \(\mathcal{N}\) be a full-dimensional and strongly connected fan in \(\mathbb{R}^d\) with associated routing scheme \((G, \alpha)\). Then inscribed virtual polytopes with underlying fan \(\mathcal{N}\) are in correspondence to trajectories for \((G, \alpha)\).

To see the special role played by the inscribed polytopes \(\mathcal{I}_+ (\mathcal{N})\), observe that a route only determines a line along the particle has to move but not a direction. Defining \(\alpha\) to take values in the space of oriented lines \(\mathbb{P}^{d-1}_+\), yields a correspondence to \(\mathcal{I}_+ (\mathcal{N})\).

6.2. Reflection groupoids. Category theory gives a natural algebraic formalism for routing schemes. Recall that a groupoid is a (small) category \(\mathfrak{G}\) in which every morphism is invertible; see [16, I.5]. Let \(G = (V, E)\) be a graph and \(\alpha : E \to \mathbb{P}^{d-1}\). As before, we denote by \(s_e\) the reflection in \(\alpha_e^{-1}\). We can now construct a groupoid \(\mathfrak{G} = \mathfrak{G}(G, \alpha)\) with objects \(\text{Ob}_{\mathfrak{G}} = V\) and morphisms
\[
\text{hom}_{\mathfrak{G}}(u, v) := \{t_W : W \text{ walk from } u \text{ to } v\}
\]
for \(u, v \in V\), where the composition of morphisms is inherited from \(O(d)\). If \(W^{-1}\) is the reverse walk, then clearly \(t_W^{-1} = t_{W^{-1}}\). As \(\mathfrak{G}(G, \alpha)\) is generated by reflections, we call \(\mathfrak{G}(G, \alpha)\) a reflection groupoid. A full-dimensional fan \(\mathcal{N}\) defines a reflection groupoid \(\mathfrak{G}(\mathcal{N}) := (G(\mathcal{N}), \alpha)\) with \(\alpha(RS) = (R \cap S)^{-1}\) and we set \(\mathfrak{G}(P) := \mathfrak{G}(\mathcal{I}(P))\) for a polytope \(P\).

For any \(v \in V\), the set \(\text{hom}_{\mathfrak{G}}(v) := \text{hom}_{\mathfrak{G}}(v, v) = \{t_W : W \text{ closed walk based at } v\}\) is a group. If \(G\) is connected, then \(\mathfrak{G}\) is a connected groupoid, that is, \(\text{hom}_{\mathfrak{G}}(u, v) \neq \emptyset\) for all \(u, v \in V\). In this case, it is well-known that \(\mathfrak{G}\) is determined up to isomorphism by \(\text{hom}_{\mathfrak{G}}(v)\) for any fixed \(v \in V\) and the set \(V\). Indeed, for any \(w \in V\) and a (non-canonical) choice \(h_w \in \text{hom}_{\mathfrak{G}}(v, w)\), we have for any \(x, y \in V\)
\[
\text{hom}_{\mathfrak{G}}(x, y) = \{h_y g h_x^{-1} : g \in \text{hom}_{\mathfrak{G}}(v)\}.
\]
In particular \(\text{hom}_{\mathfrak{G}}(u) = h_u \text{hom}_{\mathfrak{G}}(u)h_u^{-1}\) for all \(u \in V\).

The following makes an interesting connection to virtual inscribability.

Theorem 6.3. Let \(\mathcal{N}\) be a virtually inscribable fan and \(\mathfrak{G} = \mathfrak{G}(\mathcal{N})\) its reflection groupoid. Then the group \(\text{hom}_{\mathfrak{G}}(R) \subset O(d)\) is generated by reflections for any region \(R \in \mathcal{N}\).

Proof. As we stated above, \(\text{hom}_{\mathfrak{G}}(R)\) is generated by elements of the form \(t_W^{-1} t_{W'} t_W\), where \(W'\) is a closed walk and \(W\) is a path from \(R\) to the start point of \(W'\). Clearly, it suffices to
only consider closed walks which form a cycle bases, and so we can choose \( \mathcal{W}' \) to be a closed path around a cone \( C \) of codimension 2. We call \( C \) odd/even if \( \mathcal{W}' \) has odd/even length.

If \( \mathcal{N} \) is virtually inscribable, then the localization \( \mathcal{N}_C \) is virtually inscribable. Since \( \mathcal{N}_C \) is essentially a 2-dimensional fan, it follows from Proposition 3.1 that \( t_{\mathcal{W}'} \) is the identity if \( C \) is even and \( t_C \) is a reflection if \( C \) is odd. Hence \( \text{hom}_R(R) \) is generated by the reflections \( t_{\mathcal{W}'}^{-1}t_{\mathcal{W}'}t_{\mathcal{W}'} \), for any walk \( \mathcal{W}' \) around an odd codimension-2 cone.

**Corollary 6.4.** If \( \mathcal{N} \) is a virtually inscribable and even fan, then \( \text{hom}_\mathcal{R}(\mathcal{N})(R) \) is trivial for all regions \( R \in \mathcal{N} \). In particular \( \text{hom}_\mathcal{R}(\mathcal{N})(R, S) = \{s_{RS}\} \) for all adjacent regions \( R, S \in \mathcal{N} \).

Notice that the group \( \text{hom}_\mathcal{R}(\mathcal{N})(R) \) is typically infinite, even when it is generated by reflections. However, \( \text{hom}_\mathcal{R}(v) \) is trivially a subgroup of the (possibly infinite) reflection group \( \mathcal{W}(\mathcal{G}) \) generated by \( e \) for \( e \in E \).

