Theories of Linear Response in BCS Superfluids and How They Meet Fundamental Constraints

Hao Guo\textsuperscript{1,2}, Chih-Chun Chien\textsuperscript{3}, Yan He\textsuperscript{4,5}

\textsuperscript{1}Department of Physics, Southeast University, Nanjing 211189, China
\textsuperscript{2}Department of Physics, University of Hong Kong, Hong Kong 999077, China
\textsuperscript{3}Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM, 87545, USA
\textsuperscript{4}Department of Physics, University of California, Riverside, CA 92521, USA and
\textsuperscript{5}James Franck Institute, University of Chicago, Chicago, IL 60637, USA

We address the importance of symmetry and symmetry breaking on linear response theories of fermionic BCS superfluids. The linear theory of a noninteracting Fermi gas is reviewed and several consistency constraints are verified. The challenge to formulate linear response theories of BCS superfluids consistent with density and spin conservation laws comes from the presence of a broken $U(1)_{EM}$ symmetry associated with electromagnetism (EM) and we discuss two routes for circumventing this. The first route follows Nambu’s integral-equation approach for the EM vertex function, but this method is not specific for BCS superfluids. We focus on the second route based on a consistent-fluctuation-of-the order-parameter (CFOP) approach where the gauge transformation and the fluctuations of the order parameter are treated on equal footing. The CFOP approach allows one to explicitly verify several important constraints: The EM vertex satisfies not only a Ward identity which guarantees charge conservation but also a $Q$-limit Ward identity associated with the compressibility sum rule. In contrast, the spin degrees of freedom associated with another $U(1)_z$ symmetry are not affected by the Cooper-pair condensation that breaks only the $U(1)_{EM}$ symmetry. As a consequence the collective modes from the fluctuations of the order parameter only couple to the density response function but decouple from the spin response function, which reflects the different fates of the two $U(1)$ symmetries in the superfluid phase. Our formulation lays the groundwork for application to more general theories of BCS-Bose Einstein Condensation (BEC) crossover both above and below $T_c$.

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I. INTRODUCTION

Linear response theories have been an important tool for studying transport and dynamic properties of many-body systems [1–3]. Although there have been myriad successful examples in classical or non-interacting systems, developing a linear response theory for complex systems or strongly correlated systems could be quite a challenge. As summarized in Ref. [3], it is difficult to obtain consistent expressions of the compressibility of an interacting electron gas from the derivatives of thermodynamic quantities and from the correlation functions. A naive calculation of the response functions of BCS theory of conventional superconductors could have led to a violation of the charge conservation. The spontaneous symmetry breaking of the superfluid phase further complicates the treatment of any linear response theory of BCS superfluids. Thus it requires more sophisticated treatments [4, 5] to obtain the response functions that respect conservation laws.

Here we analyze two consistent linear response theories of BCS superfluids in great details and review the importance of gauge invariance, the (generalized) Ward identity, sum rules, and how they impose constraints on response functions. Also crucial is an additional $Q$-limit Ward identity [6] which is related to the compressibility sum rule. The non-interacting Fermi gas satisfies all constraints, albeit in a trivial sense. In the presence of interactions, we will focus on a linear response theory which we call consistent-fluctuation-of-the order-parameter (CFOP) theory for fermionic superfluids. Importantly, at the mean-field level, this approach is consistent with all conservation laws and associated sum rules. This will set up a solid foundation for the generalization to the more general theories of BCS-BEC crossover [7]. We also present a spin linear response theory in a systematic manner to demonstrate that fundamental constraints including the sum rules and Ward identities are also satisfied in the spin channel. One major difference between the density and spin response functions is that in the superfluid phase the order parameter, which corresponds to the condensate of Cooper pairs, breaks only the symmetry associated with the density response functions, but not the symmetry associated with the spin response function. Therefore the density response function exhibits richer structures across the superfluid phase transition. This is the base for constructing an order-parameter like quantity that measures the “spin-charge separation” in a BCS superfluid as proposed in Ref. [8].

There have been two major approaches for addressing the linear response of fermionic BCS superfluids. The first one is attributed to Nambu [4], who reformulated the problem in the two-dimensional Nambu space and pointed out that if certain types of corrections to the EM interacting vertex are considered consistently with those corrections to the
self-energy in the Green’s function, the gauge invariance can be maintained explicitly. Nambu presented a generalized
form of Ward identity (GWI) and proposed an integral equation for the full electromagnetic (EM) interacting vertex
in the Nambu space. He stated that this EM vertex must satisfy the GWI without giving a proof and we present
our own proof in Appendix C. By solving this integral equation at small frequency and momentum limits, it was
shown that the excitations of collective modes correspond to the poles of the density response function. Hence the
many-particle effects are indeed included in the corrections of EM vertex. Despite many virtues, this method is
relatively difficult to implement. Firstly, it is very hard to solve the integral equation for the vertex. If one truncates
the integral equation, one may not get a gauge invariant solution and can not even find the correct collective-mode
excitations. Secondly, this approach was formulated in the Nambu space and it is difficult to translate the results
in the two-dimensional Nambu space to their counterparts in the one-dimensional representation space of fermion
operators. This limits its applicability to more general problems such as BCS-BEC crossover. Moreover, it can be
shown that the GWI discussed in Nambu’s original paper [4] is not a unique constraint for gauge invariance of BCS
theory. In other words, Nambu’s method is not specific to BCS theory: It is more general but harder to obtain an
exact solution.

There is yet another approach which we call the CFOP approach with a totally different structure when compared
to Nambu’s method. Our goal is to explain this approach and test its results against some fundamental constraints.
Within this approach, we treat the gauge transformation and the fluctuations of order parameter on equal footing,
such that many-particle effects are also explicitly included in the EM vertex. This approach was first proposed by
Kadanoff and Martin[5] in a less complete form. They only considered the fluctuations of the order parameter, and
tried to decompose the three-particle Green’s functions in a way that can respect gauge invariance. It was then
independently formulated in several unrelated papers by Betbeder Matibet and Nozieres [9] and Kulik et al. [10]
in a more complete form where both phase and amplitude fluctuations are considered. Later on, Zha and Levin[11]
revisited it and presented three identities of response functions, which are now known to be part of the WIs for
response functions. Recently, it was again proposed in the Keldysh formalism with time-ordered Green’s functions
in Ref. [12]. A similar derivation using a kinetic-equation formalism is also discussed in Ref. [13]. In this paper, we
cast this formalism in a more systematic and covariant form. We will present more virtues of this approach, such
that the consistency with the $Q$-limit Ward identity and the $f$-sum rule. The EM vertex and its generalized form can be
both found in different representation spaces. A comparison between Nambu’s method and the CFOP theory will
also be presented. We will show that the CFOP approach is indeed a consistent and manageable linear response
theory for BCS superfluids. The generalization of the CFOP theory to relativistic BCS superfluids can be found in
Ref. [14]. Besides linear response theories discussed here, one may find more discussions on the gauge invariance of
BCS superconductors in terms of effective field theories in Ref. [15].

This paper is organized as follows. In Section II we briefly discuss the general symmetries in the theory of two-
component Fermi gases with contact interactions. In Section III we review the density and spin linear response
theories for non-interacting Fermi gases. In Section IV we spend a significant part of the paper on the CFOP linear
response theory of BCS superfluids. More specifically, Section IV A presents the derivation and some results of the
CFOP linear response theory; Section IV B addresses the (generalized) Ward identity and $Q$-limit (generalized) Ward
identity of the CFOP linear response theory; Sections IV C and IV D review Nambu’s integral-equation approach and
present a comparison of our CFOP approach with Nambu’s method. In Section V we develop a parallel formalism for
the linear response theory in the spin channel by “gauging” the $U(1)_z$ symmetry. Section VI concludes our work.

II. SYMMETRIES OF THEORIES OF FERMI GASES WITH CONTACT INTERACTIONS

Throughout this paper, we follow the convention $c = c = \hbar = 1$ and use $\sigma$ to denote the spin or pseudo-spin $\uparrow, \downarrow$
with $\uparrow$ and $\sigma$ being the opposite of $\downarrow$ and $\sigma$ respectively. The metric tensor of the Minkowski spacetime is chosen as
$\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and Einstein summation convention is adopted. For a two component Fermi gas interacting
via contact interactions, we consider the Hamiltonian

$$H = \int d^3x \psi_\uparrow \psi_\downarrow(\mathbf{x}) \left( \hat{p}^2 - \mu \right) \psi_\uparrow \psi_\downarrow(\mathbf{x}) - g \int d^3x \psi_\uparrow \psi_\downarrow(\mathbf{x}) \psi_\downarrow \psi_\uparrow(\mathbf{x}),$$

(1)

where $\psi$ and $\psi^\dagger$ are the annihilation and creation operators of fermions, $\mu$ is the chemical potential, $m$ is the fermion
mass, and $g$ is the bare coupling constant. There is an implicit summation over the pseudo-spin indices $\sigma$. The Hamiltonian (1)
has a $\text{SU}(2) \times \text{SU}(2)$ symmetry [4, 16, 17]. The first $\text{SU}(2)$ symmetry is generated by

$$\psi_\uparrow \rightarrow \cosh \chi \psi_\uparrow + \sinh \chi \psi_\downarrow^\dagger, \quad \psi_\downarrow \rightarrow \sinh \chi \psi_\downarrow + \cosh \chi \psi_\uparrow,$$

$$\psi_\uparrow \rightarrow \cosh \chi \psi_\uparrow - i \sinh \chi \psi_\downarrow^\dagger, \quad \psi_\downarrow \rightarrow i \sinh \chi \psi_\downarrow + \cosh \chi \psi_\uparrow,$$
\[ \psi_\sigma \rightarrow e^{-i\chi} \psi_\sigma, \quad \psi_\sigma^\dagger \rightarrow e^{i\chi} \psi_\sigma^\dagger, \quad \text{(2)} \]

where \( \chi \) is a continuous parameter. The transformation on the third line is the well-known U(1) symmetry associated to electromagnetism if the system is charged. The generators of theses transformations are \(-i\sigma_1, -i\sigma_2\) and \(\sigma_3\) in the space spanned by the Nambu spinor \((\psi_\uparrow, \psi_\downarrow)^T\), where \(\sigma_i\) are the Pauli matrices. Hence the symmetry (2) indeed SU(2), or more precisely, SU(1,1) with the U(1)EM being its subgroup.

The second SU(2) symmetry is given by

\[
\begin{align*}
\psi_\uparrow &\rightarrow \cos \alpha \psi_\uparrow + i \sin \alpha \psi_\downarrow, \quad \psi_\downarrow \rightarrow -i \sin \alpha \psi_\downarrow + \cos \alpha \psi_\uparrow, \\
\psi_\uparrow &\rightarrow \cos \alpha \psi_\uparrow + i \sin \alpha \psi_\downarrow, \quad \psi_\downarrow \rightarrow -\sin \alpha \psi_\downarrow + \cos \alpha \psi_\uparrow, \\
\psi_\sigma &\rightarrow e^{-iS_\sigma \alpha} \psi_\sigma, \quad \psi_\sigma^\dagger \rightarrow e^{iS_\sigma \alpha} \psi_\sigma^\dagger, \quad \text{where } S_\uparrow, S_\downarrow = \pm 1.
\end{align*}
\quad \text{(3)}
\]

The generators are \(\sigma_1\), \(\sigma_2\) and \(\sigma_3\) in the space spanned by \((\psi_\uparrow, \psi_\downarrow)^T\). The transformation on the third line is the spin rotation around the \(z\)-axis, which forms the subgroup U(1) of the second SU(2). In what follows, we will focus on the U(1)EM \(\times\) U(1)z symmetries, and we may call the theories associated with them as being in the density and spin channels respectively.

When the continuous parameter \(\chi\) become space-time dependent, the two global U(1) symmetries are “gauged” respectively. To keep the Lagrangian invariant under the gauge transformations, one needs to couple the fermion field by an external vector field \(A^\mu = (\phi, A)\) which transforms as \(\mathbf{A} \rightarrow \mathbf{A} - \nabla \chi\), \(\phi \rightarrow \phi + \partial_\mu \chi\). For the U(1)EM symmetry, the Hamiltonian (24) becomes

\[
H = \int d^3x \psi_\sigma^\dagger(x) \left( \left( \frac{\hat{p} - S_\sigma \mathbf{A}(x)}{2m} \right)^2 - \mu \right) \psi_\sigma(x) + \int d^3x \phi(x) \psi_\uparrow(x) \psi_\downarrow(x) - g \int d^3x \psi_\uparrow(x) \psi_\downarrow(x) \psi_\uparrow(x) \psi_\downarrow(x),
\]

\[
= \int d^3x \psi_\sigma^\dagger(x) \left( \left( \frac{\hat{p}^2}{2m} - \mu \right) \psi_\sigma(x) + \int d^3x \mathbf{J}(x) \cdot \mathbf{A}(x) + \int d^3x n(x) \phi(x) - g \int d^3x \psi_\uparrow(x) \psi_\downarrow(x) \psi_\uparrow(x) \psi_\downarrow(x), \quad \text{(4)}
\]

where

\[
\begin{align*}
J(x) &= -\frac{1}{2m} \left[ \psi_\uparrow(x) (\nabla \psi_\sigma(x)) - (\nabla \psi_\sigma(x)) \psi_\uparrow(x) \right] - \frac{1}{m} \mathbf{A}(x) \psi_\sigma^\dagger(x) \psi_\sigma(x), \\
n(x) &= \psi_\sigma^\dagger(x) \psi_\sigma(x).
\end{align*}
\quad \text{(5)}
\]

Here \(n(x)\) is the particle number density and \(\mathbf{J}(x)\) is the mass current. There is also an implicit summation over the repeated Greek index \(\sigma\). When the external field \(A^\mu\) is the EM field, the Noether current for the global U(1)EM symmetry is \(J^\mu = (n, \mathbf{J})\) which obeys the conservation law \(\partial_\mu J^\mu = 0\) in the Heisenberg picture, where \(\partial_\mu = (\partial_\mu, \nabla)\).

The same discussion can be implemented in the spin channel. The fermion can also couple to an effective external field \(A^\mu\). Hence the Hamiltonian with a U(1)z symmetry, \(S_{\uparrow, \downarrow} = 0\) as in (3). The external field \(A^\mu\) has a different physical meaning from that in the density channel. Since \(n_\sigma\) is the \(z\) component of spin density, the field \(\phi\) coupled to \(n_\sigma\) corresponds to \(B_z\). \(J_\uparrow\) is the difference between the spin-up and the spin-down currents, i.e., the magnetization current. Therefore the field which couples to it is the magnetizing field \(\mathbf{A} \equiv \mathbf{m}\). The effective external vector field is thus \(A^\mu = (B_z, \mathbf{m})\). The resulting Hamiltonian then describes a generalized spin-magnetic field interaction. The Noether current for the global U(1)z symmetry is \(J_\sigma^\uparrow = (n_\sigma, J_\uparrow)\), which also satisfies the conservation law \(\partial_\mu J_\uparrow^\mu = 0\).
III. LINEAR RESPONSE THEORY FOR NON-INTERACTING FERMI GASES

We begin with the non-interacting Fermi gases where \( g = 0 \). Many important identities can be verified explicitly and they provide useful hints for the development of consistent theories of linear response for interacting Fermi gases. When a non-interacting two-component Fermi gas is perturbed by a weak external gauge field, one can discuss the response in the density channel as well as in the spin channel the leading-order or linear-approximation of the perturbation.

