Generalized relation between pulsed and continuous measurements in the quantum Zeno effect

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Abstract
A relation is found between pulsed measurements of the excited-state probability of a two-level atom illuminated by a driving laser and a continuous measurement by a second laser coupling the excited state to a third state which decays rapidly and irreversibly. We find the time between pulses to achieve the same average detection time as a given continuous measurement in strong, weak or intermediate coupling regimes, generalizing the results of Schulman (1998 Phys. Rev. A 57 1509).

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The quantum Zeno effect has not ceased to attract attention since its discovery [1, 2] due to its importance as a fundamental phenomenon and more recently because of its different applications, in particular, in quantum information [3–5] or thermodynamic control [6, 7]. Basic properties of the effect are being scrutinized both theoretically and experimentally, see for recent work [8–12] and references therein. One of the relevant and recurrent issues is the role played by the repeated measurements (questioning their necessity [13]) and the possibility of achieving similar results with continuous measurements. The first papers on the quantum Zeno effect formulated it as the suppression of transitions between quantum states because of continuous observation, mimicked by repeated instantaneous measurements [1, 2]. This suppression was taken as an argument against such a discretized description of the continuous measurement by Misra and Sudarshan [2] but, in later works, many authors have emphasized instead the connections and similarities between pulsed and continuous models, see [14–19] and references therein, in which the transition is affected by the measurement in various degrees depending on the parameters characterizing the system and the coupling with the measurement device included in the Hamiltonian. Schulman found an elegant relation between discrete and continuous models. In the simplest case, the continuous model represents a laser-driven two-level atom with internal states |1⟩, |2⟩ subjected to an irreversible measurement of state |2⟩ [17] with an intense (strong) coupling or quenching laser, see figure 1 or, equivalently, a fast natural decay of level 2 which, possibly counterintuitively, hinders the 1–2 transition in spite of the 1–2 driving laser. (In this work the word ‘coupling’ is exclusively associated with the measurement, 2–3 transition, whereas the ‘driving’ refers to the 1–2 transition.) In the corresponding pulsed measurement to be compared with the continuous one, the 1–2 driving remains but, instead of the continuous measurement via effective or natural decay, an instantaneous measurement of level 2 is made every δt. Schulman noticed that if δt is chosen as four times the inverse of the effective decay rate, similar results are obtained in the continuous and pulsed cases, as it has later been observed experimentally [9]. We shall extend this result by providing a generalized connection between the continuous and pulsed measurements which is also valid in the weak coupling regime, in approximate (explicit) or exact (implicit) forms.

2. The model
2.1. Continuous measurement
For the continuous model we assume a driving laser of frequency ωL,12 for the 1–2 transition (with transition
Figure 1. (a) Energy levels 1, 2, 3; detunings of the two lasers with respect to the 1–2 and 2–3 transitions; on-resonance Rabi frequencies and decay rate of level 3; (b) the effective energy scheme after the adiabatic elimination of level 3. Note the level shift with respect to the scheme in (a).

frequency \(\omega_{12}\) and Rabi frequency \(\omega_0\) and a coupling laser of frequency \(\omega_{23}\) for the 2–3 transition (with transition frequency \(\omega_{23}\) and Rabi frequency \(\Omega\)). The atom in level 3 will emit a photon decaying irreversibly outside the original 1–2 subspace. In a recent experiment, this irreversible decay has been realized by the recoil imparted at the photon emission, which gives the atom enough kinetic energy to escape from the confining trap [9]. We shall adopt a number of standard approximations for the initial Hamiltonian: semiclassical description for the two lasers, dipole approximation and rotating-wave approximation, 

\[
H = \hbar \omega_{12} |2\rangle\langle 2| + \hbar (\omega_{12} + \omega_{23} - i\Gamma/2) |3\rangle\langle 3| + \hbar \omega |2\rangle\langle 2| e^{-i\omega_{23} t} + \text{H.c.},
\]

where H.c. means Hermitian conjugate and \(\Gamma\) is the decay rate of level 3. In a laser-adapted interaction picture, using \(H_0 = |2\rangle\langle 2| \hbar \omega_{12} + |3\rangle\langle 3| \hbar (\omega_{12} + \omega_{23})\), we may eventually write a time-independent effective Hamiltonian,

\[
H_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega & 0 \\ \omega & -2\delta_0 & i\Gamma \\ 0 & i\Gamma & -i\Gamma - 2(\Delta + \delta_0) \end{pmatrix}.
\]

