A convex solution to Psiaki’s first joint attitude and spin-rate estimation problem

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Abstract

We consider the problem of jointly estimating the attitude and spin-rate of a spinning spacecraft. Psiaki (J. Astronautical Sci., 57(1-2):73–92, 2009) has formulated a family of optimization problems that generalize the classical least-squares attitude estimation problem, known as Wahba’s problem, to the case of a spinning spacecraft. If the rotation axis is fixed and known, but the spin-rate is unknown (such as for nutation-damped spin-stabilized spacecraft) we show that Psiaki’s problem can be reformulated exactly as a type of tractable convex optimization problem called a semidefinite optimization problem. This reformulation allows us to globally solve the problem using standard numerical routines for semidefinite optimization. It also provides a natural semidefinite relaxation-based approach to more complicated variations on the problem.

1 Introduction

Spacecraft attitude estimation is a fundamental problem, arising, for instance, as a natural sub-problem whenever attitude control is required. Since spacecraft dynamics are non-linear, a typical and successful approach to attitude estimation is to employ variants of the Extended Kalman Filter (EKF) [14]. As with any method based on linearization of non-linear dynamics, EKF-based approaches can fail to converge given poor initial estimates, and can become unstable in the presence of large disturbances [19]. Many truly non-linear attitude estimation methods have also been proposed (see [5] for a survey). An important example is the static least-squares attitude estimation problem known as Wahba’s problem [29]. In Wahba’s problem we are simultaneously given a batch of vector measurements (from sun sensors, star trackers, etc.) in the body frame and corresponding reference directions in an inertial frame. The aim is to find the rotation matrix (i.e. direction cosine matrix) that minimizes the sum of the squared errors between the transformed reference directions and the observed vector measurements. Wahba’s problem, as stated, applies most naturally to a static spacecraft. Nevertheless, it has also found use as a subroutine in various recursive estimation algorithms including those that estimate the full dynamical state of the spacecraft (see, e.g., [19, 8]).

Recently Psiaki has posed a number of generalizations of Wahba’s problem to the case of a spinning spacecraft [20]. These problems aim to simultaneously estimate the initial attitude and spin-rate (or, more generally, initial angular momentum) of the spacecraft from vector measurements, without the need for gyroscope measurements. These generalizations are particularly suited

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to spin-stabilized spacecraft without gyroscopes. We describe Wahba’s problem and Psiaki’s generalizations formally in Section 2.

In this paper we focus on the simplest of Psiaki’s generalizations of Wahba’s problem. We refer to this problem as *Psiaki’s first problem*. In this problem we assume the spacecraft is spinning at a constant unknown angular velocity around a known (stable) inertia axis. This setting is relevant for nutation-damped spin-stabilized spacecraft [20]. The aim is to estimate the initial attitude and the unknown spin-rate given a sequence of noisy vector measurements obtained at certain sampling instants, together with corresponding reference directions. Wahba’s problem arises as the special case where the spin-rate is zero.

1.1 Main contribution

Our main contribution is to show that, when the sampling period is constant, Psiaki’s first problem can be reformulated exactly as a semidefinite optimization problem (see Theorem 3.2). Semidefinite optimization problems (described in Section 3) are a family of convex optimization problems that generalize linear programming and can be solved globally with provable efficiency guarantees using standard software. Reformulating Psiaki’s first problem as a semidefinite optimization problem means that it, like Wahba’s problem, can be solved efficiently and globally, to high precision, using numerical methods.

A description of Psiaki’s first problem as the solution to a semidefinite optimization problem allows us to do more than just solve the original problem as stated. It also allows us to take a semidefinite relaxation-based approach to many variants on Psiaki’s problem. We illustrate this in Section 4 by considering the example of a version of Psiaki’s first problem where explicit bounds on the measurement errors are incorporated into the formulation.

1.2 Organization of the paper

The remainder of the paper is organized as follows. In Section 1.3 we summarize notation not defined elsewhere in the paper. In Section 2 we first describe Psiaki’s generalizations of Wahba’s problem for spinning spacecraft. We then show how to write Psiaki’s first problem as an instance of a family of problems we call *trigonometric Wahba problems* (see (6)). We conclude the section with a summary of prior work on Psiaki’s problems. In Section 3 we briefly describe semidefinite optimization problems in general before presenting our semidefinite optimization-based reformulation of trigonometric Wahba problems, and in particular of Psiaki’s first problem. We defer the proofs to the Appendix. In Section 4 we describe a variant on Psiaki’s first problem that incorporates additional bounds on the measurement noise (if they are available) and show how to extend our semidefinite optimization-based reformulation of Psiaki’s first problem to a semidefinite relaxation of this variant. We also describe the results of a simple numerical experiment comparing Psiaki’s first problem and this variant. In Section 5 we discuss possible future research related to the work in this paper.

1.3 Notation

We briefly summarize notation used throughout the body of the paper. Additional notation that is used only in the Appendix is introduced separately there.

**Spaces** Denote by $\mathbb{R}^{n \times n}$ the space of $n \times n$ real matrices. If $X \in \mathbb{R}^{n \times n}$ let $X^T$ be its transpose. Let $\mathcal{S}^n$ be the space of $n \times n$ symmetric matrices (i.e. matrices for which $X = X^T$). Let $\mathcal{S}^n_+$ denote
Similarly if $S$ is the set of all convex combinations of elements of $S$. From the point of view of optimization, if $c \in \mathbb{R}^n$ and $S$ is compact then
\[
\max_{x \in S} \langle c, x \rangle = \max_{x \in \text{conv}(S)} \langle c, x \rangle
\]
so the optimal cost is the same whether we optimize the linear functional defined by $c$ over $S$ or over its convex hull [22, Theorem 32.2].

**Block matrices** If $T_0, T_1, \ldots, T_N$ are $d \times d$ matrices with $T_0$ being symmetric, define the corresponding $d(N+1) \times d(N+1)$ symmetric block Toeplitz matrix by
\[
\text{Toeplitz}(T_0, T_1, \ldots, T_N) = \begin{bmatrix} T_0 & T_1 & T_2 & \cdots & T_N \\ T_1^T & T_0 & T_1 & \cdots & \vdots \\ T_2^T & T_1 & T_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ T_N^T & \cdots & T_1^T & T_0 \end{bmatrix}. \tag{1}
\]

Similarly if $S_1, S_2, \ldots, S_{2N+1}$ are symmetric $d \times d$ matrices define the corresponding $d(N+1) \times d(N+1)$ block Hankel matrix by
\[
\text{Hankel}(S_1, S_2, \ldots, S_{2N+1}) = \begin{bmatrix} S_1 & S_2 & \cdots & S_N & S_{N+1} \\ S_2 & S_{N+1} & S_{N+2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ S_N & S_{N+1} & S_{2N} & \cdots & S_{2N+1} \\ S_{N+1} & S_{N+2} & \cdots & S_{2N} & S_{2N+1} \end{bmatrix}. \tag{2}
\]