In momentum space, the linear response theories associated with the two U(1) symmetries mentioned above can be expressed in a unified form as \( H = H_0 + H_D^1, \) where

\[
H_0 = \sum_p \psi_{\sigma p}^\dagger \xi_p \psi_{\sigma p},
\]

\[
H_D^1 = \sum_{pq} \psi_{\sigma p+q}^\dagger \gamma^\mu(p+q,p) A_{\mu q} \psi_{\sigma q}, \quad H_S^1 = \sum_{pq} \psi_{\sigma p+q}^\dagger \gamma^\sigma_{S\sigma}(p+q,p) A_{\mu q} \psi_{\sigma q}.
\]

Here \( \xi_p = \frac{p^2}{2m} - \mu \), and we have introduced the interacting vertices \( \gamma^\mu(p+q,p) = (1, \frac{p+q}{m}) \) for the density channel and \( \gamma^\sigma_{S\sigma}(p+q,p) = (S_\sigma, S_\sigma \frac{p+q}{2m}) \) for the spin channel. The latter explicitly depends on the spin (or pseudo-spin) and Fig. [is] illustrates this spin dependence. The four-currents of the density and spin channels in momentum space are given by

\[
J_{q}^\mu = \sum_p \psi_{\sigma p+q}^\dagger \gamma^\mu(p+q,p+q) \psi_{\sigma p}, \quad J_S^\mu = \sum_p \psi_{\sigma p}^\dagger \gamma^\sigma_{S\sigma}(p+q,p) \psi_{\sigma q}.
\]

Note in the perturbed Hamiltonian \( H_D^1, S^1 \) we only keep the terms up to the linear order of the external field \( A^\mu \) and higher order terms are neglected. Hence in the expressions of the currents \( J_q^\mu \) and \( J_S^\mu \) the linear terms of \( A^\mu \) are dropped, which is different from the currents given by Eqs. (5) and (7).

For the rest of this section we will focus on the density channel since the linear response theory in the spin channel has a similar structure as the former, and the response kernels can be shown to be the same. The imaginary time \( \tau = it \) is introduced here and the Heisenberg operator is defined as \( \mathcal{O}(\tau) = e^{iH\tau}Oe^{-iH\tau} \). The spacetime coordinates are \( x = (\tau, \mathbf{x}) \). Hence the Green’s function is defined by \( G_0(x, x') = \langle T[\psi_\sigma(x)\psi_\sigma^\dagger(x')] \rangle \) where \( T \) denotes the \( \tau \)-order of operators. Its expression in momentum space is \( G_0(\imath \omega_n, p) = (\imath \omega_n - \xi_p)^{-1} \). The four-momentum is defined as \( P = p_\mu = (\imath \omega_n, p) \) where \( \omega_n = (2n+1)\pi k_BT \) is the fermion Matsubara frequency. The particle density is given by

\[
n = T \sum_{\omega_n} \sum_p G_0(p) = 2 \sum_p f(\xi_p),
\]

where \( f(x) = 1/(e^{x/k_BT} + 1) \) is the fermion distribution function. By defining \( h^{\mu\nu} = -\eta^{\mu\nu}(1 - \eta^{\sigma\sigma}) \), we formally write the perturbation of the EM current \( J_\mu \) as

\[
\delta J_\mu(\tau, q) = \sum_p \langle \psi_{\sigma p}^\dagger(\tau) \gamma^\mu(p+q+p) \psi_{\sigma p+q}(\tau) \rangle + \frac{n}{m} h^{\mu\nu} A_\nu(\tau, q),
\]

The extra term \( \frac{n}{m} h^{\mu\nu} \) of \( \delta J_\mu \) arises from the second term of \( J \) or \( J_S \) (See Eqs. (5) or (7)), since \( J \) or \( J_S \) already contains a first order term of \( A^\mu \). The linear response theory is then written in the form

\[
\delta J_\mu(\tau, q) = \int d\tau' \langle Q_0^{\mu\nu}(\tau - \tau', q) + \frac{n}{m} h^{\mu\nu} A_\nu(\tau, q) \rangle A_\nu(\tau', q),
\]

where the response kernels are

\[
Q_0^{\mu\nu}(\tau - \tau', q) = -\langle T_\tau [J_\mu(\tau, q)J_\nu(\tau', -q)] \rangle
\]

\[
= -\sum_{pp'} \langle T_\tau [\psi_{\sigma p}^\dagger(\tau) \gamma^\mu(p+q+p) \psi_{\sigma p+q}(\tau) \psi_{\sigma p'}^\dagger(\tau') \gamma^\nu(p', p' + q) \psi_{\sigma p'}(\tau')] \rangle.
\]

Implementing a Fourier transform and making use of Wick’s theorem, we obtain

\[
Q_0^{\mu\nu}(i\Omega_l, q) = 2T \sum_{\omega_n} \sum_{pp'} \gamma^\mu(p+q+p) G_{0p+q+p'}(i\omega_n + i\Omega_l) \gamma^\nu(p', p' + q) G_{0p,p'}(i\omega_n)
\]
where \( q_\mu = (i\Omega_t, \mathbf{q}) \), \( \Omega_t \) is the boson Matsubara frequency, and \( G_{0p,p'}(i\omega_n) = G_{0p}(\omega_n)\delta_{p,p'} \) with \( G_{0p}(\omega_n) \equiv G_0(\mathbf{p}) \). For convenience, we have defined \( \gamma^\mu(\mathbf{p} + \mathbf{q}, P) \equiv \gamma^\mu(\mathbf{p}, \mathbf{q}, P) \). The factor 2 comes from the summation over the spin (or pseudo-spin) indices \( \sigma \). The spin response kernels have the same structure as one can see from the facts that \( \gamma^\mu_{\sigma\sigma}(\mathbf{p} + \mathbf{q}, \mathbf{p}) = S_\sigma\gamma^\mu(\mathbf{p} + \mathbf{q}, \mathbf{p}) \) and \( S_\sigma^2 = 1 \). Note there are also two spin interacting vertices inside the expression of the spin response kernels similar to (13). The detailed expressions of the response functions are listed in Appendix A, where \( \epsilon^\pm_{\mathbf{p}} = \xi_{\mathbf{p}+\mathbf{q}} \).

Due to the simple structure, one may verify the following identities explicitly: (1) Ward identities (WIs), (2) compressibility sum rule, (3) \( f \)-sum rule, and (4) \( Q \)-limit Ward identity. By direct evaluating each term, one can verify that the EM and spin interacting vertices satisfy the WIs

\[
q_\mu\gamma^\mu(\mathbf{p} + \mathbf{q}, P) = G_0^{-1}(\mathbf{p} + \mathbf{q}) - G_0^{-1}(\mathbf{p}),
\]

\[
q_\mu\gamma^\mu_{\sigma\sigma}(\mathbf{p} + \mathbf{q}, P) = S_\sigma(G_0^{-1}(\mathbf{p} + \mathbf{q}) - G_0^{-1}(\mathbf{p})).
\]

This leads to the WIs for the response kernels \( q_\mu\tilde{Q}^{\mu\nu}(Q) = 0 \) where \( \tilde{Q}^{\mu\nu} = Q_0^{\mu\nu} + \frac{\pi}{2} h^{\mu\nu} \). It can be shown as follows

\[
q_\mu\tilde{Q}^{\mu\nu}(Q) = 2\sum_p [|G_0^{-1}(\mathbf{p} + \mathbf{q}) - G_0^{-1}(\mathbf{p})][G_0(\mathbf{p} + \mathbf{q})\gamma^\nu(\mathbf{p},\mathbf{q}, P)G_0(\mathbf{p}) - \frac{n}{m}q^\nu(1 - \eta^{\mu\sigma})] = 2\sum_p G_0(\mathbf{p})[\gamma^\nu(\mathbf{p},\mathbf{q}, P) - \gamma^\nu(\mathbf{p}, P, \mathbf{q})] - \frac{n}{m}q^\nu(1 - \eta^{\mu\sigma}) = 2\sum_p G_0(\mathbf{p})q^\nu(1 - \eta^{\mu\sigma}) - \frac{n}{m}q^\nu(1 - \eta^{\mu\sigma}) = 0,
\]

where \( \sum_p \equiv T \sum_{i\omega_n} \sum_{\mathbf{p}} \). The WIs for the response kernels further lead to the conservation of the perturbed current \( q_\mu\delta J^\mu(Q) = 0 \). For the spin response, the same conclusions can be obtained.

Next we show that the response function satisfies the compressibility sum rule (14) and \( f \)-sum rule

\[
\nabla_\nu = -Q^{00}(\omega = 0, \mathbf{q} \to 0),
\]

\[
\int_{-\infty}^{+\infty} d\omega \chi_\rho(\omega, \mathbf{q}) = \frac{nq^2}{m},
\]

where \( \chi_\rho = -\frac{1}{\pi} \text{Im} Q^{00}_{\rho\rho} \) is the density susceptibility and the analytical continuation \( i\Omega_t \to \omega + i0^+ \) has been applied. Although these two sum rules can be directly proven by the explicit expressions of the response functions given in Appendix A, here we give a more instructive proof that has a nice connection with the U(1) symmetries in the Hamiltonian. For the compressibility sum rule, we have

\[
\frac{\partial n}{\partial \mu} = 2\sum_p \frac{\partial G_0(\mathbf{p})}{\partial \mu} = -2\sum_p G_0^2(\mathbf{p})\frac{\partial G_0^{-1}(\mathbf{p})}{\partial \mu} = -2\lim_{Q \to 0} \sum_p G_0(\mathbf{p} + \mathbf{q})G_0(\mathbf{p})|_{\omega = 0} = -Q^{00}(\omega = 0, \mathbf{q} \to 0),
\]

where the expression (14) and \( \gamma^0(P + Q, P) = 1 \) have been applied. In fact, this is a special case of the \( Q \)-limit WI \( \Gamma^0 = 1 - (\partial \Sigma/\partial \mu) \), where \( \Gamma^0 \) and \( \gamma^0 \) are the full and bare vertex functions, \( \Sigma \) is the self energy of fermions, and the limit \( \omega = 0, q \to 0 \) has been taken. For non-interacting Fermi gases \( \Sigma = 0 \) and \( \Gamma^0 = \gamma^0 = 1 \) so the \( Q \)-limit WI is trivially satisfied. For the \( f \)-sum rule, we need to implement the real time formalism to describe the non-equilibrium transport theory. The real time response function corresponding to Eq. (13) is

\[
Q^{\mu\nu}_0(t - t', \mathbf{q}) = -i\theta(t - t')\langle [J^\mu(t, \mathbf{q}), J^\nu(t', -\mathbf{q})] \rangle,
\]

where the time-dependent Heisenberg operator is defined by \( \mathcal{O}(t) = e^{i\mathcal{H}t} \mathcal{O}e^{-i\mathcal{H}t} \). The imaginary part of \( Q^{\mu\nu}_0 \) is

\[
\text{Im}Q^{\mu\nu}_0(t - t', \mathbf{q}) = -\frac{1}{2} i\langle [J^\mu(t, \mathbf{q}), J^\nu(t', -\mathbf{q})] \rangle.
\]

Using \( \omega \text{Im}Q^{\mu\nu}_0 = \text{Im}(\omega Q^{\mu\nu}_0) \), we have

\[
-\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im}Q^{\mu\nu}_0(\omega, \mathbf{q}) = \langle [\frac{\partial n(t, \mathbf{q})}{\partial t}, n(t, -\mathbf{q})]_\omega \rangle = \langle [\mathbf{q}, \mathbf{J}(t, \mathbf{q}), n(t, -\mathbf{q})]_\omega \rangle.
\]
\[ r_{\alpha \beta}^\mu = - r_{\mu \alpha}^\beta \]

Figure 1: The diagrams for the spin vertex function. Different spin indices have different signs.

\[
\begin{align*}
= & \sum_{pp'} \mathbf{q} \cdot \frac{p + \frac{q}{2}}{m} (e^{iHt} (\psi_{\sigma p} \{ \psi_{\sigma' p + q}, \psi_{\sigma' p'} + q \} \psi_{\sigma' p'} - \psi_{\sigma' p + q} \{ \psi_{\sigma p}, \psi_{\sigma' p'} \} \psi_{\sigma p + q}) e^{-iHt}) \omega \\
= & \sum_{p} \mathbf{q} \cdot \left( \frac{(p + \frac{q}{2}) - (p - \frac{q}{2})}{m} \langle \psi^\dagger_{\sigma p}(t) \psi_{\sigma p}(t) \rangle \right) = \frac{\mathbf{q}^2}{m},
\end{align*}
\]

where in the first line we used the conservation of the current \( \partial_\mu J^\mu = 0 \), in the last line we changed the variables \( \mathbf{p} \to \mathbf{p} - \mathbf{q} \) for the second term. The subscript \( \omega \) means the Fourier transform of the function with the argument \( \omega \). One thus see that the two sum rules are closely connected to the \( \text{U}(1) \text{EM} \) symmetry.

\[ \Delta(x) = g \langle \psi_\downarrow(x) \psi_\uparrow(x) \rangle. \]

Following the mean-field BCS approximation, the Hamiltonian without any external field becomes

\[
H = \int d^3 x \psi^\dagger_{\sigma}(x) \left( \frac{\mathbf{p}^2}{2m} - \mu \right) \psi_{\sigma}(x) - \int d^3 x \left( \Delta(x) \psi^\dagger_{\uparrow}(x) \psi^\dagger_{\downarrow}(x) + \text{h.c.} \right).
\]

One may see that the \( \text{U}(1) \text{EM} \) symmetry is spontaneously broken if \( \Delta(x) \neq 0 \) below \( T_c \) while the \( \text{U}(1)_z \) symmetry remains intact. Hence the original Hamiltonian has a \( \text{U}(1) \times \text{U}(1) \) symmetry while the BCS Hamiltonian in the broken-symmetry phase only has a \( \text{U}(1) \) symmetry. The phase with a broken \( \text{U}(1) \text{EM} \) symmetry below \( T_c \) brings challenges of how to cast its associated linear response theory in a gauge invariant form. Below \( T_c \), the breaking of \( \text{U}(1) \text{EM} \) and the unbroken \( \text{U}(1)_z \) symmetry may be called a \"spin-charge separation\" in BCS theory. We will explore this phenomenon in depth after we present the consistent linear response theories in both density and spin channels. Above \( T_c \), both symmetries are not broken and the charge and spin degrees of freedom are not separated there as they do below \( T_c \).

Here we present a \( \text{U}(1) \text{EM} \) gauge-invariant linear response theory for BCS superfluids based on the consistent fluctuation of the order-parameter (CFOP) approach. The gauge invariance is basically a one-particle problem from the point of view of quantum field theory while BCS theory is essentially a many-body theory (for a review, see [18]). This contrast highlights the importance of Ward identity (or its generalized form) since one can check the gauge invariance of a theory by verifying the corresponding WI and we will give some concrete examples.

### A. Nambu Based Notation for Linear Response

As we have seen previously, there are terminologies like the Nambu space, one-dimensional space and the \"Ward identity\" (WI) associated with different physical quantities. Here we explain them in details. The Nambu space is the space in which Nambu spinors are defined. For non-relativistic theories, it is a two-dimensional space and operators
are written as two by two matrices. The basic framework of the CFOP linear response theory for BCS superfluids is easier to develop in the Nambu space. The terminology one-dimensional space is the abbreviation of the one-dimensional representation space of fermion operators. The representation space of fermion operators is actually the four-dimensional spinor representation space of the Lorentz group. In the non-relativistic limit one may approximate it by a one-dimensional space representation if the spin is simply labeled as a subscript. Most linear response theories for normal Fermi gases are formulated in this space since the corrections to the interacting vertex between fermions and external fields are hard to be presented in the Nambu space. The Ward identity in quantum field theories refers to the relation among the response (correlation) functions that describe the effects of symmetry transformations allowed by the theory [19]. Here we use the Ward identity specifically for a diagrammatic identity between the vertex function and the fermion propagator, which reflects the symmetry from the EM gauge transformation. In the Nambu space, we will adopt Nambu’s convention to characterize each of these identities as a “generalized Ward identity” (GWI). There are seminal reviews [2, 18] on how to cast BCS theory in the Nambu space. However, there are less reviews on the one-dimensional space. As discussed in Ref. [10], the fluctuations of the order parameter as \( \Delta’ \) of the order parameter as \( \Delta \) are introduced by the perturbation of the order parameter.