(2)

Here \(\delta_0\) is the detuning (laser frequency minus transition frequency) for the 1–2 transition and \(\Delta\) is the detuning for the 2–3 transition. For a large enough \(\Gamma \gg \Omega\), level 3 is scarcely populated (low saturation) and it can be adiabatically eliminated to obtain an effective Hamiltonian for the subspace of levels 1–2,

\[
H_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega & 0 \\ \omega & -\delta & 0 \\ 0 & 0 & -i\Gamma - 2(\Delta + \delta_0) \end{pmatrix},
\]

where the effective decay rate \(\gamma\) is related to the measurement coupling parameters [9],

\[
\gamma = \frac{\Gamma\Omega^2}{\Gamma^2 + 4\Delta^2},
\]

and

\[
\delta = \delta_0 - \Delta\Omega^2 / (4\Delta^2 + \Gamma^2),
\]

(4)

and

\[
\delta' = \delta_0 - \Delta\Omega^2 / (4\Delta^2 + \Gamma^2).
\]

(5)

Figure 2. Dependence of the average lifetime \(\tau_c\) of a two-level atom continuously driven by a laser (of Rabi frequency \(\omega_0\)) on the decay rate \(\gamma\) for different values of the detuning \(\delta\). A given value of \(\tau_c\) can be generally achieved by applying weak, \((\gamma/\omega_0)\), or strong coupling, \((\gamma/\omega_0)^{**}\).

with \(\Delta = \Delta + \delta_0\). To obtain equations (4) and (5) using complex potentials see [20]. The general solution of the time-dependent Schrödinger equation corresponding to the Hamiltonian (3) is easy to find by Laplace transform. For the boundary condition \(\psi_1(t = 0) = 1\), corresponding to the atom being initially in the ground state, the ground- and excited-state amplitudes take the form

\[
\psi_1(t) = e^{-\gamma t/4} \left[ \cosh(R/2) + \frac{\gamma_0}{2R} \sinh(R/2) \right],
\]

and

\[
\psi_2(t) = -i \omega e^{-\gamma t/4} \frac{\sinh(R/2)}{R},
\]

where

\[
R = (\gamma_0^2 - 4\omega^2)^{1/2} / 2,
\]

\[
\gamma_0 = -2i\delta + \gamma,
\]

and the corresponding probabilities \(p_j\) are given as \(\psi_j^*(t) |\psi_j(t)\)^2, \(j = 1, 2\).

The detection probability per unit time in the continuous measurement is \(W(t) = \gamma |\psi_2^2(t)\|^2\), and the average lifetime before detection, \(\tau_c\), is given by \(\int_0^\infty dt W(t)\). Using equation (7),

\[
\tau_c = \frac{2}{\gamma} + \frac{\gamma_0}{\omega^2} + \frac{4\gamma_0^2}{\gamma^2}.
\]

As shown in figure 2, the upshot is that the same average lifetime \(\tau_c\) can be achieved for two different ratios of \(\gamma/\omega_0\), this is both in the strong and weak driving regimes. While in the former the system undergoes damped Rabi oscillations (figure 3(a)), and in the later the total probability approximately coincides with that in the ground state \(p_{\text{tot}}(t) \approx p_1\) (figure 3(b)). Nonetheless, in both regimes the total probability essentially obeys the same decay law (figure 3(c)).