**Unit quaternion parameterization of rotations** We make extensive use of the quadratic parameterization of $SO(3)$, the set of rotation (or direction-cosine) matrices, by unit quaternions, denoted by $\mathbb{H}$. Throughout we think of the unit quaternions geometrically as the unit sphere in $\mathbb{R}^4$ i.e. $\mathbb{H} = \{ q \in \mathbb{R}^4 : \|q\| = 1 \}$. We only ever work with a unit quaternion $q \in \mathbb{H}$ via the positive semidefinite matrix $qq^T$, avoiding the sign ambiguity that would arise if we were to try to work directly with variables $q \in \mathbb{H}$. It is enough only to consider $qq^T$ because any element of $SO(3)$ can be expressed as $A(qq^T)$ where $q \in \mathbb{H}$ and $A : \mathbb{S}^4 \rightarrow \mathbb{R}^{3 \times 3}$ is the linear map defined (following the convention in [5]) by
\[
A(Z) := \begin{bmatrix} Z_{11} - Z_{22} - Z_{33} + Z_{44} & 2Z_{12} + 2Z_{34} & 2Z_{13} - 2Z_{24} \\ 2Z_{12} - 2Z_{34} & -Z_{11} + Z_{22} - Z_{33} + Z_{44} & 2Z_{23} + 2Z_{14} \\ 2Z_{13} + 2Z_{24} & 2Z_{23} - 2Z_{14} & -Z_{11} - Z_{22} + Z_{33} + Z_{44} \end{bmatrix}. \tag{3}
\]
The adjoint of $A$ (with respect to the inner product on matrices) is $A^* : \mathbb{R}^{3 \times 3} \to \mathbb{S}^4$ defined by

$$A^*(Y) := \begin{bmatrix}
Y_{11} - Y_{22} - Y_{33} & Y_{12} + Y_{21} & Y_{13} + Y_{31} & Y_{23} - Y_{32} \\
Y_{12} + Y_{21} & -Y_{11} + Y_{22} - Y_{33} & Y_{23} + Y_{32} & -Y_{13} + Y_{31} \\
Y_{13} + Y_{31} & -Y_{23} + Y_{32} & Y_{11} - Y_{22} + Y_{33} & Y_{12} - Y_{21} \\
Y_{23} - Y_{32} & Y_{13} + Y_{31} & Y_{12} - Y_{21} & Y_{11} + Y_{22} + Y_{33}
\end{bmatrix}.$$  

(4)

In other words for any $Z \in \mathbb{S}^4$ and any $Y \in \mathbb{R}^{3 \times 3}$, we have the identity

$$\langle A(Z), Y \rangle = \langle Z, A^*(Y) \rangle.$$  

(5)

2 Psiaki’s generalizations of Wahba’s problem for spinning spacecraft

In this section we describe Wahba’s problem [29] and Psiaki’s generalizations to the case of a spinning spacecraft [20]. For reasons discussed in Section 2.2 we subsequently focus on the simplest of Psiaki’s problems: jointly estimating the attitude and spin-rate of a spacecraft spinning around a stable inertia axis at a constant unknown rate. In this case we show how to reformulate the resulting optimization problem in the general form

$$\max_{Q \in SO(3)} \langle A_0, Q \rangle + \sum_{n=1}^{N} \left[ \langle A_n, \cos(\omega n)Q \rangle + \langle B_n, \sin(\omega n)Q \rangle \right]$$  

(6)

for appropriate collections of $3 \times 3$ matrices $(A_n)^N_{n=0}$ and $(B_n)^N_{n=1}$. Throughout, we call problems in the form (6) trigonometric Wahba problems. In Section 3 to follow, we show how to reformulate trigonometric Wahba problems as semidefinite optimization problems.

2.1 Wahba’s problem

We briefly describe Wahba’s least squares attitude estimation problem posed in [29] with solutions published in [6].

Vector measurements Suppose we are given a batch of noisy unit vector measurements $y_0, y_1, \ldots, y_N$ in the body frame (obtained from star trackers, sun sensors, magnetometers, etc.) of corresponding unit reference directions $x_0, x_1, \ldots, x_N$ in the inertial frame.

Least squares objective Wahba’s problem is to find the rotation matrix $Q \in SO(3)$ that transforms the reference directions to best fit the measured vector measurements in the weighted least squares sense by solving

$$\min_{Q \in SO(3)} \sum_{n=0}^{N} \kappa_n \frac{1}{2} \| y_n - Q x_n \|^2$$  

(7)

where $\kappa_0, \kappa_1, \ldots, \kappa_N$ are non-negative scalar weights that one would take to be larger for measurements with smaller noise variance. Since $\|Qx\|^2 = \|x\|^2$ for all $x \in \mathbb{R}^3$ we can expand the squares and see that this optimization problem is equivalent to

$$\max_{Q \in SO(3)} \left\{ \sum_{n=0}^{N} \kappa_n y_n x_n^T Q \right\}$$  

(8)

where we have dropped an additive constant of $\sum_{n=0}^{N} \frac{\kappa_n}{2} (\|y_n\|^2 + \|x_n\|^2)$. 

4
2.2 Psiaki’s generalizations

We now describe Psiaki’s generalizations of Wahba’s problem, and show how Wahba’s problem arises as a special case.

Rigid body (Euler) equations Let $Q(t_0) \in SO(3)$ denote the initial attitude of the spacecraft, $\Omega(t_0) \in \mathbb{R}^3$ the initial body angular velocity, and $I_1 \geq I_2 \geq I_3$ the principal moments of inertia. Assuming the spacecraft undergoes torque-free motion about its centre of mass then for $t \geq t_0$ the attitude $Q(t)$ and the body angular velocity $\Omega(t) := \begin{bmatrix} \omega_1(t) & \omega_2(t) & \omega_3(t) \end{bmatrix}^T$ satisfy the rigid body equations:

\begin{align*}
I_1 \dot{\omega}_1(t) &= (I_2 - I_3) \omega_2(t) \omega_3(t) \\
I_2 \dot{\omega}_2(t) &= (I_3 - I_1) \omega_3(t) \omega_1(t) \\
I_3 \dot{\omega}_3(t) &= (I_1 - I_2) \omega_1(t) \omega_2(t)
\end{align*}

and $Q(t)$.

Note that for every $t \geq t_0$ and every $\Omega(t_0)$ we have that $Q(t) = \Phi(t - t_0; \Omega(t_0))Q(t_0)$ for some map $\Phi$ taking values in $SO(3)$. In particular $Q(t)$ is always linear in the initial attitude $Q(t_0)$.

Vector measurements Let $t_0, t_1, \ldots, t_N$ be a finite set of sampling instants. Assume, at sample instant $t_n$, that we are given a noisy unit vector measurement $y_n$ in the spacecraft body frame of a corresponding reference directions $x_n$ in the inertial frame.

Least squares objective Following Wahba’s least-squares-based objective, Psiaki suggests solving the following weighted least-squares problem to estimate the initial attitude and body angular velocity of the spacecraft, given only the vector measurements $(y_n)^N_{n=0}$ and the reference directions $(x_n)^N_{n=0}$:

$$
\min_{Q(t_0), \Omega(t_0)} \sum_{n=0}^N \frac{\kappa_n}{2} \|y_n - Q(t_n)x_n\|^2_2
$$

subject to $Q(t)$ satisfying (9) with initial conditions $Q(t_0)$ and $\Omega(t_0)$. Just as for Wahba’s problem, the $\kappa_n$ are non-negative scalars.

Dependence on $\Omega(t_0)$ In general, the dependence of $Q(t)$ on the initial body angular velocity $\Omega(t_0)$ is quite complicated. The relationship between $Q(t)$ and $\Omega(t_0)$ simplifies under additional assumptions on $\Omega(t_0)$ and the inertia tensor of the spacecraft. We now summarize these simplified problems and name them for later reference.

Wahba’s problem If $\Omega(t_0) = 0$, then $Q(t) = Q(t_0)$ for all $t \geq t_0$ and so the spacecraft is stationary. Adding this as a constraint we recover Wahba’s original formulation (7).

Psiaki’s first problem Suppose $\Omega(t_0)$ is aligned with the major inertia axis, and (without loss of generality) this is the first axis direction in body coordinates. Then $\Omega(t_0) = \begin{bmatrix} \omega & 0 & 0 \end{bmatrix}^T$ and so the dynamical constraints (9) reduce to

$$
Q(t) = \begin{bmatrix} 1 & 0 & 0 \\
0 & \cos(\omega t) & -\sin(\omega t) \\
0 & \sin(\omega t) & \cos(\omega t) \end{bmatrix} Q(t_0)
$$

(11)
where $\omega$ is the spin-rate (in rad/second). In this case the spacecraft is spinning with an unknown constant angular velocity $\omega$ around a known axis (fixed in body coordinates). Minimizing the least-squares objective (10) subject to the constraints (11) is the first generalization of Wahba’s problem posed in [20], and is relevant for a nutation damped spin-stabilized spacecraft.

