CONFINEMENT AND QUANTUM ANOMALY IN QUASI-1D SPINLESS FERMION CHAINS

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We consider the field renormalization group (RG) of two coupled one-spatial dimension (1D) spinless fermion chains under intraforward, interforward, interbackscattering and interumklapp interactions until two-loops order. Up to this order, we demonstrate the quantum confinement in the RG flow, where the interband chiral Fermi points reduce to single chiral Fermi points and the renormalized interaction couplings have Luttinger liquid fixed points. We show that this quasi-1D system is equivalently described in terms of one and two-color interactions and address the problem of quantum anomaly, inherent to this system, as a direct consequence of coupling 1+1 free Dirac fields to one and two-color interactions.

Keywords: renormalization group, confinement, anomaly.

1. Introduction

Luttinger liquids, characteristic of strongly correlated electron systems in one spatial dimension (1D), represent one of the most interesting examples of non-Fermi liquid behaviour, with intricate physical properties without counterpart in higher dimensions, as spin-charge separation \(^{12}\) and direct correspondence to bosonic excitations \(^{4}\). It is a fundamental problem for quantum phase transitions in strongly correlated systems the crossover between a Luttinger liquid state and a Fermi liquid. When this occurs, the degrees of freedom of the system are reduced in 1D. Important systems as the edges of quantum Hall systems, quantum wires and high-Tc superconductors are related to the typical behaviour of Luttinger liquids. Intermediate between 1D and two spatial dimensions (2D), the coupling among 1D chains forms a quasi-1D system, that can serve as a prototype for the phenomenon of confinement \(^{6}\) and the study of CuO, CuO\(_2\) compounds associated to high-Tc superconductors.\(^{6}\) As in 1D case, the Fermi surface of quasi-1D systems consist of discrete Fermi points, a set of Fermi points that form a discrete Fermi surface, in contrast to 2D and higher dimensions where the Fermi surface is continuous or a set of continuous.\(^{7}\)

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Here, we consider a quasi-1D system of spinless fermions under intraforward, interforward, interbackscattering and interumklapp interactions. For a linearized dispersion relation, the intra and interband interactions terms act as an external field perturbation and the system as a whole behaves as free 1 + 1 Dirac fields under an external field. As a consequence, the system presents an inherent chiral anomaly, as in the case of a 1 + 1 Schwinger model \[^{13,14}\], that is a result of the simultaneous requirement of vector current conservation (charge conservation) and axial current conservation (chiral symmetry) under vector and axial gauge transformations, respectively, appearing in the Ward-Takahashi identities in the one loop (1-loop) bubble polarizations with intra and interforward interactions \[^{3}\].

Applying a full field renormalization group (RG) until two loops (2-loops) order, taking into account a renormalization procedure that include the renormalized interband Fermi points in the physical prescription, we will derive the appropriate bare and renormalized quantities that lead to the set of RG flow equations including the renormalized interband Fermi points and the quasiparticle weight, considering the physical prescriptions of the one and two-particle irreducible functions until 2-loops order at the Fermi surface. In 2-loops, the role of the quasiparticle weight \(Z\) and the corresponding anomalous dimension \(\gamma\) turns to be important in both self-energy and scattering channels, leading to physically important contributions to the physical properties of the system \[^{18,2,20,21,29,22,25}\]. As a consequence we move forward in the previous results \[^{8,9}\], investigating the reduction of the renormalized interband chiral Fermi points to single renormalized chiral Fermi points. Taking into account the consistence with Ward-Takahashi relations, we will show that the quasi-1D system of two-coupled spinless fermions chains under intraforward, interforward, interbackscattering and interumklapp interactions can be equivalently described as 1 + 1 free Dirac fields coupled to one and two-color interactions, deriving the corresponding Ward-Takahashi identities \[^{15,16,12}\] and addressing the problem of quantum anomaly as a direct consequence of coupling 1+1 free Dirac fields to one and two-color interactions.

2. Two 1D coupled chains

The system of two coupled 1D spinless fermions chains is characterized by a transverse hopping \(t_\perp\) that couples the two 1D chains \[^{25,20}\] and the usual 1D chain intra-hopping term \(t\) (figure 1)

\[
\hat{H}_0 = -t \sum_{i,j} \left( \hat{c}_{i,1}^\dagger \hat{c}_{j,1} + \hat{c}_{i,2}^\dagger \hat{c}_{j,2} + \text{h.c.} \right) - t_\perp \sum_i \left( \hat{c}_{i,1}^\dagger \hat{c}_{i,2} + \text{h.c.} \right),
\]

where \(i\) and \(j\) are neighbouring sites, 1 and 2 are the chain labels, \(\hat{c}\) and \(\hat{\tilde{c}}\) are the creation and annihilation operators in the respective sites and chains. We can write the momentum space representation of the operators \(\hat{c}\) and \(\hat{\tilde{c}}\) by means of
the following Fourier transform

\[ \hat{c}_{i,1} = \frac{1}{\sqrt{N}} \sum_k e^{-ikx_i} \hat{c}_{k,1}, \]  

\[ \hat{c}_{i,2} = \frac{1}{\sqrt{N}} \sum_k e^{-ikx_i} \hat{c}_{k,2}. \]

It follows that we can write

\[ \hat{c}_{i,1}^\dagger \hat{c}_{j,1} = \frac{1}{N} \sum_{k,k'} e^{ik'x_i} e^{-ikx_j} \hat{c}_{k',1}^\dagger \hat{c}_{k,1}. \]

Considering the sum in the first neighbours \( x_j = x_i + a \) or \( x_j = x_i - a \), where \( a \) is the constant lattice spacing,

\[ \sum_{\langle i,j \rangle} \hat{c}_{i,1}^\dagger \hat{c}_{j,1} = \sum_i \left( \hat{c}_{i,1}^\dagger \hat{c}_{i+1,1} + \hat{c}_{i,1}^\dagger \hat{c}_{i-1,1} \right). \]

By summing the expressions

\[ \hat{c}_{i}^\dagger \hat{c}_{i+1} = \frac{1}{N} \sum_{k,k'} e^{-i(k-k')x_i} e^{-ika} \hat{c}_{k'}^\dagger \hat{c}_k, \]

\[ \hat{c}_{i}^\dagger \hat{c}_{i-1} = \frac{1}{N} \sum_{k,k'} e^{-i(k-k')x_i} e^{ika} \hat{c}_{k'}^\dagger \hat{c}_k \]

and using the relation

\[ \frac{1}{N} \sum_i e^{-i(k-k')x_i} = \delta_{kk'}, \]

the equation can be rewritten as

\[ \sum_{\langle i,j \rangle} \hat{c}_{i,1}^\dagger \hat{c}_{j,1} = 2 \sum_k \cos(ka) \hat{c}_{k,1}^\dagger \hat{c}_{k,1}. \]
The hamiltonian (1) is then written as
\[
\hat{H}_0 = -2t \sum_k \cos(ka) \left( \hat{c}_{k,1}^\dagger \hat{c}_{k,1} + \hat{c}_{k,2}^\dagger \hat{c}_{k,2} + \text{h.c.} \right) \\
- t_\perp \sum_i \left( \hat{\epsilon}_{i,1} \hat{\epsilon}_{i,2} + \text{h.c.} \right).
\]
(10)

Now we can treat the term due to the coupling between the two chains. From the transforms (2) and (3), we arrive at
\[
\hat{c}_{i,1}^\dagger \hat{c}_{i,2} = \frac{1}{N} \sum_{k,k'} e^{ik'x_i} e^{-ikx_i} \hat{c}_{k,1}^\dagger \hat{c}_{k,2}.
\]
(11)

Then, from (8), we have,
\[
\hat{H}_0 = -2t \sum_k \cos(ka) \left( \hat{c}_{k,1}^\dagger \hat{c}_{k,1} + \hat{c}_{k,2}^\dagger \hat{c}_{k,2} + \text{h.c.} \right) \\
- t_\perp \left( \sum_k \hat{c}_{k,1}^\dagger \hat{c}_{k,2} + \text{h.c.} \right).
\]
(12)

We see that the coupling term forbids the direct diagonalization of the hamiltonian. This problem can be solved by means of the following canonical transformation
\[
\hat{b}_k = \frac{\hat{c}_{k,1} + \hat{c}_{k,2}}{\sqrt{2}} \\
\hat{a}_k = \frac{\hat{c}_{k,1} - \hat{c}_{k,2}}{\sqrt{2}}
\]
(13)
(14)

which can also be written in the inverse form
\[
\hat{c}_{k,1} = \frac{\hat{b}_k + \hat{a}_k}{\sqrt{2}} \\
\hat{c}_{k,2} = \frac{\hat{b}_k - \hat{a}_k}{\sqrt{2}}
\]
(15)
(16)

By using (15) and (16), we have
\[
\hat{c}_{k,1}^\dagger \hat{c}_{k,1} = \frac{1}{2} \left( \hat{b}_k^\dagger \hat{b}_k + \hat{a}_k^\dagger \hat{a}_k \right) + \frac{1}{2} \left( \hat{a}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{a}_k \right) \\
\hat{c}_{k,2}^\dagger \hat{c}_{k,2} = \frac{1}{2} \left( \hat{b}_k^\dagger \hat{b}_k + \hat{a}_k^\dagger \hat{a}_k \right) - \frac{1}{2} \left( \hat{a}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{a}_k \right) \\
\hat{c}_{k,1}^\dagger \hat{c}_{k,2} = \frac{1}{2} \left( \hat{b}_k^\dagger \hat{b}_k - \hat{a}_k^\dagger \hat{a}_k \right) + \frac{1}{2} \left( \hat{a}_k^\dagger \hat{b}_k - \hat{b}_k^\dagger \hat{a}_k \right) \\
\hat{c}_{k,2}^\dagger \hat{c}_{k,1} = \frac{1}{2} \left( \hat{b}_k^\dagger \hat{b}_k - \hat{a}_k^\dagger \hat{a}_k \right) + \frac{1}{2} \left( -\hat{a}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{a}_k \right).
\]
(17)
(18)
(19)
(20)

From (17), (18), (19) and (20), we can rewrite (24) in the diagonalized form
\[
\hat{H}_0 = \sum_k \left( -2t \cos(ka) - t_\perp \right) \hat{b}_k^\dagger \hat{b}_k \\
+ \sum_k \left( -2t \cos(ka) + t_\perp \right) \hat{a}_k^\dagger \hat{a}_k.
\]
(21)
By defining the following dispersion relations
\[ \varepsilon^b_k = -2t \cos(ka) - t_\perp, \]
\[ \varepsilon^a_k = -2t \cos(ka) + t_\perp, \]
we arrive finally at the diagonalized form of the hamiltonian
\[ \hat{H}_0 = \sum_k \left( \varepsilon^a_k \hat{a}^\dagger_k \hat{a}_k + \varepsilon^b_k \hat{b}^\dagger_k \hat{b}_k \right). \]

As a consequence, the presence of the coupling term divides the energy spectrum in two bands (figure 2). We call the dispersion relation \( \varepsilon^a_k \) bonding band and \( \varepsilon^b_k \) antibonding band. Note that the energy gap between the bonding and antibonding bands is constant
\[ \Delta \varepsilon = \varepsilon^a_k - \varepsilon^b_k = 2t_\perp. \]