We define \( \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \) in the Nambu space and introduce the Nambu-Gorkov spinors

\[
\Psi_p = \begin{bmatrix} \psi^\dagger_{1p} \\ \psi^\dagger_{-1p} \end{bmatrix}, \quad \Psi^\dagger_p = [\psi^\dagger_{1p}, \psi^\dagger_{-1p}].
\]  

Here \( \sigma_i, i = 1, 2, 3 \) are the Pauli matrices. The Hamiltonian \( H \) in momentum space after the mean-field BCS approximation becomes

\[
H = \sum_p \Psi^\dagger_p \xi_p \sigma_3 \Psi_p + \sum_{pq} \sum_p \Psi^\dagger_{p+q} \left( -\frac{p+q}{m} \mathbf{A}_q + \Phi_q \sigma_3 - \Delta_q \sigma_+ - \Delta_q^* \sigma_- \right) \Psi_p,
\]

where \( \Phi_q = \xi_p \sigma_3 - \Delta \sigma_1 \) is an energy operator and \( \tilde{\gamma}^\mu(p+q,p) = (\sigma_3, \frac{p+q}{m}) \) is the bare EM vertex in the Nambu space. As discussed in Ref. [10], the fluctuations \( \Delta_1q \) and \( \Delta_2q \) introduce the amplitude mode and phase mode as the collective modes introduced by the perturbation of the order parameter.

For the equilibrium par, the quasi-particle energy is given by \( E_p = \sqrt{\xi^2_p + \Delta^2} \). The propagator in the Nambu space is

\[
\hat{G}(P) \equiv \hat{G}_p(i\omega_n) = \frac{1}{i\omega_n - \hat{E}_p} = \begin{pmatrix} G(P) & F(P) \\ F(P) & -G(-P) \end{pmatrix},
\]

where

\[
G(P) = \frac{u^2_p}{i\omega_n - \hat{E}_p} + \frac{v^2_p}{i\omega_n + \hat{E}_p}, \quad F(P) = -u_p v_p \left( \frac{1}{i\omega_n - \hat{E}_p} - \frac{1}{i\omega_n + \hat{E}_p} \right)
\]

are the single-particle Green’s function and anomalous Green’s function respectively and \( u^2_p, v^2_p = \frac{1}{2}(1 \pm \frac{\xi_p}{\hat{E}_p}) \).

The number density and the order parameter in equilibrium can be expressed by the propagator as

\[
n = \text{Tr} \sum_p (\sigma_3 \hat{G}(P)), \quad \Delta = g \text{Tr} \sum_p (\sigma_1 \hat{G}(P)),
\]

where \( g \) is the coupling constant.
which give the number and gap equations
\[ n = \sum_p \left[ 1 - \frac{\xi_p}{E_p} (1 - 2f(E_p)) \right], \quad \frac{1}{g} = \sum_p \frac{1 - 2f(E_p)}{2E_p}. \] (31)

We can also introduce the counterpart of the coefficients \( u, v \) in the Nambu space as \( \hat{u}_p, \hat{v}_p = \frac{1}{2} (1 \pm \frac{\xi_p}{E_p}) \). Using the properties \( \hat{u}_p^2 = \hat{u}_p, \hat{v}_p^2 = \hat{v}_p \) and \( \hat{u}_p \hat{v}_p = 0 \), one can show that in the Nambu space the propagator has the form
\[ \hat{G}(P) = \frac{\hat{u}_p^2}{i\omega_n - E_p} + \frac{\hat{v}_p^2}{i\omega_n + E_p}. \] (32)

This has a similar expression as the single particle Green’s function shown in Eq. (29). In fact, any function of \( i\omega_n - E_p \), for example \( F(i\omega_n - E_p) \), can be expressed as \( F(i\omega_n - E_p) = \hat{u}_p(i\omega_n - E_p) + \hat{v}_p(i\omega_n + E_p) \) [14]. It is an interesting property of BCS theory in the Nambu space and will be useful for deriving the response functions.

The interacting Hamiltonian \( H' \) in Eqs. (27) can be cast in the form
\[ H' = \sum_{pq} \bar{\Psi}_p^{\dagger}(\tau) \hat{\Sigma}(p + q, p) \Psi_{p+q}(\tau), \] (33)
where
\[ \hat{\Phi}_q = (\Delta_{1q}, \Delta_{2q}, A_{\mu q})^T, \quad \hat{\Sigma}(p + q, p) = (\sigma_1, \sigma_2, \tilde{\gamma}^{\mu}(p + q, p))^T, \] (34)
are defined as the generalized driving potential and generalized interacting vertex, respectively. By using the imaginary time formalism, we assume that under the perturbation from \( H' \), the generalized perturbed current \( \delta \hat{J} \) is given by
\[ \delta \hat{J}(\tau, q) = \sum_p (\bar{\Psi}_p^{\dagger}(\tau) \hat{\Sigma}(p + q, p) \Psi_{p+q}(\tau)) + \frac{n}{m} \delta^{ij} \hat{h}^{\mu\nu} A_{\nu}(\tau, q). \] (35)

Here \( \delta J_{3}^{\mu} \) denotes the perturbed EM current and \( \delta J_{1,2} \) denote the perturbations of the gap function. The linear response theory is written in a matrix form
\[ \delta \hat{J}(\omega, q) = \hat{Q}_0(\omega, q) \cdot \hat{\Phi}(\omega, q), \]
\[ = \begin{pmatrix} Q_{11}(\omega, q) & Q_{12}(\omega, q) & \tilde{Q}_{13}(\omega, q) \\ Q_{21}(\omega, q) & Q_{22}(\omega, q) & \tilde{Q}_{23}(\omega, q) \\ Q_{31}(\omega, q) & \tilde{Q}_{32}(\omega, q) & \tilde{Q}_{33}(\omega, q) + \frac{n}{m} \hat{h}^{\mu\nu} \end{pmatrix} \begin{pmatrix} \Delta_{1}(\omega, q) \\ \Delta_{2}(\omega, q) \\ A_{\nu}(\omega, q) \end{pmatrix}. \] (36)
The response functions \( Q_{ij} \) are
\[ Q_{ij}(\tau - \tau', q) = -\sum_{pp'} T_{\tau'} [\tilde{\Sigma}(p + q, p) \Psi_{p+q}(\tau) \bar{\Psi}_p^{\dagger}(\tau') \tilde{\Sigma}(p', p' + q) \Psi_{p'}(\tau')]. \] (37)

After applying Fourier transform and Wick’s theorem, we obtain
\[ Q_{ij}(i\Omega, q) = \text{Tr} T \sum_{\omega_n} \left( \tilde{\Sigma}(P + Q, P) \hat{G}(P + Q) \Sigma_j(P + Q, P + Q) \hat{G}(P) \right). \] (38)

For convenience, we have defined \( \tilde{\gamma}^{\mu}(P + Q, P) \equiv \tilde{\gamma}^{\mu}(p + q, p, q) \), hence \( \tilde{\Sigma}(P + Q, P) = \tilde{\Sigma}(p + q, p) \). By using the expression (32) the response functions are evaluated and shown in Appendix [A].

The perturbation of the order parameter and the EM perturbation are treated on equal footing and this will naturally lead to the gauge invariance of the CFOP linear response theory. The gap equation gives the self-consistent condition \( \delta J_{1,2} = -\frac{1}{2} \Delta_{1,2} \). Applying this relation to Eq. (36), we get
\[ \Delta_1 = -\frac{Q_{13}^\nu \tilde{Q}_{22} - Q_{23}^\nu Q_{12}}{\tilde{Q}_{11} Q_{22} - Q_{12} Q_{21}} A_{\nu}, \quad \Delta_2 = -\frac{Q_{23}^\nu \tilde{Q}_{11} - Q_{13}^\nu Q_{21}}{\tilde{Q}_{11} Q_{22} - Q_{12} Q_{21}} A_{\nu}. \] (39)

where \( \tilde{Q}_{11} \equiv \frac{2}{g} + Q_{11} \) and \( \tilde{Q}_{22} \equiv \frac{2}{g} + Q_{22} \). After substituting the results into
\[ \delta J^{\mu} = Q_{31}^{\mu} \Delta_1 + Q_{32}^{\mu} \Delta_2 + (Q_{33}^{\mu\nu} + \frac{n}{m} h^{\mu\nu}) A_{\nu}, \] (40)
we get \( \delta J^\mu = K^{\mu
u}A_\nu \), where the corrected EM response kernel \( K^{\mu
u} \) including the effects of fluctuations of the order parameter is given by

\[
K^{\mu
u} = \tilde{Q}_{33}^{\mu\nu} + \delta K^{\mu\nu}, \quad \delta K^{\mu\nu} = -\frac{\tilde{Q}_{11}Q_{23}^{\mu\nu} + \tilde{Q}_{22}Q_{21}^{\mu\nu} - Q_{12}Q_{33}^{\mu\nu} - Q_{21}Q_{32}^{\mu\nu}}{Q_{11}Q_{22} - Q_{12}Q_{21}}. \tag{41}
\]

Here \( \tilde{Q}_{33}^{\mu\nu} = Q_{33}^{\mu\nu} + \frac{n}{m} h^{\mu\nu} \). The gauge invariance condition for the perturbed current, \( q_\mu \delta J^\mu = 0 \), is satisfied if \( q_\mu K^{\mu\nu}(Q) = 0 \). This is further guaranteed by the following WIs for the response functions

\[
q_\mu Q_{31}^{\mu\nu} = -2i\Delta Q_{21}, \quad q_\mu Q_{32}^{\mu\nu} = -2i\Delta Q_{22}, \quad q_\mu \tilde{Q}_{33}^{\mu\nu} = -2i\Delta Q_{23}. \tag{42}
\]

These relations are more complicated than \( q_\mu \tilde{Q}_{33}^{\mu\nu}(Q) = 0 \) for noninteracting Fermi gases since the perturbation of the order parameter comes into the linear response theory. The gauge invariance condition of the perturbed EM current is immediately derived from Eq. (42).

\[
q_\mu K^{\mu\nu} = -2i\Delta Q_{23} + 2i\Delta Q_{23} + 2i\Delta Q_{23} = 0. \tag{43}
\]

Now we sketch the proof of the WIs (42). The bare inverse propagator in the Nambu space is \( \hat{G}^{-1}(P) = i\omega_n - \xi_p \sigma_3 \) and the self energy is \( \Sigma = -\Sigma_1 \). Hence from Eq. (28) the full inverse propagator can be expressed as \( \hat{G}^{-1}(P) = \hat{G}^{-1}(P) - \Sigma \), and one can verify that

\[
\sigma_3 \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P) \sigma_3 = i\Sigma \sigma_3 - (\xi_{p+q} - \xi_p) + 2i\Delta \sigma_2 = q_\mu \hat{\gamma}^\mu(P + Q, P) + 2i\Delta \sigma_2. \tag{44}
\]

In fact, as we will point out later, this identity is actually the GWI that connects the full EM interacting vertex and the full propagator in the Nambu space. Now we show how the WIs are derived from Eq. (41). For the first identity of the WIs, we have

\[
q_\mu Q_{31}^{\mu\nu} + 2i\Delta Q_{21} = \text{Tr} \sum_P \left[ (\sigma_3 \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P) \sigma_3) \hat{G}(P + Q + \sigma_1 \hat{G}(P) \right] = \text{Tr} \sum_P \left[ i\sigma_2 \hat{G}(P) \right] + \text{Tr} \sum_P \left[ \hat{G}(P + Q) i\sigma_2 \right] = 2\text{Tr} \sum_P \left[ i\sigma_2 \hat{G}(P) \right] = 0, \tag{45}
\]

where the cyclic property of the trace has been applied. For the second identity of WIs, we have

\[
q_\mu Q_{32}^{\mu\nu} + 2i\Delta Q_{22} = -2i\text{Tr} \sum_P \left[ \sigma_1 \hat{G}(P) \right] = -4i\text{Tr} \sum_P F(P) = -4i\Delta \frac{q'}{g}. \tag{46}
\]

Therefore \( q_\mu Q_{32}^{\mu\nu} = -2i\Delta (Q_{22} + \frac{q'}{g^2}) = -2i\Delta \hat{Q}_{22} \) and we get the second WI for the response functions. For the last WI, we have

\[
q_\mu \hat{Q}_{33}^{\mu\nu} + 2i\Delta Q_{23}^{\mu\nu} = \text{Tr} \sum_P \left[ \sigma_3 \hat{\gamma}^{\nu}(P + Q, P) \hat{G}(P) \right] - \text{Tr} \sum_P \left[ \hat{G}(P + Q) \hat{\gamma}^{\nu}(P, P + Q) \sigma_3 \right] - \frac{n}{m} q' (1 - \eta^\nu) \\
= \sum_P \left( \text{Tr} \left( \hat{G}(P) \sigma_3 \hat{\gamma}^{\nu}(P + Q, P) - \hat{\gamma}^{\nu}(P + Q, P) \right) \right) - \frac{n}{m} q' (1 - \eta^\nu) \\
= \frac{q'}{m} (1 - \eta^\nu) \sum_P \text{Tr} \left( \hat{G}(P) \sigma_3 \right) - \frac{n}{m} q' (1 - \eta^\nu) = 0, \tag{47}
\]

where the fact that \( \sigma_3 \) commutes with \( \hat{\gamma}^\mu \) has been applied and the number equation (30) has been used.