**Psiaki’s second problem** If $\Omega(t_0)$ is unconstrained and no additional assumptions are made about the moments of inertia of the spacecraft, we obtain the second generalization of Wahba’s problem posed in [20]. In this setting the dependence of $Q(t)$ on $\Omega(t_0)$ is more complicated. This case is discussed further in [21] (see Section 2.3 to follow).

In each case, Psiaki’s formulations involve solving non-convex optimization problems of the form in (10) subject to dynamical constraints.

**Focus of the paper** For the remainder of the paper we focus on Psiaki’s first problem, because in this case $Q(t)$ only depends on the initial body angular velocity through $\cos(\omega t)$ and $\sin(\omega t)$. In addition to focusing on Psiaki’s first problem, we also assume that the sampling instants $t_0, t_1, \ldots, t_N$ are equally spaced. As such we assume there is some $\tau$ such that $t_n = n\tau$ for $n = 0, 1, \ldots, N$.

This paper does not address Psiaki’s more general second problem, where the dependence of $Q(t)$ on $\Omega(t_0)$ is significantly more complicated. It would be very interesting if the techniques we develop can be extended to this more general situation.

**Aliasing** Since we only observe $\omega$ via vector measurements at time instants that are integer multiples of $\tau$, from the data alone we cannot distinguish between spin rates at different integer multiples of $2\pi/\tau$ due to aliasing. Hence we assume that $\omega \in [-\pi/\tau, \pi/\tau]$ so that it is possible to determine the unknown spin-rate from the data. (We could, alternatively, fix some $a$ rad/second and assume $\omega \in [a, a + 2\pi/\tau]$.) In a Bayesian formulation of the problem, we could interpret this as encoding prior information on the spin rate.

**Reformulation** We now reformulate Psiaki’s first problem as a trigonometric Wahba problem. Since $\|Q(t)x_n\|^2 = \|x_n\|^2$ for all $t$ and $n$, observe that with $t_n = n\tau$ the optimization problem (10) can be rewritten as

$$\min_{Q(0) \in SO(3)} \sum_{n=0}^{N-1} \frac{\kappa_n}{2} \left[\|y_n\|^2 - 2\langle y_n, Q(n\tau)x_n \rangle + \|x_n\|^2\right]$$

subject to

$$Q(n\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(n\tau \omega) & -\sin(n\tau \omega) \\ 0 & \sin(n\tau \omega) & \cos(n\tau \omega) \end{bmatrix} Q(0).$$

Putting $\omega' = \tau \omega$, we see that this is equivalent, as an optimization problem, to

$$\max_{Q \in SO(3)} \sum_{n=1}^{N} \left[\langle A_n, \cos(n\omega')Q \rangle + \langle B_n, \sin(n\omega')Q \rangle\right]$$

where

$$A_0 = \kappa_0 y_0 x_0^T + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\sum_{n=1}^{N} \kappa_n y_n x_n^T\right).$$

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and for \( n = 1, 2, \ldots, N \),

\[
A_n = \kappa_n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y_n x_n^T \quad \text{and} \quad B_n = \kappa_n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} y_n x_n^T.
\] (16)

We have now expressed Psiaki’s first problem in the general form described in (6).

2.3 Prior work and alternative solution methods for Psiaki’s problems

In this section we summarize previous approaches to Psiaki’s generalizations of Wahba’s problem for spinning spacecraft. We then briefly discuss a simple discretization-based approach, implicit in the work of Psiaki and Hinks [21], for solving Psiaki’s problems globally.

Psiaki’s original paper [20] describes a method to globally solve Psiaki’s first problem when two noise-free vector measurements (sampled at distinct times) are used. In this situation the problem reduces to finding all the solutions of the corresponding non-linear equations satisfied by the initial attitude and spin-rate. This method seems quite sensitive to measurement noise, and is unable to exploit additional measurements to mitigate the effects of noise. (The advantages of incorporating multiple measurements are demonstrated in Section 4.3.)

In subsequent work [11] Hinks and Psiaki describe an approach to Psiaki’s second problem under the assumption that the spacecraft is axially symmetric and exactly three noise-free vector measurements are used. In this case it is again possible to find an initial body angular velocity \( \Omega(t_0) \) and an initial attitude that are consistent with the measurements by solving a set of non-linear equations. They suggest different formulations of these equations, and apply Newton’s method (with possibly many different initializations) to obtain a solution to the equations. Again this approach is likely to be useful only when there is very little noise.

In later work [21] Psiaki and Hinks describe a method to find local optima of Psiaki’s first and second problems (with no additional assumptions) by a novel alternating optimization scheme. The main idea is that for fixed \( \Omega(t_0) \), each point \( Q(t_0), Q(t_1), \ldots, Q(t_N) \) on the trajectory is linear in \( Q(t_0) \). Hence if we can compute the trajectory \( (Q(t_n))_{n=0}^N \) for fixed \( \Omega(t_0) \) we can minimize the objective function of (10) over \( Q(t_0) \) by solving an instance of Wahba’s problem. To obtain the trajectory \( (Q(t_n))_{n=0}^N \) for fixed \( \Omega(t_0) \), Psiaki and Hinks suggest numerically solving the rigid body equations. For the other part of the alternating optimization scheme, they employ a trust-region method to locally optimize over \( \Omega(t_0) \) for fixed \( Q(t_0) \). As presented this problem only finds local optima for \( \Omega(t_0) \) and \( Q(t_0) \). Nevertheless this method makes very few assumptions, and can incorporate many measurements and so should behave well in the presence of measurement noise.

A simpler, but much more naive, strategy would be to discretize the space of \( \Omega(t_0) \), solve (in parallel) the corresponding instance of Wahba’s problem for each value of \( \Omega(t_0) \), then output the pair \( (\Omega(t_0), Q(t_0)) \) with the smallest cost. This is a reasonable strategy for Psiaki’s first problem since aliasing issues mean there is always an optimal \( \omega \) in the interval \([-\pi/\tau, \pi/\tau]\). A clear downside of this discretization approach when compared with the semidefinite optimization-based methods we describe in Section 3 is that it is expensive to obtain global solutions of high accuracy. Furthermore, the semidefinite optimization-based formulation easily extends to give semidefinite optimization-based formulations for more general problems (see Section 4) where the subproblems for fixed \( \Omega(t_0) \) do not reduce to instances of Wahba’s problem.
3 Semidefinite optimization reformulations

The main aim of this section is to describe how to reformulate trigonometric Wahba problems, and hence Psiaki’s first problem (which is a special case), as semidefinite optimization problems. Before doing so, we briefly explain what semidefinite optimization problems are, and what we mean by a semidefinite reformulation of an optimization problem. We illustrate this in Section 3.2 by giving a semidefinite reformulation of Wahba’s problem that can be thought of as a more flexible description of the q-method [13]. In Section 3.3 we give a semidefinite reformulation of trigonometric Wahba problems, before giving, in Section 3.4, pseudocode illustrating how to implement the semidefinite optimization problems we formulate using generic semidefinite optimization solvers.

3.1 Semidefinite optimization

Semidefinite optimization problems are convex optimization problems of the form

$$\max \langle c, x \rangle \quad \text{s.t.} \quad A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0$$

where $x \in \mathbb{R}^n$ is a vector of decision variables, $c \in \mathbb{R}^n$ represents a linear cost functional, the matrices $A_0, A_1, \ldots, A_n$ are symmetric $m \times m$ matrices. Recall that $X \succeq 0$ means that the symmetric matrix $X$ is positive semidefinite. An expression of the form

$$A(x) = A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0$$

is often called a linear matrix inequality because it is linear in the decision variable $x$.