3 For large \(\gamma/\omega_0\), level 2 can also be adiabatically eliminated, and from the remaining one-channel Hamiltonian one gets \(\tau_c = \frac{\gamma}{\omega_0} + \frac{4\gamma^2}{\gamma_0} = \frac{\gamma}{\omega_0} + \frac{4\gamma^2}{\gamma_0}\), consistent with equation (10) in that limit.
2.2. Pulsed measurement

Alternatively, a pulsed measurement of the population at level 2 may be performed on the same two-level atom described above, which involves measuring the Hamiltonian

\[ H_I(\Omega = 0) = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega \\ \omega & -2\delta t \end{pmatrix}, \]

equal to (3) when \( \Omega = 0 \). Note the level shift with respect to the Hamiltonian (3) because of the absence of the 2–3 coupling.

At intervals \( \delta t \), the population in level 2 is measured 'instantly' (in practice, in a time scale \( \tau_m \) small compared to \( \delta t \), but we shall ignore here the corrections of order \( \tau_m/\hbar \), see, e.g., [9]) and removed from the atomic ensemble. This process is represented by successive projections onto the state \( 1 \) at times \( \delta t, 2\delta t, \ldots, k\delta t, \ldots \). The probability of detecting the atom in the \( k \)th measurement instant is the probability of finding the atom in level 2 at that instant, \( P_2(k - 1 - P_2)^k \), where we have introduced \( P_2 \) to be distinguished from the more generic \( p_2 \), as the probability of finding the excited state at \( \delta t \) without coupling,

\[ P_2 \equiv |\psi(t)|^2(\Omega = 0, t = \delta t)^2, \]

when the (normalized) state is in the ground state at \( t = 0 \). The average detection time is thus calculated, exactly, as

\[ \langle \tau \rangle = \sum_{k=0}^{\infty} k\delta t P_2(1 - P_2)^{k-1} = \frac{\delta t P_2}{1 - P_2} \sum_{k=0}^{\infty} k(1 - P_2)^k = \frac{\delta t}{P_2}, \]

where the last sum can be performed by taking the derivative of the geometric series.

2.3. Connection between continuous and pulsed measurements

The objective here is to find the time between pulses, \( \delta t \), so that the average detection time in the pulsed measurement, \( \langle \tau \rangle \), equals the lifetime \( \tau_c \) of a given continuous measurement. Let us first pay attention to the short-time asymptotics for \( H_I(\Omega = 0) \). From equation (7), with \( \Omega = 0 \), the dominant term for the probability of level 2 at \( \delta t \), \( P_2 \), is independent of \( \delta \) (a remarkable result that only holds at short times), and takes the simple form

\[ P_2 \sim \delta t^2 \alpha^2/4 = (\delta t/\tau_c)^2. \]

The short-time dependence is characterized by a ‘Zeno time’ \( \tau_Z \equiv 2/\omega \) which defines the time scale for the quadratic time dependence [17]. Equating \( \tau_c \) and \( \langle \tau \rangle \), see equations (10) and (13), with the quadratic approximation (14) for \( P_2 \) we get

\[ \delta t = \frac{4\gamma}{2\omega^2 + \gamma^2 + 4\delta^2}. \]

Figure 4 shows the remarkable agreement between the lifetimes for the continuous and pulsed measurements in different coupling regimes according to equations (13) and (15). The same result may be obtained in the style of Schulman [17] as follows: after \( k \) successive pulses separated by \( \delta t \ll \tau_Z \), at \( t = k\delta t \), the probability of the ground state (which is the total remaining probability after removing \( P_2 \) in each pulse) will assume the form

\[ p(t) \approx \left(1 - \frac{\delta t}{\tau_c}\right)^k \approx \exp(-t/\tau_{EP}), \]