Expanding the Taylor series of the dispersion relations for the bonding and antibonding bands around the respective Fermi points \( k^a_F \), \( -k^a_F \), \( k^b_F \) and \( -k^b_F \), at first order, we arrive at linearized dispersion relations for the bond and antibonding bands
\[ \varepsilon^a_+ (k) = \left. \frac{d\varepsilon^a(k)}{dk} \right|_{k=k^a_F} (k - k^a_F), \]
\[ \varepsilon^a_- (k) = \left. \frac{d\varepsilon^a(k)}{dk} \right|_{k=-k^a_F} (k + k^a_F), \]
\[ \varepsilon^b_+ (k) = \left. \frac{d\varepsilon^b(k)}{dk} \right|_{k=k^b_F} (k - k^b_F), \]
\[ \varepsilon^b_- (k) = \left. \frac{d\varepsilon^b(k)}{dk} \right|_{k=-k^b_F} (k + k^b_F). \]
We can write the respective Fermi velocities

\[ v^a_F = \frac{d\varepsilon^a(k)}{dk} \bigg|_{k=k^a_F} = -\frac{d\varepsilon^a(k)}{dk} \bigg|_{k=-k^a_F}. \]

(30)

\[ v^b_F = \frac{d\varepsilon^b(k)}{dk} \bigg|_{k=k^b_F} = -\frac{d\varepsilon^b(k)}{dk} \bigg|_{k=-k^b_F}. \]

(31)

Then from (30) and (31), we can write (33), (32), (35) and (34) as follows

\[ \varepsilon^a_{+k} = v^a_F (k - k^a_F), \]

(32)

\[ \varepsilon^a_{-k} = v^a_F (-k - k^a_F), \]

(33)

\[ \varepsilon^b_{+k} = v^b_F (k - k^b_F), \]

(34)

\[ \varepsilon^b_{-k} = v^b_F (-k - k^b_F). \]

(35)

Now, we can write the hamiltonian for the linearized two coupled 1D chains in terms of fermionic fields \( \hat{\psi}_{\alpha k} \), in the following form

\[ \hat{H}_0 = \sum_{k,\alpha,s} \varepsilon^\alpha_{sk} \hat{\psi}^\dagger_{sk} \hat{\psi}_{sk}, \]

(36)

where \( \alpha = a, b \) is the color index, \( s = \pm \) is the chirality index, corresponding to the left and right-hand sides, \( a \) is the bonding and \( b \) is the antibonding band.

This quasi-1D system is then characterized by two colors \( (a \) and \( b) \) and chirality \( s \) given by left \((-\) and right-hand \((+)\) side of chiral spinless fermions. By means of a Legendre transform, we arrive at the following Lagrangean

\[ L_0 = \sum_{k,\alpha,s} \psi^\dagger_{sk} (k_0 - \varepsilon^\alpha_{sk}) \psi^\alpha_{sk}, \]

(37)

where the frequency \( k_0 \) comes from a Fourier transform on time derivative. Alternatively, we can write this in the configuration space

\[ L_0 = \int d^2 x \psi^\dagger_{s} (i \partial_t - v_F \partial_x + v_F k^2_F) \psi^+_{s}(x) \]

(38)

Taking into account the \( 1 + 1 \) gamma matrices with upper-index \( \gamma^0, \gamma^1 \) and \( \gamma^5 = \)

![Figure 3](image-url)
\( \gamma^0, \gamma^1, \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \)
\( \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \)
\( \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \)
\( \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \)

and the low-index \( \gamma_0 = \gamma^0, \gamma_5 = -\gamma^5 \) and \( \gamma_1 = -\gamma^1, \)
\( \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \)

Defining the pseudo-spinors \( \bar{\psi}^\alpha = \psi^\alpha \gamma^0, \psi^\alpha = \begin{pmatrix} \psi_+^\alpha \\ \psi_-^\alpha \end{pmatrix}, \psi^\alpha = \begin{pmatrix} \psi_+^\alpha & \psi_-^\alpha \end{pmatrix}, \)
\( \bar{\psi}^\alpha = \begin{pmatrix} \bar{\psi}_+^\alpha \\ \bar{\psi}_-^\alpha \end{pmatrix}, \psi^\alpha = \begin{pmatrix} \psi_+^\alpha & \psi_-^\alpha \end{pmatrix}, \)
where \( \alpha = a, b, \) we can write the action as a 1 + 1 Dirac field with an additional term. This can be easily seen, taking \( v_F = 1, x^0 = t e x^1 = x, \) with \( \phi = \gamma^\mu \partial_\mu, \mu = 0, 1, \) such that the free action corresponding to the quasi-1D spinless fermions chains can be written in the following form
\( S_0 = \sum_\alpha \int d^2 x \bar{\psi}^\alpha (\phi + \gamma^0 v_F k_\alpha^0) \psi^\alpha, \)
that is particularly useful for the analysis of quantum anomaly when we couple the band interaction terms.

### 3. Intraband and interband interactions

We consider all possible interactions with particles of opposite chirality that preserve both chiral symmetry and realizes all possible color exchanges \( (a \rightarrow b \text{ exchanges}), \) reducing to spinless Luttinger liquids with interactions of type \( -ig_2 \) and without interactions of type \( -ig_4 \) in the 1D limit. Under this consideration, we have four possible band interactions: intraforward, interforward, interbackscattering and interumklapp (figures 4 and 5). The intraforward interaction is an intraband interaction of type \( -ig_2. \) The interforward \(-ig_F\) consists of a interband scattering of opposite chirality without color exchange. The interbackscattering \(-ig_B\) is a scattering of opposite chirality with color exchange, where the initial colors in opposite chiralities are different. The interumklapp interaction \(-ig_U\) is a scattering of opposite chirality with color exchange, where the initial colors in opposite chiralities are the same. The interaction hamiltonian can be written as
\[ \hat{H}_{\text{int}} = \frac{1}{2} \sum_{k,k',q,s,\alpha} g_0 \hat{\psi}_s^{\alpha \dagger} \hat{\psi}_{s(k+q)}^{\alpha} - s(k-q) + \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} \left( g_F \hat{\psi}_s^{\alpha \dagger} \hat{\psi}_{s(k+q)}^{\beta} - s(k' - q) + g_B \hat{\psi}_s^{\alpha \dagger} \hat{\psi}_{s(k+q)}^{\beta} - s(k' - q) + g_U \hat{\psi}_s^{\alpha \dagger} \hat{\psi}_{s(k+q)}^{\beta} - s(k' - q) \right), \]

(45)

where \( s = \pm \) is the chirality index, \( \alpha, \beta = a, b \) are the color indexes, and \( k, k', q \) are the corresponding moments involved in the interactions. The term of \( 1/2 \) is due to the explicit introduction of the chirality term \( s \).

We can now write the total lagrangean corresponding to the model in the renormalized form, with the introduction of the counterterms \( \delta Z, \delta g_0, \delta g_F, \delta g_B, \delta g_U \) and
\( \delta g_{\ell, R} \)
\[
\mathcal{L}_R = \sum_{k, s, \alpha} \left( 1 + \delta Z \right) \hat{\psi}^{\alpha \dagger}_{sk, R} \left[ k_0 - \varepsilon^\alpha_{sk} \right] \hat{\psi}^\alpha_{sk, R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha} g_{0, R} \hat{\psi}^{\alpha \dagger}_{sk, R} \hat{\psi}^{\alpha \dagger}_{sk', R} \hat{\psi}^\beta_{s(k+q), R} \hat{\psi}^\beta_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha \neq \beta} g_{F, R} \hat{\psi}^{\alpha \dagger}_{sk, R} \hat{\psi}^{\beta \dagger}_{sk', R} \hat{\psi}^\alpha_{s(k+q), R} \hat{\psi}^\beta_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha \neq \beta} g_{B, R} \hat{\psi}^{\beta \dagger}_{sk, R} \hat{\psi}^{\alpha \dagger}_{sk', R} \hat{\psi}^\beta_{s(k+q), R} \hat{\psi}^\alpha_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha} \delta g_{0, R} \hat{\psi}^{\alpha \dagger}_{sk, R} \hat{\psi}^{\beta \dagger}_{sk', R} \hat{\psi}^\alpha_{s(k+q), R} \hat{\psi}^\beta_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha \neq \beta} \delta g_{F, R} \hat{\psi}^{\alpha \dagger}_{sk, R} \hat{\psi}^{\beta \dagger}_{sk', R} \hat{\psi}^\alpha_{s(k+q), R} \hat{\psi}^\beta_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha \neq \beta} \delta g_{B, R} \hat{\psi}^{\beta \dagger}_{sk, R} \hat{\psi}^{\alpha \dagger}_{sk', R} \hat{\psi}^\beta_{s(k+q), R} \hat{\psi}^\alpha_{s(-k'-q), R}
\]
\[
- \frac{1}{2} \sum_{k, k', q, s, \alpha \neq \beta} \delta g_{\ell, R} \hat{\psi}^{\beta \dagger}_{sk, R} \hat{\psi}^{\alpha \dagger}_{sk', R} \hat{\psi}^\beta_{s(k+q), R} \hat{\psi}^\alpha_{s(-k'-q), R}
\]
\[
(46)
\]
where the index \( R \) means renormalized and the counterterms \( \delta Z \), \( \delta g_{0, R} \), \( \delta g_{F, R} \), \( \delta g_{B, R} \) and \( \delta g_{\ell, R} \) correspond to the respective interactions, and the \( \hat{\psi} \)'s are given in terms of \( \hat{k} = (k_0, k) \).

The renormalized lagrangian (46) involves a prescription of renormalization. By including into the prescription the Fermi moments and Fermi velocities, we have to reserve counterterms also for such terms, i. e., for the Fermi velocity and Fermi moment associated to each band, leading to a correction in in the free dispersion relation \( \varepsilon^\alpha_{sk} \). The free term is then renormalized into the following form
\[
\mathcal{L}_{0R} = \sum_{k, s, \alpha} \hat{\psi}^{\alpha \dagger}_{sk, R} \left[ k_0 - v_{F, R} (sk - k^\alpha_{F, R}) \right] \hat{\psi}^\alpha_{sk, R}
\]
\[+ \sum_{k, s, \alpha} \hat{\psi}^{\alpha \dagger}_{sk, R} \left[ k_0 \delta Z - sk \delta v_{F} + v_{F, R} \delta k^\alpha_{F} \right] \hat{\psi}^\alpha_{sk, R},
\]
\[
(47)
\]
where we have counterterms associated to the Fermi velocity \( \delta v_{F, R} \) and to the Fermi moments \( \delta k^\alpha_{F} \). Note that for simplicity we consider the same Fermi velocity \( v_{F, R} \). We can describe the counterterm associated to the Fermi velocity in terms of \( \delta Z \), by
means of the relation \( \delta v_{F,R} = v_{F,R} \delta Z \). We can then write the previous lagrangean in the following form

\[
L_{0,R} = \sum_{k,s,\alpha} \hat{\psi}_{sk,R}^\dagger \left[ k_{0,R} - v_{F,R} (sk - k_{F,R}) \right] \hat{\psi}_{sk,R} \\
+ \sum_{k,s,\alpha} \hat{\psi}_{sk,R}^\dagger \left[ k_{0,R} \delta Z - skv_{F,R} \delta Z + v_{F,R} \delta k_{F,R} \right] \hat{\psi}_{sk,R}.
\]

(48)

Now it involves a renormalized term \( \varepsilon_{sk,R} = v_{F,R} (sk - k_{F,R}) \) and the counterterms associated.