Semidefinite optimization problems can be solved to any desired accuracy in time polynomial in $n$ and $m$ using standard software based on interior point methods [28]. The semidefinite optimization problems that arise in this paper have additional structure that could be exploited to obtain even more efficient algorithms (see Section 6 for further discussion of this point). For much more information about semidefinite optimization, including duality theory, numerical algorithms, and applications, see for example [28].

Semidefinite reformulations Many different optimization problems arising in a variety of contexts, including some optimization problems for which the natural formulation is not convex, can be reformulated as semidefinite optimization problems. Given an optimization problem, by a semidefinite reformulation we mean a semidefinite optimization problem such that

1. the optimal value of the semidefinite optimization problem and the original optimization problem are the same;

2. there is an efficient procedure to take an optimal solution to the semidefinite optimization problem and produce an optimal solution to the original optimization problem.

3.2 Wahba’s problem

We illustrate the basic idea of semidefinite reformulations with the example of solving Wahba’s problem. We note that there are much better ways to solve Wahba’s problem. The advantage of the semidefinite reformulation is that it can be extended to more complicated situations, such
as Psiaki’s first problem. The reformulation presented in this section appears (in a more general context) in [24] and is generalized to the analogous problem where \( SO(3) \) is replaced with \( SO(n) \) for any \( n \geq 2 \) in [25]. (See also [7] where a semidefinite relaxation of Wahba’s problem is described, as well as conditions under which it is exact.)

Wahba’s problem fits into the general form (6) where \( A_0 = \sum_{n=0}^{N} \kappa_{n} y(n \tau)x(n \tau)^{T} \) and all the other terms vanish. Using the quaternion parameterization of \( SO(3) \), Wahba’s problem can be expressed as

\[
\max_{Q \in SO(3)} \langle A_0, Q \rangle = \max_{q \in \mathbb{R}} \langle A(qq^{T}) \rangle = \max_{q \in \mathbb{R}} \langle A^*(A_0), qq^{T} \rangle. \tag{17}
\]

We now explain how to reformulate (17) as a semidefinite optimization problem following a general pattern that we use again in Section 3.3.

1. Rewrite the problem as the optimization of a linear functional over some set. In this case \[
\max_{Z} \langle A^*(A_0), Z \rangle \quad \text{s.t.} \quad Z \in \{qq^{T} : q \in \mathbb{R} \}.
\]

2. Replace the constraint set with the convex hull of the constraint set. In this case \[
\max_{Z} \langle A^*(A_0), Z \rangle \quad \text{s.t.} \quad Z \in \text{conv}\{qq^{T} : q \in \mathbb{R} \}.
\]

This optimization problem has the same optimal value as the original non-convex problem because the cost function is linear (see Section 1.3).

3. Describe the convex hull of the constraint set as the feasible region of a semidefinite optimization problem (if possible). In this case such a description is well known (see, e.g., [18, Theorem 3]) and given by \[
\text{conv}\{qq^{T} : q \in \mathbb{R} \} = \{ Z \in S^n : Z \succeq 0, \ \text{tr}(Z) = 1 \}. \]

(This holds because if \( Z \succeq 0 \) and \( \text{tr}(Z) = 1 \) then any eigendecomposition \( Z = \sum_{i=1}^{n} \lambda_i q_i q_i^{T} \) expresses \( Z \) as a convex combination of matrices of the form \( qq^{T} \) with \( ||q|| = 1 \).)

The resulting semidefinite reformulation of Wahba’s problem is

\[
\max_{Z} \langle A^*(A_0), Z \rangle \quad \text{s.t.} \quad \text{tr}(Z) = 1, \ Z \succeq 0. \tag{18}
\]

**Extracting an optimal point** Let \( Q \) be an optimal solution of Wahba’s problem (17), and suppose \( q \) is a corresponding unit quaternion, so that \( Q = A(qq^{T}) \). Then the positive semidefinite matrix \( Z = qq^{T} \) is an optimum for the semidefinite reformulation of Wahba’s problem (18). All the optima of the semidefinite reformulation of Wahba’s problem are convex combinations of points of the form \( qq^{T} \) where \( A(qq^{T}) \) is optimal for the original formulation of Wahba’s problem. Under mild assumptions (such as having access to at least two generic vector measurements) Wahba’s problem has a unique solution \( Q^* = A(qq^{T}) \). Whenever Wahba’s problem has a unique solution it follows that the semidefinite reformulation also has a unique solution \( Z^* = qq^{T} \) and we can recover the solution to Wahba’s problem from the solution of the semidefinite relaxation by taking \( A(Z^*) \).

**Relationship with the \( q \)-method** The value of the semidefinite optimization problem (18) is the largest eigenvalue of the Davenport matrix \( A^*(A_0) \). This can already be seen from (17) and the fact that \( \max_{q \in \mathbb{R}} \langle A^*(A_0), qq^{T} \rangle = \max_{q \in \mathbb{R}} q^{T}A^*(A_0)q = \lambda_{\max}(A^*(A_0)) \). If \( q \) is an eigenvector corresponding to the largest eigenvalue of \( A^*(A_0) \) then \( Z = qq^{T} \) is an optimal solution of the semidefinite reformulation (18). As such, our reformulation is closely related to the \( q \)-method for solving Wahba’s problem problem [13].
Discussion  Note that the transformations in the first and second steps above are merely formal and can be applied to essentially any optimization problem. The third step is non-trivial. In general it is not well understood which sets $S$ have the property that conv($S$) can be described as the feasible region of a semidefinite optimization problem—this is an area of active research (see, for example, [4]). One view of this paper is that it shows how to express the convex hulls of the non-convex constraint sets appearing in certain joint spin-rate and attitude estimation problems as the feasible regions of semidefinite optimization problems.

3.3 Trigonometric Wahba problems

We now show how to give semidefinite reformulations of trigonometric Wahba problems (defined in (6)). By specializing to the case where $(A_n)_{n=0}^N$ and $(B_n)_{n=1}^N$ are given by (15) and (16), we obtain a semidefinite reformulation of Psiaki’s first problem.

As in the case of Wahba’s problem we use the parameterization of $SO(3)$ in terms of unit quaternions to rewrite trigonometric Wahba problems as

$$\max_{q \in \mathbb{H}, \omega \in [-\pi, \pi)} \langle A^*(A_0), qq^T \rangle + \sum_{n=1}^N \left[ \langle A^*(A_n), \cos(n\omega) qq^T \rangle + \langle A^*(B_n), \sin(n\omega) qq^T \rangle \right]. \quad (19)$$

We can view this problem as the maximization of a linear functional over the set

$$\mathcal{M}_N := \{(qq^T, qq^T \cos(\omega), \ldots, qq^T \cos(N\omega), qq^T \sin(N\omega)) \in (\mathbb{S}^4)^{2N+1} : q \in \mathbb{H}, \omega \in [-\pi, \pi)]\}. \quad (20)$$

As such the convexified version of (19) is the following optimization problem where the decision variables are the $2N+1$ symmetric matrices $X_0, X_1, Y_1, \ldots, X_N, Y_N$:

$$\max_{(X_n)_{n=0}^N, (Y_n)_{n=1}^N} \langle A^*(A_0), X_0 \rangle + \sum_{n=1}^N \left[ \langle A^*(A_n), X_n \rangle + \langle A^*(B_n), Y_n \rangle \right]$$

subject to $(X_0, X_1, Y_1, \ldots, X_N, Y_N) \in \text{conv}(\mathcal{M}_N). \quad (21)$

This problem is certainly convex, and has the same optimal value as (6) and (19). It may not be immediately clear that the constraint set conv($\mathcal{M}_N$) has a succinct representation in terms of the feasible region of a semidefinite optimization problem. In fact conv($\mathcal{M}_N$) does have such a representation, and we now turn our attention to describing it.