The CFOP linear response theory also satisfies the \( f \)-sum rule

\[
\int_{-\infty}^{+\infty} d\omega \chi_{\nu \rho}^{\sigma}(\omega, \mathbf{q}) = n \frac{q^2}{m}. \tag{48}
\]
Here $\chi_{\mu \rho} = -\frac{1}{\pi} \text{Im} K^{00}$ with $K^{00}$ given by the 00-component of Eq. (41):

$$K^{00} = \frac{\hat{Q}_{33}^{00} - \frac{\hat{Q}_{11} Q_{22}^{00} Q_{23}^{00}}{\hat{Q}_{11} Q_{22} - \hat{Q}_{12} Q_{21}}}{Q_{11} Q_{22} - \hat{Q}_{12} Q_{21}}. \quad (49)$$

The following lemma will be useful for the proof of the f-sum rule

$$\int_{-\infty}^{+\infty} d\omega - \frac{1}{\pi} \text{Im}[q \cdot Q_{03}^{0} (\omega, q)] = \frac{n q^2}{m}. \quad (50)$$

The proof of this lemma can be found in Appendix [3] From $\omega \text{Im} K^{00} = \text{Im}(\omega K^{00})$ and the third equation of the WIs [42], we have

$$\omega K^{00} = q \cdot Q_{03}^{0} - 2i \Delta Q_{23}^{0} - \frac{1}{Q_{11} Q_{22} - \hat{Q}_{12} Q_{21}} \times (\hat{Q}_{11} q \cdot Q_{22}^{0} Q_{23}^{0} - 2i \Delta \hat{Q}_{11} \hat{Q}_{22} Q_{23}^{0} + \hat{Q}_{22} q \cdot Q_{31} Q_{13}^{0} - 2i \Delta \hat{Q}_{22} Q_{21} Q_{13}^{0} \hat{Q}_{12} q \cdot Q_{31} Q_{13}^{0} + 2i \Delta \hat{Q}_{12} Q_{21} Q_{13}^{0} \hat{Q}_{21} q \cdot Q_{32} Q_{13}^{0} + 2i \Delta \hat{Q}_{21} Q_{22} Q_{13}^{0}). \quad (51)$$

Note $Q_{12} = -Q_{21}$, $Q_{02}^{0} = -Q_{22}^{0}$, $Q_{23} = -Q_{32}$, $Q_{13}^{0} = Q_{31}^{0}$ and $Q_{33} = Q_{31}$, we have

$$\omega K^{00} = q \cdot Q_{03}^{0} - \frac{\hat{Q}_{11} q \cdot Q_{32} Q_{23}^{0} + \hat{Q}_{22} q \cdot Q_{31} Q_{13}^{0} - \hat{Q}_{12} q \cdot Q_{31} Q_{23}^{0} - \hat{Q}_{21} q \cdot Q_{32} Q_{13}^{0}}{Q_{11} Q_{22} - \hat{Q}_{12} Q_{21}}. \quad (52)$$

Using the lemma, it can be shown that proving the f-sum rule is equivalent to proving

$$\int_{-\infty}^{+\infty} d\omega \text{Im} \frac{\hat{Q}_{11} q \cdot Q_{32} Q_{23}^{0} + \hat{Q}_{22} q \cdot Q_{31} Q_{13}^{0} - \hat{Q}_{12} q \cdot Q_{31} Q_{23}^{0} - \hat{Q}_{21} q \cdot Q_{32} Q_{13}^{0}}{Q_{11} Q_{22} - \hat{Q}_{12} Q_{21}} = 0. \quad (53)$$

Note that $\hat{Q}_{11}$, $\hat{Q}_{22}$, $Q_{23}^{0}$ and $Q_{23}$ are even functions of $\omega$, while $Q_{12}$, $Q_{13}$ and $Q_{02}^{0}$ are odd functions of $\omega$. Hence the integrand is an odd function of $\omega$, then the integral indeed vanishes. Therefore, the f-sum rule [48] is satisfied by the density linear response theory. From the proof we can see that we need the explicit expressions of the response functions. In fact we can not prove the f-sum rule simply by using the WIs [42] or the conservation law of current $\partial_{\mu} J^{\mu} = 0$ alone, which is different from what we have done in the case of non-interacting Fermi gases.

### B. WI and Q-limit WI of the CFOP Linear Response Theory of BCS Superfluids

#### 1. GWI and Q-limit GWI in the Nambu space

To complete our discussions on the CFOP theory in the Nambu space, we now investigate the GWI which connects the full EM vertex and the fermion propagator. Due to different representations for the fermions, we will present the GWI in the Nambu space as well as the WI in the one-dimensional space. One can verify that the bare EM vertex and the propagator given in Sec. [41] satisfies the bare GWI in the Nambu space

$$q_{\mu} \hat{\gamma}^{\nu}(P + Q, P) = \sigma_{3} \hat{G}_{0}^{-1}(P + Q) - \hat{G}_{0}^{-1}(P) \sigma_{3}. \quad (54)$$

Hence, as pointed by Nambu [4], the GWI for full EM vertex has the similar structure

$$q_{\mu} \hat{\Gamma}^{\nu}(P + Q, P) = \sigma_{3} \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P) \sigma_{3}, \quad (55)$$

Moreover, the corrected EM response kernel $K^{\mu \nu}$ given by Eq. (41) should be evaluated by the full EM vertex and propagator in the Nambu space as

$$K^{\mu \nu}(Q) = \text{Tr} \sum_{P} (\hat{\Gamma}^{\mu}(P + Q, P) \hat{G}(P + Q) \hat{\gamma}^{\nu}(P, P + Q) \hat{G}(P)) + \frac{n}{m} h^{\mu \nu}, \quad (56)$$

a similar form in the one dimensional space will be presented later in Eq. (73). In fact, this full EM vertex $\hat{\Gamma}^{\mu}(P + Q, P)$ can be inferred from the expression of $K^{\mu \nu}$. We define

$$\Pi_{1}^{\mu} = \begin{pmatrix} Q_{31}^{0} & Q_{12}^{0} & Q_{11}^{0} & Q_{22}^{0} \\ Q_{11} & Q_{22} & Q_{21} & Q_{22} \end{pmatrix}, \quad \Pi_{2}^{\mu} = \begin{pmatrix} Q_{31}^{0} & Q_{12}^{0} & Q_{11}^{0} & Q_{22}^{0} \\ Q_{11} & Q_{22} & Q_{21} & Q_{22} \end{pmatrix}. \quad (57)$$
Hence, from Eq. (41) $K^{\mu\nu}$ can be expressed as

$$K^{\mu\nu}(Q) = Q_{33}^{\mu\nu}(Q) - \Pi_{13}^{\mu}(Q)Q_{13}^{\nu}(Q) - \Pi_{23}^{\mu}(Q)Q_{23}^{\nu}(Q) + \frac{n}{m}h^{\mu\nu}$$

$$= \text{Tr} \sum_{P} (\gamma^{\mu}(P + Q, P)\hat{G}(P + Q)\gamma^{\nu}(P, P + Q)\hat{G}(P)) - \Pi_{1}^{\mu}(Q)\text{Tr} \sum_{P} (\sigma_{1}\hat{G}(P + Q)\gamma^{\nu}(P, P + Q)\hat{G}(P))$$

$$- \Pi_{2}^{\mu}(Q)\text{Tr} \sum_{P} (\sigma_{2}\hat{G}(P + Q)\gamma^{\nu}(P, P + Q)\hat{G}(P)) + \frac{n}{m}h^{\mu\nu}$$

$$= \text{Tr} \sum_{P} ((\gamma^{\mu}(P + Q, P) - \sigma_{1}\Pi_{1}(Q) - \sigma_{2}\Pi_{2}(Q))\hat{G}(P + Q)\gamma^{\nu}(P, P + Q)\hat{G}(P)) + \frac{n}{m}h^{\mu\nu}. \tag{58}$$

where in the second line we have substituted the expression $\frac{Q^{\mu\nu}}{Q_{j}^{\mu\nu}}$ for $Q_{33}^{\mu\nu}, Q_{13}^{\nu}$ and $Q_{23}^{\nu}$. Comparing Eq. (56) with the last line of Eq. (58), one can find that the full EM vertex is given by

$$\hat{\Gamma}^{\mu}(P + Q, P) = \hat{\gamma}^{\mu}(P + Q, P) - \sigma_{1}\Pi_{1}^{\mu}(Q) - \sigma_{2}\Pi_{2}^{\mu}(Q). \tag{59}$$

By applying the WIs (42), one can see that $\Pi^{\mu}_{13}(Q)$ satisfies $q_{\mu}\Pi^{\mu}_{1}(Q) = 0$ and $q_{\mu}\Pi^{\mu}_{2}(Q) = -2i\Delta$. Hence, the full EM vertex given by Eq. (59) further satisfies

$$q_{\mu}\hat{\Gamma}^{\mu}(P + Q, P) = q_{\mu}\hat{\gamma}^{\mu}(P + Q, P) + 2i\Delta\sigma_{2} = \sigma_{3}\hat{G}_{0}^{-1}(P + Q) - \hat{G}_{0}^{-1}(P)\sigma_{3}, \tag{60}$$

where Eq. (44) as been applied. Therefore the full EM vertex indeed obeys the important GWI in the Nambu space. Having the expression of full EM vertex, one can show that it also respects the $Q$-limit GWI

$$\lim_{q \to 0} \hat{\Gamma}^{0}(P + Q, P) = \frac{\partial G^{-1}(P)}{\partial \mu} = \sigma_{3} - \frac{\partial \hat{S}(P)}{\partial \mu}. \tag{61}$$

which is a sufficient and necessary condition for the compressibility sum rule $[6]$, i.e.,

$$\frac{\partial n}{\partial \mu} = -K^{00}(\omega = 0, q \to 0). \tag{62}$$

This is proven as the following:

$$\frac{\partial n}{\partial \mu} = \text{Tr} \sum_{P} \left( \frac{\partial \hat{G}(P)}{\partial \mu} \sigma_{3} \right) = -\text{Tr} \sum_{P} \left( \hat{G}(P) \left( \sigma_{3} - \frac{\partial \hat{S}(P)}{\partial \mu} \right) \hat{G}(P) \sigma_{3} \right)$$

$$= -\text{Tr} \sum_{P} \left( \hat{\Gamma}^{0}(P, P)\hat{G}(P)\hat{\gamma}^{0}(P, P)\hat{G}(P) \right) = -K^{00}(\omega = 0, q \to 0). \tag{63}$$

The $Q$-limit GWI $[61]$ has a profound physical implication for interacting Fermi gases. The LHS of it is associated with the equation of state derivable from one-particle correlation functions, while the RHS involves the response function evaluated by two-particle correlation functions. Hence it builds a bridge connecting the one-particle and two-particle formalisms.

There is a subtlety that needs to be addressed here. It is on whether the $Q$-limit GWI $[61]$ can be derived from the GWI $[55]$, or equivalently, whether the $Q$-limit GWI serves as an independent constraint. Here we argue that the GWI imposes no constraint on the form of the $Q$-limit GWI. The reason is because in the limit $\omega = 0$ and $q \to 0$, the GWI $[55]$ becomes $q \cdot \Gamma(P, P) = \lim_{q \to 0} [\sigma_{3} \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P)\sigma_{3}]_{\omega = 0}$, where $\Gamma$ is the spatial component of the vertex function. However, the $Q$-limit GWI is an identity regarding $\hat{\Gamma}^{0}$ so the GWI does not reveal any information about the $Q$-limit GWI in the limit $\omega = 0$ and $q \to 0$. Thus one should treat the $Q$-limit GWI as an independent constraint of a linear response theory. The problem of how to obtain a consistent expression for the compressibility as discussed in Ref. $[6]$ thus can be rephrased as how a linear response theory can satisfy the compressibility sum rule, or more directly whether the $Q$-limit GWI could be satisfied.

Now we show that the $0$-th component of the $\hat{\Gamma}^{\mu}$ given by Eq. (59) satisfies the $Q$-limit GWI. Since $Q_{12}(Q) = -Q_{21}(Q) = Q_{23}(Q) = -Q_{32}(Q) = 0$ when $\omega = 0$, then $\Pi^{0}_{1}(Q) = 0$ in this limit. Hence we have

$$\lim_{q \to 0} \hat{\Gamma}^{0}(P + Q, P) = \sigma_{3} - \sigma_{1} \lim_{q \to 0} \Pi^{0}_{1}(Q) = \sigma_{3} - \sigma_{1} \frac{Q_{0}^{0}(0, q \to 0)}{Q_{11}(0, q \to 0)}, \tag{64}$$

where Eq. (41) as been applied.
where
\[ Q_{13}(0,q \to 0) = -\Delta \sum_p \frac{\xi_p}{E_p^2} \left[ 1 - 2f(E_p) + 2 \frac{\partial f(E_p)}{\partial E_p} \right], \]
\[ \hat{Q}_{11}(0,q \to 0) = \Delta^2 \sum_p \frac{1}{E_p^2} \left[ 1 - 2f(E_p) + 2 \frac{\partial f(E_p)}{\partial E_p} \right]. \] (65)

Note the self-energy operator is given by \( \hat{\Sigma} = -\sigma_1 \Delta \), hence we have
\[ \frac{\partial \hat{\Sigma}}{\partial \mu} = -\sigma_1 \frac{\partial \Delta}{\partial \mu}. \] (66)

We need to evaluate \( \frac{\partial \Delta}{\partial \mu} \) from the gap equation (31). Differentiating both sides with respect to \( \mu \) one gets
\[ \frac{\partial \Delta}{\partial \mu} = \sum_p \frac{\xi_p}{E_p^2} \left[ 1 - 2f(E_p) + 2 \frac{\partial f(E_p)}{\partial E_p} \right] = -\frac{Q_{13}^0(0,q \to 0)}{Q_{11}(0,q \to 0)} = -\lim_{q \to 0} \Pi_0^0(Q) \bigg|_{\omega=0}. \] (67)

Plug this into the right-hand side of Eq. (64), we prove the \( Q \)-limit GWI (63) for the full EM vertex.

Here we summarize our key results in the Nambu space:

- **Response kernel**:\[ K^{\mu\nu}(Q) = \text{Tr} \sum_P \left( \hat{\Gamma}^{\mu}(P + Q,P)\hat{G}(P + Q)\hat{\gamma}^{\nu}(P,P + Q)\hat{G}(P) \right) + \frac{n}{m} h^{\mu\nu}, \] (68)

- **Full EM interacting vertex**:\[ \hat{\Gamma}^{\mu}(P + Q,P) = \hat{\gamma}^{\mu}(P + Q,P) - \sigma_1 \Pi_1^0(Q) - \sigma_2 \Pi_2^0(Q), \] (69)

- **Generalized Ward identity**:\[ \sigma_3 \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P)\sigma_3 = q_\mu \hat{\Gamma}^{\mu}(P + Q,P), \] (70)

- **\( Q \)-limit generalized Ward identity**:\[ \hat{\Gamma}^0(P,P) = \sigma_3 - \frac{\partial \hat{\Sigma}(P)}{\partial \mu}, \] (71)

- **\( f \)-sum rule**:\[ \int_{-\infty}^{+\infty} d\omega \chi_{0\mu}(\omega, q) = \frac{n q^2}{m}. \] (72)

We emphasize that a consistent linear response theory for BCS superfluids should satisfy the GWI, \( Q \)-limit GWI, and \( f \)-sum rule.

2. WI and \( Q \)-limit WI in the one-dimensional space

We have shown the details of the CFOP theory for BCS superfluids in the Nambu space. Instead of formulating BCS theory in the matrix form in the Nambu space, one may use the Green’s functions (29) to formulate BCS theory. The Leggett-BCS theory of BCS-BEC crossover [20, 21] follows this path and we call this “BCS theory in the one-dimensional space”. This representation is convenient when generalized to the theories for BCS-BEC crossover. One may build the CFOP theory step by step in the one-dimensional space similar to what we have done so far. Here we
simply extract the key results from those in the Nambu space since the underlying physics is the same. We only need to find the full EM vertex $\Gamma^\mu$ in the one-dimensional space such that the EM response kernel can be evaluated by

$$K^{\mu\nu}(Q) = 2 \sum_P \Gamma^\mu(P + Q, P)G(P + Q)\gamma^\nu(P, P + Q)G(P) + \frac{n}{m}h^{\mu\nu},$$

(73)

where $\gamma^\mu$ is the bare EM vertex given in Section III. Moreover, the full EM vertex $\Gamma^\mu$ must obey the WI

$$q_\mu \Gamma^\mu(P + Q, P) = G^{-1}(P + Q) - G^{-1}(P)$$

(74)

and the $Q$-limit WI

$$\lim_{Q \to 0} \Gamma^0(P + Q, P)|_{\omega = 0} = \frac{\partial G^{-1}(P)}{\partial \mu} = 1 - \frac{\partial \Sigma(P)}{\partial \mu}.$$  

(75)

The expressions of the gauge-invariant response kernels should not depend on the space in which we evaluate the response functions. Hence $K^{\mu\nu}(Q)$ given by Eq. (73) must be the same as that in Eq. (56). We can derive $\Gamma^\mu$ by using this relation. We emphasize that the two sides of the WIs (74) have very different meanings. The left-hand side, which involves the interaction between fermions and the external field, is a single-particle process. The right-hand side, however, contains many particle effects since the Green’s function contains the self energy that represents the interactions among particles.