The total renormalized lagrangean can be written with the explicit contributions due to the counterterms

\[
\mathcal{L}_R = \sum_{k,s,\alpha} \hat{\psi}_{sk,R}^\dagger \left[ k_{0,R} - \varepsilon_{sk,R} \right] \hat{\psi}_{sk,R} \\
- \frac{1}{2} \sum_{k,k',q,s,\alpha} \delta g_{0,R} \psi_{sk,R} \psi_{sk,R}^\dagger \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \\
- \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} \delta g_{F,R} \psi_{sk,R} \psi_{sk,R}^\dagger \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \\
- \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} \delta g_{B,R} \psi_{sk,R} \psi_{sk,R}^\dagger \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \psi_{sk,R} \psi_{sk,R}^\dagger \left( sk - k_{F,R} \right) \\
+ \sum_{k,s,\alpha} \hat{\psi}_{sk,R}^\dagger \left[ k_{0,R} \delta Z - skv_{F,R} \delta Z + v_{F,R} \delta k_{F,R} \right] \hat{\psi}_{sk,R}.
\]

(49)

The procedure of renormalization group realizes a modification in the fermionic fields, relating to the \textit{bare} fields by means of the following relation

\[
\hat{\psi}_{sk,R}^\dagger = Z^{-1/2} \hat{\psi}_{sk,bare}^\dagger
\]

(50)

where \( Z = 1 + \delta Z \) is the weight of the quasiparticle.
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4. The bare quantities

The bare fields are fields invariant under the renormalization group, such that, written in terms. In terms of such fields the lagrangian takes the following form

\[ \mathcal{L}_{\text{bare}} = \sum_{k,s,\alpha} \hat{\psi}_{sk,\text{bare}}^\dagger \left[ k_0,\text{bare} - \epsilon_{sk,\text{bare}}^\alpha \right] \hat{\psi}_{sk,\text{bare}} \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha} g_{0,\text{bare}} \hat{\psi}_{sk,\text{bare}}^\dagger \hat{\psi}_{sk',\text{bare}} \]

\[ \times \hat{\psi}_{(k+q),\text{bare}}^\alpha \hat{\psi}_{-s(k' - q),\text{bare}}^\alpha \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} g_{F,\text{bare}} \hat{\psi}_{sk,\text{bare}}^\dagger \hat{\psi}_{sk',\text{bare}}^\dagger \]

\[ \times \hat{\psi}_{(k+q),\text{bare}}^\beta \hat{\psi}_{-s(k' - q),\text{bare}}^\alpha \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} g_{B,\text{bare}} \hat{\psi}_{sk,\text{bare}}^\dagger \hat{\psi}_{sk',\text{bare}}^\dagger \]

\[ \times \hat{\psi}_{(k+q),\text{bare}}^\alpha \hat{\psi}_{-s(k' - q),\text{bare}}^\alpha \]  \hspace{1cm} (51)

In this form, the lagrangian has the form of the non-renormalized model. It is in fact equivalent to the renormalized model, such that using the relation \(50\) in \(51\), we have

\[ \mathcal{L}_R = \sum_{k,s,\alpha} Z \hat{\psi}_{sk,R}^\dagger \left[ k_0,\text{bare} - \epsilon_{sk,\text{bare}}^\alpha \right] \hat{\psi}_{sk,R} \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha} Z^2 g_{0,\text{bare}} \hat{\psi}_{sk,R}^\dagger \hat{\psi}_{sk',R}^\dagger \]

\[ \times \hat{\psi}_{(k+q),R}^\alpha \hat{\psi}_{-s(k' - q),R}^\alpha \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} Z^2 g_{F,\text{bare}} \hat{\psi}_{sk,R}^\dagger \hat{\psi}_{sk',R}^\dagger \]

\[ \times \hat{\psi}_{(k+q),R}^\beta \hat{\psi}_{-s(k' - q),R}^\alpha \]

\[ - \frac{1}{2} \sum_{k,k',q,s,\alpha \neq \beta} Z^2 g_{B,\text{bare}} \hat{\psi}_{sk,R}^\dagger \hat{\psi}_{sk',R}^\dagger \]

\[ \times \hat{\psi}_{(k+q),R}^\alpha \hat{\psi}_{-s(k' - q),R}^\alpha \]  \hspace{1cm} (52)
The equation (52) is identical to the equation (49) for the renormalized lagrangean, where we have the following relations between bare and renormalized \( R \) quantities

\[ Z_{k_{0}, \text{bare}} = k_{0, R} + \delta Z_{k_{0}, R} \]  
\[ Z_{\varepsilon_{sk, \text{bare}}} = \varepsilon_{sk, R} + s k_{F, R} \delta Z - v_{F, R} \delta k_{F, R} \]  
\[ Z_{g_{0, \text{bare}}} = g_{0, R} + \delta g_{0, R} \]  
\[ Z_{g_{F, \text{bare}}} = g_{F, R} + \delta g_{F, R} \]  
\[ Z_{g_{B, \text{bare}}} = g_{B, R} + \delta g_{B, R} \]  
\[ Z_{g_{U, \text{bare}}} = g_{U, R} + \delta g_{U, R} \]

From the equation (54), we also arrive at the following relations

\[ Z_{v_{F, \text{bare}}} = v_{F, R} + v_{F, R} \delta Z = Z_{v_{F, R}}. \]  

It follows the relation \( v_{F, \text{bare}} = v_{F, R} \) and

\[ Z_{k_{F, \text{bare}}} = k_{F, R} + \delta k_{F, R}. \]  

Defining as \( \Delta k_{F} = k_{b}^{F} - k_{a}^{F} \) the difference between the Fermi points from the bands \( b \) and \( a \), we also can write such difference relating the bare and renormalized quantities

\[ Z_{\Delta k_{F, \text{bare}}} = \Delta k_{F, R} + \delta \Delta k_{F, R}, \]  

where we have defined \( \delta \Delta k_{F, R} = \delta k_{F, R}^{b} - \delta k_{F, R}^{a} \).

5. Green’s functions

We now consider the free propagators written in the form \( k_{0, R} - \varepsilon_{sk, R}^{\alpha} \) and the Green’s functions associated to them

\[ G_{s, R}^{\alpha}(\bar{k}) = \frac{\theta_{sk, R}^{\alpha, >}}{k_{0, R} - \varepsilon_{sk, R}^{\alpha} + i \delta} + \frac{\theta_{sk, R}^{\alpha, <}}{k_{0, R} - \varepsilon_{sk, R}^{\alpha} - i \delta}, \]  

where \( s = \pm \) correspond to the chirality index, \( \alpha = a, b \) correspond to the color index and

\[ \theta_{sk, R}^{\alpha, >} = \theta(s k - k_{F, R}^{\alpha}), \]  
\[ \theta_{sk, R}^{\alpha, <} = \theta(k_{F, R}^{\alpha} - s k), \]  

are the corresponding step functions.

The figure 6 gives the Feynman rules for the free propagators. The Feynman rules for the interactions couplings can be written in the figure 7.
6. Energy-Momentum conservation, color exchanges and chiral symmetry

As usual, we can represent the $\vec{p} = (p_0, p)$ energy-momentum conservation in the Feynman diagrams by means of the ingoing = outgoing relations

$$\vec{p}_{in} = \vec{p}_{out}. \quad (65)$$

On the other hand, the quasi-1D system with intraforward, interferward, interbackscattering and interumklapp interactions presents some unusual interesting properties, corresponding to color and chiral indices.

Color exchanges are mediated by interband interactions. In the interferward channel, there is no color exchange. In the interbackscattering and in the interumklapp channels there is always color exchange, where in the interumklapp the initial colors in opposite chiralities are the same and in the interbackscattering the initial colors in opposite chiralities are different (figure 8). As a consequence of this fact, the
vertex interactions, where there is no color exchange, are associated to intraforward and interferoward interactions (figure 9). The mediation of color exchanges by the

![Vertex Interactions](image)

Fig. 9. (Color online) Vertex interactions.

band interactions also leads to a color number conservation, expressed by the fact of the number of ingoing colors is equal to the number of outgoing colors.

There is no chiral exchanges, i.e., there is no interactions that mediate change of chirality, implying (figure 10)

\[ s_{\text{in}} = s_{\text{out}}. \]  

(66)

Note that the chiral symmetry do not forbid \(-ig_4\)-type interactions, but excludes interactions of type \(-ig_1\) and \(-ig_3\) that are associated to intraband umklapp and backscattering interactions.

![Chiral Symmetry](image)

Fig. 10. (Color online) Chiral symmetry.

7. One and two color polarization bubbles

One-color polarization bubbles are given by

\[ \chi_1^\alpha(\vec{q}) = -\int_{\vec{q}'} iG_\alpha^s(\vec{q}') iG_\alpha^s(\vec{q} + \vec{q}'), \]  

(67)

where the \(-1\) sign is due to the fermionic loop. By computing the polarization in \(b\) (figure 11) we arrive at
In the general case,
\[ \chi^\alpha_s(\vec{q}) = i \frac{sq}{2\pi} G^\alpha_s(q_0, q + sk_F), \]
where \( s = \pm, \alpha = a, b \).

As there is no color exchange, the one-color polarization bubble is always associated to the intraforward and interforward interactions (figure 12).

The two-color polarization bubbles are given by
\[ \chi^\alpha\beta_s(\vec{q}) = -\int_{\vec{q}'} iG^\alpha_s(\vec{q}')iG^\beta_s(\vec{q'} + \vec{q}), \]
where \( s = \pm, \alpha = a, b \) and \( \alpha \neq \beta \). For instance, by computing the \( ab \)-polarization (figure 13), in the case where the Fermi velocities are equal, \( \Delta v_{F,R} = 0 \), we arrive at
The two-color polarization bubble involves color exchange, always coming in association to interbackscattering and interumklapp interactions (figure 14).

As there is no chirality exchanges, the diagrams of two-chiralities (figure 15) are always associated to two-particle interactions of opposite sides and depend on the external legs (figure 16). These type of polarization can be defined as follows

\[
\Pi_{s's'}^{\alpha\beta}(\bar{p}) = \int \frac{iG_\alpha^s(\bar{q})iG_\beta^{s'}(\bar{p} - \bar{q})}{\frac{1}{2\pi}}
\]

(73)

\[
\chi_{s's'}^{\alpha\beta}(\bar{p}) = \int \frac{iG_\alpha^s(\bar{q})iG_\beta^{s'}(\bar{p} + \bar{q})}{\frac{1}{2\pi}}
\]

(74)

where \( s \neq s' \), \( s = \pm \), \( s' = \pm \), \( \alpha = a, b \) and \( \beta = a, b \).

By calculating the scattering channels in 1-loops in the Fermi surface, we can obtain the explicitly the polarizations of different chirality at Fermi surface, \( s = + \).
and \( s' = - \),

\[
\Pi_{+^-}^{ab}(\vec{p}_{ab}) = - \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\Pi_{+^-}^{ba}(\vec{p}_{ab}) = - \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right), \\
\Pi_{+^-}^{a}(\vec{p}_{a}) = - \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\Pi_{+^-}^{b}(\vec{p}_{a}) = - \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\chi_{+^-}^{ab}(\vec{p}'_{ab}) = \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\chi_{+^-}^{ba}(\vec{p}'_{ab}) = \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\chi_{+^-}^{a}(\vec{p}'_{a}) = \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right), \\
\chi_{+^-}^{b}(\vec{p}'_{a}) = \frac{i}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right).
\]
where $\vec{p}_a$, $\vec{p}_{ab}$, $\vec{p}'_{ab}$ and $\vec{p}'_{a}$ are associated to the ingoing external legs
\begin{align}
\vec{p}_{ab} &= \vec{p}_{in,a} + \vec{p}_{in,b}, \\
\vec{p}'_{ab} &= \vec{p}_{in,a} - \vec{p}_{in,b}, \\
\vec{p}_{a} &= \vec{p}_{in,a} + \vec{p}_{in,b}, \\
\vec{p}'_{a} &= \vec{p}_{in,a} - \vec{p}_{in,b},
\end{align}
and $\Omega = V_{F,R} \Lambda$ is the energy cutoff associated to the momentum cutoff $\Lambda$.