A linear matrix inequality description of conv($\mathcal{M}_N$)  We now describe conv($\mathcal{M}_N$) in terms of a linear matrix inequality, making use of the block matrix notation defined in Section 1.3. We establish the correctness of this description in the Appendix, by combining standard results with a novel symmetry reduction argument.

Proposition 3.1.

$$\text{conv}(\mathcal{M}_N) = \{(X_0, X_1, Y_1, \ldots, X_N, Y_N) \in (\mathbb{S}^4)^{2N+1} : \text{tr}(X_0) = 1, \quad \text{Toeplitz}(X_0, X_1, \ldots, X_N) + \text{Hankel}(Y_N, Y_{N-1}, \ldots, Y_1, 0, -Y_1, \ldots, -Y_{N-1}, -Y_N) \succeq 0\}. \quad (22)$$

Proof. We provide a proof in the Appendix.
Semidefinite reformulation in the general case  

Now that we have a semidefinite description of \( \text{conv}(M_N) \), we can give a semidefinite reformulation for all trigonometric Wahba problems. The following theorem explicitly describes this reformulation, which is obtained by replacing \( \text{conv}(M_N) \) in (21) with its semidefinite description from Proposition 3.1.

**Theorem 3.2.** Let \( A_0, A_1, \ldots, A_N, B_1, \ldots, B_N \in \mathbb{R}^{3 \times 3} \). Then the trigonometric Wahba problem

\[
\max_{Q \in \text{SO}(3)} \left\{ \langle A_0, Q \rangle + \sum_{n=1}^{N} \left[ \langle A_n, \cos(\omega n)Q \rangle + \langle B_n, \sin(\omega n)Q \rangle \right] \right\} \tag{23}
\]

and the semidefinite optimization problem

\[
\max_{(X_n)_{n=0}^{N}, (Y_n)_{n=1}^{N}} \left\{ \langle A^*(A_0), X_0 \rangle + \sum_{n=1}^{N} \left[ \langle A^*(A_n), X_n \rangle + \langle A^*(B_n), Y_n \rangle \right] \right\} \tag{24}
\]

s.t. \( \text{Toeplitz}(X_0, X_1, \ldots, X_N) + \text{Hankel}(Y_N, Y_{N-1}, \ldots, Y_1, 0, -Y_1, \ldots, -Y_{N-1}, -Y_N) \succeq 0 \)

\( \text{tr}(X_0) = 1 \)

have the same optimal value. The set of optimal points of the semidefinite reformulation is

\[
\text{conv} \{ (qq^T, qq^T \cos(\omega), qq^T \sin(\omega), \ldots, qq^T \cos(N\omega), qq^T \sin(N\omega)) : (\omega, A(qq^T)) \text{ is an optimal point for (23)} \}.
\]

**Extracting an optimal solution**  

If \( N \geq 2 \) we expect a generic trigonometric Wahba problem to have a unique optimal point \((\omega^*, Q^*)\) [20]. In that case the semidefinite reformulation (24) has a unique optimal point denoted \((X_0^*, X_1^*, Y_1^*, \ldots, X_N^*, Y_N^*)\) from which we can recover \((\omega^*, Q^*)\) via

\[
Q^* = A(X_0^*), \quad \cos(\omega^*) = \text{tr}(X_1^*) \quad \text{and} \quad \sin(\omega^*) = \text{tr}(Y_1^*). \tag{26}
\]

### 3.4 Pseudocode

In this section we describe code to implement our semidefinite optimization-based formulations (24) of trigonometric Wahba problems. Our motivation for doing this is to show that it is quite straightforward to use standard numerical routines to solve the semidefinite optimization problems that appear in this paper.

The code is expressed in a parsing language called YALMIP [16] that runs under MATLAB. Internally, YALMIP reformulates the human-readable description of the optimization problem we specify into a standard format, then calls a numerical solver for semidefinite optimization problems (we used MOSEK [2] version 7 for these experiments) to solve the optimization problem.

In what follows, we assume we have functions

- \texttt{A_map} implementing the linear map \( A \) taking a \( 4 \times 4 \) symmetric matrix and returning a \( 3 \times 3 \) matrix according to (3);

- \texttt{block_toeplitz} implementing the linear map \( (X_0, X_1, \ldots, X_N) \mapsto \text{Toeplitz}(X_0, X_1, \ldots, X_N) \) taking a \( 4 \times 4 \times (N+1) \) array and returning a \( 4(N+1) \times 4(N+1) \) matrix according to (1);

- \texttt{block_hankel} implementing the linear map \( (Y_1, Y_2, \ldots, Y_N) \mapsto \text{Hankel}(-Y_N, \ldots, -Y_1, 0, Y_1, \ldots, Y_N) \) taking a \( 4 \times 4 \times N \) array and returning a \( 4(N+1) \times 4(N+1) \) matrix according to (2).
We declare variables in YALMIP using the \texttt{sdpvar} command.

\begin{verbatim}
1: X = sdpvar(4,4,N+1,'symmetric');
2: Y = sdpvar(4,4,N,'symmetric');
\end{verbatim}

For example \( Y \) is a \( 4 \times 4 \times N \) array of variables with each slice \( Y(:,:,n) \) being a symmetric matrix. We specify constraints by constructing an array of constraints expressed in a very natural way. We express the two constraints in (24) by

\begin{verbatim}
3: K = [trace(X(:,:,1))==1, block toeplitz(X) + block hankel(Y) >= 0];
\end{verbatim}

where we have indexed from 1 following MATLAB’s conventions. Note that in YALMIP this latter inequality is automatically interpreted in the positive semidefinite sense since the matrix on the left hand side is structurally symmetric.

Suppose the variables \( A \) and \( B \) are respectively \( 4 \times 4 \times (N+1) \) and \( 4 \times 4 \times N \) arrays with \( A(:,:,n+1) \) being \( A^r(A_n) \) and \( B(:,:,n) \) being \( A^r(B_n) \). Then we can solve the semidefinite optimization problem (24) with the single line

\begin{verbatim}
4: solvesdp(K, -(A(:)'*X(:) + B(:)'*Y(:)));
\end{verbatim}

which calls a numerical solver with the constraint set \( K \) and the cost function \( -(A(:)'*X(:) + B(:)'*Y(:)) \) (with the minus sign because minimization is the default). Assuming that there is a unique solution to the non-convex problem we can extract the optimal rotation matrix \( Q \) and optimal \( \omega \) with

\begin{verbatim}
5: Q_opt = A_map(double(X(:,:,1)));
6: omega_opt = atan2(trace(double(Y(:,:,1))),trace(double(X(:,:,2))));
\end{verbatim}

\section{Variations}

In Section 3 we formulated Psiaki’s first problem as a semidefinite optimization problem by showing how to express the convex hull of \( \mathcal{M}_N \) in terms of linear matrix inequalities. This description of \( \mathcal{M}_N \) also allows us to take a semidefinite optimization-based approach to many variations on Psiaki’s first problem. In this section we illustrate the possibilities in this direction with one simple example—a variant on Psiaki’s problem where we assume the measurement errors are bounded, and incorporate this additional information into the formulation.