i.e., an effective exponential decay law \( \exp(-t/\tau_{EP}) \), where \( \tau_{EP} = \tau_Z^2/\delta t \) is the effective lifetime. By equating \( \tau_{EP} \) and \( \tau_c \) we again get equation (15). This requires \( \delta t \ll 2/\omega \) or \( \tau_Z \ll \tau_c \). These inequalities and therefore the exponential approximation are well fulfilled for weak \( \omega \) driving, \( \omega/\gamma \ll 1 \) (corresponding to strong 2–3 coupling), or strong \( \omega \) driving conditions \( \omega/\gamma \gg 1 \) (corresponding to weak 2–3 coupling), but not so well for intermediate driving conditions, see the region near the minimum at \( \gamma/\omega = [2 + 4(\delta/\omega)^2]^{1/2} \) in figure 1. For a more accurate treatment we may find the really optimal \( \delta t \) by imposing the equality between the exact expressions for the lifetimes, \( \tau_c = \langle \tau \rangle \) and solving the implicit equation (13), e.g. by iteration,

\[ \delta t = \tau_c P_2(\delta t). \]
values of $\gamma$ Rabi frequency, these are generally realized with two different means of different conditions. For a fixed detuning may thus be reproduced with continuous measurement by case. A given lifetime in a Zeno-type pulsed measurement measurement coupling and in the opposite weak coupling case. A given lifetime in a Zeno-type pulsed measurement using (15), respectively. The symbols represent the first and third iterations of equation (18), starting with the seed of equation (15).

From equation (10) we see that the average detection time in the scale of the figure.

3. Discussion

From equation (10) we see that the average detection time in the continuous measurement may be large both for strong measurement coupling and in the opposite weak coupling case. A given lifetime in a Zeno-type pulsed measurement may thus be reproduced with continuous measurement by means of different conditions. For a fixed detuning $\delta$ and Rabi frequency, these are generally realized with two different values of $\gamma$ for which the sum $p_1 + p_2$ (probability for the atom to remain undetected) is quite similar. This may have practical consequences as a control mechanism, since a strong continuous coupling could be difficult to produce or cause some undesirable effects. For example, if we want to maintain the effective two-level scenario, $\Omega$ cannot be made arbitrarily large in the scale of $\Gamma$ to avoid a significant population of level 3 which would make the adiabatic elimination invalid. One more problem could be the excitation of other levels, breaking down, again, the simple two (or even three) level picture.

We have in summary generalized Schulman’s relation between pulsed and continuous measurements in a two-level system in different ways: Schulman’s result corresponds to the case $\delta = 0$, with $\gamma/\omega \gg 1$, for which $\delta t = 4/\gamma$. Our relation (15) is not restricted to these conditions and is generally applicable. It is, however, an approximate one, since it relies on an approximate expression for the average measurement time in the pulsed case. We have also provided the exact result by solving the implicit equation (17). The present results should not be difficult to verify experimentally with very similar settings as those already used in [9]. Figure 6 shows the average continuous time $\tau_c$ versus the saturation parameter $s_0 = 2\Omega^2/\Gamma^2$. $\omega = 2\pi \times 48.5$ Hz, $\Gamma = 2\pi \times 1.74$ MHz. Solid line: $\Delta = \delta_0 = \delta = 0$; dashed line: $\Delta = 2\pi \times 3.18$ MHz, and $\delta_0$ is adjusted for each $s_0$ so that $\delta = 0$.

$$\delta t = \frac{P_2(\delta t_0) - \delta t_0 P_2'(\delta t_0)}{\tau_c^{-1} - P_2'(\delta t_0)},$$

(18)

where the $\delta t$ calculated on the left-hand side becomes the new $\delta t_0$ for the following iteration and the prime means derivative with respect to $\delta t$. A good seed for the first $\delta t_0$ may be equation (15), and $P_2'$ is very well approximated by $\delta t_0 \omega^2/2$. Applying equation (18) just once in this way provides a very good, approximate, but explicit expression of $\delta t$, and the corresponding $\langle t \rangle$, equation (13), is quite close to the continuous measurement result $\tau_c$. Figure 5 is a closeup of one of the cases in figure 1 and depicts a few iterations. Three of them are enough to produce a $\langle t \rangle$ indistinguishable from $\tau_c$ in the scale of the figure.
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