8. Self-energies and renormalized $\Gamma^{(2)}$ functions

As the 1-loop self-energy diagrams involves only one-color polarization bubbles (see figures 17(a), 17(b), 17(c) and 17(d)), they are given by the following formula
\begin{align}
- i \Sigma_{s,R}^\alpha &= g_{F,R} \int \frac{d\vec{q}}{\Omega} G_{s,R}^\beta(\vec{q}) + g_{0,R} \int \frac{d\vec{q}}{\Omega} G_{s,R}^\alpha(\vec{q}),
\end{align}
where $\alpha = a, b$, $\beta = a, b$, $\alpha \neq \beta$. By integrating in $\vec{q}$ the self-energy equations (87), including the momentum cutoff $\Lambda$, in the Fermi surface, we arrive explicitly at
\begin{align}
- i \Sigma_{-,R}^a &= i (\bar{g}_{F,R} + \bar{g}_{0,R}) \Omega, \\
- i \Sigma_{-,R}^b &= i (\bar{g}_{F,R} + \bar{g}_{0,R}) \Omega, \\
- i \Sigma_{+,R}^a &= i (\bar{g}_{F,R} + \bar{g}_{0,R}) \Omega, \\
- i \Sigma_{+,R}^b &= i (\bar{g}_{F,R} + \bar{g}_{0,R}) \Omega,
\end{align}
where $\bar{g}_{F,R} = \frac{g_{F,R}}{\pi V_{F,R}}$, $\bar{g}_{0,R} = \frac{g_{0,R}}{\pi V_{F,R}}$, and $\Omega = \Delta V_F$. Then, in 1-loop the self-energy contributions are given by intraforward and interforward interactions. If we consider the 2-loops self-energy contributions, we have additional contributions including interbackscattering and interumklapp interactions, with two-color polarization bubbles and color exchanges (figures 18 and 19).

\[ -i\Sigma_{\bar{p},R} = (-ig_{0,R})^2 \int_{\vec{q}} iG_{+,R}^a(\vec{q})\chi_{-,R}(\vec{p} - \vec{q}). \]  

(92)

At the Fermi surface, $p_0 = \omega$ and $p = k_{F,R}^d$,

\[ -i\Sigma_{+,R}(k_{F,R}^d, \omega) = \frac{ig_{0,R}^2\omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right). \]  

(93)
The self-energy contribution of the interforward interaction in 2-loops (21) is given by
\[ -i \Sigma^{ab}_{+, R}(\vec{p}) = (-ig_{F,R})^2 \int_{\vec{q}} iG^a_{+, R}(\vec{q}) \chi^b_{-, R}(\vec{p} - \vec{q}), \] (94)
that reduces, at the Fermi surface, \( p_0 = \omega \) and \( p = k^a_F \), for equal Fermi velocities \( v_{F,R} \), to the following
\[ -i \Sigma^{ab}_{+, R}(k^a_F, \omega) = \frac{ig_{F,R}^2 \omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right). \] (95)
Both interforward and intraforward selfenergy contributions are one-color polarization bubbles contributions to the selfenergy, with no color exchange.

Now, considering the calculation of the two color contributions at 2-loops to the selfenergy, we have the interbackscattering interaction contribution (figure 18),
\[ \Sigma^{bba}_{+, -} = (-ig_B)^2 \int_{\vec{q}} iG^b_{+, R}(\vec{q}) \chi^b_{-, -}(\vec{p} - \vec{q}), \] (96)
that at the Fermi surface \( \vec{p} = (\omega, k^a_{F,R}) \), in the case of same Fermi velocities, is given by
\[ -i \Sigma^{bba}_{+, -}(k^a_{F,R}, \omega) = \frac{ig_B^2 V_{F,R} \Delta k_{F,R}}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \]
\[ + \frac{ig_B^2 \omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right), \] (97)
and interumklapp interaction contribution (figure 19)

\[-i\Sigma_{L,R}^{\text{int}} = (-ig_{L,R})^2 \int_{\bar{q}} \bar{G}_{L,R}^b(\bar{q}) \chi_{L,R}^{ab}(\bar{p} - \bar{q}).\]  

(98)

that at the Fermi surface \(\bar{p} = (k_{F,R}, \omega)\), in the case of same Fermi velocities, is

\[-i\Sigma_{L,R}^{\text{int}}(k_{F,R}, \omega) = \frac{ig_{L,R}\omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right).\]  

(99)

Now, we can write the total contribution to the selfenergy in 2-loops for the right-hand side of the band \(a\), at the Fermi surface (FS),

\[-i\Sigma_{+R}^{a}(FS) = \frac{ig_{F,R}^2\omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right) + \frac{ig_{0,R}\omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right)\]

\[+ \frac{ig_{B,R}^2(V_{F,R}\Delta k_{F,R} + \omega)}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right)\]

\[+ \frac{ig_{L,R}\omega}{2(\pi V_{F,R})^2} \ln \left( \frac{\Omega}{\omega} \right)\]  

(100)

and the corresponding one-particle irreducible function \(\Gamma^{(2)}\) at Fermi surface

\[
\Gamma^{(2)}_{a+R}(\omega, k_{F}^a) = \omega - (\bar{g}_{0R} + \bar{g}_{F,R}) \Omega
\]

\[+ \omega \left[ \frac{\bar{g}_{F,R}^2}{2} + \frac{\bar{g}_{0,R}^2}{2} + \frac{\bar{g}_{L,R}^2}{2} \right] \ln \left( \frac{\Omega}{\omega} \right)\]

\[+ \omega \frac{\bar{g}_{B,R}^2}{2} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F}} \right)\]

\[+ (V_{F,R}\Delta k_{F}) \frac{\bar{g}_{B,R}^2}{2} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F}} \right)\]

\[+ \omega \delta Z - k_{F,R}^a v_{F,R} \delta Z + v_{F,R} \delta k_{F,R}^a,\]  

(101)

where \(\delta Z, \delta k_{F,R}^a\) are the corresponding counterterms, introduced in the corresponding renormalized lagrangean. In the case of the band \(b\) (figure 23).
Fig. 23. (Color online) Self-energy diagrams in 2-loops, band \( b^+ \).

\[
\Gamma_{b^+ R}^{(2)}(\omega, k^b_{F,R}) = \omega - (\tilde{g}_0 R + \tilde{g}_F R) \Omega \\
+ \omega \left[ \frac{g^2_{F,R}}{2} + \frac{\tilde{g}^2_{0,R}}{2} + \frac{\tilde{g}^2_{F,R}}{2} \right] \ln \left( \frac{\Omega}{\omega} \right) \\
+ \omega \frac{\tilde{g}^2_{B,R}}{2} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \\
- (V_{F,R} \Delta k_{F,R}) \frac{\tilde{g}^2_{B,R}}{2} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \\
+ \omega \delta Z - k^b_{F,R} v_{F,R} \delta Z + v_{F,R} \delta k^b_{F,R},
\]

(102)

where now we have the counterterm \( \delta k^b_{F,R} \) and we have introduced the notation \( \tilde{g}_R = g_R / \pi V_{F,R} \) for each corresponding coupling,

\[
\tilde{g}_{0,R} = g_{0,R}(\omega) / \pi V_{F,R}, \\
\tilde{g}_{F,R} = g_{F,R}(\omega) / \pi V_{F,R}, \\
\tilde{g}_{B,R} = g_{B,R}(\omega) / \pi V_{F,R}, \\
\tilde{g}_{U,R} = g_{U,R}(\omega) / \pi V_{F,R}.
\]

(103) (104) (105) (106)

Similar calculations are applied for the opposite chiralities \( \Gamma_{a^- R}^{(2)} \) and \( \Gamma_{b^- R}^{(2)} \) (figure 24).

Fig. 24. (Color online) Self-energy diagrams in 2-loops, bands \( a^- \) and \( b^- \).
9. Physical prescription for $\Gamma^{(2)}_R$ and RG flow equation for $\Delta k_{F,R}$

Now we consider the renormalization group prescription for $\Gamma^{(2)}_R$ at the corresponding Fermi surfaces. At each band in the corresponding Fermi surfaces it takes the value $\Gamma^{(2)}_R = \omega$, such that from $\Gamma^{(2)}_{a+R}(\omega, k^a_{F})$, eq. (101) and $\Gamma^{(2)}_{b+R}(\omega, k^b_{F})$, eq. (102), we arrive at the following counterterms corresponding to the quasiparticle weight 

$$\delta Z = -\left[\frac{\bar{g}^2_{F,R}}{2} + \frac{\bar{g}^2_{0,R}}{2} + \frac{\bar{g}^2_{H,R}}{2} + \frac{\bar{g}^2_{B,R}}{2}\right] \ln\left(\frac{\Omega}{\omega}\right),$$

(107)

and the respective Fermi momentum counterterms

$$\delta k^a_{F,R} = k^a_{F,R}\delta Z + (\bar{g}_{0,R} + \bar{g}_{F,R}) \frac{\Omega}{v_{F,R}} - \Delta k_{F,R} \bar{g}^2_{B,R} \ln\left(\frac{\Omega}{V_{F,R} \Delta k_{F,R}}\right),$$

(108)

$$\delta k^b_{F,R} = k^b_{F,R}\delta Z + (\bar{g}_{0,R} + \bar{g}_{F,R}) \frac{\Omega}{v_{F,R}} + \Delta k_{F,R} \bar{g}^2_{B,R} \ln\left(\frac{\Omega}{V_{F,R} \Delta k_{F,R}}\right).$$

(109)

By adding $k^\alpha_{F,R}$, $\alpha = a, b$, in each corresponding counterterm for the Fermi momentum and taking into account the relation $\delta Z = Z - 1$, we can write

$$\delta k^a_{F,R} + k^a_{F,R} = k^a_{F,R}Z + (\bar{g}_{0,R} + \bar{g}_{F,R}) \frac{\Omega}{v_{F,R}} - \Delta k_{F,R} \bar{g}^2_{B,R} \ln\left(\frac{\Omega}{V_{F,R} \Delta k_{F,R}}\right),$$

(110)

$$\delta k^b_{F,R} + k^b_{F,R} = k^b_{F,R}Z + (\bar{g}_{0,R} + \bar{g}_{F,R}) \frac{\Omega}{v_{F,R}} + \Delta k_{F,R} \bar{g}^2_{B,R} \ln\left(\frac{\Omega}{V_{F,R} \Delta k_{F,R}}\right).$$

(111)

Taking into account the relation between renormalized and bare Fermi moments, eq. (110),

$$k^a_{F,\text{bare}} = \frac{\delta k^a_{F,R} + k^a_{F,R}}{Z},$$

(112)

$$k^b_{F,\text{bare}} = \frac{\delta k^b_{F,R} + k^b_{F,R}}{Z},$$

(113)
we can write the relation between $\Delta k_{F,\text{bare}}$ and $\Delta k_{F,R}$, eq. (61), explicitly

$$
\Delta k_{F,\text{bare}} = k_{F,\text{bare}}^b - k_{F,\text{bare}}^a = \Delta k_{F,R} \left[ 1 + 2 \tilde{g}_{B,R}^2 \frac{\Omega}{Z V_{F,R} \Delta k_{F,R}} \right].
$$

(114)

As the bare quantities do not flow under RG, we arrive at the corresponding flow equation for $\Delta k_{F,R}$. We have

$$
\omega d \Delta k_{F,R} = \omega d \Delta k_{F,R} \left[ 1 + 2 \frac{\tilde{g}_{B,R}^2}{Z} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \right]
$$

$$
+ \Delta k_{F,R} \left[ -2 \frac{\tilde{g}_{B,R}^2}{Z^2} \frac{d\omega}{d\omega} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \right]
$$

$$
- \frac{2 \tilde{g}_{B,R}^2}{Z} \frac{\Delta k_{F,R}}{d\omega} \frac{d\Delta k_{F,R}}{d\omega}
$$

$$
= 0.
$$

(115)

Including the anomalous dimension

$$
\gamma = \frac{\omega}{Z} \frac{d\gamma}{d\omega}
$$

(116)

and multiplying both sides by $Z$, we arrive at

$$
\omega d \Delta k_{F,R} = \frac{2 \Delta k_{F,R} \tilde{g}_{B,R}^2 \gamma \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right)}{Z + 2 \tilde{g}_{B,R}^2 \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) - 1}.
$$

(117)

This can be also written in the following simplified form

$$
\omega d \Delta k_{F,R} = \frac{\gamma}{1 - \frac{1}{\Lambda(k_{F,R})} \frac{Z}{\tilde{g}_{B,R}^2}},
$$

(118)

where we take into account the cutoff $\Lambda = \Omega/V_{F,R}$.