\subsection{Psiaki’s first problem with bounded measurement errors}

Using the notation from Section 2, suppose we know that the error between the measured direction \( y_n \) and the true direction \( Q(n\tau)x_n \) is bounded in each coordinate, satisfying

\begin{equation}
-\epsilon \leq y_n - Q(n\tau)x_n \leq \epsilon.
\end{equation}

Here \( \epsilon = [\epsilon_1 \ \epsilon_2 \ \epsilon_3]^T \) is a vector of positive constants that are not necessarily equal, and the inequalities in (27) are to be interpreted element-wise. Adding these constraints to the formulation
of Psiaki’s first problem we obtain the following variant:

\[
\min_{Q(0) \in SO(3)} \sum_{n=0}^{N} \frac{\kappa_n}{2} \|y_n - Q(n\tau)x_n\|_2^2
\]

\[
\text{s.t. } -\epsilon \leq y_n - Q(n\tau)x_n \leq \epsilon \quad \text{for } n = 0, 1, \ldots, N.
\]

Here, as in Section 2, \(Q(n\tau)\) is related to \(Q(0)\) via (13) and so putting \(\omega' = \omega\tau\),

\[
Q(n\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(n\omega') & -\sin(n\omega') \\ 0 & \sin(n\omega') & \cos(n\omega') \end{bmatrix} Q
\]

\[
= \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] A(qq^T) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \cos(n\omega')) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \sin(n\omega')).
\]

Since the objective function of (28) is identical to the objective function of Psiaki’s first problem (10), using the notation and manipulations of Sections 2 and 3 we can rewrite the variant of Psiaki’s first problem as

\[
\max_{q \in H, \omega' \in [-\pi, \pi)} \langle A^*(A_0), qq^T \rangle + \sum_{n=1}^{N} \left[ \langle A^*(A_n), qq^T \cos(n\omega') \rangle + \langle A^*(B_n), qq^T \sin(n\omega') \rangle \right]
\]

\[
\text{s.t. } -\epsilon \leq y_n - \left( \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] A(qq^T) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \cos(n\omega')) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \sin(n\omega')) \right) x_n \leq \epsilon
\]

\[
\quad \text{for } n = 0, 1, \ldots, N.
\]

Observe that we have rewritten the problem as the maximization of a linear functional over the constraint set defined by

\[
S = \left\{ (qq^T, qq^T \cos(\omega), qq^T \sin(\omega), \ldots, qq^T \cos(N\omega), qq^T \sin(N\omega)) : q \in \mathbb{H}, \omega \in [-\pi, \pi), \ight. \\
\left. -\epsilon \leq y_n - \left( \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] A(qq^T) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \cos(n\omega')) + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A(qq^T \sin(n\omega')) \right) x_n \leq \epsilon \
\right. \\
\left. \quad \text{for } n = 0, 1, \ldots, N \right\}
\]

This set \(S\) is the intersection of \(\mathcal{M}_N\) with the additional constraints (31) coming from incorporating the knowledge that the measurement errors satisfy the explicit deterministic bounds described in (27).

### 4.2 A semidefinite relaxation

Recall from Section 3 that if we could exactly describe \(\text{conv}(S)\) in terms of linear matrix inequalities that are not too large, we could obtain a semidefinite reformulation of this problem that can be solved efficiently. Unfortunately we do not know of such a concise description of \(\text{conv}(S)\), and conjecture that no such concise description exists for all choices of the \(x_n\) and \(y_n\).

Instead, a natural general approach is to construct a semidefinite relaxation of \(\text{conv}(S)\). By this we mean a convex set \(C\) such that

1. \(C \supseteq \text{conv}(S) \supseteq S\) and
2. $C$ has a simple description in terms of linear matrix inequalities. One choice would be to take $C$ to be the convex set

$$C = \left\{ (X_0, X_1, Y_1, \ldots, X_N, Y_N) \in \text{conv } \mathcal{M}_N : -\epsilon \leq y_n - \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A(X_0) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A(Y_n) \right) x_n \leq \epsilon \right\ \text{for } n = 0, 1, \ldots, N \right\}. $$

One can check that while $C$ is, in general, strictly larger than conv$(S)$, it can be expressed using linear matrix inequalities (since $C$ is obtained by adding linear inequalities to conv$(\mathcal{M}_N)$ which has a linear matrix inequality description from Proposition 3.1).

By optimizing over $C$ rather than $S$ we obtain the following *semidefinite relaxation* of the optimization problem

$$\max_{(X_0)_n, (Y_n)_n} \left( \langle A^*(A_0), X_0 \rangle + \sum_{n=0}^N \left( \langle A^*(A_n), X_n \rangle + \langle A^*(B_n), Y_n \rangle \right) \right) \quad (32)$$

s.t. $(X_0, X_1, Y_1, \ldots, X_N, Y_N) \in C$.

When we solve this semidefinite relaxation, if the solution $(X_0^*, X_1^*, Y_1^*, \ldots, X_N^*, Y_N^*) \in C$, returned by the solver, is actually in $S$, then it is a solution of the original non-convex problem (30) we are trying to solve. In this case it is typical to say that the semidefinite relaxation is *exact* for this instance.

If $(X_0^*, X_1^*, Y_1^*, \ldots, X_N^*, Y_N^*) \notin S$, we have not solved the non-convex problem, but can still conclude that the value of the objective function at this point is an upper bound on the optimal value of the original non-convex maximization problem (30). Such a bound can be used, for example, to assess the quality (in terms of the objective function) of any feasible point obtained, for instance, by a local optimization method.

### 4.3 Numerical experiments

In this section we describe the results of two simple numerical experiments to illustrate solving Psiaki’s first problem using semidefinite optimization, as well as solving the semidefinite relaxation of the variant on Psiaki’s problem discussed in Sections 4.1 and 4.2.

For all experiments we use the same parameters as in Psiaki’s truth-model simulation in [20]—the true spin period is 45.32 seconds (so the true spin-rate is $\omega = 0.1386$ radians per second), the sampling period is $\tau = 7.7611$ seconds per sample, and the initial attitude is $Q(0) = I$. The attitude dynamics are described by (11).

#### Solving Psiaki’s first problem

In the first experiment we solve Psiaki’s first problem using the semidefinite reformulation (24). In particular we repeat the following experiment $T = 1000$ times:

1. Sample reference directions $x_0, x_1, \ldots, x_{10}$ uniformly on the sphere.

2. For $n = 0, 1, 2, \ldots, N$, sample measurements $y_n$ uniformly distributed on the intersection of the unit sphere and the region

$$-\epsilon \leq y_n - Q(t_n)x_n \leq \epsilon \quad (33)$$

where $\epsilon = [0.5 \ 0.5 \ 0.05]^T$ (by sampling uniformly on the sphere and rejecting those samples not in the box-shaped region). This corresponds to measurements that are very accurate along one axis, but quite inaccurate in other directions.
3. For \( N = 2, 3, \ldots, 10 \), use the reference directions \( x_0, x_1, \ldots, x_N \) and measurements \( y_0, y_1, \ldots, y_N \) and solve the semidefinite optimization reformulation of Psiaki’s first problem (24).

We note that although the data are generated from a model where the measurement errors satisfy the explicit bounds (33), we do not exploit this in our solution method. Also, to get a sense of the measurement errors introduced, we note that under the noise model we have adopted, the angle between \( y_n \) and \( Q(t_n)x_n \) over all samples was at most 41.1 degrees, and on average 16.8 degrees.

The average angular error (in degrees) between the estimate of the initial attitude and the true initial attitude is indicated by cross-shaped markers in Figure 1(a). Given an estimate \( \hat{Q} \) of the true initial attitude \( Q(0) = I \), the angular error \( \theta \) satisfies \( \text{tr}(\hat{Q}^TQ(0)) = 2\cos(\theta) + 1 \). Hence we compute the angular error via \( |\cos^{-1}[\text{tr}(\hat{Q}^TQ(0)) - 1]/2| \). The corresponding average error in the spin-rate estimate \( \hat{\omega} \) is computed by taking the mean of \( |\hat{\omega} - \omega| \) over all trials and is indicated by cross-shaped markers in Figure 1(b). It is clear that as more vector measurements are used (i.e. as \( N \) increases) the estimates improve, justifying using more than the minimum number of measurements required for the optimization problem to have a unique optimum.