The bare EM vertex in the Nambu space can be expressed as $\hat{\gamma}^\mu(P + Q, P) = \text{diag}(\gamma^\mu(P + Q, P), -\gamma^\mu(-P, -P - Q))$. We define

$$\Pi^\mu(Q) = -\Pi_1^\mu(Q) + i\Pi_2^\mu(Q), \quad \bar{\Pi}^\mu(Q) = -\Pi_1^\mu(Q) - i\Pi_2^\mu(Q),$$

which satisfy

$$q_\mu \Pi^\mu(Q) = 2\Delta, \quad q_\mu \bar{\Pi}^\mu(Q) = -2\Delta.$$  

(77)

In fact these two equations are the off-diagonal terms of the GWI (55) in the Nambu space. Hence from Eq. (56), the matrix form of the full EM vertex is given by

$$\hat{\Gamma}^\mu(P + Q, P) = \begin{pmatrix} \gamma^\mu(P + Q, P) & \Pi^\mu(Q) \\ \bar{\Pi}^\mu(Q) & -\gamma^\mu(-P, -P - Q) \end{pmatrix}.$$  

(78)

This type of expression was earlier obtained in Ref. [12]. Substituting Eqs. (78) and (29) into the expression (56), we have

$$K^{\mu\nu}(Q) = \frac{n}{m}h^{\mu\nu} + \sum_P \left[ \gamma^\mu(P + Q, P)G(P + Q)\gamma^\nu(P, P + Q)G(P) + \Pi^\mu(Q)F(P + Q)\gamma^\nu(P, P + Q)G(P) \\
- \gamma^\nu(P + Q, P)F(P + Q)\gamma^\nu(-P - Q, -P)F(P) + \Pi^\mu(Q)G(-P - Q)\gamma^\nu(-P - Q, -P)F(P) \\
+ \bar{\Pi}^\mu(Q)G(P + Q)\gamma^\nu(P, P + Q)F(P) - \gamma^\nu(-P, -P - Q)F(P + Q)\gamma^\nu(P, P + Q)F(P) \\
+ \bar{\Pi}^\mu(Q)F(P + Q)\gamma^\nu(-P - Q, -P)G(-P - Q) + \gamma^\nu(-P, -P - Q)G(-P - Q)\gamma^\nu(-P - Q, -P)G(-P - Q) \right].$$  

(79)

Changing variables by $-P \to -Q + P$ for the terms containing $\gamma_\nu(-P - Q, -P)$ and using $F(P) = F(-P)$, we get

$$K^{\mu\nu}(Q) = 2 \sum_P \left[ \gamma^\mu(P + Q, P)G(P + Q)\gamma^\nu(P, P + Q)G(P) + \Pi^\mu(Q)F(P + Q)\gamma^\nu(P, P + Q)G(P) \\
+ \bar{\Pi}^\mu(Q)G(P + Q)\gamma^\nu(P, P + Q)F(P) - \gamma^\nu(-P, -P - Q)F(P + Q)\gamma^\nu(P, P + Q)F(P) \right] + \frac{n}{m}h^{\mu\nu}.$$  

(80)

Substituting the expression $F(P) = \Delta G_0(-P)G(P)$ into Eq. (80) and comparing with Eq. (73), one can find the full EM vertex

$$\Gamma^\mu(P + Q, P) = \gamma^\mu(P + Q, P) + \Delta \Pi^\mu(Q)G_0(-P - Q) + \Delta \bar{\Pi}^\mu(Q)G_0(-P) \\
- \Delta^2 G_0(-P)\gamma^\mu(-P, -P - Q)G_0(-P - Q).$$  

(81)

The second and third terms can be shown to contain collective-mode effects [8]. The fourth term corresponds to the Maki-Thompson diagram [7]. One can verify that this full interacting vertex obeys the WI (74) in the one-dimensional space. Contracting both sides of Eq. (81) with $q^\mu$, we have

$$q_\mu \Gamma^\mu(P + Q, P) = G_0^{-1}(P + Q) - G_0^{-1}(P) - 2\Sigma(P + Q) + 2\Sigma(P) - \frac{\Sigma(P + Q)\Sigma(P)}{\Delta^2}(G_0^{-1}(-P) - G_0^{-1}(-P - Q)).$$
\[ G^{-1}_{\text{em}}(P + Q) - G^{-1}_{\text{em}}(P) - \Sigma(P + Q) + \Sigma(P) = G^{-1}(P + Q) - G^{-1}(P), \]  

(82)

where we have used the fact \( \Sigma(P) = -\Delta^2 G_0(-P) \) for BCS superfluids. Moreover, the \( f \)-sum rule can be shown to be satisfied.

From the expression [81], we can find that the many-particle effects are indeed included consistently in the correction of EM vertex. The collective modes, which correspond to the poles in the response functions, are many-particle effects. There is a gapless mode associated with the Nambu-Goldstone mode due to the spontaneous breaking of the \( U(1)_{\text{EM}} \) symmetry [8]. Because the contributions of the Nambu-Goldstone modes are properly included, gauge invariance of our theory is restored.

This has been shown in Eq. (67) so the \( Q \)-limit WI holds for BCS theory only when \( \lim_{q \to 0} \Gamma^0(P + Q, P) = 1 + \delta \). Using \( \Sigma(P) = -\Delta^2 G_0(-P) \), the RHS of Eq. (75) is

\[ 1 - \frac{\partial \Sigma(P)}{\partial \mu} = 1 + 2\Delta \frac{\partial (\Delta \Sigma)}{\partial \mu} - \Delta^2 G_0(-P) - \Delta^2 G_0(-P), \]  

(85)

where the identity \( \partial \mu G_0(-P) = -G_0^{-1}(-P) \partial \mu G_0(-P) = -G_0^2(-P) \) has been applied. Comparing Eqs. (84) and (85), we found that the \( Q \)-limit Ward identity holds for BCS theory only when

\[ \frac{\partial \Delta}{\partial \mu} = -\lim_{q \to 0} \Pi^0_0(Q)_{\omega = 0}. \]  

(86)

This has been shown in Eq. (67) so the \( Q \)-limit WI is also respected, which then guarantees the compressibility sum rule.

To compare with the results in the Nambu space, we also list our key results of the CFOP theory in the one-dimensional space

- **Response kernel:**
  \[ K^{\mu\nu}(Q) = 2 \sum_P \Gamma^{\mu}(P + Q) G(P + Q) \gamma^{\nu}(P, P + Q) G(P) + \frac{n}{m} \hbar^{\mu\nu}, \]  
  
  (87)

- **Full EM interacting vertex:**
  \[ \Gamma^{\mu}(P + Q, P) = \gamma^{\mu}(P + Q, P) + \Delta \Pi^{\mu}(Q) G_0(-P - Q) + \Delta \Pi^{\mu}(Q) G_0(-P) - \Delta^2 G_0(-P) \gamma^{\mu}(-P, -P - Q) G_0(-P - Q), \]  
  
  (88)

- **Ward identity:**
  \[ q_{\mu} \Gamma^{\mu}(P + Q, P) = G^{-1}(P + Q) - G^{-1}(P). \]  
  
  (89)

- **\( Q \)-limit Ward identity:**
  \[ \Gamma^0(P, P) = 1 - \frac{\partial \Sigma(P)}{\partial \mu}, \]  
  
  (90)
• $f$-sum rule:

$$\int_{-\infty}^{+\infty} d\omega \chi_{\rho\rho}(\omega, q) = n q^2/m.$$ \hfill (91)

The WI (or GWI) of the CFOP linear response theory for BCS superfluids guarantees that it is gauge invariant. In Appendix C the explicit gauge invariance of the CFOP theory is studied from another point of view based on a “generalized gauge transformation”.

C. Review of Nambu’s Linear Response Theory

Since we have the full EM vertex $\hat{\Gamma}^\mu$ from the CFOP theory, it would be helpful to compare it with the results from Nambu’s integral-equation approach [4]. A parallel discussion for relativistic BCS superfluids can be found in Ref. [14]. In BCS theory of conventional superconductors, the self energy is approximated by an integral equation which consists of a ladder approximation for the electron-phonon interaction. Nambu proposed that the EM vertex should be corrected in the same way as that for the self energy. Hence the EM interacting vertex follows an integral equation

$$\hat{\Gamma}^\mu(P + Q, P) = \hat{\gamma}^\mu(P + Q, P) + g \sum_K \sigma_3 \hat{G}(K) \hat{\Gamma}^\mu(K + Q, K) \hat{G}(K + Q) \sigma_3.$$ \hfill (92)

If a vertex is a solution to this equation, it automatically satisfies the GWI [55]. We give our own proof in Appendix C. Ideally, if we know how to solve this integral equation, we can further calculate the gauge-invariant response kernel $K^{\mu\nu}$ by Eq. (93). Unfortunately, very little is known about the solution. In general, one may expand the RHS of the equation as a series of $g$ by the iteration method, and then truncate it at some order. However, this will not produce a gauge-invariant solution. Moreover, the integral equation (92) is a vector equation while the GWI (55) is a scalar equation so they have different degrees of freedom. This suggests that there should not be a rigorous one-to-one correspondence between the solutions to the integral equation and the EM vertex respecting the GWI. As pointed out by Nambu [4], the integral equation is not only consistent with the GWI associated with the EM vertex but also with the GWIs associated with three other interaction vertices or gauge transformations (as shown in Eq. (4.4) of Ref. [4]).

In fact, what is equivalent to the GWI (55) is the contracted integral equation given by

$$q_{\mu} \hat{\Gamma}^{\mu}(P + Q, P) = q_{\mu} \hat{\gamma}^{\mu}(P + Q, P) + g \sum_K \sigma_3 \hat{G}(K) q_{\mu} \hat{\Gamma}^{\mu}(K + Q, K) \hat{G}(K + Q) \sigma_3.$$ \hfill (93)

The sufficient condition of this proposition is that the contracted integral equation (93) can lead to the GWI, which is proven in Appendix C. The necessary condition of this proposition is that any EM vertex obeying GWI must also satisfy Eq. (93), but not necessarily the integral equation (92). Importantly, this is pointed out by Nambu in his seminal paper [4] and by Schrieffer [18]. We briefly outline the proof here. Substituting Eq. (55) into the RHS of Eq. (93), we have

$$\text{RHS} = q_{\mu} \hat{\gamma}^{\mu}(P + Q, P) + g \sum_K \sigma_3 \hat{G}(K) (\sigma_3 \hat{G}^{-1}(K + Q) - \hat{G}^{-1}(K) \sigma_3) \hat{G}(K + Q) \sigma_3$$

$$= q_{\mu} \hat{\gamma}^{\mu}(P + Q, P) + 2i\Delta \sigma_2 = q_{\mu} \hat{\Gamma}^{\mu}(K + Q, K) = \text{LHS},$$ \hfill (94)

where Eq. (C1) has been applied. Therefore, any gauge invariant EM vertex $\hat{\Gamma}^{\mu}$ (including the vertex from the CFOP theory) must be a solution of the contracted integral equation (93), but not necessarily a solution of the integral equation (92). Furthermore, $\Gamma^{\mu\nu}$ may differ from the vertex $\hat{\Gamma}^{\mu}$ obtained from Eq. (92) by a gauge transformation $\hat{\Gamma}^{\mu} = \hat{\Gamma}^{\mu} + \hat{a}^{\mu}$, where $\hat{a}^{\mu}$ satisfies the Lorentz equation $q_{\mu} \hat{a}^{\mu} = 0$. Such a gauge transformation may not necessarily be expressed as $\hat{\Gamma}^{\mu} = \partial^{\mu} f$, where $f$ is a matrix in the Nambu space whose elements are harmonic functions. Another example is given by $\hat{\chi}^{\mu} = \Pi_{\rho}(Q) \hat{C}$, where $\hat{C}$ is an arbitrary constant matrix in the Nambu space with at least one nonzero element. $\Pi_{\rho}(Q) \hat{C}$ is given by Eqs. (57) and $q_{\mu} \Pi_{\rho}(Q) = 0$, hence $\hat{\chi}^{\mu}$ satisfies the Lorentz equation. Under the gauge transformation $\hat{\Gamma}^{\mu} \rightarrow \hat{\Gamma}^{\mu} + \hat{a}^{\mu}$, the 0-th component of the vertex and the density response function from the CFOP theory transform as

$$\hat{\Gamma}^{0}(P, P) \rightarrow \hat{\Gamma}^{0}(P, P) - \frac{\partial \Delta}{\partial \mu} \hat{C},$$

$$\chi_{\rho\rho}(Q) \rightarrow \chi_{\rho\rho}(Q) - \frac{1}{\pi} \text{Im} \text{Tr} \sum_P (\Pi_{\rho}(Q) \hat{C} \hat{G}(P + Q) \sigma_3 \hat{G}(P)).$$ \hfill (95)
Then one can verify that the \( Q \)-limit GWI \( (71) \) and the \( f \)-sum rule \( (72) \) can not be satisfied simultaneously under the above transformation even though the GWI \( (70) \) is respected anyway.

Here we have two remarks on the consistency of a linear response theory for BCS superfluids. Firstly, from the proof given in Appendix \( D \) Nambu’s approach tells us that if the self-energy of an interacting Fermi gas satisfies Eq.\( (C1) \), then the vertex given by the integral equation \( (92) \) must satisfies the GWI. However, the theory of BCS superfluids is not the only theory that satisfies Eq.\( (C1) \) and as shown in Ref.\[4\] there are theories with other types of symmetries that satisfy Eq.\( (C1) \). Hence Nambu’s vertex given by Eq.\( (92) \) may not be specific to BCS superfluids and it can be more general. Secondly, the \( Q \)-limit GWI and \( f \)-sum rule are not directly linked to the GWI. They should be imposed as separate constraints for a consistent linear response theory. The case of non-interacting Fermi gases is special because the bare EM vertex \( \gamma^\nu(P + Q, P) \) is already the full vertex and no correction is needed. If one formulates a linear response theory and finds a gauge-invariant vertex which obeys the GWI, one can further calculate the response kernel by Eq.\( (68) \). However, this response kernel may not satisfy the compressibility sum rule and \( f \)-sum rule. Therefore, as we emphasized, the GWI, \( Q \)-limit GWI, and \( f \)-sum rule should be independent constraints for a consistent linear response theory for BCS superfluids. We cannot fully check whether Nambu’s integral-equation approach satisfies all these criteria because finding an exact solution to the integral equation is the bottleneck, but the CFOP approach does satisfy all those constraints as we have demonstrated.