In order to simplify the above equation and achieve to a cutoff independent equation, we can consider the cutoff sufficiently large ($\Lambda \to \infty$) such that the above equation is simplified to

$$
\omega d \Delta k_{F,R} = \gamma,
$$

(119)

or, equivalently,

$$
\omega \frac{d\Delta k_{F,R}}{d\omega} = \gamma \Delta k_{F,R}.
$$

(120)
10. Two-particle irreducible function $\Gamma^{(4)}_R$ and the counterterms for the scattering channels

Now, we consider the scattering channels corresponding to the interactions of intraforward, interforward, interbackscattering and interumklapp, corresponding to the two-particle irreducible functions $\Gamma^{(4)}_R$ for each scattering channel.

In the intrafrontal channel (figure 25), the two-particle irreducible function in 1-loop has the following contribution at the Fermi surface

$$\Gamma^{(4)}_{0,R} = -i g_{0,R} - \frac{i g_{0,R}}{\pi V_F} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right)$$

$$+ \frac{i g_{4,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right) - i \delta g_{0,R}. \quad (121)$$

---

**Fig. 25.** (Color online) Interforward channel diagrams in 1-loop.

**Fig. 26.** (Color online) Intraforward channel diagrams in 1-loop.
where $-i\delta g_{0,R}$ is the corresponding counterterm for the intraforward channel in 1-loop. In the interfrontal channel (figure 25) the two-particle irreducible function in 1-loop has the following contribution at the Fermi surface

$$\Gamma^{(4)}_{F,R} = -ig_{F,R} + \frac{ig_{B,R}^2}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_F} \right),$$

where $-i\delta g_{F,R}$ is the corresponding counterterm for the interforward channel in 1-loop. In the interbackscattering channel (figure 27), we have the following contribution at the Fermi surface.
bution at the Fermi surface

\[
\Gamma_{B,R}^{(4)} = -ig_{B,R} + \frac{ig_{B,R}g_{F,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right)
+ \frac{ig_{F,R}g_{B,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right) - \frac{ig_{F,R}g_{0,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right)
- \frac{ig_{0,R}g_{B,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right) - i\delta g_{B,R},
\]

(123)

where \(-i\delta g_{B,R}\) is the corresponding counterterm for the interbackscattering channel in 1-loop. In the interumklapp channel (figure 28), we have the following contribution at the Fermi surface

\[
\Gamma_{U,R}^{(4)} = -ig_{U,R} + 2i g_{U,R}g_0,R \ln \left( \frac{\Omega}{\omega} \right)
- 2i g_{U,R}g_{F,R} \ln \left( \frac{\Omega}{\omega} \right) - i\delta g_{U,R},
\]

(124)

where \(-i\delta g_{U,R}\) is the corresponding counterterm for the interumklapp channel in 1-loop.

The renormalization prescription for the two-particle irreducible functions at the Fermi surface are the corresponding renormalized coupling

\[
\Gamma_{0,R}^{(4)} = -ig_{0,R},
\]
\[
\Gamma_{F,R}^{(4)} = -ig_{F,R},
\]
\[
\Gamma_{B,R}^{(4)} = -ig_{B,R},
\]
\[
\Gamma_{U,R}^{(4)} = -ig_{U,R}.
\]

(125, 126, 127, 128)

We can then calculate explicitly the counterterms in 1-loop for the intraforward

\[
\delta g_{0,R} = -\frac{g_{B,R}^2}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) + \frac{g_{U,R}^2}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right);
\]

(129)

Fig. 29. (Color online) Counterterm diagram for intraforward.
for the interforward channel (figure 30)

\[
\delta g_{F,R} = \frac{g_{B,R}^2}{\pi V_{F,R}} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right)
- \frac{g_{U,R}^2}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right),
\]  

(130)
for the interbackscattering channel (figure 31)

$$\delta g_{B,R} = \frac{g_{B,R}g_{F,R}}{\pi V_F} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) + \frac{g_{F,R}g_{B,R}}{\pi V_F} \ln \left( \frac{\Omega}{\omega} \right)$$

$$- \frac{g_{B,R}g_{0,R}}{\pi V_F} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right)$$

$$- \frac{g_{0,R}g_{B,R}}{\pi V_F} \ln \left( \frac{\Omega}{\omega} \right),$$  \hspace{1cm} (131)

and for the interumklapp channel (figure 32)

$$\delta g_{U,R} = 2 \frac{g_{0,R}g_{U,R}}{\pi V_{F,R}} \ln \left( \frac{\Omega}{\omega} \right)$$

$$- \frac{g_{U,R}g_{F,R}}{\pi V_F} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right).$$  \hspace{1cm} (132)

The 2-loops diagrams from the 1-loop counterterms (figure 33) cancel all the parquet diagrams in 2-loops (figures 34, 35, 36, 37).

Fig. 33. (Color online) Counterterm diagram in 2-loops.

By calculating the contributions of non-parquet diagrams in two-loops, including the 2-loops non-parquet contributions, we have in the intraforward channel (figure 38), the two-particle irreducible function in 2-loops, at the Fermi surface,

$$\bar{\Gamma}_{0,R}^{(4)} = -i\bar{g}_{0,R} + i\bar{g}_{0,R}^2 \ln \left( \frac{\Omega}{\omega} \right) - i\bar{g}_{B,R}^2 \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right)$$

$$- i\bar{g}_{B,R}^3 \ln \left( \frac{\Omega}{\omega} \right) - i\bar{g}_{U,R}^2 \bar{g}_{0,R} \ln \left( \frac{\Omega}{\omega} \right)$$

$$- i\bar{g}_{U,R}^2 \bar{g}_{F,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) - i\delta \bar{g}_{0,R},$$  \hspace{1cm} (133)

where $-i\delta \bar{g}_{0,R}$ is the corresponding counterterm for the intraforward channel until 2-loops, where $\bar{\Gamma}_{0,R}^{(4)} = \Gamma_{0,R}^{(4)}/\pi V_{F,R}$.

Including the non-parquet contribution to the interferward channel (figure 39) the two-particle irreducible function in 2-loops has the following contribution at the Fermi surface.
Fig. 34. (Color online) 2-loop parquet diagrams in the interforward scattering channel. Where we have $\vec{p} = \vec{p}_1 + \vec{p}_2$.

\[
\hat{\Gamma}^{(4)}_{F,R} = -i\bar{g}_{F,R} - i\bar{g}_{U,R}^2 \ln \left( \frac{\Omega}{\omega} \right) + i\bar{g}_{B,R}^2 \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) \\
- i\bar{g}_{0,R}^2 \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) - i\bar{g}_{F,R}^3 \ln \left( \frac{\Omega}{\omega} \right) \\
- i\bar{g}_{U,R}^2 \bar{g}_{0,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) \\
- i\delta \bar{g}_{F,R}, 
\]

where $-i\delta \bar{g}_{F,R} = -i\delta g_{F,R}/\pi V_{F,R}$ is the corresponding counterterm for the interforward channel until 2-loops, where $\hat{\Gamma}^{(4)}_{F,R} = \Gamma^{(4)}_{F,R}/\pi V_{F,R}$.

Including the non-parquet contribution to the interbackscattering channel (figure 40) the two-particle irreducible function in 2-loops has the following contribution at the Fermi surface.
\[ \Gamma^{(4)}_{B,R} = -i\bar{g}_{B,R} + i\bar{g}_{B,R}\bar{g}_{F,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) \]
\[ + i\bar{g}_{B,R}\bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) - i\bar{g}_{0,R}\bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) \]
\[ - i\bar{g}_{0,R}\bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) - i\bar{g}_{2,0,R}\bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) \]
\[ - 2i\bar{g}_{0,R}\bar{g}_{F,R}\bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) + i\bar{g}_{2,0,R}\bar{g}_{B,R} \ln \left( \frac{\Omega}{\omega} \right) - i\delta \bar{g}_{B,R}, \] (135)

where \(-i\delta \bar{g}_{B,R} = -i\delta g_{B,R}/\pi V_{F,R}\) is the corresponding counterterm for the interbackscattering channel until 2-loops, where \(\Gamma^{(4)}_{B,R} = \Gamma^{(4)}_{B,R}/\pi V_{F,R}\).

Including the non-parquet contribution to the interumklapp channel (figure 41), we have the following contribution at the Fermi surface
\[ \bar{\Gamma}^{(4)}_{\mathcal{U}, R} = -i\bar{g}_{\mathcal{U}, R} + 2i\bar{g}_{0, R}\bar{g}_{\mathcal{U}, R}\bar{g}_{\mathcal{B}, R} \ln \left( \frac{\Omega}{\omega} \right) - 2i\bar{g}_{\mathcal{F}, R}\bar{g}_{\mathcal{U}, R} \ln \left( \frac{\Omega}{\omega} \right) - 2i\bar{g}_{0, R}\bar{g}_{\mathcal{F}, R}\bar{g}_{\mathcal{U}, R} \ln \left( \frac{\Omega}{\omega} \right) - 2i\bar{g}_{\mathcal{U}, R}\bar{g}_{\mathcal{B}, R} \ln \left( \frac{\Omega}{V_{F,R}\Delta k_{F,R}} \right) - i\delta \bar{g}_{\mathcal{U}, R}, \]  

where \(-i\delta \bar{g}_{\mathcal{U}, R} = -i\delta \bar{g}_{\mathcal{U}, R}/\pi V_{F,R}\) is the corresponding counterterm for the interumklapp channel until 2-loops, where \(\bar{\Gamma}^{(4)}_{\mathcal{U}, R} = \bar{\Gamma}^{(4)}_{\mathcal{U}, R}/\pi V_{F,R}\).