Solving the variant with bounded measurement errors In the second experiment we use exactly the same data as for the first experiment. This time, instead of solving the semidefinite reformulation of Psiaki’s first problem, we solve the semidefinite relaxation (32) of the variant of Psiaki’s first problem with bounded measurement errors. This estimation method explicitly makes use of the fact that the measurement errors satisfy (33).

![Figure 1: Results from the experiment described in Section 4.3. Figure 1(a) shows the average (over 1000 random trials) error in estimating the initial attitude using \( N + 1 \) measurements by solving Psiaki’s first problem (cross-shaped markers) and the semidefinite relaxation of the bounded error variant described in Section 4.2 (dot-shaped markers). Similarly Figure 1(b) shows the average error in estimating the spin-rate for the same experiment.](image-url)

As discussed in Section 4.2, and unlike the case for the semidefinite reformulation of Psiaki’s first problem itself, when solving the semidefinite relaxation of the bounded error variant of Psiaki’s first problem there is no guarantee that the relaxation will be exact. In other words, we do not know, in
Table 1: $T_{\text{exact}}$ is the number of random experiments (as described in Section 4.3) for which the semidefinite relaxation for the variant on Psiaki’s first problem (32) was exact. Here $N + 1$ is the number of vector measurements used.

| $N$   | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------|----|----|----|----|----|----|----|----|----|
| $T_{\text{exact}} / 1000$ | 842 | 816 | 867 | 918 | 948 | 958 | 965 | 969 | 973 |

For this experiment, the average angular error in the initial attitude estimates and the average error in the spin-rate estimates are indicated by dot-shaped markers in Figures 1(a) and 1(b). If the semidefinite relaxation was exact we compute the errors in these quantities in the same way as for the previous experiment. If the semidefinite relaxation was not exact, to be as conservative as possible we take the error to be the maximum possible value: 180 degrees for the initial attitude error and $\pi$ radians/second for the spin-rate error.

The results in Figures 1(a) and 1(b) show that by incorporating explicit bounds on the measurement errors (if known) into the semidefinite optimization framework, significantly better estimates of the initial attitude can be obtained. Furthermore Table 1 indicates that the semidefinite relaxation was indeed exact on many of our random trials, suggesting that the semidefinite relaxation approach may be well suited to tackling at least this variant on Psiaki’s first problem, and perhaps others.

5 Future directions

We briefly comment on possible future research directions based on the work in the present paper.

5.1 Numerical algorithms

The semidefinite reformulation of Psiaki’s first problem (24) has a very specific structure. This structure could be exploited to develop numerical algorithms for its solution (as well as the solution for variants on the problem) that are much faster than the generic interior-point algorithms we used for our experiments. Indeed semidefinite optimization problems with a similar structure arise in problems related to the Kalman-Yakubovich-Popov (KYP) lemma in robust control and in that context numerous specialized algorithms have been developed for their solution (see, e.g., [15, 9, 12]). Furthermore, great gains can be made by producing optimized low-level code for a particular family of convex optimization problems. An excellent example of this is the code-generation software CVXGEN which focuses on linear and convex quadratic programs [17].

5.2 Further variants

A semidefinite reformulation of a problem is particularly useful because it can be combined in many ways with other semidefinite optimization primitives to yield more problems that can be solved in the semidefinite optimization framework. In Section 4 we discussed a variation on Psiaki’s first
problem that had bounds on the angular noise. In this case it was straightforward to formulate a semidefinite relaxation using our semidefinite reformulation of Psiaki’s first problem.

Another natural variation that could be approached this way would be to obtain semidefinite relaxations of Psiaki’s first problem that are robust to uncertainty in certain model parameters. A similar idea has been carried out in detail for Wahba’s problem by Ahmed et al. [1]. They extended the semidefinite formulation of Wahba’s problem [24] to a variant that is robust to uncertainty in certain parameters, such as the reference directions. As suggested by Ahmed et al., this could be useful when using magnetometer measurements together with a low-order magnetic field model.

5.3 Psiaki’s second problem

It would be interesting to try to take a similar approach to the one taken in the present paper to related problems, such as Psiaki’s second problem. To do so we would need to give a semidefinite description (or perhaps a relaxation) of

\[
\text{conv}_{Q(t_0) \in \text{SO}(3), \Omega(t_0)} \{(Q(t_0), Q(t_1), \ldots, Q(t_N)) : Q(t) = \Phi(t - t_0; \Omega(t_0))Q(t_0) \text{ for all } t \geq t_0\}. \quad (34)
\]

Given this, we note that the objective function (10) can be rewritten as the maximization of a linear functional over the convex hull described in (34).

A more modest goal along similar lines might be to discretize the differential equation (9) and try to compute the convex hull (over all initial conditions \(Q(t_0), \Omega(t_0)\)) of an appropriately subsampled trajectory of the associated difference equation for the attitude variables. This approach of convexifying a problem based on discretized dynamics would, in a sense, be a convex analogue of the methods proposed for Psiaki’s second problem in [21].

6 Conclusion

We have shown how Psiaki’s generalization of Wahba’s problem to the case of a spacecraft spinning around a fixed axis at an unknown rate can be exactly reformulated as a semidefinite optimization problem. Such convex optimization problems can be solved globally using standard methods for semidefinite optimization. As suggested by Psiaki when formulating his generalizations of Wahba’s problem [20], our solutions to these generalizations of Wahba’s problem could be used to initialize standard extended Kalman filter-based methods for attitude estimation.

Furthermore, we have illustrated how to use our reformulation of Psiaki’s first problem to construct semidefinite relaxations of a more complicated variant on the problem. Our numerical experiments with a variant that includes explicit bounds on the measurement errors suggest that incorporating additional information into the formulation can improve the estimation errors. Our results also suggest the semidefinite relaxation approach we propose, although not exact in general, often computes solutions to the original non-convex variations of Psiaki’s problem that we ultimately are aiming to solve.

A Proofs

In this appendix we prove Proposition 3.1. We split the proof into two parts, given by Lemmas A.2 and A.3 below. Together these clearly imply Proposition 3.1. In what follows we extend the notation \(\text{Toeplitz}(T_0, T_1, \ldots, T_N)\) defined in (1) to include the case where \(T_1, \ldots, T_N\) are \(d \times d\) complex matrices, \(T_0\) is a \(d \times d\) Hermitian matrix, and all transposes of real matrices are replaced with conjugate transposes, denoted \(A \mapsto A^*\), of complex matrices.
Lemma A.2, to follow, is a slight modification of the fact that any Hermitian positive semidefinite block-Toeplitz matrix admits a decomposition as a sum of rank one positive semidefinite block-Toeplitz matrices. This fact may be more familiar in its dual form as the matrix spectral factorization (or Fejér-Riesz) theorem (see, e.g., [23]). This classical result says that any Hermitian finite block-Toeplitz matrix admits a decomposition as a sum of rank one positive semidefinite matrices. The result can also be interpreted as saying that non-negative functions of the form \((T\)ismenetsky [27])

**Theorem A.1** (Tismenetsky [27]). If Toeplitz\((T_0, T_1, \ldots, T_N) > 0\) then there are \(u_k \in \mathbb{C}^4, \omega_k \in [-\pi, \pi)\) and \(\lambda_k > 0\) for \(k = 1, 2, \ldots, 4(N + 1)\) such that

\[
\text{Toeplitz}(T_0, T_1, \ldots, T_N) = \sum_{k=1}^{4(N+1)} \lambda_k \text{Toeplitz}(u_k^* v_k, u_k^* e^{i\omega_k}, \ldots, u_k^* v_k e^{iN\omega_k}).
\]

Consequently \(T_j = \sum_{k=1}^{4(N+1)} \lambda_k u_k^* v_k e^{j\omega_k}\) for \(j = 0, 1, \ldots, N\).