D. Comparisons between different linear response theories for BCS superfluids and Meissner Effect

Now we compare the two vertices given by the CFOP theory and Nambu’s integral-equation method. In Ref.\[4\], Nambu managed to solve Eq.\( (92) \) with the aid of Eq.\( (55) \) under the following conditions: (1) only the zeroth order of \( g \) is considered, (2) \( \omega \) and \( q \) are both small, and (3) it is at zero temperature. Here we compare the result \( (59) \) with Nambu’s under the same condition. Taking the same limits, the response functions \( Q_{23}^{ij}, Q_{12} \) and \( Q_{13}^{ij} \) vanish because of the particle-hole symmetry. Therefore \( \Pi^0(Q) = -\Pi^0(Q) = -i-Q_{22}^{ij}(Q) \) and \( \Pi(Q) = -\Pi(Q) = -Q_{22}^{ij}(Q) \). From the expressions of the response functions, one can verify that (see Appendix \( D \))

\[
Q_{23}^{ij}(Q) \approx -i-N(0)\frac{\omega}{2\Delta}, \quad Q_{23}(Q) = -i\frac{N(0)\sigma^2}{\Delta}q, \quad \tilde{Q}_{22}^{ij}(Q) = -N(0)\frac{\omega^2 - c_s^2 q^2}{2\Delta^2},
\]

where \( N(0) \) is the density of states at the Fermi energy and \( c_s = \frac{1}{\sqrt{3}}\frac{k_F}{m} = \frac{1}{\sqrt{3}}v_F \) is the speed of sound. Here \( k_F \) is the Fermi momentum defined by \( n = \frac{k_F}{\sqrt{2}\pi^2} \). Therefore we have

\[
\Pi^0(Q) = -\Pi^0(Q) \approx \frac{\Delta\omega}{\omega^2 - c_s^2 q^2}, \quad \Pi(Q) = -\Pi(Q) \approx \frac{2\Delta\sigma^2}{\omega^2 - c_s^2 q^2}q.
\]

and

\[
\hat{\Gamma}^0(P + Q, P) = \sigma_3 + 2i\sigma_2 \frac{\Delta\omega}{\omega^2 - c_s^2 q^2}, \quad \hat{\Gamma}(P + Q, P) = \frac{P + \frac{9}{m} + 2i\sigma_2 \frac{\Delta\sigma^2 q}{\omega^2 - c_s^2 q^2}}{m}. \tag{98}
\]

\( \omega = c_s q \) is the dispersion of the gapless collective mode. These results are exactly the same as those found by Nambu \[4\] by taking the same limits.

Since the CFOP vertex and Nambu’s vertex both satisfies the GWI, they are at most off by a term \( \hat{\chi}_\mu \) with \( q_\mu \hat{\chi}_\mu = 0 \). \( \hat{\chi}_\mu \) can be expressed by a harmonic matrix \( \hat{f} \) in the Nambu space of the form \( \hat{\chi}_\mu = \partial_\mu \hat{f} \). In momentum space, it is \( \hat{\chi}_\mu = iq_\mu \hat{f} \), which vanishes as \( q_\mu \to 0 \). Therefore, at zero temperature and in the low frequency and momentum limit, the CFOP linear response theory agrees with Nambu’s approach but in general they can be different.

Before closing our discussions on the density channel, we remark that the collective modes do not contribute to the Meissner effect, where one can show that a finite superfluid density leads to perfect diamagnetism \[2\]. This remark justifies the standard calculation of the Meissner effect and superfluid density \[2\], where one ignores the fluctuations from the order parameter and still obtains the correct results. Although one may use a fully gauge-invariant linear response theory to demonstrate this result \[12\], here we directly evaluate the collective-mode contribution in the Meissner effect. Following Ref.\[2\], the Meissner effect is associated with the current-current correlation functions, which can be inferred from the transverse components of the response kernel. Here we briefly sketch why the collective-mode effects do not contribute to the transverse components of \( \hat{K}^{ij}(0, q) \) as \( q \to 0 \). From Eq.\( (41) \), the response kernel \( \hat{K}^{ij} \) is given by

\[
\hat{K}^{ij} = \hat{\Pi}^{ij} = \hat{Q}_{22}^{ij}(Q_{23} - \frac{Q_{11}Q_{22}}{Q_{22} - Q_{12}Q_{21}}Q_{13} + \frac{Q_{22}Q_{13}Q_{13}^2 - 2Q_{12}Q_{13}Q_{23}}{Q_{22} - Q_{12}Q_{21}}).
\]

\( \hat{\Pi}^{ij} \) is the vertex given by Eq.\( (92) \) and \( \hat{Q}_{22}^{ij} \) is the response kernel given by Eq.\( (68) \). However, this response kernel may not satisfy the compressibility sum rule and \( f \)-sum rule. Therefore, as we emphasized, the GWI, \( Q \)-limit GWI, and \( f \)-sum rule should be independent constraints for a consistent linear response theory for BCS superfluids. We cannot fully check whether Nambu’s integral-equation approach satisfies all these criteria because finding an exact solution to the integral equation is the bottleneck, but the CFOP approach does satisfy all those constraints as we have demonstrated.
where the second term is associated with the collective modes. A tensor $P^{ij}_q$ can be decomposed into the longitudinal and the transverse parts $P_L$ and $P_T$, where $P_L = \hat{q} \cdot \hat{P} \cdot \hat{q}$, $P_T = (\sum_i \hat{P}^{ij}_i - P_L)/2$, and $\hat{q}$ is the unit vector along $\mathbf{q}$. Assuming that $\hat{q}$ is parallel to the $z$-axis, in the limit $\mathbf{q} \to 0$ one can show that $Q^{\mu}_{i1}$ and $Q^{\mu}_{i2}$ start with linear dependence on $\mathbf{q}$. Therefore $\lim_{\mathbf{q} \to 0} Q^{\mu}_{i1} \cdot Q^{\mu}_{i3} = \lim_{\mathbf{q} \to 0} \hat{q} \cdot Q^{\mu}_{i1} Q^{\mu}_{i3} \cdot \hat{q}$. When one evaluates the transverse components of Eq. (99), all of the collective-mode terms cancel in the limit $\mathbf{q} \to 0$ so the demonstration of the Meissner effect is insensitive to the collective modes.

V. SPIN LINEAR RESPONSE THEORY OF BCS SUPERFLUIDS

We now formulate a generalized spin linear response theory which is similar to its counterpart in the density channel. Using the notation of the Nambu spinor (25), the Hamiltonian (6) with the BCS approximation in momentum space is given by

$$H = \sum_p \Psi_p \hat{H}_p \Psi_p + \sum_{pq} \Psi^\dagger_{p+q} \left( -\frac{p + \bar{q}}{m} A_q \sigma_3 + \Phi_q - \Delta_q \sigma_+ - \Delta^*_q \sigma_- \right) \Psi_p. \quad (100)$$

Here the fluctuation of the order parameter $\Delta_q$ is in the spin channel and should not be related to the fluctuation in the density channel. We follow the same procedure as what we did in the density channel. The order parameter is separated into two parts with one denoting its equilibrium value and the other denoting the perturbative part. Furthermore, by introducing the spin interacting vertex $\tilde{\gamma}^\mu_\alpha(P + Q, P) \equiv \tilde{\gamma}^\mu_\alpha(p + q, p) = (1, \frac{p + \bar{q}}{m} \sigma_3)$ in the Nambu space, the Hamiltonian can be expressed as $H = H_0 + H'_S$, where

$$H_0 = \sum_p \Psi_p \hat{H}_p \Psi_p, \quad H'_S = \sum_{pq} \Psi^\dagger_{p+q} \left( \Delta_1 q \sigma_1 + \Delta_2 q \sigma_2 + A_{\mu q} \tilde{\gamma}^\mu_\alpha(p + q, p) \right) \Psi_p. \quad (101)$$

Similar to its density counterpart, the interacting Hamiltonian can also be cast into the form

$$H'_S = \sum_{pq} \Psi^\dagger_{p+q} \hat{\Phi}_q \cdot \hat{\Sigma}_S(p + q, p) \Psi_p \quad (102)$$

by introducing the generalized driving potential and generalized interacting vertex

$$\hat{\Phi}_q = (\Delta_1 q, \Delta_2 q, A_{\mu q})^T, \quad \hat{\Sigma}_S(p + q, p) = (\sigma_1, \sigma_2, \tilde{\gamma}^\mu_\alpha(p + q, p))^T. \quad (103)$$

The Heisenberg operator is defined as $\mathcal{O}(\tau) = e^{iH\tau}e^{-H\tau}$. Hence, when the external field is weak, the generalized perturbed current in the spin linear response theory is given by

$$\delta \hat{J}_S(\tau, q) = \sum_p \langle \Psi_p(\tau) \hat{\Sigma}_S(p + q, p) \Psi_p + q(\tau) \rangle + \frac{n}{m} \delta^{ij} \delta^{\mu \nu} A_{ij}(\tau, q). \quad (104)$$

Here $\delta J^\mu_{S3}$ denotes the perturbed spin current and $\delta J^\mu_{S1,2}$ denote the perturbations of the gap function. The spin linear response theory can also be written in a matrix form

$$\delta J^\mu_3(\omega, q) = \hat{J}^\mu_3(\omega, q) \cdot \hat{\Phi}(\omega, q) = \begin{pmatrix} Q^{\mu}_{S11}(\omega, q) & Q^{\mu}_{S12}(\omega, q) & Q^{\mu}_{S13}(\omega, q) \\ Q^{\mu}_{S21}(\omega, q) & Q^{\mu}_{S22}(\omega, q) & Q^{\mu}_{S23}(\omega, q) \\ Q^{\mu}_{S31}(\omega, q) & Q^{\mu}_{S32}(\omega, q) & Q^{\mu}_{S33}(\omega, q) + \frac{n}{m} h^{\mu \nu} \end{pmatrix} \begin{pmatrix} \Delta_1(\omega, q) \\ \Delta_2(\omega, q) \\ A_{ij}(\omega, q) \end{pmatrix}. \quad (105)$$

Following the same steps as what we did previously, the spin response functions are also expressed by

$$Q_{Sij}(\omega, q) = TrT \sum_{\omega_n} \sum_p \langle \hat{\Sigma}_S(P + Q, P) \hat{G}(P + Q) \hat{J}_S(P, P + Q) \hat{G}(P) \rangle. \quad (106)$$

The expressions of these response kernels are given in Appendix A. One can verify that $Q^{\mu}_{S11} = Q^{\mu}_{11}$, $Q^{\mu}_{S12} = Q^{\mu}_{12}$, $Q^{\mu}_{S21} = Q^{\mu}_{21}$ and $Q^{\mu}_{S22} = Q^{\mu}_{22}$ so the block of the matrix $Q_{Sij}$ for the fluctuations of the order parameter is exactly the same as that in the density channel. It can also be shown that $Q^{\mu}_{S13} = Q^{\mu}_{S23} = Q^{\mu}_{S31} = Q^{\mu}_{S32} = 0$. Hence
the perturbation of the order parameter decouples from the perturbation of the spin current. By applying the self-consistent condition \( \delta J_{1,2} = -\frac{2}{g} \Delta_{1,2} \), we obtain

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\delta J_{33}^\mu(\omega, q) & \tilde{Q}_{S11}(\omega, q) & \tilde{Q}_{S12}(\omega, q) \\
\end{pmatrix} = \begin{pmatrix}
Q_{S11}(\omega, q) & Q_{S12}(\omega, q) & 0 \\
Q_{S21}(\omega, q) & Q_{S22}(\omega, q) & 0 \\
0 & 0 & \tilde{Q}_{S}^{\mu
u}(\omega, q) \\
\end{pmatrix} \begin{pmatrix}
\Delta_1(\omega, q) \\
\Delta_2(\omega, q) \\
A\nu(\omega, q) \\
\end{pmatrix},
\]

where \( \tilde{Q}_{S11} = \frac{2}{g} + Q_{S11}, \) \( \tilde{Q}_{S22} = \frac{2}{g} + Q_{S22} \) and \( \tilde{Q}_{S}^{\mu
u} = Q_{S33}^{\mu
u} + \frac{n}{m} h^{\mu\nu} \). The perturbations \( \Delta_1 \) and \( \Delta_2 \) in the spin channel can be further shown to be zero, which are very different from their counterparts in the density channel. The spin response function becomes

\[
\delta J_{33}^\mu(\omega, q) = \tilde{Q}_{S}^{\mu
u}(\omega, q) A\nu(\omega, q).
\]

We emphasize that the fluctuations of the order parameter automatically decouple from the spin response and there is no contribution from the collective modes in the spin response function. The mechanism behind this decoupling is that the BCS order parameter does not break the \( U(1)_z \) symmetry. Even though one treats the spin linear response in the same way as what we did for the EM response, the unbroken \( U(1)_z \) symmetry leads to a significantly different result.

The \( U(1)_z \) invariance of the linear response theory is satisfied by the GWI

\[
q_\mu \hat{Q}_{S}^{\mu
u}(Q) = 0,
\]

which leads to the conservation of the spin current \( q_\mu \delta J_{33}^\mu = 0 \). Before proving this statement, it is important to notice that the spin interacting vertex obeys the GWI associated with the \( U(1)_z \) symmetry

\[
q_\mu \hat{\gamma}_{S}^{\mu}(P + Q, P) = \hat{G}_0^{-1}(P + Q) - \hat{G}_0^{-1}(P) = \hat{G}^{-1}(P + Q) - \hat{G}^{-1}(P).
\]

This leads to the GWI [109]. The second equation is due to the fact that the BCS self energy in the Nambu space is given by \( \Sigma = -\Delta \sigma_1 \) and is independent of the four-momentum. The proof of the conservation of the spin current is similar to the derivation shown in Eq. (47).

The spin susceptibility can be evaluated from the spin response kernel by \( \chi_{SS} = -\frac{i}{\pi} \text{Im} Q_{S33}^{00} \). The \( f \)-sum rule can be shown to be satisfied in the spin channel.

\[
\int_{-\infty}^{+\infty} d\omega \chi_{SS}(\omega, q) = n q^2/m.
\]

Following the same argument in the density channel, we have

\[
-\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im} Q_{S33}^{00} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im} (\omega Q_{S33}^{00}) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im} (q \cdot Q_{S33}^{00}).
\]

Comparing Eqs. (A19) to (A13), we see that \( Q_{S33}^{0i} = Q_{S33}^{0i} \). Hence by using the lemma [50] we get

\[
-\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im} Q_{S33}^{00} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \text{Im} (q \cdot Q_{S33}^{00}) = \frac{m}{n} q^2.
\]

In the one-dimensional space we follow the same steps in the density channel to find the \( U(1)_z \) gauge-invariant spin vertex \( \Gamma_{S\sigma}^\mu \) which satisfies the WI [118]. The spin response kernel can be expressed by

\[
\hat{Q}_{S}^{\mu
u}(Q) = \sum_{P} \sum_{\sigma} \Gamma_{S\sigma}^\mu(P + Q, P) G(P + Q) \gamma_{S\sigma}^{\nu}(P + Q) G(P) + \frac{n}{m} h^{\mu\nu}.
\]

where the bare spin interacting vertex \( \gamma_{S\sigma}^{\mu} \) is given in Sec. [111]. Importantly, Eq. (113) should give the same expression for the spin response function as the 33-component of Eq. (109) when the \( h^{\mu\nu} \) term is included

\[
\hat{Q}_{S}^{\mu
u}(Q) = \text{Tr} \sum_{P} \left( \gamma_{S\uparrow}^{\mu}(P + Q, P) \hat{G}(P + Q) \gamma_{S\downarrow}^{\nu}(P + Q) \hat{G}(P) + \frac{n}{m} h^{\mu\nu} \right).
\]

Note that the spin interacting vertex in the Nambu space can be expressed as

\[
\gamma_{S\sigma}^{\mu}(P + Q, P) = \begin{pmatrix}
\gamma_{S\uparrow}^{\mu}(P + Q, P) & 0 \\
0 & -\gamma_{S\downarrow}^{\mu}(-P, -P - Q) \\
\end{pmatrix}.
\]