Taking into account the prescriptions for the scattering channels at the Fermi surface

\[ \bar{\Gamma}^{(4)}_{0, R} = -i\bar{g}_{0, R}, \]  

\[ \bar{\Gamma}^{(4)}_{\mathcal{F}, R} = -i\bar{g}_{\mathcal{F}, R}, \]  

\[ \bar{\Gamma}^{(4)}_{\mathcal{B}, R} = -i\bar{g}_{\mathcal{B}, R}, \]  

\[ \bar{\Gamma}^{(4)}_{\mathcal{U}, R} = -i\bar{g}_{\mathcal{U}, R}. \]

We arrive at the explicit 2-loops contributions of the counterterms, including the
non-parquet contributions. For the intraforward channel until 2-loops, we have

\[
\delta \bar{g}_{0,R} = \bar{g}_{0,R} \ln \left( \frac{\Omega}{\omega} \right) - \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \\
- \bar{g}_{0,R} \ln \left( \frac{\Omega}{\omega} \right) - \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) \\
- \bar{g}_{0,R} \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) - \bar{g}_{B,R} \bar{g}_{F,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right),
\]

the interforward counterterm until 2-loops

\[
\delta \bar{g}_{F,R} = -\bar{g}_{0,R} \ln \left( \frac{\Omega}{\omega} \right) + \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \\
- \bar{g}_{0,R} \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) - \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) \\
- \bar{g}_{0,R} \bar{g}_{0,R} \ln \left( \frac{\Omega}{\omega} \right) - \bar{g}_{B,R} \bar{g}_{0,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right),
\]
interbackscattering counterterm until 2-loops order

\[ \delta \bar{g}_{B,R} = \bar{g}_{B,R} \bar{g}_{F,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) + \bar{g}_{B,R} \bar{g}_{F,R} \ln \left( \frac{\Omega}{\omega} \right) \]

\[- \bar{g}_{0,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) - \bar{g}_{0,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{\omega} \right) - 2 \bar{g}_{0,R} \bar{g}_{F,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) \]

\[- \bar{g}_{U,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right) + \bar{g}_{U,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{\omega} \right), \quad \text{(143)} \]
interumklapp counterterm in 2-loops order

\[
\delta \bar{g}_{U,R} = 2 \bar{g}_0, R \bar{g}_{U,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{\omega} \right)
- 2 \bar{g}_F, R \bar{g}_{U,R} \ln \left( \frac{\Omega}{\omega} \right)
- 2 \bar{g}_0, R \bar{g}_{U,R} \bar{g}_{B,R} \ln \left( \frac{\Omega}{V_{F,R} \Delta k_{F,R}} \right).
\] (144)

Fig. 40. (Color online) 2-loops non-parquet diagrams in the interbackscattering channel.

Fig. 41. (Color online) 2-loop non-parquet diagrams in the interumklapp scattering channel.
11. RG flow equations for the renormalized scattering couplings terms and confinement

By considering the relation between the bare coupling terms and the renormalized ones

$$ g_{\text{bare}} = Z^{-2} (g_R + \delta g_R), \quad (145) $$

we can calculate the flow of the renormalization group in the frequency scale $\omega$, described by means of the equation

$$ \omega \frac{dg_{\text{bare}}}{d\omega} = \omega \frac{dZ^{-2}}{d\omega} (g_R + \delta g_R) $$
$$ + \ Z^{-2} \left( \omega \frac{dg_R}{d\omega} + \omega \frac{d\delta g_R}{d\omega} \right) = 0, \quad (146) $$

where we take into account the fact that the bare quantities do not flow.

Taking into account the anomalous dimension

$$ \gamma = \frac{\omega}{Z} \frac{dZ}{d\omega}, $$

we can write

$$ \omega \frac{dZ^{-2}}{d\omega} = -2 \omega Z^{-1} \frac{dZ}{d\omega} = -2 \omega Z^{-2} \frac{dZ}{d\omega} $$
$$ = -2 Z^{-2} \gamma, \quad (147) $$

and obtain the following result

$$ -2 \gamma (g_R + \delta g_R) + \left( \omega \frac{dg_R}{d\omega} + \omega \frac{d\delta g_R}{d\omega} \right) = 0. \quad (148) $$

Thus, we arrive at the RG flow equation

$$ \omega \frac{dg_R}{d\omega} = 2 \gamma (g_R + \delta g_R) - \omega \frac{d\delta g_R}{d\omega}. \quad (149) $$

In the specific case of the scattering channels, we can write

$$ \omega \frac{dg_{0R}}{d\omega} = 2 \gamma (g_{0R} + \delta g_{0R}) - \omega \frac{d\delta g_{0R}}{d\omega}, \quad (150) $$
$$ \omega \frac{dg_{F,R}}{d\omega} = 2 \gamma (g_{F,R} + \delta g_{F,R}) - \omega \frac{d\delta g_{F,R}}{d\omega}, \quad (151) $$
$$ \omega \frac{dg_{B,R}}{d\omega} = 2 \gamma (g_{B,R} + \delta g_{B,R}) - \omega \frac{d\delta g_{B,R}}{d\omega}, \quad (152) $$
$$ \omega \frac{dg_{U,R}}{d\omega} = 2 \gamma (g_{U,R} + \delta g_{U,R}) - \omega \frac{d\delta g_{U,R}}{d\omega}. \quad (153) $$
Dividing both sides by $\pi V_{F,R}$, we have finally

$$\omega \frac{d\bar{g}_{0,R}}{d\omega} = 2\gamma (\bar{g}_{0,R} + \delta \bar{g}_{0,R}) - \omega \frac{d\delta \bar{g}_{0,R}}{d\omega},$$  \hspace{1cm} (154)

$$\omega \frac{d\bar{g}_{F,R}}{d\omega} = 2\gamma (\bar{g}_{F,R} + \delta \bar{g}_{F,R}) - \omega \frac{d\delta \bar{g}_{F,R}}{d\omega},$$  \hspace{1cm} (155)

$$\omega \frac{d\bar{g}_{B,R}}{d\omega} = 2\gamma (\bar{g}_{B,R} + \delta \bar{g}_{B,R}) - \omega \frac{d\delta \bar{g}_{B,R}}{d\omega},$$  \hspace{1cm} (156)

$$\omega \frac{d\bar{g}_{U,R}}{d\omega} = 2\gamma (\bar{g}_{U,R} + \delta \bar{g}_{U,R}) - \omega \frac{d\delta \bar{g}_{U,R}}{d\omega}.$$  \hspace{1cm} (157)

Using the 2-loops counterterms, eq. (141), (142), (143) and (144), we can write the RG flow equations for the renormalized scattering coupling terms

$$\omega \frac{d\bar{g}_{0,R}}{d\omega} = 2\gamma \bar{g}_{0,R} + \bar{g}_{0,R}^2 - \bar{g}_{0,R} \bar{g}_{B,R} \frac{\omega}{\Delta k_{F,R}} \frac{d\Delta k_{F,R}}{d\omega} - \bar{g}_{0,R}^3,$$  \hspace{1cm} (158)

$$\omega \frac{d\bar{g}_{F,R}}{d\omega} = 2\gamma \bar{g}_{F,R} + \bar{g}_{F,R}^2 - \bar{g}_{F,R} \bar{g}_{B,R} \frac{\omega}{\Delta k_{F,R}} \frac{d\Delta k_{F,R}}{d\omega} - \bar{g}_{F,R}^3,$$  \hspace{1cm} (159)

$$\omega \frac{d\bar{g}_{B,R}}{d\omega} = 2\gamma \bar{g}_{B,R} + \bar{g}_{B,R} \bar{g}_{F,R} \frac{\omega}{\Delta k_{F,R}} \frac{d\Delta k_{F,R}}{d\omega} + \bar{g}_{B,R} \bar{g}_{F,R},$$  \hspace{1cm} (160)

$$\omega \frac{d\bar{g}_{U,R}}{d\omega} = 2\gamma \bar{g}_{U,R} + \bar{g}_{U,R} \bar{g}_{F,R} \frac{\omega}{\Delta k_{F,R}} \frac{d\Delta k_{F,R}}{d\omega} + \bar{g}_{U,R} \bar{g}_{F,R}.$$  \hspace{1cm} (161)

Where the RG flow equation for $\Delta k_{F,R}$, eq.(118), at 2-loops can be written, for large cutoff $\Omega$, as

$$\omega \frac{d\ln \Delta k_{F,R}}{d\omega} = \gamma.$$  \hspace{1cm} (162)
Now we can calculate the flow to $Z$

$$Z = 1 + \delta Z$$

$$= 1 - \left[ \frac{\bar{g}_{fR}^2}{2} + \frac{\bar{g}_{0R}^2}{2} + \frac{\bar{g}_{uR}^2}{2} \right] \ln \left( \frac{\Omega}{\omega} \right)$$

$$- \frac{\bar{g}_{0R}^2}{2} \ln \left( \frac{\Omega}{v_F \Delta k_F} \right)$$

(163)

and the anomalous dimension

$$\gamma = \frac{\bar{g}_{fR}^2}{2} + \frac{\bar{g}_{0R}^2}{2} + \frac{\bar{g}_{uR}^2}{2} - \frac{\bar{g}_{bR}^2}{2} \frac{\omega}{\Delta k_F} \frac{d\Delta k_F}{d\omega},$$

(164)

that reduces, at 2-loops, to the following result

$$2\gamma = \bar{g}_{fR}^2 + \bar{g}_{0R}^2 + \bar{g}_{uR}^2.$$  

(165)

Taking into account only terms up to 2-loops order in the renormalized coupling terms, we can finally write the RG flow equations for the intraforward

$$\omega \frac{d\bar{g}_{0,R}}{d\omega} = 2\gamma \bar{g}_{0,R} + \bar{g}_{u,R}^2 - \bar{g}_{0,R}^3 - \bar{g}_{F,R} \bar{g}_{0,R} - \bar{g}_{u,R} \bar{g}_{F,R},$$

(166)

interforward

$$\omega \frac{d\bar{g}_{F,R}}{d\omega} = 2\gamma \bar{g}_{F,R} - \bar{g}_{u,R}^2 - \bar{g}_{F,R}^3 - \bar{g}_{0,R}^2 \bar{g}_{F,R} - \bar{g}_{u,R} \bar{g}_{F,R} + \bar{g}_{F,R} \bar{g}_{u,R},$$

(167)

interbackscattering

$$\omega \frac{d\bar{g}_{B,R}}{d\omega} = 2\gamma \bar{g}_{B,R} + \bar{g}_{u,R}^2 \bar{g}_{F,R} - \bar{g}_{0,R}^2 \bar{g}_{B,R} - \bar{g}_{u,R} \bar{g}_{B,R},$$

(168)

interumklapp

$$\omega \frac{d\bar{g}_{U,R}}{d\omega} = 2\gamma \bar{g}_{U,R} + 2\bar{g}_{0,R} \bar{g}_{u,R} - 2\bar{g}_{F,R} \bar{g}_{U,R} - 2\bar{g}_{0,R} \bar{g}_{F,R} \bar{g}_{U,R},$$

(169)

and the difference of the renormalized interband Fermi points

$$\omega \frac{d\Delta k_{F,R}}{d\omega} = \gamma \Delta k_{F,R}.$$  

(170)

By solving these RG flow equations and the corresponding RG flow equation for the quasi-particle weight $Z$, we found that in the RG flows to the confinement, where the renormalized $\Delta k_{F,R}$ vanishes, that the corresponding quasi-particle weight $Z$ flows to zero and the interaction coupling terms flow to Luttinger liquid fixed points, with intraforward, interforward and interumklapp interactions RG flowing to non-vanishing fixed points and the interbackscattering interactions RG flowing to zero
as a consequence of its relation to the confinement term $\Delta k_{F,R}$. This result in fact is applied for many different cases, positive initial renormalized couplings (figure 42), including negative initial renormalized coupling terms (except interumklapp) (figure 43). For initial negative values, except for the interumklapp interaction, the negative initial values still move towards the fixed points. As the intraforward and interforward interactions flow to opposite value fixed points, their sum RG flows to zero and the corresponding Luttinger theorem is satisfied in the fixed points.

As a consequence, in the confinement of renormalized chiral interband Fermi points, the bands do not stop communicating but they still interact under intraforward, interforward and interumklapp in a dinamical equilibrium characterized by the Luttinger liquid fixed point regimes.