The following lemma is a slight modification of Theorem A.1.

**Lemma A.2.** Let \(\mathcal{M}_N\) be defined as in (20). Then

\[
\text{conv}(\mathcal{M}_N) = \{(X_0, X_1, Y_1, \ldots, X_N, Y_N) \in (S^4)^{2N+1} : \text{tr}(X_0) = 1, \text{Toeplitz}(X_0, X_1 + iY_1, \ldots, X_N + iY_N) \succeq 0\}. \tag{35}
\]

**Proof.** Let \((qq^T, qq^T \cos(\omega), qq^T \sin(\omega), \ldots, qq^T \cos(N\omega), qq^T \sin(N\omega)) \in \mathcal{M}_N\). Then \(\text{tr}(qq^T) = \|q\|^2 = 1\) and it is straightforward to check that

\[
\text{Toeplitz}(qq^T, qq^T \cos(\omega)+iqq^T \sin(\omega), \ldots, qq^T \cos(N\omega)+iqq^T \sin(N\omega)) = \begin{bmatrix} q & qe^{-i\omega} & \cdots & qe^{-i(N-1)\omega} \\ qe^{-i\omega} & qe^{-i2\omega} & \cdots & qe^{-i(N-2)\omega} \\ \vdots & \vdots & \ddots & \vdots \\ qe^{-i(N-1)\omega} & qe^{-i(N-2)\omega} & \cdots & qe^{-iN\omega} \end{bmatrix} \succeq 0.
\]

Hence \(\mathcal{M}_N\) is a subset of the right hand side of (35). Since the right hand side of (35) is convex, it follows that \(\text{conv}(\mathcal{M}_N)\) is also a subset of the right-hand side of (35).

Now suppose \((X_0, X_1, X_2, \ldots, X_N, Y_N) \in (S^4)^{2N+1}\) satisfies

\[
\text{tr}(X_0) = 1 \quad \text{and} \quad \text{Toeplitz}(X_0, X_1 + iY_1, \ldots, X_N + iY_N) > 0.
\]

Then by Theorem A.1 there are \(u_k \in \mathbb{C}^4, \omega_k \in [-\pi, \pi)\) and \(\lambda_k > 0\) for \(k = 1, 2, \ldots, 4(N + 1)\) such that \(1 = \text{tr}(X_0) = \sum_{k=1}^{4(N+1)} \lambda_k \|u_k\|^2\) and \(X_j + iY_j = \sum_{k=1}^{4(N+1)} \lambda_k u_k^* v_k e^{j\omega_k}\) for \(j = 0, 1, \ldots, N\). Since \(X_j^T = X_j\) and \(Y_j^T = Y_j\) for all \(j\), it follows by a straightforward calculation that there are \(v_k \in \mathbb{R}^4\) and \(\lambda_k' > 0\) for \(k = 1, 2, \ldots, 8(N + 1)\), such that

\[
X_j + iY_j = \frac{1}{2} \left((X_j + iY_j) + (X_j + iY_j)^T\right) = \sum_{k=1}^{4(N+1)} \frac{\lambda_k'}{2} \left((u_k^* v_k^*) + (u_k^* v_k^*)^T\right) e^{j\omega_k} = \sum_{k=1}^{8(N+1)} \lambda_k' v_k v_k^T e^{j\omega_k}.
\]

Defining \(\mu_k = \lambda_k' \|v_k\|^2 > 0\) and \(q_k = v_k/\|v_k\| \in \mathbb{H}\) for \(k = 1, 2, \ldots, 8(N + 1)\) we have that \(\sum_{k=1}^{8(N+1)} \mu_k = 1\) and

\[
X_j = \sum_{k=1}^{8(N+1)} \mu_k q_k q_k^T \cos(j\omega_k) \quad \text{for} \quad j = 0, 1, \ldots, N \quad \text{and} \quad Y_j = \sum_{k=1}^{8(N+1)} \mu_k q_k q_k^T \sin(j\omega_k).
\]
for $j = 1, 2, \ldots, N$. This shows that $(X_0, X_1, Y_1, \ldots, X_N, Y_N) \in \text{conv}(M_N)$. Hence the relative interior of the right-hand side of (35) is a subset of $\text{conv}(M_N)$. Since $\text{conv}(M_N)$ is closed the right-hand side of (35) is also a subset of $\text{conv}(M_N)$, establishing the result.

**Lemma A.3.** If $X_0, X_1, Y_1, \ldots, X_N, Y_N \in S^d$ then

\[
\text{Toeplitz}(X_0, X_1 + iY_1, \ldots, X_N + iY_N) \succeq 0
\]

if and only if

\[
\text{Toeplitz}(X_0, X_1, \ldots, X_N) + \text{Hankel}(\ldots, -Y_1, 0, Y_1, \ldots, Y_N) \succeq 0.
\]

**Proof.** First observe that $Z = \text{Toeplitz}(X_0, X_1 + iY_1, \ldots, X_N + iY_N) \succeq 0$ if and only if the real $2d(N + 1) \times 2d(N + 1)$ symmetric matrix

\[
Z_R = \begin{bmatrix}
\Re Z & \Im Z \\
-\Im Z & \Re Z
\end{bmatrix}
\]

is positive semidefinite [10]. Here $\Re Z$ and $\Im Z$ are the real and imaginary parts of $Z$, respectively. Indeed

\[
\Re Z = \begin{bmatrix}
X_0 & X_1 & X_2 & \cdots & X_N \\
X_1 & X_0 & X_1 & & \\
X_2 & X_1 & & & \\
& & & & \\
X_N & \cdots & \cdots & X_1 & X_0
\end{bmatrix}
\]

and

\[
\Im Z = \begin{bmatrix}
0 & Y_1 & Y_2 & \cdots & Y_N \\
-Y_1 & 0 & Y_1 & & \\
-Y_2 & -Y_1 & & & \\
& & & & \\
-Y_N & \cdots & \cdots & 0 & Y_1
\end{bmatrix}
\]

where we have used the assumption that the $X_i$ and the $Y_i$ are symmetric.

Let $J$ be the $d(N + 1) \times d(N + 1)$ matrix with $d \times d$ identity blocks on the secondary (anti-) block diagonal. Note that left multiplication by $J$ reverses the block rows of a block matrix, and right multiplication by $J^T = J$ reverses the block columns. Observe that $J(\Re Z)J = \Re Z$ and $J(\Im Z) + (\Im Z)J = 0$. Let $Q$ denote the orthogonal matrix defined by

\[
Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ J & I \end{bmatrix}.
\]

A straightforward calculation shows that

\[
QZ_RQ^T = \frac{1}{2} \begin{bmatrix} I & -J \\ J & I \end{bmatrix} \begin{bmatrix} \Re Z & \Im Z \\ -\Im Z & \Re Z \end{bmatrix} \begin{bmatrix} I & J \\ -J & I \end{bmatrix} = \begin{bmatrix} \Re Z + J\Im Z & 0 \\ 0 & \Re Z + J\Im Z \end{bmatrix}.
\]

So $Z \succeq 0$ if and only if $Z_R \succeq 0$ which holds if and only if $\Re Z + J\Im Z \succeq 0$. Finally we note that

\[
\Re Z + J\Im Z = \text{Toeplitz}(X_0, X_1, \ldots, X_N) + \text{Hankel}(\ldots, -Y_1, 0, Y_1, \ldots, Y_N)
\]

(because reversing the block rows of a block Toeplitz matrix makes it block Hankel) to complete the proof.

&}
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