The spin susceptibility can be evaluated from the spin response kernel by \( \chi_{SS} = -\frac{i}{\pi} \text{Im} Q_{S33}^{00} \). The \( f \)-sum rule can be shown to be satisfied in the spin channel.
Substituting Eq. (115) and the propagator (28) into the expression (114), we get

\[ \tilde{Q}_{S}^{\mu\nu}(Q) = \frac{n}{m} h^{\mu\nu} + \sum_{P} \left( \gamma_{S|t}^{\mu}(P + Q, P)G(P + Q)\gamma_{S|t}^{\nu}(P, P + Q)G(P)\gamma_{S|l}^{\mu}_{t}(P, P + Q)F(P + Q)\gamma_{S|l}^{\nu}_{l}(P - Q, -P)F(P) \right) \]

\[ - \gamma_{S|l}^{\mu}(P, P - Q)F(P + Q)\gamma_{S|t}^{\nu}(P, P + Q)F(P) + \gamma_{S|l}^{\mu}(P, P + Q)G(P) - \gamma_{S|t}^{\nu}(P, P - Q)F(P)\gamma_{S|l}^{\mu}_{l}(P, P + Q)F(P + Q) \]

where we have changed variables by interacting vertex satisfies the Ward identity (118). Using (121) and Eq. (13), one can see that the following expression guarantees that the expressions for \( \tilde{Q}_{S}^{\mu\nu}(Q) \) are the same.

\[ \Gamma_{S|t}^{\mu}(P + Q, P) = \gamma_{S|t}^{\mu}(P + Q, P) - \Delta^{2}G_{0}(-P)\gamma_{S|l}^{\mu}_{t}(P, P - Q)G_{0}(-P) \]

for \( \sigma = \uparrow, \downarrow \). The second term corresponds to the Maki-Thompson diagram [7]. Now we show that this full spin interacting vertex satisfies the Ward identity (118). Using \( S_{s} = -S_{s} \) and Eq. (15), we have

\[ q_{\mu}\Gamma_{S|s}^{\mu}(P + Q, P) = S_{s}(G_{0}^{-1}(P + Q) - G_{0}^{-1}(P)) + S_{s}\Delta^{2}G_{0}(P)G_{0}(-P - Q)(G_{0}^{-1}(P) - G_{0}^{-1}(-P)) \]

\[ = S_{s}(G_{0}^{-1}(P + Q) - G_{0}^{-1}(P)) - S_{s}(\Sigma(P + Q) - \Sigma(P)) = S_{s}(G^{-1}(P + Q) - G^{-1}(P)). \]

Finally we summarize the central results in this section. In the Nambu space, we have

- **Response kernel:**

\[ \tilde{Q}_{S}^{\mu\nu}(Q) = \text{Tr} \sum_{P} \left( \gamma_{S|l}^{\mu}(P + Q, P)\tilde{G}(P + Q)\gamma_{S|t}^{\nu}(P, P + Q)\tilde{G}(P) \right) + \frac{n}{m} h^{\mu\nu}, \]

- **Spin interacting vertex:**

\[ \tilde{\gamma}_{S}^{\mu}(P + Q, P) = (1, \frac{P + Q}{m} \sigma_{3}), \]

- **Generalized Ward identity:**

\[ q_{\mu}\tilde{\gamma}_{S}^{\mu}(P + Q, P) = \tilde{G}^{-1}(P + Q) - \tilde{G}^{-1}(P). \]

In the one-dimensional space, we have

- **Response kernel:**

\[ \tilde{Q}_{S}^{\mu\nu}(Q) = \sum_{P} \sum_{\sigma} \Gamma_{S|\sigma}^{\mu}(P + Q, P)G(P + Q)\gamma_{S|\sigma}^{\nu}(P, P + Q)G(P) + \frac{n}{m} h^{\mu\nu}, \]

- **Full spin interacting vertex:**

\[ \Gamma_{S|\sigma}^{\mu}(P + Q, P) = \gamma_{S|\sigma}^{\mu}(P + Q, P) + \Delta^{2}G_{0}(-P)\gamma_{S|\sigma}^{\mu}(P, P - Q)G_{0}(-P - Q), \]

- **Ward identity:**

\[ q_{\mu}\Gamma_{S|\sigma}^{\mu}(P + Q, P) = S_{\sigma}(G^{-1}(P + Q) - G^{-1}(P)). \]

In both spaces, the \( f \)-sum rules are satisfied

\[ \int_{-\infty}^{+\infty} d\omega \chi_{SS}(\omega, q) = \frac{nq^{2}}{m}. \]
Below $T_c$, the U(1)$_{\text{EM}}$ symmetry is broken by the condensed Cooper pairs while the U(1)$_z$ symmetry remains intact. Hence the two linear response theories produce significantly different results below $T_c$. Importantly, the collective modes coming from the breaking U(1)$_{\text{EM}}$ symmetry only couples to the density response function but decouples from the spin response function. This indicates that the difference between the density and spin channels arises only in the presence of a condensate that only breaks the U(1)$_{\text{EM}}$ symmetry. Above $T_c$, both U(1) symmetries are respected since there is no Cooper-pair condensation. The effective Lagrangian after the BCS approximation is identical to that of a non-interacting Fermi gas when the gap $\Delta$ vanishes. By dropping $Q_{41}$ and $Q_{21}$ which are not defined above $T_c$, one can verify that the response functions of the two linear response theories give the same result above $T_c$. In the presence of pairing fluctuation effects, the amplitude of the pairs are not necessarily the order parameter since finite-momentum pairs may coexist with the condensate of Cooper pairs [21]. The difference between the two response functions could be shown to be still valid and one can use the difference between the density and spin structure factors to construct a quantity similar to an order parameter for detecting the phase coherence of atomic Fermi gases [8].

VI. CONCLUSION

We have shown that the CFOP approach of the linear response of BCS superfluid is a computational manageable scheme that satisfies important constraints including Ward identity, $f$ sum rule, $Q$-limit Ward identity, and compressibility sum rule that guarantee charge conservation and a consistent expression for the compressibility. The CFOP formalism provides a paradigm for studying linear response theories in interacting many-body systems in the presence of spontaneous symmetry breaking. The spin linear response theory complements the story of the CFOP theory and demonstrates the different roles played by the collective modes in the superfluid phase. Going beyond mean-field BCS theory requires considerations of non-condensed pairs and there have been different approaches [7]. We emphasize that linear response theories of those beyond-BCS theories should be subject to the same consistency constraints discussed here. In addition to conventional superconductors [12], our formalism may be useful in the study of ultra-cold atoms [8] [22] and nuclear physics [23] [24] where linear response theories of BCS superfluids are frequently implemented.

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Appendix A: Detailed expressions for response functions

The following are the response functions for non-interacting Fermi gases:

$$Q_{00}^0(\omega, q) = -2 \sum_p \frac{\xi_p^+ - \xi_p^-}{\omega^2 - (\xi_p^+ - \xi_p^-)^2} [f(\xi_p^+) - f(\xi_p^-)], \quad (A1)$$

$$Q_{0i}^0(\omega, q) = Q_{i0}^0(\omega, q) = -2 \omega \sum_p \frac{p_i}{m} \frac{f(\xi_p^+) - f(\xi_p^-)}{\omega^2 - (\xi_p^+ - \xi_p^-)^2}, \quad (A2)$$

$$\tilde{Q}_{0i}^0(\omega, q) = -2 \sum_p \frac{p_i p^i}{m^2} \frac{\xi_p^+ - \xi_p^-}{\omega^2 - (\xi_p^+ - \xi_p^-)^2} [f(\xi_p^+) - f(\xi_p^-)]. \quad (A3)$$

The following are the EM response functions of BCS superfluids from the CFOP theory:

$$Q_{11}(\omega, q) = \sum_p \left[ (1 + \frac{\xi_p^+ \xi_p^- - \Delta^2}{E_p^+ E_p^-}) \frac{E_p^+ + E_p^-}{\omega^2 - (E_p^+ + E_p^-)^2} [1 - f(E_p^+) - f(E_p^-)] \right.$$  
$$\left. - (1 - \frac{\xi_p^+ \xi_p^- - \Delta^2}{E_p^+ E_p^-}) \frac{E_p^+ - E_p^-}{\omega^2 - (E_p^+ - E_p^-)^2} [f(E_p^+) - f(E_p^-)] \right]. \quad (A4)$$
\[ Q_{12}(\omega, \mathbf{q}) = -Q_{21}(\omega, \mathbf{q}) = -i \omega \sum_p \left[ (\xi_p^{+} E_p^{+})^{1 - f(E_p^{+}) - f(E_p^{-})} - (\xi_p^{-} E_p^{-}) \frac{f(E_p^{+}) - f(E_p^{-})}{\omega^2 - (E_p^{+} + E_p^{-})^2} \right], \quad (A5) \]

\[ Q_{13}^0(\omega, \mathbf{q}) = Q_{31}^0(\omega, \mathbf{q}) = \Delta \sum_p \xi_p^{+} E_p^{+} \left[ \frac{(E_p^{+} + E_p^{-})[1 - f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} + E_p^{-})^2} + \frac{(E_p^{+} - E_p^{-})[f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A6) \]

\[ Q_{13}^i(\omega, \mathbf{q}) = Q_{31}^i(\omega, \mathbf{q}) = \sum_p \frac{p_{\mathbf{p}}^i}{m} \omega \left[ \frac{(E_p^{+} - E_p^{-})[1 - f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} + E_p^{-})^2} + \frac{(E_p^{+} + E_p^{-})[f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A7) \]

\[ Q_{22}(\omega, \mathbf{q}) = \sum_p \left[ (1 + \frac{\xi_p^{+} \xi_p^{-} + \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} + E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} [1 - f(E_p^{+}) - f(E_p^{-})] \right. \]
\[ - \left. (1 - \frac{\xi_p^{+} \xi_p^{-} + \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} - E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} [f(E_p^{+}) - f(E_p^{-})] \right], \quad (A8) \]

\[ Q_{23}(\omega, \mathbf{q}) = -Q_{32}^0(\omega, \mathbf{q}) = i \omega \sum_p \frac{p_{\mathbf{p}}^i \xi_p^{+} - \xi_p^{-}}{E_p^{+} E_p^{-}} \left[ \frac{(E_p^{+} + E_p^{-})[1 - f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} + E_p^{-})^2} + \frac{(E_p^{+} - E_p^{-})[f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A9) \]

\[ Q_{23}^i(\omega, \mathbf{q}) = -Q_{32}^i(\omega, \mathbf{q}) = i \omega \sum_p \frac{p_{\mathbf{p}}^i \xi_p^{+} - \xi_p^{-}}{E_p^{+} E_p^{-}} \left[ \frac{(E_p^{+} + E_p^{-})[1 - f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} + E_p^{-})^2} + \frac{(E_p^{+} - E_p^{-})[f(E_p^{+}) - f(E_p^{-})]}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A10) \]

\[ Q_{33}^{00}(\omega, \mathbf{q}) = \sum_p \left[ (1 - \frac{\xi_p^{+} \xi_p^{-} - \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} + E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} [1 - f(E_p^{+}) - f(E_p^{-})] \right. \]
\[ - \left. (1 + \frac{\xi_p^{+} \xi_p^{-} - \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} - E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} [f(E_p^{+}) - f(E_p^{-})] \right], \quad (A11) \]

\[ Q_{33}^{ij}(\omega, \mathbf{q}) = \sum_p \frac{p_{\mathbf{p}}^i p_{\mathbf{p}}^j}{m^2} \left[ (1 - \frac{\xi_p^{+} \xi_p^{-} + \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} + E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} [1 - f(E_p^{+}) - f(E_p^{-})] \right. \]
\[ - \left. (1 + \frac{\xi_p^{+} \xi_p^{-} + \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} - E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} [f(E_p^{+}) - f(E_p^{-})] \right], \quad (A12) \]

\[ Q_{33}^{0i}(\omega, \mathbf{q}) = Q_{33}^{0j}(\omega, \mathbf{q}) = \omega \sum_p \frac{p_{\mathbf{p}}^i}{m} \left[ \frac{\xi_p^{+} E_p^{+} - \xi_p^{-} E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} - \frac{\xi_p^{+} E_p^{+} - \xi_p^{-} E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A13) \]

The following are spin response functions of BCS Superfluids following the same structure of the CFOP theory

\[ Q_{811}(\omega, \mathbf{q}) = \sum_p \left[ (1 + \frac{\xi_p^{+} \xi_p^{-} - \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} + E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} [1 - f(E_p^{+}) - f(E_p^{-})] \right. \]
\[ - \left. (1 - \frac{\xi_p^{+} \xi_p^{-} - \Delta^2}{E_p^{+} E_p^{-}}) \frac{E_p^{+} - E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} [f(E_p^{+}) - f(E_p^{-})] \right], \quad (A14) \]

\[ Q_{812}(\omega, \mathbf{q}) = -Q_{821}(\omega, \mathbf{q}) = -i \omega \sum_p \left[ \frac{\xi_p^{+} E_p^{+} - \xi_p^{-} E_p^{-}}{\omega^2 - (E_p^{+} + E_p^{-})^2} - \frac{\xi_p^{+} E_p^{+} - \xi_p^{-} E_p^{-}}{\omega^2 - (E_p^{+} - E_p^{-})^2} \right], \quad (A15) \]
\[ Q_{S22}(\omega, q) = \sum_p \left( 1 + \frac{\xi_p^+ E_p^- + \Delta^2}{E_p^+ E_p^-} \right) \frac{E_p^+ + E_p^-}{\omega^2 - (E_p^+ + E_p^-)^2} \left[ 1 - f(E_p^+) - f(E_p^-) \right] \]

\[ -\left( 1 - \frac{\xi_p^- E_p^+ + \Delta^2}{E_p^- E_p^+} \right) \frac{E_p^+ - E_p^-}{\omega^2 - (E_p^+ - E_p^-)^2} \left[ f(E_p^+) - f(E_p^-) \right], \]

(A16)

\[ Q_{S13}^\mu(\omega, q) = Q_{S23}^\mu(\omega, q) = Q_{S31}^\mu(\omega, q) = Q_{S32}^\mu(\omega, q) = 0. \]

(A17)

\[ Q_{S33}^{0\mu}(\omega, q) = \sum_p \left\{ \left( 1 - \frac{\xi_p^+ E_p^- + \Delta^2}{E_p^- E_p^+} \right) \frac{E_p^+ + E_p^-}{\omega^2 - (E_p^+ + E_p^-)^2} \left[ 1 - f(E_p^+) - f(E_p^-) \right] \right. \]

\[ -
\left. \left( 1 + \frac{\xi_p^- E_p^+ + \Delta^2}{E_p^- E_p^+} \right) \frac{E_p^+ - E_p^-}{\omega^2 - (E_p^+ - E_p^-)^2} \left[ f(E_p^+) - f(E_p^-) \right] \right\}, \]

(A18)

\[ Q_{S33}^0(\omega, q) = Q_{S33}^0(\omega, q) = \omega \sum_p \sum_{m=-2}^{2} \left( \frac{\xi_p^+ - \xi_p^-}{E_p^+ E_p^-} \right) \frac{1 - f(E_p^+) - f(E_p^-)}{\omega^2 - (E_p^+ + E_p^-)^2} - \left( \frac{\xi_p^+ E_p^- + \xi_p^- E_p^+}{E_p^- E_p^+} \right) \frac{f(E_p^+) - f(E_p^-)}{\omega^2 - (E_p^+ - E_p^-)^2} \right\}, \]

(A19)

\[ Q_{S33}^{\mu\nu}(\omega, q) = \sum_p \sum_{m=-2}^{2} \left( \frac{\xi_p^+ E_p^- + \xi_p^- E_p^+}{E_p^- E_p^+} \right) \frac{1 - f(E_p^+) - f(E_p^-)}{\omega^2 - (E_p^+ + E_p^-)^2} \left[ 1 - f(E_p^+) - f(E_p^-) \right] \]

\[ -\left( 1 + \frac{\xi_p^- E_p^+ + \xi_p^- E_p^+}{E_p^- E_p^+} \right) \frac{1 - f(E_p^+) - f(E_p^-)}{\omega^2 - (E_p^+ - E_p^-)^2} \left[ f(E_p^+) - f(E_p^-) \right] \right\}. \]

(A20)

Here we outline the proof of Eq. (A17). When the index \( \mu = 0 \), from Eq. (106) we have

\[ Q_{S13}^0(\omega, q) = \sum_p \left( G(P)F(P + Q) - G(-P - Q)F(P) + G(P + Q)F(P) - G(-P)F(P) \right). \]

(A21)

Changing variables by \( P \to -P - Q \) and using the fact \( F(P) = F(-P) \), the second and fourth terms inside the bracket become \( G(P)F(P + Q) \) and \( G(P + Q)F(P) \) respectively, which cancel the first and third term inside the bracket respectively. Hence we have \( Q_{S13}^0(\omega, q) = 0 \). Similarly, if the index \( \mu = i \), we have

\[ Q_{S13}^i(\omega, q) = \sum_p \left( G(P)F(P + Q) + G(-P - Q)F(P) + G(P + Q)F(P) + G(-P)F(P) \right). \]

(A22)

Here we change variables by \( P \to -P - Q \) again so the pre-factor \( \frac{p^+ - q^+}{m} \to -\frac{p^+ + q^+}{m} \). Hence all terms inside the bracket cancel out and we conclude that \( Q_{S13}^i(\omega, q) = 0 \). Following the same steps, one can prove that \( Q_{S23}^\mu(\omega, q) = Q_{S31}^\mu(\omega, q) = 0 \).