Fig. 42. (Color online) RG flow equations for initial positive couplings.

12. Chiral transformations, Ward-Takahashi Identity and Quantum Anomaly

In $1 + 1$ spacetime coordinates, we can also represent the action of the quasi-1D system as $1 + 1$ Dirac field under external field interactions that do not change color $A_s^\alpha$ and interactions that change color $B_s^{\alpha\beta}$ (figure 44)

$$
S = \int d^2 x \sum_\alpha \bar{\psi}_s^\alpha \phi_s^{\alpha} + \sum A_s^\alpha \psi_s^\alpha + \sum_{\alpha \neq \beta} d^2 x \psi_s^{\alpha\dagger} B_s^{\alpha\beta} \psi_s^\beta
$$

(171)
where $\alpha = a, b$; $\beta = a, b$; $s = \pm$.

From the previous sections, we showed that only interumklapp and inter-backscattering interactions change color, such that these interactions represent color exchange interaction $B_{s}^{\alpha\beta}$. Note that $B_{s}^{\alpha\beta} = 0$, if $\alpha = \beta$.

As a whole, we can represent the external fields as a single external field interaction

$$A_{s}^{\alpha\beta} = A_{s}^{\alpha} + B_{s}^{\alpha\beta}$$

(172)
that couples to the $1+1$ free Dirac fields as follows

$$
\mathcal{L} = \sum_{\alpha} \bar{\psi}_{\alpha} \partial_x \psi_{\alpha} + \sum_{\alpha, \beta, s} \bar{\psi}_{s}^{\alpha} A_{s}^{\alpha \beta} \psi_{s}^{\beta},
$$

where $\alpha = a, b, \beta = a, b, s = \pm$.

As a procedure in $1+1$,[31] let us consider a chiral transformation, involving the $+$ side,

$$\psi_{+}^\dagger \rightarrow e^{i\theta(x)} \psi_{+}, \quad \psi_{+} \rightarrow (1 - i\theta(x)) \psi_{+}
$$

and the action will change as

$$S \rightarrow S + \int d^2 x J_{+}^{a} (-i \partial_{+} \theta(x)),
$$

where $J_{+}^{a} = \psi_{+}^{\dagger} \psi_{+}^{a}$ is the corresponding current in the $+$ side, and $\partial_{+} = \partial_{t} - iv_{F} \partial_{x}$.

The functional generator will change as follows[32]

$$Z'[J, \eta, \eta^\dagger] = \int \mathcal{D} (A', \psi', \psi'^\dagger) \times e^{[S + \int d^2 x (\sum_{\alpha, \beta, s} J_{+}^{\alpha \beta} A_{s}^{\alpha \beta} + \sum_{\alpha, \nu} \eta_{\alpha}^{\dagger} \psi_{\alpha}^{\nu} - \sum_{\alpha, \nu} \psi_{\alpha}^{\dagger} \eta_{\alpha}^{\nu})]}
$$

leading to the following Ward-Takahashi identity

$$\theta(x) \left( \sum_{\alpha} i\partial_{+} J_{+}^{\alpha} + \sum_{\alpha} i\eta_{+}^{\alpha} \psi_{+}^{\alpha} - \sum_{\alpha} i\psi_{+}^{\alpha} \eta_{+}^{\alpha} \right) Z = 0.
$$

As usual we will make the Ward-Takahashi associations[32]

\begin{align*}
A_{+}^{\alpha} & \rightarrow \frac{1}{i} \frac{\delta}{\delta J_{+}^{\alpha}} \\
\psi_{+}^{\alpha} & \rightarrow \frac{1}{i} \frac{\delta}{\delta \eta_{+}^{\alpha}} \\
\psi_{+}^{\dagger} & \rightarrow \frac{1}{i} \frac{\delta}{\delta \eta_{+}^{\dagger}}
\end{align*}

Such that

$$\theta(x) \left( \sum_{\alpha} i\partial_{+} J_{+}^{\alpha \beta} + \sum_{\alpha} i\eta_{+}^{\alpha} \frac{1}{i} \frac{\delta}{\delta \eta_{+}^{\alpha}} + \sum_{\alpha} i\eta_{+}^{\alpha} \frac{\delta}{\delta \eta_{+}^{\dagger}} \right) Z = 0
$$

(182)
Making the substitution $Z = e^{iW}$

\[
\theta(x) \left( \sum_\alpha i \partial_+ J^\alpha_+ + \sum_\alpha i \eta^\alpha_+ \frac{\delta W}{\delta \eta^\alpha_+} + \sum_\alpha i \eta^{\alpha\dagger}_+ \frac{\delta W}{\delta \eta_+^{\alpha\dagger}} \right) = 0
\]

(183)

As a consequence we have the Ward-Takahashi relation

\[
\sum_\alpha i \partial_+ J^\alpha_+ = - \left( \sum_\alpha i \eta^\alpha_+ \frac{\delta W}{\delta \eta^\alpha_+} + \sum_\alpha i \eta^{\alpha\dagger}_+ \frac{\delta W}{\delta \eta_+^{\alpha\dagger}} \right).
\]

(184)

Note that the current contribution is due to one-color interactions.

Now we will make an usual Legendre transform of quantum field theory

\[
\Gamma[\psi, \psi^\dagger, A] = W[\eta, \eta^\dagger, J] - \int d^2x [\psi^\dagger \eta + \eta^\dagger \psi + AJ]
\]

(185)

Where we have

\[
\frac{\delta \Gamma}{\delta A^\alpha_+} = - J^{\alpha\beta}_+,
\]

(186)

\[
\frac{\delta \Gamma}{\delta \eta^\alpha_+} = - \eta^{\alpha\dagger}_+,
\]

(187)

\[
\frac{\delta \Gamma}{\delta \psi^\alpha_+} = - \eta_+^\alpha,
\]

(188)

and

\[
\frac{\delta W}{\delta J^{\alpha\beta}_+} = A^\alpha_+ \beta,
\]

(189)

\[
\frac{\delta W}{\delta \eta^\alpha_+} = \psi^\alpha_+,
\]

(190)

\[
\frac{\delta W}{\delta \eta^{\alpha\dagger}_+} = \psi^{\alpha\dagger}_+.
\]

(191)

Then, from (184), we have the following

\[
- \sum_\alpha i \partial_+ \delta x \frac{\delta \Gamma}{\delta A^\alpha_+(x)} = \sum_\alpha i \frac{\delta \Gamma}{\delta \psi^{\alpha\dagger}_+(x)} \psi^\alpha_+(\bar{x})
\]

\[
+ \sum_\alpha i \frac{\delta \Gamma}{\delta \psi^\alpha_+(x)} \psi^{\alpha\dagger}_+(\bar{x})
\]

(192)
Taking the functional derivatives with respect to $\delta \psi_\alpha^\dagger(\vec{z})\delta \psi_\alpha^\prime(\vec{y})$, we arrive at the Ward-Takahashi identity

$$\sum_\alpha \frac{\delta^3 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta A_\alpha^a(\vec{x})} = \frac{\delta^2 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta \psi_\alpha^\prime(\vec{x})} \delta_\alpha \delta(\vec{x} - \vec{y}) - \frac{\delta^2 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta(\vec{x} - \vec{z})}.$$

It follows the two relations

$$\sum_\alpha \frac{\partial^2}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta A_\alpha^a(\vec{x})} = 0,$$

$$\sum_\alpha \frac{\delta^3 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta A_\alpha^a(\vec{x})} = \frac{\delta^2 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta(\vec{x} - \vec{y})} - \frac{\delta^2 \Gamma}{\delta \psi_\alpha^\dagger(\vec{z}) \delta \psi_\alpha^\prime(\vec{y}) \delta(\vec{x} - \vec{z})}.$$

Taking into account the corresponding Fourier transforms

$$\int dxdy e^{i(p'z - py)} \Gamma^{(2)}(z, y) = (2\pi)^2 \delta(p' - p) \Gamma^{(2)}(p),$$

$$\int dxdydz e^{i(p'z - py - qx)} \Gamma^{(2,1)}(z, y, x) = (2\pi)^2 \delta(p' - p - q) \Gamma^{(2,1)}(p, q, p, q)$$

$$\partial_z \Gamma^{(2,1)}(z, y; x) = \delta(x - z) \Gamma^{(2)}(z, y) - \delta(x - y) \Gamma^{(2)}(z, y).$$

Integrating in $dxdydz$ and $e^{i(p'z - py)}e^{-iqz}$,

$$\int dxdydz e^{i(p'z - py)} e^{-iqz} \partial_z \Gamma^{(2,1)} = -iq(2\pi)^2 \delta(p' - q - p) \Gamma^{(2,1)}(p, q, p, q)$$

(193)

$$\int dxdy_1d_1 e^{i(p'x_1 - py_1)} e^{-iqx} \delta(x - y_1) \Gamma^{(2)}(x_1, y_1) =$$

$$\int dy_1d_1 e^{i(p'x_1 - py_1)} e^{-iqy_1} \Gamma^{(2)}(x_1, y_1) =$$

$$i(2\pi) \delta(p' - q - p) \Gamma^{(2)}(p, q + q),$$

(194)

$$\int dxdy_1d_1 e^{i(p'x_1 - py_1)} e^{-iqx} \delta(x - x_1) \Gamma^{(2)}(x_1, y_1) =$$

$$\int dy_1d_1 e^{i(p' - q)x_1} e^{-ipy_1} \Gamma^{(2)}(x_1, y_1) =$$

$$i(2\pi) \delta(p' - q - p) \Gamma^{(2)}(p).$$

(195)
we arrive at the Ward-Takahashi identity

\[ q_+ \sum_{\alpha} \Gamma_{\alpha\alpha',++}^{(2,1)}(p, p + q; q) = \Gamma_{\alpha',+}^{(2)}(p + q) - \Gamma_{\alpha',+}^{(2)}(p), \]

(196)

where \( q_s = q_0 - sv_F q, \ s = \pm \). Note that this is in fact a generalization of the Ward-Takahashi identity for the Luttinger liquid \( ^{28} \).

Fig. 45. (Color online) Vertex functions in 1-loop.

Fig. 46. (Color online) 2-loops vertex function diagrams with interforward and intraforward interactions.
This WTI contains anomalous terms, due to the one color polarization bubbles, the 1 + 1 corresponding triangle anomalies terms,

\begin{align}
\Gamma_{ab,++}^{(2,1),(1\text{-loop})}(q) &= -ig_F \chi^b(q) \\
&= -g_F \frac{q}{2\pi} G^b_-(q_0, q - k_F^b), \\
\Gamma_{aa,++}^{(2,1),(1\text{-loop})}(q) &= -ig_0 R \chi^a(q) \\
&= -g_0 \frac{q}{2\pi} G^a_-(q_0, q - k_F^a), \\
\Gamma_{ba,++}^{(2,1),(1\text{-loop})}(q) &= -ig_F \chi^a(q) \\
&= -g_F \frac{q}{2\pi} G^a_-(q_0, q - k_F^a), \\
\Gamma_{bb,++}^{(2,1),(1\text{-loop})}(q) &= -ig_0 \chi^b(q) \\
&= -g_0 \frac{q}{2\pi} G^b_-(q_0, q - k_F^b),
\end{align}

that are the 1-loop expansion of the (2, 1)-vertex function \(\Gamma_{\alpha\alpha',++}^{(2,1)}(p, p+q; q)\) (figures 45), corresponding to the coupling of the one color interaction \(A_+^\alpha\).