**Corollary 6.5.** If \( P \) is an inscribed generalized permutahedron, then \( \text{hom}_\mathcal{R}(P)(v) \) is a finite reflection group for any \( v \in \text{Ob}_\mathcal{R}(P) \).

Let us look at some examples.

**Example 6.6 (Simplices).** Let \( S_d \) be the \( d \)-dimensional simplex with vertices \( 0, e_1, \ldots, e_d \). The group \( \mathcal{W}(\mathcal{G}(S_d)) \) is generated by reflections normal to \( e_i \) and \( e_i - e_j \) for \( i < j \) and hence is the finite reflection group of type \( B \). Consider the vertex \( v = 0 \). Walks around 2-faces containing 0 give rise to reflections in \( e_i - e_j \) for \( i < j \) and since these walks form a cycle basis we have that \( \text{hom}_{\mathcal{G}(S_d)}(v) \) is in fact the finite reflection group of type \( A_{d-1} \).

**Example 6.7 (Crosspolytopes).** Let \( K_d = \text{conv}(\pm e_1, \ldots, \pm e_d) \) be the \( d \)-dimensional cross-polytope. The group \( \mathcal{W}(\mathcal{G}(K_d)) \) is generated by reflections normal to \( e_i \) and \( e_i + e_j \) for \( 1 \leq i < j \leq d \) and hence is the finite reflection group of type \( D_d \). The 2-dimensional faces are of the form \( \text{conv}(\pm e_i, \pm e_j, \pm e_k) \) for distinct \( i, j, k \in [d] \). We denote the vertices by \( \pm i \) and the 2-faces by \( \pm (i, j, k) \).

Starting in the vertex \( i \), the 2-faces \( (i, j, k) \) and \( (i, -j, -k) \) give rise to a reflection in \( \{x_k = x_j\} \). Similarly, the 2-faces \( (i, -j, k) \) and \( (i, j, -k) \) yield a reflection in \( \{x_k = -x_j\} \). For \( v = e_1 \), the group \( \text{hom}_{\mathcal{G}(K_d)}(v) \) is generated by the reflections in the hyperplanes \( \{x_k = \pm x_j\} \) for \( 1 < j < k \leq d \). This is group is isomorphic to \( D_{d-1} \) fixing the line through \( e_1 \).

The group \( \text{hom}_{\mathcal{G}(G, \alpha)}(v) \) encodes the discrete holonomy of the routing scheme \( (G, \alpha) \). For a fixed \( v \in V \), we may associate a reference frame to the starting point \( T(v) \). Reflecting the reference frame along a cycle in \( G \) yields the holonomy of the trajectory.

In the stated generality, \( \text{hom}_{\mathcal{G}(G, \alpha)}(v) \) generalizes the groups of projectivities studied in [15]. Let \( \Delta \subseteq 2^V \) be a simplicial complex on a finite ground set \( V \). That is, \( \Delta \neq \emptyset \) and for every \( \tau \subseteq \sigma \in \Delta \), we have \( \tau \in \Delta \). The **dimension** of \( \Delta \) is \( (d-1) \)-dimensional if \( d = \max( |\sigma| : \sigma \in \Delta \) and \( \Delta \) is **pure** if every inclusion-maximal \( \sigma \in \Delta \) is of cardinality \( d \). Inclusion-maximal sets are called **facets** and two facets \( \sigma, \sigma' \in \Delta \) are **adjacent** if \( |\sigma \cap \sigma'| = d - 1 \). The dual graph \( G(\Delta) \) of a pure complex \( \Delta \) has nodes given by the facets of \( \Delta \) and \( \sigma \sigma' \in E(\Delta) \) if \( \sigma, \sigma' \) are adjacent. Finally \( \Delta \) is called **strongly connected** if \( G(\Delta) \) is connected. If \( e = (\sigma, \sigma') \in E(\Delta) \), then \( \sigma \setminus \sigma' = \{v\} \) and \( \sigma' \setminus \sigma = \{v'\} \). Following [15], this defines a **perspective** \( s_e : \sigma \to \sigma' \) with \( s_e(v) := v' \) and the identity on \( \sigma \cap \sigma' \). In particular, if \( \mathcal{W} = \sigma_0 \sigma_1 \cdots \sigma_k \) is a walk in \( G(\Delta) \),
then this yields a bijection \( t_W : \sigma_0 \rightarrow \sigma_k \). Thus, for fixed \( \sigma_0 \), this defines the group of projectivities

\[
\Pi(\Delta, \sigma_0) := \{ t_W : W \text{ closed walk based at } \sigma_0 \} \subseteq \text{Bij}(\sigma_0)
\]

For a strongly connected simplicial complex \( \Delta \subseteq 2^V \) let \( \mathbb{R}^V \) be the vector space with basis \( e_v \) for \( v \in V \). Let \( G = G(\Delta) \) and define \( \alpha : E(G) \rightarrow \mathbb{P}(\mathbb{R}^V) \) by \( \alpha(\sigma\sigma') := \mathbb{R}(e_v - e_{v'}) \).

**Proposition 6.8.** Let \( \Delta \) be a strongly connected simplicial complex and \( \sigma_0 \in \Delta \) a facet. Let \( (G, \alpha) \) be the routing scheme defined above. Then \( \text{hom}_{G,G(\Delta)}(\sigma_0) \subset \text{GL}(\mathbb{R}^V) \) is the permutation representation of \( \Pi(\Delta, \sigma_0) \).

It would be interesting to further explore the connections to discrete holonomy groups. Arrangements of linear hyperplanes induce fans that are polytopal. The associated reflection groupoids show even stronger similarities to groups of projectivities. In the sequel to this paper, we investigate arrangements whose fans are inscribable and their connection to reflection groups.

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