**Appendix B: Proof of the Lemma (50)**

From the expression shown in Appendix A, we have

\[ Q_{S33}^0(\omega, q) = \sum_p \frac{\omega}{2m} \sum_{m=-2}^{2} \left( \frac{\xi_p^+ - \xi_p^-}{E_p^+ E_p^-} \right) \frac{1 - f(E_p^+) - f(E_p^-)}{E_p^+ + E_p^-} \left( \frac{1}{\omega - E_p^+ - E_p^-} - \frac{1}{\omega + E_p^+ + E_p^-} \right) \]

\[ -\left( \frac{\xi_p^+ E_p^- + \xi_p^- E_p^+}{E_p^- E_p^+} \right) \frac{f(E_p^+) - f(E_p^-)}{E_p^+ - E_p^-} \left( \frac{1}{\omega - E_p^+ + E_p^-} - \frac{1}{\omega + E_p^+ - E_p^-} \right) \right\}. \]

(B1)

Hence

\[ -\int_{-\infty}^{+\infty} \frac{1}{\pi} \text{Im} \left[ q \cdot Q_{S33}^0(\omega, q) \right] \]
After inserting the Ward identity (54) for the bare EM vertex and using Eqs. (C4), we get

where the number equation (31) has been used. This proves the lemma.

To prove the proposition, we only need to show

which is equivalent to

From \( \hat{\Sigma} = \hat{\Sigma} \) we turn to the proof of Eq. (C3). By substituting Eq. (55) into its RHS and repeating the process, we get an equality

We change variables by \( p \to p + \frac{q}{2} \) in the first term, and change variables by \( p \to p - \frac{q}{2} \) in the second term to get

where the number equation (31) has been used. This proves the lemma.

**Appendix C: Integral Equation of EM Vertex and GWI**

Here we prove that the vertex determined by the integral equation (92) obeys GWI (55). We will use the following equality

To prove the proposition, we only need to show

which is equivalent to

From \( \hat{\Sigma} = \hat{\Sigma} \) we conclude that

Now we turn to the proof of Eq. (C3). By substituting Eq. (55) into its RHS and repeating the process, we get an iterative equation

After inserting the Ward identity (54) for the bare EM vertex and using Eqs. (C4), we get

RHS of Eq. (C3)

RHS of Eq. (C3)
Similarly, for the temporal component of $Q$ where in the fifth line we have used $p$

By changing the dummy index $i \rightarrow i + 1$ in the second and fourth summations, we get

\[
\text{RHS of Eq.}(C3) = -(\sigma_3 \hat{\Sigma} - \Sigma \sigma_3) + \sum_{i=1}^{\infty} g^i \sum_{P_1 \cdots P_i} \prod_{k=1}^{i} [\sigma_3 \hat{\Sigma}(P_k)] g \sum_{P_{i+1}} [\sigma_3 \hat{\Sigma}(P_{i+1}) - \hat{\Sigma}(P_{i+1} + Q) \sigma_3] \prod_{k=1}^{i} [\hat{\Sigma}(P_{i+1-k} + Q) \sigma_3] = -(\sigma_3 \hat{\Sigma} - \Sigma \sigma_3) = \text{LHS of Eq.}(C3),
\]

where Eq. (C1) has been applied. Therefore we have proved that any vertex that satisfies the integral equation must also satisfy the Ward identity and hence must be gauge invariant.

Appendix D: Evaluations of the Vertex

From the expressions given in Appendix A at $T = 0$ we have

\[
\frac{2}{g} + Q_{22}(\omega, \mathbf{q}) \simeq \frac{N(0)}{4} \int_{-\infty}^{t+\infty} d\xi_p \int_{-1}^{1} d\cos \theta \frac{\omega^2 - q^2 \cos^2 \theta}{E^2_p} \cdot \frac{2E_p}{-4E^2_p} = -\frac{N(0)}{4} \frac{2}{\Delta^2} \left(\omega^2 - \frac{2q^2 \mu}{3m}\right),
\]

where in the fifth line we have used $p^2 = 2m(\xi_p + \mu)$. Note that $\mu \simeq \epsilon_F = \frac{g^2}{2m}$. Thus

\[
\hat{Q}_{22}(\omega, \mathbf{q}) = -\frac{N(0)}{2\Delta^2} \left(\omega^2 - \frac{1}{3m} \frac{q^2 k_F^2}{m^2}\right) = -\frac{N(0)}{2\Delta^2} \left(\omega^2 - c_s^2q^2\right).
\]

Similarly, for the temporal component of $Q_{23}^\mu$ we have

\[
Q_{23}(\omega, \mathbf{q}) \simeq i \int \frac{d^3p}{(2\pi)^3} \frac{\Delta \omega}{2E_p} \frac{2E_p}{2E_p^2 - 4E_p^2} = -i \frac{N(0)\omega}{2\Delta},
\]

Therefore $\Pi^0(\omega, \mathbf{q}) = \frac{\Delta \omega}{\omega^2 - c_s^2q^2}$ and the temporal component of the full vertex $\hat{\Gamma}^\mu$ is given by

\[
\hat{\Gamma}^0(P + Q, P) \simeq \sigma_3 + 2i\sigma_2 \frac{\Delta \omega}{\omega^2 - c_s^2q^2}.
\]

For the spatial component, we have

\[
Q_{23}(\omega, \mathbf{q}) \simeq i\Delta \mathbf{q} \cdot \int \frac{d^3p}{(2\pi)^3} \frac{1}{m^2} \frac{2E_p}{E_p^2 - 4E_p^2} = -i\frac{\Delta}{3m^2} \mathbf{q} \cdot \mathbf{1} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2E_p^2} \simeq -i\frac{q}{3\Delta} \frac{2\mu N(0)}{m}.
\]
Using $\mu \simeq \frac{k^2}{2m} = \frac{3m}{2} \omega_s^2$, we have $Q_{23}(\omega, q) \simeq -i\frac{N(0)}{\Delta}q$, so

$$\Pi(\omega, q) \simeq \frac{2\Delta s^2}{\omega^2 - c_s^2q^2}q.$$  \hfill (D6)

The spatial component is then given by

$$\Gamma(p + q, p) = \left( \frac{p + \frac{q}{m}}{\Pi(\omega, q)} \right) = \frac{p + \frac{q}{m}}{\Pi(\omega, q)} + 2i\sigma_2\frac{\Delta s^2 q}{\omega^2 - c_s^2q^2}. \hfill (D7)$$

**Appendix E: Physical interpretation of the gauge-invariant linear response theory**

In the main text we have seen that the WIs for response functions are indeed satisfied even without knowing the exact form of the full EM vertex. Here we pay some attention to the gauge invariance of the CFOP linear response theory from the point of view of a gauge transformation for the BCS Lagrangian. In real space, the Lagrangian density following the BCS approximation is given by

$$\mathcal{L}_{BCS} = \Psi^\dagger \left( i \frac{\partial}{\partial t} - \frac{(-iV)^2}{2m} - \mu \right) \sigma_3 - A_\mu \hat{\gamma}^\mu + \Delta \sigma_1 ) \Psi,$$  \hfill (E1)

where $\hat{\gamma}^\mu = (\sigma_3, -i\sigma_3)$. This Lagrangian density is obviously not invariant under the infinitesimal gauge transformation $\Psi \rightarrow (1 - i\sigma_3\chi)\Psi$, $\Psi^\dagger \rightarrow \Psi^\dagger (1 + i\sigma_3\chi)$ and $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ if the fluctuations of the order parameter are not considered. Now we split the order parameter into its equilibrium and perturbative parts as $\Delta \rightarrow \Delta + \Delta'$, where the equilibrium value is $\Delta$ and the perturbation is $\Delta'$. Therefore in the CFOP theory, the Lagrangian density in real space becomes

$$\mathcal{L}_{CFOP} = \mathcal{L}_{BCS0} + \mathcal{L}' = \Psi^\dagger \left( i \frac{\partial}{\partial t} - \frac{(-iV)^2}{2m} - \mu \right) \sigma_3 + \Delta \sigma_1 ) \Psi - \Psi^\dagger (\Delta \sigma_1 + \Delta \sigma_2 + A_\mu \hat{\gamma}^\mu ) \Psi,$$  \hfill (E2)

Here the subscript “0” denotes the part in equilibrium. The gauge transformation of $\Psi$ and $\Psi^\dagger$ leads to the fluctuations of the order parameter $\delta \Delta_1 = 0$ and $\delta \Delta_2 = -2\chi \Delta$. Therefore, the following generalized infinitesimal gauge transformation leaves the Lagrangian density (E1) of the CFOP theory invariant:

$$\Psi \rightarrow (1 - i\sigma_3\chi)\Psi, \quad \Psi^\dagger \rightarrow \Psi^\dagger (1 + i\sigma_3\chi), \quad \Delta \rightarrow \Delta,$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi, \quad \Delta_1 \rightarrow \Delta_1, \quad \Delta_2 \rightarrow \Delta_2 - 2\Delta \chi.$$  \hfill (E3)

Under this generalized infinitesimal transformation the two parts of the Lagrangian density transform according to

$$\mathcal{L}_{BCS0} \rightarrow \mathcal{L}_{BCS0} + \Psi^\dagger \partial_\mu \hat{\gamma}^\mu \Psi - i\chi \Psi^\dagger \Delta \sigma_1 \Psi = \mathcal{L}_{BCS0} + \Psi^\dagger \partial_\mu \hat{\gamma}^\mu \Psi - 2\chi \Psi^\dagger \Delta \sigma_2 \Psi,$$

$$\mathcal{L}' \rightarrow \mathcal{L}' + 2\chi \Psi^\dagger \Delta \sigma_2 \Psi - \Psi^\dagger \partial_\mu \hat{\gamma}^\mu \Psi.$$  \hfill (E4)

Therefore $\mathcal{L}_{BCS}$ is indeed invariant under the generalized infinitesimal gauge transformation (E3). It is the gauge transformation of $\Delta_2$ that compensates for the effects associated with the Cooper-pair condensation and leads to the gauge invariance of the CFOP theory. The Noether current associated with this generalized gauge transformation can be deduced by introducing the “generalized gauge space”. We define the space where the generalized external potential and generalized interacting vertex (see Eq. 34) live as the generalized gauge space. Explicitly, the perturbative Lagrangian density is rewritten as $-\Psi^\dagger \hat{\Phi} T \cdot \Sigma \Psi$, where $\cdot$ denotes the inner product in this generalized gauge space. The generalized gauge transformation of the generalized external potential is

$$\hat{\Phi} \rightarrow \hat{\Phi} + \begin{pmatrix} 0 \\ -2\Delta \chi \\ \partial_\mu \chi \end{pmatrix} \text{ or in the momentum space } \hat{\Phi} + \begin{pmatrix} 0 \\ -2\Delta \chi \\ -iq_\mu \chi \end{pmatrix}. \hfill (E5)$$

We define the generalized external momentums $\hat{q} \equiv (0, 2i\Delta, q_\mu)^T$ and $\hat{q} \equiv (0, 2i\Delta, -q_\mu)^T$ in the generalized gauge space. Then the generalized gauge transformation (E5) can be written as

$$\hat{\Phi} \rightarrow \hat{\Phi} + i\hat{q} \chi.$$  \hfill (E6)
The GWI (44) can be expressed as

$$\sigma_3 \tilde{G}^{-1}(P + Q) - \tilde{G}^{-1}(P)\sigma_3 = \hat{q}^T \cdot \Sigma.$$  \tag{E7}

In fact, this is the generalized Ward identity associated with the generalized gauge transformation (E5) in the generalized gauge space.

Next we address the conserved current associated with this gauge transformation. Using the self-consistent condition $\delta J_{1,2} = -\frac{2}{g} \Delta_{1,2}$, the current in Eq. (36) can be written as

$$\begin{pmatrix} 0 \\ J^\mu \end{pmatrix} = \begin{pmatrix} \hat{Q}_{11} & Q_{12} & Q_{13}^\nu \\ Q_{21} & \hat{Q}_{22} & Q_{23}^\nu \\ Q_{31}^\mu & Q_{32}^\mu & \hat{Q}_{33}^\mu & Q_{35}^\mu \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ A_\nu \end{pmatrix}. \tag{E8}$$

We then define the generalized current $\hat{J} \equiv (0, 0, J^\mu)^T$ and three generalized response-function vectors

$$\hat{Q}_1 = \begin{pmatrix} \hat{Q}_{11} \\ Q_{21} \\ Q_{31}^\mu \end{pmatrix}, \quad \hat{Q}_2 = \begin{pmatrix} Q_{12} \\ \hat{Q}_{22} \\ Q_{32}^\mu \end{pmatrix}, \quad \hat{Q}_3^\mu = \begin{pmatrix} Q_{13}^\mu \\ Q_{23}^\mu \\ \hat{Q}_{33}^\mu \end{pmatrix}. \tag{E9}$$

Then the current equation (E8) becomes

$$\hat{J} = (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3^\mu) \cdot \hat{\Phi}. \tag{E10}$$

The GWIs (42) for the response functions can also be written as

$$\hat{q}^T \cdot \hat{Q}_i = 0, \quad \text{for } i = 1, 2, 3. \tag{E11}$$

Thus the GWIs directly lead to the conservation of the generalized current

$$\hat{q}^T \cdot \hat{J} = (\hat{q}^T \cdot \hat{Q}_1, \hat{q}^T \cdot \hat{Q}_2, \hat{q}^T \cdot \hat{Q}_3^\mu) \cdot \hat{\Phi} = 0. \tag{E12}$$

This gives the conservation law of the EM current $q_\mu J^\mu = 0$. Therefore $\hat{J}$ is indeed the Neother current associated with the generalized gauge transformation. Moreover, by noting that the GWIs (E11) in the generalized gauge space can be written as

$$(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3^\mu) \cdot \hat{q} = 0. \tag{E13}$$

Under the generalized gauge transformation (E6), the generalized current transforms as

$$\hat{J} = (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3^\mu) \cdot \hat{\Phi} \rightarrow (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3^\mu) \cdot (\hat{\Phi} + i\hat{q}\chi) = \hat{J}. \tag{E14}$$

Thus the generalized current is indeed invariant under the generalized gauge transformation.

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