On the other hand, the 2-loops order (2,1)-vertex function contributions not involving the 1-loop one-color polarization bubbles (figures 46 and 47) are identically cancelled, as predicted in the Adler-Bardeen theorem \(29\), while the 2-loops anomalous terms are parquet diagrams involving the 1-loop one-color polarization bubbles.
At higher order anomalous terms are given by the generic terms with the coupled 1-loop one-color polarization bubbles (figure 49).

The anomalous terms that leads to the quantum anomaly (figure 49), in the right-hand side, associated to the one-color interaction $A_+^\alpha$, can be written as

$$F_{\alpha\alpha',++}(p, p + q; q) = (g_0 \chi_a^\alpha + gf \chi_b^\alpha) \Gamma_{\alpha\alpha',++}^{(2,1)}(p, p + q; q)$$

and for the left-hand side (figure 50), associated to the one-color interaction $A_+^\alpha$,

$$F_{\alpha\alpha',--}(p, p + q; q) = (g_0 \chi_a^\alpha + gf \chi_b^\alpha) \Gamma_{\alpha\alpha',--}^{(2,1)}(p, p + q; q)$$

As a consequence, the anomalous Ward-Takahashi identity (17) for the right-hand side is given by

$$q_+ \sum_\alpha \left( \Gamma_{\alpha\alpha',++}^{(2,1)}(p, p + q; q) - F_{\alpha\alpha',++}(p, p + q; q) \right) = \Gamma_{\alpha',+}^{(2)}(p + q) - \Gamma_{\alpha',+}^{(2)}(p)$$

and for the left-hand side,

$$q_- \sum_\alpha \left( \Gamma_{\alpha\alpha',--}^{(2,1)}(p, p + q; q) - F_{\alpha\alpha',--}(p, p + q; q) \right) = \Gamma_{\alpha',--}^{(2)}(p + q) - \Gamma_{\alpha',--}^{(2)}(p).$$

As in the Schwinger model 1 + 1 (33), the quantum anomaly 1 + 1 here is a consequence of the coupling interaction in the free Dirac fields involving axial vectors. In the case, the polarization bubble of one-color is associated to quantum anomaly is not coincidence. As we will show, the interactions that do not change color are associated to axial vectors, while the interactions that change color (interbackscattering and interumklapp) are not. Indeed, we can write the intra and interband interactions explicitly in terms of current components

$$g_0 \psi_a^\dagger \psi^b \psi_c^\dagger \psi^a = g_0 J_a^a J_b^a$$

$$g_f \psi_a^\dagger \psi^b \psi_c^\dagger \psi^a = g_f J_a^a J_b^a$$

$$g_b \psi_a^\dagger \psi^b \psi_c^\dagger \psi^a = g_b J_a^a J_b^a$$

$$g_d \psi_a^\dagger \psi^b \psi_c^\dagger \psi^a = g_d J_a^a J_b^a$$

In 1 + 1 spacetime coordinates, the axial and vector currents

$$J_A^\mu = \bar{\psi} \gamma_5 \gamma^\mu \psi,$$

$$J_V^\mu = \bar{\psi} \gamma^\mu \psi.$$
can be written, respectively, as

$$J_A = \left( \begin{array}{c} \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \\ \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \end{array} \right) = \left( \begin{array}{c} J^0_A \\ J^1_A \end{array} \right),$$  

(211) 

$$J_V = \left( \begin{array}{c} \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \\ \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \end{array} \right) = \left( \begin{array}{c} J^0_V \\ J^1_V \end{array} \right).$$  

(212) 

The following $1+1$ relation is satisfied $J_A = \gamma^0 J_V$ and we can also write

$$J^0_A + J^1_A = 2J_+,$$  

(213) 

$$J^0_A - J^1_A = -2J_-.$$  

(214) 

As a consequence the intraforward and interforward interactions can be written in
terms of axial current components

\[ g_0 J_+^a J_-^a = -\frac{g_0}{4} \left( J_0^a + J_1^a \right) \left( J_0^a - J_1^a \right), \quad \text{(215)} \]

\[ g_F J_+^a J_-^b = -\frac{g_F}{4} \left( J_0^a + J_1^a \right) \left( J_0^b - J_1^b \right). \quad \text{(216)} \]

On the other hand, the interbackscattering and the interumklapp interactions are well defined in terms of one-side spinors defined as

\[ \psi_{C+} = \begin{pmatrix} \psi_+ \\ \psi_+^\dagger \end{pmatrix}, \quad \psi_{C-} = \begin{pmatrix} \psi_- \\ \psi_-^\dagger \end{pmatrix} \quad \text{(217)} \]

\[ \psi_{C+}^\dagger = \begin{pmatrix} \psi_+^\dagger \\ \psi_+ \end{pmatrix}, \quad \psi_{C-}^\dagger = \begin{pmatrix} \psi_-^\dagger \\ \psi_- \end{pmatrix} \quad \text{(218)} \]

with the corresponding one-side currents

\[ J_{C+}^\mu = \bar{\psi}_{C+} \gamma^\mu \gamma^5 \psi_{C+}, \quad \text{(219)} \]

\[ J_{C-}^\mu = \bar{\psi}_{C-} \gamma^\mu \gamma^5 \psi_{C-}. \quad \text{(220)} \]
We can write
\[
J_{C+} = \begin{pmatrix}
\psi_+^\dagger \psi_+^0 + \psi_-^\dagger \psi_+^0 \\
\psi_+^\dagger \psi_+^0 - \psi_-^\dagger \psi_+^0
\end{pmatrix} = \begin{pmatrix}
J_{C+}^0 \\
J_{C+}^J
\end{pmatrix},
\]
(221)
\[
J_{C-} = \begin{pmatrix}
\psi_-^\dagger \psi_-^0 + \psi_+^\dagger \psi_-^0 \\
\psi_-^\dagger \psi_-^0 - \psi_+^\dagger \psi_-^0
\end{pmatrix} = \begin{pmatrix}
J_{C-}^0 \\
J_{C-}^J
\end{pmatrix},
\]
(222)
where \( \psi' = \psi(x') \) and \( \psi = \psi(x) \). Note that, if \( x' = x \), we have that one component of the one-side current is zero.
\[
J_{C+}^{ab} = \begin{pmatrix}
\psi_+^\dagger \psi_+^b + \psi_-^\dagger \psi_+^b \\
\psi_+^\dagger \psi_+^b - \psi_-^\dagger \psi_+^b
\end{pmatrix} = \begin{pmatrix}
J_{C+}^{0ab} \\
J_{C+}^{1ab}
\end{pmatrix},
\]
(223)
\[
J_{C-}^{ab} = \begin{pmatrix}
\psi_-^\dagger \psi_-^a + \psi_+^\dagger \psi_-^a \\
\psi_-^\dagger \psi_-^a - \psi_+^\dagger \psi_-^a
\end{pmatrix} = \begin{pmatrix}
J_{C-}^{0ab} \\
J_{C-}^{1ab}
\end{pmatrix},
\]
(224)
Consequently
\[
J_{C}^{0ab} = J_{-}^{ab} + J_{-}^{ba},
\]
(225)
\[
J_{C}^{1ab} = J_{-}^{ab} - J_{-}^{ba},
\]
(226)
\[
J_{C}^{ab} = J_{+}^{ab} + J_{+}^{ba},
\]
(227)
\[
J_{C}^{1ab} = J_{+}^{ab} - J_{+}^{ba},
\]
(228)
and we can write the interbackscattering and interumklapp in the one-side currents
\[
g_{BS} J_{C}^{ab} J_{-}^{ab} = \frac{g_B}{4} \left( J_{C}^{0ab} - J_{C}^{1ab} \right) \left( J_{C}^{0ab} + J_{C}^{1ab} \right),
\]
(229)
\[
g_{BS} J_{C}^{ab} J_{-}^{ab} = \frac{g_B}{4} \left( J_{C}^{0ab} - J_{C}^{1ab} \right) \left( J_{C}^{0ab} + J_{C}^{1ab} \right).
\]
(230)
Alternatively, we can write
\[
J_{C}^{0ba} = J_{-}^{ab} + J_{-}^{ba},
\]
(231)
\[
J_{C}^{1ba} = J_{-}^{ba} - J_{-}^{ab},
\]
(232)
\[
J_{C}^{0ba} = J_{+}^{ab} + J_{+}^{ba},
\]
(233)
\[
J_{C}^{1ba} = J_{+}^{ba} - J_{+}^{ab},
\]
(234)
and then
\[
g_{BS} J_{C}^{ab} J_{-}^{ab} = \frac{g_B}{4} \left( J_{C}^{0ba} + J_{C}^{1ba} \right) \left( J_{C}^{0ba} - J_{C}^{1ba} \right),
\]
(235)
\[
g_{BS} J_{C}^{ab} J_{-}^{ab} = \frac{g_B}{4} \left( J_{C}^{0ba} + J_{C}^{1ba} \right) \left( J_{C}^{0ba} + J_{C}^{1ba} \right).
\]
(236)
Now we can also consider the term involving \( v_F k_F^a \),
\[
\sum_{\alpha, s} v_F k_F^a \psi_s^\alpha \psi_s^\alpha = \sum_{\alpha} v_F k_F^a \left( J_{+}^{\alpha} + J_{-}^{\alpha} \right).
\]
(237)
In terms of the axial vector this will be written as

\[ \sum_{\alpha,s} v_F k_F^\alpha s^\alpha \psi_{s^\alpha}^\dagger \psi_s = \sum_{\alpha} v_F k_F^\alpha J_A^\alpha, \]  

(238)

where we have used \( J_A^1 = J_+ + J_- \).

The consequence of quantum anomaly in \( 1 + 1 \) is more appropriately related to the component \( J_A^0 \) of the axial vector, that leads to the fermion number anomaly \(^{34}\) appearing in the intraforward and interforward interactions, i.e., while \( J_A^0 = \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \) is related to the chiral symmetry, the component \( J_A^1 = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \) is related to the charge conservation \(^{35}\).

From the point of view of the regularization, the renormalized theory is given in terms of the anomalous Ward-Takahashi identity, if we want require the invariance under chiral symmetry. Alternatively, a different axial vector could be formulated that would satisfy the normal Ward-Takahashi identity, but violating gauge invariance \(^{36}\).

13. Conclusion

We have considered a quasi-1D system of two-coupled spinless fermions chains under intraforward, interforward, interbackscattering and interumklapp interactions. We have showed that this system can be equivalently described in terms the one-color and two-color interactions, i.e., interactions that change color and interactions that do not change color.

By means of field renormalization group, we derived appropriately the bare and renormalized quantities that lead to the set of RG flow equations for this system. We have showed that the renormalized interband Fermi points, \( \Delta k_{F,R} \), flow to zero, leading consequently to a confinement of the bonding and antibonding bands. The interbackscattering interaction flows to zero, as a consequence its relation to \( \Delta k_{F,R} \), a result also verified in previous works \(^8\). The intraforward, interforward and interumklapp flows to Luttinger liquid fixed points, leading to interband communications in the confinement by constant interforward and interumklapp interactions.

Considering the quantum anomaly, inherent to the system, as a consequence of coupling to \( 1 + 1 \) Dirac fields to one-color and two-color interactions, we have arrived at the Ward-Takahashi identities that contains the anomalous terms and the corresponding, gauge invariant, anomalous Ward-Takahashi identities, derived first by \(^3\) using other methods.

We have showed that the quantum anomaly is associated to one-color interactions, as a consequence of the fact such interactions are related to axial vectors, while two-color interactions are associated to one-side vectors. More appropriately in \( 1 + 1 \), the zero-component of axial vector is related to the chiral symmetry and the one-component to the charge conservation.
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